

THE UNIVERSITY OF CHICAGO

CLASSIFYING HYPERBOLIC ERGODIC STATIONARY MEASURES ON COMPACT
COMPLEX SURFACES WITH LARGE AUTOMORPHISM GROUPS

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In loving memory of Patricia and Fred Samborsky

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ABSTRACT

Let X be a compact complex surface. Consider a finitely supported probability measure μ on $\text{Aut}(X)$ such that $\Gamma_\mu = \langle \text{Supp}(\mu) \rangle < \text{Aut}(X)$ is non-elementary. We do not assume that Γ_μ contains any parabolic elements. In this thesis, we study and classify hyperbolic, ergodic μ -stationary probability measures.

1 Introduction

In this thesis, we use techniques from homogeneous and smooth dynamics to further the classification of stationary measures on compact complex surfaces.

1.1 Main theorem and background

Theorem 1.1.1. *Let X be a compact complex surface, μ a finitely supported probability measure on the group of biholomorphisms $\text{Aut}(X)$ such that $\Gamma_\mu = \langle \text{Supp}(\mu) \rangle$ is non-elementary, and ν a hyperbolic ergodic μ -stationary probability measure on X . Assume there is no Γ_μ -invariant algebraic curve and that the Lyapunov exponents sum to be non-negative. Then, ν is invariant, and exactly one of the following statements hold*

1. ν is finitely supported,
2. the supports of the stable and unstable conditional measures are real 1-dimensional curves, the measures are absolutely continuous with respect to Lebesgue, and they are jointly integrable, or
3. ν is absolutely continuous with respect to volume.

By μ -stationary, we mean that

$$\int f_* \nu d\mu(f) = \nu.$$

Hyperbolicity is the assumption that there are no zero Lyapunov exponents. We assume this as it is conjectured that all measures in this setting are hyperbolic. Non-elementary means that the induced subgroup, $\Gamma^* < GL(H^2(X; \mathbb{Z}))$, acting on cohomology of X contains a non-Abelian free group. It turns out that this forces X to be Kähler and projective, see [13]. Further, the latter reference shows that X must be one of four types of objects: a K3 surface, an Enriques surface, an Abelian surface or a rational surface. In the case of K3

surfaces, Enriques surfaces, and Abelian surfaces, we always have a finite $\text{Aut}(X)$ -invariant volume form which forces the Lyapunov exponents to sum to zero. Hence these three cases are covered by Theorem 1.1.1. In the case of rational surfaces, some, such as the Coble family of examples carry non-elementary actions and a finite $\text{Aut}(X)$ -invariant volume form [13] and so Theorem 1.1.1 applies. This is not true in general for all rational surfaces, see for example the family of Lesieutre examples in [13, 32] which carry non-elementary actions but no $\text{Aut}(X)$ -invariant volume form.

In all cases of our theorem we have homogeneity, meaning that the stable and unstable conditional measures are supported on the orbit of some algebraic group and are the pushforward of the Haar measure of that algebraic group onto the orbit (under the orbit map).

Benoist and Quint [4] give a similar classification result in the homogeneous setting. When G is a connected almost simple Lie group with a probability measure μ whose support generates a Zariski-dense subgroup, they give a classification of μ -stationary measures on the homogeneous space G/Λ where Λ is a lattice. They also cover the Abelian surfaces case of our theorem.

In [14], Cantat and Dujardin classify all invariant probability measures (with some further analysis) in our setting with the added assumption that Γ_μ contains a parabolic element. Even if $\text{Aut}(X)$ contains a parabolic element, it is not a given that Γ_μ contains a parabolic; being non-elementary comes down to having two loxodromic elements that generate a non-Abelian free group. In Section 7 we demonstrate that it is even possible to have X (a K3 surface) such that $\text{Aut}(X)$ that does not contain any parabolic elements. Hence our result, which does not assume the existence of a parabolic element, extends the classification of [14] to this setting.

One can see [13] for more about what has been done to classify invariant probability measures for non-elementary groups when we do not assume the existence of a parabolic

element. The latter reference also provides an analysis of some rational surfaces that carry non-elementary actions.

1.2 Strategy

Our main work is to classify the measures using the ‘Benoist Quint’ argument developed in [4] (see also [17] and [7]). Alternatives 1 and 2 of Theorem 1.1.1 are shown using this technique given that we have already demonstrated invariance of the stationary measure. With an understanding of the notation given in Section 2.6, one can read the proof sketches of alternatives 1 and 2 in Sections 6.1 and 5.1.

To achieve the invariance of the stationary measures, we work to establish when quantitative non-integrability (QNI) conditions hold in Section 4 in order to use the results of [6].

Sections 2 and 3 establish the notation, background and tools needed to prove everything in Sections 4, 5 and 6.

2 Preliminaries

Let X be a compact Kähler surface and let $\text{Aut}(X)$ be the group of biholomorphisms of X (i.e. holomorphic diffeomorphisms). Note that a Kähler manifold is a Riemannian manifold. We fix some Riemannian metric on X and let $\|\cdot\|$ be the norm on $T_x X$ with respect to this metric.

Let μ be a finitely supported probability measure on $\text{Aut}(X)$, and ν an ergodic Borel μ -stationary measure. By μ -stationary, we mean that the measure ν is invariant on average. To be more precise, this means that

$$\int f_* \nu d\mu(f) = \nu. \tag{1}$$

By ergodic we mean that for any measurable set A such that $\nu(A\Delta f(A)) = 0$ for μ -a.e. $f \in \text{Aut}(X)$, we have that $\nu(A) = 0$ or 1 . Equivalently, ν being ergodic means that it cannot be written as a proper convex combination of two distinct Borel μ -stationary measures.

Let Γ_μ be the subgroup generated by the support of μ . We require that Γ_μ be non-elementary, which means that the induced subgroup $\Gamma^* < GL(H^2(X; \mathbb{Z}))$ acting on cohomology of X , contains a non-Abelian free group. By [13], X must be projective. Further, it is worth noting as above that the latter reference shows that amongst compact complex surfaces, only Kähler surfaces can carry non-elementary actions which is why we focus on them; [9] showed that only Kähler surfaces can carry automorphisms of positive topological entropy and [13] demonstrate that these elements are present when a surface carries a non-elementary action. In the end, [13, Theorem 1.3] gives us that only four types of objects can carry these kinds of actions: complex K3 surface, Enriques surface, Abelian surface, or rational surface.

It is worth remarking that there are two different definitions of K3 surface. Firstly, there is a more general definition of a complex analytic K3 which is a simply connected compact complex manifold of dimension 2 with a nowhere vanishing holomorphic 2-form. Not all of these are projective. There is also the notion of an algebraic K3 surface defined over $\mathbb{C}$, this set of surfaces can be embedded in the set of complex analytic K3s using Serre's GAGA principle. The image of these algebraic K3 surfaces in the set of complex analytic K3s turn out to be precisely the projective ones, and hence these are the ones we are interested in. One can see [21] as a standard reference for more details. We will work with examples of this kind in Section 7.

The elements of $\text{Aut}(X)$ are classified as loxodromic (hyperbolic), parabolic or elliptic. An element f is loxodromic if f^* , its image in $GL(H^2(X, \mathbb{Z}))$, has spectral radius strictly greater than 1. The Gromov-Yomdin Theorem (see [19, 20, 40]) implies that these are the elements of positive topological entropy. An element is elliptic if f^* has finite order. If it is

neither of those two, it is parabolic; this means that it is virtually unipotent, i.e. $(f^*)^k$ is unipotent but not the identity for some k .

One can see this classification of automorphisms of X using hyperbolic geometry [10]. Since we are in a setting where X is Kähler, the Hodge decomposition tells us that the de Rham cohomology splits into a direct sum of Dolbeault cohomology groups:

$$H^2(X; \mathbb{C}) = \bigoplus_{p+q=2} H^{p,q}(X; \mathbb{C}) = H^{2,0}(X; \mathbb{C}) \oplus H^{1,1}(X; \mathbb{C}) \oplus H^{0,2}(X; \mathbb{C}).$$

Classes in these $H^{p,q}(X; \mathbb{C})$ (Dolbeault cohomology groups) are represented by forms of type (p, q) that are $\bar{\partial}$ -closed. We take $H^{1,1}(X; \mathbb{R})$ with its natural non-degenerate intersection form, $\langle \cdot | \cdot \rangle$, of signature $(1, h^{1,1}(X) - 1)$, it is isometric to the standard quadratic form on Minkowski space of dimension $h^{1,1}(X)$. Taking the connected component of the hyperboloid $\{v \in H^{1,1}(X; \mathbb{R}) : \langle v | v \rangle = 1\}$ that intersects the Kähler cone (it is unique), we have a model of hyperbolic space $\mathbb{H}_X$. The automorphisms of X act by isometries with respect to the intersection form, preserve the Kähler cone, and hence, preserve $\mathbb{H}_X$. Viewing elements of $\text{Aut}(X)$ as elements of $\text{Isom}(\mathbb{H}_X)$ we can classify them as elliptic, parabolic and hyperbolic and this is equivalent to the definitions we gave before.

As explained before, hyperbolic elements are present when a surface carries a non-elementary action, but parabolic elements do not necessarily have to be present. We give an example of a K3 surface X where $\text{Aut}(X)$ does not contain any parabolic elements in Section 7.

2.1 Setup of the random dynamical system

Now we can setup our dynamical system, which is indeed ‘random’ as we will draw diffeomorphisms to act on X using the law given by μ . Let $\Omega = \text{Aut}(X)^{\mathbb{Z}}$. We typically write an element of Ω as $\omega = (\dots, f_{-2}, f_{-1}, f_0, f_1, f_2, \dots)$. We define the dynamics associated to a

given element of Ω by

$$f_\omega^n := f_{n-1} \circ \cdots \circ f_0 \quad \text{for } n \geq 1, \quad (2)$$

$$f_\omega^n := (f_{-n})^{-1} \circ \cdots \circ (f_{-1})^{-1} \quad \text{for } n < 0, \quad (3)$$

and f_ω^0 is the identity. We will often simply write f_ω for f_ω^1 .

Let $Y := X \times \Omega$. We use the canonical projection map $\pi_X : Y \rightarrow X$ to make the identification between $X \times \{\omega\} \subset Y$ and X itself, denoting this X_ω . Also let $\sigma : \Omega \rightarrow \Omega$ denote the two-sided shift map. We define the (invertible) skew product map by

$$F(x, \omega) = (f_\omega^1(x), \sigma(\omega)), \quad (4)$$

$$F^{-1}(x, \omega) = (f_\omega^{-1}(x), \sigma^{-1}(\omega)). \quad (5)$$

Occasionally, for $n \in \mathbb{Z}$ we will write $F_\omega^n(x)$ which denotes the X -component of $F^n(x, \omega)$.

We define the tangent bundle of Y as $TY := TX \times \Omega$ and denote DF as the tangent map

$$D_{(x, \omega)}F : \{\omega\} \times T_x X \rightarrow \{\sigma(\omega)\} \times T_{f_\omega^1 x} X,$$

induced by $D_x f_\omega^1$. Similarly we denote DF^{-1} the tangent map induced by $D_x f_\omega^{-1}$.

Additionally, we have a distance metric d_Y on Y , where

$$d_Y((x, \omega), (x', \omega')) = \max\{d_X(x, x'), d_\Omega(\omega, \omega')\}. \quad (6)$$

Here d_Ω is the distance metric on Ω given by

$$d_\Omega(\omega, \omega') = \sum_{-\infty}^{\infty} \frac{1}{2^{|n|} d(\omega_n, \omega'_n)}, \quad (7)$$

where ω_n is the n th entry of $\omega \in \Omega$ and d is the discrete metric.

For $\omega \in \Omega$, let ω^+ be the non-negative indexed part of the bi-infinite vector ω and ω^- be the negative indexed part of the vector ω . We then decompose Ω as $\Omega = \Omega^- \times \Omega^+$, where Ω^+ is the space of 1-sided infinite vectors ω^+ and Ω^- is the 1-sided infinite vectors ω^- .

Lemma 2.1.1 (Proposition 1.2, [25], [26]). *There exists a unique F -invariant Borel probability measure m on Y such that the image of m under the canonical projection map $\pi_{X \times \Omega^+} : Y \rightarrow X \times \Omega^+$ is $\nu \times \mu^{\mathbb{N}}$. Additionally, under the canonical projection maps $\pi_{\Omega} : Y \rightarrow \Omega$ and $\pi_X : Y \rightarrow X$ the image of m is $\mu^{\mathbb{Z}}$ and ν respectively. This measure m is given by the weak-* limit*

$$m = \lim_{n \rightarrow \infty} (F^n)_*(\nu \times \mu^{\mathbb{Z}}).$$

Remark 2.1.2. *If ν is invariant under $\text{Supp}(\mu)$, then $m = \nu \times \mu^{\mathbb{Z}}$.*

Remark 2.1.3. *Additionally, ν is ergodic μ -stationary if and only if $m_+ := \nu \times \mu^{\mathbb{N}}$ is ergodic under the non-invertible skew product map F on $X \times \Omega^+$. Further this happens if and only if m is ergodic under the invertible skew product map F (see [5] Propositions 1.8 and 1.9, [11] Section 7.1).*

2.2 Oseledets' theorem and the global stable and unstable manifolds

We will assume that μ satisfies the following moment condition:

let $\|f\|_{C^1} := \max_{x \in X} \|D_x f\|$, we ask that

$$\int \left(\log \|f\|_{C^1} + \log \|f^{-1}\|_{C^1} \right) d\mu(f) < \infty. \quad (8)$$

Note that the finiteness of the above integral does not depend on the choice of Riemannian metric.

Remark 2.2.1. *The above moment condition is required for Oseledets' theorem, but it also implies higher moment conditions, i.e. replacing C^1 with C^k (this norm computed in charts). This follows from the Cauchy estimates [11, Lemma 4.1]. This is needed to use Pesin theory.*

This moment condition given by Equation (8) and Kingsman's subadditive ergodic theorem give us that the following limits exist m -almost surely:

$$\lambda^+ = \lim_{n \rightarrow \infty} \frac{1}{n} \log \|D_x f_\omega^n\|, \quad \lambda^- = \lim_{n \rightarrow \infty} \frac{1}{n} \log \|(D_x f_\omega^n)^{-1}\|^{-1}. \quad (9)$$

We will assume that $\lambda^+ > 0 > \lambda^-$ (i.e. hyperbolicity). Applying Oseledets' theorem [25, Chapter III], we have that for m -a.e. $(x, \omega) \in Y$, there exists a decomposition

$$T_x X_\omega \cong T_x X = E^s(x, \omega) \oplus E^u(x, \omega),$$

such that for $v \in E^s(x, \omega) \setminus \{0\}$ we have that

$$\lim_{n \rightarrow \pm\infty} \frac{1}{n} \log \|D_x f_\omega^n(v)\| = \lambda^-, \quad (10)$$

and for $v \in E^u(x, \omega) \setminus \{0\}$ we have

$$\lim_{n \rightarrow \pm\infty} \frac{1}{n} \log \|D_x f_\omega^n(v)\| = \lambda^+. \quad (11)$$

Additionally, we have that the map $(x, \omega) \mapsto E^{s/u}(x, \omega)$ is measurable and that this decomposition is DF -equivariant in the sense that

$$D_{(x, \omega)} F(E^{s/u}(x, \omega)) = E^{s/u}(F(x, \omega)).$$

We will denote Λ to be the conull subset of Y where this theorem holds true, it is often called the Oseledets' set. In the above, λ^+ and λ^- are called Lyapunov exponents and $E^{s/u}$ are called the Oseledets' subspaces.

The work of [11] summarizes the conclusions of Pesin theory in our setting in Section 7 of their paper. We get from this the existence of local and global stable and unstable manifolds

for m -a.e. $(x, \omega) \in Y$. The global stables and unstables, denoted $W^s(x, \omega)$ and $W^u(x, \omega)$ respectively, are given by

$$W^s(x, \omega) = \{y \in X : \limsup_{n \rightarrow \infty} \frac{1}{n} \log d_X(f_\omega^n(y), f_\omega^n(x)) < 0\}, \quad (12)$$

$$W^u(x, \omega) = \{y \in X : \limsup_{n \rightarrow -\infty} \frac{1}{|n|} \log d_X(f_\omega^n(y), f_\omega^n(x)) < 0\}, \quad (13)$$

where d_X is the Riemannian distance computed in X . We view these manifolds as subsets of X_ω which means they live in the skew product. Further, [11, Proposition 7.10] gives us that these global stables and unstable manifolds are biholomorphic to $\mathbb{C}$ for m -a.e. (x, ω) . In fact, they are parameterized by injectively immersed entire curves.

Let d_X^s and d_X^u be the induced distance metric in the stable or unstable manifolds respectively. Note that if $y, z \in W^s(x, \omega)$ then $d_X^s(y, z) \geq d_X(y, z)$, similar for the unstable case.

2.3 Suspension flow

We will establish two ‘suspension’ flows; the ‘standard’ flow and the ‘time changed’ flow. The latter will be constructed in Section 3.9 where we have to use normal form coordinates. The space we will be flowing on is $Z = Y \times [0, 1[$ which is called the ‘suspension’.

The ‘standard’ flow is denoted $F^t : Z \rightarrow Z$ for $t \in \mathbb{R}$. For $t \geq 0$, we let

$$F^t(x, \omega, k) = (F^n(x, \omega), k + t \mod 1),$$

where F is as defined in Equation (4) and $n = \lfloor k + t \rfloor$. For $t < 0$ the flow is given by

$$F^t(x, \omega, k) = (F^n(x, \omega), k + t \mod 1),$$

where F^{-1} is as in Equation (5) and $n = \lfloor k + t \rfloor$ (note that for example $\lfloor -0.8 \rfloor = -1$). The

natural invariant measure for this flow is $\hat{m} := m \times dt$ where dt is the Lebesgue measure on $[0, 1[$.

Note that the $[0, 1[$ -component of an element of Z does not play a role in determining the global stable or unstable manifold, hence $W^u(x, \omega, k) := W^u(x, \omega) \subset X$.

2.4 Conditional measures and the results of [6]

The following definition is from [26, Section IV.2], but see also [17, Section B.4], [7, Section 6.3], [33, Section 9.3], and [6].

Definition 1 ([26], Section IV.2). *A measurable partition η^s of Y is subordinate to the stable manifolds if for m -a.e. (x, ω) , we have the following properties:*

1. $\eta_\omega^s(x) := \{y \in X : (y, \omega) \in \eta^s(x, \omega)\} \subset W^s(x, \omega)$,
2. $\eta_\omega^s(x)$ contains an open neighbourhood of x in $W^s(x, \omega)$ in the submanifold topology.

Where $\eta^s(x, \omega)$ is an atom of the partition. A similar definition holds for the unstable manifolds, denote this partition η^u .

Proposition 2.4.1 ([26], Proposition 2.1, [11], Section 7.6). *Such a measurable partition as described in Definition 1 exists and we can also ask that it has the following properties:*

1. $F^{-1}\eta^s \leq \eta$ where $F^{-1}\eta^s(x, \omega) := F^{-1}(\eta^s(F(x, \omega)))$,
2. $\{\{X \times \{\omega\}\} : \omega \in \Omega\} \leq \eta^s$,

where if two partitions P and Q are such that $P \leq Q$ then for all $B \in Q$, there exists $A \in P$ such that $B \subset A$. There is a similar statement for the unstable version of Definition 1.

In [11, Section 7.6], they also quote the above proposition and call this partition a stable (or unstable) Pesin partition. Associated to measurable partitions are families of conditional measures supported on the atoms of the partition. Here we let $\{m_{x, \omega}^{\eta^s}\}_{(x, \omega) \in Y}$ denote our

conditional measures supported on $\eta^s(x, \omega)$. Letting $\mathcal{A}^s$ be the σ -algebra of m -measurable subsets which are unions of the $\eta^s(x, \omega)$, then these measures are such that for any $\varphi : Y \rightarrow \mathbb{R}$

$$\mathbb{E}[\varphi | \mathcal{A}^s](x, \omega) = \int_Y \varphi dm_{x, \omega}^{\eta^s}. \quad (14)$$

Note that though the stable or unstable manifolds forms a partition of a full-measure subset of Y this partition is generally not measurable. This is why we have to work with partitions that are subordinate to the stable or unstable manifolds. Also, Property (1) in Proposition 2.4.1 tells us that $\eta^s(x, \omega) \subset F^{-1}\eta^s(F(x, \omega))$. This combined with the fact that m is F -invariant gives us that

$$F_* m_{x, \omega}^{\eta^s} \propto m_{F(x, \omega)}^{\eta^s}. \quad (15)$$

In particular, there is a constant $C_{x, \omega}$ dependent on x and ω such that

$$F_* m_{x, \omega}^{\eta^s} = C_{x, \omega} m_{F(x, \omega)}^{\eta^s}.$$

Note that Property (2) of Proposition 2.4.1 tells us that the $\eta(x, \omega)$ belong to an element of the partition $\{\{X \times \{\omega\}\} : \omega \in \Omega\}$. This tells us that

$$\eta^s(x, \omega) = \eta_{\omega}^s(x) \times \{\omega\}.$$

We define $\{m_{x, \omega}^s\}_{(x, \omega) \in Y}$ such that $m_{x, \omega}^s(\eta_{\omega}^s(x)) = 1$ and for any compact set $K \subset W^s(x, \omega)$ we take $t > 0$ such that

$$F^t(K \times \{\omega\}) = F_{\omega}^t(K) \times \{\sigma^t(\omega)\} \subset \eta^s(F^t(x, \omega)),$$

where $F_\omega^t(x)$ is the X -component of $F^t(x, \omega)$. Then, we let

$$m_{x,\omega}^s(K) := \frac{(F^{-t})_* m_{F_\omega^t x, \sigma^t(\omega)}^{\eta^s}(K \times \{\omega\})}{(F^{-t})_* m_{F_\omega^t x, \sigma^t(\omega)}^{\eta^s}(\eta^s(x, \omega))}. \quad (16)$$

This is a locally finite, infinite measure on the (entire) stable leaf which we call a stable leafwise conditional measure or just stable conditional measure. The right hand side is well-defined because of Equation (15). The measures $m_{x,\omega}^s$ are supported on the stable leaves. Note that we have the following property similar to Equation (15):

$$F_* m_{x,\omega}^s \propto m_{x,\omega}^s. \quad (17)$$

Let $\{m_{x,\omega}^u\}_{(x,\omega) \in Y}$ denote the unstable analogues which we will refer to as unstable conditional measures.

2.5 Compatible families and quantitative non-integrability

We shall follow the definitions of [6] in this section. We will also keep track of the $[0, 1[-$ component of Z in this section, however this component adds no information to many of these objects. For example, $W^{s/u}(x, \omega, k) = W^{s/u}(x, \omega)$.

As in [6, Section 4.5.1], unstable manifolds admit a ‘subresonant structure’ (see [6, Appendix C and D]). We denote $G^{ssr}(W^u(x, \omega, k))$ as the set of strictly subresonant transformations on $W^u(x, \omega, k)$. In our setting, these maps are linear transformations on $W^u(x, \omega, k)$ in normal form coordinates (see Section 3.8). The groups $G^{ssr}(W^u(x, \omega, k))$ are unipotent algebraic groups. We will study the action of connected unipotent algebraic subgroups $U^+(x, \omega, k) \subset G^{ssr}(W^u(x, \omega, k))$. We denote $U^+[q, \omega, k] := U^+(x, \omega, k)q$ as the orbit of the action of $U^+(x, \omega, k)$ on $q \in W^u(x, \omega, k)$.

We define the ‘combinatorial’ unstable and stable manifolds as follows

$$\hat{W}^u(x, \omega, k) := W^u(x, \omega, k) \times W^+(\omega, k), \quad (18)$$

$$\hat{W}^s(x, \omega, k) := W^s(x, \omega, k) \times W^-(\omega, k), \quad (19)$$

where

$$W^+(\omega, k) := \{(\omega', k') \in \Omega \times [0, 1[: k' = k, \text{ and for } n \in \mathbb{N} \text{ large, } f'_{-n} = f_{-n}\},$$

$$W^-(\omega, k) := \{(\omega', k') \in \Omega \times [0, 1[: k' = k, \text{ and for } n \in \mathbb{N} \text{ large, } f'_n = f_n\},$$

and where $\omega = (\dots, f_{-2}, f_{-1}, f_0, f_1, f_2, \dots)$.

Definition 2 (Random compatible family of subgroups). *We say that a measurable family of connected algebraic subgroups $(x, \omega) \mapsto U^+(x, \omega, k) \subset G^{ssr}(W^u(x, \omega, k))$ is compatible with $\hat{m}$ if we have the following:*

1. *The subgroups are F^t -equivariant.*
2. *For $\mu^{\mathbb{Z}}$ -almost every ω we require that the sets $U^+[x, \omega, k] \cap \eta_\omega^u(x)$ form a partition of X . Further, the conditional measures along $U^+[x, \omega, k] \cap \eta_\omega^u(x)$ are proportional to the restriction of the ‘Haar measure’ which in this setting is the unique $U^+(x, \omega, k)$ -invariant measure on $U^+[x, \omega, k]$.*
3. *Given $(x', \omega', k') \in \hat{W}^u(x, \omega, k)$, we have a natural identification of the group $G^{ssr}(x, \omega, k)$ with the group $G^{ssr}(x', \omega', k')$, viewing both as acting on $W^u(x, \omega, k) = W^u(x', \omega', k')$. Further, there is a conull set $X_0 \subset X$ such that if the points $(x, \omega, k), (x', \omega', k') \in X_0$ and $(x', \omega', k') \in U^+[x, \omega, k]$, then $U^+(x', \omega', k') = U^+(x, \omega, k)$.*
4. *$U^+(x, \omega, k)$ is ‘maximal’, i.e. it contains the largest connected subgroup of $G^{ssr}(x, \omega, k)$ that preserves the Haar measure on $U^+[x, \omega, k]$.*

Remark 2.5.1. We allow $U^+(x, \omega, k) = \{e\}$.

Now let

$$W_1^+(\omega, k) := \{(\omega', k') \in \Omega \times [0, 1[: k' = k, (\omega')^- = \omega^-\}, \quad (20)$$

$$W_1^-(\omega, k) := \{(\omega', k') \in \Omega \times [0, 1[: k' = k, (\omega')^+ = \omega^+\}. \quad (21)$$

Recall η^s from Definition 1 and the unstable counterpart η^u . We will define

$$\begin{aligned} \eta_{\omega, t}^u(x) &= F_{\sigma^t(\omega)}^{-t}(\eta_{\sigma^t(\omega)}^u(F_{\omega}^t(x))) \subset X_{\omega}, \\ \eta_{\omega, t}^s(x) &= F_{\sigma^{-t}(\omega)}^t(\eta_{\sigma^{-t}(\omega)}^s(F_{\omega}^{-t}(x))) \subset X_{\omega}, \end{aligned}$$

where $F_{\omega}^t(x)$ is the X -component of $F^t(x, \omega)$.

It is easy to see that $\eta_{\omega, t}^u(x)$ and $\eta_{\omega, t}^s(x)$ are smaller than (and live inside) $\eta_{\omega}^u(x)$ and $\eta_{\omega}^s(x)$ resp.

Now let

$$\eta_t^u[x, \omega, k] := \eta_{\omega, t}^u(x) \times W_1^+(\omega, k) \subset W^u(x, \omega, k) \times W_1^+(\omega, k) \subset \hat{W}^u(x, \omega, k), \quad (22)$$

$$\eta_t^s[x, \omega, k] := \eta_{\omega, t}^s(x) \times W_1^-(\omega, k) \subset W^s(x, \omega, k) \times W_1^-(\omega, k) \subset \hat{W}^s(x, \omega, k). \quad (23)$$

The natural measure on $W_1^+(\omega, k)$ is

$$\tilde{\mu}_{\omega, k}^u := \delta_{\omega^-} \times \mu^{\mathbb{N}} \times \delta_k,$$

on $\Omega^- \times \Omega^+ \times [0, 1[$. Similarly for W_1^- we have $\tilde{\mu}_{\omega, k}^s$. Let $\tilde{m}_{x, \omega}^u$ be the restriction of the conditional measure of $m_{x, \omega}^u$ to $U^+[x, \omega, k] \cap \eta_{\omega}^u(x)$. Note that $\tilde{m}_{x, \omega}^u$ is proportional to the ‘Haar’ measure as in (2) of Definition 2.

Consider the measures

$$\hat{m}_{x,\omega,k}^u := \tilde{m}_{x,\omega}^u \times \tilde{\mu}_{\omega,k}^u, \quad (24)$$

$$\hat{m}_{x,\omega,k}^s := m_{x,\omega}^s \times \tilde{\mu}_{\omega,k}^s. \quad (25)$$

They live on $\hat{W}^u(x, \omega, k)$ (or more specifically $U^+[x, \omega, k] \times W_1^+(\omega, k)$) and $\hat{W}^s(x, \omega, k)$ respectively.

Definition 3 (Random QNI). *Let $(x, \omega, k) \mapsto U^+(x, \omega, k)$ be a measurable family of connected algebraic subgroups satisfying Definition 2, we define*

$$\hat{U}[x, \omega, k] := U^+[x, \omega, k] \times W_1^+ \subset \hat{W}^u(x, \omega, k).$$

We say that $U^+(x, \omega, k)$ satisfies the random QNI condition if the following holds:

- there exists $\alpha_0 > 0$ and,
- for every $\delta > 0$ there exists a compact set $K \subset Z$ with $\hat{m}(K) > 1 - \delta$ and constants $C = C(\delta) > 0$ and $\ell_0 = \ell_0(\delta) > 0$ such that:

If $\ell > \ell_0$ and $\hat{x}_{1/2} = (x_{1/2}, \omega_{1/2}, k_{1/2}) \in K$, $\hat{x}_1 = F^{\ell/2}(\hat{x}_{1/2})$, $\hat{x} = F^{-\ell/2}(\hat{x}_{1/2}) \in K$ then

- there is a subset

$$S = S(\hat{x}_{1/2}, \ell) \subset \eta_{\ell/2}^u[\hat{x}_{1/2}] \cap \hat{U}^+[\hat{x}_{1/2}] \subset \hat{W}^u(\hat{x}_{1/2}),$$

such that

$$\hat{m}_{\hat{x}_{1/2}}^u(S) \geq (1 - \delta) \hat{m}_{\hat{x}_{1/2}}^u(\eta_{\ell/2}^u[\hat{x}_{1/2}] \cap \hat{U}^+[\hat{x}_{1/2}]),$$

and for each $\hat{y}_{1/2} \in S$ we have

- that there is a subset

$$S' = S'(\hat{y}_{1/2}, \ell) \subset \eta_{\ell/2}^s[\hat{x}_{1/2}] \subset \hat{W}^s(\hat{x}_{1/2}),$$

such that

$$\hat{m}_{\hat{x}_{1/2}}^s(S') \geq (1 - \delta) \hat{m}_{\hat{x}_{1/2}}^s(\eta_{\ell/2}^s[\hat{x}_{1/2}]),$$

so that if $\hat{z}_{1/2} \in S'$, then

$$d_X((U_q^+[\hat{z}_{1/2}] \cap \eta_{\omega_{1/2}, \ell/2}^u(z_{1/2})), W_q^s(\hat{y}_{1/2})) \geq Ce^{-\alpha_0 \ell}, \quad (26)$$

where $U_q^+[x]$ is $U^+[x]$ intersected with $W_q^u(x, \omega)$, i.e. the local stable manifold, see Theorem 3.5.4.

Now we state a theorem from [6]:

Theorem 2.5.2 ([6], Corollary 4.1.5). *Let μ be a finitely supported probability measure on $\text{Aut}(X)$, ν a μ -stationary ergodic probability measure, $\hat{m}$ as in Section 2.3, then there exists a measurable family of connected algebraic subgroups $(x, \omega, k) \mapsto U^+(x, \omega, k)$ satisfying Definition 2 that does not satisfy Definition 3, i.e. random QNI does not hold.*

2.6 Proof of Theorem 1.1.1

In this section, we will give the proof of Theorem 1.1.1 except for Lemma 2.6.3, Proposition 2.6.5, Lemma 2.6.7 and Proposition 2.6.9. These will be proven later in the paper.

Recall the assumptions made in Section 1, i.e. that Γ_μ is non-elementary, that $\text{Supp}(\mu) < \infty$, that there is no Γ_μ -invariant algebraic curve, and that $\lambda^+ + \lambda^- \geq 0$.

From Theorem 2.5.2, take our measurable family of connected algebraic subgroups

$$(x, \omega, k) \mapsto U^+(x, \omega, k).$$

This family of subgroups satisfies Definition 2 but does not satisfy random QNI. Let

$$d_+ := \dim_{\mathbb{R}}[U^+(x, \omega, k)].$$

This d_+ is constant a.e. by property (1) of Definition 2 combined with ergodicity. There are three cases to address and these cases correspond to the three alternatives in Theorem 1.1.1.

Define $\mathcal{L}^s(x, \omega, 0)$ to be the Zariski closure of the support of the stable conditional, $m_{x, \omega}^s$, inside $W^s(x, \omega) \subset X$, and $\mathcal{L}^u(x, \omega, 0)$ to be the unstable analogue. Note that we have no concrete notion of Zariski closure ambiently, but under normal form coordinates (discussed later in Section 3.8), you are now in $\mathbb{C}$ and you can take Zariski closure and then push it back.

Let $h(\mu)$ be the entropy of μ . Let ζ be a measurable partition of Z such that $F\zeta \leq \zeta$ and define

$$h_{\hat{m}}(F, \zeta) := H_{\hat{m}}(F^{-1}\zeta|\zeta) = - \int \log \hat{m}_{\hat{x}}^{\zeta}(F^{-1}\zeta(\hat{x})) d\hat{m}(\hat{x}),$$

where $\hat{m}_{\hat{x}}^{\zeta}$ is the conditional measure at $\hat{x} \in Z$ relative to the partition ζ . We now take ζ to be the partition of Z such that $\zeta(\hat{x}) = W_1^+(\hat{x})$, then we define

$$h_{\hat{m}}(F, W_1^+) = h_{\hat{m}}(F, \zeta). \quad (27)$$

This quantity is the entropy of the 1-sided shift on the set of futures, the entropy we get from the combinatorial unstable W_1^+ . There is a similar quantity, the entropy we get from the combinatorial stable W_1^- .

We will state the following Proposition from [6, Section 4] which is a version of the Ledrappier-Young formula for our setting:

Proposition 2.6.1 ([6], Proposition 4.3.8). *For $\hat{m}$ -a.e. $(x, \omega, k) \in Z$ we have*

1. $h_{\hat{m}}(F) \geq h_{\hat{m}}(F, W_1^+) + \lambda^+ \dim(E^u(\hat{x}) \cap T_x U^+[\hat{x}]),$
2. $h_{\hat{m}}(F, W_1^+) = h(\mu),$
3. $h_{\hat{m}}(F^{-1}) \leq h_{\hat{m}}(F^{-1}, W_1^-) - \lambda^- \dim(E^s(\hat{x}) \cap T_x \mathcal{L}^s(\hat{x})),$
4. $h_{\hat{m}}(F^{-1}, W_1^-) \leq h(\mu)$ with equality if and only if ν is μ -invariant.

Additionally, we have the following result that we take from [6] which follows immediately from [27], Theorem 3.4.

Proposition 2.6.2 ([6], Proposition 4.3.9). *For m -a.e. $(x, \omega) \in Y$, if the unstable conditional measure $m_{x, \omega}^u$ is supported on an embedded submanifold $N_{x, \omega} \subset W^u(x, \omega)$, then*

$$h_{\hat{m}}(F) \leq h(\mu) + \lambda^+ \dim(E^u(x, \omega) \cap T_x N_{x, \omega}) \quad (28)$$

with equality if and only if $m_{x, \omega}^u$ is equivalent to the Lebesgue measure on $N_{x, \omega}$ for m -a.e. $(x, \omega) \in Y$. A similar statement holds for stables.

Alternative 1

The first alternative of Theorem 1.1.1, that ν is finitely supported, corresponds to the case $d_+ = 0$. Here $U^+(x, \omega, k) = \{e\}$ a.s.

Lemma 2.6.3. *Suppose Γ_μ is non-elementary, $U^+(x, \omega, k) = \{e\}$ a.e., and the conditional measure $m_{x, \omega}^s$ along $W^s(x, \omega, k)$ is non-trivial for $\hat{m}$ -a.e. $(x, \omega, k) \in Z$. Then random QNI (Definition 3) holds.*

We will prove this in Section 4 once we establish more notation and a few lemmas.

Corollary 2.6.4. *In our setting, if $d_+ = 0$, then ν is Γ_μ invariant, and the stable conditional measures $m_{x, \omega}^s$ on $W^s(x, \omega, k)$ and the unstable conditional measures $m_{x, \omega}^u$ on $W^u(x, \omega)$ are trivial for $\hat{m}$ -a.e. (x, ω, k) .*

Proof. Since we are taking $U^+(x, \omega, k)$ as in Theorem 2.5.2, random QNI does not hold. Lemma 2.6.3 gives us that we must have that the stable conditional measure $m_{x, \omega}^s$ on $W^s(x, \omega, k)$ is trivial for $\hat{m}$ a.e. (x, ω, k) .

It remains to discuss Γ_μ -invariance of ν . This follows from [6, Proposition 4.3.11]. We will directly give their proof below:

Since μ is finitely supported, it can be written as $\sum_i p_i \delta_{f_i}$ for $f_i \in \text{Supp}(\mu)$ and for some p_i . Hence, its entropy, $h(\mu) = -\sum_i p_i \log(p_i)$ is clearly finite.

Recall W_1^+ from Equation (20); $h_{\hat{m}}(F, W_1^+)$ is the entropy of the 1-sided shift on the set of futures as in Equation (27). Note that as ν is μ -stationary, our measure m from Lemma 2.1.1 projects to the product $\nu \times \mu^{\mathbb{N}}$ and so $h_{\hat{m}}(F, W_1^+) = h(\mu)$ (see Proposition 2.6.1). Note that this is not apriori the case for $h_{\hat{m}}(F^{-1}, W_1^-)$, the analogue on the set of pasts. We work to show this in order to get invariance using (4) from Proposition 2.6.1.

We have that by Proposition 2.6.1 (1), we have the following lower bound on the entropy of our map:

$$h_{\hat{m}}(F) \geq h(\mu) + \lambda^+ \cdot \dim(T_x U^+). \quad (29)$$

On the other hand, Proposition 2.6.1 (3) gives us the lower bound:

$$h_{\hat{m}}(F^{-1}) \leq h_{\hat{m}}(F^{-1}, W_1^-) - \lambda^- \dim(T_x \mathcal{L}^s). \quad (30)$$

In our case Equations (29) and (30) simplify because the conditional measures $m_{x,\omega}^s$ are trivial and so $\mathcal{L}^s$ has dimension zero. Additionally we have assumed that $d_+ = 0$ and so $T_x U^+$ is also zero dimensional. So we have

$$h_{\hat{m}}(F) \geq h(\mu), \quad (31)$$

$$h_{\hat{m}}(F^{-1}) \leq h_{\hat{m}}(F^{-1}, W_1^-). \quad (32)$$

Also, by Proposition 2.6.1, we have that $h_{\hat{m}}(F^{-1}, W_1^-) \leq h(\mu)$. So we get that

$$0 \leq h(\mu) - h_{\hat{m}}(F^{-1}, W_1^-) \leq -h_{\hat{m}}(F^{-1}) + h_{\hat{m}}(F). \quad (33)$$

But $h_{\hat{m}}(F^{-1}) = h_{\hat{m}}(F)$ and hence $h(\mu) = h_{\hat{m}}(F^{-1}, W_1^-)$ which implies that ν is invariant

by Proposition 2.6.1.

Now, let $\check{\mu}$ be the measure defined by $\check{\mu}(g) = \mu(g^{-1})$, clearly ν is still $\check{\mu}$ -invariant. Letting $\check{\omega} = (\dots f_2^{-1}, f_1^{-1}, f_0^{-1}, f_{-1}^{-1}, f_{-2}^{-1}, \dots)$ we also have that

$$W^u(x, \omega, k) = W^s(x, \check{\omega}, k), \quad W^s(x, \omega, k) = W^u(x, \check{\omega}, k) \quad (34)$$

$$m_{x, \omega}^s = m_{x, \check{\omega}}^u, \quad m_{x, \omega}^u = m_{x, \check{\omega}}^s, \quad (35)$$

$$\mathcal{L}^u(x, \omega, 0) = \mathcal{L}^s(x, \check{\omega}, 0). \quad (36)$$

This tells us that $m_{x, \omega}^u$ is also trivial a.e. which completes the proof. $\square$

Proposition 2.6.5. *If Γ_μ is non-elementary, ν is Γ_μ -invariant and is not finitely supported, then the conditional measure $m_{x, \omega}^s$ on $W^s(x, \omega, k)$ is non-trivial a.e.*

Proposition 2.6.5 will take work and its proof is given in Section 6.

Proposition 2.6.6. *In the case $d_+ = 0$, the measure ν is finitely supported.*

Proof. This follows immediately from Corollary 2.6.4 and Proposition 2.6.5. $\square$

Alternative 2

The second alternative of Theorem 1.1.1 is the $d_+ = 1$ case.

Lemma 2.6.7. *If $\dim(\mathcal{L}^s) = 2$ a.e. then Definition 3, random QNI, holds.*

We will prove this in Section 4.

Lemma 2.6.8. *If $d_+ = 1$, then ν is Γ_μ -invariant and $\mathcal{L}^u(x, \omega, 0) = U^+[x, \omega, k]$.*

Proof. We use the proof from [6, Proposition 4.3.11] similarly to the way we did in our proof of Corollary 2.6.4.

We have that $h(\mu) < \infty$ and by Proposition 2.6.1, $h(\mu) = h_{\hat{m}}(F, W_1^+)$. We now consider the equations from (1) and (3) of Proposition 2.6.1. Lemma 2.6.7 gives us that $\dim(T_x \mathcal{L}^s) \leq 1$, also $d_+ = 1$ and so $\dim(T_x U^+) = 1$. Now our equations simplify to be

$$\begin{aligned} h_{\hat{m}}(F) &\geq h(\mu) + \lambda^+, \\ h_{\hat{m}}(F^{-1}) &\leq h_{\hat{m}}(F^{-1}, W_1^-) - \lambda^- \dim(T_x \mathcal{L}^s) \leq h_{\hat{m}}(F^{-1}, W_1^-) - \lambda^-. \end{aligned}$$

Since $h_{\hat{m}}(F^{-1}, W_1^-) \leq h(\mu)$ (Proposition 2.6.1) and additionally, $\lambda^+ + \lambda^- \geq 0$ we have that

$$0 \leq h(\mu) - h_{\hat{m}}(F^{-1}, W_1^-) + \lambda^- + \lambda^+ \leq h_{\hat{m}}(F) - h_{\hat{m}}(F^{-1}). \quad (37)$$

The fact that $h_{\hat{m}}(F) = h_{\hat{m}}(F^{-1})$ gives us that $h(\mu) = h_{\hat{m}}(F^{-1}, W_1^-)$ and so again, ν is invariant by Proposition 2.6.1.

Further, we see that

$$h(\mu) - \lambda^- \dim(T_x \mathcal{L}^s) = h_{\hat{m}}(F^{-1}) = h(\mu) + \lambda^+, \quad (38)$$

This forces $\dim(\mathcal{L}^s) = 1$.

The equality in Equation (38) also allows us to use Proposition 2.6.2 to say that $m_{x,\omega}^s$ is equivalent to Lebesgue on $\mathcal{L}^s$.

Let $\check{\mu}$ be the measure defined by $\check{\mu}(g) = \mu(g^{-1})$, clearly ν is still $\check{\mu}$ -invariant. Letting $\check{\omega} = (\dots f_2^{-1}, f_1^{-1}, f_0^{-1}, f_{-1}^{-1}, f_{-2}^{-1}, \dots)$ we also have that

$$W^u(x, \omega, k) = W^s(x, \check{\omega}, k), \quad W^s(x, \omega, k) = W^u(x, \check{\omega}, k),$$

$$m_{x,\omega}^s = m_{x,\check{\omega}}^u, \quad m_{x,\omega}^u = m_{x,\check{\omega}}^s,$$

$$\mathcal{L}^u(x, \omega, 0) = \mathcal{L}^s(x, \check{\omega}, 0).$$

So our argument also gives us that for a.e. $(x, \omega, k) \in Z$, $m_{x, \omega}^u$ is absolutely continuous to Lebesgue on $\mathcal{L}^u$ which is also 1-dimensional. Now note that our definition of subgroups compatible with the measure $\hat{m}$ item (2). of Definition 2 tells us that $U^+[x, \omega, k] \subset \mathcal{L}^u(x, \omega, 0)$, but both are 1-dimensional, thus $U^+[x, \omega, k] = \mathcal{L}^u(x, \omega, 0)$. In particular,

$$U^+[x, \check{\omega}, k] = \mathcal{L}^u(x, \check{\omega}, 0) = \mathcal{L}^s(x, \omega, 0).$$

We therefore set $U^-(x, \omega, k) := U^+(x, \check{\omega}, k)$, and then clearly $\mathcal{L}^s(x, \omega, 0) = U^-[x, \omega, k]$. This completes the proof. $\square$

In this alternative, we also have the statement of ‘joint integrability’:

Proposition 2.6.9. *For m -a.e. $(\hat{x}, \hat{\omega}) \in Y$, there exists a neighbourhood U of $\hat{x}$ in X such that for ν -a.e. $\hat{y} \in U$ we have that $U_{loc}^+[\hat{x}, \hat{\omega}] \cap U_{loc}^-[\hat{y}, \hat{\omega}] \neq \emptyset$ and $U_{loc}^-[\hat{x}, \hat{\omega}] \cap U_{loc}^+[\hat{y}, \hat{\omega}] \neq \emptyset$.*

Here the subscript *loc* means it is of size $q(x, \omega)$ as defined in Section 3.5. Proposition 2.6.9 will be proven in Section 5.

Alternative 3

The third alternative in Theorem 1.1.1 comes from the final case that $d_+ = 2$.

Proposition 2.6.10. *The stable and unstable conditional measures are absolutely continuous with respect to Lebesgue.*

Proof. We use the proof from [6, Proposition 4.3.11] as we did in the proofs of Corollary 2.6.4 and Lemma 2.6.8. Note that now $\dim(T_x U^+) = 2$ and that $\dim(\mathcal{L}^s) \leq 2$. So our equations from Proposition 2.6.1 simplify to

$$\begin{aligned} h_{\hat{m}}(F) &\geq h(\mu) + 2\lambda^+ \\ h_{\hat{m}}(F^{-1}) &\leq h_{\hat{m}}(F^{-1}, W_1^-) - 2\dim(T_x \mathcal{L}^s) \leq h_{\hat{m}}(F^{-1}, W_1^-) - 2\lambda^-. \end{aligned}$$

Since we have that $h_{\hat{m}}(F^{-1}, W_1^-) \leq h(\mu)$ (Proposition 2.6.1) and $\lambda^+ + \lambda^- \geq 0$ we have that

$$0 \geq h(\mu) - h_{\hat{m}}(F^{-1}, W_1^-) + 2\lambda^- + 2\lambda^+ \leq h_{\hat{m}}(F) - h_{\hat{m}}(F^{-1}). \quad (39)$$

Since $h_{\hat{m}}(F) = h_{\hat{m}}(F^{-1})$, we get that $h(\mu) = h_{\hat{m}}(F^{-1}, W_1^-)$ and so ν is invariant by Proposition 2.6.1. Further

$$h(\mu) - \lambda^- \dim(T_x \mathcal{L}^s) = h_{\hat{m}}(F^{-1}) = h(\mu) + 2\lambda^+, \quad (40)$$

and so $\dim(\mathcal{L}^s) = 2$. This equality allows us to use Proposition 2.6.2 to say that $m_{x,\omega}^s$ is equivalent to Lebesgue on $\mathcal{L}^s$.

Then, we run the same argument using $\check{\mu}$ as in Lemma 2.6.8 and get the same homogeneity for the unstable conditional measures and that $U^+[x, \omega, k] = \mathcal{L}^u(x, \omega, 0)$ and $U^-[x, \omega, k] = \mathcal{L}^s(x, \omega, 0)$ as before. $\square$

It follows immediately using [30], Corollary H, that ν is absolutely continuous with respect to Lebesgue.

3 Tools required for the proof of Theorem 1.1.1

3.1 Furstenberg theory

This section presents material from Sections 2 and 5 of [11] in an effort to establish Corollary 3.3.2 in Section 3.3.

Recall from Section 2 that $H^{1,1}(X; \mathbb{R})$, with the quadratic form it gets from the intersection form $\langle \cdot | \cdot \rangle$, is isometric to the standard quadratic form on Minkowski space $\mathbb{R}^{1,n}$ (where $n = h^{1,1}(X) - 1$). We say that a subspace $W \subset \mathbb{R}^{1,n}$ is of Minkowski type if the quadratic form restricted to W is non-degenerate and of signature $(1, \dim(W) - 1)$. Then a subspace of $H^{1,1}(X; \mathbb{R})$ is of Minkowski type if it is isometric to a Minkowski type subspace of $\mathbb{R}^{1,n}$.

Let Γ be a non-elementary subgroup of $O_{1,n}^+(\mathbb{R})$.

Lemma 3.1.1 (Lemma 2.6, Proposition 2.8, [11]). *One has a Γ -invariant decomposition*

$$H^{1,1}(X; \mathbb{R}) = H^{1,1}(X; \mathbb{R})_+ \oplus H^{1,1}(X; \mathbb{R})_0. \quad (41)$$

Here, $H^{1,1}(X; \mathbb{R})_+$ is of Minkowski type, and the restriction of Γ to this subspace is non-elementary. Further, $H^{1,1}(X; \mathbb{R})_0$ is such that the restriction of the quadratic form to this subspace is negative definite and the restriction of Γ to this subspace is contained in a compact subgroup of $GL(H^{1,1}(X; \mathbb{R}))$.

One can do a similar decomposition for $NS(X; \mathbb{R})$ and one has

$$H^{1,1}(X; \mathbb{R})_+ = NS(X; \mathbb{R})_+,$$

on which Γ acts non-elementary (see [11, Proposition 2.12]). We let

$$\Pi_\Gamma := H^{1,1}(X; \mathbb{R})_+.$$

The limit set of Γ (which is non-elementary), denoted $\lim(\Gamma)$, is the accumulation set of any Γ orbit on $\mathbb{P}(v)$ for $v \notin \Pi_\Gamma^\perp$. There are many equivalent definitions, see [11, 2.3.6]. In our setting, $\lim(\Gamma) \subset \mathbb{P}(\Pi_\Gamma) \cap \mathbb{P}(\text{Nef}(X))$ (see [11, Lemmas 2.16, 2.17]).

Furstenberg theory gives us that there is a unique stationary measure $\nu_{\mathbb{P}(\Pi_\Gamma)}$ on $\mathbb{P}(\Pi_\Gamma) \subset \mathbb{P}(H^{1,1}(X; \mathbb{R}))$ and it does not charge any proper projective subspace of $\mathbb{P}(\Pi_\Gamma)$. We will further describe this measure below in Proposition 3.1.3.

Fix κ_0 is a Kahler form. Let the mass of a class a be defined by

$$M(a) = \langle a | \kappa_0 \rangle. \quad (42)$$

Lemma 3.1.2 (Lemma 5.6, [11]). *For $\mu^{\mathbb{Z}}$ -a.e. $\omega \in \Omega$, there exists a unique nef class $e(\omega)$ with $M(e(\omega)) = 1$ and as $n \rightarrow \infty$ we have*

$$\frac{1}{M((f_{\omega}^n)^*a)}(f_{\omega}^n)^*a \rightarrow e(\omega), \quad (43)$$

where a is any pseudo class with $a^2 > 0$ (in particular any Kähler class). Also $e(\omega)$ is a.s. isotropic and $\mathbb{P}(e(\omega))$ is a point of $\lim(\Gamma)$.

Proposition 3.1.3 (Theorem 5.8, [11]). *The probability measure ν_{∂} is given by*

$$\nu_{\partial} = \int \delta_{\mathbb{P}(e(\omega))} d\mu^{\mathbb{N}}. \quad (44)$$

This measure is μ -stationary, ergodic and has no atoms. It is supported on $\lim(\Gamma)$ and further, if it assigns positive measure to any subset $\Lambda \subset \lim(\Gamma)$, then Λ is an uncountable set. It is also the unique stationary measure on $\mathbb{P}(H^{1,1}(X; \mathbb{R}))$ with $\nu_{\partial}(\mathbb{P}(\Pi_{\Gamma}^{\perp})) = 0$.

3.2 Stable currents

In this section we will deal with currents of type $(1, 1)$. We will briefly interlude to some basics about currents following [10, Sections 5, 6].

Let $\Lambda^{1,1}(X; \mathbb{R})$ be the set of smooth real valued $(1, 1)$ forms on X with the Fréchet topology. Currents of type $(1, 1)$ act like a dual to this space, they are continuous linear functionals on $\Lambda^{1,1}(X; \mathbb{R})$. If T is a current we write its action on some $\varphi \in \Lambda^{1,1}(X; \mathbb{R})$ as $\langle T | \varphi \rangle$. We provide two important examples:

1. Take α a continuous $(1, 1)$ form. It defines a current:

$$\langle \{\alpha\} | \varphi \rangle = \int_X \alpha \wedge \varphi.$$

2. Take $C \subset X$ a curve. Then, C (regardless of whether or not it is singular) defines a

current:

$$\langle \{C\} | \varphi \rangle = \int_C \varphi.$$

We say a current is positive if it takes non-negative values on the convex cone of positive forms (i.e. (1,1) forms φ such that $\varphi(u, \sqrt{-1}u) \geq 0$). We say that a current T is closed if it vanishes on the subset of $\Lambda^{1,1}(X; \mathbb{R})$ of exact forms. In Example (1) above, the current defined by α is closed if and only if α is a closed form. It is positive if and only if α is a positive form. Example (2) is closed and positive. For any closed current T , there is a unique cohomology class, $[T]$ such that

$$\langle T | \varphi \rangle = \langle [T] | [\varphi] \rangle.$$

If T a current is closed, then $\langle T | \varphi \rangle = \langle [T] | [\varphi] \rangle$ and we can define its mass analogously to how we did before in Equation (42) as $M(T) = \langle [T] | \kappa_0 \rangle$.

Theorem 3.2.1 ([11], Corollary 6.13). *For a.e. $\omega \in \Omega$ we have that*

1. *there exists unique closed positive current T_ω^s in the cohomology class of $e(\omega)$ (see Lemma 3.1.2),*
2. *for every Kähler form κ , as $n \rightarrow \infty$, we have*

$$\frac{1}{M((f_\omega^n)^* \kappa)} (f_\omega^n)^* \kappa \rightarrow T_\omega^s.$$

We call T_ω^s the stable current associated to ω . One can do similar constructions and get an unstable version of this result.

3.3 Ahlfors-Nevanlinna currents

Here we follow [11, Section 8]. Recall that stable and unstable manifolds are injectively immersed entire curves $\psi : \mathbb{C} \rightarrow X$; we use this to construct a current like example (2) of Section 3.2 where the curve C is taken to be $\psi(B(0, t))$ where $B(0, t)$ is the ball in $\mathbb{C}$ centered at the origin of radius t . This current is denoted $\{\psi(B(0, t))\}$, i.e. the same notation as before. Let

$$A(R) = \langle \{\psi(B(0, t))\} | \kappa_0 \rangle = \int_{\psi(B(0, R))} \kappa_0, \quad (45)$$

$$T(R) = \int_0^R A(t) \frac{dt}{t}. \quad (46)$$

Now, [8] (see [11, Proposition 8.1]) tells us that there are sequences $R_n \rightarrow \infty$ such that the sequence of currents given by

$$\frac{1}{T(R_n)} \int_0^{R_n} \{\psi(B(0, t))\} \frac{dt}{t},$$

converge to a closed positive current T . We call this closed positive current T an Ahlfors-Nevanlinna current associated to the curve ψ .

Theorem 3.3.1 ([11], Theorem 8.2/Corollary 8.3). *For X a projective complex Kähler surface, μ a probability measure on $\text{Aut}(X)$, ν a hyperbolic μ -stationary measure satisfying the moment condition in Equation (8), if there is no Γ -invariant algebraic curve then for m -a.e. (x, ω) , the only normalized Ahlfors-Nevanlinna current associated to $W^s(x, \omega)$ is T_ω^s .*

Corollary 3.3.2. *In the setting of Theorem 3.3.1, for m -a.e. (x, ω) , and $\mu^\mathbb{Z}$ -a.e. ω' , $W^s(x, \omega) \neq W^s(x, \omega')$.*

Proof. Theorem 3.3.1 tells us that in our setting we have that for a.e. (x, ω) , the only normalized Ahlfors-Nevanlinna current associated to $W^s(x, \omega)$ is T_ω^s . Combining this with

Proposition 3.1.3 and Theorem 3.2.1 we get that $e(\omega)$ is distinct from $e(\omega')$ for a.e. ω' and that T_ω^s is the unique closed positive current in the cohomology class of $e(\omega)$. This tells us that T_ω^s is in its own distinct cohomology class for a.e. ω . Relating this back to the stable manifolds, we see that two stable manifolds cannot be the same if they have different T_ω^s and so the result stands. $\square$

Remark 3.3.3. *The analogue of Corollary 3.3.2 for unstable manifolds also holds.*

Lemma 3.3.4. *There exists $r \in \mathbb{N}$ and a full measure set $\Omega_t^+ \subset \Omega^+$ such that for all $k < r$ and for a.e. $x \in X$, if $\omega_1, \omega_2 \in \Omega_t^+$, $W^s(x, \omega_1)$ and $W^s(x, \omega_2)$ are tangent to order k , but they are not tangent at order r .*

Remark 3.3.5. *A result similar to Lemma 3.3.4 holds for the unstable manifolds.*

Let Λ_s be the conull subset of Y such that Corollary 3.3.2 and Lemma 3.3.4 hold.

Theorem 3.3.1, Corollary 3.3.2 and Lemma 3.3.4 apply generally in our setting, regardless of the value of d_+ in Section 2.6. We will use this theorem for the case $d_+ = 0$ to prove Lemma 2.6.7 in Section 4 and Proposition 2.6.5 in Section 6. In the case that $d_+ = 1$, we use [11, Theorem C] which is stated below in the next section as Theorem 3.3.1. This theorem give us that the Oseledets' directions are random in the setting $d_+ = 1$, hence in Lemma 3.3.4, $r = 1$. This is because in this case we have positive fiber entropy and hence land in case c) of Theorem 3.4.1. In the case $d_+ = 0$ we have zero fiber entropy and hence are in alternative a) of Theorem 3.4.1.

3.4 Fibered entropy and Theorem C of [11]

We give the following result of [11] which gives us randomness of Oseledets' directions in the case of positive fiber entropy:

Theorem 3.4.1 ([11], Theorem C, Theorem 9.1)). *Let X be a complex projective surface and μ a finitely supported probability measure on $\text{Aut}(X)$. If the group generated by $\text{Supp}(\mu)$,*

denoted Γ_μ , is non-elementary, then any hyperbolic, ergodic μ -stationary measure ν on X satisfies (exactly) one of the following:

- a) ν is invariant and its fiber entropy, $h_\nu(X, \mu)$, vanishes,
- b) ν is supported on an algebraic curve that is Γ_μ -invariant,
- c) the field of Oseledets' stable directions, i.e. the $E^s(x, \omega)$, is not Γ_μ -invariant. That is to say, it depends on the choice of future in ω .

Case c) gives us that for almost every $(x, \omega) \in \Lambda$, a.e. $\omega' \in \Omega$ is such that $E^s(x, \omega) \neq E^s(x, \omega')$. It is similar for the unstables. Hence $r = 1$ in Lemma 3.3.4 in this $d_+ = 1$ setting.

As noted previously, in the case that $d_+ = 0$, i.e. alternative 1 from Section 2.6, our fiber entropy is zero and we are in case a) of Theorem 3.4.1. Hence we only have that the stable manifolds are a.e. distinct for different $\omega \in \Omega$, see Section 3.3, Corollary 3.3.2.

Elementary lemma in $d_+ = 1$ case

Fix $x \in X$ and consider the family of measures on $Gr(1, \mathbb{C}^2)$ given by $\rho_{(x, \omega)}^s := \delta_{E^s(x, \omega)}$. By Theorem 3.4.1 c) above (hence in the $d_+ = 1$, not necessarily the $d_+ = 0$ case), for a.e. $\omega^+ \in \Omega^+$ these measures are distinct. Let $P_x^s := \int_{\Omega^+} \rho_{(x, \omega)}^s d\mu^{\mathbb{N}}$, then this measure is non-atomic. Similarly, we define $\rho_{(x, \omega)}^u := \delta_{E^u(x, \omega)}$ and let $P_x^u := \int_{\Omega^-} \rho_{(x, \omega)}^u d\mu^{\mathbb{N}}$; this measure is also non-atomic.

Lemma 3.4.2. *Consider P a non-atomic probability measure on a compact metric space M . For every sequence $\epsilon_n \rightarrow 0$ there exists a sequence $\delta_n \rightarrow 0$ such that for all $x \in M$ and $n \in \mathbb{N}$ we have*

$$P(B(x, \epsilon_n)) < \delta_n. \tag{47}$$

This elementary lemma will be useful in Sections 4.4 and 5.

3.5 Non-uniform hyperbolicity and Pesin theory

Many of the theorems used in non-uniformly hyperbolic settings like our own were originally only written for the case of single iterate maps (with restrictions on the measure ν) and are due to Pesin ([36], [37]). Many of these theorems have versions in our setting, the ‘random dynamics’ setting, where instead of a single map f acting as the dynamics, we have f_ω^n as in Equations (2) and (3). The books written by [25] and [26] are standard references for this material.

Let $\epsilon_0 \ll \min\{1, \lambda^+/200, -\lambda^-/200\}$.

Since we have assumed that we have no zero Lyapunov exponents, we have “non-uniform hyperbolicity”: upper and lower bounds on the action of $D_x f_\omega^n$ on the Oseledets’ subspaces, as well as bounds on the angles between these subspaces. In the case of uniform hyperbolicity, these bounds are point independent, but this is not the case for non-uniform hyperbolicity. We instead get a statement of the following form:

Lemma 3.5.1 ([7], Lemma 6.2, see also [3], Lemma 3.5.7 and [26], Lemma 3.2). *Given the integrability condition, (8), there is a measurable function $L : X \times \Omega \rightarrow [1, \infty[$ such that for μ -a.e. (x, ω) (i.e. those in Λ) and $n \in \mathbb{Z}$ we have*

1. For $v \in E^s(x, \omega)$,

$$L(x, \omega)^{-1} e^{n\lambda^- - |n|\frac{1}{2}\epsilon_0} \|v\| \leq \|D_x f_\omega^n v\| \leq L(x, \omega) e^{n\lambda^- + |n|\frac{1}{2}\epsilon_0} \|v\|.$$

2. For $v \in E^u(x, \omega)$,

$$L(x, \omega)^{-1} e^{n\lambda^+ - |n|\frac{1}{2}\epsilon_0} \|v\| \leq \|D_x f_\omega^n v\| \leq L(x, \omega) e^{n\lambda^+ + |n|\frac{1}{2}\epsilon_0} \|v\|.$$

3. It holds¹ that

$$\angle(E^s[f_\omega^n(x), \sigma^n(\omega)], E^u[f_\omega^n(x), \sigma^n(\omega)]) > \frac{1}{L(x, \omega)} e^{-|n|\epsilon_0}.$$

4. It holds that

$$L(f_\omega^n(x), \sigma^n(\omega)) \leq L(x, \omega) e^{\epsilon_0|n|}.$$

Lyapunov norms

Lyapunov norms are an adapted norm meant to have the action of $D_x f$ reflect the Lyapunov exponents after one step in the random walk, see [7, Section 9.2] or [26, chapter VI, Section 3]. One can also see [29] for the single iterate case.

For $(x, \omega) \in Y$, $b \in \{s, u\}$ and $v \in E^b(x, \omega)$ we define the Lyapunov norm (two-sided) as

$$|v|_{\epsilon_0, (x, \omega)}^b := \left(\sum_{n \in \mathbb{Z}} \|Df_\omega^n v\|^2 e^{-2\lambda^b n - 2\epsilon_0|n|} \right)^{1/2}.$$

These sums converge for m -a.e. $(x, \omega) \in Y$ by Lemma 3.5.1, also for $v \in E^b(x, \omega)$ we have:

$$|v|_{\epsilon_0, (x, \omega)}^b \geq \|v\|.$$

Lemma 3.5.2 (Lemma 9.2 of [7]). *For m -a.e. $(x, \omega) \in Y$, $v \in E^b(x, \omega)$, $n \in \mathbb{Z}$ and $k \geq 0$ we have*

$$e^{n\lambda^b - |n|\epsilon_0} |v|_{\epsilon_0, (x, \omega)}^b \leq |Df_\omega^n v|_{\epsilon_0, (f_\omega^n(x), \sigma^n(\omega))}^b \leq e^{n\lambda^b + |n|\epsilon_0} |v|_{\epsilon_0, (x, \omega)}^b. \quad (48)$$

From here, we pass to Lyapunov charts which rely on the construction of the Lyapunov norm. These charts takes this norm to the standard Euclidean norm and our Oseledets'

1. Here, $\angle$ denotes Riemannian angle between the subspaces.

directions to the standard orthogonal coordinate system in $\mathbb{C}^2$.

Lyapunov charts

For every $0 < \epsilon < \epsilon_0$, there is a measurable function $r : Y \rightarrow [1, \infty[$ and a full measure set $\Lambda \subset Y$ (same as the Oseledets' set) such that

- For $(x, \omega) \in \Lambda$ there is a neighbourhood $U(x, \omega) \subset X$ containing x and a biholomorphism $\varphi_{(x, \omega)} : U(x, \omega) \rightarrow B(r(x, \omega)^{-1})$ (where $B(r)$ is a ball of radius r in $\mathbb{C}^2$) with

- $\varphi_{(x, \omega)}(x) = 0$,
- $D\varphi_{(x, \omega)}(E^u(x, \omega)) = \mathbb{C}^u := \mathbb{C} \times \{0\}$,
- $D\varphi_{(x, \omega)}(E^s(x, \omega)) = \mathbb{C}^s := \{0\} \times \mathbb{C}$.

- Let $\tilde{f}_{(x, \omega)} = \varphi_{(f_\omega(x), \sigma(\omega))} \circ f_\omega^1 \circ \varphi_{(x, \omega)}^{-1}$ and $\tilde{f}_{(x, \omega)}^{-1} = \varphi_{((f_\omega^{-1}(x), \sigma^{-1}(\omega))} \circ f_\omega^{-1} \circ \varphi_{(x, \omega)}^{-1}$ then,

- $\tilde{f}_{(x, \omega)}(0) = 0$,
- $e^{\lambda^+ - \epsilon}|v| \leq |D_0 \tilde{f}_{(x, \omega)} v| \leq e^{\lambda^+ + \epsilon}|v|$, for $v \in \mathbb{C} \times \{0\}$,
- $e^{\lambda^- - \epsilon}|v| \leq |D_0 \tilde{f}_{(x, \omega)} v| \leq e^{\lambda^- + \epsilon}|v|$ for $v \in \{0\} \times \mathbb{C}$,
- $\text{Lip}(\tilde{f}_{(x, \omega)} - D_0 \tilde{f}_{(x, \omega)}) < \epsilon$,
- $\text{Lip}(D\tilde{f}_{(x, \omega)}) < r(x, \omega)$.

- Similar properties for $\tilde{f}_{(x, \omega)}^{-1}$.
- There exists a uniform constant ψ with $\psi^{-1} \leq \text{Lip}(\varphi_{(x, \omega)}) \leq r(x, \omega)$ for all $(x, \omega) \in \Lambda$.
- $r((f_\omega^n(x), \sigma^n(\omega))) \leq r(x, \omega)e^{|n|\epsilon}$ for all $n \in \mathbb{Z}$.

Sometimes we will think in real coordinates instead of complex coordinates but work in Lyapunov charts as the reader is already familiar with the notation. We denote

$$\mathbb{R}^u := \mathbb{R} \times \mathbb{R} \times \{0\} \times \{0\},$$

for $\mathbb{C}^u$ and

$$\mathbb{R}^s := \{0\} \times \{0\} \times \mathbb{R} \times \mathbb{R},$$

for $\mathbb{C}^s$.

Let us set notation for the action of f in the Lyapunov chart. Given

$$\omega = (\dots f_{-1}, f_0, f_1, \dots),$$

define

$$\tilde{f}_{(x,\omega)} = \varphi_{(f_\omega(x), \sigma(\omega))} \circ f_\omega^1 \circ \varphi_{(x,\omega)}^{-1} = \varphi_{(f_\omega(x), \sigma(\omega))} \circ f_0 \circ \varphi_{(x,\omega)}^{-1}. \quad (49)$$

We will denote $\tilde{f}_{(x,\omega)}$ as $\tilde{f}_0$ in settings where (x, ω) has been made clear. Further, let

$$\tilde{f}_n := \tilde{f}_{(f_\omega^n(x), \sigma^n(\omega))} = \varphi_{(f_\omega^{n+1}(x), \sigma^{n+1}(\omega))} \circ f_n \circ \varphi_{(f_\omega^n(x), \sigma^n(\omega))}^{-1}. \quad (50)$$

Then we can write $\tilde{f}_{(x,\omega)}^n = \tilde{f}_{n-1} \circ \dots \circ \tilde{f}_0$.

Into charts – distortion estimates

Working in Lyapunov charts gives us control over the first derivative, $D_0 \tilde{f}_{(x,\omega)}$, by giving us uniform hyperbolicity. In this section we work to control the action of the second derivative, $D_x^2 f_\omega$, obtaining a naïve bound to use in subsequent sections. Note that we only use the standard flow in this section and we work in real coordinates in our Lyapunov charts to

demonstrate that this can be done in the smooth setting as well as holomorphic.

Let $(x, \omega) \in \Lambda$, we get that there is a neighbourhood $U(x, \omega) \subset X$ of x and a C^∞ diffeomorphism $\varphi_{(x, \omega)} : U(x, \omega) \rightarrow B(r(x, \omega)^{-1})$ with the properties defined in Section 3.5. As in 3.5, will use $|\cdot|$ to denote Euclidean norm in the chart. Let $k \in [0, 1[$. We define

$$\tilde{F}_{(x, \omega), k}^t : \mathbb{R}^4 \rightarrow \mathbb{R}^4, \quad \tilde{F}_{(x, \omega), k}^t(v) = \tilde{f}_{(x, \omega)}^{\lfloor t+k \rfloor}(v), \quad (51)$$

where $\tilde{f}_{(x, \omega)}$ is as defined in Equation (49) and t is the amount we ‘flow’ by (on Z). This is the action of ‘standard’ flow on X written in Lyapunov charts.

Taylor expanding our flow in charts, we get that there is a point w on the line between 0 and v such that

$$\tilde{F}_{(x, \omega), k}^t(v) - \tilde{F}_{(x, \omega), k}^t(0) = D\tilde{F}_{(x, \omega), k}^t(0)v + \frac{1}{2}v^T D^2\tilde{F}_{(x, \omega), k}^t(0)(w)v. \quad (52)$$

Using this Equation (52) we get the upper bound,

$$|\tilde{F}_{(x, \omega), k}^t(v) - \tilde{F}_{(x, \omega), k}^t(0)| \leq |D\tilde{F}_{(x, \omega), k}^t(0)v| + \frac{1}{2} \sup_{w \in [0, v]} |D^2\tilde{F}_{(x, \omega), k}^t(0)(w)| \cdot |v|^2. \quad (53)$$

Additionally we will need a lower bound. Using Equation (52) with the reverse triangle inequality we get

$$|\tilde{F}_{(x, \omega), k}^t(v) - \tilde{F}_{(x, \omega), k}^t(0)| \geq \left| |D\tilde{F}_{(x, \omega), k}^t(0)v| - \frac{1}{2}|v^T D^2\tilde{F}_{(x, \omega), k}^t(0)(w)v| \right|. \quad (54)$$

Now, to bound the second derivative, we look at the action of a diffeomorphism on a subspace of the space of 2-jets for $\mathbb{R}^4$ based at the point 0 (it gives a linear map). In general, the space of k -jets based at a point x_0 is denoted $J_{x_0}^k(\mathbb{R}^n, \mathbb{R}^m)$ and is essentially the space of all k th Taylor polynomials of smooth functions $f : \mathbb{R}^n \rightarrow \mathbb{R}^m$ based at a point x_0 . More precisely, it is the set of all equivalence classes of smooth functions $f : \mathbb{R}^n \rightarrow \mathbb{R}^m$ such that

$f \sim g$ if f and g , as well as all their first k partials agree at the point x_0 . We will denote $j_{x_0}^k(f)$ to be the k th Taylor polynomial of f centered at x_0 . For any $y_0 \in \mathbb{R}^m$, the real vector space $J_{x_0}^k(\mathbb{R}^n, \mathbb{R}^m)$ has the subspace,

$$J_{x_0}^k(\mathbb{R}^n, \mathbb{R}^m)_0 := \{j_{x_0}^k(f) \in J_{x_0}^k(\mathbb{R}^n, \mathbb{R}^m) : f(x_0) = 0\}. \quad (55)$$

For our purposes we take $n = m = 4$, $k = 2$, and $x_0 = 0$; we denote this $J_0^2(\mathbb{R}^4)_0$. Similarly, as above, for smooth functions $f : \mathbb{R}^4 \rightarrow \mathbb{R}^4$ we write $j_0^2(f) \in J_0^2(\mathbb{R}^4)$ for the 2nd Taylor polynomial of f at 0. Note that if $f, g : \mathbb{R}^4 \rightarrow \mathbb{R}^4$ are smooth functions mapping the origin to the origin, we have the property that

$$j_0^2(f \circ g) = j_{g(0)}^2(f) \circ j_0^2(g). \quad (56)$$

Note that if $t \in \mathbb{R}^+$ is such that $\lfloor t + k \rfloor = n + 1$ and $\omega = (\dots \tilde{f}_0, \dots \tilde{f}_n, \dots)$, then $\tilde{F}_{(x, \omega), k}^t$ is composed of the diffeomorphisms $\tilde{f}_0, \dots \tilde{f}_n$. Since we are working in charts, these diffeomorphisms always take the origin to the origin.

Further, a smooth function $f : \mathbb{R}^4 \rightarrow \mathbb{R}^4$ that maps the origin to the origin gives us a map on the space $J_0^2(\mathbb{R}^4)_0$ by pre-composition:

$$J_0^2(\mathbb{R}^4)_0 \rightarrow J_0^2(\mathbb{R}^4)_0, \quad (57)$$

$$h(z) \mapsto h \circ j_0^2(f)(z) \mod z^{\otimes 3}. \quad (58)$$

We endeavor to write out this map as a matrix. First one needs to write down a basis for $J_0^2(\mathbb{R}^4)_0$; recall that the k th Taylor polynomial for a smooth function $f : \mathbb{R}^n \rightarrow \mathbb{R}$ can be written as

$$j_{x_0}^k(f) = \sum_{0 \leq |\beta| \leq k} \frac{D^\beta f(x_0)}{\beta!} (x - x_0)^\beta, \quad (59)$$

where β is a multi-index $\beta = (\beta_1, \dots, \beta_n)$ with $\beta_i \in \{0, \dots, k\}$, $|\beta| = \sum \beta_i$, $\beta! = \prod_{i=1}^n \beta_i!$, and $(x - x_0)^\beta = \prod_{i=1}^n (x_i - (x_0)_i)^{\beta_i}$. Hence we can write down a basis for $J_0^2(\mathbb{R}^4, \mathbb{R})_0$ as

$$\{x_1, x_2, x_3, x_4, x_1^2, \dots, x_4^2, x_1x_2, \dots, x_1x_4, x_2x_3, x_2x_4, x_3x_4\}.$$

We then write $f : \mathbb{R}^n \rightarrow \mathbb{R}^n$ as $f(x_1, \dots, x_n) = (f_1(x_1, \dots, x_n), \dots, f_n(x_1, \dots, x_n))$, where $f_i : \mathbb{R}^n \rightarrow \mathbb{R}$. We see that we can write $J_0^2(\mathbb{R}^4) = J_0^2(\mathbb{R}^4, \mathbb{R}) \oplus \dots \oplus J_0^2(\mathbb{R}^4, \mathbb{R})$.

We see that the matrix representing the precomposition map (with respect to $f : \mathbb{R}^4 \rightarrow \mathbb{R}^4$) in Equation (58) needs to be 60×60 because it needs to be 15×15 for the precomposition action of $f : \mathbb{R}^4 \rightarrow \mathbb{R}^4$ on each component $J_0^2(\mathbb{R}^4, \mathbb{R})$ (given the size of the basis of $J_0^2(\mathbb{R}^4, \mathbb{R})$ is 15). To demonstrate what is going on, we will perform this computation in a simpler case for the map defined by $f : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ on $J_0^2(\mathbb{R}^2)_0$ where on each $J_0^2(\mathbb{R}^2, \mathbb{R})$ component it is only a 5×5 matrix.

Let $h \in J_0^2(\mathbb{R}^2)_0$, then it is of the form $h(x, y) = (h_1(x, y), h_2(x, y))$ where

$$h_i(z) = h_i(x, y) = a_1x + a_2y + a_3x^2 + a_4y^2 + a_5xy \in J_0^2(\mathbb{R}^2, \mathbb{R}).$$

Then

$$\begin{aligned}
j_0^2(f)(h_i(z)) &= h_i(J_0^2(f)(z)) \\
&= a_1\left(\frac{\partial f_1}{\partial x}x + \frac{\partial f_1}{\partial y}y + \frac{\partial^2 f_1}{\partial x^2}x^2 + \frac{\partial^2 f_1}{\partial y^2}y^2 + \frac{\partial^2 f_1}{\partial x \partial y}xy\right) \\
&\quad + a_2\left(\frac{\partial f_2}{\partial x}x + \frac{\partial f_2}{\partial y}y + \frac{\partial^2 f_2}{\partial x^2}x^2 + \frac{\partial^2 f_2}{\partial y^2}y^2 + \frac{\partial^2 f_2}{\partial x \partial y}xy\right) \\
&\quad + a_3\left(\frac{\partial f_1}{\partial x}x + \frac{\partial f_1}{\partial y}y + \frac{\partial^2 f_1}{\partial x^2}x^2 + \frac{\partial^2 f_1}{\partial y^2}y^2 + \frac{\partial^2 f_1}{\partial x \partial y}xy\right)^2 \\
&\quad + a_4\left(\frac{\partial f_2}{\partial x}x + \frac{\partial f_2}{\partial y}y + \frac{\partial^2 f_2}{\partial x^2}x^2 + \frac{\partial^2 f_2}{\partial y^2}y^2 + \frac{\partial^2 f_2}{\partial x \partial y}xy\right)^2 \\
&\quad + a_5\left(\frac{\partial f_1}{\partial x}x + \frac{\partial f_1}{\partial y}y + \frac{\partial^2 f_1}{\partial x^2}x^2 + \frac{\partial^2 f_1}{\partial y^2}y^2 + \frac{\partial^2 f_1}{\partial x \partial y}xy\right) \\
&\quad \left(\frac{\partial f_2}{\partial x}x + \frac{\partial f_2}{\partial y}y + \frac{\partial^2 f_2}{\partial x^2}x^2 + \frac{\partial^2 f_2}{\partial y^2}y^2 + \frac{\partial^2 f_2}{\partial x \partial y}xy\right).
\end{aligned}$$

In the above, the variable x in $h_i(x, y)$ gets replaced with the Taylor polynomial of f_1 and the variable y in $h_i(x, y)$ gets replaced by the Taylor polynomial of f_2 where

$$f(z) = f(x, y) = (f_1(x, y), f_2(x, y)).$$

Rearranging we get

$$\begin{aligned}
j_0^2(f)(h(z)) &= x(a_1\frac{\partial f_1}{\partial x} + a_2\frac{\partial f_2}{\partial x}) + y(a_1\frac{\partial f_1}{\partial y} + a_2\frac{\partial f_2}{\partial y}) \\
&\quad + x^2(a_1\frac{\partial^2 f_1}{\partial x^2} + a_2\frac{\partial^2 f_2}{\partial x^2} + a_3\left(\frac{\partial f_1}{\partial x}\right)^2 + a_4\left(\frac{\partial f_2}{\partial x}\right)^2 + a_5\left(\frac{\partial f_1}{\partial x} \cdot \frac{\partial f_2}{\partial x}\right)) \\
&\quad + y^2(a_1\frac{\partial^2 f_1}{\partial y^2} + a_2\frac{\partial^2 f_2}{\partial y^2} + a_3\left(\frac{\partial f_1}{\partial y}\right)^2 + a_4\left(\frac{\partial f_2}{\partial y}\right)^2 + a_5\left(\frac{\partial f_1}{\partial y} \cdot \frac{\partial f_2}{\partial y}\right)) \\
&\quad + xy\left(a_1\frac{\partial^2 f_1}{\partial x \partial y} + a_2\frac{\partial^2 f_2}{\partial x \partial y} + a_3\frac{\partial f_1}{\partial x}\frac{\partial f_1}{\partial y} + a_4\frac{\partial f_2}{\partial x}\frac{\partial f_2}{\partial y} + a_5\frac{\partial f_1}{\partial y}\frac{\partial f_2}{\partial x}\right).
\end{aligned}$$

So then the associated matrix (fixing the basis $\{x, y, x^2, y^2, xy\}$ in this order) is

$$\tilde{T}_f = \begin{pmatrix} \frac{\partial f_1}{\partial x} & \frac{\partial f_2}{\partial x} & 0 & 0 & 0 \\ \frac{\partial f_1}{\partial y} & \frac{\partial f_2}{\partial y} & 0 & 0 & 0 \\ \frac{\partial^2 f_1}{\partial x^2} & \frac{\partial^2 f_2}{\partial x^2} & \left(\frac{\partial f_1}{\partial x}\right)^2 & \left(\frac{\partial f_2}{\partial x}\right)^2 & \frac{\partial f_1}{\partial x} \cdot \frac{\partial f_2}{\partial x} \\ \frac{\partial^2 f_1}{\partial y^2} & \frac{\partial^2 f_2}{\partial y^2} & \left(\frac{\partial f_1}{\partial y}\right)^2 & \left(\frac{\partial f_2}{\partial y}\right)^2 & \frac{\partial f_1}{\partial y} \cdot \frac{\partial f_2}{\partial y} \\ \frac{\partial^2 f_1}{\partial x \partial y} & \frac{\partial^2 f_2}{\partial x \partial y} & \frac{\partial f_1}{\partial x} \frac{\partial f_1}{\partial y} & \frac{\partial f_2}{\partial x} \frac{\partial f_2}{\partial y} & \left(\frac{\partial f_1}{\partial x} \frac{\partial f_2}{\partial y} + \frac{\partial f_2}{\partial x} \frac{\partial f_1}{\partial y}\right) \end{pmatrix} \quad (60)$$

In the top left block (2x2) we get the Jacobian of f , in the bottom left block (3x2) we have second derivatives. In the right most bottom block (3x3) we get products of the first partials. Then overall, for the action of f on $J_0^2(\mathbb{R}^2)_0$ we get a 10×10 (2 copies of this matrix on the diagonal). We would call this T_f . A similar construction occurs for our setting with maps $f : \mathbb{R}^4 \rightarrow \mathbb{R}^4$ acting on $J_0^2(\mathbb{R}^4)_0$, again, call this matrix T_f .

Let us return to our setting now.

Lemma 3.5.3 ([18]). *Fix $(x, \omega, k) \in Z$ and let $t > 0$, consider $\tilde{F}_{(x, \omega), k}^t$, there exists C such that $|D^2 \tilde{F}_{(x, \omega), k}^t| \leq C_{t, (x, \omega, k)}^n$ where $n = \lfloor t + k \rfloor$. In particular, if $\text{Supp}(\mu) < \infty$, then there exists C such that for any $(x, \omega, k) \in Z$ and $t > 0$ we have that $|D^2 \tilde{F}_{(x, \omega), k}^t| \leq C^n$ where $n = \lfloor t + k \rfloor$.*

Proof. Fix $(x, \omega, k) \in Z$. Since X is compact, every first and second partial derivative of $f \in \text{Supp}(\mu)$ has a naïve bound on it, call the maximum of these C'_f . Entries in our matrix T_f then are bounded above by $C_f := 2(C'_f)^2$ as they may be products and/or sums of the first or second derivatives. Now if we take $t > 0$ and flow by it under ω , we have diffeomorphisms, $\tilde{f}_0, \dots, \tilde{f}_n : \mathbb{R}^4 \rightarrow \mathbb{R}^4$ where $n = \lfloor t + k \rfloor$. The map $T_{\tilde{f}_n \circ \dots \circ \tilde{f}_0}$ associated to the action of $\tilde{f}_n \circ \dots \circ \tilde{f}_0$ on $J_0^2(\mathbb{R}^4)_0$ splits into matrices $T_{\tilde{f}_n} \cdots T_{\tilde{f}_0}$. Each $T_{\tilde{f}_i}$ and $T_{\tilde{f}_n \circ \dots \circ \tilde{f}_0}$ is in a form similar to that described above in Equation (60), but as explained, ours is a 60×60 matrix. In particular, the first and second partials of $\tilde{f}_i$, resp. $\tilde{f}_n \circ \dots \circ \tilde{f}_0$ in these matrices are arranged in an analogous way to Equation (60). Since $T_{\tilde{f}_n} \cdots T_{\tilde{f}_0} = T_{\tilde{f}_n \circ \dots \circ \tilde{f}_0}$, the

first and second partials of $T_{\tilde{f}_n \circ \dots \circ \tilde{f}_0}$ are products and sums of the first and second partials of the $\tilde{f}_i$.

Given that each entry in $T_{\tilde{f}_i}$ is bounded above by $C_{\tilde{f}_i}$ and that the size of the matrix is 60×60 , multiplying any two $T_{\tilde{f}_i}$ and $T_{\tilde{f}_j}$ gives a bound on the entries of the product given by $60C_{\tilde{f}_i} \cdot C_{\tilde{f}_j}$. So for a given set of diffeomorphisms $\tilde{f}_1, \dots, \tilde{f}_n$ we let

$$C_{t,(x,\omega,k)} := \sqrt{60} \max_{0 \leq i \leq n} C_{\tilde{f}_i}. \quad (61)$$

We conclude that all second derivatives of $\tilde{f}_n \circ \dots \circ \tilde{f}_0$ are bounded above by $C_{t,(x,\omega,k)}^n$, and so $|D^2 \tilde{F}_{(x,\omega),k}^t| \leq C_{t,(x,\omega,k)}^n$. This $C_{t,(x,\omega,k)}$ can be taken to be independent of $\omega \in \Omega$ because $\text{Supp}(\mu) < \infty$ and X is compact. Hence this naïve bound is only dependent on the number of diffeomorphisms we draw. $\square$

Stable and unstable manifolds

Here we state a version of the local stable manifold theorem from [7] that turns a local stable (or unstable) manifold into a holomorphic graph in the Lyapunov charts. Let $\mathbb{C}^*(r)$ be a ball of radius r centered at the origin in $\mathbb{C}^*$ where $*$ $\in \{s, u\}$.

Proposition 3.5.4 ([7], Theorem 6.4). *For $(x, \omega) \in \Lambda$ there exists holomorphic functions*

$$h_{(x,\omega)}^s : \mathbb{C}^s(r(x, \omega)^{-1}) \rightarrow \mathbb{C}^u(r(x, \omega)^{-1}),$$

$$h_{(x,\omega)}^u : \mathbb{C}^u(r(x, \omega)^{-1}) \rightarrow \mathbb{C}^s(r(x, \omega)^{-1}),$$

(where $r(x, \omega)$ is from Section 3.5) such that, for any $*$ $\in \{s, u\}$,

1. $h_{(x,\omega)}^*(0) = 0$,

2. $D_0 h_{(x,\omega)}^* = 0$,

$$3. \|Dh_{(x,\omega)}^*\| \leq 1/3.$$

Define

$$W_{q(x,\omega)}^s(x,\omega) = \varphi_{(x,\omega)}^{-1}(\text{graph}(h_{(x,\omega)}^s)), \quad W_{q(x,\omega)}^u(x,\omega) = \varphi_{(x,\omega)}^{-1}(\text{graph}(h_{(x,\omega)}^u)), \quad (62)$$

where $q(x,\omega)$ is a measurable function dependent on $r(x,\omega)$. These are our local stable and unstable manifolds respectively. They have the following properties:

1. If $y \in W_{q(x,\omega)}^s(x,\omega)$ then for every $n \in \mathbb{N}$ we have

$$d_X(F^n(x,\omega), F^n(y,\omega)) \leq k(x,\omega) \exp((\lambda^- + \delta)n) d_X(x, y).$$

2. If $y \in W_{q(x,\omega)}^u(x,\omega)$ then for every $n \in \mathbb{N}$ we have

$$d_X(F^{-n}(x,\omega), F^{-n}(y,\omega)) \leq k(x,\omega) \exp(-(\lambda^+ - \delta)n) d_X(x, y).$$

3. We have the following inclusions:

$$\begin{aligned} F(W_{q(x,\omega)}^s(x,\omega)) &\subset W_{q(F(x,\omega))}^s(F(x,\omega)), \\ F^{-1}(W_{q(F(x,\omega))}^u(F(x,\omega))) &\subset W_{q(x,\omega)}^u(x,\omega). \end{aligned}$$

Where $k(x,\omega)$ is a measurable function dependent on $r(x,\omega)$.

For the cost of some $\epsilon_{loc} > 0$ small, one can use Lusin's theorem to replace k and q with uniform constants in the Proposition 3.5.4 on some compact set K_{loc} of measure $m(K_{loc}) > 1 - \epsilon_{loc}$.

The global stable and unstable manifolds are defined by

$$W^s(x, \omega) = \bigcup_{n \geq 0} F^{-n} \left(W_{q(x, \omega)}^s(F^n(x, \omega)) \right),$$

$$W^u(x, \omega) = \bigcup_{n \geq 0} F^n \left(W_{q(x, \omega)}^u(F^{-n}(x, \omega)) \right).$$

As explained in Section 2, $W^{s/u}(x, \omega)$ are injectively immersed holomorphic curves in X_ω that are biholomorphic to $\mathbb{C}$ for m -a.e. (x, ω) (see [11, Proposition 7.8]) and are equivalent to what is given in Equations (12) and (13) resp.

Let K_{hol} be the conull set of $(x, \omega) \in Y$ for which our global stables and unstables are biholomorphic to $\mathbb{C}$. Let $\Lambda_{loc} := K_{loc} \cap K_{hol}$ and note that it is almost conull.

3.6 Hölder continuity of the Oseledets' subspaces

For this section we largely follow the conventions and results of [26, Chapter 3, Section 4].

Given a metric space X and a family of subspaces $\{E_x\}_{x \in X}$ in a Hilbert space H , this family is called “ α -Hölder continuous in x ” for some $\alpha > 0$ if there is some $L > 0$ such that for any $x, y \in X$ we have

$$d_{Gr}(E_x, E_y) \leq L d_X(x, y)^\alpha, \quad (63)$$

where

$$d_{Gr}(V, W) = \max \left\{ \sup_{v \in V, |v|=1} \inf_{w \in W} |v - w|, \sup_{w \in W, |w|=1} \inf_{v \in V} |v - w| \right\}, \quad (64)$$

and $|v|$ is the Euclidean norm.

Now we pass to considering a family $\{E_x\}_{x \in X}$ where $E_x \subset T_x X_\omega$. In order to make sense of comparing subspaces in different tangent spaces, we use parallel transport. Let ρ_0

be such that $\exp_z$ is a C^∞ diffeomorphism on $B(z, \rho_0)$ for all $z \in X$ and let $x, y \in X$ be such that $d_X(x, y) < \rho_0$. Denote $P(x, y)$ as the isometry from $T_x X$ to $T_y X$ that is parallel transport along the geodesic between x and y , then we can define

$$d(E_x, E_y) := \begin{cases} 1 & \text{if } d_X(x, y) \geq \rho_0/4, \\ d_{Gr}(E_x, P(y, x)E_y) & \text{if } d_X(x, y) < \rho_0/4. \end{cases} \quad (65)$$

Definition 4. Given $\Delta \subset X$, a family of subspaces $\{E_x\}_{x \in \Delta}$ such that $E_x \subset T_x X$ is α -Hölder continuous on Δ if there is $L > 0$ such that for any $x, y \in \Delta$ we have

$$d(E_x, E_y) \leq L(d_X(x, y)^\alpha).$$

Next we define a form of “Pesin set”; a set on which we have uniform hyperbolicity rather than non-uniform hyperbolicity. It is on a set of this form where we will be able to achieve Hölder continuity for our Oseledets’ directions. Note that the sets in the definition may be empty depending on the choice of parameters.

Definition 5. Let $\{f_i\}_{i=0}^\infty \in \text{Supp}(\mu)^\mathbb{N}$ and let $f^n = f_{n-1} \circ \cdots \circ f_0$. Let $a < b$, $C \geq 1$ and define $\Delta_{a,b,C} \subset X$ to be the subset of points where there are splittings

$$T_x X = E_x \oplus E_x^\perp,$$

such that for $n \in \mathbb{N}$ we have that

$$\begin{aligned} \|D_x f^n v\| &\leq C e^{an} \|v\|, \quad v \in E_x, \\ \|D_x f^n v\| &\geq C^{-1} e^{bn} \|v\|, \quad v \in E_x^\perp. \end{aligned}$$

Since our sequence $\{f_i\}_{i=0}^\infty$ is given by either the non-negative indexed half of the vector

ω , denoted ω^+ , or the negative indexed half, denoted ω^- , in application we will denote $\Delta_{a,b,C}^{\omega^+}$ for where the $\{f_i\}_{i=0}^\infty$ set comes from ω^+ , and similarly as $\Delta_{a,b,C}^{\omega^-}$ for where this sequence of functions is from ω^- .

Lemma 3.6.1 ([26], Chapter 3, Section 4, Lemma 4.2). *There exists $\Omega_0 \subset \Omega$ such that $\sigma(\Omega_0) \subset \Omega_0$ and $\mu^\mathbb{Z}(\Omega_0) = 1$ and there exists measurable function $C : \Omega_0 \rightarrow]0, \infty[$ such that for $\omega \in \Omega_0$ and $n \in \mathbb{N}$ we have*

$$\prod_{i=0}^{n-1} \|f_i(\omega)\|_{C^{1,1*}} \leq C(\omega) e^{2c_0 n}, \quad (66)$$

where $f_i(\omega)$ is the function in the i th position in the vector ω and

$$c_0 = \int \log \|f\|_{C^{1,1*}} d\mu(f) < \infty. \quad (67)$$

The norm $C^{1,1*}$ is equivalent to the usual $C^{1,1}$ norm, $C^{1,1*}$ is given by

$$\|f\|_{C^{1,1*}} = \|Df\|_\infty + \text{Lip}(Df). \quad (68)$$

Equation (67) is finite by the moment condition, see Remark 2.2.1.

Remark 3.6.2. *In the same vain, a similar statement can be written for our backwards random walk where we get a σ^{-1} -invariant set $\Omega_0^- \subset \Omega$, $\mu^\mathbb{Z}(\Omega_0^-) = 1$ and measurable function $C^- : \Omega_0^- \rightarrow]0, \infty[$ such that for $\omega \in \Omega_0^-$ and $n \in \mathbb{N}$ we have*

$$\prod_{i=1}^{n-1} \|f_i^-(\omega)\|_{C^{1,1*}} \leq C^-(\omega) e^{2c_0^- n}, \quad (69)$$

where $f_i^-(\omega)$ is the inverse to the function in the $-i$ th position in the vector ω and

$$c_0^- = \int \log \|f^{-1}\|_{C^{1,1*}} d\mu(f) < \infty. \quad (70)$$

Lemma 3.6.3. *There exists an almost conull set $\Omega_\epsilon \subset \Omega$ on which $C(\omega)$ and c_0 from Lemma 3.6.1 and $C^-(\omega)$ and c_0^- from Remark 3.6.2 can be chosen uniformly.*

Proof. Choosing c_0 and c_0^- uniformly is trivial because of our moment condition (see Equation (8)) we can take both constants to be the value of the integral in Equation (8).

Now, since $C(\omega)$ is a measurable function, the set $E_n = \{\omega \in \Omega_0 : C(\omega) \leq n\}$ is a measurable set such that $E_n \rightarrow \Omega'$ as $n \rightarrow \infty$. Hence $\mu^{\mathbb{Z}}(E_n) \rightarrow \mu^{\mathbb{Z}}(\Omega') = 1$. Let $\epsilon_C > 0$, we can find n such that $\mu^{\mathbb{Z}}(E_n) > 1 - \epsilon_C$, call this set $\Omega_{\epsilon_C} \subset \Omega$ and take $C = \max\{1, n\}$. On this set Ω_{ϵ_C} we have that

$$\prod_{i=0}^{n-1} \|f_i(\omega)\|_{C^{1,1}} \leq C e^{2c_0 n}. \quad (71)$$

One can do similarly for the set Ω_0^- defined in Remark 3.6.2, one gets a set $\Omega_{\epsilon_{C^-}}$ where

$$\prod_{i=1}^{n-1} \|f_i^-(\omega)\|_{C^{1,1}_*} \leq C^- e^{2c_0^- n}. \quad (72)$$

Let $\Omega_\epsilon := \Omega_{\epsilon_C} \cap \Omega_{\epsilon_{C^-}}$. This completes the proof. $\square$

Proposition 3.6.4 ([26], Chapter 3, Section 4, Corollary 4.1). *Let $\{f_i\}_{i=0}^\infty$ be a sequence of diffeomorphisms $f_i : X \rightarrow X$ such that*

$$\prod_{i=0}^{n-1} \|f_i\|_{C^{1,1}_*} \leq C e^{cn}, \quad (73)$$

for some $C \geq 1$, $c > 0$. Fix $\hat{C} \geq 1$, and $a < b$ and consider the set $\Delta_{a,b,\hat{C}} \subset X$ associated to this sequence as in Definition 5. The family $\{E_x\}_{x \in \Delta_{a,b,\hat{C}}}$ is α -Hölder continuous in x on $\Delta_{a,b,\hat{C}}$ with constant $2\hat{C}^2 e^{b-a}$ and $\alpha = \frac{a-b}{a-d}$ where

$$d = \ln(2C^2) + 2c + |\ln(\rho_0/4)| + |a|.$$

Fix $\epsilon > 0$ and $N \in \mathbb{N}$ and define $\Lambda_{\epsilon, N}^{s, \omega}$ to be the set of all $x \in X$ such that for any $n \geq N$, $v \in E^s(x, \omega)$ we have

$$e^{n(\lambda^- - \epsilon)} \|v\| \leq \|D_x f_\omega^n v\| \leq e^{n(\lambda^- + \epsilon)} \|v\|. \quad (74)$$

and for $v \in E^u(x, \omega)$,

$$e^{n(\lambda^+ - \epsilon)} \|v\| \leq \|D_x f_\omega^n v\| \leq e^{n(\lambda^+ + \epsilon)} \|v\|. \quad (75)$$

Similarly, define $\Lambda_{\epsilon, N}^{u, \omega}$ to be the set of $x \in X$ such that for all $n \geq N$ we have that for $v \in E^s(x, \omega)$,

$$e^{-n(\lambda^- + \epsilon)} \|v\| \leq \|D_x f_\omega^{-n} v\| \leq e^{-n(\lambda^- - \epsilon)} \|v\|, \quad (76)$$

and for $v \in E^u(x, \omega)$,

$$e^{-n(\lambda^+ + \epsilon)} \|v\| \leq \|D_x f_\omega^{-n} v\| \leq e^{-n(\lambda^+ - \epsilon)} \|v\|. \quad (77)$$

Note that if $N_1 < N_2$, then $\Lambda_{\epsilon, N_1}^{s, \omega} \subset \Lambda_{\epsilon, N_2}^{s, \omega}$ and $\Lambda_{\epsilon, N_1}^{u, \omega} \subset \Lambda_{\epsilon, N_2}^{u, \omega}$.

Lemma 3.6.5. *For any $(x, \omega) \in \Lambda$ and any $\epsilon > 0$ there exists $N_s, N_u \in \mathbb{N}$ such that x belongs to $\Lambda_{\epsilon, N_s}^{s, \omega}$ and $\Lambda_{\epsilon, N_u}^{u, \omega}$.*

Proof. This follows immediately from Oseledets' theorem. $\square$

Given a fixed $\epsilon > 0$, we get a Lusin set K_ϵ such that if $(x, \omega) \in K_\epsilon$ then N_u and N_s are bounded above by $\tilde{N}$.

Note that given two points $(x, \omega), (y, \omega) \in \Lambda$ and $\epsilon > 0$, each has a corresponding N_s , we call them N_s^x and N_s^y resp. then $x, y \in \Lambda_{\epsilon, N}^{s, \omega}$ for $N = \max\{N_s^x, N_s^y\}$. Similarly, if N_u^x and N_u^y were their corresponding N_u values, then also $x, y \in \Lambda_{\epsilon, N}^{u, \omega}$.

Lemma 3.6.6. *For given N , there exists $L_u, L_s \in \mathbb{R}$ (independent of ω) such that*

$$\Lambda_{\epsilon, N}^{s, \omega} \subset \Delta_{(\lambda^- + \epsilon), (\lambda^+ - \epsilon), L_s}^{\omega^+}, \quad \Lambda_{\epsilon, N}^{u, \omega} \subset \Delta_{-(\lambda^- - \epsilon), -(\lambda^+ + \epsilon), L_u}^{\omega^-}.$$

Proof. Let $L_s = (\max_{i=0, \dots, N} \{1, \|Df_i\|\})^N$, (to make independent of ω , simply let $L'_s = \max_{\text{Supp}(\mu)} \{1, \|Df_i\|\}$ and take $L_s = (L'_s)^N$, we can do this since we assume $\text{Supp}(\mu) < \infty$ and X is compact) then, for $y \in \Lambda_{\epsilon, N}^{s, \omega}$, all $n \in \mathbb{N}$ and $v \in E^s(y, \omega)$ we have

$$\frac{1}{L_s} e^{n(\lambda^- - \epsilon)} \|v\| \leq \|D_x f_\omega^n v\| \leq L_s e^{n(\lambda^- + \epsilon)} \|v\|, \quad (78)$$

and for $v \in E^u(y, \omega)$,

$$\frac{1}{L_s} e^{n(\lambda^+ - \epsilon)} \|v\| \leq \|D_x f_\omega^n v\| \leq L_s e^{n(\lambda^+ + \epsilon)} \|v\|. \quad (79)$$

Hence $\Lambda_{\epsilon, N}^{s, \omega} \subset \Delta_{(\lambda^- + \epsilon), (\lambda^+ - \epsilon), L_s}^{\omega^+}$.

Similarly, let $L_u = (\max_{i=0, -1, \dots, -N} \{1, \|Df_i\|\})^N$, (we can make this independent of ω in the same way one does for L_s) then for $y \in \Lambda_{\epsilon, N}^{u, \omega}$ we have that for $v \in E^s(x, \omega)$,

$$\frac{1}{L_u} e^{-n(\lambda^- + \epsilon)} \|v\| \leq \|D_x f_\omega^{-n} v\| \leq L_u e^{-n(\lambda^- - \epsilon)} \|v\|, \quad (80)$$

and for $v \in E^u(x, \omega)$ we have

$$\frac{1}{L_u} e^{-n(\lambda^+ + \epsilon)} \|v\| \leq \|D_x f_\omega^{-n} v\| \leq L_u e^{-n(\lambda^+ - \epsilon)} \|v\|. \quad (81)$$

So we see that $\Lambda_{\epsilon, N}^{u, \omega} \subset \Delta_{-(\lambda^- - \epsilon), -(\lambda^+ + \epsilon), L_u}^{\omega^-}$ for this ω (or really ω^-). $\square$

If further we have $(x, \omega) \in \tilde{\Lambda} := \Lambda \cap (X \times \Omega_\epsilon)$, then we can use Proposition 3.6.4 to say that if we pick $\epsilon > 0$ and get the corresponding N_u and N_s , then on the subsets $\Lambda_{\epsilon, N_s}^{s, \omega} \subset \Delta_{(\lambda^- + \epsilon), (\lambda^+ - \epsilon), L_s}$ and $\Lambda_{\epsilon, N_u}^{u, \omega} \subset \Delta_{-(\lambda^- - \epsilon), -(\lambda^+ + \epsilon), L_u}$ we get the associated Hölder

continuity of E^s and E^u respectively.

3.7 Stable holonomies

Here we discuss stable holonomies and standard measurable connections. We follow the appendices of [6, Appendix B, D, E].

Definition 6. ([6, Appendix B.2]) *We say a smooth cocycle $\mathcal{G} : E \rightarrow E$ with vector bundle $E \rightarrow Y$ over F has smooth stable holonomies if for a.e. $(x, \omega) \in Y$ and for all $y \in W^s(x, \omega)$ we have maps*

$$\mathcal{P}_\omega^-(x, y) : E(x, \omega) \rightarrow E(y, \omega),$$

that vary smoothly in y and such that if $y' \in W^s(x, \omega)$, then

$$\mathcal{P}_\omega^-(x, y') = \mathcal{P}_\omega^-(y, y') \circ \mathcal{P}_\omega^-(x, y),$$

(where defined) and such that they are F -equivariant:

$$\mathcal{P}_{\sigma(\omega)}^-(F(x, \omega), F(y, \omega)) \circ \mathcal{G}_x = \mathcal{G}_y \circ \mathcal{P}_\omega^-(x, y).$$

From [6, Appendix B, D, E] we get (using [1, 39]) the existence of holonomies for our cocycles E^s and $E^s \oplus E^u / E^s$, which we denote $\mathcal{P}_s^-$ and $\mathcal{P}_u^-$ respectively. Combining these, we have that the stable associated graded $\text{gr}^- := (E^s \oplus E^u / E^s) \oplus E^s$ also has stable holonomies $\mathcal{P}^-$.

Now note that the $(E^s \oplus E^u / E^s)$ are (measurably) isomorphic to E^u and so we get isomorphisms $E^u \oplus E^s \rightarrow (E^s \oplus E^u / E^s) \oplus E^s$. Combining these isomorphisms with the holonomies $\mathcal{P}^-$, we get the standard measurable connection which we call

$$P^- : E^u \oplus E^s \rightarrow E^u \oplus E^s.$$

Here $P^- = P_u^- \oplus P_s^-$, where

$$P_u^- : E^u \rightarrow E^u, \quad P_s^- : E^s \rightarrow E^s,$$

come from $\mathcal{P}_u^-$ and $\mathcal{P}_s^-$ respectively combined with their respective canonical isomorphisms. By the Ledrappier invariance principle (see [2, 6, 17, 28]) these preserve measurable F -invariant subbundles like U^+ . So we have maps

$$P_\omega^-(z, z') : TU^+[z, \omega] \rightarrow TU^+[z', \omega],$$

(where defined) with properties

$$P_\omega^-(z, z'') = P_\omega^-(z'', z') \circ P_\omega^-(z, z'),$$

and $P_\omega^-(z, z) = Id$. Appendix E.2 of [6] explains how for any $\epsilon_\beta > 0$ we can find a set K_β (that we call a Pesin-Lusin set) of measure $1 - \epsilon_\beta$ on which $P^\pm$ are Hölder continuous with some exponent β and constant C_β dependent on ϵ_β . To be more precise, on K_β we have

$$\|P_\omega^-(z, z') - Id\| \leq C_\beta d_X(z, z')^\beta.$$

One can have the above construction for the unstables too. We denote these counterparts as $\mathcal{P}^+$ and P^+ .

Lemma 3.7.1. *The tangent space of the unstable support of a point (x, ω) , $TU^+[x, \omega]$, does not depend on ω^+ .*

Proof. Consider $\hat{x} = (x, \omega^-, \omega^+)$, $\hat{x}' = (x, \omega^-, (\omega')^+) \in Y$. We note that $W^u(\hat{x}) = W^u(\hat{x}')$ and consider $P^+(\hat{x}, \hat{x}')(TU^+[\hat{x}]) = TU^+[\hat{x}']$. Now we note that the backwards flag E^u and $E^s \oplus E^u/E^u$ are only dependent on the past which $\hat{x}$ and $\hat{x}'$ share. This forces $P^+(\hat{x}, \hat{x}')$ to

be the identity. □

3.8 Normal form coordinates

Normal form coordinates (NFCs) are charts on either the stable or unstable manifold, $W^s(x, \omega)$ or $W^u(x, \omega)$, that (in our case) conjugate the action of the dynamics to a linear action. This construction was originally due to [24]. Our case is simple because our stable and unstable manifolds are complex 1-dimensional. One can also find this in [23, Lemma 3.2 and Lemma 3.3], [7, Proposition 6.5] and [6].

Proposition 3.8.1 ([7, Proposition 6.5]). *For m -a.e. $(x, \omega) \in Y$ and for any $y \in W^u(x, \omega)$ (similar for the stable case), there exists a smooth diffeomorphism*

$$H_{(y, \omega)}^u : W^u(x, \omega) \rightarrow T_y W^u(x, \omega) \cong \mathbb{C}, \quad (82)$$

that satisfies

1. $D_y f_\omega \circ H_{(y, \omega)}^u = H_{F(y, \omega)}^u \circ f_\omega|_{W^u(x, \omega)},$
2. $H_{(y, \omega)}^u(y) = 0, D_y H_{(y, \omega)}^u = Id,$
3. for $z \in W^u(x, \omega)$ the change of coordinates

$$H_{(y, \omega)}^u \circ \left(H_{(z, \omega)}^u \right)^{-1} : T_z W^u(x, \omega) \rightarrow T_y W^u(x, \omega), \quad (83)$$

is an affine map.

We denote Λ' to be the subset of Y where we can use NFCs as in Proposition 3.8.1.

3.9 Time change suspension flow

For the ‘time changed’ flow we will construct a roof function that will allow us to grow the unstable manifold by exact amounts in normal form coordinates (after some slight modification to these coordinates, see Equation (97) below). More precisely, if we are in (modified) normal form coordinates (NFCs) for the unstable manifold we want flowing by time ℓ to expand the unstable manifold by e^ℓ . We will also get the analogue for the stable manifolds, i.e. that flowing forwards by ℓ in (modified) NFCs under the ‘time changed’ map will shrink distances by a factor of $e^{-\ell}$.

Defining the time changed flow (forwards)

As we see in Section 3.8, the way normal form coordinates works is that the action of f on the unstable manifold in these coordinates is entirely linear. Hence, this action is multiplication by $D_x f \omega|_{E^u(x, \omega)}$ (or $D_x f \omega|_{E^s(x, \omega)}$ if stable). Let $\theta : Y \rightarrow \mathbb{R}$ be defined by

$$\theta(x, \omega) := \log |D_x f \omega|_{E^u(x, \omega)}|. \quad (84)$$

We would use θ as our roof function for our flow, but it is not guaranteed to be positive. It is only positive on average; by Furstenberg’s formula

$$\int_{X \times \Omega} \theta(x, \omega) d\nu d\mu^{\mathbb{Z}} = \lambda^+ > 0. \quad (85)$$

We instead take a function cohomologous to θ , a positive integrable function that which we call τ ; recall that to be cohomologous means that there is a positive and a.s. finite function ϕ such that

$$\theta - \phi \circ F + \phi = \tau. \quad (86)$$

Since Equation (85) is also a finite quantity we could use Lemma 2.1 from [4] get such a τ . However, we prefer an explicit formula and so instead we give the following Lemma:

Lemma 3.9.1. *Let*

$$\tau(x, \omega) = \log |D_0 \tilde{f}_{(x, \omega)}|_{\mathbb{C}^u}|, \quad (87)$$

where $\tilde{f}$ is as in Section 3.5, then τ is positive and cohomologous to θ .

Proof. Note that we can write $\theta(x, \omega) = \log \left(\frac{\|D_x f v\|_{f(x)}}{\|v\|_x} \right)$ for any $v \in E^u(x, \omega)$. We want ϕ such that $\phi \circ F - \phi = \theta - \tau$, note that

$$\theta - \tau = \log \left(\frac{\|D_x f v\|_{f(x)}}{|D_0 \tilde{f}_{(x, \omega)}|_{\mathbb{C}^u}|} \right) - \log (\|v\|_x). \quad (88)$$

Take $w = (1, 0) \in \mathbb{C}^u$ and $v = D_0 \varphi_{(x, \omega)}^{-1}(w) \in E^u(x, \omega)$.

Then let

$$\phi(x, \omega) = \log(\|v\|_x),$$

then

$$\phi \circ F(x, \omega) = \log(\|D_0 \varphi_{F(x, \omega)}^{-1}(w)\|_{f(x)}). \quad (89)$$

Plugging in $\varphi_{F(x,\omega)}^{-1} = f_\omega^1 \circ \varphi_{(x,\omega)}^{-1} \circ \tilde{f}_{(x,\omega)}^{-1}$, we get

$$\phi \circ F(x, \omega) = \log \left(\|D_{\varphi_{(x,\omega)}^{-1} \circ \tilde{f}_{(x,\omega)}^{-1}(0)} f_\omega^1 \circ D_{\tilde{f}_{(x,\omega)}^{-1}(0)} \varphi_{(x,\omega)}^{-1} \circ D_0 \tilde{f}_{(x,\omega)}^{-1} w\|_{f(x)} \right), \quad (90)$$

$$= \log \left(\|D_x f_\omega^1 \circ D_0 \varphi_{(x,\omega)}^{-1} \circ D_0 \tilde{f}_{(x,\omega)}^{-1} w\|_{f(x)} \right), \quad (91)$$

$$= \log \left(\frac{\|D_x f_\omega^1 \circ D_0 \varphi_{(x,\omega)}^{-1} w\|_{f(x)}}{|D_0 \tilde{f}_{(x,\omega)}| \mathbb{C}^u|} \right), \quad (92)$$

$$= \log \left(\frac{\|D_x f v\|_{f(x)}}{|D_0 \tilde{f}_{(x,\omega)}| \mathbb{C}^u|} \right). \quad (93)$$

Then it is clear that for our chosen ϕ that we have $\phi \circ F - \phi = \theta - \tau$. This completes the proof. $\square$

Additionally, by the uniform hyperbolicity estimates in Lyapunov charts we have that

$$\lambda^+ - \epsilon \leq \tau \leq \lambda^+ + \epsilon. \quad (94)$$

Throughout the rest of this subsection, we are flowing forwards by $\ell > 0$. We define for $p \in \mathbb{N}$, $p \geq 1$,

$$\tau_p(x, \omega) := \tau(f_\omega^{p-1} x, \sigma^{p-1} \omega) + \dots + \tau(f_\omega^2 x, \sigma^2 \omega) + \tau(f_\omega^1 x, \sigma(\omega)) + \tau(x, \omega), \quad (95)$$

and additionally we define $\tau_0(x, \omega) = 0$. Then the amount of time we flow has to surpass $\tau_p(x, \omega)$ in order to apply p diffeomorphisms, $f_0, f_1, \dots, f_{p-1}$ to X at x .

Now we can define the ‘time changed’ flow on Z which we will denote F_{tc}^ℓ . Note that [4] simply defines the space over which they flow using the roof function, however as we have two different flows, we take the approach of [17] and simply change the speed of the flow rather than the $[0, 1[$ component of Z .

For $\ell \in \mathbb{R}_{\geq 0}$, we let $c = (x, \omega, k) \in Z$ and work to define $F_{tc}^\ell(c)$. Note that $k \in [0, 1[$,

but now we first want to move at speed $\tau(x, \omega)$ to start. If for example $k = 0.22$ and $\tau(x, \omega) = 1.2$, then if $\ell \geq 1.2 - (0.22\tau(x, \omega)) = 1.2 - 0.264 = 0.936$ then we apply at least one diffeomorphism f_0 . That is to say, we are at 22% of $\tau(x, \omega)$ to start.

Let

$$B(c, \ell) = \{p \geq 1 : k\tau(x, \omega) + \ell \geq \tau_p(x, \omega)\}.$$

Then define

$$p_\ell(c) = \begin{cases} \max\{p \in B(c, \ell)\} & \text{if } B(c, \ell) \neq \emptyset, \\ 0 & \text{if } B(c, \ell) = \emptyset. \end{cases} \quad (96)$$

We then let

$$F_{tc}^\ell(c) = (f_\omega^{p_\ell(c)}(x), \sigma^{p_\ell(c)}(\omega), \frac{(k\tau(x, \omega) + \ell) - \tau_{p_\ell(c)}(x, \omega)}{\tau(f_\omega^{p_\ell(c)}x, \sigma^{p_\ell(c)}\omega)}).$$

The time changed flow in normal form coordinates

Now take the NFCs we defined in Section 3.8, $H_{(y, \omega)}^u : W^u(x, \omega) \rightarrow T_y W^u(x, \omega)$, and define

$$H_{(y, \omega), new}^u := e^{-\phi(y, \omega)} H_{(y, \omega)}^u. \quad (97)$$

Note that $H_{(x, \omega)}(x) = 0$ so the initial distance between x and $z \in W^u(x, \omega)$ in original NFCs is

$$\|H_{(x, \omega)}^u(x) - H_{(x, \omega)}^u(z)\| = \|H_{(x, \omega)}^u(z)\|. \quad (98)$$

Then applying F we see that

$$\|H_{F(x, \omega)}^u(F_\omega(x)) - H_{F(x, \omega)}^u(F_\omega(z))\| = \|H_{F(x, \omega)}^u(F_\omega(z))\|. \quad (99)$$

The same applies replacing $H_{(x,\omega)}^u$ with $H_{(x,\omega),new}^u$.

Lemma 3.9.2. *The map $H_{(y,\omega),new}^u$ satisfies*

$$\|H_{F(x,\omega),new}^u(F_\omega(z))\| = e^{\tau(x,\omega)} \|H_{(x,\omega),new}^u(z)\|. \quad (100)$$

Proof. By condition (1) of the construction of NFCs in Proposition 3.8.1 and the definition of θ we have that

$$\|H_{F(x,\omega)}^u(F_\omega(z))\| = |D_x f_\omega|_{E^u} \cdot \|H_{(x,\omega)}^u(z)\| = e^{\theta(x,\omega)} \cdot \|H_{(x,\omega)}^u(z)\|. \quad (101)$$

Now using the definition of $H_{(y,\omega),new}^u$ we have

$$\begin{aligned} \|H_{F(x,\omega),new}^u(F_\omega(z))\| &= e^{-\phi(F(x,\omega))} \|H_{F(x,\omega)}^u(F_\omega(z))\| \\ &= e^{-\phi(F(x,\omega))} |D_x f_\omega|_{E^u} \cdot \|H_{(x,\omega)}^u(z)\| \\ &= e^{\theta(x,\omega) - \phi(F(x,\omega))} \cdot \|H_{(x,\omega)}^u(z)\|. \end{aligned} \quad (102)$$

Using Equation (86) $e^{\theta(x,\omega) - \phi(F(x,\omega))} = e^{\tau(x,\omega) - \phi(x,\omega)}$ and so Equation (102) becomes

$$\|H_{F(x,\omega),new}^u(F_\omega(z))\| = e^{\tau(x,\omega) - \phi(x,\omega)} \|H_{(x,\omega)}^u(z)\| = e^{\tau(x,\omega)} \|H_{(x,\omega),new}^u(z)\|. \quad (103)$$

This completes the proof. □

By construction if we put the unstable manifolds in these modified NFCs (as in Equation (97)) and flow forwards by ℓ under the time changed flow it expands distances by precisely e^ℓ . If we put the stable manifold in these modified NFCs, flowing forwards under the time changed flow by ℓ shrinks distances by $e^{-\ell}$.

Invertibility of the time changed flow

Lastly in this section, we write down explicitly the inverse of the ‘time changed’ flow.

We can take τ as above and define $\tau^-(x, \omega) = -\tau(f_{-1}^{-1}(x), \sigma^{-1}(\omega))$ where

$$\omega = (\dots f_{-2}, f_{-1}, f_0, f_1, f_2, \dots),$$

which is a negative function.

Define for $p \geq 1$

$$\begin{aligned} \tau_p^-(x, \omega) &= \tau^-(f_\omega^{-(p-1)}x, \sigma^{-(p-1)}\omega) + \dots \\ &\quad \tau^-((f_\omega^{-1})x, \sigma^{-1}(\omega)) + \tau^-(x, \omega), \end{aligned} \tag{104}$$

set $\tau_0^-(x, \omega) = 0$. In this scenario we want that if $\ell < 0$ surpasses $\tau_p^-(x, \omega)$ (in magnitude) we hit x with p diffeomorphisms from the past of ω , i.e. hit x with $(f_{-1})^{-1}, \dots (f_{-p})^{-1}$.

We let $\ell \in \mathbb{R}_{<0}$ and work to define the ‘time changed flow’. For $c = (x, \omega, k)$ we now view k closer to zero as being closer to applying a diffeomorphism. If, for example, $k = 0.22$, this k represents being $(1 - k) \times 100\% = 78\%$ towards applying a diffeomorphism when going backwards.

Let

$$B^-(c, \ell) = \{p \geq 1 : (1 - k)\tau^-(x, \omega) + \ell < \tau_p^-(x, \omega)\},$$

then define

$$p_\ell^-(c) = \begin{cases} \max\{p \in B^-(c, \ell)\} & \text{if } B^-(c, \ell) \neq \emptyset, \\ 0 & \text{if } B^-(c, \ell) = \emptyset. \end{cases} \tag{105}$$

Then we can define the ‘time changed flow’ for $\ell < 0$ as

$$F_{tc}^\ell(c) = (f_\omega^{-p_\ell^-(c)} x, \sigma^{-p_\ell^-(c)}(\omega), 1 - \frac{(1-k)\tau^-(x, \omega) + \ell - \tau_{p_\ell^-(c)}^-(x, \omega)}{\tau^-(f^{-p_\ell^-(c)}(x), \sigma^{-p_\ell^-(c)}(\omega))}).$$

Invariant measures

For our flow F_{tc}^t , the natural invariant measure $\hat{m}^\tau$ is given by

$$d\hat{m}^\tau(x, \omega, k) = C_\tau \tau(x, \omega) \cdot dm(x, \omega) dt(k), \quad (106)$$

where C_τ is a constant chosen such that $\hat{m}^\tau(Z) = 1$. This measure is clearly absolutely continuous with respect to $\hat{m}$. One makes a similar observation to that of [17, Lemma 15.1]:

Lemma 3.9.3 (Lemma 15.1, [17]). *For every $\epsilon_\tau > 0$ there exists $\mathcal{K} \subset Z$ a compact subset and $Q < \infty$ such that*

$$\hat{m}^\tau(\mathcal{K}) > 1 - \epsilon_\tau, \quad (107)$$

and for every $(x, \omega, k) \in \mathcal{K}$

$$\frac{d\hat{m}^\tau}{d\hat{m}}(x, \omega, k), \quad \frac{d\hat{m}}{d\hat{m}^\tau}(x, \omega, k) \leq Q. \quad (108)$$

Modified Normal Form Coordinates Extension Charts

Here, for a given (local) stable manifold $W^s(x, \omega)$ (similar for unstable) we work to build a smooth chart into $\mathbb{C}^2$ denoted $\tilde{H}_{(x, \omega)}^s$ on a ball around x in X (i.e. ambiently and not restricted to $W^s(x, \omega)$) such that $\tilde{H}^s$ restricted to the stable manifold is the modified NFCs map

$$H_{(x, \omega), new}^s : W^s(x, \omega) \rightarrow T_x W^s(x, \omega) = E^s(x, \omega).$$

We will make the identification of $E^s(x, \omega)$ with $\mathbb{C}$ by applying $D_x \varphi_{(x, \omega)}$, more specifically, we will identify $E^s(x, \omega)$ with the plane $W = \{(0, s) : s \in \mathbb{C}\} \subset \mathbb{C}^2$. In doing so $\tilde{H}_{(x, \omega)}^s$ locally takes $W^s(x, \omega)$ to a coordinate plane in $\mathbb{C}^2$. We will build this chart for (x, ω) in some good set for the purposes of Section 4.4.

We will build a function $g_{(x, \omega)}$ that will modify Lyapunov charts $\varphi_{(x, \omega)} : X \rightarrow \mathbb{C}^2$ so that

$$g_{(x, \omega)} \circ \varphi_{(x, \omega)} \circ (H_{(x, \omega), new}^s)^{-1} \circ ((D_0(\varphi_{(x, \omega)})^{-1})|_W)(s, t) = (0, 0, s, t).$$

for s, t sufficiently small. Then we let

$$\tilde{H}_{(x, \omega), new}^s := g_{(x, \omega)} \circ \varphi_{(x, \omega)}.$$

Making the natural identification between W and $\mathbb{C}$, let $\psi_{(x, \omega)} : \mathbb{C} \rightarrow \mathbb{C}^2$ be defined by

$$\psi_{(x, \omega)} = \varphi_{(x, \omega)} \circ (H_{(x, \omega), new}^s)^{-1} \circ ((D_0(\varphi_{(x, \omega)})^{-1})).$$

Note that $\psi_{(x, \omega)}(0) = (0, 0)$ and that

$$D_0 \psi_{(x, \omega)} = \begin{pmatrix} 0 \\ a \end{pmatrix}, a \neq 0.$$

Then we Taylor expand ψ_x around 0 $\psi_{(x, \omega)}(s) = (e_1(s), as + e_2(s))$. Here e_1 and e_2 are error terms of degree at least 2.

We will focus on getting rid of these error terms but we must eliminate e_2 first. Let $\pi_i : \mathbb{C}^2 \rightarrow \mathbb{C}$, $i = 1, 2$, be the natural coordinate projection map onto the i th coordinate. Note that

$$h_2(s) := \pi_2 \circ \psi_{(x, \omega)}(s) = (s + e_2(s)),$$

and that inverse function theorem gives us the existence of an inverse h_2^{-1} (on some ball)

such that $(Id, h_2^{-1}) \circ \psi_{(x,\omega)}(s) = (e_1(s), s)$

We can now do the same thing to eliminate e_1 , we get a function h_1^{-1} (defined on some ball) such that

$$(h_1^{-1}, Id) \circ (Id, h_2^{-1}) \circ \psi_{(x,\omega)}(s) = (0, s).$$

We let $g_{(x,\omega)} = (h_1^{-1}, Id) \circ (Id, h_2^{-1})$ and we are done. We now just have to construct a good set where we can control where these functions are defined and bounded.

Firstly, the radius of convergence for the Taylor series of $\psi_{(x,\omega)}$ around 0 depends on $\varphi_{(x,\omega)}$ and $H_{(x,\omega),new}^s$. The error terms e_1 and e_2 also depend on these functions, as do h_1 and h_2 . Hence on some good subset $K_{mod} \subset X \times \Omega$ we can ask that $\tilde{H}_{(x,\omega)}^s$ is defined on some ambient ball around x of size at least r_m and and that the second derivatives of this map and the change in distance between ambient coordinates and $\mathbb{C}^2$ under this map are bounded above by $\tilde{M}$ (for more on this see Lemma 3.10.1 and its proof below).

3.10 Some good sets

First of all, recall from Sections 2 and 3.5 that Λ was the conull set of points in Y where Oseledets' theorem holds and where the Lyapunov charts are defined. There is also the set Λ' where NFCs (and modified NFCs as in Equation (97)) can be used which is explained in Section 3.8. Additionally, there is the set Λ_{loc} which is almost conull; we fix this set with $\delta < \epsilon_0$, see Section 3.5. Lastly, recall $\tilde{\Lambda}$ from Section 3.6.

Now recall that we denote Lyapunov charts at a point $(x, \omega) \in Y$ as $\varphi_{(x,\omega)}$, see Section 3.5.

Lemma 3.10.1. *Let $\epsilon_r > 0$ be given, there exists a set $K_r \subset Y$ of m -measure at least $(1 - \epsilon_r)$ and a constant $M_r > 0$ such that if $(x, \omega) \in K_r$, and two points $y, z \in X$ belong to the Lyapunov chart domain of (x, ω) , $U(x, \omega)$, if $d = d_X(y, z)$ and $d' = |\varphi_{(x,\omega)}(y) - \varphi_{(x,\omega)}(z)|$, we have that $d = cd'$ for some $c \in [\frac{1}{M_r}, M_r]$. We can ask the same for distances when*

$D\varphi_{(x,\omega)}$ is applied and for the second derivatives of this chart be bounded above by M_r as well.

Proof. Each $\varphi_{(x,\omega)}$ is a diffeomorphism of compact sets and hence is Bi-Lipschitz for some constant L (dependent on (x,ω)). This means that for any $y, z \in U(x,\omega)$,

$$\frac{1}{L}d_X(y, z) \leq |\varphi_{(x,\omega)}(y) - \varphi_{(x,\omega)}(z)| \leq Ld_X(y, z). \quad (109)$$

We see that if d is a distance computed in a chart $\varphi_{(x,\omega)}$ and d' is the distance in ambient coordinates, we have that

$$\frac{1}{L}d \leq d' \leq Ld. \quad (110)$$

The inverse of $\varphi_{(x,\omega)}$ has the same Lipschitz constant as $\varphi_{(x,\omega)}$, hence swapping d' and d in Equation (110) still holds.

Of course each $\varphi_{(x,\omega)}$ will have a different Lipschitz constant. In Section 3.5, we have bounds on the Lipschitz constants of every $\varphi_{(x,\omega)}$ for $(x,\omega) \in \Lambda$: there is a uniform constant ψ such that

$$\psi^{-1} \leq \text{Lip}(\varphi_{(x,\omega)}) \leq r(x,\omega). \quad (111)$$

We only need the upper bound; since r is measurable, for the cost of an $\epsilon_r > 0$, we can ask that $r(x,\omega) \leq M_r$ for some constant $M_r > 0$ on a set K_r of m -measure $1 - \epsilon_r$. We can use M_r as L in the above argument for every $\varphi_{(x,\omega)}$ such that $(x,\omega) \in K_r$ and then we get the result.

We can ask the same for distances when $D\varphi_{(x,\omega)}$ is applied. This completes the proof. $\square$

Lemma 3.10.2. *Let $\epsilon_{NFC} > 0$ be given, there exists a set $K_{NFC} \subset \Lambda' \subset Y$ of m -measure*

at least $(1 - \epsilon_{NFC})$ and a constant M_{NFC} such that if $(x, \omega) \in K_{NFC}$, then the maps

$$H_{(x, \omega), new}^{s/u} : W^u(x, \omega) \rightarrow T_x W^u(x, \omega),$$

are defined and if $y, z \in W_1^u(x, \omega)$, $d_u = d_X^u(y, z)$ and

$$d' = \|H_{(x, \omega), new}^u(y) - H_{(x, \omega), new}^u(z)\|,$$

then $d_u = c'd'$ for $c' \in [\frac{1}{M_{NFC}}, M_{NFC}]$. Additionally, M_{NFC} can be chosen such that if $d = d_X(y, z)$ there is $c'' \in [\frac{1}{M_{NFC}}, M_{NFC}]$ such that $d = c''d_u$. Similarly if $y, z \in W_1^s(x, \omega)$.

Proof. Since ϕ is measurable, e^ϕ is measurable. Also, the Lipschitz constants for these maps $H_{(x, \omega)}^{s/u}$ restricted to the local stable/unstable manifolds are a measurable function, hence one runs the same argument as in Lemma 3.10.1. □

Lemma 3.10.3. *Let $\epsilon_{exp} > 0$ be given, there exists a set $K_{exp} \subset X$ of ν -measure at least $(1 - \epsilon_{exp})$ and a constant $M_{exp} > 1$ such that if $x \in K_{exp}$ and $y \in B(x, \rho_0/4)$ (i.e. the domain of the $\exp_x$ map) then if $v = \exp_x^{-1}(y)$, we have that*

$$\|v\| = Md_x(x, y),$$

where $M \in [\frac{1}{M_{exp}}, M_{exp}]$.

Proof. Proof is similar to those of Lemmas 3.10.1 and 3.10.2. □

Lemma 3.10.4. *Let $\epsilon_\varphi > 0$ be given, there exists a set K_φ of m -measure at least $1 - \epsilon_\varphi$ and a constant Φ on which $Lip(D\varphi_{(x, \omega)})$ is bounded above by Φ .*

Proof. Proof is similar to those of Lemmas 3.10.1 and 3.10.2. □

Lemma 3.10.5. *Let $\epsilon_s > 0$ be given, there exists a set $K_s \subset Y$ of m -measure at least $(1 - \epsilon_s)$ and a constant $k_U > 0$ such that if $(x, \omega) \in K_s$, then $\varphi_{(x, \omega)}$ is defined and the domain of this function, $U(x, \omega)$, contains a ball of radius k_U in the Riemannian distance metric d_X .*

Proof. We take steps similar to those of the proof of Lemma 3.10.1.

Again, the function r defined in Section 3.5 is measurable. This function defines the co-domain of our diffeomorphism $\varphi_{(x, \omega)} : U(x, \omega) \rightarrow B(1/r(x, \omega))$. Let $s(x, \omega) = \frac{1}{r(x, \omega)}$; this is also a measurable function. We can find a subset of Y on which $s(x, \omega) \in]0, 1]$ is bounded from below by some constant S on a set K_s of measure $1 - \epsilon_s$. One performs the following procedure: let

$$E_n = \{(x, \omega) : s(x, \omega) > 1/n\} \subset \Lambda,$$

this is a measurable set such that $E_n \rightarrow \Lambda$ as $n \rightarrow \infty$ where Λ has full measure. Hence there is $N > 0$ such that $m(E_N) > 1 - \epsilon_s$, take $S = \frac{1}{N}$. Additionally, if $s = \frac{1}{r} > S$, then $r < \frac{1}{S} = R$ on this set.

Now again, $\varphi_{(x, \omega)}^{-1}$ is a diffeomorphism of two compact sets and hence is Bi-Lipschitz for some Lipschitz constant L . To be more precise, for any $y, z \in B(s(x, \omega))$, we have that

$$\frac{1}{L}|y - z| \leq d_X(\varphi_{(x, \omega)}^{-1}(y), \varphi_{(x, \omega)}^{-1}(z)) \leq L|y - z|. \quad (112)$$

Then $\varphi_{(x, \omega)}^{-1}(B(S)) \subset U(x, \omega)$ contains the ball $B_X(x, S/L)$.

In Section 3.5, we have a uniform constant ψ^{-1} such that for every $(x, \omega) \in \Lambda$

$$\psi^{-1} \leq \text{Lip}(\varphi_{(x, \omega)}) \leq r(x, \omega). \quad (113)$$

We only need the upper bound; on K_s , $r(x, \omega) < R = \frac{1}{S}$. So we replace L with R in the above argument for every $\varphi_{(x, \omega)}$ such that $(x, \omega) \in K_s$. Then on K_s we have that $\varphi_{(x, \omega)}^{-1}(B(S)) \subset U(x, \omega)$ contains the ball $B_X(x, S/R)$. Let $k_U := S/R = S^2 < 1$. This

completes the proof. $\square$

Lemma 3.10.6. *Let $\epsilon_{nbhd} > 0$ be given. There exists $r, \tilde{r}$, and a set $K_{nbhd} \subset Y$ of measure at least $1 - \epsilon_{nbhd}$ such that for $(x, \omega) \in K_{nbhd}$, $\eta_\omega^{s/u}(x)$ contains a neighbourhood of size at least r in $W^{s/u}(x, \omega)$ (in the induced metric $d_X^{s/u}$), and the diameter of $\eta_\omega^{s/u}(x)$ is bounded above by $\tilde{r}$.*

Proof. Now note that the partition elements of X_ω , $\eta_\omega^{s/u}(x)$, defined in Section 2.4 contain a neighbourhood of $W^{s/u}(x, \omega)$ of size $r(x, \omega)$. This $r(x, \omega)$ is a measurable function. Similar to the proof of Lemma 3.10.5, for a cost of $\epsilon_{nbhd} > 0$ in measure, we can choose $r(x, \omega)$ uniformly as some r on a set which we call K_{nbhd} . Additionally the diameter of these partition elements is measurable and we add that to the definition of K_{nbhd} . $\square$

Lemma 3.10.7. *On the set $K_\xi := K_r \cap K_s \cap \Lambda \cap \Lambda_{loc}$, given $\xi > 0$, if $(x, \omega) \in K_\xi$ and $y \in W_q^s(x, \omega)$, the image of $E^s(y, \omega)$ under $\exp$ approximates $W^s(x, \omega)$ up to error ξ on a ball of radius $\mathfrak{r} = C\sqrt{\xi}$ with respect to the Riemannian distance metric d_X (where C is dependent on (x, ω)). Similar if $z \in W_q^u(x, \omega)$.*

Proof. Take a point $(x, \omega) \in K_r \cap \Lambda_{loc} \cap \Lambda \cap K_s$; we can define the Lyapunov chart $\varphi_{(x, \omega)}$ centered at this point. By Theorem 3.5.4, we know that $W_q^u(x, \omega), W_q^s(x, \omega) \subset X$ are pullbacks of graphs of holomorphic functions $h_{(x, \omega)}^u$ and $h_{(x, \omega)}^s$ respectively in $\mathbb{R}^4$ under $\varphi_{(x, \omega)}$.

Claim 3.10.8. *Let $\epsilon > 0$ be given. Let $f : B \rightarrow B'$ be holomorphic where $B, B' \subset \mathbb{R}^2$ are compact subsets. Let $L = \text{Lip}(Df)$ and let $\mathfrak{r}' = \sqrt{\frac{\epsilon}{L}}$. Take any point $x \in B$ such that $B_{\mathfrak{r}'}(x) \subset B$. On $B_{\mathfrak{r}'}(x)$, f is approximated up to error ϵ by its tangent plane at x .*

Proof. We are given $\epsilon > 0$. We let $L = \text{Lip}(Df)$ and take $y \in B_{\mathfrak{r}'}(x) \subset B$, let z be such that

$$f(y) - f(x) = Df(z)(y - x),$$

then

$$\begin{aligned}
& |f(y) - (f(x) + Df(x) \cdot (y - x))| \\
& \leq 0 + |Df(z) - Df(x)| \cdot |y - x| \\
& \leq L|z - x| \cdot |y - x| \leq L|y - x|^2 \leq L(\mathfrak{r}')^2 = \epsilon.
\end{aligned}$$

□

Using Claim 3.10.8, take $\epsilon = \xi/M_r$, take f in the collection

$$\mathcal{C} := \{h_{(x,\omega)}^s, h_{(x,\omega)}^u\}_{(x,\omega) \in \Lambda}.$$

For $\mathfrak{r}' = \sqrt{\frac{\epsilon}{L}}$, we get that for $(x, \omega) \in K_r \cap \Lambda_{loc} \cap \Lambda$, $h_{(x,\omega)}^s$ and $h_{(x,\omega)}^u$ are approximated up to error ϵ by tangent plane of any point on the graph.

Now we return to ambient coordinates and examine how well E^s and E^u approximate the local stable and unstable manifolds for points $(x, \omega) \in K_r \cap \Lambda_{loc} \cap \Lambda \cap K_s$. We see that since we are in K_r the largest the error could grow is by a factor of M_r , leaving us with an error of ξ . The ball in X around x for which this estimate is valid has radius $\mathfrak{r} = \mathfrak{r}'/M_r$. So the constant C can be taken to be $\frac{1}{M_r\sqrt{L}}$, where L depends on (x, ω) (because L is dependent on the function we choose from the collection $\mathcal{C}$). This completes the proof. □

Lemma 3.10.9. *Let $\epsilon_{\xi'} > 0$ be given. There exists a constant $H > 0$ and subset $K'_\xi \subset K_\xi$ such that $m(K'_\xi) > (1 - \epsilon_{\xi'})m(K_\xi)$ and if $(x, \omega) \in K'_\xi$ then $\text{Lip}(Dh_{(x,\omega)}^{s/u})$ is uniformly bounded above by H . In particular, on K'_ξ we have that the constant C in Lemma 3.10.7 is uniformly bounded from below by $\frac{1}{M_r\sqrt{H}}$.*

Proof. The functions

$$L^s(x, \omega) : (x, \omega) \mapsto \text{Lip}(Dh_{(x,\omega)}^s), \quad L^u(x, \omega) : (x, \omega) \mapsto \text{Lip}(Dh_{(x,\omega)}^u)$$

are measurable. Similar to the argument in Lemma 3.10.5, for a price of $\epsilon_{\xi'}$ to the measure of K_{ξ} we get a set K'_{ξ} on which $L^s(x, \omega)$ and $L^u(x, \omega)$ are bounded above by H . $\square$

Now we must give a similar lemma for the orbits $U^+[x, \omega, k]$ from our family of $U^+(x, \omega, k)$ specified in Theorem 2.5.2. Note that the orbits $U^+[x, \omega, k]$ are smooth real 1-dimensional curves in the case $d_+ = 1$ (see Section 2.6). In this setting, define the function $G^+ : Y \rightarrow Gr(1, \mathbb{R}^4)$ which takes a pair (x, ω) to the tangent line through x of $U^+[x, \omega]$.

Lemma 3.10.10. *In the case $d_+ = 1$, given $\xi > 0$, for $(x, \omega) \in K_{\xi}$ as in Lemma 3.10.7, if $y \in U^+[x, \omega]$, $G^+(y, \omega)$ approximates $U^+[x, \omega]$ up to error ξ on a ball of radius $\mathfrak{r} = C\sqrt{\xi}$ with respect to the Riemannian distance metric d_X . This C can be taken to be independent of the point (x, ω) .*

Proof. The proof is similar to that of Lemmas 3.10.7 and 3.10.9. $\square$

Lemma 3.10.11. *Let $\epsilon_G > 0$ be given, in the case $d_+ = 1$, there exists a set $K_G \subset Y$ of measure at least $(1 - \epsilon_G)$ such that the maps $G^s : (x, \omega) \rightarrow E^s(x, \omega)$ and G^+ are uniformly continuous on K_G .*

Proof. Let us note that the function $G^s : Y \rightarrow Gr(1, \mathbb{C}^2)$, $G^s : (x, \omega) \mapsto E^s(x, \omega)$ is measurable by Oseledets' theorem ($Gr(1, \mathbb{C}^2)$ has a natural metric, see Equation (64)). We have a similar notion for $G^+ : Y \rightarrow Gr(1, \mathbb{R}^4)$ which takes (x, ω) to $G^+(x, \omega)$ which we denote as the tangent line through x of $U^+[x, \omega]$, this is measurable by [6]. Then letting $\epsilon_G > 0$ be small, by Lusin's theorem let $K_G \subset Y$ be the set of size $1 - \epsilon_G$ such that G^s and G^+ are uniformly continuous in x on K_G . $\square$

Lemma 3.10.12. *Fix $\delta' > 0$ small. In the setting where $d_+ = 1$ (i.e. where Theorem 3.4.1 case c) applies), for given $\epsilon_v > 0$, there exists a set $K_{\epsilon} \subset X$ of ν -measure at least $(1 - \epsilon_v)$ and a constant $\varepsilon > 0$ such that if $x \in K_{\epsilon}$, then inside any ball of radius ε inside $Gr(1, \mathbb{C})$, the measures P_x^u and P_x^s have measure at most δ' (see Section 3.4).*

Proof. We use Theorem 3.4.1 and Lemma 3.4.2.

For each $x \in X$, we do not know that the measure P_x^u (see Section 3.4) is uniform on $Gr(1, \mathbb{C}^2)$. We only know that we can pick elements outside some small ball by Lemma 3.4.2. For given $\delta' > 0$, for each $x \in X$ let $\epsilon_{x, \delta'}$ be the size of the largest ball outside of which we have P_x^u -measure $(1 - \delta')$ and P_x^s -measure $(1 - \delta')$ in $Gr(1, \mathbb{C})$. This is a measurable function for fixed δ' . Let K_ϵ be the set of measure $(1 - \epsilon_v)$ where $\epsilon_{x, \delta}$ is bounded above by some ϵ . We do this the same way as we did in the proof of Lemma 3.10.5. $\square$

Lemma 3.10.13. *Let $\epsilon_b > 0$ be given, there exists a set $K_b \subset Y$ of m -measure at least $(1 - \epsilon_b)$ and a constant C'' such that if $(x, \omega) \in K_b$ and*

$$\psi^s : \mathbb{C} \rightarrow \varphi_{(x, \omega)}(B(x, \rho_0/4)), \quad \psi^s(z) = \vec{u}z + \vec{u}^\perp g(z),$$

is a parametrization of the holomorphic curve $\varphi_{(x, \omega)}(W^s(x, \omega))$ where $\vec{u}$ is a unit vector in $E^s(x, \omega)$ as viewed in Lyapunov charts (i.e. $\mathbb{C}^s$), and $\vec{u}^\perp$ is a unit vector perpendicular to $\vec{u}$ belonging to $(\mathbb{C}^s)^\perp = \mathbb{C}^u$, then

$$\|D^2\psi^s(z)|_v\| \leq C'',$$

for $v \in \varphi_{(x, \omega)}(B(x, \rho_0/4))$.

Proof. We are working inside the compact set $\varphi_{(x, \omega)}(B(x, \rho_0/4))$. For each (x, ω) in a conull subset of Y , $\|D^2\psi^s(x, \omega)|_v\|$ is bounded from above by a constant $C''(x, \omega)$ on the compact set $\varphi_{(x, \omega)}(B(x, \rho_0/4))$. This is a measurable function and so similar to the proof of Lemma 3.10.5, for the cost of ϵ_b , we get a set K_b of measure at least $1 - \epsilon_b$ and a constant C'' for which $\|D^2\psi^s(z)|_v\| \leq C''$ on $\varphi_{(x, \omega)}(B(x, \rho_0/4))$. $\square$

Lemma 3.10.14. *Let $\epsilon_{ang} > 0$ be given, there exists a set $K_{ang} \subset Y$ of m -measure at least $(1 - \epsilon_{ang})$ and a constant $\Theta > 0$ such that if $(x, \omega) \in K_{ang}$ then $\angle(E^u(x, \omega), E^s(x, \omega)) > \Theta$.*

Proof. Use Oseledets' theorem and an argument similar to that of Lemma 3.10.5. $\square$

The following lemma is standard and is a consequence of Birkhoff's theorem, one can see for example [16]:

Lemma 3.10.15. *For a fixed set K , for all $\epsilon, \delta > 0$ there is $S > 0$, and $E \subset Z$ with $\hat{m}(E) < \epsilon$ and such that for $c \in E^c$ for all $T > S$,*

$$|\frac{1}{T} \int_0^T \mathbf{1}_K(F^t(c)) dt - \int_Z \mathbf{1}_K d\hat{m}| < \delta.$$

Proof. This follows from Birkhoff's ergodic theorem. The skew product map is ergodic with respect to $\nu \times \mu^{\mathbb{Z}}$ (see Remark 2.1.3) and the 'standard' flow preserves the measure $\hat{m} = m \times dt$. $\square$

A nice collection of complex charts

We will give a construction of charts with some nice properties for the purposes of Sections 4.2 and 6. These sections deal with the $d_+ = 0$ case. These charts will be dependent on the value of r from Lemma 3.3.4. This will be done on a subset of $A' \subset (X \times \Omega_t^+) \cap \pi_{X \times \Omega^+}(\Lambda_{loc})$ which will be of measure at least $(1 - \epsilon_\varphi)$ for some ϵ_φ chosen appropriately small.

Let $\{\varphi_x\}_{x \in X}$ be a set of complex charts

$$\varphi_x : B(x, r_x) \subset X \rightarrow \mathbb{C}^2. \tag{114}$$

For a small cost in measure we can find a set $\tilde{X} \subset X$ for which r_x is bounded below by $\mathcal{R}$ and bound the Lipschitz constant of φ_x above by $L' > 0$. We add this to the definition of A' .

Since our points belong to Λ_{loc} , implicit function theorem allows us to view $\varphi_x(W^s(x, \omega^+))$ as a graph over a coordinate axis copy of $\mathbb{C}$,

$$\{(z, \psi_{x, \omega^+}(z)) : z \in \mathbb{D}_{s(x, \omega^+)}\},$$

in some disk $\mathbb{D}_{s(x,\omega+)} \subset \mathbb{C}$ around the origin which is defined uniquely up to rotation (i.e. an origin fixing automorphism of the disk). We can bound $s(x,\omega)$ from below by some s on a good set that we add to the definition of A' . Note that (up to rotation) for any $(\omega')^+ \in \Omega_t^+$ from Lemma 3.3.4, $\psi_{x,\omega+}$ and $\psi_{x,(\omega')^+}$ agree on the first $r-1$ derivatives and differ at the r th derivative. Now we discuss two different changes of coordinates needed for Section 4.2 and 6 the constructions we give will both belong to the definition of A' .

For Section 6:

There is a smooth change of coordinates (unique up to a rotation) that eliminates the first $r-1$ derivatives of $\psi_{x,\omega}$, that is to say, there is a function g_x such that $g_x \circ \varphi_x(W^s(x,\omega))$ can be written as a graph of the form

$$\{(z, \frac{1}{r}\psi_{x,\omega+}^{(r)}(0)z^r + h.o.t.) : z \in \mathbb{D}^s\},$$

where *h.o.t.* stands for higher order terms.

Note that g_x does not depend on $\omega \in \Omega_t^+$ because we are removing the $(r-1)$ derivatives where they all agree. This change of coordinates comes from a similar strategy as in Section 3.9. We abuse notation and ask that in the definition of A' this change of coordinates be defined over $\mathbb{D}_s$.

A rotation of this change of coordinates does not affect the Lipschitz constant, hence for x in some good set we can bound the Lipschitz constant of $g_x \circ \varphi_x$ above by $L'' > 0$ and add this to the definition of A' .

Additionally, we can bound $|\psi_{x,\omega+}^{(r)}(0)| > C_\varphi > 0$ on a good set that we add to the definition of A' . Here its important to note that the graph is unique up to rotation, but precomposing $\psi_{x,\omega+}$ with a rotation does not change the value of its r th derivative at zero.

That is to say, if $S \in \text{Aut}(\mathbb{D}_s)$ is a rotation, $S : z \mapsto e^{i\alpha}z$, $\alpha \in [0, 2\pi[$, we have that

$$(\psi_{x,\omega^+} \circ S)^{(n)}(0) = \psi_{x,\omega^+}^{(n)}(0) \cdot e^{ni\alpha}.$$

Hence,

$$|(\psi_{x,\omega^+} \circ S)^{(n)}(0)| = |\psi_{x,\omega^+}^{(n)}(0)|.$$

Overall we have that

$$|(\psi_{x,\omega^+} \circ S)^{(n)}(0)| = |\psi_{x,\omega^+}^{(n)}(0)|. \quad (115)$$

For Section 4.2:

Here we start with being able to write $\varphi_x(W^s(x, \omega))$ as a graph given by:

$$\{(z, \psi_{x,\omega^+}(z)) : z \in \mathbb{D}_s\}.$$

It is important for later to note again like in the above section that Equation (115) holds.

Here, we can write $\psi_{x,\omega^+} : \mathbb{C} \rightarrow \mathbb{C}$ as

$$\psi_{x,\omega^+}(z) = \sum_{i=1}^r \frac{\psi_{x,\omega^+}^{(i)}(0)}{i!} z^i + h_{x,\omega^+}(z) z^{r+1}.$$

Note that for varying $(\omega')^+ \in \Omega_t^+$ we have that $\psi_{x,\omega^+}^{(i)}(0) = \psi_{x,(\omega')^+}^{(i)}(0) \forall i = 1, \dots, (r-1)$, because we have tangency at order $i = 1, \dots, (r-1)$. However, we know that they are not tangent at order r .

We can ask for a change of coordinates (again, unique up to rotation) g_{x,ω^+} that sends $\varphi_x(W^s(x, \omega^+))$ to the coordinate axis $\{(z, 0) : z \in \mathbb{D}_s\}$ with bounded Lipschitz constant on A' . Call the upper bound of the Lipschitz constant $L''' > 0$. This change of coordinates

comes from a similar strategy as in Section 3.9. We abuse notation and ask that in the definition of A' this change of coordinates be defined over $\mathbb{D}_s$.

In these new coordinates, $g_{x,\omega} \circ \varphi_x(W^s(x, (\omega')^+))$ for $(\omega')^+ \in \Omega_t^+$ we now have that it is represented as a graph:

$$G' = \{(z, (\psi_{x,\omega^+}^{(r)}(0) - \psi_{x,(\omega')^+}^{(r)}(0))z^r + h.o.t.) : z \in \mathbb{D}^s\}.$$

For each (x, ω) , for a given cost of $\epsilon'_\varphi > 0$ to the measure of Ω_t^+ there is a constant $C_{\varphi,x} > 0$ such that

$$(\psi_{x,(\omega')^+}^{(r)}(0) - \psi_{x,\omega^+}^{(r)}(0)) > C_{\varphi,x} > 0.$$

We add to the definition of A' that on these $(1 - \epsilon'_\varphi)$ -measure subsets of Ω_t^+ which we call $U_{x,\omega}$ (since they are dependent on x and ω), $C_{\varphi,x} > C_\varphi > 0$. We also ask that on these sets $U_{x,\omega}$,

$$|\psi_{x,(\omega')^+}^{(r+1)}(\xi)| + |\psi_{x,\omega^+}^{(r+1)}(\xi')| \leq C^{r+1},$$

for any $\xi, \xi' \in \mathbb{D}_s$.

4 Proofs of Lemmas 2.6.3 and 2.6.7

Both lemmas in this section work to prove that under certain conditions, the QNI condition of Definition 3 holds. Recall from Section 2.6 that $\mathcal{L}^s[x, \omega, 0]$ is the Zariski closure of the support of the stable conditional measure $m_{x,\omega}^s$ defined in Section 2.4. One should also review Section 2.5 to see the definitions of the combinatorial unstables and stables (Equations (18) and (19) resp.), and objects that appear in Definition 3, i.e. the measures defined in Equations (24) and (25), and the sets defined in (22) and (23). Throughout this section, let $B^s(x, \omega, R)$ (or just $B^s(x, R)$ if it is implicit) be the ball of radius R around x in the induced metric,

d_X^s , inside $W^s(x, \omega)$. Also, since $\text{Supp}(\mu) < \infty$, let $\mathcal{L}$ be the upper bound on the Lipschitz constants of the $f \in \text{Supp}(\mu)$.

4.1 Outline of proof of Lemma 2.6.3

We are given that the stable conditional measures are non-trivial m -a.e. Also in this setting, $d_+ = 0$ a.e. Let K_+ be the set where both these conditions are true. We demonstrate that random QNI (Definition 3) holds.

Overview

We are given $\delta > 0$ and we will construct the set $K \subset Z$ (dependent on δ) of $\hat{m}$ -measure $1 - \delta$ and will determine the constants α_0 , ℓ_0 and C as in Definition 3. The initial properties of K will include being a subset of K^* which is an intersection of many good sets from Section 3.10. Other properties of K will allow S and S' (as in Definition 3) to be large enough in measure and have to their own good properties, such as belonging to K^* . Taking $\hat{x}_{1/2} = (x_{1/2}, \omega_{1/2}, k_{1/2}) \in K \subset K^*$ we will have that $U^+[\hat{x}_{1/2}] = \{x_{1/2}\}$ and so S is a subset of

$$\{x_{1/2}\} \times \{\omega_{1/2}^-\} \times \Omega^+ \times \{k_{1/2}\},$$

i.e. we only have to pick a new future (in a way that keeps us in a good set). We will have to pick this new future so that $W^s(\hat{y}_{1/2})$ is sufficiently distinct from $W^s(\hat{x}_{1/2})$. The set S' is slightly more complicated, and is a subset of

$$X \times \Omega^- \times \{\omega_{1/2}^+\} \times \{k_{1/2}\},$$

or more precisely a subset of $\eta_{\ell/2}^s[\hat{x}_{1/2}]$ (for some $\ell > \ell_0$). This means that we will be picking a new X -component in $W^s(\hat{x}_{1/2})$ close to $x_{1/2}$ (depending on ℓ) but not equal to $x_{1/2}$, and a new past vector (in a way that keeps us in a good set). The set S' is dependent on our

choice of $\hat{y}_{1/2} \in S$, but regardless of the choice of $\hat{y}_{1/2}$, we will choose $S' \subset K^*$ which gives us that if $\hat{z}_{1/2} = (z_{1/2}, (\omega'')^-, \omega_{1/2}^+, k_{1/2}) \in S'$, then $U^+[\hat{z}_{1/2}] = \{z_{1/2}\}$. The X -components of S' will have to be chosen in some annulus around $x_{1/2}$ to ensure that we can pick $z_{1/2}$ sufficiently bounded away from $W^s(\hat{y}_{1/2})$.

Then we must demonstrate that there are constants C and ℓ_0 (dependent only on δ) such that for all $\ell > \ell_0$ if we pick $\hat{x}_{1/2} \in K$ (where K is dependent only on δ), $\hat{y}_{1/2} \in S$ (where S depends on $\hat{x}_{1/2}$ and ℓ) and $\hat{z}_{1/2} \in S'$ (where S' is dependent on $\hat{y}_{1/2}$ and ℓ) we have

$$d_X((U_q^+[\hat{z}_{1/2}] \cap \eta_{\omega_{1/2}, \ell/2}^u(z_{1/2})), W_q^s(\hat{y}_{1/2})) \geq Ce^{-\alpha_0 \ell},$$

These constants C and ℓ_0 are technically chosen with K but we have to compute what they need to be at the end. The constant α_0 needs to be chosen even before δ is given, we give it at the beginning and use it in a computation at the end.

Details

As explained above, when constructing $K \subset Z$, we start out with it needing to belong to K^* which is an intersection of many good sets from Section 3.10. Of the many good sets that K^* belongs to, some had to be constructed directly in this section. For example K_η from Lemma 4.2.5 whose construction relied in Lemma 4.2.1.

Now we need the set K to have the property that any point

$$\hat{x}_{1/2} = (x_{1/2}, \omega_{1/2}, k_{1/2}) \in K,$$

is such that there is a large measure subset $V_{x_{1/2}, \omega_{1/2}^-, k_{1/2}} \subset \Omega^+$ of new futures that we can pick from and still belong to K^* . We demonstrate this in Lemma 4.2.6. This allows us to

keep the measure of S sufficiently large once we pick a point $\hat{x}_{1/2} \in K$,

$$S \subset \{\hat{x}_{1/2}\} \times \{\omega_{1/2}^-\} \times V_{\hat{x}_{1/2}} \times \{k_{1/2}\}.$$

It is only a subset because we will want to further reduce the choices of futures in $V_{\hat{x}_{1/2}}$ so that for $\hat{y}_{1/2} \in S$, $W^s(\hat{y}_{1/2})$ will be bounded away from $W^s(\hat{x}_{1/2})$. This is where the sets A' , $\tilde{V}_{x_{1/2}, \omega_{1/2}^+}$ and $U_{x_{1/2}, \omega_{1/2}^+}$ come in. We use this in Lemma 4.2.11 to get a bound independent of the choice of $\hat{x}_{1/2}$. Since $\tilde{V}_{x_{1/2}, \omega_{1/2}^+}$ and $U_{x_{1/2}, \omega_{1/2}^+}$ can be chosen to be large, we don't diminish the measure of $\{\hat{x}_{1/2}\} \times \{\omega_{1/2}^-\} \times V_{\hat{x}_{1/2}} \times \{k_{1/2}\}$ by much. This gives us our set S for a given $\hat{x}_{1/2} \in K$.

Things are slightly more complicated when dealing with S' because we have to change the X -component and the Ω^- -component and still land in the good set (e.g. one such that we belong to K^*). This must be incorporated into the definition of K . In changing the X -component we in particular must be in the set $\eta_{\omega_{1/2}, \ell/2}^s(x_{1/2}) \subset W^s(\hat{x}_{1/2})$. We then are concerned that restricting ourselves to choosing the X -component here will not allow us to be in a good set. This is where Lemma 4.2.7 comes in, it tells us that for good $\omega \in \Omega$, if one has a large ν -measure subset X' of X , there is a large ν -measure subset $X^* \subset X'$ such that for $x \in X^*$, a large $m_{x, \omega}^s$ -measure of $\eta_{\omega_{1/2}, \ell/2}^s(x_{1/2})$ belongs to X' . We define a set $X_{\omega, k}''' \subset X$ to have two properties: the first is that if $x \in X_{\omega, k}'''$ then $V_{x, \omega, k} = \{(\omega'')^- : (x, (\omega'')^-, \omega^+, k) \in K^*\}$ is large, and the second is that (x, ω, k) belongs to another good set K' . The set K' is K^* with the added property that we can still pick good futures as in the previous paragraph and belong to K^* . We apply Lemma 4.2.7 to this set $X_{\omega, k}'''$ for good choices of ω and k (after using Lemmas 4.2.8 and 4.2.9 to show that the set of good choices of ω and k such that $X_{\omega, k}'''$ is large is also large) to get sets $X_{\omega, k}^*$. We use this to define the large set K'' where each element is of the form (x, ω, k) where ω and k are 'good' and $x \in X_{\omega, k}^*$. This set K'' is big by Lemma 4.2.10.

We are not quite done but now we see that we can make S and S' have properties

of ‘good sets’. We need to ensure that for $\hat{x}_{1/2} \in K$, $x_{1/2}$ is not the only element in $\text{Supp}(m_{x_{1/2}, \omega_{1/2}}^s) \cap \eta_{\omega_{1/2}, \ell/2}^s(x_{1/2})$, this is Lemma 4.2.4. This is important because when we pick $\hat{z}_{1/2} = (z_{1/2}, (\omega'')^-, \omega_{1/2}^+, k_{1/2}) \in S'$ we have $U^+[\hat{z}_{1/2}] = z_{1/2}$ and we want $W^s(\hat{y}_{1/2})$ bounded away from it. If $z_{1/2} = x_{1/2}$, we lose because the X -component of $\hat{y}_{1/2}$ is $x_{1/2}$. Proving Lemma 4.2.4 involves Corollary 4.2.3 which allows us to keep a ball (of some radius dependent on ℓ) inside $\eta_{\omega, \ell/2}^s(x)$. It is also important that x is not an atom (so that the measure of S' can be sufficiently large when we restrict the X -component to some annulus around $x_{1/2}$). This comes from Lemma 4.4.1 and allows us to prove Lemma 4.2.12 using that $K_\eta \subset K^*$. Lemma 4.2.12 tells us that we can pick $z_{1/2}$ in an annulus without significant loss to the measure of S' . We in fact know everything about this annulus in terms of ℓ . Then when we construct S' with X -components belonging to this annulus we have some foresight in defining S . In particular we add some properties to S to use to prove Lemma 4.2.11 which allows us to know how $W^s(\hat{y}_{1/2})$ will be bounded away from some point $z \in W^s(\hat{x}_{1/2})$ given we know how far z is from $x_{1/2}$.

Combining all of this, we make a final computation demonstrating that for sufficiently large ℓ_0 there is an appropriate constant C (that can be chosen alongside K) such that for $\ell > \ell_0$,

$$d_X((U_q^+[\hat{z}_{1/2}] \cap \eta_{\omega_{1/2}, \ell/2}^u(z_{1/2})), W_q^s(\hat{y}_{1/2})) = d_X(z_{1/2}, W_q^s(\hat{y}_{1/2})) \geq Ce^{-\alpha_0 \ell}.$$

4.2 Proof of Lemma 2.6.3

Proof. (Lemma 2.6.3)

We are given that the stable conditional measures $m_{x, \omega}^s$ are non-trivial m -a.e. Further, we are given that $U^+(x, \omega, k) = \{e\}$ a.e. and so $U^+[x, \omega, k] = x$ a.e. As above, let $K_+ \subset Z$ be the set on which both these conull conditions are true; this is some full measure set.

On the set K_+ , the stable conditional measures $m_{x, \omega}^s$ are non-trivial, so there is some

$z \in \text{Supp}(m_{x,\omega}^s) \subset W^s(x, \omega)$ such that $z \neq x$. For each (x, ω) there exists $\mathfrak{r}(x, \omega)$ such that $z \in B^s(x, \mathfrak{r}(x, \omega)) \setminus B^s(x, \frac{1}{\mathfrak{r}(x, \omega)})$, i.e. it lives in some annulus around x . For a small price to the measure of K_+ , we can ask that $\mathfrak{r}(x, \omega)$ be bounded above by some $\mathfrak{r} > 0$ for $(x, \omega) \in K_+$.

Write $\mathcal{L}$ as the upper bound for Lipschitz constant for the $f \in \text{Supp}(\mu)$ and let $\mathcal{K}$ be such that $\mathcal{L} = e^{\mathcal{K}}$.

Let $\delta > 0$ be given. We will determine ℓ_0 and C as in Definition 3 (the QNI definition). Take $i > j > 2$ such that

$$(1 - 2\delta^j)(1 - \frac{\delta^i/2}{\delta^j - \delta^i/2}) > 1 - \delta, \quad (1 - \sqrt{2}\delta^{j/2})(1 - \delta) > 1 - \delta/4.$$

Recall the Oseledets' set Λ (conull), Λ_{loc} from Section 3.5, and the set Λ' from 3.8. Also recall K_r , K_s , and K_{nbhd} from Section 5.3 from Lemmas 3.10.1, 3.10.5, and 3.10.6 respectively.

Let $\{\varphi_x\}_{x \in X}$ be the set of complex charts

$$\varphi_x : B_X(x, r_x) \subset X \rightarrow \mathbb{C}^2. \tag{116}$$

from Section 3.10. Let A' be as in that section with appropriately large measure. Note that the Lipschitz constant of φ_x is bounded above by L' on A' . Points in $(x, \omega) \in A'$ also come with sets $U_{x,\omega}$ of $\mu^{\mathbb{N}}$ -measure at least $(1 - \epsilon'_\varphi)$.

We state the following Lemma from [6]. It will be important to the proof of Lemma 2.6.7 in this form, however for the purposes of the proof of Lemma 2.6.3 it simply tells us that there are no atoms (i.e. take $d = 0$ in the below).

Lemma 4.2.1. *For m -a.e. $(x, \omega) \in Y$ and for all $R > 0$, $m_{x,\omega}^s|_{B^s(x,\omega,R)}$ does not assign mass to any algebraic curve of degree d .*

Proof. See [6], Section 4, Lemma 4.4.10. □

Let K_{alg} be the conull subset of Y we get from Lemma 4.2.1. Now define

$$\begin{aligned}\tilde{K} = & ((\Lambda \cap \Lambda_{loc} \cap \Lambda' \cap K_+ \cap K_s \cap K_r \cap \\ & K_{nbhd} \cap K_{ang} \cap K_{NFC} \cap K_{alg} \cap (A' \times \Omega^-)) \times [0, 1]).\end{aligned}$$

For this set $\tilde{K}$, let ϵ be small, and $\delta' = \frac{1}{100}\hat{m}_\tau(\tilde{K})$ apply Lemma 3.10.15 to ϵ and $\delta = \delta'$ and get a set $E^c \subset Z$ and a constant $s > 0$ where its conclusion holds: For all $c \in E^c$ and $T > s$,

$$\left| \frac{1}{T} \int_0^T \mathbb{1}_{\tilde{K}}(F_{tc}^t(c)) dt - \int_Z \mathbb{1}_{\tilde{K}} d\hat{m}_\tau \right| < \delta', \quad (117)$$

where F_{tc}^t is the ‘time changed’ flow.

Take $\tilde{K}^* = \tilde{K} \cap E^c$.

Note that the definition of QNI only need hold for ℓ such that

$$F^{\ell/2}(\hat{x}_{1/2}), F^{-\ell/2}(\hat{x}_{1/2}) \in K,$$

once we pick $\hat{x}_{1/2} \in K$.

Lemma 4.2.2. *For $(x, \omega, k) \in \tilde{K}^*$, given ℓ such that*

$$F^{-\ell/2}(x, \omega, k), F^{\ell/2}(x, \omega, k) \in \tilde{K}^*,$$

we have that $\eta_{\omega, \ell/2}^s(x)$ contains a ball of radius r' in $W^s(x, \omega)$, i.e. $B^s(x, r')$. This r' can be taken to be

$$r' = \frac{r}{M_{NFC}^2} e^{-t},$$

where r is as in in Lemma 3.10.6, M_{NFC} is as in Lemma 3.10.2, and t is such that $F_{tc}^t(F^{-\ell/2}(x, \omega, k)) = (x, \omega, k)$ and F_{tc} is the ‘time changed’ flow.

Proof. We assume that ℓ is such that $F^{-\ell/2}(x, \omega, k) \in \tilde{K}$. Then $\eta_{\sigma^{-\ell/2}\omega}^s(F_\omega^{-\ell/2}(x))$ contains a ball $B^s(F_\omega^{-\ell/2}(x), r)$ inside $W^s(F^{-\ell/2}(x, \omega, k))$ because $F^{-\ell/2}(x, \omega, k)$ belongs to K_{nbhd} as in Lemma 3.10.6.

Now we pass to (modified) NFCs (see Equation (97), Lemma 3.10.2) since we are working in $W^s(F^{-\ell/2}(x, \omega, k))$ and $F^{-\ell/2}(x, \omega, k) \in \tilde{K}^*$. Since we have that $F^{-\ell/2}(x, \omega, k) \in K_{NFC}$, we have that the image of $B^s(F_\omega^{-\ell/2}(x), r)$ contains a ball of radius

$$r_{new} = \frac{r}{M_{NFC}}.$$

Now let t be such that $F_{tc}^t(F^{-\ell/2}(x, \omega, k)) = (x, \omega, k)$. We know that in (modified) NFCs flowing the stable forwards by t will shrink our ball of radius r_{new} by a factor of e^{-t} precisely. Hence, in (modified) NFCs for $W^s(x, \omega, k)$ we have that the image of our ball of radius r_{new} is a ball of radius $\frac{r}{M_{NFC}}e^{-t}$. When we return to $W^s(x, \omega)$, the image of our ball of radius $\frac{r}{M_{NFC}}e^{-t}$ contains a ball of radius $r' = \frac{r}{M_{NFC}^2}e^{-t}$. This tells us that

$$F^{\ell/2}(B^s(F_\omega^{-\ell/2}(x), r)) \subset F^{\ell/2}(\eta_{\sigma^{-\ell/2}\omega}^s(F_\omega^{-\ell/2}(x))) = \eta_{\omega, \ell/2}^s(x),$$

contains a ball $B^s(x, r')$ where $r' = \frac{r}{M_{NFC}^2}e^{-t}$. This completes the proof. $\square$

Corollary 4.2.3. *For $(x, \omega, k) \in \tilde{K}^*$, given ℓ such that*

$$F^{-\ell/2}(x, \omega, k), F^{\ell/2}(x, \omega, k) \in \tilde{K}^*,$$

then $\eta_{\omega, \ell/2}^s(x)$ contains a ball of radius r' in $W^s(x, \omega)$, i.e. $B^s(x, r')$, where

$$r' \geq \frac{r}{M_{NFC}^2}e^{-(\ell/2+2)(\lambda^++\epsilon)},$$

and M_{NFC} and r are as in Lemmas 3.10.2 and 3.10.6 respectively.

Proof. In Lemma 4.2.2 we found $\eta_{\omega, \ell/2}^s(x)$ always contains a ball $B^s(x, r')$ of radius

$$r' = \frac{r}{M_{NFC}^2} e^{-t}.$$

We can compute the dependence of t on ℓ .

Note that the roof function $\tau(F^{-\ell/2}(x, \omega, k))$, though point dependent, is bounded: for any (x, ω) , $\tau(x, \omega) \in [\lambda^+ - \epsilon, \lambda^+ + \epsilon]$ (see Equation (94)). Flowing forwards by $\ell/2$ applies $n = \lfloor \ell/2 + k \rfloor$ diffeomorphisms by definition of the standard flow. Taking $k = 0$, the smallest number of diffeomorphisms being applied is $n = \lfloor \ell/2 \rfloor$, taking $k = 1$ the largest number of diffeomorphisms being applied is $\lfloor \ell/2 + 1 \rfloor = n + 1$; this independent of the point $(x, \omega, k) \in \tilde{K}^*$. Hence,

$$t \in [n(\lambda^+ - \epsilon), (n + 1)(\lambda^+ + \epsilon)].$$

We then see that in terms of ℓ the smallest r' can be is

$$r' = \frac{r}{M_{NFC}^2} e^{-(n+1)(\lambda^+ + \epsilon)} = \frac{r}{M_{NFC}^2} e^{-(\lfloor \ell/2 \rfloor + 1)(\lambda^+ + \epsilon)}.$$

To remove the floor function we can go further and see that

$$r' \geq \frac{r}{M_{NFC}^2} e^{-(\ell/2 + 2)(\lambda^+ + \epsilon)}.$$

This is independent of the point $(x, \omega, k) \in \tilde{K}^*$. This completes the proof. $\square$

Lemma 4.2.4. *For $(x, \omega, k) \in \tilde{K}^*$ and for given ℓ such that*

$$F^{\ell/2}(x, \omega, k), F^{-\ell/2}(x, \omega, k) \in \tilde{K}^*,$$

the intersection $\eta_{\omega, \ell/2}^s(x) \cap \text{Supp}(m_{x, \omega}^s)$ is infinite.

Proof. Since (x, ω, k) is chosen in $E^c \subset \tilde{K}^*$, we can unpack Lemma 3.10.15, Equation (117):

for $T > s$,

$$(-\delta' + \hat{m}_\tau(\tilde{K}))T < \text{Leb}(\{t \in [0, T] : F_{tc}^t(\hat{x}_{1/2}) \in \tilde{K}\}) < (\delta' + \hat{m}_\tau(\tilde{K}))T,$$

which works out to be

$$0.99\hat{m}_\tau(\tilde{K})T < \text{Leb}(\{t \in [0, T] : F_{tc}^t(\hat{x}_{1/2}) \in \tilde{K}\}) < 1.01\hat{m}_\tau(\tilde{K})T.$$

Note that by Corollary 4.2.3 the set $\eta_{\omega, \ell/2}^s(x) \cap W^s(x, \omega, k)$ contains a neighbourhood of x in $W^s(x, \omega, k)$, $B^s(x, r')$ (which depends only on ℓ).

Take $\mathfrak{r}$ as above. If the point $F_{tc}^{-T}(x, \omega, k) \in \tilde{K}^*$ then there is a point

$$z \in (\text{Supp}(m_{F_{tc}^{-T}(x, \omega, k)}^s) \cap B^s(F_{tc}^{-T}(x, \omega, k), \mathfrak{r})) \setminus B^s(F_{tc}^{-T}(x, \omega, k), \frac{1}{\mathfrak{r}}).$$

We will work in (modified) normal form coordinates (NFCs) as in Equation (97) and Lemma 3.10.2. Let z' represent z in (modified) NFCs. Then the annulus that z' belongs to has outer diameter $M_{NFC}\mathfrak{r}$ and inner diameter $\frac{1}{M_{NFC}\mathfrak{r}}$.

Firstly, distances in (stable modified) NFCs shrink by a factor of e^{-T} when we flow by T under the time change flow F_{tc}^T . So take T_0 large such that

$$M_{NFC}\mathfrak{r}e^{-T_0} < \frac{r'}{M_{NFC}},$$

and such that $T_0 > s$ (where s is chosen such that Equation (117) holds). Let x' represent x in (modified) NFCs.

From the application of Lemma 3.10.15, since $(x, \omega, k) \in \tilde{K}^* \subset E^c$, we get that about 99% of choices of $T \in [0, 2T_0]$ will land us in the set $\tilde{K}$. Hence there is a positive Lebesgue measure choices of $T \in [T_0, 2T_0]$ such that $F_{tc}^{-T}(x, \omega, k)$ lands in $\tilde{K}$. For such T , we get

(ambiently) a point

$$z_0 \in (\text{Supp}(m_{F_{tc}^{-T}(x, \omega, k)}^s) \cap B^s(F_{tc}^{-T}(x, \omega, k), \mathfrak{r})) \setminus B^s(F_{tc}^{-T}(x, \omega, k), \frac{1}{\mathfrak{r}}).$$

Denote z'_0 as z_0 in (modified) NFCs, then let $F_{tc}^T(z'_0)$ be the result of flowing forwards z'_0 by F_{tc}^T under the Ω -component of $F_{tc}^{-T}(x, \omega, k)$. We have that $F_{tc}^T(z'_0)$ belongs to the ball of radius r'/M_{NFC} around x' by construction (which means ambiently that $F_{tc}^T(z_0) \in B^s(x, r')$). Also, we have that the closest $F^T(z'_0)$ can be to x' is $\frac{1}{M_{NFC}\mathfrak{r}}e^{-2T_0}$.

Now take $T_1 > 2T_0 + \ln(M_{NFC}^2\mathfrak{r}^2)$. Then

$$M_{NFC}\mathfrak{r}e^{-T_1} < \frac{1}{M_{NFC}\mathfrak{r}}e^{-2T_0}. \quad (118)$$

Now we pick $T^{(1)} \in [T_1, 2T_1]$ such that $F_{tc}^{-T^{(1)}}(x, \omega, k)$ lands in $\tilde{K}$ using Lemma 3.10.15. We get its corresponding point

$$z_1 \in (\text{Supp}(m_{F_{tc}^{-T^{(1)}}(x, \omega, k)}^s) \cap B^s(F_{tc}^{-T^{(1)}}(x, \omega, k), \mathfrak{r})) \setminus B^s(F_{tc}^{-T^{(1)}}(x, \omega, k), \frac{1}{\mathfrak{r}}).$$

Let z'_1 be z_1 in (modified) NFCs. The farthest away from x' that $F_{tc}^{T^{(1)}}(z'_1)$ can be is $M_{NFC}\mathfrak{r}e^{-T_1}$, but that is still closer than the point $F_{tc}^T(z'_0)$ could be to x' by construction (i.e. by Equation (118)). The closest $F_{tc}^{T^{(1)}}(z'_1)$ could be to x' is $\frac{1}{M_{NFC}\mathfrak{r}}e^{-2T_1}$.

We continue in this way, taking $T_2 > 2T_1 + \ln(M_{NFC}^2\mathfrak{r}^2)$ and $T^{(2)} \in [T_2, 2T_2]$, we get a point z'_2 such that $F_{tc}^{T^{(2)}}(z'_2)$ is closer to x' than $F_{tc}^{T^{(1)}}(z'_1)$ is, in fact the farthest $F_{tc}^{T^{(2)}}(z'_2)$ can be from x' is $M_{NFC}\mathfrak{r}e^{-T_2}$ where

$$M_{NFC}\mathfrak{r}e^{-T_2} < \frac{1}{M_{NFC}\mathfrak{r}}e^{-2T_1}.$$

We get a sequence of choices of $\{T^{(i)}\}_{i \geq 0}$ such that $F^{T^{(i)}}(x, \omega, k) \in \tilde{K}$ and there is a

point

$$z_i \in (\text{Supp}(m_{F_{tc}^{-T(i)}(x, \omega, k)}^s) \cap B^s(F_{tc}^{-T(i)}(x, \omega, k), \mathfrak{r})) \setminus B^s(F_{tc}^{-T(i)}(x, \omega, k), \frac{1}{\mathfrak{r}}).$$

For z'_i corresponding to z_i in (modified) NFCs, we have that $F_{tc}^{T(i)}(z'_i)$ is strictly closer to x' than $F_{tc}^{T(j)}(z'_j)$ for all $j < i$, but is strictly farther away from x' than $F_{tc}^{T(p)}(z'_p)$ for all $p > i$. For each i , $F_{tc}^{T(i)}(z'_i)$ belongs to the (modified) NFCs chart based at (x, ω, k) which is a diffeomorphism. When returning to stable ambient coordinates, we have a sequence of distinct points, approaching x (but not equal to x) that belong to $\text{Supp}(m_{x, \omega}^s) \cap B^s(x, \omega, r')$.

This completes the proof. $\square$

Lemma 4.2.5. *Fix $\eta > 0$ and $\delta > 0$, and let $\epsilon > 0$ be given. There exists a constant $\epsilon_\eta > 0$ and a set K_η of measure at least $(1 - \epsilon)$, such that if $(x, \omega) \in K_\eta$, then in a ball of radius η inside $W^s(x, \omega)$ (denoted $B^s(x, \omega, \eta)$), if one removes a ball of radius ϵ_η the resulting annulus has measure at least $(1 - \delta/2)m_{x, \omega}^s(B^s(x, \omega, \eta))$.*

Proof. It follows from Lemma 4.2.1 that because x is not an atom, for each (x, ω) there is a $\epsilon_\eta(x, \omega)$ such that by removing a ball of radius $\epsilon_\eta(x, \omega)$ the resulting annulus has measure at least $(1 - \delta/2)m_{x, \omega}^s(B(x, \omega, \eta))$. This $\epsilon_\eta(x, \omega)$ is a measurable function because η is fixed. We can then lower bound $\epsilon_\eta(x, \omega)$ by some constant ϵ_η on a set K_η of measure at least $(1 - \epsilon)$. This completes the proof. $\square$

Let $\alpha_0 > 0$ be defined by $\alpha_0 = \frac{1}{2}R(\lambda^+ + \epsilon)$ where R is r as in Lemma 3.3.4. Note that this choice is independent of δ because R is constant a.e.

Let $\eta < \frac{r}{M_{NFC}^4}$ be fixed and consider the set K_η from Lemma 4.2.5. Also recall the set A' from Section 3.10.

Take ϵ_{loc} , ϵ_{nbhd} , ϵ_{ang} , ϵ_φ and all other ϵ -costs to be such that for

$$K^* = \tilde{K}^* \cap (K_\eta \times [0, 1]) \cap (\pi_{X \times \Omega^+}(A') \times \Omega^- \times [0, 1]).$$

we have that $\hat{m}(K^*) > 1 - \delta^{i+1}/32$.

For each $(x, \omega^-, k) \in X \times \Omega^- \times [0, 1[$, let

$$V_{x, \omega^-, k} = \{(\omega')^+ \in \Omega^+ : (x, \omega^-, (\omega')^+, k) \in K^*\},$$

and define

$$G := \{(x, \omega^-, k) \in X \times \Omega^- \times [0, 1[: \mu^{\mathbb{N}}(V_{x, \omega^-, k}) > 1 - \delta/8\}. \quad (119)$$

Lemma 4.2.6. *For the set G defined above, $\nu \times \mu^{\mathbb{N}} \times dt(G) > 1 - \delta^i/4$.*

Proof. Consider G^c which is the set of (x, ω^-, k) such that the subset $W \subset \Omega^+$, defined by the property that if $(\omega')^+ \in W$ then $(x, \omega^-, (\omega')^+, k) \in (K^*)^c$, has μ -measure at least $\delta/8$.

Now let $E \subset (K^*)^c$ be defined by

$$\begin{aligned} E := \{(x, \omega^-, (\omega')^+, k) \in X \times \Omega^- \times \Omega^+ \times [0, 1[: \\ (x, \omega^-, k) \in G^c, (x, \omega^-, (\omega')^+, k) \in (K^*)^c\} \end{aligned}$$

Now using Fubini, note that

$$\begin{aligned} \hat{m}((K^*)^c) &\geq \hat{m}(E) = \int_{X \times \Omega^- \times [0, 1[} \int_{\Omega^+} d\mu^{\mathbb{N}} d(\nu \times \mu^{\mathbb{N}} \times dt) \\ &= \mu^{\mathbb{N}}(W) \cdot (\nu \times \mu^{\mathbb{N}} \times dt)(G^c) \\ &\geq \theta \cdot (\nu \times \mu^{\mathbb{N}} \times dt)(G^c). \end{aligned}$$

Hence $(\nu \times \mu^{\mathbb{N}} \times dt)(G^c) \leq \frac{\hat{m}((K^*)^c)}{\delta/8}$. This gives us our result. $\square$

For each $(x, \omega^+) \in X \times \Omega^+$ let

$$\tilde{V}_{x, \omega^+} := \{(\omega')^+ \in \Omega^+ : (x, \omega^+, (\omega')^+) \in A'\}.$$

Let

$$H' = \{(x, \omega^+) \in X \times \Omega^+ : \mu^{\mathbb{N}}(\tilde{V}_{x, \omega^+}) > 1 - \delta/8\}.$$

By the same argument as in the proof of Lemma 4.2.6 we have that

$$\nu \times \mu^{\mathbb{N}}(H') > 1 - \frac{\epsilon_e}{\delta/8} > 1 - \delta^i/4.$$

Note that the $\eta_\omega^{s/u}(x)$ (defined m -a.e.) are measurable partitions of X . We apply [17, Lemma 3.11], which we state below:

Lemma 4.2.7 ([17], Lemma 3.11). *Let $\delta > 0$, for a.e. $\omega \in \Omega$ given $X'_\omega \subset X_\omega$ with $\nu(X') > 1 - \delta^2$, there exists $X_\omega^* \subset X'$ with $\nu(X_\omega^*) > 1 - \delta$ and such that for $x \in X_\omega^*$ and any $\ell > 0$ we have that*

$$m_{x, \omega}^s(X'_\omega \cap \eta_{\omega, \ell}^s(x)) \geq (1 - \delta)m_{x, \omega}^s(\eta_{\omega, \ell}^s(x)). \quad (120)$$

One can do the same for the unstable partitions.

Let Ω' be the conull subset of Ω where Lemma 4.2.7 holds.

Now let $(x, \omega^+, k) \in X \times \Omega^+ \times [0, 1[$, let

$$V_{x, \omega^+, k} = \{(\omega'')^- \in \Omega^- : (x, (\omega'')^-, \omega^+, k) \in K^*\},$$

and define for a given (ω, k) :

$$X''_{\omega, k} = \{x \in X : \mu^{\mathbb{N}}(V_{x, \omega^+, k}) > 1 - \delta^j\}. \quad (121)$$

Lemma 4.2.8. *There exists a subset $W' \subset \Omega \times [0, 1[$ of measure at least $1 - \delta^{i-2j}/2$ such that for $(\omega, k) \in W'$, $X''_{\omega, k}$ has measure at least $1 - \delta^j$.*

Proof. Using Lemma 4.2.6 we get that

$$Y'' = \{(x, \omega^+, k) : \mu^{\mathbb{N}}(V_{x, \omega^+, k}) > 1 - \delta^j\},$$

has measure $\hat{m}(Y'') > 1 - \delta^{i-j}/2$

Note that $X''_{\omega, k} = \{x \in X : (x, \omega^+, k) \in Y''\}$; let

$$W' = \{(\omega, k) : \nu(X''_{\omega, k}) > 1 - \delta^j\},$$

by Lemma 4.2.6 the set W' has measure $1 - \delta^{i-2j}/2$. □

Let $W = W' \cap (\Omega' \times [0, 1])$. Let

$$K' = ((K_\xi \cap K_+ \cap K_{nbhd}) \times [0, 1]) \cap (G \times \Omega^+) \cap (H' \times \Omega^- \times [0, 1]).$$

Note that $\hat{m}(K') > 1 - \delta^i/2$.

For $(\omega, k) \in W$ let

$$X'_{\omega, k} = \{x \in X : (x, \omega, k) \in K'\} \subset X_\omega \tag{122}$$

$$G' = \{(\omega, k) \in W : \nu(X'_{\omega, k}) > 1 - \delta^j\}. \tag{123}$$

Lemma 4.2.9. *The measure of G' is $\mu^{\mathbb{Z}} \times dt(G') > 1 - \frac{\delta^{i/2}}{(\delta^j - \delta^i/2)}$.*

Proof. Consider $H := W \setminus G'$. Define $U_{(\omega, k)} \subset X$ as the set of x such that $(x, \omega, k) \in (K')^c$. The set H is the set of $(\omega, k) \in W$ such that $U_{(\omega, k)}$ has measure at least δ^j inside W , (hence has measure $\delta^j(1 - \delta^{i-2j}/2) = (\delta^j - \delta^{i-j}/2)$ where $i > j$).

$$E = \{(x, \omega, k) : (\omega, k) \in H, (x, \omega, k) \in (K')^c\} \subset (K')^c.$$

$$\begin{aligned}
\hat{m}((K')^c) &\geq \hat{m}(E) = \int_{\Omega \times [0,1[} \int_X \mathbb{1}_E d\nu d(\mu^{\mathbb{Z}} \times dt) = \int_H \int_{U_{(\omega,k)}} d\nu d(\mu^{\mathbb{Z}} \times dt) \\
&= \int_H \nu(U_{(\omega,k)}) \mu^{\mathbb{Z}} \times dt \geq (\delta^j - \delta^{i-j}/2) \cdot (\mu^{\mathbb{Z}} \times dt)(H)
\end{aligned}$$

Note that $\hat{m}((K')^c) < \delta^i/2$, and so $(\mu^{\mathbb{Z}} \times dt)(H) \leq \frac{\delta^i/2}{(\delta^j - \delta^{i-j}/2)}$. Hence

$$\mu^{\mathbb{Z}} \times dt(G') > 1 - \frac{\delta^i/2}{(\delta^j - \delta^{i-j}/2)}.$$

This completes the proof. □

Let $X'''_{\omega,k} = X'_{\omega,k} \cap X''_{\omega,k}$ and use this in Lemma 4.2.7 to get a subset $X^*_{\omega,k}$ where Equation (120) holds. We know that $X'''_{\omega,k}$ has measure at least $1 - 2\delta^j$. Take

$$K = \left(\bigcup_{(\omega,k) \in G'} X^*_{\omega,k} \times \{\omega\} \times \{k\} \right) \subset K'.$$

Lemma 4.2.10. *Let (X_1, η_1) and (X_2, η_2) be σ -finite measure spaces. Consider $V \subset X_1$, a set such that $\eta_1(V) > 1 - \theta$, and for each point $a \in V$ consider a set $W_a \subset X_2$ such that $\eta_2(W_a) > 1 - \theta'$. Then $K := \bigcup_{a \in V} W_a \times \{a\}$ has measure at least $(1 - \theta')(1 - \theta)$ with respect to the $\eta_1 \times \eta_2$ on $X_1 \times X_2$.*

Proof. Writing $K = \{(w, a) : w \in W_a, a \in V\}$, we apply Fubini's theorem:

$$\eta_1 \times \eta_2(K) = \int_V \int_{W_a} d\eta_2 d\eta_1 = \int_V \eta_2(W_a) d\eta_1 > (1 - \theta')\eta_1(V) > (1 - \theta')(1 - \theta).$$

This completes the proof. □

By Lemma 4.2.10, the measure of K is at least

$$\hat{m}(K) > (1 - 2\delta^j)(1 - \frac{\delta^i/2}{\delta^j - \delta^i/2}) > 1 - \delta.$$

Now take $\hat{x}_{1/2} = (x_{1/2}, \omega_{1/2}, k_{1/2}) \in K$, then $U^+[\hat{x}_{1/2}] = \{x_{1/2}\}$, so

$$\hat{U}^+[\hat{x}_{1/2}] = \{x_{1/2}\} \times \{\omega_{1/2}^-\} \times \Omega^+ \times \{k_{1/2}\}.$$

For any $\ell > 0$,

$$\eta_{\ell/2}^u[\hat{x}_{1/2}] = \eta_{\omega_{1/2}, \ell/2}^u(x_{1/2}) \times \{\omega_{1/2}^-\} \times \Omega^+ \times \{k\}.$$

Hence,

$$\hat{U}^+[\hat{x}_{1/2}] \cap \eta_{\ell/2}^u[\hat{x}_{1/2}] = \hat{U}^+[\hat{x}_{1/2}].$$

Now since $\{x_{1/2}\} \times \{\omega_{1/2}^-\} \times \{k_{1/2}\} \in G$ by construction, we have that the set

$$V_{x_{1/2}, \omega_{1/2}^-, k_{1/2}} \subset \Omega^+,$$

has $\mu^{\mathbb{N}}$ -measure greater than $(1 - \delta/8)$. Also $(x_{1/2}, \omega_{1/2}^+) \in H'$ so $\tilde{V}_{x_{1/2}, \omega_{1/2}^+} \subset \Omega^+$ has $\mu^{\mathbb{N}}$ -measure greater than $(1 - \delta/8)$.

Take

$$S = \{x_{1/2}\} \times \{\omega_{1/2}^-\} \times (V_{x_{1/2}, \omega_{1/2}^-, k_{1/2}} \cap \tilde{V}_{x_{1/2}, \omega_{1/2}^+} \cap U_{x_{1/2}, \omega_{1/2}^+}) \times \{k_{1/2}\},$$

then

$$\hat{m}_{\hat{x}_{1/2}}^u(S) \geq (1 - \delta/4)\hat{m}_{\hat{x}_{1/2}}^u(\hat{U}^+[\hat{x}_{1/2}]).$$

Note that by construction, $S \subset K$ and that this set is clearly dependent on the choice of

$\hat{x}_{1/2} \in K$.

Lemma 4.2.11. *For ℓ large enough, and for $\hat{y}_{1/2} \in S$, we have that if $z_{1/2} \in W^s(\hat{x}_{1/2})$ is such that $d_X(z_{1/2}, x_{1/2}) > \delta'$, for δ' appropriately small, then*

$$d_X(z_{1/2}, W_q^s(\hat{y}_{1/2})) > C'(\delta')^R,$$

(where R is equal to r as in Lemma 3.3.4 and the definition of the set A' in Section 3.10).

Proof. Recall that the X -component of $\hat{y}_{1/2}$ is $x_{1/2}$. Now $z_{1/2}$ is at least δ' away from $x_{1/2} = y_{1/2}$. Since $\hat{x}_{1/2} \in K \subset A'$ and $\hat{y}_{1/2} \in S \subset A'$ we have that $W^s(\hat{x}_{1/2})$ and $W^s(\hat{y}_{1/2})$ are tangent at order $k < R$ and not tangent at order R where R is r from Lemma 3.3.4 and Section 3.10. We take the collection of charts $\{\varphi_x\}_{x \in X}$ from Section 3.10 where we take ℓ large enough so that the scale we work in is appropriate. The Lipschitz constants of the φ_x are bounded above by L' . Here we have that for any $(\omega')^+ \in \Omega_t^+$, $\varphi_{x_{1/2}}(W^s(x_{1/2}, (\omega')^+))$ can be written as a graph

$$\{(z, \psi_{x_{1/2}, (\omega')^+}(z)) : z \in \mathbb{D}_s\}.$$

In particular we can do this for $\varphi_{x_{1/2}}(W^s(\hat{y}_{1/2}))$ and $W^s(\hat{x}_{1/2})$ writing the graphs respectively as

$$\{(z, \psi_{\hat{y}_{1/2}}(z)) : z \in \mathbb{D}_s\}, \quad \{(z, \psi_{\hat{x}_{1/2}}(z)) : z \in \mathbb{D}_s\}.$$

Since $\hat{x}_{1/2} \in A' \times [0, 1[$, we also have the existence of a map $g_{\hat{x}_{1/2}}$ with bounded Lipschitz constant L''' such that $g_{\hat{x}_{1/2}} \circ \varphi_{x_{1/2}}(W^s(\hat{y}_{1/2}))$

can be written as the graph

$$G_{(\omega')^+} = \{(z, (\psi_{\hat{y}_{1/2}}^{(R)}(0) - \psi_{\hat{x}_{1/2}}^{(R)}(0))z^R + h.o.t.) : z \in \mathbb{D}_s\},$$

where $g_{\hat{x}_{1/2}} \circ \varphi_{x_{1/2}}(W^s(\hat{x}_{1/2}))$ is the coordinate axis

$$\{(z, 0) : z \in \mathbb{D}_s\}.$$

Now let $\mathcal{L} = L'L'''$, since $z_{1/2} \in W^s(\hat{x}_{1/2})$ we can write

$$p = g_{\hat{x}_{1/2}} \circ \varphi_{x_{1/2}}(z_{1/2}) = (z', 0),$$

and we have that

$$\frac{1}{\mathcal{L}}\delta' \leq |p| \leq \mathcal{L}\delta'.$$

Since $\pi_{\Omega^+}(S) \subset U_{x_{1/2}, \omega_{1/2}^+}$ we have that

$$(\psi_{\hat{y}_{1/2}}^{(R)}(0) - \psi_{\hat{x}_{1/2}}^{(R)}(0)) > C_\varphi.$$

Also we can take ℓ large enough so that the scale we work on is small enough (i.e. $z_{1/2}$ is closer to $x_{1/2}$) so that for the function defining the second coordinate in $G_{(\omega')^+}$,

$$|(\psi_{\hat{y}_{1/2}}^{(R)}(0) - \psi_{\hat{x}_{1/2}}^{(R)}(0))z^R + h.o.t.| \geq |\frac{1}{2}C_\varphi z^R|.$$

We can do this because of our bounds on the $(R+1)$ -derivatives defining the set $U_{x_{1/2}, \omega_{1/2}^+}$.

Now we ask how far away p is from the function $H(z) = \frac{1}{2}C_\varphi z^R$ for z such that $|z| \leq |p|$, because its clear that for $|z| > p$ that $|H(z)| > |H(p)|$ as $|H(z)|$ increases with increasing $|z|$. It suffices to examine what happens when $|p|$ is as small as possible, i.e. $|p| = |(z', 0)| = \frac{1}{\mathcal{L}}\delta'$.

The squared distance function between $H(z)$ for $|z| < \frac{1}{\mathcal{L}}\delta'$ and the point p is given by

$$d(z) = |z - z'|^2 + |\frac{1}{2}C_\varphi z^R|^2.$$

We start with the case $R > 1$.

Working over the real numbers we let $z = x + iy = (x, y)$, $z' = x' + iy' = (x', y')$ where

$$|z'| = (x')^2 + (y')^2 = \frac{1}{\mathcal{L}} \delta' > x^2 + y^2 = |z|.$$

$$\begin{aligned} d(x, y) &= (x - x')^2 + (y - y')^2 + \left(\frac{C_\varphi}{2}\right)^2 |(x + iy)|^{2R} \\ &= (x - x')^2 + (y - y')^2 + \left(\frac{C_\varphi}{2}\right)^2 (x^2 + y^2)^R. \end{aligned}$$

Let $c = \left(\frac{C_\varphi}{2}\right)^2 > 0$ and let η be such that $y' = \eta x'$. This is true for some $\eta \in]-\infty, \infty[$ (or on the line $x = 0$ in which case we consider $x' = \eta y'$ and take $\eta = 0$). Now it suffices to consider x and y on this line $y = \eta x$.

Plugging this $y = \eta x$ in we see that

$$d(x, \eta x) = (1 + \eta^2)(x - x')^2 + c(1 + \eta^2)^R x^{2R}.$$

Taking the derivative and setting it to zero we see that

$$d'(x) = 2(1 + \eta^2)(x - x') + 2Rc(1 + \eta^2)^R x^{2R-1}.$$

Setting this to zero is equivalent to the expression

$$(x - x') + c(1 + \eta^2)^{R-1} R x^{2R-1} = 0.$$

Setting $D_\eta = c(1 + \eta^2)^{R-1} > 0$, we have that x^* that is a critical point for d satisfies:

$$x^* + D_\eta (x^*)^{2R-1} = x'.$$

Now recall that x' is small because $(x')^2 + (y')^2 = \frac{1}{\mathcal{L}}\delta' \ll 1/2$. Assume first that $x' \geq 0$ and consider the function $h_\eta(x) = x + D_\eta(x)^{2R-1}$ on the interval $[0, 1/2]$. This is a continuous and strictly increasing function (hence injective) whose image contains $[0, 1/2]$ and hence contains x' . Note that

$$h_\eta(x') = x' + D_\eta(x')^{2R-1} > x',$$

Further,

$$\begin{aligned} h_\eta(x'/2) &= \frac{x'}{2} + D_\eta\left(\frac{x'}{2}\right)^{2R-1} \\ &= \frac{x'}{2} + \frac{c(1+\eta^2)^{R-1}}{2^{2R-1}}(x')^{2R-1}. \end{aligned}$$

Note $x' > 0$ and $(x')^2 + (y')^2 = (1+\eta^2)(x')^2 = \frac{\delta'}{\mathcal{L}}$ so $x' = \sqrt{\frac{\delta'}{\mathcal{L}(1+\eta^2)}}$. Plugging this in we find

$$h_\eta(x'/2) = \frac{1}{2} \left(\frac{\delta'}{\mathcal{L}(1+\eta^2)} \right)^{1/2} + \frac{c(\delta')^{(2R-1)/2}}{2^{2R-1}(1+\eta^2)^{1/2}\mathcal{L}^{(2R-1)/2}},$$

where the term

$$\frac{c(\delta')^{(2R-1)/2}}{2^{2R-1}(1+\eta^2)^{1/2}\mathcal{L}^{(2R-1)/2}} \leq \frac{c(\delta')^{(2R-1)/2}}{2^{2R-1}\mathcal{L}^{(2R-1)/2}},$$

and c and $\mathcal{L}$ are constants. Since $R > 1$, δ' can be taken small so that

$$\frac{c(\delta')^{(2R-1)/2}}{2^{2R-1}\mathcal{L}^{(2R-1)/2}} < \frac{1}{4} \left(\frac{\delta'}{\mathcal{L}(1+\eta^2)} \right)^{1/2} = x'/4.$$

Then $h_\eta(x'/2) < x'$ and by the intermediate value theorem, there is a root $x^* \in [x'/2, x']$ for $d(x) = h_\eta(x) - x' = 0$. Note that the second derivative test on d tells us that this is a local minimum on $[0, 1/2]$, let $z^* = x^* + i\eta x^*$, i.e. this minimum. We must have that

$$|H(z^*)| = |H(x^* + i\eta x^*)| > |H(x'/2 + i\eta(x'/2))| = |H(\frac{1}{2}(x' + iy'))| = \frac{1}{2^{R+1}\mathcal{L}^R} C_\varphi(\delta')^R,$$

and so the minimum distance between p and the function H is at least $\frac{1}{2^{R+1}\mathcal{L}^R}C_\varphi(\delta')^R$.

Now we do the case that $x' < 0$. In fact

$$x' = -\sqrt{\frac{\delta'}{\mathcal{L}(1+\eta^2)}}.$$

Since $2R-1$ is odd, then we can run the same argue as for $x' > 0$ except on $[-1/2, 0]$ as h_η is strictly increasing (hence injective) and its image contains x' . We get a critical point for $d(x)$ in $[x', x'/2]$ and draw the same conclusion on the distance between p and H .

Lastly, we do the case $R = 1$. Now $H(z) = \frac{1}{2}C_\varphi z$ and as before $p = (z', 0)$ with $|p| = \frac{1}{\mathcal{L}}\delta'$. Note that $H(z)$ carves out a plane passing through the origin that makes an angle $\theta = \tan^{-1}(\frac{1}{2}C_\varphi)$ with the coordinate plane $\{(z, 0) : z \in \mathbb{D}_s\}$. Now the closest p could be to $H(z)$ is the distance between p and the projection of p onto the plane carved out by H . This distance is given by $|H(z')|\cos(\theta) = \frac{1}{2}C_\varphi \frac{\cos(\theta)}{\mathcal{L}}\delta'$.

Now we let $C' = \min\{\frac{1}{2^{R+1}\mathcal{L}^R}C_\varphi, \frac{1}{2}C_\varphi \frac{\cos(\theta)}{\mathcal{L}}\}$. This completes the proof. $\square$

Now take $\hat{y}_{1/2} = (x_{1/2}, \omega_{1/2}^-, (\omega')^+, k_{1/2}) \in S$.

Since $\hat{x}_{1/2}$ was chosen in K , by construction, $x_{1/2} \in X_{\omega_{1/2}, k_{1/2}}^* \subset X_{\omega_{1/2}, k_{1/2}}'''$ where $(\omega_{1/2}, k_{1/2}) \in W$. Hence by Lemma 4.2.7 we have that

$$m_{x_{1/2}, \omega_{1/2}}^s(X_{\omega_{1/2}, k_{1/2}}''' \cap \eta_{\omega_{1/2}, \ell/2}^s(x_{1/2})) \geq (1 - \sqrt{2}\delta^{j/2})m_{x_{1/2}, \omega_{1/2}}^s(\eta_{\omega_{1/2}, \ell/2}^s(x_{1/2})). \quad (124)$$

Let

$$\tilde{G} = \bigcup_{z \in X_{\omega_{1/2}, k_{1/2}}'''} \{z\} \times V_{z, \omega_{1/2}^+, k_{1/2}} \times \{\omega_{1/2}^+\} \times \{k_{1/2}\}.$$

Take $\tilde{S}' = \eta_{\ell/2}^s[\hat{x}_{1/2}] \cap \tilde{G}$. By Lemma 4.2.10 and the conditions on i and j it has measure at least $(1 - \delta/2)$ relative to the measure of $\eta_{\ell/2}^s[\hat{x}_{1/2}]$.

Lemma 4.2.12. *There exists a constant $\tilde{\epsilon} > 0$ small such that for all $\hat{x}_{1/2} \in K$ and ℓ large*

enough, if we assume that $F^{\ell/2}(\hat{x}_{1/2}), F^{-\ell/2}(\hat{x}_{1/2}) \in K$, then removing a ball $B^s(x_{1/2}, \tilde{r}')$ from $\eta_{\ell/2, \omega_{1/2}}^s(x_{1/2})$ only reduces the measure of $\eta_{\ell/2}^s[\hat{x}_{1/2}] \cap \tilde{G}$ by at most $\delta/2$.

Proof. We know by Lemma 4.2.2 that $\eta_{\ell/2}^s(x_{1/2})$ contains a ball of radius r' , $B^s(x, r')$. We got an explicit formula for this r' as

$$r' = \frac{r}{M_{NFC}^2} e^{-t}.$$

Here, r is as in in Lemma 3.10.6, M_{NFC} is as in Lemma 3.10.2, and t is such that $F_{tc}^t(F^{-\ell/2}(\hat{x}_{1/2})) = \hat{x}_{1/2}$ and F_{tc} is the ‘time changed’ flow.

Since $F^{-\ell/2}(\hat{x}_{1/2}) \in K \subset K_\eta$ from Lemma 4.2.5, in a ball of radius η around $F^{-\ell/2}(x_{1/2})$, if we remove a ball of radius ϵ_η , then the measure of the resulting annulus is

$$(1 - \delta/2) m_{F^{-\ell/2}(\hat{x}_{1/2})}^s(B^s(F^{-\ell/2}(x_{1/2}), \eta)).$$

Since $m_{x, \omega}^s$ has the property outlined in Equation (17), the proportion (in measure) between the images of annulus and the ball of radius η stays the same under the dynamics. We now wish to flow forwards by $\ell/2$ back to the point $\hat{x}_{1/2}$ and shrink this ball of radius η to be inside our ball of radius r' around $x_{1/2}$.

When we pass to (modified) NFCs at $\hat{x}_{1/2}$, the smallest the image of $B^s(x_{1/2}, r')$ could be is a ball of radius $\frac{r'}{M_{NFC}}$. Similarly, the largest the ball of radius η in (modified) NFCs at $F^{-\ell/2}(x_{1/2})$ could be is a ball of radius $\frac{\eta}{M_{NFC}}$. Then to guarantee that after flowing by ℓ (or t under ‘time changed’ flow) that

$$F^{\ell/2}(B^s(F^{-\ell/2}(x_{1/2}), \eta)) \subset B^s(\hat{x}_{1/2}, r'),$$

we need that

$$\eta M_{NFC} e^{-t} < \frac{r'}{M_{NFC}} = \frac{r}{M_{NFC}^3} e^{-t}.$$

This holds true because η was fixed such that $\eta < \frac{r}{M_{NFC}^4}$ at the beginning of the argument.

Now the smallest that the image our ball of radius ϵ_η could be in (modified) NFCs is a ball of radius $\frac{1}{M_{NFC}}\epsilon_\eta$. We flow forwards back to the point $x_{1/2}$ by going forwards by ℓ , this ball shrinks to size $\frac{1}{M_{NFC}}\epsilon_\eta e^{-t}$.

Returning to ambient coordinates, we have that $B^s(\hat{x}_{1/2}, r')$ which is of radius $r' = \frac{r}{M_{NFC}^2}e^{-t}$ contains a ball of radius $\epsilon'_\eta = \frac{1}{M_{NFC}^2}\epsilon_\eta e^{-t}$. If we remove this ball of radius ϵ'_η , by construction this reduces the measure of $\eta_{\ell/s}^s[\hat{x}_{1/2}] \cap \tilde{G}$ by at most $\delta/2$. Let $\tilde{\epsilon} = \frac{\epsilon_\eta}{r}$, then $\tilde{\epsilon}r' = \epsilon'_\eta$ and this constant $\tilde{\epsilon}$ gives us what we wanted. This completes the proof. $\square$

Let S' be the subset of $\eta_{\ell/2}^s[\hat{x}_{1/2}] \cap \tilde{G}$ that we get from Lemma 4.2.12. It has the appropriate measure by construction.

Take a point $\hat{z}_{1/2} = (z_{1/2}, (\omega'')^-, \omega_{1/2}^+, k) \in S'$. Since $S' \subset K$, $U^+[\hat{z}_{1/2}] = z_{1/2}$ which lives on $W^s(\hat{x}_{1/2})$. Since we removed an annulus (in the induced metric d_X^s) of radius $\tilde{\epsilon}r'$ around $x_{1/2}$, we have that $d_X(z_{1/2}, x_{1/2}) > \frac{1}{M_{NFC}^2}\tilde{\epsilon}r'$ (because this is the distance in X rather than the induced metric, see Lemma 3.10.2). By Lemma 4.2.11, we have that

$$d_X(z_{1/2}, W^s(\hat{y}_{1/2})) \geq C' \left(\frac{\tilde{\epsilon}}{M_{NFC}^2} \right)^R (r')^R. \quad (125)$$

We know by Corollary 4.2.3 that $r' \geq \frac{r}{M_{NFC}^2}e^{-(\ell/2+2)(\lambda^++\epsilon)}$. Plugging that in we get

$$\begin{aligned} d_X(z_{1/2}, W^s(\hat{y}_{1/2})) &\geq C' \left(\frac{\tilde{\epsilon}}{M_{NFC}^2} \right)^R \left(\frac{r}{M_{NFC}^2}e^{-(\ell/2+2)(\lambda^++\epsilon)} \right)^R \\ &= \left(C' \left(\frac{\tilde{\epsilon}}{M_{NFC}^2} \right)^R \left(\frac{r}{M_{NFC}^2} \right)^R e^{-2(\lambda^++\epsilon)R} \right) e^{-\ell/2(\lambda^++\epsilon)R} \\ &= \left(C' \left(\frac{\tilde{\epsilon}}{M_{NFC}^2} \right)^R \left(\frac{r}{M_{NFC}^2} \right)^R e^{-2(\lambda^++\epsilon)R} \right) e^{-\ell\alpha_0}, \end{aligned} \quad (126)$$

where the last inequality is because we took $\alpha_0 = \frac{1}{2}R(\lambda^+ + \epsilon)$.

Take $C = C' \left(\frac{\tilde{\epsilon}}{M_{NFC}^2} \right)^R \left(\frac{r}{M_{NFC}^2} \right)^R e^{-2(\lambda^+ + \epsilon)R}$, this is independent of δ . This completes the proof of QNI and Lemma 2.6.3. $\square$

4.3 Outline of the proof of Lemma 2.6.7

Overview

In this subsection we are in the setting of Alternative 2, (see Section 2.6). Here, $d_+ = 1$ a.e. We assume further that $\dim(\mathcal{L}^s) = 2$.

We give $\alpha_0 > 0$ at the beginning, let $\delta > 0$, and we demonstrate that the QNI definition, i.e. Definition 3, holds in our setting. Even though ℓ_0 and C as in this definition must be given alongside K , we have to compute what they must be near the end of the section, but they do not depend on the choice of $\hat{x}_{1/2} \in K$.

We open with defining a good set K' that we want the set K to be a subset of. This set K is where we will choose $\hat{x}_{1/2} = (x_{1/2}, \omega_{1/2}, k_{1/2})$ from as in Definition 3. Note that we need to pick $S \subset \eta^u[\hat{x}_{1/2}] \cap (U^+[\hat{x}_{1/2}] \times W_1^+)$ which means that we are picking a new X -component that lives along $U^+[x, \omega, k]$ and a new future but leaving the $\Omega^- \times [0, 1]$ -component the same. Note that in the last subsection, the setting of Lemma 2.6.3 assumed that $U^+[x, \omega, k] = \{x\}$ a.e., this is no longer the case, here $\dim(U^+[x, \omega, k]) = 1$. The set S' (dependent on the choice of $\hat{y}_{1/2} \in S$) is a subset of $\eta_{\omega_{1/2}, \ell/2}^s(x_{1/2})$ meaning that we pick a new X -component belonging to $W^s(\hat{x}_{1/2})$, a new past component, and keep $\Omega^+ \times [0, 1]$ -component the same. We work to ensure that $S, S' \subset K'$ so they have good properties.

We then prove Lemma 4.4.9 which demonstrates the QNI condition, with Lemma 4.4.10 as an intermediate linear approximation step. To get the idea, first think of $W^s(\hat{y}_{1/2})$ and $W^s(\hat{x}_{1/2})$ as a fixed planes. Then think of $U^+[z_{1/2}, \omega^-]$ for $(z_{1/2}, \omega, k) \in S'$ as a straight line. Vary the X -component $z_{1/2}$ throughout some ball B centered around $x_{1/2}$ in $W^s(\hat{x}_{1/2})$ and imagine that for any $z'_{1/2}$ in this ball, $U^+[z'_{1/2}, \omega^-]$ is a straight line with the same direction

vector as $U^+[z_{1/2}, \omega^-]$. The set of $z'_{1/2} \in B$ such that $U^+[z'_{1/2}, \omega^-]$ will intersect $W^s(\hat{y}_{1/2})$ is a 1-dimensional set. However $\dim(\mathcal{L}^s) = 2$ and our measure $m_{x,\omega}^s$ cannot be supported on an algebraic curve by Lemma 4.4.1. This allows us to get rid of this 1-dimensional set (for a choice of ω^-) in the definition of S' . We then add back in the error terms we get from making these approximations and use Lemma 4.4.3 to say we can eliminate a tubular neighbourhood of an algebraic curve and maintain the necessary measure required for S' . In this way we have that for $\hat{y}_{1/2} \in S$ and $\hat{z}_{1/2} \in S'$ (where S' is dependent on $\hat{y}_{1/2}$) $U^+[\hat{x}_{1/2}]$ is sufficiently bounded away from $W^s(\hat{y}_{1/2})$ and hence have the QNI condition.

Details

Again, we are in the setting of Alternative 2, i.e. Section 2.6. Here, $\dim(\mathcal{L}^s) = 2$ and $d_+ = 1$ a.e. We will give α_0 at the beginning, let $\delta > 0$ be given and define a good set K' with good properties from good sets defined in Sections 3.7 and 3.10. We add to K' another good property that for fixed degree there is some lower bound on the size of tubular neighbourhood around an algebraic curve outside of which we still have large measure. This comes from Lemma 4.4.3. Note that C and ℓ_0 from Definition 3 need to be chosen alongside K . We compute when C and ℓ_0 need to be near the end but they will be independent of $\hat{x}_{1/2}$ and hence can be chosen alongside K .

In order to make sure that S and S' belong to K' as well, we have to do some work to construct K . We first construct K_1 and if $K \subset K_1$ then S' can be constructed to belong to K' . We employ a Lemma of [17], Lemma 4.4.5, that allows us to say for a.e. ω and for a large measure subset X''' of X , there is a large measure subset X^* such that if $x \in X^*$ then the measure of X''' inside $\eta_{\omega,\ell}^s(x)$ compared to the measure of $\eta_{\omega,\ell}^s(x)$ is large. We define a set $X''_{\omega,k}$ that is the set of X such that there is a large set of Ω^- -components to choose from such that if we keep k and ω^+ the same we belong to K' . We then demonstrate in Lemma 4.4.6 that the set of (ω, k) such that $X''_{\omega,k}$ is large in measure is also large in measure. We

take these (ω, k) and restrict to the conull subset of Ω such that Lemma 4.4.5 holds and call this W . This construction allows us to change the past and land in a good set. Now we focus on changing the X -component and landing in a good set. We then let $X'_{\omega,k} \subset X$ be the set of X -components y such that $(y, \omega, k) \in K'$. We let G be the set of $(\omega, k) \in W$ such that $X'_{\omega,k}$ is large in measure. We then demonstrate in Lemma 4.4.7 that the size of G is big. We let $X'''_{\omega,k} = X'_{\omega,k} \cap X''_{\omega,k}$ which is large for $(\omega, k) \in G$ and apply Lemma 4.4.5 to get a large measure subset $X^*_{\omega,k}$. We define K_1 such that the X -component belongs to this set $X'''_{\omega,k}$ for $(\omega, k) \in G$.

We repeat this process to ensure that S can be constructed to belong to K' . Here we use a modified version of Lemma 4.4.5 which also comes from [17, Lemma 6.3]. We state this as our Lemma 4.4.8. This allows us to say that if we have a large subset $X''' \subset X$, there is a subset X^* such that restricting to the unstable supports this subset still has relatively large measure (relative to the conditional measure denoted $|\cdot|$ on the unstable support). This is used to ensure that we can change X -components from $\hat{x}_{1/2}$ to some element on $U^+[\hat{x}_{1/2}]$ and land in a good set. We also have to change futures and we go about that in the same way as we did above in the definition of $(X''_{\omega,k})^+$. We construct the set K_2 which is analogous to K_1 and take $K = K_1 \cap K_2$. Now when we construct S and S' they will belong to K' .

Lemma 4.4.9 works to demonstrate the QNI condition directly. First we do a linear approximation step. For a fixed $(\omega')^- \in \Omega^-$, the tangent lines to the $U^+[z, (\omega')^-]$ as we move z around a small ball around $x_{1/2}$ inside $W^s(\hat{x}_{1/2})$ vary Hölder continuously. If we fix a direction vector v representing such a tangent line and drag it around a small ball inside $E^s(\hat{x}_{1/2})$ we demonstrate the the set of points inside this small ball where the line in direction v hits the tangent plane through $\varphi_{\hat{y}_{1/2}}(W^s(\hat{y}_{1/2}))$ is 1-dimensional, i.e. is some line in the ball inside the tangent plane $E^s(\hat{x}_{1/2})$. This is Lemma 4.4.10. Given that $\dim(\mathcal{L}^s) = 2$ and that $m^s_{\hat{x}_{1/2}}$ cannot be supported on some algebraic curve in modified NFCs (Lemma 4.4.1), we are almost done up to some error terms, but we must make sure that this line, and some

error around it, does not support too much measure. Further we need to ensure the QNI bound holds. Hence, after picking an appropriate scale to work in, Lemmas 4.4.13, 4.4.16, 4.4.18, 4.4.19 work to achieve this. Finally in Lemma 4.4.21 we arrive with a size of bad neighbourhood in modified NFCs charts where if we remove this bad neighbourhood we will have the QNI bound we desire. Lemmas 4.4.22 and 4.4.23 work to show that removing this bad neighbourhood doesn't change the measure by very much and so we can define S' to be sufficiently large. Then we conclude.

4.4 Proof of Lemma 2.6.7

Proof. (Lemma 2.6.7)

Fix $\alpha_0 := (\lambda^+ + \epsilon) > 0$ and let $\delta > 0$, we will determine ℓ_0 and C as in Definition 3.

We are given that $\dim(U^+[x, \omega, k]) = 1$ a.e. and that $\dim(\mathcal{L}^s) = 2$ a.e. We call the set where these two conditions are true K_+ . We restate Lemma 4.2.1 from [6, Section 4, Lemma 4.4.10].

Lemma 4.4.1. *For m -a.e. $(x, \omega) \in Y$ and for all $R > 0$, let $B(x, \omega, R)$ be a ball of radius R inside $H_{(x, \omega)}^s(W^s(x, \omega))$, $(H_{(x, \omega)}^s)_*(m_{x, \omega}^s)|_{B(x, \omega, R)}$ does not assign mass to any algebraic curve of degree d .*

Proof. See [6], Section 4, Lemma 4.4.10. □

Let the conull set where Lemma 4.4.1 holds be called $K_C \subset Y$.

Recall the set K_{nbhd} and the constant $\tilde{r}$ from Lemma 3.10.6. Also recall that $B^s(x, \omega, R)$ is a ball of radius R inside $W^s(x, \omega)$ in the induced metric.

Lemma 4.4.2. *For fixed $R > \tilde{r}$ and given $\epsilon_R > 0$, there exists a set $K_R \subset K_{nbhd} \subset Y$ of measure at least $(1 - \epsilon_R)m(K_{nbhd})$, and a constant ρ such that for $(x, \omega) \in K_R$, $\eta_\omega^s(x) \subset B^s(x, \omega, R)$ has measure at least $\rho \cdot m_{x, \omega}^s(B^s(x, \omega, R))$.*

Proof. For $(x, \omega) \in K_{nbhd}$, $\eta_\omega^s(x)$ is contained in $B^s(x, \omega, \tilde{r})$ and hence also in $B^s(x, \omega, R)$. For this fixed R , the map $(x, \omega) \mapsto \rho_{x, \omega, R}$ where $m_{x, \omega}^s(\eta_\omega^s(x)) = \rho_{x, \omega, R} m_{x, \omega}^s(B^s(x, \omega, R))$ is a measurable mapping. Then we can find a set $K_R \subset K_{nbhd}$ of measure $(1 - \epsilon_R)$ on which $\rho_{x, \omega, R}$ is bounded from below by some ρ . This completes the proof. $\square$

Lemma 4.4.3. *Fix $d \geq 1$ an integer and $R' > 0$. For any $\delta \in]0, 1[$, $(x, \omega) \in K_C$, there exists $\epsilon > 0$ such that given any algebraic curve of degree d in a ball of radius R' , $B(x, \omega, R') \subset H_{(x, \omega)}^s(W^s(x, \omega))$, outside of a tubular neighbourhood of diameter ϵ we have $(H_{(x, \omega)}^s)_* m_{x, \omega}^s$ -measure at least $(1 - \delta) \cdot (H_{(x, \omega)}^s)_* m_{x, \omega}^s(B^s(x, \omega, R'))$.*

Proof. Suppose not, then there exists δ and $(x, \omega) \in K_C$ such that for all $\epsilon > 0$, there exists an algebraic curve C of degree d in the ball of radius R' , $B(x, \omega, R') \subset H_{(x, \omega)}^s(W^s(x, \omega))$ such that outside a tubular neighbourhood of radius ϵ around C we have less than $(1 - \delta) \cdot (H_{(x, \omega)}^s)_* m_{x, \omega}^s(B(x, \omega, R'))$. Note that the space of algebraic curves of degree d is compact.

Take a sequence of $\epsilon_n \rightarrow 0$, each has an algebraic curve C_n such that inside of a ϵ_n tubular neighbourhood of C_n we have measure at greater than $(1 - \delta)(H_{(x, \omega)}^s)_* m_{x, \omega}^s(B(x, \omega, R'))$. The sequence of C_n has a convergent subsequence by compactness, and so we have some limiting curve C . This limiting curve C supports mass which is a contradiction to Lemma 4.4.1. This completes the proof. $\square$

Lemma 4.4.4. *Given $R' > R'' > 0$ large enough and $\epsilon > 0$, there exists a set $K_m \subset X \times \Omega$ of measure at least $(1 - \epsilon)$ and $\rho' > 0$ such that if $(x, \omega) \in K_m$, a ball of radius R' denoted $B(x, \omega, R') \subset H_{(x, \omega), new}^s(W^s(x, \omega))$ is such that*

$$(H_{(x, \omega), new}^s)_*(m_{x, \omega}^s)(B(x, \omega, R'')) \geq \rho' (H_{(x, \omega), new}^s)_*(m_{x, \omega}^s)(B(x, \omega, R')).$$

Proof. Since $R' > R''$ are large enough, we can assume that the ball of radius R'' carries measure.

Then fixing the $R' > R''$ given, for a.e. (x, ω) we get a $\rho'_{(x, \omega)}$ such that

$$(H^s_{(x, \omega), new})_*(m^s_{x, \omega})(B(x, \omega, R'')) \geq \rho'_{(x, \omega)}(H^s_{(x, \omega), new})_*(m^s_{x, \omega})(B(x, \omega, R')).$$

this $\rho'_{(x, \omega)}$ is a measurable function and hence for a given ϵ we can bound it from below on a set of measure $(1 - \epsilon)$ which we denote K_m .

□

Fix $i > j > 2$ such that $i \gg 2j$, and

$$(1 - 2\delta^j)(1 - \frac{\delta^i/2}{(\delta^j - \delta^i/2)}) > 1 - \delta/4,$$

$$(1 - \delta^j)(1 - \sqrt{2}\delta^{j/2}) > (1 - \delta^k),$$

where k is such that $(1 - \delta^k)^2 > 1 - \delta/4$.

Fix $R > M_{NFC}^4 \tilde{r}$ (where $\tilde{r}$ is as in Lemma 3.10.6 for some given ϵ_{nbhd} defining the set K_{nbhd}) large enough so that $R'' < \frac{R}{M_{NFC}^3}$ and $R' > M_{NFC}^3 R$ satisfy Lemma 4.4.4 with some $\rho' > 0$ and set K_m . Take this R in Lemma 4.4.2 with sufficiently small ϵ_R and take the resulting ρ . Let δ'' be such that $\frac{\delta''}{\rho} < \delta^k$ and let δ' be such that $\frac{\delta'}{\rho} < \delta''$.

Take $\delta = \delta'$, R' (as above) and $d = 1$ in Lemma 4.4.3. One can get a Lusin set $K_{C, R}$ on which the ϵ from Lemma 4.4.3 is bounded below by ϵ^* .

Recall the Oseledets' set Λ (conull) and Λ_{loc} from Section 3.5; also recall K_r , K_s and K_φ from Section 3.10 from Lemmas 3.10.1, 3.10.5, 3.10.4, and $K_\xi = \Lambda \cap \Lambda_{loc} \cap K_r \cap K_s$ from Lemma 3.10.7 and Lemma 3.10.10. Belonging to K_ξ , Lemma 3.10.7 tells us that on a ball of radius r ambiently in X , the error ξ between the stable/unstable manifolds by their tangent spaces is $\xi = r^2 M_r^3 L$ where M_r is fixed from Lemma 3.10.1, $L = \text{Lip}(Dh^{s/u}_{(x, \omega)})$, and $h^{s/u}_{(x, \omega)}$ is the holomorphic graph of a stable/unstable manifold in its Lyapunov chart $\varphi_{(x, \omega)}$.

In Lyapunov charts, tangent planes differ from these holomorphic graphs by a factor of

$\epsilon = r^2/L$ on a ball of radius r in these charts. Belonging to $K'_\xi \subset K_\xi$ as in Lemma 3.10.9 tells us that we can bound L from above by some H . Also similar for the tangent lines of $U^+[x, \omega]$ using Lemma 3.10.10. Belonging to Λ_{loc} will ensure the existence of local stable manifolds of some size q . Also recall the set K_β from Section 3.7.

Let

$$K' = \Lambda \cap \Lambda_{loc} \cap K_+ \cap K_r \cap K_s \cap K_\varphi \cap K'_\xi \cap K_\beta \cap K_m \cap K_{C,R} \cap K_{nbhd} \cap K_R$$

and be such that it has measure $\hat{m}(K') > 1 - \delta^i/2$.

Now we apply Lemma 4.2.7 as we did in the proof of 2.6.3, we recall the statement below:

Lemma 4.4.5 ([17], Lemma 3.11). *Let $\delta > 0$, for a.e. $\omega \in \Omega$ given $X'_\omega \subset X_\omega$ with $\nu(X') > 1 - \delta^2$, there exists $X_\omega^* \subset X'$ with $\nu(X_\omega^*) > 1 - \delta$ and such that for $x \in X_\omega^*$ and any $\ell > 0$ we have that*

$$m_{x,\omega}^s(X'_\omega \cap \eta_{\omega,\ell}^s(x)) \geq (1 - \delta)m_{x,\omega}^s(\eta_{\omega,\ell}^s(x)). \quad (127)$$

Let Ω' be the conull subset of Ω where Lemma 4.4.5 holds.

Let

$$V_{x,\omega,k} = \{(\omega')^- \in \Omega^- : (x, (\omega^-)', \omega^+, k) \in K'\},$$

and

$$X''_{\omega,k} = \{x \in X : \mu^\mathbb{N}(V_{x,\omega,k}) > 1 - \delta^j\}.$$

Lemma 4.4.6. *There exists a subset $W' \subset \Omega \times [0, 1[$ of measure at least $1 - \delta^{i-2j}/2$ such that for $(\omega, k) \in W'$, $X''_{\omega,k}$ has measure at least $1 - \delta^j$.*

Proof. Same proof as in Lemma 4.2.8. □

Let $W = W' \cap (\Omega' \times [0, 1])$.

For $(\omega, k) \in W$ let

$$X'_{\omega,k} = \{x \in X : (x, \omega, k) \in K'\} \subset X_\omega,$$

$$G = \{(\omega, k) \in W : \nu(X'_{\omega,k}) > 1 - \delta^j\}. \quad (128)$$

Lemma 4.4.7. *The measure of G is $\mu^{\mathbb{Z}} \times dt(G) > 1 - \frac{\delta^i/2}{(\delta^j - \delta^i/2)}$.*

Proof. This is the same as the proof of Lemma 4.2.9. □

Let $X'''_{\omega,k} = X'_{\omega,k} \cap X''_{\omega,k}$ and use this in Lemma 4.4.5, get a subset $X^*_{\omega,k}$ where Equation (120) holds. For $(\omega, k) \in G$ we know that $X'_{\omega,k}$ has measure at least $1 - \delta^j$, and that $X''_{\omega,k}$ has measure at least $1 - \delta^j$. This implies that $X'''_{\omega,k}$ has measure at least $1 - 2\delta^j$.

Then take

$$K_1 = \left(\bigcup_{(\omega,k) \in G} X^*_{\omega,k} \times \{\omega\} \times \{k\} \right) \subset K'.$$

By Lemma 4.2.10, we have that this has measure

$$\hat{m}(K_1) > (1 - 2\delta^j)(1 - \frac{\delta^i/2}{(\delta^j - \delta^i/2)}) > 1 - \delta/4.$$

We can do something very similar for the unstable supports. Let

$$\mathcal{B}_\ell[x] := U^+[x, \omega] \cap \eta_{\omega, \ell}^u(x),$$

as in Section 6 of [17]. Additionally, define $|\cdot|$ as the conditional measure on $\mathcal{B}_0[x]$.

Lemma 4.4.8 ([17], Lemma 6.3). *Let $\delta > 0$, for a.e. $\omega \in \Omega$, given $(X'_\omega)^+ \subset X_\omega$ with $\nu(K) > 1 - \delta^2$, there exists a subset $(X^*_\omega)^+ \subset (X'_\omega)^+$ with $\nu(K^*) > 1 - \delta$ such that for any*

$x \in (X_\omega^*)^+$ and $\ell > 0$ we have

$$|(X_\omega^*)^+ \cap \mathcal{B}_\ell[x]| \geq (1 - \delta)|B_\ell[x]|. \quad (129)$$

We do as before, let $(\Omega')^+$ be the conull subset of Ω where Lemma 4.4.8 holds.

Let

$$V_{x,\omega,k}^+ = \{(\omega')^+ \in \Omega^+ : (x, \omega^-, (\omega^+)', k) \in K'\},$$

then set

$$(X''_{\omega,k})^+ = \{x \in X : \mu^\mathbb{N}(V_{x,\omega,k}^+) > 1 - \delta^j\}.$$

Lemma 4.2.8 tells us that there is a subset $(W')^+ \subset \Omega \times [0, 1[$ of measure at least $1 - \delta^{i-2j}/2$ such that for $(\omega, k) \in W'$, $(X''_{\omega,k})^+$ has measure at least $1 - \delta^j$.

Let $W^+ = (W')^+ \cap ((\Omega')^+ \times [0, 1])$

For $(\omega, k) \in W^+$ let

$$(X'_{\omega,k})^+ = \{x \in X : (x, \omega, k) \in K'\} \subset X_\omega$$

$$G^+ = \{(\omega, k) \in W : \nu((X'_{\omega,k})^+) > 1 - \delta^j\}.$$

Lemma 4.2.9 gives us that $\mu^\mathbb{Z} \times dt(G^+) > 1 - \frac{\delta^i/2}{(\delta^j - \delta^i/2)}$.

Now let

$$(X'''_{\omega,k})^+ = (X'_{\omega,k})^+ \cap (X''_{\omega,k})^+,$$

use this in Lemma 4.4.8 to get a subset $(X^*_{\omega,k})^+$ where Equation (129) holds. For $(\omega, k) \in G^+$ we know that $(X'_{\omega,k})^+$ has measure at least $1 - \delta^j$ and that $(X''_{\omega,k})^+$ has measure at least $1 - \delta^j$ so $(X'''_{\omega,k})^+$ has measure at least $1 - 2\delta^j$.

Take

$$K_2 = \left(\bigcup_{(\omega,k) \in G^+} (X^*_{\omega,k})^+ \times \{\omega\} \times \{k\} \right) \subset K'.$$

By Lemma 4.2.10 we know that the measure of K_2 is

$$\hat{m}(K_2) > (1 - 2\delta^j)(1 - \frac{\delta^i/2}{(\delta^j - \delta^i/2)}) > 1 - \delta/4.$$

Let

$$K = K_1 \cap K_2, \quad \nu(K) > 1 - \delta. \quad (130)$$

Now take $\hat{x}_{1/2} = (x_{1/2}, \omega_{1/2}, k_{1/2}) \in K$, here

$$\begin{aligned} \hat{U}^+[\hat{x}_{1/2}] &= U^+[x, \omega, k] \times \{\omega_{1/2}^-\} \times \Omega^+ \times \{k_{1/2}\}, \\ \eta_{\ell/2}^u[\hat{x}_{1/2}] &= \eta_{\omega_{1/2}^-, \ell/2}^u(x_{1/2}) \times \{\omega_{1/2}^-\} \times \Omega^+ \times \{k_{1/2}\}. \end{aligned}$$

Let

$$Y' := \bigcup_{y \in (X_{\omega_{1/2}, k_{1/2}}''')^+} \{y\} \times V_{y, \omega_{1/2}, k_{1/2}}^+.$$

By Lemma 4.2.10, the measure of Y' is at least $(1 - \delta^j)(1 - \sqrt{2\delta^j}) > (1 - \delta^k)$

Let

$$\tilde{S} := (Y' \cap ((\eta_{\omega_{1/2}^-, \ell/2}^u(x_{1/2}) \cap U^+[x_{1/2}, \omega_{1/2}, k_{1/2}]) \times \Omega^+)) \times \{\omega_{1/2}^-\} \times \{k_{1/2}\}.$$

Note again that by Lemma 4.4.8, since $x_{1/2} \in (X_{\omega_{1/2}, k_{1/2}}^*)^+$,

$$\begin{aligned} m_{x_{1/2}, \omega_{1/2}}^u((X_{\omega_{1/2}, k_{1/2}}''')^+ \cap \eta_{\omega_{1/2}^-, \ell/2}^u(x_{1/2}) \cap U^+[x_{1/2}, \omega_{1/2}, k_{1/2}]) &\geq \\ (1 - \delta^{j/2})m_{x_{1/2}, \omega_{1/2}}^u(\eta_{\omega_{1/2}^-, \ell/2}^u(x_{1/2}) \cap U^+[x_{1/2}, \omega_{1/2}, k_{1/2}]). \end{aligned}$$

And so, altogether,

$$\hat{m}_{\hat{x}_{1/2}}^u(\tilde{S}) \geq (1 - \delta/2) \hat{m}_{\hat{x}_{1/2}}^u(\eta_{\ell/2}^u[\hat{x}_{1/2}] \cap \hat{U}^+[x_{1/2}, \omega_{1/2}, k_{1/2}]).$$

Additionally, $\tilde{S} \subset K'$ by construction because of how $V_{y, \omega_{1/2}, k_{1/2}}^+$ is defined.

Note that in Definition 3, S is allowed to be dependent on the choice of ℓ and $\hat{x}_{1/2}$. We now (at the cost of at most $\delta/2 \cdot \hat{m}_{\hat{x}_{1/2}}^u(\eta_{\ell/2}^u[\hat{x}_{1/2}] \cap \hat{U}^+[x_{1/2}, \omega_{1/2}, k_{1/2}])$ in the $\hat{m}_{\hat{x}_{1/2}}^u$ -measure of $\tilde{S}$) use Theorem 3.4.1 to ask that P and Q are bounded apart by some angle θ . We call this set $S \subset \tilde{S}$.

Now let $\hat{y}_{1/2} \in S \subset K'$, it is of the form $\hat{y}_{1/2} = (y_{1/2}, \omega_{1/2}^-, (\omega')^+, k_{1/2})$.

Let

$$Y'' = \bigcup_{y \in X_{\omega_{1/2}, k_{1/2}}'''} \{y\} \times V_{y, \omega_{1/2}, k_{1/2}}.$$

Again, by Lemma 4.2.10, we know that

$$\nu \times \mu^{\mathbb{N}}(Y'') \geq (1 - \delta^j)(1 - \sqrt{2\delta^j}) > (1 - \delta^k). \quad (131)$$

Let

$$\tilde{S}' = (Y'' \cap (\eta_{\omega_{1/2}, k_{1/2}}^s(x_{1/2}) \times \Omega^-)) \times \{\omega_{1/2}^+\} \times \{k_{1/2}\}. \quad (132)$$

Then as before

$$\hat{m}_{\hat{x}_{1/2}}^s(\tilde{S}') \geq (1 - \delta/2) \hat{m}_{\hat{x}_{1/2}}^s(\eta_{\ell/2}^s[\hat{x}_{1/2}]). \quad (133)$$

Note that the set S' as in Definition 3 is allowed to be dependent on the choice of $\hat{y}_{1/2}$ and ℓ .

Lemma 4.4.9. *We can find constants C and ℓ_0 that can be chosen along the set K in the*

definition of QNI (definition 3), and a set $S' \subset \tilde{S}'$ dependent on $\hat{y}_{1/2}$ and ℓ , such that

$$\hat{m}_{\hat{x}_{1/2}}^s(S') \geq (1 - \delta) \hat{m}_{\hat{x}_{1/2}}^s(\eta_{\ell/2}^s[\hat{x}_{1/2}]), \quad (134)$$

and that for $\hat{z}_{1/2} = (z_{1/2}, (\omega'')^-, \omega_{1/2}^+, k_{1/2}) \in S'$, $U^+[\hat{z}_{1/2}]$ does not intersect $W^s(\hat{y}_{1/2})$. Further, for $\ell \geq \ell_0$,

$$d_X(\mathcal{B}_{\ell/2}[\hat{z}_{1/2}], W^s(\hat{y}_{1/2})) \geq Ce^{-\alpha_0 \ell}. \quad (135)$$

Note that Equation (135) is just the QNI bound from Definition 3 using notation established in this section.

We will prove this in a series of approximation steps. First, the linear approximation.

We take ℓ_0 large enough so that for all $\ell \geq \ell_0$, $\eta_{\omega_{1/2}, \ell/2}^{s/u}(x_{1/2})$ is in the ball of radius $k_U/2$ around $x_{1/2}$ and around $y_{1/2}$. Then it lives in the domains of the Lyapunov charts $\varphi_{\hat{x}_{1/2}}$ and $\varphi_{\hat{y}_{1/2}}$ (see Section 3.5). We will work in $\varphi_{\hat{x}_{1/2}}(B_X(x_{1/2}, k_U))$, the norm in these charts is denoted $|\cdot|$, the Euclidean norm. Note that $\varphi_{\hat{x}_{1/2}}(W^s(\hat{x}_{1/2}))$ and $\varphi_{\hat{x}_{1/2}}(W^s(\hat{y}_{1/2}))$ are holomorphic. We will have to work with $\mathbb{R}^4$ instead of $\mathbb{C}^2$ in this section for our charts in order to be able to say something about a real 1-dimensional set.

Let $P = D\varphi_{\hat{x}_{1/2}}(E^s(\hat{x}_{1/2})) = \mathbb{R}^s$, i.e. the tangent plane at the origin (where the origin is $\varphi_{\hat{x}_{1/2}}(x_{1/2})$ of $\varphi_{\hat{x}_{1/2}}(W^s(\hat{x}_{1/2}))$). Let Q be the tangent plane at $\varphi_{\hat{x}_{1/2}}(y_{1/2})$ of $\varphi_{\hat{x}_{1/2}}(W^s(\hat{y}_{1/2}))$. We can assume that Q is not equal to P by Theorem 3.4.1. Note that this is where the complex nature of the problem is important even though we will have to work in real coordinates.

Generically (in the choice of $(\omega')^+, z'$), $P \cap Q = \{q\}$.

Fix a direction vector $v \in \mathbb{R}^4$. For any $z' \in P$, draw a line through z' in direction v denoted $L_{z', v}$. These lines $L_{z', v}$ are to be thought of as a linear approximation of $U^+[z, (\omega'')^-]$, which are 1-dimensional curves for most points. We ask what subset of $z' \in P$ are such that

$L_{z',v}$ does not intersect Q .

In the following lemma, we ask (for generic v) which $z' \in P$ are such that $L_{z',v} \cap Q \neq \emptyset$. We will show that these z' form a line through $q \in P \cap Q$. It is worth noting that q may not be in $\varphi_{\hat{x}_{1/2}}(B_X(x_{1/2}, k_U))$ (though it would be for a large measure portion of $(\omega')^+$ and for large ℓ). This does not concern us as this is part of the set of X -components we will wish to remove when we define S' for Lemma 4.4.9. We will assume that $v \notin Q$, it will be justified later why we can do this.

Lemma 4.4.10. *For our fixed plane Q , let $v \notin Q$ be arbitrary, if $P \neq Q$ is another plane such that $P \cap Q = \{q\}$, the set of $z' \in P$ such that $L_{z',v} \cap Q \neq \emptyset$ is a line thru q inside P .*

Proof. We will first change coordinates. Given a simple rotation and translation of frame, we can WLOG let $Q = \text{span}\{(0, 0, 1, 0), (0, 0, 0, 1)\}$ and $P \cap Q = \{(0, 0, 0, 0)\}$, and consider the direction vector $v = (l, m, n, o)$, since $v \notin Q$, at least one of l or m is non-zero. WLOG let $l \neq 0$. Any $z' \in P$ is of the form $z' = (s, t, u, v)$, if we take the line $L_{z',v}$ in direction v thru z' , we ask which values of z' will this line $L_{z',v}$ will intersect Q . Clearly, these z' will be such that both equations $s + lT = 0$ and $t + mT = 0$ hold for some T . This amount to the condition $t = \frac{m}{l}s$ on the plane P inside the ball.

Note that the condition $t = \frac{m}{l}s$ on all of $\mathbb{R}^4$ creates a 3-dimensional subspace U containing Q . Now P is a two-dimensional subspace intersecting Q at only one point and so P intersects U in a 1-dimensional subspace.

When we shift back to the original coordinates, since $P \neq Q$, we see that our solution set is a line through q . □

Proof. (of Lemma 4.4.9) We were given δ at the beginning and we need to find a set $S' \subset \tilde{S}'$ where Equation (134) holds.

By construction, $K_\epsilon \supset \pi_X(\tilde{S}')$ (see Lemma 3.10.12 for the definition of K_ϵ). We have that for our chosen X -component from an element of $\tilde{S}'$, outside of any ball of size we will call

M_δ in $Gr(1, \mathbb{C})$ (where the stable and unstable Oseledets' directions live), we have measure at least $1 - \delta^k$. Call the corresponding angle defining this ball α_{M_δ} . We will ask that for any $\hat{z}_{1/2} \in S'$, $E^u(\hat{z}_{1/2})$ is bounded away from $Q = E^s(\hat{y}_{1/2})$ in $Gr(1, \mathbb{C})$ by M_δ . In particular, any v a unit tangent vector of $U^+[\hat{z}_{1/2}]$ will not belong to Q . The subset of Ω^- where this is true will be called $(\Omega^-)'$, has measure $(1 - \delta^k)$ and is dependent on $\hat{y}_{1/2} \in S$.

Fix $\eta > 0$ small. Let t be the amount of time it takes to flow under the time changed flow from the point $F^{-\ell/2}(\hat{x}_{1/2})$ to $\hat{x}_{1/2}$. Note that

$$t \in [\lfloor \ell/2 \rfloor (\lambda^+ - \epsilon), \lfloor \ell/2 + 1 \rfloor (\lambda^+ + \epsilon)],$$

as in Corollary 4.2.3. Note that we can take ℓ large so that t is large such that

$$r' := \frac{1}{M_{NFC}^2} R e^{-t},$$

is small. Here R is as in the definition of $K_{C,R}$ as above and ϵ is as in Equation (94). Remember that ℓ must also be such that $F^{-\ell/2}(\hat{x}_{1/2}), F^{\ell/2}(\hat{x}_{1/2}) \in K \subset K_{nbhd}$.

Claim 4.4.11. *Ambiently, the ball of radius r' , $B^s(\hat{x}_{1/2}, r')$ contains $\eta_{\ell/2, \omega_{1/2}}^s(x_{1/2})$*

Proof. We know that since

$$F^{-\ell/2}(\hat{x}_{1/2}) \in K_{nbhd},$$

from Lemma 4.2.5, the diameter of $\eta_{\sigma^{-\ell/2}(\omega_{1/2})}^s(x_{1/2})$ is at most $\tilde{r}$. So

$$\eta_{\sigma^{-\ell/2}(\omega_{1/2})}^s(x_{1/2}) \subset B^s(F^{-\ell/2}(\hat{x}_{1/2}), \tilde{r}) \subset W^s(F^{-\ell/2}(\hat{x}_{1/2})).$$

Passing to modified stable NFCs we see that the largest this ball could get is of size $M_{NFC} \tilde{r}$. Flowing forwards by $F^{\ell/2}$ (in under the standard flow) is like flowing forwards by F_{tc}^t in the time change flow where $t \in [n(\lambda^+ - \epsilon), (n+1)(\lambda^+ + \epsilon)]$ as in Corollary 4.2.3, where

$n = \lfloor \ell/2 \rfloor$, and ϵ is as in Equation (94). Then after flowing by t under the time change flow, the largest our ball could be now is of size $M_{NFC} \tilde{r} e^{-n(\lambda^+ - \epsilon)}$.

Now $B^s(\hat{x}_{1/2}, r')$ in modified NFCs at its smallest is a ball of size $\frac{1}{M_{NFC}} r'$ where

$$\begin{aligned} \frac{1}{M_{NFC}} r' &= \frac{1}{M_{NFC}^3} R e^{-t'} > \frac{1}{M_{NFC}^3} (M_{NFC}^4 \tilde{r}) e^{-\lfloor \ell/2 \rfloor (\lambda^+ - \epsilon)} \\ &= M_{NFC} \tilde{r} e^{-n(\lambda^+ - \epsilon)}. \end{aligned} \quad (136)$$

Hence in modified NFCs we see that the image of $\eta_{\ell/2, \omega_{1/2}}^s(x_{1/2})$ is contained in the image $B_X^s(\hat{x}_{1/2})$. When we return to ambient coordinates this is still true. This completes the proof. $\square$

Claim 4.4.11 tells us that working in $B^s(\hat{x}_{1/2}, r')$ will suffice, i.e. thinking of the distance between $y_{1/2}$ and $x_{1/2}$ as being at most r' is acceptable. We will go a bit further for convenience later and work in $F^{\ell/2}(B^s(F^{-\ell/2}(\hat{x}_{1/2}), R))$. Firstly, we demonstrate that this set contains $B^s(\hat{x}_{1/2}, r')$.

Claim 4.4.12. *The set*

$$F^{\ell/2}(B^s(F^{-\ell/2}(\hat{x}_{1/2}), R)) \subset W^s(\hat{x}_{1/2}),$$

contains $B^s(\hat{x}_{1/2}, r')$.

Proof. Remember that ℓ is always such that $F^{\ell/2}(\hat{x}_{1/2}), F^{-\ell/2}(\hat{x}_{1/2}) \in K$.

We work in modified NFCs. Here, $F^{\ell/2}(B^s(F^{-\ell/2}(\hat{x}_{1/2}), R))$ has smallest diameter $\frac{1}{M_{NFC}} R e^{-t}$. In modified NFCs the largest $B^s(\hat{x}_{1/2}, r')$ could be is

$$M_{NFC} r' = \frac{1}{M_{NFC}} R e^{-t}.$$

This concludes the proof. $\square$

Let r'' be the diameter of the set $F^{\ell/2}(B^s(F^{-\ell/2}(\hat{x}_{1/2}), R))$. For now we will imagine working in this set and come back to $\eta_{\ell/2, \omega_{1/2}}^s(x_{1/2})$. Error estimates we do below in $F^{\ell/2}(B^s(F^{-\ell/2}(\hat{x}_{1/2})), R)$ will be true for $\eta_{\ell/2, \omega_{1/2}}^s(x_{1/2})$ too.

Now, we work in Lyapunov charts, i.e. we will work in a subset of $\varphi_{\hat{x}_{1/2}}(B_X(x_{1/2}, k_U))$. Let P and Q be as above, we can work inside a ball of radius r (appropriately small) based at the origin (which is $\varphi_{\hat{x}_{1/2}}(x_{1/2})$). We take $r = M_r^2 M_{NFC}^5 r''$, note that $M_r r''$ is the largest an ambient ball of radius r'' could be in Lyapunov charts, but we need it to be a bit larger for later.

We also ensure that ℓ is large enough so that

$$\eta_{\omega_{1/2}, \ell/2}^s(x_{1/2}) \subset F^{\ell/2}(B^s(F^{-\ell/2}(\hat{x}_{1/2})), R) \subset B_X(x_{1/2}, k_U/2).$$

A lower bound for ℓ can be found uniformly in the choice of $\hat{x}_{1/2} \in K \subset K_{nbhd}$ because of Lemma 3.10.6.

Inside this ball of radius r , we know by Claim 3.10.8 that P and Q are within of ϵ of $\varphi_{\hat{x}_{1/2}}(W^s(\hat{x}_{1/2}))$ and $\varphi_{\hat{x}_{1/2}}(W^s(y_{1/2}, (\omega')^+))$ respectively. Here $\epsilon = r^2/\tilde{L}$ and $\tilde{L}$ is the maximum between the Lipschitz constants of Dh_1 and Dh_2 , where h_1 is the function tracing out $\varphi_{\hat{x}_{1/2}}(W^s(\hat{x}_{1/2}))$ and h_2 is the function tracing out $\varphi_{\hat{x}_{1/2}}(W^s(y_{1/2}, (\omega')^+))$. Note that $\tilde{L}$ is bounded above by HM_r^4 . The factor of M_r^4 comes from the fact that the graph $h_{(\hat{y}_{1/2})}^s$ will have $\text{Lip}(Dh_{(\hat{y}_{1/2})}^s) \leq H$ because $\hat{y}_{1/2} \in K'_\xi$ by construction. However we need the Lipschitz constant for h_2 which is in the Lyapunov chart based at $\hat{x}_{1/2}$. Taking the change of charts into consideration, this forces $\tilde{L} \leq HM_r^4$.

Fix $\hat{z} = (z, (\omega'')^-, \omega_{1/2}, k_{1/2}) \in \tilde{S}'$, consider $\varphi_{\hat{x}_{1/2}}(U^+[\hat{z}])$. Let v be the direction vector of the tangent line through $\varphi_{\hat{x}_{1/2}}(z)$ of $\varphi_{\hat{x}_{1/2}}(U^+[\hat{z}])$ for this fixed $\hat{z}$. Let $L_{\varphi_{\hat{x}_{1/2}}(z), v}$ be the line through $\varphi_{\hat{x}_{1/2}}(z)$ in the direction v . Keep v fixed and vary the component in $\varphi_{\hat{x}_{1/2}}(W^s(\hat{x}_{1/2}))$, i.e. take points $\varphi_{\hat{x}_{1/2}}(z')$ where $(z', (\omega'')^-, \omega_{1/2}, k_{1/2})$ is still in $\tilde{S}'$. We

draw lines $L_{\varphi_{\hat{x}_{1/2}}(z'),v}$ through these $\varphi_{\hat{x}_{1/2}}(z')$ in the direction v .

Let ℓ_v be the line we get from Lemma 4.4.10 for P , the tangent plane of $\varphi_{\hat{x}_{1/2}}(W^s(\hat{x}_{1/2}))$ at the origin. It is important to note that P is fixed by the choice of $\hat{x}_{1/2} \in K$ and that the choice of the sets S and S' are allowed to depend on this choice of $\hat{x}_{1/2}$ and on ℓ . The set S' is further allowed to be dependent on the choice of $\hat{y}_{1/2} \in S$ which determines Q .

Lemma 4.4.13. *There exists a constant $\tilde{C}$ independent of $\hat{x}_{1/2}$ or $\hat{y}_{1/2}$, such that for any fixed direction vector $v \notin Q$, if $z \in B(0, r) \cap \varphi_{\hat{x}_{1/2}}(W^s(\hat{x}_{1/2})) \cap \varphi_{\hat{x}_{1/2}}(\tilde{S}')$ is such that $L_{z,v} \cap Q$, then z is within $\tilde{C}r^2$ of ℓ_v inside $B(0, r)$.*

Proof. Let $v \notin Q$, $v = (l, m, n, o)$ be fixed and take z as in the statement, i.e. $L_{z',v} \cap Q \neq \emptyset$. WLOG assume $l \neq 0$. Change coordinates (just rotation and translation which are isometric and conformal) so that $Q = \text{span}\{(0, 0, 1, 0), (0, 0, 0, 1)\}$ and $P \cap Q = \{(0, 0, 0, 0)\}$. Let z' represent z in this new coordinate system. Label the coordinates as (x_1, x_2, x_3, x_4) . As in the proof of Lemma 4.4.10, we must have that z' is of the form

$$z' = (s, \frac{m}{l}s, u, v),$$

for some $s, u, \in \mathbb{R}$. This is because for any point $y = (s, t, u, v)$ to be such that $L_{y,v}$ intersects Q , it must satisfy $t = \frac{m}{l}$. These points form a 3-dimensional subspace that we called U in the proof of Lemma 4.4.10.

We know that P is at most $r^2/\tilde{L}$ away from $\varphi_{\hat{x}_{1/2}}(W^s(\hat{x}_{1/2}))$ in the ball of radius r . Let $p_{z'} \in P$ be the element closest to z' .

Note that if $p_{z'}$ is a translate of z' in x_3 or x_4 directions, $p_{z'}$ belongs to ℓ_v (since it belongs to U). Then z' is at most $r^2/\tilde{L}$ away from ℓ_v .

Claim 4.4.14. *There exists a basis for P of the form $q_1 = (1, 0, u_1, v_1)$, $q_2 = (0, 1, u_2, v_2)$ such that $|q_1|, |q_2| \leq \tilde{C}$. This constant $\tilde{C}$ is independent of P and Q .*

Proof. Any basis element of P must have non-trivial x_1 and/or x_2 component, otherwise P would intersect Q in a line rather than a point. If there does not exist a basis of the proposed form, then any basis element of P would have (x_1, x_2) -component that is some scalar multiple of the (x_1, x_2) -component of any other. This means that there is a vector of the form $(0, 0, u, v)$ in P . This is a contradiction to the fact that P does not intersect Q in a line.

In the definition of S , we required that $\hat{y}_{1/2}$ is such that the angle between P and Q is bounded away from some $\theta > 0$. Now we argue that u_1, v_1, u_2, v_2 then must be bounded from above, giving us the required result.

Compare q_1 to its projection onto Q denoted q , let $\theta' \geq \theta > 0$ be the angle between q and q_1 :

$$1 > \cos(\theta') = \frac{|q|}{|q_1|} = \frac{\sqrt{u_1^2 + v_1^2}}{\sqrt{1 + u_1^2 + v_1^2}}$$

If $\cos(\theta') = 0$ then $u_1 = v_1 = 0$ and so we take $C_1 = 1$. Assume $\cos(\theta') \neq 0$.

Squaring and rearranging we see that u_1 and v_1 are such that

$$1 + \left(1 - \frac{1}{\cos(\theta')^2}\right) u_1^2 + \left(1 - \frac{1}{\cos(\theta')^2}\right) v_1^2 = 0.$$

The term $\left(1 - \frac{1}{\cos(\theta')^2}\right)$ is negative and is bounded above by $\left(1 - \frac{1}{\cos(\theta)^2}\right)$ independent of Q or $\hat{y}_{1/2}$. This gives us that

$$\left(1 - \frac{1}{\cos(\theta)^2}\right) u_1^2 + \left(1 - \frac{1}{\cos(\theta)^2}\right) v_1^2 \geq -1,$$

and hence,

$$u_1^2 + v_1^2 \leq \frac{1}{\left(\frac{1}{\cos(\theta)^2} - 1\right)}.$$

A similar argument works for q_2 . This completes the proof. $\square$

Now if $p_z = (s + \alpha, s\frac{m}{l} + \beta, u + \gamma, v + \delta)$ where $q = (\alpha, \beta, \gamma, \delta)$ is such that $|q| \leq r^2/\tilde{L}$, then in particular $|\alpha|, |\beta| \leq r^2/\tilde{L}$. Then

$$p := p_z - \alpha q_1 - \beta q_2 \in \ell_v.$$

We now compute how far away p is from z ; z is at most $r^2/\tilde{L}$ away from p_z and

$$|p_z - p| = |\alpha q_1 + \beta q_2| \leq 2C_1 r^2/\tilde{L}.$$

Hence we can take $\tilde{C} = 2C_1 + 1$ and then z is at most $\tilde{C}r^2/\tilde{L}$ away from $\ell_v \subset P$. $\square$

Recall the coordinates as in the proof of Lemma 4.4.10; i.e. let

$$Q = \text{span}\{(0, 0, 1, 0), (0, 0, 0, 1)\},$$

and $P \cap Q$ is the origin. Call these coordinates $\mathcal{C}$. Let $v = (l, m, n, o) \notin Q$, WLOG $l \neq 0$.

Let U be the subspace defined in the proof of Lemma 4.4.10, i.e.

$$U = \{(s, \frac{m}{l}s, u, v) : s, u, v \in \mathbb{R}\}.$$

Note that U is dependent on v and is the subspace of points z such that $L_{z,v} \cap Q \neq \emptyset$.

Also note that Lemma 4.4.13 tells us that in the ball of radius r that we are interested in, all points in V which is defined to be $\varphi_{\hat{x}_{1/2}}(W^s(\hat{x}_{1/2})) \cap B(0, r)$ in coordinates $\mathcal{C}$ such that

$L_{z,v} \cap Q \neq \emptyset$ are within a $\tilde{C}r^2$ of $\ell_v \subset P$. So in coordinates $\mathcal{C}$ that tells us that elements belonging to U in the shifted ball of radius r are within $\tilde{C}r^2$ of ℓ_v . In the next lemma, we work to understand how far apart $L_{z,v}$ is from Q if $z \in V$ is of a certain distance from this tubular neighbourhood of radius $\tilde{C}r^2$ of $\ell_v \subset P$.

Let $C_{D,1}$ be the lower bound for the function $\frac{b^2}{1+b^2}$, and $C_{D,2}$ be the lower bound for the function $\frac{1}{1+b^2}$, given that b is bounded. Let $C_D = \min\{C_{D,1}, C_{D,2}, 1\}$.

Remark 4.4.15. *The proof of Lemma 4.4.13 also demonstrates that for the set of points $\frac{5r^2}{\sqrt{C_D}\tilde{L}}$ away from P , there is a $\tilde{C}'$ such that the points belonging to the subspace U must be within $\tilde{C}'r^2$ of ℓ_v .*

Lemma 4.4.16. *Let $v \notin Q = \text{span}\{(0,0,1,0), (0,0,0,1)\}$ be a direction vector in $\mathbb{R}^4$ with corresponding subspace U as above. Let V be $B(0,r) \cap \varphi_{\hat{x}_{1/2}}(W^s(\hat{x}_{1/2}))$ in coordinates $\mathcal{C}$, and let $z' \in V$ be distance r^2 away from the neighbourhood of ℓ_v of size $\tilde{C}'r^2$ from Remark 4.4.15. Then there exists a constant $C'' > \frac{4}{L}$, independent of v , $\hat{y}_{1/2}$ or $\hat{x}_{1/2}$, such that*

$$d(L_{z',v}, Q) \geq C'' \cdot r^2.$$

Proof. Note that when we state a definition for the set ℓ_v , we have assumed $v \notin Q$ and hence WLOG assumed $l \neq 0$. If $l = 0$ then it must be that $m \neq 0$ and we look at the line in P of slope $l/m = 0$. Since v is a unit vector, $|m|, |l| \leq 1$ and because we have assumed that $E^u(\hat{z}_{1/2})$ (where we choose v) is bounded away from Q in the Grassmanian metric by M_δ , we must have that in coordinates $\mathcal{C}$, one of l or m is greater than or equal to $M_\delta/\sqrt{2}$. When defining the set ℓ_v we will look at

$$\begin{aligned} \ell_v &= \{(s, t, u, w) \in P : t = \frac{m}{l}s\}, \text{ if } l \geq m, \\ \ell_v &= \{(s, t, u, w) \in P : s = \frac{l}{m}t\}, \text{ if } m > l. \end{aligned}$$

Assume that $l \geq m \geq 0$ (other cases similar). Since we have that $M_\delta/\sqrt{2} \leq |l| \leq 1$, $0 \leq |m| \leq 1$, we can assume that $b := \frac{m}{l}$ is bounded from above and below.

Note that $z' \in V$ is chosen so that it is at least r^2 away from the neighbourhood of size $\tilde{C}'r^2$ of ℓ_v . This neighbourhood of size $\tilde{C}'r^2$ contains all points in the subspace U that are within $\frac{5r^2}{\sqrt{C_D}\tilde{L}}$ of P (see Remark 4.4.15, note $\sqrt{C_D} \leq 1$). Note that z' is within $r^2/\tilde{L}$ of P . This implies that z' is distance at least $d := \frac{4r^2}{\sqrt{C_D}\tilde{L}}$ from the subspace U .

We can write z' as some point

$$z' = (s + \alpha, sb + \beta, u, v), \quad (137)$$

for some s (which we will compute below in Claim 4.4.17), and some α, β such that

$$\sqrt{\alpha^2 + \beta^2} = \mathcal{D} \geq d,$$

and $\beta \neq b\alpha$ (because z' does not belong to the subspace U as defined in the proof of Lemma 4.4.10).

Claim 4.4.17. *In Equation (137), $s = \frac{x+yb}{1+b^2}$ and we must have that $|\beta - b\alpha| = \mathcal{D}$.*

Proof. Writing out point z' as $z' = (x, y, u, v)$, the closest point u in the subspace U is given by

$$\left(\frac{x+yb}{1+b^2}, b\frac{x+yb}{1+b^2}, u, v \right).$$

Then in Equation (137), $s = \frac{x+yb}{1+b^2}$. Then α and β are such that

$$x = \frac{x+yb}{1+b^2} + \alpha, \quad (138)$$

$$y = b\frac{x+yb}{1+b^2} + \beta. \quad (139)$$

Now since z' is distance $\mathcal{D}$ from U , we must have that $|\beta - b\alpha| = \mathcal{D}$. If not, i.e. if $|\beta - b\alpha| \neq \mathcal{D}$,

then writing $\beta - b\alpha = \kappa$ we see that

$$z' = \left(\frac{x + yb}{1 + b^2} + \alpha, b \frac{x + yb}{1 + b^2} + \beta, w, t \right) = \left(\frac{x + yb}{1 + b^2} + \alpha, b \frac{x + yb}{1 + b^2} + \kappa + b\alpha, u, v \right) \quad (140)$$

$$= \left(\frac{x + yb}{1 + b^2} + \alpha, b \left(\frac{x + yb}{1 + b^2} + \alpha \right) + \kappa, u, v \right) \quad (141)$$

Then z' is not distance $\mathcal{D}$ away from the subspace U because $|\kappa| \neq \mathcal{D}$.

This is a contradiction. This completes the proof. $\square$

In the cases $b = 0$ and we can ignore α as the subspace U is just the set of points with the second coordinate equal to zero, Claim 4.4.17 just reduces to the obvious: that $|\beta| = \mathcal{D} \geq d$.

For now assume $b \neq 0$, we will deal with this case at the end for notational reasons.

Now, $L_{z',v}$ can be written as

$$L_{z',v}(t) = (s + \alpha + t, sb + \beta + tb, u + t \frac{n}{l}, v + t \frac{o}{l}), \quad t \in \mathbb{R}.$$

Now we find the value of t such that $L_{z',v}(t)$ is closest to Q , we will call this t' . We have the inconsistent system given by the equations:

$$t = -(s + \alpha), \quad bt = -(sb + \beta).$$

Least squares approximation gives us that

$$t' = -s - \frac{(\alpha + b\beta)}{(1 + b^2)}.$$

Let $r = L_{z',v}(t')$, it is of the form

$$\left(\alpha - \frac{(\alpha + b\beta)}{(1 + b^2)}, \beta - b \frac{(\alpha + b\beta)}{(1 + b^2)}, u - s \frac{n}{l} - \frac{n}{l} \frac{(\alpha + b\beta)}{(1 + b^2)}, v - s \frac{o}{l} - \frac{o}{l} \frac{(\alpha + b\beta)}{(1 + b^2)} \right),$$

then q' , the closest point to r in Q is of the form

$$\left(0, 0, u - s\frac{n}{l} - \frac{n}{l}\frac{(\alpha + b\beta)}{(1 + b^2)}, v - s\frac{o}{l} - \frac{o}{l}\frac{(\alpha + b\beta)}{(1 + b^2)}\right).$$

We get that the distance between $L_{z',v}$ and Q is equal to

$$f(\alpha, \beta) = \sqrt{\left(\alpha - \frac{(\alpha + b\beta)}{(1 + b^2)}\right)^2 + \left(\beta - b\frac{(\alpha + b\beta)}{(1 + b^2)}\right)^2}.$$

We can simplify the expression for f^2 to get

$$f^2(\alpha, \beta) = \alpha^2 + \beta^2 - \frac{(\alpha + b\beta)^2}{(1 + b^2)}.$$

We lower bound f^2 given our restraints, i.e. given Claim 4.4.17.

We deal with cases to make it easier:

Case 1: $\beta - b\alpha = \mathcal{D}$

Plugging this into $\alpha^2 + \beta^2 = \mathcal{D}^2$ gives

$$\alpha^2 + (b\alpha + \mathcal{D})^2 = \mathcal{D}^2.$$

Equivalent to

$$\alpha((1 + b^2)\alpha + 2b\mathcal{D}) = 0.$$

This gives two subcases: $\alpha = 0$ or $\alpha = \frac{-2b\mathcal{D}}{(1+b^2)}$.

In the first case, this forces $\beta = \mathcal{D}$ and then

$$f^2 = \mathcal{D}^2 - \frac{b^2}{1 + b^2}\mathcal{D}^2.$$

Which is bounded away from zero by $C_{D,1}\mathcal{D}^2$ since b is bounded. Since $\mathcal{D} \geq d$ we have that $f^2 \geq C_{D,1}d^2$.

In the second subcase, we can solve for β and simplify the expression for f^2 into a function of b and $\mathcal{D}$:

$$f^2 = \mathcal{D}^2 \left(1 - \frac{b^2 + 2b^4 + b^6}{(1+b^2)^3} \right) = \mathcal{D}^2 \frac{1}{(1+b^2)}.$$

This function $\frac{1}{(1+b^2)}$ is bounded from below for bounded b . Hence again, there is $C_{D,2}$ such that $f^2 \geq C_{D,2}d^2$.

Case 2: $b\alpha - \beta = \mathcal{D}$ Again one plugs in $\beta = b\alpha - \mathcal{D}$ into the expression $\alpha^2 + \beta^2 = \mathcal{D}^2$.

One gets

$$\alpha((1+b^2)\alpha - 2b\mathcal{D}) = 0.$$

So there are again two subcases: $\alpha = 0$ and $\alpha = \frac{2b\mathcal{D}}{1+b^2}$.

In the first subcase, again $\beta = \mathcal{D}$ and again,

$$f^2 = \mathcal{D}^2 - \frac{\mathcal{D}^2 b^2}{1+b^2}.$$

As before, we have $f^2 \geq C_{D,1}d^2$.

In the second subcase, $\beta = \frac{2b^2\mathcal{D}}{1+b^2} - \mathcal{D}$. We plug this into f^2 to get an expression in terms of b and $\mathcal{D}$ as before and see that

$$f^2 = \mathcal{D}^2 \left(1 - \frac{b^2 + 2b^4 + b^6}{(1+b^2)^3} \right) = \mathcal{D}^2 \frac{1}{1+b^2}.$$

Then as before $f^2 \geq C_{D,2}d^2$.

Now, in the case that $b = m = 0$, we again note that the subspace U corresponds to the second component of a point being zero. Our point z' can be written as

$$z' = (s, \beta, u, v),$$

where we know that $|\beta| > d$. Since $m = 0$, our line can be written as

$$L_{z',v}(t) = (s + t, \beta, u + \frac{n}{l}t, v + \frac{o}{l}t).$$

This is closest to $Q = \text{span}\{(0, 0, 0, 1), (0, 0, 1, 0)\}$ when $t = -s$. At this value of t , the line $L_{z',v}$ is exactly β away from Q and is hence distance $d = \frac{4r^2}{\sqrt{C_D L}}$ away from Q .

Putting these cases together, we see that taking $C'' = \frac{4}{\sqrt{C_D L}} \max\{\sqrt{C_{D,1}}, \sqrt{C_{D,2}}, 1\}$, we have that

$$\text{dist}(L_{z',v}, Q) \geq C'' r^2.$$

Clearly $C'' \geq \frac{4}{L}$. This completes the proof. $\square$

Now we see that being distance $\tilde{C}' r^2 + r^2 = (\tilde{C}' + 1) r^2$ where $\tilde{C}'$ is from Remark 4.4.15 guarantees that for z' outside a neighbourhood of ℓ_v that $L_{z',v}$ is distance at least $C'' r^2$ from Q .

One now has to take into account that v should vary as we change z because when we switch the X -component to z' we should use the tangent vector v' for $U^+[z', (\omega'')^-, \omega_{1/2}^+, k_{1/2}]$ (viewed in Lyapunov charts). Then we examine $L_{z',v'}$ at the point $\varphi(z')$ instead of $L_{z',v}$ where v is the tangent vector for $U^+[z, (\omega'')^-, \omega_{1/2}^+, k_{1/2}]$ (viewed in Lyapunov charts).

Here we use that $\hat{x}_{1/2} \in K_\beta$ (see Section 3.7) so that along ‘good’ points in $W^s(\hat{x}_{1/2})$ we have that $P_{\omega'}^-(x_{1/2}, \cdot)$ is Hölder in the sense that for good ω' and for $z, z' \in \tilde{S}'$ good, we have that

$$\|P_{\omega'}^-(z, z') - Id\| \leq C_\beta d_X(z, z')^\beta.$$

Here $P_{\omega'}^-(z, z')$ is such that $TU^+[z', \omega'] = P_{\omega'}^-(z, z')TU^+[z, \omega']$. Since all points we consider are within r , we can say that for good points z, z'

$$\|P_{\omega'}^-(z, z') - Id\| \leq C_\beta r^\beta.$$

Note that r can be taken to be $r^\beta \ll 1$ by making ℓ large. Then when one compares $\tilde{v}$ (a unit direction vector) to $\tilde{v}' = P_{\omega'}^-(z, z')\tilde{v}$ one sees that

$$\begin{aligned} |\langle P_{\omega'}^-(z, z')\tilde{v}, \tilde{v} \rangle - \langle \tilde{v}, \tilde{v} \rangle| &= |\langle (P_{\omega'}^-(z, z') - Id)\tilde{v}, \tilde{v} \rangle| = |\tilde{v}^T (P_{\omega'}^-(z, z') - Id)\tilde{v}| \\ &\leq \|P_{\omega'}^-(z, z') - Id\| \cdot \|\tilde{v}\|^2 \leq C_\beta r^\beta \end{aligned}$$

hence,

$$\langle P_{\omega'}^-(z, z')\tilde{v}, \tilde{v} \rangle \leq C_\beta r^\beta + 1.$$

Further,

$$\|(P_{\omega'}^-(z, z') - Id)\tilde{v}\| \leq C_\beta r^\beta,$$

and hence since $\cos(\angle(P_{\omega'}^-(z, z')\tilde{v}, \tilde{v})) = \frac{\langle P_{\omega'}^-(z, z')\tilde{v}, \tilde{v} \rangle}{\|P_{\omega'}^-(z, z')\tilde{v}\|}$ we get that

$$1 \geq \cos(\angle(P_{\omega'}^-(z, z')\tilde{v}, \tilde{v})) \geq \frac{1 - C_\beta r^\beta}{1 + C_\beta r^\beta} \geq 1 - 2C_\beta r^\beta.$$

Now we need to see what happens in $\varphi_{\hat{x}_{1/2}}(W^s(\hat{x}_{1/2}))$, i.e. we want the relation between $v = D\varphi_{\hat{x}_{1/2}}\tilde{v}$ and $v' = D\varphi_{\hat{x}_{1/2}}\tilde{v}'$. Note that $\text{Lip}(D\varphi_{\hat{x}_{1/2}})$ is bounded above by Φ on the set K_φ as defined in Section 3.5 and that $K \subset K_\varphi$. We get there there is a uniform constant C'_β dependent only on C_β and Φ such that

$$1 \geq \cos(\angle(v', v)) \geq 1 - 2C'_\beta r^\beta.$$

For each v' we apply Lemma 4.4.10 to P , Q and the collection of $L_{z, v'}$ and get a line $\ell_{v'}$ in P . Now we want to argue that in P , $\ell_{v'}$ is close to ℓ_v .

Recall that in the proof of Lemma 4.4.10, in Lyapunov charts, we picked coordinates $\mathcal{C}$ as above and let

$$Q = \text{Span}\{(0, 0, 1, 0), (0, 0, 0, 1)\}.$$

We saw that if $v = (l, m, n, o)$ is the unit direction vector, $l \neq 0$, then

$$\ell_v = \{(s, t, u, w) \in P : t = \frac{m}{l}s\}. \quad (142)$$

Now if $\gamma = \angle(v, v')$, we know that $\cos(\gamma) \geq 1 - 2C'_\beta r^\beta$ so we have a upper bound on the angle γ .

We then get the following claim:

Lemma 4.4.18. *There exists a constant $C''' > 0$ not dependent on v , $\hat{y}_{1/2}$ or $\hat{x}_{1/2}$ such that the line ℓ_v can deviate from $\ell_{v'}$ by at most $C'''r^{1+\frac{\beta}{2}}$ in the ball of radius r centered around the origin in Lyapunov charts $\varphi_{\hat{x}_{1/2}}(B_X(x_{1/2}, k_U))$.*

Proof. We know that $\gamma = \angle(v, v')$ is such that $\cos(\gamma) \geq 1 - 2C'_\beta r^\beta$. Since v is a unit vector, we can use this to compute that if we think of v and v' being based at the same point in $\mathbb{R}^4$, the most v could deviate from v' is distance $\sqrt{2C'_\beta r^\beta}$. This means that the most the coordinates of v could differ from those of v' is by some amount of norm less than or equal to $\sqrt{2C'_\beta r^\beta}$.

Note that in Equation (142) where we state a definition for the set ℓ_v , we have assumed $v \notin Q$ and hence WLOG assumed $l \neq 0$. If $l = 0$ then it must be that $m \neq 0$ and we look at the line in P of slope $l/m = 0$. Since v is a unit vector, $|m|, |l| \leq 1$ and because we have assumed that $E^u(\hat{z}_{1/2})$ (where we choose v) is bounded away from Q in the Grassmanian metric by M_δ , we must have that in coordinates $\mathcal{C}$, one of l or m is greater than or equal to $M_\delta/\sqrt{2}$. When defining the set ℓ_v we will look at

$$\begin{aligned} \ell_v &= \{(s, t, u, w) \in P : t = \frac{m}{l}s\}, \text{ if } l \geq m, \\ \ell_v &= \{(s, t, u, w) \in P : s = \frac{l}{m}t\}, \text{ if } m > l. \end{aligned}$$

Assume that $l \geq m \geq 0$ (other cases similar) then in the worst case scenario, writing

$$\ell_v = \{(s, t, u, w) \in P : t = \frac{m}{l}s\},$$

we look at

$$\ell_{v'} = \{(s, t, u, w) \in P : t = \frac{m + \sqrt{2C'_\beta}r^{\beta/2}}{l - \sqrt{2C'_\beta}r^{\beta/2}}s\}.$$

We compute how far apart these lines are. Both belong to P , we can think of P as the xy -plane and change coordinates to move ℓ_v to the x axis, then we can take $m = 0$. Note that we only care how much ℓ_v and $\ell_{v'}$ differ over distance r (since we are working in $B(0, r)$).

Note that then the quantity $l - \sqrt{2C'_\beta}r^{\beta/2}$ can be bounded above universally by some constant κ for small r (i.e. this is independent of ℓ and $\hat{z}_{1/2}$). This is because we can assume that $l \geq M_\delta/\sqrt{2}$.

Then $\ell_{v'}$ can be written as the line $y = \frac{\sqrt{2C'_\beta}r^{\beta/2}}{\kappa}x$ in P which was taken to be the xy -plane. At distance r along this line $\ell_{v'}$ in these coordinates one can see that

$$x = \frac{r}{\sqrt{1 + \frac{2C'_\beta}{\kappa^2}r^\beta}}.$$

Then the distance between ℓ_v and $\ell_{v'}$ at distance r away is

$$\mathcal{D} := \frac{\sqrt{2C'_\beta}}{\kappa} \frac{r^{1+\beta/2}}{\sqrt{1 + \frac{2C'_\beta}{\kappa^2}r^\beta}}.$$

Letting $b = \frac{2C'_\beta}{\kappa^2}$ and Taylor expanding $\frac{1}{\sqrt{1+x}}$ at zero helps us see that

$$\mathcal{D} = \frac{\sqrt{2C'_\beta}}{\kappa} r^{1+\beta} \left(1 - \frac{br^\beta}{2} + \frac{3b^2r^{2/3}}{8} + \dots\right)$$

For large ℓ and hence small r , one can bound $\mathcal{D}$ above by some $C'''r^{1+\beta/2}$. This completes the proof. $\square$

Remember that we must be able to pick the QNI constant C from Definition 3 alongside K once δ is given. That is echoed in the Lemma below.

Lemma 4.4.19. *There exist constants $C^{(4)}, C^{(5)} > 0$ not dependent on $\hat{y}_{1/2}$ or $\hat{x}_{1/2}$ such that we have that if $\hat{z}_{1/2} \in \tilde{S}'$ has X -component outside a neighbourhood of ℓ_v of radius $C^{(4)}r^{1+\frac{\beta}{2}}$ when viewed in Lyapunov charts $B(0, r) \cap \varphi_{\hat{x}_{1/2}}(W_q^s(\hat{x}_{1/2}))$, then*

$$d_X((\mathcal{B}_{\ell/2}[\hat{z}_{1/2}], W^s(\hat{y}_{1/2})) \geq C^{(5)}r^2.$$

Proof. Lemma 4.4.16 tells us that for fixed $v \notin Q$, looking at the collection of $L_{z',v}$ such that $z' \in B(0, r) \cap \varphi_{\hat{x}_{1/2}}(W^s(\hat{x}_{1/2}))$ is distance $(\tilde{C}' + 1)r^2$ away from ℓ_v , $L_{z',v}$ has distance at least $C''r^2$ from Q . Its important to note that $C'' \geq \frac{4}{\tilde{L}}$. Note that Q is distance $r^2/\tilde{L}$ from $\varphi_{\hat{x}_{1/2}}(W^s(\hat{y}_{1/2}))$. Hence there is some other constant $C_1 \geq 1$, dependent only on $\tilde{L}$ and C'' such that the distance between $L_{z',v}$ and $\varphi_{\hat{x}_{1/2}}(W^s(\hat{y}_{1/2}))$ is at least C_1r^2 for $z' \in B(0, r) \cap \varphi_{\hat{x}_{1/2}}(W^s(\hat{x}_{1/2}))$ distance $(\tilde{C}' + 1)r^2$ away from ℓ_v .

Now, by Lemma 4.4.18, the lines ℓ_v vary by at most $C'''r^{1+\frac{\beta}{2}}$ for $z' \in B(0, r) \cap \varphi_{\hat{x}_{1/2}}(W^s(\hat{x}_{1/2}))$. That is to say, if v corresponds to the tangent vector in Lyapunov charts of $U^+[z, (\omega'')^-, \omega_{1/2}^+, k_{1/2}]$ and z' is some other element of $B(0, r) \cap \varphi_{\hat{x}_{1/2}}(W^s(\hat{x}_{1/2}))$, with tangent vector in Lyapunov charts of $U^+[z', (\omega'')^-, \omega_{1/2}^+, k_{1/2}]$ denoted v' , then ℓ_v and ℓ'_v are at most distance $C'''r^{1+\frac{\beta}{2}}$ apart.

Combining these, we get that for $z' \in B(0, r) \cap \varphi_{\hat{x}_{1/2}}(W^s(\hat{x}_{1/2}))$ outside a neighbourhood of $C'''r^{1+\beta} + \tilde{C}'r^2 + r^2 = C_4r^{1+\beta}$ of an ℓ_v , the distance between $L_{z',v}$ and $\varphi_{\hat{x}_{1/2}}(W^s(\hat{y}_{1/2}))$ is at least C_1r^2 .

Now the lines $L_{z',v}$ approximate $U^+[z', (\omega'')^-, \omega_{1/2}^+, k_{1/2}]$ in Lyapunov charts up to error $r^2/\tilde{L}$, hence there is a constant $C_3 \geq 1$ such that the distance between $U^+[z', (\omega'')^-, \omega_{1/2}^+, k_{1/2}]$

and $\varphi_{\hat{x}_{1/2}}(W^s(\hat{y}_{1/2}))$ is at least $C_3 r^2$ for $z' \in B(0, r) \cap \varphi_{\hat{x}_{1/2}}(W^s(\hat{x}_{1/2}))$ outside a neighbourhood of $C_4 r^{1+\beta}$ of an ℓ_v .

Now we want to get a statement in ambient coordinates, since this change is bounded, we can conclude that there are constants $C^{(4)}$ and $C^{(5)}$ such that for $\hat{z}_{1/2} \in \tilde{S}'$ outside a neighbourhood of ℓ_v of radius $C^{(4)} r^{1+\frac{\beta}{2}}$ we have

$$d_X(\mathcal{B}_{\ell/2}[\hat{z}_{1/2}], W_q^s(\hat{y}_{1/2})) \geq C^{(5)} r^2.$$

This completes the proof. $\square$

We have shown in Lemma 4.4.19 that removing a tubular neighbourhood around ℓ_v of radius $C^{(4)} r^{1+\frac{\beta}{2}}$ ensures that $U^+[\hat{z}_{1/2}]$ is separated from $W^s(\hat{y}_{1/2})$ ambiently by at least $C^{(5)} r^2$.

Claim 4.4.20. *For $\alpha_0 = \lambda^+ + \epsilon$ as defined above,*

$$C^{(5)} r^2 \geq C e^{-\alpha_0 \ell},$$

where $C := C^{(5)} \frac{M_r^2 R^2}{M_{NFC}^2} e^{-2(\lambda^+ + \epsilon)}$.

Proof. Note that by construction:

$$r^2 \geq \left(\frac{1}{M_{NFC}} R e^{-2t} \right)^2 M_r,$$

where $t \in [\lfloor \ell/2 \rfloor (\lambda^+ - \epsilon), \lfloor \ell/2 + 1 \rfloor (\lambda^+ + \epsilon)]$. This gives us that

$$\begin{aligned} C^{(5)} r^2 &\geq C^{(5)} \frac{M_r^2 R^2}{M_{NFC}^2} e^{-2(\ell/2+1)(\lambda^+ + \epsilon)} \\ &= C e^{-\ell \alpha_0}. \end{aligned}$$

This completes the proof. $\square$

The constant C computed in Claim 4.4.20 is that which we need in the definition of QNI. Its clear from the way we computed constants throughout the proof of Lemma 4.4.9 that C can be chosen alongside K and ℓ_0 as in the definition of QNI, Definition 3.

Now we have to work in modified NFCs in order to use Lemma 4.4.3. We have fixed $d = 1$ in our use of this lemma and we will use it to remove bad X -components in our construction of S' . Note that this ball $B(0, r)$ we've been working in for Lyapunov charts is a ball radius $C^{(6)}r = \frac{1}{M_r^2 M_{NFC}} r = M_{NFC} r''$ at its smallest in modified NFCs.

Lemma 4.4.21. *There exists a real line $c \subset H_{\hat{x}_{1/2}, new}^s(W^s(\hat{x}_{1/2}))$ and a constant $C' > 0$ such that for $z' \in B(0, C^{(6)}r) \cap H_{\hat{x}_{1/2}, new}^s(W^s(\hat{x}_{1/2}))$ of distance at least $C'r^{1+\frac{\beta}{2}}$ from c inside $B(0, C^{(6)}r) \cap H_{\hat{x}_{1/2}, new}^s(W^s(\hat{x}_{1/2}))$, we have that*

$$d_X(\mathcal{B}_{\ell/2}[\hat{z}_{1/2}], W_q^s(\hat{y}_{1/2})) \geq Ce^{-\alpha_0 \ell},$$

where C is as in Claim 4.4.20.

Proof. From Section 3.9, we have chart $\tilde{H}_{\hat{x}_{1/2}}^s$ on a neighbourhood of $x_{1/2} \in X$ that behaves like modified NFCs on $W^s(\hat{x}_{1/2})$. That is to say,

$$\tilde{H}_{\hat{x}_{1/2}}^s(W^s(\hat{x}_{1/2})) = D_0 \varphi_{\hat{x}_{1/2}} \circ H_{\hat{x}_{1/2}, new}^s(W^s(\hat{x}_{1/2})),$$

is a coordinate axis copy of $\mathbb{C}$ at the origin which we view as a real 2-plane. We will view $\tilde{H}_{(x, \omega)}^s$ as a map into $\mathbb{R}^4$.

Now we take the line $\ell_v \subset P \subset \varphi_{\hat{x}_{1/2}}$ where we know that for $\hat{z}_{1/2} \in \tilde{S}'$ with X -component (when viewed in Lyapunov charts) outside of a neighbourhood of size $C^{(4)}r^{1+\beta/2}$ we have

$$d_X((\mathcal{B}_{\ell/2}[\hat{z}_{1/2}], W_q^s(\hat{x}_{1/2})) \geq Ce^{-\alpha_0 \ell}.$$

We let $\tilde{\ell} = D_0(\tilde{H}_{\hat{x}_{1/2}}^s \circ \varphi_{\hat{x}_{1/2}}^{-1})(\ell_v)$. This is a line because $D_0(\tilde{H}_{\hat{x}_{1/2}}^s \circ \varphi_{\hat{x}_{1/2}}^{-1})$ is affine.

Note that because $\hat{x}_{1/2} \in K \subset K_r \cap K_{mod}$ we have that $\tilde{H}_{\hat{x}_{1/2}}^s \circ \varphi_{\hat{x}_{1/2}}^{-1}(\ell_v)$ is within $\tilde{M}M_r \cdot r^2$ of $\tilde{\ell}$ and the bad neighbourhood around $\tilde{H}_{\hat{x}_{1/2}}^s \circ \varphi_{\hat{x}_{1/2}}^{-1}(\ell_v)$ is of size at most $M_r \tilde{M} C^{(4)} r^{1+\frac{\beta}{2}}$. So now we have a real line $\tilde{\ell}$ in $\mathbb{R}^4$ and a bad neighbourhood around it of size $M_r \tilde{M} C^{(4)} r^{1+\frac{\beta}{2}}$.

Now $\tilde{\ell}$ is within $C^{(7)} r^2$ of $\tilde{H}_{\hat{x}_{1/2}}^s(W^s(\hat{x}_{1/2}))$ where $C^{(7)}$ depends only on $\tilde{L}$, M_r , $\tilde{M}$. Note that r can be chosen small so that when we project $\tilde{\ell}$ onto $\tilde{H}_{\hat{x}_{1/2}}^s(W^s(\hat{x}_{1/2}))$ and call this c , we get that there is a $C' > 0$ such that for

$$z' \in B(0, C^{(6)} r) \cap H_{\hat{x}_{1/2}, new}^s(W^s(\hat{x}_{1/2})),$$

of distance at least $C' r^{1+\frac{\beta}{2}}$ from c inside $B(0, C^{(6)} r) \cap H_{\hat{x}_{1/2}, new}^s(W^s(\hat{x}_{1/2}))$, we have that

$$d_X(\mathcal{B}_{\ell/2}[\hat{z}_{1/2}], W_q^s(\hat{y}_{1/2})) \geq C e^{-\alpha_0 \ell}.$$

This completes the proof. □

Note that $H_{\hat{x}_{1/2}, new}^s(F^{\ell/2}(B_X^s(F^{-\ell/2}(\hat{x}_{1/2}), R)))$ is contained in a ball $B(0, M_{NFC}^3 R e^{-t})$ which is a subset of

$$B(0, C^{(6)} r) = B(0, \frac{1}{M_{NFC} M_r^2} r) = B(0, M_{NFC}^4 r''),$$

because $r'' \geq \frac{1}{M_{NFC}} R e^{-t}$.

Let T' be this neighbourhood of diameter $C' r^{1+\frac{\beta}{2}}$ from Lemma 4.4.21 inside

$$H_{\hat{x}_{1/2}, new}^s(F^{\ell/2}(B_X^s(F^{-\ell/2}(\hat{x}_{1/2}), R))).$$

Let $T = (\tilde{H}_{(x, \omega)}^s)^{-1}(T')$. This T determines choices of z inside

$$F^{\ell/2}(B_X^s(F^{-\ell/2}(\hat{x}_{1/2}), R)) \subset W^s(\hat{x}_{1/2}),$$

that must be eliminated when we construct S' . We must show that this doesn't reduce the measure by much. Then we use this to say that removing this neighbourhood inside $\eta_{\ell/2, \omega_{1/2}}^s(x_{1/2})$ doesn't reduce the measure by much either. We must show that once we do so, the remaining S' (after removing the choices of $(\omega')^-$ as above and this neighbourhood) must have $\hat{m}_{x, \omega}^s$ -measure $(1 - \delta)(\hat{m}_{x, \omega}^s(\eta_{\ell/2, \omega_{1/2}}^s(x_{1/2})))$.

Now note that Definition 3 of QNI requires $\ell > \ell_0$ to be such that

$$F^{\ell/2}(\hat{x}_{1/2}), F^{-\ell/2}(\hat{x}_{1/2}) \in K.$$

Lemma 4.4.22. *For ℓ large enough, the neighbourhood T inside*

$$F^{\ell/2}(B_X^s(F^{-\ell/2}(\hat{x}_{1/2}), R)),$$

has $m_{x, \omega}^s$ -measure at most δ'' of the measure of $F^{\ell/2}(B_X^s(F^{-\ell/2}(\hat{x}_{1/2}), R))$.

Proof. The proof is similar to that of Lemma 4.2.12. We give it below.

Recall that Lemma 4.4.3 tells us that if $(x, \omega) \in K_C$ then outside of a neighbourhood of any curve of degree $d = 1$ of diameter ϵ in a ball of radius R' based at the origin, denoted $B(x, \omega, R') \subset H_{(x, \omega), new}^s(W^s(x, \omega))$ we have measure at least

$$(1 - \delta')(H_{(x, \omega), new}^s)_*(m_{x, \omega}^s)(B(x, \omega, R')).$$

Here we fixed $R' > M_{NFC}^3 R$ and δ' such that $\frac{\delta'}{\rho'} < \delta''$.

Note that

$$F := H_{\hat{x}_{1/2}}^s(F^{\ell/2}(B_X^s(F^{-\ell/2}(\hat{x}_{1/2}), R))),$$

has diameter at most $M_{NFC} r''$ In the proof of Claim 4.4.12 we saw that

$$r'' \leq M_{NFC}^2 R e^{-t}. \tag{143}$$

Where t is the amount of time it takes to flow under the time changed flow from the point $F^{-\ell/2}(\hat{x}_{1/2})$ to $\hat{x}_{1/2}$. So F is contained in a ball of size $M_{NFC}^3 R e^{-t}$. If we take a ball of size R' in modified NFCs based at $F^{-\ell/2}(\hat{x}_{1/2}) \in K$, denoted $B(F^{-\ell/2}(\hat{x}_{1/2}), R')$ and flow it forwards under $F^{\ell/2}$, it becomes a ball of size $R' e^{-t} > M_{NFC}^3 R e^{-t}$ and so in modified NFCs at $\hat{x}_{1/2}$, F is contained in this ball of radius $R' e^{-t}$. Note that $F^{\ell/2}(B_X^s(F^{-\ell/2}(\hat{x}_{1/2}), R))$, contained a ball of size r' (see Claim 4.4.12) so in modified NFCs, F contains a ball of size at least $\frac{r'}{M_{NFC}} = \frac{R}{M_{NFC}^3} e^{-t}$. Further, our ball $B(F^{-\ell/2}(\hat{x}_{1/2}), R')$ contains a ball of radius $R'' < \frac{R}{M_{NFC}^3}$, where

$$(H_{F^{-\ell/2}(\hat{x}_{1/2}), new}^s)_* m_{F^{-\ell/2}(\hat{x}_{1/2})}^s (B(F^{-\ell/2}(\hat{x}_{1/2}), R'')) > \rho' (H_{F^{-\ell/2}(\hat{x}_{1/2}), new}^s)_* m_{F^{-\ell/2}(\hat{x}_{1/2})}^s (B(F^{-\ell/2}(\hat{x}_{1/2}), R')).$$

This relation doesn't change when we flow by $F^{\ell/2}$.

Meanwhile, flowing the ball $B(F^{-\ell/2}(\hat{x}_{1/2}), R'')$ by $F^{\ell/2}$ shrinks the radius to be

$$R'' e^{-t} < \frac{R}{M_{NFC}^3} e^{-t}.$$

This tells us that $F^{\ell/2}(B_X^s(F^{-\ell/2}(\hat{x}_{1/2}), R))$, in modified coordinates satisfies:

$$(H_{\hat{x}_{1/2}, new}^s)_* m_{\hat{x}_{1/2}}^s (F) \geq \rho' \cdot (H_{\hat{x}_{1/2}, new}^s)_* m_{\hat{x}_{1/2}}^s (F^{\ell/2}(B(F^{-\ell/2}(\hat{x}_{1/2}), R'))).$$

Take our neighbourhood T and pass to modified NFCs we called the image T' as in Lemma 4.4.21. This is a neighbourhood of diameter at most

$$C' r^{1+\frac{\beta}{2}} = C' (M_r^2 M_{NFC}^5 r'')^{1+\frac{\beta}{2}},$$

around the line c (a degree 1 curve) by Claim 4.4.18. Hence, by Equation (143), we have

that

$$C'(M_r^2 M_{NFC}^7 Re^{-t})^{1+\frac{\beta}{2}},$$

is the largest possible diameter of T' .

Meanwhile, since $F^{-\ell/2}(\hat{x}_{1/2}) \in K_C$, any neighbourhood $\tilde{T}$ (around curve degree $d = 1$) of diameter ϵ in a ball of radius R' based at the origin, denoted

$$B(F^{-\ell/2}(\hat{x}_{1/2}), R') \subset H_{F^{-\ell/2}(\hat{x}_{1/2}), new}^s(W^s(F^{-\ell/2}(\hat{x}_{1/2}))),$$

has measure at least $(1 - \delta')(H_{F^{-\ell/2}(\hat{x}_{1/2}), new}^s)_*(m_{\hat{x}, \omega}^s)(B(x, \omega, R'))$. We let $\tilde{T}$ be the neighbourhood of diameter ϵ around $c' = F^{-\ell/2}(c)$. (remember flowing in modified NFCs take lines to lines) Flowing $\tilde{T}$ forwards by $F^{\ell/2}$ is equivalent to flowing forwards by t under the time changed flow F_{tc} . The result is a neighbourhood of diameter at least ϵe^{-t} . Since ϵ is fixed, we can take ℓ (and hence t) large enough so that

$$C'(M_r^2 M_{NFC}^7 Re^{-t})^{1+\frac{\beta}{2}} < \epsilon e^{-t}.$$

So $F^{\ell/2}(\tilde{T})$ contains $T' \subset F$. Now we see that since $\frac{\delta'}{\rho'} < \delta''$ we have

$$\begin{aligned} (H_{\hat{x}_{1/2}, new}^s)_* m_{\hat{x}_{1/2}}^s(F \setminus T') &\geq (H_{\hat{x}_{1/2}, new}^s)_* m_{\hat{x}_{1/2}}^s(F) - (H_{\hat{x}_{1/2}, new}^s)_* m_{\hat{x}_{1/2}}^s(T') \\ &\geq (H_{\hat{x}_{1/2}, new}^s)_* m_{\hat{x}_{1/2}}^s(F) - \delta' (H_{\hat{x}_{1/2}, new}^s)_* m_{\hat{x}_{1/2}}^s(F^{\ell/2}(B(F^{-\ell/2}(\hat{x}_{1/2}), R'))) \\ &\geq \left(1 - \frac{\delta'}{\rho'}\right) (H_{\hat{x}_{1/2}, new}^s)_* m_{\hat{x}_{1/2}}^s(F) \\ &\geq (1 - \delta'') (H_{\hat{x}_{1/2}, new}^s)_* m_{\hat{x}_{1/2}}^s(F). \end{aligned}$$

This implies that

$$m_{\hat{x}_{1/2}}^s(F^{\ell/2}(B_X^s(F^{-\ell/2}(\hat{x}_{1/2}), R)) \setminus T) \geq (1 - \delta'') m_{\hat{x}_{1/2}}^s(F^{\ell/2}(B_X^s(F^{-\ell/2}(\hat{x}_{1/2}), R))).$$

This concludes the proof. □

Again, recall that ℓ is such that $F^{\ell/2}(\hat{x}_{1/2}), F^{-\ell/2}(\hat{x}_{1/2}) \in K$.

Lemma 4.4.23. *For ℓ large enough, $T \cap \eta_{\ell/2, \omega_{1/2}}^s(x_{1/2})$ has $m_{x, \omega}^s$ -measure at most*

$$\delta^k m_{x, \omega}^s(\eta_{\ell/2, \omega_{1/2}}^s(x_{1/2})).$$

Proof. We have by Claims 4.4.11 and 4.4.12 that

$$\eta_{\ell/2, \omega_{1/2}}^s(x_{1/2}) \subset F^{\ell/2}(B_X^s(F^{-\ell/2}(\hat{x}_{1/2}), R)).$$

Further, note that

$$\hat{x}_{1/2}, F^{\ell/2}(\hat{x}_{1/2}), F^{-\ell/2}(\hat{x}_{1/2}) \in K \subset K_R,$$

from Lemma 4.4.2, so $\eta_{\sigma^{-\ell/2}(\omega_{1/2})}^s(F_{\omega_{1/2}}^{-\ell/2}(x_{1/2}))$ is such that

$$m_{F^{-\ell/2}(\hat{x}_{1/2})}^s(\eta_{\sigma^{-\ell/2}(\omega_{1/2})}^s(F_{\omega_{1/2}}^{-\ell/2}(x_{1/2}))) \geq \rho \cdot m_{F^{-\ell/2}(\hat{x}_{1/2})}^s(B_X^s(F^{-\ell/2}(\hat{x}_{1/2}), R)).$$

When we apply $F^{\ell/2}$ this proportion stays the same, i.e.

$$m_{x_{1/2}, \omega_{1/2}}^s(\eta_{\ell/2, \omega_{1/2}}^s(x_{1/2})) \geq \rho \cdot m_{x_{1/2}, \omega_{1/2}}^s(F^{\ell/2}(B_X^s(F^{-\ell/2}(\hat{x}_{1/2}), R))).$$

Now Lemma 4.4.22 gives us that inside $F^{\ell/2}(B_X^s(F^{-\ell/2}(\hat{x}_{1/2}), R))$, T has measure at most δ'' which was taken to be strictly less than $\rho \cdot \delta^k$.

We can assume the worst case, i.e. that T entirely belongs to $\eta_{\ell/2, \omega_{1/2}}^s(x_{1/2})$. Then

$$\begin{aligned}
m_{x_{1/2}, \omega_{1/2}}^s(\eta_{\ell/2, \omega_{1/2}}^s(x_{1/2}) \setminus T) &= m_{x_{1/2}, \omega_{1/2}}^s(\eta_{\ell/2, \omega_{1/2}}^s(x_{1/2})) - m_{x_{1/2}, \omega_{1/2}}^s(T) \\
&\geq m_{x_{1/2}, \omega_{1/2}}^s(\eta_{\ell/2, \omega_{1/2}}^s(x_{1/2})) - \delta'' m_{x_{1/2}, \omega_{1/2}}^s(F^{\ell/s}(B_X^s(F^{-\ell/2}(\hat{x}_{1/2}, R)))) \\
&= m_{x_{1/2}, \omega_{1/2}}^s(\eta_{\ell/2, \omega_{1/2}}^s(x_{1/2})) - \delta^k \cdot \rho m_{x_{1/2}, \omega_{1/2}}^s(F^{\ell/s}(B_X^s(F^{-\ell/2}(\hat{x}_{1/2}, R)))) \\
&\geq (1 - \delta^k) m_{x_{1/2}, \omega_{1/2}}^s(\eta_{\ell/2, \omega_{1/2}}^s(x_{1/2})).
\end{aligned}$$

□

Now recall the definition of $\tilde{S}'$ in Equation (132). Note that Y'' is in the definition and that Equation (131) gives a bound on its $\nu \times \mu^{\mathbb{N}}$ -measure. We will take S' to be a subset of $\tilde{S}'$. The measure of $\tilde{S}'$ is given in Equation (133). Consider

$$N := \eta_{\ell/2, \omega_{1/2}}^s(x_{1/2}) \setminus (T \cap \eta_{\ell/2, \omega_{1/2}}^s(x_{1/2})).$$

We know by Lemma 4.4.23 that

$$m_{x_{1/2}, \omega_{1/2}}^s(N) \geq (1 - \delta^k) m_{x_{1/2}, \omega_{1/2}}^s(\eta_{\ell/2, \omega_{1/2}}^s(x_{1/2})).$$

Recall the definition of $(\Omega^-)' \subset \Omega^-$ from the beginning of the proof of Lemma 4.4.9 which has $\mu^{\mathbb{N}}$ -measure at least $(1 - \delta^k)$. Take

$$\begin{aligned}
S' &:= (N \times (\Omega^-)' \times \{\omega_{1/2}^+\} \times \{k_{1/2}\}) \cap \tilde{S}' \\
&= (Y'' \cap (N \times (\Omega^-)')) \times \{\omega_{1/2}^+\} \times \{k_{1/2}\}.
\end{aligned}$$

Since the measure of

$$(N \times (\Omega^-)' \times \{\omega_{1/2}^+\} \times \{k_{1/2}\}),$$

is bounded below by $(1 - \delta^k)^2 m_{x_{1/2}, \omega_{1/2}}^s(\eta_{\ell/2, \omega_{1/2}}^s(x_{1/2}))$, and $(1 - \delta^k)^2 \geq (1 - \delta/4)$, we see that

$$\hat{m}_{\hat{x}_{1/2}}^s(S') \geq (1 - \delta) \hat{m}_{\hat{x}_{1/2}}^s(\eta_{\ell/2}^s[\hat{x}_{1/2}]).$$

The set S' satisfies the measure requirements of Lemma 4.4.9 as in Equation (134) and Equation (135) holds. This concludes the proof of Lemma 4.4.9. $\square$

This concludes the proof of Lemma 2.6.7. This is because given our assumption that $\dim(\mathcal{L}^s) = 2$, we have given α_0 such that for a given δ , we found the sets K , S , S' of appropriate measure, and constants C and ℓ_0 (see Equations (130), and Lemma 4.4.9) as in the definition of QNI, Definition 3. Demonstrating that QNI holds is exactly what we wanted to show. $\square$

5 Proof of Proposition 2.6.9

In order to finish the case where $d_+ = 1$ and get alternative 2 of Theorem 1.1.1, we must now turn our attention to the proof of Proposition 2.6.9.

5.1 Outline of the proof of Proposition 2.6.9

Recall from Section 2.6 that the subscript loc in $U_{loc}^\pm(x, \omega)$, $W_{loc}^{s/u}(x, \omega)$ means that they are of size $q(x, \omega)$ as defined in Section 3.5.

In order to prove Proposition 2.6.9, we start by reducing it to something slightly weaker, the following proposition:

Proposition 5.1.1. *For m -a.e. $(\hat{x}, \hat{\omega}) \in Y$, there exists a neighbourhood U of $\hat{x}$ in X such that for ν -a.e. $\hat{y} \in U$ we have that $U_{loc}^+[\hat{y}, \hat{\omega}] \cap W_{loc}^s(\hat{x}, \hat{\omega}) \neq \emptyset$.*

Now we work by contradiction to prove Proposition 5.1.1. We assume there is a m -positive measure subset of Y , which we will call A where for any $(\hat{x}, \hat{\omega}) \in A$ and for any neighbourhood of $\hat{x}$ in X , there is a ν -positive measure subset of points $\hat{y}$ such that $U_{loc}^+[\hat{y}, \hat{\omega}] \cap W_{loc}^s(\hat{x}, \hat{\omega}) = \emptyset$. We say that the pairs $(\hat{x}, \hat{\omega})$ and $(\hat{y}, \hat{\omega})$ satisfy the non-jointly integrable (NJI) condition.

The proof of Proposition 5.1.1 is based on the work of [4] and [17], and uses results and techniques from [6]. By assuming that we have this set of positive measure such that NJI holds we will contradict what we demonstrated in Lemma 2.6.8, that each a.e. stable and unstable leaf has exactly 1 (smooth) real curve as the support of the conditional measure.

We will start by finding a pair of points in $X \times \Omega$ that satisfy the NJI condition and that have a $[0, 1[$ -component that allows both points to live in some ‘good’ subset of Z . This good subset will have properties that allow us to complete other aspects of the argument. The properties of most of the ‘good’ sets used in Section 5 will be described in Section 5.3.

Next, we will adjust these starting points by flowing by some very large time T . The purpose will be to get to points where the unstable planes are close, but note that the unstable supports are distinct (because of the NJI condition). We will follow the image of these unstable supports throughout the argument. The closeness of these unstable planes will be used in a future part of the argument. An argument that uses Birkhoff’s theorem, Lemma 3.10.15, will be used to ensure these points are also in a ‘good’ set that will allow other parts of the argument to go through. Once we have made these adjustments, we choose a new Ω -component for these points. Note that when we choose this new Ω -component, we will do so in a way that ensures we are still in a (different) ‘good’ set.

We will then define a ‘V’ shape for each choice of a parameter $m \in \mathbb{R}$; the legs of the ‘V’ are defined by flowing (both forwards and backwards) from our first pair of points (after flowing by T) to other pairs of points; there are 8 points (4 pairs of points) in total in our ‘V’. From our first pair of points (after flowing by T), we go backwards by m under the ‘standard’ flow and (after picking a new future) forwards by $\ell(m)$ under the ‘time change’

flow. We also flow forwards from the first pair of points (after flowing by T) by a quantity that allows the legs of the ‘V’ to be the same length w.r.t. the ‘time changed’ flow. Let ‘V1’ and ‘V2’ denote the two components of our ‘V’ (one for each point in our pair of fixed points); both are ‘V’ shapes in their own right. Note that if one switches $\ell(m)$ under the ‘time changed’ flow to a quantity in the standard flow, this amount will differ between the two components because of the point dependency of the roof function τ of the time change flow (see Section 3.9). This is why within one component, the quantity that allows the legs of the ‘V’ to be the same ($\ell(m)$ under the time change flow) will have to differ between ‘V1’ and ‘V2’. We call this quantity $d_1(m)$ for ‘V1’ and $d_2(m)$ for ‘V2’.

We will have to argue that d_1 and d_2 are Bi-Lipschitz in order to use a Birkhoff’s theorem argument to choose m such that flowing backwards from our fixed pair of points by m we land in a good set and flowing forwards from our fixed points by $d_1(m)$ and $d_2(m)$ resp. lands us in a different good set. The good set that we land in flowing forwards by $d_1(m)$ and $d_2(m)$ will allow us to use a lemma, denoted by [4] as the ‘Law of Last Jump’ whereby we can use the Martingale convergence theorem to land our last pair of points, the ones achieved from flowing by $\ell(m)$ under the time change flow, in a good set.

Meanwhile, the good set we have landed in after flowing backwards by m allows us to keep track of the distance between the image of the supports that we will compute in Lyapunov charts; this allows us to setup a situation where we pass to normal form coordinates. Normal form coordinates give us exacting control over this when flowing by ℓ once we project onto the unstable manifold of one of the points. The parameter we flow forwards by in normal form coordinates, $\ell(m)$, is computed as the distance we need to flow in order to separate the image of the supports to a distance greater than a fixed $\epsilon_f > 0$. In the end, we will use a limiting argument to get the supports in the same plane; this plane is achieved by sending ℓ to infinity, shrinking the stable components of the distance between the image of the supports. This results in having two distinct support lines in one unstable manifold. It

turns out that this suffices, we will argue this using the sets K_{00} and $\mathcal{W}$ defined in Section 5.3.

The way we send ℓ to infinity is by shrinking the distance between the initially chosen points (this will be explained in Section 5.3). The initial points will always be chosen in a ball of radius ρ and we will send ρ to 0. This will also force T and m to grow along with ℓ .

The parameter T will be chosen such that the unstable supports are very close to being ϵ_f -apart. Also the parameter T will be very big compared to m and ℓ . The parameter m will small compared to ℓ . Flowing by m ultimately undoes some of the good positioning we have achieved when flowing by T , so T has to grow faster than m . Still, m needs to grow because we need to be able to use the Birkhoff theorem argument to land the right endpoints of the ‘V’ in the good set. This will be explained later.

5.2 Reduction of Proposition 2.6.9 to Proposition 5.1.1

Assuming one can prove Proposition 5.1.1, the same argument (run backwards) proves the following:

Proposition 5.2.1. *For m -a.e. $(\hat{x}, \hat{\omega}) \in Y$, there exists a neighbourhood U of $\hat{x}$ in X such that for ν -a.e. $\hat{y} \in U$ we have that $U_{loc}^-[\hat{y}, \hat{\omega}] \cap W_{loc}^u(\hat{x}, \hat{\omega}) \neq \emptyset$.*

Remark 5.2.2. *Proposition 5.1.1 can be ‘flipped’, i.e. the statement may also say $U_{loc}^+[\hat{x}, \hat{\omega}] \cap W_{loc}^s(\hat{y}, \hat{\omega}) \neq \emptyset$ with everything else unchanged. Remarks 5.3.8 and 5.3.15 in Section 5.3 where NJI is used explain why. This is the only place where the NJI condition is used. We can also ‘flip’ Proposition 5.2.1 to say*

$$U_{loc}^-[\hat{x}, \hat{\omega}] \cap W_{loc}^u(\hat{y}, \hat{\omega}) \neq \emptyset.$$

With Propositions 5.1.1, 5.2.1 and Remark 5.2.2 in hand, we get Proposition 2.6.9:

Proof. (Proposition 2.6.9)

Proposition 5.1.1 gives us that we have a full m -measure subset $A_1 \subset Y$ such that for $(\hat{x}, \hat{\omega}) \in A_1$ there exists a neighbourhood U_1 of $\hat{x}$ in X such that for ν -a.e. $\hat{y} \in U_1$ we have

$$U_{loc}^+[\hat{y}, \hat{\omega}] \cap W_{loc}^s(\hat{x}, \hat{\omega}) = \{p\} \neq \emptyset.$$

Also (using Proposition 5.2.1 and Remark 5.2.2) there is a full measure subset $A_2 \subset Y$ where for $(\hat{x}, \hat{\omega}) \in A_2$ there exists a neighbourhood U_2 of $\hat{x}$ in X such that for ν -a.e. $\hat{y} \in U_2$ we have

$$U_{loc}^-[\hat{x}, \hat{\omega}] \cap W_{loc}^u(\hat{y}, \hat{\omega}) = \{q\} \neq \emptyset.$$

Note that these intersections must be only a point, otherwise the points would have two different behaviours, being in the unstable forces the points apart and being in the stable forces them together. Then on $A_1 \cap A_2$ (which is still full measure) letting $U = U_1 \cap U_2$ we get both results. This implies that $p = q$ because $U_{loc}^-[\hat{x}, \hat{\omega}] \subset W_{loc}^s(\hat{x}, \hat{\omega})$ and $U_{loc}^+[\hat{y}, \hat{\omega}] \subset W_{loc}^u(\hat{y}, \hat{\omega})$. Then we have that

$$U_{loc}^-[\hat{x}, \hat{\omega}] \cap U_{loc}^+[\hat{y}, \hat{\omega}] = \{p\} \neq \emptyset.$$

The complementary statement, that

$$U_{loc}^-[\hat{y}, \hat{\omega}] \cap U_{loc}^+[\hat{x}, \hat{\omega}] \neq \emptyset,$$

is proven similarly. □

5.3 Proof of Proposition 5.1.1

As explained above, we work by contradiction, so we assume there exists a set of positive m -measure $A \subset Y$, $m(A) > \epsilon_0$, such that if $(\hat{x}, \hat{\omega}) \in A$ then any neighbourhood U of $\hat{x}$ has a positive measure subset U of points $\hat{y}$ such that $U_{loc}^+[\hat{y}, \hat{\omega}] \cap W_{loc}^s(\hat{x}, \hat{\omega}) = \emptyset$. If two

points $(\hat{x}, \hat{\omega}), (\hat{y}, \hat{\omega}) \in Y$ are such that $U_{loc}^+[\hat{y}, \hat{\omega}] \cap W_{loc}^s(\hat{x}, \hat{\omega}) = \emptyset$ we say that they satisfy the non-jointly integrable condition, abbreviated as the NJI condition.

The ‘good’ sets for the proof of Proposition 5.1.1

Now let us list the sets we will need for the proof of Proposition 5.1.1 and their properties.

We also discuss their size and how they will be used. First let

$$L := \max\{2\ln(e^{-(\lambda^- - \epsilon)} + \frac{1}{2}C) + \lambda^+ + \epsilon, \frac{1}{\lambda^+ + \epsilon}\}.$$

This quantity will be derived in Section 5.3 in the proof of Lemma 5.3.36, see Equation (220). Let $\mathcal{L}$ be upper bound on the Lipschitz constants of the automorphisms in $\text{Supp}(\mu)$ and let $\mathcal{K} > 0$ be such that $\frac{1}{\mathcal{L}} = e^{-\mathcal{K}}$. Fix $\tilde{\epsilon} > 0$ small. Let $K_{3\tilde{\epsilon}}$ be the set defined in Section 3.6 for $\epsilon = 3\tilde{\epsilon}$. This Lusin set has size $1 - \epsilon_{hol}$ for some ϵ_{hol} small. We can ask ϵ_{hol} to be as small as needed and get a corresponding $\tilde{N}$.

Also, let $C_b = \frac{(\lambda^+ - \delta)}{2\ln(\mathcal{L})}$ and fix i such that $C_b > \frac{1}{10^i}$, $\frac{C_b}{10^{i+2}} \ll (\lambda^+ + \delta)$ so that

$$\frac{\lambda^+ + \delta}{\frac{C_b}{10^{i+2}}} \gg \ln(k\epsilon_f M_r/2) + (\lambda^+ + \epsilon), \quad (144)$$

and $\frac{\frac{C_b}{10^{i+2}}}{e^{(\lambda^+ + \delta)}} < 1$ (see Equations (192), (194) and (174) respectively).

Let J be chosen such that $|\frac{C_b}{10^{i+J}}(\lambda^- - \epsilon)| < |\frac{C_b}{10^{i+2}}(\lambda^- + \epsilon)|$ (needed for Equation (206)). Let $\alpha_1 = \frac{C_b}{10^{i+J}}$ and $\alpha_2 = \frac{C_b}{10^{i+2}}$, these choices will be used in the definition of the parameter m but we can define them here which is important to the argument of Lemma 5.3.38. Fix some $\epsilon' > 0$ small such that the expression

$$\mathcal{T} = \epsilon' - \frac{(\lambda^- - \epsilon)\alpha_1}{-(\lambda^- + \epsilon)\alpha_2 - \kappa} + \frac{(\lambda^- + \epsilon)\alpha_2}{-(\lambda^- + \epsilon)\alpha_2 - \kappa},$$

is positive. Then fix $\eta < \mathcal{T}$ and let $\mathcal{S} = \left(1 - \frac{1}{1+\eta}\right)$. We use these choices to define sets we need to use Lemma 3.10.15 in Lemma 5.3.38, one can see this choices made in the definitions of $\tilde{K}_2$, E_2^c , K_3 and E_3^c . This must be fixed from the beginning.

All the ϵ terms used in the definitions below will be assumed sufficiently small for their purpose unless otherwise stated more explicitly. Sometimes the size of these sets, stated here, will be proven later.

- $\mathcal{W} \subset Z$,

$$\mathcal{W} = \{(x, \omega, k) \in Z : \mathcal{L}^u(x, \omega, k) \not\supseteq U^+[x, \omega, k]\},$$

by Lemma 2.6.8 we know that this set has measure zero.

- $K_{00} \subset Z$, arbitrary compact subset of size $\hat{m}(K_{00}) > 1 - \epsilon_{00}$. We assume that $K_{00} \subset \mathcal{W}^c$, we can do this since $\hat{m}(\mathcal{W}^c) = 1$.
- $K'_0 \subset Y$, $K'_0 \subset \Lambda \cap \Lambda_{loc} \cap K_s \cap K_r \cap K_{NFC} \cap K_\xi$, $m(K'_0) > 1 - \frac{\epsilon_0}{2}$.
- $K_0 \subset Y$, $K_0 \subset \Lambda \cap \Lambda_{loc} \cap K_s \cap K_r \cap K_{NFC} \cap K_\xi$, $m(K_0) \geq m(K'_0)$.
- $K'_1 \subset Y$, $K'_1 \subset \Lambda \cap \Lambda_{loc} \cap K_s \cap K_r \cap K_{NFC} \cap K_\xi \cap K'_\xi \cap K_{exp} \cap \tilde{\Lambda} \cap K_{3\tilde{\epsilon}} \cap K_G \cap K_\epsilon$, $m(K'_1) > 1 - \epsilon'_1$.
- $K_1 \subset Y$, $K_1 \subset \Lambda \cap \Lambda_{loc} \cap K_s \cap K_r \cap (K_{exp} \times \Omega) \cap K_\epsilon$, $m(K_1) > 1 - \epsilon_1$.
- $K'_2 \subset Y$, $K'_2 \subset \Lambda \cap \Lambda_{loc} \cap K_s \cap K_\xi \cap K_b \cap K_{ang} \cap K_{exp} \cap (K_\epsilon \times \Omega)$, $m(K'_2) > 1 - \epsilon'_2$.
- $K_2 \subset Y$, $K_2 \subset \Lambda \cap \Lambda_{loc} \cap \Lambda' \cap K_s \cap K_r \cap K_\xi \cap K_{NFC} \cap K_{new} \cap K_G \cap K_b \cap K_{exp} \cap (K_\epsilon \times \Omega)$, $m(K_2) > 1 - \epsilon_2$.
- $B' \subset X \times \Omega^-$,

$$B' := \{(x, k) : \exists V_{(x, \omega^-)} \subset \Omega^+, \mu^{\mathbb{N}}(V_{(x, \omega^-)}) > 1 - \sqrt{\epsilon},$$

such that if $\omega^+ \in V_{(x, \omega^-)}$, then $(x, \omega^-, \omega^+) \in K_2\}$,

$$\nu \times \mu^{\mathbb{N}}(B') > 1 - \sqrt{\epsilon_2}.$$

- $\tilde{K}_2 \subset Y$, $\tilde{K}_2 := (B' \times \Omega^+) \cap K'_2$, ϵ_2 and ϵ'_2 small such that the corresponding measure of $\tilde{K}_2$ which we call $\gamma_2 = 1 - \tilde{\epsilon}_2$ satisfies

$$\gamma_2 > 1 - \frac{1}{200L^4} > 3/4.$$

- $E_2^c \subset Z$, take $\delta = \alpha_2(\tilde{K}_2)$ such that $\beta_2 := 2\gamma_2(1 - \alpha_2) - 1 > \frac{1}{200L^4}$, $\epsilon \ll \epsilon_1$, $K = \tilde{K}_2 \times [0, 1[$ and apply Lemma 3.10.15, we get $S > 0$ such that the set

$$E_2^c := \{(x, \omega, k) : \forall T > S, |\frac{1}{T} \int_0^T \mathbb{1}_K(F^t(x, \omega, k))dt - \int_Y \mathbb{1}_K d\hat{m}| < \delta\},$$

has $\hat{m}(E_2^c) > 1 - \epsilon$.

- $K_3 \subset Z$, $K_3 \subset (\Lambda \cap \Lambda_{loc}) \times [0, 1[$, additionally, K_3 satisfies the conditions of Lemma 3.9.3 and belongs to the $\hat{m}$ -conull set from Lemma 5.3.43. It further satisfies the properties of the set L' in Lemma 5.3.45. The size of this set is $\gamma_3 := 1 - \epsilon_3$ where

$$\gamma_3 > (1 - \frac{1}{100L^2}) + \frac{1}{100L^2} \left(\frac{1}{1 + \eta} \right) > 3/4.$$

- $E_3^c \subset Z$, take $\delta = \alpha_3\gamma_3$ such that

$$\beta_3 = 2\gamma_3(1 - \alpha_3) - 1 > (1 - \frac{1}{100L^2}) + \frac{1}{100L^2} \left(\frac{1}{1 + \eta} \right),$$

$\epsilon \ll \epsilon_1$, $K = K_3$ and apply Lemma 3.10.15, we get $S > 0$ such that the set

$$E_3^c := \{(x, \omega, k) : \forall T > S, |\frac{1}{T} \int_0^T \mathbb{1}_K(F^t(x, \omega, k))dt - \int_Y \mathbb{1}_K d\hat{m}| < \delta\},$$

has $\hat{m}(E_3^c) > 1 - \epsilon$.

- $M := (K_1 \times [0, 1]) \cap E_2^c \cap E_3^c$, $\hat{m}(M) > 1 - \epsilon_M$.
- $B \subset X \times [0, 1]$, fix $\theta > 0$ small

$$B := \{(x, k) : \exists V_{(x,k)} \subset \Omega, \mu^{\mathbb{Z}}(V_{(x,k)}) > 1 - \theta, \\ \text{such that if } \omega \in V_{(x,k)}, \text{ then } (x, \omega, k) \in M\},$$

$$\nu \times dt(B) > 1 - \frac{\epsilon_M}{\theta}.$$

- $\tilde{K}_1 \subset Z$, $\tilde{K}_1 := (B \times \Omega) \cap (K'_1 \times [0, 1])$, ϵ_M, ϵ'_1 are small so that $\hat{m}(\tilde{K}_1) > 3/4$.
- $K_4 \subset Y$, $K_4 \subset \Lambda \cap \Lambda_{loc} \cap K_{new} \cap K_{NFC}$.
- $K'_4 \subset Z$, $K'_4 = (K_4 \times [0, 1]) \cap K_{00}$.
- $E^c \subset Z$, take $\delta = \frac{1}{100} \hat{m}((\tilde{K}_1) \cap E_2^c \cap E_3^c)$, $\epsilon \ll m(B)$, $K = \tilde{K}_1$ and apply Lemma 3.10.15, we get $S > 0$ such that the set

$$E^c := \{(x, \omega, k) : \forall T > S, |\frac{1}{T} \int_0^T \mathbb{1}_K(F^t(x, \omega, k)) dt - \int_Y \mathbb{1}_K d\hat{m}| < \delta\}.$$

These sets are not in the order they appear in the proof because the definition of the earlier sets often requires foresight from later in the proof.

A brief sketch on how good sets will be used

- $\mathcal{W}$ - we know this set has zero measure by Lemma 2.6.8 but we will show this is not the case when we assume that Proposition 5.1.1 is false. This will prove Proposition 5.1.1.
- K_{00} - We will land in this set at the end and use it to show that $\mathcal{W}$ must have positive measure when we assume that Proposition 5.1.1 is false. This will prove Proposition 5.1.1.

- $K'_0 \subseteq K_0 \subset Y$ - Our starting points, $(\hat{x}, \hat{\omega})$ and $(\hat{y}, \hat{\omega})$ will be chosen from here. A placeholder set, K'_0 will be used at the beginning of the argument in Section 5.3, but ultimately the starting points will live in K_0 .
- $K'_1 \subset Y$ - After flowing the starting points $(\hat{x}, \hat{\omega}, k_0)$ and $(\hat{y}, \hat{\omega}, k_0)$ by some time T forward under the ‘standard’ flow, we want the points $F^T(\hat{x}, \hat{\omega})$ and $F^T(\hat{y}, \hat{\omega})$ to land in this set.
- $K_1 \subset Y$ - After flowing the starting points $(\hat{x}, \hat{\omega}, k_0)$ and $(\hat{y}, \hat{\omega}, k_0)$ by T under the ‘standard’ flow we get the points $F^T(\hat{x}, \hat{\omega})$ and $F^T(\hat{y}, \hat{\omega})$. We denote the X -components as $\tilde{x} := f_\omega^n(\hat{x})$ and $\tilde{y} := f_\omega^n(\hat{y})$ where $n = \lfloor T + k_0 \rfloor$. We will change the Ω -component of $F^T(\hat{x}, \hat{\omega})$ and $F^T(\hat{y}, \hat{\omega})$ to some $\tilde{\omega} \in \Omega$ so that $(\tilde{x}, \tilde{\omega})$ and $(\tilde{y}, \tilde{\omega})$ belong to this set K_1 .
- $E_2^c, E_3^c \subset Z$ - When we take the points $F^T(\hat{x}, \hat{\omega}, k_0)$ and $F^T(\hat{y}, \hat{\omega}, k_0)$ and change the Ω -component to $\tilde{\omega}$, we need the resulting points to belong to E_2^c and E_3^c . This allows us to run arguments in Sections 5.3 and 5.3, related to Lemma 3.10.15, to get appropriate m such that $m, d_1(m)$ and $d_2(m)$ are good times to flow by from the points $(\tilde{x}, \tilde{\omega})$ and $(\tilde{y}, \tilde{\omega})$ to land in sets $\tilde{K}_2$ and K_3 resp.
- $B \subset X \times [0, 1[$ - After flowing $(\hat{x}, \hat{\omega}, k_0)$ and $(\hat{y}, \hat{\omega}, k_0)$ by some time T forward under the ‘standard’ flow, we want the $X \times [0, 1[$ -component of the resulting points to land in this set so that we will be able to pick a new ω so that $(x, \omega, k) \in M$.
- $\tilde{K}_1 \subset Z$ - We land in this set after flowing by T from the starting points. Note that $K'_1 \times [0, 1[$ is part of the definition of this set.
- $K'_2 \subset Y$ - After flowing $(\tilde{x}, \tilde{\omega})$ and $(\tilde{y}, \tilde{\omega})$ backwards by m under the ‘standard’ flow we get the points $F^{-m}(\tilde{x}, \tilde{\omega})$ and $F^{-m}(\tilde{y}, \tilde{\omega})$ which need to be in this set K'_2 .

- $K_2 \subset Y$ - After flowing by $-m$ under the ‘standard’ flow we get the points $F^{-m}(\tilde{x}, \tilde{\omega}, k_1)$ and $F^{-m}(\tilde{y}, \tilde{\omega}, k_1)$. We denote the X -components by $x := f_{\omega'}^{n'}(\hat{x})$ and $y := f_{\omega'}^{n'}(\hat{y})$ where $n' = \lfloor k_1 - m \rfloor$. We will change the Ω^+ -component of $F^{-m}(\tilde{x}, \tilde{\omega}, k_1)$ and $F^{-m}(\tilde{y}, \tilde{\omega}, k_1)$ to get some $\omega' \in \Omega$ so that (x, ω') and (y, ω') belong to the set K_2 .
- $B' \subset X \times \Omega^-$ - After flowing $(\tilde{x}, \tilde{\omega}, k_1)$ and $(\tilde{y}, \tilde{\omega}, k_1)$ by some time m backwards under the ‘standard’ flow, we want the points $f_{\omega'}^{n'}(x) = x$ and $f_{\omega'}^{-n}(y) = y$ to be such that $(x, \sigma^{n'}(\tilde{\omega}^-))$ and $(y, \sigma^{n'}(\tilde{\omega}^-))$ land in this set, where $n' = \lfloor k_1 - m \rfloor$ and $\tilde{\omega}^-$ is the Ω^- -component of $\tilde{\omega}$. This way we can pick a new future $\omega^+ \in \Omega^+$ so that $(x, (\omega^-, \omega^+)) \in K_2$.
- $\tilde{K}_2 \subset Y$ - when we flow backwards by m starting from $(\tilde{x}, \tilde{\omega}, k_1)$ and $(\tilde{y}, \tilde{\omega}, k_1) \in E_2^c$ we take $K = \tilde{K}_2 \times [0, 1[$ as the set in Lemma 3.10.15. The bound on the measure of this set is for an argument in Section 5.3, Lemma 5.3.41.
- $K_3 \subset Z$ - Starting from the points $(\tilde{x}, \tilde{\omega}, k_1)$ and $(\tilde{y}, \tilde{\omega}, k_1)$, we will flow forwards by $d(m, (\tilde{x}, \tilde{\omega}, k_1))$ and $d(m, (\tilde{y}, \tilde{\omega}, k_1))$ resp. under the ‘time change’ flow. We want the resulting points to land in the set K_3 . The lower bound on the measure of this set is for the purposes of Lemma 5.3.41 in Section 5.3.
- $K_4 \subset Y, K'_4 \subset Z$ - From the points (x, ω') and (y, ω') we will flow forwards by ℓ under the ‘time changed’ flow. We want the points $F^\ell(x, \omega')$ and $F^\ell(y, \omega')$ to land in the set K_4 . The set K'_4 just extends the properties of this set to Z .
- $E^c \subset Z$ - The original points $(\hat{x}, \hat{\omega})$ and $(\hat{y}, \hat{\omega})$ must belong to E^c for some choice of $k \in [0, 1[$ and it is demonstrated that we can do this in Section 5.3.

Notation for Section 5

Here we recall/define a few constants that are needed throughout the rest of this section. It is important to note that all of these constants can be defined before beginning the argument.

Recall that we are making the assumption that $\text{Supp}(\mu)$ is a finite set. We let $\mathcal{L}$ be the upper bound on the Lipschitz constant and we can assume $\mathcal{L}$ is at least 3. We also let C be the upper bound on the second derivative of any automorphism in $\text{Supp}(\mu)$ as in Equation (61) (given by $C_{\tilde{f}_i}$).

Recall $\tilde{\epsilon} > 0$ from Section 5.3 and define $\alpha = \frac{a-b}{a-d}$ where $a = -(\lambda^- - 3\tilde{\epsilon})$, $b = -(\lambda^+ + 3\tilde{\epsilon})$ and $d = \ln(2(C^-)^2) + 2c_0^- + |\ln(\rho_0/4)| + |a|$ where C^- and c_0^- are chosen uniformly on Ω_ϵ as in Lemma 3.6.3 and ρ_0 is as in Section 3.6.

Lastly, let ϵ_f be such that

$$\epsilon_f < \frac{1}{2M_r} \max \left\{ q/2, k_U/2, \rho_0/4, \frac{1}{2M_r}, \frac{2k_1 e^{-(\lambda^- + \epsilon)}}{CM_r}, \frac{2k_2 e^{-(\lambda^+ + \epsilon)}}{CM_r} \right\}, \quad (145)$$

where $k_1, k_2 < 1$ are fixed such that $(1 - k_1)e^{-(\lambda^- + \epsilon)} > 1$, k and q are from the definition of the set K_{loc} in Section 3.5, and M_r and k_U are from Section 3.10.

The starting points

We need to pick points $(\hat{x}, \hat{\omega}), (\hat{y}, \hat{\omega}) \in Y$ such that $(\hat{x}, \hat{\omega}) \in A$, and both $(\hat{x}, \hat{\omega})$ and $(\hat{y}, \hat{\omega})$ belong to a set with the features of K_0 as defined in Section 5.3. We will need to use an intermediate set K'_0 and enlarge it at the end to get the points in K_0 ; the points also need to satisfy the NJI condition with respect to each other. Our approach relies on the fact that m is the product measure and Fubini's theorem; we will use the following lemma:

Lemma 5.3.1. *Let (X_1, η_1) and (X_2, η_2) be σ -finite measure spaces. Consider $V \subset X_1$, a set such that $\eta_1(V) > 0$, and for each point $a \in V$ consider a set $W_a \subset X_2$ such that*

$\eta_2(W_a) > 0$. Then $S := \bigcup_{a \in V} W_a \times \{a\}$ has positive measure with respect to the $\eta_1 \times \eta_2$ on $X_1 \times X_2$.

Proof. Writing $S = \{(w, a) : w \in W_a, a \in V\}$, we apply Fubini's theorem:

$$\eta_1 \times \eta_2(S) = \int_S d\eta_1 d\eta_2 = \int_V \int_{W_a} d\eta_2 d\eta_1 = \int_V \eta_2(W_a) d\eta_1.$$

Let $V_n = \{a \in V : \eta_2(W_a) > 1/n\} \subset V$, then $V_n \rightarrow V$ as $n \rightarrow \infty$. V has positive measure, so there exists $N > 0$ such that V_N has positive measure as well. Then,

$$\eta_1 \times \eta_2(S) = \int_V \eta_2(W_a) d\eta_1 \geq \int_{V_N} \eta_2(W_a) d\eta_1 = \int_{V_N} \frac{1}{N} d\eta_1 = \eta_1(V_N) \cdot \frac{1}{N} > 0.$$

This completes the proof. □

Let K'_0 be defined as in Section 5.3. Since $m(K'_0) > 1 - \frac{\epsilon_0}{2}$ and $m(A) > \epsilon_0$, we have that $B_1 := A \cap K'_0 \subset Y$ has positive m -measure. Recall the natural projection map $\pi_\Omega : Y \rightarrow \Omega$. Since B_1 has positive m -measure and m is the product measure, we must have that $\pi_\Omega(B_1)$ has positive $\mu^\mathbb{Z}$ -measure in Ω ; if not, and $\pi_\Omega(B_1)$ has zero measure, then the set $X \times \pi_\Omega(B_1)$ has measure zero and contains B_1 . This is a contradiction.

For each $(z, \omega) \in B_1 \subset A$, every neighbourhood of z in X has a positive measure subset of points y in X such that NJI holds. Fix some $\rho > 0$, and for each $(z, \omega) \in B_1$ take a ball of radius ρ around z and let $C_{(z, \omega)}$ be the positive measure subset of this ball of radius ρ where NJI holds. Define $\tilde{B}_1 := (B_1 \times [0, 1]) \cap E^c \subset Z$.

Lemma 5.3.2. *There exists a set $D \subset Z$ is such that if we make K_0 large enough, $D \cap (K_0 \times [0, 1])$ has positive measure. Then if $(\hat{y}, \hat{\omega}, k_0) \in D \cap (K_0 \times [0, 1])$, there is a corresponding $(\hat{x}, \hat{\omega}, k_0) \in (B_1 \times [0, 1]) \cap E^c$, with $(\hat{x}, \hat{\omega}) \in A$ such that $(\hat{x}, \hat{\omega})$ has the NJI condition with $(\hat{y}, \hat{\omega})$.*

Proof. For each $(z, \omega, k) \in \tilde{B}_1$ (this set has positive $\hat{m}$ -measure by the choices we made), we

take $C_{(z,\omega)} \subset X$ and define $D_{(z,\omega,k)} = C_{(z,\omega)} \times \{(\omega, k)\} \subset Z$. Note that because $\tilde{B}_1$ has positive $\hat{m}$ -measure, the projection of $\tilde{B}_1$ onto $\Omega \times [0, 1[$ has positive measure as well. One defines $D = \bigcup_{(z,\omega,k) \in \tilde{B}_1} D_{(z,\omega,k)}$ and by Lemma 5.3.1, we have that D has positive measure. Take K_0 to have all the specified properties including that it contains K'_0 , and large enough such that $D \cap (K_0 \times [0, 1[)$ has positive measure. Take $(\hat{y}, \hat{\omega}, k_0)$ from $D \cap (K_0 \times [0, 1[)$, it comes with the corresponding $(\hat{x}, \hat{\omega}, k_0) \in (B_1 \times [0, 1[) \cap E^c$ (where E^c is as defined in Section 5.3) and naturally they have the NJI condition as needed by construction. $\square$

We use Lemma 5.3.2 to get our starting points.

Remark 5.3.3. *The NJI condition between $(\hat{x}, \hat{\omega})$ and $(\hat{y}, \hat{\omega})$ gives us that $\hat{y}$ does not belong to $W^s(\hat{x}, \hat{\omega})$.*

Remark 5.3.4. *It is worth noting that in the construction above, we can shrink ρ as small as needed and always find points $(\hat{x}, \hat{\omega})$ and $(\hat{y}, \hat{\omega})$ that satisfy our conditions. This allows us to find points of arbitrary closeness, or more precisely, find $\hat{y}$ arbitrarily close to $\hat{x}$.*

Flowing by T

Using Lemma 5.3.2, we choose our starting points, $(\hat{x}, \hat{\omega}, k_0)$ and $(\hat{y}, \hat{\omega}, k_0)$ such that $d_X(\hat{x}, \hat{y}) \ll \rho$ where

$$\rho \ll \frac{\epsilon_f}{10000},$$

(see Remark 5.3.4).

These starting points belong to K_{loc} by definition of the set K_0 (see Section 5.3), this means we can define local stable and unstable manifolds for these points of size q , (see the definition of q in Section 3.5.4) note that $\epsilon_f < q/4$ so the scale we are working in is much smaller.

In this section we will take these points $(\hat{x}, \hat{\omega}, k_0)$ and $(\hat{y}, \hat{\omega}, k_0)$ and flow them by some very large time T under this $\hat{\omega}$ (using the ‘standard’ flow) to put them in a ‘good’ position.

What we mean by this is that after flowing by T we want the image of $W_q^u(\hat{x}, \hat{\omega})$ and $W_q^u(\hat{y}, \hat{\omega})$ (i.e. $W_q^u(F^T(\hat{x}, \hat{\omega}))$ and $W_q^u(F^T(\hat{y}, \hat{\omega}))$) to be close (in some ball), but still want the image of the unstable supports, i.e. the image under F^T of $U_q^+[\hat{x}, \hat{\omega}]$ and $U_q^+[\hat{y}, \hat{\omega}]$ (i.e. $U_q^+[F^T(\hat{x}, \hat{\omega})]$ and $U_q^+[F^T(\hat{y}, \hat{\omega})]$) to be bounded apart. The point of this is to have $U_q^+[F^T(\hat{x}, \hat{\omega})]$ and $U_q^+[F^T(\hat{y}, \hat{\omega})]$ roughly in the same complex plane. This will be used later in Section 5.3.

Consider $\hat{z} \in W_q^s(\hat{x}, \hat{\omega}) \cap W_q^u(\hat{y}, \hat{\omega})$. Let $n = \lfloor T + k_0 \rfloor$ and denote $\tilde{x} = F_\omega^n(\hat{x})$, $\tilde{y} = F_\omega^n(\hat{y})$, and $\tilde{z} = F_\omega^n(\hat{z})$. We will see later in this Section that T can be chosen such that $(\tilde{x}, \sigma^n(\hat{\omega}))$ and $(\tilde{y}, \sigma^n(\hat{\omega}))$ belong to the set $\tilde{\Lambda} \subset K'_1$ as defined in Section 3.6. For now we make the following standing assumption:

Assumption 1. *The parameter T can be chosen such that $(\tilde{x}, \sigma^n(\hat{\omega}))$ and $(\tilde{y}, \sigma^n(\hat{\omega}))$ belong to the set K'_1 as defined in Section 5.3. We demonstrate that we can choose the parameter T in this way in Lemma 5.3.19.*

Take $\tilde{\epsilon} > 0$ from the beginning of Section 5.3, we get $N_u \in \mathbb{N}$ such that $\tilde{x}, \tilde{y} \in \Lambda_{\tilde{\epsilon}, N_u}^{u, \sigma^n(\hat{\omega})}$ (see Section 3.6).

Lemma 5.3.5. *The point $\tilde{z}$ belongs to $\Lambda_{3\tilde{\epsilon}, 3N_u}^{u, \sigma^n(\hat{\omega})}$.*

Proof. Since $\tilde{y}$ belongs to $\Lambda_{\tilde{\epsilon}, N_u}^{u, \sigma^n(\hat{\omega})}$ and $\tilde{z} \in W^u(\tilde{y}, \sigma^n(\hat{\omega}))$, this follows from [39], Theorem 4.1; see Theorem A.1.9 of [6]. $\square$

The choices of Lusin sets we made in Section 5.3 will give a Hölder continuity between $E^u(\tilde{x}, \sigma^n(\hat{\omega}))$ and $E^u(\tilde{z}, \sigma^n(\hat{\omega}))$ in the next lemma. The Hölder exponent and constant will be the same for any choice of starting points belonging to K_0 and any choice of T that lands us in $\tilde{\Lambda} \subset K'_1$. This uniformity is required later in the argument.

Lemma 5.3.6. *The points $\tilde{x}$ and $\tilde{z}$ belong to a set on which we have α -Hölder continuity of $E^u(\cdot, \sigma^n(\hat{\omega}))$, where α is as defined in Section 5.3. Additionally, the Holder constant can be taken to be*

$$C_\alpha := 2 \left(\max_{\text{Supp}(\mu)} \{1, \|Df_i\|\} \right)^{2\tilde{N}} e^{(\lambda^- - \lambda^+ - 6\tilde{\epsilon})},$$

where $\tilde{N}$ is defined in Section 5.3.

Proof. Since $\Lambda_{\tilde{\epsilon}, N_u}^{u, \sigma^n(\hat{\omega})} \subset \Lambda_{3\tilde{\epsilon}, 3N_u}^{u, \sigma^n(\hat{\omega})}$ we have that $\tilde{x}, \tilde{z} \in \Lambda_{3\tilde{\epsilon}, 3N_u}^{u, \sigma^n(\hat{\omega})}$.

We showed in Section 3.6 that for

$$L_u = (\max_{\text{Supp}(\mu)} \{1, \|Df_i\|\})^{N_u(3\tilde{\epsilon})},$$

we have that

$$\Lambda_{3\tilde{\epsilon}, 3N_u}^{u, \sigma^n(\hat{\omega})} \subset \Delta_{-(\lambda^- - 3\tilde{\epsilon}), -(\lambda^+ + 3\tilde{\epsilon}), L_u}.$$

We know that $\sigma^n(\hat{\omega}) \in \Omega_0^-$ (because $(\tilde{x}, \sigma^n(\hat{\omega}))$ and $(\tilde{y}, \sigma^n(\hat{\omega}))$ belong to $\tilde{\Lambda}$). Hence, by Proposition 3.6.4 we have α -Holder continuity of $E^u(\cdot, \sigma^n(\hat{\omega}))$ (with constant $C_{N_u(3(\tilde{\epsilon}))} = 2L_u^2 e^{(\lambda^- - \lambda^+ - 6\tilde{\epsilon})}$) on the set $\Delta_{-(\lambda^- - 3\tilde{\epsilon}), -(\lambda^+ + 3\tilde{\epsilon}), L_u}$ to which $\tilde{x}$ and $\tilde{z}$ belong (because $\tilde{x} \in \Lambda_{\tilde{\epsilon}, N_u}^{u, \sigma^n(\hat{\omega})} \subset \Lambda_{3\tilde{\epsilon}, 3N_u}^{u, \sigma^n(\hat{\omega})}$). We get that

$$d_{Gr}(E^u(\tilde{x}, \sigma^n(\hat{\omega})), E^u(\tilde{z}, \sigma^n(\hat{\omega}))) \leq C_{N_u(3(\tilde{\epsilon}))} d_X(\tilde{x}, \tilde{z})^\alpha. \quad (146)$$

Since $(\tilde{x}, \sigma^n(\hat{\omega})), (\tilde{y}, \sigma^n(\hat{\omega})) \in K'_1$ they belong to $K_{3\tilde{\epsilon}}$ and hence N_u is bounded above by $\tilde{N}$. This means that regardless of our starting points, the Hölder constant in Equation (146), $C_{N_u(3(\tilde{\epsilon}))}$ is bounded above by

$$C_\alpha := 2(\max_{\text{Supp}(\mu)} \{1, \|Df_i\|\})^{2\tilde{N}} e^{(\lambda^- - \lambda^+ - 6\tilde{\epsilon})}.$$

Also regardless of our starting points, we will always have α -Hölderness where α is defined at the beginning of this section. $\square$

Lemma 5.3.7. *The unstable manifold $W^u(\tilde{y}, \sigma^n(\hat{\omega}))$ is approximated by $\exp_z(E^u(\tilde{z}, \sigma^n(\hat{\omega})))$ up to error $\xi = \mathfrak{r}^2 \cdot M_r^3 \cdot H$ on the ball of radius $\mathfrak{r}$ where H is the uniform upper bound on $\text{Lip}(Dh_{(x, \omega)}^u)$ for $(x, \omega) \in K'_\xi$ (see Lemma 3.10.9).*

Proof. Since $(\tilde{y}, \sigma^n(\hat{\omega})) \in K_\xi$, we know that the local unstable, $W^u(\tilde{y}, \sigma^n(\hat{\omega}))$, is well-approximated by (the image under $\exp$ of) $E^u(\tilde{z}, \sigma^n(\hat{\omega}))$ up to error ξ on the ball of radius $\mathfrak{r}$. For given $\mathfrak{r}$, the error is given by $\xi = \mathfrak{r}^2 \cdot M_r^3 \cdot \text{Lip}(Dh_{(\tilde{y}, \sigma^n(\hat{\omega}))}^u)$ (where M_r is fixed from Section 3.10 and $h_{(\tilde{y}, \sigma^n(\hat{\omega}))}$ is as in Proposition 3.5.4). Note that $\text{Lip}(Dh_{(\tilde{y}, \sigma^n(\hat{\omega}))}^u)$ is bounded above by H as in Lemma 3.10.9 because $(\tilde{y}, \sigma^n(\hat{\omega})) \in K'_\xi$. $\square$

Lemmas 5.3.6 and 5.3.7 motivate us to contract the stable distance between $\hat{x}$ and $\hat{y}$ which is $d_X(\hat{x}, \hat{z})$. We will also be interested in increasing the unstable distance between $\hat{x}$ and $\hat{y}$ which is $d_X(\hat{y}, \hat{z})$. We will take $\mathfrak{r}$ in the above lemma to be twice the resulting unstable distance. This is the scale we want to view things on.

Remark 5.3.8. *If the NJI condition was instead written as $W^s(\hat{y}, \hat{\omega}) \cap U^+[\hat{x}, \hat{\omega}] = \emptyset$ we would instead take $\hat{z} \in W_q^s(\hat{y}, \hat{\omega}) \cap W_q^u(\hat{x}, \hat{\omega})$ and work to compare $E^u(\tilde{y}, \sigma^n(\hat{\omega}))$ and $E^u(\tilde{z}, \sigma^n(\hat{\omega}))$ in Lemma 5.3.6 (where $n = \lfloor T + k_0 \rfloor$). Then we would care about $\tilde{y}$ and $\tilde{z}$ being close. Also note that the stable distance between $\tilde{x}$ and $\tilde{y}$ would now be the distance between $\tilde{y}$ and $\tilde{z}$ and the unstable distance would now be the distance between $\tilde{x}$ and $\tilde{z}$.*

By Theorem 3.5.4 we know exactly how the distance between $\hat{z}$ and $\hat{x}$ contracts and $\hat{z}$ and $\hat{y}$ expands.

Let $T_{big} \in \mathbb{R}$ be such that $d_X(\tilde{y}, \tilde{z}) = \epsilon_f$, let $n_B = \lfloor T_{big} + k_0 \rfloor$. We know we can reach ϵ_f in the unstable distance because by Theorem 3.5.4 we have

$$d_X(\tilde{y}, \tilde{z}) \geq \frac{1}{k} d_X(\hat{y}, \hat{z}) e^{n(\lambda^+ - \delta)}. \quad (147)$$

Note also that since the original points are in Λ_{loc} we have replaced $k(\hat{x}, \hat{\omega})$ with the constant k and δ is as in Section 3.5.

Lemma 5.3.9. *The parameter T_{big} increases with decreasing $d_X(\hat{y}, \hat{z})$.*

Proof. Since the Lipschitz constant of all the diffeomorphisms in $\text{Supp}(\mu)$ is bounded above

by $\mathcal{L}$, we have

$$d_X(\tilde{y}, \tilde{z}) \leq \mathcal{L}^n d_X(\hat{y}, \hat{z}). \quad (148)$$

Rearranging, we find that

$$\frac{1}{\ln(\mathcal{L})} \ln \left(\frac{d_X(\tilde{y}, \tilde{z})}{d_X(\hat{y}, \hat{z})} \right) \leq n. \quad (149)$$

Setting $d_X(\tilde{y}, \tilde{z}) = \epsilon_f$, it is clear that the closer the original points (in unstable distance) are to each other, the larger n_B has to be, i.e. the longer we have to flow for to reach distance ϵ_f . $\square$

Choice 1. We will pick $T \in [T_{big}(1 - \theta_1), T_{big}(1 - \theta_2)] = I$ for some $\theta_1, \theta_2 > 0$ that will be determined later.

We will pick the θ_i 's such that flowing by T does not take us all the way to distance ϵ_f . It will become clear why we have to pick T in some interval shortly.

Let $\alpha^s = d_X(\tilde{x}, \tilde{z})$, $\alpha^u = d_X(\tilde{z}, \tilde{y})$ for a chosen T . Denote the X -components of $F^{T_{big}}(\hat{y}, \hat{\omega}, k_0)$ and $F^{T_{big}}(\hat{z}, \hat{\omega}, k_0)$ as y_B and z_B respectively.

Lemma 5.3.10. For fixed $\theta_1, \theta_2 > 0$ as $T_{big} \rightarrow \infty$, $\alpha^u \rightarrow 0$.

Proof. Since θ_1 and θ_2 are fixed and $\theta \in [\theta_2, \theta_1]$ is such that $T = T_{big}(1 - \theta)$, we see that $T_{big} - T = T_{big} - T_{big}(1 - \theta) = T_{big}\theta$ grows with T_{big} . We apply Theorem 3.5.4 to the points y_B and z_B and flow backwards to the points $\tilde{y}$ and $\tilde{z}$. We get that

$$\alpha^u = d(\tilde{y}, \tilde{z}) \leq ke^{-(\lambda^+ + \delta)T_{big}\theta} \epsilon_f. \quad (150)$$

Since $\lambda^+ > 0$ we have that $\alpha^u \rightarrow 0$ as $T_{big} \rightarrow \infty$. $\square$

Let $n = \lfloor T + k_0 \rfloor$.

Lemma 5.3.11. *For fixed $\theta_1, \theta_2 > 0$, as $T_{big} \rightarrow \infty$, $\alpha^s \rightarrow 0$.*

Proof. By Theorem 3.5.4 we get

$$\alpha^s = d_X(\tilde{x}, \tilde{z}) \leq k e^{n(\lambda^- + \delta)} d_X(\hat{x}, \hat{z}), \quad (151)$$

As T_{big} grows, T will also grow, so $\alpha^s \rightarrow 0$. □

Remark 5.3.12. *Given that $\lambda^- < 0$, we have that*

$$\alpha^s = d_X(\tilde{x}, \tilde{z}) \leq d_X(\hat{x}, \hat{z}) k \exp \left((\lambda^- + \delta) \cdot \left(\ln \left(\frac{d_X(\tilde{y}, \tilde{z})}{d_X(\hat{y}, \hat{z})} \right) \frac{1}{\ln(\mathcal{L})} \right) \right). \quad (152)$$

So we see that all the distances are related.

Recall that M_{exp} is as in Lemma 3.10.3 and that our points belong to K_{exp} , see Section 5.3.

Combining Lemmas 5.3.6–5.3.11 we get the following corollary:

Corollary 5.3.13. *The distance between $W^u(\tilde{x}, \sigma^n(\hat{\omega}))$ and $W^u(\tilde{y}, \sigma^n(\hat{\omega}))$ goes to zero as $T_{big} \rightarrow \infty$, i.e. as the initial points get closer.*

Proof. Since $(\tilde{x}, \sigma^n(\hat{\omega})) \in K_\xi$ and $(\tilde{y}, \sigma^n(\hat{\omega})) \in K'_\xi \subset K_\xi$, the distance between $W^u(\tilde{x}, \sigma^n(\hat{\omega}))$ and $W^u(\tilde{y}, \sigma^n(\hat{\omega}))$ is bounded above by that quantity

$$\alpha^s + 2\xi + M_{exp} d_{Gr}(E^u(\tilde{x}, \sigma^n(\hat{\omega})), E^u(\tilde{z}, \sigma^n(\hat{\omega})))$$

where ξ is the error in approximating W^u by the image of E^u under $\exp$ (see Lemma 3.10.7 and Lemma 5.3.7).

This error term $\xi = (2\alpha^u)^2 M_r^3 \text{Lip}(Dh_{\tilde{y}, \sigma^n(\hat{\omega})}) \leq (2\alpha^u)^2 M_r^3 H$ (because $(\tilde{y}, \sigma^n(\hat{\omega})) \in K'_\xi$, see Lemma 3.10.9) then since $\alpha^u \rightarrow 0$ by Lemma 5.3.10, this term goes to zero.

Additionally, $\alpha^s \rightarrow 0$ with $T_{big} \rightarrow \infty$ by Lemma 5.3.11 and by Lemma 5.3.6 we have

$$d_{Gr}(E^u(\tilde{x}, \sigma^n(\hat{\omega})), E^u(\tilde{x}, \sigma^n(\hat{\omega}))) \leq C_\alpha d_X(\tilde{x}, \tilde{z})^\alpha = C_\alpha (\alpha^s)^\alpha.$$

We explained how α and C_α do not change with our choice of starting points and hence do not change as $T_{big} \rightarrow \infty$. Hence $d_{Gr}(E^u(\tilde{x}, \sigma^n(\hat{\omega})), E^u(\tilde{x}, \sigma^n(\hat{\omega}))) \rightarrow 0$ as $T_{big} \rightarrow \infty$. $\square$

As explained in the outline of the proof, we also want to make sure that $F^T(U^+[\hat{x}, \hat{\omega}])$ and $F^T(U^+[\hat{y}, \hat{\omega}])$ are bounded away from each other.

Claim 5.3.14. *To analyse the distance between $F^T(U^+[\hat{x}, \hat{\omega}])$ and $F^T(U^+[\hat{y}, \hat{\omega}])$, it suffices to analyse the distance $d_X(\tilde{y}, \tilde{z})$.*

Proof. Lemma 3.10.11 gives us that on the set K_G we have uniform continuity of the map G^+ which takes a point (x, ω) and returns the unit tangent vector of $U^+[x, \omega]$ through x . Since $\tilde{x}, \tilde{y} \in K_G$ we have this property. Combining this with Lemmas 5.3.10 and 5.3.11 this gives us that for T_{big} large, $G^+(F^T(\hat{x}, \hat{\omega}))$ and $G^+(F^T(\hat{y}, \hat{\omega}))$ are almost parallel, hence it suffices to show that $\tilde{x}$ maintains a bounded distance from $F^T(U^+[\hat{y}, \hat{\omega}])$.

Now note that the triangle inequality gives us that for $\tilde{y}' \in F^T(U^+[\hat{y}, \hat{\omega}])$

$$d_X(\tilde{x}, \tilde{y}') \geq d_X(\tilde{y}', \tilde{z}) - d_X(\tilde{x}, \tilde{z}). \quad (153)$$

Now, by the NJI condition, $W_{loc}^s(\hat{x}, \hat{\omega}) \cap U_{loc}^+[\hat{y}, \hat{\omega}] = \emptyset$. That is to say, the point $\hat{z}$, which is the intersection of $W_q^s(\hat{x}, \hat{\omega})$ with $W_q^u(\hat{y}, \hat{\omega})$ is not an element of $U^+[\hat{y}, \hat{\omega}]$. There is positive distance from $\hat{z}$ to any point on $U_q^+[\hat{y}, \hat{\omega}]$ along $W_q^u(\hat{y}, \hat{\omega})$, this is unstable distance between $\hat{x}$ and $U_q^+[\hat{y}, \hat{\omega}]$. Using Theorem 3.5.4 one can choose T large enough such that this unstable distance grows and $d_X(\tilde{y}', \tilde{z})$ is sufficiently large. It suffices to examine $d_X(\tilde{y}, \tilde{z})$ as any other point $\tilde{y}'$ would amount to the same computation. $\square$

Remark 5.3.15. *Like in Remark 5.3.8, if instead the NJI condition was written as*

$$W^s(\hat{y}, \hat{\omega}) \cap U^+[\hat{x}, \hat{\omega}] = \emptyset,$$

we would change Claim 5.3.14 slightly too. We would say that $\hat{z}$ is not an element of $U^+[\hat{x}, \hat{\omega}]$ and we would analyse the distance between $\tilde{z}$ and $\tilde{x}$ (same computation).

By Claim 5.3.14 we will analyse $d_X(\tilde{x}, \tilde{y})$ and require that T be chosen such that we always exceed the stable distance with the unstable distance. So we use this to compute a restriction on θ .

Lemma 5.3.16. *Let x_t, y_t, z_t be the X -components resp. of $F^t(\hat{x}, \hat{\omega})$, $F^t(\hat{y}, \hat{\omega})$, $F^t(\hat{z}, \hat{\omega})$, if t is the time it takes such that $d_X(y_t, z_t) > 2d_X(x_t, z_t)$, then*

$$t \leq \frac{\ln(k)}{\ln(\mathcal{L}) - (\lambda^- + \delta)} + \frac{\ln(\mathcal{L})}{\ln(\mathcal{L}) - (\lambda^- + \delta)} T_{big}. \quad (154)$$

Proof. In the worse case scenario, the unstable distance starts out small and the stable distance starts out large. As much as we have asked that $\hat{x}$ and $\hat{y}$ are chosen such that $d_X(\hat{x}, \hat{y}) < \rho \ll \epsilon_f$, let us assume that the stable distance is close to ϵ_f (or say it is ϵ_f) and let us assume that the unstable distance is close to zero (we know that it cannot be zero by the NJI assumption).

We will compute the time t such that $d_X(y_t, z_t) = 2d_X(x_t, z_t)$.

If $d_X(\hat{x}, \hat{z}) = \epsilon_f$, then after time t we would have $d_X(\tilde{x}, \tilde{z}) \leq k\epsilon_f e^{(\lambda^- + \delta)t}$ and hence we have $2d_X(\tilde{x}, \tilde{z}) \leq 2k\epsilon_f e^{(\lambda^- + \delta)t}$. Additionally, $d_X(\tilde{y}, \tilde{z}) \geq \left(\frac{1}{\mathcal{L}}\right)^{T_{big}-t} \epsilon_f$. So we set equal

$$\left(\frac{1}{\mathcal{L}}\right)^{T_{big}-t} \epsilon_f = \epsilon_f 2k e^{(\lambda^- + \delta)t}, \quad (155)$$

and solve for t . We get

$$t = \frac{\ln(2k)}{\ln(\mathcal{L}) - (\lambda^- + \delta)} + \frac{\ln(\mathcal{L})}{\ln(\mathcal{L}) - (\lambda^- + \delta)} T_{big}. \quad (156)$$

Note that both terms in Equation (156) are positive because $\lambda^- < 0$, but also that

$$\frac{\ln(\mathcal{L})}{\ln(\mathcal{L}) - (\lambda^- + \delta)} < 1.$$

This completes the proof. □

This gives us information on how to pick θ_1 and θ_2 so that T is such that

$$d_X(\tilde{y}, \tilde{z}) > 2d_X(\tilde{x}, \tilde{z}).$$

Lemma 5.3.17. *We can take T_{big} large enough (i.e. the initial points close enough together) and find a constant $Q_0 < 1$ such that fixing $\theta_1 \ll \frac{1-Q_0}{1000}$, we always have that $T \gg t$.*

Proof. In the equation for t given by Equation (154) the term $\frac{\ln(2k)}{\ln(\mathcal{L}) - (\lambda^- + \delta)}$ is a constant and does not grow with T_{big} , we can start by taking T_{big} large enough such that

$$\frac{\ln(2k)}{\ln(\mathcal{L}) - (\lambda^- + \delta)} + \frac{\ln(\mathcal{L})}{\ln(\mathcal{L}) - (\lambda^- + \delta)} T_{big} < \left(\frac{\ln(\mathcal{L})}{\ln(\mathcal{L}) - (\lambda^- + \delta)} + \epsilon_s \right) T_{big}.$$

where ϵ_s is chosen small such that $Q_0 := \left(\frac{\ln(\mathcal{L})}{\ln(\mathcal{L}) - (\lambda^- + \delta)} + \epsilon_s \right) < 1$ (remember $\lambda^- < 0$).

We want to take θ_1 such that $Q_0 \ll (1 - \theta_1)$, that way $T \gg t$. We fix arbitrarily a θ_1 such that

$$\theta_1 \ll \frac{1 - Q_0}{1000}. \quad (157)$$

□

Choice 2. We will also require that θ_1 and θ_2 be small enough such that

$$\frac{e^{(\lambda^- + \delta)(1 - \theta_1)}}{e^{-(\lambda^+ - \delta)\theta_1}} < 1, \left(\frac{\mathcal{L}^{\frac{C_b}{10^{i+2}}\theta_2}}{e^{-(\lambda^- + \delta)(1 - \theta_2)\alpha}} \right) < 1, \quad (158)$$

where α is defined at the beginning of Section 5.3 and C_b and i are as defined in Section 5.3, i.e. $C_b = \frac{(\lambda^+ - \delta)}{2 \ln(\mathcal{L})}$ and i was fixed such that $C_b > \frac{1}{10^i}$ and $\frac{C_b}{10^{i+2}} \ll (\lambda^+ + \delta)$ so that

$$\frac{\lambda^+ + \delta}{\frac{C_b}{10^{i+2}}} \gg \ln(k\epsilon_f M_r/2) + (\lambda^+ + \epsilon). \quad (159)$$

This will be used in Lemma 5.3.24 (see Claim 5.3.25 and Equation (168)), Lemma 5.3.26 (see Equations (173), (176)) and Lemma 5.3.27 (See Equation (192) and (194)) in Section 5.3.

Recall the definitions of K_1 and K'_1 from Section 5.3; They are two sets of measure $m(K_1) > 1 - \epsilon_1$, $m(K'_1) > 1 - \epsilon'_1$ resp. We will have to initially land in K'_1 after flowing by T , i.e. $X \times \Omega$ -components of the points $F^T(\hat{x}, \hat{\omega}, k_0)$ and $F^T(\hat{y}, \hat{\omega}, k_0)$ which we denote $(\tilde{x}, \sigma^n(\hat{\omega}))$ and $(\tilde{y}, \sigma^n(\hat{\omega}))$ (where $n = \lfloor T + k_0 \rfloor$) need to be in K'_1 . Then we argue that we can change the Ω -component of $(\tilde{x}, \sigma^n(\hat{\omega}))$ and $(\tilde{y}, \sigma^n(\hat{\omega}))$ to be some $\tilde{\omega}$ such that $(\tilde{x}, \tilde{\omega})$ and $(\tilde{y}, \tilde{\omega})$ that are in K_1 . We need to change the Ω -component because in the next part of the argument we need to flow backwards by a new past vector (we change the future vector as well).

To do any of this, we actually have to work in Z because we can only define the flow on Z . Additionally, we also have to land in a subset of Z such that when we change the Ω -component our points will land in $E_2^c \subset Z$ and $E_3^c \subset Z$ (fixed from the beginning in Section 5.3). This will allow us to run the same argument as in Lemma 5.3.19 (which uses Lemma 3.10.15, a standard corollary of Birkhoff's theorem) in Sections 5.3 and 5.3.

Recall the definition of M from Section 5.3. For a point $(x, k) \in X \times [0, 1[$, define

$$V_{(x,k)} = \{\omega \in \Omega : (x, \omega, k) \in M\}.$$

Now also recall from Section 5.3 the set

$$B := \{(x, k) \in [0, 1[: \mu(V_{(x,k)}) > 1 - \theta\} \subset X \times [0, 1[.$$

Then we need to land the points $F^T(\hat{x}, \hat{\omega}, k_0)$ and $F^T(\hat{y}, \hat{\omega}, k_0)$ so that their $X \times [0, 1[$ -components live in this set B . Note that the set M has $\hat{m}$ -measure $1 - \epsilon_M$, where ϵ_M is small.

Lemma 5.3.18. *The set B defined above has $\nu \times dt$ -measure at least $1 - \frac{\epsilon_M}{\theta}$ in $X \times [0, 1[$.*

Proof. Let $W_{(x,k)} = \{\omega : (x, \omega, k) \in M^c\}$.

Note that B^c can be written as the set

$$B^c = \{(x, k) \in X \times [0, 1[: \mu(W_{(x,k)}) > \theta\}.$$

Now let

$$E := \{(x, \omega, k) \in X \times \Omega \times [0, 1[: (x, k) \in B^c, (x, \omega, k) \in M^c\} \subset M^c.$$

Now using Fubini, note that

$$\hat{m}(M^c) \geq \hat{m}(E) = \int_{X \times [0, 1[} \int_{\Omega} d\mu^{\mathbb{Z}} d(\nu \times dt) = \mu^{\mathbb{Z}}(W) \cdot (\nu \times dt)(B^c) \geq \theta \cdot (\nu \times dt)(B^c).$$

Hence $(\nu \times dt)(B^c) \leq \frac{\hat{m}(M^c)}{\theta}$. This completes the proof. □

We take $\theta = \sqrt{\epsilon_M}$, then $(\nu \times dt)(B) > 1 - \sqrt{\epsilon_M}$.

Knowing that B has $\nu \times dt$ -measure $(1 - \sqrt{\epsilon_M})$, we have that $B \times \Omega$ has $\hat{m}$ -measure $(1 - \sqrt{\epsilon_M})$. The size of K'_1 is large enough ($\epsilon'_1 > 0$ small enough) such that $\tilde{K}_1 = (B \times \Omega) \cap (K'_1 \times [0, 1])$ has $\hat{m}$ -measure like $(1 - \tilde{\epsilon}_1)$ for some $\tilde{\epsilon}_1$ small. All we really need is that $\hat{m}(\tilde{K}_1) > 1/2$ (we asked that $\hat{m}(\tilde{K}_1) > 3/4$ just for convenience in Section 5.3) which can clearly be achieved. We now use Lemma 3.10.15 to land $F^T(\hat{x}, \hat{\omega}, k_0)$ and $F^T(\hat{y}, \hat{\omega}, k_0)$ in the set $\tilde{K}_1$.

Lemma 5.3.19. *For T_{big} large enough, given the interval $[T_{big}(1 - \theta_1), T_{big}(1 - \theta_2)]$, there exists a subset of positive measure $\mathcal{G} \subset [T_{big}(1 - \theta_1), T_{big}(1 - \theta_2)]$ such that, if $T \in \mathcal{G}$, then flowing forwards by T from the points $(\hat{x}, \hat{\omega}, k_0)$ and $(\hat{y}, \hat{\omega}, k_0)$ we land in the set $\tilde{K}_1$.*

Proof. We take T_{big} such that $T_{big}(1 - \theta_1)$ is greater S where S is as in the definition of E^c in Section 5.3 (a quantity fixed at the beginning of the argument, i.e. before beginning Section 5.3). This allows us to use Lemma 3.10.15.

Take $K = \tilde{K}_1$ and apply Lemma 3.10.15. Our starting points $(\hat{x}, \hat{\omega}, k_0)$ and $(\hat{y}, \hat{\omega}, k_0)$ belong to E^c (see Section 5.3). Remember that E^c was fixed (in Section 5.3) by the choice of $\delta = \frac{1}{100}\gamma$ where $\gamma = \hat{m}(K) > 3/4$. We will denote these starting points α and β resp. Let $T_1 = T_{big}(1 - \theta_1)$ $T_0 > S$. Let $T_2 = T_{big}(1 - \theta_2)$ and note that $T_2(1.98\gamma - 1) \gg T_1$. Then let

$$\int_0^{T_2} \mathbf{1}_K(F^t(\alpha)) dt = \text{Leb}\{t \in [0, T_2] : F^t(\alpha) \in K\} =: \eta_{T_2, \alpha}.$$

Then,

$$(-\delta + \gamma)T_2 < \eta_{T_2, \alpha} < (\delta + \gamma)T_2, \tag{160}$$

$$0.99\gamma T_2 < \eta_{T_2, \alpha} < 1.01\gamma T_2. \tag{161}$$

Run the same process for β . Then, using Equation (161) we have

$$\begin{aligned} \text{Leb}(\eta_{T_2,\alpha} \cap \eta_{T_2,\beta}) &= \text{Leb}(\eta_{T_2,\alpha}) + \text{Leb}(\eta_{T_2,\beta}) - \text{Leb}(\eta_{T_2,\alpha} \cup \eta_{T_2,\beta}) \\ &\geq T_2(1.98\gamma - 1) > T_1 = \text{Leb}([0, T_1]). \end{aligned} \quad (162)$$

So then there is a positive measure subset of $[T_1, T_2]$ where both $F^t(\alpha)$ and $F^t(\beta)$ belong to K . $\square$

Remark 5.3.20. *If one takes Equation (160) and uses it to compute the inequality in Equation (162), one finds that*

$$\text{Leb}(\eta_{T,\alpha} \cap \eta_{T,\beta}) = \text{Leb}(\eta_{T,\alpha}) + \text{Leb}(\eta_{T,\beta}) - \text{Leb}(\eta_{T,\alpha} \cup \eta_{T,\beta}) \geq 2(\gamma - \delta)T - T.$$

Hence one needs $2(\gamma - \delta) - 1 > 0$. This happens if and only if $\gamma - \frac{1}{2} > \delta > 0$. This is why we needed the condition $\gamma = (\nu \times \mu^{\mathbb{Z}})(K) > 3/4$, though really $> 1/2$ suffices as we could find an appropriate δ .

Flowing by m

Convention 1. *We pick the convention that m is given as a positive real number and we will flow backwards by it and hence write F^{-m} .*

From the points $\tilde{x}$ and $\tilde{y}$ we will go in two directions. First, we will choose a new Ω -component and flow backwards by some m (under the ‘standard’ flow) from $\tilde{x}$ and $\tilde{y}$ to the points which we will denote x and y . Let $\tilde{\omega} \in \Omega$ be such that $(\tilde{x}, \tilde{\omega})$ and $(\tilde{y}, \tilde{\omega}) \in K_1$; we can choose such $\tilde{\omega}$ because we landed the points $F^T(\hat{x}, \hat{\omega}, k_0) = (\tilde{x}, \sigma^n(\hat{\omega}), k_1)$ and $F^T(\hat{y}, \hat{\omega}, k_0) = (\tilde{y}, \sigma^n(\hat{\omega}), k_1)$ (where $n = \lfloor T + k_0 \rfloor$) in the set $\tilde{K}_1$ in Section 5.3. This gives us that we have a (large) positive measure subset of Ω (the intersection of $V_{(\tilde{x}, k_1)}$ and $V_{(\tilde{y}, k_1)}$ which has large positive measure because they are both large) from which to choose $\tilde{\omega}$ from. Still, we ask

more from our choice of $\tilde{\omega}$ in the next Lemma.

Lemma 5.3.21. *We can choose $\tilde{\omega}$ such that $W^s(\tilde{x}, \tilde{\omega})$ is bounded away from $W^u(\tilde{x}, \sigma^n(\hat{\omega}))$. Similarly we ask that the stables and unstables of $\sigma^n(\hat{\omega})$ and $\tilde{\omega}$ all distinct.*

Proof. The unstable plane $W^u(\tilde{x}, \sigma^n(\hat{\omega}))$ (where $n = \lfloor T + k_0 \rfloor$) is fixed at this point for some T_{big} choice large enough. Only $W^s(\tilde{x}, \tilde{\omega})$ changes with $\tilde{\omega}$.

The tangent planes $E^s(\tilde{x}, \omega)$ and $E^u(\tilde{x}, \sigma^n(\hat{\omega}))$ approximate $W^s(\tilde{x}, \tilde{\omega})$ and $W^u(\tilde{x}, \sigma^n(\hat{\omega}))$ respectively up to error $\xi = \mathfrak{r}^2 M_r^3 H$ on a ball of radius $\mathfrak{r}$ using Lemma 5.3.7. We need to take $\mathfrak{r} = 2\alpha^u$ as this is the scale we view things on as explained in Section 5.3. We can assume we have taken T_{big} large enough such that $4\xi \ll \varepsilon$ (see the definition of K_ε in Lemma 3.10.12 and note that $(\tilde{x}, \sigma^n(\hat{\omega})) \in K'_1 \subset K_\varepsilon$ and $(\tilde{x}, \tilde{\omega}) \in K_1 \subset K_\varepsilon$).

Then we use Lemma 3.4.2 to say we can pick $\tilde{\omega}$ such that $E^s(\tilde{x}, \omega)$ and $E^u(\tilde{x}, \sigma^n(\hat{\omega}))$ are at least 4ξ apart and so $W^s(\tilde{x}, \tilde{\omega})$ is bounded away from $W^u(\tilde{x}, \sigma^n(\hat{\omega}))$ on the appropriate scale, i.e. a ball of size $\mathfrak{r}$. $\square$

Let K_2 and K'_2 be as defined in Section 5.3 (they have measure $1 - \epsilon_2$ and $1 - \epsilon'_2$ resp.). As before in Section 5.3, when we flow from the points $(\tilde{x}, \tilde{\omega}, k_1)$ and $(\tilde{y}, \tilde{\omega}, k_1)$ to the points $(x, \sigma^{n'} \tilde{\omega}, k_2)$ and $(y, \sigma^{n'} \tilde{\omega}, k_2)$ (where $n' = \lfloor k_1 - m \rfloor$, $k_2 = k_1 - m \pmod{1}$), we need to land in $K'_2 \times [0, 1[\subset Y$. In this case we additionally need that $(x, \sigma^{n'} \tilde{\omega}^-)$ and $(y, \sigma^{n'} \tilde{\omega}^-)$ are a good set we called $B' \subset X \times \Omega^-$ (see Section 5.3).

Note that the measures on Ω^+ and Ω^- are both $\mu^{\mathbb{N}}$ in the obvious way.

Lemma 5.3.22. *The set B' defined above has $\nu \times \mu^{\mathbb{N}}$ -measure $1 - \sqrt{\epsilon_2}$ in $X \times \Omega^-$.*

Proof. The proof is identical to the argument of Lemma 5.3.18. $\square$

Let $\tilde{K}_2 = (B' \times \Omega^+) \cap K'_2$ be as in Section 5.3.

Lemma 5.3.23. *We can choose m such that $F^{-m}(\tilde{x}, \tilde{\omega}, k_1)$ and $F^{-m}(\tilde{y}, \tilde{\omega}, k_1)$ belong to the set $\tilde{K}_2 \times [0, 1[$.*

Proof. Knowing that B' has $\nu \times \mu^{\mathbb{N}}$ -measure at least $(1 - \sqrt{\epsilon_2})$, we have that $B' \times \Omega^+$ has $\nu \times \mu^{\mathbb{Z}}$ -measure at least $(1 - \sqrt{\epsilon_2})$. We asked in Section 5.3 that K'_2 be large enough ($\epsilon'_2 > 0$ small enough) such that $\tilde{K}_2 = (B' \times \Omega^+) \cap K'_2$ has $\nu \times \mu^{\mathbb{Z}}$ -measure like $(1 - \tilde{\epsilon}_2)$ for some $\tilde{\epsilon}_2$ small such that

$$1 - \tilde{\epsilon}_2 > 3/4. \quad (163)$$

In Section 5.3 we showed that we could land $F^T(\hat{x}, \hat{\omega}, k_0)$ and $F^T(\hat{y}, \hat{\omega}, k_0)$ in a set $\tilde{K}_1$ where $\tilde{\omega}$ could be picked such that the resulting points $(\tilde{x}, \tilde{\omega}, k_1)$ and $(\tilde{y}, \tilde{\omega}, k_1)$ (where $k_1 = F^T(k_0)$) belong to the set $M = (K_1 \times [0, 1]) \cap E_2^c \cap E_3^c \subset Z$. The set E_2^c as defined in Section 5.3 is where we need to start from in order to use Lemma 3.10.15 to choose m such that we land in $\tilde{K}_2 \times [0, 1]$. We run the same argument as in Lemma 5.3.19 on an interval $[m_1, m_2]$ (for m_1 sufficiently large), starting from the points $(\tilde{x}, \tilde{\omega}, k_1)$ and $(\tilde{y}, \tilde{\omega}, k_1)$ in E_2^c and taking $K = \tilde{K}_2 \times [0, 1]$ (Note that ϵ and δ were fixed along with E_2^c in Section 5.3). Then we get a positive measure subset of $m \in [m_1, m_2]$ such that we land the $X \times \Omega$ -components of $F^{-m}(\tilde{x}, \tilde{\omega}, k_1)$ and $F^{-m}(\tilde{y}, \tilde{\omega}, k_1)$ in the set $\tilde{K}_2$. $\square$

We will be more explicit about the proportion of good choices of m we can choose inside some interval $[m_1, m_2]$ in Lemma 5.3.38.

We have some conditions on m , firstly, the minimum value of m in the interval we need, which we call m_1 , must be bigger than the S required from Lemma 3.10.15 corresponding to the set E_2^c . Additionally, the maximum value of m for the interval, called m_2 , is such that we do not surpass a certain amount of the remaining distance to ϵ_f . Additionally, of course the error between the unstable planes that we worked for should not be significantly changed by flowing by m . We will discuss this more later, but in Lyapunov charts.

Lastly, we want the relationship between m_1 and m_2 to be $m_2 > 2m_1$. Ensuring that $[m_1, m_2]$ is at least 50% of the interval $[0, m_2]$ allows us to ensure that the ‘bad’ set is still

small; We will use this in Section 5.3. We will also demonstrate another property in our choice of m in Section 5.3, i.e. that $d_1(m_1) \geq (1 + \eta)d_1(m_2)$ and $d_2(m_1) \geq (1 + \eta)d_2(m_2)$; see Lemma 5.3.38.

Let C_b and i be as in Section 5.3/Choice 2, and let

$$m_1 := \frac{C_b}{10^{i+J}} T_{big} \theta_2, \quad m_2 := \frac{C_b}{10^{i+2}} T_{big} \theta_2, \quad (164)$$

where J is chosen such that $m_2 > 2m_1$ and $|\frac{C_b}{10^{i+J}}(\lambda^- - \epsilon)| < |\frac{C_b}{10^{i+2}}(\lambda^- + \epsilon)|$. Also start take T_{big} large enough so that $m_1 > S$.

Lemma 5.3.24. *For $m \in [m_1, m_2]$, the distance between $\tilde{x}$ and $\tilde{y}$ does not surpass ϵ_f after flowing backwards by m under $\tilde{\omega}$.*

Proof. Lemma 5.3.21 bounded $E^s(\tilde{x}, \omega)$ away from $E^u(\tilde{x}, \sigma^n(\hat{\omega}))$, so we see that the stable distance (which is the one growing because we are flowing backwards by m) is strictly less than the unstable distance with respect to $\sigma^n(\hat{\omega})$. After flowing by T , $d_X(\tilde{x}, \tilde{y})$ is mostly in the unstable direction, but now with respect to this new ω it has some stable and unstable component by Lemma 5.3.21. Let m_{big} be the amount of time we flow backwards with respect to ω until this distance is now ϵ_f . We have

$$\left(\frac{1}{\mathcal{L}}\right)^{m_{big}} \epsilon_f \leq d_X(\tilde{x}, \tilde{y}). \quad (165)$$

We can also say that

$$d_X(\tilde{x}, \tilde{y}) \leq d_X(\tilde{y}, \tilde{z}) + d_X(\tilde{x}, \tilde{z}) \leq \epsilon_f e^{-(\lambda^+ - \delta)(T_{big} - T)} + d_X(\hat{x}, \hat{z}) e^{(\lambda^- + \delta)T}. \quad (166)$$

We simplify Equation (166) and note that we can find $C_s > 0$ such that

$$\epsilon_f e^{-(\lambda^+ - \delta)(T_{big} - T)} + d_X(\hat{x}, \hat{z}) e^{(\lambda^- + \delta)T} \leq C_s \epsilon_f e^{-(\lambda^+ - \delta)(T_{big} - T)}. \quad (167)$$

Claim 5.3.25. *We can fix C_s for all T_{big} because of our choice of θ_1 in Choice 2 in Section 5.3*

Proof. We see that for C_s to satisfy Equation (167), we require it to satisfy

$$C_s \geq 1 + \frac{d_X(\hat{x}, \hat{z})}{\epsilon_f} \left(\frac{e^{(\lambda^- + \delta)(1-\theta)}}{e^{-(\lambda^+ - \delta)\theta}} \right)^{T_{big}}. \quad (168)$$

The right-hand most term of Equation (168) goes to 0 as $T_{big} \rightarrow \infty$, so we can fix C_s for all T_{big} and ρ . $\square$

Combining the inequalities (165) and (167), we get:

$$\left(\frac{1}{\mathcal{L}} \right)^{m_{big}} \epsilon_f \leq C_s \epsilon_f e^{-(\lambda^+ - \delta)(T_{big} - T)} \quad (169)$$

We find that then

$$m_{big} \geq \frac{(\lambda^+ - \delta)}{\ln(\mathcal{L})} (T_{big} - T) - \frac{\ln(C_s)}{\ln(\mathcal{L})} \quad (170)$$

$$= \frac{(\lambda^+ - \delta)}{\ln(\mathcal{L})} (T_{big}\theta) - \frac{\ln(C_s)}{\ln(\mathcal{L})}. \quad (171)$$

Since $\frac{\ln(C_s)}{\ln(\mathcal{L})}$ is fixed and

$$m_2 = \frac{C_b}{10^{i+2}} T_{big} \theta_2 = \frac{(\lambda^+ - \delta)}{2 \ln(\mathcal{L})} \frac{1}{10^{i+2}} T_{big} \theta_2,$$

where $\theta_2 < \theta$ we have that for T_{big} large enough, $m_2 < m_{big}$. This implies that the distance between $\tilde{x}$ and $\tilde{y}$ does not surpass ϵ_f after flowing backwards by $m \leq m_2$. This completes the proof. $\square$

Let d_m be the distance between $F_{\tilde{\omega}}^{-m}(W^u(F^n(\hat{x}, \hat{\omega})))$ and $F_{\tilde{\omega}}^{-m}(W^u(F^n(\hat{y}, \hat{\omega})))$ (where $n = \lfloor T + k_0 \rfloor$). Let $\delta(m) = d_X(F_{\tilde{\omega}}^{-m}(\tilde{x}), F_{\tilde{\omega}}^{-m}(\tilde{y}))$ (i.e. represents the distance between

$F_{\tilde{\omega}}^{-m}(U^+(F^n(\hat{x}, \hat{\omega})))$ and $F_{\tilde{\omega}}^{-m}(U^+(F^n(\hat{y}, \hat{\omega})))$ by Claim 5.3.14).

Lemma 5.3.26. *The distance d_m goes to zero as $T_{big}, m \rightarrow \infty$.*

Proof. We computed that the distance between $W^u(F^n(\hat{x}, \hat{\omega}))$ and $W^u(F^n(\hat{y}, \hat{\omega}))$ as

$$\mathcal{D} := (\alpha^s + 2\xi + M_{exp}d_{Gr}(E^u(\tilde{x}, \sigma^n(\hat{\omega})), E^u(\tilde{z}, \sigma^n(\hat{\omega}))),$$

and showed it goes to zero as $T_{big} \rightarrow \infty$ in Lemma 5.3.13.

The largest m could be is $\frac{C_b}{10^{i+2}}T_{big}\theta_2$. Originally this error between the unstable planes that we computed was mostly in the stable direction for $\sigma^n(\hat{\omega})$. We have asked that $\tilde{\omega}$ be chosen such that the distance between has now stable and unstable components (see Lemma 3.4.2).

Say in the worst case scenario that the distance $\mathcal{D}$ between unstable planes that we measured is entirely in the stable direction for $\tilde{\omega}$. Let $\mathcal{D}_m$ be the distance between the images of $W^u(F^n(\hat{x}, \hat{\omega}))$ and $W^u(F^n(\hat{y}, \hat{\omega}))$ after flowing backwards by m , then

$$\mathcal{D}_m \leq \mathcal{L}^{\frac{C_b}{10^{i+2}}T_{big}\theta_2}(\alpha^s + 2\xi + M_{exp}d_{Gr}(E^u(\tilde{x}, \sigma^n(\hat{\omega})), E^u(\tilde{z}, \sigma^n(\hat{\omega}))))$$

We deal with each term in $\mathcal{D}$ one at a time.

$$\mathcal{L}^{\frac{C_b}{10^{i+2}}T_{big}\theta_2}\alpha^s \leq \frac{\mathcal{L}^{\frac{C_b}{10^{i+2}}T_{big}\theta_2}}{e^{-(\lambda^-+\delta)T_{big}(1-\theta)}}d_X(\hat{x}, \hat{z}) = \frac{\mathcal{L}^{\frac{C_b}{10^{i+2}}T_{big}\theta_2}}{e^{-(\lambda^-+\delta)(1-\theta))T_{big}}}d_X(\hat{x}, \hat{z}) \quad (172)$$

where $\theta \in [\theta_2, \theta_1]$ because we used Theorem 3.5.4 for T . Recall from Choice 2 that

$$\frac{\mathcal{L}^{\frac{C_b}{10^{i+2}\theta_2}}}{e^{-(\lambda^-+\delta)(1-\theta_2)}} < \frac{\mathcal{L}^{\frac{C_b}{10^{i+2}\theta_2}}}{e^{-(\lambda^-+\delta)(1-\theta_2)\alpha}} < 1, \quad (173)$$

So as $T_{big} \rightarrow \infty$, this quantity goes to zero.

$$\mathcal{L}^{\frac{C_b}{10^i+2}T_{big}\theta_2} 2\xi \leq \mathcal{L}^{\frac{C_b}{10^i+2}T_{big}\theta_2} k\epsilon_f e^{-(\lambda^++\delta)T_{big}\theta} \quad (174)$$

but note that by choices made in Section 5.3 that $\frac{\mathcal{L}^{\frac{C_b}{10^i+2}}}{e^{\lambda^++\delta}} < 1$ and so this goes to zero as $T_{big}, m \rightarrow \infty$.

Lastly,

$$\mathcal{L}^{\frac{C_b}{10^i+2}\theta_2} M_{exp} d_{Gr}(E^u(\tilde{x}, \sigma^n(\hat{\omega})), E^u(\tilde{z}, \sigma^n(\hat{\omega}))) \leq \mathcal{L}^{\frac{C_b}{10^i+2}\theta_2} M_{exp}(\alpha^s)^\alpha \quad (175)$$

$$\leq M_{exp} \frac{\mathcal{L}^{\frac{C_b}{10^i+2}T_{big}\theta_2}}{e^{-(\lambda^++\delta)(1-\theta)\alpha}T_{big}} d_X(\hat{x}, \hat{z})^\alpha. \quad (176)$$

Equation (173) then gives us that this goes to zero as $T_{big}, m \rightarrow \infty$. Then $\mathcal{D}_m$ goes to zero as $m \rightarrow \infty$. This completes the proof. $\square$

Let $\tilde{D}$ be the distance (in ambient coordinates) between $\tilde{x}$ and $\tilde{y}$. Note that by construction, this distance is still less than q and that the set M is contained in Λ_{loc} as defined in 5.3. The points $(\tilde{x}, \tilde{\omega}, k_1)$ and $(\tilde{y}, \tilde{\omega}, k_1)$ belong to M . Additionally, the distance $\tilde{D}$ between $\tilde{x}$ and $\tilde{y}$ is less than k_U and hence they live in each other's Lyapunov chart domain.

Originally, the distance $\tilde{D}$ was largely in the unstable direction for that choice of Ω -component, i.e. $\sigma^n(\hat{\omega})$. Now that we have switched Ω -components to $\tilde{\omega}$, generically, the distance $\tilde{D}$ has stable and unstable distance components with respect to $\tilde{\omega}$. Also we picked $\tilde{\omega}$ such that $\tilde{x}$ and $\tilde{y}$ are not on each other's stable or unstable manifolds. Also, since we are going backwards by m , the stables grow and the unstables contract.

In Lyapunov charts, the images of the subspaces $E^u(\tilde{x}, \tilde{\omega})$ and $E^s(\tilde{x}, \tilde{\omega})$ form a coordinate axis, and $\tilde{y}$ is sent to some $v \in \mathbb{C}^2$ (or $\mathbb{R}^4$) in these charts. We can decompose v into a component in the E^s -direction and the E^u -direction denoted respectively v_s and v_u , i.e.

$v = v_s + v_u$. In these charts, we know exactly v_s and v_u grow and shrink resp. under flowing by $-m$. We denote $\alpha_s = |v_s|$, $\alpha_u = |v_u|$.

Then, the distance between 0 and v after flowing by m , is

$$D_m := |\tilde{F}_{(\tilde{x}, \tilde{\omega}), k_1}^{-m}(v) - \tilde{F}_{(\tilde{x}, \tilde{\omega}), k_1}^{-m}(0)|.$$

Let $\alpha_u^m := |\tilde{F}_{(\tilde{x}, \tilde{\omega})}^{-m}(v_u) - \tilde{F}_{(\tilde{x}, \tilde{\omega})}^{-m}(0)|$ and $\alpha_s^m := |\tilde{F}_{(\tilde{x}, \tilde{\omega}), k_1}^{-m}(v_s) - \tilde{F}_{(\tilde{x}, \tilde{\omega}), k_1}^{-m}(0)|$.

Now we perform a calculation that we will need in Section 5.3.

Lemma 5.3.27. *For T_{big} large enough (hence m large enough), we have*

$$D_{m+1} \leq \sqrt{1+k^2}(e^{-(\lambda^- - \epsilon)} + \frac{1}{2}C)\alpha_s^m = (e^{-(\lambda^- - \epsilon)} + \frac{1}{2}C)D_m.$$

Further, we have $D_m < D_{m+1}$.

Proof. Write $\alpha_u = k\alpha_s$ for some $k > 0$. Note that this is dependent on the points but it will not matter for the computation we will do. In fact, by Pythagoras, $D_m = \sqrt{1+k^2}\alpha_s^m$.

We examine how this compares to $D_{m+1} = |\tilde{F}_{(\tilde{x}, \tilde{\omega}), k_1}^{-(m+1)}(v) - \tilde{F}_{(\tilde{x}, \tilde{\omega}), k_1}^{-(m+1)}(0)|$. Again by Pythagoras, $D_{m+1} = \sqrt{(\alpha_u^{m+1})^2 + (\alpha_s^{m+1})^2}$. Then using distortion estimates from Section 3.5 as before, we additionally compute that

$$\alpha_s^{m+1} \leq e^{-(\lambda^- - \epsilon)}\alpha_s^m + \frac{1}{2}C(\alpha_s^m)^2 \tag{177}$$

$$\leq (e^{-(\lambda^- - \epsilon)} + \frac{1}{2}C)\alpha_s^m, \tag{178}$$

$$\alpha_u^{m+1} \leq e^{-(\lambda^+ - \epsilon)}\alpha_u^m + \frac{1}{2}C(\alpha_u^m)^2 \tag{179}$$

$$\leq k(e^{-(\lambda^+ - \epsilon)} + \frac{1}{2}C)\alpha_s^m \tag{180}$$

$$\leq k(e^{-(\lambda^- - \epsilon)} + \frac{1}{2}C)\alpha_s^m. \tag{181}$$

Where Equations (178) and (180) hold because $\alpha_s^m > (\alpha_s^m)^2$ because $\epsilon_f < \frac{1}{M_r}$ and $\alpha_u^m = k\alpha_s^m$.

Then

$$D_{m+1} \leq \sqrt{1+k^2}(e^{-(\lambda^- - \epsilon)} + \frac{1}{2}C)\alpha_s^m = (e^{-(\lambda^- - \epsilon)} + \frac{1}{2}C)D_m. \quad (182)$$

The above bound is necessary for the proof of Lemma 5.3.36, however we also need to demonstrate the fact that $D_m < D_{m+1}$.

Again, from Section 3.5 we have that

$$\alpha_s^{m+1} \geq \alpha_s^m(e^{-(\lambda^- + \epsilon)} - \frac{1}{2}C\alpha_s^m), \quad (183)$$

$$\alpha_u^{m+1} \geq \alpha_u^m(e^{-(\lambda^+ + \epsilon)} - \frac{1}{2}C\alpha_u^m). \quad (184)$$

At the beginning of Section 5.3 we fixed $k_1 < 1$ and $k_2 < 1$ small but be such that $(1 - k_1)e^{-(\lambda^- + \epsilon)} > 1$ and added a condition to ϵ_f , we use it now to get that

$$\alpha_s^{m+1} \geq (1 - k_1)e^{-(\lambda^- + \epsilon)}\alpha_s^m, \quad (185)$$

$$\alpha_u^{m+1} \geq (1 - k_2)e^{-(\lambda^+ + \epsilon)}\alpha_u^m. \quad (186)$$

Note that everything was chosen such that $(1 - k_1)e^{-(\lambda^- + \epsilon)} > 1$ and also that naturally $(1 - k_2)e^{-(\lambda^+ + \epsilon)} < 1$.

Then to have that $D_m < D_{m+1}$ (or the square which we use below) we can show that

$$(\alpha_s^m)^2(1 - k_1)^2e^{-2(\lambda^- + \epsilon)} + (\alpha_u^m)^2(1 - k_2)^2e^{-2(\lambda^+ + \epsilon)} > (\alpha_s^m)^2 + (\alpha_u^m)^2 \quad (187)$$

Solving we see that if we call $\psi_s = (1 - k_1)^2 e^{-2(\lambda^- + \epsilon)}$ and $\psi_u = (1 - k_2)^2 e^{-2(\lambda^+ + \epsilon)}$ we need

$$\left(\frac{\alpha_s^m}{\alpha_u^m} \right)^2 > \left(\frac{1 - \psi_u}{\psi_s - 1} \right) := \Psi. \quad (188)$$

Again, note here that $\psi_s > 1$ and $\psi_u < 1$. We can assume that $\Psi > 1$ and argue that m can be taken large enough that this ratio holds true. That will complete the proof.

So we need $\alpha_u^m < \frac{1}{\sqrt{\Psi}} \alpha_s^m$. We will demonstrate that $\frac{1}{2} C^{|n'|} \alpha_s^2 < e^{-|n'|(\lambda^- + \epsilon)} \alpha_s$ (where again $n' = \lfloor k_1 - m \rfloor$), then using distortion estimates, it will suffice to show the following bound:

$$e^{-(\lambda^+ + \epsilon)|n'|} \alpha_u + \frac{1}{2} C^{|n'|} \alpha_u^2 < \frac{1}{\sqrt{\Psi}} \left(e^{-|n'|(\lambda^- + \epsilon)} \alpha_s - \frac{1}{2} C^{|n'|} \alpha_s^2 \right). \quad (189)$$

Note Ψ is completely independent of any of the other parameters, in fact could be computed at the very start of this argument. Note also that C , our bound on the second derivative is also fixed from the beginning and for ease of notation we will write $C = e^j$ and work with this. So to begin with, we want

$$e^{jm} \alpha_s \leq 2e^{-(\lambda^- + \epsilon)m}. \quad (190)$$

Note that

$$\alpha_s, \alpha_u \leq M_r d_X(\tilde{y}, \tilde{x}) \leq M_r k \epsilon_f e^{-(\lambda^+ + \delta) T_{big} \theta},$$

for $\theta \in [\theta_2, \theta_1]$ and that at its smallest, $m = \frac{C_b}{10^i + J} T_{big} \theta_2$. Then instead of Equation (190) we look at

$$e^{jm} \alpha_s \leq M_r k \epsilon_f e^{j \frac{C_b}{10^i + J} T_{big} \theta_2 - (\lambda^+ + \delta) T_{big} \theta} \leq 2e^{-(\lambda^- + \epsilon) \frac{C_b}{10^i + J} T_{big} \theta_2}. \quad (191)$$

We solve this and see that we need

$$j \leq \left(\frac{(\lambda^+ + \delta)}{\frac{C_b}{10^{i+J}}} - (\lambda^- + \epsilon) \right) - \frac{\ln(k\epsilon_f M_r)}{T_{big}\theta \frac{C_b}{10^{i+J}}}. \quad (192)$$

Note that i was chosen such that $\frac{C_b}{10^{i+J}} < \frac{C_b}{10^{i+2}} \ll (\lambda^+ + \delta)$ (see Choice 2) and we can start with T_{big} large enough so that term $\frac{\ln(k\epsilon_f M_r)}{T_{big}\theta \frac{C_b}{10^{i+J}}}$ is small enough so that Equation (192) is true for all T_{big} afterwards. Hence we establish dominance of the linear term in the Taylor expansion of α_s because we have

$$\frac{1}{2}|v_s^T D^2 \tilde{F}_{(x,\omega),k}^t(w)v_s| \leq \frac{1}{2}C^m \alpha_s^2 \leq e^{-(\lambda^- + \epsilon)m} \alpha_s \leq |D\tilde{F}_{(x,\omega),k}^t(0)v_s|. \quad (193)$$

So now we have to show Equation (189). The same argument as before shows that $\frac{1}{2}C^m \alpha_u^2 \leq e^{-(\lambda^+ + \epsilon)m} \alpha_u$, we get that

$$j \leq \left(\frac{(\lambda^+ + \delta)}{\frac{C_b}{10^{i+J}}} - (\lambda^+ + \epsilon) \right) - \frac{\ln(k\epsilon_f M_r)/2}{T_{big}\theta \frac{C_b}{10^{i+J}}}. \quad (194)$$

Again, in Choice 2, we chose everything so that this would be true because $\lambda^-, \lambda^+, M_r, \epsilon_f, k$ are all chosen at the very start.

Then Equation (189) becomes

$$2e^{-(\lambda^+ + \epsilon)m} \alpha_u < \frac{1}{2\sqrt{\Psi}} e^{-(\lambda^- + \epsilon)m} \alpha_s, \quad (195)$$

which is satisfied for m large, i.e. T_{big} large.

Then Equation (189) gives us $\alpha_u^m < \frac{1}{\sqrt{\Psi}} \alpha_s^m$. This establishes that $D_m < D_{m+1}$. $\square$

For the purposes of Lemma 5.3.38, we need the following Lemma:

Lemma 5.3.28. *The constant $0 < \eta < 1$ as fixed in Section 5.3 is such that for T_{big} large*

enough (hence m large enough) $D_{m_2}^{(1+\eta)} > D_{m_1}$.

Proof. As in the previous argument, we can replace the usual distortion bounds with

$$2e^{-(\lambda^+-\epsilon)m}\alpha_s \leq \alpha_s^m \leq \frac{1}{2}e^{-(\lambda^--\epsilon)m}\alpha_s, \quad (196)$$

$$2e^{-(\lambda^+-\epsilon)m}\alpha_u \leq \alpha_u^m \leq \frac{1}{2}e^{(\lambda^+-\epsilon)m}\alpha_u. \quad (197)$$

Again, T_{big} has to be large enough for this to hold. We assume this throughout the remainder of the computation wherever m appears, T_{big} is large. Let

$$D = \sqrt{(\alpha_s)^2 + (\alpha_u)^2},$$

i.e. the original distance we compute in Lyapunov charts, then we compute

$$D_m \leq 2\sqrt{e^{-2(\lambda^--\epsilon)m}\alpha_s^2 + e^{-2(\lambda^+-\epsilon)m}\alpha_u^2} \leq 2e^{-(\lambda^--\epsilon)m}\sqrt{\alpha_s^2 + (e^{-(\lambda^+-\epsilon)m+(\lambda^--\epsilon)m}\alpha_u)^2} \quad (198)$$

$$\leq 2e^{-(\lambda^--\epsilon)m}D. \quad (199)$$

Further we have using Lemma 3.4.2 we can ask that α_s always be at least some fixed fraction of D , let us say that $\alpha_s > \Delta D$ uniformly.

$$D_m \geq \frac{1}{2}\sqrt{e^{-2(\lambda^-+\epsilon)m}\alpha_s + e^{-2(\lambda^+-\epsilon)m}\alpha_u} \geq \frac{1}{2}e^{-(\lambda^-+\epsilon)m}\alpha_s \geq \frac{\Delta}{2}e^{-(\lambda^-+\epsilon)m}D. \quad (200)$$

Further since D is the distance $d_X(\tilde{x}, \tilde{y})$ but in Lyapunov charts, we can account for the change between ambient coordinates and Lyapunov charts and get

$$\frac{1}{M_r}\epsilon_f \left(\frac{1}{\mathcal{L}}\right)^{T_{big}\theta_2} \leq D. \quad (201)$$

Now in order to find η such that $D_{m_1} < D_{m_2}^{(1+\eta)}$ we can show the middle inequality in

$$D_{m_1} \leq 2e^{-(\lambda^- - \epsilon)m_1} D < \left(\frac{\Delta}{2}\right)^{(1+\eta)} e^{-(1+\eta)(\lambda^- - \epsilon)m_2} D^{(1+\eta)} \leq D_{m_2}^{(1+\eta)}. \quad (202)$$

Further, plugging in Equation (201) it suffices to demonstrate that

$$2e^{-(\lambda^- - \epsilon)m_1} < \left(\frac{\Delta}{2}\right)^{(1+\eta)} \left(\frac{\epsilon_f}{M_r}\right)^\eta \left(\frac{1}{\mathcal{L}}\right)^{T_{big}\theta_2\eta} e^{-(1+\eta)(\lambda^- - \epsilon)m_2}. \quad (203)$$

Note that $\Delta/2 < 1$, $\epsilon_f/M_r < 1$ and $\eta < 1$ so we can solve

$$2e^{-(\lambda^- - \epsilon)m_1} < \left(\frac{\Delta}{2}\right) \left(\frac{1}{\mathcal{L}}\right)^{T_{big}\theta_2\eta} e^{-(1+\eta)(\lambda^- - \epsilon)m_2}. \quad (204)$$

Writing $m_1 = \alpha_1 T_{big}\theta_2$, $m_2 = \alpha_2 T_{big}\theta_2$, $\mathcal{C} := \frac{4}{\Delta}$ and $\frac{1}{\mathcal{L}} = e^{-\mathcal{K}}$ (where $\mathcal{K} > 0$) we solve

$$\mathcal{C}e^{-(\lambda^- - \epsilon)\alpha_1 T_{big}\theta_2} < e^{-(1+\eta)(\lambda^- + \epsilon)\alpha_2 T_{big}\theta_2 - \mathcal{K}\eta T_{big}\theta_2}, \quad (205)$$

which gives us that we need

$$\eta < \frac{\ln(\mathcal{C})}{T_{big}\theta_2} - \frac{(\lambda^- - \epsilon)\alpha_1}{-(\lambda^- + \epsilon)\alpha_2 - \mathcal{K}} + \frac{(\lambda^- + \epsilon)\alpha_2}{-(\lambda^- + \epsilon)\alpha_2 - \mathcal{K}}. \quad (206)$$

We can ignore the constant term $\frac{\ln(\mathcal{C})}{T_{big}\theta_2}$ as T_{big} can be taken large so that this term is appropriately small. Then we note that the right hand side of Equation (206) is positive because of choices made in Section 5.3 that give us that $(\lambda^- + \epsilon)\alpha_2 > (\lambda^- - \epsilon)\alpha_1$. The parameter η defined in Section 5.3 fulfills this requirement. $\square$

Now we take a (good) choice of $m \in [m_1, m_2]$ as in Lemma 5.3.23, and denote $x := F_{\tilde{\omega}}^{-m}(\tilde{x})$ and $y := F_{\tilde{\omega}}^{-m}(\tilde{y})$. We get a positive measure subset $V \subset \Omega^+$ of futures ω'^+ to choose from that allow the points $(x, (\sigma^{-\lfloor k_1 - m \rfloor}(\tilde{\omega})^-, \omega'^+))$ and $(y, (\sigma^{-\lfloor k_1 - m \rfloor}(\tilde{\omega})^-, \omega'^+))$

where $\tilde{\omega}^- \in \Omega^-$ is the past part of $\tilde{\omega}$. Let $\omega'^+ \in V$ be arbitrary, let

$$\omega' = (\sigma^{-\lfloor k_1 - m \rfloor}(\tilde{\omega})^-, \omega'^+) \in \Omega.$$

We will have additional restrictions on choosing $(\omega')^+$, but certainly, $(\omega')^+$ will belong to V .

At the points (x, ω') and (y, ω') we will project (y, ω') along $W^s(y, \omega')$, onto $W^u(x, \omega')$ and use (modified) NFCs (see Equation (97)) to control the distance between the final points when flowing by ω' by ℓ . We want this distance to be some fixed ϵ_f .

Note that our good sets make the changes between ambient coordinates, Lyapunov charts, and (modified) NFCs, a bounded change as described in Section 5.3. Also note that $W^u(x, \omega')$ is the same as $W^u(x, \sigma^{-\lfloor k_1 - m \rfloor} \tilde{\omega})$ because the past vectors of ω' and $\sigma^{-\lfloor k_1 - m \rfloor} \tilde{\omega}$ are the same. The stable manifold changes passing from $\sigma^{-\lfloor k_1 - m \rfloor} \tilde{\omega}$ to ω' .

Recall $\tilde{D}$ represents the distance between $U^+[F^T(\hat{x}, \hat{\omega})]$ and $U^+[F^T(\hat{y}, \hat{\omega})]$ in ambient coordinates because $\tilde{D}$ is the distance between $\tilde{x}$ and $\tilde{y}$. Denote $L_x(m)$ and $L_y(m)$ as the image of $U^+[F^T(\hat{x}, \hat{\omega})]$ and $U^+[F^T(\hat{y}, \hat{\omega})]$ after flowing backwards by m . Recall that $\delta(m)$ is the distance between $x := f_{\tilde{\omega}}^{-m}(\tilde{x})$ and $y := f_{\tilde{\omega}}^{-m}(\tilde{y})$, it represents the distance between $L_x(m)$ and $L_y(m)$. Then, we let γ be the distance between x and y' where y' is the projection of y onto $W^u(x, \omega')$ along $W^s(y, \omega')$. This means that $y' = W^s(y, \omega') \cap W^u(x, \omega')$.

Note that $x, y \in K_\varepsilon$ (See Lemma 3.10.12 and Section 5.3 for the definition of K'_2 and K_2). Recall from Lemma 5.3.26 that d_m is the error between $F_{\tilde{\omega}}^{-m}(W^u(F^n(\hat{x}, \hat{\omega})))$ and $F_{\tilde{\omega}}^{-m}(W^u(F^n(\hat{y}, \hat{\omega})))$ and that $d_m \rightarrow 0$ as $T_{big}, m \rightarrow \infty$.

Lemma 5.3.29. *There exists m' and a constant $M_a \geq 1$ such that for all $m > m'$ one can choose $(\omega')^+$ such that $\gamma(m) \in [\frac{1}{M_a}\delta(m), M_a\delta(m)]$.*

Proof. First let us take m large enough (T_{big} large enough) such that

$$M_r^3 d_m / 100 < \varepsilon,$$

where ε is from Lemma 3.10.12.

Now $\delta(m) = d_X(x, y)$ and $\gamma = d_X(x, y')$. We will work in Lyapunov charts denoted $\varphi_{(y, \omega')}(B(y, k_U))$, note that our scale of interest is smaller than $k_U/2$ so this is appropriate. Let $\vec{v}$ be the vector between the origin $\varphi_{(y, \omega')}(y)$ and $\varphi_{(y, \omega')}(x)$, note that there exists $M \in [\frac{1}{M_r}, M_r]$ such that $|\vec{v}| = \delta(m)M$.

Let $\vec{w}$ be the vector in $\varphi_{(y, \omega')}(B(y, k_U)) \subset \mathbb{C}^2$ of length $|\vec{w}| = \eta$ corresponding to the intersection of $\mathbb{C}^s$ (which is $E^s(y, \omega')$ in this chart) with the tangent plane through x of $\varphi_{(y, \omega')}(W^u(x, \omega'))$ which we will denote W . Using Lemma 3.4.2 and Lemma 3.10.12 one can choose ω' such that the angle θ_1 between $\vec{v}$ and $\vec{w}$ is bounded from below by ε (so this is much bigger than the error between the tangent plane through $F^{-m}(W^u(F^n(\hat{y}, \hat{\omega})))$ at y and the tangent plane through $F^{-m}(W^u(F^n(\hat{x}, \hat{\omega})))$ at x).

Claim 5.3.30. *There exists m' , c' and d' such that for all $m > m'$,*

$$|\vec{w}| = \eta = c|\vec{v}| = c\delta(m)M, \quad |\vec{v} - \vec{w}| = d|\vec{v}| = d\delta(m)M$$

for some $c \in [\frac{1}{c'}, c']$, $d \in [\frac{1}{d'}, d']$.

Proof. Note that $(x, \sigma^{-n'}(\tilde{\omega})), (y, \sigma^{-n'}(\tilde{\omega})) \in K'_2 \subset K_{ang}$ and so

$$\angle(E^s(x, \sigma^{-n'}(\tilde{\omega})), E^u(x, \sigma^{-n'}(\tilde{\omega}))) = \angle(E^s(x, \sigma^{-n'}(\tilde{\omega})), E^u(x, \omega')) > \Theta.$$

Again, we can take m large so that the error between the tangent plane Q through $F^{-m}(W^u(F^n(\hat{y}, \hat{\omega})))$ at y and the tangent plane P through $F^{-m}(W^u(F^n(\hat{x}, \hat{\omega})))$ at x is as small as needed.

We can also choose m' large enough so that Q is as close as needed to $E^s(x, \sigma^{-n'}(\tilde{\omega}))$. Combining everything, this amounts to being able to choose m' large enough so that

$$\angle(Q, E^u(x, \omega')) > 3\Theta/4.$$

Then for large m , returning to our chart, i.e. working with W instead of $E^u(x, \omega')$, this translates to being able to say that $\vec{v}$ makes an angle θ_2 with the vector $\vec{v} - \vec{w}$ of at least $\Theta/2$.

The law of sines says that $|\vec{v} - \vec{w}| = |\vec{v}| \frac{\sin(\theta_1)}{\sin(\beta)} = \delta(m)M \frac{\sin(\theta_1)}{\sin(\beta)}$ where

$$\beta = \pi - \theta_1 - \theta_2,$$

$\theta_2 \in [\Theta/2, \pi - \Theta/2]$, and $\theta_1 \in [\varepsilon, \pi - \varepsilon]$. We let $d = \frac{\sin(\theta_1)}{\sin(\beta)}$ which is clearly bounded.

The law of sines similarly gives the required relation between $|\vec{w}|$ and $|\vec{v}|$. $\square$

Let $\vec{u}$ be a unit vector in the direction of $\vec{w}$ and take $u^\perp$ to be a unit vector perpendicular to u belonging to $E^s(y, \omega')^\perp$ viewed in this chart $\varphi_{(y, \omega')}$, i.e. $(\mathbb{C}^s)^\perp$. Let $\psi(z) = \vec{u}z + \vec{u}^\perp g(z)$ be a parametrization of $\varphi_{(y, \omega')}(W^s(y, \omega'))$. The function g is an error term $|g(z)| \leq C''|z|^2$ by Lemma 3.10.13. Let z' be such that $\psi(z')$ intersects $\varphi_{(y, \omega')}(W^u(x, \omega'))$, and z'' be such that $z''u = w$.

We do a linear approximation of our problem in Claim 5.3.32.

Claim 5.3.31. *There exists $C' > 0$, $\tilde{C} > 0$ and $m' > 0$ such that for all $m > m'$ we have*

$$|z'| \leq C'\delta(m)M, \quad |z' - z''| \leq \tilde{C}\delta(m)^2.$$

Proof. We know from Claim 5.3.30 that $|z''| = |\vec{w}| \leq c'\delta(m)M$. This represents the intersection of $\mathbb{C}^s$ which is the tangent at y of $\varphi_{(y, \omega')}(W^s(y, \omega'))$ in this chart and the tangent through x of $\varphi_{(y, \omega')}(W^u(x, \omega'))$ which is denoted by W . In a ball of radius $2c'\delta(m)M_r$ around $\varphi_{(y, \omega')}(y)$ which is the origin in this chart, the error between $\mathbb{C}^s$ and $\varphi_{(y, \omega')}(W^s(y, \omega'))$ as well as the error between W and $\varphi_{(y, \omega')}(W^u(x, \omega'))$ is at most $C''(2c'\delta(m)M_r)^2$ which can be made much smaller than $c'\delta(m)$ for large m . We will let C''' be such that $C'''\delta(m)^2 = C''(2c'\delta(m)M_r)^2$.

Then we know that $\{q\} = \varphi(W^u(x, \omega')) \cap \psi$ must be within $C''' \delta(m)^2$ of W and $\mathbb{C}^s$. We also know that β , the angle between W and $\mathbb{C}^s$ is bounded away from zero (and π) by some ϵ as in the proof of Claim 5.3.30. Let $x_1 = \text{proj}_{\mathbb{C}^s}(q)$, $x_2 = \text{proj}_W(q)$ and $\mathcal{O} = W \cap \mathbb{C}^s$, then $\text{dist}(x_1, q) = |x_1 \vec{q}| \leq C''' \delta(m)^2$ and $\text{dist}(x_2, q) = |x_2 \vec{q}| \leq C''' \delta(m)^2$. Note that necessarily $\text{dist}(\mathcal{O}, x_1) = |\vec{\mathcal{O}x_1}| = |z'' - z'|$ and $\text{dist}(y, x_1) = |y \vec{x_1}| = |z'|$. We have two right angled triangles, one formed by $\mathcal{O}$, q and x_1 , the other formed by $\mathcal{O}$, q and x_2 . The hypotenuse of both of these triangles is $\mathcal{D} := \text{dist}(\mathcal{O}, q) = |\vec{\mathcal{O}q}|$. Let θ be the angle between $\vec{\mathcal{O}q}$ and $\vec{\mathcal{O}x_1}$ and θ' be the angle between $\vec{\mathcal{O}x_2}$ and $\vec{\mathcal{O}q}$. Then $\theta + \theta' + \beta = \pi$ and at least one of θ or θ' is greater than or equal to $\frac{\pi - \beta}{2}$. Without loss of generality, assume that $\theta \geq \frac{\pi - \beta}{2}$. Then

$$\mathcal{D} = \frac{|x_1 \vec{q}|}{\sin(\theta)} \leq \frac{C''' \delta(m)^2}{\sin(\frac{\pi - \beta}{2})}.$$

Since β is bounded, we can say there is some C' such that $\mathcal{D} \leq C' \delta(m)^2$.

Then basic trigonometry tells us that $|\vec{\mathcal{O}x_1}| = |z' - z''| \leq C' \delta(m)^2$ and further,

$$|z'| = \text{dist}(y, x_1) \leq \text{dist}(y, \mathcal{O}) + \text{dist}(\mathcal{O}, x_1) \leq c' \delta(m) M + C' \delta(m)^2.$$

Since C' is independent of m , we can choose m large enough so that

$$C' \delta(m)^2 < c' \delta(m) M / 2,$$

and let $\tilde{C} = \frac{3}{2} c'$. This completes the proof. □

We will take m large enough so that $\tilde{C} \delta(m)^2 \leq \frac{1}{2d' M_r} \delta(m)$.

Now note that the distance $\gamma(m)$ in this chart $\varphi_y(B(y, k_U))$, is really the distance $|\vec{v} - \psi(z')|$. That is to say,

$$M \gamma(m) = |\vec{v} - \psi(z')| = |\vec{v} - \vec{u} z' - \vec{u}^\perp g(z')|.$$

Claim 5.3.32. *We have that*

$$\frac{1}{2d'}\delta(m)M \leq |\vec{v} - \vec{u}z'| \leq 2d'\delta(m)M.$$

Proof. Using Claim 5.3.31 we see that

$$\begin{aligned} |\vec{v} - \vec{u}z'| &= |\vec{v} - \vec{u}z' + \vec{w} - \vec{w}| = |\vec{v} - \vec{w} - \vec{u}z' + \vec{u}z''| \\ &\leq |\vec{v} - \vec{w}| + |z' - z''| \\ &\leq d'\delta(m)M + \tilde{C}\delta(m)^2 \\ &\leq 2d'\delta(m)M. \end{aligned}$$

On the other hand we have that

$$\begin{aligned} |\vec{v} - \vec{u}z'| &\geq ||\vec{v} - \vec{w}| - |\vec{w} - \vec{u}z'|| = ||\vec{v} - \vec{w}| - |z' - z''|| \\ &\geq \frac{1}{2d'}\delta(m)M. \end{aligned}$$

□

We will additionally choose m large enough so that

$$C''(C'\delta(m)M_r)^2 < \frac{1}{4d'M_r}\delta(m).$$

Then, using our claims above we have:

$$\begin{aligned} \gamma \cdot M &= |\vec{v} - \vec{u}z' - \vec{u}^\perp g(z')| \geq \left| |\vec{v} - z'\vec{u}| - |\vec{u}^\perp g(z')| \right| \\ &\geq \frac{1}{4d'M_r}\delta(m). \end{aligned}$$

Now for the upper bound we see that

$$\begin{aligned}
\gamma \cdot M &= |\vec{v} - \vec{u}z' - \vec{u}^\perp g(z')| \leq |\vec{v}| + |z'\vec{u}| + |\vec{u}^\perp g(z')| \\
&\leq M_r \delta(m) + M_r C' \delta(m) + C'' (C' \delta(m) M_r)^2 \\
&\leq M_r \delta(m) + M_r C' \delta(m) + \frac{1}{4d' M_r} \delta(m) \\
&= (M_r + M_r C' + \frac{1}{4d' M_r}) \delta(m).
\end{aligned}$$

We take $M_a = \max\{(M_r + M_r C' + \frac{1}{4d' M_r}), 4d' M_r\}$. This completes the proof. $\square$

Remark 5.3.33. *Note that γ is a good proxy for the distance between L_x and the projection of L_y onto $W^u(x, \omega')$. What we mean by projection here is that if $y_{new} \in L_y$ is stably projected on $W^u(x, \omega')$ we get some $y'_{new} = W^u(x, \omega') \cap W^s(y'_{new}, \omega')$. Note that $E^s(y_{new}, \omega')$ is a good approximation for $W^s(y'_{new}, \omega')$ and that these vary Hölder continuously. This is what makes γ a good proxy.*

Now y' and y both belong to $W^s(y, \omega')$ so as we flow forwards under ω' this distance contracts. This is true for the other stably projected points on L_y . So γ is a good proxy for δ .

Recall that we were working towards having our end points be ϵ_f apart in ambient coordinates (up to some bounded constant).

Now that we know that $\gamma \neq 0$ and is comparable to δ , we want to work with this instead of δ because this is a distance that lives in $W^u(x, \omega')$ and we can use (modified) normal form coordinates (NFCs) (see Equation (97)). When we pass to (modified) NFCs we can solve for $\ell(m)$, i.e. the amount we want to flow by from the points (x, ω') and (y, ω') regardless of the choice of ω' . Ultimately we will have to compute ℓ slightly differently, but the idea is that γ gets changed by (modified) NFCs to some $C_1 \gamma$ where $C_1 \in [\frac{1}{M_{NFC}}, M_{NFC}]$ and we would like to grow this distance to be $\epsilon_f \cdot M_{NFC}$ (so that when we go back to ambient coordinates

it is bounded from below by ϵ_f). Then we solve $C_1 \gamma e^\ell = M_{NFC} \epsilon_f$ for ℓ and get

$$\ell(m) = \ln \left(\frac{M_{NFC} \epsilon_f}{C_1 \gamma(m)} \right). \quad (207)$$

Since C_1 changes with m , we instead replace it with $\frac{1}{M_{NFC}}$ (i.e. its lower bound). Solving $\ell = \ln \left(\frac{M_{NFC} \epsilon_f}{\frac{1}{M_{NFC}} \gamma} \right)$ means we would most certainly reach ϵ_f or go past ϵ_f by a (uniformly) bounded amount since C_1 is bounded from above as well. This fact is important to Section 5.3.

To make the computation in Lemma 5.3.36 simpler, we use D_m instead of γ which would make the formula look more like this:

$$\ell(m) = \ln \left(\frac{M_{NFC} \epsilon_f}{\frac{1}{M_{NFC} M_r M_a} D_m} \right). \quad (208)$$

Note that the constants in front of D_m appear because we are thinking of D_m in relation to γ . D_m is related to $\delta(m)$ by a constant bounded from below by $\frac{1}{M_r}$ and $\delta(m)$ is related to γ by C' , the lower bound being $\frac{1}{M_a}$. In this we clearly surpass ϵ_f still the distance is (uniformly) bounded above since C' and the change between $\delta(m)$ and D_m is bounded from above as well by M_a and M_r respectively. This fact is again important to Section 5.3.

However, this definition for $\ell(m)$ is not continuous in m because D_m changes only every integer (up to taking into account k_1). We need continuity for Section 5.3. Note that we showed that D_m is increasing in m , so we define ℓ to be the piecewise linear interpolation between these points. We can write this explicitly as

$$\begin{aligned} \ell(m) = & \ln \left(\frac{M_{NFC} \epsilon_f}{\frac{1}{M_{NFC} M_r M_a} D_m} \right) \\ & + (m - \lfloor m \rfloor) \left(\ln \left(\frac{M_{NFC} \epsilon_f}{\frac{1}{M_{NFC} M_r M_a} D_{m+1}} \right) - \ln \left(\frac{M_{NFC} \epsilon_f}{\frac{1}{M_{NFC} M_r M_a} D_m} \right) \right) \end{aligned} \quad (209)$$

which simplifies to

$$\ell(m) = \ln \left(\frac{M_{NFC} \epsilon_f}{\frac{1}{M_{NFC} M_r M_a} D_m} \right) + (m - \lfloor m \rfloor) \ln \left(\frac{D_m}{D_{m+1}} \right). \quad (210)$$

We make one last modification to this definition of ℓ , to correct concerns that in the interpolation, if m is between two integers, technically $\ell(m)$ computed using Equation (208) is smaller than that of $\ell(m)$ computed using Equation (210), we tack on a last constant, let

$$M_D := \left(e^{-(\lambda^- + \delta)} + \frac{1}{2} C \right),$$

then let

$$\ell(m) = \ln \left(\frac{M_{NFC} \epsilon_f}{\frac{1}{M_{NFC} M_r M_a M_D} D_m} \right) + (m - \lfloor m \rfloor) \ln \left(\frac{D_m}{D_{m+1}} \right). \quad (211)$$

This expression, (211), is how we define ℓ . Note that the second term is negative. Additionally, take note of the fact that D_m increases with m because m is large enough; see Lemma 5.3.27.

Flowing by ℓ

Once we have made our initial adjustments (from the starting points to the points denoted $(\tilde{x}, \tilde{\omega})$, $(\tilde{y}, \tilde{\omega})$) we have a ‘V’ shape where we go backwards by m (under the standard flow) and forwards by ℓ (under the time change flow) where ℓ is determined by m and the distance ϵ_f that we reach (within some constant) at the end points using (modified) NFCs. But we need the ‘legs’ of the ‘V’, i.e. the forward and backward directions to be of the same length to run the argument in Section 5.3. This is why we will introduce extra points in this section.

We fix T and the points $(\tilde{x}, \tilde{\omega}, k_1)$ and $(\tilde{y}, \tilde{\omega}, k_1)$ from Section 5.3, and for each m we pick in Section 5.3 (from the positive measure subset of ‘good choices’ in some interval, see

Lemma 5.3.23) we get the points (x, ω', k_2) and (y, ω', k_2) where $\omega' \in \Omega$ is any vector such that $\omega'^- = \sigma^{-\lfloor k_1 - m \rfloor}(\tilde{\omega})^-$ and ω' belongs to the positive measure subset V corresponding to the points $(x, \sigma^{-\lfloor k_1 - m \rfloor}(\tilde{\omega}))$ and $(y, \sigma^{-\lfloor k_1 - m \rfloor}(\tilde{\omega}))$ which live in $\tilde{K}_2$.

Additionally, for each choice of m we can compute $\ell(m)$ as we did in Section 5.3. This is independent of the choice of future ω' that we pick from some good set. Starting from the points $(\tilde{x}, \tilde{\omega})$ and $(\tilde{y}, \tilde{\omega})$ (i.e. we are in the set K_1), we want to flow forwards by an amount that will allow us to complete our ‘V’ shape.

Let us reiterate our ‘V’ shape more definitively. Since the points $(\tilde{x}, \tilde{\omega}, k_1)$ and $(\tilde{y}, \tilde{\omega}, k_1)$ are fixed at this point, our ‘V’ is determined entirely by a choice of m (which is in a fixed interval at this point). The length of the legs is $\ell(m)$ under time change flow, with the middle point of the ‘V’ being the points (x, ω', k_2) and (y, ω', k_2) . Note that because τ in the ‘time change’ flow depends on which point you start at, the resulting pair of points after flowing by $\ell(m)$ will have different Ω and $[0, 1[$ -components compared to one another, which was never the case with the ‘standard’ flow. We have to convert m , that we flowed by under the standard flow, into the quantity we would flow by from $(x, \sigma^{n'}(\tilde{\omega}), k_2)$ and $(y, \sigma^{n'}(\tilde{\omega}), k_2)$ to $(\tilde{x}, \tilde{\omega}, k_1)$ and $(\tilde{y}, \tilde{\omega}, k_1)$ under the time change flow. This quantity will differ between points. We define $m_{\tau, (\tilde{x}, \tilde{\omega}, k_1)}$ and $m_{\tau, (\tilde{y}, \tilde{\omega}, k_1)}$ to be these quantities resp. Now to complete our ‘V’ we flow backwards from $(\tilde{x}, \tilde{\omega}, k_1)$ and $(\tilde{y}, \tilde{\omega}, k_1)$ by $d(m, (\tilde{x}, \tilde{\omega}, k_1)) := \ell(m) + (-m)_{\tau, (\tilde{x}, \tilde{\omega}, k_1)}$ and $d(m, (\tilde{y}, \tilde{\omega}, k_1)) := \ell(m) + (-m)_{\tau, (\tilde{y}, \tilde{\omega}, k_1)}$ (note that both functions d are positive). We denote the resulting points $(\bar{x}, \bar{\omega}_x, k_x)$ and $(\bar{y}, \bar{\omega}_y, k_y)$ resp. Note that k_x and k_y are not necessarily the same; neither are $\bar{\omega}_x$ and $\bar{\omega}_y$.

Remark 5.3.34. *The minus sign was included next to the m before doing the time change because we want to remember to compute it using τ^- , i.e. we are going backwards and we need to remember that as we compute the time changed value.*

For simplicity, we will take our overall ‘V’ and divide it into its two components, the ‘V’ we have at the fixed point $(\tilde{x}, \tilde{\omega}, k_1)$ will be called ‘V1’ and the ‘V’ we have at the point

$(\tilde{y}, \tilde{\omega}, k_1)$ will be called ‘V2’. This allows us to drop some notation regarding time changed m and the function d . Let us just focus on ‘V1’ as everything we have to say will hold for ‘V2’ as well. Rename $m_{\tau, (\tilde{x}, \tilde{\omega}, k_1)}$ as m_1^τ and $d(m, (\tilde{x}, \tilde{\omega}, k_0))$ as $d_1(m)$. We will have to show that d_1 is Bi-Lipschitz in m (on a good set) in order to pick m such that flowing backwards by m from $(\tilde{x}, \tilde{\omega}, k_1)$ we land in the set $\tilde{K}_2$, and flowing forwards by $d_1(m)$ we land in the set K_3 . We can use Lemma 3.10.15 to do one or the other, but we have to run a different argument to ensure we can do both at the same time. A similar argument will work for ‘V2’ where we use the notation m_2^τ and $d_2(m)$ for $m_{\tau, (\tilde{y}, \tilde{\omega}, k_1)}$ and $d(m, (\tilde{y}, \tilde{\omega}, k_0))$ respectively.

Remark 5.3.35. *It is worthwhile to recall how flowing backwards works. When one passes zero in the $[0, 1[$ -component, one applies the inverse of f_{-1} (i.e. the automorphism in the -1 spot of the Ω -component). So if one starts at $k = 0.22$ (the $[0, 1[$ -component) and flows backwards by 0.23 one has applied a single diffeomorphism.*

The time changed flow going backwards was described in Section 3.9. Being closer to 1 in the $[0, 1[$ -component is being further away from zero and hence being further away from reaching τ^- of the point (where τ^- is always negative).

Lemma 5.3.36. *Take the interval $[m_1, m_2]$ from Section 5.3. The map*

$$d_1 : [m_1, m_2] \rightarrow]-\infty, 0],$$

is Bi-Lipschitz, (similar for d_2 and we get the same Lipschitz constant works in this case).

Proof. First we want to show that there is $L > 0$ such that $|d_1(m) - d_1(m')| \leq L|m - m'|$ for arbitrary m and m' . Recall the definition of d_1 from Equation (110) and plug in:

$$\begin{aligned} |d_1(m) - d_1(m')| &= |(-m)_1^\tau - (-(m'))_1^\tau + \ell(m) - \ell(m')| \\ &\leq |(-(m'))_1^\tau - (-m)_1^\tau| + |\ell(m) - \ell(m')|. \end{aligned}$$

By Equation (211) we have that this is equal to

$$\begin{aligned} & |(-m)_1^\tau - (-m')_1^\tau| + \left| \ln \left(\frac{M_{NFC} \epsilon f}{\frac{1}{M_{NFC} M_r M_a M_D} D_{m'}} \right) + (m' - \lfloor m' \rfloor) \ln \left(\frac{D_{m'}}{D_{m'+1}} \right) \right. \\ & \quad \left. - \ln \left(\frac{M_{NFC} \epsilon f}{\frac{1}{M_{NFC} M_r M_a M_D} D_m} \right) - (m - \lfloor m \rfloor) \ln \left(\frac{D_m}{D_{m+1}} \right) \right| \end{aligned}$$

Let us control the first term in the sum above. Recall Remark 5.3.35 and Section 3.9. Note also that k_1 plays a role here, we had defined d_1 once we fixed a point that had this $[0, 1[$ -component. It will show up in our computations. Making sure to keep the convention that k_1 is written as a positive number between 0 and 1, we can write

$$(-m)_1^\tau = k_1 \tau^-(\tilde{x}, \tilde{\omega}) + \tau^-(F^{-1}(\tilde{x}, \tilde{\omega})) + \dots \tau^-(F^{\lfloor k_1 - m \rfloor + 1}(\tilde{x}, \tilde{\omega})) + a, \quad (212)$$

$$(-m')_1^\tau = k_1 \tau^-(\tilde{x}, \tilde{\omega}) + \tau^-(F^{-1}(\tilde{x}, \tilde{\omega})) + \dots \tau^-(F^{\lfloor k_1 - m' \rfloor + 1}(\tilde{x}, \tilde{\omega})) + b. \quad (213)$$

Note that the first term $k_1 \tau^-(\tilde{x}, \tilde{\omega})$ appears because once we pass 0 we hit it with the automorphism f_{-1}^{-1} . For example if $k_1 = 0.22$, then we are already 78% on our way to hitting $\tau^-(\tilde{x}, \tilde{\omega})$, the remainder that we need is that 22%. The a and b terms show up because we have not quite reached the next τ^- value, i.e. $a \in [0, \tau^-(F^{\lfloor k_1 - m \rfloor}(\tilde{x}, \tilde{\omega}))[$ and $b \in [0, \tau^-(F^{\lfloor k_1 - m' \rfloor}(\tilde{x}, \tilde{\omega}))[$. We can easily compute a and b from what we have as clearly the following relations holds:

$$\psi_m := \lfloor k_1 - m \rfloor - (k_1 - m) + 1 = \frac{a}{\tau^-(F^{\lfloor k_1 - m \rfloor}(\tilde{x}, \tilde{\omega}))} > 0, \quad (214)$$

$$\psi_{m'} := \lfloor k_1 - m' \rfloor - (k_1 - m') + 1 = \frac{b}{\tau^-(F^{\lfloor k_1 - m' \rfloor}(\tilde{x}, \tilde{\omega}))} > 0. \quad (215)$$

WLOG assume $m' > m$ (remember these are positive numbers), then

$$(-m')_1^\tau > (-m)_1^\tau.$$

We have

$$\begin{aligned}
|(-(m'))_1^\tau - (-m)_1^\tau| &= \tau^-(F^{\lfloor k_1 - m' \rfloor + 1}(\tilde{x}, \tilde{\omega})) + \dots \tau^-(F^{\lfloor k_1 - m \rfloor}(\tilde{x}, \tilde{\omega})) + b - a \\
&= |\tau^-(F^{\lfloor k_1 - m' \rfloor + 1}(\tilde{x}, \tilde{\omega})) + \dots \tau^-(F^{\lfloor k_1 - m \rfloor}(\tilde{x}, \tilde{\omega})) \\
&\quad + \psi_{m'} \cdot \tau^-(F^{\lfloor k_1 - m' \rfloor}(\tilde{x}, \tilde{\omega})) - \psi_m \cdot \tau^-(F^{\lfloor k_1 - m \rfloor}(\tilde{x}, \tilde{\omega}))|, \\
&= |\tau^-(F^{\lfloor k_1 - m' \rfloor + 1}(\tilde{x}, \tilde{\omega})) + \dots + \tau^-(F^{\lfloor k_1 - m \rfloor - 1}(\tilde{x}, \tilde{\omega})) \\
&\quad + \tau^-(F^{\lfloor k_1 - m \rfloor}(\tilde{x}, \tilde{\omega}))(1 - \psi_m) + \psi_{m'} \cdot \tau^-(F^{\lfloor k_1 - m' \rfloor}(\tilde{x}, \tilde{\omega}))|, \\
&\leq |\tau^-(F^{\lfloor k_1 - m' \rfloor + 1}(\tilde{x}, \tilde{\omega}))| + \dots + |\tau^-(F^{\lfloor k_1 - m \rfloor - 1}(\tilde{x}, \tilde{\omega}))| \\
&\quad + |\tau^-(F^{\lfloor k_1 - m \rfloor}(\tilde{x}, \tilde{\omega}))(1 - \psi_m) + \psi_{m'} \tau^-(F^{\lfloor k_1 - m' \rfloor}(\tilde{x}, \tilde{\omega}))|.
\end{aligned} \tag{216}$$

Using the bounds on τ we get that this is bounded above by

$$\begin{aligned}
& [((\lfloor k_1 - m \rfloor - 1) - (\lfloor k_1 - m' \rfloor + 1) + 1) + (1 - \psi_m) + \psi_{m'}] (\lambda^+ + \epsilon) \\
&= (\lambda^+ + \epsilon)(m' - m).
\end{aligned}$$

On the other hand, we have to lower bound this quantity too. So we consider again $m' > m$ and take Equation (216). Again, the τ^- -terms are lower bounded (in absolute value) by $\lambda^+ - \epsilon$, so we get that Equation (216) is bounded below by $(\lambda^+ - \epsilon)(m' - m)$.

Now we control the second term. It reduces to

$$\begin{aligned} & \left| \ln \left(\frac{M_{NFC} \epsilon f}{\frac{1}{M_{NFC} M_r M_a M_D} D_{m'}} \right) + (m' - \lfloor m' \rfloor) \ln \left(\frac{D_{m'}}{D_{m'+1}} \right) \right. \\ & \left. - \ln \left(\frac{M_{NFC} \epsilon f}{\frac{1}{M_{NFC} M_r M_a M_D} D_m} \right) - (m - \lfloor m \rfloor) \ln \left(\frac{D_m}{D_{m+1}} \right) \right| \end{aligned}$$

We learned in Section 5.3 how D_{m+1} and D_m are related, see Lemma 5.3.27 or Equation (182). When we apply an automorphism, we never grow bigger than a factor of $\ln(e^{-(\lambda^- - \epsilon)} + \frac{1}{2}C)$. Remember throughout that D_m is constant on an interval of length 1.

We deal with cases:

The first case is that $m' > m$ but $(m' - m) < 1$. There are two subcases, one is that $D_m = D_{m'}$, the other is that $D_{m'} < D_m$.

Let us deal with the first subcase: $D_m = D_{m'}$, then

$$\ell(m) - \ell(m') = (m' - m) \ln \left(\frac{D_{m+1}}{D_m} \right) \leq (m' - m) \ln(e^{-(\lambda^- - \epsilon)} + \frac{1}{2}C).$$

So we are done.

Now we deal with the second subcase: $D_{m'} < D_m$. Then really $D_{m'} = D_{m+1}$.

$$\begin{aligned} \ell(m) - \ell(m') &= \ln \left(\frac{D_{m+1}}{D_m} \right) - (m' - \lfloor m' \rfloor) \ln \left(\frac{D_{m+1}}{D_{m+2}} \right) - (m - \lfloor m \rfloor) \ln \left(\frac{D_{m+1}}{D_m} \right) \\ &= (1 - m + \lfloor m \rfloor) \ln \left(\frac{D_{m+1}}{D_m} \right) + (m' - \lfloor m' \rfloor) \ln \left(\frac{D_{m+2}}{D_{m+1}} \right) \\ &\leq (1 - m + \lfloor m \rfloor + m' - \lfloor m' \rfloor) \ln(e^{-(\lambda^- - \epsilon)} + \frac{1}{2}C) \\ &= (m' - m) \ln(e^{-(\lambda^- - \epsilon)} + \frac{1}{2}C). \end{aligned}$$

The last equality holds because the condition of the subcase is really that $\lfloor m \rfloor + 1 = \lfloor m' \rfloor$.

The second case is $m' > m$ but $(m' - m) \geq 1$. We could have stopped with the above and just said that, since d is continuous on a compact interval, $|d(m) - d(m')|$ is uniformly

bounded from above and hence we are done. However, we actually want that this Lipschitz constant computed in this proof is independent of the interval $[m_1, m_2]$. The only restriction we have on $[m_1, m_2]$ is that we need m large enough so that that Lemma 5.3.27 holds. Hence we do more work.

Note first that $\ell(m) = \ell(m') \leq \ell(\lfloor m \rfloor) - \ell(m')$. We will work with this quantity.

$$\frac{\ell(\lfloor m \rfloor) - \ell(m')}{m' - m} = \frac{\ln\left(\frac{D_{m'}}{D_m}\right) - (m' - \lfloor m' \rfloor) \ln\left(\frac{D_{m'}}{D_{m'+1}}\right)}{m' - m}. \quad (217)$$

Since $(m' - m) \geq 1$, we have that $(m' - m) > m' - \lfloor m' \rfloor$. Also note that $D_{m'} < D_{m'+1}$ and so $\ln\left(\frac{D_{m'}}{D_{m'+1}}\right)$ is negative.

$$\begin{aligned} &\leq \frac{\ln\left(\frac{D_{m'}}{D_m}\right)}{m' - m} + \ln\left(\frac{D_{m'+1}}{D_{m'}}\right) \\ &\leq \frac{\lfloor m' - m \rfloor}{m' - m} \ln(e^{-(\lambda^- - \epsilon)} + \frac{1}{2}C) + \ln(e^{-(\lambda^- - \epsilon)} + \frac{1}{2}C) \\ &\leq 2 \ln(e^{-(\lambda^- - \epsilon)} + \frac{1}{2}C). \end{aligned}$$

Now we do the other side of the computation. Note that d is decreasing because as m increases, $(-m)_1^\tau$ decreases and ℓ decreases (in fact both do so strictly and hence so does d). Take $m, m' \in [m_1, m_2]$ without loss of generality, $m' > m$, then

$$|d_1(m) - d_1(m')| = d_1(m) - d_1(m') \geq (-m)_1^\tau - (-m')_1^\tau + \ell(m) - \ell(m') \quad (218)$$

$$= (\lambda^+ - \epsilon)(m' - m) - \ell(m + 1) + \ell(m) \geq \lambda^+ - \epsilon. \quad (219)$$

We get Lipschitz constant to be

$$L := \max\{2 \ln(e^{-(\lambda^- - \epsilon)} + \frac{1}{2}C) + \lambda^+ + \epsilon, \frac{1}{\lambda^+ + \epsilon}\}. \quad (220)$$

□

Now that we have established that d_1 is L -BiLipschitz, a similar argument works for d_2 , we get the same Lipschitz constant; now we can use the following lemma.

Lemma 5.3.37. *Consider $d : [a, b] \rightarrow [0, \infty[$ where $0 \leq a, b < \infty$. If $E \subset [a, b]$ is such that $\text{Leb}(E) < \epsilon(b - a)$ and d is L -Lipschitz then $\text{Leb}(d(E)) < L \cdot \epsilon(b - a)$. Further, if additionally d is BiLipschitz, then for any subset $F \subset [a, b]$ such that $\text{Leb}(F) > \eta(b - a)$, we have $\text{Leb}(d(F)) > \frac{1}{L} \cdot \eta(b - a)$.*

Proof. The Lebesgue measure is regular and hence

$$\text{Leb}(E) = \inf\{\text{Leb}(A) : A \supset E, \text{ open and measurable}\}.$$

Since any open subset A can be written as a countable disjoint union of open intervals, if we can prove that for $A =]a', b'[\subset [a, b]$, $\text{Leb}(d(A)) \leq L \cdot \text{Leb}(A)$ we will be done.

Since d is Lipschitz on $[a, b]$, it is continuous on $[a, b]$. If we close the interval $]a', b'[= A$ by adding the endpoints it is now compact and d attains a max and min on $[a', b']$; call the maximizer and minimizers c_1, c_2 . Then $d([a', b']) \subset [d(c_1), d(c_2)]$ and so the measure of $d(A) = d([a', b'])$ is bounded above by $|d(c_1) - d(c_2)|$. Since d is Lipschitz, we have that

$$|d(c_1) - d(c_2)| \leq L|c_1 - c_2| \leq L|b' - a'|,$$

(since $c_1, c_2 \in [a', b']$). Hence $\text{Leb}(d(A)) \leq L \cdot \text{Leb}(A)$.

We conclude this is true for any open set A . Then by regularity, for any $A \supset E$, $\text{Leb}(d(A)) \leq L \cdot \text{Leb}(A)$, taking infimum of both sides over open sets $A \supset E$ we get $\inf_{A \supset E} \text{Leb}(d(A)) \leq L \cdot \text{Leb}(E)$, but since $d(E) \subset d(A)$ for every $A \supset E$, by monotonicity $\text{Leb}(d(E)) \leq \inf_{A \supset E} \text{Leb}(d(A))$ so we have the result.

The other argument is similar. □

Note that since d is injective (as it is strictly decreasing), it is invertible on its image. Note also that the inverse of a Bilipschitz function is Bilipschitz with the same Bilipschitz constant.

Lemma 5.3.38. *For our choice of m_1 and m_2 as in Section 5.3 Equation (164), and T_{big} large enough, we have that the interval $[d_1(m_2), d_1(m_1)]$ inside $[0, d_1(m_1)]$ has proportion at least $(1 - \frac{1}{1+\eta})$ in measure (where η is as in Section 5.3 and Lemma 5.3.28). A similar result holds for d_2 .*

Proof. For T_{big} large enough we can use Lemma 5.3.28. Let η be as in this lemma.

Claim 5.3.39. *We have that $d_1(m_1) \geq (1 + \eta)d_1(m_2)$.*

Proof. Let $M = M_{NFC}^2 M_r M_a M_D$.

We can drop the interpolation terms for a small price and simply check that

$$\ln \left(\frac{\epsilon_f M}{D_{\lfloor m_1+1 \rfloor}} \right) + \left(-\frac{C_b}{10^{i+J}} T_{big} \theta_2 \right)^\tau > (1 + \eta) \left(\ln \left(\frac{\epsilon_f M}{D_{m_2}} \right) + \left(-\frac{C_b}{10^{i+2}} T_{big} \theta_2 \right)^\tau \right).$$

Note that τ is bounded (between $\lambda^+ - \epsilon$ and $\lambda^+ + \epsilon$). Also as $T_{big} \rightarrow \infty$, the length of $[m_1, m_2]$ goes to infinity, so we have that eventually

$$\left(-\frac{C_b}{10^{i+3}} T_{big} \theta_2 \right)^\tau \geq (1 + \eta) \left(-\frac{C_b}{10^{i+2}} T_{big} \theta_2 \right)^\tau,$$

(remember both terms are negative), so it suffices to compare the remaining terms (coming from the definition of ℓ).

So we want to show $\ln \left(\frac{\epsilon_f M}{D_{\lfloor m_1+1 \rfloor}} \right) \geq (1 + \eta) \ln \left(\frac{\epsilon_f M}{D_{m_2}} \right)$ which reduces to showing

$$(1 + \eta) \ln(D_{m_2}) > \ln(D_{\lfloor m_1+1 \rfloor}) + \ln(\epsilon_f M). \quad (221)$$

Remember ϵ_f is small and $\epsilon_f M < 1$. Then this is equivalent to $D_{m_2}^{(1+\eta)} > D_{m_1} \epsilon_f M$.

This can be done for T_{big} large by Lemma 5.3.28 because it gives us that $D_{m_2}^{(1+\eta)} > D_{m_1}$. $\square$

Then, note that the Lebesgue measure of $[0, d_1(m_1)]$ is $d_1(m_1)$ while the Lebesgue measure of $[d_1(m_2), d_1(m_1)]$ is $(d_1(m_1) - d_1(m_2))$ which by our claim is at least $d_1(m_1)(1 - \frac{1}{1+\eta})$. $\square$

Lemma 5.3.40. *Let $K \subset Z$ be such that $\hat{m}(K) = 1 - \epsilon' = \gamma$ for some $\epsilon' > 0$ small. Let $\delta = \alpha\gamma$ for some $\alpha > 0$ small, and fix $\epsilon > 0$ small. Let E^c be the corresponding Birkhoff set (for this δ, ϵ and K) from applying Lemma 3.10.15 and S the corresponding parameter from this lemma. Take $b > S$ and $\eta \in]0, 1]$, and consider a such that $b > (1 + \eta)a$. Let $\xi > 0$ be small and $x, y \in E^c$. If*

$$\beta = 2\gamma(1 - \alpha) - 1 > (1 - \xi) + \xi \left(\frac{1}{1 + \eta} \right),$$

and if $G \subset [a, b]$ is such that $G = \{t \in [a, b] : F^t(x), F^t(y) \in K\}$, then G is at least $(1 - \xi) \times 100\%$ of the Lebesgue measure of the interval $[a, b]$. That is to say, G has Lebesgue measure at least $(b - a)(1 - \xi)$.

Proof. Similar to the argument in Lemma 5.3.19, we have that given $x, y \in E^c$ Lemma 3.10.15 gives us that the set of ‘good points’ in $[0, b]$,

$$G' = \{t \in [0, b] : F^t(x), F^t(y) \in K\} \subset [0, b],$$

has proportion $\beta = 2\gamma(1 - \alpha) - 1$ inside $[0, m_2]$ in (Lebesgue) measure, i.e. G' is $\beta \times 100\%$ of $[0, m_2]$ in measure.

Now we are interested in the the proportion of ‘good points’ inside $[a, b]$ where $b > (1 + \eta)a$. This means that $a < \frac{b}{(1 + \eta)}$ and so $[\frac{b}{1 + \eta}, b] \subset [a, b]$. We can analyse the proportion of good points in $[\frac{b}{(1 + \eta)}, b]$ to get a lower bound.

We know that inside $[0, b]$, a proportion of $(1 - \beta)$ is ‘bad’, i.e. belongs to $B' = [0, b] \setminus G'$. This is $(1 - \beta) \times 100\%$ of the interval $[0, b]$. Let us assume (the worst case) that all of the bad points in $[0, b]$ belong to $[\frac{b}{(1+\eta)}, b]$. Without loss of generality we can assume then that inside the interval $[\frac{b}{(1+\eta)}, b]$, the interval $[\frac{b}{(1+\eta)}, b - (1 - \beta)b]$ is all good points and the interval $[b - (1 - \beta)b, b]$ is all bad points. Then the proportion of good points in $[\frac{b}{(1+\eta)}, b]$, i.e. the proportion of G inside $[\frac{b}{(1+\eta)}, b]$ is

$$P := \frac{\text{Leb}(G)}{\text{Leb}([\frac{b}{1+\eta}, b])} = \frac{b - (1 - \beta)b - \frac{b}{(1+\eta)}}{b - \frac{b}{(1+\eta)}}.$$

Now we want to show that $P \geq (1 - \xi)$, then we are done. Solving

$$\frac{b - (1 - \beta)b - \frac{b}{(1+\eta)}}{b - \frac{b}{(1+\eta)}} \geq (1 - \xi),$$

for β gives us exactly that we require $\beta > (1 - \xi) + \xi \left(\frac{1}{1+\eta} \right)$. This completes the proof. $\square$

Recall the definitions of m_1 and m_2 as given in Equation (164) with T_{big} is as appropriately large as specified in that section. With this, we have the following lemma.

Lemma 5.3.41. *Given the interval $[m_1, m_2]$ (where m_1, m_2 as specified above), there exists a positive measure subset G such that,*

1. *if $m \in G$ then flowing backwards by m from the points $(\tilde{x}, \tilde{\omega}, k_1)$ and $(\tilde{y}, \tilde{\omega}, k_1)$ lands us in the set $\tilde{K}_2 \times [0, 1[$,*
2. *if $m \in G$ then flowing backwards by $d_1(m)$ and $d_2(m)$ from the points $(\tilde{x}, \tilde{\omega}, k_1)$ and $(\tilde{y}, \tilde{\omega}, k_1)$ resp. lands both points in the set K_3 .*

Proof. First let $\gamma_2 = m(\tilde{K}_2)$ and recall $\delta_2 = \alpha_2 \gamma_2$ in the definition of E_2^c from Section 5.3. We showed in Lemma 5.3.23 that there is some subset of positive measure of $[m_1, m_2]$, call

it G_1 , where we could choose m from in Section 5.3 to get the points $(x, \sigma^{n'} \tilde{\omega})$ and $(y, \sigma^{n'} \tilde{\omega})$ (where $n' = \lfloor k_1 - m \rfloor$) in the good set $\tilde{K}_2$ (i.e these are ‘good’ choices of m). Now we will be explicit with the size of G_1 .

Since $m_2 > 2m_1$ we can apply Lemma 5.3.40 with $\eta = 1$ and $\xi = \frac{1}{100L^4}$ since by construction $\beta > \left(1 - \frac{1}{200L^4}\right) = \left(1 - \frac{1}{100L^4}\right) + \frac{1}{100L^4} \frac{1}{1+\eta}$ (see Section 5.3). We get that G_1 as proportion $\left(1 - \frac{1}{100L^4}\right)$ inside $[m_1, m_2]$ and so $B_1 = [m_1, m_2]$ has proportion $\frac{1}{100L^4}$ inside $[m_1, m_2]$.

We take this interval $[m_1, m_2]$ and apply d_1 to it, since it is a decreasing function we get the interval $[d_1(m_2), d_1(m_1)]$ and by Lemma 5.3.37 we get that

$$\text{Leb}([d_1(m_2), d_1(m_1)]) > \frac{1}{L}(m_2 - m_1).$$

We run the same Birkhoff argument for the points $(\bar{x}, \bar{\omega}_x, k_x)$ and $(\bar{y}, \bar{\omega}_y, k_y)$ using Lemma 3.10.15 with $K = K_3$ (see Section 5.3) using this interval $[d_1(m_2), d_1(m_1)]$. We get a set G_2 which is the set of ‘good’ choices in $[d_1(m_2), d_1(m_1)]$ to flow by from the points $(\tilde{x}, \tilde{\omega})$ and $(\tilde{y}, \tilde{\omega})$ to land in the set K_3 . We can do so since $|d_1(m_2)| > S_p$ where S_p is the Birkhoff bound one gets with E_3^c when using Lemma 3.10.15. Here, we are using $\hat{m}(K_3) = \gamma_2$ and $\delta = \alpha_3 \gamma_3$ as in Section 5.3; we let

$$\beta_3 = 2\gamma_3(1 - \alpha_3) - 1.$$

By Lemma 5.3.28, we have that $d_1(m_1) > (1 + \eta)d_1(m_2)$ where η is fixed in Section 5.3. We take this to be the η in Lemma 5.3.40 with $\xi = \frac{1}{100L^2}$. Since

$$\beta_3 > \left(1 - \frac{1}{100L^2}\right) + \frac{1}{100L^2} \left(1 - \frac{1}{1 + \eta}\right),$$

we get that G_2 has proportion $(1 - \frac{1}{100L^2})$ inside $[d_1(m_2), d_1(m_1)]$. Let

$$B_2 = [d_1(m_2), d_1(m_1)] \setminus G_2,$$

it has proportion at most $\frac{1}{100L^2}$ inside $[d_1(m_2), d_1(m_1)]$.

Our ultimate goal will be to choose m in the set

$$S_1 := G_1 \cap d_1^{-1}(G_2 \cap d_1(G_1)) \subset [m_1, m_2].$$

If we can do this, then being in G_1 ensures that we land in $\tilde{K}_2$ when we flow by m , but also that such an m in G_1 will give a $d_1(m)$ that allows us to land in K_3 after flowing by $d_1(m)$. So we show that S_1 has large positive measure.

The measure of B_1 is $\frac{1}{100L^4}(m_2 - m_1)$, when we apply d_1 to this set, its measure (by Lemma 5.3.37) is at most $(m_2 - m_1)\frac{1}{100L^3}$ whereas the size of $[d_1(m_2), d_1(m_1)]$ is at least $\frac{1}{L}(m_2 - m_1)$ (also by Lemma 5.3.37). So the proportion of $d_1(B_1)$ inside $[d_1(m_2), d_1(m_1)]$ is at most $\frac{1}{100L^2}$. This means that $d_1(G_1)$ has proportion at least $1 - \frac{1}{100L^2}$ inside $[d_1(m_2), d_1(m_1)]$. Meanwhile G_2 has proportion $1 - \frac{1}{100L^2}$ inside $[d_1(m_2), d_1(m_1)]$ and so $d_1(G_1) \cap G_2$ has proportion at least $1 - \frac{2}{100L^2}$ inside $[d_1(m_2), d_1(m_1)]$. Let

$$B_3 := [d_1(m_2), d_1(m_1)] \setminus (d_1(G_1) \cap G_2),$$

it has proportion at most $\frac{2}{100L^2}$ inside $[d_1(m_2), d_1(m_1)]$.

Now we apply d_1^{-1} to $[d_1(m_2), d_1(m_1)]$ to recover $[m_1, m_2]$. By Lemma 5.3.37 we have that

$$Leb([m_1, m_2]) \geq \frac{1}{L}Leb([d_1(m_2), d_1(m_1)]),$$

and that

$$\text{Leb}(d_1^{-1}(B_3)) \leq L \cdot \frac{2}{100L^2} \text{Leb}([d_1(m_2), d_1(m_1)]) = \frac{2}{100L} \text{Leb}(d_1([m_1, m_2])).$$

Hence we have shown that $S_1 \subset [m_1, m_2]$ is at least 98% of the interval $[m_1, m_2]$.

Run the same argument for d_2 and get the corresponding analogous set S_2 . Since they are both about 98% of the interval, we take their intersection and get a subset of positive measure. We call this $G = S_1 \cap S_2$. This completes the proof. $\square$

The end points

From the points $x, y \in X$, we flow forwards by the ‘time change’ flow for time ℓ under a new future ω' to our final pair of points denoted $x', y' \in X$. First we will discuss how we pick this new future. We will need to do so in a way that ensures that we land in an appropriate set called K'_4 (see Section 5.3). Throughout this section we let $K := K'_4$.

Claim 5.3.42. *We have that $\hat{m}^\tau(K^c) < Q\epsilon_4$.*

Proof. The set K_3 belongs to the set where Lemma 3.9.3 holds and so by Lemma 3.9.3, and the fact that $\hat{m}(K^c) < \epsilon_4$, we have that

$$\hat{m}^\tau(K^c) = \int_K \frac{d\hat{m}^\tau}{d\hat{m}} d\hat{m} \leq Qm(K^c) < Q\epsilon_4. \quad (222)$$

This gives us that $\hat{m}^\tau(K) > 1 - Q\epsilon_4$. $\square$

Let $\hat{\mathcal{F}}_0^\tau$ be the sigma algebra of $\hat{m}^\tau$ -measurable functions, i.e. the $\hat{m}^\tau$ completion of the natural sigma algebra on Z . Since $\hat{m}^\tau$ is absolutely continuous with respect to $\hat{m}$, they have the same measure zero sets and so this is the same as the $\hat{m}$ -completion. Let $Q_\ell^\tau := (F_{tc}^{-\ell})^{-1}(\hat{\mathcal{F}}_0^\tau)$; for a function to be measurable with respect to this σ -algebra means

that it only depends on events after ℓ time in the past under the time changed flow with roof function τ .

We start our discussion at the points $(\bar{x}, \bar{\omega}_x, k_x)$ and $(\bar{y}, \bar{\omega}_y, k_y)$ which live in K_3 . By our definition of K_3 in Section 5.3, this means that they satisfy the following lemma from [4, Corollary 3.8]:

Lemma 5.3.43 (Law of last jump, [4], Corollary 3.8). *For any $\hat{m}$ -measurable function $\varphi : Z \rightarrow \mathbb{R}$ (e.g. $\mathbf{1}_K$), for every $\ell \geq 0$, for $\hat{m}$ -a.e. $(\bar{x}, \bar{\omega}, \bar{k}) \in Z$ we have*

$$\mathbb{E}_\tau[\varphi|Q_\ell^\tau](\bar{x}, \bar{\omega}, \bar{k}) = \int_{\Omega^+} \varphi(F_{tc}^\ell(x, (\omega^-, \omega^+), k)) d\mu^\mathbb{N}(\omega^+), \quad (223)$$

where $\mathbb{E}_\tau$ is expectation with respect to $\hat{m}^\tau$.

Consider the tail σ -algebra $Q_\infty^\tau = \bigcap Q_\ell^\tau$.

Claim 5.3.44. *We have that*

$$\int \mathbb{E}_\tau[\mathbf{1}_K|Q_\infty^\tau](x, \omega, k) d\hat{m}^\tau(x, \omega, k) > 1 - Q\epsilon_4. \quad (224)$$

Proof. Using Claim 5.3.42, we have that

$$\int \mathbb{E}_\tau[\mathbf{1}_K|Q_\infty^\tau](x, \omega, k) d\hat{m}^\tau(x, \omega, k) = \hat{m}^\tau(K) > 1 - Q\epsilon_4. \quad (225)$$

This completes the proof. □

Lemma 5.3.45. *We have that*

$$\mu^\mathbb{N}(\{\omega^+ \in \Omega^+ : F_{tc}^\ell(x, (\omega^-, \omega^+), k) \in K\}) \geq 1 - \sqrt{Q\epsilon_4}. \quad (226)$$

Proof. Using Lemma 5.3.44, there is a compact subset L' of the conull set where Lemma

5.3.43 holds with $\varphi = \mathbb{1}_K$, with $\hat{m}^\tau(L') > 1 - \sqrt{Q\epsilon_4}$, such that for $(x, \omega, k) \in L'$ we have

$$\mathbb{E}_\tau[\mathbb{1}_K|Q_\infty^\tau](x, \omega, k) > 1 - \sqrt{Q\epsilon_4}.$$

On the other hand, we apply the Martingale convergence theorem to the Lemma 5.3.43 with $\varphi = \mathbb{1}_K$:

$$\begin{aligned} \mathbb{E}_\tau[\mathbb{1}_K|Q_\infty^\tau](x, \omega, k) &= \lim_{\ell \rightarrow \infty} \mathbb{E}_\tau[\mathbb{1}_K|Q_\ell^\tau](x, \omega, k) \\ &= \lim_{\ell \rightarrow \infty} \int_{\Omega^+} \mathbb{1}_K(F_{tc}^\ell(x, (\omega^-, \omega^+), k)) d\mu^\mathbb{N}(\omega^+). \end{aligned}$$

Egorov's theorem ensures that (with some arbitrarily small loss to L') that this convergence is uniform, hence there is ℓ_0 such that for all $\ell \geq \ell_0$ we have for every $(x, \omega, k) \in L'$

$$\mathbb{E}_\tau[\mathbb{1}_K|Q_\ell^\tau](x, \omega, k) = \int_{\Omega^+} \mathbb{1}_K(F_{tc}^\ell(x, (\omega^-, \omega^+), k)) d\mu^\mathbb{N}(\omega^+) \geq 1 - \sqrt{Q\epsilon_4}.$$

This translates to the statement that

$$\mu^\mathbb{N}(\{\omega^+ \in \Omega^+ : F_{tc}^\ell(x, (\omega^-, \omega^+), k) \in K\}) \geq 1 - \sqrt{Q\epsilon_4}.$$

This completes the proof. □

So there is a large measure set of appropriate new futures to choose from. Run this argument for each of the points, $(\bar{x}, \bar{\omega}_x, k_x)$ and $(\bar{y}, \bar{\omega}_y, k_y)$ (separately).

Conclusion

From Section 5.3, Lemma 5.3.45, for each choice of T and m fixed (where m is chosen in an interval determined by T_{big}) we are given $V_{T,m}$ (ℓ is determined and we assume it is greater than ℓ_0 by taking T large), a positive measure subset of Ω^+ such that we can choose $(\omega')^+$

from.

Additionally, we got in Section 5.3, i.e. when we used Lemma 3.10.15 we landed in a set $\tilde{K}_2$ where the points have a large measure set of ω' to choose from (just the future part, the past stays the same), i.e. these sets $V_{(x,\sigma^{n'}(\tilde{\omega}^-))}, V_{(y,\sigma^{n'}(\tilde{\omega}^-))} \subset \Omega^+$ such that picking ω' from there lands us in the set K_2 . This set K_2 had properties we needed to make arguments in Section 5.3. We defined $V = V_{(x,\sigma^{n'}(\tilde{\omega}^-))} \cap V_{(y,\sigma^{n'}(\tilde{\omega}^-))}$. Now let $V' = V_{T,m} \cap V$. The parameter Q is fixed at the beginning and ϵ_4 , and ϵ_2 were chosen (at the beginning) such that V' has positive measure. We have one last restriction on ω' that we used in Lemma 5.3.29. This comes from the use of Lemma 3.4.2, the required ϵ in this lemma gets smaller as T_{big} gets bigger. So we can take T_{big} large enough so that the corresponding δ is very small and such that this restriction added to $V \cap V'$ still has (large) positive measure.

Take $\ell \rightarrow \infty$, more precisely, I want the legs of the V to go to infinity. To achieve this, we take $\rho \rightarrow 0$, in turn, T_{big} gets bigger, and since θ_1 and θ_2 are fixed, even though T increases, we see that the proportion between T and T_{big} decreases. This implies that the amount T separates the unstable component (i.e. the distance between $\tilde{y}$ and $\tilde{z}$ decreases even though we are flowing for longer) and the stable distance still goes to zero. Together this constitutes the distance between the unstable planes going to zero. The parameter m grows as $\rho \rightarrow 0$ but the error between the unstable planes still goes to zero as in Lemma 5.3.26. We also have that $d_1(m_1) > (1 + \eta)d_1(m_2)$ and $d_2(m_1) > (1 + \eta)d_2(m_2)$ as in Lemma 5.3.38 so that we can make an argument for our points along the way to stay in a good set as in Lemma 5.3.41. Being in a good set after flowing by d_1 or d_2 allows us to use Lemmas 5.3.43 and 5.3.45 to get the set $V_{T,m}$ in the previous paragraph. As ℓ goes to infinity, the image of the supports land in the same plane as the stable components for ω' shrink. We end up with two support lines in one unstable manifold separated by at least distance ϵ_f (but not more than some bounded constant multiple of ϵ_f). However, these limit points by construction belong to the compact set $K_{00} \subset \mathcal{W}^c$. This is a contradiction to the definition of $\mathcal{W}$. This

means $\mathcal{W}$ must have positive measure. This is a contradiction to Lemma 2.6.8. So we have proven Proposition 5.1.1 and hence also proven Proposition 2.6.9.

6 Proof of Proposition 2.6.5

6.1 Outline of the proof of Proposition 2.6.5

In this setting, our fiber entropy is zero and unlike in the proof of Proposition 5.1.1, we cannot use Theorem 3.4.1 to say that the Oseledets' directions are random. We only have Corollary 3.3.2 that tells us that the $W^s(x, \omega)$ are random.

We will work by contradiction and run a 'floating Benoist and Quint' argument. We will assume that ν is not finitely supported and derive a contradiction to the fact that $m_{x, \omega}^u$ is trivial a.e. in a very similar vain to the proof of Proposition 5.1.1. To be more precise, we will construct an unstable manifold with two distinct support points. This will suffice in the same way it did for the proof of Proposition 5.1.1; we will argue using the sets K_{00} and $\mathcal{W}$ as defined in Section 6.2.

In assuming that ν is not finitely supported, we will show in Section 6.2 that we can pick points $(\hat{x}, \hat{\omega}, k_0)$ and $(\hat{y}, \hat{\omega}, k_0)$ such that both belong to a good set K_0 , $\hat{y} \notin W^s(\hat{x}, \hat{\omega})$ and $\hat{x}$ and $\hat{y}$ can be chosen arbitrarily close together. This setup allows us to follow aspects of the process of the proof of Proposition 5.1.1.

We then follow the same steps as in Section 5 to choose parameters T and m . This is so we can use all the work we did in Section 5, we do not actually require the step of flowing by a large T to "get the unstable planes of the points close" which we needed for the proof of Lemma 5.3.29. It also allows us to keep almost all the sets (their definitions and purpose) the same, even if some of it is unnecessary. We then compute ℓ similarly, up to a minor caveat seen in Lemma 6.2.9. Then the argument that d (which is defined similarly as in Section 5) is Bi-Lipschitz is nearly identical to that of Lemma 5.3.36. This allows us the run the same

Martingale convergence theorem argument as in Section 5.3 and draw the conclusion that assuming that ν is not finitely supported contradicts the fact that $m_{x,\omega}^u$ is trivial a.e. in a very similar way to that of Section 5.3.

6.2 Proof of Proposition 2.6.5

The ‘good’ sets for the proof of Proposition 2.6.5

We keep the definitions of the sets as in Section 5.3 with the exception of the few listed below. We keep all the constants such as C_b , J , η , etc. except we let

$$L := \max\{2r \ln(e^{-(\lambda^- - \epsilon)} + \frac{1}{2}C) + \lambda^+ + \epsilon, \frac{1}{\lambda^+ + \epsilon}\}.$$

As before, all the ϵ terms used in the definitions below will be assumed sufficiently small for their purpose unless otherwise stated more explicitly.

- $\mathcal{W} \subset Z$, $\mathcal{W} = \{(x, \omega, k) : m_{x,\omega}^u \text{ is non-trivial}\}$, by the argument at the start of Section 6 we know this set has measure zero.
- $K_{00} \subset Z$, an arbitrary compact subset of size $\hat{m}(K_{00}) > 1 - \epsilon_{00}$. We assume $K_{00} \subset \mathcal{W}^c$, we can do this since $\hat{m}(\mathcal{W}^c) = 1$.
- $K_+ \subset Y$, defined as the conull subset where $m_{x,\omega}^u$ and $m_{x,\omega}^s$ are trivial.
- $K_0 \subset Z$, $K_0 \subset ((\Lambda \cap \Lambda_{loc} \cap K_+ \cap K_s \cap K_r \cap K_\xi \cap K_{NFC}) \times [0, 1]) \cap E^c$, has size $\hat{m}(K_0) > 1 - \epsilon_0$.
- $K'_2 \subset Y$, $K'_2 \subset \Lambda \cap \Lambda_{loc} \cap K_s \cap K_\xi \cap K_b \cap K_{ang} \cap K_{exp} \cap (K_\varepsilon \times \Omega) \cap (A' \times \Omega^-)$, $\hat{m}(K'_2) > 1 - \epsilon'_2$.

- $\tilde{K}_2 \subset Y$, $\tilde{K}_2 := (B' \times \Omega^+) \cap K'_2$, ϵ_2 and ϵ'_2 small such that the corresponding measure of $\tilde{K}_2$ which we call $\gamma_2 = 1 - \tilde{\epsilon}_2$ satisfies

$$\gamma_2 > 1 - \frac{1}{200L^4} > 3/4. \quad (227)$$

- $E_2^c \subset Z$, take $\delta = \alpha_2(\tilde{K}_2)$ such that $\beta_2 := 2\gamma_2(1 - \alpha_2) - 1 > \frac{1}{200L^4}$, $\epsilon \ll \epsilon_1$, $K = \tilde{K}_2 \times [0, 1[$ and apply Lemma 3.10.15, we get $S > 0$ such that the set

$$E_2^c := \{(x, \omega, k) : \forall T > S, \left| \frac{1}{T} \int_0^T \mathbb{1}_K(F^t(x, \omega, k)) dt - \int_Y \mathbb{1}_K d\hat{m} \right| < \delta\}, \quad (228)$$

has $\hat{m}(E_2^c) > 1 - \epsilon$.

- $K_3 \subset Z$, $K_3 \subset (\Lambda \cap \Lambda_{loc}) \times [0, 1[$, additionally, K_3 satisfies the conditions of Lemma 3.9.3 and belong to the $\hat{m}$ -conull set from Lemma 5.3.43. The size of this set is $\gamma_3 := 1 - \epsilon_3$ where

$$\gamma_3 > (1 - \frac{1}{100L^2}) + \frac{1}{100L^2} \left(\frac{1}{1 + \eta} \right) > 3/4. \quad (229)$$

- $E_3^c \subset Z$, take $\delta = \alpha_3\gamma_3$ such that $\beta_3 = 2\gamma_3(1 - \alpha_3) - 1 > (1 - \frac{1}{100L^2}) + \frac{1}{100L^2} \left(\frac{1}{1 + \eta} \right)$, $\epsilon \ll \epsilon_1$, $K = K_3$ and apply Lemma 3.10.15, we get $S > 0$ such that the set

$$E_3^c := \{(x, \omega, k) : \forall T > S, \left| \frac{1}{T} \int_0^T \mathbb{1}_K(F^t(x, \omega, k)) dt - \int_Y \mathbb{1}_K d\hat{m} \right| < \delta\}, \quad (230)$$

has $\hat{m}(E_3^c) > 1 - \epsilon$.

The sets are not necessarily in the order they appear in the proof because the definition of earlier sets is often dependent on what is needed later.

All sets are used as in Section 5.3 with the exception of some of those listed above. The sets E_2^c , E_3^c , $\tilde{K}_2$ and K_3 have the same descriptions but with this new value of L listed above.

- $\mathcal{W}$ - we know this set has full measure by Lemma 2.6.4 but we will show this is not the case when we assume that Proposition 2.6.5 is false. This will prove Proposition 2.6.5.
- K_{00} - We will land in this set at the end and use it to show that $\mathcal{W}$ must have positive measure when we assume that Proposition 2.6.5 is false. This will prove Proposition 2.6.5.
- K_+ - This set is local to Section 6 (i.e. is only conull in the zero entropy case as given by Lemma 2.6.4) and is needed to define the set K_0 where we choose our starting points.
- $K_0 \subset Z$ - Our starting points, $(\hat{x}, \hat{\omega})$ and $(\hat{y}, \hat{\omega})$ will be chosen from here.

Original points

From now onwards, we assume that ν is not finitely supported.

In this section we use a proof strategy from [16], Section 3, Lemma 3.4.

Lemma 6.2.1. *For every $\rho > 0$, we can find points $(\hat{x}, \hat{\omega}, k_0), (\hat{y}, \hat{\omega}, k_0) \in K_0$ such that $\hat{y} \notin W^s(\hat{x}, \hat{\omega})$ and $d_X(\hat{x}, \hat{y}) < \rho$.*

Proof. There are two cases when it comes to the support of ν . Either it is countable or uncountable. We follow the argument of [16] Lemma 3.4:

Claim 6.2.2. *The support of ν is uncountable.*

Proof. If the support of ν is countable, then ν is supported on a sequence of points x_n each with some mass δ_n . Note that the dynamics does not change the mass of a point, so a set of these points with the same mass is an invariant set. Using that ν is ergodic we get that all points must have the same mass. This forces ν to have finite support which is a contradiction. $\square$

Let $W_{\hat{\omega}, k_0} = \{\hat{x} \in X : (\hat{x}, \hat{\omega}, k_0) \in K_0\}$ then let

$$V = \{(\hat{\omega}, k_0) : \mu^{\mathbb{Z}} \times dt(W_{\hat{\omega}, k_0}) > 1 - \sqrt{\epsilon_0}\}.$$

Claim 6.2.3. *The set V has measure at least $1 - \sqrt{\epsilon_0}$.*

Proof. The proof is identical to that of Lemma 4.2.6. □

Now take $(\hat{\omega}, k_0) \in V$ and consider $S' = W_{\hat{\omega}, k_0} \cap \text{Supp}(\nu)$. This set S' is also uncountable and it contains uncountably many accumulation points. One takes $\hat{x} \in S' \subset X$ an accumulation point, one can find a sequence $\hat{y}_n \in S'$ converging to $\hat{x}$. Now we can pick $\hat{y}$ as one of the $\hat{y}_n$ such that $d_X(\hat{y}_n, \hat{x}) < \rho$ for the given ρ . Then since $\hat{x}$ and $\hat{y}$ belong to S' , $(\hat{x}, \hat{\omega}, k_0)$ and $(\hat{y}, \hat{\omega}, k_0)$ both belong to K_0 .

Lastly, we must argue that $\hat{y}$ can be chosen such that $\hat{y} \notin W^s(\hat{x}, \hat{\omega})$.

If not, then $\text{Supp}(\nu)$ is contained in a finite union of stable manifolds;

$$\text{Supp}(\nu) \subset \bigcup_{i=1}^n W^s(x_i, \omega_i).$$

We note that the $W^s(x_i, \omega_i)$ may not be measurable sets, but if we let $B^s(x, \omega, R)$ be the ball of radius R inside $W^s(x, \omega)$ in the induced metric, this is a measurable set.

Claim 6.2.4. *The leaves $W^s(x_i, \omega_i)$ in the union above that carry positive measure have the same measure.*

Proof. Let $f(x, \omega) = \lim_{R \rightarrow \infty} \nu(B^s(x, \omega, R))$; this limit exists along some subsequence because f is upper bounded by 1. Since the support of the probability measure is contained in this finite union of leaves, for some i , $f(x_i, \omega_i)$ is positive.

If we apply the skew product map, we know that $F(W^s(x_i, \omega_i)) = W^s(F(x_i, \omega_i))$. Further since ν -measure is preserved under the dynamics F , we have that f is invariant under F . This implies that it is constant a.e. by ergodicity of m .

This tells us that amongst the finitely many leaves containing the support of ν , those that contain positive measure have the same amount of measure. $\square$

Claim 6.2.5. *Each leaf $W^s(x_i, \omega_i)$ that contains positive measure has finitely many points in the support of ν .*

Proof. In Claim 6.2.4 we learned that under F , the leaves of positive measure must map to one another.

Take a leaf $W^s(x_i, \omega_i)$ in the finite union that contains positive measure, and let $y \in \text{Supp}(\nu)$ belong to this leaf. Let $\epsilon > 0$, and consider $B^s(y, \epsilon) \subset W^s(x_i, \omega_i)$, i.e. the ball of radius ϵ around y in the induced metric.

Running the dynamics backwards, Theorem 3.5.4 tells us that the radius of this ball grows and it also stays within the finite union of leaves. In a finite number of steps, one can grow the radius of this ball to the point which it contains most of the measure of leaf (there are finitely many leaves). Since ν is invariant and Claim 6.2.4 tells us that the measure of all the leaves is the same, this tells us that $B^s(y, \epsilon)$ contains the entire measure of the leaf. This is true for any $\epsilon > 0$ and hence y is an atom containing the entire measure of $W^s(x_i, \omega_i)$. One can do this for each positive measure containing leaf. $\square$

Claim 6.2.5 is a contradiction to our assumption that ν is not finitely supported. $\square$

Flowing by T and m

Fix $\epsilon_f > 0$ as in Section 5.3.

To start, take any $\rho > 0$ such that

$$\rho < \epsilon_f/10000,$$

and use Lemma 6.2.1 to pick a point $(\hat{x}, \hat{\omega}, k_0) \in K_0$ with its corresponding $(\hat{y}, \hat{\omega}, k_0) \in K_0$

such that $U^+[\hat{x}, \hat{\omega}] = \hat{x}$, $U^+[\hat{y}, \hat{\omega}] = \hat{y}$, $\hat{y} \notin W^s(\hat{x}, \hat{\omega})$, and $d_X(\hat{x}, \hat{y}) < \rho$. From here on out, we acknowledge that $U^+[\hat{x}, \hat{\omega}, k_0]$ and $U^+[\hat{y}, \hat{\omega}, k_0]$ are just the points $\hat{x}$ and $\hat{y}$ themselves.

We first note that for our choice of $\hat{x}$ and $\hat{y}$, $\hat{y} \notin W^s(\hat{x}, \hat{\omega})$ and so there exists a point $\hat{z} \in W_q^s(\hat{y}, \hat{\omega}) \cap W_q^u(\hat{x}, \hat{\omega})$. We call $d_X(\hat{z}, \hat{y})$ the ‘unstable distance’. As before in Section 5, the local stable manifold theorem, Theorem 3.5.4, tells us that there is number T_{big} such that if $y_B = F_{\hat{\omega}}^{T_{big}}(\hat{y})$, $z_B = F_{\hat{\omega}}^{T_{big}}(\hat{z})$, $d_X(y_B, z_B) = \epsilon_f$.

Let T be chosen as in Choice 1, i.e. $T \in [T_{big}(1 - \theta_1), T_{big}(1 - \theta_2)]$ where θ_1 and θ_2 satisfy the requirements of Choice 2.

We run the same Birkhoff arguments using Lemma 3.10.15, i.e. Lemmas 5.3.18 and 5.3.19 which allow us to pick T in the given interval such that $F^T(\hat{x}, \hat{\omega}, k_0)$ and $F^T(\hat{y}, \hat{\omega}, k_0)$ land in the set $\tilde{K}_1$. This set $\tilde{K}_1$ has the properties of the good set K'_1 and also is a subset of $(B \times \Omega)$ so we can pick a good new $\tilde{\omega}$ such that our points $(\tilde{x}, \tilde{\omega}, k_1)$ and $(\tilde{y}, \tilde{\omega}, k_1)$ (where $k_1 = \lfloor T + k_0 \rfloor$) belong to K_1 . The properties of these sets allow us to use the same arguments in Section 5.3 when we choose m .

When we choose $\tilde{\omega}$, we do so as in Lemma 5.3.21.

Convention 2. *We pick the convention that m is given as a positive real number and we will flow backwards by it and hence write F^{-m} .*

From the points $\hat{x}$ and $\hat{y}$ we will go in two directions. Firstly we flow backwards by some m (under the ‘standard’ flow) from $\hat{x}$ and $\hat{y}$ to the points which we will denote x and y respectively.

Let m_1 and m_2 be the same as in Equation (164) with T_{big} large enough so that $m_2 > 2m_1$ and $m_1 > S$ (we will put further restrictions on how large T_{big} should be).

We can demonstrate, similarly to Section 5.3, that we can choose $m \in [m_1, m_2]$ such that the $X \times \Omega$ -component of $F^{-m}(\tilde{x}, \tilde{\omega}, k_1)$ and $F^{-m}(\tilde{y}, \tilde{\omega}, k_1)$ land in the set $\tilde{K}_2 \subset K'_2$. These are good choices of m . We get that there is a large positive measure set of futures $(\omega')^+ \in \Omega^+$ we can pick such that if $x = F_{\tilde{\omega}}^{-m}(\tilde{x})$ and $y = F_{\tilde{\omega}}^{-m}(\tilde{y})$, then $(x, (\sigma^{-n'}(\tilde{\omega}))^-, (\omega')^+)$ and

$(y, (\sigma^{-n'}(\tilde{\omega}))^-, (\omega')^+)$ (where $n' = \lfloor k_1 - m \rfloor$) belong to K_2 .

We still have that flowing by m does not take our points all the way to distance ϵ_f (see Lemma 5.3.24), we also have the following statement (made obvious by Theorem 3.5.4):

Lemma 6.2.6. *As $T_{big} \rightarrow \infty$ (hence $m \rightarrow \infty$), the distance between y and $W^s(x, \sigma^{-n'}(\tilde{\omega}))$ goes to zero.*

Lemma 6.2.6 is required in the proof of Lemma 6.2.9.

Let $\tilde{D}$ be the distance (in ambient coordinates) between $\tilde{x}$ and $\tilde{y}$. Note that by construction, this distance is still less than q and that the set M is contained in Λ_{loc} as defined in 5.3. The points $(\tilde{x}, \tilde{\omega}, k_1)$ and $(\tilde{y}, \tilde{\omega}, k_1)$ belong to M . Additionally, the distance $\tilde{D}$ between $\tilde{x}$ and $\tilde{y}$ is less than k_U and hence they live in each other's Lyapunov chart domain.

Originally, the distance $\tilde{D}$ was largely in the unstable direction for that choice of Ω -component, i.e. $\sigma^n(\hat{\omega})$. Now that we have switched Ω -components to $\tilde{\omega}$, generically, the distance $\tilde{D}$ has stable and unstable distance components with respect to $\tilde{\omega}$. Also we picked $\tilde{\omega}$ such that $\tilde{x}$ and $\tilde{y}$ are not on each other's stable or unstable manifolds. Also, since we are going backwards by m , the stables grow and the unstables contract.

In Lyapunov charts, the images of the subspaces $E^u(\tilde{x}, \tilde{\omega})$ and $E^s(\tilde{x}, \tilde{\omega})$ form a coordinate axis, and $\tilde{y}$ is sent to some $v \in \mathbb{C}^2$ (or $\mathbb{R}^4$) in these charts. We can decompose v into a component in the E^s -direction and a component in the E^u -direction denoted respectively v_s and v_u , i.e. $v = v_s + v_u$. In these charts, we know exactly v_s and v_u grow and shrink resp. under flowing by $-m$. We denote $\alpha_s = |v_s|$, $\alpha_u = |v_u|$.

Then, the distance between 0 and v after flowing by m , is

$$D_m := |\tilde{F}_{(\tilde{x}, \tilde{\omega}), k_1}^{-m}(v) - \tilde{F}_{(\tilde{x}, \tilde{\omega}), k_1}^{-m}(0)|.$$

Let $\alpha_u^m := |\tilde{F}_{(\tilde{x}, \tilde{\omega})}^{-m}(v_u) - \tilde{F}_{(\tilde{x}, \tilde{\omega})}^{-m}(0)|$ and $\alpha_s^m := |\tilde{F}_{(\tilde{x}, \tilde{\omega}), k_1}^{-m}(v_s) - \tilde{F}_{(\tilde{x}, \tilde{\omega}), k_1}^{-m}(0)|$.

Lemma 6.2.7. *For T_{big} large enough (hence m large enough), we have*

$$D_{m+1} \leq \sqrt{1+k^2}(e^{-(\lambda^--\epsilon)} + \frac{1}{2}C)\alpha_s^m = (e^{-(\lambda^--\epsilon)} + \frac{1}{2}C)D_m.$$

Further, we have $D_m < D_{m+1}$.

Proof. The proof is the same as in Lemma 5.3.27. □

This is used in argument that d is bi-Lipschitz, see Lemma 6.2.11.

For the purposes of Lemma 6.2.12, we need the following Lemma:

Lemma 6.2.8. *The constant $0 < \eta < 1$ as fixed in Section 5.3 is such that for T_{big} large enough (hence m large enough) $D_{m_2}^{(1+\eta)} > D_{m_1}$.*

Proof. The proof is the same as in Lemma 5.3.28. □

For a choice of m , define $x := F_{\tilde{\omega}}^{-m}(\tilde{x})$ and $y := F_{\tilde{\omega}}^{-m}(\tilde{y})$.

For any (good) m chosen from the interval $[m_1, m_2]$, we get a positive measure subset $V \subset \Omega^+$ of futures ω'^+ to choose from that allow the points $(x, (\sigma^{-\lfloor k_1-m \rfloor}(\tilde{\omega})^-, \omega'^+))$ and $(y, (\sigma^{-\lfloor k_1-m \rfloor}(\tilde{\omega})^-, \omega'^+))$ to belong to the set K_2 (where $\tilde{\omega}^- \in \Omega^-$ is the past part of $\tilde{\omega}$). Let $\omega'^+ \in V$ be arbitrary, let $\omega' = (\sigma^{-\lfloor k_1-m \rfloor}(\tilde{\omega})^-, \omega'^+) \in \Omega$. We will have additional restrictions on choosing $(\omega')^+$, but certainly, $(\omega')^+$ will belong to V .

At the points (x, ω') and (y, ω') we will project (y, ω') along $W^s(y, \omega')$, onto $W^u(x, \omega')$ and use (modified) normal form coordinates (NFCs) (see Equation (97)) to control the distance between the final points when flowing by ω' by ℓ . We want this distance to be (approximately) some fixed ϵ_f .

Note that our good sets make the changes between ambient coordinates, Lyapunov charts and (modified) NFCs a bounded change as described in Section 6.2. Also note that $W^u(x, \omega')$ is the same as $W^u(x, \sigma^{-\lfloor k_1-m \rfloor}\tilde{\omega})$ because the past vectors of ω' and $\sigma^{-\lfloor k_1-m \rfloor}\tilde{\omega}$ are the same. It is the stable manifold that changes when passing from $\sigma^{-\lfloor k_1-m \rfloor}\tilde{\omega}$ to ω' .

Let $\delta(m)$ be the distance between $x = f_{\tilde{\omega}}^{-m}(\tilde{x})$ and $y = f_{\tilde{\omega}}^{-m}(\tilde{y})$ in ambient coordinates. Let γ be the distance between x and the projection of y along $W^s(y, \omega')$ onto $W^u(x, \omega')$ which we denote as $y' = W^s(y, \omega') \cap W^u(x, \omega')$.

Since our points belong to Λ_s there exists a natural number $r > 0$ such that for any $(\omega')^+, (\omega'')^+ \in \Omega_t^+$, $W^s(y, (\omega')^+)$ and $W^s(y, (\omega'')^+)$ are not tangent at some order $r > 0$ and tangent for any natural number $k \in \{1, \dots, r-1\}$. This means they agree on the first $r-1$ derivatives at the origin but differ at the r th derivative at the origin.

Lemma 6.2.9. *There exists m' and $M_a > 1$ such that for all $m > m'$ there is a large measure subset of $A^+ \subset \Omega_t^+$ such that for $(\omega')^+ \in A'$ we have $\gamma \neq 0$, and further, $\gamma(m) = C'\delta(m)^r$ where $C' \in [\frac{1}{M_a}, M_a]$ and r is from Lemma 3.3.4.*

Proof. There are cases based on the value of r we get from Lemma 3.3.4.

Case 1: $r = 1$

Here, all the stable manifolds at y for varying $(\omega')^+ \in \Omega_t^+$ are not tangent to order 1. Then the argument is the same as in Lemma 5.3.29 as we are in a setting of Theorem 3.4.1 c) and our setup (i.e. parameter choice) is the same as in Section 5. We get that there is a $M_a > 0$ such that

$$\gamma(m) = C_1\delta(m),$$

for some $C_1 \in [\frac{1}{M_a}, M_a]$.

Case 2: $r > 1$

We will work in the complex ambient charts established in Section 3.10, see this section for more. We will work around the point y , i.e. use the chart φ_y , note that we can work on a scale small enough (for m large enough) so that we live inside of $B_X(y, \mathcal{R})$. Here, $\varphi_y(y)$ is the origin in $\mathbb{C}^2$ and as in Section 4.2. There is a large subset A^+ of $\Omega_t^+ \subset \Omega^+$ we can choose from so that if $(\omega')^+ \in A^+$ then $(y, (\omega')^+) \in A'$.

Working in A' we have that our charts have the nice properties outlined in Section 3.10. In particular, we can view $\varphi_y(W^s(y, \omega'))$ as a graph over a coordinate axis copy of $\mathbb{C}$,

$\{(z, \psi_{y, \omega'}(z)) : z \in \mathbb{D}_s\}$ where $\mathbb{D}_s \subset \mathbb{C}$ is a disk around the origin uniquely up to rotation (i.e. an origin fixing automorphism of the disk). Then there is a smooth change of coordinates (that is not dependent on $(\omega')^+$ and whose Lipschitz constant is bounded above by L'' on A') that eliminates the first $r - 1$ derivatives of $\psi_{y, \omega'}$, that is to say, there is a function h_y such that $h_y \circ \varphi_y(W^s(y, \omega'))$ can be written as a graph of the form

$$G = \{(z, \psi^{(r)}(0)z^r + h.o.t.) : z \in \mathbb{D}^s\}.$$

We let $\varphi'_y = h_y \circ \varphi_y$. Further, $|\psi^{(r)}(0)| > C_\varphi > 0$ because we are in A' .

The points x and y are separated by distance $\delta(m)$, and we had let $\vec{v}$ be the vector in $\varphi'_y(B_x(y, \mathcal{R}))$ from the origin (i.e. $\varphi'_y(y)$) to $\varphi'_y(x)$. Then $|\vec{v}| = \delta(m)M$ where $M \in [\frac{1}{L'L''}, L'L'']$ and L' is the upper bound on the Lipschitz constant of φ_y . Let $\vec{u}$ be the unit vector in the direction $\vec{v}$.

Consider the function $g : \mathbb{C} \rightarrow \mathbb{C}^2$, $g(z) = z\vec{u}$, a parametrization of the plane complex-spanned by $\vec{u}$ in the chart $\varphi'_y(B_X(y, \mathcal{R}))$.

We can choose m' large such that $W^s(y, \sigma^{-n'}(\tilde{\omega}))$ is within an appropriately small distance of $W^s(x, \sigma^{-n'}(\tilde{\omega}))$. We can do this by Lemma 6.2.6. Since $(y, \sigma^{-n'}(\tilde{\omega})) \in A'$, $W^s(y, \sigma^{-n'}(\tilde{\omega}))$ and $W^s(y, \omega')$ are tangent at order 1, i.e. $E^s(y, \sigma^{-n'}(\tilde{\omega})) = E^s(y, \omega')$. Note that these tangent planes in $h_y \circ \varphi_y$ -coordinates are also the coordinate axis $\{(z, 0) : z \in \mathbb{D}^s\}$. Further the tangent planes $E^s(x, \sigma^{-n'}(\tilde{\omega}))$ and $E^s(y, \sigma^{-n'}(\tilde{\omega}))$ can be made sufficiently close to g .

Then, since $(x, \sigma^{-n'}(\tilde{\omega})), (y, \sigma^{-n'}(\tilde{\omega})) \in K'_2 \subset K_{ang}$ we have that

$$\angle(E^u(x, \sigma^{-n'}(\tilde{\omega})), E^s(x, \sigma^{-n'}(\tilde{\omega}))) = \angle(E^u(x, \omega'), E^s(x, \sigma^{-n'}(\tilde{\omega}))) > \Theta.$$

This means that $E^u(x, \sigma^{-n'}(\tilde{\omega}))$ lives in some cone centered around $E^s(x, \sigma^{-n'}(\tilde{\omega}))^\perp$, i.e. $E^u(x, \sigma^{-n'}(\tilde{\omega}))$ is bounded away from $E^s(x, \sigma^{-n'}(\tilde{\omega}))$ by at least some angle Θ .

Returning to our coordinate system, all of this implies that there exists $C_2 > 1$ such that the graph G intersects this cone for some $z' \in \mathbb{D}_s$ such that

$$\frac{1}{C_2}\delta(m) \leq |z'| \leq C_2\delta(m).$$

Incorporating the error in the approximation of $h_Y \circ \varphi_y(W^u(x, \omega'))$ by its tangent plane, there is $C'_2 > 1$ such that the graph G intersects $h_Y \circ \varphi_y(W^u(x, \omega'))$ for some $z'' \in \mathbb{D}_s$ such that

$$\frac{1}{C'_2}\delta(m) \leq |z''| \leq C'_2\delta(m).$$

Then since g (where x lives) is sufficiently close to the coordinate plane, we have that there is a $C''_2 > 0$ such that

$$|h_Y \circ \varphi_y(x) - h_Y \circ \varphi_y(y')| = c\delta(m)^r,$$

for some $c \in [\frac{1}{C''_2}, C''_2]$.

A bounded return to ambient coordinates allows us to conclude the proof. $\square$

Now y' and y both belong to $W^s(y, \omega')$ so as we flow forwards under ω' this distance contracts and so γ is a good proxy for δ .

Recall that we were working towards having our end points be ϵ_f apart in ambient coordinates, in reality it suffices to land within some bounded constant of ϵ_f . Ensuring this final distance is bounded from above will allow us to take limits in Section 6.2 and bounding it from below ensure our goal of distinct unstables is achieved (see the heuristic argument in Section 5).

Knowing the relationship between γ and δ is important in defining ℓ . As explained before we can use (modified) normal form coordinates (NFCs) to control how γ grows because this is a distance that lives in $W^u(x, \omega')$. We use this for a preliminary computation for what

$\ell(m)$ should be, i.e. the amount we want to flow by from the points (x, ω') and (y, ω') regardless of the choice of ω' . Ultimately we will have to compute ℓ slightly differently but the first consideration that we make is that γ gets changed by (modified) NFCs to some $C_1\gamma$ where $C_1 \in [\frac{1}{M_{NFC}}, M_{NFC}]$ and we would like to grow this distance to be $\epsilon_f \cdot M_{NFC}$ (so that when we go back to ambient coordinates it is bounded from below by ϵ_f). Then we solve $C_1\gamma e^\ell = M_{NFC}\epsilon_f$ for ℓ and get

$$\ell(m) = \ln \left(\frac{M_{NFC}\epsilon_f}{C_1\gamma(m)} \right). \quad (231)$$

Since C_1 changes with m , we instead replace it with $\frac{1}{M_{NFC}}$ (i.e. its lower bound). Solving $\ell = \ln \left(\frac{M_{NFC}\epsilon_f}{\frac{1}{M_{NFC}}\gamma} \right)$ is a good first step. If one returns to ambient coordinates, $F_{\omega'}^\ell(x)$ and $F_{\omega'}^\ell(y')$ differ by at least ϵ_f and at most $\epsilon_f \cdot M_{NFC}^2$.

To make the computation in Lemma 6.2.11 simpler, we use D_m instead of γ which would make the formula look more like this:

$$\ell(m) = \ln \left(\frac{M_{NFC}\epsilon_f}{\frac{1}{M_{NFC}M_r^rM_a}D_m^r} \right). \quad (232)$$

Note that the constants in front of D_m appear because we are thinking of D_m in relation to γ . D_m is related to $\delta(m)$ by a constant bounded from below by $\frac{1}{M_r}$ and $\delta(m)^r$ is related to γ by C' (where r is as in Lemma 6.2.9), the lower bound being $\frac{1}{M_a}$. Note however that C' is also bounded from above by M_a and the relation between D_m and $\delta(m)$ is bounded above by M_r . This means that this definition pushes us to at least distance ϵ_f ambiently, but does not take us further than $M_aM_r^rM_{NFC}^2\epsilon_f$ this is important for Section 6.2.

However, this definition for ℓ is not continuous as D_m changes only every integer (up to taking into account k_1). We will need continuity of ℓ for Section 6.2. Note also that we showed that D_m is increasing in m , so we define ℓ to be the piecewise linear interpolation

between these points. We can write this explicitly as

$$\begin{aligned} \ell(m) = & \ln \left(\frac{M_{NFC} \epsilon_f}{\frac{1}{M_{NFC} M_r^r M_a} D_m^r} \right) \\ & + (m - \lfloor m \rfloor) \left(\ln \left(\frac{M_{NFC} \epsilon_f}{\frac{1}{M_{NFC} M_r^r M_a} D_{m+1}^r} \right) - \left(\ln \left(\frac{M_{NFC} \epsilon_f}{\frac{1}{M_{NFC} M_r^r M_a} D_m^r} \right) \right) \right), \end{aligned} \quad (233)$$

which simplifies to

$$\ell(m) = \ln \left(\frac{M_{NFC} \epsilon_f}{\frac{1}{M_{NFC} M_r^r M_a} D_m^r} \right) + (m - \lfloor m \rfloor) \ln \left(\frac{D_m^r}{D_{m+1}^r} \right). \quad (234)$$

Now, to correct concerns that in the interpolation if m is between two integers, technically $\ell(m)$ computed using Equation (232) is smaller than that of $\ell(m)$ computed using Equation (234), we tack on a last constant. Let $M_D := \left(e^{-(\lambda^- + \delta)} + \frac{1}{2}C \right)$ then define

$$\ell(m) = \ln \left(\frac{M_{NFC} \epsilon_f}{\frac{1}{M_{NFC} M_r^r M_a M_D} D_m^r} \right) + (m - \lfloor m \rfloor) \ln \left(\frac{D_m^r}{D_{m+1}^r} \right). \quad (235)$$

This expression, (235), is how we define ℓ .

Additionally, take note of the fact that D_m increases with m because m is large enough; see Lemma 6.2.7.

Flowing by ℓ

Now for completeness, we repeat the setup from Section 5.3:

We have a ‘V’ shape where we go backwards by m (under the standard flow) and forwards by ℓ (under the time changed flow). Here ℓ is determined by m and the distance ϵ_f that we want to reach (within some constant) at the end points using (modified) NFCs. We need the ‘legs’ of the ‘V’, i.e. the forward and backward directions to be of the same length to

run the argument in Section 6.2. This is why we will introduce extra points in this section.

We fix T and the points $(\tilde{x}, \tilde{\omega}, k_1)$ and $(\tilde{y}, \tilde{\omega}, k_1)$ from Section 6.2, and for each m we pick in Section 6.2 (from the positive measure subset of ‘good choices’ in some interval) we get the points (x, ω', k_2) and (y, ω', k_2) . Here, $\omega' \in \Omega$ is any vector such that $\omega'^- = \sigma^{-\lfloor k_1 - m \rfloor}(\tilde{\omega})^-$ and ω' belongs to the positive measure subset V corresponding to the points $(x, \sigma^{-\lfloor k_1 - m \rfloor}(\tilde{\omega}))$ and $(y, \sigma^{-\lfloor k_1 - m \rfloor}(\tilde{\omega}))$ which live in $\tilde{K}_2$.

Additionally, for each choice of m we can compute $\ell(m)$ as we did in Section 6.2. This is independent of the choice of future ω' that we pick from some a good set. Now, starting from the points $(\tilde{x}, \tilde{\omega})$ and $(\tilde{y}, \tilde{\omega})$ (i.e. we are in the set K_1), we want to flow forwards by an amount that will allow us to complete our ‘V’ shape.

Let us reiterate our ‘V’ shape more definitively. Since the points $(\tilde{x}, \tilde{\omega}, k_1)$ and $(\tilde{y}, \tilde{\omega}, k_1)$ are fixed at this point, our ‘V’ is determined entirely by a choice of m (which is in a fixed interval at this point). The length of the legs is $\ell(m)$ under time change flow, with the middle points of the ‘V’ being the points (x, ω', k_2) and (y, ω', k_2) . Note that because τ in the ‘time change’ flow depends on which point you start at, the resulting pair of points after flowing by $\ell(m)$ will have different Ω and $[0, 1]$ -components compared to one another. This was never the case with the ‘standard’ flow. We have to convert the parameter m that we flowed by under the standard flow into the quantity we would flow by from $(x, \sigma^{n'}(\tilde{\omega}), k_2)$ and $(y, \sigma^{n'}(\tilde{\omega}), k_2)$ to $(\tilde{x}, \tilde{\omega}, k_1)$ and $(\tilde{y}, \tilde{\omega}, k_1)$ under the time changed flow. This quantity will differ between points. We define $m_{\tau, (\tilde{x}, \tilde{\omega}, k_1)}$ and $m_{\tau, (\tilde{y}, \tilde{\omega}, k_1)}$ to be these quantities resp. Now to complete our ‘V’ we flow backwards from $(\tilde{x}, \tilde{\omega}, k_1)$ and $(\tilde{y}, \tilde{\omega}, k_1)$ by $d(m, (\tilde{x}, \tilde{\omega}, k_1)) := \ell(m) + (-m)_{\tau, (\tilde{x}, \tilde{\omega}, k_1)}$ and $d(m, (\tilde{y}, \tilde{\omega}, k_1)) := \ell(m) + (-m)_{\tau, (\tilde{y}, \tilde{\omega}, k_1)}$ (note that both functions d are positive). We denote the resulting points $(\bar{x}, \bar{\omega}_x, k_x)$ and $(\bar{y}, \bar{\omega}_y, k_y)$ resp. Note that k_x and k_y are not necessarily the same; neither are $\bar{\omega}_x$ and $\bar{\omega}_y$.

Remark 6.2.10. *The minus sign was included next to the m before doing the time change because we want to remember to compute it using τ^- , i.e. we are going backwards when*

flowing by , and we need to remember that as we compute the time changed value.

For simplicity, we will take our overall ‘V’ and divide it into its two components, the ‘V’ we have at the fixed point $(\tilde{x}, \tilde{\omega}, k_1)$ will be called ‘V1’ and the ‘V’ we have at the point $(\tilde{y}, \tilde{\omega}, k_1)$ will be called ‘V2’. This allows us to drop some notation regarding time changed m and the function d . Let us just focus on ‘V1’ as everything we have to say will hold for ‘V2’ as well. Rename $m_{\tau, (\tilde{x}, \tilde{\omega}, k_1)}$ as m_1^τ and $d(m, (\tilde{x}, \tilde{\omega}, k_0))$ as $d_1(m)$. We will have to show that d_1 is Bi-Lipschitz in m in order to pick m such that flowing backwards by m from $(\tilde{x}, \tilde{\omega}, k_1)$ we land in the set $\tilde{K}_2$ and flowing forwards by $d_1(m)$ we land in the set K_3 . We can use Lemma 3.10.15 to do one or the other, but we have to run a different argument to ensure we can do both at the same time. A similar argument will work for ‘V2’ where we use the notation m_2^τ and $d_2(m)$ for $m_{\tau, (\tilde{y}, \tilde{\omega}, k_1)}$ and $d(m, (\tilde{y}, \tilde{\omega}, k_0))$ respectively.

We run the argument to demonstrate that d is Bi-Lipschitz given our new definition for ℓ in Equation (235), we see that the constant r , introduced due to Lemma 6.2.9 (rather than Lemma 5.3.29) does not change anything.

Lemma 6.2.11. *Take the interval $[m_1, m_2]$ from Section 6.2. The map $d_1 : [m_1, m_2] \rightarrow] - \infty, 0]$ is Bi-Lipschitz, (similar for d_2 and the same Lipschitz constant works in this case).*

Proof. First we want to show that there is $L > 0$ such that $|d_1(m) - d_1(m')| \leq L|m - m'|$ for arbitrary m and m' . Recall the definition of d_1 from Equation (110) and plug in:

$$\begin{aligned} |d_1(m) - d_1(m')| &= |(-m)_1^\tau - (-(m'))_1^\tau + \ell(m) - \ell(m')| \\ &\leq |(-(m'))_1^\tau - (-m)_1^\tau| + |\ell(m) - \ell(m')|. \end{aligned}$$

By Equation (235) we have that this is equal to

$$|(-m)_1^\tau - (-m')_1^\tau| + \left| \ln \left(\frac{M_{NFC} \epsilon f}{\frac{1}{M_{NFC} M_r^r M_a M_D^r} D_{m'}^r} \right) + (m' - \lfloor m' \rfloor) \ln \left(\frac{D_{m'}^r}{D_{m'+1}^r} \right) \right. \\ \left. - \ln \left(\frac{M_{NFC} \epsilon f}{\frac{1}{M_{NFC} M_r^r M_a M_D^r} D_m^r} \right) - (m - \lfloor m \rfloor) \ln \left(\frac{D_m^r}{D_{m+1}^r} \right) \right|$$

Let's control the first term in the sum above. Note that k_1 plays a role here, we had defined d_1 once we fixed a point that had this $[0, 1[$ -component. It will show up in our computations. Making sure to keep the convention that k_1 is written as a positive number between 0 and 1, we can write

$$(-m)_1^\tau = k_1 \tau^-(\tilde{x}, \tilde{\omega}) + \tau^-(F^{-1}(\tilde{x}, \tilde{\omega})) + \dots \tau^-(F^{\lfloor k_1 - m \rfloor + 1}(\tilde{x}, \tilde{\omega})) + a, \quad (236)$$

$$(-m')_1^\tau = k_1 \tau^-(\tilde{x}, \tilde{\omega}) + \tau^-(F^{-1}(\tilde{x}, \tilde{\omega})) + \dots \tau^-(F^{\lfloor k_1 - m' \rfloor + 1}(\tilde{x}, \tilde{\omega})) + b. \quad (237)$$

Note that the first term $k_1 \tau^-(\tilde{x}, \tilde{\omega})$ appears because once we pass 0 we hit it with the automorphism f_{-1}^{-1} . For example if $k_1 = 0.22$, then we are already 78% on our way to hitting $\tau^-(\tilde{x}, \tilde{\omega})$, the remainder that we need is that 22%. The a and b terms show up because we have not quite reached the next τ^- value, i.e. $a \in [0, \tau^-(F^{\lfloor k_1 - m \rfloor}(\tilde{x}, \tilde{\omega}))[$ and $b \in [0, \tau^-(F^{\lfloor k_1 - m' \rfloor}(\tilde{x}, \tilde{\omega}))[$. We can easily compute a and b from what we have as clearly the following relations holds:

$$\psi_m := \lfloor k_1 - m \rfloor - (k_1 - m) + 1 = \frac{a}{\tau^-(F^{\lfloor k_1 - m \rfloor}(\tilde{x}, \tilde{\omega}))} > 0, \\ \psi_{m'} := \lfloor k_1 - m' \rfloor - (k_1 - m') + 1 = \frac{b}{\tau^-(F^{\lfloor k_1 - m' \rfloor}(\tilde{x}, \tilde{\omega}))} > 0.$$

Without loss of generality assume $m' > m$ (remember these are positive numbers), then

$(-(m'))_1^\tau > (-m)_1^\tau$. We have

$$\begin{aligned}
|(-(m'))_1^\tau - (-m)_1^\tau| &= \tau^-(F^{\lfloor k_1 - m' \rfloor + 1}(\tilde{x}, \tilde{\omega})) + \dots \tau^-(F^{\lfloor k_1 - m \rfloor}(\tilde{x}, \tilde{\omega})) + b - a \\
&= |\tau^-(F^{\lfloor k_1 - m' \rfloor + 1}(\tilde{x}, \tilde{\omega})) + \dots \tau^-(F^{\lfloor k_1 - m \rfloor}(\tilde{x}, \tilde{\omega})) \\
&\quad + \psi_{m'} \cdot \tau^-(F^{\lfloor k_1 - m' \rfloor}(\tilde{x}, \tilde{\omega})) - \psi_m \cdot \tau^-(F^{\lfloor k_1 - m \rfloor}(\tilde{x}, \tilde{\omega}))| \\
&= |\tau^-(F^{\lfloor k_1 - m' \rfloor + 1}(\tilde{x}, \tilde{\omega})) \dots + \tau^-(F^{\lfloor k_1 - m \rfloor - 1}(\tilde{x}, \tilde{\omega})) + \tau^-(F^{\lfloor k_1 - m \rfloor}(\tilde{x}, \tilde{\omega}))(1 - \psi_m) \quad (238) \\
&\quad + \psi_{m'} \cdot \tau^-(F^{\lfloor k_1 - m' \rfloor}(\tilde{x}, \tilde{\omega}))|.
\end{aligned}$$

Using Equation (238), we then have that

$$\begin{aligned}
|(-(m'))_1^\tau - (-m)_1^\tau| &\leq |\tau^-(F^{\lfloor k_1 - m' \rfloor + 1}(\tilde{x}, \tilde{\omega}))| + \dots + |\tau^-(F^{\lfloor k_1 - m \rfloor - 1}(\tilde{x}, \tilde{\omega}))| \\
&\quad + |\tau^-(F^{\lfloor k_1 - m \rfloor}(\tilde{x}, \tilde{\omega}))|(1 - \psi_m) + \psi_{m'} |\tau^-(F^{\lfloor k_1 - m' \rfloor}(\tilde{x}, \tilde{\omega}))|.
\end{aligned}$$

Using the bounds on τ we get that this is bounded above by

$$\begin{aligned}
&[(\lfloor k_1 - m \rfloor - 1) - (\lfloor k_1 - m' \rfloor + 1) + 1] + (1 - \psi_m) + \psi_{m'} (\lambda^+ + \epsilon) \\
&= (\lambda^+ + \epsilon)(m' - m).
\end{aligned}$$

On the other hand, we have to lower bound this quantity too. So we consider again $m' > m$ and take Equation (238). Again, the τ^- -terms are lower bounded (in absolute value) by $\lambda^+ - \epsilon$, so we get that Equation (238) is bounded below by $(\lambda^+ - \epsilon)(m' - m)$.

Now we control the second term. It reduces to

$$\begin{aligned} & \left| \ln \left(\frac{M_{NFC} \epsilon f}{\frac{1}{M_{NFC} M_r^r M_a M_D^r} D_{m'}^r} \right) + (m' - \lfloor m' \rfloor) \ln \left(\frac{D_{m'}^r}{D_{m'+1}^r} \right) \right. \\ & \left. - \ln \left(\frac{M_{NFC} \epsilon f}{\frac{1}{M_{NFC} M_r^r M_a M_D^r} D_m^r} \right) - (m - \lfloor m \rfloor) \ln \left(\frac{D_m^r}{D_{m+1}^r} \right) \right| \end{aligned}$$

We know how D_{m+1} and D_m are related (see Lemma 6.2.7). When we apply an automorphism, we never grow bigger than a factor of $\ln(e^{-(\lambda^- - \epsilon)} + \frac{1}{2}C)$. Remember throughout that D_m is constant on an interval of length 1.

We deal with cases:

The first case is that $m' > m$ but $(m' - m) < 1$. There are two subcases, one is that $D_m = D_{m'}$, the other is that $D_{m'} < D_m$.

Let us deal with the first subcase: $D_m = D_{m'}$, then

$$\ell(m) - \ell(m') = (m' - m)r \ln \left(\frac{D_{m+1}}{D_m} \right) \leq (m' - m)r \ln(e^{-(\lambda^- - \epsilon)} + \frac{1}{2}C).$$

So we are done.

Now we deal with the second subcase: $D_{m'} < D_m$. Then really $D_{m'} = D_{m+1}$.

$$\begin{aligned} \ell(m) - \ell(m') &= r \ln \left(\frac{D_{m+1}}{D_m} \right) - (m' - \lfloor m' \rfloor)r \ln \left(\frac{D_{m+1}}{D_{m+2}} \right) - (m - \lfloor m \rfloor)r \ln \left(\frac{D_{m+1}}{D_m} \right) \\ &= r \left[(1 - m + \lfloor m \rfloor) \ln \left(\frac{D_{m+1}}{D_m} \right) + (m' - \lfloor m' \rfloor) \ln \left(\frac{D_{m+2}}{D_{m+1}} \right) \right] \\ &\leq r(1 - m + \lfloor m \rfloor + m' - \lfloor m' \rfloor) \ln(e^{-(\lambda^- - \epsilon)} + \frac{1}{2}C) \\ &= (m' - m)r \ln(e^{-(\lambda^- - \epsilon)} + \frac{1}{2}C). \end{aligned}$$

The last equality holds because the condition of the subcase is really that $\lfloor m \rfloor + 1 = \lfloor m' \rfloor$.

The second case is $m' > m$ but $(m' - m) \geq 1$. We could have stopped with the above

and just said that since d is continuous on a compact interval, $|d(m) - d(m')|$ is uniformly bounded from above and hence we are done. However, we actually want that this Lipschitz constant computed in this proof is independent of the interval $[m_1, m_2]$ that we choose except for the fact that we need m_1 large enough so that Lemma 6.2.7 holds. Hence we do more work.

Note first that $\ell(m) = \ell(m') \leq \ell(\lfloor m \rfloor) - \ell(m')$. We will work with this quantity.

$$\frac{\ell(\lfloor m \rfloor) - \ell(m')}{m' - m} = \frac{r \ln \left(\frac{D_{m'}}{D_m} \right) - (m' - \lfloor m \rfloor) r \ln \left(\frac{D_{m'}}{D_{m'+1}} \right)}{m' - m}.$$

Since $(m' - m) \geq 1$ we have that $(m' - m) > m' - \lfloor m \rfloor$. Also note that $D_{m'} < D_{m'+1}$ and so $\ln \left(\frac{D_{m'}}{D_{m'+1}} \right)$ is negative. Then,

$$\begin{aligned} \frac{\ell(\lfloor m \rfloor) - \ell(m')}{m' - m} &\leq r \left[\frac{\ln \left(\frac{D_{m'}}{D_m} \right)}{m' - m} + \ln \left(\frac{D_{m'+1}}{D_{m'}} \right) \right] \\ &\leq r \left[\frac{\lfloor m' - m \rfloor}{m' - m} \ln(e^{-(\lambda^- - \epsilon)} + \frac{1}{2}C) + \ln(e^{-(\lambda^- - \epsilon)} + \frac{1}{2}C) \right] \\ &\leq 2r \ln(e^{-(\lambda^- - \epsilon)} + \frac{1}{2}C). \end{aligned}$$

Now we do the other side of the computation. Note that d is decreasing because as m increases, $(-m)_1^\tau$ decreases and ℓ decreases (in fact both do so strictly and hence so does d). Take $m, m' \in [m_1, m_2]$ such that $m' > m$ without loss of generality. Then,

$$\begin{aligned} |d_1(m) - d_1(m')| &= d_1(m) - d_1(m') \geq (-m)_1^\tau - (-m')_1^\tau + \ell(m) - \ell(m') \\ &= (\lambda^+ - \epsilon)(m' - m) - \ell(m + 1) + \ell(m) \\ &\geq \lambda^+ - \epsilon. \end{aligned}$$

We get Lipschitz constant to be

$$L := \max\{2r \ln(e^{-(\lambda^- - \epsilon)} + \frac{1}{2}C) + \lambda^+ + \epsilon, \frac{1}{\lambda^+ + \epsilon}\}. \quad (239)$$

This completes the proof. $\square$

Now that we have established that d_1 is L -BiLipschitz, a similar argument works for d_2 , we get the same Lipschitz constant and a similar result to Lemma 5.3.38.

Lemma 6.2.12. *For our choice of m_1 and m_2 as in Equation (164), and T_{big} large enough, we have that the interval $[d_1(m_2), d_1(m_1)]$ inside $[0, d_1(m_1)]$ has proportion at least $(1 - \frac{1}{1+\eta})$ in measure (where η is as in Section 5.3 and Lemma 6.2.8). Similar result holds for d_2 .*

Proof. The proof is the same as in Lemma 5.3.38. $\square$

Using Lemma 6.2.12 we get the following lemma:

Lemma 6.2.13. *Given the interval $[m_1, m_2]$ (where m_1, m_2 as specified above), there exists a positive measure subset G such that,*

1. *if $m \in G$ then flowing backwards by m from the points $(\tilde{x}, \tilde{\omega}, k_1)$ and $(\tilde{y}, \tilde{\omega}, k_1)$ lands us in the set $\tilde{K}_2 \times [0, 1[$,*
2. *if $m \in G$ then flowing backwards by $d_1(m)$ and $d_2(m)$ from the points $(\tilde{x}, \tilde{\omega}, k_1)$ and $(\tilde{y}, \tilde{\omega}, k_1)$ resp. lands both points in the set K_3 .*

Proof. The proof is the same as in Lemma 5.3.41. $\square$

The end points

From the points $x, y \in X$, we flow forwards by the ‘time change’ flow for time ℓ under a new future to our final pair of points denoted $x', y' \in X$. We will need to ensure that we

land in an appropriate set called K'_4 (see Section 5.3). Throughout this section we will call $K := K'_4$.

Claims 5.3.42, 5.3.44 and Lemma 5.3.43 prove that

Lemma 6.2.14. *We have that*

$$\mu^{\mathbb{N}}(\{\omega^+ \in \Omega^+ : F_{tc}^\ell(x, (\omega^-, \omega^+), k) \in K\}) \geq 1 - \sqrt{Q\epsilon_4}. \quad (240)$$

Proof. See the proof of Lemma 5.3.45. □

So there is a large measure set of appropriate new futures to choose from. Run this argument for each of the points, $(\bar{x}, \bar{\omega}_x, k_x)$ and $(\bar{y}, \bar{\omega}_y, k_y)$ (separately).

Conclusion

From Section 6.2, Lemma 6.2.14, for each choice of T and m fixed (where m is chosen in an interval determined by T) we are given $V_{T,m}$ (ℓ is determined and we assume it is greater than ℓ_0 by taking T large), a positive measure subset of Ω^+ such that we can choose $(\omega')^+$ from.

Additionally, in Section 6.2 we landed in a set $\tilde{K}_2$ where the points had a large measure set of $(\omega')^+$ to choose from such that if we change our Ω -component to $(\omega')^+$ we belong to the set K_2 . This set K_2 had properties we needed to make arguments in Section 5.3. We define $V = V_{(x, \sigma^{n'}(\tilde{\omega}^-))} \cap V_{(y, \sigma^{n'}(\tilde{\omega}^-))}$ as the set of good $(\omega')^+$. Lastly, we had the set A^+ of good $(\omega')^+$ for Lemma 6.2.9. Now let $V' = V_{T,m} \cap V \cap A^+$. The parameter Q is fixed at the beginning and ϵ_4 , and ϵ_2 were chosen (at the beginning) such that V' has positive measure.

Take $\ell \rightarrow \infty$, more precisely, I want the legs of the V to go to infinity. To achieve this, we take $\rho \rightarrow 0$, in turn, T_{big} gets bigger, and since θ_1 and θ_2 are fixed, even though T increases, we see that the proportion between T and T_{big} decreases. This implies that the amount flowing by T separates the unstable component (the distance between $\tilde{y}$ and $\tilde{z}$) and

the stable distance (the distance between $\tilde{x}$ and $\tilde{z}$) goes to zero. Together this constitutes the distance between the unstable planes going to zero. The parameter m grows as $\rho \rightarrow 0$ but does not prevent the error between the unstable planes from going to zero. We also have that $d_1(m_1) > (1 + \eta)d_1(m_2)$ and $d_2(m_1) > (1 + \eta)d_2(m_2)$ as in Lemma 5.3.38 so that we can make an argument for our points along the way to stay in good sets as in Lemma 5.3.41. Being in a good set after flowing by d_1 or d_2 allows us to use Lemma 6.2.14 to get the set $V_{T,m}$ in the previous paragraph. As ℓ goes to infinity, the image of the supports land in the same plane as the stable components for ω' (which is changing with ℓ) shrink. We end up with two support points in one unstable manifold. However, these limit points by construction belong to the compact set $K_{00} \subset \mathcal{W}$. This means $\mathcal{W}$ must have positive measure. This is a contradiction to Lemma 2.6.8, so we have proven Proposition 2.6.5.

7 A K3 surface with non-elementary actions but no parabolic elements

We now demonstrate the existence of surfaces that fall outside of the results of [14]. Consider an algebraic K3 surface over $\mathbb{C}$, denoted X , and consider the Néron-Severi group:

$$\mathrm{NS}(X) := H^{1,1}(X; \mathbb{R}) \cap H^2(X; \mathbb{Z}).$$

In [10, Remark 2.4], it is shown that because X is projective, $\langle | \rangle$ restricts to a non-degenerate quadratic form on $\mathrm{NS}(X)$. This makes $\mathrm{NS}(X)$ an even lattice of signature $(1, \rho - 1)$ where ρ is the Picard number (see [10, 15, 34]).

Let $O^+(\mathrm{NS}(X)) = O(\mathrm{NS}(X)) \cap O^+(\mathrm{NS}(X) \otimes \mathbb{R})$ where $\mathrm{NS}(X) \otimes \mathbb{R}$ is with its induced quadratic form. Consider

$$\Delta := \{\delta \in \mathrm{NS}(X) : q_X(\delta) = -2\}.$$

We call Δ the set of roots. For each root δ we get a reflection map $s_\delta \in O^+(NS(X))$

$$s_\delta : x \mapsto \langle x | \delta \rangle \delta.$$

We define the Weyl group

$$W(X) := \{s_\delta : q_X(\delta) = -2\} \subset O^+(NS(X)) \subset O(NS(X)),$$

this is a normal subgroup of $O^+(NS(X))$, see [21, Chapter 8.2].

Below we abstractly construct X with the properties we desire. This relies heavily on the work of [35] and [34] which we will state.

Theorem 7.0.1. *There exist K3 surfaces X carrying non-elementary actions such that $Aut(X)$ contains no parabolic elements and there is no algebraic invariant curve.*

Proof. First we state a result of [38, Section 7] (see also [21, Chapter 15, Theorem 2.6]); it relies heavily on the global Torelli theorem.

Proposition 7.0.2 ([38], see also [21], Chapter 15, Theorem 2.6). *The natural map*

$$\phi : Aut(X) \rightarrow O(NS(X))/W(X),$$

given by $f \mapsto f^$ has finite kernel and cokernel.*

Now we note a corollary of [35, Theorem 1.14.4] which can be found in [34, Corollary 2.9]. It tells us that we can realize certain even lattices with small signature as Néron-Severi groups of algebraic K3s.

Proposition 7.0.3 ([35] Theorem 1.14.4, see also [34] Corollary 2.9). *For $\rho \leq 10$, every even lattice of signature $(1, \rho - 1)$ occurs as the Néron-Severi group of an algebraic K3 surface.*

In particular, we can construct an even lattice S of signature $(1,1)$ and it will be the Néron-Severi group of a K3 surface X . We will construct S such that the following is true:

1. There are no elements of self-intersection -2 ,
2. There are no null vectors,
3. There are at least 2 distinct hyperbolic elements in $O(S)$.

Condition (1) will give us that $W(X)$ is trivial for the K3 we get when we apply Proposition 7.0.3, then Proposition 7.0.2 reduces to the map

$$\phi : \text{Aut}(X) \rightarrow O(NS(X)),$$

having finite kernel and cokernel.

Condition (2) gives us that there are no parabolic elements inside $O(NS(X))$ because parabolic elements fix a null vector. Combined with the fact that the map

$$\phi : \text{Aut}(X) \rightarrow O(NS(X))$$

has finite kernel we get that $\text{Aut}(X)$ has no parabolic elements. This is because if $f \in \text{Aut}(X)$ is parabolic, then for some k , $(f^*)^k$ acting on $H^{1,1}(X; \mathbb{R})$ is unipotent, i.e. has only 1s as eigenvalues. This should be true for the action of $(f^*)^k$ on $NS(X)$ too (making f^* a parabolic element of $O(NS(X))$), the only concern would be if the action of f^* on $NS(X)$ is the identity (in which case it is not a parabolic element of $O(NS(X))$). Since the kernel of $\phi : \text{Aut}(X) \rightarrow O(NS(X))$ is finite it contains no infinite order elements, such as parabolic elements of $\text{Aut}(X)$. This means that parabolic elements $f \in \text{Aut}(X)$ map to parabolic elements $f^* \in O(NS(X))$. Since there are no parabolic elements in $O(NS(X))$ we must have that $\text{Aut}(X)$ contains no parabolic elements.

Condition (3) combined with the fact that $\phi : \text{Aut}(X) \rightarrow O(NS(X))$ has finite cokernel (i.e. $O(NS(X))/\text{Im}(\phi)$ is finite) gives us that $\text{Aut}(X)$ is non-elementary. This is because the two distinct hyperbolic elements in $O(S) = O(NS(X))$ (which are infinite order) must be trivial in the cokernel and hence belong to $\text{Im}(\phi)$. Additionally, they will still have an eigenvector with eigenvalue $\lambda > 1$ when viewed acting on $H^{1,1}(X; \mathbb{R})$. We get that $\text{Aut}(X)$ has at least two sufficiently distinct hyperbolic elements that give us a non-elementary subgroup in $\text{Aut}(X)$.

We can take $S = \mathbb{Z}^2$, $q'_X = x^2 - 2y^2$; this does not represent zero because $\pm\sqrt{2}$ is not rational and it has signature $(1, 1)$. We now take q'_X as above and let $q_X = 7q'_X$, this gives a quadratic form that does not represent -2, hence condition (1) is satisfied. Condition (3) is also satisfied.

Making sure that there is no invariant algebraic curve relies on the fact that we guaranteed that there are no elements with self-intersection -2, i.e. condition 1. To reiterate, condition (1) gives us that $\text{Aut}(X) \rightarrow O(NS(X))$ has finite kernel and cokernel by Proposition 7.0.2. However, $O(NS(X))$ (and any finite index subgroup) is Zariski dense in $O(NS(X) \otimes \mathbb{R}) \cong SO(1, 1)$ by the Borel density theorem. Hence, if $\text{Aut}(X)$ fixes a curve $C \subset X$, then $SO(1, 1)$ fixes $[C]$ which cannot happen as this group does not fix any vector. This completes the proof. $\square$

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