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FIELD THEORIES FOR NON-RECIPROCAL PHENOMENA,
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This work is dedicated to my brother and parents, without whom none of it would happen.

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ABSTRACT

For most of us, life is out of equilibrium. This fact is also true in most scientific endeavors, despite our best efforts. The canonical theories of statistical physics contain a very well-developed set of tools for understanding properties of equilibrium systems. These include the partition function, notions of various thermodynamic potentials – free energy in particular – and culminate in the ideas of universality and the techniques of the renormalization group (RG). Despite the immense richness and insight the equilibrium theories provide, the possibilities of different states or phases of matter outside of equilibrium are vast. On top of that, there is no guarantee that the same insights and methods, especially those of scaling, universality and RG will apply. Therefore, it is of great interest to explore these methods in the context of non-equilibrium systems. As paradigmatic manifestations of non-equilibrium, we consider three classes of systems in this dissertation. The first is non-reciprocal systems, where an action does *not* cause an equal and opposite reaction. The second is systems with self-sustained oscillations which are made possible by either an internal source or constant influx of energy. Finally, we consider some models of neural dynamics pertaining to epileptic seizures. Drawing from the equilibrium theories, we consider these problems in a coarse-grained framework, which allows for the use of field-theoretic methods. Some common themes arise throughout these models, despite them describing seemingly unrelated phenomena. These include dynamical phases, where the steady state of the order parameter is not time-translationally invariant, and critical exceptional points (CEPs), where the statistical fluctuations of the order parameter are greatly enhanced compared to equilibrium critical points. This work contributes to the long-standing goal of establishing a theory of universality in non-equilibrium systems.

CHAPTER 1

INTRODUCTION

In physics, we are often taught to consider problems about systems in equilibrium — the energy and momentum are conserved, the laws governing the phenomenon are constant in time, and a steady state is reached if we wait long enough. These assumptions allow us to capture many laws and properties of a given system. Even if it is slightly away from equilibrium, we assume that it is close enough that the predictions of the theories we build should still apply. However, being in equilibrium is a tight constraint, and it is more natural that most phenomena of interest and most experiments are better modeled with non-equilibrium theories. A consequence of this is the much wider span of possible phases of matter, a field which is yet to be fully categorized and fully explored. A grand goal of non-equilibrium statistical physics is exactly this classification, which inevitably raises a set of important questions. Is there a notion of universality for non-equilibrium systems? The answer as of now seems to be yes, at least in part. In that case, does it apply to all many-body non-equilibrium matter? What are the mathematical tools needed to describe it? Does the renormalization group (RG) as we know it from equilibrium theories work outside of equilibrium? If not, what do we need to change?

One of the most basic notions one has to concede is that describing non-equilibrium matter with a static statistical theory is insufficient. This approach works in canonical statistical physics because equilibrium demands a close tie between dynamics and statics. Therefore, given the notion of coarse-grained free energy $\mathcal{F}$, the necessary dynamical theory can be produced by assuming relaxational dynamics and taking into account the appropriate conservation laws. At the very least, this method is valid near a continuous phase transition, where the correlation length and time scales diverge. The end result of such an approach has been revealed in a vast and powerful classification by Hohenberg and Halperin (HH) [1–4], which yields Models A-D for purely dissipative dynamics and Models E-J with reversible

dynamics included. However, in general, there is no free energy functional outside of equilibrium that one can use to infer the static and dynamic properties of the system. One example could be a system undergoing a constant exchange of energy (or momentum) between the system and the environment. In fact, it is often more natural to first consider the dynamics of non-equilibrium many-body systems and build a statistical theory from there. This immediately allows for the possibility of features that are impossible in equilibrium to arise, often resulting in dynamical phases with the order parameter sustaining perpetual motion on average.

Some of the most powerful insights of equilibrium statistical mechanics have been the ideas of scaling and universality. The ultimate result is that the vicinity of a phase transition can be categorized using a finite set of numbers and functions – critical exponents and scaling functions, respectively. Two different systems that exhibit a phase transition where these numbers and functions are the same can therefore be described by the same theory close enough to the transition. This has immense consequences. For example, we can study the phase transition of a system by using a much simpler model that describes a different system within the same universality class. We can also predict the global behavior given the microscopic laws by using the RG machinery. We can infer the underlying statistics of a system if we understand its scaling near a phase transition.

Given the powerful insights that scaling theory and RG have brought to the study of equilibrium systems, there is a natural question of whether similar ideas can be applied outside of equilibrium. However, this is where we run into trouble. As mentioned, equilibrium is a special state of matter, and it simplifies the corresponding theories immensely. When we remove that constraint, the field of possible phases of matter, symmetries, dynamics, and other properties of our system becomes exceptionally large. But given the benefits of building a classification of non-equilibrium systems, large efforts are devoted to it. The three natural examples of non-equilibrium behavior that we explore in this dissertation are

oscillating states of matter, non-reciprocal interactions between different species of agents, and neural dynamics. Oscillations are a classical example of sustained dynamical behavior that is ubiquitous in biological rhythms, chemical reactions, population cycles, social interactions, coupled oscillator systems, non-equilibrium condensates, and many more. Non-reciprocity is most easily understood as breaking Newton’s Third Law — an action does not cause an equal and opposite reaction. As an example, the interaction between predator and prey, be they animals or cells, is certainly non-reciprocal. These phenomena inherently do not have an equivalent description in equilibrium and exhibit many novel features, phases, and transitions. Given their prevalence, there is endless use for theories that capture their statistical behavior. The final example of a system outside of equilibrium is the brain. It consumes a lot of energy to constantly sustain activity, growth, and change. Given the large number of neurons that make up our brain, it is natural to approach modeling it from a statistical perspective. But this is exactly the sort of question a non-equilibrium statistical physicist is interested in – the collective behavior and function of an active system formed by a large number of microscopic units. Therefore, non-equilibrium theories also find use in neuroscience.

To motivate the study of such systems a bit more, let us mention a few more examples. The simplest example of a system outside of equilibrium is any form of living matter. It is virtually always the case that a living “thing” consumes energy from one source and transforms it into something else. That can be growth, movement, reproduction, memory, recovery, or any other process necessary to sustain itself. Even if the total energy is conserved throughout all these processes, it usually has to change form and distribute in a manner that makes tracking it virtually impossible. It is thus useful to consider instead a model where there is an in- and outflow of energy. A more traditional physics example is the study of driven dissipative condensates, where the usual setup involves the pumping and leakage of particles within the system. In later chapters, we outline the typical setup for

producing an exciton-polariton condensate. Such condensates provide a platform for testing non-equilibrium theories but can also serve as tools for sensing, quantum computation, producing lasers, etc.

In Chapter 2 we provide a brief overview of the mathematical tools used throughout this work. We start with a lightning introduction to field theory, how it arises from microscopic models, and what we can do with it, culminating in RG. We then move on to dynamical field theory, which is the main framework utilized in the later calculations. In particular, wherever appropriate, we do not constrain the discussion to equilibrium (excluding the discussion of the Hohenberg-Halperin classification).

Following that, Chapter 3 begins with an outline of the properties that constitute an equilibrium theory. These are helpful because they can be used to directly detect a non-equilibrium system. A key insight is that the famed fluctuation-dissipation theorems [2–8] that hold in equilibrium are the Ward identities associated with a particular symmetry [9–12]. We then move on to provide a number of examples of general non-equilibrium systems and generic novel features that arise solely outside of equilibrium. One of the most important examples of the latter is Critical Exceptional Points (CEPs). CEPs parallel equilibrium critical points in the sense that they separate distinct phases of the system, but the fluctuation spectrum around them is quite different, leading to a plethora of new effects [13–22].

The following three chapters explore the novel results in non-equilibrium problems. Chapter 4 discusses results arising in theories of non-reciprocal phenomena. There are generally three pathways to build non-reciprocity. One is to have two distinct species of agents that have different interactions between them. If they are active, the interaction can be direct. If they are passive, non-reciprocity generally requires some sort of mediating field that carries the interaction, which is the second generic pathway. Finally, it is possible to have a single species with interactions modulated depending on the relative configuration of the two agents. All of these are intermingled and one can often translate one viewpoint into an-

other. Since several forms of many-body non-equilibrium systems can be described using the language of non-reciprocity, generic theories of such interactions have generated increased interest in recent times [13–15, 23–38]. We begin with a description of the general principles and many examples of non-reciprocity. Then we work out the fluctuation spectrum of a general linear non-reciprocal system, which lays the groundwork for any perturbative or RG calculation of general non-linear non-reciprocal systems. We show how non-reciprocity leads to the appearance of CEPs and dynamic phases, which are ubiquitous in non-equilibrium matter. After that, we provide some novel results pertaining to CEPs, namely what we call tangential CEPs, a few results on the real-space structure of the correlation function at a CEP, and the demonstration of why a CEP-based sensor does not offer improvement over conventional critical point sensors. The final part of the chapter focuses on non-reciprocally coupled XY models in two dimensions and on the fate of the Berezinskii-Kosterlitz-Thouless (BKT) transition in the vicinity of a CEP. We demonstrate how to formulate a dual electrodynamic theory for such a model, how an RG flow is usually extracted from a vortex pair distribution, and what novel dynamics the vortices experience near a CEP.

Chapter 5 focuses on oscillating states of matter. Even more than non-reciprocity, oscillations are ubiquitous in non-equilibrium matter. Given the breadth of this phenomenon, there is great interest in building our understanding of oscillations in biological matter [39–42], chemical reactions [41, 43–45], open physical systems [10, 11, 19, 46–49], neural dynamics [50–57], and others [14, 58–65] (see also references therein). From a dynamical systems perspective, an oscillating steady state is generically described by a limit cycle attractor in the phase space of the relevant dynamics. Therefore, idealized limit cycles are a natural platform to explore such dynamical phases. We first introduce a zero-dimensional system of coalescing limit cycles that exhibits a CEP, outline its relevance and analyze its spectrum and observables. Next, we explore a theory of birhythmicity in spatially extended systems, which is relevant for many chemical reactions, for example. We again demonstrate the appearance

of a CEP at the Gaussian level, describe the observables needed to detect it, and then show how spatial coupling changes the nature of the transition from mono- to birhythmicity.

Following the more abstract expositions in the previous chapters, we move on in Chapter 6 to models of neural dynamics. The theory of dynamical systems has been an immensely fruitful approach to the study of neural behavior [50, 51, 66, 67]. The key insight is that many properties of a neuron or a collection of neurons are best understood not as a consequence of biology and chemistry, but rather the type of dynamics underlying that property. For example, you can classify neurons into "integrators" and "resonators" depending on whether their dynamical response is maximal to inputs at broadly high or specific frequencies, respectively [50]. As a consequence, when building models of neural dynamics, one can choose varying levels of abstraction without being restricted to having to capture precise biology. This leads to many approximations and abstracted models that capture only key properties. In particular, this leads to the development of neural field theories [51, 68], which we briefly outline in Section 3.2.7. Given this approach, we investigate a model of epilepsy and promote it to a field theory. In fact, epilepsy is one of the central issues where the dynamical systems perspective is widely used [56, 57, 69–80]. We propose the idea that a transition to a seizure in focal epilepsy corresponds to the nucleation of a "seizing" state throughout the system. We then show how disorder in the natural frequency of neurons across space discourages seizures, among other results. We end with a proposed stimulation protocol for focal epilepsy, with possible experiments in neural cultures left to be pursued in future work.

We conclude Chapters 4, 5, and 6 with discussions and an outlook towards future directions of interest. As mentioned before, the grand goal is the classification of non-equilibrium phases of matter. If such a thing is possible at all, this dissertation provides a step towards it by considering paradigmatic manifestations of non-equilibrium conditions: the breaking of reciprocity of interactions and dynamical steady states. By no means does that capture the totality of possible ways a system can escape equilibrium. In fact, as a prototypical example,

our brains never reach a steady state in the canonical sense. There are many directions to pursue that build on the ideas and results outlined in the bulk of the text. In mentioning those in the final chapters, we pose them as a challenge to both ourselves and the reader.

CHAPTER 2

OVERVIEW OF CRITICAL PHENOMENA

2.1 Field Theory

2.1.1 Microscopic to Continuum

One of the most profound realizations of theoretical physics has been the idea that natural laws are organized by scale. Formal and high energy theory is concerned with fundamental interactions that govern particles and fields. Condensed matter and statistical physics are more often concerned with the collective behavior of a large number of units. At least in principle, the latter can be derived from the former, but such a derivation is impossible in practice. However, the saving grace comes from the fact that a few features and governing principles are often enough to understand a system at different scales. To illustrate, one does not have to keep track of quarks to understand an ice to water phase transition, or one does not have to understand the behavior of each individual cell to think the flocking of birds. However, the scales that are in some sense "close" to each other are connected. For example, an interaction between neighboring atoms in a lattice can govern the band structure of a collection of such atoms in a given material.

How do we know which theory is appropriate and how do we, in practice, go from one theory to the next? The answer lies again in the scale of the phenomenon of interest. In this dissertation, we are concerned with the collective phenomena of a large collection of units, often near a phase transition. Field theory turns out to be a natural language to work with collections of particles, and it also provides a natural procedure to extract the global behavior and phenomena emergent at large scales of such a collection in the form of the renormalization group (RG). Let us briefly review how one would go from a model describing microscopic interactions to a continuum description.

When we consider a statistical theory, we often find that if we look at the sets of units

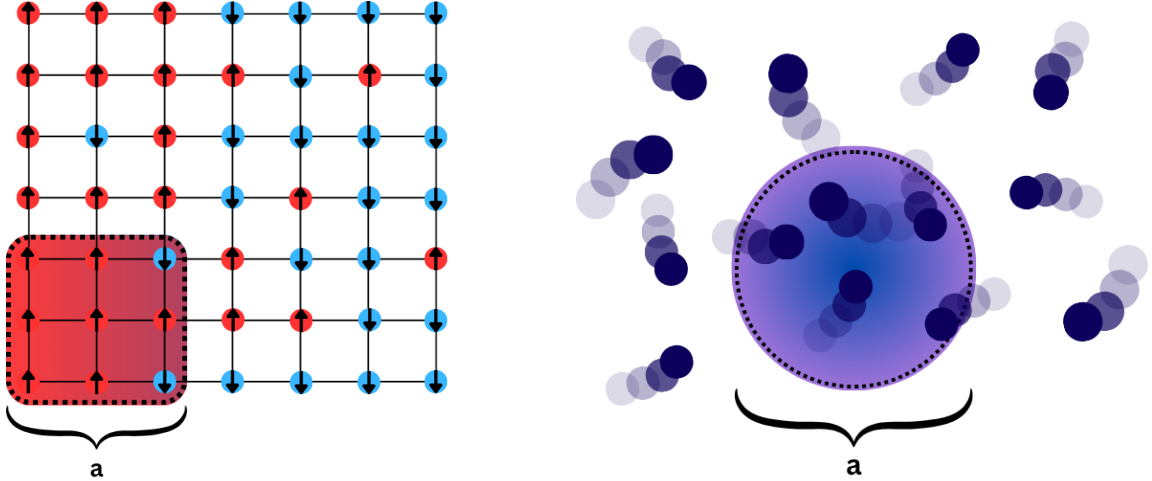


Figure 2.1: Schematic representation of coarse-graining procedures in a lattice spin model (left) and an active particle collection model (right). The coarse-graining scale a necessarily captures many objects of interest at once, but is smaller than the correlation length.

of interest that are close enough to each other, we observe that they have almost identical statistical properties. In other words, they are correlated. This usually manifests in similar density, pressure, temperature, spin alignment, velocity distribution, or any other statistical property. The length scale on which these correlations are strong is defined as the *correlation length*. This allows us to break up our system into spatial boxes of size smaller than the correlation length, and only consider the quantities averaged over these boxes. Such a procedure is usually called coarse-graining. Now, instead of isolated units, we can work with an averaged quantity $\phi(x)$ which is defined continuously in space. Since we averaged over a certain box size, any theory we build with the field $\phi(x)$ is not valid if we look at scales smaller than the box size. This is illustrated schematically in Fig. 2.1. Therefore, any statistical field theory is going to be equipped with some sort of cutoff, which in momentum space is usually denoted as $\Lambda \sim \frac{1}{\text{box size}}$. The theory is not valid at higher momenta.

The coarse-graining procedures are standard methods in equilibrium statistical mechanics and condensed matter physics [2–8]. They are applied to a plethora of phenomena, including elasticity, sound waves, heat diffusion, particle diffusion, phase transitions, quantum many-body phenomena, and critical dynamics. The translation from a microscopic theory to a

continuum theory culminates in an action, which usually corresponds to a free energy of the system:

$$\mathcal{F}[\phi] = \int d^d x f[\phi, \partial_x \phi, \dots] \quad (2.1)$$

Here and onward d is the dimensionality of space. f is a free energy density. Most of the relevant information is either encoded in this action, or can be calculated from it such as the symmetries, the correlation and response functions, other moments, critical points, etc. We briefly discuss these calculations in the next section.

In principle, this procedure is not restricted just to equilibrium. The canonical example is the theory of hydrodynamics, where the motion of a fluid is described in a continuum manner through the evolution of its mass, energy, and momentum densities. More generally, a system of equations describing motion of particles with any pair-wise interactions and noise can be written in a continuum form using the Dean equation [8]. More explicitly, we can start with a Langevin equation for a system of N particles labeled by a superscript:

$$\partial_t \vec{r}^\alpha = \sum_{\beta=1}^N \vec{F}(\vec{r}^\alpha - \vec{r}^\beta) + \vec{G}(\vec{r}^\alpha) + \vec{\xi}^\alpha \quad (2.2)$$

where $\vec{r}^\alpha$ is the position of α 'th particle, $\vec{F}$ denotes pairwise forces, $\vec{G}$ is an external force and ξ is modeling noise. Defining the suitable particle density and noise strength (which is often related to temperature), one can show that this set of equations can be related to the Dean equation of the form [8]:

$$\partial_t \rho = T \nabla^2 \rho - \vec{\nabla} \cdot \left(\vec{G} \rho + \int d^d \vec{r}' \vec{F}(\vec{r} - \vec{r}') \rho(t, \vec{r}) \rho(t, \vec{r}') + \sqrt{\rho} \vec{\eta} \right) \quad (2.3)$$

Where $\vec{\eta}$ also denotes noise. Notice that this construction does not rely on the assumption of equilibrium. For example, the forces $\vec{F}$ do not have to be reciprocal. This demonstrates the wide applicability of the continuum, or hydrodynamic, approach in the study of a large

collection of units, in or outside of equilibrium. This construction is also hinting that a Langevin equation is a natural starting point to investigate non-equilibrium matter. We discuss it in more detail throughout the text. Many of the systems considered in this dissertation can be thought of as particular cases of the Dean equation, with the density ρ representing varying quantities.

There are many other areas of study enjoying a continuum description. It turns out that many systems that have an absorbing to active phase transition belong to the directed percolation universality class [2, 8, 81–83]. Generic reaction-diffusion systems can also be analyzed with a continuum description [2]. In the realm of open quantum systems one can get continuum descriptions of an arbitrary Lindblad dynamics by using Keldysh field theory [3, 10, 11]. This is particularly useful for driven-dissipative condensates, which we discuss in Section 3.2.5. In the realm of active matter, continuum descriptions can be used to describe nematic phase transitions, polymer phases, flocking transitions, and others [8]. Finally, pattern formation has also been extensively modeled using continuous theories in many fields [20, 84–87]. We discuss these examples in their respective sections in Chapter 3.

Despite the large variety of models where a continuum description is appropriate, the basic principles of its construction are very similar. We assume that for the sake of the global behavior we do not need to keep track of all the microscopic details. We therefore construct a field-theoretic representation that should capture the long-range universal features of the model. We address why this is appropriate and how we can extract the long-range behavior in Sections 2.1.4 and 2.2.4.

2.1.2 Path Integrals and Generating Functional

In this section we lay out the general procedure and methodology of field-theoretic calculations. Most importantly, we do not restrict ourselves to an equilibrium theory. Given an action $S[\{\phi_i\}]$ which depends on all fields of interest $\{\phi_i\}$, we are usually interested in the

generating functional:

$$\mathcal{Z}[J] = \int \mathcal{D}[\phi] e^{-S[\{\phi_i\}] + J_i \phi_i} \quad (2.4)$$

The measure $\mathcal{D}[\phi]$ assumes a proper normalization due to the construction of the path integral. In equilibrium, the generating functional usually corresponds to the partition function [4, 6, 7, 88]. J are external source functions that are included for generality, but also allow us to calculate the correlation functions by taking derivatives:

$$\langle \phi_i(x_1) \phi_j(x_2) \dots \phi_k(x_n) \rangle = \frac{\delta}{\delta J_i(x_1)} \frac{\delta}{\delta J_j(x_2)} \dots \frac{\delta}{\delta J_k(x_n)} \mathcal{Z}[J] \quad (2.5)$$

Similarly, one can instead consider connected correlation functions, in which case one computes:

$$\langle \phi_i(x_1) \phi_j(x_2) \dots \phi_k(x_n) \rangle_c = \frac{\delta}{\delta J_i(x_1)} \frac{\delta}{\delta J_j(x_2)} \dots \frac{\delta}{\delta J_k(x_n)} \ln \mathcal{Z}[J] \quad (2.6)$$

The advantage lies in the various theorems, like the linked cluster theorem, that allow the calculation of the connected correlation functions only in terms of the connected Feynman diagrams [2–5, 89] (and also in the insensitivity to normalization). There are other results and techniques that assist with these calculations, especially if they are to be done perturbatively. The perturbation theory is based on separating the action into a component S_0 , the partition function of which can be calculated exactly, and everything else: S_I . Usually (but not always [89]), S_0 is taken to be quadratic in the fields, such that:

$$S_0 = \int d^d x d^d x' \phi_i(x) G_{0,ij}^{-1}(x, x') \phi_j(x') \quad (2.7)$$

The function $G_{0,ij}$ can be proved to be a Green's function of the operator $G_{0,ij}^{-1}(x, x')$ appearing in the action, and also corresponds to the connected two-point correlation function $\langle \phi_i \phi_j \rangle_c$. After that, one can do diagrammatic expansions, define vertex-generating functions, and much more. Although these formal techniques underlie the results in Chapters 4, 5 and

6, we do not spend time describing them in detail. There are many great resources that have done so [2–7]. The more relevant construction of the dynamical field theory is outlined in more detail in Section 2.2.

It is important to mention that many experimental measurements are related in some way to the correlation functions. These allow one to detect phase transitions, to understand the response of a system to external inputs, to measure scattering amplitudes, to understand the dispersion relation of the model, and so on. In many ways a model can be defined using the correlation functions instead of an underlying action or free energy. For example, the problem of reconstructing the kernel $G_{0,ij}^{-1}(x, x')$ from correlations is a huge topic of research in computational neuroscience [90].

Although we have skipped over many details, there is one more notion that is useful to mention briefly. Whenever an action exhibits a symmetry, there is an associated Ward identity that fixes some relation between several correlation functions. Instead of a general Ward identity, the derivation of which can be found in many resources [12] (and references therein), we introduce the form that deals directly with correlation functions. Notice that the fields in the path integral are dummy variables, which means we can freely perform a change of variables. Suppose that we do so infinitesimally (for one field): $\phi_0(x) \rightarrow \phi(x) = \phi_0(x) + \epsilon T(x, \phi_0)$, where ϕ_0 is the original field. If this transformation constitutes a symmetry of the action and has unit Jacobian, we get:

$$\langle \phi(x) \phi(x') \rangle = \langle (\phi_0(x) + \epsilon T(x, \phi_0)) (\phi_0(x') + \epsilon T(x', \phi_0)) \rangle \quad (2.8)$$

Since ϕ and ϕ_0 are just dummy variables in the path integral, this equation leads to:

$$\langle T(x, \phi) \phi(x') \rangle = 0 \quad (2.9)$$

This result constitutes a single Ward identity corresponding to a symmetry transforma-

tion generated by T . For any symmetry, we can generate a whole tower of identities for all possible correlation functions. We show later that the fluctuation-dissipation relations often encountered in critical dynamics are nothing but a Ward identity corresponding to the symmetry defining equilibrium.

2.1.3 Phase Transitions and Criticality

One of the most striking results of statistical physics is the notion that a continuous change of some parameter describing the system can completely change its properties in a discontinuous manner. For a system of finite number of degrees of freedom such discontinuity is impossible, since the partition function is just a finite sum. However, in the limit of infinitely many degrees of freedom, or the *thermodynamic limit*, one can observe discontinuities [5, 6]. In equilibrium, they are generally classified on the basis of which derivative of the free energy is discontinuous. That leads to the most common distinction between first-order (discontinuous) or second-order (continuous) phase transitions.

To illustrate this, we take a brief look at the Landau-Ginzburg theory of an Ising model phase transition [2–7]. The ferromagnetic Ising model describes a lattice of spins where the interactions between neighboring spins make the configurations where all spins point in the same direction energetically favorable. It is important to note that in this model the spins are generally able to take only two values – pointing "up" or "down". In this way, the model has a discrete $\mathbb{Z}_2$ symmetry, which flips all the spins. The free energy associated with the Ising model is generally written in a continuous description as:

$$\mathcal{F}_{\text{Ising}}[\phi] = \int d^d x \left[\frac{m}{2} \phi(x)^2 + \frac{D}{2} |\vec{\nabla} \phi(x)|^2 + \frac{g}{4} \phi(x)^4 - h(x) \phi(x) \right] \quad (2.10)$$

We can derive this form of the free energy from the discrete Ising model Hamiltonian [4, 5], but many terms are omitted for renormalization group reasons outlined below. However, in

the spirit of Section 2.1.1, we view $\phi(x)$ as the averaged magnetization of the system over a box of some size a centered at x . A 2d Ising model with coarse-graining was illustrated in Fig. 2.1. The parameters of the model, such as m , D , g and h can be related to the microscopic parameters, temperature in particular. However, as explained in the next section, in some sense the precise relation between microscopics and the continuum parameters is washed away. The most relevant information is how things scale at the phase transitions. Here we also assume that this free energy is dimensionless, such that it can be directly related to the action in Eq. 2.4.

Before we go to fluctuation effects, we should understand what this free energy tells us. We know from statistical physics that the probability of being in a certain configuration is proportional to $\exp(-\mathcal{F})$, which means the configurations minimizing the free energy are preferred. Clearly, spatially homogeneous states are favored, and minimizing the energy (for a spatially constant external field h) gives:

$$h = m\phi + g\phi^3 \tag{2.11}$$

Solving this equation illustrates what the first- or second-order phase transitions look like for the averaged magnetization ϕ . The solutions to the above equation of state correspond to the minima of free energy, as illustrated in Fig. 2.2. When $h = 0$, we get $\phi = 0$ for $m > 0$, often called the disordered phase. In contrast, $\phi = \pm\sqrt{-\frac{m}{g}}$ for $m < 0$, which is often called the ordered phase. ϕ grows continuously from 0 to one of the two solutions around $m = 0$. This demonstrates the schematic mechanism of continuous transitions. Similarly, for a fixed $m < 0$, one can change h and see a discontinuous jump in the magnetization ϕ , which corresponds to a first-order transition.

One also has to include thermal fluctuations in the description of the system. Following the procedure outlined in Section 2.1.2, we see that the correlation function at the linear

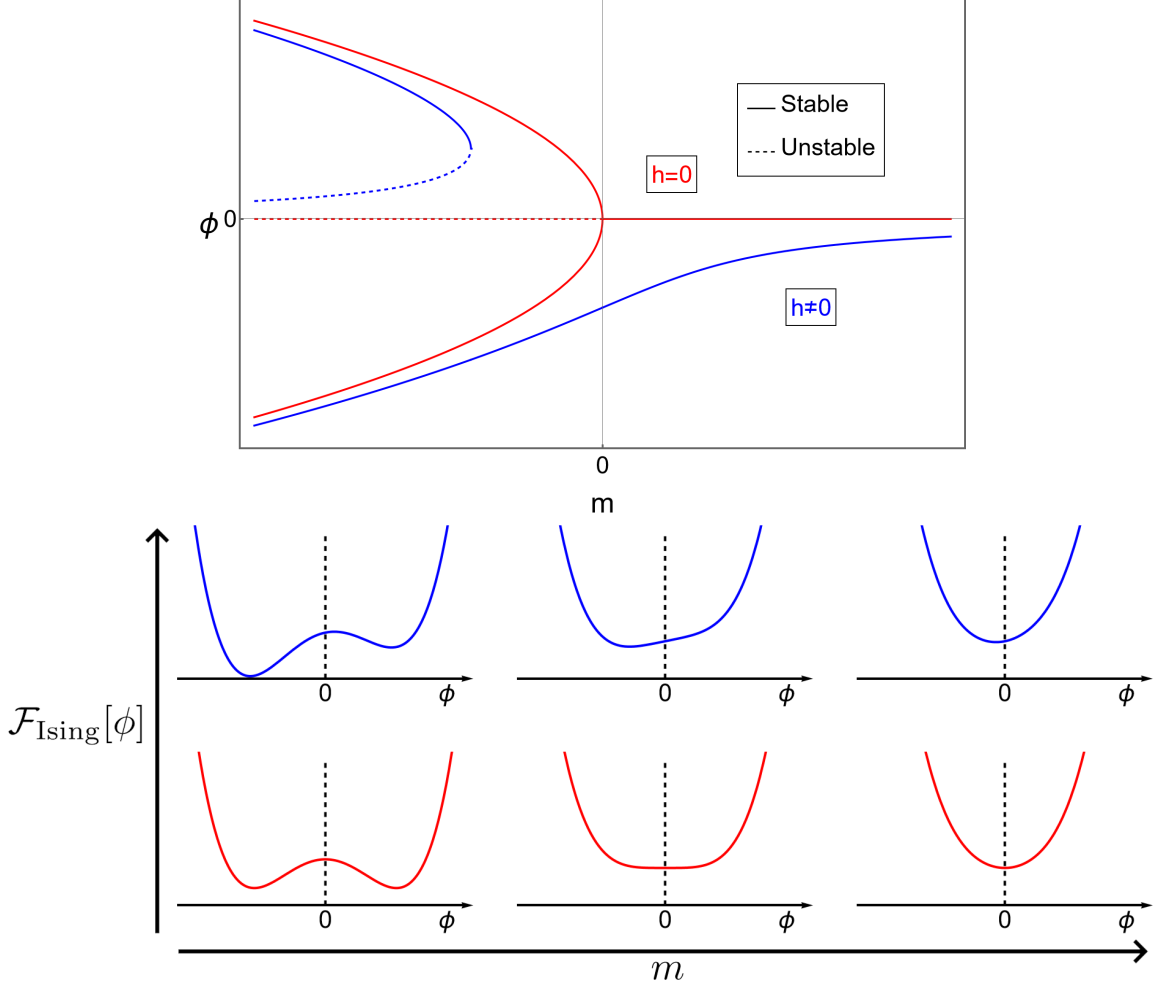


Figure 2.2: Top: the solutions ϕ to Eq. 2.11 as a function of m . Bottom: the free energy associated with a homogeneous solution ϕ . Blue curves represent $h \neq 0$, red curves represent $h = 0$.

level, in the disordered phase, in momentum space, is given by:

$$\langle \phi(k_1) \phi(k_2) \rangle_c = \frac{(2\pi)^d \delta^d(k_1 + k_2)}{m + Dk^2} \quad (2.12)$$

Here we see a correlation length emerge $\xi = \sqrt{D/m}$, which is exactly the same correlation length we mentioned in Sec. 2.1.1. At the phase transition $m = 0$, the correlation function has a pole at zero momentum and the correlation length diverges, which are fundamental properties of the long-range behavior of a system at criticality. This is an important

signature of critical points, which we use throughout the following chapters. Additionally, since in equilibrium the correlation function is related to the response function [2–7], a phase transition is also characterized by a diverging sensitivity of a system towards external perturbations. This is especially important outside of equilibrium, where the connection between response and correlation is broken, such that either can be used to detect phase transitions. This fact also allows us to connect statistical systems with dynamics in Section 2.2. Finally, we see that the field-theoretic or other continuum approaches discussed in Section 2.1.1 are more and more justified the closer the system is to criticality (at least for a continuous transition), since the correlation length diverges.

As mentioned before, after understanding the quadratic theory one can include the effects of the higher-order term $\frac{g}{4}\phi(x)^4$. This is usually done perturbatively, by looking either at the correlation functions, or at the vertex generating functions. This term also allows looking into the ordered phase. We do not go into detail on how such Landau-Ginzburg models are treated, since there are many resources for that [2–7].

We would like to mention two more important points. The first has to do with the notion of an order parameter. An order parameter is typically chosen as some measure of the phase transition, where it changes from, say, zero to a finite value on the different sides of the transition. In the paramagnet to ferromagnet transition, a natural choice is average magnetization. However, it is more broad than that. Most generally, in equilibrium, the Landau-Ginzburg theory allows one to write the free energy of a system around a phase transition in terms of the corresponding order parameter, whatever it may be. However, as mentioned before, the notion of free energy often does not extend to non-equilibrium problems. In that case, the choice of order parameters becomes a non-trivial problem, and often sits at the center of a study.

Another important notion is the symmetry of the action. We see above that in the absence of a magnetic field, the free energy does not change under transformation $\phi(x) \rightarrow -\phi(x)$.

This constitutes a $\mathcal{Z}_2$ symmetry. The phase transition then occurs when that symmetry is *spontaneously broken*. In the disordered phase the field $\phi = 0$ is symmetric under $\mathcal{Z}_2$, but the ordered states $\phi = \pm\sqrt{-\frac{m}{g}}$ flip. The notion of Spontaneous Symmetry Breaking (SSB) is fundamental to Landau-Ginzburg theory. In fact, we present in more detail in the next section why symmetry plays such an important role in understanding global behavior of systems and their phase transitions.

Associated with a continuous symmetry is the notion of a Goldstone mode [2–7]. Imagine an order parameter with an $O(2)$ symmetry, such that it is a two-dimensional vector and the free energy is the same for all its orientations. In the ordered phase, it has to spontaneously pick an orientation. However, all directions are equivalent, and the statistics of the component that corresponds to rotations around the circle have to be massless, $m = 0$. This component is the Goldstone mode, which does not change the free energy of the field configuration in the broken phase. Its spatial fluctuations govern the long-range behavior of the ordered phase. They lead to results such as the Mermin-Wagner theorem [2–7, 91], which forbids long-range order for an order parameter with a continuous symmetry in $d \leq 2$.

2.1.4 *Scaling, Universality and Renormalization Group*

With the previous three sections introducing the necessary ideas of field theory and criticality, we can finally introduce the notions of scaling at the critical points. We saw in Section 2.1.3 that a critical point is characterized by a diverging correlation length. We notice that aside from the microscopic coarse-graining length scale, the system actually does not have any other length scales left. This raises a natural hypothesis that at the critical point, the system must be *scale invariant*. In fact, it is possible to show that in real space, the correlation

function in Eq. 2.12 behaves as (notice the translation invariance) [7]:

$$\langle \phi(x)\phi(0) \rangle_c \sim \begin{cases} \frac{1}{x^{d-2}} , & x \ll \xi \\ \frac{e^{-x/\xi}}{x^{(d-1)/2}} , & x \gg \xi \end{cases} \quad (2.13)$$

We see that a critical point is characterized by a power law in the correlation function. With the use of some conformal field theory techniques, it can be demonstrated that a scale-invariant theory must have such correlations. However, without resorting to advanced tools, it was realized that one can put forward a *scaling hypothesis* valid at or near a phase transition. There are many ways to define it [2–7], but here we introduce it without resorting to a free energy. Around the critical point, we hypothesize the following scalings:

$$\begin{aligned} \langle \phi(m, h = 0) \rangle &\sim |m|^\beta , & m < 0 \\ \langle \phi(m = 0, h) \rangle &\sim h^{1/\delta} & \xi \sim |m|^{-\nu} \\ \chi_m \sim \left(\frac{\partial \phi}{\partial h} \right)_{m, h=0} &\sim |m|^{-\gamma} & \langle \phi(x)\phi(0) \rangle_c = \frac{1}{x^{d-2+\eta}} C(x/\xi) \end{aligned} \quad (2.14)$$

The critical exponents $\beta, \delta, \gamma, \nu, \eta$ are not all independent (at least in equilibrium), but characterize the phase transition. Here, we again see that around the critical point the correlation length diverges, as encompassed by the scaling relation $\xi \sim |m|^{-\nu}$. In principle, this means that the microscopic details about the system should not affect these exponents (or the form of the scaling functions such as $C(x/\xi)$). This motivates one of the most important notions of statistical physics – *universality*. Universality is the idea that near a continuous phase transition, if two distinct systems exhibit the same critical exponents and scaling functions, they can be described by the same long-range theory. This then allows for the classification of phase transitions into *universality classes* based on the exponents and scaling functions. A famous example is that the water-vapor phase transition and critical point are in the same universality class as the Ising paramagnet-ferromagnet phase transition

we have been drawing upon above. This is an incredible result, because it means that instead of studying each possible system, we can understand a much smaller number of theories very well and hopefully know which system belongs to which class. This again illustrates how nature is organized in scale – the principles governing long-range behavior of systems at criticality are described by a certain set of theories that are independent of the details at smaller scales.

However, if the critical exponents and functions are independent of the microscopic details, what determines them? In a sense, they are determined by features that persist across many spatial scales. These are:

- The spatial dimensionality of the theory – d .
- The symmetry structure of the order parameter – $\mathbb{Z}_2$, $O(N)$, $U(1)$, $SU(2)$, etc. Also the symmetry structure of the statistical theory itself.
- The conservation laws present in the model.
- The range and character of interactions in the model (although there might be less sensitivity to that since it inherently assumes a scale).

Notice that it seems as though there is nothing explicitly separating equilibrium from non-equilibrium theories in this list. However, this is not the case since both the symmetry structure and the conservation laws are integral to defining equilibrium. As a simple example we mentioned before, a system that constantly gains and loses energy to the environment certainly does not have energy conservation in its effective theory.

We can find the critical exponents and functions defined above at the level of mean-field or Gaussian theory. But how do we find them in the full theory? Here we introduce the final ingredient relevant for this dissertation: renormalization group (RG). We work through the more precise techniques in Section 2.2, and simply introduce the overarching ideas here. The following outline is based on Wilson’s momentum-shell RG.

How do we consider a theory at longer and longer length scales? We have an action that depends on the fields $\phi(x)$. However, we decided above that we would not consider fluctuations in momentum space of modes with momentum higher than the cutoff Λ . In other words, we set $\phi(k) = 0$ for $|k| > \Lambda$. We then want to zoom out from the system – i.e. coarse-grain it on bigger and bigger boxes until we only have large scale behavior. Since $\Lambda \sim \frac{1}{\text{box size}}$, we want to lower the cutoff of our theory. However, we also know that in an interacting theory (with terms in the action higher than quadratic), modes of different momenta are coupled, meaning that different scales affect each other. Therefore, we need to integrate out the high momentum modes to understand the effective behavior of low momentum modes.

We follow resources [3, 7] to introduce the basic notions of Wilson’s shell RG. In this approach, we separate $\phi(k) = \phi^-(k) + \phi^+(k)$, where:

$$\phi^-(k) = \begin{cases} \phi(k) & , k < \Lambda/b \\ 0 & , k > \Lambda/b \end{cases} ; \quad \phi^+(k) = \begin{cases} \phi(k) & , \Lambda/b < k < \Lambda \\ 0 & , \text{otherwise} \end{cases} \quad (2.15)$$

where b is infinitesimally bigger than 1. The schematic representation of this separation is depicted in Fig. 2.3. The shaded area corresponds to the $\phi^+(k)$ modes, hence the name of the RG procedure.

Since the quadratic action $S_0[\phi]$ is separable in momentum space, we get that the full action is:

$$S[\phi^-, \phi^+] = S_0[\phi^-] + S_0[\phi^+] + S_I[\phi^-, \phi^+] \quad (2.16)$$

The idea is then to perform the path integral only over the

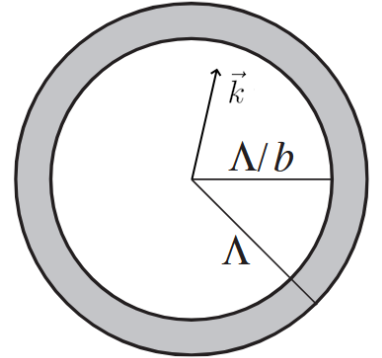


Figure 2.3: Representation of the momentum shell of the RG procedure. Figure adapted from [2].

"fast" fields $\phi^+(k)$ to get an effective action for the "slow" fields $\phi^-(k)$:

$$\int \mathcal{D}[\phi^+] e^{-S[\phi^-, \phi^+]} = e^{-S_0[\phi^-]} \int \mathcal{D}[\phi^+] e^{-S_0[\phi^+]} e^{-S_I[\phi^-, \phi^+]} = e^{-S'[\phi^-]} \quad (2.17)$$

The resulting effective action S' has a new cutoff. The integral over the fast fields hints at computing $\langle e^{-S_I[\phi^-, \phi^+]} \rangle$ perturbatively with respect to the action $S_0[\phi^+]$. By Taylor expanding, this yields:

$$\int \mathcal{D}[\phi^+] e^{-S_0[\phi^+]} e^{-S_I[\phi^-, \phi^+]} = Z_{0,+} \sum_{n=0}^{\infty} \frac{1}{n!} c_n I_n[\phi^-] \quad (2.18)$$

Here $Z_{0,+}$ is the partition function of the fast fields. The effective action is then:

$$\begin{aligned} S'[\phi^-] &= S_0[\phi^-] - \ln \left(1 + c_1 I_1 + \frac{1}{2} c_2 I_2 + \frac{1}{6} c_3 I_3 + \dots \right) \\ &\approx S_0[\phi^-] - c_1 I_1 - \frac{1}{2} \left(c_2 I_2 - c_1^2 I_1^2 \right) - \frac{1}{6} (c_3 I_3 - 3c_1 c_2 I_1 I_2 + 2c_1^3 I_1^3) + \dots \end{aligned} \quad (2.19)$$

The combinations in the second line only retain the connected contributions of the ϕ^+ diagrams [3, 7]. This procedure changes the parameters of the original action. We want to compare the new parameters with the original ones in order to understand whether they become more or less important as we consider longer and longer length scales. However, we now have two theories with different cutoffs – Λ and Λ/b . From a mathematical perspective, we should not compare two field theories where the fields have different support (so the spaces are different). That is why the last step of an RG procedure is always to rescale space such that the two theories exist with the same cutoff. Since the fields ϕ are dummy variables in the path integrals, we are free to rescale them as we choose, along with space. This choice is non-trivial, and often requires a lot of consideration, especially outside of equilibrium. For the Ising Landau-Ginzburg theory in Eq. 2.10, the choice is usually made to keep the spatial coupling D constant as we rescale. Intuitively, we don't want to change the spatial coupling

as we look at larger scales, since otherwise the model either has infinitely strong interactions or no interactions at all, both of which are trivial limits.

To make it a little more precise, suppose there is a term of the following form in the action:

$$\int d^d x g_O O(\phi(x)) \quad (2.20)$$

Under rescaling, we change momenta such that $k' = bk$ (so that $0 < |k'| < \Lambda$ when $0 < |k| < \Lambda/b$), which amounts to $x' = x/b$. Suppose after the choice of rescaling of ϕ that we made based on some considerations, the operator O changes as:

$$O(\phi(x)) = b^{-\Delta_O} O(\phi'(x')) \quad (2.21)$$

That means in the new theory, the same operator has the form:

$$\int dx' b^{d-\Delta_O} g_O O(\phi'(x')) \quad (2.22)$$

Therefore, we can assign a new coupling constant $g'_O = b^{d-\Delta_O} g_O$. Since $b > 1$, we see that the coupling constant has the following possibilities as we rescale, which then correspond to how we denote it:

- Relevant if $\Delta_O < d$ – value of the parameter grows as we zoom out. It is important for long-range behavior of the system.
- Marginal if $\Delta_O = d$ – value of the parameter stays the same as we zoom out. We then need to consider contributions coming from interactions to understand its relevance.
- Irrelevant if $\Delta_O > d$ – value of the parameter shrinks as we zoom out. We can usually describe the long-range behavior of the model without it.

Therefore, between the scaling (0-loop) contributions and the contributions coming from the interactions as we integrate out the fast fields, we get a set of coupled flow equations for

all of the constants of our theory. There are many intricacies and complications associated with such calculations, and we again refer to standard resources for more details [2–7].

Once we have the flows for all the coupling constants, we can use them by convenient choices of b and by looking at various correlation functions to extract the critical exponents. This is generally done with the use of the *renormalization group equation*, which we introduce in the following sections. The RG procedure is an incredibly powerful technique which has allowed us to find many critical exponents and scaling functions with perturbative precision. The scaling hypothesis introduced above can also be derived from the renormalization group equation. In other words, even though we postulated it before doing RG, the scaling forms of any correlation function around a critical point can be explicitly found by using RG.

With this we end the lightning introduction to field-theoretic methods as they are canonically presented in static equilibrium theories. We notice that one essential notion has been omitted – the dynamics! Obviously, any statistical system has dynamics associated with it. We can ask about correlations in time, about how fast the system relaxes towards equilibrium, etc. Additionally, as mentioned in the [Introduction](#), capturing dynamics is vital for describing non-equilibrium theories. With this in mind, the next section is dedicated to introducing field-theoretic methods that also keep information on the dynamics of a system.

2.2 Dynamical Field Theory

2.2.1 Dynamical and Stochastic Systems

Dynamics is at the center of many phenomena of interest, and not just in physics. In the last century, the theory of dynamical systems has exploded from a complicated application of calculus to an essential mathematical tool in physics, biology, neuroscience, chemistry, network science, atmospheric science, engineering, and others [39]. A key insight lies in bifurcation theory, which we introduce briefly below. Importantly, bifurcations and phase

transitions are intimately related, and as we introduce critical dynamics, that connection is made explicit.

A general dynamical system has the form:

$$\partial_t \phi_i = f_i(\{\phi_i\}, \{p_i\}) \quad (2.23)$$

Here ϕ_i are the variables (not spatial fields yet) of interest, f_i govern their evolution in time, and $\{p_i\}$ is a set of parameters, or external functions, that modulate f_i . This system is exactly solvable in the case that f_i is linear in variables ϕ_i . However, most interesting phenomena occur when f_i is non-linear – this allows for complicated phase space structure with many attractors, bifurcations and chaos.

Dynamical systems exhibiting bifurcations where there is a sudden change of state are of importance in many fields of study, ranging from biological sciences to quantum systems [39–41, 92]. We show below that critical dynamical phenomena have close connections to bifurcation theory. However, there are many other examples. The onset of polariton condensation and lasing corresponds to collective synchronization of oscillators [93–96], and the development of superconductivity and superfluidity can be mathematically described in the same way, perhaps most easily understood in the context of exactly solvable mean field Richardson-Gaudin models [97]. In biology the proximity to an instability can be used to tune the sensitivity of a sensor to an external stimulus [98], seen in the tuning of hair bundles in the cochlea [99]. Many physiological functions depend on the maintenance of synchronized patterns in time (limit cycles), including breathing [100], cardiac function, and circadian rhythms [101]. Photonic oscillators such as optically injected lasers also exhibit limit cycles, allowing for unique tunability and response properties [102, 103].

In the vicinity of the bifurcation, particular dynamical systems can be expressed in their normal forms that are generic to a class of models. These normal forms can therefore be studied to infer general information on how coupled dynamical systems with bifurcations

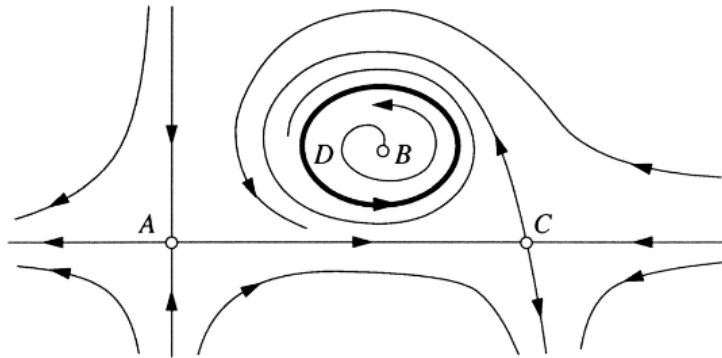


Figure 2.4: Example phase space plot for a dynamical system. Points A, B, and C are unstable fixed points, while D is a stable limit cycle. Figure adopted from [39].

behave. Another reason to study critical points is that they may have universal dynamical signatures. The existence of a scaling theory for a model system should, near the bifurcation (or critical point), provide a universality class for dynamical correlations of systems with the same underlying symmetries. Consequently, experimental systems that are microscopically much more complicated than our models may nonetheless have universal features in their correlation functions which reveal underlying dynamical transitions.

Throughout this dissertation, we are mainly concerned with two types of solutions. A fixed point is defined by a constant value $\phi_i(t) = \phi_{i,0}$ such that $f_i(\{\phi_{i,0}\}, \{p_i\}) = 0$. These points can have different stability structures, both in the linear and non-linear senses. The second type of solution are limit cycles – periodic orbits of $\phi_i(t)$. We explicitly exclude chaotic systems. Phase space is also an important notion for dynamical systems. We simply resort to plotting the flow (vector field) defined by $f_i(\{\phi_i\}, \{p_i\})$, as is customary in dynamical systems literature [39, 41, 50, 92]. In that case, the attractors can be visualized as sinks, saddles, or sources, and the limit cycles are also well defined orbits. See Fig. 2.4 for examples. Bifurcations, such as the change of stability of an attractor or a merger of several attractors, can then be viewed as qualitative changes in the geometry of the phase space for a given dynamical system [39, 92].

It is not a coincidence that we use the same language for bifurcations in dynamical systems

and phase transitions in statistical systems, an abrupt, qualitative change in behavior as some parameter is varied. To illustrate the direct relationship existing between the two, consider the following dynamical system:

$$\partial_t \phi = -m\phi + h \quad (2.24)$$

The steady state solution is given by $\partial_t \phi_0 = 0$, which gives $\phi_0 = h/m$. Notice, however, that in the absence of spatial coupling and at the Gaussian level, the states that extremized the Ising free energy in Eq. 2.10 were given by:

$$0 = \frac{\delta \mathcal{F}_{\text{Ising}}[\phi]}{\delta \phi} = m\phi - h \rightarrow \phi_0 = \frac{h}{m} \quad (2.25)$$

Therefore, we see that the minima of energy can be seen as the steady states of dynamical systems. We saw that for $h = 0$, the critical point happened at $m = 0$. For the dynamical system, this corresponds to the steady state solution ϕ_0 losing stability – a bifurcation. To get the steady states in the other phase, we simply need to add the non-linear term corresponding to the quartic coupling in the action.

In equilibrium problems, we can consider this in general terms. Given an action $\mathcal{F}[\{\phi_i\}]$ (which in equilibrium is associated with free energy, or Hamiltonian of the system, hence the denomination $\mathcal{F}$), we can consider the dynamics given by:

$$\partial_t \phi_i = -\frac{\delta \mathcal{F}[\{\phi_i\}]}{\delta \phi_i} \quad (2.26)$$

Here, we absorb the relaxation rate associated with the dynamics either into the definition of time or into the parameters of the action. The steady states of the dynamics then correspond to the extrema of the action, while bifurcations correspond to phase transitions. Notice that this dynamics is *dissipative*, meaning that the field ϕ evolves towards the minima of the action and stays there. This is not the most general construction, since one can also have reversible terms and conservation laws, for example the precession of spins in a

magnetic field. This also hints at what breaks down outside of equilibrium. There, it is not guaranteed that one can write the dynamics in the form $-\frac{\delta\mathcal{F}[\{\phi_i\}]}{\delta\phi_i}$. Additionally, especially in (open) quantum systems, one also has to consider *coherent* dynamics, which are sustained even if the system reaches a minimum of the Hamiltonian.

As was the case for Landau-Ginzburg theory, we also need to introduce fluctuations to make a parallel with statistical theories complete. This is done by explicitly adding a noise term $\xi_i(t)$ to the dynamics, such that the models of interest take the form of a Langevin equation:

$$\partial_t\phi_i = f_i(\{\phi_i\}, \{p_i\}) + \xi_i \quad (2.27)$$

where the statistical properties of ξ_i are usually defined by:

$$\begin{aligned} \langle \xi_i(t) \rangle &= 0 \\ \langle \xi_i(t)\xi_j(t') \rangle &= B_{ij}(\{\phi_i\}, t, t') \end{aligned} \quad (2.28)$$

where the averages are taken with respect to noise realizations. The second expression defines the noise kernel B_{ij} , which can depend on the variables ϕ_i and the times. Usually, the correlations are assumed to be time-translation invariant, such that $B_{ij}(\{\phi_i\}, t, t') \rightarrow B_{ij}(\{\phi_i\}, t - t')$. The noise is also often assumed to be white, which means that it is completely decorrelated in time, such that $B_{ij}(\{\phi_i\}, t - t') \rightarrow B_{ij}(\{\phi_i\})\delta(t - t')$. This is a strong assumption, and it is certainly not true for many real systems. However, both its simplicity and the time-scales that we consider systems of interest on, often make it a reasonable assumption that allows us to extract qualitatively correct information from the model.

There are many subtleties on how to handle stochastic equations of the form Eq. 2.27 [2, 104–106]. Primarily, these are associated with the discretization of time and the associated choice of definition for the noise. This is typically presented in the form of α -discretization, with the Ito and Stratonovich discretizations being two natural choices for α (see refs. above).

In this dissertation, we work with the Ito choice, since it simplifies the path integral formulation. However, somewhat surprisingly, one can formulate a path integral representation of a stochastic system without choosing a specific discretization with the use of Grassmann fields [12, 105]. Beyond a mathematical curiosity, this formulation is very convenient for deriving fluctuation-dissipation theorems and for showing how equilibrium is related to symmetry. We outline this connection in Sec. 3.1.3.

The form in Eq. 2.27 is usually first encountered in canonical physics studies in the context of Brownian motion. There, it is convenient to model the evolution of particle velocities in this manner, where the noise represents collisions. This is also a form typically encountered in equilibrium critical dynamics, as we discuss in Sec. 2.2.5, where the function f takes the form as in Eq. 2.26. However, it is not restricted to equilibrium. As we discuss throughout this dissertation, with examples, equations of such form are used to model many non-equilibrium systems. With this in mind, we take Eq. 2.27 as a starting point of the models of interest, now promoting the variables $\phi_i(t)$ to fields $\phi_i(t, \vec{x})$ and including some sort of spatial coupling in their evolution.

2.2.2 MSRJD and Keldysh Formalisms

In this section, we present how path integral formulations can be constructed starting with either a stochastic equation or Lindblad equation in open quantum systems. Although the construction is more involved than in Section 2.1, we still omit many details as there are several resources introducing the formalisms outlined below. These are referenced when appropriate.

We proceed with the construction of a field-theoretic description of stochastic dynamics according to the Martin-Siggia-Rose-Janssen-De Dominicis (MSRJD) procedure. Great references for the MSRJD construction that are used throughout the dissertation are [2–4, 12, 89, 104, 105, 107–109]. We start with a general stochastic equation for fields $\phi_\alpha(t, \vec{x})$

(from now on we explicitly write the vector $\vec{x}$ to distinguish space from time):

$$\partial_t \phi_\alpha(t, \vec{x}) = f_\alpha[\phi](t, \vec{x}) + \xi_\alpha(t, \vec{x}) \quad (2.29)$$

with the noise $\xi_\alpha(t, \vec{x})$ distributed according to some probability density $P[\xi]$. Since the dynamics is stochastic, we are interested in statistical properties of the fields $\phi_\alpha(t, \vec{x})$, which are obtained by taking averages with respect to noise. For some operator (function) of the fields $O[\phi]$, we write this as:

$$\langle O[\phi] \rangle_\xi = \int \mathcal{D}[\xi] O[\phi(\xi)] P[\xi] \quad (2.30)$$

Here we explicitly denote $\phi(\xi)$ to emphasize that the fields have to be evaluated given a particular realization of noise in the path integral. This condition can be imposed by integrating over all field configurations with a delta-function:

$$\begin{aligned} O[\phi(\xi)] &= \int \mathcal{D}[\phi] O[\phi] \prod_\alpha \prod_{x,t} \delta[\partial_t \phi_\alpha(t, \vec{x}) - f_\alpha[\phi](t, \vec{x}) - \xi_\alpha(t, \vec{x})] \\ &= \int \mathcal{D}[\phi, \bar{\phi}] O[\phi] \exp \left\{ \int dt \int d^d x \sum_\alpha i \bar{\phi}(t, \vec{x})_\alpha [\partial_t \phi_\alpha(t, \vec{x}) - f_\alpha[\phi](t, \vec{x}) - \xi_\alpha(t, \vec{x})] \right\} \end{aligned} \quad (2.31)$$

In the second line, we used a representation of a delta function through an integral (schematically $\delta(x) \sim \int dk e^{ikx}$), which introduces a new field $\bar{\phi}$. Additionally, the Jacobian associated with the delta function (due to $\partial_t \phi_\alpha$ inside) is trivial in the Ito discretization (see refs. above). The last step of the construction has to do with explicitly computing:

$$\int \mathcal{D}[\xi] \exp \left\{ - \int dt \int d^d x \sum_\alpha i \bar{\phi}_\alpha(t, \vec{x}) \xi_\alpha(t, \vec{x}) \right\} P[\xi] \quad (2.32)$$

which is equivalent to knowing the moment-generating functional for the noise explicitly.

There are not many such distributions, and one may choose to consider some that are quite exotic depending on the system of interest. In this dissertation, however, we focus on the case of *Gaussian white noise*, with the probability distribution given by:

$$P[\xi] \sim \exp \left\{ -\frac{1}{2} \int dt \int d^d x \sum_{\alpha\beta} \xi_\alpha(t, \vec{x}) (L^{-1})_{\alpha\beta} \xi_\beta(t, \vec{x}) \right\} \quad (2.33)$$

Here, the operator L can depend on the fields ϕ for the multiplicative noise case, or can be a spatial operator. It is important to notice that it is diagonal in space and time, meaning the noise is not correlated at different points. Since the Gaussian probability distribution is determined by its variance, such noise is usually defined by stating:

$$\langle \xi_\alpha(t, \vec{x}) \xi_\beta(t', \vec{x}') \rangle = L_{\alpha\beta} \delta^d(x - x') \delta(t - t') \quad (2.34)$$

The generating functional for the Gaussian distribution is known. After performing the integral over ξ , we get for the above average of interest:

$$\boxed{\begin{aligned} \langle O[\phi] \rangle &= \int \mathcal{D}[\phi, \bar{\phi}] O[\phi] \exp[-S_{\text{MSR}}] \\ S_{\text{MSR}} &= \int dt \int d^d x \left[-\sum_{\alpha} i \bar{\phi}_{\alpha} [\partial_t \phi_{\alpha}(t, \vec{x}) - f_{\alpha}[\phi](t, \vec{x})] + \sum_{\alpha, \beta} \bar{\phi}_{\alpha} \frac{L_{\alpha\beta}}{2} \bar{\phi}_{\beta} \right] \end{aligned}} \quad (2.35)$$

This concludes the formal construction of the MSRJD functional and defines the MSRJD action. Even though we arrived at the average in Eq. 2.35 rather informally, one can do it precisely by discretizing time, see refs. above. A few things are prudent to notice. First, we can choose $O[\phi] = 1$, which demonstrates that what we would normally call the partition function in field theory is nothing but identity written in a very complicated way:

$$1 = \mathcal{Z}[0] = \int \mathcal{D}[\phi, \bar{\phi}] \exp[-S_{\text{MSR}}] \quad (2.36)$$

This identity for the partition function shows that we do not have to worry about normalization. Additionally, since the number of fields doubled due to the introduction of the *response fields* $\bar{\phi}_\alpha$, there is nothing stopping us from computing averages of operators of both ϕ_α and $\bar{\phi}_\alpha$. Just like for the field theory construction outlined in Sec. 2.1, we can introduce a moment-generating functional:

$$\mathcal{Z}[J, \bar{J}] = \int \mathcal{D}[\phi, \bar{\phi}] \exp \left\{ -S_{\text{MSR}} + \int dt \int d^d x \sum_{\alpha} [J_{\alpha}(t, \vec{x}) \phi_{\alpha}(t, \vec{x}) + i \bar{J}_{\alpha}(t, \vec{x}) \bar{\phi}_{\alpha}(t, \vec{x})] \right\} \quad (2.37)$$

We outline the meaning of averages of operators involving the response fields in the next section (and also the factor of i in the $i \bar{J}_{\alpha} \bar{\phi}_{\alpha}$ term). With this, the construction of the MSRJD functional is complete, and we are now able to use all of the tools of field theory on any stochastic system of interest. In fact, one does not need to do all of the above steps for each model but can simply read off the action in Eq. 2.35 given the stochastic equation and the noise correlations.

For completeness, let us also introduce the Keldysh formalism used in constructing field-theoretic descriptions of open quantum systems. This demonstrates the connection between Keldysh and MSRJD formalisms, showing that in some sense the latter is the semi-classical limit of the former. This is not done in much detail, but useful references for this construction are [3, 4, 10, 11] (and refs. therein).

We proceed according to [10]. A typical starting point for describing open quantum systems is a quantum master equation for the reduced system density matrix:

$$\partial_t \rho = \mathcal{L} \rho = -i[H, \rho] + \sum_{\alpha} \gamma_{\alpha} \left(L_{\alpha} \rho L_{\alpha}^{\dagger} - \frac{1}{2} \{ L_{\alpha}^{\dagger} L_{\alpha}, \rho \} \right) \quad (2.38)$$

Here $\mathcal{L}$ is often called the Liouville superoperator, since it "acts" on the density matrix ρ from both sides. This equation contains the familiar evolution due to a Hamiltonian and dynamics that result from integrating out the coupling between the system and the bath.

These are encoded in the Lindblad operators L_α . The formal solution to this equation is:

$$\rho(t) = e^{(t-t_0)\mathcal{L}}\rho(t_0) \quad (2.39)$$

If we are interested in the stationary state and want to compute correlations with arbitrary time separations, we need to consider the evolution for all times $-\infty < t < \infty$. We then proceed similar to standard coherent state integrals in many-body quantum systems [4]. We break time up into infinitesimal steps such that $\rho_{n+1} = e^{\Delta t \mathcal{L}} \rho_n \approx (I + \Delta t \mathcal{L}) \rho_n$ and insert the resolutions of identity in the form $I = \int \frac{d\psi d\psi^*}{\pi} e^{-\psi^* \psi} |\psi\rangle\langle\psi|$. Since we are working with a density matrix operator, we need to insert two identities on each side of ρ_n , which generically yields two "time-contours" denoted by ψ_+ and ψ_- (depending on which side of ρ the identity was inserted):

$$\rho_n = \int \frac{d\psi_{+,n} d\psi_{+,n}^*}{\pi} \int \frac{d\psi_{-,n} d\psi_{-,n}^*}{\pi} e^{-\psi_{+,n}^* \psi_{+,n} - \psi_{-,n}^* \psi_{-,n}} \langle \psi_{+,n} | \rho_n | \psi_{-,n} \rangle |\psi_{+,n}\rangle \langle \psi_{-,n}| \quad (2.40)$$

Plugging this into the evolution equation for ρ_n , one can solve for ρ_{n+1} matrix elements in terms of the above expression. Doing this iteratively yields the partition function that is governed by an action:

$$S_{\text{Keldysh}} = \int dt [\psi_+^* i \partial_t \psi_+ - \psi_-^* i \partial_t \psi_- - i \mathcal{L}(\psi_+^*, \psi_+, \psi_-^*, \psi_-)] \quad (2.41)$$

with suitably defined $\mathcal{L}(\psi_+^*, \psi_+, \psi_-^*, \psi_-)$. The final step of the construction is the Keldysh rotation:

$$\phi_c = \frac{1}{\sqrt{2}}(\psi_+ + \psi_-) \quad , \quad \phi_q = \frac{1}{\sqrt{2}}(\psi_+ - \psi_-) \quad (2.42)$$

which usually brings the Keldysh action S_{Keldysh} to a form very similar to S_{MSR} , with the identifications $\phi_c \rightarrow \phi$ and $\phi_q \rightarrow \bar{\phi}$. There are, of course, many subtleties, both in the construction and the use of the Keldysh formalism (see refs. above). However, we see

that the structure of the theories is similar. One can often take a semi-classical limit by considering the most relevant terms in the Keldysh action to arrive at an action that can be directly thought of as the MSRJD action corresponding to a stochastic equation [10, 11]. This is relevant in our discussion of driven-dissipative condensates. This fact, however, is quite important in general, because it signifies that non-equilibrium quantum dynamics can be thought of in the same framework as classical stochastic dynamics in the regime where quantum fluctuations are irrelevant. Therefore, although we generally start the discussions in Chapters 4 and 5 from a classical standpoint, the results derived there are valid for open quantum systems in the appropriate limits.

2.2.3 *Dynamic Response and Correlation Functions*

With the MSRJD action constructed, we should consider the meaning of various correlation functions. First, we notice that the addition of the source term $i\bar{J}_\alpha\bar{\phi}_\alpha$ in Eq. 2.37 is equivalent to considering a stochastic system of the form:

$$\partial_t\phi_\alpha = f_\alpha - \bar{J}_\alpha + \xi_\alpha \quad (2.43)$$

From the generating functional, we can calculate the average of the field:

$$\langle\phi_\alpha(t, \vec{x})\rangle = \frac{\delta\mathcal{Z}[J, \bar{J}]}{\delta J_\alpha(t, \vec{x})} \Big|_{J, \bar{J}=0} \quad (2.44)$$

However, what if we want to understand the average response of the field to the external source $\bar{J}$? We then get:

$$\frac{\delta\langle\phi_\alpha(t, \vec{x})\rangle}{\delta\bar{J}_\beta(t', \vec{x}')} = \frac{\delta^2\mathcal{Z}[J, \bar{J}]}{\delta J_\alpha(t, \vec{x})\delta\bar{J}_\beta(t', \vec{x}')} \Big|_{J, \bar{J}=0} = i\langle\phi_\alpha(t, \vec{x})\bar{\phi}_\beta(t', \vec{x}')\rangle \quad (2.45)$$

This example demonstrates that the correlation functions involving the auxiliary fields

$\bar{\phi}_\alpha$ are nothing but the response functions of the stochastic process (which is why $\bar{\phi}_\alpha$ are often called response fields). Causality is automatically enforced from the MSRJD action, as we discuss later.

Just as we saw in Sec. 2.1.2, the generating functional for a stochastic process generally cannot be computed exactly, which means we often resort to perturbative methods. For that, we need a theory that we can solve exactly, around which we perturb. Just as a Gaussian theory is the basis for calculations in canonical statistical physics, it also serves as a starting point in critical dynamics. Since the MSRJD action necessarily depends on the response fields up to second order, a Gaussian theory corresponds to a linear function f . We therefore consider the equation:

$$\partial_t \phi(t, \vec{x}) = -m\phi(t, \vec{x}) + D\nabla^2 \phi(t, \vec{x}) + \xi(t, \vec{x}) \quad (2.46)$$

This is not the most general linear theory, as we could have included spatially inhomogeneous $m(t, \vec{x})$ as well as higher order spatial couplings. However, the former is excluded if we consider a translationally invariant system, while the latter is excluded based on its relevance (in the RG sense to be explored later). We also assume noise correlations of the form $\langle \xi(t, \vec{x}) \xi(t', \vec{x}') \rangle = B \delta^d(x - x') \delta(t - t')$. An extension to a many-component field is trivial.

Just like in static theories, this equation is best analyzed in Fourier space, which we define as:

$$\begin{aligned} v(t, \vec{x}) &= \int \frac{d\omega d^d k}{(2\pi)^{d+1}} v(\omega, \vec{k}) e^{i\omega t - i\vec{k} \cdot \vec{x}} \\ v(\omega, \vec{k}) &= \int dt d^d x v(t, \vec{x}) e^{-i\omega t + i\vec{k} \cdot \vec{x}} \end{aligned} \quad (2.47)$$

with the important identity $\int dt d^d x e^{-i\omega t + i\vec{k} \cdot \vec{x}} = (2\pi)^{d+1} \delta^d(k) \delta(\omega)$. From here one can proceed directly from Eq. 2.46, but for illustration, we compute the correlation functions

from the MSRJD action:

$$\begin{aligned}
S_{\text{MSR}} &= \int dt \int d^d x \left[-i\bar{\phi}(\partial_t + m - D\nabla^2)\phi + \frac{B}{2}\bar{\phi}^2 \right] \\
&= \int' \frac{d\omega_1 d^d k_1}{(2\pi)^{d+1}} \int' \frac{d\omega_2 d^d k_2}{(2\pi)^{d+1}} (2\pi)^{d+1} \delta^d(k_1 - k_2) \delta(\omega_1 - \omega_2) \phi^{*T}(\omega_1, \vec{k}_1) G_0^{-1} \phi(\omega_2, \vec{k}_2)
\end{aligned} \tag{2.48}$$

where $\phi(\omega, \vec{k}) = \begin{pmatrix} \phi(\omega, \vec{k}) & \bar{\phi}(\omega, \vec{k}) \end{pmatrix}^T$ and:

$$G_0^{-1} = \begin{pmatrix} 0 & -\omega_2 - im - iDk_2^2 \\ \omega_2 - im - iDk_2^2 & B \end{pmatrix} \tag{2.49}$$

Since the fields $\phi(t, \vec{x})$ and $\bar{\phi}(t, \vec{x})$ are real, we have $\phi^*(\omega, \vec{k}) = \phi(-\omega, -\vec{k})$. The primes on the integrals signify that we only take half the $\omega, \vec{k}$ space to avoid double-counting. Here we again encounter the inverse Green's function, and when we compute the correlation functions, we get:

$$\begin{aligned}
\langle \phi^*(\omega_1, \vec{k}_1) \phi(\omega_2, \vec{k}_2) \rangle_c &= \frac{B}{\omega_2^2 + (m + Dk_2^2)^2} (2\pi)^{d+1} \delta(\omega_1 - \omega_2) \delta^d(k_1 - k_2) \\
\langle \phi^*(\omega_1, \vec{k}_1) \bar{\phi}(\omega_2, \vec{k}_2) \rangle_c &= \frac{1}{-\omega_2 - im - iDk_2^2} (2\pi)^{d+1} \delta(\omega_1 - \omega_2) \delta^d(k_1 - k_2)
\end{aligned} \tag{2.50}$$

We most commonly write it without the complex conjugation in the following form:

$$\boxed{\langle \phi_i(\omega_1, \vec{k}_1) \phi_j(\omega_2, \vec{k}_2) \rangle_c = (2\pi)^{d+1} \delta(\omega_1 + \omega_2) \delta^d(k_1 + k_2) G_{ij}(\omega_2, \vec{k}_2)} \tag{2.51}$$

which gives the correlation structure of any stochastic theory:

$$\boxed{G(\omega, \vec{k}) = \begin{pmatrix} G^K(\omega, \vec{k}) & G^R(\omega, \vec{k}) \\ G^A(\omega, \vec{k}) & 0 \end{pmatrix} \stackrel{\text{linear}}{=} \begin{pmatrix} \frac{B}{\omega^2 + (m + Dk^2)^2} & \frac{1}{\omega - im - iDk^2} \\ \frac{1}{-\omega - im - iDk^2} & 0 \end{pmatrix}} \tag{2.52}$$

Here we defined the Keldysh and Retarded/Advanced components of the correlator. The former gives the correlations for the fluctuations of the field ϕ , while the latter clearly correspond to the response functions. The correlation structure $G(\omega, \vec{k})$ is general, and $\langle \bar{\phi} \bar{\phi} \rangle = 0$ can be shown to be an exact statement beyond the linear theory [2, 105].

The fact that $\langle \phi \bar{\phi} \rangle$ is related to the response function (or Green's functions) of the theory can be seen at the linear level by directly taking the Fourier transform of the stochastic equation:

$$i\omega\phi(\omega, \vec{k}) = -m\phi(\omega, \vec{k}) - Dk^2\phi(\omega, \vec{k}) + \xi(\omega, \vec{k}) \quad (2.53)$$

which gives:

$$\phi(\omega, \vec{k}) = \frac{1}{i\omega + m + Dk^2} \xi(\omega, \vec{k}) = -iG_0^R(\omega, \vec{k}) \xi(\omega, \vec{k}) \quad (2.54)$$

This relationship can easily be shown for a general non-linear theory, as we saw in Eq. 2.45 (we can think of ξ as $\bar{J}$), such that:

$$G^R(t, \vec{x}; t', \vec{x}') = i \langle \phi(t, \vec{x}) \bar{\phi}(t', \vec{x}') \rangle \quad (2.55)$$

Let us mention a few things about the linear theory that illustrate some useful principles. First, we see that the poles of the retarded Green's function give us the dispersion relation of the theory. This is useful in non-equilibrium models, and as we start doing perturbation theory. Additionally, the causality structure of the response functions can be shown explicitly by using the form in Eq. 2.52, see Appendix A.

We also notice that if we look at the equal-time correlation function:

$$G_0^K(t=0, \vec{k}) = \int \frac{d\omega}{2\pi} \frac{B}{\omega^2 + (m + Dk^2)^2} = \frac{B/2}{m + Dk^2} \quad (2.56)$$

we recover the correlator we saw for static Gaussian Landau-Ginzburg theory in Sec.

2.1.3. We see therefore that the static properties of a theory can be completely encoded in the dynamic functional. This is the case for any full nonlinear theory, as long as we are considering an equilibrium model. However, we now also have additional information about the dynamics. For example, at the critical point $m = 0$, the time-momentum space Keldysh Green's function is proportional to e^{-Dk^2t} , which demonstrates the phenomenon of critical slowing-down. The *correlation time* also diverges at a continuous critical point! We explore this in more detail in the next section.

With the linear theory fully understood, we are in the same position as we were in Sec. **2.1.3**. We can apply perturbation theory, define vertex functions, find effective coupling constants, and so on. We do not go into detail, since there are many wonderful resources exploring dynamical field theory specifically at great length [1–3, 10, 12, 68, 89, 105–109].

2.2.4 *Dynamic Scaling and Renormalization Group*

In Sec. **2.1.4** we introduced the idea of a scaling hypothesis in the context of static theories. We saw that for relaxational dynamic models that are based on a free energy (or Hamiltonian) description, the dynamics of interest is given by:

$$\partial_t \phi_i = -\frac{\delta \mathcal{F}[\{\phi_i\}]}{\delta \phi_i} + \xi_i \quad (2.57)$$

with the appropriate (white and Gaussian in equilibrium) noise properties. We also saw in the previous section that we recover the static properties of the system from the above dynamic theory. We can therefore make a reasonable presupposition that a *dynamic scaling hypothesis* is in order. We postulate that there might be a time-scale – correlation time t_c – that diverges at a continuous critical point. Since both the space and time correlations diverge, we can define the *dynamic critical exponent* by setting, in the vicinity of the critical

point:

$$t_c \sim \xi^z \sim |m|^{-\nu z} \quad (2.58)$$

Here, again, m is the parameter measuring the proximity to the critical point at $m = 0$. The hypothesis then postulates that the correlation functions take the form:

$$\langle \phi(t, \vec{x}) \phi(0, 0) \rangle_c = \frac{1}{x^{d-2+\eta}} C \left(\frac{x}{\xi}, \frac{t}{\xi^z} \right) \quad (2.59)$$

The other critical exponents are defined similarly to Eq. 2.14. From here we can extract the time correlations for the same point in space, $r = 0$. By demanding the correlation function remains finite and non-zero, we impose the condition $C \left(\frac{x}{\xi}, \frac{t}{\xi^z} \right) = \left(\frac{x}{\xi} \right)^{d-2+\xi} \tilde{C} \left(\frac{t}{\xi^z} \right)$. By using the fact that $\tilde{C}$ depends only on the combination t/ξ^z , we can replace the resulting pre-factor $\xi^{-(d-2+\eta)}$ with $t^{-(d-2+\eta)/z}$ to get the form:

$$\langle \phi(t, 0) \phi(0, 0) \rangle_c = \frac{1}{t^{\frac{d-2+\eta}{z}}} \tilde{C} \left(\frac{t}{\xi^z} \right) \quad (2.60)$$

However, this construction is not restricted just to equilibrium theories, which we classify in the next section. One of the most famous non-equilibrium universality classes is the Kardar-Parisi-Zhang (KPZ) dynamics [2, 3, 88]:

$$\partial_t h = D \nabla^2 h + \frac{\lambda}{2} (\vec{\nabla} h)^2 + \xi \quad (2.61)$$

The scaling hypothesis is defined in exactly the same way for this model, although usually discussed in terms of the roughening exponent $\chi = \frac{2-d-\eta}{2}$. Many other non-equilibrium universality classes still adhere to this form of the scaling hypothesis [2]. However, there are situations where this form is inappropriate, for example when a unique dynamic exponent z cannot be identified. We mention such cases throughout this dissertation.

The important takeaway of the dynamic scaling hypothesis is that at the critical point

we expect to be able to scale both space and time in a way that preserves the statistics of the system. For example, one can see from the Gaussian description of the previous section that we can expect $z = 2$ for such a linear theory. How do we find the exponents and scaling forms beyond the linear theory? The answer is the same as for static theories, except now using the MSRJD action. This also applies to the momentum-shell RG calculations, where we now also have to integrate over the frequencies of the Fourier modes. There are many resources describing momentum-shell RG in dynamical contexts [3, 4]. Here, for completeness, we introduce the slightly more systematic approach of minimal subtraction RG.

We follow [2] for this brief overview. Given an action, we can define the vertex functions through its Legendre transform. For example, one can show that for the linear theory action in 2.48:

$$\begin{aligned}\Gamma^{(1,1)}(\omega, \vec{k}) &= G^{R,-1}(-\omega, -\vec{k}) = -\omega - im - iDk^2 \\ \Gamma^{(0,2)}(\omega, \vec{k}) &= -\frac{G^K(\omega, \vec{k})}{G^R(\omega, \vec{k})G^A(\omega, \vec{k})} = -B \\ \Gamma^{(0,2)}(\omega, \vec{k}) &= 0\end{aligned}\tag{2.62}$$

The left-most equalities on each line hold for non-linear theories as well, since they come from the definitions of the vertex functions. One can similarly find vertex functions of higher order. Just like one can do perturbation theory to calculate the non-linear contributions to the correlation functions directly, one can calculate the contributions to the vertex functions. In fact, working with Γ is often more convenient, since one can isolate the N -particle irreducible contributions (see refs. above).

We see that the vertex functions, in some sense, correspond to the terms we find in the action itself. For example, in this model $m = -i\Gamma^{(1,1)}(0, 0)$. However, as discussed before, the purpose of RG is to track the change of the couplings in the theory as one changes the scale. Therefore, one can undergo a systematic RG procedure by tracking the evolution of the vertex functions.

Given the bare fields ϕ_i (which can be separate or simply components of a single field) and coupling constants c_n appearing in the path integral of the theory, we can define the renormalized fields and constants:

$$\phi_{i,R} = Z_{\phi_i}^{\frac{1}{2}} \mu^{\alpha_{\phi_i}} \phi_i \quad , \quad \bar{\phi}_{i,R} = Z_{\bar{\phi}_i}^{\frac{1}{2}} \mu^{\alpha_{\bar{\phi}_i}} \bar{\phi}_i \quad , \quad c_{n,R} = Z_{c_n} \mu^{\alpha_n} c_n \quad (2.63)$$

where μ is an arbitrary momentum scale that makes the renormalized constants dimensionless. The Z factors are there to absorb all of the UV and IR divergencies that come from the integrals one has to compute to perform perturbative calculations. This is done with the idea that, of course, when real measurements are performed on the correlation functions, no divergencies should be observed. The factors α are often called engineering dimensions of the fields and constants, which correspond to the rescaling constants Δ_O we saw in Sec. 2.1.4. From above, it can also be observed that we can define:

$$\Gamma_R^{(\bar{N},N)} \sim Z_{\bar{\phi}_i}^{-\frac{\bar{N}}{2}} Z_{\phi_i}^{-\frac{N}{2}} \Gamma^{(\bar{N},N)} \quad (2.64)$$

which comes from the fact that the vertex functions are inversely related to correlation functions. As mentioned, by demanding that the renormalized coupling constants remain finite, and with self-consistent perturbation theory, we absorb all divergences into the Z factors. It is important to understand that real measurements always happen on renormalized quantities, where the scale μ is determined externally by a measurement device. The crucial observation is that since μ is arbitrary, the unrenormalized vertex functions should not depend on it! They only depend on the momentum cutoff Λ that is the result of coarse-graining. This means that holding all the other unrenormalized parameters fixed, we must have:

$$0 = \mu \frac{d}{d\mu} \Big|_{c_n} \Gamma^{(\bar{N},N)} \quad (2.65)$$

By plugging in all of the above definitions, one can then derive the renormalization group

equation for the vertex functions – the Callan-Symanzik equation (see [2]). Its precise form is not too important for our purposes. The relevant quantities are Wilson’s flow functions defined by:

$$\begin{aligned}\gamma_{\phi_i} &= \mu \frac{\partial}{\partial \mu} \ln \frac{\phi_{i,R}}{\phi_i} = \alpha_{\phi_i} + \frac{1}{2} \mu \frac{\partial}{\partial \mu} \ln Z_{\phi_i} \\ \gamma_{\bar{\phi}_i} &= \mu \frac{\partial}{\partial \mu} \ln \frac{\bar{\phi}_{i,R}}{\bar{\phi}_i} = \alpha_{\bar{\phi}_i} + \frac{1}{2} \mu \frac{\partial}{\partial \mu} \ln Z_{\bar{\phi}_i} \\ \gamma_{c_n} &= \mu \frac{\partial}{\partial \mu} \ln \frac{c_{n,R}}{c_n} = \alpha_n + \mu \frac{\partial}{\partial \mu} \ln Z_{c_n}\end{aligned}\tag{2.66}$$

These describe how the renormalized parameters change as we change the arbitrary scale at which we consider the theory. To see their importance, we need to solve the RG equation. One can do so with a method of characteristics by defining a running scale $\tilde{\mu}(l) = \mu e^{-l}$, where $l \in [0, \infty)$. This allows the definition of running constants:

$$\frac{d\tilde{c}_{n,R}(l)}{dl} = -\tilde{c}_{n,R}(l)\gamma_{c_n}(l)\tag{2.67}$$

with $\tilde{c}_{n,R}(0) = \tilde{c}_{n,R}$ restoring the renormalized coupling constant at the original scale. With judicious choices of l and μ , we can now extract the critical exponents of interest.

Many details have been omitted in this exposition, and we refer to the resources mentioned above. The minimal subtraction scheme demonstrates how perturbative renormalization group is related to standard perturbation theory, and is completely equivalent to Wilson’s momentum-shell RG described in the static context in Sec. 2.1.4. Nothing in the above overview relied on the assumption of equilibrium. This renders the renormalization group technique exceptionally powerful in the non-equilibrium setting as well, as long as the system has some sort of scale-invariance. However, in the next section, we briefly present the powerful classification of equilibrium critical dynamics, since we often build on it and draw parallels with it.

2.2.5 Overview of Hohenberg-Halperin Classification

The seminal paper [1] introduced the classification of systems undergoing dynamics near a critical point with the names Model A through J. They are loosely split into two categories – purely relaxational Models A through D, and Models E through J with reversible dynamics. We provide a brief overview in this section, which follows the resources [1–4].

Models A and B are concerned with relaxational dynamics in the Landau-Ginzburg-Wilson Hamiltonian for an $O(N)$ order parameter:

$$\mathcal{H}_{O(N)} = \int_{t,x} \sum_i \left\{ \frac{m}{2} \phi_i(t, \vec{x})^2 + \frac{D}{2} \left[\vec{\nabla} \phi_i(t, \vec{x}) \right]^2 + \frac{g}{4} \sum_j \phi_i(t, \vec{x})^2 \phi_j(t, \vec{x})^2 - h_i(t, \vec{x}) \cdot \phi_i(t, \vec{x}) \right\} \quad (2.68)$$

Here, $\phi(t, \vec{x})$ is an N -component field. The Hamiltonian, which resembles the Ising free energy in Eq. 2.10, describes a symmetry-breaking transition from a disordered phase with $\langle \phi(t, \vec{x}) \rangle = 0$ to a symmetry broken phase with $\langle \phi(t, \vec{x}) \rangle \neq 0$, where the direction is chosen randomly in the $O(N)$ space. Model A describes purely relaxational dynamics of an order parameter in this energy landscape:

$$\partial_t \phi_i(t, \vec{x}) = - \frac{\delta \mathcal{H}_{O(N)}}{\delta \phi_i(t, \vec{x})} + \xi_i(t, \vec{x}) \quad (2.69)$$

with the noise correlator given by $\langle \xi_i(t, \vec{x}) \xi_j(t', \vec{x}') \rangle = B \delta_{ij} \delta(t - t') \delta^d(x - x')$. This Langevin equation is often referred to as the Landau-Ginzburg equation [2, 86]. This form of dynamics is valid near a continuous phase transition where all other dynamics of the system occur on much slower time scales (that presumably stay finite). Model A is a natural starting point in describing any dissipative order-to-disorder phase transition dynamics.

However, it is often the case that each component of the order parameter is conserved

in space, such that a continuity equation $\partial_t \phi_i(t, \vec{x}) = -\vec{\nabla} \cdot \vec{J}_i(t, \vec{x})$ holds. It is reasonable to assume that the probability current is proportional to the gradient of the variation of the energy, such that:

$$\vec{J}_i(t, \vec{x}) = -\vec{\nabla} \frac{\delta \mathcal{H}_{O(N)}}{\delta \phi_i(t, \vec{x})} + \xi_i(t, \vec{x}) \quad (2.70)$$

with similar noise correlations. Thus, we arrive at Model B dynamics:

$$\partial_t \phi_i(t, \vec{x}) = \nabla^2 \frac{\delta \mathcal{H}_{O(N)}}{\delta \phi_i(t, \vec{x})} + \eta_i(t, \vec{x}) \quad (2.71)$$

with $\langle \eta_i(t, \vec{x}) \eta_j(t', \vec{x}') \rangle = -B \delta_{ij} \nabla^2 \delta(t - t') \delta^d(x - x')$. The deterministic version of the above equation is the Cahn-Hilliard equation, often used to model binary fluid mixtures [33, 110–113].

For completeness, we mention that both Models A and B can be combined into one system by writing the dynamics as:

$$\partial_t \phi_i(t, \vec{x}) = -(i\vec{\nabla})^a \frac{\delta \mathcal{H}_{O(N)}}{\delta \phi_i(t, \vec{x})} + \xi_i(t, \vec{x}) \quad (2.72)$$

with $\langle \xi_i(t, \vec{x}) \xi_j(t', \vec{x}') \rangle = B \delta_{ij} (i\vec{\nabla})^a \delta(t - t') \delta^d(x - x')$, with $a = 0$ or 2 for Models A and B, respectively. Thus, all the results, such as perturbation theory, RG and expressions for the critical exponents can be found for the two models simultaneously.

It is also often the case that there are other conserved quantities, aside from the order parameter, which have to be taken into account. The simplest example is the energy density. To include some coupling between an $O(N)$ order parameter and an energy density, one has to add the following contribution for the Hamiltonian in Eq. 2.68:

$$\Delta \mathcal{H} = \frac{1}{2C} \int dt d^d x \left\{ \rho(t, \vec{x})^2 + g_\rho \rho(t, \vec{x}) \phi(t, \vec{x})^2 \right\} \quad (2.73)$$

Such coupling does not affect the static exponents, but it does affect the dynamics.

Dynamics of a non-conserved or conserved order parameter coupled to such an energy density defines Models C and D, respectively:

$$\begin{aligned}\partial_t \phi_i(t, \vec{x}) &= -(i\vec{\nabla})^a \frac{\delta \left(\mathcal{H}_{O(N)} + \Delta \mathcal{H} \right)}{\delta \phi_i(t, \vec{x})} + \xi_i(t, \vec{x}) \\ \partial_t \rho(t, \vec{x}) &= -\lambda \nabla^2 \frac{\delta \Delta \mathcal{H}}{\delta \rho(t, \vec{x})} + \eta(t, \vec{x})\end{aligned}\tag{2.74}$$

with $\langle \eta(t, \vec{x}) \eta(t', \vec{x}') \rangle = -\lambda B_\rho \nabla^2 \delta(t - t') \delta^d(x - x')$ and the order parameter noise correlations as before. Model C is often used to model structural phase transitions (for example in crystals), while Model D is used for compressible binary mixtures [1].

The generic evolution of order parameters that are also governed by reversible terms is given by:

$$\partial_t \phi_i(t, \vec{x}) = F_i^{\text{rev}}[\phi(t, \vec{x})] - (i\vec{\nabla})^a \frac{\delta \mathcal{H}}{\delta \phi_i(t, \vec{x})} + \xi_i(t, \vec{x})\tag{2.75}$$

Where $F_i^{\text{rev}}[\phi(t, \vec{x})]$ has to be suitably constructed from the Poisson brackets (commutators in the quantum setting) to satisfy the condition of equilibrium [2]. A generic construction is:

$$F_i^{\text{rev}}[\phi_i(t, \vec{x})] = - \int dt d^d x' \sum_j \left[\{ \phi_i(t, \vec{x}), \phi_j(t, \vec{x}') \} \frac{\delta \mathcal{H}}{\delta \phi_j(t, \vec{x}')} - B \frac{\delta \{ \phi_i(t, \vec{x}), \phi_j(t, \vec{x}') \}}{\delta \phi_j(t, \vec{x}')} \right]\tag{2.76}$$

Without quoting each Hamiltonian and reversible term directly, we briefly list the models in the Hohenberg-Halperin (HH) classification. Model E describes planar ferromagnets, with a two-component non-conserved order parameter coupled to a conserved scalar field (the component of the spin perpendicular to the plane). Model F describes a non-conserved two-component order parameter (often complex) coupled to two conserved scalar fields. It can describe the superfluid phase transition in Helium-4. Model G describes anti-ferromagnets, with non-conserved three-component order parameter (staggered magnetization) coupled to a three-component conserved field (total magnetization). Model H describes a conserved scalar

order parameter coupled to a conserved vector field, such as the liquid-gas critical point and phase separation in binary fluids. Finally, Model J describes isotropic ferromagnets with spin precession. See [1, 2] for overviews.

The HH universality classification covers a large variety of equilibrium systems and their dynamics. However, what makes them equilibrium? And what do we encounter when we consider non-equilibrium systems? With the overview of field theoretic ideas and methods presented in this chapter, we now proceed to discussing these questions.

CHAPTER 3

BASICS OF NON-EQUILIBRIUM

3.1 What Constitutes Equilibrium?

3.1.1 Detailed Balance and Time-Reversibility

In the previous chapter, we introduced the ideas of coarse-graining and how a plethora of systems in and out of equilibrium can be modeled as dynamical field theories. However, we did not give a consistent definition of equilibrium. The following sections are concerned with providing it, at least as far as it is relevant for this dissertation. There are many equivalent definitions, some more practically useful than others, and therefore the ones we discuss are not exhaustive. Most of them are, however, natural for dynamical systems and field-theoretic treatments.

Stepping back for a moment from dynamical systems into the realm of more general stochastic processes, we first need to introduce the notion of a *Markov process*. Given a stochastic process, we can define a many-point probability density $P_n(x_n, t_n; \dots; x_1, t_1)$ where the variable x (whatever it might be, and not just necessarily space) takes the values $x_1, \dots, x_n$ at times $t_1, \dots, t_n$, respectively. From standard probability theory [104], one can define a conditional probability $P_{m|k}(x_{k+1}, t_{k+1}; \dots; x_{k+m}, t_{k+m} | x_1, t_1; \dots; x_k, t_k)$ that describes the probability of x taking the values $x_{k+1}, \dots, x_{k+m}$ at times $t_{k+1}, \dots, t_{k+m}$ given it took the values $x_1, \dots, x_k$ at times $t_1, \dots, t_k$. $P_{1|n}$ is of particular interest, because it is concerned with the state of the process given the n states it took before. A Markov process, or chain, is defined by the conditional probability only depending on the state of the process a time step before:

$$P_{1|n}(x_{n+1}, t_{n+1} | x_1, t_1; \dots; x_n, t_n) = P_{1|1}(x_{n+1}, t_{n+1} | x_n, t_n) \quad (3.1)$$

Therefore, Markov processes contain no memory, because the state of the process at the

next time point is fully determined by its current state. They represent a combination of sufficient analytical tractability and a not-too-egregious approximation for many stochastic processes. The latter is true because even if the process is not Markovian, one can often either expand the state-space as to make it so, or "coarse-grain" time such that long-time correlations disappear. For example, if one were to consider Brownian motion of particles and look at their positions, there is certainly memory. However, you can use a Langevin equation to describe a velocity with white noise (with no correlation in time), such that the process becomes Markovian at the expense of doubling the state space. With this in mind, the problems considered in this dissertation are mostly written in a Markovian manner, and we consider the definitions of equilibrium with this in mind as well.

Now let us consider a one-point distribution $P_1(x, t)$ for a process with a continuous state space x . Given the property in Eq. 3.1, one can derive the master equation [2, 4]:

$$\partial_t P_1(x, t) = \int dx' [P_1(x', t)W(x' \rightarrow x, t) - P_1(x, t)W(x \rightarrow x', t)] \quad (3.2)$$

Here $W(x \rightarrow x', t) = \partial_\tau P_{1|1}(x', t + \tau | x, t)$ is the transition rate from state x to state x' at the time t . The master equation simply counts the "incoming" and "outgoing" contributions to the probability $P_1(x, t)$ with respect to all other states x' . There are a few details related to this equation. For instance, one can define the time-dependent entropy [2] from $P_1(x, t)$, which also helps with the forthcoming definition of equilibrium. We refer to the resources cited throughout this section for more details.

In equilibrium, we are often concerned with time-translation invariant problems. In fact, having a few macroscopic quantities, like temperature, energy, pressure, volume, particle number, etc. (depending on the choice of ensemble) that do not change in time can be taken as part of the definition of equilibrium. We therefore look for a time-independent probability density $P_s(x)$ with the assumption that the transition rates are also time-independent. A

sufficient condition for the existence of such solution is the *detailed balance* relation:

$$\boxed{P_s(x')W(x' \rightarrow x) = P_s(x)W(x \rightarrow x')} \quad (3.3)$$

We take this as our first definition of equilibrium. In particular, the entropy production rate is zero if detailed balance relation holds [2].

However, there is a more illuminating, but completely equivalent form of the detailed balance condition. It is given by the *Kolmogorov criterion*, which states that for a stationary equilibrium solution, all possible cycles of arbitrary length with intermediate states x_i have to be equally likely in the forward and backward direction:

$$W(x_0 \rightarrow x_1)W(x_1 \rightarrow x_2)\dots W(x_N \rightarrow x_0) = W(x_0 \rightarrow x_N)\dots W(x_2 \rightarrow x_1)W(x_1 \rightarrow x_0) \quad (3.4)$$

Therefore, both expressions show that equilibrium is closely related to the process being time-reversible. More broadly, the entropy production rate is given by the ratio of the two sides in Eq. 3.3 [2, 114] and is therefore zero when the forward and backward processes are balanced.

3.1.2 Fokker-Planck and Steady State Solutions

Although the definitions provided above are applicable to general stochastic processes, they are not the most practical. In some sense, we need to solve the stochastic system for the stationary distribution and the transition rates to see whether it is in equilibrium, at which point it is mostly redundant. Therefore, we want to develop an understanding of how we can tell whether the system is in equilibrium based solely on its evolution rules. The two common ways of defining such evolution is either through the Fokker-Planck (FP) equation, or through the Langevin equation. Let us introduce the former and understand how we can understand equilibrium in the FP perspective.

A generic derivation of the Fokker-Planck equation relies on the Chapman-Kolmogorov equation:

$$P_{1|1}(x', t' | x, t) = \int d\bar{x} P_{1|1}(x', t' | \bar{x}, \bar{t}) P_{1|1}(\bar{x}, \bar{t} | x, t) \quad (3.5)$$

where $t < \bar{t} < t'$ is arbitrary. Again assuming the transition probabilities are invariant with respect to time translations, one can show [2] that the evolution of the one-point probability function is described by the Kramers-Moyal expansion:

$$\partial_t P_1(x, t) = \sum_{k=1}^{\infty} \frac{(-1)^k}{k!} \partial_x^k [\alpha_k(x) P_1(x, t)] \quad (3.6)$$

where the coefficient $\alpha_k(x) = \lim_{\tau \rightarrow 0} \frac{1}{\tau} \int_{\xi} \xi^k P(x, \tau | x - \xi, 0)$ is the k -th moment of the transition probability. This hardly seems like an improvement, except for the fact that one can show that either the first two coefficients are non-zero while the rest vanish, or they are all non-zero. In the former case, we get the famed Fokker-Planck equation:

$$\partial_t P_1(x, t) = -\partial_x J_1(x, t) = -\partial_x [\alpha_1(x) P_1(x, t)] + \frac{1}{2} \partial_x^2 [\alpha_2(x) P_1(x, t)] \quad (3.7)$$

We therefore recover a local equation of the one-point probability density alone. It has a familiar form of a continuity equation, with the probability current that can be read off above. Therefore, a stationary solution is governed by $\partial_x J_1(x, t) = 0$, but that does not necessarily guarantee equilibrium! Intuitively, an equilibrium distribution should be the one where all probability currents vanish. In order to understand the relationship with thermodynamics, we should understand how the FP equation relates to Langevin equations.

Given a stochastic equation:

$$\partial_t x = f(x) + \xi(x, t) \quad (3.8)$$

With correlations $\langle \xi(x, t) \xi(x, t') \rangle = B(x) \delta(t - t')$, one can show [2–4] that the one-point

probability distribution for x satisfies:

$$\partial_t P_1(x, t) = -\partial_x [f(x)P_1(x, t)] + \partial_x^2 [B(x)P_1(x, t)] \quad (3.9)$$

thus showing that Langevin dynamics and the FP equation are intimately related. However, we know that in the canonical ensemble, a statistical system follows the Boltzmann distribution $P(x) \sim e^{-H(x)/T}$. In order for the stationary solution to the FP equation to have this form, we must have:

$$f(x) = \partial_x B(x) - \frac{\partial_x H(x)}{T} B(x) \quad (3.10)$$

However, $f(x)$ corresponds to deterministic dynamics, which should not vanish even in the absence of noise. The only way that can be achieved is if $B(x) = T$, which gives $f(x) = -\partial_x H(x)$. We therefore see that an equilibrium stochastic system corresponds to overdamped descent in an energy landscape with additive noise. Later we show that one can obtain this condition from symmetry as well.

However, we saw from the HH classification in Sec. 2.2.5 that we should be able to describe equilibrium systems having reversible terms in addition to dissipative dynamics. Let us then write $f(x) = -\partial_x H(x) + f_{\text{rev}}(x)$. The only way this is compatible with the Boltzmann distribution is if:

$$\partial_x \left[f_{\text{rev}}(x) e^{-\frac{H(x)}{T}} \right] = 0 \quad (3.11)$$

The probability current associated with the reversible terms has to be divergence-free. In a sense, reversible terms explore the landscape of equal-energy solutions, think spin precession, for example. All of the results and considerations in this section also apply to spatially-extended stochastic systems where we apply the field-theoretic treatment.

3.1.3 Equilibrium as a Symmetry and Fluctuation-Dissipation Theorems

We saw in the previous section that the noise structure most compatible with equilibrium is additive Gaussian white noise. Throughout most of this dissertation, as well as in the following sections, we assume that the noise takes that form. In Section 3.2 we provide examples of non-equilibrium systems where this is not the case.

We saw in Section 3.1.1 that equilibrium is related to time-reversibility of the stochastic process. Let us consider how this might be applied to Langevin dynamics and the MSRJD field theory. We see that for a generic stochastic equation for a single field in the form:

$$\partial_t \phi(t, \vec{x}) = f[\phi](t, \vec{x}) + \xi(t, \vec{x}) \quad (3.12)$$

we can replace $t \rightarrow -t$ and $\phi(t, \vec{x}) \rightarrow \phi(-t, \vec{x})$ and recover the same equation, since white Gaussian noise is uncorrelated in time. However, this transformation is not a symmetry of the MSRJD action in Eq. 2.35. It turns out the right transformation to look at is [9, 12, 19]:

$$\begin{aligned} \phi'(t, \vec{x}) &= \phi(-t, \vec{x}) \\ \bar{\phi}'(t, \vec{x}) &= \bar{\phi}(-t, \vec{x}) + \frac{2i}{B} \partial_t \phi(-t, \vec{x}) \end{aligned} \quad (3.13)$$

In that case, the action changes as:

$$\begin{aligned} S'_{MSR} &= \int dt \int d^d x \left[-i \left(\bar{\phi}(-t, \vec{x}) + \frac{2i}{B} \partial_t \phi(-t, \vec{x}) \right) (\partial_t \phi(-t, \vec{x}) - f[\phi](-t, \vec{x})) \right. \\ &\quad \left. + \frac{B}{2} \left(\bar{\phi}(-t, \vec{x}) + \frac{2i}{B} \partial_t \phi(-t, \vec{x}) \right)^2 \right] = S_{MSR} + \frac{2}{B} \int dt \int d^d x \partial_t \phi(t, \vec{x}) f[\phi](t, \vec{x}) \end{aligned} \quad (3.14)$$

For this transformation to be a symmetry of the action, it is sufficient to set $f[\phi](t, \vec{x}) =$

$-\frac{\delta H[\phi]}{\delta \phi(t, \vec{x})}$, in which case:

$$\int dt \int d^d x \partial_t \phi(t, \vec{x}) f[\phi](t, \vec{x}) = - \int dt \int d^d x \partial_t \phi(t, \vec{x}) \frac{\delta H[\phi]}{\delta \phi(t, \vec{x})} = - \int dt \int d^d x \partial_t H[\phi] \quad (3.15)$$

and we can ignore the surface terms at $t \rightarrow \pm\infty$. A symmetry transformation and proof for multiple-component field is easily obtained as well. Thus, we see that relaxational dynamics in an energy landscape again emerge as a signature of equilibrium. The equilibrium symmetry transformation in Eq. 3.13 can also be generalized to conserved dynamics. However, demonstrating this is a symmetry for dynamics with reversible terms is much more difficult. One has to then use a different form of the symmetry transformation where $\partial_t \phi(-t, \vec{x})$ is replaced by $\frac{\delta \mathcal{H}[\phi]}{\delta \phi(t, \vec{x})}$. In this dissertation, we consider dissipative dynamics in contexts where we draw parallels with equilibrium systems. We therefore say that dissipative Langevin dynamics with white and spatially uncorrelated noise describe an equilibrium process if and only if they are in the form:

$$\left\{ \begin{array}{l} \partial_t \phi_i(t, \vec{x}) = -\frac{\delta H[\phi]}{\delta \phi_i(t, \vec{x})} + \xi_i(t, \vec{x}) \\ \langle \xi_i(t, \vec{x}) \xi_j(t', \vec{x}') \rangle = B \delta_{ij} \delta(t - t') \delta^d(x - x') \end{array} \right. \Leftrightarrow \text{Equilibrium} \quad (3.16)$$

To reiterate, this is *not* the most general equilibrium critical dynamics form, as was seen, for example, in the HH classification in Section 2.2.5.

A few ways to go outside of equilibrium are immediately noticeable in Eq. 3.16. First, for a single component field, evolution in a general potential that depends only on powers of ϕ , and not on its gradients, can always be written in equilibrium form. Therefore, many non-equilibrium problems are written in terms of many-component fields, which often represent different species of agents. However, a famous non-equilibrium model that only has one field component but has a non-variational gradient term is the KPZ model we mentioned before. One can also imagine convective derivatives, which often occur in hydrodynamic

theories. Finally, multiplicative noise also generically (but not always) brings the system out of equilibrium and can lead to absorbing states. We discuss these possibilities, with examples, in Section 3.2.

For completeness, we mention a few details. First, we note that all of the above discussion is in the context of Ito discretization. However, there is a way to introduce this symmetry into the general α -discretization formalism by using Grassmann (or ghost) fields [12, 105]. With that construction, the theory becomes supersymmetric, which is a whole separate field of study [12, 105, 115–120]. Additionally, we mainly consider additive white noise in this dissertation, along with an overdamped limit for the Langevin dynamics. However, equilibrium constraints and the above symmetry can be formulated for more general problems, including multiplicative noise and inertial problems, see [120] and references therein.

Now, a famous property of equilibrium dynamics is the tower of fluctuation dissipation theorems (FDTs), sometimes also called Onsager relations [2–7]. They are often presented in the form of relations between the correlation and response functions. They can be derived from many perspectives, but given our discussion of symmetry, we show an elegant derivation of one of the FDTs. Recalling the discussion on symmetries at the end of Sec. 2.1.2, we consider the response function in the MSRJD formalism and apply the equilibrium symmetry transformation:

$$\begin{aligned}\langle \phi(t, \vec{x}) \bar{\phi}(t', \vec{x}') \rangle &= \left\langle \phi(-t, \vec{x}) \left[\bar{\phi}(-t', \vec{x}') + \frac{2i}{B} \partial_{t'} \phi(-t', \vec{x}') \right] \right\rangle \\ &= \langle \phi(-t, \vec{x}) \bar{\phi}(-t', \vec{x}') \rangle + \frac{2i}{B} \partial_{t'} \langle \phi(t, \vec{x}) \phi(t', \vec{x}') \rangle\end{aligned}\tag{3.17}$$

Here we used the symmetry transformation on the second term to get to the last line. Given the definition of response and correlation functions in Eq. 2.52, and the time and space translation invariance of the correlation and response functions, we immediately get:

$$\frac{iB}{2} \left[G^R(t, \vec{x}) - G^A(t, \vec{x}) \right] = \partial_t G^K(t, \vec{x})\tag{3.18}$$

Or, equivalently, in Fourier space:

$$\frac{B}{2\omega} \left[G^R(\omega, \vec{k}) - G^A(\omega, \vec{k}) \right] = G^K(\omega, \vec{k}) \quad (3.19)$$

This is an exact result that holds for any non-linear problems obeying the symmetry in Eq. 3.13. Similar results can also be derived in the quantum setting [3, 9, 10]. One can derive an infinite tower of relations similar to the above one. However, we see now that the FDTs are all related to a single symmetry (in general, they are the Ward identities), demonstrating the result that has been hinted at throughout the above chapters.

3.2 Phenomena in Non-Equilibrium

With the broad definitions and properties of equilibrium outlined in the previous section, we move on to the discussion of non-equilibrium systems. This overview is by no means exhaustive or detailed, since the theory of non-equilibrium phenomena covers an amazing span of problems, in many fields. We first provide a few examples of non-equilibrium systems, their respective models and novel phenomena, and then focus onto the specific topics explored in this dissertation. We attempt to provide both the microscopic and the hydrodynamic, or field-theoretic, descriptions for the examples we provide.

3.2.1 Active Systems

One of the conceptually simplest forms of non-equilibrium matter is a collection of microscopic units that have internal sources of energy. This is obviously the case for living things, such as bacteria or animals, where internal energy is converted into motion, growth, and reproduction. However, one can also engineer systems of self-propelling units, or microswimmers, such as active colloids (Janus particles, for example). Active matter is an area spanning a wide variety of systems, and for reviews beyond the basic illustrations presented in this sec-

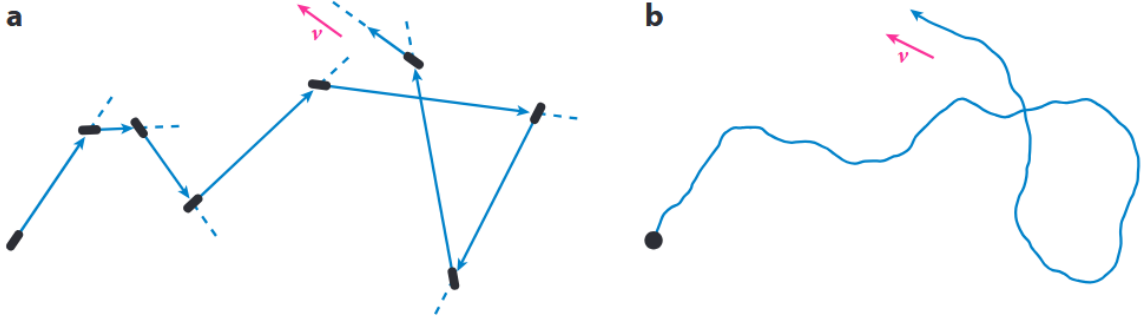


Figure 3.1: Schematic representations of a) RTPs and b) ABPs. Figure adopted from [110].

tion, we refer to the vast literature on the topic, see [8, 29, 111–113, 121–124] and references therein.

How do we model such forms of matter? In equilibrium, the simplest statistical process of particle dynamics we consider is Brownian motion. In contrast, for active matter, one can imagine a system of non-interacting particles that propel at constant speed, but where the orientation of motion changes randomly in time. This produces two models – Run-and-Tumble Particles (RTPs) and Active Brownian Particles (ABPs). For illustration of their typical motion, see Fig. 3.1. The former corresponds to motion where the particle propels in a fixed direction for some time, then stops and reorients before propelling again. The latter is a more continuous version where the particle constantly reorients, and is described by the following equations in two dimensions [8]:

$$\partial_t \vec{r}(t) = v_0 \begin{pmatrix} \cos \theta(t) \\ \sin \theta(t) \end{pmatrix}, \quad \partial_t \theta(t) = \sqrt{2D_\theta} \xi(t) \quad (3.20)$$

Here $\vec{r}$ is the position of a particle or unit and θ describes its orientation. A non-equilibrium phase transition in a collection of such particles naturally requires some sort of interaction between them. In a coarse-grained description, this leads to the so-called Active

Model B [8, 110, 125], where the density of such particles evolves according to:

$$\partial_t \rho(t, \vec{x}) = -\vec{\nabla} \cdot \left[\vec{J}[\rho](t, \vec{x}) + \sqrt{2B} \vec{\xi}(t, \vec{x}) \right] \quad (3.21)$$

where $\vec{J}$ is not necessarily related to a variation of some functional, unlike the equilibrium Model B, see Sec. 2.2.5. For example, it could be $\vec{J} = v(\rho) \vec{\nabla} [v(\rho) \rho]$ [8], where the speed of propulsion is now density dependent. If the speed decreases with particle density, this system can exhibit Motility-Induced Phase Separation (MIPS), where the particles cluster, leading to high and low density patterns forming throughout the system [8, 110, 125, 126]. It is debated whether MIPS might belong to the Ising universality class [127, 128].

Another class of active particle models are Active Ornstein-Uhlenbeck Particles (AOUPs), which undergo Brownian motion but with noise that has correlations in time. Such models are useful when, for example, the self-propulsion velocity has its own fluctuations and time evolution. These are described by the set of equations of the form [129, 130]:

$$\begin{aligned} \partial_t \vec{r}(t) &= \vec{f}(\vec{r}(t), t) + \sqrt{2D_r} \vec{\xi}(t) + \vec{v}(t) \\ \partial_t \vec{v}(t) &= -\frac{1}{\tau_v} \vec{v}(t) + \sqrt{2D_v} \vec{\eta}(t) \end{aligned} \quad (3.22)$$

such that $\langle v_i(t) v_j(0) \rangle \sim \delta_{ij} \exp\left(-\frac{t}{\tau_v}\right)$. The continuum models corresponding to a collection of such particles (obtained, for example, through the Dean equation, see Sec. 2.1.1) also have memory.

Another paradigmatic example of collective motion of active units is flocking. One can literally imagine a flock of birds or a school of fish where patterns of collective behavior form and are clearly observable. A conceptually simple model of flocking is the Vicsek model

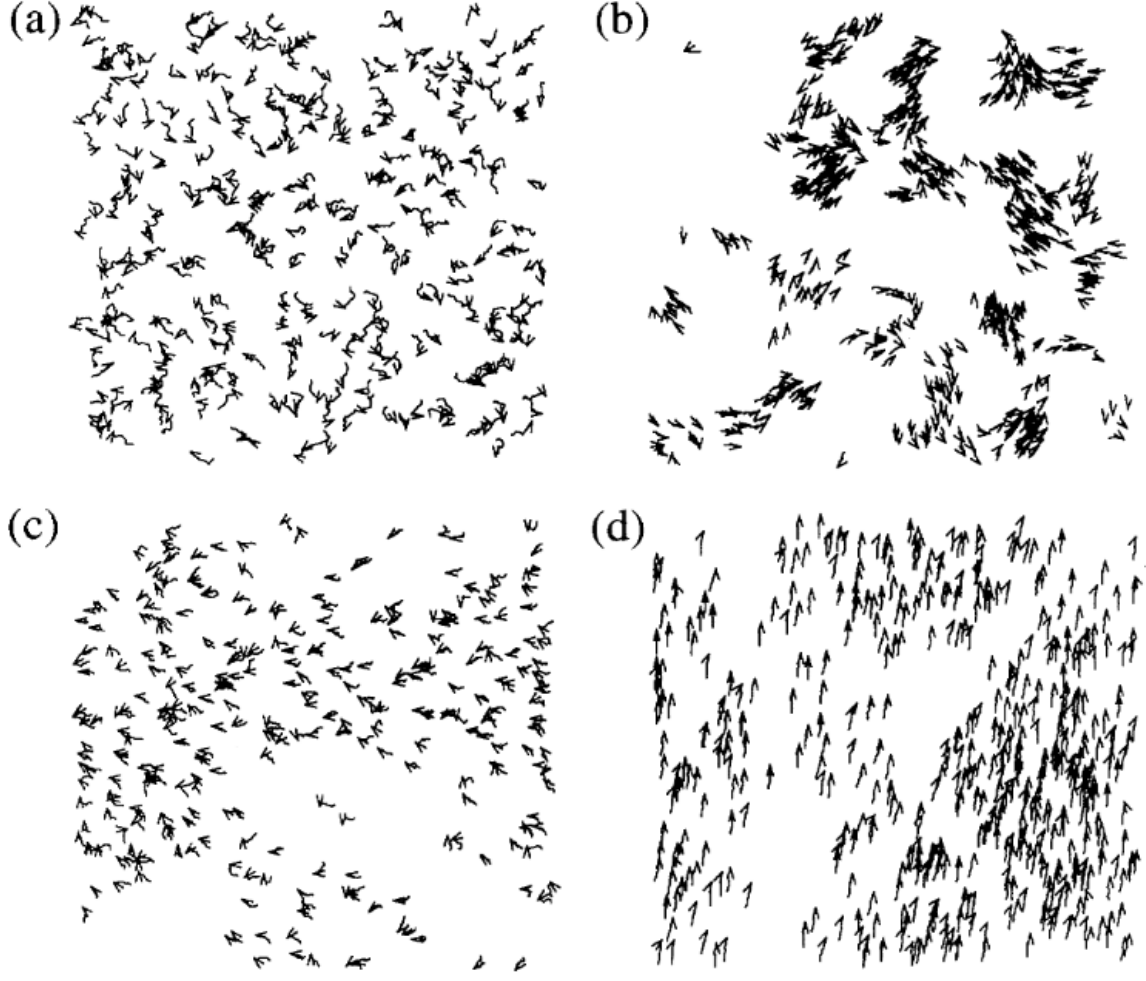


Figure 3.2: Illustration of flocking agents. a) and c) correspond to high noise and high and low densities of agents, respectively. b) is a low noise, low density phase with many smaller flocks. d) is a low noise, high density phase with a single flock. Figure adopted from [131].

[8, 131]:

$$\partial_t \vec{r}(t) = v_0 \begin{pmatrix} \cos \theta(t) \\ \sin \theta(t) \end{pmatrix}, \quad \partial_t \theta(t)_i = - \left(\theta_i(t) - \langle \theta(t) \rangle_{R,i} \right) + \sqrt{2D_\theta} \xi_i(t) \quad (3.23)$$

Here $\langle \theta(t) \rangle_{R,i}$ is the average orientation angle of particles in some neighborhood around particle i defined by the radius of interaction R . $\theta_i(t) - \langle \theta(t) \rangle_{R,i}$ is understood to take into account the compactness of θ . Therefore, a particle tends to orient itself to the average

orientation of its neighbors. The only conserved quantity is the number of particles, and the model exhibits three distinct phases. At high noise levels, the system is disordered, and no flocking emerges. At high density and low noise, a long-range order of the orientation parameter emerges, and the system is one big flock, which is referred to as the ordered homogeneous phase. At low density and low noise, many distinct flocks appear and the particle density is spatially modulated. These are genuinely non-equilibrium phases, and the ordered homogeneous phase violates the Mermin-Wagner theorem (see refs. above). For illustration, see Fig. 3.2.

The continuum description of flocking models usually takes the form of Toner-Tu theory [8, 132–134]. It describes the evolution of the coarse-grained density $\rho(t, \vec{x})$ and velocity $\vec{v}(t, \vec{x})$ fields:

$$\begin{aligned}\partial_t \rho &= -\vec{\nabla} \cdot (\rho \vec{v}) \\ \partial_t \vec{v} + \lambda \left(\vec{v} \cdot \vec{\nabla} \right) \vec{v} &= \alpha \vec{v} - \beta |\vec{v}|^2 \vec{v} - \vec{\nabla} P + D \nabla^2 \vec{v} + I(\vec{v}) + \sqrt{B} \vec{\xi}\end{aligned}\tag{3.24}$$

Here, P is the pressure and $I(\vec{v})$ includes other terms dependent on velocity. Notice here the emergence of a convective derivative, which was hinted before as one of the ways to go outside of equilibrium. The terms α , β and D remind of the already familiar Model A (or ϕ^4 theory) description; see Sect. 2.2.5, and are responsible for the ordering of the velocity. Thus, we see that the flocking transition, for example in the above Toner-Tu form, belongs to a novel, genuinely non-equilibrium universality class.

If we want to describe dynamics of living things, which are most often active, we can resort to more general population dynamics models. We know that certain stochastic rules govern the changes in population, such as birth, death, predation, and others, and we have many models for these processes. However, with the addition of spatial behavior, these fall into the broader class of reaction-diffusion systems, which we discuss next.

3.2.2 Reaction-Diffusion Systems

Imagine a system of units of different species propagating on a lattice and also interacting with each other. Such interactions are generically local and can include species transformation, annihilation, production of offspring, etc. The units can be chemicals undergoing reactions, or living creatures undergoing population dynamics. How do we describe the probabilistic evolution of such systems and construct a corresponding field-theoretic description? For a great overview, which we follow here, see [2].

Generally, the starting point in modeling reaction-diffusion systems is the master equation for the distribution $P(n_A, n_B, n_C, \dots; t)$ that captures the probability of having n_α particles of species α at time t . Given the master equation, one can consider the evolution of the average densities $\langle n_\alpha \rangle$ or correlations $\langle n_\alpha n_\beta \rangle$. Such evolution equations are by construction only mean-field, and one needs to consider the full master equation to include fluctuations in the description.

The procedure for doing so is quite lengthy, so we only outline it here. However, just like the construction of the MSRJD action, in some sense one can read off the resulting field-theoretic action from the microscopic evolution rules without having to repeat all of the steps we now outline. Generally, the master equation has the form:

$$\partial_t P(n_A, n_B, \dots; t) = \left[\sum_{k_A} \sum_{k_B} \dots \right] c_{k_A, k_B, \dots}(n_A, n_B, \dots; k_A, k_B, \dots) P(n_A + k_A, n_B + k_B, \dots; t) \quad (3.25)$$

which connects the probabilities of having different numbers of particles of each species according to the interaction rules. Then, we need to go into a Fock space representation by introducing a bosonic ladder operator algebra, which is slightly different from the one usually used in condensed matter physics. Afterwards, we can discuss an instantaneous

configuration having n_α particles of each species as a Fock state:

$$|n_A, n_B, \dots\rangle = \left(a_A^\dagger\right)^{n_A} \left(a_B^\dagger\right)^{n_B} \dots |0\rangle \quad (3.26)$$

We can use this to define the state vector:

$$|\Phi(t)\rangle = \sum_{n_\alpha} P(n_A, n_B, \dots; t) |n_A, n_B, \dots\rangle \quad (3.27)$$

And the master equation translates to the corresponding imaginary-time Schrödinger equation for this vector $\partial_t |\Phi(t)\rangle = -H |\Phi(t)\rangle$. Several subtleties then allow us to work in this adjusted quantum-mechanical setting. In particular, one can construct the coherent-state path integral representation for thermal averages, which generically yields actions similar in form to the MSRJD action. Given its similarity, we can consider them as exactly MSRJD actions corresponding to some stochastic equations.

Let us provide a few examples. Consider a single species reaction rule $kA \rightarrow lA$, where particles A either proliferate $l > k$ or die out $l < k$. The action of this process corresponds to the stochastic equation:

$$\partial_t \psi(t, \vec{x}) = D \nabla^2 \psi(t, \vec{x}) - (k - l) \lambda \psi(t, \vec{x})^k + \xi(t, \vec{x}) \quad (3.28)$$

Where D is the diffusion constant for the particles and λ is their reaction rate and ψ loosely corresponds to the particle density. The noise correlations here are not always white, and depend on k and l . For example, there is no noise for the linear processes $A \rightarrow 0$, ($k = 1, l = 0$) and $0 \rightarrow A$, ($k = 0, l = 1$), so the deterministic equation accurately describes the long-range long-time behavior. Spontaneous pair production $0 \rightarrow 2A$, ($k = 0, l = 2$) evolves under additive noise $\langle \xi \xi \rangle \sim 2\lambda$. A branching process $A \rightarrow 2A$, ($k = 1, l = 2$) is governed by multiplicative noise $\langle \xi \xi \rangle \sim 2\lambda \psi$. Here, the process has an absorbing state with

$\psi = 0$, which we discuss in the next section. The variants of multiplicative noise continue as k increases.

We can also obtain a field-theoretic description of processes with several species. For example, the action for a pair annihilation process $A + B \rightarrow 0$ corresponds to the Langevin dynamics:

$$\begin{aligned}\partial_t \psi(t, \vec{x}) &= D \nabla^2 \psi(t, \vec{x}) - \lambda \psi(t, \vec{x}) \phi(t, \vec{x}) + \xi(t, \vec{x}) \\ \partial_t \phi(t, \vec{x}) &= D \nabla^2 \phi(t, \vec{x}) - \lambda \psi(t, \vec{x}) \phi(t, \vec{x}) + \eta(t, \vec{x})\end{aligned}\tag{3.29}$$

where the noise correlations take an interesting form $\langle \xi \xi \rangle = \langle \eta \eta \rangle = 0$ and $\langle \xi \eta \rangle \sim -\lambda \psi \phi$.

We can also consider a reversible recombination reaction $2A \leftrightarrow B$ with forward rate λ and backward rate σ . This is described by dynamics of the form:

$$\begin{aligned}\partial_t \psi(t, \vec{x}) &= D \nabla^2 \psi(t, \vec{x}) - 2\lambda \psi(t, \vec{x})^2 + 2\sigma \phi(t, \vec{x}) + \xi(t, \vec{x}) \\ \partial_t \phi(t, \vec{x}) &= D \nabla^2 \phi(t, \vec{x}) + \lambda \psi(t, \vec{x})^2 - \sigma \phi(t, \vec{x})\end{aligned}\tag{3.30}$$

with $\langle \xi \xi \rangle \sim 2(\sigma \phi - \lambda \psi^2)$.

As a final example, we consider systems with predation such as the Lotka-Volterra type reaction with rules $A \rightarrow 0$ (rate κ), $B \rightarrow 0$ (rate σ) and $A + B \rightarrow 2A$ (rate λ), which translates to:

$$\begin{aligned}\partial_t \psi(t, \vec{x}) &= D \nabla^2 \psi(t, \vec{x}) - \kappa \psi(t, \vec{x}) + \lambda \psi(t, \vec{x}) \phi(t, \vec{x}) + \xi(t, \vec{x}) \\ \partial_t \phi(t, \vec{x}) &= D \nabla^2 \phi(t, \vec{x}) + \sigma \phi(t, \vec{x}) - \lambda \psi(t, \vec{x}) \phi(t, \vec{x}) + \eta(t, \vec{x})\end{aligned}\tag{3.31}$$

with noise correlations $\langle \xi \xi \rangle \sim 2\lambda \psi \phi$, $\langle \xi \eta \rangle \sim -\lambda \psi \phi$ and $\langle \eta \eta \rangle \sim 2\sigma \phi$. The two above models are the first time that we properly encounter non-reciprocity, reflected in the λ terms, for example. Non-reciprocity is a central theme of this dissertation, and is the topic of Chapter 4.

The above discussion demonstrates the virtually endless variety of models one can consider in the context of reaction-diffusion systems, spanning a plethora of phenomena and systems. This, in part, is a big motivation for the more abstract considerations in Chapters 4 and 5. We see again why the Langevin dynamics and field-theoretic descriptions are such powerful and general tools.

3.2.3 *Active to Absorbing Transitions*

An interesting class of non-equilibrium models are those with an absorbing state, which the system cannot escape once it reaches it. This clearly violates detailed balance condition in Eq. 3.3, since the rate out of the absorbing state is zero. It turns out that many phenomena with absorbing states can be described by percolation theory [2, 8, 11, 81–83, 135]. These include the motion of liquid through porous media, reaction-diffusion systems, epidemic processes, suspensions that thicken into gels, and even transitions to dark states in open quantum systems [2, 3, 8, 11].

A paradigmatic example belonging to the non-equilibrium directed percolation (DP) universality class is described by the dynamics:

$$\begin{aligned}\partial_t \phi(t, \vec{x}) &= -m\phi(t, \vec{x}) - u\phi(t, \vec{x})^2 + D\nabla^2 \phi(t, \vec{x}) + \xi(t, \vec{x}) \\ \langle \xi(t, \vec{x}) \xi(t', \vec{x}') \rangle &= u\phi(t, \vec{x}) \delta(t - t') \delta^d(x - x')\end{aligned}\tag{3.32}$$

The MSRJD action corresponding to this equation is known as the Reggeon field theory [2, 81]. One can arrive at these equations from many perspectives. For example, a reaction-diffusion system with decay $A \rightarrow 0$, branching $A \rightarrow 2A$ and pair coagulation $2A \rightarrow A$ results in the above dynamics. In porous media, the flow of liquid is generally unidirectional (produced by gravity, for example). That flow direction is what yields the axis of "time", resulting in a stochastic equation. Alternatively, epidemic processes can be modeled with recovery and diffusive spreading of the disease according to this equation.

It is clear from Eq. 3.32 that the state $\phi(t, \vec{x}) = 0$ is absorbing – it is a steady state of the deterministic dynamics, and the multiplicative noise vanishes. Consequently, this equation is usually considered in the context of $\phi(t, \vec{x}) > 0$, with the phase transition from an inactive to active phase occurring as m is tuned through zero. It is clear even at the mean field level that the critical exponents associated with this equation are different from those of the Ising class. For example (see Eq. 2.14), $\beta = 1$ as opposed to Ising $\frac{1}{2}$.

There are many ways to expand upon the basic DP dynamics [2, 8, 81–83], including multi-species DP, such as the extension of the Lotka-Volterra model, unidirectional DP, DP with long-range spatial coupling, more general epidemic models, and others.

Another example of a process with an absorbing state is the branching and annihilation random walk (BAW) [2]. It is governed by the rules $kA \rightarrow 0$ and $A \rightarrow (m+1)A$. Interestingly, the coherent state action corresponding to this reaction-diffusion system does not reduce simply to the MSRJD form. At the mean field level, the dynamics for BAW are:

$$\partial_t a(t) = -k\lambda a(t)^k + m\sigma a(t) \quad (3.33)$$

The problem is complicated by the fact that the parity of m and k play an important role in these dynamics. For example, if both k and m are even, the process cannot die out for an initial condition with an odd number of A 's! For $k = m = 2$, we can build an equivalent description in terms of continuous Langevin dynamics, which results in a voter model universality class [2]:

$$\begin{aligned} \partial_t \phi(t, \vec{x}) &= D\nabla^2 \phi(t, \vec{x}) - \left[\sigma \phi(t, \vec{x}) - \rho \phi(t, \vec{x})^3 \right] \left[1 - \phi(t, \vec{x})^2 \right] + \xi(t, \vec{x}) \\ \langle \xi(t, \vec{x}) \xi(t', \vec{x}') \rangle &= B \left[1 - \phi(t, \vec{x})^2 \right] \delta(t - t') \delta^d(x - x') \end{aligned} \quad (3.34)$$

We do not delve into more detail on systems with absorbing states beyond these examples. However, we again see that Langevin dynamics and the corresponding MSRJD (or similar)

field theories can capture yet another very broad class of non-equilibrium problems.

3.2.4 Surface Growth and Sandpiles

Growth is perhaps one of the most obvious manifestations of non-equilibrium dynamics in many contexts. Consider the critical dynamics governed by the $O(1) \simeq \mathbb{Z}_2$ (or Ising) Hamiltonian in Eq. 2.68. In the ordered phase and without an external field, there are domains of positive and negative ϕ , separated by domain walls. Now, suppose that a positive external field h is suddenly turned on. We know that when an equilibrium configuration is reached, mainly a single positive ϕ domain should remain, since we are in the ordered phase. However, what happens before that configuration is reached? The positive domains grow through the motion of the domain walls, which is a non-equilibrium process, see Fig. 3.3.

Performing the analysis of the growth of the domain wall height profile $h(t, \vec{x})$ [2, 3, 136] (in the frame co-moving with the wall), one arrives at the Kardar-Parisi-Zhang (KPZ) [88] equation:

$$\partial_t h(t, \vec{x}) = D \nabla^2 h(t, \vec{x}) + \frac{\lambda}{2} [\vec{\nabla} h(t, \vec{x})]^2 + \xi(t, \vec{x}) \quad (3.35)$$

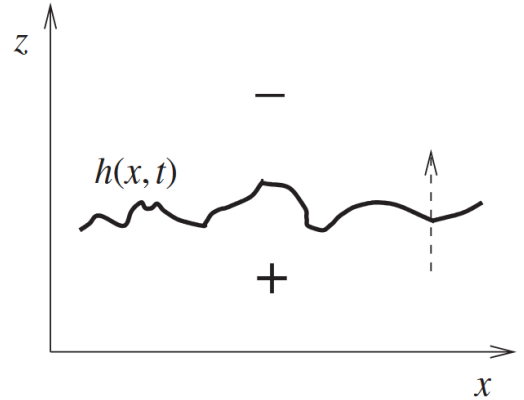


Figure 3.3: Illustration of the surface growth of a domain in an Ising model. Figure adapted from [2].

It can also be derived from a purely geometric perspective by the requirement that the rate of growth of a profile is a function only of its gradients. The simplicity of the arguments leading to the KPZ equation

also explains why it is found in so many fields: domain wall motion, flame front propagation, crystal surface growth, time-crystalline matter, driven-dissipative systems, non-reciprocal systems, and even neural dynamics; see [2–4, 11, 36, 38, 47, 83, 88, 136–140] and references therein.

Unlike the universality classes we encountered previously, the transition here is not obvious at the mean field level. It is characterized by the correlation profile, expressed in terms of the roughness exponent χ :

$$G^K(t, \vec{x}) = x^{2\chi} C\left(\frac{Dt}{x^z}\right) \quad (3.36)$$

where $\chi = \frac{2-d-\eta}{2}$, see Eq. 2.14. If $\chi > 0$, the spatial fluctuations grow with distance, and the surface becomes rough. Perturbative RG analysis shows that this model does not exhibit a transition in $d = 1$, with only the rough phase existing for any non-zero λ . For $d > 1$ there is a critical value of λ that separates the rough from the smooth phase. The theory at $\lambda = 0$ reduces to the linearized Model A dynamics with no mass term and corresponds to the Edwards-Wilkinson (EW) universality class [141].

As mentioned, KPZ dynamics are found in a lot of unexpected places, and here we want to mention one particularly relevant to this dissertation. We know that Goldstone modes corresponding to a broken continuous symmetry are necessarily massless [2–7]. Imagine, for instance, an $O(2)$ order parameter, where the Goldstone mode in the broken phase corresponds to rotations around the circle. This means that a field-theoretic description of its dynamics, at the lowest order, takes the form of the EW equation. The higher order gradient terms are less relevant in the RG sense. However, there is generally still a symmetry associated with Goldstone modes. For example, the EW equation (KPZ with $\lambda = 0$), has a $\mathbb{Z}_2$ symmetry $h \rightarrow -h$. In the $O(2)$ context this means that rotations of the order parameter clockwise and counter clockwise are equivalent.

A non-equilibrium process that breaks this paradigm is time-crystalline matter [16, 46, 47]. There, it can be shown that due to the periodicity of solutions in time, the Goldstone mode associated with time-translations (going along the periodic orbit) generically does not have a $\mathbb{Z}_2$ symmetry, and therefore the KPZ term $\sim \lambda$ is not disallowed. That means that under RG flow, unless an unnatural tuning of parameters occurred, the dynamics of the Goldstone mode obeys the KPZ equation 3.35, now with a compact field. Therefore, KPZ

scaling and periodic orbits are generically tightly related, and we demonstrate it again in Sec. 5.3.

Moving on, one can imagine a growth process where growth is balanced by some dynamics that reset the state of the system, allowing for more growth. A paradigmatic example is the sandpile model, which leads to the idea of self-organized criticality [142]. In its original form, the model was constructed as a cellular automaton on a lattice. A real field $z(x, y, t)$ describes the height of the sand at the point x, y . Each time step, a grain of sand is deposited at a random lattice site, such that $z(x, y, t + 1) = z(x, y, t) + 1$ for some x, y . The model has a critical value K , such that if $z(x, y)$ exceeds K , the sand slides to its neighbors according to the rule:

$$\begin{aligned} z(x, y, t + 1) &= z(x, y, t) - 4 \\ z(x \pm 1, y, t + 1) &= z(x \pm 1, y, t) + 1 \quad , \quad \text{if } z(x, y, t) > K \\ z(x, y \pm 1, t + 1) &= z(x, y \pm 1, t) + 1 \end{aligned} \tag{3.37}$$

This sliding can lead to an avalanche, where some of the neighbors of the original seed with $z > K$ exceed K themselves, leading to further redistribution. These avalanches are described by probability distributions of their sizes and durations. Many other models of deposition have been considered with various applications and effects, such as fractal structures. For an overview, see [136].

An important observation is that the system brings itself to a state that is "critical" in some sense by allowing growth right until the threshold. Once it crosses the threshold, the sand pile reduces its height through avalanches, allowing itself to reach it again. This is the idea behind self-organized criticality. It can also be described in a dynamical setting already

familiar to us [143]. We can write a self-organized DP dynamics in the form:

$$\begin{aligned}\partial_t \rho &= (a + \omega E)\rho - b\rho^2 \\ \partial_t E &= h - \epsilon\rho\end{aligned}\tag{3.38}$$

In the limit $h/\epsilon \rightarrow 0$, the system reaches the critical state with $\rho \rightarrow 0$ regardless of the values of other parameters. This again illustrates the feedback mechanism from the sandpile model – E brings ρ towards the critical state in a manner that depends on ρ . Self-organized criticality is a major topic of research and is suspected to be central to many phenomena, see [143] and references therein.

3.2.5 *Driven-Dissipative Systems*

In the quantum realm, a class of systems that turned out to be fruitful platforms for the experimental realization of non-equilibrium matter are driven-dissipative systems, sometimes called open quantum systems [10, 11]. We demonstrate how descriptions of open quantum matter fit into the themes of this dissertation by the example of exciton-polariton condensates. For the overviews we follow in this discussion, see [10, 11, 144–146] and references therein.

First, we need a material that can support excitons, which is typically a semiconductor such as GaAs or CdTe. An exciton is a primary excitation of such systems under external stimulation and is defined as an electron-hole pair bound due to their mutual Coulomb interaction. Since excitons are bound quasi-particles consisting of two fermions, they obey bosonic statistics as long as they are well-separated, i.e. the typical distance between them is much larger than the separation between the electron and the hole ($\sim O(10\text{nm})$). Excitons have a finite lifetime, so to support a constant density of these quasi-particles, one needs to maintain external stimulation. A material that confines the excitons in one dimension but allows them freedom in the other two is usually called a quantum well [146].

Next, the quantum well is placed in a cavity, where an external source supplies photons which bounce between (distributed) Bragg reflectors (DBRs), for example. The confinement of the photons creates an interaction between the photons and the excitons, resulting in new quasi-particles – exciton-polaritons. These particles also have bosonic statistics, and well defined dispersion relation (see above references for more details). Presumably, this means they should be able to form a condensate, such that a macroscopic number of them occupy the ground state of the collective excitation spectrum. In fact, the effective mass of the exciton-polariton is much lower than that of an electron, which allows the formation of a condensate at temperatures as high as room temperature. However, in part due to the lifetime of excitons, but also due to the leakage of photons from the cavity, exciton-polaritons have a finite lifetime. Therefore, an exciton-polariton condensate is fundamentally non-equilibrium, since it has leakage of particles and requires constant pumping to sustain itself. For a typical setup, see Fig. 3.4.

With this typical experimental realization in mind, we now proceed to discuss how driven dissipative condensates are modeled. In the language of Lindblad master equations (see Sec. 2.2.2), the dynamics of a lower polariton field is described by the Hamiltonian [10]:

$$H_{LP} = \int d^d x \left[\psi^\dagger(\vec{x}) \left(\omega_{LP} - \frac{\nabla^2}{2m_{LP}} \right) \psi(\vec{x}) + u_c |\psi(\vec{x})|^4 \right] \quad (3.39)$$

Here the term of the bracket corresponds to the energy operator of the respective mode, while u_c corresponds to the two-body coherent interaction. The dissipative part of the dynamics is described by Lindblad operators $L_p = \int_x \psi^\dagger(\vec{x})$, $L_l = \int_x \psi(\vec{x})$ and $L_d = \int_x \psi^2(\vec{x})$, with corresponding rates γ_p , γ_l and $2u_d$. These correspond to the single-particle pumping, single-particle loss and two-body loss, respectively. The Keldysh action corresponding to

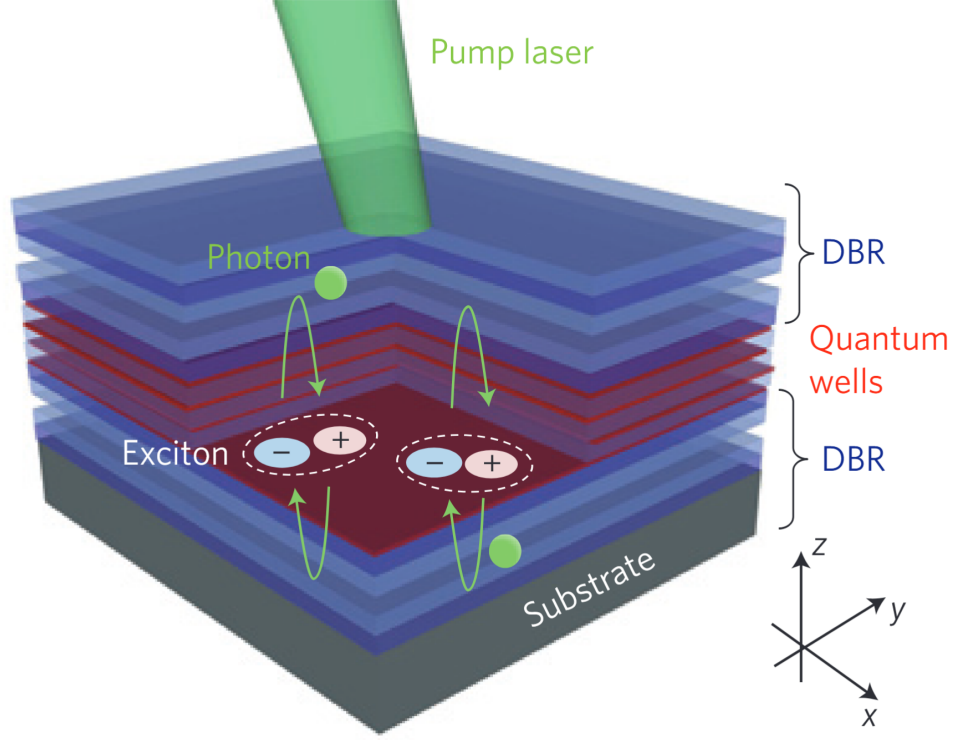


Figure 3.4: Illustration of the typical exciton-polariton condensate setup described in the text. Figure adopted from [146].

these Lindblad dynamics is [10]:

$$\begin{aligned}
 S = \int dt d^d x \Big\{ & \phi_q^* (i\partial_t + K_c \nabla^2 - r_c + i r_d) \phi_c + \text{c.c.} \\
 & - \left[(u_c - i u_d) \left(\phi_q^* \phi_c^* \phi_c^2 + \phi_q^* \phi_c^* \phi_q^2 \right) + \text{c.c.} \right] + 2i(\gamma + 2u_d \phi_c^* \phi_c) \phi_q^* \phi_q \Big\}
 \end{aligned} \tag{3.40}$$

where $K_c = 1/(2m_{LP})$, $r_c = \omega_{LP}$, $\gamma = (\gamma_l + \gamma_p)/2$ and $r_d = (\gamma_l - \gamma_p)/2$. We see here that the total rates of loss and pumping add up to an effective noise, while the difference between one-particle loss and pumping governs the "mass" of the system. It is the tuning of r_d that leads to the condensation transition. Having this action, one can directly proceed to utilizing field-theoretic tools outlined in Chapter 2.

However, it is also possible to take the semiclassical limit of this equation, where the terms most relevant in three dimensions, in the RG sense, are retained. The resulting action

corresponds to the Langevin equation [10]:

$$i\partial_t\phi_c = \left[-(K_c - iK_d)\nabla^2 + r_c - ir_d + (u_c - iu_d)|\phi_c|^2 \right] \phi_c + \xi \quad (3.41)$$

Here, an effective diffusive term $\sim K_d$ is included as it is generated under the RG flow. Therefore, we see that Langevin dynamics and the corresponding MSRJD action can also be used to describe semi-classical limits of quantum driven-dissipative many-body systems. The equation of the above form is a version of the Gross-Pitaevskii equation, which finds numerous applications [10, 11, 18, 86, 138, 147]. Note that the purely dissipative version of this equation reduces to Model A dynamics.

3.2.6 Pattern Formation

Pattern formation is yet another manifestation of non-equilibrium dynamics, occurring in many fields. A seminal review paper [84] is a great resource to start exploring this topic. In this article, Cross and Hohenberg classify patterns based on the type of linear instability a system experiences as some parameter is tuned. Here we provide a generic exposition.

Suppose a dynamical spatially-extended system supports a steady uniform solution. Linearizing the dynamics around this solution, a generic perturbation evolves as:

$$\partial_t\phi(t, \vec{x}) = A(p, \vec{\nabla})\phi(t, \vec{x}) \quad (3.42)$$

Here $\phi(t, \vec{x})$ is a multicomponent perturbation to the uniform state, and A is the matrix governing the dynamics of the perturbation, which depends on some parameters p and spatial couplings (in the form of ∇). We can write this equation in time-momentum space, where it takes the form:

$$\partial_t\phi(t, \vec{k}) = A(p, -i\vec{k})\phi(t, \vec{k}) \quad (3.43)$$

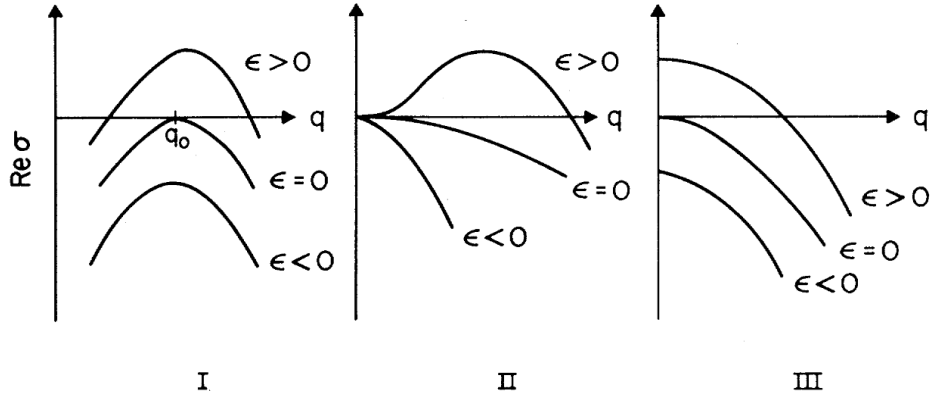


Figure 3.5: Representation of the largest eigenvalue σ as a function of momentum (here q) as some parameter ϵ is changed. The three instability types are described in the text. Figure adopted from [84].

The modes of the dynamics of $\phi(t, \vec{k})$ are determined by the eigenvalues of $A(p, -ik)$. Let us call the eigenvalue with the largest real part $\sigma(p, \vec{k})$. An instability occurs when the real part of that eigenvalue becomes positive for some $\vec{k}_c$. When that happens, the modes around $\vec{k}_c$ grow in time and form a spatially oscillatory pattern of length scale $\sim |k_c|^{-1}$.

Cross and Hohenberg made a distinction between three types of linear instabilities, as illustrated in 3.5. Type I_s has $\sigma(p, \vec{k}) = 0$ at a finite non-zero k_c . This is often called a finite-momentum instability. For an example of the patterns resulting from such an instability, see Fig. 3.6. Type II_s , often as a consequence of some conservation law, has $\sigma(p, \vec{k} = 0) = 0$, and the instability occurs when a finite band of momenta between 0 and some non-zero value are unstable. Type III_s has $k_c = 0$, in which case the pattern does not have a characteristic scale, and rather corresponds to the instability of a uniform solution. The subscripts s denote that for these instabilities $\text{Im}(\sigma(p, \vec{k}_c)) = 0$. Correspondingly, one can have spatio-temporal patterns of types I_o , II_o and III_o when $\text{Im}(\sigma(p, \vec{k}_c)) \neq 0$. We reproduce a figure from [84] to illustrate these patterns in Fig. 3.5. We discuss a type I_s pattern forming system in the section on linear non-reciprocal systems, Sec. 4.2.

However, deterministic spatio-temporal patterns are not the only avenues of pattern

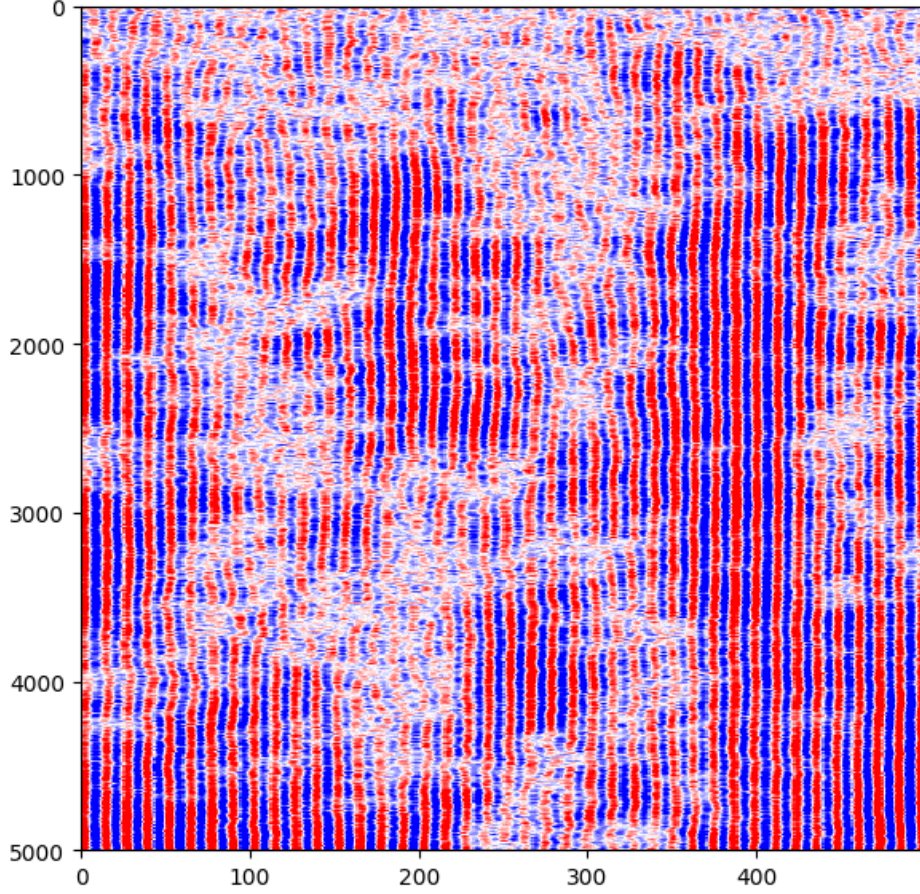


Figure 3.6: A typical finite-momentum instability pattern (type I_s) in a one-dimensional system. The vertical axis is time, the horizontal is space.

formation. For stochastic systems, one can consider a situation where the dynamics again takes the linearized form, but now with noise, such that:

$$\partial_t \phi(t, \vec{k}) = A(p, -i\vec{k}) \phi(t, \vec{k}) + \xi(t, \vec{k}) \quad (3.44)$$

If the noise is spatially uncorrelated, it constantly produces non-zero amplitudes for modes of all $\vec{k}$. Now, suppose the system is in a parameter regime where it is almost unstable, such that $\sigma(p, \vec{k}_c)$ is very close to zero. By definition of $\vec{k}_c$, modes of other $\vec{k}$ have lower $\sigma(p, \vec{k})$. Since σ , as an eigenvalue of A , corresponds to the temporal evolution of the modes, the modes with $\vec{k} \neq \vec{k}_c$ decay faster than the least stable mode. This is similar

to the critical slowing down in phase transitions, where the critical mode decays infinitely slowly. Therefore, we have a situation where modes of all $\vec{k}$ are constantly sustained by the noise, but all non-critical modes decay quicker than the critical. This allows a pattern to be sustained just below a finite-momentum instability as a result of noise. This scenario was explored in [20].

Let us provide some concrete examples, most of which are discussed in [84]. A famous pattern forming system, first introduced to describe Rayleigh-Benard convection is the Swift-Hohenberg equation [84, 85], which we write in a generalized form:

$$\partial_t \phi = \epsilon \phi - (k_0^2 + \nabla^2)^2 \phi + I(\phi) + \xi \quad (3.45)$$

Here $I(\phi)$ contains the non-linear terms. It is clear here that the critical wavenumber k_0 is explicitly present in the system, such that it allows for a finite-momentum instability.

Another interesting example is the Kuramoto-Sivashinsky equation [41, 84]:

$$\partial_t \phi = D \nabla^2 \phi - K \nabla^4 \phi + \frac{\lambda}{2} (\vec{\nabla} \phi)^2 + \xi \quad (3.46)$$

Its relevance is exemplified by the surprising fact that for any spatially continuous system with limit cycle attractors, the effective phase dynamics takes the form of the Kuramoto-Sivashinsky equation [41]. Note its similarity to the KPZ equation 3.35. Here, however, it is assumed that $D < 0$, such that the system is linearly unstable. At the linear level, it is clear that the modes in the band $0 < |k| < \sqrt{\frac{D}{K}}$ are unstable. An attempt on the momentum-shell RG investigation of the noisy Kuramoto-Sivashinsky equation has been conducted in [148].

Finally, reaction diffusion systems, and general Ginzburg-Landau equations can also exhibit patterns [84, 86]. This is especially the case for many-component systems with non-reciprocal interactions and very different diffusion constants. We explore this in more detail in Sec. 4.2, where a linearized non-reciprocal model can be viewed as a reaction-diffusion

system with an activator and inhibitor.

Pattern formation is a key phenomenon in non-equilibrium systems. A fundamental question is how to perform a sensible RG analysis of pattern-forming systems. In momentum-shell RG, as we integrate out large momenta, we inevitably reach the point where our momentum shells reach the linearly unstable band responsible for the pattern. We then encounter infrared divergencies which render the perturbative analysis impossible. As a consequence, it is possible that instead of a continuous transition to a patterned phase, a system instead experiences a fluctuation-induced first order transition [87].

3.2.7 *Neural Dynamics as a Field Theory*

Computational neuroscience is an amazing field that has developed in response to new technological advancements and experimental results. A single section cannot do it justice, so we attempt only to outline it in a way that connects with the themes of this dissertation. For an introduction to the many aspects of computational neuroscience, see [67, 149] and references therein.

Let us briefly discuss the structure of a neuron as found in a human brain [50, 67], see Fig. 3.7. A neuron is a specialized cell that maintains an electric potential difference between its inside and outside. It does so through maintaining currents in and out of its body, primarily consisting of Na^+ , K^+ , Ca^{2+} and Cl^- ions. Channels, or gates, control the flow of these currents, and respond to voltage changes and other signals received by the neuron. Simplified, a neuron receives electrical signals through its dendrites that connect to its soma (cell body), and sends out electrical signals primarily through its axon to other neurons. A non-stimulated neuron sits below approximately -70mV relative to its environment, which is usually called its resting potential. Such a neuron is called hyperpolarized. If the flow of ionic currents into the body, due to some signal to the neuron, increases its potential, it is called depolarization. If the potential increases beyond a certain threshold value, a

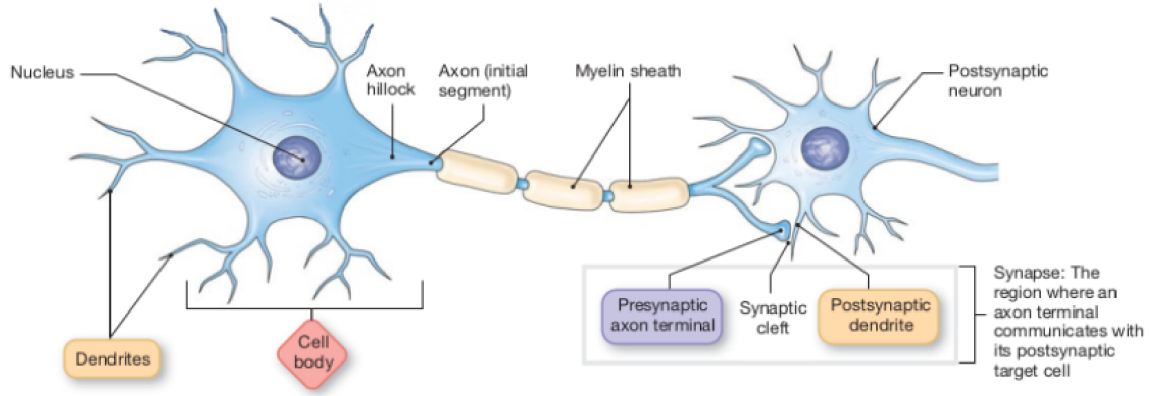


Figure 3.7: Illustration of a neuron with its major components. See description in the text. Figure adapted from [51].

positive feedback loop begins and a neuron produces a temporary large positive change of its membrane potential to around 100mV. Such an event is called action potential (or spike). The action potential is what actually propagates for long distances through the axon to other neurons. Other potential fluctuations die out as they travel through the axon. After an action potential occurs, a neuron is unable to spike again for a period of time called the refractory period. The network of neurons with these basic underlying mechanisms produces our nervous systems with all of their properties – memory, responsiveness to the environment, thinking, computation, etc.

However, a fundamental insight into neural behavior is the tight connection between the mechanics of neural spiking and dynamical systems [50]. One of the first models describing the action potential of a neuron alongside the dynamics of the gated ionic currents is the Hodgkin-Huxley model [50, 67, 150]. It has many variants that essentially consider the dynamics on different time scales and, as a result, include different ionic currents in the description. For example, a famous dimensionally-reduced variant of the Hodgkin-Huxley equations is the FitzHugh-Nagumo model [50]:

$$\begin{aligned}\partial_t V &= V(a - V)(V - 1) - w + I \\ \partial_t w &= bV - cw\end{aligned}\tag{3.47}$$

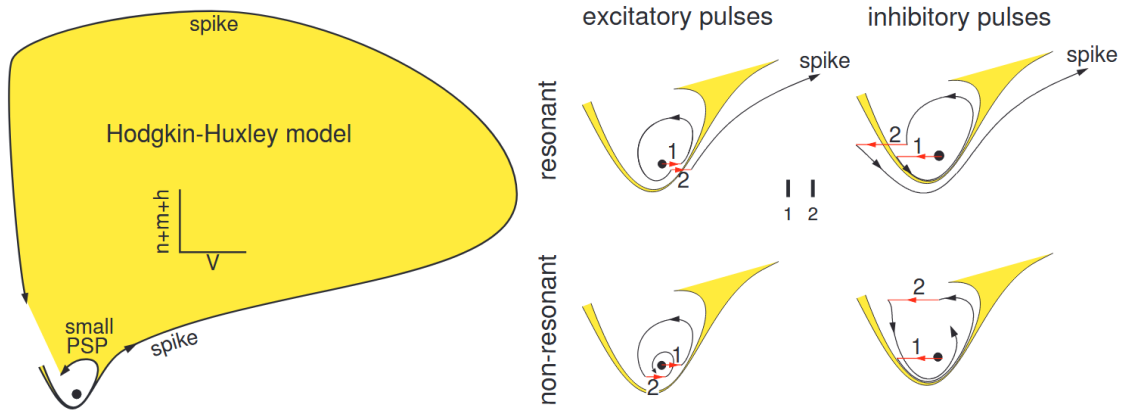


Figure 3.8: Phase space of a Hodgkin-Huxley model for a resonator neuron. The vertical axis represents the ionic currents and the horizontal the potential of the cell body. A small post-synaptic potential (PSP) coming from an external input cannot excite the neuron to spike. However, stimulation at certain frequencies can push the system into the spiking part of the phase space. Figure adapted from [50].

Here V approximates the membrane voltage and w approximates the ionic currents that help the neuron recover after a spike.

It turns out that qualitatively important features of this model come from the geometric structure of the phase space corresponding to the dynamical system. With this in mind, many computational properties could be explained by looking at the phase portraits. For example, whether the neuron is an integrator that can respond to arbitrary frequency stimulation, or a resonator that responds to stimulation of a narrow band of frequencies, is seen from the flow portraits. See Fig. 3.8 for an example resonator phase space.

However, in the spirit of this dissertation, we are less interested in the dynamics of a single neuron, and more in the coarse-grained picture, whatever form it takes. The most famous population-level description of neural dynamics is the Wilson-Cowan model [51, 151, 152]. In this model, the neurons are split into two populations – excitatory and inhibitory – depending on their action on the postsynaptic neurons. For example, a spike of an excitatory neuron makes a postsynaptic neuron connected to it more likely to spike, and the reverse effect occurs for inhibitory neurons. The averaged activity $A(t, \vec{x})$, which is usually represented as

the fraction of neurons that spike in a given time frame τ and in a given region denoted by $\vec{x}$, evolves as:

$$A_\alpha(t + \tau, \vec{x}) = R_\alpha(t, \vec{x}) S_\alpha [N_\alpha(t, \vec{x})] \quad (3.48)$$

Here $\alpha \in E, I$ corresponds to the two populations of neurons, R_α is the fraction of the population α ready to spike, S is a sigmoid function that relates the population input N to the threshold of firing for a given population. A few assumptions are necessary to produce dynamics that can be translated to a field theory [51]. The resulting dynamics is the Wilson-Cowan equation for the activity of excitatory $A_E(t, \vec{x})$ and inhibitory populations $A_I(t, \vec{x})$:

$$\begin{aligned} \tau_E \partial_t A_E &= -A_E + (1 - r_E A_E) \mathcal{S}_E [w_{EE} \otimes_x A_E - w_{EI} \otimes_x A_I + P_E(t, \vec{x})] \\ \tau_I \partial_t A_I &= -A_I + (1 - r_I A_I) \mathcal{S}_I [w_{IE} \otimes_x A_E - w_{II} \otimes_x A_I + P_I(t, \vec{x})] \end{aligned} \quad (3.49)$$

There are a few parameters that arise here. First, there are time scales τ_α associated with the dynamics of each population. These are often omitted, but in general it is believed that inhibition happens on longer time scales than excitation [50, 51, 66]. r_α correspond to refractory periods of each population. P_α are external inputs, which can be modeled as noise, or arise due to stimulation. The convolutions $w_{\alpha\beta} \otimes_x A_\beta$ correspond to the spatial connectivity structure:

$$[w_{\alpha\beta} \otimes_x A_\beta](t, \vec{x}) = \int_{x'} w_{\alpha\beta}(\vec{x} - \vec{x}') A_\beta(t, \vec{x}') \quad (3.50)$$

The issue of the spatial connectivity structure is one of the most interesting problems in neuroscience. An exact mapping of neural connections is a herculean task, but has already been achieved for the *Drosophila* fly [153]. The remarkable *Drosophila* connectome contains around 140,000 neurons, around six orders of magnitude away from the human brain. Keeping that and the coarse-grained theme of the Wilson-Cowan equations in mind, the

connectivity kernel $w_{\alpha\beta}$ is usually taken as a statistical measure. Experimental observations show a decrease in average connectivity with distance [154–156], and therefore these kernels are often assumed to have a Gaussian or decaying exponential structure [51]. Alternatively, another popular choice is random connectivity, particularly interesting for artificial neural networks.

Innumerably many other models can be built in the spirit of the Wilson-Cowan model. For example, the Amari equation [51] swaps the order of the spatial kernel convolution and the sigmoid. Another extension to such models is the finite synaptic velocity that attempts to include the finite speed of propagation of neurotransmitters and electric signals between connected neurons. If such speed is v , the convolution instead takes the form:

$$[w_{\alpha\beta} \otimes_x A_\beta](t, \vec{x}) = \int_{x'} w_{\alpha\beta}(\vec{x} - \vec{x}') A_\beta \left(t - \frac{|\vec{x} - \vec{x}'|}{v}, \vec{x}' \right) \quad (3.51)$$

In some sense, most modern field-theoretic descriptions of neural dynamics have the basic form of the Wilson-Cowan (or Amari) equations, with endless layers of additions and complications reflecting biological realities [51].

However, remembering the key insight of only the geometric structure of the phase space mattering, let us present models that are often used in neural dynamics, but are not built directly from the biological principles. First is the Kuramoto model [39, 41, 51, 157, 158], which models a transition to synchronization. Imagine a set of oscillators with phases θ_i that are attractively coupled so that they tend to align their phases. However, if their internal frequencies are too different, such an alignment is impossible. For a collection of N oscillators, Kuramoto models it in the dynamical form:

$$\partial_t \theta_i = \omega_i + \frac{K}{N} \sum_{j=1}^N \sin(\theta_j - \theta_i) \quad (3.52)$$

Here ω_i are internal frequencies, which are usually taken to be distributed in some manner,

and K is the coupling between different oscillators. Alternatively, the disordered nature of ω_i can be substituted with noise in the above equation. An analysis of this equation (see refs. above) finds that for narrow distributions of ω_i , the oscillators are synchronized, while otherwise they are not. We can define a Kuramoto order parameter $re^{i\psi} = \frac{1}{N} \sum_i e^{i\theta_i}$. For $N \rightarrow \infty$, r reaches an equilibrated value regardless of the initial condition. That value, which is sometimes denoted r_∞ [39], undergoes a continuous transition at some critical value K_c .

Synchronization is an important effect in neural dynamics [51, 54, 55, 159, 160]. In fact, there are some claims [54] that the critical brain (which we discuss below) is on the edge of a synchronization transition. Neural dynamics is by far not the only non-equilibrium setting where synchronization plays a role. It appears in physical models [23, 24, 102, 161, 162], chemical systems [44, 45, 98], biological systems, social networks, and others [58].

A useful model of neural behavior that leaves behind much of the biophysical reality while still retaining basic principles of neural dynamics is the leaky-integrate-and-fire model (LIF) [67]. In fact, the LIF model was first proposed long before the biophysics of neural spiking was understood experimentally. For the LIF model, the membrane potential dynamics is governed by an equation:

$$\tau \partial_t V = E_L - V + RI \quad (3.53)$$

Here, E_L is the resting potential of the cell, R is the resistance of the membrane, and I is the input to the neuron from other neurons or an electrode. The dynamics is supplemented by the rule that if V reaches a threshold value, the neuron is said to fire an action potential, and the membrane potential is reset to a base value V_r . This produces dynamics qualitatively similar to what is experimentally observed, see Fig. 3.9. LIF models also allow for many extensions, but have the advantage of drastic numerical simplicity as compared to the Hodgkin-Huxley model, for instance.

A model of spiking that models the refractory period by adding the dynamics of available

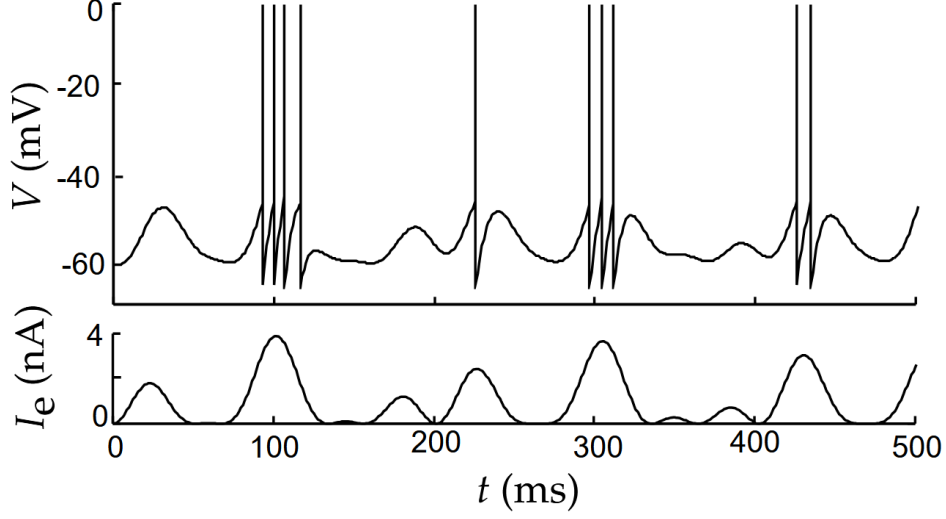


Figure 3.9: LIF model in Eq. 3.53 driven by a time varying electrode current $I = I_e$ (bottom). The membrane potential (top) is similar to experimentally observed spikes. Figure adopted from [67].

synaptic resources is the Tsodyks and Markram (TM) model [54, 163–165]:

$$\begin{aligned}\partial_t \rho &= (-a + R) \rho + b \rho^2 - \rho^3 + h \\ \tau \partial_t R &= \delta - R - R \rho\end{aligned}\tag{3.54}$$

Here ρ is the averaged activity of the neural patch, R is the available resources for neural activation, while the rest are positive parameters modulating the dynamics. This model allows for various phases of neural dynamics, including bursting. With the addition of spatial coupling, it also exhibits synchronization [54] and scale-free avalanche distributions [165].

Abstracted dynamical systems also find application in the realm of modeling neurological disorders. In Sec. 6.2 we discuss modeling epileptic seizures at length. To provide a basic description now, we can examine the following stochastic system:

$$\partial_t z = \left(\lambda + i\omega + b|z|^2 - c|z|^4 \right) z + \xi\tag{3.55}$$

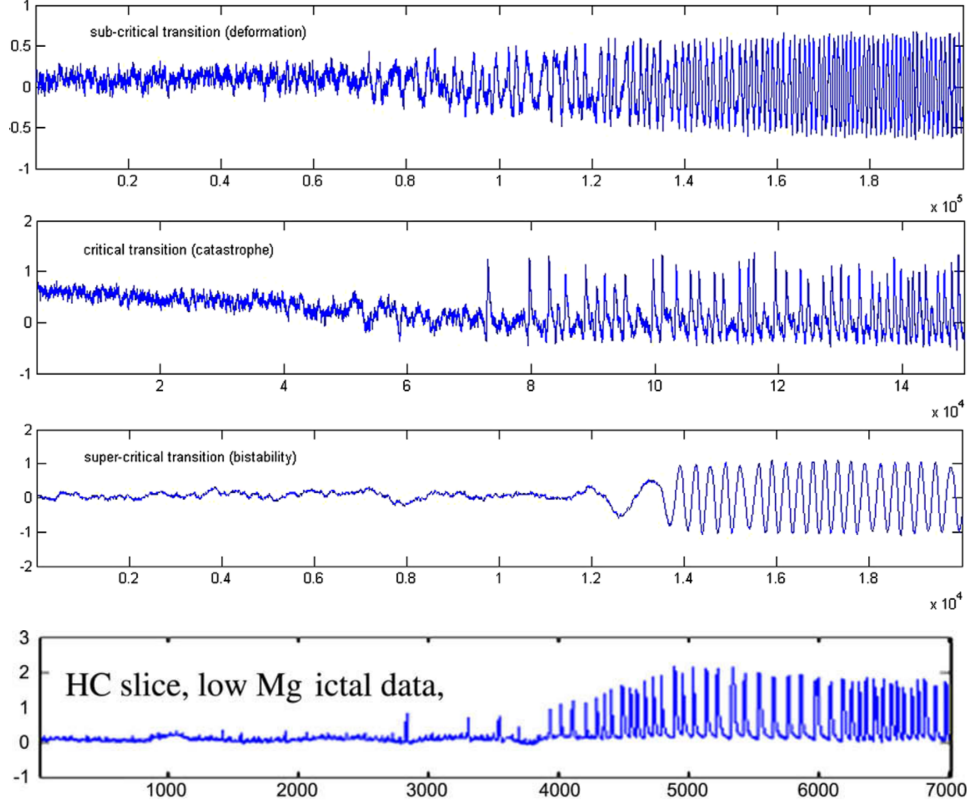


Figure 3.10: Typical EEG signals during a seizure. The top 3 plots correspond to the simulations of Eq. 3.55, where $\text{Re}(z)$ is plotted versus time. The first plot corresponds to continuously tuning λ through a bifurcation. The second corresponds to the inclusion of a symmetry breaking term, such as $|z|^2$, or external stimulation to the r.h.s. of Eq. 3.55. The third corresponds to the noise activated transition where both $z = 0$ and a limit cycle with finite $|z|$ are steady attractors. The bottom plot corresponds to real EEG data on a mouse brain slice. Figures adopted from [78].

Here z is a complex variable, the real part of which loosely models an electrical signal one would measure clinically. It can be an EEG or ECoG signal, a micro-electrode (or unit) array recording, or some other technique. All of the parameters have some biological motivations and connections. As λ changes, the system undergoes a bifurcation between having $z = 0$ as the only steady state, and having $|z| \neq 0$ as the only steady state. Alternatively, noise can activate a transition to a limit cycle with $|z| \neq 0$ in the regime where both the limit cycle and $z = 0$ are steady attractors. Finally, stimulation or other additional terms can

trigger the system to transition between attractors. Any of these can be taken as models of a transition to a seizure, which is usually observed with EEG signals as large amplitude oscillations [166–168], see Fig 3.10.

The last aspect of neural dynamics relevant to this dissertation, and also a topic of relentless debate, is the brain criticality hypothesis [51, 54, 55, 66, 140, 149, 169, 170]. We refer to the literature for more details, and present the basic idea here. It is hypothesized that a healthy brain exists at a critical point of its dynamics. This is also often mixed with the ideas of self-organized criticality (see Sec. 3.2.4), in order to avoid the need to explain the fine-tuning required for a system to sit at a critical point otherwise. This is an open question, and is by no means settled, but there are two ideas that motivate this hypothesis.

First, with the ideas of dynamical systems and statistical systems in the back of our minds, we should ask what functions a healthy brain needs to fulfill. At least two opposing functions come to mind – memory and responsiveness to external stimuli. They are opposing in the dynamical systems sense. A memory requires a deep attractor, such that noise or external perturbations cannot push the system out of it easily. Sensitivity to external stimuli requires the opposite, since shallower attractors are more sensitive to perturbations. We saw in Chapter 2 that the sensitivity of the system to external perturbations is maximized at the critical point. Thus, a critical point is hypothesized to be the necessary balance of the two functions. Our brains can store a large number of memories that we do not lose, but can also explore that variety quite freely and are able to quickly respond to stimuli.

The second motivation comes from the experimental recordings of neural avalanches (see refs. above). An avalanche is produced when a spike causes more spikes, which in turn causes more spikes, etc. Just like for sandpile models (Sec. 3.2.4), such avalanches can be studied in terms of the probability distributions of their sizes and durations. Much evidence shows that for neural dynamics, both in cultures and in brains, these distributions are scale-free and follow certain scaling laws [169, 171–176]. However, it has also been noted that subsampling

universality classes such as DP can also lead to scale-free avalanches [135]. Since brains are always subsampled in vivo, this is an important observation against the hypothesis. Additionally, scale-free avalanches are always observed at non-equilibrium critical points where one can define an avalanche in the first place. However, the opposite is not true – scale-free avalanches do not necessarily indicate a traditional critical point, see [165]. Thus, the hypothesis is still under debate.

Aside from the question of the validity of the brain criticality hypothesis, there is also a question of what phase transition the brain actually sits at. Clearly, it has to be some non-equilibrium universality class, and there are plenty of candidates, including DP, self-organized critical points, branching processes and synchronization transitions [54, 66, 135, 143, 169].

We close the discussion on neural dynamics with a note that dynamical systems (and field theory) find applications not just in the realm of biological neurons. Artificial neural networks also enjoy dynamical systems descriptions, including plain and gated recurrent neural networks [177, 178], Hopfield networks [149, 179, 180], and others.

3.2.8 *Dynamical Steady States and Non-Reciprocity*

We provide larger overviews of limit cycle phases and non-reciprocal phenomena in Chapters 4 and 5. Here we present how these phenomena generically arise in dynamical systems where the requirement of equilibrium has been removed.

Consider a many-species field that is described by a generic Hamiltonian (omitting space for now):

$$\mathcal{H} = g_a \phi_a + g_{ab} \phi_a \phi_b + g_{abc} \phi_a \phi_b \phi_c + \dots \quad (3.56)$$

The equilibrium dynamics is:

$$\begin{aligned} \partial_t \phi_\alpha &= -\frac{\delta \mathcal{H}}{\delta \phi_\alpha} = -g_\alpha - g_{\alpha b} \phi_b - g_{a\alpha} \phi_a - g_{\alpha bc} \phi_b \phi_c - g_{a\alpha c} \phi_a \phi_c - g_{ab\alpha} \phi_a \phi_b - \dots \\ &= -g_\alpha - (g_{\alpha b} + g_{b\alpha}) \phi_b - (g_{\alpha bc} + g_{b\alpha c} + g_{bc\alpha}) \phi_b \phi_c - \dots \end{aligned} \quad (3.57)$$

We see immediately that the dynamics at every total order of the fields is governed by interactions that are symmetrized. Therefore, we can immediately imagine that a way to go outside of equilibrium can be to break that symmetry, which is generically called non-reciprocal interactions [13, 23, 27, 29, 38]. In the quantum realm, such systems undergo non-Hermitian evolution [181]. Non-reciprocity is a central theme of this dissertation and is the focus of Chapter 4. There, we discuss how non-reciprocity arises in many examples and discuss novel results.

Let us illustrate the non-equilibrium nature of such systems with a zero-dimensional linear system:

$$\begin{aligned}\partial_t \phi_1 &= -m_1 \phi_1 + j_{12} \phi_2 + \xi_1 \\ \partial_t \phi_2 &= -m_2 \phi_2 + j_{21} \phi_1 + \xi_2\end{aligned}\tag{3.58}$$

We discuss, at length, this system with spatial coupling and in the field-theoretic language in Section 4.2. Here, let us consider the equilibrium symmetry transformation in Eq. 3.13 applied to this system. The transformed MSRJD action is equivalent to the original up to a term of the form:

$$\Delta S_{\text{MSR}} = -\frac{2}{B}(j_{12} - j_{21}) \int dt \phi_1 \partial_t \phi_2\tag{3.59}$$

We see that the symmetry can only be satisfied if the interactions are reciprocal $j_{12} = j_{21}$, as expected. This also shows that the difference $j_- = j_{12} - j_{21}$, which can act as a measure of non-reciprocity [13, 32], in some sense measures the distance from equilibrium. In Section 4.2 a more convenient measure emerges.

Let us next examine the linear system in the case that $m_1 = -m_2$ and $j_{12} = -j_{21}$. The eigenvalues of the normal modes associated with these dynamics are purely imaginary (see Sec. 4.2), and so the stationary states are limit cycles. Thus, non-reciprocity can also generically lead to an emergence of the so-called chiral phases [13, 27, 32].

Let us consider a basic model exhibiting chiral behavior:

$$\partial_t z = -mz + i\omega z + \xi \quad (3.60)$$

Here, z is complex. For illustration, let us explore this model in a different approach – the FP equation. By writing the FP equation, it can be shown that the stationary state is $P \sim \exp[-m|z|^2/(2B)]$. This distribution does not depend on ω , and is the same as the equilibrium case $\omega = 0$. However, the probability current in the $(x, y) = (\text{Re}(z), \text{Im}(z))$ space is:

$$\vec{J} = -\frac{\omega B}{m} \hat{z} \times \vec{\nabla} P \quad (3.61)$$

Therefore, this system has a constant flux of probability current (due to rotation at ω), and is intuitively non-equilibrium. Using the equilibrium symmetry transformation, it can also be shown that the term left after transforming the action is proportional to ω , as expected. It is also clear that this system is not time-reversible – we can definitely tell the forward from the backward time evolution from the direction of rotation of z . We can say that the ω term corresponds to reversible dynamics (see Sec. 2.2.5). For example, going into a rotating frame $z' = ze^{i\omega t}$ recovers equilibrium dynamics. However, any non-trivial dependence of ω on z produces a genuinely non-equilibrium system. See, for example, Sec. 5.3.

These basic illustrations show how non-reciprocal systems and limit cycles are very generic and broad manifestations of non-equilibrium behavior. This, in addition to the wide presence of these features in many systems, motivates the attempts to use them as building blocks towards the grand goal of classifying non-equilibrium phases of matter.

3.2.9 (Critical) Exceptional Points

As a final feature of non-equilibrium dynamics we explicitly discuss, we introduce the notions of Exceptional Points (EPs) and Critical Exceptional Points (CEPs). Let us again consider the zero-dimensional non-reciprocal system in Eq. 3.58. By an appropriate redefinition of variables, these dynamics in the non-reciprocal case can be brought to the form:

$$\partial_t \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix} = - \begin{pmatrix} m_1 & 1 \\ 0 & m_2 \end{pmatrix} \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix} = -M \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix} \quad (3.62)$$

The eigenvalues and eigenvectors of matrix M are:

$$\begin{aligned} v_1 &= \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \lambda_1 = m_1 \\ v_2 &= \begin{pmatrix} \frac{1}{m_2 - m_1} \\ 1 \end{pmatrix}, \quad \lambda_2 = m_2 \end{aligned} \quad (3.63)$$

If $m_2 = m_1 = m$, the two eigenvalues become equal, and the eigenvectors coalesce (at that point the normalized v_2 is $(1, 0)^T$). This marks an EP, and the matrix becomes non-diagonalizable. EPs possess many interesting dynamical features [14, 182]. Let us actually solve for the evolution of this system at the EP. We have $\phi_2(t) = \phi_2(0)e^{-mt}$, and $\phi_1(t) = \phi_1(0)e^{-mt} - \phi_2(0)te^{-mt}$. So, for example, if $\phi_2(0) < 0$, the evolution of $\phi_1(t)$ can first be away from zero, and only later relax back to the stationary solution $\phi_1 = \phi_2 = 0$. This is only the case at the EP.

If we also approach criticality at $m = 0$, such that the EP becomes a CEP, we can also consider fluctuations. We know that at an equilibrium critical point the fluctuation spectrum diverges as k^{-2} in momentum space, see Chapter 2. Here, however, a fluctuation away from $\phi_1 = \phi_2 = 0$ can be deterministically propagated away from the stationary point. Thus,

we can imagine that the fluctuations at a CEP are more divergent than at a conventional critical point. Indeed, it turns out to be the case, with fluctuations diverging as k^{-4} or even k^{-6} [15, 18, 19]. This enhanced fluctuation spectrum has been proposed as a sensing mechanism [21, 22, 182, 183]. However, there is debate whether CEP-based sensors actually offer any improvement over conventional sensors [184]. Once we discuss the linear system at length in Section 4.2, we delve into some of the novel properties of CEPs, including sensing, in Section 4.3.

Although the above picture shows that a fine-tuning of two parameters (m_1 and m_2) is necessary to produce a CEP, this is not always the case. For example, if one of the fields is a Goldstone mode, it has $m_i = 0$. If it is coupled to the other field, a CEP is produced by fine-tuning only one parameter [18, 36]. Alternatively, a non-equilibrium $O(N)$ system in an ordered phase can be taken through a CEP by tuning the damping term through zero, which takes it to a rotating phase [19].

We see that CEPs are a purely non-equilibrium feature. Therefore, fully classifying non-equilibrium universality necessitates a deep understanding of CEPs. We attempt to contribute to this understanding throughout this dissertation.

3.2.10 Discussion and Outlook on Non-Equilibrium Phenomena

We conclude this chapter with a few remarks. First, as mentioned at the beginning, the above examples by far do not cover the entire span of non-equilibrium problems. The choice of examples provided throughout this chapter was made to demonstrate the amazing applicability of continuum, coarse-grained models that lend themselves to the field-theoretic treatment in the MSRJD or Keldysh forms. There are, however, other interesting collective phenomena broadly related to the topics we discussed before. These include proliferating matter [123], quantum noise and measurement problems [185], learning and adaptive matter [8, 112, 186], metamaterials [8], nematic and liquid crystal phases [8], and much more.

The field-theoretic treatment and the examples presented above rely heavily on the assumption of locality of interactions. This is not always a valid view. For example, even though the brain has many more small-distance connections, there are many that span the entire size of the brain and some that extend into the rest of the body. They clearly also play a central role, and a faithful theory should not simply discard them as a statistical rarity. Situations like these hint at the use of statistical mechanics methods [149], where discrete units are considered instead of a continuum. A mix of the two, where local field theory description on small scales and a network description on large scales are both present, can be of great interest and importance in developing our understanding of neural phenomena. This is left as a call for future work.

Additionally, new tools are needed to describe some collective phenomena outside of equilibrium. For example, in our discussion on pattern formation in Section 3.2.6, we mentioned that canonical perturbative renormalization group approaches run into trouble. As the scale of the problem approaches the scale of the pattern, divergencies arise and the RG procedure fails. It would be of great use to have an RG-like tool that can detect phase transitions while also detecting patterns that cannot be scaled out.

We also note that although we work here with problems with continuous time evolution, there are many interesting phenomena and uses for discrete-map systems. For example, cellular automata [187] are now finding applications in genomics [188, 189] and medicine [190, 191]. Interestingly, many-body effects in cellular automata can lead to perturbation-resistant memories [192], which can be of use in error-correcting codes. Another example is the logistic equation, which can famously describe period doubling bifurcations and a transition to chaos [39].

To end on a cheery note, a huge problem with the models discussed here, especially as applied to living matter, is that they might not be able to describe evolution [193]. The essential problem comes in the fact that the phase space is fixed – for example in the number

of fields a theory considers. But evolution always finds new ways of creating behaviors and mechanisms that are unexpected and were not present in the original living system. So, in a sense, the phase space of evolving species always expands, and it is not clear how that can be modeled with the type of theories discussed in this dissertation. Additionally, evolution has a habit of coupling the behavior of things on largely separated scales, both in time and in space. For example, the celestial cycles, including the changes in Earth's orbit, rotation around the Sun, day-night cycles, etc., have all affected both living creatures themselves and the larger environment they live in. Thus, there is coupling between many scales that guides the evolution of a living thing, which also goes against many of the principles of coarse-grained field theories and RG. Therefore, effectively describing evolution itself calls for notions of phase space and interactions on many scales that might be drastically different from field theory.

This concludes the main overview part of this dissertation, and we now move on to novel results in non-reciprocal phenomena, limit cycle phases of matter, and a model of seizures. We provide a more detailed introduction to each in the beginning of the respective chapters.

CHAPTER 4

NON-RECIPROCAL PHENOMENA

4.1 Overview of Non-Reciprocal Phenomena

As the first class of non-equilibrium problems that we delve into in detail, we consider non-reciprocal systems. As discussed in Sec. 3.2.8, they constitute only a subset of non-equilibrium, but have a clear interpretation of having multiple subsystems, the interaction between which does not obey Newton's third law. In the context of many-body and field theories, this allows for new states or phases of matter. Examples of such systems can be found in active matter [28, 29, 35, 38, 111, 121, 194, 195], many-body classical and quantum systems [18, 25–27, 30, 31, 34, 38], dynamical phase transitions [13, 14, 18, 23, 24, 27, 30–38, 196], models of neural dynamics [14, 51, 66–68, 151, 180, 197], chemical reactions [41, 44, 45, 84], and population dynamics [14, 27, 29, 198, 199].

On the microscopic level, if all the components of the system are considered, Newton's third law of action and equal reaction is, of course, satisfied. Thus, non-reciprocity arises as an effective phenomenon when the components of the system are coupled to something "external" we would usually call the environment or the bath. The generic routes to produce non-reciprocal interactions are then either for the environment to be outside of equilibrium, such as when it moves with respect to the system particles, or when there are different species of elements within the system that interact differently with the environment. In many-body contexts, the system can be subject to both driving and dissipation through interaction with the environment, which induces non-reciprocity. And, as mentioned before, there are a plethora of other contexts where the symmetry of interaction between different elements of the system is broken. To illustrate the general idea and motivate the study of such systems, we now discuss some examples.

Before moving to more general non-equilibrium examples, it serves to demonstrate that

one can find non-reciprocity even in canonical physical theories. In order to do that, one has to again adhere to the principle outlined above – take a subpart of a given system and consider the effective interactions within that subpart while viewing the rest of the system as a mediating environment. For example, one can take two charged particles moving with perpendicular velocities and observe that the forces they exhibit on each other are non-reciprocal [38]. This is due to the magnetic interaction between them, and one can restore Newton’s third law by including electromagnetic fields that carry momentum in the description. Although in this example the origin of "non-reciprocity" is clear and can be incorporated into the description, it is not always the case. More often, analytically taking the elements external to the system of interest into account is unfeasible (those elements might not even be known a priori), and one has to resort to an effective description that has non-reciprocity.

In the realm of active matter, an example of non-reciprocal interactions is catalytic colloids that chemically interact with the surrounding solution [121, 194, 195]. Such a transport phenomenon is called diffusiophoresis. The two relevant parameters for our illustration are the surface activity function α of the colloid and the surface mobility μ . The first describes the interaction of the colloid surface with the surrounding substrate. This interaction can happen, for instance, by an expulsion from or absorption into the colloid of the substrate particles, or by a chemical reaction of the colloid surface with the substrate. This surface activity results in a concentration profile C of the substrate that is proportional to α . The second corresponds to the response of the colloid to the gradients of concentration, produced externally, by other colloids, or by the colloid of interest itself. It produces the form of slip velocity that reads $\vec{v} \sim -\mu \vec{\nabla} C$.

In principle, the two parameters α and μ can be engineered to take on different, even negative, values. Now, if the system has two species of colloids, with different activity functions and mobilities, it is quite straightforward to get non-reciprocal interactions. The

colloid of species 1 with mobility μ_1 responds to the concentration profile produced by colloid from species 2 with activity α_2 with a slip velocity $\vec{v}_1 \sim -\mu_1 \vec{\nabla} C_2 \sim \mu_1 \alpha_2$, while the colloid in species 2 responds as $\vec{v}_2 \sim \mu_2 \alpha_1$. By tuning these two relative combinations of parameters, effective non-reciprocal interactions, including those with chase and run motion, can be constructed. Of course, active colloids are not the only active matter area that observes such asymmetry [112, 113].

Moving on from active matter, non-reciprocity also arises in the context of many-body and quantum systems. As an example, we can consider a binary condensate. We already saw a discussion on the driven dissipative Gross-Pitaevskii equation describing an out-of-equilibrium BEC in Sec. 3.2.5. Now, a two species BEC would be described by a pair of GP equations [200]:

$$\begin{aligned} i\hbar\partial_t\phi_1 &= \left(-\frac{\hbar^2}{2m_1}\nabla^2 + V_1 + g_{11}|\phi_1|^2 + g_{12}|\phi_2|^2\right)\phi_1 \\ i\hbar\partial_t\phi_2 &= \left(-\frac{\hbar^2}{2m_2}\nabla^2 + V_2 + g_{22}|\phi_2|^2 + g_{21}|\phi_1|^2\right)\phi_2 \end{aligned} \tag{4.1}$$

In equilibrium, the interactions between the species are of course reciprocal $g_{12} = g_{21}$. However, in the presence of driving and dissipation, this does not have to be true. For example, the effective dynamics for the phase fluctuations of the two fields is generically non-reciprocal, which also leads to a CEP [18]. Also note that the above equations have a $\mathcal{U}(1) \times \mathcal{U}(1)$ symmetry, meaning that the symmetry for each field is preserved. In the realm of phase transitions, where the fields correspond to the order parameters of the respective transition, similar effective dynamics can be produced outside of equilibrium. For example, effective $\mathcal{Z}_2 \times \mathcal{Z}_2$ dynamics, which can, for instance, correspond to density bistability and to

antiferromagnetic transition, are explored in [30] and take the form:

$$\begin{aligned}\partial_t \phi_1 &= \left(D_1 \nabla^2 - r_1 - g_1 \phi_1^2 - g_{12} \phi_2^2 \right) \phi_1 + \xi_1 \\ \partial_t \phi_2 &= \left(D_2 \nabla^2 - r_2 - g_2 \phi_2^2 - g_{21} \phi_1^2 \right) \phi_2 + \xi_2\end{aligned}\tag{4.2}$$

where ξ_i are noises. This has also been extended to a larger $\mathcal{O}(N_1) \times \mathcal{O}(N_2)$ symmetry in [31].

Next, we can find examples of "non-reciprocal" interactions in neuroscience as well, although generally people do not refer to them as such. As discussed in Sec. 3.2.7, one of the most famous mean-field dynamical equations for the activities of the excitatory and inhibitory populations of neurons is the Wilson-Cowan equation [151]:

$$\begin{aligned}\tau_E \partial_t A_E &= -A_E + (1 - r_E A_E) \mathcal{S}_E [w_{EE} \otimes_x A_E - w_{EI} \otimes_x A_I + P_E(t, \vec{x})] \\ \tau_I \partial_t A_I &= -A_I + (1 - r_I A_I) \mathcal{S}_I [w_{IE} \otimes_x A_E - w_{II} \otimes_x A_I + P_I(t, \vec{x})]\end{aligned}\tag{4.3}$$

Clearly, the interaction between the populations is non-reciprocal. As written here, this is a population level model, but it can be extended to a network or a continuum by replacing the couplings w_{ij} with some spatial kernel. As a simplifying approximation, the sigmoid functions $\mathcal{S}_i$ can also be expanded to the necessary order which allows for simpler computational and analytic approaches [54]. Quite similarly, the Amari field equations can be written in the form [66]:

$$\mu \frac{\partial}{\partial t} u_a(t, \vec{x}) = -u_a(t, \vec{x}) + \sum_b f_b [u_b(t, \vec{x})] \otimes w_{ab}(\vec{x}) + q_a(t, \vec{x})\tag{4.4}$$

where again we have two populations that interact non-reciprocally. As mentioned above (Sec. 3.2.7), the major difference between the two is the order of applying the sigmoid and spatial kernel convolution. Nevertheless, the logic and apparent non-reciprocity are similar.

Quite similarly, in the realm of chemical reactions, non-reciprocity is quite easy to en-

counter in the context of reaction-diffusion systems. For example, a system consisting of two chemicals with concentrations $u_1(t, x)$ and $u_2(t, x)$ can be governed by a set of equations in the form [84]:

$$\begin{aligned}\partial_t u_1 &= D_1 \nabla^2 u_1 + a_1 u_1 - b_1 u_2 \\ \partial_t u_2 &= D_2 \nabla^2 u_2 - a_2 u_2 + b_2 u_1\end{aligned}\tag{4.5}$$

When the temporal rates and diffusion coefficients are very different between the two substances, such reactions can also lead to pattern formation. As a concrete example, the Belousov-Zhabotinsky reaction rate equations have a similar form, albeit with some non-linearities and an additional variable. The resulting pattern occurs through a finite-momentum instability – Turing instability [84].

It is also possible that non-reciprocity is not realized through different couplings, but rather through the couplings turning on and off depending on some condition. An example of such a system is a vision-cone XY model [201, 202]. Imagine a 2d lattice of spins, where each spin interacts only with the neighbors within a certain angle of the direction it is pointing at. Naturally, if a spin points to its neighbor, but a neighbor does not point back, the interaction is one-way. This model can exhibit long-range order in two dimensions [201, 202], therefore circumventing the Mermin-Wagner theorem due to its non-equilibrium nature. A natural application for such models could be the study of flocking, where agents use their vision to orient themselves.

This naturally brings us to the last type of system we discuss as an example – social systems. It is not difficult to imagine non-reciprocal interactions in population dynamics, where it often takes a form of chase and run motion. Predator-prey systems are perhaps the

most obvious example. A paradigmatic model is the set of Lotka-Volterra equations:

$$\begin{aligned}\frac{dA}{dt} &= \alpha A - \beta AB \\ \frac{dB}{dt} &= -\gamma B + \delta AB\end{aligned}\tag{4.6}$$

Here A and B are the prey and predator populations, respectively. The prey is consumed by the predator, thereby inducing non-reciprocal interactions. This can also be promoted to a proper field theory if spatial dynamics and fluctuations are included [2]:

$$\begin{aligned}\frac{\partial\psi}{\partial t} &= \alpha\psi + D\nabla^2\psi - \beta\psi\phi + \xi \\ \frac{\partial\phi}{\partial t} &= -\gamma\phi + D\nabla^2\phi - \delta\psi\phi + \eta\end{aligned}\tag{4.7}$$

With numerous examples, it is clear that non-reciprocal interactions constitute an important class of non-equilibrium problems. They also lead to a whole plethora of novel phenomena, many of which we do not mention here. However, attempting the problem of classifying possible non-equilibrium phases of matter necessitates studying and classifying the phases exhibited by non-reciprocal systems.

In the work laid out in the next three sections, we first describe the phenomenology of disordered phases exhibited by non-reciprocal systems at the linear level. As described in the section, a natural initial step in field-theoretic descriptions is linearization. Hence, it is relevant practically to explore linear non-reciprocal systems. We lay out more motivation for studying such systems, demonstrate how CEPs and dynamical steady states appear as a result of non-reciprocity, and discuss future directions. In the next section, we discuss more general results, such as how the geometry of the phase diagram can lead to CEPs, the generic structure of real-space correlation functions around CEPs, and whether CEPs can be used for sensing. In the final section we explore the phenomenology of non-reciprocally coupled XY models in two dimensions and fate of the Kosterlitz-Thouless transition near the CEP.

4.2 Fluctuations of Linear Non-Reciprocal Systems

This section is based on the work presented in [15]. We also provide some additional details either not included in the published version or presented in the supplemental material.

4.2.1 *Why Study Linear Systems?*

As mentioned in Section 4.1, non-reciprocal interactions have received increased interest in recent years due to their wide applicability and theoretical insights for open systems and active matter. Non-reciprocity pushes systems out of equilibrium, therefore producing novel phases of matter, critical exceptional points, and novel dynamics (see references at the beginning of Section 4.1). It is therefore useful to develop an understanding of as many exactly solvable non-reciprocal models as possible, given especially the severe additional complexity that non-reciprocity can bring to an already complicated field of noisy nonlinear models.

In equilibrium statistical field theory, one of the few solvable systems is a Gaussian ensemble, which we looked at in Chapter 2. From the perspective of dynamics, such an ensemble corresponds to a linear set of equations that define the relaxation of the order parameter to a stable steady state [1, 2, 4], as discussed in Sec. 2.1.1. Following that logic, we assume that a simple non-reciprocal model one ought to study is a set of linear stochastic equations. Statistical understanding of such a system can then be inferred either from promoting these equations to a path integral description through the MSRJD procedure (see Sec. 2.2.2) or through the direct calculation of correlation functions. The linearized theory is often the starting point for stability analysis of a full nonlinear theory, for example to determine the upper critical dimension where the system can be described as mean-field with small fluctuation corrections.

We show below that increasing non-reciprocity expands the region of stability of the system with respect to the tuning of each field's relaxation constant. This is perhaps un-

surprising: if the fields are coupled non-reciprocally, then an increase in one produces a decrease in the other, while the decrease in the other also produces the decrease in the first. That expansion of the stability region leads to the occurrence of exceptional lines (ELs) and critical exceptional points (CEPs). CEPs have been introduced in Sec. 3.2.9. In our case, exceptional lines mark the onset of oscillations between the fields. CEPs occur when the exceptional lines coincide with the critical lines, or when the point of two modes merging also happens to be critical. At these isolated points, the system experiences enhanced fluctuations. For the equilibrium Ising universality class, the critical point is marked by a k^{-2} divergence in the fluctuation spectrum [2–4]. We observe a k^{-4} divergence often associated with CEPs [18, 19], and also get a k^{-6} divergence happening at isolated points where several parameters need to be tuned.

Pattern formation or finite momentum instabilities are also prevalent in non-equilibrium systems [84]. In reaction-diffusion systems, for instance, different temporal and diffusive constants are responsible for pattern-forming behavior. The overdamped models that we consider are a generalization of such linear models. Here we demonstrate the simple mechanism of forming finite momentum instabilities in the regime of strong non-reciprocity.

Finally, we show how non-reciprocal interactions lead to the appearance of $1/f$ noise. Noise and fluctuations with $1/f$ spectra are observed in a variety of systems with various explanations as to their origin [142, 203–205]. Here we show that such spectrum of noise can be achieved for the effective dynamics of one of the fields as the other is integrated out. The $1/f$ noise is generally observed for high enough frequencies, but can be extended to appear for all the frequencies as the CEP is approached.

4.2.2 Overdamped Systems

We start with a simple linear, spatially extended and noisy dynamical system:

$$\begin{aligned}\partial_t \phi_1 &= -m_1 \phi_1 + j_{12} \phi_2 + D_1 \nabla^2 \phi_1 + \xi_1 \\ \partial_t \phi_2 &= -m_2 \phi_2 + j_{21} \phi_1 + D_2 \nabla^2 \phi_2 + \xi_2\end{aligned}\tag{4.8}$$

Here (for $i \in 1, 2$) ϕ_i are scalar fields, m_i are the relaxation constants of each field, $j_{12} \neq j_{21}$ are non-reciprocal couplings between the fields, D_i are diffusion constants, and ξ_i are Gaussian white noise variables with correlations $\langle \xi_i(t, \vec{x}) \xi_j(t'; \vec{x}') \rangle = B_{i/j} \delta_{ij} \delta(t - t') \delta^d(x - x')$. We assumed that the noises are uncorrelated and that there is no cross-diffusion for simplicity. We pick the Ito discretization [2, 3]. Although we consider it on general grounds, this dynamics can underlie more complicated non-linear dynamics in a variety of phenomena, as discussed in Sec. 4.1.

This system is $\mathcal{Z}_2$ symmetric under the transformation $(\phi_1, \phi_2) \rightarrow (-\phi_1, -\phi_2)$. We therefore expect that an appropriate system for comparison is the $\mathcal{Z}_2$ transition in the Ising model from the disordered phase. Clearly, if we remove the cross-couplings j_{12} and j_{21} , we get two copies of the equilibrium $\mathcal{Z}_2$ transition in the Gaussian approximation. In other words, this is two non-reciprocally coupled Model A systems [1, 2] in the linear approximation, which was discussed in Sec. 2.2. The k^2 IR divergence of the correlation function in Model A is put in contrast to the divergences that one gets as a consequence of non-reciprocity. This can similarly be extended to the conserved order parameter Model B variant, which only changes the correlations by an additional factor of k^2 [1, 2]. The extension to other models in the Hohenberg-Halperin classification we leave for future work. We now proceed to explore the stability and dynamics of this system.

At the mean field level where we ignore spatial extent and noise in this system, and after defining $\phi_i(t, \vec{x}) \sim \int d\omega e^{i\omega t} \hat{\phi}_i(\omega, \vec{x})$, we see that the dispersion relations for the frequencies

are:

$$i\omega_{\pm} = -\frac{m_1 + m_2}{2} \pm \frac{1}{2}\sqrt{(m_1 - m_2)^2 + 4j_{12}j_{21}} \quad (4.9)$$

The system is thus stable when:

$$m_1 m_2 > j_{12} j_{21} = \Delta \quad (4.10)$$

where we defined the Δ as a measure of non-reciprocity. That can be seen by introducing $j_{\pm} = (j_{12} \pm j_{21})/2$, see [13], which allows us to rewrite:

$$\Delta = j_+^2 - j_-^2 \quad (4.11)$$

We thus use Δ as the main control parameter to describe how much non-reciprocity there is in the system. Parameters like these, instead of the individual j_{ij} , have been observed to control the behavior of non-reciprocal systems at long ranges [28, 30, 31]. The regions of stability for $\Delta > 0$, $\Delta = 0$ and $\Delta < 0$ are depicted in Fig. 4.1. Clearly, as non-reciprocity is increased, the region of stability is expanded. We also observe that as we decrease Δ past zero, instead of a single continuous line marking the edge of stability, we pick up points that separate lines that behave differently. For example, for $\Delta = 0$, where we assume that $j_{21} \neq 0$ without loss of generality, the matrix generating the dynamics at the mean field level is:

$$\partial_t \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix} = -M \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix} = - \begin{pmatrix} m_1 & 0 \\ -j_{21} & m_2 \end{pmatrix} \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix} \quad (4.12)$$

As both $m_1, m_2 \rightarrow 0$, this matrix becomes non-diagonalizable, which marks an Exceptional Point (EP). In general, EPs occur when two modes (eigenvalues and eigenvectors) of the system coalesce. Looking at Eq. 4.9, this can only occur when the modes start to pick up an imaginary part. For $\Delta = 0$, this happens at an isolated point, since the modes don't have

an imaginary part anywhere within the stable region. For $\Delta < 0$ there are exceptional lines (Fig. 4.1) that intersect with the critical lines at the corner points. In both cases, points lying on the critical lines that also happen to be exceptional are, in fact, Critical Exceptional Points (CEPs). As discussed in Sec. 3.2.9, CEPs occur only outside of equilibrium [19], and exhibit a range of unusual dynamical and statistical properties [14, 18, 19], some of which we describe next.

The addition of spatial extent in the absence of cross-diffusion simply amounts to modifying $m_i \rightarrow m_i + D_i k^2$, where k is the momentum of the field in Fourier space. Therefore, the stability of each momentum mode can be understood by taking the (m_1, m_2) point, and then moving in a straight line of positive slope from that point, defined by the ratio of D_1 and D_2 as well as the momentum value k^2 , see Fig. 4.1. In the absence of cross-correlations in the noise, the region of stability is not altered.

In order to calculate the correlation functions and explore the type of transition to instability that each of the critical lines produces, we resort to the MSRJD procedure in Sec. 2.2.2. As we show in Fig. 4.1, for most of the lines, the frequency of oscillation of the fields as one approaches critical lines is zero, and it is therefore appropriate to look at the static, or equal time, correlation functions $\langle \phi_i(t, \vec{k}_1) \phi_j(t, \vec{k}_2) \rangle = \delta^d(\vec{k}_1 + \vec{k}_2) G_{ij}^K(t = 0, \vec{k}_1)$ in order to understand the transition. However, as shown in Fig. 4.1, the region between the exceptional lines for the $\Delta < 0$ case picks up a non-zero frequency for the oscillation of the field modes. Thus, looking at the static correlation function might be inappropriate for the critical line inside that region. We discuss that issue in more detail below. We provide the action and the full form of the correlation functions in Appendix B.

The equal-time correlation functions can be calculated exactly for all Δ :

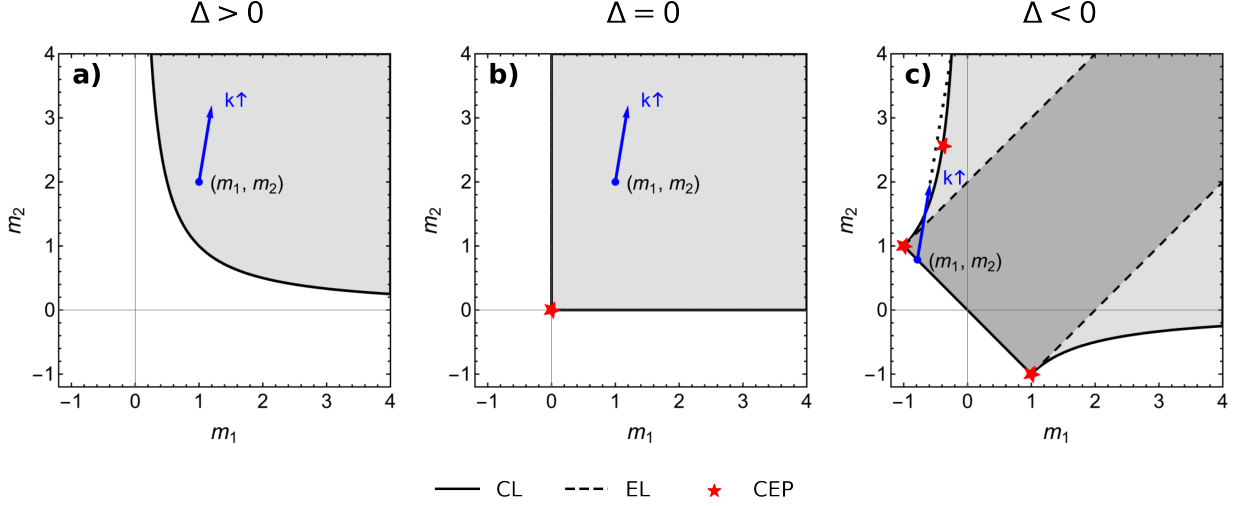


Figure 4.1: Stability regions (light grey), oscillating regions (dark grey) and CEPs (stars) of the overdamped system for various Δ . The critical lines are solid, while the dashed lines correspond to the onset of oscillations. The example points (m_1, m_2) with the shift for each mode due to momentum k are plotted as blue arrows. a) $\Delta > 0$. The critical line has the equilibrium k^{-2} divergence in the correlators for small momenta k . b) For $\Delta = 0$ the critical lines at $m_1 = 0$ and $m_2 = 0$ exhibit usual equilibrium behavior, but meet at the CEP when $m_1 = m_2 = 0$. There the correlators pick up stronger k^{-4} and k^{-6} divergences. c) For strongly non-reciprocal regime $\Delta < 0$, the region of stability is expanded, and a region with oscillations appears. The critical lines all have k^{-2} divergence in the correlators. However, the line $m_1 = -m_2$ is marked by closed rotating orbits in ϕ_1, ϕ_2 space. The CEPs where the hyperbola meets the $m_1 = -m_2$ lines also exhibit k^{-4} or k^{-6} divergence for equal diffusion constants. The dotted line is chosen with (m_1, m_2) on the critical line with increasing momenta k eventually venturing into the unstable region, thus causing a finite momentum instability. This can be achieved only for high enough D_2/D_1 . Fig. 4.2 is plotted along this line.

$$G_{11}^K(t=0, \vec{k}) \sim \frac{j_{12}^2 B_2 + B_1 \{-\Delta + (m_2 + D_2 k^2) [m_1 + m_2 + (D_1 + D_2) k^2]\}}{[m_1 + m_2 + (D_1 + D_2) k^2] [-\Delta + (m_2 + D_2 k^2) (m_1 + D_1 k^2)]} \quad (4.13a)$$

$$G_{12}^K(t=0, \vec{k}) \sim \frac{j_{12} B_2 (m_1 + D_1 k^2) + j_{21} B_1 (m_2 + D_2 k^2)}{[m_1 + m_2 + (D_1 + D_2) k^2] [-\Delta + (m_2 + D_2 k^2) (m_1 + D_1 k^2)]} \quad (4.13b)$$

$$G_{22}^K(t=0, \vec{k}) \sim \frac{j_{21}^2 B_1 + B_2 \{-\Delta + (m_1 + D_1 k^2) [m_1 + m_2 + (D_1 + D_2) k^2]\}}{[m_1 + m_2 + (D_1 + D_2) k^2] [-\Delta + (m_2 + D_2 k^2) (m_1 + D_1 k^2)]} \quad (4.13c)$$

Along the critical lines $m_1 m_2 = \Delta$, the denominator in all three becomes:

$$k^2 \left[m_1 + m_2 + (D_1 + D_2) k^2 \right] \left[D_2 m_1 + D_1 m_2 + D_1 D_2 k^2 \right] \quad (4.14)$$

Clearly, for $\Delta > 0$, where both $m_1, m_2 > 0$, neither of the brackets can be zero in the limit $k \rightarrow 0$, so the correlation functions exhibit the usual divergence of k^{-2} associated with Model A (or $O(2)$ for two fields) dynamics. Interestingly, some non-reciprocity is still allowed to have the equilibrium behavior, as long as the reciprocal coupling is larger.

When $\Delta = 0$, the critical lines are $m_1 = 0$ and $m_2 = 0$. We see that if only one of these is zero, the correlation functions again have the usual k^{-2} divergence. However, if both $m_1 = 0 = m_2$, the correlation functions pick up a k^{-4} and even k^{-6} divergencies (the numerator in G_{12}^K reduces the divergence from k^{-6} to k^{-4}). For example, if $j_{12} = 0$ with $j_{21} \neq 0$ (such that $\Delta = 0$ but there is still non-trivial coupling), it can be seen that $G_{11}^K \sim k^{-2}$, $G_{12}^K \sim k^{-4}$ and $G_{22}^K \sim k^{-6}$ as $k \rightarrow 0$. Such enhanced divergences are a signature of CEPs [18, 19]. Clearly they affect the critical dimensions of the theory from a renormalization group perspective. We see, however, that although such points can be reached without any nonlinearity in the model, it requires tuning of more than one parameter. The phases beyond the critical lines depend on the nonlinearities added to the model, and we leave them for future work.

Finally, for $\Delta < 0$ case, the two critical lines on the hyperbolas exhibit the usual k^{-2} divergence. Here, however, the relative signs of m_1 and m_2 are different, making it possible that the second bracket in the denominator also contributes to the divergence. This happens when $D_2 m_1 + D_1 m_2 = 0$. That means that depending on the values of the diffusion constants, there is a CEP along one of the branches of the hyperbola, see Fig. 4.1. In fact, this point occurs where the line with slope D_2/D_1 (which is the line along which we draw modes at different k) is tangent to the hyperbola. We describe this in more general terms in Sec. 4.3.1.

Additionally, along the $m_1 + m_2 = 0$ line, we get a different type of divergence. Plugging

this into the denominators of the correlation function gives:

$$k^2 (D_1 + D_2) \left[-\Delta + \left(-m_1 + D_2 k^2 \right) \left(m_1 + D_1 k^2 \right) \right] \quad (4.15)$$

Therefore this line also exhibits the same divergence as the other critical lines, although it is less intuitive how to interpret it. Usually, the divergence occurs due to the "flattening" of the potential landscape, and some direction becomes undamped. In this case, however, the orbits along this line are closed, see [Appendix B](#). This is a manifestation of the non-equilibrium nature of the system. The "potential" landscape is flattened in the radial direction, but slanted in the angular direction, hence the closed, rotating orbits. So, although it also exhibits a k^{-2} divergence, this critical line is clearly different from the usual equilibrium critical line, since it exhibits rotational dynamics.

At the intersections of the hyperbola and this line, i.e. where $m_1 = -m_2$ and $m_1 m_2 = \Delta$, we get the denominators:

$$k^4 (D_1 + D_2) \left[(D_2 - D_1) m_1 + D_1 D_2 k^2 \right] \quad (4.16)$$

Thus, these points are CEPs and exhibit the enhanced k^{-4} divergence. We note that the only way to approach this point from the stable region is to have $D_2 \leq D_1$ for $m_1 < 0$ and $D_2 \geq D_1$ for $m_1 > 0$, which makes this expression non-negative. Additionally, if $D_1 = D_2$, these points also coincide with the additional CEPs along the hyperbola described above, and the divergence is k^{-6} instead. We can therefore see that one can achieve CEPs by introducing non-reciprocity. Even higher order divergences can be achieved by having more than two fields. The consequences of transitions through these points vary from system to system [\[18, 19\]](#), but one needs to keep them in mind when nonlinear theories are developed.

For $\Delta < 0$, we also see that for a certain choice of (m_1, m_2) and $D_{1/2}$, the system can develop a finite momentum instability, see [Fig. 4.1](#). A typical dependence of the least

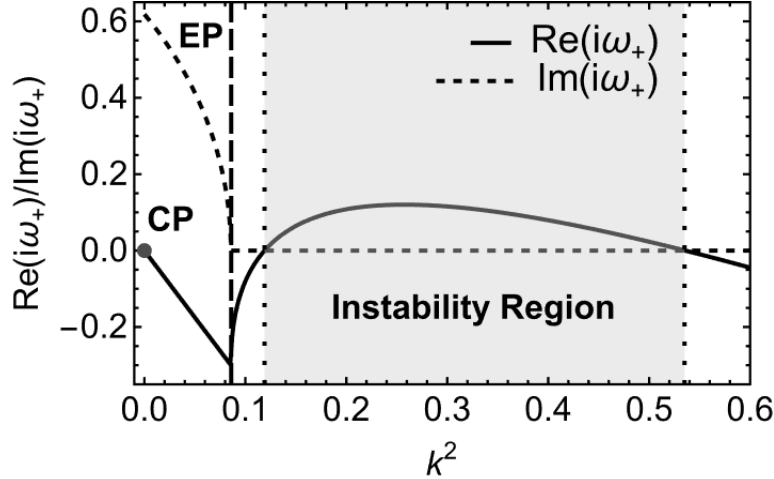


Figure 4.2: $\text{Re}(i\omega_+)/\text{Im}(i\omega_+)$ vs. k^2 for (m_1, m_2) point chosen in Fig. 4.1, c). $k^2 = 0$ marks the critical point, where the $\text{Re}(i\omega_+) = 0$, thus marking an instability. In this case the exceptional point occurs where $\text{Im}(i\omega_+) = 0$, since this is also where $i\omega_+ = i\omega_-$. As momentum is increased, the modes go into the unstable region, where they would grow indefinitely (unless stabilized by a nonlinearity). This is the finite-momentum instability and is responsible for pattern formation. Notice the similarity to the Type I_s instability by the classification in Fig. 3.5.

stable mode on momentum k is plotted in Fig. 4.2 for such a choice. Such instabilities are generally stabilized by nonlinear terms, which also determine what kind of patterns the system exhibits. Detailed analyses of pattern formation have been explored in a variety of contexts [84], including a nonlinear extension of this model to a non-reciprocal Allen-Cahn model [37]. Here we pick out several properties.

If the ratio of diffusion constants is such that the line of modes in Fig. 4.1,c) crosses the critical line at two points, we have two critical momenta $k_c^\pm$. The correlation functions at either of those points diverge as $|k - k_c^\pm|^{-1}$. The patterns exhibited by such systems consist of stripes of similar length scales defined by the values of k where instability happens. However, if the ratio of diffusion constants is such that the line of modes is tangential to the critical line for the particular values of (m_1, m_2) , the finite momentum instability happens at a CEP (see above). In that case, the divergence of the correlation functions is $|k - k_c|^{-2}$. We leave the detailed exploration of that transition to future work.

Finally, one can integrate out one of the fields in order to get the effective dynamics for the other. This can be done by using the dynamical action defined above. We perform this calculation in [Appendix B](#) by integrating out ϕ_2 . Aside from the terms influencing the deterministic behavior of ϕ_1 , we also pick up a term that alters the power spectrum of the noise. In Fourier space the noise correlations are $\langle \xi_1(k_1, \omega_1) \xi_1(k_2, \omega_2) \rangle \sim B_1 \delta(\omega_1 + \omega_2) \delta^d(k_1 + k_2)$. The effect of integrating out ϕ_2 amounts to the replacement:

$$B_1 \rightarrow B_1 + \frac{j_{12}^2 B_2}{\omega_2^2 + (m_2 + D_2 k^2)^2} \quad (4.17)$$

This is similar to the amplification of noise observed in [\[20\]](#), where it relied on the non-orthogonality of eigenvectors of the matrix generating the dynamics. The emergence of colored noise has been observed for non-reciprocal systems, see [\[28\]](#) and references therein. Here we see a similar effect in a spatially extended system. If we consider the spatially local contribution (which might be the only relevant contribution as the system undergoes some RG flow), we get that in dimensions $d \leq 3$ and for $\omega \gg m_2$, this additional term becomes:

$$\int d^d k \frac{j_{12}^2 B_2}{\omega_2^2 + (m_2 + D_2 k^2)^2} \sim \frac{j_{12}^2 B_2}{D_2^{\frac{d}{2}} \omega^{\frac{4-d}{2}}} \quad (4.18)$$

We therefore see that $1/f$ noise can be produced simply by coupling the field of interest to another field. The level of non-reciprocity does not play a role when the noise is concerned, as long as there is any coupling to the other field. Non-reciprocity plays a role for the deterministic additions to the dynamics of ϕ_1 , see [Appendix B](#). However, when $\Delta = 0$, the only contribution is the one in [Eq. 4.17](#). Thus, one can get $1/f$ noise by a one-way coupling to another Gaussian field with spatial correlations.

4.2.3 Inertial Systems

In the previous section, we explored an example overdamped system. That choice was made since often in the study of critical dynamics one approximates the decay of the order parameter towards its stationary value as relaxational [2]. However, for the sake of generality it is prudent to also include the possibility of higher order time derivatives. First of all, real systems often exhibit such dynamics, for example in neural field models [51]. Additionally, non-equilibrium conditions open up the possibility of the existence of fixed points where the ∂_t^2 terms are relevant and cannot be discarded [19].

We thus start with a natural extension of the overdamped model:

$$\begin{aligned}\partial_t^2\phi_1 + \gamma_1\partial_t\phi_1 &= -m_1\phi_1 + j_{12}\phi_2 + D_1\nabla^2\phi_1 + \xi_1 \\ \partial_t^2\phi_2 + \gamma_2\partial_t\phi_2 &= -m_2\phi_2 + j_{21}\phi_1 + D_2\nabla^2\phi_2 + \xi_2\end{aligned}\tag{4.19}$$

Here the terms are defined as in Sec. 4.2.2, with the addition of two real parameters $\gamma_{1/2}$, which we take to be positive. They can also be taken as negative, and additional terms with cross dependence on $\partial_t\phi_i$ can also be introduced. All of these possibilities can change the behavior of the system and should be addressed as needed. However, the number of possible terms that can be included even at the linear level becomes quite large very quickly. Here we restrict ourselves to the above model just to demonstrate that we observe the same effects as in the overdamped system, at least at the static level.

We proceed with the mean field description as before. In Fourier space, the frequencies obey the equation:

$$\begin{aligned}(i\omega)^4 + (\gamma_1 + \gamma_2)(i\omega)^3 + (\gamma_1\gamma_2 + m_1 + m_2)(i\omega)^2 \\ + (\gamma_1m_2 + \gamma_2m_1)(i\omega) + m_1m_2 - j_{12}j_{21} = 0\end{aligned}\tag{4.20}$$

Here we again see that the behavior of the system is controlled by the same non-reciprocity

parameter $\Delta = j_{12}j_{21}$. Additionally, we see that the relative values of $\gamma_{1/2}$ should change the dynamics as well, so one can introduce the ratio γ_1/γ_2 as another control parameter. The effect of diffusion, as before, can be thought of as moving from the point (m_1, m_2) on the stability diagram along a line with positive slope for each mode under consideration. The exact solutions to Eq. 4.20 are bulky, so we proceed with the $\gamma_1 = \gamma_2$ case analytically. The other cases can be addressed as needed with the general principles outlined below.

When $\gamma_1 = \gamma_2 = \gamma$, the least stable solutions are governed by the dispersion:

$$i\omega_{\pm} = -\frac{\gamma}{2} + \frac{1}{2}\sqrt{\gamma^2 - 2(m_1 + m_2) \pm 2\sqrt{4\Delta + (m_1 - m_2)^2}} \quad (4.21)$$

In Fig. 4.3 we plot the regions of stability as well as the regions for the presence of oscillations for a few example values of γ_1/γ_2 and Δ . For $\Delta \geq 0$ the region of stability is the same as for the overdamped case, independent of γ . For $\Delta < 0$, the region of stability is distorted depending on the values of γ_1/γ_2 . However, the key features of having three critical lines that create two isolated points still remains. We show below that these points are CEPs. The CEP in the strongly non-reciprocal regime $\Delta < 0$ along the hyperbola remains, and the system also picks up an additional CEP on the internal line. We show all of this below.

We can proceed to developing the statistical understanding of this system similarly to the overdamped case. Here we only quote the results for brevity. For $\gamma_1 = \gamma_2 = \gamma$, we again see that the equal time correlation functions are dependent on some denominator. It is given by:

$$\gamma \left[\Delta - (m_1 + D_1 k^2) (m_2 + D_2 k^2) \right] * \left[4\Delta + (m_1 - m_2 + D_1 k^2 - D_2 k^2)^2 + 2\gamma^2 (m_1 + D_1 k^2 + m_2 + D_2 k^2) \right] \quad (4.22)$$

From the first bracket it is clear that the critical lines are again at $\Delta = m_1 m_2$, which generally diverge as k^{-2} . When $\Delta = 0$, we again get k^{-2} divergences on the $m_1 = 0$ and

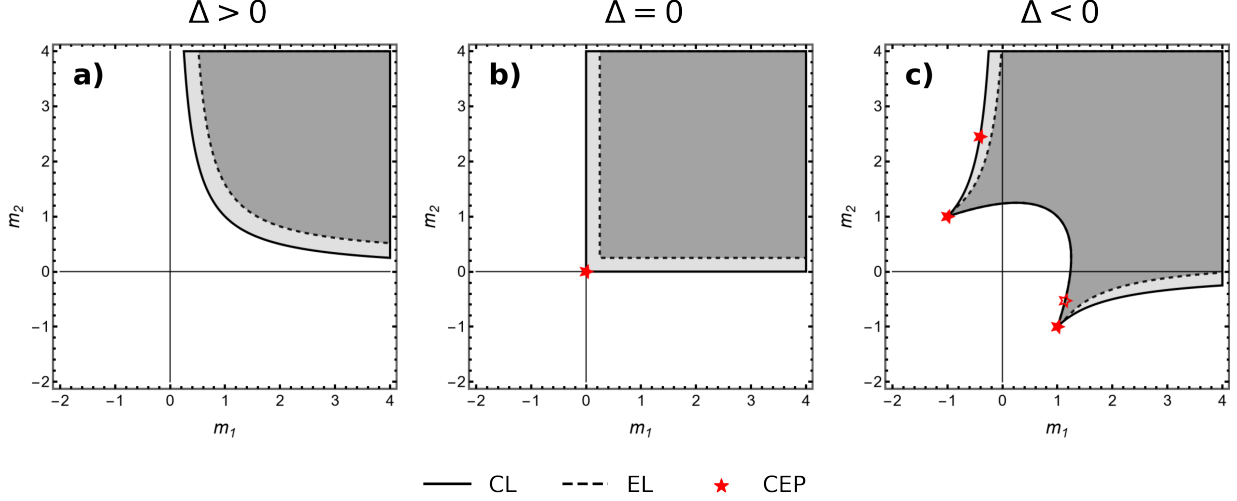


Figure 4.3: Stability regions (light grey), regions with oscillations (dark grey) and CEPs (stars) for the inertial system, Eq. 4.19 for $\gamma_1 = \gamma_2$. The critical lines are solid, the dashed lines correspond to the onset of oscillations. a) $\Delta > 0$. The critical line does not depend on γ_1/γ_2 , and the correlation functions diverge with decreasing momentum in the usual equilibrium manner as k^{-2} . b) $\Delta = 0$. The region of stability is the same as in the overdamped case, with the CEP appearing at $m_1 = m_2 = 0$. There the correlators pick up additional k^{-4} , k^{-6} divergences. c) $\Delta < 0$. Although the region of stability is distorted compared to the overdamped case and now also depends on γ_1/γ_2 , the qualitative picture is the same. There are now three critical lines, which intersect at two CEPs. The additional CEP along the hyperbola is preserved, and another CEP that appears for sufficiently different D_1 and D_2 is the hollow star. There the correlators pick up additional k^{-4} or k^{-6} divergences.

$m_2 = 0$ lines, and a CEP with k^{-4} and k^{-6} divergences at $m_1 = 0 = m_2$. And also, similarly, when $\Delta < 0$ so m_1 and m_2 are allowed to take negative values, at the point where $D_2 m_1 + D_1 m_2 = 0$, we get an enhanced divergence. The second bracket defines an additional line for $\Delta < 0$. The equation for the line is $4\Delta + (m_1 - m_2)^2 + 2\gamma^2(m_1 + m_2) = 0$. Along that line, that bracket is:

$$k^2 \left[(D_1 - D_2)^2 k^2 + 2(m_1 - m_2)(D_1 - D_2) + 2\gamma^2(D_1 + D_2) \right]. \quad (4.23)$$

For large enough D_2/D_1 , the terms not involving k^2 inside the brackets can be zero along this critical line, leading to another CEP. This happens when for some point on the internal critical line the line with slope defined by D_2/D_1 is exactly tangent. This is similar to the

CEP on the hyperbola. In the same manner, for a choice of (m_1, m_2) close enough to the cusps, and with a high enough ratio of the diffusion constants, another finite momentum instability can occur, now also at a finite frequency and therefore of type I_o , see Sec. 3.2.6.

At the intersections of the two critical lines, we have $\Delta = m_1 m_2$ and $m_1 + m_2 = 0$. In that case the denominators of the correlation functions are:

$$\gamma k^4 \left[(D_2 - D_1)m_1 + D_1 D_2 k^2 \right] \left[(D_1 - D_2)^2 k^2 + 4m_1(D_1 - D_2) + 2\gamma^2(D_1 + D_2) \right]. \quad (4.24)$$

Thus, we see CEPs with k^{-4} divergences. If the diffusion constants are equal, we see k^{-6} divergences that come from the coalescence of the CEP along the critical line with these points, as in the overdamped case. The case with $\gamma_1 \neq \gamma_2$ is not analytically tractable, but the calculations proceed in the same manner, and similar CEPs are present.

Finally, it can be shown similarly to the overdamped case that if one of the fields is eliminated, the other picks up additional noise term. This time, however, the power spectrum of that term goes as ω^{d-3} for dimensions $d \leq 3$. We see again the appearance of $1/f$ noise.

4.2.4 Conclusion

Non-reciprocity is a particular type of deviation from equilibrium behavior. A variety of systems can be thought of as obeying non-reciprocal laws, like spin models, population dynamics, neural behavior, and others. Many of these systems can be studied using field-theoretical tools. However, most of these models as well as real phenomena are nonlinear and not solvable exactly. In such cases a field theoretical approach would be to start with a linear description and continue through perturbation series or renormalization group by using that description. It is therefore of practical importance to build sufficient understanding of the linear non-reciprocal models so that they can be used as building blocks for the above-mentioned methods. It turns out that with just two coupled fields, even the linearized

dynamics is rich.

We started with a simple reaction-diffusion system with non-reciprocal interactions. However, our results extend to more complex systems, and we included a system with inertial dynamics to show that. We demonstrated that non-reciprocity as we defined it separates different qualitative sets of behaviors. For small non-reciprocity the behavior of the system is essentially equilibrium. It is interesting that some non-reciprocity, and therefore some deviation from equilibrium conditions, is allowed while still retaining equilibrium behavior, at least as far as the statistics of the fields are concerned. It is not clear at this level whether that fact persists when nonlinearities are added. That would mean that for some small deviations from equilibrium, the long-range behavior still obeys some equilibrium dynamics. In other words, the deviation from equilibrium is irrelevant in the RG sense. That has been demonstrated for some non-reciprocal systems [30, 31]. We hypothesize that even if possible, such behavior depends strongly on the other elements of the theory. There are also instances where an infinitesimal deviation from equilibrium brings the system to novel fixed points [46].

As the non-reciprocity gets larger, the system acquires isolated points in parameter space where the correlation functions diverge stronger than the usual equilibrium manner. Those are Critical Exceptional Points (CEPs), which are points where two or more modes of the system become equal, while also being critical. The existence of those points is a consequence of the non-equilibrium nature of the model. CEPs are a subject of many ongoing studies, and they display various consequences, from altering the critical dimensions of the model [18] to changing the type of transition entirely [19, 46]. It should be noted that in non-linear theories the parameters corresponding to linear couplings that we study also depend on the non-linearities through renormalization. The counting of how many parameters need to be tuned to reach the CEPs might therefore not be what would be inferred from this work. We also demonstrate how strong non-reciprocity leads to finite momentum instabilities. In

fact, we demonstrate that non-reciprocity needs to be sufficiently strong to achieve that, and that the parameters responsible for the spatial coupling need to be sufficiently different to observe pattern formation. This is in accordance with [84], but it means that in most cases CEPs are at least linearly stable to pattern formation.

Finally, we also demonstrated that coupling to a second field produces $1/f$ noise for the dynamics of the first. Various mechanisms for producing such noise are well-studied [142, 205], including the mechanism explored in this work [28]. It is possible to produce it even in equilibrium by coupling two fields to each other.

We believe that in studies of non-reciprocal systems this work should provide the linear, exactly solvable, ground that perturbative calculations can be based on. We leave the exploration of relevant models with the use of results derived here for other works.

4.3 Novel Properties of CEPs

The results in this section are not yet published.

4.3.1 *Tangential CEPs*

We briefly mentioned in Section 4.2.2 that a CEP can occur on a line where the slope defined by D_2/D_1 is tangential to the critical line. This turns out to be a general result, and we provide a proof here. Let us consider a general linear system with diffusion and noise as an extension of the two-species models we discussed in Section 4.2:

$$\partial_t \phi_i = -M_{ij} \phi_j + D_{ij} \nabla^2 \phi_j + \xi_i \quad (4.25)$$

where we now include possible cross-diffusion terms, which we take to all be positive for stability. For simplicity, let us assume there are no cross-correlations in the additive noise.

In general, the equal-time Keldysh correlation function in momentum space takes the form:

$$G^K \sim \frac{1}{f(M_{ij} + D_{ij}k^2)} \quad (4.26)$$

At a conventional critical point, the correlation function diverges at zero momentum, which sets $f(M_{ij}^c) = 0$. Expanding around small momenta near the critical point, we get:

$$G^K \sim \frac{1}{f(M_{ij}^c + D_{ij}k^2)} \approx \frac{1}{\partial_{M_{ij}}f(M_{ij}^c)D_{ij}k^2 + O(k^4)} \quad (4.27)$$

We therefore get a k^2 divergence if the first term in the denominator is not zero. However, if it is, we get a CEP with at least a k^4 divergence. The requirement:

$$\partial_{M_{ij}}f(M_{ij}^c)D_{ij} = 0 \quad (4.28)$$

is nothing but an equation for a tangent line! Therefore, any time the diffusion constants are picked such that there is a point on the critical surface where the increasing momentum mode stability line (blue lines in Fig. 4.1) is tangent to the surface, that point becomes a CEP. Interpreting this result is still a work in progress. However, finite-momentum instabilities generically occur through a point where the mode stability line is tangent to a critical surface at some finite k_c . We thus believe that there is some connection to pattern formation at these *tangential CEPs*.

It is also possible to show that such CEPs can only occur outside of equilibrium. As we discussed in Sec. 3.2.8, equilibrium dynamics for many-species problems is characterized by symmetric matrices M_{ij} and D_{ij} . Since both M_{ij} and D_{ij} are semipositive-definite symmetric matrices, the eigenvalues of the sum $M_{ij} + D_{ij}k^2$ are always greater than or equal to the eigenvalues of M_{ij} . So, higher momentum modes are always more stable. Therefore, there can be no situation in which there is a tangent line, since for such a case the mode at

a finite k_c is least stable. The only other possibility is if there is a corner or a kink in the critical surface, like that in part c of Fig. 4.1. But intuitively this does not happen for a real symmetric matrix M_{ij} . We leave the precise proof for future work.

4.3.2 Real-Space Correlations

Whenever we discuss critical phenomena, we often consider models in momentum space, since the field theory is usually greatly simplified in that view. However, it is prudent to discuss some of the novel results we find in real space representation. Both are equivalent, and often measurements can be made in either space, but some features are more apparent in real space representation.

Let us consider the generic forms of CEPs we encounter. The correlation functions around a CEP typically take one of the following forms, all other factors aside:

$$G^K(k) \sim \frac{1}{\xi^{-4} + k^4} \quad , \quad \frac{1}{k^2(\xi^{-2} + k^2)} \quad , \quad \frac{1}{(\xi_1^{-2} + k^2)(\xi_2^{-2} + k^2)} \quad (4.29)$$

First, let us illustrate a method that can be used on the second and third forms to understand their behavior in real space [7]. We are interested in $\int d^d k e^{-i\vec{k}\cdot\vec{x}} G^K(k)$. This integral can be transformed as follows:

$$G^K(x) \sim \int d^d k \frac{e^{-i\vec{k}\cdot\vec{x}}}{(\xi_1^{-2} + k^2)(\xi_2^{-2} + k^2)} = \int d^d k e^{-i\vec{k}\cdot\vec{x}} \int_0^\infty du \int_0^\infty dt e^{-u(\xi_1^{-2} + k^2)} e^{-t(\xi_2^{-2} + k^2)} \quad (4.30)$$

The integral over k is just a Gaussian integral in d dimensions, and after some manipulation of the other integrals, this can be brought to the form:

$$G^K(x) \sim \int_0^\infty dy \frac{1}{y^{\frac{d}{2}}} \frac{1}{\xi_1^{-2} - \xi_2^{-2}} \left(e^{-\xi_2^{-2}y} - e^{-\xi_1^{-2}y} \right) e^{-\frac{x^2}{4y}} \quad (4.31)$$

This integral can be computed exactly. In the case $\xi_1 \neq \xi_2$, aside from constant contri-

butions, we get:

$$\lim_{x \rightarrow 0} G^K(x) \sim \frac{\Gamma\left[\frac{d}{2} - 1\right]}{d - 4} x^{4-d} \quad (4.32)$$

This can be contrasted with the usual critical point behavior 2.13. At exactly $d = 2$, this behavior is supplemented by logarithmic correction terms. On the other hand, the behavior as $x \rightarrow \infty$ is the same as in 2.13, just with the sum of two terms of the form $x^{\frac{1}{2} - \frac{d}{2}} e^{-x/\xi_i}$.

For the case $\xi_1 = \xi_2$, we get the leading contribution:

$$\lim_{x \rightarrow 0} G^K(x) \sim \left[\frac{(d-4)\Gamma\left[\frac{d}{2} - 1\right]}{d-2} - \frac{4\Gamma\left[\frac{d}{2}\right]}{d-4} \right] x^{4-d} \quad (4.33)$$

And we are again seeing the same behavior, again with logarithmic corrections at $d = 2$. The $x \rightarrow \infty$ behavior is unchanged from before. In the second case, where, for instance, $\xi_2^{-1} = 0$, the integral only converges above 2 dimensions, as expected. However, it still yields the same short and long-range behavior.

The x^{4-d} behavior at small x raises important questions about the lower critical dimensions of non-reciprocally coupled systems. For example, if two Goldstone modes are non-reciprocally coupled, they exhibit such correlations, therefore rendering the lower critical dimension of the ordered phase to be four, instead of the Mermin-Wagner two. This is particularly important for experimental realizations, since this shows that some non-reciprocal ordered phases might not be realizable for us at all.

The first form of the CEP correlation function $\frac{1}{\xi^{-4+k^4}}$ is not so convenient to work with. However, it exhibits another interesting phenomenon. Let us illustrate in $d \leq 3$, where the

real-space correlation function may be computed exactly:

$$G^K(x) \sim \int d^d k \frac{e^{-i\vec{k} \cdot \vec{x}}}{\xi^{-4} + k^4} \sim \begin{cases} e^{-\frac{x}{\sqrt{2}\xi}} \cos\left(\frac{x}{\sqrt{2}\xi} - \frac{\pi}{4}\right) , & d = 1 \\ G\left(\frac{x^4}{256\xi^4} \middle| \begin{matrix} - \\ 0, \frac{1}{2}, \frac{1}{2}, 0 \end{matrix}\right) , & d = 2 \\ \frac{e^{-\frac{x}{\sqrt{2}\xi}} \sin\left(\frac{x}{\sqrt{2}\xi}\right)}{x/\xi} , & d = 3 \end{cases} \quad (4.34)$$

Here G is a Meijer function that exhibits both a decay and oscillations. We therefore see that this form of the CEP correlation function exhibits oscillations in real space! This is surprising, as it implies two points in space can be rendered completely independent in their fluctuations even though the system has spatial coupling. Similarly, there are regions that are anti-correlated. We leave the investigation of these results for future work.

4.3.3 Classical CEP Sensor

As discussed in Sec. 3.2.9, one of the potential applications of CEPs discussed in the literature is sensing. Let us investigate it here in the classical linear case of Sec. 4.2. Consider the case $\Delta = 0$, where there is a one-way coupling between the fields. We now add external fields:

$$\begin{aligned} \partial_t \phi_1 &= -m_1 \phi_1 + D_1 \nabla^2 \phi_1 + h_1 + \xi_1 \\ \partial_t \phi_2 &= -m_2 \phi_2 + j_{21} \phi_1 + D_2 \nabla^2 \phi_2 + h_2 + \xi_2 \end{aligned} \quad (4.35)$$

Here, h_i would be what we want to measure, and we use $\langle \phi_i \rangle$ as our sensors. The average response to external fields is $\langle \phi_1 \rangle = \frac{h_1}{m_1}$ and $\langle \phi_2 \rangle = \frac{h_2}{m_2} + \frac{j_{21} h_1}{m_1 m_2}$. As the critical point is approached, the responses diverge, as expected. However, sensing requires the readable signal to be larger than a typical fluctuation of the field due to noise, which is generically present. Therefore, we concentrate on the "signal-noise-ratio" (SNR) $\text{SNR}_i = \langle \phi_i \rangle / \sqrt{\langle \phi_i^2 \rangle_c}$.

For the usual critical point, which the dynamics of ϕ_1 exhibit, we get:

$$\text{SNR}_1 = \frac{\langle \phi_1 \rangle}{\sqrt{\langle \phi_1^2 \rangle_c}} = \frac{h_1/m_1}{\left(\int d^d k \frac{B_1/2}{m_1 + D_1 k^2} \right)^{\frac{1}{2}}} = \frac{h_1}{m_1 \left[\left(\frac{m_1}{D} \right)^{\frac{d}{2}} \frac{B_1/2}{m_1} \int d^d x \frac{1}{1+x^2} \right]^{\frac{1}{2}}} \sim h_1 m_1^{-\frac{2+d}{4}} \quad (4.36)$$

As expected, the SNR diverges at a critical point. We therefore need to compare this divergence with the one the fields experience at a CEP.

Here ϕ_1 is decoupled from ϕ_2 , so let us focus solely on the latter. We retain the most diverging contributions. Defining $m_2 = \alpha m_1 = \alpha m$, $D_2 = \beta D_1 = \beta D$ and after some manipulations, we get:

$$\begin{aligned} \text{SNR}_2 &= \frac{\langle \phi_2 \rangle}{\sqrt{\langle \phi_2^2 \rangle_c}} = \frac{\frac{h_2}{m_2} + \frac{j_{21} h_1}{m_1 m_2}}{\left[\int d^d k \left(\frac{j_{21}^2 B_1}{(m_1 + m_2 + (D_1 + D_2) k^2)(m_1 + D_1 k^2)(m_2 + D_2 k^2)} + \frac{B_2/2}{m_2 + D_2 k^2} \right) \right]^{\frac{1}{2}}} \\ &= \frac{\frac{h_2}{\alpha m} + \frac{j_{21} h_1}{\alpha m^2}}{\left[\int d^d x \left(\frac{m}{D} \right)^{\frac{d}{2}} \left(\frac{j_{21}^2 B_1}{m^3 (1 + \alpha + (1 + \beta) x^2)(\alpha + \beta x^2)(1 + x^2)} + \frac{B_2/2}{m(\alpha + \beta x^2)} \right) \right]^{\frac{1}{2}}} \\ &\rightarrow \frac{\frac{h_2}{\alpha m} + \frac{j_{21} h_1}{\alpha m^2}}{m^{\frac{d}{4} - \frac{3}{2}} (f_1 + m^2 f_2)^{\frac{1}{2}}} \end{aligned} \quad (4.37)$$

Where f_1 and f_2 are the x integrals of the respective terms. As $m \rightarrow 0$, we see that the SNR to h_2 behaves as $\sim m^{\frac{2-d}{4}}$, which is weaker than that of a regular critical point as $m \rightarrow 0$. The response to h_1 behaves as $m^{-\frac{2+d}{4}}$, which is exactly the same as the divergence at a conventional critical point. Thus, we see that in this setup, the CEP does not offer any improvement over a critical point sensor, in agreement with evidence in quantum systems [184].

4.4 Vortex Dynamics in Non-reciprocal XY Models

The results in this section are not yet published, and have been produced in collaboration with Ryo Hanai.

4.4.1 Berezinskii-Kosterlitz-Thouless Transition

The Landau-Ginzburg theory of phase transition is based on the notion of spontaneous symmetry breaking. However, the Berezinskii-Kosterlitz-Thouless (BKT) transition does not fall into that paradigm, but occurs due to the topological nature of the order parameter. We do not delve into all intricacies of the BKT transition, for which we refer to the literature [4, 206, 207]. Here we give a brief introduction.

Suppose that we have an $O(2)$ Hamiltonian in Eq. 2.68 in $d = 2$. We know from Mermin-Wagner theorem that it does not exhibit long-range order due to phase fluctuations present in the otherwise ordered phase. We can expand the order-parameter as $\phi_1 = |\phi| \cos \theta$ and $\phi_2 = |\phi| \sin \theta$. At the lowest order, the dynamics of the phase is governed by the XY model Hamiltonian:

$$\mathcal{H}_{\text{XY}} = J \int dt d^2x |\nabla \theta(t, \vec{x})|^2 \quad (4.38)$$

This model is also often considered in the discrete spin form on a lattice, where it reads:

$$\mathcal{H}_{\text{XY,discrete}} = -J \sum_{\langle i,j \rangle} \vec{S}_i \cdot \vec{S}_j = -J \sum_{\langle i,j \rangle} \cos(\theta_i - \theta_j) \quad (4.39)$$

Here J is the strength of the ferromagnetic coupling, i, j are lattice indices and $\sum_{\langle i,j \rangle}$ is the sum only over nearest neighbors. Again, it is important that the spins are restricted to rotate in the plane and the space is two-dimensional.

It can be shown [4, 207] that the static statistical theory with this Hamiltonian exhibits two different behaviors at low temperatures and at high temperatures, manifesting in the

different form of the correlation function:

$$\langle \vec{S}_0 \cdot \vec{S}_{\vec{x}} \rangle \sim \begin{cases} \exp \left[-\frac{x}{\xi} \right] & , J \rightarrow 0 \\ \left(\frac{a}{x} \right)^{\frac{1}{2\pi J}} & , J \rightarrow \infty \end{cases} \quad (4.40)$$

The low temperature phase is usually called the quasi-long-range or algebraic order phase, since the correlations are not constant, but do fall off only algebraically as opposed to exponentially. So, if no symmetry is spontaneously broken, how does this system transition between these two regimes?

The answer turns out to be hidden in the topological nature of θ . As a phase, θ is compact, here defined on $[0, 2\pi)$, with the endpoints identified. Therefore, θ can wind so that it undergoes a full rotation as some closed loop in space is traversed. We can therefore define the winding number as:

$$\Gamma_\gamma[\theta] = \oint_\gamma \vec{\nabla} \theta(\vec{x}) \cdot d\vec{l} = 2\pi n \quad (4.41)$$

Here γ is some closed loop on the two-dimensional plane.

The relationship with the XY model is revealed in the fact that vortex solutions are stationary solutions of $\mathcal{H}_{XY}$. Such solutions have to satisfy $\nabla^2 \theta(\vec{x}) = 0$. For example, two such solutions are:

$$\theta_\pm = \pm \tan^{-1} \left(\frac{y - b}{x - a} \right) \quad (4.42)$$

Where (x, y) are coordinates on the plane, and (a, b) is the vortex center. These solutions, along with their gradients, are plotted in Fig. 4.4. We also find that $\Gamma_\gamma[\theta_\pm] = \pm 2\pi$ with γ enclosing the vortex. So the winding numbers for $\theta_\pm$ are $n_\pm = \pm 1$, respectively. Since the solutions $\theta_\pm$ exhibit rotational symmetry, we can take γ to be a circle centered on the vortex, and we find that $|\vec{\nabla} \theta_\pm(\vec{r})| = \frac{|n_\pm|}{r}$. We then find that the energy of one vortex is:

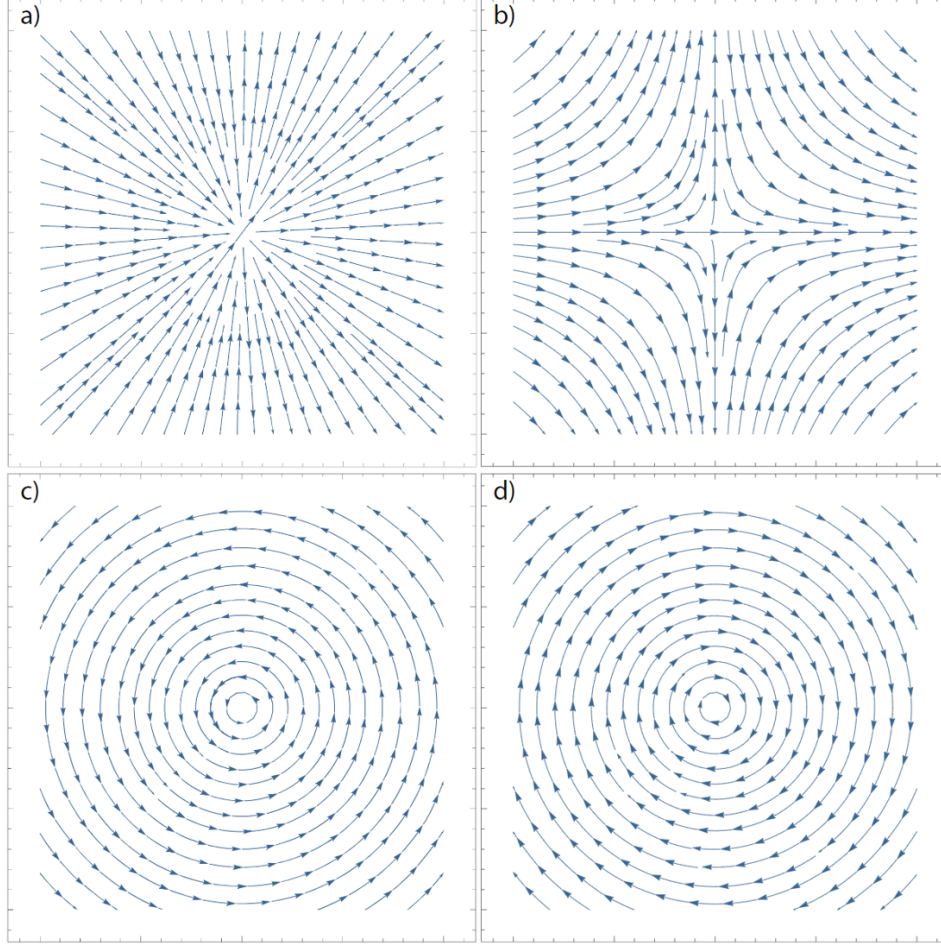


Figure 4.4: Vortex solutions $\theta_{\pm}$ in Eq. 4.42. a) and b) show the streamlines for θ_+ and θ_- , respectively. c) and d) show the gradients $\vec{\nabla}\theta_+$ and $\vec{\nabla}\theta_-$, respectively.

$$E_{vor} = J2\pi \int_a^L dr r \frac{n_{\pm}^2}{r^2} = 2\pi J \ln \left(\frac{L}{a} \right) \quad (4.43)$$

Here, L is the system size and a is the radius of the vortex core. The vortex core size turns out to be a central quantity in the study of the BKT transition, since it defines the smallest scale below which the continuous theory fails, just like the cutoff Λ we encountered before in the discussions on RG. We therefore find that although a vortex solution is a stationary solution of $\mathcal{H}_{XY}$, it has higher energy than the ground state $\theta = \text{const.}$

These simple considerations are already enough to demonstrate the presence of a tran-

sition. In a canonical ensemble, the phase of the system depends on the free energy, the change in which is given by $\Delta F = \Delta E - T\Delta S$. Let us consider the change in free energy with the addition of one vortex. We already saw that the internal energy changes by E_{vor} . The entropy is associated with the number of microstates a system can occupy, which for one vortex translates to the number of distinct positions it can occupy. This turns out to be given by the ratio of the area of the system to the area of the vortex core L^2/a^2 . The change in entropy due to one vortex is therefore $\Delta S = k_B \ln(L^2/a^2)$. This gives the change in free energy due to one vortex:

$$\Delta F = (2\pi J - 2k_B T) \ln\left(\frac{L}{a}\right) \quad (4.44)$$

Therefore, for high enough temperatures, the free energy is lowered by the appearance of vortices, and they proliferate, therefore giving a qualitatively different phase of matter. The transition happens at a critical temperature $T_c = \frac{\pi J}{k_B}$.

However, the problem is complicated by vortex interactions. It can be calculated that the interaction energy between a vortex ($n = 1$) and antivortex ($n = -1$) separated by a distance r is:

$$V(r) = 4\pi J \ln\left(\frac{r}{a}\right) \quad (4.45)$$

Thus, the interactions between vortices are given by the Coulomb interaction of a two-dimensional gas of charges! In fact, there is an exact mapping between the statistical theories of XY vortices, Coulomb gas, and the so-called Sine-Gordon theory [208]. In this view, the BKT transition is the transition between a gas with bound charge pairs (since a vortex and antivortex together annihilate into absence of vortices) and a plasma with unbound charges. One can use either mapping to perform an RG analysis of the theory to more systematically describe the transition [139, 207, 208]. The authors in [139] demonstrate how the pair interaction energy can be used to derive the BKT RG flow even in the presence of

non-equilibrium dynamics.

The dynamical description of the BKT transition turns out to be highly non-trivial. The mapping to Coulomb dynamics is possible, but requires the use of an electrodynamic duality transformation as well as other methods to prescribe the motion of the vortices explicitly. We discuss these at length below. The mapping to a dynamical Sine-Gordon theory is still an open question.

4.4.2 *Non-Reciprocal XY Model*

Keeping with the themes of this chapter, an extension of the XY model to the non-equilibrium realm is a two species non-reciprocal XY model with dynamics:

$$\begin{aligned}\partial_t \theta_x^A &= \frac{j_{AA}}{2d} \sum_{\hat{a}} \sin(\theta_{x+\hat{a}}^A - \theta_x^A) + j_{AB} \sin(\theta_x^B - \theta_x^A) + \eta_x^A \\ \partial_t \theta_x^B &= \frac{j_{BB}}{2d} \sum_{\hat{a}} \sin(\theta_{x+\hat{a}}^B - \theta_x^B) + j_{BA} \sin(\theta_x^A - \theta_x^B) + \eta_x^B\end{aligned}\tag{4.46}$$

Here j_{ij} are the inter and intra species couplings, with $j_{AA}, j_{BB} > 0$, such that the spins tend to align with the neighbors in the same species. The non-reciprocity manifests in the couplings $j_{AB} \neq j_{BA}$. The dimensionality is $d = 2$, and $\sum_{\hat{a}}$ denotes the sum over basis vectors $\pm\hat{x}, \pm\hat{y}$ such that each spin interacts with its four neighbors.

This model exhibits a transition from an aligned to a chiral phase as the difference $j_- = j_{AB} - j_{BA}$ increases relative to the average coupling $j_+ = j_{AB} + j_{BA}$ [13, 32]. In the aligned phase, where $j_- \ll j_+$, spins in species A and B tend to align with small fluctuations. In the chiral phase, one species constantly chases the other. So the question is, what happens to the topological defects (vortices, as defined above), in these phases?

In some sense, we expect that the behavior most different from the equilibrium XY picture occurs near the transition from an aligned to a chiral phase. After all, we know that such transitions are characterized by large fluctuations in some order parameter. In this case, it

turns out this transition occurs through a CEP [14, 32]. Therefore, what we are going to focus on is the behavior of topological transition as we approach the CEP from the aligned phase.

To understand the nature of this transition better, let us define $\theta_{\pm} = \frac{\theta^A \pm \theta^B}{2}$. If we were not in the aligned phase, there would be many subtleties arising from the topological nature of both θ^A and θ^B , such that these definitions have to be made more carefully. Additionally, we can, for now, make the substitution $\sum_{\hat{a}} \sin(\theta_{x+\hat{a}}^A - \theta_x^A) \approx \nabla^2 \theta$, which can be derived by expanding the sines and keeping only the linear terms. Finally, we assume $j_{AA} = j_{BB} = J$ for simplicity. With these considerations, the dynamics can instead be written as:

$$\begin{aligned}\partial_t \theta^+ &= \frac{J}{2d} \nabla^2 \theta^+ - \frac{1}{2} j^- \sin(2\theta^-) + \eta^+ \\ \partial_t \theta^- &= \frac{J}{2d} \nabla^2 \theta^- - \frac{1}{2} j^+ \sin(2\theta^-) + \eta^-\end{aligned}\tag{4.47}$$

This demonstrates that for large enough j^+ , θ^- essentially sits at zero, which leads to the aligned phase.

Since θ^- is strongly damped in the aligned phase, a natural next step in analyzing this set of equations might be integrating θ^- out to get effective dynamics of θ^+ . However, the expansion of both $\sum_{\hat{a}} \sin(\theta_{x+\hat{a}}^A - \theta_x^A)$ and $\sin(\theta_x^A - \theta_x^B)$ contains many non-linearities, which leads to many non-linear couplings between θ^+ and θ^- . It is not obvious that these couplings can be omitted, especially as the CEP is approached. Therefore, in accordance with the field-theoretic and RG ideas underlying this dissertation, we should consider the most general dynamics for θ^+ allowed by symmetries. Analyzing such a non-linear equation for the vortex dynamics is extremely difficult, and so we restrict ourselves to the most general linear form. Unlike equilibrium, where such a form would simply be the diffusion equation $\partial_t \theta^+ = D \nabla^2 \theta^+$ (no terms without derivatives are allowed since θ^+ is compact), we have to keep the non-equilibrium nature of the dynamics and especially the presence of the CEP.

Such a form turns out to be:

$$\partial_t^2 \theta + (\delta - Z \nabla^2) \partial_t \theta - v^2 \nabla^2 \theta + \eta = 0 \quad (4.48)$$

We omit the superscript in θ^+ from now on. This equation exhibits a CEP when $\delta = 0$, which can be checked by looking at the fluctuation spectrum of θ . Such an equation was first presented in [19] where generic non-equilibrium $O(N)$ models were considered, and the θ dynamics describe the Goldstone mode in the ordered and rotating phases. The rest of the parameters are related in some way to the original j_{ij} , but undergoes RG flow, and we therefore give them different, generic, names. The question of the BKT dynamics in the non-reciprocal XY model translates to the investigation of vortex dynamics in Eq. 4.48.

Let us briefly mention an important point. From the RG perspective, δ is a relevant parameter, and therefore for any non-zero δ we expect the first derivative term to dominate the dynamics [19]. Therefore, we expect to recover the usual vortex dynamics of an equilibrium XY model in the limit $\delta \rightarrow \infty$. Thus, the $\delta \rightarrow 0^+$ region is of greatest interest.

4.4.3 *Electrodynamic Duality*

Now that we really have our starting continuous Langevin equation, Eq. 4.48, how do we extract the vortex dynamics from it? In static theories, there are exact mappings that are very well developed [208]. For a dynamical theory, we have to go through the procedure of electrodynamic duality transformation. Here, we follow the mathematical approach in [139], see also references therein. We provide the outline here and the mathematical details in Appendix C.

First, we define an "electric" field $\vec{E} = -\hat{z} \times \vec{\nabla} \theta$. Intuitively, if a vortex is present at some point, $\vec{E}$ points radially away or toward it, no matter the actual orientation of θ , which is what we expect from a charge. For example, one can see that $-\hat{z} \times \vec{\nabla} \theta_+$ in Eq. 4.42 and

in Fig. 4.4 points away from the vortex center (with $\hat{z}$ out of the page). We therefore also expect that Gauss' law holds for the vortices: $\vec{\nabla} \cdot \vec{E} = \frac{2\pi}{\epsilon} n$, where ϵ is the dielectric constant associated with the field. The vortex density can be defined as $n(t, \vec{r}) = \sum_i n_i \delta(\vec{r} - \vec{r}_i(t))$, where $\vec{r}_i$ and n_i are the position and charge of vortex i , respectively. We also expect the vortex charge conservation to hold: $\partial_t n + \vec{\nabla} \cdot \vec{j} = 0$. With these considerations, and carefully introducing the vortex density and current into the equations of motion (see [Appendix C](#)), the resulting "Maxwell's" equation for the electric field is:

$$\partial_t^2 \vec{E} + (\delta - Z\nabla^2) \partial_t \vec{E} - v^2 \nabla^2 \vec{E} + \frac{2\pi v^2}{\epsilon} \vec{\nabla} n + \frac{2\pi}{\epsilon} \left[\partial_t + (\delta - Z\nabla^2) \right] \vec{j} - \hat{z} \times \vec{\nabla} \eta = 0 \quad (4.49)$$

Next, we introduce the gauge potentials ϕ and $\vec{A}$, but in a form different from usual electrodynamics, since our "Maxwell's" equation is also different:

$$\vec{E} = -\vec{\nabla} \phi - (\delta - Z\nabla^2) \vec{A} - \partial_t \vec{A} \quad (4.50)$$

The field is invariant under a gauge transformation:

$$\vec{A} \rightarrow \vec{A} + \vec{\nabla} \chi, \quad \phi \rightarrow \phi - (\delta - Z\nabla^2) \chi - \partial_t \chi \quad (4.51)$$

Finally, we impose the Lorentz gauge:

$$\partial_t \phi + v^2 \vec{\nabla} \cdot \vec{A} = 0 \quad (4.52)$$

With that, we see that the Gauss's law $\vec{\nabla} \cdot \vec{E} = \frac{2\pi}{\epsilon} n$ instead becomes:

$$\partial_t^2 \phi + (\delta - Z\nabla^2) \partial_t \phi - v^2 \nabla^2 \phi = \frac{2\pi v^2}{\epsilon} n \quad (4.53)$$

Which defines the equation of motion for the scalar potential. If we plug in the gauge

potentials into the equation of motion for $\vec{E}$, we find that the above expression guarantees ϕ drops out, and we are left with the equation of motion for the vector potential:

$$\left[\partial_t^2 + (\delta - Z\nabla^2) \partial_t - v^2 \nabla^2 \right] \left[-(\delta - Z\nabla^2) \vec{A} - \partial_t \vec{A} \right] + \frac{2\pi}{\epsilon} \left[\partial_t + (\delta - Z\nabla^2) \right] \vec{j} - \hat{z} \times \vec{\nabla} \eta = 0 \quad (4.54)$$

Equations 4.53 and 4.54 define the evolution of the gauge potential mediating the interaction between the vortices. However, to have a complete theory, we need to supplement the equation of motion for the vortices themselves in terms of the gauge fields. It turns out that, at least on a lattice, the force between two vortices even in the presence of second derivative and δ and Z terms is still Coulomb [209]. Therefore, we assume the overdamped vortex motion has the form:

$$\partial_t \vec{r}_i = \mu n_i \vec{E}(t, \vec{r}_i) + \vec{\eta}_T \quad (4.55)$$

Here μ is the vortex mobility, which we take to be sufficiently small such that the vortices are essentially stationary compared to the propagation of the electric field. η_T is white noise that incorporates thermal fluctuations in their motion.

With the vortex equation of motion supplied, we are in a position to promote these equations with the MSRJD procedure and integrate out the gauge fields to get the effective interactions between vortices. This calculation is long and requires some approximations. We provide it in Appendix C. The result for the dynamics of vortices is:

$$\begin{aligned} \partial_t \vec{r}_i(t) &= \mu n_i \vec{E}(\vec{r}_i(t)) + \vec{\xi}(t, \vec{r}_i(t)) + \vec{\eta}_i^T \\ \partial_t^2 \vec{\xi} + (\delta - Z\nabla^2) \partial_t \vec{\xi} - v^2 \nabla^2 \vec{\xi} + \hat{z} \times \vec{\nabla} \eta &= 0 \end{aligned} \quad (4.56)$$

Here $\vec{E}$ is the electric field produced by usual Coulomb interactions between vortices, while $\vec{\xi}$ is a new spatio-temporally correlated noise that carries the signature of the CEP. We expect this additional noise to be the biggest driver for changing the equilibrium BKT transition. The next step is to understand how such complicated dynamics can be analyzed

to produce an RG flow.

4.4.4 Vortex Pair Distribution and RG Flow

It turns out that in order to understand the BKT transition, or vortex unbinding transition, it is enough to look at a vortex-anti-vortex pair. An RG flow can be produced if the probability distribution for the distance between the vortex and anti-vortex is known [139]. Here, however, we demonstrate how this can be done without explicitly solving for that distribution. We do so for the equilibrium case of Coulomb force and white noise:

$$\partial_t \vec{r} = -\vec{\nabla} V(r) + \vec{\eta}^T \quad (4.57)$$

Here $\vec{r}$ is the separation between the two vortices and $V(r) = \frac{1}{\epsilon} \ln\left(\frac{r}{a}\right)$ (we absorbed μ into ϵ). Since this case is in equilibrium, we can assume that there is a stationary distribution with no probability currents (see Sec. 3.1.2), which then satisfies:

$$P(\vec{r}) \partial_{r_i} V + T \partial_{r_i} P(\vec{r}) = 0 \quad (4.58)$$

Now we have to think about the scales present in our problem. The probability distribution is dimensionless, while the only other length scale present in the system is the vortex core radius a . Therefore, in addition to rotational invariance, we require $P(\vec{r}) = P\left(\frac{r}{a}\right)$. This means that if we are to change the scale in order to perform an RG step, the probability does not generically stay invariant! So what does? We know that any physical quantity that we can actually measure in an experiment should not depend on the scale at which we observe the system. One such quantity is the susceptibility to an applied external field. For a given averaged polarization density $\langle \vec{\Pi} \rangle$, the susceptibility χ is defined as $\langle \vec{\Pi} \rangle = \chi \vec{E}$ for a small

applied field $\vec{E}$. To obtain it, we compute:

$$\langle \vec{\Pi} \rangle = \frac{1}{L^2} \int \frac{d^2 R d^2 r}{a^4} \vec{r} P(r, \vec{E}) = \int \frac{d^2 r}{a^4} \vec{r} \left(P(r, 0) + \partial_{E_i} P(r, 0) \big|_{\vec{E}=0} E_i \right) \quad (4.59)$$

Here L^2 is the system size and $\int d^2 R$ integrates over the center of mass of the vortex pairs, which simply gives back a factor of L^2 . The a^4 factor comes from breaking up space into regions separated by more than the vortex core size, since below these length scales the theory is not valid. Using rotational invariance the first term above drops out, and we therefore get:

$$\langle \vec{\Pi} \rangle = \int \frac{d^2 r}{a^4} \vec{r} \partial_{E_i} P(r, 0) \big|_{\vec{E}=0} E_i \quad (4.60)$$

It is not difficult to find that in this case $\partial_{E_i} P(r, 0) \big|_{\vec{E}=0} = \frac{P(r, 0) r_i}{T}$. This, in turn, allows us to write the expression for the linear susceptibility in terms of the stationary probability distribution:

$$\chi = \frac{\pi}{T} \int_a^\infty \frac{dr}{a^4} r^3 P(r) \quad (4.61)$$

Finally, we write $P(r) = y^2 f\left(\frac{r}{a}\right)$, where y^2 is the fugacity, which denotes the probability of having a vortex pair with separation a , and f is a dimensionless function. Since the renormalized dielectric constant is generally defined as $\epsilon_R = \epsilon + 2\pi\chi$, the flow of χ determines the flow of the dielectric constant.

Now we are finally in a position to perform an RG calculation. We increase the scale to $a' = a(1 + dl)$, where $dl \ll 1$. We thus also separate the integral as:

$$\int_a^\infty = \int_a^{a(1+dl)} + \int_{a(1+dl)}^\infty \quad (4.62)$$

Therefore, to keep the renormalized dielectric constant in the same form when we change

the scale, we absorb:

$$\epsilon' = \epsilon + \frac{2\pi^2}{T} \int_a^{a(1+dl)} \frac{dr}{a^4} r^3 P(r) = \epsilon + \frac{2\pi^2}{T} P(a) dl + O(dl^2) = \epsilon + \frac{2\pi^2}{T} y^2 dl + O(dl^2) \quad (4.63)$$

For the other part, we get:

$$\begin{aligned} & \frac{\pi y^2}{T} \int_{a'}^{\infty} \frac{dr}{a'^4 (1-dl)^4} f\left(\frac{r}{a'(1-dl)}\right) \\ &= \frac{\pi y^2}{T} (1+4dl) \int_{a'}^{\infty} \frac{dr}{a'^4} r^3 \left[f\left(\frac{r}{a'}\right) + \partial_{\frac{r}{a'}, i} f\left(\frac{r}{a'}\right) \frac{r_i}{a'} dl \right] + O(dl^2) \\ &= \frac{\pi y^2}{T} (1+4dl) \int_{a'}^{\infty} \frac{dr}{a'^4} r^3 \left[f\left(\frac{r}{a'}\right) - \frac{1}{\epsilon T} f\left(\frac{r}{a'}\right) dl \right] + O(dl^2) \end{aligned} \quad (4.64)$$

where in getting to the last line we used the FP equation. Therefore, we can assign:

$$\frac{y'^2}{T'} = \frac{y^2}{T} \left[1 + \left(4 - \frac{1}{\epsilon T} \right) dl \right] + O(dl^2) \quad (4.65)$$

By fixing our convention such that the temperature does not flow, we get the famous BKT RG flow in two parameter space:

$$\begin{aligned} \frac{d}{dl} \epsilon &= \frac{2\pi^2}{T} y^2 \\ \frac{d}{dl} y^2 &= 4 - \frac{1}{\epsilon T} \end{aligned} \quad (4.66)$$

The example of this equilibrium derivation shows how the RG flow can be derived without solving for the probability distribution explicitly. This is useful because such a solution is not possible for our case, where the dynamics of the vortex separation in the limit of slow vortices is:

$$\begin{aligned} \partial_t \vec{r} &= -\frac{1}{\epsilon} \frac{\vec{r}}{r^2} + \vec{\xi} \left(t, \frac{\vec{r}}{2} \right) + \vec{\xi} \left(t, -\frac{\vec{r}}{2} \right) + \vec{\eta}^T \\ \partial_t^2 \vec{\xi} + (\delta - Z \nabla^2) \partial_t \vec{\xi} - v^2 \nabla^2 \vec{\xi} + \hat{z} \times \vec{\nabla} \eta &= 0 \end{aligned} \quad (4.67)$$

We approach this problem by establishing an effective FP equation. We write:

$$\partial_t \vec{r} = \vec{f}(\vec{r}) + \Delta \vec{\xi} + \vec{\eta}_r \quad (4.68)$$

And we need to treat the two noise sources separately. Here:

$$\begin{aligned} B_{ij} &= \langle \Delta \xi_i(r_1, t_1) \Delta \xi_j(r_2, t_2) \rangle = \left\langle \left[\int_k \xi_i(k, t_1) \sin \left(\vec{k} \cdot \frac{\vec{r}_1}{2} \right) \right] \left[\int_{k'} \xi_j(k', t_2) \sin \left(\vec{k}' \cdot \frac{\vec{r}_2}{2} \right) \right] \right\rangle \\ &= \int_{k, k'} G_{ij}^K(k, t_1; k', t_2) \sin \left(\vec{k} \cdot \frac{\vec{r}_1}{2} \right) \sin \left(\vec{k}' \cdot \frac{\vec{r}_2}{2} \right) \end{aligned} \quad (4.69)$$

where G^K is the Keldysh correlation function one can read off from the dynamics of $\vec{\xi}$.

On the other hand:

$$\langle \eta_{r,i}(t) \eta_{r,j}(t') \rangle = T \delta_{ij} \delta(t - t') \quad (4.70)$$

In [Appendix C](#) we demonstrate how the following equation can be derived:

$$\begin{aligned} \partial_t P(\vec{r}, t) &= -\partial_{r_i} [f_i(r) P(\vec{r}, t)] + T \partial_{r_i} \partial_{r_i} P \\ &\quad + \partial_{r_i} \int_{r'} \int_{-\infty}^t dt' B_{ij}(\vec{r}, t; \vec{r}', t') J_{lj}(t, t', r') \partial_{r_l} P_2(\vec{r}, t; \vec{r}', t') \end{aligned} \quad (4.71)$$

Here $P_2(\vec{r}, t; \vec{r}', t')$ is the two-point probability distribution and

$$J_{lj}(t, t', r') = \frac{r'}{r(t)} \hat{r}_l \hat{r}_j + \frac{r(t)}{r'} (\delta_{lj} - \hat{r}_l \hat{r}_j) \quad (4.72)$$

is the Jacobian associated with the deterministic evolution of a perturbation at time t' at $\vec{r}'$ to time t . Here $\vec{r}(t) = \frac{\vec{r}'}{r'} \sqrt{r'^2 - 2\frac{1}{\epsilon}(t - t')}$. Since the Jacobian diverges when the perturbation reaches $r = 0$, the integral is only defined up to times t' in the past such that it is not divergent. It turns out to be possible to write an infinite tower of evolution equations for n -point probability functions, and each depends on the $n + 1$ -point function.

Therefore, an exact solution to the above FP equation is impossible, and we have to resort to approximations. We know at least two limits for the two-point function:

$$P_2(\vec{r}, t; \vec{r}', t') = \begin{cases} \delta(\vec{r} - \vec{r}') P(\vec{r}, t) & , t - t' \rightarrow 0 \\ P(\vec{r}, t) P(\vec{r}', t') & , t - t' \rightarrow \pm\infty \end{cases} \quad (4.73)$$

Extracting conclusive results out of these expressions is work in progress. For example, it can be shown that in the second limit above, the integral term in the FP equation goes to zero. Also, if we ignore the contribution coming from the times between the above two limits and only consider the first limit, the stationary probability distribution can be shown to obey (in polar coordinates):

$$0 = P_s \partial_r V + T \partial_r P_s + \tilde{t} \frac{\Delta\pi}{2v^2} \int_0^\Lambda dk \frac{k}{\delta + Zk^2} [1 - J_0(kr) + J_2(kr)] \partial_r P_s \quad (4.74)$$

Here J_i are Bessel functions and $\tilde{t}$ is the time scale where making the small time approximation to the two-point probability function is appropriate. One way to estimate $\tilde{t}$ is the time difference when both the deterministic evolution and the variance due to noise are smaller than the core radius, for example. Close to $\delta = 0$, the noise terms dominate, and the time-scale where this approximation is appropriate can be shown to be:

$$a^2 > T\tilde{t} - \tilde{t}^2 \ln(\delta\tilde{t}) \frac{\Delta\pi}{Zv^2} \quad (4.75)$$

Even with all these approximations, extracting the RG flow is non-trivial. For example, when $\delta = 0$, the integral of the Bessel functions above can be shown to be $\sim \ln(r/a)$, such that the FP equation has a multiplicative noise term. In that case, it is not trivial to derive the susceptibility as a function of the stationary probability distribution P_s . However, we know that δ is a highly relevant parameter, just like m in the usual Ising model. Therefore,

at large scales, as $\delta \rightarrow \infty$, we see that the additional multiplicative noise term shrinks, and we recover the equilibrium BKT physics, as expected. All of these details are described in [Appendix C](#).

Although we have not yet extracted the RG flow associated with this problem, a few things are already hinted at. For example, if we view the distribution as effectively Boltzmann of the form $\exp(-V/T_{\text{eff}})$, we see that the effective temperature increases for larger r , since the multiplicative noise term is $\sim \ln(r/a)$. Therefore, the probability distribution becomes flatter, and vortices are relatively more likely to be unbound. As $\delta \rightarrow 0$ and the noise dominates the dynamics, it is also intuitively expected that more unbound pairs would form.

As mentioned before, understanding the topological transition in such a non-equilibrium system would yield insights into the expansive field of non-equilibrium phases of matter. As such, this problem remains a work in progress.

4.5 Outlook on Non-reciprocal Phenomena

In this chapter, we provided a brief introduction into non-reciprocal phenomena, with examples, as well as some novel results. The breadth of phenomena covered with the umbrella of non-reciprocity is vast, and we conclude with some remarks on the future of non-reciprocal theories.

If we focus on collective phenomena, we can start in parallel to how phase transitions were historically analyzed in equilibrium, which is through the ideas of spontaneous symmetry breaking. In that case, at least in principle, the novel non-equilibrium phases are only restricted by our knowledge of symmetry groups, which is not much of a restriction at all. However, the presence of non-reciprocity and therefore of dynamical steady states allows for a "larger" symmetry breaking than would otherwise be possible in equilibrium. For example, this is the case for non-equilibrium $O(N)$ models [\[19\]](#), where the order parameter picks a plane in the $O(N)$ space, rather than a point. Thus, with the addition of non-reciprocity,

we have to reinvestigate the symmetry groups that are commonly used to describe phase transitions to search for new phases. As mentioned in Sec. 4.1, this is a direction that is widely pursued.

In some sense, this approach can be taken beyond spontaneous symmetry breaking. We can take a collective phenomenon or feature of interest such as SSB, disorder, boundary condition problems, first-order transitions, long-range coupling problems, etc., add non-reciprocity as another ingredient, and investigate the resulting behaviors. This has already yielded many insights and is sure to yield more. Additionally, one of the central phenomena arising due to non-reciprocity is CEPs. It might be warranted to investigate them separately, especially from the RG perspective, and not just as byproducts of transitions investigated in a given problem. There could also be new symmetries that we have not yet considered available outside of equilibrium that can yield novel behavior in non-reciprocal systems.

On the other hand, it is obviously of use to construct long-range theories from the underlying microscopic behaviors that can be confirmed experimentally. The resulting theory might not possess any of the restrictions or simplicity discussed above, but might yield novel phases that are nonetheless actually observed. Since non-reciprocity is a non-equilibrium feature, such models are likely not of a simple form of "equilibrium theory/principles + non-reciprocity", but are rather highly non-equilibrium from the start. For example, models of self-propelling agents with non-reciprocity fall into this category, and one can imagine many natural circumstances where these behaviors arise.

We should also admit that some behaviors that technically fall into the category of non-reciprocity are not best described in that language. For example, chiral behavior is a common occurrence in such systems, and yet there is a vast arsenal of other mathematical tools describing oscillations that are much more developed and insightful. Thus, we can say the transition from non-chiral to chiral behavior is best understood as a consequence of non-reciprocity, but the chiral phase itself is better described from other perspectives.

Finally, the models considered in this dissertation are always restricted to non-reciprocal interactions between two species. However, there are obvious reasons to lift that restriction, in which case many new phenomena arise [27]. In some sense, we arrive at the idea that we need an additional "dimension" describing the species space. In that case, we need to investigate how large matrix spaces (describing the inter-species couplings) interact with real spatial extent to produce new collective phenomena.

CHAPTER 5

LIMIT CYCLE PHASES

5.1 Oscillations in Matter

The next class of non-equilibrium problems we consider are systems with oscillations. In contrast to non-reciprocal matter, which seemingly breaks a fundamental law of physics, oscillations are much more intuitive. Although oscillations are easy to imagine in idealized equilibrium systems, the presence of dissipation necessarily requires an injection of energy to sustain the periodic motion. To make a simple parallel, a harmonic oscillator, which produces ongoing oscillations in an idealized setting, can only maintain an indefinite periodic motion with external forcing, if damped. We encounter periodic motion constantly, in electronic systems, lasers, waves, brain dynamics, biological rhythms, oscillations of populations, periodicity in climate, and so on. We also encountered periodic motion in non-reciprocal systems in the previous chapter. It is not obvious how to describe all of these with some general thermodynamic principle, as is the case in equilibrium statistical mechanics. This question is exactly the direction we explore in the following sections as an attempt to build a classification of oscillating phases of matter. Before going into novel results, let us motivate them with some examples.

A prototypical example of oscillators in physical systems comes in the form of the so called Lienard equation [210]:

$$\partial_t^2 x + f(x)\partial_t x + g(x) = 0 \tag{5.1}$$

It is possible to show that under certain conditions this equation has a limit cycle attractor. This equation has a wide array of applications, from electrical circuits and mechanical oscillators to biological systems. A famous example of a Lienard system is the Van der Pol

oscillator [211]:

$$\partial_t^2 x - \mu(1 - x^2)\partial_t x + x = 0 \quad (5.2)$$

In addition to the applications in physics mentioned above, the Van der Pol oscillator is also used to model heartbeat and neuronal spiking [211, 212]. In fact, the Van der Pol oscillator is a special case of the FitzHugh-Nagumo model mentioned in Sec. 3.2.7.

Of course, canonical physical systems contain a wide range of oscillating phenomena. For example, the harmonic oscillator is one of the most important foundational models in physics, with innumerable applications. More advanced examples can include lasing, mechanical oscillators, classical and quantum circuits, acoustic phonons, non-equilibrium magnetic models, Quantum Hall states, superconducting Josephson junctions, planetary motion, and many more [2, 4, 19, 39, 40, 213].

As mentioned in Section 4.1, in the realm of chemical reactions and population dynamics, a class of systems with oscillations is an activator-inhibitor reaction [84, 211] governed by the following equation:

$$\begin{aligned} \partial_t x &= f(x, y) \\ \partial_t y &= g(x, y) \end{aligned} \quad (5.3)$$

such that $\partial_x f, \partial_x g > 0$ and $\partial_y f, \partial_y g < 0$ at some fixed point. Linearizing around that fixed point shows that this system is in the strongly non-reciprocal regime, as described in Section 4.2 (except it might be unstable). The Lotka-Volterra model discussed in Section 4.1 is of this class and obviously has oscillations. Another example is the Brusselator as a model for the Beluzov-Zhabotinsky reaction [211]:

$$\begin{aligned} \partial_t x &= a + x^2 y - bx - x \\ \partial_t y &= bx - x^2 y \end{aligned} \quad (5.4)$$

One does not have to think long to find examples of oscillations in the biological context. Breathing, heartbeat, circadian rhythms, hormonal cycles, and brain waves are all oscillatory in nature [40]. If we move beyond a single organism to a collection of agents, it is also not difficult to find oscillations. For example, fireflies can synchronize their luminous phase and exhibit collective oscillations [39]. Synchronization is a huge topic of research of its own, with applications in physics, chemistry, biology, population dynamics, and neuroscience [39–41, 58, 157, 211]. A famous example is the Kuramoto model described briefly in Sec. 3.2.7.

Oscillations are also a fundamental part of brain dynamics, both healthy and afflicted by disease [51, 52, 54, 66, 151, 214, 215]. As an example, dynamic modeling of a transition to an epileptic seizure often utilizes a transition to a stable limit cycle [57, 76, 79], as we discuss in Sec. 6.1. As mentioned in Sec. 3.2.7, a synchronization transition between oscillating units has been proposed as an explanation of brain criticality [54].

Finally, there are also systems that exhibit oscillatory behavior as a result of noise. A classical example is the theory of stochastic resonance [216]. The dynamics is described by an evolution in a double well potential in an oscillating external field:

$$\partial_t \phi = m\phi - g\phi^3 + h \sin(\omega t) + \xi \quad (5.5)$$

Noise allows the system to periodically jump from the less stable to the more stable state (depending on the cycle of the external forcing), even when it would not be able to do so deterministically. The resulting oscillations can be used as a sensing mechanism [216, 217], but have also been proposed as an explanation for the transition between different climate ages [218].

We see that, unlike non-reciprocal phenomena of Chapter 4, it is much easier to find and intuitively understand oscillating systems. However, classifying them precisely is a difficult challenge, and here we introduce a few ideas relevant for that task. As discussed in Sec. 2.2.1, the classification of many-body systems with oscillating phases can have tight connections

with the corresponding dynamical bifurcation. Therefore, we should understand the common bifurcations that lead to periodic behavior.

Two such bifurcations, which are in some sense complementary, are the Hopf bifurcation and the Saddle Node on an Invariant Circle (SNIC) bifurcation [39]. The normal forms of these bifurcations can be written as:

$$\text{Hopf : } \begin{cases} \partial_t r &= -mr - gr^3 \\ \partial_t \theta &= \omega \end{cases}, \quad \text{SNIC : } \begin{cases} \partial_t r &= |m|r - gr^3 \\ \tau \partial_t \theta &= \mu - \sin \theta \end{cases} \quad (5.6)$$

These are depicted in Fig. 5.1. In the Hopf bifurcation, the radius r of the limit cycle

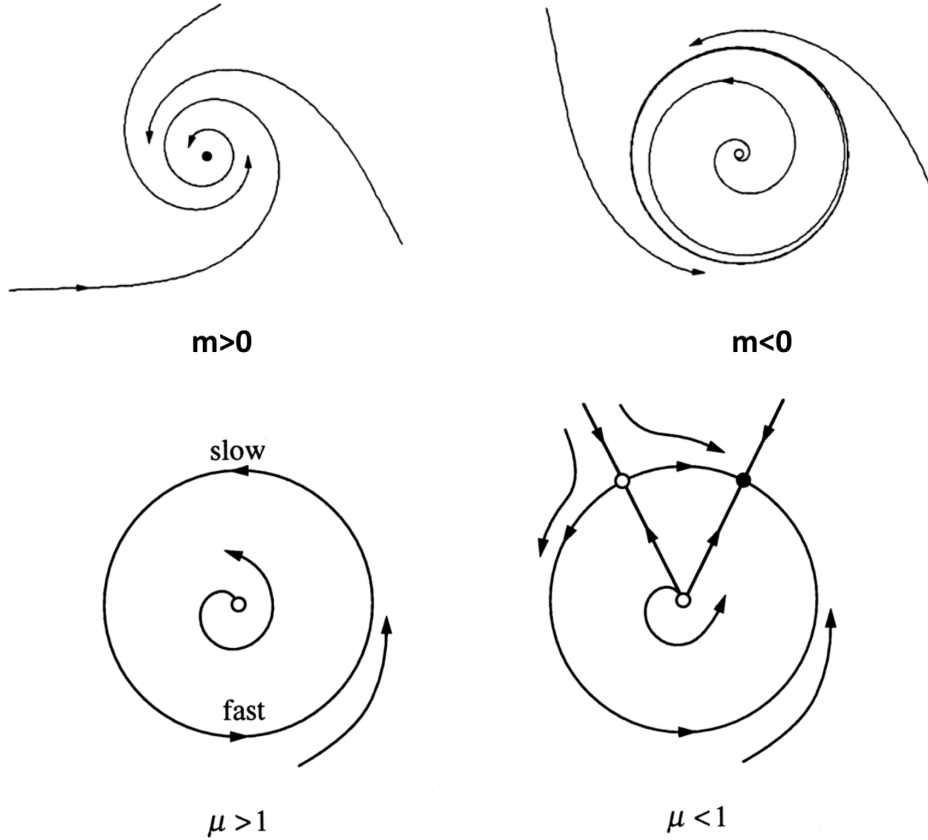


Figure 5.1: Phase space flows of the limit cycle bifurcations in Eq. 5.6. Top – Hopf bifurcation, bottom – SNIC bifurcation. Figures adapted from [39].

undergoes a pitchfork bifurcation at $m = 0$, while the phase oscillates at a constant frequency ω . In the SNIC bifurcation, the invariant circle has a finite radius $\sqrt{|m|/g}$, but the system starts undergoing oscillations when $\mu = 1$, with the frequency of oscillations increasing monotonically as μ increases. Thus, these two bifurcations describe two general pathways to oscillations – either the cycle amplitude or frequency start increasing at the bifurcation. One can, of course, imagine more non-trivial variations, such as when the frequency $\omega = \omega(r)$ in the Hopf case also depends on the radius. The Hopf bifurcation guides the work in the following sections.

Additionally, one can view periodic solutions as breaking continuous time-translation invariance down to a discrete time translation invariance by a period of oscillations. This has been denoted as time crystals, in parallel with space translation invariance breaking in physical crystals. This symmetry breaking underlies many non-equilibrium phenomena, and is a topic of intense research [46–49]. Floquet theory and Floquet systems are also exciting topics of inquiry having to do with oscillatory problems [48].

All of these examples and considerations motivate the study of limit cycle phases of matter in many-body systems. Even aside from specific systems and just like with non-reciprocal theories, limit cycles can serve as an elementary platform to explore genuinely non-equilibrium phases. We take this view in the next two sections, where we explore transitions where two limit cycles merge in the zero-dimensional and proper many-body contexts.

5.2 Coalescence of Limit Cycles in the Presence of Noise

This section is based on the work presented in [219].

5.2.1 *When Limit Cycles Merge*

Complex dynamical systems may exhibit multiple steady states, including time-periodic limit cycles, where the final trajectory depends on initial conditions. With tuning of parameters, limit cycles can proliferate or merge at an exceptional point. Here we ask how dynamics in the vicinity of such a bifurcation are influenced by noise. A pitchfork bifurcation can be used to induce bifurcation behavior. We model a limit cycle with the normal form of the Hopf oscillator, couple it to the pitchfork, and investigate the resulting dynamical system in the presence of noise. We show that the generating functional for the averages of the dynamical variables factorizes between the pitchfork and the oscillator. The statistical properties of the pitchfork in the presence of noise in its various regimes are investigated and a scaling theory is developed for the correlation and response functions, including a possible symmetry-breaking field. The analysis is done by perturbative calculations as well as numerical means. Finally, observables illustrating the coupling of a system with a limit cycle to a pitchfork are discussed and the phase-phase correlations are shown to exhibit non-diffusive behavior with universal scaling.

As has been the theme in this dissertation, we are interested in the behavior of such models in the presence of environmental fluctuations and noise. This can mean that dynamical transitions in finite systems (or infinite systems below the lower critical dimension) are blurred. However, dynamical responses can still be tuned – the classic case is stochastic resonance [216, 217] mentioned before, where a bistable system in the presence of noise is found to be maximally sensitive to A.C. fields whose frequency matches the typical rate of hopping between minima. As an example of a bistable system (at least in one regime), we take a pitchfork bifurcation (or Model A dynamics in zero dimensions 2.2), which can be

coupled to more general dynamical systems in order to imbue those with bifurcation behavior. Unlike the pitchfork itself, these coupled systems can exhibit much broader classes of dynamics, where the coalescing states can be fixed points, various stable orbits such as limit cycles, and other attractors [14]. What are the response and correlation functions of such systems in the presence of noise?

As discussed in Sec. 5.1, noisy limit cycles and oscillators are of considerable interest in many fields of study. In this section, we investigate a Hopf oscillator coupled to the pitchfork, which can describe a class of systems with coalescing limit cycles. We show that there are characteristic features in the dynamics that mix the established scaling of stochastic resonance with additional time scales associated with the recurrent limit cycle oscillations.

As discussed in Sec. 2.2.1, normal forms of bifurcations can be used to describe generic classes of models, and subsequently used to extract general results. Here, we derive a scaling theory in the vicinity of such a bifurcation and suggest methods to analyze data to explore whether the bifurcation model is appropriate for the system at hand.

The nature of the noise depends on the details of the system, and we investigate the simplest case: Gaussian white noise. One of the main results of this section, which describes how expectation values of operators factorize between the pitchfork and the rest of the dynamical system, is valid for any type of noise, as long as it is uncorrelated between the pitchfork and the rest of the system. The behavior of the pitchfork with noise has been investigated in various contexts and in various regimes, such as perturbative methods [105] and stochastic resonance [216, 217]. However, to the authors' knowledge, there is no treatment that combines the noisy pitchfork in its various regimes into one theory. Here we investigate and collect results for it in all regimes, and sew them together with scaling theory. That theory relies on the observation that the pitchfork has a very limited number of dimensionless parameters that the perturbative expansions can run over.

Hence, once the factorization result and the scaling theory of the pitchfork are explored,

our goal is to identify observables that can be measured to see whether an experimental system at hand does indeed exhibit factorization and whether it exhibits a bifurcation governed by the pitchfork. Other possible venues of exploration for future study include investigating other types of orbits that can bifurcate coupled to a pitchfork, other types of noise, and also noise that is correlated between the pitchfork and the rest of the dynamical system.

The reason to investigate an abstract model is that it may offer universal pointers for physical behavior. Slow fluctuations between different conceptual states are common in complex systems and are often loosely assumed in biology. It is almost never the case that one has a complete enough understanding of the system to produce a detailed model. We rely instead on general principles, which are necessarily mostly static. However, the statistics of fluctuations near the critical point can be indicative of the underlying dynamics. We show that fluctuations between limit cycles have a greater richness than fluctuations between fixed points (traditional stochastic resonance). Such correlations could be determined from data, agnostic of a model.

5.2.2 *Theoretical Model*

We take a Hopf oscillator as a model for a limit cycle and couple it to a pitchfork bifurcation:

$$\dot{\psi} = (a + i\Omega + b|\psi|^2)\psi + ic\psi w^n \quad (5.7)$$

$$\dot{w} = Rw - gw^3 \quad (5.8)$$

Here a, Ω, b are real parameters chosen to produce a limit cycle, c is the coupling strength between the pitchfork and the oscillator and R and $g > 0$ are parameters of the pitchfork. All parameters are of appropriate units and are taken to be constant in time. We assume that the Hopf oscillator is in a regime with a stable limit cycle ($a > 0, b < 0$), although the particular values of the parameters are not relevant for most of the analysis. This system

has been shown to exhibit bifurcation behavior and for $n = 1$ and $n = 2$ there is also an exceptional point at $R = 0$ [14]. Notice that in this case the Hopf oscillator is coupled to the pitchfork in a multiplicative way. Since ψ is complex, we can break it up into its real and imaginary components. We then add noise with the assumption that it is not correlated between the pitchfork and the oscillator:

$$\begin{aligned}\dot{\psi}_x &= a\psi_x - \Omega\psi_y + b|\psi|^2\psi_x - c\psi_y w^n + \sqrt{B_{\psi_x}(\psi)}\xi_1(t) \\ \dot{\psi}_y &= a\psi_y + \Omega\psi_x + b|\psi|^2\psi_y + c\psi_x w^n + \sqrt{B_{\psi_y}(\psi)}\xi_2(t) \\ \dot{w} &= Rw - gw^3 + \sqrt{B_w(w)}\xi_3(t)\end{aligned}$$

where the stochastic differential equations (SDEs) are taken under Ito discretization and the noise increments are defined to be independent Gaussian white noises $\langle \xi_i(t_1)\xi_j(t_2) \rangle = \delta_{ij}\delta(t_1 - t_2)$. We assume that the noises along the orthogonal directions of the Hopf oscillator are equal: $B_{\psi_x} = B_{\psi_y} = B_\psi$. Calling $\psi_x = r \cos(\theta)$ and $\psi_y = r \sin(\theta)$ we can convert the Hopf oscillator into polar coordinates, keeping the terms produced due to Ito discretization:

$$\dot{r} = ar + br^3 + \frac{B_\psi}{r} + \sqrt{B_\psi}\xi_r(t) \quad (5.9)$$

$$\dot{\theta} = \Omega + cw^n + \frac{\sqrt{B_\psi}}{r}\xi_\theta(t) \quad (5.10)$$

where $\xi_r(t)$ and $\xi_\theta(t)$ are defined through an orthogonal transformation from the original $\xi_1(t)$ and $\xi_2(t)$, and are also Gaussian white uncorrelated noises [104]. We can see here that the coupling to the pitchfork only affects the dynamics of θ , as in the corresponding classical case, and in fact the dynamics of r are completely independent of both θ and w . This is not generic, since spatial couplings lead to cross-coupling between the amplitude and phase, see Sec. 5.3.

We can now introduce the MSRJD response fields w^q , r^q and θ^q to express the problem in terms of path integrals (see Sec. 2.2.1):

$$\begin{aligned} \mathcal{Z} = & \int \mathcal{D}[w, w^q, \theta, \theta^q, r, r^q] \exp \left[\int dt i w^q (\dot{w} - R w + g w^3) - \frac{B_w}{2} w^{q2} \right] \\ & * \exp \left[\int dt i \theta^q (\dot{\theta} - \Omega - c w^n) - \frac{B_\psi}{2 r^2} \theta^{q2} \right] \exp \left[\int dt i r^q (\dot{r} - f(r)) - \frac{B_\psi}{2} r^{q2} \right] \end{aligned} \quad (5.11)$$

Here, $f(r) = ar + br^3 + B_\psi/r$ contains the deterministic part of the evolution of r . We assume unspecified initial conditions; in other words, the systems were settled into their equilibrium distributions at $t = -\infty$. Here we also note that we have not taken into account the compactness of θ , which in principle should be handled. However, the following result still holds and is confirmed numerically. We can now demonstrate that expectation values of these operators factorize into expectation values of the constituents corresponding to field theories of the Hopf oscillator and pitchfork on their own. That fact relies on the linearity of the equation for θ . It can be clearly seen that when calculating an expectation value of any operator $w^n(w^q)^m$, we can integrate out the fields corresponding to the Hopf oscillator and we'll just be left with the dynamics of the pitchfork. Therefore, the complexity comes from any parts of the operators of the form $\theta^n(\theta^q)^m r^k(r^q)^l$. These operators can be obtained from a generating functional:

$$\mathcal{Z}[\lambda_\theta, \lambda_\theta^q, \lambda_r, \lambda_r^q] = \left\langle \exp \left[\int dt \lambda_\theta \theta + \lambda_\theta^q \theta^q + \lambda_r r + \lambda_r^q r^q \right] \right\rangle \quad (5.12)$$

where all λ 's are functions of time. Employing the linearity of the equation for θ it can be shown by integrating out θ and θ^q that this generating functional factorizes into the generating functional for the Hopf oscillator in the absence of the coupling and an expectation value of an appropriate operator of the pitchfork only:

$$\mathcal{Z}[\lambda_\theta, \lambda_\theta^q, \lambda_r, \lambda_r^q] = \mathcal{Z}_{\text{Hopf}}[\lambda_\theta, \lambda_\theta^q, \lambda_r, \lambda_r^q] \left\langle \exp \left[\int dt c w^n \int_t^\infty ds \lambda_\theta(s) \right] \right\rangle_w \quad (5.13)$$

where $\langle \dots \rangle_w$ is the expectation value with respect to the pitchfork action:

$$S_w[w, w^q] = \int dt \left(-i w^q (\dot{w} - R w + g w^3) + \frac{B_w}{2} w^{q2} \right) \quad (5.14)$$

For the derivation, see [Appendix D](#). Hopf oscillators in their normal form have been studied in many contexts, with varying mathematical depths and in application to a variety of subjects, both with and without noise [\[220–224\]](#). Thus, we won't address part of the generating functional Z_{Hopf} that comes from the dynamics of the Hopf oscillator alone. However, we can look at some quantities of interest of the model and see how the coupling to the pitchfork affects them. For example, for $n = 1$ the connected angle correlation function for the full system can be shown to be:

$$\begin{aligned} \langle \theta(t_1) \theta(t_2) \rangle_c &= \frac{\delta}{\delta \lambda_\theta(t_1)} \frac{\delta}{\delta \lambda_\theta(t_2)} \ln \mathcal{Z}[\lambda_\theta, \lambda_\theta^q, \lambda_r, \lambda_r^q] \Big|_{\lambda=0} \\ &= \langle \theta(t_1) \theta(t_2) \rangle_{c, \text{Hopf}} + c^2 \int^{t_1} dt \int^{t_2} dt' \langle w(t) w(t') \rangle_{c, w} \end{aligned} \quad (5.15)$$

On the other hand, the phase response function of the coupled system is unchanged by the presence of the pitchfork:

$$\langle \theta(t_1) \theta^q(t_2) \rangle_c = \frac{\delta}{\delta \lambda_\theta(t_1)} \frac{\delta}{\delta \lambda_\theta^q(t_2)} \ln \mathcal{Z}[\lambda_\theta, \lambda_\theta^q, \lambda_r, \lambda_r^q] \Big|_{\lambda=0} = \langle \theta(t_1) \theta^q(t_2) \rangle_{c, \text{Hopf}} \quad (5.16)$$

More importantly, since the object of interest is a limit cycle, the observable of interest is the phase-phase correlations, which, due to the factorization discussed above, takes the form:

$$\langle \exp[i\theta(t_1) - i\theta(t_2)] \rangle = \langle \exp[i\theta(t_1) - i\theta(t_2)] \rangle_{\text{Hopf}} \left\langle \exp \left[-ic \int_{t_1}^{t_2} dt w^n(t) \right] \right\rangle_w \quad (5.17)$$

Another object of interest might be the phase diffusion constant [\[65\]](#), which we consider

in the absence of external fields, leaving it time-translation invariant:

$$\begin{aligned}
D &\equiv \int_{-\infty}^{\infty} d\tau \left\langle \left(\dot{\theta}(t) - \langle \dot{\theta} \rangle(t) \right) \left(\dot{\theta}(t+\tau) - \langle \dot{\theta} \rangle(t+\tau) \right) \right\rangle \\
&= \int_{-\infty}^{\infty} d\tau B_{\psi} \left\langle \frac{\delta(\tau)}{r(t)r(t+\tau)} \right\rangle + c^2 (\langle w^n(t)w^n(t+\tau) \rangle - \langle w^n(t) \rangle \langle w^n(t+\tau) \rangle)
\end{aligned} \tag{5.18}$$

where we used the independence of statistics of ξ_{θ} , w and r to arrive at the second equality. For $n = 1$, the diffusion constant is directly dependent on the connected two-point function of the pitchfork. For $n = 2$ it depends on the variance of the pitchfork distribution (which is just the two-point function at zero time separation) as well as a particular form of the four-point function. We can therefore see that to answer general questions about the coupled system, it is useful to understand the dynamics of the stochastic pitchfork bifurcation on its own, and its correlation functions.

5.2.3 Pitchfork Scaling Theory

The noisy, forced pitchfork obeys the stochastic differential equation (SDE):

$$\dot{w} = Rw - gw^3 + h(t) + \sqrt{B_w}\xi_3(t) \tag{5.19}$$

and the bifurcation occurs at $R = 0$ in the deterministic system. It is not possible to obtain the correlation and response functions for the pitchfork in all regimes exactly, so we instead employ a scaling theory and patch together different regimes. In the SDE, we assume that $g > 0$ and $B_w \geq 0$. The parameters and the functions of interest all have dimensions

restricted by the dimensions of w . We can therefore define:

$$\langle w \rangle \equiv M(R, g, B_w, h) = \left(\frac{B_w}{g} \right)^{\frac{1}{4}} F_1 \left(\frac{R}{B_w^{\frac{1}{2}} g^{\frac{1}{2}}}, \frac{h}{B_w^{\frac{3}{4}} g^{\frac{1}{4}}} \right) \quad (5.20a)$$

$$\langle w(\Delta t) w(0) \rangle_c \equiv C(R, g, B_w, h, \Delta t) = \left(\frac{B_w}{g} \right)^{\frac{1}{2}} F_2 \left(\frac{R}{B_w^{\frac{1}{2}} g^{\frac{1}{2}}}, \frac{h}{B_w^{\frac{3}{4}} g^{\frac{1}{4}}}, g^{\frac{1}{2}} B_w^{\frac{1}{2}} \Delta t \right) \quad (5.20b)$$

$$\langle w(\Delta t) w^q(0) \rangle_c \equiv \chi(R, g, B_w, h, \Delta t) = i \Theta(\Delta t) F_3 \left(\frac{R}{B_w^{\frac{1}{2}} g^{\frac{1}{2}}}, \frac{h}{B_w^{\frac{3}{4}} g^{\frac{1}{4}}}, g^{\frac{1}{2}} B_w^{\frac{1}{2}} \Delta t \right) \quad (5.20c)$$

where F_1, F_2, F_3 are dimensionless functions of dimensionless variables. Similar scaling theories have been defined for the pitchfork, especially in the context of stochastic resonance [217], but here we proceed to define it for completeness. By $\langle \dots \rangle_c$ we mean the connected correlation function, or cumulant. Several assumptions were made. First, we assume unconstrained initial conditions, so the system is time-translation invariant (at least in the absence of forcing). Second, all other dimensionless parameters that might arise from parameters implicit in the definition of $h(t)$ are present in these functions, but are not explicitly written for brevity. Third, causality is assumed for the response function χ [105, 109], and the Heaviside function $\Theta(t)$ is defined in the usual manner with the prescription that $\Theta(0) = 0$ is consistent with Ito discretization [109].

These functions are universal for any collection of parameters. Suppose some crossover is found that is intrinsic to these functions, for example in $F_2(x, 0, z)$ for some x . The universality of these functions then guarantees that in physical parameters this crossover or transition has to behave as $R \sim \sqrt{g B_w}$, with the exact proportionality constant to be determined. The only way for this relationship to be broken is through the prefactors that give these functions proper dimensions to match the physical quantities of interest. However, these prefactors are also defined universally, and thus their presence can be integrated generically into understanding of such crossovers.

We can now extract general information about the functions F_1, F_2, F_3 . No attempt was made to study these functions for all values of the parameters of the system. However, various regimes allow for use of different techniques to compute these functions. We define the effective bifurcation parameter $\tilde{R} = R/\sqrt{gB_w}$. For very large and negative $\tilde{R}$, it is possible to apply the usual field-theoretical perturbative techniques [2–5, 7, 105, 109, 136]. For large and positive $\tilde{R}$, the system is going to hop between the two wells so that we cannot just expand the field theory around one of the wells, but rarely enough so that the system has enough time to settle into either well before hopping again. In other words, the relaxation around either stable state is negligible for long-term dynamics. Here, it is possible to approach the study of the functions of interest from the perspective of Kramers’ problem and stochastic resonance [3, 216]. There are two relevant time scales. The first one is responsible for the decay within either well, and can be obtained by expanding the action around either well. The lowest order approximation of this time scale is $(2R)^{-1}$. The other one is responsible for the rate of hopping between the wells and can be obtained through standard Kramers’ problem calculation to be $\tau \approx \sqrt{2\pi}R^{-1} \exp [R^2/(2gB_w)]$. The crossover between the two is therefore governed by the competition of the two time scales. Unsurprisingly, perhaps, the ratio of these time scales is also a function of the effective bifurcation parameter: $\pi/\sqrt{2} \exp (\tilde{R}^2/2)$. The final regime centered around $R = 0$ is mainly studied numerically. We also define the effective time $\tilde{t} = \sqrt{gB_w}t$ and the effective external field $\tilde{h} = h/(B_w^{\frac{3}{4}}g^{\frac{1}{4}})$.

5.2.4 Numerical Results

To demonstrate and support the results of the scaling theory, we turn to numerical simulations of our random process. First, we consider the pitchfork itself. In Fig. 5.2, we simulate the process and plot the non-dimensionalized correlation function $F_2(\tilde{R}, 0, \tilde{t})$ for different values of $\tilde{R}$. Each line on the plot is an overlap of five results of simulations at different values of R , B_w and g that leave $\tilde{R}$ invariant. Their complete overlap demonstrates the

validity of the developed scaling theory.

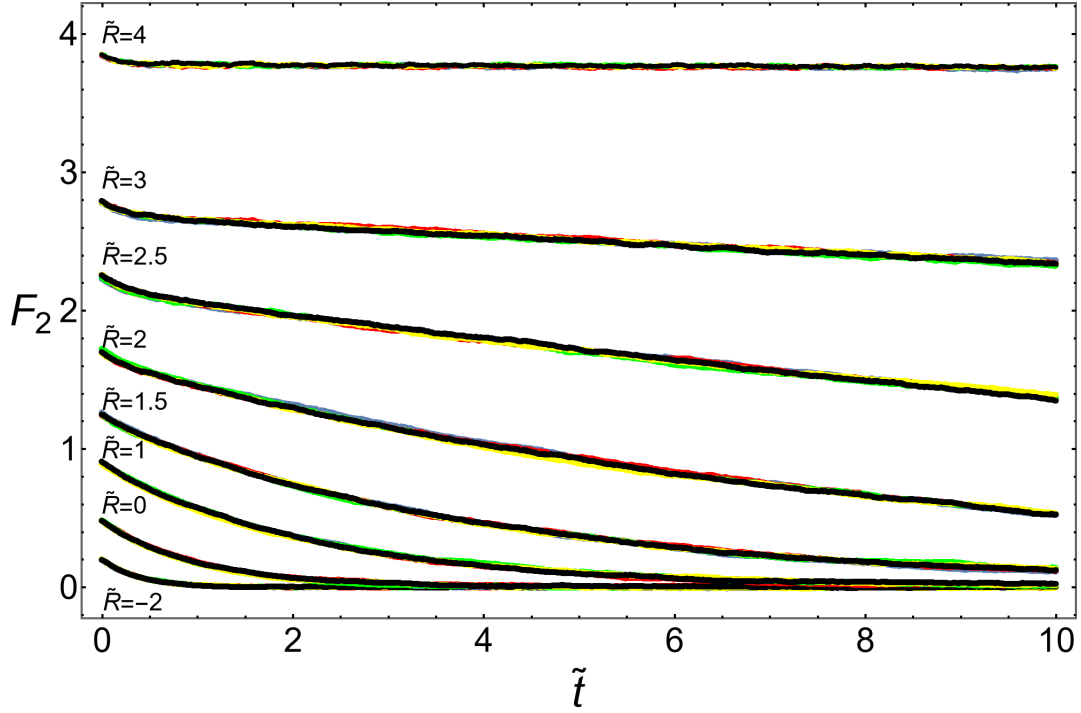


Figure 5.2: Numerical simulation of $F_2(\tilde{R}, 0, \tilde{t})$ as a function of $\tilde{t}$ at various $\tilde{R}$. Each line is an overlap of 5 simulations with different values of the individual parameters that leave $\tilde{R}$ the same.

We expect that the decay of the correlation function is governed by time scales that are described above. In Fig. 5.3 we plot the time slices of $F_2(\tilde{R}, 0, \tilde{t})$ as a function of $\tilde{R}$ for different values of $\tilde{t}$. As can be seen, there is a smaller time scale associated with the immediate decay of the correlation function, which perturbatively we expect to be R^{-1} in the $\tilde{R} \ll 0$ regime and $(2R)^{-1}$ in the $\tilde{R} \gg 0$ regime. In the $R \leq 0$ regime that is the only time scale. In $R > 0$ regime there is also Kramers' time scale which is exponential and controls the decay starting with values of smaller and smaller $\tilde{t}$ as $\tilde{R}$ increases.

Now we can return back to the original system. Since the information about the period and the diffusion due to noise of the limit cycle is contained in the phase-phase correlations, the operator of interest is the one in Eq. (5.17). This operator can also be subjected to scaling theory, and with an additional parameter – the coupling constant c – there is an additional

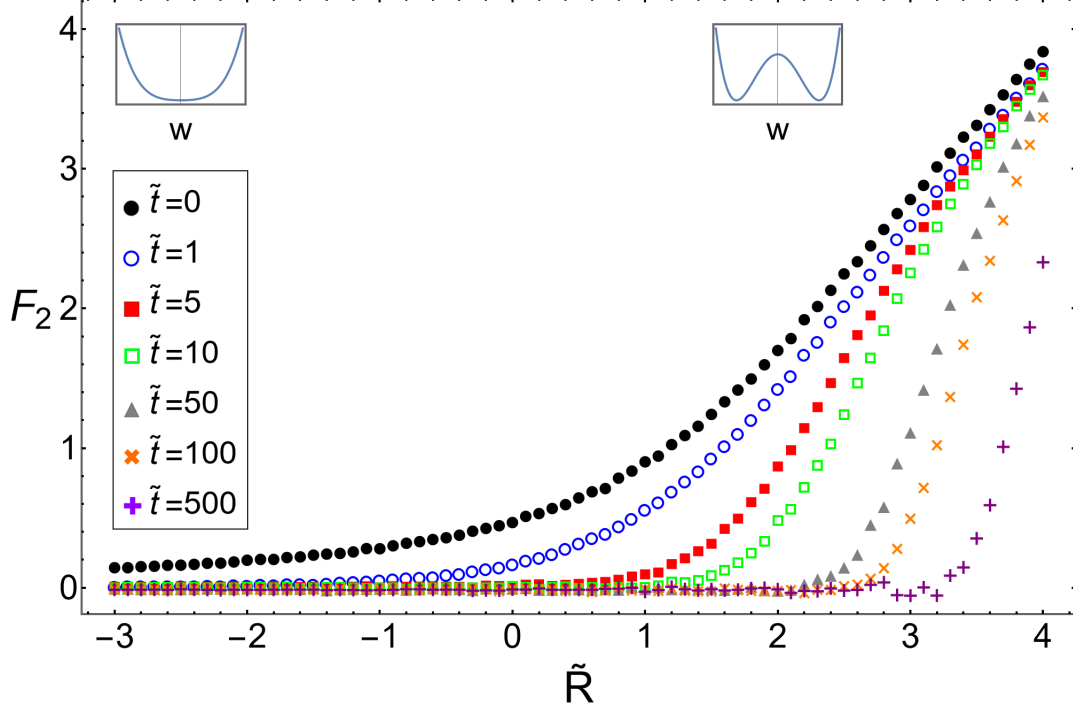


Figure 5.3: Numerical simulation of $F_2(\tilde{R}, 0, \tilde{t})$ as a function of $\tilde{R}$ at various $\tilde{t}$.

dimensionless parameter: $\tilde{c} = cB_w^{\frac{n}{4}-\frac{1}{2}}g^{-\frac{n}{4}-\frac{1}{2}}$. The phase correlations factor coming from the pitchfork is dimensionless, therefore we call this factor $F_p(\tilde{R}, \tilde{h}, t_{2,eff} - t_{1,eff}, \tilde{c})$. The scaling is demonstrated in Fig. 5.4 for $n = 1$ and in Fig. 5.5 for $n = 2$, where we plot F_p for various values of $\tilde{R}$ and $\tilde{c}$ as a function of $\tilde{t}$ (and there is no external field). With this, the two cases of $n = 1$ and $n = 2$ can be studied separately.

First, by formal expansion it can be seen that for $n = 1$, all the imaginary terms have an odd power of w . In the absence of external fields the pitchfork SDE has a Z_2 symmetry, and thus they are all zero identically, and only the real part of the phase-phase correlations survives. To formally see that the dimensionless parameter above (which is $cB_w^{-\frac{1}{4}}g^{-\frac{3}{4}}$ for $n = 1$) is indeed what controls the expansion, we can employ the scaling theory to describe the lowest order terms:

$$F_p = \left\langle \exp \left[-ic \int_{t_1}^{t_2} dt w(t) \right] \right\rangle_w \approx 1 - \frac{c^2}{2g^{\frac{3}{2}}B_w^{\frac{1}{2}}} \int_0^{\tilde{t}_2 - \tilde{t}_1} dz' \int_{-z'}^{\tilde{t}_2 - \tilde{t}_1 - z'} dz F_2(\tilde{R}, \tilde{h}, z) \quad (5.21)$$

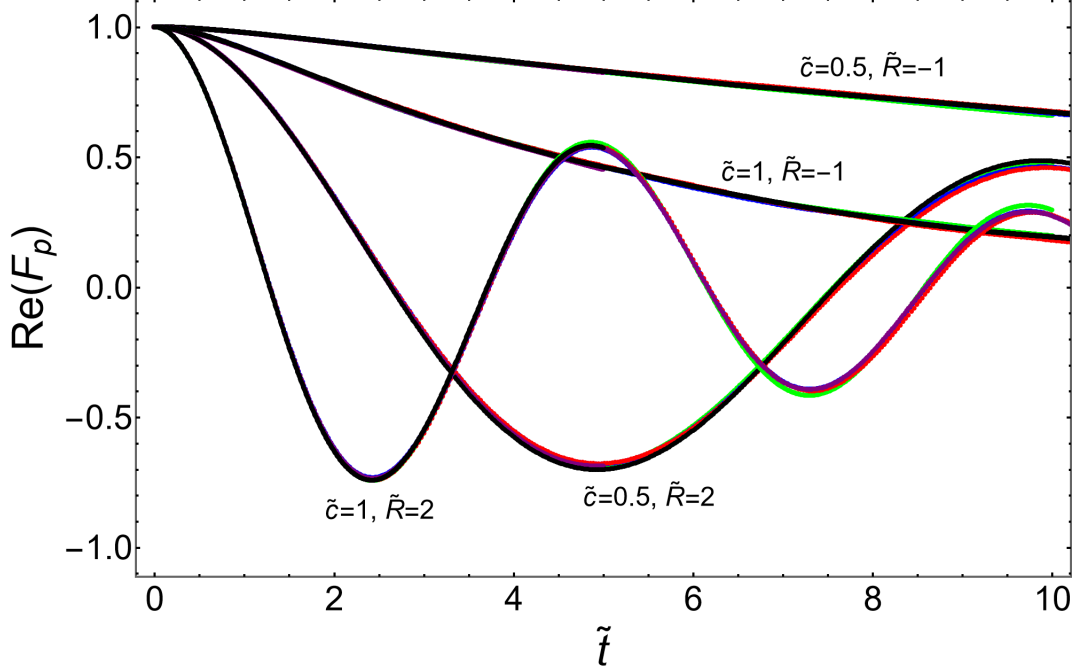


Figure 5.4: Numerical simulation of $Re[F_p(\tilde{R}, 0, \tilde{t}, \tilde{c})]$ for $n = 1$ as a function of $\tilde{t}$ at various $\tilde{R}$ and $\tilde{c}$. Each line is an overlap of 5 simulations with different values of individual parameters that leave the dimensionless variables the same.

As an example, the following limit of this function can be calculated:

$$\lim_{\tilde{R} \rightarrow -\infty} F_p(\tilde{R}, 0, \tilde{t}, \tilde{c}) = \exp \left[-\tilde{c}^2 \left(\frac{\tilde{t}}{2\tilde{R}^2} - \frac{e^{\tilde{R}\tilde{t}}}{2\tilde{R}^3} + \frac{1}{2\tilde{R}^3} \right) \right] \quad (5.22)$$

The leading corrections to $\tilde{R}$ can be calculated perturbatively [105].

For $n = 1$ the contribution of the pitchfork to the diffusion constant is also conceptually quite simple:

$$D_{p,n=1} = c^2 \int_{-\infty}^{\infty} d\tau (\langle w(t)w(t+\tau) \rangle - \langle w(t) \rangle \langle w(t+\tau) \rangle) = \frac{c^2}{g} \int_{-\infty}^{\infty} dz F_2(\tilde{R}, \tilde{h}, z) \quad (5.23)$$

Where we used the definition of the dimensionless correlation function F_2 and non-dimensionalized the integral measure $z = \sqrt{gB_w}\tau$. Although we assumed time translation invariance, this expression demonstrates the scaling of the diffusion constant in the presence

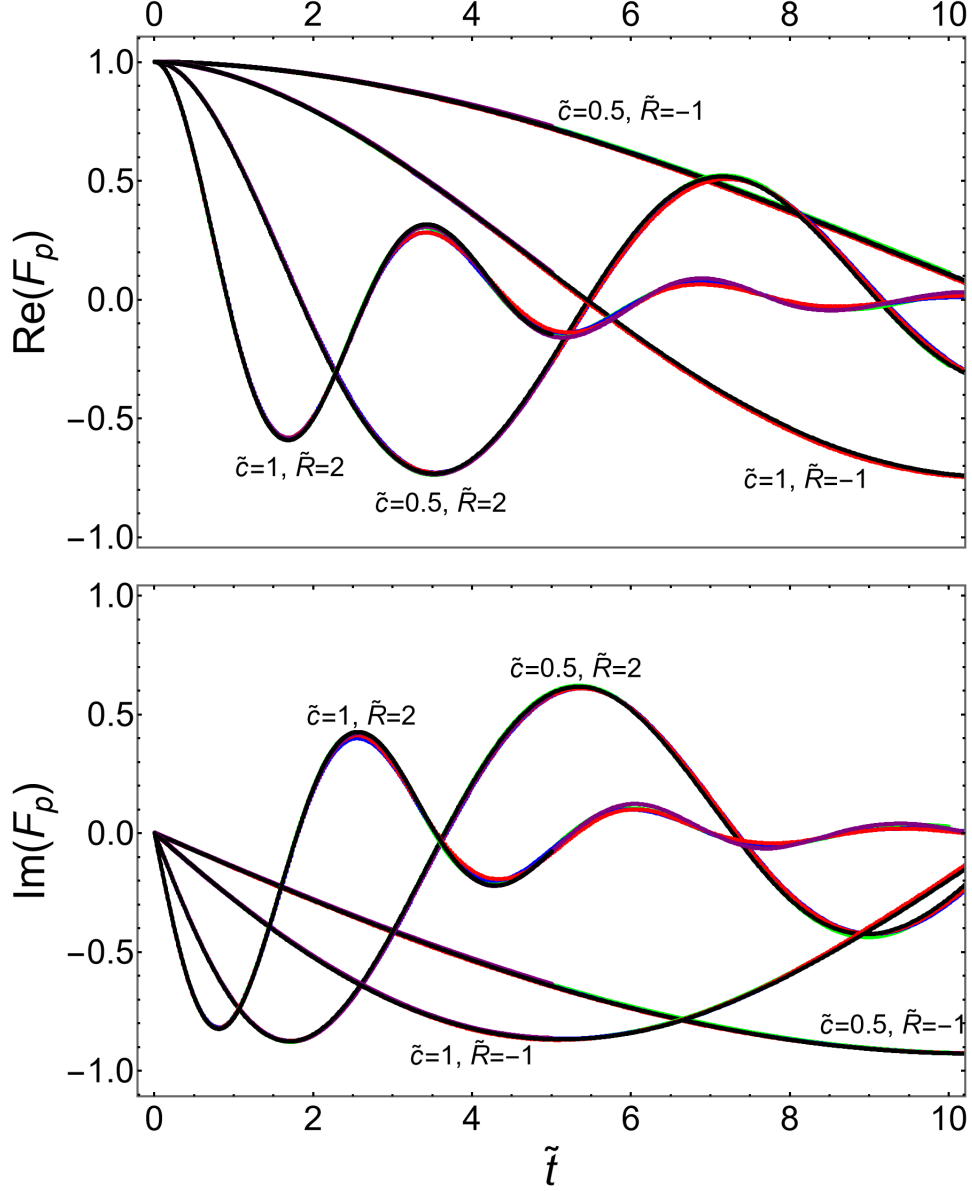


Figure 5.5: Numerical simulation of $Re[F_p(\tilde{R}, 0, \tilde{t}, \tilde{c})]$ and $Im[F_p(\tilde{R}, 0, \tilde{t}, \tilde{c})]$ for $n = 2$ as a function of $\tilde{t}$ at various $\tilde{R}$ and $\tilde{c}$. Each line is an overlap of 5 simulations with different values of individual parameters that leave the dimensionless variables the same.

of forcing as well. The numerical results for the integral are demonstrated in Fig. 5.6 for various values of $\tilde{R}$.

The $n = 2$ case is simpler conceptually. The coupling to the pitchfork changes the period of the limit cycle governed, to lowest order, by the variance of the pitchfork distribution.

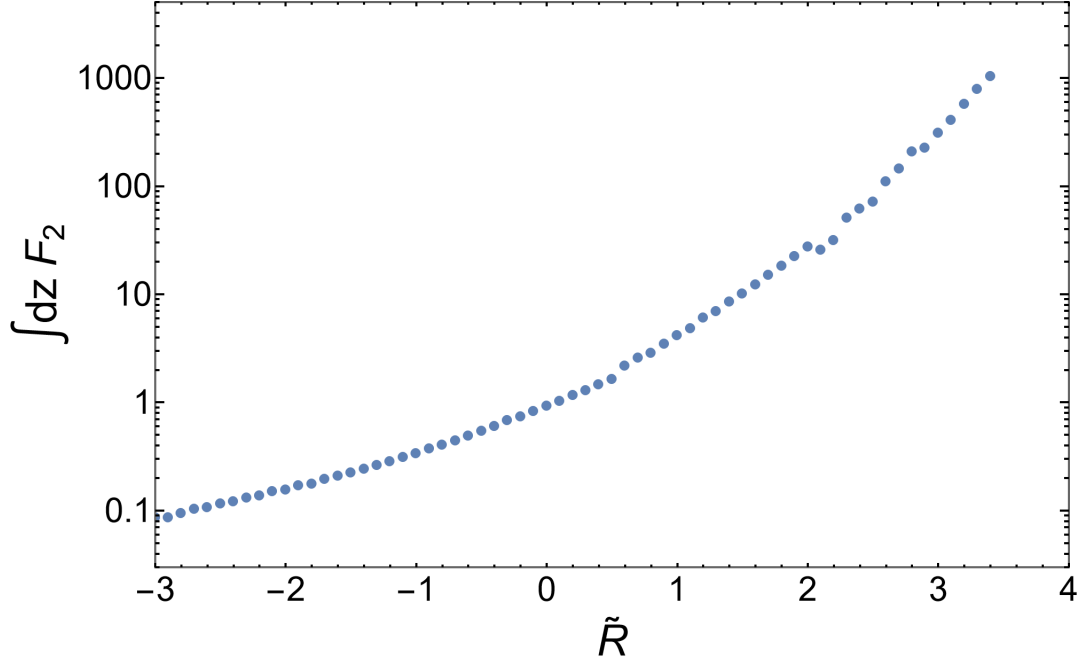


Figure 5.6: Numerical simulation of $\int_{-\infty}^{\infty} dz F_2$ as a function of $\tilde{R}$ on a log-linear scale at $\tilde{h} = 0$.

Since it is always positive, there are always oscillations in the phase-phase correlations, as well as modulation of the period. The oscillations can be understood intuitively: whatever the state of the pitchfork, it always drives the limit cycle along the positive phase direction, so at some points it is likely that on average the phase is opposite to the starting phase, hence the correlations get a negative real part.

The effective coupling constant for the $n = 2$ case is $\tilde{c} = cg^{-1}$, which is independent of pitchfork noise. Therefore, changing the noise strength of the pitchfork amounts to changing $\tilde{R}$ and thus, with corresponding tuning of R , can be scaled away (with the time axis being stretched as well).

A useful representation of the phase-phase correlations that is experimentally viable given the access to the oscillator and demonstrates the crossover is the Fourier representation. In the absence of time-dependent external fields acting on the pitchfork (so that the time translation invariance is preserved), the Fourier transform of phase-phase correlations takes

the form:

$$\int dt \left\langle \exp \left[-ic \int_0^t dt w^n(t) \right] \right\rangle_w \exp(-i\omega t) = \int dt \left\langle \exp \left[-ic \int_0^t dt \left(w^n(t) + \frac{\omega}{c} \right) \right] \right\rangle_w \quad (5.24)$$

An example of Fourier transforms for $n = 1$ is given in Fig. 5.9. The bifurcation in the pitchfork is clearly exhibited by the phase-phase correlations in frequency space. It is clear that the dominant contributions to the transform come from the most likely states that the pitchfork can be found in. To demonstrate it further, we plot in Fig. 5.7 the frequencies of the peaks of the Fourier distribution as a function of $\tilde{R}$, and see that they have the same bifurcation diagram as the pitchfork.

For $n = 1$, the mean field stable states (in the absence of an external field) of the pitchfork for $R > 0$ are $w = \pm\sqrt{R/g}$. Non-dimensionalizing the frequency of the Fourier transform as $\tilde{\omega} = \omega/\sqrt{gB_w}$, we get the scaling for the peak locations for $R > 0$: $\tilde{\omega}_{peaks} = \pm\tilde{c}\sqrt{\tilde{R}}$. Similarly, if we consider an imperfect pitchfork, the peaks approximately follow the stable branches of the bifurcation diagram, as can be seen in Fig. 5.7.

For the case $n = 2$, the expression for the mean field positions of the peaks (in the absence of an external field) takes the form $\tilde{\omega}_{peaks} = -\tilde{c}\tilde{R}$, with the replacement of $\tilde{c}$ by the corresponding definition at $n = 2$ (see above).

Notice that the peaks only appear in the negative frequencies. However, the dominant contribution to the Fourier transform comes from the factor $\langle w(t)^2 \rangle$, which means the peak locations approximately obey $\tilde{\omega}_{peaks} = -\tilde{c}F_2(\tilde{R}, \tilde{h}, 0)$. These relationships are demonstrated in Fig. 5.8.

The crossover in this case is not as clear as in the $n = 1$ case. However, in both cases the widths of the peaks undergo a broadening around $R = 0$ and are controlled by $\tilde{c}$. This can be used as an indication of a crossover.

As a final remark, the phase-phase correlations also have a term coming from the Hopf

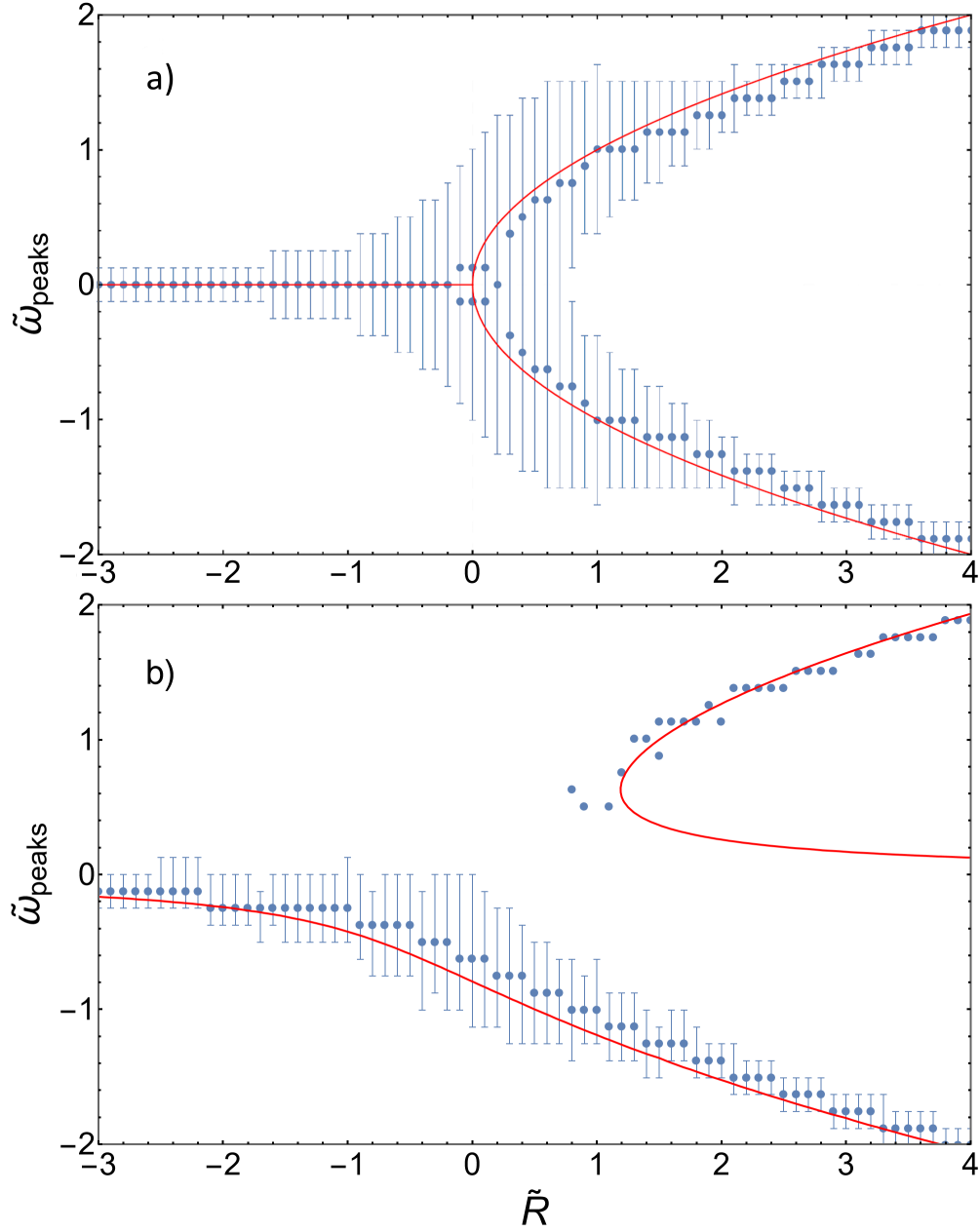


Figure 5.7: Peak frequencies (blue dots) of the absolute value of the Fourier transform of the phase-phase correlations as a function of $\tilde{R}$ for $n = 1$, $\tilde{c} = 1$ and a) $\tilde{h} = 0$, b) $\tilde{h} = 0.5$. The bars represent the widths of half maxima of the peaks. In b), the half-widths of the upper stable branch aren't shown to avoid cluttering the plot. The red solid line denotes the mean field stable states of the pitchfork – their bifurcation diagrams. The frequencies take discrete values due to the finite number of time steps in the simulations of the Ito processes. The frequency is offset by the natural frequency Ω of the oscillator.

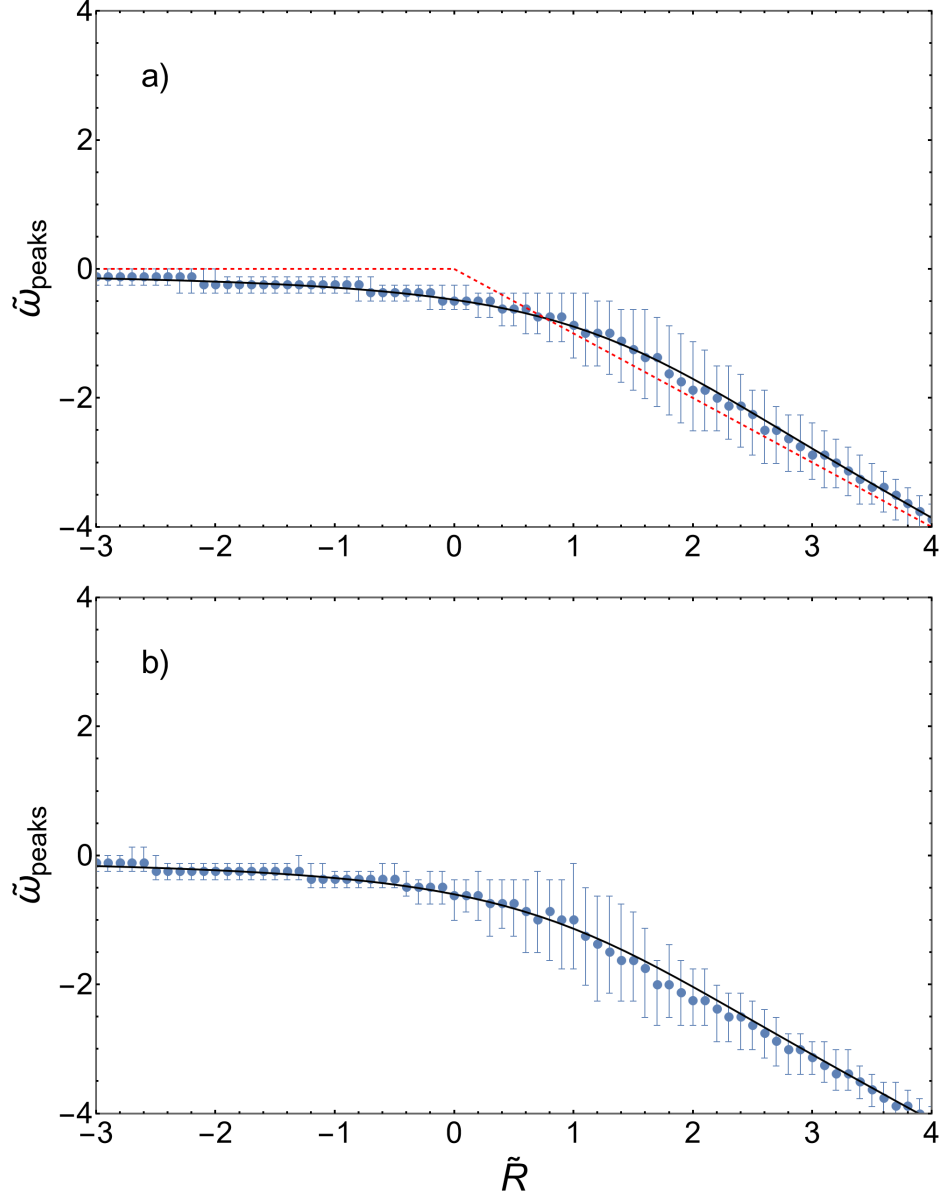


Figure 5.8: Peak frequencies (blue dots) of the absolute value of the Fourier transform of the phase-phase correlations as a function of $\tilde{R}$ for $n = 2$, $\tilde{c} = 1$ and a) $\tilde{h} = 0$, b) $\tilde{h} = 0.5$. The bars represent the widths of half maxima of the peaks. The dashed red line denotes the mean field stable states of the pitchfork. The black solid line is $-\tilde{c}F_2(\tilde{R}, \tilde{h}, 0)$. As before, the frequencies take discrete values due to the finite number of time steps in the simulations of the Ito processes. The frequency is offset by the natural frequency Ω of the oscillator.

oscillator itself that would go into the Fourier transform. However, in the limit cycle regime, the dynamics of the oscillator are simple, in that the Fourier transform of the phase-phase oscillations has a peak at $-\Omega$, the width of which depends on the noise strength. For the

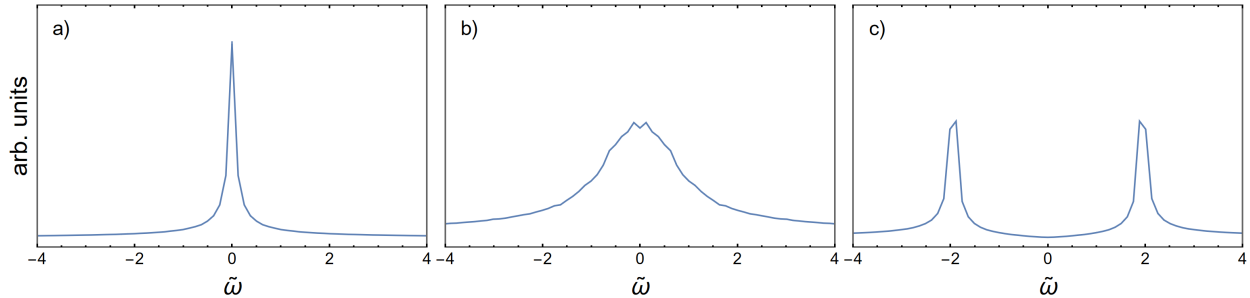


Figure 5.9: Example of absolute values of Fourier transforms of the phase-phase correlations for $n = 1$, $\tilde{h} = 0$ and $\tilde{c} = 1$ versus dimensionless frequency $\tilde{\omega} = \omega/\sqrt{gB_w}$. The vertical axis is rescaled for each plot, and is thus measured in arbitrary units (arb. units). a) $\tilde{R} = -3$, b) $\tilde{R} = 0$, c) $\tilde{R} = 4$

coupled system, it means that in the above discussion the zero frequency is shifted to $-\Omega$, and the peak widths are broader in reality. However, their locations and widths undergo the same behavior described above, with the same scaling laws.

5.2.5 Conclusion

We have taken a model of bifurcating limit cycles that we believe to be generally applicable to a wide variety of systems near a bifurcation and explored it in the presence of noise. A pitchfork bifurcation is used to imbue a limit cycle with a bifurcating behavior and the main result of the study is that this model system can be factorized into corresponding noisy dynamics of the pitchfork and the oscillator.

The factorization of the generating functional depends only on the statistical independence between the noise acting on the pitchfork and the noise acting on the limit cycle. Therefore, specifics of the noise spectrum of either part of the system only determine the details of their individual effects. For example, a temporally correlated noise added to the pitchfork adds an additional dimensionless time scale to the scaling theory, therefore changing the above analysis. However, as the noise strength decreases, the scaling functions are driven further into their asymptotic regimes, where their behavior is easier to obtain analytically, and is also believed to be less sensitive to changes in the nature of noise and to

presence of external fields.

The effect of the pitchfork with Gaussian white noise on the oscillator was then explored, and a scaling theory was applied to demonstrate how the parameters of the pitchfork must come into the functions of interest of the limit cycle. The scaling theory is supported numerically, and measurable effects of the pitchfork on the limit cycle, such as on the phase diffusion constant and on the spectral resolution of the phase-phase correlations, are demonstrated.

Our model is explicitly low-dimensional but we believe it may be a starting point for a more general approach to extended systems. The transition between silent (but excitable) and spontaneous activity has long been studied in neural cultures (e.g. [171, 172, 225]), and the transition may be accompanied by large scale activity that has been characterized as “avalanches” with a $1/f$ -like power spectrum. This is well-known as an outcome for dynamical systems that are driven very slowly close to a threshold, named self-organized criticality [142], and also for extended disordered nonlinear systems under periodic driving [226, 227]. Such states are highly correlated in space and time, but not fully periodic. Such systems are based directly on generalizations of the low dimensional model we discuss here, though extensions are not simple.

Close to the bifurcation, our model could potentially be applicable to a wide variety of other systems, such as visual perception bistability [228] and auditory perception bistability [229, 230], with suitable choices of bifurcation parameters. Scaling ideas similar to our discussion have also been explored in the context of multistable perception [231]. Although direct access to the limit cycle dynamics of such systems might not be readily available, the processing of stimuli is often inherently oscillatory (i.e. neural activity [232]), thus driving the choice to explore limit cycles. Alternatively, in perception research, the stimulus itself can be oscillatory [233]. Bifurcations of limit cycles and bistable regimes also occur in queuing theory [61], among others.

The choice of the type of bifurcation other than the pitchfork that can be coupled to

an oscillator would be based on the phenomenological knowledge of each particular system. The same also applies to the type of noise that should be used in the model, as well as its spectrum. The exploration of these ideas, in addition to increasing the dimensionality of these models is reserved for future work.

5.3 Field Theory of Birhythmicity

This section is based on the work presented in [16].

5.3.1 *Motivation for Modeling Limit Cycles in Space*

There are tremendous efforts to build tools, field theories in particular, similar to those in equilibrium, to understand non-equilibrium dynamical phases as well as the respective transitions and universality classes [2–4, 19, 46, 84]. As in equilibrium studies, it should prove valuable to develop the simplest models possible that can describe those. As mentioned in Chapter 3 a prototypical example of a non-equilibrium steady state is a dynamical phase characterized by a limit cycle – the order parameter of finite magnitude rotating or oscillating at a fixed frequency. Consequently, birhythmicity, where two stable limit cycles coexist, is a natural extension of the simpler single limit cycle phase. Both the abundance of real systems exhibiting such states (as described in Sec. 5.1) and their relevance for building our understanding of non-equilibrium phases and phase transitions are strong motivations to build and study models of such behavior. They can therefore be taken as paradigmatic states that one can use to build an understanding of non-equilibrium phases of matter. For example, a transition from both equilibrium disordered and equilibrium ordered states of $O(N)$ models to limit cycle phases is already being extensively studied [19, 46]. Field theoretic tools can be used to provide insights into either phase and the transition between them. In this Section, we explore a simple linear model of a single limit cycle phase with phase-amplitude coupling. We demonstrate how such non-equilibrium coupling affects the

fluctuation spectrum of the theory. We then extend this model to include a continuous transition to a two-cycle phase. We give various results, such as the appearance of a CEP, the destruction of the transition, enhancement of noise for the phase, and the presence of KPZ dynamics. Finally, we qualitatively demonstrate these results with numerics and discuss future directions.

Birhythmicity is a phenomenon where the system supports two stable frequencies, or limit cycles. Although the parameter space where it occurs can be small, birhythmicity is present in various fields, including physics [234–236], biology and biochemistry [42, 44, 45, 237], control theory and mathematical modeling [236, 238, 239], among many others. In many of these examples, in fact, there is a transition from a single limit cycle to a birhythmic phase.

In keeping with the themes of this dissertation, we need to include dimensionality (spatial extent) and fluctuations in order to fully characterize such a transition. Both of these ingredients can have profound effects on the behavior of limit cycles in particular. For example, spatially extended or lattice models of oscillators exhibit synchronization [64, 157], and noise plays an important role in control theory [62–64], with many other effects and applications not mentioned here. Dimensionality, of course, also plays a central role in understanding the universality class of a phase transition and changes the associated critical exponents. In order to include these ingredients, as well as for preserving the most generality, we resort to using a generalized complex Landau-Ginzburg equation. Equations of this sort have a wide variety of applications and motivations: see, e.g., [1, 84, 86] and references therein. We add Gaussian white noise for simplicity.

We recall that the symmetry group of the associated order parameter is also relevant for the study of a phase transition. A generic group that can be picked is $O(N)$, or $SO(N)$. Possible non-equilibrium phases of $O(N)$ models, as well as transitions between them, are being extensively studied in contexts similar to this work [19, 46], as mentioned above. We restrict our model to a two-component, or complex, field, which means that we stick to $O(2)$

symmetry.

The initial "toy" model describes a system that has a stable limit cycle in one phase which splits continuously into two stable limit cycles of different radii. The non-equilibrium feature is the coupling between the frequency of the cycles and their radii, which should be included for generality, but is a feature that often occurs in real systems [240–242]. We show that this coupling leads directly to the transition happening through a CEP, at least at the mean field level. We explore how the proximity to the CEP can be observed from real space and time correlations.

However, the coupling between the amplitude and the phase of oscillations that breaks the equilibrium conditions is highly relevant below six dimensions. That means that any such coupling at the microscopic level likely drives the system away from the second-order fixed point discussed in Sec. 5.3.3 and 5.3.4 through an RG flow. In addition, the inherent asymmetry coming from the transition occurring around a finite amplitude means that the $\mathbb{Z}_2$ symmetry present in continuous transitions is not guaranteed. Thus, even in the absence of non-equilibrium couplings, the system flows to a different fixed point, presumably first order, unless all the relevant couplings are finely tuned. In the language of dynamical systems, the transition instead occurs through a saddle-node bifurcation of limit cycles, which is a route that systems often take to reach birhythmicity [44, 234, 236, 238]. Despite the seeming instability of the continuous transition fixed point, the scale at which the deviations from it can be observed can be enormous by virtue of the large amplitude around which birhythmicity is born. Similar arguments apply to the non-equilibrium coupling as well, if it is suitably small at the microscopic level.

Although we approached this study from an abstract perspective, we believe it should provide a suitable framework for exploring a wide range of problems. More broadly, assuming the ideas of universality are applicable to such non-equilibrium systems, our work should provide a basis for studying any system within the appropriate universality class. Our model

is also an example of the type of field-theoretic descriptions that one can construct when considering non-equilibrium systems. Such a description allows for the use of field-theoretic tools to answer the prototypical questions of statistical physics. With these motivations, the objective of this work is thus to provide a generic model of a transition to birhythmicity and explore its non-equilibrium features, such as the correlation structure around the CEP and the appearance of $1/f$ noise. These results should be experimentally and computationally observable for systems in the appropriate class.

5.3.2 General Model of $O(2)$ Dynamics

We begin with a general Landau-Ginzburg equation for a complex field ψ that includes all the ingredients discussed in the previous section. Such a model could be written as:

$$\partial_t \psi = f_1(|\psi|)\psi + i f_2(|\psi|)\psi + D \nabla^2 \psi + \xi \quad (5.25)$$

This equation has an $O(2)$ symmetry with respect to the field $\psi = \psi_x + i\psi_y$. Here we only focus on rotations, which is the same as restricting ourselves to the $U(1)$ symmetry. The real function $f_1(|\psi|)$ is responsible for producing multistability in the radial direction, and only depends on the amplitude of the field in order to preserve the symmetry. The real function $f_2(|\psi|)$ is the coupling between the frequency of the rotation of the field and its amplitude. In the language of Langevin dynamics, f_2 is responsible for breaking the symmetry conditions that define equilibrium. This can be shown by breaking up this equation into components and directly checking for the symmetry defining thermodynamic equilibrium (see Sec. 3.1.3), or by using supersymmetry [12, 105]. Two component order parameters with $O(2)$ symmetry offer a unique simplicity in studying non-equilibrium systems. For a general $O(N)$, to go out of equilibrium, one needs to supply many additional terms that render the dynamics more complex than overdamped relaxation [19]. However, for $O(2)$, any rotation commutes with

any other rotation. Thus, complex terms (such as $if_2\psi$) that break equilibrium conditions are easy to generate while preserving the overall symmetry. It is important to note that if f_2 is constant, we can get rid of it by going into a rotating frame and effectively recovering equilibrium dynamics.

We implement spatial extent in the form of a diffusion term with strength D . This is conceptually the simplest way to do so, but diffusion is of course present in all sorts of continuous descriptions of real systems [1, 2, 84, 86]. Although generically D can be taken to have both real and imaginary parts (e.g. for models of condensate dynamics these would correspond to coherent and dissipative couplings), we take it to be real for simplicity. Finally, we add white Gaussian noise ξ with correlations $\langle \xi(t, \vec{x}) \xi(t', \vec{x}') \rangle = 0 = \langle \xi^*(t, \vec{x}) \xi^*(t', \vec{x}') \rangle$ and $\langle \xi(t, \vec{x}) \xi^*(t', \vec{x}') \rangle = 2B\delta(t-t')\delta^d(\vec{x}-\vec{x}')$, where B is the noise strength. Such a model is an example of a λ - ω system in the language of [84, 86] and has a wide range of applications (see references therein).

Defining the amplitude-phase representation through $\psi(t, \vec{x}) = r(t, \vec{x})e^{i\theta(t, \vec{x})}$, after carefully keeping track of the subtleties due to Ito discretization [3, 104, 105], the original equation can be written as:

$$\begin{aligned}\partial_t r &= f_1(r)r + D\nabla^2 r - Dr\vec{\nabla}\theta \cdot \vec{\nabla}\theta + \frac{B}{2r} + \xi_r \\ \partial_t \theta &= f_2(r) + D\nabla^2 \theta + \frac{2D}{r}\vec{\nabla}\theta \cdot \vec{\nabla}r + \frac{1}{r}\xi_\theta\end{aligned}\tag{5.26}$$

Here, the noise terms are decorrelated and each have strength B : $\langle \xi_i(t, \vec{x}) \xi_j(t', \vec{x}') \rangle = B\delta_{ij}\delta(t-t')\delta^d(\vec{x}-\vec{x}')$. We also pick up a centrifugal term, as well as two terms generated by diffusion. The former can be absorbed into $f_1(r)$ and is less relevant for our considerations below, while the latter is addressed in the following sections.

To address some of the issues arising in this model, it is useful to consider an example,

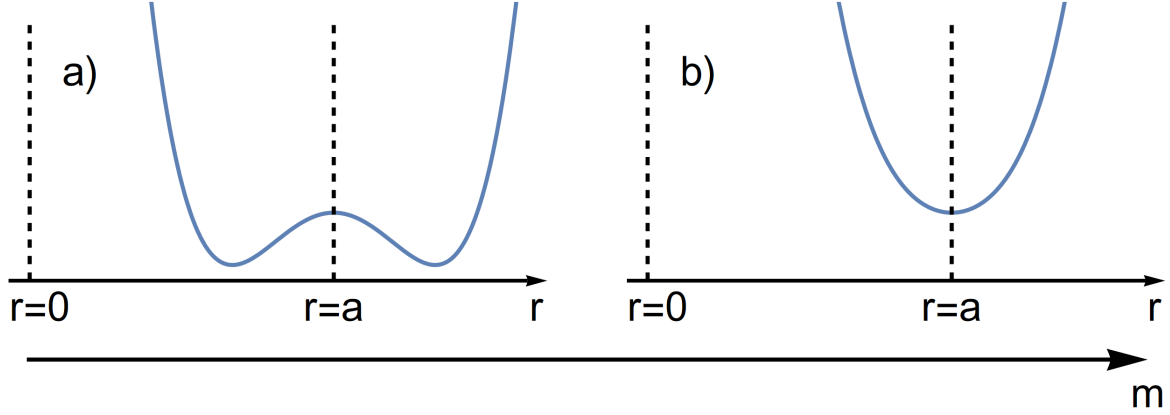


Figure 5.10: Schematic of the continuous transition model at mean field level. The curves indicate the "potential" V along which the radial degree of freedom evolves towards the minima, such that $f_1(r)r = -\partial_r V$. a) Regime with two stable radii at which the limit cycles exist ($m < 0$ here at the mean field level). b) Regime with one stable radius for the limit cycle at $r = a$. For large enough m , this can be approximated by the linear theory.

such as:

$$\partial_t \psi = \left[-m(|\psi| - a) - g(|\psi| - a)^3 \right] \psi + iF(|\psi| - a)\psi + D\nabla^2 \psi + \xi \quad (5.27)$$

A schematic illustration of this model is provided in Fig. 5.10. This model has the necessary ingredients outlined above. One can show that the transition to a phase with two stable limit cycles occurs continuously. When $F \neq 0$, it occurs through a CEP, which can be checked by looking at the Jacobian of the linearized dynamics around the limit cycles in the rotating frame [14]. However, we can see the breaking of a symmetry for a continuous transition even at the mean field level. Writing $\psi = (a + \delta r)e^{i\theta}$, we get that the dynamics of the amplitude are (ignoring diffusion and noise):

$$\partial_t \delta r = -m\delta r - g\delta r^3 - m\delta r^2 - g\delta r^4 \quad (5.28)$$

This equation does not have the $\mathcal{Z}_2$ symmetry one expects out of a Model A-like transition. However, the symmetric terms are proportional to the amplitude a of the cycle that

splits. Since we are considering the system near the transition, where $\delta r \ll a$, these terms dominate the dynamics. And although the terms breaking the symmetry can be relevant in the RG sense, their contributions alter the dynamics significantly only on scales that grow larger with increasing a . Similar logic applies to the $B/2r$ term and all the interactions it generates. Alternatively, these terms can be tuned to maintain their relative values so that the transition remains continuous.

On the other hand, once the non-equilibrium coupling F is included, it drives the system away from the continuous transition, likely to some non-equilibrium fixed point. As we show below, it can generate KPZ terms, as well as break the ratio of the terms generated by diffusion that we saw in Eq. 5.26, which is required by equilibrium. Again, however, we assume that we should be observing the signatures of the continuous transition CEP except at some large scale. We therefore have the phase diagram in Fig. 5.11 with the flow trajectories indicating that the deviations from the continuous transition are slow in RG "time".

5.3.3 Linear Theory

Suppose at this point that we are in a regime with a single limit cycle state. In that case the function $f_1(r)$ produces a single stable point for the dynamics of r . As the first intuitive step in the study of dynamical systems and field theories, we linearize around that point. Again, we call the stable radius $r_{st} = a$ and define $r = a + \delta r$. Assuming the stability is controlled by parameter m (referencing mass or reduced temperature terms in field theories, and assuming all of the constants are absorbed in m , see Eq. 5.28), we get:

$$\partial_t \delta r = -m \delta r + D \nabla^2 \delta r + \xi_r \quad (5.29)$$

We then keep linear terms for the evolution of the phase, assuming $f_2(r) \approx f_2(r_{st}) + F \delta r$,

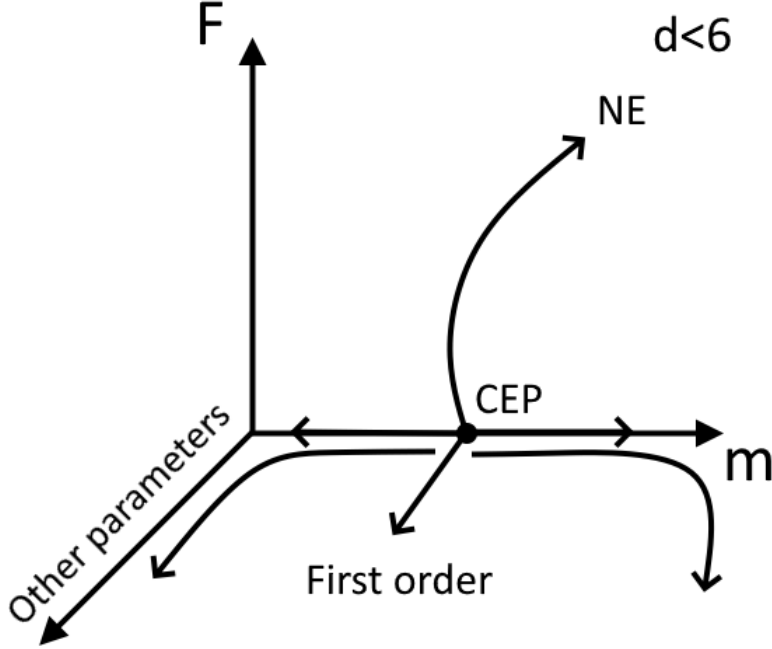


Figure 5.11: Schematic phase diagram for the model. In $d < 6$ dimensions, the non-equilibrium coupling F is highly relevant, and therefore drives the system away from the CEP towards some other fixed point labeled here as "NE". Other parameters breaking the $\mathbb{Z}_2$ symmetry might also be relevant, but with a much slower dynamics along the RG flow.

where we defined $F = \frac{df_2}{dr}(r_{st})$, and we also ignore $f_2(r_{st})$ by going into a rotating frame:

$$\partial_t \theta = F \delta r + D \nabla^2 \theta + \frac{1}{a} \xi_\theta \quad (5.30)$$

To obtain the fluctuation spectrum of such a system, we employ the MSRJD procedure (see Sec. 2.2.1) with response fields $\bar{r}$, $\bar{\theta}$. The dynamical action produced by these equations is:

$$-S_0 = \int dt d^d x \left[i \bar{r} \left(\partial_t \delta r + m \delta r - D_r \nabla^2 \delta r \right) + i \bar{\theta} \left(\partial_t \theta - F \delta r - D_\theta \nabla^2 \theta \right) - \frac{B_r}{2} \bar{r}^2 - \frac{B_\theta}{2} \bar{\theta}^2 \right] \quad (5.31)$$

where we defined $B_r = B$, $B_\theta = B/a^2$ and $D_r = D_\theta = D$ in anticipation that the renormalization group flow changes these couplings from these initial conditions. The correlators

(or propagators) in Fourier space (which we define as $v(t, \vec{x}) = \int \frac{d\omega d^d k}{(2\pi)^{d+1}} \hat{v}(\omega, \vec{k}) e^{i\omega t - i\vec{k} \cdot \vec{x}}$) are given by $G_0(\omega, \vec{k}; \omega', \vec{k}') = (2\pi)^{d+1} G_0(\omega, \vec{k}) \delta(\omega + \omega') \delta^d(\vec{k} + \vec{k}')$, with the Green's functions $G_0(\omega, \vec{k})$ similar to those quoted in [Appendix B](#).

As expected, the coupling F introduces cross-correlations between the radial and the phase degree of freedom. One way to detect a continuous phase transition is to look at the equal time correlation function. In this case, the Keldysh components of the correlator, which do not involve the response fields, are given by:

$$\begin{aligned} G_{0,rr}^K(t=0, \vec{k}) &= \int \frac{d\omega}{2\pi} \left\langle \hat{r}(\omega, \vec{k}) \hat{r}(-\omega, -\vec{k}) \right\rangle_0 = \frac{B_r/2}{m + D_r k^2} \\ G_{0,r\theta}^K(t=0, \vec{k}) &= \int \frac{d\omega}{2\pi} \left\langle \hat{r}(\omega, \vec{k}) \hat{\theta}(-\omega, -\vec{k}) \right\rangle_0 = \frac{B_r F/2}{(m + D_r k^2) [m + (D_r + D_\theta) k^2]} \\ G_{0,\theta\theta}^K(t=0, \vec{k}) &= \int \frac{d\omega}{2\pi} \left\langle \hat{\theta}(\omega, \vec{k}) \hat{\theta}(-\omega, -\vec{k}) \right\rangle_0 = \frac{B_\theta/2}{D_\theta k^2} + \frac{B_r F^2/2}{D_\theta k^2 (m + D_r k^2) [m + (D_r + D_\theta) k^2]} \end{aligned} \quad (5.32)$$

The radial degree of freedom has a correlation function that is associated with the Ising or Model A transition (see Sec. [2.2.5](#) and references therein). The phase correlation function has a k^{-2} divergence at low momenta, also typical for Goldstone modes in equilibrium [[1–7](#)]. Even though we have a non-equilibrium coupling F that changes the precise form of the correlation function, we only see deviations from the usual Goldstone mode behavior when the radial degree of freedom becomes critical at $m = 0$, or for high momenta $k^2 \gg m/D_r$. The off-diagonal term is also damped unless we are at the critical point. We discuss this point below.

We now look at the term proportional to F^2 in the phase correlations. At high momenta, when $k^2 \gg m/D_r$, that term behaves as k^{-6} . Thus, there is a length scale $l_c \sim \sqrt{D_r/m}$ below which, for $d \leq 3$ and for a system of size $L \gg l_c$, the phase correlation function behaves as:

$$\left\langle e^{i\theta(0, \vec{x}_1) - i\theta(0, \vec{x}_2)} \right\rangle \sim \exp \left[-\frac{C_{d,x} B_r F^2 l_c^{4-d}}{D_r D_\theta^2} |\vec{x}_1 - \vec{x}_2|^2 \right] \quad (5.33)$$

where $C_{d,x}$ are some numerical constants. For the correlations in time, there is no universal time scale, and the correlation functions behave differently depending on the dimension. This absence of a clear dynamical scaling exponent is reminiscent of other works on CEPs [19]. For $d \leq 3$ and equal diffusion constants, they behave in the limit $t_1 - t_2 \rightarrow 0$ as:

$$\left\langle e^{i\theta(t_1,0)-i\theta(t_2,0)} \right\rangle \sim \begin{cases} \exp \left[-C_{d,t} \frac{B_r F^2}{D^{\frac{3}{2}} m^{\frac{1}{2}}} |t_1 - t_2|^2 \right], & d = 1 \\ \exp \left[-C_{d,t} \frac{B_r F^2}{D^2} [C' - 2 \ln(m|t_1 - t_2|)] |t_1 - t_2|^2 \right], & d = 2 \\ \exp \left[-C_{d,t} \frac{B_r F^2}{D^{\frac{7}{2}}} |t_1 - t_2|^{\frac{3}{2}} \right], & d = 3 \end{cases} \quad (5.34)$$

where $C_{d,t}$ and C' are both some numerical constants. Both these scales can be understood by considering the $F\delta r$ term as an additional Ornstein-Uhlenbeck, spatially correlated noise [4]. In that case we see that it is simply another source of fluctuations that destroys phase order on the scales defined by its spatial correlation length. We demonstrate some of these effects in Sec. 5.3.6.

Another way to see this enhancement of fluctuations is to integrate out the radial degree of freedom. In that case, the phase noise is enhanced by the correlation function of the radial degree of freedom:

$$B_\theta \rightarrow B_\theta + \frac{B_r F^2}{\omega^2 + (m + D_r k^2)^2} \quad (5.35)$$

This is reminiscent of the amplification of noise observed in [20] and is the same mechanism discussed in Sec. 4.2. Thus, amplitude-phase coupling is a source of colored, e.g. $1/f$, phase noise.

The most important implication of this section is the observable correlation structures, Eqs. 5.33 and 5.34. If the behavior predicted in these results is seen in an oscillating system, it signifies the presence of a CEP, which should illuminate the underlying physics. The length scale that separates the CEP behavior from the usual Goldstone mode behavior also shows the proximity to such a transition. Additionally, the findings above show a new pathway

for a system to exhibit $1/f$ noise that seemingly does not require self-organized criticality [142].

5.3.4 Continuous Transition

With the linear theory understood, we can now explore transitions to birhythmicity. Although we saw above that there are non-linear terms in the phase-amplitude representation that produce more complex dynamics, we still believe it useful to understand a simpler toy model. We assume that the two stable limit cycles appear continuously out of the original cycle through a pitchfork bifurcation, see Fig. 5.10. This is equivalent to the dynamical Ising or Model A theory (Sec. 2.2.5) for the deviations of the radius δr , and the same phase dynamics as in the previous section:

$$\begin{aligned}\partial_t \delta r &= -m \delta r - g \delta r^3 + D_r \nabla^2 \delta r + \xi_r \\ \partial_t \theta &= F \delta r + D_\theta \nabla^2 \theta + \frac{1}{a} \xi_\theta\end{aligned}\tag{5.36}$$

where m is the control parameter taking the system through a transition, and g is some positive constant. Here, it is useful to consider this from the mean-field perspective, and again look back at the linear theory. The eigenvalues and eigenvectors of the matrix defining the linear evolution of this system when $F \neq 0$ are given by:

$$v_1 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \lambda_1 = 0 \quad v_2 = \begin{pmatrix} -\frac{m}{F} \\ 1 \end{pmatrix}, \lambda_2 = m\tag{5.37}$$

We see that as we approach $m = 0$, the eigenvalues become equal and the eigenvectors coalesce. It can be checked that the same is true approaching the critical point from the other side with respect to each stable cycle. At the critical point the matrix is non-diagonalizable, making it a CEP.

As discussed in Sec. 3.2.9 one of the signatures of a CEP is a stronger divergence of

fluctuations with respect to momenta. We saw that a common enhancement usually reveals itself in a k^4 divergence of correlators. We do see such fluctuations for the off-diagonal term $G_{0,r\theta}^K \sim k^{-4}$, where they are often present. However, the phase fluctuation term diverges as k^6 , which has not been observed before, to the authors' knowledge. The mechanism of this divergence lies in the phase being a Goldstone mode, which makes it "critical" a priori. Thus, such a divergence is a result of tuning two fields to be critical and coupling them non-reciprocally. In this case, we see that the phase correlations decay as in Eqs. 5.33 and 5.34 at all distances, which destroys the usual behavior of a Goldstone mode. Similar to the Mermin-Wagner theorem [91], this enhanced divergence implies that the ordering of the field is destroyed in dimensions below $d = 6$ at the CEP.

It can be shown that a phase with dynamics defined by Eq. 5.30 cannot exhibit a synchronization transition in low dimensions. This can be seen by defining a suitable Kuramoto order parameter [41], or by observing the angle and phase correlation functions. However, at this level of description, the frequencies of oscillations are ordered. This can be seen, for example, by observing that $\langle \partial_t \theta \rangle = F \langle \delta r \rangle$, so when the radial degree of freedom orders, so does the frequency. This is also shown in Sec. 5.3.6.

However, as mentioned before, the route to birhythmicity is often not continuous, but rather a saddle-node bifurcation [42, 44, 45, 234–236, 238]. If the mean field radial equation of motion is written as $\partial_t \delta r = -\delta V(\delta r)/\delta(\delta r)$ for some potential V , the saddle node bifurcation corresponds to the appearance of a meta-stable well and a "hill" as the control parameter is tuned, see Fig. 5.12. In our case, it can be modeled by adding an external field h_r to the r.h.s. of Eq. 5.36. Just like in usual statistical field theory, we then do not have a continuous transition between two phases, but rather a cross-over. In other words, there is no symmetry-broken phase. The problem can then be analyzed by usual means, i.e., by linearizing around the minima, using Kramers' estimation of the escape time [3], etc. However, as mentioned above, the expansion of the full equation of motion inevitably leads to terms that break the

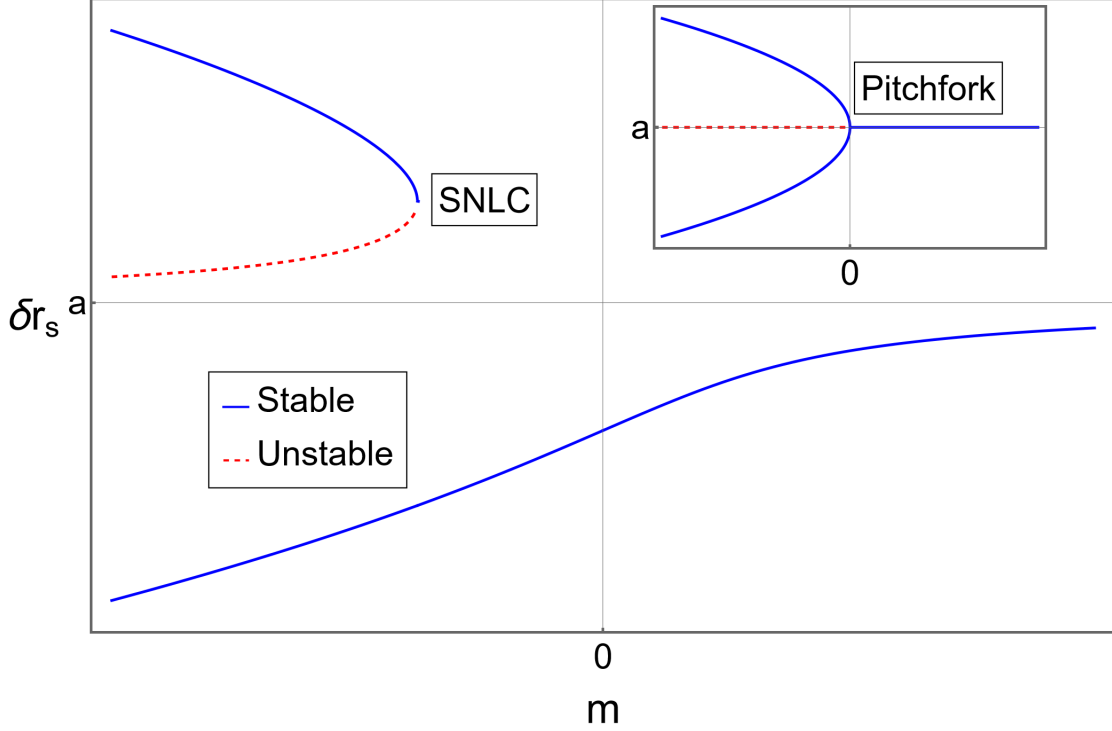


Figure 5.12: Saddle-node bifurcation scenario for the transition to birhythmicity. Solid blue lines correspond to stable radii, dashed red line corresponds to unstable radii. Either from the addition of an external field, or through the other nonlinearities described in the text, the transition happens through a saddle-node bifurcation of limit cycles (SNLC). The second stable radius (and therefore frequency) appears together with an unstable one as the control parameter m is varied. The inset shows the continuous transition scenario described in sec. 5.3.4, for comparison.

$\mathbb{Z}_2$ symmetry for δr needed for a continuous transition.

We also argue that even if those terms are ignored, the symmetry is always broken. The radial degree of freedom is defined in the domain $[0, \infty)$, or $[-a, \infty)$ for δr . That means that it is impossible to construct a function $f_1(r) = f_1(a + \delta r)$ that would not change under $\mathbb{Z}_2$ symmetry about a finite radius a , simply because it is not defined for $r < 0$. The only option is to bound it from above as well, such that $f_1(r)$ is infinite at $r = 2a$.

Although this toy model of coupling a $\mathbb{Z}_2$ order parameter to a phase does reveal the CEP and the structure of the transition, when non-linearities start to be considered, it becomes necessary to include those that also arise due to the diffusion, as we discuss in the next

section.

5.3.5 Full Theory

Now let us consider all the other spatial terms in Eq. 5.26 in addition to the toy model we had with Eq. 5.30 and 5.36. Expanding around the stable radius a , we see that the minimal action that faithfully captures the couplings between radius and phase coming from diffusion is:

$$\begin{aligned}
S &= S_0 + S_{int} \\
- S_0 &= \int dt d^d x \left[i\bar{r} \left(\partial_t \delta r + m \delta r - D_r \nabla^2 \delta r \right) + i\bar{\theta} \left(\partial_t \theta - F \delta r - D_\theta \nabla^2 \theta \right) - \frac{B_r}{2} \bar{r}^2 - \frac{B_\theta}{2} \bar{\theta}^2 \right] \\
- S_{int} &= \int dt d^d x \left[i\bar{r} \left(g \delta r^3 + u \vec{\nabla} \theta \cdot \vec{\nabla} \theta \right) + i v \bar{\theta} \vec{\nabla} \theta \cdot \vec{\nabla} \delta r \right]
\end{aligned} \tag{5.38}$$

Here we defined $u = Da$ and $v = 2D/a$, again in anticipation of the RG flow taking those couplings away from their initial values. The usual $\mathcal{Z}_2$ symmetry of the radial degree of freedom $(\delta r, \bar{r}) \rightarrow -(\delta r, \bar{r})$ is broken by the $i u \bar{r} \vec{\nabla} \theta \cdot \vec{\nabla} \theta$ term in addition to all the other terms discussed in Sec. 5.3.2. For the sake of the radial transition, it acts as an external field, which can be seen by the lowest order replacement:

$$\int dt d^d x i u \bar{r} \vec{\nabla} \theta \cdot \vec{\nabla} \theta \rightarrow \int dt d^d x i u \bar{r} \langle \vec{\nabla} \theta \cdot \vec{\nabla} \theta \rangle \tag{5.39}$$

However, this is a crude approximation, and a better way to assess the contribution of this term is through an RG procedure. Although we do not attempt to perform a complete analysis here, we can gain insight from the "zero-loop" or scaling arguments that are part of RG. Following standard dynamic RG procedures (Sec. 2.1.4 and 2.2.4), we want to keep only the slow momenta $|k| < \Lambda/b$, where $b > 1$ and Λ is some cutoff, and then scale the

coordinates and fields as:

$$\begin{aligned}
\vec{x} &\rightarrow b\vec{x} \ , \quad t \rightarrow b^z t \ , \\
r(t, \vec{x}) &\rightarrow b^{\chi_r} r(b^z t, b\vec{x}) \ , \quad \bar{r}(t, \vec{x}) \rightarrow b^{\bar{\chi}_r} \bar{r}(b^z t, b\vec{x}) \ , \\
\theta(t, \vec{x}) &\rightarrow \theta(b^z t, b\vec{x}) \ , \quad \bar{\theta}(t, \vec{x}) \rightarrow b^{\bar{\chi}_\theta} \bar{\theta}(b^z t, b\vec{x})
\end{aligned} \tag{5.40}$$

Notice that the phase field is not scaled in order to preserve its compactness and the orientation of the order parameter. We then pick the field scalings so as to preserve the free part of the Gaussian action for δr and the diffusion term for θ , which yields $z = 2$, $\chi_r = 1 - \frac{d}{2}$, $\bar{\chi}_r = -1 - \frac{d}{2}$, $\bar{\chi}_\theta = -d$. The scaling of the radial fields matches the usual second order critical point scaling [3]. With this, the couplings that are changed due to scaling are:

$$\begin{aligned}
m &\rightarrow b^2 m & g &\rightarrow b^{4-d} g \\
F &\rightarrow b^{3-\frac{d}{2}} F & u &\rightarrow b^{\frac{d}{2}-1} u \\
B_\theta &\rightarrow b^{2-d} B_\theta & v &\rightarrow b^{1-\frac{d}{2}} v
\end{aligned} \tag{5.41}$$

Although we have a symmetry-breaking term, at least at the level of scaling, it is irrelevant in $d = 1$. Similarly, the other term $iv\bar{\theta}\vec{\nabla}\theta \cdot \vec{\nabla}\delta r$ is relevant in low dimensions, although it is symmetric under $\mathcal{Z}_2$. Moreover, if they are considered without any approximations, as in Eq. 5.26, both of these terms are marginal. All this brings a possibility of a symmetry restoration for the radial field at long distances. And, of course, the term F that pushes the system out of equilibrium is highly relevant for $d < 6$. We also see that phase noise can be irrelevant in $d > 2$, which means that the phase only fluctuates through its coupling to the radius and thus might experience more ordering. Seeing also that the bare value of the phase noise is given by B_r/a^2 , it might be appropriate to set the phase noise to zero in high dimensions. However, all of the above discussion revolves around the Gaussian fixed point. Given the arguments on the breaking of the $\mathcal{Z}_2$ symmetry, there is no guarantee that this is indeed the fixed point to consider at long range.

Additionally, new terms are generated by the RG flow. For example, since we can have propagators between θ and δr : $\langle \theta \delta r \rangle_0$, we can generate a KPZ contribution to the dynamics of the angle. That can be seen by combining the u and F interactions in a suitable Feynman diagram [2–4]. Even outside of RG, if we take the part of the radial action that is linear in δr and integrate it out, we get a contribution to the phase degree of freedom that is proportional to:

$$\sim -Fu \int_{x,x',t,t'} \bar{\theta}(t, \vec{x}) (\vec{\nabla} \theta)^2(t', \vec{x}') \left[G_{0,rr}^A(t - t', \vec{x} - \vec{x}') + G_{0,rr}^R(t' - t, \vec{x}' - \vec{x}) \right] \quad (5.42)$$

Here $G_{0,rr}^{A/R}$ are the advanced and retarded propagators of the radial degree of freedom, equal to $\langle r \bar{r} \rangle$. Although in equilibrium the local part of that expression is zero [3], it might not be so outside of equilibrium, where the usual identities between correlators might not hold. Either way, we see that the existence of this term is based on non-zero F , which is expected since KPZ dynamics is non-equilibrium. In fact, since we do not have the additional $\theta \rightarrow -\theta$ (with the associated $\bar{\theta} \rightarrow -\bar{\theta}$) symmetry due to the presence of the non-equilibrium term, KPZ scaling is expected for limit cycle phases [47]. In the presence of F , other generated terms include a δr^2 non-linearity, cross-diffusions between the radial and angular degree of freedom, cross-correlated noise, etc. All of these can be observed even at a one-loop level.

To summarize, we see that the true long-range behavior of this system is highly non-trivial due to the presence of several nonlinearities as well as the generation of new ones. However, as argued in the Introduction, the scale at which a behavior different from the CEP emerges can be extremely large, especially as the radius of the cycles increases. It might therefore be of use to perform the RG analysis of a model of the type described in Sec. 5.3.4, or include only a few most relevant nonlinearities. We leave the detailed analysis of this sort and of the full problem for future work and now discuss some numerical results.

5.3.6 Simulations

To investigate the theoretical predictions outlined above, we performed lattice simulations of the respective stochastic processes. We use the TensorFlow Quant Finance package [243] and use the Euler sampling method to simulate the dynamics described in the text in the Ito discretization. Due to the stochasticity of the processes and the presence of lattice derivatives, Euler sampling should have sufficient precision [244]. In order to take into account the compactness of the phase field, we replace the spatial derivatives as follows:

$$\begin{aligned}\nabla^2\theta &\approx \sum_{\hat{a}} \sin[\theta(\vec{x} + \hat{a}) - \theta(\vec{x})] \\ (\vec{\nabla}\theta)^2 &\approx \sum_{\hat{a}} (1 - \cos[\theta(\vec{x} + \hat{a}) - \theta(\vec{x})])\end{aligned}\tag{5.43}$$

where $\vec{x}$ are the locations of the lattice points and the sum $\sum_{\hat{a}}$ goes over the nearest neighbors of $\vec{x}$, where $\hat{a}$ are lattice vectors. This is in accordance with the studies of compact KPZ equation [10], here extended to arbitrary dimensions.

In Fig. 5.13 we plot the phase autocorrelation function defined in Eq. 5.34 for one dimension. On a logarithmic scale, it exhibits a quadratic fall-off as predicted analytically. The correlations do not decay all the way to zero due to the compactness of the phase degree of freedom. In the inset, we plot the correlation function versus t^2 for small times, and it exhibits linear behavior, again confirming the prediction. In order to observe the transition from a stretched exponential decay to the usual decay of phase correlations in one dimension $\sim e^{-\sqrt{t}}$, one has to be able to discern the competition between these two regimes before the correlation function reaches the eventual constant.

To illustrate other effects described in the text, we simulate two-dimensional dynamics

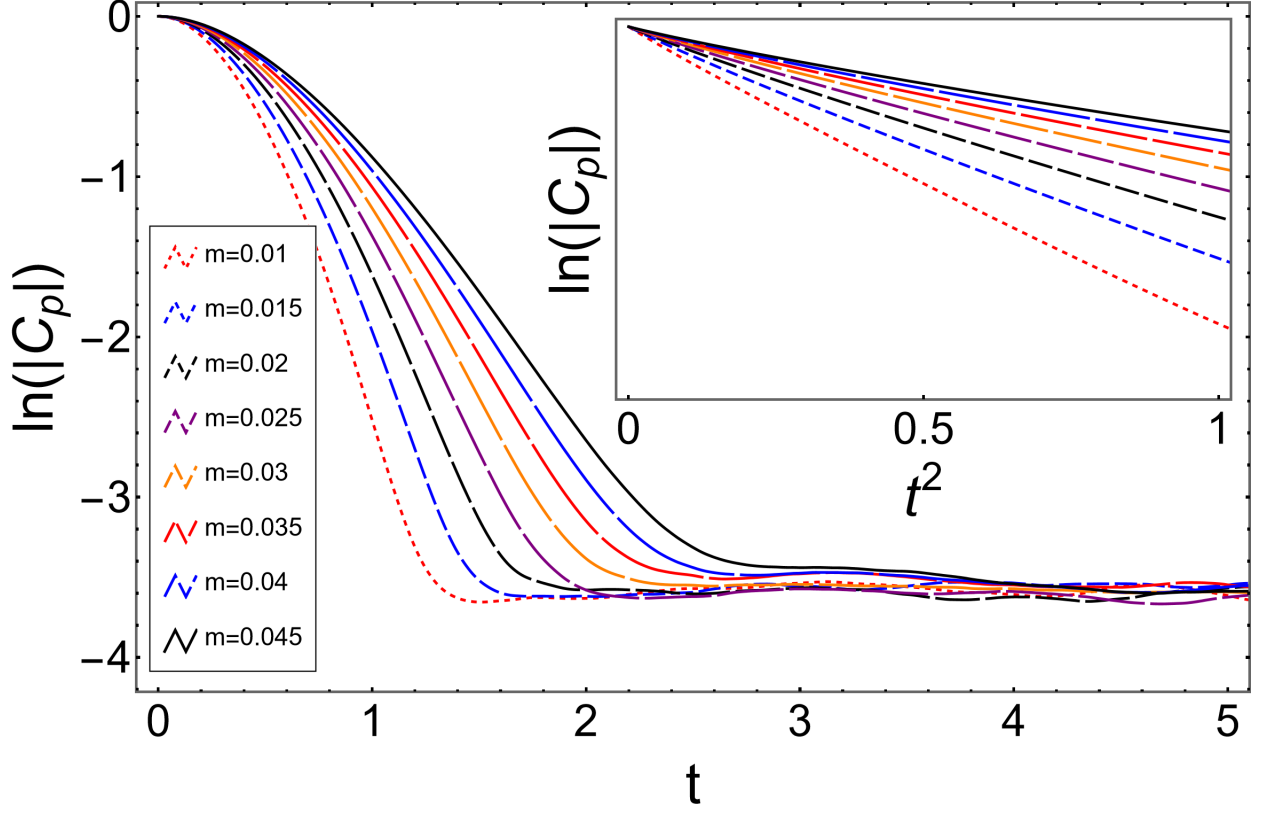


Figure 5.13: Phase autocorrelation function $C_p(t_1 - t_2) = \langle e^{i\theta(t_1,0) - i\theta(t_2,0)} \rangle$ in 1 dimension. The logarithm exhibits a quadratic falloff in time, as predicted by Eq. 5.34. The correlation function is approximately linear w.r.t. t^2 for small times as shown on the inset. The simulation parameters: number of runs $N = 1000$, length of system $L = 500$, lattice constant $\Delta L = 1$, $a = 10$, $F = 0.2$, $D = 1$, $B_r = 1$, $B_\theta = 0$.

according to the following set of equations:

$$\begin{aligned} \partial_t \delta r &= -m \delta r - g \delta r^3 - u (\vec{\nabla} \theta)^2 + D \nabla^2 \delta r + h_r + \xi_r \\ \partial_t \theta &= F \delta r + D \nabla^2 \theta + \frac{1}{a} \xi_\theta \end{aligned} \quad (5.44)$$

These produce the action in Eq. 5.38 through the MSRJD procedure with $v = 0$ (which we set that way since $u \neq 0$ is enough to demonstrate the features of interest). We also added a conjugate field h_r to detect the continuous transition. To see whether the transition

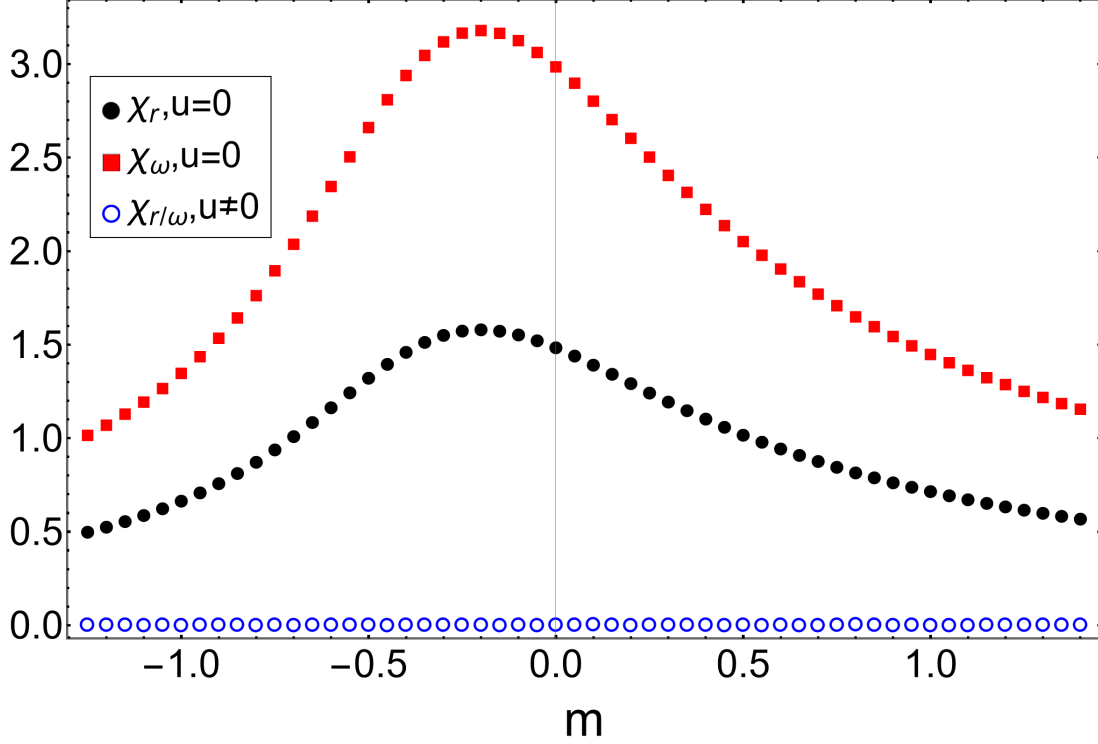


Figure 5.14: Response functions for the system in $d = 2$ dimensions evolving according to Eq. 5.44. All simulations are on a 40×40 lattice, time step is 0.01, number of runs $N=1000$. The system runs until it has thermalized. For all simulations $B = 1$, $a = 10$, $g = 0.5$, $D = 1$ and lattice constant is set to 1. For the frequency response function at $u = 0$, we set $F = 0.2$. In the presence of $u \neq 0$, the response is minimal and the continuous transition is erased.

is preserved, we can look for a peak in the response function to an external field:

$$\chi_r = \frac{\partial \langle r \rangle}{\partial h_r} \quad , \quad \chi_\omega = \frac{\partial \langle \partial_t \theta \rangle}{\partial h_r} \quad (5.45)$$

In the thermodynamic limit, the response function diverges at the phase transition [2, 4] and has a peak for a finite system. In Fig. 5.14 we plot the response functions for both radial d.o.f. and frequency in two dimensions. When $u = 0$ we have the Model A dynamics for the radius, which exhibits a phase transition in two dimensions. The frequency is proportional to the radial d.o.f., and so the frequency also orders. The peak does not necessarily align with the phase transition point exactly and depends on the size of the system. When $u \neq 0$, the fields exhibit a very weak response to external perturbations, signaling that they are in

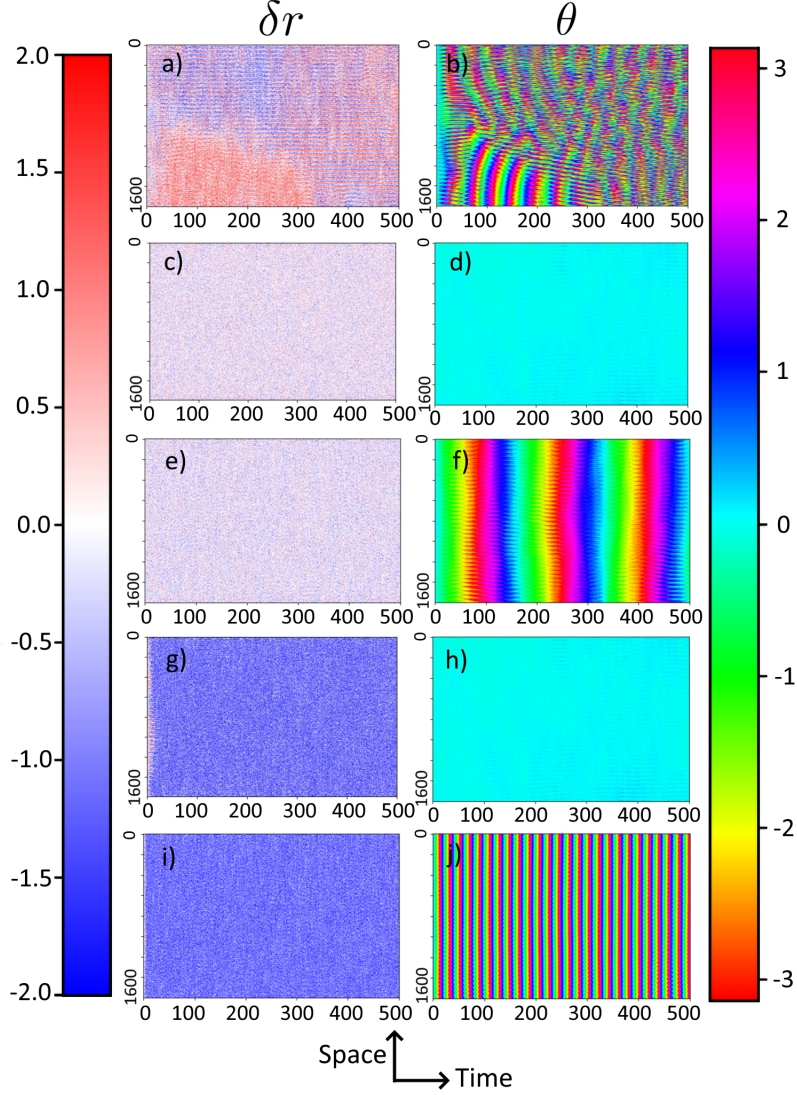


Figure 5.15: Example evolutions of δr and θ degrees of freedom according to Eq. 5.44 in $d = 2$ dimensions. The 40×40 lattice is flattened to a one-dimensional line for the plot corresponding to the vertical axis. The simulation time step is 0.01, and the time range is $[0, 500]$, corresponding to the horizontal axis. The initial condition for the radius is random and the angles are aligned. For all simulations $B = 1$, $a = 10$, $g = 0.5$, $D = 1$ and lattice constant is set to 1. a), b) $m = -0.7$, $u = 0$, $F = 0.3$. Domains of positive and negative δr are formed as in the usual Model A dynamics, with the angle oscillating with corresponding frequencies. c), d) $m = 0.5$, $u = Da$, $F = 0$. e), f) $m = 0.5$, $u = Da$, $F = 0.3$. Even with $m > 0$, where $\langle \delta r \rangle = 0$ according to mean-field theory, $\langle \delta r \rangle < 0$ due to the $u(\vec{\nabla}\theta)^2$ term. g), h) $m = -0.7$, $u = Da$, $F = 0$. i), j) $m = -0.7$, $u = Da$, $F = 0.3$. When $m < 0$, even with some initial domains with $\delta r > 0$, the eventual configuration selects the steady state with $\delta r < 0$ according to the mechanism described in the text.

a deep potential well. There is certainly no continuous transition in that case.

In Fig. 5.15 we plot example evolutions of both degrees of freedom in a $d = 2$ simulation. When $m < 0$ and $u = 0$ (Fig. 5.15a and 5.15b) the radial degree of freedom undergoes the usual $\mathcal{Z}_2$ dynamics, forming domains of positive and negative δr . The angle θ then rotates according to which configuration the domain is in. When $u \neq 0$, even in the $m > 0$ regime, the noise in the angular degree of freedom produces fluctuations which result in a non-zero $\vec{\nabla}\theta$. That produces a downward shift in the radial degree of freedom, Fig. 5.15c, d, e and f. When $m < 0$, even though there are initial domains of positive δr , they are eventually destroyed by the presence of the $u(\vec{\nabla}\theta)^2$ term, Fig. 5.15g, h. When $F \neq 0$, that destruction is even more apparent and can be understood intuitively.

At the edge of two domains, the $F\delta r$ term accumulates a phase difference between the two domains. That means that $\vec{\nabla}\theta$ is increasing along the domain wall, making the $u(\vec{\nabla}\theta)^2$ term larger. That in turn shifts the radial degree of freedom along the wall, and the negative domains eventually consume the positive through that mechanism. This growth of the domain wall is demonstrated in Fig. 5.16. It can also be seen there that as the negative domain overtakes the system, δr becomes less negative, since θ orders more on the inside, which decreases the size of $u(\vec{\nabla}\theta)^2$ term.

Similarly, one can simulate the full field equations that would satisfy all the needed properties. An example is the following equation:

$$\partial_t \psi = \left[-m \left(|\psi|^2 - a^2 \right) - g \left(|\psi|^2 - a^2 \right)^3 \right] \psi + iF \left(|\psi|^2 - a^2 \right) \psi + D\nabla^2 \psi + \xi \quad (5.46)$$

This form is chosen over the one used in Sec. 5.3.2 because it is computationally cheaper to compute $|\psi|^2$ than $|\psi|$. One can check that if the first line of this equation is seen as a gradient descent in some potential, then that potential is indeed equally deep for both minima around $r = a$ when $m < 0$ (ignoring the $B/2r$ term, which would only make the lower radius limit cycle less stable). It can also be checked by using the calculation outlined

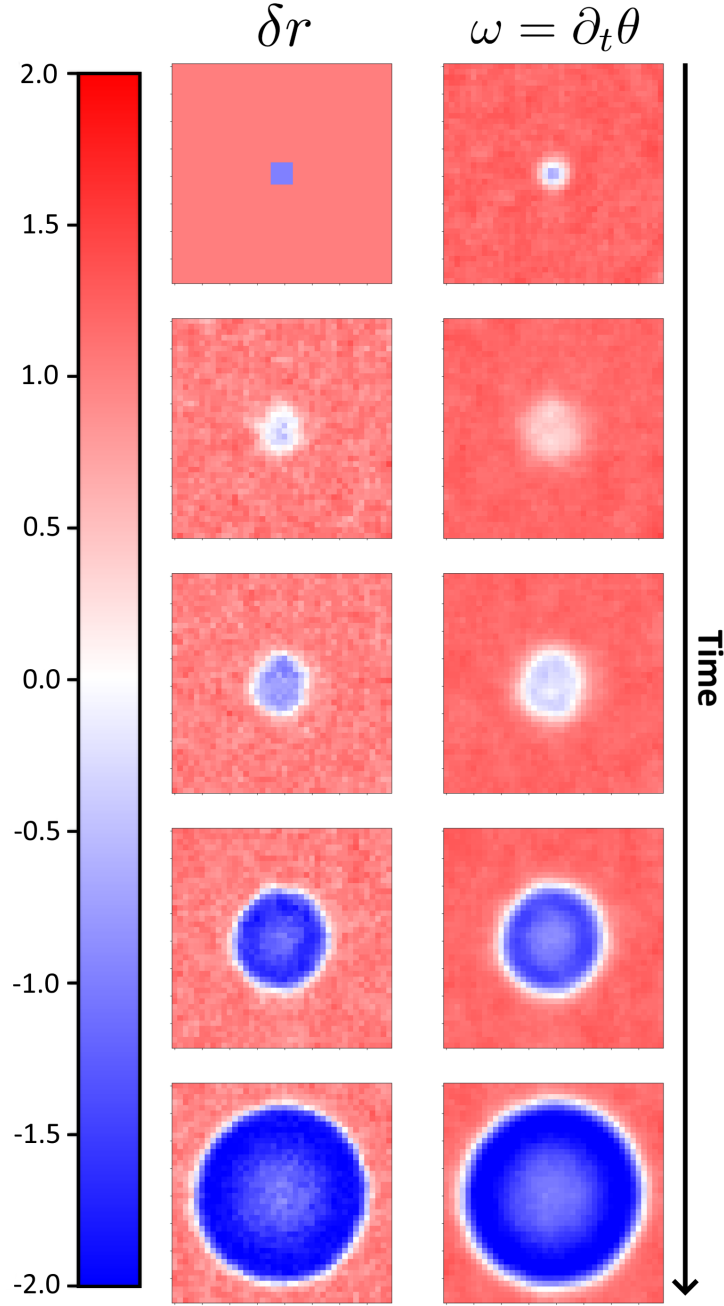


Figure 5.16: Example evolutions of δr and $\omega = \partial_t \theta$ according to Eq. 5.44 in $d = 2$ dimensions for an initial condition with a small negative domain. The simulation is on a 40×40 lattice, time step is 0.01, and the time range is $(0, 5, 7, 10, 15)$ top to bottom. Other parameters are set to $m = -0.5$, $F = 0.3$, $B = 0.3$, $a = 10$, $g = 0.5$, $D = 1$, $u = Da$ and lattice constant is set to 1. After a period of accumulating a zero domain wall between the negative and the positive domains, the negative eventually overtakes the positive through a mechanism described in the text.

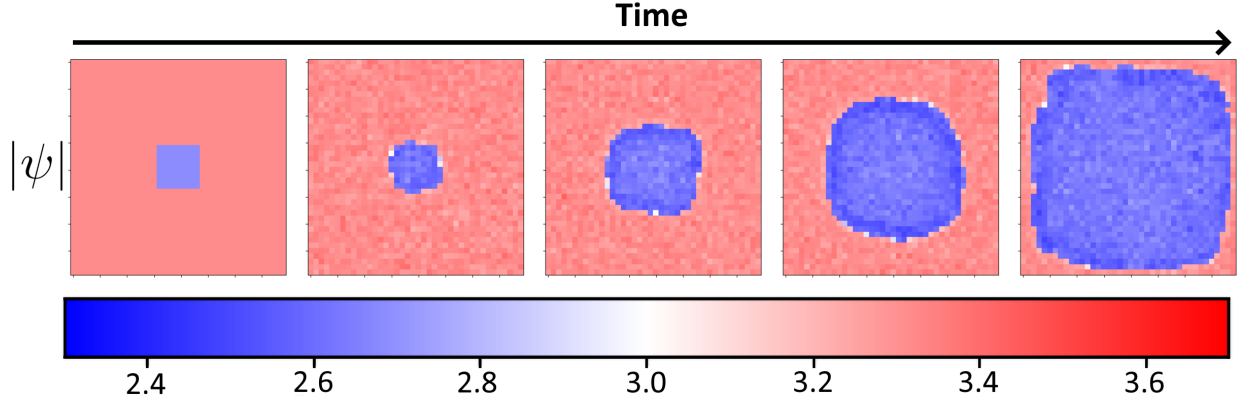


Figure 5.17: Example evolutions of the magnitude $|\psi|$ of the full field Eq. 5.38 in $d = 2$ dimensions for an initial condition with a small negative domain. The simulation is on a 40×40 lattice, time step is 0.01, and the time range is $(0, 5, 15, 25, 40)$ left to right. Other parameters are set to $m = -2$, $F = 0.3$, $B = 0.3$, $a = 3$, $g = 0.5$, $D = 1$ and lattice constant is set to 1. The smaller magnitude domain eventually overtakes the positive through a mechanism described in the text.

in [14], that the transition at $m = 0$ is going to be an exceptional point at the mean-field level given $F \neq 0$. When these equations are simulated in the two-cycle regime, we again see the same mechanism where a domain in a smaller radius state grows and consumes the entire system, see Fig. 5.17.

All of the simulations presented above are qualitative demonstrations of the analytical claims made in the previous sections. Just as a detailed perturbative or renormalization group analysis is likely necessary to make more precise analytic claims, more detailed numerical approaches can also be taken to study this system. Those could include finite-size scaling techniques, larger lattice sizes, Binder coefficient as a measure of the transition, etc. We leave both sides of the problem for future work.

5.3.7 Conclusion

Limit cycles are a phenomenon that is present in a variety of systems and models, ranging from physics to the social sciences. They are one of the most elementary steady states that a system driven out of equilibrium can settle into. Their ubiquity and relative simplicity make

them a prime candidate for exploring non-equilibrium phases of matter from a field theoretical perspective. The phenomenology of limit cycles includes the possibility of birhythmicity, a regime of a system where it supports two stable limit cycles. Birhythmicity can be found in a variety of disciplines and applications as well, such as biochemistry, biology, modeling of social behavior, and others. Through a generalized complex Ginzburg-Landau equation, we introduced a class of models with rotational symmetry that can be used to describe transitions to birhythmicity.

As a generic feature, we also included a coupling between the magnitude of the order parameter and the frequency of its oscillations. Such terms often arise in real systems and have major effects, for example, for the functioning of the brain. This feature drives the system out of equilibrium and has profound consequences for its behavior.

In order to explore a continuous onset of birhythmicity, where a single stable limit cycle splits into two stable ones, we used a pitchfork bifurcation around a finite magnitude of the complex field variable. This choice was made in order to draw a comparison with the equilibrium second-order transition of a scalar field. Despite this parallel, the symmetry-breaking terms as well as the non-equilibrium coupling are relevant interactions that should drive the system away from a continuous transition. However, we provide some arguments that the behavior of the CEP can still be observed below some very large scale where the RG flow truly escapes the CEP. With that, we described the lowest-order field-theoretic action resulting from such a model.

First, we show that, at a Gaussian level, the coupling between phase and radius leads to a stretched exponential decay of phase correlations below certain length scales and for short times separations. Such a correlation structure should provide a testable measure of the proximity to the transition point and the presence of phase-amplitude coupling. We also show that this coupling can be a source of $1/f$ noise for the phase.

Then, if the transition is indeed continuous, it occurs through a critical exceptional point.

Exceptional Points and CEPs are special points in parameter space that are responsible for a variety of unusual behavior and statistical properties of the system. They are a focus of many studies in their own right and seem to be an essential feature to be contended with when non-equilibrium models are considered. For our model, the statistical properties of the phase correlations are altered drastically at the CEP, and the fluctuations are enhanced. That is a known feature of exceptional points and hints at the appearance of new universality classes as far as the transition is concerned. We demonstrated how a coupling between the phase and the radius in such a transition would lead to a CEP.

We show that although we start with a model of a continuous transition, the fact that it occurs around a finite magnitude and the presence of the non-equilibrium coupling inevitably lead to symmetry-breaking terms that should erase it. In that case, birhythmicity onsets through a second stable limit cycle occurring through a saddle-node bifurcation, which is a pathway to birhythmicity often observed in real systems and simulations. However, we argue that for short scales, one can still observe the signatures of the CEP, since the symmetry-breaking terms might be tiny at the microscopic level. We also argue that a range of other terms are inevitably going to be produced by a renormalization group flow. As an example, we show that the non-equilibrium coupling induces KPZ dynamics for the phase degree of freedom. That is also a generic feature of the phase that is coupled to amplitude fluctuations. Here, we demonstrate it using a field-theoretic language. Although this effect breaks the $\mathbb{Z}_2$ symmetry and therefore breaks the continuous phase transition, it is possible that parameter-tuning can restore a continuous critical point by a judicious cancellation of terms; that certainly can happen at the mean-field level [19, 46, 47, 245]. We leave a more detailed exploration of this problem for future work.

Finally, we demonstrate the analytic predictions using lattice simulations in low dimensions. Most importantly, we confirm that such a transition is inevitably destroyed by the presence of the non-equilibrium coupling, which interplays with the non-linearities of the

model. We do so both by simulating the approximated dynamics developed in the text and for full field dynamics of an example Ginzburg-Landau equation. We also describe the intuitive mechanism of how droplets of negative domains eventually grow to dominate the entire system.

As discussed in Sec. 5.3.1, there is a variety of real systems exhibiting birhythmicity (and transitions to birhythmicity). In addition to providing a coarse-grained model for particular systems, this work should also give a suitable initial framework to describe any transition belonging to the appropriate universality class. To that end, we demonstrated a number of properties of the model of birhythmicity that are generic and should be observable. We leave a more detailed analysis for the broader universality class of this transition for future work.

Acknowledgments

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5.4 Outlook on Oscillating Phases of Matter

In this chapter, we introduced some basic examples of models exhibiting limit cycle behavior and then explored some models of merging limit cycles in detail. Let us conclude with a few remarks and an outlook on future problems.

In Sec. 5.1 we already mentioned a few features that generically arise as a consequence of periodic behavior. Those include, for example, KPZ scaling associated with directionality of a limit cycle state [47]. However, this is a novel field, and new insights will surely emerge as we explore limit cycle behavior. For instance, the two models discussed in the previous sections have the directionality of the limit cycle put in by hand – the field goes around the cycle in one direction but not the other. This is not always the case, and many models

exhibit phases where this directionality is picked spontaneously [19, 36, 46]. In that case, just like domain walls in the Ising model, there are regions where cycles rotating in opposite directions meet at a boundary. The phenomenology of such boundaries is likely different from equilibrium cases due to the dynamical nature of the order parameter, and many new related phenomena are being discovered [36].

There are also many situations where a system can exhibit multiple periodic behaviors simultaneously. An obvious example is living organisms, where cycles of hugely varying periods can be found coexisting. In the spirit of universality, this calls for the investigation of minimal models that can exhibit such multi-periodicity. We note that the model of birhythmicity in Sec. 5.3 can be thought of as a very basic example of such behavior, but the order parameter can pick only one cycle at a time in the birhythmic phase.

Additionally, there are periodic behaviors enabled by noise, rather than being a feature of the deterministic dynamics. Stochastic resonance discussed in Sec. 5.1 is one such system, but one can also imagine models where there is external time scale imposed on the system. This is the case in models of neural dynamics that include synaptic resource variables [54, 165]. These models exhibit periodic bursting, and one can also imagine more abstracted models with similar features. For example, something that might be called self-organized first order transition or self-organized bistability [246] can take the form:

$$\begin{aligned}\partial_t \phi &= |m|\phi - g\phi^3 + h + \xi \\ \partial_t h &= h_0 - \phi\end{aligned}\tag{5.47}$$

In that case, the field h modulates the potential for ϕ in a way that whichever well ϕ is not in always grows deeper until ϕ jumps into it due to noise. Such "bursting matter" models are of high relevance for neural dynamics.

Additionally, deterministic dynamical systems can exhibit quasi-periodic and chaotic behavior [39]. In the presence of spatial coupling and noise, it is not obvious how such

systems can be systematically distinguished from truly periodic ones. Such a distinction would be useful since it would allow us to understand the underlying dynamics (and hence mechanisms) governing a system. This is especially tricky with data-driven problems, where the underlying dynamics is not known a priori. An example of such a system that is chaotic, seems periodic, but also has spatial coupling and fluctuations, is weather or climate more broadly.

The problem of synchronization is also of interest in the many-body contexts. Although elegant models such as the Kuramoto model (see Sec. 3.2.7) provide deep insight, synchronization is hugely complicated by only having local couplings. Additionally, if an order parameter is able to change in magnitude, phase slips and domain walls become possible by setting that magnitude to zero. That discourages synchronization, and we explore that to some extent in the next chapter.

We see that dynamical steady states such as limit cycles are a huge and exciting area of inquiry. Many directions remain to be explored, with potential applications in physics, chemistry, biology, population dynamics, climate science, neuroscience, artificial neural networks, and others. Although the models discussed in this chapter are abstract, they aim at illuminating generic behaviors that limit cycle systems can exhibit. Other ideas outlined in the chapter and many more are left for future work.

CHAPTER 6

APPLICATIONS IN NEURAL DYNAMICS

6.1 Dynamical Systems View of Neural Behavior

In Sec. 3.2.7 we introduced the dynamical systems models of neural dynamics, ranging from single neural level all the way up to large neural populations. As we mentioned, dynamical systems have been a fruitful and powerful tool in understanding neural behavior. This view can be applied to both healthy neural dynamics and neurological disorders. Typically, one models the local field potential or spiking rate of neurons in some network structure, in the form:

$$\partial_t \vec{u}(t) = \vec{f}(\vec{u}(t), \vec{p}(t), \vec{\xi}(t)) \quad (6.1)$$

where the vector $\vec{u}(t)$ contains the activity of all units of the model and $\vec{f}$ encodes their time evolution. The activity can be the potentials and currents, such as in the Hodgkin-Huxley model, or spiking rates, such as in the Wilson-Cowan equation, see Sec. 3.2.7. The evolution is governed by the connection structure of the neural network that depends on some relevant set of parameters $\vec{p}(t)$, as well as noise $\vec{\xi}(t)$ that often encodes inputs into the network not captured by the model. An unhealthy behavior is then characterized by the presence of an abnormal attractor in the dynamical system or by having an attractor that is too stable or not stable enough. Transitions to abnormal dynamics can occur through a bifurcation or through activation by noise, depending on the respective clinically observed phenomena.

The next steps often include considering a network where the nodes evolve according to Eq. 6.1, but the connectivity has some structure [51, 56, 57, 66, 70, 71, 73–76, 78–80, 89, 140, 160, 197, 247–251]. That structure can be some graph, can be random, or can be based on some spatial connectivity kernel. In the latter case, the assumption is often that on the lowest spatial scales the connectivity between neurons falls off exponentially [51, 140,

247, 248]. We believe such an exponential decay in connectivity points to the possibility of a coarse-graining scheme, where the activity of neurons is averaged over scales similar to or larger than the connectivity fall-off distance. In that case, possibly very complicated networks or neural field theories can be turned into lattice theories with nearest neighbor couplings. In the spirit of coarse-graining, this amounts to replacing the vector $\vec{u}(t)$ with a spatially extended field $u_\alpha(t, \vec{x})$, where the new index α only encodes different types of neurons (for example, excitatory and inhibitory).

Given this approach, the final observation can be thought of from the perspective of the renormalization group (RG). RG postulates that the global dynamics of a model are governed by the most relevant terms in its dynamics. For spatial connectivity, the term that gives the simplest and most relevant coupling is diffusion. The final model of interest, therefore, takes the form:

$$\partial_t u_\alpha(t, \vec{x}) = f_\alpha(u_\alpha(t, \vec{x}), \vec{p}(\vec{x}), \xi_\alpha(t)) + D_{\alpha\beta} \nabla^2 u_\beta \quad (6.2)$$

Investigations of this form of neural dynamics have been conducted in many contexts, including general neural behavior [51, 66, 151, 152], synchronization transitions and epilepsy [54, 250, 252], and models of avalanching [165]. However, we put this approach forward as a way to model neurological disorders specifically. As mentioned above, the power of the dynamical systems perspective in the study of neural dynamics is the possibility of reducing the model to capture only the essential features of interest while abandoning irrelevant details. That means one can build abstracted models that are still of great use, which has been done for epilepsy, Parkinson's disease, depression, bipolar disorder, dementia, sleep disorders, and others. For an overview, see [51] and references therein. Inclusion of connectivity in the form of Eq. 6.2 opens the possibility of capturing more features of these disorders. In particular, any equation of the form of Eq. 6.2 can be analyzed using the tools of field theory, which are particularly useful for systems near phase transitions.

In the next section, we demonstrate the application of the above ideas to modeling epilepsy. We show how a field-theoretic approach to a model of epileptic seizures with diffusive coupling demonstrates several clinically-observed features. These include the emergence of synchronization and the ictal wave. We then demonstrate that incorporating quenched disorder into such a theory illuminates which generic biological parameters promote or discourage seizures. Finally, we propose a stimulation protocol for focal epilepsy specifically, based on intentionally disordering the system.

6.2 Seizure as a Dynamical Phase

6.2.1 *Modeling Epilepsy*

Around 1% of the global population has epilepsy – a disorder characterized by recurring seizures [166–168]. Given the dangers and prevalence of seizures, many resources have been devoted to studying this disorder, with approaches ranging from clinical studies to phenomenological modeling [253–255]. When it comes to the latter, there is a body of literature which considers epilepsy as a dynamical phenomenon [56, 57, 69, 70, 256], a natural approach given the typical signals observed during a seizure. These signals range from the electroencephalogram (EEG) on the largest spatial scale, to the micro-electrode array (MEA) recordings and other microscopic techniques on the smallest scales. Such techniques show various signatures of seizures, providing insight into its type and origin within a given patient. These behaviors, such as abnormal spikes, epileptiform discharges, spike-and-wave discharges, the ictal wave, etc., provide motivation to adopt models with varying levels of abstraction to capture their essential characteristics [56, 57, 69–79, 250, 257, 258].

Taking the ideas from Section 3.2.7 into account, phenomenological models are often utilized to describe seizures [56, 57, 70, 71, 73–76, 78–80, 249]. The level of connection between these models and real biological functions can vary, but as mentioned before, their

purpose is to capture the essential features, rather than the details. The hope is then to build some understanding that depends on these features, and not the details of the model, which can therefore be translated to understand the dynamics within real brains.

In this paper, we take one such phenomenological model, characterized by a coexistence of a fixed point corresponding to normal brain functioning and a large-amplitude limit cycle characterizing a seizure. We extend it to a spatiotemporal stochastic field theory, motivated by the fact that seizures are a global phenomenon that is highly dependent on the connectivity of neurons [51, 55, 56, 66, 69–71, 76, 77, 79, 80, 257–263]. With this, the usual field-theoretic tools, such as the MSRJD formalism, perturbative calculations and RG, become available, see Chapter 2. We see that the presence of noise and the form of the model makes it reminiscent of an equilibrium model describing a first-order transition in physics. As such, some of the effects observed, such as the nucleation of critical bubbles, which might correspond to the ictal wave [257, 258, 261, 262, 264], are very well understood within the context of phase transitions [2–4, 265]. However, the complex field of the model also undergoes oscillations, which makes it a non-equilibrium model, the consequences of which we attempt to understand.

The focus of this work is on the effects of disorder on the transition in question. Understanding disorder plays an important role in bridging the gap between idealized models and real systems. Within the study of phase transitions, disorder has undergone immense study, both in the context of continuous and first-order transitions, in and out of equilibrium [122, 266, 267]. It has been attacked both using precise techniques, like the replica trick and RG (see refs. above), as well as elegant scaling arguments [268–271]. Clearly, disorder drastically affects whatever transition the system undergoes, especially in low dimensions. We see this numerically, both with the disorder in excitability, which has an equivalent equilibrium theory, and the disorder in frequency, which does not. We characterize both using simulated annealing, and explore the latter through other numerical considerations, like the escape

time statistics and synchronization measures.

The model we are considering should be thought of as a model of transitions from normal brain functioning to a seizure. We show that the disorder in the frequency reduces the propensity of the system to seize, while the disorder in the excitability does the opposite. This can guide possible seizure prevention measures, whereby, for example, the focal region can be disordered to discourage seizures. In fact, stimulation has been considered within the models of the type we consider here as a pathway to stopping seizures [78].

There are, of course, many directions in which this model can be expanded. First, there should be a natural mechanism for terminating a seizure, which is often modeled by having the excitability be a dynamical variable itself [76, 78, 79]. In that case, the system is truly non-equilibrium, even when all of the frequencies are the same throughout space, and could constitute a very interesting problem from a phase transitions perspective. Such systems with the coupling to a slow variable have been and are being considered in the context of modeling neural dynamics [50, 54, 165]. Others can include homeostatic effects, other types of spatial coupling, having different spatial coupling for excitatory and inhibitory interactions, other types of noise, modeling focal regions, and considering various stimulation protocols. We leave these interesting extensions for future work.

To model the dynamics associated with normal brain rhythms as well as seizures, we employ the following stochastic system:

$$\partial_t z(t) = (\lambda + i\omega + b|z(t)|^2 - c|z(t)|^4)z(t) + \sqrt{B}\xi(t) \quad (6.3)$$

As mentioned in the Introduction, such models are often used to study the phenomenology of the transitions to seizures [71, 74, 76, 78, 79], since they capture some key aspects of interest. Namely, there is a bistability between a fixed point ($z = 0$) representing normal brain functioning and a limit cycle ($|z| \neq 0$) representing seizures. The transition between these behaviors can be driven by some change in the parameters leading to a bifurcation,

or by noise, in which case the seizures start randomly. We plot the bifurcation diagram of this model as λ is varied in Fig. 6.1. Here z is a complex variable, the real part of which can loosely represent the local field potentials measured in the brain region or culture. The other parameters can be roughly associated with biological parameters as follows: λ is the overall excitability or excitation/inhibition balance, b is the nonlinear excitability, and c is the refractory property that bounds the system [74]. In order to include other phenomena, such as recovery from a seizure, other dynamical equations can be included, such as the dynamics of λ , which often occur on different time scales [76, 79].

Starting with this model, the next steps often include considering a network in which every node evolves according to Eq. 6.3, but the connectivity has some interesting (often random) structure [76, 79]. However, here we consider a different construction, which can be thought of as a nearest-neighbor network, where the variable $z(t)$ is promoted to a field $z(t, \vec{x})$ in time and two spatial dimensions. The connectivity is then represented by diffusion of the form $D\nabla^2 z$, which represents local connections. Such a choice can be based on the assumption that on the lowest spatial scales the connectivity between neurons falls off exponentially [51, 140, 248]. From the perspective of the renormalization group (RG), this choice is motivated by assuming diffusion as the most relevant spatial coupling, although there may be regimes where higher-order spatial couplings should not be omitted [140, 172, 257, 258]. With this, the model we consider takes the form:

$$\partial_t z(t, \vec{x}) = (\lambda + i\omega + b|z(t, \vec{x})|^2 - c|z(t, \vec{x})|^4)z(t, \vec{x}) + D\nabla^2 z(t, \vec{x}) + \sqrt{B}\xi(t, \vec{x}) \quad (6.4)$$

From now on, we mostly stop explicitly writing out the dependence on $(t, \vec{x})$. In this form, the dynamics can be thought of as an $O(2)$ (or complex) order parameter z undergoing noisy relaxation in a potential landscape as well as rotation:

$$\partial_t z = -\frac{\partial V(|z|^2)}{\partial z^*} + i\omega z + D\nabla^2 z + \sqrt{B}\xi \quad (6.5)$$

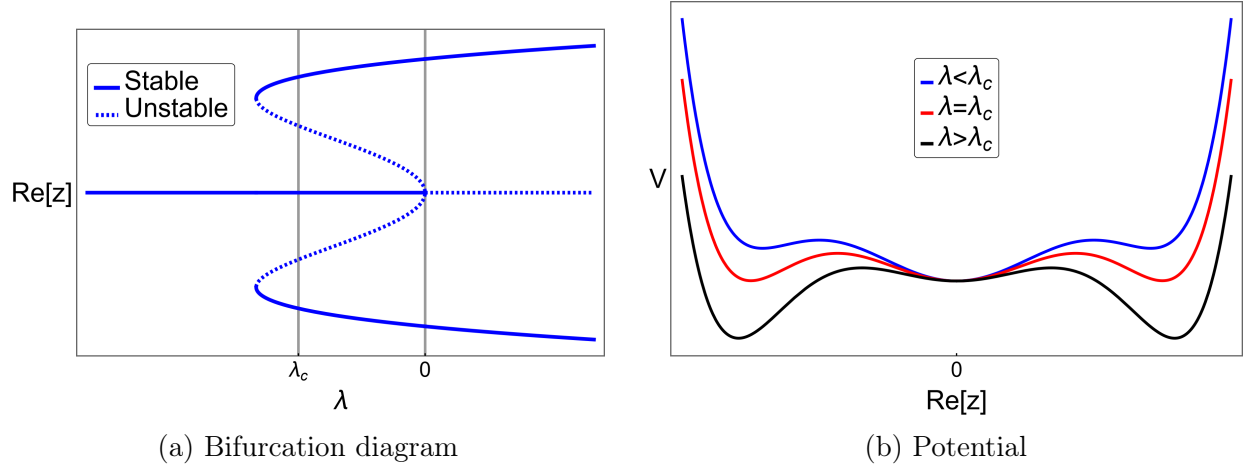


Figure 6.1: a) The bifurcation diagram at the mean field level corresponding to Eq. 6.3. The steady solutions are plotted for $Re(z)$, but exist for any rotation $z \rightarrow ze^{i\theta}$. The limit cycle is born through a saddle-node bifurcation of limit cycles (SNLC), while the stable fixed point is destroyed through a subcritical Hopf bifurcation at $\lambda = 0$. The region of interest is around the first order transition at $\lambda = \lambda_c$. b) Schematic diagram of the potential in Eq. 6.5.

with the potential $V(|z|^2) = -\lambda|z|^2 - \frac{b}{2}|z|^4 + \frac{c}{3}|z|^6$. The rotation can be eliminated by going into the rotating frame $z \rightarrow ze^{i\omega t}$, where the dynamics becomes fully equilibrium in the usual physics sense.

Before introducing the final ingredient of disorder, we briefly review the phenomenology of the disorder-free model. Given the form of the potential $V(|z|^2)$ and in the presence of noise, this model undergoes a first order transition at $\lambda_c = -\frac{3b^2}{16c}$ (at least at the mean field level). This is illustrated in Fig. 6.1, where the local potential minima representing each of the behaviors become global depending on which side of the transition the system is in. If the system starts out in a minimum that is not global, it eventually ends up in the global minimum by the process of bubble nucleation: a critical fluctuation due to noise pushes z in some spatial region to the other well, after which that region expands deterministically. Both first order transitions and nucleation are very well understood within the physics community and are standard topics within the theory of phase transitions [3, 265]. However, we believe that the wave associated with nucleation could correspond to the ictal wave which has been

observed both in patients using MEAs and in numerical simulations [257, 258, 261, 262, 264].

Now, let us consider the seizing state, which corresponds to a stable limit cycle within the model. The amplitude of the limit cycle (which we denote by subscripts s) is given by $|z_s|^2 = \frac{1}{2c} \left(b - d + \sqrt{(b-d)^2 + 4c\lambda} \right)$. Since the fluctuations in the amplitude are damped, we can write out the dynamics in the amplitude-phase decomposition $z = (|z_s| + \delta|z|)e^{i\theta}$ where the dynamics of the phase $\theta(t, \vec{x})$ takes the form:

$$\partial_t \theta = D \nabla^2 \theta + \frac{2D}{|z_s| + \delta|z|} \vec{\nabla} \theta \cdot \vec{\nabla} \delta|z| + \frac{\sqrt{B}}{|z_s| + \delta|z|} \xi_\theta \quad (6.6)$$

We see that θ is the usual undamped Goldstone mode associated with a continuous symmetry [2–4, 266]. Although there is not going to be a global long-range order, as is usually the case in two dimensions [2–4, 206, 266], the system is still going to be in the seizing state. However, synchronization plays an important role in epilepsy [159]. The order parameter that can be chosen to define the synchronization is usually called the Kuramoto index [41, 54, 157]. In the case of a discrete collection of N oscillators, it is defined as $K = \frac{1}{N} \left\langle \left| \sum_n e^{i\theta_n(t)} \right| \right\rangle$, where the sum goes over all oscillators and the average is taken over noise realizations. In our case, we can translate it to a continuum model of volume V_d in d dimensions defining the index K_1 below, which may be calculated numerically. We also define the index K_2 that is simpler for analytic calculations:

$$\begin{aligned} K_1 &= \frac{1}{V_d} \left\langle \left| \int d^d x e^{i\theta(t, \vec{x})} \right| \right\rangle \\ K_2 &= \frac{1}{V_d^2} \left\langle \int d^d x_1 e^{i\theta(t, \vec{x}_1)} \int d^d x_2 e^{-i\theta(t, \vec{x}_2)} \right\rangle \end{aligned} \quad (6.7)$$

Notice that $K_2 \neq K_1^2$, but that both are 1 for complete synchronization $\theta(t, \vec{x}) = \theta$ and 0 when the angles are totally random (equally distributed between 0 and 2π): $\int d^d r e^{i\theta(t, \vec{x})} = 0$.

Assuming large enough times t such that the system is seizing and has settled into the equilibrium distribution of θ , we can use the properties of Gaussian variables to get for our

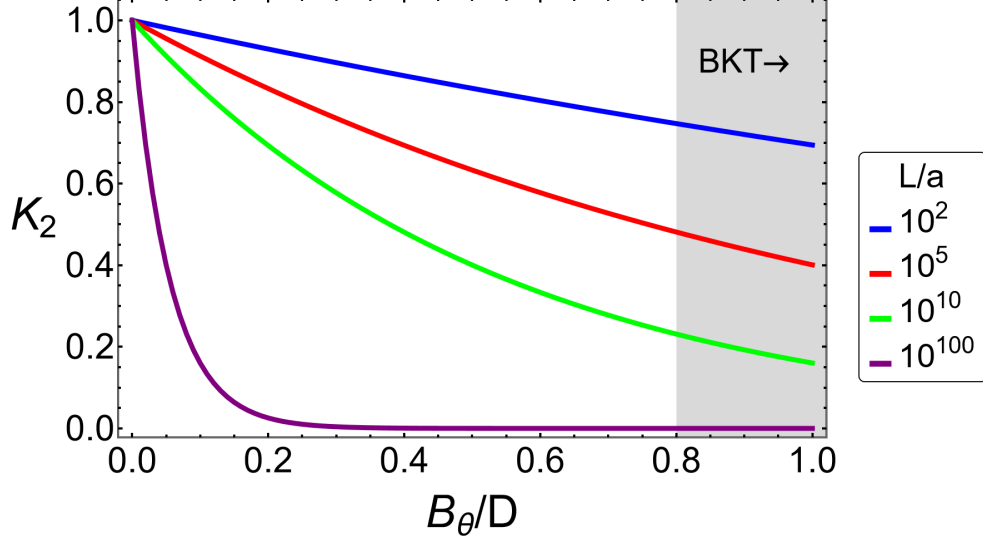


Figure 6.2: Numerical evaluation of the Kuramoto parameter K_2 defined in Eq. 6.7 assuming the correlations defined by Eq. 6.9. At the onset of the BKT transition, the correlations have a different structure, and the Kuramoto parameter is reduced. The location (loosely) of that region is shaded.

model:

$$\begin{aligned}
K_2 &= \frac{1}{V_d^2} \int d^d x_1 \int d^d x_2 e^{\langle \theta(t, \vec{x}_1) \theta(t, \vec{x}_2) \rangle - \langle \theta(t, \vec{x}_1)^2 \rangle} \\
&= \frac{1}{V_d} \int d^d \delta x e^{G_{\theta\theta}^K(t=0, \delta \vec{x}) - G_{\theta\theta}^K(t=0, \vec{x}=0)}
\end{aligned} \tag{6.8}$$

Here $G_{\theta\theta}^K$ is the correlation function for θ , which can be computed using, for example, the MSRJD formalism (see Sec. 2.2.2). In $d = 2$, the expression in the exponential for a lattice of size L and step size a is given by:

$$G_{\theta\theta}^K(t=0, \delta \vec{x}) - G_{\theta\theta}^K(t=0, \vec{x}=0) = \frac{B_\theta}{4\pi D} \int_{\frac{2\pi}{L}}^{\frac{2\pi}{a}} dk \frac{\cos(k|\delta x|) - 1}{k} \tag{6.9}$$

where $B_\theta = B/|z_s|^2$. Note that this is only valid below the BKT transition noise level $\frac{B_\theta}{D} \sim O(1)$ [206, 272]. Above the transition, the correlations decay exponentially [4], destroying the synchronization even further. Calculating this and the K_2 integrals numerically produces the dependence depicted in Fig. 6.2 for different system sizes L/a . As can be

seen, there is no first-order or continuous transition, only a drop-off in synchronization as the noise is increased. Although for the infinite system $L \rightarrow \infty$ we get $K_2 \rightarrow 0$ for any nonzero amount of noise, we see that a finite system can remain well synchronized. This fact is important not only for any real measurements, since they would always be carried out on a finite system, but also for the simulations we perform below, as well as the transitions to seizures.

Finally, there is a wide variability in neuronal behavior even within the same brain region or sample [50]. This can be represented by spatial variations of any of the parameters in the above model. For the sake of simplicity and again motivated by RG (assuming we coarse-grain on scales larger than the correlations of the parameters), we assume quenched (independent of time) spatial disorder in λ and ω of the form:

$$\begin{aligned}\overline{\lambda(\vec{x})} &= \lambda \quad , \quad \overline{\lambda(\vec{x})\lambda(\vec{x}')} = \Delta_\lambda \delta^2(\vec{x} - \vec{x}') \\ \overline{\omega(\vec{x})} &= 0 \quad , \quad \overline{\omega(\vec{x})\omega(\vec{x}')} = \Delta_\omega \delta^2(\vec{x} - \vec{x}')\end{aligned}\tag{6.10}$$

where the over-bar represents the average over the corresponding Gaussian distributions for λ and ω :

$$\begin{aligned}P[\lambda] &= \frac{1}{N_\lambda} \exp \left[- \int d^2x \frac{1}{2\Delta_\lambda} (\lambda(\vec{x}) - \lambda)^2 \right] \\ P[\omega] &= \frac{1}{N_\omega} \exp \left[- \int d^2x \frac{1}{2\Delta_\omega} \omega(\vec{x})^2 \right]\end{aligned}\tag{6.11}$$

Here N_λ and N_ω are normalization constants. The disorder in $\lambda(\vec{x})$ can be thought of as a "random-bond" problem in equilibrium statistical mechanics. As mentioned in the Introduction, disorder in equilibrium systems is a huge topic of research within the physics community, and has been widely explored in the context of both continuous and first order transitions [266, 267]. In particular, for low dimensions, a first-order transition can be smoothed out by disorder [269], which we also find numerically below. This is an important

effect, especially since the approximate value of λ where the (smoothed-out) transition occurs shifts, indicating that the disorder in λ promotes seizures. On the other hand, since the disorder in λ can be understood within equilibrium theories, we focus mainly on the disorder in frequency ω , where it is not obvious that the usual arguments (see refs. above) apply.

6.2.2 Results in Epileptic Dynamics

We simulate the dynamics of Eq. 6.4 with disorder in λ or ω on a lattice with lattice size 1. The time step is chosen to be $\Delta t = 0.01$. The values of parameters that are fixed are chosen to be $b = 2$, $c = 1$, $D = 1$. We use the lowest order lattice derivative to simulate diffusive coupling, while time evolution is performed in the Ito discretization, using a first order Euler scheme. Due to the presence of spatial derivatives and noise, low order methods should have enough precision [244]. We use periodic boundary conditions, and the simulations are performed using the TensorFlow QuantFinance package in Python.

First, we plot some visual examples of the nucleating bubbles in the dynamics of Eq. 6.4 in Fig. 6.3. As shown there, in the absence of disorder, bubbles appear and spread throughout the system, while the phase within the bubbles is aligned. Once the entire system is in the seizing state, the phases reconfigure and reach an equilibrium distribution with correlations as in Eq. 6.9. In the presence of disorder in the excitability λ , many more bubbles appear in the regions of high excitability. Their evolution is then similar to the disorder-less system, where after the entire system is consumed, the phases reconfigure. However, in the presence of the frequency disorder, even after the system is entirely in the seizing phase, the phases don't synchronize as much, and different regions form. We explore these behaviors in more detail below.

Next, we perform simulated annealing where the noise evolves as follows:

$$B(t) = B \left(1 + f e^{-\frac{t}{\tau}} \right) \quad (6.12)$$

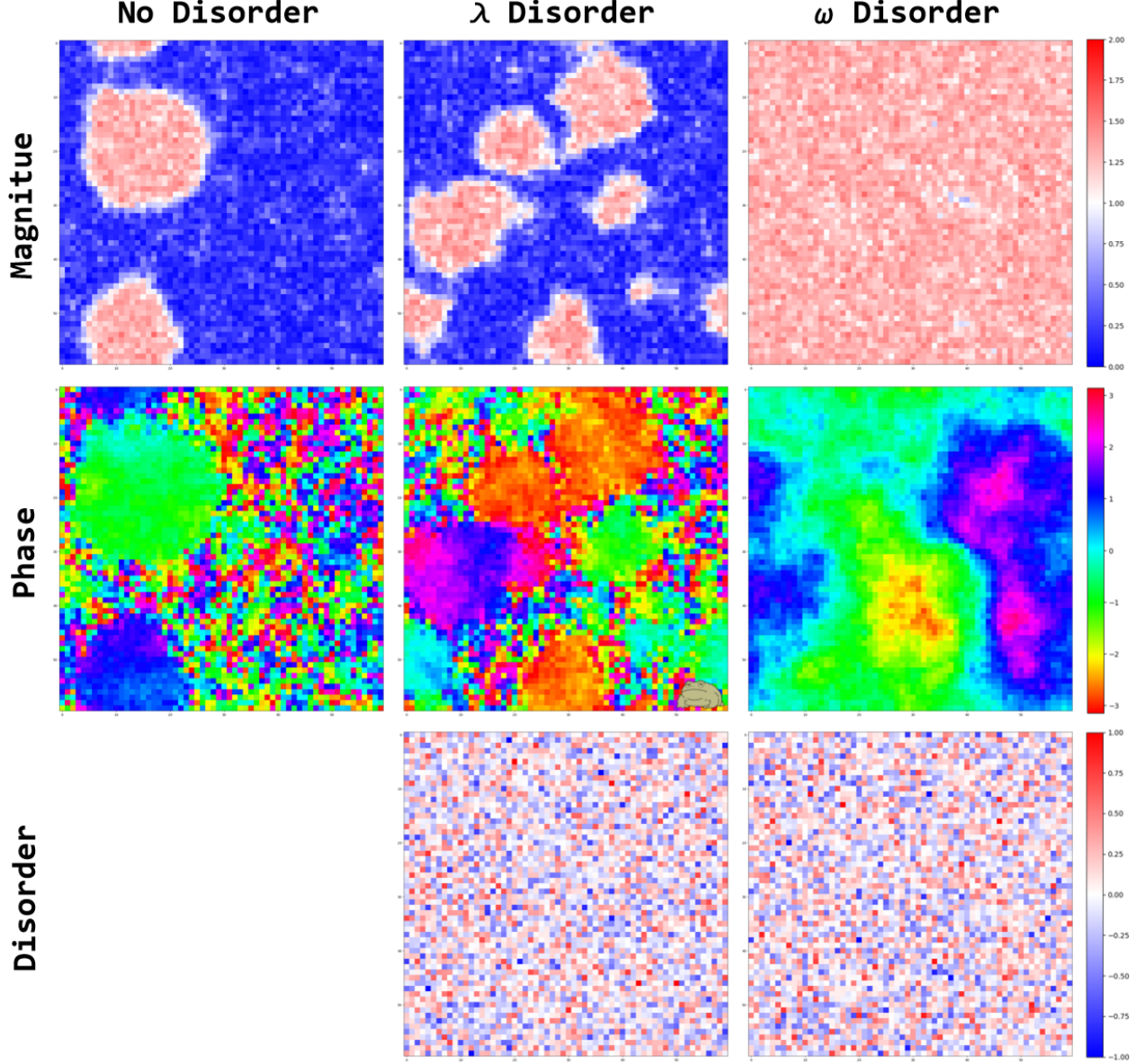


Figure 6.3: Time snaps of the evolution according to Eq. 6.4. Left column has no disorder, the nucleating bubbles appear in random regions of the system and spread virtually deterministically across the system. The phases within each one are very aligned. The middle column has disorder in λ plotted on the bottom. The bubbles appear in the regions of high excitability. The right column corresponds to disorder in ω plotted on the bottom. Even after the system is fully in the seizing state, the phases don't align and form different regions.

The noise strength is chosen to be $B = 0.02$, while $f = 20$ and $\tau = 40$. Other annealing protocols can be chosen. The purpose of such noise dynamics is similar to equilibrium annealing simulations [244]. We assume that regardless of initial conditions, large noise at

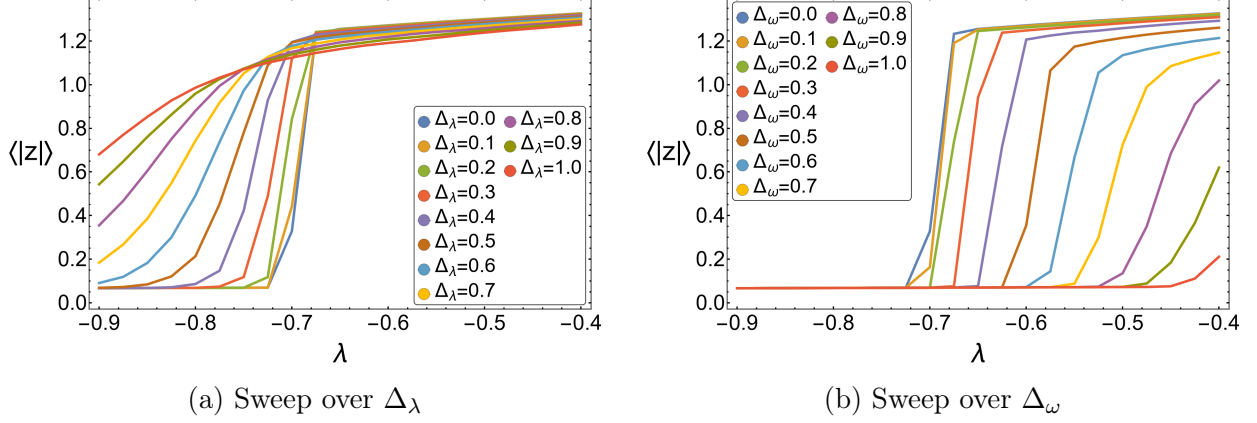


Figure 6.4: The results for simulated annealing according to protocol in Eq. 6.12, plotting the average magnitude of z versus the excitability λ . The simulation values are $T = 300$ for the total time of the simulation and performed over 100 samples for each point on the plot. The averages are taken after $t = 250$. The other parameters $b = 2$, $c = 1$, $D = 1$ (including the lattice spacing), while the lattice size is 40×40 . a) Results for the disorder in λ : first-order transition is smoothed out, and the system is more prone to seizures. b) Results for the disorder in ω : first-order transition is smoothed out, and the system is less prone to seizures.

the beginning of the simulation allow the system to explore its state space and that for longer times the system ends in one of its most likely states. The results of this simulation are plotted in Fig. 6.4. Two important features are observed. First, the sharp first-order transition that occurs in the absence of disorders $\Delta_\lambda = 0 = \Delta_\omega$ is smoothed out. This is a known effect for the case of equilibrium systems [269] (or for the disorder in λ in our case), but it is also observed for the disorder in ω , which is a non-equilibrium feature. And second, perhaps more important, is the direction of the shift of the first-order transition as the respective disorder is increased, which we discuss next.

For the case of increasing Δ_λ , we see that the system is more prone to seize, or to transition to the limit cycle. Intuitively, this is expected. Some regions on the lattice have much higher values of λ , thus being more prone to nucleation. And once the nucleation starts at those points, the bubble then spreads deterministically throughout the system. This is the case because even if a lattice point only has 1 neighbor that nucleated, this neighbor effectively adds an external field of $\frac{D}{a^2} z_s$, which can be checked to be enough to push that

point to a seizing state even if it is itself not very excitable (has lower than average λ). Of course, this effect depends on the size of the coupling. For weak enough coupling D we expect that there might be a pinning transition, where even if there is nucleation, the bubble gets stopped by regions of low excitability.

On the other hand, the disorder in the frequency of oscillation Δ_ω tends to make the system less prone to seize. There is no equivalent energy function that one can invoke to understand what happens with the transition here, so one has to understand the dynamics. Qualitatively, it can be understood as follows. With the same decomposition we used before $z = |z|e^{i\theta}$, we can look at the dynamics of $|z|$ (which is what determines which state the system is in). We get:

$$\partial_t |z| = \lambda |z| + f(|z|) - D |z| \vec{\nabla} \theta \cdot \vec{\nabla} \theta + \xi |z| , \quad (6.13)$$

where $f(|z|)$ is some function that depends only on the magnitude $|z|$ and contains all the non-linear terms. From the linearized dynamics of θ :

$$\partial_t \theta = \omega + D \nabla^2 \theta + \sqrt{B_\theta} \xi_\theta , \quad (6.14)$$

where ω is the disordered frequency, it can be shown that averaging over the disorder gives:

$$\overline{\vec{\nabla} \theta \cdot \vec{\nabla} \theta} = \vec{\nabla} \theta \cdot \vec{\nabla} \theta \Big|_{\Delta_\omega=0} + \int d^2 k \frac{2\Delta_\omega}{D^2 k^2} . \quad (6.15)$$

Thus, we see that the effect of disorder on the dynamics of $|z|$ is to shift $\lambda \rightarrow \lambda - c\Delta_\omega$ for some positive coefficient c . Therefore, λ has to be higher to reach the critical point, in agreement with the results in Fig. 6.4.

The effect of the rounding of the transition due to disorder is a more subtle problem. In equilibrium problems, there are well established results demonstrating this effect in low dimensions, which should explain the result observed in Fig. 6.4 a) [273–275]. However, a

similar effect is observed for the disorder in the non-equilibrium parameter ω . Explaining this in a way similar to [273, 274] would require the notion of a free energy, which is absent in the presence of ω . One can attempt to make similar arguments with the MSRJD action, which is an exciting direction we leave for future work.

To further demonstrate the effect of disorder, we also consider the level of synchronization of the seizing state, since it is clearly a major feature. We saw above that for a finite system and at low noise, the synchronization of this system should remain high in the seizing phase. In fact, one can observe that the system is virtually not synchronized around the normal functioning fixed point, but the region inside the nucleating bubble is. In Fig. 6.6, we simulate the system at a constant noise level $B = 0.2$ and a constant $\lambda = -0.4$ (past the first-order transition) for various Δ_ω and detect nucleations. In Fig. 6.6 a), we plot the Kuramoto index K_1 defined in Eq. 6.7, but calculated only over the nucleated region, which we assign to be the region with $|z| \geq 1$. It can be seen that this nucleated region has to be quite synchronized $K_1 \approx 0.6$. Once the bubble has consumed the entire system, the fields reconfigure to reach a steady level of synchronization possible within the system. There we see that it decreases with increasing disorder in the frequency. However, this shows an increase in Δ_ω should lead to a reduction in the rate of nucleation, as observed in Fig. 6.4: the bubble has to be somewhat synchronized for nucleation to occur, which is discouraged as the disorder increases.

One can check this also by computing the average bubble nucleation time τ_n , which we do for the same set of simulations in Fig. 6.5, along with the time it takes for 95% of the lattice to be seizing. We see that not only does it take slightly longer for the nucleation to occur, but the wave also takes longer to consume the entire system. In fact, for large enough disorder Δ_ω , it does not even consume the entire lattice, always leaving points that are not seizing. All of these results are consistent with each other and align with the intuition that a seizing state requires high synchronization. It is prudent to note that the low-amplitude

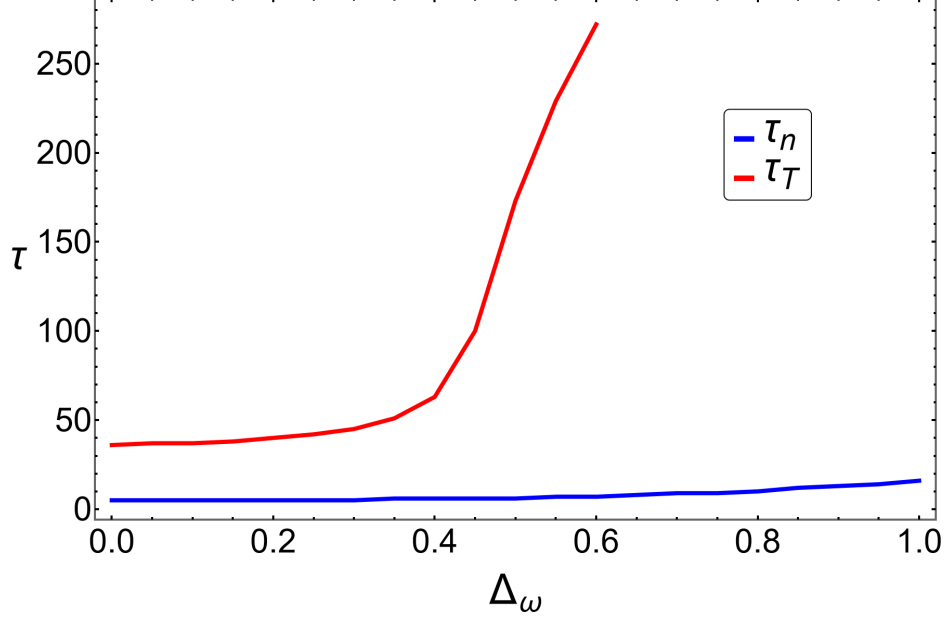


Figure 6.5: The average time to nucleation τ_n and the average time to the seizure consuming more than 95% of the lattice τ_T versus disorder in the frequency Δ_ω . Clearly, both time scales increase, showing the discouragement for the transition. τ_T increases drastically, to the point where for large enough disorder the samples of our simulation were never 95% or more in the seizing phase.

(normal brain functioning) phase does not require any synchronization to be stable, even with disorder.

Finally, we simulate a case that approximately models focal epilepsy. We start in the low amplitude phase across the entire 60×60 lattice. We assume that the entire lattice has a value of λ which puts the system in the normal-functioning state, but close enough to the first order transition. For illustration, we choose $\lambda = -0.7$. Finally, we make a small region much more excitable, setting $\lambda_{focal} = 0.3$ inside. On the deterministic level, the focal region excites the system into the seizing state, with a wave starting at the focus and propagating through the lattice. We emphasize the parallel this phenomenon has with observed ictal waves in patients [257, 258, 261, 262, 264]. Realistically, some other mechanism then terminates the seizure at a later time [76, 79]. We then disorder the frequency only inside the focal region, and observe the resulting Brain Network Ictogenicity (BNI) index [76, 79]. We define BNI

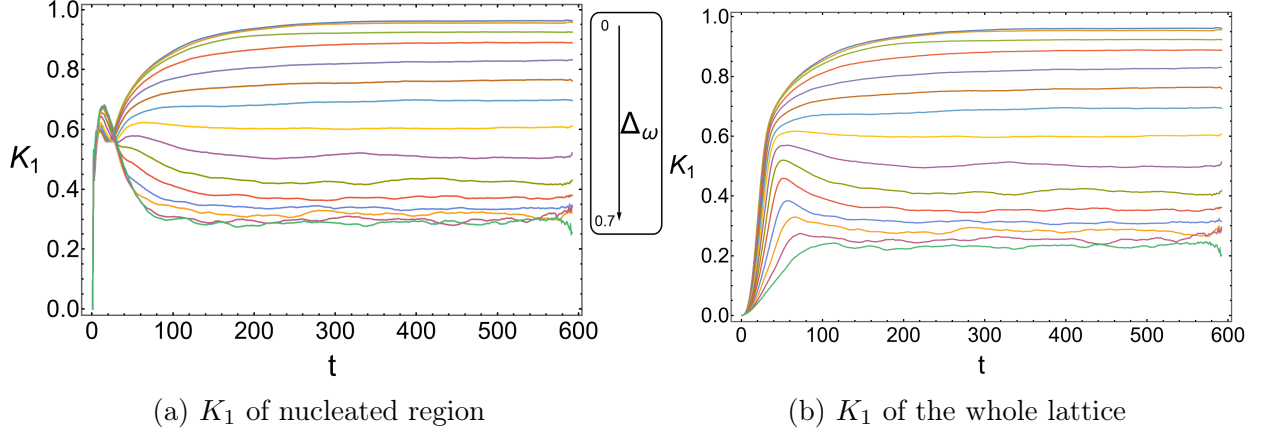
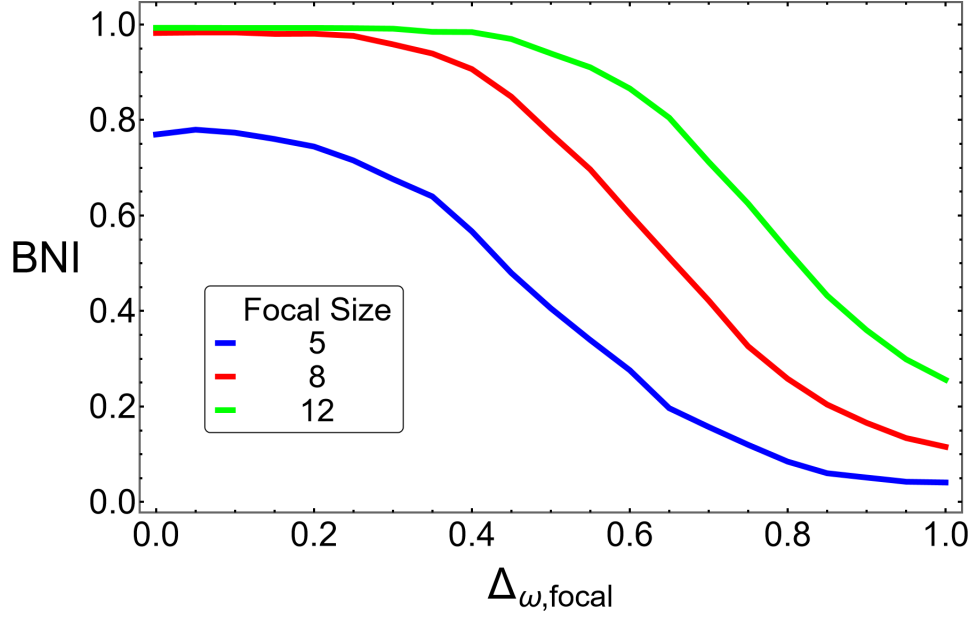


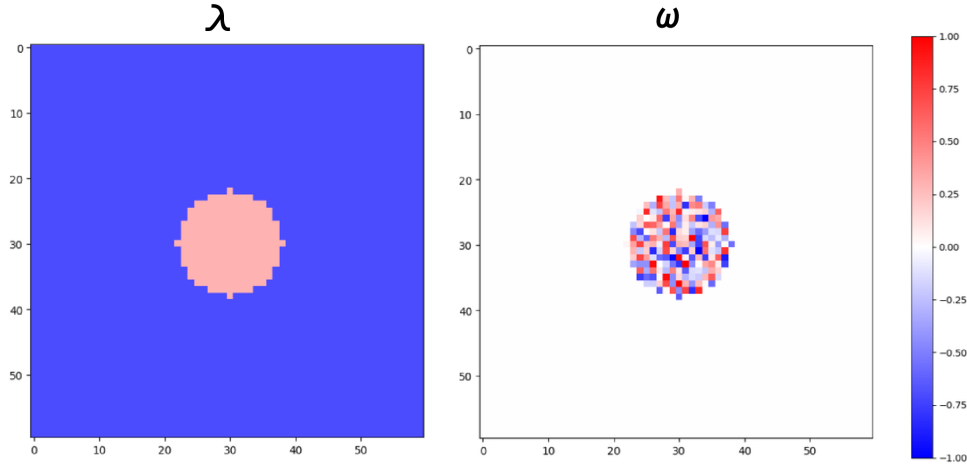
Figure 6.6: The numerical evaluation of the Kuramoto indices K_1 for a) only the region that nucleated and b) the entire lattice. The simulation values are $T = 600$ for the total time of the simulation and it is performed over 500 samples for each value of the disorder. The disorder values are between $\Delta\omega = 0$ and $\Delta\omega = 0.7$ in steps of 0.05. Higher values don't have enough nucleations to collect reliable data. The other parameters are $\lambda = -0.4$, $b = 2$, $c = 1$, $D = 1$ (including the lattice spacing) and $B = 0.2$, while the lattice size is 40×40 . The times are shifted so that $t = 0$ corresponds to the beginning of nucleation, which is detected as the first time where some point has $|z| \geq 1$.

to simply be the total fraction of lattice points in time and space that are in the seizing state $|z| > 0.8$ after an initial transient, which we set to be 400 simulation steps. The initial transient is ignored since our model does not include a mechanism to terminate seizures, and we are interested in the final state the system reaches under moderate noise $B = 0.2$. The setup and results are illustrated in Fig. 6.7. The total number of noise realizations is 200 for each point.

Clearly, in the absence of disorder, the system eventually transitions to the seizing state. However, as the disorder is increased, the BNI decreases monotonically, with a sharper inflection past a certain magnitude of disorder. This is reproduced for several focal region sizes. This brings forward the question of using stimulation-based protocols to disorder the natural frequency of the focus specifically. Electrical and magnetic stimulation is a huge topic of research, where different protocols can induce homeostatic changes in neuronal properties even after the stimulus has ceased [276–278]. Stimulation is also being intensively



(a) BNI of the system at different levels of focal frequency disorder



(b) Example configuration of excitability and frequency with a focus

Figure 6.7: The numerical evaluation of the BNI of the system past an initial transient at different sizes of the focal region versus the frequency disorder of the focal region. Increasing the disorder tends to reduce the total amount of time a system spends in the seizing state.

investigated as a treatment option for epilepsy [279–281]. Of course, these results are partly a product of the simplicity of the model. One should consider whether random disordering of the focal region is the optimal strategy, or if it is possible to design a deterministic modulation that produces a stronger reduction in BNI. However, in realistic settings, neither the shape of the focal region nor the neural connectivity around it are regular. Therefore, the basic

principle of random disorder of the natural frequency instead of designing a deterministic stimulation protocol should be more appropriate.

6.2.3 *Outlook on Epilepsy Models*

We have extended a phenomenological model of seizures introduced in the epilepsy modeling literature to a field-theoretical description by introducing an underlying spatial continuum. The simplest spatial coupling in the form of diffusion is used to model neuronal connectivity, motivated by the known distributions of the density of connections versus the distance between neurons. Although the model is highly abstract, its parameters do connect with biological functions, and the dynamics reflect the features of seizures. More concretely, a seizure is modeled as a large-amplitude attracting limit cycle, the transition to which is activated through noise. The final ingredient of variability of individual neural features, such as excitability and natural frequency, is modeled as the presence of quenched disorder. The effects of this spatial randomness on the tendency of this model to seize is the central topic of investigation.

We showed that, as is the case in epileptic seizures, synchronization of neural activity plays a central role. In fact, the propensity of the system to be in the seizing state depends on its ability to synchronize. We show numerically and provide qualitative analytical justification that the disorder in the natural frequency of oscillation across the spatial continuum makes the system less prone to seize. This is exhibited by the upward shift of the excitability necessary to invoke seizures. On the other hand, the disorder in excitability makes the system less stable and promotes seizures. This is also quite intuitive, since this disorder allows highly excitable regions to transition, and then entrain the surrounding areas. We also provide some Kuramoto index results, showing that the synchronization is drastically reduced by the presence of disorder. This is, again, in accordance to both the abstract Kuramoto models and the clinical evidence of the importance of synchronization in sustaining seizure

activity.

From a physics perspective, the system resembles a dynamical model of a first order transition. However, the sustained oscillations, especially in the presence of frequency disorder, make this a non-equilibrium system. This means that although the result of excitability disorder – namely the rounding of the transition and the shift of the critical point – can be explained by equilibrium Harris, Imry, Ma arguments, the same cannot be said for the disorder in frequency. This points to an exciting problem of non-equilibrium statistical physics, i.e. how to address the presence of disorder in a non-equilibrium parameter when there is no free energy functional? We leave this direction for future work.

Finally, we explore a case where a finite spatial region has a much higher excitability compared to the rest of the lattice. This draws a parallel to focal epilepsy. We demonstrate numerically that disordering the frequencies just in the focal region can lead to a drastic reduction in the propensity of the system to seize. Although this model is an idealized construction, we believe the general principle of intentionally disordering the frequencies of neural activity of the focal region should prove useful in clinical settings. In fact, stimulation based protocols are heavily investigated in the context of epilepsy. We therefore propose to explore this idea in the context of biologically realistic models.

As mentioned in the Introduction, there are a number of possible extensions to this model which should add other components known to be present in neural dynamics. These can include some mechanism for terminating the seizures, like the dynamics of excitability λ , some other non-triviality in the frequency aside from disorder, homeostatic effects, different spatial connectivity structures for excitation and inhibition, etc. Another interesting question is varying the correlation length of the disorder, especially relative to the correlation length of the system. This should be motivated by biological evidence on how correlated neural spiking is versus how correlated the biological parameters themselves are. Moreover, it is also natural to investigate the disorder in the structure of the spatial coupling itself. This,

of course, parallels the studies from which the model was borrowed, where the system is put on the network. However, there should be a natural gradient between a continuous theory and a random network, and it is not, a priori, evident that one should abandon the former in favor of the latter. We leave these exciting directions for future work.

We thank Stephan Ihle and Wim Van Drongelen for many insightful conversations that helped us maintain connection with experimental and clinical data.

6.3 Outlook on Neural Dynamics

The discussions in Sec. 3.2.7 and this chapter barely scratched the vast field of computational neuroscience and the applications of dynamical systems to model neural behavior. We do not attempt to address this in this section, but rather focus on ideas and problems we find of interest.

As mentioned in Sec. 6.2, modeling neural connectivity as a continuum is certainly not enough to capture the variety of neural behaviors. Even in abstracted continuum models, there is analytic evidence that long-range connectivity is relevant [140]. However, the evidence of dense local connections as described in the previous sections is also present. Thus, we need to ask a question: what models strike the perfect balance between analytical tractability and biological realism? Unfortunately, one can easily see that the change in connectivity structure is likely going to harshly change the qualitative behavior a model exhibits. For example, if one uses an Ising model to describe neural behavior, as is often done [149], the difference between a two-dimensional and an all-to-all connected (mean field) theory is drastic. The effect of randomness is also relevant, as we saw in Sec. 6.2. A reasonable proposal to investigate this problem might be to take a two-dimensional Ising model but also add connections with probability that decays with distance. As that probability distribution is changed, a new kind of phase transition might occur, which might be relevant for actual neural behavior.

A key feature of neural behavior is the ability of neurons to adapt and for the network to change its connectivity structure. Such neural plasticity usually occurs on much longer time scales than neural spiking. Therefore, a faithful theory of neural behavior that includes plasticity or learning, however abstract, runs into the realm of time-scale separated problems. However, clearly, single spikes do not affect long-term behavior of the brain. So, we now require something akin to renormalization group for neurons.

To address this point in more detail, RG for neurons is now one of the key questions of theoretical neuroscience. However, aside from the mathematical challenge of extracting the RG flow from neural models, there are conceptual questions about what such a flow yields. By construction, any perturbative RG step loses information (the high momentum modes are integrated out), and even if we are able to extract the long-range behavior of the brain, what does it actually mean? We clearly will not be able to see the infinite complexity of behaviors, memory, and computation if we look at the brain at large scales. However, some features should be observable – a seizing brain and a normal brain are different even at large scales. This is an open problem, and we suspect the way forward is to simply take a leap and start performing coarse-graining and real space RG calculations, closely guided by experiments. Neural cultures can now be grown with incredible control, forming virtually any lattice structure or connectivity one desires. We, as theorists, should utilize these tools.

Perhaps not surprisingly, artificial neural networks are now also a fruitful avenue for researchers interested in dynamical systems. Although modern artificial intelligence is based on other principles, dynamical systems can elucidate memory formation and extraction [149, 179]. Often some principles of actual neurons are applied to artificial networks, such as gating, which mimics ionic currents [178]. Even non-reciprocity has found its way into artificial neural networks [180].

CHAPTER 7

CONCLUSION

Collective phenomena truly are a wonder of nature. The fact that units obeying even simple rules can exhibit complex many-body behaviors allows for phenomena underlying many aspects of our lives, from magnetism, to liquid crystals, to flocking, to the brain. Establishing the theory of universality and classification of non-equilibrium phases of matter is therefore a worthy goal. It would not only provide explanations for an amazing array of phenomena, fulfilling the deepest scientific desire, but would also prove exceptionally useful for physics, engineering, chemistry, climate science, neuroscience, and many others. We hope that the work presented in this dissertation offers a small step towards that goal.

The outlook on specific topics presented at the ends of Chapters 4, 5, and 6 by no means exhausts the virtually unending span of problems one can pursue in relation to non-equilibrium phenomena. They merely reflect the interests and questions on the authors' minds at the time of writing, bound by our own experience and expertise. As mentioned before, we partly pose them as challenges to both ourselves and the reader.

It is an exciting time to be a statistical physicist. Experimental advances have opened the door to problems and phenomena which only a few decades ago might have been viewed as no more than abstract curiosities. The topics explored here – non-reciprocal phenomena, oscillating phases of matter, and dynamical systems applied to neuroscience, are enabled by such advances. As described in the text, they may be chemically engineered particles, driven-dissipative condensates, newly emerging large data sets of the brain, and many others. As such, we hope that the field-theoretic approach underpinning this dissertation can be of use to the scientific community.

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CHAPTER 8

APPENDIX

8.1 Appendix A: Dynamical Field Theory Details

8.1.1 Causality Structure of Response Functions

We demonstrate here the causality structure of the MSRJD response function at the linear level. This is useful in particular because perturbation series depend on the Gaussian action. The full causality structure can also be demonstrated, see [2, 89].

We saw in Eq. 2.52 that:

$$G^R(\omega, \vec{k}) = \frac{1}{\omega - im - iDk^2} \quad (8.1)$$

Taking the Fourier transform, we get:

$$G(t, \vec{k}) = \int \frac{d\omega}{2\pi} G^R(\omega, \vec{k}) e^{i\omega t} = \int \frac{d\omega}{2\pi} \frac{e^{i\omega t}}{\omega - im - iDk^2} \quad (8.2)$$

We can extend this integral to the complex plane and use the residue theorem. For $t > 0$, we close the contour in the upper complex plane. The pole at $\omega_p = im + iDk^2$ is also in the upper half plane, and we therefore get:

$$G^R(t, \vec{k}) = 2\pi i \frac{e^{i\omega_p t}}{2\pi} = ie^{-(m+Dk^2)t} \quad , \quad t > 0 \quad (8.3)$$

We see the response decays with time, as expected. However, if $t < 0$, we have to close the contour in the lower half plane which does not contain any poles. It can also be shown that in the Ito discretization the response function is zero when $t = 0$, see [89, 105]. We therefore get that $G^R(t) \neq 0$ only when $t > 0$, demonstrating causality. A similar result for $t < 0$ can be shown for the advanced Green's function $G^A(t)$.

8.2 Appendix B: Linear Non-Reciprocal Systems

8.2.1 MSRJD Computations

The most general linear non-reciprocal system for two scalar field species is given by:

$$\begin{aligned}\partial_t \phi_1 &= -m_1 \phi_1 + j_{12} \phi_2 + D_1 \nabla^2 \phi_1 + D_{12} \nabla^2 \phi_2 + \xi_1 \\ \partial_t \phi_2 &= -m_2 \phi_2 + j_{21} \phi_1 + D_2 \nabla^2 \phi_2 + D_{21} \nabla^2 \phi_1 + \xi_2\end{aligned}\tag{8.4}$$

The MSRJD action associated to these equations is given by:

$$\begin{aligned}S = \int dt d^d x & \left[-i \bar{\phi}_1 \left(\partial_t \phi_1 + m_1 \phi_1 - j_{12} \phi_2 - D_1 \nabla^2 \phi_1 - D_{12} \nabla^2 \phi_2 \right) + \right. \\ & -i \bar{\phi}_2 \left(\partial_t \phi_2 + m_2 \phi_2 - j_{21} \phi_1 - D_2 \nabla^2 \phi_2 - D_{21} \nabla^2 \phi_1 \right) \\ & \left. + \frac{B_1}{2} \bar{\phi}_1^2 + \frac{B_2}{2} \bar{\phi}_2^2 + B_{12} \bar{\phi}_1 \bar{\phi}_2 \right]\end{aligned}\tag{8.5}$$

As usual, we define the Fourier transforms:

$$v(t, \vec{x}) = \int \frac{d\omega}{2\pi} \frac{d^d k}{(2\pi)^d} \hat{v}(\omega, \vec{k}) e^{i\omega t - i\vec{k} \cdot \vec{x}}\tag{8.6}$$

Where v is any of our variables $\bar{\phi}_i, \phi_i$. After symmetrizing S through integration by parts, we get:

$$S = \int \frac{d\omega_1 d\omega_2 d^d k_1 d^d k_2}{(2\pi)^{2d+2}} \frac{1}{2} (2\pi)^{d+1} \delta(\omega_1 + \omega_2) \delta^d(\vec{k}_1 + \vec{k}_2) \phi^T(\omega_1, \vec{k}_1) G_0^{-1} \phi(\omega_2, \vec{k}_2)\tag{8.7}$$

Where:

$$\phi(\omega, \vec{k}) = \begin{pmatrix} \delta \hat{\phi}_1(\omega, \vec{k}) & \hat{\phi}_1(\omega, \vec{k}) & \hat{\phi}_2(\omega, \vec{k}) & \hat{\phi}_2(\omega, \vec{k}) \end{pmatrix}^T\tag{8.8}$$

Ignoring normalizing constants up front, and calling $\omega_2 = \omega$ with the understanding that

all the correlation functions have the form $\langle \phi_i(\omega, \vec{k}) \phi_j(\omega', \vec{k}') \rangle \sim [G_0(\omega, \vec{k})]_{ij} \delta(\omega + \omega') \delta^d(\vec{k} + \vec{k}')$, we see that the inverse correlation function is given by:

$$G_0^{-1} \sim \begin{pmatrix} 0 & -\omega - im_1 - iDk^2 & 0 & ij_{21} - iD_{21}k^2 \\ \omega - im_1 - iDk^2 & B_1 & ij_{12} - iD_{12}k^2 & B_{12} \\ 0 & ij_{12} - iD_{12}k^2 & 0 & -\omega - im_2 - iDk^2 \\ ij_{21} - iD_{12}k^2 & B_{12} & \omega - im_2 - iDk^2 & B_2 \end{pmatrix} \quad (8.9)$$

In the case $D_{12} = D_{21} = 0$ and $B_{12} = 0$ considered in the main text, the correlation functions are:

$$\begin{aligned} \langle \phi_1(\omega, \vec{k}) \phi_1(-\omega, -\vec{k}) \rangle &= \frac{j_{12}^2 B_2 + B_1(\omega^2 + (m_2 + D_2 k^2)^2)}{(j_{12} j_{21} + (\omega - im_1 - iD_1 k^2)(\omega - im_2 - iD_2 k^2))(j_{12} j_{21} + (\omega + im_1 + iD_1 k^2)(\omega + im_2 + iD_2 k^2))} \\ \langle \phi_1(\omega, \vec{k}) \phi_2(-\omega, -\vec{k}) \rangle &= \frac{j_{12} B_2(-i\omega + m_1 + D_1 k^2) + j_{21} B_1(i\omega + m_2 + D_2 k^2)}{(j_{12} j_{21} + (\omega - im_1 - iD_1 k^2)(\omega - im_2 - iD_2 k^2))(j_{12} j_{21} + (\omega + im_1 + iD_1 k^2)(\omega + im_2 + iD_2 k^2))} \\ \langle \phi_2(\omega, \vec{k}) \phi_2(-\omega, -\vec{k}) \rangle &= \frac{j_{21}^2 B_1 + B_2(\omega^2 + (m_1 + D_1 k^2)^2)}{(j_{12} j_{21} + (\omega - im_1 - iD_1 k^2)(\omega - im_2 - iD_2 k^2))(j_{12} j_{21} + (\omega + im_1 + iD_1 k^2)(\omega + im_2 + iD_2 k^2))} \end{aligned} \quad (8.10)$$

Of interest are the equal time correlation functions, which comes from integrating these over ω :

$$G_{ij}^K(t = 0, \vec{k}) = \int \frac{d\omega}{2\pi} G_{ij}^K(\omega, \vec{k}) , \quad (8.11)$$

8.2.2 Phase Space Trajectories

Here we present the phase plots of the deterministic part of the overdamped system in Eq. 3.58, without noise. For $\Delta > 0$, the typical non-critical and critical phase plots are displayed

in Fig. 8.1. When the system is not critical, the fields decay towards the fixed point at $\phi_1 = \phi_2 = 0$. At criticality, some direction becomes "flat", see 8.1b. Fig. 8.2 shows the phase plots for $\Delta = 0$. The non-critical and general critical behaviors are similar to the $\Delta > 0$ case. However, at the CEP both fields experience a force in some direction, and that direction flips across the fixed point, hence how $\phi_2 = 0$ line is also "flat". Finally, Fig. 8.3 shows the typical phase plots for $\Delta < 0$. In 8.3d, the "flat" direction becomes the radial direction, and the trajectories are closed orbits.

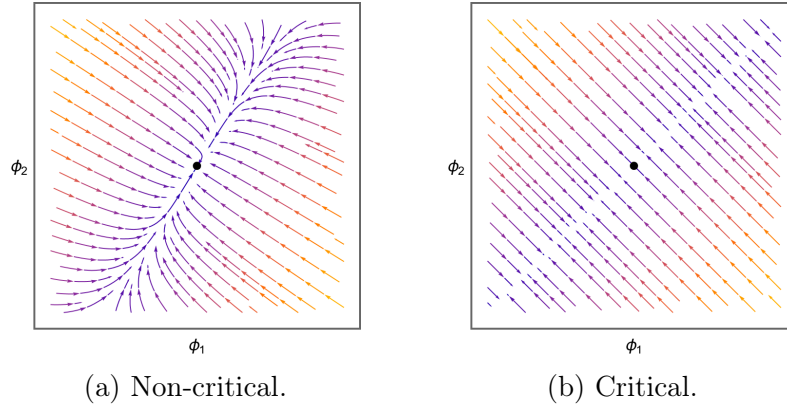


Figure 8.1: Typical phase plots for the overdamped system with $\Delta > 0$. The dot corresponds to the fixed point.

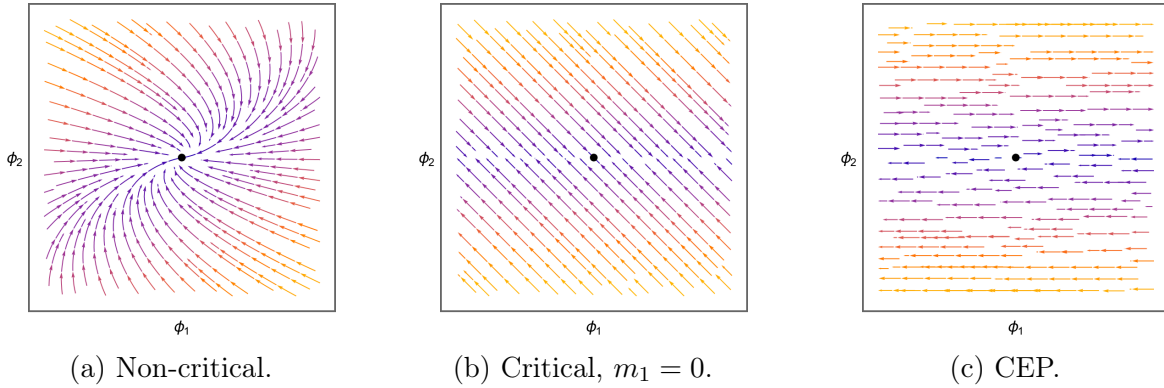
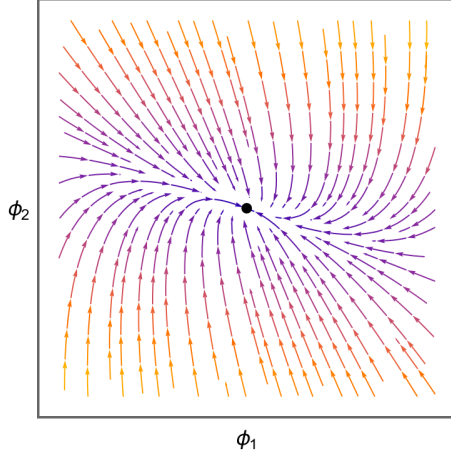
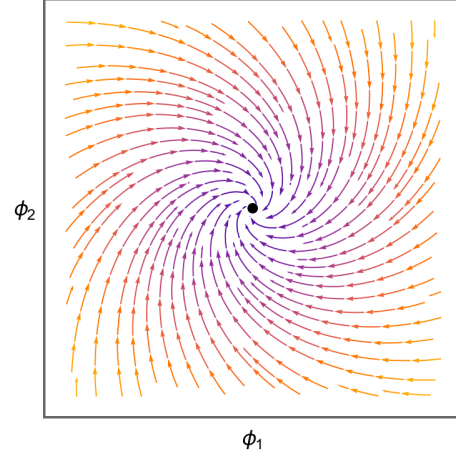


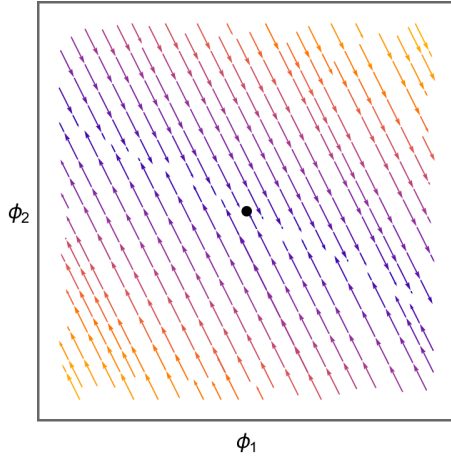
Figure 8.2: Typical phase plots for the overdamped system with $\Delta = 0$. The dot corresponds to the fixed point.



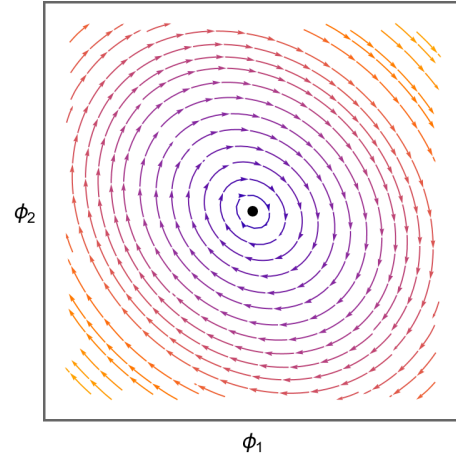
(a) Non-critical, non-rotating



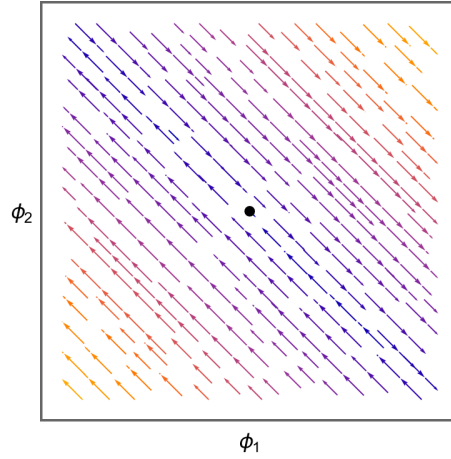
(b) Non-critical, rotating



(c) Critical, non-rotating – on the hyperbola



(d) Critical, rotating – on the $m_1 = -m_2$ line



(e) CEP

Figure 8.3: Typical phase plots for the overdamped system with $\Delta < 0$. The dot corresponds to the fixed point.

8.2.3 Effective Action of a Single Field

Without loss of generality, let us integrate out ϕ_2 . The part of the action that depends on ϕ_2 can be written as:

$$S = -\frac{1}{2} \int \frac{d\omega_1 d\omega_2 d^d k_1 d^d k_2}{(2\pi)^{2d+2}} (2\pi)^{d+1} \delta(\omega_1 + \omega_2) \delta^d(\vec{k}_1 + \vec{k}_2) \phi_2^T(\omega_1, \vec{k}_1) G_{0,2}^{-1} \phi_2(\omega_2, \vec{k}_2) \\ + \int \frac{d\omega d^d k}{(2\pi)^{d+1}} \mathbf{J}^T(-\omega, -\vec{k}) \phi_2(\omega, \vec{k}) \quad , \quad (8.12)$$

where $\phi_2(\omega, \vec{k}) = \begin{pmatrix} \hat{\phi}_2(\omega, \vec{k}) & \hat{\phi}_2(\omega, \vec{k}) \end{pmatrix}^T$ and:

$$G_{0,2}^{-1} = \begin{pmatrix} 0 & -\omega_2 - im_2 - iD_2 k_2^2 \\ \omega_2 - im_2 - iD_2 k_2^2 & B_2 \end{pmatrix} . \quad (8.13)$$

Finally:

$$\mathbf{J}(-\omega, -\vec{k}) = \begin{pmatrix} -ij_{12} \hat{\phi}_1(-\omega, -\vec{k}) \\ -ij_{21} \hat{\phi}_1(-\omega, -\vec{k}) \end{pmatrix} . \quad (8.14)$$

Integrating it out then yields:

$$\frac{1}{2} \int \frac{d\omega_1 d\omega_2 d^d k_1 d^d k_2}{(2\pi)^{2d+2}} (2\pi)^{d+1} \delta(\omega_1 + \omega_2) \delta^d(\vec{k}_1 + \vec{k}_2) J(\omega_1, \vec{k}_1) G_{0,2} J(\omega_2, \vec{k}_2) \\ = -\frac{1}{2} \int \frac{d\omega_1 d\omega_2 d^d k_1 d^d k_2}{(2\pi)^{2d+2}} (2\pi)^{d+1} \delta(\omega_1 + \omega_2) \delta^d(\vec{k}_1 + \vec{k}_2) \\ \left[j_{12}^2 \hat{\phi}_1(\omega_1, \vec{k}_1) \frac{B_2}{\omega_2^2 + (m_2 + D_2 k_2^2)^2} \hat{\phi}_1(\omega_2, \vec{k}_2) \right. \\ \left. + j_{12} j_{21} \left(\hat{\phi}_1(\omega_1, \vec{k}_1) \frac{1}{\omega_2 - im_2 - iD_2 k_2^2} \hat{\phi}_1(\omega_2, \vec{k}_2) + \hat{\phi}_1(\omega_2, \vec{k}_2) \frac{1}{-\omega_2 - im_2 - iD_2 k_2^2} \hat{\phi}_1(\omega_1, \vec{k}_1) \right) \right] \quad (8.15)$$

The deterministic terms are determined by $\Delta = j_{12}j_{21}$ and depend on the response functions. The noise term is equivalent to adding to the noise kernel of ϕ_1 an additional term:

$$\frac{j_{12}^2 B_2}{\omega^2 + (m_2 + D_2 k^2)^2} \quad (8.16)$$

8.3 Appendix C: Non-Reciprocal XY Models

8.3.1 Electrodynamic Duality

Here we present the details on the electrodynamic duality transformation leading to the vortex dynamics equations in Sec. 4.4.3. We start with Eq. 4.48:

$$\partial_t^2 \theta + (\delta - Z \nabla^2) \partial_t \theta - v^2 \nabla^2 \theta + \eta = 0 \quad (8.17)$$

We then introduce the electric field $\vec{E} = -\hat{z} \times \vec{\nabla} \theta$ and demand the Gauss' law holds for this field and the vortex density: $\vec{\nabla} \cdot \vec{E} = \frac{2\pi}{\epsilon} n$. The vortex density also obeys the continuity equation $\partial_t n + \vec{\nabla} \cdot \vec{j} = 0$. In principle, we could just apply $-\hat{z} \times \vec{\nabla}$ to the above equation and expect to get the equation of motion for the electric field. However, such an approach only works in the absence of vortices. Incorporating them requires some more care, and we notice to that end that the continuity equation can be rewritten as $\vec{\nabla} \cdot (\partial_t \vec{E} + \frac{2\pi}{\epsilon} \vec{j}) = 0$. This equation is solved by:

$$\partial_t \vec{E} = -\frac{2\pi}{\epsilon} \vec{j} + \vec{\nabla} \times \vec{u} \quad (8.18)$$

$\vec{u}$ is not arbitrary, and should be picked such that the equation for $\vec{E}$ reproduces the equation of motion in the absence of vortices. Taking another time derivative and adding it to the above equation, we get:

$$\partial_t^2 \vec{E} + (\delta - Z \nabla^2) \partial_t \vec{E} = -\frac{2\pi}{\epsilon} \left[\partial_t + (\delta - Z \nabla^2) \right] \vec{j} + \vec{\nabla} \times \left[\partial_t + (\delta - Z \nabla^2) \right] \vec{u} \quad (8.19)$$

We also notice that since we are working in a $2d$ plane, we can rewrite $-\hat{z} \times \vec{\nabla}(\nabla^2\theta) = \vec{\nabla} \times (\vec{\nabla} \times \vec{E})$. Therefore, $\vec{u}$ should satisfy:

$$\vec{\nabla} \times \left[\partial_t + (\delta - Z\nabla^2) \right] u = -v^2 \vec{\nabla} \times (\vec{\nabla} \times \vec{E}) + \hat{z} \times \vec{\nabla} \eta \quad (8.20)$$

Using $\vec{\nabla} \times (\vec{\nabla} \times \vec{E}) = \vec{\nabla}(\vec{\nabla} \cdot \vec{E}) - \nabla^2 \vec{E}$, we get the equation of motion for the electric field with the inclusion of the vortex charge density and current:

$$\partial_t^2 \vec{E} + (\delta - Z\nabla^2) \partial_t \vec{E} - v^2 \nabla^2 \vec{E} + \frac{2\pi v^2}{\epsilon} \vec{\nabla} n + \frac{2\pi}{\epsilon} \left[\partial_t + (\delta - Z\nabla^2) \right] \vec{j} - \hat{z} \times \vec{\nabla} \eta = 0 \quad (8.21)$$

As discussed in Sec. 4.4.3, introducing the appropriate gauge potentials as:

$$\vec{E} = -\vec{\nabla} \phi - (\delta - Z\nabla^2) \vec{A} - \partial_t \vec{A} \quad (8.22)$$

and supplying the equation of motion for the vortices leads to the following set of equations:

$$\begin{aligned} \partial_t \vec{r}_i &= \mu n_i \vec{E}(t, \vec{r}_i) + \vec{\eta}_T \\ \partial_t^2 \phi + (\delta - Z\nabla^2) \partial_t \phi - v^2 \nabla^2 \phi &= \frac{2\pi v^2}{\epsilon} n \\ \left[\partial_t^2 + (\delta - Z\nabla^2) \partial_t - v^2 \nabla^2 \right] \left[-(\delta - Z\nabla^2) \vec{A} - \partial_t \vec{A} \right] + \frac{2\pi}{\epsilon} \left[\partial_t + (\delta - Z\nabla^2) \right] \vec{j} - \hat{z} \times \vec{\nabla} \eta &= 0 \end{aligned} \quad (8.23)$$

If we define the modified vector potential $[\partial_t + (\delta - Z\nabla^2)] \vec{A} \rightarrow \vec{\bar{A}}$, we can use the MSRJD procedure to get the action [139] $S = S_g + S_v + S_I$, with the gauge part of the action:

$$S_0 = \frac{1}{2} \int_{t,x} \begin{pmatrix} A_\mu & \bar{A}_\mu \end{pmatrix} \begin{pmatrix} 0 & \delta_{\mu\nu} L^- \\ \delta_{\mu\nu} L^+ & \Delta_{\mu\nu} \end{pmatrix} \begin{pmatrix} A_\mu \\ \bar{A}_\mu \end{pmatrix} \quad (8.24)$$

Where $\begin{pmatrix} A_\mu & \bar{A}_\mu \end{pmatrix} = \begin{pmatrix} \phi, \vec{A}, \bar{\phi}, \vec{A} \end{pmatrix}$, $L^\pm = -i(\partial_t^2 \pm (\delta - Z\nabla^2)\partial_t - v^2\nabla^2)$ and:

$$\Delta_{\mu\nu} = \Delta(1 - \delta_{\mu,0})(1 - \delta_{\nu,0})(\nabla^2\delta_{\mu\nu} - \partial_\mu\partial_\nu) \quad (8.25)$$

The vortex and interaction parts of the action are:

$$\begin{aligned} S_v &= \sum_i \int_t -i\vec{p}_i \cdot \partial_t \vec{r}_i + \frac{\mu T}{2} p_i^2 \\ S_I &= i \int_{t,x} \left[\frac{2\pi}{\epsilon} J_\mu \bar{A}_\mu - \mu P_\mu A_\mu \right] \end{aligned} \quad (8.26)$$

Here we introduced the variables conjugate to $\vec{r}_i$ as $\vec{p}_i$ for each vortex i , and the rest of the notation is:

$$\begin{aligned} J_\mu &= \left(v^2 n, [\partial_t + (\delta - Z\nabla^2)] \vec{J} \right) \\ P_\mu &= \left(-\vec{\nabla} \cdot \vec{P}, \vec{P} \right) \\ n(t, \vec{r}) &= \sum_i n_i \delta[\vec{r} - \vec{r}_i(t)] \\ \vec{J}(t, \vec{r}) &= \sum_i n_i \partial_t \vec{r}_i \delta[\vec{r} - \vec{r}_i(t)] \\ \vec{P}(t, \vec{r}) &= \sum_i n_i \vec{p}_i \delta[\vec{r} - \vec{r}_i(t)] \end{aligned} \quad (8.27)$$

Now we can integrate out the gauge fields exactly, since the action is quadratic in them. The result is the effective action for the vortices:

$$S_{I,\text{eff}} = \frac{1}{2} \int_{t,r,t',r'} \begin{pmatrix} i\frac{2\pi}{\epsilon} J_\mu & -i\mu P_\mu \end{pmatrix}_{t,r} \begin{pmatrix} 0 & \delta_{\mu\nu} g^- \\ \delta_{\mu\nu} g^+ & g^+ \Delta_{\mu\nu} g^- \end{pmatrix}_{t,r;t',r'} \begin{pmatrix} i\frac{2\pi}{\epsilon} J_\nu \\ -i\mu P_\nu \end{pmatrix}_{t,r'} \quad (8.28)$$

where $g^\pm = (L^\pm)^{-1}$. In principle, from here we can get all the interactions, including retarded terms, that govern the dynamics of the vortices. Let us demonstrate how the

Coulomb interaction arises in this expression. For that, we need to look at the $J_0 P_0$ term, and the calculation is most easily done in Fourier space. To that end, we find:

$$\begin{aligned}
J_0(\omega, k) &= v^2 \int_{t,r} e^{-i\omega t + i\vec{k} \cdot \vec{r}} \sum_i n_i \delta(\vec{r} - \vec{r}_i) = v^2 \sum_i n_i \int_t e^{-i\omega t + i\vec{k} \cdot \vec{r}_i(t)} \\
P_0(\omega, k) &= - \int_{t,r} e^{-i\omega t + i\vec{k} \cdot \vec{r}} \sum_i n_i \vec{p}_i(t) \cdot \vec{\nabla} \delta(\vec{r} - \vec{r}_i) = \sum_i n_i \int_t i\vec{k} \cdot \vec{p}_i(t) e^{-i\omega t + i\vec{k} \cdot \vec{r}_i(t)}
\end{aligned} \tag{8.29}$$

The relevant term, aside from a pre-factor $\frac{2\pi i v^2 \mu}{2\epsilon}$, is:

$$\begin{aligned}
& \int_{\omega,k} J_0(\omega, k) \frac{1}{-\omega^2 + (\delta + Zk^2)i\omega + v^2 k^2} P_0(-\omega, -k) \\
& + P_0(\omega, k) \frac{1}{-\omega^2 - (\delta + Zk^2)i\omega + v^2 k^2} J_0(-\omega, -k) \\
& = - \sum_{ij} n_i n_j \int_{\omega,k} \int_{t,t'} \frac{e^{-i\omega t + i\vec{k} \cdot \vec{r}_i(t)} i\vec{k} \cdot \vec{p}_j(t') e^{i\omega t' - i\vec{k} \cdot \vec{r}_j(t')}}{-\omega^2 + (\delta + Zk^2)i\omega + v^2 k^2} - \frac{e^{-i\omega t + i\vec{k} \cdot \vec{r}_i(t)} i\vec{k} \cdot \vec{p}_i(t) e^{i\omega t' - i\vec{k} \cdot \vec{r}_j(t')}}{-\omega^2 - (\delta + Zk^2)i\omega + v^2 k^2}
\end{aligned} \tag{8.30}$$

Now define the central and relative times $t = t_g + \tau/2$, $t' = t_g - \tau/2$, and expand $\vec{r}_{i/j}(t/t') = \vec{r}_{i/j}(t_g) \pm \frac{\tau}{2} \dot{\vec{r}}_{i/j}(t_g)$, which gives (renaming $t_g \rightarrow t$):

$$\begin{aligned}
& - \sum_{ij} n_i n_j \int_{\omega,k} \int_{t,\tau} e^{-i\omega\tau + i\vec{k} \cdot [\vec{r}_i(t) - \vec{r}_j(t)] + i\frac{\tau}{2}\vec{k} \cdot [\dot{\vec{r}}_i(t) + \dot{\vec{r}}_j(t)]} \\
& * \left(\frac{i\vec{k} \cdot \left(\vec{p}_j(t) - \frac{\tau}{2} \dot{\vec{p}}_j(t) \right)}{-\omega^2 + (\delta + Zk^2)i\omega + v^2 k^2} - \frac{i\vec{k} \cdot \left(\vec{p}_i(t) + \frac{\tau}{2} \dot{\vec{p}}_i(t) \right)}{-\omega^2 - (\delta + Zk^2)i\omega + v^2 k^2} \right)
\end{aligned} \tag{8.31}$$

Now we integrate out τ and break up the terms. We get:

$$\begin{aligned}
& \int_{\omega,k} \int_t \delta \left(\omega - \frac{1}{2} \vec{k} \cdot [\dot{\vec{r}}_i(t) + \dot{\vec{r}}_j(t)] \right) e^{i\vec{k} \cdot [\vec{r}_i(t) - \vec{r}_j(t)]} \\
& \left(\frac{i\vec{k} \cdot \vec{p}_j(t)}{-\omega^2 + (\delta + Zk^2)i\omega + v^2k^2} - \frac{i\vec{k} \cdot \vec{p}_i(t)}{-\omega^2 - (\delta + Zk^2)i\omega + v^2k^2} \right) \\
& = \int_k \int_t e^{i\vec{k} \cdot [\vec{r}_i(t) - \vec{r}_j(t)]} \left(\frac{i\vec{k} \cdot \vec{p}_j(t)}{-\left(\frac{1}{2}\vec{k} \cdot [\dot{\vec{r}}_i(t) + \dot{\vec{r}}_j(t)]\right)^2 + (\delta + Zk^2)i\left(\frac{1}{2}\vec{k} \cdot [\dot{\vec{r}}_i(t) + \dot{\vec{r}}_j(t)]\right) + v^2k^2} \right. \\
& \quad \left. - \frac{i\vec{k} \cdot \vec{p}_i(t)}{-\left(\frac{1}{2}\vec{k} \cdot [\dot{\vec{r}}_i(t) + \dot{\vec{r}}_j(t)]\right)^2 - (\delta + Zk^2)i\left(\frac{1}{2}\vec{k} \cdot [\dot{\vec{r}}_i(t) + \dot{\vec{r}}_j(t)]\right) + v^2k^2} \right) \\
& = \int_k \int_t e^{i\vec{k} \cdot [\vec{r}_i(t) - \vec{r}_j(t)]} \left(\frac{i\vec{k} \cdot \vec{p}_j(t)}{v^2k^2} \left[1 - \frac{i\vec{k} \cdot (\dot{\vec{r}}_i(t) + \dot{\vec{r}}_j(t)) (\delta + Zk^2)}{2v^2k^2} + \frac{\left(\frac{1}{2}\vec{k} \cdot [\dot{\vec{r}}_i(t) + \dot{\vec{r}}_j(t)]\right)^2}{v^2k^2} + \dots \right] \right. \\
& \quad \left. - \frac{i\vec{k} \cdot \vec{p}_i(t)}{v^2k^2} \left[1 + \frac{i\vec{k} \cdot (\dot{\vec{r}}_i(t) + \dot{\vec{r}}_j(t)) (\delta + Zk^2)}{2v^2k^2} + \frac{\left(\frac{1}{2}\vec{k} \cdot [\dot{\vec{r}}_i(t) + \dot{\vec{r}}_j(t)]\right)^2}{v^2k^2} \dots \right] \right)
\end{aligned} \tag{8.32}$$

We see the following. The terms of order k^{-2} are nothing but the static interaction which produce the Coulomb force, since:

$$\int d^2k e^{i\vec{k} \cdot \vec{r}} \frac{1}{k^2} = 2\pi \int_{\frac{2\pi}{L}}^{\frac{2\pi}{a}} dk k \frac{J_0(kr)}{k^2} = V(r) \sim \log\left(\frac{r}{a}\right) \tag{8.33}$$

Where, in this context, J_0 is a Bessel function and we can omit other contributions since in our calculation we are taking a derivative of this term. The total resulting contribution

from this term is therefore:

$$\begin{aligned} & \frac{2\pi i v^2 \mu}{2v^2 \epsilon} \sum_{ij} n_i n_j \int_t \left(-\vec{p}_j \cdot \vec{\nabla}_j \int_k e^{i\vec{k} \cdot [\vec{r}_i(t) - \vec{r}_j(t)]} \frac{1}{k^2} - \vec{p}_i \cdot \vec{\nabla}_i \int_k e^{i\vec{k} \cdot [\vec{r}_i(t) - \vec{r}_j(t)]} \frac{1}{k^2} \right) \\ &= -i \int_t \sum_{ij} \vec{p}_i \cdot \mu n_i \vec{\nabla}_i V(\vec{r}_i - \vec{r}_j) \frac{2\pi n_j}{\epsilon} \end{aligned} \quad (8.34)$$

Which exactly recovers the 2d Coulomb interaction. For the rest of the terms, we see that they are proportional to either time derivatives of $\vec{r}$ or to something proportional to δ . In the static limit and as we approach the CEP $\delta = 0$, these terms can therefore be omitted. In fact, the expansion we performed relied on that fact, so in the case of highly mobile vortices, the results we derive here are not necessarily valid.

The signature of the CEP hides in the effective noise the vortices experience. To extract it, we look at the $P_i P_j$ terms, which are (again aside from pre-factors):

$$- \sum_{ij} n_i n_j \int_{t, t', k} e^{i\vec{k} \cdot (\vec{r}_i(t) - \vec{r}_j(t'))} p_i^\alpha(t) \Delta_{\alpha\beta} p_j^\beta(t') G^K(t - t', k) \quad (8.35)$$

Where the Keldysh correlation function is given by:

$$G^K(\omega, \vec{k}) \sim \frac{1}{(-\omega^2 + v^2 k^2)^2 + (\delta + Z k^2)^2 \omega^2} \quad (8.36)$$

The leading contribution to the noise experienced by each vortex in the above sum is the $i = j$ term. But that term is equivalent to supplying the vortex dynamics with noise which itself obeys the equation:

$$\partial_t^2 \vec{\xi} + (\delta - Z \nabla^2) \partial_t \vec{\xi} - v^2 \nabla^2 \vec{\xi} + \hat{z} \times \vec{\nabla} \eta = 0 \quad (8.37)$$

Therefore, with all these calculations and approximations, the dynamics obeyed by the

vortices are:

$$\begin{aligned}\partial_t \vec{r}_i(t) &= \mu n_i \vec{E}(\vec{r}_i(t)) + \vec{\xi}(t, \vec{r}_i(t)) + \vec{\eta}_i^T \\ \partial_t^2 \vec{\xi} + (\delta - Z \nabla^2) \partial_t \vec{\xi} - v^2 \nabla^2 \vec{\xi} + \hat{z} \times \vec{\nabla} \eta &= 0\end{aligned}\tag{8.38}$$

Where $\vec{E}$ is the electric field produced by static Coulomb interactions.

8.3.2 Fokker-Planck Equation for Vortex Separation

In this section we provide the derivation of the Fokker-Planck equation for the vortex motion and other results.

First, we had the equation of motion for $\vec{r}$ as:

$$\partial_t \vec{r} = \vec{f}(\vec{r}) + \Delta \vec{\xi} + \vec{\eta}_r\tag{8.39}$$

Here we need to treat the two noise sources separately! Here:

$$\begin{aligned}B_{ij} &= \langle \Delta \xi_i(r_1, t_1) \Delta \xi_j(r_2, t_2) \rangle = \left\langle \left[\int_k \xi_i(k, t_1) \sin \left(\vec{k} \cdot \frac{\vec{r}_1}{2} \right) \right] \left[\int_{k'} \xi_j(k', t_2) \sin \left(\vec{k}' \cdot \frac{\vec{r}_2}{2} \right) \right] \right\rangle \\ &= \int_{k, k'} G_{ij}^K(k, t_1; k', t_2) \sin \left(\vec{k} \cdot \frac{\vec{r}_1}{2} \right) \sin \left(\vec{k}' \cdot \frac{\vec{r}_2}{2} \right)\end{aligned}\tag{8.40}$$

and:

$$\langle \eta_{r,i}(t) \eta_{r,j}(t') \rangle = T \delta_{ij} \delta(t - t')\tag{8.41}$$

Now we can formally derive the FP equation. We start with:

$$\begin{aligned}
P(\vec{r}, t) &= \langle \delta(\vec{r} - \vec{r}(t)) \rangle \\
\partial_t P(\vec{r}, t) &= \langle -\partial_{(r-r(t))_i} \delta(\vec{r} - \vec{r}(t)) \partial_t r_i(t) \rangle = -\partial_{r_i} \left\langle \delta(\vec{r} - \vec{r}(t)) \frac{\partial r_i}{\partial (r_i - r_i(t))} (f_i + \Delta \xi_i + \eta_{r,i}) \right\rangle \\
&= -\partial_{r_i} [f_i(r) P(\vec{r}, t)] - \partial_{r_i} \langle \delta(\vec{r} - \vec{r}(t)) (\Delta \xi_i + \eta_{r,i}) \rangle
\end{aligned} \tag{8.42}$$

Since we have a sum of two Gaussian noises, we can use the Furutsu-Novikov identity to rewrite:

$$\begin{aligned}
\langle \delta(\vec{r} - \vec{r}(t)) (\Delta \xi_i + \eta_{r,i}) \rangle &= \int_{r'} \int_{-\infty}^t dt' \langle \Delta \xi_i(r, t) \Delta \xi_j(r', t') \rangle \left\langle \frac{\delta}{\delta \Delta \xi_j(r', t')} \delta(\vec{r} - \vec{r}(t)) \right\rangle \\
&\quad + \int_{-\infty}^t dt' \langle \eta_{r,i}(t) \eta_{r,j}(t') \rangle \left\langle \frac{\delta}{\delta \eta_{r,j}(t')} \delta(\vec{r} - \vec{r}(t)) \right\rangle
\end{aligned} \tag{8.43}$$

and we can simplify the last terms in the brackets using the original Langevin equation.

They become:

$$\begin{aligned}
\left\langle \frac{\delta}{\delta \Delta \xi_j(r', t')} \delta(\vec{r} - \vec{r}(t)) \right\rangle &= -\partial_{r_l} \left\langle \delta(\vec{r} - \vec{r}(t)) \frac{\delta \vec{r}_l(t)}{\delta \Delta \xi_j(r', t')} \right\rangle \\
\left\langle \frac{\delta}{\delta \eta_{r,j}(t')} \delta(\vec{r} - \vec{r}(t)) \right\rangle &= -\partial_{r_l} \left\langle \delta(\vec{r} - \vec{r}(t)) \frac{\delta \vec{r}_l(t)}{\delta \eta_{r,j}(t')} \right\rangle
\end{aligned} \tag{8.44}$$

We can get the derivatives from the formal solution:

$$r_i(t) = \int_{-\infty}^t ds [f_i(r(s)) + \Delta \xi_i(r(s), s) + \eta_{r,i}(s)] \tag{8.45}$$

So:

$$\begin{aligned}\frac{\delta \vec{r}_l(t)}{\delta \Delta \xi_j(r', t')} &= \int_{-\infty}^t ds \left[\partial_{r_m} f_l(r(s)) \frac{\delta r_m(s)}{\delta \Delta \xi_j(r', t')} + \frac{\delta \Delta \xi_l(r(s), s)}{\delta \Delta \xi_j(r', t')} \right] \\ \frac{\delta \vec{r}_l(t)}{\delta \eta_{r,j}(t')} &= \int_{-\infty}^t ds \left[\partial_{r_m} f_l(r(s)) \frac{\delta r_m(s)}{\delta \eta_j(t')} + \frac{\delta \eta_{r,l}(s)}{\delta \eta_{r,j}(t')} \right]\end{aligned}\quad (8.46)$$

Since $\langle \eta_{r,i}(t) \eta_{r,j}(t') \rangle$ is non-zero only for $t = t'$, we get that (using Ito calculus) that the term coming from the noise η is simply going to yield $T \partial_{r_i}^2 P$, as usual. The other term is simply the response function of $\vec{r}$ to the ξ noise. Formally, we therefore get the FP equation:

$$\begin{aligned}\partial_t P(\vec{r}, t) &= -\partial_{r_i} [f_i(r) P(\vec{r}, t)] + T \partial_{r_i} \partial_{r_i} P \\ &+ \partial_{r_i} \int_{r'} \int_{-\infty}^t dt' B_{ij}(\vec{r} - \vec{r}', t - t') \partial_{r_l} \left[\left\langle \delta(\vec{r} - \vec{r}(t)) \frac{\delta \vec{r}_l(t)}{\delta \Delta \xi_j(r', t')} \right\rangle \right]\end{aligned}\quad (8.47)$$

We need to work out how to handle the response function. From above, we had:

$$\begin{aligned}\partial_t \frac{\delta r_l(t)}{\delta \Delta \xi_j(r', t')} &= \partial_{r_m} f_l(r(t)) \frac{\delta r_m(t)}{\delta \Delta \xi_j(r', t')} + \frac{\delta \Delta \xi_l(r(t), t)}{\delta \Delta \xi_j(r', t')} \\ &= \partial_{r_m} f_l(r(t)) \frac{\delta r_m(t)}{\delta \Delta \xi_j(r', t')} + \delta_{lj} \delta^2(r(t) - r') \delta(t - t')\end{aligned}\quad (8.48)$$

Here we use the fact that $t' \leq t$ and that the response $\frac{\delta r_m(s)}{\delta \Delta \xi_j(r', t')}$ is causal. We need to understand what this response function is. Denoting $\frac{\delta r_l(t)}{\delta \Delta \xi_j(r', t')} = G_{lj}^r(r, t; r', t')$, we get the equation of motion:

$$\partial_t G_{lj}^r(r, t; r', t') = \partial_{r_m} f_l(r(t)) G_{mj}^r(r, t; r', t') + \delta_{lj} \delta^2(r(t) - r') \delta(t - t') \quad (8.49)$$

The solution is of the form:

$$G_{lj}^r(r, t; r', t') = \Theta(t - t') J_{lj}(t, t', r') \delta^2[r(t') - r'] \quad (8.50)$$

Where J satisfies:

$$\partial_t J_{lj}(t, t', r') = \partial_{r_m} f_l \left[\int_{t'}^t ds f(r(s)) \right] J_{mj}(t, t', r') \quad (8.51)$$

We can confirm this by direct substitution and remembering that this Green's function appears in an integral. We get:

$$\partial_t G_{lj}^r = \delta(t - t') J_{lj} \delta^2 [r(t') - r'] + \Theta(t - t') \partial_t J_{lj} \delta^2 [r(t') - r'] \quad (8.52)$$

In the first term, we can equate $t = t'$ everywhere, which means $J_{lj} = \delta_{lj}$ and the delta function becomes $\delta^2 [r(t) - r']$. The second term, by the definition of J , gives back $\partial_{r_m} f_l(r(t)) G_{mj}^r(r, t; r', t')$. Therefore, we just need to find J . We have the equation:

$$\partial_t r_i = f_i(r) \quad (8.53)$$

with $r(t') = r'$. Since we have a radial force, we should look at the radial equation of motion, which is:

$$\partial_t r = -\frac{1}{\epsilon r} \quad (8.54)$$

which gives:

$$\frac{1}{2}(r(t)^2 - r'^2) = -\frac{1}{\epsilon}(t - t') \rightarrow \vec{r}(t) = \frac{\vec{r}'}{r'} \sqrt{r'^2 - 2\frac{1}{\epsilon}(t - t')} \quad (8.55)$$

The matrix J then satisfies:

$$\partial_t J_{lj}(t, t', r') = -\frac{1}{\epsilon} \left[\frac{\delta_{ml}}{r(t)^2} - \frac{2r_l(t)r_m(t)}{r(t)^4} \right] J_{mj}(t, t', r') \quad (8.56)$$

The solution can be checked to be:

$$J_{lj}(t, t', r') = \frac{r'}{r(t)} \hat{r}_l \hat{r}_j + \frac{r(t)}{r'} (\delta_{lj} - \hat{r}_l \hat{r}_j) \quad (8.57)$$

Just to check:

$$\partial_t J_{lj}(t, t', r') = -\frac{r'}{r(t)^2} \partial_t r(t) \hat{r}_l \hat{r}_j + \frac{\partial_t r(t)}{r'} (\delta_{lj} - \hat{r}_l \hat{r}_j) = \frac{1}{\epsilon} \frac{r'}{r(t)^3} \hat{r}_l \hat{r}_j - \frac{1}{\epsilon r' r(t)} (\delta_{lj} - \hat{r}_l \hat{r}_j) \quad (8.58)$$

On the other hand:

$$\begin{aligned} & -\frac{1}{\epsilon} \left[\frac{\delta_{ml}}{r(t)^2} - \frac{2r_l(t)r_m(t)}{r(t)^4} \right] J_{mj}(t, t', r') \\ &= -\frac{1}{\epsilon} \left[\frac{\delta_{ml}}{r(t)^2} - \frac{2r_l(t)r_m(t)}{r(t)^4} \right] \left[\frac{r'}{r(t)} \hat{r}_m \hat{r}_j + \frac{r(t)}{r'} (\delta_{mj} - \hat{r}_m \hat{r}_j) \right] \\ &= -\frac{1}{\epsilon} \left[\frac{r'}{r(t)^3} \hat{r}_l \hat{r}_j - 2\frac{r'}{r(t)^3} \hat{r}_l \hat{r}_j + \frac{1}{r(t)r'} (\delta_{lj} - \hat{r}_l \hat{r}_j) - \frac{2}{r(t)r'} \hat{r}_l \hat{r}_j + \frac{2}{r(t)r'} \hat{r}_l \hat{r}_j \right] \\ &= \frac{1}{\epsilon} \frac{r'}{r(t)^3} \hat{r}_l \hat{r}_j - \frac{1}{\epsilon r' r(t)} (\delta_{lj} - \hat{r}_l \hat{r}_j) \end{aligned} \quad (8.59)$$

The full FP equation becomes:

$$\begin{aligned} \partial_t P(\vec{r}, t) &= -\partial_{r_i} [f_i(r) P(\vec{r}, t)] + T \partial_{r_i} \partial_{r_i} P \\ &\quad + \partial_{r_i} \int_{r'} \int_{-\infty}^t dt' B_{ij}(\vec{r} - \vec{r}', t - t') J_{lj}(t, t', r') \partial_{r_l} P_2(r, t; r', t') \end{aligned} \quad (8.60)$$

where we use a 2-point probability function:

$$\langle \delta(\vec{r} - \vec{r}(t)) \delta(\vec{r}' - \vec{r}(t')) \rangle = P_2(\vec{r}, t; \vec{r}', t') \quad (8.61)$$

This hints that we need to consider the full BBGKY hierarchy. Let us derive the equation

for the two point function:

$$\begin{aligned}
\partial_t P_2(r, t; r', t') &= - \left\langle \partial_{(r-r(t))_i} \delta(\vec{r} - \vec{r}(t)) \delta(\vec{r}' - \vec{r}(t')) \partial_t r_i(t) \right\rangle \\
&= - \partial_{r_i} \left\langle \delta(\vec{r} - \vec{r}(t)) \delta(\vec{r}' - \vec{r}(t')) \frac{\partial r_i}{\partial (r_i - r_i(t))} (f_i + \Delta \xi_i + \eta_{r,i}) \right\rangle \\
&= - \partial_{r_i} [f_i(r) P_2(r, t; r', t')] - \partial_{r_i} \left\langle \delta(\vec{r} - \vec{r}(t)) \delta(\vec{r}' - \vec{r}(t')) (f_i + \Delta \xi_i + \eta_{r,i}) \right\rangle
\end{aligned} \tag{8.62}$$

We again use the Novikov identity:

$$\begin{aligned}
&\left\langle \delta(\vec{r} - \vec{r}(t)) \delta(\vec{r}' - \vec{r}(t')) (\Delta \xi_i + \eta_{r,i}) \right\rangle \\
&= \int_{r''} \int_{-\infty}^t dt'' \langle \Delta \xi_i(r, t) \Delta \xi_j(r'', t'') \rangle \left\langle \frac{\delta}{\delta \Delta \xi_j(r'', t')} [\delta(\vec{r} - \vec{r}(t)) \delta(\vec{r}' - \vec{r}(t'))] \right\rangle \\
&+ \int_{-\infty}^t dt'' \langle \eta_{r,i}(t) \eta_{r,j}(t'') \rangle \left\langle \frac{\delta}{\delta \eta_{r,j}(t'')} [\delta(\vec{r} - \vec{r}(t)) \delta(\vec{r}' - \vec{r}(t'))] \right\rangle
\end{aligned} \tag{8.63}$$

Similarly to before, we have:

$$\begin{aligned}
&\left\langle \frac{\delta}{\delta \Delta \xi_j(r'', t'')} \delta(\vec{r} - \vec{r}(t)) \delta(\vec{r}' - \vec{r}(t')) \right\rangle = - \partial_{r_l} \left\langle \delta(\vec{r} - \vec{r}(t)) \delta(\vec{r}' - \vec{r}(t')) \frac{\delta \vec{r}_l(t)}{\delta \Delta \xi_j(r'', t'')} \right\rangle \\
&- \partial_{r'_l} \left\langle \delta(\vec{r} - \vec{r}(t)) \delta(\vec{r}' - \vec{r}(t')) \frac{\delta \vec{r}_l(t')}{\delta \Delta \xi_j(r'', t'')} \right\rangle
\end{aligned} \tag{8.64}$$

and similarly for the other term. Since we already calculated the response function, the FP equation is going to be:

$$\begin{aligned}
\partial_t P_2(r, t; r', t') &= - \partial_{r_i} [f_i P_2] + T \partial_{r_i} \left(\partial_{r_i} + \partial_{r'_i} \right) P_2 \\
&+ \partial_{r_i} \int_{r''} \int_{-\infty}^t dt'' B_{ij}(\vec{r} - \vec{r}'', t - t'') \left[J_{lj}(t, t'', r'') \partial_{r_l} + J_{lj}(t', t'', r'') \partial_{r'_l} \right] P_3(r, t; r', t'; r'', t'')
\end{aligned}$$

(8.65)

We clearly see a pattern:

$$\begin{aligned} \partial_{t_1} P_n(r_1, t_1; r_2, t_2; \dots; r_n, t_n) = & -\partial_{r_{1,i}} [f_i(r_1) P_n] + T \partial_{r_{1,i}} \sum_{\alpha=1}^n \left(\partial_{r_{\alpha,i}} \right) P_n \\ & + \partial_{r_{1,i}} \int_{\tilde{r}} \int_{-\infty}^{t_1} d\tilde{t} B_{ij}(\vec{r}_1 - \vec{r}, t_1 - \tilde{t}) \sum_{\alpha=1}^n \left[\Theta(t_\alpha - \tilde{t}) J_{lj}(t_\alpha, \tilde{t}, \tilde{r}) \partial_{r_{\alpha,l}} \right] P_{n+1}(r_1, t_1; \dots; r_n, t_n; \tilde{r}, \tilde{t}) \end{aligned} \quad (8.66)$$

In the stationary state, we have the equation:

$$0 = -f_i P_s + T \partial_{r_i} P_s + \int_{r'} \int_{-\infty}^t dt' B_{ij}(\vec{r} - \vec{r}', t - t') J_{lj}(t, t', r') \partial_{r_l} P_{2,s}(r, t; r', t') \quad (8.67)$$

In the short time approximation discussed in the text, we get:

$$0 = -f_i P_s + T \partial_{r_i} P_s + \tilde{t} B_{ij}(r, t) \partial_{r_j} P_s \quad (8.68)$$

Where $\tilde{t}$ is the time scale where such an approximation is appropriate, which is different for the cases $\delta = 0$ and $\delta \neq 0$. In our case, the correlation function is not translation invariant, but is going to be given by:

$$\begin{aligned} B_{ij}(r, t) = \langle \Delta \xi_i(r, t) \Delta \xi_j(r, t) \rangle &= \int_{k, k'} G_{ij}^K(k, t; k', t) \sin \left(\vec{k} \cdot \frac{\vec{r}}{2} \right) \sin \left(\vec{k}' \cdot \frac{\vec{r}}{2} \right) \\ &= \int_k \frac{\Delta(k^2 \delta_{ij} - k_i k_j)}{(\delta + Z k^2) v^2 k^2} \sin^2 \left(\vec{k} \cdot \frac{\vec{r}}{2} \right) \end{aligned} \quad (8.69)$$

If the angle between r and the x axis is α , we get

$$\begin{aligned}
B_{ij} &= \int_0^\Lambda dk \frac{\Delta k}{(\delta + Zk^2)v^2} \int_0^{2\pi} d\theta \sin^2 \left(\frac{kr}{2} \cos(\theta - \alpha) \right) \begin{pmatrix} \sin^2(\theta) & -\sin(\theta) \cos(\theta) \\ -\sin(\theta) \cos(\theta) & \cos^2(\theta) \end{pmatrix} \\
&= \frac{\Delta\pi}{2v^2} \int_0^\Lambda dk \frac{k}{(\delta + Zk^2)} \begin{pmatrix} 1 - J_0(kr) + \cos(2\alpha)J_2(kr) & \sin(2\alpha)J_2(kr) \\ \sin(2\alpha)J_2(kr) & 1 - J_0(kr) - \cos(2\alpha)J_2(kr) \end{pmatrix}
\end{aligned} \tag{8.70}$$

Let us write the equation in polar coordinates (and hopefully still see rotational invariance):

$$0 = \frac{\partial r}{\partial r_i} \frac{\partial}{\partial r} V P_s + T \left[\frac{\partial r}{\partial r_i} \frac{\partial}{\partial r} + \frac{\partial \alpha}{\partial r_i} \frac{\partial}{\partial \alpha} \right] P_s + B_{ij}(r, \alpha) \left[\frac{\partial r}{\partial r_j} \frac{\partial}{\partial r} + \frac{\partial \alpha}{\partial r_j} \frac{\partial}{\partial \alpha} \right] P_s \tag{8.71}$$

We need to make sure that we get:

$$\begin{aligned}
B_{ij} \frac{\partial r}{\partial r_j} &\sim \frac{\partial r}{\partial r_i} \\
B_{ij} \frac{\partial \alpha}{\partial r_j} &\sim \frac{\partial \alpha}{\partial r_i}
\end{aligned} \tag{8.72}$$

The only non-trivial part is the part of the matrix proportional to $J_2(kr)$, where it can be checked that this is still satisfied. The resulting equation in the radius becomes:

$$0 = \frac{1}{r} \partial_r V P_s \begin{pmatrix} \cos \alpha \\ \sin \alpha \end{pmatrix} + T \frac{1}{r} \partial_r P_s \begin{pmatrix} \cos \alpha \\ \sin \alpha \end{pmatrix} + \frac{1}{r} \partial_r P_s B_{ij}(r) \begin{pmatrix} \cos \alpha \\ \sin \alpha \end{pmatrix} \tag{8.73}$$

Which gives the resulting equation:

$$0 = P_s \partial_r V + T \partial_r P_s + \tilde{t} \frac{\Delta\pi}{2v^2} \int_0^\Lambda dk \frac{k}{\delta + Zk^2} [1 - J_0(kr) + J_2(kr)] \partial_r P_s \tag{8.74}$$

One way to estimate $\tilde{t}$ is the time difference when both the deterministic evolution and the variance due to noise are smaller than the core radius, for example. Close to $\delta = 0$, the noise terms dominate, and the time-scale where this approximation is appropriate should be when:

$$\begin{aligned}
a^2 &> \left\langle \left[\int_0^{\tilde{t}} ds (\eta(s) + \Delta\xi(r(s), s)) \right] \left[\int_0^{\tilde{t}} du (\eta(u) + \Delta\xi(r(u), u)) \right] \right\rangle \\
&\approx T\tilde{t} + \tilde{t} \int_0^{\tilde{t}} ds B(s) \approx T\tilde{t} + \tilde{t} \int_0^{\tilde{t}} ds \int_k G^K(k, s) \sin \left(\vec{k} \cdot \frac{\vec{r}}{2} \right)^2 > T\tilde{t} + \tilde{t} \int_0^{\tilde{t}} ds \int_k G^K(k, s) \\
&> T\tilde{t} + \tilde{t} \frac{\Delta\pi}{v^2} \int_0^{\tilde{t}} ds \int_0^\Lambda dk \frac{k}{\delta + Zk^2} e^{-\frac{1}{2}(\delta + Zk^2)s} \\
&= T\tilde{t} + \tilde{t} \frac{2\Delta\pi}{v^2} \int_0^\Lambda dk \frac{k}{(\delta + Zk^2)^2} \left[1 - e^{-\frac{1}{2}(\delta + Zk^2)\tilde{t}} \right] \\
&= T\tilde{t} + \tilde{t} \frac{2\Delta\pi}{v^2 Z^2 \Lambda^2} \int_0^1 dx \frac{x}{(\tilde{\delta} + x^2)^2} \left[1 - e^{-\frac{1}{2}(\tilde{\delta} + x^2)Z\Lambda^2\tilde{t}} \right]
\end{aligned} \tag{8.75}$$

Where r is the point of the probability distribution we are interested in and $\tilde{\delta} = \frac{\delta}{Z\Lambda^2}$. For $\delta \rightarrow 0$, the largest contribution gives:

$$a^2 > T\tilde{t} - \tilde{t}^2 \ln(\delta\tilde{t}) \frac{\Delta\pi}{Zv^2} \tag{8.76}$$

8.4 Appendix D: Coalescence of Limit Cycles

Here we provide the derivation of the factorization result in Eq. 5.13 for the generating functional of the coupled system. The partition function with respect to which all the

averages are to be taken takes the form:

$$\begin{aligned} \mathcal{Z} = \int \mathcal{D}[w, w^q, \theta, \theta^q, r, r^q] \exp & \left[\int dt i w^q (\dot{w} - R w + g w^3) - \frac{B_w}{2} w^{q2} \right] \\ & \exp \left[\int dt i \theta^q (\dot{\theta} - \Omega - c w^n) - \frac{B_\psi}{2 r^2} \theta^{q2} \right] \\ & \exp \left[\int dt i r^q (\dot{r} - f(r)) - \frac{B_\psi}{2} r^{q2} \right] . \end{aligned} \quad (8.77)$$

As mentioned in the text, the dynamics of r and w are independent of the other parts of the system, so we consider the generating functional for the observables in θ : $\mathcal{Z}[\lambda_\theta, \lambda_\theta^q] = \langle \exp [\int dt \lambda_\theta \theta + \lambda_\theta^q \theta^q] \rangle$. Assuming random initial conditions, we can employ that the action is linear in θ , so we can use $\int \mathcal{D}[\theta] e^{i q \theta} \sim \delta(q)$. Therefore the angular part of the generating functional undergoes the following transformations:

$$\begin{aligned} & \int \mathcal{D}[\theta, \theta^q] \exp \left[\int dt \left(i \theta^q (\dot{\theta} - \Omega - c w^n) - \frac{B_\psi}{2 r^2} \theta^{q2} + \lambda_\theta \theta + \lambda_\theta^q \theta_q \right) \right] \\ & = \int \mathcal{D}[\theta^q] \exp \left[\int dt \left(-i \theta^q \Omega - i \theta^q c w^n - \frac{B_\psi}{2 r^2} \theta^{q2} + \lambda_\theta^q \theta_q \right) \right] \delta(-i \dot{\theta}^q + \lambda_\theta) . \end{aligned} \quad (8.78)$$

Assuming the boundary condition that $\theta^q \rightarrow 0$ as $t \rightarrow \infty$, the delta-function ensures that $\theta^q = i \int_t^\infty ds \lambda(s)$. We call this solution θ_0^q . With this we get that the angular part of the generating functional is:

$$\exp \left[\int dt \left(-i \theta_0^q \Omega - \frac{B_\psi}{2 r^2} \theta_0^{q2} + \lambda_\theta^q \theta_0^q \right) \right] * \exp \left[\int dt (-i \theta_0^q c w^n) \right] . \quad (8.79)$$

Notice that the first factor is completely independent of c and of the dynamics of the pitchfork, and is indeed the expression one would get for the Hopf oscillator alone. The second factor, on the other hand, is independent of the dynamics of the oscillator (since we choose the source term $\lambda(t)$), and depends only on the dynamics of the pitchfork. In this manner the entire generating functional factorizes.