Supplemental Material for "Electron-boson-interaction induced particle-hole symmetry breaking of conductance into subgap states in superconductors"

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I. DERIVATION OF THE ABS-BOSON COUPLING FROM THE MICROSCOPIC ELECTRON-BOSON INTERACTION

In this section, we derive the ABS-boson coupling term in Eq. (2a) of the main text from the microscopic electron-boson interaction. We begin by writing a generic Hamiltonian for a superconductor with an ABS as

$$\hat{H}_{\rm SC} = \sum_{l,m} \int dx \int dx' h_{lm}(x,x')\hat{d}_l^\dagger(x)\hat{d}_m(x') + \left[\Delta_{lm}(x,x')\hat{d}_l^\dagger(x)\hat{d}_m^\dagger(x') + \text{H.c.}\right], \tag{S-1}$$

where $h_{l,m}$ describes the dynamics of the electrons in the superconductor with an ABS, $\Delta_{l,m}$ is the superconducting pairing potential, $d_{l,m}^\dagger$ ($d_{l,m}$) is the electron creation (annihilation) operator of the superconductor and the indices l, m represent both the orbital and spin degrees of freedom. We can diagonalize the above Hamiltonian using the Bogoliubov transformation

$$\hat{\gamma}_\alpha = \sum_m \int dx \left[u_{\alpha m}^*(x) d_m(x) + v_{\alpha m}(x) d_m^\dagger(x)\right], \tag{S-2a}$$

$$\hat{\gamma}_\alpha^\dagger = \sum_m \int dx \left[v_{\alpha m}^*(x) d_m(x) + u_{\alpha m}(x) d_m^\dagger(x)\right], \tag{S-2b}$$

which gives

$$\hat{H}_{\rm SC} = \sum_\alpha \varepsilon_\alpha \hat{\gamma}_\alpha^\dagger \hat{\gamma}_\alpha + \text{const}, \tag{S-3}$$

where the lowest energy level corresponds to the ABS energy, i.e., $\varepsilon_1 = \varepsilon_{\rm A}$.

The Hamiltonian of the electron-boson coupling is given by

$$\hat{H}_{\text{e-b}} = \sum_{lm} \int dx \int dx' g_{lm}(x,x')\hat{d}_l^\dagger(x)\hat{d}_m(x')(\hat{b}^\dagger + \hat{b}), \tag{S-4}$$

where g_{lm} is the electron-boson coupling strength, and $\hat{b}$ ($\hat{b}^\dagger$) is the boson annihilation (creation) operator. Substituting

$$d_m(x) = \sum_{\alpha>0} \left[u_{m\alpha}(x)\hat{\gamma}_\alpha + v_{m\alpha}(x)\hat{\gamma}_\alpha^\dagger\right], \tag{S-5a}$$

$$d_m^\dagger(x) = \sum_{\alpha>0} \left[v_{m\alpha}^*(x)\hat{\gamma}_\alpha + u_{m\alpha}^*(x)\hat{\gamma}_\alpha^\dagger\right], \tag{S-5b}$$

into Eq. (S-4), we have

$$\begin{aligned}\hat{H}_{\text{e-b}} &= \sum_{lm\alpha\beta} \int dx \int dx' g_{lm}(x,x') \Big[v_{l\alpha}^*(x) u_{m\beta}(x')\hat{\gamma}_\alpha\hat{\gamma}_\beta + v_{l\alpha}^*(x) v_{m\beta}(x')\hat{\gamma}_\alpha\hat{\gamma}_\beta^\dagger \\ &\qquad + u_{l\alpha}^*(x) u_{m\beta}(x')\hat{\gamma}_\alpha^\dagger\hat{\gamma}_\beta + u_{l\alpha}^*(x) v_{m\beta}(x')\hat{\gamma}_\alpha^\dagger\hat{\gamma}_\beta^\dagger\Big](\hat{b}^\dagger + \hat{b}) \\ &= \sum_{\alpha,\beta}\left(\tilde{\lambda}_{\alpha\beta}^{(c)}\gamma_\alpha^\dagger\gamma_\beta + \tilde{\lambda}_{\alpha\beta}^{(d)}\gamma_\alpha\gamma_\beta\right)(\hat{b}^\dagger + \hat{b}) + \text{H.c.},\end{aligned} \tag{S-6}$$

where we have defined

$$\tilde{\lambda}^{(c)}_{\alpha\beta} \equiv \frac{1}{2}\sum_{lm}\int dx \int dx' g_{lm}(x,x')[u^*_{l\alpha}(x)u_{m\beta}(x') - v^*_{l\beta}(x)v_{m\alpha}(x')],$$
$$\tilde{\lambda}^{(d)}_{\alpha\beta} \equiv \sum_{lm}\int dx \int dx' g_{lm}(x,x')[v^*_{l\alpha}(x)u_{m\beta}(x')]. \tag{S-7}$$

Projecting the above Hamiltonian into the lowest energy sector $\alpha = \beta = 1$ which corresponds to the ABS energy sector, we have

$$\begin{aligned}\hat{H}_{\text{e-b}} &\approx \sum_{lm}\int dx \int dx' g_{lm}(x,x')[u^*_{l1}(x)u_{m1}(x') - v^*_{l1}(x)v_{m1}(x')]\hat{\gamma}^\dagger\hat{\gamma}(\hat{b}^\dagger + \hat{b}) + \sum_{lm}\int dx \int dx' g_{lm}(x,x')(\hat{b}^\dagger + \hat{b})\\ &= \lambda\hat{\gamma}^\dagger\hat{\gamma}(\hat{b}^\dagger + \hat{b}) + \chi(\hat{b}^\dagger + \hat{b}),\end{aligned} \tag{S-8}$$

where we have defined $\hat{\gamma} \equiv \hat{\gamma}_1$ as the Bogoliubov operator for the ABS, $\lambda \equiv 2\lambda^{(c)}_{11}$ as the ABS-boson coupling strength, and $\chi \equiv \sum_{lm}\int dx \int dx' g_{lm}(x,x')$. Note that in evaluating Eq. (S-8), we have used the anticommutation relation $\{\gamma,\gamma^\dagger\} = 1$, $\{\gamma,\gamma\} = 0$, and $\{\gamma^\dagger,\gamma^\dagger\} = 0$. We can eliminate the term $\chi(\hat{b}^\dagger + \hat{b})$ in Eq. (S-8) by introducing the shift $\hat{b} \to \hat{b} - \chi/\Omega$ and $\hat{b}^\dagger \to \hat{b}^\dagger - \chi/\Omega$ which gives the ABS Hamiltonian as

$$\begin{aligned}\hat{H}_\text{A} &= \varepsilon_\text{A}\gamma^\dagger\gamma + \lambda\hat{\gamma}^\dagger\hat{\gamma}\left(\hat{b} + \hat{b}^\dagger - 2\frac{\chi}{\Omega}\right) + \chi\left(\hat{b}^\dagger + \hat{b} - 2\frac{\chi}{\Omega}\right) + \Omega\left(\hat{b}^\dagger - \frac{\chi}{\Omega}\right)\left(\hat{b} - \frac{\chi}{\Omega}\right)\\ &= \left(\varepsilon_\text{A} - 2\frac{\lambda\chi}{\Omega}\right)\gamma^\dagger\gamma + \lambda\hat{\gamma}^\dagger\hat{\gamma}\left(\hat{b}^\dagger + \hat{b}\right) + \Omega\hat{b}^\dagger\hat{b} - \frac{\chi^2}{\Omega}.\end{aligned} \tag{S-9}$$

Introducing the shift $\varepsilon_\text{A} \to \varepsilon_\text{A} + 2\lambda\chi/\Omega$ and shifting the overall energy by $\frac{\chi^2}{\Omega}$, i.e., $\hat{H}_\text{A} \to \hat{H}_\text{A} - \frac{\chi^2}{\Omega}$, we have the Hamiltonian for the boson-coupled ABS as in Eq. (2a) of the main text:

$$\hat{H}_\text{A} = \varepsilon_\text{A}\hat{\gamma}^\dagger\hat{\gamma} + \lambda\hat{\gamma}^\dagger\hat{\gamma}(\hat{b}^\dagger + \hat{b}) + \Omega\hat{b}^\dagger\hat{b}. \tag{S-10}$$

II. LANG-FIRSOV TRANSFORMATION

In this section, we follow Ref. [S1] to derive the matrix elements for the tunneling of electrons ($\hat{d}^\dagger_\text{A}$) and holes ($\hat{d}_\text{A}$) and the boson absorption or emission matrix elements $Y_{qq'}$ in Eq. (4) of the main text. We begin by writing the Hamiltonian of an ABS coupled to a one-dimensional normal lead and bosonic modes, e.g., phonons, plasmons, etc., as the sum of the Hamiltonian of a boson-coupled ABS, lead and tunnel coupling, $\hat{H} = \hat{H}_\text{A} + \hat{H}_\text{L} + \hat{H}_\text{T}$ [Eq. (2) of the main text], where

$$\hat{H}_\text{A} = \varepsilon_\text{A}\hat{\gamma}^\dagger\hat{\gamma} + \lambda\hat{\gamma}^\dagger\hat{\gamma}(\hat{b}^\dagger + \hat{b}) + \Omega\hat{b}^\dagger\hat{b}, \tag{S-11a}$$
$$\hat{H}_\text{L} = \sum_k \varepsilon_{\text{L},k}\hat{c}^\dagger_{\text{L},k}\hat{c}_{\text{L},k}, \tag{S-11b}$$
$$\hat{H}_\text{T} = t\hat{c}^\dagger_\text{L}\hat{d}_\text{A} + \text{H.c.} \tag{S-11c}$$

Here, ε_A is the ABS energy, $\hat{\gamma}$ ($\hat{\gamma}^\dagger$) is the Bogoliubov annihilation (creation) operator of the ABS, λ is the ABS-boson coupling strength, $\hat{b}$ ($\hat{b}^\dagger$) is the boson annihilation (creation) operator and Ω is the boson frequency. The operator $\hat{c}_{\text{L},k}$ ($\hat{c}^\dagger_{\text{L},k}$) annihilates (creates) the lead electron with momentum k and energy $\varepsilon_{\text{L},k}$. The tunneling Hamiltonian H_T [S2, S3] represents the electron tunneling between the normal lead and ABS, where the electron annihilation operator of the lead and ABS at the junction given by $\hat{c}_\text{L} = \int dk\hat{c}_{\text{L},k}/(2\pi)$ and $\hat{d}_\text{A}$, respectively. The operator $\hat{d}_\text{A}$ is obtained by projecting the operator $\hat{d}_1(x=0)$ [Eq. (S-5a)] to the ABS energy sector ($\alpha = 1$), where we have $\hat{d}_\text{A} = u\gamma + v\gamma^\dagger$. For notational simplicity, here we define $\gamma \equiv \gamma_1$, $u \equiv u_{11}(x=0)$ and $v \equiv v_{11}(x=0)$ where u and v are the particle and hole components of the ABS wave function at the junction ($x=0$). In this paper, we renormalize the ABS wave function such that $|u|^2 + |v|^2 = 1$. Note that since we consider only the subgap state and ignore the above-gap states, the relation $\hat{d}_\text{A} = u\gamma + v\gamma^\dagger$ is only approximate which makes $\hat{d}_\text{A}$ nonfermionic. The operator $\hat{d}_\text{A}$ becomes fermionic if all the states in the superconductor including the above-gap states are taken into account [see Eq. (S-5a)]. Our conclusion on the PHS breaking of the subgap conductance due to the ABS-boson coupling does not rely on the fermionic properties of $\hat{d}_{\mathrm{A}}$.

To eliminate the ABS-boson coupling, we can transform the Hamiltonian [Eq. (S-11)] using a canonical transformation

$$\hat{\tilde{H}} = e^{\hat{S}}\hat{H}e^{-\hat{S}}, \tag{S-12}$$

where

$$\hat{S} = \frac{\lambda}{\Omega}\hat{\gamma}^{\dagger}\hat{\gamma}(\hat{b}^{\dagger} - \hat{b}) \tag{S-13}$$

is the Lang-Firsov transformation operator [S4]. Using the relation

$$\hat{\tilde{A}} = e^{\hat{S}}\hat{A}e^{-\hat{S}} = \hat{A} + [\hat{S}, \hat{A}] + \frac{1}{2!}[\hat{S}, [\hat{S}, \hat{A}]] + \cdots, \tag{S-14}$$

we can write the transformed annihilation and creation operators for the Bogoliubov quasiparticles, electrons and bosonic modes as

$$\hat{\tilde{\gamma}} = \hat{\gamma}\hat{Y}, \tag{S-15a}$$
$$\hat{\tilde{\gamma}}^{\dagger} = \hat{\gamma}^{\dagger}\hat{Y}^{\dagger}, \tag{S-15b}$$
$$\hat{\tilde{d}}_{\mathrm{A}} = u\hat{\gamma}\hat{Y} + v\hat{\gamma}^{\dagger}\hat{Y}^{\dagger}, \tag{S-15c}$$
$$\hat{\tilde{d}}_{\mathrm{A}}^{\dagger} = u^{*}\hat{\gamma}^{\dagger}\hat{Y}^{\dagger} + v^{*}\hat{\gamma}\hat{Y}, \tag{S-15d}$$
$$\hat{\tilde{b}} = \hat{b} - \frac{\lambda}{\Omega}\hat{\gamma}^{\dagger}\hat{\gamma}, \tag{S-15e}$$
$$\hat{\tilde{b}}^{\dagger} = \hat{b}^{\dagger} - \frac{\lambda}{\Omega}\hat{\gamma}^{\dagger}\hat{\gamma}, \tag{S-15f}$$

where $\hat{Y} = \exp\left[-\frac{\lambda}{\Omega}(\hat{b}^{\dagger} - \hat{b})\right]$. Under this transformation, the number operator remains the same, i.e., $\hat{\tilde{\gamma}}^{\dagger}\hat{\tilde{\gamma}} = \hat{\gamma}^{\dagger}\hat{\gamma}\hat{Y}^{\dagger}\hat{Y} = \hat{\gamma}^{\dagger}\hat{\gamma}$ and the Hamiltonians [Eq. (S-11a) and Eq. (S-11c)] transform as

$$\begin{aligned}\hat{\tilde{H}}_{\mathrm{A}} &= \varepsilon_{\mathrm{A}}\hat{\gamma}^{\dagger}\hat{\gamma} + \lambda\left(\hat{b}^{\dagger} + \hat{b} - \frac{2\lambda}{\Omega}\hat{\gamma}^{\dagger}\hat{\gamma}\right)\hat{\gamma}^{\dagger}\hat{\gamma} + \Omega\left(\hat{b}^{\dagger} - \frac{\lambda}{\Omega}\hat{\gamma}^{\dagger}\hat{\gamma}\right)\left(\hat{b} - \frac{\lambda}{\Omega}\hat{\gamma}^{\dagger}\hat{\gamma}\right) \\ &= \left(\varepsilon_{\mathrm{A}} - \frac{\lambda^2}{\Omega}\right)\hat{\gamma}^{\dagger}\hat{\gamma} + \Omega\hat{b}^{\dagger}\hat{b},\end{aligned} \tag{S-16a}$$
$$\hat{\tilde{H}}_{\mathrm{T}} = t\hat{c}_{\mathrm{L}}^{\dagger}\hat{\tilde{d}}_{\mathrm{A}} + \mathrm{H.c.}, \tag{S-16b}$$

where $\hat{\tilde{d}}_{\mathrm{A}} = u\hat{\gamma}\hat{Y} + v\hat{\gamma}^{\dagger}\hat{Y}^{\dagger}$ is the Lang-Firsov transformation of $\hat{d}_{\mathrm{A}}$.

We can evaluate the matrix elements for the electron and hole tunneling which change the ABS occupancy number n from $0 \to 1$ and the boson occupancy from $q \to q'$ as

$$\langle 1, q'|\hat{\tilde{d}}_{\mathrm{A}}^{\dagger}|0, q\rangle = u^{*}\langle 1|\hat{\gamma}^{\dagger}|0\rangle\langle q'|\hat{Y}^{\dagger}|q\rangle = u^{*}Y_{qq'}, \tag{S-17a}$$
$$\langle 1, q'|\hat{\tilde{d}}_{\mathrm{A}}|0, q\rangle = v\langle 1|\hat{\gamma}^{\dagger}|0\rangle\langle q'|\hat{Y}^{\dagger}|q\rangle = vY_{qq'}, \tag{S-17b}$$

respectively. Using the Baker-Campbell-Haussdorf formula, we have

$$\hat{Y}^{\dagger} = e^{\frac{\lambda}{\Omega}(\hat{b}^{\dagger} - \hat{b})} = e^{-\frac{\lambda^2}{2\Omega^2}}e^{\frac{\lambda}{\Omega}\hat{b}^{\dagger}}e^{-\frac{\lambda}{\Omega}\hat{b}}, \tag{S-18}$$

and we can evaluate the boson emission or absorption matrix element as [S5]

$$\begin{aligned}Y_{qq'} \equiv \langle q'|\hat{Y}^{\dagger}|q\rangle = \langle q'|e^{\frac{\lambda}{\Omega}(b^{\dagger} - b)}|q\rangle = \langle q'|e^{-\frac{\lambda^2}{2\Omega^2}}e^{\frac{\lambda}{\Omega}\hat{b}^{\dagger}}e^{-\frac{\lambda}{\Omega}\hat{b}}|q\rangle \\ = e^{-\frac{\lambda^2}{2\Omega^2}}\sum_{m=0}^{\min(q,q')}\left(\frac{\lambda}{\Omega}\right)^{q'-m}\left(-\frac{\lambda}{\Omega}\right)^{q-m}\frac{\sqrt{q!q'!}}{m!(q-m)!(q'-m)!},\end{aligned} \tag{S-19}$$

where $|Y_{qq'}|^2$ is symmetric under the interchange $q \leftrightarrow q'$. Note that in going to the second line of Eq. (S-19), we have used the following relations:

$$e^{-\frac{\lambda}{\Omega}\hat{b}}|q\rangle = \sum_{m=0}^{\infty} \frac{1}{m!}\left(-\frac{\lambda}{\Omega}\right)^m \hat{b}^m|q\rangle = \sum_{m=0}^{q} \frac{1}{m!}\left(-\frac{\lambda}{\Omega}\right)^m \sqrt{\frac{q!}{(q-m)!}}|q-m\rangle, \tag{S-20a}$$

$$\langle q'|e^{\frac{\lambda}{\Omega}\hat{b}^\dagger} = \sum_{l=0}^{\infty} \frac{1}{l!}\left(\frac{\lambda}{\Omega}\right)^l \langle q'|(\hat{b}^\dagger)^l = \sum_{l=0}^{q'} \frac{1}{l!}\left(\frac{\lambda}{\Omega}\right)^l \sqrt{\frac{q'!}{(q'-l)!}}\langle q'-l|. \tag{S-20b}$$

III. RATE EQUATION AND TUNNELING RATES

The stationary-state rate equation satisfied by the probability P_q^n of an ABS-boson system being in the state $|n,q\rangle$, i.e., having an ABS occupation number n and boson occupation number q, is given by [S5,S6]

$$\begin{aligned} 0 &= \frac{\partial P_q^n}{\partial t} \\ &= \sum_{q'} P_{q'}^{\bar{n}}\left[R_{q'\to q}^{\bar{n}\to n;e} + R_{q'\to q}^{\bar{n}\to n;h}\right] - P_q^n \sum_{q'}\left[R_{q\to q'}^{n\to\bar{n};e} + R_{q\to q'}^{n\to\bar{n};h}\right] \\ &\quad + P_{q+1}^n \eta_{q+1;-} + P_{q-1}^n \eta_{q-1;+} - P_q^n\left(\eta_{q;+} + \eta_{q;-}\right). \end{aligned} \tag{S-21}$$

The second line in Eq. (S-21) represents the probability flux due to hopping of an electron (e) or hole (h) from the lead to the ABS which changes the ABS occupation number from $\bar{n} \equiv 1-n$ to n and the boson occupancy from q' to q and vice versa. The quantity P_q^n denotes the probability that the system is in the state $|n,q\rangle$ and $R_{q\to q'}^{n\to\bar{n}}$ denotes the transition rate from the state $|n,q\rangle$ to the state $|\bar{n},q'\rangle$. The third line of Eq. (S-21) represents the boson relaxation where the boson emission and absorption probabilities are $\eta_{q;+} = A(q+1)$ and $\eta_{q;-} = Bq$, respectively, with $A = Be^{-\Omega/k_\mathrm{B}T}$. These probability rates are consistent with the fluctuation-dissipation theorem. If the boson relaxation rate is faster than the tunneling rate $\Gamma/\hbar$ such that the bosons acquire the equilibrium distribution $P_q^\mathrm{b} = e^{-q\Omega/k_\mathrm{B}T}(1-e^{-\Omega/k_\mathrm{B}T})$, the probability P_q^n can be factorized as $P_q^n = P^n P_q^\mathrm{b}$. Summing Eq. (S-21) over q for these factorized probabilities gives

$$0 = P^{\bar{n}}(R^{\bar{n}\to n;e} + R^{\bar{n}\to n;h}) - P^n\left(R^{n\to\bar{n};e} + R^{n\to\bar{n};h}\right) + P^n\sum_q\left[P_{q+1}^\mathrm{b}\eta_{q+1;-} + P_{q-1}^\mathrm{b}\eta_{q-1;+} - P_q^\mathrm{b}\left(\eta_{q;+} + \eta_{q;-}\right)\right], \tag{S-22}$$

where $R^{n\to\bar{n};e(h)} \equiv \sum_{q,q'} P_q^\mathrm{b} R_{q\to q'}^{n\to\bar{n};e(h)}$ and $R^{\bar{n}\to n;e(h)} \equiv \sum_{q,q'} P_{q'}^\mathrm{b} R_{q'\to q}^{\bar{n}\to n;e(h)}$. Since the sum of the boson relaxation terms over q [the last four terms in Eq. (S-22)] is zero, Eq. (S-22) then reduces to Eq. (3) of the main text.

For the tunneling Hamiltonian in Eq. (S-16b), the rates of the electron and hole tunneling processes can be calculated from Fermi's Golden Rule to be

$$\begin{aligned} R_{q\to q'}^{n\to\bar{n};e} &= \frac{2\pi t^2\nu_0}{\hbar}\left|\left\langle \bar{n},q'\left|\hat{\tilde{d}}_\mathrm{A}^\dagger\right|n,q\right\rangle\right|^2 f(E_{\bar{n},q'} - E_{n,q} - eV) \\ &= \frac{\Gamma}{\hbar}|\langle\bar{n}|\hat{d}_\mathrm{A}^\dagger|n\rangle|^2|Y_{qq'}|^2 f(E_{\bar{n},q'} - E_{n,q} - eV), \end{aligned} \tag{S-23a}$$

$$\begin{aligned} R_{q\to q'}^{n\to\bar{n};h} &= \frac{2\pi t^2\nu_0}{\hbar}\left|\left\langle \bar{n},q'\left|\hat{\tilde{d}}_\mathrm{A}\right|n,q\right\rangle\right|^2 f(E_{\bar{n},q'} - E_{n,q} + eV) \\ &= \frac{\Gamma}{\hbar}|\langle\bar{n}|\hat{d}_\mathrm{A}|n\rangle|^2|Y_{qq'}|^2 f(E_{\bar{n},q'} - E_{n,q} + eV), \end{aligned} \tag{S-23b}$$

where $\langle\bar{n}|\hat{d}_\mathrm{A}^\dagger|n\rangle$ and $\langle\bar{n}|\hat{d}_\mathrm{A}|n\rangle$ are the bare tunneling matrix elements for electrons and holes, respectively, $Y_{qq'} = \langle q'|e^{-\lambda(\hat{b}^\dagger-\hat{b})/\Omega}|q\rangle$ is the boson emission or absorption matrix element, and $f(E) = [1+\exp(E/k_\mathrm{B}T)]^{-1}$ is the lead Fermi function.

IV. DETAILS ON MODEL I. TUNNELING INTO BOSON-COUPLED ABS

A. Proof for the particle-hole asymmetry of boson-coupled-ABS conductance

In this section, we will prove that, unless $|u| = |v|$, the current into a boson-coupled ABS [Eq. (5) of the main text] is in general *not* PH antisymmetric, i.e., $I(V_0) \neq I(-V_0)$ resulting in a PH *asymmetric* conductance, i.e., $\frac{dI}{dV}\big|_{V=V_0} \neq \frac{dI}{dV}\big|_{V=-V_0}$. Substituting the transition rates

$$R^{0\to 1;e} = \Gamma|u|^2 W(\tilde{\varepsilon}_{\mathrm{A},-})/\hbar, \tag{S-24a}$$
$$R^{1\to 0;e} = \Gamma|v|^2 W(-\tilde{\varepsilon}_{\mathrm{A},+})/\hbar, \tag{S-24b}$$
$$R^{0\to 1;h} = \Gamma|v|^2 W(\tilde{\varepsilon}_{\mathrm{A},+})/\hbar, \tag{S-24c}$$
$$R^{1\to 0;h} = \Gamma|u|^2 W(-\tilde{\varepsilon}_{\mathrm{A},-})/\hbar, \tag{S-24d}$$

into Eq. (4) of the main text, we can evaluate the current [Eq. (5) of the main text] as

$$I = \frac{2e}{\hbar}\frac{\Gamma|uv|^2\left[W(\tilde{\varepsilon}_{\mathrm{A},-})W(-\tilde{\varepsilon}_{\mathrm{A},+}) - W(\tilde{\varepsilon}_{\mathrm{A},+})W(-\tilde{\varepsilon}_{\mathrm{A},-})\right]}{|u|^2F(\tilde{\varepsilon}_{\mathrm{A},-}) + |v|^2F(\tilde{\varepsilon}_{\mathrm{A},+})}, \tag{S-25}$$

where

$$W(x) = \sum_{q,q'} P^{\mathrm{b}}_q |Y_{qq'}|^2 f(x - \Omega(q-q')), \tag{S-26a}$$
$$\tilde{\varepsilon}_{\mathrm{A},\pm} = \varepsilon_{\mathrm{A}} - \lambda^2/\Omega \pm eV, \tag{S-26b}$$
$$F(x) = W(x) + W(-x), \tag{S-26c}$$

with $f(x) = [1+\exp(x/k_{\mathrm{B}}T)]^{-1}$ being the Fermi function.

We will prove below that the function $F(x)$ in the denominator of Eq. (S-25) is an increasing function of x and hence the denominator in Eq. (S-25) is asymmetric with respect to the interchange $V \leftrightarrow -V$ unless $|u| = |v|$. By rewriting $W(x)$ in Eq. (S-26a) as

$$W(x) = \int_{-\infty}^{\infty} d\omega Q(\omega) f(x-\omega), \tag{S-27}$$

where

$$Q(\omega) = \sum_{q,q'} P^{\mathrm{b}}_q |Y_{qq'}|^2 \delta(\omega - \Omega(q-q')), \tag{S-28}$$

we have

$$\begin{aligned} F(x) \equiv W(x) + W(-x) &= \int_{-\infty}^{\infty}[Q(\omega)-Q(-\omega)]f(x-\omega)d\omega + \sum_{q,q'}P^{\mathrm{b}}_q|Y_{qq'}|^2 \\ &= \int_0^{\infty}[Q(\omega)-Q(-\omega)][f(x-\omega)-f(x+\omega)]d\omega + \sum_{q,q'}P^{\mathrm{b}}_q|Y_{qq'}|^2 \\ &= \frac{1}{2}\int_0^{\infty}[Q(\omega)-Q(-\omega)]\left[\tanh\left(\frac{x+\omega}{2k_{\mathrm{B}}T}\right) - \tanh\left(\frac{x-\omega}{2k_{\mathrm{B}}T}\right)\right]d\omega + \sum_{q,q'}P^{\mathrm{b}}_q|Y_{qq'}|^2. \end{aligned} \tag{S-29}$$

In the following, we will prove that $F(x)$ is a monotonic function of x. We first begin by noting that $Q(\omega)-Q(-\omega) \leq 0$ for $\omega \geq 0$. The proof is as follows

$$\begin{aligned} Q(\omega) - Q(-\omega) &= \sum_{q,q'} P^{\mathrm{b}}_q|Y_{qq'}|^2\left[\delta(\omega-\Omega(q-q')) - \delta(-\omega-\Omega(q-q'))\right] \\ &= \sum_{q,q'} P^{\mathrm{b}}_q|Y_{qq'}|^2\left[\delta(\omega-\Omega(q-q')) - \delta(\omega+\Omega(q-q'))\right] \\ &= \sum_{q,q'}(P^{\mathrm{b}}_q - P^{\mathrm{b}}_{q'})|Y_{qq'}|^2\delta(\omega-\Omega(q-q')), \end{aligned} \tag{S-30}$$

where in the third line we interchange q with q' for the second sum and use $|Y_{qq'}|^2 = |Y_{q'q}|^2$. For $\omega \geq 0$, the delta function forces $q \geq q'$ implying that $(P^{\rm b}_q - P^{\rm b}_{q'}) \propto \exp(-q\Omega/k_{\rm B}T) - \exp(-q'\Omega/k_{\rm B}T) \leq 0$. As a result, $Q(\omega) - Q(-\omega) \leq 0$.

To prove that $F(x)$ is a monotonic function of x, we take the derivative of $F(x)$ [Eq. (S-29)] with x which gives

$$F'(x) = \frac{1}{4k_{\rm B}T}\int_0^\infty [Q(\omega) - Q(-\omega)]\left[\mathrm{sech}^2\left(\frac{x+\omega}{2k_{\rm B}T}\right) - \mathrm{sech}^2\left(\frac{x-\omega}{2k_{\rm B}T}\right)\right] d\omega \geq 0. \tag{S-31}$$

So, $F(x)$ increases monotonically with x. This means that unless $|u| = |v|$, the denominator in Eq. (S-25) is PH asymmetric with respect to the interchange of $V \leftrightarrow -V$ [which amounts to interchanging $\tilde{\varepsilon}_{{\rm A},-} \leftrightarrow \tilde{\varepsilon}_{{\rm A},+}$ in Eq. (S-25)]. To prove that the boson-assisted tunneling model (model II) can also break the PHS of subgap conductances, we simply replace $|Y_{qq'}|$ by $|X_{qq'} - \lambda Y_{qq'}/\Omega|$ in the above derivation, where $X_{qq'} \equiv e^{-\frac{\lambda^2}{2\Omega^2}} \langle q'|e^{\frac{\lambda}{\Omega}\hat{b}^\dagger}(\hat{b}^\dagger + \hat{b})e^{-\frac{\lambda}{\Omega}\hat{b}}|q\rangle$ [Eq. (S-82b)]. Even though the conductance is not PH symmetric, under a simultaneous interchange of $V \leftrightarrow -V$ and $|u| \leftrightarrow |v|$, the current is antisymmetric ($I \to -I$) resulting in a symmetric conductance. This means that the conductance for $|v|^2 > |u|^2$ can be obtained from the conductance for $|v|^2 < |u|^2$ (shown in Figs. 2 and 3 of the main text) by interchanging both $|u| \leftrightarrow |v|$ and $V \leftrightarrow -V$ simultaneously. As a result, the higher and lower peaks switch sides when $|v| \leftrightarrow |u|$, which changes the sign of the PH asymmetry ζ.

The PH asymmetry of the conductance can be understood more intuitively in the limit of large positive and negative voltages $|eV| \gtrsim |\tilde{\varepsilon}_{\rm A}| + k_{\rm B}T$. In the large-positive-voltage regime ($eV \gtrsim |\tilde{\varepsilon}_{\rm A}| + k_{\rm B}T$), hole tunneling processes are energetically forbidden [$R^{0\to1;h}, R^{1\to0;h} \approx 0$ since $W(\tilde{\varepsilon}_{{\rm A},+}), W(-\tilde{\varepsilon}_{{\rm A},-}) \approx 0$]. On the other hand, in the large-negative-voltage regime where $eV \lesssim -(|\tilde{\varepsilon}_{\rm A}| + k_{\rm B}T)$, electron tunneling processes are not energetically allowed [$R^{0\to1;e}, R^{1\to0;e} \approx 0$ since $W(\tilde{\varepsilon}_{{\rm A},-}), W(-\tilde{\varepsilon}_{{\rm A},+}) \approx 0$]. In this limit, the current [Eq. (S-25)] thus reduces to Eq. (1) of the main text:

$$I = \begin{cases} 2e\dfrac{R^{0\to1;e}R^{1\to0;e}}{R^{0\to1;e} + R^{1\to0;e}} & \text{for } eV \gtrsim |\tilde{\varepsilon}_{\rm A}| + k_{\rm B}T, \\ -2e\dfrac{R^{0\to1;h}R^{1\to0;h}}{R^{0\to1;h} + R^{1\to0;h}} & \text{for } eV \lesssim -(|\tilde{\varepsilon}_{\rm A}| + k_{\rm B}T). \end{cases} \tag{S-32}$$

implying that the current at large positive and negative voltages are due to sequential tunnelings of electrons and holes, respectively (see Fig. 1 of the main text). For both the boson-coupled ABS model and boson-assisted tunneling into ABS model, the current is in general not PH antisymmetric, i.e., $I(-V_0) \neq I(V_0)$ or the conductance is PH asymmetric ($\frac{dI}{dV}\big|_{V=V_0} \neq \frac{dI}{dV}\big|_{V=-V_0}$) because of the rate asymmetry between the first and second tunneling processes of electrons and holes (i.e., $R^{0\to1;e} \neq R^{1\to0;h}$ and $R^{1\to0;e} \neq R^{0\to1;h}$). This rate asymmetry arises because the second tunneling process which happens at energy deep inside the Fermi level is energetically allowed to emit more bosons hence occurs with a larger rate than the first tunneling process. Without the ABS-boson coupling ($\lambda = 0$), $W(x) = f(x)$ where $f(x)$ is the Fermi function and the current [Eq. (S-25)] is $I = 2\frac{e}{\hbar}\Gamma|uv|^2[f(\varepsilon_{{\rm A},-}) - f(\varepsilon_{{\rm A},+})]$ which is PH antisymmetric, i.e., $I(V) = -I(-V)$. Thus, the conductance into ABSs in gapped superconductors is PH symmetric.

B. Proof for the temperature independence of the conductance peak area

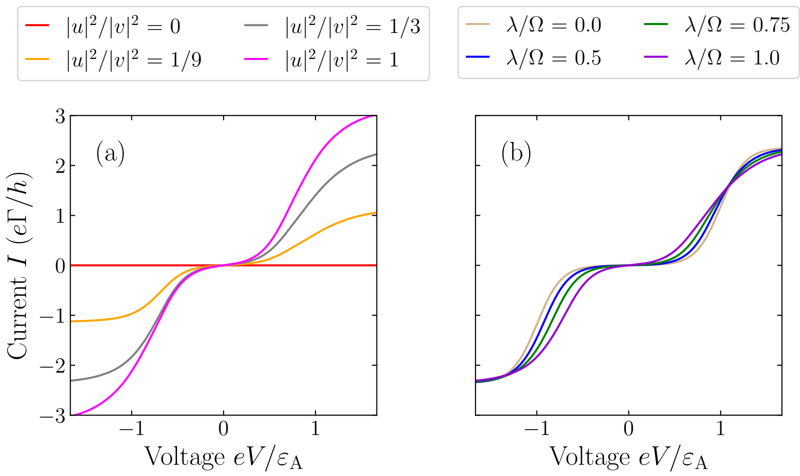
While the conductance for boson-coupled ABSs is in general PH asymmetric, the conductance peak areas calculated using the boson-coupled ABS model in Sec. III of the main text are independent of temperature and equal for both negative and positive voltages. To see this, we can calculate the current at $V = \pm\infty$ by using Eq. (S-32). Note that for $eV \gg |\tilde{\varepsilon}_{\rm A}| + k_{\rm B}T$, we have $f(\tilde{\varepsilon}_{{\rm A},-}) = f(-\tilde{\varepsilon}_{{\rm A},+}) = 1$ and for $eV \ll |\tilde{\varepsilon}_{\rm A}| + k_{\rm B}T$, $f(\tilde{\varepsilon}_{{\rm A},+}) = f(-\tilde{\varepsilon}_{{\rm A},-}) = 1$. This in turn yields $W(\tilde{\varepsilon}_{{\rm A},-}) = W(-\tilde{\varepsilon}_{{\rm A},+}) = W(\tilde{\varepsilon}_{{\rm A},+}) = W(-\tilde{\varepsilon}_{{\rm A},-}) = \sum_{q,q'} P^{\rm b}_q |Y_{qq'}|^2 = 1$. Using $|u|^2 + |v|^2 = 1$, we then have the current [Eq. (S-32)] as

$$I = \begin{cases} \dfrac{e}{2\hbar}\Gamma[1 - (|u|^2 - |v|^2)^2] & \text{for } eV \gg |\tilde{\varepsilon}_{\rm A}| + k_{\rm B}T, \\ -\dfrac{e}{2\hbar}\Gamma[1 - (|u|^2 - |v|^2)^2] & \text{for } eV \ll -(|\tilde{\varepsilon}_{\rm A}| + k_{\rm B}T). \end{cases} \tag{S-33}$$

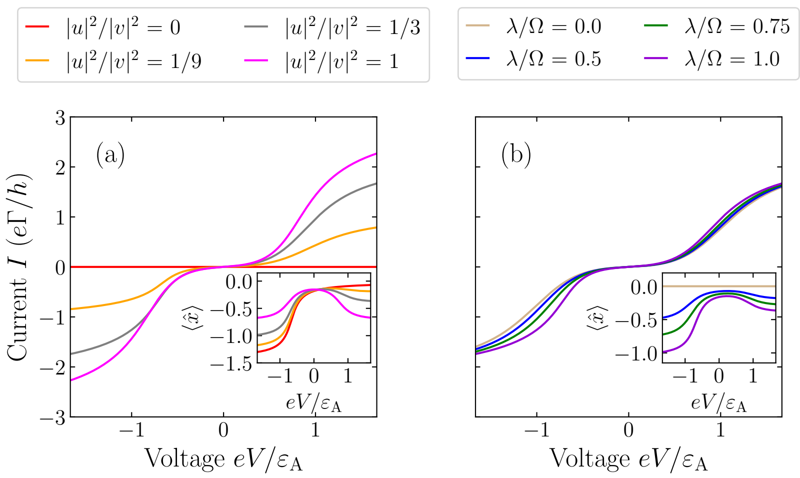
So, $I(V = \infty) = -I(V = -\infty)$ and both $I(V = \pm\infty)$ are independent of temperature. Since the current magnitude at large voltages $|I(V = \pm\infty)|$ is the area under the conductance peak, this means that the conductance peak area for positive and negative voltages are equal and independent of temperature. This fact can also be seen from the current plots in Figs. S3(c) and S4(c) which are calculated using the rate equation and Keldysh approach, respectively. Contrary to model I, the boson-assisted-tunneling model (model II) gives rise to temperature-dependent conductance peak area (see Sec. IV of the main text).

V. CURRENT CALCULATED FROM THE RATE EQUATION AND KELDYSH APPROACH

In this section, we show the current calculated from the rate equation (Fig. S1) and mean-field Keldysh approach (Fig. S2) corresponding to the conductance shown in Figs. 2 and 3 of the main text, respectively. As shown in Figs. S1(a) and S2(a), the current decreases with increasing ABS's PH content imbalance $||u|^2 - |v|^2|$ where $I = 0$ when $||u|^2 - |v|^2| = 1$. This is due to the fact that the terms $R^{0\to1;e}R^{1\to0;e}$ and $R^{0\to1;h}R^{1\to0;h}$ in the current expression [Eq. (5) of the main text] are $\propto |uv|^2 = [1 - (|u|^2 - |v|^2)^2]/4$.



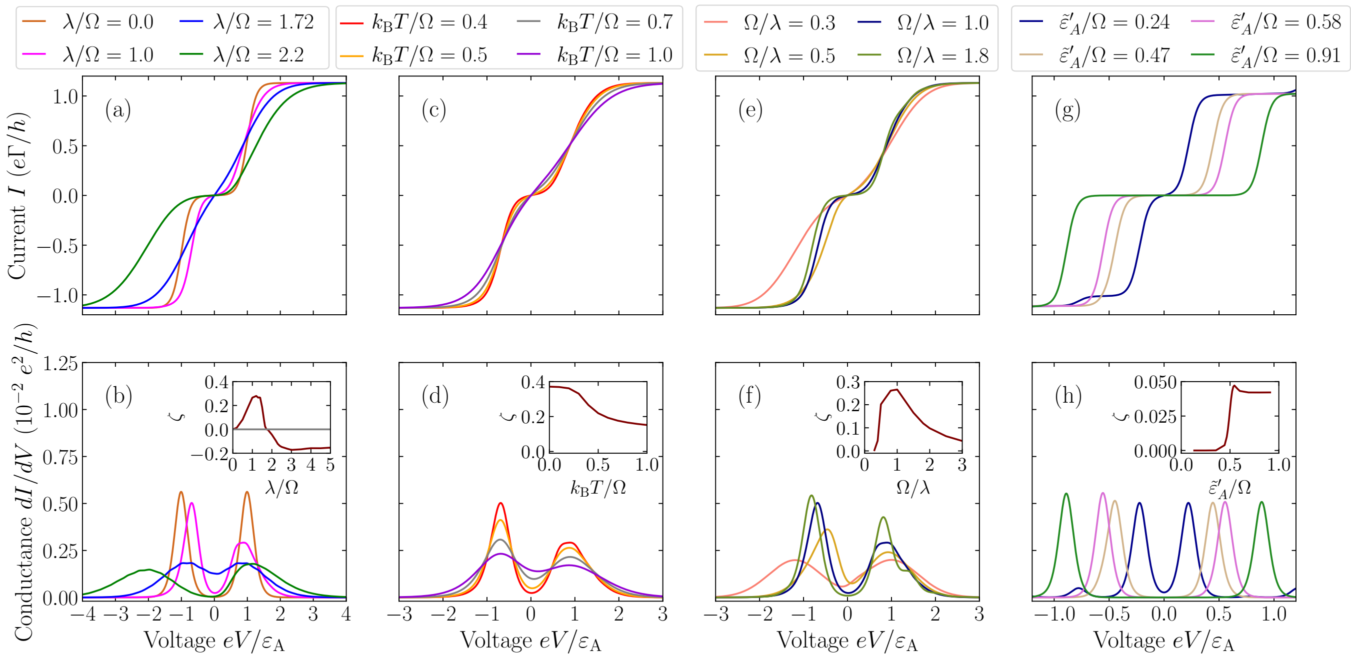
Supplemental Figure S1: Current I of boson-coupled ABSs vs voltage V calculated using the rate equation [Eq. (5) of the main text] for (a) different ratios of PH components $|u|^2/|v|^2$ with $\lambda/\Omega = 1$ and (b) different ABS-boson coupling strengths λ with $|u|^2/|v|^2 = 1/3$. The conductance calculated from the above current is shown in Fig. 2 of the main text. The parameters for all panels are: $\varepsilon_A/\Omega = 3$, $\Gamma/\Omega = 0.05/(2\pi)$, and $k_BT/\Omega = 0.4$.



Supplemental Figure S2: Current I of boson-coupled ABSs vs voltage V calculated using the mean-field Keldysh approach [Eq. (8) of the main text] for (a) different ratios of PH components $|u|^2/|v|^2$ with $\lambda/\Omega = 1$ and (b) different ABS-boson coupling strengths λ with $|u|^2/|v|^2 = 1/3$. The conductance calculated from the above current is shown in Fig. 3 of the main text. Inset: Expectation value of the boson displacement operator $\langle\hat{x}\rangle$ vs voltage V calculated self-consistently using Eq. (10) of the main text. The parameters for all panels are: $\varepsilon_A/\Omega = 3$, $\Gamma/\Omega = 2$, and $k_BT/\Omega = 0.4$.

VI. DEPENDENCE OF THE CURRENT AND CONDUCTANCE CALCULATED FROM THE RATE EQUATION ON ABS-BOSON COUPLING STRENGTH, TEMPERATURE, BOSON FREQUENCY, AND ABS ENERGY

Figure S3 shows the current (upper panels) and conductance (lower panels) of boson-coupled ABSs calculated from the rate equation [Eq. (5) of the main text] for different ABS-boson coupling strengths λ [Figs. S3(a,b)], temperatures T [Figs. S3(c,d)], boson frequencies Ω [Figs. S3(e,f)] and ABS energies $\varepsilon_{\rm A}$ [Figs. S3(g,h)]. Figure S3(b) shows that the magnitude of the conductance PH asymmetry ζ has a nonmonotonic dependence on the ABS-boson coupling strength λ. In the limit $\varepsilon_{\rm A} - \lambda^2/\Omega \gg k_{\rm B}T$ (where the two conductance peaks are well separated), the conductance PH asymmetry ζ increases with increasing λ; this corresponds to the results shown in Fig. 2(b) of the main text. As λ keeps increasing, the two conductance peaks approach each other and in the regime where $\varepsilon_{\rm A} - \lambda^2/\Omega < k_{\rm B}T$, the two peaks start to overlap with each other and ζ decreases with increasing λ. Note that in the regime where $\varepsilon_{\rm A} - \lambda^2/\Omega > 0$, the higher peak is at positive voltage for the case where $|u|^2 > |v|^2$ while for the case where $|v|^2 > |u|^2$, the higher peak is at negative voltage. When $\varepsilon_{\rm A} - \lambda^2/\Omega = 0$, the two conductance peaks merge at the zero voltage which gives a zero conductance PH asymmetry ($\zeta = 0$). Increasing λ beyond this point splits the peaks but with the low and high peaks now switching sides which in turn changes the sign of ζ. As λ increases further, the two peaks move away from each other and the PH asymmetry ζ increases in magnitude; beyond a certain value of λ, ζ becomes weakly dependent on λ as shown in the inset of Fig. S3(b). Note that for large enough λ, the position of the conductance peaks are no longer PH symmetric [see green curve in Fig. S3(b)].



Supplemental Figure S3: Current I (Upper panels) and conductance dI/dV (Lower panels) of boson-coupled ABSs vs voltage V calculated using the rate equation [Eq. (5) of the main text] for (a,b) different ABS-boson coupling strengths λ with $k_{\rm B}T = 0.4$, $\varepsilon_{\rm A} = 3$, $\Gamma = 0.05/(2\pi)$, and $\Omega = 1.0$, (c,d) different temperatures T with $\lambda = 1.0$, $\varepsilon_{\rm A} = 3.0$, $\Gamma = 0.05/(2\pi)$, and $\Omega = 1.0$, (e,f) different boson frequencies Ω with $k_{\rm B}T = 0.4$, $\varepsilon_{\rm A} = 3.0$, $\Gamma = 0.05/(2\pi)$, and $\lambda = 1.0$, and (g,h) different renormalized ABS energies $\tilde{\varepsilon}'_{\rm A} \equiv |\varepsilon_{\rm A} - \lambda^2/\Omega| + k_{\rm B}T/2$ with $k_{\rm B}T = 0.4$, $\lambda = 3.0$ and $\Gamma = 1/(40\pi)$, and $\Omega = 9.0$. Inset: (b) Conductance PH asymmetry ζ vs λ/Ω, (d) ζ vs temperature T, (f) ζ vs boson frequency Ω, and (h) ζ vs $\tilde{\varepsilon}'_{\rm A}/\Omega$. Panel (h) shows that the subgap conductances exhibit PH asymmetry only for $\tilde{\varepsilon}'_{\rm A}/\Omega \gtrsim 0.5$ where $\tilde{\varepsilon}'_{\rm A} \equiv |\varepsilon_{\rm A} - \lambda^2/\Omega| + k_{\rm B}T/2$. The parameters used for all panels are: $|u|^2/|v|^2 = 1/9$.

Figure S3(d) shows that the conductance PH asymmetry ζ decreases with increasing temperature T. This is due to the fact that temperature broadens the conductance peaks. The dependence of the ABS conductance on the boson frequency Ω is shown in Fig. S3(f). The PH asymmetry ζ has a nonmonotonic behavior with the boson frequency Ω where it first increases with increasing Ω and then after reaching its maximum, it decreases with increasing Ω.

The initial increase of ζ with increasing Ω can be attributed to the fact that the two conductance peaks move away from each other as Ω increases ($\tilde{\varepsilon}_{\rm A} = \varepsilon_{\rm A} - \lambda^2/\Omega$ increases with increasing Ω). The decrease of ζ for large Ω is due to the fact that the effective ABS-boson coupling strength λ/Ω decreases with increasing Ω. Figure S3(h) shows the dependence of the conductance on the ABS energy. As shown in the inset of panel (h), the peak conductance only exhibits the PH asymmetry for $\Omega \lesssim 2|\tilde{\varepsilon}_{\rm A}| + k_{\rm B}T$. This can be understood from the fact that the second tunneling process [whose rate is $R^{1\to 0;e}$ in Fig. 1(a) or $R^{1\to 0;h}$ in Fig. 1(b) of the main text] can transfer lead electrons or holes with an energy difference up to $\sim 2|\tilde{\varepsilon}_{\rm A}| + k_{\rm B}T$ from the subgap state, where this energy difference is transferred in form of boson energy Ω. Even though in this paper, we focus only on the regime $k_{\rm B}T \gtrsim \lambda$ where the boson sidebands vanish due to the thermal broadening [S5], the dependence of the ABS conductance peak on the above parameters also hold true in the case where there are boson sidebands. Moreover, the PH asymmetry of the boson sidebands also have similar dependences on the above parameters as that of the ABS conductance peak.

VII. DERIVATION OF THE CURRENT IN THE KELDYSH FORMALISM

In this section, we derive the current [Eq. (8) of the main text] following Refs. [S8-S9]. We begin by writing the Hamiltonian as

$$\hat{H} = \hat{H}_{\rm A} + \hat{H}_{\rm L} + \hat{H}_{\rm T}, \tag{S-34}$$

where

$$\hat{H}_{\rm A} = (\varepsilon_{\rm A} + \sqrt{2}\lambda\langle\hat{x}\rangle)\hat{\gamma}^\dagger\hat{\gamma}, \tag{S-35a}$$

$$\hat{H}_{\rm L} = \sum_k \varepsilon_{{\rm L},k}\hat{c}^\dagger_{{\rm L},k}\hat{c}_{{\rm L},k}, \tag{S-35b}$$

$$\hat{H}_{\rm T} = t\hat{c}^\dagger_{\rm L}\hat{d}_{\rm A} + {\rm H.c.} \tag{S-35c}$$

In Eq. (S-35a), $\hat{H}_{\rm A}$ is the mean-field Hamiltonian of the ABS-boson system obtained by replacing $\hat{x}$ in Eq. (7) of the main text by $\langle\hat{x}\rangle$ where $\langle\hat{x}\rangle = \frac{\langle\hat{b}+\hat{b}^\dagger\rangle}{\sqrt{2}}$ is the mean-field boson displacement. Note that we have dropped the constant in the Hamiltonian $\hat{H}_{\rm A}$ in Eq. (S-35a) since this is just a shift in the energy. To calculate the current, we first apply a gauge transformation [S10]

$$\hat{U}(\tau) = \exp\left\{\frac{i}{\hbar}\int_0^\tau d\tau'[\mu_{\rm L}(\tau')\hat{N}_{\rm L} + \mu_{\rm S}(\tau')\hat{N}_{\rm S}]\right\}, \tag{S-36}$$

to the Hamiltonian $\hat{H}$ in Eq. (S-34), where $\hat{N}_{\rm L} = \hat{c}^\dagger_{\rm L}\hat{c}_{\rm L}$ and $N_{\rm S} = \hat{d}^\dagger_{\rm A}\hat{d}_{\rm A}$ are the lead and substrate electron number, respectively, with $\hat{d}_{\rm A} = u\hat{\gamma} + v\hat{\gamma}^\dagger$. With this transformation, the single-particle energies in the lead and substrate are measured from the chemical potential of the lead ($\mu_{\rm L}$) and substrate ($\mu_{\rm S}$), respectively, where the transformed Hamiltonian is

$$\begin{aligned}\hat{\hat{H}} &= \hat{U}\hat{H}\hat{U}^\dagger - i\hbar\hat{U}\partial_\tau\hat{U}^\dagger \\ &= (\hat{H}_{\rm L} - \mu_{\rm L}\hat{N}_{\rm L}) + (\hat{H}_{\rm A} - \mu_{\rm S}\hat{N}_{\rm S}) + \hat{U}\hat{H}_{\rm T}\hat{U}^\dagger,\end{aligned} \tag{S-37}$$

with the tunneling Hamiltonian transformed as

$$\begin{aligned}\hat{\hat{H}}_{\rm T} &= \hat{U}\hat{H}_{\rm T}\hat{U}^\dagger \\ &= te^{ieV\tau/\hbar}\hat{c}^\dagger_{\rm L}\hat{d}_{\rm A} + {\rm H.c.},\end{aligned} \tag{S-38}$$

where $eV = \mu_{\rm L} - \mu_{\rm S}$.

The current operator is given by

$$\hat{I} = e\dot{\hat{N}}_{\rm L} = ie[\hat{N}_{\rm L}, \hat{\hat{H}}_{\rm T}] = i\frac{e}{\hbar}\left(te^{ieV\tau/\hbar}\hat{c}^\dagger_{\rm L}\hat{d}_{\rm A} - {\rm H.c.}\right). \tag{S-39}$$

By taking the expectation value of the current operator, we have

$$I(\tau) = \frac{e}{2h}{\rm Tr}\left\{\sigma_z[\grave{t}(\tau)G^<_{\rm AL}(\tau,\tau) - G^<_{\rm LA}(\tau,\tau)\grave{t}^\dagger(\tau)]\right\}, \tag{S-40}$$

where σ_z is the z-Pauli Matrix in the Nambu basis. In Eq. (S-40), we have introduced the hopping matrix

$$\grave{t}(\tau) = e^{ieV\tau/\hbar\sigma_z}\check{t} = t\begin{pmatrix} ue^{ieV\tau/\hbar} & ve^{ieV\tau/\hbar} \\ -v^*e^{-ieV\tau/\hbar} & -u^*e^{-ieV\tau/\hbar}\end{pmatrix}, \tag{S-41}$$

and the lesser Green's function in the Nambu space $[(G^<_{\alpha\beta})_{ij} = i\langle\Psi^\dagger_{\beta j}\Psi_{\alpha i}\rangle]$ with $i,j = \mathrm{L},\mathrm{A}$ denoting the quantities for the lead and ABS, respectively, where $\Psi_\mathrm{L} = (c_\mathrm{L}, c^\dagger_\mathrm{L})^T$ and $\Psi_\mathrm{A} = (\hat{\gamma}, \hat{\gamma}^\dagger)^T$. We can Fourier-expand the current and Green's functions in terms of the frequency $\omega_0 = eV/\hbar$, where we have

$$I(\tau) = \sum_n I_n e^{in\omega_0\tau}, \tag{S-42a}$$

$$G(\tau_1,\tau_2) = \sum_n e^{in\omega_0\tau_2}\int_{-\infty}^{\infty}\frac{d\omega}{2\pi}e^{-i\omega(\tau_1-\tau_2)}G(\omega,\omega+n\omega_0). \tag{S-42b}$$

Let us denote $G_{mn}(\omega) \equiv G(\omega + m\omega_0, \omega + n\omega_0)$ for which we have $G_{mn}(\omega) = G_{m-n,0}(\omega + n\omega_0)$.

The dc current which is the zeroth order (I_0) in the Fourier expansion of the current [Eq. (S-42a)] is given by

$$\begin{aligned} I_0 = \frac{e}{2h}t\int d\omega \Big(& ue^{i\omega_0\tau}G^{<,ee}_{\mathrm{AL},10} + ve^{i\omega_0\tau}G^{<,he}_{\mathrm{AL},10} + v^*e^{-i\omega_0\tau}G^{<,eh}_{\mathrm{AL},-1,0} + u^*e^{-i\omega_0\tau}G^{<,hh}_{\mathrm{AL},-1,0} \\ & -u^*e^{-i\omega_0\tau}G^{<,ee}_{\mathrm{LA},01} - v^*e^{-i\omega_0\tau}G^{<,eh}_{\mathrm{LA},01} - ve^{i\omega_0\tau}G^{<,he}_{\mathrm{LA},0,-1} - ue^{i\omega_0\tau}G^{<,hh}_{\mathrm{LA},0,-1}\Big), \end{aligned} \tag{S-43}$$

where the superscripts ee, eh, he, and hh denote the matrix elements in the Nambu space. Using the Langreth rule [S7]

$$G^<_\mathrm{AL} = G^r_\mathrm{A}\grave{t}^\dagger g^<_\mathrm{L} + G^<_\mathrm{A}\grave{t}^\dagger g^a_\mathrm{L}, \tag{S-44a}$$

$$G^<_\mathrm{LA} = g^r_\mathrm{L}\grave{t}G^<_\mathrm{A} + g^<_\mathrm{L}\grave{t}G^a_\mathrm{A}, \tag{S-44b}$$

where $g_\mathrm{L} = \mathrm{diag}\left(g^{ee}_\mathrm{L}, g^{hh}_\mathrm{L}\right)$, we have

$$G^{<,ee}_{\mathrm{AL},10} = t\left[\left(G^{r,ee}_{\mathrm{A},11}u^* + G^{r,eh}_{\mathrm{A},11}v^*\right)e^{-i\omega_0\tau}g^{<,ee}_{\mathrm{L},00} + \left(G^{<,ee}_{\mathrm{A},11}u^* + G^{<,eh}_{\mathrm{A},11}v^*\right)e^{-i\omega_0\tau}g^{a,ee}_{\mathrm{L},00}\right], \tag{S-45a}$$

$$G^{<,hh}_{\mathrm{AL},-1,0} = -t\left[\left(G^{r,he}_{\mathrm{A},-1,-1}v + G^{r,hh}_{\mathrm{A},-1,-1}u\right)e^{i\omega_0\tau}g^{<,hh}_{\mathrm{L},00} + \left(G^{<,he}_{\mathrm{A},-1,-1}v + G^{<,hh}_{\mathrm{A},-1,-1}u\right)e^{i\omega_0\tau}g^{a,hh}_{\mathrm{L},00}\right], \tag{S-45b}$$

$$G^{<,he}_{\mathrm{AL},10} = t\left[\left(G^{r,he}_{\mathrm{A},11}u^* + G^{r,hh}_{\mathrm{A},11}v^*\right)e^{-i\omega_0\tau}g^{<,ee}_{\mathrm{L},00} + \left(G^{<,he}_{\mathrm{A},11}u^* + G^{<,hh}_{\mathrm{A},11}v^*\right)e^{-i\omega_0\tau}g^{a,ee}_{\mathrm{L},00}\right], \tag{S-45c}$$

$$G^{<,eh}_{\mathrm{AL},-1,0} = -t\left[\left(G^{r,ee}_{\mathrm{A},-1,-1}v + G^{r,eh}_{\mathrm{A},-1,-1}u\right)e^{i\omega_0\tau}g^{<,hh}_{\mathrm{L},00} + \left(G^{<,ee}_{\mathrm{A},-1,-1}v + G^{<,eh}_{\mathrm{A},-1,-1}u\right)e^{i\omega_0\tau}g^{a,hh}_{\mathrm{L},00}\right], \tag{S-45d}$$

$$G^{<,ee}_{\mathrm{LA},01} = t\left[g^{<,ee}_{\mathrm{L},00}e^{i\omega_0\tau}\left(uG^{a,ee}_{\mathrm{A},11} + vG^{a,he}_{\mathrm{A},11}\right) + g^{r,ee}_{\mathrm{L},00}e^{i\omega_0\tau}\left(uG^{<,ee}_{\mathrm{A},11} + vG^{<,he}_{\mathrm{A},11}\right)\right], \tag{S-45e}$$

$$G^{<,hh}_{\mathrm{LA},0,-1} = -t\left[g^{<,hh}_{\mathrm{L},00}e^{-i\omega_0\tau}\left(v^*G^{a,eh}_{\mathrm{A},-1,-1} + u^*G^{a,hh}_{\mathrm{A},-1,-1}\right) + g^{r,hh}_{\mathrm{L},00}e^{-i\omega_0\tau}\left(v^*G^{<,eh}_{\mathrm{A},-1,-1} + u^*G^{<,hh}_{\mathrm{A},-1,-1}\right)\right], \tag{S-45f}$$

$$G^{<,he}_{\mathrm{LA},0,-1} = -t\left[g^{<,hh}_{\mathrm{L},00}e^{-i\omega_0\tau}\left(v^*G^{a,ee}_{\mathrm{A},-1,-1} + u^*G^{a,he}_{\mathrm{A},-1,-1}\right) + g^{r,hh}_{\mathrm{L},00}e^{-i\omega_0\tau}\left(v^*G^{<,ee}_{\mathrm{A},-1,-1} + u^*G^{<,he}_{\mathrm{A},-1,-1}\right)\right], \tag{S-45g}$$

$$G^{<,eh}_{\mathrm{LA},01} = t\left[g^{<,ee}_{\mathrm{L},00}e^{i\omega_0\tau}\left(uG^{a,eh}_{\mathrm{A},11} + vG^{a,hh}_{\mathrm{A},11}\right) + g^{r,ee}_{\mathrm{L},00}e^{i\omega_0\tau}\left(uG^{<,eh}_{\mathrm{A},11} + vG^{<,hh}_{\mathrm{A},11}\right)\right]. \tag{S-45h}$$

Substituting Eq. (S-45) into Eq. (S-43) and using $g^{ee}_{\mathrm{L},00} = g^{hh}_{\mathrm{L},00} = g_\mathrm{L}$, we have

$$\begin{aligned} I_0 = \frac{e}{2h}t^2\int d\omega \Big\{ & \Big[|u|^2(G^{r,ee}_\mathrm{A}(\omega_+) - G^{a,ee}_\mathrm{A}(\omega_+)) + uv^*(G^{r,eh}_\mathrm{A}(\omega_+) - G^{a,eh}_\mathrm{A}(\omega_+)) \\ & \qquad + u^*v(G^{r,he}_\mathrm{A}(\omega_+) - G^{a,he}_\mathrm{A}(\omega_+)) + |v|^2(G^{r,hh}_\mathrm{A}(\omega_+) - G^{a,hh}_\mathrm{A}(\omega_+))\Big]g^<_\mathrm{L}(\omega) \\ & - \Big[|v|^2(G^{r,ee}_\mathrm{A}(\omega_-) - G^{a,ee}_\mathrm{A}(\omega_-)) + uv^*(G^{r,eh}_\mathrm{A}(\omega_-) - G^{a,eh}_\mathrm{A}(\omega_-)) \\ & \qquad + u^*v(G^{r,he}_\mathrm{A}(\omega_-) - G^{a,he}_\mathrm{A}(\omega_-)) + |u|^2(G^{r,hh}_\mathrm{A}(\omega_-) - G^{a,hh}_\mathrm{A}(\omega_-))\Big]g^<_\mathrm{L}(\omega) \\ & + \Big[|u|^2G^{<,ee}_\mathrm{A}(\omega_+) + uv^*G^{<,eh}_\mathrm{A}(\omega_+) + u^*vG^{<,he}_\mathrm{A}(\omega_+) + |v|^2G^{<,hh}_\mathrm{A}(\omega_+)\Big]\left[g^a_\mathrm{L}(\omega) - g^r_\mathrm{L}(\omega)\right] \\ & - \Big[|v|^2G^{<,ee}_\mathrm{A}(\omega_-) + uv^*G^{<,eh}_\mathrm{A}(\omega_-) + u^*vG^{<,he}_\mathrm{A}(\omega_-) + |u|^2G^{<,hh}_\mathrm{A}(\omega_-)\Big]\left[g^a_\mathrm{L}(\omega) - g^r_\mathrm{L}(\omega)\right]\Big\}. \end{aligned} \tag{S-46}$$

Furthermore, by using the relation $G^< - G^> = G^a - G^r$, we obtain

$$I_0 = \frac{e}{2h}t^2 \int d\omega \Big\{ \Big[|u|^2 G_{\rm A}^{>,ee}(\omega) + uv^* G_{\rm A}^{>,eh}(\omega) + u^* v G_{\rm A}^{>,he}(\omega) + |v|^2 G_{\rm A}^{>,hh}(\omega)\Big] g_{\rm L}^<(\omega_-)$$
$$- \Big[|v|^2 G_{\rm A}^{>,ee}(\omega) + uv^* G_{\rm A}^{>,eh}(\omega) + u^* v G_{\rm A}^{>,he}(\omega) + |u|^2 G_{\rm A}^{>,hh}(\omega)\Big] g_{\rm L}^<(\omega_+)$$
$$- \Big[|u|^2 G_{\rm A}^{<,ee}(\omega) + uv^* G_{\rm A}^{<,eh}(\omega) + u^* v G_{\rm A}^{<,he}(\omega) + |v|^2 G_{\rm A}^{<,hh}(\omega)\Big] g_{\rm L}^>(\omega_-)$$
$$+ \Big[|v|^2 G_{\rm A}^{<,ee}(\omega) + uv^* G_{\rm A}^{<,eh}(\omega) + u^* v G_{\rm A}^{<,he}(\omega) + |u|^2 G_{\rm A}^{<,hh}(\omega)\Big] g_{\rm L}^>(\omega_+)\Big\}. \tag{S-47}$$

The current can be written more compactly as

$$I = \frac{e}{2h}\int d\omega {\rm Tr}\left[G_{\rm A}^>(\omega)\widetilde{\Sigma}_{\rm A}^<(\omega) - G_{\rm A}^<(\omega)\widetilde{\Sigma}_{\rm A}^>(\omega)\right], \tag{S-48}$$

where

$$\begin{aligned}\widetilde{\Sigma}_{\rm A}^{<,>}(\omega) &= \check{t}^\dagger \begin{pmatrix} g_{\rm L}^{<,>}(\omega_-) & 0 \\ 0 & -g_{\rm L}^{<,>}(\omega_+)\end{pmatrix}\check{t} \\ &= t^2 \begin{pmatrix} u^* & -v \\ v^* & -u\end{pmatrix}\begin{pmatrix} g_{\rm L}^{<,>}(\omega_-) & 0 \\ 0 & -g_{\rm L}^{<,>}(\omega_+)\end{pmatrix}\begin{pmatrix} u & v \\ -v^* & -u^*\end{pmatrix} \\ &= t^2 \begin{pmatrix} |u|^2 g_{\rm L}^{<,>}(\omega_-) - |v|^2 g_{\rm L}^{<,>}(\omega_+) & vu^*\left[g_{\rm L}^{<,>}(\omega_-) - g_{\rm L}^{<,>}(\omega_+)\right] \\ uv^*\left[g_{\rm L}^{<,>}(\omega_-) - g_{\rm L}^{<,>}(\omega_+)\right] & |v|^2 g_{\rm L}^{<,>}(\omega_-) - |u|^2 g_{\rm L}^{<,>}(\omega_+)\end{pmatrix},\end{aligned} \tag{S-49}$$

with $g_{\rm L}^<(\omega) = f(\omega)(g_{\rm L}^a(\omega) - g_{\rm L}^r(\omega))$, $g_{\rm L}^>(\omega) = -(1-f(\omega))(g_{\rm L}^a(\omega) - g_{\rm L}^r(\omega))$ and $g_{\rm L}^r(\omega_-) = g_{\rm L}^r(\omega_+) \equiv -i\pi\nu_0$, where where ν_0 is the density of states at the lead Fermi energy. Substituting the expressions for $\widetilde{\Sigma}_{\rm A}^{<,>}(\omega)$ [Eqs. (S-49)], $G_{\rm A}^<(\omega)$ [Eqs. (S-57)], and the corresponding equations for $G_{\rm A}^>(\omega)$ into Eq. (S-48), we obtain Eq. (8) of the main text:

$$I(V) = \frac{e}{h}\Gamma^2 \int d\omega \mathcal{A}(\omega)[f(\omega_-) - f(\omega_+)], \tag{S-50}$$

where

$$\mathcal{A}(\omega) = \frac{4|uv|^2}{\left[\omega - \frac{(\varepsilon_{\rm A} + \sqrt{2}\lambda\langle\hat{x}\rangle)^2}{\omega} - \frac{(\Gamma_u - \Gamma_v)^2}{4\omega}\right]^2 + (\Gamma_u + \Gamma_v)^2}, \tag{S-51}$$

with $\Gamma_u = \Gamma|u|^2$ and $\Gamma_v = \Gamma|v|^2$. Note that the term $\langle\hat{x}\rangle$ is evaluated self-consistently using

$$\langle\hat{x}\rangle = -\frac{\sqrt{2}\lambda}{\Omega}\langle\hat{\gamma}^\dagger(\tau)\hat{\gamma}(\tau)\rangle = i\frac{\sqrt{2}\lambda}{\Omega}\int\frac{d\omega}{2\pi}G_{\rm A}^{<,ee}(\omega), \tag{S-52}$$

or Eq. (10) of the main text:

$$\langle\hat{x}\rangle = -\frac{\sqrt{2}\lambda}{\Omega}\langle\hat{\gamma}^\dagger(\tau)\hat{\gamma}(\tau)\rangle = -\frac{\lambda}{\sqrt{2}\Omega}\left(1 + \langle\hat{\gamma}^\dagger(\tau)\hat{\gamma}(\tau)\rangle - \langle\hat{\gamma}(\tau)\hat{\gamma}^\dagger(\tau)\rangle\right) = -\frac{\lambda}{\sqrt{2}\Omega}\left\{1 - i\int\frac{d\omega}{2\pi}{\rm Tr}\left[G_{\rm A}^<(\omega)\sigma_z\right]\right\}. \tag{S-53}$$

VIII. EXPLICIT EXPRESSIONS FOR $G_{\rm A}^{<,>}(\omega)$

In this section, we evaluate the expressions for the ABS lesser and greater Green's functions $G_{\rm A}^{<,>}(\omega)$ which are used to calculate the current [Eq. (8) of the main text] and the expectation value of the boson displacement operator $\langle\hat{x}\rangle$ [Eq. (10) of the main text]. Using the Fourier expansion as in Eq. (S-42b), we can relate the ABS lesser and greater Green's function in the frequency domain $G_{\rm A}^{<,>}(\omega)$ to their time-domain counterparts [S7], i.e.,

$$G_{\rm A}^<(\tau_1,\tau_2) = \begin{pmatrix} G_{\rm A}^{<,ee}(\tau_1,\tau_2) & G_{\rm A}^{<,eh}(\tau_1,\tau_2) \\ G_{\rm A}^{<,he}(\tau_1,\tau_2) & G_{\rm A}^{<,hh}(\tau_1,\tau_2)\end{pmatrix} \equiv i\begin{pmatrix}\langle\gamma^\dagger(\tau_2)\gamma(\tau_1)\rangle & \langle\gamma(\tau_2)\gamma(\tau_1)\rangle \\ \langle\gamma^\dagger(\tau_2)\gamma^\dagger(\tau_1)\rangle & \langle\gamma(\tau_2)\gamma^\dagger(\tau_1)\rangle\end{pmatrix}, \tag{S-54a}$$

$$G_{\rm A}^>(\tau_1,\tau_2) = \begin{pmatrix} G_{\rm A}^{>,ee}(\tau_1,\tau_2) & G_{\rm A}^{>,eh}(\tau_1,\tau_2) \\ G_{\rm A}^{>,he}(\tau_1,\tau_2) & G_{\rm A}^{>,hh}(\tau_1,\tau_2)\end{pmatrix} \equiv -i\begin{pmatrix}\langle\gamma(\tau_1)\gamma^\dagger(\tau_2)\rangle & \langle\gamma^\dagger(\tau_1)\gamma^\dagger(\tau_2)\rangle \\ \langle\gamma(\tau_1)\gamma(\tau_2)\rangle & \langle\gamma^\dagger(\tau_1)\gamma(\tau_2)\rangle\end{pmatrix}. \tag{S-54b}$$

To evaluate $G_{\rm A}^{<,>}(\omega)$, we begin by writing the ABS Green's function in the Lehmann representation as

$$g_{\rm A}(\omega) = \frac{\Phi_+\Phi_+^\dagger}{\omega - (\varepsilon_{\rm A} + \sqrt{2}\lambda\langle\hat{x}\rangle)} + \frac{\Phi_-\Phi_-^\dagger}{\omega + (\varepsilon_{\rm A} + \sqrt{2}\lambda\langle\hat{x}\rangle)}. \tag{S-55}$$

where $\Phi_+ = (1,0)^T$ and $\Phi_- = (0,1)^T$ are the positive- and negative-energy eigenfunction of the ABS written in the Nambu basis $(\hat{\gamma}, \hat{\gamma}^\dagger)^T$. The ABS's self energy due to the lead coupling is $\Sigma_{\rm A}^r(\omega) = \check{t}^\dagger \mathrm{diag}(g_L^r(\omega_-), g_L^r(\omega_+))\check{t}$ where $\check{t} = t\begin{pmatrix} u & v \\ -v^* & -u^* \end{pmatrix}$ is the hopping matrix, and $g_{\rm L}^r(\omega_-) = g_{\rm L}^r(\omega_+) = -i\pi\nu_0$ is the lead retarded Green's function with $\omega_\pm = \omega \pm eV$. Similar relations apply for $\Sigma_{\rm A}^{a,<,>}(\omega)$. The ABS lesser Green's function is [S7]

$$\begin{aligned} G_{\rm A}^<(\omega) &= g_{\rm A}^<(\omega) + g_{\rm A}^r(\omega)\Sigma_{\rm A}^r(\omega)G_{\rm A}^<(\omega) + \left[g_{\rm A}^r(\omega)\Sigma_{\rm A}^<(\omega) + g_{\rm A}^<(\omega)\Sigma_{\rm A}^a(\omega)\right]G_{\rm A}^a(\omega) \\ &= \frac{1}{1 - g_{\rm A}^r(\omega)\Sigma_{\rm A}^r(\omega)}\left[g_{\rm A}^<(\omega)\left(1 + \Sigma_{\rm A}^a(\omega)G_{\rm A}^a(\omega)\right) + g_{\rm A}^r(\omega)\Sigma_{\rm A}^<(\omega)G_{\rm A}^a(\omega)\right], \end{aligned} \tag{S-56}$$

where $G_{\rm A}^{r,a} = g_{\rm A}^{r,a}(1 - g_{\rm A}^{r,a}\Sigma_{\rm A}^{r,a})^{-1}$, $g_j^<(\omega) = f(\omega)(g_j^a - g_j^r)$ and $g_j^>(\omega) = -(1 - f(\omega))(g_j^a - g_j^r)$ with $j = $ L, A. The explicit expressions of the matrix elements of $G_{\rm A}^<(\omega) = \begin{pmatrix} G_{\rm A}^{<,ee}(\omega) & G_{\rm A}^{<,eh}(\omega) \\ G_{\rm A}^{<,he}(\omega) & G_{\rm A}^{<,hh}(\omega) \end{pmatrix}$ can be evaluated as

$$G_{\rm A}^{<,ee}(\omega) = \frac{i}{D}\left\{\left[\Gamma_u f(\omega_-) + \Gamma_v f(\omega_+)\right]\left[\left(\omega + \varepsilon_{\rm A} + \sqrt{2}\lambda\langle\hat{x}\rangle\right)^2 + \left(\frac{\Gamma_u - \Gamma_v}{2}\right)^2\right]\right\}, \tag{S-57a}$$

$$G_{\rm A}^{<,eh}(\omega) = \frac{i}{D}\left\{\Gamma u^* v\left[\left(f(\omega_+) + f(\omega_-)\right)\left(\omega^2 - (\varepsilon_{\rm A} + \sqrt{2}\lambda\langle\hat{x}\rangle)^2 - \left(\frac{\Gamma_u - \Gamma_v}{2}\right)^2\right) + i\omega\left(f(\omega_+) - f(\omega_-)\right)\left(\Gamma_u - \Gamma_v\right)\right]\right\}, \tag{S-57b}$$

$$G_{\rm A}^{<,he}(\omega) = \frac{i}{D}\left\{\Gamma u v^*\left[\left(f(\omega_+) + f(\omega_-)\right)\left(\omega^2 - (\varepsilon_{\rm A} + \sqrt{2}\lambda\langle\hat{x}\rangle)^2 - \left(\frac{\Gamma_u - \Gamma_v}{2}\right)^2\right) - i\omega\left(f(\omega_+) - f(\omega_-)\right)\left(\Gamma_u - \Gamma_v\right)\right]\right\}, \tag{S-57c}$$

$$G_{\rm A}^{<,hh}(\omega) = \frac{i}{D}\left\{\left[\Gamma_u f(\omega_+) + \Gamma_v f(\omega_-)\right]\left[\left(\omega - (\varepsilon_{\rm A} + \sqrt{2}\lambda\langle\hat{x}\rangle)\right)^2 + \left(\frac{\Gamma_u - \Gamma_v}{2}\right)^2\right]\right\}, \tag{S-57d}$$

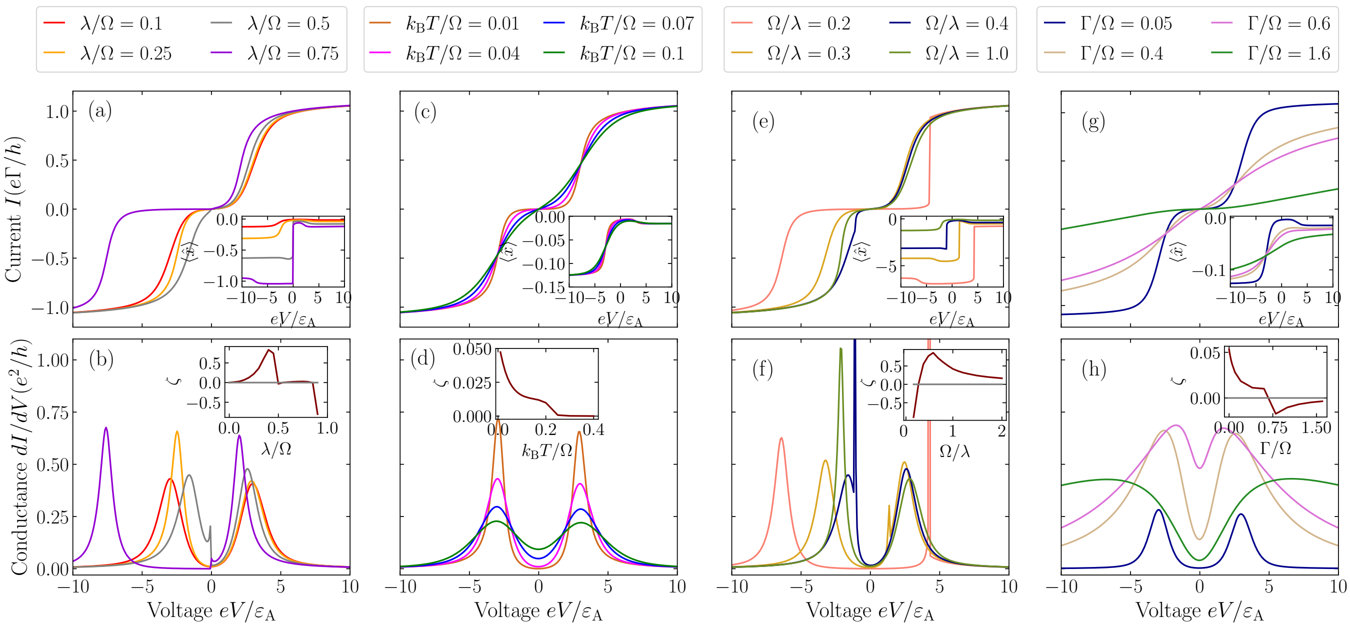
where $\omega_\pm = \omega \pm eV$ and

$$D = \left[\omega^2 - (\varepsilon_{\rm A} + \sqrt{2}\lambda\langle\hat{x}\rangle)^2 - \frac{(\Gamma_u - \Gamma_v)^2}{4}\right]^2 + \omega^2(\Gamma_u + \Gamma_v)^2, \tag{S-58}$$

with $\Gamma_u = \Gamma|u|^2$ and $\Gamma_v = \Gamma|v|^2$. The expressions for the matrix elements of the ABS greater Green's function $G_{\rm A}^>(\omega)$ can be obtained from Eq. (S-57) by using the substitutions: $f(\omega_-) \to f(\omega_-) - 1$ and $f(\omega_+) \to f(\omega_+) - 1$.

IX. DEPENDENCE OF THE CURRENT AND CONDUCTANCE CALCULATED FROM KELDYSH APPROACH ON ABS-BOSON COUPLING STRENGTH, TEMPERATURE, BOSON FREQUENCY, AND LEAD-TUNNEL COUPLING STRENGTH

Figure S4 shows the current (upper panels) and conductance (lower panels) of boson-coupled ABSs calculated from the Keldysh approach [Eq. (8) of the main text] subject to the self-consistency condition [Eq. (10) of the main text]. The plots are shown for different ABS-boson coupling strengths λ [Figs. S4(a,b)], temperatures T [Figs. S4(c,d)], boson frequencies Ω [Figs. S4(e,f)] and lead-tunnel coupling strengths Γ [Figs. S4(g,h)]. While in Fig. 3 of the main text, we have shown that the PHS breaking holds for the case of low-frequency bosons, here we will show that it also holds for the case of high-frequency bosons, i.e., $\Omega > 2\varepsilon_{\rm A} + k_{\rm B}T$. Unlike the perturbative calculation in the rate equation, the PHS breaking calculated from the Keldysh approach arises due to non-perturbative effects of tunneling, i.e., the PH asymmetry of the mean-field boson displacement value $\langle\hat{x}\rangle$. In the non-perturbative regime, electrons can tunnel from the lead into virtual states in the superconductor by emitting or absorbing bosons with high frequencies where energy violation is allowed for sufficiently large tunnel coupling ($\Gamma \gtrsim \Omega$), resulting in PHS breaking of subgap



Supplemental Figure S4: Current I (Upper panels) and conductance dI/dV (Lower panels) of boson-coupled ABSs vs voltage V calculated using the Keldysh approach [Eq. (8) of the main text] subject to the self-consistency condition [Eq. (10) of the main text]. The plots are for (a,b) different ABS-boson coupling strengths λ with $k_\mathrm{B}T = 0.4$, $\varepsilon_\mathrm{A} = 3.0$, and $\Gamma = 1.0$, (c,d) different temperatures T with $\lambda = 1.0$, $\varepsilon_\mathrm{A} = 3.0$, and $\Gamma = 1.0$, (e,f) different boson frequencies Ω with $k_\mathrm{B}T = 0.4$, $\varepsilon_\mathrm{A} = 3.0$, $\lambda = 1.0$ and $\Gamma = 1.0$, and (g,h) different lead-tunnel coupling strengths Γ with $k_\mathrm{B}T = 0.4$, $\lambda = 1.0$ and $\varepsilon_\mathrm{A} = 3.0$. Inset in upper panels: (a) Mean-field boson displacement value $\langle \hat{x} \rangle$ vs λ/Ω, (c) $\langle \hat{x} \rangle$ vs temperature T, (e) $\langle \hat{x} \rangle$ vs boson frequency Ω, and (g) $\langle \hat{x} \rangle$ vs Γ/Ω. Inset in lower panels: (b) Conductance PH asymmetry ζ vs λ/Ω, (d) ζ vs temperature T, (f) ζ vs boson frequency Ω, and (h) ζ vs Γ/Ω. The parameters used for all panels are: $|u|^2/|v|^2 = 1/9$. Note that for panels (a,b,c,d,g,h), we use high-frequency bosons ($\Omega = 10$) where $\Omega > 2\varepsilon_\mathrm{A} + k_\mathrm{B}T$.

conductances. This energy violation is allowed as long as the energy violation in the first tunneling process is negated by the second tunneling process which the conserves the total energy of a full cycle of transferring a pair of electrons in the two-step tunneling process.

Figure S4(b) shows that the magnitude of the conductance PH asymmetry ζ has a nonmonotonic dependence on the ABS-boson coupling strength λ. The conductance PH asymmetry ζ first increases with increasing λ where the two peaks approach each other until they reach a certain minimum distance. Note that for this range of λ, the higher peak is at positive voltage for the case where $|u|^2 > |v|^2$ while for the case where $|v|^2 > |u|^2$, the higher peak is at negative voltage. After the PH asymmetry reaches a maximum, it decreases to zero and stays there for a range of λ where the peaks remain more or less at the same place. As λ increases and reaches a certain value, the high and low peaks switch positions, i.e., from negative to positive voltage and vice versa. As λ keeps increasing, the two peaks move away from each other and the magnitude of the PH asymmetry increases. For large enough λ, the positions of the conductance peaks are no longer PH symmetric [see purple curve in Fig. S4(b)]. Note that the results for large λ may not be reliable as our mean-field treatment of interactions may break down in this regime.

Figure S4(d) shows that the conductance PH asymmetry ζ decreases with increasing temperature T due the temperature broadening of the conductance peaks. The dependence of the ABS conductance on the boson frequency Ω is shown in Fig. S4(f). The PH asymmetry ζ has a nonmonotonic behavior with the boson frequency Ω where its magnitude first decreases to zero with increasing Ω. This corresponds to the two conductance peaks moving towards each other as Ω increases which is due to the decrease in the effective ABS-boson coupling strength λ/Ω. After the PH asymmetry reaches zero, it switches sign which corresponds to the high and low peaks switching sides. As Ω increases, the two peaks move towards each other and the PH asymmetry increases to a certain maximum value. Having reached its maximum, the PH asymmetry ζ decreases with increasing Ω which corresponds to the decrease in the effective ABS-boson coupling strength λ/Ω. Figure S4(h) shows the dependence of the conductance on the lead tunnel coupling Γ. As shown in the inset of panel (h), the PH asymmetry ζ has a non-monotonic dependence on the lead-tunnel coupling Γ where it first decreases as Γ increases. After the PH asymmetry reaches zero, it changes sign and increases in magnitude to a certain maximum value as Γ increases. Having reached its maximum, the PH asymmetry then decreases as Γ increases. Note that unlike the rate equation, our mean-field Keldysh approach shows that the conductance in the tunneling limit ($\Gamma/\Omega \ll 1$) still exhibits PH asymmetry even for high-frequency bosons. Since the treatment of interactions within the rate equation is exact in the tunneling limit, our Keldysh results obtained using the mean-field treatment of interactions may not be correct in this tunneling limit. This is because the mean-field approximation breaks down in this limit due to the singularity in the tunneling density of states. For the case where the tunnel coupling is not too small, the mean-field approximation is valid and we can see from Fig. S4(d) that unlike the rate equation, that subgap conductance calculated from the Keldysh approach can still be PH asymmetric for high-frequency boson case. Finally we note that since we ignore the Fock term in the mean-field approximation, the conductance calculated from the Keldysh approach has no boson sidebands.

X. DETAILS ON MODEL II. BOSON-ASSISTED TUNNELING MODEL INTO ABS

In this section, we consider a boson-assisted tunneling Hamiltonian of the form

$$\hat{H}_{\mathrm{T}} = t(\hat{b} + \hat{b}^\dagger)\hat{c}_{\mathrm{L}}^\dagger \hat{d}_{\mathrm{A}} + \mathrm{H.c.} \tag{S-59}$$

This tunneling Hamiltonian can be obtained by first projecting the microscopic Hamiltonian [Eq. (S-6)] onto the lowest and second-lowest energy sector $\alpha, \beta = 1, 2$ and followed by integrating out the second-lowest Bogoliubov operator γ_2 from the total Hamiltonian of the system.

Projecting the ABS and tunneling Hamiltonian onto the lowest and second-lowest energy state gives

$$\hat{H}_{\mathrm{A}} = \sum_{\alpha=1}^{2} \varepsilon_\alpha \gamma_\alpha^\dagger \gamma_\alpha + \sum_{\alpha,\beta=1}^{2} \left(\tilde{\lambda}^{(c)}_{\alpha\beta}\gamma_\alpha^\dagger\gamma_\beta + \tilde{\lambda}^{(d)}_{\alpha\beta}\gamma_\alpha\gamma_\beta + \mathrm{H.c.}\right)(\hat{b}^\dagger + \hat{b}) + \Omega(\hat{b}^\dagger + \hat{b}), \tag{S-60a}$$

$$\hat{H}_{\mathrm{T}} = \sum_{\alpha=1}^{2} t_\alpha c_{\mathrm{L}}^\dagger \left(\sum_{\beta=1}^{2} u_{\alpha\beta}(0)\hat{\gamma}_\beta + v_{\alpha\beta}(0)\hat{\gamma}_\beta^\dagger\right) + \mathrm{H.c.} \tag{S-60b}$$

For simplicity, we will choose parameters such that the tunneling term into the lowest Bogoliubov (ABS) operator (γ_1 and $\gamma_1^\dagger$) vanishes where we only have the tunneling into the second lowest Bogoliubov operator (γ_2 and $\gamma_2^\dagger$), i.e.,

$$\hat{H}_{\mathrm{T}} = \tilde{t} c_{\mathrm{L}}^\dagger \left(\tilde{u}\hat{\gamma}_2 + \tilde{v}\hat{\gamma}_2^\dagger\right) + \mathrm{H.c.} \tag{S-61}$$

Note that in the above, we have defined $\tilde{t}\tilde{u} \equiv t_2 u_{22}(0) + t_1 u_{12}(0)$, $\tilde{t}\tilde{v} \equiv t_2 v_{22}(0) + t_1 v_{12}(0)$, and we have also chosen parameters such that the tunneling term into the ABS vanishes, i.e., $t_2 v_{21}(0) + t_1 v_{11}(0) = 0$ and $t_2 u_{21}(0) + t_1 u_{11}(0) = 0$. We note that we choose these parameters only for simplicity and our results on PHS breaking hold in general even without this simplification.

In the following, we will see that the tunneling Hamiltonian [Eq. (S-61)] and the electron-boson interaction Hamiltonian [Eq. (S-60a)] give rise to a boson-assisted tunneling into the lowest energy ABS. To this end, we will use the path integral formalism to integrate out γ_2. We begin by writing down the partition function of the system as

$$\begin{aligned} Z &= \int_{-\infty}^{\infty} \prod_{\alpha=1}^{2} \mathcal{D}\bar{\gamma}_\alpha \mathcal{D}\gamma_\alpha \mathcal{D}\bar{c}_{\mathrm{L},k}\mathcal{D}c_{\mathrm{L},k}\mathcal{D}\bar{b}\mathcal{D}b\Big(\exp[i(S_\alpha + S_{\mathrm{b}} + S_{\mathrm{L}} + S_{\text{e-b}} + S_{\mathrm{T}})]\Big) \\ &= \int_{-\infty}^{\infty} \prod_{\alpha=1}^{2} \mathcal{D}\bar{\gamma}_\alpha \mathcal{D}\gamma_\alpha \mathcal{D}\bar{c}_{\mathrm{L},k}\mathcal{D}c_{\mathrm{L},k}\mathcal{D}\bar{b}\mathcal{D}b\Bigg(\exp\Bigg[i\int_{-\infty}^{\infty} d\tau \Bigg(\hbar\bar{\gamma}_\alpha\partial_\tau\gamma_\alpha + \hbar\bar{b}\partial_\tau b + \hbar\bar{c}_{\mathrm{L},k}\partial_\tau c_{\mathrm{L},k} - \varepsilon_\alpha\bar{\gamma}_\alpha\gamma_\alpha - \Omega\bar{b}b \\ &\qquad - \sum_k \varepsilon_{\mathrm{L},k}\bar{c}_{\mathrm{L},k}c_{\mathrm{L},k} - \sum_{\beta=1,2}(\tilde{\lambda}^{(c)}_{\alpha\beta}\bar{\gamma}_\alpha\gamma_\beta + \tilde{\lambda}^{(d)}_{\alpha\beta}\gamma_\alpha\gamma_\beta + \mathrm{h.c.})(\bar{b} + b) - \tilde{t}(\tilde{u}\bar{c}_{\mathrm{L}}\gamma_2 + \tilde{v}\bar{c}_{\mathrm{L}}\bar{\gamma}_2 + \mathrm{H.c.})\Bigg)\Bigg]\Bigg), \end{aligned} \tag{S-62}$$

where the actions are given by

$$S_\alpha = \int d\tau \bar{\gamma}_\alpha(\tau)(\hbar\partial_\tau - \varepsilon_\alpha)\gamma_\alpha(\tau), \tag{S-63a}$$

$$S_{\rm b} = \int d\tau \bar{b}(\tau)(\hbar\partial_\tau - \Omega)b(\tau), \tag{S-63b}$$

$$S_{\text{e-b}} = -\int d\tau \sum_{\alpha,\beta=1}^{2} (\tilde{\lambda}^{(c)}_{\alpha\beta}\bar{\gamma}_\alpha(\tau)\gamma_\beta(\tau) + \tilde{\lambda}^{(d)}_{\alpha\beta}\gamma_\alpha(\tau)\gamma_\beta(\tau) + \text{H.c.})(\bar{b}(\tau) + b(\tau)), \tag{S-63c}$$

$$S_{\rm L} = \sum_k \int d\tau \bar{c}_{{\rm L},k}(\tau)(\hbar\partial_\tau - \varepsilon_{{\rm L},k})c_{{\rm L},k}(\tau), \tag{S-63d}$$

$$S_{\rm T} = -\int d\tau \Big[\tilde{t}(\tilde{u}\bar{c}_{\rm L}(\tau)\gamma_2(\tau) + \tilde{v}\bar{c}_{\rm L}(\tau)\bar{\gamma}_2(\tau)) + \text{H.c.}\Big]. \tag{S-63e}$$

We assume $\lambda_{22} \ll \varepsilon_2$, so that we can integrate out γ_2 and $\bar{\gamma}_2$ by using the Gaussian integral

$$\begin{aligned}
&\int_{-\infty}^{\infty} \mathcal{D}\gamma_2\mathcal{D}\bar{\gamma}_2 \exp\Big\{i\varepsilon_2\bar{\gamma}_2\gamma_2 - i\Big[\big(\tilde{\lambda}^{(c)}_{12}\bar{\gamma}_1 + \tilde{\lambda}^{(d)}_{12}\gamma_1 + \tilde{t}(\tilde{u}\bar{c}_{\rm L} + \tilde{v}^* c_{\rm L})\big)\gamma_2(b+\bar{b}) + \text{H.c.}\Big]\Big\} \\
&= \pi \exp\big\{-(b+\bar{b})\big(\tilde{\lambda}^{(c)}_{12}\bar{\gamma}_1(\tau') + \tilde{\lambda}^{(d)}_{12}\gamma_1(\tau') + \tilde{t}(\tilde{u}\bar{c}_{\rm L}(\tau') + \tilde{v}^* c_{\rm L}(\tau'))\big) i(G^{\mathbb{T}}_{22}(\tau-\tau'))^{-1} \\
&\qquad \times (b+\bar{b})\big((\tilde{\lambda}^{(c)}_{12})^*\gamma_1 + (\tilde{\lambda}^{(d)}_{12})^*\bar{\gamma}_1(\tau) + \tilde{t}(\tilde{u}^* c_{\rm L}(\tau) + \tilde{v}\bar{c}_{\rm L}(\tau))\big)\big\},
\end{aligned} \tag{S-64}$$

Substituting Eq. (S-64) into Eq. (S-62) and ignoring the λ_{22} term, we have

$$\begin{aligned}
Z = \text{const} \times &\int_{-\infty}^{\infty} \mathcal{D}\bar{\gamma}_1\mathcal{D}\gamma_1\mathcal{D}\bar{c}_{{\rm L},k}\mathcal{D}c_{{\rm L},k}\mathcal{D}\bar{b}\mathcal{D}b \\
\times \exp\Big\{i&\int_{-\infty}^{\infty} d\tau \Big[\hbar\bar{\gamma}_1\partial_\tau\gamma_1 + \hbar\bar{b}\partial_\tau b + \hbar\bar{c}_{{\rm L},k}\partial_\tau c_{{\rm L},k} - \varepsilon_1\bar{\gamma}_1\gamma_1 - \Omega\bar{b}b - 2\tilde{\lambda}^{(c)}_{11}\bar{\gamma}_1\gamma_1(\bar{b}+b) - \chi(\bar{b}+b) - \sum_k \varepsilon_{{\rm L},k}\bar{c}_{{\rm L},k}c_{{\rm L},k} \\
&- \int_{-\infty}^{\infty} d\tau' \Big((b+\bar{b})\big[\tilde{\lambda}^{(c)}_{12}\bar{\gamma}_1(\tau') + \tilde{\lambda}^{(d)}_{12}\gamma_1(\tau') + \tilde{t}(\tilde{u}\bar{c}_{\rm L}(\tau') + \tilde{v}^* c_{\rm L}(\tau'))\big]\Big) i(G^{\mathbb{T}}_{22}(\tau-\tau'))^{-1} \\
&\qquad \times (b+\bar{b})\big[(\tilde{\lambda}^{(c)}_{12})^*\gamma_1(\tau) + (\tilde{\lambda}^{(d)}_{12})^*\bar{\gamma}_1(\tau) + \tilde{t}(\tilde{u}^* c_{\rm L}(\tau) + \tilde{v}\bar{c}_{\rm L}(\tau))\big]\Big]\Big\},
\end{aligned} \tag{S-65}$$

where $\chi = \sum_{lm}\int dx dx' g_{lm}(x,x')$ and $G^{\mathbb{T}}_{\alpha\alpha}$ is the time-ordered Green's function [S11] defined by

$$iG^{\mathbb{T}}_{\alpha\alpha}(\tau-\tau') \equiv \langle\gamma_\alpha(\tau)\gamma_\alpha(\tau')\rangle = \theta(\tau-\tau')iG^{>}_{\alpha\alpha}(\tau-\tau') + \theta(\tau'-\tau)iG^{<}_{\alpha\alpha}(\tau-\tau'), \tag{S-66}$$

with

$$iG^{<}_{\alpha\alpha}(\tau-\tau') = -f(\varepsilon_\alpha)\exp(-i\varepsilon_\alpha(\tau-\tau')), \tag{S-67a}$$

$$iG^{>}_{\alpha\alpha}(\tau-\tau') = (1-f(\varepsilon_\alpha))\exp(-i\varepsilon_\alpha(\tau-\tau')), \tag{S-67b}$$

and $\theta(\tau)$ being the Heaviside step function. We consider $\varepsilon_2 \gg k_{\rm B}T$, where we have the Fermi function $f(\varepsilon_2) = 1$ which gives $iG_{22}(\tau-\tau') = -\theta(\tau'-\tau)\exp(-i\varepsilon_2(\tau-\tau'))$. Using this, we can then write the partition function as

$$\begin{aligned}
Z = \text{const} \times &\int_{-\infty}^{\infty} \mathcal{D}\bar{\gamma}_1\mathcal{D}\gamma_1\mathcal{D}\bar{c}_{{\rm L},k}\mathcal{D}c_{{\rm L},k}\mathcal{D}\bar{b}\mathcal{D}b \\
\times \exp\Big\{i&\int_{-\infty}^{\infty} d\tau \Big[\hbar\bar{\gamma}_1\partial_\tau\gamma_1 + \hbar\bar{b}\partial_\tau b + \hbar\bar{c}_{\rm L}\partial_\tau c_{\rm L} - \varepsilon_1\bar{\gamma}_1\gamma_1 - \Omega\bar{b}b - 2\lambda^{(2)}_{11}\bar{\gamma}_1\gamma_1(\bar{b}+b) - \chi(\bar{b}+b) - \sum_k \varepsilon_{{\rm L},k}\bar{c}_{{\rm L},k}c_{{\rm L},k}\Big] \\
&+ \int_{-\infty}^{\infty} d\tau \int_{\tau}^{\infty} d\tau' \Big[\Big((b+\bar{b})\big[\tilde{\lambda}^{(d)}_{12}\gamma_1(\tau') + \tilde{\lambda}^{(c)}_{12}\bar{\gamma}_1(\tau') + \tilde{t}(\tilde{u}\bar{c}_{\rm L}(\tau') + \tilde{v}^* c_{\rm L}(\tau'))\big]\Big) \exp(i\varepsilon_2(\tau-\tau')) \\
&\qquad \times (b+\bar{b})\big[(\tilde{\lambda}^{(c)}_{12})^*\gamma_1(\tau) + (\tilde{\lambda}^{(d)}_{12})^*\bar{\gamma}_1(\tau) + \tilde{t}(\tilde{u}^* c_{\rm L}(\tau) + \tilde{v}\bar{c}_{\rm L}(\tau))\big]\Big]\Big\}.
\end{aligned} \tag{S-68}$$

Defining $\tau_1 \equiv (\tau+\tau')/2$ and $\tau_2 \equiv \tau'-\tau$, we have $\int_{-\infty}^{\infty} d\tau \int_{\tau}^{\infty} d\tau' = \int_{-\infty}^{\infty} d\tau_1 \int_0^{\infty} d\tau_2$ and

$$\begin{aligned}
Z = \text{const} \times &\int_{-\infty}^{\infty} \mathcal{D}\bar{\gamma}_1 \mathcal{D}\gamma_1 \mathcal{D}\bar{c}_{\text{L},k} \mathcal{D}c_{\text{L},k} \mathcal{D}\bar{b}\mathcal{D}b \\
\times \exp\Big\{ i \int_{-\infty}^{\infty} d\tau \Big[&\hbar\bar{\gamma}_1\partial_\tau\gamma_1 + \hbar\bar{b}\partial_\tau b + \hbar\bar{c}_\text{L}\partial_\tau\hat{c}_{\text{L},k} - \varepsilon_1\bar{\gamma}_1\gamma_1 - \Omega\bar{b}b - 2\tilde{\lambda}_{11}^{(c)}\bar{\gamma}_1\gamma_1(\bar{b}+b) - \chi(\bar{b}+b) - \sum_k \varepsilon_{\text{L},k}\bar{c}_{\text{L},k}c_{\text{L},k} \Big] \\
&+ \int_{-\infty}^{\infty} d\tau_1 \int_0^{\infty} d\tau_2 \Big[\Big((b+\bar{b})\big[\tilde{\lambda}_{12}^{(d)}\gamma_1(\tau_1+\frac{\tau_2}{2}) + \tilde{\lambda}_{12}^{(c)}\bar{\gamma}_1(\tau_1+\frac{\tau_2}{2}) + \tilde{t}(\tilde{u}\bar{c}_\text{L}(\tau_1+\frac{\tau_2}{2}) + \tilde{v}^* c_\text{L}(\tau_1+\frac{\tau_2}{2}))\big] \Big) \\
&\times \exp(-i\varepsilon_2\tau_2)(b+\bar{b})\big[(\tilde{\lambda}_{12}^{(c)})^*\gamma_1(\tau_1-\frac{\tau_2}{2}) + (\tilde{\lambda}_{12}^{(d)})^*\bar{\gamma}_1(\tau_1-\frac{\tau_2}{2}) + \tilde{t}(\tilde{u}^* c_\text{L}(\tau_1-\frac{\tau_2}{2}) + \tilde{v}\bar{c}_\text{L}(\tau_1-\frac{\tau_2}{2}))\big] \Big] \Big\}.
\end{aligned} \tag{S-69}$$

Assuming a slow variation of c_L, $\bar{c}_\text{L}$, γ_1 and $\bar{\gamma}_1$, we obtain

$$\begin{aligned}
&\int_{-\infty}^{\infty} d\tau_1 \int_0^{\infty} d\tau_2 \Big\{ \Big[\tilde{t}(\tilde{u}\bar{c}_\text{L}(\tau_1+\frac{\tau_2}{2}) + \tilde{v}^* c_\text{L}(\tau_1+\frac{\tau_2}{2}))\exp(-i\varepsilon_2\tau_2)(b+\bar{b})\big[(\tilde{\lambda}_{12}^{(c)})^*\gamma_1(\tau_1-\frac{\tau_2}{2}) + (\tilde{\lambda}_{12}^{(d)})^*\bar{\gamma}_1(\tau_1-\frac{\tau_2}{2})\big] \Big] \\
&\qquad + \Big[(b+\bar{b})\big[\tilde{\lambda}_{12}^{(c)}\bar{\gamma}_1(\tau_1+\frac{\tau_2}{2}) + \tilde{\lambda}_{12}^{(d)}\gamma_1(\tau_1+\frac{\tau_2}{2})\big]\exp(-i\varepsilon_2\tau_2)\tilde{t}(\tilde{u}^* c_\text{L}(\tau_1-\frac{\tau_2}{2}) + \tilde{v}\bar{c}_\text{L}(\tau_1-\frac{\tau_2}{2})) \Big] \Big\} \\
&\approx -i\int_{-\infty}^{\infty} d\tau_1 \Big\{ \frac{\tilde{t}}{\varepsilon_2} \Big[(\tilde{u}\bar{c}_\text{L}(\tau_1) + \tilde{v}^* c_\text{L}(\tau_1))(b+\bar{b})\big[(\tilde{\lambda}_{12}^{(c)})^*\gamma_1(\tau_1) + (\tilde{\lambda}_{12}^{(d)})^*\bar{\gamma}_1(\tau_1)\big] \Big] + \text{H.c.} \Big\},
\end{aligned} \tag{S-70}$$

and

$$\begin{aligned}
&\int_{-\infty}^{\infty} d\tau_1 \int_0^{\infty} d\tau_2 \Big\{ (b+\bar{b})^2 \Big[\tilde{\lambda}_{12}^{(d)}\gamma_1(\tau_1+\frac{\tau_2}{2}) + \tilde{\lambda}_{12}^{(c)}\bar{\gamma}_1(\tau_1+\frac{\tau_2}{2})\exp(-i\varepsilon_2\tau_2)\big[(\tilde{\lambda}_{12}^{(c)})^*\gamma_1(\tau_1-\frac{\tau_2}{2}) + (\tilde{\lambda}_{12}^{(d)})^*\bar{\gamma}_1(\tau_1-\frac{\tau_2}{2})\big] \Big] \\
&\qquad + \Big[\tilde{t}(\tilde{u}\bar{c}_\text{L}(\tau_1+\frac{\tau_2}{2}) + \tilde{v}^* c_\text{L}(\tau_1+\frac{\tau_2}{2}))\exp(-i\varepsilon_2\tau_2)\tilde{t}(\tilde{u}^* c_\text{L}(\tau_1-\frac{\tau_2}{2}) + \tilde{v}\bar{c}_\text{L}(\tau_1-\frac{\tau_2}{2})) \Big] \Big\} \\
&= -i\int_{-\infty}^{\infty} d\tau_1 \Big\{ \frac{(b+\bar{b})^2}{\varepsilon_2} \Big[\Big(|\tilde{\lambda}_{12}^{(c)}|^2 - |\tilde{\lambda}_{12}^{(d)}|^2 \Big)\bar{\gamma}_1\gamma_1 + |\tilde{\lambda}_{12}^{(d)}|^2 \Big] + \frac{|\tilde{t}|^2}{\varepsilon_2} \Big[\Big((|\tilde{u}|^2 - |\tilde{v}|^2)\bar{c}_\text{L}(\tau_1)c_\text{L}(\tau_1) + |\tilde{v}|^2 \Big) \Big] \Big\},
\end{aligned} \tag{S-71}$$

where we have ignored the boundary term at $\tau_2 = \infty$ since it is highly oscillating and thus averages to zero. The terms containing $(b+\bar{b})^2$ in Eq. (S-71) can be ignored since they are of second order in $\lambda_{12}^{(c,d)}$ where we assume $\lambda_{12}^{(c,d)}/\varepsilon_2 \ll 1$. The terms proportional to $|\tilde{t}|^2$ renormalize the lead electrons' energies as well as their wave functions and can thus be subsumed into the lead Hamiltonian $\hat{H}_\text{L}$. As a result, we have

$$\begin{aligned}
Z = \text{const} \times &\int_{-\infty}^{\infty} \mathcal{D}\bar{\gamma}_1 \mathcal{D}\gamma_1 \mathcal{D}\bar{c}_{\text{L},k} \mathcal{D}c_{\text{L},k} \mathcal{D}\bar{b}\mathcal{D}b \\
\times \exp\Big\{ i \int_{-\infty}^{\infty} d\tau \Big[&\hbar\bar{\gamma}_1\partial_\tau\gamma_1 + \hbar\bar{b}\partial_\tau b + \hbar\bar{c}_\text{L}\partial_\tau\hat{c}_{\text{L},k} - \varepsilon_\text{A}\bar{\gamma}_1\gamma_1 - \Omega\bar{b}b - \lambda\bar{\gamma}_1\gamma_1(\bar{b}+b) - \chi(\bar{b}+b) - \sum_k \varepsilon_{\text{L},k}\bar{c}_{\text{L},k}c_{\text{L},k} \Big] \\
&- i\int_{-\infty}^{\infty} d\tau_1 \Big\{ \frac{\tilde{t}}{\varepsilon_2} \Big[(\tilde{u}\bar{c}_\text{L}(\tau_1) + \tilde{v}^* c_\text{L}(\tau_1))(b+\bar{b})\big[(\tilde{\lambda}_{12}^{(c)})^*\gamma_1(\tau_1) + (\tilde{\lambda}_{12}^{(d)})^*\bar{\gamma}_1(\tau_1)\big] \Big] + \text{H.c.} \Big\},
\end{aligned} \tag{S-72}$$

where we have defined $\varepsilon_\text{A} \equiv \varepsilon_1$ and $\lambda \equiv 2\tilde{\lambda}_{11}^{(c)}$. From Eq. (S-72), we can identify the effective Hamiltonian for the tunnel coupling as

$$\begin{aligned}
\hat{H}_\text{T} &= \frac{\tilde{t}}{\varepsilon_2} \Big[(\tilde{u}\hat{c}_\text{L}^\dagger + \tilde{v}^*\hat{c}_\text{L})(\hat{b}+\hat{b}^\dagger)\big[(\tilde{\lambda}_{12}^{(c)})^*\hat{\gamma} + (\tilde{\lambda}_{12}^{(d)})^*\hat{\gamma}^\dagger\big] \Big] + \text{H.c.} \\
&= t(\hat{b}+\hat{b}^\dagger)\hat{c}_\text{L}^\dagger(u\hat{\gamma} + v\hat{\gamma}^\dagger) + \text{H.c.}
\end{aligned} \tag{S-73}$$

where we have defined $\hat{\gamma} \equiv \hat{\gamma}_1$ as well as redefined

$$t \equiv \tilde{t}\frac{\tilde{\lambda}_{12}}{\varepsilon_2}, \tag{S-74a}$$

$$u \equiv \tilde{u}\frac{(\tilde{\lambda}_{12}^{(c)})^*}{\tilde{\lambda}_{12}} - \tilde{v}\frac{\tilde{\lambda}_{12}^{(d)}}{\tilde{\lambda}_{12}}, \tag{S-74b}$$

$$v \equiv \tilde{u}\frac{(\tilde{\lambda}_{12}^{(d)})^*}{\tilde{\lambda}_{12}} - \tilde{v}\frac{\tilde{\lambda}_{12}^{(c)}}{\tilde{\lambda}_{12}}, \tag{S-74c}$$

with $\tilde{\lambda}_{12} \equiv \sqrt{\left|\tilde{u}(\tilde{\lambda}_{12}^{(c)})^* - \tilde{v}\tilde{\lambda}_{12}^{(d)}\right|^2 + \left|\tilde{u}(\tilde{\lambda}_{12}^{(d)})^* - \tilde{v}\tilde{\lambda}_{12}^{(c)}\right|^2}$ which is chosen such that $|u|^2 + |v|^2 = 1$. So, the lead-ABS tunnel strength for the boson-assisted tunneling model is renormalized according to Eq. (S-74a), and the particle-(u) as well as the hole-component (v) of the ABS wave function seen by the electrons or holes tunneling from the lead are renormalized according to Eqs. (S-74b) and (S-74c), respectively. Note that the ABS Hamiltonian $\hat{H}_\mathrm{A}$ is the same as Eq. (S-11a). By using the Lang-Firsov transformation as in Sec. II, we can eliminate the electron-boson interaction term from $\hat{H}_\mathrm{A}$. As a result, the tunneling Hamiltonian for the boson-assisted tunneling model transforms into

$$\hat{\bar{H}}_\mathrm{T} = t(\hat{\bar{b}} + \hat{\bar{b}}^\dagger)\hat{c}_\mathrm{L}^\dagger \hat{\bar{d}}_\mathrm{A} + \mathrm{H.c.}, \tag{S-75}$$

where $\hat{\bar{b}}$ and $\hat{\bar{d}}_\mathrm{A}$ [Eq. (S-15)] are the Lang-Firsov transformation of the operators $\hat{b}$ and $\hat{d}_\mathrm{A} = u\gamma + v\gamma^\dagger$.

We now evaluate the matrix elements for the electron and hole tunneling which change the ABS occupancy number n from $0 \to 1$ and the boson occupancy from $q \to q'$ using the Baker-Campbell-Haussdorf formula

$$\hat{Y}^\dagger = e^{\frac{\lambda}{\Omega}(\hat{b}^\dagger - \hat{b})} = e^{-\frac{\lambda^2}{2\Omega^2}} e^{\frac{\lambda}{\Omega}\hat{b}^\dagger} e^{-\frac{\lambda}{\Omega}\hat{b}}, \tag{S-76}$$

which gives

$$\langle 1, q'|(\hat{b} + \hat{b}^\dagger - 2\frac{\lambda}{\Omega}\hat{\gamma}^\dagger\hat{\gamma})\hat{\bar{\gamma}}^\dagger|0, q\rangle = e^{-\frac{\lambda^2}{2\Omega^2}}\langle 1|\hat{\gamma}^\dagger|0\rangle\langle q'|e^{\frac{\lambda}{\Omega}\hat{b}^\dagger}(\hat{b}^\dagger + \hat{b})e^{-\frac{\lambda}{\Omega}\hat{b}}|q\rangle - \frac{\lambda}{\Omega}\langle 1|\hat{\gamma}^\dagger|0\rangle\langle q'|\hat{Y}^\dagger|q\rangle, \tag{S-77a}$$

$$\langle 0, q'|\hat{\bar{\gamma}}(\hat{b} + \hat{b}^\dagger - 2\frac{\lambda}{\Omega}\hat{\gamma}^\dagger\hat{\gamma})|1, q\rangle = \langle 1, q|(\hat{b} + \hat{b}^\dagger - 2\frac{\lambda}{\Omega}\hat{\gamma}^\dagger\hat{\gamma})\hat{\bar{\gamma}}^\dagger|0, q'\rangle^*. \tag{S-77b}$$

The explicit expressions of Eq. (S-77) can be obtained from Eq. (S-19) and the following equations:

$$\langle q'|e^{\frac{\lambda}{\Omega}\hat{b}^\dagger}\hat{b}e^{-\frac{\lambda}{\Omega}\hat{b}}|q\rangle = \sum_{m=0}^{\min(q', q-1)}\left(\frac{\lambda}{\Omega}\right)^{q'-m}\left(-\frac{\lambda}{\Omega}\right)^{q-m-1}\frac{\sqrt{q'!q!}}{m!(q'-m)!(q-m-1)!}, \tag{S-78a}$$

$$\langle q'|e^{\frac{\lambda}{\Omega}\hat{b}^\dagger}\hat{b}^\dagger e^{-\frac{\lambda}{\Omega}\hat{b}}|q\rangle = \sum_{m=0}^{\min(q'-1, q)}\left(\frac{\lambda}{\Omega}\right)^{q'-m-1}\left(-\frac{\lambda}{\Omega}\right)^{q-m}\frac{\sqrt{q'!q!}}{m!(q'-m-1)!(q-m)!}. \tag{S-78b}$$

In evaluating Eq. (S-78), we have used Eq. (S-20) and the following relations:

$$\hat{b}e^{-\frac{\lambda}{\Omega}\hat{b}}|q\rangle = \sum_{m=0}^{\infty}\frac{1}{m!}\left(-\frac{\lambda}{\Omega}\right)^m\hat{b}^{m+1}|q\rangle = \sum_{m=0}^{q-1}\frac{1}{m!}\left(-\frac{\lambda}{\Omega}\right)^m\sqrt{\frac{q!}{(q-m-1)!}}|q-m-1\rangle, \tag{S-79a}$$

$$\langle q'|e^{\frac{\lambda}{\Omega}\hat{b}^\dagger}\hat{b}^\dagger = \sum_{l=0}^{\infty}\frac{1}{l!}\left(\frac{\lambda}{\Omega}\right)^l\langle q'|(\hat{b}^\dagger)^{l+1} = \sum_{l=0}^{q'-1}\frac{1}{l!}\left(\frac{\lambda}{\Omega}\right)^l\sqrt{\frac{q'!}{(q'-l-1)!}}\langle q'-l-1|. \tag{S-79b}$$

For the tunneling Hamiltonian in Eq. (S-75), the rates of the boson-assisted electron and hole tunneling processes can be calculated from Fermi's Golden Rule to be

$$R_{q\to q'}^{n\to\bar{n};e} = \frac{2\pi t^2\nu_0}{\hbar}\left|\left\langle \bar{n}, q'\left|(\hat{\bar{b}} + \hat{\bar{b}}^\dagger)\hat{\bar{d}}_\mathrm{A}^\dagger\right|n, q\right\rangle\right|^2 f(E_{\bar{n},q'} - E_{n,q} - eV), \tag{S-80a}$$

$$R_{q\to q'}^{n\to\bar{n};h} = \frac{2\pi t^2\nu_0}{\hbar}\left|\left\langle \bar{n}, q'\left|(\hat{\bar{b}} + \hat{\bar{b}}^\dagger)\hat{\bar{d}}_\mathrm{A}\right|n, q\right\rangle\right|^2 f(E_{\bar{n},q'} - E_{n,q} + eV), \tag{S-80b}$$

where $\langle \bar{n}|\hat{d}_{\rm A}^\dagger|n\rangle$ and $\langle \bar{n}|\hat{d}_{\rm A}|n\rangle$ are the bare tunneling matrix elements for electrons and holes, respectively, and $f(E) = [1+\exp(E/k_{\rm B}T)]^{-1}$ is the lead Fermi function. Using Eqs. (S-77) and (S-78), we can evaluate the rates as

$$R^{0\to 1;e}_{q\to q'} = \frac{\Gamma|u|^2}{\hbar}\left|X_{qq'} - \frac{\lambda}{\Omega}Y_{qq'}\right|^2 f(E_{1,q'} - E_{0,q} - eV), \tag{S-81a}$$

$$R^{0\to 1;h}_{q\to q'} = \frac{\Gamma|v|^2}{\hbar}\left|X_{qq'} - \frac{\lambda}{\Omega}Y_{qq'}\right|^2 f(E_{1,q'} - E_{0,q} + eV), \tag{S-81b}$$

$$R^{1\to 0;e}_{q\to q'} = \frac{\Gamma|v|^2}{\hbar}\left|X_{q'q} - \frac{\lambda}{\Omega}Y_{q'q}\right|^2 f(E_{0,q'} - E_{1,q} - eV), \tag{S-81c}$$

$$R^{1\to 0;h}_{q\to q'} = \frac{\Gamma|u|^2}{\hbar}\left|X_{q'q} - \frac{\lambda}{\Omega}Y_{q'q}\right|^2 f(E_{0,q'} - E_{1,q} + eV), \tag{S-81d}$$

with the boson matrix elements given by

$$Y_{qq'} \equiv \langle q'|\hat{Y}^\dagger|q\rangle = \langle q'|e^{\lambda(\hat{b}^\dagger - \hat{b})/\Omega}|q\rangle, \tag{S-82a}$$

$$X_{qq'} \equiv e^{-\frac{\lambda^2}{2\Omega^2}}\langle q'|e^{\frac{\lambda}{\Omega}\hat{b}^\dagger}(\hat{b}^\dagger + \hat{b})e^{-\frac{\lambda}{\Omega}\hat{b}}|q\rangle, \tag{S-82b}$$

where the explicit expressions for $Y_{qq'}$ and $X_{qq'}$ can be obtained from Eqs. (S-19) and (S-78), respectively.

For the boson-assisted tunneling model, we can also show that the conductance is in general PH antisymmetric unless $|u| = |v|$. To this end, we replace $|Y_{qq'}|$ by $|X_{qq'} - \lambda Y_{qq'}/\Omega|$ in the derivation for the proof given in Sec. IV A. Furthermore, we note that in contrast to the tunneling into boson-coupled ABS model where the peak area of the conductance vs voltage curve is constant with temperature (see Sec. IV B), for the boson-assisted tunneling model [Eq. (S-75)] the conductance peak area increases with increasing temperature.

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