THE UNIVERSITY OF CHICAGO

SEMISIMPLICITY, RIGIDITY, AND UNIQUE ERGODICITY IN PRODUCTS AND BRANCHED COVERS OF TRANSLATION SURFACES

A DISSERTATION SUBMITTED TO
THE FACULTY OF THE DIVISION OF THE PHYSICAL SCIENCES
IN CANDIDACY FOR THE DEGREE OF
DOCTOR OF PHILOSOPHY

DEPARTMENT OF MATHEMATICS

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CHICAGO, ILLINOIS
JUNE 2026

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To my grandmother Galina, who first taught me to love mathematics.

To my grandfather Victor, whose example inspired me to become a mathematician.

To both of them for their unwavering support along the way.

"Measure what is measurable, and make measurable what is not so."

Galileo Galilei

"The road up and the road down are one and the same."

Heraclitus

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ACKNOWLEDGMENTS

I owe my deepest debt of gratitude to my advisor, Simion Filip. Without his guidance, support, and belief in me, this dissertation would not have been possible. I am profoundly thankful for everything he has taught me, both through mathematics and through his example as a researcher.

I am very thankful to the Department of Mathematics at the University of Chicago for giving me a place to grow as a mathematician. I am especially grateful to the department staff, including John Zekos, Laurie Wail, Andrea Rosalez, Judy Garza, Meagan Cusumano, and many others, for their kindness, patience, and help with the many practical matters that make academic life possible. I am also grateful to the instructional faculty, from whom I learned how to teach. In particular, I am especially grateful to Sarah Ziesler and Subhadip Chowdhury for their mentorship as I grew into a better educator, and for the time, effort, and generosity they devoted to that growth.

I deeply thank Jon Chaika, Alex Eskin, Pascal Hubert, and Howard Masur for many enriching discussions that guided and encouraged me along my mathematical path. I would also like to thank the postdocs who organized the dynamics seminar for creating a lively and welcoming mathematical community. I am grateful to Frederik Benirschke, Karen Butt, Aaron Calderon, and Daniil Rudenko for the wisdom about academic life that they shared with me over the years.

I am immensely grateful to Elizaveta Shuvaeva for a wonderful collaboration and for being my closest friend through so many years. The results presented in Chapter 3 of this thesis were obtained in co-authorship with her. I am also thankful for the support, help, and companionship of many other graduate students in the department, who became my friends as well as my mentors. A non-exhaustive list includes Megan Roda, Seung uk Jang, Rosemary Elliott Smith, Yulia Kotelnikova, Meg Doucette, DeVon Ingram, Olga Medrano Martin del Campo, Chloé Postel-Vinay, Chris Wilson, Subhasish Mukherjee, Aadrita Laha,

and Altan Erdnigor.

My mathematical journey would have been impossible without the people who first set me on this path. I am grateful to the teachers and educators of School No. 57, who taught me the foundations of mathematical thinking, and in particular to Lev Altshuler, my wonderful classroom teacher. I thank the faculty of the Department of Mathematics at the Higher School of Economics for shaping me into an aspiring mathematician with vigorous curiosity. I am also deeply grateful to Sasha Skripchenko, my undergraduate thesis advisor, who first showed me the beauty of dynamical systems.

Research is never done in isolation from life. My journey over the years 2020–2026 was not easy, and I am deeply thankful to the people who made it gentler. I am grateful to Anna Ginovker, Anna Meschansky, Michael Rozenblit, and Emma Cai for supporting me, a very distant relative they had never met before, through the years in which I was making North America my new home. Most of all, I thank my family. I am immensely grateful to my brother Denis Baron, my mother Natalia Buchstaber, my father Gleb Baron, my grandmother Galina Buchstaber, my grandfather Victor Buchstaber, my grandmother Maya Levinskaya, and my late grandfather Rudolf Baron for their unwavering love, support, and faith in me.

ABSTRACT

This dissertation studies two aspects of translation-surface dynamics: the structure of natural cocycles over products of strata, and the dynamics of explicit branched-cover constructions.

In the first part, I study the Kontsevich–Zorich cocycle and related tensor-power cocycles over products of strata of Abelian differentials, equipped with ergodic actions of subgroups of $(\mathrm{SL}_2\mathbb{R})^n$. I prove Deligne-type semisimplicity results in this setting: the relevant Hodge bundles admit decompositions into G-invariant pieces of the form $V_i \otimes_{A_i} W_i$, where the V_i carry compatible variations of Hodge structure, and every G-invariant subbundle is obtained from this decomposition by choosing appropriate A_i-submodules. I also prove a rigidity result for measurable invariant subbundles on affine invariant manifolds, showing that in affine coordinates the corresponding projection operators are jointly polynomial of homogeneous degree 0 in the coordinates and the reciprocal area function.

The second part studies slit-induced branched double covers of translation surfaces with uniquely ergodic vertical flow. Together with Elizaveta Shuvaeva, we prove a geometric criterion for unique ergodicity based on the existence of embedded neighborhoods of a branch point along a sequence in the Teichmüller orbit. We then show that such neighborhoods occur for almost every slit endpoint under a subsequential embedded-radius hypothesis, and we develop geometric tools based on Delaunay triangulations and pipe cylinders to verify this condition. This yields the following: for almost every choice of a slit, the resulting branched double cover is uniquely ergodic.

CHAPTER 1

INTRODUCTION

Translation surfaces [Zor06; Wri14; Mat18; Mas21; Fil24] occupy a rather unusual position in contemporary mathematics. They can be introduced by a concrete cut-and-paste construction in the Euclidean plane, but they quickly lead to questions in Teichmüller theory, algebraic geometry, ergodic theory, Hodge theory, and representation theory. A translation surface may be described geometrically as a surface obtained by gluing finitely many polygons in $\mathbb{R}^2$ by parallel side identifications via translations. Equivalently, it may be described as a compact topological surface endowed away from finitely many singularities with an atlas whose transition functions are translations. Equivalently again, it may be described as a pair (X, ω) consisting of a compact Riemann surface X and a nonzero holomorphic 1-form ω on X. In local coordinates obtained by integrating ω, the flat metric is Euclidean away from the zeroes of ω, and a zero of order k produces a cone point of total angle $2\pi(k+1)$. Thus even the most basic objects of the theory already sit simultaneously in Euclidean geometry, complex analysis, and topology.

This multiplicity of viewpoints is the reason the subject is so rich. The polygonal model makes examples explicit and supports a direct geometric intuition. The interpretation in terms of Abelian differentials places the theory naturally inside moduli spaces and algebraic geometry. The flat-geometric viewpoint makes directional flows, geodesics, saddle connections, and cylinder decompositions visible at once. Finally, once one passes from a single surface to a stratum of Abelian differentials, one acquires the action of $\mathrm{GL}_2^+(\mathbb{R})$ and in particular of $\mathrm{SL}_2(\mathbb{R})$ and the Teichmüller geodesic flow, and these ambient dynamical systems become the organizing mechanisms behind many problems about the original surface [Mas06; FM14; Wri14; Fil24]. One of the central lessons of the field is that the asymptotic behavior of a directional flow on one surface is very often best understood only after embedding that surface into a moduli problem and studying the larger renormalization dynamics around it.

The modern subject arose from a long historical development. Its remote origins lie in nineteenth-century work on Abelian integrals and Abelian differentials. In particular, Riemann's 1857 memoir *Theorie der Abel'schen Functionen* introduced a language of periods, moduli, and complex curves that now sits at the foundation of the theory of translation surfaces [Rie57]. From the modern point of view, the passage from a curve to a pair (X, ω) is exactly the passage from moduli of curves to moduli of curves endowed with a distinguished holomorphic differential, so the present-day stratification of the moduli space of Abelian differentials should be understood as a direct descendant of the classical theory rather than as an artificial later reformulation. The subject therefore has a classical algebro-geometric ancestry even when its contemporary questions are dynamical.

A second foundational source is Teichmueller's work on quasiconformal deformations and quadratic differentials. His 1940 chapter *Extremale quasikonforme Abbildungen und quadratische Differentiale* made quadratic differentials central to the geometry of moduli and introduced ideas that remain fundamental in Teichmüller theory [Tei40]. While the present thesis is concerned primarily with Abelian rather than quadratic differentials, the conceptual framework is the same: the Teichmüller geodesic flow, the use of flat structures with singularities, and the interpretation of moduli in terms of deformation of charts all belong to that circle of ideas. Translation surfaces, in their modern form, sit exactly at the intersection of Riemann's theory of holomorphic differentials and Teichmueller's dynamical geometry of moduli.

Another important source of problems for the modern theory came from mathematical physics, through what is now called Novikov's problem [Nov82; Nov86] (the first standard dedicated paper explicitly presenting the semiclassical-electron version of the problem under Novikov's name is Zorich's 1984 note [Zor84]). Originating in the study of the semiclassical motion of electrons in a normal metal placed in a strong magnetic field, the problem asks for the geometry of plane sections of triply periodic surfaces, or equivalently for the behavior

of level sets of quasiperiodic functions on the plane with three quasiperiods. In the three-dimensional torus, this may be reformulated as the study of the foliation induced on a compact surface by a constant closed 1-form. Although this problem is not usually formulated in the language of translation surfaces, it belongs to the same general circle of ideas: one studies foliations on compact surfaces, their asymptotic behavior, and the way global topology constrains the geometry of long trajectories. In this sense, Novikov's problem should be regarded as one of the important historical bridges between flat-surface dynamics, the theory of measured foliations, and questions coming from physics [Zor06; Zor99]. See [NM98] for a survey of the topological theory behind the semiclassical motion of electrons on Fermi surfaces in strong magnetic fields, which explains how the geometry of noncompact plane sections of periodic surfaces governs observable magnetotransport phenomena in normal metals.

The connection is not only historical. One of the recurring insights of the subject is that foliations defined by closed 1-forms can often be studied by passing to suitable first-return dynamics on transversals, where one encounters interval-exchange-type phenomena and renormalization procedures closely parallel to those appearing in the theory of translation surfaces. From this point of view, the methods developed for flat surfaces provide part of the natural dynamical language in which it is studied. The work of Zorich, Dynnikov, and later authors showed that generic open trajectories in Novikov's problem exhibit strong asymptotic regularity, while exceptional chaotic regimes lead to subtler one-dimensional return dynamics and related generalizations of interval exchange transformations [Zor06; Dyn08; AHS16]. This interaction among topology, dynamics, and physics is another reason the theory of translation surfaces developed in such close contact with neighboring problems about surface foliations and long trajectories.

The subject of translation surfaces acquired its distinctly dynamical identity in the 1970s. One of the key observations is that first-return maps of directional flows on translation surfaces are interval exchange transformations. Keane's work established interval exchange

transformations as a serious class of measure-preserving maps and formulated basic minimality questions for them [Kea75]. Rauzy's induction then provided an indispensable renormalization procedure [Rau79]. At roughly the same time, rational polygonal billiards were brought into the picture through the unfolding construction: for a polygon with angles rational multiples of π, the billiard flow can be unfolded into straight-line motion on a flat surface, so billiard dynamics becomes a special case of directional flow on a translation or half-translation surface [ZK75; MT02; Dav26]. Hubbard and Masur, in a different but closely related direction, clarified the relation between quadratic differentials and measured foliations, thereby giving a fundamental structural result about how foliations are realized by flat geometry [HM79]. By the end of that decade, the central players of the modern theory were already present: flat surfaces, interval exchanges, measured foliations, billiards, and renormalization.

The decisive breakthroughs of the early 1980s established the subject on a new level. In independent works, Masur and Veech proved the almost-everywhere unique ergodicity of interval exchange transformations in the irreducible setting [Mas82; Vee82]. Shortly afterward, Kerckhoff, Masur, and Smillie proved that for every translation surface the directional flow is uniquely ergodic in almost every direction [KMS86]. These results are hard to overstate in importance. They showed that the generic behavior of directional flow is rigid enough to be described globally, and they made precise the idea that renormalization in moduli space controls the asymptotic behavior of individual trajectories. In the same period, Masur's work on closed trajectories and billiards established existence results for periodic geodesics and clarified the geometry of flat surfaces through saddle connections and cylinder decompositions [Mas86]. This cluster of results created what is now the standard paradigm: straight-line flow on a translation surface is not an isolated dynamical system, but part of a renormalization-theoretic structure governed by Teichmüller dynamics.

A parallel development revealed the arithmetic and geometric rigidity of special translation surfaces. Veech's 1989 chapter on Teichmüller curves and triangular billiards introduced

what are now called Veech surfaces: translation surfaces whose stabilizer in $\mathrm{SL}_2(\mathbb{R})$ is a lattice and which consequently enjoy extraordinarily rigid dynamical behavior [Vee89]. These examples made clear that some orbit closures in moduli space are far from generic and instead reflect arithmetic structure. McMullen's work in genus two, particularly on billiards and Teichmüller curves on Hilbert modular surfaces and on the dynamics of the $\mathrm{SL}_2(\mathbb{R})$-action in genus two, showed that such orbit closures can often be understood via real multiplication, Jacobians, and algebraic curves inside moduli space [McM03; McM07]. Translation surfaces thus emerged not only as dynamical objects, but also as a remarkably explicit testing ground for interactions among flat geometry, arithmetic geometry, and homogeneous dynamics.

The next major phase of the theory centered on the global geometry of strata and on the linear-algebraic data carried by the Hodge bundle over them. Kontsevich and Zorich classified the connected components of strata of Abelian differentials, producing the basic topological decomposition of the ambient parameter spaces in which translation surfaces live [KZ03]. In closely related work, Kontsevich connected Lyapunov exponents of natural dynamical cocycles to Hodge-theoretic quantities, thereby opening a bridge between Teichmüller dynamics and algebraic geometry [Kon97]. The resulting Kontsevich–Zorich cocycle on cohomology has become one of the central objects of the subject. Its Lyapunov exponents govern quantitative phenomena in the dynamics of flows on individual surfaces, and the cocycle itself encodes subtle geometric information about orbit closures and invariant subbundles.

Forni's 2002 theorem on the deviation of ergodic averages [For02]. shows that for typical translation flows, the deviation of ergodic integrals is governed by the nontrivial Lyapunov exponents of the Kontsevich–Zorich cocycle. In other words, the rate at which Birkhoff averages fail to behave trivially is not an ad hoc analytic phenomenon, but rather controlled by a precise cohomological cocycle over the Teichmüller flow. Later, Avila and Viana proved simplicity of the nontrivial Lyapunov spectrum, confirming a central conjectural picture and reinforcing the idea that the cocycle has strong rigidity properties despite living over a

highly nontrivial dynamical system [AV07]. At this stage, the field had already moved well beyond qualitative statements such as minimality or unique ergodicity, beginning to control quantitative asymptotics, linear response, and fine statistical structure.

A revolution in the 2010s reshaped the subject at the level of moduli space itself. Eskin, Mirzakhani, and Mohammadi proved that $\mathrm{GL}_2^+(\mathbb{R})$-orbit closures in strata are affine invariant submanifolds, cut out locally by linear equations in period coordinates, and established strong equidistribution and isolation results for the action [EMM15]. This theorem is often compared to Ratner-type rigidity phenomena in homogeneous dynamics: it transformed the study of translation surfaces by showing that orbit closures, though a priori potentially wild, are in fact governed by rigid linear structure. Wright's survey is a standard reference for this point of view and makes clear that many apparently local geometric or dynamical questions on a surface can be reinterpreted as questions about the affine invariant submanifold generated by that surface [Wri14].

The orbit-closure theorem was followed by a sequence of results revealing that affine invariant submanifolds possess unexpectedly strong algebro-geometric and Hodge-theoretic structure. Filip proved that affine invariant submanifolds are algebraic and established deep semisimplicity and rigidity properties for the Kontsevich–Zorich cocycle over them [Fil16a; Fil16b]. Among other things, invariant subbundles are forced to respect Hodge structure, and measurable invariance can be upgraded to rigid algebraic behavior. These results showed that the geometry of orbit closures and the algebra of the natural cohomological cocycles over them are constrained in a manner closely analogous to classical semisimplicity phenomena in Hodge theory. Translation surfaces thereby became one of the most vivid settings in which ideas from dynamics and Hodge theory genuinely inform one another.

At the same time, the geometry of the boundary of strata and orbit closures became increasingly central. Mirzakhani and Wright analyzed the boundary of affine invariant submanifolds, providing a precise framework for degenerations and tangent spaces at the

boundary [MW17]. Bainbridge, Chen, Gendron, Grushevsky, and Möller constructed a compactification of strata of Abelian differentials and gave a detailed description of stable differentials appearing in the boundary [Bai+18]. This boundary theory is indispensable in modern work: degeneration arguments are now basic tools for classification, finiteness statements, and the study of how orbit closures sit inside strata. In particular, the geometry of degenerations is no longer a peripheral topic but part of the central structure of the field.

The current subject is therefore both broad and remarkably coherent. On one side stand the classical dynamical questions: minimality, unique ergodicity, invariant measures, closed geodesics, billiards, interval exchanges, and quantitative deviation of ergodic averages [MT02; Mas06; Zor06; FM14; AM24]. On the other side stand the modern structural questions: orbit closures, affine invariant submanifolds, Hodge-theoretic rigidity, algebraicity, degenerations, and the algebraic hull of the Kontsevich–Zorich cocycle [EMM15; Fil16a; Fil16b; MW17; Fil24]. These are two manifestations of the same central fact: translation surfaces admit explicit geometric constructions, but they live in highly structured moduli spaces whose dynamics feeds back into the study of each individual example.

This combination is one of the main reasons the field is so compelling. Very few subjects allow one to start from a polygon drawn in the plane and arrive, by a chain of natural constructions, at questions about Lyapunov exponents, algebraic subvarieties of moduli spaces, semisimple variations of Hodge structure, or rigidity of measurable cocycles. Just as importantly, the subject remains concrete enough that individual examples often matter. One can compute on a specific surface, cut along a saddle connection, pass to a cover, examine cylinder decompositions, and then ask how these concrete operations appear from the point of view of cohomology, orbit closures, or degeneration in moduli. This persistent dialogue between explicit geometry and global structure is, to my mind, one of the defining mathematical virtues of the theory.

The present thesis is organized around two themes that reflect these connections. The first

theme belongs to the modern theory of affine invariant geometry and the Kontsevich–Zorich cocycle. There, the goal is to understand semisimplicity and rigidity phenomena for cocycles over products of strata. Although ergodic theory may at first produce only measurable invariant subbundles or splittings for the cocycle over the moduli-space dynamics, in this setting one expects much more regularity: Hodge theory and the algebraicity of orbit closures force strong structural constraints. The second theme returns to a more classical dynamical problem: unique ergodicity for branched covers of translation surfaces obtained by explicit slit constructions. Here the central objects are straight-line flows and their invariant probability measures, but the methods and motivations are still informed by the broader moduli-space picture. In both parts, the central mathematical question is the same: how much rigidity is forced by the geometry of translation surfaces and their moduli, and how far can one push that rigidity in settings that are not covered by the classical generic theorems?

More specifically, the first part of the thesis studies invariant subbundles of natural cocycles arising over products of strata and establishes semisimplicity statements in that setting. From a historical point of view, this belongs to the line of work running from the introduction of the Kontsevich–Zorich cocycle and its Lyapunov exponents [Kon97; For02; AV07] through orbit-closure rigidity [EMM15] to Hodge-theoretic algebraicity and semisimplicity [Fil16a; Fil16b; Fil24]. The novelty in the product setting is that one must simultaneously manage the geometry of several factors and the representation-theoretic behavior of groups acting diagonally or more generally projective-surjectively on the product. The resulting structure turns out to be rigid enough to admit a Deligne-style semisimplicity theory in this context.

The second part of the thesis studies unique ergodicity for branched covers of translation surfaces obtained by cutting and regluing along slits. Although covering constructions are among the most elementary ways to manufacture new translation surfaces from old ones, their dynamical behavior can be surprisingly subtle. Even when the base surface is uniquely ergodic, the cover may support several natural invariant probability measures, and the

problem becomes one of understanding when the explicit geometry of the cover still forces uniqueness. This belongs to the classical line of investigation initiated by the theory of interval exchanges, the Masur–Veech unique ergodicity theorems, and the study of directional flows on flat surfaces [Mas82; Vee82; KMS86; Mas06; Zor06; AM24]. The aim here is to push beyond generic existence theorems and obtain results tailored to a concrete and geometrically natural family of surfaces.

Taken together, these two parts of the thesis are meant to illustrate a single picture of the subject. Translation surfaces are interesting not only because they support beautiful individual dynamical systems, and not only because their moduli spaces exhibit striking rigidity, but because these two aspects continually constrain one another. The most explicit constructions often raise the deepest structural questions, and the most abstract structural theorems often acquire meaning only when brought back to a concrete surface or family of surfaces. This is the perspective from which the present work is written.

For the reader's convenience, I conclude with a brief roadmap. Chapter 2, Part 2.2 develops the background on translation surfaces, strata, directional flows, and the $\mathrm{SL}_2(\mathbb{R})$-action that will be used throughout. Chapter 2 then turns to products of strata and the cocycles naturally associated to them, culminating in the semisimplicity and rigidity results that form the first main contribution of the thesis. Chapter 3 returns to explicit covering constructions: after developing the relevant geometric and ergodic preliminaries for slit-induced branched covers, it establishes the unique ergodicity results that form the second main contribution. In this way the thesis moves from ambient structure to explicit dynamics, and the underlying theme remains constant: rigidity on translation surfaces is most visible precisely where geometry and dynamics are forced to interact.

CHAPTER 2

SEMISIMPLICITY AND RIGIDITY OF THE KONTSEVICH–ZORICH COCYCLE FOR PRODUCTS OF STRATA OF TRANSLATION SURFACES

2.1 Introduction

A foundational breakthrough of Eskin–Mirzakhani–Mohammadi shows that orbit closures for the $\mathrm{SL}_2\mathbb{R}$-action on a moduli space of translation surfaces are *affine invariant submanifolds*: immersed suborbifolds locally cut out by real linear equations in period coordinates (and more generally, closed invariant sets are finite unions of such manifolds) [EMM15]. This structural result has shifted many questions in Teichmüller dynamics to the study of affine invariant manifolds and the linear-algebraic constraints they impose on the geometry of the surface. Further tools for analyzing affine invariant manifolds include Wright's Cylinder Deformation Theorem and its consequences for rank, field of definition, and cylinder configurations [Wri14].

The principal linear object over strata is the (cohomological) Hodge bundle, equipped with the Gauss–Manin connection. Parallel transport along $\mathrm{SL}_2\mathbb{R}$-orbits defines the Kontsevich–Zorich (KZ) cocycle. Its Lyapunov exponents govern deviation of ergodic averages for translation flows and related problems (beginning with works of Zorich and Forni, and in particular Forni's non-uniform hyperbolicity and spectral gap phenomena [For02]). They also satisfy strong global constraints: for any regular $\mathrm{SL}_2\mathbb{R}$-invariant suborbifold, the sum of the nonnegative Lyapunov exponents admits an explicit expression in terms of the Siegel–Veech constant [EKZ14].

A key Hodge-theoretic input to the modern study of the KZ cocycle is Filip's work on semisimplicity and rigidity [Fil16a]. Among other consequences, invariant subbundles of tensor powers of the Hodge bundle respect the Hodge structure, admit Deligne-type semisimplicity statements, and exhibit strong regularity (real-analyticity along stable/unstable leaves and

polynomial dependence in period coordinates) [Fil16a]. In a complementary direction, Filip proved that affine invariant manifolds are algebraic varieties via a splitting statement for certain mixed Hodge structures [Fil16b]. These results have since been sharpened and extended in several directions, including a precise computation of the algebraic hull of the KZ cocycle over any $\mathrm{GL}_2^+\mathbb{R}$-invariant subvariety and finiteness consequences [EFW18] and boundary/compactification techniques (starting with Mirzakhani–Wright and continuing through more recent compactification frameworks surveyed by Dozier) [MW17; Doz25].

Products of strata and diagonal-type actions. This chapter studies Hodge-theoretic constraints for group actions on *products* of strata

$$X = \mathcal{H}(\alpha_1) \times \cdots \times \mathcal{H}(\alpha_n),$$

equipped with an ergodic invariant measure for a closed subgroup $G \subset (\mathrm{SL}_2\mathbb{R})^n$ that projects surjectively to each factor. Up to permuting factors, such groups decompose into products of full factors $(\mathrm{SL}_2\mathbb{R})^{n_0}$ and block-indecomposable factors, and each block-indecomposable factor is conjugate to a signed diagonal group Δ_F (Section 2.2.3). Thus, even though the ambient action is a product action, the groups we consider impose nontrivial *couplings* between factors, and invariant bundles for the induced cocycles encode genuinely multi-factor phenomena.

Main results. Our work extends Filip's rigidity mechanism from single strata to the product setting, and provides an explicit algebraic form for invariant projections that is well-adapted to multi-factor applications. Let $\mathrm{E}_{\mathbb{F}}$ be the Hodge bundle over X (or a tensor power), $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$, and let Γ be the cocycle over (X, μ) induced by the Gauss–Manin connection and the G-action. Our first result is

Theorem 2.A (Deligne semisimplicity).

1. There exist G-invariant bundles $V_i \subset \mathrm{E}_{\mathbb{F}}$, $\mathbb{F} = \mathbb{R}$ or $\mathbb{F} = \mathbb{C}$, and vector spaces W_i

equipped with Hodge structures and an isomorphism

$$\mathrm{E}_{\mathbb{F}} \cong \bigoplus_i E_i \quad \text{with} \quad E_i \cong V_i \otimes_{A_i} W_i. \tag{2.1.1}$$

Each of V_i carries a variation of Hodge structure making the above isomorphism compatible. Here, A_i are division algebras that act on V_i and are compatible with Hodge structures. They also act compatibly on W_i.

2. Furthermore, any G-invariant bundle $V \subset \mathrm{E}_{\mathbb{F}}$ has the form

$$V = \bigoplus_i V_i \otimes_{A_i} W_i',$$

where $W_i' \subset W_i$ are A_i-submodules.

See Theorem 2.6.5.

A crucial intermediate step is a *fixed-part* statement (Theorem 2.6.3) showing that if a measurable section of a VHS is flat along almost every G-orbit, then each of its Hodge components is also flat. This implies that the Deligne torus acts on the poset of invariant subbundles: if $V \subset \mathrm{E}_{\mathbb{C}}$ is G-invariant and $a \in \mathbb{S}$ is a Deligne torus element, then $a \cdot V$ is again G-invariant (Theorem 2.6.4). Together with reductivity of the relevant algebraic hulls (Section 2.4), this yields the semisimplicity classification.

Our second main result is a rigidity statement for invariant subbundles on affine invariant manifolds in X.

Theorem 2.B (Joint polynomiality). On an affine invariant manifold in a product of strata, a G-invariant measurable subbundle of the Hodge bundle, or of one of its tensor powers, is polynomial in linear period coordinates after allowing denominators given by the factorwise area functions.

More precisely, writing the real period coordinates in the i-th factor as (x_i, y_i), and writing

$A_i(x_i, y_i)$ for the corresponding area pairing, the matrix coefficients of the projection operator lie in the multihomogeneous degree-zero part of

$$\mathbb{R}[x_1, \dots, x_n, y_1, \dots, y_n, A_1^{-1}, \dots, A_n^{-1}].$$

Equivalently, each coefficient has the form

$$\frac{P(x, y)}{\prod_i A_i(x_i, y_i)^{D_i}},$$

where P is polynomial and is bihomogeneous of degree (D_i, D_i) in the variables (x_i, y_i).

See Theorem 2.7.6.

Organization of the chapter. Section 2.2 recalls background on translation surfaces, strata, and products of strata, including the structure of projection-surjective subgroups of $(\mathrm{SL}_2\mathbb{R})^n$. Section 2.3 reviews cocycles and algebraic hulls in the sense of [Fil16a] and establishes reductivity for the cocycles considered here. Section 2.4 summarizes the Hodge-theoretic and curvature identities used later. Section 2.5 records the random-walk and subharmonicity input used to pass from subharmonicity to constancy. Section 2.6 proves the semisimplicity results, culminating in the decomposition and classification theorem. Finally, Section 2.7 proves analyticity and polynomiality of invariant subbundles and the joint polynomiality statement.

2.2 Flat surfaces

This section is largely based on surveys by Filip [Fil24], Matheus [Mat18], and Wright [Wri14]. We also recommend the surveys by Zorich [Zor06] and Massart [Mas21] for more introductory-level reading.

2.2.1 Definitions and motivation

Denote by $\mathcal{M}_g$ the moduli space of genus $g \geq 1$ Riemann surfaces (an orbifold). Let $\Omega\mathcal{M}_g$ be the (orbifold) Hodge bundle over $\mathcal{M}_g$, whose fiber over $X \in \mathcal{M}_g$ is the complex vector space $H^0(X, \Omega^1_X)$ of holomorphic 1–forms on X. A point of $\Omega\mathcal{M}_g$ is a pair (X, ω) with $\omega \not\equiv 0$, called an *Abelian differential*.

Fix a topological surface S_g of genus g. Let $\mathrm{Diff}^+(S_g)$ be the group of orientation-preserving diffeomorphisms and let $\mathrm{Diff}^+_0(S_g)$ be its identity component. The (classical) Teichmüller space is

$$\mathcal{T}_g := \{\text{complex structures on } S_g\}/\mathrm{Diff}^+_0(S_g),$$

and the mapping class group is $\mathrm{MCG}_g := \mathrm{Diff}^+(S_g)/\mathrm{Diff}^+_0(S_g)$. Let $\Omega\mathcal{T}_g$ be the pullback of $\Omega\mathcal{M}_g$ to $\mathcal{T}_g$; equivalently, $\Omega\mathcal{T}_g$ is the space of *marked* Abelian differentials. Then MCG_g acts properly discontinuously on $\Omega\mathcal{T}_g$ and

$$\Omega\mathcal{M}_g = \Omega\mathcal{T}_g/\mathrm{MCG}_g.$$

For notational convenience we write $\mathcal{H}_g := \Omega\mathcal{M}_g$ and $\mathcal{T}_g := \Omega\mathcal{T}_g$.

$$\mathcal{T}_g \xrightarrow{\cdot/\mathrm{MCG}_g} \mathcal{H}_g$$

Remark 2.2.1. Let X be a translation surface with a holomorphic 1-form ω, and let $\Sigma = \mathrm{div}(\omega) = \{P_1, \ldots, P_m\}$ be the zero set. Write $\mathrm{ord}_{P_i}(\omega) = a_i \geq 0$, so that $\sum_{i=1}^m a_i = 2g - 2$. The tuple $\alpha = (a_1, \ldots, a_m)$ determines the *stratum* $\mathcal{H}(\alpha) \subset \mathcal{H}_g$ containing (X, ω).

The form ω defines a singular flat metric on X by declaring the length of a piecewise C^1 path γ to be

$$\ell_\omega(\gamma) = \int_\gamma |\omega|,$$

and the distance between two points to be the infimum of the lengths of paths joining them. Away from Σ, one can choose local coordinates in which $\omega = dz$, so this metric is locally Euclidean. If $P_i \in \Sigma$ is a zero of order a_i, then in a local coordinate centered at P_i one has

$$\omega = z^{a_i}\, dz,$$

and the induced metric is a Euclidean cone metric of total angle $2\pi(a_i + 1)$ at P_i.

Remark 2.2.2. The projection $\mathcal{H}_g \to \mathcal{M}_g$ is the Hodge orbifold vector bundle. Its nontriviality comes from the Gauss–Manin monodromy: after choosing a marking of the underlying topological surface S_g, one identifies the cohomology of each fiber with $H^1(S_g; \mathbb{C})$, and passing around a loop in $\mathcal{M}_g$ acts by the corresponding mapping class on cohomology. Thus the bundle acquires the natural MCG_g-monodromy, equivalently the symplectic monodromy through

$$\mathrm{MCG}_g \longrightarrow \mathrm{Sp}(H^1(S_g; \mathbb{Z})).$$

Since this monodromy is nontrivial, the bundle is not globally trivial.

The space $\mathcal{H}_g$ admits a stratification by the zero orders:

$$\mathcal{H}_g = \bigsqcup_{\alpha} \mathcal{H}(\alpha), \qquad \alpha = (a_1, \ldots, a_m),\ \sum a_i = 2g - 2.$$

Each stratum $\mathcal{H}(\alpha)$ is a complex-analytic orbifold of (complex) dimension $2g + m - 1$. A stratum need not be connected; for example, $\mathcal{H}(4)$ and $\mathcal{H}(2, 2)$ in genus 3 have two connected components (see [KZ02]).

Points of $\mathcal{H}_g$ are also called *translation surfaces* or *flat surfaces*. Indeed, on $X \setminus \Sigma$ the 1–form ω provides local coordinates by integrating ω, and the transition maps between these coordinates are translations. Thus the induced metric is flat on $X \setminus \Sigma$ with conical singularities at Σ. As a consequence, one can think of a translation surface as a polygon with

pairwise equal and parallel sides, where the coordinate change maps glue those sides together and the choice of ω is defined by choosing a directional vector.

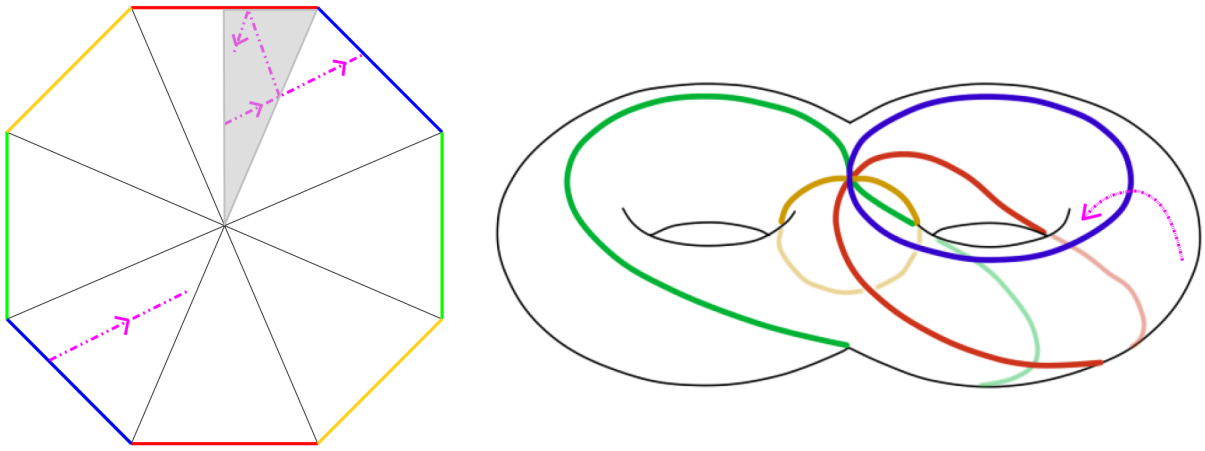


Figure 2.1: Unfolded billiard in the triangle with angles $(\pi/2, 3\pi/8, \pi/8)$ and the flow on the resulting surface.

Remark 2.2.3. One of the motivations for studying flows on surfaces comes from *polygonal billiards* with rational angles. Given such a rational polygon P with the set of vertices V, we can glue a (topologically) sphere S from it and its mirror copy by identifying corresponding sides. This sphere has a metric induced by the flat metric on the polygon, the holonomy of S/V naturally embeds into $SO(2,\mathbb{R})$. The holonomy representation $\text{rep} : \pi_1(S/V) \to SO(2,\mathbb{R})$ induces the representation of the first homology $\text{rep} : H_1(S/V,\mathbb{R}) \to SO(2,\mathbb{R})$. Moreover, since all angles of P are rational, the generators of the first homology are represented by rational angle rotation matrices. Then the image of this representation is a finite subgroup $G_P \subset SO(2,\mathbb{R})$, so we can construct a finite G_P-cover S_{G_P} of S, on which the holonomy is trivial. Thus we can "lift" a billiard trajectory into a straight line on S_{G_P} unless it passes through one of the vertices.

A simple example for a triangular rational billiard with the right angle is shown on the pictures.

Let $(X,\omega) \in \mathcal{H}(\alpha)$ and set $\Sigma = \text{div}(\omega)$ with $|\Sigma| = m$. Choose a basis $\{\gamma_1, \ldots, \gamma_{2g+m-1}\}$ of the relative homology group $H_1(X,\Sigma;\mathbb{Z})$. The *period coordinates* in a neighbourhood of

(X, ω) are given by

$$\Phi(X, \omega) = \left(\int_{\gamma_1} \omega, \ldots, \int_{\gamma_{2g+m-1}} \omega \right) \in \mathbb{C}^{2g+m-1}.$$

Changes of basis in $H_1(X, \Sigma; \mathbb{Z})$ give transition maps in $GL(2g + m - 1, \mathbb{Z})$. Identifying $\mathbb{C} \cong \mathbb{R}^2$ via $x + iy \mapsto (x, y)$, the natural $\mathrm{SL}_2\mathbb{R}$–action on period coordinates is given by the standard action on the $\mathbb{R}^2$ factor:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x_1 & \ldots & x_{2g+m-1} \\ y_1 & \ldots & y_{2g+m-1} \end{pmatrix} = \begin{pmatrix} ax_1 + by_1 & \ldots & ax_{2g+m-1} + by_{2g+m-1} \\ cx_1 + dy_1 & \ldots & cx_{2g+m-1} + dy_{2g+m-1} \end{pmatrix}.$$

The *Teichmüller geodesic flow* is the action of the diagonal subgroup $g_t = \mathrm{diag}(e^t, e^{-t})$ on each stratum.

The *(cohomological) Hodge bundle* over $\mathcal{M}_g$ (with coefficients $k = \mathbb{R}$ or $k = \mathbb{C}$) is the local system $H^1_k \to \mathcal{M}_g$ whose fiber over X is $H^1(X; k)$. It carries the Gauss–Manin connection, i.e. parallel transport in cohomology along paths in $\mathcal{M}_g$. Pulling back to $\mathcal{H}_g$ via the projection $\mathcal{H}_g \to \mathcal{M}_g$ gives a bundle (still denoted $H^1_k \mathcal{H}_g$).

The $\mathrm{SL}_2\mathbb{R}$–action on strata induces the Teichmüller flow g_t on $\mathcal{H}(\alpha)$. Lifting g_t to $H^1_k \mathcal{H}_g$ by Gauss–Manin parallel transport defines the *Kontsevich–Zorich cocycle* (see Subsection 2.2.2).

2.2.2 The Kontsevich–Zorich cocycle

Let us consider in more detail the action of $\mathrm{SL}_2\mathbb{R}$ on the space $\mathcal{H}_g$. Clearly, each stratum $\mathcal{H}(\alpha)$ is invariant under this action. The $\mathrm{SL}_2\mathbb{R}$–action preserves the *area* (or volume) function

$$A(X, \omega) := \frac{i}{2} \int_X \omega \wedge \bar{\omega},$$

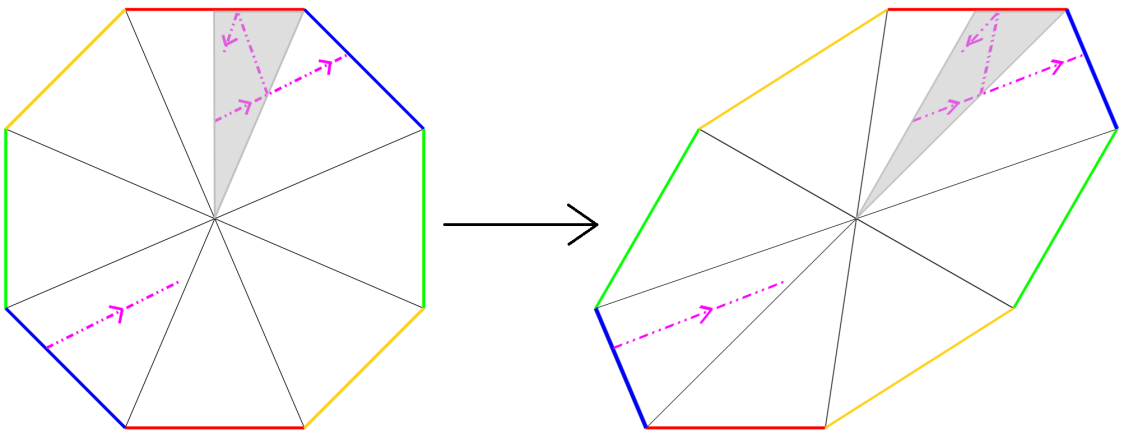


Figure 2.2: Action of the matrix $\begin{pmatrix} 1 & \frac{1}{\sqrt{3}} \\ 0 & 1 \end{pmatrix}$ on the surface from the previous example.

so it is often convenient to restrict to the area-one locus $\mathcal{H}^{(1)}(\alpha) := \mathcal{H}(\alpha) \cap \mathcal{H}_g^{(1)}$, $\mathcal{H}_g^{(1)} := A^{-1}(1)$.

On the marked space $\mathcal{T}_g = \Omega\mathcal{T}_g$, the cohomology bundle is canonically trivial:

$$H^1_{\mathbb{R}}\mathcal{T}_g \;\cong\; \mathcal{T}_g \times H^1(S_g;\mathbb{R}).$$

The $\mathrm{SL}_2\mathbb{R}$–action on $\mathcal{T}_g$ lifts to an action on $H^1_{\mathbb{R}}\mathcal{T}_g$ by acting trivially on the fiber. Passing to the quotient $\mathcal{H}_g = \mathcal{T}_g/\mathrm{MCG}_g$ identifies fibers using the mapping class group action on $H^1(S_g;\mathbb{R})$. The resulting fiberwise linear maps define the *Kontsevich–Zorich cocycle*; equivalently, it is Gauss–Manin parallel transport along $\mathrm{SL}_2\mathbb{R}$–orbits in $\mathcal{H}_g$.

Since $\mathcal{H}^{(1)}(\alpha)$ is an orbifold, one may pass to a torsion-free finite-index subgroup of MCG_g (a finite manifold cover) to avoid orbifold stabilizers; this does not affect Lyapunov-exponent computations.

2.2.3 Products of strata

Consider an n-tuple of natural numbers $(g)_n = (g_1, \ldots, g_n)$, $g_i \geq 1$. Consider Cartesian products $\mathcal{M}^n = \mathcal{M}_{(g)_n} = \mathcal{M}_{g_1} \times \ldots \times \mathcal{M}_{g_n}$ and $\mathcal{H}^n = \mathcal{H}_{(g)_n} = \mathcal{H}_{g_1} \times \ldots \times \mathcal{H}_{g_n}$. Let

$H^1_k\mathcal{M}^n$, where $k = \mathbb{R}$ or $k = \mathbb{C}$, be the real (complex) Hodge bundle over $\mathcal{M}^n$ (which is just the product $H^1_k\mathcal{M}_{g_1} \times \ldots \times H^1_k\mathcal{M}_{g_n}$), and let $H^1_k\mathcal{H}^n$ be its pullback under the natural projection $\mathcal{H}^n \to \mathcal{M}^n$. Like in the case of one stratum, we are interested in acting on $H^1_k\mathcal{H}^n$ by subgroups of $(\mathrm{SL}_2\mathbb{R})^n = \prod_{i=1}^n \mathrm{SL}_2\mathbb{R}$.

Remark 2.2.4. A map T on a finite-measure space (X, μ) is weakly mixing if and only if $T \times T$ is ergodic, but this is far from the only reason to study products. Strata of Abelian differentials admit compactifications by nodal curves, which arise as limits where one or more (possibly separating) closed curves shrink to length 0. Such nodal surfaces can be viewed as configurations of lower-genus components attached at nodes. A survey of results on this topic is given by Dozier [Doz25].

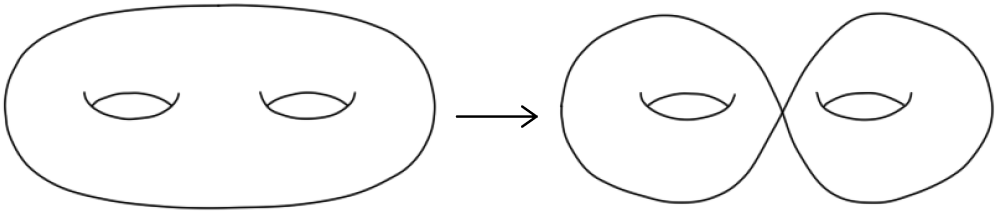


Figure 2.3: Deformation of a genus 2 surface into a nodal genus $(1, 1)$ surface at the boundary of the compactification of $\mathcal{H}_2$.

Given the motivations of our interest, we would like to consider only those subgroups of $(\mathrm{SL}_2\mathbb{R})^n$ that project surjectively on each of its factors. We will call such subgroups *projection-surjective*.

Lemma 2.2.5. *Let $G \subsetneq (\mathrm{SL}_2\mathbb{R})^n$ be a closed projection-surjective subgroup. Assume that G is* block-indecomposable*: there is no nontrivial partition $\{1, \ldots, n\} = I \sqcup J$ with $I, J \neq \varnothing$ such that, after permuting coordinates,*

$$G \subset (\mathrm{SL}_2\mathbb{R})^I \times (\mathrm{SL}_2\mathbb{R})^J \quad \textit{and} \quad G = (G \cap (\mathrm{SL}_2\mathbb{R})^I) \times (G \cap (\mathrm{SL}_2\mathbb{R})^J).$$

Then there exist $g_1, \dots, g_n \in \mathrm{GL}_2\mathbb{R}$ *and a subgroup* $F \leq \{\pm 1\}^n$ *such that*

$$(g_1, \dots, g_n)\, G\, (g_1, \dots, g_n)^{-1} = \{(\varepsilon_1 h, \dots, \varepsilon_n h) \ : \ h \in \mathrm{SL}_2\mathbb{R},\ (\varepsilon_1, \dots, \varepsilon_n) \in F\}.$$

Proof. Let

$$\pi : \mathrm{SL}_2\mathbb{R} \longrightarrow \mathrm{PSL}_2\mathbb{R}$$

be the quotient map. Set

$$\overline{G} := \pi^{\times n}(G) \subset (\mathrm{PSL}_2\mathbb{R})^n.$$

Then $\overline{G}$ is a closed projection-surjective subgroup of $(\mathrm{PSL}_2\mathbb{R})^n$, and it is still block-indecomposable.

We first show, by induction on n, that there exist $a_1, \dots, a_n \in \mathrm{PGL}_2\mathbb{R}$ such that

$$(a_1, \dots, a_n)\, \overline{G}\, (a_1, \dots, a_n)^{-1} = \Delta_n := \{(\bar{h}, \dots, \bar{h}) : \bar{h} \in \mathrm{PSL}_2\mathbb{R}\}.$$

Base case $n = 2$. By Goursat's lemma, since $\mathrm{PSL}_2\mathbb{R}$ is simple and centerless, $\overline{G}$ is the graph of an automorphism

$$\varphi : \mathrm{PSL}_2\mathbb{R} \longrightarrow \mathrm{PSL}_2\mathbb{R}.$$

Now

$$\mathrm{Aut}(\mathrm{PSL}_2\mathbb{R}) = \mathrm{PGL}_2\mathbb{R},$$

so there exists $a \in \mathrm{PGL}_2\mathbb{R}$ such that

$$\varphi(\bar{h}) = a\bar{h}a^{-1}.$$

Hence, after conjugating the second factor by a^{-1}, the graph of φ becomes the diagonal Δ_2.

Induction step. Assume the claim for $n = K$, and let $n = K + 1$. Let $\overline{H} \subset (\mathrm{PSL}_2\mathbb{R})^K$ be the projection of $\overline{G}$ to the first K factors. Then $\overline{H}$ is closed, projection-surjective, and

block-indecomposable. By the induction hypothesis, after conjugating the first K factors by suitable elements of $\mathrm{PGL}_2\mathbb{R}$, we may assume

$$\overline{H} = \Delta_K.$$

Thus

$$\overline{G} \subset \Delta_K \times \mathrm{PSL}_2\mathbb{R}.$$

Since $\Delta_K \simeq \mathrm{PSL}_2\mathbb{R}$ is simple and centerless, Goursat's lemma again implies that $\overline{G}$ is the graph of an automorphism

$$\varphi : \Delta_K \longrightarrow \mathrm{PSL}_2\mathbb{R}.$$

Identifying Δ_K with $\mathrm{PSL}_2\mathbb{R}$ via any coordinate projection, we again have

$$\varphi(\bar{h}) = a\bar{h}a^{-1} \qquad \text{for some } a \in \mathrm{PGL}_2\mathbb{R}.$$

Conjugating the last factor by a^{-1}, we obtain

$$\overline{G} = \Delta_{K+1}.$$

This completes the induction.

Now choose lifts $g_i \in \mathrm{GL}_2\mathbb{R}$ of the elements $a_i \in \mathrm{PGL}_2\mathbb{R}$, and set

$$G' := (g_1, \ldots, g_n)\, G\, (g_1, \ldots, g_n)^{-1}.$$

Then

$$\pi^{\times n}(G') = \Delta_n,$$

so

$$G' \subset \widetilde{\Delta}_n := (\pi^{\times n})^{-1}(\Delta_n).$$

Explicitly,

$$\widetilde{\Delta}_n = \{(\varepsilon_1 h, \ldots, \varepsilon_n h) : h \in \mathrm{SL}_2\mathbb{R},\ \varepsilon_i \in \{\pm 1\}\}.$$

Every element of $\widetilde{\Delta}_n$ can be written uniquely as

$$(h, \eta_2 h, \ldots, \eta_n h) \qquad (h \in \mathrm{SL}_2\mathbb{R},\ \eta_i \in \{\pm 1\}),$$

so

$$\widetilde{\Delta}_n \simeq \mathrm{SL}_2\mathbb{R} \times \{\pm 1\}^{n-1}.$$

Under this identification, let

$$p : \widetilde{\Delta}_n \longrightarrow \{\pm 1\}^{n-1}$$

be the projection to the discrete factor.

Because G is projection-surjective, so is G', and in particular the projection

$$\pi_1 : G' \to \mathrm{SL}_2\mathbb{R}$$

is surjective. Let $K := \ker(\pi_1|_{G'})$. Then $K \subset \{\pm 1\}^{n-1}$, and the image of G' in the finite group $\{\pm 1\}^{n-1}/K$ is a continuous homomorphic image of

$$G'/K \simeq \mathrm{SL}_2\mathbb{R}.$$

Since $\mathrm{SL}_2\mathbb{R}$ is connected and $\{\pm 1\}^{n-1}/K$ is discrete, this image is trivial. Hence

$$G' = \mathrm{SL}_2\mathbb{R} \times F'$$

for some subgroup $F' \leq \{\pm 1\}^{n-1}$.

Translating back to the original coordinates, there is a subgroup $F \leq \{\pm 1\}^n$ such that

$$G' = \{(\varepsilon_1 h, \ldots, \varepsilon_n h) \ : \ h \in \mathrm{SL}_2\mathbb{R},\ (\varepsilon_1, \ldots, \varepsilon_n) \in F\}.$$

This is exactly the desired conclusion. □

Remark 2.2.6. Let $G \subset (\mathrm{SL}_2\mathbb{R})^n$ be a projection-surjective subgroup acting on a product of strata

$$\mathcal{H}^{(1)}(\alpha_1, \ldots, \alpha_n) = \mathcal{H}^{(1)}(\alpha_1) \times \cdots \times \mathcal{H}^{(1)}(\alpha_n).$$

Up to permuting factors, G decomposes as a product

$$G \cong (\mathrm{SL}_2\mathbb{R})^{n_0} \times G_1 \times \cdots \times G_k, \tag{2.2.1}$$

where each $G_i \subsetneq (\mathrm{SL}_2\mathbb{R})^{n_i}$ is a strict subgroup with $n_i \geq 2$ that looks as in Lemma 2.2.5.

Ergodic reduction. In our setting we work with a G-invariant ergodic measure μ on a G-invariant subset of a product of strata. After replacing (G, μ) by a conjugate system, we may assume that each noncompact factor G_i in (2.2.1) is in the normal form of Lemma 2.2.5, i.e. a signed diagonal group Δ_F.

If, moreover, the ambient $(\mathrm{SL}_2\mathbb{R})^n$-action on (X, μ) is ergodic, then any closed subgroup

$$H \leq (\mathrm{SL}_2\mathbb{R})^n$$

whose projection to each factor $\mathrm{SL}_2\mathbb{R}$ is noncompact acts ergodically on (X, μ). This is the Moore ergodicity theorem (equivalently, it follows from the Mautner phenomenon together with the Howe–Moore theorem).

Note, however, that this criterion does *not* apply to an arbitrary noncompact subgroup of $(\mathrm{SL}_2\mathbb{R})^n$. In particular, a single factor G_i in (2.2.1), viewed as a subgroup of the full ambient

product $(\mathrm{SL}_2\mathbb{R})^n$, is generally trivial on the complementary blocks, so one cannot conclude from ambient ergodicity alone that the G_i-action is ergodic.

2.3 Properties of cocycles and the algebraic hull

Throughout this section, we assume that X is a space endowed with a Borel probability measure μ and G is a Polish group (i,e, a topological group that is homeomorphic to a separable complete metric space) whose left action on X is ergodic and measure-preserving.

Definition 2.3.1 (See [Fil16a], Section 2.1, p. 625)**.** A *linear cocycle* on the n-dimentional real (complex) vector bundle V over X is the lift of the action of the Lie group G on X to the action of G on V. More specifically, each element $g \in G$ induces a linear map of fibers $g_x : V_x \to V_{gx}$ for any $x \in X$. We obtain a map $\alpha : G \times X \to \mathrm{GL}(\mathbb{F}^n)$, where $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$ respectively.

Two cocycles α, β are *cohomologous* (or *conjugate*) if there exists a measurable map $C : X \to \mathrm{GL}(\mathbb{F}^n)$ such that

$$\alpha(g, x) = C(gx)^{-1}\, \beta(g, x)\, C(x) \qquad \text{for all } g \in G \text{ and } \mu\text{-a.e. } x \in X.$$

2.3.1 Algebraic hulls

Suppose that H is an $\mathbb{F}$-algebraic group (i.e. an algebraic variety endowed with a compatible endowed with a group structure) over a local field of characteristic 0 (in our case, $\mathbb{F} = \mathbb{R}$ or $\mathbb{F} = \mathbb{C}$). Denote by $H(\mathbb{F})$ the set of $\mathbb{F}$-rational points of (the variety) H. In this setup, we can introduce the following

Theorem 2.3.2 (See [Zim83], Chapter 9.2, pp. 166–167)**.** *Given a cocycle $\alpha : G \times X \to H(\mathbb{F})$, there exists a (unique up to conjugacy) $\mathbb{F}$-algebraic subgroup $L \subset H$ such that α can be conjugated into a cocycle α' taking values in $L(\mathbb{F})$, and cannot be conjugated into a cocycle*

α'' taking values in $M(\mathbb{F})$ for any $\mathbb{F}$-algebraic subgroup $M \subsetneq L$.

Intuitively, we want to use the trivial fact that conjugating a cocycle does not change its dynamical properties and conjugate it into the minimal possible image subgroup for convenience.

Definition 2.3.3. The equivalence class of the subgroup L from Theorem 2.3.2 is called the *algebraic hull* of α.

Sketch of proof. (After [Zim83]) The Zariski topology on algebraic varieties over an algebraically closed field is noetherian and therefore Artinian. Therefore, algebraic groups satisfy the descending chain condition (i.e. there doesn't exist an infinite descending chain of algebraic groups). This implies the existence of a minimal subgroup L that satisfies all required conditions. It remains to prove its uniqueness under conjugacy. Assume that we have two non-conjugate minimal groups L and M. Then we can construct α-invariant functions $\phi_L : X \to H(\mathbb{F})/L(\mathbb{F})$ and $\phi_M : X \to H(\mathbb{F})/M(\mathbb{F})$. Then we can consider the function $\phi = (\phi_L, \phi_M) : X \to H(\mathbb{F})/L(\mathbb{F}) \times H(\mathbb{F})/M(\mathbb{F})$, which is also α-invariant. The action of $H(\mathbb{F})$ on $H(\mathbb{F})/L(\mathbb{F}) \times H(\mathbb{F})/M(\mathbb{F})$ is smooth. □

2.3.2 Reductivity of cocycles

Lemma 2.3.4. *[See [Fil16a], Section 2.3, pp. 627–629] Algebraic hulls are preserved under respresentations. More specifically, let $\rho : H \to H'$ be a representation of a cocycle $\alpha : G \times X \to H(\mathbb{F})$, and let L be the algebraic hull of α. Then the algebraic hull of $\rho \circ \alpha$ coincides with $\rho(L)$.*

Definition 2.3.5. A group is called *reductive* if all its representations are semisimple. In other words, any invariant subspace has an invariant complement. We call a cocycle reductive if its algebraic hull is reductive.

Since images of reductive groups stay reductive, we obtain

Corollary 2.3.6. *Reductivity of cocycles is preserved under algebraic representations (in particular, tensor products of representations).*

2.3.3 Reductivity for Kontsevich–Zorich and products of strata

Let $\mathcal{H}^{(1)}(\alpha)$ be a stratum and let μ be an $\mathrm{SL}_2\mathbb{R}$-invariant ergodic probability measure on $\mathcal{H}^{(1)}(\alpha)$ (or on an affine invariant submanifold). Recall that the Kontsevich–Zorich cocycle is a cocycle

$$\alpha^{KZ} : \mathrm{SL}_2\mathbb{R} \times \mathcal{H}^{(1)}(\alpha) \to \mathrm{Sp}(2g, \mathbb{R})$$

on the absolute cohomology bundle.

Remark 2.3.7. Let $\pi : \widetilde{\mathcal{M}} \to \mathcal{M}$ be a finite manifold cover of the orbifold stratum (or affine invariant submanifold), and let $\widetilde{\alpha}^{KZ}$ be the pullback of the Kontsevich–Zorich cocycle to $\widetilde{\mathcal{M}}$. In general, the algebraic hull of $\widetilde{\alpha}^{KZ}$ need not coincide with that of α^{KZ}, because passing to a finite cover amounts to restricting the monodromy representation to a finite-index subgroup of the orbifold fundamental group, and this can change the finite component group.

However, the identity component is unchanged. Indeed, if

$$\rho : \pi_1^{\mathrm{orb}}(\mathcal{M}) \to \mathrm{Sp}(2g, \mathbb{R})$$

is the monodromy representation on absolute cohomology and

$$\widetilde{\rho} = \rho|_{\pi_1(\widetilde{\mathcal{M}})},$$

then $\pi_1(\widetilde{\mathcal{M}})$ has finite index in $\pi_1^{\mathrm{orb}}(\mathcal{M})$, so $\widetilde{\rho}(\pi_1(\widetilde{\mathcal{M}}))$ has finite index in $\rho(\pi_1^{\mathrm{orb}}(\mathcal{M}))$. Hence their Zariski closures have the same identity component.

For the absolute cohomology bundle, the algebraic hull is the stabilizer of the tautological plane inside the Zariski closure of monodromy. Therefore the algebraic hulls before and after

passing to the finite cover have the same identity component. In particular, reductivity is unchanged, since a linear algebraic group is reductive if and only if its identity component is reductive.

Theorem 2.3.8 ([EM18, Appendix A]; [AEM17, Theorem 1.5])**.** *The Kontsevich–Zorich cocycle α^{KZ} is reductive.*

Sketch of proof. Consider an invariant subbundle U of the Hodge bundle $H^1_{\mathbb{R},\alpha}$ under the action of the Kontsevich-Zorich cocycle. Define its symplectic complement by

$$U^\dagger((S_g,\,\omega)) = \{v : v \wedge u = 0 \text{ for all } u \in U((S_g,\,\omega))\} \quad \text{for all}(S_g,\,\omega) \in \mathcal{H}^{(1)}(\alpha),$$

where $U((S_g,\,\omega))$ stands for the fiber of the subbundle U over the translation surface $(S_g,\,\omega)$. Then $U^\dagger$ is also an invariant subbundle. Clearly, $K = (U \cap U^\dagger)$ is also invariant and isotropic (the standard symplectic form vanishes on it). One can prove that all the Lyapunov exponents of the restriction of the Kontsevich-Zorich cocycle to an isotropic invariant bundle K vanish. Moreover, the action of the Kontsevich-Zorich cocycle on K is isometric with respect to the Hodge inner product, and the orthogonal complement $K^\perp$ with respect to the Hodge inner product is also an invariant subbundle. Then

$$U = K \oplus \left(U \cap K^\perp\right), \quad U^\dagger = K \oplus \left(U^\dagger \cap K^\perp\right)$$

and

$$H^1(M, \mathbb{F}) = K \oplus \left(U \cap K^\perp\right) \oplus \left(U^\dagger \cap K^\perp\right)$$

Thus, $U^\dagger \cap K^\perp$ is an invariant complement to U. □

Now let

$$X = \mathcal{H}^{(1)}(\alpha_1, \ldots, \alpha_n) = \mathcal{H}^{(1)}(\alpha_1) \times \cdots \times \mathcal{H}^{(1)}(\alpha_n),$$

let $G \subset (\mathrm{SL}_2\mathbb{R})^n$ be a projection-surjective subgroup, and let μ be a G-invariant ergodic probability measure on X. Let Γ denote the cocycle on the (absolute) cohomology bundle

$$\mathrm{H}^1_{\mathbb{R}}(X) \;=\; \bigoplus_{i=1}^{n} \pi_i^* \mathrm{H}^1_{\mathbb{R}}(\mathcal{H}^{(1)}(\alpha_i))$$

induced by the Gauss–Manin connection; equivalently, Γ is the restriction to G of the product cocycle for $(\mathrm{SL}_2\mathbb{R})^n$.

Lemma 2.3.9. *The cocycle Γ has reductive algebraic hull. Consequently, any tensor power cocycle induced from $\mathrm{H}^1_{\mathbb{R}}(X)$ is reductive as well.*

Proof. Let $L \subset \prod_{i=1}^{n} \mathrm{Sp}(2g_i, \mathbb{R})$ be the algebraic hull of Γ (Theorem 2.3.2). For each i, let $\rho_i : \prod_j \mathrm{Sp}(2g_j, \mathbb{R}) \to \mathrm{Sp}(2g_i, \mathbb{R})$ be the projection to the i-th factor, and set

$$\Gamma_i := \rho_i \circ \Gamma.$$

By Lemma 2.3.4, the algebraic hull of Γ_i is $\rho_i(L)$.

Let $\pi_i : X \to \mathcal{H}^{(1)}(\alpha_i)$ be the projection. Since G is projection-surjective, $p_i(G) = \mathrm{SL}_2\mathbb{R}$, and $\pi_i(gx) = p_i(g)\,\pi_i(x)$ for all $g \in G$. Moreover, $\mu_i := (\pi_i)_*\mu$ is $\mathrm{SL}_2\mathbb{R}$-invariant. By construction, $\Gamma_i(g, x)$ depends only on $(p_i(g), \pi_i(x))$ and coincides with the Kontsevich–Zorich cocycle on the i-th factor. Hence Γ_i is (a pullback of) α^{KZ}, so by Theorem 2.3.8 the algebraic hull of Γ_i is reductive. Therefore each $\rho_i(L)$ is reductive.

Let $R_u(L)$ be the unipotent radical of L. Its image $\rho_i(R_u(L))$ is a normal unipotent subgroup of the reductive group $\rho_i(L)$, hence must be trivial for every i. Thus $R_u(L)$ lies in the intersection of the kernels of all ρ_i, which is trivial inside $\prod_i \mathrm{Sp}(2g_i, \mathbb{R})$. Therefore $R_u(L) = \{e\}$ and L is reductive.

The last statement follows from Corollary 2.3.6, since tensor constructions are algebraic representations. □

2.4 Hodge theory: crash-course

2.4.1 Setup

Definition 2.4.1. Let $\mathrm{H}_{\mathbb{R}}$ be a finite-dimensional vector space over $\mathbb{R}$, and let $\mathrm{H}_{\mathbb{C}}$ denote its complexification $\mathrm{H}_{\mathbb{R}} \otimes_{\mathbb{R}} \mathbb{C}$. A *Hodge structure of weight w on* $\mathrm{H}_{\mathbb{R}}$ is a direct sum decomposition of the complexification $\mathrm{H}_{\mathbb{C}}$,

$$\mathrm{H}_{\mathbb{C}} = \bigoplus_{0 \le p \le w} \mathrm{H}^{p,\, w-p} \qquad \text{such that} \quad \mathrm{H}^{w-p,\, p} = \overline{\mathrm{H}^{p,\, w-p}} \quad \text{for all} \quad 0 \le p \le w. \tag{2.4.1}$$

The *Hodge filtration* defined by this Hodge structure is

$$F^p \mathrm{H}_{\mathbb{C}} = \bigoplus_{i \ge p} \mathrm{H}^{i,\, w-i}, \qquad \text{for all} \quad 0 \le p \le w. \tag{2.4.2}$$

This filtration is decreasing: $F^p \mathrm{H}_{\mathbb{C}} \supseteq F^{p+1} \mathrm{H}_{\mathbb{C}}$. Note that $F^0 \mathrm{H}_{\mathbb{C}} = \mathrm{H}_{\mathbb{C}}$ and $\mathrm{H}^{p,q} = F^p \mathrm{H}_{\mathbb{C}} \cap \overline{F^p \mathrm{H}_{\mathbb{C}}}$.

Remark 2.4.2. The idea of the Hodge structure is inspired by Hodge decomposition theorem for De Rham and Dolbeault cohomologies. While cohomologies are indeed the object whose Hodge structures we want to study in this chapter, the following paragraph works in the generalized setting of a finite-dimensional vector space over $\mathbb{R}$, as the Hodge-theoretical results in this paragraph do not depend on any properties specific to cohomology, and therefore this extra data is simply unneeded.

Definition 2.4.3. The *Weil operator C* associated to a Hodge structure on $\mathrm{H}_{\mathbb{R}}$ is the operator that acts on each of the components $\mathrm{H}^{p,q}$ by multiplication by $(\sqrt{-1})^{p-q}$. This operator is real and therefore descends to $\mathrm{H}_{\mathbb{R}}$.

Definition 2.4.4. A *polarization* of a Hodge structure of weight w on $\mathrm{H}_{\mathbb{R}}$ is a nondegenerate

bilinear form $I : \mathrm{H}_{\mathbb{R}} \times \mathrm{H}_{\mathbb{R}} \to \mathbb{R}$ which is $(-1)^w$-symmetric and such that the Hermitian form

$$Q(x, y) := I(Cx, \overline{y}) \qquad (x, y \in \mathrm{H}_{\mathbb{C}})$$

is positive definite. The induced norm $\|x\| := \sqrt{Q(x, x)}$ is called the *Hodge norm*.

Definition 2.4.5. A *morphism* of Hodge structures of the same weight w is an $\mathbb{R}$-linear map $\phi : \mathrm{H}_{\mathbb{R}} \to \tilde{\mathrm{H}}_{\mathbb{R}}$ such that its complexification satisfies $\phi_{\mathbb{C}}(\mathrm{H}^{p,q}) \subseteq \tilde{\mathrm{H}}^{p,q}$ for all $p + q = w$ (equivalently, $\phi_{\mathbb{C}}(\mathrm{F}^p \mathrm{H}_{\mathbb{C}}) \subseteq \tilde{\mathrm{F}}^p \tilde{\mathrm{H}}_{\mathbb{C}}$ for all p).

Remark 2.4.6 (Hodge structures via the Deligne torus). Let $\mathbb{S} := \mathrm{Res}_{\mathbb{C}/\mathbb{R}} \mathbb{G}_m$ be the *Deligne torus*. Then $\mathbb{S}(\mathbb{R}) = \mathbb{C}^\times$ (via $a + ib \mapsto \begin{pmatrix} a & -b \\ b & a \end{pmatrix}$), and $\mathbb{S}(\mathbb{C}) \cong \mathbb{C}^\times \times \mathbb{C}^\times$; complex conjugation on $\mathbb{C}$ exchanges the two factors.

A pure Hodge structure of weight w on a real vector space $\mathrm{H}_{\mathbb{R}}$ is equivalent to an algebraic homomorphism $h : \mathbb{S} \to GL(\mathrm{H}_{\mathbb{R}})$ such that the restriction to the embedded $\mathbb{G}_{m,\mathbb{R}} \hookrightarrow \mathbb{S}$ (given on real points by $r \mapsto r \in \mathbb{C}^\times$) acts by $r \mapsto r^w \mathrm{Id}$. After complexifying, the representation of $\mathbb{S}(\mathbb{C}) \cong \mathbb{C}^\times \times \mathbb{C}^\times$ yields a decomposition

$$\mathrm{H}_{\mathbb{C}} = \bigoplus_{p+q=w} \mathrm{H}^{p,q}, \qquad \mathrm{H}^{p,q} = \{v \in \mathrm{H}_{\mathbb{C}} : \ h(z_1, z_2)v = z_1^p z_2^q v\},$$

and complex conjugation satisfies $\overline{\mathrm{H}^{p,q}} = \mathrm{H}^{q,p}$. Conversely, a decomposition $\mathrm{H}_{\mathbb{C}} = \bigoplus_{p+q=w} \mathrm{H}^{p,q}$ determines h by $h(z)v = z^p \bar{z}^q v$ for $z \in \mathbb{C}^\times$ and $v \in \mathrm{H}^{p,q}$.

With this convention the Weil operator is $C = h(i)$, so $C|_{\mathrm{H}^{p,q}} = i^{p-q}$. A morphism of Hodge structures is equivalently an $\mathbb{S}$-equivariant $\mathbb{R}$-linear map. The Tate twist $\mathrm{H}(m)$ corresponds to multiplying h by the character $(z\bar{z})^{-m}$ and shifts the weight by $-2m$. Finally, the induced Hodge structure on $\mathrm{End}(\mathrm{H}_{\mathbb{R}})$ has weight 0 and

$$\mathrm{End}(\mathrm{H}_{\mathbb{C}})^{r,-r} = \{T : \mathrm{H}^{p,q} \to \mathrm{H}^{p+r,q-r} \text{ for all } p, q\}, \qquad \mathrm{End}(\mathrm{H}_{\mathbb{R}})^{0,0} = \mathrm{End}_{\mathrm{HS}}(\mathrm{H}_{\mathbb{R}}).$$

Definition 2.4.7. Let A be an $\mathbb{R}$-algebra. A Hodge structure on A is *compatible with the algebra structure* if the multiplication map $m : A \otimes A \to A$ is a morphism of Hodge structures. In particular, the weight must be 0.

It is easy to see that $\mathrm{End}(\mathrm{H}_{\mathbb{R}})$ is always compatible with a Hodge structure, and $\mathrm{End}(\mathrm{H}_{\mathbb{R}})^{0,0}$ is the algebra of Hodge-structure-preserving endomorphisms of $\mathrm{H}_{\mathbb{R}}$.

Lemma 2.4.8. *Let H be a finite-dimensional real vector space. A Hodge structure on* $\mathrm{End}(H)$ *which is compatible with the algebra structure comes from a Hodge structure on H, unique up to shift of weight.*

The same statement is true for H a module over a division algebra A. Namely, a Hodge structure on $\mathrm{End}_A(H)$ *compatible with the algebra structure gives a Hodge structure on H, compatible with the A-action (i.e. such that A acts by endomorphisms of the Hodge structure on H).*

See [Fil16a], Lemma 3.9, for a proof.

Definition 2.4.9. Let M be a complex manifold, and consider a finite-dimensional flat vector bundle $\mathrm{H} \longrightarrow M$, which is equipped with a flat *Gauss–Manin* connection ∇^{GM}. It is called a *variation of Hodge structures of weight w on M* if the complexification $\mathrm{H}_{\mathbb{C}}$ can be decomposed into a decreasing sequence of holomorphic subbundles

$$\mathrm{H}_{\mathbb{C}} = \mathrm{F}^0 \supset \mathrm{F}^1 \supset \cdots \supset \mathrm{F}^w \supset \mathrm{F}^{w+1} = \{0\},$$

such that

1. For each point $s \in M$ the induced filtration

$$\mathrm{H}_{\mathbb{C}}(s) = \mathrm{F}^0(s) \supset \mathrm{F}^1(s) \supset \cdots \supset \mathrm{F}^w(s) \supset \{0\},$$

is a Hodge filtration of weight w. Moreover, the dimensions $h^{p,w-p} = \dim \mathrm{H}^{p,w-p}(s)$, $\mathrm{H}^{p,w-p}(s) = \mathrm{F}^p(s) \cap \overline{\mathrm{F}^{w-p}(s)}$, are independent of s.

2. The flat connection ∇^{GM} satisfies the *Griffiths transversality* condition for every p: if σ is a local section of F^p, then $\nabla^{GM}\sigma$ lies in $\mathrm{F}^{p-1} \otimes \Omega^1_X$.

The variation of Hodge structures is called *polarized* if there exists a nondegenerate bilinear form

$$I \colon \mathrm{H}_{\mathbb{C}} \times \mathrm{H}_{\mathbb{C}} \to \mathbb{C},$$

which is flat with respect to ∇^{GM} and such that, for each point $s \in M$, the pairing Q polarizes the Hodge structure on $\mathrm{H}(s)$.

2.4.2 Differential-geometric properties of Hodge bundles

Consider a holomorphic vector bundle E over an arbitrary complex manifold M. The bundle is endowed with a (pseudo-)hermitian metric $\langle \cdot, \cdot \rangle$.

Definition 2.4.10. The *Chern connection* on E-valued smooth differential forms is the unique connection $\nabla : \mathcal{A}^\bullet(E) \to \mathcal{A}^{\bullet+1}(E)$ that satisfies the following two conditions:

1. For any α, $\beta \in \mathcal{A}^\bullet(E)$,

$$d\langle \alpha, \beta \rangle = \langle \nabla\alpha, \beta \rangle + (-1)^{\deg \alpha} \langle \alpha, \nabla\beta \rangle; \tag{2.4.3}$$

2. For any holomorphic section s of E,

$$\nabla(s) \in \mathcal{A}^{1,0}(E). \tag{2.4.4}$$

The Chern connection can be given explicitly if we fix a local holomorphic basis of E. Let h be the matrix of the Hermitian metric in this basis, and let A be the matrix of $(1,0)$-forms.

Note that $A = h^{-1}\partial h$. The Chern connection is then (locally) given by the form

$$\nabla = d + A.$$

Let Ω denote the curvature of the Chern connection ∇.In the same local trivialization,

$$\Omega = dA + A \wedge A.$$

Remark 2.4.11. In some references, the curvature appears as $dA - A \wedge A$ rather than $dA + A \wedge A$. This depends on how one treats the sign in $\nabla^2 f = (dA)f \mp A(Af)$. Once A is viewed as a matrix of one-forms, it naturally appears with a plus sign in $dA + A \wedge A$.

We begin with a second variation formula for the logarithm of the norm of a holomorphic section. We then obtain curvature identities for quotient and subbundles, later specializing to Hodge bundles. The overall treatment follows the reference [Fil16a] with very minor differences and is included for the sake of convenience only.

2.4.3 A Second Variation Formula

Lemma 2.4.12. *Let ϕ be a holomorphic section of E. Then*

$$\overline{\partial}\partial \log \|\phi\|^2 = \frac{\langle \Omega\phi,\, \phi\rangle}{\|\phi\|^2} + \frac{\langle \nabla\phi,\, \phi\rangle \langle \phi,\, \nabla\phi\rangle - \|\phi\|^2 \langle \nabla\phi,\, \nabla\phi\rangle}{\|\phi\|^4}. \tag{2.4.5}$$

Proof. Since ϕ is holomorphic, $\partial\|\phi\|^2$ is simply the $(1,0)$-part of $d\|\phi\|^2$. By definition, for any $\alpha,\, \beta \in \mathcal{A}^\bullet(E)$,

$$d\langle \alpha, \beta\rangle = \partial\langle \alpha, \beta\rangle + \overline{\partial}\langle \alpha, \beta\rangle.$$

From this and the properties of the Chern connection, we obtain

$$\partial\|\phi\|^2 = \langle \nabla\phi, \phi\rangle.$$

Therefore

$$\partial \log \|\phi\|^2 = \frac{1}{\|\phi\|^2} \cdot \partial \|\phi\|^2 = \frac{\langle \nabla\phi, \phi\rangle}{\|\phi\|^2}.$$

By the product rule,

$$\overline{\partial}(\partial \log \|\phi\|^2) = (\overline{\partial}\frac{1}{\|\phi\|^2})\langle \nabla\phi, \phi\rangle + \frac{1}{\|\phi\|^2}\overline{\partial}\langle \nabla\phi, \phi\rangle.$$

Calculating the first term is straightforward:

$$\overline{\partial}\frac{1}{\|\phi\|^2} = \frac{-1}{\|\phi\|^4} \cdot \overline{\partial}\|\phi\|^2 = -\frac{\langle \phi, \nabla\phi\rangle}{\|\phi\|^4}.$$

To calculate the second term, note that

$$\overline{\partial}\langle \nabla\phi, \phi\rangle = d\langle \nabla\phi, \phi\rangle.$$

Indeed, the $(2,0)$-part of $d\langle \nabla\phi, \phi\rangle$ can be identified with $\partial\partial\langle \nabla\phi, \phi\rangle$, which always vanishes. Therefore, $d\langle \nabla\phi, \phi\rangle$ only has a $(1,1)$-part. By the properties of the Chern connection,

$$d\langle \nabla\phi, \phi\rangle = \langle \Omega\phi, \phi\rangle - \langle \nabla\phi, \nabla\phi\rangle.$$

Combining, obtain

$$\begin{aligned}\overline{\partial}\partial \log \|\phi\|^2 &= -\frac{\langle \phi, \nabla\phi\rangle}{\|\phi\|^4}\langle \nabla\phi, \phi\rangle + \frac{1}{\|\phi\|^2}\langle \Omega\phi, \phi\rangle - \frac{1}{\|\phi\|^2}\langle \nabla\phi, \nabla\phi\rangle = \\ &= \frac{\langle \Omega\phi, \phi\rangle}{\|\phi\|^2} + \frac{\langle \nabla\phi, \phi\rangle \cdot \langle \phi, \nabla\phi\rangle - \|\phi\|^2 \cdot \langle \nabla\phi, \nabla\phi\rangle}{\|\phi\|^4}.\end{aligned}$$

□

Remark 2.4.13. The second term of this expression is a negative $(1,1)$-form by the Cauchy–Schwartz inequality. In the case of Hodge bundles, one can fine-tune the first term to also be

negative.

Lemma 2.4.14. *Additionally,*

$$\overline{\partial}\partial\|\phi\|^2 = d\langle\nabla\phi,\phi\rangle = \langle\Omega\phi,\phi\rangle - \langle\nabla\phi,\nabla\phi\rangle.$$

2.4.4 Curvature of Quotients and Subbundles

Given a short exact sequence of holomorphic vector bundles

$$0 \;\to\; S \;\hookrightarrow\; E \;\overset{\pi}{\twoheadrightarrow}\; Q \;\to\; 0,$$

suppose E carries a non-degenerate hermitian form $\langle\cdot,\cdot\rangle$ (possibly indefinite). Assume $\langle\cdot,\cdot\rangle|_S$ is also non-degenerate, so that $S \cap S^\perp = \{0\}$ and $S^\perp \simeq Q$.

Definition 2.4.15. The *second fundamental form* is a tensor $B \in \mathrm{Hom}(S,Q)\otimes\mathcal{A}^{1,0}$ such that

$$B : S \xrightarrow{\nabla} E\otimes\mathcal{A}^{1,0} \xrightarrow{\pi\otimes\mathbf{id}} Q\otimes\mathcal{A}^{1,0}.$$

More specifically, $B : S \to Q\otimes\mathcal{A}^{1,0}$ is a linear tensor, i.e. $B(f\cdot s) = f\cdot B(s)$ for any function f and any section s of S.

We can define the *adjoint* $B^\dagger \in \mathrm{Hom}(Q,S)\otimes\mathcal{A}^{0,1}$ by taking the adjoint in the Hom-space using the hermitian form and complex conjugation on the second factor.

Lemma 2.4.16. *The subbundle S and the quotient Q have curvatures*

$$\Omega_S \;=\; \Omega_E\Big|_S + B^\dagger\wedge B, \quad \Omega_Q \;=\; \Omega_E\Big|_Q + B\wedge B^\dagger.$$

Here, $\Omega_E\Big|_\bullet$ means the restriction of Ω_E to S or Q.

Proof. In a local frame, write the connection matrix of E as

$$A_E \;=\; \begin{pmatrix} A_S & -B^\dagger \\ B & A_Q \end{pmatrix}.$$

Since $\Omega_E = dA_E + A_E \wedge A_E$, we can compute

$$\Omega_E \;=\; \begin{pmatrix} \Omega_S - B^\dagger \wedge B & * \\ * & \Omega_Q - B \wedge B^\dagger \end{pmatrix}.$$

Comparing the block components gives the desired identities. □

2.4.5 Conventions on Positivity

We would like to adopt the following conventions on which forms are considered positive and negative.

Definition 2.4.17. A purely imaginary $(1,1)$-form ω is *positive* if it can be written as

$$\omega \;=\; \sum h_{ij}\, dz^i \wedge \overline{dz^j},$$

with the coefficients h_{ij} forming a positive-definite hermitian matrix. For example, $dz \wedge \overline{dz}$ is positive.

A purely real $(1,1)$-form ω is *positive* if $\sqrt{-1}\,\omega$ is a positive form. For example, $dx \wedge dy$ is positive because $dx \wedge dy = \frac{\sqrt{-1}}{2}\, dz \wedge \overline{dz}$.

If $\Omega \in \mathcal{A}^{1,1} \otimes End(E)$, then Ω is *positive* if $\langle \Omega\, e, e \rangle$ is a positive $(1,1)$-form for every section e of E.

Remark 2.4.18. Equivalently, an imaginary $(1,1)$-form ω is positive if $\omega(\xi, \overline{\xi}) \geq 0$ for any tangent vector ξ.

Lemma 2.4.19. *In the setup of the previous section, assume that the hermitian metric on E is positive definite. Then $B \wedge B^\dagger$ is positive and $B^\dagger \wedge B$ is negative.*

Proof. Recall that $B \in \mathrm{Hom}(S, Q) \otimes \mathcal{A}^{1,0}$ and $B^\dagger \in \mathrm{Hom}(Q, S) \otimes \mathcal{A}^{0,1}$. For a local section $q \in Q$,

$$\langle B \wedge B^\dagger q, q\rangle \;=\; -\langle B^\dagger q, B^\dagger q\rangle,$$

which is a positive form of the indicated type. Similarly,

$$\langle B^\dagger \wedge Bs, s\rangle \;=\; -\langle Bs, Bs\rangle,$$

showing that $B^\dagger B$ is negative-definite. □

2.4.6 Curvature of Hodge Bundles

Let us go back to variations of polarized Hodge structures of weight w over a complex manifold, as in Definition 2.4.9. The polarization provides an indefinite form $\langle \cdot, \cdot\rangle_i := I(\cdot, \cdot)$ that is flat for the Gauss–Manin connection ∇^{GM}. A definite metric is introduced via

$$\langle \cdot, \cdot\rangle \;:=\; Q(\cdot, \cdot) \;=\; \langle C\,(\cdot), \cdot\rangle_i,$$

where C is the Weil operator (or Hodge star). Note that $\mathrm{H}^{p,w-p}$ as subbundles of $\mathrm{H}_{\mathbb{C}}$ are not holomorphically embedded into it. On each $\mathrm{H}^{p,w-p}$, this definite metric differs from the indefinite one by a sign factor, so the curvature computations coincide in either metric.

Remark 2.4.20. The bundle $\mathrm{H}_{\mathbb{C}}$ carries two different holomorphic structures and metrics. On the one hand, we have the flat structure (inducing a holomorphic one) and indefinite metric. On the other, we have a direct sum of holomorphic bundles, the $\mathrm{H}^{p,w-p}$, each equipped with a definite metric.

The Gauss–Manin connection is the Chern connection of $(\mathrm{H}_{\mathbb{C}}, \langle \cdot, \cdot\rangle_i)$ when viewed with

its flat complex structure. Alternatively, one can regard $\mathrm{H}_{\mathbb{C}}$ as $\bigoplus \mathrm{H}^{p,w-p}$ with the definite metric and define the Hodge connection ∇^{Hg} as a direct sum of Chern connections.

Let $\mathrm{F}^p \subset \mathrm{H}_{\mathbb{C}}$ be the Hodge filtration. Since F^p is a holomorphic subbundle and ∇^{GM} is the flat connection on $\mathrm{H}_{\mathbb{C}}$, the *second fundamental form* of F^p is the bundle map

$$\sigma_p : \mathrm{F}^p \longrightarrow (\mathrm{H}_{\mathbb{C}}/\mathrm{F}^p) \otimes \mathcal{A}^{1,0}$$

defined as follows: for a local smooth section s of F^p,

$$\sigma_p(s) := \left[\nabla^{GM} s\right] \mod \mathrm{F}^p.$$

Equivalently,

$$\sigma_p = \mathrm{pr}_{\mathrm{H}_{\mathbb{C}}/\mathrm{F}^p} \circ \nabla^{GM}\big|_{\mathrm{F}^p}.$$

Because F^p is holomorphic, the $(0,1)$-part of $\nabla^{GM} s$ remains in F^p, so the quotient above is of type $(1,0)$. By Griffiths transversality, the image actually lies in

$$\mathrm{F}^{p-1}/\mathrm{F}^p \otimes \mathcal{A}^{1,0} \cong \mathrm{H}^{p-1,w-p+1} \otimes \mathcal{A}^{1,0}.$$

Denote the adjoint of σ_p for the indefinite metric by $\sigma_p^\dagger$. We have

Theorem 2.4.21 (See [CMP17], Theorem 13.1.7)**.**

$$\nabla^{GM} = \nabla^{Hg} + \sigma_\bullet + \sigma_\bullet^\dagger.$$

Here, $\sigma_\bullet$ denotes $\oplus_p \sigma_p$.

Note that the curvature of $\mathrm{H}^{p,w-p}$ for either metric is the same, since the metrics agree up to a sign.

Lemma 2.4.22 (See [CMP17],(13.1)). *The curvature of each Hodge bundle* $\mathrm{H}^{p,w-p}$ *can be written as*

$$\Omega_{\mathrm{H}^{p,w-p}} \;=\; \sigma_p^\dagger \wedge \sigma_p \;+\; \sigma_{p+1} \wedge \sigma_{p+1}^\dagger.$$

A convenient way to view the above is via sections:

Lemma 2.4.23. *Let* e, e' *be smooth sections of* $\mathrm{H}^{p,w-p}$. *Then with respect to the definite metric, one has:*

$$\langle \Omega_{\mathrm{H}^{p,w-p}}\, e,\; e' \rangle \;=\; \langle \sigma_p\, e,\; \sigma_p\, e' \rangle \;+\; \langle \sigma_{p+1}^\dagger\, e,\; \sigma_{p+1}^\dagger\, e' \rangle.$$

Proof. On $\mathrm{H}^{p,w-p}$ itself, the indefinite metric is $(-1)^p$-times the definite one. Collecting sign factors carefully when exchanging σ and $\sigma^\dagger$ terms gives the stated identity. □

Corollary 2.4.24. *The Hodge bundle* $\mathrm{H}^{0,w}$ *is negatively curved.*

Proof. Since σ_0 vanishes, $\Omega_{\mathrm{H}^{0,w}}$ has contributions only from $\sigma_1^\dagger$, which is negative. □

2.5 Random walks on groups

Throughout this section, we are working with a probability space $(X,\, \mu)$ and a locally compact group G with a compactly supported probability measure ν such that $\mathrm{supp}(\nu)$ generates G. We assume that G acts on X on the left in such a way that (1) this action preserves μ, (2) it is ergodic. Additionally, we fix a continuous proper length function $\|\cdot\| : G \to [0, \infty)$ such that $\|e\| = 0$, $\|g^{-1}\| = \|g\|$, and $\|gh\| \leq \|g\| + \|h\|$ for all $g, h \in G$.

All results presented in this section are well-known and are included for the sake of convenience.

2.5.1 Random Ergodic Theorem

Theorem 2.5.1 ([Fur02], Theorem 3.1)**.** *Let $f \in L^1(X, \mu)$. Then for $\mu \times \nu^{\mathbb{N}}$-a.e. $(x, \omega) \in X \times G^{\mathbb{N}}$,*

$$\lim_{N\to\infty} \frac{1}{N} \sum_{i=0}^{N-1} f\Big(g_i(\omega) \cdots g_1(\omega)\, x\Big) = \int_X f \, d\mu.$$

Corollary 2.5.2. *In the setting of the previous theorem, if $f : X \to \mathbb{R}_{\geq 0}$ is measurable, then the same conclusion holds even if $\int_X f \, d\mu = +\infty$.*

Proof. For $M \in \mathbb{N}$ set $f_M := \min(f, M)$. Then $f_M \in L^1(X, \mu)$ and $f_M \uparrow f$ pointwise. Apply Theorem 2.5.1 to each f_M, and let $M \to \infty$. □

2.5.2 Harmonic and subharmonic functions

Definition 2.5.3. Consider a μ-measurable function $h : X \to \mathbb{R}$. The averaging operator $Av_G(h)$ is

$$Av_G(h)(x) = (\nu * h)(x) = \int_G h(gx) d\nu(g).$$

The function h is called *ν-harmonic* if, for μ-a.e. $x \in X$,

$$h(x) = Av_G(h)(x).$$

The function h is called ν-subharmonic if, for μ-a.e. $x \in X$,

$$h(x) \leq Av_G(h)(x).$$

The *Laplacian* is

$$Lh = Av_G(h) - h.$$

Note that $Av_G(h)(x)$ is not automatically well-defined.

Definition 2.5.4. A measurable function $u : G \to \mathbb{R}$ is called *tame* if there exists $C > 0$ such that

$$|u(g)| \leq C(1 + \|g\|) \qquad \text{for all } g \in G.$$

A μ-measurable function $h : X \to \mathbb{R}$ is called *G-tame* if for μ-a.e. $x \in X$, the function $h_x(g) := h(gx)$ is tame (with a constant $C = C(x)$).

Note that if h is G-tame and ν has compact support, then $Av_G(h)(x)$ is well-defined for μ-a.e. x. G-tameness is used only to guarantee that $Av_G(h)(x)$ is finite (since ν has compact support); constancy of ν-harmonic functions will be invoked under L^1 (or L^∞) assumptions.

Subharmonicity in particular is a very strong condition.

Lemma 2.5.5. *1. If $f \in L^1(X, \mu)$ is ν-subharmonic, then f is constant μ-almost everywhere.*

2. If $h : X \to \mathbb{R}_{\geq 0}$ is G-tame and ν-harmonic, then h is constant μ-almost everywhere.

Proof. 1. Since μ is G-invariant,

$$\int_X Av_G(f)\, d\mu = \int_G \int_X f(gx)\, d\mu(x)\, d\nu(g) = \int_X f\, d\mu.$$

Because $f \leq Av_G(f)$ μ-a.e. and both functions are integrable, it follows that

$$Av_G(f) = f \qquad \mu\text{-a.e.}$$

Thus f is ν-harmonic.

Now apply the Random Ergodic Theorem to f. For $\mu \times \nu^{\mathbb{N}}$-a.e. (x, ω),

$$\frac{1}{N} \sum_{i=0}^{N-1} f\big(g_i(\omega) \cdots g_1(\omega)x\big) \longrightarrow \int_X f\, d\mu.$$

On the other hand, since f is ν-harmonic, the process

$$M_n(x,\omega) := f\Big(g_n(\omega)\cdots g_1(\omega)x\Big)$$

is a stationary martingale, hence $M_n = M_0$ for all n almost surely. Therefore the averages above are identically equal to $f(x)$, and so

$$f(x) = \int_X f\, d\mu \qquad \text{for } \mu\text{-a.e. } x.$$

Hence f is constant μ-a.e.

2. For each $M \in \mathbb{N}$, define

$$h_M := \min(h, M).$$

Then $h_M \in L^\infty(X,\mu) \subset L^1(X,\mu)$. Since $t \mapsto \min(t, M)$ is concave on $\mathbb{R}_{\geq 0}$, Jensen's inequality gives

$$Av_G(h_M) = Av_G(\min(h, M)) \leq \min(Av_G(h), M).$$

Because h is ν-harmonic,

$$\min(Av_G(h), M) = \min(h, M) = h_M,$$

so h_M is ν-superharmonic. Equivalently,

$$f_M := M - h_M$$

is ν-subharmonic and belongs to $L^1(X,\mu)$. By part (1), f_M is constant μ-a.e., hence so is h_M.

Since $h_M \uparrow h$ pointwise as $M \to \infty$, it follows that h is constant μ-a.e.

□

2.5.3 Sublinear growth

Definition 2.5.6. A function $h : X \to \mathbb{R}$ is said to be of *sublinear growth* if for μ-a.e. x the estimate

$$|h(g_n(\omega) \cdots g_1(\omega)x)| = o(n)$$

holds for $\nu^{\mathbb{N}}$-a.e. $\omega \in G^{\mathbb{N}}$.

Note that the estimate might (and often does) depend on x and even on ω.

Lemma 2.5.7 (Sublinear ν-subharmonic functions are constant)**.** *Let (X, μ) be a probability space with an ergodic, μ-preserving left action of a locally compact group G. Let ν be a compactly supported probability measure on G whose support generates G, and let Av_G and ν-(sub)harmonicity be as in Definition 2.5.3.*

1. *Let $h : X \to \mathbb{R}_{\geq 0}$ be μ-measurable. Assume:*

   (a) *h is G-tame (hence $Av_G(h)(x) < \infty$ for μ-a.e. x, since ν has compact support);*

   (b) *h has sublinear growth along ν-random walks;*

   (c) *h is ν-subharmonic: $h(x) \leq Av_G(h)(x)$ for μ-a.e. x.*

   Then h is μ-almost everywhere constant.

2. *Let $h : X \to \mathbb{R}$ be μ-measurable and set $h^+ := \max(0, h)$. Assume:*

   (a) *h is ν-subharmonic (so $Av_G(h)(x)$ is understood wherever it is well-defined);*

   (b) *h^+ is G-tame (hence $Av_G(h^+)(x) < \infty$ for μ-a.e. x);*

   (c) *h^+ has sublinear growth along ν-random walks.*

Then h is μ-almost everywhere constant. In particular, if h^+ is G-tame and ν has compact support, then $Av_G(h^+)(x)$ is finite for μ-a.e. x.

As in the case of Lemma 2.5.5, the result is standard. We omit the proof.

2.5.4 Geodesic approximation

This section briefly explains an important variation of the multiplicative ergodic theorem proved by Kaimanovich.

Definition 2.5.8. Let G be a real semisimple Lie group, and let K be its maximal compact subgroup.Set $Y := G/K$. A sequence $\{y_n\} \subset Y$ is called *regular* if there exist a geodesic ray $\gamma : [0, \infty) \to Y$ and $C \geq 0$ such that

$$d\Big(y_n, \gamma(Cn)\Big) = o(n).$$

Theorem 2.5.9 ([Kai89]; see [Fil19] for more details)**.** *Let G be a real semisimple Lie group with the Lie algebra $\mathfrak{g}$, let K be its maximal compact subgroup, and let $\mathfrak{a} \subset \mathfrak{g}$ be the maximal abelian subalgebra. For $y \in Y$ let $r(y) \in \mathfrak{a}^+$ denote the Cartan projection of y: the unique element such that $k \exp(r(y)) = y$ for some $k \in K$. A sequence of points $\{y_n\}$ in $Y = G/K$ is regular if and only if the following two conditions are satisfied:*

1. *Small steps:* $\mathrm{d}\,(y_n,\, y_{n+1}) = o(n)$
2. *Distances converge:* $\alpha := \lim \frac{r(y_n)}{n}$ *exists in* $\mathfrak{a}^+$.

2.6 Semisimplicity

Throughout this section, we continue working with a projective-surjective group $G \subset (\mathrm{SL}_2\mathbb{R})^n$ acting on a product of strata $X = \mathcal{H}^{(1)}(\alpha_1, \ldots, \alpha_n) = \mathcal{H}^{(1)}(\alpha_1) \times \ldots \times \mathcal{H}^{(1)}(\alpha_n)$. Let μ be an ergodic G-invariant measure on X, and let Γ be an integrable cocycle over μ that gives a

variation of Hodge structures $\mathrm{H}_{\mathbb{F}} = \mathrm{H}_{\mathbb{F}}(\alpha_1) \times \ldots \times \mathrm{H}_{\mathbb{F}}(\alpha_n)$ on products of Teichmüller discs, where $\mathbb{F} = \mathbb{R}$ or $\mathbb{F} = \mathbb{C}$. Note that this cocycle and, therefore, the measure μ are invariant under the action of $(SO_2\mathbb{R})^n$ (see Remark 2.2.1). We will need the following

Lemma 2.6.1. *There exist constants C_1, C_2 such that*

$$\log ||\Gamma(g,\, x)|| \leq C_1 \log ||g|| + C_2 \tag{2.6.1}$$

for all $g \in G$, $x \in X$.

Proof. We first treat the Kontsevich–Zorich cocycle KZ on one stratum, equipped with the Hodge norm. Let $a_t = \mathrm{diag}(e^t, e^{-t})$. By [FMZ14, Lem. 2.3], there exist constants c_1, c_2 such that for all $t \in \mathbb{R}$ and all x,

$$\log \|\mathrm{KZ}(a_t, x)\| \leq c_1|t| + c_2. \tag{2.6.2}$$

Now write an arbitrary $g \in \mathrm{SL}_2\mathbb{R}$ in Cartan form $g = k_1 a_t k_2$ with $k_1, k_2 \in SO_2\mathbb{R}$. Since the Hodge norm is $SO_2\mathbb{R}$-invariant, the cocycle over k_1, k_2 acts by uniformly bounded operators, hence (2.6.2) implies

$$\log \|\mathrm{KZ}(g, x)\| \leq c_1 t + c_2' \qquad \text{for all } g \in \mathrm{SL}_2\mathbb{R},\ x.$$

Finally, for any fixed operator norm on $\mathrm{SL}_2\mathbb{R}$ we have $\|a_t\| = e^t$ and $t \asymp \log \|g\|$, so

$$\log \|\mathrm{KZ}(g, x)\| \leq C_1 \log \|g\| + C_2.$$

For a product of strata and the action of $(\mathrm{SL}_2\mathbb{R})^n$, apply the previous argument in each factor (using the product Cartan decomposition), and note that passing to tensor powers only changes the constants. Since Γ is the restriction of this cocycle from $(\mathrm{SL}_2\mathbb{R})^n \times X$ to

$G \times X$, the same inequality holds for Γ. □

We will only consider the case when G is a diagonal subgroup Δ (see Lemma 2.2.5 and Remark 2.2.6 for details), as group product operations obtain all other possibilities. Finally, we will introduce the auxiliary group notation

$$K_G = \{g \times h_1^{-1} g h_1 \times \ldots \times h_{n-1}^{-1} g h_{n-1} \mid g \in SO_2\mathbb{R}\} \times F(s), \quad F(s) \subset \prod_{i=1}^{n} \{Id, -Id\}.$$

The subgroup K_G relates to G in the same way $SO_2\mathbb{R}$ relates to $SL_2\mathbb{R}$.

Lemma 2.6.2. *Let $f : X \to \mathbb{R}$ be a μ-measurable K_G-invariant function. Assume that for μ-almost every product leaf L (i.e. product of Teichmüller discs) the restriction $f|_L$ is* (scalar) subharmonic *with respect to the product hyperbolic/Kähler metric on L, namely*

$$i\,\partial\bar{\partial}(f|_L) \wedge \omega_L^{n-1} \;\geq\; 0$$

in the sense of currents, where ω_L is the Kähler form on L and $n = \dim_{\mathbb{C}} L$.

Let $f^+ = \max(0, f)$. (Then $f^+|_L$ is also subharmonic on μ-a.e. leaf L.) Suppose moreover that:

1. *f^+ grows sublinearly along μ-almost every product of Teichmüller geodesics;*
2. *There exists $C > 0$ such that for every $g \in G$ and μ-a.e. $x \in X$,*

$$|f^+(x) - f^+(gx)| \leq C\|g\|.$$

Then f is μ-almost surely constant.

Proof. One can pick an $SO_2\mathbb{R}$-bi-invariant finite measure on $\mathrm{SL}_2\mathbb{R}$ with compact support that generates the whole $\mathrm{SL}_2\mathbb{R}$. Using this measure, we construct a K_G-bi-invariant finite

measure on G with compact support that generates the whole G. Note that $K_G\backslash G$ is just a product of the hyperbolic planes.

We can generate a random walk on G by sampling from the distribution of ν. According to the Kaimanovich's characterization of regularity from Theorem 2.5.9, there exists a rate $\delta > 0$ (strictly positive since G is generated by $\operatorname{supp}\nu$) and a random geodesic $\gamma_\bullet(\omega)$ on $K_G\backslash G$ such that

$$d([g_n(\omega)\dots g_1(\omega)x],\, [\gamma_{\delta n}(\omega)x]) = o(n).$$

Here $[\cdot]$ stands for the equivalence class in $K_G\backslash X$.

Therefore, f grows sublinearly along the random walk that we have constructed. According to Definition 2.5.3, f is ν-subharmonic. This means that f satisfies Lemma 2.5.7-2. □

Theorem 2.6.3. *Recall that we have an integrable cocycle Γ, which gives a variation of Hodge structures $\mathrm{H}_{\mathbb{C}}$ of weight w satisfying the boundedness assumption 2.6.1. Suppose that ϕ is a measurable section of $\mathrm{H}_{\mathbb{C}}$ that is flat along μ-a.e. G-orbit.*

Then each $(p, w-p)$-component of ϕ is also flat along a.e. μ-a.e. G-orbit.

Proof. We will prove this theorem by inductive exhaustion. Let p be the smallest number such that ϕ is a holomorphic section of F^p. We can decompose ϕ into (possibly empty) components:

$$\phi = \phi^{w,0} + \dots + \phi^{p,w-p}.$$

If we can prove the flatness of the highest component, we can subtract it from ϕ and then repeat again until there are no components left.

Consider the highest component $\phi^{p,w-p}$, which we'll denote by ψ for simplicity. It is the projection of ϕ onto $\mathrm{H}^{p,w-p} = \mathrm{F}^p/\mathrm{F}^{p+1}$. This is a quotient of holomorphic bundles, so the projection is also holomorphic.

Applying Formula 2.4.5 with an order switch in $\overline{\partial}\partial$, we obtain

$$\partial\overline{\partial}\log\|\psi\|^2 = -\frac{\langle\Omega_{\mathrm{H}^{p,w-p}}\psi,\ \psi\rangle}{\|\psi\|^2} + \frac{-\langle\nabla^{GM}\psi,\ \psi\rangle\,\langle\psi,\ \nabla^{GM}\psi\rangle + \|\psi\|^2\,\langle\nabla^{GM}\psi,\ \nabla^{GM}\psi\rangle}{\|\psi\|^4}.$$

As discussed before, the second term is positive. Recall Theorem 2.4.21. As ϕ is flat in the Gauss-Manin connection, we have $\sigma_p\psi = 0$ in products of Teichmüller discs. Then, according to Lemma 2.4.23, the second term is

$$-\frac{1}{\|\psi\|^2}\langle\Omega_{\mathrm{H}^{p,w-p}}\,\psi,\ \psi\rangle \ = \ -\frac{1}{\|\psi\|^2}\langle\sigma^{\dagger}_{p+1}\,\psi,\ \sigma^{\dagger}_{p+1}\,\psi\rangle.$$

This is also positive, so $\partial\overline{\partial}\log\|\psi\|^2$ is subharmonic. From the boundedness assumption 2.6.1, the positive part of $\log\|\phi\|^2$, and therefore the positive part of $\log\|\psi\|^2$, grow sublinearly along μ-almost every product of Teichmüller geodesics. Therefore, from the previous lemma, $\log\|\psi\|^2$ is constant in a.e. product of Teichmüller discs. Then $\overline{\partial}\partial\|\psi\|^2 = 0$. From Lemma 2.4.14,

$$\overline{\partial}\partial\|\psi\|^2 = d\langle\nabla\psi,\psi\rangle = \langle\Omega_{\mathrm{H}^{p,w-p}}\psi,\psi\rangle - \langle\nabla^{Hg}\psi,\nabla^{Hg}\psi\rangle.$$

Since both of the terms are negative, both must vanish. Therefore, $\sigma^{\dagger}_{p+1}\psi = 0$ and $\nabla^{Hg}\psi = 0$. Once again recalling Theorem 2.4.21, we conclude ψ is flat for the Gauss-Manin connection. □

Denote by $\mathrm{E}_{\mathbb{F}}$ the Hodge bundle or some tensor power of it. Let a be an element of the Deligne torus $\mathbb{S}$.

Theorem 2.6.4. *Suppose that $V \subset \mathrm{E}_{\mathbb{C}}$ is a G-invariant subbundle. Then $a \cdot V$ is also G-invariant.*

Proof. We know that any G-invariant subbundle of $\mathrm{E}_{\mathbb{F}}$ has an invariant complement because the cocycle induced by $\mathrm{E}_{\mathbb{F}}$ is reductive (see Lemma 2.3.9). Let us denote it by $V^{\perp}$. Then the projection π_V to V along $V^{\perp}$ must also be G-invariant. Consider the projection $\pi_{a\cdot V}$ to $a \cdot V$ along $(a \cdot V)^{\perp}$. Since a is an element of a Deligne torus, its action respects the

Hodge decomposition, therefore the projection $\pi_{a\cdot V}$, which is essentially a section, is also G-invariant as a consequence of Theorem 2.6.3. Therefore $a \cdot V$ is G-invariant. □

We now state the core result:

Theorem 2.6.5 (Deligne semisimplicity)**.** *1. There exist G-invariant bundles $V_i \subset \mathrm{E}_{\mathbb{F}}$, $\mathbb{F} = \mathbb{R}$ or $\mathbb{F} = \mathbb{C}$, and vector spaces W_i equipped with Hodge structures and an isomorphism*

$$\mathrm{E}_{\mathbb{F}} \cong \bigoplus_i E_i \quad \text{with} \quad E_i \cong V_i \otimes_{A_i} W_i. \tag{2.6.3}$$

Each of V_i carries a variation of Hodge structure making the above isomorphism compatible. Here, A_i are division algebras that act on V_i and are compatible with Hodge structures. They also act compatibly on W_i.

2. Furthermore, any G-invariant bundle $V \subset \mathrm{E}_{\mathbb{F}}$ has the form

$$V = \bigoplus_i V_i \otimes_{A_i} W_i',$$

where $W_i' \subset W_i$ are A_i-submodules.

Proof. 1. Assume first that V has the smallest possible fiber dimension among all G-invariant subbundles of $\mathrm{E}_{\mathbb{F}}$. Every G-invariant endomorphism of V must be either zero or an isomorphism (else the kernel of this endomorphism would yield a nontrivial invariant subbundle of V). Therefore, all nonzero elements of the algebra of endomorphisms $A = \mathrm{End}(V)^{op}$ are invertible, which means that A is a division algebra. We view V as a *right* A-module by $v \cdot a := a(v)$ (where $a \in A$ is the same endomorphism of V, but viewed in the opposite algebra), and we view the space

$$W := \mathrm{Hom}_G(V, \mathrm{E}_{\mathbb{F}})$$

of morphisms of G-invariant bundles from V to E (note that they might not respect the Hodge structures) as a *left* A-module by precomposition:

$$a \cdot f := f \circ a \qquad (a \in A,\ f \in W).$$

Define the evaluation map

$$ev : V \otimes_A W \longrightarrow \mathrm{E}_{\mathbb{F}}, \qquad ev(v \otimes f) := f(v).$$

This is well-defined since

$$ev((v \cdot a) \otimes f) = f(a(v)) = (f \circ a)(v) = ev(v \otimes (a \cdot f)).$$

Denote its image by E'. Because V is of minimal dimension, all elements of W must be injective (or zero). Thus $ev : V \otimes_A W \to \mathrm{E}'$ is an isomorphism. Furthermore, since V is of minimal dimension,

$$\mathrm{End}_G(\mathrm{E}') \cong \mathrm{End}_G(V \otimes_A W) \cong \mathrm{End}_A(W).$$

Lemma 2.4.8 allows us to induce a Hodge structure on $\mathrm{End}_G(\mathrm{E}')$, and this on $\mathrm{End}_A(W)$, from E'. The same Lemma thus gives us an A-compatible Hodge structure on W.

By Theorem 2.6.4, E' is G-invariant and, by Theorem 2.6.3, it is a (sub)-variation of Hodge structures. It remains to endow V with a Hodge structure that would make the isomorphism $ev : V \otimes_A W \to \mathrm{E}'$ Hodge-compatible. To do this, we identify V with a subbundle of $\mathrm{Hom}(W, \mathrm{E}')$: each element $v \in V$ induces a natural evaluation map $ev_v : W \to \mathrm{E}'$, $ev_v(f) = f(v)$. Note that this subbundle is equivariant under the action of $\mathrm{End}_A(W)$, for every $a \in \mathrm{End}_A(W)$ and $\phi \in \mathrm{Hom}(W, \mathrm{E}')$, $(a \cdot \phi)(f) = \phi(a(f))$. Thus we can endow V with a Hodge structure that is compatible with the ones on W and E'. Note that, as a consequence of using Lemma 2.4.8, the Hodge structures on V and W are unique up to a simultaneous

opposite shift of weight.

The proof is completed by inducting on the orthogonal complement of E' (which is the same in both the definite and the indefinite metric).

2. We now have the decomposition 2.6.3 of $\mathrm{E}_{\mathbb{F}}$. Consider an arbitrary G-invariant bundle $V \subset \mathrm{E}_{\mathbb{F}}$. Denote by π_i its projection to the i-th summand in the decomposition 2.6.3. We aim to prove that $V = \pi_i V \oplus (1-\pi_i)V$. If this is not true, then $\ker \pi_i$ and $\ker(1-\pi_i)$ do not span V. Then there exists their non-trivial invariant complement $V' \subsetneq V$. It has to be G-invariant and isomorphic to itself under the action of both π_i and $1-\pi_i$. But this contradicts the minimality condition of the subbundle decomposition.

We complete the proof by iterating by i. Note that any invariant subbundle of $V_i \otimes_{A_i} W_i$ has to be of the form $V_i \otimes_{A_i} W_i'$ for some A_i-submodule $W_i' \subset W_i$ because of the minimality condition on V_i. □

2.7 Rigidity

Recall that a translation surface is a pair (X, ω), where X is a genus g Riemann surface and ω is a holomorphic 1–form. Let $\Sigma = \mathrm{div}(\omega)$ and let $\mathcal{H}^{(1)}(\alpha)$ be the corresponding stratum. Fix a basis $\{\xi_1, \dots, \xi_n\}$ of the relative homology group $H_1(X, \Sigma; \mathbb{Z})$. The map

$$\Phi : \mathcal{H}^{(1)}(\alpha) \supset U \longrightarrow \mathbb{C}^n, \qquad \Phi(X,\omega) = \left(\int_{\xi_1} \omega, \dots, \int_{\xi_n} \omega \right)$$

defines local period coordinates on a neighbourhood U of (X, ω). Restricting to the $\mathrm{SL}_2\mathbb{R}$–orbit of (X, ω) yields the (possibly immersed) Teichmüller disc through (X, ω). The absolute periods are the integrals of ω over a basis of $H_1(X; \mathbb{Z})$. Period coordinates endow each stratum $\mathcal{H}^{(1)}(\alpha)$ with an atlas with values in $\mathbb{C}^n$ and transition maps in $\mathrm{GL}_n\mathbb{Z}$.

An *affine invariant submanifold* in a stratum $\mathcal{H}^{(1)}(\alpha)$ is an immersed manifold $\mathcal{M} \hookrightarrow \mathcal{H}^{(1)}(\alpha)$ such that in period coordinates it is locally cut out by real linear equations. By

[EMM15], $\mathcal{M}$ is an orbifold (and in particular has only mild self-intersections); for notational simplicity we treat $\mathcal{M}$ as embedded on a full-measure subset.

Likewise, we can define an *affine invariant submanifold* of any product of strata.

As before, we work with a projective-surjective group $G \subset (\mathrm{SL}_2\mathbb{R})^n$ acting on a product of strata $X = \mathcal{H}^{(1)}(\alpha_1, \ldots, \alpha_n) = \mathcal{H}^{(1)}(\alpha_1) \times \ldots \times \mathcal{H}^{(1)}(\alpha_n)$ with a G-invariant measure μ. Denote by $\mathrm{E}_{\mathbb{C}}$ ($\mathrm{E}_{\mathbb{R}}$) a tensor power of the (real) Hodge bundle of X. In the cases where the distinction is not of any consequence, we will abuse the notation by writing simply E. According to the previous section, E has a finite decomposition into $\mathrm{SL}_2\mathbb{R}$-invariant bundles E_i. By construction, each of the bundles E_i varies measurably in directions that are transversal to the $\mathrm{SL}_2\mathbb{R}$-action. We first show that invariant subbundles of E must vary analytically, then that the rate is jointly polynomial in all directions.

Before we begin, it is important to keep track of which cocycle we work with in which case. There are, largely, two available options. The first one is the tensor power of the KZ cocycle (the Gauss–Manin connection) on the tensor power of the Hodge bundle on the product of strata — or, in our case, its restriction to the G-induced cocycle. To avoid needlessly complicated notation, we will denote the Teichmüller flow under the action of this cocycle by g_t. Note that, when we consider the positive part of the Lyapunov spectrum of the Kontsevich-Zorich cocycle for one stratum, we have

$$1 = \lambda_1 > \lambda_2 \geq \lambda_3 \cdots \geq \lambda_g \geq 0,$$

where spectral gap inequality $1 > \lambda_2$ is due to Forni [For02]. For tensor operations on the KZ cocycle and for the G-cocycle the spectral gap persists: the difference between the top two Lyapunov exponents remains $1 - \lambda_2 > 0$ (up to the natural renormalization coming from the tensor operation).

The second cocycle $\mathrm{d}g_t$ on the tangent space of the products of strata is induced by the action of the G-cocycle on the product of strata. Under the flow, the $(\mathbb{R}^2)^n$ factor carries the

standard representation of $\mathrm{SL}_2\mathbb{R}$ with exponents ± 1, the first cohomology factor carries the KZ exponents, and the relative part feels only the ± 1. Consequently, the Lyapunov spectrum of $\mathrm{d}g_t$ is obtained by adding the exponents. The positive part of its Lyapunov spectrum is

$$1 + \lambda_1 > 1 + \lambda_2 \geq \cdots \geq 1 + \lambda_g \geq 1 \geq \cdots \geq 1 \geq 1 - \lambda_g \geq \cdots \geq 1 - \lambda_1 = 0.$$

Note that the subspace corresponding to the zero exponent comes from the centralizer of g_t inside G and does not appear when restricted to area one surfaces. More specifically, the centralizer contains the scalar subgroup, which changes the area and commutes with g_t, so the Lyapunov exponent is 0. On area 1 locus that scalar direction is not tangent anymore. The only neutral direction left from the centralizer is the flow direction itself. Therefore, from Forni's result mentioned above, on unstable leaves, the cocycle $\mathrm{d}g_t$ expands uniformly at rate equal or higher than $1 - \lambda_2 > 0$. See survey [Zor06], Section 5.8, for details.

2.7.1 Analyticity along almost every leaf

Theorem 2.7.1 (Oseledets Multiplicative Ergodic Theorem)**.** *Suppose $V \to (\Omega, \mathcal{B}, \mu, T)$ is a cocycle over an ergodic probability measure-preserving system. Assume that V is equipped with a metric $\| - \|$ on each fiber such that*

$$\int_\Omega \log^+ \|T_\omega\|_{op} \, d\mu(\omega) < \infty.$$

Here, $\log^+(x) := \max(0, \log x)$ and $\| - \|_{op}$ denotes the operator norm of a linear map between normed vector spaces.

Then there exist real numbers $\lambda_1 > \lambda_2 > \cdots > \lambda_k$ (with perhaps $\lambda_k = -\infty$) and T-invariant subbundles of V defined for a.e. $\omega \in \Omega$:

$$0 \subsetneq V^{\leq \lambda_k} \subsetneq \cdots \subsetneq V^{\leq \lambda_1} = V$$

such that for vectors $v \in V_\omega^{\leq \lambda_i} \backslash V_\omega^{\leq \lambda_{i+1}}$ *we have*

$$\lim_{N\to\infty} \frac{1}{N} \log \left\| T^N v \right\| \to \lambda_i.$$

Let F stand for a bundle E or one of the subbundles E_i in its invariant decomposition. The the *forward Lyapunov filtration*, denoted by

$$\mathrm{F} = \mathrm{F}^{\leq \lambda_1} \supsetneq \mathrm{F}^{\leq \lambda_2} \supsetneq \cdots \supsetneq \mathrm{F}^{\leq \lambda_n} \supsetneq \{0\},$$

is defined by satisfying

$$f \in \mathrm{F}^{\leq \lambda_i} \text{ iff } \limsup_{t\to+\infty} \frac{1}{t} \log \|g_t f\| \leq \lambda_i.$$

The *backward Lyapunov filtration* $\mathrm{F}^{\geq \lambda_i}$ is defined analogously, with $t \to -\infty$. This gives the subbundles $\mathrm{F}^{\lambda_i} := \mathrm{F}^{\leq \lambda_i} \cap \mathrm{F}^{\geq \lambda_i}$ and the g_t-invariant decomposition

$$\mathrm{F} = \bigoplus_i \mathrm{F}^{\lambda_i}.$$

In period coordinates, the unstable foliation is given by the real subspaces $H^1_{\mathrm{rel}}(X;\mathbb{R})$ and the stable foliation by $\sqrt{-1}\, H^1_{\mathrm{rel}}(X;\mathbb{R})$. Since $\mathcal{M}$ is locally cut out by real linear equations, these foliations restrict to $\mathcal{M}$. We denote the (local) stable leaf through $x \in \mathcal{M}$ by $W^-(x)$ and the unstable leaf by $W^+(x)$. Along stable leaves, the Gauss–Manin connection identifies the fibers, and the forward Lyapunov filtration is locally constant: $\mathrm{E}^{\leq \lambda_i}(x) = \mathrm{E}^{\leq \lambda_i}(y)$ whenever $x \in W^-(y)$ (and likewise for each E_j). Thus, the forward Lyapunov filtration exists for almost every stable leaf and is locally constant on the said stable leaf.

Lemma 2.7.2 ([**Eskin-Mirzakhani2018**], Corollary 4.5, p. 120; based on [Led86], Theorem 1, p. 59)**.** *Suppose that* V *is a* g_t*-invariant subbundle of* $H^1_{rel}(S_g;\mathbb{R})$ *and that* $\mathrm{E}^{\leq \lambda_k} \subseteq \mathrm{V} \subseteq \mathrm{E}^{\leq \lambda_{k-1}}$*. Then* V *is locally flat along the unstable leaves.*

Since this statement is only dependent on time, it holds for the action of our projective-surjective group G on the product of strata X, as well as the tensor powers of the Hodge bundle E of X.

Theorem 2.7.3. *Let $\mathrm{E}_j \subset \mathrm{E}$ be a g_t-invariant subbundle, and let*

$$\lambda_{j,1} > \lambda_{j,2} > \cdots > \lambda_{j,m_j}$$

be the Lyapunov exponents which occur in E_j.

Then the bundles

$$\mathrm{E}_j^{\leq \lambda_{j,k}}$$

in the forward Lyapunov filtration of E_j vary real-analytically on μ-almost every stable leaf W^-.

Similarly, the bundles

$$\mathrm{E}_j^{\geq \lambda_{j,k}}$$

in the backward Lyapunov filtration of E_j vary real-analytically on μ-almost every unstable leaf W^+.

In particular,

$$\mathrm{E}_j = \mathrm{E}_j^{\leq \lambda_{j,1}} = \mathrm{E}_j^{\geq \lambda_{j,m_j}}.$$

Therefore each invariant subbundle E_j varies real-analytically on μ-almost every stable leaf and on μ-almost every unstable leaf.

Proof. The second claim is analogous to the first one with the time flow reversed.

We will prove the first claim by a descending induction on the index of the exponent i, starting from n (recall that $\lambda_1 > \ldots > \lambda_n$).

Let us prove the base case first, using the fact that $\mathrm{F}^{\leq \lambda_n} = \mathrm{F}^{\lambda_n}$ whether $\mathrm{F} = \mathrm{E}$ or $\mathrm{F} = \mathrm{E}_i$.

For all j, there exists a g_t-invariant sequence of bundles

$$\{0\} \subseteq \mathrm{E}_j^{\lambda_n} \subseteq \mathrm{E}^{\lambda_n}.$$

Then $\mathrm{E}_j^{\lambda_n}$ is flat along stable leaves for all j by the above lemma.

Let us now perform the induction step. We assume that for all j we know that $\mathrm{E}_j^{\leq \lambda_{i+1}}$ vary real-analytically. We want to prove that $E_j^{\leq \lambda_i}$ also varies real-analytically. Fix some index j and consider

$$B := \mathrm{span}\left(\mathrm{E}_j^{\lambda_i} + \mathrm{E}^{\leq \lambda_{i+1}}\right).$$

This bundle is invariant under the diagonal action of the Teichmüller flow on X. Moreover, it satisfies

$$\mathrm{E}^{\leq \lambda_{i+1}} \subseteq B \subseteq \mathrm{E}^{\leq \lambda_i}.$$

Then B is flat along stable leaves by the above lemma, and we can write

$$B = \left(\bigoplus_{k \neq j} \mathrm{E}_k^{\leq \lambda_{i+1}}\right) \oplus \mathrm{E}_j^{\leq \lambda_i}.$$

This sum is clearly Hodge-orthogonal. All the summands except the last one all vary real-analytically on unstable leaves by inductive assumption. We obtain that $\mathrm{E}_j^{\leq \lambda_i}$ is Hodge-orthogonal of real-analytic $\bigoplus_{k \neq j} \mathrm{E}_j^{\leq \lambda_{i+1}}$ inside flat B. Thus, $\mathrm{E}_j^{\leq \lambda_i}$ has no choice but to be real-analytic.

By definition of the forward Lyapunov filtration, the first filtration piece is the whole bundle:

$$\mathrm{E}_j^{\leq \lambda_{j,1}} = \mathrm{E}_j,$$

because $\lambda_{j,1}$ is the largest Lyapunov exponent occurring in E_j. The first part of the theorem therefore implies that E_j varies real-analytically on μ-almost every stable leaf W^-.

On the other hand, the backward filtration is ordered in the opposite direction. The whole

bundle is obtained at the smallest exponent:

$$\mathrm{E}_j^{\geq \lambda_{j,m_j}} = \mathrm{E}_j.$$

The second part of the theorem therefore implies that E_j varies real-analytically on μ-almost every unstable leaf W^+.

Combining the two statements proves the final assertion. □

2.7.2 Polynomiality along almost every leaf

Fix the index j of the bundle E_j for the duration of this subsection. Consider a point x in our affine manifold $\mathcal{M}$ and a vector v in an unstable direction in $\mathcal{M}$ (note that v lives in the tangent space of $\mathcal{M}$, so the action of the KZ cocycle on it is given by $\mathrm{d}g_t$). Consider the projection $\pi(x,\, v) : \mathrm{E}_x \to \mathrm{E}_x$, which we define by projecting onto E_j at the point $x + v$ and then transporting the result back to E_x via the Gauss–Manin connection. All the used bundles are equivariant, therefore

$$g_{-t}\pi(x,\, v) = \pi(g_{-t}x,\, \mathrm{d}g_{-t}v), \quad \text{or} \quad \pi(x,\, v) = g_t\pi(g_{-t}x,\, \mathrm{d}g_{-t}v). \tag{2.7.1}$$

Lemma 2.7.4. *Suppose that v belongs to a small neighbourhood of x along the unstable leaf, and suppose that, for μ-almost every x, the operator $\pi(x,\, v)$ varies real-analytically in v (note that this follows as a consequence of the previous section). Then, for μ-almost every x, the operator $\pi(x,\, v)$ varies polynomially in v. Moreover, the degrees of the polynomials are uniformly bounded.*

Proof. Suppose that the Lyapunov filtration varies real-analytically in x, and the Oseledets theorem holds. In a fixed coordinate chart of a small neighbourhood of $(x,\, v)$, we have

$$\pi(x, v) = \sum_{\alpha} c_\alpha(x) v^\alpha,$$

where α is a multi-index and $c_\alpha \in \mathrm{End}(\mathrm{E}_x)$ vary measurably varying in x. We can find a subset S of $\mathcal{M}$ of positive measure where the above equality holds.

By Poincaré recurrence theorem, μ-almost every trajectory in this set returns to the set infinitely many times. In each of these times, we can use the decomposition (2.7.1) to obtain two different Taylor expansions:

$$\pi(x,\, v) = \sum_\alpha c_\alpha(x) v^\alpha = \sum_\alpha g_t c_\alpha\left(g_{-t}x\right)\left(dg_{-t}v\right)^\alpha . \tag{2.7.2}$$

Consider the largest Lyapunov exponent Λ of $\mathrm{End}(\mathrm{E})$. Fix a subset $K \subset S$ of positive measure where c_α is bounded above. We can consider only those times t when $g_{-t}x \in K$. Thus, as $t \to \infty$, for any $\epsilon > 0$

$$\left\| g_t c_\alpha\left(g_{-t}x\right)\right\| = o\left(e^{(\Lambda+\epsilon)t}\right).$$

However, for the cocycle $\mathrm{d}g_t$, for any $\epsilon_1 < 1 - \lambda_2$,

$$\left\| dg_{-t}v\right\| = o\left(e^{-t\epsilon_1}\right).$$

Thus

$$\left\| g_t c_\alpha\left(g_{-t}x\right)\left(dg_{-t}v\right)^\alpha\right\| = o\left(e^{t(\Lambda+\epsilon-|\alpha|\epsilon_1)}\right).$$

If $\Lambda + \epsilon - |\alpha|\epsilon_1 < 0$, then along the chosen return times t the right-hand side tends to 0, hence the corresponding coefficient in the Taylor expansion must vanish, i.e. $c_\alpha(x) = 0$. Therefore $c_\alpha(x) = 0$ for all multi-indices with $|\alpha| > \frac{\Lambda+\epsilon}{\epsilon_1}$. Letting $\epsilon \downarrow 0$ and $\epsilon_1 \uparrow (1 - \lambda_2)$ shows that only finitely many c_α are nonzero, so $\pi(x, v)$ is a polynomial in v (with uniformly bounded degree). □

2.7.3 Joint polynomiality

We will require the following

Lemma 2.7.5. *Let $U_i \subset \mathbb{R}n_i$ with $i \in \{1, 2\}$ be connected open products of intervals. Let $f : U_1 \times U_2 \to \mathbb{R}$ be a measurable function. Assume that for a.e. $x_1 \in U_1$, the function $f(x_1, -)$ agrees a.e. with a polynomial in the variable $x_2 \in U_2$. Assume that the same holds for variables swapped.*

Then the function f agrees a.e. with a polynomial in x_1 and x_2.

This result goes back at least to Margulis (see the appendix to the Russian translation of Raghunathan [Rag77]. A proof can be found in [Zim83, p. 55, Theorem 3.4.4] or in [Fil16a, p. 657, Lemma 7.6].

Combining the above Lemma 2.7.5 with Lemma 2.7.4, we prove the next

Theorem 2.7.6. *Let*

$$\mathcal{X} = \mathcal{H}(\alpha_1) \times \cdots \times \mathcal{H}(\alpha_n)$$

be a product of strata, and let $\mathcal{M} \subset \mathcal{X}$ be an affine invariant submanifold. Let $G \subset ({}_2\mathbb{R})^n$ be a closed projection-surjective subgroup, and let μ be an ergodic G-invariant probability measure on $\mathcal{M}$.

Let V be a G-invariant measurable subbundle of the Hodge bundle over $\mathcal{M}$, or of one of its tensor powers, and let Π denote the corresponding Hodge-orthogonal projection operator. In local flat coordinates write

$$(x, y) = ((x_1, \ldots, x_n), (y_1, \ldots, y_n)),$$

where (x_i, y_i) are real period coordinates on the i-th factor. Denote by

$$A_i(x_i, y_i)$$

the area pairing on the i-th factor.

Then, in a flat trivialization, each matrix coefficient of Π agrees a.e. with an element of the localized polynomial ring

$$\mathbb{R}[x_1, \dots, x_n, y_1, \dots, y_n, A_1^{-1}, \dots, A_n^{-1}].$$

Equivalently, for every matrix coefficient π, there exist nonnegative integers $D_1, \dots, D_n$ and a polynomial $P(x, y)$ such that

$$\pi(x, y) = \frac{P(x, y)}{\prod_{i=1}^{n} A_i(x_i, y_i)^{D_i}}$$

away from the divisor $\prod_i A_i = 0$.

Moreover, the numerator P has degree zero with respect to the scaling symmetries actually available from the G-action. More explicitly, if

$$\{1, \dots, n\} = \bigsqcup_{\beta \in \mathcal{B}} B_\beta$$

is the block decomposition of G, so that on each nontrivial block B_β the determinant-one part of G acts diagonally, then P satisfies

$$P\Big((c_i r_\beta x_i)_{i \in B_\beta}, (c_i r_\beta^{-1} y_i)_{i \in B_\beta}\Big)_\beta = \left(\prod_{i=1}^{n} c_i^{2D_i}\right) P(x, y).$$

Here the parameters c_i correspond to the scalar rescaling of the i-th Abelian differential, while r_β corresponds to the Teichmuller geodesic direction on the diagonal block B_β.

In particular, if the action is the full product action, so that the blocks are singletons, then P is bihomogeneous of bidegree (D_i, D_i) in each pair (x_i, y_i). In that case the matrix

coefficients lie in the multihomogeneous degree-zero part of

$$\mathbb{R}[x_1, \ldots, x_n, y_1, \ldots, y_n, A_1^{-1}, \ldots, A_n^{-1}].$$

For a point $x \in \mathbb{R}^N$, denote the "symplectic orthogonal" subspace by $x^\perp = \left\{ y \in \mathbb{R}^N \mid A(x,y) = 0 \right\}$. The function π satisfies $\pi(\lambda x, \nu y) = \pi(x,y)$ for any independent constants λ, ν since the projection to an $\mathrm{SL}_2\mathbb{R}$-invariant subbundle is $\mathrm{SL}_2\mathbb{R}$-equivariant. From Lemma 2.7.4, for a.e. (x,y), the function $\pi(x+v,\, y)$ is a polynomial in $v \in y^\perp$, and the function $\pi(x,\, y+w)$ is a polynomial in $w \in x^\perp$.

Let us illustrate how projection operators work; in particular, how non-trivial projection operators can arise:

Example 2.7.7 (Projection onto the product tautological bundle)**.** We describe the projection onto the tautological subbundle in a product of strata. Write

$$(x,y) = ((x_1, \ldots, x_n), (y_1, \ldots, y_n)),$$

where (x_i, y_i) are the real and imaginary period coordinates in the i-th factor. Let

$$A_i(x_i, y_i)$$

denote the area pairing in the i-th factor.

For each i, the tautological plane in the i-th cohomology bundle is

$$\mathcal{T}_i(x_i, y_i) = \mathrm{span}\{[x_i], [y_i]\} \subset H_i^1.$$

The product tautological subbundle is therefore

$$\mathcal{T}(x,y) = \bigoplus_{i=1}^{n} \mathcal{T}_i(x_i,y_i) \subset \bigoplus_{i=1}^{n} H_i^1.$$

Let

$$v = (v_1,\dots,v_n) \in \bigoplus_{i=1}^{n} H_i^1.$$

The projection $\pi_{\mathcal{T}}(x,y)$ onto $\mathcal{T}(x,y)$, along the symplectic orthogonal complement in each factor, is given by

$$\pi_{\mathcal{T}}(x,y)(v) = \left(\frac{A_1(v_1,y_1)}{A_1(x_1,y_1)}[x_1] + \frac{A_1(x_1,v_1)}{A_1(x_1,y_1)}[y_1], \dots, \frac{A_n(v_n,y_n)}{A_n(x_n,y_n)}[x_n] + \frac{A_n(x_n,v_n)}{A_n(x_n,y_n)}[y_n] \right).$$

Equivalently,

$$\pi_{\mathcal{T}}(x,y)(v) = \sum_{i=1}^{n} \left(\frac{A_i(v_i,y_i)}{A_i(x_i,y_i)}[x_i] + \frac{A_i(x_i,v_i)}{A_i(x_i,y_i)}[y_i] \right),$$

where each summand is understood as lying in the i-th direct summand of $\bigoplus_i H_i^1$.

This operator satisfies

$$\pi_{\mathcal{T}}^2 = \pi_{\mathcal{T}}, \qquad \operatorname{Im} \pi_{\mathcal{T}} = \mathcal{T}(x,y).$$

Its matrix coefficients lie in

$$\mathbb{R}[x_1,\dots,x_n,y_1,\dots,y_n,A_1^{-1},\dots,A_n^{-1}],$$

and the denominators are the factorwise area functions $A_i(x_i,y_i)$, rather than a single total area function.

The formula is invariant under the natural rescalings of the Abelian differential in each factor and under the $\mathrm{SL}_2\mathbb{R}$-action on the corresponding tautological plane. If several factors belong to the same diagonal block of G, the same formula still defines the projection onto the

direct sum of the tautological planes in those factors. The equivariance is then with respect to the common $\mathrm{SL}_2\mathbb{R}$-action on the block, not with respect to independent $\mathrm{SL}_2\mathbb{R}$-actions on the individual factors.

We are now ready to prove Theorem 2.7.6.

Proof. It is enough to prove the claim for one matrix coefficient of the projection operator. We denote such a coefficient by $\pi(x, y)$.

For each factor define

$$y_i^{\perp_i} := \{v_i : A_i(v_i, y_i) = 0\}, \qquad x_i^{\perp_i} := \{w_i : A_i(x_i, w_i) = 0\}.$$

For the product we write

$$y_{\text{prod}}^{\perp} := \bigoplus_{i=1}^{n} y_i^{\perp_i}, \qquad x_{\text{prod}}^{\perp} := \bigoplus_{i=1}^{n} x_i^{\perp_i}.$$

Thus the local stable and unstable leaves are

$$W^+(x, y) = \{(x + v, y) : v_i \in y_i^{\perp_i} \text{ for all } i\},$$

and

$$W^-(x, y) = \{(x, y + w) : w_i \in x_i^{\perp_i} \text{ for all } i\}.$$

These conditions preserve the area in each factor:

$$A_i(x_i + v_i, y_i) = A_i(x_i, y_i), \qquad A_i(x_i, y_i + w_i) = A_i(x_i, y_i).$$

By Lemma 2.7.4, for μ-a.e. (x, y), the restriction of π to $W^+(x, y)$ is polynomial in v, and the restriction of π to $W^-(x, y)$ is polynomial in w. The degree is uniformly bounded.

Choose a point

$$(x^0, y^0) = ((x_1^0, \dots, x_n^0), (y_1^0, \dots, y_n^0))$$

in the full-measure set on which these leafwise polynomiality statements hold, and normalize it so that

$$A_i(x_i^0, y_i^0) = 1 \qquad \text{for every } i.$$

By Fubini, we may choose (x^0, y^0) so that the required leafwise polynomiality statements hold for almost every point in the corresponding stable and unstable families.

Define the product slice

$$S = \left\{(x^0 + v, y^0 + w) : v_i \in (y_i^0)^{\perp_i},\ w_i \in (x_i^0)^{\perp_i} \text{ for all } i\right\}.$$

Equivalently,

$$(x, y) \in S \quad \Longleftrightarrow \quad A_i(x_i, y_i^0) = 1 \ \text{ and } \ A_i(x_i^0, y_i) = 1 \quad \text{for all } i.$$

Fix

$$q = (q_1, \dots, q_n) \in y^0 + (x^0)^{\perp}_{\text{prod}}.$$

Then $A_i(x_i^0, q_i) = 1$ for all i, and the stable leaf through (x^0, q) is

$$W^+(x^0, q) = \{(x^0 + v, q) : A_i(v_i, q_i) = 0 \text{ for all } i\}.$$

On the slice leaf

$$S^+(q) := \{(p, q) \in S\},$$

the inverse to the factorwise normalization map is

$$(p, q) \longmapsto \left(\left(\frac{p_1}{A_1(p_1, q_1)}, \dots, \frac{p_n}{A_n(p_n, q_n)}\right), q\right).$$

Therefore the restriction of π to $S^+(q)$ is polynomial in the variables

$$\frac{p_i}{A_i(p_i, q_i)}, \qquad i = 1, \dots, n.$$

Since the leafwise degrees are uniformly bounded, there exist integers $D_1, \dots, D_n$, independent of q, such that

$$\left(\prod_{i=1}^{n} A_i(p_i, q_i)^{D_i} \right) \pi(p, q)$$

is polynomial in p for almost every q.

The same argument on unstable leaves shows that, for almost every p, the same function

$$F(p, q) := \left(\prod_{i=1}^{n} A_i(p_i, q_i)^{D_i} \right) \pi(p, q)$$

is polynomial in q. By Lemma 2.7.5, F agrees a.e. on S with a polynomial in the joint variables (p, q). Hence, on S,

$$\pi(p, q) = \frac{P_S(p, q)}{\prod_{i=1}^{n} A_i(p_i, q_i)^{D_i}}$$

for some polynomial P_S.

Transporting this expression along the scaling symmetries available in the G-block under consideration gives

$$\pi(x, y) = \frac{P(x, y)}{\prod_{i=1}^{n} A_i(x_i, y_i)^{D_i}}$$

for some polynomial P. The numerator P has precisely the homogeneity forced by those scaling symmetries.

More explicitly, scalar rescaling of the i-th Abelian differential sends

$$(x_i, y_i) \mapsto (c_i x_i, c_i y_i), \qquad A_i(x_i, y_i) \mapsto c_i^2 A_i(x_i, y_i).$$

On a diagonal block B_β, the Teichmuller geodesic direction sends

$$(x_i, y_i) \mapsto (r_\beta x_i, r_\beta^{-1} y_i), \qquad i \in B_\beta,$$

and preserves each A_i. Since π is invariant under these rescalings, the numerator must satisfy

$$P\Big((c_i r_\beta x_i)_{i\in B_\beta}, (c_i r_\beta^{-1} y_i)_{i\in B_\beta}\Big)_\beta = \left(\prod_{i=1}^{n} c_i^{2D_i}\right) P(x, y).$$

This is exactly the asserted degree-zero condition.

If all blocks are singletons, then the parameters $c_i r_i$ and $c_i r_i^{-1}$ are independent. Hence

$$P(\ldots, \lambda_i x_i, \ldots, \nu_i y_i, \ldots) = (\lambda_i \nu_i)^{D_i} P(x, y),$$

so P is bihomogeneous of bidegree (D_i, D_i) in each pair (x_i, y_i). This proves the theorem. $\square$

CHAPTER 3

UNIQUE ERGODICITY OF BRANCHED DOUBLE COVERS OF TRANSLATION SURFACES

This chapter is based on joint work with Elizaveta Shuvaeva.

3.1 Introduction

A translation surface can have one or more ergodic measures for the straight-line flow in the vertical direction. Since the straight-line flow preserves the natural Lebesgue measure, unique ergodicity of the flow implies that the unique invariant probability measure is the normalized Lebesgue measure. Masur [Mas82] and Veech [Vee82] independently proved that almost all (with respect to the Euclidean Lebesgue measure in period coordinates, restricted to the unit-area hypersurface $\mathcal{H}_1(\alpha)$ and normalized to be finite) translation surfaces are uniquely ergodic. Later, Masur [Mas92] came up with a geometric criterion that guaranteed unique ergodicity of a translation surface. This criterion was improved by Cheung–Eskin [CE07b] and Trevino [Tre14]. Cheung–Masur [CM06] construct a quadratic differential whose vertical foliation is uniquely ergodic but whose Teichmüller geodesic diverges, proving that the Masur criterion is not necessary, only sufficient.

For any quadratic differential, the set of non-ergodic directions and the set of non-uniquely ergodic directions each have Hausdorff dimension at most $\frac{1}{2}$. In particular, if $c(\alpha)$ denotes the Hausdorff dimension of the set of non-uniquely ergodic directions for a Masur–Veech generic surface in the stratum $\mathcal{H}(\alpha)$ (called the Masur–Smillie constant), then

$$c(\alpha) \leq \tfrac{1}{2}$$

for every stratum $\mathcal{H}(\alpha)$ [Mas92; MS91]. Cheung [Che03] and Cheung–Hubert–Masur [CHM08] constructed examples where the set of non-ergodic directions has Hausdorff dimension $\frac{1}{2}$,

showing that Masur's upper bound is sharp. Athreya–Chaika [AC15] later proved that in $\mathcal{H}(2)$ the Masur–Smillie constant is $c(2) = \frac{1}{2}$. Chaika–Masur [CM20] showed that the set of non-uniquely ergodic d-IETs has Hausdorff codimension $\frac{1}{2}$.

For the vertical flow on a genus-g translation surface, if the flow is minimal, then it has at most g ergodic invariant probability measures [Kat73]. Note that, for a fixed direction, a translation surface decomposes into minimal components and periodic cylinders. In particular, if the directional flow has a periodic cylinder, then it carries uncountably many ergodic invariant probability measures, one on each periodic orbit. This is why the finiteness theorem of Katok applies not to arbitrary directional flows, but to minimal translation flows. This bound is sharp [Sat75]. Early examples of minimal but non-uniquely ergodic straight-line flows (and interval exchange transformations) are due to Veech [Vee69] (not phrased as an IET), Keane [Kea77], Sataev [Sat75], and Keynes–Newton [KN76] (noticed that Veech's example is an IET). The example of Veech (1969) is of particular interest to us, as it inspired the constructions used in the current chapter. Consider a torus $[0, 1] \times [0, 1]$ and take two

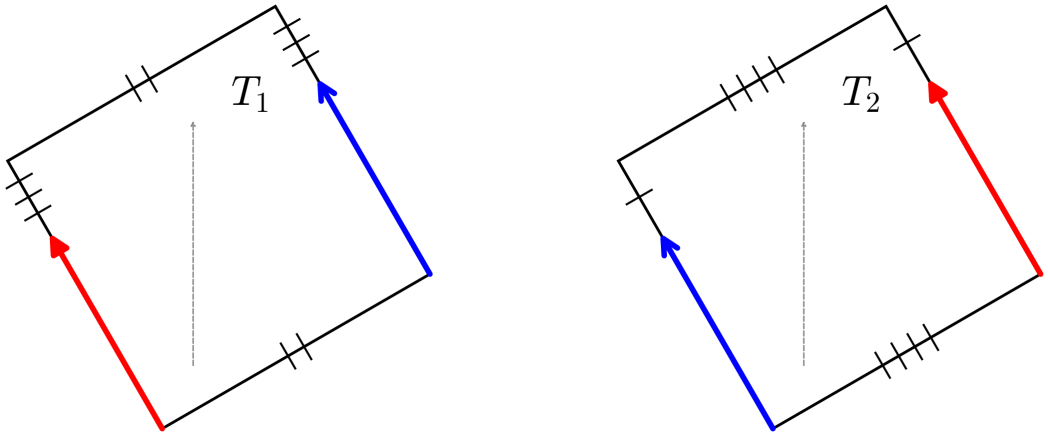


Figure 3.1: Veech's example of a NUE flat surface construction.

copies (sheets). Cut a *slit* from $(0, 0)$ to $(0, \beta)$, where $0 < \beta < 1$. Identify the slit sides across two surfaces. On each sheet, consider the flow generated by the vector $(\alpha, 1)$, where α is *well-approximable*: $\liminf_{n\to\infty} n \cdot d(n\alpha, \mathbb{Z}) = 0$. Veech gives a condition on β (given a fixed α) such that the resulting translation surface is minimal but not uniquely ergodic,

with exactly 2 ergodic invariant measures μ^-, μ^+ and where Lebesgue measure $\mathcal{L}$ satisfies $\mathcal{L} = \frac{1}{2}\left(\mu^- + \mu^+\right)$. This construction was later generalized by Ferenczi and Hubert [FH24], who give a condition for unique ergodicity of $\mathbb{Z}/n\mathbb{Z}$ extensions of a rotation of angle α when the initial interval exchange transformation is linearly recurrent, and there are one or two marked points.

To cut a given surface along a slit and reglue several copies cyclically along the cut is a standard way to build new translation surfaces. This construction is simple to describe, but its dynamical effect is subtle: even when the original surface has uniquely ergodic vertical flow, the resulting branched cover may in principle acquire additional invariant measures, like in the example of Veech. The goal of this chapter is to understand when this does not happen.

To be more precise, let X be a finite-area translation surface with uniquely ergodic vertical flow, and let s be a straight non-vertical slit joining two nonsingular points $P, Q \in X$. Gluing n copies of (X, s) cyclically along the slit produces a branched cyclic n-cover $\widetilde{X}^{(n)}$ of X, branched at P and Q.

Conjecture (Jon Chaika). For Lebesgue-almost every slit on the surface X, the branched cyclic n-cover $\widetilde{X}^{(n)}$ is uniquely ergodic.

This conjecture is proven where $n = 2$ and X is a torus: Cheung and Eskin [CE07a] give a complete characterisation of the set of nonergodic directions in this case. We prove Chaika's conjecture for $n = 2$ and any surface genus. Here, the deck involution exchanging the two sheets strongly constrains the possible ergodic invariant measures. If the cover is not uniquely ergodic, then the involution exchanges the two ergodic measures and, correspondingly, exchanges the sets of generic points for those measures (see Lemma 3.2.1 for details). This observation turns the problem into a quantitative comparison of ergodic averages along carefully chosen orbit segments. To carry this out, in Section 3.3.1, we develop a framework of forward, backward, and total genericity, together with an "almost generic" variant that is

uniform over all 1-Lipschitz test functions.

The main dynamical result of the chapter is a geometric criterion for unique ergodicity of the branched cover (Theorem 3.4.1).

Theorem 3.A. If for only one of the slit endpoints $Q \in X$, there exists a $R > 0$ and a sequence of moments in time $\{t_k\}_{k\in\mathbb{N}_0}$, $\lim_{k\to\infty} t_k = +\infty$, such that the R-neighborhood $\mathrm{U}_{t_k} \subset X_{t_k}$ of $Q_{t_k} = g_{t_k}Q$ is an embedded disk not containing $P_{t_k} = g_{t_k}P$. Then the surface $\widetilde{X}$ is uniquely ergodic.

The proof uses what we call the circle argument: one lifts the embedded disk to the branched cover, constructs symmetric points and long vertical segments inside the lifted neighborhood, and shows that the corresponding time averages must have the same limit. If two distinct ergodic measures existed, the deck symmetry would force these same points to be generic for different measures, producing a contradiction.

We next show that such disks arise for almost every slit endpoint under a natural subsequential embedded-radius hypothesis (see Theorem 3.5.2).

Theorem 3.B. Let X be a finite-area translation surface whose vertical flow is uniquely ergodic, and fix a marked point $P \in X$. Suppose that there exist $r > 0$ and a sequence $t_k \to \infty$ such that

$$r_{\mathrm{emb}}(X_{t_k}) \geq r, \qquad X_{t_k} := g_{t_k}X,$$

where $r_{\mathrm{emb}}(X_{t_k})$ is the lowest upper bound on the radii of disks that can be embedded into X_{t_k}. Then for Lebesgue-almost every point $Q \in X$ there exist a number $R(Q) > 0$ and an infinite subsequence $t_{k_j} \to \infty$ such that, for every j, the surface $X_{t_{k_j}}$ contains an embedded Euclidean disk of radius at least $R(Q)$ centered at $g_{t_{k_j}}(Q)$, and this disk does not contain $g_{t_{k_j}}(P)$.

The remaining task is to understand when the embedded-radius hypothesis holds, which we do in Section 3.5.2. For this, we introduce *pipe cylinders*, maximal flat cylinders whose

height is greater than their circumference. These cylinders are exactly the geometric features that may force long Delaunay edges and obstruct a direct construction of embedded disks. We analyze three complementary regimes. If no pipe cylinders are present (Lemma 3.5.11), Delaunay triangulations immediately provide embedded disks of uniform radius. If pipe cylinders occupy uniformly positive total area along a sequence (Lemma 3.5.12), one can flow one of them forward until it becomes a nearly round cylinder, which again contains a large embedded disk. If pipe cylinders persist but their total area tends to zero (Lemma 3.5.13), then the complement of the singular set and the pipe region still contains points with uniformly controlled embedded neighborhoods. Together, these cases yield a practical geometric route to the hypotheses of our almost-every-slit theorem.

Theorem 3.C. Let X be a finite-area translation surface whose vertical flow is uniquely ergodic. For almost every pair of points (P, Q) on X and for every slit from P to Q, the resulting branched double cover construction is uniquely ergodic.

In other words, unique ergodicity of slit-induced branched covers is governed by a local geometric condition near one branch point along the Teichmüller orbit, while the existence of that local geometry can be verified globally using embedded radius, Delaunay triangulations, and the distribution of pipe cylinders.

This chapter is organized as follows. Section 3.2 introduces the slit-induced branched double-cover construction and the associated invariant-measure dichotomy on the double cover. Section 3.3 develops the theory of generic and almost generic points and records the symmetry properties imposed by the involution. Section 3.4 contains the circle argument and the proof of the main unique ergodicity criterion, first for the double cover and then for cyclic covers of arbitrary degree. Section 3.5 proves the almost-every-slit theorem from a subsequential lower bound on embedded radius and then studies pipe cylinders as a mechanism for producing such lower bounds.

3.2 The setup

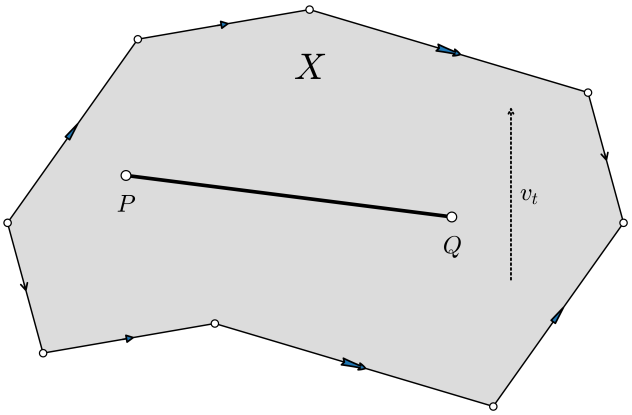


Figure 3.2: An example of a translation surface with a slit.

For the rest of the chapter, let X be a translation surface of finite area with the set of singularities denoted by Σ and with a uniquely ergodic vertical flow v_t, and let s be a slit on the surface X with endpoints P and Q (see Figure 3.5). The slit has to be a straight line, but can have any slope except for the vertical one and is allowed to be of any length. Take two copies $(X_{\mathrm{I}}, s_{\mathrm{I}})$ and $(X_{\mathrm{II}}, s_{\mathrm{II}})$ of the surface (X, s) (see Figure 3.3) and glue them along the slit into a surface $\widetilde{X}$ in the following manner. A point on $\widetilde{X}$ can be written as (S, i), where S is a position on the initial surface X and $i \in \{\mathrm{I}, \mathrm{II}\}$ is the index of the copy. The vertical flow on $\widetilde{X}$ changes the position of the coordinate S as normal and switches the index i of the copy every time the flow hits either of the slits. In other words, the surface $\widetilde{X}$ is a double cover of the surface X branched at two points P and Q. We will denote by $\widetilde{X}_{\mathrm{I}}$ and $\widetilde{X}_{\mathrm{II}}$ the natural projections of X_{I} and X_{II} respectively onto $\widetilde{X}$. See Figure 3.4 for a visual explanation of the concept.

In the future, we will denote by $\pi : \widetilde{X} \to X$ the natural projection from $\widetilde{X}$ to X (note that $\pi(\widetilde{X}_{\mathrm{I}}) = \pi(\widetilde{X}_{\mathrm{II}}) = X$) and by $\varsigma : \widetilde{X} \to \widetilde{X}$ the involution that leaves intact the X-coordinate of any point on $\widetilde{X}$ but exchanges the copy: $\varsigma(\widetilde{X}_{\mathrm{I}}) = \widetilde{X}_{\mathrm{II}}$. This involution has two fixed points, which are the endpoints $\widetilde{P} = \pi^{-1}P$ and $\widetilde{Q} = \pi^{-1}Q$ of the slit. We will also denote the forward vertical flow on $\widetilde{X}$ by $\tilde{v}_t$ and the backward vertical flow on $\widetilde{X}$ by $\tilde{v}_t^- = \tilde{v}_{-t}$.

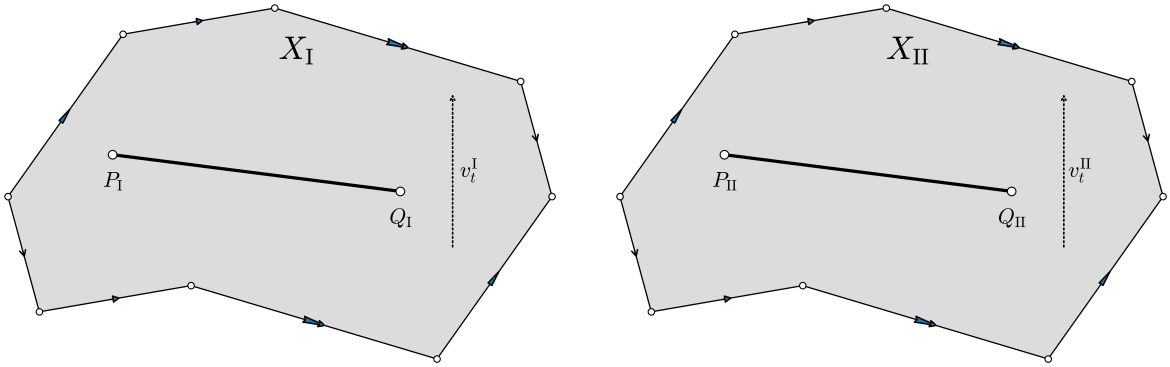


Figure 3.3: Two copies X_{I} and X_{II} of the translation surface X.

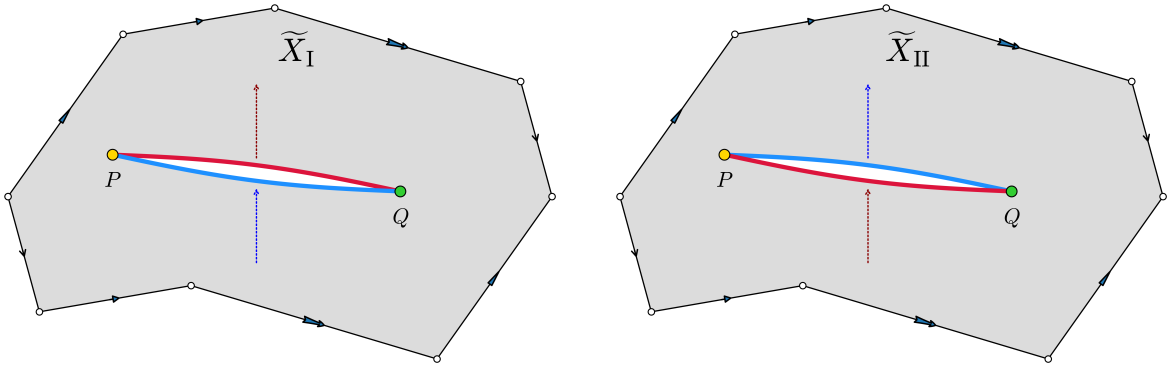


Figure 3.4: Construction of the double-cover surface $\widetilde{X}$ and visualization of its parts $\widetilde{X}_{\mathrm{I}}$ and $\widetilde{X}_{\mathrm{II}}$.

In local flat coordinates $z = x + iy$ coming from ω, the *Teichmüller flow* (g_t) is defined by post-composing all charts with the linear map

$$(x, y) \longmapsto (e^t x,\, e^{-t} y).$$

Equivalently, g_t stretches the horizontal direction by e^t and contracts the vertical direction by e^{-t}, preserving area. Note that ergodic properties of translation surfaces are invariant under the action of g_t. We set $X_t = g_t(X)$ and $\widetilde{X}_t = g_t(\widetilde{X})$.

Lemma 3.2.1. *The double-cover surface $\widetilde{X}$ either has only one probability ergodic measure $\mathcal{L}_{\widetilde{X}}$ (the Lebesgue measure), or it has two probability measures μ and ν such that*

1. *Both μ and ν are absolutely continuous with respect to $\mathcal{L}_{\widetilde{X}}$;*

2. $\mu(A) = \nu(\varsigma(A))$ *for any Borel subset* $A \subset \widetilde{X}$;

3. $\mu + \nu = 2\mathcal{L}_{\widetilde{X}}$.

Proof. First, we will show that any ergodic probability measure μ on $\widetilde{X}$ projects to an ergodic probability measure on X, that is, its pushforward $\pi_*\mu$ coincides with the Lebesgue measure $\mathcal{L}_X$ on X. Consider $B \subset X$ invariant under the vertical flow (so we have $v_{-t}B = B$ for all t). Clearly, the preimage $\pi^{-1}B \subset \widetilde{X}$ is $\widetilde{v}_t$-invariant:

$$\widetilde{v}_{-t}(\pi^{-1}B) = \pi^{-1}(v_{-t}B) = \pi^{-1}B.$$

Since μ is ergodic on $\widetilde{X}$, the measure $\mu(\pi^{-1}B)$ is either 0 or 1. By the definition of a pushforward $\pi_*\mu(B) = \mu(\pi^{-1}B)$. So $\pi_*\mu(B)$ is also either 0 or 1, which means exactly that $\pi_*\mu$ is ergodic with respect to v_t on X. There is only one ergodic probability measure on X, so $\pi_*\mu$ has to coincide with $\mathcal{L}_X$.

Now we will prove that μ is absolutely continuous with respect to $\mathcal{L}_{\widetilde{X}}$. If a Borel subset $A \subset \widetilde{X}$ has $\mathcal{L}_{\widetilde{X}}(A) = 0$, then

$$\mu(A) \leq \mu(\pi^{-1}(\pi(A))) = \pi_*\mu(\pi(A)) = \mathcal{L}_X(\pi(A)),$$

which is clearly zero, being a projection of a set of Lebesgue measure zero.

Suppose now for contradiction that there at least three ergodic probability measures μ_1, μ_2 and μ_3 on $\widetilde{X}$. It is well-known that distinct ergodic probability measures have to be mutually singular. In particular, there exists a Borel subset $A_1 \subset \widetilde{X}$ such that $\mu_1(A_1) = 1$ while $\mu_2(A_1) = 0$, and there exists a Borel subset $B_1 \subset \widetilde{X}$ such that $\mu_1(B_1) = 1$ while $\mu_3(B_1) = 0$. The intersection set $E_1 = A_1 \cap B_1$ has $\mu_1(E_1) = 1$ and $\mu_2(E_1) = \mu_3(E_1) = 0$. We can find a similar set for the second measure μ_2: a set G_2 with $\mu_2(G_2) = 1$ and $\mu_1(G_2) = \mu_3(G_2) = 0$. The complement $E_2 = G_2 \setminus E_1$ will satisfy the same equalities and be disjoint from E_1. Finally, the set $E_3 = \widetilde{X} \setminus (E_1 \cup E_2)$ has $\mu_3(E_3) = 1$ and $\mu_1(E_3) = \mu_2(E_3) = 0$. So we have

built three pairwise disjoint sets, each of which carries all of one measure, and is invisible to the other two.

Each of the three projections $\pi(E_1)$, $\pi(E_2)$, $\pi(E_3)$ has full Lebesgue measure on X, since we know that all three measures project to the Lebesgue measure on X: $\pi_*\mu_1 = \pi_*\mu_2 = \pi_*\mu_3 = \mathcal{L}_X$:

$$1 = \mu_j(E_j) \leq \mu_j(\pi^{-1}(\pi(E_j))) = \pi_*\mu_j(\pi(E_j)) = \mathcal{L}_X(\pi(E_j)).$$

Therefore

$$\mathcal{L}_X(\pi(E_1) \cap \pi(E_2) \cap \pi(E_3)) = 1.$$

In particular, this intersection is nonempty, so there is some point $x \in \pi(E_1) \cap \pi(E_2) \cap \pi(E_3)$ such that π^{-1} contains at least three distinct points, which is not possible. So there can be at most two invariant ergodic probability measures on $\widetilde{X}$.

Suppose now that there are two distinct measures μ and ν (as above) on $\widetilde{X}$. Since the involution ς commutes with the vertical flow, the pushforward $\varsigma_*\mu$ of an invariant ergodic probability measure μ is again an invariant ergodic probability measure. So either $\varsigma_*\mu = \mu$ or $\varsigma_*\mu = \nu$.

If $\varsigma_*\mu = \mu$, then for every ς-invariant Borel $A \subset \widetilde{X}$ we have $\mu(A) = \mathcal{L}_{\widetilde{X}}(A)$. Such a set coincides with the full preimage of its projection, so

$$\mu(A) = \mu(\pi^{-1}(\pi(A))) = \pi_*\mu(\pi(A)) = \mathcal{L}_X(\pi(A)).$$

Since the normalized Lebesgue measure $\mathcal{L}_{\widetilde{X}}$ also projects to the normalized Lebesgue measure $\mathcal{L}_X$, we get $\mu(A) = \mathcal{L}_{\widetilde{X}}(A)$. So μ agrees with the Lebesgue measure upstairs on all ς-invariant Borel subsets of $\widetilde{X}$. Since both μ and $\mathcal{L}_{\widetilde{X}}$ are ς-invariant, they have to agree on all Borel subsets of $\widetilde{X}$. Indeed, let us consider an arbitrary Borel $B \subset \widetilde{X}$. Both $B \cup \varsigma(B)$ and $B \cap \varsigma(B)$ are ς-invariant, so μ and $\mathcal{L}_{\widetilde{X}}$ coincide on both of these subsets. A simple exclusion-inclusion

formula then gives us

$$2\mu(B) = \mu(B \cup \varsigma(B)) + \mu(B \cap \varsigma(B)) = \mathcal{L}_{\widetilde{X}}(B \cup \varsigma(B)) + \mathcal{L}_{\widetilde{X}}(B \cap \varsigma(B)) = 2\mathcal{L}_{\widetilde{X}}(B). \quad (3.2.1)$$

So if $\varsigma_*\mu = \mu$, the measure μ has to coincide with $\mathcal{L}_{\widetilde{X}}$, and so does ν. This contradicts the assumption that μ and ν are distinct. Thus, if $\widetilde{X}$ has two distinct invariant ergodic probability measures, the involution ς has to exchange them.

Finally, we observe that the measure $\frac{\mu+\nu}{2}$ is another ς-invariant probability measure on $\widetilde{X}$ that is also invariant under the vertical flow, so by following the same as above, we conclude that it has to be the same as $\mathcal{L}_{\widetilde{X}}$. □

This leads to a natural question:

When does $\widetilde{X}$ have one ergodic measure and when does it have two?

3.3 Measure-generic and almost measure-generic points on $\widetilde{X}$

Lemma 3.3.1. *Consider the inverse vertical flow $\widetilde{v}_t^- = \widetilde{v}_{-t}$ on the translation surface $\widetilde{X}$. Any measure on $\widetilde{X}$ that is ergodic and invariant under the action of the flow $\widetilde{v}_t^-$ is ergodic and invariant under the action of the flow $\widetilde{v}_t$, and vice versa.*

Proof. The argument in both directions is essentially identical, so it suffices to give a proof of only one of them. Suppose that a measure μ is ergodic and invariant on $\widetilde{X}$ under the action of the flow $\widetilde{v}_t$. We will show that it is also invariant and ergodic under the action of the flow $\widetilde{v}_t^-$. Let A be a Borel subset of $\widetilde{X}$, and let us fix a time $t > 0$. To prove invariance, we note that

$$\mu(\widetilde{v}_t^- A) = \mu(\widetilde{v}_t(\widetilde{v}_t^- A)) = \mu(A). \quad (3.3.1)$$

To prove ergodicity, consider a Borel $A \subset \widetilde{X}$ that is invariant under the inverse vertical flow $\widetilde{v}_t^-$. This means that $\widetilde{v}_t^- A = A$, that is, $\widetilde{v}_{-t}(A) = A$. Applying $\widetilde{v}_t$ to both sides, we get

$A = \widetilde{v}_t A$, so A is also invariant under the original vertical flow. Since μ is ergodic for $\widetilde{v}_t$, the measure $\mu(A) \in \{0, 1\}$. Hence μ is ergodic for $\widetilde{v}_t^-$ as well. □

3.3.1 Measure-generic points on $\widetilde{X}$

Let $\widetilde{x}$ be a non-singular point on $\widetilde{X}$, let f be a measurable function on $\widetilde{X}$, and let $t > 0$. We introduce the following notation:

$$I_t^{\uparrow}(f, \widetilde{x}) := \frac{1}{t}\int_0^t f(\widetilde{v}_s\widetilde{x})\,\mathrm{d}s \qquad \text{and} \qquad I_t^{\downarrow}(f, \widetilde{x}) := \frac{1}{t}\int_0^t f(\widetilde{v}_{-s}\widetilde{x})\,\mathrm{d}s. \tag{3.3.2}$$

Let m be an ergodic $\widetilde{v}$-invariant probability measure on $\widetilde{X}$. Suppose that the function f is m-integrable on $\widetilde{X}$. Denote

$$\mathrm{Av}_m(f) := \int_{\widetilde{X}} f(\widetilde{x})\,\mathrm{d}m(\widetilde{x}). \tag{3.3.3}$$

Let $\mathcal{C}(\widetilde{X})$ be the space of all real-valued continuous functions on $\widetilde{X}$. Since $\widetilde{X}$ is compact, any $f \in \mathcal{C}(\widetilde{X})$ is bounded, hence integrable with respect to any finite regular Borel measure m on $\widetilde{X}$. Thus, any such measure produces a well-defined linear functional

$$\begin{aligned} \Lambda_m : \mathcal{C}(\widetilde{X}) &\longrightarrow \mathbb{R} \\ f &\longmapsto \mathrm{Av}_m(f) \end{aligned} \tag{3.3.4}$$

If $\widetilde{X}$ carries two distinct ergodic measures, then these averages can be used to distinguish between them.

Lemma 3.3.2. *Suppose that the translation surface $\widetilde{X}$ has two distinct ergodic invariant measures μ and ν. Then there exists a 1-Lipschitz function $\mathfrak{h}$ on $\widetilde{X}$ such that $\mathrm{Av}_\mu(\mathfrak{h}) \neq \mathrm{Av}_\nu(\mathfrak{h})$.*

Proof. Note that $\widetilde{X}$ is a compact metric space, so every Lipschitz function on $\widetilde{X}$ is bounded, and, consequently, integrable with respect to both probability measures μ and ν. Moreover,

since μ and ν are distinct finite regular Borel measures on $\widetilde{X}$, the Riesz–Markov–Kakutani Representation Theorem implies that they cannot define the same continuous linear functional on $\mathcal{C}(\widetilde{X})$. Therefore, there exists an $\mathfrak{f} \in \mathcal{C}(\widetilde{X})$ such that

$$\mathrm{Av}_\mu(\mathfrak{f}) \neq \mathrm{Av}_\nu(\mathfrak{f}). \tag{3.3.5}$$

Let $\mathrm{Lip}(\widetilde{X})$ and $\mathrm{Lip}_1(\widetilde{X})$ be the spaces of all Lipschitz functions and 1-Lipschitz functions on $\widetilde{X}$, respectively. Clearly, $\mathrm{Lip}_1(\widetilde{X}) \subset \mathrm{Lip}(\widetilde{X}) \subset \mathcal{C}(\widetilde{X})$. We now recall the Stone–Weierstrass Theorem.

Proposition 3.3.3 (The Stone–Weierstrass Theorem — Theorem 8.1, p. 145, in [Con90])**.** *Let $\mathcal{A}$ be a subalgebra of the space of continuous real-valued functions $\mathcal{C}(K)$ on a compact Hausdorff space K. If $\mathcal{A}$ contains all the constant functions and separates points (that is, for every distinct pair $x, y \in K$ there is some $f \in \mathcal{A}$ with $f(x) \neq f(y)$), then $\mathcal{A}$ is dense in $\mathcal{C}(K)$ in the standard supremum norm.*

It follows from the definition of a Lipschitz function that $\mathrm{Lip}(\widetilde{X})$ is closed under addition and scalar multiplication, and since $\widetilde{X}$ is compact, a product of Lipschitz functions is also Lipschitz. Hence, $\mathrm{Lip}(\widetilde{X})$ is a subalgebra in $\mathcal{C}(K)$. Obviously, all constant functions belong to $\mathrm{Lip}(\widetilde{X})$. In addition, $\mathrm{Lip}(\widetilde{X})$ separates points: the function that measures the distance from any point to a fixed point $\widetilde{x}$ is Lipschitz and gives different values on $\widetilde{x}$ itself and any other point $\widetilde{y}$. Therefore, $\mathrm{Lip}(\widetilde{X})$ is dense in $\mathcal{C}(K)$ in the standard supremum norm. Thus, there is a sequence $\{h_j\}_{j\in\mathbb{N}_+}$ with $h_j \in \mathrm{Lip}(\widetilde{X})$ (with Lipschitz constants $\{C_j\}_{j\in\mathbb{N}_+}$) that converges uniformly to $\mathfrak{f}$.

Let

$$\Delta = |\,\mathrm{Av}_\mu(\mathfrak{f}) - \mathrm{Av}_\nu(\mathfrak{f})| > 0 \tag{3.3.6}$$

Since $h_j \to \mathfrak{f}$ uniformly, there exists some $\mathfrak{j} \in \mathbb{N}_+$ such that for all $j \geq \mathfrak{j}$,

$$||h_j - \mathfrak{f}||_\infty < \frac{\Delta}{4}. \tag{3.3.7}$$

Since μ and ν are probability measures on $\widetilde{X}$, we have

$$\begin{aligned} |\operatorname{Av}_\mu(h_j) - \operatorname{Av}_\mu(\mathfrak{f})| = |\int_{\widetilde{X}} (h_j - \mathfrak{f})\, \mathrm{d}\mu| \leq \\ \leq \int_{\widetilde{X}} |(h_j - \mathfrak{f})|\, \mathrm{d}\mu \leq \\ \leq ||(h_j - \mathfrak{f})||_\infty. \end{aligned} \tag{3.3.8}$$

Similarly,

$$|\operatorname{Av}_\nu(h_j) - \operatorname{Av}_\nu(\mathfrak{f})| \leq ||(h_j - \mathfrak{f})||_\infty \tag{3.3.9}$$

By the triangle inequality,

$$\begin{aligned} |\operatorname{Av}_\mu(h_j) - \operatorname{Av}_\nu(h_j)| \geq \\ \geq |\operatorname{Av}_\mu(\mathfrak{f}) - \operatorname{Av}_\nu(\mathfrak{f})| - |\operatorname{Av}_\mu(h_j) - \operatorname{Av}_\mu(\mathfrak{f})| - |\operatorname{Av}_\nu(h_j) - \operatorname{Av}_\nu(\mathfrak{f})| \geq \\ \geq \Delta - 2||(h_j - \mathfrak{f})||_\infty > \frac{\Delta}{2} > 0. \end{aligned} \tag{3.3.10}$$

Thus, for any $j \geq \mathfrak{j}$ we have

$$\operatorname{Av}_\mu(h_j) \neq \operatorname{Av}_\nu(h_j). \tag{3.3.11}$$

Fix any $j \geq \mathfrak{j}$, for example, consider the function $h_\mathfrak{j}$. Its Lipschitz constant is some number $C_\mathfrak{j}$. Let us rescale $h_\mathfrak{j}$ and look at

$$\mathfrak{h} := \frac{h_\mathfrak{j}}{\max\{1, C_\mathfrak{j}\}}. \tag{3.3.12}$$

The function $\mathfrak{h}$ is 1-Lipschitz, and

$$\mathrm{Av}_\mu(\mathfrak{h}) - \mathrm{Av}_\nu(\mathfrak{h}) = \frac{1}{\max\{1, C_{\mathfrak{j}}\}}(\mathrm{Av}_\mu(h_{\mathfrak{j}}) - \mathrm{Av}_\nu(h_{\mathfrak{j}})) \neq 0. \tag{3.3.13}$$

So, $\mathfrak{h} \in \mathrm{Lip}_1(\widetilde{X})$ and $\mathrm{Av}_\mu(\mathfrak{h}) \neq \mathrm{Av}_\nu(\mathfrak{h})$. □

We now recall one of the fundamental ergodic theorems.

Theorem 3.3.4 (Birkhoff–Khinchin Pointwise Ergodic Theorem for Flows — Corollary 25.9, p. 517, in [Kal21])**.** *Let $(X, \mathcal{B}, m)$ be a probability space and let $\{\varphi_t\}_{t\in\mathbb{R}}$ be a jointly-measurable, m-preserving flow (i.e. $\varphi_{t+s} = \varphi_t \circ \varphi_s$, $\varphi_0 = \mathrm{id}$, and $m(\varphi_t^{-1}A) = m(A)$ for all $A \in \mathcal{B}$ and $t \in \mathbb{R}$). Then, for m-almost every $x \in X$, the integrals*

$$\frac{1}{T}\int_0^T f(\varphi_t(x))\,\mathrm{d}t$$

converge to the conditional expectation of f with respect to the σ-algebra $\mathcal{I}$ of $\{\varphi_t\}$-invariant sets

$$\mathbb{E}(f \,|\, \mathcal{I})\,(x),$$

and the convergence also holds in $L^1(m)$. In particular, if the flow is ergodic, then for every $f \in L^1(m)$,

$$\lim_{T\to\infty} \frac{1}{T}\int_0^T f(\varphi_t(x))\,\mathrm{d}t \;=\; \mathrm{Av}_m(f) \quad \text{for } m\text{-almost every } x.$$

In our notation (if we apply this to the vertical flow),

$$\lim_{T\to\infty} I_T^{\uparrow}(f, \widetilde{x}) = \mathrm{Av}_m(f).$$

As an immediate consequence of Lemma 3.3.1, we obtain the following.

Corollary 3.3.5. *Let m be an ergodic invariant measure on $\widetilde{X}$. Then the conclusion of Theorem 3.3.4 holds for both the forward and backward vertical flows on $\widetilde{X}$.*

We now define the relevant notions of generic behavior of points under the vertical flow.

Definition 3.3.6 (Generic points)**.** Let m be an ergodic invariant measure on $\widetilde{X}$, and let $\widetilde{x} \in \widetilde{X}$ be a non-singular point whose backward and forward vertical trajectories are defined for all times $t \in \mathbb{R}$.

We say that

- $\widetilde{x}$ is *forward m-generic* if for any $f \in \mathcal{C}(\widetilde{X})$ we have

$$\lim_{T\to\infty} I_T^{\uparrow}(f, \widetilde{x}) = \mathrm{Av}_m(f); \tag{3.3.14}$$

- $\widetilde{x}$ is *backward m-generic* if for any $f \in \mathcal{C}(\widetilde{X})$ we have

$$\lim_{T\to\infty} I_T^{\downarrow}(f, \widetilde{x}) = \mathrm{Av}_m(f); \tag{3.3.15}$$

- $\widetilde{x}$ is *forward generic* if $\widetilde{X}$ has multiple ergodic measures and $\widetilde{x}$ is forward m-generic for at least one of them;
- $\widetilde{x}$ is *backward generic* if $\widetilde{X}$ has multiple ergodic measures and $\widetilde{x}$ is backward m-generic for at least one of them;
- $\widetilde{x}$ is *totally m-generic* if it is both forward and backward m-generic for an ergodic invariant measure m on $\widetilde{X}$;
- $\widetilde{x}$ is *totally generic* if $\widetilde{X}$ has multiple ergodic measures and the point is both backward and forward m-generic for at least one of them.

Corollary 3.3.7. *If a non-critical point $\widetilde{x}$ on $\widetilde{X}$ is backwards/forwards/totally μ-generic, then it cannot be backwards/forwards/totally ν-generic, and vice versa.*

Proof. We first prove the forwards statement. Suppose, for contradiction, that $\widetilde{x}$ is both forwards μ-generic and forwards ν-generic. By Lemma 3.3.2, there exists a 1-Lipschitz function $h\colon \widetilde{X} \to \mathbb{R}$ such that

$$\mathrm{Av}_\mu(h) \neq \mathrm{Av}_\nu(h).$$

Since $\widetilde{X}$ is compact, every Lipschitz function on $\widetilde{X}$ is bounded, hence $h \in L^1(\mu) \cap L^1(\nu)$.

Because $\widetilde{x}$ is forwards μ-generic, we have

$$\lim_{T\to\infty} I_T^{\uparrow}(h, \widetilde{x}) = \mathrm{Av}_\mu(h).$$

Because the same point $\widetilde{x}$ is also forwards ν-generic, we likewise have

$$\lim_{T\to\infty} I_T^{\uparrow}(h, \widetilde{x}) = \mathrm{Av}_\nu(h).$$

The left-hand sides are the same limit, so we must have

$$\mathrm{Av}_\mu(h) = \mathrm{Av}_\nu(h),$$

contrary to the choice of h. This contradiction shows that no point can be forwards μ-generic and forwards ν-generic simultaneously.

The backwards statement is proved in exactly the same way, replacing the forwards averages $I_T^{\uparrow}(h, \widetilde{x})$ by the backwards averages $I_T^{\downarrow}(h, \widetilde{x})$.

Finally, if $\widetilde{x}$ were totally μ-generic and totally ν-generic, then in particular it would be forwards μ-generic and forwards ν-generic, which we have just shown to be impossible. Hence $\widetilde{x}$ cannot be totally μ-generic and totally ν-generic at the same time.

The statement with μ and ν interchanged is identical. □

Corollary 3.3.8. *If a non-critical point $\widetilde{x} \in \widetilde{X}$ is forward, backward, or totally μ-generic, then $\varsigma(\widetilde{x})$ is, respectively, forward, backward, or totally ν-generic, and vice versa.*

Proof. Assume that $\widetilde{x}$ is forward μ-generic. Fix $f \in \mathcal{C}(\widetilde{X})$. Since the involution ς commutes with the vertica flow, for every $T > 0$ we have

$$I_T^{\uparrow}(f, \varsigma(\widetilde{x})) = I_T^{\uparrow}(f \circ \varsigma, \widetilde{x}) \tag{3.3.16}$$

Since $f \circ \varsigma \in \mathcal{C}(\widetilde{X})$ (and $\widetilde{x}$ is forward μ-generic),

$$\lim_{T\to\infty} I_T^{\uparrow}(f \circ \varsigma, \widetilde{x}) = \mathrm{Av}_\mu(f \circ \varsigma), \tag{3.3.17}$$

and

$$\lim_{T\to\infty} I_T^{\uparrow}(f, \varsigma(\widetilde{x})) = \mathrm{Av}_\mu(f \circ \varsigma). \tag{3.3.18}$$

Since, $\varsigma_*\mu = \nu$,

$$\mathrm{Av}_\mu(f \circ \varsigma) = \mathrm{Av}_\nu(f). \tag{3.3.19}$$

So, $\varsigma(\widetilde{x})$ is forward ν-generic.

The proof for the backward direction is identical, and the conclusion for totally generic points follows immediately. □

Corollary 3.3.9. *The set of points in $\widetilde{X}$ that are either simultaneously forward μ-generic and backward ν-generic, or simultaneously forward ν-generic and backward μ-generic, has Lebesgue measure zero.*

Proof. Let $F_\mu, F_\nu \subset \widetilde{X}$ be the sets of forward μ-generic and forward ν-generic points, respectively. Likewise, let $B_\mu, B_\nu \subset \widetilde{X}$ be the sets of backward μ-generic and backward ν-generic points, respectively. By the Birkhoff–Khinchin Pointwise Ergodic Theorem for flows

$$\mu(F_\mu) = \mu(B_\mu) = 1 \text{ and } \nu(F_\nu) = \nu(B_\nu) = 1. \tag{3.3.20}$$

Then

$$\mu(F_\mu \cap B_\mu) = 1 \text{ and } \nu(F_\nu \cap B_\nu) = 1. \tag{3.3.21}$$

So if

$$G = \{\widetilde{x} \in \widetilde{X} | \widetilde{x} \in F_\mu \cap B_\mu \text{ or } \widetilde{x} \in F_\nu \cap B_\nu\}, \tag{3.3.22}$$

then $\mathcal{L}_{\widetilde{X}}(G) = 1$, and $\mathcal{L}_{\widetilde{X}}(\widetilde{X} \setminus G) = 0$. □

Corollary 3.3.10. *Let m be an ergodic invariant measure on $\widetilde{X}$. The set of totally m-generic points in $\widetilde{X}$ has full m-measure.*

Lemma 3.3.11. *Fix $\tau \in \mathbb{R}$. Let $\widetilde{X}_\tau = g_\tau \widetilde{X}$ be the image of $\widetilde{X}$ under the Teichmüller flow, and let $\widetilde{v}_s^{(\tau)}$ denote the vertical flow on $\widetilde{X}_\tau$. Let m be an ergodic $\widetilde{v}_s$-invariant probability measure on $\widetilde{X}$, and define*

$$m_\tau := (g_\tau)_* m.$$

Then m_τ is an ergodic $\widetilde{v}_s^{(\tau)}$-invariant probability measure on $g_\tau \widetilde{X}$.

Moreover, if $\widetilde{x} \in \widetilde{X}$ is non-critical, then

$$\widetilde{x} \text{ is forward/backward/totally } m\text{-generic on } \widetilde{X}$$

if and only if

$$g_\tau \widetilde{x} \text{ is forward/backward/totally } m_\tau\text{-generic on } \widetilde{X}_\tau.$$

Proof. Since the Teichmüller map

$$g_\tau = \begin{pmatrix} e^\tau & 0 \\ 0 & e^{-\tau} \end{pmatrix}$$

contracts the vertical direction by the factor $e^{-\tau}$, the vertical flows on $\widetilde{X}$ and on $\widetilde{X}_\tau$ are

related by

$$\widetilde{v}_s^{(\tau)} \circ g_\tau = g_\tau \circ \widetilde{v}_{e^\tau s} \qquad \text{for all } s \in \mathbb{R}. \tag{3.3.23}$$

In particular, g_τ sends non-critical points to non-critical points.

We first show that m_τ is invariant under $\widetilde{v}_s^{(\tau)}$. Let $A \subset \widetilde{X}_\tau$ be Borel. Then, using (3.3.23),

$$m_\tau\Big((\widetilde{v}_s^{(\tau)})^{-1}A\Big) = m\Big(g_\tau^{-1}((\widetilde{v}_s^{(\tau)})^{-1}A)\Big) = m\Big((\widetilde{v}_{e^\tau s})^{-1}(g_\tau^{-1}A)\Big) = m(g_\tau^{-1}A) = m_\tau(A),$$

because m is $\widetilde{v}_s$-invariant.

Next we show ergodicity. Let $A \subset \widetilde{X}_\tau$ be a Borel set such that

$$(\widetilde{v}_s^{(\tau)})^{-1}(A) = A \qquad \text{for all } s \in \mathbb{R}.$$

Then (3.3.23) implies

$$\widetilde{v}_t^{-1}(g_\tau^{-1}A) = g_\tau^{-1}A \qquad \text{for all } t \in \mathbb{R},$$

since every $t \in \mathbb{R}$ can be written as $t = e^\tau s$. Thus $g_\tau^{-1}A$ is $\widetilde{v}_t$-invariant on $\widetilde{X}$. By ergodicity of m,

$$m(g_\tau^{-1}A) \in \{0, 1\}.$$

Therefore

$$m_\tau(A) = m(g_\tau^{-1}A) \in \{0, 1\},$$

so m_τ is ergodic.

Now suppose that $\widetilde{x}$ is forward m-generic, and let $f \in L^1(m_\tau)$. Then $f \circ g_\tau \in L^1(m)$, and by (3.3.23),

$$\frac{1}{T}\int_0^T f\Big(\widetilde{v}_s^{(\tau)}(g_\tau \widetilde{x})\Big)\, ds = \frac{1}{T}\int_0^T f\Big(g_\tau(\widetilde{v}_{e^\tau s}\widetilde{x})\Big)\, ds.$$

Making the change of variables $u = e^{\tau}s$, so that $ds = e^{-\tau}du$, we get

$$\frac{1}{T}\int_0^T f\left(\widetilde{v}_s^{(\tau)}(g_\tau \widetilde{x})\right) ds = \frac{1}{e^{\tau}T}\int_0^{e^{\tau}T} (f \circ g_\tau)(\widetilde{v}_u \widetilde{x})\, du.$$

Since $\widetilde{x}$ is forward m-generic, the right-hand side converges, as $T \to \infty$, to

$$\int_{\widetilde{X}} f \circ g_\tau \, dm = \int_{g_\tau \widetilde{X}} f \, d(g_\tau)_* m = \int_{g_\tau \widetilde{X}} f \, dm_\tau.$$

Hence $g_\tau \widetilde{x}$ is forward m_τ-generic.

The backward case is identical:

$$\frac{1}{T}\int_0^T f\left(\widetilde{v}_{-s}^{(\tau)}(g_\tau \widetilde{x})\right) ds = \frac{1}{e^{\tau}T}\int_0^{e^{\tau}T} (f \circ g_\tau)(\widetilde{v}_{-u} \widetilde{x})\, du,$$

so backward m-genericity of $\widetilde{x}$ implies backward m_τ-genericity of $g_\tau \widetilde{x}$.

Thus total m-genericity of $\widetilde{x}$ implies total m_τ-genericity of $g_\tau \widetilde{x}$.

Finally, the converse implications follow by applying the same argument to the inverse Teichmüller map $g_{-\tau} : \widetilde{X_\tau} \to \widetilde{X}$. □

Measure-generic points are clearly invariant under the action of the vertical flow. This fact is useful in many ways.

Lemma 3.3.12. *For any horizontal interval I on our translation surface $\widetilde{X}$, the set of forward generic points in I has full Lebesgue measure $\mathcal{L}_I$. The same holds for backward generic points and totally generic points.*

Proof. Suppose for contradiction that the set $\widetilde{X} \setminus (F_\mu \cup F_\nu)$ of points that are not forward generic intersects I in a set of positive $\mathcal{L}_I$-measure. Since the set of singularities $\widetilde{\Sigma}$ is finite, we can consider the set $I^\circ = I \setminus \widetilde{\Sigma}$. Clearly, I° has full $\mathcal{L}_I$-measure. Since I° contains no singular point, there exists some number $\varepsilon^\circ > 0$ such that the vertical trajectory V° of I°

up to time ε° contains no points of $\widetilde{\Sigma}$. Since the set $\widetilde{X} \setminus (F_\mu \cup F_\nu)$ is invariant under the vertical flow,

$$\mathcal{L}_{\widetilde{X}}(V^\circ) = \varepsilon^\circ \cdot \mathcal{L}_I(I^\circ). \tag{3.3.24}$$

and

$$\mathcal{L}_{\widetilde{X}}(V^\circ \cap (\widetilde{X} \setminus (F_\mu \cup F_\nu))) = \varepsilon^\circ \cdot \mathcal{L}_I(I^\circ \cap (\widetilde{X} \setminus (F_\mu \cup F_\nu))). \tag{3.3.25}$$

So if $\mathcal{L}_I(I^\circ \cap (\widetilde{X} \setminus (F_\mu \cup F_\nu))) \neq 0$, then $\mathcal{L}_{\widetilde{X}}(V^\circ \cap (\widetilde{X} \setminus (F_\mu \cup F_\nu))) \neq 0$, contradicting the fact that $\mathcal{L}_{\widetilde{X}}(\widetilde{X} \setminus (F_\mu \cup F_\nu)) = 0$.

The proof for the backward direction is identical. The set of totally generic points in I is then an intersection of two sets of full $\mathcal{L}_I$-measure, so it has full $\mathcal{L}_I$-measure as well. □

Lemma 3.3.13. *Consider a horizontal interval I on our translation surface $\widetilde{X}$. There exists a pair of points, symmetric with respect to the center of I, that are both generic. The same holds for totally generic points.*

Proof. Suppose not. The reflection $\mathfrak{r}$ across the center of the interval preserves the Lebesgue measure $\mathcal{L}_I$. If G_I is the subset of generic points on I, then $\mathfrak{r}(G) = I \setminus G$ and thus $\mathcal{L}_I(I \setminus G) = \mathcal{L}_I(G)$. This is a contradiction, since $\mathcal{L}_I(I \setminus G) = 0$ and $\mathcal{L}_I(G) = \mathcal{L}_I(I)$ by Lemma 3.3.12. □

3.3.2 Almost measure-generic points on $\widetilde{X}$

Let $0 < \epsilon < 1 = \mathcal{L}_{\widetilde{X}}(\widetilde{X})$ and $C > 0$ be some real constants. Define

$$
\begin{aligned}
G^{\uparrow}_{\epsilon,C} &= \left\{ \widetilde{x} \in \widetilde{X} \;\middle|\; \begin{array}{r} \exists \text{ erg. inv. meas. } m \text{ on } \widetilde{X} \text{ s.t. } \forall T > C \text{ and } \forall h \in \mathrm{Lip}_1(\widetilde{X}) : \\ \left\| I^{\uparrow}_T(h,\, \widetilde{x}) - \mathrm{Av}_m(h) \right\| < \epsilon \end{array} \right\}, \\
G^{\downarrow}_{\epsilon,C} &= \left\{ \widetilde{x} \in \widetilde{X} \;\middle|\; \begin{array}{r} \exists \text{ erg. inv. meas. } m \text{ on } \widetilde{X} \text{ s.t. } \forall T > C \text{ and } \forall h \in \mathrm{Lip}_1(\widetilde{X}) : \\ \left\| I^{\downarrow}_T(h,\, \widetilde{x}) - \mathrm{Av}_m(h) \right\| < \epsilon \end{array} \right\}, \\
G^{\uparrow\downarrow}_{\epsilon,C} &= G^{\uparrow}_{\epsilon,C} \cap G^{\downarrow}_{\epsilon,C}, \\
G_{\epsilon,C} &= \left\{ \widetilde{x} \in \widetilde{X} \;\middle|\; \begin{array}{r} \exists \text{ erg. inv. meas. } m \text{ on } \widetilde{X} \text{ s.t. } \forall T > C \text{ and } \forall h \in \mathrm{Lip}_1(\widetilde{X}) : \\ \left\| I^{\uparrow}_T(h,\, \widetilde{x}) - \mathrm{Av}_m(h) \right\| < \epsilon \text{ and } \left\| I^{\downarrow}_T(h,\, \widetilde{x}) - \mathrm{Av}_m(h) \right\| < \epsilon \end{array} \right\}.
\end{aligned}
\tag{3.3.26}
$$

Theorem 3.3.14. *For any constant $0 < \epsilon < 1 = \mathcal{L}_{\widetilde{X}}(\widetilde{X})$, there exists a positive constant T_ϵ such that*

$$
\mathcal{L}_{\widetilde{X}}(G_{\epsilon,T_\epsilon}) > 1 - \epsilon. \tag{3.3.27}
$$

Proof. Let

$$
G^{\mu}_{\epsilon,C} = \left\{ \widetilde{x} \in \widetilde{X} \;\middle|\; \begin{array}{r} \forall T > C \text{ and } \forall h \in \mathrm{Lip}_1(\widetilde{X}) : \\ \left\| I^{\uparrow}_T(h,\, \widetilde{x}) - \mathrm{Av}_\mu(h) \right\| < \epsilon \text{ and } \left\| I^{\downarrow}_T(h,\, \widetilde{x}) - \mathrm{Av}_\mu(h) \right\| < \epsilon \end{array} \right\}, \tag{3.3.28}
$$

and let

$$
G^{\nu}_{\epsilon,C} = \left\{ \widetilde{x} \in \widetilde{X} \;\middle|\; \begin{array}{r} \forall T > C \text{ and } \forall h \in \mathrm{Lip}_1(\widetilde{X}) : \\ \left\| I^{\uparrow}_T(h,\, \widetilde{x}) - \mathrm{Av}_\nu(h) \right\| < \epsilon \text{ and } \left\| I^{\downarrow}_T(h,\, \widetilde{x}) - \mathrm{Av}_\nu(h) \right\| < \epsilon \end{array} \right\}. \tag{3.3.29}
$$

Note that

$$G_{\epsilon,C} = G^{\mu}_{\epsilon,C} \cup G^{\nu}_{\epsilon,C}. \tag{3.3.30}$$

First, suppose that we know the estimate μ and ν individually. That is, suppose that the following holds.

Lemma 3.3.15. *For any constant $0 < \epsilon < 1$ there exists a positive constant T^{μ}_{ϵ} and a positive constant T^{ν}_{ϵ} such that*

$$\mu(G^{\mu}_{\epsilon,T^{\mu}_{\epsilon}}) > 1-\epsilon \text{ and } \nu(G^{\nu}_{\epsilon,T^{\nu}_{\epsilon}}) > 1-\epsilon. \tag{3.3.31}$$

Let $T_\epsilon = \max\{T^{\mu}_{\epsilon}, T^{\nu}_{\epsilon}\}$. Then

$$G^{\mu}_{\epsilon,T^{\mu}_{\epsilon}} \subseteq G^{\mu}_{\epsilon,T_\epsilon} \text{ and } G^{\nu}_{\epsilon,T^{\nu}_{\epsilon}} \subseteq G^{\nu}_{\epsilon,T_\epsilon}. \tag{3.3.32}$$

Hence,

$$\mu(G^{\mu}_{\epsilon,T_\epsilon}) > 1-\epsilon \text{ and } \nu(G^{\nu}_{\epsilon,T_\epsilon}) > 1-\epsilon. \tag{3.3.33}$$

By (3.3.30) we have

$$2\mathcal{L}_{\widetilde{X}}(\widetilde{X} \setminus G_{\epsilon,T_\epsilon}) = \mu(\widetilde{X} \setminus G_{\epsilon,T_\epsilon}) + \nu(\widetilde{X} \setminus G_{\epsilon,T_\epsilon}) \leq \mu(\widetilde{X} \setminus G^{\mu}_{\epsilon,T_\epsilon}) + \nu(\widetilde{X} \setminus G^{\nu}_{\epsilon,T_\epsilon}) < 2\epsilon. \tag{3.3.34}$$

So

$$\mathcal{L}_{\widetilde{X}}(G_{\epsilon,T_\epsilon}) > 1-\epsilon.$$

Now in order to complete the proof of Theorem 3.3.14 we need to prove Lemma 3.3.15. We will prove it for the measure μ, as the proof for ν will be identical.

Once again we can reduce the proof to two twin proofs, by considering the one-sided sets

$$G^{\mu,\uparrow}_{\epsilon,C} = \left\{ \widetilde{x} \in \widetilde{X} \;\middle|\; \forall T > C \text{ and } \forall h \in \mathrm{Lip}_1(\widetilde{X}) : \left\| I^{\uparrow}_T(h,\, \widetilde{x}) - \mathrm{Av}_\mu(h) \right\| < \epsilon \right\} \tag{3.3.35}$$

and

$$G^{\mu,\downarrow}_{\epsilon,C} = \left\{ \widetilde{x} \in \widetilde{X} \;\middle|\; \forall T > C \text{ and } \forall h \in \mathrm{Lip}_1(\widetilde{X}) : \left\| I^{\downarrow}_T(h,\, \widetilde{x}) - \mathrm{Av}_\mu(h) \right\| < \epsilon \right\}. \tag{3.3.36}$$

By definition

$$G^{\mu}_{\epsilon,C} = G^{\mu,\uparrow}_{\epsilon,C} \cap G^{\mu,\downarrow}_{\epsilon,C}. \tag{3.3.37}$$

Assume that the following holds.

Lemma 3.3.16. *For any $0 < \xi < 1$ there exist constants $C_\uparrow$ and $C_\downarrow$ such that*

$$\mu(G^{\mu,\uparrow}_{\xi,C_\uparrow}) > 1 - \xi \text{ and } \mu(G^{\mu,\downarrow}_{\xi,C_\downarrow}) > 1 - \xi. \tag{3.3.38}$$

Take $\xi = \frac{\epsilon}{2}$. Let $T^\mu_\epsilon = \max\{C_\uparrow, C_\downarrow\}$. Then

$$\mu(G^{\mu,\uparrow}_{\xi,T^\mu_\epsilon}) > 1 - \frac{\epsilon}{2} \tag{3.3.39}$$

and

$$\mu(G^{\mu,\downarrow}_{\xi,T^\mu_\epsilon}) > 1 - \frac{\epsilon}{2}. \tag{3.3.40}$$

Hence

$$\mu(G^{\mu,\uparrow}_{\xi,T^\mu_\epsilon} \cap G^{\mu,\downarrow}_{\xi,T^\mu_\epsilon}) > 1 - \frac{\epsilon}{2}. \tag{3.3.41}$$

.

Finally, this intersection is contained in $G^{\mu}_{\epsilon, T^{\mu}_{\epsilon}}$, since $\frac{\epsilon}{2} < \epsilon$. We have now reduced the proof of Theorem 3.3.14 to proving Lemma 3.3.16. We will now give a proof for the upward direction, and the downward direction will follow by a similar argument once more.

Proof. Let $\widetilde{x}_0 \in \widetilde{X}$ be some non-singular point on $\widetilde{X}$. Let us consider all 1-Lipschitz functions on $\widetilde{X}$ that vanish at $\widetilde{x}_0$:

$$\mathcal{F}_0 = \left\{ f \in \mathrm{Lip}_1(\widetilde{X}) \;\middle|\; f(\widetilde{x}_0) = 0 \right\}.$$

We will discuss a couple of properties of the set $\mathcal{F}_0$ that will be useful in the proof.

Proposition 3.3.17. *There exists a constant D such that every $f \in \mathcal{F}_0$ is uniformly bounded by D (note that the constant here is the same for all the functions in $\mathcal{F}_0$).*

Proof. For all $f \in \mathcal{F}_0$ and for all $\widetilde{x} \in \widetilde{X}$ we have

$$|f(\widetilde{x})| = |f(\widetilde{x}) - 0| = |f(\widetilde{x}) - f(\widetilde{x}_0)| \leq d(\widetilde{x}, \widetilde{x}_0) \leq \mathrm{diam}(\widetilde{X}) =: D. \tag{3.3.42}$$

□

Proposition 3.3.18. *The set $\mathcal{F}_0 \subset \mathcal{C}(\widetilde{X})$ is compact in the supremum norm $\|\cdot\|_\infty$.*

Proof. First, we will show that $\mathcal{F}_0$ is closed in the supremum norm topology. Suppose that a sequence of functions $\{f_n\}$ from $\mathcal{F}_0$ converges uniformly on $\widetilde{X}$ to some other function f. We will prove that $f \in \mathcal{F}_0$.

Since convergence is uniform, pointwise convergence is present at each point. So

$$f(\widetilde{x}_0) = \lim_{n\to\infty} f_n(\widetilde{x}_0) = \lim_{n\to\infty} 0 = 0, \tag{3.3.43}$$

since every $f_n(\widetilde{x}_0) = 0$.

We now check that f is 1-Lipschitz. Fix any pair of points $\widetilde{x}, \widetilde{y} \in \widetilde{X}$. We have for any fixed f_n

$$|f_n(\widetilde{x}) - f_n(\widetilde{y})| \leq d_{\widetilde{X}}(\widetilde{x}, \widetilde{y}). \tag{3.3.44}$$

Since $f_n(\widetilde{x}) \to f(\widetilde{x})$ and $f_n(\widetilde{y}) \to f(\widetilde{y})$, we have

$$|f(\widetilde{x}) - f(\widetilde{y})| = \lim_{n\to\infty} |f_n(\widetilde{x}) - f_n(\widetilde{y})| \leq d_{\widetilde{X}}(\widetilde{x}, \widetilde{y}). \tag{3.3.45}$$

Now we will show that the closure of $\mathcal{F}_0$ is compact by applying the famous Arzelà–Ascoli Theorem.

Remark 3.3.19 (Arzelà–Ascoli Theorem). Recall that if

- K is a compact metric space;
- $\mathcal{F} \subset \mathcal{C}(K)$ is a subset of continuous functions from K to $\mathbb{R}$.

Then $\mathcal{F}$ is has compact closure in $\mathcal{C}(K)$ with the uniform topology if the two conditions below are met:

1. $\mathcal{F}$ is equicontinuous (that is, for all $x, y \in K$ and for any $\varepsilon > 0$ there exists a $\delta > 0$ such that for any $f \in \mathcal{F}$

$$\text{if } d_K(x, y) < \delta \Rightarrow |f(x) - f(y)| < \varepsilon);$$

2. For every $x \in K$ the set $\Big\{ f(x) \;\Big|\; f \in \mathcal{F} \Big\}$ has compact closure in $\mathbb{R}$.

We do a straightforward check of all the conditions in our case. First, note that $\widetilde{X}$ with its standard flat metric is compact, since it is a finite-area translation surface. Equicontinuity is a consequence of all the functions being 1-Lipschitz for the same constant 1. We can simply take $\delta = \varepsilon$, and then if $d_{\widetilde{X}}(\widetilde{x}, \widetilde{y}) < \delta$ we have by the definition of a 1-Lipschitz function:

$$|f(\widetilde{x}) - f(\widetilde{y})| \leq d_{\widetilde{X}}(\widetilde{x}, \widetilde{y}) < \delta = \varepsilon.$$

Note that the same δ works for all functions in $\mathcal{F}_0$.

By Proposition 3.3.17, for every $\widetilde{x} \in \widetilde{X}$ we have the set

$$\left\{ f(\widetilde{x}) \;\middle|\; f \in \mathcal{F}_0 \right\}$$

bounded from above by the diameter D of $\widetilde{X}$. It is then a bounded set in $\mathbb{R}$, and bounded sets in $\mathbb{R}$ have compact closures. So $\mathcal{F}_0$ is closed and has compact closure, which means it is compact (in the sup-norm $\|\cdot\|_\infty$). □

After establishing compactness of $\mathcal{F}_0$, we can build a finite subcover. Fix any positive number $\delta > 0$. For every $f \in \mathcal{F}_0$ consider its δ-neighborhood $U_\delta(f)$ in the uniform metric. The collection of all such neighborhoods is an open cover of the set $\mathcal{F}_0$. Since $\mathcal{F}_0$ is compact, it has a finite subcover, so there is a finite subset of functions $\{f_1, f_2, \ldots, f_{M_\delta}\} \subset \mathcal{F}_0$, that are centers of this finite subset of neighborhoods $U_\delta(f_1), U_\delta(f_2), \ldots, U_\delta(f_{M_\delta})$ that cover the entire $\mathcal{F}_0$. In other words, for any $f \in \mathcal{F}_0$ there is at least one $f_j \in \{f_1, f_2, \ldots, f_{M_\delta}\}$ such that

$$||f - f_j||_\infty \leq \delta. \tag{3.3.46}$$

By the Birkhoff–Khinchin Pointwise Ergodic Theorem for flows, for μ-almost every point in $\widetilde{X}$ the discrete sequence of the time-average integrals at integer points in time converges

to the space average for any fixed f_j :

$$\lim_{N\to\infty} I_N^{\uparrow}(f_j,\, \widetilde{x}) = \mathrm{Av}_\mu(f_j). \tag{3.3.47}$$

Fix one of the neighborhood-center f_j-s. Then fix N – the integer moment in time. Only the point $\widetilde{x}$ is now varying in the expsession $I_N^{\uparrow}(f_j,\, \widetilde{x})$, so view that average of f_j at time N as a function of $\widetilde{x}$.

Proposition 3.3.20. *This function $I_N^{\uparrow}(f_j,\, \cdot)$ is μ-measurable on $\widetilde{X}$.*

Proof. Since the function $(s, \widetilde{x}) \mapsto f_j(\widetilde{v}_s\widetilde{x})$ is continuous on $[0, N] \times \widetilde{X}$, its integral over the compact interval $[0, N]$ is continuous, hence is μ-measurable. □

The space-average $\mathrm{Av}_\mu(f_j)$ does not depend on any point, but can be viewed as a constant function on $\widetilde{X}$ that just takes the value $\mathrm{Av}_\mu(f_j)$ at all points $\widetilde{x} \in \widetilde{X}$ (j is fixed). Being a constant function, it is obviously μ-measurable on $\widetilde{X}$.

We will now prove this.

Proposition 3.3.21. *For any $0 < \zeta < 1$ there exists a μ-measurable set $E(\zeta) \subset \widetilde{X}$ such that $\mu(E(\zeta)) \geq 1 - \zeta$ and such that for any $j \in \{1, \ldots, M_\delta\}$*

$$\lim_{N\to\infty} I_N^{\uparrow}(f_j,\, \widetilde{x}) = \mathrm{Av}_\mu(f_j) \tag{3.3.48}$$

uniformly on $E(\zeta)$.

Proof. This proof amounts to an application of another classical theorem.

Remark 3.3.22 (Egorov-Severini Theorem). Suppose that (Y, m) is a finite measure space. Let $\psi_n : Y \to \mathbb{R}$ be measurable functions with $\lim_{n\to\infty} \psi_n(y) = \psi(y)$ for m-almost every $y \in Y$. Then for every $\eta > 0$ there exists a measurable set $E \subset Y$ such that $m(Y \setminus E) < \eta$ and $\psi_n \to \psi$ uniformly on E.

Fix one f_j from the finite set $\{f_1, f_2, \ldots, f_{M_\delta}\}$ obtained above. Take $Y = \widetilde{X}$, $m = \mu$ and $\psi_N(\widetilde{x}) = I_N^{\uparrow}(f_j, \widetilde{x})$ (for a fixed f_j). Note that $I_N^{\uparrow}(f_j, \widetilde{x})$ is μ-measurable by Proposition 3.3.20, and

$$\lim_{N\to\infty} I_N^{\uparrow}(f_j, \widetilde{x}) = \mathrm{Av}_\mu(f_j) \tag{3.3.49}$$

for μ-almost every $\widetilde{x} \in \widetilde{X}$ by the Birkhoff–Khinchin Pointwise Ergodic Theorem for flows.

Egorov-Severini Theorem tells us that (in particular) for $\eta = \frac{\zeta}{M_\delta}$ there exists a measurable set $E_j(\frac{\zeta}{M_\delta}) \subset \widetilde{X}$ with $\mu(E_j(\frac{\zeta}{M_\delta})) \geq 1 - \frac{\zeta}{M_\delta}$ and with

$$I_N^{\uparrow}(f_j, \widetilde{x}) \to \mathrm{Av}_\mu(f_j)$$

uniformly on $E_j(\frac{\zeta}{M_\delta})$.

Since this can be applied to each f_j individually, we can consider the intersection

$$E(\zeta) = \cap_{j=1}^{M_\delta} E_j\left(\frac{\zeta}{M_\delta}\right). \tag{3.3.50}$$

Then

$$\mu(\widetilde{X} \setminus E(\zeta)) \leq \sum_{j=1}^{M_\delta} \mu\left(\widetilde{X} \setminus E_j\left(\frac{\zeta}{M_\delta}\right)\right) < M_\delta \cdot \frac{\zeta}{M_\delta} = \zeta, \tag{3.3.51}$$

and

$$\mu(E(\zeta)) \geq 1 - \zeta. \tag{3.3.52}$$

Since for each j we have $E(\zeta) \subset E_j(\frac{\zeta}{M_\delta})$, we have

$$I_N^{\uparrow}(f_j, \widetilde{x}) \to \mathrm{Av}_\mu(f_j)$$

uniformly on $E(\zeta)$ for every $j \in \{1, ..., M_\delta\}$. So we have proved Proposition 3.3.21. $\square$

We will now combine these j separate uniform convergence statements into one using that there are only finitely many j-s. Expanding the main statement of Proposition 3.3.21, we can write that for every $\rho > 0$ and for each f_j we have some natural $N_j(\delta, \zeta, \rho)$ such that for any $n > N_j(\delta, \zeta, \rho)$ we have

$$\sup_{\widetilde{x} \in E(\zeta)} |I_n^{\uparrow}(f_j, \widetilde{x}) - \mathrm{Av}_\mu(f_j)| < \rho. \tag{3.3.53}$$

Let $N(\delta, \zeta, \rho) = \max_{j \in \{1,...,M_\delta\}} N_j(\delta, \zeta, \rho)$. This will work for all f_j-s simultaneously. To sum this up, we have shown the following.

Lemma 3.3.23. *For any $0 < \zeta < 1$ there exists a μ-measurable set $E(\zeta) \subset \widetilde{X}$ with $\mu(E(\zeta)) \geq 1 - \zeta$ such that for any $\rho > 0$ there exists a natural number $N(\delta, \zeta, \rho)$ such that for any $n > N(\delta, \zeta, \rho)$, for any $\widetilde{x} \in E(\zeta)$ and for any $j \in \{1, \ldots, M_\delta\}$*

$$|I_n^{\uparrow}(f_j, \widetilde{x}) - \mathrm{Av}_\mu(f_j)| < \rho. \tag{3.3.54}$$

We will now show that the statement of Lemma 3.3.23 holds for all functions in $\mathcal{F}_0$. To be more precise, we will prove the following.

Lemma 3.3.24. *For any $0 < \zeta < 1$ and for any $\rho > 0$ there exists a μ-measurable set $E(\zeta) \subset \widetilde{X}$ with $\mu(E(\zeta)) \geq 1 - \zeta$ such that there exists a natural number $N(\zeta, \rho)$ such that for any $n > N(\zeta, \rho)$, for any $\widetilde{x} \in E(\zeta)$ and for any $f \in \mathcal{F}_0$*

$$|I_n^{\uparrow}(f, \widetilde{x}) - \mathrm{Av}_\mu(f)| < \rho. \tag{3.3.55}$$

Proof. Let $f \in \mathcal{F}_0$. Take $\delta = \frac{\rho}{3}$ and build the corresponding finite cover and the functions $f_1, \ldots, f_{M_\delta}$. There is some $f_j \in \{f_1, \ldots, f_{M_\delta}\}$ such that

$$||f - f_j||_\infty < \delta. \tag{3.3.56}$$

Let us compare the time and space averages of f and f_j. Using elementary properties of norms we get:

$$|I_n^\uparrow(f,\, \widetilde{x}) - \mathrm{Av}_\mu(f)| < |I_n^\uparrow(f,\, \widetilde{x}) - I_n^\uparrow(f_j,\, \widetilde{x})| + |I_n^\uparrow(f_j,\, \widetilde{x}) - \mathrm{Av}_\mu(f_j)| + |\,\mathrm{Av}_\mu(f_j) - \mathrm{Av}_\mu(f)|.$$

We will now estimate each term in the expression above individually. The first term

$$|I_n^\uparrow(f,\, \widetilde{x}) - I_n^\uparrow(f_j,\, \widetilde{x})| = |\frac{1}{n}\int_0^n (f - f_j)(\widetilde{v}_t\widetilde{x})dt| \leq |\frac{1}{n}\int_0^n ||f - f_j||_\infty dt| \leq \frac{1}{n}\int_0^n \delta dt = \delta$$

For the second term, since $\widetilde{x} \in E(\epsilon)$, and $n > N(\delta, \zeta, \rho)$ we have

$$|I_n^\uparrow(f_j,\, \widetilde{x}) - \mathrm{Av}_\mu(f_j)| < \delta.$$

Finally, the third term

$$|\,\mathrm{Av}_\mu(f_j) - \mathrm{Av}_\mu(f)| = |\int_{\widetilde{X}} (f_j - f)d\mu| \leq \int_{\widetilde{X}} d\mu \cdot ||f - f_j||_\infty = 1 \cdot ||f - f_j||_\infty < \delta.$$

Combining these we get

$$|I_n^\uparrow(f,\, \widetilde{x}) - \mathrm{Av}_\mu(f)| < 3\delta = \rho. \tag{3.3.57}$$

□

Now that we have proved Lemma 3.3.24, we will upgrade it again and show that a similar

statement holds for all moments in time, not only for integer ones.

Lemma 3.3.25. *For any $0 < \zeta < 1$ and for any $\eta > 0$ there exists a μ-measurable set $E(\zeta) \subset \widetilde{X}$ with $\mu(E(\zeta)) \geq 1 - \zeta$ such that there exists a real number $T(\zeta, \eta) > 0$ such that for any $t > T(\zeta, \eta)$, for any $\widetilde{x} \in E(\zeta)$ and for any $f \in \mathcal{F}_0$*

$$|I_t^{\uparrow}(f, \widetilde{x}) - \mathrm{Av}_\mu(f)| < \eta. \tag{3.3.58}$$

Proof. Fix $1 > \zeta > 0$ and $\eta > 0$. Take $\rho = \frac{\eta}{2}$. By Lemma 3.3.24 there is a set $E(\zeta)$ with $\mu(E(\zeta)) \geq 1 - \zeta$ and an $N(\zeta, \rho) > 0$ such that for any $\widetilde{x} \in E(\zeta)$, for every integer $n > N(\zeta, \rho)$ and for any $f \in \mathcal{F}_0$ we have

$$|I_n^{\uparrow}(f, \widetilde{x}) - \mathrm{Av}_\mu(f)| < \rho. \tag{3.3.59}$$

Now fix $t > 0$, $\widetilde{x} \in \widetilde{X}$ and $f \in \mathcal{F}_0$. Let $n := \lfloor t \rfloor$. We want to estimate $|I_t^{\uparrow}(f, \widetilde{x}) - \mathrm{Av}_\mu(f)|$. We can write

$$|I_t^{\uparrow}(f, \widetilde{x}) - \mathrm{Av}_\mu(f)| \leq |I_t^{\uparrow}(f, \widetilde{x}) - I_n^{\uparrow}(f, \widetilde{x})| + |I_n^{\uparrow}(f, \widetilde{x}) - \mathrm{Av}_\mu(f)|. \tag{3.3.60}$$

If $n > N(\zeta, \rho)$, then the second term is smaller than $\rho = \frac{\eta}{2}$.

To estimate the first term, we write:

$$\begin{aligned}
|I_t^{\uparrow}(f, \widetilde{x}) - I_n^{\uparrow}(f, \widetilde{x})| = \left| \frac{1}{t} \int_0^t f(\widetilde{v}_s(\widetilde{x}))ds - \frac{1}{n} \int_0^n f(\widetilde{v}_s(\widetilde{x}))ds \right| = \left| \frac{1}{t} \int_0^n f(\widetilde{v}_s(\widetilde{x}))ds + \right. \\
\left. + \frac{1}{t} \int_n^t f(\widetilde{v}_s(\widetilde{x}))ds - \frac{1}{n} \int_0^n f(\widetilde{v}_s(\widetilde{x}))ds \right| = \left| \left(\frac{1}{t} - \frac{1}{n} \right) \int_0^n f(\widetilde{v}_s(\widetilde{x}))ds + \frac{1}{t} \int_n^t f(\widetilde{v}_s(\widetilde{x}))ds \right| \leq \\
\leq \left| \left(\frac{1}{t} - \frac{1}{n} \right) \right| nD + \frac{1}{t}(t - n)D = \\
= \left| \left(\frac{n}{t} - 1 \right) \right| D + \frac{1}{t}(t - n)D = 2\frac{t - n}{t} D
\end{aligned} \tag{3.3.61}$$

by Proposition 3.3.17. Since $t \in [n, n+1]$, we have $0 \leq t - n \leq 1$. So

$$2\frac{t-n}{t}D \leq \frac{2D}{t} \leq \frac{2D}{n}$$

So combining all of that we get

$$|I_t^{\uparrow}(f, \widetilde{x}) - \mathrm{Av}_\mu(f)| \leq \frac{2D}{n} + \frac{\eta}{2}$$

Now choose $T(\zeta, \eta)$ large enough so that whenever $t > T(\zeta, \eta)$ and $n = \lfloor t \rfloor$ we have both $n > N(\zeta, \rho)$ and $\frac{2D}{n} < \frac{\eta}{2}$.

Note that this threshold only depends on ζ, η and the constant D. This immediately leads to

$$|I_t^{\uparrow}(f, \widetilde{x}) - \mathrm{Av}_\mu(f)| \leq \frac{\eta}{2} + \frac{\eta}{2} = \eta.$$

This proves Lemma 3.3.25. □

Finally, to prove Lemma 3.3.16, we note that for any constant c

$$I_t^{\uparrow}(f + c, \widetilde{x}) = I_t^{\uparrow}(f, \widetilde{x}) + c \text{ and } I_t^{\downarrow}(f + c, \widetilde{x}) = I_t^{\downarrow}(f, \widetilde{x}) + c, \tag{3.3.62}$$

and also

$$\mathrm{Av}_\mu(f + c) = \mathrm{Av}_\mu(f) + c. \tag{3.3.63}$$

This means that

$$I_t^{\uparrow}(f + c, \widetilde{x}) - \mathrm{Av}_\mu(f + c) = I_t^{\uparrow}(f, \widetilde{x}) - \mathrm{Av}_\mu(f),$$

and

$$I_t^{\downarrow}(f + c, \widetilde{x}) - \mathrm{Av}_\mu(f + c) = I_t^{\downarrow}(f, \widetilde{x}) - \mathrm{Av}_\mu(f).$$

Thus the statement of Lemma 3.3.16 holds for all $f \in \mathrm{Lip}_1(\widetilde{X})$. This completes the proof of Theorem 3.3.14.

□

□

Definition 3.3.26 (Almost generic points)**.** We call a point $\widetilde{x} \in \widetilde{X}$ *ϵ-almost totally generic* if $\widetilde{x} \in G_{\epsilon, T_\epsilon}$.

Lemma 3.3.27. *Let $0 < \epsilon < 1$ and $C > 0$. If a non-critical point $\widetilde{x} \in \widetilde{X}$ is forward/backward/totally (ϵ, C)-almost μ-generic, then $\varsigma(\widetilde{x})$ is forward/backward/totally (ϵ, C)-almost ν-generic, respectively. The converse statement also holds, with μ and ν interchanged.*

Proof. We prove the forward statement. The backward statement is identical, and the totally (ϵ, C)-generic statement follows by combining the forward and backward cases.

Assume that $\widetilde{x}$ is forward (ϵ, C)-almost μ-generic. We must show that $\varsigma(\widetilde{x})$ is forward (ϵ, C)-almost ν-generic.

Let $h \in \mathrm{Lip}_1(\widetilde{X})$. Since ς is an isometry of $\widetilde{X}$, the function $h \circ \varsigma$ also belongs to $\mathrm{Lip}_1(\widetilde{X})$.

By Lemma 3.2.1, the involution ς pushes μ forward to ν, i.e.

$$\varsigma_* \mu = \nu.$$

Therefore

$$\mathrm{Av}_\mu(h \circ \varsigma) = \int_{\widetilde{X}} h(\varsigma(y))\, d\mu(y) = \int_{\widetilde{X}} h(z)\, d\nu(z) = \mathrm{Av}_\nu(h).$$

Next, by symmetry of the construction, ς commutes with the vertical flow:

$$\widetilde{v}_t \circ \varsigma = \varsigma \circ \widetilde{v}_t \qquad \text{for all } t \in \mathbb{R}.$$

Hence for every $T > 0$,

$$I_T^{\uparrow}(h,\, \varsigma(\widetilde{x})) = \frac{1}{T}\int_0^T h(\widetilde{v}_t(\varsigma(\widetilde{x})))\, dt = \frac{1}{T}\int_0^T h(\varsigma(\widetilde{v}_t(\widetilde{x})))\, dt = \frac{1}{T}\int_0^T (h\circ\varsigma)(\widetilde{v}_t(\widetilde{x}))\, dt = I_T^{\uparrow}(h\circ\varsigma,\, \widetilde{x}).$$

Since $\widetilde{x}$ is forward (ϵ, C)-almost μ-generic, it follows that for every $T > C$,

$$\left| I_T^{\uparrow}(h \circ \varsigma,\, \widetilde{x}) - \mathrm{Av}_\mu(h \circ \varsigma) \right| < \epsilon.$$

Using the two identities above, we obtain

$$\left| I_T^{\uparrow}(h,\, \varsigma(\widetilde{x})) - \mathrm{Av}_\nu(h) \right| < \epsilon \qquad \text{for all } T > C.$$

Since this holds for every $h \in \mathrm{Lip}_1(\widetilde{X})$, the point $\varsigma(\widetilde{x})$ is forward (ϵ, C)-almost ν-generic.

The backward case is proved in exactly the same way, using

$$I_T^{\downarrow}(h,\, \varsigma(\widetilde{x})) = I_T^{\downarrow}(h \circ \varsigma,\, \widetilde{x}),$$

which again follows from $\widetilde{v}_{-t} \circ \varsigma = \varsigma \circ \widetilde{v}_{-t}$.

If $\widetilde{x}$ is totally (ϵ, C)-almost μ-generic, then it is both forward and backward (ϵ, C)-almost μ-generic, so the argument above shows that $\varsigma(\widetilde{x})$ is both forward and backward (ϵ, C)-almost ν-generic. Hence $\varsigma(\widetilde{x})$ is totally (ϵ, C)-almost ν-generic.

The converse statement follows by interchanging μ and ν, since ς is an involution. □

Lemma 3.3.28. *Let I be a horizontal interval on $\widetilde{X}$, and let $0 < \epsilon < 1$. Let $F_I : I \to I$ be the first return map of the vertical flow to I.*

Then there exists a constant $C = C(I, \epsilon) > 0$ such that

$$\mathcal{L}_I\Big(G^I_{\epsilon, C}\Big) > (1 - \epsilon)\mathcal{L}_I(I).$$

Proof. Consider the first return map $F_I : I \to I$ of the vertical flow to I. This is an interval exchange transformation preserving the Lebesgue measure $\mathcal{L}_I$ on I.

Apply the quantitative almost-genericity Theorem 3.3.14 to the measure-preserving system $(I, F_I, \mathcal{L}_I)$. It yields a constant $C = C(I, \epsilon)$ such that the set of ϵ-almost totally generic points for F_I has $\mathcal{L}_I$-measure greater than $(1 - \epsilon)\mathcal{L}_I(I)$. □

Lemma 3.3.29. *Let I be a horizontal interval on $\widetilde{X}$, and let $0 < \epsilon < \frac{1}{2}$. Then there exists a constant $C = C(I, \epsilon) > 0$ and a pair of points in I, symmetric with respect to the center of I, that are both ϵ-almost totally generic for the first return map on I.*

Proof. Let $R : I \to I$ be reflection across the center of I. Since R preserves $\mathcal{L}_I$, for every measurable subset $A \subset I$ one has

$$\mathcal{L}_I(R(A)) = \mathcal{L}_I(A).$$

By Lemma 3.3.28, there exists $C = C(I, \epsilon)$ such that

$$\mathcal{L}_I\Big(G^I_{\epsilon, C}\Big) > (1 - \epsilon)\mathcal{L}_I(I).$$

Since $\epsilon < \frac{1}{2}$, this implies

$$\mathcal{L}_I\Big(G^I_{\epsilon, C}\Big) > \frac{1}{2}\,\mathcal{L}_I(I).$$

Therefore the two sets $G^I_{\epsilon, C}$ and $R(G^I_{\epsilon, C})$ cannot be disjoint, because otherwise

$$\mathcal{L}_I\Big(G^I_{\epsilon, C}\Big) + \mathcal{L}_I\Big(R(G^I_{\epsilon, C}))\Big) \leq \mathcal{L}_I(I),$$

contradicting the fact that each of these two sets has measure $\mathcal{L}_I(G^I_{\epsilon, C}) > \frac{1}{2}\,\mathcal{L}_I(I)$.

Hence there exists $\widetilde{x} \in G^I_{\epsilon, C} \cap R(G^I_{\epsilon, C})$. Then both $\widetilde{x}$ and $R(\widetilde{x})$ belong to $G^I_{\epsilon, C}$, and these two points are symmetric with respect to the center of I. □

3.4 The circle argument

As before, let $\pi : \widetilde{X} \to X$ be the slit double-cover with ramification points $\widetilde{P} = \pi^{-1}P$ and $\widetilde{Q} = \pi^{-1}Q$. For an $r > 0$, let $\mathrm{U}(x, r)$ denote the r-neighborhood of the point x in the flat metric. Denote the Teichmüller flow by g_t, and write $X_t = g_t(X)$, and $\widetilde{X}_t = g_t(\widetilde{X})$. Similarly, we set $P_t = g_t P$, $Q_t = g_t Q$.

Our main theorem in this section is the following geometric criterion for unique ergodicity.

Theorem 3.4.1. *Consider one of the slit endpoints on X (without loss of generality, call it Q). Suppose that the following key assumption holds: there exist a number $R > 0$ and a sequence of moments in time $\{t_k\}_{k\in\mathbb{N}_0}$, with $\lim_{k\to\infty} t_k = +\infty$, such that the R-neighborhood $\mathrm{U}_{t_k}(Q_{t_k}, R) \subset X_{t_k}$ of Q_{t_k} is an embedded disk that does not contain $P_{t_k} = g_{t_k}P$. Then the surface $\widetilde{X}$ is uniquely ergodic.*

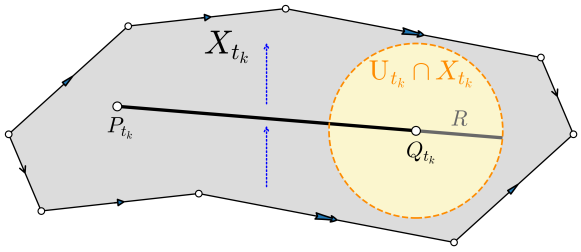


Figure 3.5: Example of the key assumption of Theorem 3.4.1.

Without loss of generality, we may assume that $t_0 = 0$ and choose the local coordinates around the slit endpoint $\widetilde{Q}$ so that it is the origin for the purposes of applying the Teichmüller flow. Suppose that $\widetilde{X}$ is non-uniquely ergodic, and denote the two normalized ergodic measures on it by μ and ν. We will prove the theorem by contradiction.

Proof. Denote by $\mathcal{U}_{t_k}$ the biggest neighborhood of $\widetilde{Q}_{t_k} = g_{t_k}\widetilde{Q}$ such that $\pi(\mathcal{U}_{t_k}) = \mathrm{U}_{t_k}$ (see Figure 3.6 for the visual representation).

I. Passing to a subsequence of times $\{t_{k_i}\}_{i\in\mathbb{N}_0}$.

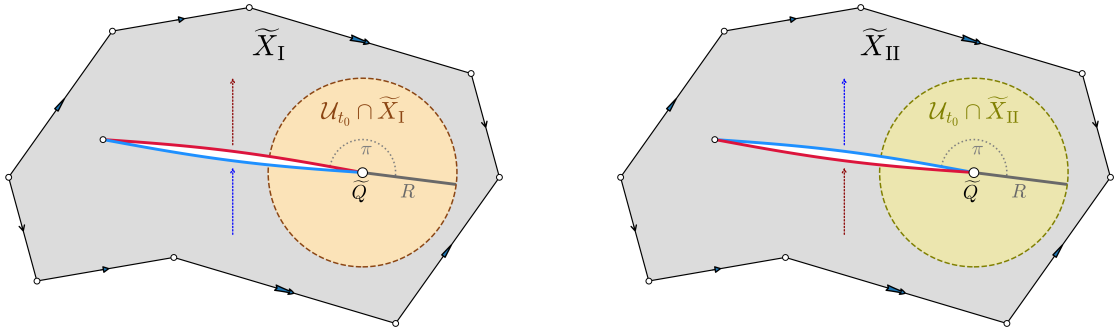


Figure 3.6: Example of the R-neighborhood $\mathcal{U}_{t_0}$ of the endpoint $\widetilde{Q}$ of a slit on the surface $\widetilde{X}_t$.

First, we pick a certain subsequence $\{t_{k_i}\}_{i\in\mathbb{N}_0}$ of the sequence $\{t_{k_i}\}_{i\in\mathbb{N}_0}$. Let us consider the sequence $\{\epsilon_i\}_{i\in\mathbb{N}}$, $\epsilon_i = 1/(i+2)$ of reciprocals of natural numbers. The first element we pick is simply $t_0 = 0$. Starting from $i = 1$, for each ϵ_i, we pick a moment in time t_{k_i} such that

$$\frac{\sqrt{3}}{2} e^{t_{k_i}} R \;>\; \max\{T_{\epsilon_i},\, T_{\epsilon_{i+1}}\}, \tag{3.4.1}$$

where T_{ϵ_i} and $T_{\epsilon_{i+1}}$ are the constants from Theorem 3.3.14 corresponding to $\epsilon = \epsilon_i$ and $\epsilon = \epsilon_{i+1}$ respectively (the motivation from this, admittedly somewhat obscure, choice will become apparent in Part V of the proof; more specifically, in Lemma 3.4.3). Moreover, we pick our new elements t_{k_i} in such a way that the subsequence remains strictly increasing.

For the sake of simplicity, we will abuse our notation and denote the resulting subsequence of times t_{k_i} by $\{t_i\}_{i\in\mathbb{N}_0}$. We will also sometimes write $\widetilde{X}_{t_i} = \widetilde{X}_i$.

II. The neighborhood $\mathcal{U}_{t_i}$.

Take a point $A \neq \widetilde{Q}_{t_i}$ on the intersection of $\mathcal{U}_{t_i}$ with the slit. Consider the clockwise path around $\widetilde{Q}_{t_i}$ on $\widetilde{X}_{t_i}$ starting from A. It is clear that, after rotating by 2π, we will arrive at $C = \varsigma(A)$, and after rotating by the angle 4π, we will return back to A. In this process, we may cross the slit again some number of times. Every time we do so, we jump between the sheets of $\widetilde{X}_{t_i}$: from $\widetilde{X}_i^{\mathrm{I}}$ to $\widetilde{X}_i^{\mathrm{II}}$, or vice versa. However, since $\mathcal{U}_{t_i} \cap \widetilde{X}_i^{\mathrm{I}}$ and $\mathcal{U}_{t_i} \cap \widetilde{X}_i^{\mathrm{II}}$ are clearly symmetric, inside $\mathcal{U}_{t_i}$, we can reglue the stripes formed by the slit. Therefore, $\mathcal{U}_{t_i}$ is

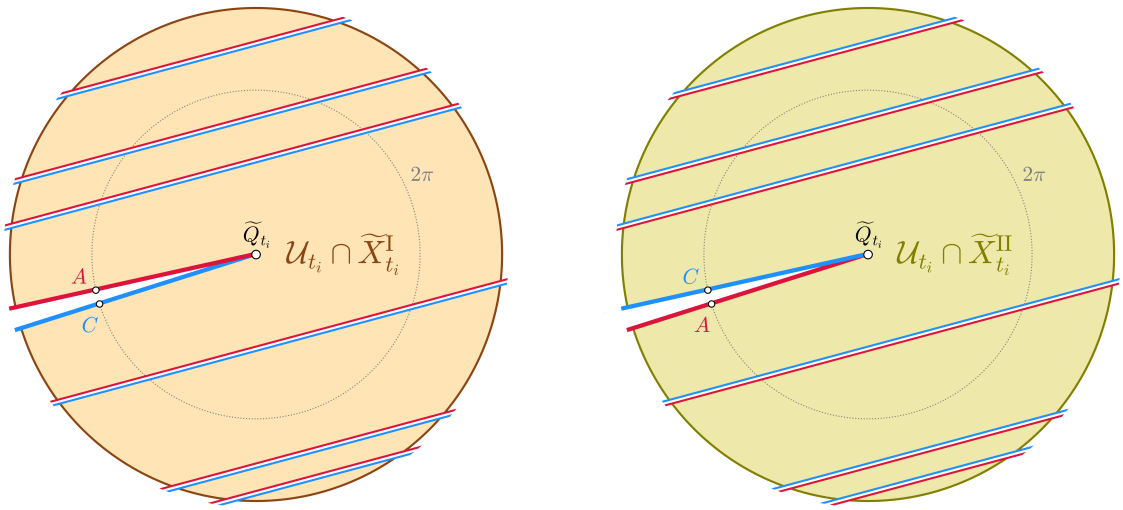


Figure 3.7: Neighborhood of the endpoint O of radius r inside $\widetilde{X}_{t_i}$.

simply a glueing of two radius r_i circles centered at $\widetilde{Q}_{t_i}$ and cut along their parallel radii. We will denote those two circles by $\mathcal{U}^+_{t_i}$ and $\mathcal{U}^-_{t_i}$ (indexing can be chosen, for example, by fixing a side of the slit and stating that $\mathcal{U}^+_{t_i}$ corresponds to counterclockwise rotation by at most 2π and $\mathcal{U}^-_{t_i}$ corresponds to clockwise rotation by at most 2π). See Figure 3.7 for the visual representation.

III. Geometry inside $\mathcal{U}^+_{t_i}$.

In this part, we aim to build a special sequence of pairs of vertical intervals on the initial surface $\widetilde{X}$.

For a fixed $i \in \mathbb{N}_0$, consider the neighborhood $\mathcal{U}_{t_i}$ of the slit endpoint $\widetilde{Q}_{t_i}$. As before, we will represent this neighborhood as two disks, $\mathcal{U}^+_{t_i}$ and $\mathcal{U}^-_{t_i}$, cut and glued along a symmetric radius segment determined by the slit. Let us first describe all our geometric constructions for the forward vertical flow on $\mathcal{U}^+_{t_i}$; from symmetry, geometric constructions on $\mathcal{U}^-_{t_i}$, as well as those for the backward vertical flow, will repeat these. See Figure 3.8 for the visual reference.

Prolong the slit to a diameter of $\mathcal{U}^+_{t_i}$. More specifically, starting from the "top" part of

the slit, we rotate it counterclockwise inside $\mathcal{U}_{t_i}^+$ around the endpoint $\widetilde{Q}_{t_i}$ in and consider the separatrix segment that goes in the direction of the slit and is contained in $\mathcal{U}_{t_i}^+$.

By Lemma 3.3.13, Corollary 3.3.8, as well as Theorem 3.3.14 and Lemmas 3.3.28, 3.3.29, we can find points $\widehat{A}_i$ on the top side of the slit, $\widehat{C}_i = \varsigma(\widehat{A}_i)$ on the bottom side of the slit, and $\widehat{B}_i$ on the separatrix such that they are all totally generic (possibly for different measures), ϵ_i-almost totally generic (from Theorem 3.3.14, since the area of the neighborhood $\mathcal{U}_{t_i}$ is strictly bigger than $2\epsilon_i$ by assumption), and satisfy $0 < \widehat{r}_i := |\widetilde{Q}_{t_i}\widehat{A}_i| = |\widetilde{Q}_{t_i}\widehat{B}_i| < R/2$.

Let $\widehat{A}_i'$ and $\widehat{B}_i'$, respectively, be the intersections of the vertical projections on the horizontal diameter of $\mathcal{U}_{t_i}$ (note that $\widehat{A}_i'$ may belong to $\mathcal{U}_{t_i}^-$ dependent on the angle between the slit and the vertical flow). We will denote by $\widehat{H}_i$ the signed vertical distance from $\widehat{B}_i$ to $\widehat{B}_i'$ (i.e. $\widehat{H}_i$ is positive if $\widehat{B}_i' = v_{|\widehat{H}_i|}\widehat{B}_i$ and negative if $\widehat{B}_i = v_{|\widehat{H}_i|}\widehat{B}_i'$). Clearly, the signed vertical distance from $\widehat{A}_i$ to $\widehat{A}_i'$ is $-\widehat{H}_i$.

Let $\widehat{M}_i^{+,top}$ and $\widehat{N}_i^{+,top}$, respectively, be the points of first intersection with the boundary of $\mathcal{U}_i^+$ of the forward vertical trajectories starting at the points $\widehat{A}_i$ and $\widehat{B}_i$. We will denote by $\widehat{L}_i$ the vertical distance from $\widehat{B}_i'$ to $\widehat{M}_i^{+,top}$. Note that $\frac{\sqrt{3}}{2}r_i \leq \widehat{L}_i < r_i$.

Evidently, $\widehat{L}_i = |\widehat{B}_i'\widehat{N}_i^{+,top}| = |\widehat{A}_i'\widehat{M}_i^{+,top}|$. Moreover, $|\widehat{B}_i\widehat{N}_i^{+,top}| = \widehat{L}_i + \widehat{H}_i$ and $|\widehat{A}_i\widehat{M}_i^{+,top}| = \widehat{L}_i - \widehat{H}_i$.

Next, we describe the preimage of this picture on $\mathcal{U}_{t_0}^+ \subset \widetilde{X}$. See Figure 3.10 for visual reference.

Let $A_i = g_{t_i}^{-1}\widehat{A}_i$, $B_i = g_{t_i}^{-1}\widehat{B}_i$, $A_i' = g_{t_i}^{-1}\widehat{A}_i'$, $B_i' = g_{t_i}^{-1}\widehat{B}_i'$, $M_i^{+,top} = g_{t_i}^{-1}\widehat{M}_i^{+,top}$, and $N_i^{+,top} = g_{t_i}^{-1}\widehat{N}_i^{+,top}$ be the respective preimages of the points $\widehat{A}_i$, $\widehat{B}_i$, $\widehat{A}_i'$, $\widehat{B}_i'$, $\widehat{M}_i^{+,top}$, and $\widehat{N}_i^{+,top}$. Note that A_i' and B_i' remain equidistant from $\widetilde{Q}$ and are still totally generic points on $\widetilde{X}$ (as the Teichmüller flow preserves measure-generality of non-critical points, see Lemma 3.3.11). The same is true for A_i' and B_i', as well as $M_i^{+,top}$ and $N_i^{+,top}$. By construction, the distance $2R_i = |A_iB_i| = 2e^{-t_i}\widehat{r}_i < e^{-t_i}r_i$. We can find a subsequence $\{t_{i_j}\}_{j\in\mathbb{N}}$ of the sequence of times $\{t_i\}_{i\in\mathbb{N}}$ such that the distances r_{i_j} are strictly decreasing. For simplicity,

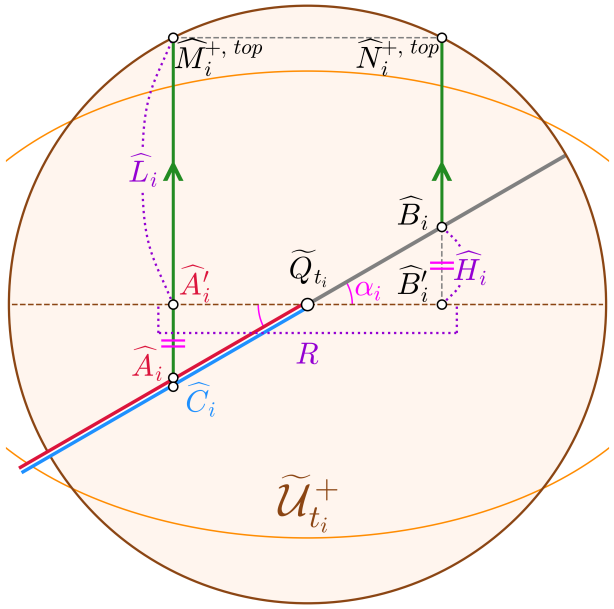


Figure 3.8: Vertical interval sequence construction on $\mathcal{U}_{t_i}^{+}$.

we will abuse our notation and denote this subsequence of times by $\{t_i\}_{i\in\mathbb{N}}$ too.

For each t_i in the subsequence, consider the parallel intervals $[A'_i,\ M_i^{+,\,top}]$ and $[B'_i,\ N_i^{+,\,top}]$ on the surface $\widetilde{X}$. Clearly, they are both of length $L_i = e^{t_i}\widehat{L}_i$, and

$$T_{\epsilon_i} \leq \frac{\sqrt{3}}{2} e^{t_i} r_i \leq L_i < e^{t_i} r_i, \qquad \text{therefore} \qquad \lim_{i\to\infty} L_i \geq \lim_{i\to\infty} \frac{\sqrt{3}}{2} e^{t_i} r_i = +\infty. \tag{3.4.2}$$

Recall that the value of T_{ϵ_i} comes from Theorem 3.3.14.

Moreover, the signed distance $H_i = e^t \widehat{H}_i$ from B_i to B'_i clearly tends to 0 as i goes to infinity (thus so does the distance $|\widetilde{Q}B'_i|$).

IV. Geometry inside the entire $\mathcal{U}_{t_i}$.

Since $\mathcal{U}_{t_i}^{+} = \varsigma(\mathcal{U}_{t_i}^{-})$, so the exact same geometric construction can be repeated in $\mathcal{U}_{t_i}^{-}$. Note that $\widehat{A}_i,\ \widehat{C}_i \in \mathcal{U}_{t_i}^{+} \cap \mathcal{U}_{t_i}^{-}$. The point $\widehat{D}_i = \varsigma(\widehat{B}_i)$ has to be totally generic since $\widehat{B}_i$ is totally generic and not critical. Moreover, $\widehat{D}_i$ has to be ϵ_i-almost totally generic from symmetry (see Lemma 3.3.27). The exact same extension construction as above can be done in the backward direction of the vertical flow on $\widetilde{X}$. The only slight issue is keeping track of the

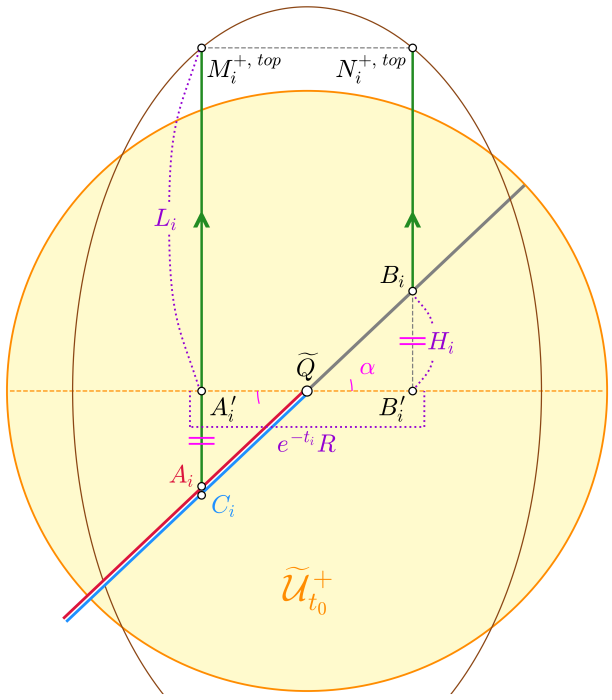


Figure 3.9: Construction of two sequences of vertical intervals, $[A_i,\, M_i^{+,\,top}]$ and $[B_i,\, N_i^{+,\,top}]$, on $\mathcal{U}_{t_0}$.

relative positions of rays and points, as demonstrated by Figure 3.11.

V. Forward flow limits.

Let f be a function on $\widetilde{X}$, and let $t \geq 0$. For simplicity, slightly change the notation of the integrals from Equation (3.3.2):

$$I^{\uparrow}(f,\, A_i) := I^{\uparrow}_{L_i - H_i}(f,\, A_i) = \frac{1}{L_i - H_i} \int_0^{L_i - H_i} f(v_s A_i)\, \mathrm{d}s, \tag{3.4.3}$$

$$I^{\uparrow}(f,\, A_i') := I^{\uparrow}_{L_i}(f,\, A_i') = \frac{1}{L_i} \int_0^{L_i} f(v_s A_i')\, \mathrm{d}s, \tag{3.4.4}$$

$$I^{\uparrow}(f,\, B_i) := I^{\uparrow}_{L_i + H_i}(f,\, B_i) = \frac{1}{L_i + H_i} \int_0^{L_i + H_i} f(v_s B_i)\, \mathrm{d}s, \tag{3.4.5}$$

$$I^{\uparrow}(f,\, B_i') := I^{\uparrow}_{L_i}(f,\, B_i') = \frac{1}{L_i} \int_0^{L_i} f(v_{-s} B_i')\, \mathrm{d}s. \tag{3.4.6}$$

Lemma 3.4.2. *If* $\lim_{i\to\infty} I^{\uparrow}(f,\, B_i')$ *exists and is finite, then so does* $\lim_{i\to\infty} I^{\uparrow}(f,\, B_i)$*, and*

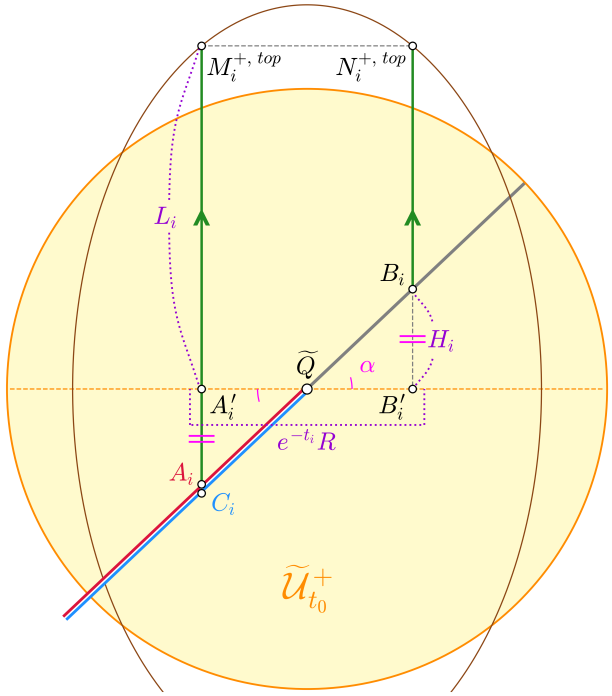


Figure 3.10: Construction of two sequences of vertical intervals, $[A_i,\, M_i^{+,\,top}]$ and $[B_i,\, N_i^{+,\,top}]$, on $\mathcal{U}_{t_0}$.

vice versa. Moreover, these two limits are equal. The same holds for $\lim_{i\to\infty} I^{\uparrow}(f,\, A_i')$ *and* $\lim_{i\to\infty} I^{\uparrow}(f,\, A_i)$.

Proof. It is enough to prove the statement for B_i' and B_i. There are two cases to consider.

First, assume that B_i' is an image of B_i under the forward vertical flow: $B_i' = g_{|H_i|}B_i$. Then H_i is positive, and

$$\begin{aligned}\left\|I^{\uparrow}(f,\, B_i) - I^{\uparrow}(f,\, B_i')\right\| &= \left\|\frac{1}{L_i+H_i}\int_0^{L_i+H_i} f(v_s B_i)\,\mathrm{d}s - \frac{1}{L_i}\int_0^{L_i} f(v_s B_i')\,\mathrm{d}s\right\| = \\ &= \left\|\frac{1}{L_i+H_i}\int_0^{H_i} f(v_s B_i)\,\mathrm{d}s + \left(\frac{1}{L_i+H_i} - \frac{1}{L_i}\right)\int_0^{L_i} f(v_s B_i')\,\mathrm{d}s\right\| = \\ &= \frac{H_i}{L_i+H_i}\left\|I^{\uparrow}_{H_i}(f,\, B_i) - I^{\uparrow}(f,\, B_i')\right\|. \quad (3.4.7)\end{aligned}$$

As i goes to infinity, L_i goes to infinity and H_i goes to 0. Therefore $I^{\uparrow}_{H_i}(f,\, B_i)$ goes to 0. Therefore, if $\lim_{i\to\infty}\|I^{\uparrow}(f,\, B_i')\|$ exists and is finite, then $\lim_{i\to\infty}\|I^{\uparrow}(f,\, B_i)\|$ exists and is

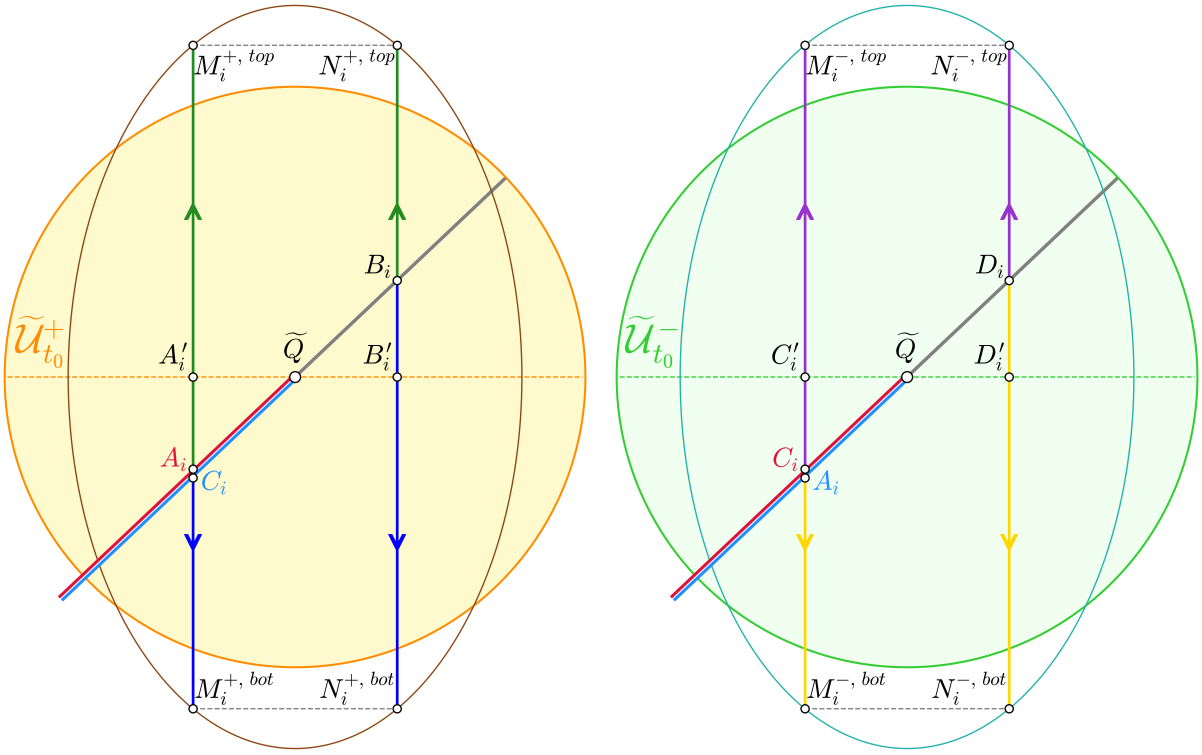


Figure 3.11: The neighborhood $\mathcal{U}_{t_0}$.

finite, and these two limits are equal. The other direction (from existence of limits for B_i to existence of limits for B_i') follows analogously from considering:

$$\left\|I^{\uparrow}(f,\,B_i) - I^{\uparrow}(f,\,B_i')\right\| = \left\|\frac{1}{L_i+H_i}\int_0^{L_i+H_i} f(v_s B_i)\,\mathrm{d}s - \frac{1}{L_i}\int_0^{L_i} f(v_s B_i')\,\mathrm{d}s\right\| =$$
$$= \left\|\left(\frac{1}{L_i+H_i} - \frac{1}{L_i}\right)\int_0^{L_i+H_i} f(v_s B_i) \;+\; \frac{1}{L_i}\int_0^{H_i} f(v_s B_i)\,\mathrm{d}s\right\| = \frac{H_i}{L_i}\left\|I^{\uparrow}_{H_i}(f,\,B_i) - I^{\uparrow}(f,\,B_i).\right\| \tag{3.4.8}$$

Calculations for the case when B_i' is a preimage of B_i under the forward vertical flow ($B_i = g_{|H_i|}B_i'$) are done analogously.

□

Consider an arbitrary 1-Lipschitz function h on $\widetilde{X}$. Since $\widetilde{X}$ is a compact metric space,

and since both measures μ and ν are absolutely continuous with respect to the Lebesgue measure $\mathcal{L}_{\widetilde{X}}$ on $\widetilde{X}$ (see Lemma 3.2.1), the function h is going to be both μ- and ν-integrable on $\widetilde{X}$.

Lemma 3.4.3. *The sequence $\{I^{\uparrow}(h,\, A_i)\}_{n\in\mathbb{N}_+}$ converges to a finite limit, and so does the sequence $\{I^{\uparrow}(h,\, B_i)\}_{n\in\mathbb{N}_+}$. Moreover, these two limits are equal to each other and to $\mathrm{Av}_m(h)$, where either $m=\mu$ or $m=\nu$.*

Proof. We first prove that the sequence $\{I^{\uparrow}(h,\, A_i)\}_{n\in\mathbb{N}_+} = \{I^{\uparrow}_{L_i-H_i}(h,\, A_i)\}_{n\in\mathbb{N}_+}$ converges (the proof for the sequence $\{I^{\uparrow}(h,\, B_i)\}_{n\in\mathbb{N}_+} = \{I^{\uparrow}_{L_i+H_i}(h,\, B_i)\}_{n\in\mathbb{N}_+}$ is analogous). Then we will prove that the limits are equal.

Step 1: From the previous lemma, we can instead prove convergence of the sequence $\{I^{\uparrow}_{L_i}(h,\, A'_i)\}_{n\in\mathbb{N}_+}$. Recall that each point A_i, $i\geq 1$, is ϵ_i-almost totally generic w.r.t. an ergodic invariant measure m on $\widetilde{X}$, where $m=\mu$ or $m=\nu$. While the same is not necessarily true for points A'_i, $i\geq 1$, we can denote by d_{A_i} the difference $\left\|I^{\uparrow}(f,\, A_i) - I^{\uparrow}(f,\, A'_i)\right\|$, implying that a point A'_i is at most d_{A_i}-away from being ϵ_i-almost totally generic. Keep in mind that d_{A_i} tends to 0 as i tends to ∞.

We want to show that there exists $N\in\mathbb{N}_+$ such that the measure m is the same for all A'_i with $i>N$.

Suppose not. Then, for any $N\in\mathbb{N}_+$, there exist two consecutive points A_i and A_{i+1}, $i\geq N$, such that A_i is ϵ_i-almost totally μ-generic and A_{i+1} is ϵ_{i+1}-almost totally ν-generic. By the construction from the main proof, we have $L_i\geq T_{\epsilon_i}$, $L_i\geq T_{\epsilon_{i+1}}$, and $L_{i+1}\geq T_{\epsilon_{i+1}}$. Therefore

$$\left\|I^{\uparrow}_{L_i}(h,\, A'_i) - \mathrm{Av}_{\mu}(h)\right\| < \epsilon_i + d_{A_i} \quad \text{and} \quad \left\|I^{\uparrow}_{L_{i+1}}(h,\, A'_{i+1}) - \mathrm{Av}_{\nu}(h)\right\| < \epsilon_{i+1} + d_{A_{i+1}}. \tag{3.4.9}$$

Moreover, given Condition (3.4.1) on the sequence of times t_i, we have

$$\left\| I^{\uparrow}_{L_i}(h,\, A'_{i+1}) - \mathrm{Av}_{\nu}(h) \right\| < \epsilon_{i+1} + d_{A_{i+1}}. \tag{3.4.10}$$

Since h is a 1-Lipschitz function, A'_{i+1} is positioned between O and A'_i by construction, and $|OA'_i| \leq \frac{1}{2}R$, we have

$$\begin{aligned}\left\| I^{\uparrow}_{L_i}(h,\, A'_i) - I^{\uparrow}_{L_i}(h,\, A'_{i+1}) \right\| &= \frac{1}{L_i} \left\| \int_0^{L_i} h(v_s A'_i)\, \mathrm{d}s - \int_0^{L_i} h(v_s A'_{i+1})\, \mathrm{d}s \right\| \leq \\ &\leq \frac{1}{L_i} \int_0^{L_i} \left\| h(v_s A'_i) - h(v_s A'_{i+1}) \right\| \, \mathrm{d}s \leq \frac{1}{L_i} \int_0^{L_i} e^{-t_i} \frac{R}{2} \, \mathrm{d}s = e^{-t_i} \frac{R}{2}. \end{aligned} \tag{3.4.11}$$

Combining (3.4.9), (3.4.10) and (3.4.11), we get

$$\begin{aligned}\left\| \mathrm{Av}_{\nu}(h) - \mathrm{Av}_{\mu}(h) \right\| \leq \left\| I^{\uparrow}_{L_i}(h,\, A'_i) - \mathrm{Av}_{\mu}(h) \right\| + d_{A_i} + \left\| I^{\uparrow}_{L_i}(h,\, A'_{i+1}) - I^{\uparrow}_{L_i}(h,\, A'_i) \right\| + d_{A_i} + d_{A_{i+1}} + \\ + \left\| \mathrm{Av}_{\nu}(h) - I^{\uparrow}_{L_i}(h,\, A'_{i+1}) \right\| + d_{A_{i+1}} < \epsilon_i + e^{-t_i} \frac{R}{2} + \epsilon_{i+1} + 2d_{A_i} + 2d_{A_{i+1}}.\end{aligned}$$

Since the sum on the right clearly converges to 0 as i tends to ∞, so, even though the measures are different, their averages $\mathrm{Av}_{\mu}(h)$ and $\mathrm{Av}_{\nu}(h)$ are the same. Since we did not specify the function h beyond the fact that it is 1-Lipschitz, we can take the function $\mathfrak{h}$ from Lemma 3.3.2. We have obtained a contradiction.

A completely analogous proof shows that the sequence $\{I^{\uparrow}(h,\, B_i)\}_{n \in \mathbb{N}_+}$ also converges to $\mathrm{Av}_{\bar{m}}(h)$ for either $\bar{m} = \mu$ or $\bar{m} = \nu$.

Step 2: If $\bar{m} = m$, the proof is complete. It remains to prove that we cannot have $\bar{m} \neq m$.

Recall that h is 1-Lipschitz, $|A'_i B'_i| = |v_s A'_i v_s B'_i|$ for any $s \in [0,\, L_i]$, and $|A'_i B'_i| \leq e^{-t_i} R$

for all $i > 0$. Therefore, for any $i > 0$,

$$\left\| I^{\uparrow}(h,\, B_i') - I^{\uparrow}(h,\, A_i') \right\| = \left\| \frac{1}{L_i} \int_0^{L_i} h(v_s B_i')\, \mathrm{d}s - \frac{1}{L_i} \int_0^{L_i} h(v_s A_i')\, \mathrm{d}s \right\| \leq$$
$$\leq \frac{1}{L_i} \int_0^{L_i} \left\| h(v_s B_i') - h(v_s A_i') \right\| \, \mathrm{d}s \leq \frac{1}{L_i} \int_0^{L_i} e^{-t_i} R \, \mathrm{d}s = e^{-t_i} R.$$

Therefore $\{I^{\uparrow}(h,\, A_i)'\}_{n \in \mathbb{N}_+}$ and $\{I^{\uparrow}(h,\, B_i')\}_{n \in \mathbb{N}_+}$ have to converge to the same limit. Then the same is true for $\{I^{\uparrow}(h,\, A_i)\}_{n \in \mathbb{N}_+}$ and $\{I^{\uparrow}(h,\, B_i)\}_{n \in \mathbb{N}_+}$ from Lemma 3.4.2. □

Part VI. Backward flow limits.

Note that the exact same construction as above can be repeated for the backward vertical flow and the sequence of pairs of points $\{(C_i,\, B_i)\}_{i \in \mathbb{N}_0}$, where $C_i = \varsigma(A_i)$ for all $i \geq 0$. Analogously to Lemma 3.4.3, we can obtain

Lemma 3.4.4. *The sequence $\{I^{\downarrow}(h,\, C_i)\}_{n \in \mathbb{N}_+}$ converges to a finite limit, and so does the sequence $\{I^{\downarrow}(h,\, B_i)\}_{n \in \mathbb{N}_+}$. Moreover, these two limits are equal to each other and to $\mathrm{Av}_{\bar{m}}(h)$, where either $\bar{m} = \mu$ or $\bar{m} = \nu$.*

Moreover, since A_i, B_i, and $C_i = \varsigma(A_i)$ are, by construction, both totally generic and ϵ_i-generic for every $i \geq 1$, we obtain

Lemma 3.4.5. *The sequences $\{I^{\uparrow}(h,\, B_i)\}_{n \in \mathbb{N}_+}$ and $\{I^{\downarrow}(h,\, B_i)\}_{n \in \mathbb{N}_+}$ converge to the same limit. The same statement holds for the sequences $\{I^{\uparrow}(h,\, A_i)\}_{n \in \mathbb{N}_+}$ and $\{I^{\downarrow}(h,\, A_i)\}_{n \in \mathbb{N}_+}$, as well as the sequences $\{I^{\uparrow}(h,\, C_i)\}_{n \in \mathbb{N}_+}$ and $\{I^{\downarrow}(h,\, C_i)\}_{n \in \mathbb{N}_+}$.*

From Lemmas 3.4.3, 3.4.4, and 3.4.5, we conclude that, for every $h \in \mathrm{Lip}_1(\widetilde{X})$,

$$\lim_{i \to \infty} I^{\uparrow}(h,\, A_i) \;=\; \lim_{i \to \infty} I^{\downarrow}(h,\, A_i) \;=\; \lim_{i \to \infty} I^{\uparrow}(h,\, \varsigma(A_i)) \;=\; \lim_{i \to \infty} I^{\downarrow}(h,\, \varsigma(A_i)) \;=\; \mathrm{Av}_m(h), \tag{3.4.12}$$

where either $m = \mu$ or $m = \nu$.

Since A_i, and $\varsigma(A_i)$ are, by construction, totally generic and ϵ_i-generic for every $i \geq 1$, we obtain a contradiction with Corollary 3.3.8. □

3.5 The proof of the main conjecture

3.5.1 Embedded disks

Definition 3.5.1. The *embedded radius* of X is

$$r_{\text{emb}}(X) := \sup\Big\{ r > 0 : \text{there exists an isometric embedding } B_r(0) \hookrightarrow X \Big\},$$

where $B_r(0) \subset \mathbb{R}^2$ is the open Euclidean disk of radius r.

Theorem 3.5.2. *Let X be a finite-area translation surface whose vertical flow is uniquely ergodic, and fix a marked point $P \in X$. Suppose that there exist $r > 0$ and a sequence $t_k \to \infty$ such that*

$$r_{\text{emb}}(X_{t_k}) \geq r, \qquad X_{t_k} := g_{t_k} X.$$

Then for Lebesgue-almost every point $Q \in X$ there exist a number $R(Q) > 0$ and an infinite subsequence $t_{k_j} \to \infty$ such that, for every j, the surface $X_{t_{k_j}}$ contains an embedded Euclidean disk of radius at least $R(Q)$ centered at $g_{t_{k_j}}(Q)$, and this disk does not contain $g_{t_{k_j}}(P)$.

Proof. Fix R with $0 < R < r$.

For each k, choose an isometric embedding

$$\iota_k : B_R(0) \hookrightarrow X_{t_k},$$

and write

$$D_k^R := \iota_k(B_R(0)).$$

Let

$$D_k^{R/2} := \iota_k(B_{R/2}(0)) \subset D_k^R.$$

We now define a measurable subset $A_k \subset D_k^{R/2}$ of uniformly positive area, all of whose points stay a definite Euclidean distance away from $g_{t_k}(P)$ inside the chosen chart.

If $g_{t_k}(P) \notin D_k^R$, set

$$A_k := D_k^{R/2}.$$

If $g_{t_k}(P) \in D_k^R$, let

$$p_k := \iota_k^{-1}(g_{t_k}(P)) \in B_R(0),$$

and set

$$A_k := \iota_k\Big(B_{R/2}(0) \setminus B_{R/4}(p_k)\Big).$$

In either case, $A_k \subset D_k^{R/2}$ is measurable. Moreover,

$$\mathcal{L}(A_k) \geq \pi(R/2)^2 - \pi(R/4)^2 = \frac{3\pi R^2}{16}.$$

Indeed, if $g_{t_k}(P) \notin D_k^R$, then $\mathcal{L}(A_k) = \pi(R/2)^2$. If $g_{t_k}(P) \in D_k^R$, then removing the set

$$B_{R/2}(0) \cap B_{R/4}(p_k)$$

can decrease the area by at most $\pi(R/4)^2$.

Now define

$$F_k := g_{-t_k}(A_k) \subset X.$$

Since the Teichmüller flow preserves Lebesgue measure,

$$\mathcal{L}_X(F_k) = \mathcal{L}(A_k) \geq \frac{3\pi R^2}{16}.$$

Let

$$U_N := \bigcup_{k \geq N} F_k, \qquad S := \bigcap_{N=1}^{\infty} U_N = \limsup_{k \to \infty} F_k.$$

The sets U_N decrease with N, and $U_N \supset F_N$ for every N. Therefore

$$\mathcal{L}_X(S) = \lim_{N \to \infty} \mathcal{L}_X(U_N) \geq \frac{3\pi R^2}{16} > 0.$$

Now apply the Birkhoff–Khinchin pointwise ergodic Theorem 3.3.4 for the vertical flow to the indicator function $\mathbf{1}_S$. Since the vertical flow is uniquely ergodic, it is ergodic for Lebesgue measure. Hence for Lebesgue-almost every point $Q \in X$,

$$\lim_{T \to \infty} \frac{1}{T} \int_0^T \mathbf{1}_S(v_t Q)\, dt = \mathcal{L}_X(S) > 0.$$

In particular, for Lebesgue-almost every Q, the forward vertical orbit of Q meets S infinitely often. Fix such a point Q, and choose one time $s(Q) \geq 0$ such that

$$v_{s(Q)} Q \in S.$$

Since $v_{s(Q)} Q \in S = \limsup_{k \to \infty} F_k$, there exists an infinite subsequence t_{k_j} such that

$$v_{s(Q)} Q \in \bigcap_{j=1}^{\infty} F_{k_j}.$$

Equivalently,

$$x_j := g_{t_{k_j}}(v_{s(Q)} Q) \in A_{k_j}.$$

Because $x_j \in D_{k_j}^{R/2}$, its distance to the boundary of $D_{k_j}^{R}$ is at least $R/2$. Also, if

$g_{t_{k_j}}(P) \in D^R_{k_j}$, then by construction of A_{k_j},

$$d_{D^R_{k_j}}\Big(x_j,\, g_{t_{k_j}}(P)\Big) \geq \frac{R}{4},$$

where $d_{D^R_{k_j}}$ denotes Euclidean distance in the chart $D^R_{k_j}$. If $g_{t_{k_j}}(P) \notin D^R_{k_j}$, then every subset of $D^R_{k_j}$ automatically avoids $g_{t_{k_j}}(P)$.

Now use the relation

$$g_t \circ v_s = v_{e^{-t}s} \circ g_t.$$

Set

$$q_j := g_{t_{k_j}}(Q).$$

Then q_j and x_j lie on the same vertical orbit segment in $X_{t_{k_j}}$, and

$$d(q_j, x_j) = e^{-t_{k_j}}\, s(Q) \longrightarrow 0.$$

After discarding finitely many terms, we may assume that

$$d(q_j, x_j) < \frac{R}{8} \qquad \text{for every } j.$$

We claim that the open metric disk

$$B_{R/8}(q_j)$$

is contained in $D^R_{k_j}$ and does not contain $g_{t_{k_j}}(P)$.

Let $y \in B_{R/8}(q_j)$. Then

$$d(y, x_j) \leq d(y, q_j) + d(q_j, x_j) < \frac{R}{8} + \frac{R}{8} = \frac{R}{4}.$$

Since x_j is at distance at least $R/2$ from $\partial D^R_{k_j}$, it follows that $y \in D^R_{k_j}$. Thus

$$B_{R/8}(q_j) \subset D^R_{k_j}.$$

If $g_{t_{k_j}}(P) \notin D^R_{k_j}$, then this disk certainly avoids $g_{t_{k_j}}(P)$. So suppose instead that

$$g_{t_{k_j}}(P) \in D^R_{k_j}.$$

Since both x_j and y lie in the embedded chart $D^R_{k_j}$, all distances between them are Euclidean chart distances. Therefore

$$d_{D^R_{k_j}}\left(y, g_{t_{k_j}}(P)\right) \geq d_{D^R_{k_j}}\left(x_j, g_{t_{k_j}}(P)\right) - d_{D^R_{k_j}}(y, x_j) > \frac{R}{4} - \frac{R}{4} = 0.$$

Hence $y \neq g_{t_{k_j}}(P)$. Therefore

$$g_{t_{k_j}}(P) \notin B_{R/8}(q_j).$$

We have shown that $B_{R/8}(q_j)$ is contained in the embedded Euclidean disk $D^R_{k_j}$, so it is itself an embedded Euclidean disk in $X_{t_{k_j}}$. This proves the theorem, with

$$R(Q) := \frac{R}{8}.$$

□

Remark 3.5.3. If P is nonsingular and Q lies on the same vertical trajectory as P, then the conclusion generally fails. Indeed, if

$$Q = v_{-a}(P)$$

for some $a \in \mathbb{R}$, then

$$g_t(Q) = v_{-e^{-t}a}(g_t(P)),$$

so

$$d(g_t(Q), g_t(P)) \to 0 \qquad \text{as } t \to \infty.$$

Hence no fixed positive-radius disk centered at $g_t(Q)$ can avoid $g_t(P)$ for all large t. This does not affect the theorem, since a single vertical trajectory has Lebesgue measure zero.

Corollary 3.5.4. *Let X be a finite-area translation surface whose vertical flow is uniquely ergodic, and fix a marked point $P \in X$. Suppose that there exist $r > 0$ and a sequence $t_k \to \infty$ such that*

$$r_{\text{emb}}(X_{t_k}) \geq r, \qquad X_{t_k} := g_{t_k} X.$$

Then for Lebesgue-almost every point $Q \in X$, the double-cover construction determined by the slit from P to Q is uniquely ergodic.

Proof. By Theorem 3.5.2, for Lebesgue-almost every point $Q \in X$ there exist a number $R(Q) > 0$ and an infinite subsequence $t_{k_j} \to \infty$ such that, for every j, the surface $X_{t_{k_j}}$ contains an embedded Euclidean disk of radius at least $R(Q)$ centered at $g_{t_{k_j}}(Q)$, and this disk does not contain $g_{t_{k_j}}(P)$.

Therefore, the conditions of Theorem 3.4.1 are satisfied. □

Note that, apart from finite area and unique ergodicity of the vertical flow, the only additional requirement is the existence of a sequence $t_k \to \infty$ along which the embedded radius of X_{t_k} is bounded below.

3.5.2 Cylinders

Definition 3.5.5 (Flat cylinder)**.** A *flat cylinder* (in direction θ) is an open subset

$$C \subset X \setminus \Sigma$$

isometric, in translation coordinates, to

$$(\mathbb{R}/c\mathbb{Z}) \times (0, h)$$

with the Euclidean metric, for some $c, h > 0$, in such a way that the curves $\{y = \text{const}\}$ map to simple closed geodesics in direction θ on X.

We write:

$$c(C) := \text{circumference}, \qquad h(C) := \text{height}, \qquad a(C) := c(C)h(C), \qquad m(C) := \frac{h(C)}{c(C)}.$$

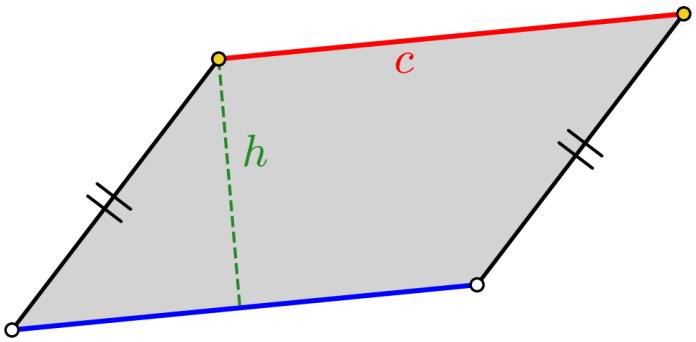


Figure 3.12: A maximal cylinder.

Definition 3.5.6 (Maximal flat cylinder)**.** A flat cylinder C is *maximal* if it is not properly contained in any larger flat cylinder in the same direction. Equivalently, it is a connected component of the set of points whose geodesic in direction θ is periodic. We call two maximal cylinders *disjoint* if they have disjoint interiors.

By [MT02, Lemma 1.6], the closure $\overline{C}$ has two boundary components, each of which is a finite union of saddle connections in direction θ.

Definition 3.5.7. A maximal flat cylinder C is called a *pipe cylinder* if

$$c(C) < h(C),$$

and a *perfect cylinder* if

$$c(C) = h(C).$$

Lemma 3.5.8. *For a fixed translation surface X, any two distinct maximal pipe cylinders are disjoint.*

Proof. Let C_1 and C_2 be two distinct maximal pipe cylinders. If they are parallel, then maximality implies that either they coincide or their interiors are disjoint. So only the non-parallel case needs to be studied.

Assume, for contradiction, that C_1 and C_2 intersect and are not parallel. Then a core curve of C_1 crosses C_2 from one boundary component of C_2 to the other, so its length is at least the height of C_2:

$$c(C_1) \geq h(C_2).$$

Similarly,

$$c(C_2) \geq h(C_1).$$

But since both cylinders are pipes,

$$h(C_1) > c(C_1), \qquad h(C_2) > c(C_2).$$

Combining these inequalities gives

$$c(C_1) \geq h(C_2) > c(C_2) \geq h(C_1) > c(C_1),$$

a contradiction. Hence the interiors of C_1 and C_2 are disjoint. □

Lemma 3.5.9. *For a fixed stratum, the number of maximal pipe cylinders on any surface in that stratum is bounded above by a constant depending only on the stratum.*

Proof. Choose one core curve from each maximal pipe cylinder. By Lemma 3.5.8, these core curves are pairwise disjoint.

Moreover, two such core curves cannot be homotopic in $X \setminus \Sigma$. Indeed, any two simple closed geodesics on a translation surface that avoid the singularities and are freely homotopic lie in the closure of the same maximal cylinder. Hence distinct maximal cylinders give non-homotopic core curves.

Thus the core curves form a collection of pairwise disjoint, pairwise non-homotopic, essential simple closed curves on the punctured surface $X \setminus \Sigma$. The size of such a collection is bounded above by the usual topological constant

$$M = 3g - 3 + n,$$

where g is the genus and $n = |\Sigma|$ is the number of singularities. This bound depends only on the stratum. □

Proposition 3.5.10 (Masur–Smillie, [MS91], Proposition 5.4)**.** *The Delaunay triangulation of an area-1 translation surface X consists of edges which either have length at most $2\sqrt{2/\pi}$ or cross a pipe cylinder C. If an edge crosses C, then its length lies in the interval*

$$[h(C),\ \sqrt{h(C)^2 + c(C)^2}].$$

Lemma 3.5.11. *Suppose there exists a sequence $t_k \to \infty$ such that $X_{t_k} = g_{t_k} X$ contains no pipe cylinders. Then there exists a constant $R > 0$ such that every X_{t_k} contains an embedded disk of radius at least R.*

Proof. Fix the stratum of X, and let T be the number of triangles in a Delaunay triangulation of a surface in that stratum. This number depends only on the stratum.

Since the total area is 1, at least one Delaunay triangle Δ has area

$$A_\Delta \geq \frac{1}{T}.$$

If X_{t_k} contains no pipe cylinders, then Proposition 3.5.10 implies that every Delaunay edge has length at most $2\sqrt{2/\pi}$. Therefore the perimeter of Δ satisfies

$$P_\Delta \leq 6\sqrt{2/\pi}.$$

The inradius of a Euclidean triangle is

$$r_\Delta = \frac{2A_\Delta}{P_\Delta},$$

so

$$r_\Delta \geq \frac{2/T}{6\sqrt{2/\pi}} = \frac{\sqrt{\pi}}{3\sqrt{2}\,T}.$$

Since the interior of each Delaunay triangle is isometrically embedded in the surface, the inscribed Euclidean disk in Δ is an embedded disk in X_{t_k}. Thus one may take

$$R := \frac{\sqrt{\pi}}{3\sqrt{2}\,T}.$$

□

Let $\mathfrak{P}_t$ denote the union of the closures of all pipe cylinders on $X_t = g_t X$, and let $A(P_t)$ be its area.

Lemma 3.5.12. *Assume that the condition of Lemma 3.5.11 fails. Suppose there exist a*

constant $K > 0$ and a sequence $t_k \to \infty$ such that

$$A(\mathfrak{P}_{t_k}) > K \qquad \textit{for all } k.$$

Then there exist a constant $R > 0$ and a sequence $t'_k \to \infty$ such that each $X_{t'_k}$ contains an embedded disk of radius at least R.

Proof. By Lemmas 3.5.8 and 3.5.9, for each k the pipe cylinders on X_{t_k} are pairwise disjoint and there are at most M of them, where M depends only on the stratum. Hence at least one pipe cylinder $C_k \subset X_{t_k}$ has area

$$a(C_k) \geq \frac{K}{M}.$$

Let $c_k(s)$ and $h_k(s)$ be the circumference and height of the image of C_k after flowing forward for time $s \geq 0$. The area is preserved, so

$$c_k(s)\, h_k(s) = a(C_k) \geq \frac{K}{M}.$$

At time $s = 0$, C_k is a pipe cylinder, so $c_k(0) < h_k(0)$. As $s \to +\infty$, the circumference in its periodic direction tends to $+\infty$, so by continuity there exists $s_k \geq 0$ such that

$$c_k(s_k) = h_k(s_k) = \sqrt{a(C_k)} \geq \sqrt{\frac{K}{M}}.$$

Thus, on the later surface

$$X_{t'_k} := X_{t_k + s_k},$$

the image of C_k is a perfect cylinder of height and circumference at least $\sqrt{K/M}$. Any flat cylinder of height L and circumference L contains an embedded Euclidean disk of radius $L/2$,

so $X_{t'_k}$ contains an embedded disk of radius at least

$$R := \frac{1}{2}\sqrt{\frac{K}{M}}.$$

Since $t'_k \geq t_k \to \infty$, we have $t'_k \to \infty$. □

Lemma 3.5.13. *Assume that the conditions of Lemmas 3.5.11 and 3.5.12 both fail. In other words, assume that for all sufficiently large t the surface $X_t = g_t X$ contains at least one pipe cylinder, and that*

$$A(\mathfrak{P}_t) \longrightarrow 0 \qquad \text{as } t \to \infty.$$

Then there exist $R > 0$ and $\tau > 0$ such that for every $t > \tau$, the surface X_t contains an embedded disk of radius at least R.

Proof. Let $\Sigma_t = \{g_t(s_1), \ldots, g_t(s_n)\}$ be the singular set of X_t, and let

$$\Sigma_t(\delta) := \{x \in X_t : d(x, \Sigma_t) \leq \delta\}.$$

If the cone angle at $g_t(s_i)$ is $2\pi(k_i + 1)$, then the Euclidean area of the radius-δ neighborhood of $g_t(s_i)$ is $\pi(k_i + 1)\delta^2$. Hence

$$\mathcal{L}(\Sigma_t(\delta)) \leq \pi\Big(\sum_{i=1}^{n}(k_i + 1)\Big)\delta^2 = \pi(2g - 2 + n)\delta^2.$$

Let

$$C_{\text{str}} := \pi(2g - 2 + n).$$

Choose $\varepsilon > 0$ and $\delta > 0$ so that

$$1 - \varepsilon - C_{\text{str}}\delta^2 > 0.$$

Since $A(\mathfrak{P}_t) \to 0$, there exists $\tau > 0$ such that

$$A(\mathfrak{P}_t) < \varepsilon \qquad \text{for all } t > \tau.$$

For such t, the set

$$F_\delta(t) := X_t \setminus \Big(\mathfrak{P}_t \cup \Sigma_t(\delta)\Big)$$

has positive area, hence is nonempty. Fix $x \in F_\delta(t)$.

Because $d(x, \Sigma_t) > \delta$, there is a well-defined local developing map

$$\iota_x : B_\delta(0) \to X_t$$

which is a local isometry and satisfies $\iota_x(0) = x$.

We claim that the restriction of ι_x to $B_{\delta/4}(0)$ is injective. Suppose not. Then there exist distinct points $u, v \in B_{\delta/4}(0)$ such that

$$\iota_x(u) = \iota_x(v).$$

In translation coordinates, the two local branches differ by a translation

$$T(z) = z + w, \qquad w := v - u.$$

Since

$$|w| = |v - u| < \frac{\delta}{2},$$

we have $w \in B_\delta(0)$, and therefore

$$\iota_x(w) = \iota_x(0) = x.$$

So the straight segment from 0 to w projects to a simple closed geodesic γ based at x of length

$$c := |w| < \frac{\delta}{2}.$$

Let C be the maximal cylinder containing γ. Let a and b be the distances from γ to the two boundary components of C, so that

$$h(C) = a + b.$$

Each boundary component has total length c. Therefore on each boundary component one can find a singularity whose tangential displacement from the perpendicular through x is at most $c/2$. It follows that there are singularities at Euclidean distance at most

$$\sqrt{a^2 + (c/2)^2} \qquad \text{and} \qquad \sqrt{b^2 + (c/2)^2}$$

from x. Since $x \notin \Sigma_t(\delta)$, every singularity is at distance bigger than δ from x, hence

$$a > \sqrt{\delta^2 - (c/2)^2}, \qquad b > \sqrt{\delta^2 - (c/2)^2}.$$

Therefore

$$h(C) = a + b > 2\sqrt{\delta^2 - (c/2)^2}.$$

Because $c < \delta/2$, the right-hand side is strictly larger than c, so

$$h(C) > c.$$

Thus C is a pipe cylinder. But $x \in \gamma \subset C \subset \mathfrak{P}_t$, contradicting the choice $x \in X_t \setminus \mathfrak{P}_t$.

This contradiction proves that $\iota_x|_{B_{\delta/4}(0)}$ is injective. Hence X_t contains an embedded disk of radius $\delta/4$ centered at x.

Since $t > \tau$ was arbitrary, the conclusion holds with

$$R = \frac{\delta}{4}.$$

□

BIBLIOGRAPHY

[AC15] Jayadev S. Athreya and Jon Chaika. "The Hausdorff dimension of non-uniquely ergodic directions in $\mathcal{H}(2)$ is almost everywhere 1/2". In: *Geometry & Topology* 19.6 (2015), pp. 3537–3563. DOI: 10.2140/gt.2015.19.3537.

[AM24] Jayadev S. Athreya and Howard Masur. *Translation Surfaces*. Vol. 242. Graduate Studies in Mathematics. Providence, RI: American Mathematical Society, 2024. ISBN: 978-1-4704-7655-7.

[AEM17] Artur Avila, Alex Eskin, and Martin Möller. "Symplectic and isometric $\mathrm{SL}(2,\mathbb{R})$-invariant subbundles of the Hodge bundle". In: *J. Reine Angew. Math.* 732 (2017), pp. 1–20. DOI: 10.1515/crelle-2014-0142.

[AHS16] Artur Avila, Pascal Hubert, and Alexandra Skripchenko. "Diffusion for chaotic plane sections of 3-periodic surfaces". In: *Inventiones Mathematicae* 206.1 (2016), pp. 109–146. DOI: 10.1007/s00222-016-0650-z.

[AV07] Artur Avila and Marcelo Viana. "Simplicity of Lyapunov spectra: proof of the Zorich-Kontsevich conjecture". In: *Acta Mathematica* 198.1 (2007), pp. 1–56. DOI: 10.1007/s11511-007-0012-1.

[Bai+18] Matt Bainbridge et al. "Compactification of strata of Abelian differentials". In: *Duke Math. J.* 167.12 (2018), pp. 2347–2416. DOI: 10.1215/00127094-2018-0012.

[CMP17] James Carlson, Stefan Müller-Stach, and Chris Peters. *Period Mappings and Period Domains*. 2nd ed. Cambridge Studies in Advanced Mathematics. Cambridge: Cambridge University Press, 2017.

[CM20] Jon Chaika and Howard Masur. "The set of non-uniquely ergodic d-IETs has Hausdorff codimension 1/2". In: *Inventiones mathematicae* 222.3 (2020), pp. 749–832. DOI: 10.1007/s00222-020-00978-3.

[Che03] Yitwah Cheung. “Hausdorff dimension of the set of nonergodic directions (with an Appendix by M. Boshernitzan)”. In: *Annals of Mathematics* 158.2 (2003), pp. 661–678. DOI: 10.4007/annals.2003.158.661.

[CE07a] Yitwah Cheung and Alex Eskin. *Slow Divergence and Unique Ergodicity*. Preprint. 2007. arXiv: 0711.0240 [math.DS]. URL: https://arxiv.org/abs/0711.0240.

[CE07b] Yitwah Cheung and Alex Eskin. “Unique Ergodicity of Translation Flows”. In: *Partially Hyperbolic Dynamics, Laminations, and Teichmüller Flow*. Ed. by Giovanni Forni et al. Vol. 51. Fields Institute Communications. Providence, RI: American Mathematical Society, 2007, pp. 213–221.

[CHM08] Yitwah Cheung, Pascal Hubert, and Howard Masur. “Topological dichotomy and strict ergodicity for translation surfaces”. In: *Ergodic Theory and Dynamical Systems* 28.6 (2008), pp. 1729–1748. DOI: 10.1017/S0143385708000126.

[CM06] Yitwah Cheung and Howard Masur. “A divergent Teichmüller geodesic with uniquely ergodic vertical foliation”. In: *Israel Journal of Mathematics* 152.1 (2006), pp. 1–15. ISSN: 0021-2172. DOI: 10.1007/BF02771972.

[Con90] John B Conway. *A Course in Functional Analysis*. 2nd. Vol. 96. Graduate Texts in Mathematics. New York: Springer-Verlag, 1990. ISBN: 978-0-387-97245-9. DOI: 10.1007/978-1-4757-4383-8.

[Dav26] Diana Davis. *Billiards, Surfaces, and Geometry: A Problem-Centered Approach*. Providence, RI: American Mathematical Society, 2026. ISBN: 978-1-4704-8379-1.

[Doz25] Benjamin Dozier. “Compactifications of strata of differentials”. In: *Bulletin of the London Mathematical Society* 57.12 (2025), pp. 3621–3636. DOI: 10.1112/blms.70153.

[Dyn08] I. A. Dynnikov. “Interval identification systems and plane sections of 3-periodic surfaces”. In: *Proceedings of the Steklov Institute of Mathematics* 263 (2008), pp. 65–77. DOI: 10.1134/S0081543808040068.

[EFW18] Alex Eskin, Simion Filip, and Alex Wright. “The algebraic hull of the Kontsevich–Zorich cocycle”. In: *Annals of Mathematics* 188.1 (2018), pp. 281–313. DOI: 10.4007/annals.2018.188.1.5. arXiv: 1702.02074 [math.DS].

[EKZ14] Alex Eskin, Maxim Kontsevich, and Anton Zorich. “Sum of Lyapunov exponents of the Hodge bundle with respect to the Teichmüller geodesic flow”. In: *Publications Mathématiques de l’IHÉS* 120 (2014), pp. 207–333. DOI: 10.1007/s10240-013-0060-3.

[EM18] Alex Eskin and Maryam Mirzakhani. “Invariant and stationary measures for the $\mathrm{SL}_2(\mathbb{R})$-action on moduli space”. In: *Publ. Math. Inst. Hautes Études Sci.* 127.1 (2018), pp. 95–324. ISSN: 1618-1913. DOI: 10.1007/s10240-018-0099-2.

[EMM15] Alex Eskin, Maryam Mirzakhani, and Amir Mohammadi. “Isolation, equidistribution, and orbit closures for the $\mathrm{SL}(2,\mathbb{R})$ action on moduli space”. In: *Ann. of Math. (2)* 182.2 (2015), pp. 673–721. DOI: 10.4007/annals.2015.182.2.7.

[FH24] Sébastien Ferenczi and Pascal Hubert. “Minimality and unique ergodicity of Veech 1969 type interval exchange transformations”. In: *Geometriae Dedicata* 218 (2024), p. 77. DOI: 10.1007/s10711-024-00888-1.

[Fil16a] Simion Filip. “Semisimplicity and rigidity of the Kontsevich–Zorich cocycle”. In: *Invent. Math.* 205.3 (2016), pp. 617–670. DOI: 10.1007/s00222-015-0643-3.

[Fil16b] Simion Filip. “Splitting mixed Hodge structures over affine invariant manifolds”. In: *Annals of Mathematics* 183.2 (2016), pp. 681–713. DOI: 10.4007/annals.2016.183.2.5.

[Fil19] Simion Filip. “Notes on the multiplicative ergodic theorem”. In: *Ergodic Theory and Dynamical Systems* 39.5 (2019), pp. 1153–1189. DOI: 10.1017/etds.2017.68.

[Fil24] Simion Filip. “Translation surfaces: Dynamics and Hodge theory”. In: *EMS Surveys in Mathematical Sciences* 11.1 (2024).

[For02] Giovanni Forni. “Deviation of ergodic averages for area-preserving flows on surfaces of higher genus”. In: *Ann. of Math. (2)* 155.1 (2002), pp. 1–103. DOI: 10.2307/3062150.

[FM14] Giovanni Forni and Carlos Matheus. “Introduction to Teichmueller theory and its applications to dynamics of interval exchange transformations, flows on surfaces and billiards”. In: *Journal of Modern Dynamics* 8.3–4 (2014), pp. 271–436. DOI: 10.3934/jmd.2014.8.271.

[FMZ14] Giovanni Forni, Carlos Matheus, and Anton Zorich. “Lyapunov spectrum of invariant subbundles of the Hodge bundle”. In: *Ergodic Theory and Dynamical Systems* 34.2 (2014), pp. 353–408. DOI: 10.1017/etds.2012.148.

[Fur02] Alex Furman. “Chapter 12 Random walks on groups and random transformations”. In: ed. by B. Hasselblatt and A. Katok. Vol. 1. Handbook of Dynamical Systems. Elsevier Science, 2002, pp. 931–1014. DOI: 10.1016/S1874-575X(02)80014-5.

[HM79] John Hubbard and Howard Masur. “Quadratic differentials and foliations”. In: *Acta Mathematica* 142 (1979), pp. 221–274. DOI: 10.1007/BF02395062.

[Kai89] Vadim A. Kaimanovich. “Lyapunov exponents, symmetric spaces, and a multiplicative ergodic theorem for semisimple Lie groups”. In: *Journal of Soviet Mathematics* 47 (1989), pp. 2387–2398.

[Kal21] Olav Kallenberg. *Foundations of Modern Probability*. 3rd ed. Vol. 99. Probability Theory and Stochastic Modelling. Cham: Springer, 2021. ISBN: 978-3-030-61871-1. DOI: 10.1007/978-3-030-61871-1.

[Kat73] Anatole B. Katok. “Invariant measures of flows on orientable surfaces”. In: *Soviet Math. Dokl.* 14 (1973). Translated from Dokl. Akad. Nauk SSSR 211 (1973), 775–778, pp. 1104–1108.

[Kea75] Michael Keane. “Interval exchange transformations”. In: *Mathematische Zeitschrift* 141 (1975), pp. 25–31. DOI: 10.1007/BF01236981.

[Kea77] Michael Keane. “Non-ergodic interval exchange transformations”. In: *Israel Journal of Mathematics* 26 (1977), pp. 188–196. DOI: 10.1007/BF03007668.

[KMS86] Steven Kerckhoff, Howard Masur, and John Smillie. “Ergodicity of billiard flows and quadratic differentials”. In: *Ann. of Math. (2)* 124.2 (1986), pp. 293–311. DOI: 10.2307/1971280.

[KN76] Harvey B. Keynes and Dan Newton. “A “minimal”, non-uniquely ergodic interval exchange transformation”. In: *Mathematische Zeitschrift* 148.2 (1976), pp. 101–105. DOI: 10.1007/BF01214699.

[Kon97] Maxim Kontsevich. “Lyapunov exponents and Hodge theory”. In: *The Mathematical Beauty of Physics (Saclay, 1996)*. Vol. 24. Adv. Ser. Math. Phys. River Edge, NJ: World Scientific, 1997, pp. 318–332.

[KZ02] Maxim Kontsevich and Anton Zorich. “Connected components of the moduli spaces of Abelian differentials with prescribed singularities”. In: *Inventiones mathematicae* 153 (2002), pp. 631–678.

[KZ03] Maxim Kontsevich and Anton Zorich. “Connected components of the moduli spaces of Abelian differentials with prescribed singularities”. In: *Inventiones Mathematicae* 153 (2003), pp. 631–678. DOI: 10.1007/s00222-003-0303-x.

[Led86] François Ledrappier. “Positivity of the exponent for stationary sequences of matrices”. In: *Lyapunov Exponents*. Ed. by Ludwig Arnold and Volker Wihstutz.

Vol. 1186. Lecture Notes in Mathematics. Berlin, Heidelberg: Springer, 1986, pp. 56–73. DOI: 10.1007/BFb0076833.

[Mas21] Daniel Massart. "A Short Introduction to Translation Surfaces, Veech Surfaces, and Teichmüller Dynamics". In: *Surveys in Geometry I*. 2021, pp. 343–388.

[Mas82] Howard Masur. "Interval exchange transformations and measured foliations". In: *Ann. of Math. (2)* 115.1 (1982), pp. 169–200. DOI: 10.2307/1971341.

[Mas86] Howard Masur. "Closed trajectories for quadratic differentials with an application to billiards". In: *Duke Mathematical Journal* 53.2 (1986), pp. 307–314. DOI: 10.1215/S0012-7094-86-05319-6.

[Mas92] Howard Masur. "Hausdorff dimension of the set of nonergodic foliations of a quadratic differential". In: *Duke Math. J.* 66.3 (1992), pp. 387–442. DOI: 10.1215/S0012-7094-92-06613-0.

[Mas06] Howard Masur. "Ergodic theory of translation surfaces". In: *Handbook of Dynamical Systems, Vol. 1B*. Amsterdam: Elsevier, 2006, pp. 527–547. DOI: 10.1016/S1874-575X(06)80032-9.

[MS91] Howard Masur and John Smillie. "Hausdorff dimension of sets of nonergodic measured foliations". In: *Ann. of Math. (2)* 134.3 (1991), pp. 455–543. DOI: 10.2307/2944356.

[MT02] Howard Masur and Serge Tabachnikov. "Chapter 13 Rational billiards and flat structures". In: ed. by B. Hasselblatt and A. Katok. Vol. 1. Handbook of Dynamical Systems. Elsevier Science, 2002, pp. 1015–1089. DOI: 10.1016/S1874-575X(02)80015-7.

[Mat18] Carlos Matheus Silva Santos. *Dynamical aspects of Teichmüller theory : $\mathrm{SL}_2\mathbb{R}$-action on moduli spaces of flat surfaces*. 2018.

[McM03] Curtis T. McMullen. "Billiards and Teichmueller curves on Hilbert modular surfaces". In: *Journal of the American Mathematical Society* 16.4 (2003), pp. 857–885. DOI: 10.1090/S0894-0347-03-00432-6.

[McM07] Curtis T. McMullen. "Dynamics of $SL_2(\mathbb{R})$ over moduli space in genus two". In: *Ann. of Math. (2)* 165.2 (2007), pp. 397–456. DOI: 10.4007/annals.2007.165.397.

[MW17] Maryam Mirzakhani and Alex Wright. "The boundary of an affine invariant submanifold". In: *Invent. Math.* 209.3 (2017), pp. 927–984. DOI: 10.1007/s00222-017-0722-8.

[Nov82] S. P. Novikov. "The Hamiltonian formalism and a many-valued analogue of Morse theory". In: *Russian Mathematical Surveys* 37.5 (1982). Originally published in Russian as: Uspekhi Mat. Nauk 37(5) (1982), 3–49, pp. 1–56.

[Nov86] S. P. Novikov. "Critical points and level surfaces of multivalued functions". In: *Proceedings of the Steklov Institute of Mathematics* 166 (1986). Originally published in Russian as: Trudy Mat. Inst. Steklov. 166 (1984), 201–209, pp. 223–232.

[NM98] S. P. Novikov and A. Ya. Mal'tsev. "Topological phenomena in normal metals". In: *Physics-Uspekhi* 41.3 (1998). Originally published in Russian as: Uspekhi Fizicheskikh Nauk 168(3) (1998), 249–258, pp. 231–239. DOI: 10.1070/PU1998v041n03ABEH000373.

[Rag77] Madabusi Santanam Raghunathan. *Diskretnye podgruppy grupp Li. Translated from English by O. V. Svarcman, Edited by E. B. Vinberg, with a supplement "Arithmeticity of irreducible lattices in semisimple groups of rank greater than 1" by G. A. Margulis.* Moscow: Izdat. "Mir", 1977.

[Rau79] Gerard Rauzy. “Echanges d’intervalles et transformations induites”. In: *Acta Arithmetica* 34.4 (1979), pp. 315–328. DOI: 10.4064/aa-34-4-315-328.

[Rie57] Bernhard Riemann. “Theorie der Abel’schen Functionen”. In: *Journal fuer die reine und angewandte Mathematik* 54 (1857), pp. 115–155. DOI: 10.1515/crll.1857.54.115.

[Sat75] E. A. Sataev. “On the number of invariant measures for flows on orientable surfaces”. In: *Mathematics of the USSR-Izvestija* 9.4 (1975). English translation of: Izv. Akad. Nauk SSSR Ser. Mat. 39(4):860–878 (1975), pp. 813–830. DOI: 10.1070/IM1975v009n04ABEH001499.

[Tei40] Oswald Teichmueller. “Extremale quasikonforme Abbildungen und quadratische Differentiale”. In: *Abhandlungen der Preussischen Akademie der Wissenschaften, Mathematisch-Naturwissenschaftliche Klasse* 22 (1940), pp. 1–197.

[Tre14] Rodrigo Treviño. “On the ergodicity of flat surfaces of finite area”. In: *Geometric and Functional Analysis* 24.1 (2014), pp. 360–386. DOI: 10.1007/s00039-014-0269-4.

[Vee69] William A. Veech. “Strict ergodicity in zero dimensional dynamical systems and the Kronecker–Weyl theorem mod 2”. In: *Transactions of the American Mathematical Society* 140 (1969), pp. 1–33. DOI: 10.1090/S0002-9947-1969-0240056-X.

[Vee82] William A. Veech. “Gauss measures for transformations on the space of interval exchange maps”. In: *Ann. of Math. (2)* 115.2 (1982), pp. 201–242. DOI: 10.2307/1971391.

[Vee89] William A. Veech. “Teichmüller curves in moduli space, Eisenstein series and an application to triangular billiards”. In: *Inventiones mathematicae* 97.3 (1989), pp. 553–583. DOI: 10.1007/BF01388890.

[Wri14] Alex Wright. *Translation surfaces and their orbit closures: An introduction for a broad audience*. 2014. arXiv: 1411.1827 [math.DS].

[ZK75] A. N. Zemljakov and A. B. Katok. "Topological transitivity of billiards in polygons". In: *Mat. Zametki* 18.2 (1975), pp. 291–300.

[Zim83] Robert J. Zimmer. *Ergodic Theory and Semisimple Groups*. Boston: Birkhäuser, 1983. DOI: 10.1007/978-1-4684-9488-4.

[Zor84] A. V. Zorich. "A problem of Novikov on the semiclassical motion of an electron in a uniform almost rational magnetic field". In: *Russian Mathematical Surveys* 39.5 (1984). Originally published in Russian as: Uspekhi Mat. Nauk 39(5) (1984), 235–236, pp. 287–288.

[Zor99] Anton Zorich. "How do the leaves of a closed 1-form wind around a surface?" In: *Pseudoperiodic Topology*. Vol. 197. American Mathematical Society Translations: Series 2. Providence, RI: American Mathematical Society, 1999, pp. 135–178.

[Zor06] Anton Zorich. "Flat surfaces". In: *Frontiers in number theory, physics, and geometry I*. 2006. URL: https://math.uchicago.edu/~masur/zorich_leshouches.pdf.