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To my father, and my family.

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ABSTRACT

We present a collection of six stories elucidating the applications of field theories across diverse topics in condensed matter theory. These stories are interconnected by an underlying hidden field theoretical framework, characterized by symmetries, diffeomorphisms, and noncommutativity, which are also interlinked and form the foundational structure of this thesis. The first story describes the nonlinear bosonization approach which offers both perturbative and non-perturbative techniques for studying Fermi and non-Fermi liquids. This approach naturally leads to a nonlinear multidimensional bosonized description, with nonlinear corrections fixed by the geometry of the Fermi surface. The second story highlights the critical role of volume-preserving diffeomorphisms in nonabelian higher-rank gauge theories. The third story explores the noncommutative field theoretical framework of the fractional quantum Hall effect, emphasizing the interplay between volume-preserving diffeomorphisms and noncommutative field theory. Moving forward, the fourth story extends this exploration to a noncommutative field theory describing the collective oscillation mode of a vortex lattice in a two-dimensional rotating Bose-Einstein condensate. Building on the fourth story, the fifth story examines broader applications, exploring potential connections with other condensed matter systems and generalizing concepts to higher dimensions. Finally, the sixth story investigates dual formulations of the previously discussed theories and examines topological crystalline defects and quantum melting transitions. Together, these stories provide a comprehensive view of how field theoretical structure underpins and interconnects diverse phenomena in condensed matter physics.

CHAPTER 1

INTRODUCTION

Condensed matter physics has significantly advanced through the application of quantum field theory (QFT) to strongly interacting many-body systems. QFT serves as a comprehensive theoretical framework that extends the principles of quantum mechanics and classical field theory to describe the collective behaviors within these systems Anderson [1972]. It facilitates understanding of a wide range of phenomena, including superconductivity, magnetism, and quantum phase transitions. By employing methods such as path integrals, Feynman diagrams, and renormalization group analysis, QFT provides a robust toolkit for exploring the intricate dynamics of topological phases and unconventional superconductors. Its ability to incorporate both relativistic and non-relativistic effects enables it to bridge the gap between microscopic particle interactions and macroscopic emergent phenomena, thereby offering profound insights into the principles governing the behavior of quantum matter.

This PhD thesis elucidates the application of quantum field theories to various condensed matter systems, interconnected by an underlying field-theoretical framework, particularly through symmetries, diffeomorphisms, and noncommutativity. The foundational principles of these theories serve as crucial sources of inspiration and building blocks throughout this thesis. The following chapter will briefly discuss the three distinct categories of phases of matter—gapped, gapless, and very gapless—and their field-theoretical connections in condensed matter theory. Additionally, new findings introduced in this thesis will be explored in greater depth in the subsequent chapters.

In condensed matter physics, a phase of matter refers to a distinct state characterized by the collective behavior of particles in many-body systems, exhibiting unique physical properties. An organizing principle involves quantifying the emergent low-energy degrees of freedom that dictate system behavior, with the concept of an energy gap classifying many-body systems into three categories: gapped, gapless, and very gapless. Gapped phases lack propagating low-energy degrees of freedom and exhibit topological characteristics described by topological quantum field theories. Recently, a new class of these phases, known as fractons Du et al. [2022, 2024a,b], has been discovered, which are not described by conventional topological field theories Nandkishore and Hermele [2019], Pretko et al. [2020]. Conversely, gapless phases possess propagating low-energy degrees of freedom, often manifesting in critical phenomena and metallic behavior. Very gapless phases feature infinitely many low-energy degrees of freedom, characterized by a finite density of states at zero energy, as seen in Fermi surfaces. These phases accommodate low-energy excitations distributed throughout the Fermi surface, but classifying them within Fermi surface physics remains challenging due to potential Fermi surface instabilities. These phases include both Fermi liquids and non-Fermi liquids, the latter having a well-defined Fermi surface but lacking well-defined fermionic quasiparticles around it.

The interconnected quantum field-theoretical framework underlying these distinct phases of quantum matter is particularly characterized by symmetries, diffeomorphisms, and noncommutativity. Noncommutative quantum field theory Douglas and Nekrasov [2001], Rubakov [2005], Du et al. [2021], or quantum field theory on noncommutative space, exemplifies the application of noncommutative structures within the framework of quantum field theory. Notably, the semiclassical limit of the noncommutative gauge group is isomorphic to the group of volume-preserving diffeomorphisms. These transformations preserve the volume element in phase space, ensuring that the total measure of the phase space remains invariant Arnold and Khesin [2013]. These foundational structures inspire and inform the works presented in this thesis.

The chapters of this thesis are organized as follows: Chapter 2, titled "The First Story," explores the nonlinear bosonization approach, offering both perturbative and non-perturbative methodologies for Fermi and non-Fermi liquids. This approach leads to a multidimensional bosonized description with nonlinear corrections determined by the geometry of the Fermi surface. Chapter 3, encompassing "The Second and Third Stories," delves into the significance of volume-preserving diffeomorphisms and their interaction with noncommutative field theory. The second story discusses the role of volume-preserving diffeomorphisms in non-abelian higher-rank gauge theories, while the third story presents a noncommutative field theoretical depiction of the fractional quantum Hall effect. Chapter 4, titled "The Fourth to Sixth Stories," comprehensively explores (noncommutative-)exotic field theories, with a focus on nonlinear Lifshitz theories. The fourth story introduces a noncommutative Lifshitz theory elucidating the oscillation mode of a vortex lattice in a two-dimensional rotating Bose-Einstein condensate, featuring a noncommutative Nambu-Goldstone boson. The fifth story extends this scope to other condensed matter systems and higher dimensions, and the sixth story examines dual formulations of the preceding field theory and topological crystalline defects. The thesis concludes with a forward-looking perspective on potential future research directions.

CHAPTER 2

STORY 1: NONLINEAR BOSONIZATION OF FERMI SURFACES

2.1 Introduction: Fermi and Non-Fermi liquids, Bosonization, and Noncommutativity

2.1.1 (Non-)Fermi liquids

Understanding gapless phases of matter in condensed matter physics is crucial. Among these phases, those characterized by a Fermi surface present particularly intricate challenges for theoretical frameworks. These phases include Fermi liquids (FLs) and, conceivably, non-Fermi liquids (NFLs) Lee [2018]. NFLs represent a distinct phase of matter describing fermionic systems, such as electrons and quarks, which are subject to strong, long-range interactions at finite density. NFLs manifest as gapless phases with a well-defined Fermi surface and distinctive metallic properties, yet they lack conventional quasiparticles. This behavior arises by coupling the theory to a gapless order parameter or gauge field. Such phenomena are anticipated to emerge across a spectrum of physical systems, including high-temperature superconductors and the half-filled Landau level.

However, understanding NFLs is hindered by the lack of a comprehensive quantum field theory concerning Fermi liquid theory, which serves as a simpler analogue governed by short-range interactions. The success of Fermi liquid theory spans from condensed matter to high-energy and astrophysical systems, covering conventional metals to neutron stars and white dwarfs. The foundational framework for Fermi liquids was originally proposed by Landau, known as Landau's Fermi liquid theory Landau [1957]. At its core, Landau's theory focuses not on an effective action but on a kinetic equation governing the phase-space distribution of quasiparticles.

While this kinetic equation enables the computation of certain response functions at leading order for low momenta, it lacks the standard machinery of effective field theory necessary for a systematic calculation of physical quantities to all orders in momentum expansion Weinberg [1979].

2.1.2 Nonlinear Bosonization

In a recent publication Delacrétaz et al. [2022], we introduced a novel formalism employing the mathematical concept of the coadjoint orbit Wiegmann [1989], Alekseev et al. [1988] to systematically address these challenges. Nonlinear bosonization offers an exact reformulation of Landau's kinetic theory into a field theory that is local in space. This innovative approach provides both perturbative and non-perturbative methods for investigating Fermi and non-Fermi liquids. The methodology naturally yields a multidimensional bosonized description Haldane [1994], Castro Neto and Fradkin [1994a], Houghton et al. [2000] with nonlinear corrections fixed by the geometry of the Fermi surface.

The resulting local effective field theory successfully incorporates both linear and nonlinear effects of Landau's Fermi liquid theory, including nonlinear responses characterized by diagrams with one fermion loop and current insertions. In our framework, these fermion loops are mapped to tree-level vertices within a local effective field theory of the bosons. The intricate cancellations observed in the diagrammatic calculations of fermion-loop diagrams Holder and Metzner [2015a] are replicated through straightforward power counting within the bosonic theory. Such nonlinear effects are anticipated to play a crucial role in the infrared behavior of non-Fermi liquids, particularly concerning diagrams involving fermion loops with multiple insertions of the critical boson Holder and Metzner [2015b].

Our approach obviates the need for decomposing the Fermi surface into patches, facilitating a systematic analysis of Fermi surface curvature and dispersion relation nonlinearities in NFLs. Understanding NFLs represents a significant advancement in formal quantum field theory, given their manifestation of various poorly understood phenomena, such as large emergent symmetries and UV/IR mixing. In the long term, this understanding holds potential practical importance in the realm of material science, particularly in the study of unconventional superconductors.

A notable aspect of our exploration is the adept incorporation of Berry curvature within the nonlinear bosonization paradigm. This integration not only enriches our understanding of the underlying physics but also establishes meaningful connections with phenomena such as the anomalous Hall effect. By elucidating how Berry curvature manifests within the nonlinear bosonization framework, we deepen our comprehension of the interplay between geometric properties and emergent physical phenomena in condensed matter systems.

2.1.3 Noncommutativity

Within the nonlinear bosonization framework, we have embarked upon a novel approach to bosonizing the Fermi surface. This method begins with a semiclassical depiction of the many-body system, characterized by a dense clustering of particles within their respective single-particle phase spaces. These particles collectively form a region of constant density enclosed within an interconnected domain, referred to as the droplet. According to Liouville's theorem, those particles exhibiting a density distribution within phase space undergoes evolution through an area-preserving diffeomorphism. Consequently, a droplet characterized by constant density transforms into a variant shape while retaining its uniform density. By quantizing the remaining phase space coordinates at the classical level, we transition to a description of noncommutative field theory Douglas and Nekrasov [2001], Rubakov [2005], Du et al. [2021]. The discussion on noncommutative quantum field theory, or quantum field theory on noncommutative spacetime, will be meticulously explored in subsequent chapters. Conceptually, the Fermi surface emerges as a droplet enmeshed within noncommutative space.

The systemic foundation behind this framework is well-grounded. Phase space volume-preserving diffeomorphisms correspond to unitary transformations, which at the semiclassical level reduce to canonical transformations Marsden and Weinstein [1982]. Consequently, area-preserving diffeomorphisms are isomorphic to canonical transformations of the phase space of a particle in 1+1 dimensions. Fermi liquid theory can essentially be thought of as a non-trivial representation of the Lie group of canonical transformations, bringing it within the effective theories whose structure is determined by symmetries. Similar principles extend to higher dimensions, with a caveat: canonical transformations, which represent the classical manifestation of noncommutative gauge transformations, constitute only a symplectic subgroup within the full volume-preserving diffeomorphism group.

2.1.4 Coherent states

From the above discussion, we expand our discussion of nonlinear bosonization to incorporate the concept of noncommutativity. Upon quantizing the phase space coordinates in terms of noncommutative space, we have the following commutation relation:

$$[x^i,\ k^j] = i\theta^{ij} \ , \tag{2.1.1}$$

where θ plays a role analogous to that of $\hbar$. In this framework, the physics of Fermi liquids, characterized by an inherent fuzziness, can be visualized as a droplet in noncommutative space. At the semiclassical level, the boundary of this droplet corresponds to the Fermi surface.

Our objective is to introduce the coherent-state path integral formulation for Fermi liquids to construct effective theories that describe particle-hole excitations above the Fermi surface ground state. We start by introducing fermion bilinears (particle-hole pairs), which act as generators of a deformed Lie algebra in the presence of noncommutativity:

$$\mathcal{O}(\mathbf{x},\mathbf{k}) = \int_{\mathbf{q}} \psi^{\dagger}_{\mathbf{k}+\frac{\mathbf{q}}{2}} \psi_{\mathbf{k}-\frac{\mathbf{q}}{2}} e^{-i\mathbf{q}\cdot\mathbf{x}} \ , \tag{2.1.2}$$

where ψ is the fermion operator. Let us define a spherical Fermi surface with Fermi momentum k_F. The ground state, describing a filled Fermi sea, is given by:

$$|\mathrm{GS}\rangle = \prod_{|\mathbf{k}|\leq k_F} \psi^{\dagger}_{\mathbf{k}} |\mathrm{empty}\rangle \ , \tag{2.1.3}$$

where $|\mathrm{empty}\rangle$ is the vacuum. The distribution function $f(\mathbf{x},\mathbf{k})$ is the expectation value of the fermion bilinear

$$f_0(\mathbf{x},\mathbf{k}) = \langle GS|\mathcal{O}(\mathbf{x},\mathbf{k})|GS\rangle = \Theta(k_F - |\mathbf{k}|) \ , \tag{2.1.4}$$

which represents the shape of the Fermi surface. We define the coherent state using the generator of the Moyal product-deformed Lie algebra $\mathcal{O}(\mathbf{x},\mathbf{k})$:

$$|\phi\rangle = \exp i \int_{\mathbf{x},\mathbf{k}} \phi(\mathbf{x},\mathbf{k}) \mathcal{O}(\mathbf{x},\mathbf{k}) |GS\rangle \ , \tag{2.1.5}$$

where $\phi(\mathbf{x},\mathbf{k})$ is a phase space bosonic variable. The Moyal product of functions, an essential feature in noncommutative field theory, is defined by:

$$f \star g = f \exp\left(\frac{i}{2}\theta^{ij} \overleftarrow{\partial}_i \overrightarrow{\partial}_j\right) g \ . \tag{2.1.6}$$

The distribution function, modified by the deformation of the Fermi surface, is then:

$$f(\mathbf{x},\mathbf{k}) = \langle\phi|\mathcal{O}(\mathbf{x},\mathbf{k})|\phi\rangle \ . \tag{2.1.7}$$

From the noncommutative version of the relation $f \star f = f$, inherent in the droplet picture, we introduce the Moyal-unitary field U given by the Moyal-exponential of a real field ϕ as

$$U \star U^{-1} = 1, \quad U = \exp_{\star}(i\phi) \equiv 1 + i\phi - \frac{1}{2}\phi \star \phi + \cdots \ . \tag{2.1.8}$$

As a consequence, the states are related by:

$$f = U \star f_0 \star U^{-1} \ . \tag{2.1.9}$$

This coherent state approach to encoding a sum over the histories of the deformation of the Fermi surface relates to the coadjoint-orbit nonlinear bosonization method at the semiclassical level. A more detailed discussion of the bosonization, coherent state, and noncommutativity will be deferred to future studies.

In the forthcoming chapter, we aim to provide a comprehensive introduction to the novel formalism of nonlinear bosonization. Additionally, we will conduct analyses aimed at elucidating their physical applications for gapless phases of matter in condensed matter systems. This exploration promises to advance our understanding of both Fermi and non-Fermi liquids, with potential implications for material science and superconductivity.

2.2 Nonlinear bosonization of Fermi surfaces: The method of coadjoint orbits

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Abstract

We develop a new method for bosonizing the Fermi surface based on the formalism of the coadjoint orbits. This allows one to parametrize the Fermi surface by a bosonic field that depends on the spacetime coordinates and on the position on the Fermi surface. The Wess-Zumino-Witten term in the effective action, governing the adiabatic phase acquired when the Fermi surface changes its shape, is completely fixed. As an effective field theory the action also involves a Hamiltonian which contains, beside the kinetic energy and the Landau interaction, terms with arbitrary number of derivatives and fields. We show that the resulting local effective field theory captures both linear and nonlinear effects in Landau's Fermi liquid theory. The approach can be extended to incorporate spin degrees of freedom and the charge-2 fields corresponding to the BCS order parameter.

2.2.1 Introduction

Understanding gapless phases of matter is an important problem of condensed matter physics. Among the gapless phases, those with a Fermi surface present a particularly difficult challenge for theoretical approaches. These phases include Fermi liquids and presumably a non-empty set of so-called "non-Fermi liquids"—gapless phases with a well-defined Fermi surface, but around which there are no well-defined fermionic quasiparticles. One concrete example where a non-Fermi liquid should appear is the problem of a half-filled Landau level of electrons with short-range interaction, which is dual to composite fermions interacting through an emergent U(1) gauge field. Currently, there is no systematic understanding of the gapless phases with a Fermi surface, or at least one cannot claim to understand these phases as well as, e.g., the Wilson-Fisher fixed points of the Ising and the $O(N)$ models.

One of the difficulties is the lack of a field-theoretical language to describe these phases. Philosophically, Landau's Fermi liquid theory (LFLT) Landau [1957] could be called one of the first low-energy effective theories—in proposing it, Landau took a modern view that the low-energy dynamics can be written down without a complete knowledge of the physics in the ultraviolet. Ideological similarities notwithstanding, the original, and still standard, formulation of Landau's Fermi liquid theory is far in form from what one would now call an effective field theory (EFT). The central object of Landau's Fermi liquid theory is not an effective action, but a kinetic equation satisfied by the phase-space distribution of the quasiparticles. While this kinetic equation allows one to compute certain response functions to leading order at low momenta, the standard machinery of effective field theory, which should allow a systematic calculation of any physical quantity, theoretically to any order in momentum expansion Weinberg [1979], is lacking. This translates into the lack of an understanding of non-Fermi liquid "fixed points" at same level of detail as our understanding of fixed points in relativistic quantum field theory.

At zero temperature, the structure of the Landau kinetic theory can be simplified by considering quasiparticle distribution functions that have a jump from 1 to 0 at a Fermi surface. LFLT is thus a theory describing the evolution of the shape of the Fermi surface in space and time. In non-Fermi liquids it is believed that the Fermi surface continues to be sharp—though the physics around it is governed by a theory different from Fermi liquid theory. Understanding how to parametrize the Fermi surface in terms of field-theoretic degrees of freedom is thus likely a necessary first step toward understanding non-Fermi liquids.

Important work aiming at a reformulation of Landau Fermi liquid theory in a form more similar to a EFT has been made in the past. Polchinski Polchinski [1992] and Shankar Shankar [1994], carefully analyzing the scaling near the Fermi surface, came up with a field-theoretical way of understanding why only the forward scattering of quasiparticles, parametrized by the Landau parameters, need to be kept—these can be thought of exactly marginal interaction terms. The BCS channel in this picture corresponds to the only other marginal (but now not exactly marginal) interaction. The approach taken by Polchinski and Shankar is however unwieldy as their theories are formulated in momentum space. Another approach known under the name "bosonization" has the goal of arriving at a local quantum field theory where the degree of freedom would be a field that lives not only in space but also on the Fermi surface Haldane [1994], Castro Neto and Fradkin [1994a], Houghton et al. [2000]. One can write down effective field theories that reproduce the linear response of fermionic systems to long-wavelength external probes, giving answers identical to those of the Landau Fermi liquid theory.

In this work, we propose a new approach to writing down an effective field theory of the Fermi surface. The approach is based on the mathematical formalism of <u>coadjoint orbits</u>, developed by Kirillov and others as a tool to understand representations of Lie groups Kirillov [2004], and in particular on actions on coadjoint orbits Wiegmann [1989], Alekseev et al. [1988]. A technical advantage of our approach is its simplicity and straightforwardness: for example, it dispenses with the need to divide the Fermi surface into patches, which is a necessary nuisance of previous approaches. But more importantly, the coadjoint-orbit method allows us to encode, in a simple and transparent manner, the nonlinear effects of LFLT. We recall here that Landau's Fermi liquid theory allows one to compute nonlinear response, which in perturbation theory corresponds to graphs with one fermion loop and an arbitrary number of current insertions. Previous works on bosonization have focused on reproducing the linear response, or two-point functions, while nonlinear response encoded in higher-points functions have received relatively scant attention. These nonlinear effects are expected to an play important role in the infrared behavior of non-Fermi liquids—diagrams that are important in the infrared involve fermion loops with multiple insertions of the critical boson (see e.g., Ref. Holder and Metzner [2015b]). In our approach, these fermion loops map to tree-level vertices in a local effective field theory of the bosons. The subtle cancellation observed in the diagrammatic calculation of the fermion-loop diagrams Holder and Metzner [2015a] is reproduced by straightforward power counting in the bosonic theory.

Recently, Else, Senthil and Thorngren proposed to characterize Fermi liquids (and possibly non-Fermi liquids) in terms of an emergent loop group symmetry $LU(1)$ with an 't Hooft anomaly Else et al. [2021]. This anomaly is closely related to the linearized algebra of densities commonly used in bosonization Haldane [1994], Castro Neto and Fradkin [1994a], Houghton et al. [2000]. As we will see, both the $LU(1)$ anomaly and this bosonization algebra are linearized approximations to a nonabelian algebra that controls the nonlinear structure of Fermi liquids.

Coadjoint orbits have been considered in connection with 1d Luttinger liquids Das et al. [1992], Dhar et al. [1993a,b], Khveshchenko [1994], and to a certain extent in higher dimensions Khveshchenko [1995]; our work shows that, beyond an elegant mathematical formulation, coadjoint orbits provide a powerful practical tool for systematic perturbative studies of Fermi liquids.

2.2.2 Canonical transformations

The fundamental degree of freedom in Landau's Fermi liquid theory is the quasiparticle distribution function $f(t, \mathbf{x}, \mathbf{p})$. Let us restrict ourselves to free fermions for the moment and turn to the general Fermi liquid case later. Given a dispersion relation $\epsilon(\mathbf{p})$, the dynamics of this free Fermi gas in external potential $V(\mathbf{x})$ can be entirely described by a kinetic equation—the Boltzmann, or Liouville, equation

$$\partial_t f + \nabla_{\mathbf{p}}\epsilon(\mathbf{p}) \cdot \nabla_{\mathbf{x}} f - \nabla_{\mathbf{x}} V \cdot \nabla_{\mathbf{p}} f = 0 \,. \tag{2.2.1}$$

Our task is to derive an action that would yield Eq. (2.2.1). We will do that with the help of a mathematical formalism that involves the group of canonical transformations. The importance of canonical transformations is related to the fact that Eq. (2.2.1) describes the evolution of a "swarm" of particles, each of which moves in phase space according to the

Hamiltonian equation of motion. But Hamiltonian evolution is a continuous sequence of canonical transformations. To make this idea concrete, we need to develop the formalism of the coadjoint orbits of canonical transformations.

We note here in passing the analogy of our problem with incompressible hydrodynamics, which can be thought of as a dynamical system on the group of volume preserving diffeomorphisms Arnold and Khesin [2013].

Group and algebra of canonical transformations

The single-particle phase space is a $2d$-dimensional space with the coordinates x^i, p_i, where $i = 1, 2, \ldots d$. The Lie algebra of canonical transformations, $\mathfrak{g}$ is a space of functions on phase space $F(\mathbf{x}, \mathbf{p})$, which vanish at infinity. We will use upper-case letters for elements of $\mathfrak{g}$. An element of $\mathfrak{g}$ that corresponds to a function $F(\mathbf{x}, \mathbf{p})$ generates an infinitesimal canonical transformation

$$\mathbf{x} \to \mathbf{x}' = \mathbf{x} - \epsilon \nabla_{\mathbf{p}} F \,, \tag{2.2.2}$$

$$\mathbf{p} \to \mathbf{p}' = \mathbf{p} + \epsilon \nabla_{\mathbf{x}} F \,. \tag{2.2.3}$$

It is straightforward to verify that this is indeed a canonical transformation: $\mathbf{x}'$ and $\mathbf{p}'$ still have canonical Poisson brackets. Eq. (2.2.2) is the Hamiltonian evolution under the Hamiltonian F during an infinitesimal time interval ϵ.

The commutator of two such coordinate transformations, parametrized by functions F

and G is a transformation parametrized by the Poisson bracket in phase space [1]

$$\{F, G\} = \nabla_{\mathbf{x}} F \cdot \nabla_{\mathbf{p}} G - \nabla_{\mathbf{p}} F \cdot \nabla_{\mathbf{x}} G \,. \tag{2.2.4}$$

We thus define the Lie bracket in $\mathfrak{g}$ as

$$[F,\, G] \equiv \{F,\, G\} \,. \tag{2.2.5}$$

From here onwards, we will use curly braces $\{,\}$ to denote the Lie bracket in $\mathfrak{g}$, which is otherwise conventionally denoted by square brackets $[,]$.

The group of canonical transformations is obtained from the algebra by exponentiation: $U = \exp F$ transforms $(\mathbf{x},\, \mathbf{p})$ into new phase-space coordinates $(\mathbf{x}',\, \mathbf{p}')$ that are the result of time evolution under the Hamiltonian F during unit time (exponentiation of the elements of the algebra should not be confused with the exponential of the function $e^{F(\mathbf{x},\mathbf{p})}$). This group will be denoted by $\mathcal{G}$.

The dual space

The dual space $\mathfrak{g}^*$ of the Lie algebra is the space of linear functionals on $\mathfrak{g}$, i.e., linear maps from $\mathfrak{g}$ to $\mathbb{R}$

$$f :\; F \in \mathfrak{g} \mapsto \langle f,\, F\rangle \in \mathbb{R} \,, \tag{2.2.6}$$

so that $\langle f,\, \alpha F + \beta G\rangle = \alpha\langle f,\, F\rangle + \beta\langle f,\, G\rangle$. The symbol $\langle f,\, F\rangle$ will be called the scalar product of f and F.

In the case of the algebra of canonical transformations, the elements of $\mathfrak{g}^*$ are functions

1. A quick way to see this is to observe that Eq. (2.2.2) tells us that each element of $\mathfrak{g}$ corresponds to a vector field, or a linear operator $X_F = \nabla_{\mathbf{x}} F \cdot \nabla_p - \nabla_p F \cdot \nabla_x$ This type of vector field is known as "Hamiltonian". The commutator (Lie bracket) of two Hamiltonian vector fields is again a Hamiltonian vector field, $[X_F, X_G] = X_{\{F,G\}}$ where $\{F,\, G\} = \nabla_{\mathbf{x}} F \cdot \nabla_{\mathbf{p}} G - \nabla_{\mathbf{p}} F \cdot \nabla_{\mathbf{x}} G$ is the Poisson bracket in phase space.

$f(\mathbf{x}, \mathbf{p})$, denoted by lower case letters, on the phase space. The scalar product is defined as

$$\langle f, F \rangle \equiv \int \frac{d^d x d^d p}{(2\pi)^d} F(\mathbf{x}, \mathbf{p}) f(\mathbf{x}, \mathbf{p}) \,. \tag{2.2.7}$$

We will interpret an element $F(\mathbf{x}, \mathbf{p})$ of the Lie algebra as an observable (more precisely, a one-particle observable), while an $f(\mathbf{x}, \mathbf{p})$ of the dual space will be interpreted as the phase-space distribution function characterizing a state. The scalar product $\langle f,\, F \rangle$ is interpreted as the expectation value of the observable F in the state given by f. Equation (2.2.7) is similar to the expression for the expectation value of an operator $\hat{A}$ in a state given by a density matrix $\hat{\rho}$ in quantum mechanics: $\langle A \rangle = \mathrm{Tr}(\hat{\rho}\hat{A})$. (A difference is that the "trace" of f is not 1 but the total number of particles.)

Alternately, one can think of the Lie algebra elements as linear functionals on $\mathfrak{g}^*$, which act on $\mathfrak{g}^*$ in the following way

$$F[f] \equiv \langle f,\, F \rangle \,. \tag{2.2.8}$$

Adjoint and coadjoint action

Here we introduce a few further mathematical notions.

Lie algebra adjoint action: The adjoint action of a Lie algebra element G on another Lie algebra element F is given simply by the Lie bracket

$$\mathrm{ad}_G F = \{G, F\}. \tag{2.2.9}$$

This adjoint action furnishes a representation of the Lie algebra on itself,

$$\mathrm{ad}_G \, \mathrm{ad}_H - \mathrm{ad}_H \, \mathrm{ad}_G = \mathrm{ad}_{\{G,\, H\}}. \tag{2.2.10}$$

Group adjoint action: The group adjoint action of a group element $U = \exp G \in \mathcal{G}$

on a Lie algebra element F is given by

$$\mathrm{Ad}_U F = e^{\mathrm{ad}_G} F = F + \{G, F\} + \frac{1}{2!}\{G, \{G, F\}\} + \cdots . \tag{2.2.11}$$

The group adjoint action forms a representation of the group on its Lie algebra

$$\mathrm{Ad}_{UV} = \mathrm{Ad}_U \mathrm{Ad}_V . \tag{2.2.12}$$

For convenience, we will sometimes denote the adjoint action as matrix conjugation

$$\mathrm{Ad}_U F = UFU^{-1} . \tag{2.2.13}$$

This form of notation is useful as it immediately allows us to use the intuition developed in quantum mechanics, but is strictly speaking not necessary and will be avoided when it may lead to confusion.

Lie algebra coadjoint action: The adjoint action of the Lie algebra on itself also defines a coadjoint action of $\mathfrak{g}$ on the dual space $\mathfrak{g}^*$ by mapping $f \in \mathfrak{g}^*$ to $\mathrm{ad}^*_F f \in \mathfrak{g}^*$ defined by requiring

$$\langle \mathrm{ad}^*_F f, \mathrm{ad}_F G \rangle = \langle f, G \rangle . \tag{2.2.14}$$

In the case of canonical transformations, We see that the Lie algebra coadjoint action maps the distribution f to

$$\mathrm{ad}^*_F f = \{F, f\} , \tag{2.2.15}$$

which takes the same form as that for the Lie algebra elements.

Group coadjoint action: This is the exponentiation of the Lie algebra coadjoint action, and defined the action of group elements on the dual algebra

$$\mathrm{Ad}^*_{\exp F} f = f + \{F, f\} + \frac{1}{2!}\{F, \{F, f\}\} + \cdots . \tag{2.2.16}$$

The distinction between the Lie algebra $\mathfrak{g}$ and its dual space $\mathfrak{g}^*$ may appear pedantic at this moment, but it will turn out to be useful later.

Boltzmann equation for free particles as Hamiltonian evolution

Consider a system of particles with dispersion $\epsilon(\mathbf{p})$ in an external field $V(\mathbf{x})$. The Hamiltonian for free fermions is an element of $\mathfrak{g}$,

$$H = \epsilon(\mathbf{p}) + V(\mathbf{x}) \in \mathfrak{g}\,. \tag{2.2.17}$$

The Boltzmann equation

$$\partial_t f + \nabla_{\mathbf{p}}\epsilon(\mathbf{p}) \cdot \nabla_{\mathbf{x}} f - \nabla_{\mathbf{x}} V(\mathbf{x}) \cdot \nabla_{\mathbf{p}} f = 0 \tag{2.2.18}$$

can be rewritten as

$$\partial_t f - \mathrm{ad}^*_H f = 0\,, \tag{2.2.19}$$

and can therefore be thought of as a dynamical system on the Lie group of canonical transformations Marsden and Weinstein [1982].

Going beyond free particles, one can generalize the Hamiltonian above to a nonlinear functional $\mathscr{H}[f]$ of $\mathfrak{g}^*$. Since the functional derivative $\delta\mathscr{H}/\delta f$ naturally lives in $\mathfrak{g}$, one can use its action on f to generalize the equation of motion (2.2.19) to a nonlinear equation of motion (see App. 2.3.6 for an example). In this paper, we will mostly work with the effective action, rather than the equation of motion. Nonlinear actions for Fermi liquids will be studied systematically in Sec. 2.2.3.

2.2.3 Fermi liquids as a coadjoint orbit

While the collisionless Boltzmann equation describes the time evolution for any and all distributions $f(\mathbf{x}, \mathbf{p})$, not all of these are relevant for the system at hand. At zero temperature, a generic state of a Fermi liquid is described by a closed Fermi surface in momentum space of arbitrary shape and fixed volume (Fig. 2.1). The distribution $f(\mathbf{x}, \mathbf{p})$ takes the value 1 inside the Fermi surface and 0 outside. Thanks to Liouville's theorem, time evolution under the collisionless Boltzmann equation preserves this form of the distribution and only changes the shape of the Fermi surface. So we need to restrict our space of states further from all of $\mathfrak{g}^*$.

Another way to motivate this restriction is to look at the entropy of the system

$$S = \int \frac{d\mathbf{x}\, d\mathbf{p}}{(2\pi)^d} \left[-f \ln f - (1-f) \ln(1-f) \right] . \tag{2.2.20}$$

If one restricts oneself to zero-entropy states, then f at any given point on the phase space can be only 0 or 1. Such zero-entropy states can be characterized by the surface separating the phase-space region with $f = 0$ from that with $f = 1$. In principle, this surface may not be connected or may have nontrivial topology, but for simplicity we will not consider that situation and assume that the Fermi surface is connected and is topologically equivalent to $S^{d-1} \times \mathbb{R}^d$.

The condition $f = 0$ or $f = 1$ can be concisely written as

$$f^2 = f \, . \tag{2.2.21}$$

This form bears resemblence to the condition in quantum mechanics for a density matrix $\hat{\rho}$ to correspond to a pure state: $\hat{\rho}^2 = \hat{\rho}$.

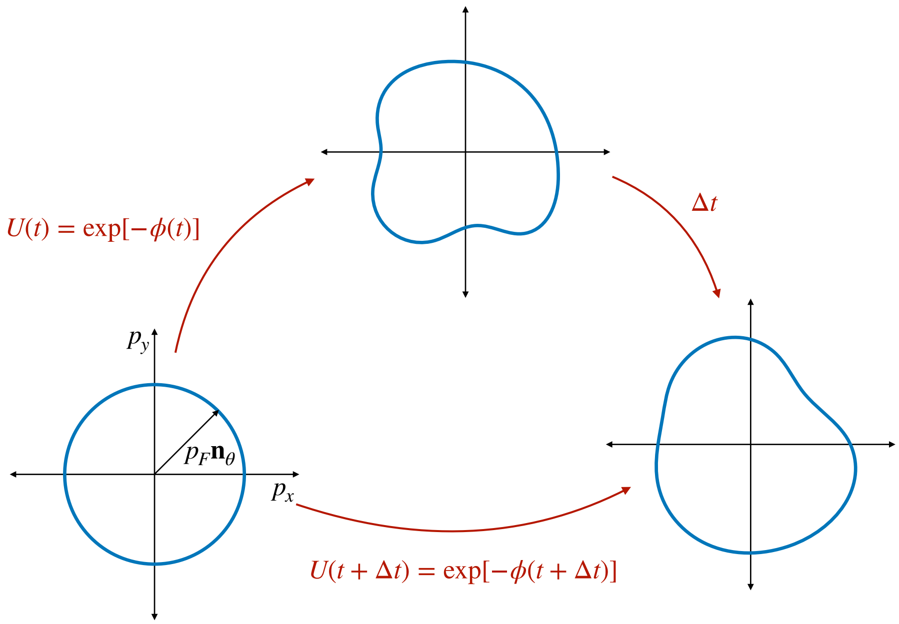

Figure 2.1: Fermi surface states from canonical transformations

Coadjoint orbits

Given the coadjoint action of the group $\mathcal{G}$ on the dual space $\mathfrak{g}^*$, we can define the coadjoint orbit $\mathcal{O}_{f_0}$ of any element $f_0 \in \mathfrak{g}^*$ as the set of all elements $f \in \mathfrak{g}^*$ that can be obtained by the coadjoint action of canonical transformations on f_0,

$$\mathcal{O}_{f_0} = \{ f \mid \exists U \in \mathcal{G} : f = \mathrm{Ad}^*_U f_0 \} \,. \tag{2.2.22}$$

In Fermi liquids, any state is a droplet in momentum space of arbitrary shape and fixed total volume. Any such state can be obtained by the coadjoint action of a canonical transformation on the spherical, translationally invariant droplet

$$f_0(\mathbf{p}) = \Theta(p_F - |\mathbf{p}|) \,. \tag{2.2.23}$$

Therefore, the relevant space of states is the coadjoint orbit $\mathcal{O}_{f_0}$ of the spherical Fermi surface. This is our desired restriction of the space of states. The description of the Fermi liquid will not depend on the specific seed state f_0 used to generate the coadjoint orbit. However, we will often be interested in the small fluctuations around the ground state, which for an isotropic Fermi liquid takes the form (2.2.23), and more generally (say, in metals) can be a more complicated function $f_0(\mathbf{p})$. The theory of these fluctuations of course depend on the state around which one is expanding.

Multiple canonical transformations can map f_0 to the same state $f = \mathrm{Ad}^*_U f_0$, since if we consider any group element V that stabilizes f_0

$$\mathrm{Ad}^*_V f_0 = f_0 \,, \tag{2.2.24}$$

then right multiplication of U by V preserves f:

$$\mathrm{Ad}^*_{UV} f_0 = \mathrm{Ad}^*_U \mathrm{Ad}^*_V f_0 = \mathrm{Ad}^*_U f_0 = f \,. \tag{2.2.25}$$

Such group elements V form a subgroup $\mathcal{H}$, called the stabilizer subgroup of f_0. The coadjoint orbit $\mathcal{O}_{f_0}$ is thus the left coset space $\mathcal{G}/\mathcal{H}$.

Parametrizing the coadjoint orbit

In order to parametrize the coadjoint orbit, we write the canonical transformation in terms of a Lie algebra element $\phi(\mathbf{x}, \mathbf{p})$:

$$U = \exp\left(-\phi\right) \,. \tag{2.2.26}$$

Recall that the exponent map used above is not the exponent of the function $\phi(\mathbf{x}, \mathbf{p})$, but rather the exponentiation of the Lie algebra element ϕ. The minus sign in the exponent is a convention which we find more convenient for later formulas. A state parametrized by ϕ

has the following distribution function,

$$
\begin{aligned}
f &= f_0 - \{\phi, f_0\} + \frac{1}{2}\{\phi, \{\phi, f_0\}\} - \frac{1}{3!}\{\phi, \{\phi, \{\phi, f_0\}\}\} + \cdots \\
&= \Theta(p_F - |\mathbf{p}|) + \delta(|\mathbf{p}| - p_F)\mathbf{n}_\theta \cdot \nabla_{\mathbf{x}}\phi + \cdots ,
\end{aligned}
\tag{2.2.27}
$$

where $\mathbf{n}_\theta \equiv \mathbf{p}/|\mathbf{p}|$ is the unit vector normal to the Fermi surface, see Fig. 2.1. We have expanded around a spherical Fermi surface (2.2.23), as we will focus on the study of fluctuations around spherical Fermi surfaces. However, our formalism can be straightforwardly adapted to fluctuations around Fermi surfaces of arbitrary shapes, by expanding around the appropriate $f_0(\mathbf{p})$. The object $\phi(\mathbf{x}, \mathbf{p})$ will be the bosonized degree of freedom, in terms of which the effective field theory is formulated.

As discussed above, one can multiply U by any element V of the stabilizer group $\mathcal{H}$ and the new group element UV still corresponds to the same state. This corresponds to a gauge equivalence between different $\phi(\mathbf{x}, \mathbf{p})$ configurations: First, we parametrize an element in $\mathcal{H}$ as $\exp \alpha$ where $\alpha \in \mathfrak{h}$. This condition translates to

$$
\begin{aligned}
&\mathrm{ad}^*_\alpha f_0 = \{\alpha, f_0\} = 0 \\
\implies &\mathbf{n}_\theta \cdot \nabla_{\mathbf{x}}\alpha\big|_{|\mathbf{p}|=p_F} = 0 \,.
\end{aligned}
\tag{2.2.28}
$$

Equivalence under right multiplication by $\mathcal{H}$ leads to the identification

$$
\begin{aligned}
&\exp(-\phi) \sim \exp(-\phi)\exp\alpha \\
\implies &\phi \sim \phi - \alpha + \frac{1}{2}\{\phi, \alpha\} + \cdots .
\end{aligned}
\tag{2.2.29}
$$

where the Baker-Campbell-Hausdorff formula has been used. By imposing a gauge-fixing condition, one can pick one representative from the equivalence class. For example, one convenient parametrization is to require that $\phi(\mathbf{x}, \mathbf{p})$ depend only only on the direction of the vector $\mathbf{p}$, but not its magnitude:

$$\phi = \phi(\mathbf{x}, \theta)\,. \tag{2.2.30}$$

The angles θ parametrize the Fermi surface. To linear order, any function $\phi(\mathbf{x}, \mathbf{p})$ can be brought to the form (2.2.30) by a gauge transformation (2.2.29) by choosing α to be the difference between ϕ and its value at the Fermi surface

$$\alpha = \phi(\mathbf{x}, \mathbf{p}) - \phi(\mathbf{x}, \theta, |\mathbf{p}| = p_F)\,. \tag{2.2.31}$$

One can check that $\{\alpha, f_0\} = 0$. Equation (2.2.30) is by no means the only possible gauge-fixing condition. One can impose, e.g.,

$$\phi = g\left(\frac{|\mathbf{p}|}{p_F}\right)\phi(\mathbf{x}, \theta)\,, \tag{2.2.32}$$

with any function $g(x)$ which satisfies the condition $g(1) = 1$. To linear order one can again easily find an α that would bring an arbitrary $\phi(\mathbf{x}, \mathbf{p})$ into the form (2.2.32). Any of the gauge-fixing choice reduces the bosonized degree of freedom to functions living on the Fermi surface; different choices will lead to equivalent EFTs for ϕ which differ by field redefinitions.

Wess-Zumino-Witten term and Fermi liquid action

To write down the action for a Fermi liquid, one needs one more mathematical ingredient: the Kirillov-Kostant-Souriau (KKS) symplectic form on the coadjoint orbit. This symplectic form is defined via its action on two tangent vectors at a point in $\mathcal{O}_{f_0}$, which are naturally identified as elements of $\mathfrak{g}^*$. Consider two such tangent vectors g, h at the point f in the coadjoint orbit. Being tangents to the coadjoint orbit, there must exist two Lie algebra elements $G, H \in \mathfrak{g}$ such that

$$g = \mathrm{ad}^*_G f, \qquad h = \mathrm{ad}^*_H f\,. \tag{2.2.33}$$

G and H are not unique, but rather representatives of equivalence classes. Given two such representatives, the action of the coadjoint orbit symplectic form on the two tangents g and h is given by

$$\omega(g, h) = \langle f, \{G, H\}\rangle\,. \tag{2.2.34}$$

The statement of the KKS theorem is that (i) ω is independent of the choice of representatives G and H and (ii) ω is closed. The proof of (i) is rather simple. Consider two different Lie algebra elements G and G' such that

$$g = \mathrm{ad}^*_G f = \mathrm{ad}^*_{G'} f\,. \tag{2.2.35}$$

Their difference is an element of the stabilizer of f, i.e.,

$$\mathrm{ad}^*_{G-G'} f = 0\,. \tag{2.2.36}$$

Therefore the difference between the two possible RHS expressions is

$$\langle f, \{G', H\}\rangle - \langle f, \{G, H\}\rangle = \langle f, \mathrm{ad}_{G'-G} H\rangle = \langle \mathrm{ad}^*_{G-G'} f, H\rangle = 0\,. \tag{2.2.37}$$

The same holds for different possible H's as well and the KKS form is hence well-defined. The closedness of the KKS form will be illustrated in our construction of the WZW action.

The KKS symplectic form allows us to write down the first of the two parts of the effective action: the WZW term, which encodes the adiabatic phase acquired when a Fermi surface evolves in time. To write this term, one adds an extra dimension, parametrized by a variable

s which takes values on the unit interval $[0, 1]$ and extrapolates our degree of freedom $f(t, s)$ into the extra dimension such that

$$f(t,0) = \text{const}, \qquad f(t,1) = f(t). \tag{2.2.38}$$

The Wess-Zumino-Witten term has the form

$$S_{\text{WZW}} = \int dt \int_0^1 ds\ \omega\left(\frac{\partial f}{\partial t}, \frac{\partial f}{\partial s}\right). \tag{2.2.39}$$

To write this term more explicitly, we use the definition of the symplectic form. Given that $f = \text{Ad}^*_U f_0$, it is easy to show that

$$\partial_t f = \text{ad}^*_{\partial_t U U^{-1}} f, \qquad \partial_s f = \text{ad}^*_{\partial_s U U^{-1}} f\,, \tag{2.2.40}$$

which tells us that

$$\begin{aligned} \omega(\partial_t f, \partial_s f) &= \left\langle f,\, \{\partial_t U U^{-1}, \partial_s U U^{-1}\}\right\rangle \\ &= \left\langle f_0,\, \{U^{-1}\partial_t U, U^{-1}\partial_s U\}\right\rangle. \end{aligned} \tag{2.2.41}$$

At this point, one can again check that the KKS symplectic form is independent of how one chooses to parametrize points on coadjoint orbit by group elements: if one make a small gauge transformation $U \to U(1 + \epsilon\alpha)$, with $\alpha \in \mathfrak{h}$, then the change of ω is zero by the virtue of $\text{ad}^*_{\mathfrak{h}} f_0 = 0$. Also, the KKS form is closed (and even exact) because, for any coordinate system x^i on the coadjoint orbit, one can write

$$\{U^{-1}\partial_i U,\, U^{-1}\partial_j U\} = -\partial_i(U^{-1}\partial_j U) + \partial_j(U^{-1}\partial_i U)\,. \tag{2.2.42}$$

This allows us to express, up to boundary terms,

$$S_{\text{WZW}} = \int dt \langle f_0, U^{-1}\partial_t U\rangle \,. \tag{2.2.43}$$

The total action will also involve a Hamiltonian part. This Hamiltonian part is in general a functional of the distribution function

$$S_H = -\int dt \mathscr{H}[f] \,, \tag{2.2.44}$$

and will be studied in detail in Sec. 2.2.3. For the free fermion, the Hamiltonian is

$$\mathscr{H}[f] = \langle f, \epsilon(\mathbf{p})\rangle = \langle f_0, U^{-1}\epsilon(\mathbf{p})U\rangle \,. \tag{2.2.45}$$

The action for the free Fermi gas then takes the form

$$S = \int dt \left\langle f_0, U^{-1}\left[\partial_t - \epsilon(\mathbf{p})\right] U\right\rangle \,. \tag{2.2.46}$$

This action can be expanded order by order in ϕ by writing $U = \exp(-\phi)$. Let us see how the Boltzmann equation arises as the equation of motion for this action. We vary our degree of freedom U as follows

$$U \to U' = \exp\alpha \cdot U \,, \tag{2.2.47}$$

where $\alpha(t, \mathbf{x}, \mathbf{p}) \in \mathfrak{g}$ is the variation. To linear order in α, we have

$$\delta[U^{-1}\partial_t U] = U^{-1}\partial_t \alpha U, \qquad \delta[U^{-1}\epsilon U] = U^{-1}\{\epsilon, \alpha\}U \,, \tag{2.2.48}$$

so that the variation of the action is

$$\begin{aligned}\delta S &= \int dt \left\langle f_0, U^{-1} \left[\partial_t \alpha - \{\epsilon, \alpha\}\right] U \right\rangle \\ &= \int dt \, \langle f, \partial_t \alpha - \{\epsilon, \alpha\}\rangle \\ &= -\int dt \, \langle \partial_t f + \{f, \epsilon\}, \alpha\rangle \, .\end{aligned} \tag{2.2.49}$$

The equation of motion is then the Boltzmann equation with no external potential

$$\partial_t f + \{f, \epsilon\} = \partial_t f + \nabla_{\mathbf{p}} \epsilon(\mathbf{p}) \cdot \nabla_{\mathbf{x}} f = 0 \, . \tag{2.2.50}$$

The action for a coadjoint orbit can be thought of as a special case of the CCWZ coset construction for nonlinearly realized symmetries Coleman et al. [1969], Callan et al. [1969], where the order parameter is adjoint-valued (or rather coadjoint-valued). For this class of 'symmetry-breaking patterns', there always exists a nontrivial element of the relative Lie algebra cohomology $H^2(\mathfrak{g}, \mathfrak{h})$ (see e.g. Goon et al. [2012] for its definition), which provides a WZW 2-form; this is the KKS form.

Coadjoint orbits for the group of canonical transformation are also relevant in the study of quantum Hall phases Iso et al. [1992], Karabali and Nair [2004], Polychronakos [2005].

Systematic higher order corrections to the EFT

So far we have considered only free fermions. Interactions between fermions can also be accounted for by appropriate modifications of the Hamiltonian functional $\mathscr{H}[f]$. The effective field theory approach is to consider the most general form for such a Hamiltonian functional, as an expansion in $\delta f(\mathbf{x}, \mathbf{p}) = f(\mathbf{x}, \mathbf{p}) - f_0(\mathbf{x}, \mathbf{p})$ and its derivatives. These terms can be systematically organized into a double expansion in derivatives and non-linearities

$$
\begin{aligned}
\mathscr{H}[f] = &\int_{\mathbf{x},\mathbf{p}} \epsilon(\mathbf{p}) f(\mathbf{x},\mathbf{p}) \\
&+ \int_{\mathbf{x},\mathbf{p},\mathbf{p}'} F^{(2,0)}_{\text{int}}(\mathbf{p},\mathbf{p}')\delta f(\mathbf{x},\mathbf{p})\delta f(\mathbf{x},\mathbf{p}') + \mathbf{F}^{(2,1)}_{\text{int}}(\mathbf{p},\mathbf{p}')\cdot\nabla_{\mathbf{x}}\delta f(\mathbf{x},\mathbf{p})\delta f(\mathbf{x},\mathbf{p}') + \dots \\
&+ \int_{\mathbf{x},\mathbf{p},\mathbf{p}',\mathbf{p}''} F^{(3,0)}_{\text{int}}(\mathbf{p},\mathbf{p}',\mathbf{p}'')\delta f(\mathbf{x},\mathbf{p})\delta f(\mathbf{x},\mathbf{p}')\delta f(\mathbf{x},\mathbf{p}'') + \dots\,,
\end{aligned}
\tag{2.2.51}
$$

where $F^{(m,n)}_{\text{int}}(\mathbf{p}_1,\dots,\mathbf{p}_n)$ are Wilsonian coefficients (or rather, functions) for terms involving m powers of δf with n derivatives, and parametrize our ignorance of the underlying microscopics. For simplicity we have assumed that the system is translationally invariant, so that the coefficients in the EFT do not depend on $\mathbf{x}$. An additional source of higher gradient terms that are of similar importance as the $F^{(m,n\geq 1)}_{\text{int}}$ above come from corrections to the semiclassical limit that produced the Poisson algebra (2.2.5) to leading order in gradients, see App. 2.3.6.

The full action is then

$$
S = S_{\text{WZW}} - \int dt \mathscr{H}[f]\,. \tag{2.2.52}
$$

It does not contain terms with extra time-derivatives, as these can be removed using the leading equations of motion with field redefinitions. Eq. (2.2.52) is the most general EFT describing Fermi liquids. It captures known Fermi liquid phenomenology—as will be studied at length in the remainder of this paper—and parametrizes all possible corrections to Fermi liquids through a tower of irrelevant interactions. It can be written in terms of the distribution function f, and its form is therefore independent of the background f_0 that one expands around. Although it is not manifest, the WZW term can also be shown to be independent of f_0 (this is clear from the equation of motion (2.2.50), which only depends on f). When studying fluctuations around a given background, say the spherical Fermi surface (2.2.23), it is convenient to write the action in terms of $\delta f = f - f_0$, and expand in ϕ, as will be done in the following sections.

In Sec. 2.2.4, this action will be linearized in fields. To leading order in derivatives, we will find that among all the terms in the EFT above, only $v_F(\theta) \equiv \partial_p \epsilon(\mathbf{p}_F(\theta))$ and $F^{(2,0)}_{\text{int}}(\mathbf{p}_F(\theta), \mathbf{p}_F(\theta'))$ appear in this Gaussian action—these are the familiar Fermi velocity and Landau parameters of LFLT. The cubic action, studied in Sec. 2.2.5, involves several additional parameters to leading order in derivatives: $\partial_p^2 \epsilon(\mathbf{p}_F(\theta))$, $\partial_p F^{(2,0)}_{\text{int}}(\mathbf{p}_F(\theta), \mathbf{p}_F(\theta'))$ and $F^{(3,0)}_{\text{int}}(\mathbf{p}_F(\theta), \mathbf{p}_F(\theta'), \mathbf{p}_F(\theta''))$. To our knowledge, the latter two have not appeared previously in the literature. Like the Landau parameters, they are nonuniversal observable parameters characterizing a Fermi liquid; they affect its nonlinear response, contributing in particular to the three-point function of charge density.

A nonlinear action for Fermi liquids of the form (2.2.46) appeared previously in Ref. Khveshchenko [1995], albeit without its completion into an EFT (2.2.51) capturing general interacting Fermi liquids. However, the nonlinearities were not properly treated in that work [2].

Coupling to background gauge fields

The EFT (2.2.52) has a number of global symmetries, whose currents may be coupled to background gauge fields. In this section, we focus on the global $U(1)$ symmetry leading to conservation of particle number, and couple more general symmetries to background gauge fields in App. 2.3.1.

The global $U(1)$ symmetry simply acts through constant functions $\lambda(\mathbf{x}, \mathbf{p}) = \text{const} \in \mathfrak{g}$. To gauge it (or rather couple the theory to background fields), we would like to make the action invariant under spacetime dependent transformations $\lambda(t, \mathbf{x})$. Let us start with time-

2. Indeed, the nonlinear WZW action in Khveshchenko [1995] is written as $S = \int_{tx\theta} \frac{1}{2} \partial_t \phi \partial_\theta \vec{k}_F \times \vec{k}_F$, where the degree of freedom is taken to satisfy $\vec{k}_F(t, x, \theta) = \vec{k}^0_F(\theta) + \nabla\phi(t, x, \theta)$. This gives rise to commutation relations $[\vec{k}_F(x, \theta), \rho(x', \theta')] \propto \nabla\delta(x-x')\delta(\theta-\theta')$ that are incompatible with the nonlinear algebra of densities (2.2.76).

independent gauge transformations: these manifest themselves as a subgroup of canonical transformations, namely the transformations generated by functions $\lambda(\mathbf{x}) \in \mathfrak{g}$, that are independent of momentum $\mathbf{p}$. Under such canonical transformations, elements $F(\mathbf{x}, \mathbf{p})$ of the Lie algebra transform to

$$(\mathrm{Ad}_{\exp\lambda} F)(\mathbf{x}, \mathbf{p}) = F(\mathbf{x}, \mathbf{p} + \nabla_{\mathbf{x}}\lambda)\,. \tag{2.2.53}$$

Under the time dependent version of this subgroup of canonical transformations, characterized by Lie algebra elements $\lambda(t, \mathbf{x})$, the distribution changes to

$$f \to \mathrm{Ad}^*_{\exp\lambda} f\,. \tag{2.2.54}$$

This amounts to the following transformation on the coset element U or equivalently the field ϕ

$$\begin{aligned} U &\to \exp\lambda \cdot U\,, \\ \phi &\to \phi - \lambda + \frac{1}{2}\{\lambda, \phi\} + \cdots\,. \end{aligned} \tag{2.2.55}$$

Defining $W = \exp\lambda$, under the transformation $U \to WU$, the nonlinear action eq. (2.2.46) transforms to

$$S' = \int dt \left\langle f_0, U^{-1}\partial_t U + U^{-1}\left[W^{-1}\partial_t W - W^{-1}\epsilon(\mathbf{p})W\right] U\right\rangle\,. \tag{2.2.56}$$

The action is evidently not invariant under this transformation. In order to make it gauge invariant we need to couple it to a background field $A_\mu(t, \mathbf{x})$ which transforms like

$$\delta_\lambda A_\mu(t, \mathbf{x}) = \partial_\mu \lambda\,. \tag{2.2.57}$$

To figure out how to couple the action to the background fields, we observe that

$$W^{-1}\partial_t W - W^{-1}\epsilon(\mathbf{p})W = \partial_t\lambda - \epsilon(\mathbf{p} - \nabla\lambda)\,. \tag{2.2.58}$$

Therefore the action, when coupled to a background gauge field is given by

$$S = \int dt \left\langle f_0, U^{-1}\left[\partial_t - A_0 - \epsilon(\mathbf{p} + \mathbf{A})\right] U \right\rangle\,. \tag{2.2.59}$$

It is easy to see that this action is invariant under the gauge transformation

$$U \to \exp\lambda \cdot U, \qquad A_\mu \to A_\mu + \partial_\mu\lambda\,. \tag{2.2.60}$$

The general interacting action (2.2.51) can also similarly be gauged by writing it in a slightly different form. While f transforms covariantly under the gauge transformation $U \to \exp\lambda \cdot U$, δf does not. We rewrite the interacting Hamiltonian as an expansion in f instead of δf with different Wilson coefficients $\tilde{F}^{(m,n)}_{\text{int}}(\mathbf{p}_1, \ldots \mathbf{p}_n)$

$$\begin{aligned}\mathscr{H}[f] = &\int_{\mathbf{x},\mathbf{p}} \epsilon(\mathbf{p}) f(\mathbf{x},\mathbf{p}) \\ &+ \int_{\mathbf{x},\mathbf{p},\mathbf{p}'} \tilde{F}^{(2,0)}_{\text{int}}(\mathbf{p},\mathbf{p}') f(\mathbf{x},\mathbf{p}) f(\mathbf{x},\mathbf{p}') + \tilde{\mathbf{F}}^{(2,1)}_{\text{int}}(\mathbf{p},\mathbf{p}') \cdot \nabla_{\mathbf{x}} f(\mathbf{x},\mathbf{p}) f(\mathbf{x},\mathbf{p}') + \cdots\,,\end{aligned} \tag{2.2.61}$$

Since the gauge transformation $\exp\lambda$ acts on f and its gradients as

$$\begin{aligned} f(\mathbf{x},\mathbf{p}) &\to (\mathrm{Ad}^*_{\exp\lambda} f)(\mathbf{x},\mathbf{p}) = f(\mathbf{p} + \nabla_{\mathbf{x}}\lambda)\,, \\ (\nabla_{\mathbf{x}} f)(\mathbf{x},\mathbf{p}) &\to (\nabla_{\mathbf{x}} f)(\mathbf{x},\mathbf{p} + \nabla_{\mathbf{x}}\lambda) + \{\nabla_{\mathbf{x}}\lambda, f\}(\mathbf{x},\mathbf{p} + \nabla_{\mathbf{x}}\lambda)\,, \end{aligned} \tag{2.2.62}$$

its effect can be canceled by simply replacing the modified Wilson coefficients with

$$\tilde{F}^{(m,n)}_{\text{int}}(\mathbf{p}_1 + \mathbf{A}, \dots \mathbf{p}_n + \mathbf{A})\,, \tag{2.2.63}$$

as well as making the gradients covariant

$$\nabla_{\mathbf{x}} f \to D_{\mathbf{x}} f \equiv \nabla_{\mathbf{x}} f - \{\mathbf{A}, f\}\,, \tag{2.2.64}$$

where $D_{\mathbf{x}}$ is the covariant derivative. The gauge invariant Hamiltonian is then

$$\begin{aligned}\mathscr{H}_A[f] = &\int_{\mathbf{x},\mathbf{p}} \epsilon(\mathbf{p}+\mathbf{A}) f(\mathbf{x},\mathbf{p}) \\ &+ \int_{\mathbf{x},\mathbf{p},\mathbf{p}'} \tilde{F}^{(2,0)}_{\text{int}}(\mathbf{p}+\mathbf{A}, \mathbf{p}'+\mathbf{A}) f(\mathbf{x},\mathbf{p}) f(\mathbf{x},\mathbf{p}') \\ &+ \int_{\mathbf{x},\mathbf{p},\mathbf{p}'} \tilde{\mathbf{F}}^{(2,1)}_{\text{int}}(\mathbf{p}+\mathbf{A}, \mathbf{p}'+\mathbf{A}) \cdot D_{\mathbf{x}} f(\mathbf{x},\mathbf{p}) f(\mathbf{x},\mathbf{p}') + \cdots\,.\end{aligned} \tag{2.2.65}$$

The interacting, gauge invariant action is finally

$$S = \int dt \left\langle f_0, U^{-1}[\partial_t - A_0] U \right\rangle - \int dt\; \mathscr{H}_A[f]\,. \tag{2.2.66}$$

In the case of free fermions, the equation of motion obtained from the gauge invariant action is the gauged Boltzmann equation. One can see this as follows: vary first the action using $U \to \exp \delta\alpha \cdot U$:

$$\delta S = -\int dt\;\; \langle \partial_t f + \{f, \epsilon(\mathbf{p}+\mathbf{A}) + A_0\}, \delta\alpha\rangle + \mathcal{O}(\delta\alpha^2)\,, \tag{2.2.67}$$

which leads to the equation of motion

$$\partial_t f + \{f, \epsilon(\mathbf{p}+\mathbf{A}) + A_0\} = 0\,. \tag{2.2.68}$$

Expanding the Poisson bracket, we find that the equation of motion takes the form

$$\partial_t f + \mathbf{v}_{\mathbf{p}} \cdot \nabla_{\mathbf{x}} f - v^i_{\mathbf{p}} \partial_j A_i \partial^j_{\mathbf{p}} f + \nabla_{\mathbf{x}} A_0 \cdot \nabla_{\mathbf{p}} f = 0\,, \tag{2.2.69}$$

where we have defined $\mathbf{v}_{\mathbf{p}} = \nabla_{\mathbf{p}} \epsilon(\mathbf{p} + \mathbf{A})$.

This is not yet the Boltzmann equation, because it involves the canonical momentum $\mathbf{p}$ instead of the gauge-invariant one $\mathbf{p} + \mathbf{A}$. In order to reduce it to the usual form, recall that the distribution that goes into the gauged Boltzmann equation is gauge invariant. Our distribution f, however, isn't, since it transforms nontrivially under canonical transformations $W = \exp \lambda$ with $\lambda(t, \mathbf{x})$ independent of $\mathbf{p}$. This explains the explicit appearance of the gauge field in the above equation–the equation is invariant under the simultaneous transformation $f \to \mathrm{Ad}^*_W f$ and $A_\mu \to A_\mu + \partial_\mu \lambda$, but not under either of those separately.

A gauge invariant distribution is obtained from $f(t, \mathbf{x}, \mathbf{p})$ via a field redefinition. Defining $\mathbf{k} = \mathbf{p} + \mathbf{A}$ as the gauge invariant momentum, the gauge invariant distribution, which we will denote by f_A, is defined as

$$f_A(t, \mathbf{x}, \mathbf{k}) = f(t, \mathbf{x}, \mathbf{k} - \mathbf{A}(t, \mathbf{x})). \tag{2.2.70}$$

Equation (2.2.69) implies that the equation of motion for f_A is

$$\partial_t f_A + \mathbf{v}_{\mathbf{k}} \cdot \nabla_{\mathbf{x}} f_A + \left(\mathbf{E} \cdot \nabla_{\mathbf{k}} + F_{ij} v^i_{\mathbf{k}} \partial^j_{\mathbf{k}} \right) f_A = 0, \tag{2.2.71}$$

where $\mathbf{v}_{\mathbf{k}} = \nabla_{\mathbf{k}} \epsilon(\mathbf{k})$ is the gauge invariant group velocity. The above is the usual form of the gauged Boltzmann equation, with the third term being the Lorentz form term in general dimension.

2.2.4 Linearized approximation

The action (2.2.52) provides a nonlinear effective field theory for Fermi liquids, in terms of a continuous family of bosonic fields $\phi(t, \mathbf{x}, \theta)$ labeled by a point on the Fermi surface θ. The main novelty of this work is to properly and systematically treat these nonlinearities. Before studying some of their consequences in Sec. 2.2.5, we show in this section that in the linear approximation, our approach reduces to the one commonly used in multidimensional bosonization Haldane [1994], Castro Neto and Fradkin [1994a], Houghton et al. [2000], and therefore reproduces the well-known linear response of Fermi liquids.

Linearized algebra of densities

One common starting point in multidimensional bosonization is the algebra of densities Castro Neto and Fradkin [1994b,a]

$$[\rho(\mathbf{x}, \theta), \rho(\mathbf{x}', \theta')] = -i\frac{p_F^{d-1}}{(2\pi)^d}\mathbf{n}_\theta \cdot \nabla\delta^d(\mathbf{x} - \mathbf{x}')\delta^{d-1}(\theta - \theta')\,. \tag{2.2.72}$$

where the square bracket is the usual commutator of operators. In $d = 1$ spatial dimension, it reduces to the familiar algebra of a Luttinger liquid or chiral boson, which are discussed in detail in Appendix 2.3.3. In higher dimensions $d > 1$, the dimensionful factor of p_F^{d-1} indicates that this algebra is a linearized approximation: a nonlinear theory of a Fermi surface would have a dynamical $p_F(\mathbf{x}, \theta)$, for the same reason that $\rho(\mathbf{x}, \theta)$ is dynamical. The c-number in the right-hand side of (2.2.72) is really the expectation value of a dynamical field. In this section, we show how the algebra (2.2.72) arises in our approach as a linearized approximation to the algebra $\mathfrak{g}$ of canonical transformations.

In the coadjoint orbit construction, densities are elements of the algebra $\rho(\bar{\mathbf{x}}, \bar{\theta}) \in \mathfrak{g}$.

Representing elements of $\mathfrak{g}$ as functions in phase space as in Sec. 2.2.2, the densities are

$$\rho(\bar{\mathbf{x}},\bar{\theta}) = \delta^d(\mathbf{x}-\bar{\mathbf{x}})\delta^{d-1}(\theta-\bar{\theta}) \in \mathfrak{g}\,. \tag{2.2.73}$$

It will be more useful to represent the densities as operators acting on the Hilbert space of the EFT. The density can be expanded in terms of the dynamical field ϕ by evaluating it in the state $f_\phi \equiv \mathrm{ad}^*_{e^\phi} f_0 \in \mathfrak{g}^*$:

$$\begin{aligned}\rho[\phi](\bar{\mathbf{x}},\bar{\theta}) \equiv \langle f_\phi, \rho(\bar{\mathbf{x}},\bar{\theta})\rangle &= \int \frac{d^d\mathbf{x}d^d\mathbf{p}}{(2\pi)^d} f_\phi(\mathbf{x},\mathbf{p})\delta^d(\mathbf{x}-\bar{\mathbf{x}})\delta^{d-1}(\theta-\bar{\theta}) \\ &= \int \frac{p^{d-1}dp}{(2\pi)^d} f_\phi(\bar{\mathbf{x}},p,\bar{\theta})\,.\end{aligned} \tag{2.2.74}$$

Expanding f_ϕ in terms of ϕ around a spherical Fermi surface f_0 as in Eq. (2.2.27), one finds

$$\rho[\phi](\mathbf{x},\theta) = \frac{p_F^d}{d(2\pi)^d} + \frac{p_F^{d-1}}{(2\pi)^d}\mathbf{n}_\theta\cdot\nabla\phi + \cdots \tag{2.2.75}$$

where the ellipses denote nonlinear terms $O(\phi^2)$.

The commutator of two densities, viewed as operators on the EFT Hilbert space, is inherited from their Lie bracket in $\mathfrak{g}$:

$$\begin{aligned}\left[\rho[\phi](\bar{\mathbf{x}},\bar{\theta}),\rho[\phi](\bar{\mathbf{x}}',\bar{\theta}')\right] &= i\langle f_\phi, \{\rho(\bar{\mathbf{x}},\bar{\theta}),\rho(\bar{\mathbf{x}}',\bar{\theta}')\}\rangle \\ &= i\int \frac{d^d\mathbf{x}d^d\mathbf{p}}{(2\pi)^d} f_\phi(\mathbf{x},\mathbf{p})\{\delta(\mathbf{x}-\bar{\mathbf{x}})\delta(\theta-\bar{\theta}),\delta(\mathbf{x}-\bar{\mathbf{x}}')\delta(\theta-\bar{\theta}')\} \\ &= i\nabla_{\bar{\mathbf{x}}}\delta^d(\bar{\mathbf{x}}-\bar{\mathbf{x}}')\cdot\int\frac{d^d\mathbf{p}}{(2\pi)^d}\nabla_{\mathbf{p}}f_\phi(\bar{\mathbf{x}},\mathbf{p})\delta^{d-1}(\theta-\bar{\theta})\delta^{d-1}(\theta-\bar{\theta}') \\ &\quad - i\delta^d(\bar{\mathbf{x}}-\bar{\mathbf{x}}')\int\frac{d^d\mathbf{p}}{(2\pi)^d}\nabla_{\bar{\mathbf{x}}}f_\phi(\bar{\mathbf{x}},\mathbf{p})\delta^{d-1}(\theta-\bar{\theta})\nabla_{\mathbf{p}}\delta^{d-1}(\theta-\bar{\theta}')\,,\end{aligned} \tag{2.2.76}$$

where in the last step we integrated by parts. Expanding the right-hand side in ϕ and keeping only the constant term $f_\phi(\bar{\mathbf{x}}, \mathbf{p}) \simeq f_0(p) = \Theta(p_F - p)$ leads to the linearized bosonization algebra, Eq. (2.2.72).

Gaussian action

The action (2.2.52) can be expanded in terms of the dynamical field ϕ, similar to how the densities were expanded above. Let us start with the Wess-Zumino-Witten term:

$$\begin{aligned} S_{\text{WZW}} &= \int dt \, \langle f_0, U^{-1}\partial_t U \rangle \\ &= \int dt \, \langle f_0, -\dot{\phi} + \frac{1}{2}\{\dot{\phi}, \phi\} + \cdots \rangle \, . \end{aligned} \tag{2.2.77}$$

The linear term is a total time derivative. The quadratic term can be computed by evaluating the Poisson bracket—after integration by parts it is

$$\begin{aligned} S_{\text{WZW}} &= \frac{1}{2}\int \frac{dt d^d\mathbf{x} d^d\mathbf{p}}{(2\pi)^d} \dot{\phi} \nabla_{\mathbf{x}}\phi \cdot \nabla_{\mathbf{p}} f_0(p) + \cdots \\ &= -\frac{p_F^{d-1}}{2}\int \frac{dt d^d\mathbf{x} d^{d-1}\theta}{(2\pi)^d} \, \dot{\phi} \, \mathbf{n}_\theta \cdot \nabla\phi + \cdots \, , \end{aligned} \tag{2.2.78}$$

where we again expanded around a spherical Fermi surface $f_0(x,p) = \Theta(p_F - p)$.

We now turn to the Hamiltonian part of the action (2.2.52). We start by only considering the term linear in f_ϕ, and then generalize. This term is

$$S_H = -\int dt \, \langle f_0, U^{-1}\epsilon U \rangle = -\int dt \, \langle f_0, \epsilon + \{\phi, \epsilon\} + \frac{1}{2}\{\phi, \{\phi, \epsilon\}\} + \cdots \rangle \, , \tag{2.2.79}$$

Here $\epsilon(p)$ is an arbitrary function—we will see that the coefficients of its Taylor expansion around the Fermi surface $\epsilon'(p_F)$, $\epsilon''(p_F)$, etc. will become Wilsonian coefficients in the EFT. For free fermions, $\epsilon(p)$ corresponds to the single particle dispersion relation. Let us study the terms in the expansion in (2.2.79): the $O(\phi^0)$ term is a constant contribution to the action which we ignore. The $O(\phi^1)$ term is a total spatial derivative, since

$$\{\phi, \epsilon\} = \nabla_{\mathbf{x}}\phi \cdot \nabla_{\mathbf{p}}\epsilon = \epsilon'(p)\mathbf{n}_\theta \cdot \nabla\phi \,. \tag{2.2.80}$$

The leading term is therefore the quadratic term, which may be written

$$S_H = \frac{1}{2}\int dt\, \langle\{\phi, f_0\}, \{\phi, \epsilon\}\rangle + \cdots \,. \tag{2.2.81}$$

Since

$$\{\phi, f_0\} = \nabla_{\mathbf{x}}\phi \cdot \nabla_{\mathbf{p}} f_0 = -\delta(p - p_F)\mathbf{n}_\theta \cdot \nabla\phi \,, \tag{2.2.82}$$

we obtain the quadratic term in S_H

$$S_H = -\frac{p_F^{d-1}}{2}\int \frac{dt d^d\mathbf{x} d^{d-1}\theta}{(2\pi)^d}\epsilon'(p_F)(\mathbf{n}_\theta \cdot \nabla\phi)^2 + \cdots \,. \tag{2.2.83}$$

Notice that only $v_F \equiv \epsilon'(p_F)$ enters the quadratic action.

Finally, let us generalize to include the remaining terms in the EFT (2.2.52). Since $\delta f_\phi = f_\phi - f_0$ is already linear in ϕ, only terms that are quadratic in δf_ϕ will contribute here; these are denoted by $F_{\text{int}}^{(2,n)}$ in (2.2.51). Furthermore, to leading order in gradients only $F_{\text{int}}^{(2,0)}$ contributes. Expanding again around a spherical Fermi surface one finds that the most general Gaussian action for a Fermi liquid to leading order in gradients is

$$S^{(2)} = -\frac{p_F^{d-1}}{2}\int \frac{dt d^d\mathbf{x} d^{d-1}\theta}{(2\pi)^d}\, \mathbf{n}_\theta \cdot \nabla\phi \left(\dot{\phi} + v_F \mathbf{n}_\theta \cdot \nabla\phi + v_F \int d^{d-1}\theta' F_{\text{int}}^{(2,0)}(\theta, \theta')\, \mathbf{n}_{\theta'} \cdot \nabla\phi'\right) \,, \tag{2.2.84}$$

with $\phi = \phi(t, \mathbf{x}, \theta)$ and $\phi' = \phi(t, \mathbf{x}, \theta')$. The interaction term in (2.2.51) has been rescaled to be dimensionless, and is evaluated at the Fermi surface: $F_{\text{int}}^{(2,0)}(\theta, \theta') \equiv F_{\text{int}}^{(2,0)}(\mathbf{p}_F(\theta), \mathbf{p}_F(\theta'))$; these are the usual Landau parameters. This action first appeared in Haldane [1994], and has been widely used since, see e.g. Houghton et al. [2000].

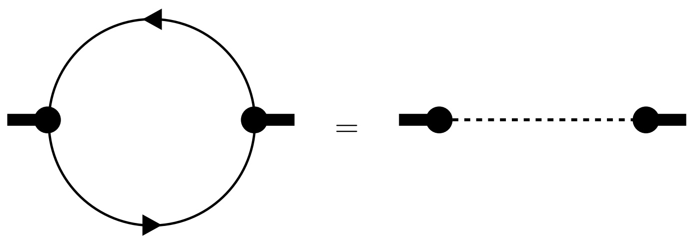

Figure 2.2: The density two-point function, which involves a loop in the fermion description, is captured by a tree level diagram in the boson description.

Landau damping

One computational advantage of a bosonized description is that fermion loops are reproduced by tree diagrams in terms of the boson field. A simple observable that illustrates this is the density two-point function (Fig. 2.2). To linear order in ϕ, the density operator can be obtained from (2.2.75) and is

$$\rho(t,\mathbf{x}) = \frac{p_F^{d-1}}{(2\pi)^d}\int d^{d-1}\theta\,\mathbf{n}_\theta\cdot\nabla\phi(t,\mathbf{x},\theta)+\cdots\,. \tag{2.2.85}$$

Using the scalar two-point function from (2.2.84) (we are setting the Landau parameters to zero for simplicity)

$$\langle\phi\phi'\rangle(\omega,\mathbf{q}) = i\frac{(2\pi)^d}{p_F^{d-1}}\frac{\delta^{d-1}(\theta-\theta')}{\mathbf{n}_\theta\cdot\mathbf{q}(\omega-v_F\mathbf{n}_\theta\cdot\mathbf{q})} \tag{2.2.86}$$

one therefore finds that the density two-point function is

$$\begin{aligned}\langle\rho\rho\rangle(\omega,q) &= i\frac{p_F^{d-1}}{(2\pi)^d}\frac{1}{v_F}\int d^{d-1}\theta\frac{\cos\theta_1}{-\frac{\omega}{v_F|q|}+\cos\theta_1}\\ &= i\frac{p_F^{d-1}}{(2\pi)^d}\frac{1}{v_F}\frac{\pi^{d/2}}{\Gamma(d/2)}\frac{2-\delta_{d,1}}{1+|s|}\left({}_2F_1(1,\tfrac{d+1}{2},d,\tfrac{2}{1+|s|})-{}_2F_1(1,\tfrac{d-1}{2},d-1,\tfrac{2}{1+|s|})\right)\,,\end{aligned} \tag{2.2.87}$$

where ${}_2F_1$ is a hypergeometric function [3] and $s \equiv \frac{\omega}{v_F q}$, with $q = |\mathbf{q}|$. The $d+1$ loop integrals in the fermion description (one frequency integral and d momentum integrals) have been replaced by $d-1$ integrals over the boson 'species', parametrized by the angles θ_i. In $d = 1$ this reduces to the density two-point function of a Luttinger liquid

$$\langle \rho\rho \rangle(\omega, q) = i\frac{1}{\pi}\frac{v_F q^2}{-\omega^2 + v_F^2 q^2} \,. \tag{2.2.88}$$

In higher dimensions $d > 1$ Eq. (2.2.87) has a branch cut for $|\omega| < v_F q$ due to the particle-hole continuum. One recovers well-known expressions in $d = 2$ spatial dimensions,

$$\langle \rho\rho \rangle(\omega, q) = i\frac{p_F}{2\pi}\frac{1}{v_F}\left(1 - \frac{|\omega|}{\sqrt{\omega^2 - v_F^2 q^2}}\right) \,, \tag{2.2.89}$$

and in $d = 3$

$$\langle \rho\rho \rangle(\omega, q) = i\frac{p_F^2}{2\pi^2}\frac{1}{v_F}\left(1 + \frac{1}{2}\frac{|\omega|}{v_F q}\log\frac{|\omega| - v_F q}{|\omega| + v_F q}\right) \,, \tag{2.2.90}$$

see, e.g., Ref. Chubukov and Maslov [2003].

2.2.5 Nonlinear response

A conspicuous aspect of the EFT (2.2.52) is its unavoidable nonlinear structure coming from the WZW term. While nonlinearities in the action also arise from the Hamiltonian (2.2.51), e.g. through nonlinear terms in the dispersion $\epsilon(p)$ familiar from one-dimensional bosonization, the nonlinearities in the WZW term have no counterpart in $d = 1$; they are tied to the geometry and, in particular, the curvature of the Fermi surface.

That such terms are necessary to reproduce even free fermion physics in $d > 1$, but not in $d = 1$, can be anticipated as follows: in $d = 1$, cancellations in fermion loops Dzyaloshinskii

3. The nonanalytic part of $\langle \rho\rho \rangle(\omega, q)/(ip_F^{d-1})$ can be simplified to $\frac{is(1-s^2)^{\frac{d-3}{2}}}{2^{d-1}\pi^{\frac{d-1}{2}}\Gamma(\frac{d-1}{2})}$ for d even Castellani et al. [1994], and $\frac{s(1-s^2)^{\frac{d-3}{2}}}{2^{d-1}\pi^{\frac{d+1}{2}}\Gamma(\frac{d-1}{2})}\log\frac{s+1}{s-1}$ for d odd.

and Larkin [1974] lead to the vanishing of connected density n-point functions with $n > 2$ for linearly dispersing fermions, making possible the representation of a relativistic fermion as a free boson. Nonlinear response only arises when the fermions have nonlinear dispersion relations $\epsilon(p)$; this introduces interactions in the bosonic description (see App. 2.3.3). In higher dimensions, these cancellations are only approximate, and even linearly dispersing fermions exhibit connected density higher-point functions.

The approximate cancellations in fermion loops in $d > 1$ render scaling analyses of Fermi liquids difficult Kim et al. [1994], Metzner et al. [1997]. In Sec. 2.2.5, we show how the correct scaling of density n-point functions is immediately captured in our approach. Next, as a quantitative check of the nonlinearities in the EFT, we compute the density three-point function by expanding the action up to cubic order in the field ϕ—the diagrams contributing are shown in Fig. 2.3.

The connection between non-vanishing of fermion loops and nonlinearities in the bosonized description was anticipated in Ref. Kopietz et al. [1995]. In the approach followed there, the two are tied because an effective action for a bosonic degree of freedom is obtained by coupling the fermion densities to Hubbard-Stratonovich fields and integrating the fermions out; a drawback of that approach is that the resulting effective action need not be local, so that a systematic generalization of the form (2.2.51) is not possible and one is limited to studying systems for which the fermion loop can be evaluated directly. In contrast, we derive a local effective action (2.2.52) from general principles and show that it correctly reproduces nonlinear response.

General scaling of n-point functions

The action (2.2.52) has the following schematic expansion in fields

$$S \sim p_F^{d-1} \int_{t,\mathbf{x},\theta} \dot{\phi} \left(\nabla\phi + \frac{1}{p_F}(\nabla\phi)^2 + \frac{1}{p_F^2}(\nabla\phi)^3 + \cdots \right) + v_F \nabla\phi \left(\nabla\phi + \frac{1}{p_F}(\nabla\phi)^2 + \cdots \right), \tag{2.2.91}$$

where the two terms come from the expansion of the WZW term and Hamiltonian respectively. We have set all dimensionful parameters to v_F or p_F and dropped $O(1)$ numerical factors for the purposes of this section. Some terms may also involve derivatives with respect to the angles θ (see Eq. (2.2.107) for an example), or nonlocal terms in θ like the Landau parameters in Eq. (2.2.84), but these will not affect the scaling argument below. Finally, we have dropped higher gradient corrections (such as $F_{\text{int}}^{(2,1)}$ in (2.2.51)); these will only give q/p_F-suppressed corrections to observables, whereas the nonlinear terms in (2.2.91) give the leading contribution to certain nonlinear observables. For the purposes of scaling it is useful to define a canonically normalized field as $\phi_c \equiv \phi/p_F^{(d-1)/2}$ so that

$$S \sim \int_{t,\mathbf{x},\theta} \dot{\phi}_c \left(\nabla\phi_c + \frac{(\nabla\phi_c)^2}{p_F^{(d+1)/2}} + \frac{(\nabla\phi_c)^3}{p_F^{d+1}} + \cdots \right) + v_F \nabla\phi_c \left(\nabla\phi_c + \frac{(\nabla\phi_c)^2}{p_F^{(d+1)/2}} + \cdots \right). \tag{2.2.92}$$

Like any EFT, S is an expansion around a Gaussian theory in terms of irrelevant operators suppressed by the UV cutoff, here p_F. The density operator similarly has an expansion

$$\rho \sim p_F^{(d-1)/2} \int_\theta \nabla\phi_c + \frac{(\nabla\phi_c)^2}{p_F^{(d+1)/2}} + \frac{(\nabla\phi_c)^3}{p_F^{d+1}} + \cdots . \tag{2.2.93}$$

The scalar propagator has the form

$$\langle \phi_c \phi_c' \rangle(\omega, \mathbf{q}) \sim \frac{\delta^{d-1}(\theta - \theta')}{\mathbf{n}_\theta \cdot \mathbf{q}(\omega - v_F \mathbf{n}_\theta \cdot \mathbf{q})}. \tag{2.2.94}$$

Several diagrams contribute to the n-point function at tree level (see Fig. 2.3 for $n = 3$); using Eqs. (2.2.92–2.2.94) one finds that they all scale as

$$\langle \rho(\omega_1, \mathbf{q}_1)\rho(\omega_2, \mathbf{q}_2) \cdots \rho(\omega_n, \mathbf{q}_n)\rangle = \frac{p_F^{d+1-n}}{v_F^{n-1}} g_n(\{\omega_i/\omega_j, v_F \mathbf{q}_i/\omega_j\}) \,, \tag{2.2.95}$$

where g_n is a function of dimensionless ratios of frequencies and momenta (we have removed the momentum conserving delta-function on the right-hand side). This result holds to leading order in q_i/p_F and $\omega_i/(v_F p_F)$; higher-order terms will be sensitive to higher-gradient corrections in the EFT. The scaling agrees with the two-point function ($n = 2$) found earlier (2.2.87).

Note that this scaling is also transparent from the kinetic theory approach to computing nonlinear response, discussed in App. 2.3.7. In contrast, this scaling is highly non-obvious from a fermionic approach: taking fermionic propagators $\langle \psi\psi^\dagger\rangle \sim \frac{1}{\omega - v_F q_\parallel}$ and only scaling momentum towards the Fermi surface $\omega \sim q_\parallel$, with $q_\perp \sim 1$, the fermion loop is estimated as

$$\langle \rho(\omega_1, \mathbf{q}_1)\rho(\omega_2, \mathbf{q}_2) \cdots \rho\rangle \sim \int d\omega dq_\parallel d^{d-1}q_\perp \langle \psi\psi^\dagger\rangle^n \sim \frac{p_F^{d-1}}{q_\parallel^{n-2}} \,. \tag{2.2.96}$$

This only agrees with (2.2.95) for $n = 2$. For $n = 3$, the fact that (2.2.96) overestimates the 3-point function by a factor of $\sim 1/q$ comes from the approximate cancellations in fermion loops after antisymmetrization of external legs, see, e.g., Ref. Metzner et al. [1997]. Our scaling result (2.2.95) shows more generally that n-point functions have $n - 2$ such cancellations.

Finally, the structure of the EFT makes it clear that, for free fermions, the dimensionless function g_n only depends on the first $n-1$ derivatives of the dispersion relation $\epsilon(p)$ evaluated at the Fermi surface:

$$g_n = g_n\left(\epsilon'(p_F), \epsilon''(p_F), \cdots, \epsilon^{(n-1)}(p_F)\right) \,. \tag{2.2.97}$$

This follows from the fact that terms in the Hamiltonian that involve n powers of ϕ have at most $n-1$ derivatives on $\epsilon(p)$ (the case $n=3$ is treated in detail below). For interacting Fermi liquids, the n-point function will depend on the additional Landau-like parameters discussed in Sec. 2.2.3.

Cubic action

Let us expand the action (2.2.46) up to cubic order in the field ϕ. The WZW term is

$$S_{\mathrm{WZW}} = \int dt\, \langle f_0, U^{-1}\partial_t U\rangle = \int dt\, \langle f_0, \frac{1}{2}\{\dot{\phi}, \phi\} - \frac{1}{3!}\{\{\dot{\phi}, \phi\}, \phi\} + \cdots \rangle\,, \tag{2.2.98}$$

where we dropped the constant piece. The quadratic term was computed in Sec. 2.2.4; the cubic term is

$$S^{(3)}_{\mathrm{WZW}} = -\frac{1}{3!}\int dt\, \langle \{\phi, f_0\}, \{\dot{\phi}, \phi\}\rangle. \tag{2.2.99}$$

Now

$$\{\phi, f_0\} = \nabla_{\mathbf{x}}\phi \cdot \nabla_{\mathbf{p}} f_0 = -\delta(p-p_F)\mathbf{n}_\theta \cdot \nabla\phi\,, \tag{2.2.100}$$

and

$$\begin{aligned}\{\dot{\phi}, \phi\} &= \nabla_{\mathbf{x}}\dot{\phi}\cdot\nabla_{\mathbf{p}}\phi - \nabla_{\mathbf{x}}\phi\cdot\nabla_{\mathbf{p}}\dot{\phi} \\ &= \frac{1}{p}\mathbf{s}^i_\theta\cdot\nabla\dot{\phi}\partial_{\theta^i}\phi - \frac{1}{p}\mathbf{s}^i_\theta\cdot\nabla\phi\partial_{\theta^i}\dot{\phi}\,,\end{aligned} \tag{2.2.101}$$

so that

$$S^{(3)}_{\mathrm{WZW}} = -\frac{p_F^{d-1}}{3!}\int_{t,\mathbf{x},\theta}\frac{1}{p_F}\mathbf{n}_\theta\cdot\nabla\phi\left(\mathbf{s}^i_\theta\cdot\nabla\phi\partial_{\theta^i}\dot{\phi} - \mathbf{s}^i_\theta\cdot\nabla\dot{\phi}\partial_{\theta^i}\phi\right)\,, \tag{2.2.102}$$

with $\int_{t,\mathbf{x},\theta} \equiv \int \frac{dt d^d\mathbf{x} d^{d-1}\theta}{(2\pi)^d}$. The $\mathbf{s}^i_\theta$, with $i=1,\ldots,d-1$, are $d-1$ unit vectors that are tangent to the Fermi surface. For example, parametrizing the sphere S_d with $\theta_1,\ldots,\theta_{d-2}\in$

$[0,\pi]$, $\theta_{d-1} \in [0,2\pi]$, and $\mathbf{n}_\theta = (\cos\theta_1, \ldots, \sin\theta_1 \cdots \sin\theta_{n-2}\cos\theta_{n-1}, \sin\theta_1 \cdots \sin\theta_{n-1})$, one can choose $\mathbf{s}^i_\theta = \frac{1}{\sin\theta_1 \cdots \sin\theta_{i-1}} \partial_{\theta^i} \mathbf{n}_\theta$. In these coordinates, the Jacobian is $d^{d-1}\theta = \sin^{d-2}\theta_1 \cdots \sin\theta_{d-2} d\theta_1 \cdots d\theta_{d-1}$. For $d = 2$ one simply has $\mathbf{n}_\theta = \binom{\cos\theta}{\sin\theta}$ and $\mathbf{s}_\theta = \binom{-\sin\theta}{\cos\theta}$.

We turn to the Hamiltonian term in (2.2.46) :

$$S_H = -\int dt\, \langle f_0, U^{-1}\epsilon U\rangle = -\int dt\, \langle f_0, \frac{1}{2}\{\phi,\{\phi,\epsilon\}\} + \frac{1}{3!}\{\phi,\{\phi,\{\phi,\epsilon\}\}\} + \cdots\rangle\,, \quad (2.2.103)$$

where we again dropped a constant contribution to the action. The cubic term can be written as

$$S_H^{(3)} = \frac{1}{3!}\int dt\, \langle \{\phi, f_0\}, \{\phi,\{\phi,\epsilon\}\}\rangle\,. \quad (2.2.104)$$

Using Eq. (2.2.100) and

$$\{\phi,\{\phi,\epsilon\}\} = \{\phi, \epsilon'\mathbf{n}_\theta \cdot \nabla\phi\} = \nabla_{\mathbf{x}}\phi \cdot \nabla_{\mathbf{p}}(\epsilon'\mathbf{n}_\theta \cdot \nabla\phi) - \nabla_{\mathbf{p}}\phi \cdot \nabla_{\mathbf{x}}(\epsilon'\mathbf{n}_\theta \cdot \nabla\phi), \quad (2.2.105)$$

one finds, after several integrations by parts,

$$S_H^{(3)} = -\frac{p_F^{d-1}}{3!}\int_{t,\mathbf{x},\theta} \frac{1}{p_F}\left(\frac{d-1}{2}\epsilon' + \epsilon'' p_F\right)(\mathbf{n}_\theta \cdot \nabla\phi)^3\,. \quad (2.2.106)$$

Collecting these results, the full action up to cubic order is

$$\begin{aligned}
S &= S_{\mathrm{WZW}} + S_H\,, \qquad (2.2.107)\\
S_{\mathrm{WZW}} &= -p_F^{d-1}\int_{t,\mathbf{x},\theta} \frac{1}{2}\dot\phi(\mathbf{n}_\theta \cdot \nabla\phi) + \frac{1}{3!}\frac{1}{p_F}(\mathbf{n}_\theta \cdot \nabla\phi)\left(\mathbf{s}^i_\theta \cdot \nabla\phi\partial_{\theta^i}\dot\phi - \mathbf{s}^i_\theta \cdot \nabla\dot\phi\partial_{\theta^i}\phi\right) + \cdots\,,\\
S_H &= -p_F^{d-1}\int_{t,\mathbf{x},\theta} \frac{1}{2}\epsilon'(\mathbf{n}_\theta \cdot \nabla\phi)^2 + \frac{1}{3!}\frac{1}{p_F}\left(\frac{d-1}{2}\epsilon' + \epsilon'' p_F\right)(\mathbf{n}_\theta \cdot \nabla\phi)^3 + \cdots\,.
\end{aligned}$$

The density can be similarly expanded. After removing the constant piece $\rho = \frac{p_F^d}{(4\pi)^{d/2}\Gamma(1+\frac{d}{2})} +$

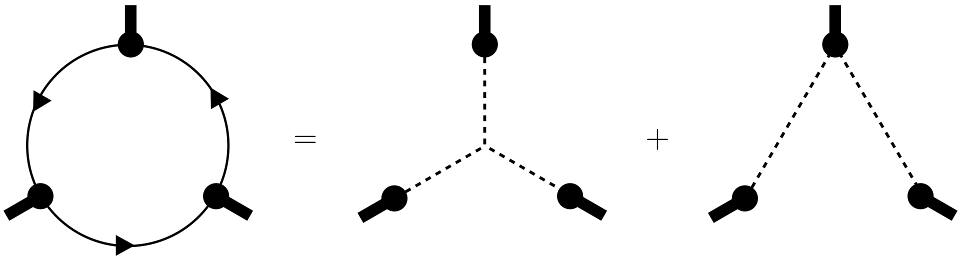

Figure 2.3: The density three-point function in the fermionic and bosonic descriptions.

$\delta\rho$ (see Eq. (2.2.75)), one finds

$$\begin{aligned} \delta\rho &= \int \frac{d^d\mathbf{p}}{(2\pi)^d}\delta f \equiv \nabla\cdot\mathbf{d}\,, \\ \mathbf{d} &= \frac{p_F^{d-1}}{(2\pi)^d}\int d^{d-1}\theta\, \mathbf{n}_\theta\phi + \frac{1}{p_F}\frac{1}{2}\mathbf{s}^i_\theta\partial_{\theta^i}\phi(\mathbf{n}_\theta\cdot\nabla\phi) + \cdots\,. \end{aligned} \tag{2.2.108}$$

Notice that $\delta\rho$ can be written as a total divergence $\delta\rho = \nabla\cdot\mathbf{d}$ as above to all orders in ϕ, since

$$\delta Q = \int_{\mathbf{x}}\delta\rho = \int_{\mathbf{x},\mathbf{p}} f - f_0 = 0\,, \tag{2.2.109}$$

where the last step follows from the fact that the phase space integral of a Poisson bracket vanishes after integration by parts, $\int_{\mathbf{x},\mathbf{p}}\{A,B\} = 0$.

Density three-point function

The cubic action (2.2.107) can be used to obtain the density three-point function. The scalar propagator is

$$\langle\phi_\theta\phi_{\theta'}\rangle(\omega,\mathbf{q}) = i\frac{(2\pi)^d}{p_F^{d-1}}\frac{1}{q_n}\frac{1}{\omega - v_F q_n}\delta^{d-1}(\theta-\theta')\,, \tag{2.2.110}$$

where we use the shorthand notation $q_n = \mathbf{n}_\theta\cdot q$. We will often write $p = (\omega,\mathbf{q})$ below.

The $S_H^{(3)}$ piece — Start by considering the 'star' diagram coming from the a single insertion of the cubic Hamiltonian term in (2.2.107) $iS_H^{(3)}$:

$$\begin{aligned}\langle\rho(p)\rho(p')\rho\rangle_{S_H^{(3)}} &= -i\left(\frac{p_F^{d-1}}{(2\pi)^d}\right)^3 q_n q'_n (q+q')_n \langle (iS_H^{(3)})\phi(p)\phi(p')\phi\rangle \\ &= -\frac{p_F^{d-2}}{(2\pi)^d}\left(\epsilon'' p_F + \frac{d-1}{2}v_F\right)\int d^{d-1}\theta \frac{q_n}{\omega - v_F q_n}\frac{q'_n}{\omega' - v_F q'_n}\frac{(q+q')_n}{\omega+\omega' - v_F(q+q')_n}\,.\end{aligned} \tag{2.2.111}$$

This is the only contribution proportional to ϵ''.

The $\rho^{(2)}$ piece — Let us now consider the 'triangle' contributions – these come from the nonlinear part of the density $\rho^{(2)}$ in (2.2.108). This diagram is given by

$$\begin{aligned}&\langle\rho(p)\rho(p')\rho\rangle_{\rho^{(2)}} \\ &= \left(\frac{p_F^{d-1}}{2\pi}\right)^3 \frac{1}{2p_F}\int d\theta_{1,2,3}\,(iq_{n_1})(iq'_{n_2})(-i(q+q')_{s_3^i})\left[-iq_{n_3}\langle\phi_3\phi_1\rangle(p)\partial_{\theta_3^i}\langle\phi_3\phi_2\rangle(p')\right] + 5\text{ perm} \\ &= -\frac{p_F^{d-2}}{2(2\pi)^d}\int d^{d-1}\theta\,\frac{q_n(q+q')_{s^i}}{\omega - v_F q_n}\partial_{\theta^i}\left(\frac{1}{\omega' - v_F q'_n}\right) + 5\text{ perm}\,.\end{aligned} \tag{2.2.112}$$

The total of six permutations are obtained by rotating the diagram by $2\pi/3$ and $4\pi/3$, i.e. sending $\{p \to p', p' \to -p - p'\}$ once and twice respectively, and then for each three terms, adding the same term with $p \leftrightarrow p'$.

The $S^{(3)}_{\rm WZW}$ piece — Finally, the star diagram with the WZW cubic vertex gives:

$$
\begin{aligned}
&\langle\rho(p)\rho(p')\rho\rangle_{S^{(3)}_{\rm WZW}} = i\left(\frac{p_F^{d-1}}{(2\pi)^d}\right)^3 q_n q'_n(q+q')_n \int_{\theta_{1,2,3}} \langle (iS^{(3)}_{\rm WZW})\phi_1(p)\phi_2(p')\phi_3\rangle \qquad (2.2.113)\\
&= i\frac{p_F^{d-2}}{3!}\left(\frac{p_F^{d-1}}{(2\pi)^d}\right)^3 \int_{\theta,\theta_{1,2,3}} q_n^2 q'_n q'_{si}(q+q')_n(\omega+2\omega')\langle\phi_1\phi\rangle(p)\langle\phi_2\phi\rangle(p')\\
&\qquad\qquad\qquad\qquad \cdot\partial_{\theta^i}\langle\phi_3\phi\rangle(-p-p') + 5\text{ perm.}\\
&= \frac{p_F^{d-2}}{3!(2\pi)^d}\int d^{d-1}\theta \frac{q_n}{\omega - vq_n}\frac{q'_{si}}{\omega'-vq'_n}\partial_{\theta^i}\frac{\omega+2\omega'}{\omega+\omega'-v(q+q')_n} + 5\text{ perm.}
\end{aligned}
$$

Comparison to kinetic theory —

Density n-point functions for free fermions were computed in Refs. Feldman et al. [1998], Neumayr and Metzner [1998] by directly evaluating the fermion loop integral, for the special case of a dispersion relation $\epsilon(p) = p^2/2m$, i.e. $\epsilon'' = p_F\epsilon'$ in the formulas above. This approach features substantial cancellations upon symmetrizing over diagrams, after which the scaling (2.2.95) is obtained Neumayr and Metzner [1998], Kopper and Magnen [2001]. To compare our results for a general dispersion relation $\epsilon(p)$, we computed instead the general three-point function using kinetic theory in App. 2.3.7. The piece proportional to ϵ'' is simplest to compare, see Eqs. (2.2.111) and (2.3.113). The remaining part is more difficult to compare, but can be also shown to match, namely:

$$
(2.2.111) + (2.2.112) + (2.2.113) = (2.3.113)\,. \qquad (2.2.114)
$$

2.2.6 Further applications and extensions

An alternative approach to NFL

The nonlinear effective field theory (2.2.46) provides an alternative formulation of Fermi liquids, and therefore offers a new starting point to study deformations of Fermi liquids by relevant interactions. In analogy with the solution to the Schwinger model from bosonization Zinn-Justin [2021], one may expect that nonlinear bosonization in higher dimensions simplifies the study of Fermi liquids coupled to a gapless boson. This possibility was already explored in the early days of multidimensional bosonization, in particular in Refs. Kwon et al. [1994], Khveshchenko [1995], Lawler et al. [2006], using the Gaussian approximation to the Fermi liquid EFT (2.2.84); as will be reviewed below, in this approximation one finds that the free bosonized description sums a class of diagrams in the fermion description corresponding to the RPA approximation, leading in particular to the dynamic critical exponent $z = 3$. The important cancellations in fermion loops discussed in Sec. 2.2.5 suppress corrections to the RPA approximation, pointing to the advantage of the bosonized approach to address non-Fermi liquids (NFLs) [4].

Non-Fermi liquid in the Gaussian approximation— Let us couple the Fermi liquid EFT, in the Gaussian approximation (2.2.84), to a bosonic field $\Phi(t, x)$; setting $d = 2$ and turning off Landau parameters for simplicity one has

$$\mathcal{L} = -\left[\frac{p_F}{8\pi^2}\int_\theta \mathbf{n}_\theta \cdot \nabla\phi(\dot{\phi} + v_F \mathbf{n}_\theta \cdot \nabla\phi)\right] - \left[\frac{1}{2}\nabla\Phi^2 + \frac{1}{2}k_o^2\Phi^2\right] + \left[\lambda\Phi\frac{p_F}{4\pi^2}\int_\theta(\mathbf{n}_\theta \cdot \nabla)\phi\right]. \tag{2.2.115}$$

An irrelevant kinetic term $(\partial_t\Phi)^2$ was omitted. The bare mass k_o will be tuned to make the field Φ gapless—this field could be an emergent gauge field, or an order parameter tuned to criticality. We have assumed for simplicity that it couples to the fermion density

4. For this same reason, RPA propagators are often used in the patch theory approach to NFLs Lee [2018]

$\rho \simeq \frac{p_F}{4\pi^2}\int d\theta\, \mathbf{n}_\theta \cdot \nabla\phi$, but one can straightforwardly generalize: for example a field coupling to the spin-ℓ harmonic of the Fermi surface would instead have

$$\mathcal{L}_{\text{int}} = \lambda\Phi \int d\theta\, e^{i\ell\theta}\mathbf{n}_\theta \cdot \nabla\phi + \text{c.c.}\ . \tag{2.2.116}$$

Since the entire theory (2.2.115) is Gaussian, correlators can be readily obtained; the Φ propagator is given by

$$\langle\Phi\Phi\rangle(\omega, q) = \frac{i}{q^2 + k_o^2 - i\lambda^2\langle\rho\rho\rangle_0(\omega, q)}\,, \tag{2.2.117}$$

with the bare density two-point function $\langle\rho\rho\rangle_0$ given by (2.2.89). In the limit $\omega \ll q$, this expression is only singular as $\omega, q \to 0$ if one tunes the bare boson mass to

$$k_o^2 = -\frac{p_F}{2\pi v_F}\lambda^2\,. \tag{2.2.118}$$

The boson correlator then becomes

$$\langle\Phi\Phi\rangle(\omega, q) \simeq \frac{1}{q^2 - i\frac{p_F\lambda^2}{2\pi v_F^2}\frac{|\omega|}{|q|}}\,, \qquad (\omega \ll v_F q) \tag{2.2.119}$$

and produces $z = 3$ from Landau damping as anticipated.

Thermodynamic properties— The Gaussian theory (2.2.115) has a specific heat $c_V \propto T^{1/z} = T^{2/3}$ [5]. This can be seen by computing the thermal partition function

$$Z(\beta) = \int D\phi D\Phi\, e^{-S_E}\,, \tag{2.2.120}$$

5. We thank Aavishkar Patel and Ilya Esterlis for discussions on how the $T^{2/3}$ specific heat arises in a large-N model of non-Fermi liquids Esterlis et al. [2021].

where Euclidean action can be obtained from (2.2.115) by rotating $t = -i\tau$ with $\tau \in [0, \beta]$. The partition function can be evaluated by first integrating over ϕ, then Φ:

$$\begin{aligned} Z &= \det\left[\mathbf{n}_\theta \cdot q(-i\omega_n + v_F \mathbf{n}_\theta \cdot q)\right]^{-1/2} \int D\Phi\, e^{-\sum_n \int_q \frac{1}{2}\{q^2 + k_o^2 - i\lambda^2 \langle\rho\rho\rangle_0(i\omega_n, q)\}|\Phi_{q,n}|^2} \\ &= \det\left[\mathbf{n}_\theta \cdot q(-i\omega_n + v_F \mathbf{n}_\theta \cdot q)\right]^{-1/2} \det\left[q^2 + \tilde{\lambda}^2 \frac{|\omega_n|}{\sqrt{\omega_n^2/v_F^2 + q^2}}\right]^{-1/2}, \end{aligned} \tag{2.2.121}$$

where $\omega_n = 2\pi T n$ are Matsubara frequencies and $\tilde{\lambda}^2 \equiv \frac{p_F}{2\pi v_F^2}\lambda^2$. The free energy, or pressure, is therefore a sum of a Fermi liquid contribution and a Landau-damped boson contribution

$$\begin{aligned} P = \frac{T}{V}\log Z = &-\frac{1}{2}T\sum_n \int_q \int_\theta \log\left[\mathbf{n}_\theta \cdot q(-i\omega_n + v_F \mathbf{n}_\theta \cdot q)\right] \\ &-\frac{1}{2}T\sum_n \int_q \log\left[q^2 + \tilde{\lambda}^2 \frac{|\omega_n|}{\sqrt{\omega_n^2/v_F^2 + q^2}}\right]. \end{aligned} \tag{2.2.122}$$

The Fermi liquid free energy will be discussed further in Sec. 2.2.9; here we focus on the contribution from the Landau-damped boson, which dominates at low temperatures. The integrals and Matsubara sum are dominated at low temperatures by the region where $v_F q \gg \omega_n \sim q^3$. The integral over q can therefore be simplified to

$$\int_q \log\left(q^2 + \tilde{\lambda}^2 \frac{|\omega_n|}{q}\right) = \tilde{\lambda}^{4/3} \frac{|\omega_n|^{2/3}}{2\sqrt{3}}, \tag{2.2.123}$$

where a temperature-independent UV divergence was dropped. The Matsubara sum is also divergent. Cutting it off at $n \leq N$ and using

$$\sum_{n=0}^{N} n^{2/3} = H_{N,-\frac{2}{3}} = \frac{3}{5}N^{5/3} + \frac{1}{2}N^{2/3} + \zeta(-\tfrac{2}{3}) + O(1/N^{1/3}), \tag{2.2.124}$$

one obtains a UV divergent contribution to the zero temperature pressure and entropy density; removing these one is left with a finite thermal piece leading to the expected NFL specific heat (note that $\zeta(-\frac{2}{3}) < 0$)

$$P = -\frac{\zeta(-\frac{2}{3})}{4\sqrt{3}}\tilde{\lambda}^{4/3}T^{5/3} \quad \Rightarrow \quad T\frac{ds}{dT} = T\frac{d^2P}{dT^2} = -\frac{5\zeta(-\frac{2}{3})}{18\sqrt{3}}\tilde{\lambda}^{4/3}T^{2/3}\,. \tag{2.2.125}$$

Note that this NFL contribution to the specific heat may be difficult to observe in models where an instability— superconducting or other—arises at a similar scale as NFL fluctuations Grossman et al. [2021].

Nonlinear terms— Within the perturbative expansion in the fermionic patch theory description Lee [2009], Metlitski and Sachdev [2010], the dynamic critical exponent of a NFL is expected to deviate from the RPA value $z = 3$ at four-loops Holder and Metzner [2015b]. The corresponding diagram only involves two loops in the nonlinear bosonized description; moreover the transparent scaling of fermion loops in this approach (see Sec. 2.2.5) may make an evaluation of $z-3$ more tractable. The importance of accounting for nonlinearities in the bosonization approach in this context was emphasized in Ref. Chubukov and Khveshchenko [2006]. We leave this for future work.

2.2.7 Spinful Fermi surfaces

Coadjoint orbit

The formalism of coadjoint orbits can be extended to describe Fermi surfaces with spin. In addition to the charge distribution function $f(\mathbf{x}, \mathbf{p})$, the low-energy degrees of freedom also involve the spin distribution function—for free fermions this is related to fermion bilinears

$$f^i(\mathbf{x}, \mathbf{p}) \sim i\int d^d\mathbf{y}\,\psi^\dagger(\mathbf{x}+\tfrac{\mathbf{y}}{2})T_i\psi(\mathbf{x}-\tfrac{\mathbf{y}}{2})e^{i\mathbf{p}\cdot\mathbf{y}}\,, \tag{2.2.126}$$

where $T_i = \frac{1}{2}\sigma_i$ acts on the spin indices. We focus here on $SU(2)$, but more general internal groups can accommodated for with minor changes. The algebra $\mathfrak{g}$ and corresponding coadjoint orbits is now enlarged: for free fermions it is the low-momentum limit of the algebra of fermion bilinears $\psi^\dagger\psi$, $\psi^\dagger T_i\psi$—it is derived from this perspective in Appendix 2.3.6. Here instead we derive it directly in the semiclassical limit. The elements of the algebra can be parametrized as

$$F^a(x,p) \in \mathfrak{g}\,, \qquad a = 0,1,2,3\,, \tag{2.2.127}$$

with F^0 corresponding to an element of the (spinless) Poisson algebra. Consider the infinitesimal action of this algebra on a function $\mathcal{O}(\mathbf{x},\mathbf{p})$ of phase space, that transforms in some representation of the $SU(2)$ group:

$$\mathbf{x} \to \mathbf{x}' = \mathbf{x} - \nabla_{\mathbf{p}} F^0(\mathbf{x},\mathbf{p}) \tag{2.2.128a}$$

$$\mathbf{p} \to \mathbf{p}' = \mathbf{p} + \nabla_x F^0(\mathbf{x},\mathbf{p}) \tag{2.2.128b}$$

$$\mathcal{O}(\mathbf{x},\mathbf{p}) \to \mathcal{O}'(\mathbf{x}',\mathbf{p}') = (1 + F^i(\mathbf{x},\mathbf{p})T_i)\mathcal{O}(\mathbf{x},\mathbf{p})\,. \tag{2.2.128c}$$

The commutator of two such transformations is

$$[F,G]^0 = \{F^0, G^0\}\,, \tag{2.2.129a}$$

$$[F,G]^k = \{F^0, G^k\} + \{F^k, G^0\} - F^i G^j f_{ij}{}^k\,, \tag{2.2.129b}$$

where $\{\cdot,\cdot\}$ still denotes the Poisson bracket and $f_{ij}{}^k$ are the structure factors of $\mathfrak{su}(2)$.

We parametrize the state again as an element of the dual space $f^a(\mathbf{x},\mathbf{p}) \in \mathfrak{g}^*$. The ground state is

$$f_0^0(\mathbf{x},\mathbf{p}) = \Theta(p_F - p)\,, \qquad f_0^i(\mathbf{x},\mathbf{p}) = 0\,. \tag{2.2.130}$$

The degree of freedom of the EFT is the coadjoint orbit:

$$\begin{aligned} f_\phi(\mathbf{x},\mathbf{p}) &= U^{-1} f_0 U \\ &\simeq f_0 + [\phi, f_0] + \frac{1}{2}[\phi,[\phi,f_0]] + \cdots\,, \end{aligned} \tag{2.2.131}$$

where $U = e^{-\phi}$. As in the spinless case (Eq. (2.2.29)), the phase is an equivalence class: one identifies $\phi \sim \phi + \alpha$ for α an element of the stabilizer of f_0. Using the algebra (2.2.129), one finds that $[\alpha, f_0] = 0$ implies

$$0 = \{\alpha^a, \Theta(p - p_F)\} = \mathbf{n}_\theta \cdot \nabla \alpha^a(\mathbf{x}, \mathbf{p}) \delta(p - p_F) \,. \tag{2.2.132}$$

Each component α^a, $a = 0, 1, 2, 3$ therefore satisfies the same constraint as in the spinless case (2.2.28), and one can use the redundancy $\phi \sim \phi + \alpha$ to chose the following representatives

$$\phi_a(\mathbf{x}, \theta), \qquad a = 0, 1, 2, 3 \,. \tag{2.2.133}$$

The EFT will therefore contain three additional low-energy degrees of freedom compared to a spinless fermi surface. That these degrees of freedom constitue the low-lying excitations of spinful Fermi surfaces is well known in the conventional Fermi liquid approach (see, e.g., Vollhardt and Wölfle [1990]), but appears to be less well-appreciated in the bosonization literature Houghton et al. [2000].

Effective field theory

To obtain the general EFT for Fermi liquids with spin, one proceeds as in Secs. 2.2.2 and 2.2.3, using instead the algebra (2.2.129). The free fermion part of the action reads again

$$S = \int dt \, \langle f_0, U^{-1} \left(\partial_t - \epsilon\right) U \rangle \,, \tag{2.2.134}$$

the only difference with the spinless case being the algebra. For the Hamiltonian to commute with the stabilizer $\mathfrak{h}$, we need

$$\epsilon^a(\mathbf{x}, \mathbf{p}) = \delta^a_0 \epsilon(p) \,. \tag{2.2.135}$$

Eq. (2.2.134) should be supplemented with nonlinear terms in f^a as in Eq. (2.2.51), which will in particular contain spin-asymmetric Landau parameters. We leave a more general analysis including these terms for future work, and focus on the action (2.2.134) in this section.

Let us expand (2.2.134) to study the dynamics of the theory, following Secs. 2.2.4 and 2.2.5. Consider first the quadratic action: the WZW term gives

$$S^{(2)}_{\mathrm{WZW}} = \frac{1}{2}\int dt\,\langle[\phi, f_0], \dot{\phi}\rangle = -\frac{p_F^{d-1}}{2}\int_{t,\mathbf{x},\theta} \dot{\phi}_a \mathbf{n}_\theta \cdot \nabla\phi_a\,, \tag{2.2.136}$$

with $\int_{t,\mathbf{x},\theta} \equiv \frac{dt d^d\mathbf{x} d^{d-1}\theta}{(2\pi)^d}$. In the last step, we used (2.2.129) which implies

$$[\phi, f_0]^a = -\mathbf{n}_\theta \cdot \nabla\phi^a \delta(p - p_F)\,. \tag{2.2.137}$$

The Hamiltonian piece gives

$$S^{(2)}_H = \frac{1}{2}\int dt\,\langle[\phi, f_0], [\phi, \epsilon]\rangle = -\frac{p_F^{d-1}}{2}\epsilon'(p_F)\int_{t,\mathbf{x},\theta}(\mathbf{n}_\theta \cdot \nabla\phi_a)^2. \tag{2.2.138}$$

The last step follows from (2.2.137) and

$$[\phi, \epsilon]^a = \mathbf{n}_\theta \cdot \nabla\phi^a \epsilon'(p)\,. \tag{2.2.139}$$

The quadratic action is therefore simply the sum of four copies of the spinless action, and all fields have the same propagator

$$\langle\phi^a_\theta \phi^b_{\theta'}\rangle(\omega, \mathbf{q}) = i\frac{(2\pi)^d}{p_F^{d-1}}\frac{\delta^{d-1}(\theta - \theta')\delta_{ab}}{\mathbf{n}_\theta \cdot \mathbf{q}(\omega - v_F \mathbf{n}_\theta \cdot \mathbf{q})}\,. \tag{2.2.140}$$

Let us now turn to cubic terms in the action. The WZW term is

$$
\begin{aligned}
S^{(3)}_{\text{WZW}} &= \frac{1}{3!}\int dt\,\langle[\phi,f_0],[\dot\phi,\phi]\rangle \\
&= \frac{p_F^{d-1}}{3!}\int_{t,\mathbf{x},\theta} \frac{1}{p_F}\nabla_n\phi^0\left(\nabla_s\dot\phi^0\partial_\theta\phi^0-\nabla_s\phi^0\partial_\theta\dot\phi^0\right) \\
&\qquad + \frac{1}{p_F}\nabla_n\phi^i\left(\nabla_s\dot\phi^0\partial_\theta\phi^i-\nabla_s\phi^i\partial_\theta\dot\phi^0+\nabla_s\dot\phi^i\partial_\theta\phi^0-\nabla_s\phi^0\partial_\theta\dot\phi^i\right) \\
&\qquad - \nabla_n\phi^i\dot\phi^j\phi^k f_{ijk}\,.
\end{aligned}
\tag{2.2.141}
$$

The terms in the first two lines are similar to the WZW term for spinless Fermi liquids, see Eq. (2.2.107). However the last term is different: it arises from a non-abelian algebra ($f_{ijk}\neq 0$) and has a different scaling in derivatives. The Hamiltonian can be shown not to have such a cubic term. Similar nonlinearities however appear in the spin density operators. Writing

$$
\rho^a(t,\mathbf{x}) = \int\frac{d^d\mathbf{p}}{(2\pi)^d}f^a_\phi = \int\frac{d^d\mathbf{p}}{(2\pi)^d}f^a_0+[\phi,f_0]^a+\frac{1}{2}[\phi,[\phi,f_0]]^a+\cdots\,, \tag{2.2.142}
$$

and evaluating the commutators using (2.2.129), one finds that the spin densities have the expansion

$$
\rho^i = -\frac{p_F^{d-1}}{(2\pi)^d}\int_\theta \nabla_n\phi^i - f_{ijk}\phi^j\nabla_n\phi^k+\cdots\,. \tag{2.2.143}
$$

The nonlinear term has one less gradient compared to the one in the charge density (2.2.108) .

Nonlinear response

The enhanced scaling of nonlinearities in the spin sector (2.2.141), (2.2.142) leads to a different scaling of spin density n-point functions than the one found in Sec. 2.2.5 for charge densities. The quadratic action (2.2.136), (2.2.138) leads to the scaling $\omega\sim q$ and

$$
\phi^a(t,\mathbf{x},\theta)\sim q^{(d-1)/2}\,. \tag{2.2.144}
$$

Since the linear part of the densities is $\rho \sim q\phi$, the spin density two-point function scales as

$$\langle \rho_i \rho_j \rangle(\omega, q) \sim \delta_{ij} \,, \tag{2.2.145}$$

as for charge density. In fact, since the entire quadratic action is unchanged, the two-point function for spin density is identical that of charge density (2.2.87). The new nonabelian vertices scale as $f_{ijk} q^{(d-1)/2}$. The three-point function only involves a single such vertex (see Fig. 2.3), so that

$$\langle \rho_i(\omega, \mathbf{q}) \rho_j(\omega', \mathbf{q}') \rho_k \rangle \sim f_{ijk} q^{(d-1)/2} q^3 \langle \phi\phi\phi \rangle \sim \frac{1}{q} \,. \tag{2.2.146}$$

The triangle diagram coming from the nonlinear term in the density produces the same scaling. Barring cancellations, we find that $\langle \rho\rho\rho \rangle \sim 1/q$.

We now generalize to higher point functions. Contributions to n-points function from diagrams involving only cubic vertices have $n-2$ such vertices, so that

$$\langle \rho_{i_1}(\omega_1, \mathbf{q}_1) \rho_{i_2}(\omega_2, \mathbf{q}_2) \cdots \rho_{i_n} \rangle \sim f_{ijk}^{n-2} q^{(n-2)(d-1)/2} q^n \langle \phi \cdots \phi \rangle \sim \frac{1}{q^{n-2}} \,. \tag{2.2.147}$$

Higher-point vertices do not change this scaling. Indeed, every additional commutator $[\phi, \cdot]$ has at most one derivative fewer than in the spinless case, where commutators are Poisson brackets. Therefore, using for example a 4-point vertex once or a 3-point vertex twice produces the same scaling.

The scaling (2.2.147) of spin density n-point functions is perhaps less surprising than that of charge density (2.2.95). Indeed, the scaling (2.2.147) could be guessed from a fermion description, see Eq. (2.2.96). The subtle cancellations in the fermionic approach to computing charge density n-point functions, arising upon antisymmetrization of external legs in the fermion loop and which invalidate the guess (2.2.96) for charge density n-point function, do not occur here due to the non-abelian nature of spin density. The scaling (2.2.147) also controls gluon n-point functions in dense QCD Braaten and Pisarski [1992], Frenkel and Taylor [1992].

2.2.8 Charged operators, BCS, and large momentum processes

In one-dimensional bosonization, charged operators—including fermions—are realized by vertex operators $e^{i\alpha\phi}$. Their correlation functions are therefore entirely fixed from those of the phase ϕ. An effort to mirror this correspondence in higher dimensions is often made in multidimensional bosonization Houghton et al. [2000], although, ultimately, a bosonic theory (without a Chern-Simons field) cannot produce fermion statistics in dimensions larger than one. However, one may be interested in studying the bosonic charged operators of the theory, which in a fermion description would include $\psi(\mathbf{x})\psi(\mathbf{y})$, $\psi^\dagger(\mathbf{x})\psi^\dagger(\mathbf{y})$, etc. These can be captured in our formalism by extending the algebra, similarly to how spinful Fermi surfaces were studied in Sec. 2.2.7. It is particularly interesting to focus on the charge ± 2 operators shown above, as these form a closed algebra with $\psi^\dagger(\mathbf{x})\psi(\mathbf{y})$, leading to a simple extension of the Poisson algebra, see App. 2.3.6. The degrees of freedom will now include, in a addition to the distribution function $f^0(\mathbf{x},\mathbf{p})$, the charged distribution functions

$$f^2(\mathbf{x},\mathbf{p})\,, \qquad\qquad f^{-2}(\mathbf{x},\mathbf{p})\,. \tag{2.2.148}$$

One difference with the spin extension discussed in Sec. 2.2.7 is that the stabilizer of the state $f_0^c(\mathbf{x},\mathbf{p}) = \delta_0^c\Theta(p_F - p)$ is not enlarged. As a result, the quotient space consists of the entire functions of phase space $f^{\pm 2}(\mathbf{x},\mathbf{p})$, in addition to $\phi(\mathbf{x},\theta)$. The Hamiltonian (2.2.51) should be generalized to a functional of all distributions $H = H[f, f^2, f^{-2}]$; the first new term is

$$H[f, f_2, f_{-2}] = \int_{\mathbf{x},\mathbf{p}} \epsilon(\mathbf{p}) f(\mathbf{x},\mathbf{p}) + \frac{1}{2}\int_{\mathbf{x},\mathbf{p},\mathbf{p}'} V_{\mathrm{BCS}}(\mathbf{p},\mathbf{p}') f_2(\mathbf{x},\mathbf{p}) f_{-2}(\mathbf{x},\mathbf{p}') + \cdots . \quad (2.2.149)$$

One can obtain an equation of motion for the charged distribution functions (see App. 2.3.6 for details)

$$\dot{f}_2(\mathbf{x},\mathbf{p},t) = -2i\epsilon(p) f_2(\mathbf{x},\mathbf{p},t) + i\int_{\mathbf{p}'} f_2(\mathbf{x},\mathbf{p}',t) V_{\mathrm{BCS}}(\mathbf{p}',\mathbf{p}) . \quad (2.2.150)$$

Integrating this equation over x produces the well-known equation of motion for a Cooper pair, leading to the Cooper instability (see Coleman [2015] for a textbook treatment).

We finally briefly mention another interesting extension of the Poisson algebra. The Poisson algebra was used in this work to obtain an EFT for the low-energy and low-momentum dynamics of Fermi liquids. Fermi liquids also have low-energy particle-hole excitations at large momentum $q \sim k_F$ (as long as $q < 2k_F$, for a spherical Fermi surface). At these high momenta, the higher gradient terms in (2.2.51) become important, and perturbative control in the EFT is lost. However, correlation functions of high momentum operators can be obtained in a different approach: one considers a distribution function $f_{\mathbf{P}}(\mathbf{x},\mathbf{p})$ for momentum $\mathbf{P}+\mathbf{p}$ particle-hole pairs (with $P \sim p_F$ and $p \ll P$). Extending the algebra as in App. 2.3.6, these will become additional degrees of freedom similar to $f^{\pm 2}$ above; their equation of motion can be used to capture low-energy correlators with large momentum P. This approach makes it clear that these correlators will not be uniquely fixed in terms of the Wilsonian parameters appearing in (2.2.51), since the Hamiltonian can involve new terms $\sim f_{\mathbf{P}} f_{-\mathbf{P}}$ similar to the BCS interaction in (2.2.149).

2.2.9 Conclusion

To summarize, in this paper we have used the algebra of canonical transformations and its coadjoint action to construct a nonlinear effective field theory of Fermi liquids in terms of bosonized degrees of freedom. In this approach, the states of a Fermi liquid form a coadjoint orbit of the group of canonical transformations; and the dynamical degrees of freedom parametrize the shape of the Fermi surface at each spacetime point. For free fermions, the resulting equation of motion coincides with the collisionless Boltzmann equation, restricted to configurations with a sharp Fermi surface. More generally, the effective action (2.2.52) describes interacting Fermi liquids: it contains Landau parameters as well as further interactions that can be systematically organized in an expansion in gradients and fluctuations. Fluctuations around a ground state f_0 can be studied by expanding the action in $f - f_0$, or, more conveniently, $\phi(t, \mathbf{x}, \theta)$; this is done up to cubic order in (2.2.107). Our approach reduces to standard constructions of bosonized Fermi liquids Haldane [1994], Castro Neto and Fradkin [1994a], Houghton et al. [2000] upon linearization.

As a check of the nonlinear structure of the theory, we showed in Sec. 2.2.5 that the density three-point function of a Fermi gas is reproduced. Even for a free Fermi gas, this calculation is substantially simpler to perform in a bosonized description, either using the EFT as in Sec. 2.2.5, or in kinetic theory as in App. 2.3.7. The general scaling of density n-point functions, which is obscured in a fermionic description and plays an important role in the study of non-Fermi liquids, is entirely manifest in the EFT. Thus we hope that the formalism will be useful for the understanding of non-Fermi liquids.

While fermion loops are reproduced by tree level diagrams in the bosonized description, we have not discussed loop corrections in the bosonized theory itself. These are expected to give suppressed but interesting non-analytic corrections to Fermi liquids correlation functions Chubukov and Maslov [2003]. One troubling aspect of the Fermi liquid EFT is that it features UV/IR mixing: certain UV divergences in loop diagrams come with non-analytic IR structures and cannot be absorbed with counterterms. A related issue arises when computing the specific heat of the Fermi liquid (the first line in Eq. (2.2.122)): the coefficient of the linear in T specific heat involves a cutoff $\sim k_F$. We leave a more careful study of these issues for future work. Note however that they may not be important for studies of non-Fermi liquids; for example, the $T^{2/3}$ specific heat obtained in Eq. (2.2.125) is not UV-sensitive.

2.3 Supplemental Material

2.3.1 Background gauge fields for canonical transformations

In Sec. 2.2.3, we saw how to couple the nonlinear theory to a background $U(1)$ gauge field by identifying gauge transformations as a subset of canonical transformations and making the action invariant under that subset.

We can in fact push further and make the action invariant under *all* time dependent canonical transformations $\lambda(t, \mathbf{x}, \mathbf{p}) \in \mathfrak{g}$. Let $W = \exp \lambda$ be the corresponding group element. One can show that the coadjoint transformation of a dual space element $f(\mathbf{x}, \mathbf{p})$ takes the form

$$(\mathrm{Ad}_W f)(\mathbf{x}, \mathbf{p}) = f(\mathbf{x}^W, \mathbf{p}^W), \tag{2.3.1}$$

where $\mathbf{x}^W$ and $\mathbf{p}^W$ are transformed coordinates

$$\begin{aligned} \mathbf{x}^W &= \mathbf{x} + W \nabla_{\mathbf{p}} W^{-1}, \\ \mathbf{p}^W &= \mathbf{p} - W \nabla_{\mathbf{x}} W^{-1}. \end{aligned} \tag{2.3.2}$$

These are the nonlinear versions of Eq. (2.2.2). In order to make any arbitrary function invariant under such a transformation, we turn on background gauge fields $(\mathbf{A}_{\mathbf{x}}(t, \mathbf{x}, \mathbf{p}), \mathbf{A}_{\mathbf{p}}(t, \mathbf{x}, \mathbf{p}))$ in phase space (with position and momentum components) and consider the new function

$$f_A(\mathbf{x}, \mathbf{p}) = f(\mathbf{x} - \mathbf{A}_{\mathbf{p}}, \mathbf{p} + \mathbf{A}_{\mathbf{x}}). \tag{2.3.3}$$

We will often refer to these phase space gauge fields collectively as A_I, using a phase space index $I = (\mathbf{x}, \mathbf{p})$. One can see that f_A is invariant under canonical transformations if we also demand that A_I transforms in the following way

$$A_I \to \tilde{A}_I = W^{-1}\left(A_I - \partial_I\right)W = A_I(t, \mathbf{x}^{W^{-1}}, \mathbf{p}^{W^{-1}}) - W^{-1}\partial_I W, \tag{2.3.4}$$

since the canonical transformation of f_A supplemented by the transformation of the gauge fields is now

$$f_A(\mathbf{x}, \mathbf{p}) \to f_{\tilde{A}}(\mathbf{x}^W, \mathbf{p}^W) = f\left(\mathbf{x}^W - \tilde{\mathbf{A}}_{\mathbf{p}}(\mathbf{x}^W, \mathbf{p}^W), \mathbf{p}^W + \tilde{\mathbf{A}}_{\mathbf{x}}(\mathbf{x}^W, \mathbf{p}^W)\right), \tag{2.3.5}$$

where we have suppressed the time dependence of the gauge field for notational simplicity. Using the fact that

$$\tilde{A}_I(\mathbf{x}^W, \mathbf{p}^W) = W\tilde{A}_I(\mathbf{x}, \mathbf{p})W^{-1} = A_I(\mathbf{x}, \mathbf{p}) - \partial_I W W^{-1}, \tag{2.3.6}$$

we can see that

$$\begin{aligned}
\mathbf{x}^W - \tilde{\mathbf{A}}_{\mathbf{p}}(\mathbf{x}^W, \mathbf{p}^W) &= \mathbf{x}^W - \mathbf{A}_{\mathbf{p}}(\mathbf{x}, \mathbf{p}) + \nabla_{\mathbf{x}} W W^{-1} = \mathbf{x} - \mathbf{A}_{\mathbf{p}}(\mathbf{x}, \mathbf{p}), \\
\mathbf{p}^W - \tilde{\mathbf{A}}_{\mathbf{x}}(\mathbf{x}^W, \mathbf{p}^W) &= \mathbf{p}^W + \mathbf{A}_{\mathbf{x}}(\mathbf{x}, \mathbf{p}) - \nabla_{\mathbf{p}} W W^{-1} = \mathbf{p} + \mathbf{A}_{\mathbf{x}}(\mathbf{x}, \mathbf{p}),
\end{aligned} \tag{2.3.7}$$

so the transformation of the phase space gauge fields cancels the canonical transformation.

These phase space gauge can be viewed as semi-classical limits of non-commutative gauge fields in phase space. It is a known fact that non-commutative gauge transformations in the limit of small non-commutativity (which in our case would be $\hbar$) reproduce infinitesimal canonical transformations of the non-commutative space Douglas and Nekrasov [2001]. The transformation law, eq. (2.3.4), is just the non-linear version of these under finite canonical transformations.

The modified coordinates

$$\mathbf{X} = \mathbf{x} - \mathbf{A}_{\mathbf{p}}, \qquad \mathbf{P} = \mathbf{p} + \mathbf{A}_{\mathbf{x}}, \tag{2.3.8}$$

will be referred to as covariant coordinates from here onwards. The distribution function $f_A(\mathbf{x}, \mathbf{p})$ evaluated on these covariant coordinates is invariant under all canonical transformations supplemented by the transformations in Eq. (2.3.4). Finally, we also turn on a time component for this gauge field $A_0(t, \mathbf{x}, \mathbf{p})$ which transforms in a similar manner,

$$A_0 \to W^{-1} \left(A_0 - \partial_0 \right) W, \tag{2.3.9}$$

to make terms with time derivatives invariant.

Equipped with these gauge fields, we can make the free fermion action invariant under all time dependent canonical transformations. The Wess-Zumino-Witten term gets modified to

$$S_{\text{WZW}} = \int dt \left\langle f_0, U^{-1} \left[\partial_t - A_0 \right] U \right\rangle, \tag{2.3.10}$$

and the free fermion Hamiltonian gets modified to

$$S_H = -\int dt \left\langle f_A, \epsilon(\mathbf{p}) \right\rangle = -\int dt \left\langle f_0, U^{-1} \epsilon(\mathbf{p} - \mathbf{A}_{\mathbf{x}}) U \right\rangle. \tag{2.3.11}$$

One can see that both terms are separately invariant under time dependent canonical transformations $W = \exp \lambda$

$$U \to WU, \qquad A_0 \to W^{-1} \left(A_0 - \partial_0 \right) W, \qquad A_I \to W^{-1} \left(A_I - \partial_I \right) W. \tag{2.3.12}$$

The total free fermion action is

$$S[\phi, A_\mu] = \int dt \left\langle f_0, U^{-1} \left[\partial_t - A_0 - \epsilon(\mathbf{p} - \mathbf{A_x})\right] U \right\rangle, \tag{2.3.13}$$

where $A_\mu = (A_0, \mathbf{A_x})$ refers to the space-time components of the phase-space gauge field. The momentum components $\mathbf{A_p}$ don't enter the action for free fermions since the dispersion relation is translationally invariant.

Gauging the interacting theory (2.2.51) also follows from the above considerations. Since f transforms covariantly under the transformation $U \to WU$, while δf does not, we instead expand the interacting Hamiltonian in f to obtain an expression of the form

$$\begin{aligned}\mathscr{H}[f] = &\int_{\mathbf{x},\mathbf{p}} \epsilon(\mathbf{p}) f(\mathbf{x},\mathbf{p}) \\ &+ \int_{\mathbf{x},\mathbf{p},\mathbf{p}'} \tilde{F}^{(2,0)}_{\text{int}}(\mathbf{p},\mathbf{p}') f(\mathbf{x},\mathbf{p}) f(\mathbf{x},\mathbf{p}') + \tilde{\mathbf{F}}^{(2,1)}_{\text{int}}(\mathbf{p},\mathbf{p}') \cdot \nabla_{\mathbf{x}} f(\mathbf{x},\mathbf{p}) f(\mathbf{x},\mathbf{p}') + \dots \\ &+ \int_{\mathbf{x},\mathbf{p},\mathbf{p}',\mathbf{p}''} \tilde{F}^{(3,0)}_{\text{int}}(\mathbf{p},\mathbf{p}',\mathbf{p}'') f(\mathbf{x},\mathbf{p}) f(\mathbf{x},\mathbf{p}') f(\mathbf{x},\mathbf{p}'') + \dots ,\end{aligned} \tag{2.3.14}$$

with modified Wilson coefficients $\tilde{F}^{(m,n)}(\mathbf{p}_1, \dots, \mathbf{p}_m)$ which can be straightforwardly related to those in Eq. (2.2.51) by rearranging the expansion.

While f transforms covariantly under canonical transformations, its phase space gradient $\partial_I f$ does not. However, one show that the covariant derivative

$$D_I f \equiv \partial_I f - \{A_I, f\} \tag{2.3.15}$$

does, i.e.,

$$(D_I f) \to \mathrm{Ad}^*_W (D_I f). \tag{2.3.16}$$

Hence, to make the interacting Hamiltonian invariant, we replace partial derivatives of f by covariant derivatives, and then evaluate the distribution function and its covariant derivatives on covariant coordinates

$$[D_I \dots D_J f](\mathbf{x}, \mathbf{p}) \to [D_I \dots D_J] f(\mathbf{x} - \mathbf{A_p}(t, \mathbf{x}, \mathbf{p}), \mathbf{p} + \mathbf{A_x}(t, \mathbf{x}, \mathbf{p})) \tag{2.3.17}$$

where the expression $[D_I \dots D_J f]$ stands for any number of covariant derivatives acting on f. The gauged Hamiltonian is then

$$\begin{aligned}
\mathscr{H}_A[f] = & \int_{\mathbf{x},\mathbf{p}} \epsilon(\mathbf{p}) f(\mathbf{x} - \mathbf{A_p}, \mathbf{p} + \mathbf{A_x}) \\
& + \int_{\mathbf{x},\mathbf{p},\mathbf{p}'} \tilde{F}^{(2,0)}_{\text{int}}(\mathbf{p}, \mathbf{p}') f(\mathbf{x} - \mathbf{A_p}, \mathbf{p} + \mathbf{A_x}) f(\mathbf{x} - \mathbf{A}'_\mathbf{p}, \mathbf{p}' + \mathbf{A}'_\mathbf{x}) \\
& + \int_{\mathbf{x},\mathbf{p},\mathbf{p}'} \tilde{\mathbf{F}}^{(2,1)}_{\text{int}}(\mathbf{p}, \mathbf{p}') \cdot D_\mathbf{x} f(\mathbf{x} - \mathbf{A_p}, \mathbf{p} + \mathbf{A_x}) f(\mathbf{x} - \mathbf{A}'_\mathbf{p}, \mathbf{p}' + \mathbf{A}'_\mathbf{x}) + \dots,
\end{aligned} \tag{2.3.18}$$

where we have written $\mathbf{A}'_\mathbf{x}$ and $\mathbf{A}'_\mathbf{p}$ as shorthand for $\mathbf{A_x}(t, \mathbf{x}, \mathbf{p}')$ and $\mathbf{A_p}(t, \mathbf{x}, \mathbf{p}')$ respectively. Note that unlike for the free fermion case, the momentum components $\mathbf{A_p}$ of the phase space gauge field do enter the action through the non-linear terms. The "maximally gauged" action is then

$$S[\phi, A_\mu] = S_{\text{WZW}}[\phi, A_0] - \int dt\ \mathscr{H}_A[f[\phi]]\,. \tag{2.3.19}$$

2.3.2 Ward Identity

Coupling the action to background gauge fields for canonical transformations naturally comes with a Ward identity. To derive it we look at the linearization of the transformation Eq. (2.3.4) in λ.

$$\delta_\lambda A_M = \partial_M \lambda + \{\lambda, A_M\} + \mathcal{O}(\lambda^2). \tag{2.3.20}$$

where the index M stands for time, space and momentum components. The variation of the action under this transformation must take the form

$$\delta_\lambda S = -\int dt \left\langle \mathcal{J}^M, \delta_\lambda A_M \right\rangle . \tag{2.3.21}$$

This equation defines the current $\mathcal{J}^M$ for canonical transformations. Its components are given by

$$\mathcal{J}^0 = -\frac{\delta S_{\text{WZW}}}{\delta A_0} = f, \qquad \mathcal{J}^{x^i} = \frac{\delta \mathscr{H}_A}{\delta A_{x^i}} = f \frac{\partial}{\partial p_i} \epsilon(\mathbf{p} - \mathbf{A_x}) + \ldots, \qquad \mathcal{J}^{p^j} = 0 + \ldots, \tag{2.3.22}$$

where the ellipses denote terms form the variation of the nonlinear-in-f terms in the Hamiltonian (2.3.18). Consequently, the Ward identity takes the form

$$\partial_M \mathcal{J}^M + \{\mathcal{J}^M, A_M\} = 0. \tag{2.3.23}$$

This is not a continuity relation in the usual sense, because the index M runs over both spacetime t, x^i and momentum p^j indices. Even in the absence of background fields $A_M = 0$, one can therefore not in general define conserved charges by integrating this equation over space, because $\partial_{p^j} \mathcal{J}^{p^j}$ is not a total spatial gradient. One exception is for free fermions, where $\mathcal{J}^{p^j} = 0$ (2.3.22)—the conserved charges $Q(\mathbf{p}) = \int d^d x\, \mathcal{J}^0(t, \mathbf{x}, \mathbf{p})$ are then the occupation numbers at each wavevector. Interestingly, in this situation one can linearize the covariant conservation law with background sources (2.3.23) around the finite density state $\langle \mathcal{J}^0 \rangle = f_0$ to obtain approximate Ward identities that resemble anomaly equations, as we discuss further below. In $1 + 1d$, we find chiral symmetries at every Fermi point (see Appendix 2.3.3) and in $2 + 1d$, the linearization results in the loop group symmetry of Else et al. [2021] (see Appendix 2.3.4).

2.3.3 (1+1)d Luttinger liquid

In this section we show that the coadjoint orbit formalism reproduces the bosonized theory of Luttinger liquids, at both the linear and non-linear level. In particular, the mixed anomaly between the emergent chiral $U(1)$ symmetries at the Fermi points can be understood as a linearization of the non-anomalous covariant conservation law (2.3.23).

Luttinger liquids have been extensively studied in the literature, see in particular Refs. Stone [1989], Das et al. [1992], Dhar et al. [1993a,b], Khveshchenko [1994] for constructions using coadjoint orbits.

Linearized Action

We begin with a review of the construction of the bosonized action for Luttinger liquids from the algebra of densities. Fermi 'surfaces' in 1+1 dimensions are a collection of discrete points in momentum space. Assuming that the dispersion relation $\epsilon(p)$ is an even function that monotonically increases with positive momentum, the Fermi surface consists of exactly two points at momentum values $p = \pm p_F$. Each Fermi point hosts a chiral mode whose chirality is given by $\text{sgn}[\partial_p \epsilon]$. Denoting the chiral modes at the points $+p_F$ and $-p_F$ by the subscripts R and L (for 'right' and 'left') respectively, the particle number densities obey the following equal time commutation relations (see Eq. (2.2.72))

$$\begin{aligned}
[\rho_R(x), \rho_R(x')] &= -\frac{i}{2\pi}\partial_x \delta(x - x'), \\
[\rho_L(x), \rho_L(x')] &= \frac{i}{2\pi}\partial_x \delta(x - x'), \\
[\rho_R(x), \rho_L(x')] &= 0\,.
\end{aligned} \tag{2.3.24}$$

The so-called Schwinger terms on the right-hand side of the first two lines are indicative of the chiral anomalies carried by each chiral fermion. $\rho_{R,L}$ are the charge densities corresponding to two copies of $U(1)$ symmetry, which we will refer to as $U(1)_R$ and $U(1)_L$. The chiral algebra can be realized in terms of bosonic fields $\phi_{R,L}$ by defining the densities as

$$\rho_R = \frac{1}{2\pi}\partial_x\phi_R, \qquad \rho_L = -\frac{1}{2\pi}\partial_x\phi_L. \tag{2.3.25}$$

The commutators of the densities with the bosonic fields are then

$$\begin{aligned} [\phi_R(x), \rho_R(x')] &= -i\delta(x-x'), \\ [\phi_L(x), \rho_L(x')] &= -i\delta(x-x')\,, \end{aligned} \tag{2.3.26}$$

which tells us that the $U(1)_{R.L}$ symmetries are non-linearly realized on the bosonic fields as

$$\phi_R \to \phi_R - \lambda_R, \qquad \phi_L \to \phi_L - \lambda_L\,. \tag{2.3.27}$$

An action that produces the algebra (2.3.26) is

$$\begin{aligned} S &= \frac{1}{2}\int dtdx\,\dot{\phi}_R\rho_R + \dot{\phi}_L\rho_L \\ &= -\frac{1}{4\pi}\int dtdx\,\partial_x\phi_R\dot{\phi}_R - \partial_x\phi_L\dot{\phi}_L\,. \end{aligned} \tag{2.3.28}$$

The factor of $\frac{1}{2}$ in the first line comes from the fact this is a constrained system: using the appropriate Dirac brackets one recovers the commutation relation (2.3.26) as desired.

This action corresponds to the WZW term in the coadjoint orbit construction. Consider Eq. (2.2.78) for $d = 1$: the integral over the Fermi surface angle θ becomes a sum over two points $\theta = 0,\ \pi$, so that one finds

$$\begin{aligned} S_{\text{WZW}} &= -\frac{1}{4\pi}\sum_{\sigma=\pm}\sigma\int dtdx\,\partial_x\phi_\sigma\dot{\phi}_\sigma \\ &= -\frac{1}{4\pi}\int dtdx\,\partial_x\phi_R\dot{\phi}_R - \partial_x\phi_L\dot{\phi}_L\,, \end{aligned} \tag{2.3.29}$$

in agreement with (2.3.28). Nonlinearities in the WZW term—present for any $d > 1$, see

Eq. (2.2.107)—entirely vanish in $d = 1$. These nonlinearities are associated with the curvature of the Fermi surface, which are absent in one dimension. For the same reason, the relation between ρ and ϕ (2.3.25) does not receive nonlinear corrections.

In $d = 1$, all nonlinearities in the bosonized description of a Luttinger liquid come from the Hamiltonian, in particular from nonlinearities in the dispersion relation. These will be discussed in Sec. 2.3.3. The Hamiltonian part of the action also produces a term in the quadratic action: taking again $d = 1$ in (2.2.84) one obtains

$$\begin{aligned} S^{(2)} &= -\frac{1}{4\pi}\sum_{\sigma=\pm}\int dtdx\,\partial_x\phi_\sigma\left(\sigma\dot{\phi}+v_F\partial_x\phi\right) \\ &= -\frac{1}{4\pi}\int \partial_x\phi_R\left(\partial_0\phi_R+v_F\partial_x\phi_R\right)-\partial_x\phi_L\left(\partial_0\phi_L-v_F\partial_x\phi_L\right)\,, \end{aligned} \tag{2.3.30}$$

which is the well-known Gaussian action for a Luttinger liquid.

Chiral anomaly as a linear approximation

When coupled to background gauge fields, both chiral symmetries are anomalous with opposite anomalies. If A_μ^R and A_μ^L are the background fields for the two global symmetries, the anomalous conservation laws are

$$\begin{aligned} \partial_\mu j_R^\mu &= -\frac{1}{4\pi}\epsilon^{\mu\nu}F_{\mu\nu}^R, \\ \partial_\mu j_L^\mu &= \frac{1}{4\pi}\epsilon^{\mu\nu}F_{\mu\nu}^L. \end{aligned} \tag{2.3.31}$$

In the coadjoint orbit formalism, the chiral anomalies appear as a linearized approximation to the invariance of the maximally gauged action (2.3.13) under all canonical transformations. To see this, we begin with the Ward identity (2.3.23) for free fermions, that have $\mathcal{J}_{p^j} = 0$ (see (2.3.22)) :

$$\partial_\mu\mathcal{J}^\mu + \{\mathcal{J}^\mu, A_\mu\} = 0\,. \tag{2.3.32}$$

Turning off A_x for simplicity, the conservation law takes the form

$$\partial_0 \mathcal{J}^0 + \partial_x \mathcal{J}^x + \partial_x \mathcal{J}^0 \partial_p A_0 = \partial_p \mathcal{J}^0 \partial_x A_0. \tag{2.3.33}$$

Recall that $\mathcal{J}^0$ is simply the phase space distribution f. Hence, it has a nonzero expectation value in the ground state

$$\langle \mathcal{J}^0 \rangle = f_0. \tag{2.3.34}$$

If we now linearize the equation around the two Fermi points by writing

$$\mathcal{J}^0 = f_0 + \delta \mathcal{J}^0, \qquad \mathcal{J}^x = \delta \mathcal{J}^x, \tag{2.3.35}$$

and treat $A_0(t, x, p)$ to be of the same order as $\delta \mathcal{J}^\mu$, we find that the equation takes the form

$$\partial_0 \delta \mathcal{J}^0 + \partial_x \delta \mathcal{J}^x = (\partial_x A_0^L)\delta(p + p_F) - (\partial_x A_0^R)\delta(p - p_F). \tag{2.3.36}$$

Integrating over either $p > 0$ or $p < 0$ and using the expressions for the chiral density and current

$$\begin{aligned} \rho_R &= \int_0^\infty \frac{dp}{2\pi}\, \delta \mathcal{J}^0, \qquad & j_R &= \int_0^\infty \frac{dp}{2\pi}\, \delta \mathcal{J}^x, \\ \rho_L &= \int_{-\infty}^0 \frac{dp}{2\pi}\, \delta \mathcal{J}^0, \qquad & j_L &= \int_{-\infty}^0 \frac{dp}{2\pi}\, \delta \mathcal{J}^x, \end{aligned} \tag{2.3.37}$$

we find that the Ward identity takes the form of the anomalous conservation laws for the chiral anomalies

$$\begin{aligned} \partial_t \rho_R + \partial_x j_R &= -\frac{1}{2\pi} \partial_x A_0^R, \\ \partial_t \rho_L + \partial_x j_L &= \frac{1}{2\pi} \partial_x A_0^L. \end{aligned} \tag{2.3.38}$$

The equivalent version with A_x not set to zero can also be obtained by performing a field redefinition similar to Eq. (2.2.70).

The chiral anomaly is therefore a linear approximation to the non-abelian Ward identity (2.3.23), or a covariant conservation law, around a state with nonzero charge density $\langle \mathcal{J}^0 \rangle \neq 0$ [6].

Density 3-point function

We have seen that nonlinearities in Luttinger liquids only arise from the Hamiltonian, and not the WZW term. In the special case of free fermions, these will come from the dispersion relation $\epsilon(p)$. Setting $d = 1$ in the cubic action (2.2.107) gives

$$S = -\frac{1}{4\pi} \sum_{\sigma=\pm} \int dtdx\, \partial_x \phi_\sigma \left(\sigma \dot{\phi}_\sigma + v_F \partial_x \phi_\sigma \right) + \frac{1}{3} \epsilon'' \left(\partial_x \phi_\sigma \right)^3 + \cdots , \tag{2.3.39}$$

where ϵ'' is the second derivative of the dispersion relation, evaluated at $p = p_F$. See Ref. Haldane [1981] for an early discussion of this action.

This nonlinear term in the action produces nonlinear response: $d = 1$ is a simple special case of 3-point function computed for general dimension in Sec. 2.2.5, where only the first diagram in Fig. 2.3 contributes. Setting $d = 1$ in Eq. (2.2.111), one finds that it gives

$$\langle \rho(\omega, q) \rho(\omega', q') \rho \rangle = -\frac{\epsilon''}{2\pi} \sum_{\sigma=\pm} \sigma \frac{q}{\omega - v_F \sigma q} \frac{q'}{\omega' - v_F \sigma q'} \frac{q + q'}{\omega + \omega' - v_F \sigma (q + q')} . \tag{2.3.40}$$

One can verify that this result agrees with the low frequency and momentum limit $\omega, v_F q \ll v_F k_F$ of the density 3-point function obtained from a free fermion description by computing a fermion loop Dzyaloshinskii and Larkin [1974], Neumayr and Metzner [1999], Pirooznia et al. [2008]. When density n-point functions are studied in the fermion description, the dispersion relation is typically taken to have a simple form, e.g. $\epsilon(p) = p^2/2m$. Our approach shows

6. For free fermions with linear dispersion relation $\varepsilon(p) \propto p$, this linear approximation is in fact exact, consistent with the fact that the 1+1d chiral anomaly exactly captures Dirac fermions

that at low frequencies and momenta, only $n-1$ first derivatives of the dispersion relation at the Fermi surface, ϵ', ϵ'', $\cdots$, $\epsilon^{(n-1)}$, can enter in the n-point function.

2.3.4 $LU(1)$ anomaly as a linear approximation

Else, Thorngren and Senthil recently proposed Else et al. [2021] that Fermi liquids (and possibly non-Fermi liquids) in 2+1 dimensions possess an emergent $LU(1)$ symmetry with an 't Hooft anomaly, mirroring the 1+1d chiral anomaly for the emergent $U(1)_L \times U(1)_R$ symmetry in Luttinger liquids (see Appendix 2.3.3).

The $LU(1)$ symmetry corresponds to conservation of particle number at each point on the Fermi surface. Coupling the symmetry to background gauge fields $A_M(t, \mathbf{x}, \theta)$ with $M = t, \mathbf{x}, \theta$, the anomalous conservation law for the $LU(1)$ current $j^M(t, \mathbf{x}, \theta)$ is

$$\partial_M j^M = \frac{\kappa}{8\pi^2} \epsilon^{ABCD} \partial_A A_B \partial_C A_D \,. \tag{2.3.41}$$

Since this symmetry is emergent, its associated background field can get activated against one's will. This is in fact what happens for Fermi liquids, where Else et al. [2021]

$$A_M(t, \mathbf{x}, \theta) = \delta^i_M p_{Fi}(\theta). \tag{2.3.42}$$

The angular component A_θ corresponds to the Berry connection in momentum space, which we will set to zero for simplicity.

We start by showing how the linearized density algebra (2.2.72) familiar in bosonization follows from the anomaly. A similar derivation appeared in Else and Senthil [2021]. Differentiating Eq. (2.3.41) with respect to A_t:

$$i\langle T\{\partial_A j^A(t, x, \theta)\rho(t', x', \theta')\}\rangle = \delta(t - t')\frac{\kappa}{4\pi^2} \epsilon^{AtCD} \partial_A(\delta(\theta - \theta')\delta^2(x - x'))\partial_C A_D \,. \tag{2.3.43}$$

The correlator is time-ordered. Integrating this equation $\int_{t'-\epsilon}^{t'+\epsilon} dt$ and taking $\epsilon \to 0$ gives

$$i\langle[\rho(x,\theta),\rho(x',\theta')]\rangle = \frac{\kappa}{4\pi^2}\epsilon^{AtCD}\partial_A(\delta(\theta-\theta')\delta^2(x-x'))\partial_C A_D\,, \tag{2.3.44}$$

where we have suppressed the dependence on time, since all operators and fields are now evaluated at equal time t'. The background (2.3.42) will now play an important role. In addition, one can also consider a background magnetic field $B = \nabla \times \vec{A}$. Taking both of these into account, this leads to the algebra

$$\langle[\rho(x,\theta),\rho(x',\theta')]\rangle = \frac{i\kappa}{4\pi^2}\left(\partial_\theta p_i^F(\theta)\epsilon^{ij}\partial_j + B\partial_\theta\right)\delta(\theta-\theta')\delta^2(x-x')\,. \tag{2.3.45}$$

This form of the algebra holds for Fermi surfaces of arbitrary shapes. For a circular Fermi surface $p_i^F = p_F\hat{n}_i(\theta)$, it reduces to (2.2.72) (setting $d=2$ there).

As explained in Sec. 2.2.4, the algebra (2.3.45) is only a linearized approximation to the full, necessarily nonlinear, density algebra (2.2.76)—it is this full nonabelian algebra that gives Fermi liquids a unique nonlinear structure, studied in this paper. Similarly, the $LU(1)$ anomaly (2.3.41) arises in our approach from the covariant conservation law (2.3.23) after linearizing. We show this below, proceeding analogously to the one-dimensional case discussed in Sec. 2.3.3.

We start from the Ward identity for canonical transformations, Eq. (2.3.23), in 2+1 dimensions, for free fermions:

$$\partial_\mu\mathcal{J}^\mu + \{\mathcal{J}^\mu, A_\mu\} = 0 \tag{2.3.46}$$

with $\mathcal{J}^0 = f$. Turning off $\mathbf{A}_\mathbf{x}$ for simplicity, the Ward identity reduces to

$$\partial_t\mathcal{J}^0 + \partial_i\mathcal{J}^i + \partial_{x_i}\mathcal{J}^0\partial_{p_i}A_0 = \partial_{p_i}\mathcal{J}^0\partial_{x_i}A_0, \tag{2.3.47}$$

where $A_0(t,\mathbf{x},\mathbf{p})$ is the time component of the phase space gauge field. Linearizing this equation around the circular Fermi surface

$$\mathcal{J}^0 = f_0 + \delta\mathcal{J}^0, \qquad \mathcal{J}^i = \delta\mathcal{J}^i, \tag{2.3.48}$$

and treating A_0 to be of the same order as $\delta\mathcal{J}^\mu$, the Ward identity reduces to

$$\partial_t \delta\mathcal{J}^0 + \partial_{x_i}\delta\mathcal{J}^i = -\delta(|\mathbf{p}| - p_F)(\mathbf{n}_\theta \cdot \nabla_{\mathbf{x}} A_0). \tag{2.3.49}$$

To turn this into the anomalous $LU(1)$ conservation law, we simply integrate over the radial component of the momentum $p = |\mathbf{p}|$ and identify the $LU(1)$ current and gauge field in the following way

$$\begin{gathered} j^0(t,\mathbf{x},\theta) = \int \frac{dp}{4\pi^2}\; p\; \delta\mathcal{J}^0(t,\mathbf{x},\mathbf{p}), \qquad j^i(t,\mathbf{x},\theta) = \int \frac{dp}{4\pi^2}\; p\; \delta\mathcal{J}^i, \\ A_0(t,\mathbf{x},\theta) = A_0(t,\mathbf{x},\mathbf{p})|_{|p|=p_F}. \end{gathered} \tag{2.3.50}$$

The linearized Ward identity then becomes

$$\partial_\mu j^\mu = -\frac{1}{4\pi^2} p_F(\mathbf{n}_\theta \cdot \nabla_{\mathbf{x}} A_0), \tag{2.3.51}$$

which agrees with Eq. (2.3.41) with $\kappa = -1$.

Paralleling the $1+1d$ chiral anomaly discussed previously, the $2+1d$ $LU(1)$ anomaly is a linearized approximation to the Ward identity (2.3.23) for free fermions. However, in more general situations including Fermi liquids, the Ward identity has a right-hand side $-\partial_{p^j}\mathcal{J}^{p^j}$ that is not a total spatial derivative. These systems therefore do not have, a priori, an extended set of conserved charges (of course, the Gaussian theory (2.2.84) that captures their linear approximation does certainly have such an extended set of charges).

Reference Else et al. [2021] explored the 'kinematic' consequences of the anomaly (2.3.41), which must hold for any dynamics realizing it. The Gaussian bosonized action (2.2.84) is a preferred (nonlinear) realization of the $LU(1)$ symmetry and its anomaly, but this does not exclude the interesting possibility of other realizations of this symmetry. At the nonlinear level, one could more generally study consequences of the covariant Ward identity (2.3.23), which may hold in states beyond Fermi liquids.

2.3.5 Boost symmetry in the EFT

Invariance under Galilean boosts is known to constrain the dispersion relation of free fermions to be quadratic in momentum, as well as relate the effective mass of quasiparticles in the interacting theory to Landau parameters. In this section, we show how these constraints arise in our approach. Galilean boosts are implemented on the one-particle phase space as a time-dependent canonical transformation $W = \exp B_v$, where the Lie algebra element B_v, parametrized by a boost velocity $\mathbf{v}$ is defined as

$$B_v = \mathbf{v} \cdot (\mathbf{p}t - m\mathbf{x}) \,. \tag{2.3.52}$$

The action of the boost on elements of $\mathfrak{g}$ and $\mathfrak{g}^*$ can be shown to take the following form

$$\begin{aligned} F(\mathbf{x},\mathbf{p}) &\to (\mathrm{Ad}_W F)(\mathbf{x},\mathbf{p}) = F(\mathbf{x}-\mathbf{v}t, \mathbf{p}-m\mathbf{v}) \,, \\ f(\mathbf{x},\mathbf{p}) &\to (\mathrm{Ad}^*_W f)(\mathbf{x},\mathbf{p}) = f(\mathbf{x}-\mathbf{v}t, \mathbf{p}-m\mathbf{v}) \,. \end{aligned} \tag{2.3.53}$$

Let us illustrate how to implement invariance under Galilean boosts in the example of the free theory. Recall that the action is given by

$$S = -\int dt \left\langle f_0, U^{-1} \left(\partial_t - \epsilon\right) U \right\rangle \,. \tag{2.3.54}$$

The boost acts on U as $U \to WU$. Under this transformation, the action changes to

$$S \to -\int dt \left[\left\langle f_0, U^{-1} \partial_t U \right\rangle + \left\langle f, W^{-1}(\partial_t - \epsilon) W \right\rangle \right] . \tag{2.3.55}$$

Boost invariance of the action then relies upon the following constraint,

$$\left\langle f, W^{-1}(\partial_t - \epsilon)W \right\rangle = -\langle f, \epsilon \rangle, \tag{2.3.56}$$

for every state f in the coadjoint orbit. This is only possible if

$$W^{-1}\partial_t W = W^{-1}\epsilon W - \epsilon. \tag{2.3.57}$$

Using the fact that $W^{-1}\epsilon W = \epsilon(\mathbf{p} + m\mathbf{v})$ and the expansion for $W^{-1}\partial_t W$, the above equation, for infinitesimal boost velocity, reduces to

$$\mathbf{v} \cdot \mathbf{p} = m\mathbf{v} \cdot \nabla_{\mathbf{p}}\epsilon. \tag{2.3.58}$$

Stripping off the factor of $\mathbf{v}$, we can integrate the equation to find that the dispersion relation must be quadratic

$$\epsilon(\mathbf{p}) = \frac{p^2}{2m} + \text{const.} \tag{2.3.59}$$

The constant term gives a constant contribution to the Hamiltonian which does not affect the dynamics. We stress that (2.3.59) is a constraint on the whole function $\epsilon(p)$. Viewing the derivatives of the dispersion relation at the Fermi surface $\epsilon'(p_F)$, $\epsilon''(p_F)$, etc. as Wilsonian coefficients, these are <u>all</u> fixed by boost invariance, for the case of free fermions. In the interacting case, one similarly obtains an infinite tower of constraints relating $\epsilon^{(n)}(p_F)$ and the other interaction terms in (2.2.51). We derive the leading constraint in this tower below.

Let us now consider the interacting case, limiting ourselves to the leading order in gradients and to terms up to $O(\delta f^2)$ in the action, i.e. we consider the constraints of boost invariance on the leading quadratic interaction $F^{(2,0)}_{\text{int}}(\mathbf{p}, \mathbf{p}')$ in (2.2.51). Recall that this function contains the Landau parameters, so we expect to obtain the quasiparticle effective mass as a result of the boost constraint.

Since the free theory with quadratic dispersion is already invariant under boosts, boost invariance reduces to the invariance of the shifted Hamiltonian

$$\tilde{\mathscr{H}}[f] = \int_{\mathbf{x},\mathbf{p}} \left(\epsilon(\mathbf{p}) - \frac{p^2}{2m}\right) f(\mathbf{x},\mathbf{p}) + \int_{\mathbf{x},\mathbf{p},\mathbf{p}'} F^{(2,0)}_{\text{int}}(\mathbf{p},\mathbf{p}')\delta f(\mathbf{x},\mathbf{p})\delta f(\mathbf{x},\mathbf{p}') + O(\delta f^3) \tag{2.3.60}$$

under the transformation

$$f(t,\mathbf{x},\mathbf{p}) \to (\mathrm{Ad}^*_W f)(t,\mathbf{x},\mathbf{p}) = f(t,\mathbf{x}-\mathbf{v}t,\mathbf{p}-m\mathbf{v}). \tag{2.3.61}$$

The ellipses in $\tilde{\mathscr{H}}$ denote the higher order terms. For infinitesimal boosts, the linearization of this transformation in $\mathbf{v}$ suffices

$$f \to f + \{B_v, f\} = f - t\mathbf{v}\cdot\nabla_{\mathbf{x}} f - m\mathbf{v}\cdot\nabla_{\mathbf{p}} f\,. \tag{2.3.62}$$

The fluctuation $\delta f = f - f_0$ transforms nonlinearly

$$\delta f \to \delta f - t\mathbf{v}\cdot\nabla_{\mathbf{x}}\delta f - m\mathbf{v}\cdot\nabla_{\mathbf{p}}\delta f - m\mathbf{v}\cdot\nabla_{\mathbf{p}} f_0\,. \tag{2.3.63}$$

We therefore see that the transformation can either leave the number of δf's in a given term unchanged, or at most reduce it by one. Under this transformation, the shifted Hamiltonian (2.3.60) transforms to the following

$$\begin{aligned}\tilde{\mathscr{H}}[f] \to \tilde{\mathscr{H}}[f] &- m\mathbf{v}\cdot\int_{\mathbf{x},\mathbf{p}}\left(\epsilon - \frac{p^2}{2m}\right)\nabla_{\mathbf{p}} f_0 \\ &+ m\mathbf{v}\cdot\int_{\mathbf{x},\mathbf{p}}\left(\nabla_{\mathbf{p}}\epsilon - \frac{\mathbf{p}}{m} - 2\int_{\mathbf{p}'} F^{(2,0)}_{\text{int}}(\mathbf{p},\mathbf{p}')\nabla_{\mathbf{p}'} f_0(\mathbf{p}')\right)\delta f(\mathbf{x},\mathbf{p}) \\ &+ O(\delta f^2).\end{aligned} \tag{2.3.64}$$

In particular, terms with $\nabla_{\mathbf{x}}\delta f$ vanish by virtue of being total derivatives. The $O(\delta f^3)$ that we have not kept track of in (2.3.60) have become $O(\delta f^2)$ due to the nonlinear transformation (2.3.63). The first line above vanishes due to rotational invariance. The second line gives us a new constraint that relates the dispersion to the quadratic interaction

$$\nabla_{\mathbf{p}} \epsilon = \frac{\mathbf{p}}{m} + 2 \int_{\mathbf{p}'} F_{\text{int}}^{(2,0)}(\mathbf{p}, \mathbf{p}') \nabla_{\mathbf{p}'} f_0(\mathbf{p}') \,, \qquad \text{for } |\mathbf{p}| = p_F. \tag{2.3.65}$$

This equation is evaluated at $|\mathbf{p}| = p_F$ because δf is localized on the Fermi surface. This can be simplified by using the expression for $f_0(\mathbf{p}')$, and by using rotation symmetry to expand $F_{\text{int}}^{(2,0)}$ as

$$F_{\text{int}}^{(2,0)}(\mathbf{p}, \mathbf{p}') = \frac{4\pi^2 v_F}{p_F^2} \sum_{l \geq 0} F_l \cos[l(\theta - \theta')] \,, \tag{2.3.66}$$

where θ and θ' are the angles that $\mathbf{p}$ and $\mathbf{p}'$ respectively make with the p_x axis in momentum space (we have set $d = 2$ for simplicity). Eq. (2.3.65) simplifies to

$$\nabla_{\mathbf{p}} \epsilon = \frac{\mathbf{p}}{m} - v_F F_1 \mathbf{n}_\theta \,, \qquad \text{for } |\mathbf{p}| = p_F. \tag{2.3.67}$$

Finally, evaluating the above at the Fermi surface $|\mathbf{p}| = p_F$, and defining the effective mass as $\nabla_{\mathbf{p}} \epsilon|_{|\mathbf{p}|=p_F} \equiv p_F/m^*$, we find

$$\frac{1}{m^*} = \frac{1}{m} - \frac{F_1}{m^*} \,, \tag{2.3.68}$$

which gives us the quasiparticle effective mass

$$m^* = (1 + F_1) m, \tag{2.3.69}$$

as expected.

Keeping higher order terms $O(\delta f^3)$ above leads to a tower of additional constraints, relating higher derivatives of the dispersion relation to Landau parameters and higher interaction terms such as $F_{\text{int}}^{(n,0)}$ in (2.2.51).

Constraints from the underlying Galilean boost invariance of the system can also be implemented in the fermionic EFTs for Fermi liquids Polchinski [1992], Shankar [1994], see Ref. Rothstein and Shrivastava [2018]. For relativistic systems, Lorentz invariance should similarly constrain the Wilsonian coefficients in (2.2.51)—this is less straightforward to implement in our nonrelativistic approach, so we leave it as an interesting extension for future work. See Refs. Alberte and Nicolis [2020], Komargodski et al. [2021] for a discussion of constraints from Lorentz boost symmetry on Fermi liquids.

2.3.6 Algebras from fermion bilinears

The Poisson algebra $\mathfrak{g}$ used throughout the paper, and derived semiclassically in (2.2.2), can also be obtained from a free fermion description as the low momentum limit of the algebra of fermion bilinears

$$\mathcal{O}(x,y) \equiv \psi^\dagger(x)\psi(y)\,. \tag{2.3.70}$$

Using the free fermion algebra

$$\{\psi(x), \psi^\dagger(y)\} = \delta(x-y)\,, \tag{2.3.71}$$

one finds that the bilinears satisfy

$$[\mathcal{O}(x,y), \mathcal{O}(x',y')] = \delta(y-x')\mathcal{O}(x,y') - \delta(x-y')\mathcal{O}(x',y)\,. \tag{2.3.72}$$

Consider a Wigner-like transform Das et al. [1992]

$$\mathcal{O}^W(q,y) \equiv \int_x \mathcal{O}(x+\tfrac{y}{2}, x-\tfrac{y}{2})e^{iqx}\,. \tag{2.3.73}$$

(this is really the Fourier-transform of the usual Wigner function). Physically, x is the center of mass coordinate, so q is the total momentum of the operator. We will be interested in low momentum operators with small point splitting, so that $qy \ll 1$. These operators satisfy the algebra

$$[\mathcal{O}^W(q,y), \mathcal{O}^W(q',y')] = -2i \sin \frac{1}{2}(q'y - qy')\, \mathcal{O}^W(p+p', y+y')\,. \tag{2.3.74}$$

In the limit $yq \ll 1$, this realizes the Poisson algebra:

$$[\mathcal{O}^W(q,y), \mathcal{O}^W(q',y')] \simeq -i(q'y - qy')\, \mathcal{O}^W(q+q', y+y')\,. \tag{2.3.75}$$

To connect with a more familiar representation (2.2.4) of the Poisson algebra, Fourier transform twice to obtain the following basis of generators:

$$T(x,p) = i \int_{qy} e^{iqx} e^{ipy} \mathcal{O}^W(q,y)\,, \qquad \mathfrak{g} = \mathrm{span}\{T(x,p)\}\,, \tag{2.3.76}$$

the factor of i makes $T(x,p)$ antihermitian, to match with our conventions in the main text. These satisfy the algebra

$$[T(x,p), T(x',p')] = \Big(\partial_x \partial_{p'} - \partial_p \partial_{x'}\Big) \left[\delta(x-x')\delta(p-p')T(x,p)\right]\,. \tag{2.3.77}$$

A general element of the algebra $F = \int_{xp} F(x,p)T(x,p)$ then satisfies

$$[F,G] = \int_{xp} \{F,G\}(x,p)T(x,p)\,, \tag{2.3.78}$$

with the Poisson bracket defined as usual as

$$\{F,G\} = \partial_x F \partial_p G - \partial_p F \partial_x G\,. \tag{2.3.79}$$

If one instead used the algebra (2.3.74) without taking the small momentum limit, one would have obtained (2.3.78) with instead the Moyal bracket

$$\{F, G\}_{\text{Moyal}} = 2\, F(x,p) \sin \frac{1}{2} \left(\overleftarrow{\partial}_x \overrightarrow{\partial}_p - \overleftarrow{\partial}_p \overrightarrow{\partial}_x \right) G(x,p)\,. \tag{2.3.80}$$

This will lead to higher derivative corrections $\sim \frac{1}{k_F}\partial_x$ to the EFT (2.2.52), and in particular to the WZW term Dhar et al. [1993a], Khveshchenko [1995], which adds to the other higher gradient corrections mentioned in Sec. 2.2.3. In this paper, we focus on observables to leading nontrivial order in q/k_F, so mostly do not make use of these higher derivative corrections.

Including spin

This approach can be straightforwardly extended to obtain the appropriate algebra for Fermi surfaces with spin, discussed in Sec. 2.2.7. One now considers the following fermion bilinears

$$\mathcal{O}_a(x,y) \equiv \psi^\dagger_\sigma(x) T_a \psi_{\sigma'}(y)\,. \tag{2.3.81}$$

The matrices T_a are defined as $T_a = \frac{1}{2}\sigma_a$ for $a = 1,2,3$ and $T_0 = \mathbb{1}$. Using the fermion algebra

$$\{\psi_\sigma(x), \psi^\dagger_{\sigma'}(y)\} = \delta_{\sigma\sigma'}\delta(x-y)\,, \tag{2.3.82}$$

one finds that the Wigner transforms $\mathcal{O}^W_a(q,y) \equiv \int_x \mathcal{O}_a(x+\frac{y}{2}, x-\frac{y}{2})e^{iqx}$ satisfy

$$\begin{aligned} &[\mathcal{O}^W_a(q,y), \mathcal{O}^W_{a'}(q',y')] \\ &= -\left([T_a, T_{a'}]^{\sigma\sigma'} \cos\frac{qy'-q'y}{2} + i\{T_a, T_{a'}\}^{\sigma\sigma'} \sin\frac{qy'-q'y}{2} \right) \mathcal{O}^W_{\sigma\sigma'}(q+q', y+y')\,. \end{aligned} \tag{2.3.83}$$

In the semiclassical limit these become

$$[\mathcal{O}_0^W(q,y),\mathcal{O}_0^W(q',y')] \simeq -i(q'y-qy')\mathcal{O}_0^W(q+q',y+y'), \tag{2.3.84a}$$

$$[\mathcal{O}_0^W(q,y),\mathcal{O}_i^W(q',y')] \simeq -i(q'y-qy')\mathcal{O}_i^W(q+q',y+y'), \tag{2.3.84b}$$

$$[\mathcal{O}_i^W(q,y),\mathcal{O}_j^W(q',y')] \simeq if_{ijk}\mathcal{O}_k^W(q+q',y+y')\,. \tag{2.3.84c}$$

One can verify that this truncation of the Taylor expansion of the commutator still satisfies the Jacobi identity. This will induce an algebra on the distribution functions $f_a(x,p)$, which can be thought of as an extension of the Poisson bracket. Taking $T_a(x,p) = i\int_{qy} e^{iqx}e^{ipy}\mathcal{O}_a^W(q,y)$ as before and letting $F \equiv \int_{xp} F^a(x,p)T_a(x,p)$, with similar expressions for G, H, one finds

$$[F,G] = H, \tag{2.3.85}$$

with

$$H^0 = \{F^0, G^0\}\,, \tag{2.3.86a}$$

$$H^k = \{F^0, G^k\} + \{F^k, G^0\} - F^iG^jf_{ij}{}^k\,, \tag{2.3.86b}$$

where $\{\cdot,\cdot\}$ still denotes the Poisson bracket. This agrees with the algebra found directly in the semiclassical limit in Eq. (2.2.129).

Including charge ±2 bilinears

Finally, a similar extension of the algebra can be used to study charge ±2 operators

$$\mathcal{O}_2(x,y) \equiv \psi^\dagger(x)\psi^\dagger(y)\,, \qquad \mathcal{O}_{-2}(x,y) \equiv \psi(x)\psi(y) = -\mathcal{O}_2^\dagger(x,y)\,. \tag{2.3.87}$$

If a higher charge operators were included (say $\mathcal{O}_4 \sim \psi^\dagger\psi^\dagger\psi^\dagger\psi^\dagger$), all even charge operators would have to be included for the algebra to close; instead, the charge ± 2 operators above form a closed algebra with the charge neutral bilinear (2.3.70). The Poisson/Moyal algebra is linearly realized on these operators as

$$\begin{aligned}
[\mathcal{O}_0^W(p,y), \mathcal{O}_2^W(p',y')] &= e^{\frac{i}{2}(py'-p'y)}\mathcal{O}_2^W(p+p',y+y') + e^{-\frac{i}{2}(py'+p'y)}\mathcal{O}_2^W(p+p',y'-y)\,, \\
[\mathcal{O}_0^W(p,y), \mathcal{O}_{-2}^W(p',y')] &= -e^{\frac{i}{2}(p'y-py')}\mathcal{O}_{-2}^W(p+p',y+y') - e^{\frac{i}{2}(py'+p'y)}\mathcal{O}_{-2}^W(p+p',y'-y)\,,
\end{aligned} \tag{2.3.88}$$

where we have taken the Wigner transform as before. The remaining nontrivial commutator is

$$\begin{aligned}
[\mathcal{O}_2^W(p,y), \mathcal{O}_{-2}^W(p',y')] &= e^{-\frac{i}{2}(p'y+py')}\mathcal{O}^W(p+p',y-y') - e^{-\frac{i}{2}(p'y-py')}\mathcal{O}^W(p+p',y+y') \\
&\quad - e^{\frac{i}{2}(p'y-py')}\mathcal{O}^W(p+p',-y-y') + e^{\frac{i}{2}(p'y+py')}\mathcal{O}^W(p+p',y'-y) \\
&\quad - \delta(p+p')\left(\delta_{yy'} - \delta_{y,-y'}\right)\,.
\end{aligned} \tag{2.3.89}$$

For the purposes of obtaining the long-wavelength dynamics of the charge 2 distribution function in Sec. 2.3.6, it will be sufficient to take the strict small momentum limit of the commutators above, taking $e^{\frac{i}{2}(p'y+py')} \simeq 1$, etc. In this limit, the Fourier transformed generators satisfy:

$$[T_0(x,p), T_0(x',p')] \simeq 0, \tag{2.3.90a}$$

$$[T_0(x,p), T_{\pm 2}(x',p')] \simeq \pm i\delta(x-x')\big(\delta(p-p') - \delta(p+p')\big)T_{\pm 2}(x,p), \tag{2.3.90b}$$

$$[T_2(x,p), T_{-2}(x',p')] \simeq i\delta(x-x')\big(\delta(p+p') - \delta(p-p')\big)\big(T_0(x,p) + T_0(x,-p) - 1\big)\,. \tag{2.3.90c}$$

Equation of motion for Cooper pair

As discussed in Sec. 2.2.8, the algebra in Eq. (2.3.90) above can be used to extend the EFT to capture the dynamics of charge ± 2 operators. The algebra (2.3.90) is the direct sum of three parts that have the dimensionality of phase space, and a 1-dimensional central extension

$$\mathfrak{g} + \mathfrak{g}_2 + \mathfrak{g}_{-2} + \mathfrak{u}(1)_c \,. \tag{2.3.91}$$

A state in the dual space is therefore labeled by three phase space functions, and a number

$$f = \{f^0(x,p), f^2(x,p), f^{-2}(x,p), f^c\} \,. \tag{2.3.92}$$

The ground state has

$$f_0^0 = \Theta(p_F - p) \,, \qquad f_0^{\pm 2} = 0 \,, \qquad f_0^c = 1 \,. \tag{2.3.93}$$

In a fermion picture, this can be obtained by evaluating the operators $\psi^\dagger\psi$, $\psi^\dagger\psi^\dagger$, $\psi\psi$ and 1 in the Fermi sea state. The stabilizer of this state is $\mathfrak{h} + \mathfrak{u}(1)$, where $\mathfrak{h} \subset \mathfrak{g}$ is the usual stabilizer of the Fermi surface, see Eq. (2.2.28). The degrees of freedom in the coadjoint orbit are therefore $\phi(x,\theta)$ (or $f^0(x,\theta)$), and $f^{\pm 2}(x,p)$. The Hamiltonian can now be a general functional

$$H = H[f^0, f^2, f^{-2}] \,. \tag{2.3.94}$$

We could obtain the equation of motion for $f^{\pm 2}$ by writing out the entire action, including the WZW term which now also involves $f^{\pm 2}$. Here we will show another equivalent method, which only requires the Hamiltonian. First note that functionals can be 'differentiated' to obtain elements of the algebra

$$\frac{\delta G}{\delta f} \equiv \int_{xp} \frac{\delta G}{\delta f_\alpha(x,p)} T_\alpha(x,p) \,, \tag{2.3.95}$$

where the summation runs over $\alpha = 2, 0, -2$. This allows one to define the Lie-Poisson structure of two functionals Kirillov [2004], Arnold and Khesin [2013]

$$[F, G]_{\mathrm{LP}} \equiv \left\langle f, \left[\frac{\delta F}{\delta f}, \frac{\delta G}{\delta f}\right]\right\rangle . \tag{2.3.96}$$

One can show that the equation of motion of another functional of f is

$$\dot{F} = [F, H]_{\mathrm{LP}} , \tag{2.3.97}$$

where H is the Hamiltonian. Taking say $F = f_2(x,p)$, one can obtain the equation of motion for the distribution function. Consider for example the Hamiltonian

$$H[f, f_2, f_{-2}] = \int_{xp} \epsilon(p) f(x,p) + \frac{1}{2}\int_{xpp'} V_{\mathrm{BCS}}(p,p') f_2(x,p) f_{-2}(x,p') . \tag{2.3.98}$$

One has

$$\frac{\delta H}{\delta f} = \int_{xp} \epsilon(p) T(x,p) + \frac{1}{2}\int_{xpp'} V_{\mathrm{BCS}}(p,p') \left(T_2(x,p) f_{-2}(x,p') + f_2(x,p) T_{-2}(x,p')\right) . \tag{2.3.99}$$

The equation of motion for the charge-2 distribution function is therefore

$$\begin{aligned} \dot{f}^2(x,p,t) &= \left\langle f, \left[T_2(x,p), \frac{\delta H}{\delta f}\right]\right\rangle \\ &= -2i\epsilon(p) f^2(x,p,t) \\ &\quad - i\left(f^0(x,p,t) + f^0(x,-p,t) - 1\right)\int_{p'} f^2(x,p',t) V_{\mathrm{BCS}}(p',p) . \end{aligned} \tag{2.3.100}$$

This nonlinear equation describes the dynamics of Cooper pairs, or the charge-2 distribution function, coupled to a dynamical Fermi surface. Linearizing, i.e. setting $f^0(x,p,t) \to f_0^0(p) =$

$\Theta(p - p_F)$ in the RHS and choosing $p > p_F$ gives

$$\dot{f}^2(x,p,t) = -2i\epsilon(p) f^2(x,p,t) + i \int_{p'} f^2(x,p',t) V_{\rm BCS}(p',p)\,. \tag{2.3.101}$$

2.3.7 Nonlinear response from kinetic theory

Nonlinearities in the effective field theory (2.2.52) are crucial to capture nonlinear response of Fermi liquids, as illustrated through the density three-point function in Sec. 2.2.5. Nonlinear response even occurs for free fermions, where it can be studied in kinetic theory. This section presents this alternative approach, inspired by the derivation of hard dense loops from kinetic theory Manuel [1996] (following a similar result for hard thermal loops Blaizot and Iancu [1993], Kelly et al. [1994]), as a consistency check of the results in Sec. 2.2.5. This approach to computing density n-point functions in a Fermi gas is substantially simpler than the direct evaluation of a fermion loop with n insertions Feldman et al. [1998], Neumayr and Metzner [1998].

We consider a free Fermi gas, described by the collisionless Boltzmann equation. In order to generate correlation functions of densities $\rho(x,t)$, we turn on an arbitrary background field $A_0(x,t)$. The Boltzmann equation then reads

$$0 = \frac{d}{dt} f(x,p,t) = \left(\partial_t + v^i(p)\partial_{x^i} + E_i \partial_{p_i}\right) f(x,p,t)\,. \tag{2.3.102}$$

We take the distribution function f to be given by a sharp but fluctuating Fermi surface

$$f(x,p,t) = \Theta(p_F(x,t,\theta) - p)\,, \tag{2.3.103}$$

and expand around a spherical Fermi surface $p_F(x,t,\theta) = p_F + \delta p_F(x,t,\theta)$. The density is given by

$$\rho(x,t) = \int \frac{d^d p}{(2\pi)^d} f(x,p,t) = \frac{S_{d-1}}{d(2\pi)^d} p_F^d + \delta\rho(x,t)\,, \tag{2.3.104}$$

using (2.3.103) one finds that it is related to $\delta p_F(x,t,\theta)$ as

$$\delta\rho(x,t) = \frac{p_F^{d-1}}{(2\pi)^d}\int d^{d-1}\theta\left(\delta p_F + \frac{1}{p_F}\frac{d-1}{2}(\delta p_F)^2 + \cdots\right). \tag{2.3.105}$$

We wish to compute n-point functions $\langle\rho(x_1,t_1)\cdots\rho(x_n,t_n)\rangle$, limiting ourselves to $n=2,3$ here. We will expand the fluctuations in terms of their response to the background $A_0(t,x)$ as

$$\delta\rho = \delta\rho^{(1)} + \delta\rho^{(2)} + \cdots\,, \qquad \delta p_F = \delta p_F^{(1)} + \delta p_F^{(2)} + \cdots\,, \tag{2.3.106}$$

with $\delta\rho^{(n)}$, $\delta p_F^{(n)} = O((A_0)^n)$. Inserting (2.3.103) in the kinetic equation leads to

$$\left(\partial_t + v^i(p_F+\delta p_F,\theta)\partial_i + \frac{E\cdot\hat{s}^i}{p_F+\delta p_F}\partial_{\theta^i}\right)\delta p_F = E\cdot\hat{n}\,, \tag{2.3.107}$$

which will govern nonlinear response. To leading order, we find

$$\delta p_F^{(1)}(\omega,q,\theta) = \frac{\hat{n}\cdot q}{\omega - v_F\hat{n}\cdot q}A_0(\omega,q). \tag{2.3.108}$$

We therefore obtain the density two-point function by differentiating with respect to A_0

$$\langle\rho\rho\rangle(\omega,q) = -i\frac{p_F^{d-1}}{(2\pi)^d}\int d^{d-1}\theta\frac{\hat{n}(\theta)\cdot q}{\omega - v_F\hat{n}(\theta)\cdot q}\,. \tag{2.3.109}$$

This agrees with what was found from the EFT (2.2.87), where the remaining integral over angles was evaluated. Let us now obtain $\delta p_F^{(2)}$ to determine the three-point function $\langle\rho\rho\rho\rangle$. The $O((A_0)^2)$ terms in the kinetic equation (2.3.102) are

$$\left(\partial_t + v_F\hat{n}\cdot\nabla\right)\delta p_F^{(2)} + \epsilon''(p_F)\delta p_F^{(1)}\hat{n}\cdot\nabla\delta p_F^{(1)} + \frac{E\cdot\hat{s}^i}{p_F}\partial_{\theta^i}\delta p_F^{(1)} = 0\,. \tag{2.3.110}$$

We see that there are two sources of nonlinearities – one from the curvature of the dispersion relation (which vanishes for a relativistic fermion, and equals $\epsilon''(p_F) = v'(p_F) = \frac{1}{m}$ for a non-relativistic fermion), and one from the last term which is always there. Let us start by setting $\epsilon''(p_F) = 0$. Solving (2.3.110) for $\delta p_F^{(2)}$ we have

$$\begin{aligned}
\delta\rho^{(2)} &= \frac{p_F^{d-1}}{(2\pi)^d} \int d^{d-1}\theta \left(\delta p_F^{(2)} + \frac{1}{p_F} \frac{d-1}{2} (\delta p_F^{(1)})^2 \right) \\
&= \frac{p_F^{d-2}}{(2\pi)^d} \int d^{d-1}\theta \, \frac{d-1}{2} \left(\frac{\hat{n}\cdot\nabla}{\partial_t + v_F \hat{n}\cdot\nabla} A_0 \right)^2 \\
&\qquad - \frac{1}{\partial_t + v_F \hat{n}\cdot\nabla} \hat{s}^i \nabla A_0 \partial_{\theta^i} \left(\frac{\hat{n}\cdot\nabla}{\partial_t + v_F \hat{n}\cdot\nabla} A_0 \right) .
\end{aligned} \tag{2.3.111}$$

Differentiating with respect to A_0 leads to the response function

$$\begin{aligned}
\langle \rho(\omega,q)\rho(\omega',q')\rho \rangle = \frac{p_F^{d-2}}{(2\pi)^d} \int d^{d-1}\theta \, \frac{1}{\omega+\omega' - v_F \hat{n}\cdot(q+q')} \left[\hat{s}^i \cdot q' \partial_{\theta^i} \left(\frac{\hat{n}\cdot q}{\omega - v_F \hat{n}\cdot q} \right) + (\omega, q \leftrightarrow \omega', q') \right] \\
- (d-1) \frac{\hat{n}\cdot q}{\omega - v_F \hat{n}\cdot q} \frac{\hat{n}\cdot q'}{\omega' - v_F \hat{n}\cdot q'} .
\end{aligned} \tag{2.3.112}$$

This manifestly vanishes when q or $q' = 0$ as required by charge conservation; although it is not manifest it also vanishes when $q = -q'$. In fact it is possible (but tedious) to show that after integration over θ, the integral satisfies the permutation symmetry of the correlator, generated by $\{\omega, q\} \leftrightarrow \{\omega', q'\}$ and $\{\omega, q\} \to \{-\omega-\omega', -q-q'\}$. The contribution proportional to $v'(p_F)$ can be obtained in a similar manner. One finds

$$\langle \rho(\omega,q)\rho(\omega',q')\rho \rangle = (2.3.112) - \frac{p_F^{d-1}}{(2\pi)^d} \epsilon'' \int d^{d-1}\theta \frac{\hat{n}\cdot(q+q')}{\omega+\omega' - v_F \hat{n}\cdot(q+q')} \frac{\hat{n}\cdot q}{\omega - v_F \hat{n}\cdot q} \frac{\hat{n}\cdot q'}{\omega' - v_F \hat{n}\cdot q'} , \tag{2.3.113}$$

which manifestly satisfies permutation symmetry and charge conservation.

The (partial) static limit $\lim_{\omega' \to 0}$ of the three-point function is related to the dependence of the two-point function on a background static potential $\mu(x)$. If one further takes the subsequent limit $\lim_{q' \to 0}$, it is then related to the variation of the two-point function under

$\delta\mu = v_F \delta p_F$:

$$\lim_{q' \to 0} \lim_{\omega' \to 0} \langle \rho(\omega, q)\rho(\omega', q')\rho \rangle = \frac{-i}{v_F} \frac{\partial}{\partial p_F} \langle \rho(\omega, q)\rho \rangle \,, \tag{2.3.114}$$

which provides an additional consistency check of the three-point function. Taking this limit of Eqs. (2.3.112) and (2.3.113) one finds indeed

$$\begin{aligned} \lim_{q' \to 0} \lim_{\omega' \to 0} \langle \rho(\omega, q)\rho(\omega', q')\rho \rangle &= \frac{p_F^{d-2}}{(2\pi)^d v_F} \int d^{d-1}\theta \left[(d-1) \frac{\hat{n} \cdot q}{\omega - v_F \hat{n} \cdot q} + \epsilon'' p_F \left(\frac{\hat{n} \cdot q}{\omega - v_F \hat{n} \cdot q} \right)^2 \right] \\ &= \frac{1}{v_F} \frac{\partial}{\partial p_F} \left[\frac{p_F^{d-1}}{(2\pi)^d} \int d^{d-1}\theta \frac{\hat{n} \cdot q}{\omega - v_F(p_F) \hat{n} \cdot q} \right] \\ &= \frac{-i}{v_F} \frac{\partial}{\partial p_F} \langle \rho(\omega, q)\rho \rangle \,. \end{aligned} \tag{2.3.115}$$

CHAPTER 3

INTRODUCTION: NONCOMMUTATIVE FIELD THEORY MEETS VOLUME-PRESERVING DIFFEOMORPHISM

In the previous chapter, we introduced a novel bosonization technique applied to Fermi surfaces, characterized by a semiclassical depiction of the many-body system. This approach considers a dense aggregation of particles within their individual single-particle phase spaces, collectively forming a region of constant density within a connected domain, referred to as the droplet. By quantizing the remaining phase space coordinates at the classical level, we transition to a description of noncommutative field theory. Conceptually, the Fermi surface emerges as a droplet embedded within noncommutative space.

It is established that phase space volume-preserving diffeomorphisms (VPDs) correspond to unitary transformations, which at the semiclassical level reduce to canonical transformations. Consequently, area-preserving diffeomorphisms (APDs) are isomorphic to canonical transformations of the phase space of a particle in 1+1 dimensions. Similar principles extend to higher dimensions, albeit with a caveat: canonical transformations, which represent the classical manifestation of noncommutative gauge transformations, solely constitute a symplectic subgroup within the full volume-preserving diffeomorphism.

3.0.1 Noncommutative quantum field theory

Noncommutativity represents a longstanding theme deeply embedded within both mathematics and physics. Noncommutative quantum field theory Douglas and Nekrasov [2001], Rubakov [2005], Du et al. [2021], or quantum field theory on noncommutative spacetime, represents a compelling application of noncommutative geometry to the framework of quantum field theory. This emerges as a natural outgrowth of noncommutative structure, where coordinates x^i with commutators yielding c-numbers:

$$[x^i,\ x^j] = i\theta^{ij} \ , \tag{3.0.1}$$

with θ is real. Given this Lie algebra, the objective is to deform the algebra of functions onto a noncommutative, associative algebra such that:

$$f \star g = f \exp\left(\frac{i}{2}\theta^{ij}\overleftarrow{\partial}_i\overrightarrow{\partial}_j\right) g \ . \tag{3.0.2}$$

It defines what is called the Moyal bracket of functions, describing the deformation. One notable characteristic of noncommutative field theories is the UV/IR mixing phenomenon, wherein high-energy physics influences low-energy phenomena, a phenomenon absent in quantum field theories where coordinates commute.

A concrete physical picture directly leading to noncommutativity arises from the drift motion of a single electron within a constant magnetic field B, particularly when formulated within the limit of a single Landau level, such as the lowest Landau level. This phenomenon is extensively studied in the theory of the quantum Hall effect Du et al. [2021].

More recently, it has been proposed that the quantum field theory governing the fractional quantum Hall droplets necessitates a noncommutative field theory framework Susskind [2001], Polychronakos [2001], Hellerman and Van Raamsdonk [2001], Fradkin et al. [2002], Cappelli and Rodriguez [2009]. In this context, in the semiclassical limit, the noncommutative theory nature reduces to area-preserving diffeomorphisms (or volume-preserving diffeomorphisms in two spatial dimensions). The exploration of the area-preserving diffeomorphisms and its associated APD algebra has led to a profound understanding of the long-wavelength limit encapsulated within the Girvin-MacDonald-Platzman (GMP) algebra Girvin et al. [1986], also known as the (w_∞ algebra). The APD algebra is characterized by the commutation relation:

$$[X_F\ X_G] = X_{\{F,\ G\}_\theta},\ \{F,\ G\}_\theta = \theta^{ij}\partial_i F \partial_j G\ , \tag{3.0.3}$$

where the operator X_F is defined as:

$$X_F = i\int_{\mathbf{x}} F(\mathbf{x})\rho(\mathbf{x})\ . \tag{3.0.4}$$

Here, $\rho(\mathbf{x})$ represents the generator of the APD group, which corresponds to the projected densities. Additionally, the noncommutative parameter θ is intricately related to the magnetic field B as $\theta = \frac{1}{B}$. Through this framework, one can establish the w_∞ algebra as

$$[\rho(\mathbf{k}),\ \rho(\mathbf{q})] = i\theta(\mathbf{k}\times\mathbf{q})\rho(\mathbf{k}+\mathbf{q})\ , \tag{3.0.5}$$

as part of a broader understanding of the underlying mathematical structure.

3.0.2 Volume-preserving diffeomorphisms

It is widely acknowledged that the semiclassical limit of the noncommutative $U(1)$ gauge group is isomorphic to the group of area-preserving diffeomorphisms in two spatial dimensions. However, in higher spatial dimensions, the noncommutative gauge groups correspond to volume-preserving diffeomorphisms. Unlike the Poisson structure characterizing area-preserving diffeomorphisms, the higher-dimensional generalization VPD structure finds its definition through the Nambu-Poisson bracket, which serves as a generalization of Poisson brackets based on a canonical multiple Nambu [1973], Takhtajan [1994]. One example for generalization of Poisson brackets by a canonical triple x^1, x^2, x^3, is defined by

$$\{f, g, h\} = \epsilon^{abc}(x)\partial_a f(x)\partial_b g(x)\partial_c h(x)\,. \tag{3.0.6}$$

A fascinating correlation emerges between the Ward identity for volume-preserving diffeomorphisms and phenomena such as exotic field theories and fractons Du et al. [2022, 2024a,b]. These findings impose notably restrictive mobility constraints on local excitations within the theory: isolated charges exhibit immobility, while dipoles are constrained to motion solely orthogonal to their dipole moment. This exotic nature finds its connection with the broader exploration of nonabelian higher-rank gauge theories within the framework of volume-preserving diffeomorphisms.

In the forthcoming chapter, we shall explore the significance of volume-preserving diffeomorphisms and their interrelation with noncommutative field theory. Furthermore, we will elucidate their relevance to physical applications across condensed matter systems.

3.1 Story 2: Volume-preserving diffeomorphism as nonabelian higher-rank gauge symmetry

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Yi-Hsien Du, Umang Mehta, Dung Xuan Nguyen and Dam Thanh Son. Volume-preserving diffeomorphism as nonabelian higher-rank gauge symmetry. SciPost Phys. 12, 050 (2022), Feb 2022.

Abstract

We propose nonabelian higher-rank gauge theories in 2+1D and 3+1D. The gauge group is constructed from the volume-preserving diffeomorphisms of space. We show that the intriguing physics of the lowest Landau level (LLL) limit can be interpreted as the consequences of the symmetry. We derive the renowned Girvin-MacDonald-Platzman (GMP) algebra as well as the topological Wen-Zee term within our formalism. Using the gauge symmetry in

2+1D, we derive the LLL effective action of vortex crystal in rotating Bose gas as well as Wigner crystal of electron in an applied magnetic field. We show that the nonlinear sigma models of ferromagnets in 2+1D and 3+1D exhibit the higher-rank gauge symmetries that we introduce in this paper. We interpret the fractonic behavior of the excitations on the lowest Landau level and of skyrmions in ferromagnets as the consequence of the higher-rank gauge symmetry.

3.1.1 Introduction

Recently, considerable interest has been drawn to "higher-rank gauge theories," i.e., theories where the gauge potential is not a one-form, but a tensor of higher rank Pretko [2017a,d], Slagle and Kim [2017], Prem et al. [2018], Zhai and Radzihovsky [2021], Wang and Yau [2020], Brauner [2021], Dubinkin et al. [2020], Qi et al. [2021], Seiberg [2020], Seiberg and Shao [2021], **?**], Gromov [2019b,a]. The physical motivation of the higher-rank gauge theories is the discovery of a new class of topological matter known as *fractons* Chamon [2005], Bravyi et al. [2011], Haah [2011], Vijay et al. [2015], Pretko et al. [2020], one feature of which is the existence of excitations with restricted mobility. The excitations either cannot move at all or can only move in lower-dimensional sub-spaces, while composites of elementary excitations can move freely. The restricted mobility of the fractonic excitations can be interpreted as the consequence of the higher-rank theories' conservation laws. In particular, the tensor Gauss's law leads to the conservation of not only the electric charge but also the electric dipole moment and, in some cases, higher moments of the charge distribution Pretko [2017a,d], Pretko et al. [2020]. This has the physical effect of rendering the electric charge immobile but leaving the dipoles mobile or partially mobile. One representative example is the so-called "traceless scalar charge theory," in which the electric charges are immobile and the electric dipole can move only in the direction perpendicular to the dipole moment Pretko [2017d]. The higher-rank gauge theories have also been applied to describe defects in solids Pretko and Radzihovsky [2018a], Cvetkovic et al. [2006], Beekman et al. [2017], Kleinert [1982, 1983], supersolids Pretko and Radzihovsky [2018b], superfluid vortices Doshi and Gromov [2021], Nguyen et al. [2020], smectics Nguyen et al. [2020], Zhai and Radzihovsky [2021], Radzihovsky [2020], and other systems.

All the higher-rank gauge symmetries in the physical models mentioned above are abelian. There were attempts to generalize these symmetries to nonabelian symmetries Wang and Yau [2020], Wang and Xu [2021], Wang et al. [2021], **?**; however, physical systems that realize those symmetries have not been explicitly proposed. In this paper, we propose nonabelian higher-rank symmetry theories for condensed matter systems of physical relevance. We reformulate the tensor gauge transformation in the traceless scalar charge theory in 2+1D Pretko [2017a,d], showing that the gauge transformation is nothing but the volume-preserving diffeomorphism (VPD) in the linearized form. We then construct nonlinear higher-rank gauge theories with VPDs as the symmetry group. The nonabelian nature of the gauge symmetry has a nontrivial consequence: the charge density operators at different positions do not commute. Instead, they form the long-wavelength limit of the Girvin-MacDonald-Platzman (GMP) algebra Girvin et al. [1986], Cappelli et al. [1993c], Iso et al. [1992], which reveals a connection to the lowest Landau level (LLL).

It is easy to notice that many features of the physics on the LLL bear a close resemblance to that of the field-theory models with higher rank symmetries Cappelli and Randellini [2016]: for example, electric charges are pinned to one place by the large magnetic field, and neutral excitations (e.g., the composite fermion in the half-filled Landau level Nguyen et al. [2018]) carry an electric dipole moment and can move in the direction perpendicular to the direction of motion. In this paper, we show that this resemblance is not accidental; in fact, the nonlinear higher-rank symmetry is realized as a symmetry of the problem of charged particles on the LLL. The relation between the dipole moment and momentum of an excitation in models with higher-rank symmetry was noticed in Refs. **?**Doshi and Gromov

[2021].

We also will present several physical systems that enjoy the nonabelian higher-rank symmetry. In 2+1D systems, the symmetry originates from the lowest-Landau-level limit, where the volume-preserving nature of the diffeomorphisms comes from the restriction that the transformations should preserve the background magnetic field. In addition to the derivation of the GMP algebra, we draw a connection between the topological Wen-Zee term Wen and Zee [1992b] with the Chern-Simons term in a higher-rank gauge theory.

Furthermore, we will use the gauge symmetry to derive the effective theories of the Wigner crystal of electrons in a strong magnetic field and the vortex crystal in a rotating Bose gas.

Finally, we find that the nonlinear sigma models describing ferromagnetism in 2+1D and 3+1D also exhibit the global higher-rank symmetry. The higher rank gauge symmetry provides a new interpretation of the conservation of multipole moments in ferromagnets Papanicolaou and Tomaras [1991]; it also explains the close resemblance between the behaviors of skyrmions in ferromagnets and charged particles in a magnetic field Stone [1996].

3.1.2 Review of the traceless scalar charge theory

For the paper to be self-contained, in this Section we will review the symmetric tensor gauge theory proposed by Pretko Pretko [2017a,d]. We consider a higher-rank gauge theory called "traceless scalar charge theory," where the gauge potential is a symmetric rank-2 tensor A_{ij}: $A_{ij} = A_{ji}$. Its conjugate momentum is the electric field E_{ij}. They satisfy the canonical commutation relation

$$[E_{ij}(\mathbf{x}),\, A_{kl}(\mathbf{y})] = i(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk})\delta(\mathbf{x} - \mathbf{y}). \tag{3.1.1}$$

One imposes the Gauss law and the traceless constraint:

$$\partial_i \partial_j E_{ij} = \rho, \tag{3.1.2}$$

$$E \equiv E_{ii} = 0, \tag{3.1.3}$$

which lead to the conservation of the following charges:

$$\int d\mathbf{x}\, \rho(\mathbf{x}), \quad \int d\mathbf{x}\, \mathbf{x} \rho(\mathbf{x}), \quad \int d\mathbf{x}\, x^2 \rho(\mathbf{x}). \tag{3.1.4}$$

The conservation of the quantities listed in in Eq. (3.1.4) imply that a charge cannot move, and a dipole can move only along the direction perpendicular to the dipole moment Pretko [2017a].

The constraints (3.1.2) and (3.1.3) generate the gauge transformations on the gauge potential

$$A_{ij} \to A_{ij} + \partial_i \partial_j \lambda, \tag{3.1.5}$$

$$A_{ij} \to A_{ij} + \delta_{ij} \mu. \tag{3.1.6}$$

For our purpose, it is convenient to use the second gauge transformation (3.1.6) to explicitly fix the gauge $A \equiv A_{ii} = 0$ and eliminate the trace of A_{ij} from the set of dynamical degrees of freedom. Since the two constraints $E = 0$ and $A = 0$ do not commute according to the commutation relation (3.1.1), following Dirac one should modify the commutators, replacing them by the Dirac brackets Dirac [1950], which in our case is

$$[O_1,\, O_2] \to [O_1,\, O_2]_{\mathrm{D}} = [O_1,\, O_2] + [O_1,\, E][E,\, A]^{-1}[A,\, O_2] - [O_1,\, A][E,\, A]^{-1}[E,\, O_2] \tag{3.1.7}$$

The new commutator is then

$$[E_{ij}(\mathbf{x}),\, A_{kl}(\mathbf{y})] = i\left(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk} - \frac{2}{d}\delta_{ij}\delta_{kl}\right)\delta(\mathbf{x}-\mathbf{y}). \tag{3.1.8}$$

The Gauss constraint $\partial_i\partial_j E_{ij} = \rho$ generates now the gauge transformation

$$A_{ij} \to A_{ij} + \partial_i\partial_j\lambda - \frac{1}{d}\delta_{ij}\partial^2\lambda, \qquad \partial^2 \equiv \partial^k\partial_k. \tag{3.1.9}$$

To construct a gauge-invariant Lagrangian, we introduce the field strengths. The electric field

$$E_{ij} = \partial_i\partial_j A_0 - \frac{1}{d}\delta_{ij}\partial^2 A_0 - \partial_t A_{ij}, \tag{3.1.10}$$

is obviously gauge invariant. One notices that

$$\omega_i = -\partial_j A_{ij} \tag{3.1.11}$$

transforms like a $U(1)$ vector potential,

$$\omega_i \to \omega_i - \partial_i\tilde{\lambda}, \qquad \tilde{\lambda} = \left(1 - \frac{1}{d}\right)\partial^2\lambda, \tag{3.1.12}$$

using which one can define the magnetic field,

$$H_{ij} = \partial_i\omega_j - \partial_j\omega_i = -\partial_i\partial_k A_{jk} + \partial_j\partial_k A_{ik}, \tag{3.1.13}$$

that is manifestly gauge invariant. The simplest Lagrangian for the gauge field is then the "Maxwell theory,"

$$L = c_1 E_{ij}^2 - c_2 H_{ij}H_{ij}. \tag{3.1.14}$$

However, as noticed in Ref. Pretko [2017c], in (2+1)D, another possible term in the Lagrangian is the Chern-Simons term which, up to an overall coefficient, reads

$$L_{\mathrm{CS}} = \varepsilon^{ij}(A_0 H_{ij} - A_{ik}\dot{A}_{jk}). \tag{3.1.15}$$

As for a Chern-Simons term, the Lagrangian density is gauge-invariant only up to a total derivative. This term is more relevant than the Maxwell term. The higher-rank Chern-Simons theory in 3 spatial dimensions was also discussed previously in Ref. You et al. [2020].

Note that one does not have to require the theory to contain dynamical gauge fields in order to have the conserved quantities (3.1.4). A theory coupled to background gauge fields (A_0, A_{ij}) and is invariant under the gauge symmetry under which the gauge fields transform as

$$A_0 \to A_0 + \dot{\lambda}, \qquad A_{ij} + \partial_i\partial_j\lambda - \frac{1}{d}\delta_{ij}\partial^2\lambda, \tag{3.1.16}$$

will have the conservation law

$$\partial_t\rho - \partial_i\partial_j J_{ij} = 0, \tag{3.1.17}$$

where ρ and J_{ij} are the operators that couple to A_0 and A_{ij}, respectively, and $J_{ii} = 0$. This is sufficient to derive the conservation of the quantities (3.1.4). In fact, in most of this paper, we will consider theories coupled to nondynamical background gauge fields.

3.1.3 Nonlinear higher-ranked symmetry

Traceless scalar charge theory in (2+1)D as a theory of linearized gravity

We now show that the theory presented in the previous section can be interpreted as a theory of linearized gravity, and the higher-rank symmetry is the linearized version of VPD. Instead of A_{ij} we introduce an equivalent field h_{ij} defined as

$$h_{ij} = -\ell^2(\varepsilon_{ik}A_{jk} + \varepsilon_{jk}A_{ik}), \tag{3.1.18}$$

where ℓ is some constant of the dimension of length[1]. Note that h_{ij} is also symmetric and traceless. The gauge transformation for h_{ij} is inherited from (3.1.16)

$$h_{ij} \to h_{ij} - \ell^2(\varepsilon_{ik}\partial_j\partial_k + \varepsilon_{jk}\partial_i\partial_k)\lambda. \tag{3.1.19}$$

If we define

$$\xi^i = \ell^2\varepsilon^{ik}\partial_k\lambda, \tag{3.1.20}$$

then the transformation law of h_{ij} can be reformulated as

$$h_{ij} \to h_{ij} - \partial_i\xi_j - \partial_j\xi_i. \tag{3.1.21}$$

The transformation (3.1.21) is nothing but the transformation of the metric under the volume-preserving (or in 2D, area-preserving) diffeomorphism $x^i \to x^i + \xi^i$, since $\partial_i\xi^i = 0$ due to the definition (3.1.20). The connection between a higher-rank gauge theory and a linearized gravity theory was proposed previously in Ref. Xu [2006].

Nonlinear higher-rank symmetry

The fact that the gauge symmetry resembles the transformation law of a metric under VPDs allows one to devise a nonlinear version of the gauge symmetry. Namely, in our nonlinear theory, instead of a gauge field h_{ij} (or A_{ij} for that matter), the degree of freedom is the metric g_{ij}. The tracelessness of h_{ij} translates into the statement that the metric is unimodular: $\det g = 1$. The linear theory is restored when one expands the metric around the flat metric: $g_{ij} = \delta_{ij} + h_{ij} + O(h^2)$.

1. We assume A_0 has dimension 1 and A_{ij} is of dimension 2, so h_{ij} is dimensionless.

Under an infinitesimal VPD $x^i \to x^i + \xi^i = x^i + \ell^2 \varepsilon^{ij} \partial_j \lambda$, the metric transforms as

$$\delta_\lambda g_{ij} = -\xi^k \partial_k g_{ij} - g_{kj} \partial_i \xi^k - g_{ik} \partial_j \xi^k = -\ell^2 \varepsilon^{kl} (\partial_k g_{ij} + g_{kj} \partial_i + g_{ik} \partial_j) \partial_l \lambda. \tag{3.1.22}$$

Now we need to write down the nonlinear version of the transformation laws for A_0. One notices that the VPDs do not commute: from Eq. (3.1.22) one reads

$$[\delta_\alpha, \delta_\beta] = \delta_{[\alpha,\beta]}, \tag{3.1.23}$$

with $[\alpha, \beta] = \ell^2 \varepsilon^{ij} \partial_i \alpha \partial_j \beta$. This means that our gauge symmetry is nonabelian; in this paper, we will use "nonlinear" and "nonabelian" interchangeably. The transformation of A_0 must satisfy the commutation relation (3.1.23). One can check that this can be accomplished by the following simple modification

$$\delta_\lambda A_0 = \partial_t \lambda - \xi^k \partial_k A_0 = \partial_t \lambda - \ell^2 \varepsilon^{kl} \partial_k A_0 \, \partial_l \lambda. \tag{3.1.24}$$

The transformation of A_0 in Eq. (3.1.24) was motivated by the symmetries of the lowest Landau level that will be discussed subsequently. Nonetheless, it is the unique nonlinear generalization of (3.1.16) given that the transformation is at most linear in A_0 and respects rotational invariance. Furthermore, the second term of (3.1.24) means that A_0 transforms as a scalar field under time-independent spatial diffeomorphism, which is expected. We leave the detailed discussion on the uniqueness of the nonlinear transformation (3.1.24) to Appendix 3.1.8. Equations (3.1.24) and (3.1.22) give the transformation laws of a nonlinear higher-rank symmetry, which we collect here for convenience:

$$\delta_\lambda A_0 = \partial_t \lambda - \ell^2 \varepsilon^{kl} \partial_k A_0 \, \partial_l \lambda, \tag{3.1.25}$$

$$\delta_\lambda g_{ij} = -\ell^2 \varepsilon^{kl} (\partial_k g_{ij} + g_{kj} \partial_i + g_{ik} \partial_j) \partial_l \lambda. \tag{3.1.26}$$

A nonlinear transformation similar to Eq. (3.1.26) was considered in Ref. Gromov [2019a] within a dynamical gauge model of the traceless scalar charge theory. One can derive the Ward identity from Eqs. (3.1.25) and (3.1.26). Let us define the charge density ρ and the stress tensor T^{ij} by varying the logarithm of the partition function

$$\delta \ln Z = \int d^3x \left(\rho \delta A_0 + \frac{1}{2} T^{ij} \delta h_{ij} \right). \tag{3.1.27}$$

The Ward identity is then

$$\dot{\rho} - \ell^2 \varepsilon^{kl} \partial_l \left[\rho \partial_k A_0 + \tfrac{1}{2} T^{ij} \partial_k g_{ij} + \partial_i (T^{ij} g_{jk}) \right] = 0 \tag{3.1.28}$$

In the presence of the background field, only the total charge is conserved, but not the higher multipoles in (3.1.4).

Since the total charge is conserved, it is possible to introduce a vector potential A_i so that the theory is invariant under the usual $U(1)$ gauge symmetry $A_\mu \to A_\mu + \partial_\mu \alpha$. In this case A_0 plays a double role: it is the temporal component of a U(1) gauge field (A_0, A_i), and also as the scalar component of the gauge potential of a higher-spin symmetry, (A_0, g_{ij}). The two sets of gauge potentials share one scalar component, see Fig. 3.1. We will see an example when we consider ferromagnets (Sec. 4.2.6).

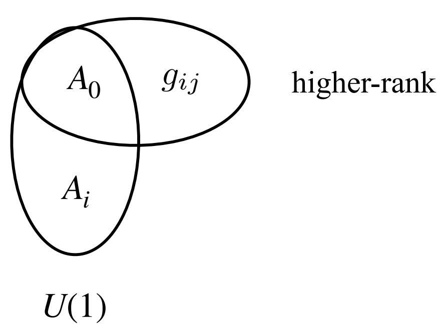

Figure 3.1: A_0 is shared by two sets of gauge potentials.

The nonabelian nature of the gauge symmetry, in some cases, allows us to derive the algebra satisfied by the charge density of the matter coupled to the gauge field. Imagine that the action $S_{\mathrm{m}}(\psi, A_0, g_{ij})$ describing the coupling of the matter fields ψ with the gauge fields (A_0, g_{ij}) does not contain the time derivatives of any fields, A_0, g_{ij}, or ψ. In this case, if one promotes the gauge fields to dynamical fields by adding to the action a pure gauge action S_{g},

$$S = S_{\mathrm{g}}[A_0, g_{ij}] + S_{\mathrm{m}}[\psi, A_0, g_{ij}], \tag{3.1.29}$$

then upon quantization, the canonical commutation relations in the gauge sector are set by S_{g} and in the matter sector by S_{m}. The left-hand side of the Gauss constraint,

$$\frac{\delta S_{\mathrm{g}}}{\delta A_0} + \frac{\delta S_{\mathrm{m}}}{\delta A_0} = 0, \tag{3.1.30}$$

is the generator that generates gauge transformations. In particular, in the matter sector, the commutator of the charge density $\rho(\mathbf{x})$ with any matter field $O(\mathbf{x})$ will give the change of O under infinitesimal gauge transformation:

$$\left[\int d\mathbf{y}\, \lambda(\mathbf{y})\rho(\mathbf{y}),\, O(\mathbf{x})\right] = i\delta_\lambda O(\mathbf{x}). \tag{3.1.31}$$

But diffeomorphisms do not commute, so we conclude that the charge density at different points does not commute in our theory. We find

$$[\rho(\mathbf{x}),\, \rho(\mathbf{y})] = i\ell^2 \varepsilon^{ij} \partial_i \rho(\mathbf{x}) \partial_j \delta(\mathbf{x} - \mathbf{y}). \tag{3.1.32}$$

Here we recover the long-wavelength version of the Girvin–MacDonald–Platzman (GMP) algebra Girvin et al. [1986] (or the w_∞ algebra), which suggests that the symmetry described here is related to the physics of the LLL.

3.1.4 Connection to quantum Hall effect

To establish the connection with the physics of the LLL, we recall the symmetry of the problem. A system of particles interacting with an electromagnetic field can also be put in curved space:

$$S = \int dt\, d\mathbf{x}\, \sqrt{g} \left(\frac{i}{2} \psi^\dagger \overleftrightarrow{\partial}_t \psi + A_0 \psi^\dagger \psi - \frac{g_{ij}}{2m} D_i \psi^\dagger D_j \psi + \cdots \right), \tag{3.1.33}$$

where $\cdots$ includes interaction terms. (Strictly speaking, the discussion here corresponds to the $g = 2$, $s = 1$ version of the LLL symmetry Geracie et al. [2015].) One can check that the classical action is invariant with respect to time-dependent spatial diffeomorphisms (i.e., all diffeomorphism transformations which preserve the time slices) Hoyos and Son [2012], Geracie et al. [2015]:

$$\delta A_0 = -\xi^k \partial_k A_0 - A_k \dot{\xi}^k, \tag{3.1.34}$$

$$\delta A_i = -\xi^k \partial_k A_i - A_k \partial_i \xi^k - m g_{ik} \dot{\xi}^k, \tag{3.1.35}$$

$$\delta g_{ij} = -\xi^k \partial_k g_{ij} - g_{kj} \partial_i \xi^k - g_{ik} \partial_j \xi^k. \tag{3.1.36}$$

The LLL limit corresponds to taking $m \to 0$. The term proportional to m disappears from the transformation law for A_i; now A_μ simply transforms like a one-form under spatial diffs:

$$\delta A_\mu = -\xi^k \partial_k A_\mu - A_\lambda \partial_\mu \xi^\lambda, \qquad \xi^\lambda = (0, \xi^i). \tag{3.1.37}$$

The metric g_{ij} also transforms like a covariant tensor. The nontrivial feature of the states on the LLL that sets it apart from other states in a magnetic field is that, although a time-varying diffeomorphism generates the g_{0i} components of the metric tensor from g_{ij}: $\delta g_{0i} = \cdots + g_{ij} \dot{\xi}^j$, the partition function of the theory does not depend at all on g_{0i}.

The fractional quantum Hall effect exists in a finite magnetic field $B = \partial_1 A_2 - \partial_2 A_1$.

Suppose one is not interested in computing the electric current by varying the partition function with respect to A_i. In that case, one can assume that A_i has some fixed value, for example, $A_i = -\frac{1}{2}B\varepsilon_{ij}x^j$, and only consider A_0 and g_{ij} as external backgrounds. Then it is natural to ask if the partition function of the theory is symmetric under any gauge transformation that touches only A_0 and g_{ij}, and explore the Ward-Takahashi identities that follow. To keep B unchanged, we need to restrict ourselves to VPDs. These correspond to $\xi^k = \ell^2\varepsilon^{kl}\partial_l\lambda$, where we chose ℓ to be the magnetic length $\ell = 1/\sqrt{B}$. For VPDs, the change of the spatial components of gauge potential A_i,

$$\delta A_i = -\ell^2\varepsilon^{kl}\partial_l\lambda\partial_k A_i - \ell^2 A_k\varepsilon^{kl}\partial_i\partial_l\lambda = -\ell^2\varepsilon^{kl}\partial_l\lambda(\partial_k A_i - \partial_i A_k) - \ell^2\partial_i(\varepsilon^{kl}A_k\partial_l\lambda)$$
$$= -\partial_i(\lambda + \ell^2\varepsilon^{kl}A_k\partial_l\lambda), \quad (3.1.38)$$

can be compensated by a gauge transformation $A_\mu \to A_\mu + \partial_\mu\alpha$ with $\alpha = \lambda + \ell^2\varepsilon^{kl}A_k\partial_l\lambda$. Under this combination of coordinates and gauge transformations, A_0 transforms as

$$\delta A_0 = -\ell^2\varepsilon^{kl}\partial_l\lambda\partial_k A_0 - \ell^2 A_k\partial_t(\varepsilon^{kl}\partial_l\lambda) + \dot{\lambda} + \ell^2\partial_t(\varepsilon^{kl}A_k\partial_l\lambda) = \dot{\lambda} - \ell^2\varepsilon^{kl}\partial_k A_0\partial_l\lambda. \quad (3.1.39)$$

We see that the transformation law for A_0 has exactly the form that we have postulated in Eq. (3.1.24). The metric, of course, transforms as in Eq. (3.1.22).

Higher-rank symmetry in the lowest Landau level limit

Within the context of the LLL, it is possible to give an intuitive interpretation of the higher-rank conservation law. We write down the current conservation

$$\frac{\partial\rho}{\partial t} + \boldsymbol{\nabla}\cdot\mathbf{j} = 0, \quad (3.1.40)$$

and the law of conservation of momentum,

$$\frac{\partial \pi_i}{\partial t} + \partial_j T_{ij} = E_i \rho + \varepsilon_{ik} j_k B, \tag{3.1.41}$$

where π_i is the momentum density. In a Galilean-invariant theory with particles of mass m, the momentum density is proportional to the particle number flux $\pi_i = m j_i$, and vanishes in the LLL limit $m \to 0$. Now the conservation of momentum becomes simply the equation of balance of force, which in the absence of the electric field simply reads

$$\partial_j T_{ij} = \varepsilon_{ik} j_k B, \tag{3.1.42}$$

and can be solved to yield for the current

$$j_i = -\frac{1}{B} \varepsilon_{ij} \partial_k T_{jk}. \tag{3.1.43}$$

The equation for the conservation of charge is now

$$\frac{\partial \rho}{\partial t} - \frac{1}{2B} \partial_i \partial_j (\varepsilon_{ik} T_{kj} + \varepsilon_{jk} T_{ik}) = 0. \tag{3.1.44}$$

One notices that the conservation law (3.1.44) is in the same form as (3.1.17) in the earlier version of the symmetric tensor gauge theory of fracton. Thus, the conservation of charge has a "higher-rank" form due to the fact that, on the LLL, the current density is no longer independent but can be expressed through the derivative of the stress tensor. The same conservation law was derived previously in Ref. Nguyen et al. [2014] using a LLL field theory formalism. The connection between volume-preserving diffeomorphism and quantum Hall physics was also noticed in Refs. Gromov [2019a], Doshi and Gromov [2021], Cappelli et al. [1993c], Iso et al. [1992], Cappelli et al. [1993b,a, 1994], KOGAN [1994].

Some comments are in order. We began with two independent Ward's identities, (3.1.40) and (3.1.41). The conservation law (3.1.44) can be considered as the linear combination of the charge conservation (3.1.40) and the massless limit of (3.1.41). One then recognizes that we end up with two independent conservation laws, one being (3.1.44) and the other being the original charge conservation. It is the same conclusion that we arrived at in Section 3.1.3.

The Wen-Zee term

One possible term in the effective action for the fractional quantum Hall is the Wen-Zee term Wen and Zee [1992b]. To introduce this term, we need to define the Newton-Cartan geometry and the spin connection that comes with it. We only give the relevant formulas here; for details, see, e.g., Refs. Son [2013], Geracie et al. [2015]. The Newton-Cartan geometry structure is given by a one-form n_μ (in the simplest version of the geometry $dn = 0$), a vector v^μ, and a symmetric contravariant metric tensor $g^{\mu\nu}$ satisfying $n_\mu v^\mu = 1$, $g^{\mu\nu} n_\nu = 0$. In our case,

$$n_\mu = (1, \mathbf{0}), \qquad v^\mu = \begin{pmatrix} 1 \\ v^i \end{pmatrix}, \qquad g^{\mu\nu} = \begin{pmatrix} 0 & 0 \\ 0 & g^{ij} \end{pmatrix}. \tag{3.1.45}$$

where

$$v^i = \ell^2 \varepsilon^{ij} \partial_j A_0, \tag{3.1.46}$$

and g^{ij} is the inverse matrix of g_{ij}. One then defines the covariant metric tensor $g_{\mu\nu}$ so that $g_{\mu\nu} v^\nu = 0$ and $g_{\mu\nu} g^{\nu\lambda} = \delta_\mu^\lambda - n_\mu v^\lambda$,

$$g_{\mu\nu} = \begin{pmatrix} g_{ij} v^i v^j & -v_j \\ -v_i & g_{ij} \end{pmatrix}, \tag{3.1.47}$$

where $v_i \equiv g_{ij}v^j$, together with the Christoffel symbol which can be used to define covariant derivatives

$$\Gamma^{\mu}_{\nu\lambda} = v^{\mu}\partial_{\nu}n_{\lambda} + \frac{1}{2}g^{\mu\rho}(\partial_{\nu}g_{\rho\lambda} + \partial_{\lambda}g_{\rho\nu} - \partial_{\rho}g_{\nu\lambda}). \tag{3.1.48}$$

One further defines the vielbein e^a_{μ} so that

$$g^{\mu\nu} = e^{a\mu}e^{a\nu}, \tag{3.1.49}$$

and the spin connection is defined as

$$\omega_{\mu} = \frac{1}{2}\varepsilon^{ab}e^{a\nu}\nabla_{\mu}e^{b}_{\nu}, \tag{3.1.50}$$

which, in components, reads

$$\omega_0 = \frac{1}{2}(\varepsilon^{ab}e^{aj}\partial_0 e^b_j + \varepsilon^{ij}\partial_i v_j), \tag{3.1.51}$$

$$\omega_i = \frac{1}{2}(\varepsilon^{ab}e^{aj}\partial_i e^b_j - \varepsilon^{jk}\partial_j g_{ik}). \tag{3.1.52}$$

The spin connection transforms like the gauge potential under the local O(2) rotation of the vielbein: $e^a(x) \to e^a(x) + \alpha(x)\varepsilon^{ab}e^b(x)$. The Wen-Zee term Wen and Zee [1992b] is given by

$$\frac{\kappa}{4\pi}\varepsilon^{\mu\nu\lambda}\omega_{\mu}\partial_{\nu}A_{\lambda} = \frac{\kappa}{4\pi}\left(\omega_0 B + A_0\varepsilon^{ij}\partial_i\omega_j - \varepsilon^{ij}\omega_i\partial_0 A_j\right) = \frac{\kappa}{4\pi}\left(\frac{\omega_0}{\ell^2} + \frac{1}{2}A_0 R\right), \tag{3.1.53}$$

where we have put $\varepsilon^{ij}\partial_i A_j = \ell^{-2}$ and $\partial_0 A_i = 0$. The coefficient κ is related to the filling fraction ν and the Wen-Zee shift $\mathcal{S}$ of a fraction quantum Hall (FQH) system by the relation

$$\kappa = \nu\mathcal{S}. \tag{3.1.54}$$

Up to quadratic order and ignoring the total derivative terms, we can rewrite the Wen-Zee term as

$$\frac{\kappa}{8\pi}\left(A_0 R - \frac{1}{4\ell^2}\varepsilon^{ij} h_{ik}\dot{h}_{jk}\right), \tag{3.1.55}$$

which is exactly the Chern-Simons term (3.1.15).

A remark can be made here. We demonstrated that the Wen-Zee term in the FQH literature is nothing but the Chern-Simon term in the higher-rank gauge theory. One can think of this in the reversed order. The higher-rank gauge symmetry dictates the relationship between the Wen-Zee shift $\mathcal{S}$ and the Hall viscosity η_H, which enter two separated components of the Chern-Simons Lagrangian in the higher-rank gauge theory (3.1.55).

3.1.5 Generalization to (3+1) dimensions

Construction of the (3+1)D nonlinear higher-rank symmetry

This section will generalize the nonabelian higher-rank symmetry to (3+1) dimensions. To do that, we imagine a three-dimensional version of the LLL. Instead of a background magnetic field, we imagine a background Kalb-Ramond field. Concretely, we imagine a nonrelativistic theory living in background metric g_{ij} and a Kalb-Ramond field $B_{\mu\nu} = -B_{\nu\mu}$. The field strength of the latter is

$$H_{\mu\nu\lambda} = \partial_\mu B_{\nu\lambda} + \partial_\nu B_{\lambda\mu} + \partial_\lambda B_{\mu\nu}, \tag{3.1.56}$$

and we assume that there is gauge symmetry with one-form gauge parameter α_μ under which

$$\delta B_{\mu\nu} = \partial_\mu \alpha_\nu - \partial_\nu \alpha_\mu. \tag{3.1.57}$$

and $H_{\mu\nu\lambda}$ is invariant. Most crucially, we assume that our theory is invariant symmetry under time-dependent spatial diffeomorphisms,

$$\delta g_{ij} = -\xi^k \partial_k g_{ij} - g_{kj}\partial_i \xi^k - g_{ik}\partial_j \xi^k, \tag{3.1.58}$$

$$\delta B_{ij} = -\xi^k \partial_k B_{ij} - B_{kj}\partial_i \xi^k - B_{ik}\partial_j \xi^k, \tag{3.1.59}$$

$$\delta B_{i0} = -\xi^k \partial_k B_{i0} - B_{k0}\partial_i \xi^k - B_{ik}\dot{\xi}^k. \tag{3.1.60}$$

which is a 3D version of the $m \to 0$ (i.e., LLL) limit of the nonrelativistic diffeomorphism (3.1.34). We do not have a concrete example of a well-defined theory with the symmetry (3.1.58), (3.1.59) and (3.1.60). As the LLL can be thought of as the massless limit for particles in a magnetic field, one can imagine a theory of massless strings coupled to a Kalb-Ramond field. The details (or even the existence) of such a theory are not important for our further discussion.

Following our discussion of the LLL in (2+1)D, we assume that our system lives in a finite Kalb-Ramond field $H_{ijk} = \ell^{-3}\varepsilon_{ijk}$, and restrict ourselves to VPDs with $\partial_k \xi^k = 0$ or

$$\xi^k = \ell^3 \varepsilon^{klm} \partial_l \lambda_m. \tag{3.1.61}$$

The change of B_{ij} under this VPD,

$$\delta B_{ij} = -\ell^3 \varepsilon^{klm} (\partial_k B_{ij} + B_{kj}\partial_i + B_{ik}\partial_j)\partial_l \lambda_m, \tag{3.1.62}$$

again can be compensated by a gauge transformation (3.1.57) with the gauge parameter

$$\alpha_i = \lambda_i - \ell^3 \varepsilon^{klm}\, B_{ik}\partial_l \lambda_m. \tag{3.1.63}$$

The combined VPD and gauge transformation changes only the B_{0i} components of the Kalb-

Ramond potential and the metric. This is a higher-rank symmetry that transforms our set of gauge fields (B_{0i}, g_{ij}) like

$$\delta B_{0i} = \dot{\lambda}_i - \ell^3 \varepsilon^{klm}(\partial_k B_{0i} + B_{0k}\partial_i)\partial_l \lambda_m, \tag{3.1.64a}$$

$$\delta g_{ij} = -\ell^3 \varepsilon^{klm}(\partial_k g_{ij} + g_{kj}\partial_i + g_{ik}\partial_j)\partial_l \lambda_m. \tag{3.1.64b}$$

In addition, we also inherit from the gauge symmetry (3.1.57) those transformations which leave B_{ij} invariant. These correspond to gauge parameters with vanishing spatial components: $\alpha_\mu = (\alpha_0, \mathbf{0})$. Under these gauge transformations,

$$\delta B_{0i} = -\partial_i \alpha_0, \tag{3.1.65a}$$

$$\delta g_{ij} = 0. \tag{3.1.65b}$$

Equations (3.1.64) and (3.1.65) represent the full group of higher-rank symmetries. Though we have used an analogy with the LLL physics in 2D as a motivation, the resulting transformation laws do not require any LLL-type microscopic physics. In fact, we will see that the higher-rank symmetry appears in the context of 3D ferromagnets.

As in 2D, it is possible to "complete" B_{0i} by adding the spatial components B_{ij} so that $B_{\mu\nu}$ form a set of Kalb-Ramond gauge potentials. This would make B_{0i} the shared components of the two sets of gauge potentials: the Kalb-Ramond gauge potentials and the gauge potentials of the higher-rank symmetry of VPDs (see Fig. 3.2).

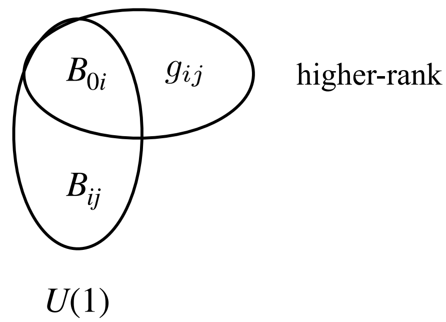

Figure 3.2: B_{0i} is shared by two sets of gauge potentials.

Linearized higher-ranked symmetries and conservation laws

At the linearized level, the higher-rank gauge invariance is

$$\delta B_{0i} = \dot{\lambda}_i - \partial_i \alpha_0, \tag{3.1.66}$$

$$\delta h_{ij} = -\ell^3 (\varepsilon_{jkl}\partial_i + \varepsilon_{ikl}\partial_j)\partial_k \lambda_l. \tag{3.1.67}$$

As far as we know, this linear higher-rank symmetry has not been considered previously in the literature.

Let us now assume that the currents coupled to B_{0i} and h_{ij} are J^i and T^{ij},

$$\delta \ln Z = \int d^4x \left(J^i \delta B_{0i} + \frac{1}{2} T^{ij} \delta h_{ij} \right). \tag{3.1.68}$$

Two conservation laws follow from Eqs. (3.1.66) and (3.1.67). First, the gauge invariance with gauge parameter α_0 implies that the "current" J^i is divergence-free:

$$\boldsymbol{\nabla} \cdot \mathbf{J} = 0. \tag{3.1.69}$$

On the other hand the VPD invariance, generated by λ_i, leads to

$$\frac{\partial J_i}{\partial t} - \ell^3 \varepsilon_{ijk} \partial_j \partial_l T_{kl} = 0. \tag{3.1.70}$$

This means the following quantities are conserved

$$\mathbf{I}_1 = \int d\mathbf{x}\, \mathbf{J}, \tag{3.1.71}$$

$$I_2 = \int d\mathbf{x}\, (\mathbf{x}\cdot\mathbf{J}), \quad \mathbf{I}_3 = \int d\mathbf{x}\, (\mathbf{x}\times\mathbf{J}), \quad I_4^{ij} = \int d\mathbf{x}\, x^{\{i} J^{j\}}, \tag{3.1.72}$$

$$\mathbf{I}_5 = \int d\mathbf{x}\, \mathbf{x}(\mathbf{x}\cdot\mathbf{J}), \qquad \mathbf{I}_6 = \int d\mathbf{x}\, \frac{x^2}{2}\mathbf{J}, \tag{3.1.73}$$

$$I_7^{ijk} = \int d\mathbf{x}\, x^{\{i} x^j J^{k\}}. \tag{3.1.74}$$

where the $\{ij\cdots\}$ denotes symmetrization over the indices $i, j, \cdots$ [2]. On the other hand,

$$I_8^{ij} = \int d\mathbf{x}\, x^{\{i} (\mathbf{x}\times\mathbf{J})^{j\}} \tag{3.1.77}$$

is not conserved; its time derivative is proportional to $\int d\mathbf{x}\, T^{ij}$.

Assume that the "current" $\mathbf{J}$ is nonzero only in a finite region of space, then because it is divergence-free, one can express it as the curl of a vector field: $\mathbf{J} = \boldsymbol{\nabla}\times\boldsymbol{\mu}$, where $\boldsymbol{\mu}$ vanishes at infinity. Then among the conserved quantities, only the following are nonzero: $\mathbf{I}_3$, $\mathbf{I}_5$, and $\mathbf{I}_6$, and the last two quantities are not independent:

$$\mathbf{I}_3 = \int d\mathbf{x}\, \boldsymbol{\mu}, \tag{3.1.78}$$

$$\mathbf{I}_5 = -\mathbf{I}_6 = \int d\mathbf{x}\, (\mathbf{x}\times\boldsymbol{\mu}). \tag{3.1.79}$$

One can think of $\boldsymbol{\mu}$ as the magnetic moment density and $\mathbf{J}$ as the magnetization current. Let us assume that there is a quasiparticle that carries a magnetic moment. Then the

2. Explicitly

$$A^{\{i} B^{j\}} = \frac{1}{2}\left(A^i B^j + A^j B^i\right), \tag{3.1.75}$$

and

$$A^{\{i} B^j C^{k\}} = \frac{1}{3}\left(A^i B^{\{j} C^{k\}} + B^i C^{\{j} A^{k\}} + C^i A^{\{j} B^{k\}}\right). \tag{3.1.76}$$

conservation of $\mathbf{I}_3$ means that the total magnetic moment does not change its value. The conservation of $\mathbf{I}_5$ implies that a particle can move only along the direction of its magnetic moment, but not along the two perpendicular directions.

Note that the (3+1)D higher-rank symmetry presented above, even in the linearized version, differs from that of the vector charge theory proposed in Ref. Pretko [2017d]. Our motivation was to generalize the area-preserving diffeomorphism of the LLL in (2+1)D to volume-preserving diffeomorphism of (3+1)D. We have generalized the charge density ρ, the one-form gauge potential A_μ and the constant magnetic field B to the vector charge density J^i, the two-form gauge potential $B_{\mu\nu}$ and the constant Kalb-Ramond field strength H_{ijk}. We then arrive at a different gauge transformation rather than the one in Ref. Pretko [2017d]. One can define the electric field and magnetic field that are invariant under the gauge transformations and write down a generalized Maxwell action. We will not do that here, as the aim of this paper is to investigate the conservation laws and their physical consequences[3].

The conservation law (3.1.70), though similar to the one of vector charge theory in Ref Pretko [2017d], is not the same because of the different gauge transformations. Recall that the gauge transformation of the symmetry tensor gauge in the vector charge version is Pretko [2017d,c]

$$\delta A_{ij} = \partial_i \lambda_j + \partial_j \lambda_i, \quad \delta \phi_i = \partial_t \lambda_i \tag{3.1.80}$$

which leads to a conservation law with only one spatial derivative acting on the current density J^{ij}

$$\partial_t \rho^i + \partial_j J^{ij} = 0 \tag{3.1.81}$$

instead of an equation with two spatial derivative. Furthermore, inherited from a system

3. Since the gauge transformations (3.1.66) and (3.1.67) differ from the ones in Ref. Pretko [2017d], the definitions of the gauge-invariant electric field and magnetic field should be modified accordingly, namely $E^{ij} = (\varepsilon_{jkl}\partial_i + \varepsilon_{ikl}\partial_j)\,\partial_k B_{0l} + \partial_t h_{ij}$ and $H = \partial_i \partial_j h_{ij}$. We will not discuss them further.

of vector matter field coupled with the Kalb-Ramond field, we have an *extra* "conservation law" $\vec{\nabla} \cdot \vec{J} = 0$, which does not have a counterpart in the vector charge theory proposed in Ref. Pretko [2017d].

We will show in Section 4.2.6 that the symmetry that has been proposed is realized in the nonlinear sigma model describing 3D ferromagnets.

3.1.6 Examples of theories with volume-preserving diffeomorphism invariance

For the abelian higher-rank symmetry, one of the simplest ways to couple a matter field to the gauge field (A_0, h_{ij}) is to introduce a Goldstone boson φ which transforms under the gauge transformation as $\varphi \to \varphi + \lambda$ and write

$$\mathcal{L} = \frac{c_1}{2}(\partial_0\varphi - A_0)^2 - \frac{c_2}{2}(\partial_i\partial_j\varphi - h_{ij})^2. \tag{3.1.82}$$

However, we were not able to find a nonlinear version of this transformation. For example, if one postulates

$$\delta_\lambda\varphi = \lambda - \ell^2\varepsilon^{kl}\partial_k\varphi\partial_l\lambda, \tag{3.1.83}$$

then one can check by direct calculation that $[\delta_\alpha, \delta_\beta]\varphi \neq \delta_{[\alpha,\beta]}\varphi$, so such transformation law would be inconsistent. We have to devise other ways to couple the matter field to the gauge field.

Composite fermions

In the fractional quantum Hall effect at filling fractions $\nu = 1/2$, $\nu = 1/4$ etc., the quasiparticle is electrically neutral. One can consistently couple such a particle to the higher-rank gauge field. For example, a Lagrangian for a nonrelativistic particle with dispersion relation

$\omega = k^2/2m$ would be

$$L = \frac{i}{2} v^\mu \psi^\dagger \overset{\leftrightarrow}{\partial}_\mu \psi - \frac{1}{2m} g^{\mu\nu} \partial_\mu \psi^\dagger \partial_\nu \psi = \frac{i}{2} \psi^\dagger \overset{\leftrightarrow}{\partial}_0 \psi + \frac{i}{2} \ell^2 \varepsilon^{ij} \partial_j A_0 \, \psi^\dagger \overset{\leftrightarrow}{\partial}_i \psi - \frac{g^{ij}}{2m} \partial_i \psi^\dagger \partial_j \psi. \quad (3.1.84)$$

One can interpret the coupling of the electric field $\partial_i A_0$ with the particle momentum as a dipole moment, perpendicular to the direction of its motion. This is consistent with the constraints that follow from conservation laws. In fact, that is how a composite fermion in the fractional quantum Hall state at $\nu = 1/2$ or $\nu = 1/4$ should couple to the external potential: the composite fermion is neutral but possesses an electric dipole moment.

Crystal on the lowest Landau level

Another way to realize the higher-rank symmetry is through an effective theory of a solid. Such a solid may be realized as a Wigner crystal, which is expected to be the ground state of electrons on the LLL at small filling fractions. A solid is parametrized by a map from the physical coordinates x^i to the coordinate system X^a frozen into the solid: $X^a = X^a(x^i)$ Soper [1976]. In the ground state $X^a = x^i \delta_{ai}$, the displacement u^a are defined as $X^a = x^a - u^a$. On the LLL the coordinates of a lattice site do not commute: $[x, y] = -i\ell^2$. The Lagrangian should thus contain the following term

$$S_{\text{Berry}} = \frac{n_0}{2\ell^2} \varepsilon^{ab} X^a \partial_t X^b, \quad (3.1.85)$$

where n_0 is the equilibrium particle number density. Under a VPD, X^a transforms as

$$\delta_\lambda X^a = -\ell^2 \varepsilon^{ij} \partial_i X^a \partial_j \lambda, \quad (3.1.86)$$

and so

$$\delta_\lambda S_{\text{Berry}} = -\frac{n_0}{2} \varepsilon^{ij} \varepsilon^{ab} X^a \partial_i X^b \partial_j \dot{\lambda} = -\frac{n_0}{2} \dot{\lambda} \varepsilon^{ij} \varepsilon^{ab} \partial_i X^a \partial_j X^b. \quad (3.1.87)$$

This change of the action can be compensated by including a term proportional to A_0 into the Lagrangian. The full Lagrangian is then

$$\mathcal{L} = \frac{n_0}{2\ell^2}\varepsilon^{ab}X^a\partial_t X^b + \frac{n_0}{2}A_0\varepsilon^{ab}\varepsilon^{ij}\partial_i X^a\partial_j X^b - \varepsilon(O^{ab}), \tag{3.1.88}$$

with $O^{ab} = g^{ij}\partial_i X^a\partial_j X^b$ and $\varepsilon(O^{ab})$ is the energy associated with elastic deformations. The spectrum of this theory can be obtained by expanding the action to quadratic order over the displacement u^a. The presence of a term with a first time derivative implies that the dispersion relation of the lattice sound wave has the quadratic form $\omega \sim q^2$ rather than the linear form.

Thus, we have been able to construct the effective theory of a Wigner crystal on the LLL starting from the higher-rank symmetry.

Vortex crystal

Another type of matter is the "vortex crystal," which is realized, for example, in a rotating Bose gas **?**. The crystal is formed by the zeros of the condensate wave function. In this case, the lattice fields X^a do not couple to A_0 directly, but through a dynamical gauge field a_μ, which is the dual of the superfluid phonon. Under VPD a_μ transforms as a one-form,

$$\delta a_\mu = -\ell^2\varepsilon^{kl}(\partial_k a_\mu - a_k\partial_\mu)\partial_l\lambda. \tag{3.1.89}$$

The field a_μ couples to the background A_0 through the following term

$$\frac{1}{2\pi}\int d^3x\left(\frac{a_0}{\ell^2} + A_0 b\right), \tag{3.1.90}$$

where $b = \varepsilon^{ij}\partial_i a_j$ is the emergent magnetic field. One can check directly that this term is invariant under VPDs. It is also obviously invariant under gauge transformations $a_\mu \to$

$a_\mu + \partial_\mu \alpha$. In fact, (3.1.90) can be obtained from the Chern-Simons term $\frac{1}{2\pi}\varepsilon^{\mu\nu\lambda}a_\mu\partial_\nu A_\lambda$ by setting A_i to be the static background with $\varepsilon^{ij}\partial_i A_j = \ell^{-2}$.

The Lagrangian of the vortex crystal is then determined by the symmetry and reads

$$\mathcal{L} = -\varepsilon^{\mu\nu\lambda}\varepsilon^{ab}a_\mu\partial_\nu X^a\partial_\lambda X^b - \varepsilon(O^{ab}) - \varepsilon_b(b) + \frac{1}{2\pi}\left(\frac{a_0}{\ell^2} + A_0 b\right), \tag{3.1.91}$$

where $\varepsilon(O^{ab})$ and $\varepsilon_b(b)$ represent the energies of the lattice and the condensate, respectively. Assuming a_μ transforms like a one-form under VPD, it can be checked that the Lagrangian above is invariant with respect to this symmetry. The eigenmode of this theory is again a Tkachenko mode with a quadratic dispersion relation **?**.

Note that the above Lagrangian contains only leading-derivative terms and does not include next-to-leading terms considered in Ref. **?**.

Ferromagnets

Ferromagnets in 2+1 D – We now show that the ferromagnet in 2+1D secretly possesses a higher-rank symmetry similar to the models with particles on the LLL [4]. At the long-wavelength limit, a ferromagnet is described by a nonlinear sigma (NLS) model Fradkin [2013], written in terms of an $O(3)$ unit vector n^a, $n^a n^a = 1$, with the action

$$S = S_{\text{Berry}} + S_{\text{NLS}} = S_0 \int_0^1 d\sigma \int dt\, d\mathbf{x}\, \varepsilon^{abc} n^a \partial_t n^b \partial_\sigma n^c - \frac{J}{2}\int dt\, d\mathbf{x}\, \delta^{ij}\partial_i n^a \partial_j n^a. \tag{3.1.92}$$

The first term is a Wess-Zumino topological term in spin's action, which is induced by the Berry phase Fradkin [2013]. The second term is the energy term of the nonlinear sigma model and can be made invariant under VPD by replacing δ^{ij} with g^{ij}. The first term is,

4. In fact, one can show that the dynamical equation of a single skyrmion in a 2-dimensional ferromagnet is the same as the equation of motion of a charged particle in a constant magnetic field Han [2017].

however not invariant:

$$\delta_\lambda S_{\text{Berry}} = -S_0\ell^2 \int_0^1 d\sigma \int dt\, d\mathbf{x}\, \varepsilon^{abc}\varepsilon^{ij} n^a \partial_i n^b \partial_\sigma n^c \partial_j \dot{\lambda}. \tag{3.1.93}$$

Integrating by parts, taking into account that $\epsilon^{abc}\partial_j n^a \partial_i n^b \partial_\sigma n^c = 0$ (this is because all the three O(3) vectors $\partial_i n^a$, $\partial_j n^a$, $\partial_\sigma n^a$ are perpendicular to n^a and hence are linearly dependent) we find

$$\delta_\lambda S_{\text{Berry}} = S_0\ell^2 \int_0^1 d\sigma \int dt\, d\mathbf{x}\, \varepsilon^{abc}\varepsilon^{ij} n^a \partial_i n^b \partial_j \partial_\sigma n^c \dot{\lambda}$$
$$= \frac{S_0}{2}\ell^2 \int_0^1 d\sigma \int dt\, d\mathbf{x}\, \varepsilon^{abc}\varepsilon^{ij} \partial_\sigma (n^a \partial_i n^b \partial_j n^c)\dot{\lambda} = \frac{S_0}{2}\ell^2 \int dt\, d\mathbf{x}\, \varepsilon^{abc}\varepsilon^{ij} n^a \partial_i n^b \partial_j n^c \dot{\lambda}. \tag{3.1.94}$$

We now choose

$$\ell^2 = \frac{1}{4\pi S_0}\,, \tag{3.1.95}$$

and add the following term to the action

$$S_{A_0} = -\frac{1}{8\pi}\int dt\, d\mathbf{x}\, A_0 \varepsilon^{abc}\varepsilon^{ij} n^a \partial_i n^b \partial_j n^c. \tag{3.1.96}$$

Then $\delta_\lambda(S_{\text{Berry}} + S_{A_0}) = 0$. Thus if we couple the ferromagnetic order parameter with the gauge fields A_0, g_{ij} in the following way

$$S = S_{\text{Berry}} + S_{A_0} - \frac{J}{2}\int dt\, d\mathbf{x}\, g^{ij} \partial_i n^a \partial_j n^a, \tag{3.1.97}$$

then the action is invariant under VPD with ℓ^2 defined in Eq. (3.1.95).

One can further introduce into the theory the vector gauge potential A_i, promoting

Eq. (3.1.96) to

$$S_{A_\mu} = -\frac{1}{8\pi}\int dt\, d\mathbf{x}\, A_\mu \varepsilon^{abc}\varepsilon^{\mu\nu\lambda} n^a \partial_\nu n^b \partial_\lambda n^c. \tag{3.1.98}$$

with A_i transforming as a one-form under VPDs. In this case, the potential A_0 is simultaneously the scalar component of the U(1) gauge field (A_0, A_i) and the scalar component of the gauge potential of a higher-spin symmetry, (A_0, g_{ij}) (Fig. 3.1).

Now the scalar potential A_0 is coupled to the topological charge density $\rho(\mathbf{x})$. This means that a skyrmion behaves like a particle in an effective magnetic field with the magnitude

$$B_{\text{eff}} = -4\pi S_0 q, \tag{3.1.99}$$

where $q = \int d\mathbf{x}\, \rho(\mathbf{x})$ is the topological charge of the skyrmion. This fact can be derived by calculating the Berry phase associated with the motion of a skyrmion Stone [1996]. One also finds that the following quantities

$$\int d\mathbf{x}\, \rho, \quad \int d\mathbf{x}\, x^i \rho, \qquad \int d\mathbf{x}\, x^2 \rho, \tag{3.1.100}$$

are conserved. This fact is again well known Papanicolaou and Tomaras [1991].[5]

It is instructive to rewrite the ferromagnet in the $\mathbb{CP}^1$ parametrization, where

$$n^a = z^\dagger \sigma^a z, \qquad z = \begin{pmatrix} z_1 \\ z_2 \end{pmatrix}, \qquad z^\dagger z = 1. \tag{3.1.101}$$

The action of the ferromagnet is then Han [2017]

$$S = S_{Berry} + S_{NLS} = 2iS_0 \int dt\, d\mathbf{x}\, z^\dagger \partial_t z - 2J \int dt\, d\mathbf{x}\, D_i z^\dagger D_i z, \tag{3.1.102}$$

5. In the presence of the Dyaloshinskii-Morya interaction, which breaks the higher-rank symmetry, only the first two quantities are conserved Komineas and Papanicolaou [2015].

where $D_i z \equiv (\partial_i - i a_i) z$, and $a_i = -i z^\dagger \partial_i z$ is promoted to a dynamical field.

Now we couple the $\mathbb{CP}^1$ model to the external probes (g_{ij}, A_0). We assume that under VPD, z transforms as

$$\delta_\lambda z = -\ell^2 \varepsilon^{kl} \partial_k z \partial_l \lambda, \tag{3.1.103}$$

therefore,

$$\delta_\lambda S_{\text{Berry}} = -2iS_0 \ell^2 \int dt\, d\mathbf{x}\, \varepsilon^{ij} z^\dagger \partial_i z \partial_j \dot{\lambda} = -2iS_0 \ell^2 \int dt\, d\mathbf{x}\, \dot{\lambda} \varepsilon^{ij} \partial_i z^\dagger \partial_j z. \tag{3.1.104}$$

We now add the following term to the action:

$$S_{A_0} = 2iS_0 \ell^2 \int dt\, d\mathbf{x}\, A_0 \varepsilon^{ij} \partial_i z^\dagger \partial_j z. \tag{3.1.105}$$

Then $\delta_\lambda (S_{Berry} + S_{A_0}) = 0$, therefore the coupling of the ferromagnetic model with the gauge field g_{ij}, A_0 is

$$S = S_{\text{Berry}} + S_{A_0} - 2J \int dt\, d\mathbf{x}\, g^{ij} D_i z^\dagger D_j z, \tag{3.1.106}$$

and respects the higher-rank symmetry.

Ferromagnets in 3+1 D – For a ferromagnet in (3+1)D, the term S_{A_0} that couples A_0 to the topological charge density is replaced by the coupling of B_{0i} to the density of a one-form current,

$$S_{B_{0i}} = -\frac{1}{8\pi} \int dt\, d\mathbf{x}\, \varepsilon^{abc} \varepsilon^{ijk} B_{0i} n^a \partial_j n^b \partial_k n^c, \tag{3.1.107}$$

and this has the (3+1)D version of VPD invariance with $\ell^3 = (4\pi S_0)^{-1}$. Again one can promote (3.1.107) to

$$S_{B_{\mu\nu}} = -\frac{1}{8\pi} \int dt\, d\mathbf{x}\, \varepsilon^{abc} \varepsilon^{\mu\nu\lambda\rho} B_{\mu\nu} n^a \partial_\lambda n^b \partial_\rho n^c. \tag{3.1.108}$$

In this case, B_{0i} are the shared components of a Kalb-Ramond gauge field and the set of gauge potentials of a higher-rank gauge symmetry (B_{0i}, g_{ij}) (see Fig. 3.2).

It was found in Ref. Papanicolaou and Tomaras [1991] that $\mathbf{I}_3$ and $\mathbf{I}_5 = -\mathbf{I}_6$ are conserved. We have given this fact a new interpretation in terms of a hidden higher-rank symmetry.

3.1.7 Conclusion

We have presented a nonlinear version of a higher-rank gauge symmetry. The symmetry is basically that of volume-preserving diffeomorphism. We show several examples of coupling of matter with the higher-rank gauge potential that respects the symmetry. Many examples are taken from the physics of the LLL, which we show to naturally have volume-preserving diffeomorphism invariance. We also show that the nonlinear sigma models of ferromagnetism also exhibit this symmetry if one couples a gauge potential of the higher-rank symmetry with the topological charge density.

We have shown that, under certain conditions, the charge densities satisfy the commutation relation of the VPD, i.e., the w_∞ algebra. In this way, one can easily understand why this algebra is realized in the bimetric model of the FQH effect Gromov and Son [2017], without explicit calculations. One interesting question is whether the w_∞ symmetry can be "upgraded" to the quantum W_∞ symmetry Cappelli et al. [1993c], Iso et al. [1992]. This question, relevant for the fractional quantum Hall effect, is deferred to future work.

3.1.8 Supplemental Material

The uniqueness of the nonlinear transformation

In this Appendix, we will briefly argue that the nonlinear transformation (3.1.24) is the unique generalization of (3.1.16), given the following assumptions:

- The nonlinear transformation satisfies the area-preserving diffeomorphism algebra (3.1.23).

- In the background (A_0, g_{ij}), A_0 is a scalar, therefore the transformation $\delta_\lambda A_0$ should be a scalar.
- The transformation of (A_0, g_{ij}) is at most linear in (A_0, g_{ij}).
- The rotational symmetry is preserved.

Due to the algebra (3.1.23), the transformation $\delta_\lambda A_0$ has to be linear in λ. We consider the general transformation that is linear in A_0 with the form [6]

$$\delta_\lambda A_0 \sim (\partial_{i_1} \cdots \partial_{i_m}) A_0 (\partial_{j_1} \cdots \partial_{j_n}) \lambda, \tag{3.1.110}$$

with ∂_k is the derivative in the spartial directions. By counting the number of derivative in the spartial directions, in order to satisfy the algebra (3.1.23), both m and n have to be 1. Furthermore, $\delta_\lambda A_0$ has to be a scalar and invariant under rotation. We need to contract all the spatial indices of the derivatives, ε^{ij} is the only rotational invariant two indices tensor that helps us to satisfy the algebra (3.1.23)

$$\delta_\lambda A_0 = -\ell^2 \varepsilon^{ij} \partial_i A_0 \partial_j \lambda. \tag{3.1.111}$$

We also consider the term that is independent of A_0

$$\delta_\lambda A_0 = \alpha^{i_1 \cdots i_s} (\partial_t)^u (\partial_{i_1} \cdots \partial_{i_s}) \lambda - \ell^2 \varepsilon^{ij} \partial_i A_0 \partial_j \lambda, \tag{3.1.112}$$

with ∂_t is the time derivative and $\alpha^{i_1 \cdots i_s}$ is a rotational symmetric tensor. By counting the derivatives, to satisfy (3.1.23) then $u + s = 1$. There is no rotational symmetric tensor with

6. One can even generalize the argument with the more general transformation

$$\delta_\lambda A_0 \sim \sum_{m,n} c_{mn} (\partial_{i_1} \cdots \partial_{i_m}) A_0 (\partial_{j_1} \cdots \partial_{j_n}) \lambda \tag{3.1.109}$$

and ends up the same conclusion. By couting the number of time derivatives, we also see that one can't add time derivatives to (3.1.110) in order to satisfy the algbera (3.1.23).

just one spartial index, thereofre $s = 0$ and $u = 1$. We then arrive at the transformation (3.1.24).

3.2 Story 3: Noncommutative gauge symmetry in the fractional quantum Hall effect

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Yi-Hsien Du, Umang Mehta, and Dam Thanh Son. Noncommutative gauge symmetry in the fractional quantum Hall effect. arXiv:2110.13875.

Abstract

We show that a system of particles on the lowest Landau level can be coupled to a probe $U(1)$ gauge field $\mathcal{A}_\mu$ in such a way that the theory is invariant under a noncommutative $U(1)$ gauge symmetry. While the temporal component $\mathcal{A}_0$ of the probe field is coupled to the projected density operator, the spatial components $\mathcal{A}_i$ are best interpreted as quantum displacements, which distort the interaction potential between the particles. We develop a Seiberg-Witten-type map from the noncommutative $U(1)$ gauge symmetry to a simpler version, which we call "baby noncommutative" gauge symmetry, where the Moyal brackets are replaced by the Poisson brackets. The latter symmetry group is isomorphic to the group of volume preserving diffeomorphisms. By using this map, we resolve the apparent contradiction between the noncommutative gauge symmetry, on the one hand, and the particle-hole symmetry of the half-filled Landau level and the presence of the mixed Chern-Simons terms in the effective Lagrangian of the fractional quantum Hall states, on the other hand. We outline the general procedure which can be used to write down effective field theories which respect the noncommutative $U(1)$ symmetry.

3.2.1 Introduction

The problem of the fractional quantum Hall effect is often formulated in the limit of a single Landau level (e.g., the lowest Landau level (LLL)), in which the nontriviality of the problem becomes most stark. In this limit, the drift motion of a single electron in an external potential can be captured by a Hamiltonian formalism in which the two Cartesian coordinates x and y of the electrons do not commute with each other. It has been suggested that the quantum field theory describing the fractional quantum Hall fluids has to be a noncommutative (NC) field theory Susskind [2001], Polychronakos [2001], Hellerman and Van Raamsdonk [2001], Fradkin et al. [2002], Cappelli and Rodriguez [2009]. However, noncommutativity of space in the first-quantized description does not translate directly to a noncommutativity of space in the second-quantized formalism. Some progress in deriving noncommutative field theories for the quantum Hall effect has recently been made for bosonic quantum Hall states near filling factor $\nu = 1$ Dong and Senthil [2020], based on previous works by Pasquier and Haldane Pasquier and Haldane [1998] and Read Read [1998].

In this paper we try to address the question of whether the quantum Hall states are governed by a noncommutative field theory. Instead of following a constructive route, we will base our approach on symmetry constraints. We show that the LLL electrons can be coupled to an external probe which can be identified with a NC $U(1)$ gauge potential $\mathcal{A}_\mu$ in a way that partition function of the theory is invariant under NC $U(1)$ gauge transformation. However, $\mathcal{A}_\mu$ is not the usual electromagnetic probe. While the scalar potential $\mathcal{A}_0$ is coupled simply to the projected density operator, the vector components $\mathcal{A}_i$ do not couple to the charge current. In fact, $\mathcal{A}_i$ is best interpreted as a displacement, disturbing the shape of the potential between two electrons. So in addition to the NC gauge symmetry, there is an additional symmetry where $\mathcal{A}_i$ is shifted by a functions of time which are independent of space [Eq. (3.2.34) below].

Armed with the symmetry information, one can then proceed to a construction of effective field theories describing various fractional quantum Hall states. The simplest way to guarantee the invariance of the theory with respect to the NC $U(1)$ symmetry is to try to promote field theories of the fractional quantum Hall effect to noncommutative field theories. However, one immediately encounters serious problems along the way. First, one sees that the naive noncommutative version of the Dirac composite fermion theory now breaks particle-hole symmetry. Furthermore, the mixed Chern-Simons term, crucial for the construction of the effective field theory for many quantum Hall states (the "hierarchy states") Wen and Zee [1992a] , does not have an obvious noncommutative extension.

We solve this problem by developing a map, inspired by the Seiberg-Witten (SW) map Seiberg and Witten [1999], which we term the "SW' map," which maps the noncommutative probe field A_μ into what we call "baby noncommutative" (bNC) gauge field A^b_μ. In the transformation laws of the bNC gauge field, the Moyal brackets are replaced by the Poisson brackets. The bNC $U(1)$ gauge group is isomorphic to the group of volume-preserving diffeomorphisms (VPDs). Thus the task of writing down an action invariant under the gauge symmetry reduces to ensuring volume-preserving diffeomorphism invariance, which can be accomplished by using an appropriate geometric formalism, for example, the Newton-Cartan formalism Son [2013], Du et al. [2022]. In this way one can write down a particle-hole-symmetric effective field theory for the half-filled Landau level as well as a Chern-Simons theory with a general K-matrix.

The paper is organized as follows. In Sec. 3.2.2 we review the basic formulas of noncommutative space and noncommutative gauge theory. In Sec. 3.2.3 we couple the electrons on a single Landau level to a NC $U(1)$ gauge field. In Sec. 3.2.4 we outline the ways one can construct effective field theories which respect the NC $U(1)$ gauge symmetry. Section 3.2.5 contains final remarks.

3.2.2 Noncommutative space and noncommutative gauge symmetry

For convenience and further reference, in this Section we collect some formulas related to noncommutative space and noncommutative gauge symmetry.

Consider particles in a magnetic field $B = \partial_1 A_2 - \partial_2 A_1$. The gauge-invariant momenta are

$$\hat{p}_i = -i(\partial_i - iA_i) = \hat{k}_i - A_i(\hat{x}). \tag{3.2.1}$$

They satisfy the commutation relation

$$[\hat{p}_i,\, \hat{p}_j] = i\varepsilon_{ij} B. \tag{3.2.2}$$

Projecting to the LLL effectively sets $\hat{p}_i \approx 0$ (the kinetic energy is $\hat{p}^2/2m$, LLL projection corresponds to taking $m \to 0$). This introduces the Dirac brackets between x_i:

$$[\hat{x}^i,\, \hat{x}^j]_{\mathrm{D}} = -[\hat{x}^i,\, \hat{p}_k]([\hat{p},\, \hat{p}])^{-1}_{kl}[\hat{p}_l,\, \hat{x}^j] = -i\ell^2\varepsilon^{ij}. \tag{3.2.3}$$

So effectively the particle lives in a noncommutative space, with the noncommutative parameter $\theta^{ij} = \theta\varepsilon^{ij}$ with

$$\theta = -\ell^2. \tag{3.2.4}$$

Any operator in the Heisenberg algebra (i.e., can be expanded in Taylor series over $\hat{x}_i$) can be put into correspondence it <u>Weyl symbol</u> by the following rule

$$e^{i\hat{q}_i\hat{x}^i} \to e^{iq_i x^i}. \tag{3.2.5}$$

Taking all possible linear combinations of the exponential one can put any function $f(x)$ into correspondence with an operator $\hat{f}(\hat{x})$ in a unique way. (We will use the same letter for the operator and its Weyl symbol, putting a hat on top of the the symbol for the operator.)

The Moyal product between two functions $f(x)$ and $g(x)$, $f \star g$, is defined so that the Weyl symbol of $\hat{f}\hat{g}$ is $f \star g$.

$$f \star g(x) = \exp\left(\frac{i}{2}\theta\varepsilon^{ij}\frac{\partial}{\partial x^i}\frac{\partial}{\partial y^j}\right) f(x)g(y)\bigg|_{y\to x} = f(x)\exp\left(\frac{i}{2}\theta\varepsilon^{ij}\overleftarrow{\partial}_i\overrightarrow{\partial}_j\right) g(x). \qquad (3.2.6)$$

The Moyal bracket is defined as

$$\{\!\{f,\, g\}\!\} = \frac{1}{i}(f \star g - g \star f) = 2f\sin\left(\frac{1}{2}\theta\varepsilon^{ij}\overleftarrow{\partial}_i\overrightarrow{\partial}_j\right) g, \qquad (3.2.7)$$

and is (up to the factor $-i$) the Weyl symbol of the commutator $[\hat{f},\, \hat{g}]$. The factor of $-i$ makes sure that if f and g are real functions then $\{f,\, g\}$ is also real. To leading order in θ the Moyal bracket becomes the Poisson bracket

$$\{\!\{f,\, g\}\!\} = \theta\{f,\, g\} + O(\theta^2), \qquad (3.2.8)$$

where

$$\{f,\, g\} = \varepsilon^{ij}\partial_i f \partial_j g. \qquad (3.2.9)$$

We now consider noncommutative gauge symmetry. For the purposes of this paper, we can limit ourselves to the $U(1)$ case and only to adjoint fields. An adjoint field ψ transforms under a noncommutative $U(1)$ gauge transformation as

$$\delta_\lambda\psi = \{\!\{\psi,\, \lambda\}\!\}, \qquad (3.2.10)$$

with λ being the parameter of the infinitesimal gauge transform. The covariant derivative is defined as

$$D_\mu\psi = \partial_\mu\psi + \{\!\{\mathcal{A}_\mu,\, \psi\}\!\}, \qquad (3.2.11)$$

and the gauge potential transforms as

$$\delta_\lambda \mathcal{A}_\mu = \partial_\mu \lambda + \{\!\{\mathcal{A}_\mu,\, \lambda\}\!\}. \tag{3.2.12}$$

The gauge invariant gauge field is

$$\mathcal{F}_{\mu\nu} = \partial_\mu \mathcal{A}_\nu - \partial_\nu \mathcal{A}_\mu + \{\!\{\mathcal{A}_\mu,\, \mathcal{A}_\nu\}\!\}. \tag{3.2.13}$$

3.2.3 Symmetry of the Landau level problem

Introduction of external probe fields

The starting point of our discussion is the Hamiltonian describing a system of N particles on a single Landau level, interacting with each other through a two-body potential $V(\mathbf{x}-\mathbf{y})$,

$$H = \sum_{\langle ab\rangle} \hat{\mathcal{V}}(\hat{\mathbf{x}}_a - \hat{\mathbf{x}}_b) = \sum_{\langle ab\rangle} \int \frac{d\mathbf{q}}{(2\pi)^2}\, \mathcal{V}(\mathbf{q}) e^{i\mathbf{q}\cdot(\hat{\mathbf{x}}_a - \hat{\mathbf{x}}_b)}, \tag{3.2.14}$$

where $\mathcal{V}(\mathbf{q})$ is the projected version of the inter-particle potential $V(\mathbf{q})$

$$\mathcal{V}(\mathbf{q}) = V(\mathbf{q}) e^{-q^2\ell^2/2} L_n^2\left(\frac{q^2\ell^2}{2}\right), \tag{3.2.15}$$

with L_n being the nth Laguerre polynomial and n is the quantum number of the Landau level on which the electrons live (for a quick derivation of Eq. (3.2.14) see, e.g., Ref. Yang [2013]).

We will couple the theory to a set of probe fields. These probes are functions in spacetime and act on all particles in the same ways. The first probe is the scalar potential, or the temporal component of the electromagnetic field $A_0(\mathbf{x})$. The Hamiltonian now contains a one-body term

$$H = \sum_{\langle ab \rangle} \hat{\mathcal{V}}(\hat{\mathbf{x}}_a - \hat{\mathbf{x}}_b) - \sum_a \hat{\mathcal{A}}_0(\hat{\mathbf{x}}_a), \tag{3.2.16}$$

where, on the lowest Landau level, $\mathcal{A}_0(\mathbf{q}) = e^{-q^2\ell^2/4} A_0(\mathbf{q})$.

Now we introduce two more probe fields which we call $u^x(\mathbf{x})$ and $u^y(\mathbf{x})$, forming a vector $\mathbf{u}(\mathbf{x})$. These probes replace the coordinates of the particles in the interaction potential $\mathcal{V}(\mathbf{x}_a, \mathbf{x}_b)$ by new coordinates

$$\mathbf{x} \to \mathbf{X} = \mathbf{x} + \mathbf{u}(\mathbf{x}). \tag{3.2.17}$$

The perturbed Hamiltonian is now

$$\hat{H} = \sum_{\langle ab \rangle} \hat{\mathcal{V}}(\hat{\mathbf{X}}_a - \hat{\mathbf{X}}_b) - \sum_a \hat{\mathcal{A}}_0(\hat{\mathbf{x}}_a) = \sum_{\langle ab \rangle} \hat{\mathcal{V}}\left(\hat{\mathbf{x}}_a + \hat{\mathbf{u}}(\hat{\mathbf{x}}_a) - \hat{\mathbf{x}}_b - \hat{\mathbf{u}}(\hat{\mathbf{x}}_b)\right) - \sum_a \hat{\mathcal{A}}_0(\hat{\mathbf{x}}_a). \tag{3.2.18}$$

Note that we do not shift the coordinates in the one-body term $\hat{\mathcal{A}}_0$. The probes can also be made time-dependent: $\mathcal{A}_0 = \mathcal{A}_0(t, \mathbf{x})$, $\mathbf{u} = \mathbf{u}(t, \mathbf{x})$.

Detour: time dependent unitary transformation

Consider a system described by a Hamiltonian $\hat{H}$ (which, in general can be time-dependent, $\hat{H} = \hat{H}(t)$). Suppose we know how to find all solutions $|\psi(t)\rangle$ to the time-dependent Schrödinger equation

$$i\frac{\partial}{\partial t}|\psi\rangle = \hat{H}(t)|\psi\rangle. \tag{3.2.19}$$

If now $\hat{U}$ is a (time-independent) unitary operator, and $\hat{H}_U = \hat{U}\hat{H}\hat{U}^{-1}$, then $|\psi\rangle_U = U|\psi\rangle$ is the solution to the Schrödinger equation with $\hat{H}$ replaced by $\hat{H}_U$.

But one can also perform a time-dependent unitary transformation with time-dependent

$\hat{U}$. It is easy to see that if one sets

$$\hat{H}_U(t) = \hat{U}\hat{H}\hat{U}^{-1} + i\partial_t\hat{U}\hat{U}^{-1}, \tag{3.2.20}$$

then the state $|\psi\rangle_U = U(t)|\psi\rangle$ still solves the time-dependent Schrödinger equation. Therefore the new Hamiltonian is completely equivalent to the old Hamiltonian. In particular, if $\hat{U}(t)$ is not equal to 1 only in a finite time interval $t_i < t < t_f$, then the evolution operator $T\exp\left[-i\int_{t_i}^{t_f} dt\,\hat{H}(t)\right]$ does not change when $\hat{H}(t)$ is replaced by $\hat{H}_U(t)$.

We now consider an infinitesimal unitary transformation with $\hat{U} = e^{i\hat{\lambda}}$ where $\hat{\lambda} \ll 1$. The transformed Hamiltonian is then

$$\hat{H}_\lambda = \hat{H} + i[\hat{\lambda},\, \hat{H}] - \partial_t\hat{\lambda}. \tag{3.2.21}$$

Unitary transformation in the Landau-level problem

We now act on our Hamiltonian describing N particles noncommutative plane with the following infinitesimal unitary transformation

$$\hat{U} = \exp\left[i\sum_a \hat{\lambda}(t,\hat{\mathbf{x}}_a)\right], \tag{3.2.22}$$

where $\hat{\lambda}(t,\hat{\mathbf{x}})$ is an infinitesimal function of t and $\hat{\mathbf{x}}$. Note that all particles are subjected to the same unitary transformation.

The transformed Hamiltonian is

$$\hat{H}^\lambda = \hat{U}\hat{H}\hat{U}^{-1} + i\partial_t\hat{U}\hat{U}^{-1} = \sum_{\langle ab\rangle}\mathcal{V}(\hat{\mathbf{X}}_a^\lambda - \hat{\mathbf{X}}_b^\lambda) - \sum_a \hat{\mathcal{A}}_0^\lambda(t,\hat{\mathbf{x}}_a), \tag{3.2.23}$$

where

$$\hat{\mathbf{X}}^{\lambda} = \hat{\mathbf{X}} + i[\hat{\lambda},\, \hat{\mathbf{X}}], \tag{3.2.24}$$

$$\hat{\mathcal{A}}_0^{\lambda} = \hat{\mathcal{A}}_0 + i[\hat{\lambda}, \hat{\mathcal{A}}_0] + \partial_t \hat{\lambda}. \tag{3.2.25}$$

So the new Hamiltonian has the same form as the old Hamiltonian but with the new values of the probe fields. The transformation laws of the corresponding Weyl symbols are

$$\mathbf{X}^{\lambda} = \mathbf{X} + \{\!\{\mathbf{X},\, \lambda\}\!\}, \tag{3.2.26}$$

$$\mathcal{A}_0^{\lambda} = \mathcal{A}_0 + \{\!\{\mathcal{A}_0,\, \lambda\}\!\} + \partial_t \lambda. \tag{3.2.27}$$

Knowing how X^i transform we can find how u^i transforms. We have $\mathbf{X} = \mathbf{x} + \mathbf{u}$, so Eq. (3.2.26) reads

$$x^i + (u^i)^{\lambda} = x^i + u^i + \{\!\{x^i + u^i,\, \lambda\}\!\}. \tag{3.2.28}$$

But

$$\{\!\{x^i,\, \lambda\}\!\} = -\ell^2 \varepsilon^{ij} \partial_j \lambda, \tag{3.2.29}$$

which means

$$(u^i)^{\lambda} = u^i + \{\!\{u^i,\, \lambda\}\!\} - \ell^2 \varepsilon^{ij} \partial_j \lambda. \tag{3.2.30}$$

Now if we introduce $\mathcal{A}_i$ so that

$$u^i = -\ell^2 \varepsilon^{ij} \mathcal{A}_j, \tag{3.2.31}$$

then $\mathcal{A}_i$ transforms as

$$\mathcal{A}_i^{\lambda} = \mathcal{A}_i + \{\!\{\mathcal{A}_i,\, \lambda\}\!\} + \partial_i \lambda. \tag{3.2.32}$$

That means we can combine $\mathcal{A}_0$ with $\mathcal{A}_i$ into $\mathcal{A}_\mu$, all components of which transform in the same way

$$\delta_\lambda \mathcal{A}_\mu = \{\!\{\mathcal{A}_\mu,\, \lambda\}\!\} + \partial_\mu \lambda. \tag{3.2.33}$$

This is exactly the gauge transformation of the noncommutative $U(1)$ gauge symmetry.

So the system of particles on a Landau level can be coupled to a set of gauge potentials $\mathcal{A}_\mu$ so that the physics is invariant under the noncommutative $U(1)$ gauge transformation. It follows that any low-energy effective theory should also couple to the $\mathcal{A}_\mu$ probe in a way that respects this invariance. Note that while $\mathcal{A}_0$ is related to the temporal component of the electromagnetic field, $\mathcal{A}_i$ is not at all the usual vector potential of electromagnetism.

Beside the noncommutative $U(1)$ gauge symmetry, translational invariance of the two-body potential term leads to a symmetry

$$\mathcal{A}_i(t, \mathbf{x}) \to \mathcal{A}_i(t, \mathbf{x}) + \alpha_i(t), \tag{3.2.34}$$

where $\alpha_i(t)$ are functions of time only. This additional symmetry should also be respected by any low-energy effective field theory.

3.2.4 Noncommutative symmetry in effective field theory

Problem with particle-hole symmetry

The effective field theory describing the half-filled Landau level Halperin et al. [1993], Son [2015] is a $U(1)$ gauge theory which involves, as dynamical degree of freedom, a composite fermion ψ and an emergent gauge field a_μ. The effective action has a $U(1)_a$ emergent gauge symmetry, and also the conventional $U(1)_{\text{em}}$ gauge symmetry, encoded in the coupling of the theory to external background gauge field A_μ.

One now asks how one can couple the theory to the noncommutative probe field $\mathcal{A}_\mu$. One may expect that the field theory should be promoted to a noncommutative field theory.

In this scenario the $U(1)_a$ gauge symmetry would become a $U(1)_a$ noncommutative gauge symmetry. In order to ensure the time-dependent shift symmetry (3.2.34), one may require all fields to transform in the adjoint representation of the $U(1)$ NC gauge symmetry.

Such noncommutative field theory of the half-filled Landau level can be constructed (see, e.g., Ref. Govcanin et al. [2021]) but one sees an immediate problem. Namely, the NC $U(1)_a$ gauge symmetry is in conflict with the particle-hole (PH) symmetry (or $\mathcal{CT}$ symmetry) of the physics on a single Landau level. To show that, let us write down the transformation laws for the gauge fields under gauge transformations Son [2015],

$$U(1)_a^{\rm NC}: \qquad \delta_\alpha a_\mu = \{\!\{a_\mu,\, \alpha\}\!\} + \partial_\mu \alpha, \tag{3.2.35}$$

$$U(1)_{\rm em}^{\rm NC}: \qquad \delta_\lambda \mathcal{A}_\mu = \{\!\{\mathcal{A}_\mu,\, \lambda\}\!\} + \partial_\mu \lambda. \tag{3.2.36}$$

Under particle-hole symmetry the fields transform as follows:

$$A_0(t,\mathbf{x}) \to -A_0(-t,\mathbf{x}), \qquad A_i(t,\mathbf{x}) \to \quad A_i(-t,\mathbf{x}), \tag{3.2.37}$$

$$a_0(t,\mathbf{x}) \to \quad a_0(-t,\mathbf{x}), \qquad a_i(t,\mathbf{x}) \to -a_i(-t,\mathbf{x}). \tag{3.2.38}$$

For this symmetry to be consistent with the noncommutative gauge symmetry, we need to be able to assign transformation laws to the gauge transformation parameters λ and α under PH conjugation. While we can do this for λ by postulating that $\lambda(t,\mathbf{x}) \to \lambda(-t,\mathbf{x})$, we cannot do the same for α: the two terms in the transformation law for a_0 in Eq. (3.2.35) always transform differently under PH, no matter which PH conjugation rule for α. Thus one concludes that emergent noncommutative $U(1)$ gauge symmetry is fundamentally incompatible with the particle-hole symmetry of the half-filled Landau level.

There is one more obstacle in writing down a fully noncommutative description of quantum Hall states. The general Chern-Simons description of an abelian quantum Hall state involves, in general, multiple internal gauge fields with mutual Chern-Simons terms Wen and

Zee [1992a]

$$L \sim \sum_{I,J} K_{IJ} \varepsilon^{\mu\nu\lambda} a_\mu^I \partial_\nu a_\lambda^J. \tag{3.2.39}$$

However, since the NC $U(1)$ symmetry is nonabelian, it is not possible to write a mutual CS term involving two different NC $U(1)$ gauge fields. This problem has been noted recently revisited by Goldman and Senthil Goldman and Senthil [2021], where a partial solution, valid to next order in the expansion over the noncommutativity parameter θ is presented. Here we present a different approach, capable of solving the problem of the mutual Chern-Simons terms and the problem of particle-hole symmetry in one scoop, to all orders in the noncommutativity parameter.

Volume preserving diffeomorphism as a "baby" version of noncommutative gauge symmetry

To solve the problems identified above, we first note a close analogy between the noncommutative gauge symmetry $U(1)_\mathcal{A}$ and volume preserving diffeomorphism Du et al. [2022]. We recall (see Ref. Du et al. [2022] for details) that the fractional quantum Hall problem has invariance under time-dependent diffeomorphisms that preserve the spatial volume:

$$x^i \to x^i + \xi^i(t, \mathbf{x}), \qquad \xi^i = \ell^2 \epsilon^{ij} \partial_j \lambda, \tag{3.2.40}$$

where $\ell^2 = 1/B$ is the magnetic length, if it is coupled to a scalar potential A_0 and a metric g_{ij} that transform as follows:

$$\delta_\lambda A_0 = \partial_0 \lambda + \theta \{A_0, \lambda\}, \qquad \{A_0, \lambda\} = \epsilon^{ij} \partial_i A_0 \partial_j \lambda, \tag{3.2.41}$$

$$\delta_\lambda g_{ij} = -\xi^k \partial_k g_{ij} - g_{ik} \partial_j \xi^k - g_{kj} \partial_i \xi^k, \qquad \xi^k = \ell^2 \epsilon^{kl} \partial_l \lambda. \tag{3.2.42}$$

A_0 and g_{ij} are the background fields for volume preserving diffeomorphisms. Note that the transformation law for A_0 is the long-wavelength limit of that for the noncommutative potential $\mathcal{A}_0$, Eq. (3.2.12) Furthermore, one can limit oneself to flat metrics (which obviously remain flat under diffeomorphisms), which can be parametrized in terms of the coordinates $\mathsf{X}^a(\mathbf{x})$, defined as the coordinates in which the metric tensor is δ_{ab}:

$$g_{ij} = \delta_{ab}\, \partial_i \mathsf{X}^a \partial_j \mathsf{X}^b. \tag{3.2.43}$$

The transformation law for X^a is

$$\delta \mathsf{X}^a = -\xi^i \partial_i \mathsf{X}^a = \theta\{\mathsf{X}^a,\, \lambda\}. \tag{3.2.44}$$

If we now write the covariant coordinates as $\mathcal{O}(\theta)$ perturbations around the background coordinates

$$\mathsf{X}^i = x^i + \theta \epsilon^{ij} \mathsf{A}_j, \tag{3.2.45}$$

then the transformation law for the metric induces a transformation law for the background fields A_j^b which look like

$$\delta_\lambda \mathsf{A}_i = \partial_i \lambda + \theta\{\mathsf{A}_i,\, \lambda\}. \tag{3.2.46}$$

Once again, this is the long-wavelength limit of the transformation law for the spatial components $\mathcal{A}_i$ of the noncommutative gauge field. Combining A_0 and A_i into a spacetime vector A_μ (with $\mathsf{A}_0 = A_0$), the transformation law of the latter,

$$\delta_\lambda \mathsf{A}_\mu = \partial_\mu \lambda + \theta\{\mathsf{A}_\mu,\, \lambda\}, \tag{3.2.47}$$

is the long wavelength limit of the noncommutative gauge transformation law where the Moyal brackets are replaced by the Poisson brackets. We will call this type of gauge symmetry the "baby noncommutative" (bNC) gauge symmetry.

There exists a well-defined procedure to write down field theories that are VPD-invariant. In these theories, each field transforms in a certain representation (scalar, vector, tensor, etc.) of the VPD group. For example, in the Dirac composite fermion theory, the composite fermion field transforms as

$$\delta\psi = \theta\{\psi,\, \lambda\} \tag{3.2.48}$$

(here we ignore the possible coupling of ψ to the spin connection) and the emergent gauge field transforms as a one-form under VPD,

$$\delta_\lambda a_\mu = -\xi^k \partial_k a_\mu - a_k \partial_\mu \xi^k = \theta\{a_\mu,\, \lambda\} + \theta\epsilon^{kl} a_k \partial_\mu \partial_l \lambda. \tag{3.2.49}$$

We note that this is <u>not</u> what one would expect for the transformation law of an adjoint field. Under the internal $U(1)_a$ gauge symmetry, the transformation law is simply

$$\delta_\alpha a_\mu = \partial_\mu \alpha. \tag{3.2.50}$$

The absence of the term $\theta\{a_\mu,\, \alpha\}$ allows the transformation law to be consistent with PH symmetry.

For completeness, we write down the simplest Lagrangian of the effective field theory of the Dirac composite fermion,

$$\mathcal{L} = \frac{i}{2} v^\mu (\psi^\dagger D_\mu \psi - D_\mu \psi^\dagger \psi) + \frac{i}{2} v_F \sigma^a e^i_a (\psi^\dagger D_i \psi - D_i \psi^\dagger \psi) - \frac{a_0}{4\pi\ell^2} + A_0 \left(\frac{1}{4\pi\ell^2} - \frac{b}{2\pi} \right), \tag{3.2.51}$$

where (neglecting the coupling of ψ to the spin connection) $D_\mu \psi = (\partial_\mu - i a_\mu)\psi$, $v^\mu = (1, \ell^2 \varepsilon^{ij} \partial_j A_0)$, and e^i_a is the vielbein, which can be defined by $e^i_a \partial_i \mathsf{X}^b = \delta^b_a$. One can check that this theory is invariant under VPDs.

SW' map

We now would like to use the insights obtained from the last Section to find a way to write down a composite fermion theory that is both PH-symmetric and can be coupled to an external NC $U(1)$ gauge field $\mathcal{A}_\mu$.

Our method is inspired by the Seiberg-Witten map between the NC and commutative gauge symmetries. We want to derive an analogous map that maps the NC gauge field $\mathcal{A}_\mu$ to a new gauge field A^b_μ

$$A^b_\mu = A^b_\mu[\mathcal{A}_\mu], \tag{3.2.52}$$

such that the field A^b_μ transforms under gauge transformations as a bNC gauge field, similarly to the field A_μ discussed in the previous Section:

$$\delta A^b_\mu = \partial_\mu \lambda^b + \theta\{A^b_\mu,\, \lambda^b\}. \tag{3.2.53}$$

The gauge parameter λ^b can depend on the gauge field

$$\lambda^b = \lambda^b[\lambda, \mathcal{A}_\mu]. \tag{3.2.54}$$

If such a map can be found, then the rules for writing down the coupling of a composite fermion theory to the NC probe $\mathcal{A}_\mu$ is as follows. We first couple the theory to the probe A^b_μ in a way that respect the bNC gauge symmetry. Then we restrict ourselves to flat metrics and parametrize the metric through the coordinates X^a, or equivalently A^b_i. Now we have an action coupled to the baby-NC gauge fields A^b_μ and is invariant under baby-NC $U(1)$ gauge transformations. We now use the SW' map (3.2.52) to replace the probe field A^b_μ by the NC gauge field $\mathcal{A}_\mu$. The result is an effective theory coupled to the $\mathcal{A}_\mu$ field with NC $U(1)$ gauge symmetry.

One may wonder if one can use the original Seiberg-Witten map to map the NC field $\mathcal{A}_\mu$

to a commutative gauge field A^c_μ, which can be coupled easily to any EFT of the FQHE. The reason we cannot do so is the time-dependent shift symmetry (3.2.34). The map from $\mathcal{A}_\mu$ to A^c_μ involves $\mathcal{A}_i$ without spatial derivatives, and a generic coupling of the effective theory to the commutative field A^c_μ, when rewritten in terms of $\mathcal{A}_\mu$, does not generally have the shift symmetry (3.2.34).

To find the map from the NC theory to the bNC theory, we first develop a map from the bNC theory to the commutative theory, then use the known Seiberg-Witten map to map the commutative theory to the NC theory. It is possible that there is a more efficient algorithm to find the map between bNC and NC theories, but we leave finding it to future work.

Map between baby-NC theory and commutative theory

Suppose we have a commutative theory A^c_μ which gauge transforms as

$$\delta A^c_\mu = \partial_\mu \lambda^c. \tag{3.2.55}$$

We wish to construct a field and a gauge parameter:

$$A^b_\mu = A^b_\mu[A^c_\mu], \quad \lambda^b = \lambda^b[\lambda^c, A^c_\mu], \tag{3.2.56}$$

so that A^b_μ transforms as a baby-NC gauge field with parameter λ^b,

$$\delta A^b_\mu = \partial_\mu \lambda^b + \theta\{A^b_\mu,\, \lambda^b\}, \tag{3.2.57}$$

To find these, we will promote θ to a parameter which we will call τ and try to find a family of gauge field $A_\mu(\tau)$ ($= A_\mu(\tau)[A^c_\mu]$) and $\lambda(\tau)$ ($= \lambda(\tau)[\lambda^c, A^c_\mu]$) so that for each value of τ

$$\delta A_\mu(\tau) = \partial_\mu \lambda(\tau) + \tau\{A_\mu(\tau),\, \lambda(\tau)\}. \tag{3.2.58}$$

Then the commutative and the bNC fields are just $A_\mu(\tau)$ at two special values of τ: $A^c_\mu = A_\mu(0)$ and $A^b_\mu = A_\mu(\theta)$. We differentiate Eq. (3.2.58) over τ to find that $A_\mu(\tau)$ and $\lambda(\tau)$ must depend on τ in such away that the following condition is satisfied:

$$\delta \dot{A^b}_\mu(\tau) = \partial_\mu \dot{\lambda^b}(\tau) + \{A_\mu(\tau),\, \lambda(\tau)\} + \tau\{\dot{A}_\mu(\tau),\, \lambda(\tau)\} + \tau\{A_\mu(\tau),\, \dot{\lambda}(\tau)\}, \tag{3.2.59}$$

where dot denotes derivative with respect to τ. One can check that this is satisfied if one requires

$$\dot{A}_\mu(\tau) = \frac{1}{2}\varepsilon^{ij}\big[\partial_i A_\mu(\tau) + F_{i\mu}(\tau)\big]A_j(\tau), \tag{3.2.60a}$$

$$\dot{\lambda}(\tau) = \frac{1}{2}\varepsilon^{ij}\partial_i\lambda(\tau)A_j(\tau), \tag{3.2.60b}$$

where $F_{i\mu}(\tau) = \partial_i A_\mu(\tau) - \partial_\mu A_i(\tau) + \tau\{A_i(\tau),\, A_\mu(\tau)\}$. To verify that, one just needs to replace, in Eq. (3.2.59), $\dot{A}_\mu(\tau)$ and $\dot{\lambda}(\tau)$ by the expression given in Eqs. (3.2.60), then use Eq. (3.2.58) to convince oneself that two sides of the equation coincide.

To express the baby-NC field and gauge parameter A^b_μ and λ^b in terms of the commutative field A_μ and gauge parameter λ, one solves Eq. (3.2.60a) with the initial condition $A_\mu(0) = A^c_\mu$, $\lambda(0) = \lambda^c$. Then at $\tau = \theta$ we find the bNC field and gauge parameter: $A_\mu(\theta) = A^b_\mu$, $\lambda(\theta) = \lambda^b$. Inversely, to express the commutative field A^c_μ in terms of the bNC field A^b_μ, one solves this equation with the initial condition at $\tau = \theta$ and extract the solution at $\tau = 0$.

Seiberg-Witten map between NC theory and commutative theory

For convenience we review here the original Seiberg-Witten map between the NC and commutative field and gauge parameter:

$$\mathcal{A}_\mu = \mathcal{A}_\mu[A^c_\mu], \qquad \lambda = \lambda[\lambda^c, A^c_\mu]. \tag{3.2.61}$$

We now construct a one-parameter family of gauge field and gauge parameter $\mathcal{A}_\mu(\tau)$ and $\lambda(\tau)$ so that the gauge transformation law is that of a NC theory with the noncommutativity parameter τ:

$$\delta\mathcal{A}_\mu(\tau) = \partial_\mu\lambda(\tau) + \{\!\{\mathcal{A}_\mu(\tau),\, \lambda(\tau)\}\!\}_\tau, \tag{3.2.62}$$

where we introduce the notations

$$(f \star g)_\tau = f \exp\left(\frac{i}{2}\tau\varepsilon^{ij}\overleftarrow{\partial}_i\overrightarrow{\partial}_j\right) g, \tag{3.2.63}$$

$$\{\!\{f,\, g\}\!\}_\tau = \frac{1}{i}\left[(f\star g)_\tau - (g\star f)_\tau\right], \quad \{\!\{f,\, g\}\!\}^+_\tau = \frac{1}{2}\left[(f\star g)_\tau + (g\star f)_\tau\right]. \tag{3.2.64}$$

Differentiating Eq. (3.2.62) over τ, we find

$$\delta\dot{\mathcal{A}}_\mu(\tau) = \partial_\mu\dot{\lambda}(\tau) + \varepsilon^{ij}\{\!\{\partial_i\mathcal{A}_\mu(\tau),\, \partial_j\lambda(\tau)\}\!\}^+_\tau + \{\!\{\dot{\mathcal{A}}_\mu(\tau),\, \lambda(\tau)\}\!\}_\tau + \{\!\{\mathcal{A}_\mu(\tau),\, \dot{\lambda}(\tau)\}\!\}_\tau. \tag{3.2.65}$$

This condition is satisfied if one requires that $\mathcal{A}_\mu(\tau)$ and $\lambda(\tau)$ satisfy the evolution equations

$$\dot{\mathcal{A}}_\mu(\tau) = \frac{1}{2}\varepsilon^{ij}\{\!\{\partial_i\mathcal{A}_\mu(\tau) + \mathcal{F}_{i\mu}(\tau),\, \mathcal{A}_j(\tau)\}\!\}^+_\tau, \tag{3.2.66a}$$

$$\dot{\lambda}(\tau) = \frac{1}{2}\varepsilon^{ij}\{\!\{\partial_i\lambda(\tau),\, \mathcal{A}_j(\tau)\}\!\}^+_\tau, \tag{3.2.66b}$$

where $\mathcal{F}_{i\mu}(\tau) = \partial_i\mathcal{A}_\mu(\tau) - \partial_\mu\mathcal{A}_i(\tau) + \{\!\{\mathcal{A}_i(\tau),\, \mathcal{A}_\mu(\tau)\}\!\}_\tau$. Solving this system of equations between $\tau = 0$ and $\tau = \theta$ one can then establish a mapping between the NC gauge field and a commutative gauge field.

From NC to bNC

To express the bNC field A^b_μ and gauge parameter λ^b in terms of the NC field $\mathcal{A}_\mu$ and gauge parameter λ, one can do a two-step process: first one solves Eqs. (3.2.60) for τ running from

θ to 0 to A^c_μ and λ^c, and then solves Eqs. (3.2.66) with τ running back from 0 to θ. The result is an expression relating A^b_μ and λ^b with $\mathcal{A}_\mu$ and λ. Since Eqs. (3.2.60) and (3.2.60) differ from each other only by terms of order θ^2 and higher on the right-hand sides, the difference between and A^b_μ and $\mathcal{A}_\mu$ starts at order θ^3

$$A^b_\mu = \mathcal{A}_\mu + \frac{\theta^3}{48}\varepsilon^{ij}\varepsilon^{i_1 j_1}\varepsilon^{i_2 j_2}\partial_{i_1}\partial_{i_2}(2\partial_i \mathcal{A}_\mu - \partial_\mu \mathcal{A}_i)\partial_{j_1}\partial_{j_2}\mathcal{A}_j + O(\theta^4), \tag{3.2.67}$$

$$\lambda^b = \lambda + \frac{\theta^3}{48}\varepsilon^{ij}\varepsilon^{i_1 j_1}\varepsilon^{i_2 j_2}\partial_{i_1}\partial_{i_2}\partial_i \lambda\, \partial_{j_1}\partial_{j_2}\mathcal{A}_j + O(\theta^4). \tag{3.2.68}$$

Since the difference between A^b_μ and $\mathcal{A}_\mu$ involves spatial derivatives of A_i, time-dependent shifts on $\mathcal{A}_i$ [Eq. (3.2.34)] simply become time-dependent shifts of A^b_i.

One can now require that fields in the effective field theories of the FQHE transform as as tensors (or spinor) under VPD with parameter λ^b. Written in terms of λ and $\mathcal{A}$ these transformation laws are rather complicated and unnatural from the point of view of noncommutative field theory. But as the SW' map demonstrates, there is no requirement that the correct theory of the FQHE should be a simple noncommutative field theory.

3.2.5 Conclusion

In this paper we show that the fact that the fact that the quantum Hall state lives on a single Landau level implies that such a system can be coupled to an external noncommutative $U(1)$ gauge field in a way that preserves a NC $U(1)$ gauge symmetry. We show that the task of writing down such a theory can be simplified by transforming the NC $U(1)$ gauge symmetry into a "baby-NC" $U(1)$ gauge symmetry, which is isomorphic to volume-preserving diffeomorphism. The task of enforcing VPD invariance in an effective field theory can be accomplished quite easily with existing tools, for example using the Newton-Cartan formalism Son [2013], Du et al. [2022].

It appears from the discussion above that the NC $U(1)$ gauge symmetry does not place any additional constraint on the effective field theory of the FQHE besides those which can be seen from the VPD.

It would be nice to find a direct map between the NC and bNC theories, bypassing the need to go through the intermediate commutative theory. Another remaining open question is on the precise relationship between the $U(1)$ noncommutative gauge symmetry of the effective field theory and the GMP algebra Girvin et al. [1986]. We leave these questions to future work.

CHAPTER 4

INTRODUCTION: EXOTIC FIELD THEORIES IN CONDENSED MATTER SYSTEMS

In Chapter 3, we delve into the profound significance of volume-preserving diffeomorphisms and their intricate interrelation with noncommutative field theory Douglas and Nekrasov [2001], Rubakov [2005], Du et al. [2021]. Our exploration extends to the connection between volume-preserving diffeomorphisms and nonabelian higher-rank gauge theories Du et al. [2022]. The intriguing fractonic nature from these theories elucidates notably restrictive mobility constraints on local excitations, an aspect often encountered in exotic quantum field theories Pretko [2017a,b], Pretko and Radzihovsky [2018a], Seiberg and Shao [2021, 2020], Gorantla et al. [2020, 2022, 2023]. In this chapter, we synthesize the preceding findings to formulate a noncommutative exotic field theory.

The exotic field theories exhibit exotic symmetries that frequently transcend the constraints of ordinary Lorentz-invariant theories. A prominent example embodying many features typical of such theories is the “Lifshitz scalar” theory, characterized by a Lagrangian containing terms involving four spatial derivatives of the field, while the conventional two-derivative term is absent. Recently, a nonlinear Lifshitz scalar theory, endowed with nonlinear dipole and multipole symmetries, has been identified as the descriptor of the Tkachenko mode, which serves as the oscillation in rotating two-dimensional superfluids Du et al. [2024a].

In this system, the dipole symmetry corresponds to the symmetry of magnetic translations, while the higher multipole symmetry pertains to magnetic rotation. As these symmetries are exact at the microscopic level, the nonlinear quantum Lifshitz theory of the Tkachenko mode necessitates no fine tuning Lake et al. [2022]. This theory finds a convenient formulation as a noncommutative field theory. The cubic self-interaction of the Tkachenko scalar is determined by these symmetries and governs the decay rate of the Tkachenko mode.

The higher-dimensional generalizations of nonlinear Lifshitz theories encompass higher-dimensional analogs of the Tkachenko mode observed in two spatial dimensions. In three spatial dimensions, the resultant theory characterizes a $U(1)$ gauge boson with a quadratic dispersion relation, termed the "Lifshitz photon Du et al. [2024b]." This photon can be interpreted as the Nambu-Goldstone boson of a spontaneously broken one-form symmetry. Further exploration navigates the connections between Lifshitz theories and other condensed matter systems, with a specific emphasis on quantum spin ice and ferromagnets.

Moreover, progress has been made in identifying a class of $U(1)$ fractonic gauge theories as duals of quantum crystals featuring topological defects. This connection has demystified and clarified several aspects of fractonic orders, demonstrating their physical realization through crystalline topological defects. Moreover, it has revealed that generalized fracton models can be understood as duals of well-established elastic theories, such as quantum smectics and supersolids. These generalized gauge theories also offer a compact effective field theory description of quantum crystals, their defects, and the associated quantum melting transitions Du et al. [2023]. We anticipate that generalizing the duality to nonlinear Lifshitz theory in higher spatial dimensions Du et al. [2024b], as well as to other condensed matter systems, is feasible. However, we defer these explorations to future studies.

In the forthcoming chapter, we aim to comprehensively examine the various facets of (noncommutative-)exotic field theories, placing particular emphasis on nonlinear Lifshitz theories in arbitrary dimensions. Additionally, we will conduct analyses to elucidate their physical applications within condensed matter systems.

4.1 Story 4: Noncommutative field theory of the Tkachenko mode: Symmetries and decay rate

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Yi-Hsien Du, Sergej Moroz, Dung Xuan Nguyen and Dam Thanh Son. Noncommutative field theory of the Tkachenko mode: Symmetries and decay rate. Phys. Rev. Research 6, L012040, Feb 2024.

Abstract

We construct an effective field theory describing the collective Tkachenko oscillation mode of a vortex lattice in a two-dimensional rotating Bose-Einstein condensate in the long-wavelength regime. The theory has the form of a noncommutative field theory of a Nambu-Goldstone boson, which exhibits a noncommutative version of dipole symmetry. From the effective field theory, we show that, at zero temperature, the decay width Γ of the Tkachenko mode scales with its energy E as $\Gamma \sim E^3$ in the low-energy limit. We also discuss the width of the Tkachenko mode at a small temperature.

4.1.1 Introduction

When a superfluid rotates, a lattice of quantized vortices forms. The oscillations of the vortex lattice, the so-called Tkachenko mode Tkachenko [1965, 1966, 1969] (for a recent review, see Ref. Sonin [2014]), has many distinctive properties. Unlike ordinary sound waves in a solid, at low momenta, the Tkachenko wave has a quadratic dispersion relation $\omega \sim q^2$ and only one polarization Sonin [1976], Volovik and Dotsenko Jr [1979], Baym [2003]. The Tkachenko mode is a consequence of a rather intricate realization of spontaneous symmetry breaking: there are many symmetries broken by the superfluid vortex lattice, but only one Nambu-Goldstone boson (NGB) Watanabe and Murayama [2013]. The Tkachenko mode should exist in rotating superfluid ^{4}He, but it has been observed most conclusively in the rotating Bose-Einstein condensate of ultracold atoms Coddington et al. [2003]. At a much larger length scale, the Tkachenko mode has been suggested to be the source of an oscillation mode of the Crab pulsar Ruderman [1970].

As the Tkachenko mode is the only low-energy degree of freedom, one expects that it can be described by an effective field theory (EFT) which involves a single field. However, up to now, a complete understanding of the structure of such a theory has yet to be achieved. At the quadratic level, the effective Lagrangian Watanabe and Murayama [2013] coincides with that for a Lifshitz scalar Grinstein [1981], but the form of the interaction terms in the Lagrangian and how they are constrained by symmetries are not known. These interaction terms are needed to calculate the decay rate of the Tkachenko mode Matveenko and Shlyapnikov [2011].

In this Letter, we show that noncommutative field theory (see, e.g., Refs. Douglas and Nekrasov [2001], Rubakov [2005]) provides a convenient framework for constructing the effective field theory of the Tkachenko mode. That noncommutative field theory (NCFT) may be applicable to the problem is intuitively understandable—rotating a nonrelativistic system is formally equivalent to placing it in a magnetic field, and on the lowest Landau level (LLL) the guiding-center coordinates do not commute. Because of that, NCFT has often been invoked in the context of the quantum Hall effect Susskind [2001], Polychronakos [2001], Hellerman and Van Raamsdonk [2001], Fradkin et al. [2002], Pasquier [2007], Dong and Senthil [2020], Du et al. [2021]. Vortex lattices can also be realized on the LLL Ho [2001], Sinova et al. [2002], Sonin [2005]. As we will see, in the case of the Tkachenko mode, NCFT provides a way to organize terms in the Lagrangian consistent with symmetries. Following the formalism, we are able to determine the general structure of the interacting Lagrangian, and from there, that the decay rate of a Tkachenko mode (at zero temperature) scales like the cube of its energy,

$$\Gamma \sim E^3. \tag{4.1.1}$$

This implies, in particular, that the Tkachenko mode becomes a more and more well-defined quasiparticle (i.e., $\Gamma/E \to 0$) as the energy E approaches zero.

We will also establish a connection between the Tkachenko mode and the "dipole" symmetry, which recently became a popular topic (see, e.g., Refs Pretko [2017a], Pretko and Radzihovsky [2018a], Nandkishore and Hermele [2019], Gromov [2019b], Wang et al. [2021], Pretko et al. [2020], Gromov et al. [2020], Seiberg and Shao [2021], Nguyen et al. [2020], Lake et al. [2022], Gorantla et al. [2022], Kapustin and Spodyneiko [2022], Gorantla et al. [2023], Radzihovsky [2022], Gromov and Radzihovsky [2022]). The Tkachenko mode realizes a more complex version of dipole symmetry: the magnetic translations, which form a nonabelian group.

4.1.2 Tkachenko mode as a noncommutative Nambu-Goldstone boson

One can arrive at the theory of the Tkachenko mode from microscopic considerations, taking, for example, as the starting point the microscopic theory of bosons with short-range repulsive interactions and then eliminating all redundant degrees of freedom Watanabe and Murayama [2013]. It is instructive, however, to derive the most general form of the effective Lagrangian, relying solely on symmetries. Such an approach has the advantage of being applicable for strongly correlated rotating superfluids where microscopic calculations are not reliable, e.g., close to a quantum melting transition of the vortex crystal Cooper [2008].

We first note that the lattice of vortices can be described, as a two-dimensional solid, by two ($a = 1, 2$) scalar fields $X^a(t, x^i)$; they present the coordinates frozen in the solid Leutwyler [1997], Soper [1976]. In this description, in Cartesian coordinates the lattice displacement u^a is related to X^a by $X^a = x^a - u^a$. The vortex current is related to X^a by

$$j_v^\mu = \frac{1}{2} n_0 \epsilon^{\mu\nu\lambda} \epsilon^{ab} \partial_\nu X^a \partial_\lambda X^b, \tag{4.1.2}$$

where n_0 is the equilibrium vortex density. For a superfluid under rotation with angular velocity Ω, $n_0 = \frac{1}{2\pi} B$, where the effective magnetic field $B = 2m\Omega$ with m being the mass of the elementary boson.

In a superfluid, the vortices carry charge with respect to the $u(1)$ dynamical gauge field a_μ dual to the superfluid Nambu-Goldstone boson Peskin [1978], Dasgupta and Halperin [1981].

The boson particle number current is expressed as $j^\mu = \frac{1}{2\pi}\epsilon^{\mu\nu\lambda}\partial_\nu a_\lambda$. The Lagrangian of the system contains terms that describe the coupling of the vortex current with the dual $u(1)$ gauge field and the kinetic and potential terms of the latter,

$$\mathcal{L} = -j_v^\mu a_\mu + \frac{m}{4\pi b}\mathbf{e}^2 - \epsilon\left(\frac{b}{2\pi}\right) + \frac{1}{2\pi}\epsilon^{\mu\nu\lambda}A_\mu\partial_\nu a_\lambda, \tag{4.1.3}$$

where $e_i = \partial_0 a_i - \partial_i a_0$, and $b = \epsilon^{ij}\partial_i a_j$. In the above equation, $m\mathbf{e}^2/(4\pi b)$ represents the kinetic energy of the superfluid condensate, $\epsilon(b/2\pi)$ is the internal energy as a function of the density, and A_μ is the gauge potential of the external effective magnetic field B. In the lowest Landau level limit $m \to 0$, the kinetic term vanishes. In fact, this term can be dropped if one is eventually interested in the limit $\omega \sim q^2$: in this regime $\mathbf{e}^2 \ll b^2$. Without the $\mathbf{e}^2$ term, variation with respect to a_0 give a constraint

$$\frac{1}{2}\epsilon_{ab}\epsilon^{ij}\partial_i X^a \partial_j X^b = 1. \tag{4.1.4}$$

That means the map from x^i to X^a is area-preserving. To linear order in the displacement u^i, Eq. (4.1.4) implies $\partial_i u^i = 0$, i.e., the displacement is divergence free: the Tkachenko mode is a transverse sound.

Here we would like to resolve the constraint (4.1.4) at the nonlinear level. Here we take a more elegant approach: on the LLL, one expects the spatial coordinates x and y to become noncommutative (see e.g., Refs. Susskind [2001], Pasquier [2007]),

$$[\hat{x}, \hat{y}] = i\theta, \quad \theta = -\ell^2. \tag{4.1.5}$$

where $\ell = 1/\sqrt{B}$ is the magnetic length. The quantum version of Eq. (4.1.4) then can be written as

$$[\hat{X}, \hat{Y}] = i\theta. \tag{4.1.6}$$

We then conclude that $\hat{X}^a$ and $\hat{x}^a$ are related by a unitary transformation,

$$\hat{X}^a = e^{i\hat{\phi}} \hat{x}^a e^{-i\hat{\phi}}, \tag{4.1.7}$$

where the operator $\hat{\phi}$ is an arbitrary function of the two noncommuting coordinates $\hat{x}$ and $\hat{y}$. In noncommutative field theories Douglas and Nekrasov [2001], Rubakov [2005], any operator corresponds to a Weyl symbol which is a function in space, and the above equation becomes

$$X^a(x) = e^{i\phi(x)} \star x^a \star e^{-i\phi(x)}. \tag{4.1.8}$$

Here the star product is defined as $f \star g \equiv f(x) \exp(\frac{i}{2}\theta\epsilon^{ij} \overleftarrow{\partial}_i \overrightarrow{\partial}_j) g(x)$. To linear order in ϕ the displacement u^a is then

$$u^a = \{\{\phi,\, x^a\}\} = -\theta\epsilon^{ab}\partial_b\phi, \tag{4.1.9}$$

where $\{\{\cdot, \cdot\}\}$ denotes the Moyal bracket, $\{\{f,\, g\}\} = 2f \sin(\frac{1}{2}\theta\epsilon^{ij} \overleftarrow{\partial}_i \overrightarrow{\partial}_j) g$. As expected, to this order, the displacement is purely transverse. To all orders in ϕ, Eq. (4.1.8) can be written as

$$X^a = x^a + i\{\{U,\, x^a\}\} \star U^{-1} = x^a + \theta\epsilon^{ai} D_i\phi, \tag{4.1.10}$$

where $U = e^{i\phi}$, $U^{-1} = e^{-i\phi}$ and

$$D_i\phi \equiv -i\partial_i U \star U^{-1}. \tag{4.1.11}$$

Thus, we identify the Tkachenko mode with a Nambu-Goldstone boson of a noncommutative field theory. We now show that this field is a compact scalar that shifts under the particle number U(1) symmetry.

4.1.3 Magnetic translations as noncommutative dipole symmetry

On the LLL, translations are magnetic translations and do not commute:

$$[\hat{P}_x,\, \hat{P}_y] = -\frac{i}{\theta}\hat{Q}, \tag{4.1.12}$$

where $\hat{Q}$ denotes the boson particle number operator. In our case, the Tkachenko mode is the only low-energy degree of freedom, so it should provide a nontrivial representation of magnetic translations. In the noncommutative theory, translations are realized as a special class of unitary transformations that are exponents of a linear function of coordinate. Acting on $\hat{X}^a$, such a transformation changes the Weyl symbol of the latter as

$$X^a \to e^{i\alpha_i x^i} \star X^a \star e^{-i\alpha_i x^i} = X^a(\vec{x} - \vec{\xi}), \tag{4.1.13}$$

with $\xi^i = -\theta\epsilon^{ij}\alpha_j$. This is a spatial translation by $\vec{\xi}$. Viewing X^a as fields in a field theory, such a translation is supposed to be generated by $\hat{X}^a \to e^{-i\vec{\xi}\cdot\hat{\vec{P}}}\hat{X}^a e^{i\vec{\xi}\cdot\hat{\vec{P}}}$. Thus, we can identify the magnetic translation as [1]

$$\hat{P}_i = \frac{1}{\theta}\epsilon_{ij}\hat{x}^j. \tag{4.1.14}$$

According to Eqs. (4.1.8) and (4.1.13) and the associativity of the star product, magnetic translations by $\vec{c}$ act on the Tkachenko field ϕ as multiplication on the left

$$e^{i\phi} \to \exp\left(\frac{i}{\theta}\epsilon_{ij}c^i x^j\right) \star e^{i\phi}. \tag{4.1.15}$$

1. The hat symbol ˆ on the two sides of Eq. (4.1.14) have different meanings: on the left-hand side, it denotes an operator of field theory, on the right-hand side, an operator in the sense of a function of noncommutative coordinates.

This allows us to interpret the action of magnetic translations on a Tkachenko field as a noncommutative version of a dipole symmetry. Expanding in ϕ, Eq. (4.1.15) reads

$$\phi \to \phi + \frac{1}{\theta}\epsilon_{ij}c^i x^j - \frac{1}{2}c^i\partial_i\phi + \cdots . \tag{4.1.16}$$

To leading order, these are simply a dipole symmetry transformation $\phi \to \phi + \alpha_i x^i$ with the parameter $\alpha_i = \theta^{-1}\epsilon_{ij}c^j$, but in addition, there are an infinite number of terms composed of derivatives acting on fields ϕ. These terms make the magnetic translations noncommuting, as in Eq. (4.1.12).

Knowing the transformation law for ϕ under magnetic translations, we can find the transformation law for ϕ under particle number $U(1)$ symmetry. Apply four translations on $e^{i\phi}$, one after another: $e^{-i\beta\hat{P}_y}e^{-i\alpha\hat{P}_x}e^{i\beta\hat{P}_y}e^{i\alpha\hat{P}_x}$, from Eq. (4.1.15) we see that $e^{i\phi}$ becomes $e^{i(\phi+\frac{\alpha\beta}{\ell^2})}$. But we also know from Eq. (4.1.12) that the product of the four translations is $e^{i\frac{\alpha\beta}{\ell^2}\hat{Q}}$. Thus, under U(1) charge, the Tkachenko field transforms exactly as the phase of the superfluid condensate: $\phi \to \phi + c$. The Tkachenko field, therefore, has a dual role: it is the condensate phase, but at the same time, its gradient is the lattice displacement. Such a dual role is possible, of course, because at low energy, the condensate phase is entirely determined by the configuration of the vortex lattice. By the "condensate phase," one should have in mind the regular part of the phase where the singular contributions from the vortices have been subtracted away.

As a condensate phase, ϕ then should be a compact scalar field with periodicity 2π: $\phi \sim \phi + 2\pi$. The periodicity of ϕ can also be seen from the following argument. Let us put the system on a torus of size $L_x \times L_y$. Then the magnetic field breaks translation symmetry along the x direction to a discrete group of finite translations generated by $x \to x + 2\pi\ell^2/L_y$ (which can be seen by computing the Wilson line of the gauge field along a curve wrapping the torus along the y direction at fixed x). This discrete translation is generated by the operator $e^{2\pi i\hat{y}/L_y}$ under which $\phi \to \phi + 2\pi y/L_y + \cdots$. This is allowed only when the identification $\phi \sim \phi + 2\pi$ is valid.

4.1.4 Ingredients for a Lagrangian

We now write down the most general Lagrangian consistent with symmetries for the field $U = e^{i\phi}$. The symmetries include global $U(1)$ phase rotations $U \to e^{i\alpha}U$ and global magnetic translations $U \to e^{i\vec{\alpha}\cdot\vec{x}} \star U$ (noncommutative dipole symmetry). The structures that are covariant (i.e., transforming like $O \to e^{i\vec{\alpha}\cdot\vec{x}} \star O \star e^{-i\vec{\alpha}\cdot\vec{x}}$) under these transformations are

$$D_0\phi \equiv -i\partial_t U \star U^{-1}, \tag{4.1.17a}$$

$$D_{ij}\phi \equiv \frac{1}{2}(\partial_i D_j\phi + \partial_j D_i\phi), \tag{4.1.17b}$$

where $D_i\phi$ is defined as in Eq. (4.1.11). These can be expanded in powers of ϕ as

$$D_0\phi = \dot{\phi} + \frac{1}{2}\{\{\dot{\phi}, \phi\}\} + \cdots , \tag{4.1.18a}$$

$$D_{ij}\phi = \partial_i\partial_j\phi + \frac{1}{2}\{\{\partial_i\partial_j\phi, \phi\}\} + \cdots . \tag{4.1.18b}$$

So each of the structures expands to an infinite series over ϕ. These series have the property that, at the order ϕ^n with a given integer n, the leading terms (in derivatives) have $2n$ derivatives if one count ∂_t as two derivatives, $\partial_t \sim \partial_i^2$. This counting is natural as the Tkachenko mode, which is the only low-energy degree of freedom, has a quadratic dispersion. To keep at each power of ϕ only terms with the minimal number of derivatives, one needs to replace in Eqs. (4.1.18) the Moyal brackets with the magnetic Poisson brackets, $\{f, g\}_\theta = \theta\epsilon^{ij}\partial_i f\partial_j g$. In this way, we can write

$$D_0\phi = \dot{\phi} + \frac{\theta}{2}\epsilon^{ij}\partial_i\dot{\phi}\partial_j\phi + \cdots , \tag{4.1.19a}$$

$$D_{ij}\phi = \partial_i\partial_j\phi + \frac{\theta}{2}\epsilon^{kl}\partial_i\partial_j\partial_k\phi\partial_l\phi + \cdots . \tag{4.1.19b}$$

Note that D_{ij} is defined in Eq. (4.1.17b) to be symmetric in the indices. The antisymmetric version of Eq. (4.1.17b) can be shown to be related to $D_{ij}\phi$ by using the relationship (valid in the Poisson-bracket approximation) $\det|\partial_i X^j| = 1$, where $X^i = x^i + \theta\epsilon^{ij}D_j\phi$. We find

$$\epsilon^{ij}\partial_i D_j\phi = \frac{2}{\theta}\left[\sqrt{1+\frac{\theta^2}{4}\Big[(D_{ij}\phi)^2 - (D_{ii}\phi)^2\Big]} - 1\right], \tag{4.1.20}$$

and hence is not an independent structure.

4.1.5 Effective Lagrangian

We can now write down the Lagrangian of the Tkachenko mode, keeping at each power of ϕ terms with the minimal number of derivatives, counting each occurrence of ∂_t as two derivatives. This Lagrangian would allow one to compute the rate of any scattering process to leading order over the momenta of the particles involved, similarly to the nonlinear Lagrangians for superfluids Greiter et al. [1989] or solids Leutwyler [1997], Soper [1976]. We explicitly derive a nonlinear effective theory of the Tkachenko field ϕ from the leading-order effective theory of a vortex lattice introduced in Refs. Moroz et al. [2018], Moroz and Son [2019].

The most general Lagrangian consistent with the $U(1)$ and magnetic translation symmetries is a function of the invariant structures defined above:

$$L = L\big(D_t\phi, D_{ij}\phi\big). \tag{4.1.21}$$

The form of the Lagrangian can be restricted further by imposing additional symmetries. In particular, assuming the vortex lattice is a triangular lattice, one should expect the C_6 group of rotations by angles multiple of $\frac{2\pi}{6}$. Introducing the complex coordinate $z = x + iy$, the rotationally invariant structures are now

$$D_0\phi,\ D_{ii}\phi,\ (D_{ij}\phi)^2,\ \mathrm{Re}(D_{zz}\phi)^3,\ \mathrm{Im}(D_{zz}\phi)^3. \tag{4.1.22}$$

A system of particles in a magnetic field has an antiunitary RT symmetry that combines spatial reflection (R) and time reversal (T):

$$x \to x, \quad y \to -y, \quad t \to -t, \quad i \to -i. \tag{4.1.23}$$

Under this symmetry, $\phi \to -\phi$, which can be seen from its connection to the displacement u^a in Eq. (4.1.9). Among the C_6 invariants in Eq. (4.1.22), $D_{ii}\phi$ and $\mathrm{Re}(D_{zz}\phi)^3$ are odd, while the rest are even. Thus the Lagrangian has to be a function of the following invariants: $D_0\phi$, $(D_{ij}\phi)^2$, $(D_{ii}\phi)^2$, $\mathrm{Im}(D_{zz}\phi)^3$, $D_{ii}\phi\,\mathrm{Re}(D_{zz}\phi)^3$, $\big(\mathrm{Re}(D_{zz}\phi)^3\big)^2$. Finally, rotational symmetry implies the invariance of the action with respect to $U \to e^{\frac{i}{2}\omega x^2} \star U$. This means $(D_{ij}\phi)^2$ and $D_{ii}\phi$ enter the Lagrangian only through the combination $(D_{ij}\phi)^2 - \frac{1}{2}(D_{ii}\phi)^2$. Thus the most general effective Lagrangian is a function of four arguments,

$$L = L\Big(D_0\phi,\ (D_{ij}\phi)^2 - \tfrac{1}{2}(D_{ii}\phi)^2, \mathrm{Im}(D_{zz}\phi)^3, \big(\mathrm{Re}(D_{zz}\phi)^3\big)^2\Big). \tag{4.1.24}$$

4.1.6 The Girvin-MacDonald-Platzman (GMP) algebra

The NCFT construction realizes a key feature of the LLL—the GMP algebra Girvin et al. [1986]. Indeed, upon canonical quantization, the particle number density $n = -\delta S/\delta(D_0\phi)$ realizes the NC U(1) gauge transformation, i.e.,

$$\left[\int d^2y\,\lambda(y)n(y),\ O(x)\right] = i\delta_\lambda O(x), \tag{4.1.25}$$

where $\delta_\lambda O$ is the infinitesimal change of O under the gauge transformation, under which $e^{i\phi} \to e^{i\lambda} \star e^{i\phi}$. But the gauge transformations do not commute: $[\delta_\alpha, \delta_\beta] = \delta_{\{\{\alpha, \beta\}\}}$. From this, one derives the GMP algebra satisfied by $n(x)$.

4.1.7 Quadratic Lagrangian

The only terms that contribute to the quadratic Lagrangian are $(D_0\phi)^2$ and $(D_{ij}\phi)^2 - \frac{1}{2}(D_{ii}\phi)^2$. Modulo a total derivative, the quadratic Lagrangian is that of the quantum Lifshitz model Grinstein [1981]

$$\mathcal{L}_2 = \frac{c_0}{2}(\partial_0\phi)^2 - \frac{c_1}{2}(\nabla^2\phi)^2, \tag{4.1.26}$$

which corresponds to a quadratic dispersion relation $\omega \sim q^2$. This quadratic dispersion relation is protected by the magnetic translation symmetry Watanabe and Murayama [2013]. From Eq. (4.1.26), one easily reproduces the power-law behavior of the correlation function of the superfluid order parameter at large distances, first found in Ref. Sinova et al. [2002] (see also Refs. Pitaevskii and Stringari [1993], Lake et al. [2022], Kapustin and Spodyneiko [2022]).

4.1.8 Decay width of the Tkachenko mode

The quadratic dispersion relation of the Tkachenko mode allows a decay of one Tkachenko quantum into two quanta. To find the rate of such decay, we need to determine the interaction vertices cubic in the field ϕ. It is easy to see that, even as cubic terms appear when one expands the "quadratic" terms $(D_0\phi)^2$ and $(D_{ij}\phi)^2 - \frac{1}{2}(D_{ii}\phi)^2$ to cubic order in ϕ, these terms are total derivatives. The real cubic interaction appears from the following terms in the Lagrangian: $(D_0\phi)^3$, $D_0\phi[(D_{ij}\phi)^2 - \frac{1}{2}(D_{ii}\phi)^2]$, and $\mathrm{Im}(D_{zz}\phi)^3$. Up to a total derivative, the cubic Lagrangian has the form

$$\mathcal{L}_3 = g_1(\partial_0\phi)^3 + g_2(\partial_0\phi)(\nabla^2\phi)^2 + g_3\,\mathrm{Im}(\partial_z\partial_z\phi)^3. \tag{4.1.27}$$

From this, one easily finds the energy dependence of the decay width of the Tkachenko mode. All the cubic interaction terms scale the same way in the scaling scheme with $\partial_0 \sim \partial_i^2$. In this scheme, ϕ is dimensionless and the g's have dimension $p^{-2} \sim E^{-1}$. The decay width Γ is proportional to g^2, and to have the correct dimension, Γ should scale as $\sim g^2E^3$. This can be confirmed by writing down the decay rate of the Tkachenko mode:

$$\Gamma_{\mathbf{q}} = \frac{1}{2\epsilon_{\mathbf{q}}}\frac{1}{2}\int\frac{d^2\mathbf{p}}{(2\pi)^2 2\epsilon_{\mathbf{p}}2\epsilon_{\mathbf{q}-\mathbf{p}}}\,|\mathcal{M}(\mathbf{q}\to\mathbf{p},\mathbf{q}-\mathbf{p})|^2\times(2\pi)\delta(\epsilon_{\mathbf{q}}-\epsilon_{\mathbf{p}}-\epsilon_{\mathbf{q}-\mathbf{p}}). \tag{4.1.28}$$

Estimating the integral with $p\sim q$, $\mathcal{M}\sim gq^6$, we get $\Gamma_{\mathbf{q}}\sim g^2q^6\sim g^2E^3$. The presence of an anisotropic cubic vertex means that the decay rate depends on the direction of the momentum of the decaying particle.

At small but finite temperature T, the $U(1)$ condensate phase disappears, but the order parameter of translation symmetry breaking $\partial_i\phi$ has a logarithmic correlation at long distances Baym [2004] (see also Refs. Lake et al. [2022], Kapustin and Spodyneiko [2022]). Below the Berezinskii-Kosterlitz-Thouless phase transition where the lattice melts, the Tkachenko mode should still exist. The $1\to 2$ decay rate (4.1.28) is modified for modes with energy much less than T by the factor $(1+f_{\mathbf{p}}+f_{\mathbf{q}-\mathbf{p}})$ where $f_{\mathbf{p}}$ and $f_{\mathbf{q}-\mathbf{p}}$ are the occupation numbers in the final state. For $E\ll T$, this factor is of order T/E, hence the $1\to 2$ rate is now g^2TE^2 for $E\ll T$. However, the dominant contribution to the width is now a different process: the "Landau damping" process, i.e., the absorption of the soft Tkachenko quantum by a hard thermal Tkachenko photon in the medium:

$$\Gamma_{\mathbf{q}} = \frac{1}{2\epsilon_{\mathbf{q}}}\int\frac{d^2\mathbf{p}}{(2\pi)^2 2\epsilon_{\mathbf{p}}2\epsilon_{\mathbf{q}+\mathbf{p}}}\,|\mathcal{M}(\mathbf{q},\mathbf{p}\to\mathbf{q}+\mathbf{p})|^2\times(f_{\mathbf{p}}-f_{\mathbf{q}+\mathbf{p}})(2\pi)\delta(\epsilon_{\mathbf{q}}+\epsilon_{\mathbf{p}}-\epsilon_{\mathbf{q}+\mathbf{p}}). \tag{4.1.29}$$

The width of the Tkachenko mode due to this process is $g^2(TE)^{3/2}$, which means that the Tkachenko mode is still a well-defined resonance.

The estimate above assumes that the hard Tkachenko quanta participating in the scattering process has no width and is valid only when the energy of the Tkachenko mode under consideration is larger than the width of a typical thermal mode, which is, by dimensional analysis, g^2T^3. Thus the estimate $\Gamma(E) \sim g^2(TE)^{3/2}$ is valid in the interval $g^2T^3 \ll E \ll T$. The regime $E \ll g^2T^3$ is the hydrodynamic regime, the analysis of which we defer to future work.

We note that our formulas for the width of the Tkachenko mode, both at zero and nonzero temperature, are in conflict with a previous result obtained from a microscopic calculation Matveenko and Shlyapnikov [2011]. For bosons on the LLL, the authors of Ref. Matveenko and Shlyapnikov [2011] found that at zero temperature ratio of the width to the energy of the Tkachenko mode is a constant independent of the energy (which depends only on the filling factor), and at nonzero temperature the mode is overdamped. The results are untypical for a NGB, and we cannot reconcile them with the symmetries of the system. This discrepancy needs to be investigated further.

4.1.9 Conclusion

In this Letter, we have provided a new interpretation of the Tkachenko mode in a rotating superfluid: it is a noncommutative Nambu-Goldstone boson that arises from the breaking of $U(1)$ and translation symmetries. Noncommutative field theory provides a convenient way to impose the invariance of the theory with respect to $U(1)$ and magnetic translations, and the resulting theory gives us a prediction for the decay width of the Tkachenko mode at low momentum.

4.2 Story 5: Nonlinear Lifshitz Photon Theory in Condensed Matter Systems

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Abstract

We present an interacting theory of a $U(1)$ gauge boson with a quadratic dispersion relation, which we call the "nonlinear Lifshitz photon theory." The Lifshitz photon is a three-dimensional generalization of the Tkachenko mode in rotating superfluids. Starting from the Wigner crystal of charged particles coupled to a dynamical $U(1)$ gauge field, after integrating out gapped degrees of freedom, we arrive at the Lagrangian for the nonlinear Lifshitz photon. The symmetries of the theory include a global $U(1)$ 1-form symmetry and nonlinearly realized "magnetic" translation and rotation symmetries. The interaction terms in the theory lead to the decay of the Lifshitz photon, the rate of which we estimate. We show that the Wilson loop, which plays the role of the order parameter of the spontaneous breaking of the 1-form global symmetry, deviates from the perimeter law by an additional logarithmic factor. We explore potential connections to other condensed matter systems, with a particular focus on quantum spin ice and ferromagnets. Finally, we generalize our theory to higher dimensions.

4.2.1 Introduction

Recent years have seen a significant surge of interest in exotic quantum field theories Pretko [2017a,b], Pretko and Radzihovsky [2018a], Seiberg and Shao [2021, 2020], Gorantla et al.

[2020, 2022, 2023]. These field theories frequently have unusual symmetries that are not possible in ordinary Lorentz-invariant theories. One example that exhibits many features typical of these theories is the so-called "Lifshitz scalar" theory, whose Lagrangian contains a term involving four spatial derivatives of the field, while the usual term with two spatial derivative is absent. These theories manifest exotic symmetries that limit the mobility of a single charge (the "dipole symmetry" or, in some cases, its generalization called "higher multipole symmetries"). Recently, a nonlinear Lifshitz scalar theory, characterized by nonlinear dipole and multipole symmetries, has been found to describe the Tkachenko mode of a rotating two-dimensional superfluids Du et al. [2024a]. In this system, the dipole symmetry is simply the symmetry of magnetic translations, while the higher multipole symmetry is that of the magnetic rotation. Since these are exact symmetries of the system at the microscopic level, the nonlinear quantum Lifshitz theory of the Tkachenko mode does not require fine tuning Lake et al. [2022]. The theory of the Tkachenko mode allows a convenient formulation as a noncommutative field theory Douglas and Nekrasov [2001], Rubakov [2005], Du et al. [2021]. The form of the cubic self-interaction of the Tkachenko scalar is fixed by the symmetries and determines the decay rate of the Tkachenko mode.

In this paper, we consider three-dimensional analogs of the Tkachenko mode in two dimensions. We show that such an extension is indeed possible, and the resulting theory describes a $U(1)$ gauge boson with a quadratic dispersion relation—the "Lifshitz photon." This photon can be interpreted as the Nambu-Goldstone boson of the spontaneously broken one-form symmetry.

The paper is organized as follows. We first analyze in detail in Sec. 4.2.2 a prototypical example—a Wigner crystal of charged particles, immersed in a fixed uniformed compensating background, interacting with a dynamical $U(1)$ gauge field. We show that the low-energy mode in this model is a Lifshitz photon. We then (in Sec. 4.2.3) show that the model exhibits a nonlinear version of the dipole symmetry, which can be used to constrain the form of the self-interaction of the Lifshitz photon, and which allows us to determine the decay rate of the latter. In Sec. 4.2.4 we comment on one peculiar feature of the higher-form symmetry in our system: a logarithmic correction to the perimeter law for the 't Hooft loop. In Sec. 4.2.5 we discuss some spin systems where the Lifshitz photon may be realized, and in Sec. 4.2.6 we speculate on a possible deconfined phase of ferromagnetism in which the symmetries of the ferromagnetic nonlinear sigma model are realized through the Lifshitz photon.

4.2.2 Prototype: Wigner crystal coupled to dynamical $U(1)$ gauge field

The prototypical system that we will consider is the system of charged particles ("ions") forming a lattice, interacting with a $U(1)$ gauge field A_μ. At this moment we do not specify the nature of A_μ; it can be the physical electromagnetic field or an emergent $U(1)$ gauge field of, e.g., a three-dimensional spin liquid. We assume that there is a neutralizing background with charge of the opposite sign, and that this background has no dynamics of its own.

In this Section we will construct a linear theory of excitations and show that the lowest mode is a "Lifshitz photon," i.e., a quasiparticle with two transverse polarizations and a quadratic dispersion. One can describe the lattice of the ions in terms of the displacement field of $\mathbf{u}(\mathbf{x})$. Let n_0 be the equilibrium density of the ions, M and e be the mass and the charge of the ion. The Lagrangian of the system is

$$\mathcal{L} = \frac{Mn_0}{2}\dot{\mathbf{u}}^2 - \frac{\mu}{4}\left(\partial_i u_j + \partial_j u_i - \frac{2}{3}\delta_{ij}\partial_k u_k\right)^2 - \frac{K}{2}(\partial_i u_i)^2 + en_0\mathbf{u}\cdot\mathbf{E} + \frac{\mathbf{E}^2}{2} - \frac{\mathbf{B}^2}{2}\,, \quad (4.2.1)$$

where $\mathbf{E} = \boldsymbol{\nabla} A_0 - \partial_t \mathbf{A}$, $\mathbf{B} = \boldsymbol{\nabla} \times \mathbf{A}$, and μ and K are the shear and bulk moduli of the lattice (for simplicity we assume rotational invariance). We will find later that our mode has quadratic dispersion $\omega \sim q^2 \ll q$, so at low momenta the terms $\dot{u}^2$ and E^2 can be omitted since $\dot{u}^2 \ll (\partial u)^2$, and $E^2 \ll B^2$. The Lagrangian now becomes

$$\mathcal{L} = -\frac{\mu}{4}\left(\partial_i u_j + \partial_j u_i - \frac{2}{3}\delta_{ij}\partial_k u_k\right)^2 - \frac{K}{2}(\partial_i u_i)^2 + en_0\mathbf{u}\cdot\mathbf{E} - \frac{\mathbf{B}^2}{2}\,. \tag{4.2.2}$$

Variation of the action with respect to A_0 gives rise to condition

$$\boldsymbol{\nabla}\cdot\mathbf{u} = 0\,, \tag{4.2.3}$$

which means that the displacement is constrained to be transverse. This condition can be solved:

$$\mathbf{u} = \frac{1}{gn_0}\boldsymbol{\nabla}\times\mathbf{a}\,, \tag{4.2.4}$$

where for future convenience we have introduced a for now unspecified constant g. We will now integrate out A_i. There are two terms in the Lagrangian that contain A_i,

$$-\frac{e}{g}(\boldsymbol{\nabla}\times\mathbf{a})\cdot\partial_t\mathbf{A} - \frac{\mathbf{B}^2}{2} = \frac{e}{g}\partial_t\mathbf{a}\cdot\mathbf{B} - \frac{\mathbf{B}^2}{2} + \text{total derivative}\,. \tag{4.2.5}$$

It is tempting to integrate over $\mathbf{B}$ here to obtain $\dot{\mathbf{a}}^2$, but that would be incorrect: the three components of $\mathbf{B}$ are not independent but satisfies the condition $\boldsymbol{\nabla}\cdot\mathbf{B} = 0$. We introduce a Lagrange multiplier enforcing this constraint:

$$\frac{e}{g}\partial_t\mathbf{a}\cdot\mathbf{B} - \frac{\mathbf{B}^2}{2} + \frac{e}{g}a_0(\boldsymbol{\nabla}\cdot\mathbf{B})\,. \tag{4.2.6}$$

Now we can integrate over $\mathbf{B}$. Setting $\mathbf{B}$ at the saddle point located at

$$\mathbf{B} = \frac{e}{g}(\partial_t\mathbf{a} - \boldsymbol{\nabla}a_0)\,. \tag{4.2.7}$$

we get

$$\frac{e^2}{2g^2}(\boldsymbol{\nabla} a_0 - \partial_t \mathbf{a})^2 \,. \tag{4.2.8}$$

Together with the elastic energy, the Lagrangian is

$$\mathcal{L} = \frac{e^2}{2g^2} e_i e_i - \frac{\mu}{2g^2 n_0^2}(\partial_i b_j)^2 \,, \tag{4.2.9}$$

where $e_i = \partial_i a_0 - \partial_t a_i$, and $b_i = \epsilon_{ijk}\partial_j a_k$. The theory now has a $U(1)$ gauge invariance under $a_\mu \to a_\mu + \partial_\mu \lambda$; the excitation of the model is a "Lifshitz photon" with the quadratic dispersion relation

$$\omega = \frac{\sqrt{\mu}}{n_0} q^2 \,. \tag{4.2.10}$$

That the dispersion relation is quadratic is due to the absence of the usual term $\mathbf{b}^2$ from the Lagrangian, which can be traced back to the translational invariance. Under translation the lattice displacement is shifted by the constant vector: $\mathbf{u} \to \mathbf{u} + \mathbf{c}$. Taking into account Eq. (4.2.4) this is implies a shift symmetry acting on $\mathbf{b}$, which forbids the term $\mathbf{b}^2$ in the Lagrangian.

The photon here is a dual to the physical photon. That can be seen from Eq. (4.2.7), which equates, up to a proportionality coefficient, the Lifshitz electric field with the physical magnetic field. One can also see this duality by introducing an external electric charge e located at $\mathbf{x} = \mathbf{0}$ into the system. This is done by adding to Eq. (4.2.2) a term $e\delta(\mathbf{x})A_0(\mathbf{x})$. Then variation over A_0 gives rise to the equation

$$e n_0 \boldsymbol{\nabla} \cdot \mathbf{u} = e\delta(\mathbf{x}) \Rightarrow \boldsymbol{\nabla} \cdot \mathbf{b} = g\delta(\mathbf{x}) \,, \tag{4.2.11}$$

which means that electric charge is a magnetic monopole in the dual theory, with the previously introduced parameter g playing the role of the monopole charge.

Vice versa, if we have a magnetic monopole of charge g then we have to modified the last term in Eq. (4.2.6) as

$$\frac{e}{g} a_0 (\boldsymbol{\nabla} \cdot \mathbf{B} - g\delta(\mathbf{x}))\,. \tag{4.2.12}$$

That means introduction of a point-like electric charge $-e$ in Lifshitz photon theory.

The Lifshitz photon can be interpreted as the result of a hybridization between two vector modes: the transverse phonon, carried by the transverse components of the displacement $\mathbf{u}$, and the photon A_μ. The mixing between the two mode is due to the dipole coupling $\mathbf{u} \cdot \mathbf{E}$.

Let us also note here that the situation described in our model should occur exactly at the quantum critical point of the ferroelectric phase transition Kittel [2004]. Such a phase transition can be described by the Lagrangian (4.2.1) where $\mathbf{u}$ is interpreted as the electric polarization density of the material, with the addition a potential energy term,

$$\mathcal{L}_{\text{pot}} = -\frac{Mn_0}{2}\omega_0^2 \mathbf{u}^2 - O(u^4). \tag{4.2.13}$$

Here ω_0 is the frequency of the transverse optical phonon (at zero wavenumber). It is known that the dielectric constant diverges as one approaches the critical point, $\omega_0 \to 0$, hence the velocity of light in the material goes to zero. Exactly at the phase transition point, $\omega_0 = 0$, the photon velocity vanishes, and hence it is natural for the dispersion relation for the photon to become quadratic.

Note that the quadratically dispersing photon can only be seen very close to the ferroelectric phase transition, i.e., when the photon velocity is smaller than the phonon velocity. This requires the dielectric constant larger than 10^{10}. To compare, the largest value of the dielectric constant that has been reported in isotope-substituted $SrTiO_3$ is $\lesssim 2 \times 10^5$ Itoh and Wang [2000].

4.2.3 Nonlinear Lifshitz photon theory

In the previous Section, we have presented a model in three spatial dimensions that reduces to the Lifshitz photon theory at low energy in Eq. (4.2.9). The model is a Wigner crystal coupled to a dynamical $U(1)$ gauge field. We now investigate this model at the nonlinear level. We will find that the model possesses a nonlinear version of the dipole symmetry as well as a nonlinear multipole symmetry. Our construction can be considered as a three-dimensional generalization of the theory of the Tkachenko mode constructed in Ref. Du et al. [2024a].

A state of a solid can be described by a map between the external coordinate x^a and the coordinate frozen into the solid $X^a(t, x^i)$, where $a = 1, 2, 3$ in three spatial dimensions. The lattice displacement is $u^a = \delta^a_i x^i - X^a$. The particle number current is a topological current,

$$J^\mu = \frac{n_0}{6}\epsilon^{\mu\nu\lambda\rho}\epsilon^{abc}\partial_\nu X^a \partial_\lambda X^b \partial_\rho X^c\,, \tag{4.2.14}$$

where n_0 is the unperturbed density. In particular, the particle number density Son [2005],

$$J^0 = n_0 \det|\partial_i X^a| = \frac{n_0}{6}\epsilon^{ijk}\epsilon^{abc}\partial_i X^a \partial_j X^b \partial_k X^c\,, \tag{4.2.15}$$

is proportional to the Jacobian of the transformation from the external spatial coordinate x^a to the coordinates frozen within the solid X^a.

The particle number current is coupled to the $U(1)$ gauge field A_μ through a term $A_\mu J^\mu$. To introduce a nondynamical neutralizing background with the background charge density $-n_0$, we will turn on a background Kalb-Ramond gauge field $B_{\mu\nu} = -B_{\nu\mu}$ with a nonzero field strength

$$H_{ijk} = \partial_i B_{jk} + \partial_j B_{ki} + \partial_k B_{ij} = \ell^{-3}\epsilon_{ijk}, \tag{4.2.16}$$

with the "magnetic length" ℓ related to n_0 by $n_0 = \frac{1}{2\pi}\ell^{-3}$ and couple it to the electromagnetic field A_μ through a "BF" term $-\frac{1}{2\pi}\epsilon^{\mu\nu\lambda\sigma}B_{\mu\nu}\partial_\lambda A_\sigma$. The Lagrangian for our model can then be written as follows:

$$\mathcal{L} = \frac{1}{6\pi\ell^3}A_\mu\epsilon^{\mu\nu\lambda\rho}\epsilon^{abc}\partial_\nu X^a\partial_\lambda X^b\partial_\rho X^c - \frac{B_iB_i}{2e^2} - \epsilon(O^{ab}) - \frac{1}{2\pi}\epsilon^{\mu\nu\lambda\sigma}B_{\mu\nu}\partial_\lambda A_\sigma\,, \quad (4.2.17)$$

where $O^{ab} = \partial_i X^a\partial_i X^b$, $B^i = \epsilon^{ijk}\partial_i A_k$ (and should be distinguished from the components of the Kalb-Ramond two-index tensor $B_{\mu\nu}$), and $\epsilon(O^{ab})$ is the elastic energy of the lattice. Note that terms involving the time derivatives of X^a or A_i are dropped because we expect the low energy modes will have quadratic dispersion $\omega\sim q^2$.

Introducing the compensating background through the Kalb-Ramond field allows the background to carry a nonzero divergenceless electric current, parameterized through the B_{0i} components. Indeed, the BF term in the action can be expanded as

$$-\frac{1}{2\pi}\epsilon^{\mu\nu\lambda\sigma}B_{\mu\nu}\partial_\lambda A_\sigma = -\frac{1}{\pi}B_{0i}B^i - \frac{A_0}{2\pi\ell^3}\,. \quad (4.2.18)$$

The first term, up to a total derivative, can be written as $J^i_{\text{ext}}A_i$ where the external current $J^i_{\text{ext}} = -\frac{1}{\pi}\epsilon^{ijk}\partial_j B_{0k}$. This current is divergenceless, consistent with the constant density of the neutralizing background.

Upon varying the Lagrangian with respect to A_0, we obtain a constraint:

$$\frac{1}{6}\epsilon^{ijk}\epsilon^{abc}\partial_i X^a\partial_j X^b\partial_k X^c = 1\,, \quad (4.2.19)$$

which means that the transformation from x^i to X^a is volume-preserving. To the linear order of displacement $\mathbf{u}$, it follows from Eq. (4.2.19) that $\boldsymbol{\nabla}\cdot\mathbf{u} = 0$, indicating that the displacement is divergence-free. Here we would like to address the constraint (4.2.19) at the nonlinear level. One knows that a volume-preserving diffeomorphism (VPD) can be obtained by exponentiating an infinitesimal VPD [2]; the latter being given by a differential operator $-\xi^i\partial_i$ where ξ^i is a divergence-free vector field, $\partial_i\xi^i = 0$. We can solve the constraint on ξ^i by introducing a gauge potential a_i: $\xi^i = \ell^3\epsilon^{ijk}\partial_j a_k \equiv \ell^3 b^i$. Thus we have

$$\begin{aligned} X^a &= e^{-\xi^i\partial_i}x^a = x^a - \xi^a + \frac{1}{2}\xi^k\partial_k\xi^a - \frac{1}{6}\xi^l\partial_l(\xi^k\partial_k\xi^a) + \cdots \\ &= x^a - \ell^3 b^a + \frac{\ell^6}{2}b^k\partial_k b^a + \cdots . \end{aligned} \tag{4.2.23}$$

In order to explore the linear regime and find the quadratic Lagrangian, we initiate our analysis by Eq. (4.2.23). One can then expand the Lagrangian and reach the linearized Lifshitz photon theory in Eq. (4.2.9). This corresponds to a quadratic dispersion relation $\omega \sim q^2$ which is protected by the "magnetic" translation symmetry, as described in the subsequent section.

Nonlinear multipole symmetries in (3+1) dimensions

In this Section, our focus is directed toward the exploration of nonlinear dipole and higher multiple symmetries in (3+1) dimensions.

We first consider translation. The operation of translation of the whole lattice by a

2. Another way to express the mapping from x^a to X^a is by the Nambu-Poisson bracket. The Nambu-Poisson bracket Nambu [1973], Takhtajan [1994] serves as a generalization of Poisson brackets based on a canonical triple x^1, x^2, x^3, is defined by

$$\{f, g, h\} = \epsilon^{abc}(x)\partial_a f(x)\partial_b g(x)\partial_c h(x) . \tag{4.2.20}$$

Equipped with the generalized Poisson bracket, we can represent the VPD constraint in the following form:

$$\{X^a, X^b, X^c\} = \epsilon^{abc} . \tag{4.2.21}$$

Then, the Nambu bracket description of the mapping from x^a to X^a is given by

$$X^a = x^a - \ell^3\{x^a, x^c, a_c\} + \frac{\ell^6}{2!}\{\{x^a, x^c, a_c\}, x^k, a_k\} + \cdots . \tag{4.2.22}$$

The utilization of the Nambu bracket implies that there is no noncommutative field theory description applicable to spatial dimensions higher than two.

spatial vector c^i is given by [3]

$$X^a \to X^a(\vec{x} - \vec{c}) = e^{-c^i \partial_i} X^a \,. \tag{4.2.26}$$

But since X^a is related to a_i by Eq. (4.2.23), one reaches

$$e^{-\ell^3 (\boldsymbol{\nabla}\times\mathbf{a})^i \partial_i} \to e^{-c^i \partial_i} \; e^{-\ell^3 (\boldsymbol{\nabla}\times\mathbf{a})^i \partial_i} \,. \tag{4.2.27}$$

From this we derive the action of translation on the Lifshitz photon a_i, it can be expressed as

$$a_i \to a_i + \frac{1}{\ell^3}\epsilon_{ijk} c^j x^k - \frac{1}{2}\epsilon_{ijk} c^j (\boldsymbol{\nabla} \times \mathbf{a})_k + \cdots \,. \tag{4.2.28}$$

This nontrivial form of the transformation law has the consequence that two translations do not commute:

$$[P_i, P_j] = i\ell^{-3} \varepsilon_{ijk} Q_k \,, \tag{4.2.29}$$

where under Q_k the Lifshitz photon transforms as $a_k \to a_k + c_k$. Thus Q_k can be identified with a conserved charge of a $U(1)^{(1)}$ one-form symmetry, which is spontaneously broken, giving rise the Lifshitz photon is a Nambu-Goldstone boson (NGB).

Equation (4.2.29) should remind one of algebra of the translations on a two-dimensional plane in the presence of a magnetic field—the "magnetic translations." There the commutator of the two magnetic translations is proportional to the $U(1)$ charge, with the coefficient of

3. An alternative method for determining the nonlinear dipole symmetry can be achieved through the application of the subsequent algebraic approach. Let us introduce the diveregence-free vector fields $\boldsymbol{\xi_1} = \ell^3 \boldsymbol{\nabla} \times \mathbf{a}'$, $\boldsymbol{\xi_2} = \ell^3 \boldsymbol{\nabla} \times \bar{\mathbf{a}}$ and $\boldsymbol{\xi_3} = \ell^3 \boldsymbol{\nabla} \times \tilde{\mathbf{a}}$. The Lie bracket is given by

$$[\xi_1^i, \xi_2^i] = \{\xi_1, \xi_2\}^i = \xi_1^j \partial_j \xi_2^i - \xi_2^j \partial_j \xi_1^i = \xi_3^i \,, \tag{4.2.24}$$

from which we find

$$[a'_i, \bar{a}_j] = -\ell^3 \epsilon_{ijk} (\boldsymbol{\nabla} \times \mathbf{a}')_i (\boldsymbol{\nabla} \times \bar{\mathbf{a}})_j \,. \tag{4.2.25}$$

By employing this algebraic approach in conjunction with Eq. (4.2.23), one can derive the nonlinear dipole symmetry as expressed in Eq. (4.2.28).

proportionality being the magnetic field.

Thus, the Lifshitz photon is a three-dimensional generalization of the Tkachenko mode propagating on a vortex lattice of a rotating two-dimensional superfluid. As in the latter case, the quadratic dispersion relation is a consequence of the symmetry. The Lifshitz photon here is also a NGB "shared" between the spontaneously broken $U(1)^{(1)}$ and translational symmetries, with the consequence that the number of broken generators is not equal to the number of NGBs, which is two in the case of the Lifshitz photon Nielsen and Chadha [1976], Schäfer et al. [2001], ?], Watanabe and Murayama [2014].

In Appendix 4.2.8 we generalize the Lifshitz photon theory to d spatial dimensions. The algebraic formulation presented in Eq. (4.2.29) becomes associated with a $(d-2)$-form charge $Q^{(d-2)}$.

The system also realizes rotations as nonlinear higher multipole symmetry, in analogy with the magnetic rotation in the theory of the Tkachenko mode. From

$$e^{-\ell^3(\boldsymbol{\nabla}\times\mathbf{a})^i\partial_i} \to e^{\epsilon^{ijk}\omega_i x_j \partial_k}\; e^{-\ell^3(\boldsymbol{\nabla}\times\mathbf{a})^i\partial_i}\,, \tag{4.2.30}$$

we find that rotation is realized as a higher multipole symmetry

$$a_i \to a_i + \frac{1}{\ell^3}\omega_i x^2 + \cdots\,. \tag{4.2.31}$$

Interaction and decay rate of the Lifshitz photon

The nonlinear nature of the parameterization (4.2.23) and the dipole symmetry (4.2.28) imply that the effective Lagrangian of the Lifshitz photon must contain nonlinear terms describing its self-interaction.

We will limit ourselves to finding the cubic vertices of interaction. We first compute the current J^i in Eq. (4.2.14) by substituting Eq. (4.2.23),

$$J^i = \dot{b}^i + \frac{\ell^3}{2}(\dot{b}^k \partial_k b^i - b^k \partial_k \dot{b}^i) + O(a^3)\,. \tag{4.2.32}$$

Since $\partial_i J^i = 0$, one expects that one can write $J^i = \epsilon^{ijk}\partial_j \Pi_k$. From Eq. (4.2.32) one obtaines

$$\Pi_i = \dot{a}_i + \frac{\ell^3}{2}\epsilon_{ijk}\dot{b}^j b^k + O(a^3)\,. \tag{4.2.33}$$

This allows us to rewrite

$$J^i A_i - \frac{B^2}{2e^2} = B^i \Pi_i - \frac{B^2}{2e^2} + \text{total derivative}\,. \tag{4.2.34}$$

Integrating over B_i after introducing a Lagrange multiplier enforcing the constraint $\partial_i B^i = 0$, one then obtains in the action the term

$$2\pi\alpha \left(e_i - \frac{\ell^3}{2}\epsilon_{ijk}\dot{b}^j b^k \right)^2 , \tag{4.2.35}$$

where $\alpha = e^2/(4\pi)$. The cubic interaction that emerges from this term, after integration by part, can be written as

$$2\pi\alpha\ell^3 e_i e_j \partial_i b_j\,. \tag{4.2.36}$$

Another source of interaction is in the elastic energy. This includes the quadratic term in the elastic energy, expanded to cubic order in b according to Eq. (4.2.22), as well as in the cubic term of the elastic energy. The terms that one obtains are (suppressing spatial indices)

$$(\partial b)^3, \quad bdb\partial^2 b\,. \tag{4.2.37}$$

Using the power counting scheme appropriate for the Lifshitz photon theory (in which momentum has dimension 1 and energy has dimension 2), one finds that the coupling constants in front of the terms (4.2.36) and (4.2.37) have dimension $-\frac{5}{2}$. Since the decay rate is proportional to the square of the coupling constants, the energy dependence of the decay width of the Lifshitz photon is

$$\Gamma(E) \sim E^{\frac{7}{2}} \,. \tag{4.2.38}$$

Compared with the Tkachenko mode in two dimensions, the decay width of the Lifshitz photon tends to zero at a faster rate.

4.2.4 Prospects of higher form symmetry

As we mentioned before, the gauge field $\mathbf{a}$ is the dual gauge field of the physical EM field $\mathbf{A}$. The Wilson loop of $\mathbf{a}$, i.e. $W_{\mathbf{a}} = \exp(\mathrm{i} \oint_{\mathcal{C}} d\mathbf{l} \cdot \mathbf{a})$ is the order parameter of the magnetic 1-form symmetry $U(1)^{(1)}$ of the physical EM field, where $\mathcal{C}$ is a closed loop in space. And $W_{\mathbf{a}}$ is also the 't Hooft loop of the EM field. Based on the Lagrangian Eq. 4.2.1, the scaling dimension of $\mathbf{a}$ is $[\mathbf{a}] = 1/2$, this leads to the observation that the Wilson loop of $\mathbf{a}$ scales as

$$\langle W_{\mathbf{a}} \rangle \sim \exp(-cL \log L), \tag{4.2.39}$$

where L is the perimeter of $\mathcal{C}$. This is a qualitatively different scaling from that of the 't Hooft loop of an ordinary EM field in the vacuum, which should decay with a perimeter law. A suppressed scaling of the 't Hooft loop of the EM field can usually be attributed to the screening from the electric charges. The strongest screening of the EM field is the condensation of the electric charges, which drives the EM field into a Higgs phase and render the 't Hooft loop decay with an area law. In our case, the EM field is screened by the fluctuation $\mathbf{u}$ of the Wigner crystal of the electric charges, which is a much weaker screening compared with the Higgs machanism, but still leads to a different scaling of the 't Hooft loop. Simple power-counting suggests that, in higher dimensional generalizations of the theory where $\mathbf{a}$ become a $(d-2)-$form gauge field (d is the spatial dimension rather than space-time dimension), the "Wilson-membrane" of $\mathbf{a}$ should always violate the perimeter law with an extra logarithmic factor.

The logarithmic correction to the 't Hooft loop also occurs in another more familiar system: the $(3+1)d$ QED with massless Dirac fermions. First of all, let us assume that there is no Dirac monopoles in the QED, hence there is a strict magnetic $U(1)^{(1)}$ 1-form symmetry. With massive matter fields, the 't Hooft loop should decay with a perimeter law, and the coefficient of the perimeter law is proportional to $1/\alpha$, where α is the fine-structure constant:

$$\log\langle W_{\mathbf{a}}\rangle \sim -\frac{1}{\alpha}L. \tag{4.2.40}$$

However, when the EM field A_μ is coupled to the massless Dirac fermions, α will be marginally irrelevant in the infrared due to screening from the massless fermions, hence we expect the 't Hooft loop to receive an extra factor of $\log L$ for large L:

$$\log\langle W_{\mathbf{a}}\rangle \sim -\frac{1}{\alpha_0}L\log L. \tag{4.2.41}$$

This is because the fine-structure constant $\alpha(\mu)$ at energy scale μ is $\alpha(\mu) \sim \alpha_0/\log(1/\mu)$, and $\mu \sim 1/L$.

4.2.5 Potential connection to other lattice gauge systems

In this section we explore a potential realization of the physics discussed in the previous sections in the context of other systems with a description of lattice gauge theories. One class of such systems is the well-known quantum spin-ice Hermele et al. [2004]. The quantum spin ice materials usually have quantum Ising spins from the rare-earth elements that live on the Pyrochlore lattice, which is dual to the diamond lattice in the sense that the sites of the Pyrochlore lattice are the links of the diamond lattice. The largest term of the Hamiltonian is the following:

$$H_0 = \sum_t J_z (\sum_{i=1}^{4} \mathbf{S}^z_{t,i})^2 \,. \tag{4.2.42}$$

The subscript "t" labels each tetrahedron of the Pyrochlore lattice; and $\mathbf{S}^z_{t,i}$ labels the i-th spin-1/2 degree of freedom (the z-component only) on the tetrahedron. Notice that H_0 is simply a nearest neighbor antiferromagnetic interaction on the Pyrochlore lattice. The ground states of H_0 consist of all configurations of $\mathbf{S}^z_{t,i}$ which satisfy $\sum_i \mathbf{S}^z_{t,i} = 0$ for each tetrahedron t, i.e. each tetrahedron has two up-spins and two down-spins.

$\mathbf{S}^z$ is mapped to the (discrete) electric field; and the condition $\sum_i \mathbf{S}^z_{t,i} = 0$ is mapped to the Gauss law constraint $\boldsymbol{\nabla}\cdot\mathbf{E} = 0$. Now suppose we turn on spin exchange $\sum_{\langle i,j\rangle} J_\perp (S^x_i S^x_j + S^y_i S^y_j)$, at the third order perturbation of $J_\perp/U$, we are going to generate a ring exchange term

$$H_r \sim \sum_{\hexagon} J_r S^+_1 S^-_2 S^+_3 S^-_4 S^+_5 S^-_6 \,. \tag{4.2.43}$$

This term will map to the term $H_r \sim J_r \cos(\boldsymbol{\nabla}\times\mathbf{A})$, where $J_r \sim J^3_\perp / J^2_z$.

Now we organize the entire low energy effective Hamiltonian as

$$H = \frac{U}{2}\mathbf{E}^2 - J_r \cos(\boldsymbol{\nabla}\times\mathbf{A}) \,. \tag{4.2.44}$$

Notice that now we allow $\mathbf{E}$ to take all half-integer values, and the first U term will constrain the low energy Hilbert space to $\mathbf{E} = \pm 1/2$.

We can now consider polarizing a fraction of the Ising spin $\mathbf{S}^z$. If one $\mathbf{S}^z$ is flipped from $\mathbf{S}^z = -1/2$ to $\mathbf{S}^z = +1/2$, it amounts to create an electric field on a site of the Pyrochlore lattice (or the link of the diamond lattice), which violates the constraint $\boldsymbol{\nabla}\cdot\mathbf{E} = 0$ on a pair of nearest neighbor sites of the diamond lattice, i.e. this is equivalent to creating a dipole of gauge charges. More precisely it creates a positive gauge charge on the sublattice A of the diamond lattice, and a negative gauge charge on the sublattice B of the diamond lattice. In

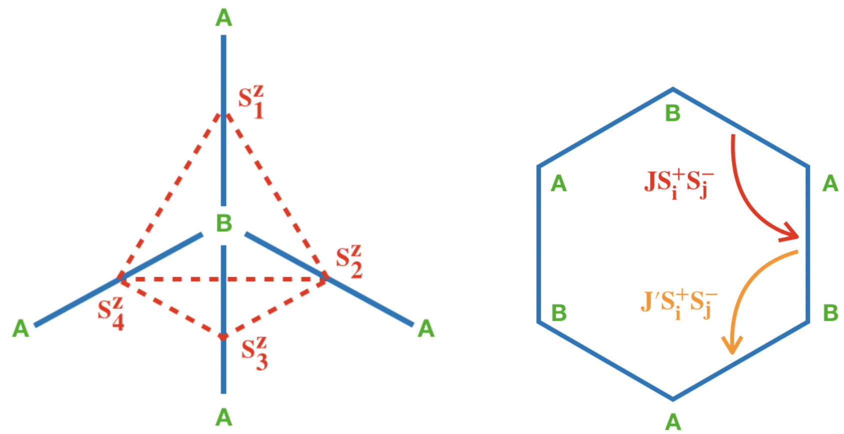

Figure 4.1: (a) shows the basic structure of the Pyrochlore lattice, which is built with corner-sharing tetrahedrons (red); the sites of the tetrahedrons are the links of a dual diamond lattice; the two sublattices of the diamond lattice are labelled A and B. The purpose of (b) is to show that the dynamics of defects (gauge charges) on A and B sublattices can be very different, as long as there is no symmetry that connects A and B sublattices. The dynamics of gauge charges on sublattice A is controlled by spin exchange J' in the sketch, while the dynamics of gauge charges on B is controlled by spin exchange J.

general we have the following relation:

$$\begin{aligned} \oint_{\mathcal{A}} \mathbf{E} \cdot d\sigma &= n_A - n_B \,, \\ M &\sim \sum n_A + n_B \,. \end{aligned} \tag{4.2.45}$$

Here n_A and n_B count the number of "defects" that violate the original constraint $\sum_i \mathbf{S}^z_{t,i} = 0$ on sublattice A and B of the diamond lattice respectively, and their total number can be controlled by the magnetization M of the system. These defects become mobile with the $J_\perp$ spin exchange term, but $J_\perp$ always hops defects within the same sublattice. Defects on sublattice A and B should have the same total number, but they do not necessarily have the same dynamics unless there is a symmetry that connects these two sublattices. Hence it is conceivable that the defects on sublattice A form a Wigner crystal that is incommensurate with the lattice and hence lead to gapless excitations that correspond to $\mathbf{u}$, while the defects on sublattice B are pinned by the lattice or disorder. Then the WC of defects on sublattice

A will lead to physics discussed in this paper. We leave more detailed construction in the context of quantum spin ice to future study.

4.2.6 An alternative candidate: ferromagnets

In this Section, we speculate on another possible candidate for a physical system featuring Lifshitz photons: a "deconfined" ferromagnet. First we recall some facts about the nonlinear sigma-model of ferromagnets. The action of the sigma-model is Du et al. [2022], Fradkin [2013]:

$$S = S_0 \int_0^1 d\sigma \int dt\, d\mathbf{x}\, \varepsilon^{abc} n^a \partial_t n^b \partial_\sigma n^c - \frac{J}{2} \int dt\, d\mathbf{x}\, \delta^{ij} \partial_i n^a \partial_j n^a \,, \tag{4.2.46}$$

where S_0 is a parameter related to the volume density of spin. The model has a 2-form current

$$J^{\mu\nu} = \frac{1}{8\pi} \varepsilon^{\mu\nu\lambda\rho} \varepsilon_{abc} n^a \partial_\lambda n^b \partial_\rho n^c \,, \tag{4.2.47}$$

which is by construction conserved, $\partial_\mu J^{\mu\nu} = 0$. The location where the 2-form density J^{0i} is nonzero can be identified with the Skyrmions, which are one-dimensional lines (or loops) in three-dimensional space.

There is an alternative description of the $\mathbb{CP}^1$ parametrization of the spin vector: $n^a = z^\dagger \sigma^a z$, where

$$z = \begin{pmatrix} z_1 \\ z_2 \end{pmatrix}, \qquad z^\dagger z = 1 \,. \tag{4.2.48}$$

The phase of z is redundant, giving rise to an emergent gauge field α_μ. The effective action for the ferromagnet can be written as

$$\mathcal{L} = 2iS_0 z^\dagger \partial_t z - 2J D_i z^\dagger D_i z + \lambda(|z|^2 - 1) \,, \tag{4.2.49}$$

where D_μ is the covariant derivative, $D_\mu z = (\partial_\mu - i\alpha_\mu) z$, and λ is a Lagrange multiplier enforcing the condition $|z|^2 = 1$. The equation of motion for a_μ can be solved to yield,

$$\alpha_\mu = -\frac{i}{2}\left[z^\dagger(\partial_\mu z) - (\partial_\mu z^\dagger)z\right] . \tag{4.2.50}$$

The 2-form current (4.2.47) can be expressed in terms of the photon as

$$J^{\mu\nu} = \frac{1}{2\pi}\epsilon^{\mu\nu\rho\lambda}\partial_\rho \alpha_\lambda . \tag{4.2.51}$$

For example, the Skyrmion density vector is

$$J_i \equiv J^{0i} = \frac{1}{8\pi}\epsilon^{ijk}\epsilon_{abc}n^a\partial_j n^b \partial_k n^c = \frac{1}{2\pi}\epsilon^{ijk}\partial_j \alpha_k , \tag{4.2.52}$$

hence a Skyrmion loop is the magnetic flux loop of α_μ.

It is also known Papanicolaou and Tomaras [1991] that the ferromagnetic nonlinear sigma model possesses nontrivial conservation laws: in addition to the conservation of the one-form charge

$$Q_i = \int d\mathbf{x}\ J_i . \tag{4.2.53}$$

all the first moments

$$I_{ij} = \int d\mathbf{x}\ x_i J_j , \tag{4.2.54}$$

are also conserved. In addition, one higher moment, the angular momentum is expressed by

$$l_i = \frac{1}{2}\int d\mathbf{x}\ x^2 J_i , \tag{4.2.55}$$

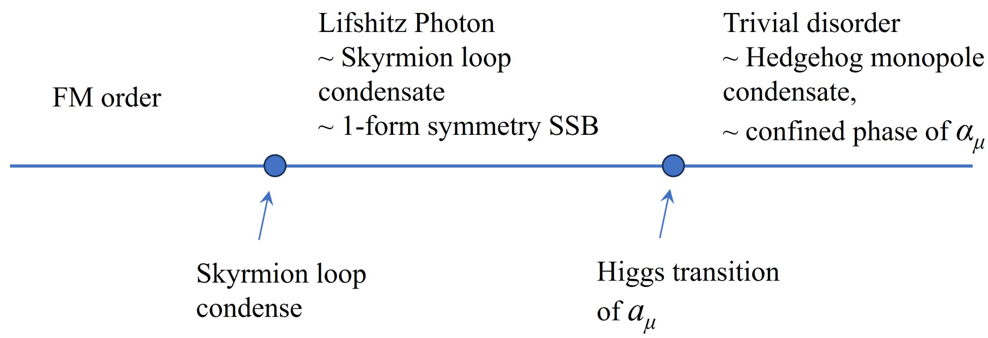

Figure 4.2: We speculate that starting with a ferromagnet order, one can enter a Lifshitz photon phase by proliferating the Skyrmion loops of the FM order $\vec{n}$. By condensing the hedgehog monopole of the FM order $\vec{n}$ one drives the system into a trivial disordered phase. A hedgehog monopole is the Dirac monopole of gauge field α_μ, and the charge of the Lifshitz photon gauge field a_μ.

is also conserved. The quantities

$$p_i = \int d\mathbf{x}\ \epsilon_{ijk} x_j J_k, \qquad l_i = \frac{1}{2}\int d\mathbf{x}\ x^2 J_i\,, \tag{4.2.56}$$

are similar to the conservation laws that follow from the dipole and the higher multipole symmetries of the nonlinear Lifshitz photon theory. In particular, there is a nontrivial commutation relation Papanicolaou and Tomaras [1991]

$$[p_x,\ p_y] \sim iQ_z\,. \tag{4.2.57}$$

We now speculate that there is a phase where Skyrmions proliferate and condense, destroying the ferromagnetic order. The magnetic flux loop of α_μ is dual to the electric flux loop of a_μ, and in this phase the gauge field a_μ becomes the gapless Nambu-Goldstone boson as a result of the condensation of the 1-form symmetry charge, i.e. the electric field flux. Based on the similarity between the symmetries of the ferromagnetic nonlinear sigma model and that of the nonlinear Lifshitz theory, one can expect that the low energy degree of freedom of such a "deconfined" ferromagnet is a Lifshitz photon. The gauge field describing this

Lifshitz photon should be the electromagnetic dual of the gauge field in the $\mathbb{CP}^1$ formulation of the sigma model; its Lagrangian

$$\mathcal{L} = c_1 e_i e_i - c_2 \left(\partial_i b_j + \partial_j b_i\right)^2 + g_1 e_i e_j \left(\partial_i b_j + \partial_j b_i\right) + \cdots , \tag{4.2.58}$$

should have the nonlinear dipole and higher multipole symmetries. One notices the noncommutative momentum algebra in Eq. (4.2.57) results in the nonlinear version of the Lifshitz photon theory. (Note that if one takes into account the the Dzyaloshinskii-Morya interaction Komineas and Papanicolaou [2015], then $\frac{1}{2}\int d\mathbf{x}\, x^2 J_i$ is not conserved, which means that the higher multipole symmetry is absent.)

The Lifshitz photon phase is one type of "exotic" quantum disordered state of the magnetic system. Of course there could be another completely trivial disordered phase with fully gapped spectrum and no spontaneous symmetry breaking. This trivial disordered phase is allowed in a quantum spin system unless there is a Lieb-Shultz-Matthis theorem that excludes it. The Lifshitz photon phase is actually an intermediate disordered phase sandwiched between the FM order and the trivial disordered phase, and both these disordered phases can be constructed using ingredients of the FM order. The Lifshitz photon phase is driven by proliferating the Skyrmion loops of the FM order $\vec{n}$, while the trivial disordered phase is driven by the condensation of the hedgehog monopole of $\vec{n}$. The hedgehog monopole of $\vec{n}$ is nothing but the Dirac monopole of α_μ, hence in a phase where the Skyrmion loops proliferate while the hedgehog monopoles do not condense, the gauge field α_μ is in its deconfined Lifshitz photon phase. If we further condense the Dirac monopole of α_μ, the system enters a trivial disordered phase. Notice that α_μ is the dual gauge field of a_μ, hence the condensation of the Dirac monopole is a Higgs transition of the Lifshitz gauge field a_μ (Fig. 4.3).

4.2.7 Conclusion

In this work, we present a nonlinear version of the Lifshitz photon theory applied to diverse condensed matter systems in (3+1) dimensions. Lifshitz photon, which is dual to the physical photon, emerges as a quasiparticle with two transverse polarizations and a quadratic dispersion relation. Our primary focus lies on the Wigner crystal of charged particles coupled to a dynamical $U(1)$ gauge field, which serves as a prototypical model for our study. We formulate the Lagrangian for the nonlinear Lifshitz photon theory, which is consistent with a global $U(1)$ 1-form symmetry and nonlinearly realized "magnetic" translation and rotation symmetries. Moreover, our analysis reveals the energy-dependent decay rate of the Lifshitz photon through the nonlinear theory. Additionally, we explore the generalization of the nonlinear Lifshitz theory to higher dimensions.

In (3+1)-dimensions, we uncover the presence of a 1-form global symmetry. From the dual of the Lifshitz photon, we know that the Wilson loop of $\mathbf{a}$ serves as the order parameter of the magnetic 1-form symmetry $U(1)^{(1)}$ of the electromagnetic field $\mathbf{A}$. Notably, the scaling of the Wilson loop of $\mathbf{a}$ is determined as $\exp(-cL\log L)$. We know the electromagnetic field is screened due to the fluctuation $\mathbf{u}$ of the Wigner crystal of the electric charges. This screening effect, although weaker than the Higgs mechanism, induces a distinct scaling for the 't Hooft loop. Additionally, when extending the theory to higher dimensions, we observe a deviation of the "Wilson-membrane" associated with $\mathbf{a}$ from the perimeter law, featuring an additional logarithmic factor.

We investigate the potential realization of other lattice gauge systems. Specifically, we focus on quantum spin ice, where quantum Ising spins live on the Pyrochlore lattice, which is dual to the diamond lattice. By considering two sublattices of the diamond lattice, with defects either pinned by the lattice or subjected to disorder on one of the sublattices. We observe the emergence of intriguing physics discussed in this paper by the Wigner crystal of defects on another sublattice. We defer a more comprehensive construction within the context of quantum spin ice to future investigations.

Moreover, we explore the "deconfined" ferromagnet as an alternative candidate for establishing connections to the Lifshitz photon theory, employing Skyrmion condensation. The Lifshitz photon phase represents one type of "exotic" quantum disordered state of the magnetic system, serving as an intermediate disordered phase situated between the FM order and the trivial disordered phase. Our study highlights the intriguing possibility of exploring the interconnection between the nonlinear Lifshitz theory Du et al. [2024a] and systems of quantum Hall ferromagnets Sondhi et al. [1993], MacDonald et al. [1996], Else [2021] or twisted bilayer graphene. It is found in the superconductivity mechanism of magic-angle twisted bilayer graphene, while elementary Skyrmions condensing brings out a superconducting state, which breaks the $U(1)$ symmetry spontaneously Khalaf et al. [2021]. Similar facts of the Skyrmion condensation happen in the antiferromagnet system; condensing Skyrmions restores the spin rotational symmetry through a deconfined quantum critical point (DQCP) Senthil et al. [2004b,a], Wang et al. [2017], Xu [2012], Senthil [2023], the Landau-forbidden quantum phase transitions. We defer these questions to future research.

4.2.8 Supplemental Material

Generalizations to higher dimensions

Now, we aim to extend the nonlinear Lifshitz theory to (d+1)-dimensions with d ($d > 2$) as the spatial dimension. To begin with, the generalized Lagrangian of the prototypical system (4.2.2) is given by

$$\mathcal{L} = -\frac{\mu}{4}\left(\partial_i u_j + \partial_j u_i - \frac{2}{d}\delta_{ij}\partial_k u_k\right)^2 - \frac{K}{2}(\partial_i u_i)^2 + e n_0 \mathbf{u}\cdot\mathbf{E} - \frac{\mathbf{B}^2}{2}\,. \tag{4.2.59}$$

We can then derive the generalization of the linearized Lifshitz theory. To obtain the generalized nonlinear Lifshitz theory, let us introduce the neutralizing background with background density $n_0 = \frac{1}{2\pi}\ell^{-d}$ and include a (d-1)-form field $B_{\mu_1\mu_2\ldots\mu_{d-1}}$ with its field strength

$$H_{\mu_1\mu_2\cdots\mu_d} = \partial_{[\mu_1} B_{\mu_2\cdots\mu_d]} = \ell^{-d}\epsilon_{\mu_1\mu_2\cdots\mu_d}\,. \tag{4.2.60}$$

The gauge symmetry accompanied by a (d-2)-form gauge parameter $\Lambda_{\mu_1\cdots\mu_{d-2}}$:

$$\delta_\Lambda B_{\mu_1\mu_2\ldots\mu_{d-1}} \equiv \partial_{[\mu_1}\Lambda_{\mu_2\cdots\mu_{d-1}]}\,. \tag{4.2.61}$$

It is straightforward to generalize the particle number current which is a topological current as

$$J^{\mu_1} = \frac{n_0}{d!}\epsilon^{\mu_1\cdots\mu_{d+1}}\epsilon^{i_1\cdots i_d}\partial_{\mu_2}X^{i_1}\cdots\partial_{\mu_{d+1}}X^{i_d}\,. \tag{4.2.62}$$

Henceforth, we can establish the generalized Lagrangian in d spatial dimensions as follows:

$$\mathcal{L} = \frac{1}{d!\pi\ell^d}A_{\mu_1}\epsilon^{\mu_1\cdots\mu_{d+1}}\epsilon^{i_1\cdots i_d}\partial_{\mu_2}X^{i_1}\cdots\partial_{\mu_{d+1}}X^{i_d} - \epsilon(B^{(d-2)},O^{ab}) - \frac{1}{2\pi}\epsilon^{\mu_1\cdots\mu_{d+1}}B_{\mu_1\cdots\mu_{d-1}}\partial_{\mu_d}A_{\mu_{d+1}}\,. \tag{4.2.63}$$

Following the same procedure upon varying the Lagrangian with respect to A_0, one finds the VPD constraint [4]

$$\frac{1}{d!}\epsilon^{i_1\cdots i_d}\epsilon^{a_1\cdots a_d}\partial_{i_1}X^{a_1}\cdots\partial_{i_d}X^{a_d} = 1\,. \tag{4.2.66}$$

We know the constraint can be obtained by exponentiating an infinitesimal VPD transformation in Eq. (4.2.23) where ξ^i is the divergenceless vector field $\xi^{i_1} = \ell^d\epsilon^{i_1\cdot i_d}\partial_{i_2}\phi_{i_3\cdots i_d}$.

4. This constraint can be rewritten as the condition on the generalized Nambu-Poisson bracket of X^a,

$$\{X^{a_1}, X^{a_2}, \cdots, X^{a_d}\} = 1\,. \tag{4.2.64}$$

Recall that the generalized Nambu-Poisson bracket is defined as

$$\{f_1, f_2, \ldots, f_d\} = \epsilon^{i_1 i_2\ldots i_d}\partial_{i_1}f_1\partial_{i_2}f_2\ldots\partial_{i_d}f_d\,. \tag{4.2.65}$$

Generalized multipole symmetries— One can find the generalized "magnetic" translation and rotation symmetries by Eq. (4.2.23). It is important to note that these symmetries are termed generalized "magnetic" translations and rotations due to the presence of the d-form field strength $H_{\mu_1\mu_2\cdots\mu_d} = \ell^{-d}\epsilon_{\mu_1\mu_2\cdots\mu_d}$.

The generalized "magnetic" translation is the generalized dipole symmetry, can be expressed by

$$\phi_{i_1\cdots i_{d-2}} \to \phi_{i_1\cdots i_{d-2}} + \frac{1}{\ell^d}\epsilon_{i_1\cdots i_d}c^{i_{d-1}}x^{i_d} - \frac{1}{2}\epsilon_{i_1\cdots i_d}c^{i_{d-1}}\epsilon^{i_d\cdot i_{2d-1}}\partial_{i_{d+1}}\phi_{i_{d+2}\cdots i_{2d-1}} + \cdots\,, \tag{4.2.67}$$

which satisfies the "magnetic" translations algebra:

$$[P_{i_1}, P_{i_2}] = i\ell^{-d}\epsilon_{i_1\cdots i_d}Q_{i_3\cdots i_d}\,, \tag{4.2.68}$$

with a (d-2)-form charge $Q_{i_3\cdots i_d}$ of the $U(1)^{(d-2)}$ symmetry. We find the generalized "magnetic" rotations is the generalized higher multipole symmetry, which is given by

$$\phi_{i_1\cdots i_{d-2}} \to \phi_{i_1\cdots i_{d-2}} + \frac{1}{\ell^d}\omega^{i_1\cdots i_{d-2}}x^2 + \cdots\,. \tag{4.2.69}$$

Generalized Lifshitz theory— In the linear regime, one gets the divergenceless condition $\nabla \cdot \mathbf{u} = 0$ and it can be solved by

$$u^{i_1} = \ell^d\epsilon^{i_1\cdot i_d}\partial_{i_2}\phi_{i_3\cdots i_d}\,, \tag{4.2.70}$$

where $\phi_{i_3\cdots i_d}$ is the (d-2) higher-form field. Thus, one gets the linearized theory by adding the Lagrange multiplier

$$\mathcal{L} = c_1\left(\dot{\phi}_{i_1\cdots i_{d-2}} - \partial_{i_1}\cdots\partial_{i_{d-2}}\phi_0\right)^2 - c_2\left(\epsilon^{i_1\cdot i_d}\partial_j\partial_{i_2}\phi_{i_3\cdots i_d}\right)^2\,. \tag{4.2.71}$$

The general Lagrangian for the generalized nonlinear Lifshitz theory at higher dimensions that are consistent with $U(1)^{(d-2)}$ symmetry and "magnetic" translations (4.2.67), contains the combination of the terms

$$\left(\dot{\phi}_{i_1\cdots i_{d-2}} - \partial_{i_1}\cdots\partial_{i_{d-2}}\phi_0\right), \left(\epsilon^{i_1\cdots i_d}\partial_j\partial_{i_2}\phi_{i_3\cdots i_d}\right) . \tag{4.2.72}$$

Finally, the energy dependence of the decay rate is

$$\Gamma(E) \sim E^{\frac{d+4}{2}} . \tag{4.2.73}$$

4.3 Story 6: Quantum vortex lattice: Lifshitz duality, topological defects and multipole symmetries

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Yi-Hsien Du, Ho Tat Lam and Leo Radzihovsky. Quantum vortex lattice: Lifshitz duality, topological defects and multipole symmetries. arXiv:2310.13794.

Abstract

We study an effective field theory of a vortex lattice in a two-dimensional neutral rotating superfluid. Utilizing particle-vortex dualities, we explore its formulation in terms of a $U(1)$ gauge theory coupled to elasticity, that at low energies reduces to a compact Lifshitz theory augmented with a Berry phase term encoding the vortex dynamics in the presence of a superflow. Utilizing elasticity- and Lifshitz-gauge theory dualities, we derive dual formulations of the vortex lattice in terms of a traceless symmetric scalar-charge theory and demonstrate low-energy equivalence of our dual gauge theory to its elasticity-gauge theory dual. We further discuss a multipole symmetry of the vortex lattice and its dual gauge theory's multipole one-form symmetries. We also study its topological crystalline defects, where

the multipole one-form symmetry plays a prominent role. It classifies the defects, explains their restricted mobility, and characterizes descendant vortex phases, which includes a novel vortex supersolid phase. Using the dual gauge theory, we also develop a mean-field theory for the quantum melting transition from a vortex crystal to a vortex supersolid.

4.3.1 Introduction

Motivated by numerous direction – robust quantum memory Haah [2011], Brown et al. [2016], Roberts and Bartlett [2020] and computing, beyond-QFT class of quantum liquids, quantum elasticity and melting Beekman et al. [2017], Pretko and Radzihovsky [2018a,b], generalized symmetries Gaiotto et al. [2015], Qi et al. [2021] – recently there has been an explosion of research on models with quasi-particles characterized by restricted mobility Nandkishore and Hermele [2019], Pretko et al. [2020], Du et al. [2022], Gromov and Radzihovsky [2022], Du et al. [2024a]. Significant progress was made in identifying a class of $U(1)$ fractonic gauge theories as duals of quantum crystals with topological defects Pretko and Radzihovsky [2018a,b], Pretko et al. [2019], Radzihovsky and Hermele [2020]. This connection demystified and clarified some aspects of fractonic orders, provided their in-principle physical realization as crystalline topological defects, and uncovered generalized fractonic models as duals of well-understood elastic theories, such as e.g., quantum smectics Radzihovsky [2020], Zhai and Radzihovsky [2021] and supersolids Pretko and Radzihovsky [2018b], Pretko et al. [2019], Kumar and Potter [2019]. Complementarily, such generalized gauge theories provided a compact effective field theory description of quantum crystals, their defects and associated quantum melting transitions.

The duality was subsequently extended to time-reversal broken crystals such as for example a Wigner crystal Pretko et al. [2019], Kumar and Potter [2019] and rotating superfluids hosting a vortex lattice Nguyen et al. [2020]. Although there is a strong similarity between resulting models and their fractonic tensor-gauge theory duals, a vortex crystal is strongly coupled to its superfluid sector, that encodes superflow-mediated vortex lattice incompressibility, that on dualizing the bosonic sector, couples elasticity to a $U(1)$ vector gauge field.

Upon a proliferation of mobile vortex vacancies and/or interstitials a vortex crystal is expected to undergo a quantum phase transition to a non-superfluid form, where the underlying superfluid order is destroyed ($U(1)$ symmetry of boson number is restored), a state that is the vortex lattice analog of a supersolid – a superfluid crystal of bosons that exhibits ODLRO Pretko and Radzihovsky [2018b], Pretko et al. [2019]. Fundamentally, such "normal" (i.e., a non-superfluid) vortex lattice state is a rotated Mott insulating crystal of bosons. In this "vortex supersolid" the superflow is absent, with the aforementioned dual $U(1)$ gauge sector Higgs'ed and one expects the state to reduce to a time-reversal broken Wigner crystal with its guiding-center dynamics and previously studied gauge-theory dual Pretko et al. [2019], Kumar and Potter [2019]. We also expect that the crystal can further disorder through quantum melting into a smectic, hexatic or nematic vortex states Radzihovsky [2020], Zhai and Radzihovsky [2021], Gromov and Radzihovsky [2022]. These properties can be equivalently captured both in the direct and dual tensor-gauge theory descriptions Nguyen et al. [2020], though the analysis of phase transitions is particularly well accessible on the gauge-dual side, where they correspond to various types of generalized Higgs transitions Radzihovsky [2020], Zhai and Radzihovsky [2021], Gromov and Radzihovsky [2022]. The classical analogs of such states have been previously studied in three-dimensional vortex states in type-II superconductors Frey et al. [1994], Marchetti and Radzihovsky [1999].

It was recently observed Du et al. [2024a] that in the superfluid vortex phase, the coupling to the superfluid flow ($U(1)$ vector gauge field in the dual description) leads to vortex lattice incompressibility. This constrains lattice distortions to be divergenceless, reducing its description to a generalized Lifshitz theory (studied previously in a number of other contexts Moessner et al. [2001], Vishwanath et al. [2004], Fradkin et al. [2004], Ardonne et al. [2004], Gorantla et al. [2022], Lake et al. [2022], Gorantla et al. [2023]) of a single superfluid phase-like Goldstone mode. Duality of the Lifshitz theory has been analyzed in earlier works Radzihovsky [2022], Gorantla et al. [2023] and we thus extend it here to duality of a vortex crystal.

We summarize the main results of this paper below. First, we revisit the derivation of the low-energy effective Lifshitz theory for the vortex lattice, paying special attention to the compactness of the Lifshitz field. This allows us to uncover a rotation-induced Berry phase in the effective Lifshitz theory. Second, we derive a dual gauge theory description of the vortex lattice in terms of traceless symmetric scalar-charge theory. Third, using the effective Lifshitz theory and the dual gauge theory, we systematically analyze a hidden multipole (one-form) symmetry and the topological defects of the vortex lattice, which are associated with the winding of the Lifshitz field and the Wilson defects in the tensor gauge theory. Fourth, we characterize the neighboring quantum phases of the vortex lattice and formulate a mean-field theory for the transitions between the vortex lattice phase and these neighboring phases using the Higgs transitions of the dual gauge theory. These phases are illustrated schematically in Fig. 4.3.

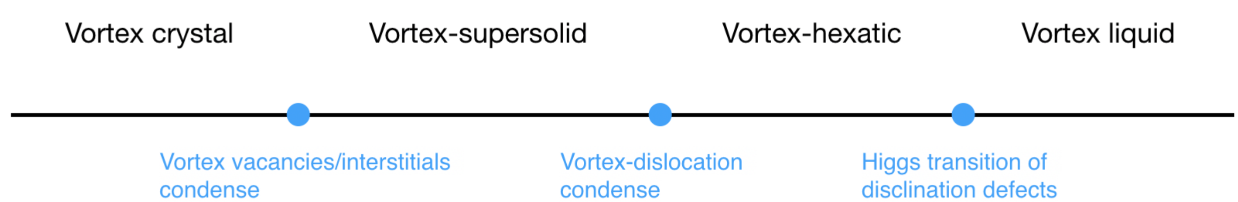

Figure 4.3: A schematic phase diagram for a vortex crystal and its associated phases studied here. With increasing quantum fluctuations (e.g., rate of rotation), vortex-crystal undergoes a quantum phase transition into vortex-supersolid, a non-superfluid but periodic vortex state, driven by condensation of vortex vacancies/interstitials. Upon further increase in quantum fluctuations, vortex-supersolid undergoes a quantum melting transition to a vortex-hexatic (or, depending on microscopic interaction details, to a vortex-nematic or vortex-smectic), driven by condensation of vortex-dislocations Marchetti and Radzihovsky [1999], Frey et al. [1994], Radzihovsky [2020], Zhai and Radzihovsky [2021]. At higher rotation, the system quantum melts into a vortex liquid phase.

The manuscript is organized as follows. In Sec. 4.3.2 we review superfluids and their $U(1)$ boson-vortex duality. We then introduce in Sec. 4.3.3 a rotating superfluid and its linearized Lifshitz theory formulation. Then utilizing Lifshitz gauge duality, in Sec. 4.3.4 we describe vortex lattice in terms of traceless symmetric scalar-charge gauge theory and discuss its multipole symmetries in Sec. 4.3.5, its defects, multipole one-form symmetries in Sec. 4.3.6, and its descendant vortex phases and their phase transitions in Sec. 4.3.7.

4.3.2 Superfluid and its duality

At low-energies, a superfluid is characterized by a local superfluid phase $\varphi(\mathbf{x}) \sim \varphi(\mathbf{x})+2\pi$ and canonically conjugate boson density $n(\mathbf{x})$, with boson operator $a \sim e^{i\varphi}$ satisfying $[e^{i\varphi}, n] = e^{i\varphi}$. Its universal properties are well-captured by a Hamiltonian density (taking $\hbar = 1$),

$$\mathcal{H}_{sf} = \frac{n_s}{2}(\boldsymbol{\nabla}\varphi)^2 + \frac{\chi^{-1}}{2}(\delta n)^2 \, , \tag{4.3.1}$$

with $\delta n = n - n_0$ the fluctuation of boson density n away from its equilibrium value n_0, and n_s and χ the superfluid stiffness and compressibility (proportional to the inverse of the boson interaction), respectively. A corresponding Lagrangian density is given by

$$\mathcal{L}_{sf} = \frac{\chi}{2}(\partial_t\varphi)^2 - \frac{n_s}{2}(\boldsymbol{\nabla}\varphi)^2 \, . \tag{4.3.2}$$

The bosonic particle 3-current is given by

$$J_\mu = (\delta n, J_i)_\mu = (\chi\partial_t\varphi, n_s\partial_i\varphi)_\mu \, , \tag{4.3.3}$$

with the Euler-Lagrange equation encoding boson conservation via continuity relation

$$\partial^\mu J_\mu = 0 \, , \tag{4.3.4}$$

where $\mu = (t, x, y)$. In addition to these low-energy Goldstone mode (GM) excitations, a superfluid admits vortex excitations, corresponding to singular configurations of the compact phase φ,

$$\frac{1}{2\pi}\epsilon_{\mu\nu\gamma}\partial^{\nu}\partial^{\gamma}\varphi = j_{\mu} \ , \tag{4.3.5}$$

with the integer-valued vortex 3-current j_{μ}, enabling superfluid rotation. Note that naively, the left-hand side of the equation vanishes due to the antisymmetric properties of the Levi-Civita tensor. However, it can be non-zero for singular, nonsingle-valued φ configurations corresponding to vortices.

Following the standard boson-vortex duality Peskin [1978], Dasgupta and Halperin [1981], Fisher and Lee [1989], we express the bosonic Lagrangian (4.3.2) in terms of the bosonic current $J_{\mu} \equiv (\partial \times a)_{\mu}$, with $(\partial \times a)_{\mu} \equiv \epsilon_{\mu\nu\gamma}\partial^{\nu}a^{\gamma}$, whose continuity (4.3.4) is solved by expressing it in terms of $U(1)$ gauge potential a_{μ}. This thereby maps the bosonic 3-current J onto a dual electromagnetic field, $b = \boldsymbol{\nabla} \times \mathbf{a}$, $\mathbf{e} = \partial_t \mathbf{a} - \boldsymbol{\nabla} a_t$,

$$J_{\mu} = (\delta n, J_i)_{\mu} \equiv \frac{1}{2\pi}(\delta b, \epsilon_{ij}e_j)_{\mu}, \tag{4.3.6}$$

with $\delta b = b - b_0$ (and $b_0/2\pi = n_0$, corresponding to the background boson density) and equivalently the Hodge-dual $^*J$ maps onto the electromagnetic field strength $f_{\mu\nu} = 2\pi\epsilon_{\mu\nu\gamma}J^{\gamma}$. With this the bosonic continuity (4.3.4) and circulation (4.3.5) equations map onto the Faraday and Ampere law, respectively, and the Lagrangian (4.3.2) to the Maxwell action

$$\mathcal{L} = \frac{1}{8\pi^2 n_s}\mathbf{e}^2 - \frac{1}{8\pi^2\chi}(\delta b)^2 - a_{\mu}j^{\mu} \ , \tag{4.3.7}$$

coupled to charged matter representation of vortices. Correspondingly, the Hamiltonian density (4.3.1), transforms into $\mathcal{H} = \frac{1}{8\pi^2 n_s}\mathbf{e}^2 + \frac{1}{8\pi^2\chi}(\boldsymbol{\nabla} \times \delta\mathbf{a})^2 - \mathbf{a} \cdot \mathbf{j}$, with electric field

e_i and gauge vector potential a_i canonically conjugate, and supplemented by a Gauss's law $\boldsymbol{\nabla} \cdot \mathbf{e} = j_t$. The latter encodes vortex circulation density, corresponding to the temporal component of (4.3.5).

4.3.3 Rotating superfluid: vortex lattice

An imposed rotation with angular velocity $\boldsymbol{\Omega} = \Omega\hat{\mathbf{z}}$ is conveniently treated in a rotating reference frame, where a particle of mass m experiences the Coriolis force $\mathbf{F} = 2m\mathbf{v} \times \boldsymbol{\Omega}$. It can be effectively viewed as the Lorentz force acting on a unit-charged particle moving in an effective magnetic field $B = 2m\Omega$. At the level of a Lagrangian, Ω appears as a Lagrange multiplier $-\boldsymbol{\Omega}\cdot\mathbf{L} = -\mathbf{A}\cdot\mathbf{J}$, enforcing a nonzero angular momentum density $\mathbf{L} = n_s m\mathbf{x}\times\boldsymbol{\nabla}\varphi$, with the effective external vector potential $\mathbf{A} = m\boldsymbol{\Omega} \times \mathbf{x}$ inducing a particle current $\mathbf{J}$. In the dual language, the rotation is encoded via the external gauge potential $\mathbf{A}$ coupled to the particle current $\mathbf{J} = \frac{1}{2\pi}\mathbf{e} \times \hat{\mathbf{z}}$ leading to the coupling $-\mathbf{A} \cdot \mathbf{J} = \frac{1}{2\pi} A_i \epsilon_{ij} \partial_j a_t = \frac{1}{2\pi} a_t B$ (having integrated by parts under an implicit integral sign and dropped the $\partial_t a_i$ component of e_i that is a total derivative for a time-independent A_i). As a consequence, a two-dimensional superfluid rotating at angular velocity Ω gives rise to the formation of a triangular lattice of quantized vortices with equilibrium vortex density

$$n_v = \frac{2m\Omega}{2\pi} \equiv \frac{B}{2\pi} \; . \tag{4.3.8}$$

The elementary Goldstone-mode excitations of the vortex lattice, known as Tkachenko modes Tkachenko [1965, 1966, 1969], Sonin [2014], Du et al. [2024a,b], possess several unconventional characteristics compared to ordinary sound waves in solids. Most notably, they are characterized by a quadratic dispersion, $\omega \sim q^2$ at low momentum, with only one polarization.

Effective vortex-lattice field theory

A recent discovery has revealed that the Tkachenko modes can be described by a non-linear Lifshitz scalar theory, which incorporates non-linear dipole and higher multipole symmetries arising from magnetic translations and rotations Du et al. [2024a] and will be the starting point of our analysis. The non-linear theory of the Tkachenko mode allows a convenient formulation as a noncommutative field theory Douglas and Nekrasov [2001], Rubakov [2005], Du et al. [2021]. Here, we begin with a review of this effective field theory of a vortex lattice and then derive the linearized Lagrangian by expanding the effective field theory to quadratic order Du et al. [2024a].

As any two-dimensional crystal, a vortex lattice can be described by two scalar fields frozen into the lattice $X^a(t, x^i)$ with $a = x, y$, which is related to the lattice displacement u^a by $X^a(t, x^i) = \delta^a_i x^i - u^a(t, x^i)$, in the Eulerian (laboratory) coordinates x^i. The Lagrangian density consists of the elastic component

$$\mathcal{L}_{el} = \frac{1}{2}\rho(\partial_t u_i)^2 - \frac{1}{2}K_{ij;kl}u_{ij}u_{kl} \tag{4.3.9}$$

and the superfluid sector $\mathcal{L}_{sf}$, (4.3.2), coupled through the superflow (J_μ) and vortex (j_μ) currents. In (4.3.9), ρ is the effective vortex mass density, $K_{ij;kl}$ is the elastic tensor, $u_{ij} = \frac{1}{2}(\partial_i u_j + \partial_j u_i)$ is the linearized symmetric strain tensor.

The crucial coupling of the two sectors is most transparently implemented in the superfluid's gauge-dual description (4.3.7), via $-a_\mu j^\mu$, with the vortex current given by

$$j_\mu = \frac{1}{2}n_v \epsilon_{\mu\nu\gamma}\epsilon_{ab}\partial^\nu X^a \partial^\gamma X^b \ , \tag{4.3.10}$$

that is a "space-time Jacobian" between the reference and vortex lattice coordinates. In terms of the phonon distortions, the temporal and spatial currents take the form

$$
\begin{aligned}
j_t &= n_v - n_v \partial_i u^i + \frac{1}{2} n_v \epsilon_{ij} \epsilon_{ab} \partial_i u_a \partial_j u_b \ , \\
j_i &= -n_v \partial_t u_i - n_v \epsilon_{ij} \epsilon_{ab} \partial_j u_a \partial_t u_b \ ,
\end{aligned}
\tag{4.3.11}
$$

that have a natural and transparent interpretation. Combining these inside $-a_\mu j^\mu$, together with the elastic $\mathcal{L}_{el}$ and electromagnetic (superfluid) $\mathcal{L}_{sf}$ sectors and the rotation-imposed external vector potential, $-\mathbf{A} \cdot \mathbf{J}$, we obtain the vortex lattice Lagrangian density, that, at harmonic level and dropping the subdominant kinetic $\frac{1}{2}\rho(\partial_t u_i)^2$ and electric $\frac{1}{8\pi^2 n_s}\mathbf{e}^2$ energies takes the form

$$
\begin{aligned}
\mathcal{L} = &- \mu u_{ij}^2 - \frac{\lambda}{2} u_{ii}^2 - \frac{1}{8\pi^2 \chi} (\delta b)^2 \\
&+ a_t \left(\frac{B}{2\pi} - n_v \right) + n_v e_i u_i + \frac{1}{2} n_v b_0 \epsilon_{ij} u_i \partial_t u_j \ .
\end{aligned}
\tag{4.3.12}
$$

Above, for simplicity we have taken isotropic elasticity with elastic tensor

$$
K_{ij;kl} = \mu(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}) + \lambda \delta_{ij}\delta_{kl} \ , \tag{4.3.13}
$$

characterized by two Lamé elastic constants, μ (shear modulus) and λ (valid for a triangular lattice) and integrated by parts in the second and last terms of the second line. We also restricted our analysis to a harmonic approximation, with b replaced by $b_0 = 2\pi n_0$ in the last term, set by the equilibrium particle number density n_0. This last term crucially encodes guiding center vortex dynamics, that "sees" background boson density n_0 as an effective magnetic field $b_0 = 2\pi n_0$, equivalently corresponding to the Magnus force on a moving vortex. It encodes noncommutativity of components of the phonon operator, but with the commutation relation modified by the superflow encoded in the gauge field a_i. The Lagrangian explicitly breaks the spatial reflection (**R**) and time-reversal (**T**) symmetry but is still invariant under the combined **RT** transformation Moroz et al. [2018], **?**], Du et al. [2024a].

Functionally integrating over a_t in the partition function, locks the equilibrium vortex density n_v to the flux density B (equivalently to the imposed angular velocity Ω, with critical velocity Ω_c vanishing in the thermodynamic limit) with one unit of flux quantum per vortex lattice unit cell according to (4.3.8). At long wavelengths, this then introduces a crucial incompressibility constraint on the vortex lattice

$$\partial_i u_i = 0 \ , \tag{4.3.14}$$

forcing a transversal form of the vortex lattice phonon $u_i(\mathbf{x})$. We emphasize that this incompressibility constraint is for the vortex lattice, not the underlying superfluid [5]. We also note that keeping the subdominant electric energy term $\frac{1}{8\pi^2 n_s}\mathbf{e}^2$ in the above Lagrangian, would allow for nonzero vortex lattice comprehensibility. At long scales, the incompressibility constraint reduces the Lagrangian (4.3.12) to

$$\mathcal{L} = -\mu u_{ij}^2 - \frac{1}{8\pi^2\chi}(\delta b)^2 - n_v a_i \partial_t u_i + \frac{1}{2} n_v b_0 \epsilon_{ij} u_i \partial_t u_j \ , \tag{4.3.15}$$

with the Lamé elastic constant λ dropping out due to the incompressibility condition (4.3.14). It is straightforward to see (via, e.g., the equation of motion) that (4.3.15) encodes a single-polarization Tkachenko modes with a quadratic dispersion relation $\omega \sim q^2$. We emphasize, that, because of the Berry phase coupling of phonon velocity $\partial_t u_i$ to the superfluid gauge-field a_i, the commutation relations are distinct from that of a pure guiding-center dynamics (in the absence of the gauge field) of e.g., a Wigner crystal in a magnetic field, though the quadratic dispersion is the same.

5. For an incompressible fluid, the dispersion is linear instead of quadratic. It can be understood as follows. For vanishing fluid compressibility, $\chi = 0$ and the kinetic term $\frac{\chi}{2}(\partial_t \phi)^2$ in the effective Lifshitz theory (4.3.23) vanishes. It is then necessary to retain the subdominant term $\frac{\rho}{2}(\partial_t u_i)^2$ in (4.3.9), which under (4.3.19) becomes $\frac{\rho}{2B^2}(\partial_t \partial_i \phi)^2$. The dispersion is then modified to a linear one.

The Euler-Lagrange equation for u_i is given by

$$n_v e_i + n_v b_0 \epsilon_{ij} \partial_t u_j + 2\mu \partial_j u_{ij} + \lambda \partial_i u_{kk} = 0 \ . \tag{4.3.16}$$

It encodes a local equilibrium force balance in the vortex lattice. The first and second terms are the dual "electric" and "magnetic" forces (dual "Lorentz" force), corresponding to a Magnus force experienced by a moving vortex due to a superflow $J_i = \frac{1}{2\pi}\epsilon_{ij} e_j$. It is equivalent to Kelvin's condition that in the absence of other forces, a vortex moves with a local superfluid velocity. The third and fourth term is a force associated with elastic stresses. Equations (4.3.14) and (4.3.16) will be important for our subsequent analyses.

We take the vortex lattice to be triangular, corresponding to the expected lowest energy state for repulsive vortex interaction. A triangular unit cell has area $\sqrt{3}l^2/2$, with the equilibrium vortex density n_v related to the lattice constant l by $n_v = 2/\sqrt{3}l^2$. Our discussions so far have neglected vortex lattice topological defects – disclinations, dislocations, vacancies and interstitials. These enter the low-energy linearized effective field theory (4.3.12) through the compactness of u_i, associated with lattice periodicity and captured by the identifications on the displacement field u_i,

$$u_x \sim u_x + k_x l + \frac{1}{2} k_y l \ , \quad u_y \sim u_y + \frac{\sqrt{3}}{2} k_y l \ , \tag{4.3.17}$$

where k_i are integers. Similarly, a $2\pi/6$ rotation around a lattice site leaves the lattice invariant so the bond angle θ should be $2\pi/6$-periodic. In the linearized elasticity theory, the bond angle is approximated by $\theta \approx \frac{1}{2}\epsilon_{ij}\partial_i u_j$ which together with the $2\pi/6$-periodicity of the bond angle implies the following identification on u_i

$$u_i \sim u_i + \frac{2\pi k}{6} \epsilon^{ij} x_j \ , \tag{4.3.18}$$

where k is an integer. Above thereby allow singular configurations, with nonzero $\boldsymbol{\nabla} \times \boldsymbol{\nabla} u_i$

and $\boldsymbol{\nabla} \times \boldsymbol{\nabla}\theta$, which will be vitally important for the defect analysis in Sec. 4.3.6.

Compact Lifshitz theory with a Berry phase

The Lifshitz field theory arises from the linearized effective field theory (4.3.15) of the vortex lattice by integrating out the $U(1)$ vector gauge field a_μ representing the bosonic matter component and the associated superflow. As we saw in the previous section, the integration over a_t imposed the incompressibility condition (4.3.14) and transformed (4.3.12) into (4.3.15). The condition (4.3.14) is straightforwardly solved using a phase $\phi(\mathbf{x})$, defined by

$$u_i = \frac{1}{B}\epsilon_{ij}\partial_j\phi \ . \tag{4.3.19}$$

A priori, at this stage, $\phi(\mathbf{x})$ is unrelated to the superfluid phase $\varphi(\mathbf{x})$ of the previous section. However, as we will see below, $\phi(\mathbf{x})$ is a deviation from $\varphi_{VL}(\mathbf{x})$ describing a triangular vortex lattice of single vortex configurations, $\varphi_{vortex}(\mathbf{x}) = \arctan[(y-y_i)/(x-x_i)]$. The linearized symmetric strain tensor can then be expressed as

$$u_{ij} = \frac{1}{2}\left(\partial_i u_j + \partial_j u_i\right) = \frac{1}{B}\epsilon_{jk}D_{ik}\phi \ , \tag{4.3.20}$$

where we defined a differential operator

$$D_{ij}\phi \equiv \left(\partial_i\partial_j - \frac{1}{2}\delta_{ij}\partial^2\right)\phi \ . \tag{4.3.21}$$

Next, integrating out a_i gives

$$\frac{1}{2\pi}\delta b = -\chi\partial_t\phi \ , \tag{4.3.22}$$

which together with (4.3.19) reduces the Lagrangian (4.3.15) to a *linearized* (quadratic) Lifshitz theory,

$$\mathcal{L} = \frac{\chi}{2}\left(\partial_t\phi\right)^2 - \frac{\mu}{B^2}\left(D_{ij}\phi\right)^2 + \frac{b_0}{4\pi B}\epsilon_{ij}\partial_i\phi\partial_j\partial_t\phi \ , \tag{4.3.23}$$

supplemented by a Berry phase, that is a distinctive characteristic of a quantum vortex lattice, crucially contrasting it from the prevously-studied Lifshitz theory description Watanabe and Murayama [2013]. The Berry phase is the only term in the Lagrangian that breaks the **T** and **R** symmetry while preserves the combined **RT** symmetry.

Naively, one may hastily neglect the Berry phase term, arguing that as a total derivative, it is unimportant in the bulk. However, since $\phi(\mathbf{x})$ is a compact angular field, it can wind, encoding singular vortex configurations for which $\frac{1}{2\pi}\epsilon_{ij}\partial_i\partial_j\phi = \frac{1}{2\pi}\boldsymbol{\nabla}\times\boldsymbol{\nabla}\phi = \delta j_t(\mathbf{x}) = j_t(\mathbf{x}) - j_t^{VL}(\mathbf{x}) \neq 0$, where $j_t^{VL}(\mathbf{x}) = \boldsymbol{\nabla}\times\boldsymbol{\nabla}\varphi_{VL}$ is the vortex density corresponding to a vortex lattice in $\varphi(\mathbf{x})$, with average n_v. For such vortex configurations the Berry phase term contributes nontrivially in the bulk,

$$\mathcal{L}_{Berry} = \frac{b_0}{4\pi B}\epsilon_{ij}(\partial_i\partial_j\phi)\partial_t\phi = \frac{1}{2}\nu\delta j_t\partial_t\phi \ , \tag{4.3.24}$$

where $\nu \equiv b_0/B$ is the filling fraction of boson per vortex.

Not unrelated, it is tempting to rewrite the second "elastic" term as simply $\frac{\mu}{2B^2}(\nabla^2\phi)^2$ through integration by parts. However, in the presence of vortices there is an obstruction to integration by parts as derivatives on ϕ_{vortex} do not commute (see Pretko and Radzihovsky [2018a,b], Pretko et al. [2019], Gorantla et al. [2023] for related discussions).

The compactness of ϕ follows from the identifications on u_i, listed in (4.3.17) and (4.3.18), and the relationship between u_i and ϕ in (4.3.19), which lead to the following identifications:

$$\phi(\mathbf{x}) \sim \phi(\mathbf{x}) + k_x yBl + \frac{1}{2}k_y(y-\sqrt{3}x)Bl + \frac{2\pi k}{12}Br^2 \ , \tag{4.3.25}$$

with $r = \sqrt{x^2+y^2}$ the radial coordinate and integer k_x, k_y, k. In addition to these, there is another constant identification on ϕ,

$$\phi(\mathbf{x}) \sim \phi(\mathbf{x}) + 2\pi \ , \tag{4.3.26}$$

following from the 2π periodicity of the superfluid phase $\varphi(\mathbf{x}) \sim \varphi(\mathbf{x}) + 2\pi$.

The Tkachenko modes are made explicit in the linearized Lifshitz theory (4.3.23). They are the plane wave excitations of the phase field ϕ, that manifestly have only one polarization and a quadratic dispersion with dynamical exponent $z = 2$, characteristic of the Lifshitz symmetry $(t, x_i) \to (\xi t, \xi^2 x_i)$, with the scaling factor ξ **?**.

For later use, we summarize the operator maps between the effective field theory (4.3.15) and the linearized Lifshitz theory (4.3.23):

$$
\begin{aligned}
u_i \quad &\longleftrightarrow \quad \frac{1}{B}\epsilon_{ij}\partial_j\phi \ , \\
u_{ij} \quad &\longleftrightarrow \quad \frac{1}{B}\epsilon^{jk}D_{ik}\phi \ , \\
\frac{1}{2\pi}\delta b \quad &\longleftrightarrow \quad -\chi\partial_t\phi \ , \\
\frac{1}{2\pi}e_i \quad &\longleftrightarrow \quad \frac{b_0}{2\pi B}\partial_i\partial_t\phi - \frac{\mu}{B^2}\epsilon_{ij}\partial_j\partial^2\phi \ .
\end{aligned}
\tag{4.3.27}
$$

The last relation can be derived by substituting (4.3.19) into the Euler-Lagrange equation (4.3.16).

4.3.4 Duality

In this section we derive a number of equivalent formulations of the vortex lattice starting from the Lifshitz field theory presented in (4.3.23) and the effective field theory presented in (4.3.15). We apply a generalized Lifshitz duality Gorantla et al. [2023], Radzihovsky [2022] to the Lifshitz field theory Du et al. [2024a], obtaining the main new result of this section – a *traceless symmetric scalar-charge gauge theory* description of the vortex lattice.

We then demonstrate the low-energy equivalence of this dual gauge theory to the vortex lattice description via fracton-elasticity duality Pretko and Radzihovsky [2018a,b], Pretko et al. [2019], derived in an earlier work Nguyen et al. [2020].

Lifshitz duality to traceless symmetric scalar-charge gauge theory

We now derive another equivalent description of the vortex lattice by dualizing the linearized Lifshitz theory to a traceless symmetric scalar-charge gauge theory, generalizing the Lifshitz duality studied in Gorantla et al. [2023], Radzihovsky [2022].

To this end we introduce Hubbard-Stratonovich (HS) fields $\hat{\pi}_t, \hat{\pi}_{ij}$ and u_i into Lagrangian $\mathcal{L}[\phi]$ (4.3.23) so that ϕ appears linearly in the transformed Lagrangian

$$\begin{aligned}\mathcal{L}[\phi, \hat{\pi}, u_i] = & -\frac{1}{2\chi}\hat{\pi}_t^2 + \hat{\pi}_t \partial_t \phi + \frac{B^2}{4\mu}\hat{\pi}_{ij}^2 + \hat{\pi}_{ij} D_{ij} \phi \\ & - \frac{b_0 B}{4\pi}\epsilon_{ij} u_i \partial_t u_j - \frac{b_0}{2\pi} u_i \partial_t \partial_i \phi \ ,\end{aligned} \tag{4.3.28}$$

where $\hat{\pi}_{ij}$ is a traceless symmetric tensor, with a hat symbol emphasizing its tracelessness. The original Lifshitz Lagrangian is straightforwardly recovered integrating out in the path integral the $\hat{\pi}_t$, $\hat{\pi}_{ij}$, u_i fields. At the harmonic level this establishes relations,

$$\hat{\pi}_t = \chi \partial_t \phi \ , \quad \hat{\pi}_{ij} = -\frac{2\mu}{B^2} D_{ij} \phi \ , \quad u_i = \frac{1}{B} \epsilon_{ij} \partial_j \phi \ , \tag{4.3.29}$$

which reveals that $\hat{\pi}_t$, $\hat{\pi}_{ij}$ and u_i play the role of the boson density, vortex lattice stress, and the displacement (phonon) field, respectively. We note that although we used the same symbol u_i, there was no a priori relationship with the vortex lattice phonon field appearing in (4.3.9). However, the third equation above, relating u_i and ϕ is exactly the one that appeared in (4.3.19) thereby identifying HS field u_i with the vortex displacement field.

To obtain the dual Lagrangian, we integrate out ϕ, that appears linearly and thereby gives the following constraint,

$$\partial_t \hat{\pi}_t - D_{ij}\hat{\pi}_{ij} + \frac{b_0}{2\pi}\partial_t\partial_i u_i = 0 \ . \tag{4.3.30}$$

We note that through the last term, the dynamics here is qualitatively modified by the Berry phase, encoding a coupling between superfluid current $\partial_i\phi$, vortex density $\partial_i u_i$ and boson density $\hat{\pi}_t$, that physically goes back to the Kelvin's condition for vortex motion in a superflow. To solve the constraint, it is convenient to define dual magnetic and electric fields (that respectively encode a sum of boson and vortex densities, and vortex-lattice stress),

$$\hat{\mathcal{B}} \equiv 2\pi\hat{\pi}_t + b_0\partial_i u_i \ , \quad \hat{\mathcal{E}}_{ij} \equiv 2\pi\epsilon_{jk}\hat{\pi}_{ik} \ , \tag{4.3.31}$$

in terms of which constraint transforms into Faraday-like law,

$$\partial_t\hat{\mathcal{B}} - \epsilon_{ik}D_{jk}\hat{\mathcal{E}}_{ij} = 0 \ . \tag{4.3.32}$$

Here $\hat{\mathcal{E}}_{ij}$ is a traceless symmetric tensor following from the fact that $\hat{\pi}_{ij}$ is traceless symmetric, respectively. The constraint can be solved by introducing a tensor gauge field

$$\hat{\mathcal{A}}_t \sim \hat{\mathcal{A}}_t + \partial_t\hat{\lambda} \ , \quad \hat{\mathcal{A}}_{ij} \sim \hat{\mathcal{A}}_{ij} + D_{ij}\hat{\lambda} \ , \tag{4.3.33}$$

with $\hat{\mathcal{A}}_{ij}$ a traceless symmetric tensor and identifying $\hat{\mathcal{E}}_{ij}$, $\hat{\mathcal{B}}$ with the gauge invariant field strengths of the tensor gauge field

$$\hat{\mathcal{E}}_{ij} = \partial_t\hat{\mathcal{A}}_{ij} - D_{ij}\hat{\mathcal{A}}_t \ , \quad \hat{\mathcal{B}} = \epsilon_{ik}D_{jk}\hat{\mathcal{A}}_{ij} \ . \tag{4.3.34}$$

This then gives our main result, the dual traceless tensor gauge theory Lagrangian

$$\mathcal{L} = \frac{B^2}{16\pi^2\mu}\hat{\mathcal{E}}_{ij}^2 - \frac{1}{8\pi^2\chi}(\hat{\mathcal{B}} - b_0\partial_i u_i)^2 - \frac{b_0 B}{4\pi}\epsilon_{ij}u_i\partial_t u_j \ . \tag{4.3.35}$$

As u_i appears quadratically in the Lagrangian, we can further integrate it out and reduce the Lagrangian to a functional depending only on the traceless symmetric tensor gauge fields. However, because u_i is gapless, this will result in a long-range interaction and thus will obscure the locality of the resulting Lagrangian. Thus, we keep the u_i field and present the Lagrangian in the current form. In Appendix 4.3.9, for comparison, we study the duality of the Lifshitz theory (4.3.23) without the Berry phase.

This tensor gauge theory description of the vortex lattice has the advantage of making the topological crystalline defects and their mobility explicit (see Sec. 4.3.6). It will also serve as the starting point for our study of the vortex phases that can appear after melting of the vortex lattice.

For later use, we summarize the operator maps between the Lifshitz theory (4.3.23) and the tensor gauge theory (4.3.35):

$$\begin{aligned} \partial_t\phi \qquad &\longleftrightarrow \qquad \frac{1}{2\pi\chi}(\hat{\mathcal{B}} - b_0\partial_i u_i) \ , \\ D_{ij}\phi \qquad &\longleftrightarrow \qquad \frac{B^2}{4\pi\mu}\epsilon_{jk}\hat{\mathcal{E}}_{ik} \ , \end{aligned} \tag{4.3.36}$$

that follow from (4.3.29) and (4.3.31).

Vortex-lattice dual through fracton-elasticity duality

In a distinct approach, authors of Ref. Nguyen et al. [2020] dualized a vortex lattice by applying fracton-elasticity duality Pretko and Radzihovsky [2018a,b], Pretko et al. [2019] to the elastic sector of the effective field theory (4.3.12), obtaining a symmetric scalar charge theory coupled to a $U(1)$ vector gauge field of the superfluid. We review their derivation below and show that our traceless symmetric tensor gauge theory (4.3.35) consistently emerges in the low energy limit.

Following fracton-elasticity duality Pretko and Radzihovsky [2018a,b], Pretko et al. [2019], we linearize the appearance of the phonon field u_i in (4.3.12) by respectively introducing the lattice momentum, stress tensor and bond angle HS fields $\pi_i, \sigma_{ij}, \theta$, that transforms the effective field theory to

$$\begin{aligned}\mathcal{L} =& \frac{1}{2}K^{-1}_{ij;kl}\sigma_{ij}\sigma_{kl} + \sigma_{ij}(\partial_i u_j - \theta\epsilon_{ij}) \\ & - \frac{1}{2n_v b_0}\epsilon_{ij}\pi_i\partial_t\pi_j + \pi_i\partial_t u_i + n_v e_i u_i - \frac{1}{8\pi^2\chi}(\delta b)^2 \ ,\end{aligned} \tag{4.3.37}$$

with $K^{-1}_{ij;kl}$ the inverse of the elastic tensor (4.3.13),

$$K^{-1}_{ij;kl} = \frac{2\mu\delta_{ik}\delta_{jl} + \lambda(2\delta_{ik}\delta_{jl} - \delta_{ij}\delta_{kl})}{4\mu(\lambda+\mu)} \ . \tag{4.3.38}$$

The role of the bond angle field θ is to relax the symmetrization of the strain tensor u_{ij} to $\partial_i u_j - \theta\epsilon_{ij}$, with θ effectively Higgs'ing the antisymmetric component of the unsymmetrized strain $\partial_i u_j$ Radzihovsky and Hermele [2020]. The original Lagrangian is recovered by integrating over $\theta, \pi_i, \sigma_{ij}$ in the partition function, which, because the model is quadratic, gives the relations

$$\pi_i = -n_v b_0 \epsilon_{ij} u_j \ , \quad \sigma_{ij} = -K_{ij;kl}u_{kl} \ , \quad \theta = \frac{1}{2}\epsilon_{ij}\partial_i u_j \ . \tag{4.3.39}$$

To derive the dual Lagrangian, we instead integrate out (exactly) u_i and θ, which gives rise to the following constraints

$$\partial_t \pi_i + \partial_j \sigma_{ji} = n_v e_i \ , \quad \epsilon_{ij}\sigma_{ij} = 0 \ . \tag{4.3.40}$$

The first constraint is the continuity equation for the vortex lattice stress-energy tensor "broken" by a dual electric field associated with the superflow, that exerts a force on the lattice. The second constraint symmetrizes the stress tensor σ_{ij}. To solve the constraint, it is convenient to define dual magnetic and electric fields,

$$\begin{aligned}\mathcal{B}_i &\equiv -\frac{1}{n_v}\epsilon_{ij}(\pi_j - n_v \delta a_j) \ , \\ \mathcal{E}_{ij} &\equiv -\frac{1}{n_v}\epsilon_{ik}\epsilon_{jl}(\sigma_{kl} + n_v \delta a_t \delta_{kl}) \ ,\end{aligned} \tag{4.3.41}$$

with $\delta a_\mu = a_\mu - a_{0,\mu}$ the fluctuation around the equilibrium gauge field $a_{0,\mu}$, whose magnetic field is b_0 and electric field vanishes. In terms of these new fields, the constraint reduces to a Faraday-like law Pretko and Radzihovsky [2018a,b], Pretko et al. [2019], Radzihovsky and Hermele [2020] and symmetrization of the electric field tensor,

$$\partial_t \mathcal{B}_i - \epsilon_{jk}\partial_j \mathcal{E}_{ki} = 0 \ , \quad \epsilon_{ij}\mathcal{E}_{ij} = 0 \ . \tag{4.3.42}$$

These can be solved by introducing a symmetric tensor gauge field

$$\mathcal{A}_t \sim \mathcal{A}_t + \partial_t \lambda \ , \quad \mathcal{A}_{ij} \sim \mathcal{A}_{ij} + \partial_i \partial_j \lambda \ , \tag{4.3.43}$$

and setting $\mathcal{B}_i$, $\mathcal{E}_{ij}$ to be its gauge invariant field strengths

$$\mathcal{B}_i = \epsilon_{jk}\partial_j \mathcal{A}_{ki} \ , \quad \mathcal{E}_{ij} = \partial_t \mathcal{A}_{ij} - \partial_i \partial_j \mathcal{A}_t \ . \tag{4.3.44}$$

We emphasize that, unlike (4.3.33), this symmetric tensor gauge field is not traceless. With this, utilizing this gauge field description, we arrive at the dual Lagrangian derived in

Ref. Nguyen et al. [2020],

$$
\begin{aligned}
\mathcal{L} = &\frac{1}{2} n_v^2 \tilde{K}^{-1}_{ij;kl} (\mathcal{E}_{ij} + \delta a_t \delta_{ij})(\mathcal{E}_{kl} + \delta a_t \delta_{kl}) \\
&- \frac{B}{4\pi b_0} \epsilon_{ij} (\mathcal{B}_i - \epsilon_{ik} \delta a_k) \partial_t (\mathcal{B}_j - \epsilon_{jl} \delta a_l) - \frac{1}{8\pi^2 \chi} (\delta b)^2 \ ,
\end{aligned}
\tag{4.3.45}
$$

with $\tilde{K}^{-1}_{ij;kl} = \epsilon_{ii'} \epsilon_{jj'} \epsilon_{kk'} \epsilon_{ll'} K^{-1}_{i'j';k'l'}$. It describes a symmetric tensor gauge theory coupled to a vector gauge theory. For the Lagrangian to be gauge invariant, $\mathcal{A}_{ij}$ must transform under the gauge redundancy of δa_μ as

$$
\delta a_\mu \sim \delta a_\mu + \partial_\mu \beta \ , \quad \mathcal{A}_{ij} \sim \mathcal{A}_{ij} + \beta \delta_{ij} \ , \tag{4.3.46}
$$

in additional to its own gauge redundancy (4.3.43).

We now relate this description of the vortex lattice with our *traceless* symmetric tensor gauge theory (4.3.35). To this end we define the following fields identifications

$$
\begin{aligned}
u_i &\equiv \frac{1}{b_0} (\mathcal{B}_i - \epsilon_{ij} \delta a_j) \ , \\
(\hat{\mathcal{A}}_t, \hat{\mathcal{A}}_{ij}) &\equiv \left(\mathcal{A}_t, \mathcal{A}_{ij} - \frac{1}{2} \delta_{ij} \mathcal{A}_{kk} \right) \ .
\end{aligned}
\tag{4.3.47}
$$

Here, u_i is the displacement field with the definition consistent with (4.3.39) and (4.3.41), and $(\hat{\mathcal{A}}_t, \hat{\mathcal{A}}_{ij})$ is a traceless symmetric tensor gauge field which share the same gauge redundancy as (4.3.33) with the gauge parameter $\hat{\lambda} = \lambda$ and is invariant under the gauge transformation (4.3.46) associated with β. The field strength of $(\hat{\mathcal{A}}_t, \hat{\mathcal{A}}_{ij})$ is related to the field strength of $(\mathcal{A}_t, \mathcal{A}_{ij})$ by

$$
\hat{\mathcal{E}}_{ij} = \mathcal{E}_{ij} - \frac{1}{2} \delta_{ij} \mathcal{E}_{kk} \ , \quad \hat{\mathcal{B}} = -\partial_i \mathcal{B}_i \ . \tag{4.3.48}
$$

With this, we can reduce the Lagrangian (4.3.45) into a functional of these new fields. To this end, we integrate out δa_t which sets a relation

$$\delta a_t = -\frac{1}{2}\mathcal{E}_{kk} \ . \tag{4.3.49}$$

The first term of the Lagrangian (4.3.45) then transforms into

$$\frac{1}{2}n_v^2 \tilde{K}^{-1}_{ij;kl}\hat{\mathcal{E}}_{ij}\hat{\mathcal{E}_{kl}} = \frac{B^2}{16\pi^2\mu}\hat{\mathcal{E}}_{ij}\hat{\mathcal{E}}_{ij} \ , \tag{4.3.50}$$

which simplifies significantly due to the tracelessness of $\hat{\mathcal{E}}_{ij}$. Expressing the remaining components of the Lagrangian in terms of the newly defined fields gives

$$-\frac{b_0 B}{4\pi}\epsilon_{ij}u_i\partial_t u_j - \frac{1}{8\pi^2\chi}(\hat{\mathcal{B}} - b_0\partial_i u_i)^2 \ , \tag{4.3.51}$$

which together with (4.3.50), reduces to our Lagrangian (4.3.35) of the *traceless* symmetric tensor gauge theory.

4.3.5 Global symmetries

Microscopic symmetries

In this section, we analyze the global symmetries of the effective field theory (4.3.12) and track them across dualities. These global symmetries include the $U(1)$ symmetry associated with boson number conservation and the translation and rotation symmetry of the vortex lattice.

The microscopic boson number conservation is realized in the effective field theory (4.3.12) as a $U(1)$ symmetry generated by the current

$$J_\mu = \frac{1}{2\pi}\epsilon_{\mu\nu\rho}\partial_\nu a_\rho - n_0\delta_{\mu,t} \ . \tag{4.3.52}$$

In our definition about we subtracted the equilibrium boson density n_0 in the temporal component of the current J_t, so that J_t represents the fluctuation of the boson density around its equilibrium value. This current obeys a continuity equation

$$\partial^{\mu} J_{\mu} = 0 \ , \tag{4.3.53}$$

corresponding to the local boson number conservation. The quantized conserved charge is

$$Q = \int d^2x \ J_t \ , \tag{4.3.54}$$

which measures the fluctuations of the boson number around its equilibrium value.

The translation and rotation symmetry of the vortex lattice acts on the displacement fields u_i as

$$u_i \to u_i + c_x \delta_{i,x} + \frac{1}{2} c_y (\delta_{i,x} + \sqrt{3}\delta_{i,y}) + \alpha \epsilon^{ij} x_j \ , \tag{4.3.55}$$

where c_x and c_y are respectively the translations along the x axis and along the axis $2\pi/6$ counter-clockwise to the x axis, and α is the rotation angle. We parametrized the symmetry transformation in such a way so that the parameters have the periodicity $c_i \sim c_i + l$ and $\alpha \sim \alpha + 2\pi/6$. The rotation symmetry parametrized by α takes the form of an infinitesimal rotation, although it remains a symmetry of the effective field theory (4.3.12) at nonzero α due to linearized-elasticity of the Lagrangian. We note that these symmetries should not be confused with the translation and rotation symmetry of the full system.

The current $J_{\mu\nu}$ associated with translational and rotational global symmetries is the symmetric rank-2 stress-energy tensor of the vortex lattice

$$\begin{aligned} J_{it} &\equiv p_i = n_v a_i + n_v b_0 \epsilon_{ij} u_j \ , \\ J_{ij} &= n_v a_t \delta_{ij} - 2\mu u_{ij} - \lambda u_{kk} \delta_{ij} \ , \end{aligned} \tag{4.3.56}$$

with p_i denoting the momentum density. It obeys the current conservation equation

$$\partial_t J_{it} = \partial_j J_{ij} \ , \tag{4.3.57}$$

that follows from the Euler-Lagrange equation (4.3.16) of u_i. The conserved charges are the momentum P_i and the orbital angular momentum L of the vortex lattice

$$P_i = \int d^2x \, p_i \ , \quad L = \int d^2x \, \epsilon_{ij} x_i p_j \ . \tag{4.3.58}$$

The conjugate momenta of u_i and a_i are $\pi_i = p_i$ and $\Pi_i = n_v u_i$ respectively. Upon quantization,

$$\begin{aligned}
[\pi_i(x), u_j(x')] &= i\delta_{ij}\delta^2(x - x') \ , \\
[\pi_i(x), a_j(x')] &= 0 \ , \\
[\Pi_i(x), u_j(x')] &= 0 \ , \\
[\Pi_i(x), a_j(x')] &= i\delta_{ij}\delta^2(x - x') \ .
\end{aligned} \tag{4.3.59}$$

These commutation relations reduce to more basic ones

$$\begin{aligned}
[u_i(x), u_j(x')] &= 0 \ , \\
[u_i(x), a_j(x')] &= \frac{i}{n_v}\delta_{ij}\delta^2(x - x') \ , \\
[a_i(x), a_j(x')] &= -\frac{ib_0}{n_v}\epsilon_{ij}\delta^2(x - x') \ .
\end{aligned} \tag{4.3.60}$$

Using these commutation relations, we obtain the following algebra of the momentum density of the vortex lattice

$$[p_i(x), p_j(x')] = in_v b_0 \epsilon_{ij}\delta^2(x - x') \ , \tag{4.3.61}$$

where $b_0 = 2\pi n_p$. The momentum operator thus obeys

$$[P_i, P_j] = iN_v b_0 \epsilon_{ij} \tag{4.3.62}$$

where N_v is the total number of vortices. This non-trivial algebra is a consequence of the Berry phase term in (4.3.12). It is similar to the algebra obeyed by the magnetic translation operator $\hat{P}_i = -i\partial_i - A_i$ in a background magnetic field $B = \epsilon^{ij}\partial_i A_j$

$$[\hat{P}_i, \hat{P}_j] = iB\epsilon_{ij} \ . \tag{4.3.63}$$

This is not a coincidence, as the vortices are coupled to the vector gauge field a_μ of the superfluid and experience an effective magnetic field b_0, proportional to boson number density.

The algebra (4.3.62) has a c-number on the right-hand side, which can also be expressed as $iBN_0\epsilon_{ij}$ with N_0 the total boson number in equilibrium. Physically, this algebra captures a mixed anomaly between the orthogonal lattice translation symmetries when the boson number is nonzero in equilibrium. It is to be contrasted with the noncommutative algebra discussed in Du et al. [2024a] where the boson number on the right-hand side is promoted to an operator. Such a noncommutative algebra implies that the full symmetry group is an extension of the translation symmetry by the $U(1)$ symmetry associated to boson number conservation. The algebra reduces to our algebra (4.3.62) by approximating the boson number operator by its expectation value.

Multipole symmetry of the Lifshitz theory

The Lifshitz theory (4.3.23) exhibits a quadrupole symmetry that shifts

$$\phi \to \phi + c + c_x By + \frac{1}{2}c_y B(y - \sqrt{3}x) + \frac{1}{2}\alpha Br^2. \tag{4.3.64}$$

The associated current is

$$J_t^\phi = \chi\partial_t\phi - \frac{b_0}{B}\delta j_t \ , \quad J_{ij}^\phi = \frac{2\mu}{B^2}D_{ij}\phi + \frac{b_0}{2\pi B}\epsilon_{ij}\partial_t\phi \ , \tag{4.3.65}$$

where

$$\delta j_t = \frac{1}{2\pi}\epsilon_{ij}D_{ij}\phi = \frac{1}{2\pi}\boldsymbol{\nabla}\times(\boldsymbol{\nabla}\phi) \tag{4.3.66}$$

is the density of vortex interstitials. It obeys the following current conservation equation corresponding to the Euler-Lagrange equation for ϕ,

$$\partial_t J_t^{\phi} + D_{ij}J_{ij}^{\phi} = 0 \ . \tag{4.3.67}$$

Similar quadrupole symmetry has also featured in classical vortex systems in a two dimensional incompressible fluid Doshi and Gromov [2021].

We can construct four conserved charges from this current. They are respectively the integral, the first-moment integral and the second-moment integral of J_t^{ϕ},

$$\begin{aligned}
Q &= -\int d^2x \ J_t^{\phi} \ , \\
P_i &= \int d^2x \ B\epsilon_{ij}x_j J_t^{\phi} \ , \\
L &= -\int d^2x \ \frac{1}{2}Br^2 J_t^{\phi} \ .
\end{aligned} \tag{4.3.68}$$

The monopole charge Q generates the constant shift on $\phi \to \phi + c$. Through the operator correspondence (4.3.27), it is mapped to the charge Q in (4.3.54) that measures the fluctuation of the total boson number around the equilibrium. Since Q is an integer, the constant identification on ϕ is fixed to be $\phi \sim \phi + 2\pi$.

The dipole charge P_i generates the linear shift on $\phi \to \phi + B\epsilon_{ij}x_j$, which under the relation (4.3.19) acts on the displacement field u_i as lattice translation $u_j \to u_j + \delta_{ij}$. It is therefore identified with the lattice momentum operator P_i in (4.3.58). Indeed, one can check that the two charges are related by the operator map (4.3.27) and integration by parts.

The conjugate momentum of ϕ is J_t^ϕ so, upon quantization, we have

$$[J_t^\phi(x), \phi(x')] = i\delta^2(x - x') \ . \tag{4.3.69}$$

This, together with (4.3.66), leads to the following algebra of the current J_t^ϕ,

$$[J_t^\phi(x), J_t^\phi(x')] = -\frac{ib_0}{2\pi B}\boldsymbol{\nabla} \times (\boldsymbol{\nabla}\delta^2(x - x')) \ , \tag{4.3.70}$$

which can be understood as a consequence of the Berry term in (4.3.23). Using integration by parts, the dipole charge P_i reproduces the algebra of momentum operators in (4.3.62).

Similarly, the quadrupole charge L generates the quadratic shift on $\phi \to \phi - \frac{1}{2}Br^2$ that acts on u_i as the lattice rotation $u_i \to u_i - \epsilon_{ij}x_j$ under the relation (4.3.19). It is thus identified with the angular momentum operator (4.3.58) of the vortex lattice.

By Lifshitz duality encoded in (4.3.36), the traceless symmetric tensor gauge theory is manifestly invariant under the quadrupole symmetry (4.3.64). The corresponding currents (4.3.65), expressed in terms of $\partial_t\phi$ and $D_{ij}\phi$, can be expressed in terms of the dual field strength $\hat{\mathcal{B}}, \hat{\mathcal{E}}_{ij}$ and the displacement field u_i,

$$\begin{aligned} J_t^\phi &= \frac{1}{2\pi}(\hat{\mathcal{B}} - b_0\partial_i u_i) - \frac{b_0 B}{8\pi^2\mu}\epsilon_{jk}\hat{\mathcal{E}}_{ik} \ , \\ J_{ij}^\phi &= \frac{1}{2\pi}\epsilon_{jk}\hat{\mathcal{E}}_{ik} + \frac{b_0}{4\pi^2\chi B}\epsilon_{ij}(\hat{\mathcal{B}} - b_0\partial_i u_i) \ . \end{aligned} \tag{4.3.71}$$

The operator charged under the quadrupole symmetry are then the displacement field u_i and the magnetic monopole operators $e^{i\phi}$ of the tensor gauge field, where the latter has no local representation on the dual side.

4.3.6 Topological crystalline defects

One common way to characterize phases and phase transitions (complementary to Landau's approach of breaking symmetry of a disordered symmetric state) is through a proliferation of topological defects that restore symmetry of the symmetry-broken state Kosterlitz and Thouless [1973], Halperin and Nelson [1978], Young [1979], as have been recently applied to quantum melting of crystals and its descendant states, particularly utilizing fracton-elasticity dualities Pretko and Radzihovsky [2018a,b], Pretko et al. [2019], Radzihovsky and Hermele [2020], Gromov and Radzihovsky [2022], Zhai and Radzihovsky [2021], Radzihovsky [2020], Pretko and Radzihovsky [2018b], Pretko et al. [2019], Kumar and Potter [2019].

A vortex lattice disorders similarly by proliferating topological defects that we analyze below, as was first implemented in a 3d classical vortex lattice in Ref. Marchetti and Radzihovsky [1999] and recently in a 2d quantum vortex crystal in Ref. Nguyen et al. [2020]. The distinction from a crystal of bosonic atoms is that in a vortex lattice, time reversal symmetry is explicitly broken and vortices are nontrivially coupled to superfluidity (superflow) encoded in the dual $U(1)$ gauge theory, as detailed in Sec. 4.3.3. As we have seen, important consequences of this are the incompressibility constraint encoded in the transversality of the phonon displacement $\partial_i u_i = 0$ and a nontrivial Berry phase of the Lifshitz theory (4.3.23) and its dual that we derived in (4.3.35).

There are three types of topological crystalline defects exhibited by a vortex lattice: disclinations, dislocations and vortex vacancies/interstitials [6]. Below we discuss and formulate these vortex lattice topological defects in terms of Lifshitz and its dual traceless symmetric tensor gauge theory, and use them to summarize possible descendant phases and discuss their properties, leaving a detailed analysis of corresponding Higgs transitions to future research.

To this end we note that there are a number of consistent disordering transition routes

6. For topological crystalline defects, we are referring to defects associated with field configurations with non-trivial topology. This should not be confused with the symmetry operators/defects that implement the crystalline symmetries.

Vortex lattice	Lifshitz theory $\phi(\mathbf{x})$	Traceless symmetric tensor gauge theory
Disclination	$-\frac{B}{12}\vartheta r^2$	$\exp\left[\frac{iB}{6}\int \hat{\mathcal{A}}_t\, dt\right]$
Dislocation with Burgers vector $\mathbf{b}$	$\frac{B}{4\pi}(\mathbf{b}\cdot\mathbf{x}+2\vartheta\mathbf{b}\times\mathbf{x})$	$\exp\left[-\frac{iB}{2\pi}\int \mathbf{b}\times\boldsymbol{\nabla}\hat{\mathcal{A}}_t\, dt\right]$
Vortex interstitial	ϑ	$\exp\left[-\frac{i}{2}\int \boldsymbol{\nabla}^2\hat{\mathcal{A}}_t\, dt\right]$

Table 4.1: The topological crystalline defects of the vortex lattice in the Lifshitz theory description (4.3.23) and in its dual traceless symmetric scalar-charge theory description (4.3.35). The phase $\phi(\mathbf{x})$ of the Lifshitz theory winds around the defect placed at the origin, as listed in the second column. In the tensor gauge theory, the defects are realized as Wilson defects, as listed in the third column. These defects are charged under the multipole one-form symmetry of (4.3.84).

of a fully ordered commensurate superfluid vortex crystal. However, these are constrained by the interrelation of the associated topological defects, most importantly that dislocations and vacancies/interstitials are dipoles and quadrupoles of the elementary disclination defects, respectively. This is encoded in e.g., dislocation ($\psi_b^\dagger$) – disclination ($\psi_s^\dagger$) coupling, $\psi_s^\dagger(\mathbf{x}-\mathbf{d}/2)\psi_s(\mathbf{x}+\mathbf{d}/2)\psi_b(\mathbf{x})$, that demands that when the latter condenses, so does, necessarily the former. As such, phases in which some defects have proliferated and condensed, but their multipoles have not, are inconsistent and thus physically impossible Pretko and Radzihovsky [2018b], Zhai and Radzihovsky [2021], Kumar and Potter [2019]. With this we now construct these defects in Lifshitz and its gauge-dual representations, as summarized in Table 4.1, and discuss the corresponding phases that will generically appear as descendants of a vortex crystal.

Vortex-lattice defects via winding defects in Lifshitz theory

In the Lifshitz theory, the crystalline defects are realized as point singularities around which the phase $\phi(\mathbf{x})$ winds. For such defects at the origin, the most general winding of $\phi(\mathbf{x})$ is fixed by the identification (4.3.25), (4.3.26) and is given by

$$\begin{aligned} &\phi(\vartheta+2\pi,r) \\ =&\phi(\vartheta,r)+2\pi\mathbb{Z}+yBl\mathbb{Z}+\frac{1}{2}(y-\sqrt{3}x)Bl\mathbb{Z}+\frac{2\pi}{12}Br^2\mathbb{Z}\ , \end{aligned} \tag{4.3.72}$$

where (ϑ, r) is the polar coordinate related to the Cartesian coordinate by $(x, y) = (r\cos\vartheta, r\sin\vartheta)$. Below, we relate crystalline defects of the vortex lattice to the winding defects in phase ϕ of the Lifshitz theory.

Disclination – A disclination defect is a singularity of the local lattice bond angle $\theta = \frac{1}{2}\epsilon_{ij}\partial_i u_j$, which, utilizing (4.3.19) is given by

$$\theta = -\frac{1}{2B}\partial^2\phi\ , \tag{4.3.73}$$

in the Lifshitz theory under the operator map (4.3.27). As the vortex lattice is a triangular lattice, the bond angle θ winds by $2\pi/6$ around an elementary disclination defect leading to the following singularity equation for ϕ

$$-\frac{1}{2B}\epsilon_{ij}\partial_i\partial_j\partial^2\phi = \epsilon_{ij}\partial_i\partial_j\theta = \frac{2\pi}{6}\delta^2(x)\ . \tag{4.3.74}$$

Solving it gives

$$\phi = -\frac{B}{12}\vartheta r^2\ , \tag{4.3.75}$$

which corresponds to the quadratic winding in (4.3.72), with ϑ the polar angle singular at the defect location $r = 0$. The energy of a disclination

$$E_{disc} = \int_a^L rdr\int_0^{2\pi} d\theta\frac{\mu}{B^2}\left(D_{ij}\phi\right)^2 = \frac{2\pi\mu}{144}L^2\ , \tag{4.3.76}$$

diverges quadratically with the system size L, consistent with the studies of 2d classical crystals Halperin and Nelson [1978], Young [1979].

Dislocation – A dislocation defect is characterized by its Burgers vector $\mathbf{b}$ – the shift by a lattice vector when going around the defect, possible because of the compactness of $\mathbf{u}(\mathbf{x})$. It can be decomposed into a dipole of the elementary disclinations separated by the vector $\frac{6}{2\pi}\hat{\mathbf{z}} \times \mathbf{b}$. As a result, the ϕ configuration around it is related to the one in (4.3.75) around a disclination by the action of a differential operator $-\frac{6}{2\pi}\mathbf{b} \times \boldsymbol{\nabla}$, which gives

$$\phi = \frac{B}{4\pi}\epsilon_{ij}b_i\partial_j(\vartheta r^2) = \frac{B}{4\pi}b_i(x_i + 2\vartheta\epsilon_{ij}x_j) \ . \tag{4.3.77}$$

As ϑ increases by 2π, $\phi(\mathbf{x})$ winds around by

$$\begin{aligned} \phi(\vartheta + 2\pi) &= \phi(\vartheta) + B\epsilon_{ij}b_i x_j \\ &= \phi(\vartheta) + yBl\mathbb{Z} + \frac{1}{2}(y - \sqrt{3}x)Bl\mathbb{Z} \ , \end{aligned} \tag{4.3.78}$$

where we use the fact that the Burgers vector are integer linear combinations of the triangular lattice vectors

$$\mathbf{b} = \left\{\hat{\mathbf{x}}\mathbb{Z} + \frac{1}{2}(\hat{\mathbf{x}} + \sqrt{3}\hat{\mathbf{y}})\mathbb{Z}\right\} l \ . \tag{4.3.79}$$

This ϕ winding around a dislocation corresponds to the linear winding in the most general identification in (4.3.72). The energy of a dislocation is given by

$$E_v = \int_a^L rdr \int_0^{2\pi} d\theta \frac{\mu}{B^2}\left(D_{ij}\phi\right)^2 = \frac{\mu}{2\pi}\mathbf{b}^2 \log\left(\frac{L}{a}\right) \ , \tag{4.3.80}$$

diverging logarithmically with the system size L as in a conventional 2d classical crystal Halperin and Nelson [1978], Young [1979].

Vacancy and interstitial – A vacancy/interstitial of the vortex lattice can be created by removing/inserting a vortex into the lattice. In the effective field theory (4.3.23), a vortex is represented by a Wilson line $W = \exp(i \int a_t \, dt)$ of the superfluid gauge field a_μ. Consequently, a vortex vacancy and interstitial are described by $W^\dagger$ and W, respectively. Without loss of generality, we will focus on the interstitial defect in the discussion below.

Inserting a vortex interstitial modifies the Euler-Lagrangian equation of a_t, resulting in a singularity in the incompressibility condition, modifying (4.3.14) to be

$$n_v \partial_i u_i = \frac{1}{2\pi} \epsilon_{ij} \partial_i \partial_j \phi = \delta^2(x) \ . \tag{4.3.81}$$

Here we employed the operator map in (4.3.27) to replace u_i with ϕ. Solving this equation gives the $\phi(\mathbf{x})$ configuration around the interstitial defect,

$$\phi(r, \vartheta) = \vartheta \ , \tag{4.3.82}$$

which corresponds to the constant winding component in (4.3.72). This ϕ configuration is related to the one around a disclination defect in (4.3.75) by the action of the differential operator $-\frac{3}{B}\partial^2$. This relationship arises from the fact that an interstitial defect can be decomposed into a group of disclination defects. The energy of a vacancy/interstitial is given by

$$E_v = \int_a^L r dr \int_0^{2\pi} d\theta \frac{\mu}{B^2} \left(D_{ij}\phi\right)^2 = \frac{2\pi\mu}{B^2 a^2} \ , \tag{4.3.83}$$

finite in the infrared limit, but, as expected depending sensitively on the lattice cutoff a.

Multipole one-form global symmetry and defect mobility

An interesting characteristic of these crystalline defects is that they could have restricted mobility, similar to fractons. To systematically organize them and study their mobility, we utilize the generalized global symmetries that they are charged under.

In the Lifshitz theory, the relevant symmetry is the winding symmetry with the winding conserved current

$$J_t^{ij} = -\frac{1}{2\pi B} D_{ij}\phi \ , \quad J = -\frac{1}{2\pi B}\partial_t \phi \ , \tag{4.3.84}$$

where J_t^{ij} is a traceless symmetric tensor. In the absence of topological defects, this winding current obeys conservation equations

$$\partial_t J_t^{ij} = D_{ij} J \ , \tag{4.3.85}$$

and a differential condition relating different temporal components of the current

$$\epsilon_{jk} D_{ij} J_t^{ik} = 0 \ . \tag{4.3.86}$$

These equations, (4.3.85) and (4.3.86), are the Lifshitz theory analogs akin to the spatial and temporal components of (4.3.5) for vortices in the superfluid. Here too, defects violate these homogeneous continuity equations, appearing as sources in it. In particular, a static defect generates only sources to the differential condition (4.3.86).

Using the winding current, we can construct several conserved charges, e.g. $Q_i^{xy} = \int J_t^{xy} dx_i$, that act on the winding operators and the winding states of ϕ. Their conservation follows from the current conservation equation (4.3.85). Notably, these conserved charges act only on operators inserted at a fixed time but not on static defects extending in the time direction. In the language of Gorantla et al. [2022], these charges generate a space-like symmetry while the defects are charged under a time-like symmetry. As defects are sources to the differential conditions (4.3.86), the time-like charges can be built from the latter.

As a warm-up, we recall the simplest example of space-like and time-like symmetries is the $U(1)$ electric one-form symmetry in Maxwell's electrodynamics, generated by a conserved two-form current

$$J_{\mu\nu} = \frac{1}{g^2} F_{\mu\nu} \ , \tag{4.3.87}$$

obeying current conservation equations (Faraday's law)

$$\partial_t J_{ti} = \partial_j J_{ji} \ , \tag{4.3.88}$$

and a differential condition relating different temporal components of the current (Gauss's law)

$$\partial_i J_{it} = 0 \ . \tag{4.3.89}$$

The conservation and differential condition can be summarized concisely as $\partial^\mu J_{\mu\nu} = 0$. The $U(1)$ one-form symmetry comprises a $U(1)$ space-like symmetry that acts on Wilson loop operators inserted at a fixed time and a $U(1)$ time-like symmetry that acts on Wilson line defects oriented in the time direction. The time-like charge for the time-like symmetry is an integral of the differential condition (4.3.89) over an open volume $\mathcal{V}$,

$$\begin{aligned} Q &= \int_{\mathcal{V}} \partial_i J_{it} \, dx \wedge dy \wedge dz \\ &= \oint_{\Sigma} \frac{1}{2} \epsilon^{ijk} J_{it} \, dx_j \wedge dx_k \\ &= \oint_{\Sigma} \mathbf{E} \cdot d\mathbf{S} \ , \end{aligned} \tag{4.3.90}$$

which can be expressed as the Gauss's law operator localized on the boundary surface $\Sigma = \partial\mathcal{V}$. A Wilson defect induces a source to the differential condition (4.3.89) and thus carries charge under the time-like charges or equivalently the Gauss's law operators surrounding it. This is essentially the content of the Gauss's law. The time-like charge is conserved in time if there is no Wilson line operators crossing it

$$\partial_t Q = \int_{\mathcal{V}} \partial_i \partial_t J_{it} \, dv = \int_{\mathcal{V}} \partial_i \partial_j J_{ij} \, dv = 0 \tag{4.3.91}$$

where $dv = dx \wedge dy \wedge dz$ denotes the volume form. In the second equality, we use the current conservation equation (4.3.88) while in the last equality, we use the fact that J_{ij} is antisymmetric and single-valued, namely there are defects so that derivatives on it commute.

In relativistic systems, the space-like and time-like symmetry share the same symmetry group and together form a higher-form symmetry since space and time are on equal footings but in non-relativistic systems, they are generically distinct (see the examples discussed in Gorantla et al. [2022]).

After the warm-up, we are now ready to explore the time-like winding symmetry in the Lifshitz theory. The time-like symmetry is a multipole symmetry with in total three types of time-like winding charges, monopole, dipole and quadrupole charges, paralleling our classification of winding defects in the previous subsection, and in particular the three – constant, linear and quadratic – components of (4.3.72).

The monopole charge Q is the integral of the differential condition (4.3.86) over an open surface area Σ,

$$\begin{aligned} Q &= \int_\Sigma \epsilon_{jk} D_{ij} J_t^{ik}\, dx \wedge dy \\ &= \oint_\gamma \partial_i J_t^{ij} dx_j \\ &= -\oint_\gamma \frac{1}{4\pi B} d(\partial^2 \phi) = \oint_\gamma \frac{1}{2\pi} d\theta\ , \end{aligned} \tag{4.3.92}$$

which can be expressed as an operator localized on the boundary curve $\gamma = \partial\Sigma$. In the last two equalities, we used the explicit form of the winding current (4.3.84) and the relation (4.3.73) between the bond angle θ and the Lifshitz scalar field ϕ. This reveals the physical meaning of the monopole charge: it measures the quadratic winding of ϕ in (4.3.72) or equivalently the bond angle θ winding around the curve γ, nonzero for an enclosed disclination defect.

The dipole charge Q_n is the first-moment integral of the differential condition (4.3.86) over an open area Σ,

$$\begin{aligned} Q_n &= \int_\Sigma 2\pi\epsilon_{nm}x_m \times \epsilon_{jk}D_{ij}J_t^{ik}\, dx\wedge dy \\ &= \oint_\gamma 2\pi\epsilon_{n\bar{n}}\left[x_{\bar{n}}\left(\partial_n J_t^{xy} + 2\partial_{\bar{n}}J_t^{\bar{n}\bar{n}}\right)dx_{\bar{n}} + \left(x_{\bar{n}}\partial_{\bar{n}}J_t^{xy} - J_t^{xy}\right)dx_n\right] , \\ &= \oint_\gamma \frac{1}{B}\epsilon_{n\bar{n}}d\left(\partial_{\bar{n}}\phi - x_{\bar{n}}\partial_{\bar{n}}^2\phi\right) = \oint_\gamma d\left(u_n + \theta\epsilon_{n\bar{n}}x_{\bar{n}}\right) . \end{aligned} \tag{4.3.93}$$

where in the last line the indices are not summed over; instead, $\bar{n}$ denotes the index distinct from n. It can be expressed as an operator localized on the boundary $\partial\Sigma = \gamma$, which measures the linear winding of ϕ in (4.3.72), corresponding to a Burgers displacement $\mathbf{b}$ associated with an enclosed dislocation defect. The dipole charge is not simply an integral of $d(\partial_{\bar{n}}\phi)$; rather, it has an additional term $-d(x_{\bar{n}}\partial_{\bar{n}}^2\phi)$ that removes the contributions from the quadratic winding. In the last equality, we expressed the dipole charge in terms of the displacement field u_n and the bond angle θ using (4.3.19) and (4.3.73). This allows us to interpret the linear in x winding of ϕ physically as measuring the variation of the displacement field u_n around the curve γ (dislocation), modulo the contribution $\theta\epsilon_{n\bar{n}}x_{\bar{n}}$ from the winding of the bond angle θ (disclination).

The quadratic charge $\mathcal{Q}$ is the second-moment integral of the differential condition (4.3.86) over an open area Σ,

$$\begin{aligned} \mathcal{Q} &= \int_\Sigma -\frac{1}{2}Br^2 \times \epsilon_{jk}D_{ij}J_t^{ik}\, dx\wedge dy \\ &= \oint_\gamma \frac{1}{2}B\left[\left(2yJ_t^{xy} - r^2\partial_y J_t^{xy} - 2x^2\partial_x J_t^{xx}\right)dx + \left(2xJ_t^{xy} - r^2\partial_x J_t^{xy} - 2y^2\partial_y J_t^{yy}\right)dy\right] , \\ &= \oint_\gamma \frac{1}{4\pi}d(x^2\partial_x^2\phi - 2x\partial_x\phi + y^2\partial_y^2\phi - 2y\partial_y\phi + 2\phi) = \int_\Sigma \delta j_t - n_v\oint_\gamma d\left(\epsilon_{ij}u_i x_j + \frac{1}{2}r^2\theta\right) . \end{aligned} \tag{4.3.94}$$

It can be expressed as an operator localized on the boundary, $\partial\Sigma = \gamma$ which measures only the constant winding of ϕ in (4.3.72). Physically, the quadrupole charge can be interpreted as the number of vortices enclosed by the curve γ modulo the contributions from the underlying vortex lattice and the change of lattice area $\epsilon_{ij}u_ix_j + \frac{1}{2}r^2\theta$ due to dislocations and disclinations, namely it measures vortex vacancies and interstitial defects added to the underlying vortex lattice.

We can systematically organize the winding defects or equivalently the crystalline defects according to their time-like winding charges. Let us first consider only defects placed at the origin. Then, (i) defects charged under the monopole charge Q are the disclination defects with quantized $Q \in \frac{1}{6}\mathbb{Z}$ measuring the winding of the bond angle θ, (ii) defects charged under the dipole charges Q_n are the dislocation defects with $Q_n = b_n$ measuring the Burgers vector of the dislocation defects $\mathbf{b}$, and (iii) defects charged under the quadrupole charge $\mathcal{Q}$ are vortex vacancies/interstitials with quantized $\mathcal{Q} \in \mathbb{Z}$ measuring the constant winding of ϕ, or equivalently the constant winding of the superfluid phase φ over and above the underlying winding due to the vortex lattice.

For defects away from the origin, the ones charged under the dipole and quadrupole time-like symmetry are no longer just the dislocations and vortex vacancies/interstitials respectively. Comparing the singularity equation (4.3.74) with the differential condition (4.3.86), we learn that a disclination defects generate a δ-function source to the differential condition

$$\epsilon_{jk}D_{ij}J_t^{ik} = \frac{1}{6}\delta^2(\mathbf{x} - \mathbf{x}') \;, \tag{4.3.95}$$

where $\mathbf{x}'$ is the position of the disclination defect. Inserting (4.3.95) into (4.3.93) and (4.3.94) we see that such disclination carries both a dipole charge $Q_n = \frac{2\pi}{6}\epsilon_{nm}x'_m$ and a quadrupole charge $\mathcal{Q} = \frac{1}{12}Br'^2$, respectively. Conservation of these charges requires that $\mathbf{x}'$ is time independent, and thus a disclination is completely immobile. Physically, it corresponds to the fact that moving a disclination creates dislocations Radzihovsky and Hermele [2020]. Similarly, a dislocation carries a quadrupole charge $\mathcal{Q} = \frac{1}{2\pi}B\mathbf{b}\times\mathbf{x}'$ which freezes its position transverse to its Burgers vector $\mathbf{b}$, forbidding its "climb". We thereby recover the glide-only constraint on a dislocation motion. Physically, this corresponds to a creation of vortex vacancies and interstitials in the climb process, that is forbidden by vortex conservation, as first explicitly demonstrated in Refs. Pretko and Radzihovsky [2018a,b], Pretko et al. [2019], Radzihovsky and Hermele [2020].

Vortex-lattice defects via Wilson defects in tensor gauge theory

The topological crystalline defects of the vortex lattice are realized as Wilson defects in the traceless symmetric tensor gauge theory (4.3.35). A systematic way to match these defects is to utilize the generalized global symmetry that acts on them. In the Lifshitz theory, the relevant symmetry is the winding symmetry whose conserved current under the operator correspondence (4.3.36) is mapped to

$$J_t^{ij} = -\frac{B}{8\pi^2\mu}\epsilon_{jk}\hat{\mathcal{E}}_{ik} \ , \quad J = -\frac{1}{4\pi^2 B\chi}(\hat{\mathcal{B}} - b_0\partial_i u_i) \ . \tag{4.3.96}$$

The differential condition (4.3.86) of this current now is a consequence of the Euler-Lagrange equation of $\hat{\mathcal{A}}_t$ in the traceless symmetric tensor gauge theory, namely the electric Gauss's law for $\hat{\mathcal{E}}_{ik}$. The sources to the differential conditions are the Wilson lines built out of $\hat{\mathcal{A}}_t$. Suppose we insert such a Wilson line $\exp(iC\int \hat{\mathcal{A}}_t\, dt)$ with an undetermined coefficient C at the origin. The Euler-Lagrangian equation or relatedly the differential condition is modified into

$$\epsilon_{jk}D_{ij}J_t^{ik} = \frac{B}{8\pi^2\mu}D_{ij}\hat{\mathcal{E}}_{ij} = \frac{C}{B}\delta^2(x) \ , \tag{4.3.97}$$

with Gauss's law encoding disclinations as the (dual) electric charges. Comparing it with the differential condition (4.3.95) around the elementary disclination defect, we learn that a elementary disclination defect is represented by the Wilson line

$$\text{vortex-disclination:} \quad \exp\left[\frac{iB}{6}\int \hat{\mathcal{A}}_t\, dt\right] \ . \tag{4.3.98}$$

The exponent of the Wilson line is dimensionless despite the appearance of the effective background magnetic field B in it. It is because the gauge parameter $\hat{\lambda}$ has mass dimension -2. Since this Wilson line must be the minimal properly quantized Wilson line, the gauge parameter $\hat{\lambda}$ has the following identification

$$\hat{\lambda} \sim \hat{\lambda} + \frac{12\pi}{B}\mathbb{Z} \ . \tag{4.3.99}$$

As dislocations and vortex interstitials are composed of disclinations, we can infer their representations by acting differential operators on the exponent of Wilson line representation (4.3.98) of disclinations. With this, we deduce that the dislocation with Burgers vector $\mathbf{b}$

$$\mathbf{b} = \left\{\hat{\mathbf{x}}\mathbb{Z} + \frac{1}{2}(\hat{\mathbf{x}} + \sqrt{3}\hat{\mathbf{y}})\mathbb{Z}\right\} l \tag{4.3.100}$$

and the vortex interstitial are respectively represented by the following properly quantized Wilson defects

$$\text{vortex-dislocation:} \quad \exp\left[-\frac{iB}{2\pi}\int \mathbf{b} \times \boldsymbol{\nabla}\hat{\mathcal{A}}_t \, dt\right] \ , \tag{4.3.101}$$

$$\text{vortex-interstitial:} \quad \exp\left[-\frac{i}{2}\int \boldsymbol{\nabla}^2\hat{\mathcal{A}}_t \, dt\right] \ . \tag{4.3.102}$$

The quantization of these defects demands that the gauge parameter $\hat{\lambda}$ obey the following additional identifications

$$\hat{\lambda} \sim \hat{\lambda} + \frac{8\pi^2}{\sqrt{3}Bl}[x\mathbb{Z} + \frac{1}{2}(x + \sqrt{3}y)\mathbb{Z}] + \pi(x^2 + y^2)\mathbb{Z} \ . \tag{4.3.103}$$

The mobility of these defects is reflected in whether there exists gauge invariant Wilson defects that can be deformed in spatial directions. If such defects exist, they represent the worldlines of the mobile defects. In the case of disclinations and dislocations, such gauge invariant defects do *not* exist, so they are completely immobile, consistent with our earlier conclusion in the previous subsection and with general considerations based on fracton-elasticity duality Pretko and Radzihovsky [2018a,b], Pretko et al. [2019], Radzihovsky and Hermele [2020]. On the other hand, a vortex interstitial can move freely, and its motion is represented by the gauge invariant Wilson defect

$$\exp\left[-i\int_\gamma\left(\frac{1}{2}\partial^2\hat{\mathcal{A}}_t\,dt+\partial_i\hat{\mathcal{A}}_{ij}\,dx_j\right)\right]\ , \tag{4.3.104}$$

where γ is the worldline of the vortex interstitial. The Wilson defect is invariant under the gauge transformation (4.3.33), as the phase Φ generated by the transformation,

$$\Phi=\int_\gamma\left(\frac{1}{2}\partial^2\partial_t\hat{\lambda}\,dt+\partial_i D_{ij}\hat{\lambda}\,dx_j\right)=\int_\gamma\frac{1}{2}d(\partial^2\hat{\lambda}), \tag{4.3.105}$$

is the integral of an exact form and thereby vanishes.

The Wilson defect descriptions of the topological crystalline defects allow us to compute the potentials between them. Consider a disclination defect sitting at the origin. It modifies the equation of motion of $\mathcal{A}_t$, $\mathcal{A}_{ij}$ and u_i to

$$\begin{aligned}-\frac{B^2}{8\pi^2\mu}D_{ij}\hat{\mathcal{E}}_{ij}+\frac{B}{6}\delta^2(x)=0\ ,\\ -\frac{B^2}{4\pi^2\mu}\partial_t\hat{\mathcal{E}}_{ij}-\frac{1}{2\pi^2\chi}\epsilon_{ik}D_{jk}(\hat{\mathcal{B}}-b_0\partial_i u_i)=0\ ,\\ -\frac{b_0}{4\pi^2\chi}\partial_i(\hat{\mathcal{B}}-b_0\partial_i u_i)-\frac{b_0 B}{2\pi}\epsilon_{ij}\partial_t u_j=0\ .\end{aligned} \tag{4.3.106}$$

Solving them gives the solution

$$\mathcal{A}_t=-\frac{\pi\mu}{3B}r^2(\log r-1)\ ,\quad \mathcal{A}_{ij}=0\ ,\quad u_i=0\ . \tag{4.3.107}$$

The potential between a disclination and an anti-disclination is then given by

$$V_d(r) = -\frac{B}{6}\mathcal{A}_t = \frac{\pi\mu}{18}r^2(\log r - 1) \ , \tag{4.3.108}$$

which describes a confining potential. Since the dislocations and vacancies/interstitials are compositions of disclinations, it is straightforward to derive their potentials. For example, the potentials between a vacancy and interstitial is given by

$$V_v(r) = \left(\frac{3}{B}\boldsymbol{\nabla}^2\right)^2 V_d(r) = 0 \ , \tag{4.3.109}$$

which vanishes. The potential gets corrected if the subdominant kinetic and electric energies were included in the effective theory (4.3.12).

4.3.7 Vortex phases and phase transitions

We now turn to a discussion of phases whose general structure directly follows from various schemes of proliferation of topological defects: vacancies/interstitials, dislocations and disclinations.

Vortex-supersolid

Per our comments above on generic unbinding the highest multipole defects first, we consider proliferating disclination quadrupoles, i.e., vacancies and interstitials, as discussed above, corresponding to vortices in ϕ. In the dual traceless symmetric tensor gauge theory picture, this condensation corresponds to Higgs transition of the vortex crystal to the "vortex-supersolid" Nguyen et al. [2020] (a novel, non-superfluid vortex lattice phase without ODLRO of the underlying bosons, that has been previously extensively studied in the context of type-II superconductors Frey et al. [1994], Marchetti and Radzihovsky [1999]). These quadrupole "charges" couple to an ordinary Wilson line $W = \exp(i\int_\gamma a_\mu dx^\mu)$ built from a composite vector gauge field in (4.3.104):

$$a_\mu = (a_t, a_i)_\mu = -\Big(\frac{1}{2}\partial^2 \hat{\mathcal{A}}_t, \partial_j \hat{\mathcal{A}}_{ji}\Big)_\mu \ . \tag{4.3.110}$$

The fluctuations of the quadrupole charges can be represented by a complex field Ψ, with the gauge symmetry

$$\Psi \to e^{\frac{i}{2}\partial^2 \alpha}\Psi \ . \tag{4.3.111}$$

Then, the vortex crystal-to-vortex supersolid melting transition is captured by the following quantum Landau-Ginzburg theory

$$\begin{aligned}\mathcal{L} = &\frac{B^2}{16\pi^2\mu}\hat{\mathcal{E}}_{ij}^2 - \frac{1}{8\pi^2\chi}(\hat{\mathcal{B}} - b_0\partial_i u_i)^2 - \frac{b_0 B}{4\pi}\epsilon_{ij}u_i\partial_t u_j \\ &+ K_t\big|\big(\partial_t - \frac{i}{2}\partial^2\hat{\mathcal{A}}_t\big)\Psi\big|^2 + K\big|\big(\partial_i - i\partial_j\hat{\mathcal{A}}_{ji}\big)\Psi\big|^2 \\ &+ m^2|\Psi|^2 + g|\Psi|^4 \ .\end{aligned} \tag{4.3.112}$$

When m^2 is positive, Ψ is massive and the theory is the vortex lattice phase. When m^2 is negative, Ψ condenses and develops a vacuum expectation value $\langle\Psi\rangle = v$. It can be parametrized as

$$\Psi = (v + \sigma)e^{i\mathcal{A}} \ . \tag{4.3.113}$$

The phase $\mathcal{A}$ transforms under the gauge symmetry as a gauge field,

$$\mathcal{A} \to \mathcal{A} + \frac{1}{2}\partial^2\hat{\lambda} \ . \tag{4.3.114}$$

Combining this gauge field $\mathcal{A}$ with the traceless symmetric gauge field $(\hat{\mathcal{A}}_t, \hat{\mathcal{A}}_{ij})$, we can define a *traceful* symmetric tensor gauge field as

$$\mathcal{A}_t = \hat{\mathcal{A}}_t \ , \quad \mathcal{A}_{ij} = \hat{\mathcal{A}}_{ij} + \delta_{ij}\mathcal{A} \ , \tag{4.3.115}$$

which transforms under the gauge symmetry as

$$\mathcal{A}_t \to \mathcal{A}_t + \partial_t \hat{\lambda} \ , \quad \mathcal{A}_{ij} \to \mathcal{A}_{ij} + \partial_i \partial_j \hat{\lambda} \ . \tag{4.3.116}$$

The gauge invariant field strength are

$$\mathcal{B}_i = \epsilon_{jk} \partial_j \mathcal{A}_{ki} \ , \quad \mathcal{E}_{ij} = \partial_t \mathcal{A}_{ij} - \partial_i \partial_j \mathcal{A}_t \ . \tag{4.3.117}$$

They are related to the field strength of $(\hat{\mathcal{A}}_t, \hat{\mathcal{A}}_{ij})$ by

$$\hat{\mathcal{E}}_{ij} = \mathcal{E}_{ij} - \frac{1}{2}\delta_{ij}\mathcal{E}_{kk} \ , \quad \hat{\mathcal{B}} = -\partial_i \mathcal{B}_i \ . \tag{4.3.118}$$

In terms of this new symmetric tensor gauge field, the vortex supersolid phase is described by the following Lagrangian

$$\begin{aligned} \mathcal{L} = & \frac{B^2}{16\pi^2\mu}(\mathcal{E}_{ij} - \frac{1}{2}\delta_{ij}\mathcal{E}_{kk})^2 + \frac{1}{4}K_t\mathcal{E}_{kk}^2 + K\mathcal{B}_i^2 \\ & - \frac{1}{8\pi^2\chi}(\partial_i\mathcal{B}_i + b_0\partial_i u_i)^2 - \frac{b_0 B}{4\pi}\epsilon_{ij}u_i\partial_t u_j \ . \end{aligned} \tag{4.3.119}$$

The Higgs transition thus leads to the dual *traceful* symmetric tensor gauge theory of the vortex-supersolid state (as opposed to a *traceless* one in (4.3.35)). Interestingly, this dual tensor gauge theory resembles that of a time-reversal broken Wigner crystal (e.g., bosonic crystal in an effective magnetic field) Pretko and Radzihovsky [2018b].

In this vortex-supersolid phase, the condensate of vortex vacancies and interstitials are encoded in the traces of $\mathcal{A}_{ij}$ and $\mathcal{E}_{ij}$. They break the $U(1)$ vortex conservation symmetry and thereby lift the glide-only constraint on the dislocation mobility, akin to its time-reversal symmetric analog of bosonic crystals Pretko and Radzihovsky [2018b], Pretko et al. [2019], Radzihovsky and Hermele [2020], Kumar and Potter [2019]. The motions of these mobile vortex-dislocations are represented by the gauge invariant Wilson defect of the symmetric tensor gauge field

$$\exp\left[-\frac{iB}{2\pi}\int \epsilon_{ij}b_i(\partial_j\mathcal{A}_t\,dt+\mathcal{A}_{jk}dx_k)\right]\ . \tag{4.3.120}$$

Physically, vortex-dislocations become fully mobile in the presence of vortex condensate because the latter absorbs the vortex vacancies/interstitials created by the motions of dislocations. Contrary to the vortex-dislocations, vortex-disclinations remain immobile in the vortex-supersolid phase Marchetti and Radzihovsky [1999].

Vortex-hexatic, -smectic and -nematic

Vortex-hexatic: Next, a simultaneous proliferation of all dislocations $\mathbf{b}_\alpha$ (dipoles) in (4.3.100) melts the vortex-supersolid into vortex-hexatic, fully restoring the translational symmetry while retaining bond orientational order. Like its vortex-supersolid parent, this state lacks ODLRO of bosons, i.e., it is not a superfluid. Extending arguments from melting of time-reversal invariant crystals of bosons to here, we find that the corresponding hexatic vortex-liquid state is characterized by mobile disclinations. Physically, this is due to a condensate of dislocations now able to absorb dislocations produced by disclination motion, and characterized by a space component of the disclination Wilson loop in Table I.

Vortex-smectic: If in contrast to vortex-hexatic, only one of the three vortex-dislocations in (4.3.100) unbinds, the condensation of such dislocation $\mathbf{b}$ melts the vortex-crystal into a vortex-smectic, characterized by a wavevector transverse to the condensed $\mathbf{b}$ and a vanishing shear modulus for shear along $\mathbf{b}$. The Higgs transition gaps out gauge fields associated with the phonon displacement along $\mathbf{b}$ and lead to restricted lineon mobility of the corresponding disclinations, immobile only along smectic layers.

Vortex-nematic: Unbinding the remaining two elementary dislocations (they must unbind in pairs), leads to a non-superfluid vortex-nematic state with fully mobile disclinations, quite similar to a hexatic, but with C_2 symmetry.

Both the hexatic and nematics can then quantum melt into an isotropic vortex liquid via a Higgs condensation transition of disclination defects. We leave the detailed derivation and the resulting field theory to later studies.

4.3.8 Conclusion

In this manuscript we formulated and studied an effective quantum field theory of a two-dimensional zero-temperature vortex lattice in a neutral rotating superfluid, utilizing a combination of particle-vortex, elasticity-fracton and Lifshitz-gauge dualities, augmented with a Berry phase term that encodes vortex dynamics in the presence of a superflow, demonstrating consistency between these different formulations. We discussed a hidden multipole symmetry of the vortex lattice, used its dual traceless symmetric tensor gauge theory to explore vortex lattice dynamics, characterize its topological vortex crystalline defects using a multipole one-form symmetry and generalized Wilson lines, uncovering and detailing their restricted mobility. This also allowed us to outline a number of novel descendant vortex phases separated by generalized-Higgs transitions, with detailed analysis left for future studies.

It would be interesting to generalize our vortex lattice duality to nonlinear Lifshitz theory. We also expect that the duality can be extended to higher-dimensional theories, e.g. the Lifshitz photon theory Du et al. [2024b] in three dimensions, which is a gauge theory with Lifshitz scaling symmetry.

4.3.9 Supplemental Material

Duality of Lifshitz theory without Berry phase

In Sec. 4.3.4, we study the duality of Lifshitz theory with a Berry phase presented in (4.3.23). In this appendix, we instead consider Lifshitz theory without the Berry phase term and revisit the Lifshitz duality in the presence of higher-rank background probe fields Du et al. [2022].

Coupling to the higher-rank background probe fields

The Lagrangian is given by

$$\mathcal{L} = \frac{\chi}{2}\left(\partial_t\phi - \hat{C}_t\right)^2 - \frac{K}{2}\left(D_{ij}\phi - \hat{C}_{ij}\right)^2 \ , \tag{4.3.121}$$

where $(\hat{C}_t, \hat{C}_{ij})$ is the traceless symmetric background gauge field with background gauge transformation

$$\phi \sim \phi + \gamma \ , \quad \hat{C}_t \sim \hat{C}_t + \partial_t\hat{\gamma} \ , \quad \hat{C}_{ij} \sim \hat{C}_{ij} + D_{ij}\hat{\gamma} \ . \tag{4.3.122}$$

We dualize the theory (4.3.121) by introducing Hubbard-Stratonovich fields $\hat{\pi}_t$ and $\hat{\pi}_{ij}$, which is symmetric and traceless, and rewrite the Lifshitz theory Lagrangian as

$$\mathcal{L} = -\frac{1}{2\chi}\hat{\pi}_t^2 + \frac{1}{2K}\hat{\pi}_{ij}^2 + \hat{\pi}_t(\partial_t\phi - \hat{C}_t) + \hat{\pi}_{ij}(D_{ij}\phi - \hat{C}_{ij}) \ . \tag{4.3.123}$$

Integrating out ϕ, that appears linearly, gives the following constraint,

$$\partial_t\hat{\pi}_t - D_{ij}\hat{\pi}_{ij} = 0 \ . \tag{4.3.124}$$

To solve the constraint, it is convenient to define dual magnetic and electric fields,

$$\hat{\mathcal{B}} \equiv 2\pi\hat{\pi}_t \ , \quad \hat{\mathcal{E}}_{ij} \equiv 2\pi\epsilon_{jk}\hat{\pi}_{ik} \ , \tag{4.3.125}$$

in terms of which constraint transforms into Faraday-like law,

$$\partial_t\hat{\mathcal{B}} - \epsilon_{ik}D_{jk}\hat{\mathcal{E}}_{ij} = 0 \ . \tag{4.3.126}$$

Here $\hat{\mathcal{E}}_{ij}$ is a traceless symmetric tensor, following from the fact that $\hat{\pi}_{ij}$ is traceless symmetric. The constraint can be solved by introducing a traceless symmetric tensor gauge field

$$\hat{\mathcal{A}}_t \sim \hat{\mathcal{A}}_t + \partial_t \hat{\lambda} \ , \quad \hat{\mathcal{A}}_{ij} \sim \hat{\mathcal{A}}_{ij} + D_{ij} \hat{\lambda} \ , \tag{4.3.127}$$

and identifying $\hat{\mathcal{E}}_{ij}$, $\hat{\mathcal{B}}$ with the gauge invariant field strengths of the tensor gauge field

$$\hat{\mathcal{E}}_{ij} = \partial_t \hat{\mathcal{A}}_{ij} - D_{ij} \hat{\mathcal{A}}_t \ , \quad \hat{\mathcal{B}} = \epsilon_{ik} D_{jk} \hat{\mathcal{A}}_{ij} \ . \tag{4.3.128}$$

This then gives the dual tensor gauge theory Lagrangian

$$\mathcal{L} = \frac{1}{8\pi^2 K} \hat{\mathcal{E}}_{ij}^2 - \frac{1}{8\pi^2 \chi} \hat{\mathcal{B}}^2 - \frac{1}{2\pi} \hat{C}_t \hat{\mathcal{B}} + \frac{1}{2\pi} \epsilon_{jk} \hat{C}_{ij} \hat{\mathcal{E}}_{ik} \ , \tag{4.3.129}$$

where the higher-rank background probe field $(\hat{C}_t, \hat{C}_{ij})$ couples to the tensor gauge theory via a generalized Chern-Simons term for the tensor gauge fields. The background gauge invariance of the coupling follows from the Faraday-like law in (4.3.126).

Connections with linearized gravity

The space components of the background traceless symmetric gauge field $\hat{C}_{ij}$ introduced in the previous section can be naturally interpreted in terms of a background linearized traceless symmetric metric $\hat{H}_{ij}$ with the relation Du et al. [2022],

$$\hat{H}_{ij} = -\bar{\ell}^2 \left(\epsilon_{ik} \hat{C}_{jk} + \epsilon_{jk} \hat{C}_{ik} \right) \ , \tag{4.3.130}$$

where $\bar{\ell}$ is a constant with dimension of length. The gauge transformation on $\hat{C}_{ij}$ (4.3.122) implies transformation of the metric,

$$\hat{H}_{ij} \to \hat{H}_{ij} - \bar{\ell}^2 \left(\epsilon_{ik} \partial_j \partial_k + \epsilon_{jk} \partial_i \partial_k \right) \hat{\gamma} \ , \tag{4.3.131}$$

which can be reformulated by linearized version of the transformation of the metric under area-preserving diffeomorphism

$$\hat{H}_{ij} \to \hat{H}_{ij} - \partial_i \xi_j - \partial_j \xi_i \ , \tag{4.3.132}$$

with the definition of $\xi_i = \bar{\ell}^2 \epsilon_{ik} \partial_k \hat{\gamma}$. Similarly the traceless symmetric tensor gauge field $(\hat{\mathcal{A}}_t, \hat{\mathcal{A}}_{ij})$, can be equivalently interpreted as a dynamical metric

$$\hat{h}_{ij} = -\bar{\ell}^2 \left(\epsilon_{ik} \hat{\mathcal{A}}_{jk} + \epsilon_{jk} \hat{\mathcal{A}}_{ik} \right) \ , \tag{4.3.133}$$

in linearized gravity, which transforms as (4.3.127)

$$\hat{h}_{ij} \to \hat{h}_{ij} - \bar{\ell}^2 \left(\epsilon_{ik} \partial_j \partial_k + \epsilon_{jk} \partial_i \partial_k \right) \hat{\lambda} \ . \tag{4.3.134}$$

This establishes an area-preserving diffeomorphism, represented as $x^i \to x^i + \xi^i$, subject to the constraint $\partial_i \xi^i = 0$, with the parameter ξ^i defined as $\xi^i = \bar{\ell}^2 \epsilon^{ik} \partial_k \hat{\lambda}$. The gauge invariant field strength $\hat{\mathcal{E}}_{ij}, \hat{\mathcal{B}}$ in (4.3.128) can be written in terms of this equivalent expression $(\hat{\mathcal{A}}_t, \hat{h}_{ij})$ as

$$\bar{\ell}^2 \left(\epsilon_{ik} \hat{\mathcal{E}}_{jk} + \epsilon_{jk} \hat{\mathcal{E}}_{ik} \right) = \partial_i \hat{v}_j + \partial_j \hat{v}_i + \dot{\hat{h}}_{ij}, \ \bar{\ell}^2 \hat{\mathcal{B}} = \frac{1}{2} \hat{\mathcal{R}} = \frac{1}{2} \partial_i \partial_j \hat{h}_{ij} \ , \tag{4.3.135}$$

where the shift vector is expressed as $\hat{v}_i = \bar{\ell}^2 \epsilon_{ij} \partial_j \hat{\mathcal{A}}_t$ and $\hat{\mathcal{R}}$ as the linearized Ricci scalar. Therefore, the Lagrangian for the dual tensor gauge theory (4.3.129) can be expressed as a "Maxwell" Chern-Simons theory within the framework of linearized gravity.

CHAPTER 5

CONCLUSION AND OUTLOOK

In this thesis, we have embarked on a journey to explore the applications of field theories in quantum matter, as presented through six interconnected stories. These stories highlight the significance of theoretical techniques such as symmetries, diffeomorphisms, and noncommutativity, which are also interlinked. Through this exploration, we uncover hidden structures across diverse condensed matter systems, including (Non-)Fermi liquids and various other condensed matter phenomena.

5.0.1 Nonlinear bosonization

In Chapter 2, we introduced nonlinear bosonization as a novel formalism of Fermi surfaces. This approach reformulates Landau's kinetic theory into a field theory localized in space, leading to a multidimensional bosonized description with nonlinear corrections fixed by the geometry of the Fermi surface. The resulting local effective field theory successfully incorporates both linear and nonlinear effects of Landau's Fermi liquid theory. Notably, within this formalism, the nonlinear response with fermionic loop diagrams is mapped to tree-level vertices from the bosonic description. Such nonlinear effects are anticipated to play a crucial role in further studies of non-Fermi liquids (NFLs).

Our approach obviates the need to decompose the Fermi surface into patches, facilitating a systematic analysis of Fermi surface curvature and dispersion relation nonlinearities in NFLs. Understanding NFLs represents a significant advancement in formal quantum field theory, given their manifestation of various poorly understood phenomena, such as large emergent symmetries and UV/IR mixing. In the long term, this understanding holds potential practical importance in material science, particularly in the study of unconventional superconductors. Moreover, by incorporating Berry curvature, we deepen our understanding of the underlying physics and establish connections with phenomena such as the anomalous Hall effect. These directions are deferred for future investigation.

This method begins with a semiclassical depiction of a dense clustering of particles within their respective single-particle phase spaces, referred to as the droplet. This geometric structure represents a Fermi liquid state. It is found that the space of states can be thought of as a non-trivial representation of the Lie group of canonical transformations, placing it within effective theories determined by symmetries. The systemic foundation behind this framework is robust. At the semiclassical level, it is established that canonical transformations of a particle in 1+1 dimensions correspond to phase space area-preserving diffeomorphisms, revealing the particle conservation nature of the system.

By quantizing the remaining phase space coordinates of the geometric structure, we transition to a picture of noncommutative field theory Douglas and Nekrasov [2001], or quantum field theory on noncommutative spacetime. Conceptually, the Fermi surface emerges as a droplet embedded within noncommutative space. It is established that phase space volume-preserving diffeomorphisms correspond to unitary transformations in 1+1 dimensions. Similar principles extend to higher dimensions, with canonical transformations representing the classical manifestation of noncommutative gauge transformations, constituting only a symplectic subgroup within the full volume-preserving diffeomorphism group. In this framework, the geometric structure encapsulates the fuzziness properties that naturally arise from noncommutative geometry, where coordinates are noncommutative. Given that this Lie algebra is deformed into a noncommutative associative algebra (Moyal algebra), the deformation is described by the Moyal bracket. This noncommutative structure and its Moyal algebra highlight the UV/IR mixing phenomenon characteristic of noncommutative field theories, wherein high-energy physics influences low-energy phenomena. The generalizations of noncommutativity with the Moyal bracket also underscore the importance of $2k_F$ physics, BCS, and non-analytic corrections. These questions are deferred for future detailed investigations.

5.0.2 Noncommutativity meets volume-preserving diffemorphism

Expanding upon this foundation, Chapter 3 explores the interplay between noncommutative field theory and volume-preserving diffeomorphisms (VPDs). It is known that the semiclassical limit of the noncommutative gauge group is isomorphic to the group of volume-preserving diffeomorphisms, which finds its structure through the (Nambu-)Poisson bracket. This VPD structure and its algebra illuminate their significance in condensed matter systems, particularly in the contexts of the fractional quantum Hall effect, Wigner crystal, ferromagnets, and vortex lattices.

The exploration of area-preserving diffeomorphisms (or equivalently, volume-preserving diffeomorphisms in two spatial dimensions) and their associated algebra has led to a profound understanding of the long-wavelength limit of the Girvin-MacDonald-Platzman (GMP) algebra in quantum Hall physics. Furthermore, a fascinating correlation emerges between volume-preserving diffeomorphisms and exotic field theories. An intriguing future direction involves exploring the noncommutative nature of a broader range of condensed matter systems.

5.0.3 Exotic field theories

Chapter 4 further expands our horizons by investigating exotic quantum field theories in condensed matter systems. It is found that field theories with exotic symmetry emerge from the structure of volume-preserving diffeomorphisms. By delving deeper into the profound interrelation between VPDs and noncommutative field theory, we develop a noncommutative description of exotic field theories in two spatial dimensions. The utilization of the Nambu bracket, inherent in the VPD nature, implies that there is no noncommutative exotic field theory description applicable to spatial dimensions higher than two.

In particular, we focus on the discussion of exotic nature in nonlinear Lifshitz theories, characterized by a Lagrangian containing terms involving four spatial derivatives of the field, while the conventional two-derivative term is absent. We investigate the potential realization of nonlinear Lifshitz theories in various condensed matter systems, specifically focusing on rotating superfluids, quantum spin ice, and ferromagnets. A more comprehensive construction within the context of quantum spin ice and "deconfined" ferromagnet is deferred to future investigations.

Our study also highlights the intriguing possibility of exploring the interconnection between nonlinear Lifshitz theory and systems such as quantum Hall ferromagnets or twisted bilayer graphene. In the superconductivity mechanism of magic-angle twisted bilayer graphene, elementary Skyrmions condensing brings about a superconducting state, which spontaneously breaks the $U(1)$ symmetry. Similar phenomena occur in antiferromagnet systems, where condensing Skyrmions restore spin rotational symmetry through a deconfined quantum critical point (DQCP), leading to Landau-forbidden quantum phase transitions. Furthermore, the field theoretical structure has the potential to provide insights into understanding the spectrum and Goldstone mode counting within the framework of anomalous Hall crystal Dong et al. [2023], Soejima et al. [2024] and Skyrmion crystal. These questions are also deferred to future research.

In conclusion, this thesis comprehensively explores quantum phases of matter, emphasizing the theoretical frameworks for understanding nonlinear bosonization, noncommutativity, and exotic field theories. Our work not only advances the theoretical framework but also paves the way for potential practical applications in material science and condensed matter physics. By uncovering the rich tapestry of field theoretical structures underlying various condensed matter phenomena, we offer insights into the interplay between geometry, symmetries, and emergent phenomena in phases of quantum matter. Moving forward, these insights hold promise for unlocking new frontiers in condensed matter physics and beyond.

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