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Abstract

This dissertation consists of two chapters, each representing an individual research paper. In the first chapter, I analyze how a firm's labor market power shapes, and is shaped by, its workforce, and I evaluate the implications for welfare and inequality. Using matched worker-firm panel data from Norway (1995-2018), I develop, identify, and estimate an equilibrium model of the labor market where firms compete with one another for workers who are heterogeneous in both their skills and preferences over wages versus non-wage job amenities. I allow the wage-amenity trade-offs to be correlated with skills, while also varying among equally skilled workers. When a firm adjusts its wages, the composition of its workforce shifts, and these compositional changes, in turn, affect the labor supply curve to the firm. As a result, the firm's wage-setting power varies based on which types of workers it employs. I find that this variation leads to large allocative inefficiency, with welfare losses from imperfect competition estimated at 9.5% relative to the competitive benchmark.

In the second chapter, I explore how identity influences group behavior through social interactions. I study a discrete choice model where people wish to conform to the actions of some members of their network, while deviating from the actions of others. Under this generalized framework, I explore what aggregate outcomes arise from noncooperative decisionmaking. I characterize the uniqueness and stability of equilibria, and I discuss implications of negative spillovers for welfare and inefficiency. Additionally, I demonstrate how the model may be taken to data. I introduce a novel identification strategy that accounts for unobserved network effects by leveraging within-network variation in individual characteristics. I also construct internal instruments to overcome the issue of measurement error, which is a primary source of endogeneity in network-based models with incomplete information. Finally, I apply this methodology using data from the large-scale education experiment Project STAR, where I find robust evidence of gender differences in peer effects.

Chapter 1

Job Preferences, Labor Market Power, and Inequality

I. Introduction

The role of firms in shaping wage inequality is a central topic for labor policy. Three growing empirical literatures examine how the characteristics of firms influence workers' earnings. One body of research (Manning, 2021; Card, 2022; Azar & Marinescu, 2024) studies the role of imperfect competition in the labor market, where firms exert wage-setting power due to facing upward-sloping labor supply curves. A second literature—building on work by Abowd, Kramarz, & Margolis (1999)—focuses on sorting, using matched employer-employee data to uncover the relationship between worker and firm productivity. A third literature examines workers' willingness to trade higher wages for better non-wage aspects of a job, finding that the trade-offs vary substantially among workers (Mas & Pallais, 2017; Wiswall & Zafar, 2018).

While often studied separately, the issues explored by these literatures are interconnected. In a labor market with differentiated employers, a firm's wages and non-wage amenities determine which workers it attracts. Furthermore, if workers make different trade-offs between wages and amenities, then the composition of a firm's workforce could matter for the elasticity of labor supply that it faces, which in turn impacts the firm's wage-setting power. These factors

prompt a key question: *How is a firm's wage-setting power linked to its workforce?* Addressing this question would offer deeper insight into how workers sort and how market power varies within the labor market, which has important implications for welfare and inequality.

This paper develops an empirical framework to examine how workers choose jobs based on heterogeneous wage-amenity trade-offs, and the consequences of this behavior for a firm's wage-setting power. I show that the market power at a firm—as measured by wage markdowns and rents—directly depends on the composition of its workforce, which is influenced by the wages and amenities the firm provides. Moreover, by failing to account for this relationship, one would overlook a key source of allocative inefficiency that arises from differences in wage markdowns among heterogeneous firms. I show how to recover economic quantities under my framework using matched worker-firm panel data, while allowing for unobserved heterogeneity in worker skills, firm productivity, technology, and amenities. I then apply my method to administrative data from Norway, where I draw inference about the determinants of worker sorting, the concentration of market power, and the welfare impacts of imperfect competition.

I begin my analysis by presenting an equilibrium model of the labor market. Building on previous models of job differentiation, such as Card et al. (2018), Lamadon et al. (2022), and Azar et al. (2022), I assume that workers' idiosyncratic tastes are not fully priced into wages, leading firms to face imperfectly elastic labor supply curves. However, unlike previous work, I allow workers to have varying marginal rates of substitution between wages and amenities. Drawing on the literature about the willingness to pay for amenities, I allow the wage-amenity trade-offs to be correlated with skills, while also differing among equally skilled workers.

This extension has key implications for the shape of the labor supply curve faced by a firm. It establishes a link between worker and firm characteristics, such that the elasticity of labor to the firm depends on the composition of its workforce. Consequently, the elasticities

differ among firms, and they are also endogenous to changes in the environment: if a firm adjusts its wages, then the composition of its workforce would shift, and these compositional changes would, in turn, influence the elasticities. This variation contrasts with the standard assumption of homogenous and isoelastic labor supply curves, which leads to uniform wage markdowns. Moreover, to the extent that workers’ wage-amenity trade-offs are correlated with their skills, wage markdowns may also differ within a firm—among workers with varying skills—as well as across firms. These implications do not rely on firms competing strategically for workers, as in Berger et al. (2022). Even when firms are strategically small within the labor market, their wage-setting power would still vary based on *who* is employed at each firm.

To account for many potential sources of worker sorting, I allow firms to be differentiated along several key dimensions. Specifically, I allow firms to vary in productivity, such that a worker’s skills may be valued differently at different firms. I also allow for heterogeneous technology in the production function, where firms can exhibit decreasing returns to scale and imperfect substitutability between labor inputs. Moreover, I assume that firms have distinct non-wage amenities, which also differ within a firm based on workers’ skills.¹ I demonstrate how each of these firm characteristics interacts with workers’ preferences in equilibrium, and I discuss the consequences of these interactions for wage markdowns, rents, and misallocation.

I then turn to the question of identification, demonstrating how to learn about imperfect competition under my model using matched worker-firm data. In my analysis, identification is challenging for two reasons. First, as is typical in most contexts, a worker’s wage markdown is not observed in the data. In particular, observationally similar workers could earn different

¹Recent empirical work provides mixed evidence about which non-wage job characteristics are most important to workers; for discussion, see Maestas et al. (2017). In my analysis, I do not take a stance on this issue. Instead, I assume that firms are endowed with a fixed set of amenities that are unobserved by the analyst; or, more precisely, I restrict amenities to be fixed over the estimation window. This assumption neither imposes nor precludes the possibility that firms initially choose amenities to maximize profits. Indeed, it is straightforward to show that permitting firms to choose their amenities initially would have no impact on any of my estimates.

wages due to unobserved skill differences—not market power.² Wages may also not fully reflect a worker’s compensation, as firms may have amenities that researchers do not observe. Indeed, as Rosen (1986) discusses, compensating differentials can exist even in competitive markets. A second identification challenge stems from the generality of my framework, which allows workers’ wage markdowns to vary both within and across firms, and to be endogenous to wage changes within a firm. This added feature complicates my analysis because instead of recovering a single markdown, I need to recover entire functions of firm characteristics.

I show how to overcome these challenges to achieve identification of the model through the use of panel data by leveraging an economic shock that affects the productivity of multiple firms in the economy. These productivity shocks can be recovered either from existing data using internal instruments or from external data sources. One novelty of my approach is that I allow the productivity shocks to shift wages in a way that is reflected in workers’ wage indices. Therefore, I allow there to be spillover effects on the labor supply curves faced by *all* firms, including those that do not experience a productivity shock. These spillovers would prevent me from recovering the elasticity of labor supply to a firm by directly comparing a firm’s outcomes before and after the shocks. Nevertheless, I prove that—under weak conditions on the instrument—the elasticities are identified from a difference-in-differences comparison of outcomes. This method involves comparing mean changes in wages and labor between firms with different exposures to productivity shocks, while controlling for a firm’s initial wages.

An additional contribution of my analysis is that I draw inference about the differences in workers’ trade-offs between wages and amenities. This approach is based on a revealed preference argument: holding all else fixed, a firm that offers better wages (for a given skill group) is more likely to attract workers who value wages more highly. This compositional effect is captured by the labor supply elasticity faced by a firm. Given this property, I prove that there

²See, e.g., Murphy & Topel (1990), Gibbons & Katz (1992), Abowd et al. (1999), and Gibbons et al. (2005).

exists a one-to-one mapping between the density function governing workers' wage-amenity trade-offs and a firm's labor supply curve. I then develop a method to estimate this density nonparametrically, which can be implemented separately for different population subgroups.

Using this identification strategy, I demonstrate how to recover the structural parameters in the model that characterize firm productivity, technology, and amenities, as well as workers' preferences. From these parameters, I can calculate the rents for both workers and firms, and conduct counterfactual analyses to explore sources of allocative inefficiency and inequality.

Finally, I estimate the model using a matched employer-employee panel dataset, covering the universe of workers and firms in Norway for the period 1995-2018. My results yield three key findings. First, I find that many firms have substantial wage-setting power. On average, workers' wages constitute only 86% of their marginal products. Workers and firms also earn large rents: on average, workers are willing to forgo 17% of their wages to remain at their current job, while the average firm derives 29% of its profits by exerting wage-setting power.³

Second, I find that labor market power varies significantly within and across firms, with the wage-to-marginal-product ratio ranging from 0.70 to 0.95. I also find that market power is more concentrated among firms that pay lower wages for a given skill group. This finding contrasts with the traditional view that larger, more productive firms should have more wage-setting power over workers.⁴ However, it supports a key prediction of my model. Specifically, it suggests that lower-paying firms are more likely to employ workers who are less responsive to wage changes (less elastic), thereby enabling these firms to exert greater wage-setting power.

³Empirically, it is hard to credibly differentiate between ex ante and ex post firm rents. This task necessitates further information—or assumptions—about how firms choose and pay for the amenities provided to workers.

⁴This view traces back to the early research on monopsony (Robinson, 1933), which examines a labor market with a single buyer of labor. However, if labor markets contain many firms, then market power need not increase with a firm's size (Manning, 2021). Indeed, my estimates reveal that less productive firms tend to have greater wage-setting power because of they are more likely to attract workers who are less responsive to wage changes.

Third, I find that imperfect competition generates substantial misallocation of workers to firms. By eliminating labor wedges, I estimate that total welfare would increase by 9.5%, measured in dollar terms. Moreover, in a competitive (Walrasian) labor market, more workers would be employed at firms with lower productivity. This key source of inefficiency would not be captured by frameworks that do not account for heterogeneity in workers’ wage markdowns.

My paper contributes to three different literatures. First, it relates to a large literature on imperfect competition in the labor market. Manning (2021), Sokolova & Sorensen (2021), Card (2022), and Azar & Marinescu (2024) provide recent surveys of this research.⁵ Within this literature, my paper builds on a class of equilibrium models studied by Card et al. (2018), Kroft et al. (2021), Lamadon et al. (2022), Azar et al. (2022), and Chan et al. (2024), where imperfect competition arises from workers’ idiosyncratic tastes for jobs that are not fully priced into earnings. This work, which Manning (2021) terms “new classical monopsony,” uses differentiated demand models that are commonly employed in Industrial Organization. A limitation of these approaches, however, is that they typically specify utility functions where worker and firm characteristics do not interact. As a result, they place strong restrictions on the way that workers sort, which limits how firm wage-setting power varies in the economy.⁶ I relax these restrictions by extending the model to allow workers to exhibit heterogeneous wage-amenity trade-offs. This extension is similar to one that is already well-established in the context of product markets, based on work developed by Berry et al. (1995). Moreover, as I demonstrate, it significantly broadens the scope of worker behavior in the labor market. In doing so, it leads to a richer characterization of the determinants of firm wage-setting power.

Second, my paper contributes to an empirical literature on sorting, drawing on the addi-

⁵For additional discussion, I refer to Boal & Ransom (1997), Bhaskar et al. (2002), and Manning (2011).

⁶Abstracting from strategic interactions, it leads to uniform wage markdowns. As discussed in Section II.E, this restriction can be relaxed by using a nested logit model (Lamadon et al., 2022; Azar et al., 2022) or by letting markdowns vary within a firm based on worker observables (Chan et al., 2024; Deb et al., 2024). Yet, these approaches still cannot capture sorting patterns arising from unobserved variation in wage-amenity trade-offs.

tive worker and firm effects wage model introduced by Abowd, Kramarz, & Margolis (1999). This work has largely focused on skill-based sorting, arising, e.g., from worker-firm production complementarities (Song et al., 2018). While my model allows for sorting on workers' skills, I also study a second type of sorting based on workers' wage-amenity trade-offs. I show that this feature introduces a firm-skill interaction term—the wage markdown—in a worker's wage equation, such that the same firm can have a larger markdown for one skill group and a smaller markdown for another. In doing so, my paper provides additional insight into assortativeness in the allocation of workers to firms, e.g., Andrews et al. (2008) and Card et al. (2013).

Third, my paper contributes to a growing literature on the willingness to pay for non-wage amenities. Both experimental studies and survey data offer strong evidence that workers value a wide variety of non-wage job attributes (Hamermesh, 1999; Pierce, 2001; Maestas et al., 2017). Additionally, the willingness to pay for these amenities differs among workers (Mas & Pallais, 2017; Wiswall & Zafar, 2018), contributing to wage inequality (Maestas et al., 2023). My paper takes this research area in a new direction, exploring how differences in the valuations of amenities affect the wage-setting power of firms in imperfectly competitive labor markets. I demonstrate that workers' wage-amenity trade-offs are reflected in the labor supply elasticities faced by firms. Moreover, by recovering these elasticities, I can draw inference about differences in wage-amenity trade-offs among workers. This method provides a new way to learn about variation in the willingness to pay for amenities using revealed preference, contributing to a recent literature in this area, e.g., Sorkin (2018) and Hall & Mueller (2018).

This paper is organized as follows. Section II presents the model and explores equilibrium properties related to sorting, imperfect competition, and welfare. Section III discusses the data sources and sample selection. Section IV outlines the identification strategy. Section V describes the estimation procedure, parameter estimates, and fit. Section VI presents the empirical findings about rents, misallocation, and inequality. Lastly, Section VII concludes.

II. Model of Imperfect Competition in the Labor Market

In this section, I develop a theoretical framework to explain how workers sort into jobs and the implications of this sorting for employers' wage-setting power. My framework extends prior equilibrium models of the labor market, used by Kroft et al. (2021), Lamadon et al. (2022), and Azar et al. (2022), by allowing workers to exhibit varying marginal rates of substitution with respect to wages and non-wage job characteristics. I show that this extension leads employers to face heterogeneous labor supply elasticities, which reflect differences in the composition of workers at different firms. As a result, the model generates variable wage markdowns and rents, thus introducing a potential source of allocative inefficiency. Throughout my analysis, I allow for unobserved heterogeneity in worker skills, firm production, and non-wage amenities.

II.A. Environment

The economy comprises a unit measure of workers, indexed by i , and a finite number of employers, indexed by $j \in \{1, \dots, J\}$. Each employer operates within a local labor market $m(j)$, and $\mathcal{J}_m$ denotes the set of employers in market m . In this economy, workers are differentially productive, and they exhibit heterogeneous tastes for jobs. Meanwhile, employers provide differentiated work environments, and they also vary in their productivity and technology.

Each worker i has a vector of skills $X_i = (\chi_i, \varphi_i)$. These skills contain two components: a categorical *skill type* $\chi_i \in \mathcal{X}$ and a continuous *skill level* $\varphi_i \in \mathbb{R}_+$. The worker's skill type χ_i captures the non-ordinal aspects of human capital such as training and specialization, which are valued differently at different firms. The worker's skill level φ_i , which is unobservable to the analyst, measures individual productivity within a given skill type. By characterizing skills as a composite of types and levels, the model adopts approaches from a growing literature about the importance of multidimensional skills, e.g., Lise & Postel-Vinay (2020), while also allowing human capital to vary in ways that researchers cannot directly observe.

Throughout my analysis, I maintain two assumptions about the information structure. First, I abstract from search frictions by assuming that workers are fully informed about job opportunities and that, given posted wages, a worker freely chooses where to work. Second, I assume that employers observe workers' skills $\{X_i\}_i$, but they only know the distribution of workers' preferences conditional on X_i . Therefore, while employers can assign wages based on skills, they cannot further price-discriminate with respect to individual preferences.⁷

II.B. Worker Preferences

Workers exhibit heterogeneous preferences over wages and non-wage amenities. Let $W_j(X)$ be the wage that a firm j offers to workers with skills X , and let $a_j(X)$ be a firm's skill-specific amenity. For any worker i with skills X_i , the indirect utility from working at firm j is:

$$u_{ij}(W_j(X_i), a_j(X_i)) = \beta_i \log W_j(X_i) + a_j(X_i) + \epsilon_{ij}. \quad (1.1)$$

In this utility function, ϵ_{ij} represents the worker's idiosyncratic taste for firm j , reflecting non-pecuniary considerations such as distance from work and personal preferences over work environment. I assume that $\{\epsilon_{ij}\}_{i,j}$ are independent and identically distributed (i.i.d.) random variables, each following a continuous distribution function F_ϵ with positive density over $\mathbb{R}$.

The coefficient β_i is the worker's marginal utility of log earnings. This parameter governs the marginal rate of substitution between wages and amenities, thus capturing the trade-off between higher pay and better working conditions. A worker with a lower β_i is more willing to sacrifice an increase in earnings in exchange for improved non-wage job characteristics.⁸ In Appendix A.8, I provide an explicit micro-foundation for utility specification (2.1) where I re-interpret β_i as the worker's marginal rate of substitution between consumption and leisure.

⁷This assumption reflects information asymmetries between workers and firms. It may also account for laws against pay discrimination, which are common in many labor markets. See Bhaskar et al. (2002) for discussion.

⁸Wiswall & Zafar (2018) and Maestas et al. (2023) examine how variation in the wage coefficient within a worker's utility function directly reflects differences in the willingness to trade earnings for non-wage amenities.

In my analysis, I allow the coefficient β_i to vary freely among workers in the economy. Specifically, I treat β_i as a random variable that is distributed according to a density f_β . I do not place any parametric restrictions on this density, such as assuming a normal distribution for β_i . One important aspect of this framework is that I allow β_i to vary conditionally within each skill group X_i . As a result, equally skilled workers could exhibit different marginal rates of substitution with respect to wages and amenities. This level of variation is economically meaningful as it suggests that the trade-offs workers make when choosing jobs would reflect a variety of considerations beyond productivity. For example, β_i can vary based on contextual factors such as age, gender, household characteristics, and wealth. It may also be correlated among workers in the same household, neighborhood, or peer group. In my framework, I also allow β_i to be correlated with workers' skills X_i . I assume that $P(\beta_i > 0 | X_i = X) = 1$ for all X .

II.C. Firm Production

Let $D_j(\chi, \varphi)$ denote the mass of workers with skills $X = (\chi, \varphi)$ that is demanded by a firm j . For each skill type $\chi \in \mathcal{X}$, the total efficiency units of labor at the firm are defined as follows:

$$L_j^{\text{eff}}(\chi) = \int \varphi D_j(\chi, \varphi) d\varphi. \quad (1.2)$$

Define Y_j to be the firm's value added from hiring labor. The firm's production function is a constant elasticity of substitution (CES) composite of the effective labor for different skill types. This formulation permits imperfect substitutability between skill types and allows for decreasing returns to scale, so that output need not increase in equal proportion to labor inputs.

$$Y_j = T_j \left(\sum_{\chi \in \mathcal{X}} \theta_{j\chi} \left(L_j^{\text{eff}}(\chi) \right)^{\rho_j} \right)^{\frac{1-\alpha_j}{\rho_j}}. \quad (1.3)$$

In this production function, $T_j \in \mathbb{R}_{++}$ is the firm's total factor productivity (TFP), $\theta_{j\chi} \in \mathbb{R}_{++}$ is

the firm-specific efficiency of skill type χ , $(1 - \rho_j)^{-1} \in [0, \infty)$ is the elasticity of substitution between skill types at the firm, and $1 - \alpha_j \in (0, 1]$ is the firm's returns to scale. For simplicity, my specification abstracts from capital and intermediate inputs, as well as the structure of the product market. Nevertheless, as I show in Appendix A.9, this production function could be rationalized under either a perfectly competitive or an imperfectly competitive product market.

By allowing the productivity and technology parameters $(T_j, \{\theta_{j\chi}\}_{\chi \in \mathcal{X}}, \rho_j, \alpha_j)$ to vary among firms, this framework is able to accommodate a wide range of employer characteristics across different industries. Specifically, firms can be differentially productive, as captured by variation in T_j . In addition, as $\{\theta_{j\chi}\}_{\chi \in \mathcal{X}}$ is heterogeneous, the relative efficiencies of various skill types can differ among firms. For example, certain employers may place more emphasis on a college degree than others. Meanwhile, the amount of substitutability between these skill types can also vary, as reflected by differences in ρ_j . For example, employers may have specific job requirements that affect the ability to interchange high- and low-educated workers. Finally, due to variation in α_j , firms can exhibit different returns to scale. Importantly, my analysis does not impose restrictions on the relationship between amenities, productivity, and technology. Rather, I allow these fundamentals to vary freely across firms in the economy.

To better understand the relationship between skill types and skill levels in the production function, it will be useful to define the standardized unit of labor at the firm to be N_j , where:

$$N_j = \left(\sum_{\chi \in \mathcal{X}} \theta_{j\chi} \left(\int \varphi D_j(\chi, \varphi) d\varphi \right)^{\rho_j} \right)^{1/\rho_j}. \quad (1.4)$$

For any combination of skills $X = (\chi, \varphi)$, the elasticity of N_j with respect to $D_j(\chi, \varphi)$ equals:

$$\frac{\partial \log N_j}{\partial \log D_j(\chi, \varphi)} = \underbrace{\frac{\theta_{j\chi} \left(L_j^{\text{eff}}(\chi) \right)^{\rho_j}}{\sum_{\chi' \in X} \theta_{j\chi'} \left(L_j^{\text{eff}}(\chi') \right)^{\rho_j}}}_{\partial \log N_j / \partial \log L_j^{\text{eff}}(\chi)} \times \underbrace{\frac{\varphi D_j(\chi, \varphi)}{\int \varphi' D_j(\chi, \varphi') d\varphi'}}_{\partial \log L_j^{\text{eff}}(\chi) / \partial \log D_j(\chi, \varphi)}. \quad (1.5)$$

This framework nests both efficiency units of labor and CES specifications. Within any skill type χ , workers exhibit varying productivity, as measured by φ . In contrast, workers that have different skill types are viewed as imperfect substitutes, and their productivities vary in a way that is specific to each firm. Under this setup, two skill levels φ and φ' are only comparable among workers in the same skill type, but they are not directly comparable across skill types.

II.D. Characterization of an Equilibrium

In equilibrium, each worker selects an employer to maximize utility, as defined in equation (2.1). Given a set of wage offers $\{W_k(X)\}_{k=1}^J$ and non-wage amenities $\{a_k(X)\}_{k=1}^J$, a worker with a preference parameter β and a skill profile X chooses to work for firm j with probability:

$$P(j(i) = j | \beta, X) = \int_{-\infty}^{\infty} \prod_{k \neq j} F_{\epsilon} \left(\beta \log \left(\frac{W_j(X)}{W_k(X)} \right) + a_j(X) - a_k(X) + \tilde{\epsilon} \right) f_{\epsilon}(\tilde{\epsilon}) d\tilde{\epsilon}. \quad (1.6)$$

Averaging across the realizations of β , the labor supply at firm j for workers with skills X is:

$$S_j(X) = \int \left(\int_{-\infty}^{\infty} \prod_{k \neq j} F_{\epsilon} \left(\beta \log \left(\frac{W_j(X)}{W_k(X)} \right) + a_j(X) - a_k(X) + \tilde{\epsilon} \right) f_{\epsilon}(\tilde{\epsilon}) d\tilde{\epsilon} \right) f_{\beta, X}(\beta, X) d\beta. \quad (1.7)$$

Given the firm-specific labor supply curves in equation (2.7), each firm optimally assigns wages to workers in order to maximize profit. Since the firm cannot discriminate based on individual preferences (β_i, ϵ_{ij}) , its wage-setting decisions are informed by the distributions

of these parameters, given by $\{F_{\beta|X}\}_X$ and F_ϵ . The firm's profit maximization problem is:

$$\max_{\{W_j(\chi, \varphi)\}_{\chi, \varphi}} T_j \left(\sum_{\chi \in \mathcal{X}} \theta_{j\chi} \left(\int \varphi D_j(\chi, \varphi) d\varphi \right)^{\rho_j} \right)^{\frac{1-\alpha_j}{\rho_j}} - \sum_{\chi \in \mathcal{X}} \left(\int W_j(\chi, \varphi) D_j(\chi, \varphi) d\varphi \right), \quad (1.8)$$

subject to the constraint that labor demand equals supply: $D_j(X) = S_j(X)$ for all $X = (\chi, \varphi)$.

I consider an equilibrium in which firms view themselves to be strategically small in the economy. Consequently, a firm does not internalize the effect of changing its own wages on the labor that is supplied to other firms.⁹ It is important to note that this assumption does not mean that workers regard firms as infinitesimal. On the contrary, there is a finite number of firms, and every worker chooses his/her job by considering the full set of wage offers and amenities. However, my analysis assumes that each firm sees itself as a marginal player, thus ruling out strategic interactions in firm wage-setting.¹⁰ An equilibrium is defined as follows:

Definition. Given any set of worker skill and preference distributions $(F_{\chi, \varphi}, \{F_{\beta|\chi, \varphi}\}_{\chi, \varphi}, F_\epsilon)$, firm amenities $\{a_j(\chi, \varphi)\}_{\chi, \varphi}$ and production parameters $(T_j, \{\theta_{j\chi}\}_\chi, \rho_j, \alpha_j)$, an equilibrium is defined by the job decisions $j(i)$, labor supply curves $S_j(\chi, \varphi)$, and wages $W_j(\chi, \varphi)$ where:

- (i) A worker chooses a firm that maximizes his/her utility, as specified by equation (2.1).
- (ii) Labor supply curves $S_j(\chi, \varphi)$ are consistent with workers' optimal choices, as in (2.7).
- (iii) Each firm posts wages $W_j(\chi, \varphi)$ for all $\chi \in \mathcal{X}$ and $\varphi \in \mathbb{R}_+$ to maximize profit as defined in equation (2.8), ensuring that labor demand $D_j(\chi, \varphi)$ equals labor supply $S_j(\chi, \varphi)$.

In Lemma 1 of Appendix B.2, I establish the existence of an equilibrium involving strictly

⁹This assumption is also made by Card et al. (2018), Kroft et al. (2021), Lamadon et al. (2022), and Azar et al. (2022). Alternatively, Berger et al. (2022), Deb et al. (2024), and Chan et al. (2024) study models of job differentiation where firms internalize market share effects, and Jarosch et al. (2024) studies a search model with large firms. Identification is very difficult with strategic interactions, as it relies on more exclusion restrictions.

¹⁰Roussille & Scuderi (2023) provide a framework to test between different models of firm conduct, finding that models where firms ignore strategic interactions outperform models which incorporate such interactions.

positive wages and employment. Lemma 2 further shows that this equilibrium is unique with probability one. Additionally, I analyze properties such as the concavity of the profit function and the conditions required for dynamic stability of the equilibrium. These properties will be important for running counterfactual analyses because they provide me with a way to compute equilibrium outcomes based on the underlying structural parameters in the model.

II.E. Properties of the Labor Supply Curve to a Firm

A primary motivation for incorporating random coefficients in the worker's utility function is that it enables the model to capture rich substitution patterns that reflect sorting on individual heterogeneity. This subsection examines how this heterogeneity expands the scope of worker behavior in the model and what it implies about the shape of the labor supply curve to a firm.

Substitution Patterns under Homogeneous Effects

The standard assumption in models of job differentiation is that β is homogeneous among workers. Under this assumption, employer wages and amenities do not interact with worker-level preference parameters in the utility function. This additive separability between firm and worker characteristics makes the model more tractable. Yet, it imposes a strong form of homogeneity on aggregate substitution patterns that can lead to unrealistic market outcomes. This critique has previously been raised in the context of product markets (e.g., Berry et al., 1995). However, as I will demonstrate, it also holds significant implications for labor markets.

To illustrate how the homogeneous effects assumption restricts economic behavior, consider a scenario where β is constant across workers. In this case, the only reason that different workers rank firms differently is that their taste shocks ϵ_{ij} vary. Whenever these taste shocks are i.i.d., a worker's second choice firm does not depend on where that worker is employed.

This aspect of the model matters when analyzing a change in the economic environment. Specifically, if wages were to shift in the market, then a firm's employment response would

not depend on that firm's characteristics. One way to formalize this idea is to consider two arbitrary firms, j and j' . I then analyze how the labor supplied to a third set of firms, defined as $S_{\mathcal{J}^*}(X) = \sum_{k \in \mathcal{J}^*} S_k(X)$ where $j, j' \notin \mathcal{J}^*$, is affected by the wages at j and j' , respectively. When β is homogeneous, these labor supply responses are identical. In particular, I find that:

$$\frac{\partial \log S_{\mathcal{J}^*}(X)}{\partial \log W_j(X)} = \frac{\partial \log S_{\mathcal{J}^*}(X)}{\partial \log W_{j'}(X)}, \quad \text{for any } \mathcal{J}^* \in \{1, \dots, J\} \text{ where } j, j' \notin \mathcal{J}^*. \quad (1.9)$$

This restriction implies that the workers at firms j and j' would be just as likely to substitute toward a third firm k , regardless of which of the two firms is more similar to firm k . This contradicts the intuition that firms with similar attributes should exhibit larger substitution effects. To see how this restriction affects a firm's labor supply curve, consider the following property.

Property 1. If β is homogeneous, then firms encounter identical labor supply elasticities:

$$\frac{\partial \log S_j(X)}{\partial \log W_j(X)} = \frac{\partial \log S_{j'}(X)}{\partial \log W_{j'}(X)}, \quad \text{for all } j, j' \in \{1, \dots, J\}.$$

Property 1 underscores the limitations of using utility specification (2.1) with homogeneous effects. For example, consider two firms that offer different wages and amenities. It would be natural to expect that these firms attract different types of workers who respond differently to changes in their earnings. So, the elasticities of labor supply would likely differ between these firms. Nevertheless, if β is homogeneous, then the model cannot account for this pattern. In particular, firms would always encounter the same labor supply elasticities, even when they exhibit vastly different technologies and work environments. These implications do not depend on the distribution of the idiosyncratic taste shocks, e.g., logit or probit.

One way to relax these restrictions on substitution patterns is to assume that $\{\epsilon_{ij}\}_{i,j}$ are not i.i.d., but rather that ϵ_{ij} is correlated among firms within local labor markets, e.g., via a

nested logit structure. This method is taken by Azar et al. (2022) and Lamadon et al. (2022), among others. It implies that workers are more willing to substitute toward firms within markets than across markets. However, within each market, substitution effects would still be independent of firm characteristics, leading to constant labor supply elasticities across firms.¹¹

A second extension could be to allow β to vary based on observable worker characteristics such as education or experience.¹² However, a concern with this strategy is that firms are also likely to observe these characteristics and would thus factor them into a worker's skills X . As long as β is constant conditional on skills, the implications of Property 1 remain unaffected. In particular, for any skill group X , the labor supply elasticities would be uniform across firms.

Substitution Patterns under Heterogeneous Effects

If β is heterogeneous, then the labor supply curve to a firm will reflect differences in workers' preferences over wages and amenities. For example, firms offering higher wages and fewer amenities are more likely to attract workers who place a greater value on wages, as indicated by larger values of β . Conversely, firms that provide lower wages but better amenities will tend to attract workers who prioritize amenities, as indicated by smaller values of β . Thus, firms that share similar characteristics are more likely to attract workers with similar preferences over wages and amenities, which would lead these firms to exhibit larger substitution effects.

This variation in worker composition affects the elasticities of labor supply to a firm. To see how, note that any firm-specific elasticity can be represented as the average of β -specific elasticities for workers employed at the firm. The next property formalizes this relationship.

¹¹In principle, nests can be defined more broadly so they do not coincide with markets. Yet, these definitions still rely on factors observable to the analyst, which is often undesirable. See Nevo (2000) for more discussion.

¹²Examples of this approach include Chan et al. (2024) and Deb et al. (2024). The former uses a multinomial logit discrete choice model and the latter uses a representative agent CES model. As shown in Verboven (1996), Proposition 2, both modeling approaches lead to demand functions that are isomorphic at the market level.

Property 2. The elasticity of labor supply faced by a firm j has the following representation:

$$\frac{\partial \log S_j(X)}{\partial \log W_j(X)} = E \left(\frac{\partial \log P(j(i) = j | \beta_i, X_i)}{\partial \log W_j(X_i)} \middle| X_i = X, j(i) = j \right).$$

As this property indicates, the elasticity of labor supply to a firm is shaped by the composition of the firm's workforce, which is determined by the wages and amenities that the firm provides. Therefore, when β is heterogeneous, the labor supply elasticities are directly linked to firm characteristics in a way that would not be captured under a model with homogeneous effects.

Another implication of heterogeneous effects is that, even if firms are strategically small within the economy, the labor supply elasticities are endogenous to wage changes at a firm. When a firm posts a higher wage, it attracts workers who are more responsive to wage changes, i.e., those with larger β 's. As a result, the elasticity of labor supply to the firm would increase. To formalize this dynamic, I analyze the second derivative of the labor supply curve to a firm.

Property 3. The second derivative of log labor supply with respect to the log wage at a firm is:

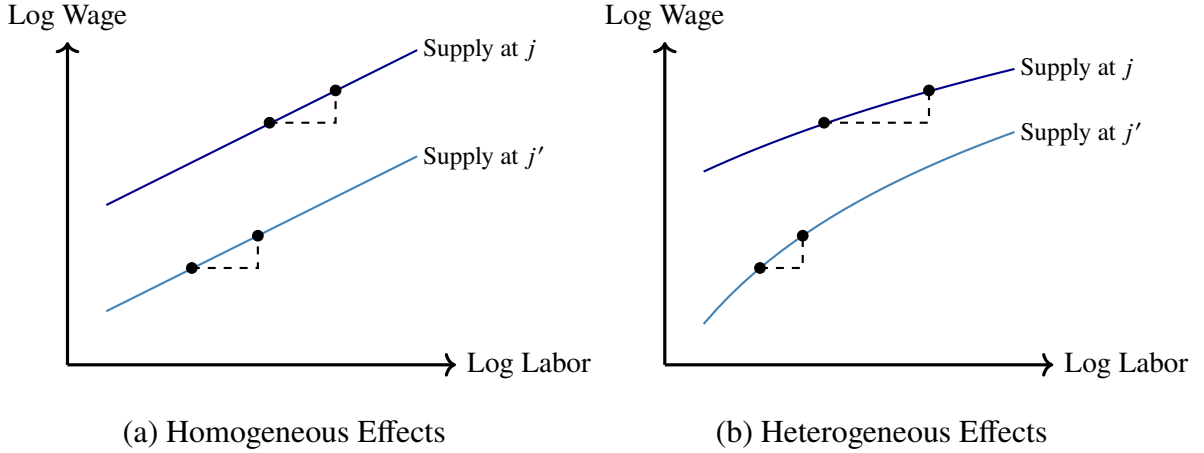
$$\frac{\partial^2 \log S_j(X)}{\partial [\log W_j(X)]^2} = \text{Var} \left(\frac{\partial \log P(j(i) = j | \beta_i, X_i)}{\partial \log W_j(X_i)} \middle| X_i = X, j(i) = j \right).$$

This second derivative is strictly positive if and only if the variance $\text{Var}(\beta_i | X_i = X)$ is nonzero.

Figure 1 illustrates how heterogeneous effects influence the labor supply curve to a firm. Consider two firms, j and j' , where firm j has high wages but unfavorable amenities, while firm j' has low wages but favorable amenities. In the homogeneous effects case (depicted in Figure 1.a), both firms would face the same labor supply responses to internal wage changes, meaning that an $X\%$ wage change results in the same percent change in labor at either firm. Under heterogeneous effects (depicted in Figure 1.b), these two firms would face different labor supply responses due to differences in workforce composition. For the same $X\%$ wage change at each firm, the high-wage, low-amenity firm j would see a larger employment re-

sponse, while the low-wage, high-amenity firm j' would see a smaller employment response.

Figure 1.1: Illustration of Firm-Specific Labor Supply Responses



Notes. This figure depicts the labor supply curves $\log S_j(X)$ and $\log S_{j'}(X)$ for two firms, j and j' , respectively. Figure 1.a shows the case where $\text{Var}(\beta_i | X_i = X) = 0$. Figure 1.b shows the case where $\text{Var}(\beta_i | X_i = X) > 0$.

II.F. Wage Determination and Sorting in Equilibrium

In equilibrium, firms face upward-sloping labor supply curves, which enable them to set their wages below a worker's marginal product of labor. Let $L_j(X)$ represent the amount of labor from skill group X that is employed at firm j in equilibrium, and define $\varepsilon_j(X) = \frac{\partial \log L_j(X)}{\partial \log W_j(X)}$ to be the elasticity of labor supply faced by the firm. A worker's wage is determined as follows:

$$W_j(\chi, \varphi) = \underbrace{\frac{\varepsilon_j(\chi, \varphi)}{1 + \varepsilon_j(\chi, \varphi)}}_{\text{Markdown}} \times \underbrace{\varphi T_j (1 - \alpha_j) \theta_{j\chi} \left(L_j^{\text{eff}}(\chi) \right)^{\rho_j - 1} \left(\sum_{\chi' \in \mathcal{X}} \theta_{j\chi'} \left(L_j^{\text{eff}}(\chi') \right)^{\rho_j} \right)^{\frac{1 - \alpha_j}{\rho_j} - 1}}_{\text{Marginal Product of Labor}}. \quad (1.10)$$

There are three main factors that drive workers to systematically sort into different firms. One factor is that a worker's marginal product varies across firms. This variation arises from differences in firm-specific productivity parameters $(T_j, \{\theta_{j\chi}\}_{\chi})$ and technology (ρ_j, α_j) . In

particular, productivity differences imply that the same worker will be more or less efficient depending on the firm, while technology differences affect how a worker's marginal product is influenced by the firm's existing labor inputs. Together, these features have the potential to generate large differences in worker productivity distributions both within and across firms.

A second factor influencing sorting is that firms have distinct non-wage amenities $a_j(X)$, which are allowed to potentially vary within a firm based on worker skills. These amenities imply that firms are vertically differentiated, meaning that some firms exhibit superior amenities compared to others. They also lead firms to be horizontally differentiated, since workers at the same firm may receive varying levels of benefit from the same amenity. Note that a firm's amenities may depend on its production parameters $(T_j, \{\theta_{jX}\}_X, \rho_j, \alpha_j)$, which means that these non-wage job characteristics could be directly linked to a worker's marginal product.

A third factor affecting sorting is the worker's wage markdown $\frac{\varepsilon_j(X)}{1+\varepsilon_j(X)}$. These markdowns differ across firms due to variation in workforce composition, reflecting an interaction between worker preferences and firm characteristics. For example, a low-productivity, high-amenity firm is likely to attract workers with smaller β 's, who place less emphasis on wages relative to amenities. As such, the firm's workforce would be less responsive to wage changes. This aspect enables the firm to exert more wage-setting power, leading to larger markdowns. In my model, markdowns are specific to each skill group X , which reflects the possibility that skill groups have systematically different preferences over wages and amenities. Consequently, workers with different skills may experience different markdowns even within the same firm.

One notable implication of sorting in my model is that a worker's wage markdown tends to amplify the impact of firm productivity on earnings. To understand this point, consider that a firm's wage-setting power diminishes as its labor supply becomes more elastic. Also, as Property 3 demonstrates, the elasticity of labor supply to a firm increases with the firm's wage. Consequently, when a demand shock drives up wages at a firm, it also shifts the composition

of the firm's workforce, making the labor supply more elastic. This larger elasticity reduces the firm's wage-setting power, resulting in even higher wages for workers. By this process, wage changes at a firm can become self-reinforcing.¹³ Importantly, as I demonstrate in Appendix B, this feature of the model does not interfere with the uniqueness of the equilibrium.¹⁴

II.G. Rents, Pass-throughs, and Allocative Inefficiency

I conclude this section by examining how the structural parameters in the model correspond to different economic quantities, such as the rents earned by workers and firms, the pass-through of productivity shocks to earnings, and measures of welfare loss due to allocative inefficiency.

To simplify the derivation of these quantities, it will be useful to specify a functional form for the distribution of workers' taste shocks $\{\epsilon_{ij}\}_{i,j}$. To do so, I analyze a special case of the model where ϵ_{ij} has a Type 1 Extreme Value (Logit) distribution: $F_\epsilon(\epsilon) = \exp(-\exp(-\epsilon))$. This assumption does not place meaningful restrictions on the properties discussed in Section II.E, as these properties are primarily driven by the parameter β_i , not by ϵ_{ij} . Nevertheless, it allows me to derive clearer expressions for the firm's labor supply curve and its elasticities.

¹³Another way to visualize this effect is to analyze the wage elasticity of the markdown $\frac{\epsilon_j(X)}{1+\epsilon_j(X)}$, which equals:

$$\frac{\partial \log \frac{\epsilon_j(X)}{1+\epsilon_j(X)}}{\partial \log W_j(X)} = \frac{\frac{\partial^2 \log L_j(X)}{\partial [\log W_j(X)]^2}}{\epsilon_j(X)[1 + \epsilon_j(X)]}, \quad \text{where} \quad \frac{\partial^2 \log L_j(X)}{\partial [\log W_j(X)]^2} \geq 0.$$

This elasticity is positive, indicating that fluctuations in wages at a firm are magnified through the markdown.

¹⁴In principle, the spillover effect of wages on markdowns can lead to multiple fixed points for the equilibrium wage equation (2.11). This multiplicity suggests that the firm's profit function might have two local optima: one characterized by lower wages, reduced employment, and larger markdowns; and the other marked by higher wages, increased employment, and smaller markdowns. This phenomenon is discussed by Manning (2010) in the context of spatial agglomeration. Fortunately, I find that multiplicity does not occur in my model as long as the conditional variance $\text{Var}(\beta|X)$ is bounded in relation to the returns to scale $1 - \alpha_j$. I also prove that, even if multiple fixed points exist, there is, with probability one, a unique point where profit is globally maximized.

Under the logit model, a worker's choice probability, defined in equation (2.6), becomes:

$$P(j(i) = j | \beta, X) = \frac{\exp(\beta \log W_j(X) + a_j(X))}{\sum_{k=1}^J \exp(\beta \log W_k(X) + a_k(X))}. \quad (1.11)$$

Let $I(\beta, X) = \sum_{k=1}^J \exp(\beta \log W_k(X) + a_k(X))$ denote the wage index for any worker with preference parameter β and skills X . This index varies across individuals, reflecting the fact that workers have different reservation wages. The labor supply curve to a firm j is given by:

$$S_j(X) = \int \frac{1}{I(\beta, X)} \exp(\beta \log W_j(X) + a_j(X)) f_{\beta, X}(\beta, X) d\beta. \quad (1.12)$$

One key advantage of using this specification is that it leads to a clearer economic interpretation of the random coefficient β . In particular, under the logit error structure, β can be expressed as an individual-specific labor supply elasticity, defined as: $\frac{\partial \log P(j(i)=j | \beta, X)}{\partial \log W_j(X)} = \beta$. This elasticity measures how a worker's probability of choosing a job at firm j is affected by variations in firm j 's own wage. By Property 2, it follows that the labor supply elasticity faced by the firm represents a firm-wide average of the realizations of β . Specifically, I find:

$$\frac{\partial \log S_j(X)}{\partial \log W_j(X)} = E(\beta_i | X_i = X, j(i) = j). \quad (1.13)$$

Moreover, by Property 3, the curvature of a firm's labor supply curve—represented by its second derivative—can be defined in terms of the variance of β among workers at the firm:

$$\frac{\partial^2 \log S_j(X)}{\partial [\log W_j(X)]^2} = \text{Var}(\beta_i | X_i = X, j(i) = j). \quad (1.14)$$

In Appendix A.3, I also derive the third, fourth, and fifth derivatives of the labor supply curve to a firm under the logit model. I demonstrate that these derivatives are characterized by the higher-order moments of the distribution of β . In each of the derivations in the Appendix, I

carefully highlight which parts of the analysis are reliant on utilizing the logit error structure.

Worker Rents

In this economy, employers cannot observe workers' individual preferences, β_i and ε_{ij} , which means that they are unable to price-discriminate based on each worker's reservation wage. As a result, an equilibrium involves surpluses—or rents—for workers who are inframarginal. These surpluses represent the additional earnings that workers are willing to forgo to remain indifferent between their current job and their next-best alternative. Let R_i^w denote the rents that a worker i receives from his or her current job. A worker's rents are defined as follows:

$$u_{ij}(W_j(X_i) - R_i^w, a_j(X_i)) = \max_{j' \neq j(i)} u_{ij'}(W_{j'}(X_i), a_{j'}(X_i)). \quad (1.15)$$

To illustrate how worker rents systematically vary in the economy, I derive an expression for the expected rents $\bar{R}_{jX}^w = E(R_i^w | j(i) = j, X_i = X)$ for workers with skills X employed at firm j .

$$\bar{R}_{jX}^w = W_j(X) \times E\left(\frac{1}{1 + \beta_i} \mid j(i) = j, X_i = X\right). \quad (1.16)$$

In a model with homogeneous effects, where $\beta_i = \beta$ for all i , a worker's expected rents are a fixed proportion of his or her earnings, specifically: $\bar{R}_{jX}^w / W_j(X) = (1 + \beta)^{-1}$. As a result, the model predicts that all workers, irrespective of their employer j or their skills X , would expect to receive the same share of their wages as rents. This restriction runs counter to the intuition that workers at different jobs would benefit in different ways from their wage contracts.

When β is heterogeneous, the share of wages that workers obtain as rents systematically varies across firms and skill groups. To see how this variation occur across firms, I examine how the ratio $\bar{R}_{jX}^w / W_j(X)$ —which measures expected rents as a share of a worker's pay—is driven by firm characteristics. I find that this ratio is a decreasing function of a firm's wage:

$$\frac{\partial [\bar{R}_{jX}^w / W_j(X)]}{\partial \log W_j(X)} = \text{Cov} \left(\beta_i, \frac{1}{1 + \beta_i} \mid j(i) = j, X_i = X \right) \leq 0. \quad (1.17)$$

This negative relationship reflects how workers sort based on variation in β . Within each skill group X , workers who choose jobs at higher-paying firms are more likely to have higher reservation wages. Thus, even while these workers earn more, they tend to receive a smaller share of their wages as rents. As I show empirically, neglecting this source of heterogeneity leads to an inaccurate assessment of where worker rents are concentrated within the economy.

Firm Rents

Firms may also capture rents in equilibrium. These rents come in the form of excess profits that firms obtain by exploiting their wage-setting power. To quantify these rents, I analyze a counterfactual setting where a firm is a price-taker in the economy, facing a perfectly elastic labor supply curve. For any firm j , I construct the counterfactual profits $\Pi_j^{\text{price-taker}}$ by solving:

$$\max_{\{D_j^{\text{pt}}(\chi, \varphi)\}_{\chi, \varphi}} T_j \left(\sum_{\chi \in \mathcal{X}} \theta_{j\chi} \left(\int \varphi D_j^{\text{pt}}(\chi, \varphi) d\varphi \right)^{\rho_j} \right)^{\frac{1-\alpha_j}{\rho_j}} - \sum_{\chi \in \mathcal{X}} \left(\int W_j^{\text{pt}}(\chi, \varphi) D_j^{\text{pt}}(\chi, \varphi) d\varphi \right), \quad (1.18)$$

subject to the labor supply constraint: $D_j^{\text{pt}}(X) = S_j^{\text{pt}}(X)$ for all $X = (\chi, \varphi)$. In this counterfactual, the only difference in the firm's behavior is that it does not exert wage-setting power. Therefore, this counterfactual differs from one where all firms in the economy are price-takers.

I define a firm j 's rents as the difference between actual profits and counterfactual profits at the firm, expressed as $R_j^e = \Pi_j - \Pi_j^{\text{price-taker}}$. In Appendix A.5, I show that these rents equal:

$$R_j^e = Y_j \times \left[\sum_{\chi \in \mathcal{X}} \left[\int \left(\frac{1 + \alpha_j \varepsilon_j(\chi, \varphi)}{1 + \varepsilon_j(\chi, \varphi)} \right) \frac{\partial \log N_j}{\partial \log L_j(\chi, \varphi)} d\varphi \right] - \alpha_j \left(\frac{Y_j^{\text{pt}}}{Y_j} \right) \right], \quad (1.19)$$

where Y_j^{pt} represents the firm's value added as a price-taker. In my model, Y_j^{pt} does not appear

to have a closed-form expression. Nevertheless, I show how to solve for this counterfactual quantity from the structural parameters in the model using a gradient descent algorithm. This method relies on the existence of a unique solution to the price-taker's first-order condition and the concavity of the firm's profit function, both of which are established in Appendix B.2.

Under my framework, the relationship between a firm's productivity and its rents R_j^e is ambiguous. On the one hand, a more productive firm generates greater value added, thereby increasing profitability. On the other hand, increased productivity typically leads to higher wages, which in turn results in higher labor supply elasticities $\{\varepsilon_j(X)\}_X$ faced by the firm. As these labor supply elasticities become larger, the firm's market power declines, making the labor inputs even more costly. In particular, as $\varepsilon_j(X) \rightarrow \infty$ for all X , the firm's profits will converge to those of a price-taking firm, which causes the rents to approach zero: $R_j^e \rightarrow 0$.

A firm's rents are also influenced by its technology. Specifically, as the firm approaches constant returns to scale, i.e., $\alpha_j \rightarrow 0$, the profits it could earn as a price-taker tend to zero. So, any profits that the firm manages to capture will come in the form of rents: $\lim_{\alpha_j \rightarrow 0} \Pi_j = R_j^e$.

In my analysis, it is hard to empirically distinguish between a firm's ex-ante and ex-post rents. This task would require knowing more about how firms select amenities, which I do not observe in the data. Therefore, for identification, I assume that firms are unable to manipulate their amenities—at least during the time frame that the model is estimated. Importantly, this assumption does not restrict how amenities correlate with a firm's productivity or technology. It also does not rule out the possibility that firms initially choose amenities to maximize profits.

Pass-through of TFP Shocks to Wages

Next, I examine the impact of a hypothetical shock to firm productivity—specifically, the TFP parameter T_j —on workers' earnings. This reduced form quantity is frequently used to learn about rent-sharing, as it indicates how changes in a firm's value added are passed on to

workers in the form of pay adjustments.¹⁵ I derive the pass-through of TFP shocks to wages as:

$$\begin{aligned} \frac{\partial \log W_j(\chi, \varphi)}{\partial \log T_j} = & \left[1 - \left(\frac{1}{\varepsilon_j(\chi, \varphi)[1 + \varepsilon_j(\chi, \varphi)]} \right) \times \frac{\partial^2 \log L_j(\chi, \varphi)}{\partial [\log W_j(\chi, \varphi)]^2} \right. \\ & + (1 - \rho_j) \int \varepsilon_j(\chi, \varphi') \frac{\partial \log L_j^{\text{eff}}(\chi)}{\partial \log L_j(\chi, \varphi')} d\varphi' \\ & \left. - (1 - \alpha_j - \rho_j) \sum_{\chi' \in \mathcal{X}} \left(\int \varepsilon_j(\chi', \varphi') \frac{\partial \log N_j}{\partial \log L_j(\chi', \varphi')} d\varphi' \right) \right]^{-1}. \quad (1.20) \end{aligned}$$

To interpret the pass-through, first consider a case where β is homogenous across workers. In this setting, the firm's labor supply curve is isoelastic, which implies that $\varepsilon_j(\chi, \varphi) = \varepsilon$ for a scalar $\varepsilon \in \mathbb{R}_+$ and $\frac{\partial^2 \log L_j(\chi, \varphi)}{\partial [\log W_j(\chi, \varphi)]^2} = 0$. Under these restrictions, the pass-through reduces to:

$$\frac{\partial \log W_j(\chi, \varphi)}{\partial \log T_j} = \frac{1}{1 + \alpha_j \varepsilon}. \quad (1.21)$$

This pass-through varies across firms in the economy due differences in the returns to scale parameters $\{\alpha_j\}_j$, with firms that have higher returns to scale exhibiting larger pass-throughs.

When β is heterogeneous among workers, the pass-through changes in two key aspects. First, since the labor supply elasticity $\varepsilon_j(\chi, \varphi)$ may differ by skill type χ , the pass-through will depend on the elasticity of substitution between skill types at a firm, specifically $(1 - \rho_j)^{-1}$. This feature introduces an additional source of variation across firms. Second, since β varies among equally-skilled workers, the wage markdown at a firm will be endogenous to changes in the firm's productivity. Specifically, a positive TFP shock that increases wages will alter the firm's workforce composition in a way that reduces its wage-setting power, resulting in even higher wages at the firm. This effect is captured by the positive second derivative of the labor supply curve, given by $\frac{\partial^2 \log L_j(X)}{\partial [\log W_j(X)]^2}$. Note that, in principle, the pass-through $\frac{\partial \log W_j(X)}{\partial \log T_j}$

¹⁵For example, Card et al. (2018) explicitly interprets the pass-through $\frac{\partial \log W_j(X)}{\partial \log T_j}$ as a rent-sharing elasticity.

can be greater than one, although my empirical results show that it is less than one in practice.

Welfare and Allocative Inefficiency

I consider a measure of social welfare that incorporates firm profits. Specifically, I assume that profits are redistributed to workers in proportion to their wages. This approach ensures that the redistribution does not distort workers' employment decisions. Welfare is defined as:

$$\mathcal{W} = \mathbb{E} \left(\max_j \{u_{ij}(W_j(X_i), a_j(X_i))\} \right) + \log \left(\sum_{j=1}^J \Pi_j \right). \quad (1.22)$$

To evaluate the welfare cost of imperfect competition, I compare welfare in the monopsonistic economy to that which would be obtained by a social planner who optimally allocates labor to firms by solving: $\mathcal{W}^* = \max_{\{j(i)\}_i} \mathcal{W}$. I assume that the social planner has complete knowledge of worker and firm fundamentals but that firms continue to set wages based on their limited information of (β_i, ϵ_{ij}) . As I prove in Appendix A.7, the planner's optimal allocation is equivalent to what would be achieved in a competitive (Walrasian) economy, where firms act as price takers facing perfectly elastic labor supply curves. Moreover, I show that it is possible to solve for welfare under this optimal allocation from the model's structural parameters.

In an economy with constant markdowns, there would be no misallocation of labor due to imperfect competition, since workers would sort into the same firms that they would have chosen in the absence of monopsony power. However, if markdowns vary across firms, then there is a potential for allocative inefficiency. This inefficiency occurs, for example, when a worker who is most productive at one firm j earns a higher wage at another firm k because firm j has a larger markdown than firm k . These discrepancies can alter workers' rankings of firms and distort their labor supply decisions. In this way, heterogeneous markdowns introduce a welfare cost of imperfect competition that is not present in the case of constant markdowns.¹⁶

¹⁶This conclusion assumes that there is no nonemployment margin. In a model involving nonemployment,

III. Data and Empirical Context

I now describe the data sources and the key variables that are used in my empirical analysis, as well as the criteria for sample selection. Additional details are provided in the Appendix.

III.A. Data Sources

In my analysis, I use a matched employer-employee panel dataset from Norway, containing information about workers and firms over the period 1995-2018. The data is constructed by linking annual business tax records to a national employment register from the Norwegian Labour and Welfare Administration. It encompasses all workers who are not self-employed and it covers the universe of firms that are required to file a tax return. Further information on worker demographics and financial assets come from individual tax filings and social security registers. All administrative data is maintained by Statistics Norway and is subject to comprehensive quality control measures. This data is considered to be highly reliable.¹⁷

At the worker-level, the data contains information on annual earnings, scheduled work hours, and the duration of a job spell. Earnings account for a worker's salary, bonuses, and other payments offered by an employer such as overtime, vacation, and severance pay.¹⁸ All monetary quantities are adjusted for inflation and are expressed in 2014 US dollars. The data also includes details about a worker's education, experience, and specialization (e.g., college major and vocational training). These variables are used to define a worker's skill type χ . In the baseline specification, I define χ as a composite of education and experience, where a worker's education level is divided into three categories (no high school degree, high school degree but no college degree, and college degree) and experience is divided into two cate-

misallocation can exist even with constant markdowns, as the trade-off between labor and leisure is distorted.

¹⁷See Fagereng et al. (2021) for discussion. Bruun et al. (2021) also give an independent quality evaluation.

¹⁸In-kind benefits are also included in earnings; however, parental leave and sickness benefits are excluded.

gories (<10 years in the workforce and ≥ 10 years in the workforce). Alternate definitions of χ are considered in the Appendix. Lastly, the data contains rich demographic information on workers and their families. These data will be used for making cross-sectional comparisons.

In the data, workers are linked to establishments, which represent sub-units (or branches) of a firm. To address this dimension, I measure employment at the establishment-level, and I assume that a firm's production function aggregates labor in a way that allows for imperfect substitutability between establishments.¹⁹ This approach enables greater flexibility in sorting and wage determination. For example, if an oil and gas firm operates a processing plant in rural Norway and has a corporate office in Oslo, then these two establishments are treated as distinct entities—both by workers and by the managing firm. Establishments are assigned to local labor markets based on industry and geographic location. The Central Register of Establishments and Enterprises identifies 10 broad industries and 46 commuting zones, as defined in Bhuller (2009). A market is constructed as the intersection of these two identifiers.

Accounting data is collected at the firm-level based on corporate income statements and balance sheets. A firm's value-added is defined as the difference between revenue and non-labor expenses, which include operating costs, spending on intermediate inputs, and capital depreciation. Profit is calculated to be revenue minus total costs, where total costs account for the wages paid to workers. All these quantities are measured prior to interest and taxes.

III.B. Sample Construction

Because my framework studies labor supply on the extensive margin, I restrict the estimation sample to full-time workers, defined as those who work at least 30 hours per week at a given

¹⁹Specifically, if a firm j operates multiple establishments $k \in \mathcal{K}_j$, then I adapt the production function so that $Y_j = T_j [\sum_{k \in \mathcal{K}_j} \sum_{\chi \in \mathcal{X}} \theta_{jk\chi} (L_{jk}^{\text{eff}}(\chi))^{\rho_j}]^{(1-\alpha_j)/\rho_j}$, where $L_{jk}^{\text{eff}}(\chi) = \int \varphi D_{jk}(\chi, \varphi) d\varphi$ is the total efficiency units of labor demanded by firm j for establishment k . This extension does not meaningfully change the theoretical results or the identification analysis. Therefore, to ease notation, the ongoing discussion will primarily focus on a setting where each firm operates only one establishment, i.e., $|\mathcal{K}_j| = 1$. As an additional robustness check in the Appendix, I consider the sensitivity of my estimates to restricting the sample to single-establishment firms.

establishment. Throughout my analysis, labor will be measured as full-time employment. To calculate earnings, I further restrict the sample to individuals aged 25 to 55 who work at the same establishment for an entire year and who do not have a higher-paying job elsewhere. Following Song et al. (2018), I also require that a worker's earnings are above a minimum threshold. In the baseline sample, this threshold is set to \$15,000 per year. In the Appendix, I show that my empirical results are robust to changes in this minimum earnings threshold.

Table 1.1: Summary Statistics—Main Estimation Sample

	Unique Identifiers	Unique Observations
<i>Panel A. Population Counts</i>		
Number of Full-time Workers	1,325,863	8,303,075
Number of Establishments	31,252	475,614
Number of Firms	23,066	274,687
Number of Local Markets	457	10,563
	Share of Workforce	Mean Log Earnings
<i>Panel B. Worker Skill Types</i>		
Lower Secondary, <10 Years Exp.	6.87%	10.91
Upper Secondary, <10 Years Exp.	18.98%	11.11
College Graduate, <10 Years Exp.	14.74%	11.33
Lower Secondary, ≥10 Years Exp.	21.30%	11.09
Upper Secondary, ≥10 Years Exp.	23.67%	11.30
College Graduate, ≥10 Years Exp.	14.43%	11.59
	Mean	Std. Dev.
<i>Panel C. Accounting Variables</i>		
Log Revenues	18.05	2.20
Log Total Costs	18.01	2.19
Log Value Added	16.86	2.18
Log Profit	14.97	2.54

Notes. This table presents descriptive statistics for the baseline estimation sample. All monetary quantities are adjusted for inflation and are expressed in log 2014 US dollars.

Part of my analysis relies on utilizing a balanced panel of firms. To achieve this objective, I limit the sample to firms that remain operational for nine consecutive years. During these years, each firm must consistently report positive revenue and maintain at least five full-time employees. In addition, each firm must continue operating the same set of establishments, and each establishment must stay in the same industry and commuting zone for the full nine years.

Table 1 provides descriptive statistics for the main estimation sample. More information about the distribution of wages, labor, value added, and profits can be found in the Appendix.

IV. Identification of Economic Quantities

In this section, I demonstrate how the model can be taken to data. Specifically, I show how to learn about key economic quantities, such as wage markdowns, rents, and allocative inefficiency, from the matched employer-employee dataset described in Section III. While some of these quantities can be recovered without full knowledge of the model’s structural parameters, others—particularly those defined by counterfactual outcomes—do rely on the entire set of parameters. For this reason, my objective is to achieve full point identification of the model.

My identification analysis follows two main steps. First, I demonstrate how to recover the firm-specific labor supply elasticities $\varepsilon_j(\chi, \varphi)$ through the use of panel data by leveraging an economic shock that affects the productivity of multiple firms in the economy. This strategy relies on instrumental variables, as well as assumptions about the unobserved skill levels φ that enable me to distinguish between the labor from different skill groups at a given firm. Second, I prove the identification of firms’ production parameters $(T_j, \{\theta_{j\chi}\}_{\chi}, \rho_j, \alpha_j)$ and amenities $\{a_j(\chi, \varphi)\}_{\chi, \varphi}$, as well as the conditional densities $\{f_{\beta|\chi, \varphi}\}_{\chi, \varphi}$ corresponding to the distribution of workers’ preferences over wages and amenities. As I demonstrate, the ability to determine these structural quantities relies on credibly recovering the elasticities $\varepsilon_j(\chi, \varphi)$.

IV.A. Identification of Firm-Specific Labor Supply Elasticities

Ideal Experiment

I begin by laying out the comparative statics that will enable the identification of labor supply elasticities. Consider a hypothetical productivity shock that shifts a firm j ’s TFP in logs from $\log(T_j)$ to $\log(T_j) + \delta_j$. Assume that all other parameters in the economy remain unchanged.

For each skill group X , let $(W_j(X), L_j(X))$ and $(W'_j(X), L'_j(X))$ represent the equilibrium wage and employment levels at the firm before and after the TFP shock, respectively. As δ_j approaches zero, these equilibrium quantities can be used to recover the elasticity $\epsilon_j(X)$. In particular, as I demonstrate in the Appendix, the equilibrium outcomes satisfy the condition:

$$\lim_{\delta_j \rightarrow 0} \left[(\log L'_j(X) - \log L_j(X)) - \epsilon_j(X) \times (\log W'_j(X) - \log W_j(X)) \right] = 0. \quad (1.23)$$

This condition follows from two key properties of the model. First, in the equilibrium characterization, firms view each other as strategically small. Therefore, they do not react to wage changes that are confined to a single firm in the economy. As a result, the labor supply $L_j(X)$ to a firm j would only be affected by an exogenous shock to T_j through changes in firm j 's own wage $W_j(X)$. The second property of the model is that the equilibrium is locally stable, which means that iterating on the equilibrium conditions (2.11) converges to an equilibrium for all nearby initial values. This property is important for comparative statics as it implies that a small productivity shock does not lead to large jumps in wages or employment.

From equation (2.20), an ideal experiment to recover the elasticities $\{\epsilon_j(X)\}_{j,X}$ would involve inducing independent and infinitesimal shocks to each firm's TFP and then measuring how wages and labor respond. However, such an experiment is not practically feasible. In particular, I cannot independently manipulate each firm's productivity. Moreover, even if it were possible to do so, I cannot directly observe a worker's skill level φ . Thus, I would be unable to fully differentiate between labor inputs at a firm. In the remainder of this subsection, I explain how to overcome these two challenges using panel data and instrumental variables.

Econometric Model and Assumptions

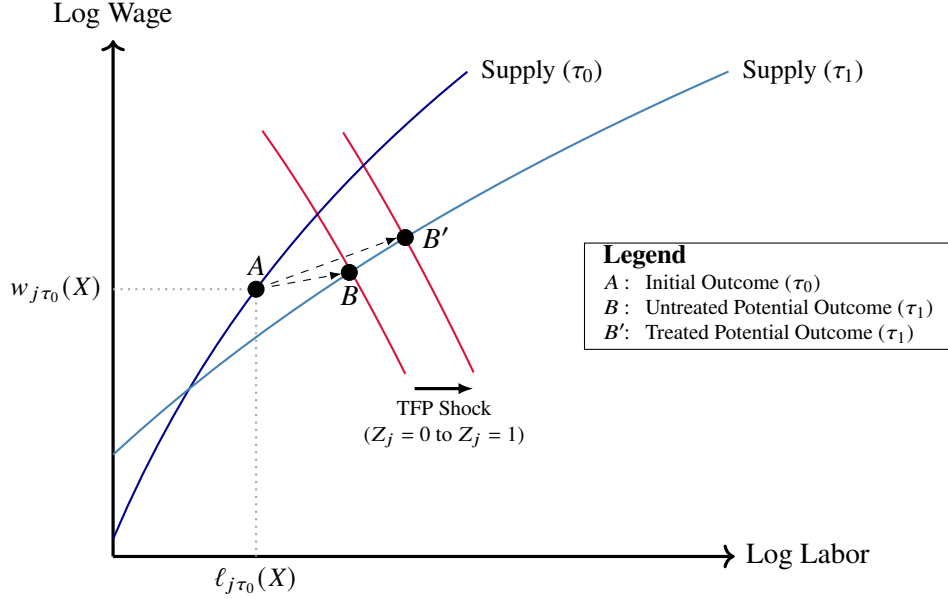
I now describe the empirical framework and key assumptions that form the basis for my identification results. Throughout this discussion, I will utilize the logit error structure, where I assume $F_\epsilon(\epsilon) = \exp(-\exp(-\epsilon))$. Consequently, the firm-specific labor supply curves are defined by equation (2.10). To ease notation, I use lowercase letters to denote quantities that are in log form, specifically defining $w_j(X) = \log W_j(X)$, $\ell_j(X) = \log L_j(X)$, and $y_j = \log Y_j$.

Description of Empirical Framework for a Generic Instrument

Consider the evolution of equilibrium outcomes over two time periods: τ_0 and τ_1 . After the initial period τ_0 , an event occurs whereby a subset of firms in the economy experiences a TFP shock, while the productivities of the other firms remain unchanged. Following this event, wages and labor adjust to a new equilibrium by the second period τ_1 . Let $\{w_{j\tau_0}(X), \ell_{j\tau_0}(X)\}_X$ denote the log wages and labor for a firm j in period τ_0 , and let $\{w_{j\tau_1}(X), \ell_{j\tau_1}(X)\}_X$ be the corresponding values in τ_1 . Additionally, let Z_j be a binary variable indicating whether firm j experiences a TFP shock. Firms are classified as *treated* if $Z_j = 1$ and *untreated* if $Z_j = 0$.

Figure 2 illustrates the evolution of wages and labor for a given firm j , delineating which aspects of the model are constant and which vary over time. Crucially, between periods τ_0 and τ_1 , wages and labor adjust for all firms, including those that are untreated. To understand why, note that the TFP shocks are not limited to a single firm, but rather they occur at multiple firms in the economy. Consequently, wage changes among the treated firms would shift the wage indices $I(\beta, X)$ of workers, which in turn reshapes the labor supply curve to every firm. Therefore, the TFP shocks generate spillover effects that impact all employers in the economy.

Figure 1.2: Illustration of Potential Outcomes for a Firm



Notes. This figure depicts how a firm's potential outcomes evolve in response to an event that shifts the TFP of firms in the economy. At τ_0 , worker wage indices, skills and preferences are characterized by $\{I_{\tau_0}(\beta, X)\}_{\beta, X}$ and $\{F_{\beta, X|\tau_0}\}_X$, respectively. At τ_1 , they are characterized by $\{I_{\tau_1}(\beta, X)\}_{\beta, X}$ and $\{F_{\beta, X|\tau_1}\}_X$, respectively. Aside from the TFP parameter T_j , which shifts if firm j is treated, the firm's characteristics $(\{\theta_{jX}\}_X, \rho_j, \alpha_j, \{a_j(X)\}_X)$ are constant over time.

Contributing to these dynamics, I allow the skills and preferences of workers to change over time in response to the productivity shocks affecting firms. This variation implies that the joint distribution of workers' skills $F_{\chi, \varphi}$, along with the conditional distributions $\{F_{\beta|\chi, \varphi}\}_{\chi, \varphi}$ that govern workers' preferences over wages and amenities, can evolve freely between periods τ_0 and τ_1 due to shifting economic factors arising from the previously described event. While I also allow workers' idiosyncratic tastes to evolve, I assume that the cross-sectional distribution of these tastes retains a logit structure in each period, i.e., $F_{\epsilon}(\epsilon_{ij\tau}) = \exp(-\exp(-\epsilon_{ij\tau}))$. To achieve this, I assume that $\epsilon_{ij\tau}$ is a Markov process with independent innovations across i .

Given the evolution of workers' wage indices, skills, and preferences over time, firms face different labor supply curves in periods τ_0 and τ_1 . Thus, in order to measure the elasticity of labor supply to a firm j , it is necessary to compare the firm's potential outcomes within

the same time period. In period τ_0 , the treated and untreated potential outcomes are equal, since no TFP shocks have yet occurred. However, in period τ_1 , the potential outcomes differ, as illustrated by points B and B' in Figure 2. This difference reflects the comparative statics from the ideal experiment, indicating how firm j 's wages and labor would respond to a TFP shock that is confined only to that firm. If it were possible to compare both potential outcomes for a firm in period τ_1 , then one could determine the firm's labor supply elasticity in that period.

Identification Assumptions

Throughout my identification analysis, I impose two key assumptions. First, I restrict how the exposure to a TFP shock, denoted by Z_j , depends on the inherent characteristics of a firm.

Assumption I (*Instrument Independence*). $Z_j \perp (\rho_j, \alpha_j, \{\theta_{j\chi}\}_\chi, \{a_j(X) - a_j(X')\}_{X,X'})$.

Assumption I states that the firm's treatment status Z_j is statistically independent of its technology (ρ_j, α_j) , the relative efficiencies $\{\theta_{j\chi}\}_\chi$ of each skill type, and the differences in amenities $\{a_j(X) - a_j(X')\}_{X,X'}$ between skill groups at the firm. Importantly, this assumption does not require Z_j to be fully exogenous. In particular, I allow it to depend on a firm's productivity through the baseline TFP T_j . Additionally, while Z_j is independent of amenity differences within a firm, it may still be affected by a firm's average amenity $E(a_j(X)|j)$. Consequently, treated and untreated firms can exhibit systematically different wages and labor.

Next, I make an assumption about the relationship between a worker's skill level φ and the other structural parameters in the model. This assumption consists of three main restrictions.

Assumption II (*Restrictions on Unobserved Skill Levels*).

II.1. $\varphi \perp \beta|\chi, \tau$.

II.2. $a_j(\chi, \varphi) = a_{j\chi} + a_{\chi\varphi}$.

II.3. $E(\log \varphi|\chi, \tau) = E(\log \varphi|\chi)$.

Assumption II.1 asserts that, given a worker's skill type χ and time period τ , preferences over wages and amenities, as captured by the parameter β , do not depend on the skill level φ . This assumption does not imply that β is fully independent of worker skills, as I still allow β to depend on the skill type χ . Additionally, it does not rule out the possibility that β varies along unobserved skill dimensions; however, it does preclude systematic variation based on φ .

Assumption II.2 states that the non-wage amenity $a_j(\chi, \varphi)$ can be expressed as the sum of two components: one that is specific to the firm and the skill type, $a_{j\chi}$, and another that is specific to the skill level, $a_{\chi\varphi}$. This assumption ensures that amenities do not vary across firms with respect to unobserved aspects of worker skills. To better interpret the two amenity components $a_{j\chi}$ and $a_{\chi\varphi}$, I impose the normalization $E(a_{\chi\varphi}|\chi) = 0$ without loss of generality. This normalization allows me to define $a_{j\chi}$ as the average amenity at firm j for skill type χ .

Assumption II.3 requires that the average log skill level $E(\log \varphi|\chi, \tau)$ among workers in a given skill type χ is invariant over time τ . Note that this assumption does not restrict how the variance of these skill levels, or other higher-order moments, evolve. Moreover, it does not restrict the evolution of the distribution $F_{\chi|\tau}$, which governs workers' observable skill types. In my subsequent analysis, it will be useful to normalize φ by setting $E(\log \varphi|\chi) = 0$ for each $\chi \in \mathcal{X}$. This normalization is without loss of generality, as it is always possible to redefine the relative efficiencies $\{\theta_{j\chi}\}_{j,\chi}$ of each skill type to include the values of $\{E(\log \varphi|\chi)\}_{\chi}$.²⁰

Identification Results

I now provide results about the identification of firm-specific labor supply elasticities. First, I establish the identification of unobserved skill levels. Next, I outline the IV design used to recover the elasticities. Finally, I discuss the sources of instruments employed in my analysis.

Identification of Worker Skills

²⁰Specifically, if $E(\log \varphi|\chi) \neq 0$ for some skill type χ , then I can redefine $\log \theta_{j\chi}$ as $\log \theta_{j\chi} + \rho_j E(\log \varphi|\chi)$.

In order to draw inferences about the labor supply elasticities, I need to distinguish between workers with different skills. These skills are only partially observed in the data, however, as I do not directly measure a worker's skill level φ . To overcome this challenge, I rely on the restrictions implied by Assumption II. I demonstrate that, under this assumption, the skill levels φ are identified up to scale from observed differences in wages among workers at a given firm.

To establish identification, I first examine the implications of Assumption II for workers' labor supply decisions and the determination of wages at a firm. The key implication of this assumption is that the labor supply elasticities faced by a firm j do not vary along unobserved skill dimensions, i.e., $\varepsilon_{j\tau}(\chi, \varphi) = \varepsilon_{j\tau}(\chi)$. Thus, the wage markdowns at a firm j differ by workers' skill types χ , but not by their skill levels φ . This property implies the following result.

Proposition 1. Under Assumption II, the wage can be written as: $W_{j\tau}(\chi_i, \varphi_i) = \varphi_{i\tau} \times \psi_{j\chi\tau}$, where the term $\psi_{j\chi\tau}$ is uniform across workers in a given firm j , skill type χ , and period τ .

This result provides a way to compute workers' skill levels from data about workers' earnings. In particular, I can recover φ from cross-sectional wage differences in each firm and skill type:

$$\log \varphi_{i\tau} = \log W_{j\tau}(\chi_i, \varphi_i) - E [\log W_{j\tau}(\chi_i, \varphi_i) \mid j, \chi, \tau]. \quad (1.24)$$

Going forward, it will be useful to measure wages in terms of efficiency units. I define the effective wage for workers with skill type χ at firm j in period τ as $W_{j\tau}^{\text{eff}}(\chi) = W_{j\tau}(\chi, \varphi) / \varphi$.²¹ Using this notation, I can re-interpret the elasticity of labor supply to a firm as a function of the firm's effective wage. Specifically, I demonstrate that $\varepsilon_{j\tau}(\chi)$ takes the following form:

$$\varepsilon_{j\tau}(\chi) = \int \beta \times \frac{[\exp(\beta \log W_{j\tau}^{\text{eff}}(\chi)) / I_{\tau}^{\text{eff}}(\beta, \chi)] f_{\beta|\chi, \tau}(\beta|\chi, \tau)}{\int [\exp(\beta' \log W_{j\tau}^{\text{eff}}(\chi)) / I_{\tau}^{\text{eff}}(\beta', \chi)] f_{\beta|\chi, \tau}(\beta'|\chi, \tau) d\beta'} d\beta. \quad (1.25)$$

²¹ A firm's effective wages can be recovered in logs as follows: $\log W_{j\tau}^{\text{eff}}(\chi) = E [\log W_{j\tau}(\chi, \varphi) \mid j, \chi, \tau]$.

In this expression, $I_{\tau}^{\text{eff}}(\beta, \chi) = \sum_{k=1}^J \exp(\beta \log W_{k\tau}^{\text{eff}}(\chi) + a_{k\chi})$ can be interpreted as the effective wage index for workers with preference parameter β and skill type χ during time period τ .

Difference-in-Differences Estimand

I now present the main identification result of this subsection, which demonstrates how to recover the elasticity of labor supply to a firm using a difference-in-differences (DiD) research design. To motivate my approach, it is first important to recognize that I would not be able to determine the elasticities through either a cross-sectional comparison of treated and untreated firms or a comparison of outcomes within a firm over time. A cross-sectional comparison is not suitable because treated and untreated firms face different labor supply curves due to systematic variation in amenities. Moreover, a within-firm panel regression fails because of the spillover effects of TFP shocks on workers' wage indices and preferences. These spillovers imply that the same firm faces different labor supply curves in the pre- and post-periods.

I propose to recover the elasticity $\epsilon_{j\tau_1}(\chi)$ from a ratio of DiD estimands, which control for a worker's skill type and a firm's effective wage in the pre-period. I define this estimand below.

Definition. Let $(\ell_{j\tau_0}(\chi), w_{j\tau_0}^{\text{eff}}(\chi))$ and $(\ell_{j\tau_1}(\chi), w_{j\tau_1}^{\text{eff}}(\chi))$ be the log labor and effective wage for skill type χ at firm j in periods τ_0 and τ_1 , respectively. The estimand $\text{DiD}_{\tau_0, \tau_1}(w|\chi)$ equals:

$$\frac{\mathbb{E} \left[\ell_{j\tau_1}(\chi) - \ell_{j\tau_0}(\chi) | Z_j = 1, w_{j\tau_0}^{\text{eff}}(\chi) = w \right] - \mathbb{E} \left[\ell_{j\tau_1}(\chi) - \ell_{j\tau_0}(\chi) | Z_j = 0, w_{j\tau_0}^{\text{eff}}(\chi) = w \right]}{\mathbb{E} \left[w_{j\tau_1}^{\text{eff}}(\chi) - w_{j\tau_0}^{\text{eff}}(\chi) | Z_j = 1, w_{j\tau_0}^{\text{eff}}(\chi) = w \right] - \mathbb{E} \left[w_{j\tau_1}^{\text{eff}}(\chi) - w_{j\tau_0}^{\text{eff}}(\chi) | Z_j = 0, w_{j\tau_0}^{\text{eff}}(\chi) = w \right]}.$$

The numerator of this expression is a DiD estimand corresponding to changes in log labor at a firm, while the denominator is a DiD estimand for changes in effective log earnings at a firm.

To show that this estimand corresponds to the labor supply elasticity, I first prove that the common trends assumption holds. Specifically, I show that—after controlling for a firm j 's

initial effective wage $w_{j\tau_0}^{\text{eff}}(\chi)$ —the change in untreated potential outcomes over time, denoted by $w_{j\tau_1,0}^{\text{eff}}(\chi) - w_{j\tau_0,0}^{\text{eff}}(\chi)$ and $\ell_{j\tau_1,0}(\chi) - \ell_{j\tau_0,0}(\chi)$, can be written as a deterministic function of the firm-level structural parameters Γ_j , where $\Gamma_j = (\rho_j, \alpha_j, \{\theta_{j\chi}\}_\chi, \{a_j(X) - a_j(X')\}_{X,X'})'$. Assumption I ensures that a firm j 's treatment status Z_j does not depend on the parameters Γ_j . Therefore, under this assumption, treated and untreated firms with the same initial effective wages would, on average, exhibit the same change in their untreated potential outcomes.

Proposition 2. Suppose that Assumptions I and II hold. Then, given a firm's effective wage at τ_0 , the change in untreated potential outcomes is mean independent of the treatment status:

$$\begin{aligned} \mathbb{E} \left[\ell_{j\tau_1,0}(\chi) - \ell_{j\tau_0,0}(\chi) | Z_j = 1, w_{j\tau_0}^{\text{eff}}(\chi) = w \right] &= \mathbb{E} \left[\ell_{j\tau_1,0}(\chi) - \ell_{j\tau_0,0}(\chi) | Z_j = 0, w_{j\tau_0}^{\text{eff}}(\chi) = w \right]. \\ \mathbb{E} \left[w_{j\tau_1,0}^{\text{eff}}(\chi) - w_{j\tau_0,0}^{\text{eff}}(\chi) | Z_j = 1, w_{j\tau_0}^{\text{eff}}(\chi) = w \right] &= \mathbb{E} \left[w_{j\tau_1,0}^{\text{eff}}(\chi) - w_{j\tau_0,0}^{\text{eff}}(\chi) | Z_j = 0, w_{j\tau_0}^{\text{eff}}(\chi) = w \right]. \end{aligned}$$

Proposition 2 is central to my analysis. It implies that, by using a DiD research design, I can compare the treated and untreated potential outcomes for a firm in the post-period, thereby capturing the comparative statics from the ideal experiment. By this result, I demonstrate that the elasticity $\varepsilon_{j,\tau_1}(\chi)$, as defined in equation (2.22), is point identified from the estimand $\text{DiD}_{\tau_0,\tau_1}(w_{j\tau_0}^{\text{eff}}(\chi)|\chi)$ for any firm j and skill type χ . Moreover, by evaluating these estimands over the effective wage distribution, I can recover the labor supply elasticities faced by each firm in the economy. As I discuss in Section V, this task relies on an *overlap condition*, asserting that $\mathbb{P}(Z_j = 1 | w_{j\tau_0}^{\text{eff}}(\chi) = w) \in (0, 1)$ for all w in the support of $w_{j\tau_0}^{\text{eff}}(\chi)$. This condition guarantees that there are treated and untreated firms across the effective wage distribution.

Sources of Instruments

Up to this point, I have not specified how to construct the instrument Z_j . In my analysis, I consider both internal instruments—derived from existing data based on the model's implied restrictions—and external instruments, which draw on outside data sources. Both types of

instruments may be used within my framework. However, they rely on different assumptions.

My main specification utilizes internal instruments, which are constructed by specifying a process by which the firm experiences productivity shocks. This approach follows Lamadon et al. (2022) by assuming that the firm's TFP evolves according to an autoregressive process, which has a unit root. Specifically, for each time period τ , I assume that log TFP is given by:

$$t_{j\tau} = \bar{t}_j + \tilde{t}_{j\tau}, \quad \text{where} \quad \tilde{t}_{j\tau} = \tilde{t}_{j\tau-1} + u_{j\tau}. \quad (1.26)$$

To ensure that the productivity shocks do not lead to large jumps in wages and labor, I assume that $u_{j\tau}$ is small relative to the log wage levels at firm j . I also make the following assumption.

Assumption III (*Identification Assumptions for Internal Instruments*).

III.1. (*Instrument Relevance*) $\text{Var}(u_{j\tau}) > 0$.

III.2. (*Instrument Independence*) $u_{j\tau} \perp (\rho_j, \alpha_j, \{\theta_{j\chi}\}_\chi, \{a_j(X) - a_j(X')\}_{X,X'})$.

Assumption III.1 asserts that productivity shocks vary across firms, which will imply that the internal instrument is relevant. Assumption III.1 guarantees that the internal instrument will satisfy Assumption I, which is sufficient to ensure that the common trends assumption holds.

Given this setup, I can construct the internal instruments from differences in the log value added of firms, $\Delta y_{j\tau}$, within a given calendar year τ . For integers $p, p' \geq 2$, I define the pre-period as $\tau_0 = \tau - p$, representing p years before τ , and the post-period as $\tau_1 = \tau + p'$, representing p' years after τ . Under Assumption III, I derive the following moment condition:

$$\mathbb{E} \left[\Delta y_{j\tau} \left[(\ell_{j\tau_1}(\chi) - \ell_{j\tau_0}(\chi)) - \text{DiD}_{\tau_0, \tau_1}(w|\chi)(w_{j\tau_1}^{\text{eff}}(\chi) - w_{j\tau_0}^{\text{eff}}(\chi)) \right] \mid w_{j\tau_0}^{\text{eff}}(\chi) = w \right] = 0. \quad (1.27)$$

This moment condition reflects an IV regression of long differences in log employment on long differences in log effective wages, instrumented by short differences in log value added,

while controlling for effective wages in the pre-period. When implementing this procedure, I will define treatment status Z_j by separating firms at the median of the distribution of $\Delta y_{j\tau}$.²²

To support the analyses using internal instruments, I also consider an external instrument. This instrument is based on plausibly exogenous product demand shocks arising from public procurement auction outcomes in Norway. This supplementary analysis follows the research design used by H. de Frahan et al. (2024), where a firm is classified as treated ($Z_j = 1$) in the year τ when it wins its first public procurement contract. Since this instrument is defined from outside data sources, it does not require making assumptions about the evolution of firm TFP. However, the main limitation of this instrument is that data on auction outcomes is only available for a subset of firms in the sample and for a limited time period, from 2003 to 2018.

IV.B. Identification of Technology, Amenities, and Preferences

I now demonstrate how to recover the structural parameters that characterize firm productivity, technology, and non-wage firm amenities, as well as the distribution of workers' preferences. These parameters are needed to determine rents for workers and firms, and they also allow me to run counterfactual analyses exploring sources of allocative inefficiency and inequality.

Firm Productivity and Technology

I begin by proving identification of the production parameters $(T_{j\tau}, \{\theta_{j\chi}\}_\chi, \rho_j, \alpha_j)$ for any arbitrary firm j and any time period τ . Consistent with my previous analysis, I assume that the firm's TFP parameter $T_{j\tau}$ can evolve over time, while the parameters $(\{\theta_{j\chi}\}_\chi, \rho_j, \alpha_j)$ remain constant. In addition, I normalize the firm-specific efficiencies $\{\theta_{j\chi}\}_\chi$ by setting $\theta_{j\chi^*} = 1$ for some skill type $\chi^* \in \mathcal{X}$, as I can only recover these terms up to scale relative to the firm's TFP.

I demonstrate that identification relies on the ability to recover the labor supply elasticities

²²In the Appendix, I show how to extend this framework to allow for transitory measurement error in value added and labor. I require that the errors are mutually independent and also independent of firm-level parameters.

faced by the firm at two different time periods. This task is feasible in my context because I have access to a 28-year panel, enabling me to replicate the analysis from Section IV.A over multiple years. For the time periods τ and τ' , respectively, I define $\varepsilon_{j\tau}(\chi)$ and $\varepsilon_{j\tau'}(\chi)$ as the labor supply elasticities for firm j corresponding to skill type χ . Using these elasticities, I can evaluate the marginal products of labor at the firm. Specifically, for $\tilde{\tau} \in \{\tau, \tau'\}$, I define:

$$\text{MPL}_{j\tilde{\tau}}^{\text{eff}}(\chi) = \frac{1 + \varepsilon_{j\tilde{\tau}}(\chi)}{\varepsilon_{j\tilde{\tau}}(\chi)} \times W_{j\tilde{\tau}}^{\text{eff}}(\chi) \quad (1.28)$$

as the effective marginal product for skill type χ at firm j , adjusting for workers' skill levels φ .

Using the structural wage equation (2.11), I can recover the substitution parameter ρ_j by analyzing how the relative effective marginal products and effective labor shares of different skill types change over time at a firm j . This result is formalized by the following proposition.

Proposition 3. For each firm j , the elasticity of substitution between different skill types is:

$$(1 - \rho_j)^{-1} = \frac{\log \left(\frac{L_{j\tau}^{\text{eff}}(\chi)}{L_{j\tau}^{\text{eff}}(\chi')} \right) - \log \left(\frac{L_{j\tau'}^{\text{eff}}(\chi)}{L_{j\tau'}^{\text{eff}}(\chi')} \right)}{\log \left(\frac{\text{MPL}_{j\tau}^{\text{eff}}(\chi)}{\text{MPL}_{j\tau}^{\text{eff}}(\chi')} \right) - \log \left(\frac{\text{MPL}_{j\tau'}^{\text{eff}}(\chi)}{\text{MPL}_{j\tau'}^{\text{eff}}(\chi')} \right)}, \quad \text{for any } \chi, \chi' \in \mathcal{X}.$$

Proposition 3 provides a new expression for the elasticity of substitution in terms of identified quantities in the model. This expression reflects how the relative demands for two labor inputs depend on their relative productivities, as measured by the elasticity of $L_{j\tau}^{\text{eff}}(\chi)/L_{j\tau}^{\text{eff}}(\chi')$ with respect to $\text{MPL}_{j\tau}^{\text{eff}}(\chi)/\text{MPL}_{j\tau}^{\text{eff}}(\chi')$. In my framework, this relationship is specific to each firm.

Once the substitution parameter ρ_j is recovered, I prove that $\{\theta_{j\chi}\}_{\chi}$, α_j , and T_j can be computed sequentially from the effective marginal product, labor, and value added at the firm.

Proposition 4. For any firm j , the parameters $\{\theta_{j\chi}\}_{\chi}$, α_j , and T_j are defined by the equations:

$$\begin{aligned}\theta_{j\chi} &= \exp \left[\log \left(\frac{\text{MPL}_{j\tau}^{\text{eff}}(\chi)}{\text{MPL}_{j\tau}^{\text{eff}}(\chi^*)} \right) + (1 - \rho_j) \log \left(\frac{L_{j\tau}^{\text{eff}}(\chi)}{L_{j\tau}^{\text{eff}}(\chi^*)} \right) \right] \\ 1 - \alpha_j &= \exp \left[\log \text{MPL}_{j\tau}^{\text{eff}}(\chi) - y_{j\tau} - \log(\theta_{j\chi}) + (1 - \rho_j) \log L_{j\tau}^{\text{eff}}(\chi) + \log \sum_{\chi' \in \mathcal{X}} \theta_{j\chi'} \left(L_{j\tau}^{\text{eff}}(\chi') \right)^{\rho_j} \right] \\ T_{j\tau} &= \exp \left[y_{j\tau} - \frac{1 - \alpha_j}{\rho_j} \log \sum_{\chi \in \mathcal{X}} \theta_{j\chi} \left(L_{j\tau}^{\text{eff}}(\chi) \right)^{\rho_j} \right].\end{aligned}$$

Note that recovering the substitution parameter ρ_j is the only step that requires knowledge of labor supply elasticities for two different time periods. If this parameter is already identified, or if one assumes that there is perfect substitutability between skill types (i.e., $\rho_j = 1$), then the other parameters $\{\theta_{j\chi}\}_{\chi}$, α_j , and T_j can be recovered from a single time period of data.

Firm Amenities

Next, I prove identification of the parameters $\{a_{j\chi}\}_{j,\chi}$, which represent the average amenities at each firm j for any skill type χ . In my analysis, these terms are only identified up to scale, as they can be shifted in a way that does not impact workers' labor supply decisions. Therefore, I will normalize the amenities by defining a reference firm j^* such that $a_{j^*\chi} = 0$ for all $\chi \in \mathcal{X}$.

To recover firms' amenities, I use a revealed preference approach. This method is based on the premise that, after accounting for the differences in workers' wages, a firm that has superior amenities for a given skill type χ would attract more workers of that type. Consequently, by comparing the employment levels among firms in a counterfactual setting where all firms offer the same wages, I could draw inference about the differences in firms' amenities.

To make these comparisons, I utilize the labor supply elasticity curves $\varepsilon_{j\tau}(\chi, w)$, which are identified for each skill type χ by the estimand $\text{DiD}_{\tau_0, \tau_1}(w|\chi)$. By integrating along these

curves, I can infer the labor supply to any firm j if it were to offer a different wage, holding fixed all other wages in the economy. In doing so, I can evaluate the amenities at firm j by comparing j 's labor supply to that of the reference firm j^* in the case where j and j^* pay the same wages. To illustrate this identification strategy, consider the following proposition.

Proposition 5. Under Assumptions I and II, the average amenity for skill type χ at firm j is:

$$a_{j\chi} = \ell_{j\tau}(\chi) + \int_{w_{j\tau}^{\text{eff}}(\chi)}^{w_{j^*\tau}^{\text{eff}}(\chi)} \varepsilon_{j\tau}(\chi, w) dw - \ell_{j^*\tau}(\chi).$$

Worker Preferences

I conclude this section by establishing the identification of the densities $\{f_{\beta|\chi, \tau}\}_{\chi}$ that characterize the distribution of workers' marginal rates of substitution between wages and amenities. This identification is achieved through revealed preference, establishing a direct link between the distribution of workers' preferences and the shape of the labor supply curve faced by a firm.

To understand the source of identification, consider how the distribution of β influences a firm's labor supply curve. When β is homogeneous, this curve is isoelastic, and the elasticity of labor supply is equal to β . However, when β is heterogeneous, the curve is nonlinear, and the labor supply elasticity depends on the firm's wage. As shown by (2.12) and (2.13), the nature of this dependence carries information about the moments of the distribution of β .

In the following proposition, I demonstrate that this relationship extends even further. In particular, the elasticity curve $\varepsilon_{j\tau}(\chi, w)$ uniquely corresponds to the density $f_{\beta|\chi, \tau}(\beta|\chi, \tau)$.

Proposition 6. Given wages and labor $\{W_{j\tau}^{\text{eff}}(\chi), L_{j\tau}(\chi)\}_{j=1}^J$, there is a one-to-one mapping between the labor supply elasticity curve $\varepsilon_{j\tau}(\chi, w)$ and the density function $f_{\beta|\chi, \tau}(\beta|\chi, \tau)$.

Proposition 6 implies that the density $f_{\beta|\chi, \tau}$ is point identified under the same assumptions that are needed to recover the labor supply elasticity curve $\varepsilon_{j\tau}(\chi, w)$. The proof leverages

the fact that a firm's labor supply curve, as defined by (2.10), can be expressed as a Laplace transform. Moreover, I show that the inverse of this transform is proportional to the density of β . This property establishes the invertibility of a firm's labor supply curve to obtain $f_{\beta|\chi,\tau}$.²³

V. Estimation Strategy and Empirical Results

In this section, I describe my estimation procedure and present my empirical results, including estimates of firm-specific labor supply elasticities and structural parameters. My results show significant heterogeneity in the labor supply elasticities, indicating that wage markdowns for workers vary considerably both within and across firms. Additionally, I find that firms offering lower wages to workers in a given skill group are more likely to face smaller elasticities. This pattern aligns with the model's predictions, suggesting that firms with lower effective wages tend to exert greater wage-setting power. When comparing across skill groups, I find that the labor supply to firms is generally less elastic for more experienced workers. Finally, my estimates indicate that there is substantial variation in firm productivity, technology, and non-wage amenities, suggesting that these factors all contribute to systematic worker sorting.

V.A. Estimation of Firm-Specific Labor Supply Elasticities

This subsection addresses the estimation of labor supply elasticities. I begin by describing my methodology and my specification choices, followed by a discussion of the empirical findings.

Estimation Procedure and Specifications

My estimation procedure is based on the identification strategy outlined in Section IV.A. This approach consists of two key steps. First, I estimate workers' skill levels by implementing a two-way fixed effects estimator, regressing log wages on a worker fixed effect γ_i and a fixed effect $\psi_{\chi(i,\tau),j(i,\tau),\tau}$ for the skill type, firm, and year. From this regression, I define a worker's

²³Any two functions share the same Laplace transform only if they differ on a set of Lebesgue measure zero.

skill level φ_i as $\exp(\gamma_i)$ and the effective wage $W_{j\chi\tau}^{\text{eff}}$ as $\exp(\psi_{j(i,\tau),\chi(i,\tau),\tau})$. Table 2 presents basic summary statistics on workers' skill levels, with more details provided in the Appendix.

Next, I estimate the labor supply elasticities for each firm by computing a sample analogue of the estimand $\text{DiD}_{\tau_0,\tau_1}(w|\chi)$. This estimand corresponds to a ratio of DiD estimands, controlling for a firm's effective wage in the pre-period. Importantly, because there is a finite number of firms in the data, it is unlikely that I would observe multiple firms offering exactly the same effective wages. To overcome this issue, I employ a nonparametric Kernel estimator:

$$\frac{\sum_j K_{1,j}(w) \mathbf{1}\{Z_j = 1\} (\ell_{j\tau_1}(\chi) - \ell_{j\tau_0}(\chi)) - \sum_j K_{0,j}(w) \mathbf{1}\{Z_j = 0\} (\ell_{j\tau_1}(\chi) - \ell_{j\tau_0}(\chi))}{\sum_j K_{1,j}(w) \mathbf{1}\{Z_j = 1\} (w_{j\tau_1}^{\text{eff}}(\chi) - w_{j\tau_0}^{\text{eff}}(\chi)) - \sum_j K_{0,j}(w) \mathbf{1}\{Z_j = 0\} (w_{j\tau_1}^{\text{eff}}(\chi) - w_{j\tau_0}^{\text{eff}}(\chi))}, \quad (1.29)$$

where $K_{1,j}(w)$ and $K_{0,j}(w)$ denote the Kernel weights corresponding to treated and untreated populations, respectively. This estimator allows me to compare the outcomes of firms with similar, though not identical, initial wages, thereby ensuring that the method is tractable for finite sample sizes. In Appendix C.6, I prove that this estimator is consistent. Additionally, I derive the asymptotic properties of the estimator, demonstrating that it is capable of inference.

In my baseline specification, I utilize a Gaussian kernel function to construct the weights $K_{z,j}(w)$.²⁴ As I show in the Appendix, the point estimates are robust to alternative kernel functions, including the Uniform kernel, which corresponds to a standard nonparametric binning estimator. I also investigate the sensitivity of the estimates to the bandwidth choice, finding that the results remain relatively stable across varying bandwidths. Collectively, these robustness analyses provide confidence in the estimator's ability to approximate $\text{DiD}_{\tau_0,\tau_1}(w|\chi)$.

My primary estimates are based on internal instruments, following the same identifica-

²⁴Specifically, I set $K_{z,j}(w) = \frac{1}{\sqrt{2\pi}} \exp\left[-\frac{1}{2}\left(\frac{1}{h}(w_{j\tau_0}^{\text{eff}}(\chi) - w)\right)^2\right]$ where the parameter h governs the bandwidth.

tion arguments laid out in Section IV.A. For each year τ that TFP shocks occur, I define treatment status Z_j by separating firms at the median of the distribution of $\Delta y_{j\tau}$. I then define the pre- and post-periods to be $\tau_0 = \tau - 2$ and $\tau_1 = \tau + 3$, respectively. I estimate the model repeatedly using all years of available data, with the post-period τ_1 spanning from 2002 to 2017. As a robustness check, I also calibrate the model using estimates obtained with external instruments based on public procurement contract wins in Norway. These calibration results draw on estimates from H. de Frahan et al. (2024), who analyze the same Norwegian administrative dataset. I present my findings in the Appendix, along with a description of the data sources used in the original paper. Both internal and external instruments yield similar results.

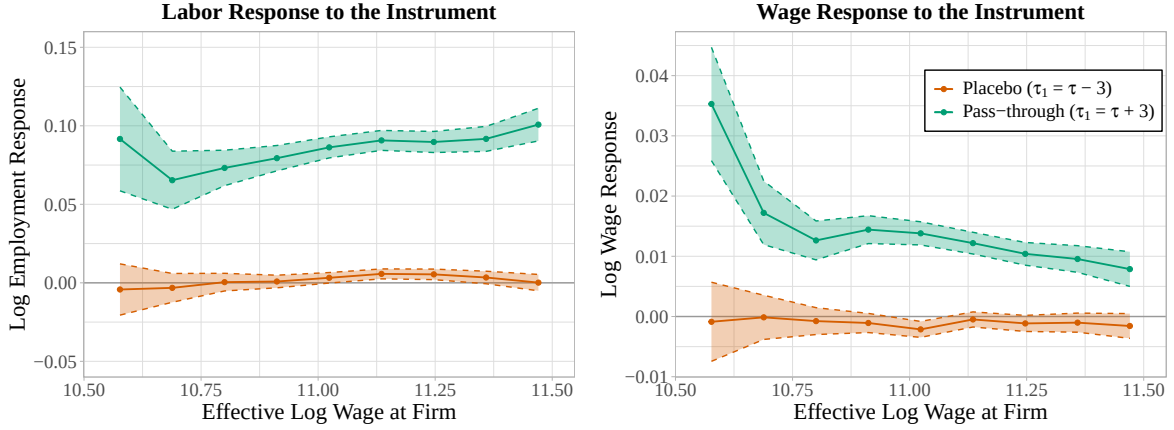
When implementing my approach, I include fixed effects for local labor markets, which represent the interactions between industries and commuting zones as defined in Section III. For comparison, I also estimate the model without fixed effects, treating Norway as one single labor market. Those results can be found in the Appendix. I average my estimates across all available years of data. Since my framework allows the labor supply elasticities to vary over time, my estimates may be interpreted as an average of these quantities across multiple years.

Description of Empirical Findings

Figure 3 presents the reduced form estimates, averaged across workers' skill types and plotted over the distribution of effective wages. These estimates correspond to the numerator and denominator of the DiD estimator that is defined in equation (1.29). In this figure, the green line displays estimates for $\tau_1 = \tau + 3$, while the orange line depicts the difference in pre-period trends between treated and untreated firms, with τ_1 set to $\tau - 3$. Notably, the placebo effect does not significantly differ from zero, lending further support to the validity of the IV design.

The reduced form estimates reveal two distinct patterns. First, the log labor response to the instrument consistently ranges between 5% and 10%, suggesting that employment growth

Figure 1.3: Average Pass-Through Rates by Effective Wage



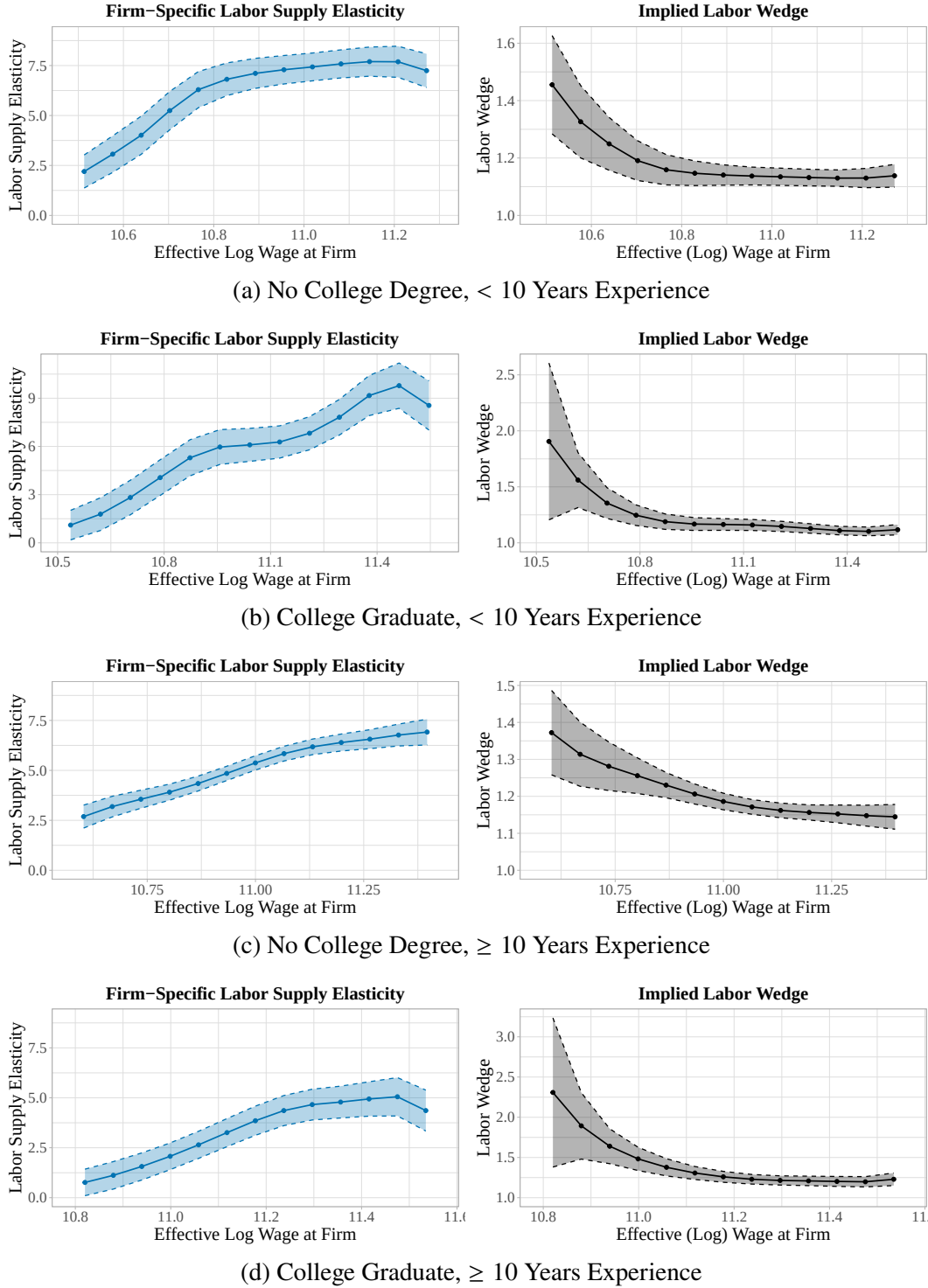
Notes. This figure plots the reduced form estimates, averaged across skill types and years $\tau_1 \in \{2002, \dots, 2017\}$. This specification includes market fixed effects. Standard errors are bootstrapped using 500 bootstrap samples.

is relatively independent of firm size. This observation aligns with Gibrat's law. Second, the log wage response ranges from 1% to 4%, and it decreases as a firm's effective wage rises. Based on equation (2.17), this pattern may suggest that higher-paying firms have lower returns to scale. Additionally, it could indicate that these firms face larger labor supply elasticities.

Figure 4 presents the IV estimates of firm-specific labor supply elasticities, disaggregated by workers' college attainment and experience. The figure also plots the implied labor wedge at each firm, defined as the ratio of the marginal product to a worker's wage. These estimates reveal that the labor supply elasticities differ substantially across firms, ranging between 1.5 and 10.0 for certain skill types. Moreover, they indicate that firms with higher effective wages face larger elasticities. This pattern is generally similar across worker skill types, and it implies that market power is more concentrated at the lower end of the effective wage distribution.

Although the estimated labor supply elasticities are broadly consistent across skill types, I observe notable differences based on a worker's experience. Specifically, I find that workers who have less job experience tend to exhibit larger labor supply elasticities. In addition, these elasticities are considerably more variable across firms. Under my model, these differences

Figure 1.4: IV Estimates of Labor Supply Elasticities and Labor Wedges



Notes. This figure plots IV estimates of firm-specific labor supply elasticities and labor wedges. These estimates are averaged across all years of available data, with $\tau_1 \in \{2002, \dots, 2017\}$. The subgroup “No College Degree” represents a worker-weighted average of two different skill types: “Lower Secondary” and “Upper Secondary”. This specification includes market fixed effects. Standard errors are bootstrapped using 500 bootstrap samples.

imply that less experienced workers are more likely to prioritize higher wages when choosing a job, and that there is also more dispersion in wage-amenity trade-offs among these workers.

To further validate my approach, I conduct two additional robustness checks. First, I test whether the instrument Z_j impacts labor on the intensive margin by influencing work hours. This test addresses a potential concern that firms respond to productivity shocks by adjusting workers' scheduled hours instead of their wages. If this were the case, then it would threaten the validity of my IV design, which assumes that the entire pass-through of demand shocks occurs in the form of wage adjustments, not changes in work hours. I find no evidence that these shocks significantly affect workers' hours, which reinforces the credibility of my assumptions.

Second, I test the assumption that firms are strategically small in the economy. A testable implication of my model is that firms offering identical wages face the same labor supply elasticities. However, this property need not hold if firms are strategically large. In a concentrated labor market, firms that offer the same wages could still face different labor supply elasticities if they have different market shares. Given this property, I test whether a firm's labor supply elasticity varies with respect to the market share, after controlling for the effective wages. I find no evidence of a market share effect, which lends support to the validity of my framework.

V.B. Estimation of Structural Parameters in the Model

Next, I present estimates of the structural parameters in the model, including the production parameters $(T_{j\tau}, \{\theta_{j\chi}\}_{\chi}, \rho_j, \alpha_j)$, the amenities $a_{j\chi}$, and the densities $f_{\beta|\chi, \tau}$ that characterize workers' preferences. These estimates are summarized in Table 2, as well as Figures 5 and 6.

Firm Productivity, Technology, and Amenities

To estimate the production parameters, I apply the formulas provided in Propositions 3 and 4, which rely on the elasticity estimates, the effective wages, and the value added for each firm.

I perform these calculations separately for every firm j and year τ , giving me a unique set of estimates for each combination of j and τ in the sample.²⁵ In Figure 5, I plot the distributions of firms' estimated TFPs $T_{j\tau}$, returns to scale $1 - \alpha_j$, and substitution parameters ρ_j , averaged across all years of data. The means and variances of these quantities are reported in Table 2. I observe significant variation in firms' returns to scale and substitution parameters, suggesting that differences in technology across firms may play an important role in wage determination.

Table 1.2: Estimates of Structural Parameters—Means and Variances

	Notation	Mean	Variance
<i>Panel A. Firm Production</i>			
Log Total Factor Productivity	$t_{j\tau}$	12.92 (0.05)	1.19 (0.01)
Returns to Scale	$1 - \alpha_j$	0.70 (0.01)	0.04 (0.01)
Substitution Parameter	ρ_j	0.72 (0.04)	0.54 (0.13)
<i>Panel B. Firm Amenities</i>			
	$a_{j\chi}$		
No College, < 10 Years Exp.		—	1.54 (0.06)
College, < 10 Years Exp.		—	2.02 (0.02)
No College, $\geq$ 10 Years Exp.		—	1.63 (0.11)
College, $\geq$ 10 Years Exp.		—	1.60 (0.04)
<i>Panel C. Worker Skill Levels</i>			
	φ_i		
No College, < 10 Years Exp.		—	0.062 (0.01)
College, < 10 Years Exp.		—	0.073 (0.02)
No College, $\geq$ 10 Years Exp.		—	0.064 (0.00)
College, $\geq$ 10 Years Exp.		—	0.076 (0.01)
<i>Panel D. Worker Preferences</i>			
	β_i		
No College, < 10 Years Exp.		6.30 (0.76)	13.76 (1.01)
College, < 10 Years Exp.		7.23 (1.05)	16.25 (2.71)
No College, $\geq$ 10 Years Exp.		5.96 (0.42)	15.71 (0.87)
College, $\geq$ 10 Years Exp.		4.28 (0.68)	15.11 (2.25)

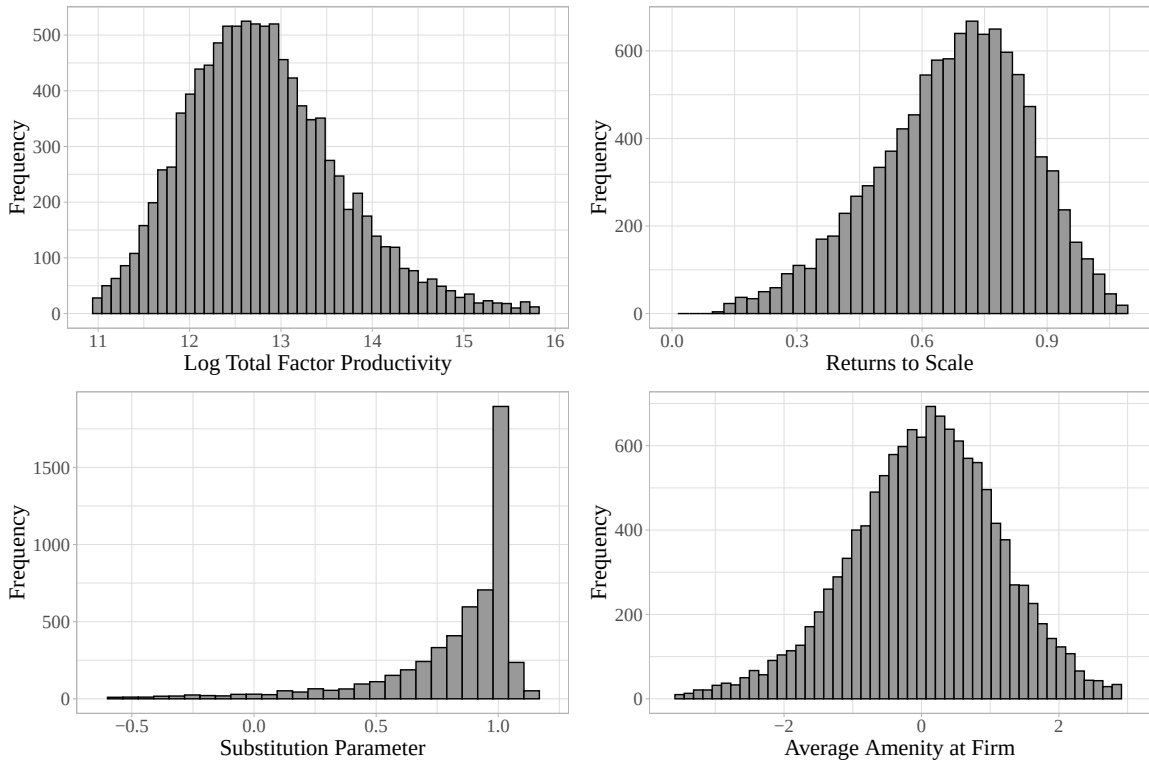
Notes. This table presents estimates of cross-sectional means and variances of model parameters, averaged across years. The means are omitted for parameters that are only identified up-to-scale.

To estimate non-wage amenities, I follow the procedure laid out in Proposition 5, which

²⁵Recall that computing the substitution parameter ρ_j for a firm requires me to compare estimates over two different time periods: τ and τ' . In my implementation, I define τ' as $\tau - 1$, corresponding to the previous year.

involves integrating the estimated labor supply elasticities with respect to a firm's effective wage.²⁶ This approach allows me to compare labor across firms in a counterfactual setting where firms offer identical wages. I plot the estimated distribution of firms' average amenities in Figure 5, with variances by subtype shown in Table 2. I find that the amount of variation in amenities is generally similar across worker types. However, for less experienced workers, it appears that those with a college education face greater variation in amenities across firms.

Figure 1.5: Histograms of Estimated Firm Productivity, Technology, and Amenities



Notes. This figure plots histograms of the estimated (log) total factor productivity $\log T_{j\tau}$, the returns to scale $1 - \alpha_j$, the substitution parameter ρ_j , and the average amenities $E(a_{j\chi}|j)$ across firms in the estimation sample.

Worker Preferences

To estimate the density function $f_{\beta|\chi,\tau}$, I draw on Proposition 6. This result establishes that the density is uniquely determined from the labor supply curve faced by an individual firm.

²⁶For each skill type χ , amenities are normalized relative to a “reference firm” j^* , where $a_{j^*\chi} = 0$. In my analysis, j^* is defined as the firm offering an average wage equal to the median of the distribution in the sample.

Consequently, $f_{\beta|\chi,\tau}$ may be recovered as the unique solution to the following equations:

$$L_{j\tau}(\chi, W) = \int \frac{\exp(\beta \log W + a_{j\chi})}{I_{\tau}^{\text{eff}}(\beta, \chi)} f_{\beta|\chi,\tau}(\beta|\chi, \tau) f_{\chi|\tau}(\chi|\tau) d\beta, \quad \text{for } W > 0. \quad (1.30)$$

Note that the labor supply curve $L_{j\tau}(\chi, W)$, the effective wage indices, $I_{\tau}^{\text{eff}}(\beta, \chi)$, the density of skill types $f_{\chi|\tau}(\chi|\tau)$, and the non-wage amenities $a_{j\chi}$ can all be recovered from the data.

To make the estimation problem tractable, I introduce two important modifications. First, given that the density function $f_{\beta|\chi,\tau}$ is an infinite-dimensional parameter, I will approximate it using a finite-dimensional basis expansion. Specifically, I replace $f_{\beta|\chi,\tau}$ with $\hat{f}_{\beta|\chi,\tau}$, where:

$$\hat{f}_{\beta|\chi,\tau}(b) = \sum_{k=1}^{d_{\phi}} \phi_k \hat{f}_{\beta|\chi,\tau}^{(k)}(b), \quad \text{for } \phi = (\phi_1, \dots, \phi_{d_{\phi}}) \in \Phi. \quad (1.31)$$

In this expression, ϕ is a d_{ϕ} -dimensional vector of unknown coefficients with support Φ , and $\{\hat{f}_{\beta|\chi,\tau}^{(k)}\}_k$ are basis functions known to the researcher. By defining the basis in this way, I can approximate $f_{\beta|\chi,\tau}$ with a function $\hat{f}_{\beta|\chi,\tau}$ that is parametrized by a finite-dimensional vector ϕ .

Second, rather than imposing an infinite number of constraints, I evaluate equation (1.30) on a finite grid of effective wages $\mathbf{W} \subset \mathbb{R}_+$, noting that identification is achieved as $|\mathbf{W}| \rightarrow \infty$. This modification allows me to recast (1.30) as a convex optimization problem, minimizing the ℓ^2 -distance between the estimated choice probabilities and predicted probabilities under the density $\hat{f}_{\beta|\chi,\tau}$. In particular, my estimation procedure involves choosing ϕ^* by solving:

$$\phi^* = \underset{\phi}{\operatorname{argmin}} \sum_{W \in \mathbf{W}} [\phi' \hat{h}_{\tau}(\chi, W) - \hat{L}_{j\tau}(\chi, W)]^2. \quad (1.32)$$

where $\hat{L}_{j\tau}(\chi, W)$ is the estimated labor supply for $W_{j\tau}^{\text{eff}}(\chi) = W$ and $\hat{h}_{\tau}(\chi, W) \in \mathbb{R}^{d_{\phi}}$ is a vector with entries $\hat{h}_{\tau,k}(\chi, W) = \int [\exp(\beta \log W + \hat{a}_{j\chi}) / \hat{I}_{\tau}^{\text{eff}}(\beta, \chi)] \hat{f}_{\beta|\chi,\tau}^{(k)}(\beta|\chi, \tau) \hat{f}_{\chi|\tau}(\chi|\tau) d\beta$.²⁷

²⁷Note that (1.32) may be written in matrix form as $\phi^* = \underset{\phi}{\operatorname{argmin}} \{\phi^{\top} \hat{h}^{\top} \hat{h} \phi - 2\phi^{\top} \hat{h}^{\top} \hat{L} + \hat{L}^{\top} \hat{L}\}$, where

In my baseline specification, I construct a basis using mixtures of $d_\phi = 18$ Gamma density functions. This type of density have been shown to be particularly effective for nonparametric estimation of functions that have positive support; e.g., see Wiper et al. (2001) for discussion. As a robustness check, I investigate the sensitivity of my resulting estimates to the choice of basis functions, the basis size d_ϕ , and the number of points $|\mathbf{W}|$ in the wage grid.²⁸ I find that the estimates remain robust to variation in each of these elements. Moreover, I demonstrate that the earnings and employment distributions predicted by the estimated densities $\hat{f}_{\beta|\chi,\tau}$ closely match those in the sample, which suggests that my estimates are a good fit for the data.

Figure 1.6: Density of Workers' Individual Labor Supply Elasticities



Notes. This figure plots the estimated density f_β , which corresponds to the distribution of worker's preferences over wages and amenities. The confidence region is bootstrapped using estimates from 500 bootstrap samples.

Figure 6 plots the estimated density of β among workers in the population. I also compare moments of this distribution for different worker types in Table 2. I find that more experienced workers tend to have lower values of β , which is associated with a greater willingness to pay for amenities. However, there is substantial variation in preferences within each worker type.

$\hat{\mathbf{L}} \in \mathbb{R}^{|\mathbf{W}|}$ and $\hat{\mathbf{h}} \in \mathbb{R}^{|\mathbf{W}| \times d_\phi}$. This framing of the problem is more convenient for computational implementation.

²⁸For example, I also approximate $f_{\beta|\chi,\tau}$ using a linear sieve constructed from Laguerre polynomials. These functions are capable of approximating any square-integrable function with positive support (see Chen, 2007).

VI. Implications for Rent Sharing, Misallocation, and Inequality

I now summarize the main empirical findings from my estimates, quantifying imperfect competition in the labor market and assessing its implications for welfare and earnings inequality.

VI.A. Quantifying Imperfect Competition

Table 3 presents the estimates of wage markdowns, along with worker and employer rents. On average, I find that workers are paid approximately 86% of their marginal product. However, these markdowns vary both within and across firms, ranging from approximately 70% to 95% at the 1st and 99th percentiles. I estimate that about 60% of the variation in markdowns takes place within firms, while the remaining 40% is due to differences between firms. The variation within firms is explained by systematic differences in workers' wage-amenity trade-offs across different skill types χ . Meanwhile, the variation between firms arises from differences in wage-amenity trade-offs for equally-skilled workers. Note that this variation would be overlooked without accounting for unobserved heterogeneity in workers' wage-amenity trade-offs.

Table 1.3: Estimates of Wage Markdowns and Rents

	Mean	Std. Dev.	Decomposition of Variation	
			Within Firm	Between Firm
<i>Wage Markdown</i>	0.864 (0.011)	0.035 (0.025)	61.9%	38.1%
<i>Worker Rents</i>				
Per-worker Dollars	\$13,469 (788)	\$2,859 (2,662)	70.5%	29.5%
Share of Earnings	17.1% (0.9)			
<i>Firm Rents</i>				
Per-worker Dollars	\$9,403 (1,451)	\$4,763 (3,370)	0.0%	100.0%
Share of Profits	29.2% (2.6)			

Notes. This table presents estimates of wage markdowns and rents. Variance decompositions are calculated via the Law of Total Variance. Standard errors, shown in parentheses, are bootstrapped using 500 bootstrap samples.

I find that workers and firms capture substantial rents, with worker rents being larger, on average, than firm rents. I estimate that the average worker is willing to pay approximately \$13,500—or 17% of their earnings—to avoid working for a different firm. In contrast, the average firm obtains about \$9,400 per worker—or 29% of its profit—by exerting wage-setting power. Figure 17 of the Appendix plots the cross-sectional densities of estimated worker and firm rents, averaged across years. I find that these rents vary considerably in the economy.

VI.B. *Misallocation and Inequality*

To evaluate the welfare impact of imperfect competition, I compare the equilibrium outcomes in the monopsonistic economy to counterfactual outcomes that would arise in a competitive (Walrasian) economy, where firms do not exert any wage-setting power. In my framework, misallocation occurs because firms vary in their wage-setting power. This variation can lead workers to sort into firms that they would not have chosen in a competitive labor market.

Table 1.4: Estimates of Counterfactual Comparisons

	Monopsonistic Labor Market	Competitive Labor Market	% Difference in Dollars
Total Welfare (log dollars)	12.71 (0.21)	12.81 (0.23)	9.58%
Mean Earnings (log dollars)	11.29 (0.00)	11.46 (0.01)	15.23%
<u>Variance in Log Earnings</u>			
Total	0.112 (0.00)	0.110 (0.01)	-2.29%
Between Firms	0.028 (0.00)	0.025 (0.01)	-11.12%
Within Firms	0.084 (0.00)	0.085 (0.00)	0.63%

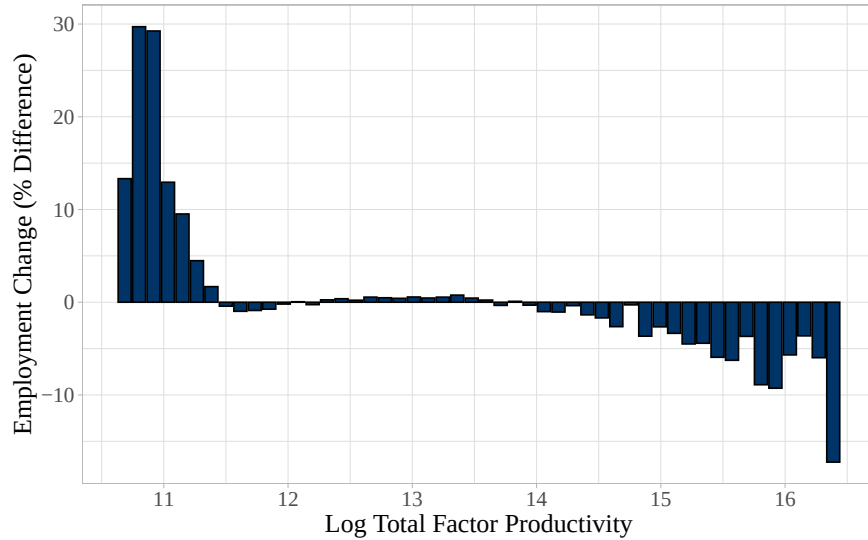
Notes. This table compares outcomes in the monopsonistic economy to counterfactual outcomes in a competitive (Walrasian) economy. The standard errors, shown in parentheses, are bootstrapped using 500 bootstrap samples.

Table 4 presents estimates for counterfactual comparisons. These estimates suggest that imperfect competition produces large misallocation of workers to firms, resulting in a 9.58% reduction in total welfare in dollar terms. Furthermore, in a competitive labor market, average earnings would increase by 15.23%. However, despite these changes, the overall impact on

wage inequality, as measured by the total variance in log earnings, appears to be minimal.

To understand how workers would be reallocated in a competitive labor market, recall that I estimate larger markdowns for firms that have lower effective wages. This suggests that, relative to the competitive benchmark, workers are being allocated away from less productive firms. Indeed, I find that the correlation between a firm's log labor $\ell_{j\tau}$ and its log TFP $t_{j\tau}$ is higher (0.155) in a monopsonistic market than it would be in a competitive market (0.089). Moreover, in Figure 6, I plot the changes in estimated labor shares that would occur if labor wedges were eliminated across the distribution of firm TFP. This figure illustrates that, under the competitive allocation, more workers would be employed at firms with lower productivities. Meanwhile, fewer workers would be employed at firms with higher productivities.

Figure 1.7: Estimated Reallocation of Labor by Eliminating Labor Wedges



Notes. This figure depicts the reallocation of labor that results from eliminating labor wedges. It plots the estimated percent change in employment across different groups of firms, organized by their estimated log TFPs.

VII. Conclusions

In this paper, I examine how an employer's wage-setting power depends on the characteristics of its workforce, and I assess the consequences of this relationship for welfare and inequality.

To this end, I develop a structural model that integrates labor market frictions with individual heterogeneity in workers' trade-offs between wages and non-wage job characteristics. I show that these two factors, taken together, cause employers to face varying labor supply elasticities, resulting in differences in labor market power both within and across firms. Moreover, I show that this framework is empirically tractable and can be used to draw inferences about sorting, imperfect competition, and rent sharing. I estimate the model using a matched worker-firm panel dataset that covers the universe of workers and firms in Norway between 1995 and 2018.

One key empirical finding from my estimates is that, after accounting for workers' skills, firms with lower wages tend to have more labor market power. My model offers an explanation for this finding. That is, when workers make different trade-offs over job attributes, they tend to sort accordingly, with workers who prioritize amenities (e.g., flexible work hours, fewer physical demands, and/or more societal impact) choosing jobs that pay less. In a labor market that is imperfectly competitive, these sorting patterns lead lower-paying firms to attract labor that is less elastic (i.e., less sensitive to wage changes), thereby increasing wage-setting power. This phenomenon has important welfare implications, because the disparities in market power between firms generate misallocation of labor relative to the competitive benchmark. In a perfectly competitive labor market, more individuals would be employed at low-wage firms.

While this paper offers an initial examination of the relationship between workers' wage-amenity trade-offs and labor market power, there is much room for future work. For example, my framework does not consider the possibility that firms strategically compete for workers by internalizing their market shares. I justify this assumption by testing whether a firm's market share impacts its labor supply elasticities after controlling for wages, finding no significant effect. However, this test does not definitively rule out the existence of strategic interactions. A meaningful extension of my framework would be to examine how a firm's wage-setting power is affected not only by the composition of its workforce but also by its market share.

Appendix

A. Derivation of Equilibrium Quantities and Model Extensions

A.1. Firm Labor Supply

Given a set of wage offers $\mathbf{W}(X) = \{W_k(X)\}_{k=1}^J$, a worker with skills X whose marginal utility of (log) earnings equals β would choose to work for an employer j with probability:

$$\begin{aligned}
 P(j(i) = j | \beta, X) &= P\left(u_{ij}(W_j(X_i), a_j(X_i)) > \max_{k \neq j} \{u_{ik}(W_k(X_i), a_k(X_i))\} \mid \beta_i = \beta, X_i = X\right) \\
 &= P\left(\beta \log W_j(X) + a_j(X) + \epsilon_{ij} > \beta \log W_k(X) + a_k(X) + \epsilon_{ik}, \forall k \neq j\right) \\
 &= P\left(\epsilon_{ik} < \beta \log \left(\frac{W_j(X)}{W_k(X)}\right) + a_j(X) - a_k(X) + \epsilon_{ij}, \forall k \neq j, \forall k \neq j\right) \\
 &= \int_{-\infty}^{\infty} P\left(\epsilon_{ik} < \beta \log \left(\frac{W_j(X)}{W_k(X)}\right) + a_j(X) - a_k(X) + \epsilon_{ij}, \forall k \neq j \mid \epsilon_{ij} = \tilde{\epsilon}\right) f_{\epsilon_{ij}}(\tilde{\epsilon}) d\tilde{\epsilon} \\
 &= \int_{-\infty}^{\infty} \prod_{k \neq j} P\left(\epsilon_{ik} < \beta \log \left(\frac{W_j(X)}{W_k(X)}\right) + a_j(X) - a_k(X) + \epsilon_{ij} \mid \epsilon_{ij} = \tilde{\epsilon}\right) f_{\epsilon_{ij}}(\tilde{\epsilon}) d\tilde{\epsilon} \\
 &= \int_{-\infty}^{\infty} \prod_{k \neq j} F_{\epsilon}\left(\beta \log \left(\frac{W_j(X)}{W_k(X)}\right) + a_j(X) - a_k(X) + \tilde{\epsilon}\right) f_{\epsilon}(\tilde{\epsilon}) d\tilde{\epsilon}.
 \end{aligned}$$

Here, the second-to-last equality holds because $\{\epsilon_{ij}\}_{i,j}$ are independent and the final equality holds because ϵ_{ij} is identically distributed. The mass of workers with skills X at firm j is:

$$S_j(X) = \int \left(\int_{-\infty}^{\infty} \prod_{k \neq j} F_{\epsilon}\left(\beta \log \left(\frac{W_j(X)}{W_k(X)}\right) + a_j(X) - a_k(X) + \tilde{\epsilon}\right) f_{\epsilon}(\tilde{\epsilon}) d\tilde{\epsilon} \right) f_{\beta, X}(\beta, X) d\beta.$$

Under a logit error structure, where $F_{\epsilon}(\epsilon) = \exp(-\exp(-\epsilon))$, the choice probability becomes:

$$P(j(i) = j | \beta, X) = \frac{\exp(\beta \log W_j(X) + a_j(X))}{\sum_{k=1}^J \exp(\beta \log W_k(X) + a_k(X))}.$$

Define $I(\beta, X) = \sum_{k=1}^J \exp(\beta \log W_k(X) + a_k(X))$ as the wage index for this type of worker.

For workers with skills $X_i = X$, the total mass of workers supplied to an employer j equals:

$$S_j(X) = \int \frac{1}{I(\beta, X)} \exp(\beta \log W_j(X) + a_j(X)) f_{\beta, X}(\beta, X) d\beta.$$

A.2. Equilibrium Wage Equation

Each employer j in the market chooses a set of wages $\{W_j(\chi, \varphi)\}_{\chi, \varphi}$ to maximize profit:

$$\Pi_j = T_j \left(\sum_{\chi \in \mathcal{X}} \theta_{j\chi} \left(\int \varphi D_j(\chi, \varphi) d\varphi \right)^{\rho_j} \right)^{\frac{1-\alpha_j}{\rho_j}} - \sum_{\chi \in \mathcal{X}} \left(\int W_j(\chi, \varphi) D_j(\chi, \varphi) d\varphi \right).$$

In this expression, $D_j(\chi, \varphi)$ is the labor demand for skills $X = (\chi, \varphi)$, which equals the labor supply curve $S_j(\chi, \varphi)$ in equilibrium. Plugging in these curves, the first order condition is:

$$\begin{aligned} \frac{\partial \Pi_j}{\partial W_j(\chi, \varphi)} &= \varphi T_j (1 - \alpha_j) \theta_{j\chi} \left(L_j^{\text{eff}}(\chi) \right)^{\rho_j - 1} \left(\sum_{\chi' \in \mathcal{X}} \theta_{j\chi'} \left(L_j^{\text{eff}}(\chi') \right)^{\rho_j} \right)^{\frac{1-\alpha_j}{\rho_j} - 1} \frac{\partial L_j(\chi, \varphi)}{\partial W_j(\varphi, \chi)} \\ &\quad - L_j(\chi, \varphi) - W_j(\chi, \varphi) \frac{\partial L_j(\chi, \varphi)}{\partial W_j(\varphi, \chi)} = 0, \quad \text{for all } (\chi, \varphi) \in \mathcal{X} \times \mathbb{R}, \end{aligned}$$

where $L_j^{\text{eff}}(\chi) = \int \varphi L_j(\chi, \varphi) d\varphi$ denotes the efficiency units of labor for a given skill type χ .

Let $\varepsilon_j(X) = \partial \log L_j(X) / \partial \log W_j(X)$ be the labor supply elasticity for workers with skills $X = (\chi, \varphi)$. As $\varepsilon_j(X) = \partial L_j(X) W_j(X) / \partial W_j(X) L_j(X)$, the first-order condition is:

$$W_j(\chi, \varphi) = \frac{\varepsilon_j(\chi, \varphi)}{1 + \varepsilon_j(\chi, \varphi)} \times \varphi T_j (1 - \alpha_j) \theta_{j\chi} \left(L_j^{\text{eff}}(\chi) \right)^{\rho_j - 1} \left(\sum_{\chi' \in \mathcal{X}} \theta_{j\chi'} \left(L_j^{\text{eff}}(\chi') \right)^{\rho_j} \right)^{\frac{1-\alpha_j}{\rho_j} - 1}.$$

For workers with skills X , the wage markdown at a firm j is $\frac{W_j(X)}{\partial Y_j / \partial L_j(X)} = \varepsilon_j(X) / (1 + \varepsilon_j(X))$.

Lemmas 1 and 2 establish the existence of an equilibrium corresponding to a unique set of profit-maximizing wages $\{W_j(X)\}_X$ for each firm j . In this equilibrium, the firm's problem has an interior solution, so wages satisfy the first-order condition. In logs, this condition is:

$$\begin{aligned} w_j(\chi, \varphi) &= \log \varphi + \log T_j + \log(1 - \alpha_j) + \log \theta_{j\chi} - \log(1 + \varepsilon_j^{-1}(\chi, \varphi)) \\ &\quad - (1 - \rho_j) \log L_j^{\text{eff}}(\chi) + \frac{1 - \alpha_j - \rho_j}{\rho_j} \log \sum_{\chi' \in \mathcal{X}} \theta_{j\chi'} \left(L_j^{\text{eff}}(\chi') \right)^{\rho_j}. \end{aligned}$$

A.3. Firm Labor Supply Elasticity and Higher-Order Derivatives

I now derive the firm-specific elasticity of labor supply, as well as higher derivatives of the labor supply curves, from the perspective of an employer that views itself as strategically

small. In particular, I assume that the firm does not internalize the impact of its own wage on a worker i 's wage index, i.e., $\partial I(\beta, X)/\partial W_j(X) = 0$. As a first step, I establish two claims.

Claim A.3.1. For any function $g : \mathbb{R}_+ \rightarrow \mathbb{R}$ taking values in the support of β , it follows that:

$$\frac{\partial E_X(g(\beta) \exp(\beta w_j(X) + a_j(X))/I_i(X))}{\partial w_j(X)} = E_X(\beta g(\beta) \exp(\beta w_j(X) + a_j(X))/I(\beta, X)).$$

Proof. The derivative of $E_X(g(\beta) \exp(\beta w_j(X) + a_j(X))/I(\beta, X))$ with respect to $w_j(X)$ is:

$$\begin{aligned} \frac{\partial E_X(g(\beta) \exp(\beta w_j(X) + a_j(X))/I_i(X))}{\partial w_j(X)} &= \int \frac{\partial}{\partial w_j(X)} \left[\frac{g(\beta) \exp(\beta w_j(X) + a_j(X)) f_{\beta|X}(\beta|X)}{I(\beta, X)} \right] d\beta \\ &= \int \frac{\beta g(\beta) \exp(\beta w_j(X) + a_j(X)) f_{\beta|X}(\beta|X)}{I_i(\beta, X)} d\beta \\ &= E_X(\beta g(\beta) \exp(\beta w_j(X) + a_j(X))/I(\beta, X)). \end{aligned}$$

□

Claim A.3.2. For any function $g : \mathbb{R}_+ \rightarrow \mathbb{R}$ taking values in the support of β , it follows that:

$$\frac{E_X(g(\beta) \exp(\beta w_j(X) + a_j(X))/I(\beta, X))}{E_X(\exp(\beta w_j(X) + a_j(X))/I(\beta, X))} = E_X(g(\beta)|j(i) = j).$$

Proof. Given that $P(j(i) = j|\beta, X)$ is equal to $\exp(\beta \log W_j(X) + a_j(X))/I(\beta, X)$, I write:

$$\frac{E_X(g(\beta) \exp(\beta w_j(X) + a_j(X))/I(\beta, X))}{E_X(\exp(\beta w_j(X) + a_j(X))/I(\beta, X))} = \frac{E_X(g(\beta) P(j(i) = j|\beta, X))}{P(j(i) = j|X)}.$$

Using the Law of Iterated Expectations, I can decompose this quantity in the following way:

$$\begin{aligned} \frac{E_X(g(\beta) P(j(i) = j|\beta, X))}{P(j(i) = j|X)} &= \frac{E_X(g(\beta) P(j(i) = j|\beta, X)|j(i) = j)}{P(j(i) = j|X)} \times P(j(i) = j|X) \\ &\quad + \frac{E_X(g(\beta) P(j(i) = j|\beta, X)|j(i) \neq j)}{P(j(i) = j|X)} \times [1 - P(j(i) = j|X)] \\ &= E_X(g(\beta) \times 1|j(i) = j) + \frac{E_X(g(\beta) \times 0|j(i) \neq j)}{P(j(i) = j|X)} \times [1 - P(j(i) = j|X)] \\ &= E_X(g(\beta)|j(i) = j). \end{aligned}$$

□

To simplify notation going forward, I define the function $\tau_{jX,s} : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ such that:

$$\tau_{jX,s}(w_j(X)) = E_X(\beta^s \exp(\beta w_j(X) + a_j(X)) / I(\beta, X)).$$

Claim A.3.1 implies that: $\partial \tau_{jX,s}(w_j(X)) / \partial w_j(X) = \tau_{jX,s+1}(w_j(X))$. Claim A.3.2 implies that: $\tau_{jX,s}(w_j(X)) / \tau_{jX,0}(w_j(X)) = E_X(\beta^s | j(i) = j)$. Using these properties, I derive the first five derivatives of the (log) labor supply curve $\ell_j(X)$ with respect to (log) wage $w_j(X)$.

First Derivative. The labor supply elasticity for a firm is $\epsilon_j(X) = \partial \ell_j(X) / \partial w_j(X)$, where:

$$\begin{aligned} \frac{\partial \ell_j(X)}{\partial w_j(X)} &= \frac{\partial \log(\tau_{jX,0}(w_j(X)) f_X(X))}{\partial w_j(X)} \\ &= \frac{\partial \log(\tau_{jX,0}(w_j(X)))}{\partial w_j(X)} \\ &= \frac{\tau_{jX,1}(w_j(X))}{\tau_{jX,0}(w_j(X))} \\ &= E_X(\beta_i | j(i) = j). \end{aligned}$$

As a consequence of Claim A.3.2, this elasticity can be decomposed in the following way:

$$\begin{aligned} \frac{\partial \ell_j(X)}{\partial w_j(X)} &= \frac{E_X(\beta \exp(\beta w_j(X) + a_j(X)) / I(\beta, X))}{E_X(\exp(\beta w_j(X) + a_j(X)) / I(\beta, X))} \\ &= E_X(\beta) + \frac{\text{Cov}_X(\beta, \exp(\beta w_j(X) + a_j(X)) / I(\beta, X))}{E_X(\exp(\beta w_j(X) + a_j(X)) / I(\beta, X))}. \end{aligned}$$

Second Derivative. The second derivative of the labor supply curve $\partial^2 \ell_j(X) / \partial w_j^2(X)$ is:

$$\begin{aligned} \frac{\partial^2 \ell_j(X)}{\partial w_j^2(X)} &= \frac{\partial(\tau_{jX,1}(w_j(X)) / \tau_{jX,0}(w_j(X)))}{\partial w_j(X)} \\ &= \frac{\tau_{jX,2}(w_j(X)) \tau_{jX,0}(w_j(X)) - \tau_{jX,1}^2(w_j(X))}{\tau_{jX,0}^2(w_j(X))} \\ &= \frac{\tau_{jX,2}(w_j(X))}{\tau_{jX,0}(w_j(X))} - \left(\frac{\tau_{jX,1}(w_j(X))}{\tau_{jX,0}(w_j(X))} \right)^2 \\ &= E_X(\beta_i^2 | j(i) = j) - E_X(\beta_i | j(i) = j)^2 \\ &= \text{Var}_X(\beta_i | j(i) = j). \end{aligned}$$

Third Derivative. The third derivative of the labor supply curve $\partial^3 \ell_j(X)/\partial w_j^3(X)$ is:

$$\begin{aligned}
\frac{\partial^3 \ell_j(X)}{\partial w_j^3(X)} &= \frac{\partial}{\partial w_j(X)} \left(\frac{\tau_{jX,2}(w_j(X))\tau_{jX,0}(w_j(X)) - \tau_{jX,1}^2(w_j(X))}{\tau_{jX,0}^2(w_j(X))} \right) \\
&= \frac{\tau_{jX,3}(w_j(X))\tau_{jX,0}^3(w_j(X)) - 3\tau_{jX,2}(w_j(X))\tau_{jX,1}(w_j(X))\tau_{jX,0}^2(w_j(X)) + 2\tau_{jX,1}^3(w_j(X))\tau_{jX,0}(w_j(X))}{\tau_{jX,0}^4(w_j(X))} \\
&= \frac{\tau_{jX,3}(w_j(X))}{\tau_{jX,0}(w_j(X))} - 3 \left(\frac{\tau_{jX,2}(w_j(X))}{\tau_{jX,0}(w_j(X))} \right) \left(\frac{\tau_{jX,1}(w_j(X))}{\tau_{jX,0}(w_j(X))} \right) + 2 \left(\frac{\tau_{jX,1}(w_j(X))}{\tau_{jX,0}(w_j(X))} \right)^3 \\
&= E_X(\beta_i^3 | j(i) = j) - 3 E_X(\beta_i | j(i) = j) E_X(\beta_i^2 | j(i) = j) + 2 E_X(\beta_i | j(i) = j)^3 \\
&= E_X([\beta_i - E_X(\beta_i | j(i) = j)]^3 | j(i) = j).
\end{aligned}$$

Fourth Derivative. The fourth derivative of the labor supply curve $\partial^4 \ell_j(X)/\partial w_j^4(X)$ is:

$$\begin{aligned}
\frac{\partial^4 \ell_j(X)}{\partial w_j^4(X)} &= \frac{\partial}{\partial w_j(X)} \left(\frac{\tau_{jX,3}(w_j(X))\tau_{jX,0}^3(w_j(X)) - 3\tau_{jX,2}(w_j(X))\tau_{jX,1}(w_j(X))\tau_{jX,0}^2(w_j(X)) + 2\tau_{jX,1}^3(w_j(X))\tau_{jX,0}(w_j(X))}{\tau_{jX,0}^4(w_j(X))} \right) \\
&= \frac{\tau_{jX,4}(w_j(X))\tau_{jX,0}^7(w_j(X)) - 4\tau_{jX,3}(w_j(X))\tau_{jX,1}(w_j(X))\tau_{jX,0}^6(w_j(X)) - 3\tau_{jX,2}^2(w_j(X))\tau_{jX,0}^6(w_j(X))}{\tau_{jX,0}^8(w_j(X))} \\
&\quad + \frac{12\tau_{jX,2}(w_j(X))\tau_{jX,1}^2(w_j(X))\tau_{jX,0}^5(w_j(X)) - 6\tau_{jX,1}^4(w_j(X))\tau_{jX,0}^4(w_j(X))}{\tau_{jX,0}^8(w_j(X))} \\
&= \frac{\tau_{jX,4}(w_j(X))}{\tau_{jX,0}(w_j(X))} - 4 \left(\frac{\tau_{jX,3}(w_j(X))}{\tau_{jX,0}(w_j(X))} \right) \left(\frac{\tau_{jX,1}(w_j(X))}{\tau_{jX,0}(w_j(X))} \right) - 3 \left(\frac{\tau_{jX,2}(w_j(X))}{\tau_{jX,0}(w_j(X))} \right)^2 \\
&\quad + 12 \left(\frac{\tau_{jX,2}(w_j(X))}{\tau_{jX,0}(w_j(X))} \right) \left(\frac{\tau_{jX,1}(w_j(X))}{\tau_{jX,0}(w_j(X))} \right)^2 - 6 \left(\frac{\tau_{jX,1}(w_j(X))}{\tau_{jX,0}(w_j(X))} \right)^4 \\
&= E_X(\beta_i^4 | j(i) = j) - 4 E_X(\beta_i | j(i) = j) E_X(\beta_i^3 | j(i) = j) - 3 E_X(\beta_i^2 | j(i) = j)^2 \\
&\quad + 12 E_X(\beta_i | j(i) = j)^2 E_X(\beta_i^2 | j(i) = j) - 6 E_X(\beta_i | j(i) = j)^4 \\
&= E_X([\beta_i - E_X(\beta_i | j(i) = j)]^4 | j(i) = j) - 3 \text{Var}_X(\beta_i | j(i) = j)^2.
\end{aligned}$$

Fifth Derivative. The fifth derivative of the labor supply curve $\partial^5 \ell_j(X)/\partial w_j^5(X)$ is:

$$\begin{aligned}
\frac{\partial^5 \ell_j(X)}{\partial w_j^5(X)} &= \frac{\partial}{\partial w_j(X)} \left(\frac{\tau_{jX,4}(w_j(X))\tau_{jX,0}^7(w_j(X)) - 4\tau_{jX,3}(w_j(X))\tau_{jX,1}(w_j(X))\tau_{jX,0}^6(w_j(X)) - 3\tau_{jX,2}^2(w_j(X))\tau_{jX,0}^6(w_j(X))}{\tau_{jX,0}^8(w_j(X))} \right. \\
&\quad \left. + \frac{12\tau_{jX,2}(w_j(X))\tau_{jX,1}^2(w_j(X))\tau_{jX,0}^5(w_j(X)) - 6\tau_{jX,1}^4(w_j(X))\tau_{jX,0}^4(w_j(X))}{\tau_{jX,0}^8(w_j(X))} \right) \\
&= \frac{\tau_{jX,5}(w_j(X))\tau_{jX,0}^{15}(w_j(X))}{\tau_{jX,0}^{16}(w_j(X))} \\
&\quad - \frac{5\tau_{jX,4}(w_j(X))\tau_{jX,1}(w_j(X))\tau_{jX,0}^{14}(w_j(X)) - 20\tau_{jX,3}(w_j(X))\tau_{jX,1}^2(w_j(X))\tau_{jX,0}^{13}(w_j(X))}{\tau_{jX,0}^{16}(w_j(X))} \\
&\quad - \frac{10\tau_{jX,3}(w_j(X))\tau_{jX,2}(w_j(X))\tau_{jX,0}^{14}(w_j(X)) - 30\tau_{jX,2}^2(w_j(X))\tau_{jX,1}(w_j(X))\tau_{jX,0}^{13}(w_j(X))}{\tau_{jX,0}^{16}(w_j(X))} \\
&\quad - \frac{60\tau_{jX,2}(w_j(X))\tau_{jX,1}^3(w_j(X))\tau_{jX,0}^{12}(w_j(X)) - 24\tau_{jX,1}^5(w_j(X))\tau_{jX,0}^{11}(w_j(X))}{\tau_{jX,0}^{16}(w_j(X))} \\
&= \frac{\tau_{jX,5}(w_j(X))}{\tau_{jX,0}(w_j(X))} - 5 \left(\frac{\tau_{jX,4}(w_j(X))}{\tau_{jX,0}(w_j(X))} \right) \left(\frac{\tau_{jX,1}(w_j(X))}{\tau_{jX,0}(w_j(X))} \right) + 20 \left(\frac{\tau_{jX,3}(w_j(X))}{\tau_{jX,0}(w_j(X))} \right) \left(\frac{\tau_{jX,1}(w_j(X))}{\tau_{jX,0}(w_j(X))} \right)^2 \\
&\quad - 10 \left(\frac{\tau_{jX,3}(w_j(X))}{\tau_{jX,0}(w_j(X))} \right) \left(\frac{\tau_{jX,2}(w_j(X))}{\tau_{jX,0}(w_j(X))} \right) + 30 \left(\frac{\tau_{jX,2}(w_j(X))}{\tau_{jX,0}(w_j(X))} \right)^2 \left(\frac{\tau_{jX,1}(w_j(X))}{\tau_{jX,0}(w_j(X))} \right) \\
&\quad - 60 \left(\frac{\tau_{jX,2}(w_j(X))}{\tau_{jX,0}(w_j(X))} \right) \left(\frac{\tau_{jX,1}(w_j(X))}{\tau_{jX,0}(w_j(X))} \right)^3 + 24 \left(\frac{\tau_{jX,1}(w_j(X))}{\tau_{jX,0}(w_j(X))} \right)^5 \\
&= E_X(\beta_i^5 | j(i) = j) - 5 E_X(\beta_i | j(i) = j) E_X(\beta_i^4 | j(i) = j) + 20 E_X(\beta_i | j(i) = j)^2 E_X(\beta_i^3 | j(i) = j) \\
&\quad - 10 E_X(\beta_i^2 | j(i) = j) E_X(\beta_i^3 | j(i) = j) + 30 E_X(\beta_i | j(i) = j) E_X(\beta_i^2 | j(i) = j)^2 \\
&\quad - 60 E_X(\beta_i | j(i) = j)^3 E_X(\beta_i^2 | j(i) = j) + 24 E_X(\beta_i | j(i) = j)^5 \\
&= E_X([\beta_i - E_X(\beta_i | j(i) = j)]^5 | j(i) = j) - 10 \text{Var}_X(\beta_i | j(i) = j) E_X([\beta_i - E_X(\beta_i | j(i) = j)]^3 | j(i) = j).
\end{aligned}$$

A.4. Worker Rents

The average rents for workers with skills X at firm j are $R_{jX}^w = E_X(R_i^w | j(i) = j)$, where:

$$u_{ij}(W_j(X_i) - R_i^w, a_j(X_i)) = \max_{j' \neq j(i)} u_{ij'}(W_{j'}(X_i), a_{j'}(X_i)).$$

Let $W_j(X)$ be the wage that firm j provides to workers with skills X . For any $W \leq W_j(X)$, the density of these workers who would be willing to accept their current job at wage W is:

$$L'_j(X, W) = \left. \frac{\partial L_j(X)}{\partial W_j(X)} \right|_{W_j(X)=W} \times \frac{1}{L_j(X)}.$$

Average worker rents are computed by integrating $W_j(X) - W$ with respect to this density:

$$\begin{aligned}
R_{jX}^w &= \int_0^{W_j(X)} (W_j(X) - W) L'_j(X, W) dW \\
&= \int_0^{W_j(X)} (W_j(X) - W) \left(\frac{\partial L_j(X)}{\partial W_j(X)} \Big|_{W_j(X)=W} \times \frac{1}{L_j(X)} \right) dW \\
&= \frac{W_j(X)}{L_j(X)} \times \int_0^1 (1 - \omega) \left(\frac{\partial}{\partial \omega} \int \frac{\exp(\beta_i \log(\omega W_j(X)) + a_j(X)) f_{\beta|X}(\beta_i|X) f_X(X)}{I_i(X)} d\beta_i \right) d\omega \\
&= \frac{W_j(X)}{L_j(X)} \times \int_0^1 (1 - \omega) \left(\int \frac{\beta_i \omega^{\beta_i-1} \exp(\beta_i \log W_j(X) + a_j(X)) f_{\beta|X}(\beta_i|X) f_X(X)}{I_i(X)} d\beta_i \right) d\omega.
\end{aligned}$$

The final equality relies on the assumption that firms view themselves as infinitesimal in the market. By changing the order of integration, the average worker rents can be re-written as:

$$\begin{aligned}
R_{jX}^w &= \frac{W_j(X)}{L_j(X)} \times \int \frac{1}{I_i(X)} \exp(\beta_i \log W_j(X) + a_j(X)) f_{\beta|X}(\beta_i|X) f_X(X) \left(\int_0^1 (1 - \omega) \beta_i \omega^{\beta_i-1} d\omega \right) d\beta_i \\
&= \frac{W_j(X)}{L_j(X)} \times \int \frac{1}{I_i(X)} \left(\frac{1}{1 + \beta_i} \right) \exp(\beta_i \log W_j(X) + a_j(X)) f_{\beta|X}(\beta_i|X) f_X(X) d\beta_i \\
&= W_j(X) \times \frac{E_X \left(\frac{1}{1 + \beta_i} \times \exp(\beta_i \log W_j(X) + a_j(X)) / I_i(X) \right)}{E_X \left(\exp(\beta_i \log W_j(X) + a_j(X)) / I_i(X) \right)} \\
&= W_j(X) \times E \left(\frac{1}{1 + \beta_i} \mid j(i) = j, X_i = X \right).
\end{aligned}$$

By averaging over worker skills X , I can compute the mean rents for all workers at firm j :

$$R_j^w = E(R_i^w | j(i) = j) = \int W_j(X) E \left(\frac{1}{1 + \beta_i} \mid j(i) = j, X_i = X \right) f_X(X) dX.$$

Note that the conditional expectation $E \left(\frac{1}{1 + \beta_i} \mid j(i) = j, X_i = X \right)$ can be decomposed such that:

$$E \left(\frac{1}{1 + \beta_i} \mid j(i) = j, X_i = X \right) = E_X \left(\frac{1}{1 + \beta_i} \right) + \frac{\text{Cov}_X \left(\frac{1}{1 + \beta_i}, \exp(\beta_i \log W_j(X) + a_j(X)) / I_i(X) \right)}{E_X \left(\exp(\beta_i \log W_j(X) + a_j(X)) / I_i(X) \right)}.$$

The elasticity of average worker rents with respect to the wage $W_j(X)$ is $\frac{\partial \log R_{jX}^w}{\partial \log W_j(X)}$, where:

$$\frac{\partial \log R_{jX}^w}{\partial \log W_j(X)} = 1 + \frac{\frac{\partial E \left(\frac{1}{1 + \beta_i} \mid j(i) = j, X_i = X \right)}{\partial \log W_j(X)}}{E \left(\frac{1}{1 + \beta_i} \mid j(i) = j, X_i = X \right)} = 1 + \frac{\text{Cov} \left(\beta_i, \frac{1}{1 + \beta_i} \mid j(i) = j, X_i = X \right)}{E \left(\frac{1}{1 + \beta_i} \mid j(i) = j, X_i = X \right)}.$$

If $\text{Var}(\beta_i | X_i = X) > 0$, then the second term is negative, which means that $\frac{\partial R_{jX}^w W_j(X)}{\partial W_j(X) R_{jX}^w} < 1$.

A.5. Employer Rents

Employer rents come in the form of excess profits that firms obtain by exploiting their wage-setting power. To calculate these rents, I consider a counterfactual setting where firms are price-takers in the market, facing perfectly-elastic labor supply curves. I define the rents at firm j to be the difference between the true and counterfactual profits $R_j^e = \Pi_j - \Pi_j^{\text{price-taker}}$.

For any employer j , the profit Π_j that is realized in the monopsonistic labor market is:

$$\begin{aligned}\Pi_j &= Y_j - \sum_{\chi \in \mathcal{X}} \left(\int \left(\frac{\varepsilon_j(\chi, \varphi)}{1 + \varepsilon_j(\chi, \varphi)} \right) \left(\frac{\partial Y_j}{\partial L_j(\chi, \varphi)} \right) L_j(\chi, \varphi) d\varphi \right) \\ &= Y_j - \sum_{\chi \in \mathcal{X}} \left(\int \left(\frac{\varepsilon_j(\chi, \varphi)}{1 + \varepsilon_j(\chi, \varphi)} \right) \varphi T_j (1 - \alpha_j) \theta_{j\chi} \left(L_j^{\text{eff}}(\chi) \right)^{\rho_j - 1} \left(\sum_{\chi' \in \mathcal{X}} \theta_{j\chi'} \left(L_j^{\text{eff}}(\chi') \right)^{\rho_j} \right)^{\frac{1 - \alpha_j}{\rho_j} - 1} L_j(\chi, \varphi) d\varphi \right) \\ &= Y_j \times \left(1 - (1 - \alpha_j) \sum_{\chi \in \mathcal{X}} \left(\frac{\theta_{j\chi} \left(L_j^{\text{eff}}(\chi) \right)^{\rho_j}}{\sum_{\chi' \in \mathcal{X}} \theta_{j\chi'} \left(L_j^{\text{eff}}(\chi') \right)^{\rho_j}} \right) \left(\int \left(\frac{\varepsilon_j(\chi, \varphi)}{1 + \varepsilon_j(\chi, \varphi)} \right) \left(\frac{\varphi L_j(\chi, \varphi)}{\int \varphi' L_j(\chi, \varphi') d\varphi'} \right) d\varphi \right) \right).\end{aligned}$$

To simplify the expression above, define $\omega_j(\chi, \varphi)$ as the share of effective labor that workers with skills $X = (\chi, \varphi)$ contribute to firm j . These effective labor shares are defined so that:

$$\omega_j(\chi, \varphi) = \frac{\partial \log N_j}{\partial \log L_j(\chi, \varphi)} = \frac{\theta_{j\chi} \left(L_j^{\text{eff}}(\chi) \right)^{\rho_j}}{\sum_{\chi' \in \mathcal{X}} \theta_{j\chi'} \left(L_j^{\text{eff}}(\chi') \right)^{\rho_j}} \times \frac{\varphi L_j(\chi, \varphi)}{\int \varphi' L_j(\chi, \varphi') d\varphi'}.$$

These shares aggregate to one, since: $\sum_{\chi \in \mathcal{X}} \int \omega_j(\chi, \varphi) d\varphi = 1$. Using this property, I write:

$$\begin{aligned}\Pi_j &= Y_j \times \left(1 - (1 - \alpha_j) \sum_{\chi \in \mathcal{X}} \int \left(\frac{\varepsilon_j(\chi, \varphi)}{1 + \varepsilon_j(\chi, \varphi)} \right) \omega_j(\chi, \varphi) d\varphi \right) \\ &= Y_j \times \sum_{\chi \in \mathcal{X}} \int \left[1 - (1 - \alpha_j) \left(\frac{\varepsilon_j(\chi, \varphi)}{1 + \varepsilon_j(\chi, \varphi)} \right) \right] \omega_j(\chi, \varphi) d\varphi \\ &= Y_j \times \sum_{\chi \in \mathcal{X}} \int \left(\frac{1 + \alpha_j \varepsilon_j(\chi, \varphi)}{1 + \varepsilon_j(\chi, \varphi)} \right) \omega_j(\chi, \varphi) d\varphi.\end{aligned}$$

If an employer j is a price-taker in the market, then its profit $\Pi_j^{\text{price-taker}}$ equals:

$$\Pi_j^{\text{price-taker}} = \max_{\{D_j^{\text{pt}}(\chi, \varphi)\}_{\chi, \varphi}} T_j \left(\sum_{\chi \in \mathcal{X}} \theta_{j\chi} \left(\int \varphi D_j^{\text{pt}}(\chi, \varphi) d\varphi \right)^{\rho_j} \right)^{\frac{1 - \alpha_j}{\rho_j}} - \sum_{\chi \in \mathcal{X}} \left(\int W_j^{\text{pt}}(\chi, \varphi) D_j^{\text{pt}}(\chi, \varphi) d\varphi \right).$$

Taking first-order conditions with respect to labor demand yields the wage equation:

$$W_j^{\text{pt}}(\chi, \varphi) = \varphi T_j (1 - \alpha_j) \theta_{j\chi} \left(\int \varphi D_j^{\text{pt}}(\chi, \varphi) d\varphi \right)^{\rho_j - 1} \left(\sum_{\chi \in \mathcal{X}} \theta_{j\chi} \left(\int \varphi D_j^{\text{pt}}(\chi, \varphi) d\varphi \right)^{\rho_j} \right)^{\frac{1 - \alpha_j}{\rho_j} - 1}.$$

In equilibrium, the labor demand $D_j^{\text{pt}}(X)$ equals the labor supply $L_j^{\text{pt}}(X)$, which is given by:

$$L_j^{\text{pt}}(X) = \int \frac{1}{I(\beta, X)} \exp(\beta \log W_j^{\text{pt}}(X) + a_j(X)) f_{\beta|X}(\beta|X) f_X(X) d\beta.$$

Given counterfactual wages and labor supply curves, the profit $\Pi_j^{\text{price-taker}}$ can be written as:

$$\begin{aligned} \Pi_j^{\text{price-taker}} &= T_j \left(\sum_{\chi \in \mathcal{X}} \theta_{j\chi} \left(\int \varphi L_j^{\text{pt}}(\chi, \varphi) d\varphi \right)^{\rho_j} \right)^{\frac{1 - \alpha_j}{\rho_j}} \times \left(1 - (1 - \alpha_j) \sum_{\chi \in \mathcal{X}} \int \omega_j(\chi, \varphi) d\varphi \right) \\ &= \alpha_j T_j \left(\sum_{\chi \in \mathcal{X}} \theta_{j\chi} \left(\int \varphi L_j^{\text{pt}}(\chi, \varphi) d\varphi \right)^{\rho_j} \right)^{\frac{1 - \alpha_j}{\rho_j}}. \end{aligned}$$

For any firm j , define $Y_j^{\text{pt}} = T_j \left(\sum_{\chi \in \mathcal{X}} \theta_{j\chi} \left(\int \varphi L_j^{\text{pt}}(\chi, \varphi) d\varphi \right)^{\rho_j} \right)^{\frac{1 - \alpha_j}{\rho_j}}$. The employer rents are:

$$\Pi_j^* - \Pi_j^{\text{price-taker}} = Y_j \times \left[\sum_{\chi \in \mathcal{X}} \int \left(\frac{1 + \alpha_j \varepsilon_j(\chi, \varphi)}{1 + \varepsilon_j(\chi, \varphi)} \right) \omega_j(\chi, \varphi) d\varphi - \alpha_j \left(\frac{Y_j^{\text{pt}}}{Y_j} \right) \right].$$

A.6. Pass-through of TFP Shocks to Wages

The elasticity of the wage $W_j(X)$ with respect to a firm's total factor productivity T_j is:

$$\begin{aligned}
\frac{\partial \log W_j(X)}{\partial \log T_j} &= 1 + \frac{\partial \log \left(\frac{\varepsilon_j(X)}{1+\varepsilon_j(X)} \right)}{\partial \log T_j} - (1 - \rho_j) \frac{\partial \log L_j^{\text{eff}}(\chi)}{\partial \log T_j} + \frac{1 - \alpha_j - \rho_j}{\rho_j} \frac{\partial \log \sum_{\chi' \in \mathcal{X}} \theta_{j\chi'} (L_j^{\text{eff}}(\chi'))^{\rho_j}}{\partial \log T_j} \\
&= 1 + \frac{\partial \log W_j(X)}{\partial \log T_j} \times \left(\frac{1}{\varepsilon_j(\chi, \varphi) [1 + \varepsilon_j(\chi, \varphi)]} \right) \times \frac{\partial^2 \log L_j(\chi, \varphi)}{\partial [\log W_j(\chi, \varphi)]^2} \\
&\quad - \frac{\partial \log W_j(X)}{\partial \log T_j} \times (1 - \rho_j) \int \varepsilon_j(\chi, \varphi') \frac{\partial \log L_j^{\text{eff}}(\chi)}{\partial \log L_j(\chi, \varphi')} d\varphi' \\
&\quad + \frac{\partial \log W_j(X)}{\partial \log T_j} \times (1 - \alpha_j - \rho_j) \sum_{\chi' \in \mathcal{X}} \left(\int \varepsilon_j(\chi', \varphi') \frac{\partial \log N_j}{\partial \log L_j(\chi', \varphi')} d\varphi' \right) \\
&= \left[1 - \left(\frac{1}{\varepsilon_j(\chi, \varphi) [1 + \varepsilon_j(\chi, \varphi)]} \right) \times \frac{\partial^2 \log L_j(\chi, \varphi)}{\partial [\log W_j(\chi, \varphi)]^2} \right. \\
&\quad + (1 - \rho_j) \int \varepsilon_j(\chi, \varphi') \frac{\partial \log L_j^{\text{eff}}(\chi)}{\partial \log L_j(\chi, \varphi')} d\varphi' \\
&\quad \left. - (1 - \alpha_j - \rho_j) \sum_{\chi' \in \mathcal{X}} \left(\int \varepsilon_j(\chi', \varphi') \frac{\partial \log N_j}{\partial \log L_j(\chi', \varphi')} d\varphi' \right) \right]^{-1}.
\end{aligned}$$

A.7. Allocative Inefficiency

In this economy, aggregate social welfare is defined as $\mathcal{W}$, where:

$$\mathcal{W} = \mathbb{E} \left(\max_j \{u_{ij}(W_j(X_i), a_j(X_i))\} \right) + \log \sum_{j=1}^J \Pi_j.$$

Using the formula for the expectation of a maximum over TIEV random variables, I write:¹

$$\begin{aligned}
\mathcal{W} &= \mathbb{E} \left(\max_j \left\{ \beta_i \log W_j(X_i) + a_j(X_i) + \epsilon_{ij} \right\} \right) + \log \sum_{j=1}^J \Pi_j \\
&= \int \mathbb{E} \left[\max_j \left\{ \beta_i \log W_j(X_i) + a_j(X_i) + \epsilon_{ij} \right\} \middle| X_i = X \right] f_{\beta, X}(\beta_i, X) d(\beta_i, X) + \log \sum_{j=1}^J \Pi_j \\
&= \int \left(\log \sum_{j=1}^J \exp [\beta_i \log W_j(X) + a_j(X)] + \gamma \right) f_{\beta, X}(\beta_i, X) d(\beta_i, X) + \log \sum_{j=1}^J \Pi_j \\
&= \int \log I(\beta_i, X) f_{\beta, X}(\beta_i, X) d(\beta_i, X) + \gamma + \log \sum_{j=1}^J \Pi_j,
\end{aligned}$$

where $\gamma \approx 0.5772$ is the Euler-Mascheroni constant. The social planner seeks to maximize welfare by solving $\mathcal{W}^* = \max_{\{j(i)\}_i} \mathcal{W}$. The optimality condition of the planner's problem is:

$$\frac{\partial \mathcal{W}}{\partial L_j(X)} = 0,$$

for all skills X and employers j . By evaluating these derivatives, I obtain the following:

$$\begin{aligned}
\frac{\partial \mathcal{W}}{\partial L_j(X)} &= \frac{\partial \left(\int \log I(\beta_i, X) f_{\beta|X}(\beta_i|X) f_X(X) d\beta_i \right)}{\partial L_j(X)} + \frac{\partial \log \sum_{j=1}^J \Pi_j}{\partial L_j(X)} \\
&= \int \frac{\partial \log I(\beta_i, X)}{\partial L_j(X)} f_{\beta|X}(\beta_i|X) f_X(X) d\beta_i + \frac{\partial \Pi_j}{\partial L_j(X)} \left(\sum_{j=1}^J \Pi_j \right)^{-1} \\
&= \int \frac{\beta_i \exp (\beta_i \log W_j(X) + a_j(X))}{I(\beta_i, X)} \left(\frac{\partial \log W_j(X)}{\partial L_j(X)} \right) f_{\beta|X}(\beta_i|X) f_X(X) d\beta_i + \frac{\partial \Pi_j}{\partial L_j(X)} \left(\sum_{j=1}^J \Pi_j \right)^{-1}.
\end{aligned}$$

¹This property is proven in Small & Rosen (1981). Even without TIEV errors, expected maximal utility is:

$$\begin{aligned}
\mathcal{W} &= \sum_{j=1}^J \left(\int \mathbb{E} \left[\beta_i \log W_j(X_i) + a_j(X_i) + \epsilon_{ij} \middle| j(i) = j, X_i = X \right] L_j(X) dX \right) \\
&= \mathbb{E} [\epsilon_{ij} | j(i) = j] + \sum_{j=1}^J \left(\int \left(\epsilon_j(X) \log W_j(X) + a_j(X) \right) L_j(X) dX \right).
\end{aligned}$$

Let $\epsilon_j^*(X)$ be the elasticity of wage w.r.t. labor in the solution to the social planner's problem.

$$\begin{aligned}
\frac{\partial \mathcal{W}}{\partial L_j(X)} &= \frac{\epsilon_j(X)}{\epsilon_j^*(X)} + \frac{\partial \Pi_j}{\partial L_j(X)} \left(\sum_{j=1}^J \Pi_j \right)^{-1} \\
&= \frac{\epsilon_j(X)}{\epsilon_j^*(X)} + \frac{\partial [Y_j - \int W_j(X) L_j(X) dX]}{\partial L_j(X)} \left(\sum_{j=1}^J \Pi_j \right)^{-1} \\
&= \frac{\epsilon_j(X)}{\epsilon_j^*(X)} + \left(\sum_{j=1}^J \Pi_j \right)^{-1} \left(\frac{\partial Y_j}{\partial L_j(X)} - \frac{\partial W_j(X) L_j(X)}{\partial L_j(X)} - W_j(X) \right) \\
&= \frac{\epsilon_j(X)}{\epsilon_j^*(X)} + \frac{\frac{\partial Y_j}{\partial L_j(X)} - W_j(X)}{\sum_{j=1}^J \Pi_j} - \frac{W_j(X)}{\epsilon_j^*(X) \times \sum_{j=1}^J \Pi_j}.
\end{aligned}$$

The solution to the social planner's problem is to set the elasticity of wages with respect to labor to zero for all X and j . This involves adjusting wages so that the markdowns are zero. To implement the first-best policy, a planner can give wage-specific wage subsidies to workers, where the shape of the subsidy curve depends on the distribution of preferences.

In the monopsonistic economy, without any policy intervention, welfare is given by:

$$\mathcal{W} = \int \log I(\beta_i, X) f_{\beta, X}(\beta_i, X) d(\beta_i, X) + \gamma + \log \sum_{j=1}^J Y_j \left(\int \left(\frac{1 + \alpha_j \epsilon_j(X)}{1 + \epsilon_j(X)} \right) \omega_j(X) dX \right)$$

Under the first-best optimal allocation that solves the planner's problem, welfare is given by:

$$\mathcal{W}^* = \int \log I(\beta_i, X) f_{\beta, X}(\beta_i, X) d(\beta_i, X) + \gamma + \log \sum_{j=1}^J \alpha_j Y_j^*.$$

The first-best optimal allocation is achieved in a competitive (Walrasian) economy. To compute optimal welfare $\mathcal{W}^*$, I consider a counterfactual setting where all firms are price-takers.

A.8. Micro-foundation for the Worker's Indirect Utility Function

I now present a simple micro-foundation for the indirect utility function of the worker, where each worker i chooses a firm j to maximize utility from consuming goods and leisure.

Let C_{ij} denote the worker's expected consumption from working at the firm and let H_{ij} denote the worker's expected time spent working. A worker i 's utility from choosing a firm j equals:

$$u_{ij} = f_i(C_{ij}, H_{ij}).$$

Let $\bar{H}_j(X_i)$ denote the scheduled work hours for employees with skills X_i at firm j , accounting for paid time off, overtime, vacation leave, and other benefits. Let $\tilde{H}_{ij}$ denote the idiosyncratic component of hours worked, accounting for commuting time and worker-firm match factors. Assume that $f_i : \mathbb{R}_+^2 \rightarrow \mathbb{R}$ is log-additive in consumption and scheduled hours, such that:

$$f_i(C_{ij}, H_{ij}) = a_i + \kappa_i \log C_{ij} - \eta \log \bar{H}_j(X_i) - \tilde{f}(\tilde{H}_{ij}).$$

The marginal rate of substitution between log consumption and log work hours equals $-\kappa_i/\eta$. Define the worker's budget constraint as $C_{ij} = W_j(X_i)$, where $W_j(X_i)$ denotes total earnings. Additionally, define $a_j(X) = -\log \bar{H}_j(X_i)$, $\beta_i = \kappa_i/\eta$, and $\epsilon_{ij} = -\eta^{-1} \tilde{f}(\tilde{H}_{ij})$. It follows that:

$$u_{ij} = \frac{a_i}{\eta} + \beta_i \log W_j(X_i) + a_j(X_i) + \epsilon_{ij},$$

which corresponds to utility specification (1) in the paper.² Note that this utility function can be extended to include non-labor income. For example, let V_i denote the worker's other forms of income. The budget constraint becomes $C_{ij} = W_j(X_i) + V_i$. If V_i is observed in data, then I can re-define earnings to be $\tilde{W}_{ij}(X_i) = W_j(X_i) + V_i$, and the same utility specification applies.

A.9. *Extension: Capital and Monopolistic Competition in the Product Market*

I now give an extension of the model that includes capital and monopolistic competition in the product market. Consider a monopolistic firm j with the following production function:

$$Q_j = T_j K_j^{\eta_j} N_j^{1-\alpha_j},$$

²The term $\frac{a_i}{\eta}$ does not vary across firms. Therefore, it does not impact the worker's employment decisions.

where K_j denotes capital. In a monopolistic product market, the revenue curve is $Y_j = Q_j^{1-\kappa_j}$. For each skill vector X , labor is hired according to the labor supply curve $L_j(X)$ and capital is rented at some fixed price r_j . The firm's profit function has the following representation:

$$\begin{aligned}\Pi_j &= Q_j^{1-\kappa_j} - \sum_{\chi \in \mathcal{X}} \left(\int W_j(\chi, \varphi) L_j(\chi, \varphi) d\varphi \right) - r_j K_j \\ &= \tilde{T}_j K_j^{\tilde{\eta}_j} N_j^{1-\tilde{\alpha}_j} - \sum_{\chi \in \mathcal{X}} \left(\int W_j(\chi, \varphi) L_j(\chi, \varphi) d\varphi \right) - r_j K_j,\end{aligned}$$

where $\tilde{T}_j = T_j^{1-\kappa_j}$, $\tilde{\eta}_j = \eta_j(1 - \kappa_j)$, and $\tilde{\alpha}_j = \alpha_j + \kappa_j(1 - \alpha_j)$. I now show that both perfect and monopolistic competition in the product market yield the same profit function. As a first step, I derive the first order condition of the firm's problem with respect to capital K_j . I write:

$$K_j = \left(\frac{r_j}{\tilde{\eta}_j \tilde{T}_j N_j^{1-\tilde{\alpha}_j}} \right)^{\frac{1}{\tilde{\eta}_j-1}}.$$

Plugging this condition into the firm's profit function, I obtain the following:

$$\begin{aligned}\Pi_j &= \tilde{T}_j \left(\frac{r}{\tilde{\eta}_j \tilde{T}_j N_j^{1-\tilde{\alpha}_j}} \right)^{\frac{\tilde{\eta}_j}{\tilde{\eta}_j-1}} N_j^{1-\tilde{\alpha}_j} - \sum_{\chi \in \mathcal{X}} \left(\int W_j(\chi, \varphi) L_j(\chi, \varphi) d\varphi \right) - r \left(\frac{r}{\tilde{\eta}_j \tilde{T}_j N_j^{1-\tilde{\alpha}_j}} \right)^{\frac{1}{\tilde{\eta}_j-1}} \\ &= \left[\tilde{T}_j \left(\frac{r}{\tilde{\eta}_j \tilde{T}_j} \right)^{\frac{\tilde{\eta}_j}{\tilde{\eta}_j-1}} - r \left(\frac{r}{\tilde{\eta}_j \tilde{T}_j} \right)^{\frac{1}{\tilde{\eta}_j-1}} \right] N_j^{-\frac{1-\tilde{\alpha}_j}{\tilde{\eta}_j-1}} - \sum_{\chi \in \mathcal{X}} \left(\int W_j(\chi, \varphi) L_j(\chi, \varphi) d\varphi \right).\end{aligned}$$

Note that this is just a reinterpretation of the original problem, where the firm's profit equals:

$$\Pi_j = \hat{T}_j N_j^{1-\hat{\alpha}_j} - \sum_{\chi \in \mathcal{X}} \left(\int W_j(\chi, \varphi) L_j(\chi, \varphi) d\varphi \right),$$

where I define the terms $(\hat{T}_j, \hat{\alpha}_j)$ such that: $\hat{T}_j = \tilde{T}_j \left(\frac{r}{\tilde{\eta}_j \tilde{T}_j} \right)^{\frac{\tilde{\eta}_j}{\tilde{\eta}_j-1}} - r \left(\frac{r}{\tilde{\eta}_j \tilde{T}_j} \right)^{\frac{1}{\tilde{\eta}_j-1}}$ and $\hat{\alpha}_j = \frac{\tilde{\eta}_j - \tilde{\alpha}_j}{\tilde{\eta}_j - 1}$.

B. Properties of the Model: Proofs and Discussion

B.1. Preference Heterogeneity and Substitution Patterns

Consider a version of the utility specification (1) where the taste shocks $\{\epsilon_{ij}\}_{i,j}$ are i.i.d. and the coefficient β is constant across workers. In this case, preferences can be written as:

$$u_{ij}(W_j(X_i), a_j(X_i)) = \delta_j X_i + \epsilon_{ij},$$

where $\delta_{jX_i} = \beta \log W_j(X_i) + a_j(X_i)$ is a deterministic function of $(W_j(X_i), a_j(X_i))$. Given posted wages $\{W_k(X)\}_{k=1}^J$, a worker with skills X will work for employer j with probability:

$$P(j(i) = j | X_i = X) = \int_{-\infty}^{\infty} \prod_{k \neq j} F_{\epsilon}(\delta_{jX} - \delta_{kX} + \tilde{\epsilon}) f_{\epsilon}(\tilde{\epsilon}) d\tilde{\epsilon}.$$

Proof of Property 1.

Define $P_{jX} = P(j(i) = j | X_i = X)$ as the share of workers with skills X at firm j . If the terms $\{\epsilon_{ij}\}_{i,j}$ are i.i.d. and $\beta_i = \beta$ for all i , then the own-wage elasticity of labor supply is:

$$\begin{aligned} \frac{\partial P_{jX} W_j(X)}{\partial W_j(X) P_{jX}} &= \frac{W_j(X)}{P_{jX}} \times \frac{\partial}{\partial W_j(X)} \int_{-\infty}^{\infty} \prod_{k \neq j} F_{\epsilon}(\delta_{jX} - \delta_{kX} + \tilde{\epsilon}) f_{\epsilon}(\tilde{\epsilon}) d\tilde{\epsilon} \\ &= \frac{W_j(X)}{P_{jX}} \times \int_{-\infty}^{\infty} \frac{\partial}{\partial W_j(X)} \prod_{k \neq j} F_{\epsilon}(\delta_{jX} - \delta_{kX} + \tilde{\epsilon}) f_{\epsilon}(\tilde{\epsilon}) d\tilde{\epsilon} \\ &= \frac{W_j(X)}{P_{jX}} \times \int_{-\infty}^{\infty} \left[\sum_{k \neq j} \left(\prod_{\ell \notin \{j,k\}} F_{\epsilon}(\delta_{jX} - \delta_{\ell X} + \tilde{\epsilon}) f_{\epsilon}(\delta_{jX} - \delta_{kX} + \tilde{\epsilon}) \frac{\partial \delta_{jX}}{\partial W_j(X)} \right) f_{\epsilon}(\tilde{\epsilon}) d\tilde{\epsilon} \right] \\ &= \frac{W_j(X)}{P_{jX}} \times \frac{\partial \delta_{jX}}{\partial W_j(X)} \times \sum_{k \neq j} \left[\int_{-\infty}^{\infty} \prod_{\ell \notin \{j,k\}} F_{\epsilon}(\delta_{jX} - \delta_{\ell X} + \tilde{\epsilon}) f_{\epsilon}(\delta_{jX} - \delta_{kX} + \tilde{\epsilon}) f_{\epsilon}(\tilde{\epsilon}) d\tilde{\epsilon} \right] \\ &= \frac{\beta}{P_{jX}} \times \sum_{k \neq j} \left[\int_{-\infty}^{\infty} \prod_{\ell \notin \{j,k\}} F_{\epsilon}(\delta_{jX} - \delta_{\ell X} + \tilde{\epsilon}) f_{\epsilon}(\delta_{jX} - \delta_{kX} + \tilde{\epsilon}) f_{\epsilon}(\tilde{\epsilon}) d\tilde{\epsilon} \right]. \end{aligned}$$

In addition, the cross-wage elasticity of labor supply with respect to any firm $k \neq j$ equals:

$$\begin{aligned} \frac{\partial P_{jX} W_k(X)}{\partial W_k(X) P_{jX}} &= \frac{W_k(X)}{P_{jX}} \times \frac{\partial}{\partial W_k(X)} \int_{-\infty}^{\infty} \prod_{\ell \neq j} F_{\epsilon}(\delta_{jX} - \delta_{\ell X} + \tilde{\epsilon}) f_{\epsilon}(\tilde{\epsilon}) d\tilde{\epsilon} \\ &= \frac{W_k(X)}{P_{jX}} \times \int_{-\infty}^{\infty} \frac{\partial}{\partial W_k(X)} \prod_{\ell \neq j} F_{\epsilon}(\delta_{jX} - \delta_{\ell X} + \tilde{\epsilon}) f_{\epsilon}(\tilde{\epsilon}) d\tilde{\epsilon} \\ &= -\frac{W_k(X)}{P_{jX}} \times \frac{\partial \delta_{kX}}{\partial W_k(X)} \times \int_{-\infty}^{\infty} \prod_{\ell \notin \{j,k\}} F_{\epsilon}(\delta_{jX} - \delta_{\ell X} + \tilde{\epsilon}) f_{\epsilon}(\delta_{jX} - \delta_{kX} + \tilde{\epsilon}) f_{\epsilon}(\tilde{\epsilon}) d\tilde{\epsilon} \\ &= -\frac{\beta}{P_{jX}} \times \int_{-\infty}^{\infty} \prod_{\ell \notin \{j,k\}} F_{\epsilon}(\delta_{jX} - \delta_{\ell X} + \tilde{\epsilon}) f_{\epsilon}(\delta_{jX} - \delta_{kX} + \tilde{\epsilon}) f_{\epsilon}(\tilde{\epsilon}) d\tilde{\epsilon}. \end{aligned}$$

Suppose that $P_{jX} = P_{j'X}$ for firms $j, j' \in \{1, \dots, J\}$. Then $\delta_{jX} = \delta_{j'X}$, which means that:³

$$\begin{aligned} \frac{\partial P_{jX} W_j(X)}{\partial W_j(X) P_{jX}} &= \frac{\beta}{P_{j'X}} \times \sum_{k \neq j} \left[\int_{-\infty}^{\infty} \prod_{\ell \notin \{j, k\}} F_{\epsilon}(\delta_{j'X} - \delta_{\ell X} + \tilde{\epsilon}) f_{\epsilon}(\delta_{j'X} - \delta_{kX} + \tilde{\epsilon}) f_{\epsilon}(\tilde{\epsilon}) d\tilde{\epsilon} \right] \\ &= \frac{\beta}{P_{j'X}} \times \left[\int_{-\infty}^{\infty} \prod_{\ell \notin \{j, j'\}} F_{\epsilon}(\delta_{j'X} - \delta_{\ell X} + \tilde{\epsilon}) f_{\epsilon}^2(\tilde{\epsilon}) d\tilde{\epsilon} \right. \\ &\quad \left. + \sum_{k \notin \{j, j'\}} \left[\int_{-\infty}^{\infty} \prod_{\ell \notin \{j, j', k\}} F_{\epsilon}(\delta_{j'X} - \delta_{\ell X} + \tilde{\epsilon}) F_{\epsilon}(\tilde{\epsilon}) f_{\epsilon}(\delta_{j'X} - \delta_{kX} + \tilde{\epsilon}) f_{\epsilon}(\tilde{\epsilon}) d\tilde{\epsilon} \right] \right] \\ &= \frac{\beta}{P_{j'X}} \times \sum_{k \neq j'} \left[\int_{-\infty}^{\infty} \prod_{\ell \notin \{j', k\}} F_{\epsilon}(\delta_{j'X} - \delta_{\ell X} + \tilde{\epsilon}) f_{\epsilon}(\delta_{j'X} - \delta_{kX} + \tilde{\epsilon}) f_{\epsilon}(\tilde{\epsilon}) d\tilde{\epsilon} \right]. \end{aligned}$$

This final expression corresponds to $\frac{\partial P_{j'X} W_j'(X)}{\partial W_j'(X) P_{j'X}}$. Moreover, for any $k \notin \{j, j'\}$, I can write:

$$\begin{aligned} \frac{\partial P_{jX} W_k(X)}{\partial W_k(X) P_{jX}} &= -\frac{\beta}{P_{j'X}} \times \int_{-\infty}^{\infty} \prod_{\ell \notin \{j, k\}} F_{\epsilon}(\delta_{j'X} - \delta_{\ell X} + \tilde{\epsilon}) f_{\epsilon}(\delta_{j'X} - \delta_{kX} + \tilde{\epsilon}) f_{\epsilon}(\tilde{\epsilon}) d\tilde{\epsilon} \\ &= -\frac{\beta}{P_{j'X}} \times \int_{-\infty}^{\infty} \prod_{\ell \notin \{j, j', k\}} F_{\epsilon}(\delta_{j'X} - \delta_{\ell X} + \tilde{\epsilon}) F_{\epsilon}(\tilde{\epsilon}) f_{\epsilon}(\delta_{j'X} - \delta_{kX} + \tilde{\epsilon}) f_{\epsilon}(\tilde{\epsilon}) d\tilde{\epsilon} \\ &= -\frac{\beta}{P_{j'X}} \times \int_{-\infty}^{\infty} \prod_{\ell \notin \{j', k\}} F_{\epsilon}(\delta_{j'X} - \delta_{\ell X} + \tilde{\epsilon}) f_{\epsilon}(\delta_{j'X} - \delta_{kX} + \tilde{\epsilon}) f_{\epsilon}(\tilde{\epsilon}) d\tilde{\epsilon}. \end{aligned}$$

This expression equals $\frac{\partial P_{j'X} W_k(X)}{\partial W_k(X) P_{j'X}}$, the cross-wage elasticity at firm j' with respect to firm k . □

Consider the special case where ϵ_{ij} follows a Type I extreme value distribution. The labor shares in this case have the following closed form expressions: $P_{jX} = \exp(\delta_{jX}) / \sum_{k=1}^J \exp(\delta_{kX})$.

Under this specification, it is easy to see that Property 1 holds since the wage elasticities are:

$$\frac{\partial P_{jX} W_k(X)}{\partial W_k(X) P_{jX}} = \begin{cases} \beta(1 - P_{jX}) & \text{if } j = k \\ -\beta P_{kX} & \text{if } j \neq k. \end{cases}$$

Next, suppose that β_i is heterogeneous across workers. Then the choice probabilities are:

$$P(j(i) = j | X_i = X) = \int P(j(i) = j | \beta_i = \beta, X_i = X) f_{\beta|X}(\beta | X) d\beta.$$

Equivalently, this probability can be expressed as $E[P(j(i) = j | \beta_i, X_i) | X_i = X]$, which equals the conditional expectation of β -specific choice probabilities among workers with skills X .

³The labor shares P_{jX} are strictly increasing in (and thus are uniquely determined by) the parameters δ_{jX} .

Proof of Property 2.

To ease notation, let $P_{jX} = P(j(i) = j|X_i = X)$ and $P_{jX}(\beta_i) = P(j(i) = j|\beta_i, X_i = X)$ be the aggregate and β -specific labor shares, respectively, for a firm. The wage elasticities are:

$$\begin{aligned}\frac{\partial P_{jX}W_k(X)}{\partial W_k(X)P_{jX}} &= \frac{W_k(X)}{P_{jX}} \times \frac{\partial}{\partial W_k(X)} \left(\int P_{jX}(\beta) f_{\beta|x}(\beta|X) d\beta \right) \\ &= P_{jX}^{-1} \int \frac{\partial P_{jX}(\beta)}{\partial W_k(X)} W_k(X) f_{\beta|x}(\beta|X) d\beta \\ &= P_{jX}^{-1} \int \frac{\partial P_{jX}(\beta) W_k(X)}{\partial W_k(X) P_{jX}(\beta)} P_{jX}(\beta) f_{\beta|x}(\beta|X) d\beta,\end{aligned}$$

for any firm $k \in \{1, \dots, J\}$. These elasticities can also be expressed in terms of expectations:

$$\begin{aligned}\frac{\partial P_{jX}W_k(X)}{\partial W_k(X)P_{jX}} &= P_{jX}^{-1} E \left(\frac{\partial P_{jX}(\beta_i) W_k(X)}{\partial W_k(X) P_{jX}(\beta_i)} P_{jX}(\beta_i) \right) \\ &= P_{jX}^{-1} \left[E \left(\frac{\partial P_{jX}(\beta_i) W_k(X)}{\partial W_k(X) P_{jX}(\beta_i)} \times 1 \mid j(i) = j \right) P_{jX} + E \left(\frac{\partial P_{jX}(\beta_i) W_k(X)}{\partial W_k(X) P_{jX}(\beta_i)} \times 0 \mid j(i) \neq j \right) (1 - P_{jX}) \right] \\ &= E \left(\frac{\partial P_{jX}(\beta_i) W_k(X)}{\partial W_k(X) P_{jX}(\beta_i)} \mid j(i) = j \right).\end{aligned}$$

So, the aggregate elasticity is the average of β -specific elasticities among workers at the firm. □

Proof of Property 3.

To ease notation, let $\varepsilon_j(X) = \frac{\partial P_{jX}W_j(X)}{\partial W_j(X)P_{jX}}$ and $\varepsilon_j(\beta, X) = \frac{\partial P_{jX}(\beta_i)W_k(X)}{\partial W_k(X)P_{jX}(\beta_i)}$ be the aggregate and β -specific own-wage elasticities of labor supply, respectively, for a firm. It follows that:

$$\begin{aligned}\frac{\partial \varepsilon_j(X)}{\partial \log W_j(X)} &= \frac{\partial}{\partial \log W_j(X)} \left[P_{jX}^{-1} \int \varepsilon_j(\beta, X) P_{jX}(\beta) f_{\beta|x}(\beta|X) d\beta \right] \\ &= P_{jX}^{-1} \int \frac{\partial [\varepsilon_j(\beta, X) P_{jX}(\beta)]}{\partial \log W_j(X)} f_{\beta|x}(\beta|X) d\beta - P_{jX}^{-2} \left(\frac{\partial P_{jX}W_j(X)}{\partial W_j(X)} \right) \int \varepsilon_j(\beta, X) P_{jX}(\beta) f_{\beta|x}(\beta|X) d\beta \\ &= P_{jX}^{-1} \int \frac{\partial [\varepsilon_j(\beta, X) P_{jX}(\beta)]}{\partial \log W_j(X)} f_{\beta|x}(\beta|X) d\beta - \varepsilon_j^2(X) \\ &= P_{jX}^{-1} \left[\int \frac{\partial \varepsilon_j(\beta, X)}{\partial \log W_j(X)} P_{jX}(\beta) f_{\beta|x}(\beta|X) d\beta + \int \frac{\partial P_{jX}(\beta)}{\partial \log W_j(X)} \varepsilon_j(\beta, X) f_{\beta|x}(\beta|X) d\beta \right] - \varepsilon_j^2(X) \\ &= P_{jX}^{-1} \int \frac{\partial \varepsilon_j(\beta, X)}{\partial \log W_j(X)} P_{jX}(\beta) f_{\beta|x}(\beta|X) d\beta + P_{jX}^{-1} \int \varepsilon_j^2(\beta, X) P_{jX}(\beta) f_{\beta|x}(\beta|X) d\beta - \varepsilon_j^2(X).\end{aligned}$$

By the same reasoning that is used in the proof of Property 2, I can re-write this quantity as:

$$\begin{aligned}\frac{\partial \varepsilon_j(X)}{\partial \log W_j(X)} &= E_X \left(\frac{\partial \varepsilon_j(\beta_i, X)}{\partial \log W_j(X)} \mid j(i) = j \right) + E_X \left(\varepsilon_j^2(\beta_i, X) \mid j(i) = j \right) - E_X \left(\varepsilon_j(\beta_i, X) \mid j(i) = j \right)^2 \\ &= E \left(\frac{\partial \varepsilon_j(\beta_i, X)}{\partial \log W_j(X)} \mid X_i = X, j(i) = j \right) + \text{Var} \left(\varepsilon_j(\beta_i, X) \mid X_i = X, j(i) = j \right).\end{aligned}$$

The final expression above is additively separable into two terms. The first term is the average derivative of $\varepsilon_j(\beta_i, X)$ taken with respect to $\log W_j(X)$ among workers i at the firm. The second term represents the conditional variance of $\varepsilon_j(\beta_i, X)$ among workers i at the firm.

□

B.2. Existence and Uniqueness of an Equilibrium

In order to prove the existence and uniqueness of an equilibrium, it is first necessary to establish some basic properties about the firm-specific labor supply curves, as well as the production and profit functions. The properties that I discuss below will guide my analysis.

Firm-Specific Labor Supply Curves

The total mass of workers with skills X employed at a firm j is given by $S_j(X)$, where:

$$S_j(X) = \int \left(\int_{-\infty}^{\infty} \prod_{k \neq j} F_{\epsilon} \left(\beta \log \left(\frac{W_j(X)}{W_k(X)} \right) + a_j(X) - a_k(X) + \tilde{\epsilon} \right) f_{\epsilon}(\tilde{\epsilon}) d\tilde{\epsilon} \right) f_{\beta, X}(\beta, X) d\beta.$$

Throughout my analysis, I assume that firms perceive themselves to be strategically small within the economy. Thus, a firm j sets its wages $\{W_j(X)\}_X$ without considering the impact of changing its own wages on the labor that is supplied to other firms. By Property 2, I write:

$$\begin{aligned}\frac{\partial \log S_j(X)}{\partial \log W_j(X)} &= E \left(\frac{\partial \log P(j(i) = j \mid \beta_i, X_i)}{\partial \log W_j(X_i)} \mid X_i = X, j(i) = j \right) \\ &= E \left(\frac{\beta_i}{P_{jX_i}} \times \sum_{k \neq j} \left[\int_{-\infty}^{\infty} \prod_{\ell \notin \{j, k\}} F_{\epsilon} (\delta_{jX_i} - \delta_{\ell X_i} + \tilde{\epsilon}) f_{\epsilon} (\delta_{jX_i} - \delta_{kX_i} + \tilde{\epsilon}) f_{\epsilon} (\tilde{\epsilon}) d\tilde{\epsilon} \right] \mid X_i = X, j(i) = j \right),\end{aligned}$$

where I define $\delta_{jX_i} = \beta \log W_j(X_i) + a_j(X_i)$. Using this formula, I prove the following claim.

Claim B.2.1. The labor supplied to a firm $S_j(X)$ is strictly-increasing in the wage $W_j(X)$.

Proof. Because $\log(\cdot)$ is a strictly-increasing transformation, it is sufficient to show that the derivative $\partial \log S_j(X) / \partial \log W_j(X)$ is strictly positive for any wage $W_j(X) \in \mathbb{R}_{++}$. First, note that $0 < F_\epsilon(\epsilon_{ij}) < 1$ and $0 < f_\epsilon(\epsilon_{ij}) < 1$ for all $\epsilon_{ij} \in \mathbb{R}$ since the taste shocks ϵ_{ij} take positive density everywhere on $\mathbb{R}$. In addition, because $P(\beta_i > 0 | X_i = X) > 0$ for every X , it must be that $\partial \log P(j(i) = j | \beta_i, X_i) / \partial \log W_j(X_i)$ is strictly positive for all values of (β_i, X_i) . From this property, I conclude that the derivative $\partial \log S_j(X) / \partial \log W_j(X)$ is strictly positive. $\square$

This claim ensures that, for any values of $\{W_k(X)\}_{k \neq j}$, the labor supply $S_j(X)$ for firm j is uniquely defined by firm j 's wage $W_j(X)$. Thus, any equilibrium is uniquely characterized by the wages that maximize the firms' profit function subject to the labor supply constraint. The goal of the rest of this section is to demonstrate that such wages do exist and are unique.

Firm Production Functions

In equilibrium, labor demand $D_j(X)$ equals labor supply $S_j(X)$. A firm j 's output is:

$$Y_j = T_j \left(\sum_{\chi \in \mathcal{X}} \theta_{j\chi} \left(\int \varphi L_j(\chi, \varphi) d\varphi \right)^{\rho_j} \right)^{\frac{1-\alpha_j}{\rho_j}}.$$

Among workers with skills $X = (\chi, \varphi)$, the marginal product of labor at the firm is given by:

$$\begin{aligned} \frac{\partial Y_j}{\partial L_j(\chi, \varphi)} &= \varphi T_j (1 - \alpha_j) \theta_{j\chi} \left(\int \varphi' L_j(\chi, \varphi') d\varphi' \right)^{\rho_j - 1} \left(\sum_{\chi' \in \mathcal{X}} \theta_{j\chi'} \left(\int \varphi' L_j(\chi', \varphi') d\varphi' \right)^{\rho_j} \right)^{\frac{1-\alpha_j}{\rho_j} - 1} \\ &= \varphi T_j (1 - \alpha_j) \theta_{j\chi} \left(L_j^{\text{eff}}(\chi) \right)^{\rho_j - 1} \left(\sum_{\chi' \in \mathcal{X}} \theta_{j\chi'} \left(L_j^{\text{eff}}(\chi') \right)^{\rho_j} \right)^{\frac{1-\alpha_j}{\rho_j} - 1}. \end{aligned}$$

I prove the next three claims by deriving the Hessian matrix H_{Y_j} of the production function.

Claim B.2.2. If $\alpha \in (0, 1)$ and $\rho_j < 1$, then $\frac{\partial^2 Y_j}{\partial L_j(\chi, \varphi) \partial L_j(\chi, \varphi')} < 0$ for $\chi \in \mathcal{X}$ and $\varphi, \varphi' \in \mathbb{R}_{++}$.

Proof. Fix $\chi \in \mathcal{X}$ and $\varphi, \varphi' \in \mathbb{R}$. The derivative of $\frac{\partial Y_j}{\partial L_j(\chi, \varphi)}$ with respect to $L_j(\chi, \varphi')$ is:

$$\begin{aligned} \frac{\partial^2 Y_j}{\partial L_j(\chi, \varphi) \partial L_j(\chi, \varphi')} &= \varphi T_j (1 - \alpha_j) \theta_{j\chi} \left[\varphi' (\rho_j - 1) \left(\int \varphi L_j(\chi, \varphi) d\varphi \right)^{\rho_j - 2} \left(\sum_{\chi' \in \mathcal{X}} \theta_{j\chi'} \left(\int \varphi L_j(\chi, \varphi) d\varphi \right)^{\rho_j} \right)^{\frac{1 - \alpha_j}{\rho_j} - 1} \right. \\ &\quad \left. + \varphi' (1 - \alpha_j - \rho_j) \theta_{j\chi} \left(\int \varphi L_j(\chi, \varphi) d\varphi \right)^{2\rho_j - 2} \left(\sum_{\chi' \in \mathcal{X}} \theta_{j\chi'} \left(\int \varphi L_j(\chi, \varphi) d\varphi \right)^{\rho_j} \right)^{\frac{1 - \alpha_j}{\rho_j} - 2} \right]. \end{aligned}$$

Define $\zeta_{j\chi\varphi\varphi'} = \varphi \varphi' T_j (1 - \alpha_j) \theta_{j\chi} \left(L_j^{\text{eff}}(\chi) \right)^{\rho_j - 2} \left(\sum_{\chi' \in \mathcal{X}} \theta_{j\chi'} \left(L_j^{\text{eff}}(\chi') \right)^{\rho_j} \right)^{\frac{1 - \alpha_j}{\rho_j} - 1}$. This term is positive since $\varphi, \varphi', T_j, \theta_{j\chi} > 0$ and employment is positive. Using this notation, I write:

$$\begin{aligned} \frac{\partial^2 Y_j}{\partial L_j(\chi, \varphi) \partial L_j(\chi, \varphi')} &= \zeta_{j\chi\varphi\varphi'} \left[(\rho_j - 1) + (1 - \alpha_j - \rho_j) \theta_{j\chi} \left(L_j^{\text{eff}}(\chi) \right)^{\rho_j} \left(\sum_{\chi' \in \mathcal{X}} \theta_{j\chi'} \left(L_j^{\text{eff}}(\chi') \right)^{\rho_j} \right)^{-1} \right] \\ &= \zeta_{j\chi\varphi\varphi'} \left[(\rho_j - 1) + (1 - \alpha_j - \rho_j) \frac{\theta_{j\chi} \left(L_j^{\text{eff}}(\chi) \right)^{\rho_j}}{\sum_{\chi' \in \mathcal{X}} \theta_{j\chi'} \left(L_j^{\text{eff}}(\chi') \right)^{\rho_j}} \right]. \end{aligned}$$

As $\zeta_{j\chi\varphi\varphi'} > 0$, it must be that $\frac{\partial^2 Y_j}{\partial L_j(\chi, \varphi) \partial L_j(\chi, \varphi')} < 0$ if and only if $\frac{1}{\zeta_{j\chi\varphi\varphi'}} \times \frac{\partial^2 Y_j}{\partial L_j(\chi, \varphi) \partial L_j(\chi, \varphi')} < 0$.

The effective labor share $\theta_{j\chi} \left(L_j^{\text{eff}}(\chi) \right)^{\rho_j} / \sum_{\chi' \in \mathcal{X}} \theta_{j\chi'} \left(L_j^{\text{eff}}(\chi') \right)^{\rho_j}$ for skill type χ is bounded between 0 and 1. Given these properties, I conclude that $\frac{\partial^2 Y_j}{\partial L_j(\chi, \varphi) \partial L_j(\chi, \varphi')} < 0$ if and only if:

$$\begin{aligned} 0 &> \frac{1}{\zeta_{j\chi\varphi\varphi'}} \times \frac{\partial^2 Y_j}{\partial L_j(\chi, \varphi) \partial L_j(\chi, \varphi')} = -(1 - \rho_j) \left[1 - \frac{\theta_{j\chi} \left(L_j^{\text{eff}}(\chi) \right)^{\rho_j}}{\sum_{\chi' \in \mathcal{X}} \theta_{j\chi'} \left(L_j^{\text{eff}}(\chi') \right)^{\rho_j}} \right] - \alpha_j \frac{\theta_{j\chi} \left(L_j^{\text{eff}}(\chi) \right)^{\rho_j}}{\sum_{\chi' \in \mathcal{X}} \theta_{j\chi'} \left(L_j^{\text{eff}}(\chi') \right)^{\rho_j}} \\ &= -(1 - \rho_j) - \alpha_j \frac{\theta_{j\chi} \left(L_j^{\text{eff}}(\chi) \right)^{\rho_j}}{\sum_{\chi' \neq \chi} \theta_{j\chi'} \left(L_j^{\text{eff}}(\chi') \right)^{\rho_j}}. \end{aligned}$$

Re-arranging terms, this inequality becomes: $\rho_j < 1 + \alpha_j \times \theta_{j\chi} \left(L_j^{\text{eff}}(\chi) \right)^{\rho_j} / \sum_{\chi' \neq \chi} \theta_{j\chi'} \left(L_j^{\text{eff}}(\chi') \right)^{\rho_j}$.

Whenever $\alpha_j \in (0, 1)$ and $\rho_j < 1$, this inequality holds trivially. It is worth noting that values of ρ_j above unity may also satisfy this inequality, particularly if the returns to scale parameter $1 - \alpha_j$ is small and/or if the effective labor share for the skill type χ is large within this firm.

□

Claim B.2.3. Suppose that $\alpha \in (0, 1)$. For any $\chi, \chi' \in \mathcal{X}$, where $\chi \neq \chi'$, and $\varphi, \varphi' \in \mathbb{R}_{++}$, the derivative $\frac{\partial^2 Y_j}{\partial L_j(\chi, \varphi) \partial L_j(\chi', \varphi')}$ is positive when $\rho_j < 1 - \alpha_j$ and negative when $\rho_j > 1 - \alpha_j$.

Proof. Fix $\chi, \chi' \in \mathcal{X}$, where $\chi \neq \chi'$, and $\varphi, \varphi' \in \mathbb{R}$. The derivative $\frac{\partial^2 Y_j}{\partial L_j(\chi, \varphi) \partial L_j(\chi', \varphi')}$ is:

$$\frac{\partial^2 Y_j}{\partial L_j(\chi, \varphi) \partial L_j(\chi', \varphi')} = \varphi \varphi' T_j (1 - \alpha_j) (1 - \alpha_j - \rho_j) \theta_{j\chi} \theta_{j\chi'} \left(L_j^{\text{eff}}(\chi) L_j^{\text{eff}}(\chi') \right)^{\rho_j - 1} \left(\sum_{\chi \in \mathcal{X}} \theta_{j\chi} \left(L_j^{\text{eff}}(\chi) \right)^{\rho_j} \right)^{\frac{1 - \alpha_j}{\rho_j} - 2}.$$

The term $\zeta_{j\chi\chi'\varphi\varphi'} = \varphi \varphi' T_j (1 - \alpha_j) \theta_{j\chi} \theta_{j\chi'} \left(L_j^{\text{eff}}(\chi) L_j^{\text{eff}}(\chi') \right)^{\rho_j - 1} \left(\sum_{\chi \in \mathcal{X}} \theta_{j\chi} \left(L_j^{\text{eff}}(\chi) \right)^{\rho_j} \right)^{\frac{1 - \alpha_j}{\rho_j} - 2}$

is always greater than zero. Therefore, the sign of the derivative is pinned down by $1 - \alpha_j - \rho_j$.

This quantity will be positive whenever $\rho_j < 1 - \alpha_j$ and will be negative whenever $\rho_j > 1 - \alpha_j$.

□

Claim B.2.4. Suppose that $\alpha \in (0, 1)$ and $\rho_j < 1$. The production function Y_j is concave.

Proof. The effective labor $L_{j\chi}^{\text{eff}} = \int \varphi L_j(\chi, \varphi) d\varphi$ in each skill type χ is a concave, strictly increasing function of $\{L_j(\chi, \varphi)\}_\varphi$. Define the mappings $g : \mathbb{R}_+^{|\mathcal{X}|} \rightarrow \mathbb{R}_+$ and $h : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ so that: $g(L_j^{\text{eff}}) = \sum_{\chi \in \mathcal{X}} \theta_{j\chi} (L_j^{\text{eff}}(\chi))^{\rho_j}$ and $h(x) = T_j x^{(1 - \alpha_j)/\rho_j}$. If $0 < \rho_j \leq 1$, then $g(L_j^{\text{eff}})$ has a diagonal Hessian matrix with negative eigenvalues $-\rho_j(1 - \rho_j) \theta_{j\chi} (L_j^{\text{eff}}(\chi))^{\rho_j - 2}$. Also, since $h(x)$ is monotonically increasing, $Y_j = h(g(L_j^{\text{eff}}))$ is quasiconcave in L_j^{eff} . Now suppose that $\rho_j < 0$. Then the Hessian matrix of $g(L_j^{\text{eff}})$ is positive definite, which means that $g(L_j^{\text{eff}})$ is convex. In addition, the function $h(x)$ is monotonically decreasing, so $Y_j = h(g(L_j^{\text{eff}}))$ is quasiconcave in L_j^{eff} . Any positive, quasiconcave function is concave if it is homogeneous of degree $k \in (0, 1]$. It follows that Y_j is concave in L_j^{eff} . So, it is also concave in $\{L_j(\chi, \varphi)\}_{\chi, \varphi}$.

□

Claim B.2.2 establishes conditions under which firms face decreasing marginal returns to hiring labor of the same skill type. Claim B.2.3 presents conditions under which different skill types are treated as substitutes (or complements) in the firm's production. Claim B.2.4 demonstrates that the output of a firm is a concave function of its labor inputs $\{L_j(\chi, \varphi)\}_{\chi, \varphi}$.

Firm Profit Functions

After plugging in the labor supply constraint, the profit function for any firm j equals:

$$\Pi_j = Y_j - \int W_j(X) L_j(X) dX.$$

The derivative of this function with respect to the wage $W_j(X)$ for workers with skills X is:

$$\begin{aligned} \frac{\partial \Pi_j}{\partial W_j(X)} &= \frac{\partial Y_j}{\partial L_j(X)} \times \frac{\partial L_j(X)}{\partial W_j(X)} - L_j(X) - W_j(X) \frac{\partial L_j(X)}{\partial W_j(X)} \\ &= L_j(X) \times \left(\frac{\partial Y_j}{\partial L_j(X)} \times \frac{\epsilon_j(X)}{W_j(X)} - 1 - \epsilon_j(X) \right). \end{aligned}$$

Next, I derive the Hessian matrix H_{Π_j} of the firm's profit function. The diagonal entries are:

$$\begin{aligned} \frac{\partial^2 \Pi_j}{\partial W_j^2(X)} &= \frac{\partial^2 Y_j}{\partial L_j^2(X)} \times \left(\frac{\partial L_j(X)}{\partial W_j(X)} \right)^2 + \frac{\partial Y_j}{\partial L_j(X)} \times \frac{\partial^2 L_j(X)}{\partial W_j^2(X)} - \frac{\partial L_j(X)}{\partial W_j(X)} - \frac{\partial L_j(X)}{\partial W_j(X)} - W_j(X) \frac{\partial^2 L_j(X)}{\partial W_j^2(X)} \\ &= \frac{\partial^2 Y_j}{\partial L_j^2(X)} \times \left(\frac{\partial L_j(X)}{\partial W_j(X)} \right)^2 + \left(\frac{\partial Y_j}{\partial L_j(X)} - W_j(X) \right) \frac{\partial^2 L_j(X)}{\partial W_j^2(X)} - 2 \frac{\partial L_j(X)}{\partial W_j(X)}. \end{aligned}$$

If the first-order condition binds, then $\frac{\partial Y_j}{\partial L_j(X)} - W_j(X) = W_j(X)/\epsilon_j(X)$, which implies that:

$$\begin{aligned} \frac{\partial^2 \Pi_j}{\partial W_j^2(X)} &= \frac{\partial^2 Y_j}{\partial L_j^2(X)} \times \left(\frac{\partial L_j(X)}{\partial W_j(X)} \right)^2 + \frac{W_j(X)}{\epsilon_j(X)} \times \frac{\partial^2 L_j(X)}{\partial W_j^2(X)} - 2 \frac{\partial L_j(X)}{\partial W_j(X)} \\ &= \frac{L_j(X)}{W_j(X)} \times \left[\frac{\partial^2 Y_j}{\partial L_j^2(X)} \times \frac{\partial L_j(X)}{\partial W_j(X)} \times \epsilon_j(X) + \frac{W_j^2(X)}{\epsilon_j(X) L_j(X)} \times \frac{\partial^2 L_j(X)}{\partial W_j^2(X)} - 2 \epsilon_j(X) \right] \\ &= \frac{L_j(X)}{W_j(X)} \times \left[\frac{\partial^2 Y_j}{\partial L_j^2(X)} \times \frac{\partial L_j(X)}{\partial W_j(X)} \times \epsilon_j(X) + \frac{\frac{\partial \epsilon_j(X)}{\partial \log W_j(X)} + \epsilon_j^2(X) - \epsilon_j(X)}{\epsilon_j(X)} - 2 \epsilon_j(X) \right] \\ &= \frac{L_j(X)}{W_j(X)} \times \left[\frac{\partial^2 Y_j}{\partial L_j^2(X)} \times \frac{\partial L_j(X)}{\partial W_j(X)} \times \epsilon_j(X) + \frac{\frac{\partial \epsilon_j(X)}{\partial \log W_j(X)}}{\epsilon_j(X)} - \epsilon_j(X) - 1 \right]. \end{aligned}$$

where the second-to-last equality above relies on the observation that $\frac{\partial \epsilon_j(X)}{\partial \log W_j(X)}$ is equal to:

$$\begin{aligned} \frac{\partial \frac{\partial L_j(X) W_j(X)}{\partial W_j(X) L_j(X)} W_j(X)}{\partial W_j(X)} &= W_j(X) \times \left[\frac{W_j(X)}{L_j(X)} \times \frac{\partial^2 L_j(X)}{\partial W_j^2(X)} - \left(\frac{\partial L_j(X)}{\partial W_j(X)} \right)^2 \times \frac{W_j(X)}{L_j^2(X)} + \frac{\partial L_j(X)}{\partial W_j(X)} \times \frac{1}{L_j(X)} \right] \\ &= \frac{W_j^2(X)}{L_j(X)} \times \frac{\partial^2 L_j(X)}{\partial W_j^2(X)} - \left(\frac{\partial L_j(X)}{\partial W_j(X)} \right)^2 \times \frac{W_j^2(X)}{L_j^2(X)} + \frac{\partial L_j(X)}{\partial W_j(X)} \times \frac{W_j(X)}{L_j(X)} \\ &= \frac{W_j^2(X)}{L_j(X)} \times \frac{\partial^2 L_j(X)}{\partial W_j^2(X)} - \epsilon_j^2(X) + \epsilon_j(X). \end{aligned}$$

The off-diagonal entries of the Hessian matrix are given by $\frac{\partial^2 \Pi_j}{\partial W_j(X) \partial W_j(X')}$, for $X \neq X'$, where:

$$\frac{\partial^2 \Pi_j}{\partial W_j(X) \partial W_j(X')} = \frac{\partial^2 Y_j}{\partial L_j(X) \partial L_j(X')} \times \frac{\partial L_j(X)}{\partial W_j(X)} \times \frac{\partial L_j(X')}{\partial W_j(X')}.$$

Taken together, the Hessian of Π_j has the form $H_{\Pi_j} = A^\top (H_{Y_j}) A + B$, where H_{Y_j} is the Hessian matrix of the firm's production function, and where A and B are both diagonal matrices with entries $A_{XX} = \frac{\partial L_j(X)}{\partial W_j(X)}$ and $B_{XX} = \frac{L_j(X)}{W_j(X)} \left[\left(\frac{\partial \epsilon_j(X)}{\partial \log W_j(X)} \right) / \epsilon_j(X) - (\epsilon_j(X) + 1) \right]$, respectively.⁴

To interpret the properties of the Hessian matrix H_{Π_j} , I will consider two special cases:

Special Case 1. Assume β is constant for workers with skills X , i.e., $\text{Var}(\beta_i | X_i = X) = 0$. In this case, $\partial \epsilon_j(X) / \partial \log W_j(X) = \text{Var}(\beta_i | X_i = X, j(i) = j) = 0$. It follows that H_{Π_j} equals:

$$H_{\Pi_j} = A^\top (H_{Y_j}) A + B, \quad \text{where} \quad B = -\text{diag} \left(\left[\frac{L_j(X) (\epsilon_j(X) + 1)}{W_j(X)} \right]_X \right).$$

This matrix is negative definite, which means that the profit function Π_j is concave. To see why, note that B has strictly negative eigenvalues and that H_{Y_j} is always negative definite when $\alpha_j \in (0, 1)$ and $\rho_j < 1$. Thus, H_{Π_j} must also be negative definite, since, for any $v \neq 0$:

$$\begin{aligned} v^\top H_{\Pi_j} v &= v^\top (A^\top (H_{Y_j}) A + B) v \\ &= v^\top A^\top (H_{Y_j}) A v + v^\top B v \\ &= (A v)^\top (H_{Y_j}) (A v) + v^\top B v < 0. \end{aligned}$$

Special Case 2. Assume that firms do not exercise wage-setting power, i.e., $W_j(X) = \frac{\partial Y_j}{\partial L_j(X)}$. In this setting, the Hessian matrix of the firm's profit function equals $H_{\Pi_j} = A^\top (H_{Y_j}) A + B$, where $B = -2 \text{diag} \left(\left[\frac{\partial L_j(X)}{\partial W_j(X)} \right]_X \right)$. Just as in the previous case, this simplification ensures that this matrix H_{Π_j} is negative definite. Therefore, the firm's profit function must be concave.

Both special cases lead to concavity of the profit function, which is a useful property for proving that a unique equilibrium exists. However, this property does not apply in general, which means that I cannot use it in the proof. To understand why, consider the matrix B . The

⁴Since $\frac{\partial L_j(X)}{\partial W_j(X)} > 0$ for all X , the entries of the matrix $A^\top (H_{Y_j}) A$ share the same signs as the entries of H_{Y_j} .

diagonal entries of B , which also correspond to its eigenvalues, are negative as long as:

$$\frac{\partial \varepsilon_j(X)}{\partial \log W_j(X)} < \varepsilon_j(X)(\varepsilon_j(X) + 1), \quad \text{for all } X.$$

Under the logit model, $\varepsilon_j(X)$ equals $E(\beta_i|X_i = X, j(i) = j)$ and $\partial \varepsilon_j(X)/\partial \log W_j(X)$ equals $\text{Var}(\beta_i|X_i = X, j(i) = j)$. So, the inequality above is more likely to hold if workers at firm j have higher marginal utilities of (log) earnings and/or if the dispersion of these marginal utilities is low. Since the elasticity $\varepsilon_j(X)$ increases in a firm's wage $W_j(X)$, the inequality is more likely to hold when firms offer higher wages. Note that it is difficult to draw general conclusions about the signs of the entries of B without placing restrictions on the distribution of β . This ambiguity makes it hard to establish when H_{Π_j} is negative definite.

Lemma 1. There exists an equilibrium involving strictly positive wages and employment.

Proof. I only consider equilibria where wages are positive, i.e., $W_j(X) > 0$ for every X . Also, since the taste shocks ϵ_{ij} take positive density on $\mathbb{R}$, it follows that $P(j(i) = j|\beta_i, X_i) > 0$ for all (β_i, X_i) . Thus, $L_j(X) = f_X(X) \times E[P(j(i) = j|\beta_i, X_i)|X_i = X] > 0$ for any X , which means that any equilibrium involves strictly positive employment. For a firm to be profitable, its wage cannot exceed the revenue that it receives per unit of labor. Because $\frac{\partial^2 Y_j}{\partial L_j^2(X)} < 0$, this restriction guarantees that there exists a strict upper bound on the wage $W_j(X)$ at each firm. Therefore, for any firm j , the set of feasible wages $\{W_j(X)\}_X$ is contained within a convex, compact subset of the Euclidean space. Moreover, any equilibrium must lie in the interior of this subspace since $\lim_{W_j(X) \rightarrow 0} \partial \Pi_j / \partial W_j(X) > 0$ and $\lim_{W_j(X) \rightarrow \infty} \partial \Pi_j / \partial W_j(X) < 0$ for all X , implying that a firm is always able to increase profit by deviating from a corner solution.

Given this reasoning, I restrict attention to wages that satisfy the first-order condition:

$$W_j(X) = \frac{\varepsilon_j(X)}{1 + \varepsilon_j(X)} \times \frac{\partial Y_j}{\partial L_j(X)}, \quad \text{for all } X.$$

This condition takes the form of a continuously differentiable system of equations, where the right-hand-side is bounded within the set of feasible wages.⁵ By Brouwer's fixed point theorem, there is a solution to this system of equations, which corresponds to an equilibrium. $\square$

Note. Not every critical point of the profit function is necessarily an equilibrium. For the wages $\{W_j(X)\}_X$ to exist in equilibrium, they must be a global maximizer of the firm's profit function Π_j . I now prove that there is almost always a unique global maximizer of Π_j .

Lemma 2. There is a unique solution to the firm's problem for almost all values of (α_j, ρ_j) .

Proof. Let $W_j(\mathbf{X}) = [W_j(X)]_X$ denote the vector of wages at firm j . The first-order condition requires that $g_j(W_j(\mathbf{X}))$ equals zero, where $g_j(W_j(\mathbf{X}))$ is a multi-valued function satisfying:

$$g_{jX}(W_j(\mathbf{X})) = W_j(X) - \frac{\varepsilon_j(X)}{1 + \varepsilon_j(X)} \times \frac{\partial Y_j(W_j(\mathbf{X}))}{\partial L_j(X)}, \quad \text{for all } X.$$

I examine the properties of g_j by deriving the Jacobian J_{g_j} . This matrix has diagonal entries:

$$\begin{aligned} \frac{\partial g_{jX}(W_j(\mathbf{X}))}{\partial W_j(X)} &= 1 - (1 + \varepsilon_j(X))^{-2} \times \frac{\partial \varepsilon_j(X)}{\partial W_j(X)} \times \frac{\partial Y_j(W_j(\mathbf{X}))}{\partial L_j(X)} - \frac{\varepsilon_j(X)}{1 + \varepsilon_j(X)} \times \frac{\partial^2 Y_j(W_j(\mathbf{X}))}{\partial L_j^2(X)} \times \frac{\partial L_j(X)}{\partial W_j(X)} \\ &= 1 - \frac{1}{\varepsilon_j(X)(1 + \varepsilon_j(X))} \times \frac{\partial \varepsilon_j(X)}{\partial \log W_j(X)} - \frac{\varepsilon_j(X)}{1 + \varepsilon_j(X)} \times \frac{\partial^2 Y_j(W_j(\mathbf{X}))}{\partial L_j^2(X)} \times \frac{\partial L_j(X)}{\partial W_j(X)}, \end{aligned}$$

where the second equality uses the first-order condition. The off-diagonal entries of J_{g_j} are:

$$\frac{\partial g_{jX}(W_j(\mathbf{X}))}{\partial W_j(X')} = -\frac{\varepsilon_j(X)}{1 + \varepsilon_j(X)} \times \frac{\partial^2 Y_j(W_j(\mathbf{X}))}{\partial L_j(X) \partial L_j(X')} \times \frac{\partial L_j(X')}{\partial W_j(X')}, \quad \text{for } X \neq X'.$$

Thus, this Jacobian matrix has the form $J_{g_j} = I - (\text{diag}(\frac{\varepsilon_j(X)}{1 + \varepsilon_j(X)})(H_{Y_j})A + C)$, where H_{Y_j} is the Hessian of the production function, A is a diagonal matrix with entries $A_{XX} = \frac{\partial L_j(X)}{\partial W_j(X)}$, C is a diagonal matrix with entries $C_{XX} = \varepsilon_j^{-1}(X)(1 + \varepsilon_j(X))^{-1} \times [\partial \varepsilon_j(X) / \partial \log W_j(X)]$, I is the identity matrix, and $\text{diag}(\frac{\varepsilon_j(X)}{1 + \varepsilon_j(X)})$ is a diagonal matrix of skill-specific wage markdowns.

⁵Specifically, $0 < \left(\frac{\varepsilon_j(X)}{1 + \varepsilon_j(X)}\right) \left(\frac{\partial Y_j}{\partial L_j(X)}\right) < \frac{\partial Y_j}{\partial L_j(X)}$, where $\frac{\partial Y_j}{\partial L_j(X)}$ is finite for all $L_j(X) > 0$. The labor supply curve $L_j(X)$ is bounded from below by 0: $L_j(X|W_j(X) = 0) = f_X(X) \int \frac{1}{L_i(X)} \exp(a_j(X)) f_{\beta_i|X}(\beta_i|X) d\beta_i > 0$.

Consider any vector of wages $W_j(\mathbf{X})$ that satisfies the first-order condition. Given firm amenities and the distribution of workers' preference parameters, the matrices $\text{diag}(\frac{\varepsilon_j(X)}{1+\varepsilon_j(X)})$, A , and C are fully determined by these wages. Moreover, for any $\alpha_j \in (0, 1)$ and $\rho_j < 1$, the Hessian matrix H_{Y_j} of the production function is always nonsingular for any wage vector that is realized. Given this property, the determinant of the Jacobian matrix J_{g_j} , when evaluated at the wage vector $W_j(\mathbf{X})$, is nonzero for almost all values of $(\alpha_j, \rho_j) \in (0, 1) \times (-\infty, 1)$. Therefore, the matrix J_{g_j} will be nonsingular with probability one at the wage vector $W_j(\mathbf{X})$. By the inverse function theorem, it is further guaranteed that $W_j(\mathbf{X})$ is a locally unique fixed point solution to the first-order condition with probability one. Furthermore, given that the set of fixed points is compact, it must have finitely-many elements with probability one.

Lastly, suppose that there are two equilibria. Since they are almost always locally stable, we may use the implicit function theorem to define the marginal effect of $\frac{1-\alpha_j}{\rho_j}$ on the firm's profit at each equilibrium. Since these marginal effects are different, any slight change in $\frac{1-\alpha_j}{\rho_j}$ causes profit to differ at the resulting equilibria. So, there is almost always one equilibrium. $\square$

C. Identification Proofs and Estimation Details

C.1. Identification of Worker Skills

Proof of Proposition 1.

Suppose that $\varphi \perp \beta | \chi, \tau$ and $a_j(\chi, \varphi) = a_{j\chi} + a_{\chi\varphi}$. The labor supplied to a firm j is:

$$L_{j\tau}(\chi, \varphi) = \int \frac{\exp(\beta \log W_{j\tau}^{\text{eff}}(\chi, \varphi) + a_{j\chi})}{\sum_{k=1}^J \exp(\beta \log W_{k\tau}^{\text{eff}}(\chi, \varphi) + a_{k\chi})} f_{\beta|\chi, \tau}(\beta | \chi) f_{\chi, \varphi|\tau}(\chi, \varphi) d\beta,$$

where $W_{j\tau}^{\text{eff}}(\chi, \varphi) = W_{j\tau}(\chi, \varphi)/\varphi$ is the effective wage of the skill type χ at firm j . Given any density $f_{\beta|\chi, \tau}$ and firm amenities $\{a_{k\chi}\}_k$, the firm-specific labor supply elasticity, defined as

$\varepsilon_{j\tau}(\chi, \varphi) = \partial \log L_{j\tau}(\chi, \varphi) / \partial \log W_{j\tau}(\chi, \varphi)$, equals a χ -specific function of $\{W_{k\tau}^{\text{eff}}(\chi, \varphi)\}_k$:

$$\varepsilon_{j\tau}(\chi, \varphi) = \int \beta \times \frac{[\exp(\beta \log W_{j\tau}^{\text{eff}}(\chi)) / I_{\tau}^{\text{eff}}(\beta, \chi, \varphi)] f_{\beta|\chi, \tau}(\beta|\chi, \tau)}{\int [\exp(\beta' \log W_{j\tau}^{\text{eff}}(\chi)) / I_{\tau}^{\text{eff}}(\beta', \chi, \varphi)] f_{\beta|\chi, \tau}(\beta'|\chi, \tau) d\beta'} d\beta,$$

where $I_{\tau}^{\text{eff}}(\beta, \chi, \varphi) = \sum_{k=1}^J \exp(\beta \log W_{k\tau}^{\text{eff}}(\chi, \varphi) + a_{k\chi})$. The equilibrium wage equations are:

$$W_{j\tau}^{\text{eff}}(\chi, \varphi) = \left(\frac{\varepsilon_{\chi}(W_{j\tau}^{\text{eff}}(\chi, \varphi) \mid \{W_{k\tau}^{\text{eff}}(\chi, \varphi)\}_{k \neq j})}{1 + \varepsilon_{\chi}(W_{j\tau}^{\text{eff}}(\chi, \varphi) \mid \{W_{k\tau}^{\text{eff}}(\chi, \varphi)\}_{k \neq j})} \right) \times T_j (1 - \alpha_j) \theta_{j\chi} \left(L_j^{\text{eff}}(\chi) \right)^{\rho_j - 1} \left(\sum_{\chi' \in \mathcal{X}} \theta_{j\chi'} \left(L_j^{\text{eff}}(\chi') \right)^{\rho_j} \right)^{\frac{1 - \alpha_j}{\rho_j} - 1},$$

for any firm j and vector of skills $(\chi, \varphi) \in \mathcal{X} \times \mathbb{R}_{++}$. By construction, the right-hand-side of these equations is a (j, χ) -specific function of the effective wages $\{W_{k\tau}^{\text{eff}}(\chi, \varphi)\}_k$. As shown by Lemma 2, there is almost always a unique, profit-maximizing solution to these equations. Thus, with probability one, the effective wages satisfy $W_{j\tau}^{\text{eff}}(\chi, \varphi) = W_{j\tau}^{\text{eff}}(\chi)$ in equilibrium. $\square$

C.2. Identification of Labor Supply Elasticities

My identification strategy relies on a *common trends* assumption, which asserts that the difference in untreated potential outcomes over time is the same, on average, between treated and untreated firms in the economy. To assess when this assumption is valid, it is necessary to understand how untreated potential outcomes evolve. In my setting, these outcomes evolve due to labor supply shifts resulting from the TFP shocks at treated firms. These shifts occur through a common channel: workers' wage indices $\{I(\beta, X)\}_{\beta, X}$ and the joint distribution $F_{\beta, X}$. Therefore, for the common trends assumption to hold, I require that treated and untreated firms are affected in the same way, on average by any change in $\{I(\beta, X)\}_{\beta, X}$ and $F_{\beta, X}$.

Proof of Proposition 2.

To begin, I consider the evolution of labor supply and effective wages for any untreated firm j between time periods τ_0 and τ_1 . This firm does not experience a TFP shock. Yet, its wages and labor shift due to changes in $\{I(\beta, X)\}_{\beta, X}$ and $F_{\beta, X}$. The proof follows two steps.

Step 1. Derive a sufficient statistic for $w_{j\tau_1,0}^{\text{eff}}(\chi) - w_{j\tau_0,0}^{\text{eff}}(\chi)$ and $\ell_{j\tau_1,0}(\chi) - \ell_{j\tau_0,0}(\chi)$.

Before proceeding, I introduce some new notation. Let $f_{\beta|\chi,\tau_0}(\beta_i|\chi)$ and $f_{\beta|\chi,\tau_1}(\beta_i|\chi)$ denote the conditional densities of workers' preference parameters β_i given their skill type χ at time periods τ_0 and τ_1 , respectively. I also define $f_{\chi|\tau_0}(\chi)$ and $f_{\chi|\tau_1}(\chi)$ as the corresponding densities of worker skill types. Next, for $\Gamma_j = (\rho_j, \alpha_j, \{\theta_{j\chi}\}_{\chi}, \{a_{j\chi} - a_{j\chi^*}\}_{\chi,\chi'})'$, I define:

$$g_{\chi,\tau}(w|\Gamma_j) = \log(1 - \alpha_j) + \log \theta_{j\chi} - (1 - \rho_j)h_{\chi,\tau}(w|\Gamma_j) + \frac{1 - \alpha_j - \rho_j}{\rho_j} \log \sum_{\chi' \in \mathcal{X}} \theta_{j\chi'} (h_{\chi',\tau}(w|\Gamma_j))^{\rho_j}$$

$$h_{\chi,\tau}(w|\Gamma_j) = \int \int \varphi \left(\frac{\exp(\beta w + a_{j\chi} - a_{j\chi^*})}{I_{\tau}^{\text{eff}}(\beta, \chi)} \right) f_{\beta|\chi,\tau}(\beta|\chi) f_{\varphi|\chi,\tau}(\varphi) f_{\chi|\tau}(\chi) d\beta d\varphi.$$

for $\tau \in \{\tau_0, \tau_1\}$, where χ^* denotes some “reference skill type” at firm j . I assume that each firm j takes the effective wage index $I_{\tau}^{\text{eff}}(\beta, \chi)$ as given. I can interpret $g_{\chi,\tau}(w|\Gamma_j)$ as the equilibrium wage equation for a price-taking firm (with zero markdowns) where $\log(T_j) = 0$ and $a_{j\chi} = a_{j\chi} - a_{j\chi^*}$. As shown in Appendix B.2, there is a unique fixed point solution to this system since the Jacobian matrix is negative definite. So, I can write: $g_{\chi,\tau}(w^*|\Gamma_j) = g_{\chi,\tau}^*(\Gamma_j)$.

By construction, the effective labor of skill type χ for an untreated firm j in period τ_1 equals $L_{j\tau_1,0}^{\text{eff}}(\chi) = \exp(a_{j\chi^*}) \times h_{\chi,\tau}(w_{j\tau_1,0}^{\text{eff}}(\chi)|\Gamma_j)$. So, the potential outcome $w_{j\tau_1,0}^{\text{eff}}(\chi)$ is:

$$w_{j\tau_1,0}^{\text{eff}}(\chi) = \log T_j + \log(1 - \alpha_j) + \log \theta_{j\chi} - \log \left(1 + \frac{1}{\varepsilon_{j\tau_1,0}(\chi)} \right)$$

$$- (1 - \rho_j) \log \left(L_{j\tau_1,0}^{\text{eff}}(\chi) \right) + \frac{1 - \alpha_j - \rho_j}{\rho_j} \log \sum_{\chi' \in \mathcal{X}} \theta_{j\chi'} \left(L_{j\tau_1,0}^{\text{eff}}(\chi') \right)^{\rho_j}$$

$$= \log T_j - \alpha_j a_{j\chi^*} - \log \left(1 + \frac{1}{\varepsilon_{j\tau_1,0}(\chi)} \right) + g_{\chi,\tau_1}^*(\Gamma_j).$$

Since the wage changes are infinitesimal, I can write $\varepsilon_{j\tau_1,0}(\chi) = \varepsilon_{\tau_1,0}(\chi, w_{j\tau_0,0}^{\text{eff}}(\chi))$, where $\varepsilon_{j\tau_1,0}(\chi, w_{j\tau_0,0}^{\text{eff}}(\chi))$ represents the labor supply elasticity for an untreated firm j during time period τ_1 evaluated at the pre-period effective wage $w_{j\tau_0,0}^{\text{eff}}(\chi)$. By this property, I can write:

$$w_{j\tau_1,0}^{\text{eff}}(\chi) = \underbrace{w_{j\tau_0,0}^{\text{eff}}(\chi) - \log \left(1 + \frac{1}{\varepsilon_{\tau_0,0}(\chi, w_{j\tau_0,0}^{\text{eff}}(\chi))} \right) + \log \left(1 + \frac{1}{\varepsilon_{\tau_1,0}(\chi, w_{j\tau_0,0}^{\text{eff}}(\chi))} \right) + g_{\chi,\tau_1}^*(\Gamma_j) - g_{\chi,\tau_0}^*(\Gamma_j)}_{q_w(w_{j\tau_0,0}^{\text{eff}}(\chi)|\Gamma_j, \tau_0, \tau_1, \chi)}.$$

Additionally, I can write the change in untreated labor outcomes $\ell_{j\tau_1,0}(\chi) - \ell_{j\tau_0,0}(\chi)$ to be:

$$\ell_{j\tau_1,0}(\chi) - \ell_{j\tau_0,0}(\chi) = \log \left(\underbrace{\frac{\int \frac{\exp(\beta q_w(w_{j\tau_0,0}^{\text{eff}}(\chi)|\Gamma_j, \tau_0, \tau_1, \chi) + a_{j\chi} - a_{j\chi^*})}{I_{\tau_1}(\beta, X)} f_{\beta|\chi, \tau_1}(\beta|\chi) f_{\chi, \varphi|\tau_1}(\chi, \varphi) d\beta}{\int \frac{\exp(\beta' w_{j\tau_0,0}^{\text{eff}}(\chi) + a_{j\chi} - a_{j\chi^*})}{I_{\tau_0}(\beta', X)} f_{\beta|\chi, \tau_0}(\beta'|\chi) f_{\chi, \varphi|\tau_0}(\chi, \varphi) d\beta'}}_{q\ell(w_{j\tau_0,0}^{\text{eff}}(\chi)|\Gamma_j, \tau_0, \tau_1, \chi)} \right).$$

Step 2. Demonstrate that the common trends assumption holds under Assumption I.

To begin, I write down the difference-in-differences estimand in the following way:

$$\begin{aligned} \text{DiD}_{\tau_0, \tau_1}(w|\chi) &= \frac{\mathbb{E}[\ell_{j\tau_1}(\chi) - \ell_{j\tau_0}(\chi)|Z_j = 1, w_{j\tau_0}^{\text{eff}}(\chi) = w] - \mathbb{E}[\ell_{j\tau_1}(\chi) - \ell_{j\tau_0}(\chi)|Z_j = 0, w_{j\tau_0}^{\text{eff}}(\chi) = w]}{\mathbb{E}[w_{j\tau_1}^{\text{eff}}(\chi) - w_{j\tau_0}^{\text{eff}}(\chi)|Z_j = 1, w_{j\tau_0}^{\text{eff}}(\chi) = w] - \mathbb{E}[w_{j\tau_1}^{\text{eff}}(\chi) - w_{j\tau_0}^{\text{eff}}(\chi)|Z_j = 0, w_{j\tau_0}^{\text{eff}}(\chi) = w]} \\ &= \frac{\mathbb{E}[\ell_{j\tau_1}(\chi) - \ell_{j\tau_0}(\chi)|Z_j = 1, w_{j\tau_0}^{\text{eff}}(\chi) = w] - \mathbb{E}[\ell_{j\tau_1}(\chi) - \ell_{j\tau_0}(\chi)|Z_j = 0, w_{j\tau_0}^{\text{eff}}(\chi) = w]}{\mathbb{E}[w_{j\tau_1}^{\text{eff}}(\chi)|Z_j = 1, w_{j\tau_0}^{\text{eff}}(\chi) = w] - \mathbb{E}[w_{j\tau_1}^{\text{eff}}(\chi)|Z_j = 0, w_{j\tau_0}^{\text{eff}}(\chi) = w]}. \end{aligned}$$

In terms of the potential outcomes in the model, this estimand may be re-written to be:

$$\text{DiD}_{\tau_0, \tau_1}(w|\chi) = \frac{\mathbb{E}[\ell_{j\tau_1,1}(\chi) - \ell_{j\tau_0,1}(\chi)|Z_j = 1, w_{j\tau_0}^{\text{eff}}(\chi) = w] - \mathbb{E}[\ell_{j\tau_1,0}(\chi) - \ell_{j\tau_0,0}(\chi)|Z_j = 0, w_{j\tau_0}^{\text{eff}}(\chi) = w]}{\mathbb{E}[w_{j\tau_1,1}^{\text{eff}}(\chi)|Z_j = 1, w_{j\tau_0}^{\text{eff}}(\chi) = w] - \mathbb{E}[w_{j\tau_1,0}^{\text{eff}}(\chi)|Z_j = 0, w_{j\tau_0}^{\text{eff}}(\chi) = w]}.$$

Using the functions $q_w(w_{j\tau_0,0}^{\text{eff}}(\chi)|\Gamma_j, \tau_0, \tau_1, \chi)$ and $q\ell(w_{j\tau_0,0}^{\text{eff}}(\chi)|\Gamma_j, \tau_0, \tau_1, \chi)$, I can write:

$$\begin{aligned} \mathbb{E}[w_{j\tau_1,0}^{\text{eff}}(\chi)|Z_j = 1, w_{j\tau_0}^{\text{eff}}(\chi) = w] &= \mathbb{E}[q_w(w_{j\tau_0,0}^{\text{eff}}(\chi)|\Gamma_j, \tau_0, \tau_1, \chi)|Z_j = 0, w_{j\tau_0}^{\text{eff}}(\chi) = w] \\ &= \mathbb{E}[q_w(w_{j\tau_0,0}^{\text{eff}}(\chi)|\Gamma_j, \tau_0, \tau_1, \chi)|Z_j = 1, w_{j\tau_0}^{\text{eff}}(\chi) = w] \\ &= \mathbb{E}[w_{j\tau_1,0}^{\text{eff}}(\chi)|Z_j = 1, w_{j\tau_0}^{\text{eff}}(\chi) = w] \\ \mathbb{E}[\ell_{j\tau_1,0}(\chi) - \ell_{j\tau_0,0}(\chi)|Z_j = 0, w_{j\tau_0}^{\text{eff}}(\chi) = w] &= \mathbb{E}[q\ell(w_{j\tau_0,0}^{\text{eff}}(\chi)|\Gamma_j, \tau_0, \tau_1, \chi)|Z_j = 0, w_{j\tau_0}^{\text{eff}}(\chi) = w] \\ &= \mathbb{E}[q\ell(w_{j\tau_0,0}^{\text{eff}}(\chi)|\Gamma_j, \tau_0, \tau_1, \chi)|Z_j = 1, w_{j\tau_0}^{\text{eff}}(\chi) = w] \\ &= \mathbb{E}[\ell_{j\tau_1,0}(\chi) - \ell_{j\tau_0,0}(\chi)|Z_j = 1, w_{j\tau_0}^{\text{eff}}(\chi) = w]. \end{aligned}$$

In both these equations, the second equality directly follows from Assumption I. Next, I write:

$$\begin{aligned} \text{DiD}_{\tau_0, \tau_1}(w|\chi) &= \frac{\mathbb{E}[\ell_{j\tau_1,1}(\chi) - \ell_{j\tau_0,1}(\chi)|Z_j = 1, w_{j\tau_0}^{\text{eff}}(\chi) = w] - \mathbb{E}[\ell_{j\tau_1,0}(\chi) - \ell_{j\tau_0,0}(\chi)|Z_j = 0, w_{j\tau_0}^{\text{eff}}(\chi) = w]}{\mathbb{E}[w_{j\tau_1,1}^{\text{eff}}(\chi)|Z_j = 1, w_{j\tau_0}^{\text{eff}}(\chi) = w] - \mathbb{E}[w_{j\tau_1,0}^{\text{eff}}(\chi)|Z_j = 0, w_{j\tau_0}^{\text{eff}}(\chi) = w]} \\ &= \frac{\mathbb{E}[\ell_{j\tau_1,1}(\chi) - \ell_{j\tau_1,0}(\chi)|Z_j = 1, w_{j\tau_0}^{\text{eff}}(\chi) = w]}{\mathbb{E}[w_{j\tau_1,1}^{\text{eff}}(\chi) - w_{j\tau_1,0}^{\text{eff}}(\chi)|Z_j = 1, w_{j\tau_0}^{\text{eff}}(\chi) = w]}. \end{aligned}$$

Letting $\Delta\ell_{j,\tau_1}(\chi) = \ell_{j\tau_1,1}(\chi) - \ell_{j\tau_1,0}(\chi)$ and $\Delta w_{j\tau_1}^{\text{eff}}(\chi) = w_{j\tau_1,1}^{\text{eff}}(\chi) - w_{j\tau_1,0}^{\text{eff}}(\chi)$, I write:

$$\begin{aligned} \mathbb{E} \left[\Delta\ell_{j,\tau_1}(\chi) | Z_j = 1, w_{j\tau_0}^{\text{eff}}(\chi) = w \right] &= \mathbb{E} \left[\varepsilon_{j\tau_1}(\chi) \times \Delta w_{j\tau_1}^{\text{eff}}(\chi) | Z_j = 1, w_{j\tau_0}^{\text{eff}}(\chi) = w \right] \\ &= \mathbb{E} \left[\varepsilon_{\tau_1}(\chi, w_{j\tau_0,0}^{\text{eff}}(\chi)) \times \Delta w_{j\tau_1}^{\text{eff}}(\chi) | Z_j = 1, w_{j\tau_0}^{\text{eff}}(\chi) = w \right] \\ &= \varepsilon_{\tau_1}(\chi, w) \times \mathbb{E} \left[\Delta w_{j\tau_1}^{\text{eff}}(\chi) | Z_j = 1, w_{j\tau_0}^{\text{eff}}(\chi) = w \right] \end{aligned}$$

where the second equality uses the fact that $\varepsilon_{\tau_1,0}(\chi, \cdot)$ is a continuous function of the (log) wage. Since the TFP shocks are infinitesimal, the difference $\Delta w_{j\tau_1}^{\text{eff}}(\chi)$ is also infinitesimal for any firm j . By continuity, the difference $\varepsilon_{j\tau_1}(\chi) - \varepsilon_{\tau_1}(\chi, w_{j\tau_0}^{\text{eff}}(\chi))$ is also infinitesimal. Thus, from the equation above, I conclude that $\varepsilon_{\tau_1}(\chi, w) = \text{DiD}_{\tau_0, \tau_1}(w)$ for any wage $w \in \mathbb{R}$. $\square$

C.3. Identification of Technology

Proof of Proposition 3.

Consider two skill types χ and χ' . For any time period $\tilde{\tau} \in \{\tau, \tau'\}$, equation (10) implies:

$$\log \text{MPL}_{j\tilde{\tau}}^{\text{eff}}(\chi) - \log \text{MPL}_{j\tilde{\tau}}^{\text{eff}}(\chi') = \log \theta_{j\chi} - \log \theta_{j\chi'} - (1 - \rho_j) \left[\log L_{j\tilde{\tau}}^{\text{eff}}(\chi) - \log L_{j\tilde{\tau}}^{\text{eff}}(\chi') \right].$$

Because ρ_j and $\{\theta_{j\chi}\}_{\chi}$ are fixed over time, the elasticity of substitution may be recovered by computing inter-temporal shifts in the relative marginal products and effective labor shares:

$$(1 - \rho_j)^{-1} = \frac{\log \left(\frac{L_{j\tau}^{\text{eff}}(\chi)}{L_{j\tau}^{\text{eff}}(\chi')} \right) - \log \left(\frac{L_{j\tau'}^{\text{eff}}(\chi)}{L_{j\tau'}^{\text{eff}}(\chi')} \right)}{\log \left(\frac{\text{MPL}_{j\tau}^{\text{eff}}(\chi)}{\text{MPL}_{j\tau}^{\text{eff}}(\chi')} \right) - \log \left(\frac{\text{MPL}_{j\tau'}^{\text{eff}}(\chi)}{\text{MPL}_{j\tau'}^{\text{eff}}(\chi')} \right)}.$$

$\square$

Proof of Proposition 4.

I normalize the firm-specific efficiencies $\{\theta_{j\chi}\}_\chi$ by setting $\theta_{j\chi^*} = 1$ for skill type $\chi^* \in \mathcal{X}$. Under this normalization and given knowledge of ρ_j , these parameters may be computed as:

$$\theta_{j\chi} = \exp \left[\log \left(\frac{\text{MPL}_{j\tau}^{\text{eff}}(\chi)}{\text{MPL}_{j\tau}^{\text{eff}}(\chi^*)} \right) + (1 - \rho_j) \log \left(\frac{L_{j\tau}^{\text{eff}}(\chi)}{L_{j\tau}^{\text{eff}}(\chi^*)} \right) \right].$$

A firm's returns to scale and total factor productivity can then be recovered from the effective marginal products, the effective labor shares, and (log) value added at the firm. I write:

$$1 - \alpha_j = \exp \left[\log \text{MPL}_{j\tau}^{\text{eff}}(\chi) - y_{j\tau} - \log(\theta_{j\chi}) + (1 - \rho_j) \log L_{j\tau}^{\text{eff}}(\chi) + \log \sum_{\chi' \in \mathcal{X}} \theta_{j\chi'} \left(L_{j\tau}^{\text{eff}}(\chi') \right)^{\rho_j} \right].$$

$$T_{j\tau} = \exp \left[y_{j\tau} - \frac{1 - \alpha_j}{\rho_j} \log \sum_{\chi \in \mathcal{X}} \theta_{j\chi} \left(L_{j\tau}^{\text{eff}}(\chi) \right)^{\rho_j} \right].$$

□

C.4. Identification of Non-Wage Amenities

Proof of Proposition 5.

For some firm j^* , set $a_{j^*\chi} = 0$. Under this normalization, the amenities $a_{j\chi}$, $j \neq j^*$, are:

$$a_{j\chi} = \log L_{j\tau}(\chi, w_{k\tau}^{\text{eff}}(\chi)) - \log L_{j^*\tau}(\chi),$$

where $L_{j^*\tau}(\chi)$ is the labor supplied to firm j^* by skill type χ at time τ , and $L_{j\tau}(\chi, w_{k\tau}^{\text{eff}}(\chi))$ is the labor supplied to firm j if it posts the same log effective wage as firm k . If the elasticity curve $\varepsilon_{j\tau}(\chi, w)$ is known to the researcher, then $\log L_j(\chi, w_{k\tau}^{\text{eff}})$ may be recovered via:

$$a_{j\chi} = \log L_{j\tau}(\chi) + \int_{w_{j\tau}^{\text{eff}}(\chi)}^{w_{j^*\tau}^{\text{eff}}(\chi)} \varepsilon_{j\tau}(\chi, w) dw - \log L_{j^*\tau}(\chi).$$

Note that Proposition 2 establishes that $\varepsilon_{j\tau}(\chi, w)$ is point-identified from $\text{DiD}_{\tau_0, \tau_1}(w_{j\tau_0}^{\text{eff}}(\chi) | \chi)$.

□

C.5. Identification of Worker Preferences

Proof of Proposition 6.

Suppose that the elasticity curve $\varepsilon_{j\tau}(\chi, w)$ is known to the researcher for skill type χ . Then the firm-specific labor supply curves $\{L_{j\tau}(\chi, w)\}_j$ can be recovered through integration:

$$\log L_{j\tau}(\chi, w) = \log L_{j\tau}(\chi) + \int_{w_{j\tau}^{\text{eff}}(\chi)}^w \varepsilon_{j\tau}(\chi, \tilde{w}) d\tilde{w}.$$

Each labor supply curve $L_{j\tau}(\chi, w)$ may be expressed as a Laplace transform $\mathcal{L}\{g\}(s)$, where:

$$s = -w$$

$$\mathcal{L}\{g\}(s) = \int_0^\infty g(t) \exp(-st) dt, \quad \text{such that} \quad t = \beta$$

$$g(t) = \frac{\exp(a_{j\chi})}{\sum_{k=1}^J \exp(t w_{k\tau}^{\text{eff}}(\chi) + a_{k\chi})} f_{\beta|\chi, \tau}(t|\chi) f_{\chi|\tau}(\chi).$$

The transform $\mathcal{L}\{g\}(s)$ is a one-to-one mapping of $g(t)$. Specifically, any two functions $g(t)$ can only share the same Laplace transform if they differ on a set of Lebesgue measure zero. Therefore, when the elasticity $\varepsilon_{j\tau}(\chi, w)$ is point-identified, so is the function $g(\beta)$, where:

$$g(\beta) = \frac{\exp(a_{j\chi})}{\sum_{k=1}^J \exp(\beta w_{k\tau}^{\text{eff}}(\chi) + a_{k\chi})} f_{\beta|\chi, \tau}(\beta|\chi) f_{\chi|\tau}(\chi).$$

If the amenities $\{a_{j\chi}\}_j$ are identified up-to-scale, i.e., relative to some reference amenity $a_{j^*\chi}$, then the density $f_{\beta|\chi}(\beta|\chi)$ is point-identified for any β -value from the following formula:

$$f_{\beta|\chi}(\beta|\chi) = g(\beta) \times \frac{\sum_{k=1}^J \exp(\beta w_k^{\text{eff}}(\chi) + \tilde{a}_{k\chi} - \tilde{a}_{j^*\chi})}{\exp(a_{j\chi} - a_{j^*\chi}) f_{\chi|\tau}(\chi)}.$$

□

C.6. Details on the Estimation of Firm-Specific Labor Supply Elasticities

In my estimation procedure, I consider the following nonparametric Kernel estimator:

$$\widehat{\text{DiD}}_{\tau_0, \tau_1}(w|\chi) = \frac{\sum_j K_{1,j}(w) \mathbf{1}\{Z_j = 1\} (\ell_{j\tau_1}(\chi) - \ell_{j\tau_0}(\chi)) - \sum_j K_{0,j}(w) \mathbf{1}\{Z_j = 0\} (\ell_{j\tau_1}(\chi) - \ell_{j\tau_0}(\chi))}{\sum_j K_{1,j}(w) \mathbf{1}\{Z_j = 1\} (w_{j\tau_1}^{\text{eff}}(\chi) - w_{j\tau_0}^{\text{eff}}(\chi)) - \sum_j K_{0,j}(w) \mathbf{1}\{Z_j = 0\} (w_{j\tau_1}^{\text{eff}}(\chi) - w_{j\tau_0}^{\text{eff}}(\chi))},$$

where I define $K_{z,j}(w)$ to be the Kernel weight for firm j with treatment status $z \in \{0, 1\}$.

Examples of kernel functions include the Gaussian and Uniform kernel, defined as follows:

$$\begin{aligned} \text{Gaussian: } K_{z,j}(w) &= \frac{1}{\sqrt{2\pi}} \exp \left[-\frac{1}{2} \left(\frac{w_{j\tau_0}^{\text{eff}}(\chi) - w}{h} \right)^2 \right]. \\ \text{Uniform: } K_{z,j}(w) &= \frac{\mathbb{1}\{w - h \leq w_{j\tau_0}^{\text{eff}}(\chi) \leq w + h\}}{\sum_j \mathbb{1}\{w - h \leq w_{j\tau_0}^{\text{eff}}(\chi) \leq w + h\}}. \end{aligned}$$

In each case, the tuning parameter h determines the bandwidth. As $h \rightarrow 0$, I find that:

$$\lim_{h \rightarrow 0} \widehat{\text{DiD}}_{\tau_0, \tau_1}(w|\chi) = \frac{\frac{1}{N_1} \sum_j \mathbf{1}\{Z_j = 1\}(\ell_{j\tau_1}(\chi) - \ell_{j\tau_0}(\chi)) - \frac{1}{N_0} \sum_j \mathbf{1}\{Z_j = 0\}(\ell_{j\tau_1}(\chi) - \ell_{j\tau_0}(\chi))}{\frac{1}{N_1} \sum_j \mathbf{1}\{Z_j = 1\}(w_{j\tau_1}^{\text{eff}}(\chi) - w_{j\tau_0}^{\text{eff}}(\chi)) - \frac{1}{N_0} \sum_j \mathbf{1}\{Z_j = 0\}(w_{j\tau_1}^{\text{eff}}(\chi) - w_{j\tau_0}^{\text{eff}}(\chi))},$$

where $N_0 = \sum_j \mathbf{1}\{Z_j = 0\}$ and $N_1 = \sum_j \mathbf{1}\{Z_j = 1\}$. Also, by the weak law of large numbers:

$$\begin{aligned} \frac{1}{N_z} \sum_j \mathbf{1}\{Z_j = z\}(\ell_{j\tau_1}(\chi) - \ell_{j\tau_0}(\chi)) &\xrightarrow{P} \mathbb{E} \left[\ell_{j\tau_1}(\chi) - \ell_{j\tau_0}(\chi) | Z_j = z, w_{j\tau_0}^{\text{eff}}(\chi) = w \right], \quad \text{and:} \\ \frac{1}{N_z} \sum_j \mathbf{1}\{Z_j = z\}(w_{j\tau_1}^{\text{eff}}(\chi) - w_{j\tau_0}^{\text{eff}}(\chi)) &\xrightarrow{P} \mathbb{E} \left[w_{j\tau_1}^{\text{eff}}(\chi) - w_{j\tau_0}^{\text{eff}}(\chi) | Z_j = z, w_{j\tau_0}^{\text{eff}}(\chi) = w \right], \end{aligned}$$

for $z \in \{0, 1\}$. Furthermore, using the continuous mapping theorem, I obtain the property:

$$\lim_{h \rightarrow 0} \widehat{\text{DiD}}_{\tau_0, \tau_1}(w|\chi) \xrightarrow{P} \text{DiD}_{\tau_0, \tau_1}(w|\chi) \quad \text{as } J \rightarrow \infty.$$

D. Data Preparation and Robustness Analyses

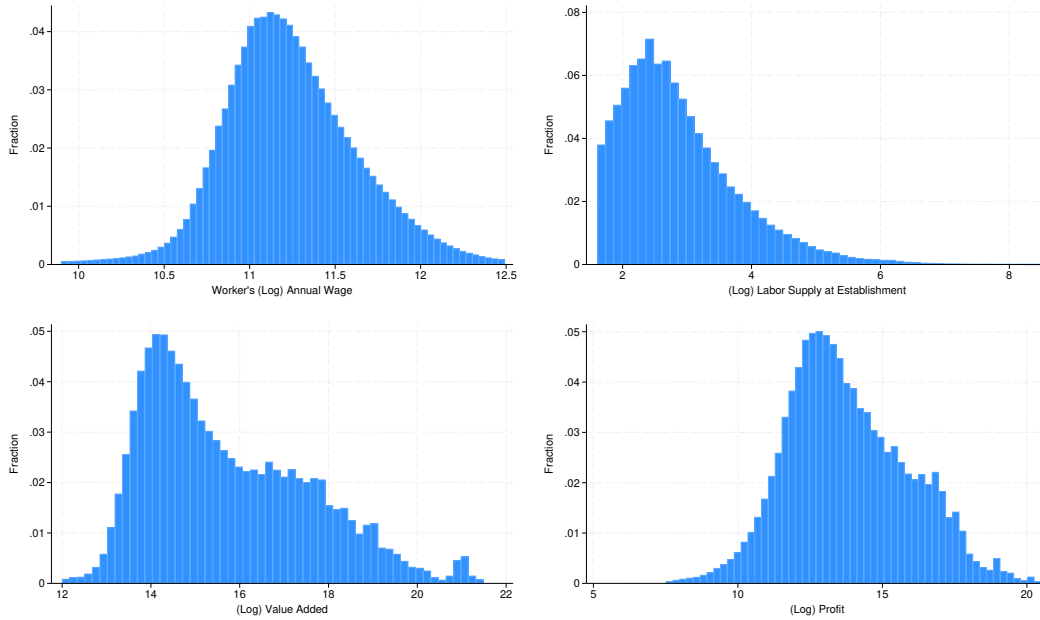
D.1. Details about the Estimation Sample

Figure 8 plots the empirical distributions of log wages, log labor, log value added, and log profits for the main estimation sample. Note that the distribution of log labor is truncated, as the estimation sample is restricted to firms that maintain at least five full-time employees.

Additional Data Sources: External Instruments

Public procurement contracts in Norway are enforced by the Public Procurement Act and associated regulations. The laws apply to procurement of products, services, and construction contracts valued at over NOK 100,000, and they cover all public sector entities in Norway. All

Figure 1.8: Empirical Distributions of Wages, Labor, Value Added, and Profits



Notes. This figure plots histograms of wages, labor, value added, and profits (in logs) for the estimation sample.

public sector procurement is competitive, and procurement auctions are publicly announced on Doffin.no. The number and scale of announcements depends on the size of each contract.

H. de Frahan et al. (2024) collect data on contract award announcements from Doffin.no during the period from 2003 to 2018, encompassing over 60,000 announcements. The data collection process involves three main steps. First, the researchers obtain HTML files through the Doffin IT department, which contain the full history of contract awards for the specified period. Second, they employ standard scraping techniques to extract information from these HTML files. For each contract, various characteristics are collected, including the name of the winning firm, the size and date of the contract, and the product specifications. Notably, the Doffin files do not provide information on the names of losing bidders. Also, for 90% of the contract award announcements, the entire tax identifier of the winning firm is missing.

In the third step, the researchers utilize a fuzzy string matching procedure, following Raffo and Lhuillery (2009), to link firm names in the Doffin data with government registers

that include both the names and tax identifiers of all Norwegian firms. This matching process yields results for approximately 30,000 contracts. The researchers assess the accuracy of the fuzzy matching procedure by analyzing the 10% of contract awards for which both the name and tax identifier of the winning firm are available. They find that the algorithm correctly assigns the tax identifier in 93% of cases. The accuracy of this matching procedure does not appear to be correlated with contract characteristics, such as the value or number of bidders.

The estimation sample is defined using the same data and following similar protocols as those used to analyze the internal instrument. Firms are classified as “treated” when they win their first procurement contracts through Doffin. All firms that receive treatment within the same calendar year τ are categorized into a “cohort”. The control group for a given cohort consists of firms that win their first procurement contract in year $\tau' > \tau + 3$ or never. Future winner control firms are drawn from the Doffin data, and never winners are drawn from the full firm sample. Just as with the main sample, the data is restricted to full-time workers and firms that remain operational, maintaining at least five employees, for nine consecutive years. In total, the Doffin sample is composed of 25,714 unique firms and 901,811 unique workers.

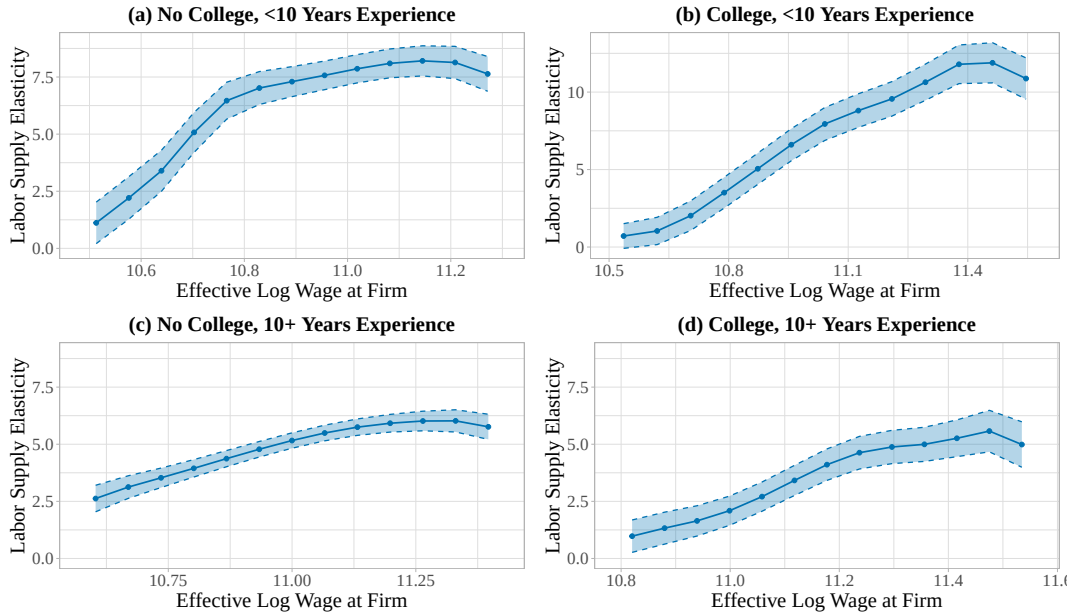
D.2. Robustness Analyses

The first set of robustness analyses replicates the main IV estimates presented in Figure 4 of the paper using alternative specifications. In Figure 9, I show a version of the estimates that exclude market fixed effects, effectively treating Norway as one single labor market. In Figure 10, I report estimates based on a restricted sample where firms only operate a single establishment. In Figures 11 and 12, I show estimates computed using different types of Kernel estimators—Uniform and Epanechnikov kernel functions, respectively. In Figure 13, I provide estimates for a restricted sample of firms with market shares below the median in their local labor markets. Note that these estimates aim to address potential concerns about firms leveraging their market shares to compete strategically for workers. Finally, in Figure

14, I present estimates calibrated using the external instruments described in Appendix D.1.

These exercises reveal that the main IV estimates are robust to specification changes. In particular, restricting the sample to firms with a single establishment or below-median market shares does not significantly affect the results. Additionally, alternative types of estimators, including the nonparametric binning estimator with a Uniform Kernel, yield similar estimates. Furthermore, all the estimates remain relatively stable with and without market fixed effects.

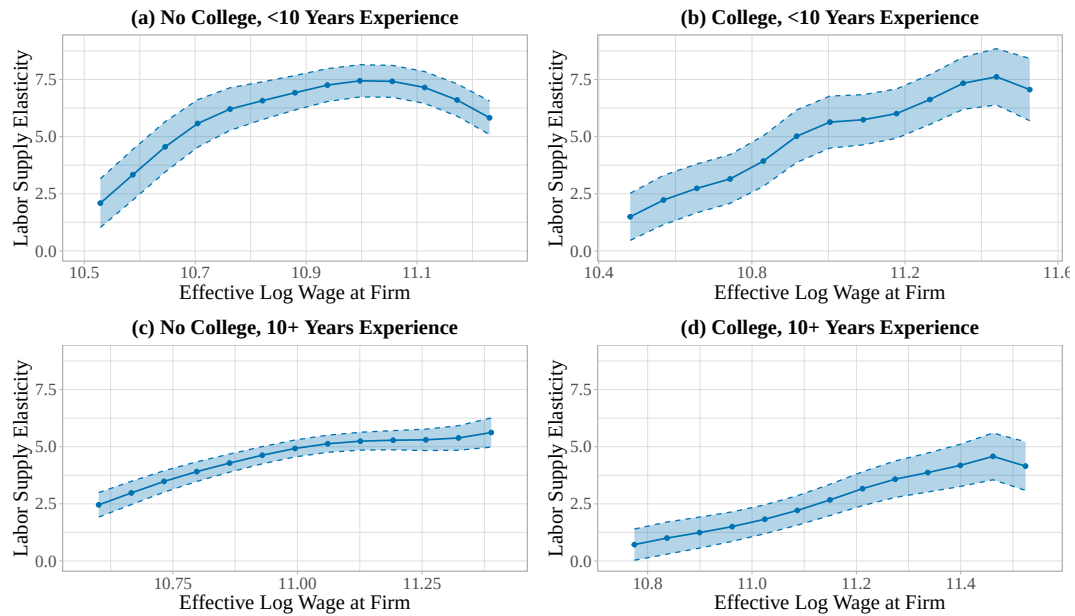
Figure 1.9: IV Estimates of Labor Supply Elasticities—No Market Fixed Effects



Notes. This figure presents IV estimates of firm-specific labor supply elasticities, excluding market fixed effects. Otherwise, the estimates are based on the same specification choices used to generate Figure 4 in the main text.

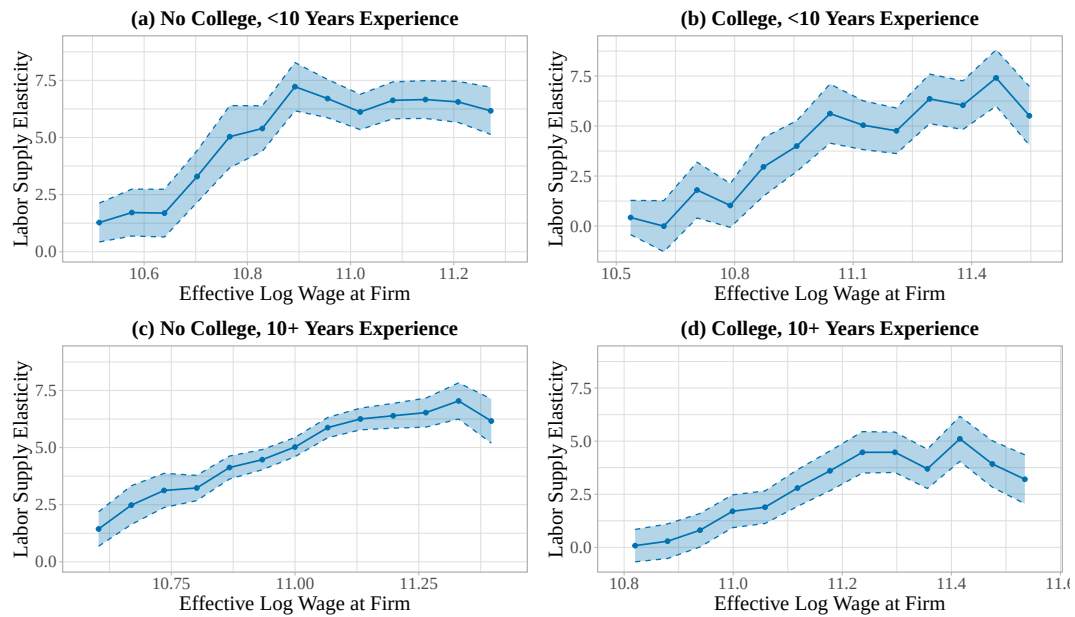
The estimates calibrated using external instruments are not directly comparable to those presented in Figure 4 due to differences in specification choices between the two papers. In addition to imposing different sample restrictions, H. de Frahan et al. (2024) do not account for skill differences among workers. Rather, they estimate wage and employment responses to the instrument using aggregate firm-level measures, such as average wages and total labor. Additionally, these wage and labor responses are estimated at the firm level rather than at the establishment level. Perhaps even more importantly, their DiD estimators for computing wage

Figure 1.10: IV Estimates of Labor Supply Elasticities—Single-Establishment Firms



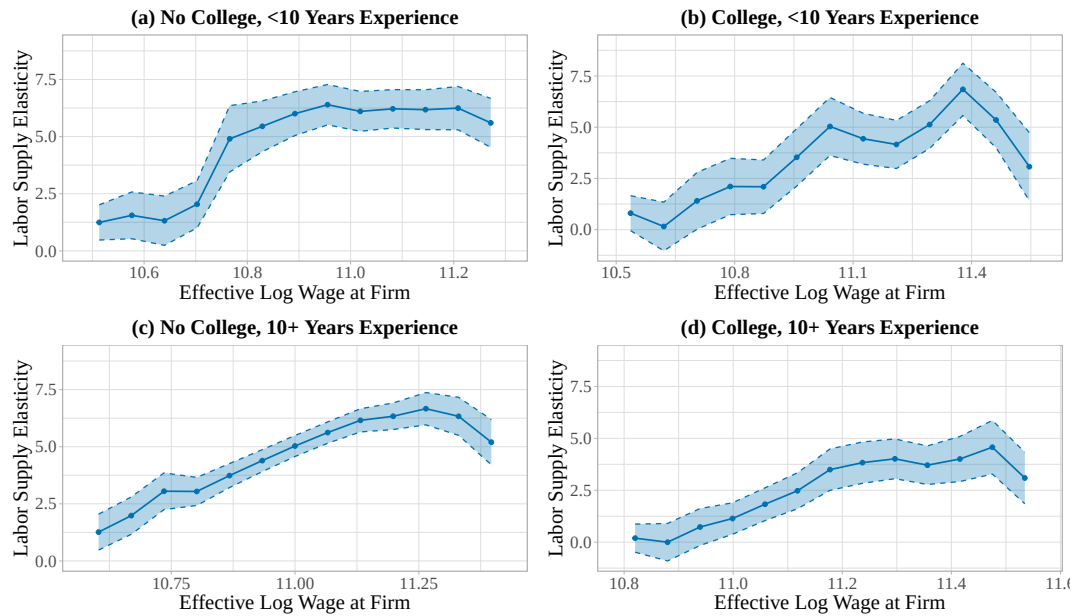
Notes. This figure plots IV estimates of firm-specific labor supply elasticities with the sample restricted to firms with one establishment. Aside from this change, all other specifications are unchanged, aligning with Figure 4.

Figure 1.11: IV Estimates of Labor Supply Elasticities—Uniform Kernel Estimator



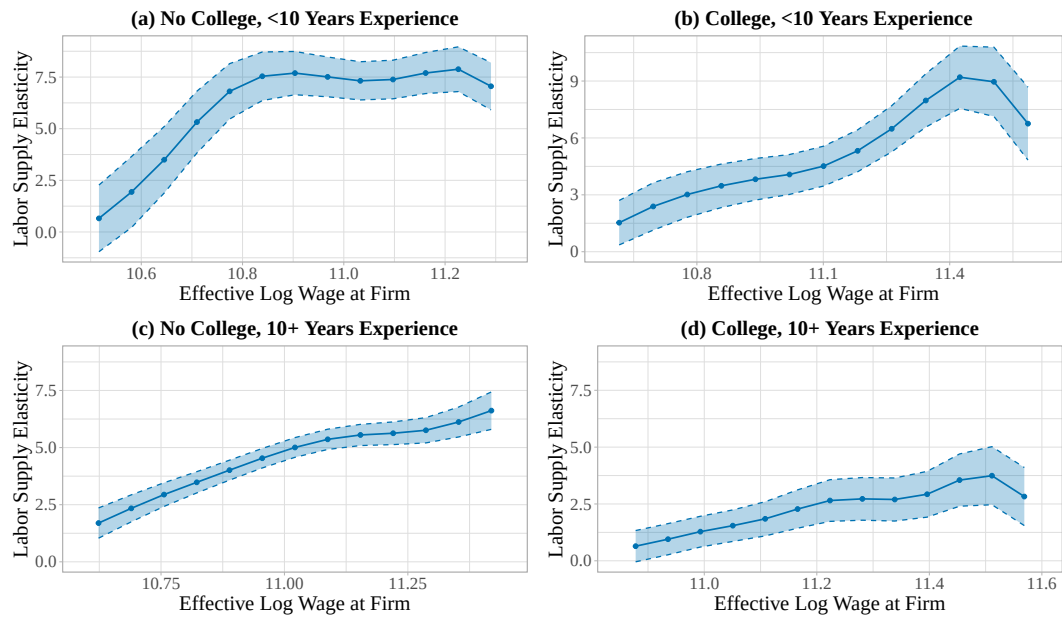
Notes. This figure plots IV estimates of firm-specific labor supply elasticities using a Uniform Kernel estimator. Otherwise, the estimates are based on the same specification choices used to generate Figure 4 in the main text.

Figure 1.12: IV Estimates of Labor Supply Elasticities—Epanechnikov Kernel Estimator



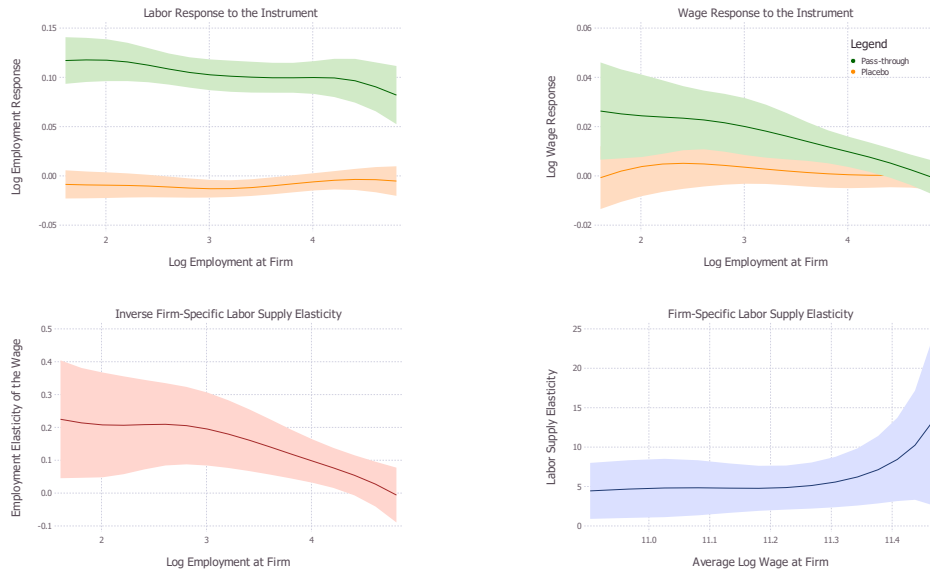
Notes. This figure plots IV estimates of firm-specific labor supply elasticities using an Epanechnikov Kernel estimator. Otherwise, the estimates rely on the same specifications used to generate Figure 4 in the main text.

Figure 1.13: IV Estimates of Labor Supply Elasticities—Small Market Shares



Notes. This figure plots IV estimates of firm-specific labor supply elasticities with the sample restricted to firms with below-median market shares. Otherwise, all other specifications are unchanged and align with Figure 4.

Figure 1.14: Calibrated Estimates for External Instruments



Notes. This figure plots the reduced form estimates for the external instrument, averaged across skill types and years. The point estimates and standard errors in the first two figures come from H. de Frahan et al. (2024). The bottom two figures show the estimated firm-specific labor supply elasticities calibrated to my framework. Note that elasticities are measured at the firm level, and they do not account for skill differences between workers.

and employment responses differ from mine. Whereas I control for a firm's initial effective wages, the estimator used by H. de Frahan et al. (2024) controls for a firm's initial labor share. This approach would be invalid under my framework, as I allow for the possibility that two firms with identical labor shares face different labor supply elasticities. Taken together, these differences in methodology could lead to notable variations in the IV estimates.

Despite these differences, when I calibrate my model using the wage and labor responses estimated by H. de Frahan et al. (2024), I observe a pattern that closely aligns with the main IV estimates in my paper. Specifically, I find that labor supply elasticities vary significantly across firms, with higher elasticities being associated with higher average wages at the firm. Any discrepancies between the two sets of estimates may be attributed to the specification differences outlined above. Yet, it is reassuring that the same patterns emerge in both cases.

The second set of robustness analyses are misspecification tests. First, I test whether the

internal instrument Z impacts labor on the intensive margin by influencing work hours. I find that there is no significant effect. Next, I test whether the IV estimates (averaged across years, firms, and skill types) differs between firms with below-median and above-median market shares in their local labor market. I find no significant market share effect, which supports my assumption that firms do not internalize their market shares when setting workers' wages.

Table 1.5: Two Misspecification Tests

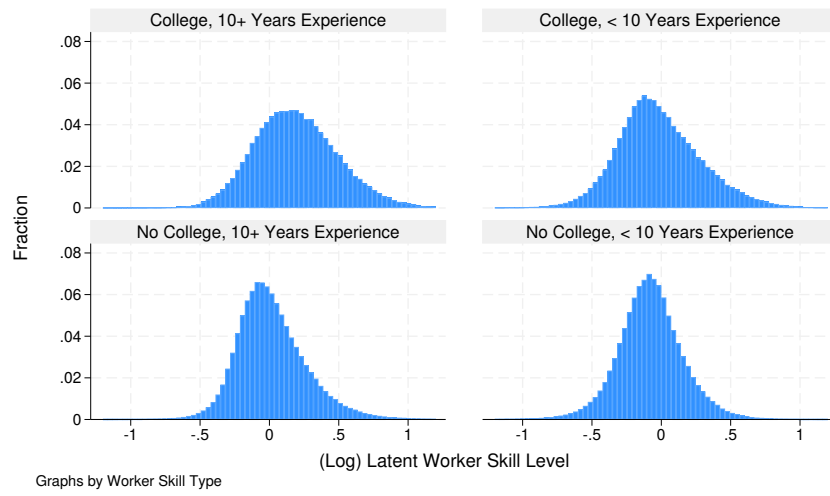
	Effect of Z on Work Hours		Market Share Effect	
	Estimated Effect	p -value	Estimated Effect	p -value
Market FEs	-0.011 (0.007)	0.105	-0.380 (0.773)	0.311
No Market FEs	-0.003 (0.007)	0.624	-0.073 (0.774)	0.462

Notes. This table presents two misspecification tests. The first two columns report the estimated effect of Z on worker's scheduled hours. The third and fourth columns report the effect of having an above-median local market share on the labor supply elasticities. Both tests are conducted with and without market fixed effects.

D.3. Additional Tables and Figures

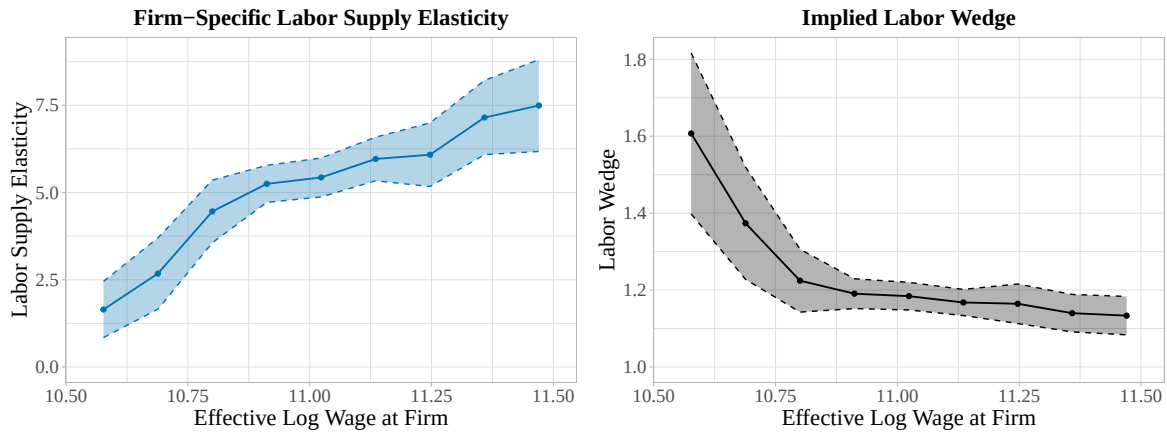
This Appendix section provides additional tables and figures that are not included in the main text of the paper. In Figure 15, I plot the estimated distributions of worker skill levels, disaggregated by college attainment and experience. In Figure 16, I plot the IV estimates for my main specification, averaged across all years and skill types. In Figure 17, I plot the estimated distributions of worker and firm rents across firms in the main estimation sample.

Figure 1.15: Estimated Distributions of Worker Skill Levels



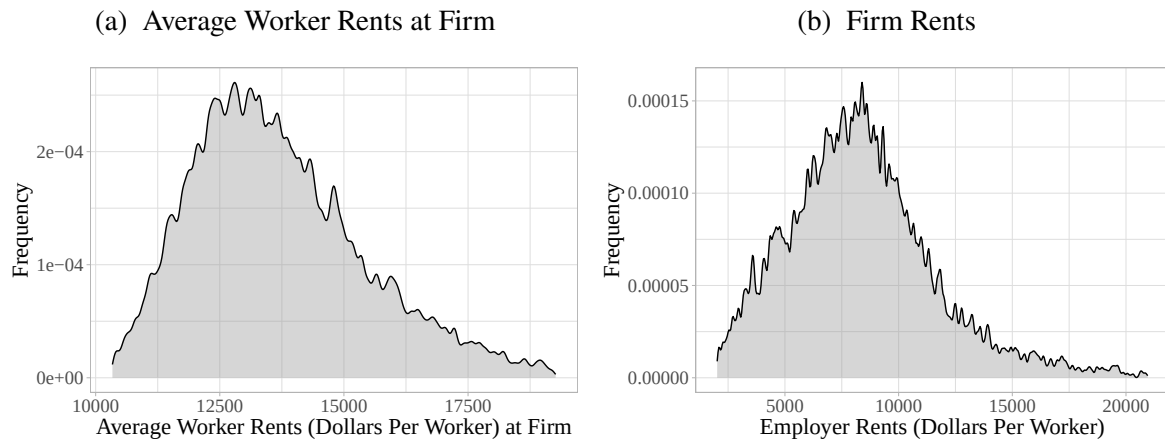
Notes. This figure plots the the estimated distributions of worker skill levels by college attainment and experience.

Figure 1.16: Average IV Estimates by Effective Wage



Notes. This figure plots firm-specific labor supply elasticities and labor wedges, averaged across all subgroups and years. Market fixed effects are included. All standard errors are bootstrapped using 500 bootstrap samples.

Figure 1.17: Estimated Distributions of Worker and Firm Rents



Notes. This figure plots the distributions of estimated firm rents and estimated average worker rents at a firm.

Chapter 2

Discrete Choice with Generalized Social Interactions

I. Introduction

The role of social interactions in individual decisionmaking has received widespread attention in economics. This work has largely focused on settings with uniform strategic complementarities, where agents seek to conform to the average action in the rest of their network. Meanwhile, less attention has been given to settings with nonuniform interactions, where agents conform to the behavior of some people, while deviating from others. These types of interactions are more challenging to study, particularly when agents make discrete choices, since the nonlinear dependence between outcomes and the potential for multiple equilibria may complicate the theoretical understanding of behavioral outcomes and also pose barriers to identification and estimation. Nevertheless, nonuniform interactions appear in many real-world contexts. Consider, for example, the impact of gender on peer effects in schools. Male and female students often face pressure to conform to members of their own gender, while there may be less pressure to conform across genders. In some cases, students might even wish to distinguish themselves from the opposite gender—a preference that can be modeled by a negative interaction effect. As Hoxby (2000) and others discuss, these types

of peer effects may contribute to gender differences in behavior and academic achievement. More generally, as Akerlof & Kranton (2000) argue, social identity (e.g., gender, race, class, or politics) helps to determine a wide range of phenomena, and it is relevant for most everyday decisions that people make. Given these considerations, it is worth understanding the impact of nonuniform, and even negative, social interaction effects on economic outcomes.

In this paper, I develop a framework for studying generalized social interactions, where agents are affected differently by different people in their network. I focus on a model with binary choices where agents experience positive and negative spillover effects. I characterize what aggregate outcomes can occur in equilibrium, and I show how the model is empirically tractable. This work may be especially useful for understanding the impact of identity on social and economic behavior, as well as the emergence of polarized or segregated networks.

I begin my analysis by modeling a network of agents who make binary choices subject to social influences and private, idiosyncratic preferences. Following Brock & Durlauf (2001), I assume that agents act with incomplete information, such that they are influenced by the expected average outcomes in the network, rather than by every person's realized outcome. However, unlike previous work, I assume that the social interaction effects differ, and might even be negative, across agents. To achieve this objective, I categorize the agents into different social groups (or "identities"). These classifications play a role in determining private payoffs, and they also form the basis for nonuniform social interactions. In particular, the interaction effects can be expressed in terms of a matrix $\mathbf{J}$, where each entry $J_{k\ell}$ indicates how an agent in group k is influenced by the expected average action of people in group ℓ .

I examine equilibrium properties of the model under noncooperative decisionmaking. These properties are well-studied in settings with uniform complementarities, where the spillover effects are all positive; e.g., consider Glaeser, Sacerdote & Scheinkman (1996, 2003), Brock & Durlauf (2001), Calvo-Armengol, Patacchini & Zenou (2009), and Cabrales,

Calvo-Armengol & Zenou (2011), among others. In these settings, the interactions produce social multipliers, which can lead to large differences in aggregate outcomes across similar populations. Models with uniformly positive interactions may exhibit multiple, locally stable equilibria, which are generally well-ordered, in the sense that they form a complete lattice.¹

Overall, there is less known about the equilibrium properties of models with both positive and negative interaction effects. One issue is that the tools economists use to analyze settings with complementarities do not readily transfer to settings with strategic substitutabilities, i.e., negative spillovers. For example, models involving negative interaction effects may fail to possess a pure strategy Nash equilibrium. Moreover, even when equilibria do exist, it may be that none of them are dynamically stable, which means that they are unlikely to be observed by researchers.² In much of the networks literature, e.g., Ballester et al. (2006), Galeotti et al. (2010), Bramoullé et al. (2014), contraction mapping arguments are used to prove that there exists a unique, stable equilibrium if the spillover effects are sufficiently weak. I derive a similar, albeit slightly more general, result for my model by using an index theorem. However, the bulk of my theoretical analysis focuses on cases with strong interaction effects where unstable equilibria arise and where aggregate outcomes become harder to interpret.

I identify two conditions—one strictly weaker than the other—that each guarantees the existence of a locally stable equilibrium in contexts with strong interaction effects. Both conditions have meaningful economic interpretations. In either a strict or weak sense, they require that agents are not repelled by their own social group. Using these two conditions, I show how key results in the complementarities literature extend to a broader class of models

¹The microeconomic foundations for models of complementarity are largely developed by Donald Topkis (see Topkis (1998) for a summary), Vives (1990), and Milgrom & Roberts (1990). This work leverages key results from lattice theory, e.g., Tarski's fixed point theorem, to study equilibrium behavior in settings with uniform complementarities. Cooper & John (1988) also provide an early discussion about the role of complementarity in economics. For related work, see Milgrom & Shannon (1994) and Athey (2001, 2002).

²Jackson & Zenou (2015) describe several unresolved questions regarding how and why network-based models with substitutabilities behave differently from models with uniform complementarities. Of course, strategic substitutabilities appear frequently in economics, albeit outside of the social interactions literature; see Bramoullé & Kranton (2007), Bramoullé et al. (2014), and Elliott & Golub (2019) for recent contributions.

that involve substitutabilities. I start by establishing necessary and sufficient conditions for the existence of multiple equilibria. I show that multiplicity depends on a single parameter: the spectral radius of the Jacobian matrix of the equilibrium system. This number measures the intensity of the cycles in a network, thereby quantifying the cumulative strength of the interaction effects. I find that a unique, stable equilibrium exists if the spectral radius is below a certain threshold, while multiple, locally stable equilibria exist if the spectral radius is above this threshold. Next, I consider the social welfare of agents at different equilibria. I find that negative interaction effects can introduce welfare trade-offs, such that it is impossible to jointly maximize the expected utility of agents in different social groups. Specifically, if two agents who otherwise prefer the same action are negatively influenced by one another, then the equilibrium that maximizes welfare for one may not maximize welfare for the other. This result is instructive for policy analysis as it carries implications about social inefficiency.

In the second half of the paper, I turn to the question of identification. That is, given data on individual choices across multiple networks, what can be said about the role of social interactions? The identification of network-based discrete choice models with incomplete information is carefully studied by Brock & Durlauf (2001, 2007), and more recently by Bhattacharya et al. (2023). Paula (2017) and Kline & Tamer (2020) also give recent reviews of the literature.³ My analysis builds on this work by tackling two key identification problems.

First, I address the issue of unobservable network effects. These contextual factors may prevent researchers from uncovering the role of social interactions. For example, when comparing student achievement across different classrooms, it is hard to distinguish the impact of peer effects from unobserved teacher quality or unknown class characteristics. Much of the applied literature tends to rule out these unknown network factors.⁴ Alternatively, some

³Blume et al. (2011) also review issues about identification of linear and nonlinear network-based models.

⁴For example, Brock & Durlauf (2001, 2007) assume that the network effect is a constant linear function of observed variables, whereas Bhattacharya et al. (2023) assume that it exhibits a specific linear factor structure.

papers have used a panel to difference-out the fixed effects between time periods; e.g., see Hoxby (2000) and Brock & Durlauf (2007). However, this approach relies on having access to panel data, as well as the assumption that the model parameters do not change over time.

To handle the issue of unobservable network effects, I introduce a novel approach that allows for the partial identification of social interactions. Importantly, this method places no restrictions on the network-level determinants of people's actions. Rather, it leverages a panel structure that is inherent in the model, whereby members of different social groups interact in the same network. Since these agents face the same contextual factors, I can control for network effects by comparing the outcomes of different social groups in a given network. Using this approach, I recover the differences between any two social interaction effects, which I can then use to measure the amount of polarization in a network.⁵ This technique of exploiting within-network variation in individual-level characteristics is new to the social interactions literature. However, similar approaches have been proposed in other areas of economics. For example, Berry & Haile (2022) show how to use data about heterogeneity within markets to reduce the number of instruments needed to estimate systems of demand.

Next, I address an issue that is common to most network-based models with incomplete information, which is that the expected average outcomes in a network are never directly observed. Instead, a researcher only sees the average actions among finitely-many agents. Since these observed averages converge to the true expectations as the networks grow large, previous research has treated this issue as an estimation problem rather than as a barrier to identification; e.g., see Bhattacharya et al. (2023). These estimation strategies rely on double asymptotic arguments, such that the number of networks and the size of each network must both tend to infinity. I argue that these approaches are not advisable given that replacing

⁵My definition of polarization contributes to a burgeoning literature documenting how people's choices are divided along cultural or political lines; e.g., see Boxell et al. (2022) and Bertrand & Kamenica (2023). In this literature, there is still no clear consensus on how to measure polarization as it is a relatively abstract concept. One benefit of my approach is that I measure polarization in a way that is motivated by an economic model.

expectations with observed averages always leads to measurement error. Indeed, I show that, in any setting with finite networks, the observed average outcomes are endogenous in the model. Moreover, by failing to account for the endogeneity, most common estimation methods would produce biased estimates—even as the number of networks tends to infinity.

I approach the issue as a classical measurement error problem (Wooldridge, 2013, sec. 15.4). Specifically, I note that the observed average outcomes are noisy approximations of the true expectations and are therefore endogenous. To correct for this endogeneity, I use internal instruments. This procedure involves randomly partitioning the network into two parts and then computing the sample average action within each part. Using these sample averages, I treat one as the endogenous variable and the other as an instrument. Since each network is partitioned randomly, this IV strategy ensures that the model parameters are identified even for small networks. I then define an IV estimator and prove it is consistent. I also demonstrate the efficacy of the estimation strategy by running Monte Carlo simulations.

Finally, I provide an empirical application of the model and identification strategy using data from the class size reduction experiment Project STAR. Prior research has used this data to measure peer effects in classrooms, e.g., see Boozer & Cacciola (2001) and Graham (2008). However, I leverage the generalized interactions framework to study a new question: How do peer effects differ by gender? For this application, I find evidence of significant gender differences in peer effects, where male and female students are both more likely to conform to members of their own gender than to members of the opposite gender. This finding is notable as it contributes to a longstanding literature about the impacts of gender and social pressure on academic achievement, e.g., see Hoxby (2000), Lavy & Schlosser (2011), and Bostwick & Weinberg (2022). While previous work has primarily focused on the effects of gender composition, I explore how students are directly affected by the expected achievement of their male and female peers. Moreover, I perform this analysis while accounting for unobserved

classroom-level determinants of student outcomes. Overall, my empirical findings illustrate the advantages of using a generalized interactions framework to learn about systematic differences in social interactions. They also contribute to a growing literature on identity in economics; see Charness & Chen (2020) and Shayo (2020) for discussions.

This paper proceeds as follows. Section II describes the binary choice model. Section III explores the equilibrium properties of the model, such as existence, uniqueness, dynamic stability, and social welfare. Section IV considers alternative specifications and also explores how the equilibrium properties generalize to a broader class of models. Section V explains the identification strategy, along with the estimation procedure and the simulation results. Section VI presents the empirical application and key findings. Lastly, Section VII concludes.

II. A Model with Generalized Social Interaction Effects

I study a binary choice model with interaction effects that vary on the basis of group identity. This model extends Brock & Durlauf’s (2001) binary choice framework by assuming that individuals are influenced differently—perhaps even negatively—by different people. For example, a person might seek to conform to the average behavior in certain groups, while distinguishing herself from others. Unlike previous work, I allow the distribution of idiosyncratic preferences to be nonparametric. This feature ensures that the equilibrium properties of the model will be robust under relatively loose functional form assumptions.

II.A. Individual Preferences

Consider a network of I agents, where each agent i belongs to one of K social groups. Every agent chooses a binary action $\omega_i \in \{0, 1\}$ at a common time. Let $\bar{\omega}^k$ be the average action in group k , and let $\bar{\omega}_{-i}^k$ be the average action among members of group k excluding i .

When forming decisions, people are influenced by the expected behavior of everyone

else in their network. For any person i in group k , the utility from choosing an action ω_i is:

$$U_i(\omega_i|k) = v_k(\omega_i) + J_{kk}\omega_i E_i(\bar{\omega}_{-i}^k) + \sum_{\ell \neq k} J_{k\ell}\omega_i E_i(\bar{\omega}^\ell) + \epsilon_i(\omega_i). \quad (2.1)$$

Here, $E_i(\bar{\omega}_{-i}^k)$ and $E_i(\bar{\omega}^\ell)$ represent i 's subjective expectations about $\bar{\omega}_{-i}^k$ and $\bar{\omega}^\ell$, respectively. The term $v_k(\omega_i)$ is the private utility associated with a choice. This utility may vary by group membership. Finally, $\epsilon_i(\omega_i)$ is an idiosyncratic preference that is independent across agents.

Under this framework, utility exhibits proportional spillovers, so there is a multiplicative interaction between an agent's choice and the expected average choice in every group.⁶ Each term $J_{k\ell}$ captures how much members of group k seek to conform to the mean behavior in ℓ .

Because the action is binary, I can write $v_k(\cdot)$ as an affine function: $v_k(\omega_i) = h_k\omega_i + \eta_k$. Also, I can write $\epsilon_i(\omega_i) = \varepsilon_i\omega_i + \xi_i$ without loss of generality, where ε_i and ξ_i are random coefficients in the model. Note that h_k parameterizes the deterministic private utility bias toward $\omega_i = 1$ for an agent in group k , while ε_i captures the agent's own idiosyncratic payoff.

I assume that the idiosyncratic payoffs ε_i may be distributed differently in every group:

$$P(\varepsilon_i \leq z|k) = F_{\varepsilon|k}(z), \quad \text{for } k = 1, \dots, K, \quad (2.2)$$

where $F_{\varepsilon|k}(\cdot)$ is continuously differentiable, symmetric about zero, and has positive density everywhere. I make no further parametric assumptions about these distributions. So, this framework applies for a variety of empirical specifications, e.g., logistic or Gaussian errors.

Three quantities are especially important for characterizing the model. First, there is a vector $h = (h_1, \dots, h_K)'$ of private utility terms, which specifies each group's intrinsic preference over the two actions. Second, there is a vector of distribution functions $\{F_{\varepsilon|k}\}_{k=1}^K$,

⁶The spillover term $J_{k\ell}\omega_i E_i(\bar{\omega}^\ell)$ may also be generated from a quadratic conformity effect $-\frac{1}{2}J_{k\ell}[\omega_i - E_i(\bar{\omega}^\ell)]^2$ as studied by Bernheim (1994). See Brock & Durlauf (2001) and Blume et al. (2015) for discussion.

which determine how likely it is that any idiosyncratic payoff is realized in each group. Third, there is a matrix $\mathbf{J} \in \mathbb{R}^{K \times K}$ containing all the social interaction effects:

$$\mathbf{J} = \begin{bmatrix} J_{11} & J_{12} & \cdots & J_{1K} \\ J_{21} & J_{22} & \cdots & J_{2K} \\ \vdots & \vdots & \ddots & \vdots \\ J_{K1} & J_{K2} & \cdots & J_{KK} \end{bmatrix}. \quad (2.3)$$

Throughout this paper, I will refer to $\mathbf{J}$ as the *interaction matrix*. It may also be interpreted as the adjacency matrix of a directed graph $(\mathcal{K}, \mathbf{J})$, where the nodes $\mathcal{K} = \{1, \dots, K\}$ represent different groups of individuals.⁷ The entries of $\mathbf{J}$ specify the nature and intensity of the relationships between groups. Note that the interaction effects may not be symmetric, so $J_{k\ell}$ need not equal $J_{\ell k}$ for $k, \ell \in \mathcal{K}$. In addition, any of the interaction effects could be negative.

II.B. Equilibrium under Noncooperative Decisionmaking

When analyzing this model, I focus on (pure strategy) Bayesian Nash equilibria where agents act noncooperatively. In other words, agents do not coordinate with one another when making decisions. Each agent i in group k chooses the action $\omega_i = 1$ with probability:

$$P(\omega_i = 1|k) = F_{\varepsilon|k} \left(h_k + J_{kk} E_i(\bar{\omega}_{-i}^k) + \sum_{\ell \neq k} J_{k\ell} E_i(\bar{\omega}^\ell) \right). \quad (2.4)$$

Since ω_i takes values in the set $\{0, 1\}$, the expected action $E(\omega_i|k)$ also equals $P(\omega_i = 1|k)$.

I assume that everyone has rational expectations about other people's choices. So, while agents cannot directly observe the actions of others, they do correctly infer these actions in expectation, i.e., $E_i(\omega_j|k) = E(\omega_j|k)$ for all agents i and j , and all groups k . By symmetry of the conditional expected choices, it follows that $E(\omega_i|k) = E(\omega_j|k)$ for all i, j , and k .

⁷For linear-in-means models, this matrix is sometimes called a “sociomatrix” (e.g., see Blume et al., 2015).

An equilibrium is defined by the expected average choices $\{E(\bar{\omega}^k)\}_{k=1}^K$ that are consistent with individually optimal decisions. Letting m^{k*} denote $E(\bar{\omega}^k)$, I can write this condition as:

$$m^{k*} = F_{\varepsilon|k} \left(h_k + \sum_{\ell=1}^K J_{k\ell} m^{\ell*} \right), \quad \text{for } k = 1, \dots, K. \quad (2.5)$$

Any fixed point solution $m^* = (m^{1*}, \dots, m^{K*})$ to this system of equations is an equilibrium.

III. Equilibrium Properties of the Model

I now examine various properties of an equilibrium, such as existence, dynamic stability, and uniqueness. I also explore the implications of equilibrium outcomes for social welfare.

III.A. Existence

As the distributions $\{F_{\varepsilon|k}\}_{k=1}^K$ are continuous and the support of m^* is $[0, 1]^K$, Brouwer's fixed point theorem guarantees that there is at least one solution to the equilibrium system.

Property 1. There exists at least one equilibrium m^* in the binary choice model.

III.B. Dynamic Stability

Another key equilibrium property is dynamic stability. This property asserts that any iteration on best response dynamics would converge to an equilibrium. Formally, consider a dynamic analogue of the equilibrium system: $m_t^k = F_{\varepsilon|k} \left(h_k + \sum_{\ell=1}^K J_{k\ell} m_{t-1}^\ell \right)$ for $k = 1, \dots, K$. An equilibrium m^* is defined to be *locally stable* if it is a limiting solution to this dynamical system, where the initial iterate m_0 lies within some sufficiently small neighborhood of m^* . Alternatively, an equilibrium is *unstable* if there is a neighborhood of m^* such that, for any m_0 arbitrarily close to m^* , there is some eventual iterate m_t that lies outside of the neighborhood.

The question of local stability is fundamental to comparative statics. If a locally stable equilibrium is slightly perturbed, then the average outcomes in a network would return to that

equilibrium. Meanwhile, if an equilibrium is unstable, then nearby outcomes would diverge from it. In practice, locally stable equilibria are the ones that a researcher observes, while unstable equilibria represent tipping points between different equilibrium outcomes.

To assess when an equilibrium is locally stable, I need to introduce some terminology. First, I define the *Jacobian matrix* of the right-hand-side of the equilibrium system (2.5) as:

$$\mathbf{D}(m^*) = \boldsymbol{\beta}(m^*)\mathbf{J}. \quad (2.6)$$

Here, $\boldsymbol{\beta}(m^*) = \text{diag} [f_{\varepsilon|1}(h_1 + \sum_{\ell=1}^K J_{1\ell}m^{\ell*}), \dots, f_{\varepsilon|K}(h_K + \sum_{\ell=1}^K J_{K\ell}m^{\ell*})]$ is a diagonal matrix of densities $f_{\varepsilon|k}(h_k + \sum_{\ell=1}^K J_{k\ell}m^{\ell*})$, each representing the relative likelihood that an agent in group k is close to indifferent between the two actions at an equilibrium m^* . So, $\mathbf{D}(m^*)$ equals the interaction matrix $\mathbf{J}$ where each row k is weighted by the expected fraction of group k that is near indifferent at m^* . As all density functions $f_{\varepsilon|k}$ are strictly positive, the entries $D_{k\ell}(m^*)$ of the Jacobian matrix have the same signs as the interaction effects $J_{k\ell}$.

Next, I define the *spectral radius* of the Jacobian matrix $\rho(\mathbf{D}(m^*))$ as the largest eigenvalue of this matrix in absolute value. Formally, for any square matrix $\mathbf{A}$, this quantity is:

$$\rho(\mathbf{A}) = \max \{|\lambda| : \lambda \text{ is an eigenvalue of } \mathbf{A}\}. \quad (2.7)$$

A matrix $\mathbf{A}$ is convergent, in the sense that $\lim_{t \rightarrow \infty} \mathbf{A}^t = \mathbf{0}$, if and only if $\rho(\mathbf{A}) < 1$. So, a larger spectral radius corresponds to a more expansive matrix. In this model, $\rho(\mathbf{D}(m^*))$ allows me to measure the collective strength of the social interaction effects within and across groups. As seen by the next property, this quantity governs the local stability of equilibria.

Property 2. If $\rho(\mathbf{D}(m^*)) < 1$, then m^* is locally stable. If $\rho(\mathbf{D}(m^*)) > 1$, then m^* is unstable.

This property almost gives a necessary and sufficient condition for local stability. It falls short only when $\rho(\mathbf{D}(m^*))$ equals one. However, this case is not especially relevant in this

model, since it is not economically meaningful and it occurs with probability measure zero.⁸

III.C. Uniqueness

This model has the potential to exhibit multiple equilibria. In other words, for some fixed parameter values, there may be multiple solutions to the equilibrium equations. In this subsection, I examine what social environments lead to uniqueness versus multiplicity.

Social Environments with a Unique Equilibrium

When is there only one equilibrium? This situation arises whenever the social interaction effects are not strong enough to generate any unstable equilibria. To see why, consider the next property, which follows from Sard's Theorem and the Poincaré-Hopf Index Theorem.

Property 3. For almost all distributions $\{F_{\varepsilon|k}\}_{k=1}^K$, the number of equilibria is finite and odd. Also, if there are d_s locally stable equilibria, then there are at least $d_s - 1$ unstable equilibria.

Using Property 3, I recover a sufficient condition for uniqueness. Namely, if all equilibria are locally stable, then there can only be one. This reasoning leads to the following corollary.

Corollary. If $\rho(\mathbf{D}(m^*)) < 1$ at all equilibria m^* , then there is a unique, locally stable equilibrium.

Recall that $\rho(\mathbf{D}(m^*))$ measures the intensity of social interactions, weighted by the likelihood that agents are indifferent between the two choices. So, a unique equilibrium exists if the social interactions are relatively weak and/or if agents have strong private preferences.⁹

⁸When $\rho(\mathbf{D}(m^*)) = 1$, the stability of m^* is governed by the Hessian matrices of the equilibrium equations. However, for almost all $\{F_{\varepsilon|k}\}_{k=1}^K$, the Jacobian $\mathbf{D}(m^*)$ evaluated at any equilibrium m^* has no unit eigenvalues.

⁹By using the Poincaré-Hopf Index Theorem, I obtain a sufficient condition for uniqueness that is weaker than the conditions that are adopted in much of the literature, i.e., those that are implied by the Banach contraction mapping theorem. Note that Property 3 is consistent with a longstanding literature about the oddness of Nash equilibria in finite games; for example, see Wilson (1971) Harsanyi (1973), and Kohlberg & Mertens (1986).

Social Environments with Multiple Equilibria

When is there more than one equilibrium? By Property 3, multiplicity can only occur in settings with strong interaction effects where unstable equilibria exist. However, strong interactions do not always imply multiplicity. In some cases, there is a unique equilibrium that is unstable.¹⁰ To interpret equilibrium behavior when there are strong interaction effects, I must first examine the role of strategic complementarity and substitutability in networks.

Negative Spillovers and Global Instability of Equilibria

If all the entries of the interaction matrix $\mathbf{J}$ are non-negative, then the model exhibits strategic complementarity between all agents. In this case, each person's utility is a super-modular function of individual choices, which means that the marginal payoff from one's action (weakly) increases when anyone else chooses that same action. This type of model has important properties. Most notably, it almost always has a locally stable equilibrium.¹¹

If the interaction matrix has negative entries, then these same equilibrium properties do not generally apply. In particular, some interaction effects are incompatible with locally stable equilibria. This instability has an economic interpretation. It arises when agents are strongly repelled by the choices in their own group. In these cases, the group can never settle on one average action because its members will always be discontented with the outcome.

To better understand this point, assume there is one group in the network, i.e. $K = 1$, and suppose, for simplicity, that there is no private utility bias, i.e. $h = 0$. In this case, an equilibrium m^* is defined as the fixed point solution to $m^* = F_\varepsilon(Jm^*)$ (group subscripts are removed for notational convenience). When $J < 0$, agents are repelled by the expected average action in the population. Therefore, if $E(\bar{\omega})$ is high, then people tend to prefer the

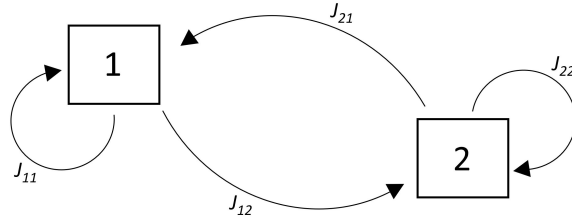
¹⁰One example of this phenomenon occurs when the interaction matrix $\mathbf{J}$ is symmetric and only has eigenvalues with non-positive real parts. For such interactions, the model always has a unique equilibrium m^* , which is locally stable when $\rho(\mathbf{D}(m^*)) < 1$ and is unstable when $\rho(\mathbf{D}(m^*)) > 1$. See the Appendix for details.

¹¹Milgrom & Roberts (1990, 1994) and Milgrom & Shannon (1994) give properties of supermodular games.

low action. Conversely, if $E(\bar{\omega})$ is low, then most people prefer the high action. When $J < 0$ is sufficiently large in magnitude, the equilibrium behavior in the model becomes unstable.¹²

When there are multiple groups, the role of negative interactions is more complicated. To illustrate why, suppose that there are now two groups, i.e. $K = 2$. The interactions in this case are depicted in Figure 1, where arrows indicate the direction of each effect. By the same reasoning as before, a stable equilibrium cannot exist if the within-group effects J_{11} and J_{22} are very negative. Additionally, consider what happens when $J_{12} > 0$ and $J_{21} < 0$. Members of group 1 seek to conform to the mean behavior in group 2, while members of group 2 want to distinguish themselves from group 1. These social influences, if they are strong enough, lead to self-contradictory behavior: if $E(\bar{\omega}^2)$ is high, then $E(\bar{\omega}^1)$ is also high, which means that $E(\bar{\omega}^2)$ is low, and so forth. In this setting, equilibrium outcomes again become unstable.

Figure 2.1: Social Interactions with Two Groups



Social Interactions that Maintain Stable Equilibria

Going forward, it will be useful to classify what types of social interactions are conducive to stable behavioral outcomes. The discussion above suggests that global instability arises when preferences are self-contradictory in the sense that a population responds negatively to its own average choice. So, it makes sense to rule out these cases. The following condition captures the idea that all groups are overall positively affected by their own average choices.

Assumption A. There is an invertible matrix $\mathbf{B}$ such that $\mathbf{B}\mathbf{J}\mathbf{B}^{-1}$ has all non-negative entries.

¹²Specifically, global dynamic instability occurs when $J < -f_{\varepsilon}^{-1}(0)$. For an explanation, see the Appendix.

This assumption states that the interaction matrix $\mathbf{J}$ is similar to a non-negative matrix. When two matrices are similar, they represent the same linear operator under different bases. In this way, Assumption A ensures that the model behaves similarly to one where all the interactions are non-negative, i.e. where there is strategic complementarity between all agents.

On its face, Assumption A is hard to interpret. Nevertheless, it covers a range of contexts where agents do not wish to deviate from their own group. I highlight two special cases.

Example 1. The way that agents are affected by their own group depends on the cycles in a network. These cycles dictate how an agent’s choice is reflected back onto itself via social interactions. Suppose that the product of interaction effects along any cycle is non-negative.

Assumption A.1. For any k and integer M , $J_{kj_1}J_{j_1j_2} \cdots J_{j_Mk} \geq 0$ for all $j_1, \dots, j_M \in \mathcal{K}$.

Under this condition, agents would never be repelled by their own group. For example, with two groups, A.1 implies: $J_{11} \geq 0$, $J_{22} \geq 0$, and $J_{12}J_{21} \geq 0$. This restriction does not rule out negative interactions, but it requires that J_{12} and J_{21} have the same sign. In other words, between-group relations are mutual: agents either “agree to agree” or “agree to disagree”.

More broadly, Assumption A.1 holds if the indirect spillover effect $J_{km}J_{m\ell}$ has the same sign as the direct spillover effect $J_{k\ell}$, i.e. if $\text{sgn}(J_{k\ell}) = \text{sgn}(J_{km}J_{m\ell})$ for all $k, \ell, m \in \mathcal{K}$.¹³ In this way, A.1 invokes the phrases “the friend of my friend is my friend” and “the enemy of my friend is my enemy”. This condition is rooted in balance theory (Cartwright & Harary, 1956).

To see how Assumptions A and A.1 are related, consider the following lemma. It reveals that A.1 is a special case of Assumption A where the change-of-basis matrix $\mathbf{B}$ is diagonal.

Lemma 1. Assumption A.1 holds if and only if $\mathbf{B}\mathbf{J}\mathbf{B}^{-1}$ is non-negative for a diagonal matrix $\mathbf{B}$.

¹³To motivate this restriction, consider a random effects version of the model where the interaction effects are unweighted. In particular, let $J_{ij} \in \{-1, 1\}$, where J_{ij} varies across all agents i and j . Interpret $\mathbf{J}$ as a matrix of average interactions $\mathbf{J}_{k\ell} = \mathbb{E}(J_{ij} | i \in k, j \in \ell)$. In this case, the restriction must only hold in an average sense, such that $\mathbb{P}(J_{i_0i_1}J_{i_1i_2} = J_{i_0i_2} | i_0 \in k, i_1 \in m, i_2 \in \ell) \geq 0.5$ for all $k, \ell, m \in \mathcal{K}$. See the Appendix for justification.

Example 2. Suppose that the interaction matrix is symmetric, positive semi-definite. In this case, it satisfies Assumption A since it can be diagonalized so that $\mathbf{J} = \mathbf{B}^{-1}\mathbf{A}\mathbf{B}$, where $\mathbf{B}$ is orthogonal and $\mathbf{A}$ is a diagonal matrix of the eigenvalues of $\mathbf{J}$, which are all non-negative.

One example in this class of matrices is the *diagonally dominant* matrix, which satisfies:

$$J_{kk} \geq \sum_{\ell \neq k} |J_{k\ell}|, \quad \text{for } k = 1, \dots, K. \quad (2.8)$$

This condition may be interpreted as saying that the level of cohesion in a group is strong relative to the between-group influences. Hence, agents tend to conform to their own group.

Deriving a Sufficient Condition for Multiple Equilibria

In settings with strong interaction effects, Assumption A is instrumental in characterizing equilibrium behavior. This assumption ensures that there is almost always one locally stable equilibrium. Also, the locally stable equilibria outnumber the unstable equilibria.

Property 4. Suppose that Assumption A is satisfied. Then, for almost all distributions $\{F_{\varepsilon|k}\}_{k=1}^K$, there is exactly one more locally stable equilibrium than there are unstable equilibria.

To justify Property 4, I first give a proof in the case where $\mathbf{J}$ is non-negative. This proof relies on the Perron-Frobenius theorem, as well as other mathematical results related to non-negative matrices. I then show how the proof extends to settings where Assumption A holds (where $\mathbf{J}$ is similar to a non-negative matrix). These arguments in the Appendix.

Under Assumption A, I obtain a sufficient condition for multiplicity. Specifically, multiple equilibria arise when agents have a strong desire to conform to the average action in their own group. In these social environments, aggregate behaviors become self-reinforcing, which leads to multiple locally stable equilibrium outcomes from the same fundamentals.

Corollary. Suppose that Assumption A is satisfied. If $\rho(\mathbf{D}(m^*)) > 1$ at some equilibrium m^* ,

then the model has multiple equilibria, and at least two equilibria are locally stable.

III.D. Comparisons of Aggregate Welfare across Social Groups

In this model, there is no strict Pareto ranking across equilibria since extreme realizations of the random payoff ε_i can dominate an agent's utility. So, to learn about social welfare, I consider the expected utility of agents in a group. In doing so, I can assess which equilibrium makes agents better off on average. In group k , the expected utility at an equilibrium m^* is:

$$\mathbb{E} \left(\max_{\omega_i} U_i(\omega_i|k)|m^* \right) = \mathbb{E} \left(\max_{\omega_i} \left\{ h_k \omega_i + \eta_k + \sum_{\ell=1}^K J_{k\ell} \omega_i m^{\ell*} + \varepsilon_i \omega_i + \xi_i \right\} \right). \quad (2.9)$$

To evaluate this expected utility, it helps to rescale the choices so that $\omega_i \in \{-1, 1\}$. This modification has no impact on equilibrium behavior; however, it makes the welfare calculations easier to interpret. For example, it implies that—in absence of private preferences, i.e., when $h_k = \varepsilon_i = 0$ for all i, k —an agent's realized payoff when everyone selects the high action is the same as it would be when everyone selects the low action. In this way, there is no negative externality inherent to an equilibrium in settings where all agents are ambivalent between their choices. In general, welfare analysis in models of social interactions is highly sensitive to the way that utility is specified, even when different specifications yield the same expected choice functions. The reason is that welfare depends on the exact mechanisms that give rise to spillover effects (e.g., social learning, pressure to conform, or free-riding), not just the spillover effects themselves; see Bhattacharya et al. (2023) for a detailed explanation.

When the action takes values in the set $\{-1, 1\}$, the expected utility in group k equals:

$$\mathbb{E} \left(\max_{\omega_i} U_i(\omega_i|k)|m^* \right) = \mathbb{E} \left(\left| h_k + \sum_{\ell=1}^K J_{k\ell} m^{\ell*} + \varepsilon_i \right| \right) + \eta_k + \mathbb{E}(\xi_i|k). \quad (2.10)$$

As seen through the next result, the equilibrium that generates the highest expected utility is

the one where most group members choose the same action, i.e., where $E(\bar{\omega}^k)$ is largest in magnitude. In addition, if agents are privately biased toward the high (low) action, then they tend to maximize their expected utility at the equilibrium where $E(\bar{\omega}^k)$ is highest (lowest).

Property 5. Let $\mathcal{M}^*$ be the set of equilibria. For any group k , $\operatorname{argmax}_{m^* \in \mathcal{M}^*} E(\max_{\omega_i} U_i(\omega_i|k)|m^*)$ equals $\operatorname{argmax}_{m^* \in \mathcal{M}^*} |E(\bar{\omega}^k)|$. Also, there always exists some threshold T_k for which:

(i) If $h_k > T_k$, then $\operatorname{argmax}_{m^* \in \mathcal{M}^*} E(\max_{\omega_i} U_i(\omega_i|k)|m^*) = \operatorname{argmax}_{m^* \in \mathcal{M}^*} E(\bar{\omega}^k)$.¹⁴

(ii) If $h_k < T_k$, then $\operatorname{argmax}_{m^* \in \mathcal{M}^*} E(\max_{\omega_i} U_i(\omega_i|k)|m^*) = \operatorname{argmin}_{m^* \in \mathcal{M}^*} E(\bar{\omega}^k)$.

When do different social groups favor the same equilibrium? To answer this question, it will be useful to characterize when two groups are positively or negatively influenced by one another. These notions are generally hard to define because social influences reflect the sum of direct spillover effects (e.g., $J_{k\ell}$) and indirect spillover effects (e.g., $J_{km}J_{m\ell}$), which arise through interactions with other groups. Fortunately, using Assumption A.1, I can more easily interpret the nature of relations between groups. Consider the following definitions.

Definition 1. Group k is *connected* to group ℓ if $J_{kj_1}J_{j_1j_2}\dots J_{j_M\ell} \neq 0$ for some $j_1, j_2, \dots, j_M \in \mathcal{K}$.

Definition 2. Group k is *positively (negatively) influenced* by group ℓ if, for any positive integer M ,

$$J_{kj_1}J_{j_1j_2}\dots J_{j_M\ell} \geq 0 (\leq 0) \text{ for all } j_1, j_2, \dots, j_M \in \mathcal{K}, \text{ with at least one inequality strict.}$$

Whenever group k is positively (negatively) influenced by group ℓ , its members seek to conform to (deviate from) the average behavior in ℓ . If all groups are connected, i.e., if $\mathbf{J}$ is an irreducible matrix, then Assumption A.1 ensures: (1) every group is positively influenced by itself and (2) any two groups are either positively or negatively influenced by one another.

When Assumption A.1 holds, the equilibria in the model are ordered in a distinctive way.

¹⁴Why would $T_k \neq 0$? In any group k , the payoff from choosing ω_i depends on h_k , as well as on h_ℓ for each group ℓ that influences k . So, even if $h_k > 0$, most agents in group k may still prefer the low action if (1) $h_\ell < 0$ for some ℓ that attracts k or if (2) $h_\ell > 0$ for some ℓ that repels k . Only if h_k is strong enough to overcome external influences (if h_k lies above some T_k) is $E(\max_{\omega_i} U_i(\omega_i|k)|m^*)$ maximized where $E(\bar{\omega}^k)$ is highest.

This ordering determines how the relative welfare of equilibria varies across social groups.

Property 6. Suppose that Assumption A.1 is satisfied, and consider any two social groups k and ℓ .

- (i) Suppose that k and ℓ are positively influenced by one another. Then the equilibrium where $E(\bar{\omega}^k)$ is highest (lowest) is the same equilibrium where $E(\bar{\omega}^\ell)$ is highest (lowest).
- (ii) Suppose that k and ℓ are negatively influenced by one another. Then the equilibrium where $E(\bar{\omega}^k)$ is highest is the same equilibrium where $E(\bar{\omega}^\ell)$ is lowest, and vice versa.

To understand the implications of Property 6, consider any social environment where A.1 applies and there are multiple equilibria. In this setting, there are two *extremal equilibria* (call them $\underline{m}^*$ and $\bar{m}^*$), where $E(\bar{\omega}^k)$ is either maximized or minimized for all groups k . As I prove in the Appendix, both $\underline{m}^*$ and $\bar{m}^*$ are always locally stable. So, under appropriate initial conditions, any fixed-point iteration on (2.5) will converge to an extremal equilibrium.

Taken together, Properties 5 and 6 reveal that social interactions can introduce welfare trade-offs, such that it is impossible to maximize aggregate welfare jointly in every group. In particular, if two groups are biased toward the same action, i.e., if $h_k > T_k$ and $h_\ell > T_\ell$, and if they are negatively influenced by one another, then they will maximize their expected utility at different equilibria. A similar trade-off arises when two groups are biased toward different actions, i.e., if $h_k > T_k$ and $h_\ell < T_\ell$, and are positively influenced by one another.

IV. Extensions and Alternative Network-Based Models

IV.A. Games on Networks

The properties in Section 3 are not exclusive to the binary choice framework. They also have implications for a much broader class of models where agents interact in a network. To see how, consider a game with K players. Each player k chooses a_k from a compact action space $A_k \in \mathbb{R}$. Given a profile of actions $a \in A_1 \times A_2 \times \cdots \times A_K$, the player's best response is

$a_k^* = q_k(\sum_{\ell=1}^K J_{k\ell} a_\ell)$, where $q_k(\cdot)$ is some non-decreasing function that maps from $\mathbb{R}$ to A_k .

This game encompasses an abundance of economic models. For example, an action a_k could represent a person's investment into a public good, with everyone benefiting from how much their neighbors contribute.¹⁵ Alternatively, a_k could be the output of a firm that competes in an oligopoly, where each firm's action influences the market price. These types of models are well-studied, and they both involve strategic substitutabilities between agents.

Another interpretation of this game is that each player represents a community of individuals. In the binary choice model, the players are social groups, where the members of each group make one of two choices subject to social influences and idiosyncratic biases. Agents act noncooperatively, and a_k refers to the average choice within group k . This framework would also apply to a different type of model, in which the residents of a country or local institution make a collective decision. For example, consider modeling spillover effects in US state policy, where voters support more liberal or conservative agendas based on the laws enacted in other states. Here, a_k would represent the collective action taken in state k .

I focus on pure strategy Nash equilibria, which are defined by the action profiles a^* where no player k wishes to deviate from a_k^* . It is well known that an equilibrium exists if there are continuous best responses and compact, convex action spaces. However, even without these restrictions, an equilibrium would still exist if the interaction matrix $\mathbf{J}$ satisfies Assumption A.1. To prove this result, I use Tarski's fixed point theorem, which ensures existence if the best responses are increasing, i.e., if $\mathbf{J}$ is a non-negative matrix. I then show that this property extends to settings where A.1 holds.¹⁶ Crucially, this approach does not rely on continuous

¹⁵Bramoullé et al. (2014) study a type of public goods game that is nested by my framework. In their paper, players choose from an interval $[0, 1]$ and the best responses are $a_k^* = \max\{0, 1 - \delta \sum_{\ell \neq k} g_{k\ell} a_\ell\}$, where $\delta \in [0, 1]$ and $g_{k\ell} \in \{0, 1\}$ indicates whether two players k and ℓ are linked. Notice that their paper focuses only on contexts with pure strategic substitutes, whereas my analysis permits both positive and negative spillovers.

¹⁶Specifically, under A.1, the best response functions map to an alternate system of equations with a non-negative interaction matrix. Any equilibrium in the original model corresponds to an equilibrium in a different model that has supermodular payoffs. Given this mapping, I use Tarski's theorem to prove that equilibria exist.

best responses. So, in the binary choice model, Assumption A.1 is enough to ensure existence even when the random utility component ε_i is not continuously distributed.

If the best responses are differentiable at each equilibrium a^* , then the spectral radius of the Jacobian matrix is all that is needed to determine uniqueness. In particular, let $\tilde{a}^*$ be the equilibrium at which $\rho(\mathbf{D}(a^*))$ is greatest. If $\rho(\mathbf{D}(\tilde{a}^*)) < 1$, then $\tilde{a}^*$ is the unique equilibrium. Additionally, when Assumption A holds, there are multiple equilibria if $\rho(\mathbf{D}(\tilde{a}^*)) > 1$.

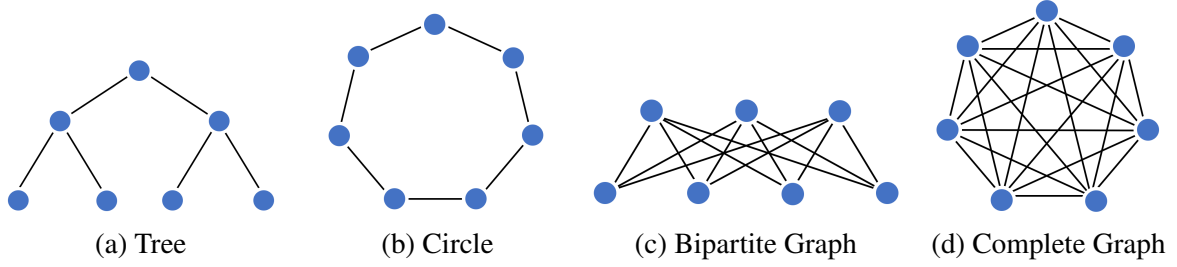
The equilibria of this game are generally not well-ordered. For example, they often do not form a lattice structure, which is useful for drawing conclusions about social welfare. As seen through Property 6, ensuring well-ordered equilibria can be achieved by imposing Assumption A.1. This condition guarantees that there always exist extremal equilibria. It also implies that there are trade-offs between different players in the network: if k and ℓ are negative influenced by one another, then a_k^* is highest wherever a_ℓ^* is lowest, and vice versa.

One convenient feature of this game is that each player's best response either (weakly) increases or decreases monotonically in another player's action. So, if the best responses are continuously differentiable, then the entries of the Jacobian matrix $\mathbf{D}$ retain the same sign for every value of a . However, this monotonicity assumption may be weakened. For example, suppose that a player's current action determines whether they conform to or deviate from someone else. In this case, characterizing equilibrium properties would require evaluating whether Assumptions A and A.1 are satisfied locally in certain regions of the support of a .

IV.B. Stable Network Structures

Given the implications of Assumptions A and A.1 for equilibrium behavior, it is worth understanding what network structures would satisfy these conditions. Namely, what types of exclusion restrictions (entries of 0 in the interaction matrix) are associated with stable equilibrium outcomes? I explore this question by examining four canonical types of networks.

Figure 2.2: Graphs with Seven Nodes



Example 1 (Tree). If the links between agents are sparse, then Assumptions A and A.1 are likely to hold even in very large networks. To see why, consider a tree with K nodes. This network is used to study peer effects in social hierarchies or firm interactions in vertical production networks. Trees also encompass two common types of network structures: lines and stars. Since a tree has no cycles, any walk to and from the same node requires retracing the same edges. So, if the interactions are symmetric or even *weakly mutual*, in the sense that either $J_{k\ell}, J_{\ell k} \geq 0$ or $J_{k\ell}, J_{\ell,k} \leq 0$ for all $k, \ell \in \mathcal{K}$, then Assumption A.1 is always satisfied.¹⁷

Example 2 (Circle). Consider a circle with K nodes. This network is used to study domino effects that arise when agents only interact with close contacts, rather than with the entire population; e.g., see Ellison (1993). Suppose that the interactions are weakly mutual, and let d_{edge} be the number of edges involving negative interactions. If d_{edge} is even, then A.1 always holds. If d_{edge} is odd, then each agent is negatively influenced by herself. In this case, Assumptions A and A.1 both fail. So, the stability of equilibria for this type of network depends on the number of agents, which determines the parity of negative interactions.

Example 3 (Bipartite Graphs). Consider an environment with pure strategic substitutes, i.e., let $J_{k\ell} \leq 0$ for any $k, \ell \in \mathcal{K}$. In this setting, A.1 holds if and only if the corresponding graph is bipartite. More generally, A.1 applies if and only if the agents can be partitioned into

¹⁷If the interactions are not weakly mutual, then $J_{k\ell}J_{\ell k} < 0$ for some $k, \ell \in \mathcal{K}$, which means that A.1 fails. This scenario is strategically similar to a matching pennies game, which has no pure strategy Nash equilibrium.

two teams, such that negative interactions only exist between members of different teams. Note that, even if A.1 fails, Assumption A may still be used to ensure equilibrium stability.¹⁸

Example 4 (Complete Graphs). Consider a network where all agents are linked. As there are no exclusion restrictions, Assumption A.1 is unlikely to hold if there are many agents in the network. For example, if the interactions are weakly mutual, then the fraction of complete graphs for which A.1 holds is $1/2^{\gamma_K}$, where $\gamma_K = (K-1)(K-2)/2$ for $K \geq 2$. With three agents, this fraction is $1/2$. With seven agents (depicted in Figure 2d), it is $1/32,768$. So, Assumption A.1 is more likely to apply in settings with very few agents and/or where the interaction effects are fairly uniform. In the binary choice model, I take the latter approach by partitioning the network into a few subgroups, in which the spillover effects are constant.

IV.C. Preferences over Network Composition

Until now, I have assumed that agents are influenced by the expected average action in each group, regardless of which group comprises a larger share of the total population. I now consider an alternative setup, where utility depends on the expected composition of people who choose an action. Namely, suppose agents care about $E(k|\omega_i)$ instead of $E(\omega_i|k)$.

This reformulation may be used to study how social influences affect network selection. For example, suppose that agents are choosing whether to enter a new environment, such as a school, and they care about what types of people they are likely to encounter there. This scenario invariably leads to negative interaction effects, since a preference that one group is in the majority is equivalent to a preference that other groups are in the minority. These behaviors can be analyzed using the same techniques that I outlined in the previous sections.

¹⁸Assumption A typically holds if the within-group spillovers J_{kk} are positive and large relative to the between-group spillovers $J_{k\ell}$, $\ell \neq k$. For example, let the interaction matrix $\mathbf{J}$ and the matrix $\mathbf{B}$ equal:

$$\mathbf{J} = \begin{bmatrix} \delta & 1 \\ -1 & 1 \end{bmatrix} \quad \text{and} \quad \mathbf{B} = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

If $\delta \geq 3$, then $\mathbf{B}\mathbf{J}\mathbf{B}^{-1}$ is a non-negative matrix, which means that the model will have a locally stable equilibrium.

Consider a model with two social groups: a and b .¹⁹ Let λ_a be the share of people in group a , and let $s_a(\omega_i)$ be the probability of being in group a given that someone chooses ω_i . The utility from a choice ω_i depends on the composition of agents who make that choice. Let:

$$U_i(\omega_i|k) = v_k(\omega_i) + J_k s_a(\omega_i) + \epsilon_i(\omega_i), \quad \text{for } k \in \{a, b\}. \quad (2.11)$$

Under this framework, the parameter J_k indicates how much the people in group k benefit from associating with members of group a . Both $v_k(\cdot)$ and $\epsilon_i(\cdot)$ are specified as before.

In equilibrium, the expected composition of agents making a choice must be consistent with individually optimal decisionmaking. By Bayes' rule, any equilibrium should satisfy:

$$s_a(\omega_i) = \frac{\lambda_a P(\omega_i|a)}{\lambda_a P(\omega_i|a) + (1 - \lambda_a) P(\omega_i|b)}, \quad \text{for } \omega_i \in \{0, 1\}, \quad (2.12)$$

where $P(\omega_i = 1|k) = F_{\epsilon|k}(h_k + J_k(s_a(1) - s_a(0)))$. By Brouwer's FPT, an equilibrium exists.

Uniqueness and dynamic stability of equilibria depend on the Jacobian matrix $\mathbf{D} \in \mathbb{R}^{2 \times 2}$ of the system (2.12). Letting $\beta_k = f_{\epsilon|k}(h_k + J_k(s_a(1) - s_a(0)))$, I define this matrix so that:

$$\mathbf{D} = \begin{bmatrix} D_{11} & -D_{11} \\ -D_{22} & D_{22} \end{bmatrix}, \quad \text{where: } \begin{cases} D_{11} = s_a(0)[1 - s_a(0)] \left(\frac{J_a \beta_a}{P(0|a)} - \frac{J_b \beta_b}{P(0|b)} \right) \\ D_{22} = s_a(1)[1 - s_a(1)] \left(\frac{J_a \beta_a}{P(1|a)} - \frac{J_b \beta_b}{P(1|b)} \right) \end{cases} \quad (2.13)$$

Suppose that agents prefer to associate with their own group, i.e., let $J_a \geq 0$ and $J_b \leq 0$. In this case, the matrix $\mathbf{D}$ satisfies Assumption A.1. Hence, there is almost always a locally stable equilibrium—even if the interaction effects are strong enough to generate multiplicity.

If agents prefer not to associate with their own group, i.e., if $J_a < 0$ or $J_b > 0$, then the model may not have any locally stable equilibria. This global instability occurs when agents' behavior is self-undermining, such that a group always seeks to deviate from its own action.

¹⁹I study the two-group case for simplicity, but this framework may be generalized to cases with many groups.

As an example, consider Allende (2019), who studies how peer effects drive student sorting in Peru. In her model, all parents prefer to send their children to schools that are made up of wealthy, high-achieving peers. If the peer effects are sufficiently weak, then a unique, locally stable equilibrium would exist. However, if low-performing students overwhelmingly seek to associate with their high-achieving peers, then a locally stable equilibrium may not exist.

In this model, the amount of diversity in a network is tied to stability of an equilibrium. To see why, note that the rows of the Jacobian matrix $\mathbf{D}$ are scaled by $s_a(\omega_i)[1 - s_a(\omega_i)]$. This term grows larger as $s_a(\omega_i)$ approaches $1/2$, and it tends to zero as $s_a(\omega_i)$ approaches 0 or 1. So, if the agents who choose an action are more diverse, then the Jacobian is more expansive, and the realized equilibrium is more likely to be unstable. Conversely, if one group makes up the majority of people choosing an action, then the equilibrium is more likely to be locally stable. Of course, this relationship only holds for a static model, where group membership is fixed. In the school choice example, some characteristics like student achievement are likely to change over time, while other traits, such as racial identity, may be constant. One area for future work is to explore how the dynamics of identity affect the formation of networks.

V. Identification and Estimation of Endogenous Interaction Effects

I now discuss how the model can be brought to data. Specifically, I show how to use data on individual choices to learn about the social interaction effects $\{J_{k\ell}\}_{k,\ell}$. These effects are of primary interest to applied researchers as they have major implications for public policy.

V.A. Empirical Setting

Suppose that a researcher has data pertaining to many networks. In practice, a network could represent a household, a school, a neighborhood, or a workplace. Agents only interact with other people in their network, and these interactions may differ on the basis of social identity, e.g., by gender or occupation. In the data, the researcher either observes the entire

network or a random sample that is drawn from each network. In addition, the researcher knows which network n an agent belongs to, as well as an agent's social identity k . Let $I_{n,k}$ denote the total number of agents in the sample belonging to network n and social group k .

To ensure a broad scope for application of the model, I assume that an agent's private utility is heterogeneous across networks. So, for any agent i with social identity k who resides in a network n , the private payoff from an action ω_i depends on: (1) individual-level factors ε_i , (2) identity-specific factors h_k , and (3) contextual network effects α_n . Importantly, each of these features is unobserved by the researcher. Hence, an agent i 's action ω_i equals:

$$\omega_i = \mathbb{1} \left\{ h_k + \alpha_n + J_{kk} E_i(\bar{\omega}_{n,-i}^k) + \sum_{\ell \neq k} J_{k\ell} E_i(\bar{\omega}_n^\ell) + \varepsilon_i \geq 0 \right\}, \quad (2.14)$$

where h_k , α_n , ε_i , and $\{J_{k\ell}\}_{k,\ell}$ are unknown coefficients. The model can also be adapted to incorporate observed individual-level covariates. However, since including covariates does not meaningfully change my analysis, I omit them for now and give details in the Appendix.

Throughout this section, I make two assumptions about the error structure in the model. First, I assume that the idiosyncratic payoffs ε_i and ε_j are independent for any two agents i and j within and across networks. Hence, there is no covariation between the error terms. Second, I assume that ε_i , conditional on group membership k , is independent of the network effect α_n . This condition implies that there is no self-selection into networks based on agents' unobserved idiosyncratic payoffs.²⁰ Both of these assumptions are standard in the literature on social interactions, and they could also be relaxed to allow for selection on observables.²¹

Assumption B.1. (i) the errors $\{\varepsilon_i\}_i$ are pairwise independent; (ii) $P(\varepsilon_i \leq z | k, \alpha_n) = F_{\varepsilon|k}(z)$.

As before, agents act with incomplete information and consistent beliefs. While they do

²⁰This restriction is stronger than needed to prove identification. As shown by Manski (1988) and elaborated on by Horowitz (2009), full conditional independence could be replaced by a quantile independence restriction.

²¹Suppose that $P(\varepsilon_i \leq z | W_n, \alpha_n, k) = F_{\varepsilon|W_n,k}(z)$ for some observed network-level variable W_n . Even if $F_{\varepsilon|W_n,k}$ does not equal $F_{\varepsilon|k}$, identification still follows by comparing networks with the same observables.

not see everyone's idiosyncratic payoff ε_i , they know about the generic parameters h_k , α_n , and $\{J_{k\ell}\}_{k,\ell}$, as well as the distribution functions $\{F_{\varepsilon|k}\}_{k=1}^K$. Therefore, they would accurately infer the expected average choices $\{E(\bar{\omega}_n^k)\}_{k=1}^K$. In equilibrium, the expected outcomes equal:

$$m_n^{k*} = F_{\varepsilon|k} \left(h_k + \alpha_n + \sum_{\ell=1}^K J_{k\ell} m_n^{\ell*} \right), \quad \text{for } k = 1, \dots, K \text{ and } n = 1, \dots, N. \quad (2.15)$$

As shown in Section 3, multiple equilibria can arise if the social interaction effects $\{J_{k\ell}\}_{k,\ell}$ are large enough in magnitude. When multiple equilibria exist, I assume that agents know which one is realized, so there is no coordination involved in selecting an equilibrium.

V.B. Overview of the Identification Strategy

The model is point identified if there is only one set of parameter values $\{h_k\}_k$, $\{\alpha_n\}_n$, $\{J_{k\ell}\}_{k,\ell}$, and $\{F_{\varepsilon|k}\}_k$ that is consistent with the data under the equilibrium equations. Even if there is no self-selection into networks based on ε_i , two obstacles to identification remain. In the rest of this section, I describe both of these obstacles and explain how to overcome them.

Network-Level Unobservables

The first barrier to identification is the presence of unobservable network effects α_n , which impede my ability to learn about social interaction effects. To account for this issue, I propose a new technique that allows for the partial identification of social interactions while imposing no added assumptions on the network-level determinants of agent's choices. Specifically, I am able to difference-out the network fixed effects for members of two social groups residing in the same network. I outline this procedure in detail in the next subsection.

Using my approach, I show that I can recover the differences between two interaction effects, i.e., $\{J_{k_1\ell} - J_{k_2\ell}\}_{\ell=1}^K$ for any groups k_1 and k_2 . These parameters are economically meaningful since they specify how the members of any two groups k_1 and k_2 differ in their

desire to conform to another group ℓ . Furthermore, I can use these parameters to construct an economic measure of polarization. Specifically, for any groups k_1 and k_2 , I can compute $\delta_{k_1 k_2} = J_{k_1 k_1} + J_{k_2 k_2} - J_{k_1 k_2} - J_{k_2 k_1}$. This term quantifies how much agents want to resemble their own group plus how much they want to distinguish themselves from the other group.

Unobserved Expected Average Choices

The second barrier to identification is that the expected average choices $\{m_n^{k*}\}_{k,n}$ are not actually observed. Instead, a researcher only sees the average choices among a finite number of agents in a network. This consideration is an inherent feature of network-based models with incomplete information. However, in the literature, identification proofs often assume that $\{m_n^{k*}\}_{k,n}$ are already known; e.g., see Brock & Durlauf (2001) and Blume et al. (2015). The basis for this assumption is that the expectation m_n^{k*} may be consistently estimated by the observed average choice $\bar{\omega}_n^k$ as the size of each network tends to infinity. So, the researcher's inability to see m_n^{k*} is treated as an estimation problem that is separate from identification.

I claim that it is not desirable to treat the expected average actions as known quantities. In many applications, the network sizes are small, which makes the observed averages be poor approximations for the true expectations. Even in large networks, $\bar{\omega}_n^k$ is a noisy measure of m_n^{k*} . So, by replacing m_n^{k*} with $\bar{\omega}_n^k$ when estimating the model, the estimates are biased due to measurement error. This bias exists even as the number of networks tends to infinity.

I explain how to correct for the bias with internal instruments. In particular, I randomly split each network into two subsamples, and I use the average choice in one subsample as an instrument for the (endogenous) average choice in the other subsample. By construction, these averages are both noisy measures of m_n^{k*} , and the measurement errors do not depend on one another. Hence, this IV procedure is valid. I demonstrate that this approach leads to consistent estimates of the social interaction effects, even as the network sizes remain small.

Remarks on Reflection and Multiple Equilibria

In linear simultaneous equation models, a primary threat to identification is the *reflection problem* (Manski, 1993). This issue arises whenever the expectation m_n^{k*} is linearly dependent on α_n such that—even if α_n were a deterministic function of observed variables—it would be impossible to disentangle the role of social interactions from contextual network effects. To overcome the reflection problem, researchers typically rely on exclusion restrictions, which are variables that only affect some agents in a network, while leaving others unaffected.

In my setting, the reflection problem does not arise. Indeed, Brock & Durlauf (2001, 2007) show that this issue is not a threat to identification in the binary choice framework because the data always uncovers a nonlinear relationship between m_n^{k*} and α_n . This nonlinearity is inherent to the model and it acts like an exclusion restriction, which allows me to distinguish between the social and contextual effects. So, after accounting for the two issues above, I find that the parameters in the model are identified without additional restrictions.

Lastly, the potential for multiple equilibria would not interfere with identification. The model is identified as long as there is a many-to-one mapping from the data to the preference parameters. Moreover, when estimating the model, I do not need to solve the equilibrium equations (2.15) directly. So, my analysis is robust to any issues that stem from nonuniqueness; see Bhattacharya et al. (2023), sec. 5.5, for more discussion.²² In most cases, it makes sense to assume that the realized equilibrium is locally stable; otherwise, it is unlikely to be observed by a researcher. However, no part of my identification strategy relies on dynamic stability.

V.C. Identification with Known Expected Average Choices

As a first step, suppose that the expected average choices $\{m_n^{k*}\}_{k,n}$ are observed. In this case, the main obstacle to identification is the unobserved network effect α_n . To handle this

²²In discrete choice models with full information, nonuniqueness can affect identification (see Tamer, 2003).

issue, I propose to difference-out the fixed effects by contrasting the outcomes of two social groups in the same network. The intuition for this strategy is contained in the lemma below.

I include the proof along with the result so that my approach can be more clearly interpreted.

Lemma 2. (*Sufficiency Property.*) For any network $n \in \{1, \dots, N\}$ and social group $k \in \mathcal{K}$:

$$E(\omega_i|k, \alpha_n, \{m_n^{\ell*}\}_{\ell=1}^K) = E(\omega_i|k, \{m_n^{\ell*}\}_{\ell=1}^K). \quad (2.16)$$

Proof. Choose some social group $\tilde{k}$ such that $\tilde{k} \neq k$. The expected average choice $m_n^{\tilde{k}*}$ is defined according to equation (2.15). Since $F_{\varepsilon|\tilde{k}}$ is strictly increasing, it is invertible. So:

$$\alpha_n = F_{\varepsilon|\tilde{k}}^{-1}(m_n^{\tilde{k}*}) - h_{\tilde{k}} - \sum_{\ell=1}^K J_{\tilde{k}\ell} m_n^{\ell*}. \quad (2.17)$$

By plugging this expression for α_n into the definition of $E(\omega_i|k, \alpha_n, \{m_n^{\ell*}\}_{\ell=1}^K)$, I find that:

$$E(\omega_i|k, \alpha_n, \{m_n^{\ell*}\}_{\ell=1}^K) = F_{\varepsilon|k} \left(h_k - h_{\tilde{k}} + \sum_{\ell=1}^K (J_{k\ell} - J_{\tilde{k}\ell}) m_n^{\ell*} + F_{\varepsilon|\tilde{k}}^{-1}(m_n^{\tilde{k}*}) \right). \quad (2.18)$$

The network effect cancels, and $E(\omega_i|k, \alpha_n, \{m_n^{\ell*}\}_{\ell=1}^K)$ is now a constant function of $\{m_n^{\ell*}\}_{\ell=1}^K$.

□

This lemma shows how the observed differences between individuals in a network can be used to control for unknown contextual effects. Consider any two agents i and j with different social identities (k and $\tilde{k}$, respectively) who both reside in the same network n . Since these agents share the same context, all the network-level determinants of i 's choice are captured by j 's decision. Any difference between ω_i and ω_j is driven by idiosyncratic preferences (i.e., ε_i versus ε_j), as well as factors that relate to social identity (i.e., k versus $\tilde{k}$). This framework offers a natural panel structure, which allows me to control for contextual effects by comparing the expected outcomes of different types of agents in the same network.

I give two versions of my identification result. First, I provide conditions for semiparametric identification, where the error distributions $\{F_{\varepsilon|k}\}_{k=1}^K$ are known by the researcher. Then, I give conditions for nonparametric identification, where $\{F_{\varepsilon|k}\}_{k=1}^K$ are unknown. While the nonparametric version allows for greater flexibility, it also requires that there is a lot of variation in the data. In both versions, identification is achieved without α_n being observed.

Conditions for Semiparametric Identification

Before stating the identification result, I first write down the following assumption:

Assumption B.2. The network effect α_n is a continuously distributed random variable on $\mathbb{R}$.

This assumption corresponds to condition A.4 in Brock & Durlauf (2007), which is based on Manski (1988). It requires that α_n varies across networks and takes infinitely-many values. By implication, the equilibrium outcomes $\{m_n^{k*}\}_{k=1}^K$ must be heterogeneous across networks. This heterogeneity is crucial for uncovering the nonlinear relationship between the expected average choices. It ensures there is ample variation to isolate the role of social interactions.

Theorem 1. Suppose that Assumptions B.1 & B.2 hold, and assume that m_n^{k*} is observed for all networks n and all social groups k . If the distribution functions $\{F_{\varepsilon|k}\}_{k=1}^K$ are known, then:

- (i) Without further assumptions, $\{h_{k_1} - h_{k_2}\}_{k_1, k_2}$ and $\{J_{k_1\ell} - J_{k_2\ell}\}_{k_1, k_2, \ell}$ are identified.
- (ii) If $\alpha_n = W_n' d$ for some observed vector W_n , then d , $\{h_k\}_k$, and $\{J_{k\ell}\}_{k, \ell}$ are identified.

This theorem has two parts. Part (i) leverages Lemma 2 by showing that the model is partially identified even if researchers have no prior knowledge about the network effects. In particular, I can recover the differences in identity fixed effects $h_{k_1} - h_{k_2}$, as well as the differences between the interaction effects $\{J_{k_1\ell} - J_{k_2\ell}\}_{\ell=1}^K$ for any social groups $k_1, k_2 \in \mathcal{K}$. This finding is new to the literature, and it is the main contribution of the theorem. Part (ii) considers a special case—studied by Brock & Durlauf (2001, 2007), among others—where α_n

is a constant linear function of observed variables. In this case, the model is fully identified.

Remark 1. Subtracting $J_{k_1 k_2} - J_{k_2 k_2}$ from $J_{k_1 k_1} - J_{k_2 k_1}$, I obtain a measure of polarization:

$$\delta_{k_1 k_2} = J_{k_1 k_1} + J_{k_2 k_2} - J_{k_1 k_2} - J_{k_2 k_1}. \quad (2.19)$$

This term specifies how much agents in social groups k_1 and k_2 prefer conforming to their own group over the other group. In the current literature, polarization is often viewed as an abstract concept, and there is still no clear consensus on how define it. One benefit of my approach is that I can measure polarization in a way that is motivated by an economic model.²³

Remark 2. If the error distributions are known (e.g., if $\{F_{\varepsilon|k}\}_{k=1}^K$ are standard Gaussian or logistic), then—under appropriate normalizations—all the interaction effects $\{J_{k\ell}\}_{k,\ell}$ are identified up to scale. To see how, assume $J_{\ell\ell} = 1$ for some group ℓ . Since $\{J_{k\ell} - J_{\ell\ell}\}_{k=1}^K$ are point identified, every coefficient $J_{k\ell}$ is known relative to $J_{\ell\ell}$. Hence, by setting the diagonal entries of the interaction matrix $\mathbf{J}$ to one, each element $J_{k\ell}$ may be recovered from the data.²⁴

Conditions for Nonparametric Identification

For nonparametric identification, I need an extra assumption. Specifically, there must be an exogenous, individual-level covariate that varies continuously over an unbounded support. Using this variation, I can recover each of the error distributions $\{F_{\varepsilon|k}\}_{k=1}^K$, which I then use to identify the rest of the model. This strategy closely follows Brock & Durlauf (2007).

To set ideas, I first modify the choice equation to allow for exogenous covariates $X_i \in \mathbb{R}^r$. I define $\omega_i = \mathbb{1}\{X_i'c + h_k + \alpha_n + \sum_{\ell=1}^K J_{k\ell}m_n^{\ell*} + \varepsilon_i \geq 0\}$, such that $P(\varepsilon_i \leq z|X_i, k, \alpha_n) = F_{\varepsilon|k}(z)$ and $P(X_i \leq x|k, \alpha_n) = P(X_i \leq x|k)$. Following Manski (1988), I impose the next assumption:

²³Under my framework, polarization is defined with respect to a particular choice ω_i . So, if social identity is more salient for some choices than others, then $\{\delta_{k_1 k_2}\}_{k_1, k_2}$ will depend on the decision that agents are making.

²⁴In many practical contexts, $\{F_{\varepsilon|k}\}_{k=1}^K$ are only known up to a scale parameter, e.g., the variance. In these cases, $\{J_{k_1 \ell} - J_{k_2 \ell}\}_{k_1, k_2, \ell}$ is already identified up to scale, and additional normalizations would not be advisable.

Assumption B.3. For any $k \in \mathcal{K}$, $\text{supp}(X|k)$ is not contained in a proper linear subspace of $\mathbb{R}^r$; also, there is some component x_j of X —with a nonzero coefficient c_j —such that, for almost all values of $x_{-j|k}$, the distribution of $x_{j|k}$ given $x_{-j|k}$ has positive density everywhere on $\mathbb{R}$.

Assumption B.3 ensures that there is enough variation in the data to recover the model parameters even if the distributions $\{F_{\varepsilon|k}\}_{k=1}^K$ are unknown. Consider the following result.

Theorem 2. Suppose that Assumptions B.1, B.2, & B.3 hold, and assume that m_n^{k*} is observed for all networks n and social groups k . Then $(\{F_{\varepsilon|k}\}_{k=1}^K, c)$ is identified up to scale. Also:

- (i) With no more assumptions, $(\{h_{k_1} - h_{k_2}\}_{k_1, k_2}, \{J_{k_1\ell} - J_{k_2\ell}\}_{k_1, k_2, \ell})$ is identified up to scale.
- (ii) If $\alpha_n = W'_n d$ for a known vector W_n , then $(d, \{h_k\}_k, \{J_{k\ell}\}_{k, \ell})$ is identified up to scale.

V.D. Identification with Unknown Expected Average Choices

In practice, researchers do not see the expected average choice m_n^{k*} . Instead, they would only see the average outcome $\bar{\omega}_n^k$ among finitely-many individuals. This quantity may be interpreted as a noisy measure of the true expectation, such that $\bar{\omega}_n^k = m_n^{k*} + u_{nk}$ where u_{nk} has mean zero and $\sqrt{I_{n,k}} \times u_{nk}$ converges in distribution to $N(0, m_n^{k*}(1 - m_n^{k*}))$ as $I_{n,k} \rightarrow \infty$.

If a researcher simply replaces m_n^{k*} with $\bar{\omega}_n^k$ without accounting for measurement error, then the previous identification arguments break down. To understand this point, it helps to re-write an agent's choice equation in terms of the quantities that are observed in the data.

$$\omega_i = \mathbb{1} \left\{ h_k + \alpha_n + \sum_{\ell=1}^K J_{k\ell} \bar{\omega}_n^\ell + \tilde{\varepsilon}_i \geq 0 \right\}, \quad \text{where } \tilde{\varepsilon}_i = \varepsilon_i - \sum_{\ell=1}^K J_{k\ell} u_{n\ell}. \quad (2.20)$$

In this equation, the sample averages $\{\bar{\omega}_n^\ell\}_{\ell=1}^K$ are correlated with the idiosyncratic term $\tilde{\varepsilon}_i$. Indeed, even if agent i 's choice ω_i were excluded from the sample mean $\bar{\omega}_n^k$, it is still the case that $\text{Cov}(\bar{\omega}_n^\ell, \tilde{\varepsilon}_i) = -J_{k\ell} \times \text{Var}(u_{n\ell})$ for every $\ell \in \{1, \dots, K\}$. Hence, $\{\bar{\omega}_n^\ell\}_{\ell=1}^K$ is endogenous in the model, which further implies that conditions (i) and (ii) of Assumption B.1 are violated.

To correct for this endogeneity, I propose an IV strategy that uses internal instruments. This procedure involves carrying out two steps. First, in each network n and social group k , I randomly split the sample into two subsets: a and b . In practice, there are many ways to form these subsets (e.g., by drawing Bernoulli random variables), and their sizes will not matter for identification. Second, I compute the average actions $\bar{\omega}_{n,a}^k$ and $\bar{\omega}_{n,b}^k$ in each subset.

By construction, the two sample averages $\bar{\omega}_{n,a}^k$ and $\bar{\omega}_{n,b}^k$ are both noisy measures of m_n^{k*} , where the measurement errors ($u_{nk,a}$ and $u_{nk,b}$, respectively) are independent of one another. This independence is a consequence of the incomplete information setting, in which agents only respond to expectations rather than to the realized choices of others. Given this property, I can account for endogeneity in the model through IV estimation, where $\bar{\omega}_{n,b}^k$ is used as an instrument for $\bar{\omega}_{n,a}^k$. Before formalizing my result, I will first motivate it with an example.

Example (Brock & Durlauf, 2001). In their paper, Brock & Durlauf (2001) study a model:

$$m_n^* = \tanh(h + W_n' d + J m_n^*), \quad \text{for } n = 1, \dots, N, \quad (2.21)$$

where W_n is a vector of observed contextual factors and (h, d', J) are unknown parameters.²⁵ In the case where m_n^* is known, the authors prove that the model is identified as long as there is sufficient variation in W_n for uncovering the nonlinear dependence between m_n^* and W_n .

However, the expectation m_n^* is generally unknown. So, any researcher who wishes to apply the model to data would instead be relying on the observed average choice $\bar{\omega}_n$. Unless one accounts for measurement error, many common estimation strategies (e.g., maximum likelihood estimation or OLS regression) will lead to biased estimates. To illustrate this point, suppose that someone wants to estimate the model via OLS using the observed means $\{\bar{\omega}_n\}_n$.

²⁵Brock & Durlauf (2001) have individual-level covariates X_i in their model, which I omit to ease notation.

Defining $u_n = \bar{\omega}_n - m_n^*$ and $v_n = \tanh^{-1}(\bar{\omega}_n) - \tanh^{-1}(m_n^*)$, I can re-write equation (2.21) as:

$$\tanh^{-1}(\bar{\omega}_n) = \tilde{h} + W'_n d + J\bar{\omega}_n + \tilde{\xi}_n, \quad \text{where:} \quad \begin{cases} \tilde{h} &= h + E(v_n) \\ \tilde{\xi}_n &= -Ju_n + v_n - E(v_n) \end{cases} \quad (2.22)$$

To ease notation, I define $\tilde{m}_n^*$, $\tilde{\omega}_n$, and $\tilde{Y}$ to be the residuals from a least squares regression of m_n^* , $\bar{\omega}_n$, and $\tanh^{-1}(\bar{\omega}_n)$, respectively, on the vector $(1, W'_n)$. The OLS estimand for J equals:

$$J^{\text{OLS}} = \frac{\text{Cov}(\tilde{\omega}_n, \tilde{Y})}{\text{Var}(\tilde{\omega}_n)} = J \times \frac{\text{Var}(\tilde{m}_n^*)}{\text{Var}(\tilde{m}_n^*) + \text{Var}(u_n)} + \frac{\text{Cov}(u_n, v_n)}{\text{Var}(\tilde{m}_n^*) + \text{Var}(u_n)}. \quad (2.23)$$

Observe that this estimand is “doubly-biased” because both the explanatory variable and the outcome variable are measured with error. As a result, $\text{Var}(u_n) \neq 0$ and $\text{Cov}(u_n, v_n) \neq 0$ in any setting with finite network sizes. So, the estimand J^{OLS} generally does not equal J .

To overcome this issue, I can use two-stage least squares. To do so, I start by partitioning each network into two parts (a and b), and then I compute the average choices $\bar{\omega}_{n,a}$ and $\bar{\omega}_{n,b}$. By construction, $\bar{\omega}_{n,b}$ is a valid instrument for $\bar{\omega}_{n,a}$ since $\bar{\omega}_{n,b} \perp u_{n,a}$. In addition, $\bar{\omega}_{n,b}$ will satisfy instrument relevance because $\text{Cov}(\bar{\omega}_{n,a}, \bar{\omega}_{n,b}) = \text{Var}(m_n^*) \neq 0$. Given these properties, I can recover J via TSLS. This strategy involves: (1) regressing $\bar{\omega}_{n,a}$ on $(1, W'_n, \bar{\omega}_{n,b})$ in the first stage to estimate all the fitted values $\mathbf{L}(\bar{\omega}_{n,a} | 1, W'_n, \bar{\omega}_{n,b})$, and then (2) regressing $\tanh^{-1}(\bar{\omega}_{n,a})$ on $(1, W'_n, \mathbf{L}(\bar{\omega}_{n,a} | 1, \bar{\omega}_{n,b}))$ in the second stage. By the usual IV arguments, this procedure yields consistent estimates of the parameters d and J even for small networks.

An IV Estimator to Recover Endogenous Interaction Effects

To formalize this identification strategy, I begin by defining $\bar{\omega}_{n,a}^k$ and $\bar{\omega}_{n,b}^k$, which are the average outcomes in each randomly-generated subset of a network n and social group k .²⁶ I

²⁶Here, an implicit assumption is that there are at least two agents within any network n and social group k . Otherwise, it will be impossible to further partition the sample, which means that $\bar{\omega}_{n,a}^k$ and $\bar{\omega}_{n,b}^k$ are undefined.

also define $u_{nk,a} = \bar{\omega}_{n,a}^k - m_n^{k*}$ and $v_{nk,a} = F_{\varepsilon|k}^{-1}(\bar{\omega}_{n,a}^k) - F_{\varepsilon|k}^{-1}(m_n^{k*})$ to be the measurement error that enters the model when a researcher tries to approximate m_n^{k*} and $F_{\varepsilon|k}^{-1}(m_n^{k*})$, respectively, using the average outcome $\bar{\omega}_{n,a}^k$. In the following theorem, I show that the social interaction effects are recovered from linear IV estimation, where $\bar{\omega}_{n,b}^k$ is used as an instrument for $\bar{\omega}_{n,a}^k$.

Theorem 3. Suppose that Assumptions B.1 & B.2 hold. Then the following two statements apply.

(i) For any two social groups $k_1, k_2 \in \{1, \dots, K\}$, define the IV estimand β_{k_1, k_2}^{IV} so that:

$$\beta_{k_1, k_2}^{IV} = E(Z_n X_n')^{-1} E(Z_n Y_n), \quad \text{where: } \begin{cases} Y_n &= F_{\varepsilon|k_1}^{-1}(\bar{\omega}_{n,a}^{k_1}) - F_{\varepsilon|k_2}^{-1}(\bar{\omega}_{n,a}^{k_2}) \\ X_n &= (1, \bar{\omega}_{n,a}^1, \dots, \bar{\omega}_{n,a}^K)' \\ Z_n &= (1, \bar{\omega}_{n,b}^1, \dots, \bar{\omega}_{n,b}^K)' \end{cases} \quad (2.24)$$

This estimand is $\beta_{k_1, k_2}^{IV} = [h_{k_1} - h_{k_2} + E(v_{nk_1,a}) - E(v_{nk_2,a}), J_{k_1 1} - J_{k_2 1}, \dots, J_{k_1 K} - J_{k_2 K}]'$.

(ii) Let $\alpha_n = W_n' d$ for some observed vector W_n , and define the IV estimand β_k^{IV} so that:

$$\beta_k^{IV} = E(Z_n X_n')^{-1} E(Z_n Y_n), \quad \text{where: } \begin{cases} Y_n &= F_{\varepsilon|k}^{-1}(\bar{\omega}_{n,a}^k) \\ X_n &= (1, W_n, \bar{\omega}_{n,a}^1, \dots, \bar{\omega}_{n,a}^K)' \\ Z_n &= (1, W_n, \bar{\omega}_{n,b}^1, \dots, \bar{\omega}_{n,b}^K)' \end{cases} \quad (2.25)$$

For any social group k , the estimand β_k^{IV} is equal to $[h_k + E(v_{nk,a}), d', J_{k 1}, \dots, J_{k K}]'$.

By the sample analogue principle, I can construct estimators for β_{k_1, k_2}^{IV} and β_k^{IV} that will converge in probability to the desired parameters as the number of networks grows large.²⁷ By standard arguments, these estimators are capable of inference, and they may be used for hypothesis testing. Importantly, this estimation strategy yields consistent estimates even when the size of each network remains small. Therefore, I do not rely on double asymptotics.

²⁷Specifically, the estimator $(\frac{1}{N} \sum_{n=1}^N Z_n X_n')^{-1} (\frac{1}{N} \sum_{n=1}^N Z_n Y_n)$ converges to $E(Z_n X_n')^{-1} E(Z_n Y_n)$ as $N \rightarrow \infty$.

To evaluate the efficacy of the estimation method, I conduct Monte Carlo simulations. In Table 1, I compare the performance of the IV estimates as I vary the size of each network, as well as the number of networks, in the simulated data. I perform this analysis with both IV estimators: (i) computing $\{\hat{\beta}_{k_1, k_2}^{IV}\}_{k_1, k_2}$ when α_n is unknown, and then (ii) computing $\{\hat{\beta}_k^{IV}\}_k$ when $\alpha_n = W_n' d$ for some known vector W_n . Throughout all these simulations, I fix $K = 2$.

The simulation results in Table 1 illustrate two key properties of the IV estimators. First, the estimators perform better in settings with larger networks. Indeed, as the network sizes grow, the sample average choices $\{\bar{\omega}_{n,a}^k\}_{k,n}$ will better approximate the expectations $\{m_n^{k*}\}_{k,n}$, reducing the amount of noise in the model. Second, the estimators become more precise as the number of networks N increases. This pattern is implied by Theorem 3, which shows that, for any fixed network sizes, I can consistently estimate the social interaction effects.²⁸

VI. Application: Differences in Classroom Peer Effects by Gender

In this section, I present an empirical application of my model and identification strategy. This application uses data from Project STAR, a large-scale education experiment that randomly assigned students to classrooms of different sizes and subsequently measured their test scores. Within this context, I study the role of peer effects on academic performance, and I use the generalized interactions framework to assess how these spillovers differ by gender.

VI.A. Description of the Data and Specification Choices

The Project STAR experiment took place among students in 79 Tennessee public schools who entered kindergarten in 1985. Within each school, students and teachers were randomly assigned to three types of classrooms: small (13 to 17 students), large (22 to 25 students),

²⁸When implementing this IV strategy, the researcher has flexibility in choosing how to partition the networks. Namely, one can select how much of the sample to use to define $\{\bar{\omega}_{n,a}^k\}_{k,n}$ and $\{\bar{\omega}_{n,b}^k\}_{k,n}$, as long as this selection process does not depend on the individual-level determinants of choices. Given this freedom, there may be a way to split the sample in an “optimal way” in order to improve estimator performance. Also, if data on individual choices is available, then the researcher may obtain more precise estimates via maximum likelihood estimation. These issues are worth considering when estimating the model, and I leave them for future work.

Table 2.1: Performance of IV Estimators for Different Network Sizes

Number of Networks	Agents per Network	Mean Squared Error					
		IV Estimator (i)		IV Estimator (ii)			
		$\widehat{J}_{11} - \widehat{J}_{12}$	$\widehat{J}_{22} - \widehat{J}_{21}$	$\hat{J}_{11}$	$\hat{J}_{12}$	$\hat{J}_{21}$	$\hat{J}_{22}$
N = 50	100	0.249	0.922	2.740	4.264	1.858	2.226
	500	0.041	0.102	0.153	0.610	0.154	0.377
	1000	0.017	0.044	0.070	0.254	0.067	0.155
	2000	0.008	0.020	0.032	0.112	0.032	0.068
N = 500	100	0.066	0.122	0.188	0.989	0.155	0.859
	500	0.006	0.013	0.017	0.076	0.016	0.060
	1000	0.002	0.005	0.007	0.028	0.007	0.020
	2000	0.001	0.002	0.003	0.012	0.003	0.008
N = 5000	100	0.049	0.042	0.023	0.152	0.014	0.064
	500	0.003	0.003	0.003	0.022	0.002	0.025
	1000	0.001	0.001	0.001	0.006	0.001	0.006
	2000	0.000	0.000	0.000	0.002	0.000	0.002

Notes. MSE's are computed across $M = 5000$ simulation draws. For each specification, I assume equal network sizes, and I construct X_n using 2/3 of the network sample, while using the remaining 1/3 to define the instrument Z_n . I assume that agents have logistic preferences, i.e., $\varepsilon_i|k \stackrel{\text{i.i.d.}}{\sim} \text{Logistic}(0, 1)$ for $k \in \{1, 2\}$. For more details about the data generating process, see the Online Appendix and replication code.

and large with a teacher's aide. At the end of the school year, students took the *Stanford Achievement Tests in Math and Reading*, and the scores from these exams were recorded as part of the study. In total, the experiment encompassed 6,325 students across 325 classrooms.²⁹ For more details about the design and implementation of the program, see Word et al. (1990).

The data reports each student's raw test score, which I transform into a binary outcome to make the setting suitable for my model. Since there was no pass/fail threshold for these tests, I study a variety of outcomes, which include: scoring in the top 25%, 50%, and 75% among Tennessee kindergarten students on both the math and reading exams. I estimate the model separately for each of these outcomes to assess the impact of peer effects on each one.

²⁹The public-use data does not have classroom identifiers. However, following Boozer & Cacciola (2001) and Graham (2008), I can uniquely assign students to classrooms by matching on observed classroom characteristics. I recover a sample of 6,248 students—among which 5,801 have non-missing test scores—across 321 classrooms.

Under the protocols of the experiment, students within a given school are unable to self-select into classrooms. So, Assumption B.1 holds conditional on a school. I account for this conditional independence by including school fixed effects when estimating the model. Following Graham (2008), I also give a version of the estimates where I restrict the sample to schools that have three classrooms (one of each type). This exercise addresses the unlikely possibility that there was nonrandom assignment of students to classrooms of the same type.

To evaluate how peer effects differ by gender, I focus on a version of the model with two social identities: male (m) and female (f). The Project STAR data is well-suited for studying these gender differences because male and female students are both well-represented within and across classrooms. Among participating students, there were 3,250 boys and 3,075 girls.

Student outcomes are specified according to equation (2.14). This framework allows for unknown gender effects h_f and h_m , which may reflect prior socialization or developmental differences between girls and boys. It also incorporates peer effects ($J_{ff}, J_{fm}, J_{mf}, J_{mm}$) that vary based on gender identity. Furthermore, the model allows for unobserved classroom characteristics α_n , which could include anything from teacher quality to the furnishings and layout of the room. Throughout my analysis, I make no assumptions about α_n —only that it is a continuously-distributed random variable. Therefore, my approach allows for a wide range of classroom-level determinants that are typically not accounted for in applied work.

When estimating the model, I assume that the idiosyncratic payoff terms follow logistic distributions, i.e., $\varepsilon_i|k \stackrel{\text{i.i.d.}}{\sim} \text{Logistic}(0, 1)$ for $k \in \{f, m\}$. This assumption relieves the burden of recovering these distributions nonparametrically, a task that can be quite challenging in practice. In my setting, the data is simply not rich enough for nonparametric estimation.³⁰

³⁰Recall from Theorem 2 that recovering $\{F_{\varepsilon|k}\}_k$ requires significant variation of individual-level covariates in each network. For small to moderate sample sizes, these functions may be poorly approximated, which causes the rest of the estimates to be imprecise. If $\{F_{\varepsilon|k}\}_k$ are known a priori, then this step is entirely avoided. Hence, there is a trade-off between making fewer parametric assumptions and having more precise estimates.

VI.B. Empirical Results

Table 2 reports the IV estimates for $J_{ff} - J_{mf}$ and $J_{mm} - J_{fm}$ using the Project STAR data. To interpret these results, I first consider the descriptive evidence for gender differences in test scores. Averaging across all Project STAR classrooms, girls performed 1.01% better than boys on their combined math and reading scores. However, this difference does not reflect student outcomes at the classroom-level, where the gender gap favored boys nearly as many times as it favored girls. On average, the gender gap within a classroom was 2.83%—nearly three times higher than it was across classrooms. This disparity may suggest that certain factors at the classroom-level were contributing to gender differences in student achievement. By examining these patterns under the generalized interactions framework, I can isolate the role of differential classroom peer effects from other classroom factors that affect test scores.

The estimates in Table 2 indicate the presence of strong classroom peer effects that differ by gender. Both $J_{ff} - J_{mf}$ and $J_{mm} - J_{fm}$ are estimated to be positive, which implies that the pressure to conform is higher among peers of the same gender than it is for peers of opposite genders. In particular, a 1% increase in the expected fraction of girls scoring in the top 50% in math is estimated to raise the log-odds of achieving this outcome by 0.047 more for a female student than for a male student, on average. These estimates are roughly consistent across all six outcome variables. Moreover, there is no evidence to reject the hypothesis that $J_{ff} - J_{mf}$ equals $J_{mm} - J_{fm}$, which suggests that the peer effect differences may be symmetric.

To defend the validity of my approach, I perform two types of robustness checks. First, I test whether any of the gender-specific parameters ($h_f, h_m, J_{ff}, J_{fm}, J_{mf}, J_{mm}$) depend on observed classroom characteristics, such as the share of minority students, the poverty rate, the location of a school (urban versus rural), as well as a teacher's education and experience. Such dependence would suggest that the model is misspecified, which could interfere with identification. Although I cannot test for variation along unobserved dimensions, I find no

Table 2.2: IV Estimates for Math and Reading Test Scores

	<i>Outcome Variable:</i>					
	Math			Reading		
	Top 25%	Top 50%	Top 75%	Top 25%	Top 50%	Top 75%
$J_{ff} - J_{mf}$	4.219 (3.109)	4.710*** (0.866)	5.116 (3.530)	4.659*** (0.230)	4.886*** (0.920)	4.444*** (0.151)
$J_{mm} - J_{fm}$	4.310 (2.747)	4.845*** (0.879)	5.509 (4.958)	4.556*** (0.302)	4.924*** (0.957)	4.674*** (0.175)
<i>Intercept</i>	0.002 (0.605)	0.220 (0.313)	0.024 (0.280)	-0.008 (0.085)	0.059 (0.301)	0.166 (0.108)
Number of Classrooms	321	321	321	321	321	321
School Fixed Effects	Yes	Yes	Yes	Yes	Yes	Yes
$F_{(df1, df2)}$ 1st-Stage ($\bar{\omega}_n^m$)	8.17 _(2,60)	9.09 _(2,83)	4.71 _(2,55)	7.27 _(2,50)	6.43 _(2,86)	5.73 _(2,44)
$F_{(df1, df2)}$ 1st-Stage ($\bar{\omega}_n^f$)	5.41 _(2,60)	11.25 _(2,83)	9.53 _(2,55)	14.12 _(2,50)	9.22 _(2,86)	5.06 _(2,44)

Notes. Estimates are obtained by computing $\hat{\beta}_{f,m}^{IV}$, which corresponds to the estimand in equation (2.24). For implementation, I randomly split each classroom so that half the sample is used to form endogenous variables X_n , and the remaining half is used to form instruments Z_n . *p<0.1; **p<0.05; ***p<0.01.

evidence to indicate that gender-specific effects differ by observed classroom characteristics.

Second, I investigate how sensitive the IV estimates are to the way that the classrooms are partitioned. Specifically, I produce histograms that show how each of the estimates differ across a variety of random classroom partitions. I show that, under alternative partitions, the parameter estimates would not deviate too much from the results reported in Table 2. So, the estimates reliably demonstrate strong evidence of differential peer effects by gender.

VII. Conclusion

The goal of this paper is to extend the theory of discrete choice and social interactions to more general settings, where agents are affected differently by different people. I analyze how aggregate outcomes depend on the features of a network, and I consider externalities that arise under both positive and negative spillovers. Finally, I show how data may be used to recover the social interaction effects in the model, and I propose a new estimation strategy

that can be applied in any context with finite networks and unobserved contextual effects.

One contribution of this work is to classify when models with negative interactions have the same types of equilibrium properties as models with uniformly positive interactions. I present two conditions (Assumptions A and A.1), which both imply that agents are not repelled by their own actions. These conditions ensure that there is almost always a locally stable equilibrium, and they allow me to derive a sufficient condition for multiple equilibria. Under these conditions, I also draw conclusions about welfare that extend over a broad class of network-based models. To my knowledge, both these conditions are new to the literature.

In the second part of the paper, I explain how to tackle two key obstacles to identification. First, I show that the generalized interactions framework offers a panel structure, which may be leveraged to isolate the role of social interactions from unobserved network effects. Then, I demonstrate how to use internal instruments to correct for measurement error that enters the model when the expected average choices are not observed. This method assures that the model will be empirically tractable even in settings with small to moderate network sizes.

I implement my framework using data from Project STAR, and I find evidence that peer effects differ systematically on the basis of gender. These differences would not be captured by a model that assumes uniform interaction effects. Hence, this application highlights the advantages of the generalized interactions framework. Note that my analysis only scratches the surface on the economic implications of heterogeneous interactions. One could take the model further by allowing for full heterogeneity in spillover effects between all individuals. Therefore, while this paper contributes to the discussion about heterogeneous interactions in discrete choice environments, there is still much room for future research on this topic.

Appendix

A. Proofs of Main Results

Proof of Property 1

Proof. Define $Q : [0, 1]^K \rightarrow [0, 1]^K$ so $Q_k(m) = F_{\varepsilon|k}(h_k + \sum_{\ell=1}^K J_{k\ell} m^\ell)$ for $k = 1, \dots, K$. Since Q is a continuous, self-mapping function on a non-empty, compact, convex set in $\mathbb{R}^K$, Brouwer's fixed point theorem ensures that there exists a vector m^* that solves $m^* = Q(m^*)$. □

Proof of Property 2

Proof. Suppose that $\rho(\mathbf{D}(m^*)) < 1$. For any $\epsilon > 0$, there is a matrix norm $\|\cdot\|$ for which $\|\mathbf{D}(m^*)\| \leq \rho(\mathbf{D}(m^*)) + \epsilon$. By defining ϵ so that $\epsilon < 1 - \rho(\mathbf{D}(m^*))$, it follows that $\|\mathbf{D}(m^*)\| < 1$ for some matrix norm. The system (2.5) is a contraction at m^* under this norm:

$$\|m_t - m^*\| = \|Q(m_{t-1}) - Q(m^*)\| \leq \kappa \|m_{t-1} - m^*\|,$$

where $\kappa = [0, 1)$ for any vector m_{t-1} that lies in some sufficiently small neighborhood of m^* . Iterating on the inequality above ensures that $\|m_t - m^*\| \leq \kappa^t \|m_0 - m^*\|$, where $\lim_{t \rightarrow \infty} \kappa^t = 0$.

Next, suppose that $\rho(\mathbf{D}(m^*)) > 1$. Since $\|\mathbf{D}(m^*)\| \geq \rho(\mathbf{D}(m^*))$ for any matrix norm $\|\cdot\|$, it must be that $\|\mathbf{D}(m^*)\| > 1$. By Henry (1981), Theorem 5.1.5., there exists some $u > 0$ such that, for any $\delta > 0$, there is an initial iterate m_0 where $\|m_0 - m^*\| < \delta$ for which some future iterate m_t , where $t \geq 1$, satisfies $\|m_t - m^*\| \geq u$. So m^* is an unstable equilibrium. □

Proof of Property 3

Proof. Define the mapping $\mathcal{H} : [0, 1]^K \rightarrow \mathbb{R}^K$ so that $\mathcal{H}_k(m) = m^k - Q_k(m)$ for $k = 1, \dots, K$. By definition, m^* is an equilibrium if and only if $\mathcal{H}(m^*) = \mathbf{0}_K$. Since no equilibrium lies on the boundary of $[0, 1]^K$, I restrict attention to the interior $(0, 1)^K$.¹ Let

¹ Since each $F_{\varepsilon|k}$ has positive density everywhere, $\mathcal{H}_k(m) < 0$ when $m_k = 0$ and $\mathcal{H}_k(m) > 0$ when $m_k = 1$.

$\mathbf{D}_{\mathcal{H}}$ denote the Jacobian matrix of $\mathcal{H}$, and define a set $C = \{m \in (0, 1)^K : \det(\mathbf{D}_{\mathcal{H}}(m)) = 0\}$. By Sard's Theorem, $\mathcal{H}(C)$ has Lebesgue measure zero. Therefore, for almost all $\mathcal{H}(m)$ evaluated on the domain $(0, 1)^K$, the matrix $\mathbf{D}_{\mathcal{H}}(m)$ is invertible. Moreover, for any fixed $y = \mathcal{H}(m)$, almost all distributions $\{F_{\varepsilon|k}\}_{k=1}^K$ would satisfy $y \notin \mathcal{H}(C)$. So, when applied to the case where $\mathcal{H}(m) = \mathbf{0}_K$, Sard's Theorem ensures that $\mathbf{D}_{\mathcal{H}}(m^*)$ is almost always invertible over the full set of equilibria m^* . By the inverse function theorem, each equilibrium is locally unique. Finally, since the set of equilibria is compact, it must have a finitely-many elements.

To prove that there is an odd number of equilibria, I rely on a version of the Poincaré-Hopf Index Theorem that is proven in Milnor (1965), Ch. 6. This theorem is presented below.

Poincaré-Hopf Theorem. Let v be a smooth vector field on the disk D^K that points outward on the boundary and has a finitely many isolated zeros $\{x_{(j)}^*\}_{j=1}^M$ satisfying $\det(\mathbf{D}_v(x_{(j)}^*)) \neq 0$. Then $\sum_{j=1}^M \text{index}_{x_{(j)}^*}(v) = 1$ where $\text{index}_{x_{(j)}^*}(v)$ equals 1 if $\det(\mathbf{D}_v(x_{(j)}^*)) > 0$ and -1 otherwise.

This theorem applies within this context because the set of equilibria is defined on $(0, 1)^K$, which is diffeomorphic to an open disk. In addition, $\mathcal{H}$ is a smooth vector field pointing outward at all boundary points, since $\lim_{m_k \rightarrow 0} \mathcal{H}(m) < 0$ and $\lim_{m_k \rightarrow 1} \mathcal{H}(m) > 0$ for all k . Finally, for almost all $\{F_{\varepsilon|k}\}_{k=1}^K$, it is the case that $\mathcal{H}$ has finitely-many isolated zeros and that, at each of these zeros, the Jacobian has a nonzero determinant. Therefore, the index theorem ensures that $\sum_{j=1}^M \text{index}_{m_{(j)}^*}(\mathcal{H}) = 1$, which means there is an odd number of equilibria.

Finally, suppose there are d_s locally stable equilibria. At each one of these equilibria m^* , the eigenvalues of $\mathbf{D}(m^*)$ all lie below unity in absolute value with probability one. Hence:

$$\det(\mathbf{D}_{\mathcal{H}}(m^*)) = \det(I - \mathbf{D}(m^*)) = \prod_{k=1}^K (1 - \lambda_k(m^*)) > 0,$$

where $\{\lambda_k(m^*)\}_{k=1}^K$ denote the eigenvalues of $\mathbf{D}(m^*)$. By the index theorem, there must also be at least $d_s - 1$ equilibria m^* at which $\det(\mathbf{D}_{\mathcal{H}}(m^*)) < 0$. Each of these equilibria have at least one eigenvalue that exceeds unity. Therefore, there are at least $d_s - 1$ unstable equilibria.

□

Proof of Lemma 1

Proof. “ $\Rightarrow$ ” Suppose that $\mathbf{J}$ is similar to a non-negative matrix $\mathbf{A}$ by way of a diagonal change-of-basis matrix $\mathbf{B}$. That is, there exists some diagonal $\mathbf{B}$ for which $\mathbf{A} = \mathbf{B}\mathbf{J}\mathbf{B}^{-1} \geq \mathbf{0}$. Since $\mathbf{B}$ is diagonal, the elements of $\mathbf{A}$ can be expressed as $A_{k\ell} = B_{kk}J_{k\ell}/B_{\ell\ell}$, for all k, ℓ . Thus, for any selection of indices k and $\ell_1, \dots, \ell_M$ in the set $\mathcal{K} = \{1, \dots, K\}$, it must be that:

$$\begin{aligned} 0 &\leq A_{k\ell_1}A_{\ell_1\ell_2} \cdots A_{\ell_M k} \\ &= \left(\frac{B_{kk}}{B_{\ell_1\ell_1}}J_{k\ell_1}\right)\left(\frac{B_{\ell_1\ell_1}}{B_{\ell_2\ell_2}}J_{\ell_1\ell_2}\right) \cdots \left(\frac{B_{\ell_M\ell_M}}{B_{kk}}J_{\ell_M k}\right) \\ &= J_{k\ell_1}J_{\ell_1\ell_2} \cdots J_{\ell_M k} \end{aligned}$$

“ $\Leftarrow$ ” Suppose that A.I holds. I first restrict attention to the case where $\mathbf{J}$ is irreducible. For any k , I define $\{\gamma_\ell^k\}_{\ell=1}^K$ so that $\gamma_\ell^k = 1$ if k is positively influenced (see Definition 2) by ℓ , and $\gamma_\ell^k = -1$ otherwise. Next, fixing some index $k_0 \in \mathcal{K}$, I construct the matrix $\mathbf{B}$ such that:

$$\mathbf{B} = \text{diag} \begin{bmatrix} \gamma_1^{k_0} \\ \vdots \\ \gamma_K^{k_0} \end{bmatrix}$$

Notice that $\mathbf{B}$ is involutory, i.e. $\mathbf{B}^{-1} = \mathbf{B}$. Thus, for all (g, ℓ) , Assumption A.1 ensures that:

$$[\mathbf{B}\mathbf{J}\mathbf{B}^{-1}]_{k,\ell} = [\mathbf{B}\mathbf{J}\mathbf{B}]_{k,\ell} = \gamma_k^{k_0}\gamma_\ell^{k_0}J_{k\ell} = \gamma_{k_0}^k\gamma_\ell^{k_0}J_{k\ell} = \gamma_\ell^k J_{k\ell} = |J_{k\ell}|$$

It follows that $\mathbf{B}\mathbf{J}\mathbf{B}^{-1}$ equals the absolute value of $\mathbf{J}$. Therefore, $\mathbf{B}\mathbf{J}\mathbf{B}^{-1}$ is non-negative. Finally, if $\mathbf{J}$ is not irreducible, then this same reasoning applies to all irreducible blocks of $\mathbf{J}$. So, it must always be true that $\mathbf{B}\mathbf{J}\mathbf{B}^{-1}$ is non-negative for some diagonal matrix $\mathbf{B} \in \mathbb{R}^{K \times K}$.

□

Proof of Property 4

Proof. First, suppose that $\mathbf{J}$ —and therefore $\mathbf{D}(m)$ for every $m \in [0, 1]^K$ —is a non-negative matrix. Moreover, assume there is an equilibrium $m^* \in (0, 1)^K$ with $\rho(\mathbf{D}(m^*)) > 1$.

If $\mathbf{D}(m^*)$ is an irreducible matrix, then the Perron-Frobenius theorem guarantees that:

$$\mathbf{D}(m^*)x = \rho(\mathbf{D}(m^*))x > x,$$

for some strictly positive vector $x \in \mathbb{R}_{++}^K$. It follows that $\mathbf{D}(m^*)\delta x > \delta x$ for any scalar $\delta > 0$.

By taking the first-order Taylor approximation of $Q(m^* + \delta x)$ about m^* , I obtain the following:

$$Q(m^* + \delta x) = \underbrace{Q(x^*)}_{=x^*} + \underbrace{\mathbf{D}(x^*)\delta x}_{>\delta x} + h_1(m^* + \delta x)\delta x, \quad \text{where } \lim_{\delta \rightarrow 0} h_1(m^* + \delta x) = 0$$

For sufficiently small δ , the vector $a = m^* + \delta x$, where $a \in (0, 1)^K$, will satisfy $Q(a) > a > m^*$. By an analogous argument, there also exists some $b = m^* - \delta x$, which satisfies $Q(b) < b < m^*$. Since no equilibrium lies on the boundary of $[0, 1]^K$, the following holds:

$$\mathbf{0}_K < Q(\mathbf{0}_K) < Q(b) < b < m^* < a < Q(a) < Q(\mathbf{1}_K) < \mathbf{1}_K$$

Brouwer's fixed point theorem ensures that Q has two more fixed points $\underline{m}^*$ and $\overline{m}^*$, such that $\mathbf{0}_K < \underline{m}^* < m^*$ and $m^* < \overline{m}^* < \mathbf{1}_K$. Moreover, if either of these equilibria is unstable, i.e., if either $\rho(\mathbf{D}(\underline{m}^*)) > 1$ or $\rho(\mathbf{D}(\overline{m}^*)) > 1$, then these same arguments can be used to show that there exist two more equilibria: one that lies between the two unstable equilibria and another that lies between the unstable equilibrium and the boundary. Since there is a finite number of equilibria with probability one, I may conclude that there are almost always more stable equilibria than unstable equilibria. Taken together with Property 3, this result implies that there is almost always exactly one more stable equilibrium than there are unstable equilibria.

Next, consider the case where $\mathbf{D}(m^*)$ is a reducible matrix. If $\rho(\mathbf{D}(m^*)) > 1$, then the same must hold for some irreducible block of $\mathbf{D}(m^*)$. Let $\mathcal{B}$ denote the set of indices within this block. Applying the Perron-Frobenius theorem to that block, there exists some vector x , satisfying $x_\ell > 0$ for $\ell \in \mathcal{B}$ and $x_\ell = 0$ otherwise, so that $Q(m^* + \delta x) > m^* + \delta x$ for $\delta > 0$ sufficiently small. Setting $a = m^* + \delta x$, it follows that $m^* < a < Q(a) < Q(\mathbf{1}_K) < \mathbf{1}_K$. By Brouwer's theorem, there must exist a fixed point of Q between m^* and $\mathbf{1}_K$, and (by analogous

arguments) a fixed point of Q between $\mathbf{0}_K$ and m^* . So, just as in the previous case, there is always exactly one more stable equilibrium than there are unstable equilibria.

Finally, I explain how this fixed point property, which is specific to monotone mappings, i.e., where $\mathbf{J}$ is non-negative, can be extended to a certain class of non-monotone mappings. Suppose that Assumption A is satisfied, i.e., let there be an invertible matrix $\mathbf{B}$ such that $\mathbf{B}\mathbf{J}\mathbf{B}^{-1}$ is non-negative. For $\mathcal{I} = [0, 1]^K$, define the mapping $\hat{Q} : \mathbf{B}\mathcal{I} \rightarrow \mathbf{B}\mathcal{I}$ in such a way that $\hat{Q}(m) = \mathbf{B}Q(\mathbf{B}^{-1}m)$. This mapping has a Jacobian matrix of $\mathbf{D}_{\hat{Q}}(m) = \mathbf{B}\mathbf{D}(\mathbf{B}^{-1}m)\mathbf{B}^{-1}$. Note that $\phi : \mathbb{R}^K \rightarrow \mathbb{R}^K$, where $\phi(m) = \mathbf{B}m$, is a bijective linear map. Therefore, $\phi(\cdot)$ is a *homeomorphism*, and it preserves interior points.² It follows that $\mathbf{B}\mathbf{D}(m)\mathbf{B}^{-1}$ is non-negative on $\text{int}(\mathcal{I})$ if and only if $\mathbf{D}_{\hat{Q}}(m) = \mathbf{B}\mathbf{D}(\mathbf{B}^{-1}m)\mathbf{B}^{-1}$ is non-negative on $\mathbf{B}\text{int}(\mathcal{I})$, which equals $\text{int}(\mathbf{B}\mathcal{I})$. So, under Assumption A, the matrix $\mathbf{D}_{\hat{Q}}(m)$ is non-negative for every $m \in \text{int}(\mathbf{B}\mathcal{I})$.

To summarize, Assumption A implies that $\hat{Q}(x) = \mathbf{B}Q(\mathbf{B}^{-1}x)$ is monotonic on $\text{int}(\mathbf{B}\mathcal{I})$. Note also that m^* is a fixed point of Q if and only if $\mathbf{B}m^*$ is a fixed point of $\hat{Q}$. To see why, write $\hat{Q}(\mathbf{B}m^*) = \mathbf{B}Q(m^*) = \mathbf{B}m^*$. In addition, note that $\mathbf{D}(m)$ and $\mathbf{D}_{\hat{Q}}(\mathbf{B}m)$ are similar matrices, since $\mathbf{D}_{\hat{Q}}(\mathbf{B}m) = \mathbf{B}\mathbf{D}(m)\mathbf{B}^{-1}$, which implies that they share the same eigenvalues. In particular, their spectral radii are equivalent: $\rho(\mathbf{D}_Q(m)) = \rho(\mathbf{D}_{\hat{Q}}(\mathbf{B}m))$ for all $m \in \text{int}(\mathcal{I})$.

By this relationship, I can assess the existence, uniqueness, and local stability of fixed points of the mapping Q by focusing on the (non-decreasing) mapping $\hat{Q}$. While the fixed points of Q and $\hat{Q}$ may be ordered differently on their respective domains, their number and local stability are the same. So, this property extends to models where Assumption A holds.

□

Proof of Property 5

Proof. Since the random payoffs ε_i are distributed symmetrically about zero, the expected utility $E(\max_{\omega_i} U_i(\omega_i|k)|m^*)$ defined in equation (2.10) strictly increases in $|h_k +$

²A *homeomorphism* is a continuous bijection between two topological spaces that has a continuous inverse.

$\sum_{\ell=1}^K J_{k\ell} m^{\ell*}$. In addition, since $|E(\bar{\omega}^k)|$ is a strictly increasing function of $|h_k + \sum_{\ell=1}^K J_{k\ell} m^{\ell*}|$ in equilibrium, $E(\max_{\omega_i} U_i(\omega_i|k)|m^*)$ is highest at the equilibrium where $|E(\bar{\omega}^k)|$ is highest. Both (i) and (ii) follow from the fact that $\lim_{h_k \rightarrow \infty} E(\bar{\omega}^k) = 1$ and $\lim_{h_k \rightarrow -\infty} E(\bar{\omega}^k) = -1$. $\square$

Proof of Property 6

Proof. Suppose $\mathbf{J}$ satisfies A.1. By Lemma 1, there exists some diagonal matrix $\mathbf{B}$ for which $\mathbf{B}\mathbf{J}\mathbf{B}^{-1}$ is non-negative. Without loss of generality, let $\mathbf{B}$ be the matrix that is constructed in the “ $\Leftarrow$ ” part of the proof of Lemma 1. That is, fix some group k , and define $\mathbf{B} = \text{diag}[\gamma_1^k, \dots, \gamma_K^k]$, where $\gamma_\ell^k = 1$ if k is positively influenced by ℓ , and $\gamma_\ell^k = -1$ otherwise.

As shown in the proof of Property 4, the mapping $\hat{Q}(m) = \mathbf{B}Q(\mathbf{B}^{-1}m)$ is non-decreasing on $\text{int}(\mathbf{B}\mathcal{I})$. Also, because $(\mathbf{B}\mathcal{I}, \leq)$ is a complete lattice, Tarski’s fixed point theorem ensures that the set of fixed points of $\hat{Q}$ forms a complete lattice. In particular, $\hat{Q}$ has greatest and least fixed points. Let $\mathbf{B}\bar{m}^*$ denote the greatest fixed point of $\hat{Q}$. Then $E(\bar{\omega}^\ell)$ is maximal at the equilibrium $\bar{m}^*$ if k is positively influenced by ℓ , and $E(\bar{\omega}^\ell)$ is minimal at $\bar{m}^*$ otherwise. Let $\mathbf{B}\underline{m}^*$ be the lowest fixed point of $\hat{Q}$. Then $E(\bar{\omega}^\ell)$ is minimal at $\underline{m}^*$ whenever k is positively influenced by ℓ , and $E(\bar{\omega}^\ell)$ is maximal at $\underline{m}^*$ otherwise. This argument holds for any $k, \ell \in \mathcal{K}$. $\square$

Proof of Theorem 1

Proof. Pick any two social groups k_1 and k_2 , and define the function $v_{k_1, k_2} : \mathbb{R}^K \rightarrow \mathbb{R}$ such that $v_{k_1, k_2}(m) = (h_{k_1} - h_{k_2}) + \sum_{\ell=1}^K (J_{k_1\ell} - J_{k_2\ell})m^\ell + F_{\varepsilon|k_2}^{-1}(m^{k_2})$. By equation (2.18), the expected average action $m_n^{k_1*}$ equals $F_{\varepsilon|k_1}(v_{k_1, k_2}(m_n^*))$. Additionally, since the cumulative distribution function $F_{\varepsilon|k_1} : \mathbb{R} \rightarrow [0, 1]$ is strictly increasing over $\mathbb{R}$, the following holds:

$$\begin{aligned} m_n^{k_1*} &= F_{\varepsilon|k_1}(h_{k_1} - h_{k_2} + \sum_{\ell=1}^K (J_{k_1\ell} - J_{k_2\ell})m_n^{\ell*} + F_{\varepsilon|k_2}^{-1}(m_n^{k_2*})) \\ &= F_{\varepsilon|k_1}(\widehat{h_{k_1} - h_{k_2}} + \sum_{\ell=1}^K \widehat{(J_{k_1\ell} - J_{k_2\ell})} m_n^{\ell*} + F_{\varepsilon|k_2}^{-1}(m_n^{k_2*})) \end{aligned}$$

if and only if $(h_{k_1} - h_{k_2}) - (\widehat{h_{k_1} - h_{k_2}}) = \sum_{\ell=1}^K [(J_{k_1\ell} - J_{k_2\ell}) - (J_{k_1\ell} - J_{k_2\ell})] m_n^{\ell*}$. This property holds for all networks n . Also, since each of the functions $\{F_{\varepsilon|k}\}_{k=1}^K$ is nonlinear, $\{m_n^{k*}\}_{k=1}^K$ are nonlinear functions of one another. Enough variation in $\{m_n^{k*}\}_{k=1}^K$ across networks means:

$$h_{k_1} - h_{k_2} = \widehat{h_{k_1} - h_{k_2}} \quad \text{and} \quad J_{k_1\ell} - J_{k_2\ell} = \widehat{J_{k_1\ell} - J_{k_2\ell}},$$

for every $\ell \in \mathcal{K}$. Also, since k_1 and k_2 are chosen arbitrarily, this result holds for all $k_1, k_2 \in \mathcal{K}$. The proof in the case where $\alpha_n = W'_n d$ relies on an analogous argument, and it also follows the proof of Brock & Durlauf (2007), Proposition 1. I exclude it for this reason. $\square$

Proof of Theorem 2

Proof. To demonstrate that c , $\{h_{k_1} - h_{k_2}\}_{k_1, k_2}$, and $\{J_{k_1\ell} - J_{k_2\ell}\}_{k_1, k_2, \ell}$ are identified, I apply Corollary 5 of Proposition 2 in Manski (1988). This result not only allows me to recover the parameters of interest, but it also guarantees that the error distributions $\{F_{\varepsilon|k}\}_{k=1}^K$ are identified up to scale. Ordinarily, recovering the error distributions would be unnecessary. However, in this setting, the presence of endogenous interaction effects means that $\{m_n^{\ell*}\}_{\ell=1}^K$ is functionally dependent on $\{F_{\varepsilon|k}\}_{k=1}^K$. To handle this issue, I take a two-step approach to identification: first I recover c and $\{F_{\varepsilon|k}\}_{k=1}^K$, then I use these quantities to recover the rest.

To start, I show that c and $\{F_{\varepsilon|k}\}_{k=1}^K$ are identified up to scale. Consider any group $k \in \mathcal{K}$. By Assumption B.3, there is some element x_j of X that varies continuously across $\mathbb{R}$. Without loss of generality, let $x_j = x_1$, and normalize the coefficient c_1 to one. In addition, fix some network n , and define the quantity $\zeta_n^k = h_k + \alpha_n + \sum_{\ell=1}^K J_{k\ell} m_n^{\ell*}$. For any agent in group k who resides in network n and has individual-level observables X_i , the expected choice is:

$$E(\omega_i | X_i, k, \alpha_n, \{m_n^{\ell*}\}_{\ell=1}^K) = F_{\varepsilon|k}(\zeta_n^k + X_i' c)$$

To recover $\{c, F_{\varepsilon|k}\}$, I must show $F_{\varepsilon|k}(\zeta_n^k + X_i' c) = \hat{F}_{\varepsilon|k}(\hat{\zeta}_n^k + X_i' \hat{c})$ implies $c = \hat{c}$ and $F_{\varepsilon|k} = \hat{F}_{\varepsilon|k}$ for any $X_i \in \text{supp}(X|k)$. This property holds by Manski's (1988) corollary. Since this holds for all $k \in \mathcal{K}$, I can conclude that c and $\{F_{\varepsilon|k}\}_{k=1}^K$ are identified up to scale.

Having shown that c and $\{F_{\varepsilon|k}\}_{k=1}^K$ can be recovered, the rest of the proof follows by the same arguments used to prove Theorem 1. So, the rest of the parameters are also identified. $\square$

Proof of Theorem 3

Proof. For part (i), I begin by re-writing equation (2.18) in terms of the observed quantities:

$$F_{\varepsilon|k_1}^{-1}(\bar{\omega}_{n,a}^{k_1}) - F_{\varepsilon|k_2}^{-1}(\bar{\omega}_{n,a}^{k_2}) = h_{k_1} - h_{k_2} + E(v_{nk_1,a}) - E(v_{nk_2,a}) + \sum_{\ell=1}^K (J_{k_1\ell} - J_{k_2\ell})\bar{\omega}_{n,a}^{\ell} + \tilde{\xi}_n,$$

where $\tilde{\xi}_n = -\sum_{\ell=1}^K (J_{k_1\ell} - J_{k_2\ell})u_{n\ell,a} + v_{nk_1,a} - v_{nk_2,a} - E(v_{nk_1,a}) + E(v_{nk_2,a})$ and $E(\tilde{\xi}_n) = 0$. Note that: (1) $(\bar{\omega}_{n,b}^1, \dots, \bar{\omega}_{n,b}^K) \perp (u_{n1,a}, \dots, u_{nK,a})$ and (2) $\text{Cov}(\bar{\omega}_{n,a}^{\ell}, \bar{\omega}_{n,b}^{\ell}) = \text{Var}(m_n^{\ell*}) \neq 0$ for all social groups $\ell \in \mathcal{K}$. Therefore, both exogeneity and instrument relevance are satisfied. By standard IV arguments, $\beta_{k_1,k_2}^{IV} = [h_{k_1} - h_{k_2} + E(v_{nk_1,a}) - E(v_{nk_2,a}), J_{k_11} - J_{k_21}, \dots, J_{k_1K} - J_{k_2K}]'$.

For part (ii), I again re-write the equilibrium equation in terms of the observed quantities:

$$F_{\varepsilon|k}^{-1}(\bar{\omega}_{n,a}^k) = h_k + E(v_{nk,a}) + W_n' d + \sum_{\ell=1}^K J_{k\ell} \bar{\omega}_{n,a}^{\ell} + \tilde{\xi}_n,$$

where $\tilde{\xi}_n = -\sum_{\ell=1}^K J_{k\ell} u_{n\ell,a} + v_{nk,a} - E(v_{nk,a})$. By the same reasoning as above, the instruments satisfy exogeneity and relevance, which guarantees that $\beta_k^{IV} = [h_k + E(v_{nk,a}), d', J_{k1}, \dots, J_{kK}]'$. $\square$

B. Justification for the Remarks in the Footnotes

Remark in Footnote 10

Suppose $\mathbf{J}$ is a symmetric matrix and that its eigenvalues all have non-positive real parts. Since $\beta(m^*)$ is a diagonal matrix with positive elements, $\mathbf{J}$ is congruent to $\beta^{1/2}(m^*)\mathbf{J}\beta^{1/2}(m^*)$, which is similar to $\beta^{1/2}(m^*)[\beta^{1/2}(m^*)\mathbf{J}\beta^{1/2}(m^*)]\beta^{-1/2}(m^*) = \beta(m^*)\mathbf{J} = \mathbf{D}(m^*)$. By Sylvester's law of inertia, it follows that $\mathbf{D}(m^*)$ also has eigenvalues with non-positive real

parts for any equilibrium m^* . Given this property, $0 < \prod_{k=1}^K (1 - \lambda_k(m^*)) = \det(I - \mathbf{D}(m^*)) = \det(\mathbf{D}_{\mathcal{H}}(m^*))$ at every equilibrium m^* . By the index theorem, there is a unique equilibrium.

Remark in Footnote 12

Consider the equilibrium equation $m^* = F_{\varepsilon}(Jm^*)$, where $J < 0$. By the previous remark, there is always a unique equilibrium. Moreover, since $h = 0$ and F_{ε} is symmetric about zero, the equilibrium equals $m^* = 0$. It is unstable if $\frac{\partial F_{\varepsilon}(Jm)}{\partial m} \Big|_{m=0} = J \times f_{\varepsilon}(0) < -1 \Leftrightarrow J < -f_{\varepsilon}^{-1}(0)$.

Remark in Footnote 13

Claim. Suppose $\text{sgn}(J_{k\ell}) = \text{sgn}(J_{km}J_{m\ell})$ for all $k, \ell, m \in \mathcal{K}$. Then Assumption A.1 holds.

Proof. Suppose that $\text{sgn}(J_{k\ell}) = \text{sgn}(J_{km}J_{m\ell})$ for all $k, \ell, m \in \mathcal{K}$. I will now prove through induction that $\text{sgn}(J_{j_0j_1}J_{j_1j_2} \cdots J_{j_Mj_0}) \geq 0$ for any arbitrary indices $j_0, j_1, \dots, j_M \in \mathcal{K}$. First, note that $\text{sgn}(J_{j_0j_2}) = \text{sgn}(J_{j_0j_1}J_{j_1j_2})$. Next, assume $\text{sgn}(J_{j_0j_m}) = \text{sgn}(J_{j_0j_1}J_{j_1j_2} \cdots J_{j_{m-1}j_m})$ for some $m \in \mathbb{N}$, and write: $\text{sgn}(J_{j_0j_{m+1}}) = \text{sgn}(J_{j_0j_m}J_{j_mj_{m+1}}) = \text{sgn}(J_{j_0j_1}J_{j_1j_2} \cdots J_{j_{m-1}j_m}J_{j_mj_{m+1}})$. By induction, it follows that $\text{sgn}(J_{j_0j_M}) = \text{sgn}(J_{j_0j_1}J_{j_1j_2} \cdots J_{j_M\ell})$ for any index $\ell \in \mathcal{K}$. Setting $\ell = j_0$, I obtain $\text{sgn}(J_{j_0j_1}J_{j_1j_2} \cdots J_{j_Mj_0}) = \text{sgn}(J_{j_0j_0})$, where $\text{sgn}(J_{j_0j_0}) = \text{sgn}(J_{j_0j_0}J_{j_0j_0}) \geq 0$. $\square$

Claim. Let $\mathbf{J}_{k\ell}$ be $E(J_{ij}|i \in k, j \in \ell)$, where $J_{ij} \in \{-1, 1\}$. Then $\text{sgn}(\mathbf{J}_{k\ell}) = \text{sgn}(\mathbf{J}_{km}\mathbf{J}_{m\ell})$ for all $k, \ell, m \in \mathcal{K}$ when $P(J_{i_0i_1}J_{i_1i_2} = J_{i_0i_2}|i_0 \in k, i_1 \in m, i_2 \in \ell) \geq 0.5$ for all $k, \ell, m \in \mathcal{K}$.

Proof. For any indices $k, \ell, m \in \mathcal{K}$, the product $\mathbf{J}_{km}\mathbf{J}_{m\ell}$ equals:

$$\begin{aligned} \mathbf{J}_{km}\mathbf{J}_{m\ell} &= E(J_{ij}|i \in k, j \in m) \times E(J_{i'j'}|i' \in m, j' \in \ell) \\ &= E(J_{ij} \times E(J_{i'j'}|i' \in m, j' \in \ell)|i \in k, j \in m) \\ &= E(J_{ij}J_{i'j'}|i \in k; j, i' \in m; j' \in \ell) \\ &= 2P(J_{ij}J_{i'j'} = 1|i \in k; j, i' \in m; j' \in \ell) - 1 \end{aligned}$$

Suppose that $P(J_{i_0i_1}J_{i_1i_2} = J_{i_0i_2}|i_0 \in k, i_1 \in m, i_2 \in \ell) \geq 0.5$ for all $k, \ell, m \in \mathcal{K}$. To simplify notation, I define: $\gamma = P(J_{ij}J_{ji'} = J_{ii'}|i \in k; j, i' \in m)$, $\delta = P(J_{ii'}J_{i'j'} = J_{ij}|i \in k; i' \in m; j' \in \ell)$.

ℓ), and $\kappa = P(J_{ji'} = 1 | j, i' \in m)$. Next, I decompose $P(J_{ij}J_{i'j'} = 1 | i \in k; j, i' \in m; j' \in \ell)$ as:

$$\begin{aligned}
P(J_{ij}J_{i'j'} = 1 | i \in k; j, i' \in m; j' \in \ell) &= \kappa \times P(J_{ij}J_{ji'}J_{i'j'} = 1 | i \in k; j, i' \in m; j' \in \ell, J_{ji'} = 1) \\
&\quad + (1 - \kappa) \times P(J_{ij}J_{ji'}J_{i'j'} = -1 | i \in k; j, i' \in m; j' \in \ell, J_{ji'} = -1) \\
&= \kappa \times \left[\gamma \times P(J_{ii'}J_{i'j'} = 1 | i \in k; j, i' \in m; j' \in \ell, J_{ij}J_{ji'} = J_{ii'}) \right. \\
&\quad \left. + (1 - \gamma) \times P(J_{ii'}J_{i'j'} \neq 1 | i \in k; j, i' \in m; j' \in \ell, J_{ij}J_{ji'} \neq J_{ii'}) \right] \\
&\quad + (1 - \kappa) \times \left[\gamma \times P(J_{ii'}J_{i'j'} \neq 1 | i \in k; j, i' \in m; j' \in \ell, J_{ij}J_{ji'} = J_{ii'}) \right. \\
&\quad \left. + (1 - \gamma) \times P(J_{ii'}J_{i'j'} = 1 | i \in k; j, i' \in m; j' \in \ell, J_{ij}J_{ji'} \neq J_{ii'}) \right] \\
&= \kappa \times \left[\gamma \times \left[\delta \times P(J_{ij'} = 1 | i \in k; j' \in \ell) + (1 - \delta) \times P(J_{ij'} \neq 1 | i \in k; j' \in \ell) \right] \right. \\
&\quad \left. + (1 - \gamma) \times \left[\delta \times P(J_{ij'} \neq 1 | i \in k; j' \in \ell) + (1 - \delta) \times P(J_{ij'} = 1 | i \in k; j' \in \ell) \right] \right] \\
&\quad + (1 - \kappa) \times \left[\gamma \times \left[\delta \times P(J_{ij'} \neq 1 | i \in k; j' \in \ell) + (1 - \delta) \times P(J_{ij'} = 1 | i \in k; j' \in \ell) \right] \right. \\
&\quad \left. + (1 - \gamma) \times \left[\delta \times P(J_{ij'} = 1 | i \in k; j' \in \ell) + (1 - \delta) \times P(J_{ij'} \neq 1 | i \in k; j' \in \ell) \right] \right] \\
&= (1 - \kappa) + (1 - \gamma)(2\kappa - 1) + (1 - \delta)(2\gamma - 1)(2\kappa - 1) \\
&\quad + (2\kappa - 1)(2\gamma - 1)(2\delta - 1)P(J_{ij'} = 1 | i \in k; j' \in \ell) \\
&= \frac{1}{2} \left[1 - (2\kappa - 1)(2\gamma - 1)(2\delta - 1) \right] + (2\kappa - 1)(2\gamma - 1)(2\delta - 1)P(J_{ij'} = 1 | i \in k; j' \in \ell)
\end{aligned}$$

Note that $\kappa, \gamma, \delta \in [0.5, 1]$ by assumption. So $(2\kappa - 1)(2\gamma - 1)(2\delta - 1)$ is bounded somewhere between 0 and 1. It follows from the last equality above that $P(J_{ij'} = 1 | i \in k; j' \in \ell) \geq 0.5$ if and only if $P(J_{ij}J_{i'j'} = 1 | i \in k; j, i' \in m; j' \in \ell) \geq 0.5$. This statement further implies that $\mathbf{J}_{k\ell}$ has the same sign as $\mathbf{J}_{km}\mathbf{J}_{m\ell}$. As this result applies for any $k, m, \ell \in \mathcal{K}$, the claim is true. $\square$

Remark in Paragraph 4 of Page 13

I now justify that “both $\underline{m}^*$ and $\overline{m}^*$ are always locally stable”. Consider the mapping $\hat{\mathcal{Q}}(m) = \mathbf{B}\mathcal{Q}(\mathbf{B}^{-1}m)$, which is defined in the proof of Property 6. Tarski’s fixed point theorem implies that $\hat{\mathcal{Q}}$ has a least fixed point $\mathbf{B}\underline{m}^*$ and a greatest fixed point $\mathbf{B}\overline{m}^*$. Suppose, for sake of contradiction, that $\overline{m}^*$ is unstable. Then, as argued in the proof of Property 4, $\hat{\mathcal{Q}}$ must have another fixed point, which is strictly greater than $\mathbf{B}\overline{m}^*$. Arriving at a contradiction in this

case, I conclude that $\overline{m}^*$ is a locally stable. By the same reasoning, $\underline{m}^*$ is also locally stable.

C. Adapting the Identification Results to Allow for Covariates

Suppose that the choice equation is modified to allow for exogenous covariates $X_i \in \mathbb{R}^r$. I define $\omega_i = \mathbb{1}\{X_i'c + h_k + \alpha_n + \sum_{\ell=1}^K J_{k\ell}m_n^{\ell*} + \varepsilon_i \geq 0\}$, with $P(\varepsilon_i \leq z|X_i, k, \alpha_n) = F_{\varepsilon|k}(z)$ and $P(X_i \leq x|k, \alpha_n) = F_{X|k}(x)$. In equilibrium, $m_n^{k*} = \int E(\omega_i|X_i, k, \alpha_n, \{m_n^{\ell*}\}_{\ell=1}^K) dF_{X|k}$, where:

$$E(\omega_i|X_i, k, \alpha_n, \{m_n^{\ell*}\}_{\ell=1}^K) = F_{\varepsilon|k}\left(h_k + \alpha_n + X_i'c + \sum_{\ell=1}^K J_{k\ell}m_n^{\ell*}\right), \quad \text{for } k = 1, \dots, K.$$

Proof of Lemma 2 (Version with Covariates)

In the presence of covariates, Lemma 2 must be adapted. I do so in the following way.

Lemma 2. (*Sufficiency Property.*) For any agent i in group k_1 and any agent j in group k_2 residing

in network n : $E(\omega_i|X_i, k_1, \alpha_n, \{m_n^{\ell*}\}_{\ell=1}^K) = E(\omega_i|X_i, X_j, k_1, \{m_n^{\ell*}\}_{\ell=1}^K, E(\omega_j|X_j, k_2, \alpha_n, \{m_n^{\ell*}\}_{\ell=1}^K))$.

Proof. As $F_{\varepsilon|k_2}$ is strictly increasing, its inverse $F_{\varepsilon|k_2}^{-1}$ exists. By this property, I write:

$$\alpha_n = F_{\varepsilon|k_2}^{-1}\left(E(\omega_j|X_j, k_2, \alpha_n, \{m_n^{\ell*}\}_{\ell=1}^K)\right) - h_{k_2} - X_j'c - \sum_{\ell=1}^K J_{k_2\ell}m_n^{\ell*}$$

Plugging this expression for α_n into the definition of $E(\omega_i|X_i, k_1, \alpha_n, \{m_n^{\ell*}\}_{\ell=1}^K)$, I find:

$$\begin{aligned} E(\omega_i|X_i, k_1, \alpha_n, \{m_n^{\ell*}\}_{\ell=1}^K) &= F_{\varepsilon|k_1}\left(h_{k_1} - h_{k_2} + (X_i - X_j)'c \right. \\ &\quad \left. + \sum_{\ell=1}^K (J_{k_1\ell} - J_{k_2\ell})m_n^{\ell*} + F_{\varepsilon|k_2}^{-1}\left(E(\omega_j|X_j, k_2, \alpha_n, \{m_n^{\ell*}\}_{\ell=1}^K)\right)\right) \end{aligned}$$

□

Proof of Theorem 1 (Version with Covariates)

For semiparametric identification (Theorem 1), I must make an additional assumption. In particular, I require that there exists some $k \in \mathcal{K}$ such that $\text{supp}(X|k)$ is not contained in a proper linear subspace of $\mathbb{R}^r$. Note that this condition is weaker than Assumption B.3.

Nevertheless, using Proposition 5 in Manski (1988), it is enough to show that c , $\{h_{k_1} - h_{k_2}\}_{k_1, k_2}$ and $\{J_{k_1\ell} - J_{k_2\ell}\}_{k_1, k_2, \ell}$ are identified. The proof of Theorem 1 is adapted as follows.

Theorem 1. Suppose that Assumptions B.1 & B.2 hold, and assume that m_n^{k*} is observed for all networks n and social groups k . Also, suppose that there is some $k \in \mathcal{K}$ where $\text{supp}(X|k)$ is not contained in a proper linear subspace of $\mathbb{R}^r$. Whenever $\{F_{\varepsilon|k}\}_{k=1}^K$ are known:

- (i) Without further assumptions, $\{h_{k_1} - h_{k_2}\}_{k_1, k_2}$, $\{J_{k_1\ell} - J_{k_2\ell}\}_{k_1, k_2, \ell}$, and c are identified.
- (ii) If $\alpha_n = W'_n d$ for some observed vector W_n , then d , $\{h_k\}_k$, $\{J_{k\ell}\}_{k, \ell}$, and c are identified.

Proof. Fixing some network n and defining the term $\zeta_n^k = h_k + \alpha_n + \sum_{\ell=1}^K J_{k\ell} m_n^{\ell*}$, I write:

$$E(\omega_i | X_i, k, \alpha_n, \{m_n^{\ell*}\}_{\ell=1}^K) = F_{\varepsilon|k}(\zeta_n^k + X'_i c)$$

To recover c , I must show that $F_{\varepsilon|k}(\zeta_n^k + X'_i c) = F_{\varepsilon|k}(\hat{\zeta}_n^k + X'_i \hat{c})$ implies $c = \hat{c}$. Since $F_{\varepsilon|k}$ is known, this property holds by Proposition 5 of Manski (1988). It follows that c is identified.

Having demonstrated that c can be recovered, I now focus on the other parameters. Consider any two social groups k_1 and k_2 , and define the function $\phi : \mathbb{R} \rightarrow [0, 1]$ so that:

$$\phi(v) = \int F_{\varepsilon|k_1}((X_i - X_j)'c + v) dF_{X|k_1},$$

where X_j is chosen from $\text{supp}(X|k_2)$ and where the integral is evaluated over the conditional support of X_i given k_1 . By construction, $\phi(\cdot)$ is nonlinear and monotonically increasing in v .

For any network n , let $v_n = h_{k_1} - h_{k_2} + \sum_{\ell=1}^K (J_{k_1\ell} - J_{k_2\ell}) m_n^{\ell*} + F_{\varepsilon|k_2}^{-1}(E(\omega_j | X_j, k_2, \alpha_n, \{m_n^{\ell*}\}_{\ell=1}^K))$.

By Lemma 2, the expected average choice $m_n^{k_1*}$ is equal to $\phi(v_n)$. Because $\phi(\cdot)$ is monotonic:

$$\begin{aligned} m_n^{k_1*} &= \int F_{\varepsilon|k_1} \left(h_{k_1} - h_{k_2} + (X_i - X_j)'c + \sum_{\ell=1}^K (J_{k_1\ell} - J_{k_2\ell}) m_n^{\ell*} + F_{\varepsilon|k_2}^{-1}(E(\omega_j | X_j, k_2, \alpha_n, \{m_n^{\ell*}\}_{\ell=1}^K)) \right) dF_{X|k_1} \\ &= \int F_{\varepsilon|k_1} \left(\widehat{h_{k_1} - h_{k_2}} + (X_i - X_j)'c + \sum_{\ell=1}^K (\widehat{J_{k_1\ell} - J_{k_2\ell}}) m_n^{\ell*} + F_{\varepsilon|k_2}^{-1}(E(\omega_j | X_j, k_2, \alpha_n, \{m_n^{\ell*}\}_{\ell=1}^K)) \right) dF_{X|k_1} \end{aligned}$$

is satisfied if and only if $\sum_{\ell=1}^K [(\widehat{J_{k_1\ell} - J_{k_2\ell}}) - (J_{k_1\ell} - J_{k_2\ell})] m_n^{\ell*} = (h_{k_1} - h_{k_2}) - (\widehat{h_{k_1} - h_{k_2}})$ for all networks $n \in \{1, \dots, N\}$. Since the expected average choices are nonlinear functions

of one another, sufficient variation in these equilibria across different networks ensures that:

$$h_{k_1} - h_{k_2} = \widehat{h_{k_1} - h_{k_2}} \quad \text{and} \quad J_{k_1\ell} - J_{k_2\ell} = \widehat{J_{k_1\ell} - J_{k_2\ell}},$$

for $\ell \in \mathcal{K}$. Also, since k_1 and k_2 are chosen arbitrarily, this result holds for all $k_1, k_2 \in \mathcal{K}$. $\square$

For nonparametric identification (Theorem 2), there is nothing to modify, since these results already rely on individual-level covariates. Also, for identification in contexts with unknown expected average choices (Theorem 3), all modifications will follow directly from Theorems 1 and 2. Specifically, the IV estimands may be adapted to incorporate covariates.

D. Additional Details about the Monte Carlo Simulations

To perform simulations, I draw observations from the following data generating process:

$$\omega_i = \mathbb{1} \left\{ h_k + \alpha_n + J_{k1}m_n^{1*} + J_{k2}m_n^{2*} + \varepsilon_i \geq 0 \right\},$$

where $m_n^{k*} = F_{\varepsilon|k}(h_k + \alpha_n + J_{k1}m_n^{1*} + J_{k2}m_n^{2*})$ for $k \in \{1, 2\}$ and $n = 1, \dots, N$. The idiosyncratic payoffs follow logistic distributions, i.e., $\varepsilon_i|k \stackrel{\text{i.i.d.}}{\sim} \text{Logistic}(0, 1)$ for $k \in \{1, 2\}$. The contextual effect α_n is a continuous random variable that is evenly distributed over the interval $[-2, 2]$. The identity-specific effects are: $h_1 = 0$ and $h_2 = 1$. Finally, $\mathbf{J}$ equals:

$$\mathbf{J} = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}.$$

For case (ii), i.e., where $\alpha_n = W_n' d$ for some observed W_n , I set $W_n = \alpha_n$ and $d = 2$. Note that these parameters are chosen arbitrarily and that alternative DGP's produce similar results.

To draw agent's choices ω_i in equilibrium, I run a fixed point iteration on the equilibrium condition. This procedure leverages the fact that $\mathbf{J}$ satisfies Assumption A, which guarantees there exists at least one locally stable equilibrium (Property 4). Note that it is not necessary

that Assumption A holds for this estimation procedure to be valid. However, this condition is helpful for conducting simulations as it facilitates the computation of an equilibrium. For all additional details about specifications for the simulations, I refer to the replication code.

E. Robustness Analyses

1. Verifying Random Assignment to Classrooms

Under the experimental protocols, students in each school were randomly assigned into classrooms of three different types. However, of the 79 schools in the study, 48 schools had more than three kindergarten classrooms. Since these schools had more than one classroom of each type, it is conceivable that there may have been nonrandom sorting within the same class type. As Graham (2008) argues, this scenario is unlikely. Indeed, he finds no evidence of within-class-type sorting. Nevertheless, I present a version of the IV estimates where I restrict the sample to the 31 schools that have only three classrooms. In doing so, I rule out any possibility of nonrandom assignment to classrooms of the same type. These estimates are presented below, and they appear to be consistent with the results that use the full sample.

2. Misspecification Tests

I conduct a variety of hypothesis tests to determine whether any of the gender-specific parameters in the model depend on the observed classroom-level variables. Specifically, I test the null hypotheses that $J_{ff} - J_{mf}$, $J_{mm} - J_{fm}$, and the intercept (respectively) differ for:

1. high poverty classrooms ($\geq 50\%$ FRPL) and low poverty classrooms ($< 50\%$ FRPL)
2. high minority classrooms ($< 75\%$ white) and low minority classrooms ($\geq 75\%$ white)
3. more experienced teachers (> 10 years) and less experienced teachers (≤ 10 years)
4. more educated teachers (graduate deg.) and less educated teachers (no graduate deg.)

Table 2.3: IV Estimates for Schools with Fewer than 4 Classrooms

	<i>Outcome Variable:</i>					
	Math			Reading		
	Top 25%	Top 50%	Top 75%	Top 25%	Top 50%	Top 75%
$J_{ff} - J_{mf}$	3.494 (2.826)	4.215*** (1.039)	3.961** (1.444)	3.843 (3.951)	5.265** (1.952)	6.425 (4.974)
$J_{mm} - J_{fm}$	3.251 (3.133)	4.088*** (1.251)	4.104*** (1.111)	3.048 (5.612)	5.117*** (1.552)	6.522 (5.430)
<i>Intercept</i>	-0.280 (0.443)	-0.097 (0.201)	0.081 (0.273)	-0.008 (1.282)	-0.045 (0.218)	1.088 (2.872)
Number of Classrooms	89	89	89	89	89	89
School Fixed Effects	Yes	Yes	Yes	Yes	Yes	Yes
$F_{(df1,df2)}$ 1st-Stage ($\bar{\omega}_n^m$)	2.23 _(2,10)	3.06 _(2,19)	2.78 _(2,17)	2.58 _(2,12)	2.82 _(2,17)	8.35 _(2,13)
$F_{(df1,df2)}$ 1st-Stage ($\bar{\omega}_n^f$)	2.01 _(2,10)	2.26 _(2,19)	2.18 _(2,17)	5.71 _(2,12)	2.02 _(2,17)	3.61 _(2,13)

Notes. Estimates are obtained by computing $\hat{\beta}_{f,m}^{IV}$, which corresponds to the estimand in equation (24). For implementation, I randomly split each classroom so that half the sample is used to form endogenous variables X_n , and the remaining half is used to form instruments Z_n . *p<0.1; **p<0.05; ***p<0.01.

5. rural classrooms (in rural district) and urban classrooms (in urban or suburban district)

The p -values from each of these one-dimensional hypothesis tests are reported in Tables A.2 and A.3. When conducting these tests, I consider two different outcomes variables: (1) scoring in the top 50% on the math exam and (2) scoring in the top 50% on the reading exam. Recall that these percentiles are all calculated relative to the full sample of Tennessee kindergarten students who participated in the study. For both outcome variables, I find no evidence to reject the hypothesis that any of the gender-specific parameters differ across networks. These results help to justify the model specification and identification strategy.

3. Sensitivity of IV Estimates to the Partitioning of Classrooms

To estimate the model, internal instruments are defined by randomly partitioning each classroom into two equal (or almost equal if there is an odd number of students) subsamples. In general, the estimates will be sensitive to the way in which the classrooms are partitioned.

Table 2.4: Tests for Misspecification (*Outcome*: Top 50% on Math Exam)

	Large Share Poverty	Large Share Minority	Teacher Has >10yrs Experience	Teacher Has Higher Degree	Rural District
$J_{ff} - J_{mf}$	0.948	0.35	0.525	0.754	0.927
$J_{mm} - J_{fm}$	0.634	0.77	0.611	0.893	0.945
<i>Intercept</i>	0.568	0.198	0.767	0.606	0.92

Notes. This table reports p -values corresponding to the one-dimensional hypothesis tests for whether $J_{ff} - J_{mf}$, $J_{mm} - J_{fm}$, and the intercept (respectively) differ by observed classroom features.

Table 2.5: Tests for Misspecification (*Outcome*: Top 50% on Reading Exam)

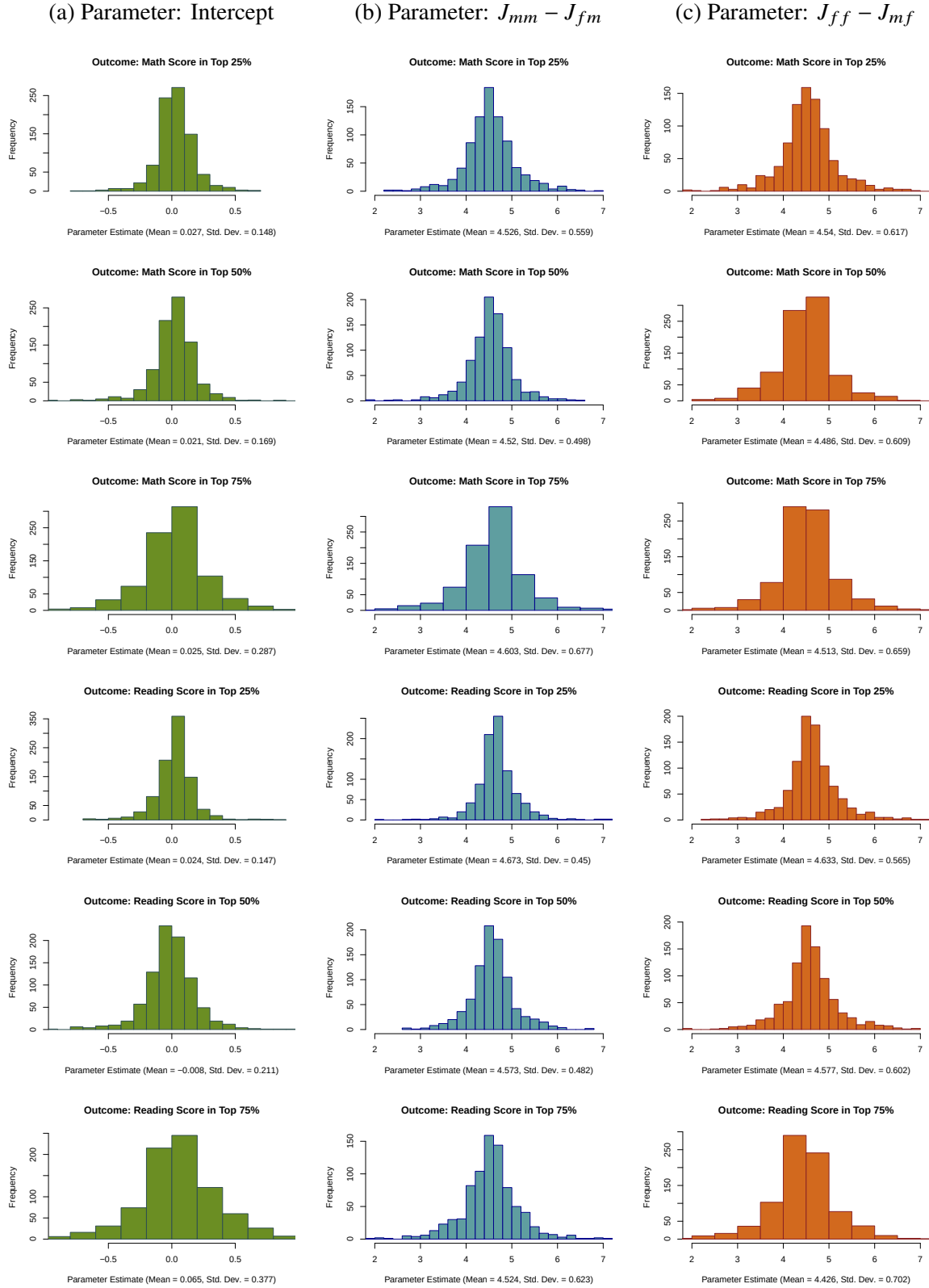
	Large Share Poverty	Large Share Minority	Teacher Has >10yrs Experience	Teacher Has Higher Degree	Rural District
$J_{ff} - J_{mf}$	0.762	0.984	0.911	0.996	0.662
$J_{mm} - J_{fm}$	0.626	0.979	0.967	0.875	0.886
<i>Intercept</i>	0.532	0.977	0.898	0.905	0.321

Notes. This table reports p -values corresponding to the one-dimensional hypothesis tests for whether $J_{ff} - J_{mf}$, $J_{mm} - J_{fm}$, and the intercept (respectively) differ by observed classroom features.

Nevertheless, as long as the model is correctly specified, this IV strategy always generates consistent estimates regardless of which partitions are realized. To justify the efficacy of this approach, I re-estimate the model $M = 1000$ times, each time randomly choosing a different partition of classrooms when constructing the instruments. Through this procedure, I can evaluate how sensitive the IV estimates are to the way that internal instruments are defined.

I report histograms of the parameter estimates for each outcome variable in Figure A.1. Observe that the point estimates appear approximately normally distributed, and the amount of dispersion is not large enough to invalidate any of the qualitative findings in the paper. Moreover, the main IV estimates reported in Table 2 do not seem to be outliers, which indicates that most alternative partitions of classrooms would give similar results. So, I interpret Figure A.1 as evidence that the main results in the empirical application are robust.

Figure 2.3: Densities of IV Estimates over Random Classroom Partitions



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