THE UNIVERSITY OF CHICAGO

BESICOVITCH SETS, DISTANCE SETS AND EMBEDDINGS INTO EUCLIDEAN SPACES WITHOUT SHRINKING

A DISSERTATION SUBMITTED TO
THE FACULTY OF THE DIVISION OF THE PHYSICAL SCIENCES
IN CANDIDACY FOR THE DEGREE OF
DOCTOR OF PHILOSOPHY

DEPARTMENT OF MATHEMATICS

BY
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CHICAGO, ILLINOIS
JUNE 2025

Dedicated to my brother Shujaat Ishaq, our *Aeda*, to Palestine, and to Kashmir.

ابھی عشق کے امتحاں اور بھی ہیں -
اقبال

TABLE OF CONTENTS

ACKNOWLEDGMENTS

First and foremost, all praise and thanks are to Allah, God, the most high and the most knowledgeable, and peace and blessings upon his Prophet ﷺ who taught us, *"[h]e who does not thank people has not truly thanked God."*

I would like to express my sincere and deepest gratitude to my advisor Marianna Csörnyei, without her unparalleled guidance, teaching and patience this thesis would not have been possible. She taught me all the mathematics needed to write this thesis, shaped my way of thinking about mathematics and also taught me mathematical communication and writing. She has been a pillar of strength and support for me since the first day of graduate school. I am blessed to be her student, for life.

I would like to thank Carlos Kenig for being the second reader of my thesis. I would also like to thank David Preiss and Raanan Schul for reading my work and supporting me in my last year of graduate school.

I would like to thank Bobby Wilson for hosting me at the University of Washington in the summer of 2022 and for his continued collaboration and teaching.

I would like to thank Ryan Bushling and Kornélia Héra for their generous collaboration. I would like to thank Donald Stull, Chris Wilson and all the participants of the Geometric measure theory seminar over the past six years for their valuable talks, discussions and insights.

I would like to thank Amin Sofi and the entire department of mathematics at Kashmir University for inviting me to their department many times and for their work in promoting math education across Kashmir.

I am also extremely grateful to all the mathematics teachers I have had in my life including Parasar Mohanty, who introduced me to the beauty of analysis as an undergraduate. My first mathematics teacher was my mother, Nasreena Jan. Her passion, sincerity and love for both learning and teaching is infectious. I owe everything in life to her, my first teacher in mathematics and life.

My second teacher in mathematics and life was my brother, Shujaat Ishaq. He was also my friend and confidant. He was a beautiful human being, dedicated to academic excellence, perseverance and also had a great sense of humour. He was an inspiration for me since childhood. His memory is a gift and will continue to be a source of strength for me.

My warmest thanks go to family, to my grandfather Ghulam Mustafa Latoo for his honesty and steadfastness, and the positive impact he has had on me and everyone else around him, to my father, Altaf Hussain and my brother, Asif Bashir, for providing me with the best possible education, to my siblings Shazia Malik, Sabahat Altaf, Aasima Ishaq, Wajid Ishaq and Irtiqa Altaf, and my sister-in-law Fozia Ishaq for their selfless love, support and encouragement. Irtiqa aka Chiku, also loves math and we have many other shared interests as well. Talking to her is the best part of my day. I am also grateful to my entire extended family, my Uncles, Aunts and cousins for their prayers and care.

I have also been blessed with beautiful friendships from primary school to UChicago who have supported me during the ups and downs of graduate school. I would like to offer profound thanks to Deqa Aden, Farah Akhtar, Alaa Alaghbari, Nada Ali, Arwa Awan, Asma Ayoub, Zeeshan Bandey, Xhesika Bardhi, Umar Bashir, Abdul Basit, Adiba Chowdhary, Huma Dar, Pratiti Deb, Ahmed Fouad, Samra Jamil, Mohib Khan, Sherif Mabrouk, Izza Malik, Shafiya Malik, Sumeed Manzoor, Kiran Nizamani, Fatima Omar, Sankalp Patel, Anisa Rashid, Isha Abibatu Sesay, Sameena Shabir, Sahila Shafi, Mufti Taha Shah and Iffat Siddiqi.

I am like to offer my sincere thanks to Irfan Malik and Zahoor Wani, and their families for welcoming me in their homes and extending their support and kindness. Their dedication to serving the Kashmiri community is inspiring.

I would like to thank Suhail Ahmed Malik for his kindness and support during the initial stages of my academic journey outside Kashmir.

I am thankful to my cat, Zooni, who is a source of incredible joy and bliss in my life. She is a beautiful creature and I deeply love her, and her company. She is often the first one, to listen to my mathematical ideas. I am also thankful to our cat friend, Kiwi.

I would like to offer my thanks and appreciation to Tarik Aougab, Denis Hirschfeldt and Ila Verma for their efforts to shift the global mathematics community towards justice, and making a path for others to do the same.

Stephen Jay Gould famously said "I am, somehow, less interested in the weight and convolutions of Einstein's brain than in the near certainty that people of equal talent have lived and died in cotton fields and sweatshops." We live in a deeply unequal world, where access to knowledge and the freedom to pursue it, is unjustly limited to very few. Therefore, I acknowledge the deep and utmost privilege of being able to pursue mathematical research while having the financial security and a safe environment to do so.

ABSTRACT

In this thesis, we study problems related to scaled oscillation, Besicovitch sets, Falconer distance problem and Sard's theorem.

In chapter 2 (joint work with Marianna Csörnyei and Bobby Wilson), we study the size and regularity properties of level sets of continuous functions with bounded lower-scaled oscillation.

In chapter 3 (joint work with Marianna Csörnyei and Kornélia Héra), we consider the Hausdorff dimension of planar Besicovitch sets for rectifiable sets Γ, i.e. sets that contain a rotated copy of Γ in each direction. We show that for a large class of Cantor sets C and Cantor-graphs Γ built on C, the Hausdorff dimension of any Γ-Besicovitch set must be at least $\min(2 - s^2, 1/s)$ where $s = \dim_{\mathrm{H}} C$.

In chapter 4, we construct a non-trivial 1-rectifiable set Γ in the plane, for which there exists a 1-dimensional Γ-Besicovitch set.

In chapter 5 (joint work with Ryan Bushling and Bobby Wilson), we use techniques of algorithmic complexity theory to study distance sets. We prove that for every norm on $\mathbb{R}^d$ and every $E \subseteq \mathbb{R}^d$, the Hausdorff dimension of the distance set of E with respect to that norm is at least $\dim_{\mathrm{H}} E - (d-1)$. An explicit construction follows, demonstrating that this bound is sharp for every polyhedral norm on $\mathbb{R}^d$.

In chapter 6 (joint work with Ryan Bushling and Bobby Wilson), we prove that, for every polyhedral or C^1 norm on $\mathbb{R}^d$ and every set $E \subseteq \mathbb{R}^d$ of packing dimension s, the packing dimension of the distance set of E with respect to that norm is at least s/d; likewise for upper box dimension.

In chapter 7 (joint work with Marianna Csörnyei) we study which spaces (X, ρ) can be embedded into $\mathbb{R}^d$ without decreasing any of the distances in X. This work is motivated by the following generalization of the classical Sard theorem in the plane. Let f be a function defined on a subset $A \subset \mathbb{R}^2$. If f has modulus of continuity $\omega(r) \lesssim r^2$, then $f(A) \subset \mathbb{R}$ has

Lebesgue measure zero. Choquet claimed in [18] that this was a full characterization, i.e. for every ω for which $\omega(r)/r^2$ converges to ∞ as $r \to 0$, there is a counterexample. We disprove this by showing that the correct characterization, in $\mathbb{R}^d$, is $\int_0^1 \omega(r)^{-1/d} = \infty$.

CHAPTER 1

INTRODUCTION

1.1 Scaled Oscillation

Lipschitz functions hold central importance in mathematical analysis. Their differentiability properties are widely studied. A very well-known result called Rademacher's theorem says that a Lipschitz function $f : \mathbb{R}^n \to \mathbb{R}^m$ is differentiable almost everywhere. One of the generalizations of Rademacher's theorem was done by Cheeger [17] who introduced a notion of derivative in the setting of metric spaces and generalized Rademacher's theorem to doubling metric spaces. Keith [43], [44] introduced a scaled oscillation, a generalization of the Lipschitz property, to give sufficient conditions in a metric space X for which the Cheeger-Rademacher theorem of differentiability holds. For a metric space (X, d) and a continuous function $f : X \to \mathbb{R}$, he defined the upper and lower scaled oscillation functions of f at x as

$$L_f(x) = \limsup_{r\to 0} \frac{\sup_{d(x,y)\le r} |f(y) - f(x)|}{r} \qquad \text{and}$$
$$l_f(x) = \liminf_{r\to 0} \frac{\sup_{d(x,y)\le r} |f(y) - f(x)|}{r}$$

respectively.

Another generalization of Rademacher's theorem is Stepanov's theorem which states that for a function on $\mathbb{R}^n$, if $L_f(x) < \infty$ for a.e $x \in \mathbb{R}^n$ then the function is differentiable almost everywhere. Balogh and Csörnyei in [9] proved that upper-scaled oscillation cannot be replaced by lower-scaled oscillation. They constructed a nowhere differentiable continuous function $f : [0, 1] \to \mathbb{R}$ such that $l_f(x) = 0$ for a.e $x \in [0, 1]$. However, they were able to show that if the $l_f(x) < \infty$ for all x except countably many points of $[0, 1]$, then f is differentiable on a set of positive measure, though not necessarily differentiable almost everywhere. Csörnyei and Wilson in [24] observed that the graph of such a function has σ-

finite $\mathcal{H}^1$-measure, and they also showed that if a closed, continuous simple curve in $\mathbb{R}^n$ has σ-finite $\mathcal{H}^1$-measure then C has a tangent on a set of positive measure. As a consequence, they extended the earlier result of Balogh and Csörnyei to higher dimensions, by showing that any function $f : [0,1]^n \to \mathbb{R}$ with $l_f(x) < \infty$ for all x except countably many points, is differentiable on a set of positive measure.

If the graph of a continuous function $f : [0,1] \to \mathbb{R}$ has σ-finite $\mathcal{H}^1$-measure then it easily follows that almost all level sets are countable. Moreover, if f is Lipschitz or $L_f(x) < \infty$ everywhere then almost all level sets are finite. In chapter 2 (joint work with Csörnyei and Wilson) we show that for functions with finite lower-scaled oscillation, all level sets can be infinite. Our result is the following.

Theorem 1. *There exists a continuous function $f : [0,1] \to [0,1]$ such that $l_f(x) < \infty$ for all $x \in [0,1]$ and $f^{-1}(y)$ is infinite for every $y \in [0,1]$.*

For $n \geq 2$, there exist functions $f : [0,1]^n \to \mathbb{R}$ with finite $L_f(x) < \infty$ with all level sets having infinite $\mathcal{H}^{n-1}$-measure. However the almost everywhere differentiability property implies that almost all level sets are rectifiable. We also show that this fact isn't true in case of functions with finite lower-scaled oscillation.

Theorem 2. *For $m \geq 2$ there exists a continuous function $f : [0,1]^m \to [0,1]$ such that $l_f(x) < \infty$ for all $x \in [0,1]^m$, and $f^{-1}(y)$ is not $(m-1)$-rectifiable for any $y \in [0,1]$.*

1.2 Γ-Besicovitch sets

A Besicovitch set or a Kakeya set in $\mathbb{R}^n$ is a set of measure zero which contains a unit line segment in every direction. In [11], Besicovitch proved the existence of such sets for $n \geq 2$. The famous Kakeya conjecture asserts that Besicovitch sets cannot be much smaller than a set of measure zero.

Conjecture 3 (Kakeya conjecture)**.** *Every Besicovitch set in $\mathbb{R}^n$ has Hausdorff dimension n.*

The conjecture is true in dimension 2 and 3 and remains an important open problem for higher dimensions. The Kakeya conjecture has connections to many areas including harmonic analysis, incidence geometry and additive combinatorics. For example: one important conjecture in Fourier analysis, the restriction conjecture implies the Kakeya conjecture.

It is an interesting question to ask what happens if the line segment is replaced by other rectifiable sets? Csörnyei and Chang in [16] show that for any rectifiable planar set Γ, there exists a measure zero set which contains a rotated copy of a full $\mathcal{H}^1$-measure subset of Γ in every direction. This raises the question what the Hausdorff dimension of such sets can be.

For a rectifiable set Γ, a Γ-Besicovitch set is a set that contains a rotated copy of Γ in every direction. In a joint work with Héra and Csörnyei [7], we considered the Hausdorff dimension of planar Besicovitch sets for a large class of rectifiable sets which are graphs over self-similar sets. These graphs resemble the graph of the devil staircase function over the middle third Cantor set. For details see chapter 3.

We prove that for such a graph Γ over an s-dimensional Cantor set C, every Γ-Besicovitch set E has

$$\dim_{\mathrm{H}} E \geq \min(2 - s^2, \frac{1}{s}).$$

In [7], we only got a lower bound on the dimension of such Γ-Besicovitch sets. It left open the problem, whether there exist a non-trivial Γ-Besicovitch set of dimension less than 2? There are trivial examples of rectifiable sets for which the corresponding Γ-Besicovitch sets have dimension less than 2. For example, for a circular arc the complete circle containing it is its Besicovitch set. More interestingly, it has also been mentioned in [16] that if Γ is an arbitrary countable union of circles, then there exists a Γ-Besicovitch set of Hausdorff dimension 1.

Every 1-rectifiable set in the plane has a tangent field which is defined almost everywhere.

We want to understand the standard Besicovitch set with lines. The construction of measure zero sets by Csörnyei and Chang in [16] depended upon the tangent field of Γ. For a straight line, all the tangent lines are parallel. This suggested that it is interesting to look at the Γ-Besicovitch sets for rectifiable sets with the same tangent field as that of a line i.e. rectifiable sets whose tangents are parallel almost everywhere. Marianna Csörnyei asked the question, does there exist a 1-rectifiable set with parallel tangents almost everywhere for which there exists a Γ-Besicovitch set of dimension less than 2? In chapter 4, we answer this question affirmatively.

Theorem 4. *There exists a 1-rectifiable set Γ with $\mathcal{H}^1(\Gamma) > 0$ and parallel tangents almost everywhere, for which there is a 1-dimensional Γ-Besicovitch set.*

We construct the set Γ as the graph of a monotone function whose domain is a Cantor set of measure 0 and thus the tangents of Γ are vertical almost everywhere.

1.3 Distance Sets and the Complexity Methods

Jack Lutz and Neil Lutz in [50] gave an effective characterization of the Hausdorff and packing dimensions of a set in terms of the constructs from computability theory which they call the point-to-set principle. It has been the key to building the bridges between the theory of computing and geometric measure theory. It has enabled the theory of computing to not only answer open questions in geometric measure theory but also give short and elegant proofs to some well-known classical results. For example, Lutz and Stull used the complexity methods to give better bounds than the (then) known bounds on the dimension of Fürstenberg sets, see [54]. In [71], Stull improved the bound on the dimension of pinned distance sets for small values of $\dim_{\mathrm{H}}(E)$. For more such applications of complexity theory, see [53], [49], [52] and [51].

The *Kolmogorov complexity of a binary string σ* denoted by $K(\sigma)$ is the length of the shortest program that gives output σ. For $x \in \mathbb{R}^n$ and $r \in \mathbb{N}$, the *Kolmogorov complexity of*

x *at precision* r denoted by $K_r(x)$ is the Kolmogorov complexity of the string of length nr which is obtained by taking the first r digits of the binary expansion of each coordinate of x.

The *effective Hausdorff dimension* and the *effective packing dimension* of a point $x \in \mathbb{R}^n$ are defined by

$$\dim_{EH}(x) := \liminf_{r\to\infty} \frac{K_r(x)}{r} \text{ and } \dim_{EP}(x) := \limsup_{r\to\infty} \frac{K_r(x)}{r},$$

respectively. We note that these definitions are different than the usual Hausdorff and packing dimensions, both of which are zero for a point. However, the concepts of complexity and dimension defined above can be relativized to an oracle A. The *Kolmogorov complexity of a binary string* σ *relative to the oracle* A denoted by $K^A(\sigma)$ is the shortest program having access to A that gives output σ. Similarly, the concepts of complexity and dimension defined above can be *relativized to an oracle* A and are denoted by $K^A(\sigma), \dim^A(x)$, for details see [50]. The point-to-set principle by Jack Lutz and Neil Lutz [50] uses the (relative) dimension of points in the set.

Theorem 5 (Point-to-Set Principle)**.** *For every set* $E \subseteq \mathbb{R}^n$*,*

$$\dim_{\mathrm{H}} E = \min_{A\subseteq\mathbb{N}} \sup_{x\in E} \dim^A_{EH}(x) \quad \textit{and} \quad \dim_{\mathrm{P}} E = \min_{A\subseteq\mathbb{N}} \sup_{x\in E} \dim^A_{EP}(x).$$

Given a set $E \subset \mathbb{R}^n$, the distance set of E is

$$\Delta(E) := \{\|x-y\| : \ x, y \in E\},$$

where $\|\cdot\|$ is the Euclidean norm. The *Falconer distance problem* is to find a lower bound on $\dim \Delta(E)$, the Hausdorff dimension of the distance set, in terms of the Hausdorff dimension of E.

Conjecture 6 (Falconer's distance conjecture)**.** *For any Borel set $E \subset \mathbb{R}^n$ with $\dim_{\mathrm{H}}(E) > n/2$, $\Delta(E)$ has positive measure.*

The conjecture remains open for all $n \geq 2$. The current best-known bound is due to Guth, Iosevich, Ou and Wang [37], who proved that if a planar set has $\dim_{\mathrm{H}} E > 5/4$ then $\Delta(E)$ has positive measure.

An interesting variant of the Falconer distance problem is to find dimension bounds on distance sets relative to a different norm $\|\cdot\|$. Falconer [28] proved that for any set $E \subseteq \mathbb{R}^n$, the Hausdorff dimension of $\Delta(E)$ is at least $\dim_{\mathrm{H}} E - (n-1)$. The proof is simple and easily extends to all norms. Moreover it also generalizes to the pinned distance set which is defined as $\Delta^x(E) := \{\|x-y\| \in [0,\infty) : \ y \in E\}$. In chapter 5 (joint work with Ryan Bushling and Bobby Wilson) we used the theory of algorithmic complexity in [6] to give an alternate proof of this theorem. Using the same methods, we proved that it is sharp for polyhedral norms.

Theorem 7. *Let $\|\cdot\|_P$ be a polyhedral norm on $\mathbb{R}^n$. For any $s \in [n-1, n]$, there exist a compact set $E \subset \mathbb{R}^n$ with $\dim_{\mathrm{H}} E = s$ such that*

$$\dim_{\mathrm{H}} \Delta(E) = s - (n-1).$$

In chapter 6 (joint work with the same co-authors) we replace the Hausdorff dimension with the packing dimension and prove the following.

Theorem 8. *Let $\|\cdot\|_*$ be either a C^1 or polyhedral norm on $\mathbb{R}^d$ and let $E \subseteq \mathbb{R}^d$. For every $\epsilon > 0$, there exists $x \in E$ such that*

$$\dim_{\mathrm{P}} \Delta_x^*(E) \geq \frac{1}{d} \dim_{\mathrm{P}} E - \epsilon. \tag{1.1}$$

Just as Theorem 7 was sharp for every polyhedral norm in a strong way, (1.1) is also sharp for polyhedral norms in that same manner.

Theorem 9. *Let $\|\cdot\|_P$ be a polyhedral norm such that the normal vectors to the faces of the polyhedron are rationally dependent and let $s \in [0, d]$. There exists a compact set $E \subset \mathbb{R}^d$ with $\dim_{\mathrm{P}} E = s$ such that*

$$\dim_{\mathrm{P}} \Delta^P(E) = \frac{s}{d}.$$

1.4 On the modulus of continuity of functions whose image has positive measure, and metric embeddings into $\mathbb{R}^d$ without shrinking

The classical Morse-Sard Theorem says that if a function $f : \mathbb{R}^n \to \mathbb{R}^m$ is C^k-differentiable with $k \geq \max\{n - m + 1, 1\}$, then f maps the set of critical points to a set of Lebesgue measure 0. A generalization of this theorem in the case $n = 2$ and $m = 1$ is the following. Let $f \in C^{1,1}(\mathbb{R}^2, \mathbb{R})$, and let A be the critical set of f, that is, the set of points at which the gradient of f is zero. Then the restriction of f onto A has modulus of continuity $\omega(r) \lesssim r^2$. Using Lebesgue density theorem we can easily show that, for every $d \geq 2$ and for every $A \subset \mathbb{R}^d$, if a function $f\colon A \to \mathbb{R}$ has modulus of continuity $\omega(r) \lesssim r^d$ then $f(A)$ is null.

Choquet showed the following statement (see page 53 in [18]). If $\omega(r)/r^2$ converges to infinity as $r \to 0$ then there exist a set $A \subset \mathbb{R}^2$ and a function $f\colon A \to \mathbb{R}^2$ with modulus of continuity ω s.t. $|f(A)| > 0$. It should be noted that this is not the main result from the paper rather it was a quick remark. We show that Choquet's statement is not true and obtain the following result.

Theorem 10. *Assuming that $\omega(r)$ is convex and $\omega(r)/r^n$ is decreasing, the following are equivalent:*

1. *For every $A \subset \mathbb{R}^n$ and $f\colon A \to \mathbb{R}$ with modulus of continuity ω, $|f(A)| = 0$.*

2. *$\int_0^1 \omega(r)^{-1/n}\, dr = \infty$.*

We obtain this as a special case of a more general result. We study which spaces (X, ρ) can be embedded into $\mathbb{R}^d$ without decreasing any of the distances in X. That is, we ask the question whether there is an $f : X \to \mathbb{R}^d$ such that $\|f(x) - f(y)\| \geq \rho(x, y)$ for every $x, y \in X$. We study this problem for some very general distance functions ρ (we do not even assume that it is a metric space, in particular, we do not assume that ρ satisfies the triangle inequality), and find quantitative necessary and sufficient conditions under which such a mapping exists. We will obtain the characterization mentioned above as a special case of our metric embedding results, by choosing X to be an interval in $\mathbb{R}$, and defining ρ by putting $\rho(x, y) = r$ if $\|x - y\| = \omega(r)$.

CHAPTER 2

SCALED OSCILLATION AND LEVEL SETS

This chapter is joint work with Marianna Csörnyei and Bobby Wilson and originally appeared in Proceedings of the American Mathematical Society, 150(07), 2913-2924, 2022.

2.1 Introduction

Let $f : A \to \mathbb{R}^k$, $A \subset [0,1]^m$, be a Lipschitz continuous function. According to Rademacher's theorem, a Lipschitz function is differentiable almost everywhere in its domain. In fact, Stepanov's theorem implies that it suffices to assume that the upper-scaled oscillation (defined below) of a function is finite almost everywhere in order to conclude that it is differentiable almost everywhere. Assuming $m \geq k$, another well-known regularity property of a Lipschitz function, $f : A \to \mathbb{R}^k$, is that for Lebesgue almost every $z \in \mathbb{R}^k$, the level set $f^{-1}(z)$ is countably $(m-k)$-rectifiable and $\mathcal{H}^{m-k}(f^{-1}(z)) < \infty$. In fact, it suffices to prescribe that the graph of f has finite $\mathcal{H}^{m-k}$ measure then almost all of its level sets have finite $\mathcal{H}^{m-k}$ measure. For the more regular class of C^n functions one has the stronger theorem, known as Sard's theorem, which states that if $f : \mathbb{R}^m \to \mathbb{R}^k$ is n-times continuously differentiable for $n \geq \max\{m-k+1, 1\}$ and

$$S = \{x \in \mathbb{R}^m : \operatorname{rank}(\nabla f(x)) < k\},$$

then $f(S)$ has Lebesgue measure zero.

In order to demonstrate necessary and sufficient conditions on the vector field of the continuity equation that would ensure uniqueness of solutions, Alberti, Bianchini, and Crippa [3] define a notion of a "weak Sard" property. Specifically, they showed in [3] that if the vector field, $b : [0,T] \times \mathbb{R}^2 \to \mathbb{R}^2$, is divergence-free and bounded then the following Cauchy problem

$$\begin{cases} \partial_t u = -\operatorname{div}(bu) \\ u(0) = u_0 \in W^{1,\infty}(\mathbb{R}^2) \end{cases}$$

has a unique weak solution in $C([0,T]; L^\infty)$ if and only if the potential of b, f (defined by satisfying $b = \nabla^\perp f$), satisfies the weak Sard property. The function f is said to satisfy the weak Sard property if any measure μ, supported on $f(S \cap E)$, is mutually singular to the Lebesgue measure on $\mathbb{R}^k$, where E is "the union of all connected components with positive length of all level sets." Moreover, in [2], Alberti, et. al. show that the minimal regularity for the weak Sard property to hold is $f \in W^{2,1}(\mathbb{R}^2)$. Initially, Alberti [1], taking into account that $W^{2,1}$ functions are not necessarily differentiable, proved that the size of the image of $S \cup N$, where

$$N = \{x \in \mathbb{R}^2 : \nabla f \text{ does not exist}\},$$

is Lebesgue negligible. Further analysis discussed in [2] and [3] demonstrates that more regularity for f is necessary to achieve equivalent results for the higher-dimensional case, $f : \mathbb{R}^m \to \mathbb{R}^{m-1}$. Specifically, the authors show that one needs $f \in C^{1,1/2}$. In this paper, we consider the characterization of $f(S \cup N)$ and the characterization of the level sets of f when f satisfies the following "local Lipschitz" conditions. For $x \in A$, the upper-scaled and lower-scaled oscillation of f are defined as

$$L_f(x) = \limsup_{r \to 0} \frac{\sup_{d(x,y) \le r} |f(y) - f(x)|}{r} \qquad \text{and}$$
$$l_f(x) = \liminf_{r \to 0} \frac{\sup_{d(x,y) \le r} |f(y) - f(x)|}{r}$$

respectively. The differentiability properties of functions satisfying either $L_f(x) < \infty$ or

$l_f(x) < \infty$ on some suitably sized subset of the domain of f were studied initially in [9] and then further studied in [24]. In [9], the authors show that L^p boundedness of the function, l_f, implies Sobolev regularity for the function, f. Moreover, one simply needs the boundedness condition, $l_f(x) < \infty$, in order to prove the existence of a set of positive measure on which the function, $f : [0,1] \to \mathbb{R}$, is differentiable. In [24], among other results, the authors show that for higher-dimensional domains, $f : [0,1]^m \to \mathbb{R}$, $l_f(x) < \infty$ implies the existence of positive measure sets of differentiability for f.

One can ask to what extent the weak Sard property fails for functions with such low regularity. In consideration of this question, we construct continuous functions with finite lower-scaled oscillation at each x whose level sets are "large". First, we present our result in dimension 1.

Theorem 11. *There exists a continuous function $f : [0,1] \to [0,1]$ such that $l_f(x) < \infty$ for all $x \in [0,1]$ and $f^{-1}(y)$ is infinite for every $y \in [0,1]$.*

It is clear that for every $y \in [0,1]$ in the above construction, $f^{-1}(y) \cap (N \cup S) \neq \emptyset$. One should also note that, due to the argument in [24] demonstrating regularity of such functions, countably infinite is the most that should be expected for almost every level set. For the higher dimensional case, we obtain the following result:

Theorem 12. *For $m \geq 2$ there exists a continuous function $f : [0,1]^m \to [0,1]$ such that $l_f(x) < \infty$ for all $x \in [0,1]^m$, and $f^{-1}(y)$ is not $(m-1)$-rectifiable for every $y \in [0,1]$.*

It is important to emphasize that the level sets in the above construction are not purely unrectifiable although they are unrectifiable. In fact, the existence of positive measure subsets on which such functions are differentiable, [24], implies that generic level sets will have rectifiable parts.

Trivially, if we take the construction, $f : [0,1]^m \to [0,1]$, from Theorem 11 or Theorem 12 and construct the function, $g : [0,1]^{\ell+m} \to [0,1]^{\ell+1}$, defined by $g(z,x) := (z, f(x))$ for $x \in [0,1]^m$ and $z \in [0,1]^\ell$, we have the following corollary:

Corollary 13. *1. For $m = k \geq 1$, there exists a continuous function $f : [0,1]^m \to [0,1]^k$ such that $l_f(x) < \infty$ for all $x \in [0,1]^m$, and $f^{-1}(y)$ is infinite for all $y \in [0,1]^k$.*

2. For $m > k \geq 1$, there exists a continuous function $f : [0,1]^m \to [0,1]^k$ such that $l_f(x) < \infty$ for all $x \in [0,1]^m$, and $f^{-1}(y)$ is not $(m-k)$-rectifiable for all $y \in [0,1]^k$.

Regarding functions with $L_f(x) < \infty$ for every $x \in \mathbb{R}^m$, one can divide $\mathbb{R}^m$ into countably many parts, on each of which the function is L-Lipschitz for some L. Thus almost all level sets of such functions are rectifiable and have σ-finite $\mathcal{H}^{m-1}$ measure. However, there is a discrepancy between the characterization of level sets of functions with finite upper-scaled oscillation defined on $\mathbb{R}$, and in higher dimension. A simple construction (Remark 18) demonstrates that the level sets of functions, $f : [0,1]^m \to \mathbb{R}$, satisfying $L_f(x) < \infty$ for all $x \in [0,1]^m$ may have infinite $\mathcal{H}^{m-1}$ measure, while almost every level set must be finite when $m = 1$.

The structure of this chapter is as follows: we begin with a discussion of the preliminary tools necessary to analyze functions with finite upper/lower scaled oscillation. In this preliminary section, we will also discuss our simple results concerning functions with finite upper-scaled oscillations. In the main part of our paper we will construct functions with finite lower-scaled oscillation that will prove Theorems 11 and 12.

2.2 Preliminaries

Throughout this chapter, we denote by $B(x,r) \subset \mathbb{R}^m$, the open ball of radius r and center x. The Lebesgue measure on $\mathbb{R}^m$ is denoted by $\mathcal{L}^m$, and the s-dimensional Hausdorff measure is denoted by $\mathcal{H}^s$. Recall that the density of A in the ball $B(x,r)$ is defined by

$$D^s(A,x,r) = \frac{\mathcal{H}^s(A \cap B(x,r))}{(2r)^s},$$

and the s-density of A at the point x is defined by

$$D^s(A,x) = \lim_{r\to 0} D^s(A,x,r)$$

if the limit exists.

A set $A \subset \mathbb{R}^m$ is called s-rectifiable if there exist $f_i : \mathbb{R}^s \to \mathbb{R}^m, i \in \mathbb{N}$, such that

$$\mathcal{H}^s\big(A \setminus \bigcup_{i=1}^{\infty} f_i(\mathbb{R}^s)\big) = 0.$$

An unrectifiable set is a set which is not rectifiable. One characterization of rectifiability of sets is the density theorem of Marstrand and Mattila:

Theorem 14 (Theorem 16.2 from [57])**.** *Let A be an $\mathcal{H}^s$ measurable subset of $\mathbb{R}^m$ such that $\mathcal{H}^s(A) < \infty$. Then A is s-rectifiable if and only if $D^s(A,x) = 1$ for $\mathcal{H}^s$ almost all $x \in A$.*

Let $f : \mathbb{R}^m \to \mathbb{R}$ be a function and let E be a subset of $\mathbb{R}^m$. We denote the graph of f on E by $G_f(E)$.

Lemma 15. *Let $f : [0,1]^m \to \mathbb{R}$ be a continuous function such that $l_f(x) < \infty$, for all $x \in [0,1]$. Then*
(i) $G_f([0,1]^m)$ has σ-finite $\mathcal{H}^m$-measure.
(ii) $\mathcal{H}^m\big(G_f(H)\big) = 0$ for every set $H \subset [0,1]^m$ with $\mathcal{H}^m(H) = 0$.

We refer to [24] for the proof of this statement.

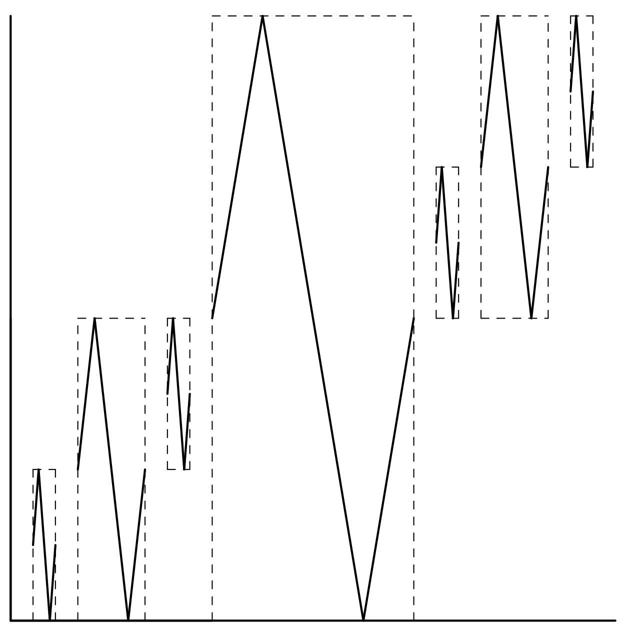
As discussed in the introduction, it follows from part (i) of Lemma 15 that if $f : [0,1]^m \to \mathbb{R}$ is a function such that $l_f(x) < \infty$ for all $x \in [0,1]^m$ then almost all level sets of f have σ-finite $\mathcal{H}^{m-1}$ measure. For $m = 1$, part (ii) of Lemma 15 implies that f satisfies the Lusin's condition. This leads to the following result.

Theorem 16. *Let $f : [0,1] \to \mathbb{R}$ be a continuous function such that $L_f(x) < \infty$ for all except countably many $x \in [0,1]$. Then $f^{-1}(y)$ is finite for almost every $y \in \mathbb{R}$.*

This statement follows by first observing that if a level set were to have infinitely many points then it has at least one accumulation point. At this point the function either has a zero derivative or the derivative does not exist. Since the function is differentiable almost everywhere, Lusin's condition implies that the image of the set of points at which the function is not differentiable is measure zero. Furthermore, a standard argument implies that the set of points at which the derivative is zero also has null image.

However, if we replace the countable exceptional set in Theorem 16 with other exceptional sets, we cannot conclude that almost all level sets are finite. In fact, as described in the following remark, countability of the exceptional set is necessary.

Remark 17. Let E be an uncountable Borel set. There exists a continuous function $f : [0,1] \to [0,1]$ such that $L_f(x) < \infty$ for all $x \in [0,1] \setminus E$ and the level set $f^{-1}(y)$ has infinite cardinality for each $y \in [0,1]$.



Construction of the function when E is the middle third cantor set.

In order to prove the remark, we first consider the case where E is the middle-third Cantor set and consider the Cantor function over that set. For each of the 2^{n-1} intervals of size 3^{-n} in the complement of the Cantor set we replace the flat piece of the function defined on this interval with a piecewise linear function whose image is an interval of length $2^{-(n-1)}$ so that the intervals of this size cover the interval $[0,1]$. One can do this in a continuous way.

For an arbitrary Cantor set E, we can find a homeomorphism $h : \mathbb{R} \to \mathbb{R}$ that maps E to the middle-third Cantor set, and which is linear on each component of the complement of E, and then compose this homeomorphism h with the function f we have constructed for the middle-third Cantor set. Finally, the statement of Remark 17 follows since every uncountable Borel set contains a Cantor set.

As discussed in the introduction, the characteristics of level sets of functions satisfying $L_f(x) < \infty$ is heavily determined by the dimension of the domain. Although $L_f(x) < \infty$ for every x guarantees that almost every level set will have finite cardinality in the one-dimensional case, in the two-dimensional case and higher, there is no such phenomenon as displayed in the following example.

Remark 18. There exists a continuous function $f : [0,1]^m \to [0,1]$ such that $L_f(x) < \infty$ for all $x \in [0,1]^m$ and the level sets $f^{-1}(y)$ have infinite $\mathcal{H}^{m-1}$-measure for all $y \in [0,1]$.

Proof of Remark 18. It suffices to construct an example for $m = 2$. Consider the function $g : [0,1] \to [0,1]$ defined by

$$g(x) = \begin{cases} x \sin^2(x^{-1}) & x \in (0,1] \\ 0 & x = 0. \end{cases}$$

It is clear that $L_g(x) < \infty$ for all $x \in [0,1]$ and the graph of g has infinite $\mathcal{H}^1$ measure. Now define

$$f(x,y)=\begin{cases}1 & y-g(x)\geq 1\\ 0 & y-g(x)\leq 0\\ y-g(x) & \text{otherwise.}\end{cases}$$

Then for every $z\in[0,1]$, $f^{-1}(z)=\{(x,y)\in[0,1]\ :\ y=g(x)-z\}$ so the level sets have infinite length. Finally,

$$f(x_1,y_1)-f(x_2,y_2)=y_1-y_2-(g(x_1)-g(x_2))$$

and thus $L_f(x,y)\leq 2\max(1,L_g(x))<\infty$ for all (x,y). □

2.3 Lower-Scaled Oscillation

In this section, we present the proofs to Theorems 11 and 12.

Proof of Theorem 11. Our construction of the function is based on an iterative argument on rectangles. Assume we have a collection of rectangles with sides parallel to the coordinate axes. We will divide each rectangle into two parts vertically. We will define the function piece-wise linearly on the right rectangle. We will then divide the diagonal of the left rectangle into countably many line segments and consider the rectangles with these line segments as diagonals and sides parallel to the coordinate axes. These will form a new collection of rectangles. We will explain the details below.

Let $\mathbb{P}_x$ and $\mathbb{P}_y$ be the projection maps onto the x-axis and y-axis respectively. For a rectangle Q with sides parallel to the coordinate axes, we denote by $l(Q)$ and $h(Q)$ the length of its horizontal and vertical sides, respectively. We start with the square $Q_{0,1}:=[0,1]^2$ and the function $f_0(x)=x$.

Assume that we have completed $n\in\mathbb{N}$ steps of the construction, and we have (at most) countably many rectangles $(Q_{n,i})_{i\in\mathbb{N}}$ and a function f_n, such that inside each $Q_{n,i}$ the graph of f_n is the linear line segment connecting its bottom left and its top right corners. We will define f_{n+1} by replacing this line segment inside each $Q_{n,i}$ by another continuous graph. Outside $\bigcup_{i=1}^{\infty}\mathbb{P}_x(Q_{n,i})$ we define $f_{n+1} = f_n$.

Consider one of the rectangles $Q = Q_{n,i}$, and denote $Q = [x, x + l(Q)] \times [y, y + h(Q)]$. We divide Q vertically into two rectangles,

$$Q' = [x, x + (1 - a_n)l(Q)] \times [y, y + h(Q)]$$

and

$$Q'' = [x + (1 - a_n)l(Q), x + l(Q)] \times [y, y + h(Q)],$$

where $0 < a_n \ll 1$ is to be chosen later. In Q'', we define f_{n+1} to be an arbitrary piecewise linear function connecting its top left and top right corners, such that the graph of the function lies inside Q'', and it has a horizontal piece lying on the top and on the bottom side of Q'', respectively. Therefore the level sets $f_{n+1}^{-1}(y)$ and $f_{n+1}^{-1}(y + h(Q))$ are infinite, and the pre-image of each point in the open interval $(y, y + h(Q))$ has at least two points.

In Q', we define f_{n+1} such that its graph is the diagonal of Q' connecting its bottom left and top right corners. Then we apply the 1-dimensional Whitney decomposition theorem (see e.g, J.1 in Grafakos, [36]) to divide this diagonal into countably many line segments, such that the length of each of these line segment is comparable to its distance from the closest endpoint of the diagonal of Q'.

Corresponding to each line segment, we consider a rectangle with the line segment as its diagonal and sides parallel to the coordinate axes. We call these rectangles the child rectangles of $Q_{n,i}$.

We repeat this construction for each $Q_{n,i}$, $i \in \mathbb{N}$ and enumerate the countably many rectangles thus obtained as $\{Q_{n+1,j}\}_{j\in\mathbb{N}}$. This completes the $(n+1)th$ step.

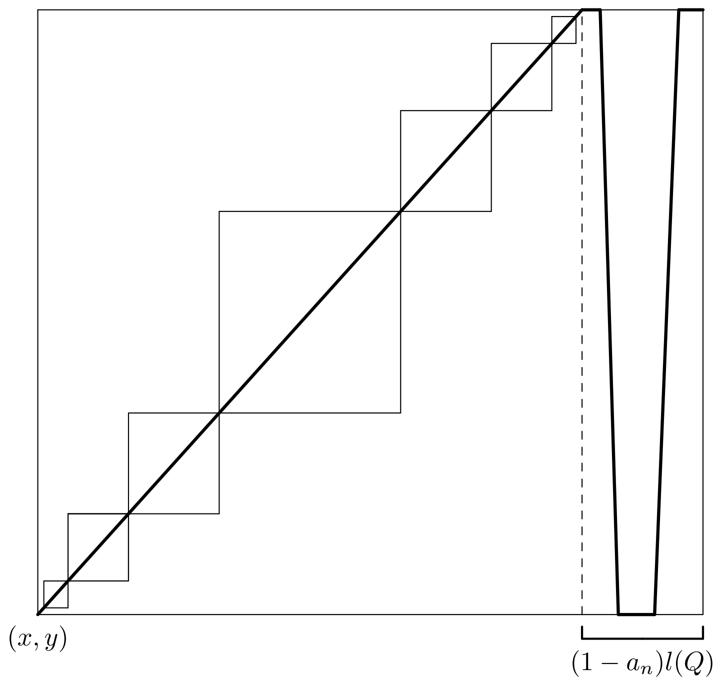


Figure 1 : $(n+1)^{th}$ step on $Q_{n,i}$

It is clear from our construction that the functions f_n converge to a continuous function f, and that f takes every value y_0 between 0 and 1 at infinitely many points. Indeed, if y_0 is the y coordinate of one of the vertices of one of the rectangles Q of our construction, then it takes this value at infinitely many points in its right rectangle Q''. And if it is not the y coordinate of any of the vertices of any of the rectangles, then for each n we can choose a $Q_{n,i}$ s.t. $y_0 \in \mathbb{P}_y(Q_{n,i})$. Then $f^{-1}(y_0)$ has at least two points in $\mathbb{P}_x(Q''_{n,i})$. Since this is true for each n, and the sets $\mathbb{P}_x(Q''_{n,i})$ are pairwise disjoint, therefore $f^{-1}(y_0)$ is infinite.

It remains to prove that the lower-scalled oscillation $l_f(x)$ is finite for each $x \in [0,1]$. The restriction of f onto the interval $\mathbb{P}_x(Q''_{n,i})$ is Lipschitz because f is piecewise linear inside each rectangle $Q''_{n,i}$.

Inside each of the rectangles $Q'_{n,i}$, the graph of f is covered by its child rectangles.

We observe that for each fixed $n \in \mathbb{N}$, all the rectangles $Q_{n,i}$ are similar. Moreover the height/length ratio at the $(n+1)th$ step increases by a factor of $1/(1-a_n)$. Thus for all the rectangles, the height/length ratio is bounded by $\prod_{i=0}^{\infty}(1-a_n)^{-1}$. We choose a_n small enough such that $\prod_{i=0}^{\infty}(1-a_n)^{-1} < \infty$.

Then selecting the diagonals of the rectangles by applying the Whitney decomposition theorem implies that there is a constant c satisfying the following: if $(x, f(x))$ is one of the vertices of $Q'_{n,i}$, and $(x', f(x'))$ is an interior point of $Q'_{n,i}$, then $|f(x) - f(x')| \leq c|x - x'|$. This shows that the lower scaled oscillation is finite at x. Putting $r := |x' - x|$, it also implies that for any $x'' \in \mathbb{P}_x(Q'_{n,i})$, if $|x'' - x'| \leq r$ then

$$|f(x'')-f(x')| \leq |f(x'')-f(x)|+|f(x')-f(x)| \leq c|x''-x|+c|x'-x| \leq 3c|x'-x| = 3cr. \quad (2.1)$$

If $(x', f(x'))$ is not a vertex of any rectangle of our construction, then for each n it is in the interior of a rectangle $Q'_{n,i}$. By selecting $(x, f(x))$ to be the endpoint of its diagonal closest to x' and putting $r = |x' - x|$, (2.1) holds for every x'' with $|x'' - x'| \leq r$. Since this estimate holds for an arbitrary small r, therefore $l_f(x') \leq 3c$.

□

Now we turn to the proof of Theorem 12.

Proof of Theorem 12. Our construction of the function is based on an iterative argument on cuboids in $\mathbb{R}^{m+1}$. By a cuboid, Q, in $\mathbb{R}^{m+1}$, we mean a Cartesian product of an axis-parallel cube in $\mathbb{R}^m$ and an interval. For a cube, C, in $\mathbb{R}^m$ we denote its side-length by $l(C)$, and for a cuboid $Q = C \times I$, we define the length, $l(Q)$, by $l(Q) := l(C)$ and the height, $h(Q)$, by $h(Q) := l(I)$. We write an element in $\mathbb{R}^{m+1}$ as $(x, z) = (x_1, .., x_m, z)$. If (x, z) lies on the graph of the function we are going to construct, then the first m coordinates denote the domain and z denotes the value of the function. We will also denote y as $y = (x_2, .., x_m)$. The direction of $(0, .., 0, 1)$ is called the z-axis. The projection maps onto the m-dimensional plane $z = 0$, the $(m-1)$-dimensional plane $x_1 = z = 0$ and the z-axis will be denoted by $\mathbb{P}_x$, $\mathbb{P}_y$ and $\mathbb{P}_z$ respectively.

We start with the cube $[0,1]^{m+1}$, $E_1 = \emptyset$, and a sequence $(a_n)_{n=0}^{\infty}$ satisfying $\prod_{n=0}^{\infty}(1 - a_n)^{-1} < 2$. Assume that we have completed n steps of our iterative argument and we have countably many cuboids $(Q_{n,i})_{i\in\mathbb{N}}$ in $\mathbb{R}^{m+1}$ and that f is defined on $E_n = [0,1]^m \setminus \mathbb{P}_x(\bigcup_{i=1}^{\infty} Q_{n,i})$.

Let $Q = Q_{n,i} = C \times [z, z+h]$ be such a cuboid. Let $\alpha = h(Q)/l(Q) = h/l(C)$ and let D be its diagonal plane parallel to the plane $z = \alpha x_1$. We apply our iterative argument on Q whose key parts are the following:

Part (1) We apply the Whitney decomposition theorem to C and decompose it into countably many Whitney cubes $\{C_j\}_{j\in\mathbb{N}}$ such that the diameter of each cube is less than the distance to the boundary of C and at least a constant times the distance to the boundary of C.

Part (2) Let C_j be a Whitney cube obtained in Part (1) with length $l(C_j)$. We partition the smaller cube with the same center as C_j and length $(1-a_n)^{1/2}l(C_j)$ into N^m equal cubes of length $(1-a_n)^{1/2}l(C_j)/N$ each, where N will be chosen later. Inside each of these N^m cubes, we consider a smaller cube with the same center and length $(1-a_n)l(C_j)/N$. Let these smaller cubes be denoted as $C_{j,k}$, $k \in \{1, 2, ..., N^m\}$. We chose N large enough so that we can label $C_{j,k}$ with N different labels such that

(i) For any cube with a label, ω, and any point x in the cube, the ball of centre x and radius $2\sqrt{m}\cdot l(C_{j,k})$ intersects at most one cube of each label. Moreover, we choose N to be large enough so that this ball lies in the cube C_j.

(ii) For every label, ω, the union of cubes of label, ω, has full $\mathbb{P}_y$ projection. In other words, we require that $\mathbb{P}_y(C_j) = \mathbb{P}_y(\bigcup_{k\in L_\omega} C_{j,k})$ where L_ω is the set of all indices k such that $C_{j,k}$ has label ω.

Thus we divided C_j into N^m labeled cubes and narrow strips, i.e, the part of C_j which is not covered by labeled cubes. (See Figure 3 for $m = 2$).

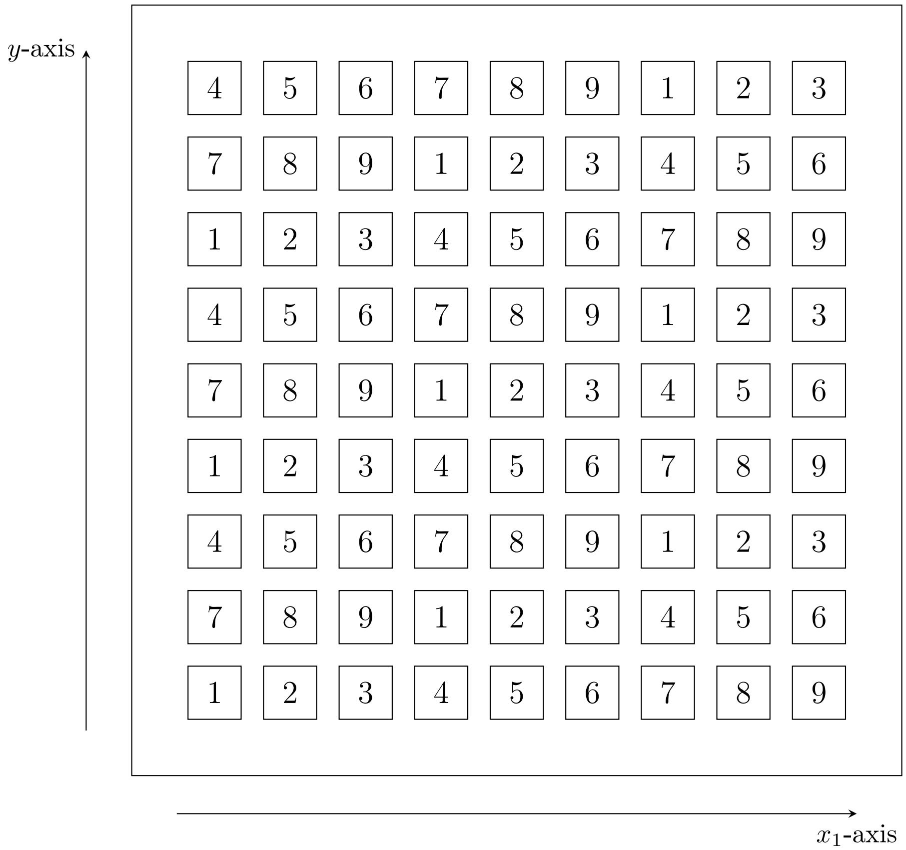


Figure 4 : Decomposition of C_j into cubes with 9 labels and narrow strips.

Part (3) Let $D_j \subset D$ be the intersection of D and the pre-image of C_j under the projection $\mathbb{P}_x$. Consider the cuboid R_j in $\mathbb{R}^{m+1}$ with sides parallel to the coordinate axes and D_j as its diagonal plane. We denote $R_j = C_j \times [z_j, z_j + h_j]$. We call the cuboids $\{R_j\}_{j\in\mathbb{N}}$ the child cuboids of Q. (See Figure 4 for $m = 2$).

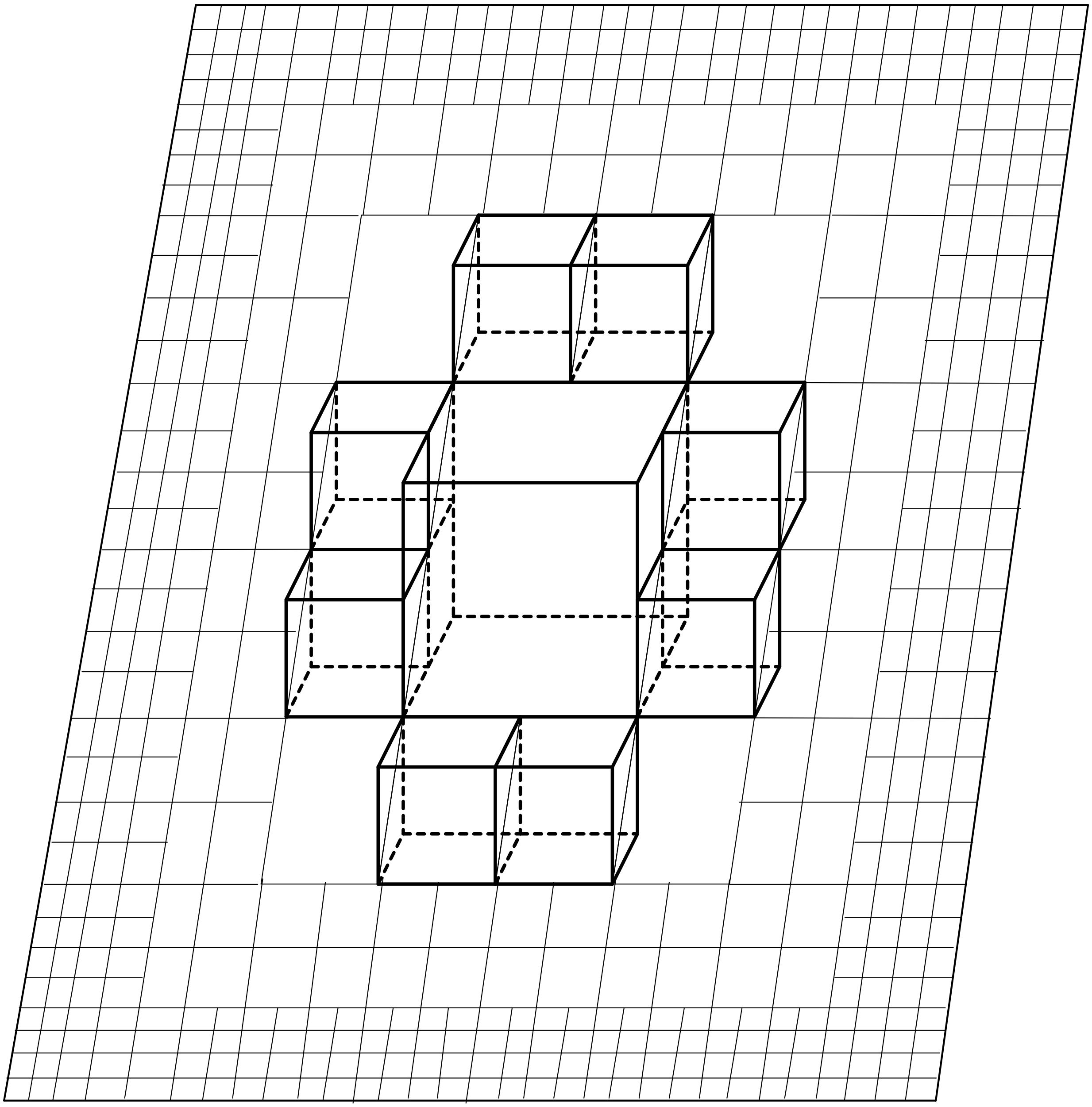


Figure 4 : Construction of child cuboids of Q.

Part (4) Divide $[z_j, z_j + h_j]$ into N equal parts and to each of these intervals assign a label from the N labels in Part 2 so that no two intervals have the same label. Let the interval with label, ω, be $I_{j,\omega}$. If the label of a cube $C_{j,k}$ is also ω, then we consider the corresponding cuboid

$$S_{j,k} = C_{j,k} \times I_{j,\omega}.$$

Part (5) Let $D_{j,k}$ be the diagonal plane of $S_{j,k}$ parallel to the $(1-a_n)z = \alpha x_1$ plane. On the boundary of $C_{j,k}$, we define f so that its graph coincides with the boundary of $D_{j,k}$, i.e if x lies on the boundary of $C_{j,k}$, then $(x, f(x))$ lies on $D_{j,k}$. Similarly, we define the graph of f over the boundary of C_j to be the boundary of D_j. We then extend f piece-wise linearly on the narrow strips defined in Part (2) so that it is continuous, and imitating the construction of Theorem 11, we define f to be constant near the top and bottom faces of $S_{j,k}$ so that the Hausdorff dimension of each of the level sets containing these faces is m. This will ensure that the level sets associated to the top and bottom faces are not $(m-1)$-rectifiable.

Part (6) The countable collection of cuboids thus obtained, i.e, $\{S_{j,k} : Q = Q_{n,i}, i, j \in \mathbb{N}, k \in \{1, 2.., N^m\}\}$ will form our new collection of cuboids. We re-enumerate them as $\{Q_{n+1,j}\}_{j\in\mathbb{N}}$.

This completes the $(n+1)^{th}$ step.

For $x \in [0,1]^m \setminus \bigcup_{n=1}^{\infty} E_n$, we define the point $(x, f(x))$ to be the intersection of all the cuboids $Q_{n,i}$ whose projection onto the plane $z = 0$ contains x. Thus the function is continuous for every $[0,1]^m$. We claim that the function thus obtained has $l_f(x) < \infty$ and each level set is not rectifiable.

Clearly if $(x, f(x))$ lies on the piece-wise linear parts of the graph of f then $l_f(x) < \infty$. It remains to prove so for $(x, f(x))$ which lie either (1) on the boundary of a diagonal plane or (2) in infinitely many cuboids $Q_{n,i}$.

We observe that for each $n \in \mathbb{N}$, the height/length ratio of the cuboids, $Q_{n,i}$, increases by a factor of $1/(1-a_n)$ after the $(n+1)$th step. Since we start with the cube $[0,1]^{m+1}$, the height/length ratio for all the cuboids is bounded by $\prod_{n=0}^{\infty}(1-a_n)^{-1}$. We recall that we've chosen $(a_n)_{n=0}^{\infty}$ such that $\prod_{n=0}^{\infty}(1-a_n)^{-1} < 2$.

Now for the first case, let $(x, f(x))$ lie on the boundary of the diagonal plane of a cuboid $Q = Q_{n,i}$ and let $(y, f(y)) \in Q$. The graph of the function inside Q lies in its child cuboids. Let $(y, f(y)) \in R_j$, where R_j is a child cuboid of Q and let $(w, f(w))$ be a point on the diagonal plane of R_j closest to $(x, f(x))$. By the properties of the Whitney decomposition and the diagonal planes, the length of the interval $I_j = \mathbb{P}_z(R_j)$ is less than the distance between the boundary of $I = \mathbb{P}_z(Q)$ and I_j. Thus the height of the cuboid R_j is less than $|f(w) - f(x)|$. This implies $|f(y) - f(w)| \leq |f(w) - f(x)|$ and

$$\frac{|f(y) - f(x)|}{|y - x|} \leq \frac{|f(y) - f(w)| + |f(w) - f(x)|}{|y - x|} \leq \frac{2|f(w) - f(x)|}{|y - x|}.$$

Since we have taken the closest vertex to $(x, f(x))$ we have $|y - x| \geq |w - x|$. Thus

$$\frac{2|f(w) - f(x)|}{|y - x|} \leq \frac{2|f(w) - f(x)|}{|w - x|} < c,$$

for some constant c. Thus $l_f(x) < \infty$.

For the second case, we estimate $l_f(y)$ where $(y, f(y))$ lies inside infinitely many cuboids $Q_n := Q_{n,i}$ of arbitrarily small length. Let $(x_n, f(x_n))$ be a point on the diagonal plane of Q_n such that if $s_n := |y - x_n|$, then $B(y, s_n) \subset \mathbb{P}_x(Q_n)$. From above, we know that if $(x_n, f(x_n))$ lies on the boundary of Q_n, then $\frac{|f(x_n) - f(v)|}{|x_n - v|} < c$ for every $v \in \mathbb{P}_x(Q_n)$. Thus for $v \in B(y, s_n) \subset \mathbb{P}_x(Q_n)$ we have

$$\begin{aligned}
|f(y) - f(v)| &\leq |f(y) - f(x_n)| + |f(x_n) - f(v)| \\
&< c|y - x_n| + c|x_n - v| \\
&< c|y - x_n| + c|x_n - y| + c|y - v| \leq 3cs_n.
\end{aligned}$$

Therefore, $l_f(y) < \infty$ and we have shown that the lower-scaled oscillation of f is bounded in every case.

It remains to prove that the level sets are not rectifiable. Define

$$F = \{(x, f(x)) : (x, f(x)) \text{ lies in infinitely many cuboids } Q_{n,i}\}$$

and let $F_z = f^{-1}(z) \cap F$. By property (ii) in Part (2) of the cuboid construction, we see that F_z has full $\mathbb{P}_y$-projection. Thus $\mathcal{H}^{m-1}(F_z) > 0$. We claim that F_z is unrectifiable and thus $f^{-1}(z)$ is unrectifiable.

Consider $x \in F_z$. Then there exists some N such that $n > N$ implies that $x \in Q_n$ for some cuboid, $Q_n := Q_{n,i}$, constructed at step n. By the construction of Q_n and Q_{n+1} (Part (2)), we can say that for $r_n = \sqrt{m} \cdot l(Q_{n+1})$, $\mathbb{P}_x(Q_{n+1}) \subset B(x, r_n) \subset B(x, 2r_n) \subset \mathbb{P}_x(Q_n)$. In addition, the labeling argument of Part (4) and Part (5) imply that

$$F_z \cap B(x, 2r_n) = F_z \cap B(x, r_n).$$

Thus,

$$\begin{aligned} D^{m-1}(F_z', x, 2r_n) &= \frac{\mathcal{H}^{m-1}(F_z' \cap B(x, 2r_n))}{(4r_n)^{m-1}} \\ &= 2^{-m+1} \frac{\mathcal{H}^{m-1}(F_z' \cap B(x, r_n))}{(2r_n)^{m-1}} = 2^{-m+1} D^{m-1}(F_z', x, r_n). \end{aligned}$$

Since this holds for a sequence of scales, r_n, that converges to zero, both densities, $D^{m-1}(F_z', x, 2r_n)$ and $D^{m-1}(F_z', x, r_n)$, cannot converge to 1. Therefore, the Hausdorff density of F_z is not 1 at any point in F_z which implies that F_z is not rectifiable by Theorem 14. This proves that $f^{-1}(z)$ is unrectifiable for all $z \in [0, 1] \setminus E$, where E is set of the level sets intersecting the top and bottom faces of the cuboids $\{Q_{n,i}\}_{n,i \in \mathbb{N}}$. Finally, according to Part (5) of the cuboid construction, for every $z \in E$, $f^{-1}(z)$ has dimension m. $\square$

CHAPTER 3

HAUSDORFF DIMENSION OF BESICOVITCH SETS OF CANTOR GRAPHS

This chapter is joint work with Marianna Csörnyei and Kornélia Héra and originally appeared in Mathematika, Volume 70: e12241, 2024.

3.1 Introduction

It is well known that Besicovitch sets in the plane - sets containing a unit line segment in every direction - can have zero Lebesgue measure, and must have Hausdorff dimension 2. It is natural to ask what happens for other curves in place of the line segment. For circular arcs, the problem is trivial, as we can rotate the circular arc around the center of the circle to obtain a Besicovitch set for that circular arc, of dimension 1. For other curves Γ, it is worthwhile to ask first whether it is possible to obtain Γ-Besicovitch sets - sets containing a rotated (and translated) copy of Γ in every direction - of zero Lebesgue measure. In [16], A. Chang and the second author showed that for any rectifiable planar curve Γ, there exists an "almost-Γ-Besicovitch set" - a set containing a rotated (and translated) copy of a full $\mathcal{H}^1$-measure subset of Γ in each direction - of zero Lebesgue measure. In fact, they showed that Γ can be continuously turned around covering zero Lebesgue measure, if at each time moment, we are allowed to remove a set of $\mathcal{H}^1$-null measure of Γ. For closely related results about the problem of moving curves continuously while covering arbitrarily small measure, see [23] and [40].

The existence of the above mentioned "almost-Γ-Besicovitch sets" of measure zero for rectifiable curves Γ suggests that it is interesting to consider the Hausdorff dimension of Γ-Besicovitch sets (or of "almost-Γ-Besicovitch sets" - one naturally expects that in terms of dimension, there is no difference between the two). It turns out that for sufficiently smooth curves, the answer is very satisfactory. As observed by A. Chang, it follows from the results of J. Zahl in [74] that for any C^∞ curve Γ that is not a circular arc, any Γ-Besicovitch set must have Hausdorff dimension 2. In fact, the same holds under the relaxed assumption that Γ is a C^3 curve, see [63]. Curvature plays an important role here. Another interesting fact, which has been mentioned in [16], is that if Γ is a countable union of (not necessarily concentric) circles, then there exists a Γ-Besicovitch set of Hausdorff dimension 1. The next natural question could be: what happens for non-smooth curves that behave differently from circular arcs? In fact, it would be interesting to consider objects whose behaviour is much closer to the behaviour of line segments than of circular arcs. Two immediately visible reasons for the special role of circular arcs are rotational symmetry and constant curvature. There is a third, probably most relevant aspect: the geometry of normal lines. As a special case of the results in [16], it is known that if Γ is a rectifiable set possessing a continuous tangent field, then Γ can be continuously turned around covering zero Lebesgue measure, if at each time moment t, we are allowed to choose a line ℓ_t and delete the points of Γ at which the normal line is equal to ℓ_t. When Γ is a line segment, all of its normal lines are parallel, while in case Γ is a cirular arc, all of its normal lines point in different directions. Based on this, it would be interesting to consider rectifiable curves that, at almost every point, have parallel normal lines.

As a first attempt in this direction, we consider Besicovitch sets of Cantor graphs. More precisely, we consider a large class of (nearly self-similar) Cantor sets C (see Section 3.2.1), and build rectifiable sets Γ on C that resemble Cantor-graphs (see Section 3.2.2). This is a generalization of the usual Cantor function, a.k.a. Devil's staircase built on the standard middle third Cantor set, noting that naturally, we only consider the Cantor-part of the graph - the horizontal line segments are not included. We show that the Hausdorff dimension of Γ-Besicovitch sets must be at least $\min\left(2 - s^2, \frac{1}{s}\right)$, where $s = \dim C$. As it turns out, it is important that our objects possess some structure.

This chapter is organized as follows. In Section 3.2 we introduce the main setting: we define Besicovitch sets of general Cantor graphs built on Cantor sets, and we state our main result, Theorem 19. Section 3.3 contains the main body of our proof: the proof of Proposition 23. Finally, in Section 3.4 we include a standard argument that deduces the Hausdorff dimension bound of Theorem 19 from Proposition 23.

3.1.1 Notation

The 2-dimensional Lebesgue measure is denoted by $\mathcal{L}$, and the 1-dimensional Lebesgue measure by λ. The open ball of center x and radius r is denoted by $B(x,r)$. For a set $U \subset \mathbb{R}^n$, $U(\delta) = \bigcup_{x\in U} B(x,\delta)$ denotes the open δ-neighborhood of U, and $\operatorname{diam}(U)$ denotes the diameter of U. For any $s \geq 0$, $\delta \in (0,\infty]$, the s-dimensional Hausdorff δ-premeasure is denoted by $\mathcal{H}^s_\delta$, and the s-dimensional Hausdorff measure is denoted by $\mathcal{H}^s$. We use the notation $\dim A$ for the Hausdorff dimension of $A \subset \mathbb{R}^n$. For the well known properties of Hausdorff measures and dimension, see e.g. [57].

For a finite set A, let $|A|$ denote its cardinality. We write $x \lesssim y$ for $x \leq Cy$ where C is an absolute constant or a constant depending only on a and b, where a, b are fixed constants, see the the next section. We write $x \approx y$ if $x \lesssim y$ and $y \lesssim x$. Let pr_1, pr_2 denote the projections onto the x and y coordinate axes, respectively.

3.2 The setup and our main result

3.2.1 Cantor sets

We describe a general Cantor set construction, where the ratio of removed intervals will be fixed throughout the process. Let $a \in \mathbb{N}, a \geq 3$, $b \in \mathbb{N}, 2 \leq b < a$, and let $J = \{d_0, d_1, \dots, d_{b-1}\} \subset \{0, 1, \dots a-1\}$ be a digit set such that $0 \leq d_0 \leq d_1 \leq \cdots \leq d_{b-1} \leq a-1$. Partition $[0,1]$ into a closed intervals $I_0, \dots, I_{a-1}$ of equal length (denoted this way from left to right). We keep $I_{d_0}, \dots, I_{d_{b-1}}$, and remove the interiors of the remaining $a-b$ intervals. Let $C_1 = I_{d_0} \cup I_{d_1} \cup \cdots \cup I_{d_{b-1}}$. We will iterate this process below to define C_n, $n = 1, 2, \dots,$ such that the next digit set (of cardinality b) will be chosen independently in each subinterval in each step.

For each $x_1 \in J$, let $J_{(x_1)} \subset \{0, 1, \dots a-1\}$ with $|J_{(x_1)}| = b$. Whenever $(x_1 \dots x_n)$ is a sequence of length n such that $J_{(x_1 \dots x_n)}$ is already defined and $x_{n+1} \in J_{(x_1 \dots x_n)}$, let $J_{(x_1 \dots x_n x_{n+1})} \subset \{0, 1, \dots a-1\}$ with $|J_{(x_1 \dots x_n x_{n+1})}| = b$. For notational convenience we wright $J = J_{(x_1 x_0)}$. Let

$$C_n = \bigcup_{\substack{x_j \in J_{(x_1 \dots x_{j-1})},\\ j=1,\dots,n}} \left[\sum_{j=1}^{n} \frac{x_j}{a^j}, \sum_{j=1}^{n} \frac{x_j}{a^j} + \frac{1}{a^n} \right].$$

By construction, C_n is a union of b^n intervals of length $1/a^n$, and the lengths of the gaps between the intervals in C_n are integer multiples of $1/a^n$. Let $C = \bigcap_{n=1}^{\infty} C_n$, our Cantor set. By construction, $C = \left\{ \sum_{j=1}^{\infty} \frac{x_j}{a^j} : x_j \in J_{(x_1 \dots x_{j-1})} \ \forall\ j \right\}$. Let $s = \log b / \log a = \dim C$. In the special case when each $J_{(x_1 \dots x_n)} = J$, C is a self-similar set.

3.2.2 Cantor graphs

Now we define a general Cantor graph construction built on C. Whenever $J_{(x_1 \dots x_n)}$ is defined, let $\sigma_{(x_1 \dots x_n)} : J_{(x_1 \dots x_n)} \to \{0, 1, \dots, b-1\}$ be an arbitrary bijection. Define

$$\Gamma = \left\{ \left(\sum_{j=1}^{\infty} \frac{x_j}{a^j}, \sum_{j=1}^{\infty} \frac{\sigma_{(x_1 \dots x_{j-1})}(x_j)}{b^j} \right) : x_j \in J_{(x_1 \dots x_{j-1})} \ \forall\ j \right\}.$$

Note that in the special case when each $\sigma_{(x_1 \dots x_n)}$ is order preserving, Γ is the graph of a monotone function, in fact, it is the classical devil's staircase graph built on C. We also note that in some special cases Γ is not precisely a graph, since at the points x where the representation $x = \sum_{j=1}^{\infty} \frac{x_j}{a^j}$ is not unique we might have two points above x, but this is irrelevant for our purposes.

We list some well-known or easily verifiable properties of our Cantor graphs.

i) $\dim \Gamma = 1$, and $\mathcal{H}^1 \restriction \Gamma = \lambda \restriction \mathrm{pr}_2(\Gamma)$.

ii) Γ is rectifiable, and for $\mathcal{H}^1$-almost all $p \in \Gamma$, the tangent to Γ at p is vertical.

We will consider the natural approximation to Γ consisting of b^n rectangles of size $\frac{1}{a^n} \times \frac{1}{b^n}$:

$$\bigcup_{\substack{x_j \in J_{(x_1 \dots x_{j-1})},\\ j=1,\dots,n}} \left[\sum_{j=1}^{n} \frac{x_j}{a^j}, \sum_{j=1}^{n} \frac{x_j}{a^j} + \frac{1}{a^n} \right] \times \left[\sum_{j=1}^{n} \frac{\sigma_{(x_1 \dots x_{j-1})}(x_j)}{b^j}, \sum_{j=1}^{n} \frac{\sigma_{(x_1 \dots x_{j-1})}(x_j)}{b^j} + \frac{1}{b^n} \right].$$

For technical reasons, we will also consider the 3 times larger rectangles:

$$R_n = \bigcup_{\substack{x_j \in J_{(x_1 \dots x_{j-1})},\\ j=1,\dots,n}} \left[\sum_{j=1}^{n} \frac{x_j}{a^j} - \frac{1}{a^n}, \sum_{j=1}^{n} \frac{x_j}{a^j} + \frac{2}{a^n} \right] \times \left[\sum_{j=1}^{n} \frac{\sigma_{(x_1 \dots x_{j-1})}(x_j)}{b^j} - \frac{1}{b^n}, \sum_{j=1}^{n} \frac{\sigma_{(x_1 \dots x_{j-1})}(x_j)}{b^j} + \frac{2}{b^n} \right]. \tag{3.1}$$

3.2.3 Besicovitch sets of Cantor graphs

We say that $E \subset \mathbb{R}^2$ is a Γ-Besicovitch set, if E contains a rotated (and translated) copy of Γ in every direction. That is, for every $\theta \in [0, \pi]$ there exists $\omega_\theta \in \mathbb{R}^2$ such that $E = \bigcup_{\theta \in [0,\pi]} \tau_\theta(\Gamma)$, where $\tau_\theta(z) = e^{i\theta} z + \omega_\theta$ for $z \in \mathbb{C}$. Here we used the identification $z = (z_1, z_2) \leftrightarrow z = z_1 + iz_2$ between $\mathbb{R}^2$ and $\mathbb{C}$ which will be used occasionally in the paper for convenience. Denote $\Gamma_\theta = \tau_\theta(\Gamma)$. Recall that $s = \dim C = \log b / \log a$. Our main result is the following.

Theorem 19. *Let $E \subset \mathbb{R}^2$ be a Borel Γ-Besicovitch set. Then*

$$\dim E \geq \min\left(2 - s^2, \frac{1}{s}\right).$$

3.3 Main proof

Let $\delta = \frac{1}{a^n}$ for some $n \in \mathbb{N}$, and write $R = R_n = \bigcup_{i=1}^{b^n} T_i$ for the union of renctagles defined in (3.1), where the rectangles are indexed using their vertical order. That is, $\min \mathrm{pr}_2(T_1) < \min \mathrm{pr}_2(T_2) < \cdots < \min \mathrm{pr}_2(T_{b^n})$, and T_i is a rectangle of size $3\delta \times 3\delta^s$ for each i. By construction, it is easy to see that $\Gamma(\delta) \subset R$, and that $\mathcal{L}(R) \leq 9\delta$. Write $R_\theta = \tau_\theta(R)$.

First we show that each fixed rectangle of R can intersect at most 10 rectangles of R_θ.

Lemma 20. *Let $\theta \in [0, \pi]$ and $i \in \{1, \ldots, b^n\}$. Then*

$$|\left\{j \in \{1, \ldots, b^n\} : T_i \cap T_{j,\theta} \neq \emptyset\right\}| \leq 10.$$

Proof. We have $\operatorname{diam} T_i = 3\sqrt{\delta^2 + \delta^{2s}} < 6\delta^s$. Let $j, j' \in \{1, \ldots, b^n\}$ with $j' - j > 10$. Since $\min(\mathrm{pr}_2(T_{j+1})) - \min(\mathrm{pr}_2(T_j)) = \delta^s$ and $\lambda(\mathrm{pr}_2(T_j)) = 3\delta^s$, we get that

$$d(T_{j,\theta}, T_{j',\theta}) = d(T_j, T_{j'}) \geq 10\delta^s - 3\delta^s > \operatorname{diam} T_i.$$

This means that $T_{j,\theta}$ and $T_{j',\theta}$ can not simultaneously intersect T_i. Therefore, if $T_{j-k,\theta}, T_{j-k+1,\theta}, \ldots, T_{j,\theta}, \ldots, T_{j+l-1,\theta}, T_{j+l,\theta}$ all intersect T_i, then we must have $j + l - (j - k) = l + k \leq 10$, and so

$$|\left\{j \in \{1, \ldots, b^n\} : T_i \cap T_{j,\theta} \neq \emptyset\right\}| \leq 10.$$

$\square$

We introduce a notation for the number of intersecting pairs of rectangles of R and R_θ, let

$$L(\delta,\theta) = |\{(i,j) \in \{1,\ldots,b^n\} : T_i \cap T_{j,\theta} \neq \emptyset\}|. \tag{3.2}$$

We note that by Lemma 20, $L(\delta,\theta) \lesssim |\{i : \text{ there is } j \text{ such that } T_i \cap T_{j,\theta} \neq \emptyset\}|$, and the constant in $\lesssim$ is independent of δ and θ. Therefore, we trivially have $L(\delta,\theta) \lesssim \delta^{-s}$, which is the number of all the rectangles in R.

We also need the following very easy observation about intersecting rectangles.

Lemma 21. *For any $\theta \in (0,1], i, j$, we have $\mathcal{L}(T_i \cap T_{j,\theta}) \lesssim \delta \cdot \delta^s$, and $\mathcal{L}(T_i \cap T_{j,\theta}) \lesssim \frac{\delta^2}{\theta}$.*

Proof. Recall that T_i and $T_{j,\theta}$ are $3\delta \times 3\delta^s$ rectangles with angle θ between their longer side, the first inequality is trivial as $\mathcal{L}(T_i) \lesssim \delta \cdot \delta^s$. The second statement is also trivial noting that the area of the intersection of two infinite strips of width δ with angle θ is at most $\frac{\delta^2}{\sin\theta} \lesssim \frac{\delta^2}{\theta}$. □

Remark 22. We have that $\frac{\delta^2}{\theta} > \delta \cdot \delta^s$ if and only if $\theta < \delta^{1-s}$, so the first bound in Lemma 21, although trivial, is actually useful for some angles.

Now we let $\delta_0 \leq 1$ be small enough (to be specified later), and fix $\delta \leq \delta_0$. Without loss of generality, we can assume that we only have rotated copies of Γ with angle in $[0,1]$. Let A be a maximal δ-separated subset of $[0,1]$, it is easy to see that $|A| \approx \frac{1}{\delta}$.

The main result of this section is the following.

Proposition 23. *For any $\delta \leq \delta_0$, we have*

$$\sum_{\theta,\theta' \in A} \mathcal{L}(R_\theta \cap R_{\theta'}) \lesssim \max\left(\delta^{-s^2}, \delta^{1/s-2}\right) \cdot \log\left(\frac{1}{\delta}\right). \tag{3.3}$$

Using the Cauchy-Schwarz inequality, Proposition 23 will easily yield the bound of Theorem 19 for (lower) Minkowski dimension, we include the details below. A standard pigeonholing argument can be used to prove the corresponding Hausdorff dimension bound, this will be carried out in Section 3.4.

Proof of Theorem 19 for (lower) Minkowski dimension . Let $d = \min\left(2 - s^2, \frac{1}{s}\right)$. Using $\bigcup_{\theta\in A}\Gamma_\theta(\delta) \subset E(\delta)$ and $\Gamma_\theta(\delta) \subset R_\theta$ for each $\theta \in A$, we have

$$\sum_{\theta\in A} \mathcal{L}(\Gamma_\theta(\delta)) \leq \mathcal{L}\left(\bigcup_{\theta\in A}\Gamma_\theta(\delta)\right)^{1/2} \cdot \left(\sum_{\theta,\theta'\in A} \mathcal{L}(\Gamma_{\theta'}(\delta)\cap\Gamma_\theta(\delta))\right)^{1/2} \leq \tag{3.4}$$

$$\leq \mathcal{L}\left(E(\delta)\right)^{1/2} \cdot \left(\sum_{\theta,\theta'\in A} \mathcal{L}(R_{\theta'}\cap R_\theta)\right)^{1/2}.$$

For the left hand side, it is easy to see that $\sum\limits_{\theta\in A} \mathcal{L}\left(\Gamma_\theta(\delta)\right) \gtrsim \frac{1}{\delta}\cdot\delta = 1$, and by Proposition 23, $\sum\limits_{\theta,\theta'\in A} \mathcal{L}(R_\theta \cap R_{\theta'}) \lesssim \delta^{d-2}\cdot\log(\frac{1}{\delta})$. This yields $\mathcal{L}\left(E(\delta)\right) \gtrsim \frac{\delta^{2-d}}{\log(1/\delta)}$, therefore the Minkowski dimension of E is at least d. □

We now turn to the proof of Proposition 23.

Proof of Proposition 23. We have

$$\sum_{\theta,\theta'\in A} \mathcal{L}(R_\theta\cap R_{\theta'}) = \sum_{\theta,\theta'\in A, |\theta-\theta'|\geq\delta} \mathcal{L}(R_\theta\cap R_{\theta'}) + \sum_{\theta'\in A}\mathcal{L}(R_{\theta'}).$$

For the second term, we get

$$\sum_{\theta'\in A}\mathcal{L}(R_{\theta'}) \lesssim \frac{1}{\delta}\cdot\delta = 1. \tag{3.5}$$

Now we fix θ', and for simplicity, we assume without loss of generality that $\theta' = 0$ and $R_{\theta'} = R$. So we need to give an upper bound for

$$\sum_{\theta\in A,\theta\geq\delta} \mathcal{L}(R_\theta\cap R). \tag{3.6}$$

Remark 24. Using the observation above Lemma 21, we can give an easy upper bound for

(3.6). Namely, we get that

$$\sum_{\theta\in A,\theta\geq\delta}\mathcal{L}(R_\theta\cap R)\lesssim\sum_{\theta\in A,\theta\geq\delta}L(\delta,\theta)\cdot\frac{\delta^2}{\theta}\lesssim\delta^{1-s}\log\left(\frac{1}{\delta}\right). \tag{3.7}$$

This yields

$$\sum_{\theta,\theta'\in A}\mathcal{L}(R_\theta\cap R_{\theta'})\lesssim\delta^{-s}\cdot\log\left(\frac{1}{\delta}\right).$$

Using the same standard argument as in the proof of Theorem 19, it follows that the Hausdorff dimension of Γ-Besicovitch sets is at least $2-s$.

To improve this easy bound, we will improve the bound in (3.7) to

$$\sum_{\theta\in A,\theta\geq\delta}\mathcal{L}(R_\theta\cap R)\lesssim\max\left(\delta^{1-s^2},\delta^{1/s-1}\right)\cdot\log\left(\frac{1}{\delta}\right). \tag{3.8}$$

In order to do this, we will consider two substantially different cases: the small angle and the large angle cases. We say that

$$\theta\text{ is}\begin{cases}\text{small if} & \delta\leq\theta\leq\min\left(\delta^s,\delta^{1-s}\right)=:\beta(\delta)=:\beta,\\ \text{large if} & \beta\leq\theta,\end{cases} \tag{3.9}$$

$$\theta\text{ is very large if }\gamma:=\gamma(\delta):=\max\left(\delta^s,\delta^{1-s}\right)\leq\theta. \tag{3.10}$$

We have the following estimates for the number of intersecting pairs of rectangles.

Lemma 25. *If θ is small, then*

$$L(\delta,\theta)\lesssim\frac{1}{\delta^s}\cdot\max\left(\frac{\delta}{\theta},\theta^s\right).$$

Lemma 26. *If θ is large, then*

$$L(\delta, \theta) \lesssim \frac{1}{\delta^s} \cdot \max\left(\frac{r_0}{\theta}, \theta^s\right),$$

moreover, if $\theta \geq \delta^{1-s}$, then

$$L(\delta, \theta) \lesssim \frac{1}{\theta^s} \cdot \frac{1}{\delta^{s^2}} \cdot \max\left(\frac{r_0}{\theta}, \theta^s\right),$$

where $r_0 = \max\left(\theta^{1/s}, \theta^{1/(1-s)}\right)$.

We start with the proof of Lemma 25.

Proof of Lemma 25. Fix θ such that $\theta \in A, \delta \leq \theta \leq \beta$, and choose the unique $m \in \{0, \ldots, n\}$ such that $\theta \in (1/a^{m+1}, 1/a^m]$. Recall that $R = \bigcup_{i=1}^{b^n} T_i, R_\theta = \bigcup_{i=1}^{b^n} T_{i,\theta}$. Fix T_i and $T_{j,\theta}$ such that they intersect. For simplicity, let x,y denote the bottom left corners of T_i and T_j, respectively, and let y_θ be the bottom left corner of $T_{j,\theta}$. We will use the following simple geometric lemma.

Lemma 27. *Let $\theta \in (0, 1]$ and assume that $T_i \cap T_{j,\theta} \neq \emptyset$. Then*

1. $|\operatorname{pr}_2(x - y_\theta)| \leq 10\delta^s$.
2. *If $\theta \leq \delta^{1-s}$, then* $|\operatorname{pr}_1(x - y_\theta)| \leq 10\delta$.

Proof of Lemma 27. Clearly, $|\operatorname{pr}_2(x - y_\theta)| \leq \lambda(\operatorname{pr}_2(T_i)) + \lambda(\operatorname{pr}_2(T_{j,\theta})) = 3\delta^s + \lambda(\operatorname{pr}_2(T_{j,\theta}))$. Similarly, $|\operatorname{pr}_1(x - y_\theta)| \leq \lambda(\operatorname{pr}_1(T_i)) + \lambda(\operatorname{pr}_1(T_{j,\theta})) = 3\delta + \lambda(\operatorname{pr}_1(T_{j,\theta}))$. By simple planar geometry, we have that $\lambda(\operatorname{pr}_2(T_{j,\theta})) \leq 3\delta^s |\cos\theta| + 3\delta|\sin\theta| \leq 6\delta^s$. Similarly, if $\theta \leq \delta^{1-s}$, then $\lambda(\operatorname{pr}_1(T_{j,\theta})) \leq 3\delta^s|\sin\theta| + 3\delta|\cos\theta| \leq 3\delta^s\theta + 3\delta \leq 6\delta$ and the claim of Lemma 27 follows.

□

By Lemma 27, since θ is small, see (3.9), we have

$$|\operatorname{pr}_1(x-y_\theta)| \le 10\delta,\ |\operatorname{pr}_2(x-y_\theta)| \le 10\delta^s. \tag{3.11}$$

For $x = \left(\sum_{j=1}^{n} \frac{x_j}{a^j} - \frac{1}{a^n}, \sum_{j=1}^{n} \frac{\sigma_{(x_1 \dots x_{j-1})}(x_j)}{b^j} - \frac{1}{b^n}\right)$, we write $x = x_{\text{large}} + x_{\text{small}}$, where

$$x_{\text{large}} = \left(\sum_{j=1}^{m} \frac{x_j}{a^j}, \sum_{j=1}^{m} \frac{\sigma_{(x_1 \dots x_{j-1})}(x_j)}{b^j}\right), \tag{3.12}$$

$$x_{\text{small}} = \left(\sum_{j=m+1}^{n} \frac{x_j}{a^j} - \frac{1}{a^n}, \sum_{j=m+1}^{n} \frac{\sigma_{(x_1 \dots x_{j-1})}(x_j)}{b^j} - \frac{1}{b^n}\right).$$

Similarly, write $y = y_{\text{large}} + y_{\text{small}}$. Note that by construction,

$$\operatorname{pr}_2(x_{\text{large}}), \operatorname{pr}_2(y_{\text{large}}) \text{ are integer multiples of } 1/b^m, \tag{3.13}$$

$$\operatorname{pr}_2(x_{\text{small}}), \operatorname{pr}_2(y_{\text{small}}) \text{ are integer multiples of } 1/b^n = \delta^s, \tag{3.14}$$

$$\operatorname{pr}_1(x_{\text{large}}), \operatorname{pr}_1(y_{\text{large}}) \text{ are integer multiples of } 1/a^m, \tag{3.15}$$

$$\operatorname{pr}_1(x_{\text{small}}), \operatorname{pr}_1(y_{\text{small}}) \text{ are integer multiples of } 1/a^n = \delta. \tag{3.16}$$

Now we fix $\xi \in \{x_{\text{small}} : x_{m+1}, \dots, x_n \in J\}$, we have b^{n-m} choices for ξ. Let M_θ denote the number of pairs (x, y) such that the corresponding rectangles T_i and $T_{j,\theta}$ intersect and such that $x_{\text{small}} = \xi$. Clearly,

$$L(\delta, \theta) \lesssim b^{n-m} \cdot M_\theta \lesssim \frac{\theta^s}{\delta^s} \cdot M_\theta. \tag{3.17}$$

Claim 28. *We have*

$$M_\theta \lesssim \max\left(\frac{\delta}{\theta^{1+s}}, 1\right).$$

Proof of Claim 28. Suppose now that (x, y) satisfy (3.11) and that $x_{\text{small}} = \xi$. We will need the following simple geometric lemma:

Lemma 29. *For any $z \in \mathbb{C}$, we have*

1. $|\operatorname{pr}_1(e^{i\theta}z) - \operatorname{pr}_1 z| \leq |\theta z|$.

2. $|\operatorname{pr}_2(e^{i\theta}z) - \operatorname{pr}_2 z| \leq |\theta z|$.

3. $||\operatorname{pr}_1(e^{i\theta}z) - \operatorname{pr}_1 z| - |\theta \operatorname{pr}_2 z|| \leq 2|\theta^2 z|$.

Proof. The simple proof is left to the reader. □

By (3.11), we have $|\operatorname{pr}_2 y_\theta - \operatorname{pr}_2 x| \leq 10\delta^s$. By (2) of Lemma 29 applied for y and using that $|y| \leq 2$ (as $y \in R \subset [-\delta, 1+\delta] \times [-\delta^s, 1+\delta^s]$) as well as $y_\theta = e^{i\theta}y + \omega_\theta$, we have $|\operatorname{pr}_2(e^{i\theta}y) - \operatorname{pr}_2 y| \leq 2\theta$, and so

$$|\operatorname{pr}_2 y - \operatorname{pr}_2 x + \operatorname{pr}_2(\omega_\theta)| = |\operatorname{pr}_2 y - \operatorname{pr}_2(e^{i\theta}y) + \operatorname{pr}_2 y_\theta - \operatorname{pr}_2 x| \leq 2\theta + 10\delta^s.$$

Using $\theta \leq \beta \leq \delta^s$, we get that

$$|\operatorname{pr}_2 y - \operatorname{pr}_2 x + \operatorname{pr}_2(\omega_\theta)| \leq 12\delta^s.$$

By (3.13), $\operatorname{pr}_2(y_{\text{small}}) \equiv \operatorname{pr}_2 y \pmod{1/b^m}$ and $\operatorname{pr}_2(x_{\text{small}}) \equiv \operatorname{pr}_2 x \pmod{1/b^m}$. Therefore, since $x_{\text{small}} = \xi$, we have

$$\begin{aligned}\operatorname{pr}_2(y_{\text{small}}) \in (\operatorname{pr}_2 x - \operatorname{pr}_2(\omega_\theta) + [-12\delta^s, 12\delta^s] \pmod{1/b^m}) = \\ (\operatorname{pr}_2 \xi - \operatorname{pr}_2(\omega_\theta) + [-12\delta^s, 12\delta^s] \pmod{1/b^m}). \qquad (3.18)\end{aligned}$$

Using (3.14) and the fact that $\mathrm{pr}_2(y_{\mathrm{small}}) \in [0, 1/b^m) - 1/b^n$, (3.18) implies that $\mathrm{pr}_2(y_{\mathrm{small}})$ can only take at most 25 different values. Consequently, since by construction, $\mathrm{pr}_2(y_{\mathrm{small}})$ uniquely determines y_{small},

$$y_{\mathrm{small}} \text{ can only take at most 25 different values.} \tag{3.19}$$

Fix z to be one of these 25 different values.

Suppose now that (x, y), and also (x', y') satisfy (3.11) and that $x_{\mathrm{small}} = \xi = x'_{\mathrm{small}}$, $y_{\mathrm{small}} = z = y'_{\mathrm{small}}$. By (3.11), we have $|\,\mathrm{pr}_1(x - x' - y_\theta + y'_\theta)| \le 20\delta$. We also have

$$x - x' - y_\theta + y'_\theta = x - x' - e^{i\theta} y + e^{i\theta} y' = x - x' - e^{i\theta}(y - y').$$

By (3) of Lemma 29 applied to $y - y'$, we get

$$||\,\mathrm{pr}_1(e^{i\theta}(y - y')) - \mathrm{pr}_1(y - y')| - |\theta\,\mathrm{pr}_2(y - y')|| \le 2\theta^2 |y - y'|.$$

Since

$$|\,\mathrm{pr}_1(x - x' - y + y') - \mathrm{pr}_1(x - x' - y_\theta + y'_\theta)| = |\,\mathrm{pr}_1(e^{i\theta}(y - y')) - \mathrm{pr}_1(y - y')|,$$

we also get

$$||\,\mathrm{pr}_1(x - x' - y + y') - \mathrm{pr}_1(x - x' - y_\theta + y'_\theta)| - |\theta\,\mathrm{pr}_2(y - y')|| \le 2\theta^2 |y - y'|. \tag{3.20}$$

Since $|\,\mathrm{pr}_1(x - x' - y_\theta + y'_\theta)| \le 20\delta \le 20/a^m$, $\theta|\,\mathrm{pr}_2(y - y')| \le \theta \le 1/a^m$, and $2\theta^2|y - y'| \le 4\theta \le 4/a^m$, we must have $|\,\mathrm{pr}_1(x - x' - y + y')| \le 25/a^m$.

Now we utilize $\mathrm{pr}_1(x - x' - y + y') = \mathrm{pr}_1(x - x' - y + y')_{\mathrm{large}} + \mathrm{pr}_1(x - x' - y + y')_{\mathrm{small}}$.

Since by definition, $y_{\text{small}} = y'_{\text{small}}$ and $x_{\text{small}} = x'_{\text{small}}$, we have

$$\operatorname{pr}_1(x - x' - y + y')_{\text{small}} = 0.$$

Therefore,

$$|\operatorname{pr}_1(x - x' - y + y')_{\text{large}}| = |\operatorname{pr}_1(x - x' - y + y')| \leq 25/a^m.$$

Finally, since by (3.15), $\operatorname{pr}_1(x - x' - y + y')_{\text{large}}$ is an integer multiple of $1/a^m$, we have that

$$\operatorname{pr}_1(x - x' - y + y') \text{ can only take at most 51 different values.} \tag{3.21}$$

Now, using (3.20), $|\operatorname{pr}_1(x - x' - y_\theta + y'_\theta)| \leq 20\delta$, $\theta^2 \leq \delta^s \delta^{1-s} = \delta$, and $|y - y'| \leq 2$, we get that

$$\begin{aligned} |\theta \operatorname{pr}_2(y - y')| &\leq |\operatorname{pr}_1(x - x' - y + y')| + |\operatorname{pr}_1(x - x' - y_\theta + y'_\theta)| + 2\theta^2 |y - y'|) \leq \\ &\leq |\operatorname{pr}_1(x - x' - y + y')| + 24\delta. \end{aligned}$$

Similarly,

$$\begin{aligned} |\theta \operatorname{pr}_2(y - y')| &\geq |\operatorname{pr}_1(x - x' - y + y')| - |\operatorname{pr}_1(x - x' - y_\theta + y'_\theta)| - 2\theta^2 |y - y'| \geq \\ &\geq |\operatorname{pr}_1(x - x' - y + y')| - 24\delta. \end{aligned}$$

Combining these with (3.21), we obtain that $|\theta \operatorname{pr}_2(y - y')|$ is contained in the union of at most 51 intervals of length 48δ. By construction, since $y_{\text{small}} = y'_{\text{small}}$, $\operatorname{pr}_2(y - y') = \frac{l}{b^m}$ for some integer l. We get that l is contained in the union of at most 51 intervals of length $\frac{48\delta \cdot b^m}{\theta} \lesssim \frac{\delta}{\theta^{s+1}}$. Since l is an integer, this means that the number of possible values of $\operatorname{pr}_2(y - y')$ is $\lesssim \max\left(\frac{\delta}{\theta^{s+1}}, 1\right)$. Since $\operatorname{pr}_2(y)$ uniquely determines y, this means that the number of y such that $y_{\text{small}} = z$ and (3.11) holds for y and some x, is $\lesssim \max\left(\frac{\delta}{\theta^{s+1}}, 1\right)$.

Taking all 25 values of z and using Lemma 20, by the definition of M_θ we get

$$M_\theta \lesssim \max\left(\frac{\delta}{\theta^{s+1}}, 1\right)$$

and the proof of Claim 28 concludes. $\square$

Using (3.17) and Claim 28, we get that

$$L(\delta,\theta) \lesssim \frac{\theta^s}{\delta^s} \cdot M_\theta \lesssim \frac{\theta^s}{\delta^s} \cdot \max\left(\frac{\delta}{\theta^{s+1}}, 1\right) = \frac{1}{\delta^s} \cdot \max\left(\frac{\delta}{\theta}, \theta^s\right), \tag{3.22}$$

and the proof of Lemma 25 concludes.

$\square$

Proof of Lemma 26. Fix θ such that $\theta \in A, \beta < \theta \leq 1$. For clarity, we will consider the $s \leq 1/2$ and $s > 1/2$ cases separately. Note that

$$\beta = \begin{cases} \delta^s & \text{when } s \geq 1/2 \\ \delta^{1-s} & \text{when } s \leq 1/2. \end{cases} \tag{3.23}$$

Let

$$r_0 = \begin{cases} \theta^{1/s} & \text{if } s \geq 1/2 \\ \theta^{1/(1-s)} & \text{if } s \leq 1/2. \end{cases} \tag{3.24}$$

Choose the unique $t \in \mathbb{N}$ such that $r_0 \in (1/a^{t+1}, 1/a^t]$, and let $r = 1/a^t$. We have

$$\theta = \min\left(r_0^s, r_0^{1-s}\right) \leq \min\left(r^s, r^{1-s}\right) =: \beta(r). \tag{3.25}$$

The main idea is the following: We will apply the previous "small angle" case for r in place of δ, that is, for the approximation of Γ consisting of rectangles of size $3r \times 3r^s$ (note that r is larger than δ). Moreover, we will show that inside each intersecting pair of rectangles of size $3r \times 3r^s$, there can not be too many intersecting rectangles of size $3\delta \times 3\delta^s$.

First, using Lemma 25 for r in place of δ which is valid by (3.25), we obtain that

$$L(r,\theta) \lesssim \frac{1}{r^s} \cdot \max\left(\frac{r}{\theta}, \theta^s\right). \tag{3.26}$$

Second, using that each $3r \times 3r^s$ rectangle contains r^s/δ^s many $3\delta \times 3\delta^s$ rectangles in our construction, we trivially have that

$$L(\delta,\theta) \lesssim \frac{r^s}{\delta^s} \cdot L(r,\theta) \lesssim \frac{1}{\delta^s} \cdot \max\left(\frac{r}{\theta}, \theta^s\right) \lesssim \frac{1}{\delta^s} \cdot \max\left(\frac{r_0}{\theta}, \theta^s\right), \tag{3.27}$$

which is the first claim of Lemma 26. We will use this estimate when $s > 1/2$ and θ is large but not very large, see (3.9) and (3.10).

Assume now that $\theta \geq \delta^{1-s}$. This holds when $s \leq 1/2$ and θ is large, and when $s > 1/2$ and θ is very large. In this case, we will use the following lemma.

Lemma 30. *Let $\theta \geq \delta^{1-s}$. Then $L(\delta,\theta) \lesssim \frac{r^s}{(\theta\delta^s)^s} \cdot L(r,\theta)$.*

Proof of Lemma 30. We introduce another scale. Let $\rho_0 = \theta \cdot \delta^s$, choose the unique $k \in \mathbb{N}$ such that $\rho_0 \in (1/a^{k+1}, 1/a^k]$, and let $\rho = 1/a^k$. One can easily check that $\delta \leq \rho_0 \leq r_0$, and then $\delta \leq \rho \leq r$. Recall that $T_i, T_{j,\theta}$ denote rectangles of size $3\delta \times 3\delta^s$ in R, R_θ, respectively.

Lemma 31. *Let S be a rectangle of R_k and let S'_θ be a rectangle of $R_{k,\theta}$ such that $S \cap S'_\theta \neq \emptyset$. Then*

$$|\left\{(i,j) : T_i \subset S, T_{j,\theta} \subset S'_\theta, T_i \cap T_{j,\theta} \neq \emptyset\right\}| \leq 220a.$$

Proof of Lemma 31. We will start by showing the following claim.

Claim 32. *Assume that i, i', j, j' such that $T_i \cap T_{j,\theta} \neq \emptyset, T_{i'} \cap T_{j',\theta} \neq \emptyset$, $T_i, T_{i'} \subset S, T_{j,\theta}, T_{j',\theta} \subset S'_\theta$. Then $|i'-i| \leq 10a$ or $|j'-j| \leq 10a$.*

Proof of Claim 32. Let $v \in \mathbb{R}^2$ be a vector such that $|\operatorname{pr}_1 v| \leq 3\rho$, $|\operatorname{pr}_2 v| \geq 8a\delta^s$. Let α

denote the angle of v and the vertical axis. We show that $\alpha < \theta/2$:

$$\alpha \leq \tan\alpha = \frac{|\operatorname{pr}_1 v|}{|\operatorname{pr}_2 v|} \leq \frac{3\rho}{8a\delta^s} \leq \frac{3a\rho_0}{8a\delta^s} = \frac{3\theta\cdot\delta^s}{8\delta^s} < \frac{\theta}{2}. \tag{3.28}$$

Assume now for contradiction that $|i'-i| > 10a$ and $|j'-j| > 10a$, $p \in T_i \cap T_{j,\theta}$, $q \in T_{i'} \cap T_{j',\theta}$, and let $v = p - q$. By construction, $|\operatorname{pr}_2 v| \geq (10a+1)\delta^s - 3\delta^s \geq 8a\delta^s$, and since $p, q \in S$, we have $|\operatorname{pr}_1 v| \leq 3\rho$. So by (3.28), $\alpha < \frac{\theta}{2}$.

Now let ℓ denote the vertical axis, $\ell_\theta = e^{i\theta}\ell$, let α_θ denote the angle of v and ℓ_θ, and let $\operatorname{proj}_{\ell_\theta}, \operatorname{proj}_{\ell_{\theta\perp}}$ denote the orthogonal projection onto ℓ_θ and $\ell_{\theta\perp}$, respectively. By construction, similarly above, $|\operatorname{proj}_{\ell_\theta}(v)| \geq 8a\delta^s$. Moreover, since $p, q \in S'_\theta$, $|\operatorname{proj}_{\ell_{\theta\perp}}(v)| \leq 3\rho$. Similarly as in (3.28) above, we obtain that $\alpha_\theta < \frac{\theta}{2}$. But clearly, $\theta = \alpha + \alpha_\theta < \theta/2 + \theta/2$, so we get a contradiction and the statement of Claim 32 follows.

$\square$

Let i, i', j, j' as in the statement of Claim 32. Then without loss of generality, $|i'-i| \leq 10a$. Therefore, the number of T_i's contained in S that intersect some $T_{j,\theta} \subset S'_\theta$ is at most $22a$. By Lemma 20, we know that for each T_i there are at most 10 $T_{j,\theta}$ such that $T_i \cap T_{j,\theta} \neq \emptyset$. So we get that

$$|\left\{(i,j) : T_i \subset S, T_{j,\theta} \subset S'_\theta, T_i \cap T_{j,\theta} \neq \emptyset\right\}| \leq 10 \cdot 22a = 220a$$

and the proof of Lemma 31 concludes. $\square$

We are ready to finish the proof of Lemma 30. By Lemma 31, $L(\delta,\theta) \lesssim L(\rho,\theta)$. Since the number of $3\rho \times 3\rho^s$-rectangles in our construction inside a fixed rectangle of size $3r \times 3r^s$ is $\lesssim r^s/\rho^s$, clearly, we have $L(\rho,\theta) \lesssim r^s/\rho^s \cdot L(r,\theta) \leq r^s/(\theta\delta^s)^s \cdot L(r,\theta)$. Consequently, $L(\delta,\theta) \lesssim \frac{r^s}{(\theta\delta^s)^s} \cdot L(r,\theta)$ and the proof of Lemma 30 concludes.

$\square$

Combining (3.26) and Lemma 30, we obtain that

$$L(\delta,\theta) \lesssim \frac{r^s}{(\theta\delta^s)^s}\cdot\frac{1}{r^s}\cdot\max\left(\frac{r}{\theta},\theta^s\right) \lesssim \frac{1}{\theta^s}\cdot\frac{1}{\delta^{s^2}}\cdot\max\left(\frac{r_0}{\theta},\theta^s\right), \tag{3.29}$$

which is the second statement in Lemma 26.

□

Using the upper bounds for $L(\delta,\theta)$ obtained in Lemma 25 and 26, we now give an upper bound for $\mathcal{L}(R_\theta\cap R)$ and eventually for $\sum\limits_{\theta\in A,\theta\geq\delta}\mathcal{L}(R_\theta\cap R)$.

Let θ be small. Then by Lemma 21 and Lemma 25, we have

$$\mathcal{L}(R_\theta\cap R) \lesssim \frac{1}{\delta^s}\cdot\max\left(\frac{\delta}{\theta},\theta^s\right)\cdot\delta^{s+1} = \delta\cdot\max\left(\frac{\delta}{\theta},\theta^s\right). \tag{3.30}$$

Summing up for small angles θ, we obtain

$$\begin{aligned}\sum_{\theta\in A,\delta\leq\theta\leq\beta}\mathcal{L}(R_\theta\cap R) &\lesssim \sum_{\theta\in A,\delta\leq\theta\leq\beta}\delta\cdot\max\left(\frac{\delta}{\theta},\theta^s\right) = \\ &= \delta^2\cdot\sum_{\theta\in A,\delta\leq\theta\leq\delta^{1/(s+1)}}\frac{1}{\theta}+\delta\cdot\sum_{\theta\in A,\delta^{1/(s+1)}\leq\theta\leq\beta}\theta^s.\end{aligned}$$

Easy computation gives that

$$\sum_{\theta\in A,\delta\leq\theta\leq\delta^{1/(s+1)}}\frac{1}{\theta}\lesssim\frac{1}{\delta}\cdot\log\left(\frac{1}{\delta}\right),\quad \sum_{\theta\in A,\delta^{1/(s+1)}\leq\theta\leq\beta}\theta^s\lesssim\frac{1}{\delta}\cdot\beta^{s+1}. \tag{3.31}$$

Therefore, using the definition of β, see (3.23), easy computation gives that

$$\sum_{\theta\in A,\delta\leq\theta\leq\beta}\mathcal{L}(R_\theta\cap R)\lesssim\delta\cdot\log\left(\frac{1}{\delta}\right)+\beta^{s+1}\lesssim\begin{cases}\delta\cdot\log\left(\frac{1}{\delta}\right) & \text{if } s\geq s_0,\\ \delta^{s(s+1)}\cdot\log\left(\frac{1}{\delta}\right) & \text{if } \frac{1}{2}\leq s\leq s_0,\\ \delta^{1-s^2}\cdot\log\left(\frac{1}{\delta}\right) & \text{if } s\leq\frac{1}{2},\end{cases}$$

where $s_0 = \frac{\sqrt{5}-1}{2}$. Since $s(s+1) \geq 1 - s^2$ when $s \geq 1/2$, we obtain

$$\sum_{\theta \in A, \delta \leq \theta \leq \beta} \mathcal{L}(R_\theta \cap R) \lesssim \delta^{1-s^2} \cdot \log\left(\frac{1}{\delta}\right) \tag{3.32}$$

for each s.

Now we consider the large angles. Assume first that $s \leq 1/2$. In this case, θ large implies that $\theta \geq \delta^{1-s}$, and $r_0 = \theta^{1/(1-s)}$, see (3.9), (3.23), and (3.24). By Lemma 21 and the second bound from Lemma 26 and using that $\theta^{1/(1-s)} \leq \theta^{s+1}$, we obtain

$$\mathcal{L}(R_\theta \cap R) \lesssim \frac{1}{\delta^{s^2}} \cdot \max\left(\frac{\theta^{1/(1-s)}}{\theta^{s+1}}, 1\right) \cdot \frac{\delta^2}{\theta} = \frac{\delta^{2-s^2}}{\theta}.$$

Summing up for θ, we get that

$$\sum_{\theta \in A, \theta > \beta} \mathcal{L}(R_\theta \cap R) \lesssim \delta^{2-s^2} \sum_{\theta \in A, \theta > \beta} \frac{1}{\theta} \lesssim \delta^{1-s^2} \cdot \log\left(\frac{1}{\delta}\right).$$

Now let $s > 1/2$, so $r_0 = \theta^{1/s}$, see (3.24). For $\theta \geq \delta^{1-s}$, we use the same bound from Lemma 26 as in the $s \leq 1/2$ case, and obtain that

$$\mathcal{L}(R_\theta \cap R) \lesssim \frac{1}{\delta^{s^2}} \cdot \max\left(\frac{\theta^{1/s}}{\theta^{s+1}}, 1\right) \cdot \frac{\delta^2}{\theta} = \frac{\delta^{2-s^2}}{\theta} \cdot \max\left(\frac{\theta^{1/s}}{\theta^{s+1}}, 1\right). \tag{3.33}$$

For $\beta = \delta^s \leq \theta \leq \delta^{1-s}$, we use the first bound from Lemma 26, and the first trivial bound from Lemma 21. Therefore,

$$\mathcal{L}(R_\theta \cap R) \lesssim \frac{1}{\delta^s} \cdot \max\left(\frac{\theta^{1/s}}{\theta}, \theta^s\right) \cdot \delta^{s+1} = \delta \cdot \theta^s \cdot \max\left(\frac{\theta^{1/s}}{\theta^{s+1}}, 1\right) \tag{3.34}$$

Easy computation gives that $\frac{\theta^{1/s}}{\theta^{s+1}} \leq 1$ if and only if $s \leq s_0 = \frac{\sqrt{5}-1}{2}$.

In the $s \le s_0$ case, from (3.33) and (3.34) we obtain

$$\mathcal{L}(R_\theta \cap R) \lesssim \begin{cases} \frac{\delta^{2-s^2}}{\theta} & \text{for } \theta \ge \delta^{1-s}, \\ \delta \cdot \theta^s & \text{for } \delta^s \le \theta \le \delta^{1-s}, \end{cases}$$

and then

$$\sum_{\theta \in A, \theta > \beta} \mathcal{L}(R_\theta \cap R) \lesssim \sum_{\theta \in A, \theta > \delta^{1-s}} \frac{\delta^{2-s^2}}{\theta} + \sum_{\theta \in A, \delta^s \le \theta \le \delta^{1-s}} \delta \cdot \theta^s \lesssim \delta^{1-s^2} \log\left(\frac{1}{\delta}\right). \tag{3.35}$$

Combining (3.32) with (3.35), we obtain the statement of (3.8) for $s \le s_0$.

In the $s \ge s_0$ case, we get

$$\mathcal{L}(R_\theta \cap R) \lesssim \begin{cases} \frac{\delta^{2-s^2}}{\theta} \cdot \frac{\theta^{1/s}}{\theta^{s+1}} & \text{for } \theta \ge \delta^{1-s}, \\ \delta \cdot \theta^s \cdot \frac{\theta^{1/s}}{\theta^{s+1}} & \text{for } \delta^s \le \theta \le \delta^{1-s}. \end{cases}$$

Summing up for θ, easy computation using that $1/s - s - 1 \le 0$ as $s \ge s_0$ yields

$$\sum_{\theta \in A, \theta > \beta} \mathcal{L}(R_\theta \cap R) \lesssim \tag{3.36}$$
$$\sum_{\theta \in A, \theta > \delta^{1-s}} \delta^{2-s^2} \theta^{1/s-s-2} + \sum_{\theta \in A, \delta^s \le \theta \le \delta^{1-s}} \delta \cdot \theta^{1/s-1} \lesssim \delta^{1/s-1}.$$

Combining (3.32) with (3.36), we obtain the statement of (3.8) for $s \ge s_0$ as well.

Finally, using (3.8), summing up for $\theta' \in A$ as well as accounting for (3.5), we get the statement of Proposition 23:

$$\sum_{\theta, \theta' \in A} \mathcal{L}(R_\theta \cap R_{\theta'}) \lesssim \frac{1}{\delta} \cdot \sum_{\theta \in A, \theta \ge \delta} \mathcal{L}(R_\theta \cap R) + 1 \lesssim \max\left(\delta^{-s^2}, \delta^{1/s-2}\right) \cdot \log\left(\frac{1}{\delta}\right).$$

□

3.4 Proof of Theorem 19 from Proposition 23

The proof is based on a standard pigeonholing argument. For the sake of completeness, we include the proof. Let $d = \min\left(2 - s^2, \frac{1}{s}\right)$, fix $0 < \varepsilon < d$, and write $u = d - \varepsilon$. Let K be a positive integer such that $\sum_{l=K}^{\infty} 1/l^2 < 1$ and $2a^{-K} \leq \delta_0$. Moreover, let $0 < \eta \leq 2a^{-K}$. We will show that $\mathcal{H}^u_\eta(E) \gtrsim_\varepsilon 1$ (independently of η), so $\dim E \geq d$.

Let $E \subset \bigcup_{i=1}^{\infty} B(x_i, r_i)$ be any countable cover of E consisting of disks with $2r_i \leq \eta \leq 2a^{-K}$ for all i. For any $l \geq K$, let

$$J_l = \left\{i : a^{-l-1} < r_i \leq a^{-l}\right\}, \text{ and let } E_l = \left(\bigcup_{i \in J_l} B(x_i, r_i)\right) \cap E.$$

Then $E = \bigcup_{l=K}^{\infty} E_l$, and E_l is covered by disks of radius $\approx a^{-l}$. Using a pigeonholing argument, we prove the following:

Lemma 33. *There exists an integer $l \geq K$ such that*

$$\lambda\left(\theta \in [0,1] : \mathcal{H}^1_\infty(E_l \cap \Gamma_\theta) \geq 1/l^2\right) \geq 1/l^2.$$

Proof. Assume for contradiction that no such l exists. Then for each l,

$$\lambda\left(\theta \in [0,1] : \mathcal{H}^1_\infty(E_l \cap \Gamma_\theta\right) \geq 1/l^2) < 1/l^2,$$

and $\sum_{l=K}^{\infty} 1/l^2 < 1 = \lambda([0,1])$, so these sets can not cover $[0,1]$. We note that the measurability of the above sets can be checked similarly as in the proof of Lemma 3.1, [39]. Therefore, there exists θ such that for all $l \geq K$, $\mathcal{H}^1_\infty(E_l \cap \Gamma_\theta) < 1/l^2$. But then, using that by construction, $1 \leq \mathcal{H}^1_\infty([0,1]) \leq \mathcal{H}^1_\infty(\Gamma_\theta) \leq 1$, we have

$$1 = \mathcal{H}^1_\infty(\Gamma_\theta) \leq \sum_{l=K}^{\infty} \mathcal{H}^1_\infty(E_l \cap \Gamma_\theta) < \sum_{l=K}^{\infty} 1/l^2 < 1,$$

which is a contradiction. $\square$

Fix an l obtained from Lemma 33, and let

$$A_0 = \left\{\theta \in [0,1] : \mathcal{H}^1_\infty(E_l \cap \Gamma_\theta) \geq 1/l^2\right\},$$

so we have $\lambda(A_0) \geq 1/l^2$. Letting $\widetilde{\Gamma_\theta} = \Gamma_\theta \cap E_l$ for $\theta \in A_0$, by construction we have $\mathcal{H}^1_\infty(\widetilde{\Gamma_\theta}) \geq 1/l^2$. Let $\widetilde{E} = \bigcup_{\theta \in A_0} \widetilde{\Gamma_\theta} \subset E_l \subset E$.

Claim 34. *Let $\delta = a^{-l}$, where l is fixed as above. Then $\lambda(\widetilde{E}(\delta)) \gtrsim \frac{\delta^{2-d}}{\log^9(1/\delta)}$.*

Assuming Claim 34 and using that by construction, $\widetilde{E}(\delta) \subset \bigcup_{i \in J_l} B(x_i, r_i + \delta) \subset \bigcup_{i \in J_l} B(x_i, 2\delta)$, we get the following:

$$\frac{\delta^{2-d}}{\log^9(1/\delta)} \lesssim \lambda(\widetilde{E}(\delta)) \leq \lambda\left(\bigcup_{i \in J_l} B(x_i, 2\delta)\right) \lesssim |J_l| \cdot \delta^2,$$

so $|J_l| \gtrsim \frac{\delta^{-d}}{\log^9(1/\delta)}$. Then we have

$$\sum_{i=1}^{\infty} (2r_i)^u \geq \sum_{i \in J_l} (2r_i)^u \gtrsim \sum_{i \in J_l} \left(a^{-l}\right)^u = |J_l| \cdot \delta^u \gtrsim \frac{\delta^{-d+d-\varepsilon}}{\log^9(1/\delta)} = \frac{\delta^{-\varepsilon}}{\log^9(1/\delta)} \gtrsim_\varepsilon 1.$$

This means that $\mathcal{H}^u_\eta(B) \gtrsim_\varepsilon 1$, and the proof concludes.

Proof of Claim 34. Let $\widetilde{R_\theta} = \widetilde{\Gamma_\theta}(\delta) \subset \Gamma_\theta(\delta) \subset R_\theta$ for $\theta \in A_0$. Then $\bigcup_{\theta \in A_0} \widetilde{R_\theta} = \widetilde{E}(\delta)$. Let $\widetilde{A}$ be a maximal δ-separated subset of A_0. Then clearly, $A_0 \subset \bigcup_{\theta \in \widetilde{A}} (\theta - \delta, \theta + \delta)$. Therefore we get $\lambda(A_0) \lesssim |\widetilde{A}| \cdot \delta$, and by the definition of A_0 and δ, we obtain that

$$|\widetilde{A}| \gtrsim \lambda(A_0) \cdot \delta^{-1} \geq \frac{1}{l^2} \cdot \delta^{-1} \gtrsim \frac{1}{\log^2(1/\delta) \cdot \delta}. \tag{3.37}$$

By the Cauchy-Schwarz inequality, and using that trivially, $\bigcup_{\theta \in \widetilde{A}} \widetilde{R_\theta} \subset \widetilde{E}(\delta)$ and $\widetilde{R_\theta} \subset$

R_θ, we get the following:

$$\sum_{\theta\in\widetilde{A}} \mathcal{L}(\widetilde{R_\theta}) \le \mathcal{L}\left(\bigcup_{\theta\in\widetilde{A}} \widetilde{R_\theta}\right)^{1/2} \cdot \left(\sum_{\theta,\theta'\in\widetilde{A}} \mathcal{L}(\widetilde{R_{\theta'}} \cap \widetilde{R_\theta})\right)^{1/2} \le \tag{3.38}$$

$$\le \mathcal{L}\left(\widetilde{E}(\delta)\right)^{1/2} \cdot \left(\sum_{\theta,\theta'\in\widetilde{A}} \mathcal{L}(R_{\theta'} \cap R_\theta)\right)^{1/2}.$$

By Proposition 23 applied for the above δ and $A = \widetilde{A}$, we have

$$\sum_{\theta,\theta'\in\widetilde{A}} \mathcal{L}(R_\theta \cap R_{\theta'}) \lesssim \delta^{d-2} \cdot \log(\frac{1}{\delta}). \tag{3.39}$$

It remains to prove a lower bound for the left hand side of the Cauchy-Schwarz inequality. Let $N(X,\delta)$ denote the smallest number of δ-disks needed to cover $X \subset \mathbb{R}^2$. Then $\mathcal{H}^1_\infty(X) \le N(X,\delta)\cdot 2\delta$ and $\mathcal{L}(X(\delta)) \gtrsim N(X,\delta)\cdot\delta^2$. Therefore we have

$$\mathcal{L}(\widetilde{R_\theta}) = \mathcal{L}(\widetilde{\Gamma_\theta}(\delta)) \gtrsim N(\widetilde{\Gamma_\theta},\delta)\cdot\delta^2 \gtrsim \mathcal{H}^1_\infty(\widetilde{\Gamma_\theta})\cdot\delta \gtrsim \frac{\delta}{\log^2(1/\delta)}.$$

Summing up for $\theta \in \widetilde{A}$ and using (3.37), we get

$$\sum_{\theta\in\widetilde{A}} \mathcal{L}(\widetilde{R_\theta}) \gtrsim |\widetilde{A}| \cdot \frac{\delta}{\log^2(1/\delta)} \gtrsim \frac{1}{\log^4(1/\delta)}. \tag{3.40}$$

Finally, combining (3.38), (3.39) and (3.40) we get

$$\mathcal{L}(\widetilde{E}(\delta)) \gtrsim \frac{1}{\log^8(1/\delta)} \cdot \frac{1}{\log(1/\delta)} \cdot \delta^{2-d} = \frac{\delta^{2-d}}{\log^9(1/\delta)},$$

and this was the statement of Claim 34. □

CHAPTER 4

A ONE-DIMENSIONAL PLANAR BESICOVITCH-TYPE SET

I have submitted this chapter as a paper to a journal.

4.1 Introduction

In the last chapter, we only got a lower bound on the dimension of such Γ-Besicovitch sets. It left open the problem, whether there exist a non-trivial Γ-Besicovitch set of dimension less than 2?

It was a well known open problem asked by Marianna Csörnyei whether there exists a non-trivial Γ-Besicovitch set of dimension less than 2. Until now, the only known Γ-Besicovitch sets of dimension less than 2 were sets covered by lower dimensional union of circles. The construction of measure zero sets by Csörnyei and Chang in [16] depended upon the tangent field of Γ. It is well-known that every 1-rectifiable set in the plane has a tangent field which is defined almost everywhere. For a circular arc all the tangent lines have different directions and for a straight line, all the tangent lines are parallel. This suggested that it is interesting to look at the Γ-Besicovitch sets for rectifiable sets with the same tangent field as that of a line i.e. rectifiable sets whose tangents are parallel almost everywhere.

Marianna Csörnyei asked, in particular, does there exist a 1-rectifiable set with parallel tangents almost everywhere for which there exists a Γ-Besicovitch set of dimension less than 2? Apart from being an interesting question in itself, it may help us in understanding the (usual) Besicovitch set. In this chapter we are answering this question affirmatively. Our main theorem is the following.

Theorem 35. *There exists a* 1*-rectifiable set* Γ *with the following properties.*

1. *The 1-dimensional Hausdorff measure of* Γ *is positive.*

2. *There exists a 1-dimensional* Γ*-Besicovitch set.*

3. *The set* Γ *is the graph of a monotone function whose domain is a Cantor set of measure* 0 *and thus the tangents of* Γ *are vertical almost everywhere.*

For any $s \in [0, 1]$*, we can construct the set* Γ *such that the Cantor set in (iii) has Hausdorff dimension* s*.*

Remark 36. As the set Γ has positive length and parallel tangents almost everywhere, it is straightforward to see that it cannot be covered by a union of concentric circles whose dimension is less than 2.

Notation For $d \in \{1, 2\}$ and a set $E \subset \mathbb{R}^d$, $|E|$ denotes the d-dimensional Lebesgue measure of E. For $z \in \mathbb{R}^2$, $P_x(z)$ and $P_y(z)$ denote the x and the y coordinates of z respectively, and $\|z\|$ denotes the Euclidean norm of z. For simplicity of notation we identify the complex plane $\mathbb{C}$ with $\mathbb{R}^2$, i.e. $(x, y) \cong x + iy$. For every $a, b \in \mathbb{R}$, $a \lesssim b$ implies that there is an absolute constant C such that $a \leq Cb$, and $a \simeq b$ implies that $a \lesssim b$ and $b \lesssim a$.
Throughout the paper, dimension refers to Hausdorff dimension.

4.2 Construction of Γ

We will construct the 1-rectifiable set Γ mentioned in Theorem 35 as the countable intersection of the sets $\{\mathcal{S}_n\}_{n\in\mathbb{N}}$. The sets $\mathcal{S}_n$ will be constructed inductively such that they are nested, i.e. $\mathcal{S}_{n+1} \subset \mathcal{S}_n$. For each $n \in \mathbb{N}$, $\mathcal{S}_n$ will be a union of finitely many rectangles with sides parallel to the coordinate axes. We will introduce two positive sequences $\{\delta_n\}_{n\in\mathbb{N}}$ and $\{\Delta_n\}_{n\in\mathbb{N}}$ such that for each rectangle in $\mathcal{S}_n$, its x-projection has length δ_n and its y-projection has length close to Δ_n.

Let $c < 1$ be a small constant to be specified later. Let $\{\delta_n\}_{n\in\mathbb{N}}$ and $\{\Delta_n\}_{n\in\mathbb{N}}$ be positive decreasing sequences converging to 0 with the following properties.

$$\delta_1 = \Delta_1 = 1. \tag{4.1}$$

$$\Delta_{n+1} \leq c\delta_n^{n+1}. \tag{4.2}$$

$$\delta_{n+1} \leq c\Delta_{n+1}\delta_n. \tag{4.3}$$

$$\Delta_2^{-1},\ \Delta_{n+1}^{-1}\delta_n^{-1}\Delta_n\delta_{n-1} \in \mathbb{N},\ \text{ for all } n \geq 2. \tag{4.4}$$

We see from (4.3) that $\delta_n\Delta_n^{-1} \to 0$. This means the rectangles in $\mathcal{S}_n$ get narrower, in the sense that the ratio of the length of the side parallel to the x-axis to the length of the side parallel to the y-axis will go to 0 as $n \to \infty$.

We define $\theta_1 := c$ and $\theta_{n+1} := c\Delta_{n+1}\delta_n$, for all $n \geq 1$. The property (4.4) above implies that

$$\theta_{n+1}^{-1}\theta_n \in \mathbb{N},\ \text{ for all } n \geq 1. \tag{4.5}$$

For our construction of Γ, we will later need to choose points α_{n+1} and q_{n+1} on a circular arc C_n such that α_{n+1} is below the x-axis and is the center of C_n, and the following hold. (See Figure 1)

1. The arc C_n lies in the first quadrant of $\mathbb{R}^2$ and joins the points (0,0) and (δ_n, Δ_n).

2. The point q_{n+1} is the intersection of C_n with the line $y = \Delta_{n+1}$.

3. The sub-arc of C_n joining (0,0) and q_{n+1} has angle $\theta_{n+1} = c\Delta_{n+1}\delta_n$.

To see why we can choose such C_n, α_{n+1} and q_{n+1}, let $\mathcal{C}$ be any circular arc in the first quadrant, joining (0,0) and (δ_n, Δ_n). Let q be the intersection of $\mathcal{C}$ with the line $y = \Delta_{n+1}$ and let θ be the angle of the arc joining $(0,0)$ and q. For any circular arc to pass through the points $(0,0)$ and (δ_n, Δ_n), its center must lie on the perpendicular bisector of the line segment joining these two points, call it I_n. Let $\alpha \in I_n$ be the center of the circular arc

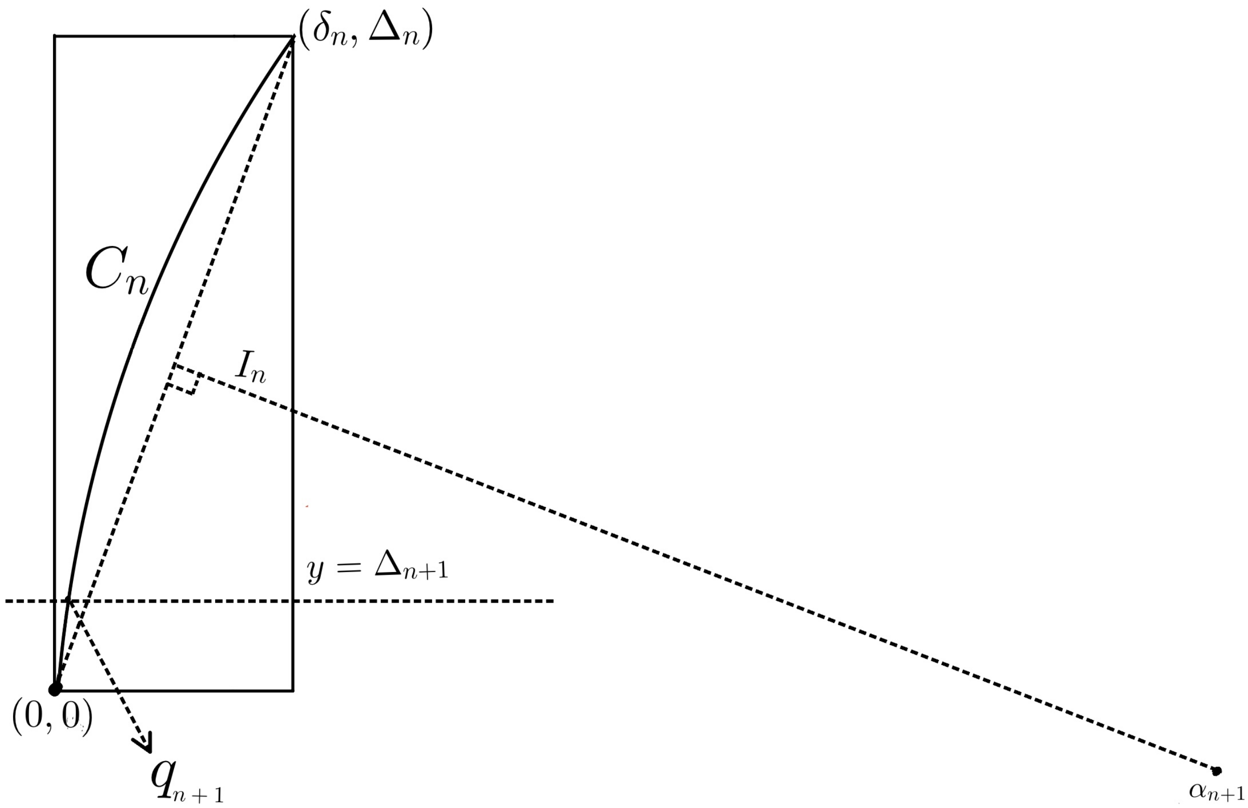


Figure 4.1

$\mathcal{C}$. If $\|\alpha\| \to \infty$, the arc $\mathcal{C}$ converges to the line segment joining $(0,0)$ and (δ_n, Δ_n) and the angle of $\mathcal{C}$ converges to 0. An elementary calculation shows that if α lies on the x-axis, then $\alpha = (\frac{\delta_n^2+\Delta_n^2}{2\delta_n}, 0)$. Since the radius of $\mathcal{C}$ is $\|\alpha\|$, the length of the arc joining (0,0) and q is $\|\alpha\|\theta$. Using $\delta_n \leq \Delta_n$ we have

$$\Delta_{n+1} \leq \|q\| \leq \|\alpha\|\theta = \Big(\frac{\delta_n^2 + \Delta_n^2}{2\delta_n}\Big)\theta \leq \frac{\Delta_n^2}{\delta_n}\theta.$$

Thus in this case $\theta \geq \Delta_{n+1}\Delta_n^{-2}\delta_n$.

Since $\Delta_n \leq 1$ and $c < 1$ we have $0 < c\Delta_{n+1}\delta_n \leq \Delta_{n+1}\Delta_n^{-2}\delta_n$. So we can choose $\alpha_{n+1} \in I_n$ below the x axis such that $\theta = \theta_{n+1} = c\Delta_{n+1}\delta_n$. We denote the corresponding $\mathcal{C}$ and q by C_n and q_{n+1}. (See Figure 1).

We are now ready to construct Γ. We start the inductive process by defining $\mathcal{S}_1 :=$

$Q_{1,1} := [0,1] \times [0,1]$. Assume we have constructed $\mathcal{S}_n = \bigcup_j Q_{n,j}$, where the union is over finitely many axis parallel rectangles $Q_{n,j}$, whose horizontal sides are disjoint and vertical sides are non-overlapping. We denote the bottom-left corner of $Q_{n,j}$ as $p_{n,j}$. We also assume that $Q_{n,j}$ satisfies the following properties.

$$P_x(p_{n,j}) < P_x(p_{n,j+1}) \text{ and } P_y(p_{n,j}) < P_y(p_{n,j+1}) \tag{4.6}$$

$$Q_{n,1} = [0, \delta_n] \times [0, \Delta_n]. \tag{4.7}$$

$$|P_x(Q_{n,j})| = \delta_n. \tag{4.8}$$

For each j, we will construct finitely many rectangles $Q_{n,j,k}$ contained in $Q_{n,j}$. We first define the bottom-left corners of the rectangles $Q_{n,j,k}$ which we call $p_{n,j,k}$. For $k \leq 2\pi\theta_{n+1}^{-1}$, let the point $p_{n,j,k}$ be

$$p_{n,j,k} = \alpha_{n+1}(1 - e^{-i(k-1)\theta_{n+1}}) + e^{-i(k-1)\theta_{n+1}} p_{n,j}. \tag{4.9}$$

Putting $k = 1$ in the above equation, we see $p_{n,j,1} = p_{n,j}$. From (4.7), we see that $p_{n,1} = (0,0)$. This means that for $j = 1$ we have

$$p_{n,1,k} = \alpha_{n+1}(1 - e^{-i(k-1)\theta_{n+1}}). \tag{4.10}$$

Thus the point $p_{n,1,k}$ is the image of $(0,0)$ under rotation by $-(k-1)\theta_{n+1}$ around the point α_{n+1} (see Figure 2) and $p_{n,1,2} = q_{n+1}$. From (4.9) we also get

$$\begin{aligned} p_{n,j,k+1} - p_{n,j,k} &= \alpha_{n+1}(e^{-i(k-1)\theta_{n+1}} - e^{-ik\theta_{n+1}}) + \left(e^{-ik\theta_{n+1}} - e^{-i(k-1)\theta_{n+1}}\right) p_{n,j} \\ &= e^{-i(k-1)\theta_{n+1}}(1 - e^{-i\theta_{n+1}})(\alpha_{n+1} - p_{n,j}). \end{aligned} \tag{4.11}$$

Thus we have

$$|p_{n,j,k+1} - p_{n,j,k}| = |(1 - e^{-i\theta_{n+1}})(\alpha_{n+1} - p_{n,j})|.$$

Therefore for any fixed j, the points $p_{n,j,k}$ are equidistant, as the modulus of their difference is independent of k. We also have from (4.9)

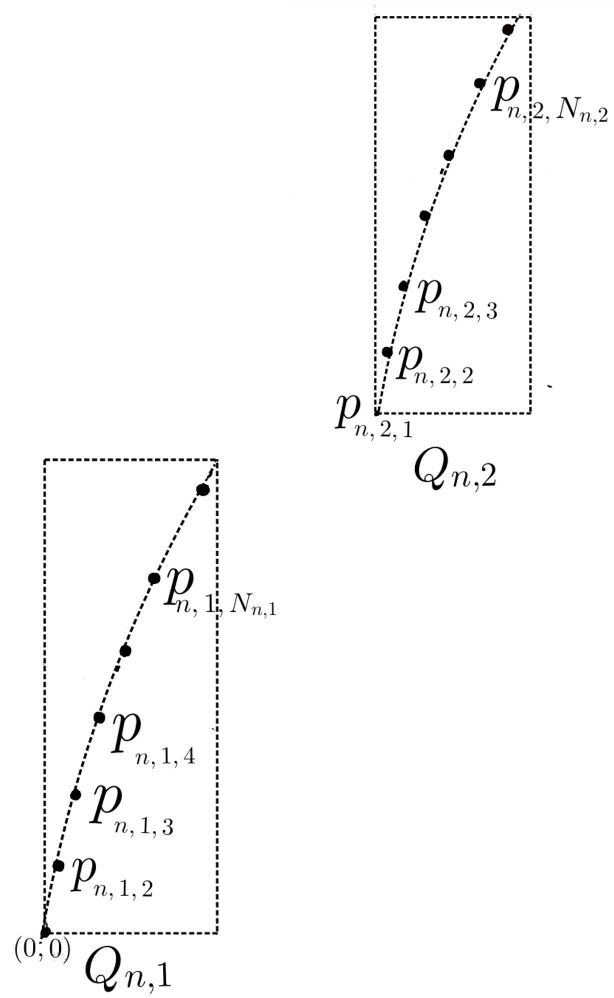


Figure 4.2

$$p_{n,j,k+l} - e^{-il\theta_{n+1}} p_{n,j,k} = \alpha_{n+1}(1 - e^{-il\theta_{n+1}}) = p_{n,1,l+1}. \tag{4.12}$$

Equation (4.12) implies that the point $p_{n,j,k+l}$ is the image of $p_{n,j,k}$ under rotation about the origin by an angle of $-l\theta_{n+1}$ and translation by $p_{n,1,l+1}$. We will use this fact to define the Γ-Besicovitch set later.

If $p_{n,j,k+1} \in Q_{n,j}$ we define the rectangle $Q_{n,j,k}$ with sides parallel to the x and the y axes such that

$$P_x(Q_{n,j,k}) = [P_x(p_{n,j,k}), P_x(p_{n,j,k}) + \delta_{n+1}] \tag{4.13}$$

and

$$P_y(Q_{n,j,k}) = [P_y(p_{n,j,k}), P_y(p_{n,j,k+1})]. \tag{4.14}$$

For two rectangles $Q_{n,j,k}$ and $Q_{n,j',k'}$ with $j \neq j'$, their y-projections are non-overlapping and their x-projections are disjoint, because that is the case for the larger rectangles $Q_{n,j}$ and $Q_{n,j'}$. However, we need to justify this property for rectangles $Q_{n,j,k}$ and $Q_{n,j,k'}$ which lie in the same larger rectangle $Q_{n,j}$. It will follow from Lemma 37 that if $p_{n,j,k} \in Q_{n,j}$ then

$$P_y(p_{n,j,k+1}) > P_y(p_{n,j,k}) \text{ and } P_x(p_{n,j,k+1}) > P_x(p_{n,j,k}) + \delta_{n+1}.$$

Therefore the condition $p_{n,j,k+1} \in Q_{n,j}$, along with the above inequalities, implies that the rectangles $Q_{n,j,k}$ lie inside $Q_{n,j}$ and have disjoint x-projections and non-overlapping y-projections.

We establish some technical properties of the points $p_{n,j,k}$ which lie inside the unit square.

Lemma 37. *Let $n \geq 1$ and $p_{n,j,k} \in [0,1] \times [0,1]$. Then for an absolute constant c_1, we have*

1. $|\Delta_{n+1} - \big(P_y(p_{n,j,k+1}) - P_y(p_{n,j,k})\big)| \leq c_1\theta_{n+1}$.
2. $|\delta_n \Delta_n^{-1} \Delta_{n+1} - \big(P_x(p_{n,j,k+1}) - P_x(p_{n,j,k})\big)| \leq 3c_1\theta_{n+1}$.

Proof. By (4.12) we have $p_{n,1,2} = p_{n,j,k+1} - e^{-i\theta_{n+1}} p_{n,j,k}$. Therefore

$$\begin{aligned} p_{n,1,2} - (p_{n,j,k+1} - p_{n,j,k}) &= p_{n,j,k+1} - e^{-i\theta_{n+1}} p_{n,j,k} - (p_{n,j,k+1} - p_{n,j,k}) \\ &= (1 - e^{-i\theta_{n+1}}) p_{n,j,k}. \end{aligned} \tag{4.15}$$

As $|1-e^{-i\theta_{n+1}}| \lesssim \theta_{n+1}$ and $|p_{n,j,k}| \lesssim 1$, we have

$$|p_{n,1,2} - (p_{n,j,k+1} - p_{n,j,k})| = |(1-e^{-i\theta_{n+1}})p_{n,j,k}| \le c_1\theta_{n+1}. \tag{4.16}$$

Notice that c_1 is an absolute constant and it does not depend on c. Recall that $P_y(p_{n,1,2}) = \Delta_{n+1}$. Therefore (i) holds.

Recall that the circular arc C_n is in the first quadrant with center below the x-axis. So it is the graph of a concave function. Moreover, the line segment joining (0,0) and (δ_n, Δ_n) makes an angle greater than $\pi/4$ with the x-axis. Thus C_n is also the graph of a monotonic function and

$$0 < P_y(p_{n,1,k+1}) - P_y(p_{n,1,k}) \le P_y(p_{n,1,2}) = \Delta_{n+1}.$$

Thus the number of points $p_{n,1,k}$ on C_n is at least $\Delta_n\Delta_{n+1}^{-1} - 1$. By (4.2) and (4.3), we have $\Delta_n\Delta_{n+1}^{-1} \ge c^{-1}\Delta_n\delta_n^{-(n+1)} \ge c^{-1}$. We choose c very small, in particular $c < 1/10$. Thus there are at least 9 points $p_{n,1,k}$ on C_n. By concavity, the slope of the line joining the points $p_{n,1,k}$ and $p_{n,1,k+1}$ decreases as k increases. The slope of the vector $p_{n,1,2}$ is greater than $\Delta_n\delta_n^{-1}$. If $p_{n,1,k+1}$ is the closest point to the corner (δ_n, Δ_n), then $\operatorname{dist}(p_{n,1,k}, (\delta_n, \Delta_n)) < \operatorname{dist}(p_{n,1,k}, (0,0))$. Thus the slope of $p_{n,1,k+1} - p_{n,1,k}$ will be less than $\Delta_n\delta_n^{-1}$. Therefore, using (4.16) we have the following.

$$\frac{\Delta_{n+1} - c_1\,\theta_{n+1}}{P_x(p_{n,1,2}) + c_1\,\theta_{n+1}} = \frac{P_y(p_{n,1,2}) - c_1\,\theta_{n+1}}{P_x(p_{n,1,2}) + c_1\,\theta_{n+1}} \le \frac{P_y(p_{n,1,k+1} - p_{n,1,k})}{P_x(p_{n,1,k+1} - p_{n,1,k})} \le \frac{\Delta_n}{\delta_n}.$$

Thus,

$$\delta_n\Delta_n^{-1}\Delta_{n+1} - P_x(p_{n,1,2}) \le c_1\,\theta_{n+1}\delta_n\Delta_n^{-1} + c_1\,\theta_{n+1} \le 2c_1\theta_{n+1}. \tag{4.17}$$

Combining this with (4.16) gives (ii). $\square$

It follows from Lemma (37) (i) that

$$P_y(p_{n,j,k+1}) - P_y(p_{n,j,k}) \geq \Delta_{n+1} - c_1\theta_{n+1} = \Delta_{n+1} - cc_1\Delta_{n+1}\delta_n = \Delta_{n+1}(1 - cc_1\delta_n).$$

From Lemma 37 (ii), we also have

$$P_x(p_{n,j,k+1}) - P_x(p_{n,j,k}) \geq \delta_n\Delta_n^{-1}\Delta_{n+1} - 3c_1\theta_{n+1} > \delta_n\Delta_n^{-1}\Delta_{n+1}(1 - 3cc_1\Delta_n).$$

We choose c to be small enough such that $1/2 < 1 - 3cc_1 < 1 - 3cc_1\Delta_n$ and $c < 1/10$. Therefore the y-projections of the rectangles $Q_{n,j,k}$ are non-overlapping. Using (4.3) we have

$$P_x(p_{n,j,k+1}) - P_x(p_{n,j,k}) \geq \delta_n\Delta_n^{-1}\Delta_{n+1}/2 \geq 3c\delta_n\Delta_n^{-1}\Delta_{n+1} \geq 3c\delta_n\Delta_{n+1} \geq 3\delta_{n+1}. \quad (4.18)$$

Recall that $|P_x(Q_{n,j,k})| = \delta_{n+1}$. Thus the x-projections of the $Q_{n,j,k}$ are disjoint.

We define $N_{n,j}$ to be the number of rectangles $Q_{n,j,k}$ inside the bigger rectangle $Q_{n,j}$, i.e.

$$N_{n,j} = \#\{k : Q_{n,j,k} \subset Q_{n,j}\} \text{ and } N_n = \min_j N_{n,j}. \quad (4.19)$$

Define

$$\mathcal{S}_{n+1} = \bigcup_j \bigcup_{k \leq N_n} Q_{n,j,k}. \quad (4.20)$$

We relabel the rectangles $Q_{n,j,k}$ as $Q_{n+1,i}$ in the order of distance from the origin. Notice that the rectangles $Q_{n+1,i}$ satisfy (4.6), (4.7) and (4.8) with n replaced by $n+1$. We continue the iterative process for all $n \in \mathbb{N}$ and define

$$\Gamma = \bigcap_{n=1}^{\infty} \mathcal{S}_n. \quad (4.21)$$

The next lemma gives an estimate on N_n and on the 1-dimensional Hausdorff measure of Γ.

Lemma 38. *For all $n \geq 1$, we have the following.*

1. $\Delta_n \Delta_{n+1}^{-1}(1 - c_2\delta_{n-1}) \leq N_n \leq \Delta_n \Delta_{n+1}^{-1}(1 + c_2\delta_{n-1})$*, where* $c_2 = 4c(c_1 + 1)$.

2. $\#\{j : Q_{n,j} \subset \mathcal{S}_n\} \gtrsim \Delta_n^{-1}$ *and* $|P_y(\mathcal{S}_n)| \gtrsim 1$.

Proof. By the construction of the rectangles, we have

$$\#\{k : Q_{n,j,k} \subset Q_{n,j}\} \geq \min\left\{\frac{|P_y(Q_{n,j})|}{\max_k\left(|P_y(Q_{n,j,k})|\right)}, \frac{|P_x(Q_{n,j})|}{\max_k\left(|P_x(p_{n,j,k+1} - p_{n,j,k})|\right)}\right\} - 1.$$

From Lemma 37 we have that $|P_y(Q_{n,j}) - \Delta_n| \lesssim \theta_n$, for all n and j. Using Lemma 37 (i) and (ii), we have

$$\#\{k : Q_{n,j,k} \subset Q_{n,j}\} \geq \min\left\{\frac{\Delta_n - c_1\,\theta_n}{\Delta_{n+1} + c_1\theta_{n+1}}, \frac{\delta_n}{\delta_n \Delta_n^{-1}\Delta_{n+1} + 3c_1\,\theta_{n+1}}\right\} - 1.$$

Recall $\theta_n = c\Delta_n\delta_{n-1}$ and $\theta_{n+1} = c\Delta_{n+1}\delta_n$. Therefore,

$$\frac{\Delta_n - c_1\,\theta_n}{\Delta_{n+1} + c_1\theta_{n+1}} = \frac{\Delta_n\Delta_{n+1}^{-1}(1 - cc_1\delta_{n-1})}{1 + cc_1\delta_n} \quad \text{and} \quad \frac{\delta_n}{\delta_n\Delta_n^{-1}\Delta_{n+1} + 3c_1\,\theta_{n+1}} = \frac{\Delta_n\Delta_{n+1}^{-1}}{1 + 3cc_1\Delta_n}.$$

Now we use the inequality that $\frac{1}{1+x} > 1 - x$ to obtain

$$\frac{(1 - cc_1\delta_{n-1})}{1 + cc_1\delta_n} \geq (1 - cc_1\delta_{n-1})(1 - cc_1\delta_n) \geq 1 - 2cc_1\delta_{n-1},$$

and

$$\frac{1}{1 + 3cc_1\Delta_n} \geq 1 - 3cc_1\Delta_n.$$

Finally using (4.2) we have $\Delta_n \leq \delta_{n-1}$ and

$$\begin{aligned} N_{n,j} \geq \Delta_n \Delta_{n+1}^{-1}(1 - 3cc_1\delta_{n-1}) - 1 &\geq \Delta_n \Delta_{n+1}^{-1}(1 - 3cc_1\delta_{n-1} - \Delta_n^{-1}\Delta_{n+1}) \\ &\geq \Delta_n \Delta_{n+1}^{-1}(1 - 3cc_1\delta_{n-1} - c\Delta_n^n) \\ &\geq \Delta_n \Delta_{n+1}^{-1}(1 - 3cc_1\delta_{n-1} - c\delta_{n-1}) \\ &\geq \Delta_n \Delta_{n+1}^{-1}(1 - 4c(c_1 + 1)\delta_{n-1}). \end{aligned}$$

Since the above lower bound holds for all j, we have the left hand side of (i). Similarly,

$$N_{n,j} \leq \frac{|P_y(Q_{n,j})|}{\min_k(|P_y(Q_{n,j,k})|)} \leq \frac{\Delta_n + c_1\,\theta_n}{\Delta_{n+1} - c_1\theta_{n+1}} = \frac{\Delta_n \Delta_{n+1}^{-1}(1 + cc_1\delta_{n-1})}{1 - cc_1\delta_n}.$$

Now we use the inequality that $\frac{1}{1-x} < 1 + 2x$, for small x, to obtain

$$\frac{1 + cc_1\delta_{n-1}}{1 - cc_1\delta_n} \leq (1 + cc_1\delta_{n-1})(1 + 2cc_1\delta_n) \leq 1 + 4cc_1\delta_{n-1}.$$

Therefore we have the right hand side of (i).

By (4.20), the number of rectangles obtained in $(n+1)^{st}$ step $\#\{(j,k) : Q_{n,j,k} \subset \mathcal{S}_{n+1}\}$ is $\#\{j : Q_{n,j} \subset \mathcal{S}_{n+1}\}N_n$. From (i), we have

$$\Delta_n \Delta_{n+1}^{-1}(1 - c_2\delta_{n-1})\#\{j : Q_{n,j} \subset \mathcal{S}_n\} \leq \#\{(j,k) : Q_{n,j,k} \subset \mathcal{S}_{n+1}\}.$$

Rewriting this equation,

$$\#\{j : Q_{n,j} \subset \mathcal{S}_n\}\Delta_n(1 - c_2\delta_{n-1}) \leq \#\{(j,k) : Q_{n,j,k} \subset \mathcal{S}_{n+1}\}\Delta_{n+1}.$$

Recall $\#\{Q_{1,1}\}\Delta_1 = 1$, thus we have

$$\prod_{l=1}^{n-1}(1 - c_2\delta_l) \leq \#\{(j,k) : Q_{n,j,k} \subset \mathcal{S}_{n+1}\}\Delta_{n+1}. \tag{4.22}$$

The product $\prod_{l=1}^{n}(1 - c_2\delta_l)$ converges to a positive number as $n \to \infty$. This is because $\sum_l \delta_l < \infty$. Therefore

$$\#\{j : Q_{n,j} \subset \mathcal{S}_n\} \gtrsim \Delta_n^{-1}, \text{ for all } n \in \mathbb{N}.$$

From Lemma 37 (i) we have $|P_y(Q_{n,j})| \simeq \Delta_n$. Thus

$$|P_y(\mathcal{S}_n)| = \sum_j |P_y(Q_{n,j})| \gtrsim \Delta_n^{-1}\Delta_n = 1.$$

This completes the proof of (ii). □

It follows from Lemma 38 (ii) that

$$|P_y(\Gamma)| = \lim_{n\to\infty} |P_y(\mathcal{S}_n)| \simeq 1. \tag{4.23}$$

Thus Γ has positive 1-dimensional Hausdorff measure. Clearly Γ is a subset of a graph of a monotonous continuous function and thus is 1-rectifiable. Moreover,

$$|P_x(\Gamma)| \leq \delta_n \#\{j : Q_{n,j} \subset \mathcal{S}_n\} \simeq \delta_n\Delta_n^{-1}. \tag{4.24}$$

Since $\delta_n\Delta_n^{-1} \to 0$ the domain is a Cantor set of measure 0 and therefore Γ has vertical tangents almost everywhere. Thus we have shown property (i) and (iii) of Theorem 35.

We end this section with another estimate on N_n.

Lemma 39. *For all $n \in \mathbb{N}$, we have $N_n < \theta_n\theta_{n+1}^{-1}$.*

Proof. From Lemma 38 (i) we have $N_n \leq \Delta_n \Delta_{n+1}^{-1}(1 + c_2\delta_{n-1})$. Recall $\theta_{n+1} = c\Delta_{n+1}\delta_n$, for all $n \geq 1$. Therefore

$$N_n\theta_{n+1} \leq \Delta_n\Delta_{n+1}^{-1}(1 + c_2\delta_{n-1})c\Delta_{n+1}\delta_n < c\Delta_n\delta_n(1 + c_2\delta_{n-1}).$$

We use $\delta_n \leq c\Delta_n\delta_{n-1} = \theta_n$ from (4.3) to get

$$N_n\theta_{n+1} < c\Delta_n\delta_n(1 + c_2\delta_{n-1}) \leq c\Delta_n\theta_n(1 + c_2\delta_{n-1}) \leq c\theta_n(1 + c_2).$$

We have $c(1+c_2) = c(1+4c(c_1+1))$. We choose c to be small enough such that $c(1+4c(c_1+1)) < 1$. Thus $N_n\theta_{n+1} < \theta_n$ and this completes the proof. □

4.3 Construction of the Γ-Besicovitch Set

We will construct a 1-dimensional set which contains a translated copy of $e^{-i\theta}\Gamma$, for all $\theta \in [0, 1]$. The translated copy of $e^{-i\theta}\Gamma$ will be denoted as Γ_θ. A union of finitely many rotated copies of this 1-dimensional set will contain a copy of Γ in every direction.

First we define Γ_θ, for $\theta \in [0, 1]$ that are multiples of θ_n for some $n \in \mathbb{N}$. We will do this inductively. We start by defining Γ_θ for multiples of $\theta_1 = c$ by

$$\Gamma_{j\theta_1} = e^{-ij\theta_1}\Gamma, \text{ for } 0 \leq j \leq \theta_1^{-1}.$$

Assume that for some $n \in \mathbb{N}$, we have defined Γ_θ, for angles $\theta \leq 1$ that are integer multiples of θ_n. We will now define Γ_θ for angles that are integer multiples of θ_{n+1}.

We start by defining Γ_θ for $\theta < N_n$. Define

$$\Gamma_{k\theta_{n+1}} := e^{-ik\theta_{n+1}}\Gamma + p_{n,1,k+1}, \text{ for } 0 \leq k < N_n. \tag{4.25}$$

The above definition is motivated by (4.12). Since the point $p_{n,j,k+l} \in \Gamma$ is the same as $e^{-ik\theta_{n+1}} p_{n,j,l} + p_{n,1,k+1} \in \Gamma_{k\theta_{n+1}}$, there will be a lot of intersection between the θ_{n+1} neighborhoods of Γ and $\Gamma_{k\theta_{n+1}}$, for $k < N_n$. We will show that the Lebesgue measure of the θ_{n+1} neighborhood of the set $\bigcup_{k<N_n} \Gamma_{k\theta_{n+1}}$ is comparable to the Lebesgue measure of the θ_{n+1} neighborhood of Γ which is around θ_{n+1}. This will be a key property to show that the Γ-Besicovitch set we will construct indeed has dimension 1.

Now let $\theta = K\theta_{n+1} < \theta_n$. We have from Lemma (39) that $N_n < \theta_n \theta_{n+1}^{-1}$. Thus for some non-negative integer q and for $k < N_n$, we have $K = (qN_n + k)$. Define

$$\Gamma_{K\theta_{n+1}} := e^{-iqN_n\theta_{n+1}} \Gamma_{k\theta_{n+1}}. \tag{4.26}$$

We then have that

$$\bigcup_{K<\theta_n\theta_{n+1}^{-1}} \Gamma_{K\theta_{n+1}} \subseteq \bigcup_{qN_n<\theta_n\theta_{n+1}^{-1}} e^{-iqN_n\theta_{n+1}} \Big(\bigcup_{k<N_n} \Gamma_{k\theta_{n+1}} \Big). \tag{4.27}$$

By the induction hypothesis, we have already defined $\Gamma_{j\theta_n}$. Let the point $v_{j\theta} \in \mathbb{R}^2$ be such that

$$\Gamma_{j\theta_n} = e^{-ij\theta_n} \Gamma + v_{j\theta_n}. \tag{4.28}$$

To define Γ_θ for mutiples of θ_{n+1} in $[j\theta_n, (j+1)\theta_n)$, we rotate the set $\bigcup_{K<\theta_n\theta_{n+1}^{-1}} \Gamma_{K\theta_{n+1}}$ by $-j\theta_n$ and place this rotated set at $v_{j\theta_n}$.

$$\bigcup_{K<\theta_n\theta_{n+1}^{-1}} \Gamma_{j\theta_n+K\theta_{n+1}} = e^{-ij\theta_n} \Big(\bigcup_{K<\theta_n\theta_{n+1}^{-1}} \Gamma_{K\theta_{n+1}} \Big) + v_{j\theta_n}. \tag{4.29}$$

Thus we have

$$\Gamma_{j\theta_n+K\theta_{n+1}} := e^{-ij\theta_n} \Gamma_{K\theta_{n+1}} + v_{j\theta_n}, \text{ for all } K < \theta_n\theta_{n+1}^{-1}. \tag{4.30}$$

Observe that if $K=0$, then the above definition agrees with (4.28).

Recall from (4.5) that $\theta_n\theta_{n+1}^{-1}\in\mathbb{N}$. Thus we have defined Γ_θ for all θ which are multiples of θ_{n+1}. Denote the points v_θ such that $\Gamma_\theta := e^{-i\theta}\Gamma+v_\theta$. From (4.25), (4.26) and (4.30) we have

$$v_\theta=\begin{cases} p_{n,1,k+1}, & \text{for } \theta=k\theta_{n+1} \text{ and } k<N_n \\ e^{-iqN_n\theta_{n+1}}v_{k\theta_{n+1}}, & \text{for } \theta=(qN_n+k)\theta_{n+1}<\theta_n \\ e^{-ij\theta_n}v_{\theta-j\theta_n}+v_{j\theta_n}, & \text{for } \theta\in[j\theta_n,(j+1)\theta_n) \text{ and } \theta\theta_{n+1}^{-1}\in\mathbb{N}. \end{cases} \tag{4.31}$$

We continue the inductive process for all $n\in\mathbb{N}$. This defines Γ_θ for all θ that are multiples of θ_n, $n\in\mathbb{N}$.

We will now introduce rectangles $\mathcal{Q}_{n,j}$ centered at $p_{n,j}$ that contain $Q_{n,j}$. Let

$$\mathcal{Q}_{n,j}=p_{n,j}+[-C\theta_n,C\theta_n]\times[-C\Delta_n,C\Delta_n], \tag{4.32}$$

where $C>1$ is a large enough absolute constant to be specified later. We also let

$$T_n=\bigcup_j \mathcal{Q}_{n,j} \text{ and } T_{n,l\theta_n}=e^{-il\theta_n}T_n+v_{l\theta_n}. \tag{4.33}$$

We define

$$B=\bigcap_{n=1}^{\infty}\bigcup_{l\le\theta_n^{-1}}T_{n,l\theta_n}. \tag{4.34}$$

We claim that B is a 1-dimensional Γ-Besicovitch set.

For a set $U\subset\mathbb{R}^2$, let $U(\delta)$ denote the open δ-neighborhood of U.

Lemma 40. *The set B defined in (4.34) is a 1-dimenionsal set.*

Proof. It is clear from definition (4.34) that $B\subset\bigcup_{l\le\theta_{n+1}^{-1}}T_{n+1,l\theta_{n+1}}$. We will estimate the measure of the set $\bigcup_{l\le\theta_{n+1}^{-1}}T_{n+1,l\theta_{n+1}}(\theta_{n+1})$.

Let ${\mathcal{Q}'}_{n,j,k}$ be rectangles centered at $p_{n,j,k}$ with size $4C\theta_{n+1} \times 4C\Delta_{n+1}$ i.e.

$$ {\mathcal{Q}'}_{n,j,k} = p_{n,j,k} + [-2C\theta_{n+1}, 2C\theta_{n+1}] \times [-2C\Delta_{n+1}, 2C\Delta_{n+1}]. $$

From definitions (4.32) and (4.33), we see that T_{n+1} is a union of rectangles centered at points $p_{n,j,k}$ with size $2C\theta_{n+1} \times 2C\Delta_{n+1}$. Therefore

$$ T_{n+1}(\theta_{n+1}) \subset \bigcup_{j} \bigcup_{k \le N_n} {\mathcal{Q}'}_{n,j,k}. $$

We let

$$ T'_{n+1} := \bigcup_{j} \bigcup_{k \le N_n} {\mathcal{Q}'}_{n,j,k} \quad \text{and} \quad T'_{n+1,l\theta_{n+1}} := e^{-il\theta_{n+1}} T'_{n+1} + v_{l\theta_{n+1}}. $$

Therefore

$$ B(\theta_{n+1}) \subset \bigcup_{l \le \theta_{n+1}^{-1}} T_{n+1,l\theta_{n+1}}(\theta_{n+1}) \subset \bigcup_{l \le \theta_{n+1}^{-1}} T'_{n+1,l\theta_{n+1}}. \tag{4.35} $$

From (4.12) we have

$$ p_{n,j,k+l} = e^{-il\theta_{n+1}} p_{n,j,k} + p_{n,1,l+1} = e^{-i(l-1)\theta_{n+1}} p_{n,j,k+1} + p_{n,1,l} $$

and $v_{l\theta_{n+1}} = p_{n,1,l+1}$ from definition (4.25), for $l < N_n$. Thus for l, $k < N_n$, the center of the two rectangles $e^{-il\theta_{n+1}} {\mathcal{Q}'}_{n,j,k} + v_{l\theta_{n+1}}$ and $e^{-i(l-1)\theta_{n+1}} {\mathcal{Q}'}_{n,j,k+1} + v_{(l-1)\theta_{n+1}}$ is $p_{n,j,k+l}$ and the angle between them is θ_{n+1}. For any such pair we have

$$ \begin{aligned} \left| \left(e^{-i(l-1)\theta_{n+1}} {\mathcal{Q}'}_{n,j,k+1} + v_{(l-1)\theta_{n+1}} \right) \setminus \left(e^{-il\theta_{n+1}} {\mathcal{Q}'}_{n,j,k} + v_{l\theta_{n+1}} \right) \right| &\lesssim \theta_{n+1} (\max(\theta_{n+1}, \Delta_{n+1}))^2 \\ &\le \Delta_{n+1}^2 \theta_{n+1}. \end{aligned} $$

Thus we have from Lemma 38 (ii) that

$$\begin{aligned}
|T'_{n+1,(l-1)\theta_{n+1}} \setminus T'_{n+1,l\theta_{n+1}}| &\lesssim \Delta_{n+1}^2 \theta_{n+1} (\#\{(j,k) : Q_{n,j,k} \subset \mathcal{S}_{n+1},\, k < N_n\} \\
&\qquad + \Delta_{n+1}\theta_{n+1}(\#\{j : Q_{n,j,N_n} \subset \mathcal{S}_{n+1}\}) \\
&\lesssim \Delta_{n+1}^2 \theta_{n+1} \#\{(j,k) : Q_{n,j,k} \subset \mathcal{S}_{n+1}, k < N_n\} \\
&\qquad + \Delta_{n+1}\theta_{n+1}(\#\{j : Q_{n,j} \subset \mathcal{S}_n\}) \\
&\lesssim \Delta_{n+1}\Delta_n^{-1}\theta_{n+1}.
\end{aligned}$$

Summing over $l < N_n$ and using Lemma 38 again we get

$$\begin{aligned}
\Big| \bigcup_{l<N_n} T'_{n+1,l\theta_n} \Big| &\lesssim |T'_{n+1}| + \Delta_{n+1}\Delta_n^{-1}\theta_{n+1}N_n \\
&\lesssim \Delta_{n+1}\theta_{n+1}(\#\{(j,k) : Q_{n,j,k} \subset \mathcal{S}_{n+1}\}) + \Delta_{n+1}\Delta_n^{-1}\theta_{n+1}N_n \\
&\lesssim \theta_{n+1}. \qquad (4.36)
\end{aligned}$$

Let $K < \theta_n \theta_{n+1}^{-1}$ and let $K = qN_n + l$ and $l < N_n$. From (4.31), we have

$$\begin{aligned}
T'_{n+1,K\theta_{n+1}} &= e^{-iK\theta_{n+1}} T'_{n+1} + v_{K\theta_{n+1}} \\
&= e^{-iqN_n\theta_{n+1}} e^{-il\theta_{n+1}} T'_{n+1} + e^{-iqN_n\theta_{n+1}} v_{l\theta_{n+1}} \\
&= e^{-iqN_n\theta_{n+1}} (e^{-il\theta_{n+1}} T'_{n+1} + v_{l\theta_{n+1}}) = e^{-iqN_n\theta_{n+1}} T'_{n+1,l\theta_{n+1}}.
\end{aligned}$$

Therefore

$$\bigcup_{K<\theta_n\theta_{n+1}^{-1}} T'_{n+1,K\theta_{n+1}} \subset \bigcup_{qN_n \le \theta_n\theta_{n+1}^{-1}} \bigcup_{l<N_n} T'_{n+1,l\theta_{n+1}}.$$

Thus using the estimate (4.36), we obtain

$$\Big| \bigcup_{K<\theta_n\theta_{n+1}^{-1}} T'_{n+1,l\theta_{n+1}} \Big| \lesssim \theta_{n+1} N_n^{-1} \theta_n \theta_{n+1}^{-1} \leq N_n^{-1}\theta_n. \tag{4.37}$$

We will now sum over all $\theta \leq 1$ which are multiples of θ_{n+1}. Let $\theta = j\theta_n + K\theta_{n+1}$ such that $K\theta_{n+1} < \theta_n$. From (4.31) we have

$$\begin{aligned} T'_{n+1,\theta} &= e^{-i\theta} T'_{n+1} + v_\theta \\ &= e^{-ij\theta_n} e^{-i(\theta - j\theta_n)} T'_{n+1} + e^{-ij\theta_n} v_{\theta-j\theta_n} + v_{j\theta_n}. \\ &= e^{-ij\theta_n}(e^{-iK\theta_{n+1}} T'_{n+1} + v_{K\theta_{n+1}}) + v_{j\theta_n} = e^{-ij\theta_n} T'_{n+1,K\theta_{n+1}} + v_{j\theta_n}. \end{aligned}$$

Therefore from Lemma 38 (i) we have that

$$\Big| \bigcup_{l\leq\theta_{n+1}^{-1}} T'_{n+1,l\theta_{n+1}} \Big| \leq \Big| \bigcup_{l<\theta_n\theta_{n+1}^{-1}} T'_{n+1,l\theta_{n+1}} \Big| \theta_n^{-1} \lesssim N_n^{-1}\theta_n\theta_n^{-1} \lesssim \Delta_{n+1}\Delta_n^{-1}. \tag{4.38}$$

Thus from (4.35) and (4.38) we have that

$$|B(\theta_{n+1})| \lesssim \Delta_{n+1}\Delta_n^{-1}.$$

From (4.2), we see that $\Delta_{n+1}^{\frac{1}{n+1}} \leq c^{\frac{1}{n+1}}\delta_n \leq \delta_n \leq \Delta_n$. Thus $B(c\Delta_{n+1}^{1+\frac{1}{n+1}}) \subset B(c\Delta_{n+1}\delta_n) = B(\theta_{n+1})$ and we have

$$|B(c\Delta_{n+1}^{1+\frac{1}{n+1}})| \lesssim \Delta_{n+1}\Delta_n^{-1} \lesssim \Delta_{n+1}^{1-\frac{1}{n+1}} \leq \left(\Delta_{n+1}^{1+\frac{1}{n+1}}\right)^{1-\frac{2}{n+1}}.$$

Therefore for all $m > n$

$$|B(c\Delta_{m+1}^{1+\frac{1}{m+1}})| \lesssim \left(\Delta_{m+1}^{1+\frac{1}{m+1}}\right)^{1-\frac{2}{m+1}} \lesssim \left(\Delta_{m+1}^{1+\frac{1}{m+1}}\right)^{1-\frac{2}{n+1}}.$$

Since $\Delta_{m+1} \to 0$ as $m \to \infty$ we conclude that

$$\dim_H B \leq 2 - (1 - \frac{2}{n+1}) = 1 + \frac{2}{n+1}.$$

Taking $n \to \infty$, we see that $\dim_H B = 1$. □

It remains to prove that B contains a rotated copy of Γ in every direction. Let $D = \{j\theta_n : \text{for some } n \in \mathbb{N} \text{ and } j \leq \theta_n^{-1}\}$, the dense subset of $[0,1]$ for which we have already defined v_θ. We will now define v_θ for all $\theta \in [0,1]$ as the following left limit.

$$v_\theta = \lim_{\substack{\theta' \to \theta^- \\ \theta' \in D}} v_{\theta'}. \tag{4.39}$$

It will follow from Lemma 41 below that the above limit exists. We define Γ_θ for all $\theta \in [0,1]$ as before

$$\Gamma_\theta = e^{-i\theta}\Gamma + v_\theta.$$

Therefore we will have the following.

$$\Gamma_\theta = e^{-ij\theta_n}(\Gamma_{\theta - j\theta_n}) + v_{j\theta_n}, \text{ for all } \theta \in [j\theta_n, (j+1)\theta_n). \tag{4.40}$$

Lemma 41. *The limit in (5.11) exists. Moreover for $\theta < \theta_n$, we have*

$$|P_x(v_\theta)| \lesssim \theta_n \text{ and } |P_y(v_\theta)| \lesssim \Delta_n, \tag{4.41}$$

where the implicit constant is absolute and independent of n.

Proof. If $\theta = k\theta_{n+1}$ and $k < N_n$, then we have from (4.25) that $v_\theta = p_{n,1,k+1}$. By definition of N_n (see (4.19)) this implies that v_θ lies in the rectangle $Q_{n,1} = [0, \delta_n] \times [0, \Delta_n]$. Therefore $|v_\theta| \leq 2\Delta_n$.

If $\theta = K\theta_{n+1} < \theta_n$ and $K = qN_n + k$ with $k < N_n$, then we have from (4.26) that

$$v_\theta = e^{-iqN_n\theta_{n+1}}(v_{k\theta_{n+1}}).$$

We are rotating a vector of length at most $2\Delta_n$ around origin by angle of magnitude at most θ_n. Therefore we get

$$\|v_\theta - v_{k\theta_{n+1}}\| = \|v_{k\theta_{n+1}}\|\theta \le 2\Delta_n\theta_n$$

and

$$|P_x(v_\theta)| \le |P_x(v_{k\theta_{n+1}})| + 2\Delta_n\theta_n \le \delta_n + 2\Delta_n\theta_n \tag{4.42}$$

and

$$|P_y(v_\theta)| \le |P_y(v_{k\theta_{n+1}})| + 2\Delta_n\theta_n \le \Delta_n + 2\Delta_n\theta_n. \tag{4.43}$$

From (4.31), we have that if $\theta \in [j\theta_n, (j+1)\theta_n)$ and is a multiple of θ_{n+1}, then

$$v_\theta - v_{j\theta_n} = e^{-ij\theta_n}(v_{\theta - j\theta_n}).$$

We are rotating a vector of length at most $2\Delta_n$ around origin by angle of magnitude at most 1. Therefore we get

$$\|v_\theta - v_{j\theta_n}\| = \|v_{\theta - j\theta_n}\|j\theta_n \le 2\Delta_n, \text{ for all } \theta \in [j\theta_n, (j+1)\theta_n) \text{ and } \theta\theta_{n+1}^{-1} \in \mathbb{N}. \tag{4.44}$$

Now let $\theta \in [0,1]$. Let $\Theta_m = \lfloor \theta\theta_m^{-1} \rfloor \theta_m$, where $\lfloor \cdot \rfloor$ denotes the greatest integer function. Then Θ_m denotes the greatest integer multiple of θ_m less than θ. From (4.44) we have

$$\|v_{\Theta_{m+1}} - v_{\Theta_m}\| \le 2\Delta_m, \text{ for all } m \in \mathbb{N}.$$

Therefore

$$\|v_{\Theta_{m+l}} - v_{\Theta_m}\| = \Big\| \sum_{j=m}^{m+l-1} (v_{\Theta_{j+1}} - v_{\Theta_j})\Big\| \leq \sum_{j=m}^{m+l-1} \|v_{\Theta_{j+1}} - v_{\Theta_j}\| \leq \sum_{j=m}^{m+l-1} 2\Delta_j \lesssim \Delta_m. \tag{4.45}$$

We have used that $\Delta_{m+1} \leq c\delta_m^{m+1} \leq \Delta_m^{m+1}$ which follows from (4.2). As $\Delta_m \to 0$, we conclude that $\{v_{\Theta_m}\}_{m\in\mathbb{N}}$ converges. Thus the limit v_θ as defined in (4.39) exists.

Now we again let $\theta < \theta_n$. We have from (4.45) that $\|v_{\Theta_{n+1+m}} - v_{\Theta_{n+1}}\| \lesssim \Delta_{n+1}$. Therefore we have

$$\|v_\theta - v_{\Theta_{n+1}}\| \lesssim \Delta_{n+1}.$$

Combining the above equation with (4.42) and (4.43), we get for $\theta < \theta_n$

$$|P_x(v_\theta)| \leq |P_x(v_{\Theta_{n+1}})| + \|v_\theta - v_{\Theta_{n+1}}\| \lesssim \delta_n + 2\Delta_n\theta_n + \Delta_{n+1} \lesssim \Delta_n\delta_{n-1} = \theta_n.$$

and

$$|P_y(v_\theta)| \leq |P_y(v_{\Theta_{n+1}})| + \|v_\theta - v_{\Theta_{n+1}}\| \lesssim \Delta_n + 2\Delta_n\theta_n + \Delta_{n+1} \lesssim \Delta_n.$$

□

We are now ready to prove that B is a Γ-Besicovitch set.

Lemma 42. *For all $n \in \mathbb{N}$ and $\theta \in [0,1]$, we have*

$$\Gamma_\theta \subset \bigcup_{i \leq \theta_n^{-1}} T_{n,i\theta_n}. \tag{4.46}$$

Proof. For any $\theta < \theta_n$ and $p_{n,j} \in \Gamma \subset [0,1]^2$, we have

$$\|e^{-i\theta}p_{n,j} - p_{n,j}\| \leq \|p_{n,j}\|\theta \lesssim \theta_n.$$

The rectangle $Q_{n,j}$ has sides parallel to the coordinate axes. Therefore,

$$|P_x(e^{-i\theta}Q_{n,j})| \lesssim |P_x(Q_{n,j})| + \Delta_n\theta \lesssim \delta_n + \Delta_n.\theta_n \lesssim \theta_n$$

and

$$|P_y(e^{-i\theta}Q_{n,j})| \leq |P_y(Q_{n,j})| + \Delta_n\theta \leq 2\Delta_n + 2\delta_n\theta_n \lesssim \Delta_n.$$

Let $z \in \Gamma_\theta$. Since $\Gamma_\theta \subset e^{-i\theta}(\bigcup Q_{n,j}) + v_\theta$, the point $z - v_\theta$ lies in the rectangle $e^{-i\theta}Q_{n,j}$ for some j. Thus from the above discussion and Lemma 41, we get

$$|P_x(z - p_{n,j})| \leq |P_x(z - v_\theta - e^{-i\theta}(p_{n,j}))| + |P_x(e^{-i\theta}(p_{n,j}) - p_{n,j})| + |P_x(v_\theta)| \lesssim \theta_n.$$

and

$$|P_y(z - p_{n,j})| \leq |P_y(z - v_\theta - e^{-i\theta}(p_{n,j}))| + |P_y(e^{-i\theta}(p_{n,j}) - p_{n,j})| + |P_y(v_\theta)| \lesssim \Delta_n.$$

We take C in (4.32) large enough such that $\Gamma_\theta \subset \bigcup \mathcal{Q}_{n,j} = T_n$, for all $\theta < \theta_n$. Now let $\theta \in [j\theta_n, (j+1)\theta_n))$ then by (4.40), we obtain

$$\Gamma_\theta = e^{-ij\theta_n}\Gamma_{\theta - j\theta_n} + v_{j\theta_n} \subset e^{-ij\theta_n}T_n + v_{j\theta_n} = T_{n,j\theta_n}. \tag{4.47}$$

This completes the proof. □

Thus $\Gamma_\theta \subset B$ for every $\theta \in [0,1]$ and so we have proved Theorem 35 (ii).

4.4 Dimension of the Domain of Γ

So far we have proved that if $\{\delta_n\}_{n\in\mathbb{N}}$ and $\{\Delta_n\}_{n\in\mathbb{N}}$ satisfy (4.1)-(4.4), and if Γ and B are constructed as in Section 4.2 and Section 4.3 respectively, then Γ has property (i), (ii) and (iii) of Theorem 35. It remains to prove that the dimension of the x-projection of Γ can take any value $s \in [0, 1]$.

Let $s \in [0, 1]$ and let $\{\delta_n\}_{n\in\mathbb{N}}$ and $\{\Delta_n\}_{n\in\mathbb{N}}$ satisfy conditions (4.1), (4.2), (4.4) and

$$\delta_n = c\Delta_n^{\frac{1}{s_n}}, \tag{4.48}$$

where

$$s_n = \begin{cases} \frac{1}{n}, & \text{if } s = 0 \\ \frac{sn}{n+1}, & \text{if } 0 < s \leq 1. \end{cases}$$

We will show that (4.3) holds and the corresponding Γ we constructed in Section 4.2 satisfies

$$\dim\left(P_x(\Gamma)\right) = s. \tag{4.49}$$

From (4.2) ($\Delta_{n+1}^{\frac{1}{n+1}} \leq c^{\frac{1}{n+1}}\delta_n$) and (4.48) we have

$$\delta_{n+1} = c\Delta_{n+1}^{\frac{1}{s_{n+1}}} = c\Delta_{n+1}\Delta_{n+1}^{\frac{1}{s_{n+1}}-1} \leq c\Delta_{n+1}(c\delta_n^{n+1})^{\frac{1}{s_{n+1}}-1}.$$

If $s = 0$ then

$$(n+1)(s_{n+1}^{-1} - 1) = (n+1)n \geq 1.$$

If $s > 0$ then

$$(n+1)(s_{n+1}^{-1} - 1) = (n+1)(s^{-1}(n+2)/(n+1) - 1) \geq 1.$$

Hence in both cases

$$\delta_{n+1} \leq c\Delta_{n+1}(c\delta_n^{n+1})^{\frac{1}{s_{n+1}}-1} \leq c^{1+\frac{1}{n+1}}\Delta_{n+1}\delta_n \leq c\Delta_{n+1}\delta_n.$$

So indeed (4.3) holds.

Proof of (4.49). We first show that $\dim(P_x(\Gamma)) \leq s$. By construction, for all $p \in \mathbb{N}$, $\bigcup_j P_x(Q_{p,j})$ is a cover of $P_x(\Gamma)$.

Assume $s = 0$ and let $s' > 0$. Using Lemma 38 (ii) and (4.48) we have

$$\sum_j |P_x(Q_{p,j})|^{s'} = \delta_p^{s'} \#\{j : Q_{p,j} \subset \mathcal{S}_p\} \lesssim \delta_p^{s'} \Delta_p^{-1} \lesssim \delta_p^{s' - \frac{1}{p}}.$$

Hence for any $p \in \mathbb{N}$ such that $1/p < s'$, we have

$$\sum_j |P_x(Q_{p,j})|^{s'} \lesssim \delta_p^{s' - \frac{1}{p}} < 1.$$

As $p \to \infty$, $\delta_p \to 0$. Therefore $\dim(P_x(\Gamma)) \leq s'$. Since $0 < s'$ was arbitrary we conclude that $\dim(P_x(\Gamma)) = 0$.

Now assume that $s > 0$. We again use Lemma 38 (ii) and (4.48) to get

$$\sum_j |P_x(Q_{p,j})|^{s} = \delta_p^{s} \#\{j : Q_{p,j} \subset \mathcal{S}_p\} \lesssim \delta_p^{s} \Delta_p^{-1} \lesssim \delta_p^{s} \delta_p^{-s_p} = \delta_p^{\frac{s}{p+1}} \leq 1.$$

Therefore $\dim(P_x(\Gamma)) \leq s$.

It remains to prove the opposite inequality for $s > 0$. Fix $m \in \mathbb{N}$, and let $\bigcup_i I_i$ be a cover of $P_x(\Gamma)$ by open intervals I_i such that $|I_i| < \delta_m$, for all i. We can assume it is a finite cover since $P_x(\Gamma)$ is compact. Let $p \in \mathbb{N}$ be such that $\delta_p < \min_i |I_i|$.

Now fix an arbitrary i and let $n \in \mathbb{N}$ be such that $\delta_{n+1} \leq |I_i| < \delta_n$. We will show that

$$|I_i|^{s_m} \gtrsim \sum_{\{(j,k) \colon P_x(Q_{n,j,k}) \cap I_i \neq \emptyset\}} |P_x(Q_{n,j,k})|^{s_{n+1}}. \tag{4.50}$$

Recall from (4.18) that $3\delta_n \leq P_x(p_{n,j+1}) - P_x(p_{n,j})$. Since the distance between the left endpoints of $P_x(Q_{n,j})$ and $P_x(Q_{n,j+1})$ is greater than $3\delta_n = 3|P_x(Q_{n,j})|$, we can assume that I_i intersects $P_x(Q_{n,j})$ for only one rectangle $Q_{n,j}$. Let

$$\lambda = \frac{|I_i|}{\delta_n \Delta_n^{-1} \Delta_{n+1}}. \tag{4.51}$$

We see from (4.18) that $P_x(p_{n,j,k+1}) - P_x(p_{n,j,k}) \gtrsim \delta_n \Delta_n^{-1} \Delta_{n+1}$. Therefore we have

$$\#\{Q_{n,j,k} : P_x(Q_{n,j,k}) \cap I_i \neq \emptyset\} \lesssim \lambda + 1.$$

We look at two cases.

Case 1: $\lambda < 1$.

In this case I_i intersects $P_x(Q_{n,j,k})$ for at most constant many rectangles $Q_{n,j,k}$. Since $\delta_{n+1} \leq |I_i| < \delta_m$ we have $m < n+1$. Therefore $s_m < s_{n+1}$ and

$$|I_i|^{s_m} > |I_i|^{s_{n+1}} \geq \delta_{n+1}^{s_{n+1}} \gtrsim \sum_{\{(j,k) \colon P_x(Q_{n,j,k}) \cap I_i \neq \emptyset\}} |P_x(Q_{n,j,k})|^{s_{n+1}}.$$

So (4.50) holds.

Case 2: $\lambda \geq 1$.

To conclude (4.50) we need to verify that

$$|I_i|^{s_m} = (\lambda \delta_n \Delta_n^{-1} \Delta_{n+1})^{s_m} \gtrsim (\lambda + 1)\delta_{n+1}^{s_{n+1}} \simeq \lambda \delta_{n+1}^{s_{n+1}}.$$

Rewriting the above inequality we get

$$(\delta_n \Delta_n^{-1} \Delta_{n+1})^{s_m} \gtrsim \lambda^{1-s_m} \delta_{n+1}^{s_{n+1}}. \tag{4.52}$$

Using $s_m \leq s_n$ and (4.48) we have that

$$\delta_n^{s_m} \geq \delta_n^{s_n} \simeq \Delta_n \simeq \Delta_n \Delta_{n+1}^{-1} \delta_{n+1}^{s_{n+1}}.$$

Thus

$$(\delta_n \Delta_n^{-1} \Delta_{n+1})^{s_m} \gtrsim (\Delta_n \Delta_{n+1}^{-1})^{1-s_m} \delta_{n+1}^{s_{n+1}}.$$

Since $|I_i| < \delta_n$ we have $\lambda < \Delta_n \Delta_{n+1}^{-1}$. As $s_m < 1$, we have $\lambda^{1-s_m} < (\Delta_n \Delta_{n+1}^{-1})^{1-s_m}$. Therefore we obtain (4.52) from the above inequality. Thus (4.50) holds in Case 2.

Since $\delta_p < |I_i| < \delta_n$ we have that $p \geq n+1$. By the definition (4.19) of N_n and Lemma 3 (i) we have

$$\#\{Q_{p,l} : Q_{p,l} \subset Q_{n,j,k}\} \leq \prod_{q=n+1}^{p-1} \Delta_q \Delta_{q+1}^{-1} (1 + c_2 \delta_{q-1}) = \Delta_{n+1} \Delta_p^{-1} \prod_{q=n+1}^{p-1} (1 + c_2 \delta_{q-1}).$$

As noted earlier below (4.22), the product $\prod_{q=n+1}^{p} (1 + c_2 \delta_{q-1})$ converges to an absolute constant. Therefore $\prod_{q=n+1}^{p} (1 + c_2 \delta_{q-1}) \simeq 1$ and

$$\sum_{\{l : Q_{p,l} \subset Q_{n,j,k}\}} |P_x(Q_{p,l})|^{s_p} \lesssim \delta_p^{s_p} \Delta_{n+1} \Delta_p^{-1} \simeq \Delta_{n+1} \simeq \delta_{n+1}^{s_{n+1}} = |P_x(Q_{n,j,k})|^{s_{n+1}}. \tag{4.53}$$

Hence we obtain from (4.53) and (4.50) that

$$\begin{aligned} |I_i|^{s_m} \gtrsim \sum_{\{(j,k)\colon P_x(Q_{n,j,k}) \cap I_i \neq \emptyset\}} |P_x(Q_{n,j,k})|^{s_{n+1}} &\gtrsim \sum_{\{(j,k)\colon P_x(Q_{n,j,k}) \cap I_i \neq \emptyset\}} \sum_{\{l\colon Q_{p,l} \subset Q_{n,j,k}\}} |P_x(Q_{p,l})|^{s_p} \\ &\geq \sum_{\{l\colon P_x(Q_{p,l}) \cap E_i \neq \emptyset\}} |P_x(Q_{p,l})|^{s_p}. \end{aligned}$$

Summing over all i we get

$$\sum_i |I_i|^{s_m} \gtrsim \sum_i \sum_{\{l\colon P_x(Q_{p,l}) \cap I_i \neq \emptyset\}} |P_x(Q_{p,l})|^{s_p} \geq \sum_l |P_x(Q_{p,l})|^{s_p} \gtrsim \delta_p^{s_p} \Delta_p^{-1} \simeq 1. \tag{4.54}$$

This proves $\dim(P_x(\Gamma)) \geq s_m$. Taking $m \to \infty$ we get (4.49). $\square$

CHAPTER 5

DISTANCE SETS BOUNDS FOR POLYHEDRAL NORMS VIA EFFECTIVE DIMENSION

This chapter is joint work with Ryan Bushling and Bobby Wilson and it has been accepted as a paper by Real Analysis Exchange.

5.1 Introduction

Given a set $E \subseteq \mathbb{R}^d$, let

$$\Delta(E) := \{\|x-y\| \in [0,\infty): \ x, y \in E\}$$

be the *distance set* of E, where $\|\cdot\|$ is the Euclidean norm. The *Falconer distance problem* is to find a lower bound on $\dim_{\mathrm{H}} \Delta(E)$, the Hausdorff dimension of the distance set, in terms of the Hausdorff dimension of E itself. A frequent simplification is to fix a point $x \in \mathbb{R}^d$ and instead investigate the *pinned distance set*

$$\Delta_x(E) := \{\|x-y\| \in [0,\infty): \ y \in E\}.$$

Since $\Delta_x(E) \subseteq \Delta(E)$ whenever $x \in E$, any lower bound on the size of $\Delta_x(E)$ immediately implies the same bound on the size of $\Delta(E)$. An interesting variant of the Falconer distance problem is to seek dimension bounds on distance sets relative to a different norm $\|\cdot\|_*$. In this case, the corresponding distance and pinned distance sets are denoted $\Delta^*(E)$ and $\Delta^*_x(E)$, respectively.

Falconer [28] proves a simple estimate in the Euclidean case:

Theorem 43 (Falconer [28])**.** *If $E \subseteq \mathbb{R}^d$ is any set, then*

$$\dim_{\mathrm{H}} \Delta(E) \geq \dim_{\mathrm{H}} E - (d-1).$$

The elementary proof presented by Falconer generalizes to other norms with little modification. However, the techniques of algorithmic complexity provided in [50] are highly applicable here, and we use this theory to give an alternate proof for pinned distance sets of the general case.

Theorem 44. *Let $\|\cdot\|_*$ be any norm on $\mathbb{R}^d$ and let $E \subseteq \mathbb{R}^d$. Then*

$$\dim_{\mathrm{H}} \Delta_x^*(E) \geq \dim_{\mathrm{H}} E - (d-1)$$

for all $x \in \mathbb{R}^d$.

Remark 45. The proof of Theorem 44 does not use the symmetry of $\|\cdot\|_*$, so the theorem statement can be generalized to so-called *asymmetric norms*. The same is true of Theorem 47 below.

Polyhedral norms provide a class of finite-dimensional normed spaces which can be considered the "worst-case scenario" for the distance set problem, as the curvature of the norm ball plays a major role in precluding the possibility of a large number of points with equal pairwise distances. The relevance of non-vanishing curvature manifests itself in [37]: the authors provide the best known bound in the planar Euclidean case, and it is demonstrated that the essential characteristic of the Euclidean ball required for their argument is that it is a smooth norm ball with strictly positive curvature. A construction of Falconer [29] illustrates the marked differences in the context of polyhedral norms.

Proposition 46 (Falconer [29])**.** *Let $\|\cdot\|_P$ be a polyhedral norm on $\mathbb{R}^d$. Then there exists a compact set $F \subset \mathbb{R}^d$ with $\dim_{\mathrm{H}} F = d$ such that $\mathcal{L}(\Delta^P(F)) = 0$.*

The relevance of curvature to the distance set conjecture is further explored in [12], in which Falconer's construction is generalized to the case in which at most countably many points of the surface of the unit ball do not lie on any open line segment contained in the surface of the unit ball (all but finitely many points on the surface of a polyhedral ball satisfy this condition) or for certain convex sets for which the measure representing the curvature field is singular with respect to surface measure. For more state-of-the-art estimates and information on the history of the Euclidean distance set conjecture, see [25] and [26].

Our second result shows that polyhedral norms witness the sharpness of the dimension bound in Theorem 44. This, too, will be proven using complexity theoretic tools.

Theorem 47. *Let $\|\cdot\|_P$ be a polyhedral norm on $\mathbb{R}^d$. For any $s \in [d-1, d]$, there exists a compact set $E \subset \mathbb{R}^d$ with $\dim_{\mathrm{H}} E = s$ such that*

$$\dim_{\mathrm{H}} \Delta^P(E) = s - (d-1). \tag{5.1}$$

Remark 48. The proof of Theorem 44 also works with *Packing dimension* in place of Hausdorff dimension. However, the resulting statement for packing dimension is far from sharp.

Under the hypotheses of Theorem 47, it follows from Theorem 44 that the pinned distance sets of E have minimal dimension $s-(d-1)$ at every pin in E: that is,

$$\dim_{\mathrm{H}} \Delta_x^P(E) = s - (d-1) \quad \text{for all} \quad x \in E.$$

Following a brief synopsis of the background material, we prove Theorem 44 in §5.3. In §5.4 we construct the examples described in Theorem 47.

5.2 Preliminaries

This section summarizes the requisite terminology and results from algorithmic complexity theory. See [50] for a more thorough exposition of this material. We denote by

$$\{0,1\}^* := \bigcup_{n\in\mathbb{N}} \{0,1\}^n$$

the set of all (finite) binary strings, including the empty string $\lambda \in \{0,1\}^0$.

Definition 49 (Kolmogorov Complexity of a String)**.** *Let $\sigma, \tau \in \{0,1\}^*$. The conditional Kolmogorov complexity of σ given τ is the length of the shortest program π that will output σ given τ. More precisely,*

$$K(\sigma|\tau) := \min_{\pi\in\{0,1\}^*} \{\ell(\pi) :\ U(\pi,\tau) = \sigma\},$$

where U is a fixed universal prefix-free Turing machine and $\ell(\pi)$ is the length of π. The Kolmogorov complexity of σ is simply the conditional Kolmogorov complexity of σ given the empty string:

$$K(\sigma) := K(\sigma|\lambda),$$

where λ is the empty string.

Throughout, we work with a fixed *encoding* $\{0,1\}^* \to \bigcup_{d\in\mathbb{N}} \mathbb{Q}^d$, under which the definitions above extend from strings to vectors over the rationals.

Definition 50 (Kolmogorov Complexity of a Point)**.** *Let $x \in \mathbb{R}^d$ and $r \in \mathbb{Z}_+$. The Kolmogorov complexity of x at precision r is the length of the shortest program that outputs a point in $\mathbb{Q}^d$ that approximates x to r bits of precision:*

$$K_r(x) := \min\{K(p) :\ p \in B(x, 2^{-r}) \cap \mathbb{Q}^d\}.$$

If also $y \in \mathbb{R}^{d'}$ and $s \in \mathbb{Z}_+$, then the conditional Kolmogorov complexity of x at precision r given y at precision s is defined by

$$K_{r,s}(x|y) := \max\left\{\min\{K(p|q) : p \in B(x, 2^{-r}) \cap \mathbb{Q}^d\} : q \in B(y, 2^{-s}) \cap \mathbb{Q}^{d'}\right\}.$$

Note that, for $x \in \mathbb{R}^d$, $K_r(x)$ is always less than equal to $dr + O(\log r)$. To simplify notation, we also denote $K_{r,r}(x|y)$ by $K_r(x|y)$, $K_{r,s}(x|x)$ by $K_{r,s}(x)$, and, when $r \in (0, \infty)$, $K_{\lceil r \rceil}(x)$ by $K_r(x)$.

Lemma 51 (Symmetry of Information [54])**.** *For all $m, n \in \mathbb{N}$, $x \in \mathbb{R}^m$, $y \in \mathbb{R}^n$, and $r, s \in \mathbb{N}$ with $r \geq s$:*

1. $\left|K_r(x|y) + K_r(y) - K_r(x, y)\right| \leq O(\log r) + O(\log\log \|y\|)$.

2. $\left|K_{r,s}(x|x) + K_s(x) - K_r(x)\right| \leq O(\log r) + O(\log\log \|x\|)$.

In the remainder of the article, we will use the $O(\log r)$ error term to encapsulate all errors accumulated from computing terms that do not change as the precision, r, increases to infinity. This would include the $\log\log \|y\|$ and $\log\log \|x\|$ terms from Lemma 51 as well as $\log m$ and $\log n$ terms already suppressed in the statement of Lemma 51. Using the notion of Kolmogorov complexity, one defines the concept of effective dimension as the asymptotic Kolmogorov complexity of a given point.

Definition 52. *The effective Hausdorff dimension of a point $x \in \mathbb{R}^d$ is given by*

$$\dim(x) := \liminf_{r \to \infty} \frac{K_r(x)}{r}.$$

The effective Packing dimension of a point $x \in \mathbb{R}^d$ is given by

$$Dim(x) := \limsup_{r \to \infty} \frac{K_r(x)}{r}.$$

In computability theory, a set $A \subseteq \mathbb{N}$ is called an *oracle*. Heuristically, a Turing machine T can be said to *have access to* A if, in addition to its usual operations, it can inquire whether the number currently printed on the work tape belongs to A. This operation takes one step, and the answer to the question is recorded as a state. The machine derived from T by allowing it access to A is denoted T^A.

The concepts of complexity and dimension defined above can be *relativized to an oracle* A by replacing the fixed universal Turing machine U with U^A. These relativized concepts are denoted K^A, $\dim^A$, etc. For us, the utility of oracles is that they allow us to compute the Hausdorff dimension of a set from the effective dimensions the points it contains.

Theorem 53 (Point-to-Set Principle [50]). *For every set $E \subseteq \mathbb{R}^d$,*

$$\dim_{\mathrm{H}} E = \min_{A\subseteq\mathbb{N}} \sup_{x\in E} \dim^A(x) \quad \textit{and} \quad \dim_{\mathrm{P}} E = \min_{A\subseteq\mathbb{N}} \sup_{x\in E} Dim^A(x).$$

One can infer from the proof of the point-to-set principle that, for any oracle A and $s \in (0, d]$, the set of points x that satisfy $\dim^A(x) < s$ has s-dimensional Hausdorff measure zero. In particular, $\dim^A(x) = d$ for all x outside of a Lebesgue-null set.

As in [69], we describe the complexity of a point $x \in \mathbb{R}^d$ relative to a point $y \in \mathbb{R}^{d'}$ as the complexity of x relative to an oracle set A_y that encodes the binary expansion of y in a standard way. We define

$$K^y_r(x) := K^{A_y}_r(x).$$

Throughout the paper, we will often invoke the notion of (Martin–Löf) "randomness." A point $x \in \mathbb{R}^d$ is *random* with respect to an oracle A if

$$\left|K^A_r(x) - rd\right| \leq O(\log r).$$

This also implies that each of the coordinates of x is random with respect to the others and has effective Hausdorff dimension 1 with respect to the oracle A. More quantitatively, for $t < r$, the symmetry of information (Lemma 51) implies that

$$\begin{aligned} rd + O(\log r) = K_r^A(x) &= K_{r,t}^A(x) + K_t(x) + O(\log r) \\ &\leq K_{r,t}^A(x) + td + O(\log r), \end{aligned}$$

whence

$$|K_{r,t}^A(x) - (r-t)d| \leq O(\log r).$$

This means that the first t digits of the binary expansion of x "do not help much" in the calculating the remaining $r - t$ digits, and that we would approximately need $(r - t)d$ bits to calculate them. In the proof of Theorem 47, we will have occasion to choose collections of strings of digits that are random with respect to each other, but we can see from this discussion that such a choice is always possible.

5.3 Distance set bounds for arbitrary norms

We begin with a "trivial" lemma for the benefit of the reader new to the theory of Kolmogorov complexity.

Lemma 54. *Let $A \subseteq \mathbb{N}$. For all $x, y \in \mathbb{R}^d$ and $r \in \mathbb{N}$,*

$$K_r^{A,x}(y) \leq K_r^{A,x}(x+y) + O(\log r). \tag{5.2}$$

Proof. Let $T^{A,x}$ be an oracle Turing machine that operates as follows: given a rational point $p \in \mathbb{Q}^d$, a precision level $r \in \mathbb{N}$, and a "corrector" string $\sigma = \sigma_1 \dots \sigma_{d'} \in \{0,1\}^{d'}$ of length $d' := \lceil d^{1/2} \rceil$, it first computes the "small" dyadic rational number

$$q_{\sigma,r} := \frac{\sigma_1}{2^{-(r+1)}} + \cdots + \frac{\sigma_{d'}}{2^{-(r+d')}}.$$

The machine then returns the first $r + d' + 1$ bits of $p + q_{r,\sigma} - x$ as its output:

$$T^{A,x}(p, r, \sigma) = (p + q_{r,\sigma} - x) \upharpoonright (r + d' + 1).$$

Because the oracle encodes x, this is a computable operation relative to (A, x); hence, $T^{A,x}$ exists.

Now let $p \in B(x + y, 2^{-r}) \cap \mathbb{Q}^d$ be a point with $K^{A,x}(p) = K_r^{A,x}(x + y)$. We use p and $T^{A,x}$ to compute a precision-r approximation of y. To do this, take $\sigma \in \{0,1\}^{d'}$ such that $p + q_{r,\sigma} \in B(x + y, 2^{-(r+1)})$. (This explains the choice of d': it is just large enough to encode a rational number that "corrects" p by a factor of up to $\frac{1}{2}$, upgrading the precision-r approximation to a precision-$(r + 1)$ approximation.) Since $\|(z \upharpoonright (t + d')) - z\| < 2^{-t}$ for all $z \in \mathbb{R}^d$ and $t \in \mathbb{N}$,

$$\begin{aligned}
&\big\|\big((p + q_{r,\sigma}) - x) \upharpoonright (r + d' + 1)\big) - y\big\| \\
&\qquad \leq \big\|(p + q_{r,\sigma} - x) - y\big\| + \big\|\big((p + q_{r,\sigma} - x) \upharpoonright (r + d' + 1)\big) - (p + q_{r,\sigma} - x)\big\| \\
&\qquad < \|(p + q_{r,\sigma}) - (x + y)\| + 2^{-r-1} < 2^{-(r+1)} + 2^{-(r+1)} = 2^{-r},
\end{aligned}$$

so $T^{A,x}(p, r, \sigma) \in B(y, 2^{-r})$.

If μ is a string that encodes $T^{A,x}$ in the language of $U^{A,x}$, so that $U^{A,x}(\mu, q, r, \sigma) = T^{A,x}(q, r, \sigma)$ for all q, r, and σ, then

$$\begin{aligned}
K_r^{A,x}(y) &\leq K^{A,x}(\mu, p, r, \sigma) \leq |\mu| + K^{A,x}(p) + K^{A,x}(r) + K^{A,x}(\sigma) + O(1) \\
&= K^{A,x}(p) + O(\log r) = K_r^{A,x}(x + y) + O(\log r).
\end{aligned}$$

(The value $|\mu|$ is a constant independent of r, and the additional $O(1)$ represents the complexity required to concatenate μ, p, r, and σ into a single string in the language of $U^{A,x}$.) $\square$

This computations of this sort are routine, arithmetic properties such as that in (5.2) will be used freely in what follows.

Proof of Theorem 44. The unit sphere in the norm $\|\cdot\|_*$ has Packing dimension $d-1$. Therefore, by the point-to-set principle, there exists an oracle $A \subseteq \mathbb{N}$ such that

$$\sup_{z \in \partial B_*(0,1)} \mathrm{Dim}^A(z) = d - 1.$$

For any $y \in E$, if we write $y = x + \|x-y\|_* z$ for some $z \in \partial B_*(0,1)$, then the following holds: for any oracle $B \subseteq \mathbb{N}$,

$$\begin{aligned} K_r^{A,B,x}(y) &\leq K_r^{A,B,x}(y-x) + O(\log r) \\ &= K_r^{A,B,x}(\|x-y\|_* z) + O(\log r) \\ &\leq K_r^{A,B,x}(\|x-y\|_*) + K_r^{A,B,x}(z) + O(\log r). \end{aligned}$$

Therefore,

$$\liminf_{r\to\infty} \frac{K_r^{A,B,x}(y)}{r} \leq \liminf_{r\to\infty} \frac{K_r^{A,B,x}(\|x-y\|_*)}{r} + \limsup_{r\to\infty} \frac{K_r^{A,B,x}(z)}{r} + \limsup_{r\to\infty} \frac{O(\log r)}{r}$$

and so

$$\begin{aligned} \dim^{A,B,x}(y) &\leq \dim^{A,B,x}(\|x-y\|_*) + \mathrm{Dim}^{A,B,x}(z) \\ &\leq \dim^{A,B,x}(\|x-y\|_*) + d - 1. \end{aligned}$$

In particular, for every $\epsilon > 0$, the point-to-set principle gives a point $y \in E$ such that

$$\dim_{\mathrm{H}} E \leq \dim^{A,B,x}(y) + \epsilon.$$

Thus, for every ϵ, there exists a point $t = \|x-y\|_* \in \Delta^*_x(E)$ such that

$$\dim_{\mathrm{H}} E - (d-1) - \epsilon \leq \dim^{A,B,x}(t).$$

Taking the minimum over all oracles and letting $\epsilon \to 0$ then gives the desired inequality. □

5.4 Sharp examples

In this section we construct the set $E \subset \mathbb{R}^d$ proving Theorem 47. Fundamentally, the construction adapts [30] Example 7.8, which in turn works on the same principles as the "Venetian blinds" constructions that have been studied widely in geometric measure theory for the extreme features they exhibit, e.g., in the context of orthogonal projections, see [21]. As such, the use of complexity in *building* our set E is secondary to its use in actually *proving* that it has the asserted properties.

Proof of Theorem 47. The case $s = d$ is trivial, so let $s \in [d-1, d)$ and $\alpha := s - (d-1)$. Since $\|\cdot\|_P$ is a polyhedral norm, there exists a finite set $\{v^1, ..., v^N\} \subset \mathbb{R}^d$ of vectors—one for each face of the $\|\cdot\|_P$-ball—such that

$$\|x\|_P = \max\{|x \cdot v^\ell|\}_{\ell=1}^N \tag{5.3}$$

for all $x \in \mathbb{R}^d$. Let c be a large constant to be specified later, and let $\{m_k\}_{k=1}^\infty$ be a sequence of positive integers satisfying $m_1 = 1$, $1 + \frac{2c}{1-\alpha} < m_2$, and, for all $k \in \mathbb{Z}_+$,

$$km_k \leq m_{k+1}. \tag{5.4}$$

We also define a sequence $\{n_k\}_{k=1}^\infty$ by

$$n_k := m_k + \alpha(m_{k+1} - m_k).$$

Note that, since $\alpha \in [0,1)$ and $m_2 > 1 + \frac{2c}{1-\alpha}$, we have $n_k + c < m_{k+1} - c$ for all k.

We construct $E \subset [0,1]^d$ in blocks of N steps. Define

$$F_k := \begin{Bmatrix} x \in [0,1]^d : \lfloor 2^j (x \cdot v^\ell) \rfloor = 0 \text{ for all} \\ j \in (n_k + c, m_{k+1} - c], \text{ where } \ell \equiv k \pmod N \end{Bmatrix} \tag{5.5}$$

and

$$E := \bigcap_{k=1}^{\infty} F_k.$$

These definitions prescribe that the binary expansions of the projections $x \cdot v^\ell$ of the points in $x \in E$ alternate between long strings of free digits and long strings of zeros, with the lengths of the strings rapidly approaching infinity. When a real number has two different binary expansions, we associate to it the expansion terminating in an infinite string of ones. This makes the sets F_k and thus E to be closed.

Claim: For every oracle $A \subseteq \mathbb{N}$, there exists $x \in E$ such that

$$\liminf_{r \to \infty} \frac{K_r^A(x)}{r} \geq s. \tag{5.6}$$

Moreover, for all $\tau \in \Delta^P(E)$,

$$\liminf_{r \to \infty} \frac{K_r^A(\tau)}{r} \leq \alpha. \tag{5.7}$$

By the definition of α, this will imply that $\dim_{\mathrm{H}} \Delta^P(E) \leq s - (d-1) \leq \dim_{\mathrm{H}} E - (d-1)$, and an application of Theorem 44 will entail that these hold with equality.

To obtain this point x and ensure that it belongs to E, we iteratively select the digits in the binary expansions of its coordinates. In particular, assuming we have already chosen the digits in the places before the m_kth place, the digits in the places $[m_k, m_{k+1})$ will be chosen in two steps. For each $\ell = 1, ..., N$, fix any index $1 \leq i_\ell \leq d$ such that the i_ℓth coordinate of v^ℓ is nonzero: $v^\ell_{i_\ell} \neq 0$. Let $\ell \equiv k \pmod N$, $1 \leq \ell \leq d$.

1. We define the digits in the binary expansion of x_i in the places $[m_k, m_{k+1})$ for $i \neq i_\ell$, and in the places $[m_k, n_k)$ for $i = i_\ell$, such that the corresponding d strings (of which $d-1$ have length $m_{k+1} - m_k$ and one has length $n_k - m_k$) are random with respect to each other, the oracle A and the first m_k digits of x. In particular, this implies that

$$K^A_{r,m_k}(x) \geq d(r - m_k) - O(\log r), \text{ for all } r \in [m_k, n_k).$$

After we choose the digits of x_{i_ℓ} in the places $[n_k, m_{k+1})$ in Step 2, these random choices will allow us to conclude that

$$K^A_{r,n_k}(x) \geq (d - \alpha)(r - n_k) - O(\log r), \text{ for all } r \in [n_k, m_{k+1}).$$

2. We want to define the digits in the binary expansion of x_{i_ℓ} in the places $[n_k, m_{k+1})$ such that

$$\left\lfloor 2^j \sum_{i=1}^{d} 2^{-m_{k+1}} \lfloor 2^{m_{k+1}} x_i \rfloor v_i^\ell \right\rfloor \equiv \begin{cases} 0 & \text{if } j \in (n_k + c, m_{k+1} - c], \\ 1 & \text{if } j = m_{k+1} - c + 1, \\ 0 & \text{if } j = m_{k+1} - c + 2, \end{cases} \tag{5.8}$$

where the equivalence holds modulo 2. Before showing that this is possible, we prove that Equation (5.8) implies

$$\left\lfloor 2^j (x \cdot v^\ell) \right\rfloor \equiv 0 \pmod 2 \text{ for every } j \in (n_k + c, m_{k+1} - c], \tag{5.9}$$

provided c is chosen large enough. It will then follow from (5.5) that the point x satisfying (5.8) belongs to F_k. In order to derive Equation (5.9) from Equation (5.8), we first observe that

$$x \cdot v^\ell = \sum_{i=1}^{d} 2^{-m_{k+1}} \lfloor 2^{m_{k+1}} x_i \rfloor v_i^\ell + \sum_{i=1}^{d} \left(x_i - 2^{-m_{k+1}} \lfloor 2^{m_{k+1}} x_i \rfloor \right) v_i^\ell. \tag{5.10}$$

Notice that, if (5.8) holds, then for some $M \in \mathbb{Z}_+$ and $t \leq \frac{1}{4}$, we have

$$\sum_{i=1}^{d} 2^{-m_{k+1}} \lfloor 2^{m_{k+1}} x_i \rfloor v_i^\ell = 2^{-n_k - c} M + 2^{-m_{k+1}+c} \left(\tfrac{1}{2} + t \right). \tag{5.11}$$

Choose c large enough such that

$$\max \left(\sum_{i=1}^{d} |v_i^\ell|, \, \left| v_{i_\ell}^\ell \right|^{-1} \right) \leq 2^{c-3} \quad \text{for all } \ell, \tag{5.12}$$

from which it follows that

$$\begin{aligned} \left| \sum_{i=1}^{d} (x_i - 2^{-m_{k+1}} \lfloor 2^{m_{k+1}} x_i \rfloor) v_i^\ell \right| &\leq \sum_{i=1}^{d} \left| x_i - 2^{-m_{k+1}} \lfloor 2^{m_{k+1}} x_i \rfloor \right| |v_i^\ell| \\ &\leq 2^{-m_{k+1}} \sum_{i=1}^{d} |v_i^\ell| \\ &\leq 2^{-m_{k+1}+c-3}. \end{aligned} \tag{5.13}$$

Combining (5.10), (5.11), and (5.13), we see that

$$2^j (x \cdot v^\ell) = 2^{j - n_k - c} M + 2^{j - (m_{k+1} - c)} \left(\tfrac{1}{2} + t + t' \right),$$

where $|t'| \leq \frac{1}{8}$. Hence, for $j \in (n_k + c, m_{k+1} - c]$, the second term above lies in the interval $(0, 1)$:

$$0 < 2^{j - (m_{k+1} - c)} \left(\tfrac{1}{2} + t + t' \right) < 1.$$

Therefore, we have shown that Equation (5.8) implies

$$\left\lfloor 2^j \sum_{i=1}^{d} 2^{-m_{k+1}} \lfloor 2^{m_{k+1}} x_i \rfloor v_i^\ell \right\rfloor = \lfloor 2^j (x \cdot v^\ell) \rfloor \tag{5.14}$$

for $j \in (n_k + c, m_{k+1} - c]$, which immediately entails Equation (5.9).

It remains to see that we can choose the digits of x_{i_ℓ} in the places $[n_k, m_{k+1})$ such that (5.8) holds. The condition (5.8) determines the digits of $x_{i_\ell} \cdot v_{i_\ell}^\ell$ in places $[n_k, m_{k+1} - c + 2)$, and we see from (5.12) that

$$2^{-c+3} \leq v_{i_\ell}^\ell \leq 2^{c-3}.$$

Therefore, we can indeed make such a choice for the digits of x_{i_ℓ}.

Repeating this process for all $k \in \mathbb{Z}_+$ produces the desired point $x \in E$.

Now we prove that $\dim^A(x) \geq s$. Let $r \in (m_k, m_{k+1}]$ and $\ell \equiv k \pmod N$. We estimate $K_r^A(x)$ in two cases.

Case 1: $r \in (m_k, n_k]$

By symmetry of information, we have

$$\begin{aligned} K_r^A(x) &= K_{r,m_k}^A(x) + K_{m_k}^A(x) + O(\log r) \\ &\geq (r - m_k)d + K_{m_k}^A(x) - O(\log r). \end{aligned} \tag{5.15}$$

The inequality is true because the digits of x in the m_kth through rth places are chosen randomly with respect to A and the digits in the places 1 through m_k.

Case 2: $r \in (n_k, m_{k+1}]$

By the symmetry of information, we have

$$K_r^A(x) = K_{r,n_k}^A(x) + K_{n_k}^A(x) - O(\log r).$$

It follows from (5.15) of Case 1 that

$$\begin{aligned} K^A_{n_k}(x) &\geq (n_k - m_k)d + K^A_{m_k}(x) - O(\log r) \\ &= \alpha(m_{k+1} - m_k)d + K^A_{m_k}(x) - O(\log r). \end{aligned}$$

Thus we have

$$\begin{aligned} K^A_r(x) &\geq K^A_{r,n_k}(x) + \alpha(m_{k+1} - m_k)d + K^A_{m_k}(x) - O(\log r) \\ &\geq K^A_{r,n_k}(x_1, ..., x_{i_\ell - 1}, x_{i_\ell + 1}, ..., x_d) + \alpha(m_{k+1} - m_k)d + K^A_{m_k}(x) - O(\log r) \\ &\geq (r - n_k)(d-1) + \alpha(m_{k+1} - m_k)d + K^A_{m_k}(x) - O(\log r) \\ &= (r - m_k)(d-1) + \alpha(m_{k+1} - m_k) + K^A_{m_k}(x) - O(\log r). \end{aligned} \tag{5.16}$$

The third inequality above also follows from randomness. We put $r = m_{k+1}$ in (5.16) to get

$$\begin{aligned} K^A_{m_{k+1}}(x) &= (m_{k+1} - m_k)s + K^A_{m_k}(x) - O(\log r) \\ &\geq m_{k+1}\left(1 - \frac{1}{k}\right)s + K^A_{m_k}(x) - O(\log r), \end{aligned}$$

where the inequality follows from (5.4). Thus for any arbitrarily small $t > 0$, we have

$$\frac{K^A_{m_{k+1}}(x)}{m_{k+1}} \geq s - t \quad \text{for large enough } k.$$

We will now justify that the expression $K^A_r(x)/r$ is minimized when $r = m_k$ for some k. By

(5.15) of Case 1,

$$\begin{aligned}\frac{K^A_r(x)}{r} &\geq \left(1-\frac{m_k}{r}\right)d + \frac{K^A_{m_k}(x)}{r} - \frac{O(\log r)}{r}\\ &= \left(1-\frac{m_k}{r}\right)d + \frac{K^A_{m_k}(x)}{m_k}\left(\frac{m_k}{r}\right) - \frac{O(\log r)}{r}\\ &\geq \left(1-\frac{m_k}{r}\right)d + (s-t)\frac{m_k}{r} - \frac{O(\log r)}{r}\\ &= d-(\alpha-1+t)\frac{m_k}{r} - \frac{O(\log r)}{r}\end{aligned} \tag{5.17}$$

and similarly, by (5.16), we have in Case 2

$$\begin{aligned}\frac{K^A_r(x)}{r} &\geq \left(1-\frac{m_k}{r}\right)(d-1) + \frac{\alpha(m_{k+1}-m_k)}{r} + \frac{K^A_{m_k}(x)}{r} - \frac{O(\log r)}{r}\\ &\geq d-1+\alpha\frac{m_{k+1}}{r} + \frac{K^A_{m_k}(x)-sm_k}{r} - \frac{O(\log r)}{r}\\ &\geq d-1+\alpha\frac{m_{k+1}}{r} - \frac{tm_k}{r} - \frac{O(\log r)}{r}.\end{aligned} \tag{5.18}$$

Recalling $\frac{m_k}{r} \leq 1$ and $\frac{m_{k+1}}{r} \geq 1$, we conclude (5.6) from (5.17) and (5.18), and therefore we have proved that $\dim_{\mathrm{H}} E \geq s$.

Now, for any fixed $z, y \in E$, let $\tau := \|z-y\|_P \in \Delta^P(E)$. Thus, $\tau = |(z-y)\cdot v^\ell|$ for some ℓ. Hence, for any $k \equiv \ell \pmod N$, we have

$$\lfloor 2^j((z-y)\cdot v^\ell)\rfloor \equiv 0 \pmod 2 \quad \text{for all} \quad j \in [n_k+c+1, m_{k+1}-c-1]$$

or

$$\lfloor 2^j((z-y)\cdot v^\ell)\rfloor \equiv 1 \pmod 2 \quad \text{for all} \quad j \in [n_k+c+1, m_{k+1}-c-1].$$

Therefore,

$$K^A_{m_{k+1}-c,n_k+c}(\tau) \leq O(\log m_{k+1}).$$

This is because specifying a string of 000's or 111's having length less than m_{k+1} requires a program of length at most $O(\log m_{k+1})$. We consequently have

$$\begin{aligned} K^A_{m_{k+1}-c}(\tau) &\leq K^A_{m_{k+1}-c,n_k+c}(\tau) + K^A_{n_k}(\tau) + O(\log m_{k+1}) \\ &\leq n_k + O(\log m_{k+1}) \\ &\leq m_{k+1}\alpha\left(1 - \frac{1}{k}\right) + O(\log m_{k+1}), \end{aligned}$$

where the last inequality follows from (5.4). Taking $k \to \infty$ gives (5.7), and, since this holds for all $\tau \in \Delta^P(E)$, we conclude that $\dim_{\mathrm{H}} \Delta^P(E) \leq \alpha$. The desired upper bound on $\dim_{\mathrm{H}} E$ then follows immediately from Theorem 44:

$$s \leq \dim_{\mathrm{H}} E \leq \dim_{\mathrm{H}} \Delta^P(E) + (d-1) \leq \alpha + (d-1) = s.$$

As such, all the inequalities above are actually equalities, so Equation (5.1) holds. $\square$

CHAPTER 6

ON THE PACKING DIMENSION OF DISTANCE SETS WITH RESPECT TO C^1 AND POLYHEDRAL NORMS

This chapter is joint work with Ryan Bushling and Bobby Wilson, and we will be submitting it to a journal as a paper.

6.1 Introduction

For a norm $\|\cdot\|_*$ on $\mathbb{R}^d$ and set $E \subseteq \mathbb{R}^d$, let

$$\Delta^*(E) := \big\{\|y-x\|_* \in [0,\infty) \colon x, y \in E\big\}$$

be the *distance set* of E with respect to that norm. Given a fixed $x \in \mathbb{R}^d$, we also let

$$\Delta^*_x(E) := \big\{\|y-x\|_* \in [0,\infty) \colon y \in E\big\}$$

be the *pinned distance set* of E at x. The classical *Falconer distance conjecture* [28] posits that, for E (say, Borel),

$$\dim_{\mathrm{H}} E > \frac{d}{2} \qquad \Longrightarrow \qquad \dim_{\mathrm{H}} \Delta(E) = 1,$$

where $\dim_{\mathrm{H}}$ denotes Hausdorff dimension and Δ the distance set operator for the Euclidean norm. More quantitatively, one frequently seeks lower bounds on $\dim_{\mathrm{H}} \Delta(E)$ or $\dim_{\mathrm{H}} \Delta_x(E)$ in terms of $\dim_{\mathrm{H}} E$. For recent progress on the distance set conjecture, see [26, 33, 34]. In particular, [26] contains a clear and thorough history of the problem.

This chapter may be seen as a follow-up to chapter 5, in which the authors treat the Falconer distance problem for general norms, emphasizing in particular the case of polyhedral norms. The main result therein is the following.

Theorem 55 ([6] Theorems 1.1 & 1.4)**.** *Let $\|\cdot\|_*$ be a norm on $\mathbb{R}^d$ and let $E \subseteq \mathbb{R}^d$. Then*

$$\dim_{\mathrm{H}} \Delta_x^*(E) \geq \dim_{\mathrm{H}} E - (d-1) \tag{6.1}$$

for all $x \in \mathbb{R}^d$. This bound is sharp for polyhedral norms in the sense that, if $\|\cdot\|_P$ is a polyhedral norm on $\mathbb{R}^d$, then for any $s \in [d-1, d]$, there exists a compact set $E \subset \mathbb{R}^d$ with $\dim_{\mathrm{H}} E = s$ such that

$$\dim_{\mathrm{H}} \Delta^P(E) = s - (d-1).$$

6.1.1 Main results

Distance sets with respect to polyhedral norms have also been considered in [48, 29, 47, 12], and in addition [12] presents strictly convex C^1 norms with pathological distance sets. The works [60, 45, 33, 34] address problems concerning the *packing* dimension $\dim_{\mathrm{P}}$ of distance sets, and here we consider the packing dimension of distance sets with respect to polyhedral and C^1 norms. Our main theorem is the following.

Theorem 56. *Let $\|\cdot\|_*$ be either a C^1 or polyhedral norm on $\mathbb{R}^d$ and let $E \subseteq \mathbb{R}^d$. For every $\varepsilon > 0$, there exists $x \in E$ such that*

$$\dim_{\mathrm{P}} \Delta_x^*(E) \geq \frac{1}{d} \dim_{\mathrm{P}} E - \varepsilon. \tag{6.2}$$

In particular,

$$\dim_{\mathrm{P}} \Delta^*(E) \geq \frac{1}{d} \dim_{\mathrm{P}} E.$$

In both the C^1 and the polyhedral case, the normal vector field to the surface the unit ball is regular enough to establish reasonable transversality conditions on the pinned distance functions $y \mapsto \|x-y\|_*$. Just as (6.1) was sharp for every polyhedral norm in a strong way, (6.2) is also sharp for a large class of polyhedral norms.

Theorem 57. *Let $\|\cdot\|_P$ be a polyhedral norm such that the unit normal vectors to the faces of the polyhedron are rationally dependent and let $s \in [0, d]$. There exists a compact set $E \subset \mathbb{R}^d$ with $\dim_{\mathrm{P}} E = s$ such that*

$$\dim_{\mathrm{P}} \Delta^P(E) = \frac{s}{d}. \tag{6.3}$$

A main tool in the proof of Theorem 56 is a nonlinear projection estimate extending a result of Järvenpää [41]. Let $\mathbf{p} = (\mathbf{p}_i)_{i=1}^d$ be a family of functions $\mathbf{p}_i \colon E \to \mathbb{R}$ defined on a set $E \subseteq \mathbb{R}^d$ and let $\overline{\dim}_{\mathrm{B}}$ denote upper box dimension.

Definition 58 (Weak transversality)**.** *We say that $\mathbf{p}$ is weakly transversal if, for every bounded set $F \subseteq E$, its restriction $\mathbf{p}|_F$ is bounded and has fibers of upper box dimension* 0*. That is:*

1. *$\mathbf{p}(F)$ is bounded; and*
2. *for every $\delta \in (0, 1)$, there exists a natural number $M(\delta) \lesssim_\varepsilon \delta^{-\varepsilon}$ for every $\varepsilon > 0$ such that the following holds: for every $\xi \in \mathbb{R}^n$, there exist $m \le M(\delta)$ many points $x_1, \ldots, x_m \in \mathbb{R}^n$ such that*

$$\mathbf{p}^{-1}\big(B(\xi, \delta)\big) \cap F \subseteq \bigcup_{k=1}^{m} B(x_k, \delta). \tag{6.4}$$

The first condition simply ensures that $\overline{\dim}_{\mathrm{B}}\, \mathbf{p}(F)$ is defined whenever $\overline{\dim}_{\mathrm{B}}\, F$ is. The second condition ensures that the fibers of the functions $\mathbf{p}_i$ intersect fairly transversely at most points; it is always satisfied, for example, when $\mathbf{p}$ is locally invertible with Lipschitz inverses. Either δ (or both) in (6.4) can be replaced with a number $\sim_\varepsilon \delta$.

With this terminology, we have the following result.

Proposition 59. *Let* $\mathbf{p} = \big(\mathbf{p}_i \colon E \to \mathbb{R}\big)_{i=1}^{d}$ *be a weakly transversal family, let* $n \in \{1, \dots, d\}$*, and for every string of indices* $i_1, \dots, i_n$ *let* $\mathbf{p}_{i_1,\dots,i_n} := \big(\mathbf{p}_{i_j} : E \to \mathbb{R}\big)_{j=1}^{n}$*. Then for every set* $F \subseteq E$*,*

$$\frac{n}{d}\,\overline{\dim}_{\mathrm{B}} F \le \max_{1 \le i_1 < \dots < i_n \le d} \overline{\dim}_{\mathrm{B}}\, \mathbf{p}_{i_1,\dots,i_n}(F) \tag{6.5}$$

provided F *is bounded, and*

$$\frac{n}{d}\dim_{\mathrm{P}} F \le \max_{1 \le i_1 < \dots < i_n \le d} \dim_{\mathrm{P}}\, \mathbf{p}_{i_1,\dots,i_n}(F). \tag{6.6}$$

We only require the result for $n = 1$, and it is possible to prove this directly by a simple covering argument. However, we have included the full statement in the interest of generality and elected a proof that leverages Järvenpää's result on orthogonal projections.

If $\|\cdot\|_*$ is a strictly convex norm on $\mathbb{R}^2$, the "projection maps" $\mathbf{p}_i(x) := \|x - x_i\|_*$ are weakly transversal for any two distinct $x_1, x_2 \in \mathbb{R}^2$, whereby the distance set bound of Theorem 56 follows immediately from Proposition 59. In higher dimensions, one can establish a similar result for strictly convex norms for d pins $x_1, \dots, x_d$ in general position. Things are not so simple for norms with more degenerate curvature. Establishing transversality for *some* set of pins on a large portion of E becomes the main hurdle in proving distance sets estimates for such norms.

6.1.2 Outline

After §6.2 lays out the necessary background on the upper box and packing dimensions, §6.3 presents the proof of Proposition 59 and §6.4 gives the proof of Theorem 56 for polyhedral norms. Next, the transversality conditions necessary to apply Proposition 59 for projections defined by C^1 norms are established in §6.5, and in §6.6 the proof of Theorem 56 for C^1 norms is completed. Finally, the sharp example of Theorem 57 is constructed in §6.7.

6.2 Preliminaries

Let $E \subset \mathbb{R}^d$ be any bounded set and $N(E, \delta)$ its *covering number* at scale δ—the minimal number of δ-balls required to cover E. The *upper box dimension* of a bounded set $E \subset \mathbb{R}^d$ is defined by

$$\overline{\dim}_{\mathrm{B}} E := \limsup_{\delta \to 0} \frac{\log N(E, \delta)}{-\log \delta}.$$

Note that upper box dimension is finitely stable not but countably stable: that is, if $E = \bigcup_{i=1}^{m} E_i$, then $\overline{\dim}_{\mathrm{B}} E = \max_{1 \le i \le m} \overline{\dim}_{\mathrm{B}} E_i$ provided $m < \infty$, whereas it can happen that $\overline{\dim}_{\mathrm{B}} E > \sup_{1 \le i < \infty} \overline{\dim}_{\mathrm{B}} E_i$. Packing dimension can be defined simply by "forcing" upper box dimension to be countably stable: given any set $E \subseteq \mathbb{R}^d$ (not necessarily bounded), we let

$$\dim_{\mathrm{P}} E := \inf \left\{ \sup_{1 \le i < \infty} \overline{\dim}_{\mathrm{B}} E_i : E \subseteq \bigcup_{i=1}^{\infty} E_i \right\}, \tag{6.7}$$

where the infimum is taken over all countable covers of E by bounded sets. It follows immediately that $\overline{\dim}_{\mathrm{B}} E \ge \dim_{\mathrm{P}} E$ for every bounded set E, and it is also not difficult to show that $\dim_{\mathrm{P}} E \ge \dim_{\mathrm{H}} E$.

We sometimes use the standard notation $A \lesssim_\alpha B$ to indicate that $A \le c_\alpha B$ for some constant c_α depending only on the parameter (or collection of parameters) α; these may be suppressed from the notation where the parameters are not especially important. The notation $A \sim_\alpha B$ means that both $A \lesssim_\alpha B$ and $B \lesssim_\alpha A$. Importantly, if for some $E_1, E_2 \subset \mathbb{R}^n$ the relation $N(E_1, \delta) \lesssim_\alpha N(E_2, \delta)$ holds for all $\delta \in (0, 1)$, where α is independent of δ, then $\overline{\dim}_{\mathrm{B}} E_1 \le \overline{\dim}_{\mathrm{B}} E_2$.

Finally, we remark here that §6.5 makes heavy use of tools from convex analysis, but these will be introduced as needed. For more thorough references, see [19, 20].

6.3 Proof of Proposition 59

First we recall an orthogonal projection theorem of Järvenpää [41]. Given linearly independent vectors $e_1, \dots, e_n \in \mathbb{R}^d$, let $V = V(e_1, \dots, e_n) := \mathrm{Span}\{e_1, \dots, e_n\}$ and let $P_V : \mathbb{R}^d \to \mathbb{R}^n$ be the orthogonal projection of $\mathbb{R}^d$ onto V.

Theorem 60 (Järvenpää [41] Theorem 3.4 & Corollary 3.5)**.** *If $(e_1, \dots, e_d)$ is a basis for $\mathbb{R}^d$ and $1 \le n \le d$, then for all $E \subseteq \mathbb{R}^d$ there exist $1 \le i_1 < \cdots < i_n \le d$ such that*

$$\frac{n}{d} \overline{\dim}_{\mathrm{B}} E \le \overline{\dim}_{\mathrm{B}} P_{V(e_{i_1}, \dots, e_{i_n})}(E)$$

provided E is bounded, and

$$\frac{n}{d} \dim_{\mathrm{P}} E \le \dim_{\mathrm{P}} P_{V(e_{i_1}, \dots, e_{i_n})}(E).$$

The statement in Järvenpää [41] is for $1 \le n \le d-1$, but it also holds trivially for $n = d$ as stated above.

Proof of Proposition 59. We first consider the case (6.5) of upper box dimension. Let $F \subseteq E$ be a bounded set and $1 \le n \le d$. By Theorem 60, there exist indices $1 \le i_1 < \cdots < i_n \le d$ such that

$$\frac{n}{d} \overline{\dim}_{\mathrm{B}} \mathbf{p}(F) \le \overline{\dim}_{\mathrm{B}} P_{V(e_{i_1}, \dots, e_{i_n})}(\mathbf{p}(F)),$$

where $(e_1, \dots, e_d)$ denotes the standard basis of $\mathbb{R}^d$. Observe that $P_{V(e_{i_1}, \dots, e_{i_n})}(\mathbf{p}(F)) = \mathbf{p}_{i_1, \dots, i_n}(F)$, so it suffices to show that $\overline{\dim}_{\mathrm{B}} \mathbf{p}(F) \ge \overline{\dim}_{\mathrm{B}} F$. Let $\varepsilon > 0$ and $\delta \in (0,1)$, and let $\{B_i\}_{i=1}^N$ be a cover of $\mathbf{p}(F)$ by $N := N(\mathbf{p}(F), \delta)$-many δ-balls. By the definition of weak transversality, for each i there exists a collection $\{B_{i,j}\}_{j=1}^{N_i}$ of δ-balls such that

$$\mathbf{p}^{-1}(B_i) \cap F \subseteq \bigcup_{j=1}^{N_i} B_{i,j},$$

where $N_i \lesssim \delta^{-\varepsilon}$ for every $\varepsilon > 0$. Hence, $\{B_{i,j} : 1 \le i \le N,\, 1 \le j \le N_i\}$ is a cover of F by

$$\sum_{i=1}^{N} N_i \lesssim N\delta^{-\varepsilon}$$

balls of radius δ. It follows that

$$\begin{aligned}
\overline{\dim}_{\mathrm{B}} F &= \limsup_{\delta\to 0} \frac{\log N(F,\delta)}{-\log\delta} \le \limsup_{\delta\to 0} \frac{\log(N\delta^{-\varepsilon})}{-\log\delta} \\
&= \limsup_{\delta\to 0} \frac{\log N(\mathbf{p}(F),\delta)}{-\log\delta} + \lim_{\delta\to 0} \frac{\log\delta^{-\varepsilon}}{-\log\delta} \\
&= \overline{\dim}_{\mathrm{B}}\, \mathbf{p}(F) + \varepsilon.
\end{aligned}$$

Since $\varepsilon > 0$ was arbitrary, the result follows and (6.5) holds.

We now prove the packing dimension case (6.6). Let $\varepsilon > 0$ and, for each string of indices $1 \le i_1 < \cdots < i_n \le d$, let $\{F^{(j)}_{i_1,\ldots,i_n}\}_{j=1}^{\infty}$ be a cover of $\mathbf{p}_{i_1,\ldots,i_n}(F)$ by bounded sets such that

$$\sup_{1\le j<\infty} \overline{\dim}_{\mathrm{B}} F^{(j)}_{i_1,\ldots,i_n} < \dim_{\mathrm{P}} \mathbf{p}_{i_1,\ldots,i_n}(F) + \frac{\varepsilon}{2}.$$

The collection of all sets of the form

$$\bigcap_{1\le i_1<\cdots<i_n\le d} \mathbf{p}^{-1}_{i_1,\ldots,i_n}\Big(F^{(j_{i_1,\ldots,i_n})}_{i_1,\ldots,i_n}\Big), \quad 1 \le j_{i_1,\ldots,i_n} < \infty \tag{6.8}$$

is a cover of F, but not necessarily by bounded sets. Therefore, we refine this to a new cover $\{G^{(k)}\}_{k=1}^{\infty}$ consisting of all sets formed by intersecting the elements (6.8) with unit cubes $[n_1, n_1+1) \times \cdots \times [n_d, n_d+1)$, $n_j \in \mathbb{Z}$.

Taking an index k such that $\overline{\dim}_{\mathrm{B}} G^{(k)} > \dim_{\mathrm{P}} F - \frac{d}{2n}\varepsilon$, we have by (6.5) a string $1 \le i'_1 < \cdots < i'_n \le d$ such that

$$\frac{n}{d}\dim_{\mathrm{P}} F - \frac{\varepsilon}{2} < \frac{n}{d}\,\overline{\dim}_{\mathrm{B}} G^{(k)} \le \overline{\dim}_{\mathrm{B}}\, \mathbf{p}_{i'_1,\ldots,i'_n}\big(G^{(k)}\big) \tag{6.9}$$

We note that for every k,

$$G^{(k)} \subset \bigcap_{1 \le i_1 < \cdots < i_n \le d} \mathbf{p}^{-1}_{i_1,\ldots,i_n}\Big(F^{(j_{i_1,\ldots,i_n})}_{i_1,\ldots,i_n}\Big)$$

for some collection of $j_{i_1,\ldots,i_n}$. Thus for our given string, $G^{(k)} \subset \mathbf{p}^{-1}_{i'_1,\ldots,i'_n}(F^{(j)}_{i'_1,\ldots,i'_n})$ for some j. Therefore, $\mathbf{p}_{i'_1,\ldots,i'_n}\big(G^{(k)}\big) \subseteq F^{(j)}_{i_1,\ldots,i_n}$, which implies

$$\begin{aligned}\frac{n}{d}\dim_{\mathrm{P}} F - \frac{\varepsilon}{2} &< \overline{\dim}_{\mathrm{B}}\, \mathbf{p}_{i'_1,\ldots,i'_n}\big(G^{(k)}\big) \\ &\le \sup_{1\le j<\infty} \overline{\dim}_{\mathrm{B}}\, F^{(j)}_{i'_1,\ldots,i'_n} < \dim_{\mathrm{P}} \mathbf{p}_{i'_1,\ldots,i'_n}(F) + \frac{\varepsilon}{2},\end{aligned} \tag{6.10}$$

This string $i'_1,\ldots,i'_n$ depends on ε, but since there are only $\binom{d}{n}$ such strings, there is at least one string for which (6.10) holds for arbitrarily small $\varepsilon > 0$; hence, this concludes the proof. $\square$

6.4 Proof of Theorem 56 – The polyhedral case

Let $\|\cdot\|_*$ be a norm defined by a collection $\{v_i\}_{i=1}^M$ of vectors so that

$$\|x\|_* := \max\{|\langle x, v_i\rangle|\}_{i=1}^M.$$

We note that we must have $M \ge d$. For each $x \in \mathbb{R}^d$ and $v \in \mathbb{R}^d$, we define the cone

$$C(x,v) := \Big\{ y \in \mathbb{R}^d \ : \ \|x-y\|_* = |\langle x-y, v\rangle| \Big\}.$$

Obviously, for each $x \in \mathbb{R}^d$,

$$\bigcup_{j=1}^M C(x, v_i) = \mathbb{R}^d.$$

In addition, let $\{z_j\}_{j=1}^{\infty} \subset \mathbb{R}^d$ be a countable set, so that

$$\bigcap_{j=1}^{\infty} \bigcup_{i=1}^{M} C(z_j, v_i) = \mathbb{R}^d = \bigcup_{\substack{\{z_{j_1},\ldots,z_{j_M}\} \\ \{v_{i_1},\ldots,v_{i_M}\}}} \bigcap_{\ell=1}^{M} C(z_{j_\ell}, v_{i_\ell})$$

where in the union we allow $j_s = j_t$ and $i_s = i_t$ for $s \neq t$. For $\alpha > 0$ and a countable set $G \subset \mathbb{R}^d$, define

$$k(E, G, \alpha) := \max_{\substack{\{z_{j_1},\ldots,z_{j_M}\} \subset G \\ \{v_{i_1},\ldots,v_{i_M}\}}} \Big\{ \dim\left(\mathrm{span}(\{v_{i_1}, \ldots, v_{i_M}\})\right) \; : \; \dim_{\mathrm{P}} \left(E \cap \textstyle\bigcap_{\ell=1}^{M} C(z_{j_\ell}, v_{i_\ell}) \right) \geq \dim_{\mathrm{P}} E - \alpha \Big\}.$$

This is the largest number of cones based at points of G that “see" a large (relative to α) portion of E through independent faces of the norm polyhedra $\{y \in \mathbb{R}^d \colon \|y - z_j\|_* = 1\}$.

Lemma 61. *Let $E \subseteq \mathbb{R}^d$, and $\alpha > 0$, and let $G \subset \mathbb{R}^d$ be countable. Then there exist $k \leq k(E, G, \alpha)$ and $F \subseteq E$ with $\dim_{\mathrm{P}} E = \dim_{\mathrm{P}} F$ such that the following hold:*

1. *If $F \cap \bigcap_{\ell=1}^{M} C(z_\ell, v_{i_\ell}) \neq \varnothing$ for any $\{z_1, \ldots, z_M\} \subset G$, then*

$$\dim\left(\mathit{span}\{v_{i_1}, \ldots, v_{i_M}\}\right) \leq k. \tag{6.11}$$

2. *There exist $\{z_1, z_2, \ldots, z_k\} \subset G$ and $\{v_{i_1}, v_{i_2}, \ldots, v_{i_k}\} \subseteq \{v_1, \ldots, v_M\}$ such that*

$$\dim\left(\mathit{span}\{v_{i_1}, \ldots, v_{i_k}\}\right) = k$$

and

$$\dim_{\mathrm{P}}\left(F \cap \bigcap_{\ell=1}^{k} C(z_{j_\ell}, v_{i_\ell})\right) \geq \dim_{\mathrm{P}} E - \alpha.$$

Proof. Let $s = \dim_{\mathrm{P}} E$. The sequence $k(E, G, 2^{-m}\alpha)$ for $m \in \mathbb{Z}_+$ is a monotonically decreasing sequence of integers between 1 and d, so pigeonhole principle implies that there exists $m \geq 1$ such that

$$k(E, G, \alpha) \geq k\big(E, G, 2^{-(m+1)}\alpha\big) = k(E, G, 2^{-m}\alpha) =: k.$$

For each $\bar{z} \in G^M = G \times G \times \cdots \times G$ and $\bar{v} \in \{v_1, \ldots, v_M\}^M$, define

$$E_{\bar{z},\bar{v}} := E \cap \bigcap_{\ell=1}^{M} C(z_{j_\ell}, v_{i_\ell}).$$

If $\dim\left(\operatorname{span}\{v_{i_1}, \ldots, v_{i_M}\}\right) > k$, then

$$\dim_{\mathrm{P}}(E_{\bar{z},\bar{v}}) \leq s - 2^{-m}\alpha$$

for all $\bar{z} \in G^k$. If we let $E_{\bar{v}} := \bigcup_{\bar{z} \in G^M} E_{\bar{z},\bar{v}}$, then $\dim_{\mathrm{P}} E_{\bar{v}} \leq s - 2^{-m}\alpha$. Therefore, if $V_k := \{\bar{v} \in \{v_1, \ldots, v_M\}^M \ : \ \dim\left(\operatorname{span}\{v_{i_1}, \ldots, v_{i_M}\}\right) > k\}$ then

$$F := E \setminus \bigcup_{\bar{v} \in V_k} E_{\bar{v}}$$

has $\dim_{\mathrm{P}} F = \dim_{\mathrm{P}} E$ and the desired condition holds.

For the second part of the statement, if $k\big(E, G, 2^{-(m+1)}\alpha\big) = k$, then by definition there exist $\{z_1, z_2, \ldots, z_k\} \subset G$ and $\{v_{i_1}, v_{i_2}, \ldots, v_{i_k}\} \subseteq \{v_1, \ldots, v_M\}$ such that $\dim\left(\operatorname{span}\{v_{i_1},\right.$

$\ldots, v_{i_k}\}) = k$ and

$$\dim_{\mathrm{P}}\left(E \cap \bigcap_{\ell=1}^{M} C(z_{j_\ell}, v_{i_\ell})\right) \geq \dim_{\mathrm{P}} E - 2^{-(m+1)}\alpha.$$

Finally, $\dim_{\mathrm{P}} E_{\bar{v}} \leq s - 2^{-m}\alpha$ implies

$$\dim_{\mathrm{P}}\left(F \cap \bigcap_{\ell=1}^{M} C(z_{j_\ell}, v_{i_\ell})\right) \geq \dim_{\mathrm{P}} E - 2^{-(m+1)}\alpha \geq \dim_{\mathrm{P}} E - \alpha. \qquad \square$$

Proposition 62. *Let $E \subset \mathbb{R}^d$, $\alpha > 0$, and $G \subset E$ be a countable, dense subset of E. Then there exists $k \in \{1, \ldots, d\}$, $\{z_1, \ldots, z_k\} \subset G$, $\{w_1, w_2, \ldots, w_k\} \subset \{v_1, \ldots, v_M\}$, and $F \subset E$, such that $\dim_{\mathrm{P}} F \geq \dim_{\mathrm{P}} E - \alpha$, and*

$$F \subset \bigcap_{\ell=1}^{k} C(z_\ell, w_\ell).$$

Moreover, $V :=$ span$\{w_1, \ldots, w_k\} \in G(d,k)$ and for each $x, y \in F$

$$\|x - y\|_* = \|P_V(x - y)\|_*.$$

Proof. Lemma 61 implies that there exist an integer $k \in \{1, \ldots, d\}$, $\{z_1, z_2, \ldots, z_k\} \subset G$, $\{w_1, w_2, \ldots, w_k\} \subset \{v_1, \ldots, v_M\}$, and $F_0 \subset E$ such that $\dim_{\mathrm{P}} F_0 = \dim_{\mathrm{P}} E$,

$$\dim\left(\mathrm{span}\{w_1, w_2, \ldots, w_k\}\right) = k$$

and

$$\dim_{\mathrm{P}}\left(F_0 \cap \bigcap_{\ell=1}^{k} C(z_{j_\ell}, w_\ell)\right) \geq \dim_{\mathrm{P}} E - \alpha.$$

Moreover, inequality (6.11) implies that if $F_0 \cap \bigcap_{\ell=1}^{M} C(z_\ell, w_\ell) \neq \varnothing$ for any $\{z_1, \ldots, z_M\} \subset G$ and $(v_{j_1}, \ldots, v_{j_M}) \in \{v_1, \ldots, v_M\}^M$, then

$$\dim\left(\operatorname{span}\{v_{j_1}, \ldots, v_{j_M}\}\right) \leq k.$$

Let $F := F_0 \cap \bigcap_{\ell=1}^{k} C(z_{j_\ell}, w_\ell)$ and $V := \operatorname{span}\{w_1, \ldots, w_k\}$. If $x \in F$, $z \in G$, and $v \in \{v_1, \ldots, v_M\}$ satisfy $x \in C(z, v)$ then

$$\dim\left(\operatorname{span}\{w_1, \ldots, w_k, v\}\right) \leq k,$$

which implies $v \in \operatorname{span}\{w_1, \ldots, w_k\}$. This further implies that for any $x \in F$ and $z \in G$, there is a $v \in \{v_1, \ldots, v_M\}$ such that

$$\begin{aligned}
\|P_V(z-x)\|_* \leq \|z-x\|_* &= |\langle z-x, v\rangle| \\
&= |\langle P_V(z-x), v\rangle + \langle P_V^{\perp}(z-x), v\rangle| = |\langle P_V(z-x), v\rangle| \\
&\leq \|P_V(z-x)\|_*.
\end{aligned}$$

Since G is dense in E, G is dense in F and $\|\cdot\|$ and P_V are continuous, we can conclude that for every $x, y \in F$,

$$\|P_V(y-x)\|_* = \|y-x\|_*. \qquad \square$$

We are now ready to prove Theorem 56 in the case of polyhedral norms.

Proof of Theorem 56 in the polyhedral case. For $E \subset \mathbb{R}^d$, $\alpha > 0$, and $G \subset E$ with $\dim_{\mathrm{P}} E = s$ and with G countable and dense in E, let $F \subset E$, k, $\{z_1, \ldots, z_k\} \subset G$ and $\{w_1, \ldots, w_k\} \subset \{v_1, \ldots, v_M\}$ be the sets given by Proposition 62. Let $V = \operatorname{span}\{w_1, \ldots, w_k\}$. Then Proposition 62 implies $\|x-y\|_* = \|P_V(x-y)\|_*$ for all $x, y \in F$. Therefore, the projection onto V restricted to the set F is a bi-Lipschitz map. Thus, $\dim_{\mathrm{P}} F = \dim_{\mathrm{P}}(P_V F) \geq s - \alpha$.

For each $j \in \{1, \ldots, k\}$, let $P_j(x) := \langle w_j, x\rangle \frac{w_j}{\|w_j\|}$ be the orthogonal projection onto the line spanned by w_j. Then since $F \subset C(z_j, w_j)$, we have $P_j(x - z_j) = \|x - z_j\|_* \frac{w_j}{\|w_j\|}$ whenever $x \in F$. Therefore,

$$\dim_{\mathrm{P}} P_j(F - z_j) = \dim_{\mathrm{P}} \Delta^*_{z_j}(F).$$

Now Theorem 60 implies that for some $j \in \{1, ..., k\}$

$$\dim_{\mathrm{P}} P_j(P_V F) \geq \frac{1}{k} \dim_{\mathrm{P}} P_V F \geq \frac{s - \alpha}{k}.$$

Since $P_V P_j = P_j P_V$ and packing dimensions are stable under translations, $\dim_{\mathrm{P}} P_j(P_V F) = \dim_{\mathrm{P}} P_V(P_j F) = \dim_{\mathrm{P}} P_V(P_j(F - z_j)) = \dim_{\mathrm{P}} P_j(F - z_j) = \dim_{\mathrm{P}} \Delta^*_{z_j}(F)$. In conclusion,

$$\Delta^*_{z_j}(F) \geq \frac{s - \alpha}{k} \geq \frac{s - \alpha}{d}$$

for arbitrary $\alpha > 0$. □

6.5 Establishing weak transversality in the C^1 case

Let $(\mathbb{R}^n, \|\cdot\|_*)$ be represented by X as a Banach space with dual space X^*. For $x \in X$ and $f \in X^*$ we have the pairing $f(x) = \langle f, x\rangle$. Throughout this section, $B(x, r) := \{y \in \mathbb{R}^d : \|x - y\|_* < r\}$ denotes a ball with respect to the norm $\|\cdot\|_*$ and $N_r(E) := \{y \in \mathbb{R}^d : \|x - y\|_* < r \text{ for some } x \in E\}$ denotes that r-neighborhood of a set E.

We first define the subdifferential:

Definition 63 (Subdifferential)**.** *For any $x \in \Omega \subset X$ and any $u : \Omega \to \mathbb{R}$, the subdifferential*

$\partial u(x)$ *of* u *at* x *is defined by*

$$\partial u(x) := \left\{ p \in X^* : \liminf_{y \to x} \frac{u(y) - u(x) - \langle p, y - x \rangle}{\|y - x\|_*} \geq 0 \right\}.$$

Of course, for a convex function, the $\liminf_{y \to x}$ can be removed without affecting the definition. The following appears in the text by Cioranescu [19].

Definition 64 (Duality mapping)**.** *We define the duality mapping as a function from* X *to the power set of* X^**,* $J : X \to 2^{X^*}$*, by*

$$Jx := \left\{ f \in X^* \ : \ \langle f, x \rangle = \|f\|_{X^*} \|x\|_X, \, \|f\|_{X^*} = \|x\|_X \right\}$$

Theorem 65 (Asplund [19] Theorem I.4.4)**.** *For each* $x \in X$*,*

$$Jx = \partial \left(\tfrac{1}{2} \|x\|_X^2 \right) = \|x\|_X \cdot \partial \|x\|_X.$$

For a finite collection of vectors $\{v_1, \ldots, v_m\} \subset \mathbb{R}^d$ with $m \leq d$ we denote the $\mathcal{H}^m$-measure of the parallelepiped defined by $\{v_1, \ldots, v_m\}$ by

$$\left| \left(v_1 \mid v_2 \mid \cdots \mid v_m \right) \right|.$$

We also introduce the cone notation with respect to orthogonal projection P_V for $V \in G(d, k)$, the Grassmannian of k-dimensional subspaces of $\mathbb{R}^d$. With $a \in \mathbb{R}^d$, $s \in (0, 1)$, and $V \in G(d, k)$ we denote

$$X(a, V, s) := \left\{ x \in \mathbb{R}^d \ : \ \|P_{V^\perp}(x - a)\| < s\|x - a\| \right\},$$

where we recall that $\| \cdot \|$ represents the Euclidean norm.

Lemma 66. *Let $z \in \mathbb{R}^d$. There exists $\eta = \eta(d, \|\cdot\|_*)$ such that*

$$z \in X(0, span(\nabla \|z\|_*), \eta).$$

Proof. By Asplund's Theorem 65, $\langle z, \nabla\|z\|_*\rangle = \|z\|_*$. If $\langle \cdot, \cdot \rangle$ is the Euclidean inner product and $\|\cdot\|$ is the Euclidean norm, then since all norms are equivalent in finite-dimensional normed spaces there is a constant $\Lambda = \Lambda(d, \|\cdot\|_*) \in [0, 1/2]$ such that

$$\Lambda\|z\| \leq \langle z, \nabla\|z\|_*\rangle \leq \Lambda^{-1}\|z\|$$

and thus we can let $\eta = \sqrt{1 - \Lambda^2}$. $\square$

Lemma 67. *Let $\varepsilon > 0$. There exists a function, $h : (0, \infty) \to (0, \infty)$ such that $\lim_{\varepsilon \to 0} \frac{h(\varepsilon)}{\varepsilon} = 0$ and, for all $x \in \mathbb{R}^d$ satisfying $\|x - y\|_* < \varepsilon$,*

$$\big|\|x\|_* - \|y\|_*\big| + h(\varepsilon) \geq \big|\langle \nabla\|x\|_*, x - y\rangle\big|.$$

Proof. Follows from the definition of gradient. $\square$

Now we define transversality with respect to this norm. For $x, y \in \mathbb{R}^d$ and $m \in \mathbb{Z}_+$, define

$$p_y(x) := \|x - y\|_*.$$

Definition 68 (***k*-transversality**)**.** *Let $k \in \{1, \ldots, d\}$. We say a set $E \subseteq \mathbb{R}^d$ is k-transversal with respect to $G \subseteq \mathbb{R}^d$ if there exist $L \in (0, 1)$, $\{z_1, \ldots, z_k\} \subseteq G$,*

$$\big|\big(\nabla p_{z_1}(x) \mid \nabla p_{z_2}(x) \mid \cdots \mid \nabla p_{z_k}(x)\big)\big| \geq L$$

for all $x \in E$.

Definition 69 (($\boldsymbol{\alpha, k}$)-transversality). *Let $k \in \{1, \ldots, d\}$ and $\alpha > 0$. We say a set $E \subseteq \mathbb{R}^d$ is (α, k)-transversal with respect to $G \subseteq \mathbb{R}^d$ if there exists $F \subseteq E$ such that $\dim_{\mathrm{P}} F \geq \dim_{\mathrm{P}} E - \alpha$ and F is k-transversal with respect to G.*

Lemma 70. *If $E \subseteq \mathbb{R}^d$ is not (α, k)-transversal with respect to a countable set $G \subset \mathbb{R}^d$, then there exists $F \subseteq E$ (with $\dim_{\mathrm{P}} E = \dim_{\mathrm{P}} F$) such that*

$$\left|\left(\nabla p_{z_1}(x) \mid \nabla p_{z_2}(x) \mid \cdots \mid \nabla p_{z_k}(x)\right)\right| = 0$$

for all $\{z_1, \ldots, z_k\} \subset G$ and $x \in F$.

Proof. Let $s = \dim_{\mathrm{P}} E$ and suppose E is not (α, k)-transversal. Let $\varepsilon_j := 2^{-j}$. For each j, and $\bar{z} \in G^k = G \times G \times \cdots \times G$ define

$$E_{j,\bar{z}} := \left\{x \in E \ : \ \left|\left(\nabla p_{z_1}(x) \mid \nabla p_{z_2}(x) \mid \cdots \mid \nabla p_{z_k}(x)\right)\right| \geq 2^{-j} \text{ for } \bar{z} = (z_1, \ldots, z_k)\right\}.$$

Since E is not (α, k)-transversal,

$$\dim_{\mathrm{P}}(E_{j,\bar{z}}) \leq s - \alpha$$

for all $\bar{z} \in G^k$. If we let $E_j := \bigcup_{\bar{z} \in G^k} E_{j,\bar{z}}$, then $\dim_{\mathrm{P}}(E_j) \leq s - \alpha$. Therefore, if $F_j := E \backslash E_j$, then for every $x \in F_j$ and $\{z_1, \ldots, z_k\} \subset G$ we have

$$\left|\left(\nabla p_{z_1}(x) \mid \nabla p_{z_2}(x) \mid \cdots \mid \nabla p_{z_k}(x)\right)\right| \leq 2^{-j}.$$

Since $\cup_j E_j$ has dimension less than $s - \alpha$, $F := \cap_j F_j$ has dimension s and the proof is complete. □

Corollary 71. *Let $E \subseteq \mathbb{R}^d$, $\alpha > 0$ and let $G \subseteq \mathbb{R}^d$ be countable. Then there exist an integer $k \in \{1, \ldots, d\}$, and $F \subset E$ such that $\dim_{\mathrm{P}} F \geq \dim_{\mathrm{P}} E - \alpha$, F is k-transversal, and for* *each* $\{x_1, \ldots, x_{k+1}\} \subset G$

$$\left|\left(\nabla p_{x_1}(x) \mid \nabla p_{x_2}(x) \mid \cdots \mid \nabla p_{x_{k+1}}(x)\right)\right| = 0.$$

for all $x \in F$.

Proof. For any $x, y \in \mathbb{R}^d$, $\|\nabla p_y(x)\| = 1$, thus every set is $(\alpha/d, 1)$-transversal with respect to G. By definition, there exists a constant $c_1 > 0$, $z \in G$ and $F_1 \subset E$ (namely $F_1 = E$) such that $\dim_{\mathrm{P}} F_1 \geq \dim_{\mathrm{P}} E - \frac{1}{d}$ and $\left|\left(\nabla p_z(x)\right)\right| \geq c_1$ for all $x \in F_1$.

Now F_1 is either $(\alpha/d, 2)$-transversal or not. If not, then Lemma 70 implies that there is a set $\tilde{F}_1 \subset F_1$ such that $\dim_{\mathrm{P}} \tilde{F}_1 = \dim_{\mathrm{P}} F_1 \geq \dim_{\mathrm{P}} E - \alpha/d$. The corollary is then proven with $\tilde{F}_1 =: F$.

If F_1 is $(\alpha/d, 2)$-transversal then there exists $F_2 \subset F_1 \subset E$, $c_2 > 0$ and $z_1, z_2 \in G$ such that $\dim_{\mathrm{P}} F_2 \geq \dim_{\mathrm{P}} F_1 - \alpha/d \geq \dim_{\mathrm{P}} E - 2\alpha/d$ and $\left|\left(\nabla p_{z_1}(x) \mid \nabla p_{z_2}(x)\right)\right| \geq c_2$ for all $x \in F_2$. Induction completes the argument. $\square$

Lemma 72. *Let* $x \in E \subset \mathbb{R}^d$, $0 < \gamma < \delta \ll L < 1$, *and* $k \in \mathbb{Z}_+$. *Let* $G \subseteq E$ *be a countable, dense subset of* E. *Suppose that for each* $\{x_1, \ldots, x_{k+1}\} \subset G$,

$$\left|\left(\nabla p_{x_1}(x) \mid \nabla p_{x_2}(x) \mid \cdots \mid \nabla p_{x_{k+1}}(x)\right)\right| = 0 \tag{6.12}$$

and there exists $\{z_1, \ldots, z_k\} \subseteq G$ *such that*

$$\left|\left(\nabla p_{z_1}(x) \mid \nabla p_{z_2}(x) \mid \cdots \mid \nabla p_{z_k}(x)\right)\right| \geq L. \tag{6.13}$$

If $\mathbf{p} = (p_{z_i})_{i=1}^k$, *then there exists* $C = C(L)$ *and* $W_x \in G(d,k)$ *such that*

$$\mathbf{p}^{-1}(B(\xi, \gamma)) \cap E \cap B(x, \delta) \subset N_{C\cdot(\gamma + h(\delta))}(W_x^{\perp} + x) \cap X(x, W_x, \eta)$$

where $\xi = \mathbf{p}(x)$.

Proof. Define

$$W_{\bar{z}} := \operatorname{span}\left\{\nabla p_{z_1}(x), \nabla p_{z_2}(x), \ldots, \nabla p_{z_k}(x)\right\}.$$

By inequality (6.13), $W_{\bar{z}} \in G(d,k)$. By inequality (6.12), for any $y \in G$, $\nabla p_y(x) \in W_{\bar{z}}$. Therefore, Lemma 66 implies that, for all $y \in G$,

$$y - x \in X(0, W_{\bar{z}}, \eta)$$

and, since G is dense in E,

$$E \subset X(x, W_{\bar{z}}, \eta).$$

Let $\xi = \mathbf{p}(x)$. Now

$$\mathbf{p}^{-1}(B(\xi,\gamma)) \subset \bigcap_{i=1}^{k} \left\{y \in \mathbb{R}^d \mid |\|y - z_i\|_* - \xi_i| < \gamma\right\}.$$

For each i, if $y_0 \in \left\{y \in \mathbb{R}^d \mid |\|y - z_i\|_* - \xi_i| < \gamma\right\}$ and $\|y_0 - x\|_* < \delta$, then

$$\left|\|y_0 - z_i\|_* - \|x - z_i\|_*\right| < \gamma$$

and by Lemma 67,

$$\left|\|y_0 - z_i\|_* - \|x - z_i\|_*\right| + h(\delta) \geq |\langle \nabla \|x - z_i\|_*, y_0 - x\rangle|.$$

Therefore,

$$\bigcap_{i=1}^{k}\left\{y\in\mathbb{R}^d \mid |\|y-z_i\|_*-\xi_i|<\gamma\right\}\subset\bigcap_{i=1}^{k}\left\{y\in\mathbb{R}^d \mid |\langle\nabla p_{z_i}(x),x-y\rangle|<\gamma+h(\delta)\right\}.$$

Now, inequality (6.13) implies that there exists a constant $C=C(L)$ satisfying $C(L)\to\infty$ as $L\to 0$ such that

$$\bigcap_{i=1}^{k}\left\{y\in\mathbb{R}^d \mid |\langle\nabla p_{z_i}(x),x-y\rangle|<\gamma+h(\delta)\right\}\subset N_{C\cdot(\gamma+h(\delta))}(W_{\bar{z}}^{\perp}+x)$$

This completes the proof with $W_x:=W_{\bar{z}}$. □

Proposition 73. *Let $E\subset B(0,1)\subset\mathbb{R}^d$, $\alpha>0$, and $G\subset E$ be a countable, dense subset of E. Then there exists $k\in\{1,\dots,d\}$, $\{z_1,\dots,z_k\}\subset G$, $F\subset E$, and $\delta_0>0$ such that $\dim_{\mathrm{P}}E-\alpha\leq\dim_{\mathrm{P}}F$ and for every $\delta\in(0,\delta_0)$ and $\xi\in\mathbb{R}^d$, there exist m points $x_1,\dots,x_m\in\mathbb{R}^d$ such that $m=m(\|\cdot\|_*,d)$ and*

$$\mathbf{p}^{-1}\big(B(\xi,\delta)\big)\cap F\subseteq\bigcup_{n=1}^{m}B\big(x_n,\delta\big) \tag{6.14}$$

where $\mathbf{p}=(p_{z_i})_{i=1}^{k}$.

Proof. Corollary 71 implies that there exist an integer $k\in\{1,\dots,d\}$, and $F\subset E$ such that $\dim_{\mathrm{P}}F\geq\dim_{\mathrm{P}}E-\alpha$, F is k-transversal, and for each $\{x_1,\dots,x_{k+1}\}\subset G$

$$\big|\big(\nabla p_{x_1}(x)\mid\nabla p_{x_2}(x)\mid\cdots\mid\nabla p_{x_{k+1}}(x)\big)\big|=0$$

for all $x\in F$. Since F is k-transversal with respect to G, there exist $L\in(0,1)$, $\{z_1,\dots,z_k\}\subseteq G$, such that

$$\big|\big(\nabla p_{z_1}(x)\mid\nabla p_{z_2}(x)\mid\cdots\mid\nabla p_{z_k}(x)\big)\big|\geq L$$

for all $x \in F$.

Let $x \in F$ and $\xi = \mathbf{p}(x)$. Lemma 72 implies that there exists $W_x \in G(d,k)$ and $C = C(L)$ such that

$$\mathbf{p}^{-1}(B(\xi,\gamma)) \cap F \cap B(x,\delta) \subset N_{C\cdot(\gamma+h(\delta))}(W_x^{\perp}+x) \cap X(x,W_x,\eta).$$

Recall, $\Lambda = \Lambda(\|\cdot\|_*, d)$ for the proof of Lemma 66. Let $\delta_0 > 0$ be small enough so that $h(\delta) \leq \frac{\Lambda^2\sqrt{1-\eta^2}}{100C}\delta$ for all $\delta \leq \delta_0$ and let $\gamma \leq \frac{\Lambda^2\sqrt{1-\eta^2}}{100C}\delta$.

If $y \in N_{C\cdot(\gamma+h(\delta))}(W_x^{\perp}+x)$, then

$$\|P_{W_x}(y-x)\| \leq \Lambda^{-1}\|P_{W_x}(y-x)\|_* \leq \Lambda^{-1}C\cdot(\gamma+h(\delta)).$$

Also, if $y \in X(x,W_x,\eta)$, then $\|P_{W_x^{\perp}}(y-x)\| < \eta\|y-x\|$ which implies $(1-\eta^2)^{1/2}\|x-y\| < \|P_{W_x}(y-x)\|$. Thus if $y \in N_{C\cdot(\gamma+h(\delta))}(W_x^{\perp}+x) \cap X(x,W_x,\eta)$, then

$$\|x-y\|_* \leq \Lambda^{-1}\|x-y\| < \Lambda^{-2}C\cdot(\gamma+h(\delta))\cdot(1-\eta^2)^{-1/2}.$$

Therefore,

$$N_{C\cdot(\gamma+h(\delta))}(W_x^{\perp}+x) \cap X(x,W_x,\eta) \subset B\left(x, C\cdot\frac{(\gamma+h(\delta))}{\Lambda^2\sqrt{1-\eta^2}}\right) \subset B(x,\tfrac{1}{50}\delta).$$

This further implies that for $\gamma = \frac{\Lambda^2\sqrt{1-\eta^2}}{100C}\delta$

$$\mathbf{p}^{-1}(B(\xi,\gamma)) \cap F \cap B(x,\delta) \subset B(x,\tfrac{1}{50}\delta) \subset \bigcup_i B(y_i,\gamma) \tag{6.15}$$

for at most approximately $(2C/\Lambda^2\sqrt{1-\eta^2})^d$ balls of the form $B(y_i,\gamma)$.

Now let $\gamma \leq \frac{1}{100}\frac{\Lambda^2\sqrt{1-\eta^2}}{100C}\delta_0$ and cover F with approximately δ_0^{-d} balls, $\{B_\ell\}$ of radius $\frac{1}{100}\delta_0$. If $\mathbf{p}^{-1}(B(\xi,\gamma)) \cap F \cap B_\ell \neq \emptyset$, let $x_\ell \in \mathbf{p}^{-1}(B(\xi,\gamma)) \cap F \cap B_\ell$ and $\xi_\ell = \mathbf{p}(x_\ell)$. Then triangle inequality implies

$$\mathbf{p}^{-1}(B(\xi,\gamma)) \cap F \cap B_\ell \subset \mathbf{p}^{-1}(B(\xi_\ell, 100\gamma)) \cap F \cap B(x_\ell, \delta_0)$$

Now (6.15) implies that for each ℓ, there is a collection, $\{B_i^\ell\}$, of approximately $\left(100\frac{2C}{\Lambda^2\sqrt{1-\eta^2}}\right)^d$ balls of radius γ such that

$$\mathbf{p}^{-1}(B(\xi_\ell, 100\gamma)) \cap F \cap B(x_\ell, \delta_0) \subset \bigcup_i B_i^\ell$$

which implies

$$\begin{aligned}\mathbf{p}^{-1}(B(\xi,\gamma)) \cap F &= \bigcup_\ell \mathbf{p}^{-1}(B(\xi,\gamma)) \cap F \cap B_\ell \\ &= \bigcup_\ell \mathbf{p}^{-1}(B(\xi_\ell, 100\gamma)) \cap F \cap B(x_\ell, \delta_0) \subset \bigcup_\ell \bigcup_i B_i^\ell\end{aligned}$$

Thus inequality (6.14) is established with

$$m \sim \delta_0^{-d}\left(\frac{200C}{\Lambda^2\sqrt{1-\eta^2}}\right)^d. \qquad \square$$

6.6 Proof of Theorem 56 – The C^1 case

Let $E \subset \mathbb{R}^d$, let $\varepsilon > 0$, and let $\|\cdot\|_*$ be a C^1 norm on $\mathbb{R}^d$. Let $s = \dim_{\mathrm{P}} E$ and without loss of generality, let $E \subset B(0,1)$.

Proposition 73 implies that there exists $k \in \{1,\dots,d\}$, $F \subset E$ and $\{z_1,\dots,z_k\} \subset E$ such that $\dim_{\mathrm{P}} F \geq \dim_{\mathrm{P}} E - \varepsilon$ and if $\mathbf{p}_i : F \to \mathbb{R}$ is defined by

$$\mathbf{p}_i(x) := \|x - z_i\|_*$$

for all $i \in \{1, \ldots, k\}$, then $\mathbf{p} := (\mathbf{p}_i)_{i=1}^k$ is weakly transversal. Therefore, Proposition 59 implies

$$\frac{1}{k} \dim_{\mathrm{P}} F \leq \max_{1 \leq i \leq k} \dim_{\mathrm{P}} \mathbf{p}_i(F).$$

Since $\mathbf{p}_i(F) = \Delta^*_{z_i}(F)$,

$$\frac{1}{d} \dim_{\mathrm{P}} E - \varepsilon \leq \frac{1}{k} \dim_{\mathrm{P}} E - \varepsilon \leq \frac{1}{k} \dim_{\mathrm{P}} F \leq \max_{1 \leq i \leq k} \dim_{\mathrm{P}} \Delta^*_{z_i}(F) \leq \max_{1 \leq i \leq k} \dim_{\mathrm{P}} \Delta^*_{z_i}(E). \square$$

6.7 Proof of Theorem 57

Let $\|x\|_P = \max \{|x \cdot v^\ell|\}_{\ell=1}^N$ be a polynomial norm such that

$$\{v^1, \ldots, v^N\} \subset \mathbb{Q}^d.$$

Then there exists $q \in \mathbb{Z}^+$ such that the following holds: for each $v^j = (v^j_1, \ldots, v^j_d) \in \{v^1, \ldots, v^N\}$ and for each $\ell \in \{1, \ldots, d\}$, there is a $p^j_\ell \in \{0, \ldots, q-1\}$ such that

$$v^j_\ell = \frac{p^j_\ell}{q}.$$

Consider two sequences of integers $\{m_k\}_{k=1}^\infty$ and $\{M_k\}_{k=1}^\infty$ such that $M_k - m_k \geq 2^k$ for all k and

$$\limsup_{N \to \infty} \frac{\# \left([0, N] \cap \bigcup_{k=1}^\infty [m_k, M_k] \right)}{N} = \frac{s}{d}.$$

Let $A := \bigcup_{k=1}^\infty [m_k, M_k]$ and define $F \subset \mathbb{R}$ to be the compact set

$$F := \left\{ x = \sum_{m=1}^\infty \frac{x_m}{q^m} \ : \ x_m \in \{0, 1, \ldots, q-1\} \, \forall m \text{ and } x_m = 0 \text{ if } m \notin A \right\}.$$

Then $\dim_{\mathrm{P}} F = \frac{s}{d}$, so if we let

$$E := \underbrace{F \times F \times \cdots \times F}_{d \text{ times}} \subset \mathbb{R}^d,$$

then we have $\dim_{\mathrm{P}} E = s$. Now for every $j \in \{1, \ldots, N\}$,

$$\begin{aligned}
&\left\{(x-y)\cdot v^j \ : \ x, y \in E\right\} \\
&= \bigcup_{\ell=1}^{d} \left\{ \sum_{m=1}^{\infty} \frac{(x_m - y_m)p_\ell^j}{q^{m+1}} : x_m, y_m \in \{0, 1, \ldots, q-1\} \ \forall m \text{ and } x_m = 0 \text{ if } m \notin A \right\} \\
&\subseteq \left\{ \sum_{m=1}^{\infty} \frac{x_m}{q^m} : x_m \in \{-q+1, \ldots, q-1\} \ \forall m \text{ and } x_m = 0 \text{ if } m \notin \bigcup_{k=1}^{\infty} [m_k, M_k + 2 + \log d] \right\}.
\end{aligned}$$

Therefore,

$$\dim_{\mathrm{P}} \left\{ |(x-y)\cdot v_\ell^j| \ : \ x \in E \right\} \le \limsup_{N \to \infty} \frac{\#\left([0, N] \cap \bigcup_{k=1}^{\infty} [m_k, M_k + 2 + \log d]\right)}{N} = \frac{s}{d}$$

and thus

$$\dim_{\mathrm{P}} \Delta^*(E) \le \frac{s}{d}. \qquad \square$$

We do not know whether the hypothesis of rational dependence in Theorem 57 can be omitted. Schematically, producing an example of the sort constructed above without this dependence between the faces would require solving an overdetermined system of linear equations. As such, a hypothetical example could not rely on this “prescribed projection" approach. Such constructions are common in the Hausdorff dimension regime and were previously used in [6], but prescribing projections at *all* scales rather than simply a sequence of scales is fundamentally more challenging.

CHAPTER 7

ON THE MODULUS OF CONTINUITY OF FUNCTIONS WHOSE IMAGE HAS POSITIVE MEASURE, AND METRIC EMBEDDINGS INTO $\mathbb{R}^d$ WITHOUT SHRINKING

This chapter is joint work with Marianna Csörnyei and we will be submitting it to a journal as a paper.

7.0.1 Sard: a brief history

Let $f\colon \mathbb{R}^n \to \mathbb{R}^m$ be a differentiable function. The critical points of f are the points at which the Jacobian matrix of f has rank less than m, and the critical values are the values of f at the critical points. The classical Sard theorem (also known as the Sard lemma or Morse-Sard theorem) says that if a function f is C^k differentiable with $k \geq \max(n-m+1, 1)$ then the set of the critical values of f in $\mathbb{R}^m$ has Lebesgue measure 0. The case $m = 1$ was shown by Morse [58], and the general case by Sard [65]. Later Sard in [66] generalized his theorem to show that if $k \geq \max(n-m+1, 1)$, f is C^k and X_d is the set of points at which the rank of the Jacobian matrix is strictly less than d, then the d-dimensional Hausdorff measure of $f(X_d)$ is zero.

Bates [10] lowered the regularity assumption in the original Sard theorem by showing that if $n > m$ and $f \in C^{n-m,1}(\mathbb{R}^n, \mathbb{R}^m)$ then the critical values of f has Lebesgue measure zero. It does not hold for $f \in C^{n-m,\alpha}(\mathbb{R}^n, \mathbb{R}^m)$ with $\alpha < 1$.

Dubovitskii and Federer [27], [31] independently gave another generalization of Sard theorem as the following result. Let $n, m, k, d \in \mathbb{N}$, $d \leq \min\{n, m\} - 1$, $v = d + \frac{n-d}{k}$ and $f \in C^k(\mathbb{R}^n, \mathbb{R}^m)$. Then the v-dimensional Hausdorff measure of $f(X_{d+1})$ is zero. (Putting $d = m - 1$ and $k = n - m + 1$ gives the classical Sard theorem.) Recently in [32] this has been generalized to $C^{k,\alpha}$ functions by Ferone, Korobkov and Roviello. They showed that if

n, m, $k \in \mathbb{N}$, $0 \le \alpha \le 1$, $d < m$, $f \in C^{k,\alpha}(\mathbb{R}^n, \mathbb{R}^m)$, $q \in (d, \infty)$ and $v = n - d - (k+\alpha)(q-d)$, then

$$\mathcal{H}^v(X_d \cap f^{-1}(y)) = 0 \text{ for } \mathcal{H}^q\text{-a.e. } y \in \mathbb{R}^m.$$

In this theorem with $\alpha = 0$ and $q = d + \frac{n-d}{k}$ we get the above-mentioned result of Dubovitskii and Federer. And with $\alpha = 1$, $k = n - m$, $q = m$ and $d = m - 1$ we obtain Bates theorem for $C^{n-m,1}(\mathbb{R}^n, \mathbb{R}^m)$ functions.

Sard theorem has also been generalized in the context of Sobolov spaces. In [61], Pascale showed that if $n < p$ and $k = n - m + 1$ then Sard theorem holds for $W^{k,p}_{\mathrm{loc}}(\mathbb{R}^n, \mathbb{R}^m)$. For many other generalizations of Sard theorem see [1], [14], [15], [32], [35], [59], [62], [64] and [68].

7.0.2 Our generalization, modulus of continuity

Let $f \in C^{1,1}(\mathbb{R}^2, \mathbb{R})$, and let A be the critical set of f, that is, the set of points at which the gradient of f is zero. Then the restriction of f onto A has modulus of continuity $\omega(r) \lesssim r^2$. It is an easy exercise to show that, for every $d \ge 2$ and for every $A \subset \mathbb{R}^d$, if a function $f\colon A \to \mathbb{R}$ has modulus of continuity $\omega(r) \lesssim r^d$ then $f(A)$ is null.

One of our aims in this paper is to study which functions r^d can be replaced by so that the conclusion of Sard theorem remains true. We ask the following question:

Problem 74. *For which ω can one find a set $A \subset \mathbb{R}^d$ and a function $f\colon A \to \mathbb{R}$ with modulus of continuity ω such that $f(A)$ has positive measure?*

Based on the discussion above, it is natural to consider only modulus of continuities $\omega(r)$ for which $\omega(r)/r^d \to \infty$ as $r \to 0$. Choquet showed (see page 52 in [18]) the following statement: If $\omega(r)/r^2$ converges to infinity as $r \to 0$, then there exist a set $A \subset \mathbb{R}^2$ and a function $f\colon A \to \mathbb{R}$ with modulus of continuity ω s.t. $f(A)$ has positive measure. The second author announced in several conferences (see e.g. [22]) that Choquet's argument was incorrect and, in fact, there was a counterexample for his statement: for every $A \subset \mathbb{R}^2$ and $f \colon A \to \mathbb{R}$ with modulus of continuity $\omega(r) = r^2|\log r|$ the measure of $f(A)$ is zero.

In this paper we give the following characterization:

Theorem 75. *Assuming that $\omega(r)$ is strictly increasing, convex, $\omega(0) = 0$ and $\omega(r)/r^d$ is monotone decreasing, the following are equivalent:*

(i) For every $A \subset \mathbb{R}^d$ and for every $f \colon A \to \mathbb{R}$ with modulus of continuity ω, the measure of $f(A)$ is zero.

(ii) $\int_0^1 \omega(r)^{-1/d}\, dr = \infty$.

We will obtain Theorem 75 as a special case of some general metric embedding results that we discuss in the next sections. The precise statements of these embedding results are quite technical (see Section 7.2).

In Section 7.1 we discuss the metric properties that we need to state and prove our general results. We will focus in Section 7.1 on two interesting special cases: the first one is the Sard example (i.e. Theorem 75). The other special case, which we call 'Cantor example' can be thought of as a sort of discretized version of the continuous Sard problem.

The Cantor case is easier than the Sard case. For instance, in the Cantor case many of the general constants used in our proof will be 1 (or 0); and while in the Sard case we need to estimate how certain sets overlap, in the Cantor case they will be disjoint. We hope that seeing how our proof works in the simpler Cantor example will also help the reader to understand some of the more delicate details of the Sard (and of the general) proof.

Section 7.3 and Section 7.4 contain the proof of our main result. In Section 7.3 we prove the non-trivial direction; in Section 7.4 we present a very easy construction showing the sharpness of our results. This construction has a very similar flavour to the one that Choquet was using in his paper – we show that, under the correct assumptions, it indeed gives a function whose image has positive measure.

7.0.3 Embeddings into $\mathbb{R}^d$.

We study the following question. Suppose that we are given a set X, and a $\rho\colon X \times X \to \mathbb{R}^+$. (We do not assume that ρ is a metric, in particular, it does not need to satisfy the triangle inequality.)

Problem 76. *Does there exist a mapping $F\colon X \to \mathbb{R}^d$ s.t. $F(X)$ is bounded, and*

$$\rho(x,y) \le |F(x) - F(y)| \tag{7.1}$$

for every $x, y \in X$?

In Problem 74 without loss of generality we can assume that the set A is bounded. We can also assume that $f(A)$ is an interval, since every set of positive measure in $\mathbb{R}$ can be mapped onto an interval by a contraction. Therefore Problem 74 becomes a special case of Problem 76 by choosing X to be an interval, and

$$\rho(x,y) := r \quad \text{if} \quad |x-y| = \omega(r). \tag{7.2}$$

Another interesting special case to keep in mind is the following. Suppose that X is a Cantor set, and we consider a 'distance function' on the Cantor set given by a sequence $r_n \searrow 0$ as follows: $\rho(x,y) = r_n$ if x and y were separated in the n^{th} step of the standard Cantor set construction. For which sequence $\{r_n\}$ can this Cantor set be embedded into $\mathbb{R}^d$ without decreasing any of the distances, i.e. s.t. (7.1) holds?

Another way of thinking about the Cantor set example is the following. As usual, we identify the 2^n subsets in the Cantor set construction by the 2^n sequences of 0's and 1's of length n, which we denote by $\mathcal{I}_n$. We are looking for mappings s.t. $|F(x) - F(y)|$ is at least r_n if x and y don't belong to the same $I \in \mathcal{I}_n$.

More generally, let $\mathcal{I}$ denote the set of all finite sequences of 0's and 1's. For every space

X, we can:

1. Choose a 'distance function' $\rho(I, J)$ for every $I, J \in \mathcal{I}$.

2. For every $I \in \mathcal{I}$, choose a set $X_I \subset X$.

3. Ask the question whether there is an $F\colon X \to \mathbb{R}^n$ s.t. $F(X)$ is bounded, and

$$\operatorname{dist}\left(F(X_I), F(X_J)\right) \geq \rho(I, J)$$

   for every $I, J \in \mathcal{I}$.

In the Sard problem, when X is an interval, we can assume this interval contains $(0, 1)$, and then choose X_I to be the open dyadic subinterval of $(0, 1)$ encoded by the sequence I.

Also in the general case, it is convenient to keep the notation coming form the dyadic intervals idea, i.e. we denote $J \subset I$ if the sequence J is a continuation of the sequence I, and $I \cap J = \emptyset$ if they are not continuations of each other (for each I, J, either $I \subset J$ or $J \subset I$ or $I \cap J = \emptyset$). Using this notation, we also assume:

4. $X_I \subset X_J$ if $I \subset J$.

5. $X_I \cap X_J = \emptyset$ if $I \cap J = \emptyset$. (We do not assume $\rho(X_I, X_J) > 0$.)

We can now forget about the underlying space X and study ρ by itself. We assume that we are given a 'distance function' $\rho\colon \mathcal{I} \times \mathcal{I} \to \mathbb{R}$ s.t. for every $I, J \in \mathcal{I}$:

- $\rho(I, J) = 0$ if $I \subset J$ or $J \subset I$, and $\rho(I, J) \geq 0$ for every I, J;

- $\rho(I, J) = \rho(J, I)$;

- $\rho(I, K) \leq \rho(J, K)$ if $J \subset I$.

Our main question is the following:

Problem 77. *Can one find for every $I \in \mathcal{I}$ a bounded set $A_I \subset \mathbb{R}^d$ s.t.*

(i) $A_I \subset A_J$ *if* $I \subset J$*;*

(ii) $A_I \cap A_J = \emptyset$ *if* $I \cap J = \emptyset$*;*

(iii) $\operatorname{dist}(A_I, A_J) \geq \rho(I, J)$ *if* $I \cap J = \emptyset$*?*

Remark 78. If for every $x, y \in X$,

$$\rho(x, y) = \sup\{\rho(I, J) \colon x \in X_I, y \in X_J\},$$

then of course by choosing for each $x \in X$ a point

$$F(x) \in \bigcap_{x \in X_I} \operatorname{cl}(\mathrm{A_I}),$$

(7.1) will hold.

We cannot answer this question in its full generality, but we will try to answer it under some 'bounded growth' assumptions, which we will discuss in the next section. These assumptions will be satisfied in the Sard problem (i.e. when ρ is defined by (7.2), with a modulus of continuity whose properties satisfy the assumptions in Theorem 75). And our bounded growth assumptions will also be automatically satisfied in the Cantor set example discussed above, when ρ is given by a decreasing sequence $\{r_n\}$.

Notation: For any set $A \subset \mathbb{R}^d$, we denote by $|A|$ the Lebesgue measure and by $\mathcal{H}^s(A)$ the s-dimensional Hausdorff measure of A, respectively. We denote by $B(A, r)$ the open r-neighborhood of A. We use the standard notation $a \lesssim b$ for the existence of an absolute contant c such that $a \leq cb$, and $a \sim b$ stands for the existence of an absolute contant c such that both $a \leq cb$ and $b \leq ca$ hold.

7.1 Bounded growth conditions

7.1.1 Properties of ω

Recall that we assumed in Theorem 75 that $\omega\colon [0,\infty) \to [0,\infty)$ is a continuous, strictly increasing, convex function such that $\omega(0) = 0$ and $\omega(r)/r^d$ is decreasing. In this section we state some important corollaries that we will use in the coming sections to show that our 'bounded growth assumptions' are automatically satisfied in the Sard example. The first part of Lemma 79 says that ω is doubling. The heart of the matter in our proof in the Sard case will be part (ii) of Lemma 79 which we will refer to as the 'Lipschitz' condition. We will explain its geometric meaning in Section 7.1.3.

Lemma 79. *Let $\omega\colon [0,\infty) \to [0,\infty)$ be a continuous, strictly increasing, convex function such that $\omega(0) = 0$ and $\omega(r)/r^d$ is decreasing. Then:*

(i) $\omega(2r) \lesssim \omega(r)$ for every $r > 0$.

(ii) $\omega(r + \alpha R) - \omega(r) \lesssim \alpha\omega(R)$ for every $R \geq r > 0$ and $\alpha \in [0, 1]$.

Proof. Using the notation $\omega(r) = r^d u(r)$, the function $u(r)$ is decreasing. Therefore

$$\omega(2r) = 2^d r^d u(2r) \leq 2^d r^d u(r) = 2^d \omega(r),$$

so indeed (i) holds. For every $R \geq r > 0$, we have

$$\begin{aligned} \omega(R) - \omega(r) &= R^d u(R) - r^d u(r) = (R^d - r^d)u(R) + r^d(u(R) - u(r)) \\ &\leq (R^d - r^d)u(R) \leq d(R - r)R^{d-1}u(R) = d(R - r)\omega(R)/R. \end{aligned} \tag{7.3}$$

And by convexity, we have

$$\omega(r + \alpha R) - \omega(r) \leq \omega(R + \alpha R) - \omega(R). \tag{7.4}$$

By replacing R by $R+\alpha R$ and replacing r by R in (7.3), we get

$$\omega(R+\alpha R)-\omega(R)\leq d\alpha R\omega(R+\alpha R)/(R+\alpha R)\lesssim \alpha\omega(2R). \tag{7.5}$$

Combining (7.4), (7.5) and (i) gives (ii). $\square$

7.1.2 The sequence $\{r_n\}$ and the sets C_I

Now consider the general case. For each $J\subset I$ denote

$$\rho_{J,I}:=\inf\{\rho(J,K)\colon K\cap I=\emptyset\}$$

and

$$\rho_I:=\sup_{J\subset I}\rho_{J,I}.$$

In the Cantor set example, if $I\in\mathcal{I}_n$ then everything in I has distance (at least) r_n from the complement of I, so in this case $\rho_{J,I}=\rho_I=r_n$ for every $J\subset I$. In the Sard example (identifying the sequence I with the corresponding dyadic interval $I\subset(0,1)$), $\rho_{J,I}=r$ is defined by $\omega(r)=\mathrm{dist}\,(J,I^c)$, and therefore $\rho_I=r$ is given by $\omega(r)=1/2^{n+1}$. In both examples, ρ_I depends only on the length of the sequence I. We will also assume in the general case that $r_n:=\rho_I$ depends only on n, for $I\in\mathcal{I}_n$.

Now we are ready to state our first bounded growth condition: There are some absolute constants s.t. for each $I\in\mathcal{I}_n$ and for each $m\geq n$, and for all except at most constant many $J\in\mathcal{I}_m$ outside I,

$$\rho(I,J)\gtrsim r_m.$$

Heuristically, this captures the fact that our dyadic intervals have a linear structure, they are not, say, dyadic squares. More precisely, their boundary is '0 dimensional': in every 'annulus' of width $\sim r_m$ there are only constant many intervals of size $\sim r_m$. (In the Sard example, this means we are looking at the real valued, not the vector valued case.)

This 0 dimensional boundary condition is obviously true when our intervals are indeed intervals, and ω is doubling (see property (i) in Lemma 79), i.e. our bounded growth assumption holds trivially in the Sard case. And it also obviously holds in the Cantor set example, with no exceptional J's, because of our special distance function ρ we chose in that case.

Why are we happy about this? It has a geometric meaning. Remember that we are looking for embeddings into $\mathbb{R}^d$, satisfying the conditions in Problem 77. If such a mapping exists, denote

$$C_I := \bigcup_{J \subset I} B(A_J, c\rho_{I,J}) \subset \mathbb{R}^d \tag{7.6}$$

with some sufficiently small constant c, given by the bounded growth assumption. By the definition of $\rho_{J,I}$, if $c \leq 1/2$ then these sets are pairwise disjoint. But thanks to the bounded growth assumption, we can say more: since for every interval $I \in \mathcal{I}_n$, for every $m \geq n$, and for all but constantly many $I' \in \mathcal{I}_m$, and for any $J \subset I$, $J' \subset I'$ there holds $\rho(J, J') \gtrsim \max(\rho_{J,I}, \rho_{J',I'}, r_m)$, therefore for these intervals and for c sufficiently small enough $\rho(J, J') \geq c(\rho_{J,I} + \rho_{J',I'} + 2r_m)$. Hence:

Property 1. *For each $I \in \mathcal{I}_n$ and for each $m \geq n$, there are at most constant many $I' \in \mathcal{I}_m$ s.t. $B(C_I, cr_m) \cap B(C_{I'}, cr_m) \neq \emptyset$.*

In our main result we will characterize for which ρ can such sets C_I with small overlaps of their neighborhoods exist in $\mathbb{R}^d$. We will not actually use how the sets C_I were defined, and we will forget what ρ was, we will only assume that the sets C_I have some nice properties implied by ρ. One of the key assumptions we will use is that Property 1 holds.

Another key property we will use (which follows immediately from the definition of our C_I above) is that the sets C_I are not too small:

Property 2. *For each $I \in \mathcal{I}_n$, C_I contains a ball of radius $\sim r_n$.*

7.1.3 The sets $E_{I,j}$ and their annuli $D_{I,j}$

Consider first the Sard problem. Recall our 'Lipschitz' condition (ii) in Lemma 79:

$$\omega(r+\alpha R)-\omega(r)\lesssim \alpha\omega(R) \tag{7.7}$$

for every $r\leq R$ and $\alpha\in[0,1]$.

This again has a very nice geometric corollary, which we now explain. In the Cantor set example we won't have an analogue of the Lipschitz condition (7.7), but the same geometric corollary will hold also in that case. As in the previous section, we will later care only about this corollary and not about how our sets were defined.

In the Sard case it is natural to consider every interval $I\subset[0,1]$, not only dyadic intervals. For arbitrary intervals $J\subset I$ we can define $\rho_{J,I}$ to be the ρ distance of J from the complement of I, and then define C_I by (7.6). In (7.6), I is now not necessarily a dyadic interval, but the subintervals J are dyadic (we haven't defined the sets A_J for non-dyadic intervals).

By exactly the same argument as in the previous section, if $I\cap I'=\emptyset$ then $C_I\cap C_I'=\emptyset$, and moreover, using the same argument we will show that the following generalization of Property 1 hold: if I is arbitrary, $J\in\mathcal{I}_m$, $|I|\geq|J|$, and the ρ-distance between I and I' is at least r_m, then

$$B(C_I,cr_m)\cap B(C_J,cr_m)=\emptyset. \tag{7.8}$$

In order to show this we need to check that for every $J\subset I$ and $J'\subset I'$, the distance of A_J, A_J' is at least $c\rho_{J,I}+c\rho_{J',I'}+2cr_m$, which we indeed have since $\rho(J,J')\geq\max(\rho_{J,I},\rho_{J',I'},r_m)$.

So far nothing happened, everything is exactly the same as in the previous section. But (7.7) allows us to say something more delicate: for every $J\subset I\subset I'$, the Lipschitz condition (7.7) tells us that there is an absolute constant L s.t. $\rho_{J,I'}-\rho_{J,I}\geq\alpha R$ if $\omega(\rho_{J,I'})-\omega(\rho_{J,I})\geq L\alpha\omega(R)$ and $\rho(J,I)\leq R$. In particular, if the Euclidean distance of I from the complement of I' is $\geq L\alpha\omega(R)$ and $\rho(J,I) \leq R$ then $\rho_{J,I'} - \rho_{J,I} \geq \alpha R$, for all $J \subset I$.

For any $d > 0$, let I_d denote the Euclidean d-neighbourhood of I. Then with $I' = I_d$, $d = L\alpha\omega(R)$, the above discussion tells us that $\rho_{J,I} + \alpha R \leq \rho_{J,I'}$, i.e.,

$$B(C_I, \alpha R) \subset C_{I_{L\alpha\omega(R)}} \tag{7.9}$$

for every $\alpha \leq 1$, provided that $\rho_{J,I} \leq R$ for every $J \subset I$.

Now let $I \in \mathcal{I}_n$, $m \geq n$, and for each $j \geq 0$ denote

$$E_{I,j} := C_{I_{j2^{-m}}}, \quad D_{I,j} = B(E_{I,j}, cr_m) \setminus E_{I,j}.$$

(Slightly abusing the notation, we put $I_0 = I$, i.e. $E_{I,0} = C_I$. Also, the sets $E_{I,j}$, $D_{I,j}$ depend not only on I and j but also on m. We will always fix an m before using them therefore we hope that the ambiguity of our notation will not cause any confusion.) For future reference, note that

Property 3. *$C_I \subset E_{I,j}$ for every $I \in \mathcal{I}_n$ and for every j.*

From (7.9), with $R \sim r_n$, $\omega(R) \sim 2^{-n}$, and $\alpha R = cr_m$, i.e. $\alpha = cr_m/R$ we get

$$D_{I,j} \subset B(E_{I,j}, cr_m) = B(C_{I_{j2^{-m}}}, cr_m) \subset C_{I_{j2^{-m}+L\alpha\omega(R)}} \subset C_{I_{j'2^{-m}}}, \tag{7.10}$$

if

$$j' - j \geq L\alpha\omega(R)2^m \sim 2^{m-n} r_m/r_n.$$

In order for this calculation to be valid we need $\alpha \leq 1$ (which we have if c is small enough), but we also need that for every $J \subset I_{j2^{-m}}$ there holds $\rho_{J,I_{j2^{-m}}} \leq R$. So (using the doubling property of ω) it holds for every $0 \leq j < k_{m,n}$ with some

$$k_{n,m} \sim 2^{m-n}.$$

Note that since ω is convex, therefore $\omega(r_m)/r_m \le \omega(r_n)/r_n$, and since $\omega(r_m) \sim 2^{-m}$, $\omega(r_n) \sim 2^{-n}$, therefore

$$k_{n,m} r_m / r_n \sim 2^{m-n} r_m / r_n \gtrsim 1.$$

We will show the following two properties:

Property 4. *There is an absolute constant s.t. for each $J \in \mathcal{I}_m$ there are only constant many pairs (I, j) with $I \in \mathcal{I}_n$, $j < k_{n,m}$ s.t. $D_{I,j} \cap C_J \neq \emptyset$.*

Property 5. *For every $m > n$, $\sum_{I \in \mathcal{I}_n, j < k_{n,m}} \chi_{D_{I,j}(x)} \lesssim k_{n,m} r_m / r_n$ for every x.*

Property 5 follows from the observations that:

(i) For $j' - j \gtrsim 2^{m-n} r_m / r_n$, we have $C_{I_{j2^{-m}}} \cap D_{I,j} = \emptyset$ and $D_{I,j} \subset C_{I_{j'2^{-m}}}$. Therefore $D_{I,j} \cap D_{I,j'} = \emptyset$ and for each I and $x \in \mathbb{R}^d$, $\sum_{j < k_{n,m}} \chi_{D_{I,j}(x)} \lesssim k_{n,m} r_m / r_n$.

(ii) For every I, j, I', j', we have $D_{I,j} \subset C_{I_{2^{-n}}}$ and $D_{I',j'} \subset C_{I'_{2^{-n}}}$, therefore if $\operatorname{dist}(I, I') > 2/2^{-n}$, then $D_{I,j} \cap D_{I',j'} = \emptyset$. And for each fixed I there are only constant many I' with $\operatorname{dist}(I, I') \le 2/2^{-n}$.

In order to prove Property 4, we use (7.8). If $D_{I,j} \cap C_J \neq \emptyset$, then the distance of $E_{I,j} = C_{I_{j2^{-m}}}$ and J is at most 2^{-m}. Clearly, there are only constantly many such I, and for each I there is only one j for which $D_{I,j} \cap C_J \neq \emptyset$.

Now consider the Cantor case. In that example we don't have any non-dyadic intervals, and we don't have any analogue of the Lipschitz condition (7.7). We cannot repeat the same construction, we can instead find sets $E_{I,j}$ and their annuli $D_{I,j}$ satisfying Properties 3-5 using a much more trivial construcion. We will show that in the Cantor case we can choose $k_{n,m}$ to be as large as we like.

Fix $n \le m$ and a $k_{n,m} \gtrsim r_n / r_m$. Recall that in the Cantor case we have $C_I = \bigcup_{J \subset I} B(A_J, c r_n)$ for $I \in \mathcal{I}_n$, with some sufficiently small constant c. We assume $c < 1/6$, and for every $j < k_{n,m}$ define

$$E_{I,j} := B(C_I, cjr_n/k_{n,m}), \quad D_{I,j} = B(E_{I,j}, cr_m) \setminus E_{I,j}.$$

Since $D_{I,j} \subset B(E_{I,j}, cr_m) \subset B(C_I, 2cr_n) \subset B(A_I, 3cr_n) \subset B(A_I, r_n/2)$, therefore $D_{I,j}$ cannot intersect any C_J, so Property 4 holds trivially. Moreover each $D_{I,j}$ intersects $\lesssim k_{n,m}r_m/r_n$ many other annuli, so indeed Property 5 holds.

Remark 80. Heuristically, the fact that we can choose $k_{n,m}$ to be arbitrarily large says that the Cantor set exhibits properties that a 0-dimensional set has. The fact that for the interval we have $k_{n,m} \sim 2^{m-n}$ corresponds to the fact that the interval is 1-dimensional. In fact, our proof will only use that $k_{n,m}$ increases faster than $2^{(m-n)/d}$; heuristically this says that (that part of) our proof works as long as our set is less than d-dimensional.

7.1.4 Boundary

In this section we will describe two more properties, which are very easy to check both in the Sard and in the Cantor case. They will again state a sort of '0-dimensional boundary' type condition.

We will now consider three scales, $m > n > k$. We fix a large enough absolute constant q, and assume that $n \geq k + q$, $m \geq n + q$. We fix k, and also fix an interval $I \in \mathcal{I}_k$. Our claim is that for every $n \geq k + q$ there is an $\mathcal{I}'_n \subset \{I' \in \mathcal{I}_n : I' \subset I\}$, s.t. $\mathcal{I}'_n \neq \emptyset$, and the following hold.

Remark 81. We want to think of this as intervals I' in I not too close to the complement of I. In the Cantor set example, nothing in I is close to the complement of I, and therefore $\mathcal{I}'_n$ will be the set of all intervals $I' \in \mathcal{I}_n$ contained in I. And we can take $q = 1$. In the Sard example, we will need to throw away intervals close to each of the endpoints. The statement will be true with $q = 3$.

The properties we will show: for every $n \geq k+q$ and for every $m \geq n+q$, the sets $D_{J,j}$ in the previous section satisfy the following.

Property 6. *$D_{I',j} \subset C_I$ for every $I' \in \mathcal{I}'_n$, $j < k_{n,m}$.*

Property 7. *If $D_{I',j} \cap C_{I''} \neq \emptyset$ for some $I' \in \mathcal{I}'_n$, $j < k_{n,m}$ and $I'' \in \mathcal{I}_m$, then $I'' \in \mathcal{I}'_m$.*

In the Cantor case this is obvious. In the Sard case, we divide I into 2^{n-k} subintervals and throw away the first two and the last two. With $q \geq 3$ some intervals will be left, i.e. $\mathcal{I}'_n \neq \emptyset$.

We have already seen in the last section that $D_{I',j} \subset C_{I_{2^{-n}}}$, therefore $D_{I',j} \subset C_I$ for all except possibly the first and the last I'. And we have also seen that if $D_{I',j} \cap C_{I''} \neq \emptyset$, then $\operatorname{dist}(I', I'') \leq 2^{-n}$. So I'' is not contained in the first or last I', and then of course it cannot be the one of the first or last two I''.

7.2 Main result

7.2.1 Main geometric result in $\mathbb{R}^d$

For the convenience of the reader, first we re-list Properties 1-7, with cr_n replaced by r_n everywhere. The absolute constant in Property 4 we denote by C. Using this new notation, the bounded growth conditions are telling us that:

- For every n and $I \in \mathcal{I}_n$ there is a set $C_I \subset \mathbb{R}^d$ s.t.

  (P1) For each $m \geq n$, there are at most constant many $I' \in \mathcal{I}_m$ s.t. $B(C_I, r_m) \cap B(C_{I'}, r_m) \neq \emptyset$.

  (P2) C_I contains a ball of radius $\sim r_n$.

- For every $m \geq n$ and $j \geq 0$ there are sets $E_{I,j}$ and their annuli $D_{I,j} = B(E_{I,j}, r_m) \setminus E_{I,j}$, and $k_{n,m} \gtrsim r_n/r_m$, satisfying the following:

(P3) $C_I \subset E_{I,j}$ for every $I \in \mathcal{I}_n$ and for every j.

(P4) For each $m \geq n$ and $J \in \mathcal{I}_m$ there are at most C many pairs (I, j) with $I \in \mathcal{I}_n$, $j < k_{n,m}$ s.t. $D_{I,j} \cap C_J \neq \emptyset$.

(P5) For every $m > n$, $\sum_{I \in \mathcal{I}_n, j < k_{n,m}} \chi_{D_{I,j}(x)} \lesssim k_{n,m} r_m / r_n$ for every x.

- If $q \in \mathbb{N}$ is large enough, then for every $n \geq k + q, m \geq n + q$ and $I \in \mathcal{I}_k$ there is an $\mathcal{I}'_n \subset \{I' \in \mathcal{I}_n : I' \subset I\}$, s.t. $\mathcal{I}'_n \neq \emptyset$, and the following hold:

(P6) $D_{I',j} \subset C_I$ for every $I' \in \mathcal{I}'_n$, $j < k_{n,m}$.

(P7) If $D_{I',j} \cap C_{I''} \neq \emptyset$ for some $I' \in \mathcal{I}'_n$, $j < k_{n,m}$ and $I'' \in \mathcal{I}_m$, then $I'' \in \mathcal{I}'_m$.

Proposition 82 below is a characterization for the existence of these sets. Note that it is a purely geometric statement about sets in $\mathbb{R}^d$; it has nothing to do with any modulus of continuity ω or distance function ρ. We do not assume in Proposition 82 that the sets $C_I, E_{I,j}$ were defined via any embedding of any space into $\mathbb{R}^d$.

For our characterization we need $k_{n,m} \gg 2^{(m-n)/d}$, more precisely, the assumptions we use are:

$$\sup_n \sum_{m \geq n} 2^{(m-n)/d} k_{n,m}^{-1} < \infty \tag{7.11}$$

and

$$2^{(m-n)/d} k_{n,m}^{-1} < (4C)^{-1} \text{ if } m - n \text{ is large enough.} \tag{7.12}$$

Proposition 82. *Let $C, q \in \mathbb{N}$, let $\{r_n\}$ be a decreasing sequence converging to 0, and suppose that for every $m \geq n$ we are given a $k_{n,m} \gtrsim r_n / r_m$ satisfying (7.11) and (7.12). Then the following are equivalent:*

(I) There exist some sets for which (P1)-(P7) hold.

(II) $\sum_n 2^{n/d} r_n < \infty$.

Remark 83. Note that (II) holds for a sequence $\{r_n\}$ if and only if it holds for $\{cr_n\}$ with a constant c. Also all other properties listed in this section remain valid when we replace each set $S \subset \mathbb{R}^d$ by cS and we replace r_n by cr_n. There wasn't any mathematical reason for replacing cr_n by r_n at the beginning of this section – we did this only to simplify our formulas and notation (in this and in all subsequent sections).

Remark 84. We do not actually need the implication (II) $\Longrightarrow$ (I) in Proposition 82 for Theorem 85 and Theorem 86. We will get this for free, without any additional work, in Section 7.4. Since it is an interesting statement, we included this for the sake of completeness.

7.2.2 Our answer to Problem 77

We verified in Section 7.1 that in the Sard and in the Cantor case part (I) of Proposition 82 holds with some $C, q \in \mathbb{N}$. In the Sard case we had $\omega(r_n) = 1/2^{n+1}$ (using our old definition of r_n), and $k_{n,m} \sim 2^{n-m}$. In the Cantor case $\{r_n\}$ was an arbitrary sequence decreasing to 0, and we could choose $k_{n,m}$ to be arbitrary large.

It is an easy exercise to check that with the choice $\omega(r_n) = 1/2^{n+1}$, (II) of Proposition 82 is equivalent to $\int_0^1 \omega(r)^{-1/d}\, dr < \infty$. Since (I) implies (II) in Proposition 82, therefore indeed (ii) implies (i) in Theorem 75. Similarly, in the Cantor case, the implication (I) $\Longrightarrow$ (II) in Proposition 82 shows that if we have a positive answer to Problem 77 then (II) holds.

In the general case, we say that a distance function ρ has *bounded growth*, if the following is true: if the answer to Problem 77 is positive, i.e. the sets A_I exist, then (I) of Proposition 82 holds (with the sequence $\{r_n\}$ defined by ρ in Section 7.1.2 and with some appropriately chosen $C, q, k_{n,m}$ satisfying the assumptions in Proposition 82). We know that in the Sard and in the Cantor example we always have bounded growth.

Now let ρ be an arbitrary distance function and let $\{r_n\}$ be the sequence defined in Section 7.1.2. Then it follows from (I) $\Longrightarrow$ (II) in Proposition 82 that:

Theorem 85. *Suppose that the distance function ρ has bounded growth. Then a positive answer to Problem 77 implies $\sum_n 2^{n/d} r_n < \infty$.*

For the reverse implication, we need to introduce one more notation. Let r_n^* denote the supremum of $\rho(J_1, J_2)$, where the supremum is taken over all sequences $I_1, J_2 \in \mathcal{I}$ whose length is at least n and whose first $n-1$ terms agree and the n^{th} term differs. That is, n is the first index for which they are contained in two different intervals $I_1, I_2 \in \mathcal{I}_n$. (In the Sard case $\omega(r_n^*) = 1/2^{n-1}$ and in the Cantor case $r_n^* = r_n$. So in both cases $r_n^* \sim r_n$.)

For the following theorem we don't need to assume that ρ has bounded growth. Our construction in Section 7.4 will show that:

Theorem 86. *If $\sum_n 2^{n/d} r_n^* < \infty$, then there is a positive answer to Problem 4.*

A corollary of this is that:

Corollary 87. *If ρ has bounded growth, and $r_n \sim r_n^*$, then the answer to Problem 4 is positive if and only if $\sum_n 2^{n/d} r_n < \infty$.*

In particular, this implies Theorem 75.

In Section 7.3 we prove the implication (I) $\Longrightarrow$ (II) in Proposition 82, and hence also Theorem 85. In Section 7.4 we will prove Theorem 86 and also the reverse implication in Proposition 82.

7.3 Proof of (I) $\Longrightarrow$ (II) in Proposition 82

7.3.1 Beginning of the proof

Suppose that (I) of Proposition 82 holds and (II) fails. Our aim is to find a contradiction.

Let C be the absolute constant in (P4). Choose $q \in \mathbb{N}$ large enough so that (P6) and (P7) hold for q and moreover (7.12) holds if $m - n \geq q$.

If we split the divergent sum $\sum 2^{n/d} r_n$ into q sums, at least one of them will be divergent. Without loss of generality we assume

$$\sum_n 2^{qn/d} r_{qn} = \infty. \tag{7.13}$$

Let $p_n^{d/(d-1)}$ denote the minimal measure among the $|C_I|$, where $I \in \mathcal{I}_n$.

Remark 88. The motivation for our strange notation is that later in some sense p_n will play the role of perimeter.

To simplify our notation, denote

$$Q_n := 2^{qn/d},\ R_n := r_{qn},\ P_n := p_{qn},\ K_{n,m} := k_{qn,qm},\ S_{n,m} = \sum_{k=n+1}^{m} Q_k R_k.$$

By (P2), we have $R_n \lesssim P_n^{1/(d-1)}$ for every n. We will contradict (7.13), i.e. contradict $\sum Q_n R_n = \infty$ by showing that:

Lemma 89. *For every n there is $m > n$ s.t.*

$$S_{n,m} \lesssim Q_n P_n^{1/(d-1)} - Q_m P_m^{1/(d-1)}. \tag{7.14}$$

Finding a contradiction using Lemma 89. We take $n_1 = 1$ and using Lemma 89 inductively choose a sequence $\{n_k\}_{k\in\mathbb{N}}$ such that

$$S_{n_k,n_{k+1}} \lesssim Q_{n_k} P_{n_{k+1}}^{1/(d-1)} - Q_{n_k} P_{n_{k+1}}^{1/(d-1)}.$$

Summing over all k, we have a telescopic sum on the right hand side. We have

$$\sum_{i=1}^{n_{k+1}} Q_i R_i \lesssim Q_1 P_1^{1/(d-1)}.$$

Since the left hand side goes to infinity, we get a contradiction. □

Our aim is to prove Lemma 89. Fix n, and fix an $I \in \mathcal{I}_{qn}$ with $|C_I| = P_n^{d/(d-1)}$. Note that for $m = n+1$, the left hand side of (7.14) is $Q_{n+1}R_{n+1} \le Q_{n+1}R_n \lesssim Q_n P_n^{1/(d-1)}$. Since also $Q_{n+1} \le Q_n$, therefore if $P_{n+1} \ll P_n$ then (7.14) holds with $m = n+1$. From now on, we assume that

$$P_{n+1} \gtrsim P_n \quad \text{(for our fixed } n\text{)}. \tag{7.15}$$

Under this assumption, we will show a stronger statement than Lemma 89. Since $x^d - y^d \lesssim x^{d-1}(x-y)$ for every $x \ge y \ge 0$, we have

$$\begin{aligned} |C_I \setminus \bigcup_{J \in \mathcal{I}_{qm}} C_J| &\le P_n^{d/(d-1)} - 2^{q(m-n)} P_m^{d/(d-1)} \\ &\lesssim P_n(P_n^{1/(d-1)} - 2^{q(m-n)/d} P_m^{1/(d-1)}) \\ &= P_n Q_n^{-1}(Q_n P_n^{1/(d-1)} - Q_m P_m^{1/(d-1)}). \end{aligned}$$

We will show that there is an $m > n$ s.t.

$$P_n Q_n^{-1} S_{n,m} \lesssim |C_I \setminus \bigcup_{J \in \mathcal{I}_{qm}} C_J|. \tag{7.16}$$

Remark 90. We will decompose the set in the right hand side of (7.16), using the annuli described in the bounded growth sections. And then we will estimate its measure using a generalization of the isoperimetric inequality, which we describe in the next section.

7.3.2 Perimeters

By the isoperimetric inequality,

$$|B(S,r) \setminus S| \gtrsim r|S|^{(d-1)/d}$$

for every set $S \subset \mathbb{R}^d$ and every $r > 0$. A trivial corollary of this is that

$$|B(T,r) \setminus T| \gtrsim r|T|^{(d-1)/d} \geq r|S|^{(d-1)/d} \quad \text{for every} \quad T \supset S.$$

As a refinement of this, we introduce in this section a sort of 'minimal perimeter' of a bounded set S.

Let S be an arbitrary bounded set, and let u be the distance function from S. Then u is Lipschitz with Lipschitz constant 1, hence it is differentiable almost everywhere, with $|\nabla u| \leq 1$. Moreover, it is easy to check that at every $x \notin \mathrm{cl}(S)$, the directional derivative of u is 1 along the line segment joining x to its closest point $y \in \mathrm{cl}(S)$. Therefore $|\nabla u| = 1$ at a.e. $x \notin \mathrm{cl}(S)$, thus by the coarea formula we have

$$|B(S,r) \setminus S| \geq \int_{B(S,r)\setminus \mathrm{cl}(S)} |\nabla u| = \int_0^r \mathcal{H}^{d-1}(u^{-1}(t))\, dt = \int_0^r \mathcal{H}^{d-1}(\partial B(S,t))\, dt.$$

Motivated by this, denote

$$P_0(S) := \text{ess inf}_{t>0} \mathcal{H}^{d-1}(\partial B(S,t))$$

and

$$P(S) := \inf_{T \supset S} P_0(T).$$

Then, clearly,

$$P_0(S) = \inf_{r_1,r_2>0} |B(S,r_1+r_2) \setminus B(S,r_1)|/r_2, \tag{7.17}$$

where, by the isoperimetric inequality,

$$|B(S,r_1+r_2) \setminus B(S,r_1)|/r_2 \gtrsim |B(S,r_1)|^{(d-1)/d} \geq |S|^{(d-1)/d}.$$

Therefore $P_0(S) \gtrsim |S|^{(d-1)/d}$, and moreover, for every $T \supset S$, $P_0(T) \gtrsim |T|^{(d-1)/d} \geq$

$|S|^{(d-1)/d}$. So

$$P(S) \gtrsim |S|^{(d-1)/d}. \tag{7.18}$$

By (7.17), for every $\varepsilon > 0$ there exist $T \supset S$ and $r_1, r_2 > 0$ such that

$$|B(T, r_1 + r_2) \setminus B(T, r_1)|/r_2 \leq P_0(T) + \varepsilon/2 \leq P(S) + \varepsilon.$$

Taking $T' = B(T, r_1)$, we have that $P(S) \leq |B(T', r_2) \setminus T'|/r_2 \leq P(S) + \varepsilon$. Since ε is arbitrary, we have

$$P(S) = \inf_{T \supset \mathrm{cl}(S), r > 0} |B(T, r) \setminus T|/r, \tag{7.19}$$

and for future reference let us note also its trivial corollary

$$|B(S, r) \setminus S| \geq rP(S). \tag{7.20}$$

Further properties of $P(S)$ that we will be using are the following.

Lemma 91. *For every bounded sets $S_1, S_2 \subset \mathbb{R}^d$ and $r > 0$:*

(i) $P(S_1 \cup S_2) \leq P(S_1) + P(S_2)$*;*

(ii) $|B(S_1, r) \setminus (S_1 \cup S_2)| \geq rP(S_1) - rP(S_2)$*.*

Using induction, an immediate corollary is: for every bounded sets $S_j \subset \mathbb{R}^d$ and $r > 0$:

$$|B(S_0, r) \setminus \bigcup_{j=0}^{n} S_n| \geq rP(S_0) - r\sum_{j=1}^{n} P(S_j). \tag{7.21}$$

Remark 92. The key properties we will need for the proof of our main theorem are (7.18), (7.20) and (7.21). The reason for considering $P(S)$ instead of the simpler notions $|S|^{(d-1)/d}$ or $P_0(S)$ is that Lemma 91, and hence also (7.21) do not hold with P replaced by $|S|^{(d-1)/d}$ or P_0.

Proof of Lemma 91. Property (i) follows easily from the observation that if $T_1 \supset S_1$, $T_2 \supset S_2$, and (say) $r_1 \geq r_2$, then with $T := B(T_1, r_1 - r_2) \cup T_2$,

$$\mathcal{H}^{d-1}(\partial B(T, r_2)) \leq \mathcal{H}^{d-1}(\partial B(T_1, r_1)) + \mathcal{H}^{d-1}(\partial B(T_2, r_2)). \tag{7.22}$$

Indeed, for each $\varepsilon > 0$ there are positively many r_1 and positively many r_2 s.t.

$$\mathcal{H}^{d-1}(\partial B(T_1, r_1)) \leq P_0(T_1) + \varepsilon, \quad \mathcal{H}^{d-1}(\partial B(T_2, r_2)) \leq P_0(T_2) + \varepsilon, \tag{7.23}$$

and without loss of generality we can assume that there are positively many r_1, r_2 satisfying (7.23) for which $r_1 \geq r_2$. Then we can fix an $r > 0$ for which there are positively many r_1, r_2 satisfying (7.23) with $r = r_1 - r_2$. With $T = B(T_1, r) \cup T_2$, from (7.22) and (7.23) we get $P(S_1 \cup S_2) \leq P_0(T) \leq P_0(T_1) + P_0(T_2) + 2\varepsilon$. Since this holds for every $T_1 \supset S_1$, $T_2 \supset S_2$ and $\varepsilon > 0$, therefore (i) follows.

In order to prove (ii), again fix an $\varepsilon > 0$. Then by (7.19) and by the co-area formula there is a $T' \supset S_2$ and $r > 0$ s.t.

$$\frac{1}{r} \int_0^r \mathcal{H}^{d-1}(\partial B(T', t))\, dt = \frac{1}{r} |B(T', r) \setminus \mathrm{cl}(T')| < P(S_2) + \varepsilon/2.$$

Let $t' \in (0, r)$ to be a Lebesgue density point of $t \to \mathcal{H}^{d-1}(\partial B(T', t))$ for which $\mathcal{H}^{d-1}(\partial B(T', t')) < \varepsilon/2 + \operatorname{ess\,inf}_{t \in (0,r)} \mathcal{H}^{d-1}(\partial B(T', t))$. Then with $T_2 := B(T', t')$ and an r_ε small enough we have

$$\frac{1}{r} \int_0^r \mathcal{H}^{d-1}(\partial B(T_2, t))\, dt < P(S_2) + \varepsilon \text{ for all } r < r_\varepsilon.$$

Then for any $T \supset S_1$ and $r < r_\varepsilon$, we have

$$\begin{aligned} P(T) \le P(T \cup T_2) &\le \frac{1}{r} \int_0^r \mathcal{H}^{d-1}(\partial B(T \cup T_2, t))\, dt \\ &\le \frac{1}{r} \left(\int_0^r \mathcal{H}^{d-1}(\partial B(T, t) \setminus T_2)\, dt + \int_0^r \mathcal{H}^{d-1}(\partial B(T_2, t))\, dt \right) \\ &\le \frac{1}{r} \int_0^r \mathcal{H}^{d-1}(\partial B(T, t) \setminus T_2)\, dt + P(S_2) + \varepsilon. \end{aligned}$$

Since $T_2 \supset \mathrm{cl}(S_2)$ we have

$$\begin{aligned} P(S_1) - P(S_2) - \varepsilon \le P(T) - P(S_2) - \varepsilon &\le \frac{1}{r} \int_0^r \mathcal{H}^{d-1}(\partial B(T, t) \setminus T_2)\, dt \\ &\le \frac{1}{r} \int_0^r \mathcal{H}^{d-1}(\partial B(T, t) \setminus \mathrm{cl}(S_2))\, dt \end{aligned}$$

for all $r < r_\varepsilon$. This holds for every $T \supset S_1$, in particular it holds for $T = B(S_1, t_0)$ with every $t_0 > 0$. Taking infimum for all $t_0 > 0$ and all $r < r_\varepsilon$ we can see that

$$P(S_1) - P(S_2) - \varepsilon \le \operatorname{ess\,inf}_{t>0} \mathcal{H}^{d-1}(\partial B(S_1, t) \setminus \mathrm{cl}(S_2)).$$

Applying the co-area formula for the distance function from S_1, on the domain $B(S_1, r) \setminus (\mathrm{cl}(S_1) \cup \mathrm{cl}(S_2))$, we obtain

$$\begin{aligned} r(P(S_1) - P(S_2) - \varepsilon) &\le \int_0^r \mathcal{H}^{d-1}(\partial B(S_1, t) \setminus \mathrm{cl}(S_2))\, dt \\ &= |B(S_1, r) \setminus (\mathrm{cl}(S_1) \cup \mathrm{cl}(S_2))| \le |B(S_1, r) \setminus (S_1 \cup S_2)|. \end{aligned}$$

Since this holds for every $\varepsilon > 0$, the proof is finished. $\square$

7.3.3 *Estimating the right hand side of* (7.16)

Denote

$$T_m := R_m \sum_{J \in \mathcal{I}'_{qm}} P(C_J).$$

By (7.20) this is a lower estimate for $\sum |B(C_J, R_m) \setminus C_J|$, and by (P1) the sets $B(C_J, R_m)$ have bounded overlap.

Therefore $T_m \lesssim 1$ for every m. On the other hand, for $n+1$ we have from (7.15) the lower estimate

$$T_{n+1} \gtrsim R_{n+1} \sum_{J \in \mathcal{I}'_{q(n+1)}} |C_J|^{(d-1)/d} \geq R_{n+1} P_{n+1} \gtrsim R_{n+1} P_n \gtrsim P_n Q_n^{-1} S_{n,n+1}.$$

So (7.16) would hold, with $m = n+1$, if we could replace its right hand side by T_{n+1}.

Remark 93. In the Cantor example we can indeed replace it by T_{n+1}. Which makes everything much easier. However, in the Sard, and in the general case, we cannot, we will prove only a (somewhat) weaker lower bound, with an error term. See (7.24) below. In the next section our goal will be to make the error term small enough (compared to the main term).

Note that in all the above formulas we sum only for $\mathcal{I}'_{qm}$ and not for $\mathcal{I}_{qm}$. We apply in this section the results of Section 7.1.4 for the interval I we fixed.

By (P5) and (P6), for each $M > N > n$ we have

$$\frac{r_M}{r_N} \cdot K_{N,M} |C_I \setminus \bigcup_{J \in \mathcal{I}_M} C_J| \gtrsim \sum_{\substack{I_0 \in \mathcal{I}'_N \\ 0 \leq j < K_{N,M}}} |D_{I_0,j} \setminus \bigcup_{J \in \mathcal{I}_M} C_J|,$$

and by (7.21)

$$|D_{I_0,j} \setminus \bigcup_{J \in \mathcal{I}_M} C_J| \geq r_M P(E_{I_0,j}) - r_M \sum_{\substack{J \in \mathcal{I}_M \\ C_J \cap D_{I_0,j} \neq \emptyset}} P(C_J).$$

By (P3), we have $P(E_{I_0,j}) \geq P(C_{I_0})$, and by (P4) and (P7),

$$\sum_{\substack{I_0 \in \mathcal{I}_N \\ 0 \leq j < K_{N,M}}} \sum_{\substack{J \in \mathcal{I}_M \\ C_J \cap D_{I_0,j} \neq \emptyset}} P(C_J) \leq C \sum_{\substack{J \in \mathcal{I}_M \\ \exists (I_0,j)\colon C_J \cap D_{I_0,j} \neq \emptyset}} P(C_J) \leq C \sum_{J \in \mathcal{I}'_M} P(C_J).$$

where C is the absolute constant in (P4).

Putting these estimates together we get

$$\begin{aligned} |C_I \setminus \bigcup_{J \in \mathcal{I}_M} C_J| &\gtrsim r_N(\sum_{J \in \mathcal{I}'_N} P(C_J) - C K_{N,M}^{-1} \sum_{J \in \mathcal{I}'_M} P(C_J)) \\ &= T_N - C(K_{N,M} r_M / r_N)^{-1} T_M. \end{aligned}$$

Recall that our aim is to show that (7.16) holds for some (carefully chosen) m. We will prove this by showing that there is $n < N < M$ s.t.

$$P_n Q_n^{-1} S_{n,M} \lesssim T_N - C(K_{N,M} r_M / r_N)^{-1} T_M. \tag{7.24}$$

We have already seen that

$$P_n Q_n^{-1} S_{n,n+1} \lesssim T_{n+1}. \tag{7.25}$$

7.3.4 Finding N and M

To further simplify notation, for every $m \geq n$ denote

$$a_m := P_n Q_n^{-1} Q_m R_m.$$

Using this notation, $a_{m+1} \lesssim a_m$ for every m, $\sum_m a_m = \infty$, and our aim is to show that

$$\sum_{k=n+1}^{M} a_k \lesssim T_N - C K_{N,M}^{-1} 2^{q(M-N)/d} T_M a_N / a_M \tag{7.26}$$

for some $n < N < M$. For simplicity, also denote

$$L_{N,M} := 2CK_{N,M}^{-1}2^{q(M-N)/d}$$

for every $n < N < M$. We know from (7.11) and (7.12) that $\sum_M L_{N,M} \lesssim 1$ for every N, and $L_{N,M} \leq 1/2$ for every N, M.

We have $\limsup_{M\to\infty} L_{N,M}^{-1} a_M = \infty$ (otherwise $\infty = \sum_M a_M \lesssim \sum_M L_{N,M} < \infty$, which is a contradiction). Therefore, inductively, we can choose a subsequence, starting from $n_0 = n+1$, and then by choosing, after each a_{n_k}, the first $a_{n_{k+1}}$ for which

$$a_{n_{k+1}} > L_{n_k,n_{k+1}} a_{n_k}.$$

Since $a_l \leq L_{n_k,l} a_{n_k} < a_{n_k}$ for $n_k < l < n_{k+1}$, and $a_{n_{k+1}} \lesssim a_{n_{k+1}-1}$, therefore

$$a_{n_{k+1}} \lesssim a_{n_k}. \tag{7.27}$$

Also

$$\sum_{l=n_k}^{n_{k+1}-1} a_l = a_{n_k} + \sum_{l=n_k+1}^{n_{k+1}-1} a_l \leq a_{n_k} + (\sum_{l=n_k+1}^{n_{k+1}-1} L_{n_k,l}) a_{n_k} \sim a_{n_k},$$

so

$$\sum_{l=n_k}^{n_{k+1}-1} a_l \sim a_{n_k}. \tag{7.28}$$

Inequalities (7.27) and (7.28), together with

$$\begin{aligned} a_{n_m}/a_{n_{k-1}} &= (a_{n_m}/a_{n_{m+1}}) \cdot (a_{n_{m+1}}/a_{n_{m+2}}) \cdots (a_{n_{k-2}}/a_{n_{k-1}}) \\ &\lesssim (L_{n_m,n_{m+1}} L_{n_{m+1}1,n_{m+2}} \ldots L_{n_{k-2},n_{k-1}})^{-1} \end{aligned}$$

give

$$\sum_{l=n_0}^{n_k} a_l \lesssim \sum_{m=0}^{k} a_{n_m} \lesssim \sum_{m=0}^{k-1} a_{n_m} = a_{n_{k-1}} \sum_{m=0}^{k-1} a_{n_m}/a_{n_{k-1}}$$
$$\lesssim a_{n_{k-1}} \sum_{m=0}^{k-1} (L_{n_m,n_{m+1}} L_{n_{m+1},n_{m+2}} \dots L_{n_{k-2},n_{k-1}})^{-1}.$$

By our assumption $L_{N,M} \leq 1/2$, in the last sum each term is at most half of the previous term. Therefore we can estimate the sum by its first term, and we get

$$\sum_{l=n_0}^{n_k} a_l \lesssim a_{n_{k-1}} (L_{n_0,n_1} L_{n_1,n_2} \dots L_{n_{k-2},n_{k-1}})^{-1} := s_{n_{k-1}}.$$

Since the left hand side of this converges to infinity, therefore $s_{n_k} \to \infty$ as $k \to \infty$.

We defined $s_{n_{m-1}}$ as an upper bound of the left hand side of (7.26) (when we replace M by n_m in (7.26)). We will prove (7.26) by showing that there is an $m \geq 1$ s.t. with $N = n_{m-1}$, $M = n_m$,

$$s_N \lesssim T_N - 2^{-1} L_{N,M} T_M a_N / a_M. \tag{7.29}$$

Using (7.25) we have $s_{n_0} = a_{n_0} \lesssim T_{n_0}$. The fact that s_{n_k} goes to infinity and T_{n_k} is bounded allows us to choose m to be the first index for which this inequality fails. That is, with some absolute constant c_0 and some $m > 0$, $N = n_{m-1}$, $M = n_m$ we have $s_N \leq c_0 T_N$ and $s_M > c_0 T_M$. In particular, $T_M/T_N < s_M/s_N = L_{N,M}^{-1} a_M/a_N$, i.e. $L_{N,M} T_M a_N / a_M < T_N$. Using this, the right hand side of (7.29) is

$$T_N - 2^{-1} L_{N,M} T_M a_N / a_M > T_N/2.$$

This together with $s_N \leq c_0 T_N$ finishes the proof of (7.29) and hence finishes the proof of Lemma 89.

7.4 Construction

Let $\{v_n\}$ be an arbitrary sequence decreasing to 0, and assume that $\sum_n 2^{n/d} v_n < \infty$. We split it into d subsequences $\{v_{dn+m}\}_{n=1}^{\infty}$, $m = 0, 1, \ldots, d-1$. For each m, take a Cantor set lying on the m^{th} coordinate axes of $\mathbb{R}^d$, whose gaps have length v_{dn+m} in the n^{th} stage of the standard Cantor set construction (since it has 2^{n-1} gaps of length v_{dn+m}, and $\sum_n 2^{n-1} v_{dn+m} < \infty$, therefore such a Cantor set exists).

Now consider a finite sequence of $0's$ and $1's$, say, $I = x_1 x_2 \ldots x_N$. We can split this into d subsequences, by putting into the m^{th} subsequence those x_j for which $j \equiv m \pmod{d}$. For the m^{th} subsequence, let I_m denote the corresponding construction interval of the Cantor set on the m^{th} coordinate axis. And let $R_I = R_{x_1 x_2 \ldots x_N}$ denote the rectangle $R_I = I_0 \times I_1 \times \cdots \times I_{m-1}$. It is immediate to see that if we have another sequence $I' = x'_1 x'_2 \ldots x'_N$, and $x_k \neq x'_k$ for some $1 \leq k \leq N$, then the distance of R_I and $R_{I'}$ is at least v_k: indeed, if $k \equiv m \pmod{d}$, then the sets $R_I, R_{I'}$ project onto two different intervals in the m^{th} direction and these two intervals are separated by a gap of length at least v_k.

Proof of Theorem 86. Put $v_n := r_n^*$ and $A_I := R_I$ in the above construction. □

Proof of (II) $\Longrightarrow$ (I) in Proposition 82. Let $\{r_n\}$ be the sequence in (II) of Proposition 82. Consider the Cantor case, with this sequence $\{r_n\}$. By Theorem 86 in this case we have a positive answer to Problem 77. And since in the Cantor case we have bounded growth, therefore we know that there are sets satisfying (P1)-(P7) with some $C, q, k_{n,m}$. Moreover, in the Cantor case we can choose C to be anything we like (since in (P4), instead of 'constant many pairs (I, j)', there are none), we can also choose q to be anything we like (since we have seen that (P6)-(P7) hold with $q = 1$), and we have also seen that we can choose $k_{n,m}$ to be arbitrary. This shows that indeed for any given $C, q, k_{n,m}$ there are sets satisfying (P1)-(P7), i.e. (I) of Proposition 82 holds. □

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