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GEOMETRIC AND PROBABILISTIC PERSPECTIVES ON LIOUVILLE QUANTUM GRAVITY AND LOEWNER ENERGY

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ABSTRACT

This dissertation gathers five interrelated articles on random conformal geometry.

Part I concerns results on Liouville quantum gravity (LQG), which is a conformally covariant model of random two-dimensional geometry. Chapter 2, which is joint work with Ewain Gwynne, demonstrates that the LQG area measure is the Minkowski content measure with respect to the LQG metric in the subcritical phase. Chapter 3, which is joint work with Manan Bhatia and Ewain Gwynne, studies rigorously the heuristic that LQG in the supercritical phase has infinite spikes at a dense set of locations. Concretely, we exhibit that a discrete model of supercritical LQG, when conditioned to be finite, converges to a continuum random tree in the scaling limit.

Part II concerns results on Loewner energy, which is a conformally invariant functional on crosscuts of a disk that appears naturally in the semiclassical limit of Schramm–Loewner evolution (SLE). Chapter 4 investigates the reversibility of Loewner energy and gives a deterministic proof of this basic property through commutation relations. Chapter 5, which is joint work with Yilin Wang, studies the first variation of Loewner energy under quasiconformal deformations. Chapter 6, which is joint work with Shuo Fan, shows that the Weil–Petersson Teichmüller space, which consists of loops on the Riemann sphere with finite Loewner energy modulo Möbius transformations, has a quasi-invariant action on the SLE loop measure defined naturally via conformal welding. This gives strong evidence in support of the idea that the Weil–Petersson Teichmüller space is the Cameron–Martin space for SLE loops.

To my grandmother

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CHAPTER 1

INTRODUCTION

1.1 Liouville quantum gravity

Liouville quantum gravity (LQG) is a canonical one-parameter family of random geometries on orientable surfaces with given topology. Heuristically speaking, an LQG surface with matter central charge $\mathbf{c} \in (-\infty, 25)$ is a random two-dimensional Riemannian manifold sampled from "the uniform measure on the space of surfaces with given topology weighted by $(\det \Delta)^{-\mathbf{c}/2}$," where Δ is the Laplace–Beltrami operator of the surface [206]. One way of making this heuristic rigorous is via discrete approximations, such as with the Gromov–Hausdorff scaling limit of a uniform triangulation with the given topology, weighted by the corresponding power of the graph Laplacian, as the number of triangles tends to infinity [174, 192, 2, 137].

In the subcritical phase where $\mathbf{c} \in (-\infty, 1]$, the corresponding scaling limit in the simply connected case has been constructed directly in the continuum. An LQG surface with central charge $\mathbf{c}$ can be constructed as a metric measure space starting from the formal Riemannian metric tensor $e^{\gamma h}(dx^2 + dy^2)$, where h is a GFF, $\gamma \in (0, 2)$ is the coupling constant satisfying $\mathbf{c} = 25 - 6(\frac{2}{\gamma} + \frac{\gamma}{2})^2$, and $dx^2 + dy^2$ is the Euclidean metric tensor on the given domain. In practice, the LQG measure and the LQG metric are constructed under separate renormalization procedures called Gaussian multiplicative chaos [99, 213, 37] and Liouville first passage percolation [85, 93, 129], respectively. Similar constructions are available in the critical phase of $\mathbf{c} = 1$ [96, 97, 87, 88, 205]. In these phases, an LQG surface almost surely has the topology of a 2d manifold but its Hausdorff dimension is the γ-dependent quantity $d_\gamma > 2$ [93, 132].

In contrast, mathematics and physics literature present seemingly contradictory descriptions of the underlying geometry of LQG in the supercritical phase $\mathbf{c} \in (1, 25)$. On one hand, while the definition of the LQG metric extends to the supercritical phase [87, 88, 205], the resulting surface almost surely has a dense set of points which are each at infinite distances from all other points, consistent with the physics viewpoint [223] that the surface is torn apart by densely located "infinite spikes." On the other hand, an approach using random planar maps predicts a tree-like structure with no interior (called the branched polymer in physics) [64, 59, 30, 65, 14, 81]. A possible resolution suggested by [121, 21] is that a supercritical LQG surface, which a priori has singularities at a dense set of points, should look like a continuum random tree when conditioned to have finite volume. Indeed, previous planar map approaches to supercritical LQG considered maps with large but finite numbers of faces, edges, or vertices. It is also not obvious what the area of supercritical LQG surface should be: the Gaussian multiplicative chaos corresponding to a complex value of γ yields a white noise containing further randomness than the GFF h [164, 162, 163], but there is also a claim in the physics literature [110, 111, 107] that special integrable algebraic structures exist at certain values of $\mathbf{c}$ which allows these authors to define finite local operators which give rise to a so-called "area element."

1.1.1 LQG area measure

In Chapter 2, based on the joint work [133] with Ewain Gwynne, we study how the area of an LQG surface should be understood as the fractal measure corresponding to its metric structure. In the subcritical phase, we show that the LQG measure corresponds to the Minkowski content measure of the LQG metric. It is not obvious that the two objects are connected directly without a prior knowledge of the Euclidean parameterization of the domain, since their definitions rely on a regularization of the GFF at a Euclidean scale.

Theorem 2.1.1. *For each $\gamma \in (0,2)$, there exists a deterministic sequence $(\mathfrak{b}_\varepsilon)_{\varepsilon>0}$ with $\mathfrak{b}_\varepsilon \asymp \varepsilon^{-d_\gamma}$ such that the following is true. Let h be a GFF on some domain U and let μ_h and D_h be the corresponding γ-LQG measure and metric. For any deterministic bounded Borel* *set $A \subset U$ such that ∂A has zero Lebesgue measure, if $N_\varepsilon(A; D_h)$ denotes the minimum number of D_h-balls of radius $\varepsilon > 0$ required to cover A, then*

$$\lim_{\varepsilon \to 0} \mathfrak{b}_\varepsilon^{-1} N_\varepsilon(A; D_h) = \mu_h(A) \quad \textit{in probability.}$$

Combined with the results of [130, 19], this theorem implies that the LQG metric D_h almost surely determines the domain U and the GFF h up to affine transformations (Corollary 2.1.2). The proof of Theorem 2.1.1 utilizes the mating-of-trees theory of Duplantier, Miller, and Sheffield [95]. In particular, we used the infinitely divisible metric measure space structure offered by a space-filling SLE_κ curve exploring a certain γ-LQG surface called the quantum cone when $\kappa = 16/\gamma^2$.

In Chapter 3, which is based on the joint work [44] with Manan Bhatia and Ewain Gwynne, we consider the supercritical phase of LQG. In this case, we show that there does not exist a locally finite area measure. This result is consistent with the fact that the Hausdorff dimension of a supercritical LQG metric space is almost surely infinite [88, 205]. Here, the key insight is to consider, instead of the coupling parameter γ, the background charge Q as the fundamental parameter that corresponds to the central charge through the identity $\mathbf{c} = 25 - 6Q^2$. This parameter appears in the coordinate change rule for subcritical and critical LQG surfaces: the pullback of μ_h and D_h under a conformal transformation f is given by $\mu_{h \circ f + Q \log |f'|}$ and $D_{h \circ f + Q \log |f'|}$, respectively [99, 230, 128]. The background charge Q takes a real value in $(0, 2)$ within the supercritical phase $\mathbf{c} \in (1, 25)$ of LQG.

Theorem 3.1.5. *Fix the background charge $Q \in (0, 2)$. Let $\mathfrak{m}_h$ be a random Borel measure determined locally by the GFF h which, for any fixed conformal map $f : \widetilde{U} \to U$, almost surely has*

$$\mathfrak{m}_{h \circ f + Q \log |f'|}(A) = \mathfrak{m}_h(f(A))$$

for every Borel subset $A \subset \widetilde{U}$. Then, almost surely, $\mathfrak{m}_h$ is the zero measure or it assigns *infinite measure to every Euclidean open set.*

This theorem extends to the case where $\mathfrak{m}_h$ is a complex-valued measure. The proof of Theorem 3.1.5 is based on the coupling of supercritical LQG disk with conformal loop ensemble of $\kappa = 4$ (CLE_4) [18].

1.1.2 Tree-like structure of supercritical LQG

By considering a natural random planar map model of the aforementioned coupling between a supercritical LQG disk and CLE_4, we give the first reconciliation between the "infinite spikes" picture (as in the case of supercritical LQG metric) and the "branched polymer" picture (as in the planar map calculations). Our planar map model of supercritical LQG disk can be considered as the supercritical analog of the loop $O(n)$ model [57].

Theorem 3.1.4. *For each positive integer p, let $M^{(p)}$ be a random planar map of boundary length p modeling a supercritical LQG disk. There exists a constant $\theta > 0$ such that if we condition $M^{(p)}$ to have finitely many vertices and rescale its graph distance by a factor of $\theta/\sqrt{p}$, then the corresponding metric space converges in law as $p \to \infty$ to the continuum random tree with respect to the Gromov–Hausdorff distance.*

The proof of Theorem 3.1.4 is based on the heuristic that a supercritical branching process becomes subcritical when conditioned to terminate at finite time. In our model, conditioning the supercritical map to be finite reduces the lengths of discrete loops corresponding to CLE_4 loops to be exponentially smaller, and the gaskets between these discrete loops become subcritical Boltzmann maps. It is known that subcritical Boltzmann maps with boundary have the continuum random tree as the scaling limit [42, 160, 189].

1.2 Loewner energy and Schramm–Loewner evolution

Given a chord η in the upper half-plane $\mathbb{H}$ from 0 to ∞, we can parameterize it by $\mathbb{R}_+$ such that there exists a unique conformal map $g_t : \mathbb{H} \setminus \eta(0,t) \to \mathbb{H}$ with the expansion $g_t(z) = z + 2t/z + O(1/z^2)$ at ∞ for each $t \in (0,\infty)$. The Loewner transform encodes the chord η by its driving function defined as $W_t := g_t(\eta(t))$. SLE_κ is a random non-crossing curve [218] with the driving function $W = \sqrt{\kappa}B$ where B is the standard Brownian motion. In the semiclassical limit of SLE_κ as $\kappa \to 0^+$, a large deviation principle has been established [244, 204] with a good rate function given by Loewner energy, defined as the Dirichlet energy of the driving function: $I(\eta) = \frac{1}{2}\int_0^\infty |\dot{W}_t|^2\, dt$. In line with the conformal invariance of the law of SLE, Loewner energy is defined to be a conformally invariant quantity.

The Loewner transform extends to a mapping from certain continuous driving functions $\mathbb{R} \to \mathbb{R}$ to rooted oriented loops on the Riemann sphere considered up to Möbius transformations. The loop Loewner energy I^L is again defined as the Dirichlet energy of the driving function; it is root-invariant and orientation-invariant [244, 219]. The (equivalence classes of) loops with finite Loewner energy comprise the Weil–Petersson Teichmüller space $T_0(1)$ [245], which carries a rich geometric structure of a connected topological group with right-invariant Kähler metric [239] that can be parameterized in terms of Beltrami coefficients on the unit disk or quasisymmetric homeomorphisms of the unit circle. The work [245] of Yilin Wang further showed that the loop Loewner energy corresponds to a constant multiple of the universal Liouville action, which was previously identified as a Kähler potential on $T_0(1)$ [239]. There are various further geometric and probabilistic characterizations of Loewner energy and the Weil–Petersson class of loops [52, 243, 246, 191, 60]. A new active area of research is to investigate the algebraic and geometric structures on SLE in relation to those on $T_0(1)$ [63, 113, 32, 33, 101].

1.2.1 Reversibility of chordal Loewner energy

A fundamental property of Loewner energy is its invariance under reversing the orientation of the chord despite the directionality of the Loewner transform.

Theorem 4.1.1. *Let η and $\hat{\eta}$ be chords in the upper half-plane $\mathbb{H}$ from 0 to ∞ such that $\hat{\eta}$ is the image of η under the inversion map $z \mapsto -1/z$. Then, their Loewner energies $I(\eta)$ and $I(\hat{\eta})$ are equal.*

It does not follow directly from the definition of Loewner energy as the relationship between a chord and the corresponding driving function is difficult to obtain explicitly. This reversibility of Loewner energy was previously established in [244] based on the reversibility of SLE_κ for $\kappa \in (0, 4]$ [254, 195, 173]. A deterministic proof was given in [245] based on the identification of Loewner energy with universal Liouville action.

In Chapter 4, which is based on the work [237], we give an elementary proof of this fact using the commutation relation for Loewner energy [244, Corollary 4.4] to establish that piecewise hyperbolic geodesic chords satisfy Theorem 4.1.1, which are dense in the space of all chords. This deterministic proof can be considered as the $\kappa \to 0^+$ limit of the SLE_κ reversibility proof in [254] and explains the geometric meaning of Zhan's martingale calculations.

In fact, our proof of Theorem 4.1.1 can be extended to show, in terms of Loewner energy minimizers, the existence of C^1 piecewise hyperbolic geodesics in each isotopy class of chords relative to a finite set of fixed points that they traverse through (Theorem 4.1.2). Combined with the proof of uniqueness in [55], the space of C^1 piecewise hyperbolic geodesics can be identified with the Teichmüller space of a disk with marked interior points, analogously to the loop case considered in that article.

1.2.2 First variation of Loewner energy under quasiconformal deformations

The Kähler structure of the Weil–Petersson Teichmüller space $T_0(1)$ is described naturally in terms of the Beltrami differentials, which are complex-valued, bounded functions on the Riemann sphere $\widehat{\mathbb{C}}$ describing infinitesimal quasiconformal deformations of $\widehat{\mathbb{C}}$ via the measurable Riemann mapping theorem. In Chapter 5, which is based on the joint work [238] with Yilin Wang, we give a formula describing the first variation of the loop driving function in terms of the Beltrami differential (Theorem 5.1.1). This is achieved through concrete calculations starting from an identification of the driving function of a two-sided Loewner chain $(g_t)_{t\in\mathbb{R}}$ in terms of the "Schwarzian derivative of g_t at ∞." It further leads to the following formula for the first variation of loop Loewner energy under infinitesimal quasiconformal deformations.

Theorem 5.1.4. *Let η be a Jordan curve in $\mathbb{C}$ and $\mu \in L^\infty(\mathbb{C})$ be a Beltrami differential with compact support in $\widehat{\mathbb{C}} \setminus \eta$. For $t \in \mathbb{R}$ with $\|t\mu\|_\infty < 1$, let $w^{t\mu}$ be any quasiconformal self-map of $\widehat{\mathbb{C}}$ with Beltrami coefficient $t\mu$. Let f be a conformal map taking the bounded and unbounded components of $\widehat{\mathbb{C}} \setminus \eta$ to those of $\widehat{\mathbb{C}} \setminus S^1$, respectively, where S^1 is the unit circle. Then,*

$$\frac{d}{dt}\bigg|_{t=0} I^L(w^{t\mu}(\eta)) = -\frac{4}{\pi} \operatorname{Re} \int_{\mathbb{C}} \mu(z)\, Sf(z)\, d^2z$$

where $Sf = f'''/f' - (3/2)(f''/f')$ is the Schwarzian derivative and d^2z is the Euclidean area measure.

In other words, the holomorphic stress-energy tensor of the Loewner energy is given by a constant multiple of the Schwarzian derivative of the uniformizing maps of the complementary components of the curve. This formula for first variation of Loewner energy coincides with that of the universal Liouville action — a key step in [239] from which a straightforward computation identifies the universal Liouville action as a Kähler potential on $T_0(1)$ — giving another explanation of the equivalence between these quantities.

1.2.3 Quasi-invariance of SLE welding measures

A series of recent works [245, 63, 113, 32] identified a close relationship between the Schramm–Loewner evolution (SLE) loop measure and the Weil–Petersson Teichmüller space $T_0(1)$, suggesting that the former measure is described in terms of a path integral formulation via Loewner energy. In Chapter 6, which is based on the joint work [133] with Shuo Fan, we further investigate this connection, focusing on the role of the group structure on the Weil–Petersson Teichmüller space. The group structure on $T_0(1)$ is given naturally by composition when it is modeled on circle homeomorphisms via conformal welding. The Weil–Petersson homeomorphisms of $\mathbb{S}^1$ are characterized by the condition $\log f' \in H^{1/2}(\mathbb{S}^1)$ [232], and Loewner energy is the unique right-invariant Kähler potential on this space [239].

In particular, we give the following Cameron–Martin type result for the SLE loop measure. For $\kappa \in (0, 4)$, we define $\mathrm{SLE}_\kappa^{\mathrm{weld}}$ to be a probability measure on the space of circle homeomorphisms obtained from the SLE_κ loop measure via conformal welding. We consider its quasi-invariance when composed with a fixed Weil–Petersson circle homeomorphism.

Theorem 6.1.2. *For $\kappa \in (0, 4)$, sample ψ_κ from $\mathrm{SLE}_\kappa^{\mathrm{weld}}$. We have the following for any fixed $\varphi \in \mathrm{WP}(\mathbb{S}^1)$ with $\varphi(1) = 1$.*

- *The law of $\psi_\kappa \circ \varphi$ is mutually absolutely continuous with respect to $\mathrm{SLE}_\kappa^{\mathrm{weld}}$.*
- *The law of $\varphi^{-1} \circ \psi_\kappa$ is mutually absolutely continuous with respect to $\mathrm{SLE}_\kappa^{\mathrm{weld}}$ if the log-ratio $\log |(\varphi(\cdot) - \varphi(1))/(\cdot - 1)|$ belongs to $H^{1/2}(\mathbb{S}^1)$.*

Our proof is based on the characterization of the composition action in terms of Hilbert–Schmidt operators on the Cameron–Martin space of the log-correlated Gaussian field. We also rely on the description of the SLE loop measure as the welding of two independent Liouville quantum gravity disks [19].

Part I

Liouville quantum gravity

CHAPTER 2

THE MINKOWSKI CONTENT MEASURE FOR THE LIOUVILLE QUANTUM GRAVITY METRIC

This chapter presents material first published in [133].

2.1 Introduction

2.1.1 Overview

Fix $\gamma \in (0, 2)$ and let $U \subset \mathbb{C}$ be an open domain. Let h be the Gaussian free field (GFF) on U or a minor variant thereof. The γ-Liouville quantum gravity (LQG) surface described by (U, h) is formally the two-dimensional Riemannian manifold with metric tensor

$$e^{\gamma h}(dx^2 + dy^2) \tag{2.1.1}$$

where $dx^2 + dy^2$ is the Euclidean metric tensor. LQG was introduced by Polyakov [206] in the physics literature as a canonical model of two-dimensional random geometry. Since then, LQG has been identified as the scaling limit of various types of random planar maps, as described in surveys [117, 123, 193, 229].

The expression (2.1.1) does not make literal sense because the GFF h does not admit pointwise values; it is defined only as a random distribution. Nevertheless, it is possible to rigorously construct the area measure and distance function corresponding to (2.1.1) through regularization and renormalization techniques.

The γ-LQG volume measure μ_h is the Gaussian multiplicative chaos measure associated with the GFF h, which can be constructed via various regularization methods [23, 37, 99, 146, 213]. In the context of LQG, it was first defined by Duplantier and Sheffield [99] to be the almost sure weak limit

$$\mu_h := \lim_{\varepsilon \to 0} \varepsilon^{\gamma^2/2} e^{\gamma h_\varepsilon(z)}\, d^2 z, \tag{2.1.2}$$

where $h_\varepsilon(z)$ is the average of h on the circle $\partial B_\varepsilon(z)$ and $d^2 z$ is the Lebesgue measure on U. Note that for a smooth function $f : U \to \mathbb{R}$, the volume form associated with the Riemannian metric tensor $e^f(dx^2 + dy^2)$ is $e^f d^2 z$. With probability one, μ_h is mutually singular with respect to the Lebesgue measure but has no point masses and assigns a positive mass to any open set. The circle average approximation can be replaced by other mollification methods for the GFF; see [23, 37, 146, 213] for further details. Notably, let $p_{\varepsilon^2/2}(z) = \frac{1}{\pi\varepsilon^2}\exp(-|z|^2/\varepsilon^2)$ be the heat kernel at time $\varepsilon^2/2$ and let

$$h^*_\varepsilon(z) := (h * p_{\varepsilon^2/2})(z) = \int_U h(w) p_{\varepsilon^2/2}(z - w)\, d^2 w. \tag{2.1.3}$$

Then $\lim_{\varepsilon \to 0} \varepsilon^{\gamma^2/2} e^{\gamma h^*_\varepsilon(z)} d^2 z = \mu_h$ in probability with respect to the topology of weak convergence of measures.[1]

More recently, the γ-LQG metric D_h was constructed as the scaling limit of Liouville first passage percolation (LFPP) in the case that $U = \mathbb{C}$ and h is a whole-plane GFF. For $z, w \in \mathbb{C}$ and $\varepsilon > 0$, the ε-LFPP metric is defined by

$$D^\varepsilon_h(z, w) := \inf_{P: z \to w} \int_0^1 e^{\xi h^*_\varepsilon(P(t))} |P'(t)|\, dt, \tag{2.1.4}$$

where the infimum is taken over all piecewise continuously differentiable paths P from z to w. Note that for a smooth function $f : \mathbb{C} \to \mathbb{R}$, the distance between z and w corresponding

1. Most works on Gaussian multiplicative chaos require GFF to be regularized using compactly supported mollifiers. The heat kernel mollification of GFF is considered in [213] (which calls it "white noise decomposition"), where it is shown that $\varepsilon^{\gamma^2/2} e^{\gamma h^*_\varepsilon(z)} d^2 z$ converges to μ_h in law. One can extend this to convergence in probability by applying existing methods, for instance by checking [37, Lemma 3.5] using the covariance formula $\mathrm{Cov}(h^*_\varepsilon(z), h^*_r(w)) = \pi \int_{(\varepsilon^2 + r^2)/2}^\infty p_t(z - w)\, dt$.

to the Riemannian metric tensor $e^f(dx^2+dy^2)$ is $\inf_{P:z\to w}\int_0^1 e^{f(P(t))/2}|P'(t)|\,dt$. Here,

$$\xi := \frac{\gamma}{d_\gamma}, \tag{2.1.5}$$

where $d_\gamma > 2$ is the dimension of γ-LQG. The constant d_γ is obtained a priori as a scaling exponent corresponding to various approximations of the γ-LQG metric space [86, 90]. Ding, Dubédat, Dunlap, and Falconet [85] showed that for a suitable choice of deterministic scaling constants $\{\mathfrak{a}_\varepsilon\}_{\varepsilon>0}$, the family of rescaled LFPP metrics $\{\mathfrak{a}_\varepsilon^{-1}D_h^\varepsilon\}_{\varepsilon\in(0,1)}$ is tight. Building furthermore on [93, 125, 126], Gwynne and Miller [129] showed that the subsequential limit is unique and defined γ-LQG metric as the limit

$$D_h := \lim_{\varepsilon\to 0}\mathfrak{a}_\varepsilon^{-1}D_h^\varepsilon \tag{2.1.6}$$

in probability with respect to the local uniform topology on $\mathbb{C}\times\mathbb{C}$. In particular, D_h almost surely induces the Euclidean topology. A posteriori, d_γ was identified as both the Hausdorff dimension [132] and the Minkowski dimension [16] of the γ-LQG metric space $(\mathbb{C}, D_h)$.

It is natural to ask how the measure μ_h and the metric D_h are related. In this paper, we show that μ_h is almost surely equal to the Minkowski content measure with respect to D_h (Theorem 2.1.1). This answers [129, Problem 7.10]. Our result can be viewed as an LQG analog of [169], which constructed the Minkowski content for Schramm–Loewner evolution (SLE) curves and showed that it is equivalent to the so-called natural parameterization of SLE [171].

A particular consequence of our result is that D_h almost surely determines μ_h. It was shown in [16, Theorem 1.3] that the pointed metric measure space $(\mathbb{C}, 0, D_h, \mu_h)$ almost surely determines h up to rotation and scaling. Therefore, our result shows that the pointed metric space $(\mathbb{C}, 0, D_h)$ almost surely determines h modulo rotation and scaling (Corollary 2.1.2).

The primary tool in our proofs is the mating-of-trees theorem of Duplantier, Miller, and

Sheffield [95]. This theorem says that the left/right boundary length process for a space-filling SLE curve η on an LQG surface is a correlated two-dimensional Brownian motion. Roughly speaking, this result is useful for two reasons. First, it gives a source of exact independence since the LQG surfaces traced by the curve η during disjoint time intervals are independent. In particular, this leads to a short proof of a lower bound for the number of LQG metric balls needed to cover a given set (Proposition 2.4.1) without needing a separate two-point estimate. Second, the mating-of-trees theory provides a convenient way of decomposing space into regions of equal LQG mass, namely the segments $\eta([x-\varepsilon, x])$ for $x \in \varepsilon\mathbb{Z}$, where η is parameterized by μ_h-mass. See Section 2.2.6 for more details on the proof method.

The mating-of-trees theory has many applications in the study of random conformal geometry, including LQG and SLE; see [122] for a survey of these applications. This is the first paper to use this theory to prove properties of the LQG metric for general $\gamma \in (0,2)$. We expect that there will be more applications of the mating-of-trees theory to the LQG metric in the future. In the course of our proof, we obtain some estimates for space-filling SLE on an LQG surface, which are of independent interest. We especially highlight the Hölder continuity result (Theorem 2.1.4), which is already used in the paper [56] to prove a result about random permutations.

Previously, Le Gall [175] showed that the volume measure of the Brownian sphere is equal to a constant multiple of the Hausdorff measure associated with the gauge function $r^4 \log\log(1/r)$. This implies that D_h almost surely determines μ_h for $\gamma = \sqrt{8/3}$, due to the equivalence between the Brownian sphere and the $\sqrt{8/3}$-LQG sphere established by Miller and Sheffield [197, 198, 199]. See also [129, Corollary 1.4] for the fact that the Miller–Sheffield metric agrees with the limit of LFPP. In a sense, our result is a generalization of Le Gall's result to general $\gamma \in (0,2)$, but with Minkowski content instead of Hausdorff measure. Our proof and Le Gall's have some superficial similarities, but the main techniques are fundamentally different.

2.1.2 Main results

We say that a random distribution h on $\mathbb{C}$ is a whole-plane GFF plus a continuous function if there is a coupling of h with a random continuous function $f : \mathbb{C} \to \mathbb{R}$ such that $h - f$ has the law of a whole-plane GFF (see Section 2.2.2 for the definition of the whole-plane GFF). Our main theorem states that for each $\gamma \in (0, 2)$, the Minkowski content measure (see Definition 2.2.4) exists on the γ-LQG metric space $(\mathbb{C}, D_h)$ and is equal to the corresponding γ-LQG measure μ_h.

Theorem 2.1.1. *Fix $\gamma \in (0, 2)$. There exists a deterministic sequence $\{\mathfrak{b}_\varepsilon\}_{\varepsilon>0}$ such that the following is true.*

Let h be a whole-plane GFF plus a continuous function and D_h be the corresponding γ-LQG metric. Suppose $A \subset \mathbb{C}$ is either:

(i) a deterministic bounded Borel set; or

(ii) a random compact set, measurable with respect to the Borel σ-algebra induced by the Hausdorff distance on $\mathbb{C}$, that is coupled with the random distribution h

such that $\mu_h(\partial A) = 0$ almost surely. Let $N_\varepsilon(A; D_h)$ be the minimum number of D_h-balls of radius $\varepsilon > 0$ required to cover A. Then,

$$\lim_{\varepsilon\to 0} \mathfrak{b}_\varepsilon^{-1} N_\varepsilon(A; D_h) = \mu_h(A) \quad \textit{in probability.} \tag{2.1.7}$$

We do not exclude the possibility that the limit (2.1.7) may hold almost surely. However, our methods are insufficient to prove how fast the sequence $\mathfrak{b}_\varepsilon^{-1} N_\varepsilon(A; D_h)$ converges. We expect that stronger LQG metric estimates than those currently available are necessary to prove quantitative bounds required for almost sure convergence.

A corollary of Theorem 2.1.1 is that the metric space structure of a γ-LQG surface is sufficient information to determine not only its metric measure space structure but also its conformal structure, in the sense that it almost surely determines the field h.

Corollary 2.1.2. *Let $\gamma \in (0,2)$. Suppose h is a whole-plane GFF normalized to have a mean zero on the unit circle. Let D_h and μ_h be the corresponding γ-LQG metric and measure, respectively. Then the random pointed metric space $(\mathbb{C}, 0, D_h)$ almost surely determines the random pointed metric measure space $(\mathbb{C}, 0, D_h, \mu_h)$, and moreover the field h up to rotation and scaling of the complex plane.*

By considering $(\mathbb{C}, 0, D_h)$ as a random pointed metric space, we forget the parameterization of D_h in the complex plane. More precisely, we consider it as a random element in the space of isometry classes of complete and locally compact length spaces endowed with the local Gromov–Hausdorff topology (as defined by Gromov in [114]). Similarly, the random pointed metric measure space $(\mathbb{C}, 0, D_h, \mu_h)$ in Corollary 2.1.2 is measurable with respect to the local Gromov–Hausdorff–Prokhorov topology [1]. It was known previously that the random pointed metric measure space $(\mathbb{C}, 0, D_h, \mu_h)$ almost surely determines the field h up to rotation and scaling. For $\gamma = \sqrt{8/3}$, this fact was first established in [199] as a key component of the equivalence between the Brownian sphere and the $\sqrt{8/3}$-LQG sphere. An explicit way of reconstructing h from the pointed metric measure space $(\mathbb{C}, 0, D_h, \mu_h)$ was given in [130], which was extended to all $\gamma \in (0,2)$ in [16].

Our choice of scaling constants $\{\mathfrak{b}_\varepsilon\}_{\varepsilon>0}$ has the following description which depends only on $\gamma \in (0,2)$. Let $(\mathbb{C}, h^\gamma, 0, \infty)$ be a γ-quantum cone under the circle average embedding (see Section 2.2.5 for definitions). Let η be a whole-plane space-filling $\mathrm{SLE}_{\kappa'}$ curve from ∞ to ∞ with $\kappa' = 16/\gamma^2$ that is sampled independently of h^γ. Then parameterize η by the γ-LQG measure: that is, $\eta(0) = 0$ and $\mu_h(\eta[s,t]) = t - s$ for all real $s \le t$. (We review this setup further in Section 2.2.5.) Let D_{h^γ} be the γ-LQG metric associated with h^γ. For $\varepsilon > 0$, define

$$\mathfrak{b}_\varepsilon := \mathbb{E}[N_\varepsilon(\eta[0,1]; D_{h^\gamma})]. \tag{2.1.8}$$

While we do not have an exact formula for $\mathfrak{b}_\varepsilon$, the following properties justify calling the limit in (2.1.7) the d_γ-Minkowski content of A with respect to the metric D_h (also see

Section 2.2.1).

Proposition 2.1.3. *Let $\mathfrak{b}_\varepsilon$ be as in* (2.1.8). *There exist constants* $0 < c_1 < c_2 < \infty$ *such that for all* $\varepsilon \in (0,1)$,

$$c_1 \varepsilon^{-d_\gamma} \leq \mathfrak{b}_\varepsilon \leq c_2 \varepsilon^{-d_\gamma}. \tag{2.1.9}$$

Moreover, the function $\varepsilon \mapsto \mathfrak{b}_\varepsilon$ is regularly varying with index $-d_\gamma$; i.e., for every $r > 0$ we have

$$\lim_{\varepsilon \to 0} \frac{\mathfrak{b}_{r\varepsilon}}{\mathfrak{b}_\varepsilon} = r^{-d_\gamma}. \tag{2.1.10}$$

As a byproduct of our proof of Theorem 2.1.1, we obtain a Hölder continuity result for the space-filling SLE curve on an LQG metric space.

Theorem 2.1.4. *Let $\gamma \in (0,2)$ and $\kappa' \in (4,\infty)$ be constants, which do not necessarily satisfy $\kappa' = 16/\gamma^2$. Let h^γ be the field of a γ-quantum cone under the circle average embedding, and let η be a whole-plane space-filling $SLE_{\kappa'}$ from ∞ to ∞ which is sampled independently of h^γ and then parameterized by μ_{h^γ}. Almost surely, η on the metric space $(\mathbb{C}, D_{h^\gamma})$ is locally Hölder continuous with any exponent less than $1/d_\gamma$ and is not locally Hölder continuous with any exponent greater than $1/d_\gamma$.*

Theorem 2.1.4 is used in [56, Section 4] to show that the dimensions of the supports of certain random permutons defined in terms of SLE-decorated LQG are almost surely equal to one. Since LQG decorated by space-filling SLE is related to many other mathematical objects [123], we expect that the theorem will have more applications in the future.

2.1.3 Notations

We use the following notations throughout the paper.

- The constant $\gamma \in (0,2)$ is fixed, and we do not consider multiple values of γ simultaneously. When we do not specify the value of γ (e.g., in expressions such as "LQG metric ball" or "quantum dimension"), we refer to the corresponding γ-LQG quantities.

- We write d_γ for the dimension of γ-LQG. We also use the γ-dependent constants

$$Q = \frac{\gamma}{2} + \frac{2}{\gamma} \quad \text{and} \quad \xi = \frac{\gamma}{d_\gamma}. \tag{2.1.11}$$

- Given a random distribution h on $\mathbb{C}$, we denote the associated γ-LQG metric as μ_h and the γ-LQG measure as D_h.

- We denote by $B_r(z)$ the Euclidean ball of radius r centered at z. $\mathcal{B}_r(z; D_h) := \{w \in \mathbb{C} : D_h(z,w) < r\}$ is the D_h-metric ball of radius r centered at z. For a set $A \subset \mathbb{C}$, we denote by $B_r(A) := \bigcup_{z \in A} B_r(z)$ and $\mathcal{B}_r(A; D_h) := \bigcup_{z \in A} \mathcal{B}_r(z; D_h)$ its r-neighborhoods in the Euclidean and LQG metrics, respectively.

- We say that an event E_ε indexed by $\varepsilon > 0$ occurs with superpolynomially high probability if, for each $p > 0$, we have $\mathbb{P}(E_\varepsilon) \geq 1 - \varepsilon^p$ for sufficiently small $\varepsilon > 0$.

- Let X and Y be random variables coupled on a probability space, taking values in measurable spaces $\mathcal{X}$ and $\mathcal{Y}$, respectively. We say that Y is almost surely determined by X if there is a measurable function $F : \mathcal{X} \to \mathcal{Y}$ such that $Y = F(X)$ almost surely.

2.1.4 Outline

We review the necessary preliminaries in Section 2.2. With these preliminaries in hand, we present a detailed overview of our proof of the main results in Section 2.2.6. Section 2.3 is dedicated to showing the tightness of the Minkowski content approximations. In Section 2.4, we prove a key stepping stone towards the main results: the convergence in (2.1.7) of the normalized covering number $\mathfrak{b}_\varepsilon^{-1} N_\varepsilon(A; D_h)$ to the γ-LQG area $\mu_h(A)$ when A is a space-filling SLE segment of fixed LQG area. We extend this convergence to general sets A and fields h and complete the proofs of the main results in Section 2.5.

2.2 Preliminaries

We review a few preliminaries, including the Minkowski content measure on a metric space (Section 2.2.1), the axiomatic characterization of the LQG metric (Section 2.2.3), and the continuum mating-of-trees theory (Section 2.2.5).

We also prove some extensions of known LQG results. In Section 2.2.4, we show that the conformal coordinate change rule for the LQG metric [128] extends to certain random scalings and translations. In Section 2.2.5, we use these results to prove that the mating-of-trees theorem is, in a certain precise sense, compatible with the LQG metric.

In Section 2.2.6, we give an overview of the proof of Theorem 2.1.1 and a comparison with Le Gall's proofs in [175].

2.2.1 Minkowski content measure

Let (X, d) be a metric space. Given a set $A \subset X$, let $N_\varepsilon(A; d)$ be the minimum number of metric balls with radius $\varepsilon > 0$ required to cover A. The Minkowski dimension of a set A is defined as

$$\dim_{\mathrm{M}}(A; d) = \lim_{\varepsilon \to 0} \frac{\log N_\varepsilon(A; d)}{\log \varepsilon^{-1}} \tag{2.2.1}$$

if the limit exists. There are several equivalent descriptions of the Minkowski dimension. For instance, we can replace the covering number $N_\varepsilon(A; d)$ with the packing number $N_\varepsilon^{\mathrm{pack}}(A; d)$, which is the maximum possible number of disjoint metric balls with radius ε whose centers all lie in A. These two definitions are equivalent because $N_\varepsilon^{\mathrm{pack}}(A; d) \leq N_\varepsilon(A; d) \leq N_{\varepsilon/2}^{\mathrm{pack}}(A; d)$ for every $\varepsilon > 0$.

The Minkowski dimension of the γ-LQG metric is the Minkowski dimension of any open set with respect to the γ-LQG metric. In [16], this quantity was shown to be equal to d_γ. The proof was based on the estimate (2.2.2) on the volume of LQG metric balls, which we utilize prominently throughout our paper.

Theorem 2.2.1 ([16, Theorem 1.1])**.** *Let h be a whole-plane GFF normalized so that $h_1(0) = 0$. For any compact set $K \subset \mathbb{C}$ and $\zeta > 0$, almost surely,*

$$\sup_{\varepsilon \in (0,1)} \sup_{z \in K} \frac{\mu_h(\mathcal{B}_\varepsilon(z; D_h))}{\varepsilon^{d_\gamma - \zeta}} < \infty \quad \textit{and} \quad \inf_{\varepsilon \in (0,1)} \inf_{z \in K} \frac{\mu_h(\mathcal{B}_\varepsilon(z; D_h))}{\varepsilon^{d_\gamma + \zeta}} > 0. \tag{2.2.2}$$

Moreover, for any bounded Borel measurable set $A \subset \mathbb{C}$ containing an open set, almost surely,

$$\lim_{\varepsilon \to 0} \frac{\log N_\varepsilon(A; D_h)}{\log \varepsilon^{-1}} = d_\gamma. \tag{2.2.3}$$

The Minkowski content is a method of assigning sizes to subsets of a metric space by using the quantities used to find their Minkowski dimensions. It has several different definitions in the literature which are not equivalent. One such definition is the following: if the limit computed in (2.2.1) equals δ, then the Minkowski content of A is

$$\mathrm{Cont}_\delta(A; d) = \lim_{\varepsilon \to 0} \varepsilon^\delta N_\varepsilon(A; d) \tag{2.2.4}$$

if the limit exists. Replacing $N_\varepsilon(A; d)$ with other quantities that give rise to equivalent definitions of the Minkowski dimension, such as $N_\varepsilon^{\mathrm{pack}}(A; d)$, do not necessarily give identical values for the Minkowski content. As such, it makes sense to introduce the following general notion of Minkowski content.

Definition 2.2.2. For a constant $\delta > 0$ and a family of constants $b = \{b_\varepsilon\}_{\varepsilon > 0}$, we say that b is a sequence of δ-dimensional rescaling coefficients if the following two conditions are satisfied.

(i) There exist constants $0 < c_1 < c_2 < \infty$ and $\varepsilon_0 > 0$ such that $c_1 \varepsilon^{-\delta} \le b_\varepsilon \le c_2 \varepsilon^{-\delta}$ for every $\varepsilon \in (0, \varepsilon_0)$.

(ii) The function $\varepsilon \mapsto b_\varepsilon$ is regularly varying at 0 with index $-\delta$. That is, $\lim_{\varepsilon \to 0} b_{r\varepsilon}/b_\varepsilon = r^{-\delta}$ for every $r > 0$.

For instance, $\{\varepsilon^{-\delta}\}_{\varepsilon>0}$ is a trivial sequence of δ-dimensional rescaling coefficients. Proposition 2.1.3 says that $\{\mathfrak{b}_\varepsilon\}_{\varepsilon>0}$ as defined in (2.1.8) is a sequence of d_γ-dimensional rescaling coefficients. In Definition 2.2.2, we require condition (ii) so that $\varepsilon^\delta b_\varepsilon$ does not fluctuate arbitrarily as $\varepsilon \to 0$.

Definition 2.2.3. Let (X, d) be a metric space and $b = \{b_\varepsilon\}_{\varepsilon>0}$ be a sequence of δ-dimensional rescaling coefficients. For $A \subset X$ with $\dim_{\mathrm{M}}(A; d) = \delta$, the Minkowski content of A with respect to coefficients b is the limit

$$\mathrm{Cont}_b(A; d) = \lim_{\varepsilon\to 0} b_\varepsilon^{-1} N_\varepsilon(A; d). \tag{2.2.5}$$

if it exists. In that case, we say that A is Minkowski measurable with respect to coefficients b.

Theorem 2.1.1 states that with $\mathfrak{b} = \{\mathfrak{b}_\varepsilon\}_{\varepsilon>0}$ as in (2.1.8), for every bounded Borel set $A \subset \mathbb{C}$ with $\mu_h(\partial A) = 0$, the Minkowski content of A with respect to $\mathfrak{b}$ exists and is almost surely equal to $\mu_h(A)$. The condition $\mu_h(\partial A) = 0$ is natural; if an open set $A \subset X$ is Minkowski measurable with respect to coefficients b, then $\mathrm{Cont}_b(A; d) = \mathrm{Cont}_b(\overline{A}; d)$. This is since $N_\varepsilon(A; d) \le N_\varepsilon(\overline{A}; D) \le N_{(1-\zeta)\varepsilon}(A; d)$ for every $\varepsilon > 0$ and $\zeta \in (0, 1)$, and we require b_ε to vary regularly.

In general, it is difficult to a priori find the correct coefficients $\{b_\varepsilon\}_{\varepsilon>0}$ such that the limit (2.2.5) exists. Indeed, we will use (2.2.4) as an ansatz for the Minkowski content in γ-LQG, and we show only in a later stage of the proof that (2.1.8) is the correct rescaling coefficient to use.

One reason for considering the Minkowski content for the LQG metric space is that the Minkowski content is extremely useful in the context of random fractal subsets. For instance, the Minkowski content has been used to construct natural measures on the Schramm–Loewner evolution (SLE) curve and its subsets (e.g., [9, 136, 168, 169, 256]). However, the

Minkowski content has been considered traditionally in the context of fractal subsets of Euclidean spaces. When A is a fractal subset of a Euclidean space $\mathbb{R}^n$, the Minkowski dimension of A can be defined equivalently as

$$\dim_{\mathrm{M}}(A) = n - \lim_{\varepsilon \to 0} \frac{\log(\mathrm{vol}\, B_\varepsilon(A))}{\log \varepsilon}. \tag{2.2.6}$$

This is why the δ-dimensional Minkowski content of $A \subset \mathbb{R}^n$ for $\delta < n$ is defined usually as

$$\mathrm{Cont}_\delta(A) = \lim_{\varepsilon \to 0} \frac{\mathrm{vol}\, B_\varepsilon(A)}{\varepsilon^{n-\delta}}. \tag{2.2.7}$$

We eventually wish to construct a Borel measure from the Minkowski content on an LQG metric space and compare it with the LQG measure in Corollary 2.1.2. However, Minkowski content is not countably additive in general. Thus, we define a Minkowski content measure to be any Borel measure that is compatible with some version of the Minkowski content. The following definition aligns with various definitions of similar concepts in the literature (e.g., [140, 256]).

Definition 2.2.4. Let (X, d) be a locally compact metric space whose Minkowski dimension is $\delta > 0$: i.e., $\dim_{\mathrm{M}}(U; d) = \delta$ for every totally bounded open set $U \subset X$. A Minkowski content measure on X is a Borel measure μ on X that satisfies the following conditions.

(i) The measure μ is finite on all compact subsets of X.

(ii) There is a sequence $b = \{b_\varepsilon\}_{\varepsilon > 0}$ of δ-dimensional rescaling coefficients such that $\mathrm{Cont}_b(K; d)$ exists and equals $\mu(K)$ for every compact set $K \subset X$ with $\mu(\partial K) = 0$.

In other words, Theorem 2.1.1 states that the LQG measure μ_h is a Minkowski content measure on the LQG metric space $(\mathbb{C}, D_h)$. In the proof of Corollary 2.1.2, we give an explicit method to recover μ_h as a Minkowski content measure on the LQG metric space $(\mathbb{C}, D_h)$. In particular, we give a random π-system of compact sets coupled with the GFF h which a.s. generates the Borel σ-algebra, so that the values of Minkowski content $\mathrm{Cont}_{\mathfrak{d}}(K; D_h)$ for the sets K in this π-system a.s. uniquely determine the Minkowski content measure μ_h.

2.2.2 Whole-plane GFF

We give a brief introduction to the whole-plane GFF insofar as it is relevant to the rest of the paper. We refer the reader to the introductory sections of [95, 196] and the expository articles [38, 225, 251] for further details.

The whole-plane Gaussian free field (GFF) h is a centered Gaussian process on $\mathbb{C}$ with covariances

$$\mathrm{Cov}(h(z), h(w)) = G(z,w) := \log \frac{(\max\{|z|,1\})(\max\{|w|,1\})}{|z-w|}, \quad \forall z, w \in \mathbb{C}. \tag{2.2.8}$$

This definition does not make literal sense since $\lim_{z\to w} G(z,w) = \infty$, but we can make sense of the whole-plane GFF as a random distribution (i.e., generalized function). Let $\mathcal{M}$ be the collection of signed Borel measures ρ satisfying $\int_{\mathbb{C}\times\mathbb{C}} G(z,w)\, |\rho|(dz)|\rho|(dw) < \infty$ with compact support on $\mathbb{C}$. We define [251, Section 3.1] the whole-plane GFF as the centered Gaussian process indexed by $\mathcal{M}$ with

$$\mathrm{Cov}((h,\rho_1),(h,\rho_2)) = \int_{\mathbb{C}\times\mathbb{C}} G(z,w)\, \rho_1(dz)\rho_2(dw). \tag{2.2.9}$$

A random distribution h on $\mathbb{C}$ is called a whole-plane GFF plus a continuous function if there exists a coupling of h with a random continuous function $f : \mathbb{C} \to \mathbb{R}$ such that $h - f$ has the law of a whole-plane GFF.

Given a whole-plane GFF h, we define the circle average on the circle $\partial B_r(z)$ as the pairing $h_r(z) := (h, \lambda_{z,r})$ where $\lambda_{z,r}$ is the uniform probability measure on the circle $\partial B_r(z)$. The following properties of the circle average process were given in [99, Section 3.1].

Lemma 2.2.5. *There exists a version of whole-plane GFF h such that the map $(z,r) \mapsto$*

$h_r(z)$ is a.s. continuous on $\mathbb{C} \times (0, \infty)$. Moreover, for each $z \in \mathbb{C}$, the process $\{B_t\}_{t \in \mathbb{R}}$, $B_t := h_{e^{-t}}(z)$, is a two-sided standard Brownian motion with the initial value $B_0 = h_1(z)$.

Let h be a whole-plane GFF plus a continuous function. By the circle average part of h, we refer to the function $f(z) = h_{|z|}(0)$. By the lateral part of h, we refer to the distribution $g = h - f$. The following is a key property of the whole-plane GFF used in this paper.

Lemma 2.2.6 ([95, Lemma 4.9])**.** *The circle average and lateral parts of the whole-plane GFF are independent.*

We note that the term whole-plane GFF also refers to the random distribution considered modulo additive constant. Our choice of covariance kernel in (2.2.8) corresponds to fixing the additive constant so that the average $h_1(0)$ of h over the unit circle is zero [241, Section 2.1.1]. The law of the whole-plane GFF is invariant under deterministic complex affine transformations of $\mathbb{C}$, modulo additive constant. That is, if $a \in \mathbb{C} \setminus \{0\}$ and $b \in \mathbb{C}$, then

$$h(a \cdot +b) - h_{|a|}(b) \stackrel{d}{=} h. \tag{2.2.10}$$

In the few instances where we refer to the whole-plane GFF in this sense, we always write explicitly that we are considering it modulo additive constant.

Remark 2.2.7. We close our discussion of the planar GFF with a brief discussion of the free-boundary GFF on the upper half-plane $\mathbb{H} = \{z : \operatorname{Im} z > 0\}$, which will be relevant in Section 2.2.5. This is a random distribution $\tilde{h}$ on $\mathbb{H}$ which has the law of $(h(z) + h(\bar{z}))/2$ where h is a whole-plane GFF; it can be rigorously defined as a centered Gaussian process indexed by signed Borel measures in $\mathcal{M}$ which are supported on $\mathbb{H}$, with $\operatorname{Cov}(\tilde{h}(\rho_1), \tilde{h}(\rho_2)) = \int_{\mathbb{H}\times\mathbb{H}}[G(z,w) + G(z,\bar{w})]\rho_1(dz)\rho_2(dw)$ where G is the whole-plane Green's function given in (2.2.8). For $x \in \mathbb{R}$ and $r > 0$, we define $\tilde{h}_r(x)$ to be the paring $\tilde{h}(\tilde{\lambda}_{x,r})$ where $\tilde{\lambda}_{x,r}$ is the uniform probability measure on the semicircle $\partial B_r(x) \cap \mathbb{H}$. Define $\tilde{h}_{\|}(z) := \tilde{h}(z) - \tilde{h}_{|z|}(0)$ be the projection of $\tilde{h}$ onto the space of functions that have average zero on all semi-circles centered at the origin ("lateral part"). Then, similarly to Lemma 2.2.6, the semicircle averages $\{\tilde{h}_r(0)\}_{r>0}$ and the lateral part $\tilde{h}_{\|}$ are independent [95, Lemma 4.2].

2.2.3 LQG metric axioms

Due to its variational formulation, it is difficult to work with the definition (2.1.6) of the LQG metric. The axiomatic characterization of the LQG metric given in [129] is often a more tractable means of studying the LQG metric.

Before we state the LQG metric axioms, we recall the following definitions regarding metric spaces. Let (X, D) be a metric space. A curve in X is a continuous function $P : [a, b] \to X$. The D-length of P is

$$\mathrm{len}(P; D) = \sup_T \sum_{i=1}^{|T|} D(P(t_{i-1}), P(t_i)) \tag{2.2.11}$$

where the supremum is over all partitions $T : a = t_0 < t_1 < \cdots < t_{|T|} = b$. For $Y \subset X$, the internal metric of D on Y is defined as

$$D^Y(x, y) = \inf_{P \subset Y} \mathrm{len}(P; D) \tag{2.2.12}$$

where the infimum is over all paths P in Y from x to y.

We say that (X, D) is a length space if for each $x, y \in X$ and $\varepsilon > 0$, there exists a curve P in X from x to y with $\mathrm{len}(P; D) < D(x, y) + \varepsilon$. We say that a metric D is continuous metric on an open domain $U \subset \mathbb{C}$ if it induces the Euclidean topology on U. In the following, we equip the space of continuous metrics on U with the local uniform topology for functions $U \times U \to [0, \infty)$.

Definition 2.2.8. For $\gamma \in (0, 2)$, a γ-LQG metric is a Borel measurable function $h \mapsto D_h$ from the space of distributions on $\mathbb{C}$ to the space of the continuous metrics on $\mathbb{C}$ such that the following are true whenever h is a whole-plane GFF plus a continuous function. Here, Q and ξ are constants defined in (2.1.11).

I. *Length space.* $(\mathbb{C}, D_h)$ is almost surely a length space. That is, the D_h-distance between any two points in $\mathbb{C}$ is the infimum over the D_h-lengths of curves between these two points.

II. *Locality.* Let $U \subset \mathbb{C}$ be a deterministic set. Then the internal metric D_h^U is almost surely determined by (cf. Section 2.1.3) the restriction $h|_U$.[2]

III. *Weyl scaling.* For a continuous function $f : \mathbb{C} \to \mathbb{R}$, define

$$(e^{\xi f} \cdot D_h)(z, w) := \inf_{P:x \to y} \int_0^{\text{len}(P;D_h)} e^{\xi f(P(t))} \, dt \quad \forall z, w \in \mathbb{C} \tag{2.2.13}$$

where the infimum is over all curves from z to w parameterized by D_h-length. The following holds almost surely: we have $D_{h+f} = e^{\xi f} \cdot D_h$ for every continuous function $f : \mathbb{C} \to \mathbb{R}$.

IV. *Coordinate change for translation and scaling.* For each fixed deterministic $r > 0$ and $z \in \mathbb{C}$, almost surely

$$D_h(ru + z, rv + z) = D_{h(r\cdot+z)+Q\log r}(u, v) \quad \forall u, v \in \mathbb{C}. \tag{2.2.14}$$

In [129], it was shown that the random metric defined in (2.1.6) using LFPP is a γ-LQG metric as in the sense of the above definition, and each γ-LQG metric is a deterministic constant multiple of it. Hence, it makes sense to refer to (2.1.6) as the γ-LQG metric. The paper [93] contains an extensive list of estimates for the LQG metric deduced from the axiomatic definition, which we introduce as necessary.

2. The restriction $h|_U$ of the whole-plane GFF h to U can be defined precisely as the process $\{(h, \rho)\}_{\rho \in \mathcal{M}_U}$ where the index set $\mathcal{M}_U$ comprises signed Borel measures $\rho \in \mathcal{M}$ (recall Section 2.2.2) with $\text{supp}(\rho) \subset U$.

Finally, we refer the reader to [125, Remark 1.2] for the definition of the γ-LQG metric on a proper subdomain of $\mathbb{C}$.

2.2.4 Quantum surfaces

Recall from (2.1.11) that $Q = 2/\gamma + \gamma/2$. Consider the pair (U, h) where $U \subset \mathbb{C}$ is an open set and h is a distribution on U. A γ-quantum surface (or a γ-LQG surface) is an equivalence class of such pairs where $(U, h) \sim (\tilde{U}, \tilde{h})$ if there exists a conformal transformation $\phi : \tilde{U} \to U$ such that

$$\tilde{h} = h \circ \phi + Q \log |\phi'|. \tag{2.2.15}$$

An embedding of a quantum surface is a choice of representative (U, h) from the equivalence class.

We often consider quantum surfaces with additional structures. As before, let $U \subset \mathbb{C}$ be an open set and h be a distribution on U. Let $z_1, \ldots, z_k$ be points in $\overline{U}$. A γ-quantum surface with k marked points is an equivalence class of the tuples $(U, h, z_1, \ldots, z_k)$ where $(U, h, z_1, \ldots, z_k) \sim (\tilde{U}, \tilde{h}, \tilde{z}_1, \ldots, \tilde{z}_k)$ if there exists a conformal transformation $\phi : \tilde{U} \to U$ such that $z_j = \phi(\tilde{z}_j)$ for $j = 1, \ldots, k$ in addition to (2.2.15). If η is a curve in $\overline{U}$, then a curve-decorated γ-quantum surface is an equivalence class of triples (U, h, η) where $(U, h, \eta) \sim (\tilde{U}, \tilde{h}, \tilde{\eta})$ if there exists a conformal transformation $\phi : \tilde{U} \to U$ such that $\eta = \phi \circ \tilde{\eta}$ in addition to (2.2.15). We define a curve-decorated γ-quantum surface with k marked points by combining these definitions.

The equivalence relation (2.2.15) of a γ-LQG surface is chosen so that the γ-LQG measure and metric transform naturally between different embeddings of the same quantum surface. Suppose h is a GFF plus a continuous function on an open set $U \subset \mathbb{C}$.[3] Let $\phi : \tilde{U} \to U$ be

3. We say that a random distribution h on an open subset $U \subset \mathbb{C}$ is a GFF plus a continuous function if it can be coupled with a random continuous function $f : U \to \mathbb{C}$ such that $h - f$ has the law of a zero-boundary GFF on U (or a whole-plane GFF if $U = \mathbb{C}$). Note the definition of the whole-plane GFF plus a continuous function above Theorem 2.1.1.

a fixed conformal transformation. It was established in [99] that, almost surely,

$$\mu_h(A) = \mu_{h\circ\phi+Q\log|\phi'|}(\phi^{-1}(A)) \quad \text{for every Borel } A \subset U \tag{2.2.16}$$

and in [128] that, almost surely,

$$D_h(z,w) = D_{h\circ\phi+Q\log|\phi'|}(\phi^{-1}(z),\phi^{-1}(w)) \quad \text{for every } z,w \in U. \tag{2.2.17}$$

While not sufficiently emphasized in the early literature on LQG, it is necessary to include random conformal transformations in the definition of quantum surfaces to be able to compare their laws under a canonical embedding rule. For instance, there are uncountably many ways to embed a quantum surface with the disk topology into the unit disk $\mathbb{D}$ if it is unmarked or has only one or two marked boundary points. To specify a canonical embedding, we need to use information about the GFF. This requires a random conformal change of coordinates since the field is random.

Nevertheless, this random change in coordinates does not present an issue for the LQG measure. The following theorem implies that if ϕ is the random conformal transformation that maps a given embedding (U,h) of a quantum surface to its canonical embedding $(\tilde{U},\tilde{h})$, then it is a.s. the case that the LQG measure $\mu_{\tilde{h}}$ is well-defined and equal to the pushforward measure $\phi_*\mu_h$.

Theorem 2.2.9 ([230, Theorem 1.4])**.** *Let $U \subset \mathbb{C}$ be a simply connected domain and h be a GFF plus a continuous function on U. Let Λ be the collection of all conformal maps $\phi : \tilde{U} \to U$ where $\tilde{U} \subset \mathbb{C}$ is any simply connected domain. It is almost surely the case that for all $\phi \in \Lambda$, the measures $\mu_{h\circ\phi+Q\log|\phi'|}$ are well-defined and the transformation rule* (2.2.16) *holds simultaneously for all $\phi \in \Lambda$.*

Coordinate change for random translation and scaling

An analog of Theorem 2.2.9 for the LQG metric is expected to be true but has not yet been established. (After the acceptance of this article, it was proven in [84] that, almost surely, (2.2.17) holds for all complex affine transformations simultaneously.) In the following two lemmas, we show that the transformation rule (2.2.17) holds almost surely for a certain subset of random conformal maps from $\mathbb{C}$ to itself. These are random translations (Lemma 2.2.10) and random scalings where the scaling factor is almost surely determined by the circle average part of the field (Lemma 2.2.11). These correspond exactly to the random transformations that determine a canonical embedding, which appears in the continuum mating-of-trees theory, which we present in the next section.

Lemma 2.2.10. *Suppose h is a whole-plane GFF plus a continuous function and $z \in \mathbb{C}$ is any random point (not necessarily independent from h). If $h(\cdot + z)$ has the law of a whole-plane GFF plus a continuous function, then, almost surely,*

$$D_{h(\cdot+z)}(u,v) = D_h(u+z, v+z) \quad \textit{for all } u, v \in \mathbb{C}. \tag{2.2.18}$$

Proof. Denote $\hat{h} := h(\cdot + z)$. Let $\varepsilon > 0$ and suppose $P : [0,1] \to \mathbb{R}$ is a piecewise continuously differentiable path. Letting $\hat{P}(t) := P(t) + z$, almost surely,

$$\int_0^1 e^{\xi \hat{h}^*_\varepsilon(P(t))} |P'(t)|\, dt = \int_0^1 e^{\xi h^*_\varepsilon(\hat{P}(t))} |\hat{P}'(t)|\, dt \tag{2.2.19}$$

for all such paths P. From the definition (2.1.4) of the ε-LFPP metric, almost surely, $D^\varepsilon_{\hat{h}}(u,v) = D^\varepsilon_h(u+z, v+z)$ for all $u, v \in \mathbb{C}$. Since the rescaled ε-LFPP metric $\mathfrak{a}_\varepsilon^{-1} D^\varepsilon_h$ converges in probability with respect to the local uniform topology on $\mathbb{C} \times \mathbb{C}$ to D_h [129, Theorem 1.1], so does $\mathfrak{a}_\varepsilon^{-1} D^\varepsilon_{\hat{h}}$. If $\hat{h}$ is a whole-plane GFF plus a continuous function, then the LQG metric $D_{\hat{h}}$ is the limit. $\square$

We remark that we can use (2.2.18) to define $D_{h(\cdot+z)}$ even when $h(\cdot+z)$ does not have the law of a whole-plane GFF plus a continuous function.

As for random scaling, we only consider the case for which the scaling factor r is measurable with respect to the circle average part of the field (recall the discussion just above Lemma 2.2.6).

Lemma 2.2.11. *Let h be a whole-plane GFF plus a continuous function whose circle average and lateral parts are independent (by Lemma 2.2.6; this is the case for the whole-plane GFF). If $r > 0$ is a random scaling factor which is almost surely determined by the circle average part of h, then almost surely*

$$D_h(ru, rv) = D_{h(r\cdot)+Q\log r}(u, v) \quad \forall\, u, v \in \mathbb{C}. \tag{2.2.20}$$

Proof. Let us denote the circle average part and the lateral part of h as h^{circ} and h^{lat}, respectively. Let $\tilde{h}^{\mathrm{circ}}$ be an independent and identically distributed copy of h^{circ}, which is also independent of h^{lat}. Let $\tilde{h} := \tilde{h}^{\mathrm{circ}} + h^{\mathrm{lat}}$, which is a whole-plane GFF plus a continuous function independent of h^{circ}. Let $f = h - \tilde{h} = h^{\mathrm{circ}} - \tilde{h}^{\mathrm{circ}}$. From Lemma 2.2.5, $t \mapsto h_{e^{-t}}(0)$ and $t \mapsto \tilde{h}_{e^{-t}}(0)$ are independent random continuous functions on $\mathbb{R}$. Hence, f is almost surely a continuous function on $\mathbb{C} \setminus \{0\}$.

We claim that the Weyl scaling axiom holds for f even though it may be discontinuous at 0. That is, almost surely,

$$D_h = e^{\xi f} \cdot D_{\tilde{h}} \tag{2.2.21}$$

as random continuous metrics on $\mathbb{C}$. To make sense of the right-hand side, we first define $e^{\xi f} \cdot D_{\tilde{h}}$ between points in $\mathbb{C} \setminus \{0\}$ as in (2.2.13) except that we take the infimum over all curves that stay in $\mathbb{C} \setminus \{0\}$. We then extend $e^{\xi f} \cdot D_{\tilde{h}}$ to a continuous metric on all of $\mathbb{C}$ if possible; this is how the LQG metric is defined for a field with logarithmic singularities in the discussion preceding [93, Theorem 1.10]. Since D_h is a.s. a continuous length metric on $\mathbb{C}$, it suffices to check that, almost surely, the D_h-lengths and the $e^{\xi f} \cdot D_{\tilde{h}}$-lengths agree for all curves in $\mathbb{C} \setminus \{0\}$. For $\delta > 0$, let f_δ be a random function which is almost surely continuous on $\mathbb{C}$ and agrees with f on $\mathbb{C} \setminus B_\delta(0)$. By the locality axiom, almost surely, the D_h-lengths and the $D_{\tilde{h}+f_\delta}$-lengths agree for all curves in $\mathbb{C} \setminus B_\delta(0)$. By the Weyl scaling axiom, $D_{\tilde{h}+f_\delta} = e^{\xi f_\delta} \cdot D_{\tilde{h}}$ almost surely. Letting $\delta \to 0$ proves the claim.

By the same reasoning, $D_{h(r\cdot)+Q\log r} = e^{\xi f(r\cdot)} \cdot D_{\tilde{h}(r\cdot)+Q\log r}$ almost surely. Since $r > 0$ is independent of $\tilde{h}$, by the coordinate change axiom (2.2.14) for deterministic scaling, we almost surely have

$$D_{\tilde{h}}(ru, rv) = D_{\tilde{h}(r\cdot)+Q\log r}(u, v) \quad \forall\, u, v \in \mathbb{C}. \tag{2.2.22}$$

The lemma now follows by combining (2.2.21) and (2.2.22). $\square$

It is straightforward to check that Lemmas 2.2.10 and 2.2.11 are also valid when h is equal to a whole-plane GFF plus a continuous function plus a finite number of logarithmic singularities of the form $-\alpha \log|\cdot - z|$ for $z \in \mathbb{C}$ and $\alpha < Q$. In particular, they can be applied to a γ-quantum cone.

2.2.5 Mating-of-trees theory

The continuum mating-of-trees theorem is the central tool we utilize in this paper. We first review the setup for the theorem, in particular the definitions and properties of the γ-quantum cone and the whole-plane space-filling SLE$_{\kappa'}$ curve. We state the mating-of-trees theorem afterward.

Quantum cone

In the previous section, we discussed that for some quantum surfaces, we must use information about the field to define a canonical embedding of the surface. In this paper, we consider doubly-marked quantum surfaces parameterized by the Riemann sphere: i.e., those with embeddings of the form $(\mathbb{C}, h, 0, \infty)$. Fixing the two marked points at 0 and ∞ gives an embedding of the quantum surface that is unique only up to scaling. One choice of canonical embedding for such a quantum surface is called the circle average embedding.

Definition 2.2.12. We say that an embedding $(\mathbb{C}, h, 0, \infty)$ of a doubly-marked γ-LQG surface is a circle average embedding if

$$\sup\{r > 0 : h_r(0) + Q \log r = 0\} = 1 \tag{2.2.23}$$

where $h_r(0)$ is the circle average of h on $\partial B_r(0)$.

That is, given an embedding $(\mathbb{C}, h, z, \infty)$ of a γ-LQG surface where one marked point is at $z \in \mathbb{C}$ and the other marked point is at ∞, the circle average embedding of this LQG surface is $(\mathbb{C}, h(R \cdot -z) + Q \log R, 0, \infty)$ where

$$R := \sup\{r > 0 : h_r(z) + Q \log r = 0\}. \tag{2.2.24}$$

From the perspective of circle average embedding, the quantum surface induced by a whole-plane GFF is unnatural in that it does not satisfy scale invariance. That is, if h is a whole-plane GFF and C is a nonzero constant, the circle average embeddings of $(\mathbb{C}, h, 0, \infty)$ and $(\mathbb{C}, h+C, 0, \infty)$ do not agree in law. The natural scale-invariant analog of such a surface is called the quantum cone.

Definition 2.2.13. Let $\{A_t\}_{t\in\mathbb{R}}$ be a real-valued stochastic process with the following distribution.

- For $t < 0$, $A_t = \widehat{B}_{-t} + \gamma t$ where $\{\widehat{B}_s\}_{s\geq 0}$ is a standard Brownian motion with $\widehat{B}_0 = 0$ conditioned so that $\widehat{B}_s + (Q - \gamma)s > 0$ for every $s > 0$.
- For $t \geq 0$, $A_t = B_t + \gamma t$ where $\{B_s\}_{s\geq 0}$ is a standard Brownian motion with $B_0 = 0$ that is sampled independently of $\{\widehat{B}_s\}_{s\geq 0}$.

A γ-quantum cone is a doubly-marked quantum surface whose circle average embedding $(\mathbb{C}, h^\gamma, 0, \infty)$ has the following law.

- The circle average process $t \mapsto h^\gamma_{e^{-t}}(0)$ has the same law as the process A.

- The lateral part $h^\gamma - h^\gamma_{|\cdot|}(0)$ of h^γ agrees in law with the lateral part of a whole-plane GFF.

- The circle average and the lateral parts of h^γ are independent.

A γ-quantum cone has the following scale invariance property.

Proposition 2.2.14 ([95, Proposition 4.13(i)]). *Let $(\mathbb{C}, h^\gamma, 0, \infty)$ be the circle average embedding of a γ-quantum cone. Let C be a real constant. Then the quantum surfaces represented by $(\mathbb{C}, h^\gamma, 0, \infty)$ and $(\mathbb{C}, h^\gamma + C, 0, \infty)$ agree in law. That is, let*

$$R_C := \sup \left\{ r : h^\gamma_r(0) + Q \log r + C = 0 \right\} \tag{2.2.25}$$

so that $(\mathbb{C}, h^\gamma(R_C \cdot) + Q \log R_C + C, 0, \infty)$ is the circle average embedding of the γ-LQG surface represented by $(\mathbb{C}, h^\gamma + C, 0, \infty)$. Then $h^\gamma \stackrel{d}{=} h^\gamma(R_C \cdot) + Q \log R_C + C$.

Since $\mu_{h^\gamma + C} = e^{\gamma C} \mu_{h^\gamma}$, this property means that the law of a γ-quantum cone is invariant under scaling its LQG measure by a positive constant. Similarly, the fact that $D_{h^\gamma + C/\gamma} = e^{\xi C} D_{h^\gamma}$ implies that the law of a quantum cone is invariant under scaling its LQG metric by a constant. Note that the spatial scaling factor (2.2.25) is a random variable that depends on the circle averages of h^γ. Hence, it is only due to Lemma 2.2.11 that the scale invariance of the quantum cone extends to the LQG metric.

Another reason to consider the γ-quantum cone is that it is the γ-LQG surface one obtains by starting with a generic γ-LQG surface, choosing a marked point on it from the γ-LQG measure, and then "zooming in" near the marked point. More precisely, we have the following lemma, which is helpful for transferring results about a whole-plane GFF to a γ-quantum cone and is used in Section 2.4.1.

Lemma 2.2.15. *Let dh denote the law of a whole-plane GFF with $h_1(0) = 0$. Let let (z, h) be a pair sampled from the probability measure $Z^{-1}\mathbf{1}_{\mathbb{D}}(z)\mu_h(dz)dh$ where $Z = \mathbb{E}[\mu_h(\mathbb{D})]$ is the normalization constant. Then, the field under the circle average embedding of the quantum surface $(\mathbb{C}, h + C, z, \infty)$ — i.e., $h(R_C\cdot) + Q\log R_C + C$ where R_C is defined as in* (2.2.25) *— converges locally in total variation distance as $C \to \infty$ to h^γ, the field of the γ-quantum cone under circle average embedding.*

Proof. By [95, Lemma A.10], we can sample $(z, \tilde{h})$ from $Z^{-1}\mathbf{1}_{\mathbb{D}}(z)\mu_h(dz)dh$ by first sampling (z, h) from $Z^{-1}\mathbf{1}_{\mathbb{D}}(z)dzdh$, then letting $\tilde{h} = h - \gamma\log|\cdot - z| + \gamma\log\max(|\cdot|, 1)$. Conditioned on $z \in \mathbb{D}$, the field under the circle average embedding of $(\mathbb{C}, \tilde{h} + C, z, \infty)$ converges locally in total variation distance to h^γ as $C \to \infty$ as in [95, Proposition 4.13(ii)]. It follows that the desired convergence also holds without conditioning on z. □

Space-filling SLE

The whole-plane space-filling $\mathrm{SLE}_{\kappa'}$ curve from ∞ to ∞, defined for $\kappa' > 4$, is a random space-filling continuous curve which intersects itself but does not cross itself. Such a curve was initially constructed for the chordal version (0 to ∞ in $\mathbb{H}$) in [196, Theorem 1.16], and extended to the whole-plane version from ∞ to ∞ in [95, Footnote 4]. We refer to these sources and [123, Section 3.6] for more detailed descriptions of the curve. Here, we summarize the construction and structure of the space-filling SLE curve as given in these references.

The space-filling $\mathrm{SLE}_{\kappa'}$ curve is defined in terms of flow lines, which are $\mathrm{SLE}_{16/\kappa'}$-type curves coupled with a GFF introduced in the context of imaginary geometry [194, 196]. In particular, the whole-plane space-filling $\mathrm{SLE}_{\kappa'}$ curve is the Peano curve tracing between dual space-filling trees formed by flow lines.

Here is the detailed construction. Let $\kappa = 16/\kappa'$. Starting from a whole-plane GFF h, for each $z \in \mathbb{C}$ and $\theta \in \mathbb{R}$, we can define the flow line starting from z with angle θ as a random curve a.s. determined by h with the law of a whole-plane $\mathrm{SLE}_\kappa(2-\kappa)$ curve from z to ∞ [196, Theorems 1.4]. We consider the angles $\pm\frac{\pi}{2}$; denote the flow line starting from z with angle $\pm\frac{\pi}{2}$ as $\eta_z^\pm$, and orient the flow line from z to ∞. For every $z, w \in \mathbb{C}$, almost surely, η_z^+ and η_w^+ merge and so do η_w^- and η_w^-. Moreover, for each $z \in \mathbb{C}$, η_z^+ and η_z^- almost surely do not cross each other [196, Theorem 1.7]. Thus, given a dense countable subset $\{z_j\}_{j\in\mathbb{N}}$ of $\mathbb{C}$, the unions of flow lines $\{\eta_{z_j}^+\}_{j\in\mathbb{N}}$ and $\{\eta_{z_j}^-\}_{j\in\mathbb{N}}$ form dual trees rooted at ∞ with leaves $\{z_j\}_{j\in\mathbb{N}}$ [196, Theorem 1.10]. To define concretely the Peano curve between these dual trees, define a total order on $\{z_j\}_{j\in\mathbb{N}}$ by saying that z_j comes before z_k if z_k lies in the connected component of $\mathbb{C} \setminus (\eta_{z_j}^+ \cup \eta_{z_j}^-)$ whose boundary consists of the left side of $\eta_{z_j}^-$ and the right side of $\eta_{z_j}^+$ (when orienting these flow lines from z_j to ∞). There almost surely exists a unique space-filling curve η which traces the points $\{z_j\}_{j\in\mathbb{N}}$ in this order, which is visualized in Figure 2.1. The curve η is continuous when parameterized by the Lebesgue measure on $\mathbb{C}$. Moreover, η almost surely does not depend on the choice of the dense countable set $\{z_j\}_{j\in\mathbb{N}}$; it is a measurable function of the GFF which generates the flow lines [196, Theorem 1.16]. This η is the whole-plane space-filling $\mathrm{SLE}_{\kappa'}$ curve from ∞ to ∞. Here are a few basic properties of this curve following from the definition, which were collected in [123, Section 3.6.4].

- For each fixed $z \in \mathbb{C}$, almost surely, η visits z only once. If $z = \eta(t)$, then $\partial\eta(-\infty, t] = \partial\eta[t, \infty) = \eta_z^+ \cup \eta_z^-$.
- Let η be parameterized by Lebesgue measure with $\eta(0) = 0$. Then $\eta^R : t \mapsto \eta(-t)$ has the same law as η. This property is referred to as reversibility.
- Let ϕ be a deterministic conformal transformation of the Riemann sphere $\mathbb{C} \cup \{\infty\}$ which fixes ∞: i.e., a composition of scaling, rotation, and translation. Then, $\phi \circ \eta$ has

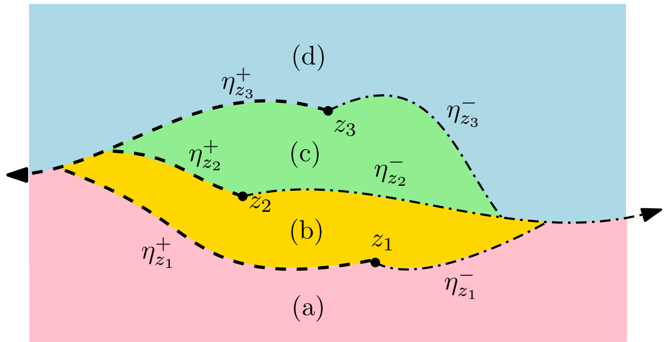

Figure 2.1: Definition of whole-plane space-filling SLE. Almost surely, the flow lines $\{\eta^+_{z_j}\}_{j\in\mathbb{N}}$ and $\{\eta^-_{z_j}\}_{j\in\mathbb{N}}$ each merge to form two trees. The flow lines in this illustration determine the ordering $z_1 < z_2 < z_3$, the order in which the space-filling SLE curve η visits these three points. In particular, η fills the four colored regions in the order (a)→(b)→(c)→(d).

the same law as η up to reparameterization.

- For $\kappa' \geq 8$, $\eta(-\infty, 0]$ and $\eta[0,\infty)$ are both homeomorphic to the closed half-plane $\overline{\mathbb{H}}$. Note that $\eta^-_0 \cup \eta^+_0 = \eta(-\infty,0] \cap \eta[0,\infty)$. Conditioned on $\eta^-_0 \cup \eta^+_0$, the conditional law of $\eta|_{[0,\infty)}$ is that of a chordal SLE_κ curve from 0 to ∞ in $\eta[0,\infty)$, and the conditional law of the time reversal of $\eta|_{(-\infty,0]}$ is that of a chordal SLE_κ curve from 0 to ∞ in $\eta(-\infty, 0]$. Moreover, the two curves are conditionally independent given $\eta^-_0 \cup \eta^+_0$. See also [95, Footnote 4].

- For $\kappa' \in (4, 8)$, the interiors of $\eta(-\infty, 0]$ and $\eta[0,\infty)$ are both infinite chains of Jordan domains. Again $\eta^-_0 \cup \eta^+_0 = \eta(-\infty,0] \cap \eta[0,\infty)$. Conditioned on $\eta^-_0 \cup \eta^+_0$, the curve $\eta|_{[0,\infty)}$ and the time reversal of $\eta|_{(-\infty,0]}$ are conditionally independent concatenations of chordal SLE_κ curves in the connected components of $\eta[0,\infty)$ and $\eta(-\infty,0]$, respectively. The two curves are conditionally independent given $\eta^-_0 \cup \eta^+_0$. See also [95, Footnote 4].

The following proposition states that every segment of a whole-plane space-filling $\mathrm{SLE}_{\kappa'}$ curve contains a Euclidean ball of comparable size with high probability. In [119], this estimate was used to show that η parameterized according to the Lebesgue measure is locally

Hölder continuous with any exponent less than 1/2 and is not locally Hölder continuous with any exponent greater than 1/2 with respect to the Euclidean metric (cf. Theorem 2.1.4). This is a key estimate in the proof of Theorem 2.1.1.

Proposition 2.2.16 ([119, Proposition 3.4 and Remark 3.9])**.** *Fix $\kappa' > 4$ and let η be a whole-plane space-filling SLE$_{\kappa'}$ curve from ∞ to ∞. For each $r \in (0,1)$ and $R > 0$, the following happens with superpolynomially high probability as $\varepsilon \to 0$: for each $\delta \in (0, \varepsilon]$ and every $a < b$ such that $\eta[a,b] \subset B_R(0)$ and $\operatorname{diam} \eta[a,b] \geq \delta^{1-r}$, the set $\eta[a,b]$ contains a Euclidean ball of radius at least δ.*

Translation and scale invariance of quantum cone decorated with space-filling SLE

Let $\gamma \in (0,2)$ and $\kappa' \in (4,\infty)$. Let $(\mathbb{C}, h^\gamma, 0, \infty, \eta)$ be the circle average embedding of a γ-quantum cone decorated by an independent whole-plane space-filling SLE$_{\kappa'}$ curve η from ∞ to ∞. Reparameterize η by the γ-LQG measure μ_{h^γ}. That is, $\eta(0) = 0$ and $\mu_{h^\gamma}([s,t]) = t - s$ for all $s < t$.[4] This is the default parameterization of η that we consider in the rest of this paper.

We already saw in Proposition 2.2.14 that the circle average embedding of a γ-quantum cone is invariant under adding a deterministic constant to the field. This operation also preserves the law of the independent space-filling SLE$_{\kappa'}$ curve η decorating the quantum cone.

Lemma 2.2.17. *For each fixed constant $C \in \mathbb{R}$, the circle average embedding of $(\mathbb{C}, h^\gamma + C/\gamma, 0, \infty, \eta(e^C \cdot))$ agrees in law with $(\mathbb{C}, h^\gamma, 0, \infty, \eta)$.*

Proof. Let R_C be as in (2.2.25), so that for $\tilde{h} := h^\gamma(R_C \cdot) + Q \log R_C + C/\gamma$, the circle average embedding of $(\mathbb{C}, h^\gamma + C/\gamma, 0, \infty, \eta)$ is $(\mathbb{C}, \tilde{h}, 0, \infty, R_C^{-1}\eta)$. Recall from proposition 2.2.14 that

4. The a.s. continuity of the reparameterized curve follows from Proposition 2.2.16 combined with the fact that every bounded open subset of $\mathbb{C}$ a.s. has positive μ_{h^γ}-mass.

$\tilde{h} \stackrel{d}{=} h^\gamma$. Since η modulo reparameterization is independent from h^γ, it is also independent from R_C. By the scale invariance of whole-plane space-filling $\mathrm{SLE}_{\kappa'}$ (see previous section), $R_C^{-1}\eta$ modulo reparameterization agrees in law with η modulo reparameterization and is independent of $\tilde{h}$. Therefore, the joint law of $(\tilde{h}, R_C^{-1}\eta)$ agrees with that of (h^γ, η) with the curves viewed modulo reparameterization. Since $\mu_{\tilde{h}} = e^C \mu_{h^\gamma}(R_C \cdot)$, the $\mu_{\tilde{h}}$-parameterization of $R_C^{-1}\eta$ is given by $R_C^{-1}\eta(e^C \cdot)$. $\square$

Another important property of $(\mathbb{C}, h^\gamma, 0, \infty, \eta)$ is that for each fixed $t \in \mathbb{R}$, the law of the circle average embedding is invariant under re-centering this quantum surface at $\eta(t)$ (i.e., translating by $-\eta(t)$).

Lemma 2.2.18 ([95, Lemma 8.3])**.** *The law of* $(\mathbb{C}, h^\gamma, 0, \infty, \eta)$ *as a path-decorated quantum surface is invariant under shifting a fixed amount of* γ*-LQG area. That is, for each* $t \in \mathbb{R}$*, the circle average embedding of* $(\mathbb{C}, h^\gamma, \eta(t), \infty, \eta(\cdot + t))$ *agrees in law with* $(\mathbb{C}, h^\gamma, 0, \infty, \eta)$*.*

We emphasize that Lemmas 2.2.17 and 2.2.18 hold for any $\gamma \in (0, 2)$ and $\kappa' > 4$, including when $\kappa' \neq 16/\gamma^2$. The key fact behind both the proof of Lemma 2.2.17 presented above and the proof of Lemma 2.2.18 in [95] is Lemma 2.2.15, which implies that we get a γ-quantum cone if we zoom in at a point sampled according to the γ-LQG measure on a space-filling SLE_κ' curve η. The only property of η used here is that it is almost surely a continuous space-filling curve. With the additional property that the law of η is scale-invariant, we have that the law of η is preserved when zooming in according to an independent field. Both properties are true regardless of the value of $\kappa' > 4$. (The condition $\kappa' = 16/\gamma^2$ is necessary to identify the law of $\eta(-\infty, t]$ and $\eta[t, \infty)$ as independent quantum surfaces for each $t \in \mathbb{R}$; see Proposition 2.2.22.)

Notice that both lemmas state the invariance in law of the circle average embedding of a path-decorated quantum surface (either after adding a constant to the field or re-centering the quantum cone). In our applications of these results, we consider the laws of random variables defined almost surely in terms of the pointed curve-decorated metric measure space structure of these quantum surfaces: e.g., $N_\varepsilon(\eta[s,t]; D_{h^\gamma})$ where $s < t$ are fixed numbers. The scaling and stationarity properties of such random variables do not follow trivially from Lemmas 2.2.17 and 2.2.18 because mapping one embedding of a quantum surface to its circle average embedding involves a random scaling factor that is determined by the circle average part of the field. Nevertheless, we can use Lemmas 2.2.10 and 2.2.11 to translate Lemmas 2.2.17 and 2.2.18 in terms of the laws of pointed curve-decorated metric measure spaces.

Remark 2.2.19. Consider the equivalence relation on pointed curve-decorated metric measure spaces where $(X, x, d, \mu, \eta) \sim (X', x', d', \mu', \eta')$ if there exists an isometry $f : X \to X'$ such that $f(x) = x'$, $f_*\mu = \mu'$, and $f \circ \eta = \eta'$. We identify a pointed curve-decorated metric measure space with the equivalence class that it belongs to. The pointed Gromov–Hausdorff–Prokhorov–uniform (GHPU) metric introduced in [124] is a natural choice of metric on the space of above equivalence classes of noncompact pointed curve-decorated metric measure spaces. The precise definition of this metric is not essential for our purposes; instead, we will only need the corresponding Borel σ-algebra.

Let h and $\tilde{h}$ be two instances of whole-plane GFF plus a continuous function, and let η and $\tilde{\eta}$ be random continuous space-filling curves. Suppose $(h, \eta) \stackrel{d}{=} (\tilde{h}, \tilde{\eta})$ with respect to the product of the following two topologies: the weak-* topology for distributions on $\mathbb{C}$ with respect to smooth and compactly supported test functions[5], and the local uniform topology on functions $\mathbb{C} \to \mathbb{C}$. Note that μ_h is almost surely determined by h [99, 213] and D_h is a.s. determined by h [129]; analogous statements hold for $\tilde{h}$, $\mu_{\tilde{h}}$, and $D_{\tilde{h}}$. Hence, if η (resp. $\tilde{\eta}$) is parameterized by μ_h (resp. $\mu_{\tilde{h}}$) and $\eta(0) = \tilde{\eta}(0)$ a.s., then $(D_h, \mu_h, \eta) \stackrel{d}{=} (D_{\tilde{h}}, \mu_{\tilde{h}}, \tilde{\eta})$ with respect to the product of the local uniform topology on functions $\mathbb{C} \times \mathbb{C} \to \mathbb{R}$, the weak-*

5. This is equivalent to the topology corresponding to considering the whole-plane GFF as a stochastic process indexed by signed Borel measures $\mathcal{M}$ (recall Section 2.2.2) by Itô's isometry for the GFF [38, Section 1.7].

topology on signed Borel measures on $\mathbb{C}$, and the local uniform topology on functions $\mathbb{R} \to \mathbb{C}$. It is straightforward to check from the definition of the local GHPU metric that the following map is continuous: the map from the tuple (d, μ, η) of a continuous metric, a signed Borel measure, and a continuous curve whose space is assigned the above product topology, to the pointed curve-decorated metric measure space $(\mathbb{C}, 0, d, \mu, \eta)$ whose space is assigned the local GHPU topology. Therefore, $(\mathbb{C}, 0, D_h, \mu_h, \eta) \stackrel{d}{=} (\mathbb{C}, 0, D_{\tilde{h}}, \mu_{\tilde{h}}, \tilde{\eta})$ w.r.t. the local GHPU topology. This conclusion continues to hold when h has finitely many singularities of the form $-\alpha \log|\cdot - z|$ with $\alpha < Q$ since D_h is almost surely a continuous metric determined by h [93, Theorem 1.10]. (For $\alpha > Q$, almost surely, $D_h(z, w) = \infty$ for every $w \in \mathbb{C} \setminus \{z\}$.)

Proposition 2.2.20. *Let $(\mathbb{C}, h^\gamma, 0, \infty, \eta)$ be a γ-quantum cone in the circle average embedding that is decorated with an independent whole-plane space-filling $SLE_{\kappa'}$ curve, which is then parameterized so that $\eta(0) = 0$ and $\mu_{h^\gamma}(\eta[a, b]) = b - a$ for every $a < b$. The following statements hold w.r.t the local GHPU topology on pointed curve-decorated metric measure spaces.*

(i) For each fixed $s > 0$,

$$(\mathbb{C}, 0, s^{1/d_\gamma} D_{h^\gamma}, s\mu_{h^\gamma}, \eta(s\cdot)) \stackrel{d}{=} (\mathbb{C}, 0, D_{h^\gamma}, \mu_{h^\gamma}, \eta). \tag{2.2.26}$$

(ii) For each fixed $t \in \mathbb{R}$,

$$(\mathbb{C}, \eta(t), D_{h^\gamma}, \mu_{h^\gamma}, \eta(\cdot + t)) \stackrel{d}{=} (\mathbb{C}, 0, D_{h^\gamma}, \mu_{h^\gamma}, \eta). \tag{2.2.27}$$

Proof. Given a conformal map $\phi : C \to \mathbb{C}$ and an embedding $(\mathbb{C}, h, x, \infty, \eta)$ of a path-decorated quantum surface, denote

$$\phi_* h := h \circ \phi^{-1} + Q \log|(\phi^{-1})'| \tag{2.2.28}$$

and

$$\phi_*(\mathbb{C}, h, x, \infty, \eta) := (\mathbb{C}, \phi_* h, \phi(x), \infty, \phi \circ \eta). \tag{2.2.29}$$

That is, $\phi_*(\mathbb{C}, 0, \infty, h, \eta)$ is the pushforward of the embedding $(\mathbb{C}, 0, \infty, h, \eta)$ under ϕ using (2.2.15). For a continuous metric d on $\mathbb{C}$ and a Borel measure μ on $\mathbb{C}$, denote their pushforwards under ϕ as $\phi_* d$ and $\phi_* \mu$, respectively.

(i) Let $s > 0$ be fixed and denote $\tilde{h} := h^\gamma + (\log s)/\gamma$. Let $r := R_{\log s}$ be defined as in (2.2.25) and define $\phi : \mathbb{C} \to \mathbb{C}$ by $\phi(z) = r^{-1} z$ so that $\phi_*(\mathbb{C}, \tilde{h}, 0, \infty, \eta(s\cdot))$ is the circle average embedding. Setting $C = \log s$ in Lemma 2.2.17 gives $(\mathbb{C}, \phi_* \tilde{h}, 0, \infty, \phi \circ \eta(s\cdot)) \stackrel{d}{=} (\mathbb{C}, h^\gamma, 0, \infty, \eta)$. As discussed in Remark 2.2.19, this implies

$$(\mathbb{C}, 0, D_{\phi_* \tilde{h}}, \mu_{\phi_* \tilde{h}}, \phi \circ \eta(s\cdot)) \stackrel{d}{=} (\mathbb{C}, 0, D_{h^\gamma}, \mu_{h^\gamma}, \eta) \tag{2.2.30}$$

w.r.t. the local GHPU topology.

Since r is a.s. determined by the circle average part of $\tilde{h}$, Lemma 2.2.11 implies $D_{\phi_* \tilde{h}} = \phi_* D_{\tilde{h}}$ almost surely. By Theorem 2.2.9, $\mu_{\phi_* \tilde{h}} = \phi_*(\mu_{\tilde{h}})$ almost surely. Hence,

$$\begin{aligned} (\mathbb{C}, 0, D_{\phi_* \tilde{h}}, \mu_{\phi_* \tilde{h}}, \phi \circ \eta(s\cdot)) &= (\mathbb{C}, 0, \phi_* D_{\tilde{h}}, \phi_* \mu_{\tilde{h}}, \phi \circ \eta(s\cdot)) \\ &= (\mathbb{C}, 0, D_{\tilde{h}}, \mu_{\tilde{h}}, \eta(s\cdot)) \\ &= (\mathbb{C}, 0, s^{1/d_\gamma} D_{h^\gamma}, s\mu_{h^\gamma}, \eta(s\cdot)) \end{aligned} \tag{2.2.31}$$

almost surely as pointed curve-decorated metric measure spaces. We obtain (2.2.26) by combining (2.2.30) and (2.2.31).

(ii) The proof is similar to part (i). Let $t \in \mathbb{R}$ be fixed. Let $r > 0$ be the random constant such that the circle average embedding of $(\mathbb{C}, h^\gamma, \eta(t), \infty, \eta(\cdot + t))$ is its pushforward under $\phi(z) := r(z - \eta(t))$. Lemma 2.2.18 gives $(\mathbb{C}, \phi_* h^\gamma, 0, \infty, \phi \circ \eta(\cdot + t)) \stackrel{d}{=}$

$(\mathbb{C}, h^\gamma, 0, \infty, \eta)$, which implies

$$(\mathbb{C}, 0, D_{\phi_* h^\gamma}, \mu_{\phi_* h^\gamma}, \phi \circ \eta(\cdot + t)) \stackrel{d}{=} (\mathbb{C}, 0, D_{h^\gamma}, \mu_{h^\gamma}, \eta) \tag{2.2.32}$$

w.r.t. the local GHPU topology.

Since r is a.s. determined by the circle average part of $h^\gamma(\cdot + \eta(t))$, Lemmas 2.2.10 and 2.2.11 imply $D_{\phi_* h^\gamma} = \phi_* D_{h^\gamma}$ almost surely. (It follows from the proof of Lemma 2.2.18 in [95] that $h^\gamma(\cdot + \eta(t))$ is a whole-plane GFF plus a continuous function plus $-\gamma \log|\cdot|$.) By Theorem 2.2.9, $\mu_{\phi_* h^\gamma} = \phi_* \mu_{h^\gamma}$ almost surely. Hence,

$$\begin{aligned}(\mathbb{C}, 0, D_{\phi_* h^\gamma}, \mu_{\phi_* h^\gamma}, \phi \circ \eta(\cdot + t)) &= (\mathbb{C}, 0, \phi_* D_{h^\gamma}, \phi_* \mu_{h^\gamma}, \phi \circ \eta(\cdot + t)) \\ &= (\mathbb{C}, \eta(t), D_{h^\gamma}, \mu_{h^\gamma}, \eta(\cdot + t))\end{aligned} \tag{2.2.33}$$

almost surely as pointed curve-decorated metric measure spaces. We obtain (2.2.27) by combining (2.2.32) and (2.2.33). □

From Proposition 2.2.20, we immediately obtain the following stationarity and scaling result for the number of LQG metric balls needed to cover a space-filling $\mathrm{SLE}_{\kappa'}$ segment.

Corollary 2.2.21. *Let $(\mathbb{C}, h^\gamma, 0, \infty, \eta)$ be as in Proposition 2.2.20. For any $s > 0$, $t \in \mathbb{R}$, and $\varepsilon > 0$,*

$$N_\varepsilon(\eta[t, t+s]; D_{h^\gamma}) \stackrel{d}{=} N_\varepsilon(\eta[0, s]; D_{h^\gamma}) \stackrel{d}{=} N_{\varepsilon s^{-1/d_\gamma}}(\eta[0, 1]; D_{h^\gamma}). \tag{2.2.34}$$

Mating-of-trees theorem

The above results are valid for any choices of $\gamma \in (0, 2)$ and $\kappa' > 4$. In contrast, the following independence property requires an exact relation between γ and κ'. As in Proposition 2.2.20, let $(\mathbb{C}, h^\gamma, 0, \infty, \eta)$ be the circle average embedding of a γ-quantum cone decorated with an independent whole-plane space-filling $\mathrm{SLE}_{\kappa'}$ curve, which is then parameterized so that $\eta(0)=0$ and $\mu_{h^\gamma}(\eta[a,b]) = b-a$ for every $a<b$.

Proposition 2.2.22 ([95, Theorem 1.9]). *Suppose $\kappa' = 16/\gamma^2$. Denote the interiors of $\eta(-\infty,0]$ and $\eta[0,\infty)$ as U_- and U_+, respectively. Then the γ-LQG surfaces represented by $(U_-, h^\gamma|_{U_-}, 0, \infty)$ and $(U_+, h^\gamma|_{U_+}, 0, \infty)$ are independent and identically distributed quantum surfaces called $\frac{3\gamma}{2}$-quantum wedges (also known as quantum wedges with weight $2-\frac{\gamma^2}{2}$).*

As in Lemmas 2.2.17 and 2.2.18, what Proposition 2.2.22 means is that the fields under the canonical embeddings of $(U_-, h^\gamma|_{U_-}, 0, \infty)$ and $(U_+, h^\gamma|_{U_+}, 0, \infty)$ as described in [95, Sections 4.2 and 4.4] are independent and identically distributed. Likewise, we can rephrase this statement in terms of curve-decorated metric measure spaces, stated precisely in Proposition 2.2.23. The key idea is that canonical embedding of a $\frac{3\gamma}{2}$-quantum wedge is defined in terms of the field average, similarly to the circle average embedding of a quantum cone. The subtlety lies in choosing the correct topology for the $\frac{3\gamma}{2}$-quantum wedge considered as a metric measure space. Once this is done, we can extend Lemma 2.2.11 to show that the LQG metric is preserved when we reparameterize the quantum wedge to its canonical embedding.

Below, we introduce the precise definition of the $\frac{3\gamma}{2}$-quantum wedge and its canonical embedding as well as the topology on the metric measure space necessary to establish Proposition 2.2.23, but we only need the proposition itself for the proofs of our main results. Upon first reading, we suggest that the reader skip to the statement of Proposition 2.2.23.

Recall from Section 2.2.5 the two regimes for the topology of $U_\pm$ depending on the value of γ. For $\gamma \in (0, \sqrt{2}]$, the two domains $U_\pm$ are each almost surely homeomorphic to the upper half plane $\mathbb{H}$. The canonical embedding of the $\frac{3\gamma}{2}$-quantum wedge in this case is $(\mathbb{H}, h, 0, \infty)$ where $\sup\{r>0 : h_r(0) + Q\log r = 0\} = 1$, similar to Definition 2.2.12 of the circle average embedding of a quantum cone. Under this embedding, the Gaussian field h for the $\frac{3\gamma}{2}$-quantum cone has the following law defined in terms of the semicircle averages and the lateral part (recall Remark 2.2.7).

- The semicircle average process $t \mapsto h_{e^{-t}}(0)$ has the same law as the following process A.

  - For $t < 0$, $A_t = \widehat{B}_{-2t} + \frac{3\gamma}{2}t$ where $\{\widehat{B}_s\}_{s\geq 0}$ is a standard Brownian motion with $\widehat{B}_0 = 0$ conditioned so that $\widehat{B}_{2s} + (Q - \frac{3\gamma}{2})s > 0$ for every $s > 0$.
  - For $t \geq 0$, $A_t = B_{2t} + \frac{3\gamma}{2}t$ where $\{B_s\}_{s\geq 0}$ is a standard Brownian motion with $B_0 = 0$ that is sampled independently of $\{\widehat{B}_s\}_{s\geq 0}$.

- The lateral part $h - h_{|\cdot|}(0)$ of h agrees in law with the lateral part of a free-boundary GFF on $\mathbb{H}$.

- The semicircle average and the lateral parts of h are independent.

When $\gamma \in (\sqrt{2}, 2)$, a $\frac{3\gamma}{2}$-quantum wedge is a concatenation of countably many connected components, each of which is a quantum disk (i.e., a simply connected quantum surface) with two marked points on its boundary. They are attached to other components via their marked points. We call each component of this quantum wedge a bead, and a quantum surface with the same topology as the quantum wedge a beaded quantum surface. In the setup of Proposition 2.2.22, the space-filling $\mathrm{SLE}_{\kappa'}$ curve η fills up one component of the domain $U_\pm$ at a time, inducing a chronological order on them. We define the canonical embedding of the $\frac{3\gamma}{2}$-quantum cone in this regime by specifying the embedding of each component. We embed each bead to $(\mathbb{H}, h, 0, \infty)$ so that $h_r(0) + Q \log r$ achieves its maximum at $r = 1$.[6] Under this embedding, the $\frac{3\gamma}{2}$-quantum wedge has the following law.

1. Sample a Poisson point process Λ with intensity measure $du \otimes dh$, where du is the Lebesgue measure on $(0, \infty)$ and dh is an infinite measure on distributions on $\mathbb{H}$ with the following description.

6. We cannot use the circle average embedding because $h_r(0) + Q \log r \to -\infty$ almost surely as $r \to 0$ and as $r \to \infty$. The canonical embedding in [95, Section 4.4] is given on the strip $\mathbb{R} + [0, i\pi]$; to avoid introducing additional notations, we give an equivalent description under the LQG coordinate change rule corresponding to the conformal map $z \mapsto e^z$.

- Let $d\tilde{h}$ be the probability measure on the space of distributions on $\mathbb{H}$ corresponding to the Gaussian field $\tilde{h}$ sampled in the following way.
  - The semicircle average process $t \mapsto \tilde{h}_{e^{-t}}(0)$ has the same law as the following process A.
    - For $t < 0$, $A_t = \widehat{B}_{-2t} + \frac{3\gamma}{2}t$ where $\{\widehat{B}_s\}_{s \geq 0}$ is a standard Brownian motion with $\widehat{B}_0 = 0$ conditioned so that $\widehat{B}_{2s} + (Q - \frac{3\gamma}{2})s > 0$ for every $s > 0$.
    - For $t \geq 0$, $A_t = B_{2t} + \frac{3\gamma}{2}t$ where $\{B_s\}_{s \geq 0}$ is a standard Brownian motion with $B_0 = 0$ that is sampled independently of $\{\widehat{B}_s\}_{s \geq 0}$.
  - The lateral part $\tilde{h} - \tilde{h}_{|\cdot|}(0)$ of $\tilde{h}$ agrees in law with the lateral part of a free-boundary GFF on $\mathbb{H}$.
  - The semicircle average and the lateral parts of $\tilde{h}$ are independent.
- Sample (m, h) from the infinite law $m^{1-4/\gamma^2} dm \otimes d\tilde{h}$, where dm is the Lebesgue measure on $(0, \infty)$. Then, dh agrees with the infinite law of the distribution $h = \tilde{h} + (2/\gamma)\log m$. Here, the sum $\tilde{h} + (2/\gamma)\log m$ refers to the distribution on $\mathbb{H}$ obtained by adding the constant function $(2/\gamma)\log m$ to the distribution $\tilde{h}$. The number $(2/\gamma)\log m$ corresponds to the value of the semicircle average $h_1(0)$ under the canonical embedding described above.

2. To each point (u, h) in the p.p.p. Λ, correspond to it the quantum surface $(\mathbb{H}, h, 0, \infty)$. We concatenate these components according to the first coordinate u in increasing order to sample the $\frac{3\gamma}{2}$-quantum wedge. They are concatenated at the marked points 0 and ∞ so that removing the point corresponding to 0 (resp. ∞) of the component $(u, h) \in \Lambda$ disconnects it from all components $(u', h') \in \Lambda$ with $u' < u$ (resp. $u < u'$).

We deduce from these definitions that reparameterizing the $\frac{3\gamma}{2}$-quantum wedge appearing in Proposition 2.2.22 by its canonical embedding preserves the LQG metric. First, we map the quantum wedge (or its bead) conformally to $\mathbb{H}$, sending the two marked points to 0 and

∞. We can choose this conformal map only using the flow lines that cut out the quantum wedge, which then preserves the LQG metric a.s. because these flow lines are independent of the quantum wedge (recall (2.2.17)). Now, we merely need to scale each component by a random factor that is almost surely determined by the semicircle averages of the field. Since the semicircle average part of the quantum wedge is a continuous function that is independent of the lateral part, we can apply the Weyl scaling axiom as in the proof of Lemma 2.2.11 to conclude that the LQG metric is preserved under this scaling.

Therefore, we can restate Proposition 2.2.22 in the language of random curve-decorated measure metric spaces as below. We omit a detailed proof, which is analogous to that of Proposition 2.2.20 except for the additional consideration that conditioned on $\eta(-\infty,0] \cap \eta[0,\infty) = \eta_0^+ \cup \eta_0^-$, the curve $\eta|_{[0,\infty)}$ and the time reversal of $\eta|_{(-\infty,0]}$ are conditionally independent and identically distributed (as discussed in Section 2.2.5). Recall that D_h^U denotes the internal metric on U, which was defined in (2.2.12).

Proposition 2.2.23. *Let $\kappa' = 16/\gamma^2$ and $U_\pm$ be as in Proposition 2.2.22. Then the random curve-decorated metric measure spaces represented by $(U_-, D_{h^\gamma}^{U_-}, \mu_{h^\gamma}|_{U_-}, \eta|_{(-\infty,0]})$ and $(U_+, D_{h^\gamma}^{U_+}, \mu_{h^\gamma}|_{U_+}, \eta|_{[0,\infty)})$ are independent and identically distributed.*

Remark 2.2.24. When $\gamma \in (0, \sqrt{2}]$, the $\frac{3\gamma}{2}$-quantum wedge decorated with an independent space-filling SLE curve is simply connected. Hence, it is measurable with respect to the Borel σ-algebra generated by the pointed Gromov–Hausdorff–Prokhorov–uniform topology pointed at 0. (The LQG metric extends continuously to the boundary of the quantum wedge [142, Proposition 1.6].) We use this σ-algebra in the above proposition as we did for the quantum cone in Proposition 2.2.20.

We require an alternative σ-algebra for $\gamma \in (\sqrt{2}, 2)$, since the $\frac{3\gamma}{2}$-quantum wedge is a beaded surface in this regime. The different beads lie at an infinite distance from each other by the definition of the internal metric. We use the Borel σ-algebra with respect to the following metric topology on the space of equivalence classes of curve-decorated beaded metric measure spaces (with the property that the curve enters the beads in chronological order and does not re-enter any bead after entering a subsequent bead) modulo measure-and-curve-preserving isometries. It is given by an extension of the Prokhorov–uniform metric on curve-decorated beaded domains defined in [127, Section 2.2.5], where we use the GHPU metric instead.

1. Let $\mathbb{M}^{\mathrm{GHPU}}$ be the space of equivalence classes of compact metric measure spaces decorated with a continuous curve modulo measure-preserving, curve-preserving isometries. Let d^{GHPU} be the GHPU metric on $\mathbb{M}^{\mathrm{GHPU}}$ defined in [124, Section 1.3].

2. Given a curve-decorated beaded metric measure space $\mathcal{S}$, for $t \geq 0$, let $\mathcal{K}_t \in \mathbb{M}^{\mathrm{GHPU}}$ be the bead of $\mathcal{S}$ with the property that the sum of the measures of the previous beads (not including the bead itself) is at least t, equipped with the curve restricted to this bead. We view $\mathcal{K}$ as a function $[0,\infty) \to \mathbb{M}^{\mathrm{GHPU}}$ defined for almost every t. to it a function $\mathcal{K} : [0,\infty) \to \mathbb{M}^{\mathrm{GHPU}}$ defined as

3. Let $\mathbb{M}^{\mathrm{GHPU}}_{\mathrm{bead}}$ be the set of all Borel measurable functions $\mathcal{K} : [0,\infty) \to \mathbb{M}^{\mathrm{GHPU}}_{\mathrm{bead}}$ which are defined almost everywhere. Define a metric on $\mathbb{M}^{\mathrm{GHPU}}_{\mathrm{bead}}$ by

$$d^{\mathrm{GHPU}}_{\mathrm{bead}}(\mathcal{K}, \widetilde{\mathcal{K}}) = \int_0^\infty e^{-t} \left(1 \wedge d^{\mathrm{GHPU}}(\mathcal{K}_t, \widetilde{\mathcal{K}}_t)\right) dt. \tag{2.2.35}$$

Observe that the total contribution to $d^{\mathrm{GHPU}}_{\mathrm{bead}}$ of beads of LQG measure less than ε is bounded and tends to zero as $\varepsilon \to 0$.

When $\gamma^2 = \kappa = 16/\kappa'$, define the process $\{L_t\}_{t\in\mathbb{R}}$ (resp. $\{R_t\}_{t\in\mathbb{R}}$) to be the change in the left (resp. right) quantum boundary length of $\eta(-\infty, t]$ with respect to 0. Here is the precise construction of this process. Recall the countable dense set of points $\{z_k\}_{k\in\mathbb{N}}$ that we used to define the whole-plane space-filling $\mathrm{SLE}_{\kappa'}$ curve η. If $z_k = \eta(t)$, then the left (resp. right) boundary of $\eta(-\infty, t]$ is $\eta^+_{z_k}$ (resp. $\eta^-_{z_k}$), which is an SLE_κ-type curve. Given a

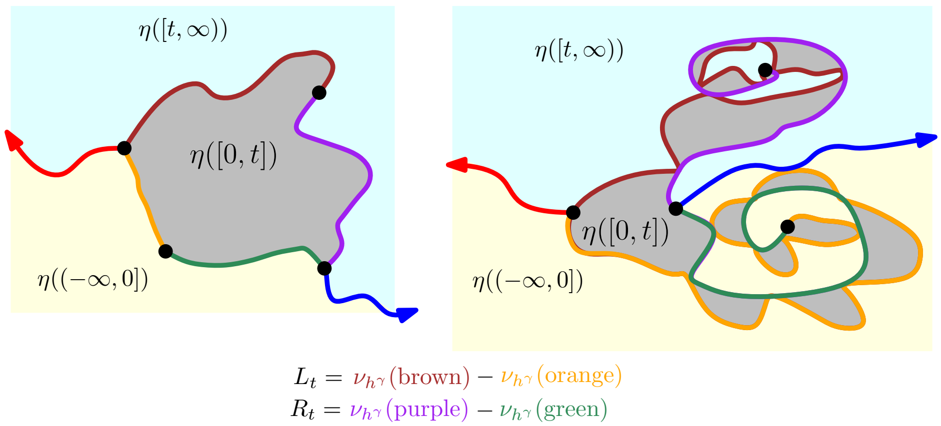

Figure 2.2: Illustration of the quantum boundary length process (L, R) when $\kappa' \geq 8$ (left) and $\kappa' \in (4, 8)$ (right). Suppose $z_k = \eta(t)$, where $t > 0$. The left (resp. right) boundary of $\eta(-\infty, 0]$, i.e., the flow line η_0^+ (resp. η_0^-), is the union of red and orange (resp. green and blue) curves. The left (resp. right) boundary of $\eta(-\infty, t]$, i.e., the flow line $\eta_{z_k}^+$ (resp. $\eta_{z_k}^-$), is the union of red and brown (resp. purple and blue) curves.

GFF-type field h and an independent SLE$_\kappa$-type curve $\tilde{\eta}$, we can define the γ-LQG length measure ν_h on $\tilde{\eta}$ [99, 228, 35]. The quantum length of the entire flow line $\eta_{z_k}^+$ is infinite, but it a.s. merges with η_0^+, so it makes sense to define the difference L_t between the quantum boundary lengths of $\eta_{z_k}^+$ and η_0^+ as illustrated in Figure 2.2. R_{τ_k} is defined analogously using $\eta_{z_k}^-$ and η_0^-. A priori, this defines the processes L and R on a countable dense subset of (random) times t.

The following main theorem of mating-of-trees theory states that, almost surely, the processes L and R can be continuously extended to all real times and that they together give a complete description of the quantum surface $(\mathbb{C}, h^\gamma, 0, \infty, \eta)$. The term mating-of-trees comes from the observation that the collections of flow lines $\{\eta_z^+\}_{z\in\mathbb{C}}$ and $\{\eta_z^-\}_{z\in\mathbb{C}}$ are trees that we can "mate" to obtain the curve-decorated quantum surface (h^γ, η).

Theorem 2.2.25 ([95, Theorems 1.9 and 1.11], [120]). *Let $(\mathbb{C}, h^\gamma, 0, \infty, \eta)$ be the circle average embedding of a γ-quantum cone decorated by an independent space-filling SLE$_{\kappa'}$ curve, which is then reparemeterized according to the γ-LQG measure μ_{h^γ}. Let L_t (resp. R_t) denote the γ-LQG length ν_{h^γ} of the left (resp. right) boundary of $\eta(-\infty, t]$ minus that* *of $\eta(-\infty, 0]$. Then (L, R) evolves as a correlated two-dimensional Brownian motion. In particular, there is a non-random constant $a > 0$ such that*

$$\mathrm{Var}(L_t) = a|t|, \quad \mathrm{Var}(R_t) = a|t|, \quad \textit{and} \quad \mathrm{Cov}(L_t, R_t) = -a\cos\left(\frac{4\pi}{\kappa'}\right)|t| \quad \textit{for} \quad t \in \mathbb{R}. \tag{2.2.36}$$

Moreover, the pair (L, R) almost surely determines both h^γ and η up to a rigid rotation of $\mathbb{C}$ about the origin.

2.2.6 Proof strategy

For the proof of Theorem 2.1.1, we consider the configuration $(\mathbb{C}, h^\gamma, 0, \infty, \eta)$ of a γ-quantum cone decorated with an independent whole-plane space-filling $\mathrm{SLE}_{\kappa'}$, where $\kappa' = 16/\gamma^2$. The proof follows the following outline.

1. In Section 2.3, we prove that $\{\mathfrak{b}_\varepsilon^{-1} N_\varepsilon(\eta[s,t]; D_{h^\gamma})\}_{\varepsilon\in(0,1)}$ is tight for each $s < t$ using estimates for the space-filling SLE and the LQG metric. Hence, the infinite-dimensional random vector $(\mathfrak{b}_\varepsilon^{-1} N_\varepsilon(\eta[s,t]; D_{h^\gamma}) : s, t \in \mathbb{Q}, s < t)$ has a weak subsequential limit. Denote it as $(X_{[s,t]} : s < t)$.

2. In Section 2.4.1, we show that $X_{[s,t]}$ is additive: i.e., $X_{[r,t]} = X_{[r,s]} + X_{[s,t]}$ a.s. for every rational $r < s < t$. Hence, we can define a "Minkowski content process" $\{Y_t\}_{t\in\mathbb{Q}}$ where $X_{[s,t]} = Y_t - Y_s$. Roughly speaking, this step holds because the Minkowski dimension of $\partial\eta[s,t]$ is less than d_γ.

3. In Section 2.4.2, we prove that $\{Y_t\}_{t\in\mathbb{Q}}$ can be extended a continuous process on $\mathbb{R}$ with independent and stationary increments. We also show that $X_{[s,t]} = Y_t - Y_s > 0$ a.s. for every $s < t$. By Blumenthal's zero-one law, a strictly increasing Brownian motion cannot be random. Thus, $t \mapsto Y_t$ must be a deterministic linear function.

4. Our choice (2.1.8) of $\mathfrak{b}_\varepsilon$ ensures that the slope of Y_t is exactly 1. Any subsequence

of $\mathfrak{b}_\varepsilon^{-1} N_\varepsilon(\eta[s,t]; D_{h^\gamma})$ has a further subsequence which converges weakly to $t-s$, so $\lim_{\varepsilon\to 0} \mathfrak{b}_\varepsilon^{-1} N_\varepsilon(\eta[s,t]; D_{h^\gamma}) = t-s$ in probability (Proposition 2.4.3).

5. In Section 2.5, we extend to general sets and other GFF-type fields.

We remark that our proof is similar in a superficial sense to Le Gall's proof in [175] that the volume measure on the Brownian sphere $\mathbf{m}_\infty$ is a Hausdorff measure. In his proof, Le Gall considers the natural projection map $\mathbf{p} : [0,1] \to \mathbf{m}_\infty$ and computes the Hausdorff measures of segments $\mathbf{p}[s,t]$ of this space-filling curve. However, the curve $\mathbf{p}$ is distinct from the space-filing $\mathrm{SLE}_{\kappa'}$ curve η appearing in our proof, and correspondingly, the two proofs use a different set of tools (Brownian snake vs. mating-of-trees).

As for the possibility of constructing the γ-LQG measure as a Hausdorff measure for general $\gamma \in (0,2)$, a key step in such a construction would be to identify the suitable gauge function. This comes down to finding an up-to-constant estimate for the volume of LQG metric balls for general $\gamma \in (0,2)$. In [175], Le Gall proves the estimate $\mathrm{Vol}(\mathcal{B}_r(x)) = \Theta(h(r))$ as $r \to 0$ for the Brownian sphere, where $h(r) = r^4 \log\log(1/r)$; from this estimate, $h(r)$ is identified as the correct gauge function. On the other hand, for general $\gamma \in (0,2)$, the best available estimate for the γ-LQG volume of γ-LQG metric balls is from [16], which states $\mu_h(\mathcal{B}_r(z; D_h)) = r^{d_\gamma + o(1)}$ as $r \to 0$, (We have used this fact extensively in our paper.) Le Gall's proof of the up-to-constant metric ball volume estimate strongly relies on the Brownian snake encoding of the Brownian sphere; currently, we do not have an alternative method to improve the estimate for general γ.

2.3 Tightness of Minkowski content approximations

Let $(\mathbb{C}, h^\gamma, 0, \infty)$ be the circle average embedding of a γ-quantum cone and let η be an independent whole-plane space-filling $\mathrm{SLE}_{\kappa'}$ curve from ∞ to ∞. Let η be parameterized by

μ_{h^γ}. As discussed in Section 2.2.1, we use

$$\lim_{\varepsilon \to 0} \varepsilon^{d_\gamma} N_\varepsilon(\eta[s,t]; D_{h^\gamma}) \tag{2.3.1}$$

as an ansatz for the Minkowski content of $\eta[s,t]$ with respect to the γ-LQG metric D_{h^γ}. The first step is to show that this limit exists along subsequences.

The goal of this section is to show that for each fixed $s < t$, the family of random variables

$$\{\varepsilon^{d_\gamma} N_\varepsilon(\eta[s,t]; D_{h^\gamma})\}_{\varepsilon \in (0,1)} \tag{2.3.2}$$

is tight, so that a subsequential limit in distribution exists for $\varepsilon^{d_\gamma} N_\varepsilon(\eta[s,t]; D_{h^\gamma})$ as $\varepsilon \to 0$. Moreover, the tightness of (2.3.2) is also used to show $\mathfrak{b}_\varepsilon \asymp \varepsilon^{d_\gamma}$ (Corollary 2.4.2); this is why we first use ε^{d_γ} in (2.3.1) rather than $\mathfrak{b}_\varepsilon$ as in the statement of Theorem 2.1.1.

Remark. All results in this section hold for any fixed $\gamma \in (0,2)$ and $\kappa' \in (4,\infty)$, even when $\gamma^2 \neq 16/\kappa'$.

2.3.1 Exit time of space-filling SLE from an LQG metric ball

The key estimate in showing the tightness of (2.3.2) is an upper bound for the probability that the D_{h^γ}-diameter of a space-filling $\mathrm{SLE}_{\kappa'}$ segment $\eta[s,t]$ is large (Proposition 2.3.12). More precisely, we prove an equivalent estimate, which is an upper bound for the probability that the exit time

$$\tau_1 := \inf\{t > 0 : \eta(t) \notin \mathcal{B}_1(0; D_{h^\gamma})\} \tag{2.3.3}$$

is small.

Proposition 2.3.1. *The lower tail of the exit time τ_1 is superpolynomially small. That is, for each $p > 0$,*

$$\mathbb{P}\{\tau_1 \leq \varepsilon\} = o(\varepsilon^p) \quad \textit{as } \varepsilon \to 0^+. \tag{2.3.4}$$

The key input for the proof of Proposition 2.3.1 is the following relation between the LQG distance and the circle averages of GFF. Recall that $\xi = \gamma/d_\gamma$ is the γ-dependent factor which appears in the definition (2.1.4) of the LQG metric.

Lemma 2.3.2 ([93, Proposition 3.15])**.** *Let h be a whole-plane GFF normalized so that $h_1(0) = 0$. Then, with superpoynomially high probability as $C \to \infty$, at a rate which is uniform in the choice of $z \in \mathbb{C} \setminus \{0\}$,*

$$D_{h-\gamma \log|\cdot|}(0, z; B_{4|z|}(0)) \leq C \int_{-\log \frac{|z|}{2}}^{\infty} \left(e^{\xi h_{e^{-t}}(0) - \xi(Q-\gamma)t} + e^{\xi h_{e^{-t}}(z) - \xi Q t} \right) dt. \qquad (2.3.5)$$

Let us first give a heuristic argument for Proposition 2.3.1. To begin, $\partial \mathcal{B}_1(0; D_{h^\gamma})$ is macroscopic with high probability (i.e., it has constant-order Euclidean diameter). Since $D_{h^\gamma}(0, \eta(\tau_1)) = 1$, there has to be some not too small $r > 0$ such that either $h_r(0)$ or $h_r(\eta(\tau_1))$ is large, along the lines of Lemma 2.3.2. By Proposition 2.2.16, $\eta[0, \tau_1] \cap B_r(0)$ and $\eta[0, \tau_1] \cap B_r(\eta(\tau_1))$ each contain a macroscopic Euclidean ball. By comparing the LQG volumes of these Euclidean balls with $e^{\gamma h_r(0)}$ and $e^{\gamma h_r(\eta(\tau_1))}$, we conclude that with high probability, $\tau_1 = \mu_{h^\gamma}(\eta[0, \tau_1])$ cannot be too small.

There are three main points in making this heuristic rigorous. First, the laws of h^γ and $h - \gamma \log|\cdot|$ agree only when restricted to the unit disk $\mathbb{D}$. To this end, we analyze

$$\tau_\varepsilon := \inf\{t > 0 : \eta(t) \notin \mathcal{B}_\varepsilon(0; D_{h^\gamma})\} \qquad (2.3.6)$$

instead of τ_1, because $\mathcal{B}_\varepsilon(0; D_{h^\gamma}) \subset \mathbb{D}$ with increasingly high probability as $\varepsilon \to 0$. Observe from Proposition 2.2.20 that for each $\varepsilon > 0$,

$$\tau_\varepsilon \overset{d}{=} \varepsilon^{d_\gamma} \tau_1. \qquad (2.3.7)$$

Second, the point z in Lemma 2.3.2 has to be deterministic, whereas the point $\eta(\tau_1)$ is random. The idea to get around this issue is to take a union bound over $z \in \mathbb{D} \cap \varepsilon^s \mathbb{Z}^2$ for a large constant s and show that there is some point in $\mathbb{D} \cap \varepsilon^s \mathbb{Z}^2$ which is close to $\eta(\tau_\varepsilon)$ in both Euclidean and LQG distances. Third, we can compare $\mu_{h^\gamma}(B_r(w))$ and $h_r(w)$ only for fixed w and r. We again need to take a union bound over Euclidean balls $B_r(w)$, polynomially many in ε, that are possibly contained in $\eta[0, \tau_1]$.

Proof of Proposition 2.3.1. If we show that $\tau_\varepsilon = \mu_{h^\gamma}(\eta[0, \tau_\varepsilon]) > \varepsilon^{d_\gamma + 1}$ with superpolynomially high probability as $\varepsilon \to 0$, then the proposition follows by (2.3.7).

Below, we describe an event consisting of six steps on which $\mu_{h^\gamma}(\eta[0, \tau_\varepsilon]) > \varepsilon^{d_\gamma+1}$. These events are stated using constants $0 < a < b < s$, $0 < N < s - b$, $\zeta \in (0,1)$, and $q > 2$. Eventually, these constants will be chosen in terms of a single constant $p > 0$, which we will eventually allow to be arbitrarily large. We shall then verify that the stated event holds with probability $1 - O(\varepsilon^{p/3})$ as $\varepsilon \to 0$ using the lemmas stated and proven just after the main body of the proof.

Throughout, h refers to the field $h^\gamma + \gamma \log|\cdot|$, whose restriction to the unit disk agrees in law with the corresponding restriction of a whole-plane GFF.

1. The points which lie at D_{h^γ}-distance ε from the origin — i.e., $\partial \mathcal{B}_\varepsilon(0; D_{h^\gamma})$ — are within the Euclidean annulus $\overline{B_{\varepsilon^a}(0)} \setminus B_{\varepsilon^b}(0)$. In particular, $\varepsilon^b \le |\eta(\tau_\varepsilon)| \le \varepsilon^a$. (Figure 2.3(a))

2. For $\varepsilon^b \le |\eta(\tau_\varepsilon)| \le \varepsilon^a$, there exists a point $z \in \varepsilon^s \mathbb{Z}^2$ satisfying $\varepsilon^b \le |z| \le \varepsilon^a$ which is close to $\eta(\tau_\varepsilon)$ in both Euclidean and LQG distances: $|\eta(\tau_\varepsilon) - z| < \varepsilon^s$ and $D_{h^\gamma}(\eta(\tau_\varepsilon), z) \le \varepsilon/2$ precisely. The latter condition implies $D_{h^\gamma}(0, z) \ge \varepsilon/2$ since $D_{h^\gamma}(0, \eta(\tau_\varepsilon)) = \varepsilon$. (Figure 2.3(b))

3. Given the point z found in the previous step, let $k_0 = \lfloor -\log(|z|/2) \rfloor$. There exists an integer $k \in [k_0, k_0 + N \log \varepsilon^{-1}]$ such that at least one of the circle averages $h_{e^{-k}}(0)$ or $h_{e^{-k}}(z)$ is bounded below in terms of the logarithm of the LQG distance $D_{h^\gamma}(0, z)$. Specifically, at least one of the following two cases is true.

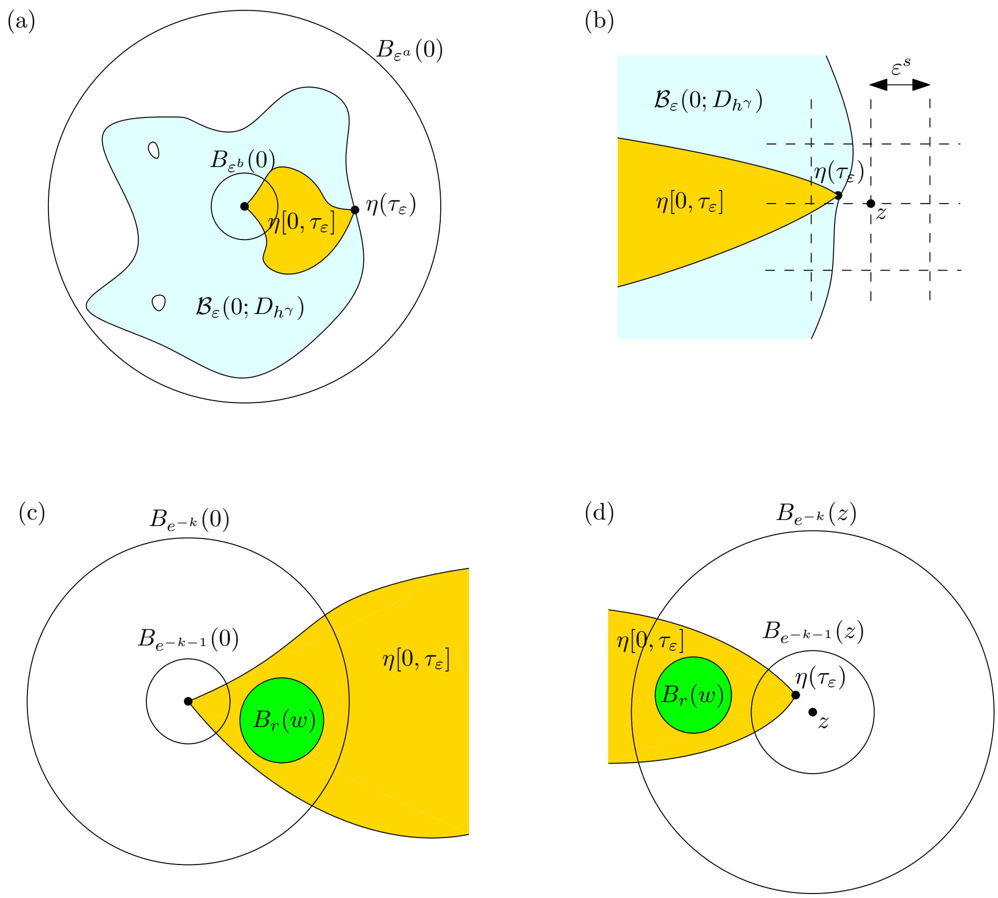

Figure 2.3: Proof of Proposition 2.3.1. Let $h = h^\gamma + \gamma \log|\cdot|$ and $p > 0$. The constants a, b, s, N, q, ζ are defined in (2.3.14) and (2.3.15) in terms of p. **(a) Step 1:** With probability $1 - O(\varepsilon^{p/3})$, the boundary of $\mathcal{B}_\varepsilon(0; D_{h^\gamma})$ is contained within the Euclidean annulus $B_{\varepsilon^a}(0) \setminus \overline{B_{\varepsilon^b}(0)}$. In particular, $\varepsilon^b < |\eta(\tau_\varepsilon)| < \varepsilon^a$. **(b) Step 2:** With probability $1 - O(\varepsilon^{p-Q/\xi})$, there exists $z \in \varepsilon^s \mathbb{Z}^2$ for which $|\eta(\tau_\varepsilon) - z| < \varepsilon^s$ and $D_{h^\gamma}(\eta(\tau_\varepsilon), z) < \varepsilon/2$. Let $k_0 = \lfloor -\log(|z|/2) \rfloor$ and $k_1 = k_0 + N \log \varepsilon^{-1}$. Conditioned on the previous events, with superpolynomially high probability, at least one of the following two cases holds. **(c) Steps 3 and 4, Case 1:** There exists an integer $k \in [k_0, k_1]$ such that $\gamma h_{e^{-k}}(0) \geq \gamma(Q - \gamma)k + (d_\gamma + \frac{1}{3}) \log \varepsilon$. For this k, there exists a Euclidean ball $B_{e^{-k(1+\zeta)}}(w)$ contained in the intersection of $\eta[0, \tau_\varepsilon]$ and $B_{e^{-k}}(0) \setminus \overline{B_{e^{-k-1}}(0)}$ with $w \in e^{-k(1+\zeta)}\mathbb{Z}^2$. **(d) Steps 3 and 4, Case 2:** There exists an integer $k \in [k_0, k_1]$ such that $\gamma h_{e^{-k}}(z) \geq \gamma Q k + (d_\gamma + \frac{1}{3}) \log \varepsilon$. For this k, there exists a Euclidean ball $B_{e^{-k(1+\zeta)}}(w)$ contained in the intersection of $\eta[0, \tau_\varepsilon]$ and $B_{e^{-k}}(z) \setminus \overline{B_{e^{-k-1}}(z)}$ with $w \in e^{-k(1+\zeta)}\mathbb{Z}^2$. **Steps 5 and 6 (not visualized):** We obtain a lower bound of $\mu_{h^\gamma}(B_r(w))$ in terms of $h_r(w)$, which we compare with either $h_{e^{-k}}(0)$ (Case 1) or $h_{e^{-k}}(z)$ (Case 2). For sufficiently large p, on the event that all of the previous conditions hold, $\tau_\varepsilon \geq \mu_{h^\gamma}(B_r(w)) \geq \varepsilon^{d_\gamma + 1}$. By (2.3.7), this implies $\mathbb{P}\{\tau_1 \geq \varepsilon\} = 1 - O(\varepsilon^p)$ for any given $p > 0$.

- Case 1: There exists $k \in [k_0, k_0 + N \log \varepsilon^{-1}]$ such that

$$e^{\gamma h_{e^{-k}}(0)} \geq 2^{d_\gamma} \varepsilon^{\frac{1}{3}} e^{\gamma(Q-\gamma)k} \left(D_{h^\gamma}(0,z)\right)^{d_\gamma}. \tag{2.3.8}$$

- Case 2: There exists $k \in [k_0, k_0 + N \log \varepsilon^{-1}]$ such that

$$e^{\gamma h_{e^{-k}}(z)} \geq 2^{d_\gamma} \varepsilon^{\frac{1}{3}} e^{\gamma Q k} \left(D_{h^\gamma}(0,z)\right)^{d_\gamma}. \tag{2.3.9}$$

Since $D_{h^\gamma}(0,z) \geq \varepsilon/2$ by our choice of z, we have

$$e^{\gamma h_{e^{-k}}(0)} \geq e^{\gamma(Q-\gamma)k} \varepsilon^{d_\gamma + \frac{1}{3}} \quad \text{in Case 1,} \quad e^{\gamma h_{e^{-k}}(z)} \geq e^{\gamma Q k} \varepsilon^{d_\gamma + \frac{1}{3}} \quad \text{in Case 2.}$$

4. Given the integer k found in the previous step, let $r = e^{-k(1+\zeta)}$. There exists a point $w \in r\mathbb{Z}^2$ such that:
   - In Case 1, $B_r(w) \subset \eta[0, \tau_\varepsilon] \cap (B_{e^{-k}}(0) \setminus \overline{B_{e^{-k-1}}(0)})$. (Figure 2.3(c))
   - In Case 2, $B_r(w) \subset \eta[0, \tau_\varepsilon] \cap (B_{e^{-k}}(z) \setminus \overline{B_{e^{-k-1}}(z)})$. (Figure 2.3(d))
5. For z, k, and w as in the previous three steps, we have the following comparison of circle averages.
   - Case 1: $|h_r(w) - h_{e^{-k}}(0)| \leq kq\zeta$. Therefore, $e^{\gamma h_r(w)} \geq e^{\gamma(Q-\gamma-q\zeta)k} \varepsilon^{d_\gamma + \frac{1}{3}}$.
   - Case 2: $|h_r(w) - h_{e^{-k}}(z)| \leq kq\zeta$. Therefore, $e^{\gamma h_r(w)} \geq e^{\gamma(Q-q\zeta)k} \varepsilon^{d_\gamma + \frac{1}{3}}$.
6. The LQG measure of the Euclidean ball $B_r(w)$ is bounded below by

$$\mu_h(B_r(w)) \geq \varepsilon^{\frac{1}{3}} r^{\gamma Q} e^{\gamma h_r(w)}. \tag{2.3.10}$$

These are the six steps comprising the event on which $\tau_\varepsilon > \varepsilon^{d_\gamma + 1}$ with the choice of constants stated below ((2.3.14) and (2.3.15)). Let us first verify that on the stated event, we have the correct lower bound on τ_ε as claimed. We begin by substituting the lower bound for $h_r(w)$ claimed in Step 5 into (2.3.10). Recall $r = e^{-k(1+\zeta)}$ and $h^\gamma = h - \gamma \log|\cdot|$.

- Case 1: Since $B_r(w) \subset B_{e^{-k}}(0)$, we have $\mu_{h^\gamma}(B_r(w)) \geq e^{\gamma^2 k}\mu_h(B_r(w))$. Hence,

$$\mu_{h^\gamma}(B_r(w)) \geq e^{\gamma^2 k}\mu_h(B_r(w)) \geq e^{\gamma^2 k}\varepsilon^{\frac{1}{3}}(e^{-k(1+\zeta)})^{\gamma Q} e^{\gamma(Q-\gamma-q\zeta)k}\varepsilon^{d_\gamma+\frac{1}{3}} \tag{2.3.11}$$

- Case 2: Since $B_r(w) \subset \mathbb{D}$, we have $\mu_{h^\gamma}(B_r(w)) \geq \mu_h(B_r(w))$. Hence,

$$\mu_{h^\gamma}(B_r(w)) \geq \mu_h(B_r(w)) \geq \varepsilon^{\frac{1}{3}}(e^{-k(1+\zeta)})^{\gamma Q} e^{\gamma(Q-q\zeta)k}\varepsilon^{d_\gamma+\frac{1}{3}}. \tag{2.3.12}$$

Collecting exponents, we conclude in both cases that

$$\tau_\varepsilon \geq \mu_{h^\gamma}(B_r(w)) \geq \varepsilon^{d_\gamma+\frac{2}{3}} e^{-\gamma(Q+q)\zeta k}. \tag{2.3.13}$$

We now set

$$a = \frac{1}{\xi^2 p} \asymp p^{-1}, \quad b = \frac{p}{(Q-\gamma)^2} \asymp p, \quad s = \left(2 + \frac{\gamma Q}{Q^2/2 - 2}\right) b \asymp p \tag{2.3.14}$$

where $\asymp$ means equality up to γ-dependent multiplicative constants when p is large. We also choose

$$N = \sqrt[4]{1 + \frac{6s}{p}}\sqrt{\xi p} \asymp p^{1/2}, \quad q = N^7 \asymp p^{7/2}, \quad \zeta = N^{-10} \asymp p^{-5}. \tag{2.3.15}$$

Note that $s - b > b = \frac{p}{(Q-\gamma)^2} > N$ for sufficiently large p. Since we only consider $k \leq (b+N)\log \varepsilon^{-1} = O(N^2 \log \varepsilon^{-1})$ in Step 3,

$$e^{-\gamma(Q+q)\zeta k} \geq \varepsilon^{O(N^{-1})} \geq \varepsilon^{1/3} \tag{2.3.16}$$

for all $0 < \varepsilon < 1/2$ given that p is sufficiently large. Hence, the lower bound for τ_ε from (2.3.13) satisfies

$$\tau_\varepsilon > \varepsilon^{d_\gamma + \frac{2}{3}} e^{-\gamma(Q+q)\zeta k} \geq \varepsilon^{d_\gamma + 1} \tag{2.3.17}$$

for all large p.

It remains to estimate the probability of the above event. We compute this probability step-by-step given the constants in (2.3.14) and (2.3.15).

1. In Lemma 2.3.4, we show that for sufficiently large p, Step 1 occurs with probability $1 - O(\varepsilon^{p/3})$ as $\varepsilon \to 0$.

2. In Lemma 2.3.5, we show that given our choices of a, b, and s, truncated on the event $\varepsilon^b \leq |\eta(\tau_\varepsilon)| \leq \varepsilon^a$, the probability that Step 2 fails is bounded above by $O(\varepsilon^{p-Q/\xi})$ as $\varepsilon \to 0$.

3. In Lemma 2.3.6, we show that for each fixed $z \in B_{1/4}(0) \setminus \{0\}$, Step 3 occurs with superpolynomially high probability as $\varepsilon \to 0$ at a rate uniform in the choice of z. A union bound over $O(\varepsilon^{-2s})$-many points of $z \in \varepsilon^s \mathbb{Z}^2$ with $\varepsilon^b \leq |z| \leq \varepsilon^a$ gives a superpolynomially high probability for this step.

4. Consider $\varepsilon > 0$ sufficiently small so that $\varepsilon^s < \frac{1}{2e}\varepsilon^{b+N}$ (this holds for all small enough ε since we chose our parameters so that $s > b + N$). If $\varepsilon^b \leq |z| \leq \varepsilon^a$ and $\eta(\tau_\varepsilon) \in B_{\varepsilon^s}(z)$, then $\eta[0, \tau_\varepsilon] \cap (B_{e^{-k}}(0) \setminus \overline{B_{e^{-k-1}}(0)})$ and $\eta[0, \tau_\varepsilon] \cap (B_{e^{-k}}(z) \setminus \overline{B_{e^{-k-1}}(z)})$ are both nonempty and the Euclidean diameters of the two intersections are at least e^{-k-1}. For the latter set, this is true because $\eta[0, \tau_\varepsilon]$ is a connected set and

$$\eta(0) = 0 \notin B_{2e^{-k_0-1}}(z) \quad \text{and} \quad \eta(\tau_\varepsilon) \in B_{\varepsilon^s}(z) \subset B_{e^{-k_0 + N\log\varepsilon - 1}}(z),$$

where we used $\varepsilon^s < \frac{1}{2e}\varepsilon^{b+N} < \frac{|z|}{2e}\varepsilon^N < e^{-k_0-1}\varepsilon^N$. A similar argument works for the former set.

Hence, Proposition 2.2.16 implies that for each lattice point $z \in \varepsilon^s\mathbb{Z}^2 \cap (B_{\varepsilon^a}(0) \setminus \overline{B_{\varepsilon^b}(0)})$ and integer $k \in [k_0, k_0 + N\log\varepsilon^{-1}]$, the following holds with superpolynomially high probability: on the event that $\eta(\tau_\varepsilon) \in B_{\varepsilon^s}(z)$, the sets $\eta[0,\tau_\varepsilon] \cap (B_{e^{-k}}(0) \setminus \overline{B_{e^{-k-1}}(0)})$ and $\eta[0,\tau_\varepsilon] \cap (B_{e^{-k}}(z) \setminus \overline{B_{e^{-k-1}}(z)})$ each contain a Euclidean ball of radius $e^{-(k+1)(1+\frac{\zeta}{2})}$. For sufficiently small ε (hence sufficiently large k_0), we can always find within each of these Euclidean balls a smaller Euclidean ball $B_r(w)$ where $r = e^{-k(1+\zeta)}$ and $w \in r\mathbb{Z}^2$. Now take a union bound over all such pairs (z,k). The number of these pairs is at most some negative power of ε, so Step 4 holds with superpolynomially high probability in ε.

5. In Lemma 2.3.7, we show that given an integer $k_0 \geq 2$, for each

$$z \in \varepsilon^s\mathbb{Z}^2 \cap \{z \in \mathbb{C} : |z| \in (2e^{-k_0-1}, 2e^{-k_0}]\}, \tag{2.3.18}$$

the following event holds with probability $1 - O(e^{-(q^2/2-2)\zeta k_0})$: the comparisons in Step 5 hold for all possible choices of k and w. There are $O(\varepsilon^{-2s}e^{2k_0})$-many points z satisfying (2.3.18) for each positive integer k_0. Since $|z| \in [\varepsilon^a, \varepsilon^b]$ and $k_0 = \lfloor -\log(|z|/2) \rfloor$, we only need to consider k_0 satisfying $a\log\varepsilon^{-1} + \log(2/e) \leq k_0 \leq b\log\varepsilon^{-1} + \log 2$. Taking a union bound over these k_0 and z, Step 5 holds with probability

$$1 - O\left(\sum_{k_0=\lfloor a\log\varepsilon^{-1}\rfloor}^{\lceil b\log\varepsilon^{-1}\rceil} \varepsilon^{-2s}e^{2k_0}e^{-(q^2/2-2)\zeta k_0}\right) = 1 - O\left(\varepsilon^{((\frac{q^2}{2}-2)\zeta-2)a-2s}\right) \tag{2.3.19}$$

provided that $(\frac{q^2}{2} - 2)\zeta - 2 > 0$. The constants q and ζ chosen in (2.3.15) satisfy this condition for all sufficiently large p. Moreover, with the further choice of constants a

and s in (2.3.14), the error term in (2.3.19) satisfies

$$1 - O\left(\varepsilon^{((\frac{q^2}{2}-2)\zeta-2)a-2s}\right) \geq 1 - O\left(\varepsilon^{\frac{N^4}{3}a-2s}\right) \geq 1 - O(\varepsilon^{p/3}). \tag{2.3.20}$$

6. In Lemma 2.3.8, we show that for each possible choice of z, k, and w, Step 6 holds with superpolynomially high probability as $\varepsilon \to 0$ with rate uniform over all such choices. Now take a union bound over these z, k, and w, of which there are only polynomially many in ε.

To summarize, the probabilities of the events we truncate on each step are:

- Steps 1 and 5: $1 - O(\varepsilon^{p/3})$;
- Step 2: $1 - O(\varepsilon^{p-Q/\xi})$;
- Steps 3, 4, and 6: superpolynomially high in ε.

Since the choice of $p > 0$ was arbitrary, we conclude that $\tau_\varepsilon > \varepsilon^{d_\gamma+1}$ holds with superpolynomially high probability in ε as $\varepsilon \to 0$. $\square$

We now state and prove the lemmas used in the proof of Proposition 2.3.1, which are all based on standard LQG estimates. First, we record a property of whole-plane GFF that we will use repeatedly. It is a straightforward consequence of Lemmas 2.2.5 and 2.2.6.

Lemma 2.3.3 ([89, Lemma 3.4])**.** *Let h be a whole-plane GFF normalized so that $h_1(0) = 0$. For each deterministic $r > 0$, the field $(h - h_r(0))|_{B_r(0)}$ is independent from the circle average $h_r(0)$.*

The following lemma calculates the probability of the event $\{\varepsilon^b \leq |\eta(\tau_\varepsilon)| \leq \varepsilon^a\}$ appearing in Step 1.

Lemma 2.3.4. *Let $\mathcal{B}_\varepsilon(0; D_{h^\gamma})$ be the LQG metric ball of radius ε and $B_r(0)$ be the Euclidean ball of radius r, both centered at the origin.*

(i) *For each $p > 0$, there exists a constant $C_p > 0$ depending only on p such that for every $\varepsilon, r \in (0,1)$,*

$$\mathbb{P}\left\{\mathcal{B}_\varepsilon(0; D_{h^\gamma}) \subset B_r(0)\right\} \geq 1 - C_p \varepsilon^p r^{-(Q-\gamma)\xi p - \frac{1}{2}\xi^2 p^2}. \tag{2.3.21}$$

In particular, for each sufficiently large p, as $\varepsilon \to 0$,

$$\mathbb{P}\left\{\mathcal{B}_\varepsilon(0; D_{h^\gamma}) \subset B_{\varepsilon^{1/(\xi^2 p)}}(0)\right\} \geq 1 - C_p \varepsilon^{p/2-(Q-\gamma)/\xi} = 1 - O(\varepsilon^{p/3}). \tag{2.3.22}$$

(ii) *There exists a constant $C > 0$ such that for every $\varepsilon \in (0,1)$ and $r \in (0, \frac{1}{2})$,*

$$\mathbb{P}\left\{B_r(0) \subset \mathcal{B}_\varepsilon(0; D_{h^\gamma})\right\} \geq 1 - C\varepsilon^{-\frac{Q-\gamma}{\xi}} r^{\frac{(Q-\gamma)^2}{2}}. \tag{2.3.23}$$

In particular, for each sufficiently large p, as $\varepsilon \to 0$,

$$\mathbb{P}\left\{B_{\varepsilon^{p/(Q-\gamma)^2}}(0) \subset \mathcal{B}_\varepsilon(0; D_{h^\gamma})\right\} \geq 1 - C\varepsilon^{p/2-(Q-\gamma)/\xi} = 1 - O(\varepsilon^{p/3}). \tag{2.3.24}$$

Proof. Let h refer to the field $h^\gamma + \gamma \log|\cdot|$, whose restriction to $\mathbb{D}$ agrees in law with the corresponding restriction of a whole-plane GFF.

(i) If $\mathcal{B}_\varepsilon(0; D_{h^\gamma}) \not\subset B_r(0)$, then $D_{h^\gamma}(0, \partial B_r(0)) < \varepsilon$. By the Markov inequality,

$$\mathbb{P}\{\mathcal{B}_\varepsilon(0; D_{h^\gamma}) \not\subset B_r(0)\} \leq \mathbb{P}\left\{D_{h^\gamma}(0, \partial B_r(0)) < \varepsilon\right\} \leq \varepsilon^p\, \mathbb{E}\left[(D_{h^\gamma}(0, \partial B_r(0)))^{-p}\right]. \tag{2.3.25}$$

By the Weyl scaling and locality properties of the LQG metric, $e^{-\xi h_r(0)} D_{h^\gamma}(0, \partial B_r(0))$ is a.s. determined by $(h - h_r(0))|_{B_r(0)}$, which is independent of $h_r(0)$ by Lemma 2.3.3. By this independence, [93, Proposition 3.12], and the fact that $h_r(0) \sim N(0, \log(1/r))$, there exists a constant $C_p > 0$ such that

$$\begin{aligned}
&\mathbb{E}\left[(D_{h^\gamma}(0, \partial B_r(0)))^{-p}\right] \\
&= r^{(\gamma - Q)\xi p}\, \mathbb{E}\left[e^{-\xi p h_r(0)}\right] \mathbb{E}\left[\left(r^{\xi(\gamma-Q)} e^{-\xi h_r(0)} D_{h^\gamma}(0, \partial B_r(0))\right)^{-p}\right] \\
&\le C_p r^{-(Q-\gamma)\xi p - \frac{1}{2}\xi^2 p^2}.
\end{aligned} \tag{2.3.26}$$

(ii) Let $0 < u < (\frac{4}{\gamma^2} - 1)d_\gamma$. If $B_r(0) \not\subset \mathcal{B}_\varepsilon(0; D_{h^\gamma})$, then $\sup_{w \in B_r(0)} D_{h^\gamma}(0, w) > \varepsilon$, which implies $\sup_{w \in B_r(0)} D_{h^\gamma}^{B_{2r}(0)}(0, w) > \varepsilon$. (Recall the definition (2.2.12) of the internal metric.) From [93, Proposition 3.17], there exists a constant $\widetilde{C}_u > 0$ depending only on u such that

$$\mathbb{E}\left[\left((2r)^{(\gamma-Q)\xi} e^{-\xi h_{2r}(0)} \sup_{w \in B_r(0)} D_h^{B_{2r}(0)}(0, w)\right)^u\right] \le \widetilde{C}_u. \tag{2.3.27}$$

From the independence of $h_{2r}(0)$ and $(h - h_{2r}(0))|_{B_{2r}(0)}$ (Lemma 2.3.3) and the fact that $h_{2r}(0) \sim N(0, \log(1/(2r)))$, we have

$$\begin{aligned}
&\mathbb{P}\left\{\sup_{w \in B_r(0)} D_{h^\gamma}^{B_{2r}(0)}(0, w) > \varepsilon\right\} \le \varepsilon^{-u}\, \mathbb{E}\left[\left(\sup_{w \in B_r(0)} D_h^{B_{2r}(0)}(0, w)\right)^u\right] \\
&= \varepsilon^{-u}\, \mathbb{E}\left[\left((2r)^{(Q-\gamma)\xi} e^{\xi h_{2r}(0)}\right)^u\right] \mathbb{E}\left[\left((2r)^{(\gamma-Q)\xi} e^{-\xi h_{2r}(0)} \sup_{w \in B_r(0)} D_h^{B_{2r}(0)}(0, w)\right)^u\right] \\
&\le (2^u \widetilde{C}_u)\varepsilon^{-u} r^{(Q-\gamma)\xi u - \frac{1}{2}\xi^2 u^2}.
\end{aligned} \tag{2.3.28}$$

We obtain the lemma by choosing $u = \frac{1}{\xi}(Q - \gamma) = \frac{1}{2}(\frac{4}{\gamma^2} - 1)d_\gamma$. □

The following lemma computes the probability of the event in Step 2, in which we find a point z on the lattice $\varepsilon^s \mathbb{Z}^2$ which is close to $\eta(\tau_\varepsilon)$ in both Euclidean and LQG distances. Again, recall the definition of the internal metric in (2.2.12).

Lemma 2.3.5. *Suppose $0 < a < b$ are given. For $s > b$, let $E_{\varepsilon,s}$ be the event that for all*

$z \in \varepsilon^s \mathbb{Z}^2$ *with* $\varepsilon^b \le |z| \le \varepsilon^a$, *we have*

$$\sup_{w \in B_{\varepsilon^s}(z)} D_{h^\gamma}^{B_{2\varepsilon^s}(z)}(z,w) \le \frac{\varepsilon}{2}. \tag{2.3.29}$$

Then, as $\varepsilon \to 0$,

$$\mathbb{P}(E_{\varepsilon,s}) = 1 - O\big(\varepsilon^{(\frac{Q^2}{2}-2)s - \gamma Q b - \frac{Q}{\xi}}\big). \tag{2.3.30}$$

Proof. Recall the notation $h = h^\gamma + \gamma \log|\cdot|$. Consider a fixed $z \in \varepsilon^s\mathbb{Z}^2$ with $\varepsilon^b \le |z| \le \varepsilon^a$ for now. For $\varepsilon > 0$ is small enough that $|z| - 2\varepsilon^s \ge \varepsilon^b - 2\varepsilon^s \ge \varepsilon^b/2$, by the Weyl scaling axiom, we have

$$\begin{aligned}\sup_{w \in B_{\varepsilon^s}(z)} D_{h^\gamma}^{B_{2\varepsilon^s}(z)}(z,w) &\le (|z| - 2\varepsilon^s)^{-\xi\gamma} \sup_{w \in B_{\varepsilon^s}(z)} D_h^{B_{2\varepsilon^s}(z)}(z,w) \\ &\le \left(\frac{\varepsilon^b}{2}\right)^{-\xi\gamma} \sup_{w \in B_{\varepsilon^s}(z)} D_h^{B_{2\varepsilon^s}(z)}(z,w).\end{aligned} \tag{2.3.31}$$

Let $p \in (0, 4d_\gamma/\gamma^2)$. By Lemma 2.3.3 and the translation invariance of the law of the whole-plane GFF modulo the additive constant as in (2.2.10), $e^{-\xi h_{2\varepsilon^s}(z)} \sup_{w \in B_{\varepsilon^s}(z)} D_h^{B_{2\varepsilon^s}(z)}(z,w)$ is independent of $h_{2\varepsilon^s}(z)$. By this independence and the fact that $h_{2\varepsilon^s}(z) \sim N(0, \log(2\varepsilon^s))$, we have

$$\begin{aligned}&\mathbb{P}\left\{\sup_{w \in B_{\varepsilon^s}(z)} D_{h^\gamma}^{B_{2\varepsilon^s}(z)}(z,w) > \frac{\varepsilon}{2}\right\} \le \mathbb{P}\left\{\sup_{w \in B_{\varepsilon^s}(z)} D_h^{B_{2\varepsilon^s}(z)}(z,w) > \frac{\varepsilon}{2}\left(\frac{\varepsilon^b}{2}\right)^{\xi\gamma}\right\} \\ &\le \left(\frac{2^{1+\xi\gamma}}{\varepsilon^{1+\xi\gamma b}}\right)^p \mathbb{E}\left[\left(\sup_{w \in B_{\varepsilon^s}(z)} D_h^{B_{2\varepsilon^s}(z)}(z,w)\right)^p\right] \\ &= \left(\frac{2^{1+\xi\gamma}}{\varepsilon^{1+\xi\gamma b}}\right)^p \mathbb{E}\left[\left((2\varepsilon^s)^{\xi Q} e^{\xi h_{2\varepsilon^s}(z)}\right)^p\right] \mathbb{E}\left[\left((2\varepsilon^s)^{-\xi Q} e^{-\xi h_{2\varepsilon^s}(z)} \sup_{w \in B_{\varepsilon^s}(z)} D_h^{B_{2\varepsilon^s}(z)}(z,w)\right)^p\right] \\ &= \left(\frac{2^{1+\xi\gamma}}{\varepsilon^{1+\xi\gamma b}}\right)^p (2\varepsilon^s)^{\xi Q p - \frac{1}{2}\xi^2 p^2} \mathbb{E}\left[\left((2\varepsilon^s)^{-\xi Q} e^{-\xi h_{2\varepsilon^s}(z)} \sup_{w \in B_{\varepsilon^s}(z)} D_h^{B_{2\varepsilon^s}(z)}(z,w)\right)^p\right].\end{aligned} \tag{2.3.32}$$

By [93, Proposition 3.9], there exists a constant $C_p > 0$ depending only on p and γ such that

$$\mathbb{E}\left[\left((2\varepsilon^s)^{-\xi Q} e^{-\xi h_{2\varepsilon^s}(z)} \sup_{w \in B_\varepsilon^s(z)} D_h^{B_{2\varepsilon^s}(z)}(z,w)\right)^p\right] \le C_p. \tag{2.3.33}$$

By plugging this into the previous inequality, we get that there exists another constant $\widetilde{C}_p > 0$ depending only on p and γ such that

$$\mathbb{P}\left\{\sup_{w \in B_\varepsilon^s(z)} D_{h^\gamma}^{B_{2\varepsilon^s}(z)}(z,w) > \frac{\varepsilon}{2}\right\} \le \widetilde{C}_p \varepsilon^{p(-1-\xi\gamma b+\xi Q s-\frac{1}{2}\xi^2 sp)}. \tag{2.3.34}$$

The constant $\widetilde{C}_p$ does not depend on the choice of $z \in \overline{B_{\varepsilon^a}(0)} \setminus B_{\varepsilon^b}(0)$. Choose $p = Q/\xi = (2+\frac{\gamma^2}{2})\frac{d_\gamma}{\gamma^2} < \frac{4d_\gamma}{\gamma^2}$. The lemma follows by taking a union bound over $O(\varepsilon^{-2s})$-many points $z \in \varepsilon^s\mathbb{Z}^2$ such that $\varepsilon^b \le |z| \le \varepsilon^a$. □

The following lemma is used in Step 3. This is the key step in the proof of Proposition 2.3.1, in which we combine Lemma 2.3.2 with $D_{h^\gamma}(0,\tau_\varepsilon) = \varepsilon$ to find an integer k such that either $h_{e^{-k}}(0)$ or $h_{e^{-k}}(z)$ is large.

Lemma 2.3.6. *Let $N > 0$ be given. Let $z \in B_{1/4}(0) \setminus \{0\}$ be a fixed point and let k_0 be the positive integer such that $2e^{-k_0-1} < |z| \le 2e^{-k_0}$. Denote $k_1 = \lfloor k_0 + N\log C\rfloor$ and $h = h^\gamma + \gamma\log|\cdot|$. Then, with superpolynomially high probability as $C \to \infty$, at a rate uniform over the choice of z, there exists an integer $k \in [k_0, k_1]$ such that either*

$$e^{\gamma h_{e^{-k}}(0)} \ge C^{-1} e^{\gamma(Q-\gamma)k} (D_{h^\gamma}^{\mathbb{D}}(0,z))^{d_\gamma} \quad \textit{or} \quad e^{\gamma h_{e^{-k}}(z)} \ge C^{-1} e^{\gamma Q k} (D_{h^\gamma}^{\mathbb{D}}(0,z))^{d_\gamma}. \tag{2.3.35}$$

Proof. From Lemma 2.3.2, it holds with superpolynomially high probability as $C \to \infty$, at a rate which is uniform in the choice of $z \in B_{1/4}(0) \setminus \{0\}$, that

$$D_{h^\gamma}^{\mathbb{D}}(0,z) \le D_{h^\gamma}^{B_{4|z|}(0)}(0,z) \le C \int_{-\log\frac{|z|}{2}}^{\infty} \left(e^{\xi h_{e^{-t}}(0)-\xi(Q-\gamma)t} + e^{\xi h_{e^{-t}}(z)-\xi Q t}\right) dt. \tag{2.3.36}$$

The main idea of the proof is to bound the above integral by a Riemann sum approximation.

By Lemma 2.2.5, $t \mapsto h_{e^{-t}}(z)$ is a standard Brownian motion for $t \geq k_0$ given the initial value $h_{e^{-k_0}}(z)$. For each integer $k \in [k_0, k_1 - 1]$, by the reflection principle,

$$\mathbb{P}\left\{ \sup_{t \in [k,k+1]} |h_{e^{-t}}(z) - h_{e^{-k}}(z)| > \xi^{-1} \log C \right\} = O\left(\frac{e^{-\frac{(\log C)^2}{2\xi^2}}}{\log C} \right) \tag{2.3.37}$$

as $C \to \infty$. The constant here is uniform over z and k. On the complementary event,

$$e^{\xi h_{e^{-t}}(z) - \xi Q t} \leq C e^{\xi h_{e^{-t}}(z) - \xi Q k} \quad \text{for all } t \in [k, k+1]. \tag{2.3.38}$$

Since $Q - \gamma = (2/\gamma) - (\gamma/2) > 0$, the same lower bound applies to the probability that

$$e^{\xi h_{e^{-t}}(0) - \xi(Q-\gamma)t} \leq C e^{\xi h_{e^{-t}}(0) - \xi(Q-\gamma)k} \quad \text{for all } t \in [k, k+1]. \tag{2.3.39}$$

On the intersection of the events (2.3.38) and (2.3.39),

$$\int_k^{k+1} \left(e^{\xi h_{e^{-t}}(0) - \xi(Q-\gamma)t} + e^{\xi h_{e^{-t}}(z) - \xi Q t} \right) dt \leq C \left(e^{\xi h_{e^{-k}}(0) - \xi(Q-\gamma)k} + e^{\xi h_{e^{-k}}(z) - \xi Q k} \right). \tag{2.3.40}$$

We now deal with the integral from k_1 to infinity. Since $s \mapsto h_{e^{-s-k_1}}(z) - h_{e^{-k_1}}(z)$ for $s \geq 0$ is a standard Brownian motion,

$$\int_0^\infty e^{\xi(h_{e^{-s-k_1}}(z) - h_{e^{-k_1}}(z)) - \xi Q s} \, ds \overset{d}{=} \frac{2}{\xi^2} Z_{\frac{2Q}{\xi^2}}, \tag{2.3.41}$$

where Z_ν is a Gamma random variable of index ν [94, 252]. That is, for $c \geq 0$,

$$\mathbb{P}\{Z_\nu > c\} = \int_c^\infty \frac{s^{\nu-1} e^{-s}}{\Gamma(\nu)} \, ds. \tag{2.3.42}$$

In particular, there exists a number $u > 0$ depending only on γ such that

$$\mathbb{P}\left\{\int_{k_1}^{\infty} e^{\xi h_{e^{-t}}(z)-\xi Q t}\,dt \le C e^{\xi h_{e^{-k_1}}(z)-\xi Q k_1}\right\} = 1-\mathbb{P}\left\{Z_{\frac{2Q}{\xi^2}} > \frac{\xi^2}{2}C\right\} = 1-O\left(e^{-uC}\right) \tag{2.3.43}$$

as $C \to \infty$ with the rate uniform on z. Similarly,

$$\begin{aligned}\mathbb{P}\left\{\int_{k_1}^{\infty} e^{\xi h_{e^{-t}}(0)-\xi(Q-\gamma)t}\,dt \le C e^{\xi h_{e^{-k_1}}(z)-\xi(Q-\gamma)k_1}\right\} &= 1-\mathbb{P}\left\{Z_{\frac{2(Q-\gamma)}{\xi^2}} > \frac{\xi^2}{2}C\right\}\\ &= 1-O\left(e^{-uC}\right)\end{aligned} \tag{2.3.44}$$

as $C \to \infty$ with the rate uniform on z.

Let $u > 0$ be a constant to be determined later. Now take a union bound over the events (2.3.38) and (2.3.39) for all integers $k \in [k_0, k_0 + uN \log C)$ as well as the events in (2.3.43) and (2.3.44). Recalling (2.3.36) and (2.3.40), we find that with superpolynomially high probability as $C \to \infty$, at a rate uniform in z,

$$\begin{aligned}D_{h^\gamma}^{\mathbb{D}}(0,z) &\le C\int_{-\log\frac{|z|}{2}}^{\infty}\left(e^{\xi h_{e^{-t}}(0)-\xi(Q-\gamma)t}+e^{\xi h_{e^{-t}}(z)-\xi Q t}\right)dt\\ &\le C^2\sum_{k=k_0}^{k_0+uN\log C}\left(e^{\xi h_{e^{-k}}(0)-\xi(Q-\gamma)k}+e^{\xi h_{e^{-k}}(z)-\xi Q k}\right).\end{aligned} \tag{2.3.45}$$

Given (2.3.45), there must exist an integer $k \in [k_0, k_0 + uN \log C]$ satisfying either

$$e^{\gamma h_{e^{-k}}(0)} \ge C^{-2d_\gamma} e^{\gamma(Q-\gamma)k}\left(\frac{D_{h^\gamma}^{\mathbb{D}}(0,z)}{N\log C}\right)^{d_\gamma} \quad\text{or}\quad e^{\gamma h_{e^{-k}}(z)} \ge C^{-2d_\gamma} e^{\gamma Q k}\left(\frac{D_{h^\gamma}^{\mathbb{D}}(0,z)}{N\log C}\right)^{d_\gamma}. \tag{2.3.46}$$

We now let $u = 3d_\gamma$ and replace C in (2.3.46) by $C^{1/u}$. For sufficiently large C, since $N > 0$ is a fixed constant, the right hand sides of (2.3.46) are bounded below by $C^{-1}e^{\gamma(Q-\gamma)k}(D_{h^\gamma}^{\mathbb{D}}(0,z))^{d_\gamma}$ and $C^{-1}e^{\gamma Q k}(D_{h^\gamma}^{\mathbb{D}}(0,z))^{d_\gamma}$, respectively. Hence, the probability that there exists an integer $k \in [k_0, k_0 + N \log C]$ satisfying (2.3.35) is bounded below by

$1 - C^{\rho/u}$ as $C \to \infty$ for every $\rho > 0$ as claimed. ☐

The final two lemmas are standard estimates on the circle averages of GFF and the LQG measure. In Step 5, we obtain a lower bound on the circle average $h_r(w)$ on the boundary of the Euclidean ball $B_r(w)$ found in Step 4. This is done by comparing $h_r(w)$ with either $h_{e^{-k}}(0)$ (Case 1) or $h_{e^{-k}}(z)$ (Case 2) using the following lemma.

Lemma 2.3.7. *Let h be a whole-plane GFF normalized so that $h_1(0) = 0$. Let $\zeta \in (0,1)$ and $q > 2$ be given constants. Let $z \in B_{2e^{-2}}(0)$ and denote $k_0 = \lfloor -\log(|z|/2) \rfloor$. For all $C > C_0(q,\zeta)$, the following event is true with probability $1 - O(e^{-(q^2/2-2)\zeta k_0})$, where the rate is otherwise uniform over all choices of z. For every integer $k \geq k_0$, writing $r = e^{-k(1+\zeta)}$:*

(i) For every $w \in r\mathbb{Z}^2$ such that $B_r(w) \subset B_{e^{-k}}(0) \setminus \overline{B_{e^{-k-1}}(0)}$, we have

$$|h_r(w) - h_{e^{-k}}(0)| \leq kq\zeta. \tag{2.3.47}$$

(ii) For every $w \in r\mathbb{Z}^2$ such that $B_r(w) \subset B_{e^{-k}}(z) \setminus \overline{B_{e^{-k-1}}(z)}$, we have

$$|h_r(w) - h_{e^{-k}}(z)| \leq kq\zeta. \tag{2.3.48}$$

Proof. [16, Lemma 4.5] proves that the event (i) holds for every $k \geq k_0$ with probability

$$1 - O\left(\sum_{k=k_0}^{\infty} e^{k\zeta(2-q^2/2)}\right) = 1 - O(e^{-(q^2/2-2)\zeta k_0}). \tag{2.3.49}$$

The random variable $\left|h_r(w) - h_{e^{-k}}(z)\right|$ depends on h viewed modulo additive constant. By the translation invariance (2.2.10) of the law of the GFF modulo additive constant, it follows that (ii) also holds with the same probability computed in (2.3.49) regardless of the choice of z. ☐

In Step 6, we give a lower bound on the LQG measure $\mu_h(B_r(w))$ in terms of the circle average $h_r(w)$.

Lemma 2.3.8. *Let h be a whole-plane GFF normalized so that $h_1(0) = 0$. For each fixed $w \in \mathbb{C}$ and $r > 0$, it holds with superpolynomially high probability as $C \to \infty$, with the rate uniform over w and r, that*

$$\mu_h(B_r(w)) \geq C^{-1} r^{\gamma Q} e^{\gamma h_r(w)}. \tag{2.3.50}$$

Proof. The case when $w = 0$ and $r = 1$ is a standard estimate for the LQG measure. See, e.g., [99, Lemma 4.6] or [213, Theorem 2.12]. The case of a general $w \in \mathbb{C}$ and $r > 0$ follows from the case when $w = 0$ and $r = 1$ since $r^{-\gamma Q} e^{-\gamma h_r(w)} \mu_h(B_r(w)) \stackrel{d}{=} \mu_h(B_1(0))$ due to the scale and translation invariance of the law of h, (2.2.10), and the LQG coordinate change formula (2.2.16). $\square$

2.3.2 Proof of tightness

From Proposition 2.3.1, we can not only deduce that (2.3.2) is tight, but also that it is uniformly bounded in L^p for every $p \geq 1$. This stronger result is necessary later, specifically for Proposition 2.4.9. Recall the notation $N_\varepsilon(A) = N_\varepsilon(A; D_{h^\gamma})$ for the number of D_{h^γ}-balls of radius ε needed to cover the set A.

Proposition 2.3.9. *For every $p \geq 1$, there exists a constant $C_p < \infty$ such that for every $s < t$,*

$$\sup_{0<\varepsilon<|t-s|^{1/d_\gamma}} \mathbb{E}[(\varepsilon^{d_\gamma} N_\varepsilon(\eta[s,t]))^p] \leq C_p |t-s|^p.$$

In particular, for each fixed $s < t$, the random variables $\varepsilon^{d_\gamma} N_\varepsilon(\eta[s,t])$ for $\varepsilon \in (0,1)$ are tight.

Proposition 2.3.9 is a consequence of the following lemma together with a scaling argument.

Lemma 2.3.10. *For each $p \geq 1$, the number of LQG balls of radius 1 needed to cover $\eta[0,1]$ satisfies*

$$\mathbb{E}[(N_1(\eta[0,1]))^p] < \infty. \tag{2.3.51}$$

Proof. Let K be a positive integer. For each integer $0 \leq k < K$, define the event

$$E_{k,K} = \left\{ \eta\Big[\frac{k}{K}, \frac{k+1}{K}\Big] \subset \mathcal{B}_1\big(\eta(k/K); D_{h^\gamma}\big) \right\}. \tag{2.3.52}$$

Clearly,

$$\bigcap_{k=0}^{K-1} E_{k,K} \subset \{N_1(\eta[0,1]) \leq K\}. \tag{2.3.53}$$

By Proposition 2.2.20,

$$\mathbb{P}(E_{k,K}) = \mathbb{P}(E_{0,K}) = \mathbb{P}\{\tau_1 > 1/K\} \quad \text{for every } k. \tag{2.3.54}$$

Then

$$\mathbb{P}\{N_1(\eta[0,1]) > K\} \leq \sum_{k=0}^{K-1} \mathbb{P}(E_{k,K}^{\complement}) = K \cdot \mathbb{P}\{\tau_1 \leq 1/K\}. \tag{2.3.55}$$

By Proposition 2.3.1, it follows that $\mathbb{P}\{N_1(\eta[0,1]) > K\}$ decays faster than any negative power of K as $K \to \infty$. Therefore,

$$\mathbb{E}[(N_1(\eta[0,1]))^p] \leq \sum_{K=1}^{\infty} ((K+1)^p - K^p) \cdot \mathbb{P}\{N_1(\eta[0,1]) \geq K\} < \infty. \tag{2.3.56}$$

$\square$

Proof of Proposition 2.3.9. It suffices to find a constant $C_p < \infty$ so that we have the estimate $\sup_{0<\varepsilon<1} \mathbb{E}[(\varepsilon^{d_\gamma} N_\varepsilon(\eta[0,1])^p] \leq C_p$. Indeed, by Corollary 2.2.21,

$$\sup_{0<\varepsilon<|t-s|^{1/d_\gamma}} \mathbb{E}[(\varepsilon^{d_\gamma} N_\varepsilon(\eta[s,t]))^p] \stackrel{d}{=} |t-s|^p \left(\sup_{0<u<1} \mathbb{E}[(u^{d_\gamma} N_u(\eta[0,1])^p] \right). \tag{2.3.57}$$

We thus restrict our attention to uniform L^p-bounds for $\{\varepsilon^{d_\gamma} N_\varepsilon(\eta[0,1])\}_{0<\varepsilon<1}$.

For each $\varepsilon \in (0,1)$,

$$N_\varepsilon(\eta[0,1]) \leq \sum_{k=0}^{\lfloor \varepsilon^{-d_\gamma} \rfloor} N_\varepsilon(\eta[k\varepsilon^{d_\gamma}, (k+1)\varepsilon^{d_\gamma}]). \tag{2.3.58}$$

By Corollary 2.2.21, for every integer k,

$$N_1(\eta[0,1]) \stackrel{d}{=} N_\varepsilon(\eta[k\varepsilon^{d_\gamma}, (k+1)\varepsilon^{d_\gamma}]). \tag{2.3.59}$$

Hence, by Jensen's inequality,

$$\begin{aligned} \mathbb{E}\left[(N_\varepsilon(\eta[0,1]))^p\right] &\leq (\varepsilon^{-d_\gamma}+1)^{p-1} \sum_{k=0}^{\lfloor \varepsilon^{-d_\gamma} \rfloor} \mathbb{E}\left[(N_\varepsilon(\eta[k\varepsilon^{d_\gamma}, (k+1)\varepsilon^{d_\gamma}]))^p\right] \\ &\leq (\varepsilon^{-d_\gamma}+1)^p \mathbb{E}[(N_1(\eta[0,1]))^p]. \end{aligned} \tag{2.3.60}$$

By Lemma 2.3.10, $\mathbb{E}[(N_1(\eta[0,1]))^p]$ is a finite constant depending only on p and γ. Multiplying both sides of (2.3.60) by $\varepsilon^{d_\gamma p}$, we obtain $\mathbb{E}[(\varepsilon^{d_\gamma} N_\varepsilon(\eta[0,1]))^p] \leq 2\mathbb{E}[(N_1(\eta[0,1]))^p] =: C_p$. $\square$

2.3.3 Further LQG metric estimates for space-filling SLE

We record two consequences of Proposition 2.3.1, which we do not use elsewhere in this paper but are of independent interest. We first show that for each $s < t$, the D_{h^γ}-diameter of $\eta[s,t]$, which we denote by $\operatorname{diam}(\eta[s,t]; D_{h^\gamma})$, has finite positive moments of all orders. The other result is the first half of Theorem 2.1.4, that for every exponent less than $1/d_\gamma$, the space-filling $\mathrm{SLE}_{\kappa'}$ curve η parameterized by μ_{h^γ} is almost surely locally Hölder continuous with respect to D_{h^γ}.

For completeness, we also prove that all negative moments of $\operatorname{diam}(\eta[s,t]; D_{h^\gamma})$ are finite and that η is almost surely not locally Hölder continuous for exponents greater than $1/d_\gamma$. These complementary results follow from the following extension of Theorem 2.2.1 to the γ-quantum cone.

Lemma 2.3.11. *Let h^γ be the field of a γ-quantum cone under the circle average embedding. For $\zeta \in (0,1)$ and $k \geq 1$, there exists a constant $C_{k,\zeta}$ such that for all $r \in (0,1)$,*

$$\mathbb{E}[(\mu_{h^\gamma}(\mathbb{D} \cap \mathcal{B}_r(0; D_{h^\gamma})))^k] \leq C_{k,\zeta} r^{kd_\gamma - \zeta}. \tag{2.3.61}$$

Moreover, for any compact set $K \subset \mathbb{D}$ and $\zeta > 0$, we almost surely have that

$$\sup_{r \in (0,1)} \sup_{z \in K} \frac{\mu_{h^\gamma}(\mathbb{D} \cap \mathcal{B}_r(z; D_{h^\gamma}))}{r^{d_\gamma - \zeta}} < \infty. \tag{2.3.62}$$

Proof. Recall that if h is a whole-plane GFF with $h_1(0) = 0$, then $(h - \gamma \log|\cdot|)|_{\mathbb{D}}$ agrees in law with $h^\gamma|_{\mathbb{D}}$. Hence, (2.3.61) follows from the proofs of [16, Lemmas 3.16 and 3.18]. The proof of (2.3.62) from (2.3.61) is analogous to that of (2.2.2) in [16]. $\square$

We now show that the D_{h^γ}-diameter of $\eta[s,t]$ has finite moments of all orders.

Proposition 2.3.12. *Let $s < t$. For each $p \in \mathbb{R}$, there exists a constant $C_p > 0$ only depending on p such that*

$$\mathbb{E}[(\operatorname{diam}(\eta[s,t]; D_{h^\gamma}))^p] \leq C_p |t-s|^{p/d_\gamma}. \tag{2.3.63}$$

Proof. By Proposition 2.2.20, for each $s < t$,

$$\operatorname{diam}(\eta[s,t]; D_{h^\gamma}) \overset{d}{=} \operatorname{diam}(\eta[0,t-s]; D_{h^\gamma}) \overset{d}{=} (t-s)^{1/d_\gamma} \operatorname{diam}(\eta[0,1]; D_{h^\gamma}). \tag{2.3.64}$$

Hence, it suffices to show that $\mathbb{E}[(\operatorname{diam}(\eta[0,1]; D_{h^\gamma}))^p] < \infty$ for all p.

Suppose $p > 0$. For $r > 0$, let τ_r be as in (2.3.6). Note that for $r > 0$, if $\tau_r > 1$ then $\eta[0,1] \subset \mathcal{B}_r(0; D_{h^\gamma})$ and hence $\mathrm{diam}(\eta[0,1]; D_{h^\gamma}) < 2r$. By (2.3.7),

$$\begin{aligned}\mathbb{E}[((\mathrm{diam}(\eta[0,1]; D_{h^\gamma}))^p] &= \int_0^\infty \mathbb{P}\{\mathrm{diam}(\eta[0,1]; D_{h^\gamma}) \geq x^{1/p}\}\, dx \\ &\leq \int_0^\infty \mathbb{P}\{\tau_{(x^{1/p})/2} \leq 1\}\, dx = \int_0^\infty \mathbb{P}\{\tau_1 \leq 2^{-d_\gamma} x^{d_\gamma/p}\}\, dx\end{aligned} \tag{2.3.65}$$

This integral is finite since by Proposition 2.3.1, $\mathbb{P}\{\tau_1 \leq 2^{-d_\gamma} x^{d_\gamma/p}\}$ decays superpolynomially as $x \to 0$.

Now suppose $p < 0$. If $\mathrm{diam}(\eta[0,1]; D_{h^\gamma}) < r$, then $\mu_{h^\gamma}(\mathcal{B}_r(0; D_{h^\gamma})) \geq 1$. Then, by Proposition 2.2.20,

$$\begin{aligned}&\mathbb{E}[((\mathrm{diam}(\eta[0,1]; D_{h^\gamma}))^p] = \int_0^\infty \mathbb{P}\{((\mathrm{diam}(\eta[0,1]; D_{h^\gamma}))^p > x\}\, dx \\ &= \int_0^\infty \mathbb{P}\{\mathrm{diam}(\eta[0,1]; D_{h^\gamma}) < x^{1/p}\}\, dx \leq \int_0^\infty \mathbb{P}\{\mu_{h^\gamma}(\mathcal{B}_{x^{1/p}}(0; D_{h^\gamma})) \geq 1\}\, dx \\ &\leq 1 + \int_1^\infty \mathbb{P}\{\mu_{h^\gamma}(\mathbb{D} \cap \mathcal{B}_{x^{1/p}}(0; D_{h^\gamma})) \geq 1\}\, dx + \int_1^\infty \mathbb{P}\{\mathcal{B}_{x^{1/p}}(0; D_{h^\gamma}) \not\subset \mathbb{D}\}\, dx.\end{aligned} \tag{2.3.66}$$

From (2.3.61) and Lemma 2.3.4 (with $r = 1/2$, say, $\varepsilon = x^{1/p}$, and k in place of p), for each $k \geq 1$ and each $\zeta > 0$, there are constants $C_{k,\zeta}, C_k > 0$ such that for all $x > 1$,

$$\mathbb{P}\{\mu_{h^\gamma}(\mathbb{D} \cap \mathcal{B}_{x^{1/p}}(0; D_{h^\gamma})) \geq 1\} \leq C_{k,\zeta} x^{(kd_\gamma - \zeta)/p} \quad \text{and} \quad \mathbb{P}\{\mathcal{B}_{x^{1/p}}(0; D_{h^\gamma}) \not\subset \mathbb{D}\} \leq C_k x^{k/p}. \tag{2.3.67}$$

We conclude that the right-hand side of (2.3.66) is finite by choosing a sufficiently large k. □

Proposition 2.3.12 gives a quick proof of Theorem 2.1.4, on the Hölder continuity of the whole-plane space-filling $\mathrm{SLE}_{\kappa'}$ curve η with respect to the LQG metric D_{h^γ}.

Proof of Theorem 2.1.4. Let us first show that almost surely, η is locally Hölder continuous for any exponent less than $1/d_\gamma$. We proved in Proposition 2.3.12 that for each $p > 0$, there exists a constant $C_p > 0$ such that for all $s < t$,

$$\mathbb{E}[(D_{h^\gamma}(\eta(s),\eta(t)))^p] \le \mathbb{E}[(\operatorname{diam}(\eta[s,t];D_{h^\gamma}))^p] \le C_p|t-s|^{p/d_\gamma}. \tag{2.3.68}$$

The claim now follows from the Kolmogorov continuity theorem as we let $p \to \infty$.[7]

On the other hand, for every $\zeta \in (0, d_\gamma)$, Lemma 2.3.11 (with the fact that η is parameterized by μ_{h^γ}) implies that there a.s. exists a random $c_\zeta > 0$ such that $\operatorname{diam}(\eta[s,t];D_{h^\gamma}) > c_\zeta(t-s)^{1/(d_\gamma-\zeta)}$ for all $s < t$ such that $\eta[s,t] \subset \mathbb{D}$. In particular, η is almost surely not Hölder continuous with exponent greater than $1/d_\gamma$ in any neighborhood of 0. By Proposition 2.2.20, η is almost surely not Hölder continuous with any exponent greater than $1/d_\gamma$ in any neighborhood of $t \in \mathbb{Q}$, and therefore not in any bounded open interval. $\square$

2.4 Minkowski content of space-filling SLE segments

Remark. We assume $\kappa' = 16/\gamma^2$ throughout this section, as the results in this section rely on Proposition 2.2.23. Recall the shorthand $N_\varepsilon(\eta[s,t]) = N_\varepsilon(\eta[s,t];D_{h^\gamma})$.

We showed in Proposition 2.3.9 that for each $s < t$, the random variables $\varepsilon^{d_\gamma}N_\varepsilon(\eta[s,t])$ admit subsequential limits in law as $\varepsilon \to 0$. We need to rule out further the possibility that the subsequential limit is zero, which is the purpose of Proposition 2.4.1. Unlike in many results regarding fractal dimensions, where the lower bound is more difficult to prove than the upper bound, this proposition has a much shorter proof than Proposition 2.3.9. This is thanks to the independence of the metric measure space structure on disjoint space-filling SLE segments, which comes from Proposition 2.2.23.

7. The Kolmogorov continuity theorem is usually stated for a stochastic process X_t taking values on a fixed metric space (S,d). However, the Hölder continuity part of the theorem merely requires that the real-valued random variables $d(X_s,X_t)$ are measurable and have appropriate uniform moments.

Proposition 2.4.1. *There exists a deterministic constant $c = c(\gamma) > 0$ such that*

$$\lim_{\varepsilon \to 0} \mathbb{P}\{\varepsilon^{d_\gamma} N_\varepsilon(\eta[0,1]) > c\} = 1. \tag{2.4.1}$$

Proof. For $r > 0$, let p_r be the probability that $\eta[0,1]$ contains a D_{h^γ}-ball of radius r. By Proposition 2.2.16, $\eta[0,1]$ almost surely contains a Euclidean ball, which in turn contains a D_h-ball since D_h is a continuous metric. Hence, $p_r \to 1$ as $r \to 0$.

Fix $r > 0$ such that $p_r > 0$. For this r, define for each positive integer n and integer $k \in [1, n]$ the event

$$E_{n,k} := \left\{ \eta \left[\frac{k-1}{n}, \frac{k}{n} \right] \text{ contains a } D_{h^\gamma}\text{-ball of radius } rn^{-1/d_\gamma} \right\}. \tag{2.4.2}$$

By Proposition 2.2.20, $\mathbb{P}(E_{n,k}) = p_r$ for every n and k. Furthermore, $E_{n,1}, \ldots, E_{n,n}$ are independent by Proposition 2.2.23. To see this, for fixed $t \in \mathbb{R}$, define U_{t-} and U_{t+} to be the interiors of $\eta(-\infty, t]$ and $\eta[t, \infty)$, respectively. Then, $E_{n,k}$ is almost surely determined by $(U_{t-}, D_{h^\gamma}^{U_{t-}}, \mu_{h^\gamma}|_{U_{t-}}, \eta|_{(-\infty,t]})$ if $t \geq \frac{k}{n}$ and by $(U_{t+}, D_{h^\gamma}^{U_{t+}}, \mu_{h^\gamma}|_{U_{t+}}, \eta|_{[t,\infty)})$ if $t \leq \frac{k-1}{n}$. These two curve-decorated metric measure spaces are independent by Proposition 2.2.23 combined with the translation invariance of Proposition 2.2.20. Choosing $t = k/n$, we see that the random vectors $(\mathbf{1}_{E_{n,1}}, \ldots, \mathbf{1}_{E_{n,k}})$ and $(\mathbf{1}_{E_{n,k+1}}, \ldots, \mathbf{1}_{E_{n,n}})$ are independent. Since this is true for every k, we conclude that $\mathbf{1}_{E_{n,1}}, \ldots, \mathbf{1}_{E_{n,n}}$ are i.i.d. Bernoulli random variables with success probability p_r.

We now argue that for every n,

$$N_{rn^{-1/d_\gamma}}(\eta[0,1]) \geq \sum_{k=1}^{n} \mathbf{1}_{E_{n,k}}. \tag{2.4.3}$$

Indeed, if $\sum_{k=1}^{n} \mathbf{1}_{E_{n,k}} = m$, then there are distinct points $z_1, \ldots, z_m \in \eta[0,1]$ such that $D_{h^\gamma}(z_j, z_k) \geq 2rn^{-1/d_\gamma}$ for every $j \neq k$. Such z_j and z_k cannot be within a single D_{h^γ}-ball of radius rn^{-1/d_γ}, so $N_{rn^{-1/d_\gamma}}(\eta[0,1]) \geq m$. Hence, (2.4.3) holds. By the law of large numbers,

$$\lim_{n\to\infty} \mathbb{P}\left\{n^{-1} N_{rn^{-1/d_\gamma}}(\eta[0,1]) \geq p_r\right\} = 1. \tag{2.4.4}$$

Let $\varepsilon \in (0,1)$ and set $n_\varepsilon := \lfloor \varepsilon^{-d_\gamma} \rfloor$. Since $N_{r\varepsilon}(\eta[0,1])$ increases as ε decreases to 0,

$$(r\varepsilon)^{d_\gamma} N_{r\varepsilon}(\eta[0,1]) \geq r^{d_\gamma}(n_\varepsilon \varepsilon^{d_\gamma}) \frac{N_{rn_\varepsilon^{-1/d_\gamma}}(\eta[0,1])}{n_\varepsilon}. \tag{2.4.5}$$

Since d_γ is positive, we have $n_\varepsilon \varepsilon^{d_\gamma} \to 1$ as $\varepsilon \to 0$. By combining this with (2.4.4) and (2.4.5), we get

$$\lim_{\varepsilon\to 0} \mathbb{P}\{(r\varepsilon)^{d_\gamma} N_{r\varepsilon}(\eta[0,1]) \geq r^{d_\gamma} p_r\} = 1.$$

Since r is a constant which depends only on γ, this implies (2.4.1) with $c = r^{d_\gamma} p_r$. $\square$

Another possibility that we have yet to rule out is that different subsequential limits of $\varepsilon^{d_\gamma} N_\varepsilon(\eta[s,t])$ may take different values. To this end, we replace the rescaling coefficients from ε^{-d_γ} to

$$\mathfrak{b}_\varepsilon := \mathbb{E}[N_\varepsilon(\eta[0,1])] \tag{2.4.6}$$

as introduced in (2.1.8), so that any subsequential limit of $\mathfrak{b}_\varepsilon^{-1} N_\varepsilon(\eta[0,1])$ has mean 1. Indeed, $\{\mathfrak{b}_\varepsilon\}_{\varepsilon>0}$ is a sequence of d_γ-dimensional rescaling coefficients (recall Definition 2.2.2).

Corollary 2.4.2. *There exists a deterministic constant $C > 0$ such that for every $\varepsilon \in (0,1)$,*

$$C^{-1}\varepsilon^{-d_\gamma} < \mathfrak{b}_\varepsilon < C\varepsilon^{-d_\gamma}.$$

The upper bound follows immediately from Proposition 2.3.9 and the lower bound from Proposition 2.4.1. Note that this corollary is the first part of Proposition 2.1.3.

The following proposition is the main result of this section.

Proposition 2.4.3. *Let $\kappa' = 16/\gamma^2$. Then, for each fixed $s < t$,*

$$\lim_{\varepsilon \to 0} \mathfrak{b}_\varepsilon^{-1} N_\varepsilon(\eta[s,t]) = t - s \quad \textit{in probability.} \tag{2.4.7}$$

Here is an overview of the proof of Proposition 2.4.3. Let

$$\mathcal{I}_{\mathbb{Q}} = \{[s,t] : s,t \in \mathbb{Q}, s < t\} \tag{2.4.8}$$

be the collection of all closed intervals with rational endpoints. Suppose we are given an arbitrary sequence of ε-values decreasing to 0.

1. By Proposition 2.3.9, we can find a subsequence ε_n for which the sequence of $\mathbb{R}^{\mathcal{I}_{\mathbb{Q}}}$-valued random variables $(\mathfrak{b}_{\varepsilon_n}^{-1} N_{\varepsilon_n}(\eta(I)) : I \in \mathcal{I}_{\mathbb{Q}})$ converges in distribution with respect to the product topology on $\mathbb{R}^{\mathcal{I}_{\mathbb{Q}}}$. Denote this subsequential weak limit by $(X_I : I \in \mathcal{I}_{\mathbb{Q}})$.

2. We show in Proposition 2.4.6 that X_I is finitely additive: i.e., for each rational $r < s < t$, $X_{[r,s]} + X_{[s,t]} = X_{[r,t]}$ almost surely. We are thus able to construct the "Minkowski content process" $\{Y_t\}_{t \in \mathbb{Q}}$ where $Y_t = X_{[0,t]}$ (resp. $-X_{[t,0]}$) for $t \geq 0$ (resp. $t < 0$), so that $X_{[s,t]} = Y_t - Y_s$ for every $[s,t] \in \mathcal{I}_{\mathbb{Q}}$.

3. We show in Proposition 2.4.9 that $\{Y_t\}_{t \in \mathbb{Q}}$ can be extended to a continuous process on $\mathbb{R}$ with independent and stationary increments: i.e., a Brownian motion with drift.

4. From Proposition 2.4.1, almost surely, $Y_s < Y_t$ for all rational $s < t$. Hence, $Y_t = at$ for some deterministic constant $a > 0$. In fact, $a = 1$ because we chose $\mathfrak{b}_\varepsilon$ so that $\mathbb{E}[\mathfrak{b}_\varepsilon^{-1} N_\varepsilon(\eta[0,1])] = 1$ for every $\varepsilon > 0$.

5. In conclusion, we have the following convergence in distribution:

$$\lim_{n \to \infty} \mathfrak{b}_{\varepsilon_n}^{-1} N_{\varepsilon_n}(\eta[s,t]) = t - s \quad \text{for all } [s,t] \in \mathcal{I}_{\mathbb{Q}}$$

Since $t-s$ is a deterministic constant, the convergence holds in probability. We started with an arbitrary sequence of ε decreasing to 0, so (2.4.7) holds for every rational $s < t$. This extends to all real $s < t$ by the following simple observation: if $[s_1, t_1] \subset [s_2, t_2]$, then $N_\varepsilon(\eta[s_1, t_1]) \leq N_\varepsilon(\eta[s_2, t_2])$ for all $\varepsilon > 0$.

2.4.1 Finite additivity

Given a closed and bounded set $A \subset \mathbb{C}$, for $\varepsilon > 0$, define

$$\partial_\varepsilon A := \{z \in A : D_{h^\gamma}(z, \partial A) < \varepsilon\} \tag{2.4.9}$$

to be the the intersection of A with the ε-neighborhood of ∂A in the LQG metric. Denote

$$N_\varepsilon^\circ(A) := N_\varepsilon(A \setminus \partial_\varepsilon A). \tag{2.4.10}$$

Note that every ball counted in $N_\varepsilon^\circ(A)$ is centered at a point in the interior of A. Also,

$$N_\varepsilon^\circ(A) \leq N_\varepsilon(A) \leq N_\varepsilon^\circ(A) + N_\varepsilon(\partial_\varepsilon A). \tag{2.4.11}$$

The following lemma is the main technical input in the proof of finite additivity.

Lemma 2.4.4. *For each fixed $s < t$,*

$$\lim_{\varepsilon \to 0} \varepsilon^{d_\gamma} N_\varepsilon(\partial_\varepsilon \eta[s,t]) = 0 \quad \textit{almost surely.} \tag{2.4.12}$$

Consequently,

$$\lim_{\varepsilon \to 0} |\varepsilon^{d_\gamma} N_\varepsilon(\eta[s,t]) - \varepsilon^{d_\gamma} N_\varepsilon^\circ(\eta[s,t])| = 0 \quad \textit{almost surely.} \tag{2.4.13}$$

We first explain how Lemma 2.4.4 implies that the Minkowski content is additive over finitely many disjoint space-filling SLE segments, and then prove the lemma in the rest of the subsection. We need the following classic result on weak convergence in the proof of finite additivity.

Lemma 2.4.5 ([48, Theorems 2.7 and 3.1])**.** *Suppose (S,d) is a metric space and (Y_n, Z_n) is a sequence of $S \times S$-valued Borel-measurable random variables. If $Y_n \xrightarrow{d} Y$ and $d(Y_n, Z_n) \xrightarrow{p} 0$ as $n \to \infty$, then $(Y_n, Z_n) \xrightarrow{d} (Y, Y)$.*

Proposition 2.4.6 (Finite additivity)**.** *Suppose that for some sequence $\{\varepsilon_n\}_{n\in\mathbb{N}}$ of positive numbers decreasing to zero, $(\mathfrak{b}_{\varepsilon_n}^{-1} N_{\varepsilon_n}(\eta(I)) : I \in \mathcal{I}_{\mathbb{Q}})$ converges weakly to $(X_I : I \in \mathcal{I}_{\mathbb{Q}})$ as $n \to \infty$. Then, for every rational $r < s < t$, we have $X_{[r,s]} + X_{[s,t]} = X_{[r,t]}$ almost surely.*

Proof. Assume that Lemma 2.4.4 holds. Then, by Corollary 2.4.2, we have

$$\lim_{n\to\infty} |\mathfrak{b}_{\varepsilon_n}^{-1} N_{\varepsilon_n}(\eta(I)) - \mathfrak{b}_{\varepsilon_n}^{-1} N_{\varepsilon_n}^{\circ}(\eta(I))| = 0 \quad \text{almost surely for all } I \in \mathcal{I}_{\mathbb{Q}}. \tag{2.4.14}$$

Combining this with Lemma 2.4.5, we obtain

$$((\mathfrak{b}_{\varepsilon_n}^{-1} N_{\varepsilon_n}(\eta(I)), \mathfrak{b}_{\varepsilon_n}^{-1} N_{\varepsilon_n}^{\circ}(\eta(I))) : I \in \mathcal{I}_{\mathbb{Q}}) \xrightarrow{d} ((X_I, X_I) : I \in \mathcal{I}_{\mathbb{Q}}) \quad \text{as } n \to \infty. \tag{2.4.15}$$

Let $r < s < t$ be any triple of rationals. Fix $\varepsilon > 0$ for now. Note that every metric ball counted in $N_\varepsilon^\circ(\eta[r,t])$ has its center in $\eta[r,t]$. The idea is to classify these balls based on whether their centers are in $\eta[r,s]$ or $\eta[s,t]$. Let $m = N_\varepsilon^\circ(\eta[r,t])$ and let $z_1, \ldots, z_k \in \eta[r,s]$, $z_{k+1}, \ldots, z_m \in \eta[s,t]$ be any collection of points such that $\bigcup_{j=1}^m \mathcal{B}_\varepsilon(z_j; D_{h^\gamma})$ covers $\eta[r,t] \setminus \partial_\varepsilon \eta[r,t]$. If $j \le k$, then $\mathcal{B}_\varepsilon(z_j; D_{h^\gamma}) \cap (\eta[s,t] \setminus \partial_\varepsilon \eta[s,t]) = \varnothing$. Hence, $\bigcup_{j=k+1}^m B_\varepsilon(z_j; D_{h^\gamma})$ covers $\eta[s,t] \setminus \partial_\varepsilon \eta[s,t]$ and $N_\varepsilon^\circ(\eta[s,t]) \le m - k$. Similarly, $N_\varepsilon^\circ(\eta[r,s]) \le k$. Therefore,

$$N_\varepsilon^\circ(\eta[r,s]) + N_\varepsilon^\circ(\eta[s,t]) \le N_\varepsilon^\circ(\eta[r,t]) \le N_\varepsilon(\eta[r,t]) \le N_\varepsilon(\eta[r,s]) + N_\varepsilon(\eta[s,t]). \tag{2.4.16}$$

Since this inequality holds for every $\varepsilon > 0$, we have from (2.4.15) that $X_{[r,t]} = X_{[r,s]} + X_{[s,t]}$

almost surely. $\square$

We now begin the proof of Lemma 2.4.4. The first step in the proof of (2.4.12) is to show that the Minkowski dimension of $\partial\eta[s,t]$ is strictly less than d_γ. In fact, we prove that it is at most $d_\gamma/2$.

Lemma 2.4.7. *For each fixed $s<t$ and $\delta > d_\gamma/2$,*

$$\lim_{\varepsilon\to 0} \varepsilon^\delta N_\varepsilon(\partial\eta[s,t]) = 0 \quad \textit{almost surely.} \tag{2.4.17}$$

Proof. By Proposition 2.2.20, for each $s<t$ and $\varepsilon>0$,

$$N_\varepsilon(\partial\eta[s,t]) \stackrel{d}{=} N_{\varepsilon(t-s)^{-1/d_\gamma}}(\partial\eta[0,1]). \tag{2.4.18}$$

Hence, it suffices to prove the lemma for $[s,t]=[0,1]$ only. To prove the lemma in this case, we will first use Theorem 2.2.25 and an elementary Brownian motion estimate to bound the number of ε^{-d_γ}-length space-filling SLE segments needed to cover $\partial\eta[0,1]$. We will then conclude by combining this bound with Proposition 2.3.9.

Without loss of generality, we may re-scale the LQG boundary length measure by a γ-dependent constant factor so that for the boundary length process (L,R) appearing in Theorem 2.2.25, the variance parameter a equals 1. Let ∂^L and ∂^R denote the left and right boundaries of $\eta[0,1]$, respectively. Further decompose ∂^L and ∂^R into

$$\partial_0^q := \partial^q \cap \eta(-\infty,0] \quad \text{and} \quad \partial_1^q := \partial^q \cap \eta[1,\infty) \quad \text{where } q\in\{L,R\}. \tag{2.4.19}$$

That is, in Figure 2.2 with $t=1$, the orange boundary corresponds to ∂_0^L, the green boundary to ∂_0^R, the brown boundary to ∂_1^L, and the purple boundary to ∂_1^R.

Let us first show that $\lim_{\varepsilon\to 0}\varepsilon^\delta N_\varepsilon(\partial_0^L)=0$ almost surely. Let K be a positive integer, which shall be determined later. Denote $I_k = [\frac{k}{K}, \frac{k+1}{K}]$. We start with the following trivial inequality:

$$N_\varepsilon(\partial_0^L) \le \sum_{k=0}^{K-1} N_\varepsilon(\partial_0^L \cap \eta(I_k)) \le \sum_{k=0}^{K-1} \mathbf{1}\{\partial_0^L \cap \eta(I_k) \neq \varnothing\} \cdot N_\varepsilon(\eta(I_k)). \tag{2.4.20}$$

We now consider the following geometric encoding of η by (L, R) described in [123, Section 4.2.3]: the times $t \in [0,1]$ such that $\eta(t) \in \partial_0^L$ (resp. ∂_0^R) are precisely those at which L_t (resp. R_t) attains a running minimum. Recall that L_t is a standard Brownian motion. A well-known result of P. Lévy states that the process $L_t - \min_{0\le s\le t} L_s$ is a reflected Brownian motion, whereas the set $\{t \in [0,1] : \eta(t) \in \partial_0^L\}$ has the law of the zero set of Brownian motion on $[0,1]$ (e.g., see [201, Theorem 2.34]). Then, by the arcsine law for the last time that a Brownian motion B_t changes sign, for any nonnegative integer k regardless of K,

$$\begin{aligned}\mathbb{P}\{\partial_0^L \cap \eta(I_k) \neq \varnothing\} &= \mathbb{P}\{B_t = 0 \text{ for some } t \in I_k\} = \mathbb{P}\left\{B_t = 0 \text{ for some } t \in \left[\frac{k}{k+1}, 1\right]\right\} \\ &= 1 - \frac{2}{\pi}\arcsin\sqrt{\frac{k}{k+1}} = 1 - \frac{2}{\pi}\arctan\sqrt{k}.\end{aligned} \tag{2.4.21}$$

For $p > 1$, by Hölder's inequality followed by Corollary 2.2.21 and (2.4.21),

$$\begin{aligned}\mathbb{E}[N_\varepsilon(\partial_0^L)] &\le \sum_{k=0}^{K-1} \mathbb{P}(\partial_0^L \cap \eta(I_k) \neq \varnothing)^{1-\frac{1}{p}} \cdot \mathbb{E}\big[\big(N_\varepsilon(\eta(I_k))\big)^p\big]^{\frac{1}{p}} \\ &= \sum_{k=0}^{K-1} \left(1 - \frac{2}{\pi}\arctan\sqrt{k}\right)^{1-\frac{1}{p}} \cdot \frac{K}{\varepsilon^{d_\gamma}} \mathbb{E}\left[\left((\varepsilon K^{-1/d_\gamma})^{d_\gamma} N_{\varepsilon K^{-1/d_\gamma}}(\eta[0,1])\right)^p\right]^{\frac{1}{p}}.\end{aligned} \tag{2.4.22}$$

We now take

$$K = \lceil \varepsilon^{-d_\gamma} \rceil. \tag{2.4.23}$$

From Proposition 2.3.9, there exists a constant $C_p > 0$ such that

$$\sup_{0<\varepsilon<1} \mathbb{E}\left[\left(\varepsilon K^{1/d_\gamma})^{d_\gamma} N_{\varepsilon K^{-1/d_\gamma}}(\eta[0,1])\right)^p\right]^{\frac{1}{p}} \le C_p. \tag{2.4.24}$$

Since $1 - \frac{2}{\pi}\arctan\sqrt{k} = O(1/\sqrt{k})$ as $k \to \infty$,

$$\begin{aligned}\mathbb{E}[\varepsilon^\delta N_\varepsilon(\partial_0^L)] &\le 2C_p \varepsilon^\delta \sum_{k=0}^{K} \left(1 - \frac{2}{\pi}\arctan\sqrt{k}\right)^{1-\frac{1}{p}} \\ &= O\left(\varepsilon^\delta K^{1-\frac{1}{2}(1-\frac{1}{p})}\right) = O\left(\varepsilon^{\delta - \frac{d_\gamma}{2}(1+\frac{1}{p}))}\right) \quad \text{as } \varepsilon \to 0.\end{aligned} \tag{2.4.25}$$

Since $\delta > d_\gamma/2$, the right-hand side tends to 0 as $\varepsilon \to 0$ for sufficiently large p. This proves the claim that $\lim_{\varepsilon\to 0} \varepsilon^\delta N_\varepsilon(\partial_0^L) = 0$ almost surely.

Replacing L_t with R_t gives $\lim_{\varepsilon\to 0} \varepsilon^\delta N_\varepsilon(\partial_0^R) = 0$ almost surely. By the reversibility of the whole-plane space-filling SLE η and Proposition 2.2.20,

$$(\mathbb{C}, \eta(1), D_{h^\gamma}, \mu_{h^\gamma}, \eta(1-\cdot)) \stackrel{d}{=} (\mathbb{C}, 0, D_{h^\gamma}, \mu_{h^\gamma}, \eta). \tag{2.4.26}$$

Consequently, $N_\varepsilon(\partial_0^L) \stackrel{d}{=} N_\varepsilon(\partial_1^L)$ and $N_\varepsilon(\partial_0^R) \stackrel{d}{=} N_\varepsilon(\partial_1^R)$. Since $\partial\eta[0,1] = \partial_0^L \cup \partial_0^R \cup \partial_1^L \cup \partial_1^R$, we therefore obtain the lemma statement. $\square$

Let us first sketch how to deduce (2.4.12) from Lemma 2.4.7. First, note that it suffices to show that for some fixed $\zeta \in (0, d_\gamma/2)$, with probability tending to 1 as $\varepsilon \to 0$, we have $N_\varepsilon(\partial_\varepsilon \eta[s,t]) \le \varepsilon^{-\zeta} N_\varepsilon(\partial\eta[s,t])$. The idea is to sample a collection $\mathcal{X}_\varepsilon$ of $\lfloor \varepsilon^{-\zeta} N_\varepsilon(\partial\eta[s,t]) \rfloor$ i.i.d. points in $\partial_{3\varepsilon/2}\eta[s,t]$ from the LQG measure μ_{h^γ}, and show that $\bigcup_{x\in\mathcal{X}_\varepsilon} \mathcal{B}_\varepsilon(x; D_{h^\gamma})$ covers $\partial_\varepsilon\eta[s,t]$.

To this end, we sample another set of i.i.d. points $\widetilde{W}_\varepsilon$ from μ_{h^γ}, conditionally independent of $\mathcal{X}_\varepsilon$ given h^γ and η, such that $\partial_\varepsilon\eta[s,t] \subset \bigcup_{w\in\widetilde{\mathcal{W}}_\varepsilon} \mathcal{B}_{\varepsilon/2}(w; D_{h^\gamma}) \subset \partial_{2\varepsilon}\eta[s,t]$. (Because the Minkowski dimension of γ-LQG is finite, we only need polynomially many points in ε in $\widetilde{\mathcal{W}}_\varepsilon$ [118, Theorem A.3].) For each $w \in \widetilde{\mathcal{W}}_\varepsilon$, the conditional probability given h^γ, η, and $x \in \mathcal{X}_\varepsilon$ that $\mathcal{B}_{\varepsilon/2}(w; D_{h^\gamma}) \subset \mathcal{B}_\varepsilon(x; D_{h^\gamma})$ is given by $\mu_{h^\gamma}(\mathcal{B}_{\varepsilon/2}(x; D_{h^\gamma}))/\mu_{h^\gamma}(\partial_{2\varepsilon}\eta[s,t])$. Using Theorem 2.2.1 (the volume estimate for LQG metric balls), we show that this number is no more than $\varepsilon^{-\zeta}/|\mathcal{X}_\varepsilon|$. Since the points in $\mathcal{X}_\varepsilon$ are sampled conditionally i.i.d., the conditional probability that $\mathcal{B}_{\varepsilon/2}(w; D_{h^\gamma}) \subset \bigcup_{x\in\mathcal{X}_\varepsilon} \mathcal{B}_\varepsilon(x; D_{h^\gamma})$ is no less than $1-(1-\varepsilon^{-\zeta}/|\mathcal{X}_\varepsilon|)^{|\mathcal{X}_\varepsilon|}$, which tends to 1 superpolynomially fast as $\varepsilon \to 0$. Since the cardinality of $\widetilde{\mathcal{W}}_\varepsilon$ is polynomial in ε, by taking a union bound over points in this set, we conclude that $\partial_\varepsilon \eta[s,t] \subset \bigcup_{w\in\widetilde{\mathcal{W}}_\varepsilon} \mathcal{B}_{\varepsilon/2}(w; D_{h^\gamma}) \subset \bigcup_{x\in\mathcal{X}_\varepsilon} \mathcal{B}_\varepsilon(x; D_{h^\gamma})$ with probability increasing to 1 as $\varepsilon \to 0$.

What we have above are essentially the statement and the proof of Lemma 2.4.8, except that they are for the whole-plane GFF h instead of the γ-quantum cone field h^γ. The reason that we prove (2.4.27) for h first is that most results about the Minkowski dimension of γ-LQG, such as Theorem 2.2.1, are stated in terms of h instead of h^γ. After proving Lemma 2.4.8, we transfer this result to the setting of a γ-quantum cone using Lemma 2.2.15 to complete the proof of Lemma 2.4.4. Recall the notation $\mathcal{B}_\varepsilon(A; D_h) := \bigcup_{z\in A} \mathcal{B}_\varepsilon(z; D_h)$.

Lemma 2.4.8. *Let h be a whole-plane field whose law is absolutely continuous with that of the whole-plane GFF normalized to have mean zero on $\partial\mathbb{D}$. Let $K \subset \mathbb{C}$ be a fixed compact set. Let $A \subset K$ be a random set, not necessarily independent from h. Then, for each $\zeta > 0$,*

$$N_\varepsilon(\mathcal{B}_\varepsilon(A; D_h); D_h) \le \varepsilon^{-\zeta} N_\varepsilon(A; D_h) \tag{2.4.27}$$

holds with probability tending to 1 as $\varepsilon \to 0$.

Proof. Let $\mathcal{X}_\varepsilon$ be a collection of $\lfloor \varepsilon^{-\zeta} N_\varepsilon(A; D_h) \rfloor$ points in $\mathcal{B}_{2\varepsilon}(A; D_h)$ sampled, given h and A, conditionally i.i.d. from $\mu_h|_{\mathcal{B}_{2\varepsilon}(A;D_h)}$ normalized to be a probability measure. We claim that $\bigcup_{x\in\mathcal{X}_\varepsilon} \mathcal{B}_\varepsilon(x; D_h)$ covers $\mathcal{B}_\varepsilon(A; D_h)$ with probability increasing to 1 as $\varepsilon \to 0$.

To show that $\bigcup_{x\in\mathcal{X}_\varepsilon} \mathcal{B}_\varepsilon(x; D_h)$ covers $\mathcal{B}_\varepsilon(A; D_h)$, we consider an $\frac{\varepsilon}{2}$-cover of $\mathcal{B}_\varepsilon(A; D_h)$ and then check that $\bigcup_{x\in\mathcal{X}_\varepsilon} \mathcal{B}_\varepsilon(x; D_h)$ contains this $\frac{\varepsilon}{2}$-cover. Let us first compute the conditional probability that an LQG metric ball $\mathcal{B}_{\varepsilon/2}(w; D_h)$ appearing in the $\frac{\varepsilon}{2}$-cover is contained in $\bigcup_{x\in\mathcal{X}_\varepsilon} \mathcal{B}_\varepsilon(x; D_h)$. If $\mathcal{B}_{\varepsilon/2}(w; D_h) \cap \mathcal{B}_\varepsilon(A; D_h) \neq \varnothing$, then $w \in \mathcal{B}_{3\varepsilon/2}(A; D_h)$ and

$\mathcal{B}_{\varepsilon/2}(w; D_h) \subset \mathcal{B}_{2\varepsilon}(A; D_h)$. Conditioned on h, A, and $w \in \mathcal{B}_{3\varepsilon/2}(A; D_h)$, each $x \in \mathcal{X}_\varepsilon$ has

$$\mathbb{P}\left(\mathcal{B}_{\varepsilon/2}(w; D_h) \subset \mathcal{B}_\varepsilon(x; D_h) \middle| h, A, w\right) \geq \mathbb{P}\left(x \in \mathcal{B}_{\varepsilon/2}(w; D_h) \middle| h, A, w\right) = \frac{\mu_h(\mathcal{B}_{\varepsilon/2}(w; D_h))}{\mu_h(\mathcal{B}_{2\varepsilon}(A; D_h))}. \tag{2.4.28}$$

Since the points in $\mathcal{X}_\varepsilon$ are conditionally i.i.d. given h and A,

$$\mathbb{P}\left(\mathcal{B}_{\varepsilon/2}(w; D_h) \subset \bigcup_{x \in \mathcal{X}_\varepsilon} \mathcal{B}_\varepsilon(x; D_h) \middle| h, A, w\right) \geq 1 - \left(1 - \frac{\mu_h(\mathcal{B}_{\varepsilon/2}(w; D_h))}{\mu_h(\mathcal{B}_{2\varepsilon}(A; D_h))}\right)^{\lfloor \varepsilon^{-\zeta} N_\varepsilon(A; D_h) \rfloor}. \tag{2.4.29}$$

We now specify the $\frac{\varepsilon}{2}$-cover of $\mathcal{B}_\varepsilon(A; D_h)$. Let $U := B_1(K)$ and $W := B_2(K)$. Let $\mathcal{W}_\varepsilon$ be a collection of $\lfloor (\varepsilon/2)^{-d_\gamma - \zeta} \rfloor$ points sampled conditionally i.i.d. from $\mu_h|_W$ normalized to be a probability measure, where $\mathcal{W}_\varepsilon$ and $\mathcal{X}_\varepsilon$ are conditionally independent given h. By [118, Lemma A.3], the event

$$E_1 := \left\{ U \subset \bigcup_{w \in \mathcal{W}_\varepsilon} \mathcal{B}_{\varepsilon/2}(w; D_h) \right\} \tag{2.4.30}$$

occurs with probability tending to 1 as $\varepsilon \to 0$. Also, since D_h is almost surely a continuous metric, the event

$$E_2 := \{D_h(K, \partial U) \geq \varepsilon\} \cap \{D_h(\partial U, \partial W) \geq \varepsilon\} \tag{2.4.31}$$

occurs with probability tending to 1 as $\varepsilon \to 0$. Truncating on the event $E_1 \cap E_2$, we have $\mathcal{B}_\varepsilon(A; D_h) \subset U$ and hence

$$\mathcal{B}_\varepsilon(A; D_h) \subset \bigcup_{w \in \widetilde{\mathcal{W}}_\varepsilon} \mathcal{B}_{\varepsilon/2}(w; D_h) \quad \text{where} \quad \widetilde{\mathcal{W}}_\varepsilon := \mathcal{W}_\varepsilon \cap \mathcal{B}_{3\varepsilon/2}(A; D_h). \tag{2.4.32}$$

This is the $\frac{\varepsilon}{2}$-cover we choose for $\mathcal{B}_\varepsilon(A; D_h)$, which holds on the event $E_1 \cap E_2$. By [16, Theorem 1.1], there exists a random $c > 0$ which depends only on W and ζ such that a.s. for every $\varepsilon \in (0, 1)$,

$$\frac{\inf_{z \in W} \mu_h(\mathcal{B}_{\varepsilon/2}(z; D_h))}{\sup_{z \in W} \mu_h(\mathcal{B}_{3\varepsilon}(z; D_h))} \geq c\varepsilon^{\zeta/2}. \tag{2.4.33}$$

Note that

$$\mu_h(\mathcal{B}_{2\varepsilon}(A;D_h)) \le N_\varepsilon(A;D_h)\left(\sup_{z\in\mathcal{B}_\varepsilon(A;D_h)} \mu_h(\mathcal{B}_{3\varepsilon}(z;D_h))\right) \tag{2.4.34}$$

since we can cover $\mathcal{B}_{2\varepsilon}(A;D_h)$ by first choosing an ε-cover of A and then blowing up the radius of every metric ball in this cover to 3ε. Combining (2.4.33) and (2.4.34), we almost surely have

$$\inf_{w\in\mathcal{B}_{3\varepsilon/2}(A;D_h)} \frac{\mu_h(\mathcal{B}_{\varepsilon/2}(w;D_h))}{\mu_h(\mathcal{B}_{2\varepsilon}(A;D_h))} \ge \frac{1}{N_\varepsilon(A;D_h)}\cdot\frac{\inf_{z\in W}\mu_h(\mathcal{B}_{\varepsilon/2}(z;D_h))}{\sup_{z\in W}\mu_h(\mathcal{B}_{3\varepsilon}(z;D_h))} \ge \frac{c\varepsilon^{\zeta/2}}{N_\varepsilon(A;D_h)} \tag{2.4.35}$$

truncated on the event E_2 so that $\mathcal{B}_{2\varepsilon}(A;D_h)\subset W$. Combining (2.4.29) with (2.4.35), for each $w\in\mathcal{W}_\varepsilon$ we have

$$\begin{aligned}&\mathbb{P}\left(\mathcal{B}_{\varepsilon/2}(w;D_h)\subset\bigcup_{x\in\mathcal{X}_\varepsilon}\mathcal{B}_\varepsilon(x;D_h)\text{ or } w\notin\mathcal{B}_{3\varepsilon/2}(A;D_h)\,\middle|\, h,A,E_2\right)\\&\ge 1-\left(1-\frac{c\varepsilon^{-\zeta/2}}{\varepsilon^{-\zeta}N_\varepsilon(A;D_h)}\right)^{\varepsilon^{-\zeta}N_\varepsilon(A;D_h)}.\end{aligned} \tag{2.4.36}$$

Since $\varepsilon^{-\zeta}N_\varepsilon(A;D_h)\ge\varepsilon^{-\zeta}$, the conditional probability in (2.4.36) becomes superpolynomially high as $\varepsilon\to 0$ at a rate uniform on h and A. Since there are $\varepsilon^{-d_\gamma-\zeta}$ points in $\mathcal{W}_\varepsilon$, an intersection of (2.4.36) over the points $w\in\mathcal{W}_\varepsilon$ implies that truncated on the event E_2, we have $\bigcup_{w\in\widetilde{W}_\varepsilon}\mathcal{B}_{\varepsilon/2}(w;D_h)\subset\bigcup_{x\in\mathcal{X}_\varepsilon}\mathcal{B}_\varepsilon(x;D_h)$ with superpolynomially high probability as $\varepsilon\to 0$. Recalling our choice of the $\frac{\varepsilon}{2}$-cover of $\mathcal{B}_\varepsilon(A;D_h)$ in (2.4.32), we conclude the proof of the lemma by observing that $\mathbb{P}\left\{\mathcal{B}_\varepsilon(A;D_h)\subset\bigcup_{x\in\mathcal{X}_\varepsilon}\mathcal{B}_\varepsilon(x;D_h)\right\}$ is bounded below by

$$\mathbb{P}\left(E_1\cap E_2\cap\left\{\bigcup_{w\in\widetilde{\mathcal{W}}_\varepsilon}\mathcal{B}_{\varepsilon/2}(w;D_h)\subset\bigcup_{x\in\mathcal{X}_\varepsilon}\mathcal{B}_\varepsilon(x;D_h)\right\}\right), \tag{2.4.37}$$

which tends to 1 as $\varepsilon\to 0$. $\square$

We are now ready to complete the proof of Lemma 2.4.4.

Proof of Lemma 2.4.4. It suffices to show $\lim_{\varepsilon\to 0}\varepsilon^{d_\gamma}N_\varepsilon(\partial_\varepsilon\eta[0,1])=0$ almost surely, due to Proposition 2.2.20. Let (z,h) be sampled from $\mathbf{1}_{\mathbb{D}}(z)\mu_h(dz)dh$ normalized to be a probability measure as described in Lemma 2.2.15. For each constant C, let ϕ_C be the random translation and scaling such that $h^C := (h+C)\circ\phi_C + Q\log|\phi_C'|$ is the field under the circle average embedding of $(\mathbb{C}, h+C, z, \infty)$. Given $\beta>0$, choose a large constant $R>0$ such that the event

$$E_3 := \{\eta[0,1]\subset B_R(0) \text{ and } D_{h^\gamma}(\partial B_R(0), \partial B_{2R}(0))\geq 1\} \tag{2.4.38}$$

has probability at least $1-\beta/3$. Then, choose a large constant $C>0$ such that the following two conditions are satisfied.

(i) The total variation distance between the laws of the fields h^γ and h^C restricted to $B_{2R}(0)$ is at most $\beta/3$.

(ii) The event $E_4 := \{\phi_C(B_R(0))\subset B_2(0)\}$ has probability at least $1-\beta/3$.

For sufficiently large C, the first item holds by Lemma 2.2.15 and the second item holds by the conditional law of $h(\cdot - z)$ as described in [95, Lemma A.10].

Fix $\zeta\in(0,d_\gamma/2)$. Couple h^C and h^γ so that their restrictions to $B_{2R}(0)$ agree on an event of probability $1-\beta/3$, which we call E_5. Substitute $A=\phi_C(\partial\eta[0,1])$ and $K=\overline{B_2(0)}$ in Lemma 2.4.8. Then, truncated on $E_3\cap E_4\cap E_5$, the event

$$N_{\varepsilon C^\xi}(\mathcal{B}_{\varepsilon C^\xi}(\phi_C(\partial\eta[0,1]); D_h); D_h) \leq (\varepsilon C^\xi)^{-\zeta} N_{\varepsilon C^\xi}(\phi_C(\partial\eta[0,1]); D_h) \tag{2.4.39}$$

occurs with probability increasing to the full probability of $E_3\cap E_4\cap E_5$ as $\varepsilon\to 0$. By

Lemmas 2.2.10 and 2.2.11, we almost surely have

$$\begin{aligned} N_{\varepsilon C^{\xi}}(\mathcal{B}_{\varepsilon C^{\xi}}(\phi_C(\partial\eta[0,1]); D_h); D_h) &= N_{\varepsilon}(\mathcal{B}_{\varepsilon}(\phi_C(\partial\eta[0,1]); D_{h+C}); D_{h+C}) \\ &= N_{\varepsilon}(\mathcal{B}_{\varepsilon}(\partial\eta[0,1]; D_{h^C}); D_{h^C}) \end{aligned} \tag{2.4.40}$$

on the left-hand side of (2.4.39) and

$$\begin{aligned} (\varepsilon C^{\xi})^{-\zeta} N_{\varepsilon C^{\xi}}(\phi_C(\partial\eta[0,1]); D_h) &= (\varepsilon C^{\xi})^{-\zeta} N_{\varepsilon}(\phi_C(\partial\eta[0,1]); D_{h+C}) \\ &= (\varepsilon C^{\xi})^{-\zeta} N_{\varepsilon}(\partial\eta[0,1]); D_{h^C}) \end{aligned} \tag{2.4.41}$$

on the right-hand side. On the event $E_3 \cap E_5$, since $D_{h^C}(\partial\eta[0,1], \partial B_{2R}(0)) \geq 1$, both $N_{\varepsilon}(\partial\eta[0,1]; D_{h^C})$ and $N_{\varepsilon}(\mathcal{B}_{\varepsilon}(\partial\eta[0,1]; D_{h^C}); D_{h^C})$ for $\varepsilon \in (0, 1/2)$ are almost surely determined by $h^C|_{B_{2R}(0)}$. Hence, (2.4.39) implies

$$\varepsilon^{d_\gamma} N_{\varepsilon}(\partial_{\varepsilon}\eta[0,1]; D_{h^\gamma}) \leq C^{-\xi\zeta} \varepsilon^{d_\gamma - \zeta} N_{\varepsilon}(\partial\eta[0,1]; D_{h^\gamma}). \tag{2.4.42}$$

Since $d_\gamma - \zeta > d_\gamma/2$, we deduce from Lemma 2.4.7 that, truncated on $E_3 \cap E_4 \cap E_5$, the right-hand side of (2.4.42) tends to 0 almost surely as $\varepsilon \to 0$. This proves the lemma since $E_3 \cap E_4 \cap E_5$ occurs with probability at least $1 - \beta$, and our choice of β was arbitrary. $\square$

2.4.2 Identifying the Minkowski content process

For the remainder of this section, let $(X_I : I \in \mathcal{I}_{\mathbb{Q}})$ be a weak limit of $(\mathfrak{b}_{\varepsilon_n}^{-1} N_{\varepsilon_n}(\eta(I)) : I \in \mathcal{I}_{\mathbb{Q}})$ for some sequence ε_n that tends to 0. As described in the proof overview for

Proposition 2.4.3, we define the “Minkowski content process” $\{Y_t\}_{t\in\mathbb{Q}}$ by

$$Y_t := \begin{cases} X_{[0,t]} & t > 0, \\ 0 & t = 0, \\ -X_{[t,0]} & t < 0. \end{cases} \tag{2.4.43}$$

By Proposition 2.4.6, $X_{[s,t]} = Y_t - Y_s$ a.s. for all $[s,t] \in \mathcal{I}_{\mathbb{Q}}$. The next step is to show that $\{Y_t\}_{t\in\mathbb{Q}}$ extends to a two-sided Brownian motion with drift.

Proposition 2.4.9. *The process* $\{Y_t\}_{t\in\mathbb{Q}}$ *defined in* (2.4.43) *extends to a Lévy process with almost surely continuous paths: i.e., a two-sided Brownian motion with drift.*

Proof. We first show that $\{Y_t\}_{t\in\mathbb{Q}}$ extends to a continuous process defined on $\mathbb{R}$. Let $p > 1$. For every rational $s < t$, by Fatou's lemma, Corollary 2.4.2, and Proposition 2.3.9,

$$\begin{aligned} \mathbb{E}[|Y_t - Y_s|^p] = \mathbb{E}[|X_{[s,t]}|^p] &\le \liminf_{n\to\infty} \mathbb{E}[|\mathfrak{b}_{\varepsilon_n}^{-1} N_{\varepsilon_n}(\eta[s,t])|^p] \\ &\le C_1 \liminf_{n\to\infty} \mathbb{E}[|\varepsilon_n^{d_\gamma} N_{\varepsilon_n}(\eta[s,t])|^p] \le C_2|t-s|^p \end{aligned} \tag{2.4.44}$$

for constants $C_1, C_2 > 0$ depending only on γ. We deduce from the Kolmogorov continuity criterion that $\{Y_t\}_{t\in\mathbb{Q}}$ can be extended to a process $\{Y_t\}_{t\in\mathbb{R}}$ whose paths are almost surely continuous.

It now suffices to check that $\{Y_t\}_{t\in\mathbb{Q}}$ has stationary and independent increments since these properties extend to $\{Y_t\}_{t\in\mathbb{R}}$ by continuity.

- *Stationary increments.* We found in Corollary 2.2.21 that $N_\varepsilon(\eta[s,t]) \overset{d}{=} N_\varepsilon(\eta[0,t-s])$ for every fixed $s < t$ and $\varepsilon > 0$. Hence, $Y_t - Y_s = X_{[s,t]}$ and $Y_{t-s} = X_{[0,t-s]}$ agree in law for every rational $s < t$.

- *Independent increments.* For each $s \in \mathbb{R}$, denote by U_{s-} and U_{s+} the interiors of $\eta(-\infty, s]$ and $\eta[s,\infty)$, respectively. We claim that for any $\varepsilon > 0$ and any fixed interval

$I \in \mathcal{I}_{\mathbb{Q}}$, if $I \subset (-\infty, s]$ (resp. $I \subset [s, \infty)$), then the quantity $N_\varepsilon^\circ(\eta(I))$ is almost surely determined by the curve-decorated metric measure space $(U_{s-}, D_h^{U_{s-}}, \mu_h|_{U_{s-}}, \eta|_{(-\infty,s]})$ (resp. $(U_{s+}, D_h^{U_{s+}}, \mu_h|_{U_{s+}}, \eta|_{[s,\infty)]}))$.

Let us prove the claim. Without loss of generality, assume $I \subset (-\infty, s]$. Recall the definition

$$N_\varepsilon^\circ(\eta(I)) = N_\varepsilon(\eta(I) \setminus \partial_\varepsilon \eta(I); D_{h^\gamma}) \tag{2.4.45}$$

where $\eta(I) \setminus \partial_\varepsilon \eta(I) := \{z \in \eta(I) : D_{h^\gamma}(z, \partial\eta(I)) \geq \varepsilon\}$. Suppose $\mathcal{B}_\varepsilon(w; D_{h^\gamma})$ is a ball counted in $N_\varepsilon^\circ(\eta(I))$. Then $\mathcal{B}_\varepsilon(w; D_{h^\gamma})$ has a nonempty intersection with $\eta(I) \setminus \partial_\varepsilon \eta(I)$, so w must be in the interior of $\eta(I)$. Now suppose z is any point within the intersection of $\mathcal{B}_\varepsilon(w; D_{h^\gamma})$ and $\eta(I) \setminus \partial_\varepsilon \eta(I)$. Then, the D_{h^γ}-geodesic from z to w must be contained in the interior of $\eta(I)$, or otherwise its D_{h^γ}-length would be at least

$$D_{h^\gamma}(z, \partial\eta(I)) + D_{h^\gamma}(w, \partial\eta(I)) \geq \varepsilon. \tag{2.4.46}$$

Since $\mathrm{int}(\eta(I)) \subset U_{s-}$, we have $D_{h^\gamma}(z, w) = D_{h^\gamma}^{U_{s-}}(z, w)$. Since this holds for any z in the intersection of $\mathcal{B}_\varepsilon(w; D_{h^\gamma})$ and $\eta(I) \setminus \partial_\varepsilon \eta(I)$, we have

$$\mathcal{B}_\varepsilon(w; D_{h^\gamma}^{U_{s-}}) \cap (\eta(I) \setminus \partial_\varepsilon \eta(I)) = \mathcal{B}_\varepsilon(w; D_{h^\gamma}) \cap (\eta(I) \setminus \partial_\varepsilon \eta(I)). \tag{2.4.47}$$

Since (2.4.47) holds for any w in the interior of $\eta(I)$, we obtain

$$N_\varepsilon^\circ(\eta(I)) = N_\varepsilon(\eta(I) \setminus \partial_\varepsilon \eta(I); D_{h^\gamma}^{U_{s-}}). \tag{2.4.48}$$

Observe that

$$\eta(I) \setminus \partial_\varepsilon \eta(I) = \{z \in \eta(I) : \mathcal{B}_\varepsilon(z; D_{h^\gamma}^{U_{s-}}) \subset \eta(I)\}. \tag{2.4.49}$$

Hence, the right-hand side of (2.4.48) is determined by $(U_{s-}, D_h^{U_{s-}}, \mu_h|_{U_{s-}}, \eta|_{(-\infty,s]})$

almost surely.

Let us now deduce from the claim that $\{Y_t\}_{t\in\mathbb{Q}}$ has independent increments. Let $t_0 < t_1 < \cdots < t_m$ be a fixed collection of finitely many rational times. Setting $s = t_k$, we see from the claim that for each ε_n, the random vector $(\mathfrak{b}_{\varepsilon_n}^{-1} N_{\varepsilon_n}^{\circ}(\eta[t_{j-1}, t_j]) : 1 \le j \le k)$ is a.s. determined by $(U_{s-}, D_h^{U_{s-}}, \mu_h|_{U_{s-}}, \eta|_{(-\infty,s]})$, and $(\mathfrak{b}_{\varepsilon_n}^{-1} N_{\varepsilon_n}^{\circ}(\eta[t_{j-1}, t_j]) : k+1 \le j \le m)$ is a.s. determined by $(U_{s+}, D_h^{U_{s+}}, \mu_h|_{U_{s+}}, \eta|_{[s,\infty)})$. These two random vectors are thus independent by Proposition 2.2.23 combined with the translation invariance of Proposition 2.2.20. Recalling (2.4.15), we conclude that $(X_{[t_{j-1},t_j]} : 1 \le j \le k)$ and $(X_{[t_{j-1},t_j]} : k+1 \le j \le m)$ are independent. This works for any $1 \le k \le m$, so the random variables $X_{[t_0,t_1]}, \ldots, X_{[t_{m-1},t_m]}$ are independent. $\square$

We conclude the proof of Proposition 2.4.3 by checking that the Minkowski content process has almost surely positive increments.

Proposition 2.4.10 (Positive increments)**.** *Almost surely, $X_{[s,t]} > 0$ for every rational $s < t$.*

Proof. By Proposition 2.4.1, we can find a deterministic constant $c > 0$ such that with probability tending to 1 as $\varepsilon \to 0$, we have $N_\varepsilon(\eta[0,1]) \ge c\varepsilon^{-d_\gamma}$. By Corollary 2.2.21, for any fixed rational times $s < t$, it also holds with probability tending to 1 as $\varepsilon \to 0$ that $N_\varepsilon(\eta[s,t]) \ge c\varepsilon^{-d_\gamma}(t-s)$. Combining this with Corollary 2.4.2, we deduce that $X_{[s,t]} > 0$ almost surely for every rational $s < t$. $\square$

Proof of Proposition 2.4.3. We proved that $\{\varepsilon^{d_\gamma} N_\varepsilon(\eta[s,t])\}_{0<\varepsilon<(t-s)^{1/d_\gamma}}$ is tight for any $s < t$ in Proposition 2.3.9. By Corollary 2.4.2, $\{\mathfrak{b}_\varepsilon^{-1} N_\varepsilon(\eta[s,t])\}_{0<\varepsilon<(t-s)^{1/d_\gamma}}$ is also tight. Therefore, given any sequence of ε decreasing to 0, we can find a subsequence ε_n for which the corresponding sequence of $\mathbb{R}^{\mathcal{I}_\mathbb{Q}}$-valued random variables $(\mathfrak{b}_{\varepsilon_n}^{-1} N_{\varepsilon_n}(\eta(I)) : I \in \mathcal{I}_\mathbb{Q})$ converges in distribution w.r.t. the product topology on $\mathbb{R}^{\mathcal{I}_\mathbb{Q}}$.

Now suppose ε_n is any sequence decreasing to 0 such that $(\mathfrak{b}_{\varepsilon_n}^{-1} N_{\varepsilon_n}(\eta(I)) : I \in \mathcal{I}_{\mathbb{Q}})$ has a weak limit $(X_I : I \in \mathcal{I}_{\mathbb{Q}})$. Define the process $\{Y_t\}_{t\in\mathbb{Q}}$ as in (2.4.43). By Proposition 2.4.9, $\{Y_t\}_{t\in\mathbb{Q}}$ extends to a Brownian motion with drift. Since $\{Y_t\}_{t\in\mathbb{Q}}$ has positive increments (Proposition 2.4.10), the variance of the Brownian motion must be zero. That is, there exists a deterministic constant $a > 0$ such that almost surely, $Y_t = at$ for all t. In other words, for every rational $s < t$, the subsequence $\mathfrak{b}_{\varepsilon_n}^{-1} N_{\varepsilon_n}(\eta[s,t])$ converges in distribution to $a(t-s)$ as $n \to \infty$. Since $a(t-s)$ is a deterministic constant, the convergence is in probability. In fact, for every $p \geq 1$, the convergence is also in L^p since $\sup_{0<\varepsilon<|t-s|^{1/d_\gamma}} \mathbb{E}[|\mathfrak{b}_{\varepsilon}^{-1} N_{\varepsilon}(\eta[s,t])|^p] < \infty$ (Proposition 2.3.9). By the definition (2.1.8) of $\mathfrak{b}_\varepsilon$, we have $\mathbb{E}[\mathfrak{b}_{\varepsilon}^{-1} N_{\varepsilon}(\eta[0,1])] = 1$ for all $\varepsilon > 0$. Hence, $a = 1$ regardless of the choice of the subsequence ε_n.

In summary, we have that given any sequence of ε decreasing to 0, there exists a subsequence ε_n such that $\lim_{n\to\infty} \mathfrak{b}_{\varepsilon_n}^{-1} N_{\varepsilon_n}(\eta[s,t]) = t - s$ in probability for all rational $s < t$. This proves (2.4.7) for rational $s < t$. Now suppose $s < t$ are any fixed real numbers. If $s_1 < s_0 < t_0 < t_1$ are rational numbers such that $[s_0, t_0] \subset [s,t] \subset [s_1, t_1]$, then

$$N_\varepsilon(\eta[s_0,t_0]) \leq N_\varepsilon(\eta[s,t]) \leq N_\varepsilon(\eta[s_1,t_1]). \tag{2.4.50}$$

By taking s_1, s_0 close to s and t_0, t_1 close to t, we conclude that $\lim_{n\to\infty} \mathfrak{b}_{\varepsilon}^{-1} N_{\varepsilon}(\eta[s,t]) = t-s$ in probability. □

2.5 Generalization to other sets and fields

In this section, we show that the LQG measure μ_h can be recovered from D_h in terms of Minkowski content with respect to $\mathfrak{b}_\varepsilon$, not only for the γ-quantum cone but also for other variants of the whole-plane GFF.

In general, Minkowski content is not a measure. Accordingly, we do not expect that the Minkowski content of an arbitrary bounded Borel set $A \subset \mathbb{C}$ is equal to its LQG measure

$\mu_h(A)$. The sufficient condition we impose is that A is a random bounded Borel set such that

$$\mu_h(\partial A) = 0 \quad \text{a.s.} \tag{2.5.1}$$

Since ∂A is closed, this condition is equivalent to

$$\lim_{\delta \to 0} \mu_h(B_\delta(\partial A)) = 0 \quad \text{a.s.} \tag{2.5.2}$$

Since the LQG metric induces the Euclidean topology, another equivalent condition is

$$\lim_{\delta \to 0} \mu_h(\mathcal{B}_\delta(\partial A; D_h)) = 0 \quad \text{a.s.} \tag{2.5.3}$$

Remark 2.5.1. One sufficient condition for (2.5.1) is that the LQG Minkowski dimension of A is bounded above by $d_\gamma - \zeta$ for some deterministic constant $\zeta > 0$. That is, $\lim_{\varepsilon \to 0} \varepsilon^{d_\gamma - \zeta} N_\varepsilon(\partial A; D_h) = 0$ almost surely. Note that this condition does not reference the LQG measure μ_h. Indeed, suppose that h is a whole-plane GFF with $h_1(0) = 0$. Theorem 2.2.1 states that

$$\sup_{\varepsilon \in (0,1)} \sup_{z \in B_r(0)} \frac{\mu_h(\mathcal{B}_\varepsilon(z; D_h))}{\varepsilon^{d_\gamma - \frac{\zeta}{2}}} < \infty \quad \text{a.s.} \tag{2.5.4}$$

Then, on the event $A \subset B_r(0)$,

$$\mu_h(\partial A) \leq \limsup_{\varepsilon \to 0} \left(N_\varepsilon(\partial A; D_h) \cdot \sup_{z \in B_r(0)} \mu_h(\mathcal{B}_\varepsilon(z; D_h)) \right) = 0 \quad \text{a.s.} \tag{2.5.5}$$

Letting $r \to \infty$, we obtain (2.5.1) since A is almost surely bounded. This implication holds even when we replace h with a whole-plane GFF plus continuous function because μ_{h+f} is almost surely absolutely continuous with respect to μ_h for any random continuous function $f : \mathbb{C} \to \mathbb{R}$.

Remark 2.5.2. Let $A \subset \mathbb{C}$ be a deterministic bounded Borel set whose boundary has zero

Lebesgue measure, and let h be a whole-plane GFF plus continuous function. We claim that A satisfies (2.5.1). Indeed, let $r > 0$ such that $\overline{A} \subset B_r(0)$. Let h^0 be a zero-boundary GFF on $B_r(0)$. From [99, Proposition 1.2], we have $\mathbb{E}[\mu_{h^0}(\partial A)] = \int_{\partial A} \operatorname{crad}(z; B_r(0))^{\gamma^2/2}\, d^2 z$ where $\operatorname{crad}(z; B_r(0))$ is the conformal radius of $B_r(0)$ viewed from z. Since the Lebesgue measure of ∂A is zero, we get $\mathbb{E}[\mu_{h^0}(\partial A)] = 0$ and $\mu_{h^0}(\partial A) = 0$ almost surely. By the Markov property of the whole-plane GFF [196, Proposition 2.8], we can couple h^0 and h so that $(h - h^0)|_{B_r(0)}$ is a continuous function. Since adding a continuous function to the field results in weighting the Liouville measure by a continuous function, we also get the desired statement for h.

2.5.1 General sets on the quantum cone

In Proposition 2.4.3, we found that if $s < t$ are any fixed real numbers, $\mathfrak{b}_\varepsilon^{-1} N_\varepsilon(\eta[s,t]; D_{h^\gamma})$ converges in probability to $t - s = \mu_{h^\gamma}(\eta[s,t])$. The goal of this subsection is to replace $\eta[s,t]$ with more general subsets of $\mathbb{C}$; in particular, we consider bounded Borel sets satisfying (2.5.1). We can also consider random subsets of $\mathbb{C}$ coupled with the field h^γ. The idea is to bound such a set from inside and outside by unions of finitely many SLE cells — i.e., $\bigcup_{I \subset \mathcal{I}} \eta(I)$ where $\mathcal{I} \subset \mathcal{I}_\mathbb{Q} := \{[s,t] : s,t \in \mathbb{Q}, s < t\}$ is a random collection of finitely many closed intervals with dyadic rational endpoints.

Proposition 2.5.3. *Let h^γ be the field of a γ-quantum cone in the circle average embedding. Let $A \subset \mathbb{C}$ be either a deterministic bounded Borel set or a random compact set coupled with h^γ and η (as in Theorem 2.1.1) such that $\mu_{h^\gamma}(\partial A) = 0$ almost surely. Then,*

$$\lim_{\varepsilon \to 0} \mathfrak{b}_\varepsilon^{-1} N_\varepsilon(A; D_{h^\gamma}) = \mu_{h^\gamma}(A) \tag{2.5.6}$$

in probability.

Proof. As introduced in the above proof idea, we first consider the case $A = \bigcup_{I \in \mathcal{I}} \eta(I)$

where $\mathcal{I}$ is a random collection of finitely many closed bounded intervals with dyadic rational endpoints. Concretely, fix a positive integer k and consider a random collection $\mathcal{I} \subset \{[j/2^k, (j+1)/2^k] : j \in \mathbb{Z}\}$ coupled with h^γ and η which a.s. has finitely many elements. We work in the same way we proved the finite additivity of Minkowski content of SLE cells (Proposition 2.4.6). Recall that $\partial_\varepsilon \eta(I) = \{z \in \eta(I) : D_{h^\gamma}(z, \partial\eta(I)) < \varepsilon\}$. For such $\mathcal{I}$, for each $\varepsilon > 0$, we almost surely have

$$\begin{aligned}\sum_I \mathbf{1}_{\{I\in\mathcal{I}\}} \mathfrak{b}_\varepsilon^{-1}[N_\varepsilon(\eta(I); D_{h^\gamma}) - N_\varepsilon(\partial_\varepsilon\eta(I); D_{h^\gamma})] \\ \leq \mathfrak{b}_\varepsilon^{-1} N_\varepsilon\left(\bigcup_{I\in\mathcal{I}} \eta(I); D_{h^\gamma}\right) \leq \sum_I \mathbf{1}_{\{I\in\mathcal{I}\}} \mathfrak{b}_\varepsilon^{-1} N_\varepsilon(\eta(I); D_{h^\gamma})\end{aligned} \tag{2.5.7}$$

where the two sums are over all intervals I of the form $[j/2^k, (j+1)/2^k]$. Both sums converge in probability as $\varepsilon \to 0$ to $\sum_{I\in\mathcal{I}} \mu_{h^\gamma}(\eta(I)) = \mu_{h^\gamma}(\bigcup_{I\in\mathcal{I}} \eta(I))$ since each term in these sums converges in probability to $\mathbf{1}_{\{I\in\mathcal{I}\}}\eta(I)$ as seen in Proposition 2.4.3 and Lemma 2.4.4. Therefore,

$$\lim_{\varepsilon\to 0} \mathfrak{b}_\varepsilon^{-1} N_\varepsilon\left(\bigcup_{I\in\mathcal{I}} \eta(I); D_{h^\gamma}\right) = \mu_{h^\gamma}\left(\bigcup_{I\in\mathcal{I}} \eta(I)\right) \quad \text{in probability.} \tag{2.5.8}$$

Let $A \subset \mathbb{C}$ now be a deterministic bounded Borel set or a random compact set coupled with h^γ and η such that $\mu_{h^\gamma}(\partial A) = 0$ almost surely. For each integer k, let

$$\mathcal{I}_k := \left\{\left[j/2^k, (j+1)/2^k\right] : j \in \mathbb{Z}, \eta\left(\left[j/2^k, (j+1)/2^k\right]\right) \subset A\right\} \tag{2.5.9}$$

and

$$\mathcal{J}_k := \left\{\left[j/2^k, (j+1)/2^k\right] : j \in \mathbb{Z}, \eta\left(\left[j/2^k, (j+1)/2^k\right]\right) \cap A \neq \varnothing\right\}. \tag{2.5.10}$$

That is, $\bigcup_{I\in\mathcal{I}_k}\eta(I)$ and $\bigcup_{I\in\mathcal{J}_k}\eta(I)$ are approximations of A from inside and outside, respectively. For each j and k, note that $\eta([j/2^k,(j+1)/2^k])$ is a random compact subset of $\mathbb{C}$ that is measurable w.r.t. the Borel σ-algebra generated by the Hausdorff distance on compact subsets of $\mathbb{C}$; hence, $\mathcal{I}_k$ and $\mathcal{J}_k$ are measurable. Almost surely, $\mathcal{I}_k$ and $\mathcal{J}_k$ contain finitely many intervals since A is bounded and $\lim_{t\to\pm\infty}\eta(t)=\infty$. Since $\bigcup_{I\in\mathcal{I}_k}\eta(I)\subset A\subset\bigcup_{I\in\mathcal{J}_k}\eta(I)$, we have

$$\mathfrak{b}_\varepsilon^{-1}N_\varepsilon\left(\bigcup_{I\in\mathcal{I}_k}\eta(I);D_{h^\gamma}\right)\le\mathfrak{b}_\varepsilon^{-1}N_\varepsilon(A)\le\mathfrak{b}_\varepsilon^{-1}N_\varepsilon\left(\bigcup_{I\in\mathcal{J}_k}\eta(I);D_{h^\gamma}\right)\tag{2.5.11}$$

almost surely for each $\varepsilon>0$. As in (2.5.8), we have

$$\lim_{\varepsilon\to0}\mathfrak{b}_\varepsilon^{-1}N_\varepsilon\left(\bigcup_{I\in\mathcal{I}_k}\eta(I);D_{h^\gamma}\right)=\mu_{h^\gamma}\left(\bigcup_{I\in\mathcal{I}_k}\eta(I)\right)\tag{2.5.12}$$

and

$$\lim_{\varepsilon\to0}\mathfrak{b}_\varepsilon^{-1}N_\varepsilon\left(\bigcup_{I\in\mathcal{J}_k}\eta(I);D_{h^\gamma}\right)=\mu_{h^\gamma}\left(\bigcup_{I\in\mathcal{J}_k}\eta(I)\right)\tag{2.5.13}$$

in probability for each k.

We claim

$$\lim_{k\to\infty}\mu_{h^\gamma}\left(\bigcup_{I\in\mathcal{I}_k}\eta(I)\right)=\mu_{h^\gamma}(A)=\lim_{k\to\infty}\mu_{h^\gamma}\left(\bigcup_{I\in\mathcal{J}_k}\eta(I)\right)\tag{2.5.14}$$

almost surely, which, in combination with (2.5.12) and (2.5.13), completes the proof of the proposition. Note that for each integer k,

$$\mu_{h^\gamma}\left(\bigcup_{I\in\mathcal{I}_k}\eta(I)\right)\le\mu_{h^\gamma}(A)\le\mu_{h^\gamma}\left(\bigcup_{I\in\mathcal{J}_k}\eta(I)\right).\tag{2.5.15}$$

Because $\bigcup_{I\in\mathcal{I}_k} I$ increases and $\bigcup_{I\in\mathcal{J}_k} I$ decreases as k increases, it suffices to prove

$$\lim_{k\to\infty}\left[\mu_{h^\gamma}\left(\bigcup_{I\in\mathcal{J}_k}\eta(I)\right)-\mu_{h^\gamma}\left(\bigcup_{I\in\mathcal{I}_k}\eta(I)\right)\right]=\lim_{k\to\infty}\mu_{h^\gamma}\left(\bigcup_{I\in\mathcal{J}_k\setminus\mathcal{I}_k}\eta(I)\right)=0 \quad (2.5.16)$$

almost surely. Note that

$$\mathcal{J}_k\setminus\mathcal{I}_k=\left\{\left[j/2^k,(j+1)/2^k\right]:j\in\mathbb{Z},\eta\left(\left[j/2^k,(j+1)/2^k\right]\right)\cap\partial A\neq\varnothing\right\}. \quad (2.5.17)$$

Since A is bounded and $t\mapsto\eta(t)$ is continuous,

$$\lim_{k\to\infty}\max_{I\in\mathcal{J}_k}\operatorname{diam}(\eta(I);D_{h^\gamma})=0 \quad (2.5.18)$$

with probability one. Hence,

$$\lim_{k\to\infty}\mu_{h^\gamma}\left(\bigcup_{I\in\mathcal{J}_k\setminus\mathcal{I}_k}\eta(I)\right)\leq\lim_{\delta\to0}\mu_{h^\gamma}(\mathcal{B}_\delta(\partial A;D_{h^\gamma})). \quad (2.5.19)$$

almost surely. As discussed in the beginning of this section, because D_{h^γ} induces the Euclidean topology, $\mathbb{P}\{\mu_{h^\gamma}(\partial A)=0\}=1$ implies $\lim_{\delta\to0}\mu_{h^\gamma}(\mathcal{B}_\delta(\partial A;D_{h^\gamma}))=0$ almost surely and thus our claim. □

2.5.2 Generalization to other GFF variants

The final technical detail that allows us to replace h^γ in Proposition 2.5.3 with any whole-plane GFF plus a continuous function is that $\mathfrak{b}_\varepsilon$ is regularly varying with index $-d_\gamma$.

Proof of Proposition 2.1.3. We proved (2.1.9) in Corollary 2.4.2. It remains to show (2.1.10). Recall from Corollary 2.2.21 that $N_\varepsilon(\eta[0,r^{-d_\gamma}];D_{h^\gamma})\overset{d}{=}N_{r\varepsilon}(\eta[0,1];D_{h^\gamma})$ for every $\varepsilon>0$.

Hence, by Proposition 2.4.3,

$$r^{-d_\gamma} = \frac{\lim_{\varepsilon\to 0} \mathfrak{b}_\varepsilon^{-1} N_\varepsilon(\eta[0, r^{-d_\gamma}]; D_{h^\gamma})}{\lim_{\varepsilon\to 0} \mathfrak{b}_{r\varepsilon}^{-1} N_{r\varepsilon}(\eta[0,1]; D_{h^\gamma})} = \frac{\lim_{\varepsilon\to 0} \mathfrak{b}_\varepsilon^{-1} N_{r\varepsilon}(\eta[0,1]; D_{h^\gamma})}{\lim_{\varepsilon\to 0} \mathfrak{b}_{r\varepsilon}^{-1} N_{r\varepsilon}(\eta[0,1]; D_{h^\gamma})} = \lim_{\varepsilon\to 0} \frac{\mathfrak{b}_{r\varepsilon}}{\mathfrak{b}_\varepsilon}. \tag{2.5.20}$$

□

Proof of Theorem 2.1.1. The proof proceeds by standard arguments based on the fact that the Minkowski content depends locally on h and behaves nicely under adding a constant to h. Specifically, if f is a deterministic smooth function with compact support on $\mathbb{C}$ and h is a whole-plane GFF with circle average normalization $h_r(z) = 0$ on a fixed circle $\partial B_r(z)$ disjoint from the support of f, then the laws of $h+f$ and h are mutually absolutely continuous [196, Proposition 2.9]. Using this fact, we transfer our results from a γ-quantum cone to a whole-plane GFF in steps, introducing more generality in our choice of the field h and the deterministic bounded Borel or random compact set A. In each stage, we assume $\mu_h(\partial A) = 0$ almost surely.

1. *The field h is a whole-plane GFF with $h_1(0) = 0$, and $A \subset B_{1/4}(1/2)$ almost surely.*

   Note that $\mu_h(A)$, $\mu_h(\partial A)$, and $\lim_{\varepsilon\to 0} N_\varepsilon(A; D_h)$ are a.s. determined by $h|_{B_{1/3}(1/2)}$. Since $h - \gamma \log|\cdot|$ and h^γ agree in law when restricted to $\mathbb{D}$ (Definition 2.2.13), the laws of h^γ and h restricted to $B_{1/3}(1/2)$ are mutually absolutely continuous. We deduce from Proposition 2.5.3 that $\lim_{\varepsilon\to 0} \mathfrak{b}_\varepsilon^{-1} N_\varepsilon(A; D_h) = \mu_h(A)$ in probability.

2. *The field h is a whole-plane GFF with normalization $h_{4r}(-2r) = 0$, and $A \subset B_r(0)$ a.s. for some fixed $r > 0$.*

   Denote $\tilde{h} := h(4r \cdot -2r)$. By (2.2.10), $\tilde{h}$ is a whole-plane GFF with $\tilde{h}_1(0) = 0$. Since $A \subset B_r(0)$, we have $(4r)^{-1}A + 1/2 \subset B_{1/4}(1/2)$. We saw in Step 1 that if

$\mu_{\tilde{h}}((4r)^{-1}A + 1/2) = 0$ almost surely, then

$$\lim_{\varepsilon \to 0} \mathfrak{b}_\varepsilon^{-1} N_\varepsilon((4r)^{-1}A + 1/2; D_{\tilde{h}}) = \mu_{\tilde{h}}((4r)^{-1}A + 1/2) \tag{2.5.21}$$

in probability. From the coordinate change axiom (2.2.14) for deterministic translation and scaling, we almost surely have

$$D_h(4ru - 2r, 4rv - 2r) = D_{\tilde{h} + Q\log(4r)}(u, v) = (4r)^{\xi Q} D_{\tilde{h}}(u, v) \quad \forall u, v \in \mathbb{C}. \tag{2.5.22}$$

Then, almost surely,

$$N_{\varepsilon(4r)^{\xi Q}}(A; D_h) = N_\varepsilon((4r)^{-1}A + 1/2; D_{\tilde{h}}) \quad \forall \varepsilon > 0. \tag{2.5.23}$$

On the other hand, from the coordinate change rule (2.2.16) for the LQG measure, almost surely,

$$\mu_h(A) = \mu_{\tilde{h} + Q\log(4r)}((4r)^{-1}(A + 2r)) = (4r)^{\gamma Q} \mu_{\tilde{h}}((4r)^{-1}A + 1/2). \tag{2.5.24}$$

One consequence of (2.5.24) is that $\mu_h(\partial A) = 0$ implies $\mu_{\tilde{h}}((4r)^{-1}(\partial A) + 1/2) = 0$. Hence, we deduce from (2.5.21) combined with (2.5.23) and (2.5.24) that

$$\lim_{\varepsilon \to 0} \mathfrak{b}_\varepsilon^{-1} N_{\varepsilon(4r)^{\xi Q}}(A; D_h) = (4r)^{-\gamma Q} \mu_h(A) \tag{2.5.25}$$

in probability. Since we have

$$\lim_{\varepsilon \to 0} \frac{\mathfrak{b}_{\varepsilon(4r)^{\xi Q}}}{\mathfrak{b}_\varepsilon} = (4r)^{-\gamma Q} \tag{2.5.26}$$

from Proposition 2.1.3, we conclude that $\lim_{\varepsilon \to 0} \mathfrak{b}_\varepsilon^{-1} N_\varepsilon(A; D_h) = \mu_h(A)$ in probability.

3. *The field h is any whole-plane GFF plus a continuous function, but $A \subset B_r(0)$ for some fixed $r > 0$.*

We can write $h = \hat{h} + f$ where $\hat{h}$ is a whole-plane GFF with normalization $\hat{h}_{4r}(-2r) = 0$ and f is a random continuous function (which is not necessarily independent from $\hat{h}$). Again, the goal is to show that for any Borel $A \subset B_r(0)$ with $\mu_{\hat{h}+f}(\partial A) = 0$ almost surely,

$$\lim_{\varepsilon \to 0} \mathfrak{b}_\varepsilon^{-1} N_\varepsilon(A; D_{\hat{h}+f}) = \mu_{\hat{h}+f}(A) \tag{2.5.27}$$

in probability. Since f is a.s. bounded on $\overline{B_r(0)}$, we have $\mu_{\hat{h}}(\partial A) = 0$ almost surely. The idea is to use that f is a.s. locally uniformly continuous. To this end, consider the collection of dyadic squares $\mathcal{S}_k = \{[\frac{m}{2^k}, \frac{m+1}{2^k}] + [\frac{n}{2^k}, \frac{n+1}{2^k}]i \subset \overline{B_r(0)} : m, n \in \mathbb{Z}\}$ and define

$$\mathcal{A}_k = \{S \in \mathcal{S}_k : S \subset A\} \quad \text{and} \quad \mathcal{A}^k = \{S \in \mathcal{S}_k : S \cap A \neq \varnothing\} \tag{2.5.28}$$

to be the subcollections whose unions approximate A from inside and outside, respectively. For each dyadic square $S = [\frac{m}{2^k}, \frac{m+1}{2^k}] + [\frac{n}{2^k}, \frac{n+1}{2^k}]i \in \mathcal{S}_k$, denote its union with all adjacent dyadic squares by $\hat{S} = [\frac{m-1}{2^k}, \frac{m+2}{2^k}] + [\frac{n-1}{2^k}, \frac{n+2}{k}]i$. Denote

$$m_S = \min_{\hat{S}} f \quad \text{and} \quad M_S = \max_{\hat{S}} f. \tag{2.5.29}$$

In Step 2, we established (2.5.27) when $f = 0$. By Remark 2.5.2, $\mu_{\hat{h}}(\partial S) = 0$ a.s. for all $S \in \bigcup_{k \geq 1} \mathcal{S}_k$. Also note $\lim_{\varepsilon \to 0} \mathfrak{b}_{\varepsilon \exp(-\xi M_S)} / \mathfrak{b}_\varepsilon = e^{\gamma M_S}$ from (2.1.10). Given any sequence of ε decreasing to 0, we can choose a subsequence ε_n such that almost surely,

$$e^{\gamma M_S} \mu_{\hat{h}}(S) = \lim_{n \to \infty} \mathfrak{b}_{\varepsilon_n}^{-1} N_{\varepsilon_n \exp(-\xi M_S)}(S; D_{\hat{h}}) \quad \text{for all } S \in \bigcup_{k \geq 1} \mathcal{S}_k. \tag{2.5.30}$$

For $S = [\frac{m}{2^k}, \frac{m+1}{2^k}] + [\frac{n}{2^k}, \frac{n+1}{2^k}]i \in \mathcal{S}_k$ and $j \in \mathbb{N}$, denote

$$S_j := \left[\frac{m}{2^k} + \frac{1}{2^{k+j}}, \frac{m+1}{2^k} - \frac{1}{2^{k+j}}\right] + \left[\frac{n}{2^k} + \frac{1}{2^{k+j}}, \frac{n+1}{2^k} - \frac{1}{2^{k+j}}\right] i. \tag{2.5.31}$$

We again have $\mu_{\hat{h}}(\partial S_j) = 0$ by Remark 2.5.2. Take a further subsequence of ε_n such that, almost surely,

$$e^{\gamma m_S} \mu_{\hat{h}}(S_j) = \lim_{n\to\infty} \mathfrak{b}_{\varepsilon_n}^{-1} N_{\varepsilon_n \exp(-\xi M_S)}(S_j; D_{\hat{h}}) \quad \text{for all } S \in \bigcup_{k\geq 1} \mathcal{S}_k,\ j \in \mathbb{N}. \tag{2.5.32}$$

By the Weyl scaling axiom (2.2.13), for each $S \in \mathcal{S}_k$ and $j \in \mathbb{N}$, we almost surely have

$$N_{\varepsilon \exp(-\xi m_S)}(S_j; D_{\hat{h}}) \leq N_\varepsilon(S_j; D_h) \leq N_\varepsilon(S; D_h) \leq N_{\varepsilon \exp(-\xi M_S)}(S; D_{\hat{h}}) \tag{2.5.33}$$

for all sufficiently small $\varepsilon > 0$. (The threshold is random; it depends on $\hat{h}$ and f.) On one hand, from (2.5.30) and (2.5.33), we almost surely have

$$\begin{aligned} \sum_{S\in\mathcal{A}^k} e^{\gamma M_S} \mu_{\hat{h}}(S) &= \sum_{S\in\mathcal{A}^k} \lim_{n\to\infty} \mathfrak{b}_{\varepsilon_n}^{-1} N_{\varepsilon_n \exp(-\xi M_S)}(S; D_{\hat{h}}) \geq \sum_{S\in\mathcal{A}^k} \lim_{n\to\infty} \mathfrak{b}_{\varepsilon_n}^{-1} N_{\varepsilon_n}(S; D_h) \\ &= \lim_{n\to\infty} \mathfrak{b}_{\varepsilon_n}^{-1} \sum_{S\in\mathcal{A}^k} N_{\varepsilon_n}(S; D_h) \geq \limsup_{n\to\infty} \mathfrak{b}_{\varepsilon_n}^{-1} N_{\varepsilon_n}(A; D_h). \end{aligned} \tag{2.5.34}$$

On the other hand, for each k and j, we almost surely have

$$\sum_{S\in\mathcal{A}_k} N_\varepsilon(S_j; D_h) \leq N_\varepsilon(A; D_h) \tag{2.5.35}$$

for all sufficiently small $\varepsilon > 0$, since $\inf_{S,\tilde{S}\in\mathcal{S}_k, S\neq\tilde{S}} D_h(S_j, \tilde{S}_j) > 0$. Combining this with (2.5.32) and (2.5.33), we almost surely have

$$\begin{aligned}\sum_{S\in\mathcal{A}_k} e^{\gamma m_S}\mu_{\hat{h}}(S_j) &= \sum_{S\in\mathcal{A}_k} \lim_{n\to\infty} \mathfrak{b}_{\varepsilon_n}^{-1} N_{\varepsilon_n \exp(-\xi m_S)}(S_j; D_{\hat{h}}) \\ &\le \sum_{S\in\mathcal{A}_k} \lim_{n\to\infty} \mathfrak{b}_{\varepsilon_n}^{-1} N_{\varepsilon_n}(S_j; D_h) \\ &= \lim_{n\to\infty} \mathfrak{b}_{\varepsilon_n}^{-1} \sum_{S\in\mathcal{A}_k} N_{\varepsilon_n}(S_j; D_h) \le \liminf_{n\to\infty} \mathfrak{b}_{\varepsilon_n}^{-1} N_{\varepsilon_n}(A; D_h).\end{aligned} \tag{2.5.36}$$

For each $S \in \mathcal{S}_k$, since $\mu_{\hat{h}}(\partial S) = 0$, we have $\lim_{j\to\infty} \mu_{\hat{h}}(S_j) = \mu_{\hat{h}}(S)$ almost surely. Combining (2.5.34) and (2.5.36) and letting $j \to \infty$, we almost surely have that

$$\begin{aligned}\sum_{S\in\mathcal{A}_k} e^{\gamma m_S}\mu_{\hat{h}}(S) &\le \liminf_{n\to\infty} \mathfrak{b}_{\varepsilon_n}^{-1} N_{\varepsilon_n}(A; D_h) \\ &\le \limsup_{n\to\infty} \mathfrak{b}_{\varepsilon_n}^{-1} N_{\varepsilon_n}(A; D_h) \le \sum_{S\in\mathcal{A}^k} e^{\gamma M_S}\mu_{\hat{h}}(S).\end{aligned} \tag{2.5.37}$$

Since $\mathcal{A}^k \setminus \mathcal{A}_k = \{S \in \mathcal{S}_k : S \cap \partial A \neq \varnothing\}$, it follows from the equivalence between (2.5.1) and (2.5.2) that $\lim_{k\to\infty} \sum_{S\in\mathcal{A}^k\setminus\mathcal{A}_k} \mu_{\hat{h}}(S) = 0$. In addition,

$$M := \sup_{k\in\mathbb{N}} \max_{S\in\mathcal{S}_k} M_S \le \sup_{B_{r+1}(0)} f < \infty \quad \text{and} \quad \lim_{k\to\infty} \max_{S\in\mathcal{S}_k} (\exp(\gamma M_S) - \exp(\gamma m_S)) = 0 \tag{2.5.38}$$

almost surely because f is and uniformly continuous on $B_R(0)$. Hence, the right-hand side of the inequality

$$\begin{aligned}&\sum_{S\in\mathcal{A}^k} e^{\gamma M_S}\mu_{\hat{h}}(S) - \sum_{S\in\mathcal{A}_k} e^{\gamma m_S}\mu_{\hat{h}}(S) \\ &\le e^{\gamma M} \sum_{S\in\mathcal{A}^k\setminus\mathcal{A}_k} \mu_{\hat{h}}(S) + \sum_{S\in\mathcal{A}_k} (e^{\gamma M_S} - e^{\gamma m_S})\mu_{\hat{h}}(S).\end{aligned} \tag{2.5.39}$$

converges almost surely to 0 as $k \to \infty$. By this together with (2.5.37), since ε_n is a subsequence of an arbitrary sequence of ε decreasing to 0,

$$\lim_{k\to\infty}\sum_{S\in\mathcal{A}_k} e^{\gamma m_S}\mu_{\hat{h}}(S) = \lim_{\varepsilon\to 0}\mathfrak{b}_\varepsilon^{-1}N_\varepsilon(A;D_h) = \lim_{k\to\infty}\sum_{S\in\mathcal{A}^k} e^{\gamma M_S}\mu_{\hat{h}}(S) \quad \text{in probability.} \tag{2.5.40}$$

On the other hand, since $\mu_h(\partial S) = 0$ and thus $\mu_h = \mu_{\hat{h}+f}$ is finitely additive for $S \in \mathcal{S}_k$,

$$\sum_{S\in\mathcal{A}_k} e^{\gamma m_S}\mu_{\hat{h}}(S) \le \sum_{S\in\mathcal{A}_k}\mu_h(S) \le \mu_h(A) \le \sum_{S\in\mathcal{A}^k}\mu_h(S) \le \sum_{S\in\mathcal{A}^k} e^{\gamma M_S}\mu_{\hat{h}}(S) \tag{2.5.41}$$

almost surely. We found in (2.5.40) the leftmost and the rightmost terms of this inequality converge to the same limit in probability as $k\to\infty$, which must be equal to $\mu_h(A)$ almost surely. Therefore, we conclude that

$$\mu_h(A) = \lim_{\varepsilon\to 0}\mathfrak{b}_\varepsilon^{-1}N_\varepsilon(A;D_h) \tag{2.5.42}$$

in probability.

4. *The field h is any whole-plane GFF plus a continuous function and $A \subset \mathbb{C}$ is any random bounded Borel set.*

   Let E_k be the event $\{A \subset B_k(0)\}$. Given any sequence of ε decreasing to 0, using a diagonal argument we can find a subsequence ε_n such that $\lim_{n\to\infty}\mathfrak{b}_{\varepsilon_n}^{-1}N_{\varepsilon_n}(A;D_h) = \mu_h(A)$ a.s. on E_k for all integer k. Since $\mathbb{P}(\bigcup_{k\in\mathbb{N}}E_k) = 1$, we conclude that the limit $\lim_{n\to\infty}\mathfrak{b}_\varepsilon^{-1}N_\varepsilon(A;D_h) = \mu_h(A)$ holds in probability.

This completes the proof of (2.1.7) for general Gaussian field h and set A. $\square$

We finally show that the pointed metric space $(\mathbb{C},0,D_h)$ almost surely determines the marked quantum surface $(\mathbb{C},h,0)$.

Proof of Corollary 2.1.2. Let $\mathbb{L}$ (resp. $\mathbb{M}$) be the space of isometry classes of complete, locally compact length spaces with a marked point (resp. a marked point and a locally finite Borel measure), endowed with the local Gromov–Hausdorff (resp. local Gromov–Hausdorff–Prokhorov) topology. We consider $\mathbb{L}$ (resp. $\mathbb{M}$) as a *complete* probability space endowed with the probability measure $\mathbb{P}_h^{\mathbb{L}}$ (resp. $\mathbb{P}_h^{\mathbb{M}}$) corresponding to $(\mathbb{C}, 0, D_h)$ (resp. $(\mathbb{C}, 0, D_h, \mu_h)$). Let $f : \mathbb{M} \to \mathbb{L}$ be the natural projection. We claim that f is injective on a set $E \subset \mathbb{M}$ with $\mathbb{P}_h^{\mathbb{M}}(E) = 1$. If so, since $\mathbb{M}$ is a Polish space [1], by the measurable selection theorem (see, e.g., [54, Theorem 6.9.1]), there is a measurable function $g : \mathbb{L} \to \mathbb{M}$ such that $g \circ f$ is the identity map on E. This means that $(\mathbb{C}, 0, D_h)$ almost surely determines $(\mathbb{C}, 0, D_h, \mu_h)$.

To this end, consider a sequence $A_1, A_2, \ldots$ of measurable functions from continuous metrics on $\mathbb{C}$ to compact subsets of $\mathbb{C}$. Let $\Pi(D) := \{A_1(D), A_2(D), \ldots\}$ be the collection of these sets without ordering: i.e., Π is a function from the set of continuous metrics on $\mathbb{C}$ to the power set of compact subsets of $\mathbb{C}$. For now, we prescribe the following set of properties that these functions should satisfy; we shall construct an explicit sequence of such maps at the end of the proof.

- Almost surely, $\mu_h(\partial A_j(D_h)) = 0$ for every j.
- Almost surely, $\Pi(D_h)$ is a π-system which generates the Borel σ-algebra on $\mathbb{C}$.
- If D and D' are continuous metrics on $\mathbb{C}$ such that $\phi : (\mathbb{C}, D) \to (\mathbb{C}, D')$ is an isometry preserving 0, then $\phi(\Pi(D)) = \Pi(D')$.

By Theorem 2.1.1 and a standard diagonalization argument, the first property implies that there is a sequence ε_n decreasing to 0 such that, almost surely,

$$\lim_{n\to\infty} \mathfrak{b}_{\varepsilon_n}^{-1} N_{\varepsilon_n}(A_j(D_h); D_h) = \mu_h(A_j(D_h)) \quad \text{for every } j. \tag{2.5.43}$$

Hence, there exists a Borel subset $\widetilde{E}$ of the product space of continuous metrics on $\mathbb{C}$ and Borel measures on $\mathbb{C}$ with $\mathbb{P}\{(D_h, \mu_h) \in \widetilde{E}\} = 1$ on which the first two bulleted properties as well as (2.5.43) hold. By the π-λ theorem, if two Borel measures agree on a π-system, then they must be identical. Hence, the projection $(D, \mu) \mapsto D$, where D is a continuous metric on $\mathbb{C}$ and μ is a Borel measure on $\mathbb{C}$, is injective on $\widetilde{E}$.

Let E be the image of $\widetilde{E}$ under the "forget the embedding" map: i.e., $(D, \mu) \mapsto (\mathbb{C}, 0, D, \mu)$ where the latter is a pointed metric measure space.[8] Let $(D, \mu), (D', \mu') \in \widetilde{E}$ be any two elements which are mapped into the same pointed metric space $(\mathbb{C}, 0, D) = (\mathbb{C}, 0, D')$ under the natural embedding $(D, \mu) \mapsto (\mathbb{C}, 0, D)$. That is, there is an isometry $\phi : (\mathbb{C}, D) \to (\mathbb{C}, D')$ fixing 0. By the last bulleted property, $\Pi(D') = \phi(\Pi(D))$ and

$$\mu'(\phi(A)) = \lim_{n\to\infty} \mathfrak{b}_{\varepsilon_n}^{-1} N_{\varepsilon_n}(\phi(A); D') = \lim_{n\to\infty} \mathfrak{b}_{\varepsilon_n}^{-1} N_{\varepsilon_n}(A; D) = \mu(A) \tag{2.5.44}$$

for every $A \in \Pi(D)$. Since $\Pi(D')$ is a π-system, we conclude that $\mu' = \phi_* \mu$. That is, the pointed metric measure spaces $(\mathbb{C}, 0, D, \mu)$ and $(\mathbb{C}, 0, D', \mu')$ are identical. Therefore, the natural projection $f : \mathbb{M} \to \mathbb{L}$, $(\mathbb{C}, 0, D, \mu) \mapsto (\mathbb{C}, 0, D)$ is injective on E.

It remains to show the existence of measurable maps $A_1, A_2, \ldots$ with the bulleted properties. For each continuous metric D on $\mathbb{C}$, define

$$\mathcal{Z}(D) = \{z \in \mathbb{C} : \text{there are exactly three distinct } D\text{-geodesics from } 0 \text{ to } z\}. \tag{2.5.45}$$

Let $\mathcal{K}(D)$ be the collection of all compact $K \subset \mathbb{C}$ that are of the following form:

1. Let $z_1, z_2, \ldots, z_n$ be any finite subset of $\mathcal{Z}(D)$;
2. For $k = 1, \ldots, n$, let η_k be any D-geodesic between z_{k-1} and z_k (where $z_0 = z_n$);
3. Let $\eta = \eta_1 \cup \eta_2 \cup \cdots \cup \eta_n$ be the closed curve formed by concatenating the geodesics;
4. Let K be the union of η and all bounded components of $\mathbb{C} \setminus \eta$.

8. It is straightforward to check that the natural embedding $(D, \mu) \mapsto (\mathbb{C}, 0, D, \mu)$ is a continuous map whose domain is a Polish space, so E is an analytic set. Since $\mathbb{P}_h^{\mathbb{M}}$ is a complete probability measure, E is $\mathbb{P}_h^{\mathbb{M}}$-measurable (see, e.g., [153, Theorem 21.10]).

Finally, let $\Pi(D)$ consist of any finite intersections of sets in $\mathcal{K}(D)$. These sets are defined only using the pointed metric structure $(\mathbb{C}, 0, D)$ so that if $\phi : (\mathbb{C}, D) \to (\mathbb{C}, D')$ is an isometry fixing 0, then $\phi(\mathcal{Z}(D)) = \mathcal{Z}(D')$, $\phi(\mathcal{K}(D)) = \mathcal{K}(D')$, and $\phi(\Pi(D)) = \Pi(D')$.

For the γ-LQG metric D_h, almost surely, the set $\mathcal{Z}(D_h)$ is countable and dense [118, Theorem 1.2], and any two points in $\mathcal{Z}(D_h)$ are joined by finitely many D_h-geodesics [118, Theorem 1.7]. Moreover, since D_h is a.s. a continuous metric, we can a.s. find for every $z \in \mathbb{Q}^2$ and $r \in \mathbb{Q}_{>0}$ a set $K \in \mathcal{K}(D_h)$ such that K contains the Euclidean ball $B_r(z)$ and is contained in the Euclidean ball $B_{2r}(z)$. Hence, $\Pi(D_h)$ is a.s. a countable π-system generating the Borel σ-algebra on $\mathbb{C}$. Using the measurable selecction theorem inductively, we can find a sequence of measurable functions $A_1, A_2, \dots$ such that $\Pi(D) = \{A_1(D), A_2(D), \dots\}$ almost surely.

Finally, we claim that the quantum Minkowski dimension of $\partial A_j(D_h)$ is a.s. no more than 1 for every j. Since D_h is a continuous metric on $\mathbb{C}$, the lengths of D_h-geodesics between all points in $\mathcal{Z}(D_h)$ (which are equal to the D_h-distances between these points) are finite. Because $(\mathbb{C}, D_h)$ is a length space, a curve η with D_h-length $L < \infty$ satisfies $N_\varepsilon(\eta; D_h) \leq L/\varepsilon$. This is since if η is parameterized by D_h-length, then $\{\mathcal{B}_\varepsilon(P(\varepsilon k); D_h) : k = 1, 2, \dots, \lfloor L/\varepsilon \rfloor\}$ covers η. For every j, since $\partial A_k(D_h)$ is a finite union of D_h-geodesics between points in $\mathcal{Z}(D_h)$, its LQG Minkowski dimension is at most 1 as claimed. Therefore, by Remark 2.5.1, we have that $\mu_h(\partial A_j(D_h)) = 0$ a.s. for every j.

As these functions $A_1, A_2, \dots$ satisfy all three bulleted properties stated above, we conclude that $(\mathbb{C}, 0, D_h)$ a.s. determines $(\mathbb{C}, 0, D_h, \mu_h)$. The rest of the corollary follows since the pointed metric measure space $(\mathbb{C}, 0, D_h, \mu_h)$ a.s. determines the field h up to rotation and scaling of the complex plane centered at the origin [16, Theorem 1.3]. $\square$

CHAPTER 3

AREA MEASURES AND BRANCHED POLYMERS IN SUPERCRITICAL LIOUVILLE QUANTUM GRAVITY

This chapter presents material first announced in [44].

3.1 Introduction

Liouville quantum gravity (LQG) is a canonical one-parameter family of random geometries on an orientable surface with given topology. Conceived by Polyakov [206], LQG was studied initially in the physics literature in the context of string theory and two-dimensional quantum gravity and since has become an active area of mathematical research.

LQG is indexed by the parameter $\mathbf{c}_{\mathrm{L}} > 1$ called the (Liouville) central charge. (Some works on LQG alternatively consider the matter central charge $\mathbf{c}_{\mathrm{M}} = 26 - \mathbf{c}_{\mathrm{L}}$.) Heuristically speaking, an LQG surface of given topology is described by a random metric g sampled from the "uniform measure on the space of Riemannian metric tensors weighted by $(\det \Delta_g)^{-(26-\mathbf{c}_{\mathrm{L}})/2}$," where Δ_g is the Laplace–Beltrami operator of the surface with the metric g. While this definition does not make literal sense, there are various ways to define LQG rigorously that we discuss below. These definitions utilize alternative parameters, with common choices being the background charge $Q > 0$ and the coupling constant $\gamma \in (0,2] \cup \{z \in \mathbb{C} : |z| = 2\}$ related by the formulae

$$\mathbf{c}_{\mathrm{L}} = 1 + 6Q^2 \quad \text{and} \quad Q = \frac{2}{\gamma} + \frac{\gamma}{2}. \tag{3.1.1}$$

The geometric properties of an LQG surface depend heavily on the value of its central charge, which can be classified into subcritical, critical, and supercritical phases as summarized in Table 3.1. In the physics literature, LQG is said to be *weakly coupled* in the subcritical and

	Central charge $\mathbf{c}_{\mathrm{L}}$	Background charge Q	Coupling constant γ
Subcritical	$(25, \infty)$	$(2, \infty)$	$(0, 2)$
Critical	25	2	2
Supercritical	$(1, 25)$	$(0, 2)$	$\gamma \in \mathbb{C} \setminus \mathbb{R}$ with $\lvert\gamma\rvert = 2$

Table 3.1: The phases of LQG and their parameter ranges

critical phases ($\mathbf{c}_{\mathrm{L}} \geq 25$) and *strongly coupled* in the supercritical phase ($\mathbf{c}_{\mathrm{L}} \in (1, 25)$). In this paper, we are primarily interested in the latter case.

Over the past two decades, the subcritical and critical phases of LQG have been studied at length in the mathematical community. In these cases, the so-called DDK ansatz [80, 91] states that, when defined on a domain $U \subseteq \mathbb{C}$, the metric tensor associated with LQG should take the form

$$g = e^{\gamma h}(dx^2 + dy^2) \tag{3.1.2}$$

where $dx^2 + dy^2$ is the Euclidean metric tensor and h is a variant of the Gaussian free field (GFF) on U. (We assume that the reader is familiar with the GFF and we refer to the surveys [225, 251, 38] for detailed introductions to it.) A GFF is rigorously defined only as a random generalized function and thus the metric tensor (3.1.2) does not make a priori sense. Nevertheless, the geometry associated with (3.1.2) can be made rigorous through a renormalization procedure approximating h by a sequence of continuous functions. Indeed, as the limit of such approximations, the LQG area measure was constructed in [99] as a Gaussian multiplicative chaos measure (see also [146, 212]). Furthermore, the distance function associated with LQG (or simply, the LQG metric) was constructed in a series of works culminating in [85, 129] again by a renormalization procedure involving smooth approximations to the GFF h. We also note the vast literature on the deep connections between LQG in the subcritical and critical phases and various mathematical objects including Schramm–Loewner evolution (SLE), random planar maps, conformal field theory, and random permutations; we direct the reader to the surveys [117, 123, 229, 38, 115] as starting points.

3.1.1 *Physics predictions on the behavior of supercritical LQG and the main contributions of this paper*

In the context of string theory, it is the supercritical phase of LQG which is expected to be physically relevant: indeed, the regime $\mathbf{c}_{\mathrm{L}} \in \{1, \ldots, 25\}$ is the one that was originally considered by Polyakov [206] to model the worldsheets swept out by strings moving in $\mathbb{R}^{26-\mathbf{c}_{\mathrm{L}}}$. The reader may refer to [18, Remark 1.4] for further explanation on the above-mentioned string theory connection. Moreover, given the recent works [68, 143, 66, 67, 182, 183, 62] which have shown that Wilson loop observables in lattice gauge theory can be represented as sums over surfaces in $\mathbb{R}^d$, it is plausible that supercritical LQG may be relevant to Yang–Mills theory. To be specific, the surfaces appearing in the surface-sum representation for $U(N)$ lattice gauge theory in the $N \to \infty$ limit might be related to supercritical LQG; we note such connections are still speculative and refer the reader to [62, Section 7] for more on this.

However, when compared to the subcritical and critical phases, the supercritical phase of LQG has stayed much more mysterious. There are a number of works in the physics literature painting the underlying geometry of supercritical LQG in different lights.

(a) One line of thought (e.g., [223], see also [112, Figure 10]) suggests that when extending the conformal field theory to the regime $\mathbf{c}_{\mathrm{L}} \in (1, 25)$, the geometry of the surface is torn apart by "infinite spikes" at a dense set of points. In fact, this picture of supercritical LQG is consistent with the mathematical literature [121, 87, 88, 205, 18], where it is represented as a random geometry constructed from a variant of the GFF with a dense set of singular points which are at infinite distance from all other points.

(b) A drastically different viewpoint is taken in the works [100, 13, 64, 59, 30, 65, 14, 81], where, by often using numerical and heuristic analyses of random planar maps weighted by $(\det \Delta_{\mathrm{graph}})^{(\mathrm{c}_{\mathrm{L}}-26)/2}$ as an approximation of LQG, they suggest that LQG in the supercritical regime should look like a continuum random tree, or as it is often called in the physics literature, a branched polymer.

(c) Yet another picture arises in the works of Gervais et al. [108, 109, 110, 111, 74, 105, 106, 107], where, based on an algebraic study of the path integral formulation of LQG, it is claimed that there are some special integrable algebraic structures associated with supercritical LQG at the specific values $\mathbf{c}_{\mathrm{L}} = 7, 13, 19$. Moreover, for these values of $\mathbf{c}_{\mathrm{L}}$, these authors define finite local operators which gives rise to a so-called "area element" (e.g., (4.3) in [110]), which could conceivably be interpreted as a measure associated with LQG.

(d) There are also purely algebraic approaches to supercritical LQG based on conformal field theory. See, e.g., [214] for an introduction to this approach.

While the branched polymer picture seems to have the most support in the physics literature, there have been no mathematically rigorous results in support of this view prior to this work. In the works [121, 21], a possible reconciliation of the branched polymer and the infinite spikes pictures was suggested. Namely, it was suggested that, while a supercritical LQG surface a priori has infinite spikes at a dense set of points, it degenerates to a branched polymer when "conditioned" on the zero-probability event of having no infinite spikes. However, previous to our work, it was unclear whether this heuristic explanation is correct. Furthermore, it was not clear how to make this idea rigorous, especially given that we need to condition on the zero-probability event of a supercritical LQG surface having no infinite spikes at all.

Our goal in this work is to present a rigorous formulation of a supercritical LQG surface reconciling the two viewpoints ((a)) and ((b)). Moreover, in disagreement with the claim in ((c)), we establish that it is not possible to construct a measure associated with supercritical LQG for any value of $\mathbf{c}_{\mathrm{L}} \in (1, 25)$: this was unclear from the previous mathematical literature (see [121, Questions 6.1–2]). We now give a quick overview of the main results of this paper.

1. We establish a link between the branched polymer and infinite spikes pictures via a discrete model of supercritical LQG. We consider a model of random planar maps with boundary (see Section 3.1.2) which can be expected to converge to supercritical LQG in the scaling limit (see Theorem 3.1.2). While a map in this model has a finite perimeter $p \in \mathbb{N}$, it has infinitely many vertices (and thereby "infinite spikes") with positive probability. In fact, the probability of the map with perimeter p being infinite increases to 1 as $p \to \infty$ (Proposition 3.1.3). We establish that upon *conditioning* this planar map on the rare event that it contains finitely many vertices, we obtain a continuum random tree in the scaling limit as $p \to \infty$ (Theorem 3.1.4). The physics papers referenced in ((b)) considered *finite* planar maps (e.g., with a fixed number of edges), so they were implicitly conditioning the surface to be finite.

2. One interpretation of the "area element" described in ((c)) above is that for the values $\mathbf{c}_{\mathrm{L}} = 7, 13, 19$, a supercritical LQG surface has a locally finite volume form associated to it. In other words, given a GFF h, one would expect to find a locally finite Borel measure $\mathfrak{m}_h$ which is locally determined by and compatible with the geometry of a supercritical LQG surface associated with h. Nonetheless, in Theorem 3.1.5, we show that there does not exist any natural notion of a locally finite and locally determined volume measure for the entire range $\mathbf{c}_{\mathrm{L}} \in (1, 25)$ of supercritical LQG.

The central idea of our proofs is to exploit the multiplicative cascade structure of supercritical LQG, which comes from its coupling with nested CLE_4 [18]. This is closely related to the scaling limit of the analogous structure for the loop $O(n)$ model identified in [70] based on the gasket decomposition of [57]. The crucial difference is that in our model, even in the discrete version introduced in the next subsection, there are infinitely many loops of macroscopic length. This is the first paper to use multiplicative cascades to prove properties of objects related to supercritical LQG. We expect that this idea will have many additional applications in the future.

We now introduce our results in further detail, first giving a precise description of our random planar map model of supercritical LQG. This is followed by statements of the main results of this paper and the ideas of their proofs.

3.1.2 A discrete model of supercritical LQG

As mentioned above, we give a rigorous meaning to "conditioning a supercritical LQG surface to have no infinite spikes" by considering a discrete model in the universality class of supercritical LQG, for which there is a natural notion of conditioning it to have finite size. In particular, we consider a random loop-decorated planar map which is a discrete analog of the supercritical LQG disk decorated with a conformal loop ensemble of parameter $\kappa = 4$ (CLE_4) as constructed recently in [18]; see Section 3.2 for an introduction to this continuum model. This discrete model is a variant of planar maps decorated by the loop $O(2)$ model where we force the loops to have discrepancies between their outer and inner lengths. This is a slight modification of the model suggested in [18], which is constructed iteratively from the gasket decomposition of the loop $O(n)$ model [57]. We refer the reader to Remark 3.4.5 for a further discussion on the relations between these discrete models.

We begin with a few basic definitions. A planar map is a planar graph embedded into the Riemann sphere considered up to orientation-preserving homeomorphisms. Our planar map M will have a designated root face, which we view as the "outside" of a planar map which has the "topology" of a disk. The boundary ∂M of the map M is the subgraph of M consisting of vertices and edges incident to the root face. The non-root faces of M are its interior faces, denoted as $\mathfrak{F}(M)$. Given a face f, we use $\mathrm{per}(f)$ to denote the half-perimeter of f. That is, $2\,\mathrm{per}(f)$ is equal to the number of edges incident to f, where an edge is counted twice if it is incident to f on both of its sides (see, e.g., the root face f in Figure 3.2(a) with $\mathrm{per}(f) = 13$).

We shall construct the model inductively, with two types of random planar maps inserted into the existing planar map one after another.

- The first kind is a rooted bipartite[1] Boltzmann map conditioned on a fixed boundary length, which should be understood as the discrete counterpart of the CLE_4 gasket. We fix a critical non-generic weight sequence $\mathbf{q} = (q_i)_{i\in\mathbb{N}}$ of type $a = 2$, and choose our maps from the probability measure $\mathbb{P}_{\mathbf{q}}^{(p)}$ on rooted bipartite planar maps with boundary perimeter $2p$. For example, we can take $\mathbf{q}$ to be a weight sequence corresponding to the gasket of a version of the critical $O(2)$ loop model-decorated quadrangulation (see the case $g = h/2$ in [6, Appendix B]).

  We define these precisely in Section 3.4.1, and at this point we simply note that a Boltzmann map sampled from the law $\mathbb{P}_{\mathbf{q}}^{(p)}$ is expected to converge in the $p \to \infty$ scaling limit to the gasket of a CLE_4 on an independent critical LQG disk. (See Lemma 3.4.14 for a rigorous result on the convergence of perimeters of faces to length of CLE_4 loops in support of this). We also emphasize that these Boltzmann maps are not required to have simple boundaries; for example, we may sample from $\mathbb{P}_{\mathbf{q}}^{(p)}$ a tree with p edges, which, we note, has no interior faces ($\mathfrak{F}(M) = \varnothing$).

- The second kind of random planar map that we need is a "ring" with two distinguished faces — an outer face f_{out} and an inner face f_{in} — such that both f_{out} and f_{in} have even degrees. See Figure 3.1 for an example of a ring. These rings are inserted into each interior face of a Boltzmann map. They are analogous to the loops in the loop $O(n)$ model (see [57]) on the dual map, where the two faces of a ring correspond to the inside and outside of a loop.

  Let $Q \in (0, 2)$ be the background charge associated with the given universality class of

1. A graph is called bipartite if all of its faces have even degrees. We choose to work with bipartite Boltzmann maps so as to utilize the random walk bijections of [58] and [144] (see Proposition 3.4.7). We expect our results to hold in a greater generality without assuming the planar maps to be bipartite.

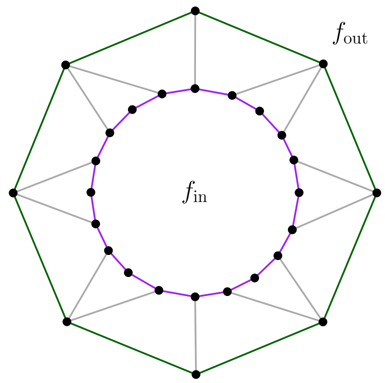

Figure 3.1: An example of a ring sampled from $\mathbb{P}^{(4)}_{\text{ring}}$ with the outer boundary colored in green and the inner boundary in purple. For this ring, $\text{per}(f_{\text{out}}) = 4$ and $\text{per}(f_{\text{in}}) = 10$.

supercritical LQG and define

$$\beta_Q := \frac{\pi\sqrt{4-Q^2}}{Q}. \tag{3.1.3}$$

We fix a sequence of probability measures $\left(\mathbb{P}^{(p)}_{\text{ring}}\right)_{p\in\mathbb{N}}$ of rings such that under $\mathbb{P}^{(p)}_{\text{ring}}$, we have $\text{per}(f_{\text{out}}) = p$ a.s. and the ratio $\text{per}(f_{\text{in}})/\text{per}(f_{\text{out}})$ is approximately equal in distribution to $\exp(\beta_Q Y)$ where Y is a Rademacher random variable. This is the law of the ratio of inner and outer LQG lengths of CLE_4 loops on supercritical LQG in the setting of [18] (see (3.2.11)).

For concreteness, we can choose the law of $\text{per}(f_{\text{in}})$ under $\mathbb{P}^{(p)}_{\text{ring}}$ to be equal to the law of $\lfloor p\exp(\beta_Q Y)\rfloor$, and let $\mathbb{P}^{(p)}_{\text{ring}}$ to be supported on rings where all vertices are on the boundary of either the inner or outer face and all faces other than f_{out} and f_{in} are quadrilaterals. However, we can allow a more general class of distributions of rings, and this is specified later in Definition 3.4.4.

With these building blocks at hand, we now iteratively construct a discrete model of supercritical LQG. See Figure 3.2 for an illustration of the following construction.

Definition 3.1.1. Fix $p \in \mathbb{N}$. Let $\mathbb{P}^{(p)}_{\infty}$ denote the law of the Markov chain $\{(M_i, \mathfrak{F}_i)\}_{i\in\mathbb{N}\cup\{0\}}$ defined below, where M_i is a rooted bipartite planar map whose root face has degree $2p$ and $\mathfrak{F}_i \subseteq \mathfrak{F}(M_i)$. As can be seen below, $\mathfrak{F}_i$ denotes the set of interior faces of M_i where Boltzmann

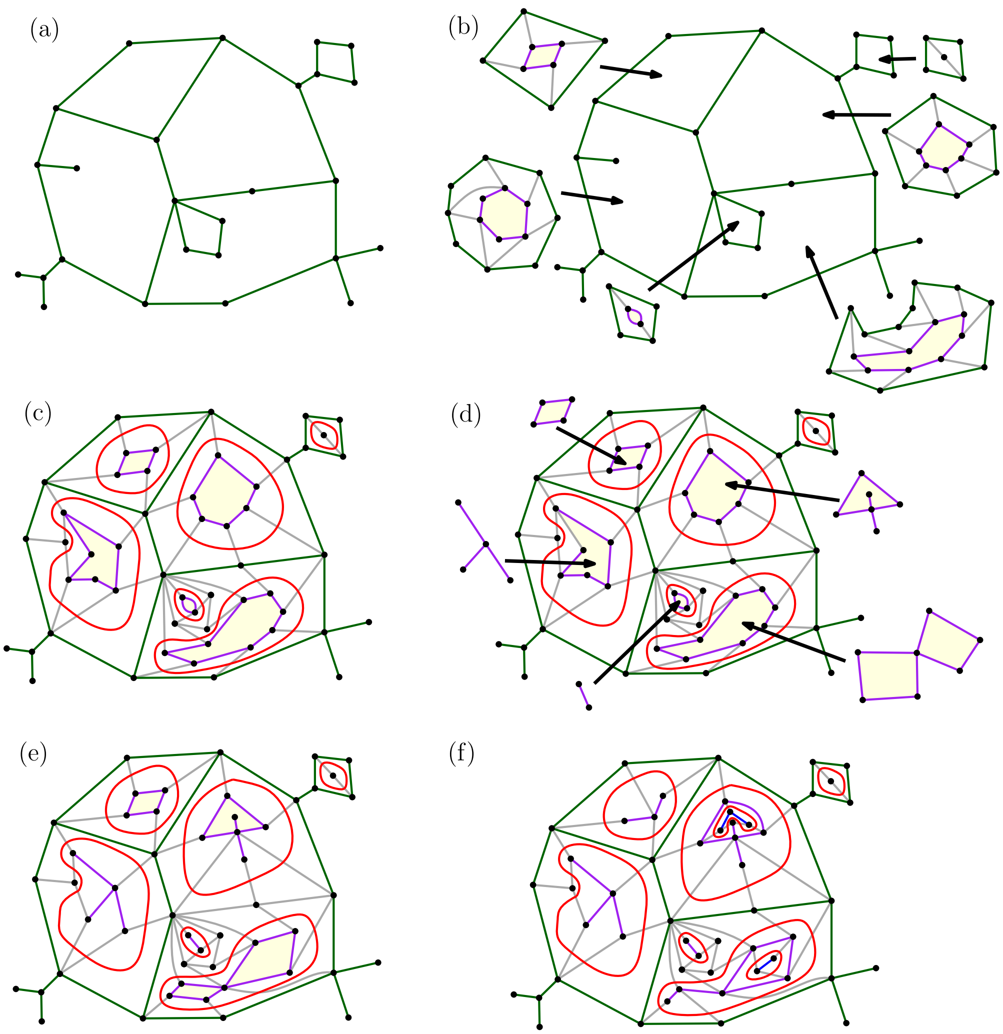

Figure 3.2: An illustration of the iterative construction in Definition 3.1.1 with $p = 13$. (a) The outermost gasket M_0 is sampled from $\mathbb{P}_{\mathbf{q}}^{(p)}$. The collection $\mathfrak{F}_0$ comprises the inner faces of M_0. (b) Given M_0, a ring $R(f)$ is sampled conditionally independently for each face $f \in \mathfrak{F}_0$. The outer/inner boundaries of the rings are colored in green/purple, respectively. The ring in the top-right of the figure has inner perimeter zero. (c) The rings are attached to the corresponding faces, with the possible rotations chosen uniformly. The rings are identified with red loops that separate inner and outer boundaries. These loops are the discrete analogs of CLE_4 loops in the coupling of supercritical LQG disk with CLE_4. (d) Given the previous figure, for each ring $R(f)$, sample conditionally independent Boltzmann maps from $\mathbb{P}_{\mathbf{q}}^{\mathrm{per}_{\mathrm{in}}(f)}$ where $\mathrm{per}_{\mathrm{in}}(f)$ is the inner half-perimeter of the ring $R(f)$. (e) The Boltzmann maps are glued to the inner boundaries of the rings, with the possible rotations chosen uniformly. These comprise the map M_1 colored purple and their inner faces $\mathfrak{F}_1$ colored yellow. (f) The map after another iteration, with M_0 colored in green, M_1 in purple, and M_2 in blue. In this case, $\mathfrak{F}_2 = \varnothing$ as the Boltzmann maps comprising M_2 do not have non-root (inner) faces. The construction terminates at this stage, giving us a finite map.

maps are to be inserted in the subsequent step.

- We start with M_0 sampled from the law $\mathbb{P}_{\mathbf{q}}^{(p)}$ and set $\mathfrak{F}_0 = \mathfrak{F}(M_0)$.
- The induction step consists of two phases.
  (i) Given $(M_i, \mathfrak{F}_i)$, sample rings $\{R(f)\}_{f \in \mathfrak{F}_i}$ conditionally independently from the measure $\mathbb{P}_{\text{ring}}^{(\text{per}(f))}$. For each $f \in \mathfrak{F}_i$, glue the outer boundary of the ring $R(f)$ to the edges encircling f with a uniform choice among the possible rotations of $R(f)$ relative to f. Define the inner perimeter $\text{per}_{\text{in}}(f)$ of f to be that of $R(f)$ (i.e., $\text{per}(f_{\text{in}})$ with f_{in} being the inner face of $R(f)$).

  (ii) Conditioned on the rings $\{R(f)\}_{f \in \mathfrak{F}_i}$ glued to the marked faces $\mathfrak{F}_i \subseteq \mathfrak{F}(M_i)$ in the previous phase, for each $f \in \mathfrak{F}_i$, sample an independent Boltzmann map $M(f)$ from $\mathbb{P}_{\mathbf{q}}^{(\text{per}_{\text{in}}(f))}$. (If $\text{per}_{\text{in}}(f) = 0$, then we declare $M(f)$ to be a single point.) Glue its outer boundary to the inner boundary of the ring $R(f)$, again choosing uniformly among the possible rotations of $M(f)$ relative to the inner boundary of $R(f)$.

  The new map M_{i+1} is the one resulting from gluing in all of the rings $\{R(f)\}_{f \in \mathfrak{F}_i}$ and the Boltzmann maps $\{M(f)\}_{f \in \mathfrak{F}_i}$ into each $f \in \mathfrak{F}_i$ as described above. The new marked faces are $\mathfrak{F}_{i+1} := \bigcup_{f \in \mathfrak{F}_i} \mathfrak{F}(M(f))$.
- There is a positive probability that the above iterative construction terminates after a finite number of steps. For example, it could be that M_0 is a tree, in which case $\mathfrak{F}_0 = \varnothing$. Also, given $\mathfrak{F}_i \neq \varnothing$, we may have $\mathfrak{F}_{i+1} = \varnothing$ if for every $f \in \mathfrak{F}_i$, either the inner boundary of the ring $R(f)$ is a single vertex (that is, $\text{per}(f_{\text{in}}) = 0$) or $\mathfrak{F}(M(f)) = \varnothing$ (i.e., the Boltzmann map $M(f)$ is a tree). These events can happen with positive probabilities since for every $p \in \mathbb{N}$, the Boltzmann map with perimeter $2p$ has a positive chance of having no interior faces.

If $\mathfrak{F}_i = \varnothing$ for some $i \in \mathbb{N} \cup \{0\}$, we define $(M_j, \mathfrak{F}_j) := (M_i, \varnothing)$ for all $j \geq i$ and set $M := M_i$. In this case, we say that the resulting map M is finite, as it has finitely many vertices.

- Otherwise, we define $M = \bigcup_{i=0}^{\infty} M_0$, where we note that we have the natural inclusions $M_0 \subseteq M_1 \subseteq \cdots$. More formally, we define M to be the projective limit of M_i as $i \to \infty$. That is, M is the unique (infinite) planar map with an embedding of the whole chain $(M_i, \mathfrak{F}_i)_{i \in \mathbb{N} \cup \{0\}}$ into M such that every vertex and edge of M appears in some M_i under this embedding. We say that the resulting map M is infinite, as it has infinitely many vertices.

We shall refer to the map M_0 in the above process as the outermost gasket of the discrete supercritical map M. Also, for any $f \in \bigcup_{i \in \mathbb{N} \cup \{0\}} \mathfrak{F}_i$, we will use $M|_f$ to denote the submap of M inside the inner boundary of the ring $R(f)$. Finally, let $\mathbb{P}_{\mathtt{F}}^{(p)}$ be the probability measure obtained by conditioning $\mathbb{P}_{\infty}^{(p)}$ to output a finite map M.

3.1.3 Supercritical LQG disk conditioned to be finite is a branched polymer

Our first main result (Theorem 3.1.4) is that our discrete model of supercritical LQG disk given in Definition 3.1.1, conditioned on the event that it is finite, degenerates into a continuum tree as the perimeter of its boundary tends to infinity. We expect that a planar map sampled from $\mathbb{P}_{\infty}^{(p)}$, when scaled appropriately, would converge to a supercritical LQG surface with central charge $\mathbf{c}_{\mathrm{L}} = 1 + 6Q^2$. We establish this scaling limit for the loops appearing in our discrete and continuum models, observing that the joint law of the lengths of the discrete loops scaled by $1/p$ converges to the joint law of the supercritical LQG lengths of the CLE_4 loops. We refer the reader to [18, Section 3.2] for a further discussion on the relation between our discrete and continuum models of supercritical LQG.

Theorem 3.1.2 (See Proposition 3.4.13 for a precise statement)**.** *Consider the loop-decorated planar map sampled from the unconditioned law $\mathbb{P}_\infty^{(p)}$ defined in Definition 3.1.1. For each $n \in \mathbb{N}$, let $(\chi_i^n)_{i\in\mathbb{N}}$ be the inner half-perimeters of the nth generation rings $\{\mathrm{per}_{\mathrm{in}}(f) : f \in \mathfrak{F}_{n-1}\}$ arranged in a decreasing order and padded by zeros. Let $(Z_i^n)_{i\in\mathbb{N}}$ be the inner supercritical LQG lengths of the nth generation CLE_4 loops in the coupling of [18], arranged in a decreasing order. Then, as $p \to \infty$, we have*

$$(p^{-1}\chi_i^n)_{i,n\in\mathbb{N}} \xrightarrow{d} (Z_i^n)_{i,n\in\mathbb{N}} \tag{3.1.4}$$

jointly with respect to the product topology on $\mathbb{R}^{\mathbb{N}\times\mathbb{N}}$.

When we condition on the event that the map sampled from $\mathbb{P}_\infty^{(p)}$ is finite, we are looking at an event that becomes increasingly rarer as the outermost half-perimeter p tends to ∞. Using Theorem 3.1.2 as a key input, we show that this probability decays exponentially in p.

Proposition 3.1.3. *For $p \in \mathbb{N}$, define*

$$F(p) = \mathbb{P}_\infty^{(p)}(M \text{ is finite}). \tag{3.1.5}$$

Then, there exists a constant $\alpha \in (0,\infty)$ depending only on $Q \in (0,2)$ such that as $p \to \infty$,

$$\lim_{p\to\infty} \frac{-\log F(p)}{p} = \alpha. \tag{3.1.6}$$

Note that Proposition 3.1.3 implies that a map sampled from $\mathbb{P}_\infty^{(p)}$ is actually infinite with positive probability (i.e., $F(p) < 1$) for large p, which is not a priori obvious. In fact, we show that $F(p) < 1$ for all $p \in \mathbb{N}$ (Lemma 3.5.2). The value of the constant α is unknown as it is shown to exist via subadditivity, nor do we know whether this α depends on the particular random planar map model or just on $\mathbf{c}_{\mathrm{L}}$.

We now give the first main result of this paper after stating a few preliminaries. We use the abbreviation CRT to refer to the law of Aldous's continuum random tree [10], which is a canonical random tree constructed out of a Brownian excursion. We describe the scaling limit using the Gromov–Hausdorff distance, which is a canonical metric on the space of compact metric spaces defined up to isometry. (These concepts are reviewed in Section 3.4.4.) Also, given a planar map M with vertex set V_M and graph distance d_M, for $r \in \mathbb{R}$, let rM denote the metric space (V_M, rd_M).

Theorem 3.1.4. *For each $p \in \mathbb{N}$, let $M^{(p)}$ be the planar map sampled from $\mathbb{P}^{(p)}_{\mathtt{F}}$ — i.e., by conditioning a map drawn from $\mathbb{P}^{(p)}_{\infty}$ to be finite. Then, there is a constant $\theta_{\mathtt{F}}$ depending on* $\mathbf{q}$ *such that*

$$\frac{\theta_{\mathtt{F}}}{\sqrt{p}} M^{(p)} \xrightarrow{d} \mathrm{CRT} \tag{3.1.7}$$

with respect to the Gromov–Hausdorff distance.

As discussed earlier in Section 3.1.1, this result provides a rigorous reconciliation of the "infinite spikes" picture ((a)) and the "branched polymer" description ((b)) of supercritical LQG. There are various settings in which one could attempt to prove versions of the statement that "supercritical LQG surfaces conditioned to be finite converge to the CRT." For instance, one could look at the approximation scheme used to construct the supercritical LQG metric in [87, 88] and condition on the event that the approximating metrics have bounded diameter. Or, one could consider the dyadic tiling model studied in [121, 21] and condition on the (rare event) that there are only finitely many dyadic squares in some bounded region. These approaches have been tried unsuccessfully in the past. A key difficulty is that it is hard to describe what effect the conditioning has on the underlying GFF. The advantage of the random planar map approximations considered in Theorem 3.1.4 is that the operation of conditioning to be finite is in some sense compatible with the solvability of the random planar map model.

The heuristic reasoning behind Theorem 3.1.4 is that the iterative construction of the map in Definition 3.1.1 can be thought of, in some sense, as a supercritical branching process. As is the case for the Bienaymé–Galton–Watson (BGW) processes, it turns out that our planar map becomes "subcritical" when conditioned to be finite, and we use this subcriticality to show that the planar map sampled from $\mathbb{P}_{\mathtt{F}}^{(p)}$ degenerates to a branched polymer in the scaling limit. Implementing this heuristic requires a combination of several ideas, including results from [42, 144, 160, 189] on the convergence of rescaled subcritical Boltzmann maps to the CRT, combinatorial encodings of Boltzmann maps in terms of heavy-tailed random walks [58, 144, 70], and tricky estimates to show that the submaps surrounded by the outermost loops on $M^{(p)}$ are all microscopic (Section 3.6.3). We refer the reader to Section 3.1.5 for a more detailed overview of the proof.

Finally, we note that it was shown in [103] that random planar maps decorated by the Fortuin–Kasteleyn (FK) cluster model with parameter $q > 4$ converge to the CRT in the Gromov–Hausdorff sense (using completely different techniques from the ones in this paper). FK-decorated random planar maps for $q \in [0, 4]$ are expected to converge to γ-LQG where $\gamma \in [\sqrt{2}, 2]$ satisfies $q = 2 + 2\cos(\pi\gamma^2/2)$ [227], but we currently have no reason to believe that such maps for $q > 4$ are related to supercritical LQG.

3.1.4 Non-existence of supercritical LQG volume measure

In the subcritical and critical phases, an LQG surface with the disk topology is associated with two natural measures: a length measure on the boundary and an area measure in the bulk. In saying that these measures are "natural," we mean that they are intrinsic to the LQG surface in the following sense: (i) they are locally determined by the background GFF h and (ii) they transform appropriately under a conformal change of coordinates (see (3.1.8) and (3.2.2) for the precise condition). In the recent work [18], it was shown that for a supercritical LQG disk, there exists a natural notion of a boundary length measure.

Given this result, one may wonder if it is also possible to construct a natural volume measure in the bulk of a supercritical LQG surface. In this work, we establish that there does not exist any such locally finite, nontrivial measure. It is not at all obvious how to prove this directly from the above two axioms. One reason for this is that there are a huge number of possible ways to construct random measures (see the discussion just after Theorem 3.1.5). The key insight in our proof is that the exact self-similarity provided by the coupling of supercritical LQG with CLE_4 from [18] prevents a measure satisfying the conditions of Theorem 3.1.5 from being locally finite.

To state the result, we require the following notion of local absolute continuity with respect to the GFF. Let $\tilde{h}$ be a Dirichlet GFF on an open set $U \subseteq \mathbb{C}$. A random generalized function h on U is said to be locally absolutely continuous to a GFF if for every $z \in U$, there exists a open neighborhood $V \subsetneq U$ containing z with a compact closure in U such that the law of $h|_V$ is absolutely continuous to that of $\tilde{h}|_V$. We are now ready to state the second main result of this article.

Theorem 3.1.5. *Fix the background charge $Q \in (0,2)$. Suppose that for each domain $U \subset \mathbb{C}$, we have a measurable mapping $h \mapsto \mathfrak{m}_h$ from generalized functions on U to Borel measures on U. Assume further that the following conditions hold whenever h is a random generalized function on a domain $U \subset \mathbb{C}$ which is locally absolutely continuous to a GFF.*

- *Locality: Almost surely, for any fixed $V \subseteq U$, we have $(\mathfrak{m}_h)|_V = \mathfrak{m}_{(h|_V)}$.*
- *Coordinate change: For any fixed conformal map $f : \widetilde{U} \to U$, we almost surely have*

$$\mathfrak{m}_{h \circ f + Q \log |f'|}(A) = \mathfrak{m}_h(f(A)) \tag{3.1.8}$$

for every Borel subset $A \subseteq \widetilde{U}$.

Then, any such mapping $\mathfrak{m}$ must be trivial in the following sense: for any random generalized function h on a domain $U \subseteq \mathbb{C}$ which is locally absolutely continuous to a GFF, the measure

$\mathfrak{m}_h$ *is either a.s. equal to the zero measure or a.s. assigning infinite mass to every Euclidean open subset of* U*.*

In the above statement, we do not care how the mapping $h \mapsto \mathfrak{m}_h$ is defined when h is not absolutely continuous with respect to a GFF. For instance, we could require it to be identically zero for concreteness. (A similar setup appears in the definition of LQG metric in [93, 129, 205, 88].)

While the above theorem is stated with $\mathfrak{m}_h$ being a positive measure, we note that by the usual decomposition of signed measures into positive and negative parts, it follows immediately that there does not exist any natural signed (or even complex) bulk measure in the supercritical phase of LQG. This rules out some possible constructions for the volume measure of supercritical LQG, including some versions of Gaussian multiplicative chaos (GMC): e.g., supercritical GMC in either the dual phase [31] or the glassy phase [181], as well as some version of complex GMC [164]. It also rules out a construction based on finding a natural operator other than the exponential of the field h such as in the works of Gervais et al. reviewed in Section 3.1.1.

Finally, we note that all of the three conditions for $\mathfrak{m}_h$ — a random measure, locally determined by h, and satisfying the coordinate change rule — are necessary in Theorem 3.1.5. Here are examples of functionals of the field h which satisfy a strict subset of these conditions.

(a) A clear candidate for a random measure locally determined by a GFF h is the β-LQG measure μ_h where $\beta \in (0,2]$. However, this does not satisfy the coordinate-change rule (3.1.8) for any $Q \in (0,2)$.

(b) Since a GFF h in U is an element of the Sobolev space $H^{-\varepsilon}(U)$ for $\varepsilon > 0$ (see, e.g., [251]), Δh can be defined as a generalized function in $H^{-(2+\varepsilon)}(U)$ for $\varepsilon > 0$. While Δh is not a random measure, it is a local functional of h and satisfies

$$\Delta(h \circ f + Q \log |f'|) = (\Delta h) \circ f \tag{3.1.9}$$

trivially for any conformal f and $Q \in \mathbb{R}$ since $\Delta \log|f'| = 0$.

(c) Let h be a Gaussian field on $U \subsetneq \mathbb{C}$. Fix $\beta \in (0,2]$ and let $\alpha = Q - (\frac{2}{\beta} + \frac{\beta}{2})$. Recall that the conformal radius of a domain U viewed from $z \in U$ is defined as $\mathrm{CR}(z;U) := |g'(0)|$ where $g : \mathbb{D} \to U$ is a conformal map sending 0 to z. Consider

$$\mathfrak{m}_h(dz) := \mathrm{CR}(z;U)^\alpha \mu_h^{(\beta)}(dz) \tag{3.1.10}$$

where $\mu_h^{(\beta)}$ is the β-LQG measure on U. Then, for a conformal map $f : U \to \widetilde{U}$ and a GFF $\tilde{h}$ on $\widetilde{U}$, the pullback of $\mathfrak{m}_{\tilde{h}}$ under f is given by

$$\begin{aligned} f^*\mathfrak{m}_{\tilde{h}}(dz) &= \mathrm{CR}(f(z);\widetilde{U})^\alpha \mu_{\tilde{h}}^{(\beta)}(df(z)) \\ &= |f'(z)|^\alpha \mathrm{CR}(z;U)^\alpha \mu^{(\beta)}_{\tilde{h}\circ f + (\frac{2}{\beta}+\frac{\beta}{2})\log|f'|}(dz) = \mathfrak{m}_{\tilde{h}\circ f + Q\log|f'|}(dz). \end{aligned} \tag{3.1.11}$$

Hence, this $\mathfrak{m}_h$ is a random Borel measure satisfying the coordinate change rule (3.1.8) with $Q \in (0,2)$. Nevertheless, $\mathfrak{m}_h$ does not satisfy the locality condition of Theorem 3.1.5 since the measure $\mathfrak{m}_{(h|_V)}$ on a domain $V \subseteq U$ also depends on the background domain U via the conformal radius factor appearing in (3.1.10).

(d) In Section 3.3.3, we give a natural family of random measures on supercritical LQG disks constructed via the multiplicative cascade associated to the coupling of supercritical LQG with nested CLE_4. These measures satisfy the coordinate change rule (3.1.8), but they are not local.

Remark 3.1.6. For $\mathbf{c}_{\mathrm{L}} \in (1,25)$, the LQG metric does not induce the Euclidean topology. In particular, the α-thick points for $\alpha > Q$ lie at infinite distance from every other point (see [205, Proposition 1.11]), and consequently LQG metric balls have empty Euclidean interior. Could there be a measure satisfying the hypotheses of Theorem 3.1.5 which assigns finite mass to LQG metric balls, but infinite mass to Euclidean open sets?

We expect that no such measure exists. It is shown in [205, Proposition 1.14] that almost surely for every $s < r$, a supercritical LQG metric ball of radius r cannot be covered by finitely many supercritical metric balls of radius s. This means that if we have a measure $\mathfrak{m}_h$ associated with a supercritical LQG surface with field h, then either $\mathfrak{m}_h$ assigns very small mass to most supercritical LQG metric balls, or $\mathfrak{m}_h$ assigns infinite mass to every supercritical LQG metric ball. In particular, there cannot be any measure $\mathfrak{m}_h$ which is "compatible with the supercritical LQG metric D_h" in the same sense that the LQG measure is compatible with the metric in the subcritical case.

3.1.5 Proof outline

This article is divided into two main parts: continuous (Sections 3.2–3.3) and discrete (Sections 3.4–3.7). Throughout both parts, the central idea is to consider the multiplicative cascade of lengths of loops on a supercritical LQG disk or its discrete analog. The linchpin between the two parts is Theorem 3.1.2, in which we establish the convergence of the discrete cascade of loop lengths in Definition 3.1.1 to the continuum cascade of loop lengths in the CLE_4-coupled supercritical LQG disk of [18]. This is the supercritical analog of [70], which considered the convergence of the discrete cascade originating from the gasket decomposition [57] of loop $O(n)$ model decorated planar maps with $n \in (0, 2)$ to the continuum cascade of subcritical γ-LQG disks coupled with independent CLE_κ with $\kappa = \gamma^2 \in (8/3, 4)$ in [200].

The first part of this work concentrates on the proof of Theorem 3.1.5. In Section 3.2, we recall from [18] the definition of a supercritical LQG disk and its coupling with (nested) CLE_4. We identify the exact law of the multiplicative cascade on the boundary lengths of CLE_4 loops in this coupling (Corollary 3.2.8). In Section 3.3.1, we draw classical facts from the branching random walk literature to analyze the law of the multiplicative cascade and show, in particular, that the number of CLE_4 loops of any given size tends to infinity as we look at further generations of loops (Proposition 3.3.1). Given these results on the lengths of CLE_4 loops, the proof of Theorem 3.1.5 in Section 3.3.2 follows straightforwardly from the fact the supercritical LQG disks enclosed by CLE_4 loops of the same generation are conditionally independent given their boundary lengths (Proposition 3.2.7). Section 3.3.3 gives the construction of a natural family of (non-local) measures on supercritical LQG disks using its coupling with nested CLE_4.

In the second part, we analyze the behavior of our random planar map model of supercritical LQG and prove Theorem 3.1.4. Section 3.4 recalls from the random planar maps literature the two key tools for our analysis: the connection between Boltzmann maps and random walks (Proposition 3.4.7) and the convergence of the boundaries of large subcritical Boltzmann maps to the CRT (Proposition 3.4.17).

In Section 3.5, we consider the probability $F(p)$ that a map sampled from our discrete model $\mathbb{P}_{\infty}^{(p)}$ of supercritical LQG with boundary length $2p$ is finite. Recalling the convergence (Theorem 3.1.2) of the discrete perimeter cascade of our model to the continuum perimeter cascade of the CLE_4-coupled supercritical LQG disk, we use the existence of large loops in the latter (Proposition 3.3.1 again) to conclude that $F(p) \to 0$ as $p \to \infty$ (Lemma 3.5.1). We then prove Proposition 3.1.3 based on a slightly modified form of the standard subadditivity argument which allows for logarithmic errors, which we establish using the random walk description of Boltzmann maps.

In Section 3.6, we analyze the behavior of our discrete model $\mathbb{P}_{\mathrm{F}}^{(p)}$ of a supercritical LQG disk conditioned to be finite. This is the most difficult part of the paper. We first identify that the marginal law on the outermost gasket $M_0^{(p)}$ under this conditioning is that of a subcritical Boltzmann map (Proposition 3.6.2). From here, we extrapolate the subcriticality of the entire map $M^{(p)}$ sampled from $\mathbb{P}_{\mathrm{F}}^{(p)}$ in the sense that the expected size of maps added in subsequent generations decays exponentially (Corollary 3.6.7). In particular, the size of the submaps of $M^{(p)}$ inside each face of the outermost gasket $M_0^{(p)}$ is of smaller order than the diameter of the boundary $\partial M^{(p)}$ (Proposition 3.6.13). Establishing this requires intricate estimates for the branching process which describes the lengths of the loops in each generation. Throughout this analysis, we find another use for Proposition 3.1.3 in that it allows us to use p and $-\alpha^{-1}\log F(p)$ interchangeably for large values of p in many of our estimates. This is useful due to the relation

$$F(p) = \mathbb{E}\left[\prod_{f\in\mathfrak{F}_0} F(\mathrm{per_{in}}(f))\right], \tag{3.1.12}$$

where $\mathfrak{F}_0$ is the collection of interior faces of the outermost gasket $M_0^{(p)}$.

We finally prove Theorem 3.1.4 in Section 3.7. Applying the results of Section 3.6 and the convergence of the entire subcritical Boltzmann map to the CRT (Proposition 3.4.17), we see that a map $M^{(p)}$ sampled from $\mathbb{P}_{\mathbf{F}}^{(p)}$ is "thin" in the sense that the maximum graph distance from an interior vertex of $M^{(p)}$ to the boundary $\partial M^{(p)}$ is of smaller order than the diameter of $\partial M^{(p)}$. Thus, the problem at hand reduces to showing the convergence to the CRT when we equip $\partial M^{(p)}$ with the restriction of the graph distance $d_{M^{(p)}}$ on the entire map (Proposition 3.7.1). We adapt the proof of the convergence of $\partial M^{(p)}$ with the graph distance $d_{\partial M^{(p)}}$ in [160] by showing that the two distances $d_{M^{(p)}}$ and $d_{\partial M^{(p)}}$ are equivalent on $\partial M^{(p)}$ (Proposition 3.7.10). This follows from a law of large numbers argument based on the spine decomposition of the critical looptree structure of $\partial M^{(p)}$ identified in [77, 215].

3.1.6 Notational comments

Throughout the paper, we shall have multiple occasions to use the Bienaymé-Galton-Watson (or Galton–Watson) tree, which which refer to as a BGW tree. For any planar map M, we denote its vertex set as V_M and use $d_M(x,y)$ to denote the graph distance between vertices x and y in M. Also, we will use $\mathrm{Vol}(M)$ to denote the total number of vertices present in a planar map M. Let $\mathbb{N}$ denote the set $\{1,2\ldots\}$ of positive integers and $\mathbb{N}^\# = \mathbb{N}\cup\{0\}$ denote the nonnegative integers. For $a<b\in\mathbb{R}$, the double interval $[\![a,b]\!]$ refers to $[a,b]\cap\mathbb{Z}$.

3.2 Background on the supercritical LQG disk

3.2.1 Supercritical LQG disk

We begin with a general definition of an LQG surface, which is a two-dimensional domain equipped with a generalized function considered up to a conformal change of parameterization. The following is a generalization of the definitions in [99, 228, 95], etc., to the supercritical case $Q \in (0, 2)$.

Definition 3.2.1. Let $\mathbf{c}_{\mathrm{L}} > 1$ and let $Q = \sqrt{(\mathbf{c}_{\mathrm{L}} - 1)/6} > 0$ be the corresponding background charge. Consider the set of tuples (U, h, A) where $U \subset \mathbb{C}$ is open, h is a generalized function on U, and $\mathcal{A}$ a sequence of compact subsets of U (this is some sort of "decoration" which could be a collection of curves, points, etc.). Let $\sim_Q$ be an equivalence relation on such tuples where we define

$$(U, h, \mathcal{A}) \sim_Q (\widetilde{U}, \tilde{h}, \widetilde{\mathcal{A}}) \tag{3.2.1}$$

if there is a conformal map $f : \widetilde{U} \to U$ such that

$$\tilde{h} = h \circ f + Q \log |f'| \quad \text{and} \quad f(\widetilde{\mathcal{A}}) = \mathcal{A}. \tag{3.2.2}$$

A Liouville quantum gravity (LQG) surface of central charge $\mathbf{c}_{\mathrm{L}}$, which we often call a Q-LQG surface, is an equivalence class under $\sim_Q$. A representative of an LQG surface is called an embedding.

In the theory of LQG, the generalized function h is always a Gaussian free field (GFF) or some variant of it, and we now give a quick introduction to the GFF. A GFF on a domain $U \subset \mathbb{C}$ is a centered Gaussian generalized function h with the covariance kernel G given by a Green's function[2] corresponding to the Laplacian on U. That is, if we denote the paring

2. We assume that the Green's function G is normalized so that $G(z, w) \sim -\log|z - w|$ as $w \to z$.

of h with a smooth function $f \in C^\infty(U)$ as (h, f), then this is a Gaussian random variable with

$$\mathbb{E}[(h, f_1)(h, f_2)] = \int_{U \times U} f_1(z_1) G(z_1, z_2) f_2(z_2)\, dz_1 dz_2. \tag{3.2.3}$$

for any smooth functions $f_1, f_2 \in C^\infty(U)$. We note that the GFF is said to have Dirichlet/zero boundary conditions if the Green's function G is defined with zero boundary conditions. The same is true with Neumann/free boundary conditions.

We note that the definition of the equivalence relation $\sim_Q$ in (3.2.2) is given to reflect the conformal covariance of the LQG measure [99, 230] and the LQG metric [128] in the subcritical regime ($\mathbf{c}_\mathrm{L} > 25$, $Q > 2$). Thus, if h is a GFF on U and μ_h and D_h are the LQG measure and the metric corresponding to the field h with central charge $\mathbf{c}_\mathrm{L}$, then for a conformal map $f : \widetilde{U} \to U$ with $\tilde{h}$ given by (3.2.2), we have $f_* \mu_{\tilde{h}} = \mu_h$ and $f_* D_{\tilde{h}} = D_h$ almost surely, where, as usual, f_* is used to denote the push-forward with respect to f. This result extends to the critical case ($\mathbf{c}_\mathrm{L} = 25$, $Q = 2$) for the LQG measure [97, Theorem 13] and is expected to hold for the LQG metric in the critical/supercritical regime ($\mathbf{c}_\mathrm{L} \in (1, 25]$, $Q \in (0, 2]$) but is so far known only for complex affine f [88, Proposition 1.9] (see also [84, Theorem 1.4]). This is the context in which the analogous conformal covariance condition (3.1.8) is required in our definition of a supercritical LQG measure in Theorem 3.1.5.

We now present the definition of the supercritical LQG disk in [18], which is based on the critical LQG disk defined in [24]. Let ν_h be the *critical* LQG boundary measure corresponding to the field h; this was constructed in [97] as

$$\nu_h := \lim_{\varepsilon \to 0} \sqrt{\log(1/\varepsilon)}\, \varepsilon e^{h_\varepsilon(z)} |dz|, \tag{3.2.4}$$

where h_ε is a mollification of h of size ε and $|dz|$ denotes the Lebesgue measure on the boundary of the domain. Also see [26, Section 4.1.2] for its construction as a limit of subcritical LQG boundary measures. As observed in [17, Lemma 2.1], the conformal covariance relation

$f_*\nu_{\tilde{h}} = \nu_h$ given (3.2.2) follows from this limiting construction thanks to the analogous result for subcritical boundary measures [99, Section 6].

Definition 3.2.2 (Critical LQG disk)**.** Let $\mathcal{S} = \mathbb{R} \times (0, 2\pi)$. Let $\mathcal{B} : \mathbb{R} \to (-\infty, 0]$ be a random function such that $(\mathcal{B}_s/\sqrt{2})_{s\geq 0}$ and $(\mathcal{B}_{-s}/\sqrt{2})_{s\geq 0}$ are independent 3-dimensional Bessel processes started at 0, and let $h^{|} : \mathcal{S} \to \mathbb{R}$ be the random function which is identically equal to $\mathcal{B}_s$ on each vertical segment $\{s\} \times (0, 2\pi)$. Let $h^{\dagger}$ be the lateral part of a free-boundary GFF on $\mathcal{S}$ (i.e., the orthogonal projection to the subspace of generalized functions on $\mathcal{S}$ which has zero mean on every vertical segment) sampled independently from $h^{|}$. Define $h := h^{|} + h^{\dagger}$ and $\hat{h} := h - \log \nu_h(\partial\mathcal{S})$. For $L > 0$, the critical LQG disk with boundary length L is the 2-LQG surface $(\mathcal{S}, \hat{h} + \log L)/\sim_2$.

Definition 3.2.3 (Supercritical LQG disk)**.** Fix $Q \in (0, 2)$ and $L > 0$. Let (U, h^C) be an embedding of the critical LQG disk with boundary length L given in Definition 3.2.2. Let h^D be an independent zero-boundary GFF on U. Then, the supercritical LQG disk with central charge $\mathbf{c}_{\mathrm{L}} = 1 + 6Q^2$ and boundary length L (also called the Q-LQG disk with boundary length L) is the Q-LQG surface $(U, h)/\sim_Q$ with

$$h := \frac{Q}{2} h^C + \frac{\sqrt{4 - Q^2}}{2} h^D. \tag{3.2.5}$$

We note that for any $\tilde{L} > 0$, if (U, h) is an embedding of a Q-LQG disk with boundary length L, then $(U, h + \frac{2}{Q}\log(\tilde{L}/L))/\sim_Q$ is a Q-LQG disk with boundary length $\tilde{L}$. The supercritical LQG length measure on the boundary of a Q-LQG disk is defined as

$$\mathfrak{n}_h := \nu_{h^C} \quad \text{``} = e^{h^C}|dz| = e^{\frac{2}{Q}h}|dz|\text{''} \tag{3.2.6}$$

where ν_{h^C} is the critical LQG boundary length measure associated with the field h^C given in Definition 3.2.3. The second half of (3.2.6) is an informal expression based on the following heuristic reasoning: since h^D is a zero-boundary GFF, we have $h^D|_{\partial U} = 0$ and hence $h^C|_{\partial U} =$

$\frac{2}{Q}h|_{\partial U}$. Thus, if (U, h) is an embedding of a Q-LQG disk, $f : \widetilde{U} \to U$ is a conformal map, and $\tilde{h} = h \circ f + Q \log |f'|$, then we have $f_* \mathfrak{n}_{\tilde{h}} = \mathfrak{n}_h$ almost surely [18, Lemma 2.8].

3.2.2 Coupling with CLE$_4$

For $\kappa \in (8/3, 4]$, the conformal loop ensemble (CLE$_\kappa$) on a simple connected domain $U \subsetneq \mathbb{C}$ is a random countable collection of disjoint non-nested Jordan curves in U which look locally like SLE$_\kappa$. It can be defined via branching SLE curves [226] or as the outer boundaries of the outermost clusters of a Brownian loop soup on U [231]. We are interested in the critical case $\kappa = 4$, which corresponds to the Brownian loop soup of intensity $1/2$.

The coupling of CLE$_4$ with a supercritical LQG disk in [18] is given by combining two different couplings. First, we record the following integrable description of CLE$_4$ on an independent critical LQG disk proved in [17] (see [41, 70, 200] for the subcritical analog).

Proposition 3.2.4. *Let (U, h^C) be an embedding of the critical LQG disk with boundary length $L > 0$. Let Γ be a (non-nested) CLE$_4$ in U sampled independently from h^C. For each loop $\ell \in \Gamma$, let $U_\ell \subset U$ be the open region enclosed by ℓ and $Z(\ell) := \nu_{h^C}(\ell)$ denote the critical LQG length of the loop ℓ.*

(a) The 2-LQG surfaces $\{(U_\ell, h^C|_{U_\ell})/\sim_2\}_{\ell \in \Gamma}$ are conditionally independent critical LQG disks given their boundary lengths $\{Z(\ell)\}_{\ell \in \Gamma}$.

(b) Let $(\zeta(t))_{t \geq 0}$ be a 3/2-stable Lévy process with no downward jumps[3] and $\tau_L := \inf\{t \geq 0 : \zeta(t) = -L\}$. Denote by $(\Delta\zeta)^{\downarrow}_L$ the sizes of upward jumps of ζ in $[0, \tau_L]$ enumerated in a decreasing order. Define the probability distribution $\rho^{(L)}$ on $(\mathbb{R}_+)^{\mathbb{N}}$ so that for

3. We note that this is defined up to the choice of the scale parameter $C > 0$, where the Lévy measure of the process is equal to $Cx^{-5/2}\,\mathbf{1}_{\{x>0\}}\,dx$. The precise choice of C is irrelevant for us, since, as discussed in the introduction of [70], any choice of C yields the same distribution $\rho^{(L)}$.

each measurable function $F : (\mathbb{R}_+)^{\mathbb{N}} \to \mathbb{R}$*, we have*

$$\int F\, d\rho^{(L)} = \frac{\mathbb{E}[(\tau_L)^{-1} F((\Delta\zeta)^{\downarrow}_L)]}{\mathbb{E}[(\tau_L)^{-1}]}. \tag{3.2.7}$$

The sequence of boundary lengths $\{Z(\ell)\}_{\ell\in\Gamma}$ *arranged in a decreasing order has the law* $\rho^{(L)}$*. Note that by the scaling relation for stable processes,*

$$(\Delta\zeta)^{\downarrow}_L \overset{d}{=} L \cdot (\Delta_\zeta)^{\downarrow}_1. \tag{3.2.8}$$

The second coupling that we recall is the level-line coupling of CLE_4 with zero-boundary GFF due to Miller and Sheffield (see [27] for a proof).

Proposition 3.2.5. *Given a domain* $U \subsetneq \mathbb{C}$*, there is a coupling of the zero-boundary GFF* h^D *and the (non-nested)* CLE_4 *in* U *such that* Γ *is almost surely determined by* h^D *and the following holds: There exists a sequence of random variables* $\{Y_\ell\}_{\ell\in\Gamma}$ *such that, conditioned on* Γ*,* $\{Y_\ell\}_{\ell\in\Gamma}$ *are i.i.d. Rademacher random variables and* $\{h^D|_{U_\ell} - \pi Y_\ell\}_{\ell\in\Gamma}$ *are independent zero-boundary GFFs on the respective domains* $\{U_\ell\}_{\ell\in\Gamma}$*.*

The coupling of a supercritical LQG disk with CLE_4 in [18] is obtained by sampling independent fields h^C and h^D on a domain $U \subsetneq \mathbb{C}$ such that $(U, h^C)/\sim_Q$ is a critical LQG disk with boundary length $L > 0$ and h^D is a zero-boundary GFF, and then defining the field h as in (3.2.5) and the CLE_4 Γ as a function of h^D as in Proposition 3.2.5.

It is straightforward to adapt Proposition 3.2.4 to this supercritical coupling except that, since h^D has a "height gap" of size πY_ℓ across the loop $\ell \in \Gamma$ as in Proposition 3.2.5, the supercritical field h has a corresponding gap of size $\beta_Q Y_\ell$ across the loop ℓ where $\beta_Q := \pi\sqrt{4-Q^2}/Q$ as defined in (3.1.3). This means that the length of loop ℓ measured using the Q-LQG length measure (3.2.6) is different based on whether we use the field to the inside or outside of ℓ. More precisely, define the outer boundary length of ℓ as

$$\mathfrak{n}_h^{\mathrm{out}}(\ell) := \nu_{h^C}(\ell) \tag{3.2.9}$$

and the inner boundary length of ℓ as

$$\mathfrak{n}_h^{\mathrm{in}}(\ell) = \mathfrak{n}_{h|_{U_\ell}}(\ell) := \nu_{h^C+\beta_Q Y_\ell}(\ell). \tag{3.2.10}$$

In particular, the two lengths are related by ratio

$$\frac{\mathfrak{n}_h^{\mathrm{in}}(\ell)}{\mathfrak{n}_h^{\mathrm{out}}(\ell)} = e^{\beta_Q Y_\ell}. \tag{3.2.11}$$

The notation $\mathfrak{n}_{h|_{U_\ell}}(\ell)$ makes sense since if we let $f_\ell : \mathbb{D} \to U_\ell$ be a conformal map, then combining the coordinate change rule (3.2.2) of a Q-LQG surface and the definition (3.2.6) of the Q-LQG boundary length of a domain, we have $\mathfrak{n}_h^{\mathrm{in}}(\ell) = \mathfrak{n}_{(h|_{U_\ell})\circ f_\ell + Q\log|f_\ell'|}(\partial\mathbb{D})$.

Proposition 3.2.6 ([18, Theorem 2.9])**.** *Let $Q \in (0,2]$ and (U,h) be an embedding of a Q-LQG disk with boundary length $L > 0$. There exists a coupling (h,Γ) of h with a (non-nested) CLE$_4$ Γ on U such that the following is true. For each loop $\ell \in \Gamma$, let $U_\ell \subset U$ be the open region enclosed by ℓ and $Z_Q(\ell) := \mathfrak{n}_h^{\mathrm{in}}(\ell)$ be the supercritical LQG length of the loop ℓ measured using the field $h|_{U_\ell}$ within the loop.*

(a) The Q-LQG surfaces $\{(U_\ell, h|_{U_\ell})/\sim_Q\}_{\ell\in\Gamma}$ are conditionally independent Q-LQG disks given their boundary lengths $\{Z_Q(\ell)\}_{\ell\in\Gamma}$.

(b) Let $\{Z(\ell)\}_{\ell\in\Gamma}$ be the sequence with the law $\rho^{(L)}$ given in (3.2.7), and let $\{Y_\ell\}_{\ell\in\Gamma}$ be independent Rademacher random variables which are also independent from $\{Z(\ell)\}_{\ell\in\Gamma}$. Recall the constant $\beta_Q := \pi\sqrt{4-Q^2}/Q$. Then, $\{Z_Q(\ell)\}_{\ell\in\Gamma}$ has the same law as $\{Z(\ell)\exp(\beta_Q Y_\ell)\}_{\ell\in\Gamma}$ if both are arranged in decreasing orders. Let us denote this law as $\rho_Q^{(L)}$.

In this coupling, Γ is neither independent from nor a.s. determined by h.

3.2.3 Coupling with nested CLE$_4$ and the perimeter cascade

For a simply connected domain $U \subsetneq \mathbb{C}$, the nested CLE$_4$ $\overline{\Gamma}$ on U is obtained from the non-nested CLE$_4$ by the following inductive procedure. Let $\Gamma^0 := \{\partial U\}$. Given collections of loops $\{\Gamma^i\}_{i \in [\![0,n]\!]}$, sample a conditionally independent CLE$_4$ Γ_ℓ for each open region U_ℓ enclosed in a loop $\ell \in \Gamma^n$ and let $\Gamma^{n+1} = \bigcup_{\ell \in \Gamma^n} \Gamma_\ell$. The nested CLE$_4$ is defined as $\overline{\Gamma} := \bigcup_{n \in \mathbb{N}^\#} \Gamma^n$, where we recall that $\mathbb{N}^\# = \mathbb{N} \cup \{0\}$. We call the elements of Γ^n as $\boldsymbol{n}$th generation loops.

Given a zero-boundary GFF h^D in a simply connected domain $U \subsetneq \mathbb{C}$, we can produce the CLE$_4$ Γ_ℓ in this iterative construction as a function of the zero-boundary GFF $h^D|_{U_\ell} - \pi Y_\ell$ as in Proposition 3.2.5. Sampling an embedding (U, h^C) of a critical LQG disk independently from h^D and again defining h as a linear combination of h^C and h^D as in (3.2.5), we have the following coupling of a supercritical LQG disk and a nested CLE$_4$.

Proposition 3.2.7 ([18, Theorem 2.13])**.** *Let $Q \in (0, 2]$ and let (U, h) be an embedding of a Q-LQG disk with boundary length $L > 0$. There exists a coupling of $(h, \overline{\Gamma})$ of h with a nested CLE$_4$ $\overline{\Gamma}$ on U such that the following is true. For each loop $\ell \in \overline{\Gamma}$, define $U_\ell \subset U$ to be the subdomain enclosed by ℓ and $Z_Q(\ell) = \nu_h^{\mathrm{in}}(\ell)$ to be the inner boundary length of the loop ℓ measured using $h|_{U_\ell}$ as defined in* (3.2.10)*. Let $\overline{\Gamma}(\ell) := \{\tilde{\ell} \in \overline{\Gamma} : \tilde{\ell} \subset U_\ell\}$ be the subcollection of CLE$_4$ loops $\overline{\Gamma}$ which are inside the loop ℓ.*

(a) For each $n \in \mathbb{N}$, if we condition on the inner boundary lengths $\{Z_Q(\ell)\}_{\ell \in \Gamma^n}$ of the nth level loops, then the Q-LQG surfaces $\{(U_\ell, h|_{U_\ell}, \overline{\Gamma}(\ell))/\sim_Q\}_{\ell \in \Gamma^n}$ are conditionally independent nested-CLE$_4$-decorated Q-LQG disks with given boundary lengths.

(b) In particular, conditioned on the inner boundary lengths $\{Z_Q(\ell)\}_{\ell \in \Gamma^n}$ of the nth level loops, those of $(n+1)$th level loops in each $\ell \in \Gamma^n$ — i.e., $\{Z_Q(\tilde{\ell})\}_{\tilde{\ell} \in \Gamma_\ell}$ — are condi-

tionally independent with distribution $\rho_Q^{(Z_Q(\ell))}$ defined in Proposition 3.2.6.

In this coupling, $\overline{\Gamma}$ is neither independent from nor a.s. determined by h.

Consequently, the inner boundary lengths $\{Z_Q(\ell)\}_{\ell\in\overline{\Gamma}}$ form a multiplicative cascade. To specify its law, let us index the loops of nested CLE_4 by the Ulam tree

$$\mathcal{U} := \bigcup_{n\in\mathbb{N}^{\#}} \mathbb{N}^n \qquad \text{where} \quad \mathbb{N}^0 := \{\emptyset\}. \tag{3.2.12}$$

That is, each element of $\mathcal{U}$ other than $\emptyset$ is a word consisting of finitely many positive integers. For $u \in \mathcal{U}$, let $|u|$ denote the length of the word ("generation"): e.g., if $u = u_1u_2\dots u_n$, then $|u| = n$. Define $|\emptyset| = 0$. We assign a partial order $\prec$ on $\mathcal{U}$ where $u \prec v$ if $|u| < |v|$ and the first $|u|$ letters of v is equal to u. If $u \prec v$, then we call u an ancestor of v and v a descendant of u. If $u \prec v$ and $|u|+1 = |v|$, then u is the parent of v and v is a child of u. We define $\emptyset$ to be the parent of every element of $\mathcal{U}$ with length 1. Now, given the coupling $(h, \overline{\Gamma})$ as in Proposition 3.2.7, let us inductively build a bijection between $\overline{\Gamma}$ and $\mathcal{U}$ so that Γ^n is mapped to $\mathcal{U}^n := \{u \in \mathcal{U} : |u| = n\}$. In particular, if $\ell \in \Gamma^n$ is indexed by $u = u_1u_2\dots u_n \in \mathcal{U}$, then the loops in Γ_ℓ (i.e., the non-nested CLE_4 in U_ℓ consisting of the outermost loops of $\overline{\Gamma}(\ell)$) are indexed by the children of u. Moreover, for $k \in \mathbb{N}$, let $uk = u_1u_2\dots u_nk \in \mathcal{U}$ be the index of the loop $\ell_k \in \Gamma_\ell$ with the kth largest value of the *critical* LQG length $\nu_{h^C}(\ell_k)$ among such loops. The following statement is a rephrasing of the law of $\{Z_Q(\ell)\}_{\ell\in\overline{\Gamma}}$ described in Proposition 3.2.7 given this setup.

Corollary 3.2.8. *Fix $Q \in (0, 2]$ and recall $\beta_Q := \pi\sqrt{4-Q^2}/Q$. Let $\{X_k^u\}_{u\in\mathcal{U},k\in\mathbb{N}}$ be a collection of random variables such that for each fixed $u \in \mathcal{U}$, the sequence $\{X_k^u\}_{k\in\mathbb{N}}$ has the law $\rho^{(1)}$ defined in Proposition 3.2.4. Furthermore, as u is varied in $\mathcal{U}$, the sequences $\{X_k^u\}_{k\in\mathbb{N}}$ are mutually independent. Let $\{Y_u\}_{u\in\mathcal{U}}$ be i.i.d. Rademacher random variables sampled in-*

dependently of $\{X_k^u\}_{u\in\mathcal{U},k\in\mathbb{N}}$. *Define* $Z(\emptyset) = Z_Q(\emptyset) = 1$ *and, for* $u = u_1u_2\dots u_n \in \mathcal{U}$,

$$Z(u) = \prod_{i=1}^{n} X_{u_i}^{u_1\dots u_{i-1}} \quad \text{and} \quad Z_Q(u) = Z(u)\prod_{i=1}^{n} e^{\beta_Q Y_{u_1\dots u_i}} = \prod_{i=1}^{n}\left(X_{u_i}^{u_1\dots u_{i-1}} e^{\beta_Q Y_{u_1\dots u_i}}\right). \tag{3.2.13}$$

Then, under the coupling of a unit boundary length Q*-LQG disk with nested* CLE_4 *in Proposition 3.2.7,* $\{Z_Q(\ell)\}_{\ell\in\overline{\Gamma}}$ *has the same law as* $\{Z_Q(u)\}_{u\in\mathcal{U}}$ *with the above bijection between* $\overline{\Gamma}$ *and* $\mathcal{U}$.

3.3 Non-existence of the supercritical volume measure

In this section, we analyze the multiplicative cascade $Z_Q = (Z_Q(u))_{u\in\mathcal{U}}$ of inner boundary lengths in the coupling of supercritical LQG with CLE_4 presented in Section 3.2. The key observation is Proposition 3.3.1, which states that there almost surely exist infinitely many quantum-macroscopic CLE loops in this coupling. This shall eventually lead to a proof of Theorem 3.1.5, and this is the main content of this section.

Another consequence of Proposition 3.3.1 is that the probability $F(p) = \mathbb{P}_\infty^{(p)}(M \text{ is finite})$ tends to 0 as the outermost boundary length p increases to ∞ (Lemma 3.5.1), which is proved in Section 3.5. This lemma is the first step in the proof of Proposition 3.1.3, where we strengthen this first estimate for $F(p)$ to an exponential decay in p.

3.3.1 The perimeter process contains infinitely many macroscopic loops

Recall the multiplicative cascade $Z_Q = (Z_Q(u))_{u\in\mathcal{U}}$ given in (3.2.13). In Corollary 3.2.8, we saw that its law agrees with that of the inner boundary lengths of the CLE_4 loops coupled with a unit boundary length Q-LQG disk. The main property of the multiplicative cascade Z_Q used in this paper is the following.

Proposition 3.3.1. *For a Borel subset* $A \subset (0,\infty)$*, denote by* $N_n(A; Z_Q)$ *the cardinality* *of the set* $\{u \in \mathcal{U} : |u| = n, Z_Q(u) \in A\}$*. If the Lebesgue measure of* A *is positive, then* $N_n(A; Z_Q) \to \infty$ *almost surely as* $n \to \infty$*.*

Our analysis of the multiplicative cascade Z_Q relies on rephrasing it in terms of a branching random walk. Recall its law given in Corollary 3.2.8 and observe that the process $S_Q = (S_Q(u))_{u\in\mathcal{U}}$ given by $S_Q(u) = -\log Z_Q(u)$ is a branching random walk, which almost surely does not go extinct since each element in the multiplicative cascade Z_Q has infinitely many descendants. We rephrase here the results from the branching random walk literature in terms of the multiplicative cascade Z_Q.

The main property of the branching random walk S_Q that we use is the location of its extreme positions. This is a well-studied topic with precise asymptotic estimates [3, 138, 7, 5]. For our purposes, however, the following cruder result on the velocity of the extreme position suffices.

Lemma 3.3.2. *For* $Q \in (0, 2)$*, define the Biggins transform of the multiplicative cascade* Z_Q *as the function*

$$\phi_Q(\theta) := \mathbb{E}\Bigg[\sum_{|u|=1} \big(Z_Q(u)\big)^\theta\Bigg]. \tag{3.3.1}$$

Then,

$$\lim_{n\to\infty} \frac{1}{n} \log\left(\sup_{|u|=n} Z_Q(u)\right) = \mu_Q := \inf_{\theta>0} \frac{1}{\theta} \log \phi_Q(\theta) > 0 \quad \textit{almost surely.} \tag{3.3.2}$$

Proof. This lemma is an immediate consequence of [45] (see also [233, Theorem 1.3]). All we need to show is $\mu_Q \in (0, \infty)$.

We can compute $\phi_Q(\theta)$ explicitly using the Biggins transform of the multiplicative cascade

Z from Corollary 3.2.8, which was calculated[4] in equation (17) of [70] to be

$$\phi(\theta) := \mathbb{E}\Bigg[\sum_{|u|=1} \big(Z(u)\big)^{\theta}\Bigg] = \begin{cases} \sec(\pi\theta) & \text{if } \theta \in (\frac{3}{2}, \frac{5}{2}), \\ +\infty & \text{otherwise.} \end{cases} \tag{3.3.3}$$

Recall from (3.2.13) that if $\{Y_i\}_{i\in\mathbb{N}}$ are i.i.d. Rademacher random variables which are independent of Z, then for each $i \in \mathbb{N}$, we have $Z_Q(i) = Z(i)\exp(\beta_Q Y_i)$. Hence,

$$\phi_Q(\theta) = \mathbb{E}\Bigg[\sum_{i\in\mathbb{N}} \big(Z(i)\big)^{\theta} e^{\beta_Q Y_i \theta}\Bigg] = \frac{\phi(\theta)(e^{\beta_Q\theta} + e^{-\beta_Q\theta})}{2} = \begin{cases} \frac{\cosh(\beta_Q\theta)}{\cos(\pi\theta)} & \text{if } \theta \in (\frac{3}{2}, \frac{5}{2}), \\ +\infty & \text{otherwise.} \end{cases} \tag{3.3.4}$$

Hence, $\mu_Q < \infty$. Note that $\phi_Q(\theta) \to \infty$ as $\theta \downarrow 3/2$ and $\theta \uparrow 5/2$, so $\theta^{-1}\log\phi_Q(\theta)$ attains its minimum μ_Q in $(3/2, 5/2)$. Moreover, $\phi_Q(\theta) > 1$ for all $\theta \in (3/2, 5/2)$, so $\mu_Q > 0$. □

Lemma 3.3.2 tells us that, almost surely, $\sup_{|u|=n} Z_\infty^{\mathrm{in}}(u)$ grows exponentially fast. We extrapolate Proposition 3.3.1 from this fact by showing that these large loops have many macroscopic children. This requires the following estimate on the offspring distribution of the cascade Z_Q.

Lemma 3.3.3. *Let $A \subset (0, \infty)$ be a Borel subset with positive Lebesgue measure. Recall that we denote by*

$$N_1(A; Z_Q) = \sum_{|u|=1} \mathbf{1}\{Z_Q(u) \in A\} \tag{3.3.5}$$

the number of first-generation elements of Z_Q whose sizes are in A. Then, for any fixed $k \in \mathbb{N}$, there exists a constant $c > 0$ such that

$$\mathbb{P}\big(N_1(L^{-1}A; Z_Q) \le k\big) = O\big(e^{-cL^{15/16}}\big) \tag{3.3.6}$$

4. The analysis in [70] for the multiplicative cascade generated by the jumps of a spectrally positive α-stable process applies to any $\alpha = a - 1/2 \in (1, 2)$.

as $L \to \infty$.

Proof. From the law of Z_Q in (3.2.13), if $(Z(i))_{i\in\mathbb{N}}$ are sampled from the law $\rho_Q^{(1)}$ given in Proposition 3.2.6 and $\{Y_i\}_{i\in\mathbb{N}}$ are i.i.d. Rademacher random variables independent from $\{Z(i)\}_{i\in\mathbb{N}}$, then $(Z_Q(i))_{i\in\mathbb{N}} \overset{d}{=} (Z(i)\exp(\beta_Q Y_i))_{i\in\mathbb{N}}$. Hence, with

$$\tilde{c} = \log(2/(1+e^{-1})),$$

we have

$$\begin{aligned}
\mathbb{E}\Big[e^{-N_1(L^{-1}A;Z_Q)}\Big] &= \mathbb{E}\Big[\prod_{i\in\mathbb{N}}\mathbb{E}\Big[e^{-\mathbf{1}\{Z(i)e^{\beta_Q Y_i}\in L^{-1}A\}}\Big|Z\Big]\Big]\\
&= \mathbb{E}\Big[\prod_{i\in\mathbb{N}}\frac{e^{-\mathbf{1}\{Z(i)\in L^{-1}e^{\beta_Q}A\}}+e^{-\mathbf{1}\{Z(i)\in L^{-1}e^{-\beta_Q}A\}}}{2}\Big]\\
&\le \mathbb{E}\Big[\prod_{i\in\mathbb{N}}e^{-\tilde{c}\mathbf{1}\{Z(i)\in L^{-1}(e^{\beta_Q}A\cup e^{-\beta_Q}A)\}}\Big]\\
&= \mathbb{E}\Big[e^{-\tilde{c}N_1(L^{-1}(e^{\beta_Q}A\cup e^{-\beta_Q}A);Z)}\Big],
\end{aligned} \tag{3.3.7}$$

where $N_1(\cdot;Z)$ used in the last term above is defined by simply replacing Z_Q by Z in (3.3.5). For convenience, let us define

$$\widetilde{A} = e^{\beta_Q}A \cup e^{-\beta_Q}A$$

for the remainder of this proof.

Let us now recall the law of $(Z(i))_{i\in\mathbb{N}}$ from Proposition 3.2.4. Let ζ be a 3/2-stable Lévy process with no negative jumps started at 0, and let τ denote the hitting time of -1 of this process. That is, for some scale parameter $C>0$, with $\Gamma(\cdot)$ denoting the Gamma function, the Lévy measure of ζ is $(C/\Gamma(-3/2))\mathbf{1}_{\{x>0\}}x^{-5/2}dx$ and $\mathbb{E}[e^{-\lambda\tau}] = \exp(-(\lambda/C)^{2/3})$ for $\lambda \ge 0$. As mentioned in Footnote 3, the scale parameter C does not change the law of the multiplicative cascade Z; hence, we choose $C=1$ for simplicity. Let $(\Delta\zeta)^{\downarrow}$ denote the sequence consisting of the sizes of the jumps of ζ up to the stopping time τ. Then,

$$\mathbb{E}\Big[e^{-\tilde{c}N_1(L^{-1}\widetilde{A};Z)}\Big] = \frac{\mathbb{E}\big[\tau^{-1}\exp\big(-\tilde{c}\sum_{x\in(\Delta\zeta)^\downarrow}\mathbf{1}\{x\in L^{-1}\widetilde{A}\}\big)\big]}{\mathbb{E}\big[\tau^{-1}\big]}. \tag{3.3.8}$$

For fixed $t_0 > 0$, the number of jumps of ζ in the time interval $[0, t_0]$ whose sizes are in $L^{-1}\widetilde{A}$ is a Poisson random variable with rate given by t_0 times the Lévy measure

$$\Pi(L^{-1}\widetilde{A}) := \frac{1}{\Gamma(-3/2)}\int_{L^{-1}\widetilde{A}} s^{-5/2}\,ds = \frac{L^{3/2}}{\Gamma(-3/2)}\int_{\widetilde{A}} s^{-5/2}\,ds = L^{3/2}\Pi(\widetilde{A}). \tag{3.3.9}$$

On the event $\{\tau > t_0\}$, this gives a lower bound for $\sum_{x\in(\Delta\zeta)^\downarrow}\mathbf{1}\{x\in L^{-1}\widetilde{A}\}$. Hence,

$$\mathbb{E}\Big[\tau^{-1}e^{-\tilde{c}\sum_{x\in(\Delta\zeta)^\downarrow}\mathbf{1}\{x\in L^{-1}\widetilde{A}\}}\cdot\mathbf{1}\{\tau>t_0\}\Big] \le (t_0)^{-1}e^{-(1-e^{-\tilde{c}})\Pi(\widetilde{A})L^{3/2}t_0}. \tag{3.3.10}$$

It is well known that τ is a positive 2/3-stable random variable. We also have the tail estimate

$$\mathbb{P}(\tau\in dt) = \exp(-(1+o(1))t^{-5/3})\,dt \quad \text{as} \quad t\downarrow 0 \tag{3.3.11}$$

from, e.g., [257, Chapter 2.5]. Letting $c_A = (1-e^{-\tilde{c}})\Pi(\widetilde{A})$, for each fixed $\alpha\in(0,3/2)$, we thus have

$$\begin{aligned}
&\mathbb{E}\Big[\tau^{-1}e^{-\tilde{c}\sum_{x\in(\Delta\zeta)^\downarrow}\mathbf{1}\{x\in L^{-1}\widetilde{A}\}}\Big]\\
&\le \mathbb{E}\big[\tau^{-1}\cdot\mathbf{1}\{\tau\le L^{-\alpha}\}\big] + \mathbb{E}\Big[\tau^{-1}e^{-\tilde{c}\sum_{x\in(\Delta\zeta)^\downarrow}\mathbf{1}\{x\in L^{-1}\widetilde{A}\}}\cdot\mathbf{1}\{\tau>L^{-\alpha}\}\Big]\\
&\le \int_0^{L^{-\alpha}} t^{-1}\exp(-(1+o(1))t^{-5/3})\,dt + L^{\alpha}\exp(-c_A L^{\frac{3}{2}-\alpha})\\
&= \frac{5}{3}\int_{L^{5\alpha/3}}^{\infty}\frac{1}{s}e^{-(1+o(1))s}\,ds + L^{\alpha}\exp(-c_A L^{\frac{3}{2}-\alpha})\\
&= O\big(e^{-c(L^{5\alpha/3}\wedge L^{3/2-\alpha})}\big)
\end{aligned} \tag{3.3.12}$$

as $L \to \infty$. Optimizing over α, we have a constant $c > 0$ depending on A such that

$$\mathbb{E}\Big[e^{-N_1(L^{-1}\widetilde{A};Z)}\Big] = O(e^{-cL^{15/16}}) \quad \text{as} \quad L \to \infty. \tag{3.3.13}$$

The lemma now follows by applying the Markov inequality to (3.3.7) and (3.3.13). $\square$

Proof of Proposition 3.3.1. Fix $\varepsilon \in (0, \mu_Q)$, where μ_Q is the constant defined in (3.3.2). By Lemma 3.3.3,

$$\mathbb{P}(N_{n+1}(A; Z_Q) \le k | N_n([e^{n(\mu_Q - \varepsilon)}, \infty); Z_Q) \ge 1) \le O(e^{-ce^{(15/16)(\mu_Q-\varepsilon)n}}).$$

Moreover, we saw in Lemma 3.3.2 that with probability one, $N_n([e^{n(\mu_Q-\varepsilon)}, \infty); Z_Q) \ge 1$ for all sufficiently large n. Since the above upper bound is summable, we conclude that for every positive integer k, we have $\mathbb{P}(N_n(A; Z_Q) \le k$ for infinitely many $n) = 0$. The positive integer k here can be chosen arbitrarily large, so we conclude $\lim_{n\to\infty} N_n(A; Z_Q) = \infty$ almost surely. $\square$

3.3.2 Proof of non-existence

The proof of Theorem 3.1.5 is straightforward given our analysis of the multiplicative cascade Z_Q in the previous section.

Proof of Theorem 3.1.5. Suppose $\mathfrak{m}$ is a functional satisfying the conditions given in the theorem. For each $L > 0$, let $P^{(L)}$ denote the law of the total volume $\mathfrak{m}_h(\mathbb{D})$ where $(\mathbb{D}, h)$ is an embedding of the Q-LQG disk with boundary length L.

Let $(\mathbb{D}, h, \overline{\Gamma})$ be an embedding of a unit boundary length Q-LQG disk coupled with nested CLE_4 as in Proposition 3.2.7. Let $(\ell_u)_{u\in\mathcal{U}}$ be the indexing of the nested CLE_4 $\overline{\Gamma}$ as described immediately above Corollary 3.2.8. For $u \in \mathcal{U}$, let D_u denote the open set enclosed by ℓ_u, so that D_u contains ℓ_v for every $v \succ u$. Then,

$$\mathfrak{m}_h(\mathbb{D}) = \mathfrak{m}_h\left(\mathbb{D} \setminus \bigcup_{|u|=1} D_u\right) + \sum_{|u|=1} \mathfrak{m}_h(D_u) \tag{3.3.14}$$

almost surely. Recall that conditioned on the inner boundary length $\mathfrak{n}_h^{\text{in}}(\ell_u)$ as defined in (3.2.10), the equivalence class $(D_u, h|_{D_u})/\sim_Q$ has the law of a Q-LQG disk with boundary length $\mathfrak{n}_h^{\text{in}}(\ell_u)$. By the assumption on the coordinate change rule (3.1.8), conditioned on $\ell \in \overline{\Gamma}$ as well as a conformal map f which takes the open disk D_ℓ enclosed by ℓ to the unit disk $\mathbb{D}$, we a.s. have $\mathfrak{m}_h(D_\ell) = \mathfrak{m}_{h\circ f + Q\log|f'|}(\mathbb{D})$ since $h \circ f + Q \log|f'|$ is locally absolutely continuous to a GFF. Therefore, the conditional law of $\mathfrak{m}_h(D_u)$ given ℓ_u and $\mathfrak{n}_h^{\text{in}}(\ell_u)$ is given by $P^{(\mathfrak{n}_h^{\text{in}}(\ell_u))}$. Moreover, given the inner boundary lengths $\{\mathfrak{n}_h^{\text{in}}(\ell_u)\}_{|u|=1}$, the supercritical LQG surfaces $\{(D_u, h|_{D_u})/\sim_Q\}_{|u|=1}$ are conditionally independent and hence so are the volumes $\{\mathfrak{m}_h(D_u)\}_{|u|=1}$. This implies that the family of conditional laws $\{P^{(L)}\}_{L\in\mathbb{R}_+}$ satisfies exactly one of the following scenarios.

(1) There exists a set $A \subset \mathbb{R}_+$ of zero Lebesgue measure such that for every $L \in \mathbb{R}_+ \setminus A$, we have $P^{(L)}(\mathfrak{m}_h(\mathbb{D}) = 0) = 1$. Since the Lévy measure of a spectrally positive stable process is absolutely continuous with respect to the Lebesgue measure on $\mathbb{R}_+$, in this case, by Proposition 3.2.6, the inner boundary lengths $\{\mathfrak{n}_h^{\text{in}}(\ell_u)\}_{|u|=1}$ of the outermost CLE_4 loops are all contained in A with probability one. Hence, this condition is equivalent to the case that, almost surely, the CLE_4 gasket $\mathbb{D} \setminus \bigcup_{|u|=1} D_u$ accounts for all of the $\mathfrak{m}_h$-volume of the supercritical LQG disk: i.e., $\mathfrak{m}_h(D_u) = 0$ for every $|u| = 1$.

(2) On the complement of the first case, there exists a set $A \subset \mathbb{R}_+$ with a positive Lebesgue measure and a constant $\varepsilon > 0$ such that if $L \in A$, then $P^{(L)}(\mathfrak{m}_h(\mathbb{D}) > \varepsilon) > \varepsilon$.

We claim that in the first case, $P^{(L)}(\mathfrak{m}_h(\mathbb{D}) = 0) = 1$ for every $L \in (0, \infty)$ and not just on a subset of full Lebesgue measure. For this, let us choose a specific embedding $(\mathbb{D}, h^{(L)})$

of a Q-LQG disk with boundary length L. As in Definition 3.2.3, we let

$$h^{(L)} = \frac{Q}{2}(h^C + \log L) + \frac{\sqrt{4-Q^2}}{2} h^D = \frac{Q}{2} h^C + \frac{\sqrt{4-Q^2}}{2} \left(h^D + \frac{Q}{\sqrt{4-Q^2}} \log L \right) \quad (3.3.15)$$

where h^D is a zero boundary GFF on $\mathbb{D}$ independent from $h^C = \hat{h} \circ f + \log|f'|$ where $\hat{h}$ is the field on the strip $\mathcal{S} = \mathbb{R} \times (0, 2\pi)$ in Definition 3.2.2 and f is a conformal map from $\mathbb{D}$ to $\mathcal{S}$. Now choose any $\widetilde{L} \in (0,\infty)$ such that $P^{(\widetilde{L})}(\mathfrak{m}_h(\mathbb{D}) = 0) = 1$. Since $h^D + (Q/\sqrt{4-Q^2}) \log(L/\widetilde{L})$ is absolutely continuous with respect to h^D away from the boundary $\partial\mathbb{D}$ [194, Proposition 3.4], the two Gaussian fields are absolutely continuous when restricted to the open ball $B_r(0)$ for any $r \in (0,1)$. Then, since $\mathfrak{m}_{h^{(\widetilde{L})}}(B_r(0)) = 0$ a.s., we must have $\mathfrak{m}_{h^{(L)}}(B_r(0)) = 0$ almost surely. Letting $r \uparrow 1$, we have $P^{(L)}(\mathfrak{m}_h(\mathbb{D}) = 0) = 1$ for every $L \in (0,\infty)$ as claimed.

In the second case, considering the disks D_u cut out by the nth generation CLE_4 loops $\ell_u \in \Gamma^n$, we have

$$\mathfrak{m}_h(\mathbb{D}) \geq \sum_{|u|=n} \mathfrak{m}_h(D_u) \quad (3.3.16)$$

for every positive integer n. Proposition 3.3.1 implies that almost surely, by choosing a sufficiently large n, we can find an arbitrarily large number of disks D_u with $|u| = n$ such that $\mathfrak{n}_h^{\mathrm{in}}(\ell_u) \in A$. By Proposition 3.2.7, conditioned on the boundary lengths $\{\mathfrak{n}_h^{\mathrm{in}}(\ell_u)\}_{|u|=n}$, each of these disks with $\mathfrak{n}_h^{\mathrm{in}}(\ell_u) \in A$ has an independent chance of size $\varepsilon > 0$ such that $\mathfrak{m}_h(D_u) > \varepsilon$. Thus, $\mathfrak{m}_h(\mathbb{D}) = \infty$ almost surely. Moreover, since $(D_u, h_{D_u})/\sim_Q$ is a Q-LQG disk conditioned on its boundary length, we almost surely have $\mathfrak{m}_h(D_u) = \infty$ for every $u \in \mathcal{U}$. Now, this implies $\mathfrak{m}_h(U) = \infty$ a.s. for any open subset U of $\mathbb{D}$, since there almost surely exists $u \in \mathcal{U}$ such that $D_u \subset U$ (this follows from the local finiteness of nested CLE_4 [25, Lemma 2.1]). Now, note that the field $\hat{h}$ in Definition 3.2.2, when restricted to compact subsets of $\mathcal{S} = \mathbb{R} \times (0, 2\pi)$ away from $\{0\} \times (0, 2\pi)$, is mutually absolutely continuous with respect to a free-boundary GFF. Recalling the conformal map $f : \mathbb{D} \to \mathcal{S}$, we conclude the proof by noting that for any domain $U \subset \mathbb{D}$ at a positive distance away from $\partial\mathbb{D} \cup f^{-1}(\{0\} \times (0, 2\pi))$, the field $h|_U$ is mutually absolutely continuous with respect to a free boundary GFF on U since both $h^C|_U$ and $h^D|_U$ are. For general domains $U \subset \mathbb{C}$, we apply an affine transformation paired with the coordinate change rule (3.1.8). □

3.3.3 Measure constructed from the multiplicative cascade

An important condition in Theorem 3.1.5 was that the supercritical LQG measure $\mathfrak{m}_h$ is locally determined by the field h. We conclude this section by describing in Proposition 3.3.5 a natural family of random measures on the CLE_4-coupled supercritical LQG disk $(\mathbb{D}, h, \overline{\Gamma})/\sim_Q$ arising from its multiplicative cascade structure. While they do not satisfy the locality condition as the CLE_4 in this coupling is not determined by the field h, they satisfy the Q-LQG coordinate change formula (3.1.8) and almost surely assign finite and strictly positive values to each Euclidean open set.

Recall from (3.3.4) the notation $\phi_Q(\theta) = \mathbb{E}[\sum_{|u|=1}(Z_Q(u))^\theta]$, which, for $\theta \in (3/2, 5/2)$, is finite and takes the value $\cosh(\beta_Q\theta)/\cos(\pi\theta)$. For θ in this range, observe that

$$M_n^{(\theta)} := \frac{1}{\big(\phi_Q(\theta)\big)^n} \sum_{|u|=n} \big(Z_Q(u)\big)^\theta \tag{3.3.17}$$

is a martingale with respect to the natural filtration of the multiplicative cascade Z_Q. In the branching random walk literature, $M_n^{(\theta)}$ is called the additive martingale of the branching random walk $S_Q(u) = -\log Z_Q(u)$, which was introduced in [158, 46]. Since $M_n^{(\theta)}$ is nonnegative for all n, it converges almost surely to a nonnegative random variable $M_\infty^{(\theta)}$. A fundamental result regarding the additive martingale is the Biggins martingale convergence theorem, which describes the conditions under which the limit $M_\infty^{(\theta)}$ almost surely does not vanish. We describe this condition for the multiplicative cascade Z_Q. Recall the notation

$$\mu_Q := \inf_{\theta>0} \frac{1}{\theta} \log \phi_Q(\theta)$$

from Lemma 3.3.2.

Lemma 3.3.4. *For each $Q \in (0,2)$, there exists a unique positive number θ^* such that $(1/\theta^*) \log \phi_Q(\theta^*) = \mu_Q$. Moreover, the following are equivalent.*

(i) $\theta \in (\frac{3}{2}, \theta^*)$

(ii) $\{M_n^{(\theta)}\}_{n\in\mathbb{N}\#}$ *is uniformly integrable.*

(iii) $M_\infty^{(\theta)} > 0$ *a.s.*

Proof. We first show that $\mu_Q = (1/\theta^*) \log \phi_Q(\theta^*)$ is achieved at a unique value of $\theta^* \in (\frac{3}{2}, \frac{5}{2})$. At θ^*, we have

$$\frac{d}{d\theta}\left(\frac{1}{\theta} \log \phi_Q(\theta)\right)\bigg|_{\theta=\theta^*} = \frac{1}{(\theta^*)^2}\left(\theta^* \frac{\phi_Q'(\theta^*)}{\phi_Q(\theta^*)} - \log \phi_Q(\theta^*)\right) = 0. \tag{3.3.18}$$

Using the explicit formula $\phi_Q(\theta) = \cosh(\beta_Q\theta)/\cos(\pi\theta)$ from (3.3.4), we furthermore have

$$\frac{d}{d\theta}\left(\theta\frac{\phi_Q'(\theta)}{\phi_Q(\theta)} - \log \phi_Q(\theta)\right) = \theta\left(\frac{\pi^2}{\cos^2(\pi\theta)} + \frac{(\beta_Q)^2}{\cosh^2(\beta_Q\theta)}\right) > 0 \tag{3.3.19}$$

for all $\theta \in (\frac{3}{2}, \frac{5}{2})$. Hence, there can be at most one solution to the equation (3.3.18). Moreover, we have $\theta\phi_Q'(\theta)/\phi_Q(\theta) < \log \phi_Q(\theta)$ if $\theta \in (\frac{3}{2}, \theta^*)$.

The Biggins martingale convergence theorem [46, Theorem 1] states that the following three conditions are equivalent:

(i) $M_\infty^{(\theta)} > 0$ a.s.

(ii) $\mathbb{E}M_\infty^{(\theta)} = 1$

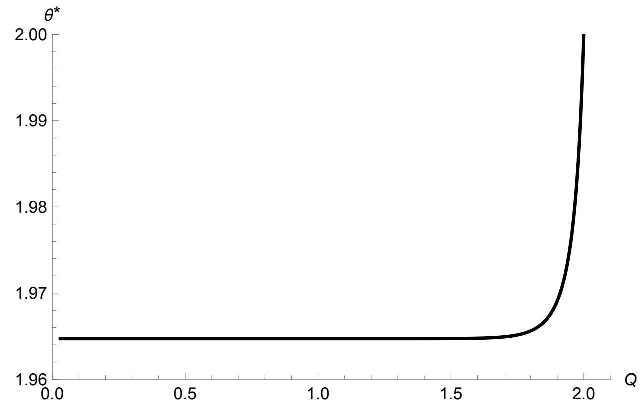

Figure 3.3: The unique value of θ^* satisfying $\mu_Q = (1/\theta^*)\log\phi_Q(\theta^*)$, obtained by solving (3.3.21) numerically. We observe $\theta^* \approx 1.9647$ as $Q \to 0$.

(iii) $\theta(\phi_Q)'(\theta)/\phi_Q(\theta) < \log\phi_Q(\theta)$ and $\mathbb{E}\big[M_1^{(\theta)}\log_+ M_1^{(\theta)}\big] < \infty$.

We verify the third condition for each $\theta \in (3/2, \theta^*)$ using the moment estimates for the critical multiplicative cascade Z in [70, Lemma 13] (see Footnote 4). This result states that $\mathbb{E}\big[(\sum_{|u|=1} Z(u)^\theta)^p\big] < \infty$ for every $p < \frac{5}{2\theta}$. Recall from (3.2.13) that $Z_Q(i) = Z(i)\exp(\beta_Q Y_i)$ for $i \in \mathbb{N}$ where $\{Y_u\}_{i\in\mathbb{N}}$ are i.i.d. Rademacher random variables that are also independent from the cascade Z. Therefore,

$$\begin{aligned}\mathbb{E}\big[M_1^{(\theta)}\log_+ M_1^{(\theta)}\big] \le \mathbb{E}\big[\big(M_1^{(\theta)}\big)^p\big] &= \mathbb{E}\Bigg[\bigg(\sum_{|u|=1}(Z(u))^\theta e^{\theta\beta_Q Y_u}\bigg)^p\Bigg] \\ &\le e^{p\theta\beta_Q}\,\mathbb{E}\Bigg[\bigg(\sum_{|u|=1} Z(u)^\theta\bigg)^p\Bigg] < \infty\end{aligned} \tag{3.3.20}$$

choosing any $p \in (1, \frac{5}{2\theta})$. □

Using the explicit formula (3.3.4) for ϕ_Q, the equation (3.3.18) satisfied by θ^* is equivalent to

$$\pi\tan(\pi\theta^*) + \beta_Q\tanh(\beta_Q\theta^*) = \frac{1}{\theta^*}\log\frac{\cosh(\beta_Q\theta^*)}{\cos(\pi\theta^*)}. \tag{3.3.21}$$

The numerical solution of this equation for the range $Q \in (0, 2)$ is displayed in Figure 3.3.

For $\theta \in (\frac{3}{2}, \theta^*)$ and $u \in \mathcal{U}$, define

$$M_n^{(\theta)}(u) := \frac{1}{\big(\phi_Q(\theta)\big)^n} \sum_{|v|=n, v \succ u} \big(Z_Q(v)\big)^\theta \quad \text{and} \quad W_u^{(\theta)} := \lim_{n\to\infty} M_n^{(\theta)}(u), \tag{3.3.22}$$

where $W_u^{(\theta)}$ is the a.s. limit of the uniformly integrable martingale $\{M_n^{(\theta)}(u)\}_{n\geq |u|}$. Note that, for every $u \in \mathcal{U}$ and $n \geq k > |u|$, we have

$$M_n^{(\theta)}(u) = \frac{\sum_{|v'|=n, v'\succ u} \big(Z_Q(v')\big)^\theta}{\big(\phi_Q(\theta)\big)^n} = \sum_{|v|=k, v\succ u} \frac{\sum_{|v'|=n, v'\succ v} \big(Z_Q(v')\big)^\theta}{\big(\phi_Q(\theta)\big)^n} = \sum_{|v|=k, v\succ u} M_n^{(\theta)}(v). \tag{3.3.23}$$

Thus, we have

$$W_u^{(\theta)} = \lim_{n\to\infty} \sum_{|v|=k, v\succ u} M_k^{(\theta)}(v) \geq \sum_{|v|=k, v\succ u} \lim_{n\to\infty} M_k^{(\theta)}(v) = \sum_{|v|=k, v\succ u} W_v^{(\theta)} \tag{3.3.24}$$

almost surely by Fatou's lemma. On the other hand, since $\{M_n^{(\theta)}(v)\}_{n\geq |v|}$ is uniformly integrable for each $v \in \mathbb{N}$,

$$\mathbb{E}\big[W_u^{(\theta)}\big] = \mathbb{E}\big[M_k^{(\theta)}(u)\big] = \sum_{|v|=k, v\succ u} \mathbb{E}\big[M_k^{(\theta)}(v)\big] = \sum_{|v|=k, v\succ u} \mathbb{E}\big[W_v^{(\theta)}\big] = \mathbb{E}\Big[\sum_{|v|=k, v\succ u} W_v^{(\theta)}\Big]. \tag{3.3.25}$$

Hence,

$$W_u^{(\theta)} = \sum_{|v|=k, v\succ u} W_v^{(\theta)} \quad \text{a.s.} \tag{3.3.26}$$

and thus $W_u^{(\theta)}$ defines a random measure on $\mathcal{U}$ with respect to the product σ-algebra.

The analog of Lemma 3.3.4 holds for the derivative martingale

$$M_n^* := \sum_{|u|=n} \frac{(Z_Q(u))^{\theta^*}}{(\phi_Q(\theta^*))^n} \left(n \frac{\phi_Q'(\theta^*)}{\phi_Q(\theta^*)} - \log Z_Q(u) \right) = -\frac{dM_n^{(\theta)}}{d\theta}\bigg|_{\theta=\theta^*} \tag{3.3.27}$$

defined in [47]. This fact can be verified by checking the conditions in [8] again using the formulas in [70]. Hence,

$$W_u^* := \lim_{n\to\infty} \sum_{|v|=n, v\succ n} \frac{Z_Q(v)^{\theta^*}}{\phi_Q(\theta^*)^n}\left(n\frac{\phi_Q'(\theta^*)}{\phi_Q(\theta^*)} - \log Z_Q(v)\right) \tag{3.3.28}$$

also defines a random measure for the product σ-algebra on $\mathcal{U}$.

The following proposition gives a construction for a random finite measure ξ which is almost surely determined by the CLE$_4$-coupled supercritical LQG disk $(\mathbb{D}, h, \overline{\Gamma})/\sim_Q$ through the multiplicative cascade Z_Q.

Proposition 3.3.5. *Let $(\ell_u)_{u\in\mathcal{U}}$ be the loops of a nested CLE$_4$ in $\mathbb{D}$ indexed so that $u \prec v$ if and only if ℓ_u encloses ℓ_v. Denote the open set enclosed by ℓ_u as D_u. Suppose $\{W_u\}_{u\in\mathcal{U}}$ is a collection of nonnegative random variables which satisfy $W_u = \sum_{k\in\mathbb{N}} W_{uk}$ for every $u \in \mathcal{U}$. Then, there exists an almost surely unique random Borel measure ξ on $\overline{\mathbb{D}}$ such that $\xi(\overline{D_u}) = W_u$. Moreover, if $\xi^{(n)}$ is any sequence of random Borel measures on $\overline{\mathbb{D}}$ such that $\xi^{(n)}(\overline{D_u}) = W_u$ almost surely for every $|u| \le n$, then, almost surely, $\xi^{(n)}$ converges weakly to ξ as $n \to \infty$.*

Proof. Let $\mathcal{C}$ be a countable collection of nonnegative continuous functions on $\overline{\mathbb{D}}$ such that two (deterministic) Borel measures μ and $\tilde{\mu}$ on $\overline{\mathbb{D}}$ are the same if $\int_{\overline{\mathbb{D}}} f\, d\mu = \int_{\overline{\mathbb{D}}} f\, d\tilde{\mu}$ for every $f \in \mathcal{C}$.

Uniqueness. Suppose ξ is a random Borel measure on $\overline{\mathbb{D}}$ that almost surely satisfies $\xi(\overline{D_u}) = W_u$ for every $u \in \mathcal{U}$. For each $f \in \mathcal{C}$ and $n \in \mathbb{N}$, define

$$W_{f-}^{(n)} := \sum_{|u|=n} W_u \cdot \inf_{z\in\overline{D_u}} f(z) \quad \text{and} \quad W_{f+}^{(n)} := \sum_{|u|=n} W_u \cdot \sup_{z\in\overline{D_u}} f(z). \tag{3.3.29}$$

Note that since $\xi(\overline{D_u}) = \sum_{k\in\mathbb{N}} \xi(\overline{D_{uk}})$, we have $\xi(\overline{D_u} \setminus (\bigcup_{k\in\mathbb{N}} \overline{D_{uk}})) = 0$ a.s for every $u \in \mathcal{U}$.

Hence,

$$W_{f-}^{(n)} = \sum_{|u|=n} \xi(\overline{D_u}) \cdot \inf_{z \in \overline{D_u}} f(z) \le \int_{\overline{\mathbb{D}}} f \, d\xi \le \sum_{|u|=n} \xi(\overline{D_u}) \cdot \sup_{z \in \overline{D_u}} f(z) = W_{f+}^{(n)} \quad \text{a.s.} \tag{3.3.30}$$

Also, since $\overline{D_{uk}} \subset \overline{D_u}$ for every $k \in \mathbb{N}$,

$$W_{f-}^{(n)} = \sum_{|u|=n} \xi(\overline{D_u}) \cdot \inf_{z \in \overline{D_u}} f(z) \le \sum_{|u|=n} \sum_{k \in \mathbb{N}} \xi(\overline{D_{uk}}) \cdot \inf_{z \in \overline{D_{uk}}} f(z) = W_{f-}^{(n+1)} \quad \text{a.s.} \tag{3.3.31}$$

and similarly $W_{f+}^{(n)} \ge W_{f+}^{(n+1)}$ a.s. for every n. By [25, Lemma 2.1],

$$\lim_{n \to \infty} \sup_{|u|=n} \operatorname{diam}(\overline{D_u}) \to 0 \quad \text{a.s.} \tag{3.3.32}$$

Hence, for each $\varepsilon > 0$, there exists a positive integer N such that

$$\mathbb{P}\left(\sup_{|u| \ge N} \sup_{x,y \in \overline{D_u}} |f(x) - f(y)| \le \varepsilon \right) \ge 1 - \varepsilon. \tag{3.3.33}$$

On this event, $0 \le W_{f+}^{(n)} - W_{f-}^{(n)} \le \varepsilon \sum_{|u|=n} W_u = \varepsilon W_\emptyset$ for every $n \ge N$. Taking $\varepsilon \downarrow 0$, we have that $\lim_{n\to\infty} W_{f\pm}^{(n)} = \int_{\overline{\mathbb{D}}} f \, d\xi$ almost surely. Since $W_{f\pm}^{(n)}$ does not depend on the choice of ξ, we conclude that this random measure is almost surely unique.

Existence. Suppose $\xi^{(n)}$ is a sequence of random Borel measures on $\overline{\mathbb{D}}$ such that, for each $|u| \le n$, we have $\xi^{(n)}(\overline{D_u}) = W_u$ almost surely. For instance, we may choose $\xi^{(n)}$ where, if Leb is the Lebesgue measure on $\overline{\mathbb{D}}$, then

$$\frac{d\xi^{(n)}}{d\mathrm{Leb}}(z) = \sum_{|u|=n} \frac{W_u}{\mathrm{Leb}(D_u)} \mathbf{1}\{z \in \overline{D_u}\}. \tag{3.3.34}$$

We have $W_{f-}^{(n)} \le \int f \, d\xi^{(n)} \le W_{f+}^{(n)}$ a.s. for continuous $f : \overline{\mathbb{D}} \to [0, \infty)$, in an analogous way to (3.3.30). Hence,

$$\lim_{n\to\infty}\int_{\overline{\mathbb{D}}} f\,d\xi^{(n)} = \lim_{n\to\infty} W^{(n)}_{f\pm} =: W_f \quad \text{a.s.} \tag{3.3.35}$$

The convergence $\int_{\overline{\mathbb{D}}} f\,d\xi^{(n)} \xrightarrow{d} W_f$ for all nonnegative continuous f on $\overline{\mathbb{D}}$ implies the existence of a random measure ξ such that $(\int_{\overline{\mathbb{D}}} f\,d\xi)_{f\in\mathcal{C}} \stackrel{d}{=} (W_f)_{f\in\mathcal{C}}$ jointly [147, Lemma 5.1]. Recall that W_f is a measurable function of $(W_u,\ell_u)_{u\in\mathcal{U}}$. Since $(\int_{\overline{\mathbb{D}}} f\,d\xi)_{f\in\mathcal{C}}$ determines ξ uniquely at each point in the probability space, there exists a coupling of ξ with $(W_u,\ell_u)_{u\in\mathcal{U}}$ such that $\int_{\overline{\mathbb{D}}} f\,d\xi = W_f$ a.s. for every $f\in\mathcal{C}$, and hence for every continuous f on $\overline{\mathbb{D}}$.

Let us now check that $\xi(\overline{D_u}) = W_u$ a.s. for every $u\in\mathcal{U}$. Fixing $u\in\mathcal{U}$ and choosing a sequence of random continuous functions f_k approximating $\mathbf{1}_{D_u}$ from below, we have

$$\xi(\overline{D_u}) \geq \sup_k W_{f_k} = \sup_k \sup_n W^{(n)}_{f_k-} \quad \text{a.s.} \tag{3.3.36}$$

Since $\xi^{(n)}$ is supported on $\bigcup_{|v|=n}\overline{D_v}$, which almost surely does not intersect $\eta_u = \partial D_u$ when $n > |u|$, we have $\sup_k W^{(n)}_{f_k-} = \xi^{(n)}(\overline{D_u}) = W_u$ almost surely. Hence, $\xi(\overline{D_u}) \geq W_u$ a.s. for every $u\in\mathcal{U}$. Since the loops of nested CLE_4 are a.s. disjoint, we have

$$\sum_{|u|=n} W_u \leq \sum_{|u|=n} \xi(\overline{D_u}) \leq \xi(\overline{\mathbb{D}}) = \sum_{|u|=n} W_u \quad \text{a.s.} \tag{3.3.37}$$

Therefore, $\xi(\overline{D_u}) = W_u$ almost surely for every $u\in\mathcal{U}$. $\square$

A natural choice for the process $W = (W_u)_{u\in\mathcal{U}}$ is to set either $W = W^{(\theta)}$ for $\theta\in(\frac{3}{2},\theta^*)$ or $W = W^*$. For any of these choices, since the Q-LQG lengths $(\mathfrak{n}_h(\ell_u))_{u\in\mathcal{U}}$ are invariant under different choices of embedding the nested-CLE_4-decorated supercritical LQG disk, the measure ξ that we constructed also satisfies the Q-LQG coordinate change rule (3.1.8). We do not have a suggestion for a value of θ that has a special meaning within the context of Liouville theory with central charge $\mathbf{c}_{\mathrm{L}}\in(1,25)$.

Remark 3.3.6. The measure ξ violates the locality condition of Theorem 3.1.5; that is,

$\xi(U)$ is not determined by the domain $U \subset \mathbb{D}$ and the local field $h|_U$. Indeed, given the CLE_4 loop $\ell_u \in \overline{\Gamma}$ and the field $h|_{D_u}$ inside the loop ℓ_u, the value of $\xi(D_u)$ still depends on $|u|$ — i.e., the generation of the loop ℓ_u within the nested CLE_4 $\overline{\Gamma}$ — since $\phi_Q(\theta) > 1$ for any $Q \in (0,2)$ and $\theta \in (3/2, \theta^*)$.

Note that ξ is almost surely supported on the set of points which are surrounded by infinitely many CLE_4 loops. Using the many-to-one formula for the branching random walk (see, e.g., [233, Theorem 1.1]), one can check that if $W = W^{(\theta)}$ (resp. W^*), then the corresponding random measure ξ is a.s. supported on the points $z \in \mathbb{D}$ such that if ℓ_{u_n} is the nth generation CLE_4 loop enclosing z, then $\frac{1}{n} \log Z_Q(u_n) \to \phi'_Q(\theta)/\phi_Q(\theta)$ (resp. $\phi'_Q(\theta^*)/\phi_Q(\theta^*)$) as $n \to \infty$. It would be interesting to investigate how these points are related with the thick points of the field.

3.4 Background on random planar maps

We now move on to the second part of this article, which studies the random planar map model of a supercritical LQG disk introduced in Section 3.1.2. We begin this section by giving the precise definitions for the laws of planar maps involved in the construction of this discrete model. Its connection to the continuum model described in Section 3.2 is given in Proposition 3.4.13, which states that the perimeters of marked faces in our discrete model converge in the scaling limit to the lengths of CLE_4 loops on the supercritical LQG disk.

We also introduce the random walk representation of the perimeters of faces in a Boltzmann map (Proposition 3.4.7), which is a key tool in the proof of Proposition 3.1.3 in Section 3.5. We finally survey the convergence results for subcritical Boltzmann maps (Propositions 3.4.17 and 3.4.18) that form the key input for the proof of Theorem 3.1.4 in Section 3.7.

3.4.1 Definitions of Boltzmann maps and ring distributions

There were two kinds of planar maps needed in the construction of Definition 3.1.1: Boltzmann maps and rings. We first give the definition of a Boltzmann map with a fixed perimeter for the root face. Recall that $\mathrm{per}(f)$ refers to the half-perimeter of the face f.

Definition 3.4.1. For $p \in \mathbb{N}$, let $\mathcal{M}^{(p)}$ denote the set of all finite bipartite rooted planar maps with the degree of the root face equal to $2p$. Given a weight sequence $\mathbf{q} = (q_i)_{i\in\mathbb{N}}$ of nonnegative real numbers, define the Boltzmann weight of a map $M \in \bigcup_{p\in\mathbb{N}} \mathcal{M}^{(p)}$ as

$$w_{\mathbf{q}}(M) = \prod_{f\in\mathfrak{F}(M)} q_{\mathrm{per}(f)}, \tag{3.4.1}$$

where $\mathfrak{F}(M)$ refers to the collection of inner (non-root) faces of M. We say that the weight sequence $\mathbf{q}$ is admissible if $W_{\mathbf{q}}^{(p)} := \sum_{M\in\mathcal{M}^{(p)}} w_{\mathbf{q}}(M) < \infty$ for all $p \in \mathbb{N}$. In this case, we define for each $p \in \mathbb{N}$ the corresponding probability measure

$$\mathbb{P}_{\mathbf{q}}^{(p)}(M) = \frac{1}{W_{\mathbf{q}}^{(p)}} w_{\mathbf{q}}(M), \quad M \in \mathcal{M}^{(p)} \tag{3.4.2}$$

of Boltzmann maps with boundary length $2p$.

Given a weight sequence $\mathbf{q}$, define

$$f_{\mathbf{q}}(x) := \sum_{k=1}^{\infty} \binom{2k-1}{k-1} q_k x^{k-1}, \quad x \geq 0. \tag{3.4.3}$$

By [184, Proposition 1], $\mathbf{q}$ is admissible if and only if the equation $f_{\mathbf{q}}(x) = 1 - 1/x$ has a positive solution. Let $Z_{\mathbf{q}}$ denote the smallest such solution, so that the measure $\mu_{\mathbf{q}}$ in Definition 3.4.2 below is a probability measure. It arises as the offspring distribution of a Galton–Watson tree associated to a Boltzmann map with weight sequence $\mathbf{q}$ via the Bouttier–Di Francesco–Guitter bijection [58] with the Janson–Stefánsson trick [144]; see [70, Proposition 4] for further details.

Definition 3.4.2. If $\mathbf{q}$ is an admissible weight sequence, we associate to it a probability measure $\mu_{\mathbf{q}}$ on nonnegative integers given by

$$\mu_{\mathbf{q}}(k) = (Z_{\mathbf{q}})^{k-1}\binom{2k-1}{k-1}q_k, \quad k \geq 1 \tag{3.4.4}$$

and $\mu_{\mathbf{q}}(0) = 1/Z_{\mathbf{q}}$, where $Z_{\mathbf{q}}$ is the smallest positive solution to the equation $f_{\mathbf{q}}(x) = 1-1/x$.

The measure $\mu_{\mathbf{q}}$ allows us to classify the weight sequence $\mathbf{q}$ depending on the large-scale behavior of Boltzmann maps sampled from $\mathbb{P}_{\mathbf{q}}^{(p)}$. The following classification is drawn from [70, Definition 1] and [76, Section 5.3].

Definition 3.4.3. An admissible weight sequence $\mathbf{q} = (q_i)_{i\in\mathbb{N}}$ is said to be critical if the associated probability measure $\mu_{\mathbf{q}}$ in (3.4.4) has mean 1 and subcritical if its mean is strictly less than 1.

A critical weight sequence $\mathbf{q}$ is said to be generic critical if $\mu_{\mathbf{q}}$ has a finite variance. A critical weight sequence $\mathbf{q}$ is non-generic critical of type $a \in (3/2, 5/2)$ if $k^{a+1/2}\mu_{\mathbf{q}}(k)$ converges to a positive constant as $k \to \infty$.

In the critical generic case, there are no macroscopic faces and the scaling limit of the Boltzmann maps as the number of total vertices tends to infinity is the celebrated Brownian map [184, 174, 188]. The analogous scaling limit in the critical non-generic case depends on the value of a and is called a stable map [176]. The scaling limit in the subcritical case is the continuum random tree; see Section 3.4.4 for further details.

For the remainder of this work, we fix $\mathbf{q} = (q_i)_{i\in\mathbb{N}^\#}$ to be a non-generic critical weight sequence of type $\boldsymbol{a = 2}$. An important example of such a weight sequence is one corresponding to the gasket of a rigid $O(2)$ loop model-decorated quadrangulation.[5] More generally, a

5. The works [6, 149] show that the gasket of a rigid $O(2)$ loop-decorated bipartite map is a non-generic

Boltzmann map M sampled from the law $\mathbb{P}_{\mathbf{q}}^{(p)}$ associated with such a weight sequence can be thought of as a discrete version of the CLE_4 gasket, in the sense that the set of perimeters $\{\mathrm{per}(f_i)\}_{i\in\mathfrak{F}(M)}$ of the non-boundary faces converges in the scaling limit to the corresponding set of lengths of loops in a critical LQG disk decorated with independent CLE_4 (see Lemma 3.4.14).

We now define the distribution of rings which puts our discrete model in the universality class of supercritical LQG with given central charge. As promised earlier in the introduction, we allow the following more general distribution of rings in the rest of this article.

Definition 3.4.4. Let $\left(\mathbb{P}_{\mathrm{ring}}^{(p)}\right)_{p\in\mathbb{N}}$ be a sequence of probability measures on planar maps R with two distinguished faces — an outer face f_{out} and an inner face f_{in} — such that $\mathrm{per}(f_{\mathrm{out}}) = p$ and $\mathrm{per}(f_{\mathrm{in}}) \in \mathbb{N}^{\#}$. Let $\mathrm{Rat}(R)$ denote the ratio $\mathrm{per}(f_{\mathrm{in}})/\mathrm{per}(f_{\mathrm{out}})$ of the ring R. We assume that this sequence of distributions satisfy the following conditions.

(1) *Background charge:* Recall the background charge $Q \in (0,2)$ associated with supercritical LQG. The law of $\mathrm{Rat}(R)$ under $\mathbb{P}_{\mathrm{ring}}^{(p)}$ converges in distribution to $\exp(\beta_Q Y)$ as $p \to \infty$, where Y is a Rademacher random variable and

$$\beta_Q := \frac{\pi\sqrt{4-Q^2}}{Q}. \tag{3.4.5}$$

(2) *Non-thickness:* Let $\mathrm{Vol}(R)$ denote the total number of vertices in R. There is a constant $C > 0$ such that, for any $p \in \mathbb{N}$, a ring R sampled from $\mathbb{P}_{\mathrm{ring}}^{(p)}$ satisfies $\mathrm{Vol}(R) \leq$

critical Boltzmann map of type $a = 2$ in a more general sense than Definition 3.4.3, where a slowly varying correction is allowed for the tail of $\mu_{\mathbf{q}}$ as in [215, Definition 2.2] and [78, Definition 1]. For the gaskets of rigid $O(2)$ loop-decorated quadrangulations, Aïdékon, Da Silva, and Hu verify that $k^{5/2}\mu_{\mathbf{q}}(k)$ either asymptotic to a constant or grows logarithmically as $k \to \infty$ [6, Equation B.8], and gives an explicit condition for the former case.
We believe that Theorem 3.1.4 would continue to hold for random planar map models of supercritical LQG constructed using this more general class of non-generic critical Boltzmann maps of type $a = 2$. In this case, with slowly varying corrections to the estimates for $T_S^{(p)}$ and $L_S^{(p)}$ in Lemma 3.4.8, our results continue to hold prior to Lemma 3.6.9. However, we then no longer have the exponential decay in (3.6.24), so our proof of Lemma 3.6.9 is not valid with this generalization.

$C(\mathrm{per}(f_{\mathrm{out}}) + \mathrm{per}(f_{\mathrm{in}}))$ almost surely.

(3) *Non-triviality:* $\mathbb{P}^{(p)}_{\mathrm{ring}}(\mathrm{Rat}(R) > 0) > 0$ for all $p \in \mathbb{N}$.

(4) *Lower tail:* For each $\lambda > 0$, there exists a constant $c > 0$ such that $\mathbb{E}^{(p)}_{\mathrm{ring}}[e^{-\lambda p \,\mathrm{Rat}(R)}] \le e^{-cp}$ for all $p \in \mathbb{N}$.

(5) *Upper tail:* There exists a constant $\delta > 0$ such that $\sup_{p\in\mathbb{N}} \mathbb{E}^{(p)}_{\mathrm{ring}}[(\mathrm{Rat}(R))^{(2+\delta)}] < \infty$.

(6) *Gluing of boundaries:* There exists a constant $K < \infty$ such that, for any $p \in \mathbb{N}$, a ring R sampled from $\mathbb{P}^{(p)}_{\mathrm{ring}}$ satisfies the following property almost surely: each vertex on the outer boundary of R is at most distance K from its inner boundary, and vice versa.[6]

The main condition that we require is (1). We emphasize that the dependence of our planar map model on the central charge $\mathbf{c}_{\mathrm{L}}$ is precisely through the constant β_Q appearing in the law of $\mathrm{Rat}(R)$. In terms of the continuum picture in [18], the limiting law in the first condition corresponds to the ratio between the inner and outer perimeters of a CLE_4 loop in a CLE_4-decorated supercritical LQG disk (see (3.2.11)). The remaining conditions are technical hypotheses needed in our proofs which we do not claim to be optimal. For instance, with a more restrictive law for the rings (e.g., $\mathrm{per}(f_{\mathrm{in}}) \overset{d}{=} \lfloor p \exp(\beta_Q Y) \rfloor$ for all $p \in \mathbb{N}$ as in [18]), many of the estimates in our work can be improved. We refer the reader to Remark 3.6.5 for a further discussion of this point.

Remark 3.4.5. Our construction of the planar map in Definition 3.1.1 has two main differences from that in [18]. First, in place of gaskets of the fully-packed $O(2)$ loop-decorated triangulations, we use critical non-generic Boltzmann maps of type $a = 2$. This change allows us to extend and apply the results of [70], which provided an exact solvability of the

6. This condition is not needed in our proofs. However, it is imposed so that planar maps sampled from the *unconditioned* law $\mathbb{P}^{(p)}_{\infty}$ can be reasonably expected to converge in the scaling limit to the supercritical LQG disk decorated by CLE_4. For instance, this condition excludes a ring where the inner and outer boundaries are connected through a single edge, since this would yield an outsize impact to the distances between the vertices to the inside and the outside of the ring.

perimeter process of the loop $O(n)$ model for $n \in (0,2)$ via an analysis of the gaskets in terms of the corresponding Boltzmann maps. We note that, for $n \in (0,2)$, the distribution of the gasket of a loop $O(n)$ model-decorated planar map has been identified in [57] with a critical non-generic Boltzmann map with $a = 2 + (1/\pi)\arccos(n/2)$. Furthermore, the above was extended to the case $n = 2$ independently in the works [6] and [149].

The other difference in our model from that of [18] is the general class of rings that we allow in Definition 3.4.4. Since Definition 3.1.1 outputs an infinite map with positive probability (see Lemma 3.5.2), it is a priori conceivable that minor changes in the sequence $\big(\mathbb{P}^{(p)}_{\text{ring}}\big)_{p\in\mathbb{N}}$ may drastically affect the law $\mathbb{P}^{(p)}_{\mathbf{F}}$. Our main result Theorem 3.1.4 implies that for any reasonable choice of ring distribution, its effect on the global geometry of the planar map, when conditioned to be finite, becomes negligible as $p \to \infty$.

3.4.2 The connection between Boltzmann maps and random walks

An important ingredient in this work is the understanding of the distribution of the perimeters of the faces of a Boltzmann map sampled from $\mathbb{P}^{(p)}_{\overline{\mathbf{q}}}$ — i.e., the probability measure on rooted bipartite maps with boundary perimeter $2p$ which is proportional to the Boltzmann weight (3.4.1). Given a Boltzmann map M sampled from $\mathbb{P}^{(p)}_{\overline{\mathbf{q}}}$, let $\boldsymbol{\chi}_M = (\chi_M(i))_{i\in\mathbb{N}}$ encode the half-perimeters $\operatorname{per}(f)$ for $f \in \mathfrak{F}(M)$ listed in a non-increasing order with duplicity, padded by $\chi_M(i) = 0$ for $i > |\mathfrak{F}(M)|$. A key tool in our proof is the encoding of the law of $\boldsymbol{\chi}_M$ is the following random walk.

Definition 3.4.6. Let $\overline{\mathbf{q}}$ be an admissible weight sequence. Consider the random walk $S_n = \sum_{i=1}^n X_i$ starting at 0 and with steps X_i sampled independently from the distribution

$$\mathbb{P}\{X_i = k\} = \mu_{\overline{\mathbf{q}}}(k+1), \quad k \in \{-1, 0, 1, \dots\}, \tag{3.4.6}$$

where $\mu_{\overline{\mathbf{q}}}$ is the probability measure associated with the weight sequence $\mathbf{q}$ defined in (3.4.4).

Let $T_S^{(p)} = \inf\{n \geq 0 : S_n = -p\}$ be the first time that this walk hits $-p$. Let $L_S^{(p)} = \sum_{i=1}^{T_S^{(p)}} \mathbf{1}_{\{X_i=-1\}}$ be the total number of negative steps (always -1) taken by the walk up to time $T_S^{(p)}$. Let $\boldsymbol{\chi}_S^{(p)} = (\chi_S^{(p)}(i))_{i\in\mathbb{N}}$ be the non-increasing sequence of integers satisfying

$$\{\chi_S^{(p)}(i)\}_{i\in[\![1,T_S^{(p)}]\!]} = \{X_n + 1\}_{n\in[\![1,T_S^{(p)}]\!]} \tag{3.4.7}$$

as multi-sets and $\chi_S^{(p)}(i) = 0$ for $i > T_S^{(p)}$.

The following identity is given in [70, Section 2.3.1] based on the bijections of [58, 144].

Proposition 3.4.7. *Let* $\mathbf{q}$ *be an admissible weight sequence and let* M *be a Boltzmann map sampled from* $\mathbb{P}_{\mathbf{q}}^{(p)}$ *for any* $p \in \mathbb{N}$. *Then, the law of* $\boldsymbol{\chi}_M$ *is equal to that of* $\boldsymbol{\chi}_S^{(p)}$ *weighted by* $(L_S^{(p)}+1)^{-1}$. *That is, for any positive measurable function* $F\colon \mathbb{R}^{\mathbb{N}} \to \mathbb{R}$, *we have*

$$\mathbb{E}_{\mathbf{q}}^{(p)}[F(\boldsymbol{\chi}_M)] = \frac{\mathbb{E}[(L_S^{(p)}+1)^{-1}F(\boldsymbol{\chi}_S^{(p)})]}{\mathbb{E}[(L_S^{(p)}+1)^{-1}]}. \tag{3.4.8}$$

Our proofs of Theorems 3.1.3 and 3.1.4 in Sections 3.5 and 3.6 make heavy uses of this random walk encoding of Boltzmann maps and its basic properties that we now collect. We first consider properties of the random walk associated with our critical non-generic weight sequence $\mathbf{q}$ of type $a = 2$. These results can be stated with slight modification for other values of $a \in (3/2, 5/2)$ as well, but we do not stray away from the $a = 2$ case that we consider exclusively in our work.

Lemma 3.4.8. *Let* $\mathbf{q}$ *be a critical non-generic weight sequence of type* $a = 2$. *Let* S_n *be the associated random walk and* $T_S^{(p)}$ *and* $L_S^{(p)}$ *be the random variables defined as in Definition 3.4.6. Then,* $p^{-3/2}T_S^{(p)}$ *converges in distribution to a positive* $2/3$*-stable random variable. Moreover, there is a constant* $c > 0$ *depending only on* $\mathbf{q}$ *such that for all* $\varepsilon > 0$

and $p \in \mathbb{N}$, we have

$$\mathbb{P}(T_S^{(p)} \leq \varepsilon p^{3/2}) \leq \mathbb{P}(L_S^{(p)} \leq \varepsilon p^{3/2}) \leq e^{-c\varepsilon^{-2}}. \tag{3.4.9}$$

Consequently, there exists a constant $C > 0$ such that $\mathbb{E}[(L_S^{(p)} + 1)^{-1}] \geq Cp^{-3/2}$ for all $p \in \mathbb{N}$.

Proof. The convergence of $p^{-3/2}T_S^{(p)}$ is established within the proof of [70, Proposition 3] and (3.4.9) is proved in Lemma 5 of the same article. To prove the last estimate, it suffices to show $\mathbb{E}[(T_S^{(p)})^{-1}] \geq Cp^{-3/2}$ since $L_S^{(p)} \leq T_S^{(p)}$. We know from (3.4.9) that $\{p^{3/2}(T_S^{(p)})^{-1}\}_{p\in\mathbb{N}}$ is uniformly integrable. Denoting by τ the positive 2/3-stable random variable that $p^{-3/2}T_S^{(p)}$ converges in law to, we conclude $p^{3/2}\mathbb{E}[1/T_S^{(p)}] \to \mathbb{E}[1/\tau]$. This limit is finite, as can be checked from well-known estimates on the density of τ (see, e.g., [257, Chapter 2.5]). □

We now move on to estimates for $T_S^{(p)}$ and $L_S^{(p)}$ corresponding to a subcritical weight sequence. Let

$$m_{\overline{\mathbf{q}}} := \sum_{k=-1}^{\infty} k\mu_{\overline{\mathbf{q}}}(k+1) \in (-1, 0) \tag{3.4.10}$$

be the mean of the step distribution (3.4.6) of the associated random walk S.

Lemma 3.4.9. *Let $\overline{\mathbf{q}}$ be a subcritical admissible weight sequence. Then,*

$$\lim_{p\to\infty} \mathbb{P}\big(T_S^{(p)} > -2p/m_{\overline{\mathbf{q}}}\big) = 0. \tag{3.4.11}$$

Furthermore, for all $p \in \mathbb{N}$, we have $p^{-1}\mathbb{E}T_S^{(p)} = \mathbb{E}T_S^{(1)} < \infty$ and

$$\mathbb{E}\big[(L_S^{(p)} + 1)^{-1}\big] \geq \mathbb{E}\big[(T_S^{(p)} + 1)^{-1}\big] \geq \big(p\,\mathbb{E}[T_S^{(1)}] + 1\big)^{-1}. \tag{3.4.12}$$

Proof. Note that $\{T_S^{(p)} > -2p/m_{\overline{\mathbf{q}}}\} \subseteq \{S_{\lfloor -2p/m_{\overline{\mathbf{q}}}\rfloor} > -p\}$. Since $\mathbb{E}[S_{\lfloor -2p/m_{\overline{\mathbf{q}}}\rfloor}] \sim -2p$, by the law of large numbers, the probability of the latter event decreases to 0 as $p \to \infty$.

To obtain (3.4.12), note that $\mathbb{E}T_S^{(p)} = p \cdot \mathbb{E}T_S^{(1)}$ since the only possible negative jumps of S are -1. We have $\mathbb{E}T_S^{(1)} < \infty$ from, e.g., [116, Theorem 2.1]. The claim then follows by Jensen's inequality since the function $x \mapsto (x+1)^{-1}$ is convex. $\square$

As a simple instance of extracting information about Boltzmann maps from the corresponding random walk, we show that (3.4.11) gives an upper bound on the number of faces in a subcritical Boltzmann map. Recall that for a rooted planar map M, we use $\mathfrak{F}(M)$ to denote the set of interior (or non-root) faces of M.

Lemma 3.4.10. *If $\overline{\mathbf{q}}$ is a subcritical weight sequence, then $\mathbb{P}_{\overline{\mathbf{q}}}^{(p)}(|\mathfrak{F}(M)| \leq -2p/m_{\overline{\mathbf{q}}}) \to 1$ as $p \to \infty$.*

Proof. By Proposition 3.4.7, it suffices to show that

$$\lim_{p\to\infty} \frac{\mathbb{E}[(L_S^{(p)}+1)^{-1}\mathbf{1}\{T_S^{(p)} > -2p/m_{\overline{\mathbf{q}}}\}]}{\mathbb{E}[(L_S^{(p)}+1)^{-1}]} = 0. \tag{3.4.13}$$

In Lemma 3.4.11 below, we show that $(L_S^{(p)}+1)^{-1}$ is a bounded and decreasing function of the sequence $(X_1, X_2, \dots)$, whence $\mathbf{1}\{T_S^{(p)} \geq -2p/m_{\overline{\mathbf{q}}}\}$ is an increasing function of the same sequence. Thus, the FKG inequality implies that the ratio in (3.4.13) is bounded above by $\mathbb{P}(T_S^{(p)} > -2p/m_{\overline{\mathbf{q}}})$, which tends to 0 as $p \to \infty$ by Lemma 3.4.9. $\square$

Lemma 3.4.11. *Let $(X_1, X_2, \dots)$ be a sequence of integers taking values in $\{-1\} \cup \mathbb{N}^{\#}$ and consider the walk $S_n = X_1 + \cdots + X_n$ for $n \in \mathbb{N}$. Let $T_S^{(p)}$ be the first time that the walk hits $-p$ and $L_S^{(p)}$ be the total number of negative steps until this hitting time. Then, $T_S^{(p)}$ and $L_S^{(p)}$ are increasing functions of the sequence $(X_1, X_2, \dots)$ for every $p \in \mathbb{N}$.*

Proof. Fix $p \in \mathbb{N}$. Suppose $(\widetilde{X}_1, \widetilde{X}_2, \dots)$ is another sequence of integers such that $X_i \leq \widetilde{X}_i$ for every $i \in \mathbb{N}$. Let $\widetilde{S}_n = \widetilde{X}_1 + \cdots + \widetilde{X}_n$ and define $T_{\widetilde{S}}^{(p)}$ and $L_{\widetilde{S}}^{(p)}$ analogously. Since $S_i \leq \widetilde{S}_i$ for all i, we have $T_S^{(p)} \leq T_{\widetilde{S}}^{(p)}$ almost surely.

Let us show $L^{(p)}_S \le L^{(p)}_{\widetilde{S}}$. It suffices to show this for the case that $X_j < \widetilde{X}_j$ at a unique index $j \in [\![1, T^{(p)}_{\widetilde{S}}]\!]$ and $X_i = \widetilde{X}_i$ for all other $i \le T^{(p)}_{\widetilde{S}}$, since we can make comparisons while changing $(X_i)_{i \le T^{(p)}_{\widetilde{S}}}$ to $(\widetilde{X}_i)_{i \le T^{(p)}_{\widetilde{S}}}$ index-by-index. In fact, since $T^{(p)}_S = T^{(p)}_{\widetilde{S}}$ if $X_i = \widetilde{X}_i$ for every $i \in [\![1, T^{(p)}_S]\!]$, so we only need to consider the case that $X_j < \widetilde{X}_j$ for some $j \in [\![1, T^{(p)}_S]\!]$ and $X_i = \widetilde{X}_i$ for all $i \ne j$ in this interval. If $X_j \ne -1$, then X_j is not counted in $L^{(p)}_S$, so $T^{(p)}_S \le T^{(p)}_{\widetilde{S}}$ implies $L^{(p)}_S \le L^{(p)}_{\widetilde{S}}$. If $X_j = -1$ and $\widetilde{X}_j \ge 0$, then $\widetilde{S}_{T^{(p)}_S} > S_{T^{(p)}_S} = -p$, so there exists at least one negative X_i among $T^{(p)}_S < i \le T^{(p)}_{\widetilde{S}}$. Hence,

$$L^{(p)}_{\widetilde{S}} = \sum_{i=1}^{T^{(p)}_S} (\mathbf{1}\{X_i = -1\} - \mathbf{1}_{i=j}) + \sum_{i=T^{(p)}_S+1}^{T^{(p)}_{\widetilde{S}}} \mathbf{1}\{\widetilde{X}_i = -1\} \ge (L^{(p)}_S - 1) + 1 = L^{(p)}_S. \quad (3.4.14)$$

Hence, $L^{(p)}_S \le L^{(p)}_{\widetilde{S}}$ whenever $X_i \le \widetilde{X}_i$ for all i; this completes the proof. □

3.4.3 Perimeter cascade of supercritical planar maps

In this subsection, we investigate the law of the perimeters of faces that appear in the iterative construction of supercritical maps in Definition 3.1.1. Let us first define a process indexed by the Ulam tree $\mathcal{U} = \bigcup_{n \in \mathbb{N}^\#} \mathbb{N}^n$ which describes these perimeters along with their genealogy. For $u \in \mathbb{N}^n \subset \mathcal{U}$, denote $|u| = n$.

Definition 3.4.12. Given $p \in \mathbb{N}$, let $\{(M_i, \mathfrak{F}_i)\}_{i \in \mathbb{N}^\#}$ be the Markov chain in Definition 3.1.1 with the law $\mathbb{P}^{(p)}_\infty$. Let M be the random planar map resulting from this chain. We define the processes $\boldsymbol{\chi}^{\text{in}}_M = (\chi^{\text{in}}_M(u))_{u \in \mathcal{U}}$ and the indexing $f : \mathcal{U} \to \{\partial M, \varnothing\} \cup (\bigcup_{i \in \mathbb{N}^\#} \mathfrak{F}_i)$ through the following inductive procedure.

- Let $f_\emptyset = \partial M$ and $\chi^{\text{in}}_M(\emptyset) = p$.
- Suppose we have $\{(f_u, \chi^{\text{in}}_M(u))\}_{|u|=i}$. Here is how we define $\{(f_u, \chi^{\text{in}}_M(u))\}_{|u|=i+1}$.

– For each $u \in \mathcal{U}$ with $|u| = i$ and $f_u \neq \varnothing$, recall that $\mathfrak{F}(M(f_u))$ are the faces of the Boltzmann map glued into the face $f_u \in \mathfrak{F}_i$. With $m = |\mathfrak{F}(M(f_u))|$, let $f_{u1}, f_{u2}, \ldots, f_{um}$ be an enumeration of the faces $\mathfrak{F}(M(f_u))$ such that, respectively, $\mathrm{per}(f_{u1}), \ldots, \mathrm{per}(f_{um})$ is in a non-increasing order. Let $\chi^{\mathrm{in}}_M(uk)$ be the inner half-perimeter $\mathrm{per}_{\mathrm{in}}(f_{uk})$ of the ring attached to the face f_{uk}.

– If $f_u = \varnothing$ or $k > |\mathfrak{F}(M(f_u))|$, let $f_{uk} = \varnothing$ and $\chi^{\mathrm{in}}_M(uk) = 0$.

The main result of this subsection is that the above perimeter process $\boldsymbol{\chi}^{\mathrm{in}}_M$ converges in distribution as $p \to \infty$ to the multiplicative cascade Z_Q of the inner boundary lengths of CLE_4 loops in a unit boundary length supercritical LQG disk.

Proposition 3.4.13. *For each $p \in \mathbb{N}$, let $\boldsymbol{\chi}^{\mathrm{in}}_{M^{(p)}}$ denote the perimeter process of Definition 3.4.12 sampled from $\mathbb{P}^{(p)}_\infty$ corresponding to the background charge $Q \in (0, 2)$. Also, let $(Z_Q(u))_{u\in\mathcal{U}}$ be the multiplicative cascade given in Corollary 3.2.8. Then, as $p \to \infty$,*

$$\left(p^{-1}\chi^{\mathrm{in}}_{M^{(p)}}(u)\right)_{u\in\mathcal{U}} \xrightarrow{d} \left(Z_Q(u)\right)_{u\in\mathcal{U}} \tag{3.4.15}$$

with respect to the product topology on $\mathbb{R}^{\mathcal{U}}$.

This result is the supercritical analog of [70, Theorem 1], which gave the convergence of the perimeter process in the loop $O(n)$ model for $n \in (0, 2)$ to a multiplicative cascade. Its proof is based on the following description of the law of perimeters of faces in the Boltzmann maps that constitute our planar map. Note the similarity between its law (3.4.8) and the law (3.2.7) of the lengths of (non-nested) CLE_4 loops on an independent unit boundary length critical LQG disk, which we denoted $\rho^{(1)}$. It was proved in [70] that, in fact, the former converges to the latter in the scaling limit.[7]

7. Though [70] assumes $a \in (\frac{3}{2}, \frac{5}{2}) \setminus \{2\}$ throughout, their proof of this proposition only requires that the step distribution of the random walk is centered and is in the domain of attraction of the totally asymmetric stable law of parameter $\alpha = a - 1/2 \in (1, 2)$. The proof for this general case can by found in [6, Proposition C.2].

Lemma 3.4.14 ([70, Proposition 3], [6, Proposition C.2])**.** *Let* $\mathbf{q}$ *be a critical non-generic weight sequence of type* $a = 2$*. For each* $p \in \mathbb{N}$*, let* $M_0^{(p)}$ *be a Boltzmann map sampled from the law* $\mathbb{P}_{\mathbf{q}}^{(p)}$ *given in Definition 3.4.1 and recall that* $\boldsymbol{\chi}_{M_0^{(p)}}$ *refers to the decreasing sequence of half-perimeters of internal faces of* $M_0^{(p)}$*. Then, as* $p \to \infty$*, the law of the sequence* $(p^{-1}\chi_{M_0^{(p)}}(i))_{i\in\mathbb{N}}$ *converges to* $\rho^{(1)}$ *with with respect to the product topology on* $\mathbb{R}^{\mathbb{N}}$*.*

Proof of Proposition 3.4.13. Comparing Definition 3.4.12 of the perimeter process $\boldsymbol{\chi}^{\text{in}}_{M^{(p)}}$ with the iterative construction of the map $M^{(p)}$ in Definition 3.1.1, it suffices to show the convergence of the first generation: i.e.,

$$\big(p^{-1}\chi^{\text{in}}_{M^{(p)}}(i)\big)_{i\in\mathbb{N}} \overset{d}{\to} \big(Z_Q(i)\big)_{i\in\mathbb{N}} \tag{3.4.16}$$

as $p \to \infty$.

Let $M_0^{(p)}$ be a Boltzmann map sampled from the law $\mathbb{P}_{\mathbf{q}}^{(p)}$ and let $f_1, f_2, \ldots, f_{|\mathfrak{F}(M_0^{(p)})|}$ be an enumeration of its interior faces so that $\chi_{M_0^{(p)}}(i) = \text{per}(f_i)$ for each i. That is, $\text{per}(f_1), \text{per}(f_2), \ldots$ is in a non-increasing order. For each face $f_i \in \mathfrak{F}(M_0^{(p)})$, sample a ring R_i conditionally independently from the law $\mathbb{P}_{\text{ring}}^{(\text{per}(f_i))}$. By our choice of the ring distribution (Definition 3.4.4), the ratio $\text{Rat}(R_i)$ of the inner and outer parameters of the ring R_i converge jointly in distribution to $\exp(\beta_Q Y_i)$ where $Y_1, Y_2, \ldots$ are i.i.d. Rademacher random variables. Hence, if we put $\text{per}(f_i)\text{Rat}(R_i) := 0$ for $i > |\mathfrak{F}_{M_0^{(p)}}|$ and let $(Z(i))_{i\in\mathbb{N}}$ be a sequence with the law $\rho^{(1)}$ sampled independently from $\{Y_i\}_{i\in\mathbb{N}}$, then by Lemma 3.4.14 we have

$$\big(p^{-1}\chi^{\text{in}}_{M^{(p)}}(i)\big)_{i\in\mathbb{N}} = \big(p^{-1}\text{per}(f_i)\text{Rat}(R_i)\big)_{i\in\mathbb{N}} \overset{d}{\to} \big(Z(i)e^{\beta_Q Y_i}\big)_{i\in\mathbb{N}} = \big(Z_Q(i)\big)_{n\in\mathbb{N}} \tag{3.4.17}$$

as $p \to \infty$ with respect to the product topology on $\mathbb{R}^{\mathbb{N}}$. □

3.4.4 *Convergence of subcritical Boltzmann maps to the continuum random tree*

We now recall the definitions of the continuum random tree (CRT) and the Gromov–Hausdorff distance that appear in our main result. Then, we survey the literature on convergence of subcritical Boltzmann maps to the CRT, which shall provide key inputs to our proof of Theorem 3.1.4.

Continuum random tree

The Brownian CRT is a random real tree defined from a Brownian excursion that arises as the scaling limit of a large class, as defined and investigated by Aldous in the pioneering works [10, 11, 12]. More recent results on the convergence of random discrete structures to the CRT are surveyed in [236].

Definition 3.4.15. Let $\mathbb{e}\colon [0,1] \to \mathbb{R}_+$ be the normalized Brownian excursion. Consider the pseudo-distance defined on the interval $[0,1]$ by

$$d_{\mathbb{e}}(s,t) = \mathbb{e}_s + \mathbb{e}_t - 2 \min_{s\wedge t \le u \le s \vee t} \mathbb{e}_u \tag{3.4.18}$$

for $s,t \in [0,1]$. Let $\sim_{\mathbb{e}}$ be an equivalence relation on $[0,1]$ given by $s \sim_{\mathbb{e}} t$ if and only if $d_{\mathbb{e}}(s,t) = 0$. The continuum random tree is the random metric space $(\mathcal{T}_{\mathbb{e}}, d_{\mathcal{T}_{\mathbb{e}}}) := ([0,1], d_{\mathbb{e}})/\sim_{\mathbb{e}}$. The equivalence class containing 0 and 1 is the root $\emptyset$ of the CRT $(\mathcal{T}, d)$. With an abuse of notation, we also use the acronym CRT to denote the law of the above object.

Gromov–Hausdorff convergence

The Gromov–Hausdorff distance measures how close two metric spaces are being isometric to each other and has been used widely to describe convergence of random discrete structures to continuum metric spaces.

Definition 3.4.16. Let $\mathcal{X} = (X, d_X)$ and $\mathcal{Y} = (Y, d_Y)$ be compact metric spaces. The following are equivalent definitions for the Gromov–Hausdorff distance $d_{\mathrm{GH}}(\mathcal{X}, \mathcal{Y})$ between $\mathcal{X}$ and $\mathcal{Y}$.

(i) Given two compact sets K_1, K_2 in a metric space $\mathcal{Z} = (Z, d)$, recall that their Hausdorff distance is given by

$$d_{\mathrm{H}}(K_1, K_2) = \left(\sup_{z_1 \in K_1} d(z_1, K_2) \right) \vee \left(\sup_{z_2 \in K_2} d(z_2, K_1) \right). \tag{3.4.19}$$

Then,

$$d_{\mathrm{GH}}(\mathcal{X}, \mathcal{Y}) = \inf_{\mathcal{Z}, \varphi_X, \varphi_Y} d_{\mathrm{H}}(\varphi_X(X), \varphi_Y(Y)) \tag{3.4.20}$$

where the infimum is over all metric spaces $\mathcal{Z} = (Z, d)$ and isometric embeddings $\varphi_X : \mathcal{X} \to \mathcal{Z}$, $\varphi_Y : \mathcal{Y} \to \mathcal{Z}$.

(ii) A correspondence between $\mathcal{X}$ and $\mathcal{Y}$ is a subset $\mathcal{R} \subset X \times Y$ such that $\mathcal{R} \cap (\{x\} \times Y) \neq \varnothing$ for every $x \in X$ and $\mathcal{R} \cap (X \times \{y\}) \neq \varnothing$ for every $y \in Y$. The distortion of the correspondence R is defined as

$$\mathrm{dis}(\mathcal{R}) = \sup\{|d_X(x_1, x_2) - d_Y(y_1, y_2)| : (x_1, y_1), (x_2, y_2) \in \mathcal{R}\}. \tag{3.4.21}$$

Then,

$$d_{\mathrm{GH}}(\mathcal{X}, \mathcal{Y}) = \frac{1}{2} \inf_{\mathcal{R}} \mathrm{dis}(\mathcal{R}) \tag{3.4.22}$$

where the infimum is taken over all correspondences $\mathcal{R}$ of $\mathcal{X}$ and $\mathcal{Y}$.

See, e.g., [61, Section 7.3] for the equivalence of the two definitions. We shall often switch between the above two equivalent formulations depending on which one is more convenient for the particular application at hand.

Previous convergence results for subcritical maps

As mentioned earlier, the behavior of Boltzmann maps sampled from $\mathbb{P}_{\overline{\mathbf{q}}}^{(p)}$ depends strongly on the choice of the weight sequence $\overline{\mathbf{q}}$. If the weight sequence is critical non-generic, the corresponding Boltzmann maps have macroscopic faces, and upon renormalizing distances appropriately, these Boltzmann maps are expected to converge to a "stable map with a boundary" as in [176]. On the other hand, in the critical generic case, there are no macroscopic faces and the scaling limit is known to be the Brownian map [184, 174, 188].

If the weight sequence is subcritical, there are again no macroscopic faces in the Boltzmann map, but in this case, the scaling limit turns out to be a CRT. This was proved in [144, 189] for the case of Boltzmann maps without boundary. We will need the analogous convergence for subcritical Boltzmann maps with boundary (i.e., sampled from $\mathbb{P}_{\overline{\mathbf{q}}}^{(p)}$) in two different flavors. First, we state a result from [160, Corollary 5] about the convergence of the outer boundary of the Boltzmann map to the CRT. Recall that for a planar map M with vertex set V_M and graph distance d_M, we let rM refer to the metric space (V_M, rd_M) for $r \in \mathbb{R}_+$.

Proposition 3.4.17. *Let $\overline{\mathbf{q}}$ be a subcritical weight sequence. For each $p \in \mathbb{N}$, let $M_0^{(p)}$ be a Boltzmann map sampled from $\mathbb{P}_{\overline{\mathbf{q}}}^{(p)}$. Then, there exists a constant $K_{\overline{\mathbf{q}}}$ such that*

$$\frac{K_{\overline{\mathbf{q}}}}{\sqrt{p}} \partial M_0^{(p)} \xrightarrow{d} \mathrm{CRT} \tag{3.4.23}$$

as $p \to \infty$ with respect to the Gromov–Hausdorff distance.

The proof of this result in [160] is based on a bijection between the outer boundary $\partial M_0^{(p)}$ to a looptree associated with a critical Bienaymé–Galton–Watson process, which we follow closely for our proof of Theorem 3.1.4. See Section 3.7.1 for further details.

With additional assumptions on the decay rate of $\mu_{\overline{\mathbf{q}}}$, the entire map $M^{(p)}$ converges to a CRT.

Proposition 3.4.18. *Let $\overline{\mathbf{q}}$ be a subcritical weight sequence such that $\mu_{\overline{\mathbf{q}}}([k,\infty)) = o(k^{-1})$ as $k \to \infty$. If $M^{(p)}$ has the law $\mathbb{P}^{(p)}_{\overline{\mathbf{q}}}$ for each $p \in \mathbb{N}$, then*

$$\frac{1}{\sqrt{2p}} M^{(p)} \xrightarrow{d} \mathrm{CRT} \tag{3.4.24}$$

as $p \to \infty$ with respect to the Gromov–Hausdorff distance. Moreover,

$$\frac{1}{\sqrt{p}} \max_{v \in V_{M^{(p)}}} d_{M^{(p)}}(v, \partial M^{(p)}) \xrightarrow{d} 0. \tag{3.4.25}$$

Proof. For the proof of (3.4.24), in view of [189, Theorem 1.3], it suffices to show that

$$\frac{1}{p^2} \sum_{f \in \mathfrak{F}(M^{(p)})} [\mathrm{per}(f)]^2 \to 0 \tag{3.4.26}$$

in distribution as $p \to \infty$. We use the connection between random walks and Boltzmann maps as explained in Section 3.4.2. Let $S_n = X_1 + \cdots + X_n$ be a random walk with step distribution (3.4.6). By Proposition 3.4.7, the convergence (3.4.26) is equivalent to

$$\lim_{p\to\infty} \frac{\mathbb{E}\Big[\frac{1}{L_S^{(p)}+1} \mathbf{1}\big\{ \sum_{i=1}^{T_S^{(p)}} (X_i+1)^2 > \varepsilon p^2 \big\}\Big]}{\mathbb{E}\Big[\frac{1}{L_S^{(p)}+1}\Big]} = 0 \qquad \text{for all } \varepsilon > 0. \tag{3.4.27}$$

By Lemma 3.4.11, the indicator $\mathbf{1}\{\sum_{i=1}^{T_S^{(p)}} (X_i+1)^2 > \varepsilon p^2\}$ is an increasing function of the sequence $(X_1, X_2, \dots)$ while $(L_S^{(p)} + 1)^{-1}$ is a decreasing function of the same sequence. Using the FKG inequality, the problem at hand reduces to checking

$$\mathbb{P}\Bigg(\sum_{i=1}^{T_S^{(p)}} (X_i+1)^2 > \varepsilon p^2 \Bigg) \to 0. \tag{3.4.28}$$

Since $\overline{\mathbf{q}}$ is subcritical, in view of Lemma 3.4.9, it suffices to show that

$$\mathbb{P}\bigg(\sum_{i=1}^{\lfloor -2p/m_{\overline{\mathbf{q}}} \rfloor} (X_i + 1)^2 > \varepsilon p^2 \bigg) \to 0. \tag{3.4.29}$$

This follows by the law of large numbers for i.i.d. heavy tailed random variables (e.g., [148, Theorem 6.17] with $p = 1/2$ therein).

In fact, the convergence (3.4.24) holds because the boundary of $(2p)^{-1/2} M^{(p)}$ converges to the CRT, whereas its interior faces disappear under the scaling limit as expressed in (3.4.25). This reasoning, first given heuristically in [42], was stated within the proof of [189] in terms of the "key bijection" between (a uniformly chosen negative pointed map version of) $M^{(p)}$ and a labelled forest combining those of [58] and [144]. In this bijection, the leaves of the forest are in bijection with the non-distinguished vertices of $M^{(p)}$. Moreover, we can check directly from its construction (see [187, Section 2.3]) that each tree in the forest contains a leaf that is mapped to a vertex on $\partial M^{(p)}$. Since the labels on the leaves correspond to the graph distance to the distinguished vertex in $M^{(p)}$, the maximum distance from any vertex in $M^{(p)}$ to $\partial M^{(p)}$ is bounded above by the maximum difference of the labels in each tree. The latter quantity is equal to $\max \widetilde{L}^{(p)} - \min \widetilde{L}^{(p)}$ in the notation of [189]. By Theorem 2.6(1) in that paper, it follows from (3.4.26) that $p^{-1/2}(\max \widetilde{L}^{(p)} - \min \widetilde{L}^{(p)}) \to 0$ in distribution as $p \to \infty$. This proves (3.4.25). $\square$

3.5 The probability that a supercritical map is finite

The goal of this section is to prove Proposition 3.1.3, which will be done via a subadditivity argument. A priori, it is not even clear if $F(p) = \mathbb{P}_\infty^{(p)}(M \text{ is finite}) < 1$ for any value of p, so we turn to this first. We saw in Proposition 3.3.1 that in the continuum limit, macroscopic loops are abundant at all stages in the supercritical multiplicative cascade Z_Q. We now employ a comparison to a supercritical Bienaymé–Galton–Watson (BGW) tree to deduce the non-triviality of F.

Lemma 3.5.1. *We have the convergence $F(p) \to 0$ as $p \to \infty$.*

Proof. For each positive integer p, let $n_q = n_q(p)$ be a constant to be determined later. Let M be a map sampled from $\mathbb{P}_{\mathtt{F}}^{(p)}$ and recall the associated perimeter cascade $\boldsymbol{\chi}_M^{\text{in}} = (\chi_M^{\text{in}}(u))_{u\in\mathcal{U}}$ from Definition 3.4.12. Consider the random variable

$$N_m = \#\{u \in \mathcal{U} : |u| = mn_q, \chi_M^{\text{in}}(u) > p\}. \tag{3.5.1}$$

Then,

$$F(p) \leq \mathbb{P}_\infty^{(p)}(N_m = 0 \text{ for some } m \in \mathbb{N}). \tag{3.5.2}$$

The goal now is to show that the right-hand side of this inequality tends to 0 as $p \to \infty$. We do so by constructing a supercritical BGW process whose size at each generation m is stochastically dominated by N_m and has a sufficiently small extinction probability.

As a consequence of Proposition 3.3.1, for each $q \in (0,1)$, we can find $n_q \in \mathbb{N}$ such that

$$\mathbb{P}(\#\{u \in \mathcal{U} : |u| = n_q, Z_Q(u) > 2\} \geq 100) \geq q. \tag{3.5.3}$$

Combining this with the distributional convergence of Proposition 3.4.13, we see that for each sufficiently large $p \in \mathbb{N}$, we can choose $q = q(p) \in (0,1)$ with q increasing to 1 as $p \to \infty$ such that for every integer $\tilde{p} \geq p$, we have

$$\mathbb{P}_\infty^{(\tilde{p})}(\#\{u \in \mathcal{U} : |u| = n_q, \chi_M^{\text{in}}(u) > p\} \geq 100) \geq q. \tag{3.5.4}$$

Now, consider a BGW process with offspring distribution μ such that $\mu(100) = q$ and $\mu(0) = 1 - q$. Let Z_m denote the number of offspring of this BGW process in generation m

and define

$$\widetilde{N}_m = \#\{u \in \mathcal{U} : |u| = mn_q, \chi_M^{\text{in}}(u_k) > p \text{ for } 1 \leq k \leq m$$
$$\text{where } u_k \text{ is the ancestor of } u \text{ with } |u_k| = kn_q\}. \tag{3.5.5}$$

Then, $(\widetilde{N}_1, \widetilde{N}_2, \dots)$ sampled from $\mathbb{P}_\infty^{(p)}$ is a BGW process whose offspring distribution $\tilde{\mu}$ satisfies $\tilde{\mu}(k) \geq \mu(k)$ for all $k > 0$, whence $(Z_1, Z_2, \dots)$ is stochastically dominated by $(\widetilde{N}_1, \widetilde{N}_2, \dots)$. Since $\widetilde{N}_m \leq N_m$ for every m,

$$(N_1, N_2, \dots) \geq_{\text{SD}} (Z_1, Z_2, \dots). \tag{3.5.6}$$

On the other hand, for the corresponding supercritical BGW process, we know that the extinction probability $\mathbb{P}(Z_m = 0 \text{ for some } m \in \mathbb{N})$ tends to 0 as $q \to 1$. By stochastic domination, we conclude $\mathbb{P}_\infty^{(p)}(N_m = 0 \text{ for some } m) \to 0$ as $p \to \infty$ as claimed. □

Lemma 3.5.2. *For every $p \in \mathbb{N}$, we have $0 < F(p) < 1$.*

Proof. Fix $p \in \mathbb{N}$ and let M be a map sampled from $\mathbb{P}_\infty^{(p)}$. Let us first show $F(p) > 0$. The construction of the map M in Definition 3.1.1 terminates right away if the gasket M_0 has no interior faces. Since M_0 has the marginal law of a Boltzmann map sampled from $\mathbb{P}_{\mathbf{q}}^{(p)}$, in terms of the random walk representation of the perimeter process χ_{M_0} (Proposition 3.4.7), this corresponds to the event that $X_i = -1$ for all $i \in [\![1, p]\!]$. Using the trivial bound $L_S^{(p)} \geq p$, we thus have

$$\begin{aligned} F(p) &\geq \frac{\mathbb{E}\big[(L_S^{(p)}+1)^{-1}\mathbf{1}\{X_i = -1 \ \ \forall i \in [\![1,p]\!]\}\big]}{\mathbb{E}\big[(L_S^{(p)}+1)^{-1}\big]} = \frac{(p+1)^{-1}(\mu_{\mathbf{q}}(-1))^p}{\mathbb{E}\big[(L_S^{(p)}+1)^{-1}\big]} \\ &\geq (\mu_{\mathbf{q}}(-1))^p = (Z_{\mathbf{q}})^{-p} > 0. \end{aligned} \tag{3.5.7}$$

We now prove $F(p) < 1$. Choose a face $f_1 \in \mathfrak{F}_0 = \mathfrak{F}(M_0)$ with the longest perimeter: that is, $\mathrm{per}(f_1) = \max_{f\in\mathfrak{F}_0} \mathrm{per}(f) = \chi_{M_0}(1)$. Note that for each $k \in \mathbb{N}$, we have $\mathbb{P}^{(p)}_{\mathbf{q}}(\mathrm{per}(f_1) > k) = \mu_{\mathbf{q}}([k,\infty)) > 0$ by the definition of the non-generic critical weight sequence $\mathbf{q}$. We now fix k to be large enough that $\sup_{m>k} F(m) < 1/2$. Then,

$$\begin{aligned} F(p) \le \mathbb{E}^{(p)}_\infty\big[F(\mathrm{per}_{\mathrm{in}}(f_1))\big] &\le \mathbb{P}^{(p)}_\infty(\mathrm{per}_{\mathrm{in}}(f_1) \le k) + \mathbb{P}^{(p)}_\infty(\mathrm{per}_{\mathrm{in}}(f_1) > k) \sup_{m>k} F(m) \\ &\le 1 - \frac{1}{2}\mathbb{P}^{(p)}_\infty(\mathrm{per}_{\mathrm{in}}(f_1) > k). \end{aligned} \tag{3.5.8}$$

Recall from Definition 3.4.4 that $\mathrm{Rat}(R)$ for a ring R sampled from $\mathbb{P}^{(m)}_{\mathrm{ring}}$ converges in distribution to $\exp(\beta_Q Y)$ where Y is a Rademacher random variable. Hence, we may assume (by choosing a larger k if necessary) that $\inf_{m>k} \mathbb{P}^{(m)}_{\mathrm{ring}}(\mathrm{Rat}(R) \ge 1) > 0$. Since $\mathrm{per}_{\mathrm{in}}(f_1)$ is the inner half-perimeter of the ring sampled from $\mathbb{P}^{(\mathrm{per}(f_1))}_{\mathrm{ring}}$, we have

$$\mathbb{P}^{(p)}_\infty(\mathrm{per}_{\mathrm{in}}(f_1) > k) \ge \mathbb{P}^{(p)}_{\mathbf{q}}(\chi_{M_0}(1) > k) \cdot \inf_{m>k} \mathbb{P}^{(m)}_{\mathrm{ring}}(\mathrm{Rat}(R) \ge 1) > 0. \tag{3.5.9}$$

This combined with (3.5.8) completes the proof. □

Proof of Proposition 3.1.3. We first show the existence of a constant $\alpha \in [0,\infty)$ such that $-p^{-1}\log F(p) \to \alpha$ as $p \to \infty$. By a standard subadditivity argument (see [83, Theorem 23]), it suffices to find a rational function $g(p,q)$ and $p_0 > 0$ such that if $p, q \ge p_0$, then

$$F(p+q) \ge g(p,q)F(p)F(q). \tag{3.5.10}$$

Note that $\alpha = \inf_p(-p^{-1}\log F(p)) < \infty$ from Lemma 3.5.2.

By the construction of supercritical planar maps from Definition 3.1.1, we have the recursive relation

$$F(p+q) = \mathbb{E}^{(p+q)}_\infty\Big[\prod_{f\in\mathfrak{F}_0} F(\mathrm{per}_{\mathrm{in}}(f))\Big] = \mathbb{E}^{(p+q)}_{\mathbf{q}}\Big[\prod_{i\in\mathbb{N}} \mathbb{E}^{(\chi_{M_0}(i))}_{\mathrm{ring}}\big[F(\chi_{M_0}(i)\mathrm{Rat}(R))\big]\Big]. \tag{3.5.11}$$

Now, we transfer the above expression to one involving the corresponding random walk in Definition 3.4.6. Let $\{X_i + 1\}_{i\in\mathbb{N}}$ be i.i.d. random variables with the law $\mu_{\mathbf{q}}$ and let $S_n = X_1 + \cdots + X_n$ be the corresponding random walk. Given the walk S, sample R_i conditionally independently for each $i \in \mathbb{N}$ from the marginal law of $\mathrm{Rat}(R)$ under $\mathbb{P}^{(X_i+1)}_{\text{ring}}$. By an application of Proposition 3.4.7 along with (3.4.7), we can rewrite (3.5.11) as

$$F(p+q) = \frac{\mathbb{E}\big[(L_S^{(p+q)}+1)^{-1}\prod_{i=1}^{T_S^{(p+q)}} F((X_i+1)R_i)\big]}{\mathbb{E}\big[(L_S^{(p+q)}+1)^{-1}\big]}. \tag{3.5.12}$$

We now consider the modified walk $\hat{S}_j := S_{j+T_S^{(p)}} + p$, which is the part of the walk S from the first hitting time of $-p$ up to the hitting time of $-(p+q)$, translated upwards by p. Also denote $\hat{X}_i = X_{i+T_S^{(p)}}$ and $\hat{R}_i = R_{i+T_S^{(p)}}$ for simplicity. Then, since the only possible negative steps of S are of unit size, $T_S^{(p+q)} = T_S^{(p)} + T_{\hat{S}}^{(q)}$ and $L_S^{(p+q)} = L_S^{(p)} + L_{\hat{S}}^{(q)}$. Moreover, rearranging $(X_{T_S^{(p)}+1}+1, \ldots, X_{T_S^{(p+q)}}+1)$ in a decreasing order and then padding zeroes to its end yields $\boldsymbol{\chi}_{\hat{S}}^{(q)}$, which is independent from $\boldsymbol{\chi}_S^{(p)}$ by the strong Markov property of the walk S. As a consequence, from (3.5.12), we obtain that for some constant C_1,

$$\begin{aligned}
F(p+q) &\geq \frac{\mathbb{E}\big[(L_S^{(p)}+1)^{-1}\prod_{i=1}^{T_S^{(p)}} F((X_i+1)R_i)\cdot(L_{\hat{S}}^{(q)}+1)^{-1}\prod_{j=1}^{T_{\hat{S}}^{(q)}} F((\hat{X}_j+1)\hat{R}_j)\big]}{\mathbb{E}\big[(L_S^{(p+q)}+1)^{-1}\big]} \\
&= \frac{\mathbb{E}\big[(L_S^{(p)}+1)^{-1}\prod_{i=1}^{T_S^{(p)}} F((X_i+1)R_i)\big]\cdot\mathbb{E}\big[(L_{\hat{S}}^{(q)}+1)^{-1}\prod_{j=1}^{T_{\hat{S}}^{(q)}} F((\hat{X}_j+1)\hat{R}_j)\big]}{\mathbb{E}\big[(L_S^{(p+q)}+1)^{-1}\big]} \\
&= \frac{\mathbb{E}\big[(L_S^{(p)}+1)^{-1}\big]\mathbb{E}\big[(L_{\hat{S}}^{(q)}+1)^{-1}\big]}{\mathbb{E}\big[(L_S^{(p+q)}+1)^{-1}\big]}F(p)F(q) \geq \frac{C_1(p+q+1)}{p^{3/2}q^{3/2}}F(p)F(q)
\end{aligned} \tag{3.5.13}$$

where, in the last line, we used $L_S^{(p+q)} \geq p+q$ along with Lemma 3.4.8. We have thus shown (3.5.10) with $g(p,q) = (p+q+1)/(pq)^2$.

We now show that, in fact, $\alpha > 0$. Starting again from from (3.5.12), since $T_S^{(p)} \geq L_S^{(p)} \geq p$, we have

$$\begin{aligned} F(p) &= \frac{\mathbb{E}\big[(L_S^{(p)}+1)^{-1}\prod_{i=1}^{T_S^{(p)}} F((X_i+1)R_i))\big]}{\mathbb{E}\big[(L_S^{(p)}+1)^{-1}\big]} \leq \frac{\mathbb{E}\big[\prod_{i=1}^{p} F((X_i+1)R_i))\big]}{\mathbb{E}\big[(L_S^{(p)}+1)^{-1}\big]} \\ &\leq Cp^{3/2}\big(\mathbb{E}[F((X_1+1)R_1)]\big)^p \end{aligned} \tag{3.5.14}$$

for some constant C; for the second inequality, we used Lemma 3.4.8 and that $\{(X_i + 1)R_i\}_{i\in[\![1,p]\!]}$ are i.i.d. Since $\mu_{\mathbf{q}}([k,\infty)) > 0$ for all k and $\mathbb{P}_{\text{ring}}^{(m)}(\text{Rat}(R) \geq 1) \to 1/2$ as $m \to \infty$, we have $\mathbb{P}((X_1+1)R_1 > 0) > 0$. Thus, $\mathbb{E}[F((X_1+1)R_1)] < 1$ by Lemma 3.5.2. This completes the proof. $\square$

3.6 Estimates on the size of a supercritical map conditioned to be finite

The broad intuition of this section is that supercritical LQG conditioned to be finite should in some sense look "subcritical," similar to what happens for BGW trees.

Recall from Definition 3.1.1 the Markov chain $(M_i, \mathfrak{F}_i)$ with the law $\mathbb{P}_\infty^{(p)}$, where M_i are the partial maps which increase to the whole map M and $\mathfrak{F}_i$ are the set of "active" faces inside with the next layer of rings and critical Boltzmann maps are inserted. The law $\mathbb{P}_{\mathtt{F}}^{(p)}$ on finite planar maps appearing in Theorem 3.1.4 can then be naturally considered as the marginal law on M when we condition the chain $(M_i, \mathfrak{F}_i)_{i\in\mathbb{N}^\#}$ to terminate at a finite time. That is, there exists $n \in \mathbb{N}$ such that $M_i = M$ and $\mathfrak{F}_i = \varnothing$ for all $i \geq n$. We thus use $\mathbb{P}_{\mathtt{F}}^{(p)}$ to also denote this conditional law of the chain terminated at a finite time.

The plan is to first prove that the marginal law of the outermost gasket M_0 for a map M sampled from $\mathbb{P}_{\mathtt{F}}^{(p)}$ is that of a subcritical Boltzmann map (Proposition 3.6.2). From here, we extrapolate that the expected total sum of the inner perimeter lengths decreases exponentially (Corollary 3.6.7). Using this estimate, we conclude that the map M sampled from $\mathbb{P}_{\mathtt{F}}^{(p)}$ looks essentially like its outermost gasket M_0 in the sense that the volumes of submaps $M|_f$ within the faces $f \in \mathfrak{F}_0 = \mathfrak{F}(M_0)$ grow slower than $p^{1/2}$ (Proposition 3.6.13).

3.6.1 Law of the gasket sampled from $\mathbb{P}_{\mathtt{F}}^{(p)}$

We first establish that the outermost gasket M_0 for a map M sampled from $\mathbb{P}_{\mathtt{F}}^{(p)}$ is a subcritical Boltzmann map. Recall the notation $\mathrm{per}_{\mathrm{in}}(f)$ for the inner half-perimeter of the ring attached to each face $f \in \mathfrak{F}_0 = \mathfrak{F}(M_0)$. Here is an explicit description of the corresponding weight sequence, which is subcritical.

Lemma 3.6.1. *Let* $\mathbf{q}$ *be the critical non-generic weight sequence with* $a = 2$ *as in Definition 3.4.3. Consider the weight sequence* $\mathbf{q}'$ *given by*

$$q'_k = \mathbb{E}_{\mathrm{ring}}^{(k)}[F(\mathrm{per}_{\mathrm{in}}(f))] \cdot q_k \tag{3.6.1}$$

where $\mathrm{per}_{\mathrm{in}}(f)$ *is the half-perimeter of the inner face of the ring sampled from the distribution* $\mathbb{P}_{\mathrm{ring}}^{(k)}$ *in Definition 3.4.4. (In particular, the outer half-perimeter of the ring is* k*.) Then, the weight sequence* $\mathbf{q}'$ *is admissible and subcritical.*

Proof. Let

$$g_{\mathbf{q}}(x) := 1 - x + x f_{\mathbf{q}}(x) = 1 - x + \sum_{k=1}^{\infty} \binom{2k-1}{k-1} q_k x^k \tag{3.6.2}$$

and define $g_{\mathbf{q}'}$ analogously. Recall from Definition 3.4.2 that $\mathbf{q}$ is admissible if and only if $f_{\mathbf{q}}(x) = 1 - 1/x$ has a positive solution and the smallest such solution is denoted $Z_{\mathbf{q}}$. Then, $g_{\mathbf{q}}(Z_{\mathbf{q}}) = 0$. For every $k \in \mathbb{N}$, since $\mathbb{P}_{\mathrm{ring}}^{(k)}(\mathrm{per}_{\mathrm{in}}(f) > 0) > 0$ by the non-triviality condition in Definition 3.4.4, we have $q'_k < q_k$ by Lemma 3.5.2 and thus $g_{\mathbf{q}'}(Z_{\mathbf{q}}) < g_{\mathbf{q}}(Z_{\mathbf{q}}) = 0$. Since $g_{\mathbf{q}'}(0) = 1$, we conclude that $\mathbf{q}'$ is admissible and $Z_{\mathbf{q}'}$, the smallest positive solution of $f_{\mathbf{q}'}(x) = 1 - 1/x$, is less than $Z_{\mathbf{q}}$.

Note from (3.4.4) that the mean of the measure $\mu_{\mathbf{q}'}$ is given by $1 + (g_{\mathbf{q}'})'(Z_{\mathbf{q}'})$. Since $0 \le q'_k < q_k$ for all k, we have $(g_{\mathbf{q}'})'(Z_{\mathbf{q}'}) \le (g_{\mathbf{q}'})'(Z_{\mathbf{q}}) < (g_{\mathbf{q}})'(Z_{\mathbf{q}})$. Since $\mathbf{q}$ is critical, $1 + (g_{\mathbf{q}'})'(Z_{\mathbf{q}'}) < 1 + (g_{\mathbf{q}})'(Z_{\mathbf{q}}) = 1$ and thus $\mathbf{q}'$ is subcritical. $\square$

Proposition 3.6.2. *For every $p \in \mathbb{N}$, the marginal law of the outermost gasket M_0 under $\mathbb{P}^{(p)}_{\mathtt{F}}$ is equal to $\mathbb{P}^{(p)}_{\mathbf{q}'}$.*

Proof. Consider the chain $(M_i, \mathfrak{F}_i)$ sampled from $\mathbb{P}^{(p)}_{\infty}$. By construction, the random variables $\{\mathrm{per}_{\mathrm{in}}(f)\}_{f \in \mathfrak{F}_0}$ are conditionally independent given the outermost gasket M_0. Now, we know that $\mathbb{P}^{(p)}_{\infty}(M \text{ is finite} \,|\, M_0, \{\mathrm{per}_{\mathrm{in}}(f)\}_{f \in \mathfrak{F}_0}) = \prod_{f \in \mathfrak{F}_0} F(\mathrm{per}_{\mathrm{in}}(f))$. Hence, for any measurable set A on the space of finite planar maps with outer boundary length p,

$$\begin{aligned}
\mathbb{P}^{(p)}_{\mathtt{F}}(M_0 \in A) &= \frac{\mathbb{P}^{(p)}_{\infty}(M_0 \in A,\, M \text{ is finite})}{\mathbb{P}^{(p)}_{\infty}(M \text{ is finite})} \\
&= \frac{\mathbb{E}^{(p)}_{\infty}\big[\mathbf{1}\{M_0 \in A\} \cdot \prod_{f \in \mathfrak{F}_0} F(\mathrm{per}_{\mathrm{in}}(f))\big]}{\mathbb{E}^{(p)}_{\infty}\big[\prod_{f \in \mathfrak{F}_0} F(\mathrm{per}_{\mathrm{in}}(f))\big]} \\
&= \frac{\mathbb{E}^{(p)}_{\mathbf{q}}[\mathbf{1}\{M_0 \in A\} \cdot \prod_{f \in \mathfrak{F}_0} \mathbb{E}^{(\mathrm{per}(f))}_{\mathrm{ring}}[F(\mathrm{per}_{\mathrm{in}}(f))]]}{\mathbb{E}^{(p)}_{\mathbf{q}}[\prod_{f \in \mathfrak{F}} \mathbb{E}[F(\mathrm{per}_{\mathrm{in}}(f))]]} \\
&= \mathbb{P}^{(p)}_{\mathbf{q}'}(M_0 \in A)
\end{aligned} \tag{3.6.3}$$

as claimed. $\square$

We now show that conditioning M to be finite leads to the removal of all macroscopic loops.

Lemma 3.6.3. *There exists a constant $\Delta > 2$ such that for any fixed $\varepsilon > 0$, we have*

$$\lim_{p \to \infty} \mathbb{P}^{(p)}_{\mathtt{F}}\left(\sup_{f \in \mathfrak{F}_0} \mathrm{per}_{\mathrm{in}}(f) \le \varepsilon p^{1/\Delta} \right) = 1. \tag{3.6.4}$$

The following lemma is needed for the proof of Lemma 3.6.3.

Lemma 3.6.4. *Let* $\mathbf{q}'$ *be the weight sequence defined by* (3.6.1)*. Then, there exists constants* $c > 0$ *such that* $\mu_{\mathbf{q}'}([k, \infty)) = O(e^{-ck})$ *as* $k \to \infty$*.*

Proof. As we saw in the proof of Lemma 3.6.1, $Z_{\mathbf{q}'} < Z_{\mathbf{q}}$. Hence,

$$\begin{aligned} \mu_{\mathbf{q}'}(k) = (Z_{\mathbf{q}'})^{k-1}\binom{2k-1}{k-1} q'_k &\leq (Z_{\mathbf{q}})^{k-1}\binom{2k-1}{k-1} \mathbb{E}^{(k)}_{\text{ring}}[F(\text{per}_{\text{in}}(f))]\, q_k \\ &= \mathbb{E}^{(k)}_{\text{ring}}[F(k\,\text{Rat}(R))]\, \mu_{\mathbf{q}}(k) \end{aligned} \tag{3.6.5}$$

where $\text{per}_{\text{in}}(f)$ is the inner half-perimeter of the ring R sampled from $\mathbb{P}^{(k)}_{\text{ring}}$.

The lower tail condition in Definition 3.4.4 implies that there exist constants $\varepsilon, c > 0$ satisfying

$$\mathbb{P}^{(k)}_{\text{ring}}(\text{Rat}(R) < \varepsilon) = O(e^{-ck}) \tag{3.6.6}$$

as $k \to \infty$. Hence, by the exponential decay of $F(p)$ proved in Proposition 3.1.3, choosing a smaller constant c if necessary, we have that for all k large enough,

$$\mathbb{E}^{(k)}_{\text{ring}}[F(k\,\text{Rat}(R))] \leq \sup_{p \geq k\varepsilon} F(p) + \mathbb{P}^{(k)}_{\text{ring}}(\text{Rat}(R) < \varepsilon) = O(e^{-ck}). \tag{3.6.7}$$

Here, we used the trivial bound $F \leq 1$. We conclude that as $k \to \infty$,

$$\mu_{\mathbf{q}'}([k, \infty)]) \leq \mu_{\mathbf{q}}([k, \infty)) \cdot \sup_{p \geq k} \mathbb{E}^{(p)}_{\text{ring}}[F(p\,\text{Rat}(R))] = O(e^{-ck}) \tag{3.6.8}$$

since $\mu_{\mathbf{q}}(k)$ decays polynomially in k by Definition 3.4.3 of a non-generic critical weight sequence. $\square$

We now use above results to bound $\sup_{f \in \mathfrak{F}_0} \text{per}_{\text{in}}(f)$ as claimed.

Proof of Lemma 3.6.3. Let $\{X_i + 1\}_{i \in \mathbb{N}}$ be i.i.d. random variables with the law $\mu_{\mathbf{q}'}$ and, given these, sample R_i conditionally independently for each $i \in \mathbb{N}$ from the marginal law of $\text{Rat}(R)$ under $\mathbb{P}^{(X_i+1)}_{\text{ring}}$. By Propositions 3.4.7 and 3.6.2, if $L^{(p)}_S$ is the number of negative steps in the random walk $S_n = X_1 + \cdots + X_n$ up to the first time $T_S^{(p)}$ that this walk visits $-p$, then

$$\mathbb{P}_{\mathtt{F}}^{(p)}\left(\sup_{f\in\mathfrak{F}_0} \mathrm{per}_{\mathrm{in}}(f) > \varepsilon p^{1/\Delta}\right) = \frac{\mathbb{E}\big[(L_S^{(p)}+1)^{-1}\mathbf{1}\big\{\sup_{i\in[\![1,T_S^{(p)}]\!]}(X_i+1)R_i > \varepsilon p^{1/\Delta}\big\}\big]}{\mathbb{E}\big[(L_S^{(p)}+1)^{-1}\big]}. \tag{3.6.9}$$

Using the trivial bound $(L_S^{(p)}+1)^{-1} \leq (p+1)^{-1}$ and the estimate $1/\mathbb{E}[(L_S^{(p)}+1)^{-1}] \leq p\,\mathbb{E}[T_S^{(1)}]+1$ from Lemma 3.4.9, we can find a constant $C > 0$ such that for all p,

$$\begin{aligned}&\frac{\mathbb{E}\big[(L_S^{(p)}+1)^{-1}\mathbf{1}\big\{\sup_{i\in[\![1,T_S^{(p)}]\!]}(X_i+1)R_i > \varepsilon p^{1/\Delta}\big\}\big]}{\mathbb{E}[(L_S^{(p)}+1)^{-1}]}\\ &\leq C\,\mathbb{P}\left(\sup_{i\in[\![1,T_S^{(p)}]\!]}(X_i+1)R_i > \varepsilon p^{1/\Delta}\right).\end{aligned} \tag{3.6.10}$$

By Lemma 3.4.10, it suffices find $\Delta > 2$ such that

$$\lim_{p\to\infty}\mathbb{P}\left(\sup_{i\in[\![1,-2p/m_{\mathbf{q}'}]\!]}(X_i+1)R_i > \varepsilon p^{1/\Delta}\right) = 0 \tag{3.6.11}$$

to complete the proof. We can pick $\Delta = 2+\delta/3$ where $\delta > 0$ is the constant appearing in the upper bound condition of Definition 3.4.4, since then, for sufficiently large p,

$$\begin{aligned}\mathbb{P}((X_1+1)R_1 > \varepsilon p^{1/\Delta}) &\leq \mathbb{P}(X_1+1 > \varepsilon(\log p)^2) + \mathbb{P}(R_1 > p^{1/(2+\delta/2)})\\ &\leq \mu_{\mathbf{q}'}((\varepsilon(\log p)^2-1,\infty)) + \sup_{k\in\mathbb{N}}\mathbb{P}_{\mathrm{ring}}^{(k)}(\mathrm{Rat}(R) > p^{1/(2+\delta/2)})\\ &= o(p^{-1})\end{aligned} \tag{3.6.12}$$

by applying Lemma 3.6.4 to the first term and the upper tail condition of Definition 3.4.4 to the second term. Since $(X_i+1)R_i$ are identically distributed, taking the union bound over

$i \in [\![1, -2p/m_{\mathbf{q}'}]\!]$ gives (3.6.11). $\square$

Remark 3.6.5. Lemma 3.6.3 can be improved if we impose further restrictions on the distribution of rings than in Definition 3.4.4. For instance, the random planar model for supercritical LQG disk suggested in [18, Section 3.2] has the property that the inner half-parameter of a ring sampled from $\mathbb{P}^{(k)}_{\text{ring}}$ is exactly equal to $\lfloor k \exp(\pm\beta_Q) \rfloor$ with probability $1/2$. In this case, we can replace $\varepsilon p^{1/\Delta}$ with $\varepsilon(\log p)^2$ in (3.6.4) since $\sup_k \mathbb{P}^{(k)}_{\text{ring}}(\mathrm{Rat}(R) > c) = 0$ for sufficiently large c.

3.6.2 Maps sampled from $\mathbb{P}^{(p)}_{\mathtt{F}}$ are subcritical

Now that we have identified that the marginal law of the outermost gasket M_0 sampled from $\mathbb{P}^{(p)}_{\mathtt{F}}$ is that of a subcritical Boltzmann map, we extrapolate from here an exponential decay in i for the total perimeter of the maps added at step i of the iterative construction.

We find it useful to consider

$$h(p) := -\log F(p) \tag{3.6.13}$$

with $h(0) = 0$ as a proxy for p, since $h(p) \asymp p$ by Proposition 3.1.3. The goal of this subsection is to obtain the following estimate. The fact that $s_0 < 1$ in the following proposition is the sense in which the iterative construction of a map sampled from $\mathbb{P}^{(p)}_{\mathtt{F}}$ behaves as a subcritical branching process.

Proposition 3.6.6. *There exists a constant $s_0 \in (0,1)$ such that for all $p \in \mathbb{N}$, we have*

$$\mathbb{E}^{(p)}_{\mathtt{F}}\Bigg[\sum_{f \in \mathfrak{F}_0} h(\mathrm{per}_{\mathrm{in}}(f))\Bigg] \le s_0 h(p). \tag{3.6.14}$$

By a simple iteration argument, we obtain from Proposition 3.6.6 the following exponential decay for the expected total perimeter of each generation for a supercritical map conditioned to be finite.

Corollary 3.6.7. *There exist constants $c > 0$ and $s_0 \in (0,1)$ such that for all $p \in \mathbb{N}$ and all $i \in \mathbb{N}$, we have*

$$\mathbb{E}_{\mathtt{F}}^{(p)}\Big[\sum_{f\in\mathfrak{F}_i} h(\mathrm{per}_{\mathrm{in}}(f))\Big] \le c(s_0)^i p. \tag{3.6.15}$$

Proof. By the iterative construction of $\mathbb{P}_\infty^{(p)}$ in Definition 3.1.1, conditioning the whole planar map M to be finite given $\{\mathrm{per}_{\mathrm{in}}(f)\}_{f\in\mathfrak{F}_i}$ is equivalent to conditioning each submap $M(f)$ glued into the inner face of the ring $R(f)$ attached in $f \in \mathfrak{F}_i$ to be finite. In particular, this implies

$$\begin{aligned}\mathbb{E}_{\mathtt{F}}^{(p)}\Big[\sum_{\tilde f\in\mathfrak{F}_{i+1}} h(\mathrm{per}_{\mathrm{in}}(\tilde f))\Big|\{\mathrm{per}_{\mathrm{in}}(f)\}_{f\in\mathfrak{F}_i}\Big] &= \sum_{f\in\mathfrak{F}_i}\mathbb{E}_{\mathtt{F}}^{(\mathrm{per}_{\mathrm{in}}(f))}\Big[\sum_{\tilde f\in\mathfrak{F}(M(f))} h(\mathrm{per}_{\mathrm{in}}(\tilde f))\Big]\\ &\le s_0\sum_{f\in\mathfrak{F}_i} h(\mathrm{per}_{\mathrm{in}}(f)).\end{aligned} \tag{3.6.16}$$

The claim follows immediately from Proposition 3.6.6 by an induction on i. $\square$

The rest of this subsection deals with the proof of Proposition 3.6.6. The argument is somewhat convoluted due to the lack of a precise estimate for $F(p)$. The following simple lemma illustrates that the main challenge in the proof of Proposition 3.6.6 is the requirement that $s_0 < 1$.

Lemma 3.6.8. *There exist constants $C, c > 0$ such that for all p and all $s > 1$, we have*

$$\mathbb{P}_{\mathtt{F}}^{(p)}\Big(\sum_{f\in\mathfrak{F}_0} h(\mathrm{per}_{\mathrm{in}}(f)) > sh(p)\Big) \le Ce^{-c(s-1)p}. \tag{3.6.17}$$

Proof. Let E_s denote the event $\{\sum_{f\in\mathfrak{F}_0} h(\mathrm{per}_{\mathrm{in}}(f)) > sh(p)\}$. By Bayes' rule, we have

$$\mathbb{P}_{\mathtt{F}}^{(p)}(E_s) = F(p)^{-1}\mathbb{P}_\infty^{(p)}(E_s)\mathbb{P}_\infty^{(p)}(M \text{ is finite}|E_s). \tag{3.6.18}$$

By Definition 3.1.1 of $\mathbb{P}_{\infty}^{(p)}$ along with the relation $F(p) = e^{-h(p)}$, we have

$$\mathbb{P}_{\infty}^{(p)}(M \text{ is finite}|E_s) \leq e^{-sh(p)}. \tag{3.6.19}$$

Using the trivial bound $\mathbb{P}_{\infty}^{(p)}(E_s) \leq 1$, we obtain $\mathbb{P}_{\mathtt{F}}^{(p)}(E_s) \leq e^{-(s-1)h(p)}$ from (3.6.18). Note that by Proposition 3.1.3 and Lemma 3.5.2, there is a constant $c > 0$ such that $h(p) \geq cp$ for all p. $\square$

Note that (3.6.19) holds for any $s > 0$. For the proof of Proposition 3.6.6, we improve upon the trivial bound $\mathbb{P}_{\infty}^{(p)}(E_s) \leq 1$ so that we can take $s \in (0,1)$. More precisely, we show the following estimate, which allows us to later replace E_s with the event $\{\sum_{f\in\mathfrak{F}_0} h(\mathrm{per}_{\mathrm{in}}(f)) \in [sh(p), rh(p)]\}$ for an appropriate choice of parameters $0 < s < 1 < r$.

Lemma 3.6.9. *For each $r > 0$, there is a constant $c_r > 0$ depending on r such that for all p, we have*

$$\mathbb{P}_{\infty}^{(p)}\bigg(\sum_{f\in\mathfrak{F}_0} h(\mathrm{per}_{\mathrm{in}}(f)) < rh(p)\bigg) \leq e^{-c_r p}. \tag{3.6.20}$$

Proof. As in the proof of Lemma 3.6.3, we use the random walk encoding (Proposition 3.4.7) of M_0 sampled from $\mathbb{P}_{\infty}^{(p)}$, which by construction is a critical non-generic Boltzmann map sampled from $\mathbb{P}_{\mathbf{q}}^{(p)}$. Let $S_n = X_1 + \cdots + X_n$ be a random walk with i.i.d. step distribution $\mathbb{P}(X_i = k-1) = \mu_{\mathbf{q}}(k)$ and, given S, let R_i be sampled conditionally independently for each $i \in \mathbb{N}$ from the marginal law of $\mathrm{Rat}(R)$ under $\mathbb{P}_{\mathrm{ring}}^{(X_i+1)}$. Since $h(p) = -\log F(p) \asymp p$ (Proposition 3.1.3), there exists a constant $a > 0$ such that

$$\mathbb{P}_{\infty}^{(p)}\bigg(\sum_{f\in\mathfrak{F}_0} h(\mathrm{per}_{\mathrm{in}}(f)) < rh(p)\bigg) \leq \mathbb{P}_{\infty}^{(p)}\bigg(\sum_{f\in\mathfrak{F}_0} \mathrm{per}_{\mathrm{in}}(f) < arp\bigg). \tag{3.6.21}$$

By Proposition 3.4.7 with the bound $1/\mathbb{E}[(L_S^{(p)}+1)^{-1}] = O(p^{3/2})$ from Lemma 3.4.8, we have

$$\mathbb{P}_\infty^{(p)}\biggl(\sum_{f\in\mathfrak{F}_0}\mathrm{per}_{\mathrm{in}}(f) < arp\biggr) = \frac{\mathbb{E}\bigl[(L_S^{(p)}+1)^{-1}\mathbf{1}\bigl\{\sum_{i=1}^{T_S^{(p)}}(X_i+1)R_i < arp\bigr\}\bigr]}{\mathbb{E}[(L_S^{(p)}+1)^{-1}]}$$
$$\leq C_0 p^{3/2}\,\mathbb{P}\biggl(\sum_{i=1}^{T_S^{(p)}}(X_i+1)R_i < arp\biggr) \tag{3.6.22}$$

for some constant $C_0 > 0$.

To complete the proof, it suffices to show that for some constant c_r depending on r, we have

$$\mathbb{P}\biggl(\sum_{i=1}^{T_S^{(p)}}(X_i+1)R_i < arp\biggr) \leq e^{-c_r p}. \tag{3.6.23}$$

The key ingredient is Lemma 3.4.8, which states that that $T_S^{(p)}$ is very unlikely to be much smaller than $p^{3/2}$. Consequently, for any constant $c > 0$, there exist positive constants C_2 and C_3 such that

$$\mathbb{P}(T_S^{(p)} \leq cp) \leq C_2 e^{-C_3 p} \tag{3.6.24}$$

for all p. Each of the i.i.d. nonnegative random variables $\{(X_i+1)R_i\}_{i\in\mathbb{N}}$ have a positive probability of being nonzero. By the standard Chernoff bound, we know that there exist positive constants C_4 and C_5 such that for all $\ell \in \mathbb{N}$, we have

$$\mathbb{P}\biggl(\sum_{i=1}^{n}(X_i+1)R_i \leq C_4 n\biggr) \leq e^{-C_5 n}. \tag{3.6.25}$$

Thus,

$$\mathbb{P}\biggl(\sum_{i=1}^{T_S^{(p)}}(X_i+1)R_i \leq arp\biggr) \leq \mathbb{P}\bigl(T_S^{(p)} < arp/C_4\bigr) + \sum_{n=\lceil arp/C_4\rceil}^{\infty}\mathbb{P}\biggl(\sum_{i=1}^{n}(X_i+1)R_i \leq C_4 n\biggr). \tag{3.6.26}$$

We complete the proof by substituting (3.6.24) and (3.6.25) to the right-hand side. $\square$

We now improve upon the result of Lemma 3.6.8.

Lemma 3.6.10. *For each $r > 1$, there is a constant $c_r > 0$ such that for any $s \in (0,1)$ and for all p large enough, we have*

$$\mathbb{P}_{\mathtt{F}}^{(p)}\Bigg(\sum_{f \in \mathfrak{F}_0} h(\mathrm{per}_{\mathrm{in}}(f)) \in [sh(p), rh(p)]\Bigg) \leq e^{(1-s)h(p)-c_r p}. \tag{3.6.27}$$

Proof. Let E denote the event $\{\sum_{f \in \mathfrak{F}_0} h(\mathrm{per}_{\mathrm{in}}(f)) \in [sh(p), rh(p)]\}$. By using Bayes's rule as in (3.6.18), we have

$$\mathbb{P}_{\mathtt{F}}^{(p)}(E) = F(p)^{-1}\mathbb{P}_{\infty}^{(p)}(E)\mathbb{P}_{\infty}^{(p)}(M \text{ is finite}|E). \tag{3.6.28}$$

From the iterative construction of M under $\mathbb{P}_{\infty}^{(p)}$, we have the inequality $\mathbb{P}_{\infty}^{(p)}(M \text{ is finite}|E) \leq e^{-sh(p)}$ just as in (3.6.19). Recalling $F(p) = e^{-h(p)}$ and applying Lemma 3.6.9, we complete the proof by choosing a slighter smaller c_r if necessary. $\square$

We are now ready to complete the proof of Proposition 3.6.6 for large p.

Lemma 3.6.11. *There exists constants $s_0 \in (0,1)$ and $p_0 \in \mathbb{N}$ such that for all $p \geq p_0$, we have*

$$\mathbb{E}_{\mathtt{F}}^{(p)}\Bigg[\sum_{f \in \mathfrak{F}_0} h(\mathrm{per}_{\mathrm{in}}(f))\Bigg] \leq s_0 h(p). \tag{3.6.29}$$

Proof. Fix $\varepsilon > 0$. By combining Proposition 3.1.3 with Lemma 3.6.10 for $s \in (0,1)$ sufficiently close to 1, we find that there exist constants $p_0 \in \mathbb{N}$, $c, C > 0$, and $s_0 \in (0,1)$ such that for all $p \geq p_0$,

$$\mathbb{P}_{\mathtt{F}}^{(p)}\Bigg(\sum_{f \in \mathfrak{F}_0} h(\mathrm{per}_{\mathrm{in}}(f)) \in [s_0 h(p), (1+\varepsilon)h(p)]\Bigg) \leq Ce^{-cp}. \tag{3.6.30}$$

By Lemma 3.6.8, the constants p_0, C, and c can be chosen such that for all $p \geq p_0$ and $\alpha \geq (1+\varepsilon)$,

$$\mathbb{P}_{\mathtt{F}}^{(p)}\bigg(\sum_{f \in \mathfrak{F}_0} h(\mathrm{per}_{\mathrm{in}}(f)) > \alpha h(p) \bigg) \leq C e^{-c(\alpha-1)p}. \tag{3.6.31}$$

Combining the two estimates, we that for all $p \geq p_0$,

$$\begin{aligned} &\mathbb{E}_{\mathtt{F}}^{(p)}\bigg[\sum_{f \in \mathfrak{F}_0} h(\mathrm{per}_{\mathrm{in}}(f)) - s_0 h(p) \bigg]_+ \\ &\leq (1+\varepsilon - s_0) h(p) \cdot \mathbb{P}_{\mathtt{F}}^{(p)}\bigg(\sum_{f \in \mathfrak{F}_0} h(\mathrm{per}_{\mathrm{in}}(f)) \in [s_0 h(p), (1+\varepsilon) h(p)] \bigg) \\ &\qquad\qquad + \mathbb{E}_{\mathtt{F}}^{(p)}\bigg[\bigg(\sum_{f \in \mathfrak{F}_0} h(\mathrm{per}_{\mathrm{in}}(f)) \bigg) \cdot \mathbf{1}\bigg\{ \sum_{f \in \mathfrak{F}_0} h(\mathrm{per}_{\mathrm{in}}(f)) > (1+\varepsilon) h(p) \bigg\} \bigg] \\ &\leq (1+\varepsilon - s_0) h(p) \cdot C e^{-cp} + \int_{1+\varepsilon}^{\infty} \mathbb{P}_{\mathtt{F}}^{(p)}\bigg(\sum_{f \in \mathfrak{F}_0} h(\mathrm{per}_{\mathrm{in}}(f)) > \alpha h(p) \bigg) h(p) \, d\alpha \\ &\leq \widetilde{C} p (e^{-cp} + e^{-c\varepsilon p}) \end{aligned} \tag{3.6.32}$$

for some constant $\widetilde{C} > 0$. Since this bound tends to 0 as $p \to \infty$, we obtain (3.6.29) by slightly increasing p_0 and s_0 if necessary. □

The proof of Proposition 3.6.6 for smaller values of p is achieved via the following lemma.

Lemma 3.6.12. *For each $p_0 \in \mathbb{N}$, there is a constant $s_0 \in (0,1)$ such that for all $p \in [\![1, p_0]\!]$, we have*

$$\mathbb{E}_{\mathtt{F}}^{(p)}\bigg[\sum_{f \in \mathfrak{F}_0} h(\mathrm{per}_{\mathrm{in}}(f)) \bigg] \leq s_0 h(p). \tag{3.6.33}$$

Proof. Let us denote

$$Z := \mathbb{P}_{\infty}^{(p)}\big(M \text{ is finite} \big| \{\mathrm{per}_{\mathrm{in}}(f)\}_{f \in \mathfrak{F}_0} \big) = \prod_{f \in \mathfrak{F}_0} F(\mathrm{per}_{\mathrm{in}}(f)). \tag{3.6.34}$$

Note that $\mathbb{E}_\infty^{(p)}[Z] = F(p)$. Moreover,

$$\begin{aligned}\mathbb{E}_{\mathtt{F}}^{(p)}\Big[\sum_{f\in\mathfrak{F}_0} h(\mathrm{per}_{\mathrm{in}}(f))\Big] &= \frac{\mathbb{E}_\infty^{(p)}\big[\big(\sum_{f\in\mathfrak{F}_0} h(\mathrm{per}_{\mathrm{in}}(f))\big)\,\mathbb{P}_\infty^{(p)}\big(M \text{ is finite}\big|\sum_{f\in\mathfrak{F}_0} h(\mathrm{per}_{\mathrm{in}}(f))\big)\big]}{F(p)} \\ &= F(p)^{-1}\mathbb{E}_\infty^{(p)}[Z\log(1/Z)].\end{aligned} \tag{3.6.35}$$

Since the function $x \mapsto x\log(1/x)$ is strictly concave for $x > 0$, by Jensen's inequality,

$$\mathbb{E}_{\mathtt{F}}^{(p)}\Big[\sum_{f\in\mathfrak{F}_0} h(\mathrm{per}_{\mathrm{in}}(f))\Big] < F(p)^{-1}(\mathbb{E}_\infty^{(p)}[Z])\log(1/(\mathbb{E}_\infty^{(p)}[Z])) = \log F(p)^{-1} = h(p). \tag{3.6.36}$$

In particular, the inequality is strict, so we can find $s_0 \in (0,1)$ as in (3.6.33) for any finite set of p. $\square$

Proof of Proposition 3.6.6. Combine Lemmas 3.6.11 and 3.6.12. $\square$

3.6.3 Estimates on the largest volume of submaps of finite supercritical maps

The goal of this subsection is to obtain the following estimate on the maximum volume of submaps inserted into each face $f \in \mathfrak{F}_0$ of the gasket M_0, where the volume of a map refers to the total number of its vertices. This result will be used in Lemma 3.7.1 to show that the submaps $\{M|_f\}_{f\in\mathfrak{F}_0}$, which are parts of M inside f, become negligible as $p \to \infty$ when we rescale so that the gasket M_0 converges to the continuum random tree.

Proposition 3.6.13. *For M sampled from $\mathbb{P}_{\mathtt{F}}^{(p)}$,*

$$\frac{1}{\sqrt{p}} \sup_{f\in\mathfrak{F}_0} \mathrm{Vol}(M|_f) \xrightarrow{d} 0 \tag{3.6.37}$$

as $p \to \infty$.

The main idea for the proof of Proposition 3.6.13 is that instead of the volume of the whole map, it suffices to keep track of the sum of the perimeters of the rings. More precisely, let us denote the total perimeter of a map M sampled from $\mathbb{P}_{\mathtt{F}}^{(p)}$ over all generations as

$$\mathrm{TPerm}(M) := p + \sum_{f \in \bigcup_{i \in \mathbb{N}^{\#}} \mathfrak{F}_i} (\mathrm{per}_{\mathrm{out}}(f) + \mathrm{per}_{\mathrm{in}}(f)) \tag{3.6.38}$$

where $\mathrm{per}_{\mathrm{out}}(f)$ and $\mathrm{per}_{\mathrm{in}}(f)$ are, respectively, the outer and inner half-perimeters of the ring $R(f)$ inserted into the face $f \in \bigcup_{i \in \mathbb{N}^{\#}} \mathfrak{F}_i$.

Lemma 3.6.14. *There is a constant $C > 1$ such that for any $p \in \mathbb{N}$, if M is a map sampled from $\mathbb{P}_{\mathtt{F}}^{(p)}$, then $\mathrm{Vol}(M) \leq C \cdot \mathrm{TPerm}(M)$ almost surely.*

Proof. Recalling the construction of supercritical maps from Definition 3.1.1, we have

$$\mathrm{Vol}(M) \leq \mathrm{Vol}(M_0) + \sum_{f \in \bigcup_{i \in \mathbb{N}^{\#}} \mathfrak{F}_i} (\mathrm{Vol}(R(f)) + \mathrm{Vol}(M(f))) \tag{3.6.39}$$

where $R(f)$ is the ring inserted into the face f and $M(f)$ is the Boltzmann map inserted into the inner face of the ring $R(f)$. By the non-thickness condition in Definition 3.4.4, there is a constant C such that for any ring $R(f)$, we have $\mathrm{Vol}(R(f)) \leq C(\mathrm{per}_{\mathrm{out}}(f) + \mathrm{per}_{\mathrm{in}}(f))$. Observe also that the outermost gasket M_0 satisfies $\mathrm{Vol}(M_0) \leq 2(p + \sum_{f \in \mathfrak{F}_0} \mathrm{per}_{\mathrm{out}}(f))$, and similarly

$$\mathrm{Vol}(M(f)) \leq 2\Bigg(\mathrm{per}_{\mathrm{in}}(f) + \sum_{\tilde{f} \in \mathfrak{F}(M(f))} \mathrm{per}_{\mathrm{out}}(\tilde{f})\Bigg) \tag{3.6.40}$$

for every Boltzmann map $M(f)$ added during the construction. Hence, we obtain

$$\mathrm{Vol}(M) \leq 2p + (C+2) \sum_{f \in \bigcup_{i \in \mathbb{N}^{\#}} \mathfrak{F}_i} (\mathrm{per}_{\mathrm{out}}(f) + \mathrm{per}_{\mathrm{in}}(f)) \leq (C+2)\mathrm{TPerm}(M) \tag{3.6.41}$$

by combining all of the above bounds. □

The first step in the proof of Proposition 3.6.13 is to show that the gasket decomposition of a map sampled from $\mathbb{P}^{(p)}_{\mathtt{F}}$ does not go on for many steps.

Lemma 3.6.15. *Let M be a map sampled from $\mathbb{P}^{(p)}_{\mathtt{F}}$. Let $T_{\text{ext}}(M) := \min\{i : \mathfrak{F}_i = \varnothing\}$ be the total number of iterations in the construction of M. Then, there exist constants $c, C > 0$ such that for every p and n, we have $\mathbb{P}^{(p)}_{\mathtt{F}}(T_{\text{ext}}(M) > n) \le Cpe^{-cn}$.*

Proof. If $T_{\text{ext}}(M) > n$, then $\sum_{f\in\mathfrak{F}_n} h(\text{per}_{\text{in}}(f)) > 0$. Since $\inf_{p'\in\mathbb{N}} h(p') > 0$ by Proposition 3.1.3 and Lemma 3.5.2, Markov's inequality implies

$$\begin{aligned}\mathbb{P}^{(p)}_{\mathtt{F}}\Bigg(\sum_{f\in\mathfrak{F}_n} h(\text{per}_{\text{in}}(f)) > 0\Bigg) &= \mathbb{P}^{(p)}_{\mathtt{F}}\Bigg(\sum_{f\in\mathfrak{F}_n} h(\text{per}_{\text{in}}(f)) \ge \inf_{p'\in\mathbb{N}} h(p')\Bigg) \\ &\le \frac{\mathbb{E}^{(p)}_{\mathtt{F}}\big[\sum_{f\in\mathfrak{F}_n} h(\text{per}_{\text{in}}(f))\big]}{\inf_{p'\in\mathbb{N}} h(p')}.\end{aligned} \tag{3.6.42}$$

The claim now follows from Corollary 3.6.7. ☐

Now, we show that with high probability, the total perimeter at is kept small at every level.

Lemma 3.6.16. *Consider the events*

$$G_t := \Bigg\{\sum_{f\in\mathfrak{F}_i} \text{per}_{\text{in}}(f) \ge t \quad \text{for some } i\Bigg\} \quad \textit{and} \quad \widetilde{G}_t := \Bigg\{\sum_{f\in\mathfrak{F}_i} \text{per}_{\text{out}}(f) \ge t \quad \text{for some } i\Bigg\}. \tag{3.6.43}$$

Then, there is a constant $c > 0$ such that $\mathbb{P}^{(p)}_{\mathtt{F}}(G_t \cup \widetilde{G}_t) \le e^{-ct}/F(p)$ for all t.

Proof. By Proposition 3.1.3 and Lemma 3.5.2, there exists $c > 0$ such that $h(p) \ge cp$ for all $p \in \mathbb{N}$. The proof again uses the Bayes's rule argument from (3.6.18). Note that we have

$$\mathbb{P}^{(p)}_{\mathtt{F}}(G_t) \le F(p)^{-1}\mathbb{P}^{(p)}_{\infty}(M \text{ is finite}|G_t). \tag{3.6.44}$$

On G_t, define $I_t := \inf\{i \in \mathbb{N}^{\#} : \sum_{f\in\mathfrak{F}_i} \text{per}_{\text{in}}(f) \ge t\}$ so that $\sum_{f\in\mathfrak{F}_{I_t}} h(\text{per}_{\text{in}}(f)) \ge ct$.

Further conditioning on I_t, $\mathfrak{F}_{I_t}$, and $\{\mathrm{per}_{\mathrm{in}}(f)\}_{f\in\mathfrak{F}_{I_t}}$, we have

$$\begin{aligned}\mathbb{P}_\infty^{(p)}(M \text{ is finite}|G_t) &= \mathbb{E}_\infty^{(p)}\left[\mathbb{P}_\infty^{(p)}(M \text{ is finite}|I_t, \mathfrak{F}_{I_t}, \{\mathrm{per}_{\mathrm{in}}(f)\}_{f\in\mathfrak{F}_{I_t}})\big|G_t\right] \\ &= \mathbb{E}_\infty^{(p)}\left[\prod_{f\in\mathfrak{F}_{I_t}} F(\mathrm{per}_{\mathrm{in}}(f))\middle|G_t\right] = \mathbb{E}_\infty^{(p)}\left[e^{-\sum_{f\in\mathfrak{F}_{I_t}} h(\mathrm{per}_{\mathrm{in}}(f))}\middle|G_t\right] \le e^{-ct}.\end{aligned} \tag{3.6.45}$$

Substituting the above into (3.6.44) gives $\mathbb{P}_{\mathtt{F}}^{(p)}(G_t) \le e^{-ct}/F(p)$.

Define $\widetilde{I}_t$ analogously on the event $\widetilde{G}_t$. Then, similarly to (3.6.45), there is a constant $\tilde{c} > 0$ such that

$$\begin{aligned}\mathbb{P}_\infty^{(p)}(M \text{ is finite}|\widetilde{G}_t) &= \mathbb{E}_\infty^{(p)}\left[\mathbb{P}_\infty^{(p)}(M \text{ is finite}|I_t, \mathfrak{F}_{I_t}, \{\mathrm{per}_{\mathrm{out}}(f)\}_{f\in\mathfrak{F}_{I_t}})\big|\widetilde{G}_t\right] \\ &= \mathbb{E}_\infty^{(p)}\left[\prod_{f\in\mathfrak{F}_{I_t}} \mathbb{E}_{\mathrm{ring}}^{\mathrm{per}_{\mathrm{out}}(f)}\left[F(\mathrm{per}_{\mathrm{in}}(f))\right]\middle|\widetilde{G}_t\right] \\ &\le \mathbb{E}_\infty^{(p)}\left[\prod_{f\in\mathfrak{F}_{I_t}} \mathbb{E}_{\mathrm{ring}}^{\mathrm{per}_{\mathrm{out}}(f)}\left[e^{-c\cdot\mathrm{per}_{\mathrm{in}}(f)}\right]\middle|\widetilde{G}_t\right] \\ &\le \mathbb{E}_\infty^{(p)}\left[e^{-\tilde{c}\sum_{f\in\mathfrak{F}_{I_t}} \mathrm{per}_{\mathrm{out}}(f)}\middle|\widetilde{G}_t\right] \le e^{-\tilde{c}t}\end{aligned} \tag{3.6.46}$$

where we used the lower tail condition in Definition 3.4.4 for the penultimate inequality. Using Bayes's rule as in (3.6.44), we obtain $\mathbb{P}_{\mathtt{F}}^{(p)}(\widetilde{G}_t) \le e^{-\tilde{c}t}/F(p)$. □

Combining the above two lemmas, we obtain a bound on $\mathrm{TPerm}(M)$ for a finite super-critical map M.

Lemma 3.6.17. *Given $\delta > 0$, there exist positive constants C and c such that for all $t \ge k^\delta$ and all $k \in \mathbb{N}$, we have*

$$\mathbb{P}_{\mathtt{F}}^{(k)}(\mathrm{TPerm}(M) \ge tk) \le Ce^{-c\sqrt{t}}. \tag{3.6.47}$$

Proof. Using Lemma 3.6.14 along with the notation of Lemmas 3.6.15 and 3.6.16, we have

$$\mathbb{P}_{\mathtt{F}}^{(k)}\big(\mathrm{TPerm}(M) \geq tk\big) \leq \mathbb{P}_{\mathtt{F}}^{(k)}\big(T_{\mathrm{ext}}(M) \geq \sqrt{t}\big) + \mathbb{P}_{\mathtt{F}}^{(k)}\big(G_{k\sqrt{t}/2} \cup \widetilde{G}_{k\sqrt{t}/2}\big). \tag{3.6.48}$$

By Lemma 3.6.15, the first term on the right-hand side is bounded above by $C_1ke^{-c_1\sqrt{t}} \leq C_1'e^{-c_1'\sqrt{t}}$ for some constants $C_1, C_1', c_1, c_1' > 0$, where we have used that $t \geq k^{\delta}$. By Lemma 3.6.16, the second term in (3.6.48) is bounded above by $C_2e^{c_2k(1-\sqrt{t})} \leq C_2'e^{-c_2'\sqrt{t}}$ for some positive constants C_2, C_2', c_2, c_2'. This completes the proof. □

We are now ready to complete the proof of Proposition 3.6.13.

Proof of Proposition 3.6.13. By Lemma 3.6.14, it suffices to show that

$$\frac{1}{\sqrt{p}} \sup_{f\in\mathfrak{F}_0} \mathrm{TPerm}(M|_f) \xrightarrow{d} 0. \tag{3.6.49}$$

By Lemmas 3.4.10 and 3.6.3, we know that for some constant $\Delta > 2$, we have

$$\lim_{p\to\infty} \mathbb{P}_{\mathtt{F}}^{(p)}\left(\sup_{f\in\mathfrak{F}_0} \mathrm{per}_{\mathrm{in}}(f) \leq p^{1/\Delta} \text{ and } |\mathfrak{F}_0| \leq -\frac{2p}{m_{\mathbf{q}'}} \right) = 1. \tag{3.6.50}$$

Denote the above event as A_p. Now, for any fixed $\varepsilon > 0$,

$$\begin{aligned}
\mathbb{P}_{\mathtt{F}}^{(p)}\left(\sup_{f\in\mathfrak{F}_0} \mathrm{TPerm}(M|_f) \geq \varepsilon\sqrt{p} \right) &\leq \mathbb{P}_{\mathtt{F}}^{(p)}\big(A_p^c\big) + \left(-\frac{2p}{m_{\mathbf{q}'}}\right) \sup_{k\leq p^{1/\Delta}} \mathbb{P}_{\mathtt{F}}^{(k)}\big(\mathrm{TPerm}(M) \geq \varepsilon\sqrt{p}\big) \\
&= \mathbb{P}_{\mathtt{F}}^{(p)}\big(A_p^c\big) + \left(-\frac{2p}{m_{\mathbf{q}'}}\right) \sup_{k\leq p^{1/\Delta}} \mathbb{P}_{\mathtt{F}}^{(k)}\big(\mathrm{TPerm}(M) \geq k(\varepsilon\sqrt{p}/k)\big).
\end{aligned} \tag{3.6.51}$$

The second term tends to 0 as $p \to \infty$ by Lemma 3.6.17 and the fact that $1/\Delta < 1/2$, thus proving (3.6.49). □

3.7 Convergence to the CRT

We finally prove Theorem 3.1.4 in this section, showing that the map $M^{(p)}$ sampled from $\mathbb{P}_{\mathtt{F}}^{(p)}$ converges to the continuum random tree after appropriate scaling. While we have so far used M to denote a sample from $\mathbb{P}_{\mathtt{F}}^{(p)}$, we will always use $M^{(p)}$ to denote such a sample from now so as to avoid confusion in the many convergence statements in this section. We denote its outermost gasket (M_0 in the iterative construction of Definition 3.1.1) as $M_0^{(p)}$. We recall the notation that if f is an internal face of $M_0^{(p)}$ (or $\partial M^{(p)}$), then $M^{(p)}|_f$ denotes the edges and vertices that bound this face in $M_0^{(p)}$ as well as those of $M^{(p)}$ that are glued into this face.

In Proposition 3.6.2, we saw that the marginal law on the outermost gasket $M_0^{(p)}$ sampled from $\mathbb{P}_{\mathtt{F}}^{(p)}$ is that of a subcritical Boltzmann map $\mathbb{P}_{\mathbf{q}'}^{(p)}$. Then, by Proposition 3.4.18, there is a a constant $\theta_{\mathbf{q}'} > 0$ such that $(\theta_{\mathbf{q}'}/\sqrt{p})M_0^{(p)}$ converges to the CRT. Our goal in this section is to upgrade this result to one for the entire map $M^{(p)}$ by showing that the submaps added into the faces of $M_0^{(p)}$ change distances by a constant factor.

To do so, instead of the outermost gasket $M_0^{(p)}$, we will work with its outer boundary $\partial M^{(p)}$, which consists of the vertices and edges that border the root face of $M^{(p)}$. The reason is that the scaling limit of $\partial M^{(p)}$ is also the CRT by Proposition 3.4.17, and it has additional integrability through its connection with BGW trees studied in the works [77, 215]. For the first step, we reduce Theorem 3.1.4 to a convergence statement for a "projection" of the whole map $M^{(p)}$ to its boundary $\partial M^{(p)}$.

For the rest of the section, given a graph M, we denote its set of vertices and the graph distance on it by V_M and d_M, respectively. We remind the reader that rM denotes the metric space (V_M, rd_M) for $r \in \mathbb{R}_+$.

Proposition 3.7.1. *Let $M^{(p)}$ be sampled from $\mathbb{P}_{\mathtt{F}}^{(p)}$. Denote $d_{M^{(p)}}^{\partial} := d_{M^{(p)}}|_{V_{\partial M^{(p)}} \times V_{\partial M^{(p)}}}$.*

Then, there exists a constant $\theta_{\mathtt{F}}$ such that

$$\mathcal{X}^{(p)} := \left(V_{\partial M^{(p)}}, \frac{\theta_{\mathtt{F}}}{\sqrt{p}} d^{\partial}_{M^{(p)}}\right) \xrightarrow{d} \mathrm{CRT} \tag{3.7.1}$$

as $p \to \infty$ in the Gromov–Hausdorff distance.

Proof of Theorem 3.1.4 assuming Proposition 3.7.1. Let

$$L^{(p)} := \max_{v \in V_{M_0^{(p)}}} \min_{w \in V_{\partial M^{(p)}}} d_{M^{(p)}}(v, w) \tag{3.7.2}$$

be the maximum distance from a vertex in $M_0^{(p)}$ to the boundary $\partial M^{(p)}$. Recalling that $M_0^{(p)}$ sampled from $\mathbb{P}_{\mathtt{F}}^{(p)}$ is a subcritical Boltzmann map, we see from Proposition 3.4.18 that $L^{(p)}/\sqrt{p} \to 0$ in distribution as $p \to \infty$. Combining this with Proposition 3.6.13,

$$\frac{1}{\sqrt{p}} \max_{f \in \mathfrak{F}(\partial M^{(p)})} \operatorname{diam}(M^{(p)}|_f) \leq \frac{1}{\sqrt{p}} \left(L^{(p)} + \max_{\tilde{f} \in \mathfrak{F}(M_0^{(p)})} \operatorname{diam}(M^{(p)}|_{\tilde{f}}) \right) \tag{3.7.3}$$

converges to 0 in distribution as $p \to \infty$.

Now consider $\mathcal{X}^{(p)}$ as naturally embedded within $(\theta_{\mathtt{F}}/\sqrt{p})M^{(p)}$. Under this isometric embedding, the Hausdorff distance between the $V_{M^{(p)}}$ and $V_{\partial M^{(p)}}$ is bounded above by the left-hand side of (3.7.3). Hence, $d_{\mathrm{GH}}(\mathcal{X}^{(p)}, (\theta_{\mathtt{F}}/\sqrt{p})M^{(p)}) \to 0$ in distribution as $p \to \infty$. $\square$

The rest of this section is thus dedicated to the proof of Proposition 3.7.1. The main idea is to study the looptree structure of $\partial M^{(p)}$ in association with Bienaymé–Galton–Watson trees. The argument proceeds along the same lines as the proof of Proposition 3.4.17 in [160], which uses the spinal decomposition to prove the convergence of critical looptrees to the CRT. The additional work we do is to show that the addition of vertices and edges going from $\partial M^{(p)}$ to $M^{(p)}$ affect the scaling limit merely by multiplying distances by a deterministic constant.

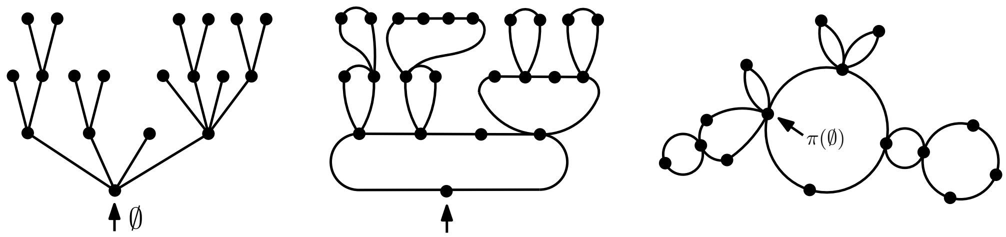

Figure 3.4: From left to right: a tree $\mathcal{T}$, the corresponding looptree $\mathrm{Loop}(\mathcal{T})$, and the contracted looptree $\overline{\mathrm{Loop}}(\mathcal{T})$. The root vertex $\emptyset \in V_{\mathcal{T}}$ and the corresponding vertices on $\mathrm{Loop}(\mathcal{T})$ and $\pi(\emptyset) \in \overline{\mathrm{Loop}}(\mathcal{T})$ are marked with arrows.

3.7.1 Boundaries of Boltzmann maps and looptrees

We begin by introducing the connection between looptrees and boundaries of Boltzmann maps. First, a few notations: $\mathcal{T}$ refers to a rooted plane tree, by which we mean a tree with a distinguished vertex (called the root and denoted by $\emptyset$) and an ordering specified among the children of any vertex. Thus, intuitively, we can think of the root of a rooted plane tree being at the bottom and the children of any vertex being located above the corresponding vertex.

Here are the definitions of the looptree and the contracted looptree associated with a rooted planar tree as given in [77].

Definition 3.7.2. Let $\mathcal{T}$ be a rooted plane tree. We define $\mathrm{Loop}(\mathcal{T})$ to be a planar map which has the same set of vertices as $\mathcal{T}$, with an edge between $v_1, v_2 \in V_{\mathrm{Loop}(\mathcal{T})}$ if and only if either of the following are true.

(1) v_1 and v_2 are the consecutive children of a common parent in $\mathcal{T}$.

(2) v_1 is the leftmost/rightmost child of v_2 in $\mathcal{T}$, or vice versa.

The contracted looptree $\overline{\mathrm{Loop}}(\mathcal{T})$ is obtained from $\mathrm{Loop}(\mathcal{T})$ by contracting every edge (u, v) for which v is the rightmost child of u in $\mathcal{T}$.

See Figure 3.4 for an illustration of this definition. Let π denote the natural projection from $V_{\mathcal{T}}$ to $V_{\overline{\mathrm{Loop}}(\mathcal{T})}$. We note that π is one-to-one on the leaves of $\mathcal{T}$ (i.e., vertices with no

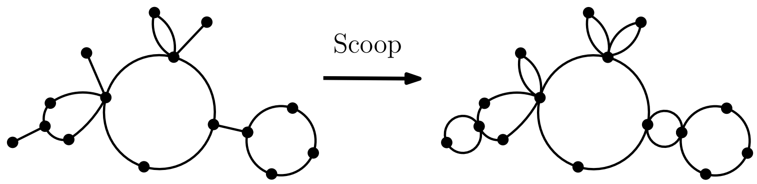

Figure 3.5: A planar map (left) and the corresponding scooped-out map (right).

children).

For $v \in V_{\mathcal{T}}$, let $f_v \in \mathfrak{F}(\overline{\mathrm{Loop}}(\mathcal{T}))$ be the face enclosed by the vertices π(children of v in $\mathcal{T}$) $\subset V_{\overline{\mathrm{Loop}}(\mathcal{T})}$. This forms a one-to-one correspondence between the faces of $\overline{\mathrm{Loop}}(\mathcal{T})$ and the non-leaf vertices of $\mathcal{T}$. Let $f_v = \varnothing$ if v is a leaf in $\mathcal{T}$.

As observed in [77, Lemma 4.3] (see also [215, Lemma 4.1] for the statement for non-triangulations), if M is a rooted planar map (i.e., with a fixed outermost face on the right of the root edge, and the root vertex $\emptyset$ given by the incident vertex of the root edge), then ∂M is almost a looptree: we just need to duplicate each single edge connecting loops into a double edge. This procedure was described in detail in [77, Section 2.3].

Definition 3.7.3. Given a rooted planar map M, define the scooped-out map $\mathrm{Scoop}(M)$ as the outer boundary ∂M modified so that every edge e in ∂M which is adjacent to the outermost face on both sides is duplicated.

Given a rooted planar map M, it is straightforward to see that there is a unique rooted plane tree $\mathcal{T}_M$ such that $\mathrm{Scoop}(M) = \overline{\mathrm{Loop}}(\mathcal{T}_M)$ and π(root vertex of $\mathcal{T}_M$) = root vertex of $\mathrm{Scoop}(M)$. We use $\emptyset$ to refer to both root vertices. Note that $\mathrm{Scoop}(M)$ and ∂M can be different planar maps but define the same metric space. Hence, (3.7.1) can be stated equivalently with $\mathrm{Scoop}(M^{(p)}) = \overline{\mathrm{Loop}}(\mathcal{T}_{M^{(p)}})$ in place of $\partial M^{(p)}$.

For $M^{(p)}$ sampled from $\mathbb{P}_{\mathtt{F}}^{(p)}$, note that $\mathcal{T}_{M^{(p)}} = \mathcal{T}_{M_0^{(p)}}$. The key fact is that since $M_0^{(p)}$ is a subcritical Boltzmann map, $\mathcal{T}_{M_0^{(p)}}$ has the distribution of a BGW tree. The following proposition is an immediate application of the relationship between the boundary of a subcritical Boltzmann map and a BGW tree described in [77, Lemma 4.3] and [215, Proposition 3.6, Lemma 4.1] given Proposition 3.6.2.

Proposition 3.7.4. *There is a finite variance measure $\nu_{\mathbf{q}'}$ on the nonnegative even integers such that for every $p \in \mathbb{N}$, the rooted tree $\mathcal{T}^{(p)} := \mathcal{T}_{M^{(p)}}$ is a critical BGW tree with offspring distribution $\nu_{\mathbf{q}'}$ conditioned to have $2p+1$ total vertices.*

We now describe how to sample $M^{(p)}$ given $\mathcal{T}^{(p)}$. First, let $\widehat{\mathbb{P}}_{\mathbf{q}'}^{(p)}$ denote the law of a Boltzmann map with weight sequence $\mathbf{q}'$ conditioned to have a simple boundary of perimeter $2p$. For $p = 1$, this includes the map consisting of one edge and no interior faces. As usual, let $\mathfrak{F}(\partial M)$ be the collection of inner faces of ∂M (i.e., excluding the root face of M).

Lemma 3.7.5 ([215, Corollary 3.7])**.** *Suppose M is a Boltzmann map with distribution $\mathbb{P}_{\mathbf{q}'}^{(p)}$. Given ∂M, the submaps $\{M|_f\}_{f \in \mathfrak{F}(\partial M)}$ are conditionally independent, with the conditional law of $M|_f$ given by $\widehat{\mathbb{P}}_{\mathbf{q}'}^{(\mathrm{per}(f))}$ for each $f \in \mathfrak{F}(\partial M)$.*

Let $\widehat{\mathbb{P}}_{\mathtt{F}}^{(p)}$ be the probability measure on the chain $(M_i^{(p)}, \mathfrak{F}_i^{(p)})_{i \in \mathbb{N}^\#}$ obtained from $\mathbb{P}_{\mathtt{F}}^{(p)}$ by conditioning on $M_0^{(p)}$ (and thus $M^{(p)}$) having a simple boundary. Since the marginal law of $\mathbb{P}_{\mathtt{F}}^{(p)}$ on the outermost gasket $M_0^{(p)}$ is that of a Boltzmann map (Proposition 3.6.2), we obtain the following variation from Lemma 3.7.5.

Lemma 3.7.6. *Let $M^{(p)}$ be sampled from $\mathbb{P}_{\mathtt{F}}^{(p)}$. The maps $\{M^{(p)}|_f\}_{f \in \mathfrak{F}(\partial M^{(p)})}$ are conditionally independent given $\partial M^{(p)}$, and the conditional distributions are equal to $\widehat{\mathbb{P}}_{\mathtt{F}}^{(\mathrm{per}(f))}$.*

We thus have a complete description of the conditional law of $M^{(p)}$ given $\mathcal{T}^{(p)}$.

3.7.2 The spine decomposition of the boundary map

In this subsection, we introduce the spine decomposition of $\mathcal{T}^{(p)}$ that we use to prove Proposition 3.7.1.

Definition 3.7.7. Given a rooted plane tree $\mathcal{T}$ with root $\emptyset$ and vertex $u \in V_{\mathcal{T}}$, we define the trunk $\mathrm{Trunk}(\mathcal{T}, u)$ as the induced subgraph obtained by considering the unique path $\emptyset, v_1, v_2, \ldots, v_{k-1}, u$ from $\emptyset$ to u in the tree $\mathcal{T}$ along with all the vertices that are the immediate children of $\emptyset, v_1, \ldots, v_{k-1}$.

The goal is to construct $\mathrm{Trunk}(\mathcal{T}, V_{\mathrm{unif}})$ when V_{unif} is a uniformly chosen random vertex of a BGW tree $\mathcal{T}$. This is done in two steps: first sample the random distance between $\emptyset$ and V_{unif}, and then construct the entire trunk given this distance. The latter law is given in terms of $\nu^*_{\mathbf{q}'}$, the sized biased version of $\nu_{\mathbf{q}'}$, defined as

$$\nu^*_{\mathbf{q}'}(i) = \frac{i \cdot \nu_{\mathbf{q}'}(i)}{\sum_{k \in \mathbb{N}^{\#}} k \cdot \nu_{\mathbf{q}'}(k)} \tag{3.7.4}$$

for $i \in \mathbb{N}^{\#}$ as first given in [156]. Since $\nu_{\mathbf{q}'}$ has a finite variance, $\nu^*_{\mathbf{q}'}$ has a finite mean.

Definition 3.7.8. Given $h \in \mathbb{N}$, let Trunk^*_h be a random plane tree of height h distributed as follows. Denoting its "spine" as $v_0, \ldots, v_{h-1}$ where v_0 is the root, each vertex v_i has an independent number of children with distribution $\nu^*_{\mathbf{q}'}$. Given the topology of the tree (that is, the tree without the orderings between its vertices), the location of v_{i+1} is chosen uniformly among the children of v_i, with this choice being conditionally independent for all the values of i.

Again, we sample $M^{(p)}$ from $\mathbb{P}^{(p)}_{\mathbf{F}}$ and define its root vertex to be the head of the root edge of the Boltzmann map $M^{(p)}_0$ (cf. Section 3.4). We denote by $\mathcal{T}^{(p)}$ the unique plane tree such that $\mathrm{Scoop}(M^{(p)}) = \overline{\mathrm{Loop}}(\mathcal{T}^{(p)})$ where its root vertex inherited from $M^{(p)}$ is equal to $\pi(\emptyset)$. We now record the spine decomposition of the BGW tree $\mathcal{T}^{(p)}$ as described in Proposition 3.7.4.

Lemma 3.7.9 ([160, Theorem 3])**.** *Let $V^{(p)}_{\mathrm{unif}}$ be a uniform vertex chosen from $\mathcal{T}^{(p)}$. There exists a constant $c > 0$ such that if X is a positive random variable with density $cxe^{-x^2}dx$, then*

$$d_{\mathrm{TV}}\big(\mathrm{Trunk}(\mathcal{T}^{(p)}, V^{(p)}_{\mathrm{unif}}), \mathrm{Trunk}^*_{\lfloor X\sqrt{p}\rfloor}\big) \to 0 \tag{3.7.5}$$

as $p \to \infty$. Here, d_{TV} denotes the total variation distance.

The following key technical lemma links the metric structure of $M^{(p)}$ to that of $\mathcal{T}^{(p)}$, allowing us to rephrase the main task in the proof of Proposition 3.7.1 in terms of the spine decomposition of the BGW tree $\mathcal{T}^{(p)}$. It is similar in its statement and proof to [160, Lemma 15(ii)], with compares distances in a subcritical Boltzmann map M to that in the associated tree $\mathcal{T}_M$. Recall that π denotes the natural projection from $V_{\mathcal{T}}^{(p)}$ to $V_{\overline{\mathrm{Loop}}(\mathcal{T}^{(p)})} = V_{\mathrm{Scoop}(M^{(p)})} = V_{\partial M^{(p)}} \subset V_{M^{(p)}}$.

Proposition 3.7.10. *Let $V_{\mathrm{unif}}^{(p)}$ be a uniformly chosen vertex on $\mathcal{T}^{(p)}$. Then, there exists a constant $\hat{\theta}_{\mathrm{F}}$ such that*

$$\frac{1}{\sqrt{p}}\big(d_{M^{(p)}}(\pi(\emptyset), \pi(V_{\mathrm{unif}}^{(p)})) - \hat{\theta}_{\mathrm{F}}\, d_{\mathcal{T}^{(p)}}(\emptyset, V_{\mathrm{unif}}^{(p)})\big) \xrightarrow{d} 0 \tag{3.7.6}$$

as $p \to \infty$.

Proof. Observe that given the unique path $(v_0, v_1, v_2, \ldots, v_h)$ from $v_0 = \emptyset$ to $v_h = V_{\mathrm{unif}}^{(p)}$ in $\mathcal{T}^{(p)}$ (hence $h = d_{\mathcal{T}^{(p)}}(\emptyset, V_{\mathrm{unif}}^{(p)})$), if we define

$$Y_i := d_{M^{(p)}}(\pi(v_i), \pi(v_{i+1})) \tag{3.7.7}$$

for each $i = 0, 1, \ldots, h-1$, then we have

$$d_{M^{(p)}}(\pi(\emptyset), \pi(V^{(p)})) = \sum_{i=0}^{h-1} Y_i. \tag{3.7.8}$$

This is because any path from $\pi(\varnothing)$ to $\pi(v_h)$ in $M^{(p)}$ must pass through the faces of $\partial M^{(p)}$ which are conjoined by $\pi(v_1), \pi(v_2), \ldots, \pi(v_{h-1})$ in this exact order.

Let p_i denote half the number of children of k_i. The following description is an immediate consequence of Proposition 3.7.4 and Lemma 3.7.6.

Given $(p_0, p_1, \ldots, p_{h-1})$, the Y_i are conditionally independent. The conditional distribution of Y_i agrees with that of $d_{M_i}(v, w)$ where M_i is sampled from $\widehat{\mathbb{P}}^{(p_i)}_{\mathtt{F}}$ and v, w are vertices sampled conditionally independently given M_i from the uniform distribution on ∂M_i.

Let $\hat{p}$ be sampled from $\nu^*_{\mathbf{q}'}$ and $\widehat{M}$ from $\widehat{\mathbb{P}}^{(\hat{p})}_{\mathtt{F}}$. Let $\widehat{Y} = d_{\widehat{M}}(v, w)$, where v and w are vertices sampled conditionally independently given $\widehat{M}$ from the uniform distribution on $\partial\widehat{M}$. Then, since $\widehat{Y} \leq \hat{p}$ and $\nu^*_{\mathbf{q}'}$ has a finite mean, $\mathbb{E}[\widehat{Y}] < \infty$. Let $\widehat{Y}_0, \widehat{Y}_1, \ldots$ be i.i.d. random variables with the law of $\widehat{Y}$.

We claim that (3.7.6) holds with $\hat{\theta}_{\mathtt{F}} := \mathbb{E}[\widehat{Y}]$. By Lemma 3.7.9, there exists a random variable X independent from $\{\widehat{Y}_i\}_{i \in \mathbb{N}^\#}$ with density $cxe^{-x^2}dx$ such that the total variation distance between the two random-length sequences $(Y_0, Y_1, \ldots, Y_{\ell-1})$ and $(\widehat{Y}_0, \widehat{Y}_1, \ldots, \widehat{Y}_{\lfloor X\sqrt{p} \rfloor})$ tends to 0 as $p \to \infty$. Hence, it suffices to check that

$$\frac{1}{\sqrt{p}}\left(\sum_{i=1}^{\lfloor X\sqrt{p}\rfloor} \widehat{Y}_i - \lfloor X\sqrt{p}\rfloor \mathbb{E}\big[\widehat{Y}\big]\right) \xrightarrow{d} 0 \tag{3.7.9}$$

as $p \to \infty$. This is an immediate consequence of the law of large numbers: note that for any $0 < x_1 < x_2 < \infty$ and $\varepsilon > 0$, we have

$$\sup_{x \in [x_1, x_2]} \mathbb{P}\left(\frac{1}{x\sqrt{p}} \sum_{i=1}^{\lfloor x\sqrt{p}\rfloor} (\widehat{Y}_i - \mathbb{E}[\widehat{Y}]) \geq \frac{\varepsilon}{x}\right) \to 0 \tag{3.7.10}$$

as $p \to \infty$. □

3.7.3 Proof of Proposition 3.7.1

We now move on to defining the various height functions on $\mathcal{T}^{(p)}$. Let $\emptyset = v_0, v_1, \ldots, v_{2p}$ be the enumeration of vertices of $\mathcal{T}^{(p)}$ in lexicographic order (i.e., depth-first search). Given a metric $d^{(p)}$ on $\mathcal{T}^{(p)}$, we define the corresponding height function $\mathrm{H} : [0,1] \to [0,\infty)$ as

$$\mathrm{H}_{\mathcal{T}^{(p)};d^{(p)}}\left(\frac{i}{2p}\right) = d^{(p)}(\emptyset, v_i) \quad \text{for } i = 0, 1, \ldots, 2p \tag{3.7.11}$$

and linearly interpolated in between. Recall that π denotes the natural projection from $V_{\mathcal{T}}$ to $V_{\overline{\mathrm{Loop}}(\mathcal{T})}$. There are three height functions that we will be particularly interested in.

- $\mathrm{H}^{(p)}_{\mathcal{T}}$ defined from the graph metric $d_{\mathcal{T}^{(p)}}$ on $\mathcal{T}^{(p)}$.
- $\mathrm{H}^{(p)}_{\mathrm{Loop}}$ defined from the pull-back of the graph metric on $\overline{\mathrm{Loop}}(\mathcal{T}^{(p)})$ by π (that is, $d^{(p)}(v,w) = d_{\overline{\mathrm{Loop}}(\mathcal{T}^{(p)})}(\pi(v), \pi(w))$).
- $\mathrm{H}^{(p)}_{M}$ defined from the pull-back of the graph metric on $M^{(p)}$ by π (that is, $d^{(p)}(v,w) = d_{M^{(p)}}(\pi(v), \pi(w))$.

We also need the contour function of $\mathcal{T}^{(p)}$. This time, let $\emptyset = w_0, w_1, \ldots, w_{4p} = \emptyset$ be the enumeration of the vertices with $\mathcal{T}^{(p)}$ (with duplicity) as we follow along the contour of $\mathcal{T}^{(p)}$. Define $\mathrm{C}^{(p)}_{\mathcal{T}} : [0,1] \to [0,\infty)$ as

$$\mathrm{C}^{(p)}_{\mathcal{T}}\left(\frac{i}{4p}\right) = d_{\mathcal{T}^{(p)}}(\emptyset, w_i) \quad \text{for } i = 0, 1, \ldots, 4p \tag{3.7.12}$$

and linearly interpolated in between.

Lemma 3.7.11. *Let $\sigma^2_{\mathbf{q}'}$ be the variance of the offspring distribution $\nu_{\mathbf{q}'}$ for the BGW tree $\mathcal{T}^{(p)}$ found in Proposition 3.7.4 and let $\hat{\theta}_{\mathrm{F}}$ refer to the constant found in Proposition 3.7.10. Denote $\theta_{\mathrm{F}} := \hat{\theta}_{\mathrm{F}}\sigma_{\mathbf{q}'}/(2\sqrt{2})$. Let $(\mathbb{e}_t : 0 \leq t \leq 1)$ be the normalized Brownian excursion. Then, as $p \to \infty$, the joint convergence*

$$\left(\frac{\sigma_{\mathbf{q}'}}{2\sqrt{2p}}\mathrm{C}^{(p)}_{\mathcal{T}}(t), \frac{\sigma_{\mathbf{q}'}}{2\sqrt{2p}}\mathrm{H}^{(p)}_{\mathcal{T}}(t), \frac{\theta_{\mathrm{F}}}{\sqrt{p}}\mathrm{H}^{(p)}_{M}(t)\right)_{0\leq t\leq 1} \xrightarrow{d} \left(\mathbb{e}_t, \mathbb{e}_t, \mathbb{e}_t\right)_{0\leq t\leq 1} \tag{3.7.13}$$

holds in the space $C([0,1])^3$ endowed with the supremum norm in each coordinate.

Proof. The proof is nearly identical to that of [160, Proposition 13]. Comparing Lemma 15(ii) in [160] with Proposition 3.7.10, all that needs to be shown is the tightness of the processes $((1/\sqrt{p})\mathrm{H}_M^{(p)})_{p\in\mathbb{N}}$. This follows from [160, Lemma 15(i)]: it states that $((1/\sqrt{p})\mathrm{H}_{\mathrm{Loop}}^{(p)})_{p\in\mathbb{N}}$ is tight[8] since we have $d_{M^{(p)}}(\pi(v),\pi(w)) \leq d_{\mathrm{Loop}(\mathcal{T}^{(p)})}(\pi(v),\pi(w))$ for every $v,w\in V_{\mathcal{T}^{(p)}}$. □

We are finally ready to complete the proof of Proposition 3.7.1 and thereby that of Theorem 3.1.4. The proof proceeds in a similar manner as [160, Section 5.4].

Proof of Proposition 3.7.1. We begin by choosing a Skorokhod embedding of $(\mathcal{T}^{(p)})_{p\in\mathbb{N}}$ and $\mathbb{e}$ in the same probability space such that the convergence in (3.7.13) holds almost surely. Again let $\emptyset = v_0^{(p)}, v_1^{(p)}, \ldots, v_{2p}^{(p)}$ be the enumeration of vertices of $\mathcal{T}^{(p)}$ in lexicographic order and let π be the projection from $\mathcal{T}^{(p)}$ onto $\overline{\mathrm{Loop}}(\mathcal{T}^{(p)})$. Let $\mathcal{T}_{\mathbb{e}}$ be the CRT defined from the Brownian excursion $\mathbb{e}$ and let $\mathbf{p}_{\mathbb{e}}$ denote the canonical projection map from $[0,1]$ to $\mathcal{T}_{\mathbb{e}}$. We now define a correspondence $\mathcal{R}^{(p)}$ between $\mathcal{X}^{(p)}$ and $\mathcal{T}_{\mathbb{e}}$ by

$$\mathcal{R}^{(p)} = \left\{ \left(\pi(v_i^{(p)}), \mathbf{p}_{\mathbb{e}}(s)\right) \in \mathcal{X}^{(p)} \times \mathcal{T}_{\mathbb{e}} : i = \lfloor 2ps \rfloor, s\in[0,1], i \in [\![0,2p]\!]. \right\}, \tag{3.7.14}$$

Recall that the distortion of this correspondence is defined as

$$\mathrm{dis}(\mathcal{R}^{(p)}) = \sup\left\{ \left| \frac{\theta_{\mathtt{F}}}{\sqrt{p}} d_{M^{(p)}}(v,w) - d_{\mathbb{e}}(s,t) \right| : (v,\mathbf{p}_{\mathbb{e}}(s)),(w,\mathbf{p}_{\mathbb{e}}(t)) \in \mathcal{R}^{(p)} \right\}. \tag{3.7.15}$$

We now show that $\mathrm{dis}(\mathcal{R}^{(p)}) \to 0$ almost surely as $p\to\infty$, arguing by contradiction. If this convergence were not true, then there would be an $\varepsilon > 0$ such that with positive probability, there exists a sequence of integers p_n with corresponding indices i_{p_n}, j_{p_n} and s_{p_n}, t_{p_n} for

8. To be precise, the statement in [160, Lemma 15] is for the height function associated with the non-contracted looptree. The proof for the contracted looptree also holds as well, as discussed in Section 6.1 of the same paper.

which $(v_{i_{p_n}}^{(p_n)}, \mathbf{p}_{\mathbb{e}}(s_{p_n})), (v_{j_{p_n}}^{(p_n)}, \mathbf{p}_{\mathbb{e}}(t_{p_n})) \in \mathcal{R}^{(p_n)}$ and

$$\left| \frac{\theta_{\mathtt{F}}}{\sqrt{p}} d_{M^{(p_n)}}(v_{i_{p_n}}^{(p_n)}, v_{j_{p_n}}^{(p_n)}) - d_{\mathbb{e}}(s_{p_n}, t_{p_n}) \right| > \varepsilon. \tag{3.7.16}$$

Passing to a subsequence if necessary, we may assume that

$$\lim_{n\to\infty} \frac{i_{p_n}}{2p_n} = \lim_{n\to\infty} s_{p_n} = s \quad \text{and} \quad \lim_{n\to\infty} \frac{j_{p_n}}{2p_n} = \lim_{n\to\infty} t_{p_n} = t \tag{3.7.17}$$

for some constants $0 \le s < t \le 1$.

For $0 \le i < j \le 2p$, let $\mathrm{m}^{(p)}(i,j) \in [\![0, 2p]\!]$ be the smallest index so that $v_{\mathrm{m}^{(p)}(i,j)}^{(p)}$ is the most recent common ancestor of $v_i^{(p)}$ and $v_j^{(p)}$ in $\mathcal{T}^{(p)}$. Let $f_i^{(p)} \in \mathfrak{F}(\partial M^{(p)})$ be the face bounded by π(children of $v_i^{(p)}$). We claim that

$$\begin{aligned} &\left| d_{M^{(p)}}(\pi(v_i^{(p)}), \pi(v_j^{(p)})) - \left(\mathrm{H}_M^{(p)}\left(\frac{i}{2p}\right) + \mathrm{H}_M^{(p)}\left(\frac{j}{2p}\right) - 2\mathrm{H}_M^{(p)}\left(\frac{\mathrm{m}^{(p)}(i,j)}{2p}\right)\right)\right| \\ &\le 2\mathrm{per}(f_{\mathrm{m}^{(p)}(i,j)}^{(p)}). \end{aligned} \tag{3.7.18}$$

This is because if w_i and w_j are the children of $\mathrm{m}^{(p)}(i,j)$ which are the ancestors of $v_i^{(p)}$ and $v_j^{(p)}$, respectively, then a geodesic from $\pi(v_i^{(p)})$ to $\pi(v_j^{(p)})$ in $M^{(p)}$ must pass through $\pi(w_i)$ and $\pi(w_j)$. Hence, the difference in the left-hand side is equal to

$$|d_{M^{(p)}}(\pi(w_i), \pi(w_j)) - d_{M^{(p)}}(\pi(w_i), \pi(v_{\mathrm{m}^{(p)}(i,j)}^{(p)})) - d_{M^{(p)}}(\pi(w_j), \pi(v_{\mathrm{m}^{(p)}(i,j)}^{(p)}))|. \tag{3.7.19}$$

Since geodesics for the all three pairs must stay in $M^{(p)}|_{f_{\mathrm{m}^{(p)}(i,j)}^{(p)}}$, each of them are bounded above by the maximal distance between two boundary points of this face, which is at most $\mathrm{per}(f_{\mathrm{m}^{(p)}(i,j)}^{(p)})$.

Recalling Proposition 3.4.17, since $p^{-1/2}\partial M^{(p)}$ converges in distribution to a multiple of the continuum random tree with respect to the Gromov–Hausdorff distance, we have

$p^{-1/2}\max_{i\in[\![0,2p]\!]}\mathrm{per}(f_i^{(p)})\to 0$ almost surely as $p\to\infty$ (see [160, Equation (15)]). Moreover, if we let $b^{(p)}(i) := 2i - \mathrm{H}_{\mathcal{T}}^{(p)}(i/(2p)) \in [\![0,4p]\!]$ denote the first time that we encounter $v_i^{(p)}$ in the enumeration of vertices of $\mathcal{T}^{(p)}$ in contour order, then

$$\mathrm{H}_{\mathcal{T}}^{(p)}\left(\frac{\mathrm{m}^{(p)}(i,j)}{2p}\right) = \mathrm{C}_{\mathcal{T}}^{(p)}\left(\frac{b^{(p)}(\mathrm{m}^{(p)}(i,j))}{4p}\right) = \inf_{\frac{b^{(p)}(i)}{4p}\le u\le\frac{b^{(p)}(j)}{4p}} \mathrm{C}_{\mathcal{T}}^{(p)}(u). \tag{3.7.20}$$

Lemma 3.7.11 implies $(\frac{b^{(p)}(2pt)}{4p})_{0\le t\le 1}\to(t)_{0\le t\le 1}$ almost surely in $C([0,1])$ and

$$\sup_{0\le t\le 1}\left|\frac{b^{(p)}(2pt)}{4p}-t\right| + \sup_{0\le t\le 1}\left|\frac{\theta_{\mathtt{F}}}{\sqrt{p}}\mathrm{H}_M^{(p)}(t) - \frac{\sigma}{2\sqrt{2p}}\mathrm{H}_{\mathcal{T}}^{(p)}(t)\right| \to 0 \tag{3.7.21}$$

almost surely as $p\to\infty$. Hence,

$$\lim_{n\to\infty}\frac{\theta_{\mathtt{F}}}{\sqrt{p_n}}d_{M^{(p_n)}}\big(\pi(v_{i_{p_n}}^{(p_n)}),\pi(v_{j_{p_n}}^{(p_n)})\big) = \mathbb{e}_s+\mathbb{e}_t-2\inf_{s\le u\le t}\mathbb{e}_u = d_{\mathbb{e}}(s,t) = \lim_{n\to\infty} d_{\mathbb{e}}(s_{p_n},t_{p_n}) \tag{3.7.22}$$

almost surely. This is a contradiction, thus proving $\mathrm{dis}(\mathcal{R}^{(p)})\to 0$ as $p\to\infty$. $\square$

Part II

Loewner energy and Schramm–Loewner evolution

CHAPTER 4

A DETERMINISTIC PROOF OF LOEWNER ENERGY REVERSIBILITY VIA LOCAL REVERSALS

This chapter presents material first announced in [237].

4.1 Introduction

The Loewner energy of a chord is defined as the Dirichlet energy of the driving function of the corresponding Loewner chain. It was introduced in independent works by Friz and Shekhar [104] and Yilin Wang [244], the latter of whom considered it in the context of large deviations for a family of random curves called the chordal Schramm–Loewner evolution[1] (SLE_κ) as $\kappa \to 0$. Loewner energy has since been identified in terms of various geometric and probabilistic quantities, hinting at a deeper connection between these fields. We refer the reader to the survey articles [247, 248] for various perspectives on Loewner energy.

Let γ be a simple curve from 0 to ∞ in the upper half-plane $\mathbb{H} = \{z \in \mathbb{C} : \operatorname{Im} z > 0\}$. We parameterize γ by $\mathbb{R}_+ = (0, \infty)$ so that, if $g_t : \mathbb{H} \setminus \gamma((0,t]) \to \mathbb{H}$ is the unique conformal map with the normalization $g_t(z) - z \to 0$ as $z \to \infty$, then the expansion $g_t(z) = z + 2a_t z^{-1} + O(z^{-2})$ as $z \to \infty$ satisfies $a_t = t$ for every $t > 0$. This is called the (half-plane) capacity parameterization of γ. By Carathéodory's theorem, we can extend g_t to the prime ends of $\mathbb{R} \cup \gamma((0,t])$. Let $\lambda_t := g_t(\gamma(t))$, which we call the driving function of γ. The Loewner energy of γ is defined as

$$I_{\mathbb{H};0,\infty}(\gamma) := \frac{1}{2} \int_0^\infty \left| \frac{d\lambda_t}{dt} \right|^2 dt \tag{4.1.1}$$

if $t \mapsto \lambda_t$ is absolutely continuous and $\dot{\lambda} \in L^2([0,\infty))$; the energy is set to be infinite otherwise.

1. The $\kappa \to 0$ limit of chordal SLE_κ was also considered in [92, Section 9.3].

Loewner energy is a conformally invariant quantity, which is natural given the conformal invariance of chordal SLE_κ. A priori, the energy $I_{\mathbb{H};0,\infty}$ is invariant under scaling the curve by a map $z \mapsto cz$ where $c > 0$. This is since the corresponding driving function under capacity parameterization enjoys the Brownian scaling property: if the driving function of γ is $t \mapsto \lambda_t$, then that of $c\gamma$ is $t \mapsto c\lambda_{c^{-2}t}$. Note that the right-hand side of (4.1.1) is invariant under this scaling. We extend the definition of Loewner energy to a general simply connected domain $D \subsetneq \mathbb{C}$ and a crosscut (henceforth called a chord) γ from a prime end a to another prime end b so that

$$I_{D,a,b}(\gamma) := I_{\mathbb{H},0,\infty}(\varphi(\gamma)) \tag{4.1.2}$$

where $I_{\mathbb{H};0,\infty}$ is defined as in (4.1.1) and $\varphi : D \to \mathbb{H}$ is a conformal transformation with $\varphi(a) = 0$ and $\varphi(b) = \infty$. This definition does not depend on the choice of such map φ due to scale invariance of $I_{\mathbb{H};0,\infty}$.

An fundamental property of Loewner energy, called reversibility, is that the Loewner energy of a chord does not depend on the choice of its orientation. It is not obvious a priori that the two quantities are related due to the directionality in the definition of Loewner energy. If γ is a chord in $\mathbb{H}$ going from 0 to ∞, let γ^{R} denote the same chord in the reverse orientation, going from ∞ to 0 in $\mathbb{H}$. If we denote $\iota : z \mapsto -1/z$, we see from (4.1.2) that $I_{\mathbb{H},\infty,0}(\gamma^{\mathrm{R}}) = I_{\mathbb{H},0,\infty}(\iota(\gamma^{\mathrm{R}})) = \frac{1}{2}\int_0^\infty |\dot{\lambda}_t^{\mathrm{R}}|^2\,dt$ where λ_t^{R} is the driving function corresponding to the curve $\iota(\gamma^{\mathrm{R}})$. The map $(\lambda_t)_{t\in[0,\infty)} \mapsto (\lambda_t^{\mathrm{R}})_{t\in[0,\infty)}$ induced by reversing of the chord is rather intricate and there does not exist an explicit formula except for extremely simple examples. Nevertheless, Wang showed in [244] that the Loewner energy of γ^{R} is equal to that of γ. The purpose of this article is to give another proof of this fact.

Theorem 4.1.1 (Reversibility)**.** *Let D be a simply connected domain and let a and b be any two distinct prime ends of D. For a chord γ in D from a to b, let γ^{R} be the same chord with its orientation reversed to trace from b to a. Then, $I_{D;a,b}(\gamma) = I_{D;b,a}(\gamma^{\mathrm{R}})$.*

Let us briefly describe previous proofs of Loewner energy reversibility and explain how ours differs from them. The original proof in [244] relied upon the reversibility of chordal SLE_κ for $\kappa \in (0,4]$, which was first established in [254] and proved subsequently with different methods in [195, 173]. By interpreting Loewner energy $I_{\mathbb{H};0,\infty}$ as the large deviation rate function as $\kappa \to 0$ for chordal SLE_κ in $\mathbb{H}$ from 0 to ∞, Wang showed that a finite-energy chord must have the same Loewner energy in the reverse orientation.

Wang gave a deterministic proof of Theorem 4.1.1 in [245] through an identification between $I_{\mathbb{H},0,\infty}(\gamma)$ and the Dirichlet energy of $\log|f'|$, where f is the uniformizing map on each component of $\mathbb{H} \setminus \gamma$. The reversibility of Loewner energy follows immediately since f does not depend on the orientation of the chord γ. Based on this result, she revealed in the same work an unexpected link between Loewner chains, conformal welding, and the Kähler geometry of the Weil–Petersson Teichmüller space. We direct the reader to [52] for various other geometric characterizations of curves with finite Loewner energy (Weil–Petersson quasicircles).

The key difference in our proof of Loewner energy reversibility from the previous deterministic one is that, rather than "lifting" γ to the space of chords without orientation, we consider a "continuous" reversal of its orientation. In particular, our proof is based on the commutation relation for Loewner energy (Theorem A), which says that the order in which we calculate the total energy of two non-intersecting Jordan arcs does not change this quantity. Despite being closely related with the commutation relation for chordal SLE [92], this identity was obtained deterministically in [244] from the definition (4.1.1) of Loewner energy through a careful calculation identifying how the driving function of one curve changes when the other curve is grown first.

4.1.1 Proof overview

Here is an overview of our proof of Theorem 4.1.1. Given a chord γ in a simply connected domain D from a to b, we approximate it with chords which pass through finite subsets of its trace and satisfy energy reversibility. In particular, given a finite subset $\underline{z} \subset \gamma$, we choose a chord $\gamma_{\underline{z}}$ with minimal energy among those which pass through $\underline{z}$ between a and b. Using an inequality version of the commutation relation for simple curves terminating at the same point (Lemma 4.3.1), we identify that $\gamma_{\underline{z}}$ is piecewise geodesic relative to $\underline{z}$. This means that $\underline{z}$ partitions γ into simple curves $\gamma_1, \ldots, \gamma_n$ such that each γ_k is a hyperbolic geodesic in the domain $D \setminus (\bigcup_{j \neq k} \gamma_j)$. The commutation relation further implies that the Loewner energy of any piecewise geodesic chord is invariant under reversing its orientation (Proposition 4.3.4). As $\underline{z}$ becomes dense in γ, the Loewner energy of $\gamma_{\underline{z}}$ increases to that of γ; this holds in both directions, thus proving that the Loewner energies of γ and its orientation reversal are the same.

Our proof technique can be applied to show that when $\underline{z}$ contains $n \geq 2$ points, there exists a Loewner energy minimizer within each isotopy class of chords relative to $\underline{z}$ and is a piecewise geodesic. Let us say that two chords γ_0, γ_1 in D from a to b passing through $\underline{z}$ are isotopic relative to $\underline{z}$ if there exists an isotopy between them that fixes $\underline{z}$ pointwise: i.e., a continuous map $F : [0,1] \times \gamma_0 \to D \cup \{a, b\}$ with $F(0, \gamma_0) = \gamma_0$ and $F(1, \gamma_0) = \gamma_1$ such that, for every $s \in [0,1]$, the set $F(s, \gamma_0)$ is a chord in D from a to b and $F(s, z) = z$ for each $z \in \underline{z}$. Note that this is a equivalence relation that is preserved naturally under conformal transformations.

Theorem 4.1.2. *Given a chord γ in D between prime ends a, b passing through a finite set of points $\underline{z} \subset D$, let $\mathcal{X}(D; a, b, \underline{z}, \gamma)$ be the set of chords in D between a and b that can be obtained through an isotopy of γ leaving $\underline{z} \cup \{a, b\}$ pointwise fixed. Then, given any chord γ and a finite subset $\underline{z} \subset \gamma$, there exists a chord $\gamma^{\min} \in \mathcal{X}(D; a, b; \underline{z}, \gamma)$ such that*

$$I_{D;a,b}(\gamma^{\min}) = \inf_{\eta \in \mathcal{X}(D;a,b;\underline{z},\gamma)} I_{D;a,b}(\eta) < \infty. \tag{4.1.3}$$

Moreover, if $\gamma^{\min}$ is any chord in $\mathcal{X}(D; a, b; \underline{z}, \gamma)$ satisfying (4.1.3)*, then it is piecewise* *geodesic relative to $\underline{z}$.*

In the case that $\underline{z}$ contains a single point z_1, the piecewise geodesic property as well as the uniqueness of the energy minimizing chord passing thorough this point was verified by an explicit identification of this chord in [244, Proposition 3.1] (see also [190, Theorem 3.3(i)]). In fact, it was shown in [185, Theorem 3.9] that this is the unique C^1 chord which is piecewise geodesic relative to z_1. By adopting the the proof of [55, Theorem 3.1] to the chordal case, it can also be shown when $\underline{z}$ contains more than one point that a C^1 piecewise geodesic chord relative to $\underline{z}$ is unique in each isotopy class $\mathcal{X}(D; a, b; \underline{z}, \gamma)$. This implies that there exists a unique Loewner energy minimizer in this isotopy class.

Remark 4.1.3. Our proof of Loewner energy reversibility is closely related to the proof of chordal SLE_κ reversibility for $\kappa \in (0, 4]$ in [254]. Zhan establishes this fact by constructing a coupling of an SLE_κ chord γ_1 going from x_1 to x_2 in the domain $\mathbb{H}$ for fixed $x_1, x_2 \in \mathbb{R}$ with another SLE_κ chord γ_2 in $\mathbb{H}$ between the same endpoints in reverse direction so that both traverse the same random set of points $\underline{z} \subset \mathbb{H}$ visited by γ_1 at stopping times $t_1 < \cdots < t_n$. In fact, the reverse chord γ_2 is constructed by replacing each curve $\gamma_1([t_{k-1}, t_k])$, as k decreases from $n+1$ to 1, with a reweighted version of chordal SLE_κ going from $\gamma_1(t_k)$ to $\gamma_1(t_{k-1})$ in the complement of $\gamma_1([0, t_{k-1}])$ and the already constructed part of γ_2 (cf. proof of Proposition 4.3.4). The random set $\underline{z}$ is taken to become denser and denser on γ_1, ultimately converging to a coupling where the two SLE_κ chords are almost surely orientation reversals of each other.

The relation to our proof of Loewner energy reversibility is that, conditioned on the points $\underline{z}$, the law of the chord γ_1 (resp. γ_2) in the above coupling is that of a chordal SLE_κ curve in $\mathbb{H}$ from x_1 to x_2 (resp. x_2 to x_1) conditioned to pass through $\underline{z}$. Since $I_{\mathbb{H};x_1,x_2}$ (resp. $I_{\mathbb{H};x_2,x_1}$) is the large deviation rate function for the chordal SLE_κ curve in $\mathbb{H}$ from x_1 to x_2 (resp. x_2 to x_1), when we take the $\kappa \to 0$ limit for this coupling conditioned on the points $\underline{z}$, the SLE_κ chord γ_1 (resp. γ_2) should converge to the energy minimizing chord $\gamma_1^{\min}$ among those going from x_1 to x_2 (resp. $\gamma_2^{\min}$ from x_2 to x_1) in $\mathbb{H}$ while passing through $\underline{z}$. Given the uniqueness of $\gamma_1^{\min}$ and $\gamma_2^{\min}$ (which we do not claim but expect to be true for a general sequence $\underline{z}$), we know from their piecewise geodesic property (Proposition 4.3.4) that these are orientation reversals of each other. That is, γ_1 and γ_2 are random perturbations of the same piecewise geodesic chord passing through $\underline{z}$. The proof of Loewner energy reversibility in [244] essentially proceeds by taking $\underline{z}$ to become dense on γ_1 and γ_2, in which case the limits of $\gamma_1^{\min}$ and $\gamma_2^{\min}$ have the same trace due to the Jordan curve theorem.

Here is an overview of the rest of this article. In Section 4.2, we define and discuss the properties of the partial Loewner energy of a simple curve, which can be considered as that of the chord obtained by completing it with a hyperbolic geodesic. We also collect some results about topologies on the space of chords. In Section 4.3, we first establish an inequality version of the commutation relation for Loewner energy and then use it to prove Theorem 4.1.1. We prove Theorem 4.1.2 in Section 4.4.

4.2 Preliminaries

Given a simply connected domain $D \subsetneq \hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ with distinct prime ends a and b, we denote by $\mathcal{X}(D; a, b)$ the space of chords in D (i.e., Jordan arcs γ in D with $\partial\gamma \subset \partial D$) from a to b. We consider these chords modulo orientation-preserving reparameterizations. In particular, $\mathcal{X}(D; a, b)$ and $\mathcal{X}(D; b, a)$ differ in the orientations of chords. Since our notation for Loewner energy already contains information about the orientation of the chord, we denote orientation reversals implicitly in the rest of this paper. That is, if $\gamma \in \mathcal{X}(D; a, b)$, then $I_{D;b,a}(\gamma)$ should be understood as the energy of the reversed chord $\gamma^{\mathrm{R}} \in \mathcal{X}(D; b, a)$.

If D is a bounded Jordan domain, we endow $\mathcal{X}(D; a, b)$ with the topology induced by the Hausdorff distance on the (closure of) chords. For a general simply connected domain D conformally equivalent to the unit disk $\mathbb{D} := \{|z| \leq 1\}$, we choose any conformal map φ taking D onto $\mathbb{D}$ and endow $\mathcal{X}(D; a, b)$ with the pullback of the Hausdorff topology on

$\mathcal{X}(\mathbb{D}; \varphi(a), \varphi(b))$. We will make heavy use of the following fact that Loewner energy is lower semicontinuous with respect to the Hausdorff topology on chords.

Lemma 4.2.1 ([204, Lemma 2.7])**.** *The map $I_{D;a,b} : \mathcal{X}(D; a, b) \to [0, \infty]$ is lower semicontinuous, and its sublevel set $\{\gamma \in \mathcal{X}(D; a, b) : I_{D;a,b}(\gamma) \le c\}$ is compact for any $c \in [0, \infty)$.*

In fact, if $(\gamma^n)_{n\in\mathbb{N}}$ is a sequence of chords in $\mathcal{X}(\mathbb{D}; -1, 1)$ with $\sup_{n\in\mathbb{N}} I_{\mathbb{D};-1,1}(\gamma^n) \le c < \infty$, then there exists a corresponding sequence of homeomorphisms $\varphi_n : \overline{\mathbb{D}} \to \overline{\mathbb{D}}$, with $\varphi_n|_{\partial\mathbb{D}}$ equal to the identity and $\gamma^n = \varphi_n([-1, 1])$ for each $n \in \mathbb{N}$, such that φ_n converges uniformly along a subsequence to some self-homeomorphism φ of $\overline{\mathbb{D}}$. Moreover, $\gamma := \varphi([-1, 1]) \in \mathcal{X}(\mathbb{D}; -1, 1)$ satisfies $I_{\mathbb{D};-1,1}(\gamma) \le c$.

The convergence in the latter half of the above lemma was established within its proof in [204].

4.2.1 Energy of a simple curve and hyperbolic geodesics

In this subsection, we expand the definition of Loewner energy to curves that end in the interior of the domain and obtain its additivity property (4.2.6). We then recall basic properties of hyperbolic geodesics that we use in our proof. The facts surveyed here can be found in [244].

Let $\gamma : [0, T] \to \mathbb{H} \cup \{0\}$ be a Jordan arc (henceforth called a simple curve) with $\gamma(0) = 0$ and $T \in (0, \infty)$. For each $t \in [0, T]$, we consider the unique conformal map $g_t : \mathbb{H}\setminus\gamma([0, t]) \to \mathbb{H}$ with hydrodynamic normalization: i.e., satisfying

$$g_t(z) - z \to 0 \quad \text{as} \quad z \to \infty. \tag{4.2.1}$$

The half-plane capacity of the curve $\gamma([0, t])$, which we denote $\operatorname{hcap}(\gamma([0, t]))$, is defined via the expansion

$$g_t(z) = z + \frac{\operatorname{hcap}(\gamma([0, t]))}{z} + O\left(\frac{1}{z^2}\right) \quad \text{as} \quad z \to \infty. \tag{4.2.2}$$

We can reparametrize γ on the interval $[0, \frac{1}{2}\mathrm{hcap}(\gamma)]$ so that $\mathrm{hcap}(\gamma([0,t])) = 2t$ for every t, which we call the (half-plane) capacity parameterization of γ. For the rest of this subsection, we assume that γ is in capacity parameterization with $T = \frac{1}{2}\mathrm{hcap}(\gamma)$.

We call the family of maps $(g_t)_{t\in[0,T]}$ the Loewner chain corresponding to the curve γ, and define the driving function of the curve γ as the continuous function $t \mapsto \lambda_t := g_t(\gamma(t))$. For every point $z \in \overline{\mathbb{H}} \setminus \{0\}$, the map $t \mapsto g_t(z)$ is continuously differentiable and satisfies the Loewner equation

$$\partial_t g_t(z) = \frac{2}{g_t(z) - \lambda_t} \tag{4.2.3}$$

on the entire interval $[0,T]$ if $z \notin \gamma$, or on the interval $[0, T_z)$ if $z = \gamma(T_z)$ for some $T_z \in (0,T]$.

Definition 4.2.2. Given a simple curve γ in $\mathbb{H}$ starting at 0, we define its Loewner energy targeted at ∞ as

$$I^{\circ}_{\mathbb{H};0,\infty}(\gamma) := \frac{1}{2}\int_0^T \left|\frac{d\lambda_t}{dt}\right|^2 dt \tag{4.2.4}$$

if the driving function $(\lambda_t)_{t\in[0,T]}$ of the curve γ (in capacity parametrization) is absolutely continuous and $\dot{\lambda} \in L^2([0,T])$. Otherwise, we define $I^{\circ}_{\mathbb{H};0,\infty}(\gamma) = \infty$.

We use the notation $I^{\circ}_{\mathbb{H};0,\infty}(\gamma)$ to clarify that γ does not reach ∞. This extends the definition (4.1.1) for the Loewner energy of a chord in $\mathcal{X}(\mathbb{H}; 0, \infty)$, which amounts to the case $T = \infty$. For a simple curve γ in an arbitrary simply connected domain D starting from a prime end a, we define its Loewner energy targeted at a prime end b so that it is conformally invariant:

$$I^{\circ}_{D;a,b}(\gamma) = I^{\circ}_{\mathbb{H};0,\infty}(\varphi(\gamma)) \tag{4.2.5}$$

for any conformal map $\varphi : D \to \mathbb{H}$ with $\varphi(a) = 0$ and $\varphi(b) = \infty$, analogously to (4.1.2).

If $\gamma : [0,T] \to \mathbb{H} \cup \{0\}$ is a simple curve with $\gamma(0) = 0$ in half-plane capacity parameterization, then, for each $s \in (0,T)$, the image $\gamma^s := g_s(\gamma([s,T])) - \lambda_s$ is also a simple curve in $\mathbb{H}$ starting at 0. Its half-plane capacity parameterization is given by $\gamma^s(t) = g_s(\gamma(s+t)) - \lambda_s$ on the interval $[0, T-s]$, and the corresponding driving function is $t \mapsto \lambda_{t+s} - \lambda_s$. We thus have

$$I^{\circ}_{\mathbb{H};0,\infty}(\gamma) = \int_0^s \big|\dot{\lambda}_t\big|^2 \, dt + \int_s^T \big|\dot{\lambda}_t\big|^2 \, dt = I^{\circ}_{\mathbb{H};0,\infty}(\gamma([0,s])) + I^{\circ}_{\mathbb{H};0,\infty}(\gamma^s)$$
$$= I^{\circ}_{\mathbb{H};0,\infty}(\gamma([0,s])) + I^{\circ}_{\mathbb{H}\setminus\gamma([0,s]);\gamma(s),\infty}(\gamma([s,T])) \tag{4.2.6}$$

for each $s \in (0,T)$, as was observed in [244]. This additivity property (4.2.6) can be stated for more general domains via conformal invariance (4.2.5), which naturally leads to the idea that the partial Loewner energy $I^{\circ}_{D;a,b}(\gamma_1)$ of a simple curve γ_1 in a simply connected domain D starting at a prime end a and ending at $\zeta \in D$ represents the Loewner energy of the chord $\gamma_1 \cup \gamma_2 \in \mathcal{X}(D;a,b)$ obtained by completing γ_1 all the way to the prime end b by a chord $\gamma_2 \in \mathcal{X}(D \setminus \gamma_1; \zeta, b)$ with zero energy. Such zero energy chords are hyperbolic geodesics and they are central to the theory of Loewner energy. Indeed, the imaginary axis $i\mathbb{R}_+$ is the hyperbolic geodesic in the upper half-plane from 0 to ∞, and the corresponding driving function is the constant zero function.

Definition 4.2.3. The hyperbolic geodesic in a simply connected domain $D \subsetneq \hat{\mathbb{C}}$ from a prime end a to another prime end b is the image $\varphi(i\mathbb{R}_+)$ of the imaginary axis under a conformal map $\varphi : \mathbb{H} \to D$ with $\varphi(0) = a$ and $\varphi(\infty) = b$. Note that this specifies a unique chord in $\mathcal{X}(D;a,b)$.

For instance, the hyperbolic geodesic in $\mathbb{H}$ between $x, y \in \mathbb{R}$ is the semicircle with the interval between x and y as its diameter. The hyperbolic geodesic γ in D from a to b has the following useful properties that follow immediately from its definition.

- The hyperbolic geodesic γ is the unique chord in $\mathcal{X}(D;a,b)$ with $I_{D;a,b}(\gamma) = 0$. Any other chord in $\mathcal{X}(D;a,b)$ has positive (or infinite) Loewner energy $I_{D;a,b}$.
- The reversed chord γ^{R} is the hyperbolic geodesic in D from b to a. Hence, it makes sense to omit the orientation and call γ the hyperbolic geodesic in D between a and b.

- Given any parameterization $\gamma : (0, T) \to D$ with $\gamma(0) = a$ and $\gamma(T) = b$, for each $t \in (0, T)$, the curve $\gamma(t, T)$ is the hyperbolic geodesic in $D \setminus \gamma([0, t])$ between $\gamma(t)$ and b.

Let us say that a chord $\gamma \in \mathcal{X}(D; a, b)$ has a geodesic tip if we can decompose it into simple curves γ_1 and γ_2 joined at $\zeta \in D$ such that γ_2 is the hyperbolic geodesic in $D \setminus \gamma_1$ between ζ and b. (Note that by the third bulleted observation above, a chord $\gamma \in \mathcal{X}(D; a, b)$ has a geodesic tip if and only if, given any parameterization $\gamma : (0, T) \to D$ with $\gamma(0) = a$ and $\gamma(T) = b$, there exists a time $s \in (0, T)$ such that the following is true: for every $t \in [s, T)$, the curve $\gamma([t, T])$ is the hyperbolic geodesic in $D \setminus \gamma([0, t])$ between $\gamma(t)$ and b.) Then,

$$I_{D;a,b}(\gamma) = I^{\circ}_{D;a,b}(\gamma_1) + I_{D \setminus \gamma_1;\zeta,b}(\gamma_2) = I^{\circ}_{D;a,b}(\gamma_1) = I^{\circ}_{D;a,b}(\gamma_1) + I_{D \setminus \gamma_1;b,\zeta}(\gamma_2) \tag{4.2.7}$$

by (4.2.6) and the first two bulleted observations above. Here, we already see that when computing the Loewner energy of a chord with a geodesic tip, we can reverse the orientation of this tip. The the key idea in our proof is to generalize this to the case that the hyperbolic geodesic is in a middle part of the chord (see Lemma 4.3.1).

4.2.2 Identifying chords via hulls

In our proof of the “commutation inequality” (Lemma 4.3.1), we will need to identify the Hausdorff limit of chords using the convergence of the hulls that they generate. For concreteness, let us consider chords in the upper half-plane $\mathbb{H}$ between two points on $\mathbb{R}$. Given $\gamma \in \mathcal{X}(\mathbb{H}; x, y)$, by the Jordan curve theorem, $\mathbb{H} \setminus \gamma$ consists of exactly two connected components: a bounded component D_γ and an unbounded component U_γ, both of which are simply connected domains. Moreover, $\partial D_\gamma = \gamma \cup [x, y]$ where $[x, y] \subset \mathbb{R}$ denotes the closed interval between x and y even when $x > y$. We define the hull generated by the chord γ as the closure $K_\gamma := \overline{D_\gamma}$ in $\overline{\mathbb{H}} = \mathbb{H} \cup \mathbb{R}$. Since $\partial K_\gamma \cap \mathbb{H} = \partial D_\gamma \cap \mathbb{H} = \gamma$, a bounded chord in $\mathbb{H}$

can be uniquely identified by the hull it generates.

Lemma 4.2.4. *Suppose $(\gamma^n)_{n\in\mathbb{N}}$ is a sequence of chords in $\mathbb{H}$ between two fixed points $x, y \in \mathbb{R}$ such that $\sup_{n\in\mathbb{N}} I_{\mathbb{H};x,y}(\gamma^n) < \infty$. Then, there exists a subsequence γ^{n_k} which converges to some chord $\gamma \in \mathcal{X}(\mathbb{H}; x, y)$ in the Hausdorff distance and, furthermore, the corresponding hulls $K_{\gamma^{n_k}}$ converge to K_γ in the Hausdorff distance.*

Proof. Without loss of generality, assume $x < y$. Fix a Möbius map $f : \mathbb{D} \to \mathbb{H}$ with $f(-1) = x$ and $f(1) = y$. By Lemma 4.2.1, for each n, there exists a homeomorphism $\varphi_n : \overline{\mathbb{D}} \to \overline{\mathbb{D}}$ such that $\varphi_n|_{\partial\mathbb{D}}$ is the identity map and $\varphi_n([-1, 1]) = f^{-1}(\gamma_n)$. Furthermore, we can find a subsequence φ_{n_k} which converges uniformly to a homeomorphism $\varphi : \overline{\mathbb{D}} \to \overline{\mathbb{D}}$. Then, γ_{n_k} converges to $\gamma := (f \circ \varphi)([-1, 1]) \in \mathcal{X}(\mathbb{H}; x, y)$ in the Hausdorff distance.

Denote by D the lower half-disk $\{z \in \mathbb{D} : \mathrm{Im}(z) < 0\}$ and note that the bounded connected component of $\mathbb{H} \setminus \gamma_n$ is $(f \circ \varphi_n)(D)$. Since the homeomorphisms $f \circ \varphi_{n_k} : \overline{\mathbb{D}} \to \overline{\mathbb{H}} \cup \{\infty\}$ converge uniformly, the hulls $K_{\gamma_{n_k}} = \overline{(f \circ \varphi_{n_k})(D_\eta)} = (f \circ \varphi_{n_k})(\overline{D})$ also converge in the Hausdorff distance to $K_\gamma = \varphi(\overline{D})$. $\square$

Our definition of the hull K_γ generated by a bounded chord γ in $\mathbb{H}$ is in agreement with the following more general definition of a hull in $\mathbb{H}$. Let $\mathcal{C}$ be the space of non-empty closed subsets of $\overline{\mathbb{H}}$. The set of compact $\mathbb{H}$-hulls is defined as

$$\mathcal{K} := \{K \in \mathcal{C} : K \text{ is bounded, } \mathbb{H} \setminus K \text{ is simply connected, and } \overline{K \cap \mathbb{H}} = K\}.$$

Indeed, since D_γ is bounded, K_γ is bounded. Moreover, $K_\gamma \cap \mathbb{H} = D_\gamma \cup (\partial D_\gamma \cap \mathbb{H}) = D_\gamma \cup \gamma$, so $\mathbb{H} \setminus K_\gamma = (\mathbb{H} \setminus \gamma) \setminus D_\gamma = U_\gamma$, which is simply connected. Also, $\overline{K_\gamma \cap \mathbb{H}} = \overline{D_\gamma} \cup \overline{\gamma} = K_\gamma \cup (\gamma \cup \{x, y\}) = K_\gamma$. Thus, the hull K_γ defined as the closure of the bounded component of $\mathbb{H} \setminus \gamma$ in $\overline{\mathbb{H}}$ is an element of $\mathcal{K}$.

For each hull $K \in \mathcal{K}$, there exists a unique conformal transformation $g_K : \mathbb{H} \setminus K \to \mathbb{H}$ such that $g_K(z) - z \to 0$ as $z \to \infty$, which we refer to as the mapping-out function of K.

This induces the following alternative topology on $\mathcal{K}$ which is natural from the perspective of Loewner chains.

Definition 4.2.5. Let $(K^n)_{n\in\mathbb{N}}$ be a sequence compact $\mathbb{H}$-hulls. We say that K^n converges to $K \in \mathcal{K}$ in the Carathéodory topology if $g_{K^n}^{-1}$ converges to g_K^{-1} uniformly away from $\mathbb{R}$. This means that for every $\varepsilon > 0$, $g_{K^n}^{-1}$ converges uniformly to g_K^{-1} on $\{z \in \mathbb{C} : \operatorname{Im} z > \varepsilon\}$.

An equivalent geometric definition is available via Carathéodory kernel theorem (see, e,g., [34, Section 3.3]), from which the following topological lemma follows. This allows us to identify the limiting chord in Lemma 4.2.1 through the Carathéodory limit of the hulls.

Lemma 4.2.6 ([204, Lemma 2.3])**.** *Let $(K^n)_{n\in\mathbb{N}}$ be a sequence in $\mathcal{K}$. If K^n converges to $K \in \mathcal{K}$ with respect to the Carathéodory topology and to a compact set $K^* \subset \overline{\mathbb{H}}$ in the Hausdorff distance, then $\mathbb{H}\setminus K$ coincides with the unbounded connected component of $\mathbb{H}\setminus K^*$. In particular, if $K^* \in \mathcal{K}$, then $K = K^*$.*

4.3 Proof of reversibility of chordal Loewner energy

In this section, we first establish the inequality version of the commutation relation for Loewner energy when two curves are joined at the same endpoint (Lemma 4.3.1). Based on this result, we show that a minimal energy chord passing through a fixed finite set of points is piecewise geodesic, and the Loewner energy of any piecewise geodesic chord is invariant under reversing its orientation (Proposition 4.3.4). We use this to give a proof of Theorem 4.1.1 at the end of this section.

4.3.1 A commutation inequality

The starting point of our proof of Theorem 4.1.1 is the following commutation relation for Loewner energy given in [244, Corollary 4.4].

Theorem A. *Let D be a simply connected domain and a, b be distinct prime ends. Suppose γ is a simple curve D from a to $\gamma(S) \in D$, and η is a simple curve in D from b to $\eta(T) \in D$ with $\gamma \cap \eta = \varnothing$. Then,*

$$I^{\circ}_{D;a,b}(\gamma) + I^{\circ}_{D\setminus\gamma;b,\gamma(S)}(\eta) = I^{\circ}_{D;b,a}(\eta) + I^{\circ}_{D\setminus\eta;a,\eta(T)}(\gamma). \tag{4.3.1}$$

In other words, we can compute the joint Loewner energy of γ and η by computing the energy of either γ or η first, then adding the energy of the other curve in the complement of the first. The main restriction in Theorem A is that the curves γ and η cannot intersect, even at their endpoints. Nevertheless, we can deduce from it a "commutation inequality" satisfied by two curves terminating at the same interior point.

Lemma 4.3.1. *Let D be a simply connected domain with distinct prime ends a and b. Suppose γ and η are disjoint simple curves in D starting from a and b, respectively, except that they both terminate at $\zeta \in D$. Then,*

$$I^{\circ}_{D;b,a}(\eta) \leq I^{\circ}_{D;a,b}(\gamma) + I_{D\setminus\gamma;b,\zeta}(\eta). \tag{4.3.2}$$

Furthermore, if η has a geodesic tip when considered as a chord in $D \setminus \gamma$ from b to ζ,[2] *then*

$$I^{\circ}_{D;b,a}(\eta) + I_{D\setminus\eta;a,\zeta}(\gamma) \leq I^{\circ}_{D;a,b}(\gamma) + I_{D\setminus\gamma;b,\zeta}(\eta). \tag{4.3.3}$$

Proof. Take any continuous parameterization $\eta : (0, T] \to D$ with $\eta(0) = b$ and $\eta(T) = \zeta$. For $t \in (0, T)$, since γ and $\eta((0, t])$ do not intersect, the commutation relation (Theorem A) states

$$I^{\circ}_{D;b,a}(\eta([0, t])) + I^{\circ}_{D\setminus\eta([0,t]);a,\eta(t)}(\gamma) = I^{\circ}_{D;a,b}(\gamma) + I^{\circ}_{D\setminus\gamma;b,\zeta}(\eta([0, t])). \tag{4.3.4}$$

2. That is, η can be partitioned into simple curves η_1, from b to an intermediate point $\xi \in \eta$, and η_2, from ξ to ζ, such that η_2 is the hyperbolic geodesic in $D \setminus (\gamma \cup \eta_1)$ between ξ and ζ.

Note from the definition of the Loewner energy of a curve (4.2.4)–(4.2.5) that

$$\lim_{t\to T^-} I^{\circ}_{D;b,a}(\eta([0,t])) = I^{\circ}_{D;b,a}(\eta) \quad \text{and} \quad \lim_{t\to T^-} I^{\circ}_{D\setminus\gamma;b,\zeta}(\eta([0,t])) = I_{D\setminus\gamma;b,\zeta}(\eta). \tag{4.3.5}$$

We thus obtain

$$\lim_{t\to T^-} I^{\circ}_{D\setminus\eta([0,t]);a,\eta(t)}(\gamma) = I^{\circ}_{D;a,b}(\gamma) + I_{D\setminus\gamma;b,\zeta}(\eta) - I^{\circ}_{D;b,a}(\eta). \tag{4.3.6}$$

The first inequality (4.3.2) follows from the simple observation that the left-hand side of (4.3.6) must be nonnegative by the definition of Loewner energy.

If we furthermore had

$$I_{D\setminus\eta;a,\zeta}(\gamma) \le \lim_{t\to T^-} I^{\circ}_{D\setminus\eta([0,t]);a,\eta(t)}(\gamma), \tag{4.3.7}$$

then we would obtain the second inequality (4.3.3) from (4.3.6). For the rest of this proof, we prove the inequality (4.3.7) in the case that $\eta \in \mathcal{X}(D\setminus\gamma; b, \zeta)$ has a geodesic tip. That is, suppose there exists $T_0 \in [0,T)$ such that $\eta([T_0,T])$ is the hyperbolic geodesic in $D \setminus (\gamma \cup \eta([0,T_0]))$ between $\eta(T_0)$ and $\zeta = \eta(T)$. By the conformal invariance (4.2.5) of Loewner energy, we may assume that $D = \mathbb{H}$, $a = 1$, and $b = 0$. Let us further assume that η is in half-plane capacity parameterization and let $(g_t)_{t\in[0,T]}$ be the Loewner chain corresponding to the curve η. As illustrated in Figure 4.1, let us denote

$$\gamma^t := g_t(\gamma \cup \eta([T,t])) \in \mathcal{X}(\mathbb{H}; x_t, \lambda_t)$$

where $\lambda_t := g_t(\eta(t))$ is the driving function of the Loewner chain and $x_t := g_t(1)$. Note that x_t and λ_t are continuous real-valued functions satisfying $\lambda_t \neq x_t$ for $t \in [0,T]$. For each t, let $\varphi_t(z) = (z-\lambda_t)/(x_t-\lambda_t)$ be the conformal automorphism of $\mathbb{H}$ sending the triple

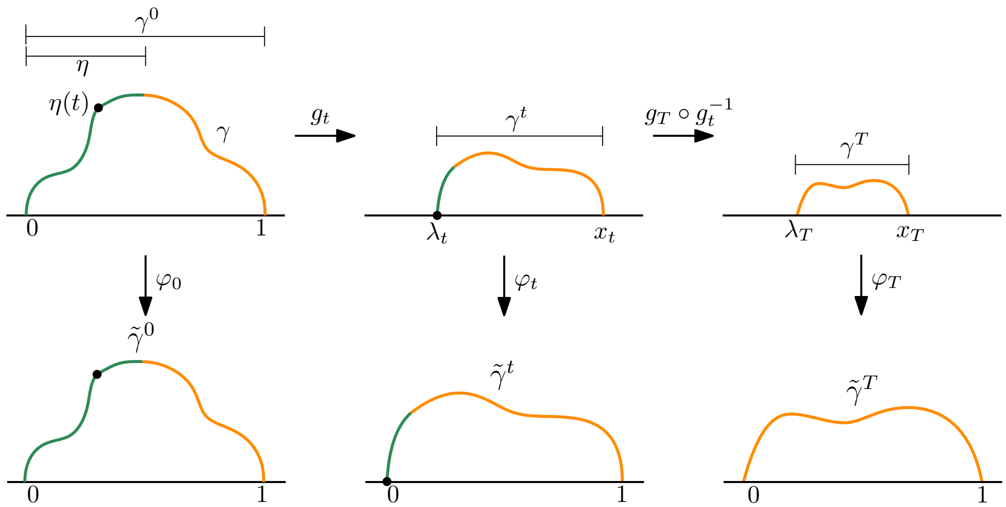

Figure 4.1: A pictorial description of the proof of Lemma 4.3.1. Let $(g_t)_{t\in[0,T]}$ be the Loewner chain mapping out η, and denote by γ^t the chord remaining after mapping out $\gamma \cup \eta$ by g_t. Let $\tilde{\gamma}^t$ be the image of γ^t under an affine transformation that maps the endpoints of the chord to 0 and 1. The hulls enclosed by γ^t converges in the Carathéodory topology to that enclosed by γ^T, and the chords $\tilde{\gamma}^t$ converge along a subsequence to some $\tilde{\gamma}^* \in \mathcal{X}(\mathbb{H}; 1, 0)$ in Hausdorff distance. We have $\tilde{\gamma}^* = \varphi_T(\gamma^T)$ from Lemma 4.2.6.

(λ_t, x_t, ∞) to $(0, 1, \infty)$ and denote

$$\tilde{\gamma}^t := \varphi_t(\gamma^t) \in \mathcal{X}(\mathbb{H}; 0, 1).$$

For $t \in [T_0, T]$, since $\eta([T, t])$ is the hyperbolic geodesic in $D \setminus (\gamma \cup \eta([0, t]))$ between $\zeta = \eta(T)$ and $\eta(t)$, we have

$$\begin{aligned} I_{\mathbb{H};1,0}(\tilde{\gamma}^t) = I_{\mathbb{H};x_t,\lambda_t}(\gamma^t) &= I_{\mathbb{H}\setminus\eta([0,t]);1,\eta(t)}(\gamma \cup \eta([T, t])) \\ &= I^{\circ}_{\mathbb{H}\setminus\eta([0,t]);1,\eta(t)}(\gamma) + I_{\mathbb{H}\setminus(\gamma\cup\eta([0,t]));\zeta,\eta(t)}(\eta([T, t])) \qquad (4.3.8) \\ &= I^{\circ}_{\mathbb{H}\setminus\eta([0,t]);1,\eta(t)}(\gamma). \end{aligned}$$

From (4.3.4), we have

$$I_{\mathbb{H};0,1}(\tilde{\gamma}^t) = I^{\circ}_{\mathbb{H}\setminus\eta([0,t]);1,\eta(t)}(\gamma) \le I^{\circ}_{\mathbb{H};1,0}(\gamma) + I_{\mathbb{H}\setminus\gamma;0,\zeta}(\eta) =: c \qquad (4.3.9)$$

for all $t \in [T_0, T]$. The lemma holds trivially if $c = \infty$, so let us assume $c < \infty$. By Lemma 4.2.1, there exists a sequence $(t_k)_{k\in\mathbb{N}}$ increasing to T and a chord $\tilde{\gamma}^* \in \mathcal{X}(\mathbb{H}; 1, 0)$ such that $\tilde{\gamma}^{t_k} \to \tilde{\gamma}^*$ and $K_{\tilde{\gamma}^{t_k}} \to K_{\tilde{\gamma}^*}$ in the Hausdorff distance.

We claim that $\tilde{\gamma}^* = \tilde{\gamma}^T$, for which it suffices to show $\gamma^* := \varphi_T^{-1}(\tilde{\gamma}^*) = \gamma^T$. Recalling that $x_{t_k} \to x_T$ and $\lambda_{t_k} \to \lambda_T$, the affine maps φ_{t_k} converge uniformly on bounded subsets of $\overline{\mathbb{H}}$ to φ_T. Thus, the rescaled hulls $K_{\gamma^{t_k}} = \varphi_{t_k}^{-1}(K_{\tilde{\gamma}^{t_k}})$ converge to $K_{\gamma^*} = \varphi_T^{-1}(K_{\tilde{\gamma}^*})$ in the Hausdorff distance. On the other hand, observe that if $f_T : \mathbb{H} \setminus K_{\gamma^T} \to \mathbb{H}$ is the mapping-out function of the hull K_{γ^T} (i.e., the unique hydrodynamically normalized conformal map), then $f_t := f_T \circ g_T \circ g_t^{-1}$ is the mapping-out function of the hull K_{γ^t}. We see from the Loewner equation (4.2.3) that $g_t \to g_T$ uniformly away from $\mathbb{R}$ as $t \to T^-$. Since $f_T \circ g_T$ is hydrodynamically normalized, we conclude that $f_t^{-1} \to f_T^{-1}$ uniformly away from $\mathbb{R}$ and thus $K_{\gamma^t} \to K_{\gamma^T}$ in the Carathéodory topology as $t \to T^-$. By Lemma 4.2.6, $K_{\gamma^*} = K_{\gamma^T}$ and hence $\gamma^* = \partial K_{\gamma^*} \cap \mathbb{H} = \partial K_{\gamma^T} \cap \mathbb{H} = \gamma^T$ as claimed. In particular,

$$I_{\mathbb{H};1.0}(\tilde{\gamma}^*) = I_{\mathbb{H};1,0}(\tilde{\gamma}^T) = I_{\mathbb{H}\setminus\eta;1,\zeta}(\gamma).$$

Finally, by Lemma 4.2.1,

$$I_{\mathbb{H}\setminus\eta;1,\zeta}(\gamma) = I_{\mathbb{H};1,0}(\tilde{\gamma}^*) \leq \liminf_{k\to\infty} I_{\mathbb{H};1,0}(\tilde{\gamma}^{t_k}) = \lim_{t\to T^-} I_{\mathbb{H};1,0}(\tilde{\gamma}^t) = \lim_{t\to T^-} I^{\circ}_{\mathbb{H}\setminus\eta([0,t]);1,\eta(t)}(\gamma). \tag{4.3.10}$$

This conclusion is equivalent to (4.3.7) by the conformal invariance of Loewner energy. □

4.3.2 Piecewise geodesic energy minimizers

Let D be a simply connected domain with two distinct prime ends a and b. Recall that $\mathcal{X}(D; a, b)$ is the set of all chords in D from a to b. Given a finite set $\underline{z} \subset D$, let

$$\mathcal{X}(D; a, b; \underline{z}) = \{\gamma \in \mathcal{X}(D; a, b) : \underline{z} \subset \gamma\} \tag{4.3.11}$$

be the set of chords in D from a to b passing through all points in $\underline{z}$. We first observe that there exists a Loewner energy minimizer on this set.

Lemma 4.3.2. *Given any finite $\underline{z} \subset D$, there exists a chord $\gamma^{\min} \in \mathcal{X}(D; a, b; \underline{z})$ such that*

$$I_{D;a,b}(\gamma^{\min}) := \inf_{\gamma \in \mathcal{X}(D;a,b;\underline{z})} I_{D;a,b}(\gamma) < \infty. \tag{4.3.12}$$

Proof. Without loss of generality, let D be a bounded Jordan domain. The existence of a finite-energy chord in $\mathcal{X}(D; a, b; \underline{z})$ is given in [244, Lemma 3.3]. Let us thus take a sequence of finite-energy chords $(\gamma^n)_{n\in\mathbb{N}}$ in $\mathcal{X}(D; a, b; \underline{z})$ whose corresponding energies decrease to $\inf_{\gamma\in\mathcal{X}(D;a,b;\underline{z})} I_{D;a,b}(\gamma) < \infty$. By Lemma 4.2.1, there exists a subsequence γ^{n_k} which converges in the Hausdorff distance to some chord $\gamma^{\min} \in \mathcal{X}(D; a, b)$. We furthermore have $I_{D;a,b}(\gamma^{\min}) \le \liminf_{k\to\infty} I_{D;a,b}(\gamma^{n_k})$. Observe that $\max_{z\in\underline{z}} \min_{w\in\gamma^{\min}} |z - w|$ is bounded above by the Hausdorff distance between γ^{n_k} and $\gamma^{\min}$ for every n_k. Since the latter converges to 0 as $k \to \infty$, we have $\underline{z} \subset \gamma^{\min}$. □

The key observation in our proof of Theorem 4.1.1 is that the energy minimizer in Lemma 4.3.2 has the following property, which was introduced in [185] for the case of Jordan curves.

Definition 4.3.3. Let $\underline{z} \subset D$ be a finite set. We say that a chord $\gamma \in \mathcal{X}(D; a, b; \underline{z})$ is piecewise geodesic relative to $\underline{z}$ if $\underline{z}$ partitions γ into simple curves $\gamma_1, \gamma_2, \ldots, \gamma_n$ such that each γ_k is a hyperbolic geodesic of the domain $D \setminus (\gamma \setminus \gamma_k)$.

In other words, a chord $\gamma : [0, 1] \to D \cup \{a, b\}$ visiting $\underline{z} \cup \{a, b\}$ at times $0 = t_0 < t_1 < \cdots < t_{n-1} < t_n = 1$ is piecewise geodesic relative to $\underline{z}$ if, for every $k = 1, \ldots, n$, the chord $\gamma([t_{k-1}, t_k])$ is the hyperbolic geodesic in $D \setminus \gamma([0, t_{k-1}] \cup [t_k, 1])$ between $\gamma(t_{k-1})$ and $\gamma(t_k)$.

Proposition 4.3.4. *Let D be a simply connected domain with distinct prime ends a, b and let $\underline{z}$ be a finite subset of D.*

1. *If γ is a chord in $\mathcal{X}(D; a, b; \underline{z})$ satisfying $I_{D;a,b}(\gamma) = \inf_{\eta \in \mathcal{X}(D;a,b,\underline{z})} I_{D;a,b}(\eta)$, then γ is piecewise geodesic relative to $\underline{z}$.*

2. *If $\gamma \in \mathcal{X}(D; a, b; \underline{z})$ is piecewise geodesic relative to $\underline{z}$, then its energy is invariant under the reversal of orientation. That is, $I_{D;a,b}(\gamma) = I_{D;b,a}(\gamma)$.*

Proof. Take an arbitrary chord $\gamma \in \mathcal{X}(D; a, b; \underline{z})$ and let $z_1, z_2, \ldots, z_n$ be the enumeration of $\underline{z}$ in the order visited by γ. For each $k = 1, 2, \ldots, n+1$, let us denote the part of the chord γ between z_{k-1} and z_k as γ_k (where $z_0 = a$ and $z_{n+1} = b$). We shall reverse the orientation of γ inductively, finishing with a chord $\eta \in \mathcal{X}(D; b, a; \underline{z})$ with $I_{D;b,a}(\eta) \leq I_{D;a,b}(\gamma)$. Furthermore, we will show that the inequality is strict if γ is not piecewise geodesic relative to $\underline{z}$. Let us denote $D_k := D \setminus (\bigcup_{i \leq k} \gamma_i)$ for the rest of this proof.

- In the first step, replace γ_{n+1} with the hyperbolic geodesic in D_n from b to z_n, which we denote η_{n+1}. Since $I_{D_n;b,z_n}(\eta_{n+1}) = 0$, we have

$$\begin{aligned} I_{D;a,b}(\gamma) &= I^{\circ}_{D;a,b}\left(\bigcup_{i \leq n} \gamma_i\right) + I_{D_n;z_n,b}(\gamma_{n+1}) \\ &\geq I^{\circ}_{D;a,b}\left(\bigcup_{i \leq n} \gamma_i\right) + I_{D_n;b,z_n}(\eta_{n+1}). \end{aligned} \tag{4.3.13}$$

  Here, the inequality is strict if γ_{n+1} is not a hyperbolic geodesic (i.e., γ_{n+1} is not the orientation reversal of η_{n+1}). Note that $(\bigcup_{i \leq n} \gamma_i) \cup \eta_{n+1} \in \mathcal{X}(D; a, b; \underline{z})$.

- Let $k \in \{1, \ldots, n\}$. Suppose we have replaced each γ_j with $j > k$ with a curve η_j between z_{j-1} and z_j so that $(\bigcup_{i \leq k} \gamma_i) \cup (\bigcup_{j \geq k+1} \eta_j)$ is a chord in $\mathcal{X}(D; a, b; \underline{z})$. Suppose, furthermore, that η_{k+1} is the hyperbolic geodesic in $D_k \setminus (\bigcup_{j \geq k+2} \eta_j)$ between z_{k+1} and z_{k+2}, and

$$I_{D;a,b}(\gamma) \geq I_{D;a,b}\left(\bigcup_{i \leq k} \gamma_i\right) + I_{D_k;b,z_k}\left(\bigcup_{j \geq k+1} \eta_j\right). \tag{4.3.14}$$

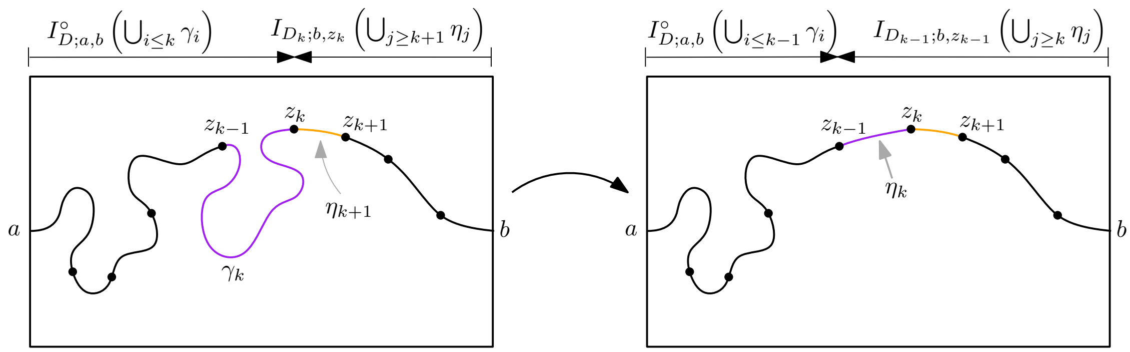

Figure 4.2: An orientation reversal step in the proof of Proposition 4.3.4. Left: The curves $\gamma_{k+1},\dots,\gamma_{n+1}$ to the right of z_k have been replaced with $\eta_{k+1},\dots,\eta_{n+1}$ in the previous steps. The orange curve η_{k+1} is a hyperbolic geodesic in the complement of black and purple parts of the chord. Right: We replace the purple curve γ_k with a hyperbolic geodesic η_k in the complement of black and orange parts of the chord. The total energy of the curves in the right figure is no more than that in the left figure. Moreover, if $\gamma_k \neq \eta_k$, then the inequality is strict.

Now, replace γ_k with the hyperbolic geodesic in $D_{k-1}\setminus(\bigcup_{j\ge k+1}\eta_j)$ from z_k to z_{k-1}, which we denote η_k; see Figure 4.2 for an illustration. Since $\bigcup_{j\ge k+1}\eta_j$ has a geodesic tip as a chord in D_k, Lemma 4.3.1 implies

$$\begin{aligned}
&I^{\circ}_{D_{k-1};z_{k-1},b}(\gamma_k) + I_{D_{k-1}\setminus\gamma_k;b,z_k}\left(\bigcup_{j\ge k+1}\eta_j\right)\\
&\ge I^{\circ}_{D_{k-1};b,z_{k-1}}\left(\bigcup_{j\ge k+1}\eta_j\right) + I_{D_{k-1}\setminus(\bigcup_{j\ge k+1}\eta_j);z_{k-1},z_k}(\gamma_k)\\
&\ge I^{\circ}_{D_{k-1};b,z_{k-1}}\left(\bigcup_{j\ge k+1}\eta_j\right) + I_{D_{k-1}\setminus(\bigcup_{j\ge k+1}\eta_j);z_k,z_{k-1}}(\eta_k) = I_{D_{k-1};b,z_{k-1}}\left(\bigcup_{j\ge k}\eta_j\right).
\end{aligned} \tag{4.3.15}$$

Note that the second inequality in (4.3.15) is strict if γ_k is not a hyperbolic geodesic (i.e., γ_k is not the orientation reversal of η_k). The new chord $(\bigcup_{i\le k-1}\gamma_k)\cup(\bigcup_{j\ge k}\eta_j)$ satisfies the induction hypothesis since it passes through all points in $\underline{z}$ and, combining

(4.3.14) and (4.3.15), we have

$$
\begin{aligned}
I_{D;a,b}(\gamma) &\geq I^{\circ}_{D;a,b}\left(\bigcup_{i\leq k}\gamma_i\right) + I_{D_k;b,z_k}\left(\bigcup_{j\geq k+1}\eta_j\right) \\
&= I^{\circ}_{D;a,b}\left(\bigcup_{i\leq k-1}\gamma_i\right) + I^{\circ}_{D_{k-1};z_{k-1},b}(\gamma_k) + I_{D_k;b,z_k}\left(\bigcup_{j\geq k+1}\eta_j\right) \qquad (4.3.16) \\
&\geq I^{\circ}_{D;a,b}\left(\bigcup_{i\leq k-1}\gamma_i\right) + I_{D_{k-1};b,z_{k-1}}\left(\bigcup_{j\geq k}\eta_j\right).
\end{aligned}
$$

After completing this above reversal procedure all the way from $k=n+1$ to $k=1$, we are left with a chord $\eta := \bigcup_{j=1}^{n+1}\eta_j \in \mathcal{X}(D;a,b;\underline{z})$ satisfying $I_{D;b,a}(\eta) \leq I_{D;a,b}(\gamma)$. By construction, if γ is not piecewise geodesic relative to $\underline{z}$, then we cannot have $\eta_k = \gamma_k$ modulo orientation for all k. In this case, we have a strict inequality $I_{D;b,a}(\eta) < I_{D;a,b}(\gamma)$. Applying the above inductive reversal procedure again to η, we obtain a chord $\tilde{\gamma} \in \mathcal{X}(D;a,b;\underline{z})$ such that $I_{D;a,b}(\tilde{\gamma}) \leq I_{D;b,a}(\eta) < I_{D;a,b}(\gamma)$. Hence, if γ minimizes the energy $I_{D;a,b}$ among chords passing through $\underline{z}$, then it must be piecewise geodesic relative to $\underline{z}$.

If γ is piecewise geodesic relative to $\underline{z}$, then $\eta_k = \gamma_k$ modulo orientation in each step of the above reversal procedure. Then, the chord η that we obtain through the reversal procedure is simply γ in reverse orientation and we have $I_{D;b,a}(\gamma) \leq I_{D;a,b}(\gamma)$. Note that the piecewise geodesic property of γ is invariant under reversing its orientation, whence $I_{D;b,a}(\gamma) \geq I_{D;a,b}(\gamma)$ as well. We conclude that $I_{D;a,b}(\gamma) = I_{D;b,a}(\gamma)$ if γ is piecewise geodesic. $\square$

We note that the loop version of Proposition 4.3.4 was given in [219, Proposition 2.13] using the root-invariance of loop Loewner energy. The latter fact was proved in the same work using the the chordal reversibility as a preliminary.

The final input needed for our proof of Theorem 4.1.1 is the following consequence of the Jordan curve theorem.

Lemma 4.3.5. *Let $D \subsetneq \mathbb{C}$ be a bounded Jordan domain and suppose γ and $\tilde{\gamma}$ are chords in D between distinct boundary points a and b. If $\gamma \subseteq \tilde{\gamma}$, then $\gamma = \tilde{\gamma}$.*

Proof. Suppose, for contradiction, that there is a point $z \in \tilde{\gamma} \setminus \gamma$. It must be in D since $\tilde{\gamma} \setminus D = \gamma \setminus D = \{a, b\}$. By the Jordan curve theorem, $D \setminus \gamma$ has two connected components, say U and V, such that $D \cap \partial U = D \cap \partial V = \gamma \setminus \{a, b\}$ (see, e.g., [207, Proposition 2.12] for a proof). Let U be the component containing z.

Now, $D \setminus \tilde{\gamma}$ also has exactly two connected components, say $\widetilde{U}$ and $\widetilde{V}$. Since $D \setminus \tilde{\gamma} \subset D \setminus \gamma$, the two components $\widetilde{U}$ and $\widetilde{V}$ are each a subset of either U or V but not both. However, $z \in \tilde{\gamma} = D \cap \partial\widetilde{U} = D \cap \partial\widetilde{V}$, which means that both $\widetilde{U}$ and $\widetilde{V}$ are subsets of U. This implies $V \subset \tilde{\gamma}$ and hence $V \subset D \cap \partial\widetilde{U}$, which is a contradiction since $V \cap (D \cap \partial\widetilde{U}) \subset V \cap (D \cap \overline{U}) = V \cap (U \cup \gamma) = \varnothing$. □

We are now ready to prove the orientation reversibility of chordal Loewner energy.

Proof of Theorem 4.1.1. Without loss of generality, assume that D is a bounded Jordan domain. Suppose γ is a chord in D from a to b with $I_{D;a,b}(\gamma) < \infty$. Choose an increasing sequence $\underline{z}^1 \subset \underline{z}^2 \subset \cdots$ of finite subsets of $\gamma \cap D$ such that $\bigcup_{n \geq 1} \underline{z}^n$ is dense in γ. For instance, we can take any continuous parameterization $\gamma : [0, 1] \to D \cup \{a, b\}$ and set $\underline{z}^n = \{\gamma(j/2^n) : j = 1, \ldots, 2^n - 1\}$. By Lemma 4.3.2, for each positive integer n, we can find chord $\gamma^n \in \mathcal{X}(D; a, b; \underline{z}^n)$ which minimizes the energy $I_{D;a,b}$ among those passing through $\underline{z}^n$. By Proposition 4.3.4, each γ^n is piecewise geodesic relative to $\underline{z}^n$ and hence

$$I_{D;b,a}(\gamma^n) = I_{D;a,b}(\gamma^n) \leq I_{D;a,b}(\gamma) < \infty. \tag{4.3.17}$$

By Lemma 4.2.1, there exists a subsequence γ^{n_k} which converges to a chord $\eta \in \mathcal{X}(D; b, a)$ in the Hausdorff distance, which furthermore satisfies

$$I_{D;b,a}(\eta) \leq \liminf_{k \to \infty} I_{D;b,a}(\gamma^{n_k}) = \liminf_{k \to \infty} I_{D;a,b}(\gamma^{n_k}) \leq I_{D;a,b}(\gamma). \tag{4.3.18}$$

We claim that $\eta = \gamma$ modulo orientation. Note that for every $z \in \underline{z}^m$ and $n_k \geq m$, since $\underline{z}^m \subset \underline{z}^{n_k}$, the distance between z and the chord η is bounded above by the Hausdorff distance between γ^{n_k} and η. Taking $n_k \to \infty$, we see that every $z \in \bigcup_{m \geq 1} \underline{z}^m$ is in η. Since $\bigcup_{m \geq 1} \underline{z}^m$ is dense in γ and η is closed, we have $\gamma \subseteq \eta$ and thus $\gamma = \eta$ by Lemma 4.3.5. Therefore, $I_{D;b,a}(\gamma) \leq I_{D;a,b}(\gamma)$ and, changing the roles of a and b, we conclude $I_{D;a,b}(\gamma) \leq I_{D;b,a}(\gamma)$ as well.

This proves the theorem if either $I_{D;a,b}(\gamma)$ or $I_{D;b,a}(\gamma)$ is finite. It is trivially satisfied when the energy is infinite in both orientations, so $I_{D;a,b}(\gamma) = I_{D;b,a}(\gamma)$ for any $\gamma \in \mathcal{X}(D; a, b)$. $\square$

4.4 Energy minimizers for isotopy classes of chords

In this section, we prove Theorem 4.1.2 on the existence of a Loewner energy minimizer for each isotopy class of chords passing through a finite set of points and its characterization as a piecewise geodesic chord.

We first note that an isotopy between two chords γ_0 and γ_1 relative to $\underline{z}$ extends to an ambient isotopy of $\overline{D}$ relative to $\underline{z}$, which refers to a continuous deformation of the domain D fixing $\underline{z} \cup \partial D$ which restricts to an isotopy between the two chords. Here, $\overline{D}$ means the union of D with the set ∂D of its prime ends. Here is a more precise statement. Let us denote by $\mathrm{Homeo}(\overline{D}, K)$ the space of self-homeomorphisms of $\overline{D}$ which restrict to the identity map on $K \subset \overline{D}$, equipped with the compact-open topology.

Lemma 4.4.1. *If two chords $\gamma_0, \gamma_1 \in \mathcal{X}(D; a, b; \underline{z})$ are isotopic relative to $\underline{z}$, then there exists a continuous map $\widetilde{F} : [0,1] \times \overline{D} \to \overline{D}$ such that $\widetilde{F}(s, \cdot) \in \mathrm{Homeo}(\overline{D}, \underline{z} \cup \partial D)$ for each $s \in [0,1]$, $\widetilde{F}(0, \cdot)$ is the identity map, and $\gamma_1 = \widetilde{F}(1, \gamma_0)$.*

The converse relationship is clear: if there exists an ambient isotopy between two chords relative to $\underline{z}$, then they are isotopic relative to $\underline{z}$.

Proof of Lemma 4.4.1. Without loss of generality, assume D is the unit disk $\mathbb{D}$ and denote $\underline{z} = \{z_1, \ldots, z_n\}$. Choose $\varepsilon > 0$ such that the closed disks $\overline{B_{2\varepsilon}(z_k)}$ are contained in $\mathbb{D}$ and disjoint from each other. It is straightforward to find a family of maps $(\phi_{\underline{w}})$, indexed by $\underline{w} = \{w_1, \ldots, w_n\}$ satisfying $\max_k |w_k - z_k| < \varepsilon$, with the following conditions: $\phi_{\underline{z}}$ is the identity map, $\phi_{\underline{w}} \in \mathrm{Homeo}(\overline{\mathbb{D}}, \partial\mathbb{D})$ for each $\underline{w}$, and $\phi_{\underline{w}}(w_k) = z_k$ for each $w_k \in \underline{w}$. Furthermore, we can choose these maps so that $\underline{w} \mapsto \phi_{\underline{w}}$ is continuous with respect to the Hausdorff distance on the domain and the uniform norm on the codomain.

Let $\gamma_0 : [0,1] \to \mathbb{D} \cup \{a,b\}$ be a continuous parameterization of a chord in $\mathcal{X}(D; a, b, \underline{z})$ with $t_k := (\gamma_0)^{-1}(z_k)$ for each $z_k \in \underline{z}$. We claim that there exists a $\delta > 0$ (depending on γ_0) so that the following is true: if $\gamma_1 : [0,1] \to \mathbb{D} \cup \{a,b\}$ is a parameterized chord with $\sup_{t\in[0,1]} |\gamma_0(t) - \gamma_1(t)| < \delta$ and $\gamma_1(t_k) = z_k$ for every k, then there exists an ambient isotopy $\widetilde{F}$ as in the statement of the lemma. By the Schoenflies theorem, we can choose $\delta > 0$ such that for any such γ_1, there is $f \in \mathrm{Homeo}(\overline{\mathbb{D}}, \partial\mathbb{D})$ with $\sup_{x\in\overline{\mathbb{D}}} |f(x) - x| < \varepsilon$ and $\gamma_1 = f \circ \gamma_0$ (see [207, Corollary 2.9 and Theorem 2.11]). Since $\mathrm{Homeo}(\overline{\mathbb{D}}, \partial\mathbb{D})$ is locally contractible [135], f extends to an isotopy $F : [0,1] \times \overline{\mathbb{D}} \to \overline{\mathbb{D}}$ from the identity map to f such that $F(s, \cdot) \in \mathrm{Homeo}(\overline{\mathbb{D}}, \partial\mathbb{D})$ and $\sup_{x\in\overline{\mathbb{D}}} |F(s,x) - x| < \varepsilon$ for every $s \in [0,1]$. Then, $\widetilde{F}(s, \cdot) := \phi_{F(s,\underline{z})} \circ F(s, \cdot)$ is the desired ambient isotopy between γ_0 and γ_1 leaving $\underline{z}$ pointwise fixed.

Now, given any $\gamma_1 \in \mathcal{X}(\mathbb{D}; a, b; \underline{z}, \gamma_0)$, choose an isotopy $H : [0,1] \times \gamma_0 \to \mathbb{D} \cup \{a,b\}$ from γ_0 to γ_1 leaving $\underline{z}$ pointwise fixed and denote $\gamma_s(t) := H(s,t)$. By the previous paragraph, there exists a $\delta_s > 0$ for every $s \in [0,1]$ such that $\gamma_{\tilde{s}}$ is ambient isotopic to γ_s through $\mathrm{Homeo}(\overline{\mathbb{D}}, \underline{z} \cup \partial\mathbb{D})$ whenever $|\tilde{s} - s| < \delta_s$. This gives an open cover of $[0,1]$, so we conclude that γ_1 is ambient isotopic to γ_0 through $\mathrm{Homeo}(\overline{\mathbb{D}}, \underline{z} \cup \partial\mathbb{D})$. $\square$

Remark 4.4.2. All chords in $\mathcal{X}(D; a, b)$ are isotopic to each other by the Jordan curve theorem. When $\underline{z}$ contains $1 \leq n < \infty$ number of points, by Lemma 4.4.1, the collection of isotopy classes of chords in $\mathcal{X}(D; a, b; \underline{z})$ relative to $\underline{z}$ correspond to the mapping class group of a disk with n punctures. This can be further identified with the braid group on n strands (see, e.g., [102, Section 9.1]). As such, there is a unique isotopy class relative to $\underline{z}$ when $\underline{z}$ consists of a single point, but the number of isotopy classes relative to $\underline{z}$ is countably infinite when $\underline{z}$ contains more than one point.

Proof of Theorem 4.1.2. If $\underline{z} = \varnothing$, then $\gamma^{\min}$ is the hyperbolic geodesic in $\mathcal{X}(D; a, b)$ and the claim is trivially satisfied. Let us thus consider the case that $\underline{z}$ contains $n \geq 1$ points and show that there exists a chord in $\mathcal{X}(D; a, b; \underline{z}, \gamma)$ with finite energy. The proof is by induction on n.

- If $\underline{z}$ contains a single point z_1, then there is a unique isotopy class of chords in $\mathcal{X}(D; a, b)$ passing through z_1 relative to this point. In this case, $\gamma^{\min}$ with finite energy was uniquely identified in [244, Proposition 3.1] using the interpretation of chordal Loewner energy as the large deviation rate function of chordal SLE. The same result was proved deterministically in [190, Theorem 3.3(i)] by an identification of the system of differential equations satisfied by the driving function of $\gamma^{\min}$.

- Let $n \geq 2$ and suppose there exists a finite energy chord passing through any set of $n - 1$ points in D with any given isotopy class. Let z_1 and z_2 be the first and second points of $\underline{z}$ visited by γ, respectively. Let γ_0 be the part of γ until it visits z_1, let γ_1 be the part between z_1 and z_2, and finally let γ_2 be the rest of γ after it visits z_2.

  Using the result for $n = 1$ case referenced above, let us replace $\gamma_0 \cup \gamma_1$ with a finite energy chord η in $\mathcal{X}(D \setminus \gamma_2; a, z_2)$ passing through z_1. Let η_0 be the part of this curve up to its visit to z_1 and η_1 be the remaining part from z_1 to z_2. Note that we can find an isotopy between $\gamma_0 \cup \gamma_1$ and η in $D \setminus \gamma_2$ fixing z_1 and extend it to one between γ and $\eta \cup \gamma_2$ in D fixing $\underline{z}$. That is, $\eta \cup \gamma_2 \in \mathcal{X}(D; a, b; \underline{z}, \gamma)$. Moreover, since $I_{D \setminus \gamma_2; a, z_2}(\eta_0) < \infty$, we have $I_{D; a, b}(\eta_0) < \infty$ by [244, Lemma 4.3].

  We now recall the induction hypothesis to choose a chord $\tilde{\eta} \in \mathcal{X}(D \setminus \eta_0; z_1, b; \underline{z} \setminus$

$\{z_1\}, \eta_1 \cup \gamma_2)$ with finite energy. Then, by the additivity (4.2.6) of Loewner energy, we have

$$I_{D;a,b}(\eta_0 \cup \tilde{\eta}) = I^{\circ}_{D;a,b}(\eta_0) + I_{D\setminus\eta_0;z_1,b}(\tilde{\eta}) < \infty. \tag{4.4.1}$$

Furthermore, $\eta_0 \cup \tilde{\eta}$ is isotopic to $\eta \cup \gamma_2$ relative to $\underline{z}$, so we have found a finite-energy chord in $\mathcal{X}(D; a, b; \underline{z}, \gamma)$ as claimed.

Let us assume $D = \mathbb{D}$, $a = -1$, and $b = 1$ and choose a sequence of finite-energy chords $(\gamma^m)_{m\in\mathbb{N}}$ in $\mathcal{X}(\mathbb{D}; -1, 1; \underline{z}, \gamma)$ whose energies decrease to the infimum energy in this isotopy class. By Lemma 4.2.1, for each m, there exists $\varphi_m \in \mathrm{Homeo}(\overline{\mathbb{D}}, \partial\mathbb{D})$ with $\varphi_m([-1, 1]) = \gamma^m$. Moreover, along a subsequence, φ_{m_k} converges uniformly to some $\varphi \in \mathrm{Homeo}(\overline{\mathbb{D}}, \partial\mathbb{D})$. Since $\overline{\mathbb{D}}$ is compact, $\varphi_{n_k}^{-1} \to \varphi^{-1}$ uniformly as well, which implies that $\varphi_{m_k}^{-1}(\underline{z})$ converges in the Hausdorff distance. Then, we can pick a sequence of homeomorphisms $\psi_{m_k} \in \mathrm{Homeo}(\overline{\mathbb{D}}, \partial\mathbb{D})$ converging uniformly to the identity map such that $\psi_{m_k}([-1, 1]) = [-1, 1]$ and $(\varphi_{m_k} \circ \psi_{m_k})^{-1}(\underline{z}) = \varphi^{-1}(\underline{z})$ for every m_k. Letting $\gamma^{\min} := \varphi([-1, 1])$, we see that $\varphi_{m_k} \circ \psi_{m_k} \circ \varphi^{-1} \in \mathrm{Homeo}(\overline{\mathbb{D}}, \underline{z} \cup \partial\mathbb{D})$ maps $\gamma^{\min}$ to γ^{m_k}, and these homeomorphisms converge uniformly to the identity map as $m_k \to \infty$. Since the identity component of $\mathrm{Homeo}(\overline{\mathbb{D}}, \underline{z} \cup \partial\mathbb{D})$ is homotopically trivial [134], we conclude $\gamma^{\min} \in \mathcal{X}(\mathbb{D}; -1, 1; \underline{z}, \gamma)$. Since $\gamma^{m_k} \to \gamma^{\min}$ in the Hausdorff distance, we obtain (4.1.3) by the lower semicontinuity of Loewner energy (Lemma 4.2.1).

The claim that an energy minimizer of $\mathcal{X}(D; a, b; \underline{z}, \gamma)$ is piecewise geodesic relative to $\underline{z}$ follows from the fact that the manipulations in the proof of Proposition 4.3.4 do not change the isotopy class. $\square$

CHAPTER 5

QUASICONFORMAL DEFORMATION OF THE CHORDAL LOEWNER DRIVING FUNCTION AND FIRST VARIATION OF THE LOEWNER ENERGY

This chapter presents material first published in [238].

5.1 Introduction

One hundred and one years ago, Loewner introduced [180] a method to encode a simple planar curve by a family of uniformize maps (called the Loewner chain) which satisfies a differential equation driven by a real-valued function. This method has become a powerful tool in geometric function theory. It was instrumental in the proof of Bieberbach conjecture by De Branges [82] (which was also the original motivation of Loewner) and was revived around 2000 as a fundamental building block in the definition of the Schramm–Loewner Evolution [221]. On the other hand, quasiconformal mapping is one of the fundamental concepts in geometric function theory and Teichmüller theory. Thus, we find it natural to investigate the interplay between quasiconformal maps and the Loewner transform. We will further comment on the motivation of this work and discuss follow-up questions in Section 5.1.1. We mention that analytic properties of the Loewner driving function have been investigated in, e.g., [218, 186, 179, 178, 104, 219].

Our first result shows how quasiconformal deformations of the ambient domain $\mathbb{H} = \{z \in \mathbb{C} \colon \operatorname{Im}(z) > 0\}$ affect the driving function of a simple chord in $\mathbb{H}$ connecting 0 to ∞.

Theorem 5.1.1. *Let η be a simple chord in $(\mathbb{H}; 0, \infty)$ under capacity parametrization and $\nu \in L^\infty(\mathbb{H})$ be an infinitesimal Beltrami differential whose support is compact and disjoint from η. For $\varepsilon \in \mathbb{R}$ such that $\|\varepsilon\nu\|_\infty < 1$, let $\psi^{\varepsilon\nu}$ be the unique quasiconformal self-map of $\mathbb{H}$ with Beltrami coefficient $\varepsilon\nu$ such that $\psi^{\varepsilon\nu}(0) = 0$ and $\psi^{\varepsilon\nu}(z) - z = O(1)$ as $z \to \infty$. Denote* *the capacity and driving functions of the parametrized chord* $\psi^{\varepsilon\nu} \circ \eta$ *in* $(\mathbb{H}, 0, \infty)$ *by* $a_\cdot^{\varepsilon\nu}$ *and* $\lambda_\cdot^{\varepsilon\nu}$*, respectively. Then,*

$$\left.\frac{\partial \lambda_t^{\varepsilon\nu}}{\partial \varepsilon}\right|_{\varepsilon=0} = -\frac{2}{\pi} \operatorname{Re} \int_{\mathbb{H}} \nu(z) \left(\frac{g_t'(z)^2}{g_t(z) - \lambda_t} - \frac{1}{z} \right) \mathrm{d}^2 z \tag{5.1.1}$$

and

$$\left.\frac{\partial a_t^{\varepsilon\nu}}{\partial \varepsilon}\right|_{\varepsilon=0} = \frac{1}{\pi} \operatorname{Re} \int_{\mathbb{H}} \nu(z) \big(g_t'(z)^2 - 1 \big) \mathrm{d}^2 z \tag{5.1.2}$$

where $\mathrm{d}^2 z$ *is the Euclidean area measure,* $\lambda_\cdot$ *is the driving function of* η*,* $g_\cdot$ *is the Loewner chain of* η*.*

Our proof relies on the simple but crucial observation that the Loewner driving function and the capacity parametrization of the curve can be expressed by the pre-Schwarzian and Schwarzian derivatives, respectively, of well-chosen maps (Lemma 5.2.1).

We extend our considerations to the Loewner driving function associated with a Jordan curve $\gamma \subset \hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$, now defined on $\mathbb{R}$ instead of $\mathbb{R}_+$. The loop driving function was defined in [245] and can be thought of as a consistent family of chordal Loewner driving functions. See Section 5.3.1 for the precise definition. We point out that for a given Jordan curve, there are a few choices we make to define its driving function $t \mapsto \lambda_t$:

- the orientation of γ;
- a point on γ called the root, which we denote by $\gamma(-\infty) = \gamma(+\infty)$ (we also use $\gamma(\pm\infty)$ when we do not emphasize the difference between the start point and the end point of the parametrization);
- another point on γ, which we call $\gamma(0)$;
- a conformal map $H_0 : \hat{\mathbb{C}} \smallsetminus \gamma[-\infty, 0] \to \mathbb{C} \smallsetminus \mathbb{R}_+$, such that $H_0(\gamma(0)) = 0$ and $H_0(\gamma(+\infty)) = \infty$, where $\gamma[-\infty, 0]$ denotes the closed subinterval of γ (as a set) going from the root to $\gamma(0)$ following the orientation of the curve.

Then, we can complete the continuous parametrization of γ on $(-\infty, 0) \cup (0, +\infty)$ in a unique way such that for each $s \in \mathbb{R}$, the chord $\gamma(\cdot + s)$ traverses the simply connected domain $\hat{\mathbb{C}} \smallsetminus \gamma[-\infty, s]$ in capacity parametrization.[1] Moreover, the chordal driving function of $\gamma(\cdot + s)$ in $\hat{\mathbb{C}} \smallsetminus \gamma[-\infty, s]$ is given by $\lambda_{\cdot + s} - \lambda_s$ (see Lemma 5.3.2).

If the orientation and the root of γ are fixed, different choices of $\gamma(0)$ and H_0 result in changes to the driving function of the form

$$\tilde{\lambda}_t = c\left(\lambda_{c^{-2}(t+s)} - \lambda_{c^{-2}s}\right) \tag{5.1.3}$$

for some $c > 0$ and $s \in \mathbb{R}$. Such transformations do not change the Dirichlet energy of λ. Rather surprisingly, the Dirichlet energy of the loop driving function does not depend on the choice of the root or the orientation either, as shown in [244, 219]. These symmetries are further explained by the following theorem.

Theorem 5.1.2 (See [245])**.** *The Loewner energy of γ, defined as*

$$I^L(\gamma) = \frac{1}{2}\int_{-\infty}^{+\infty} |\dot{\lambda}_t|^2 \, \mathrm{d}t \tag{5.1.4}$$

equals $1/\pi$ times the universal Liouville action $\mathbf{S}$ introduced by Takhtajan and Teo in [239]*, defined as*

$$\mathbf{S}(\gamma) := \int_{\mathbb{D}} \left|\frac{f''}{f'}(z)\right|^2 \mathrm{d}^2 z + \int_{\mathbb{D}^*} \left|\frac{g''}{g'}(z)\right|^2 \mathrm{d}^2 z + 4\pi \log\left|\frac{f'(0)}{g'(\infty)}\right|. \tag{5.1.5}$$

Here, $f : \mathbb{D} \to \Omega$ and $g : \mathbb{D}^* \to \Omega^*$ are conformal maps such that $g(\infty) = \infty$, Ω and Ω^* are respectively the bounded and unbounded connected components of $\mathbb{C} \smallsetminus \gamma$, and $g'(\infty) = \lim_{z\to\infty} g'(z)$. If γ passes through ∞, we replace γ by $A(\gamma)$ where A is any Möbius transformation of $\hat{\mathbb{C}}$ sending γ to a bounded curve.

Remark 5.1.3. Although it may not be so apparent from (5.1.5), it will follow immediately

1. For the simplicity of notation, we assumed here that the capacity parametrization of γ is bi-infinite. See Footnote 3 for further details regarding this assumption.

from the definition of the loop driving function and (5.1.4) that I^L is invariant under Möbius transformations of $\hat{\mathbb{C}}$. See Remark 5.3.4. A Jordan curve for which $\mathbf{S}$ is finite is called a Weil–Petersson quasicircle.

Using Theorem 5.1.1, we obtain in Section 5.3 the following first variation formula of the Loewner energy. This formula coincides with that of the universal Liouville action $\mathbf{S}$ in [239, Chapter 2, Theorem 3.8] divided by π, thus giving another explanation of the identity $I^L = \mathbf{S}/\pi$. This variational formula was crucial in [239] to show that $\mathbf{S}$ is a Kähler potential of the Weil–Petersson Teichmüller space.

Theorem 5.1.4. *Let $\mu \in L^\infty(\mathbb{C})$ be an infinitesimal Beltrami differential with compact support in $\hat{\mathbb{C}} \smallsetminus \gamma$. For $\varepsilon \in \mathbb{R}$ with $\|\varepsilon\mu\|_\infty < 1$, let $\omega^{\varepsilon\mu} : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ be any quasiconformal mapping with Beltrami coefficient $\varepsilon\mu$. Let $\gamma^{\varepsilon\mu} = \omega^{\varepsilon\mu}(\gamma)$. Then,*

$$\frac{\mathrm{d}}{\mathrm{d}\varepsilon}\bigg|_{\varepsilon=0} I^L(\gamma^{\varepsilon\mu}) = -\frac{4}{\pi}\,\mathrm{Re}\left[\int_\Omega \mu(z)\mathcal{S}[f^{-1}](z)\,\mathrm{d}^2 z + \int_{\Omega^*} \mu(z)\mathcal{S}[g^{-1}](z)\,\mathrm{d}^2 z\right], \tag{5.1.6}$$

where f and g are the conformal maps in Theorem 5.1.2 and $\mathcal{S}[\varphi] = \varphi'''/\varphi' - (3/2)(\varphi''/\varphi')^2$ is the Schwarzian derivative of φ.

In the language of conformal field theory, this theorem states that the holomorphic stress-energy tensor of the Loewner energy is given by a multiple of the Schwarzian derivative of the uniformizing map on each complementary component of the curve.

Remark 5.1.5. The Loewner energy of $\gamma^{\varepsilon\mu}$ does not depend on the choice of the solution $\omega^{\varepsilon\mu}$ to the Beltrami equation, as all such solutions are equivalent up to post-compositions by Möbius transformations of $\hat{\mathbb{C}}$. In [239], μ is an L^2-harmonic Beltrami differential supported on only one side of the curve γ. Here, we allow the support of μ to be on both sides of γ but require it to be disjoint from γ.

Since the support of μ is away from γ, there exists a (not necessarily simply connected) domain D containing γ such that $D \cap \mathrm{supp}(\mu) = \varnothing$. In particular, $\omega^{\varepsilon\mu}$ is conformal in D.

In [246, Theorem 4.1], the second author showed that the change of the Loewner energy under a conformal map in the neighborhood of the curve could be expressed in terms of the $\mathrm{SLE}_{8/3}$ loop measure introduced in [250], which is the induced measure obtained by taking the outer boundary of a loop under Brownian loop measure [166, 172]. Combining this with Theorem 5.1.4, we immediately obtain the following variational formula for the $\mathrm{SLE}_{8/3}$ loop measure.

Corollary 5.1.6. *For every domain D containing γ such that $D \cap \operatorname{supp}(\mu) = \varnothing$, we have*

$$\frac{\mathrm{d}}{\mathrm{d}\varepsilon}\bigg|_{\varepsilon=0} \mathcal{W}(\gamma^{\varepsilon\mu}, \omega^{\varepsilon\mu}(D)^c; \hat{\mathbb{C}}) = \frac{1}{3\pi} \operatorname{Re}\left[\int_{\Omega} \mu(z)\mathcal{S}[f^{-1}](z)\,\mathrm{d}^2 z + \int_{\Omega^*} \mu(z)\mathcal{S}[g^{-1}](z)\,\mathrm{d}^2 z\right] \tag{5.1.7}$$

where $\mathcal{W}(\gamma^{\varepsilon\mu}, \omega^{\varepsilon\mu}(D)^c; \hat{\mathbb{C}})$ denotes the total mass of loops intersecting both $\gamma^{\varepsilon\mu}$ and the complement of $\omega^{\varepsilon\mu}(D)$ under the $\mathrm{SLE}_{8/3}$ *loop measure on $\hat{\mathbb{C}}$.*

5.1.1 Remarks and open questions

While Loewner chains have been studied extensively due to their applications in the study of Schlicht functions and, more recently, in that of Schramm–Loewner evolution (SLE), their infinitesimal variations seem to have been overlooked. Our investigation of this topic is motivated by an effort to understand the large deviations of SLE at a deeper level. Roughly speaking, an SLE curve is a non-self-intersecting curve whose Loewner chain is driven by a constant multiple of Brownian motion. We refer the reader to the textbooks [249, 167, 154, 34] for detailed introductions to SLE and its applications.

Theorem 5.1.2 implies that the Loewner energy is at the same time the action functional of an SLE loop [244, 63] and the Kähler potential on the Weil–Petersson Teichmüller space $T_0(1)$ [245, 239], thus building a bridge between two fundamentally different perspectives on the geometry of the space of Jordan curves in $\hat{\mathbb{C}}$. On the SLE side, Jordan curves are viewed as dynamically growing slits, which are naturally described through the language of

Loewner chains. The link from SLE to Loewner energy comes from stochastic analysis and the fact that the action functional of a Brownian motion is its Dirichlet energy. On the Kähler geometry side, there are no dynamics. Instead, all geometric structures are expressed infinitesimally on the tangent spaces of $T_0(1)$. The reason behind the fact that the action functional of SLE coincides with the Kähler potential of the unique homogeneous Kähler metric on $T_0(1)$ remains a mystery. See [248] for an expository article on this link.

Our motivation lies in building tools that can be used to reconcile these two distinct viewpoints. The current work serves as the first step in one of the possible directions to this end by elucidating which infinitesimal variations of the Loewner driving function correspond to those on $T_0(1)$. Natural questions going forward are how the complex structure, the symplectic form, the Weil–Petersson metric, and the group structure on $T_0(1)$ are encoded in the Loewner driving function. Through these identifications, there is hope to build a more robust connection between random conformal geometry and Teichmüller theory and shed new light on their relationship.

There are yet other possible avenues in the study of quasiconformal deformations of SLE. For example, there is an interesting open question [49, Conj. 7.1] to identify the conformal dimension of SLE, which is the minimal Hausdorff dimension of the image of an SLE curve under quasiconformal mappings. The relevance of our work in this direction is that some of our results are sufficiently general to be applied in this context. For instance, the variational formula for the Loewner driving function (Theorem 5.1.1) does not assume any regularity on the driving function, so it can be applied even when the driving function is a Brownian motion. On the other hand, there is room for improvement in our results. The most obvious limitation is that we require the support of the Beltrami differential to be away from the curve so as not to deal with improper integrals. The deformations we considered in this work are thus conformal in a neighborhood of the curve, and in this regard, we are still in the same setup as in the conformal restriction of SLE considered in [166]. To understand general quasiconformal deformations of SLE, we need to allow the supports of Beltrami differentials to touch the curve. We think this is an interesting question that may require taking into account the stochastic nature of SLE.

5.2 Deformation of chords in the half-plane

5.2.1 Variation of the chordal Loewner driving function

Let $\eta : (0, +\infty) \to \mathbb{H}$ be a continuously parametrized simple chord from 0 to ∞. For general η, there exists a unique conformal map $\mathbb{H} \smallsetminus \eta(0,t] \to \mathbb{H}$ with the expansion

$$z + \frac{2a_t}{z} + O\left(\frac{1}{z^2}\right) \quad \text{as} \quad z \to \infty$$

where $a_t > 0$ is a constant known as (one-half of) the half-plane capacity of the curve $\eta[0,t]$. We call $t \mapsto a_t$ the capacity function of η. Let us denote $T_+ := \lim_{t\to+\infty} a_t \in (0, +\infty]$. [2] After an appropriate time change, we may assume that η is parametrized by its (one-half) half-plane capacity: i.e., $a_t = t$. The family of uniformizing maps $g_t : \mathbb{H} \smallsetminus \eta(0,t] \to \mathbb{H}$ satisfying

$$g_t(z) = z + \frac{2t}{z} + O\left(\frac{1}{z^2}\right) \quad \text{as} \quad z \to \infty$$

under this capacity parametrization is called the Loewner chain corresponding to η. The function $[0, T_+) \to \mathbb{R}$, $t \mapsto \lambda_t := g_t(\eta(t))$ is called the driving function of the curve η.

We consider deformations of η under quasiconformal self-maps of $\mathbb{H}$. Let $\nu \in L^\infty(\mathbb{H})$ be a complex-valued function with compact support in $\mathbb{H} \smallsetminus \eta$. The measurable Riemann mapping theorem states that for $\varepsilon \in \mathbb{R}$ with $|\varepsilon| < 1/\|\nu\|_\infty$, there exists a unique quasiconformal

2. See [165, Theorem 1] for an example where $T_+ < \infty$.

self-map $\psi^{\varepsilon\nu} : \mathbb{H} \to \mathbb{H}$ which solves the Beltrami equation

$$\partial_{\bar{z}}\psi^{\varepsilon\nu} = (\varepsilon\nu)\partial_z\psi^{\varepsilon\nu} \tag{5.2.1}$$

and has $\psi^{\varepsilon\nu}(0) = 0$ and $\psi^{\varepsilon\nu}(z) - z = O(1)$ as $z \to \infty$. We adopt the following notations, illustrated in Figure 5.1.

- Denote the deformed chord by $\eta^{\varepsilon\nu} := \psi^{\varepsilon\nu} \circ \eta$.
- Let $g_t^{\varepsilon\nu} : \mathbb{H} \smallsetminus \eta^{\varepsilon\nu}(0,t] \to \mathbb{H}$ be the Loewner chain associated with the deformed curve $\eta^{\varepsilon\nu}[0,t]$.
- Denote the driving function of $\eta^{\varepsilon\nu}$ by $\lambda_t^{\varepsilon\nu} := g_t^{\varepsilon\nu}(\eta^{\varepsilon\nu}(t))$.
- Note that $\eta^{\varepsilon\nu}$ is not necessarily parametrized by its half-plane capacity. Denote the capacity function of $\eta^{\varepsilon\nu}[0,t]$ by $a_t^{\varepsilon\nu}$, so that $g_t^{\varepsilon\nu}(z) = z + 2a_t^{\varepsilon\nu}z^{-1} + O(z^{-2})$ as $z \to \infty$.

This section aims to prove Theorem 5.1.1. For this, we first express $\lambda_t^{\varepsilon\nu}$ and $a_t^{\varepsilon\nu}$ in terms of the pre-Schwarzian and Schwarzian derivatives of an appropriately conjugated Loewner chain (Lemma 5.2.1). We then find the first variations of these derivatives using the measurable Riemann mapping theorem (Proposition 5.2.2).

The centered Loewner chain

$$f_t(z) := g_t(z) - \lambda_t, \tag{5.2.2}$$

satisfies $f_t(\eta(t)) = 0$ and

$$f_t(z) = z - \lambda_t + \frac{2t}{z} + O\left(\frac{1}{z^2}\right) \qquad \text{as } z \to \infty. \tag{5.2.3}$$

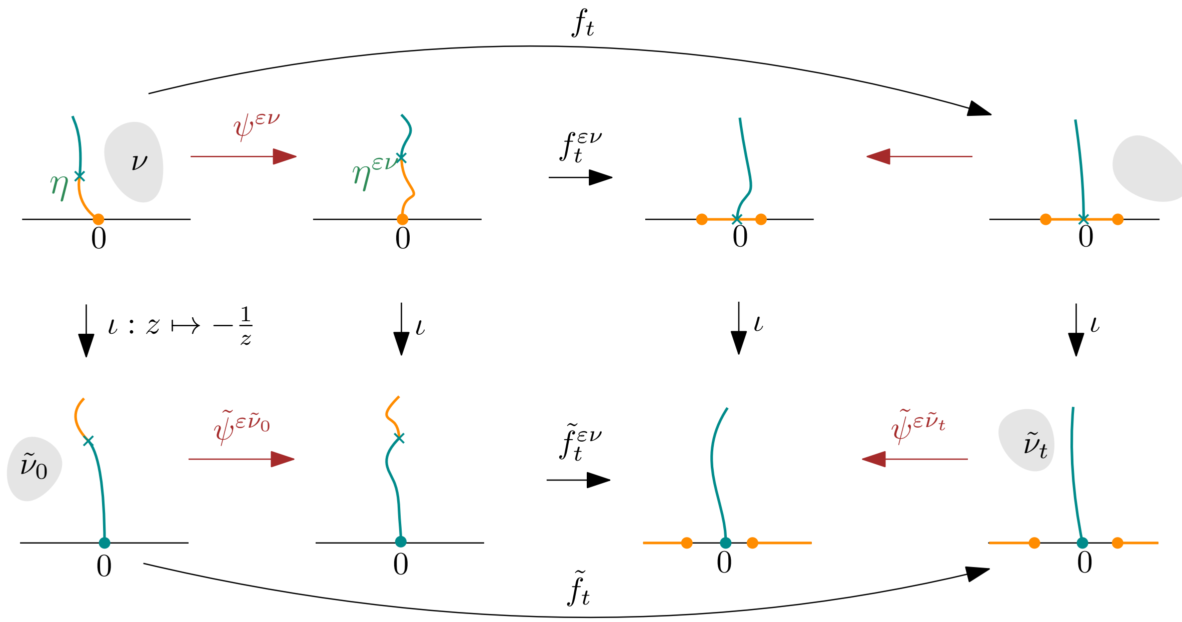

Figure 5.1: A commutative diagram illustrating the quasiconformal maps and related Loewner chains in Section 5.2.1. The gray shaded areas denote the support of the Beltrami differentials. The arrows in red are quasiconformal maps, and those in black are conformal maps.

Let $\iota(z) := -1/z$ be the inversion map. Define the inverted Loewner chain by

$$\tilde{f}_t(z) := \iota \circ f_t \circ \iota(z) = -\frac{1}{f_t(-1/z)}. \tag{5.2.4}$$

Then, $\tilde{f}_t : \mathbb{H} \setminus (\iota \circ \eta(0,t]) \to \mathbb{H}$ is the uniformizing map with normalization $\tilde{f}_t(0) = 0$, $\tilde{f}_t'(0) = 1$, and $\tilde{f}_t(\iota \circ \eta(t)) = \infty$. Combining the expansion (5.2.3) of f_t at ∞ with (5.2.4), we see that as $z \to 0$,

$$\tilde{f}_t(z) = -\frac{1}{-z^{-1} - \lambda_t - 2tz + O(z^2)} = z - \lambda_t z^2 + (\lambda_t^2 - 2t)z^3 + O(z^4). \tag{5.2.5}$$

Similarly, define

$$f_t^{\varepsilon\nu}(z) := g_t^{\varepsilon\nu}(z) - \lambda_t^{\varepsilon\nu} \quad \text{and} \quad \tilde{f}_t^{\varepsilon\nu}(z) := \iota \circ f_t^{\varepsilon\nu} \circ \iota(z). \tag{5.2.6}$$

A calculation analogous to (5.2.5) using the series expansion of $g_t^{\varepsilon\nu}$ at ∞ leads to

$$\tilde{f}_t^{\varepsilon\nu}(z) = z - \lambda_t^{\varepsilon\nu} z^2 + ((\lambda_t^{\varepsilon\nu})^2 - 2a_t^{\varepsilon\nu})z^3 + O(z^4) \quad \text{as } z \to 0. \tag{5.2.7}$$

By the Schwarz reflection principle, $\tilde{f}_t$ and $\tilde{f}_t^{\varepsilon\nu}$ extend respectively to conformal maps on $\mathbb{C} \setminus \iota(\eta(0,t] \cup \overline{\eta(0,t]})$ and $\mathbb{C} \setminus \iota(\eta^{\varepsilon\nu}(0,t] \cup \overline{\eta^{\varepsilon\nu}(0,t]})$, where $\bar{\cdot}$ denotes the complex conjugate. In particular, they are conformal in some neighborhood of 0.

Recall that the pre-Schwarzian (also known as non-linearity) and Schwarzian derivatives of a conformal map φ are, respectively,

$$\mathcal{N}\varphi = \frac{\varphi''}{\varphi'} \quad \text{and} \quad \mathcal{S}\varphi = \frac{\varphi'''}{\varphi'} - \frac{3}{2}\left(\frac{\varphi''}{\varphi'}\right)^2. \tag{5.2.8}$$

The chain rules for the pre-Schwarzian and Schwarzian derivatives are

$$\mathcal{N}[f \circ g] = ((\mathcal{N}f) \circ g)g' + \mathcal{N}g \quad \text{and} \quad \mathcal{S}[f \circ g] = ((\mathcal{S}f) \circ g)(g')^2 + \mathcal{S}g \tag{5.2.9}$$

for any conformal maps f and g such that $f \circ g$ is well-defined.

Lemma 5.2.1. *Consider $\tilde{f}_t$ and $\tilde{f}_t^{\varepsilon\nu}$ as conformal maps extended by reflection to a neighborhood of 0. Then,*

$$\lambda_t = -\frac{1}{2}\mathcal{N}\tilde{f}_t(0), \quad \lambda_t^{\varepsilon\nu} = -\frac{1}{2}\mathcal{N}\tilde{f}_t^{\varepsilon\nu}(0), \quad \textit{and} \quad a_t^{\varepsilon\nu} = -\frac{1}{12}\mathcal{S}\tilde{f}_t^{\varepsilon\nu}(0). \tag{5.2.10}$$

Proof. The lemma follows from inspecting the coefficients of (5.2.5) and (5.2.7). □

Let $\varepsilon\tilde{\nu}_t$ be the Beltrami coefficient of the quasiconformal map

$$\tilde{\psi}^{\varepsilon\tilde{\nu}_t} := \tilde{f}_t^{\varepsilon\nu} \circ \iota \circ \psi^{\varepsilon\nu} \circ \iota \circ \tilde{f}_t^{-1} = \iota \circ f_t^{\varepsilon\nu} \circ \psi^{\varepsilon\nu} \circ f_t^{-1} \circ \iota. \tag{5.2.11}$$

In particular, $\tilde{\psi}^{\varepsilon\tilde{\nu}_0} = \iota \circ \psi^{\varepsilon\nu} \circ \iota$ is the quasiconformal map which deforms $\iota \circ \eta$ to $\iota \circ \eta^{\varepsilon\nu}$.

Note that $\tilde{\psi}^{\varepsilon\tilde{\nu}_t}$ is conformal in a neighborhood of 0 and satisfies $\tilde{\psi}^{\varepsilon\tilde{\nu}_t}(0) = 0$, $(\tilde{\psi}^{\varepsilon\tilde{\nu}_t})'(0) = 1$, and $\tilde{\psi}^{\varepsilon\tilde{\nu}_t}(\infty) = \infty$.

Proposition 5.2.2. *Let $\tilde{\nu}_t$ be the Beltrami differential defined above. Then,*

$$\begin{aligned} \frac{\partial \lambda_t^{\varepsilon\nu}}{\partial\varepsilon}\bigg|_{\varepsilon=0} &= \frac{2}{\pi}\operatorname{Re}\int_{\mathbb{H}} \frac{\tilde{\nu}_t(z) - \tilde{\nu}_0(z)}{z^3}\,\mathrm{d}^2 z \\ &= \frac{2}{\pi}\operatorname{Re}\int_{\mathbb{H}} \tilde{\nu}_0(z)\left(\frac{\tilde{f}_t'(z)^2}{\tilde{f}_t(z)^3} - \frac{1}{z^3}\right)\mathrm{d}^2 z \\ &= -\frac{2}{\pi}\operatorname{Re}\int_{\mathbb{H}} \nu(z)\left(\frac{f_t'(z)^2}{f_t(z)} - \frac{1}{z}\right)\mathrm{d}^2 z \end{aligned} \tag{5.2.12}$$

and

$$\begin{aligned} \frac{\partial a_t^{\varepsilon\nu}}{\partial t}\bigg|_{\varepsilon=0} &= \frac{1}{\pi}\operatorname{Re}\int_{\mathbb{H}} \frac{\tilde{\nu}_t(z) - \tilde{\nu}_0(z)}{z^4}\,\mathrm{d}^2 z \\ &= \frac{1}{\pi}\operatorname{Re}\int_{\mathbb{H}} \tilde{\nu}_0(z)\left(\frac{\tilde{f}_t'(z)^2}{\tilde{f}_t(z)^4} - \frac{1}{z^4}\right)\mathrm{d}^2 z \\ &= \frac{1}{\pi}\operatorname{Re}\int_{\mathbb{H}} \nu(z)\big(f_t'(z)^2 - 1\big)\,\mathrm{d}^2 z. \end{aligned} \tag{5.2.13}$$

Proof of Theorem 5.1.1. It suffices to substitute $f_t(z) = g_t(z) - \lambda_t$ in Proposition 5.2.2. $\square$

Proof of Proposition 5.2.2. We can extend $\tilde{\psi}^{\varepsilon\tilde{\nu}_t}$ to a quasiconformal self-map of the Riemann sphere $\hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ by reflecting it with respect to the real axis. The Beltrami coefficient for this extension of $\tilde{\psi}^{\varepsilon\tilde{\nu}_t}$ is $\varepsilon\hat{\nu}_t$ where

$$\hat{\nu}_t(z) := \begin{cases} \tilde{\nu}_t(z) & \text{if } z \in \mathbb{H}, \\ 0 & \text{if } z \in \mathbb{R}, \\ \overline{\tilde{\nu}_t(\overline{z})} & \text{if } z \in \mathbb{H}^*. \end{cases} \tag{5.2.14}$$

Then, by the measurable Riemann mapping theorem,

$$\tilde{\psi}^{\varepsilon\tilde{\nu}_t}(\zeta) = \zeta - \frac{\varepsilon}{\pi}\int_{\mathbb{C}} \hat{\nu}_t(z)\left(\frac{1}{z-\zeta} - \frac{1}{z} - \frac{\zeta}{z^2}\right)\mathrm{d}^2 z + o(\varepsilon) \tag{5.2.15}$$

locally uniformly in $\zeta \in \mathbb{C}$ as $\varepsilon \to 0$. Moreover, since ∂_ε commutes with ∂_ζ when applied to $\tilde{\psi}^{\varepsilon\tilde{\nu}_t}$ and $\hat{\nu}_t$ has a compact support in $\mathbb{C} \setminus \{0\}$, we have

$$\frac{\partial}{\partial\varepsilon}\bigg|_{\varepsilon=0} \mathcal{N}\tilde{\psi}^{\varepsilon\tilde{\nu}_t}(0) = -\frac{2}{\pi}\int_{\mathbb{C}} \frac{\hat{\nu}_t(z)}{z^3}\,\mathrm{d}^2 z = -\frac{4}{\pi}\operatorname{Re}\int_{\mathbb{H}} \frac{\tilde{\nu}_t(z)}{z^3}\mathrm{d}^2 z, \tag{5.2.16}$$

$$\frac{\partial}{\partial\varepsilon}\bigg|_{\varepsilon=0} \mathcal{S}\tilde{\psi}^{\tilde{\nu}_t}(0) = -\frac{6}{\pi}\int_{\mathbb{C}} \frac{\hat{\nu}_t(z)}{z^4}\,\mathrm{d}^2 z = -\frac{12}{\pi}\operatorname{Re}\int_{\mathbb{H}} \frac{\tilde{\nu}_t(z)}{z^4}\,\mathrm{d}^2 z. \tag{5.2.17}$$

Since $\tilde{f}_t^{\varepsilon\nu} = \tilde{\psi}^{\varepsilon\tilde{\nu}_t} \circ \tilde{f}_t \circ (\tilde{\psi}^{\varepsilon\tilde{\nu}_0})^{-1}$ and $\tilde{\psi}^{\varepsilon\tilde{\nu}_t}(z)$, $\tilde{f}_t(z)$, and $\tilde{\psi}^{\varepsilon\tilde{\nu}_0}(z)$ all behave as $z + o(z)$ as $z \to 0$, we have from Lemma 5.2.1 and the chain rules (5.2.9) that

$$\begin{aligned} -2\lambda_t^{\varepsilon\mu} &= \mathcal{N}\tilde{f}_t^{\varepsilon\mu}(0) = \mathcal{N}\tilde{\psi}^{\varepsilon\tilde{\nu}_t}(0) + \mathcal{N}\tilde{f}_t(0) - \mathcal{N}\tilde{\psi}^{\varepsilon\tilde{\nu}_0}(0) \\ &= \mathcal{N}\tilde{\psi}^{\varepsilon\tilde{\nu}_t}(0) - 2\lambda_t - \mathcal{N}\tilde{\psi}^{\varepsilon\tilde{\nu}_0}(0), \end{aligned} \tag{5.2.18}$$

$$\begin{aligned} -12a_t^{\varepsilon\mu} &= \mathcal{S}\tilde{f}_t^{\varepsilon\mu}(0) = \mathcal{S}\psi_t^{\varepsilon\nu_t}(0) + \mathcal{S}\tilde{f}_t(0) - \mathcal{S}\psi^{\varepsilon\nu_0}(0) \\ &= \mathcal{S}\psi_t^{\varepsilon\nu_t}(0) - 12t - \mathcal{S}\psi^{\varepsilon\nu_0}(0). \end{aligned} \tag{5.2.19}$$

Combining these with (5.2.16) and (5.2.17), we obtain the first equalities in (5.2.12) and (5.2.13).

Observe that $\tilde{\psi}^{\varepsilon\tilde{\nu}_t} = \tilde{f}_t^{\varepsilon\nu} \circ \tilde{\psi}^{\varepsilon\tilde{\nu}_0} \circ \tilde{f}_t^{-1}$, where $\tilde{f}_t^{\varepsilon\nu}$ and $\tilde{f}_t^{-1}$ are conformal maps. Hence, by the composition rule for Beltrami coefficients,

$$\tilde{\nu}_t(\tilde{f}_t(z)) = \tilde{\nu}_0(z)\frac{\tilde{f}_t'(z)^2}{|\tilde{f}_t'(z)|^2}. \tag{5.2.20}$$

Substituting (5.2.20) into (5.2.16) and (5.2.17), we have that

$$\frac{\partial}{\partial\varepsilon}\bigg|_{\varepsilon=0} \mathcal{N}\tilde{\psi}^{\varepsilon\tilde{\nu}_t}(0) = -\frac{4}{\pi}\operatorname{Re}\int_{\mathbb{H}} \tilde{\nu}_0(z)\frac{\tilde{f}_t'(z)^2}{\tilde{f}_t(z)^3}\,\mathrm{d}^2 z, \tag{5.2.21}$$

$$\frac{\partial}{\partial\varepsilon}\bigg|_{\varepsilon=0} \mathcal{S}\tilde{\psi}^{\varepsilon\tilde{\nu}_t}(0) = -\frac{12}{\pi}\operatorname{Re}\int_{\mathbb{H}} \tilde{\nu}_0(z)\frac{\tilde{f}_t'(z)^2}{\tilde{f}_t(z)^4}\,\mathrm{d}^2 z. \tag{5.2.22}$$

We thus have the second equalities in (5.2.12) and (5.2.13).

Finally, recall that $\tilde{\psi}^{\varepsilon\tilde{\nu}_0} = \iota \circ \psi^{\varepsilon\nu} \circ \iota$. Since the inversion map $\iota(\zeta) = -1/\zeta$ is conformal,

$$\tilde{\nu}_0(-1/\zeta) = \nu(\zeta)\frac{|\zeta|^4}{\zeta^4}. \tag{5.2.23}$$

Recall that $\tilde{f}_t(z) = -1/f_t(-1/z)$, and hence $\tilde{f}_t'(z) = f_t'(-1/z)/(zf_t(-1/z))^2$. Substituting $z = -1/\zeta$ in (5.2.21) and (5.2.22), we obtain the final equalities in (5.2.12) and (5.2.13). □

5.2.2 Variation of chordal Loewner energy

Let $\eta : (0, T_+) \to \mathbb{H}$ be a simple chord from 0 to ∞ parametrized by half-plane capacity (i.e., $a_t = t$) and $t \mapsto \lambda_t$ be its Loewner driving function. The Loewner energy of η (resp. the partial Loewner energy of η up to time $T \in (0, T_+)$) is

$$I^C(\eta) = \frac{1}{2}\int_0^{T_+} \dot{\lambda}_t^2 \,\mathrm{d}t \qquad \text{resp.} \qquad I^C(\eta(0,T]) = \frac{1}{2}\int_0^{T} \dot{\lambda}_t^2 \,\mathrm{d}t$$

if λ is absolutely continuous and $\dot{\lambda}$ is its almost everywhere defined derivative with respect to t. We set $I^C(\eta) = \infty$ if λ is not absolutely continuous. If $I^C(\eta) < +\infty$, then $T_+ = +\infty$; see [247, Theorem 2.4].

We also define the Loewner energy of a simple chord η in a simply connected domain D connecting two distinct prime ends a, b as

$$I^C_{D;a,b}(\eta) := I^C(\varphi(\eta))$$

where φ is any conformal map $D \to \mathbb{H}$ with $\varphi(a) = 0$ and $\varphi(b) = \infty$. The partial Loewner energy in $(D; a, b)$ is defined similarly.

When λ_t is absolutely continuous, we can compute the first variations of $\dot{\lambda}_t^{\varepsilon\nu}$ and $\dot{a}_t^{\varepsilon\nu}$.

Proposition 5.2.3. *For all $\varepsilon \in (-1/\|\nu\|_\infty, 1/\|\nu\|_\infty)$, the functions $t \mapsto \lambda_t^{\varepsilon\nu} - \lambda_t$ and*

$t \mapsto a_t^{\varepsilon\nu}$ are continuously differentiable. Furthermore, if $t \mapsto \lambda_t$ is absolutely continuous, then $t \mapsto \lambda_t^{\varepsilon\nu}$ is also absolutely continuous and, for almost every t,

$$\left.\frac{\partial \dot{\lambda}_t^{\varepsilon\nu}}{\partial \varepsilon}\right|_{\varepsilon=0} = \frac{1}{\pi}\,\mathrm{Re}\int_{\mathbb{H}} \nu(z)\left(12\frac{f_t'(z)^2}{f_t(z)^3} - 2\dot{\lambda}_t \frac{f_t'(z)^2}{f_t(z)^2}\right) \mathrm{d}^2 z \tag{5.2.24}$$

and

$$\left.\frac{\partial \dot{a}_t^{\varepsilon\nu}}{\partial \varepsilon}\right|_{\varepsilon=0} = -\frac{4}{\pi}\,\mathrm{Re}\int_{\mathbb{H}} \nu(z)\frac{f_t'(z)^2}{f_t(z)^2}\,\mathrm{d}^2 z. \tag{5.2.25}$$

Proof. From the Loewner equation $\partial_t g_t(z) = 2/(g_t(z) - \lambda_t)$, we have

$$\partial_t(f_t(z) + \lambda_t) = \frac{2}{f_t(z)} \quad \text{and} \quad \partial_t f_t'(z) = -\frac{2f_t'(z)}{f_t(z)^2}.$$

Recalling $\tilde{f}_t(z) = -1/f_t(-1/z)$, it follows that $\tilde{f}_t'(z)$ is continuously differentiable in t. From (5.2.14) and (5.2.20), we see that $(\varepsilon, t) \mapsto \varepsilon\hat{\nu}_t$ is continuously differentiable. Then, $\lambda_t^{\varepsilon\nu} - \lambda_t = -\frac{1}{2}(\mathcal{N}\tilde{\psi}^{\varepsilon\tilde{\nu}_t}(0) - \mathcal{N}\tilde{\psi}^{\varepsilon\tilde{\nu}_0}(0))$ and $a_t^{\varepsilon\nu} - t = -\frac{1}{12}(\mathcal{S}\tilde{\psi}^{\varepsilon\tilde{\nu}_t}(0) - \mathcal{S}\tilde{\psi}^{\varepsilon\tilde{\nu}_0}(0))$ are continuously differentiable in the same variables [4].

We can check directly from the integral representations of $\partial_\varepsilon \mathcal{N}\tilde{\psi}^{\varepsilon\tilde{\nu}_t}(0)$ and $\partial_\varepsilon \mathcal{S}\tilde{\psi}^{\varepsilon\tilde{\nu}_t}(0)$ that they are continuously differentiable in t. If λ_t is absolutely continuous, then we have from (5.2.12) that for almost every t,

$$\begin{aligned}\left.\frac{\partial \dot{\lambda}_t^{\varepsilon\nu}}{\partial \varepsilon}\right|_{\varepsilon=0} &= -\frac{1}{2}\left.\frac{\partial^2(\mathcal{N}\tilde{\psi}^{\varepsilon\tilde{\nu}_t}(0))}{\partial\varepsilon\partial t}\right|_{\varepsilon=0} = -\frac{1}{2}\left.\frac{\partial^2(\mathcal{N}\tilde{\psi}^{\varepsilon\tilde{\nu}_t}(0))}{\partial t\partial\varepsilon}\right|_{\varepsilon=0} \\ &= -\frac{2}{\pi}\,\mathrm{Re}\int_{\mathbb{H}} \nu(z)\partial_t\left(\frac{f_t'(z)^2}{f_t(z)}\right) \mathrm{d}^2 z.\end{aligned}$$

Similarly, (5.2.13) implies

$$\left.\frac{\partial \dot{a}_t^{\varepsilon\nu}}{\partial \varepsilon}\right|_{\varepsilon=0} = \frac{1}{\pi}\,\mathrm{Re}\int_{\mathbb{H}} \nu(z)\,\partial_t\big(f_t'(z)^2\big)\,\mathrm{d}^2 z.$$

Using the formulas for $\partial_t f_t$ and $\partial_t f_t'$ above, we have

$$\partial_t\big(f_t'(z)^2\big) = -4\frac{f_t'(z)^2}{f_t(z)^2} \quad \text{and} \quad \partial_t\left(\frac{f_t'(z)^2}{f_t(z)}\right) = -6\frac{f_t'(z)^2}{f_t(z)^3} + \dot\lambda_t\frac{f_t'(z)^2}{f_t(z)^2}.$$

This completes the proof. □

Proposition 5.2.3 allows us to compute the first variation of the chordal Loewner energy for a finite portion of the chord $\eta^{\varepsilon\nu}$.

Corollary 5.2.4. *Let $T \in (0, T_+)$. Suppose λ_t is absolutely continuous on $[0,T]$ and $\dot\lambda_t \in L^2([0,T])$. Then,*

$$\frac{\partial}{\partial\varepsilon}\bigg|_{\varepsilon=0} I^C(\eta^{\varepsilon\nu}(0,T]) = \frac{12}{\pi}\operatorname{Re}\int_{\mathbb{H}}\nu(z)\int_0^T \dot\lambda_t\frac{f_t'(z)^2}{f_t(z)^3}\,\mathrm{d}t\,\mathrm{d}^2 z. \tag{5.2.26}$$

Proof. From Proposition 5.2.3, we see that $(\dot\lambda_t^{\varepsilon\nu})^2/\dot a_t^{\varepsilon\nu}$ is integrable on $[0,T]$ whenever $|\varepsilon| < 1/\|\nu\|_\infty$. Moreover, we obtain that for almost every $t\in[0,T]$,

$$\frac{\partial}{\partial\varepsilon}\bigg|_{\varepsilon=0}\frac{(\dot\lambda_t^{\varepsilon\nu})^2}{\dot a_t^{\varepsilon\nu}} = 2\dot\lambda_t\frac{\partial\dot\lambda_t^{\varepsilon\nu}}{\partial\varepsilon}\bigg|_{\varepsilon=0} - \dot\lambda_t^2\frac{\partial\dot a_t^{\varepsilon\nu}}{\partial\varepsilon}\bigg|_{\varepsilon=0} = \frac{24\dot\lambda_t}{\pi}\operatorname{Re}\int_{\mathbb{H}}\nu(z)\frac{f_t'(z)^2}{f_t(z)^3}\,\mathrm{d}^2 z.$$

Since ν is compactly supported in $\mathbb{H}\smallsetminus\eta$, the integral on the right-hand side is continuous in t and hence bounded on $[0,T]$. Using the Leibniz integral rule, we conclude that

$$\begin{aligned} I^C(\eta^{\varepsilon\nu}(0,T]) &= \frac{1}{2}\int_0^T\left|\frac{\mathrm{d}\lambda_t^{\varepsilon\nu}}{\mathrm{d}a_t^{\varepsilon\nu}}\right|^2\mathrm{d}a_t^{\varepsilon\nu} = \frac{1}{2}\int_0^T\frac{(\dot\lambda_t^{\varepsilon\nu})^2}{\dot a_t^{\varepsilon\nu}}\,\mathrm{d}t \\ &= I^C(\eta[0,T]) + \varepsilon\int_0^T\frac{12\dot\lambda_t}{\pi}\operatorname{Re}\left[\int_{\mathbb{H}}\nu(z)\frac{f_t'(z)^2}{f_t(z)^3}\mathrm{d}^2 z\right]\mathrm{d}t + o(\varepsilon) \end{aligned}$$

as $\varepsilon\to 0$. □

5.3 Deformation of Weil–Petersson quasicircles

In this section, we consider the Loewner chain for a Jordan curve by conjugating the chordal Loewner chain by $z \mapsto z^2$. This simple operation relates the integrand in (5.2.26) to a Schwarzian derivative (Proposition 5.3.8) which leads to the proof of Theorem 5.1.4.

Convention. We take the branch of the complex square root function $\sqrt{z}$ (or $z^{1/2}$) on $\mathbb{C}$ to be the one with the image in $\mathbb{H} \cup \mathbb{R}_+$.

5.3.1 Loop driving function

We first recall the definition of the Loewner driving function for a Jordan curve. See Figure 5.2 for an illustration of the maps used in Section 5.3.

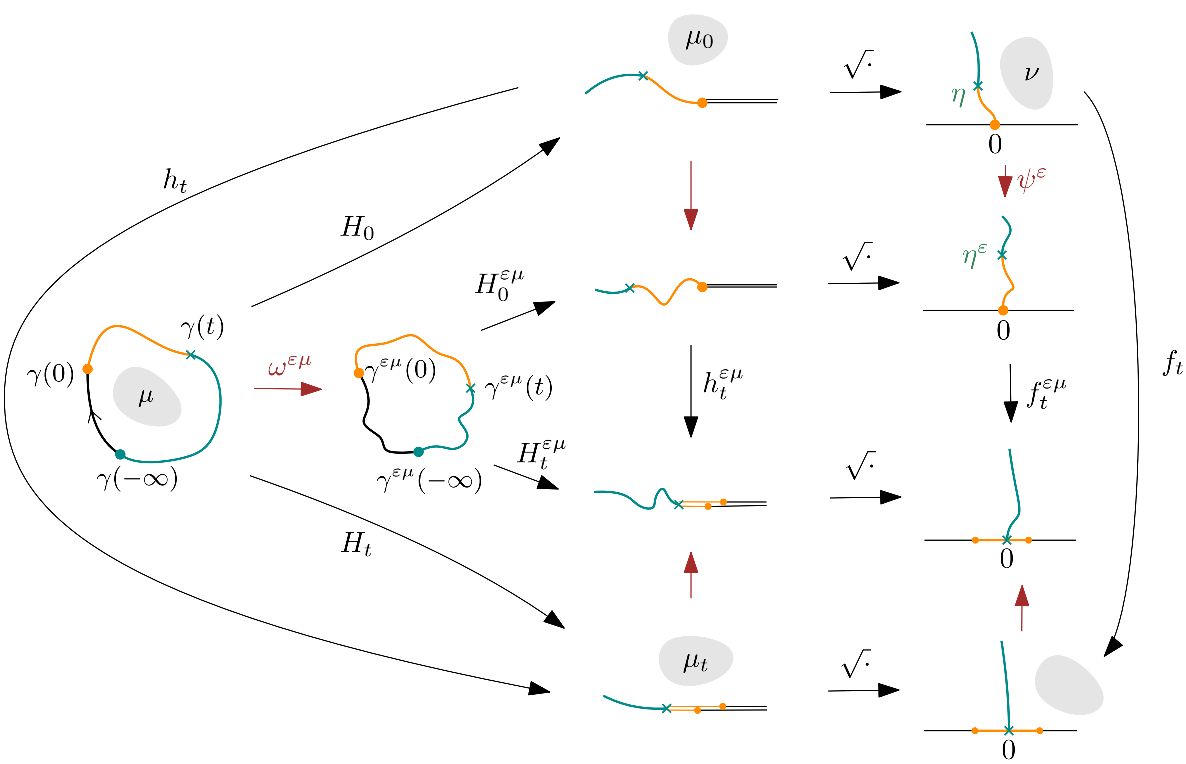

Figure 5.2: A commutative diagram illustrating the quasiconformal maps and related conformal mapping-out functions in Section 5.3. The gray shaded areas denote the support of the Beltrami differentials. The arrows in red are quasiconformal maps, and those in black are conformal maps.

Let $\gamma : [-\infty, +\infty] \to \hat{\mathbb{C}}$ be a Jordan curve where $\gamma(-\infty) = \gamma(+\infty)$. We choose a family of uniformizing maps $H_t : \hat{\mathbb{C}} \smallsetminus \gamma[-\infty, t] \to \mathbb{C} \smallsetminus \mathbb{R}_+$ such that $H_t(\gamma(t)) = 0$ and $H_t(\gamma(+\infty)) = \infty$ for each $t \in \mathbb{R}$. Note that each H_t is unique up to a real multiplicative factor. We fix a consistent normalization such that for every $s < t$, $H_t \circ H_s^{-1}(z) = z + o(z)$ as $z \to \infty$. Note that it suffices to fix the map H_0 and then normalize H_t for $t \neq 0$ so that $H_t \circ H_0^{-1}$ has the correct asymptotic behavior. This is possible since if we write

$$h_t = H_t \circ H_0^{-1} \quad \text{and} \quad f_t(z) = \sqrt{h_t(z^2)}, \tag{5.3.1}$$

then f_t is a conformal map taking a some neighborhood of ∞ in $\mathbb{H}$ to another neighborhood of ∞ in $\mathbb{H}$. Schwarz reflection applied to f_t in a neighborhood of ∞ shows that f_t is holomorphic at ∞. Normalizing h_t as above is equivalent to normalizing f_t such that $f_t(z) = z + O(1)$ as $z \to \infty$.

Definition 5.3.1. Define the (loop) driving function $t \mapsto \lambda_t$ and the (loop) capacity function $t \mapsto a_t$ from the expansion

$$f_t(z) = z - \lambda_t + \frac{2a_t}{z} + O\left(\frac{1}{|z|^2}\right) \quad \text{as } z \to \infty. \tag{5.3.2}$$

We always have $\lambda_0 = a_0 = 0$ and $t \mapsto a_t$ is continuous and strictly increasing. If $a_t \to \pm\infty$ when $t \to \pm\infty$, then we can reparameterize γ such that $a_t = t$ for every $t \in \mathbb{R}$. In this case, we say that γ is capacity-parametrized by $\mathbb{R}$.[3] We remind the reader that the capacity parametrization and the corresponding driving function depend on the choices of the orientation of γ, $\gamma(\pm\infty), \gamma(0)$, and H_0.

The reader may wonder about the different behaviors of the map f_t depending on the

3. Similar to the chordal case, a general Jordan curve γ may not have infinite capacity at either end of the root. A priori, $a_t \to T_\pm$ as $t \to \pm\infty$ where $T_+ \in (0, +\infty]$ and $T_- \in [-\infty, 0)$. In other words, γ may be capacity-parametrized by a strict sub-interval of $\mathbb{R}$. However, we show in Lemma 5.4.1 that if $I^L(\gamma) < +\infty$ (see Definition 5.3.5), then the capacity parametrization of γ must be defined on all of $\mathbb{R}$. Throughout Section 5.3, we assume that γ is capacity-parametrized by $\mathbb{R}$.

sign of t, which seem to give a different meaning to the term "capacity." This difference is not fundamental as the designation of the point of zero capacity on γ is artificial. We shall view the capacity given by f_t as the "relative capacity" with respect to our choice of the part $\gamma[-\infty, 0]$. Precisely, it means the following.

Lemma 5.3.2. *Suppose γ is a Jordan curve, capacity-parametrized by $\mathbb{R}$, with driving function $t \mapsto \lambda_t$. Then, for every $s \in \mathbb{R}$, $\sqrt{H_s \circ \gamma(\cdot + s)}$ defined on $\mathbb{R}_+$ is a simple chord in $(\mathbb{H}; 0, \infty)$ parametrized by capacity. Moreover, its chordal driving function is given by $\lambda_{\cdot+s} - \lambda_s$.*

Proof. When $s = 0$, $\eta(\cdot) = \sqrt{H_0 \circ \gamma(\cdot)}$ is a simple chord in $\mathbb{H}$ from 0 to ∞. For $t > 0$, the conformal map h_t takes $H_0(\hat{\mathbb{C}} \smallsetminus \gamma[-\infty, t])$ onto $\mathbb{C} \smallsetminus \mathbb{R}_+$. Hence, f_t is a conformal map from $\mathbb{H} \smallsetminus \eta(0, t]$ onto $\mathbb{H}$. Therefore, when $t > 0$, (5.3.2) simply means that $2a_t = 2t$ is the half-plane capacity of $\eta[0, t]$ and $(\lambda_t)_{t \geq 0}$ is the chordal driving function of η by (5.2.3).

For general $s \in \mathbb{R}$, we note that $(f_{t+s} \circ f_s^{-1})_{t \geq 0}$ is a centered Loewner chain associated with the curve $\sqrt{H_s \circ \gamma(\cdot + s)}$. Moreover, (5.3.2) implies

$$f_{t+s} \circ f_s^{-1}(z) = z - (\lambda_{t+s} - \lambda_s) + \frac{2(a_{t+s} - a_s)}{z} + O\left(\frac{1}{z^2}\right) \quad \text{as } z \to \infty$$

for all $t \geq 0$. Thus, $t \mapsto \lambda_{t+s} - \lambda_s$ and $t \mapsto a_{t+s} - a_s$ are respectively the driving function and the capacity function corresponding to the chain $(f_{t+s} \circ f_s^{-1})_{t \geq 0}$. In particular, the assumption $a_t = t$ for all $t \in \mathbb{R}$ means that the chain $(f_{t+s} \circ f_s^{-1})_{t \geq 0}$ is also in capacity parametrization. $\square$

Remark 5.3.3. The loop driving function generalizes the chordal Loewner driving function. If η is a simple chord in $(\mathbb{H}; 0, \infty)$ with driving function $\lambda : \mathbb{R}_+ \to \mathbb{R}$, then the Jordan curve $\gamma := \eta(\cdot)^2 \cup \mathbb{R}_+$ with the same orientation as η (from 0 to ∞), root ∞, $\gamma(0) = 0$, and $H_0(z) = z$, has the driving function $(\tilde{\lambda}_t)_{t \in \mathbb{R}}$ where $\tilde{\lambda}_t = \lambda_t$ if $t \geq 0$ and $\tilde{\lambda}_t = 0$ if $t \leq 0$.

Remark 5.3.4. Let γ be a Jordan curve capacity-parametrized by $\mathbb{R}$ using the conformal map H_0. If $A : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ is a Möbius transformation, then the loop $t \mapsto \tilde{\gamma}(t) := A(\gamma(t))$ has the same driving function as γ when we choose its capacity parametrization using the conformal map $\tilde{H}_0 = H_0 \circ A^{-1}$. Moreover, the conformal maps corresponding to H_t and h_t for $\tilde{\gamma}$ are $\tilde{H}_t = H_t \circ A^{-1}$ and $\tilde{h}_t = \tilde{H}_t \circ \tilde{H}_0^{-1} = H_t \circ H_0^{-1} = h_t$. Hence, the map f_t remains unchanged, and so are the capacity and driving functions.

Definition 5.3.5. The Loewner energy of a Jordan curve γ is

$$I^L(\gamma) = \frac{1}{2} \int_{-\infty}^{+\infty} \dot{\lambda}_t^2 \, \mathrm{d}t,$$

where $(\lambda_t)_{t \in \mathbb{R}}$ is the driving function of γ described above. See also Lemma 5.4.1. Theorem 5.1.2 shows that this energy does not depend on the parametrization of the curve (but we will not use this fact in our proof).

The next corollary is immediate after Lemma 5.3.2.

Corollary 5.3.6. *For all $s < t$, the partial chordal Loewner energy of $\gamma(s,t]$ in the simply connected domain $\hat{\mathbb{C}} \smallsetminus \gamma[-\infty, s]$ is given by*

$$I^C_{\hat{\mathbb{C}} \smallsetminus \gamma[-\infty,s]}(\gamma(s,t]) = \frac{1}{2} \int_s^t \dot{\lambda}_r^2 \, \mathrm{d}r$$

where the slit domain $\hat{\mathbb{C}} \smallsetminus \gamma[-\infty, s]$ is always understood with the two marked prime ends being the two ends of $\gamma[-\infty, s]$.

Proof. It suffices to notice that $\sqrt{H_s(\cdot)}$ maps $\hat{\mathbb{C}} \smallsetminus \gamma[-\infty, s]$ conformally onto $\mathbb{H}$. Hence, the partial chordal Loewner energy of $\gamma(s,t]$ in $\hat{\mathbb{C}} \smallsetminus \gamma[-\infty, s]$ equals that of $\sqrt{H_s(\gamma(s,t])}$ in $\mathbb{H}$, which has the driving function $r \mapsto \lambda_{s+r} - \lambda_s$ defined for $r \in [0, t-s]$ by Lemma 5.3.2. $\square$

5.3.2 Variation of Loewner energy for a part of the quasicircle

We now consider deformations of a Jordan curve γ. Let $\mu \in L^\infty(\hat{\mathbb{C}})$ be a complex-valued function with compact support in $\hat{\mathbb{C}} \smallsetminus \gamma$. For $\varepsilon \in \mathbb{R}$ with $\|\varepsilon\mu\|_\infty < 1$, let $\omega^{\varepsilon\mu} : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ be a quasiconformal homeomorphism which satisfies the Beltrami equation

$$\partial_{\bar{z}}\omega^{\varepsilon\mu} = (\varepsilon\mu)\partial_z\omega^{\varepsilon\mu}.$$

Denote the deformation of γ under the quasiconformal map $\omega^{\varepsilon\mu}$ as

$$\gamma^{\varepsilon\mu} = \omega^{\varepsilon\mu} \circ \gamma.$$

Again we choose a family of uniformizing maps $H_t^{\varepsilon\mu} : \hat{\mathbb{C}} \smallsetminus \gamma^{\varepsilon\mu}[-\infty, t] \to \mathbb{C} \smallsetminus \mathbb{R}_+$ with $H_t^{\varepsilon\mu}(\gamma^{\varepsilon\mu}(t)) = 0$ and $H_t^{\varepsilon\mu}(\gamma^{\varepsilon\mu}(+\infty)) = \infty$, normalized so that $H_t^{\varepsilon\mu} \circ \omega^{\varepsilon\mu} \circ H_t^{-1}(z) = z + o(z)$ as $z \to \infty$ for each $t \in \mathbb{R}$.

We define analogously the chains $(h_t^{\varepsilon\mu})_{t\in\mathbb{R}}$ and $(f_t^{\varepsilon\mu})_{t\in\mathbb{R}}$, the driving function $(\lambda_t^{\varepsilon\mu})_{t\in\mathbb{R}}$, and the capacity function $(a_t^{\varepsilon\mu})_{t\in\mathbb{R}}$. That is,

$$h_t^{\varepsilon\mu} = H_t^{\varepsilon\mu} \circ (H_0^{\varepsilon\mu})^{-1} \quad \text{and} \quad f_t^{\varepsilon\mu}(z) = \sqrt{h_t^{\varepsilon\mu}(z^2)}.$$

Then, by our choice of normalization, $h_t^{\varepsilon\mu}(z) = z + o(z)$ and $f_t^{\varepsilon\mu}(z) = z + O(1)$ as $z \to \infty$. We define $\lambda_t^{\varepsilon\mu}$ and $a_t^{\varepsilon\mu}$ from the expansion

$$f_t^{\varepsilon\mu}(z) = z - \lambda_t^{\varepsilon\mu} + \frac{2a_t^{\varepsilon\mu}}{z} + O\left(\frac{1}{z^2}\right) \quad \text{as } z \to \infty.$$

Remark 5.3.7. The map $\omega^{\varepsilon\mu}$, and hence the Jordan curve $\gamma^{\varepsilon\mu}$, is unique only up to a post-composition by some Möbius transformation. The choice of $\omega^{\varepsilon\mu}$ does not affect our analysis, because the first step in it is always to apply the appropriately normalized uniformizing map

$H_t^{\varepsilon\mu}$ from $\hat{\mathbb{C}} \smallsetminus \gamma^{\varepsilon\mu}[-\infty, t]$ onto $\mathbb{C} \smallsetminus \mathbb{R}_+$.

In this subsection, we translate Corollary 5.2.4 into an analogous formula for the Weil–Peterson curve γ and its deformation $\gamma^{\varepsilon\mu}$. The following is the main result.

Proposition 5.3.8. *Suppose $s < t$ and $I^C_{\hat{\mathbb{C}} \smallsetminus \gamma[-\infty,s]}(\gamma(s,t]) < \infty$. Then,*

$$\frac{\partial}{\partial\varepsilon}\bigg|_{\varepsilon=0} I^C_{\hat{\mathbb{C}} \smallsetminus \gamma^{\varepsilon\mu}[-\infty,s]}(\gamma^{\varepsilon\mu}[s,t]) = -\frac{4}{\pi}\operatorname{Re}\int_{\mathbb{C}\smallsetminus\gamma} \mu(z)(\mathcal{S}H_t(z) - \mathcal{S}H_s(z))\,\mathrm{d}^2 z. \tag{5.3.3}$$

The following lemma is a straightforward calculation used in the proof of Proposition 5.3.8.

Lemma 5.3.9. *Suppose λ_t is absolutely continuous. Then, for each $z \in H_0(\hat{\mathbb{C}} \smallsetminus \gamma)$, the Schwarzian $\mathcal{S}h_t(z)$ is absolutely continuous in t. Moreover, for almost every t,*

$$\frac{\partial}{\partial t}\mathcal{S}h_t(z) = -\frac{3h_t'(z)^2}{4h_t(z)^{5/2}}\dot{\lambda}_t. \tag{5.3.4}$$

Proof. From the relation

$$\mathcal{S}h_{t+u}(z) = \mathcal{S}[h_{t+u}\circ h_t^{-1}](h_t(z)) \cdot h_t'(z)^2 + \mathcal{S}h_t(z),$$

we deduce

$$\frac{\partial}{\partial t}\mathcal{S}h_t(z) = \frac{\partial}{\partial u}\bigg|_{u=0}(\mathcal{S}h_{t+u}(z) - \mathcal{S}h_t(z)) = h_t'(z)^2 \cdot \frac{\partial}{\partial u}\bigg|_{u=0}\mathcal{S}[h_{t+u}\circ h_t^{-1}](h_t(z)).$$

Hence, it suffices to show that

$$\frac{\partial}{\partial u}\bigg|_{u=0}\mathcal{S}[h_{t+u}\circ h_t^{-1}](h_t(z)) = -\frac{3\dot{\lambda}_t}{4h_t(z)^{5/2}}. \tag{5.3.5}$$

To see this, note that $\tilde{f}_u := f_{t+u} \circ f_t^{-1}$ solves the Loewner equation (see Lemma 5.3.2)

$$\partial_u \tilde{f}_u(z) = \frac{2}{\tilde{f}_u(z)} - \dot{\lambda}_{t+u}.$$

Since $\tilde{h}_u(z) := h_{t+u} \circ h_t^{-1}(z) = \tilde{f}_u(\sqrt{z})^2$, we have

$$\partial_u \tilde{h}_u(z) = 2\tilde{f}_u(\sqrt{z})\partial_u \tilde{f}_u(\sqrt{z}) = 4 - 2\dot{\lambda}_{t+u}\tilde{h}_u(z)^{1/2}.$$

Then, because $\tilde{h}_0(z) = z$,

$$\left.\frac{\partial(\mathcal{S}\tilde{h}_u(z))}{\partial u}\right|_{u=0} = \left(\partial_u \tilde{h}_u(z)\big|_{u=0}\right)''' = (4 - 2\dot{\lambda}_t z^{1/2})''' = -\frac{3\dot{\lambda}_t}{4z^{5/2}}. \tag{5.3.6}$$

Replacing z in (5.3.6) with $h_t(z)$, we obtain (5.3.5). This completes the proof. $\square$

Proof of Proposition 5.3.8. Let us first consider the case $s = 0$. Letting $\eta(t) = \sqrt{H_0 \circ \gamma(t)}$ and $\eta^\varepsilon(t) = \sqrt{H_0^{\varepsilon\mu} \circ \gamma^{\varepsilon\mu}(t)} = \sqrt{H_0^{\varepsilon\mu} \circ \omega^{\varepsilon\mu} \circ \gamma(t)}$, we have $\eta^\varepsilon = \psi^\varepsilon \circ \eta$ where $\psi^\varepsilon(z) = \sqrt{(H_0^{\varepsilon\mu} \circ \omega^{\varepsilon\mu} \circ H_0^{-1})(z^2)}$. Let $\varepsilon\nu$ be the Beltrami coefficient corresponding to the quasiconformal map $\psi^\varepsilon : \mathbb{H} \to \mathbb{H}$. Let $\varepsilon\mu_0$ denote the Beltrami coefficient of $H_0^{\varepsilon\mu} \circ \omega^{\varepsilon\mu} \circ H_0^{-1}$. Then, $\nu(\zeta) = \mu_0(\zeta^2)(|\zeta|^2/\zeta^2)$.

Substituting this ν into (5.2.26) and letting $\zeta = \sqrt{z}$, since $f_t(\zeta) = \sqrt{h_t(z)}$ and $f_t'(\zeta)/\zeta = h_t'(z)/\sqrt{h_t(z)}$, we get

$$\begin{aligned}\left.\frac{\partial}{\partial\varepsilon}\right|_{\varepsilon=0} I^C(\eta^\varepsilon(0,T]) &= \frac{12}{\pi} \operatorname{Re} \int_{\mathbb{H}} \nu(\zeta) \int_0^T \dot{\lambda}_t \frac{f_t'(\zeta)^2}{f_t(\zeta)^3} \mathrm{d}t \, \mathrm{d}^2\zeta \\ &= \frac{3}{\pi} \operatorname{Re} \int_{\mathbb{C}\smallsetminus\mathbb{R}_+} \mu_0(z) \int_0^T \dot{\lambda}_t \frac{h_t'(z)^2}{h_t(z)^{5/2}} \mathrm{d}t \, \mathrm{d}^2 z.\end{aligned}$$

Applying Lemma 5.3.9, we have

$$\frac{\partial}{\partial \varepsilon}\Big|_{\varepsilon=0} I^C(\eta^\varepsilon(0,T]) = -\frac{4}{\pi}\operatorname{Re}\int_{\mathbb{C}\smallsetminus\mathbb{R}_+} \mu_0(z)\int_0^T \frac{\partial(\mathcal{S}h_t(z))}{\partial t}\,\mathrm{d}t\,\mathrm{d}^2z$$
$$= -\frac{4}{\pi}\operatorname{Re}\int_{\mathbb{C}\smallsetminus\mathbb{R}_+} \mu_0(z)\mathcal{S}h_T(z)\,\mathrm{d}^2z.$$

Recalling our definition of $\varepsilon\mu_0$, we have $\mu_0(H_0(z)) = \mu(z)(H_0'(z)^2/|H_0'(z)|^2)$. Moreover, from $H_T = h_T \circ H_0$, we have

$$\mathcal{S}h_T(H_0(z))\cdot H_0'(z)^2 = \mathcal{S}H_T(z) - \mathcal{S}H_0(z).$$

Hence,

$$\int_{\mathbb{C}\smallsetminus\mathbb{R}_+} \varepsilon\mu_0(z)\mathcal{S}h_T(z)\,\mathrm{d}^2z = \int_{\mathbb{C}\smallsetminus\gamma} \varepsilon\mu(z)(\mathcal{S}H_T(z) - \mathcal{S}H_0(z))\,\mathrm{d}^2z.$$

Therefore, the case $s = 0$ holds. In fact, this implies (5.3.3) for any $s \in \mathbb{R}$ because the parametrization of γ is arbitrary up to translations as discussed in Lemma 5.3.2. □

5.3.3 Variation of the loop Loewner energy

The goal of this subsection is to prove Theorem 5.1.4. Let $H_{+\infty} : \hat{\mathbb{C}} \smallsetminus \gamma \to \mathbb{C} \smallsetminus \mathbb{R}$ be any conformal map which maps $\Omega \to \mathbb{H}$ and $\Omega^* \to \mathbb{H}^*$. Note that the map $H_{+\infty}$ restricted to Ω coincides with f^{-1} (as in Theorem 5.1.4) post-composed by a Möbius transformation, so $\mathcal{S}H_{+\infty}|_\Omega = \mathcal{S}[f^{-1}]$. Similarly, $\mathcal{S}H_{+\infty}|_{\Omega^*} = \mathcal{S}[g^{-1}]$. In view of Corollary 5.3.6 and Proposition 5.3.8, it suffices to show that as $s \to -\infty$ and $t \to +\infty$, we have

$$\int_{\mathbb{C}\smallsetminus\gamma} \mu(z)(\mathcal{S}H_t(z) - \mathcal{S}H_s(z))\,\mathrm{d}^2z \to \int_{\mathbb{C}\smallsetminus\gamma} \mu(z)\mathcal{S}H_{+\infty}(z)\,\mathrm{d}^2z, \tag{5.3.7}$$

and

$$\frac{\mathrm{d}}{\mathrm{d}\varepsilon}\Big|_{\varepsilon=0}\int_s^t \left|\frac{\mathrm{d}\lambda_r^{\varepsilon\mu}}{\mathrm{d}a_r^{\varepsilon\mu}}\right|^2 \mathrm{d}a_r^{\varepsilon\mu} \to \frac{\mathrm{d}}{\mathrm{d}\varepsilon}\Big|_{\varepsilon=0}\int_{-\infty}^{+\infty} \left|\frac{\mathrm{d}\lambda_r^{\varepsilon\mu}}{\mathrm{d}a_r^{\varepsilon\mu}}\right|^2 \mathrm{d}a_r^{\varepsilon\mu}. \tag{5.3.8}$$

For this, we need a few lemmas.

Lemma 5.3.10. *Suppose $\gamma(\pm\infty) = \infty$. Then, $\mathcal{S}H_s \to 0$ locally uniformly as $s \to -\infty$.*

Proof. Given any $R > 0$, there exists a large negative s_R such that for $s \le s_R$, we have $\gamma[-\infty, s] \cap \{z : |z| < R\} = \varnothing$. Then, H_s is conformal on $R\mathbb{D}$. By the Nehari bound,

$$|\mathcal{S}H_s(z)| \le \frac{6}{\big(R(1-|z|^2/R^2)\big)^2}$$

for every $z \in R\mathbb{D}$. The right-hand side of the inequality tends uniformly to 0 as $R \to +\infty$ on any compact subset of $\mathbb{C}$. □

Lemma 5.3.11. *Suppose $\gamma(\pm\infty) = \infty$. Then, $\mathcal{S}H_t \to \mathcal{S}H_{+\infty}$ locally uniformly on $\mathbb{C} \setminus \gamma$ as $t \to +\infty$.*

Proof. Choose either component of $\mathbb{C} \setminus \gamma$ and call it $U_{+\infty}$. Let us denote by $\gamma^U(s)$ the prime end of $\gamma(s)$ as viewed from $U_{+\infty}$.

Let $\gamma_t = \gamma[-\infty, t]$ and denote by $\gamma_t^U := \bigcup_{s\in(-\infty,t)} \gamma^U(s)$ the prime ends of γ_t accessible from $U_{+\infty}$. Let Γ_t be the hyperbolic geodesic in $\mathbb{C} \setminus \gamma_t$ connecting $\gamma(t)$ and $\gamma(+\infty)$. Let U_t be the component of $\mathbb{C} \setminus (\gamma_t \cup \Gamma_t)$ such that the prime ends of γ_t as viewed from U_t comprise γ_t^U. Observe that if $\mathrm{hm}(z, D; \cdot)$ is the harmonic measure on the domain D as viewed from $z \in D$, then $z \in U_t$ if and only if $\mathrm{hm}(z, \mathbb{C} \setminus \gamma_t; \gamma_t^U) > 1/2$.

We claim that if $z \in U_{+\infty}$, then $z \in \bigcup_{T\ge 0} \bigcap_{t \ge T} U_t$. Suppose $z \in U_{+\infty}$. Choose a sufficiently small constant $a \in (0,1)$ such that $\mathrm{hm}(0, \mathbb{D} \setminus [a,1], [a,1]) > 1/2$. Since $\gamma(\pm\infty) = \infty$, for all sufficiently large $t > 0$, we can find $R > 0$ such that $\gamma_t \cap B_{aR}(z) \neq \varnothing$ but $\gamma(t, +\infty) \cap B_R(z) = \varnothing$. Then,

$$\mathrm{hm}(z, \mathbb{C} \setminus \gamma_t; \gamma_t^U) \ge \mathrm{hm}(z, B_R(z) \setminus \gamma_t; \gamma_t^U) = \mathrm{hm}(z, B_R(z) \setminus \gamma_t; \gamma_t).$$

By the Beurling projection theorem,

$$\operatorname{hm}(z, B_R(z) \smallsetminus \gamma_t, \gamma_t) \geq \operatorname{hm}(0, R\mathbb{D} \smallsetminus [aR, R]; [aR, R]) = \operatorname{hm}(0, \mathbb{D} \smallsetminus [a, 1]; [a, 1]) > 1/2.$$

This completes the proof of the claim.

Note that H_t is a conformal map which sends $\mathbb{C} \smallsetminus (\gamma_t \cup \Gamma_t)$ onto $\mathbb{C} \smallsetminus \mathbb{R}$. Then, by the Carathéodory kernel theorem, H_t post-composed with an appropriate Möbius transformation converges locally uniformly on $U_{+\infty}$ to $H_{+\infty}$ as $t \to \infty$. Consequently, $\mathcal{S}H_t \to \mathcal{S}H_{+\infty}$ locally uniformly on $U_{+\infty}$ as $t \to \infty$. An analogous argument applies to the other component of $\hat{\mathbb{C}} \smallsetminus \gamma$. $\square$

Proof of Theorem 5.1.4. Given $\gamma(\pm\infty) = \infty$, the limit (5.3.7) follows by applying Lemmas 5.3.10 and 5.3.11. Otherwise, choose a Möbius map $A : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ so that $A(\infty) = \gamma(\pm\infty)$. If $H_{+\infty} : \hat{\mathbb{C}} \smallsetminus \gamma \to \mathbb{C} \smallsetminus \mathbb{R}$ is a conformal map as in the theorem statement, then $H_{+\infty} \circ A$ is a conformal map from $\hat{\mathbb{C}} \smallsetminus \gamma$ onto $\mathbb{C} \smallsetminus \mathbb{R}$ with $\mathcal{S}[H \circ A] = \mathcal{S}H \cdot (A')^2$. The pullback $A^*\mu$ of μ under φ satisfies $\mu(A(z)) = (A^*\mu)(z)(A'(z))^2/|A'(z)|^2$. Letting $\zeta = A(z)$, we have

$$\int_{\hat{\mathbb{C}} \smallsetminus \gamma} \mu(\zeta) \mathcal{S}H_{+\infty}(\zeta) \, \mathrm{d}^2\zeta = \int_{\hat{\mathbb{C}} \smallsetminus A^{-1} \circ \gamma} A^*\mu(z) \mathcal{S}[H \circ A](z) \, \mathrm{d}^2 z,$$

so we can consider the curve $A^{-1} \circ \gamma$ instead.

To show (5.3.8), it suffices to prove that we can switch between the integral over $t \in (-\infty, +\infty)$ and the derivative in ε. To this end, we prove that the following integral is absolutely convergent:

$$\int_{-\infty}^{+\infty} \left| \frac{\partial}{\partial \varepsilon}\Big|_{\varepsilon=0} \frac{(\dot{\lambda}_t^{\varepsilon\mu})^2}{\dot{a}_t^{\varepsilon\mu}} \right| \mathrm{d}t < +\infty. \tag{5.3.9}$$

Recall from the proof of Corollary 5.2.4 that for $t \geq 0$,

$$\frac{\partial}{\partial \varepsilon}\Big|_{\varepsilon=0} \frac{(\dot{\lambda}_t^{\varepsilon\mu})^2}{\dot{a}_t^{\varepsilon\mu}} = \frac{24\dot{\lambda}_t}{\pi} \operatorname{Re} \int_{\mathbb{H}} \nu(z) \frac{f_t'(z)^2}{f_t(z)^3} \, \mathrm{d}^2 z \tag{5.3.10}$$

where ν is the push-forward of the Beltrami differential μ under the map $\sqrt{H_0}$ as displayed in Figure 5.2. Using Lemma 5.3.2 and the composition rule (5.2.20) for Beltrami differentials, it is straightforward to check that (5.3.10) is true for all $t \in \mathbb{R}$. From Lemma 5.4.2, the assumption that $\int_{-\infty}^{+\infty} |\dot\lambda_t|^2 \,\mathrm{d}t < \infty$, and the Cauchy–Schwarz inequality, it follows that $\int_{-\infty}^{+\infty} |\dot\lambda_t \frac{f_t'(z)^2}{f_t(z)^3}| \,\mathrm{d}t$ is finite and locally uniform in z. This implies (5.3.9) since ν is compactly supported and $\|\nu\|_\infty < \infty$. Therefore, for any $-\infty \le s \le t \le +\infty$,

$$\frac{\partial}{\partial\varepsilon}\Big|_{\varepsilon=0} \int_s^t \left|\frac{\mathrm{d}\lambda_r^{\varepsilon\mu}}{\mathrm{d}a_r^{\varepsilon\mu}}\right|^2 \mathrm{d}a_r^{\varepsilon\mu} = \frac{\partial}{\partial\varepsilon}\Big|_{\varepsilon=0} \int_s^t \frac{(\dot\lambda_r^{\varepsilon\mu})^2}{\dot a_r^{\varepsilon\mu}} \mathrm{d}r = \int_s^t \frac{\partial}{\partial\varepsilon}\Big|_{\varepsilon=0} \frac{(\dot\lambda_r^{\varepsilon\mu})^2}{\dot a_r^{\varepsilon\mu}} \mathrm{d}r. \tag{5.3.11}$$

Letting $s \to -\infty$ and $t \to +\infty$, by the dominated convergence theorem and (5.3.9), we conclude (5.3.8). This proves Theorem 5.1.4 as explained at the beginning of the subsection.

□

5.4 Large time behavior of finite-energy Loewner chains

A priori, a Jordan curve may have finite total capacity. However, we now show that if a Jordan curve has finite Loewner energy, then its capacity must be infinite at both its initial and terminal parts.

Lemma 5.4.1. *Let $\gamma : [T_-, T_+] \to \hat{\mathbb{C}}$ be a Jordan curve parametrized by capacity, where $T_- \in [-\infty, 0)$, $T_+ \in (0, +\infty]$ and $\gamma(T_-) = \gamma(T_+)$ is the root. More precisely, this means that there is a conformal map $H_0 : \hat{\mathbb{C}} \smallsetminus \gamma[T_-, 0] \to \mathbb{C} \smallsetminus \mathbb{R}_+$ such that for a_t defined as in Definition 5.3.1, we have $a_t = t$ for all $t \in [T_-, T_+]$. If $I^L(\gamma) := \frac{1}{2}\int_{T_-}^{T_+} |\dot\lambda_t|^2 \,\mathrm{d}t < \infty$, then $T_- = -\infty$ and $T_+ = +\infty$.*

Consequently, the capacity parametrization of a finite-energy Jordan curve must take all of $\mathbb{R}$ as its domain, which justifies the formula in Definition 5.3.5.

Proof. If the Loewner energy of the chord $\gamma(0, T_+)$ traversing the domain $\hat{\mathbb{C}} \smallsetminus \gamma[T_-, 0]$ is finite, then its total capacity T_+ must be infinite by [247, Theorem 2.4]. By definition, this chordal Loewner energy is bounded above by the loop Loewner energy $I^L(\gamma)$.

Let us now consider T_-. By Carathéodory's theorem, the conformal map $\sqrt{H_0}$ from $\hat{\mathbb{C}} \smallsetminus \gamma[T_-, 0]$ onto $\mathbb{H}$ extends continuously to a bijection between the prime ends of $\hat{\mathbb{C}} \smallsetminus \gamma[T_-, 0]$ and $\mathbb{R} \cup \{\infty\}$. Note that $\sqrt{H_0(\gamma(0))} = 0$ and $\sqrt{H_0(\gamma(T_-))} = \infty$, and for each $t \in (T_-, 0)$, one of the prime ends at $\gamma(t)$ is mapped to a point $x_t \in \mathbb{R}_+$. Let f_t and λ_t be defined as in (5.3.1) and (5.3.2). By Lemma 5.3.2, the Loewner equation $\partial_t f_t(z) = 2/f_t(z) - \dot{\lambda}_t$ holds for $t \in (T_-, 0)$ as well. This implies

$$\frac{\partial}{\partial t}(\operatorname{Re} f_t(z))^2 = \frac{4(\operatorname{Re} f_t(z))^2}{|f_t(z)|^2} - 2\dot{\lambda}_t(\operatorname{Re} f_t(z)).$$

(See (5.4.3) below.) For $T \in (T_-, 0)$, we deduce from above using Schwarz reflection that

$$\frac{\mathrm{d}(f_t(x_T))^2}{\mathrm{d}t} = 4 - 2\dot{\lambda}_t f_t(x_T), \quad t \in (T, 0]. \tag{5.4.1}$$

Assume, for the sake of contradiction, that $I^L(\gamma) < +\infty$ and $T_- > -\infty$. Since $f_T(x_T) = 0$, we deduce from (5.4.1) using the Cauchy–Schwarz inequality that

$$\begin{aligned}\sup_{t\in[T,0]} |f_t(x_T)|^2 &\le 4|T| + 2\Big(\int_T^0 |\dot{\lambda}_t|^2 \,\mathrm{d}t\Big)^{\frac{1}{2}}\Big(\int_T^0 |f_t|^2 \,\mathrm{d}t\Big)^{\frac{1}{2}} \\ &\le 4|T_-| + 2\sqrt{I^L(\gamma)|T_-|}\Big(\sup_{t\in[T,0]} |f_t(x_T)|^2\Big)^{\frac{1}{2}}.\end{aligned}$$

Completing the square in the above inequality, we obtain

$$|x_T|^2 \le \sup_{t\in[T,0]} |f_t(x_T)|^2 \le |T_-|\Big(2 + \sqrt{I^L(\gamma)}\Big)^2. \tag{5.4.2}$$

By assumption, the right-hand side of (5.4.2) is a finite quantity independent of $T \in (T_-, 0)$. This is a contradiction since $t \mapsto x_t$ maps $[T_-, 0]$ onto $[0, +\infty]$. $\square$

The following lemma is used in the proof of Theorem 5.1.4 to switch the order between the derivative and the integral in (5.3.8). Note that if $\lambda_t = 0$ for all t, then $f_t(z) = \sqrt{z^2 + 4t}$. In particular, $f_t(z)^2/(4t) \to 1$ and $2|t|^{1/2}|f_t'(z)| \to 1$ as $t \to \pm\infty$.

Lemma 5.4.2. *Let γ be a Jordan curve with $I^L(\gamma) < \infty$. Let H_t and f_t be the uniformizing maps defined in Section 5.3.1. Then, as $t \to \pm\infty$, $f_t(z)^2/(4t) \to 1$ and $|f_t'(z)| = |t|^{-1/2+o(1)}$ locally uniformly for $z \in H_0(\hat{\mathbb{C}} \smallsetminus \gamma)$.*

Proof. The starting point of this proof is similar to that of Lemma 5.4.1. Let us denote $f_t(z) = u_t + iv_t$, where $u_t \in \mathbb{R}$ and $v_t > 0$. The real and imaginary parts of the Loewner equation $\partial_t f_t(z) = 2/f_t(z) - \dot{\lambda}_t$ correspond to

$$\dot{u}_t = \frac{2u_t}{u_t^2 + v_t^2} - \dot{\lambda}_t \quad \text{and} \quad \dot{v}_t = -\frac{2v_t}{u_t^2 + v_t^2}. \tag{5.4.3}$$

Let us consider the $t \to +\infty$ limit first. Given $\varepsilon > 0$, since γ has finite Loewner energy, we can choose a large T_0 so that $\int_{T_0}^{+\infty} \dot{\lambda}_t^2 \,\mathrm{d}t < \varepsilon$. Since

$$\frac{\mathrm{d}u_t^2}{\mathrm{d}t} = \frac{4u_t^2}{u_t^2 + v_t^2} - 2\dot{\lambda}_t u_t,$$

for $T_0 \le t \le T$, we have

$$\begin{aligned} u_t^2 &\le u_{T_0}^2 + 4\int_{T_0}^{T} \frac{u_t^2}{u_t^2 + v_t^2}\,\mathrm{d}t + 2\int_{T_0}^{T} |\dot{\lambda}_t u_t|\,\mathrm{d}t \\ &\le u_{T_0}^2 + 4(T - T_0) + 2\bigg(\int_{T_0}^{T} \dot{\lambda}_t^2\,\mathrm{d}t\bigg)^{1/2}\bigg(\int_{T_0}^{T} u_t^2\,\mathrm{d}t\bigg)^{1/2}. \end{aligned} \tag{5.4.4}$$

Hence,

$$\sup_{t\in[T_0,T]} u_t^2 \le u_{T_0}^2 + 4(T - T_0) + 2\bigg(\varepsilon(T - T_0)\sup_{t\in[T_0,T]} u_t^2\bigg)^{1/2}.$$

Completing the square, we obtain

$$\left(\sup_{t\in[T_0,T]} u_t^2\right)^{1/2} \leq \left(u_{T_0}^2 + (4+\varepsilon)(T-T_0)\right)^{1/2} + \left(\varepsilon(T-T_0)\right)^{1/2}.$$

Then, there exists a $T_1 \geq T_0$ such that for all $T \geq T_1$,

$$\sup_{t\in[T_0,T]} u_t^2 \leq u_{T_0}^2 + 4(1+\varepsilon)(T-T_0). \tag{5.4.5}$$

Now, from

$$\frac{\partial}{\partial t}\mathrm{Re}(f_t(z)^2) = \frac{\mathrm{d}(u_t^2 - v_t^2)}{\mathrm{d}t} = 4 - 2\dot{\lambda}_t u_t,$$

we have

$$\begin{aligned}\left|\mathrm{Re}(f_T(z)^2) - 4T\right| &\leq |\mathrm{Re}(f_{T_0}(z)^2)| + 4T_0 + 2\int_{T_0}^{T} |\dot{\lambda}_t u_t| \,\mathrm{d}t \\ &\leq |\mathrm{Re}(f_{T_0}(z)^2)| + 4T_0 + 2\left(\varepsilon(T-T_0)\sup_{t\in[T_0,T]} u_t^2\right)^{1/2}.\end{aligned}$$

Substituting (5.4.5), we can find a $T_2 \geq T_1$ such that for all $T \geq T_2$,

$$\left|\mathrm{Re}(f_T(z)^2) - 4T\right| \leq 5\sqrt{\varepsilon}T.$$

Since the choice of ε was arbitrary, we conclude $\mathrm{Re}(f_t(z)^2)/(4t) \to 1$ as $t \to +\infty$.

As for $\mathrm{Im}(f_t(z)^2) = 2u_t v_t$, note that (5.4.3) implies v_t is monotonically decreasing. Hence, (5.4.5) implies $\mathrm{Im}(f_t(z)^2)/t \to 0$ as $t \to +\infty$. Combining the limits of the real and imaginary parts, we obtain $f_t(z)^2/(4t) \to 1$ as $t \to +\infty$. Moreover, since $f_{T_0}(z) = u_{T_0} + iv_{T_0}$ depends continuously on z whereas T_0 was chosen independently of z, the limit we proved converges uniformly on each compact subset of $H_0(\hat{\mathbb{C}} \smallsetminus \gamma)$.

Let us now consider the $t \to -\infty$ limit. This time, since the Loewner energy of γ is finite, we can find a large negative $\tilde{T}_0$ such that $\int_{-\infty}^{\tilde{T}_0} \dot{\lambda}_t^2 \,\mathrm{d}t < \varepsilon$. Hence, (5.4.4) implies that there exists a $\tilde{T}_1 \le \tilde{T}_0$ such that for all $T \le \tilde{T}_1$,

$$\sup_{t \in [T, \tilde{T}_1]} u_t^2 \le u_{\tilde{T}_0}^2 + 4(1+\varepsilon)|T - T_0|.$$

Again, we have

$$\left|\mathrm{Re}(f_T(z)^2) - 4T\right| \le |\mathrm{Re}(f_{T_0}(z)^2)| + 4|\tilde{T}_0| + 2\left(\varepsilon |T - \tilde{T}_0| \sup_{t \in [T, \tilde{T}_0]} u_t^2\right)^{1/2},$$

and choosing ε to be arbitrarily small, we obtain $\mathrm{Re}(f_t(z)^2)/(4t) \to 1$ as $t \to -\infty$.

For the imaginary part of $f_t(z)^2$, we consider

$$\frac{\partial}{\partial t}\mathrm{Im}(f_t(z)^2) = \frac{\mathrm{d}(2u_t v_t)}{\mathrm{d}t} = -2\dot{\lambda}_t v_t.$$

Since $\mathrm{Re}(f_t(z)^2) = u_t^2 - v_t^2$, we have that as $T \to -\infty$,

$$\sup_{t \in [T, \tilde{T}_0]} v_t^2 \le \sup_{t \in [T, \tilde{T}_0]} |\mathrm{Re}(f_t(z)^2)| + \sup_{t \in [T, \tilde{T}_0]} u_t^2 \le (8 + o(1))|T|.$$

Hence, using the Cauchy–Schwarz inequality as above, we have

$$\left|\mathrm{Im}(f_T(z)^2)\right| \le \left|\mathrm{Im}(f_{\tilde{T}_0}(z)^2)\right| + 2\int_T^{\tilde{T}_0} |\dot{\lambda}_t v_t| \,\mathrm{d}t = o(T)$$

as $T \to -\infty$. Therefore, $f_t(z)^2/(4t) \to 1$ as $t \to -\infty$. Again, this limit converges uniformly on compact subsets of $\mathbb{H}$ since $f_{\tilde{T}_0}(z)$ depends continuously on z and $\tilde{T}_0$ can be chosen independently of z on a compact set.

Finally, to estimate $|f_t'(z)|$, consider the equation

$$\frac{\partial}{\partial t}\log|f_t'(z)| = \mathrm{Re}\left(\frac{\partial_t f_t'(z)}{f_t'(z)}\right) = -\mathrm{Re}\frac{2}{f_t(z)^2}.$$

As $t \to \pm\infty$, the right-hand side behaves as $(-1/2 + o(1))t^{-1}$. We thus obtain $|f_t'(z)| = |t|^{-1/2+o(1)}$ as claimed. $\square$

CHAPTER 6

QUASI-INVARIANCE OF SLE WELDING MEASURES

This chapter presents material first announced in [101].

6.1 Introduction

6.1.1 Background and motivation

The Schramm–Loewner evolution (SLE_κ) loop measure is a one-parameter family of σ-finite measures indexed by $\kappa \in (0, 8)$ on the space of non-self-crossing loops on the Riemann sphere. For $\kappa \in (0, 4]$, the SLE_κ loop measure is supported on Jordan curves. SLE curves first arose as interfaces in the scaling limits of critical lattice models [221, 235, 170, 234, 222, 69, 15] and the loop version of SLE was constructed and studied in [159, 250, 155, 36, 255].

The group of Weil–Petersson homeomorphisms, denoted $\mathrm{WP}(\mathbb{S}^1)$, is a subgroup of quasisymmetric circle homeomorphisms introduced in [75, 239]. The most straightforward characterization of $\mathrm{WP}(\mathbb{S}^1)$ is that $\varphi \in \mathrm{WP}(\mathbb{S}^1)$ if and only if φ is absolutely continuous and $\log |\varphi'|$ belongs to the fractional Sobolev space $H^{1/2}(\mathbb{S}^1)$ [232]. Takhtajan and Teo [239] showed that the Weil–Petersson Teichmüller space $T_0(1)$, which is the quotient space of $\mathrm{WP}(\mathbb{S}^1)$ up to post-composition by Möbius transformations, carries an essentially unique Kähler structure that is right-invariant (namely, invariant under post-composition by elements in $\mathrm{WP}(\mathbb{S}^1)$).

A close relationship between these two objects was first observed by Yilin Wang [245], who showed the equivalence between the Loewner energy — the large deviation rate function for SLE_κ as $\kappa \to 0$ [244, 219, 204] — and the universal Liouville action — the unique right-invariant Kähler potential on the Weil–Petersson Teichmüller space $T_0(1)$ found by Takhtajan and Teo [239]. Recently, the Loewner energy was further proven to be the Onsager–Machlup functional of the SLE_κ loop measure for $\kappa \in (0, 4]$ in the work of Carfagnini and Wang

[63], thus extending the connection beyond the semiclassical regime. Hence, it is natural to wonder how the Kähler and group structures on the Weil–Petersson Teichmüller space manifest on the SLE_κ loop measure. In this work, we concentrate on the latter part of the question: i.e., quasi-invariance under the composition of circle homeomorphisms given by conformal welding.

The Loewner energy of a Jordan curve is defined as the Dirichlet energy of its driving function [244, 219, 238]. The driving function of an SLE_κ loop is $\sqrt{\kappa}$ times the two-sided Brownian motion, whose Cameron–Martin space consists of functions with finite Dirichlet energy. Thus, the loops with finite Loewner energy, which are Weil–Petersson quasicircles as identified in [245], can be viewed as the Cameron–Martin space of SLE_κ loop measures for the "addition" of driving functions. Our result shows that despite the nonlinearity of the composition of welding homeomorphisms as opposed to the addition of driving functions, $\mathrm{WP}(\mathbb{S}^1)$ behaves as the Cameron–Martin space of the random welding homeomorphism corresponding to SLE_κ loops under the natural group action given by composition.

6.1.2 Main result

To describe our main result, we give a brief overview of conformal welding. Given an oriented Jordan curve η on the Riemann sphere $\hat{\mathbb{C}}$, let $f : \mathbb{D} \to \Omega$ and $g : \mathbb{D}^* \to \Omega^*$ denote any conformal maps from the unit disk $\mathbb{D} = \{z \in \hat{\mathbb{C}} : |z| < 1\}$ and the outer unit disk $\mathbb{D}^* := \hat{\mathbb{C}} \setminus \overline{\mathbb{D}}$ onto the bounded and unbounded components Ω and Ω^* of $\hat{\mathbb{C}} \setminus \eta$, respectively. A classic result of Carathéodory states that f and g extend to homeomorphisms on the closed disks $\overline{\mathbb{D}}$ and $\overline{\mathbb{D}^*}$, respectively. The map $\psi := (g^{-1} \circ f)|_{\mathbb{S}^1}$ is then an orientation-preserving homeomorphism on the unit circle $\mathbb{S}^1$ to itself. We call any homeomorphism that arises in this way a welding.

Note that for any Jordan curve η and a Möbius map $\omega \in \mathrm{Möb}(\hat{\mathbb{C}})$ of the Riemann sphere $\hat{\mathbb{C}}$, the image $\omega(\eta)$ has the same welding as η. On the other hand, we may pre-compose them by Möbius maps fixing $\mathbb{S}^1$, the space of which we denote $\mathrm{Möb}(\mathbb{S}^1)$. Hence, conformal welding is better described in terms of the correspondence

$$\text{Möb}(\hat{\mathbb{C}})\backslash\{\text{Oriented Jordan curves}\} \to \text{Möb}(\mathbb{S}^1)\backslash \operatorname{Homeo}_+(\mathbb{S}^1)/\text{Möb}(\mathbb{S}^1). \tag{6.1.1}$$

This correspondence is neither injective nor onto; see [53] and the references therein. (We note a recent article [217] that describes every circle homeomorphism as a composition of two welding homeomorphisms.) The map (6.1.1) is injective when restricting to conformally removable curves (see Section 6.4.1). For example, quasicircles (images of the unit circle under quasiconformal homeomorphisms of the Riemann sphere $\hat{\mathbb{C}}$) are conformally removable [43], and SLE_κ curves are almost surely conformally removable for $\kappa \in (0, \kappa_0)$ where $\kappa_0 \in (4, 8)$ [145, 218, 151, 152]. However, there are no known geometric or analytic characterizations that are equivalent to conformal removability [51].

In this work, we will only consider conformally removable curves, whose weldings comprise the space $\text{RM}(\mathbb{S}^1)$. We endow $\text{RM}(\mathbb{S}^1)$ with the topology of joint compact convergence for the Riemann maps f and g (i.e., Carathéodory topology for both components of $\hat{\mathbb{C}}\backslash\eta$). Let us first make the following observation, which allows us to consider the group of quasisymmetric homeomorphisms acting measurably on the random weldings corresponding to SLE_κ loops.

Proposition 6.1.1 (See Propositions 6.4.3 and 6.4.5)**.** *The pre- and post-compositions of conformally removable weldings by quasisymmetric homeomorphisms are continuous.*

Our main result concerns random weldings corresponding SLE_κ loops when $\kappa \in (0, 4)$. We focus on a specific realization of the one-to-one correspondence between weldings and loops, described in Definition 6.4.6. At this time, we note that the SLE_κ loop measure restricted to loops that separate 0 from ∞ induces a probability measure on the stabilizers of 1 in $\text{RM}(\mathbb{S}^1)$, which we denote as $\text{SLE}_\kappa^{\text{weld}}$.

Theorem 6.1.2. *For $\kappa \in (0, 4)$, sample ψ_κ from $\text{SLE}_\kappa^{\text{weld}}$. We have the following for any fixed $\varphi \in \text{WP}(\mathbb{S}^1)$ with $\varphi(1) = 1$.*

- *The law of* $\psi_\kappa \circ \varphi$ *is mutually absolutely continuous with respect to* $\mathrm{SLE}_\kappa^{\mathrm{weld}}$.

- *The law of* $\varphi^{-1} \circ \psi_\kappa$ *is mutually absolutely continuous with respect to* $\mathrm{SLE}_\kappa^{\mathrm{weld}}$ *if the log-ratio* $\log|(\varphi(\cdot) - \varphi(1))/(\cdot - 1)|$ *belongs to* $H^{1/2}(\mathbb{S}^1)$.

We expect that if φ is a quasisymmetric circle homeomorphism which is not in $\mathrm{WP}(\mathbb{S}^1)$, then neither $\psi_\kappa \circ \varphi$ nor $\varphi^{-1} \circ \psi_\kappa$ has a law equivalent to $\mathrm{SLE}_\kappa^{\mathrm{weld}}$.

Remark 6.1.3. For each $\varphi \in \mathrm{WP}(\mathbb{S}^1)$, the circle homeomorphism $\hat{\varphi}(\cdot) := \varphi(z_0 \cdot)/\varphi(z_0)$ obtained by conjugating φ with the rotation $z \mapsto z_0 z$ satisfies $\log|(\hat{\varphi}(\cdot) - \hat{\varphi}(1))/(\cdot - 1)| \in H^{1/2}(\mathbb{S}^1)$ for a.e. $z_0 \in \mathbb{S}^1$, as shown in Lemma 6.2.10. Instead of normalizing weldings corresponding to SLE_κ loops by considering those that stabilize $1 \in \mathbb{S}^1$, if we consider the post-composition of ψ_k by a uniform rotation of $\mathbb{S}^1$, then we obtain a version of Theorem 6.1.2 that is symmetric for pre- and post-compositions by elements of $\mathrm{WP}(\mathbb{S}^1)$. See Corollary 6.4.12.

As far as the authors are aware, before this work, only analytic deformations of SLE loops (hence, equivalently, of their welding homeomorphisms) have been studied, within the framework of conformal restriction in particular. This is the first time a non-smooth deformation of SLE is considered. We will comment on related literature in Section 6.1.4.

Baverez and Jego [33] have independently studied the SLE welding measure under compositions by analytic circle diffeomorphisms, and they compute the Radon–Nikodym derivatives under such deformations using an infinitesimal approach which is complementary to ours. As a consequence, they give an alternative proof of the conformal welding of quantum disks [19] which extends to the critical case ($\kappa = 4$).

6.1.3 Key ingredients of the proof

Our strategy for the proof of Theorem 6.1.2 is as follows. We first show that the log-correlated Gaussian field (LGF) on the unit circle is quasi-invariant under pullbacks by Weil–Petersson homeomorphisms. Then, we "lift" this characterization to that of the Gaussian multiplicative chaos (GMC). Finally, we obtain our main theorem by identifying the law of SLE welding in terms of compositions of two homomorphisms defined using GMC measures. Here, we explain this proof outline in further detail.

Given an integrable function h on $\mathbb{S}^1$, we define $\pi_0 h$ to be the projection to its "mean-zero part." That is, $\pi_0 h = h - c_0$ where c_0 is the average of h on $\mathbb{S}^1$. The LGF on the unit circle is defined as the formal sum

$$h(\cdot) = \sqrt{2} \sum_{n \geq 1} \xi_n h_n(\cdot), \tag{6.1.2}$$

where ξ_n is a sequence of i.i.d. standard Gaussian random variables and h_n is an orthonormal basis of $\mathcal{H}_0 := \pi_0 H^{1/2}(\mathbb{S}^1)$. Here, we write $\sqrt{2}$ to demonstrate that the covariance of $h(x)$ and $h(y)$ is $-2 \log |x - y|$.

Let $\varphi \in \mathrm{Homeo}_+(\mathbb{S}^1)$ be an orientation preserving homeomorphism. Then, φ induces a pullback operator $\Pi(\varphi)$ given by

$$\Pi(\varphi)(f) := \pi_0(f \circ \varphi) \quad \text{for } f \in \mathcal{H}_0. \tag{6.1.3}$$

The operator was introduced in [202] and [203]; the former showed that $\varphi \in \mathrm{QS}(\mathbb{S}^1)$ if and only if $\Pi(\varphi)$ is a bounded operator from $\mathcal{H}_0$ onto $\mathcal{H}_0$. We extend $\Pi(\varphi)$ to act on the LGF h using the expansion (6.1.2) and by linearity.

Theorem 6.1.4. *Let h be an LGF on the unit circle and suppose that $\varphi \in \mathrm{QS}(\mathbb{S}^1)$. Then, the law of $\Pi(\varphi)(h)$ is mutually absolutely continuous with LGF if and only if $\varphi \in \mathrm{WP}(\mathbb{S}^1)$. Otherwise, they are mutually singular.*

The proof of Theorem 6.1.4 is based on the Feldman–Hájek theorem, which gives a necessary and sufficient condition for classifying two infinite dimensional Gaussian measures on a locally convex space as either mutually absolutely continuous or mutually singular. In our context, this depends on whether $\Pi(\varphi)\Pi(\varphi)^* - I$ is Hilbert–Schmidt. We show that this condition holds if and only if $\varphi \in \mathrm{WP}(\mathbb{S}^1)$ (see Lemma 6.2.7) based on previous results [220, 239, 139] which identified certain operators associated with $\Pi(\varphi)$ to be Hilbert–Schmidt.

Our next step is to "lift" Theorem 6.1.4 to that for the Gaussian multiplicative chaos (GMC) measure $\mathcal{M}_h^\gamma$ defined from LGF h heuristically as $\exp(\frac{\gamma}{2}h)\,\mathrm{d}\theta$ for $\gamma \in (0,2]$. See Section 6.3 for the precise definition and its basic properties. We define the normalized GMC measure as $\widehat{\mathcal{M}}_h := \mathcal{M}_h/\mathcal{M}_h(\mathbb{S}^1)$ such that the total measure on the circle is equal to 1.

Proposition 6.1.5. *Let $\mathcal{M}_h = \mathcal{M}_h^\gamma$ be the GMC measure corresponding to an LGF h for $\gamma \in (0,2]$. If $\varphi \in \mathrm{WP}(\mathbb{S}^1)$, then the following coordinate change rule holds for its pull-back: almost surely,*

$$\varphi^*\mathcal{M}_h = \mathcal{M}_{h\circ\varphi + Q\log|\varphi'|} \tag{6.1.4}$$

where $Q = \frac{\gamma}{2} + \frac{2}{\gamma}$. Furthermore, for $\varphi \in \mathrm{Homeo}_+(\mathbb{S}^1)$, the law of the normalized pull-back measure $\varphi^*\widehat{\mathcal{M}}_h$ is absolutely continuous with respect to the law of $\widehat{\mathcal{M}}_h$ if and only if $\varphi \in \mathrm{WP}(\mathbb{S}^1)$.

The coordinate change formula (6.1.4) was proven to hold for 2D log-correlated field in [99]. This proof can be applied straightforwardly for the 1D LGF when φ is a diffeomorphism of $\mathbb{S}^1$ (Lemma 6.3.8). We prove the first part of Proposition 6.1.5 by approximating an arbitrary $\varphi \in \mathrm{WP}(\mathbb{S}^1)$ with diffeomorphisms and showing that the corresponding GMC measures converge (Lemma 6.3.9). To show the second part of Proposition 6.1.5, we use that the GMC measure almost surely determines the LGF as proved in [39] (for the $\gamma = 2$ case, in [242]).

The final step of our proof of Theorem 6.1.2 is based on the theory of conformal welding of Liouville quantum gravity (LQG) surfaces. In particular, Ang, Holden, and Sun [19] proved that the conformal welding of two independent LQG disks gives the SLE_κ loop measure. We use the following translation of this result in terms of the law of the welding homeomorphism.

If h is an LGF or a variant thereof, let

$$\phi_h^\gamma(z) := \exp(2\pi i \cdot \widehat{\mathcal{M}}_h^\gamma([1, z])),$$

where $[1, z] \subset \mathbb{S}^1$ denotes the arc running counterclockwise from 1 to z.

Lemma 6.1.6. *Let $\gamma \in (0, 2)$ and $\kappa = \gamma^2 \in (0, 4)$. If h_1 and h_2 are independent LGFs on the unit circle, then $\mathrm{SLE}_\kappa^{\mathrm{weld}}$ is mutually absolutely continuous with respect to the law of $(\phi^\gamma_{h_2 - \gamma \log|\cdot - 1|})^{-1} \circ \phi^\gamma_{h_1}$.*

Lemma 6.1.6 is a short version of Lemma 6.4.8. We will obtain Lemma 6.4.8 using conformal welding of quantum disks when each disk has one interior marked point, as established in [15] based on the works [228, 19]. Theorem 6.1.2 follows from combining Proposition 6.1.5 with Lemma 6.1.6. The conformal welding result for quantum disks in [19] is expected to hold for $\gamma = 2$, in which case Theorem 6.1.2 would extend to $\kappa = 4$.

6.1.4 Comments and related literature

The SLE_κ loop measure can be defined for $\kappa \in (0, 8)$ [255]. When $\kappa \in (4, 8)$, SLE_κ is not simple [218]. However, it was recently proved in [15] that the conformal welding of generalized quantum disks (which have the topology of infinitely many disks concatenated into a tree-like shape) gives the SLE_κ loop for $\kappa \in (4, 8)$. It would be interesting to consider if there is a family of homeomorphisms on the boundary of a generalized quantum disk that gives an analogous statement to Theorem 6.1.2.

Several recent works have considered conformal deformations of SLE loops. The conformal restriction covariance of chordal SLE was outlined by Lawler, Schramm, and Werner in [166]. Its loop version was postulated by Kontsevich and Suhov [159] and proved by Zhan in [255]. The work [238] computed the first variation of the driving function of a Loewner chain under quasiconformal deformations away from the curve, leading to the alternative proof of Loewner energy as a Kähler potential on the Weil–Petersson Teichmüller space. Based on this result, Gordina, Qian, and Wang [113] used the SLE_κ loop measure for $\kappa \in (0,4]$ to construct a natural representation of the Virasoro algebra of central charge $c \leq 1$. An independent work of Baverez and Jego [32] developed the conformal field theory for the SLE_κ loop measure and proved its characterization as a Malliavin–Kontsevich–Suhov measure for $\kappa \in (0,4]$. On the one hand, these results provide evidence for the idea that the Weil–Petersson Teichmüller space should be the Cameron–Martin space of SLE loop measures. On the other hand, deformations considered in these works are limited to those which are analytic in a neighborhood of the Jordan curve. In this work, we consider the full class of Weil–Petersson quasisymmetric homeomorphisms acting on SLE loops for the first time.

Our proof of Theorem 6.1.2 can be adapted to show quasi-invariance for other weldings derived from GMC measures that have appeared in the literature. The work [29] showed, using geometric function theory methods without reference to LQG theory, that ϕ_h^γ is a welding for $\gamma \in (0,2)$. The same work also showed that $\phi_{h_2}^{\gamma_2} \circ (\phi_{h_1}^{\gamma_1})^{-1}$, where h_1, h_2 are independent LGFs and $\gamma_1, \gamma_2 \in (0,2)$, is a welding. Using a similar approach, the articles [50, 161] showed that $(\phi_{h_2}^{\gamma_2})^{-1} \circ \phi_{h_1}^{\gamma_1}$, which is more alike the random homeomorphism in Lemma 6.1.6, is a welding for small $\gamma_1, \gamma_2 > 0$. In fact, 6.1.6 shows that the Jordan curve solving the welding problem for $(\phi_{h_2}^{\gamma})^{-1} \circ \phi_{h_1}^{\gamma}$ is locally mutually absolutely continuous with the chordal SLE_{γ^2} curve if we remove a neighborhood of the root (the image of $1 \in \mathbb{S}^1$) of the loop. We give the analogous quasi-invariance results for these random weldings in Corollary 6.4.13.

This paper is organized as follows. In Section 6.2, we introduce the groups of circle homeomorphisms that are of interest in this work and describe properties of the associated pull-back operator $\Pi(\varphi)$ on the function space $\mathcal{H}_0$. In Section 6.3, we introduce the LGF and GMC and prove their quasi-invariance under pullbacks by Weil–Petersson homeomorphisms. Section 6.4 completes the proof of the quasi-invariance of SLE welding, with the necessary development of the following concepts: conformally removable weldings and the continuity of the composition action on it by the quasisymmetric group, the SLE loop measure and the corresponding SLE welding measure, and the conformal welding of quantum disks.

6.2 Pullback operator associated with the Weil–Petersson class

In this section, we introduce the group of quasisymmetric circle homeomorphisms and the associated pullback operators on a function space of the unit circle and the log-ratio. The first result of this section is Lemma 6.2.7, where we find various correspondences between subgroups of circle homeomorphisms and the properties of pullback operators. The second result is Lemma 6.2.10, where we prove that the log-ratio is in the same space as the log-derivative.

6.2.1 Quasisymmetric homeomorphisms and the Weil–Petersson class

Let $\mathbb{S}^1$ denote the unit circle, which we identify with the boundary of the unit disk $\mathbb{D}$ in the complex plane. The set of orientation-preserving homeomorphisms of $\mathbb{S}^1$, which we denote $\mathrm{Homeo}_+(\mathbb{S}^1)$, has a natural group structure with the composition of functions as the group action. In this subsection, we introduce various subgroups of it. Some trivial examples are $\mathrm{Diff}_+(\mathbb{S}^1)$, $\mathrm{Möb}(\mathbb{S}^1)$, and $\mathrm{Rot}(\mathbb{S}^1)$, which consist of smooth diffeomorphisms, Möbius transformations, and rotations, respectively.

We shall pay special attention to the subgroup of quasisymmetric homeomorphisms. We briefly recall its definition and basic properties for the reader who is unfamiliar with the concept and direct to, e.g., [177] for further details.

Definition 6.2.1. We say that $\varphi \in \mathrm{Homeo}_+(\mathbb{S}^1)$ is quasisymmetric if there exists some $C_0 > 0$ such that

$$\frac{1}{C_0} \leq \left|\frac{\varphi(e^{i(\theta+t)}) - \varphi(e^{i\theta})}{\varphi(e^{i\theta}) - \varphi(e^{i(\theta-t)})}\right| \leq C_0 \tag{6.2.1}$$

for all $\theta \in \mathbb{R}$ and $t \in (0, 2\pi)$. Let $\mathrm{QS}(\mathbb{S}^1)$ denote the group of orientation preserving quasisymmetric homeomorphisms of the unit circle $\mathbb{S}^1$.

Beurling and Ahlfors [43] proved that $\varphi \in \mathrm{Homeo}_+(\mathbb{S}^1)$ is quasisymmetric if and only if there exists some quasiconformal homeomorphism ω of $\mathbb{D}$ onto itself that extends continuously to φ on $\mathbb{S}^1 = \partial\mathbb{D}$. That is, the Beltrami coefficient

$$\mu_\omega = \partial_{\bar{z}}\omega / \partial_z \omega$$

of ω is defined almost everywhere on $\mathbb{D}$ and satisfies $\|\mu_\omega\|_{L^\infty(\mathbb{D})} := \sup_{z\in\mathbb{D}} |\mu_\omega(z)| < 1$. Heuristically speaking, ω maps small circles centered at $z \in \mathbb{D}$ to ellipses with eccentricity $K_\omega(z)$, where the function

$$K_\omega = \frac{1 + |\mu_\omega|}{1 - |\mu_\omega|},$$

is called the dilatation of ω. The Teichmüller distance between two quasisymmetric homeomorphisms is defined as

$$\tau_1(\varphi_1, \varphi_2) = \inf\left\{ \frac{1}{2}\log \frac{1 + \|\frac{\mu_1-\mu_2}{1-\bar{\mu}_1\mu_2}\|_{L^\infty(\mathbb{D})}}{1 - \|\frac{\mu_1-\mu_2}{1-\bar{\mu}_1\mu_2}\|_{L^\infty(\mathbb{D})}} \middle| \varphi_1 = \omega_{\mu_1}|_{\mathbb{S}^1}, \varphi_2 = \omega_{\mu_2}|_{\mathbb{S}^1} \right\}.$$

The Teichmüller distance induces the natural topology on $\mathrm{QS}(\mathbb{S}^1)$. Let us further introduce two special subgroups of $\mathrm{QS}(\mathbb{S}^1)$.

Definition 6.2.2. We say that $\varphi \in \mathrm{QS}(\mathbb{S}^1)$ is symmetric if φ can be extended to a quasiconformal map ω on $\mathbb{D}$ whose Beltrami coefficient μ_ω satisfies $\mu_\omega(z) \to 0$ as $|z| \to 1$. Let $\mathrm{S}(\mathbb{S}^1)$ denote the sets of all symmetric orientation preserving homeomorphisms of $\mathbb{S}^1$.

The following subgroup, first introduced in [75], is our protagonist.

Definition 6.2.3. We say that $\varphi \in \mathrm{QS}(\mathbb{S}^1)$ belongs to the Weil–Petersson class if φ has a quasiconformal extension ω to the unit disk whose Beltrami coefficient μ_ω satisfies

$$\int_{\mathbb{D}} |\mu_\omega(z)|^2 (1-|z|^2)^{-2}\,\mathrm{d}A(z) < \infty, \tag{6.2.2}$$

where A denotes the area measure. Equivalently, φ is absolutely continuous (with respect to the arc-length measure) and $\log|\varphi'| \in H^{1/2}(\mathbb{S}^1)$. Here, $H^{1/2}(\mathbb{S}^1)$ is the fractional Sobolev space consisting of functions $f : \mathbb{S}^1 \to \mathbb{R}$ satisfying

$$\iint_{\mathbb{S}^1\times\mathbb{S}^1} \left|\frac{f(x)-f(y)}{x-y}\right|^2 \mathrm{d}x\,\mathrm{d}y < \infty. \tag{6.2.3}$$

Let $\mathrm{WP}(\mathbb{S}^1)$ denote the set of all quasisymmetric homeomorphisms of the unit circle that belong to the Weil–Petersson class.

The equivalence between the two definitions above is due to Yuliang Shen [232]. Moreover, it is proved there that the above two metric induces the same topology. The following is an equivalent definition of $H^{1/2}(\mathbb{S}^1)$.

$$H^{1/2}(\mathbb{S}^1) = \left\{ f : \mathbb{S}^1 \to \mathbb{R} \;\middle|\; f(e^{i\theta}) = c_0 + \sum_{n\in\mathbb{Z}\setminus\{0\}} c_n \frac{e^{in\theta}}{\sqrt{|n|}},\ \sum_{n=1}^{\infty} |c_n|^2 < \infty \right\}.$$

We discuss this space in further detail in the next subsection. There is a further multitude of equivalent definitions for the Weil–Petersson class related to various parts of mathematics: see [52] for a compilation.

Here is the relationship between the groups of circle homeomorphisms we have considered in this section so far.

$$\mathrm{Diff}_+(\mathbb{S}^1) \subsetneqq \mathrm{WP}(\mathbb{S}^1) \subsetneqq \mathrm{S}(\mathbb{S}^1) \subsetneqq \mathrm{QS}(\mathbb{S}^1) \subsetneqq \mathrm{RM}(\mathbb{S}^1) \subsetneqq \mathrm{Homeo}_+(\mathbb{S}^1).$$

Remark 6.2.4. The universal Teichmüller space $T(1)$ and the Weil–Petersson Teichmüller space $T_0(1)$ can be represented by $\mathrm{Möb}(\mathbb{S}^1)\backslash \mathrm{QS}(\mathbb{S}^1)$ and $\mathrm{Möb}(\mathbb{S}^1)\backslash \mathrm{WP}(\mathbb{S}^1)$, respectively. Identifying these coset spaces with the subgroup of homeomorphisms that fix $-1, -i$, and 1, the group structure on these spaces is given by composition. As hinted in the introduction, taking the quotient under left actions by $\mathrm{Möb}(\mathbb{S}^1)$ corresponds to considering the equivalence class of quasicircles on $\hat{\mathbb{C}}$ modulo Möbius transformations that fix 0. The norm (6.2.2) is inherited from the Weil–Petersson metric, which gives the universal Teichmüller space a Hilbert structure [239].

In our work, we shall be mostly interested in the Weil–Petersson Teichmüller curve $\mathcal{T}_0(1) \simeq \mathrm{Rot}(\mathbb{S}^1)\backslash \mathrm{WP}(\mathbb{S}^1)$, which we identify with the cosets of homeomorphisms that fix 1. By conformal welding, these homeomorphisms can be identified with quasicircles in $\mathbb{C}$ that disconnect 0 from ∞ modulo Möbius transformations that fix 0 and ∞. See [239, Section I.1] for further details.

6.2.2 *Sobolev spaces on the unit disk and its boundary*

In this subsection, we firstly introduce the symplectic Hilbert space $\mathcal{H}_0$ consisting of elements of the fractional Sobolev space $H^{1/2}(\mathbb{S}^1)$ with zero mean. Looking ahead, the significance of the space $\mathcal{H}_0$ is that it is the Cameron–Martin space of the log-correlated Gaussian field on $\mathbb{S}^1$ (see Section 6.3). Then we consider its compexification and the Poisson integral. Finally, we mention other Sobolev spaces.

Let $\mathcal{H}_0$ denote the real Hilbert space

$$\mathcal{H}_0 := \left\{ f : \mathbb{S}^1 \to \mathbb{R} \;\middle|\; f(e^{i\theta}) = \sum_{n\neq 0} c_n \frac{e^{in\theta}}{\sqrt{|n|}} \text{ with } c_{-n} = \overline{c_n},\ \sum_{n=1}^{\infty} |c_n|^2 < \infty \right\} \tag{6.2.4}$$

with the inner product

$$\left\langle \sum_{n\neq 0} c_n \frac{e^{in\theta}}{\sqrt{|n|}}, \sum_{n\neq 0} d_n \frac{e^{in\theta}}{\sqrt{|n|}} \right\rangle = \sum_{n\neq 0} c_n \overline{d_n}. \tag{6.2.5}$$

We consider the canonical symplectic form Θ on $\mathcal{H}_0$ introduced in [202] as

$$\Theta(f,g) = \frac{1}{2\pi}\int_{\mathbb{S}^1} f \; \mathrm{d}g = -i\sum_{n=1}^{\infty}(c_n\bar{d}_n - c_{-n}\bar{d}_{-n}). \tag{6.2.6}$$

for $f(e^{i\theta}) = \sum_{n\neq 0} c_n \frac{e^{in\theta}}{\sqrt{|n|}}$ and $g(e^{i\theta}) = \sum_{n\neq 0} d_n \frac{e^{in\theta}}{\sqrt{|n|}}$. Let us denote the group of bounded symplectomorphisms of $\mathcal{H}_0$ as $\mathrm{Sp}(\mathcal{H}_0)$.

By complex linearity, the symplectic form Θ defined in (6.2.6) extends to the complexification

$$\mathcal{H}_0^{\mathbb{C}} := \left\{ f:\mathbb{S}^1\to\mathbb{C} \;\middle|\; f(e^{i\theta}) = \sum_{n\neq 0} c_n\frac{e^{in\theta}}{\sqrt{|n|}},\ \sum_{n\neq 0}|c_n|^2<\infty \right\}. \tag{6.2.7}$$

With respect to Θ, the Hilbert space $\mathcal{H}_0^{\mathbb{C}}$ has a canonical decomposition into two closed isotropic subspaces

$$\mathcal{H}_0^{\mathbb{C}} = W_+ \oplus W_-, \tag{6.2.8}$$

where

$$W_+ = \left\{ f:\mathbb{S}^1\to\mathbb{C} \;\middle|\; f(e^{i\theta}) = \sum_{n=1}^{\infty} a_n\frac{e^{in\theta}}{\sqrt{n}},\ \sum_{n=1}^{\infty}|a_n|^2<\infty \right\}, \tag{6.2.9}$$

$$W_- = \left\{ f:\mathbb{S}^1\to\mathbb{C} \;\middle|\; f(e^{i\theta}) = \sum_{n=1}^{\infty} b_n\frac{e^{-in\theta}}{\sqrt{n}},\ \sum_{n=1}^{\infty}|b_n|^2<\infty \right\}. \tag{6.2.10}$$

Let

$$\left\{ e_n = \frac{e^{in\theta}}{\sqrt{n}} \right\}_{n\geq 1} \quad\text{and}\quad \left\{ f_n = \frac{e^{-in\theta}}{\sqrt{n}} \right\}_{n\geq 1} \tag{6.2.11}$$

denote the standard bases of the subspaces W_+ and W_-, respectively. Under these bases, W_+ and W_- are naturally isomorphic to $\ell^2(\mathbb{C})$.

Each element of $\mathrm{Sp}(\mathcal{H}_0)$, the group of bounded symplectomorphism of $\mathcal{H}_0$, extends to $\mathcal{H}_0^{\mathbb{C}}$ again by complex linearity. In the basis $\{e_n\}_{n\geq 1}$ and $\{f_n\}_{n\geq 1}$, they can be represented by the matrices

$$\begin{pmatrix} M & N \\ \bar{N} & \bar{M} \end{pmatrix} \text{ where } MM^* - NN^* = I, MN^t = NM^t. \tag{6.2.12}$$

Above, M^* denotes the adjoint matrix of M and M^t denotes the transpose matrix of M.

Let us now consider the relationship between $\mathcal{H}_0$ and the Dirichlet class of functions on the unit disk $\mathbb{D}$.

- The space $\mathcal{H}_0$ is naturally isomorphic to the real Hilbert space $\mathcal{D}_0$ of harmonic functions F on the unit disk $\mathbb{D}$ with $F(0) = 0$ and finite Dirichlet energy. That is,

$$\mathcal{D}_0 = \left\{ F : \mathbb{D} \to \mathbb{R} \;\middle|\; F(z) = \sum_{n>0} c_n \frac{z^n}{\sqrt{n}} + c_{-n} \frac{\bar{z}^n}{\sqrt{n}}, c_{-n} = \overline{c_n}, \sum_{n=1}^{\infty} |c_n|^2 < \infty \right\}. \tag{6.2.13}$$

  If $F \in \mathcal{D}_0$, then it is straightforward to check that the trace $f := F|_{\mathbb{S}^1}$ on $\mathbb{S}^1$ is an element of $\mathcal{H}_0$ with

$$\|f\|_{\mathcal{H}_0}^2 = \frac{1}{2\pi} \int_{\mathbb{D}} |\nabla F|^2 < \infty. \tag{6.2.14}$$

  On the other hand, let $\mathrm{P}(f)$ denote the Poisson integral of an integrable function f on the unit circle $\mathbb{S}^1$: i.e.,

$$\mathrm{P}(f)(z) = \frac{1}{2\pi i} \int_{\mathbb{S}^1} \mathrm{Re} \frac{w+z}{w-z} \frac{f(w)}{w} \, \mathrm{d}w, \text{ for } z \in \mathbb{D}. \tag{6.2.15}$$

  Then, for each $f \in \mathcal{H}_0$, we have $\mathrm{P}(f) \in \mathcal{D}_0$ with $f = \mathrm{P}(f)|_{\mathbb{S}^1}$. Clearly, this isomorphism between $\mathcal{H}_0$ and $\mathcal{D}_0$ given by the trace operator and the Poisson integral extends naturally to that between their complexifications.

- Let $H^1(\mathbb{D})$ (resp. $H_0^1(\mathbb{D})$) be the real Hilbert space given by the completion of the space of smooth functions on $\mathbb{D}$ (resp. with compact support) with respect to the Dirichlet inner product. Then, we have the decomposition

$$H^1(\mathbb{D}) = H_0^1(\mathbb{D}) \oplus \mathcal{D}_0 \oplus \mathbb{R} \tag{6.2.16}$$

as a direct sum with respect to the Dirichlet inner product. We will see later that, from the decomposition above, there exists a decomposition of the Neumann Gaussian free field into the sum of independent Dirichlet Gaussian free field and the "harmonic" Gaussian field on $\mathbb{D}$ (see Remark 6.3.1).

We conclude this subsection by giving a quick overview of fractional Sobolev spaces of general index $s \in \mathbb{R}$ on the unit disk $\mathbb{D}$ and its boundary $\mathbb{S}^1$.

First, let us consider the space $H_0^s(\mathbb{D})$ with zero boundary conditions. Let $\{g_n\}_{n\geq 1}$ be a sequence of eigenfunctions of the Laplacian $-\Delta$ on $\mathbb{D}$ with Dirichlet boundary conditions which are mutually orthogonal with respect to the Dirichlet inner product. Let $\{\lambda_n\}_{n\geq 1}$ be the corresponding eigenvalues. That is, g_n and λ_n satisfy

$$\begin{cases} -\Delta g_n = \lambda_n g_n & \text{in } \mathbb{D} \\ g_n = 0 & \text{on } \partial\mathbb{D} \end{cases} \tag{6.2.17}$$

for each n. The eigenvalues $\{\lambda_n\}_{n\geq 1}$ are positive and satisfy $\lambda_n \to \infty$ as $n \to \infty$.

For each $s \in \mathbb{R}$, we define $H_0^s(\mathbb{D})$ as the real Hilbert space obtained by taking the completion of the set of smooth, compactly supported real-valued functions on the unit disk with respect to the inner product

$$\langle f, g\rangle_s = \sum_{n\geq 1} \lambda_n^s \langle f, g_n\rangle_{L^2} \langle g, g_n\rangle_{L^2}. \tag{6.2.18}$$

When $s = 1$, this inner product is the standard Dirichlet inner product on $\mathbb{D}$.

We define the Sobolev space $H^s(\mathbb{D})$ with free boundary conditions analogously using the eigenfunctions of $-\Delta$ on $\mathbb{D}$ with Neumann boundary conditions. For the eigenvalue 0 corresponding to the constant function, we let $0^s := 1$ for all $s \in \mathbb{R}$ in (6.2.18).

Similarly, we can define the Sobolev space $H^s(\mathbb{S}^1)$ of functions on $\mathbb{S}^1$ replacing the operator $-\Delta$ by $-\partial_{\theta\theta}$. Then, $\mathcal{H}_0$ agrees with the closed subspace $H^{1/2}(\mathbb{S}^1)/\mathbb{R}$ of functions $f \in \mathcal{H}$ with $\int_{\mathbb{S}^1} f = 0$ ("mean zero"). In other words, $\mathcal{H}$ consists of functions in $\mathcal{H}_0$ plus a constant. Let us denote the natural projection $H^{1/2}(\mathbb{S}^1) \to \mathcal{H}_0$ as

$$\pi_0(f) := f - \frac{1}{2\pi} \int_{\mathbb{S}^1} f. \tag{6.2.19}$$

The equivalence between this definition of $H^{1/2}(\mathbb{S}^1)$ and the condition (6.2.3) can be found in introductory texts on fractional Sobolev spaces.

6.2.3 The pullback operator

In this subsection, we introduce the pullback operator as a right group action of $\mathrm{Homeo}_+(\mathbb{S}^1)$ on $\mathcal{H}_0$. Then we prove a key relationship (Lemma 6.2.7) with the Weil–Petersson class.

Definition 6.2.5. Given a orientation preserving homeomorphism $\varphi \in \mathrm{Homeo}_+(\mathbb{S}^1)$, we define the pullback operator $\Pi(\varphi)$ on $\mathcal{H}_0^{\mathbb{C}}$ as

$$\Pi(\varphi)(f) := \pi_0(f \circ \varphi) = f \circ \varphi - \frac{1}{2\pi} \int_{\mathbb{S}^1} f \circ \varphi \, \mathrm{d}\theta. \tag{6.2.20}$$

Let us denote the space of bounded linear operators from $\mathcal{H}_0^{\mathbb{C}}$ to itself as $\mathscr{B}(\mathcal{H}_0^{\mathbb{C}})$. Nag and Sullivan [202] proved that $\Pi(\varphi) \in \mathscr{B}(\mathcal{H}_0^{\mathbb{C}})$ if and only if $\varphi \in \mathrm{QS}(\mathbb{S}^1)$, and that the assignment $\Pi : \mathrm{QS}(\mathbb{S}^1) \to \mathscr{B}(\mathcal{H}_0^{\mathbb{C}})$ defines a right group action on $\mathcal{H}_0^{\mathbb{C}}$ by symplectomorphisms. In the basis $\{e_n\}_{n\geq 1}$ and $\{f_n\}_{n\geq 1}$ given in (6.2.11), the symplectomorphism $\Pi(\varphi)$ for $\varphi \in \mathrm{QS}(\mathbb{S}^1)$

can be represented by the matrix of the form (6.2.12), whose entries are given by

$$M_{mn}(\varphi) = \frac{1}{2\pi}\sqrt{\frac{m}{n}}\int_{\mathbb{S}^1} \left(\varphi(e^{i\theta})\right)^n e^{-im\theta}\mathrm{d}\theta, \tag{6.2.21}$$

$$N_{mn}(\varphi) = \frac{1}{2\pi}\sqrt{\frac{m}{n}}\int_{\mathbb{S}^1} \left(\varphi(e^{i\theta})\right)^{-n} e^{-im\theta}\mathrm{d}\theta. \tag{6.2.22}$$

We note that the operator $\Pi(\varphi)$ preserves the subspaces W_+ and W_- (i.e., $\Pi(\varphi)$ belongs to the unitary subgroup $\mathrm{U}(\mathcal{H}_0)$ of $\mathrm{Sp}(\mathcal{H}_0)$ consisting of bounded symplectomorphisms with $N = 0$) if and only if $\varphi \in \mathrm{Möb}(\mathbb{S}^1)$.

We now give the key result of this section, which will be used in Section 6.3 to identify the quasi-invariance of the log-correlated Gaussian field on $\mathbb{S}^1$ under pullbacks by $\mathrm{Homeo}_+(\mathbb{S}^1)$. First, let us recall the definition of a Hilbert–Schmidt operator.

Definition 6.2.6. For any bounded linear operator T from a Hilbert space H to itself, we define the Hilbert–Schmidt norm $\|T\|_{\mathrm{HS}}$ by

$$\|T\|_{\mathrm{HS}}^2 := \sum_{\mathrm{a}\in\mathcal{A}} \|Te_{\mathrm{a}}\|_H^2 = \mathrm{Tr}(T^*T), \tag{6.2.23}$$

where $\{e_{\mathrm{a}}, \mathrm{a} \in \mathcal{A}\}$ is any orthonormal basis of H and T^* is the adjoint operator of T. For a self-adjoint bounded linear operator A, the trace $\mathrm{Tr}(A)$ of A is defined by the sum of all its eigenvalues if the series is absolutely summable and set to be $+\infty$ otherwise. We say T is Hilbert–Schmidt if $\|T\|_{\mathrm{HS}} < \infty$. The collection of all Hilbert–Schmidt operators on H forms a Hilbert space $\mathrm{HS}(H)$ with respect to the norm (6.2.23).

Note that if $H^{\mathbb{C}}$ is the complexification of H, then the complex extension of T belongs to $\mathrm{HS}(H^{\mathbb{C}})$ if and only if $T \in \mathrm{HS}(H)$. If $T \in \mathrm{HS}(H)$ and $S \in \mathscr{B}(H)$, then T^t, T^*, ST, and TS belong to $\mathrm{HS}(H)$.

Lemma 6.2.7. *Assume $\varphi \in \mathrm{QS}(\mathbb{S}^1)$ and let $\Pi(\varphi) \in \mathscr{B}(\mathcal{H}_0)$ be the pullback operator defined by (6.2.20). Then, $\Pi(\varphi)\Pi(\varphi)^* - I$ is Hilbert–Schmidt if and only if $\varphi \in \mathrm{WP}(\mathbb{S}^1)$.*

Proof. Recall the matrices M and N associated with $\Pi(\varphi)$ as given by (6.2.21)–(6.2.22). Since $\Pi(\varphi) \in \mathrm{Sp}(\mathcal{H}_0)$, we have

$$\begin{pmatrix} M & N \\ \bar{N} & \bar{M} \end{pmatrix} \begin{pmatrix} M^* & -N^t \\ -N^* & M^t \end{pmatrix} = I.$$

as in (6.2.12). Thus,

$$\Pi(\varphi)\Pi(\varphi)^* - I = \begin{pmatrix} M & N \\ \bar{N} & \bar{M} \end{pmatrix} \begin{pmatrix} M^* & N^t \\ N^* & M^t \end{pmatrix} - I = 2\Pi(\varphi) \begin{pmatrix} & N^t \\ N^* & \end{pmatrix}. \tag{6.2.24}$$

In [139, Theorem 2.2], it was shown (up to the isomorphism described in the previous subsection) that N is Hilbert–Schmidt if and only if $\varphi \in \mathrm{WP}(\mathbb{S}^1)$. Combined with the fact that $\Pi(\varphi)$ and $\Pi(\varphi)^{-1} = \Pi(\varphi^{-1})$ are bounded for $\varphi \in \mathrm{QS}(\mathbb{S}^1)$, we obtain the desired result. □

Remark 6.2.8. For the same reason, the operator $\Pi(\varphi)\Pi(\varphi)^* - I$ is compact if and only if $\varphi \in \mathrm{S}(\mathbb{S}^1)$.

In fact, we will need that all characterizations of $\mathrm{WP}(\mathbb{S}^1)$ that we have discussed so far induce the same topology, which we check by retracing the proofs of equivalences among the various definitions.

Lemma 6.2.9. *If $\varphi \in \mathrm{WP}(\mathbb{S}^1)$, then there exists a sequence $\varphi_n \in \mathrm{Diff}_+(\mathbb{S}^1)$ such that $\sup_{x\in\mathbb{S}^1} |\varphi_n(x) - \varphi(x)|$, $\|\Pi(\varphi_n \circ \varphi^{-1})\Pi(\varphi_n \circ \varphi^{-1})^* - I\|_{\mathrm{HS}}$, $\|\log |(\varphi_n \circ \varphi^{-1})'|\|_{H^{1/2}(\mathbb{S}^1)}$ and $\|\Pi(\varphi_n)u - \Pi(\varphi)u\|_{H^{1/2}(\mathbb{S}^1)}$ for any $u \in H^{1/2}(\mathbb{S}^1)$ all converge to 0 as $n \to \infty$.*

Proof. It is shown in [232, Theorem 1.4] that the $H^{1/2}(\mathbb{S}^1)$ norm for $\log |\varphi'|$ induces the same topology as the Weil–Petersson metric. It is shown in [232, Proposition 4.1] that the topology for φ induced by the Teichmüller distance is finer than the strong operator topology for $\Pi(\varphi)$. From the proof of [139, Theorem 4.2], when the dilatation is bounded, this is equivalent to the Hilbert–Schmidt norm for the matrix N with entries (6.2.22) and therefore the Hilbert–Schmidt norm for the matrix $\Pi(\varphi)\Pi(\varphi)^* - I$ by (6.2.24). The uniform convergence follows from the argument using the normal family. More precisely, up to subsequence, we can choose φ_n to be the boundary of the quasiconformal self-homeomorphism ω_n of $\mathbb{D}$ that agrees with φ at $1, i, -1$ with Beltrami coefficient

$$\mu_n(z) = \mu(z)\mathbf{1}_{\{|z|<1-1/n\}}$$

where μ is the Beltrami coefficient of the Douady–Earle extension ω of φ to $\mathbb{D}$. $\square$

6.2.4 Estimates on the log-ratio

Now, we aim to informally replace the derivative with the difference quotient, but we actually use conformal welding. Looking ahead, it corresponds to adding one boundary marked point on the quantum disk, see Corollary 6.3.10. Let $H^{1/2}(\mathbb{S}^1, \mathbb{C})$ denote the complexification of $H^{1/2}(\mathbb{S}^1)$.

Lemma 6.2.10. *If $\varphi \in \mathrm{WP}(\mathbb{S}^1)$, then for a.e. $z_0 \in \mathbb{S}^1$,*

$$u_\varphi(\cdot, z_0) := \log \frac{\varphi(\cdot) - \varphi(z_0)}{\cdot - z_0} \in H^{1/2}(\mathbb{S}^1, \mathbb{C}). \tag{6.2.25}$$

Moreover, the function $z_0 \mapsto \|u_\varphi(\cdot, z_0)\|_{H^{1/2}(\mathbb{S}^1,\mathbb{C})}$ is L^2-integrable.

Proof. Recall from the introduction the conformal welding decomposition for $\varphi \in \mathrm{WP}(\mathbb{S}^1)$: there exist quasiconformal maps f and g on $\hat{\mathbb{C}}$, conformal on $\mathbb{D}$ and $\mathbb{D}^*$, respectively, such that $\varphi = (g^{-1} \circ f)|_{\mathbb{S}^1}$, which is also a direct corollary of Lemma 6.4.1 when ψ is the identity map. Since

$$u_\varphi(\cdot, z_0) = \log \frac{f(\cdot) - f(z_0)}{\cdot - z_0} - \log \frac{g \circ \varphi(\cdot) - g \circ \varphi(z_0)}{\varphi(\cdot) - \varphi(z_0)} =: u_f(\cdot, z_0) - u_g(\varphi(\cdot), \varphi(z_0))$$

and $H^{1/2}(\mathbb{S}^1,\mathbb{C})$ is invariant under pullback by a Weil–Petersson quasisymmetric map, we only need to show that $u_f = u_f(\cdot, z_0)$ and $u_g = u_g(\cdot, \varphi(z_0))$ belong to $H^{1/2}(\mathbb{S}^1,\mathbb{C})$ for almost every $z_0 \in \mathbb{S}^1$.

It is shown in [239, Chapter 2, Lemma 2.5] that $u_f(z,w) = \sum_{n,m=0}^{\infty} c_{n,m} z^n w^m$ with

$$\sum_{n,m=1}^{\infty} |nm|\, |c_{n,m}|^2, \sum_{n=1}^{\infty} |n|\, |c_{n,0}|^2, \sum_{m=1}^{\infty} |m|\, |c_{0,m}|^2 < \infty,$$

which is equivalent to the associated Grunsky operator being Hilbert–Schmidt. Define $F_n(e^{i\psi}) := \sum_{m=0}^{\infty} c_{n,m} e^{im\psi}$ such that $\|F_n\|^2_{H^{1/2}(\mathbb{S}^1,\mathbb{C})} = \sum_{m=1}^{\infty} |m|\, |c_{n,m}|^2$, then

$$\|u_f(\cdot, z_0)\|^2_{H^{1/2}(\mathbb{S}^1,\mathbb{C})} = |F_0(z_0)|^2 + \sum_{n=1}^{\infty} |n|\, |F_n(z_0)|^2 =: F(z_0).$$

It follows that

$$\begin{aligned} \|F\|_{L^1(\mathbb{S}^1)} &= \|F_0\|^2_{L^2(\mathbb{S}^1)} + \sum_{n=1}^{\infty} |n|\, \|F_n\|^2_{L^2(\mathbb{S}^1)} \\ &\le \left|c_{0,0}\right|^2 + \sum_{m=1}^{\infty} |m|\, |c_{0,m}|^2 + \sum_{n=1}^{\infty} |n| \left(\|F_n\|^2_{H^{1/2}(\mathbb{S}^1,\mathbb{C})} + |c_{n,0}|^2\right) < \infty. \end{aligned}$$

The case for g is similar, but we need to control the constant term that comes from the difference between composition and pullback Π by

$$\sum_n \left| \int_{\mathbb{S}^1} \hat{h}_n(\varphi(e^{i\theta}))\, \mathrm{d}\theta \right|^2 = \sum_n \left| \int_{\mathbb{S}^1} \hat{h}_n(e^{i\theta})\, |(\varphi^{-1})'|(e^{i\theta})\, \mathrm{d}\theta \right|^2 \le \|\log(\varphi^{-1})'\|^2_{H^{1/2}(\mathbb{S}^1,\mathbb{C})} < \infty,$$

where $\hat{h}_n$ is an orthonormal basis of $\mathcal{H}_0^{\mathbb{C}}$. □

Remark 6.2.11. When $\varphi \in C^3(\mathbb{S}^1) \supsetneq \mathrm{WP}(\mathbb{S}^1)$ is three times continuously differentiable, the proof is straightforward at every point.

Remark 6.2.12. For the same reason, plus controlling the term $c_{n,m}$ when $nm = 0$, we can show that if $\varphi \in \mathrm{WP}(\mathbb{S}^1)$, then

$$u_\varphi(e^{i\theta}, e^{i\psi}) = \sum_{n,m=-\infty}^{\infty} c_{n,m} e^{i(n\theta+m\psi)}, \tag{6.2.26}$$

$$\text{where} \sum_{n,m=-\infty}^{\infty} |nm|\,|c_{n,m}|^2, \sum_{n=-\infty}^{\infty} |n|\,\left|c_{n,0}\right|^2, \sum_{m=-\infty}^{\infty} |m|\,\left|c_{0,m}\right|^2 < \infty. \tag{6.2.27}$$

It is nothing but the condition for the covariance functions of two equivalent LGFs. So we conjecture that (6.2.26) and (6.2.27) hold if and only if $\varphi \in \mathrm{WP}(\mathbb{S}^1)$ without assuming $\varphi \in \mathrm{QS}(\mathbb{S}^1)$.

6.3 Quasi-invariance of Gaussian fields and boundary measures

In this section, we show that the law of the log-correlated Gaussian field on $\mathbb{S}^1$ is quasi-invariant under the pullback by $\varphi \in \mathrm{QS}(\mathbb{S}^1)$ if and only if φ is of the Weil–Petersson class. We then "lift" this characterization to that for the Gaussian multiplicative chaos on $\mathbb{S}^1$.

6.3.1 Preliminaries of Log-correlated Gaussian field on the unit circle

In this subsection, we survey the definition and the basic properties of the log-correlated Gaussian field on the unit circle. This is a well-researched object, and we do not aim to be comprehensive in our introduction; we direct the reader to surveys such as [98] for further information.

The (mean-zero) log-correlated Gaussian field on $\mathbb{S}^1$ (abbreviated as LGF) is a random generalized function on $\mathbb{S}^1$ which has a centered Gaussian law with Cameron–Martin space $\mathcal{H}_0$ and the Cameron–Martin norm $\|\cdot\|_{\mathcal{H}_0}$ That is,

$$h = \sum_{n\geq 1} \xi_n h_n, \tag{6.3.1}$$

where $\{h_n\}_{n\geq 1}$ is an orthonormal basis of $\mathcal{H}_0$ comprised of continuous functions and $\{\xi_n\}_{n\geq 1}$ is an i.i.d. sequence of standard normal random variables, is an instance of LGF on $\mathbb{S}^1$. For concreteness, we can take

$$h_{2n-1}(e^{i\theta}) := \frac{e_n + f_n}{\sqrt{2}}(e^{i\theta}) = \sqrt{\frac{2}{n}}\cos(n\theta), \quad h_{2n}(e^{i\theta}) := \frac{e_n - f_n}{\sqrt{2}i}(e^{i\theta}) = \sqrt{\frac{2}{n}}\sin(n\theta), \tag{6.3.2}$$

where $\{e_n, f_n\}_{n\geq 1}$ is the standard basis for $\mathcal{H}_0^{\mathbb{C}}$ introduced in (6.2.11). However, the definition (6.3.1) is independent of the choice of the orthonormal basis $\{h_n\}_{n\geq 1}$ for $\mathcal{H}_0$.

The sum (6.3.1) can be seen to converge almost surely in $H^s(\mathbb{S}^1)$ for any $s < 0$. Equivalently, we may consider h in terms of the centered Gaussian process $\{\langle h, f\rangle\}_{f\in\mathcal{H}_0}$ where $\langle\, ,\, \rangle$ is the inner product (6.2.5) on $\mathcal{H}_0$. That is,

$$\left\langle h, \sum_{n\geq 1} c_n h_n \right\rangle := \sum_{n\geq 1} \xi_n c_n \tag{6.3.3}$$

for $\sum_{n\geq 1} c_n h_n \in \mathcal{H}_0$. Note that for $f \in \mathcal{H}_0$, we have $\mathrm{Var}(\langle h, f\rangle) = \|f\|^2_{\mathcal{H}_0}$. More generally, we consider

$$(h, \rho) := \sum_{n\geq 1} \xi_n \int_{\mathbb{S}^1} h_n(e^{i\theta})\, \mathrm{d}\rho(e^{i\theta}) \tag{6.3.4}$$

with signed Borel measures ρ on $\mathbb{S}^1$ for which the above sum is almost surely absolutely convergent. Then, such pairings (h, ρ) form a continuous version of the centered Gaussian process satisfying

$$\mathrm{Cov}((h, \rho), (h, \tilde{\rho})) = \iint_{\mathbb{S}^1\times\mathbb{S}^1} G(e^{i\theta}, e^{i\tilde{\theta}})\, \mathrm{d}\rho(e^{i\theta})\, \mathrm{d}\tilde{\rho}(e^{i\tilde{\theta}}), \tag{6.3.5}$$

where

$$G(e^{i\theta}, e^{i\tilde{\theta}}) := \sum_{n\geq 1} h_n(e^{i\theta}) h_n(e^{i\tilde{\theta}}) = -2\log\left|e^{i\theta} - e^{i\tilde{\theta}}\right| \tag{6.3.6}$$

is the covariance kernel of the LGF h. Note that for $f \in \mathcal{H}_0$, we have

$$\langle h, f \rangle = \left(h, \frac{(-\partial_{\theta\theta})^{1/2} f(e^{i\theta})}{2\pi} \, \mathrm{d}\theta \right), \tag{6.3.7}$$

where $(-\partial_{\theta\theta})^{1/2}$ is the linear operator from $\mathcal{H}_0$ to $L^2(\mathbb{S}^1)$ which maps $\cos(n\theta)$ to $n\cos(n\theta)$ and $\sin(n\theta)$ to $n\sin(n\theta)$ for each positive integer n.

This definition of the LGF on $\mathbb{S}^1$ is similar to that of the Gaussian free field (GFF) on the unit disk $\mathbb{D}$, which is given by the sum (6.3.1) where $\{h_n\}_{n\geq 1}$ is chosen to be a sequence of functions on $\mathbb{D}$. The pairing of GFF with signed Borel measures ρ on $\mathbb{D}$ is defined analogously as in (6.3.4).

- The zero boundary (Dirichlet) GFF on $\mathbb{D}$ is given by choosing $\{h_n\}_{n\geq 1}$ to be any orthonormal basis of $H_0^1(\mathbb{D})$. The corresponding covariance kernel is given by the Dirichlet Green's function

  $$G^{\text{zero}}(z, w) = -\log |(z-w)/(1-z\bar{w})|. \tag{6.3.8}$$

  We use Γ to denote a zero boundary GFF.

- We obtain the harmonic Gaussian field on $\mathbb{D}$ by choosing $\{h_n\}_{n\geq 1}$ to be any orthonormal basis of $\mathcal{D}_0$. The covariance kernel is given by

  $$G(z, w) = -2\log |1 - z\bar{w}|, \tag{6.3.9}$$

  which agrees with (6.3.6) for $z, w \in \mathbb{S}^1$. By the isomorphism between $\mathcal{D}_0$ and $\mathcal{H}_0$ given in Section 6.2, we can identify it as the "harmonic extension" of an LGF to $\mathbb{D}$. We will use h to denote both an LGF on $\mathbb{S}^1$ and its harmonic extension on $\mathbb{D}$.

- The free boundary (Neumann) GFF on $\mathbb{D}$ (with mean zero on $\mathbb{S}^1$) is obtained by choosing $\{h_n\}_{n\geq 1}$ to be any orthonormal basis of $H^1(\mathbb{D})/\mathbb{R}$. The corresponding covariance kernel is given by the Neumann Green's function

$$G^{\text{free}}(z,w) = -\log|(z-w)(1-z\bar{w})|. \tag{6.3.10}$$

We use $\overline{\Gamma}$ to denote a free boundary GFF.

Remark 6.3.1. By the decomposition (6.2.16), if Γ and h are independent zero boundary GFF and harmonic Gaussian field on $\mathbb{D}$, respectively, then $\Gamma + h$ has the law of a free boundary GFF on $\mathbb{D}$. This can also be seen from the identify $G^{\text{zero}} + G = G^{\text{free}}$ satisfied by the covariance kernels (6.3.8), (6.3.9), and (6.3.10). We emphasize that the multiplicative factor of 2 in (6.3.9) (hence also in (6.3.6)) is necessary for this relationship to hold.

6.3.2 Quasi-invariance of Log-correlated Gaussian field

In this subsection, our goal is to culminate in the quasi-invariance result for pullbacks with respect to Weil-Petersson homeomorphisms (Theorem 6.3.3).

Given $\varphi \in \mathrm{Homeo}_+(\mathbb{S}^1)$, we define the pullback of an LGF $h = \sum_{n\geq 1} \xi_n h_n$ formally as

$$h \circ \varphi := \sum_{n\geq 1} \xi_n (h_n \circ \varphi). \tag{6.3.11}$$

This can be rigorously considered as the centered Gaussian process

$$(h \circ \varphi, \rho) := (h, \varphi_* \rho) \tag{6.3.12}$$

indexed by signed Borel measures ρ on $\mathbb{S}^1$ for which the right-hand side of (6.3.12) is well-defined. Here, $\varphi_*\rho$ is the pushforward of ρ under the homeomorphism φ. Note that the covariance kernel of $h \circ \varphi$ is

$$G_\varphi(e^{i\theta}, e^{i\tilde{\theta}}) := G(\varphi(e^{i\theta}), \varphi(e^{i\tilde{\theta}})) = -2\log\left|\varphi(e^{i\theta}) - \varphi(e^{i\tilde{\theta}})\right| \tag{6.3.13}$$

since

$$\begin{aligned}\operatorname{Cov}\big((h\circ\varphi,\rho),(h\circ\varphi,\tilde{\rho})\big) &= \operatorname{Cov}\big((h,\varphi_*\rho),(h,\varphi_*\tilde{\rho})\big)\\ &= \iint_{\mathbb{S}^1\times\mathbb{S}^1} G(e^{i\theta}, e^{i\tilde{\theta}})\, d(\varphi_*\rho)(e^{i\theta})\,\mathrm{d}(\varphi_*\tilde{\rho})(e^{i\tilde{\theta}})\\ &= \iint_{\mathbb{S}^1\times\mathbb{S}^1} G(e^{i\theta}, e^{i\tilde{\theta}})\, d\rho(e^{i\theta})\,\mathrm{d}\tilde{\rho}(e^{i\tilde{\theta}}).\end{aligned}$$

If $\varphi \in \mathrm{QS}(\mathbb{S}^1)$, then we define the mean-zero part of the pullback $h \circ \varphi$ of the LGF $h = \sum_{n\geq 1}\xi_n h_n$ formally as

$$\Pi(\varphi)(h) = \pi_0(h\circ\varphi) := \sum_{n\geq 1}\xi_n \Pi(\varphi)(h_n). \tag{6.3.14}$$

For quasisymmetric φ, since $\Pi(\varphi)$ is a bounded linear operator on $\mathcal{H}_0$, the (6.3.14) defines $\Pi(\varphi)(h)$ as a centered Gaussian field on $\mathbb{S}^1$ with the Cameron–Martin space $\mathcal{H}_0$ and the covariance operator $\Pi(\varphi)\Pi(\varphi)^*$. That is, $\{\langle\Pi(\varphi)(h), f\rangle = \sum_{n\geq 1}\xi_n\langle\Pi(\varphi)(h_n), f\rangle\}_{f\in\mathcal{H}_0}$ is a centered Gaussian process with

$$\operatorname{Cov}(\langle\Pi(\varphi)(h), f\rangle, \langle\Pi(\varphi)(h), g\rangle) = \langle\Pi(\varphi)^* f, \Pi(\varphi)^* g\rangle = \langle f, \Pi(\varphi)\Pi(\varphi)^* g\rangle. \tag{6.3.15}$$

Remark 6.3.2. The relationship between $h\circ\varphi$ and $\Pi(\varphi)(h)$ is the following: if

$$\iint_{\mathbb{S}^1\times\mathbb{S}^1} G_\varphi(e^{i\theta}, e^{i\tilde{\theta}})\,\mathrm{d}\theta\,\mathrm{d}\tilde{\theta} < \infty \tag{6.3.16}$$

so that $\fint_{\mathbb{S}^1} h \circ \varphi := (h \circ \varphi, (2\pi)^{-1}\,\mathrm{d}\theta)$ is an a.s. finite random variable, then

$$(\Pi(\varphi)(h), \rho) = (h \circ \varphi, \rho) - \rho(\mathbb{S}^1) \fint_{\mathbb{S}^1} h \circ \varphi \tag{6.3.17}$$

a.s. for signed Borel measures ρ on $\mathbb{S}^1$ for which $\iint_{\mathbb{S}^1 \times \mathbb{S}^1} |G_\varphi(e^{i\theta}, e^{i\tilde{\theta}})|\,\mathrm{d}\rho(e^{i\theta})\,\mathrm{d}\rho(e^{i\tilde{\theta}}) < \infty$. This can be seen directly from the decompositions (6.3.11) and (6.3.14) for $h\circ\varphi$ and $\Pi(\varphi)(h)$, respectively.

One sufficient condition for (6.3.16) is for φ to be a diffeomorphism, since then

$$\tilde{u}_\varphi(e^{i\theta}, e^{i\tilde{\theta}}) := \log \left| \frac{\varphi(e^{i\theta}) - \varphi(e^{i\tilde{\theta}})}{e^{i\theta} - e^{i\tilde{\theta}}} \right| \tag{6.3.18}$$

is a continuous function on $\mathbb{S}^1 \times \mathbb{S}^1$ with $\tilde{u}_\varphi(e^{i\theta}, e^{i\theta}) = \log|\varphi'(e^{i\theta})|$. Hence, $G_\varphi = G - 2\tilde{u}_\varphi$ is integrable with respect to the arc-length measure on $\mathbb{S}^1 \times \mathbb{S}^1$.

More generally, it suffices for φ^{-1} to be absolutely continuous with respect to the arc-length measure and satisfy $|(\varphi^{-1})'| \in H^{-1/2}(\mathbb{S}^1)$. Then,

$$\begin{aligned}
\iint_{\mathbb{S}^1\times\mathbb{S}^1} G_\varphi(e^{i\theta}, e^{i\tilde{\theta}})\,\mathrm{d}\theta\,\mathrm{d}\tilde{\theta} &= \iint_{\mathbb{S}^1\times\mathbb{S}^1} \left(\sum_{n\geq 1} (h_n \circ \varphi)(e^{i\theta})\,(h_n\circ\varphi)(e^{i\tilde{\theta}}) \right) \mathrm{d}\theta\,\mathrm{d}\tilde{\theta} \\
&= \sum_{n\geq 1} \left| \int_{\mathbb{S}^1} h_n(\varphi(e^{i\theta}))\,\mathrm{d}\theta \right|^2 = \sum_{n\geq 1} \left| \int_{\mathbb{S}^1} h_n(e^{i\theta})\,|(\varphi^{-1})'|(e^{i\theta})\,\mathrm{d}\theta \right|^2 \\
&= \| |(\varphi^{-1})'| \|^2_{H^{-1/2}(\mathbb{S}^1)/\mathbb{R}} < \infty.
\end{aligned} \tag{6.3.19}$$

In particular, if $\varphi \in \mathrm{WP}(\mathbb{S}^1)$, then $|(\varphi^{-1})'| \in L^2(\mathbb{S}^1)$; this follows from the standard argument using the VMO space that $\log|\varphi'| \in H^{1/2}(\mathbb{S}^1)$ implies $|\varphi'|^p = \exp(p\log|\varphi'|) \in L^1(\mathbb{S}^1)$ for any $p \geq 1$. Hence, we have (6.3.16) in this case.

We now give our first main result, which identifies $\mathrm{WP}(\mathbb{S}^1)$ as the class of quasisymmetric circle homeomorphisms φ for which $\Pi(\varphi)(h)$ and h have equivalent laws.

Theorem 6.3.3. *Suppose $\varphi \in \mathrm{QS}(\mathbb{S}^1)$ and f is a fixed real-valued function on $\mathbb{S}^1$. Let $\Pi(\varphi)(h)$ be the Gaussian field given in (6.3.14), where h is an LGF on $\mathbb{S}^1$. Then, the law of the random field $\Pi(\varphi)(h)+f$ is equivalent to that of an LGF on $\mathbb{S}^1$ if and only if $\varphi \in \mathrm{WP}(\mathbb{S}^1)$ and $f \in \mathcal{H}_0$. Otherwise, the two laws are mutually singular.*

Proof. The Feldman–Hájek theorem states that the laws of h and $\Pi(\varphi)(h) + f$ are either equivalent or mutually singular, and the former holds if and only if

- The Cameron–Martin space for the law of $\Pi(\varphi)(h) + f$ is $\mathcal{H}_0$;
- The difference in the means, f, lies in the common Cameron–Martin space $\mathcal{H}_0$;
- The difference in the covariance operators, $\Pi(\varphi)\Pi(\varphi)^* - I$, is a Hilbert–Schmidt operator on the common Cameron–Martin space $\mathcal{H}_0$.

See, e.g., [79, Theorem 2.25]. By Lemma 6.2.7, these conditions are satisfied if and only if $\varphi \in \mathrm{WP}(\mathbb{S}^1)$ and $f \in \mathcal{H}_0$.

Recalling from Definition 6.2.3 that $\varphi \in \mathrm{Homeo}_+(\mathbb{S}^1)$ is in the Weil–Petersson class if and only if $\log|\varphi'| \in H^{1/2}(\mathbb{S}^1)$, we immediately obtain the following corollary from Theorem 6.3.3. □

Corollary 6.3.4. *Let $\varphi \in \mathrm{QS}(\mathbb{S}^1)$ and Q be a real constant. Furthermore, let $\tilde{u}_\varphi$ be the function on $\mathbb{S}^1 \times \mathbb{S}^1$ given in (6.3.18) and let $z_0 \in \mathbb{S}^1$, $\alpha \in \mathbb{R}$ be fixed. If h is an LGF on $\mathbb{S}^1$, then the law of the random field*

$$\Pi(\varphi)(h) + \pi_0\left(Q \log\left|\varphi'\right| + \alpha\tilde{u}_\varphi(\cdot, z_0)\right)$$

is equivalent to that of h if and only if $\varphi \in \mathrm{WP}(\mathbb{S}^1)$ and either of the following holds: $\alpha = 0$ or $\tilde{u}_\varphi(\cdot, z_0) \in H^{1/2}(\mathbb{S}^1)$. Otherwise, the two laws are mutually singular.

6.3.3 Preliminaries of Gaussian multiplicative chaos

Now we shift our attention to the Gaussian multiplicative chaos (GMC) measure on the unit circle. Given $\gamma \in (0,2)$, the γ-GMC measure with respect to an LGF h on $\mathbb{S}^1$ is defined formally as

$$\mathcal{M}_h^{\gamma}(\mathrm{d}\theta) = e^{\frac{\gamma}{2}h(e^{i\theta}) - \frac{\gamma^2}{8}\mathbb{E}[h(e^{i\theta})^2]}\,\mathrm{d}\theta. \tag{6.3.20}$$

We omit the superscript γ and write $\mathcal{M}_h = \mathcal{M}_h^{\gamma}$ if there is no confusion. A rigorous study of GMC was initiated by Kahane [146] with seminal contributions by Robert and Vargas [216], Duplantier and Sheffield [99], and Shamov [224]. The GMC measure $\mathcal{M}_h$ is defined rigorously via renormalization. The following two equivalent approaches almost surely give the same measure as shown in [37].

- Let $\{h_k\}_{k\geq 1}$ be a sequence of continuous functions on $\mathbb{S}^1$ forming an orthonormal basis for $\mathcal{H}_0$. Given an LGF $h = \sum_{k\geq 1} \xi_k h_k$, we define $\mathcal{M}_h^{\gamma}$ as the almost sure weak limit of the measures

$$\exp\left(\frac{\gamma}{2}\sum_{k=1}^{n}\xi_k h_k(e^{i\theta}) - \frac{\gamma^2}{8}\sum_{k=1}^{n}\left(h_k(e^{i\theta})\right)^2\right)\mathrm{d}\theta \tag{6.3.21}$$

  as $n \to \infty$.

- Let σ be a fixed nonnegative Radon measure on the interval $(-\pi,\pi)$ with unit mass and $\sup_{x\in(-\pi,\pi)}\int_{-\pi}^{\pi}\log_+(1/|x-y|)\,\sigma(\mathrm{d}y) < \infty$. For $e^{i\theta} \in \mathbb{S}^1$ and $\varepsilon \in (0,1)$, define $\sigma_{e^{i\theta},\varepsilon}$ to be the pushforward of σ under the map $x \mapsto e^{i(\theta+\varepsilon x)}$, and denote $h_\varepsilon(e^{i\theta}) := (h, \sigma_{e^{i\theta},\varepsilon})$. Then, we define $\mathcal{M}_h^{\gamma}$ as the weak limit as $\varepsilon \to 0$ of

$$\exp\left(\frac{\gamma}{2}h_\varepsilon(e^{i\theta}) + \frac{\gamma^2}{4}\iint_{\mathbb{S}^1\times\mathbb{S}^1}\log|x-y|\,\sigma_{e^{i\theta},\varepsilon}(\mathrm{d}x)\,\sigma_{e^{i\theta},\varepsilon}(\mathrm{d}y)\right)\mathrm{d}\theta \tag{6.3.22}$$

  in probability.

  For instance, we can choose σ to be the uniform probability measure on $[-1,1]$, in which case $h_\varepsilon(e^{i\theta})$ will be average value of the LGF h on the closed interval from $e^{i(\theta-\varepsilon)}$ to $e^{i(\theta+\varepsilon)}$ on $\mathbb{S}^1$. However, the limit (6.3.22) does not depend on the choice of the measure σ.

Since the GMC measure $\mathcal{M}_h$ is almost surely determined by the LGF h, we can define the measure $\mathcal{M}_h$ using the limits (6.3.21) or (6.3.22) when h is a random field on $\mathbb{S}^1$ whose law is absolutely continuous with respect to LGF. When h is a random field on $\mathbb{S}^1$ with a decomposition $h = c + \tilde{h}$ where c is a real-valued random variable and the law of $\tilde{h}$ is absolutely continuous with respect to LGF, then we define

$$\mathcal{M}_h^\gamma := e^{(\gamma/2)c}\mathcal{M}_{\tilde{h}}^\gamma. \tag{6.3.23}$$

Conversely, the GMC measure $\mathcal{M}_h^\gamma$ almost surely determines the LGF h [39, Theorem 1.4].

Remark 6.3.5. When $\gamma = 2$, the limits (6.3.21) and (6.3.22) give the zero measure almost surely. However, a slightly modified limit gives a nontrivial measure $\mathcal{M}_h^{\mathrm{crit}}$ called the critical Gaussian multiplicative chaos [96, 97]. As described in [26, Section 4.1.2] (also see [208] and [209]), we can also obtain the critical GMC measure through the weak limit $\frac{1}{2(2-\gamma)}\mathcal{M}_h^\gamma \to \mathcal{M}_h^{\mathrm{crit}}$ in probability as $\gamma \to 2^-$. We also have that the critical GMC measure $\mathcal{M}_h^{\mathrm{crit}}$ almost surely determines the LGF h [242].

Remark 6.3.6. Our multiplicative factor of $\gamma/2$ in (6.3.21)–(6.3.22) as well as the factor of 2 in (6.3.6) differs from some of the recent works on the GMC on the circle, such as [71, 72]. In particular, they are chosen to agree with the literature on Liouville quantum gravity [99]. In the LQG theory, the quantum boundary length of a two-dimensional domain D is defined as the GMC measure on ∂D with respect to the free boundary GFF on D. More precisely, assume $D \subset \mathbb{H}$ and $\partial D \cap \mathbb{R}$ is a nonempty interval. Given free boundary GFF $\overline{\Gamma} = \Gamma + h$ where Γ is a zero boundary GFF and h is an independent harmonic Gaussian field h on D

(recall Remark 6.3.1), we define the γ-LQG boundary length $\nu_{\overline{\Gamma}}$ on $\partial D \cap \mathbb{R}$ as the almost sure weak limit

$$\lim_{\varepsilon \to 0} \varepsilon^{\gamma^2/4} e^{\frac{\gamma}{2}\overline{\Gamma}_\varepsilon(x)}\, \mathrm{d}x, \tag{6.3.24}$$

where $\overline{\Gamma}_\varepsilon(x)$ is the average value of $\overline{\Gamma}$ on the semicircle $\partial B_\varepsilon(x) \cap \mathbb{H}$. Note that for every $x \in \mathbb{R}$ and $\varepsilon > 0$, we have

$$\begin{aligned}
\mathbb{E}[\Gamma_\varepsilon(x)^2] &= \frac{1}{\pi^2} \iint_{[0,\pi]^2} \big(-\log|\varepsilon e^{i\theta} - \varepsilon e^{i\tilde{\theta}}| + \log|\varepsilon e^{i\theta} - \varepsilon e^{-i\tilde{\theta}}| \big) \mathrm{d}\theta\, \mathrm{d}\tilde{\theta} \\
&= \frac{1}{\pi^2} \iint_{[0,\pi]^2} \big(-\log|e^{i\theta} - e^{i\tilde{\theta}}| - \log|e^{i\theta} - e^{-i\tilde{\theta}}| \big) \mathrm{d}\theta\, \mathrm{d}\tilde{\theta} \\
&\qquad\qquad + \frac{2}{\pi^2} \iint_{[0,\pi]^2} \log|e^{i\theta} - e^{-i\tilde{\theta}}|\, \mathrm{d}\theta\, \mathrm{d}\tilde{\theta} \\
&= 0 + \frac{7}{\pi^2}\zeta(3) =: C \approx 0.85,
\end{aligned} \tag{6.3.25}$$

where $\zeta(3)$ is Apéry's constant. By the equivalence between the GMC defined using semicircle averages and the Karhunen–Loève expansion of GFF as explained in [37], we have that

$$\lim_{\varepsilon \to 0} e^{\frac{\gamma}{2}\overline{\Gamma}_\varepsilon(x) - \frac{\gamma^2}{8}\mathbb{E}[\overline{\Gamma}_\varepsilon(x)^2]}\, \mathrm{d}x = \lim_{\varepsilon \to 0} e^{\frac{\gamma}{2}h_\varepsilon(x) - \frac{\gamma^2}{8}\mathbb{E}[h_\varepsilon(x)^2]}\, \mathrm{d}x. \tag{6.3.26}$$

Since $\mathbb{E}[\Gamma_\varepsilon(x)^2] = \mathbb{E}[\overline{\Gamma}_\varepsilon(x)^2] - \mathbb{E}[h_\varepsilon(x)^2]$, we have

$$\nu_{\overline{\Gamma}} = \lim_{\varepsilon \to 0} \varepsilon^{\gamma^2/4} e^{\frac{\gamma}{2}h_\varepsilon(x) + \frac{\gamma^2}{8}\mathbb{E}[\Gamma_\varepsilon(x)^2]}\, \mathrm{d}x = e^{(\gamma^2/8)C} \nu_h. \tag{6.3.27}$$

By the isomorphism between $\mathcal{H}_0$ and the harmonic extension $\mathcal{D}_0$, the relation (6.3.27) holds with ν_h, the boundary LQG measure on $\partial\mathbb{D}$ defined using the harmonic field h in the unit disk, replaced by $\mathcal{M}_h$, the GMC measure on $\mathbb{S}^1$ defined using the LGF h on the unit circle. In particular, if $\overline{\Gamma}$ is a free boundary GFF on $\mathbb{D}$ and h is its trace on $\mathbb{S}^1$ (equivalently, the harmonic part of $\overline{\Gamma}$ on $\mathbb{D}$), then the normalized measures $(\nu_{\overline{\Gamma}}(\mathbb{S}^1))^{-1}\nu_{\overline{\Gamma}}$ and $(\mathcal{M}_h(\mathbb{S}^1))^{-1}\mathcal{M}_h$ agree almost surely.

6.3.4 Quasi-invariance of Gaussian multiplicative chaos

Given an LGF h on $\mathbb{S}^1$, define the corresponding normalized GMC measure as

$$\widehat{\mathcal{M}}_h^\gamma := \frac{1}{\mathcal{M}_h(\mathbb{S}^1)}\mathcal{M}_h^\gamma. \tag{6.3.28}$$

Our goal for this subsection is to show the following quasi-invariance result for normalized GMC measures.

Proposition 6.3.7. *Suppose $\varphi \in \mathrm{Homeo}_+(\mathbb{S}^1)$. Let h be the LGF on $\mathbb{S}^1$ and $\widehat{\mathcal{M}}_h = \widehat{\mathcal{M}}_h^\gamma$ be the corresponding normalized GMC measure* (6.3.28) *for $\gamma \in (0,2]$. Then, the law of the pullback $\varphi^*\widehat{\mathcal{M}}_h$ is absolutely continuous with respect to the law of $\widehat{\mathcal{M}}_h$ if and only if $\varphi \in \mathrm{WP}(\mathbb{S}^1)$. Moreover, if $\varphi \in \mathrm{WP}(\mathbb{S}^1)$, then we almost surely have*

$$\varphi^*\mathcal{M}_h^\gamma = \mathcal{M}_{h\circ\varphi+Q\log|\varphi'|}^\gamma \tag{6.3.29}$$

where $Q = \frac{2}{\gamma} + \frac{\gamma}{2}$.

It is well-known that given a 2D log-correlated Gaussian field Γ, the corresponding LQG area measure μ_Γ almost surely satisfies the conformal coordinate change rule

$$\psi^*\mu_\Gamma = \mu_{\Gamma\circ\psi+Q\log|\psi'|} \tag{6.3.30}$$

if ψ is a conformal map and $Q = 2/\gamma + \gamma/2$ [99, Proposition 2.1]. In showing (6.3.29), we extend this result to non-smooth and even non-C^1 homeomorphisms φ on $\mathbb{S}^1$. To begin our proof of Proposition 6.3.7, we verify that the coordinate change formula holds when φ is a diffeomorphism.

Lemma 6.3.8. *Let h be an LGF on the unit circle $\mathbb{S}^1$ and suppose φ is an orientation-preserving C^1 diffeomorphism of $\mathbb{S}^1$. Then, for every $\gamma \in (0,2]$ with $Q = \frac{2}{\gamma} + \frac{\gamma}{2}$, we almost surely have the coordinate change rule* (6.3.29).

We note that there exist C^1 diffeomorphisms φ of $\mathbb{S}^1$ that does not belong to the Weil–Petersson class, in which case the law of the mean-zero part of $h\circ\varphi + Q\log|\varphi'|$ is singular to that of LGF on $\mathbb{S}^1$. Nevertheless, this is a Gaussian field whose covariance kernel $G_\varphi(x,y)$ is of the form $-\log|x-y| + \tilde{u}_\varphi(x,y)$ where $\tilde{u}_\varphi$ is a continuous function on $\mathbb{S}^1\times\mathbb{S}^1$ (recall Remark 6.3.2). Hence, as proved in [37], the GMC measure $\mathcal{M}^\gamma_{h\circ\varphi+Q\log|\varphi'|}$ is well-defined via the limit (6.3.22) where the mollified field $h_\varepsilon(e^{i\theta}) = (h, \sigma_{e^{i\theta},\varepsilon})$ is replaced with $(h\circ\varphi, \sigma_{e^{i\theta},\varepsilon}) + \int_{\mathbb{S}^1} Q\log|\varphi'|\,\mathrm{d}\sigma_{e^{i\theta},\varepsilon}$.

Proof of Lemma 6.3.8. Let us first assume that $\gamma\in(0,2)$. Consider the GMC measure given by the following weak limit in probability:

$$\widetilde{\mathcal{M}}^\gamma_{h\circ\varphi} := \lim_{\varepsilon\to 0}\exp\left(\frac{\gamma}{2}(h\circ\varphi)_\varepsilon(e^{i\theta}) - \frac{\gamma^2}{8}\mathrm{Var}\big((h\circ\varphi)_\varepsilon(e^{i\theta})\big)\right)\mathrm{d}\theta,$$

where

$$(h\circ\varphi)_\varepsilon(e^{i\theta}) := \frac{1}{2\varepsilon}\int_{-\varepsilon}^{\varepsilon}(h\circ\varphi)(e^{i(\theta+t)})\,\mathrm{d}t.$$

Let us define $h_\varepsilon(e^{i\theta})$ similarly, so that $\mathrm{Var}\big(h_\varepsilon(e^{i\theta})\big) = (2\varepsilon)^{-2}\int_{-\varepsilon}^{\varepsilon}\int_{-\varepsilon}^{\varepsilon} -2\log|e^{i\theta}-e^{i\tilde{\theta}}|\,\mathrm{d}\theta\,\mathrm{d}\tilde{\theta}$. Then,

$$\mathrm{Var}\big((h\circ\varphi)_\varepsilon(e^{i\theta})\big) - \mathrm{Var}\big(h_\varepsilon(e^{i\theta})\big) = \frac{1}{4\varepsilon^2}\int_{-\varepsilon}^{\varepsilon}\int_{-\varepsilon}^{\varepsilon} -2\log\left|\frac{\varphi(e^{i(\theta+t)}) - \varphi(e^{i(\theta+s)})}{e^{i(\theta+t)} - e^{i(\theta+s)}}\right|\mathrm{d}s\,\mathrm{d}t$$

converges uniformly to $-2\log|\varphi'(e^{i\theta})|$ as $\varepsilon\to 0$. Hence, we have

$$\frac{\mathrm{d}\widetilde{\mathcal{M}}^\gamma_{h\circ\varphi}}{\mathrm{d}\mathcal{M}^\gamma_{h\circ\varphi}}(e^{i\theta}) = \big|\varphi'(e^{i\theta})\big|^{\gamma^2/4}. \tag{6.3.31}$$

Let us choose $\{h_n\}_{n\geq 1}$ to be the orthonormal basis (6.3.2) for $\mathcal{H}_0$. Suppose we have the decomposition $h = \sum_{n\geq 1}\xi_n h_n$, where $\{\xi_n\}_{n\geq 1}$ are i.i.d. standard normal random variables. Then, since $h\circ\varphi = \sum_{n\geq 1}\xi_n(h_n\circ\varphi)$ is a Karhunen–Loève expansion where each $h_n\circ\varphi$ are continuous on $\mathbb{S}^1$, we have from [37, Section 5] that for any interval $A \subset \mathbb{S}^1$,

$$
\begin{aligned}
\mathcal{M}_h^\gamma(\varphi(A)) &= \lim_{N\to\infty} \int_{\varphi(A)} e^{\sum_{n=1}^N \left(\frac{\gamma}{2}\xi_n h_n(e^{i\theta}) - \frac{\gamma^2}{8}[h_n(e^{i\theta})]^2\right)}\,\mathrm{d}\theta \\
&= \lim_{N\to\infty} \int_A e^{\sum_{n=1}^N \left(\frac{\gamma}{2}\xi_n (h_n\circ\varphi)(e^{i\theta}) - \frac{\gamma^2}{8}[(h_n\circ\varphi)(e^{i\theta})]^2\right)} |\varphi'(e^{i\theta})|\,\mathrm{d}\theta \\
&= \int_A |\varphi'(e^{i\theta})|\,\mathrm{d}\widetilde{\mathcal{M}}_{h\circ\varphi}^\gamma(e^{i\theta}) = \int_A |\varphi'(e^{i\theta})|^{1+\gamma^2/4}\,\mathrm{d}\mathcal{M}_{h\circ\varphi}^\gamma(e^{i\theta}) \\
&= \mathcal{M}^\gamma_{h\circ\varphi + Q\log|\varphi'|}(A)
\end{aligned}
\tag{6.3.32}
$$

almost surely. For the last equality, we used the identity $\mathrm{d}\mathcal{M}_{h+f}^\gamma/\mathrm{d}\mathcal{M}_h^\gamma = e^{(\gamma/2)f}$ for any continuous f on $\mathbb{S}^1$, which can be checked from the definition (6.3.22) of the GMC measure. The equality for critical GMC holds by taking the limit as $\gamma \to 2^-$ as in Remark 6.3.5. $\square$

To extend Lemma 6.3.8 to the Weil–Petersson class, we approximate φ by a sequence of diffeomorphisms which are continuous in the topology induced by the Weil–Petersson metric as in Lemma 6.2.9. We now show that the GMC measure on $\mathbb{S}^1$ is almost surely continuous under these approximations. To our knowledge, this is the first time that the GMC measure is shown to be continuous with respect to the random field. The continuity with respect to the base measure and the parameter γ has been shown in 2D in [209, Proposition 4.1].

Lemma 6.3.9. *Let $A \in \mathscr{B}(\mathcal{H}_0)$ be invertible in $\mathscr{B}(\mathcal{H}_0)$ and suppose $AA^* - I$ is Hilbert–Schmidt. Suppose $\{A_n\}_{n\geq 1}$ is a sequence in $\mathscr{B}(\mathcal{H}_0)$ such that A_n converges to A in the weak operator topology and $\|(A_nA^{-1})(A_nA^{-1})^* - I\|_{\mathrm{HS}} \to 0$. Let $\{\beta_n\}_{n\geq 1}$ be a sequence in $\mathcal{H}_0$ which converges to $\beta \in \mathcal{H}_0$. Let h be an LGF on $\mathbb{S}^1$. Then, for $\gamma \in (0,2)$, the Gaussian multiplicative chaos measures $\mathcal{M}_{A_nh+\beta_n}$ converges weakly to $\mathcal{M}_{Ah+\beta}$ in probability.*

Proof. Assume first that $A = I$ and $\beta = 0$. In this case, as outlined in [37, Section 6], it suffices to show that $\mathcal{M}_{A_nh+\beta}(S) \to \mathcal{M}_h(S)$ in probability for each fixed interval $S \subseteq \mathbb{S}^1$. The proof proceeds by considering the approximations (6.3.21) of $X := \mathcal{M}_h(S)$ using the orthonormal basis $\{h_k\}_{k\in\mathbb{N}}$ for $\mathcal{H}_0$ given in (6.3.2). That is, let

$$X_m := \int_S \exp\left(\frac{\gamma}{2}\sum_{k=1}^{m}\langle h, h_k\rangle h_k(e^{i\theta}) - \frac{\gamma^2}{8}\sum_{k=1}^{m}(h_k(e^{i\theta}))^2\right) \mathrm{d}\theta. \tag{6.3.33}$$

We know from [37, Equation (5.1)] that $X_m = \mathbb{E}[X|\mathcal{F}_m]$ where $\mathcal{F}_m$ is the σ-algebra generated by $\{\langle h, h_k\rangle\}_{1\le k\le m}$, whence $X_m \to X$ in $L^1(\mathbb{P})$. We consider the analogous approximations

$$X_m^{(n)} := \int_S \exp\left(\frac{\gamma}{2}\sum_{k=1}^{m}\langle A_n h + \beta_n, h_k\rangle h_k(e^{i\theta}) - \frac{\gamma^2}{8}\sum_{k=1}^{m}(h_k(e^{i\theta}))^2\right) \mathrm{d}\theta, \tag{6.3.34}$$

which converges in probability to $X^{(n)} := \mathcal{M}_{A_n h+\beta_n}(S)$ as $n\to\infty$ since the law of $A_n h + \beta_n$ is absolutely continuous with respect to that of h by the Feldman–Hájek theorem (cf. Theorem 6.3.3). We now claim the following.

- For each $m \in \mathbb{N}$, as $n \to \infty$,

$$X_m^{(n)} \to X_m \quad \text{in probability.} \tag{6.3.35}$$

- There exists a positive integer n_0 such that for any $\delta > 0$, we have

$$\lim_{m\to\infty}\sup_{n\ge n_0}\mathbb{P}(|X_m^{(n)} - X^{(n)}| > \delta) = 0. \tag{6.3.36}$$

Then, for any given $\delta > 0$, we can choose sufficiently large m and then n_0 such that

$$\mathbb{P}(|X^{(n)} - X_m^{(n)}| > \delta) + \mathbb{P}(|X_m^{(n)} - X_m| > \delta) + \mathbb{P}(|X_m - X| > \delta) < \delta.$$

That is, $X^{(n)} \to X$ in probability as $n \to \infty$.

Let us now show (6.3.35). Note from (6.3.2) that $|h_k(e^{i\theta})| \le \sqrt{2}$ for every k and θ.

Hence,

$$\mathbb{E}\left[\sup_{e^{i\theta}\in\mathbb{S}^1}\left|\sum_{k=1}^m\langle (A_n-I)h+\beta_n,h_k\rangle h_k(e^{i\theta})\right|\right] \le \sqrt{2}\sum_{k=1}^m\left(\mathbb{E}\left[|\langle (A_n-I)h,h_k\rangle|\right]+|\langle \beta_n,h_k\rangle|\right)$$
$$\le \sqrt{2}\sum_{k=1}^m \|(A_n-I)^*h_k\|_{\mathcal{H}_0}+2\sqrt{m}\|\beta_n\|_{\mathcal{H}_0}.$$

Note that

$$\|(A_n-I)^*h_k\|_{\mathcal{H}_0}^2 = |\langle h_k,(A_nA_n^*-I)h_k\rangle+\langle h_k,(2I-A_n-A_n^*)h_k\rangle|$$
$$\le \|A_nA_n^*-I\|_{\mathrm{HS}}+2|\langle h_k,(A_n-I)h_k\rangle|,$$

which tends to 0 as $n\to\infty$ by assumption. Combining the above two inequalities, we see that given any sequence of positive integers, we can find a further sequence $\{n_j\}_{j\ge1}$ such that $\sup_{e^{i\theta}\in\mathbb{S}^1}|\sum_{k=1}^m\langle (A_{n_j}-I)h+\beta_{n_j},h_k\rangle-\langle h,h_k\rangle h_k(e^{i\theta})|\to 0$ almost surely as $j\to\infty$. As we have

$$\inf_{e^{i\theta}\in S} e^{\frac{\gamma}{2}\sum_{k=1}^m\langle (A_{n_j}-I)h+\beta_{n_j},h_k\rangle h_k(e^{i\theta})} \le \frac{X_m^{(n_j)}}{X_m} \le \sup_{e^{i\theta}\in S} e^{\frac{\gamma}{2}\sum_{k=1}^m\langle (A_{n_j}-I)h+\beta_{n_j},h_k\rangle h_k(e^{i\theta})}$$

from the definitions of $X_m^{(n)}$ and X_m, we conclude that $X_m^{(n_j)}\to X_m$ almost surely as $j\to\infty$ and thus $X_m^{(n)}\to X_m$ in probability as $n\to\infty$.

Let us now show (6.3.36). For this, we will first check that if ρ_n is the Radon–Nikodym derivative of the law of $A_nh+\beta_n$ with respect to that of h, then $\sup_{n\ge n_0}\mathbb{E}[(\rho_n)^2]<\infty$ for sufficiently large n_0.[1] Let us fix $A\in\mathscr{B}(\mathcal{H}_0)$ with $\|AA^*-I\|_{\mathrm{HS}}^2<1/2$ and $\beta\in\mathcal{H}_0$. Choose $\{\tilde{h}_n\}_{n\ge1}$ be an orthonormal eigenbasis for $\mathcal{H}_0$ with respect to AA^*-I, with $(AA^*-I)\tilde{h}_k = a_k\tilde{h}_k$ for each k. Observe that $\{\langle h,\tilde{h}_k\rangle\}_{k\ge1}$ are i.i.d. standard normal random variables, whereas $\{\langle Ah+\beta,\tilde{h}_k\rangle\}_{k\ge1}$ are independent Gaussian with mean $\langle\beta,\tilde{h}_k\rangle =: b_k$ and variance

1. The same calculations can be used to show that $\sup_{n\ge1}\mathbb{E}[(\rho_n)^p]<\infty$ for sufficiently small $p>1$, or, for any given $p>1$, we have $\sup_{n\ge n_0}\mathbb{E}[(\rho_n)^p]<\infty$ for sufficiently large n_0.

$\langle \tilde{h}_k, AA^*\tilde{h}_k\rangle = (1 + a_k)$. Hence, the Radon–Nikodym derivative ρ of the law of $Ah + \beta$ with respect to that of h is given by

$$\rho\left(\sum_{k\geq 1} x_k \tilde{h}_k\right) = \prod_{k\geq 1} \frac{(2\pi(1+a_k))^{-1/2}\exp(-(x_k - b_k)^2/2(1+a_k))}{(2\pi)^{-1/2}\exp(-x_k^2/2)} =: \prod_{k\geq 1} F_k(x_k).$$

A straightforward calculation gives us that if Z is a standard normal random variable, then

$$\mathbb{E}[\rho^2] = \prod_{k\geq 1}\mathbb{E}[(F_k(Z))^2] = \prod_{k\geq 1}\frac{1}{(1-a_k^2)^{1/2}}\exp\left(\frac{b_k^2}{1-a_k^2}\right).$$

Now, since $\sum_{k\geq 1} a_k^2 = \|AA^* - I\|_{\mathrm{HS}}^2 < 1/2$, we have $a_k^2 < 1/2$ for all k. Hence,

$$\log \mathbb{E}[\rho^2] \leq \sum_{k\geq 1} a_k^2 + 2\sum_{k\geq 1} b_k^2 = \|AA^* - I\|_{\mathrm{HS}}^2 + 2\|\beta\|_{\mathcal{H}_0}^2.$$

Since $\|A_n A_n^* - I\|_{\mathrm{HS}} \to 0$ and $\|\beta_n\|_{\mathcal{H}_0} \to 0$ as $n \to \infty$, we conclude that $\mathbb{E}[\rho_n^2] \to 1$ as $n \to \infty$. Fix n_0 to be a positive integer such that $\sup_{n\geq n_0}\mathbb{E}[(\rho_n)^2] < \infty$. Now, given any $\delta > 0$, we have

$$\begin{aligned}\mathbb{P}(|X_m^{(n)} - X^{(n)}| > \delta) = \mathbb{E}[\rho_n \mathbf{1}_{\{|X_m - X| > \delta\}}] &\leq K\mathbb{P}(|X_m - X| > \delta) + \mathbb{E}[\rho_n \mathbf{1}_{\{\rho_n > K\}}]\\ &\leq \frac{K}{\delta}\mathbb{E}|X_m - X| + \sqrt{\frac{\mathbb{E}[(\rho_n)^2]}{K}}.\end{aligned}$$

Since $X_m \to X$ in $L^1(\mathbb{P})$, by first choosing sufficiently large K and then choosing large m, we can make $\sup_{n\geq n_0}\mathbb{P}(|X_m^{(n)} - X^{(n)}| > \delta)$ arbitrarily small. This completes the proof of (6.3.36) and therefore the lemma in the case $A = I$ and $\beta = 0$.

For general $A \in \mathscr{B}(\mathcal{H}_0)$ such that $\|AA^* - I\|_{\mathrm{HS}} < \infty$ and $\beta \in \mathcal{H}_0$, recall that the law of $\tilde{h} := Ah + \beta$ is absolutely continuous with respect to that of h by the Feldman–Hájek theorem. Let $\tilde{A}_n := A_n A^{-1} \in \mathscr{B}(\mathcal{H}_0)$ and $\tilde{\beta}_n := \beta_n - A_n A^{-1}\beta \in \mathcal{H}_0$, so that $A_n h + \beta_n = \tilde{A}_n \tilde{h} + \tilde{\beta}_n$. Then, $\tilde{A}_n \to I$ in the weak operator topology, $\|\tilde{A}_n \tilde{A}_n^* - I\|_{\mathrm{HS}} \to 0$, and $\|\tilde{\beta}_n\|_{\mathcal{H}_0} \to 0$ by the assumptions of the lemma, so we conclude that $\mathcal{M}_{\tilde{A}_n\tilde{h}+\tilde{\beta}_n} = \mathcal{M}_{A_nh+\beta_n}$ converges weakly to $\mathcal{M}_{\tilde{h}} = \mathcal{M}_{Ah+\beta}$ in probability. □

We are now ready for a proof of Proposition 6.3.7. We divide this into two parts, first showing that if $\varphi \in \mathrm{WP}(\mathbb{S}^1)$, then the laws of $\widehat{\mathcal{M}}_h$ and $\widehat{\mathcal{M}}_{h\circ\varphi+Q\log|\varphi'|}$ are absolutely continuous by checking that the coordinate change formula (6.3.29) holds.

Proof of Sufficiency. Let us first consider $\gamma \in (0, 2)$. Suppose $\varphi \in \mathrm{WP}(\mathbb{S}^1)$ and let $\varphi_n \in \mathrm{Diff}_+(\mathbb{S}^1)$ be a sequence of approximations to φ given in Lemma 6.2.9. Let $A_n = \Pi(\varphi_n)$ and $\beta_n = \pi_0(Q\log|\varphi_n'|)$ for each n, and similarly let $A = \Pi(\varphi)$ and $\beta = \pi_0(Q\log|\varphi'|)$. Then, we have $A_n \to A$ in the strong operator topology and $\|(A_nA^{-1})(A_nA^{-1})^* - I\|_{\mathrm{HS}} \to 0$ by our choice of approximations. Also, since $\log|(\varphi_n\circ\varphi^{-1})'| = (\log|\varphi_n'| - \log|\varphi'|)\circ\varphi^{-1}$, we have $\|\beta_n - \beta\|_{\mathcal{H}_0} \le Q\|\Pi(\varphi)\|_{\mathscr{B}(\mathcal{H}_0)}\|\log|(\varphi_n\circ\varphi^{-1})|'\|_{\mathcal{H}_0}$ tending to 0 as $n\to\infty$.

We have from Lemma 6.3.8 that the coordinate change formula

$$\varphi_n^*\mathcal{M}_h = \mathcal{M}_{h\circ\varphi_n+Q\log|\varphi_n'|} = e^{\frac{\gamma}{4\pi}\int_{\mathbb{S}^1}(h\circ\varphi_n+Q\log|\varphi_n'|)}\mathcal{M}_{A_nh+\beta_n}$$

holds almost surely for every n. By Lemma 6.3.9, $\mathcal{M}_{A_nh+\beta_n}$ converges weakly in probability to $\mathcal{M}_{Ah+\beta}$. Since $\sum_k \left|\int_{\mathbb{S}^1}(h_k\circ\varphi_n - h_k\circ\varphi)\right|^2 \le C\|\varphi_n' - \varphi'\|^2_{H^{-1/2}(\mathbb{S}^1)} \to 0$ as $n\to\infty$, we have $\int_{\mathbb{S}^1} h\circ\varphi_n \to \int_{\mathbb{S}^1} h\circ\varphi$ almost surely. We also have $\int_{\mathbb{S}^1}\log|\varphi_n'| \to \int_{\mathbb{S}^1}\log|\varphi'|$, so

$$\varphi_n^*\mathcal{M}_h \to e^{\frac{\gamma}{4\pi}\int_{\mathbb{S}^1}(h\circ\varphi+Q\log|\varphi'|)}\mathcal{M}_{Ah+\beta} = \mathcal{M}_{h\circ\varphi+Q\log|\varphi'|} \tag{6.3.37}$$

weakly in probability as $n\to\infty$.

On the other hand, since $\|\varphi - \varphi_n\|_\infty \to 0$, for every $F \in C(\mathbb{S}^1)$, we have

$$\lim_{n\to\infty}\int_{\mathbb{S}^1} F\,\mathrm{d}(\varphi_n^*\mathcal{M}_h) = \lim_{n\to\infty}\int_{\mathbb{S}^1}(F\circ\varphi_n)\,\mathrm{d}\mathcal{M}_h = \int_{\mathbb{S}^1}(F\circ\varphi)\,\mathrm{d}\mathcal{M}_h = \int_{\mathbb{S}^1} F\,\mathrm{d}(\varphi^*\mathcal{M}_h)$$

almost surely. That is, $\varphi_n^*\mathcal{M}_h$ converges almost surely in the weak topology to $\varphi^*\mathcal{M}_h$. We thus conclude that $\varphi^*\mathcal{M}_h = \mathcal{M}_{h\circ\varphi+Q\log|\varphi'|}$ almost surely. As described in Remark 6.3.5, we have $\frac{1}{2(2-\gamma)}\mathcal{M}_h^\gamma \to \mathcal{M}_h^{\text{crit}}$ weakly in probability as $\gamma \to 2^-$. Hence, the coordinate change rule (6.3.29) holds for the critical case $\gamma = 2$ as well.

To conclude the proof, observe that

$$\varphi^*\widehat{\mathcal{M}}_h = \frac{1}{\mathcal{M}_h(\mathbb{S}^1)}\varphi^*\mathcal{M}_h = \frac{1}{\varphi^*\mathcal{M}_h(\mathbb{S}^1)}\varphi^*\mathcal{M}_h = \frac{1}{\mathcal{M}_{Ah+\beta}(\mathbb{S}^1)}\mathcal{M}_{Ah+\beta} = \widehat{\mathcal{M}}_{Ah+\beta} \quad (6.3.38)$$

almost surely and the law of $Ah+\beta$ is equivalent to that of h by Theorem 6.3.3. Therefore, the law of $\varphi^*\widehat{\mathcal{M}}_h$ is equivalent to that of $\widehat{\mathcal{M}}_h$. □

Proof of necessity. Let $\varphi \in \mathrm{Homeo}_+(\mathbb{S}^1)$ and assume that the law of the pull-back $\varphi^*\widehat{\mathcal{M}}_h$ is absolutely continuous with respect to that of $\widehat{\mathcal{M}}_h$. Then, since the GMC almost surely determines the LGF [39, 242], there exists a random mean-zero distribution $\tilde{h}$ on $\mathbb{S}^1$ with a law absolutely continuous respect to LGF on $\mathbb{S}^1$ such that $\widehat{\mathcal{M}}_{\tilde{h}} = \varphi^*\widehat{\mathcal{M}}_h$ almost surely.

Choose a sequence of smooth $\varphi_n \in \mathrm{Diff}_+(\mathbb{S}^1)$ which converges uniformly to φ (e.g., by convolution). Since $\widehat{\mathcal{M}}_{h\circ\varphi_n+Q\log|\varphi_n'|} = \varphi_n^*\widehat{\mathcal{M}}_h \to \varphi^*\widehat{\mathcal{M}}_h$ almost surely, we have $\Pi(\varphi_n)h + \pi_0(Q\log|\varphi_n'|) \to \tilde{h}$ in law. In particular, $\tilde{h}$ is a Gaussian field whose law is absolutely continuous with respect to that of LGF on $\mathbb{S}^1$. Then, by the Feldman–Hájek theorem, there exists some $Q \in \mathscr{B}(\mathcal{H}_0)$ with $Q - I$ Hilbert–Schmidt such that for any $f, g \in \mathcal{H}_0$,

$$\begin{aligned}\lim_{n\to\infty}\langle \Pi(\varphi_n)^*f, \Pi(\varphi_n)^*g\rangle &= \lim_{n\to\infty}\mathrm{Cov}(\langle \Pi(\varphi_n)h, f\rangle, \langle \Pi(\varphi_n)h, g\rangle) \\ &= \mathrm{Cov}(\langle \tilde{h}, f\rangle, \langle \tilde{h}, g\rangle) = \langle f, Qg\rangle\end{aligned} \quad (6.3.39)$$

where $\langle\cdot,\cdot\rangle$ is the inner product on $\mathcal{H}_0$. That is, $\Pi(\varphi_n)\Pi(\varphi_n)^*$ converges in the weak operator topology to Q. In particular, for any $f \in \mathcal{H}_0$, we have

$$\lim_{n\to\infty}\|\Pi(\varphi_n)^*f\|^2_{\mathcal{H}_0} = \langle f, Qf\rangle \le \|Q\|_{\mathscr{B}(\mathcal{H}_0)}\|f\|^2_{\mathcal{H}_0}.$$

Then, by the uniform boundedness principle,

$$\sup_n \|\Pi(\varphi_n)\|_{\mathscr{B}(H)} = \sup_n \|\Pi(\varphi_n)^*\|_{\mathscr{B}(H)} < \infty.$$

For any $f \in \mathcal{H}_0$ with $\|f\|_{\mathcal{H}_0} \le 1$, by Fatou's lemma,

$$\begin{aligned}
\|\Pi(\varphi)f\|_{\mathcal{H}_0}^2 &= \iint_{\mathbb{S}^1\times\mathbb{S}^1} \frac{|f(\varphi(e^{i\theta})) - f(\varphi(e^{i\psi}))|^2}{|e^{i\theta}-e^{i\psi}|^2}\,\mathrm{d}\theta\,\mathrm{d}\psi \\
&\le \liminf_{n\to\infty} \iint_{\mathbb{S}^1\times\mathbb{S}^1} \frac{|f(\varphi_n(e^{i\theta})) - f(\varphi_n(e^{i\psi}))|^2}{|e^{i\theta}-e^{i\psi}|^2}\,\mathrm{d}\theta\,\mathrm{d}\psi = \liminf_{n\to\infty} \|\Pi(\varphi_n)f\|_{\mathcal{H}_0}^2 \\
&\le \liminf_{n\to\infty} \|\Pi(\varphi_n)\|_{\mathscr{B}(\mathcal{H}_0)}^2 < \infty.
\end{aligned}$$

Therefore, $\Pi(\varphi) \in \mathscr{B}(\mathcal{H}_0)$ and $\varphi \in \mathrm{QS}(\mathbb{S}^1)$ follows by Lemma 6.2.7.

We see from the formulas (6.2.21) and (6.2.22) that $\Pi(\varphi_n) \to \Pi(\varphi)$ entrywise with respect to the basis (6.2.11) for $\mathcal{H}_0^{\mathbb{C}}$. Hence, $Q = \Pi(\varphi)\Pi(\varphi)^*$ and $\Pi(\varphi)\Pi(\varphi)^* - I$ is Hilbert–Schmidt. We conclude $\varphi \in \mathrm{WP}(\mathbb{S}^1)$ using Lemma 6.2.7. $\square$

In the Liouville theory, we often consider Gaussian multiplicative chaos measures with respect to a log-correlated Gaussian field plus a logarithmic singularity (cf. Lemma 6.1.6). Below, we give an analog of Proposition 6.3.7 for such fields. Recall from (6.2.25) the log-ratio

$$u_\varphi(z,w) = \log \frac{\varphi(z)-\varphi(w)}{z-w}.$$

Corollary 6.3.10. *Fix $\gamma \in (0,2]$. Let h be the LGF on $\mathbb{S}^1$ and $\mathfrak{h}(z) := -\alpha \log|z-1|$ where α is a positive constant. Suppose $\varphi \in \mathrm{Homeo}_+(\mathbb{S}^1)$ fixes 1. Then, the law of the normalized pullback measure $\varphi^*\widehat{\mathcal{M}}_{h+\mathfrak{h}}$ is absolutely continuous with respect to the law of $\widehat{\mathcal{M}}_{h+\mathfrak{h}}$ if and only if $\varphi \in \mathrm{WP}(\mathbb{S}^1)$ and $u_\varphi(\cdot,1) \in H^{1/2}(\mathbb{S}^1)$.*

Proof. Suppose $\varphi \in \mathrm{WP}(\mathbb{S}^1)$ and $u_\varphi(\cdot,1) \in H^{1/2}$, in which case $\tilde{u}_\varphi(\cdot,1) = \operatorname{Re} u_\varphi(\cdot,1) = \log|(\varphi(\cdot)-1)/(\cdot-1)| \in H^{1/2}$. For each fixed C, since the law of $h + (\mathfrak{h} \wedge C)$ is absolutely continuous to that of h, we almost surely have $\varphi^* \mathcal{M}_{h+(\mathfrak{h}\wedge C)} = \mathcal{M}_{h\circ\varphi+(\mathfrak{h}\circ\varphi\wedge C)+Q\log|\varphi'|}$ by Proposition 6.3.7. Letting $C \to \infty$, since GMC is a local functional of the field, we obtain $\varphi^*\mathcal{M}_{h+\mathfrak{h}} = \mathcal{M}_{h\circ\varphi+\mathfrak{h}\circ\varphi+Q\log|\varphi'|}$ almost surely. Since $\mathfrak{h}\circ\varphi - \mathfrak{h} + Q\log|\varphi'| \in H^{1/2}$ by our assumption that $\tilde{u}_\varphi(\cdot,1) \in H^{1/2}$, the law of $\Pi(\varphi)h + \pi_0(\mathfrak{h}\circ\varphi + Q\log|\varphi'|)$ is absolutely continuous with respect to that of $h + \pi_0(\mathfrak{h})$ by Theorem 6.3.3. This implies that the law of $\varphi^*\widehat{\mathcal{M}}_{h+\mathfrak{h}}$ is absolutely continuous with respect to that of $\widehat{\mathcal{M}}_{h+\mathfrak{h}}$.

On the other hand, suppose that the law of $\varphi^*\widehat{\mathcal{M}}_{h+\mathfrak{h}}$ is absolutely continuous with respect to that of $\widehat{\mathcal{M}}_{h+\mathfrak{h}}$ for some $\varphi \in \mathrm{Homeo}_+(\mathbb{S}^1)$. Pick $\varphi_n \in \mathrm{Diff}_+(\mathbb{S}^1)$ which converges uniformly to φ. As in the proof of Proposition 6.3.7, there exists a random distribution $\tilde{h}$ on $\mathbb{S}^1$ whose law is mutually absolutely continuous with respect to that of the LGF on $\mathbb{S}^1$ such that $\tilde{h}_n := \Pi(\varphi_n)h + \pi_0(\mathfrak{h}\circ\varphi_n - \mathfrak{h} + Q\log|\varphi_n'|)$ converges in law to $\tilde{h}$ as $n \to \infty$. Noting that $\mathrm{Cov}(\langle\tilde{h}_n, f\rangle, \langle\tilde{h}_n, g\rangle) = \langle\Pi(\varphi_n)^* f, \Pi(\varphi_k)^* g\rangle$ for any $f, g \in \mathcal{H}_0$, a proof analogous to that of Proposition 6.3.7 gives $\varphi \in \mathrm{WP}(\mathbb{S}^1)$. Then, $\tilde{h}_n$ converges in law to $\Pi(\varphi)h + \pi_0(-\alpha\,\tilde{u}_\varphi(\cdot,1) + Q\log|\varphi'|)$, and the absolute continuity of its law with respect to LGF implies $u_\varphi(\cdot,1) \in H^{1/2}(\mathbb{S}^1)$. $\square$

6.4 Quasi-invariance of SLE welding

In this section, we introduce the space $\mathrm{RM}(\mathbb{S}^1)$ of conformally removable weldings. We endow it with a topology so that composition gives a continuous group action of quasisymmetric circle homeomorphisms $\mathrm{QS}(\mathbb{S}^1)$ on $\mathrm{RM}(\mathbb{S}^1)$ (Lemma 6.4.1). This is due to the definition of quasi-invariance. For a measure $\mathscr{P}$ on a measurable space $(S, \mathcal{F})$, let G be a group that acts (left or right) measurably on S. We say that $\mathscr{P}$ is quasi-invariant under the action of G if for each $g \in G$ and $A \in \mathcal{F}$, we have $\mathscr{P}(A) = 0$ if and only if $\mathscr{P}(gA) = 0$. That is, the pull-back of $\mathscr{P}$ by g is mutually absolutely continuous with respect to $\mathscr{P}$. We refer the reader to [157] for more background. In particular, we show the measurability of the group action of Weil–Petersson homeomorphisms on SLE weldings.

After that, we introduce the SLE loop shape measure and find the law of the corresponding welding homeomorphism (see Lemma 6.4.8), which leads to the proof of Theorem 6.1.2.

6.4.1 Conformal welding of Jordan curves

We say that a compact set K is (quasi)conformally removable if any homeomorphism of $\hat{\mathbb{C}}$ that is (quasi)conformal off K is (quasi)conformal on $\hat{\mathbb{C}}$. An easy application of the measurable Riemann mapping theorem shows that K is conformally removable if and only if it is quasiconformally removable. We refer the reader to the survey [253] for an in-depth look at conformal removability.

It is well known that if a Jordan curve η is conformally removable, then the corresponding welding homeomorphism is unique up to pre- and post-compositions by $\text{Möb}(\mathbb{S}^1)$. Given an oriented Jordan curve η, let $f : \mathbb{D} \to \Omega$ and $g : \mathbb{D}^* \to \Omega^*$ be any conformal maps onto the components of $\hat{\mathbb{C}} \setminus \eta$ on the left and the right of the curve, respectively. Suppose $\tilde{f} : \mathbb{D} \to \tilde{\Omega}$ and $\tilde{g} : \mathbb{D} \to \tilde{\Omega}^*$ is another pair of conformal maps such that $\tilde{\eta} := \hat{\mathbb{C}} \setminus (\tilde{\Omega} \cup \tilde{\Omega}^*)$ is a Jordan curve and $(\tilde{g}^{-1} \circ \tilde{f})|_{\mathbb{S}^1} = (g^{-1} \circ f)|_{\mathbb{S}^1}$. Then, we can define a homeomorphism of $\hat{\mathbb{C}}$ given by $\tilde{f} \circ f^{-1}$ on $\overline{\Omega}$ and $\tilde{g} \circ g^{-1}$ on $\overline{\Omega^*}$. Since this homeomorphism is conformal on $\hat{\mathbb{C}} \setminus \eta$, it must be conformal on all of $\hat{\mathbb{C}}$. Hence, $\tilde{\eta}$ is an image of η under a Möbius transformation as desired. However, there is no analytic characterization of weldings corresponding to conformally removable curves [51].

Recall that For a compact set K in $\hat{\mathbb{C}}$, Hausdorff dimension $d_H(K)$ is defined to be

$$\inf\{\epsilon \geq 0 | \inf_{D \in \mathscr{D}} \sum_{\mathrm{a} \in \mathcal{A}} r_{\mathrm{a}}^{\epsilon} = 0\},$$

where $\mathscr{D}$ is the family of finite coverings $D = \{D_{\mathrm{a}}\}_{\mathrm{a} \in \mathcal{A}}$ indexed by a set $\mathcal{A}$ of K by discs D_{a} with radius r_{a}. The following lemma is well-known; see, for example, [240, Theorem 1.4] for the first part. Here, we provide a proof based on the measurable Riemann mapping theorem.

Lemma 6.4.1. *Let* $\mathrm{RM}(\mathbb{S}^1)$ *denote the space of orientation-preserving circle homeomorphisms that are weldings of conformally removable Jordan curves. If* $\varphi \in \mathrm{QS}(\mathbb{S}^1)$ *and* $\psi \in \mathrm{RM}(\mathbb{S}^1)$*, then* $\varphi^{-1} \circ \psi$ *are* $\psi \circ \varphi$ *are in* $\mathrm{RM}(\mathbb{S}^1)$*. If we further assume that* $\varphi \in \mathrm{S}(\mathbb{S}^1)$*, then the three curves that solve the welding problem for* ψ*,* $\psi \circ \varphi$ *and* $\varphi^{-1} \circ \psi$ *share the same Hausdorff dimension.*

Proof. Given $\varphi \in \mathrm{QS}(\mathbb{S}^1)$, choose a quasiconformal extension ω of φ^{-1} to $\hat{\mathbb{C}}$ whose Beltrami coefficient $\mu = \partial_{\bar{z}}\omega/\partial_z\omega$ satisfies the symmetry condition $\mu(z) = (z/\bar{z})^2\bar{\mu}(1/\bar{z})$. For a conformally removable Jordan curve η solving the welding problem for ψ, let $f : \mathbb{D} \to \Omega$ and $g : \mathbb{D}^* \to \Omega^*$ denote the conformal maps onto the bounded and unbounded complementary components of η, respectively. Let $\tilde{\omega}$ be a quasiconformal homeomorphism of $\hat{\mathbb{C}}$ that fixes 0, 1 and ∞, whose Beltrami coefficient is given by

$$\tilde{\mu} = \begin{cases} \left(\mu f'/\bar{f}'\right) \circ f^{-1} & \text{in } \Omega, \\ 0 & \text{otherwise.} \end{cases} \tag{6.4.1}$$

We define as above so that $\tilde{f} = \tilde{\omega} \circ f \circ \omega^{-1} : \mathbb{D} \to \tilde{\omega}(\Omega)$ and $\tilde{g} = \tilde{\omega} \circ g : \mathbb{D}^* \to \tilde{\omega}(\Omega^*)$ are exactly the conformal maps corresponding to the Jordan curve $\tilde{\eta} = \tilde{\omega}(\eta)$, whose welding homeomorphism is $\tilde{\psi} := (\tilde{g}^{-1} \circ \tilde{f})|_{\mathbb{S}^1} = \psi \circ \varphi$. To see that $\tilde{\eta}$ is quasiconformally removable, note that given a homeomorphism F of $\hat{\mathbb{C}}$ which is quasiconformal off $\tilde{\eta}$, we have that $\tilde{F} = F \circ \tilde{\omega}$ is a homeomorphism of $\hat{\mathbb{C}}$ which is quasiconformal off η. Since η is (quasi)conformally removable, F is quasiconformal on $\hat{\mathbb{C}}$ and it follows that $\tilde{F}$ is quasiconformal on $\hat{\mathbb{C}}$. This shows that if $\psi \in \mathrm{RM}(\mathbb{S}^1)$ and $\varphi \in \mathrm{QS}(\mathbb{S}^1)$, then $\psi \circ \varphi \in \mathrm{RM}(\mathbb{S}^1)$.

Let us further assume that $\varphi \in \mathrm{S}(\mathbb{S}^1)$. Let d and $\tilde{d}$ denote the Hausdorff dimensions of η and $\tilde{\eta} = \tilde{\omega}(\eta)$, respectively. From classical distortion estimates for quasiconformal maps [28], we have

$$(1/d - 1/2)/K \leq 1/\tilde{d} - 1/2 \leq K(1/d - 1/2)$$

where K is the supremum of dilatation of $\tilde{\omega}$ near η. This is asymptotically equal to 1, as $\tilde{\omega}$ is asymptotically conformal near η. More precisely, we have $K_r \to 1$ as $r \uparrow 1$, where K_r is the supremum of dilatation of $\tilde{\omega}$ restricted on $\mathbb{C} \backslash f(r\mathbb{D})$. Therefore, we have $\tilde{d} = d$.

The analogous results for $\varphi^{-1} \circ \psi$ follow in a similar manner, taking $\tilde{\omega}$ to be a quasiconformal homeomorphism of $\hat{\mathbb{C}}$ where its Beltrami coefficient agrees with the pullback of μ by g in Ω^* and 0 otherwise. □

Remark 6.4.2. We may replace the Hausdorff dimension in Lemma 6.4.1 by the upper Minkowski dimension, the packing dimension, Assouad dimension, etc., as they share the similar dilatation-dependent distortion bounds. See [73] and the discussion therein.

Now we define the pre-composition $R : \mathrm{RM}(\mathbb{S}^1) \times \mathrm{QS}(\mathbb{S}^1) \to \mathrm{RM}(\mathbb{S}^1)$ by $R(\psi, \varphi) = \psi \circ \varphi$ and the post-composition $L : \mathrm{RM}(\mathbb{S}^1) \times \mathrm{QS}(\mathbb{S}^1) \to \mathrm{RM}(\mathbb{S}^1)$ by $R(\psi, \varphi) = \varphi^{-1} \circ \psi$. Recall that the topology on $\mathrm{QS}(\mathbb{S}^1)$ is induced from the Teichmüller distance. We define the topology on $\mathrm{RM}(\mathbb{S}^1)$ by the Carathéodory topology on the two conformal maps via conformal welding. More precisely, we say $\psi_n \to \psi$ if there exist welding solutions $\psi_n = g_n^{-1} \circ f_n$ and $\psi = g^{-1} \circ f$ such that $f_n \to f$ and $g_n \to g$ uniformly on compact subsets. Notably, the induced σ-algebra coincides precisely with that described in [32, Appendix B]. Moreover, the following continuity still holds.

Proposition 6.4.3. *The post-composition map R and the pre-composition map L defined above are continuous.*

Proof. Suppose $\psi_n \to \psi$ and $\varphi_n \to \varphi$ in $\mathrm{RM}(\mathbb{S}^1)$ and $\mathrm{QS}(\mathbb{S}^1)$, respectively. Let $f_n \to f$ and $g_n \to g$ uniformly on compact subsets be the conformal maps corresponding to ψ_n and ψ. Let μ_n and μ be the Beltrami coefficient of the quasiconformal extension ω_n and ω of φ_n^{-1} and φ^{-1}, respectively. We may choose these extensions, e.g., by using the Douady–Earle extension, so that $\mu_n \to \mu$ locally uniformly on $\mathbb{D}$.

Define $\tilde{\mu}_n$ as in (6.4.1) using μ_n and f_n instead. Then, $\tilde{\mu}_n \to \tilde{\mu}$ almost everywhere on $\hat{\mathbb{C}}$. Hence, if we choose ω_n and $\tilde{\omega}_n$ to fix $0, 1, \infty$, then $\omega_n^{-1} \to \omega^{-1}$ and $\tilde{\omega}_n \to \tilde{\omega}$ uniformly with respect to the spherical metric [177, Chapter 1, Theorem 4.6]. Thus, $\tilde{f}_n = \tilde{\omega}_n \circ f_n \circ \omega_n^{-1} \to \tilde{\omega} \circ f \circ \omega^{-1} = \tilde{f}$ and $\tilde{g}_n = \tilde{\omega}_n \circ g_n \to \tilde{\omega} \circ g = \tilde{g}$ uniformly on compact subsets.

The analogous results for $\varphi^{-1} \circ \psi$ follow in a similar manner as in Lemma 6.4.1. $\square$

As we will see in the next subsection, from the perspective of the SLE_κ loop measure, it is more appropriate to consider a correspondence between the following spaces.

- The space of orientation-preserving circle homeomorphisms which arise as weldings of conformally removable Jordan curves that fix 1, denoted $\mathbb{S}^1\backslash \mathrm{RM}(\mathbb{S}^1)$.
- The space of conformally removable Jordan curves η on $\hat{\mathbb{C}}$ separating 0 from ∞ with a unit conformal radius of $\hat{\mathbb{C}} \setminus \eta$ viewed from 0, denoted $\mathcal{J}_\#$.

Definition 6.4.4. Given $\eta \in \mathcal{J}_\#$, let Ω and Ω^* be the bounded and unbounded components of $\hat{\mathbb{C}} \setminus \eta$, respectively. Consider the unique pair of Riemann maps $f : \mathbb{D} \to \Omega$ and $g : \mathbb{D}^* \to \Omega^*$ satisfying $f(0) = 0$, $f'(0) = 1$, $g(\infty) = \infty$, and $f(1) = g(1)$. The corresponding element of $\mathbb{S}^1\backslash \mathrm{RM}(\mathbb{S}^1)$ is given by $(g^{-1} \circ f)|_{\mathbb{S}^1}$.

Note that this correspondence is one-to-one since the only map $\omega \in \mathrm{Möb}(\hat{\mathbb{C}})$ satisfying $\omega(0) = 0$, $\omega'(0) = 1$, and $\omega(\infty) = \infty$ is the identity. We endow $\mathcal{J}_\#$ with the Carathéodory topology for Ω and Ω^*, which is equivalent to the topology of local uniform convergence for f and g. We endow the topology on $\mathbb{S}^1\backslash \mathrm{RM}(\mathbb{S}^1)$ that makes the above correspondence a homeomorphism.

The correspondence in Definition 6.4.4, when restricted to quasisymmetric circle homeomorphisms in $\mathbb{S}^1\backslash \mathrm{RM}(\mathbb{S}^1)$ and quasicircles in $\mathcal{J}_\#$, agrees with the models of the universal Teichmüller curve $\mathcal{T}(1)$ as discussed in [239, Section I.1.2]. Let us consider $\mathbb{S}^1\backslash \mathrm{QS}(\mathbb{S}^1)$ as a right topological group [40] equipped with the topology inherited from $\mathcal{T}(1)$: i.e., the one induced from the Teichmüller distance.

Proposition 6.4.5. *The post-composition and pre-composition are continuous restricted on* $\mathbb{S}^1\backslash \mathrm{RM}(\mathbb{S}^1) \times \mathbb{S}^1\backslash \mathrm{QS}(\mathbb{S}^1)$.

Proof. We only need to take care of the normalization. Note that if $\tilde{f}_n$ and $\tilde{g}_n$ are the conformal maps corresponding to $\psi_n \circ \varphi_n$ as in Definition 6.4.4, then they differ from $\tilde{\omega}_n \circ f \circ \omega_n^{-1}$ and $\tilde{\omega}_n \circ g$ by a post-composition by $z \mapsto c_n z$ where $1/c_n = (\tilde{\omega}_n \circ f \circ \omega_n^{-1})'(0)$. Since $(\tilde{\omega}_n \circ f \circ \omega_n^{-1})'(0) \to (\tilde{\omega} \circ f \circ \omega^{-1})'(0)$, we conclude that $\tilde{f}_n \to \tilde{f}$ and $\tilde{g}_n \to \tilde{g}$ uniformly on compact subsets. The analogous results for $\varphi^{-1} \circ \psi$ follow in a similar manner as before. $\square$

6.4.2 SLE loop welding measure

For $\kappa \in (0, 4]$, the SLE_κ loop measure μ^κ is a σ-finite measure on the space of Jordan curves on $\widehat{\mathbb{C}}$ satisfying the following properties as defined by Kontsevich and Suhov [159].

- Generalized restriction property: For any simply connected domain $D \subsetneq \hat{\mathbb{C}}$, we define μ_D^κ by

$$\frac{\mathrm{d}\mu_D^\kappa}{\mathrm{d}\mu^\kappa}(\cdot) = \mathbf{1}_{\{\cdot \subset D\}} \exp\left(c(\kappa)\Lambda^*(\cdot, D^c)/2\right)$$

  where $c(\kappa) = (6-\kappa)(3\kappa-8)/2\kappa$ is the central charge of SLE_κ and $\Lambda^*(\eta, D^c)$ is the size of the normalized Brownian loop measure for loops hitting both η and D^c.

- Conformal invariance: If $f : D \to D'$ is a conformal map between two subdomains of $\widehat{\mathbb{C}}$, then the pushforward measure $f_*\mu_D^\kappa$ is agrees with $\mu_{D'}^\kappa$.

For each $\kappa \in (0, 4]$, the SLE_κ loop measure μ^κ exists and is unique up to a multiplicative constant. This was proved by Werner [250] for $\kappa = 8/3$; the measure μ^κ for the entire range of $\kappa \in (0, 4]$ was constructed by Zhan [255] and was proved to be unique by Baverez and Jego [32].

From the perspective of conformal welding, it is natural to consider the shape measure of the SLE_κ loops. First, consider the restriction $\mu_{\mathbb{C}\setminus\{0\}}^\kappa$ of the SLE_κ loop measure μ^κ to the loops that separate 0 and ∞. Given a Jordan curve η separating 0 and ∞, let $\mathrm{CR}(\eta, 0)$ denote the conformal radius $|f'(0)|$ of the bounded component D_η of $\widehat{\mathbb{C}} \setminus \eta$ viewed from 0 where $f : \mathbb{D} \to D_\eta$ is a conformal map with $f(0) = 0$. Then, we define $\mu_\#^\kappa$ to be the conditional law[2] of $\mu^\kappa_{\mathbb{C}\setminus\{0\}}$ on the set of loops η for which $\mathrm{CR}(\eta, 0) = 1$. We may choose the multiplicative constant for the SLE_κ loop measure μ^κ such that the shape measure $\mu^\kappa_\#$ is a probability measure; we assume this henceforth.

Definition 6.4.6. For $\kappa \in (0, 4]$, we define $\mathrm{SLE}^{\mathrm{weld}}_\kappa$ to be the probability measure on $\mathbb{S}^1 \setminus \mathrm{RM}(\mathbb{S}^1)$ given by the pushforward of the SLE_κ loop shape measure $\mu^\kappa_\#$ under the measurable one-to-one correspondence $\mathcal{J}^{0,\infty}_\# \to \mathbb{S}^1 \setminus \mathrm{RM}(\mathbb{S}^1)$ induced by conformal welding.

Remark 6.4.7. A property of the SLE_κ loop measure that is fundamental to our work is that it is supported on the set of conformally removable Jordan curves. For $\kappa \in (0, 4)$, this is since an SLE_κ loop η is, except on a set of measure zero, the boundary of a Hölder domain (i.e., there exists a conformal map f on $\mathbb{D}$ such that f extends to a Hölder continuous map on $\overline{\mathbb{D}}$ and $\eta = f(\partial\mathbb{D})$), which is a sufficient condition for conformal removability given in the acclaimed work of Jones and Smirnov [145].

It is a classical fact that Loewner maps uniformizing chordal SLE_κ is almost surely uniformly Hölder continuous on bounded sets [218, Theorem 5.2] (see [131, Corollary 1.8] for the optimal Hölder exponent). This implies that SLE_κ loops are Hölder domains thanks to Zhan's construction of the SLE_κ loop measure [255, Theorem 4.2]: consider a triple (η, z, w) where the marginal law of η is the SLE_κ loop measure and, conditioned on η, the points z and w are sampled independently from the Minkowski content measure of η. Let η_0 and η_1 be two components of η divided by z and w. Then, conditioned on z, w, and η_0, the law of η_1 in the domain $\hat{\mathbb{C}} \setminus \eta_0$ is that of chordal SLE_κ. By first mapping $\hat{\mathbb{C}} \setminus \eta_0$ conformally to $\mathbb{H}$, say by f_0, and then considering the conformal map f_1 taking either the left or the right component of $\mathbb{H} \setminus f_0(\eta_1)$ to $\mathbb{D}$, we see that $f := (f_1 \circ f_0)^{-1}$ is a Riemann map taking $\mathbb{D}$ to a bounded component of $\hat{\mathbb{C}} \setminus \eta$ which is uniformly Hölder continuous on subsets $A \subset \overline{\mathbb{D}}$

2. One way to give this law is to consider the restriction of $\mu^\kappa_{\mathbb{C}\setminus\{0\}}$ to loops η with $1/2 \leq \mathrm{CR}(\eta, 0) \leq 1$, which is a finite measure since it must wind around the disk $\frac{1}{8}\mathbb{D}$ and intersect the unit circle $\partial\mathbb{D}$ [255]. Scaling each loop η sampled under this measure using the map $z \mapsto z/\mathrm{CR}(\eta, 0)$ gives a loop sampled from $\mu^\kappa_\#$ by the conformal invariance of the SLE_κ loop measure.

such that $f(A)$ is a subset of η_1 with strictly positive distance from z and w. Resampling (z, w) countably many times, we see that the Riemann map f uniformizing the bounded component of $\hat{\mathbb{C}} \setminus \eta$ is uniformly Hölder continuous for all SLE_κ loops η except on a set of measure zero.

This argument shows that the uniformizing maps for SLE_4 loops satisfy the (non-Hölder) modulus of continuity given in [150] except on a set of zero measure. The conformal removability of an SLE_4 loop follows from observing that the sufficient condition for conformal removability in [151], which was verified there for chordal SLE_4, is local.

6.4.3 Conformal welding of quantum disks

Fix $\gamma \in (0, 2)$ and let $\kappa = \gamma^2$. Building upon Sheffield's quantum zipper [228], Ang, Holden, and Sun proved in [19] that conformally welding two independent γ-LQG disks gives a quantum sphere decorated with an independent SLE_κ loop. We give a translation of this result in terms of the welding homeomorphism of two independent LGFs on $\mathbb{S}^1$.

Recall that given an LGF h on $\mathbb{S}^1$, we let $\widehat{\mathcal{M}}_h$ to be the γ-GMC measure $\mathcal{M}_h = \mathcal{M}_h^\gamma$ corresponding to h normalized to have unit mass. Define $\phi_h = \phi_h^\gamma \in \mathrm{Homeo}_+(\mathbb{S}^1)$ by

$$\phi_h^\gamma(z) := \exp(2\pi i \cdot \widehat{\mathcal{M}}_h^\gamma([1, z])),$$

for each $z \in \mathbb{S}^1$ where $[1, z] \subset \mathbb{S}^1$ denotes the arc running counterclockwise from 1 to z. That is, the pushforward of the normalized GMC measure $\widehat{\mathcal{M}}_h$ under the homeomorphism ϕ_h is the uniform (arc-length) measure on $\mathbb{S}^1$. The following is the main result of this subsection.

Lemma 6.4.8. *Let $\gamma \in (0, 2)$ and $\kappa = \gamma^2 \in (0, 4)$. Suppose h_1, h_2 are independent LGFs on $\mathbb{S}^1$ and α is a uniform rotation of $\mathbb{S}^1$ independent from h_1 and h_2. Let $\mathcal{W}^\kappa$ denote the law of $(\phi_{h_2}^\gamma)^{-1} \circ \alpha \circ \phi_{h_1}^\gamma \in \mathrm{Homeo}_+(\mathbb{S}^1)$. Then, the pushforward of $\mathcal{W}^\kappa$ under the natural projection $\mathrm{Homeo}_+(\mathbb{S}^1) \to \mathbb{S}^1 \backslash \mathrm{Homeo}_+(\mathbb{S}^1)$ is equivalent to $\mathrm{SLE}_\kappa^{\mathrm{weld}}$. Furthermore, if we* *identify* $\mathbb{S}^1 \backslash \mathrm{Homeo}_+(\mathbb{S}^1)$ *with the stabilizers of 1 in* $\mathrm{Homeo}_+(\mathbb{S}^1)$*, then* $\mathrm{SLE}_\kappa^{\mathrm{weld}}$ *is equivalent to the law of* $(\phi^\gamma_{h_2-\gamma\log|\cdot-1|})^{-1}\circ\phi^\gamma_{h_1}$.

Note that $(\phi^\gamma_{h_2})^{-1}\circ\alpha\circ\phi^\gamma_{h_1}$ is the circle homeomorphism corresponding to the conformal welding of two disks with boundary length measures $\widehat{\mathcal{M}}_{h_1}$ and $\widehat{\mathcal{M}}_{h_2}$ where we make a random shift for the point on the second disk glued to 1 on the first disk according to the boundary length measure $\widehat{\mathcal{M}}_{h_2}$. This result immediately implies Theorem 6.1.2.

Proof of Theorem 6.1.2. Combine Lemma 6.4.8 with Proposition 6.3.7. □

We will obtain Lemma 6.4.8 using the conformal welding of independent quantum disks each with one interior marked point as established in [15]. To state this result, let us recall some basic definitions from the LQG literature. A γ-LQG disk with one marked point is defined as the equivalence class $(D,\Gamma,z)/\sim_\gamma$ of the tuple of a domain D conformally equivalent to the unit disk, a random distribution h, and a point $z\in D$, where $(D,\Gamma,z)\sim_\gamma(\tilde{D},\tilde{\Gamma},\tilde{z})$ if there exists a conformal map $f: D\to\tilde{D}$ such that

$$\Gamma = \tilde{\Gamma}\circ f + Q\log|f'| \tag{6.4.2}$$

for $Q=\frac{2}{\gamma}+\frac{\gamma}{2}$ and $\tilde{z}=f(z)$. We define a γ-LQG sphere with two marked points and a Jordan curve similarly as an equivalence class $(D,\Gamma,\eta,z_1,z_2)/\sim_\gamma$ up to conformal transformations where the field Γ satisfies the coordinate change rule (6.4.2).

We now define Liouville field on $\mathbb{D}$. While the Liouville theory on simply connected domains with boundary was stated originally on the unit disk $\mathbb{D}$ [141], subsequent works such as [210, 211, 22] used the upper half-plane $\mathbb{H}$ as the standard domain. Nevertheless, Proposition 3.4 and Proposition 3.7 of [141] imply that the (infinite) law we give here agrees with the definitions of Liouville fields on $\mathbb{H}$ up to a γ-dependent multiplicative factor. See [211, Section 5] for further details.

Definition 6.4.9 ([22, Lemma 4.4–4.5])**.** Let $P_{\mathbb{D}}$ be the law of the free-boundary GFF on $\mathbb{D}$ normalized to have zero mean on the boundary $\partial\mathbb{D}$. Let $P_{\mathbb{C}}$ denote the law of the whole-plane GFF with zero mean on the unit circle.[3]

1. For $\ell > 0$, define $\mathrm{LF}_{\mathbb{D}}^{(\gamma,0)}(\ell)$ to be the law of the field $\Gamma^{\ell} := \hat{\Gamma} + \frac{2}{\gamma}\log\frac{\ell}{\nu_{\hat{\Gamma}}(\partial\mathbb{D})}$, where

$$\hat{\Gamma} := \Gamma - \gamma\log|\cdot| \tag{6.4.3}$$

   for the field Γ sampled under the reweighted measure $\ell^{-4/\gamma^2}(\nu_{\hat{\Gamma}}(\partial\mathbb{D}))^{4/\gamma^2-1}P_{\mathbb{D}}$. Let $\mathcal{M}_{1,0}^{\mathrm{disk}}(\gamma;\ell)$ be the measure on quantum surfaces $(\mathbb{D},\Gamma^{\ell},0)/\sim_{\gamma}$, where Γ^{ℓ} is sampled from $\mathrm{LF}_{\mathbb{D}}^{(\gamma,0)}(\ell)$. This is a finite measure on quantum disks with boundary length ℓ and one interior marked point. Define

$$\mathrm{LF}_{\mathbb{D}}^{(\gamma,0)} = \int_0^{\infty}\mathrm{LF}_{\mathbb{D}}^{(\gamma,0)}(\ell)\,d\ell \quad\text{and}\quad \mathcal{M}_{1,0}^{\mathrm{disk}}(\gamma) = \int_0^{\infty}\mathcal{M}_{1,0}^{\mathrm{disk}}(\gamma;\ell)\,d\ell. \tag{6.4.4}$$

2. For $\ell > 0$, define $\mathrm{LF}_{\mathbb{D}}^{(\gamma,0),(\gamma,1)}(\ell)$ to be the law of the field $\Gamma^{\ell} := \hat{\Gamma} + \frac{2}{\gamma}\log\frac{\ell}{\nu_{\hat{\Gamma}}(\partial\mathbb{D})}$, where

$$\hat{\Gamma} := \Gamma - \gamma\log|\cdot| - \gamma\log|\cdot -1| \tag{6.4.5}$$

   for the field Γ sampled under the reweighted measure $\ell^{-4/\gamma^2+1}(\nu_{\hat{\Gamma}}(\partial\mathbb{D}))^{4/\gamma^2}P_{\mathbb{D}}$.

3. Let $\mathrm{LF}_{\mathbb{C}}^{(\gamma,0),(\gamma,\infty)}$ denote the law of the field $\Gamma - \gamma\log|\cdot| - 2(Q-\gamma)\log|\cdot|_+ + c$ where (Γ, c) is sampled from the law $P_{\mathbb{C}} \times [e^{2(\gamma-Q)c}dc]$ and $|z|_+ = \max\{1,|z|\}$.

We note that $\mathrm{LF}_{\mathbb{D}}^{(\gamma,0)}(\ell)$ and $\mathrm{LF}_{\mathbb{D}}^{(\gamma,0),(\gamma,1)}(\ell)$ are finite measures for each $\ell > 0$ [141, Corollary 3.10]. See also [211] for explicit formulas for their total masses. The following lemma clarifies the relation between these two measures, which follows from combining Definition 2.10, Proposition 3.4, and Proposition 3.9 of [22] and Proposition 2.21 of [15].

3. This is a centered Gaussian field with covariance kernel $G_{\mathbb{C}}(z,w) = -\log|z-w| + \log|z|_+ + \log|w|_+$.

Lemma 6.4.10. *Given a finite measure λ, let $\lambda^{\#}$ denote the this law normalized to be a probability measure. Fix $\ell > 0$.*

1. *Sample a field Γ on $\mathbb{D}$ from $\mathrm{LF}_{\mathbb{D}}^{(\gamma,0)}(\ell)^{\#}$, then sample $e^{i\theta} \in \partial\mathbb{D}$ from the normalized boundary length measure $\hat{\nu}_{\Gamma^\ell} := \nu_{\Gamma^\ell}/\nu_{\Gamma^\ell}(\partial\mathbb{D})$. Then, the field $\Gamma^\ell(e^{i\theta}\cdot)$ has the law $\mathrm{LF}_{\mathbb{D}}^{(\gamma,0),(\gamma,1)}(\ell)^{\#}$.*

2. *Sample a field Γ on $\mathbb{D}$ from $\mathrm{LF}_{\mathbb{D}}^{(\gamma,0),(\gamma,1)}(\ell)^{\#}$ and independently sample a uniform boundary point $e^{i\theta} \in \partial\mathbb{D}$ (from the normalized arc-length measure on $\partial\mathbb{D}$). Then, $\Gamma^\ell(e^{i\theta}\cdot)$ has the law $\mathrm{LF}_{\mathbb{D}}^{(\gamma,0)}(\ell)^{\#}$.*

Given quantum disks $\mathcal{D}_1$ and $\mathcal{D}_2$, let $\mathrm{Weld}(\mathcal{D}_1, \mathcal{D}_2)$ denote the uniform conformal welding of $\mathcal{D}_1$ and $\mathcal{D}_2$ as described in [15, Section 2.5]. That is, $\mathrm{Weld}(\mathcal{D}_1, \mathcal{D}_2)$ is a probability measure on the quantum sphere obtained by conformally welding the boundaries of $\mathcal{D}_1$ and $\mathcal{D}_2$ starting from matching the points sampled independently on each boundary from the normalized boundary length measures. Put otherwise, $\mathrm{Weld}(\mathcal{D}_1, \mathcal{D}_2)$ is the law of the quantum sphere $(\widehat{\mathbb{C}}, \Gamma, \eta, 0, \infty)/\sim_\gamma$ where, if D_1 and D_2 are the bounded and unbounded components of $\widehat{\mathbb{C}} \setminus \eta$, then $(\mathcal{D}_1, \mathcal{D}_2) = ((D_1, \Gamma|_{D_1}, 0)/\sim_\gamma, (D_2, \Gamma|_{D_2}, \infty)/\sim_\gamma)$ and the quantum boundary length measures of $\mathcal{D}_1$ and $\mathcal{D}_2$ agree on η, which we call the quantum length measure of η. Moreover, if we sample $p \in \eta$ uniformly from the quantum length measure normalized to be a probability measure, then the joint law of $(D_1, \Gamma|_{D_1}, 0, p)/\sim_\gamma$ and $(D_2, \Gamma|_{D_2}, 0, p)/\sim_\gamma$ is that of $\mathcal{D}_1$ and $\mathcal{D}_2$ each with a boundary point sampled independently of each other from the normalized quantum boundary length measure. Let us define the law

$$\mathrm{Weld}(\mathcal{M}_{1,0}^{\mathrm{disk}}(\gamma;\ell), \mathcal{M}_{1,0}^{\mathrm{disk}}(\gamma;\ell)) = \int \mathrm{Weld}(\mathcal{D}_1, \mathcal{D}_2)\, d[\mathcal{M}_{1,0}^{\mathrm{disk}}(\gamma;\ell) \times \mathcal{M}_{1,0}^{\mathrm{disk}}(\gamma;\ell)](\mathcal{D}_1, \mathcal{D}_2). \tag{6.4.6}$$

We now state the conformal welding theorem of [19] for the loops that separate 0 and ∞, and translate it to Lemma 6.4.8.

Lemma 6.4.11 ([15, Lemma 7.7][4])**.** *Let $\kappa = \gamma^2 \in (0,4)$. If (Γ, η) is sampled from $\mathrm{LF}_{\mathbb{C}}^{(\gamma,0),(\gamma,\infty)} \times \mu_{\#}^{\kappa}$, then the law of the loop-decorated quantum surface $(\widehat{\mathbb{C}}, \Gamma, \eta, 0, \infty)/\sim_\gamma$ is equal to $\int_0^\infty \ell \cdot \mathrm{Weld}(\mathcal{M}_{1,0}^{\mathrm{disk}}(\gamma;\ell), \mathcal{M}_{1,0}^{\mathrm{disk}}(\gamma;\ell))\, \mathrm{d}\ell$ up to a γ-dependent multiplicative constant.*

Proof of Lemma 6.4.8. Consider the disintegration

$$\mathrm{LF}_{\mathbb{C}}^{(\gamma,0),(\gamma,\infty)} \times \mu_{\#}^{\kappa} = \int_0^\infty [\mathrm{LF}_{\mathbb{C}}^{(\gamma,0),(\gamma,\infty)} \times \mu_{\#}^{\kappa}](\ell)\, \mathrm{d}\ell$$

where $[\mathrm{LF}_{\mathbb{C}}^{(\gamma,0),(\gamma,\infty)} \times \mu_{\#}^{\kappa}](\ell)$ is a measure on the pair (Γ,η) such that the quantum length of η with respect to Γ, which we denote $\nu_\Gamma(\eta)$, equals ℓ. From Definition 6.4.9, we see that a version of $[\mathrm{LF}_{\mathbb{C}}^{(\gamma,0),(\gamma,\infty)} \times \mu_{\#}^{\kappa}](\ell)$ is given by $(\Gamma + \frac{2}{\gamma}\log(\ell/\nu_\Gamma(\eta)), \eta)$ where (Γ,η) is sampled from the restriction of

$$\frac{\ell^{-4/\gamma^2}}{\int_1^\infty t^{-4/\gamma^2}\mathrm{d}t} \mathrm{LF}_{\mathbb{C}}^{(\gamma,0),(\gamma,\infty)} \times \mu_{\#}^{\kappa}$$

to the event $\{\nu_\Gamma(\eta) \geq 1\}$. In particular, note that the marginal law of η under $[\mathrm{LF}_{\mathbb{C}}^{(\gamma,0),(\gamma,\infty)} \times \mu_{\#}^{\kappa}](\ell)$ is equivalent to $\mu_{\#}^{\kappa}$.

Fix $\ell = 1$. Recall that the quantum disk $(\mathbb{D}, \Gamma, 0)/\sim_\gamma$ with Γ sampled from $\mathrm{LF}_{\mathbb{D}}^{(\gamma,0)}(1)^{\#}$ has the law $\mathcal{M}_{1,0}^{\mathrm{disk}}(\gamma;1)^{\#}$. Define $\phi_\Gamma(z) = \exp(2\pi i \hat{\nu}_\Gamma([1,z]))$, where $[1,z]$ denotes the arc from 1 to z counterclockwise. Suppose Γ_1, Γ_2 are independent fields sampled from $\mathrm{LF}_{\mathbb{D}}^{(\gamma,0)}(1)^{\#}$. Observe that if α_1, α_2 are independent samples from the uniform measure on the rotation group of $\mathbb{S}^1$ which are further independent from Γ_1, Γ_2, then $(\alpha_2 \circ \phi_{\Gamma_2|_{\partial\mathbb{D}}})^{-1} \circ (\alpha_1 \circ \phi_{\Gamma_1|_{\partial\mathbb{D}}})$ is the homeomorphism corresponding to the uniform conformal welding of quantum disks $\mathcal{D}_1 = (\mathbb{D}, \Gamma_1, 0)/\sim_\gamma$ and $\mathcal{D}_2 = (\mathbb{D}, \Gamma_2, 0)/\sim_\gamma$. That is, if η is the Jordan curve which corresponds to the welding homeomorphism $(\alpha_2 \circ \phi_{\Gamma_2|_{\partial\mathbb{D}}})^{-1} \circ (\alpha_1 \circ \phi_{\Gamma_1|_{\partial\mathbb{D}}})$, then there is a random field Γ such that $(\widehat{\mathbb{C}}, \Gamma, \eta, 0, \infty)/\sim_\gamma$ has the law $\mathrm{Weld}(\mathcal{M}_{1,0}^{\mathrm{disk}}(\gamma;1)^{\#}, \mathcal{M}_{1,0}^{\mathrm{disk}}(\gamma;1)^{\#})$.

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4. In place of $\mathbb{C}$ and $\mu_{\#}^{\kappa}$, [15] uses the infinite cylinder $\mathcal{C} = \mathbb{R} \times \mathbb{S}^1$ and the law of SLE_κ loops on $\mathcal{C}$ separating $\pm\infty$ that touch $\{0\} \times \mathbb{S}^1$ from the left but do not cross it, respectively. By the conformal invariance of SLE_κ loops and the translation invariance of $\mathrm{LF}_{\mathcal{C}}^{(\gamma,\pm\infty)}$ [20, Theorem 2.13], our statement is equivalent to [15, Lemma 7.7].

Hence, by Lemma 6.4.11, η has the law $\mu_{\#}^{\kappa}$ when considered up to rotations of $\widehat{\mathbb{C}}$ around 0. By the invariance of the law $\mu_{\#}^{\kappa}$ under such rotations, if we choose $\alpha_0 \in \partial\mathbb{D}$ uniformly yet again independently from $\Gamma_1, \Gamma_2, \alpha_1, \alpha_2$, then $\alpha_0 \cdot \eta$ has the law $\mu_{\#}^{\kappa}$. Observe that if we denote the rotation $z \mapsto \alpha_0 \cdot z$ also as α_0, then the welding homeomorphism corresponding to $\alpha_0 \cdot \eta$ is given by $(\alpha_2 \circ \phi_{\Gamma_2|_{\partial\mathbb{D}}})^{-1} \circ (\alpha_1 \circ \phi_{\Gamma_1|_{\partial\mathbb{D}}}) \circ \alpha_0^{-1}$ when both are considered as random elements of $\mathbb{S}^1 \backslash \mathrm{Homeo}_+(\mathbb{S}^1)$. From Definition 6.4.9, we see that if Γ is a sample from $\mathrm{LF}_{\mathbb{D}}^{(\gamma,0)}(1)^{\#}$, then the law of $\Gamma|_{\partial\mathbb{D}}$ is equivalent to that of $h - \frac{2}{\gamma}\nu_h(\mathbb{S}^1)$ where h an LGF on $\mathbb{S}^1$. That is, the law of $\phi_{\Gamma|_{\partial\mathbb{D}}}$ is equivalent to that of $\phi_{h-\frac{2}{\gamma}\nu_h(\mathbb{S}^1)} = \phi_h$.

Thus, to paraphrase our conclusion so far, if h_1, h_2 are LGFs on $\mathbb{S}^1$ and $\alpha_0, \alpha_1, \alpha_2$ are uniform random variables on $\mathbb{S}^1$, all mutually independent, the pushforward of the law of $\phi_{h_2}^{-1} \circ (\alpha_2^{-1} \circ \alpha_1) \circ \phi_{h_1} \circ \alpha_0^{-1}$ under the projection $\mathrm{Homeo}_+(\mathbb{S}^1) \to \mathbb{S}^1 \backslash \mathrm{Homeo}_+(\mathbb{S}^1)$ is equivalent to $\mathrm{SLE}_{\kappa}^{\mathrm{weld}}$. Since the law of LGF on $\mathbb{S}^1$ is invariant under fixed rotations of $\mathbb{S}^1$ and α_0 is independent of h_1, we have that $\phi_{h_1}(\alpha_0^{-1}(1))^{-1} \cdot \phi_{h_1} \circ \alpha_0^{-1}$ agrees in law with ϕ_{h_1}. Moreover, $\alpha_2^{-1} \circ \alpha_1$ is a uniform rotation of $\mathbb{S}^1$ independent from h_1, h_2, and α_0, so we obtain the first part of the lemma.

For the second part of the lemma, we observe that $e^{i\Theta} := (\alpha_2 \circ \phi_{\Gamma_2|_{\partial\mathbb{D}}})^{-1} \circ (\alpha_1 \circ \phi_{\Gamma_1|_{\partial\mathbb{D}}})(1) = (\phi_{\Gamma_2|_{\partial\mathbb{D}}})^{-1} \circ (\alpha_2^{-1} \circ \alpha_1)(1)$ is, conditioned on Γ_2, a point sampled uniformly from the boundary length measure ν_{Γ_2} normalized to be a probability measure. By Lemma 6.4.10, we see that $e^{-i\Theta} \cdot (\phi_{\Gamma_2|_{\partial\mathbb{D}}})^{-1} \circ (\alpha_2^{-1} \circ \alpha_1)$ agrees in law with $\phi_{\tilde{\Gamma}_2}^{-1}$ where $\tilde{\Gamma}_2$ is sampled from $\mathrm{LF}_{\mathbb{D}}^{(\gamma,0),(\gamma,1)}(1)^{\#}$. Since the law of $\tilde{\Gamma}_2|_{\partial\mathbb{D}}$ is equivalent to that of $h_2 - \gamma \log|\cdot - 1|$ from Definition 6.4.9, the second part of the lemma follows as in the previous paragraph. $\square$

6.4.4 Equivalence for other weldings

We end with a few remarks on generalizations of Theorem 6.1.2. From Lemma 6.2.10, for each $\varphi \in \mathrm{WP}(\mathbb{S}^1)$ and almost every $z_0 \in \mathbb{S}^1$, we have that

$$\hat{\varphi}(e^{i\theta}) := \frac{1}{\varphi(z_0)}\varphi(z_0 e^{i\theta}) \tag{6.4.7}$$

satisfies

$$\operatorname{Re} u_{\hat{\varphi}}(\cdot, 1) = \log\left|\frac{\hat{\varphi}(\cdot) - \hat{\varphi}(1)}{\cdot - 1}\right| \in H^{1/2}(\mathbb{S}^1). \tag{6.4.8}$$

Hence, the asymmetry in Theorem 6.1.2 can be seen to be associated with our choice of stabilizers of 1 as the representatives of $\mathbb{S}^1 \backslash \mathrm{RM}(\mathbb{S}^1)$. We record a symmetric version of Theorem 6.1.2 without the "special" point 1.

Corollary 6.4.12. *For $\kappa \in (0, 4)$, sample ψ_κ from $\mathrm{SLE}_\kappa^{\mathrm{weld}}$ and z_0 from a probability measure that is mutually absolutely continuous with respect to the arc-length measure on the unit circle. Define $\hat{\psi}_\kappa(\cdot) := z_0\psi_\kappa(\cdot)$. Then, for any $\varphi \in \mathrm{WP}(\mathbb{S}^1)$, the laws of $\hat{\psi}_\kappa \circ \varphi$ and $\varphi^{-1} \circ \hat{\psi}_\kappa$ are mutually absolutely continuous with respect to that of $\hat{\psi}_\kappa$.*

The homeomorphism ϕ_h^γ giving the normalized GMC measures $\widehat{\mathcal{M}}_h^\gamma$ of intervals was first introduced in [29], which showed using analytic methods that if h_1 and h_2 are independent LGFs on $\mathbb{S}^1$ and $\gamma_1, \gamma_2 \geq 0$ are sufficiently small, then $(\phi_{h_1}^{\gamma_1}) \circ (\phi_{h_2}^{\gamma_2})^{-1}$ is almost surely a welding. (Let ϕ_h^γ be the identity map if $\gamma = 0$.) The works [50] or [161] showed that $(\phi_{h_1}^{\gamma_1})^{-1} \circ (\phi_{h_2}^{\gamma_2})$ is almost surely a welding as well for sufficiently small γ_1, γ_2. A notable feature of these works is that the constants γ_1 and γ_2 can be distinct; however, the law of the Jordan curves that solve the welding problem for these homeomorphisms were not identified. Our analysis leads to the following quasi-invariance results for these random homeomorphisms.

Corollary 6.4.13. *Let $\varphi, \tilde{\varphi} \in \mathrm{Homeo}_+(\mathbb{S}^1)$ fix 1. For independent LGFs $h, \tilde{h}$ on $\mathbb{S}^1$ and*

$\gamma, \tilde{\gamma} \in (0, 2]$, *we have:*

- *The law of* $\phi_h^\gamma \circ \varphi$ *is mutually absolutely continuous with respect to the law of* ϕ_h^γ *if and only if* $\varphi \in \mathrm{WP}(\mathbb{S}^1)$.
- *The law of* $\varphi \circ (\phi_h^\gamma)^{-1}$ *is mutually absolutely continuous with respect to the law of* $(\phi_h^\gamma)^{-1}$ *if and only if* $\varphi \in \mathrm{WP}(\mathbb{S}^1)$.
- *The law of* $\varphi \circ (\phi_h^\gamma)^{-1} \circ \phi_{\tilde{h}}^{\tilde{\gamma}} \circ \tilde{\varphi}$ *is mutually absolutely continuous with respect to the law of* $(\phi_h^\gamma)^{-1} \circ \phi_{\tilde{h}}^{\tilde{\gamma}}$ *if* φ, $\tilde{\varphi} \in \mathrm{WP}(\mathbb{S}^1)$.

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