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TOPICS AROUND BRAID GROUPS AND SPACES OF POLYNOMIALS

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BY
PETER JAMES HUXFORD

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ABSTRACT

This dissertation comprises three papers.

In Chapter 1 we will introduce the basic setup, definitions, and notation common to these papers, including the braid group B_n on n strands, and the configuration spaces $\mathrm{UConf}_n\,\mathbb{C}$ of n unordered points in the complex plane..

In Chapter 2 we consider the two related problems of classifying the homomorphisms $B_n \to B_m$ between braid groups, and classifying the holomorphic maps $\mathrm{UConf}_n\,\mathbb{C} \to \mathrm{UConf}_m\,\mathbb{C}$. As a consequence of our work, we complete the classification of holomorphic maps $\mathrm{UConf}_n\,\mathbb{C} \to \mathrm{UConf}_m\,\mathbb{C}$ for $2 \leq m \leq n$. In particular, the map $R\colon\ \mathrm{UConf}_4\,\mathbb{C} \to \mathrm{UConf}_3\,\mathbb{C}$ that appears in Ferrari's solution to the quartic equation is characterized as the the unique holomorphic map, up to an appropriate equivalence relation, whose image is not contained in a single orbit of the affine group. The contents of this chapter are joint work with Jeroen Schillewaert, first published in Mathematische Annalen 391.3 (2025), pp. 4409–4440 [HS25].

In Chapter 3 we consider certain subgroups $B_n[m]$ of B_n, called level m congruence subgroups, associated to the integral Burau representation $\rho_n\colon B_n \to \mathrm{GL}_n(\mathbb{Z})$. For $n \geq 5$ we prove that $B_n[m]$ is generated by $\ker(\rho_n)$ and the normal closure of a single mth power of a half twist. As a consequence, we find that $B_n[m]$ is normally generated in B_n by just three elements for $n \geq 5$, each of which admits a simple topological description. The results of this chapter are joint work with Ishan Banerjee [BH24].

In Chapter 4 we work with a stratification of the configuration space $\mathrm{UConf}_n\,\mathbb{C}$, called the equicritical stratification, indexed by partitions κ of $n-1$. The fundamental group of a stratum $\mathrm{Poly}_n(\mathbb{C})[\kappa]$ is called a stratified braid group $\mathscr{B}_n[\kappa]$. If $r = |\kappa|$ is the number of parts in the partition, then there is a monodromy homomorphism $\mathscr{B}_n[\kappa] \to B_{n+r}$ which arises from a natural map $\mathrm{Poly}_n(\mathbb{C})[\kappa] \to \mathrm{UConf}_{n+r}(\mathbb{C})$. We prove that this monodromy homomorphism is not injective when $r = 2$. The results of this chapter are joint work with Nick Salter [HS24].

CHAPTER 1

INTRODUCTION

1.1 Notation

The main objects in this dissertation are configuration spaces and braid groups. For a topological space X, let

$$\mathrm{PConf}_n X \coloneqq \{(x_1, \ldots, x_n) \in X^n : x_i \neq x_j \text{ for all } i \neq j\}$$

be the *ordered configuration space* of n points in X. The symmetric group S_n acts naturally on $\mathrm{PConf}_n X$ by permuting the coordinates, and the quotient

$$\mathrm{UConf}_n X \coloneqq (\mathrm{PConf}_n X)/S_n$$

is the *unordered configuration space* of n points in X, and can be thought of as the space of n-element subsets of X. If X is a complex manifold, then so too are $\mathrm{PConf}_n X$ and $\mathrm{UConf}_n X$. We are largely concerned with the special case of $X = \mathbb{C}$. In this special case, there is an important isomorphism

$$\mathrm{UConf}_n \mathbb{C} \cong \mathrm{Poly}_n \mathbb{C} \coloneqq \{f \in \mathbb{C}[z] : f \text{ is monic and squarefree}\},$$

given by associating $\{z_1, \ldots, z_n\} \in \mathrm{UConf}_n \mathbb{C}$ to the polynomial $f(z) = (z - z_1) \cdots (z - z_n)$.

The fundamental groups

$$PB_n \coloneqq \pi_1(\mathrm{PConf}_n \mathbb{C}), \qquad B_n \coloneqq \pi_1(\mathrm{UConf}_n \mathbb{C}),$$

are the *pure braid group* and the *braid group*, respectively, on n strands.

1.2 Holomorphic rigidity

The main result of Chapter 2, which is joint work with Jeroen Schillewaert, first published in Mathematische Annalen 391.3 (2025), pp. 4409–4440 [HS25], is a classification of the holomorphic maps $\mathrm{UConf}_n \mathbb{C} \to \mathrm{UConf}_m \mathbb{C}$ for $n \geq 3$ and $m \in \{3, 4\}$. When combined with earlier work of Lin [Lin04b], and Chen and Salter [CS23], this completes the classification of such holomorphic maps for $2 \leq m \leq n$. To state this result more efficiently, we supress some of the details here, saving them for Chapter 2, and slightly weaken the statement of the theorem here.

Let the *affine group* $\mathrm{Aff} \cong \mathbb{C} \rtimes \mathbb{C}^*$ denote the group of affine automorphisms $z \mapsto az + b$ of $\mathbb{C}$. The coordinatewise action of Aff on $\mathrm{PConf}_n \mathbb{C}$ descends to $\mathrm{UConf}_n \mathbb{C}$. We say that two holomorphic maps $\mathrm{UConf}_n \mathbb{C} \to \mathrm{UConf}_m \mathbb{C}$ are *equivalent* if they define the same map to the orbit space $\mathrm{UConf}_n \mathbb{C} \to (\mathrm{UConf}_m \mathbb{C})/\mathrm{Aff}$.

We prove that for $n \geq 3$ and $m \in \{3, 4\}$, every holomorphic map $\mathrm{UConf}_n \mathbb{C} \to \mathrm{UConf}_m \mathbb{C}$ not equivalent to the identity or a constant map must be equivalent to one of the following three maps

- Ferrari's map $R\colon \mathrm{UConf}_4 \mathbb{C} \to \mathrm{UConf}_3 \mathbb{C}$ defined by

$$R(\{x_1, x_2, x_3, x_4\}) \coloneqq \{x_1x_4 + x_2x_3, x_1x_3 + x_2x_4, x_1x_2 + x_3x_4\}.$$

  When reinterpreted as a map $\mathrm{Poly}_4 \mathbb{C} \to \mathrm{Poly}_3 \mathbb{C}$, this is step one of Ferrari's solution to the quartic equation.

- An elliptic curve construction $\Psi_3\colon \mathrm{UConf}_3 \mathbb{C} \to \mathrm{UConf}_4 \mathbb{C}$ defined by

$$\Psi_3(\{x_1, x_2, x_3\}) \coloneqq \left\{ \begin{array}{l} x\text{-coordinates of the elementary 3-torsion} \\ \text{points of } y^2 = (x - x_1)(x - x_2)(x - x_3) \end{array} \right\}.$$

- The composition $\Psi_3 \circ R \colon \operatorname{UConf}_4 \mathbb{C} \to \operatorname{UConf}_4 \mathbb{C}$.

The key strategy we employ to prove this result is to reduce our theorem to a classification of particular kinds of homomorphisms between braid groups, which we also carry out.

1.3 Congruence subgroups of braid groups

The main result of Chapter 3, which is joint work with Ishan Banerjee [BH24], relates three important subgroups of the braid group.

- Let $\sigma_1, \dots, \sigma_{n-1}$ be the standard generators of B_n. A *half-twist* is any element of B_n that is conjugate to σ_1. Note that the standard generators of B_n are all half-twists. We define the *half-twist mth power subgroup* B_n^m to be the normal closure of σ_1^m in B_n:

  $$B_n^m \coloneqq \langle \sigma^m : \sigma \in B_n \text{ is a half-twist} \rangle.$$

  This conflicts with the common notation G^m, which typically denotes the subgroup of G generated by all its mth powers. As such, we do not make use of this more common notation in our paper.

- The *unreduced Burau representation* is the homomorphism $B_n \to \mathrm{GL}_n(\mathbb{Z}[t, t^{-1}])$ defined by $\sigma_i \mapsto I_{i-1} \oplus \left(\begin{smallmatrix} 1-t & 1 \\ t & 0 \end{smallmatrix}\right) \oplus I_{n-i-1}$, $1 \le i < n$, where I_j is the $j \times j$ identity matrix. The *integral Burau representation* is the homomorphism $\rho_n \colon B_n \to \mathrm{GL}_n(\mathbb{Z})$ obtained by specializing the unreduced Burau representation at $t = -1$. The *braid Torelli group* $\mathcal{BI}_n$ is the kernel

  $$\mathcal{BI}_n \coloneqq \ker\left(B_n \xrightarrow{\rho_n} \mathrm{GL}_n(\mathbb{Z})\right).$$

  The Burau representation is often defined using the transposes of the matrices we use, however this difference does not affect any of the subgroups we define.

- By post-composing the integral Burau representation with the mod-m reduction map $\mathrm{GL}_n(\mathbb{Z}) \to \mathrm{GL}_n(\mathbb{Z}/m\mathbb{Z})$, we obtain a homomorphism $B_n \to \mathrm{GL}_n(\mathbb{Z}/m\mathbb{Z})$. Its kernel $B_n[m]$ is the *level m congruence subgroup* of B_n:

$$B_n[m] \coloneqq \ker\left(B_n \xrightarrow{\rho_n} \mathrm{GL}_n(\mathbb{Z}) \xrightarrow{\mathrm{mod}\ m} \mathrm{GL}_n(\mathbb{Z}/m\mathbb{Z})\right).$$

Our theorem is for $n \geq 5$ that

$$B_n[m] = \langle B_n^m, \mathcal{BI}_n \rangle.$$

This equality is equivalent to the assertion that the normal closure of $\rho(\sigma_1)^m$ in $\rho_n(B_n)$ is equal to $\rho_n(B_n[m])$, and this is the perspective we take when proving this result.

1.4 Noninjective monodromy for equicritical strata

The main result of Chapter 4, which is joint work with Nick Salter [HS24], is that a particular monodromy homomorphism is not injective. This monodromy homomorphism comes from considering a particular stratification of $\mathrm{UConf}_n \mathbb{C} \cong \mathrm{Poly}_n \mathbb{C}$. If $\kappa = \{k_1, \ldots, k_r\}$ is a partition of $n-1$, define the *equicritical stratum* $\mathrm{Poly}_n(\mathbb{C})[\kappa]$ as the locus in $\mathrm{Poly}_n(\mathbb{C})$ consisting of those f whose critical points have multiplicities specified by κ.

Each $\mathrm{Poly}_n(\mathbb{C})[\kappa]$ admits a natural map into the space $\mathrm{UConf}_{n+r}(\mathbb{C})$ by associating to f the $(n+r)$-tuple of roots and critical points. Writing $\mathscr{B}_n[\kappa] \coloneqq \pi_1(\mathrm{Poly}_n(\mathbb{C})[\kappa])$, passing to the fundamental group induces a monodromy homomorphism

$$\rho\colon \mathscr{B}_n[\kappa] \to B_{n+r}.$$

Our main result is that for all $n \geq 3$ and all $p, q \geq 1$ such that $p+q = n-1$, the monodromy map $\rho\colon \mathscr{B}_n[p,q] \to B_{n+2}$ is not injective.

CHAPTER 2

BRAID GROUPS, ELLIPTIC CURVES, AND RESOLVING THE QUARTIC

This chapter is joint work with Jeroen Schillewaert [HS25], first published in Mathematische Annalen 391.3 (2025), pp. 4409–4440, reproduced here with permission from Springer Nature. Both authors contributed to the ideas, writing, and figures.

2.1 Introduction

In this chapter we address some interesting special cases of the following fundamental question.

Question. What are the holomorphic maps $\mathrm{UConf}_n \mathbb{C} \to \mathrm{UConf}_m \mathbb{C}$?

An answer to this question can be thought of as a classification of the geometric constructions which take n points in the complex plane, and produce m points in the complex plane.

Lin answered this question when $n \geq 5$ and $m \leq n$ [Lin04b], which was generalized by Chen and Salter to the cases with $n \geq 5$ and $m \leq 2n$ [CS23]. In order to state their results and our own, we first define a natural equivalence relation on the set such maps, whose name we take from Chen and Salter [CS23].

Definition (**Affine twist**)**.** Let $f \colon Y \to Z$ be a holomorphic map between complex manifolds Y and Z, and suppose the *affine group*

$$\mathrm{Aff} \coloneqq \{z \mapsto az + b : a \in \mathbb{C}^*, b \in \mathbb{C}\} \cong \mathbb{C} \rtimes \mathbb{C}^*.$$

has a holomorphic action on Z. If $A \colon Y \to \mathrm{Aff}$ is a holomorphic map, then the *affine twist*

$f^A \colon Y \to Z$ of f by A is the holomorphic map given by

$$f^A(y) \coloneqq A(y) \cdot f(y).$$

We say that f and f^A are *affine equivalent*.

We let Aff act on $\mathrm{UConf}_n \mathbb{C}$ element-wise, and consider the holomorphic maps $\mathrm{UConf}_n \mathbb{C} \to \mathrm{UConf}_m \mathbb{C}$ up to affine equivalence.

Chen and Salter define the following class of examples of holomorphic maps $\mathrm{UConf}_n \mathbb{C} \to \mathrm{UConf}_m \mathbb{C}$ [CS23, §3].

Definition (Root map). If $k, p \geq 1$ and $\epsilon \in \{0, 1\}$, then a *basic root map* is a map $r_{p,\epsilon} \colon \mathrm{UConf}_k \mathbb{C}^* \to \mathrm{UConf}_{kp+\epsilon} \mathbb{C}$ given by taking pth roots, and also including zero if $\epsilon = 1$. A *root map* $\mathrm{UConf}_n \mathbb{C} \to \mathrm{UConf}_{kp+\epsilon}$ is a composition of the form

$$\mathrm{UConf}_n \mathbb{C} \to \mathrm{UConf}_k \mathbb{C}^* \xrightarrow{r_{p,\epsilon}} \mathrm{UConf}_{kp+\epsilon} \mathbb{C},$$

where the first map is an affine twist of a constant map $\mathrm{UConf}_n \mathbb{C} \to \mathrm{UConf}_k \mathbb{C}^*$. To make sense of the affine twist, we let Aff act on $\mathrm{UConf}_k \mathbb{C}^*$ element-wise with translations acting trivially.

It's worth noting that constant maps are examples of root maps. In fact, Chen and Salter prove that affine twists of root maps are precisely the holomorphic maps $\mathrm{UConf}_n \mathbb{C} \to \mathrm{UConf}_m \mathbb{C}$ for which the induced map $\mathrm{UConf}_n \mathbb{C} \to \mathrm{UConf}_m \mathbb{C}/\operatorname{Aff}$ is constant [CS23, §3.3].

Now we are in a position to more precisely state what is known about holomorphic maps between unordered configuration spaces. Lin's results [Lin04a, Theorems 1.4, 4.5, 4.8] together with Chen and Salter's generalizations [CS23, Theorem 3.1] imply that if $n \geq 5$ and $m \leq 2n$, then, up to affine equivalence, the only holomorphic maps $\mathrm{UConf}_n \mathbb{C} \to \mathrm{UConf}_m \mathbb{C}$ are root maps, and the identity map when $n = m$.

It is an open question whether all holomorphic maps $\mathrm{UConf}_n \mathbb{C} \to \mathrm{UConf}_m \mathbb{C}$ with $n \geq 5$ are affine twists of either a root map or of the identity map. However, for $n = 3, 4$ there are many examples of holomorphic maps $\mathrm{UConf}_n \mathbb{C} \to \mathrm{UConf}_m \mathbb{C}$ that are not of this form.

The next map we describe is such an example, and arises from identifying a monic, square-free polynomial of degree n over $\mathbb{C}$ with the set of its roots in $\mathrm{UConf}_n \mathbb{C}$. It has some interesting history: as an 18-year-old in 1540, Lodovico Ferrari discovered a method for solving a general quartic equation, by first solving a closely related cubic equation. This method gives rise to a holomorphic map often referred to as *resolving the quartic*.

Definition (**Resolving the quartic**)**.** Let $R\colon \mathrm{UConf}_4 \mathbb{C} \to \mathrm{UConf}_3 \mathbb{C}$ be the map that lifts to $\tilde{R}\colon \mathrm{PConf}_4 \mathbb{C} \to \mathrm{PConf}_3 \mathbb{C}$ given by

$$\tilde{R}(x_1, x_2, x_3, x_4) \coloneqq (x_1x_4 + x_2x_3, x_1x_3 + x_2x_4, x_1x_2 + x_3x_4).$$

It is clear that $\tilde{R}$ defines a map $\mathrm{PConf}_4 \mathbb{C} \to \mathbb{C}^3$, but it is quite remarkable that its image is contained in $\mathrm{PConf}_3 \mathbb{C}$. Indeed, the cross ratio of the three points $\tilde{R}(x_1, x_2, x_3, x_4)$ and ∞ is equal to the cross ratio of x_1, x_2, x_3, and x_4.

Farb conjectured that, up to affine equivalence, any holomorphic map $\mathrm{UConf}_n \mathbb{C} \to \mathrm{UConf}_m \mathbb{C}$ with $n \geq 4$ and $m \geq 3$ is either a root map, the identity, or factors through the resolving quartic map $R\colon \mathrm{UConf}_4 \mathbb{C} \to \mathrm{UConf}_3 \mathbb{C}$ [Far24, Conjecture 2.1]. The aforementioned work of Lin [Lin04b], and Chen and Salter [CS23] confirms this is the case in the ranges $n \geq 5$, $m \leq 2n$.

Farb points out that this conjecture requires the assumption $n \geq 4$. This is due to the existence of the following holomorphic maps. Consider the family of elliptic curves over $\mathrm{UConf}_3 \mathbb{C}$ equipped with a planar embedding given by the equation

$$y^2 = (x - x_1)(x - x_2)(x - x_3), \qquad \{x_1, x_2, x_3\} \in \mathrm{UConf}_3 \mathbb{C} \tag{$*$}$$

For $k \geq 2$, let *Jordan's totient function* $J_2(k)$ be the number of elementary k-torsion points on an elliptic curve over $\mathbb{C}$, and let $m_k = J_2(k)/2$ if $k > 2$, and $m_2 = J_2(2) = 3$.

Definition (**Elliptic curve construction**)**.** For $k \geq 2$, let $\Psi_k \colon \mathrm{UConf}_3 \mathbb{C} \to \mathrm{UConf}_{m_k} \mathbb{C}$ be the holomorphic map given by

$$\Psi_k(\{x_1, x_2, x_3\}) \coloneqq \left\{ \begin{array}{l} x\text{-coordinates of the elementary } k\text{-torsion} \\ \text{points of } y^2 = (x - x_1)(x - x_2)(x - x_3) \end{array} \right\}.$$

Although $\Psi_2 \colon \mathrm{UConf}_3 \mathbb{C} \to \mathrm{UConf}_3 \mathbb{C}$ is just the identity map, one can show $\Psi_3 \colon \mathrm{UConf}_3 \mathbb{C} \to \mathrm{UConf}_4 \mathbb{C}$ is a holomorphic map that does not lift to $\mathrm{PConf}_3 \mathbb{C} \to \mathrm{PConf}_4 \mathbb{C}$.

We're now in a position to state our main theorem. We complete the classification of holomorphic maps $\mathrm{UConf}_n \mathbb{C} \to \mathrm{UConf}_m \mathbb{C}$ for $m \leq n$. We also partially resolve Conjecture 2.1 of Farb [Far24].

Theorem 2.1.1. *If $n \geq 3$ and $m \in \{3, 4\}$, then, up to affine equivalence, every holomorphic map $\mathrm{UConf}_n \mathbb{C} \to \mathrm{UConf}_m \mathbb{C}$ is either*

- *A root map* $\mathrm{UConf}_n \mathbb{C} \to \mathrm{UConf}_m \mathbb{C}$
- *The identity map* $\mathrm{UConf}_n \mathbb{C} \to \mathrm{UConf}_n \mathbb{C}$
- $R \colon \mathrm{UConf}_4 \mathbb{C} \to \mathrm{UConf}_3 \mathbb{C}$
- $\Psi_3 \colon \mathrm{UConf}_3 \mathbb{C} \to \mathrm{UConf}_4 \mathbb{C}$
- $\Psi_3 \circ R \colon \mathrm{UConf}_4 \mathbb{C} \to \mathrm{UConf}_4 \mathbb{C}$

It's worth noting that the composition $R \circ \Psi_3 \colon \mathrm{UConf}_3 \mathbb{C} \to \mathrm{UConf}_3 \mathbb{C}$ happens to be affine equivalent to a root map.

Combined with the results of Lin [Lin04b], and Chen and Salter [CS23], Theorem 2.1.1 completes the classification of holomorphic maps $\mathrm{UConf}_n \mathbb{C} \to \mathrm{UConf}_m \mathbb{C}$ for $m \leq n$.

Our second theorem concerns geometric constructions of elliptic curves. Let $\mathcal{M}_{g,n}$ be the moduli space of genus g Riemann surfaces with n unordered marked points. Then $\mathcal{M}_{1,1} \cong \mathrm{SL}_2 \mathbb{Z} \backslash \mathbb{H}$ is the moduli space of elliptic curves. Let the *hyperelliptic map*

$$H\colon \mathrm{UConf}_3 \mathbb{C} \to \mathcal{M}_{1,1}$$

be the holomorphic map corresponding to the family of elliptic curves $(*)$, forgetting the planar embedding. We think of $\mathcal{M}_{1,1}$ as a complex orbifold, so that holomorphic maps $Y \to \mathcal{M}_{1,1}$, where Y is a complex manifold, bijectively correspond to isomorphism classes of families of elliptic curves over Y. We may forget the orbifold structure of $\mathcal{M}_{1,1}$ by using the j-invariant $j\colon \mathcal{M}_{1,1} \to \mathbb{C}$. If $f, g\colon Y \to \mathcal{M}_{1,1}$ are holomorphic families of elliptic curves, then $j \circ f = j \circ g$ if and only if for all $y \in Y$ the fibers over y in each family are isomorphic.

Let $\Delta\colon \mathrm{UConf}_n \mathbb{C} \to \mathbb{C}^*$ be the *discriminant*

$$\Delta(\{x_1, \ldots, x_n\}) \coloneqq \prod_{i \neq j} (x_i - x_j).$$

We can affine twist by the discriminant by viewing $\mathbb{C}^*$ as a subgroup of Aff.

Theorem 2.1.2. *Let $f\colon \mathrm{UConf}_n \mathbb{C} \to \mathcal{M}_{1,1}$ be a holomorphic family of elliptic curves over $\mathrm{UConf}_n \mathbb{C}$. Let id^{Δ} be the affine twist of the identity on $\mathrm{UConf}_3 \mathbb{C}$ by the discriminant Δ. Then one of the following holds:*

- *$j \circ f$ is constant,*
- *$n = 3$ and f is H or $H \circ \mathrm{id}^{\Delta}$,*
- *$n = 4$ and f is $H \circ R$ or $H \circ \mathrm{id}^{\Delta} \circ R$.*

Although $j \circ H = j \circ H \circ \mathrm{id}^{\Delta}$, the families H and $H \circ \mathrm{id}^{\Delta}$ have different monodromy. A similar phenomenon occurs in a theorem of Chen and Salter. They prove that if $\mathcal{M}_g \coloneqq \mathcal{M}_{g,0}$,

then the hyperelliptic family $\mathrm{UConf}_n \mathbb{C} \to \mathcal{M}_{\lfloor (n-1)/2 \rfloor}$ is the only non-trivial holomorphic family $\mathrm{UConf}_n \mathbb{C} \to \mathcal{M}_g$ for $n \geq 26$ and $g \leq n-2$ up to precomposing with $\mathrm{id}^{\Delta} \colon \mathrm{UConf}_n \mathbb{C} \to \mathrm{UConf}_n \mathbb{C}$, see [CS23, §3.5].

There is also a hyperelliptic family of genus one curves over $\mathrm{UConf}_4 \mathbb{C}$, given by the equation

$$y^2 = (x - x_1)(x - x_2)(x - x_3)(x - x_4).$$

By taking the Jacobian of the underlying genus one curves in this family, we obtain a family $\mathrm{UConf}_4 \mathbb{C} \to \mathcal{M}_{1,1}$ of elliptic curves. One can show that this is isomorphic to the family $H \circ R$.

To prove Theorems 2.1.1 and 2.1.2 we first classify the homomorphisms between the relevant fundamental groups.

The *braid group on n strands of X*, where X is a topological space, is the fundamental group

$$B_n(X) \coloneqq \pi_1(\mathrm{UConf}_n X).$$

The *braid group on n strands* is $B_n \coloneqq B_n(\mathbb{C})$. Lin classified the homomorphisms $B_n \to B_m$ for $n \geq 5$ and $m \leq n$ [Lin04a]. These results were extended by Castel who handled the cases $n \geq 6$, $m \leq n+1$ [Cas16], and finally Chen, Kordek, and Margalit extended this to $n \geq 5$, $m \leq 2n$ [CKM23].

Chen, Kordek, and Margalit's classification is in terms of a natural equivalence relation on the set of homomorphisms $B_n \to B_m$ that we define in §2.4. This equivalence relation is roughly analogous to affine equivalence for holomorphic maps. They show that if $n \geq 5$ and $m \leq 2n$, then there are at most five equivalence classes of homomorphisms $B_n \to B_m$, and they give explicit representatives of these equivalence classes [CKM23, Theorem 1.1].

We prove the following theorem which stands in stark contrast to these results. The special case $m = n = 3$ is a conjecture of Castel [Cas09, Conjecture 14.2].

Theorem 2.1.3. *If $n \in \{3, 4\}$ and $m \geq 3$, then there are infinitely many equivalence classes of homomorphisms $B_n \to B_m$.*

The ubiquity of homomorphisms in these cases reflects that previously established classification methods break down in the cases we consider, both in the group theoretic and holomorphic settings.

Indeed, Chen and Salter's classification [CS23, Theorem 3.1] uses techniques from Teichmüller theory and complex analysis to show that at most two of Chen, Kordek, and Margalit's five equivalence classes can be represented by holomorphic maps. Since both of these are in fact represented by holomorphic maps, they are able to apply a theorem of Imayoshi and Shiga [IS88] to complete their classification.

Our proofs require, in addition to the techniques of Chen and Salter, a well-known consequence of work of Bers [Ber78], which concerns the monodromy of families of Riemann surfaces over a punctured disk, see Theorem 2.7.1. This fact imposes a constraint on homomorphisms $B_n \to B_m$ that can arise from a holomorphic map. When $n \geq 5$ this constraint is automatically satisfied for all known homomorphisms with non-cyclic image. However, when $n \in \{3, 4\}$ there are infinitely many inequivalent examples where it is not satisfied.

To prove Theorem 2.1.1 we classify the homomorphisms $B_n \to B_m$ for $m \in \{3, 4\}$ that satisfy the constraint imposed by Theorem 2.7.1 and the other constraints used by Chen and Salter [CS23]. Our methods rely on special properties of the braid groups B_3 and B_4. To handle homomorphisms $B_n \to B_4$ we solve a delicate equation in the free group of rank 2 — this is the sole purpose of §2.6.

To prove Theorem 2.1.2, many of the considerations for Theorem 2.1.1 also apply, since the orbifold fundamental group $\pi_1^{\mathrm{orb}}(\mathcal{M}_{1,1}) \cong \mathrm{SL}_2 \mathbb{Z}$ is a central quotient of B_3.

2.1.1 Structure of the paper

In §2.2 we classify homomorphisms $B_3 \to \mathrm{PSL}_2\,\mathbb{C}$, and use this to classify homomorphisms $B_n \to \mathrm{PSL}_2\,\mathbb{Z}$ and $B_n \to B_3$. We discuss Nielsen–Thurston theory in §2.3 and use it in §2.4 to prove Theorem 2.1.3, in particular proving a conjecture of Castel. In §2.5 we use the Nielsen–Thurston classification theorem to classify the homomorphisms $B_3 \to B_4$ and $B_4 \to B_4$. In §2.6 we prove a property of a particular automorphism of the free group of rank 2. We use many of our group theoretic results, together with results on holomorphic maps we discuss in §2.7, to prove Theorems 2.1.1 and 2.1.2.

2.2 Homomorphisms $B_n \to \mathrm{PSL}_2\,\mathbb{Z}$ and $B_n \to B_3$

We begin by establishing some relevant facts about the braid groups. Artin's presentation of B_n is

$$B_n = \langle \sigma_1, \ldots, \sigma_{n-1} \mid \sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1}, \quad 1 \leq i < n-1 \\ \sigma_i\sigma_j = \sigma_j\sigma_i, \quad 1 \leq i < j-1 < n-1\rangle.$$

The holomorphic map $R\colon \mathrm{UConf}_4\,\mathbb{C} \to \mathrm{UConf}_3\,\mathbb{C}$ induces the homomorphism $R_*\colon B_4 \twoheadrightarrow B_3$ given by $\sigma_1, \sigma_3 \mapsto \sigma_1$ and $\sigma_2 \mapsto \sigma_2$.

From Artin's presentation it can be seen that $B_3 = \langle \alpha, \beta \mid \alpha^3 = \beta^2\rangle$, where $\alpha = \sigma_1\sigma_2$ and $\beta = \sigma_1\sigma_2\sigma_1$. Hence $Z(B_3) = \langle \alpha^3\rangle = \langle \beta^2\rangle$, and there is a short exact sequence

$$1 \to Z(B_3) \to B_3 \to \mathrm{PSL}_2\,\mathbb{Z} \to 1.$$

Bearing these facts in mind, we can state the two main theorems of this section. We say a homomorphism $\varphi\colon G \to H$ is *cyclic* if the image $\varphi(G)$ is a cyclic subgroup of H.

Theorem 2.2.1. *If $\varphi\colon B_n \to \mathrm{PSL}_2\,\mathbb{Z}$ is a homomorphism, then the following hold.*

(i) If $n \geq 5$, then φ is cyclic.

(ii) If $n = 4$, then φ factors through $R_*\colon B_4 \twoheadrightarrow B_3$.

(iii) If $n = 3$ and φ is non-cyclic, then φ descends to an endomorphism of $\mathrm{PSL}_2\,\mathbb{Z}$.

Since $\mathrm{PSL}_2\,\mathbb{Z} \cong \mathbb{Z}/2\mathbb{Z} * \mathbb{Z}/3\mathbb{Z}$, it is possible to classify its many endomorphisms, see Proposition 2.2.6. This gives a classification of homomorphisms $B_n \to \mathrm{PSL}_2\,\mathbb{Z}$.

We also classify the homomorphisms $B_3 \to \mathrm{PSL}_2\,\mathbb{Z}$ which send σ_1 and σ_2 to parabolic elements of $\mathrm{PSL}_2\,\mathbb{Z}$. This additional constraint will be relevant in §2.7 where we classify holomorphic maps.

We write $\left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right]$ for the image of $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2\,\mathbb{Z}$ in $\mathrm{PSL}_2\,\mathbb{Z}$.

Theorem 2.2.2. *If $\varphi\colon B_3 \to \mathrm{PSL}_2\,\mathbb{Z}$ is a non-cyclic homomorphism with $\varphi(\sigma_1)$ and $\varphi(\sigma_2)$ parabolic, then it is given by composing the quotient map $B_3 \twoheadrightarrow B_3/Z(B_3)$, with some isomorphism $B_3/Z(B_3) \cong \mathrm{PSL}_2\,\mathbb{Z}$. In other words, up to automorphisms of $\mathrm{PSL}_2\,\mathbb{Z}$ it is given by*

$$\varphi(\sigma_1) = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \qquad \varphi(\sigma_2) = \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}.$$

Note that the quotient map $B_3 \twoheadrightarrow \mathrm{PSL}_2\,\mathbb{Z}$ lifts to a surjection $H_*\colon B_3 \twoheadrightarrow \pi_1^{\mathrm{orb}}(\mathcal{M}_{1,1}) \cong \mathrm{SL}_2\,\mathbb{Z}$ given by

$$\sigma_1 \mapsto \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \qquad \sigma_2 \mapsto \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}.$$

In §2.2.3 we will address the implications of Theorems 2.2.1 and 2.2.2 for homomorphisms $B_n \to B_3$. Throughout §2.2 we will use the following result extensively.

Lemma 2.2.3 ([For96, Corollary 2])**.** *Let $\varphi\colon B_n \to G$ be a homomorphism. Then either of the following conditions imply that φ is cyclic.*

(i) $\varphi(\sigma_i)$ commutes with $\varphi(\sigma_{i+1})$ for some $1 \leq i < n-1$.

(ii) $n \geq 5$, *and* $\varphi(\sigma_i) = \varphi(\sigma_j)$ *for some* $1 \leq i < j < n$.

2.2.1 $B_3 \to \mathrm{PSL}_2\,\mathbb{C}$ *and* $B_3 \to \mathrm{PSL}_2\,\mathbb{Z}$

Lemma 2.2.4. *If* $\varphi\colon B_3 \to \mathrm{PSL}_2\,\mathbb{C}$ *is non-cyclic, then* $Z(B_3) \leq \ker(\varphi)$. *In particular* φ *descends to a homomorphism* $\mathrm{PSL}_2\,\mathbb{Z} \to \mathrm{PSL}_2\,\mathbb{C}$.

Proof. Write $a = \varphi(\sigma_1\sigma_2)$ and $b = \varphi(\sigma_1\sigma_2\sigma_1)$, so that $a^3 = b^2$. It suffices to show that either a and b commute, or $a^3 = b^2 = I$. Non-identity powers of a given Möbius transformation have the same fixed point set in $\mathbb{CP}^1$. Hence, if $a^3 = b^2$ is not the identity, then a and b both have the same fixed point set, and thus a and b commute by [Bea83, Theorem 4.3.5(ii)]. □

Remark 2.2.5. Since $H^2(B_3; \mathbb{Z}/2\mathbb{Z}) = 0$ [Fuk70], any homomorphism $B_3 \to \mathrm{PSL}_2\,\mathbb{C}$ lifts to a representation $B_3 \to \mathrm{SL}_2\,\mathbb{C}$. Formanek classified the two-dimensional irreducible representations of B_3 [For96, Theorem 11]. From that classification one can easily check that $Z(B_3)$ has image contained in $\{\pm I\}$ under any irreducible representation $B_3 \to \mathrm{SL}_2\,\mathbb{C}$. However Lemma 2.2.4 also handles the reducible indecomposable representations $B_3 \to \mathrm{SL}_2\,\mathbb{C}$.

The following proposition classifies the endomorphisms of $\mathrm{PSL}_2\,\mathbb{Z}$.

Proposition 2.2.6. *Let* a, b *denote the images of* $\alpha = \sigma_1\sigma_2$, $\beta = \sigma_1\sigma_2\sigma_1$ *respectively under the quotient map* $B_3 \twoheadrightarrow \mathrm{PSL}_2\,\mathbb{Z}$, *so that* $\mathrm{PSL}_2\,\mathbb{Z} = \langle a \rangle * \langle b \rangle$ *and* $a^3 = b^2 = 1$. *Then the non-cyclic endomorphisms of* $\mathrm{PSL}_2\,\mathbb{Z}$ *are precisely those of the form*

$$a \mapsto g a^{\pm 1} g^{-1}, \quad b \mapsto h b h^{-1},$$

for some group elements $g, h \in \mathrm{PSL}_2\,\mathbb{Z}$.

Proof. By the free product description of $\mathrm{PSL}_2\,\mathbb{Z}$ all maps above extend to endomorphisms. Since the order of a and the order of b are coprime integers, Kurosh's theorem [Kur34] implies that any endomorphism must carry $\langle a \rangle$ into a conjugate subgroup $g\langle a \rangle g^{-1}$, and also $\langle b \rangle$ to a conjugate subgroup $h\langle b \rangle h^{-1}$. Hence these are the only possibilities. □

We now describe the non-cyclic homomorphisms $B_3 \to \mathrm{PSL}_2\,\mathbb{Z}$ where the images of the standard generators are parabolic.

Proof of Theorem 2.2.2. Without loss of generality, we may assume that $\varphi(\sigma_1) = \left[\begin{smallmatrix} 1 & x \\ 0 & 1 \end{smallmatrix}\right]$ for some integer $x > 0$. Let $\eta \in \mathbb{P}^1(\mathbb{Q}) = \mathbb{Q} \cup \{\infty\}$ be the unique fixed point of $\varphi(\sigma_2)$. Since φ is non-cyclic, $\eta \neq \infty$.

Let $T_\eta \coloneqq \begin{bmatrix} 1 & \eta \\ 0 & 1 \end{bmatrix}$, and define $\psi\colon B_3 \to \mathrm{PSL}_2\,\mathbb{Q}$ by $\psi(g) = T_\eta^{-1} \cdot \varphi(g) \cdot T_\eta$. Note that $\psi(\sigma_2)$ fixes $0 \in \mathbb{P}^1(\mathbb{Q})$, hence its top right entry is zero. We can directly compute the remaining entries in terms of $\varphi(\sigma_2) = \left[\begin{smallmatrix} a & * \\ c & d \end{smallmatrix}\right]$ to be

$$\psi(\sigma_2) = \begin{bmatrix} a - c\eta & 0 \\ c & d + c\eta \end{bmatrix}.$$

Since $\det(\psi(\sigma_2)) = 1 \in \mathbb{Z}$ and $a, d \in \mathbb{Z}$, we must have $c\eta \in \mathbb{Z}$. Hence $\psi(\sigma_2) = \begin{bmatrix} 1 & 0 \\ y & 1 \end{bmatrix}$, where $y = \pm c \in \mathbb{Z}$ is non-zero. The braid relation implies

$$\begin{bmatrix} 1 & x \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ y & 1 \end{bmatrix} \begin{bmatrix} 1 & x \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ y & 1 \end{bmatrix} \begin{bmatrix} 1 & x \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ y & 1 \end{bmatrix}$$

$$\implies xy + 1 = 0 \implies (x, y) = (1, -1).$$

Hence $c = \pm 1$, and thus $\eta \in \mathbb{Z}$. Thus $T_\eta \in \mathrm{PSL}_2\,\mathbb{Z}$, and $\psi\colon B_3 \to \mathrm{PSL}_2\,\mathbb{Z}$ has the required form, completing the proof. □

2.2.2 $B_n \to \mathrm{PSL}_2\,\mathbb{Z}$

We first prove a useful lemma about $\mathrm{PSL}_2\,\mathbb{Z}$, by viewing it as a subgroup of $\mathrm{Isom}^+(\mathbb{H}) \cong \mathrm{PSL}_2\,\mathbb{R}$.

Lemma 2.2.7. *If two elements of* $\mathrm{PSL}_2\,\mathbb{Z}$ *commute and are conjugate, then they must be either equal or inverses.*

Proof. If both are hyperbolic they have the same translation axis and translation length. If both are elliptic they have a common fixed point and the same angle of rotation. In both of these cases, it follows they must be either equal or inverses.

The remaining possibility is that the two commuting and conjugate elements are both parabolic. In this case, we claim that they are in fact equal. Note that this does not hold in the larger group $\mathrm{PSL}_2\,\mathbb{R}$, where any two parabolics are conjugate. The reason why things differ in $\mathrm{PSL}_2\,\mathbb{Z}$ ultimately follows from the following two well-known facts.

- If $A \in \mathrm{PSL}_2\,\mathbb{Z}$ is parabolic, then the unique fixed point of A in $\mathbb{P}^1(\mathbb{R})$ is rational, i.e., it lies in $\mathbb{P}^1(\mathbb{Q})$.
- If $A \in \mathrm{PSL}_2\,\mathbb{Z}$ is hyperbolic, then its two fixed points in $\mathbb{P}^1(\mathbb{R})$ are irrational, i.e., they do not lie in $\mathbb{P}^1(\mathbb{Q})$.

Now two parabolics commute if and only if they have the same fixed point in $\mathbb{P}^1(\mathbb{R})$. If $P, P' \in \mathrm{PSL}_2\,\mathbb{Z}$ are two commuting and conjugate parabolics, then the conjugating element $A \in \mathrm{PSL}_2\,\mathbb{Z}$ must also fix their common fixed point in $\mathbb{P}^1(\mathbb{Q})$. Therefore A cannot be hyperbolic. Thus A is another parabolic that commutes with both P, P', so $P = P'$. □

Proof of Theorem 2.2.1. We prove the three parts in reverse order. Note that Lemma 2.2.4 immediately implies (iii).

Suppose that $n = 4$. By Lemma 2.2.7, $\varphi(\sigma_1)$ and $\varphi(\sigma_3)$ are either equal or inverses. By Proposition 2.2.6 if φ is non-cyclic, then $\varphi(\sigma_i)$ and $\varphi(\sigma_{i+1})$ are mapped to the same generator of the abelianization $\mathbb{Z}/6\mathbb{Z}$ of $\mathrm{PSL}_2\,\mathbb{Z}$, for $i = 1, 2$. This rules out the possibility that $\varphi(\sigma_1)$ and $\varphi(\sigma_3)$ are inverses. Hence $\varphi(\sigma_1) = \varphi(\sigma_3)$, which proves (ii).

Finally, suppose that $n \geq 5$. Note that $\varphi(\sigma_1) = \varphi(\sigma_3)$ by (ii). By Lemma 2.2.3(ii), it follows that φ is cyclic, which proves (i). □

2.2.3 Lifts to $B_n \to B_3$

Throughout this paper we will denote the centralizer of a group H in a group G by $C_G(H)$, and $C_G(g) \coloneqq C_G(\langle g \rangle)$ for $g \in G$.

Definition (Transvection). Let $\varphi\colon G \to H$ be a homomorphism. If $t\colon G \to C_H(\varphi(G))$ is a cyclic homomorphism, then the *transvection of φ by t* is the homomorphism $\varphi^t\colon G \to H$ defined by

$$\varphi^t(g) = \varphi(g)t(g).$$

Theorem 2.2.8. *Let $\varphi\colon B_n \to B_3$ be a homomorphism.*

(i) If $n \geq 5$, then φ is cyclic.

(ii) If $n = 4$, then φ factors through R_*.

(iii) If $n = 3$ and φ is non-cyclic, then up to a transvection by a homomorphism $B_3 \to Z(B_3)$ and post-composing by an automorphism of B_3, the map φ has the form

$$\alpha \mapsto g\alpha g^{-1}, \quad \beta \mapsto \beta,$$

for some $g \in B_3$.

Proof. If we post-compose the above homomorphisms with the quotient map

$$B_3 \twoheadrightarrow B_3/Z(B_3) \cong \mathrm{PSL}_2\,\mathbb{Z},$$

then we obtain all homomorphisms classified in Theorem 2.2.1. Furthermore, any two homomorphisms $B_n \to B_3$ that lift the same homomorphism $B_n \to \mathrm{PSL}_2\,\mathbb{Z}$ can only differ by a transvection $B_n \to Z(B_3)$. Hence this is a complete list of homomorphisms $B_n \to B_3$. □

Dyer and Grossman [DG81] have shown that $\mathrm{Aut}(B_n) = \mathrm{Inn}(B_n) \rtimes \langle \iota \rangle$ for $n \geq 2$, where

$\iota\colon B_n \to B_n$ is the inversion automorphism given by $\sigma_i \mapsto \sigma_i^{-1}$ for $i = 1, \dots, n-1$, so this gives a complete classification of the homomorphisms $B_n \to B_3$.

In Theorem 2.2.8 the case $n \geq 5$ is a special case of a Theorem of Lin [Lin04a, Theorem 3.1], and $n = 4$ is already known by Chen, Kordek and Margalit [CKM23, Proposition 8.6]. The case $n = 3$ is stated by Orevkov [Ore24, Proposition 1.8] without proof. Our proof of Theorem 2.2.1 is independent of these results.

2.3 Nielsen–Thurston theory

The purpose of this section is twofold: to recall well-known facts about the braid group from the Nielsen–Thurston point of view, while also establishing the notation and conventions we use.

Let S be a connected, oriented, compact surface, possibly with boundary, and let P be a finite unordered set of points in the interior of S. We call the points in P *marked points*. Let $\mathrm{Homeo}^+(S, \partial S, P)$ be the group of orientation preserving homeomorphisms $f\colon S \to S$ that fix the boundary ∂S pointwise, and preserve the set P of marked points. The *mapping class group* $\mathrm{Mod}(S, P)$ is defined to be

$$\mathrm{Mod}(S, P) \coloneqq \pi_0(\mathrm{Homeo}^+(S, \partial S, P)) \cong \mathrm{Homeo}^+(S, \partial S, P)/\mathrm{isotopy}$$

Elements of $\mathrm{Mod}(S, P)$ are called *mapping classes*. We sometimes write $\mathrm{Mod}(S)$ instead of $\mathrm{Mod}(S, P)$ when it is clear what the set of marked points is. See [FM12, Chapter 2] for more on this definition.

Let $\mathbb{D}_n$ be a disk including its boundary with n unordered marked points in the interior. There is a canonical isomorphism between the braid group B_n and the mapping class group $\mathrm{Mod}(\mathbb{D}_n)$ [FM12, Chapter 9].

We make extensive use of the Nielsen–Thurston classification theorem:

Theorem 2.3.1 ([Thu88])**.** *For each mapping class in* $\mathrm{Mod}(S, P)$*, exactly one of the following possibilities holds.*

- *The mapping class is periodic.*
- *The mapping class has non-empty canonical reduction system.*
- *The mapping class is pseudo-Anosov.*

We define these terms for elements of the braid group B_n later in this section. See [FM12, Chapter 13] for more detail.

Remark 2.3.2. The Nielsen–Thurston classification theorem is often not stated as three mutually exclusive possibilities, but rather that every mapping class is periodic, reducible, or pseudo-Anosov [Thu88]. While pseudo-Anosov mapping classes are neither periodic nor reducible, it is possible for a mapping class to be both reducible and periodic. For a proof that the above are indeed mutually exclusive possibilities, see for example [Min13, §4].

Our strategy for classifying homomorphisms $B_n \to B_4$ involves considering the possible Nielsen–Thurston types of the image of a generator of $Z(B_n)$ in B_4, and its possible canonical reduction systems.

2.3.1 Periodic braids

A braid in B_n is *periodic* if it has finite order in $B_n/Z(B_n)$.

Let

$$\alpha_n \coloneqq \sigma_1 \cdots \sigma_{n-1}, \quad \beta_n \coloneqq \alpha_n \sigma_1, \quad z_n \coloneqq \alpha_n^n = \beta_n^{n-1}.$$

It is a result of Chow that $Z(B_n)$ is infinite cyclic generated by z_n [Cho48], hence α_n and β_n are periodic. We will continue to write $\alpha = \alpha_3$ and $\beta = \beta_3$.

The following proposition follows from work of Kerékjártó and Eilenberg.

Proposition 2.3.3 ([Eil34; Ker19])**.** *The periodic elements of B_n are all conjugate to a power of either α_n or β_n.*

If $\pi\colon B_n \to S_n$ is the natural homomorphism, and $n_1 + \cdots + n_\ell$ is a partition of n, then let the *mixed braid group* be

$$B_{n_1,\ldots,n_\ell} \coloneqq \pi^{-1}(S_{n_1} \times \cdots \times S_{n_\ell}) \leq B_n.$$

González-Meneses and Wiest attribute the computation of centralizers of non-central periodic braids below to Bessis, Digne and Michel [BDM02].

Proposition 2.3.4. *[GW04, Theorems 3.2 and 3.4]*

(a) Let $d = \gcd(k, n)$, and suppose that $d < n$. The centralizer of α_n^k in B_n is isomorphic to the mixed braid group $B_{d,1}$.

(b) Let $d = \gcd(k, n-1)$, and suppose that $d < n-1$. The centralizer of β_n^k in B_n is isomorphic to $B_{d,1}$.

2.3.2 Canonical reduction systems

A *simple closed curve* in $\mathbb{D}_n$ is an isotopy class of smooth maps $S^1 \to \mathbb{D}_n$ without self intersection that avoid the n marked points. For convenience, we do not consider null-homotopic curves to count as simple closed curves. The braid group B_n acts on the set of simple closed curves in $\mathbb{D}_n$.

A *multicurve* is a set of disjoint simple closed curves. For our purposes, it will be convenient to allow multicurves to contain non-essential simple closed curves, i.e., curves that are isotopic to a marked point or boundary component. We say a multicurve is *nontrivial* if it contains at least one essential component, and we say it is *essential* if all its components are essential.

A *reduction system* of a braid $g \in B_n$ is an essential multicurve preserved by g. Note that the components of a reduction system of g may be permuted by g. The *canonical reduction system* $\mathrm{crs}(g)$ of g is the intersection of all maximal reduction systems of g. In the special case of the surface $\mathbb{D}_n$ there is one additional notion we may define: the *outer canonical reduction system* $\mathrm{crs}^\circ(g)$. We define this to be the outermost loops in the canonical reduction system, together with a small loop surrounding every marked point not already encircled by a component of the canonical reduction system.

Note that $\mathrm{crs}(ghg^{-1}) = g \cdot \mathrm{crs}(h)$ and $\mathrm{crs}^\circ(ghg^{-1}) = g \cdot \mathrm{crs}^\circ(h)$. In particular commuting elements preserve each other's (outer) canonical reduction systems.

The outer canonical reduction systems in $\mathbb{D}_n$, up to the action of B_n, are in bijection with the partitions of n: the partition of $\mathrm{crs}^\circ(g)$ is obtained by recording the number of marked points each component of g encircles.

2.3.3 Reducible braids

A braid is *reducible* if it preserves a non-empty reduction system. For non-periodic braids, reducibility is equivalent to having a non-empty canonical reduction system.

Important examples of reducible braids include *Dehn twists* T_γ about a simple closed curve γ, see [FM12, Chapter 3] for a definition. The Dehn twist $z_n = T_{\partial \mathbb{D}_n}$ about a curve isotopic to the boundary component of $\mathbb{D}_n$ generates $Z(B_n)$. A *multitwist* is a product $\prod_{i=1}^k T_{\gamma_i}^{p_i}$ of Dehn twists, where $p_i \in \mathbb{Z}$ and the curves γ_i are pairwise disjoint. The canonical reduction system of such a product is the set of all γ_i that are not isotopic to a boundary component.

If $\varphi\colon G \to B_m$ is a homomorphism, we say it is *reducible* if $\varphi(G)$ preserves a non-empty reduction system. If $G = B_n$, then φ is reducible if and only if $\varphi(z_n)$ is, since $Z(B_n) = \langle z_n \rangle$. For this reason we define $\mathrm{crs}(\varphi) \coloneqq \mathrm{crs}(\varphi(z_n))$ and $\mathrm{crs}^\circ(\varphi) \coloneqq \mathrm{crs}^\circ(\varphi(z_n))$.

We will need to work with the stabilizer B_Γ in B_n of an outer canonical reduction system

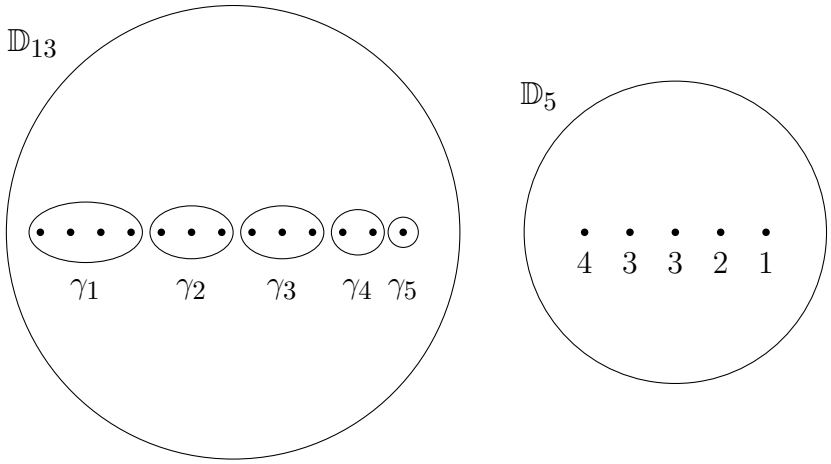


Figure 2.1: Labelling from an outer canonical reduction system

$\Gamma \coloneqq \{\gamma_1, \ldots, \gamma_m\}$ in $\mathbb{D}_n$. If we collapse the disks bounded by each γ_i to a point, then we obtain a disk $\mathbb{D}_m$ whose ith marked point can be labelled with the number n_i of marked points originally encircled by γ_i.

Given a labelling L of the marked points in $\mathbb{D}_m$, let B_L be the mixed braid subgroup of B_m preserving this labelling. If a labelling is obtained by the above construction collapsing the disks bounded by components of Γ, then denote the labelling by $[\Gamma]$. For example, if $\Gamma \coloneqq \{\gamma_1, \ldots, \gamma_5\}$ is the outer canonical reduction system in Figure 2.1, then $B_{[\Gamma]} \cong B_{1,2,1,1} < B_5$.

Lemma 2.3.5 ([KM22, Lemma 3.1])**.** *The short exact sequence*

$$1 \to (B_{n_1} \times \cdots \times B_{n_m}) \to B_\Gamma \to B_{[\Gamma]} \to 1$$

obtained by collapsing the disks bounded by γ_i to a point, has a section. We obtain a semi-direct product $B_\Gamma \cong (B_{n_1} \times \cdots \times B_{n_m}) \rtimes B_{[\Gamma]}$. The action of $B_{[\Gamma]}$ on $B_{n_1} \times \cdots \times B_{n_m}$ factors through the homomorphism $B_{[\Gamma]} \to S_m$ and is given by permuting the factors.

2.3.4 Pseudo-Anosov braids

We will not define pseudo-Anosov braids here, we refer to [FM12, Chapter 13] for a definition. By the Nielsen–Thurston classification theorem, they are precisely the non-periodic irreducible braids.

The centralizers of pseudo-Anosov braids are well understood:

Proposition 2.3.6 ([GW04, Proposition 4.1])**.** *The centralizer in B_n of a pseudo-Anosov braid is isomorphic to $\mathbb{Z}^2$.*

By Lemma 2.2.3 this implies that if $\rho\colon B_n \to B_m$ is a homomorphism and $\rho(z_n)$ is pseudo-Anosov, then ρ has cyclic image.

2.4 Equivalence classes of homomorphisms

Following [CKM23], we define an equivalence relation on the set of homomorphisms $G \to H$ as follows:

Definition (**Equivalence of homomorphisms**)**.** Two homomorphisms $G \to H$ are *equivalent* if they are related by post-composition by an automorphism of H and a sequence of transvections.

The goal of this section is to prove Theorem 2.1.3, namely that there are infinitely many equivalence classes of homomorphisms $B_n \to B_m$ when $n \in \{3, 4\}$ and $m \geq 3$.

We will also give an explicit list of the equivalence classes of homomorphisms $B_n \to B_3$. To do this, we first compute the centralizers of the non-cyclic homomorphisms from Theorem 2.2.8.

2.4.1 Centralizers

We will repeatedly use the following result of González-Meneses and Wiest.

Lemma 2.4.1 ([GW04, Propositions 3.3 and 3.5])**.**

(i) The centralizer of α_n in B_n is $\langle \alpha_n \rangle$.

(ii) The centralizer of β_n in B_n is $\langle \beta_n \rangle$.

Lemma 2.4.2. *If ρ is a non-cyclic endomorphism of B_3, then the centralizer of $\rho(B_3)$ in B_3 is the center $Z(B_3)$.*

Proof. By Theorem 2.2.8 it suffices to show that $C_{B_3}(g\alpha g^{-1}) \cap C_{B_3}(\beta) = Z(B_3)$ for all $g \in G$. The image of $g\alpha g^{-1}$ in S_3 is a 3-cycle, and the image of β is a 2-cycle. Therefore, the intersection of $C_{B_3}(g\alpha g^{-1}) = \langle g\alpha g^{-1} \rangle$ with $C_{B_3}(\beta) = \langle \beta \rangle$ is contained in, and hence equal to, $\langle (g\alpha g^{-1})^3 \rangle = \langle \beta^2 \rangle = Z(B_3)$. □

2.4.2 Reduction to endomorphisms of B_3

Next we show that Castel's conjecture, the $n = m = 3$ case of Theorem 2.1.3, implies all other cases of Theorem 2.1.3.

Lemma 2.4.3. *Inequivalent endomorphisms of B_3 remain inequivalent as homomorphisms after either one or both of the following operations*

(i) Post-composition with the standard inclusion $B_3 \hookrightarrow B_m$

(ii) Pre-composition with $R_* \colon B_4 \twoheadrightarrow B_3$.

Proof. Since R_* is surjective, (ii) doesn't collapse equivalence classes of homomorphisms. So it suffices to show (i) doesn't collapse equivalence classes of endomorphisms of B_3.

Suppose that $\varphi \colon B_3 \to B_m$ is a non-cyclic endomorphism of B_3 postcomposed with the standard inclusion $B_3 \hookrightarrow B_m$. Let $t \colon B_3 \to B_{1,m-3}$ be a cyclic homomorphism. View $B_3 \times B_{1,m-3}$ as a subgroup of B_m, namely the stabilizer of the simple closed curve γ in Figure 2.2 surrounding three marked points, cf. Lemma 2.3.5. We claim $C_{B_m}(\varphi^t(B_3)) \leq Z(B_3) \times B_{1,m-3}$.

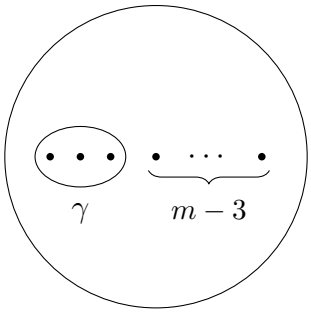


Figure 2.2: The canonical reduction system in $\mathbb{D}_m$ of $z_3 \in Z(B_3) < B_m$, where $m \geq 3$

First note that φ and φ^t agree on $[B_3, B_3]$. By the classification of endomorphisms of B_3, $\varphi(B_3) \leq B_3$ surjects onto S_3, hence $\varphi^t([B_3, B_3]) = \varphi([B_3, B_3])$ surjects onto A_3. If $g \in \varphi^t([B_3, B_3])$ has non-trivial image in A_3, then its canonical reduction system in $\mathbb{D}_m$ is $\{\gamma\}$. By a result of González-Meneses and Wiest [GW04, Theorem 1.1], $C_{B_m}(g) = C_{B_3}(g) \times B_{1,m-3}$. In particular $C_{B_m}(\varphi^t(B_3)) \leq B_3 \times B_{1,m-3}$. Now Lemma 2.4.2 implies $C_{B_m}(\varphi^t(B_3)) \leq Z(B_3) \times B_{1,m-3}$.

This implies by induction that repeated transvections of φ can be realized by one transvection in B_3 together with one transvection by a cyclic homomorphism $t\colon B_3 \to B_{1,m-3} \leq B_m$. Suppose now that $\tau \in \mathrm{Aut}(B_m)$ is such that $\tau(\varphi^t(B_3)) \leq B_3 \leq B_m$. Then in particular, $\tau(g) \in B_3$. Hence the canonical reduction system of $\tau(g)$ is also $\{\gamma\}$, and so τ preserves γ. Thus τ restricts to an automorphism of $B_3 \leq B_m$. This implies t is trivial.

Hence, if two homomorphisms $\varphi_1, \varphi_2\colon B_3 \to B_3 \hookrightarrow B_m$ are equivalent as homomorphisms $B_3 \to B_m$, then they are also equivalent as endomorphisms of B_3. □

2.4.3 Equivalence classes of endomorphisms of B_3

Let ρ_g be the endomorphism of B_3 defined by $\alpha \mapsto g\alpha g^{-1}$, $\beta \mapsto \beta$. By Theorem 2.2.8 every endomorphism of B_3 is equivalent to some ρ_g, hence Theorem 2.1.3 is equivalent to the ρ_g representing infinitely many equivalence classes. The next lemma shows that these equivalence classes are in bijective correspondence with certain double cosets in B_3.

Lemma 2.4.4. *The endomorphisms ρ_g and $\rho_{g'}$ are equivalent if and only if the double cosets $\langle\beta\rangle g\langle\alpha\rangle$ and $\langle\beta\rangle g'\langle\alpha\rangle$ are equal.*

Proof. First note that ρ_g itself only depends on the coset $g\langle\alpha\rangle$. Conjugating by β shows that its equivalence class only depends on the double coset $\langle\beta\rangle g\langle\alpha\rangle$.

Conversely, suppose that ρ_g is equivalent to $\rho_{g'}$. By Lemma 2.4.2, there exists a homomorphism $t\colon B_3 \to Z(B_3)$ and $\tau \in \mathrm{Aut}(B_3)$ such that

$$\rho_{g'} = \tau \circ \rho_g^t. \tag{2.4.5}$$

By restricting both sides of (2.4.5) to $Z(B_3)$, we can conclude that t is trivial and $\tau \in \mathrm{Inn}(B_3)$. By evaluating both sides of (2.4.5) at β, we see that τ fixes β. Hence $\tau\colon g \mapsto hgh^{-1}$ for some $h \in C_{B_3}(\beta) = \langle\beta\rangle$. By evaluating both sides of (2.4.5) at α we find that $(g')^{-1}hg \in C_{B_3}(\alpha) = \langle\alpha\rangle$. Hence $\langle\beta\rangle g\langle\alpha\rangle = \langle\beta\rangle g'\langle\alpha\rangle$. □

Everything is now in place to prove Theorem 2.1.3.

Proof of Theorem 2.1.3. By Lemmas 2.4.3 and 2.4.4 it suffices to show that there are infinitely many double cosets $\langle\beta\rangle g\langle\alpha\rangle$. Recall that $B_3/Z(B_3) \cong \langle a\rangle * \langle b\rangle \cong \mathbb{Z}/3\mathbb{Z} * \mathbb{Z}/2\mathbb{Z}$, where a and b are the images of α and β respectively. By considering the usual normal form for elements of a free product, the double cosets $\langle b\rangle(ab)^n\langle a\rangle = \{b^i(ab)^n a^j : i, j \in \mathbb{Z}\}$ are distinct as n ranges over the positive integers. Therefore, the double cosets $\langle\beta\rangle(\alpha\beta)^n\langle\alpha\rangle$ are also all distinct as n ranges over the positive integers. □

2.5 Classification of homomorphisms $B_3 \to B_4$ and $B_4 \to B_4$

A major goal of this section is to prove the following theorem.

Theorem 2.5.1. *The non-cyclic homomorphisms $B_3 \to B_4$ are, up to a transvection $B_3 \to Z(B_4)$ and post-composing by an automorphism of B_4, one of the following.*

(i) An endomorphism of B_3, followed by the standard inclusion $B_3 \hookrightarrow B_4$.

(ii) Given by $\alpha \mapsto g\beta_4 g^{-1}$, $\beta \mapsto \alpha_4^2$ for some $g \in B_4$.

We provide a proof of Theorem 2.5.1 based on the Nielsen–Thurston classification theorem and canonical reduction systems. These methods were suggested to us by Farb. Theorem 2.5.1 is also stated by Orevkov [Ore24, Proposition 1.9]. Although a proof is not provided, the methods suggested there are the same as our own.

We also prove the following theorem in §2.5.3, which classifies the homomorphisms $B_3 \to B_4$ satisfying certain conditions. In §2.7 these will be shown to be necessary conditions to be induced by a non-root holomorphic map.

Theorem 2.5.2. *The homomorphism $\Psi_{3*}\colon B_3 \to B_4$ given by*

$$\Psi_{3*}(\sigma_1) = \sigma_1\sigma_2, \qquad \Psi_{3*}(\sigma_2) = \sigma_3\sigma_2,$$

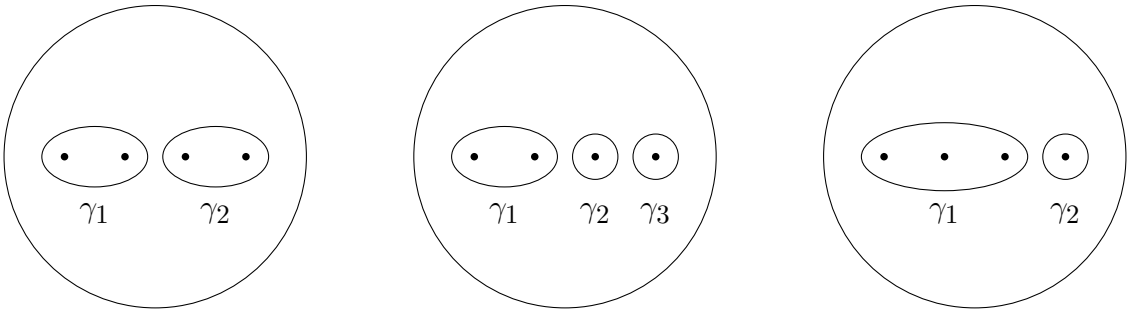
is, up to automorphisms of B_4 and transvections $B_3 \to Z(B_4)$, the only non-cyclic, irreducible homomorphism $\varphi\colon B_3 \to B_4$ such that $\varphi(\sigma_1)$ has some positive power that is a multitwist.

A complete proof of Theorem 2.5.2 will rely on the solution to an equation in the free group of rank 2 that §2.6 is dedicated to proving.

Chen, Kordek, and Margalit prove that every endomorphism of B_4 is either a transvection of the identity, or factors through the homomorphism $R_*\colon B_4 \twoheadrightarrow B_3$ [CKM23, Theorem 8.4]. Together with Theorem 2.5.1, this implies the following classification of endomorphisms of B_4.

Theorem 2.5.3. *The non-cyclic endomorphisms of B_4 are, up to a transvection $B_4 \to Z(B_4)$ and post-composing by an automorphism of B_4, one of the three following types.*

(i) The identity.



(a) (b) (c)

Figure 2.3: Nontrivial outer canonical reduction systems in B_4

(ii) The resolving quartic homomorphism $R_*\colon B_4 \twoheadrightarrow B_3$, followed by an endomorphism of B_3, followed by the standard inclusion $B_3 \hookrightarrow B_4$.

(iii) The resolving quartic homomorphism $R_*\colon B_4 \twoheadrightarrow B_3$, followed by a homomorphism $B_3 \to B_4$ given by $\alpha \mapsto g\beta_4 g^{-1}$, $\beta \mapsto \alpha_4^2$ for some $g \in B_4$.

Let $\varphi\colon B_3 \to B_4$ be a non-cyclic homomorphism, and let $Z(B_3) = \langle z_3 \rangle$. Theorem 2.5.1 follows from two lemmas that we prove in §§2.5.1 and 2.5.2.

2.5.1 The non-periodic reducible case

Lemma 2.5.4. *If φ is reducible and $\varphi(z_3)$ is not periodic, then φ belongs to case (i) of Theorem 2.5.1.*

Proof of Lemma 2.5.4. Since φ has non-empty canonical reduction system, up to the action of B_n there are three possible values of $\mathrm{crs}^\circ(\varphi)$. These are depicted in Figures 2.3a to 2.3c, and they correspond to the three non-trivial partitions of 4: $2+2$, $2+1+1$, and $3+1$. We consider these three cases in turn.

The partition $2+2$

Claim: If $\mathrm{crs}^\circ(\varphi)$ is $\Gamma \coloneqq \{\gamma_1, \gamma_2\}$ as depicted in Figure 2.3a up to the action of B_4, then φ is cyclic.

Proof. By Lemma 2.3.5, $B_\Gamma \cong (B_2 \times B_2) \rtimes B_2 \cong \mathbb{Z}^2 \rtimes \mathbb{Z}$, where $n \in \mathbb{Z}$ acts on $\mathbb{Z}^2$ by swapping coordinates if n is odd. We present the group $\mathbb{Z}^2 \rtimes \mathbb{Z}$ as $\langle x, y, z \mid [x, y] = 1,\ zx = yz,\ zy = xz \rangle$, where every element of this group can be written uniquely in the form $x^a y^b z^c$. Write

$$\varphi(\sigma_i) = x^{a_i} y^{b_i} z^{c_i}, \quad i = 1, 2.$$

Since $\langle x, y, z \rangle / \langle x, y \rangle \cong \langle z \rangle$ and since σ_1 and σ_2 are conjugate, we see that $c_1 = c_2$. If this common value c is even, then φ is cyclic since $\langle x, y, z^2 \rangle$ is abelian. So suppose that c is odd. Since $\sigma_1 \sigma_2 \sigma_1 = \sigma_2 \sigma_1 \sigma_2$, we find

$$x^{2a_1 + b_2} y^{2b_1 + a_2} z^{3c} = x^{2a_2 + b_1} y^{2b_2 + a_1} z^{3c}.$$

Hence $a_1 = a_2$ and $b_1 = b_2$, and so again φ is cyclic. □

The partition $2 + 1 + 1$

Claim: If $\mathrm{crs}^\circ(\varphi)$ is $\Gamma \coloneqq \{\gamma_1, \gamma_2, \gamma_3\}$ as depicted in Figure 2.3b up to the action of B_4, then φ is cyclic.

Proof. By Lemma 2.3.5,

$$B_\Gamma \cong (B_2 \times B_1 \times B_1) \rtimes B_{1,2} \cong \mathbb{Z} \times B_{1,2}.$$

So it suffices to show that any homomorphism $B_3 \to B_{1,2}$ is cyclic. By Theorem 2.2.8(iii), the image of any non-cyclic homomorphism $B_3 \to B_3$ surjects onto S_3. Since $B_{1,2} < B_3$

does not surject onto S_3, it follows that φ is cyclic. □

The partition $3 + 1$

Claim: If $\mathrm{crs}^\circ(\varphi)$ is $\Gamma \coloneqq \{\gamma_1, \gamma_2\}$ as depicted in Figure 2.3c up to the action of B_4, then φ belongs to case (i) of Theorem 2.5.1.

Proof. By Lemma 2.3.5,

$$B_\Gamma \cong (B_3 \times B_1) \rtimes B_{1,1} \cong B_3 \times \mathbb{Z}.$$

The B_3 factor can be identified with the standard copy of $B_3 < B_4$. The $\mathbb{Z}$ factor produced by the section in Lemma 2.3.5 is generated by $z_3 z_4^{-1}$, where $z_3 = (\sigma_1\sigma_2)^3$ generates the center of B_3 and $z_4 = (\sigma_1\sigma_2\sigma_3)^4$ generates the center of B_4. Hence $B_\Gamma = B_3 \times Z(B_4)$, and so φ is a composition of an endomorphism of B_3, followed by the standard inclusion, up to transvections by a homomorphism $B_3 \to Z(B_4)$ and automorphisms of B_4. □

In conclusion, if $\varphi\colon B_3 \to B_4$ is reducible and non-cyclic, then it belongs to case (i) of Theorem 2.5.1, which completes the proof of Lemma 2.5.4. □

2.5.2 The periodic case

Lemma 2.5.5. *If $\varphi(z_3)$ is periodic, then φ belongs to case (ii) of Theorem 2.5.1.*

Proof. We consider two cases.

Periodic but not central

Claim: If $\varphi(z_3)$ is periodic but not central, then φ is cyclic.

Proof. By Proposition 2.3.4, the centralizer of $\varphi(z_3)$ is isomorphic to $B_{d,1}$ where d is a proper divisor of either 3 or 4, i.e., $d = 1$ or $d = 2$. If $d = 1$, then $B_{d,1} \cong \mathbb{Z}$ and so φ is cyclic. If $d = 2$, then $B_{d,1} = B_{2,1}$. By Theorem 2.2.8, the image of any non-cyclic endomorphism of B_3 surjects onto S_3. Thus any homomorphism $B_3 \to B_{2,1}$ is cyclic. Therefore φ is cyclic. □

Central

Claim: If $\varphi(z_3)$ is central, then φ belongs to case (ii) of Theorem 2.5.1.

Proof. Since $\varphi(Z(B_3)) \leq Z(B_4)$, φ descends to a homomorphism

$$\psi\colon B_3/Z(B_3) \to B_4/Z(B_4).$$

Let a, b denote the images of α, β in $B_3/Z(B_3)$. If $\psi(a)$ or $\psi(b)$ is trivial, then ψ is cyclic, and hence φ is cyclic, contradiction. Hence $\psi(a)$ has order 3 and $\psi(b)$ has order 2. By Proposition 2.3.3, possibly after replacing φ with $\iota \circ \varphi$, we find that $\psi(a)$ and $\psi(b)$ are conjugate to the images of β_4 and α_4^2 in $B_4/Z(B_4)$, respectively. By the relation $\alpha^3 = \beta^2$ it follows that, up to a transvection $B_3 \to Z(B_4)$ and an automorphism of B_4, φ belongs to case (ii) of Theorem 2.5.1. □

This completes the proof of Lemma 2.5.5, and hence Theorem 2.5.1. □

2.5.3 Proof of Theorem 2.5.2

We now classify the irreducible homomorphisms $\varphi\colon B_3 \to B_4$ such that $\varphi(\sigma_1)$ and $\varphi(\sigma_2)$ have some powers which are multitwists. This will be used in §2.7. The following lemma imposes a strong constraint on the image of σ_1.

Lemma 2.5.6. *Let $\varphi\colon B_3 \to B_4$ be an irreducible, non-cyclic homomorphism such that $\varphi(\sigma_1)$ has a power that is a multitwist. Then there exist $\tau \in \mathrm{Aut}(B_4)$ and $t\colon B_3 \to Z(B_4)$ such that $\tau \circ \varphi^t$ maps $\sigma_1 \mapsto (\sigma_1\sigma_2)^{6k+1}$ for some $k \in \mathbb{Z}$.*

Proof. Since φ irreducible and non-cyclic, by Theorem 2.5.1 we may assume without loss of generality that φ is given by $\alpha \mapsto g\beta_4 g^{-1}$, $\beta \mapsto \alpha_4^2$ for some $g \in B_4$. Thus $\varphi(\sigma_1) = g\beta_4^{-1}g^{-1}\alpha_4^2$. Note that $\varphi(\sigma_1)$ has order 6 in the abelianization $\mathbb{Z}/12\mathbb{Z}$ of $B_4/Z(B_4)$, whereas periodic elements of B_4 have orders dividing 3 or 4 in $B_4/Z(B_4)$ by Proposition 2.3.3. Hence $\varphi(\sigma_1)$ is not periodic, and thus $\mathrm{crs}(\varphi(\sigma_1))$ is non-empty.

Furthermore, the image of $\varphi(\sigma_1) = g\beta_4^{-1}g^{-1}\alpha_4^2$ in S_4 is a 3-cycle, since it is the product of a 3-cycle and a double transposition. Hence, up to an inner automorphism of B_4, the canonical reduction system of $\varphi(\sigma_1)$ consists of the single curve $\gamma \coloneqq \gamma_1$ in Figure 2.3c. By Lemma 2.3.5 we have $\varphi(\sigma_1) \in B_{\{\gamma\}} = B_3 \times Z(B_4) < B_4$. Hence, by possibly applying a transvection $B_3 \to Z(B_4)$, we may assume that $\varphi(\sigma_1) \in B_3 < B_4$. Thus, some power of $\varphi(\sigma_1)$ is a power of a Dehn twist about the curve γ, i.e. $\varphi(\sigma_1)$ is periodic when viewed as an element of B_3. Since the image of $\varphi(\sigma_1)$ is a 3-cycle in S_4, it follows that $\varphi(\sigma_1)$ is conjugate to some power of $\alpha_3 = \sigma_1\sigma_2$. Note that $\varphi(\sigma_1)$ and $\sigma_1\sigma_2$ have the same image in the abelianization $\mathbb{Z}/12\mathbb{Z}$ of $B_4/Z(B_4)$, and both have order 6 in this abelianization. Hence $\varphi(\sigma_1)$ is conjugate to $(\sigma_1\sigma_2)^{6k+1}$ for some $k \in \mathbb{Z}$. □

It remains to determine the possible values of $\varphi(\sigma_2) \in B_4$ under the hypothesis that $\varphi(\sigma_1) = (\sigma_1\sigma_2)^{6k+1}$. To aid us in this we invoke the following result of Gassner [Gas61].

Theorem 2.5.7. *The kernel of $R_*\colon B_4 \to B_3$ is freely generated by*

$$x \coloneqq \sigma_1^{-1}\sigma_3, \quad \text{and} \quad y \coloneqq x^{-1}\sigma_2^{-1}x\sigma_2.$$

The action of the subgroup $B_3 < B_4$ on $\ker(R_*)$ by conjugation is given by

$$\sigma_1^{-1} x \sigma_1 = x, \quad \sigma_2^{-1} x \sigma_2 = xy, \quad \sigma_1^{-1} y \sigma_1 = yx^{-1}, \quad \sigma_2^{-1} y \sigma_2 = y.$$

Proof. Gassner shows that $\ker(R_*)$ is freely generated by $x^{-1} = \sigma_3^{-1}\sigma_1$ and $yx^{-1} = \sigma_2\sigma_1\sigma_3^{-1}\sigma_2^{-1}$ [Gas61, Theorem 7], and gives an equivalent description of the action of B_3 on these generators. □

Since the standard inclusion $B_3 \hookrightarrow B_4$ is a section of R_* the short exact sequence

$$1 \to F_2 \to B_4 \xrightarrow{R_*} B_3 \to 1$$

splits, where F_2 is the free group generated by x and y. Theorem 2.5.7 thus gives an explicit description of the semi-direct product decomposition of $B_4 = F_2 \rtimes B_3$.

Let $f \in \mathrm{Aut}(F_2)$ be defined by $w \mapsto (\sigma_1\sigma_2)^{-1} w (\sigma_1\sigma_2)$. By Theorem 2.5.7 we have

$$f(x) = xy, \qquad f(y) = x^{-1}.$$

In the proof of Theorem 2.5.2 we will show that classifying the irreducible, non-cyclic homomorphisms $\varphi\colon B_3 \to B_4$ with $\varphi(\sigma_1)$ having some power that is a multitwist, is equivalent to solving the following delicate equation in F_2 involving the automorphism f

$$f^{6k+1}(w) f^{-6k-1}(w) = w, \quad w \in F_2.$$

Solving this equation is the purpose of §2.6, see Theorem 2.6.1. There we show that if there is a non-identity solution, then $k = 0$ and $w = f^n(x)$ for some $n \in \mathbb{Z}$.

Proof of Theorem 2.5.2. Let $\varphi\colon B_3 \to B_4$ be a non-cyclic, irreducible homomorphism such that some power of $\varphi(\sigma_1)$ is a multitwist. We seek $\tau \in \mathrm{Aut}(B_4)$ and $t\colon B_3 \to Z(B_4)$ such

that $\Psi_{3*} = \tau \circ \varphi^t$.

By Lemma 2.5.6, we can assume without loss of generality that $\varphi(\sigma_1) = (\sigma_1\sigma_2)^{6k+1}$ for some integer k. By Theorem 2.2.8, $R_* \circ \varphi$ is cyclic, so there exists $w \in F_2$ such that $\varphi(\sigma_2) = w(\sigma_1\sigma_2)^{6k+1}$. The braid relation $\varphi(\sigma_1\sigma_2\sigma_1) = \varphi(\sigma_2\sigma_1\sigma_2)$ is equivalent to

$$f^{6k+1}(w)f^{-6k-1}(w) = w.$$

By Theorem 2.6.1, $k = 0$ and $w = f^n(x)$ for some $n \in \mathbb{Z}$. Thus

$$\varphi(\sigma_2) = f^n(x)(\sigma_1\sigma_2) = (\sigma_1\sigma_2)^{-n}(\sigma_1^{-1}\sigma_3)(\sigma_1\sigma_2)^{n+1} = (\sigma_1\sigma_2)^{-n}(\sigma_3\sigma_2)(\sigma_1\sigma_2)^n.$$

Hence we may take t to be trivial, and $\tau\colon g \mapsto (\sigma_1\sigma_2)^n x(\sigma_1\sigma_2)^{-n}$. □

2.6 Solution to the equation in the free group of rank 2

Let x, y be a free basis of the free group F_2 of rank 2, and let $f \in \mathrm{Aut}(F_2)$ be the automorphism defined by

$$f(x) \coloneqq xy, \qquad f(y) \coloneqq x^{-1}.$$

The goal of this section is to prove the following theorem.

Theorem 2.6.1. *If $w \in F_2$ is a non-identity word satisfying*

$$f^{6k+1}(w)f^{-6k-1}(w) = w,$$

then $k = 0$ and $w = f^n(x)$ for some $n \in \mathbb{Z}$.

2.6.1 Basic properties of f

Note that f has order 6 in $\mathrm{GL}_2\,\mathbb{Z}$. In fact:

$$f^6(w) = cwc^{-1}, \quad \text{where } c \coloneqq [x,y] = xyx^{-1}y^{-1}.$$

Lemma 2.6.2. *The set of words fixed by f is the subgroup $\langle c \rangle$.*

Proof. One can easily check that $f(c) = c$, hence f fixes every element of $\langle c \rangle$. Conversely, if w is fixed by f, then it is also fixed by f^6. Hence w must commute with c. Since c has cyclic centralizer and is not an nth power in F_2 for any $n \geq 2$, it follows that $w \in \langle c \rangle$. □

If $w \in F_2$, then let $\ell(w)$ be the length of w as a reduced word in the alphabet $\{x, x^{-1}, y, y^{-1}\}$.

Definition (**Start and end**)**.** Suppose that $w = uv$ where $u, v \in F_2$ and there is no cancellation in the product uv, i.e., $\ell(w) = \ell(u) + \ell(v)$. Then we say that w *starts with* u and *ends with* v.

We write $w = u\_\_$ to denote that w starts with u, or $w = \_\_v$ to denote that w ends with v, or $w = u\_\_v$ to denote that w starts with u and ends with v.

For example, we may write $c = xy\_\_$, or $c = \_\_y^{-1}$. It is also true that $c = xyx^{-1}\_\_x^{-1}y^{-1}$, since in the expression $w = u\_\_v$, the subwords u and v can overlap in the expression of w as a reduced word.

The following lemma gives a precise relationship between the start and end of w and $f(w)$.

Lemma 2.6.3. *Let $n \in \mathbb{Z}$. Then*

$$\begin{aligned}
w = y^n x __ &\iff f(w) = x^{-n+1} y __ \\
w = y^n x^{-1} __ &\iff f(w) = x^{-n} y^{-1} __ \\
w = __ xy^n &\iff f(w) = __ yx^{-n} \\
w = __ x^{-1} y^n &\iff f(w) = __ y^{-1} x^{-n-1}.
\end{aligned}$$

Proof. Write

$$w = y^{n_0} x^{m_1} y^{n_1} \cdots x^{m_r} y^{n_r},$$

where $r \geq 1$ and the exponents all nonzero except possibly n_0 and n_r. We will prove all four equivalences simultaneously by induction on r. We have

$$f(y^{n_0} x^{m_1} y^{n_1}) = x^{-n_0} (xy)^{m_1} x^{-n_1} = \begin{cases} x^{-n_0+1} y \_\_ yx^{-n_1} & m_1 > 0 \\ x^{-n_0} y^{-1} \_\_ y^{-1} x^{-n_1-1} & m_1 < 0 \end{cases}$$

which proves the lemma for $r = 1$. Suppose now that $r > 1$, and let

$$w_1 \coloneqq y^{n_0} x^{m_1} y^{n_1} \cdots x^{m_{r-1}} y^{n_{r-1}}, \qquad w_2 \coloneqq x^{m_r} y^{n_r},$$

so that $w = w_1 w_2$. By the inductive hypothesis

$$f(w_1) = \begin{cases} \_\_ yx^{-n_{r-1}} & m_{r-1} > 0 \\ \_\_ y^{-1} x^{-n_{r-1}-1} & m_{r-1} < 0, \end{cases}$$

and

$$f(w_2) = \begin{cases} xy \_\_ & m_r > 0 \\ y^{-1} \_\_ & m_r < 0 \end{cases}$$

We claim that if there is cancellation in the product $f(w_1)f(w_2)$, then only one letter from each factor cancels: an x^{-1} at the end of $f(w_1)$, and an x at the start of $f(w_2)$.

Note that $n_{r-1} \neq 0$ since $r > 1$. If $f(w_1)$ ends with a non-trivial power of y, then it must end with y^{-1}. However $f(w_2)$ does not start with y. Hence, if there is cancellation, then $f(w_1)$ must end with a non-trivial power of x. If $f(w_2)$ starts with a non-trivial power of x, then $m_r > 0$, in which case $f(w_2) = xy\_\_$. If there is cancellation, then $f(w_1)$ must end with x^{-1}. To cancel two letters from each factor would require $f(w_1)$ to end with $y^{-1}x^{-1}$, which is not possible since $n_{r-1} \neq 0$.

In particular, no powers of y cancel in the product $f(w_1)f(w_2)$. Hence, if we apply the inductive hypothesis to the start of w_1 and the end of w_2, then we find that all four equivalences also hold for w. □

2.6.2 Proof that $k = 0$

Suppose that $w \notin \langle c \rangle$. Let $L(w), R(w)$ be maximal in absolute value such that $w = c^{L(w)}\_\_c^{-R(w)}$, and let $m(w) \in F_2$ be the unique word satisfying

$$w = c^{L(w)}m(w)c^{-R(w)}.$$

Lemma 2.6.4. *Suppose that $w \notin \langle c \rangle$. Then there is no cancellation in the product $c^{L(w)}m(w)c^{-R(w)}$, i.e., $\ell(w) = \ell(c^{L(w)}) + \ell(m(w)) + \ell(c^{-R(w)})$.*

Proof. This is equivalent to showing that the $c^{L(w)}$ at the start of w does not overlap with the $c^{-R(w)}$ at the end of w.

Suppose for a contradiction that they do overlap. Then there exists $u \in F_2$ with $0 < \ell(u) < \ell(c)$ such that $c^{\epsilon} = \_\_u$ and $c^{\epsilon'} = u\_\_$, where $\epsilon, \epsilon' \in \{\pm 1\}$ depend on the signs of $L(w)$ and $R(w)$. If $\epsilon = -\epsilon'$, then $u = u^{-1}$ which is impossible. If $\epsilon = \epsilon'$, then $c^{\epsilon} = u\_\_u$, however the only non-trivial word u satisfying this is $u = c^{\epsilon}$. Hence there is no overlap. □

Lemma 2.6.5. *If $w_1, w_2, w_1w_2 \notin \langle c \rangle$, then either $L(w_1w_2) = L(w_1)$, or $R(w_1w_2) = R(w_2)$.*

Proof. Let $L_i = L(w_i)$, $R_i = R(w_i)$, $m_i = m(w_i)$, $L_{12} = L(w_1w_2)$, and $R_{12} = R(w_1w_2)$, so

$$w_1w_2 = c^{L_1}m_1c^{L_2-R_1}m_2c^{-R_2}.$$

Since m_1 does not start or end with c or c^{-1}, by Lemma 2.6.4 the only way that L_{12} and L_1 could differ is if the middle factor $c^{L_2-R_1}$ is trivial, so that some cancellation in the product m_1m_2 can occur. Similarly, $L_2 = R_1$ is also a necessary condition for R_{12} and R_1 to differ.

Thus, we may assume going forward that $L_2 = R_1$. This means that

$$w_1w_2 = c^{L_1}m_1m_2c^{-R_2}.$$

Note that this implies $m_1m_2 \notin \langle c \rangle$. Hence

$$L_{12} = L(c^{L_1}m_1m_2c^{-R_2}) = L_1 + L(m_1m_2),$$

and

$$R_{12} = R(c^{L_1}m_1m_2c^{-R_2}) = R(m_1m_2) + R_2.$$

Therefore, it suffices to prove that either $L(m_1m_2) = 0$ or $R(m_1m_2) = 0$.

If $L(m_1m_2) \neq 0$, then the first four letters of m_1m_2 cannot be the same as the first four letters of m_1, i.e., at most three letters of m_1 remain in the product m_1m_2 after cancellation. Similarly, if $R(m_1m_2) \neq 0$, then at most three letters of m_2 remain in the product m_1m_2 after cancellation. Therefore, if both $L(m_1m_2) \neq 0$ and $R(m_1m_2) \neq 0$, then $\ell(m_1m_2) \leq 6$. However, Lemma 2.6.4 implies that

$$4|L(m_1m_2)| + 4|R(m_1m_2)| = \ell(c^{L(m_1m_2)}) + \ell(c^{-R(m_1m_2)}) < \ell(m_1m_2).$$

Hence, either $L(m_1m_2) = 0$ or $R(m_1m_2) = 0$, as required. □

Lemma 2.6.6. *If $w \notin \langle c \rangle$ then $L(w) \leq L(f(w))$ and $R(w) \leq R(f(w))$.*

Proof. It suffices to prove the first inequality for all $w \notin \langle c \rangle$, since $R(w) = L(w^{-1})$.

If $L(w) > 0$, then

$$w = c^{L(w)-1}xyx^{-1}\_\_,$$

which by Lemmas 2.6.2 and 2.6.3 implies that $f(w) = c^{L(w)}\_\_$, so $L(w) \leq L(f(w))$.

Similarly, if $L(f(w)) < 0$, then

$$f(w) = c^{L(f(w))+1}yxy^{-1}\_\_,$$

which by Lemmas 2.6.2 and 2.6.3 implies that $w = c^{L(f(w))}\_\_$, so $L(w) \leq L(f(w))$.

The only other possibility is $L(w) \leq 0 \leq L(f(w))$, and the inequality clearly holds here. □

Lemma 2.6.7. *If there is a non-identity word w satisfying $w = f^{6k+1}(w)f^{-6k-1}(w)$, then $k = 0$.*

Proof. Note that the set of solutions to this equation is closed under the action of $\langle f^{6k+1} \rangle$. Let w be a non-identity solution to $w = f^{6k+1}(w)f^{-6k-1}(w)$. The identity is the only solution that is fixed by f, so $w \notin \langle c \rangle$ by Lemma 2.6.2.

Write $w_n = f^n(w)$, $L_n = L(w_n)$ and $R_n = R(w_n)$. Since $w_{n+6} = cw_nc^{-1}$, then for all n sufficiently large $L_{n+6} = L_n + 1$ and $R_{n+6} = R_n + 1$. Therefore, for all n sufficiently large we have both

$$L_{n+6k+1} = L_{n+1} + k, \quad \text{and} \quad R_{n-6k-1} = R_{n-1} - k.$$

Furthermore, the sequences L_n and R_n are non-decreasing by Lemma 2.6.6. It follows that

there are arbitrarily large n such that both

$$L_{n+1} = L_n, \quad \text{and} \quad R_{n-1} = R_n.$$

In particular, $L_{n+6k+1} = L_n + k$ and $R_{n-6k-1} = R_n - k$ for some integer n. By Lemma 2.6.5 and the equation $w_n = w_{n+6k+1} w_{n-6k-1}$ we can conclude that either $L_n = L_n + k$ or $R_n = R_n - k$. In either case we must have $k = 0$. □

2.6.3 The solutions when $k = 0$

Lemma 2.6.8. *Let $w \in F_2$, and suppose that both w and $f(w)$ start with u, where $\ell(u) = 4$. Then $u = c^{\pm 1}$. Similarly if both w and $f(w)$ end with v, where $\ell(v) = 4$, then $v = c^{\pm 1}$.*

Proof. Note that the second statement follows from applying the first to w^{-1}. Write

$$w = y^{n_0} x^{m_1} y^{n_1} \cdots x^{m_r} y^{n_r}$$

as before. Suppose that w and $f(w)$ both start with the same length four word u. By Lemma 2.6.3, w starts with either yx or x. Hence $r > 0$, $m_1 > 0$ and $n_0 \in \{0, 1\}$. In the arguments that follow, we will freely apply Lemma 2.6.3 in similar ways to the previous application. For instance,

$$f(y^{n_1} \cdots x^{m_r} y^{n_r}) = \begin{cases} x^{-n_1} & r = 1 \\ x^{-n_1+1} y \_\_ & m_2 > 0 \\ x^{-n_1} y^{-1} \_\_ & m_2 < 0. \end{cases}$$

Hence,

$$f(w) = \begin{cases} x^{-n_0+1}y(xy)^{m_1-1}x^{-n_1} & r = 1 \\ x^{-n_0+1}y(xy)^{m_1-1}x^{-n_1+1}y\_\_ & m_2 > 0 \\ x^{-n_0+1}y(xy)^{m_1-1}x^{-n_1}y^{-1}\_\_ & m_2 < 0. \end{cases}$$

If $n_0 = 0$, then $f(w) = xy\_\_$, hence $m_1 = 1$. If $n_0 = 1$, then $f(w) = y(xy)^{m_1-1}\_\_$, hence m_1 cannot be larger than 1. Thus, in all cases $m_1 = 1$. Hence,

$$f(w) = \begin{cases} x^{-n_0+1}yx^{-n_1} & r = 1 \\ x^{-n_0+1}yx^{-n_1+1}y\_\_ & m_2 > 0 \\ x^{-n_0+1}yx^{-n_1}y^{-1}\_\_ & m_2 < 0. \end{cases}$$

Clearly $r = 1$ leads to a contradiction. If $n_0 = 1$, then

$$f(w) = \begin{cases} yx^{-n_1+1}y\_\_ & m_2 > 0 \\ yx^{-n_1}y^{-1}\_\_ & m_2 < 0. \end{cases}$$

Since $m_1 = 1$ (and $n_0 = 1$) we must have $n_1 = -1$ and $m_2 < 0$. Thus, if $n_0 = 1$, then w starts with $c^{-1} = yxy^{-1}x^{-1}$.

Now suppose $n_0 = 0$, and assume for a contradiction that $n_1 \neq 1$. Then $f(w) = xyx^e\_\_$, where $e \neq 0$. Hence, by comparing with w we must have $n_1 = 1$.

Thus, if $n_0 = 0$, then $m_1 = n_1 = 1$, and hence $m_2 < 0$. Note that

$$f(y^{n_2} \cdots x^{n_k}y^{n_k}) = \begin{cases} x^{-n_2} & r = 2 \\ x^{-n_2+1}y\_\_ & m_3 > 0 \\ x^{-n_2}y^{-1}\_\_ & m_3 < 0. \end{cases}$$

Hence, if $n_0 = 0$, then

$$f(w) = \begin{cases} xyx^{-1}(y^{-1}x^{-1})^{-m_2-1}y^{-1}x^{-n_2-1} & r = 2 \\ xyx^{-1}(y^{-1}x^{-1})^{-m_2-1}y^{-1}x^{-n_2}y\_\_ & m_3 > 0 \\ xyx^{-1}(y^{-1}x^{-1})^{-m_2-1}y^{-1}x^{-n_2-1}y^{-1}\_\_ & m_3 < 0, \end{cases}$$

and so w starts with $c = xyx^{-1}y^{-1}$. □

In what follows, the *f-orbit of w* is the set $\{f^n(w) : n \in \mathbb{Z}\}$, i.e., the union of the forwards and backwards f-orbits of w.

Lemma 2.6.9. *Every word in F_2 not in $\langle c \rangle$ contains a word w in its f-orbit such that $f^{-1}(w)$, w, and $f(w)$ all neither end with c nor c^{-1}.*

Proof. Suppose $w \notin \langle c \rangle$. Since $f^6(w) = cwc^{-1}$, the word $f^n(w)$ ends with c^{-1} for all sufficiently large n, and ends with c for all sufficiently negative n.

For a given f-orbit not in $\langle c \rangle$, choose v in it such that v ends in c but $f(v)$ does not, and let $w \coloneqq f^2(v)$, so $f^{-1}(w)$ does not end in c. By Lemma 2.6.3, neither w nor $f(w)$ end with c. Also, by Lemma 2.6.3,

$$v = \_\_c \implies f(v) = f^{-1}(w) = \_\_yx^{-1}y^{-1} \implies w = \_\_y^{-1}.$$

Suppose for a contradiction that $f(w) = \_\_c^{-1}$. Then in particular $f(w) = \_\_y^{-1}x^{-1}$, so by Lemma 2.6.3

$$w = \_\_x^{-1},$$

which is a contradiction. Hence, none of $f^{-1}(w)$, w or $f(w)$ end in c^{-1}. □

Lemma 2.6.10. *If w is a non-identity solution of $w = f(w)f^{-1}(w)$, then $w = f^n(x)$ for some $n \in \mathbb{Z}$.*

Proof. First note the claim holds under the additional hypothesis that $\ell(w) \leq 6$, which is easily verified by a computer. We can also ignore elements of $\langle c \rangle$, since the only solution contained in there is the identity.

Suppose w neither starts nor ends with either of c or c^{-1}. Then by Lemma 2.6.8, in the product $w = f(w)f^{-1}(w)$ all but at most the first three letters of $f(w)$ and at most the last three letters of $f^{-1}(w)$ must be cancelled, hence $\ell(w) \leq 6$, so w is in the f-orbit of x.

Assume then that all words in the f-orbit of $w \notin \langle c \rangle$ either start or end with one of c or c^{-1}. By Lemma 2.6.9, we may assume without loss of generality that $f^{-1}(w)$, w and $f(w)$ do not end with either of c or c^{-1}, and hence all start with c or c^{-1}.

In particular, by Lemma 2.6.8 all but at most three letters of $f^{-1}(w)$ are cancelled in the product $f(w)f^{-1}(w)$. Hence $\ell(f^{-1}(w)) \leq 6$, so w is in the f-orbit of x. □

Proof of Theorem 2.6.1. If w is a non-identity solution of $f^{6k+1}(w)f^{-6k-1}(w)$, then $k = 0$ by Lemma 2.6.7, and $w = f^n(x)$ for some $n \in \mathbb{Z}$ by Lemma 2.6.10. □

2.7 Classification of holomorphic maps

The goal of this section is to classify the holomorphic maps $\mathrm{UConf}_n \mathbb{C} \to \mathrm{UConf}_m \mathbb{C}$ for $m \in \{3, 4\}$, and to classify the families of elliptic curves over $\mathrm{UConf}_n \mathbb{C}$.

We denote by $\mathcal{M}_{g,n}$ the moduli space of Riemann surfaces of genus g with n unordered marked points. We only consider this moduli space when $2g - 2 + n > 0$, where it can be identified with the orbifold quotient $\mathrm{Mod}_{g,n} \backslash \mathcal{T}_{g,n}$, where $\mathrm{Mod}_{g,n}$ and $\mathcal{T}_{g,n}$ are the associated mapping class group and Teichmüller spaces. Note that $\mathrm{Mod}_{g,n}$ need not act faithfully on $\mathcal{T}_{g,n}$, e.g., $\mathcal{M}_{1,1} = \mathrm{SL}_2 \mathbb{Z} \backslash \mathbb{H}$. Adopting this convention means that holomorphic maps $B \to \mathcal{M}_{g,n}$, in the orbifold sense, where B is a complex manifold, correspond to families of genus g curves having n marked points.

Following the techniques of Chen and Salter [CS23], we first constrain the possible homomorphisms that can come from holomorphic maps, and then apply a theorem of Imayoshi and

Shiga [IS88] to see that each possible equivalence class is realized by a unique holomorphic map, up to affine twists.

2.7.1 Constraining the homomorphisms

The following is a well-known consequence of work of Bers [Ber78], see, e.g., [McM00, §3].

Theorem 2.7.1. *Let $f\colon \mathbb{D}^* \to \mathcal{M}_{g,n}$ be holomorphic, where $\mathbb{D}^* = \{z \in \mathbb{C} : 0 < |z| < 1\}$ is a punctured disk and $2g - 2 + n > 0$. Then $f_*(k)$ is a multitwist for some $k \geq 1$, where $f_*\colon \mathbb{Z} \to \mathrm{Mod}_{g,n}$ is the induced map on fundamental groups.*

Proof. It is a theorem of Royden [Roy71] (see [EK74, §2.3]) that $\mathcal{M}_{g,n}$ is Kobayashi hyperbolic, with Kobayashi metric equal to the Teichmüller metric. Hence, the holomorphic map f is distance non-increasing with respect to the hyperbolic metric on $\mathbb{D}^*$ and the Teichmüller metric on $\mathcal{M}_{g,n}$. Since a loop around the puncture in $\mathbb{D}^*$ can be made arbitrarily short in the hyperbolic metric, $f_*(k_0)$ has translation length zero for all $k_0 \in \mathbb{Z}$. Let $k_0 \geq 1$ be such that $f_*(k_0)$ preserves the connected components of the complement of its canonical reduction system. By assertion (A) in the proof of Bers' Theorem 7 [Ber78], we can conclude that $f_*(k_0)$ is not pseudo-Anosov when restricted to any of these connected components. Thus, $f_*(k_0)$ is periodic on each component, hence $f_*(k)$ is a multitwist for some positive multiple k of k_0. □

As a useful consequence we have the following.

Proposition 2.7.2. *Let $f\colon \mathrm{UConf}_m \mathbb{C} \to \mathcal{M}_{g,n}$ be a holomorphic map, where $2g-2+n > 0$. Then there exists some $k \neq 0$ such that $f_*(\sigma_i^k)$ is a multitwist for $i = 1, \ldots, n-1$.*

Proof. The configuration space $\mathrm{UConf}_m \mathbb{C}$ is isomorphic to the space of monic degree m square-free polynomials over $\mathbb{C}$. This identifies $\mathrm{UConf}_m \mathbb{C}$ with $\mathbb{C}^m - \mathcal{D}_m$, where $\mathcal{D}_m$ is the discriminant locus. The smooth points of $\mathcal{D}_m$ correspond to configurations with exactly

one repeated point. Thus, a loop in the discriminant complement representing σ_i is freely homotopic to a small loop γ in the normal space of $\mathcal{D}_m$ at some smooth point. In fact, we can choose the same γ for each σ_i, since the set of smooth points of $\mathcal{D}_m$ is connected. Applying Theorem 2.7.1 to the punctured disk bounded by γ gives the result. □

We will also need the following theorem, which is a special case of a theorem of Daskalopoulos and Wentworth.

Theorem 2.7.3 ([DW07, Theorem 5.7])**.** *Let $f\colon B \to \mathcal{M}_{g,n}$ be a holomorphic map, where B is a Riemann surface of finite type. If $f_*\colon \pi_1(B) \to \mathrm{Mod}_{g,n}$ is cyclic or reducible, then f is constant.*

Chen and Salter prove the following in [CS23, §3.3].

Lemma 2.7.4. *Let $f : \mathrm{UConf}_n \mathbb{C} \to \mathrm{UConf}_m \mathbb{C}$ be a holomorphic map such that the induced map $\bar{f} : \mathrm{UConf}_n \mathbb{C} \to \mathrm{UConf}_m \mathbb{C}/\mathrm{Aff}$ is constant. Then f is an affine twist of a root map.*

2.7.2 Uniqueness

The following result is a generalization of a theorem of Imayoshi–Shiga [IS88]. We mimic the proof of [AAS18, Proposition 3.2] and invoke [CS23, Theorem 2.5].

Theorem 2.7.5. *Let M be a smooth, quasi-projective complex variety, let $\mathcal{M}'_{g,n}$ be a finite orbifold cover of $\mathcal{M}_{g,n}$, and let $f, g\colon M \to \mathcal{M}'_{g,n}$ be non-constant holomorphic or antiholomorphic maps. If f and g are homotopic as orbifold maps, then $f = g$.*

Proof. Chen and Salter's result [CS23, Theorem 2.5] handles the case where M is a Riemann surface of finite type. For the general case, note that a generic curve in M is a Riemann surface of finite type. Thus, f and g agree on a generic curve in M, and hence $f = g$. □

Note that $\mathrm{UConf}_m \mathbb{C}$ is a smooth quasi-projective variety, so Theorem 2.7.5 applies to it. Furthermore, since $\mathrm{UConf}_m \mathbb{C}$ is a $K(\pi, 1)$ space, $f, g\colon Y \to \mathcal{M}'_{g,n}$ are homotopic as orbifold

maps if and only if $f_*, g_*\colon \pi_1(Y) \to \pi_1(\mathcal{M}'_{g,n})$ are conjugate, for any covering space Y of $\mathrm{UConf}_m \mathbb{C}$.

The following consequence is proved by Chen and Salter in the special case where $m = n$ [CS23, §3.1]. Note that there is a natural holomorphic map $\mathrm{UConf}_n \mathbb{C} \to \mathcal{M}_{0,n+1}$, given by associating to $\{x_1, \ldots, x_n\} \in \mathrm{UConf}_n \mathbb{C}$ the Riemann surface $\mathbb{CP}^1 = \mathbb{C} \cup \{\infty\}$, with marked points $\{x_1, \ldots, x_n, \infty\}$.

Proposition 2.7.6. *Let $f, g\colon \mathrm{UConf}_m \mathbb{C} \to \mathrm{UConf}_n \mathbb{C}$ be holomorphic maps such that the induced maps $\bar{f}, \bar{g}\colon \mathrm{UConf}_m \mathbb{C} \to \mathcal{M}_{0,n+1}$ are not constant. Suppose that $f_*, g_*\colon B_m \to B_n$ differ by at most an automorphism of B_n and a transvection $B_m \to Z(B_n)$. Then f is an affine twist of g.*

Proof. Without loss of generality, we can apply an affine twist by a power of the discriminant $\Delta\colon \mathrm{UConf}_m \mathbb{C} \to \mathbb{C}^*$ so that f_*, g_* differ only by an automorphism of B_n.

Dyer and Grossman showed that $\mathrm{Out}(B_n) \cong \langle \iota \rangle$, where ι is given by $\sigma_i \mapsto \sigma_i^{-1}$ for $i = 1, \ldots, n-1$ [DG81]. The involution ι is induced by the coordinate-wise complex conjugation on $\mathrm{UConf}_n \mathbb{C}$. Thus, if $f_* = \tau \circ g_*$ for an outer automorphism τ of B_n, then by Theorem 2.7.5 the induced maps $\mathrm{UConf}_m \mathbb{C} \to \mathcal{M}_{0,n+1}$ only differ by complex conjugation, which contradicts both f, g being holomorphic. Hence f_*, g_* are conjugate, and thus by Theorem 2.7.5 they induce the same maps $\bar{f} = \bar{g}\colon \mathrm{UConf}_m \mathbb{C} \to \mathcal{M}_{0,n+1}$.

Let Y be the finite cover of $\mathrm{UConf}_m \mathbb{C}$ such that f, g lift to maps $\tilde{f}, \tilde{g}\colon Y \to \mathrm{PConf}_n \mathbb{C}$, and write $\tilde{f} = (f_1, \ldots, f_n)$ and $\tilde{g} = (g_1, \ldots, g_n)$, where $f_i, g_i\colon Y \to \mathbb{C}$ for $i = 1, \ldots, n$. Note that Y is also a quasi-projective variety by a result of Grauert and Remmert [GR58]. Since $\bar{f} = \bar{g}$ and f_*, g_* are conjugate, $\tilde{f}_*, \tilde{g}_*$ are also conjugate. Theorem 2.7.5 implies that $\tilde{f}, \tilde{g}$ induce identical maps $Y \to \mathrm{PConf}_n \mathbb{C}/\mathrm{Aff}$. Hence, there exists a function $\tilde{A}\colon Y \to \mathrm{Aff}$ such $\tilde{f} = \tilde{g}^{\tilde{A}}$.

If $\tilde{A} \cdot z = az + b$ for all $z \in \mathbb{C}$, where $a, b\colon Y \to \mathbb{C}$, then for all distinct $i, j \in \{1, \ldots, n\}$

we have

$$a = \frac{g_i - g_j}{f_i - f_j}, \qquad b = \frac{f_i g_j - f_j g_i}{f_i - f_j}.$$

Hence $\tilde{A}$ is holomorphic. Furthermore, the above holds for all distinct indices i, j, and $\tilde{f}, \tilde{g}$ are equivariant with respect to the group of deck transformations of the covering map $Y \to \mathrm{UConf}_m \mathbb{C}$, which is a subgroup of S_n. Hence $\tilde{A}$ descends to a holomorphic map $A\colon \mathrm{UConf}_m \mathbb{C} \to \mathrm{Aff}$ such that $f = g^A$. □

Remark 2.7.7. Grauert and Remmert's result is not needed by Chen and Salter, since in the case $m = n$ all such holomorphic maps lift to self maps of $\mathrm{PConf}_n \mathbb{C}$, which is clearly quasi-projective. However, we will apply Proposition 2.7.6 with the map Ψ_3, which does not lift to a holomorphic map $\mathrm{PConf}_3 \mathbb{C} \to \mathrm{PConf}_4 \mathbb{C}$. Neither the argument of Grauert and Remmert, nor Grothendieck's vast generalization [Gro71, XII, Théorème 5.1], give explicit equations for an arbitrary finite cover of $\mathrm{UConf}_n \mathbb{C}$. However, since Ψ_3 is also algebraic, one can write down explicit equations for the pullback $Y \to \mathrm{UConf}_3 \mathbb{C}$ of the cover $\mathrm{PConf}_4 \mathbb{C} \to \mathrm{UConf}_4 \mathbb{C}$ along Ψ_3 that express Y as a quasi-projective variety.

2.7.3 Classifying the holomorphic maps

We are finally ready to prove Theorem 2.1.1.

Proof of Theorem 2.1.1. Let $f\colon \mathrm{UConf}_n \mathbb{C} \to \mathrm{UConf}_m \mathbb{C}$ be a holomorphic map, where $n \geq 3$ and $m \in \{3, 4\}$. Assume that f is not a root map. By Proposition 2.7.2 and Theorem 2.7.3, the homomorphism $f_*\colon B_n \to B_m$ must be irreducible, non-cyclic, and $f_*(\sigma_i)$ must have a power that is a multitwist for all $i = 1, \ldots, n-1$.

If $m = 3$, then for f_* to be non-cyclic we must have $n \in \{3, 4\}$ by Theorem 2.2.8. Since R_* is surjective, $n \in \{3, 4\}$ is also necessary if $m = 4$.

Suppose $n = 3$. By Theorem 2.2.2 and Theorem 2.5.2, f_* is either the identity on B_3, or is Ψ_{3*}, up to a transvection $B_3 \to Z(B_m)$ and an automorphism of B_m.

Suppose $n = 4$. By Theorems 2.2.8 and 2.5.3 and the above, f_* is either R_*, the identity on B_4, or $\Psi_{3*} \circ R_*$, up to a transvection $B_4 \to Z(B_m)$ and an automorphism of B_m.

By Proposition 2.7.6, it follows that f is an affine twist of either the identity map, R, Ψ_3, or $\Psi_3 \circ R$. □

Next we classify the holomorphic maps $\mathrm{UConf}_n \mathbb{C} \to \mathrm{PSL}_2 \mathbb{Z} \backslash \mathbb{H}$. There is a natural map $\mathrm{SL}_2 \mathbb{Z} \backslash \mathbb{H} \to \mathrm{PSL}_2 \mathbb{Z} \backslash \mathbb{H}$ corresponding to the quotient $\mathrm{SL}_2 \mathbb{Z} \twoheadrightarrow \mathrm{PSL}_2 \mathbb{Z}$. Moreover, $\mathrm{PSL}_2 \mathbb{Z} \backslash \mathbb{H}$ is the moduli space of quotients of elliptic curves by their hyperelliptic involution. This is the same data as a four times marked sphere, where one of the four points is distinguished. Hence $\mathrm{PSL}_2 \mathbb{Z} \backslash \mathbb{H}$ is a finite orbifold cover of $\mathcal{M}_{0,4}$.

Theorem 2.7.8. *Let $f\colon \mathrm{UConf}_n \mathbb{C} \to \mathrm{PSL}_2 \mathbb{Z} \backslash \mathbb{H}$ be a holomorphic map.*

- *If $n \geq 5$, then f is constant.*
- *If $n = 4$, then $f = g \circ R$, where $g\colon \mathrm{UConf}_3 \mathbb{C} \to \mathrm{PSL}_2 \mathbb{Z} \backslash \mathbb{H}$ is holomorphic.*
- *If $n = 3$, then either f is constant, or it is the post-composition $\bar{H}$ of the hyperelliptic embedding H with the natural map $\mathcal{M}_{1,1} \cong \mathrm{SL}_2 \mathbb{Z} \backslash \mathbb{H} \to \mathrm{PSL}_2 \mathbb{Z} \backslash \mathbb{H}$.*

Proof. Suppose that f is non-constant. By Theorem 2.7.3, $f_*\colon B_n \to \mathrm{PSL}_2 \mathbb{Z}$ is non-cyclic, and hence by Theorem 2.2.1, $n < 5$, and if $n = 4$, then f_* factors through R_*. By Proposition 2.7.2, the $f_*(\sigma_i)$ are parabolic. Let $\iota \in \mathrm{Aut}(B_n)$ be the inversion automorphism which is defined by $\sigma_i \mapsto \sigma_i^{-1}, i = 1, \cdots, n-1$. By Theorem 2.2.2 $n = 3$ or $n = 4$, and f_* is H_* up to precomposition by one or both of R_* and ι.

Note that coordinate-wise complex conjugation is an antiholomorphic self map of $\mathrm{UConf}_n \mathbb{C}$ inducing ι. Hence, by Theorem 2.7.5 either $f = \bar{H}$, $f = R \circ \bar{H}$, or f is one of these maps precomposed with coordinate-wise complex conjugation.

Since complex conjugation is antiholomorphic, the only holomorphic possibilities for f are $f = H$ when $n = 3$, and $f = H \circ R$ when $n = 4$, as desired. □

Proof of Theorem 2.1.2. Let $f\colon \mathrm{UConf}_n \mathbb{C} \to \mathcal{M}_{1,1}$ be a holomorphic family such that the induced map $\bar{f}\colon \mathrm{UConf}_n \mathbb{C} \to \mathrm{PSL}_2 \mathbb{Z}\backslash\mathbb{H}$ non-constant. Since the kernel of $\mathrm{SL}_2 \mathbb{Z} \twoheadrightarrow \mathrm{PSL}_2 \mathbb{Z}$ is $Z(\mathrm{SL}_2 \mathbb{Z}) \cong \mathbb{Z}/2\mathbb{Z}$, and $H^1(B_n; \mathbb{Z}/2\mathbb{Z}) \cong \mathbb{Z}/2\mathbb{Z}$, each holomorphic map $\mathrm{UConf}_n \mathbb{C} \to \mathrm{PSL}_2 \mathbb{Z}\backslash\mathbb{H}$ lifts to at most two holomorphic families $\mathrm{UConf}_n \mathbb{C} \to \mathcal{M}_{1,1}$, whose induced homomorphisms $B_n \to \mathrm{SL}_2 \mathbb{Z}$ differ by a transvection $B_n \to Z(\mathrm{SL}_2 \mathbb{Z})$. For any holomorphic family, we can induce such a transvection by pre-composing with id^{Δ}, since $\Delta\colon \mathrm{UConf}_n \mathbb{C} \to \mathbb{C}^*$ induces the abelianization map $\Delta_*\colon B_n \to \mathbb{Z}$.

Hence, if $\bar{f}$ is non-constant, then by Theorem 2.7.8 we have $n \in \{3, 4\}$ and $f = H$ up to pre-composition by one or both of R and id^{Δ}. □

CHAPTER 3

GENERATORS FOR THE LEVEL m CONGRUENCE SUBGROUPS OF BRAID GROUPS

This chapter is joint work with Ishan Banerjee [BH24], and is a preprint on the arXiv: Ishan Banerjee and Peter Huxford. *Generators for the level m congruence subgroups of braid groups*. 2024. arXiv: `2409.09612v1 [math.GR]`. Both authors contributed to the ideas and writing.

3.1 Introduction

Let $m \geq 1$, $n \geq 2$. Our main result relates three important subgroups of the braid group on n strands B_n that we define below.

The half-twist mth power subgroup

Let $\sigma_1, \dots, \sigma_{n-1}$ be the standard generators of B_n. A *half-twist* is any element of B_n that is conjugate to σ_1. Note that the standard generators of B_n are all half-twists. We define the *half-twist mth power subgroup* B_n^m to be the normal closure of σ_1^m in B_n:

$$B_n^m \coloneqq \langle \sigma^m : \sigma \in B_n \text{ is a half-twist} \rangle.$$

This conflicts with the common notation G^m, which typically denotes the subgroup of G generated by all its mth powers. As such, we do not make use of this more common notation in our paper.

The braid Torelli group

The *unreduced Burau representation* is the homomorphism $B_n \to \mathrm{GL}_n(\mathbb{Z}[t,t^{-1}])$ defined by $\sigma_i \mapsto I_{i-1} \oplus \left(\begin{smallmatrix} 1-t & 1 \\ t & 0 \end{smallmatrix}\right) \oplus I_{n-i-1}$, $1 \leq i < n$, where I_j is the $j \times j$ identity matrix. The *integral Burau representation* is the homomorphism $\rho_n \colon B_n \to \mathrm{GL}_n(\mathbb{Z})$ obtained by specializing the unreduced Burau representation at $t = -1$. The *braid Torelli group* $\mathcal{BI}_n$ is the kernel

$$\mathcal{BI}_n \coloneqq \ker\left(B_n \xrightarrow{\rho_n} \mathrm{GL}_n(\mathbb{Z})\right).$$

The Burau representation is often defined using the transposes of the matrices we use, however this difference does not affect any of the subgroups we define.

The level m congruence subgroup

By post-composing the integral Burau representation with the mod-m reduction map $\mathrm{GL}_n(\mathbb{Z}) \to \mathrm{GL}_n(\mathbb{Z}/m\mathbb{Z})$, we obtain a homomorphism $B_n \to \mathrm{GL}_n(\mathbb{Z}/m\mathbb{Z})$. Its kernel $B_n[m]$ is the *level m congruence subgroup* of B_n:

$$B_n[m] \coloneqq \ker\left(B_n \xrightarrow{\rho_n} \mathrm{GL}_n(\mathbb{Z}) \xrightarrow{\mathrm{mod}\ m} \mathrm{GL}_n(\mathbb{Z}/m\mathbb{Z})\right).$$

We are now equipped to state our main theorem.

Theorem A. *Suppose $m \geq 1$, $n \geq 2$, and $m < 6$ if $n = 3, 4$. The level m congruence subgroup $B_n[m]$ of B_n is generated by the braid Torelli group $\mathcal{BI}_n$ and B_n^m:*

$$B_n[m] = \langle \mathcal{BI}_n, B_n^m \rangle.$$

The equality $B_n[m] = \langle \mathcal{BI}_n, B_n^m \rangle$ does not hold when $n = 3, 4$ and $m \geq 6$; in fact $\langle \mathcal{BI}_n, B_n^m \rangle$ is an infinite index subgroup of B_n in these cases (see Remark 3.1.1).

We cannot omit the group $\mathcal{BI}_n$ from Theorem 4.1.1 in general. We discuss when $B_n[m] = B_n^m$ in Remark 3.1.2.

Brendle, Margalit, and Putman prove that $\mathcal{BI}_n$ is generated by the squares of Dehn twists about curves that surround 3 or 5 points [BMP15, Theorem C]. Combining their result with Theorem 4.1.1 we obtain a natural generating set of $B_n[m]$, solving a problem of Margalit [Mar19, Problem 3.5] for braid groups on at least five strands. In particular, we find that $B_n[m]$ for $n \geq 5$ can be normally generated by three elements.

Corollary B. *Suppose $m \geq 1$ and $n \geq 2$. The level m congruence subgroup $B_n[m]$ of B_n is normally generated by the following braids:*

- σ_1^m *if* $n = 2$,
- σ_1^m, *and* $(\sigma_1\sigma_2)^6$ *if* $n = 3, 4$ *and* $m < 6$,
- σ_1^m, $(\sigma_1\sigma_2)^6$, *and* $(\sigma_1\sigma_2\sigma_3\sigma_4)^{10}$ *if* $n \geq 5$.

For $n \geq 5$, Theorem 4.1.1 is an analog of Mennicke's result [Men65, Section 10] that for $m \geq 1$ and $g \geq 2$ the principal level m congruence subgroup $\mathrm{Sp}_{2g}(\mathbb{Z})[m]$ of the integral symplectic group $\mathrm{Sp}_{2g}(\mathbb{Z})$ is generated by mth powers of transvections. Indeed, A'Campo [ACa79] showed that $\rho_{2g+1}(B_{2g+1})$ may be regarded as a proper, finite index subgroup of $\mathrm{Sp}_{2g}(\mathbb{Z})$ for $g \geq 2$, with a similar result for ρ_{2g+2}. This finite index difference is why Theorem 4.1.1 does not follow from Mennicke's result, although it does play a crucial role in our proof.

The situation for $n = 3, 4$ is more delicate; this is related to the failure of $\mathrm{SL}_2(\mathbb{Z})$ to satisfy the congruence subgroup property. Let $\bar{\rho}_3 \colon B_3 \to \mathrm{SL}_2(\mathbb{Z})$ denote the surjection defined by $\sigma_1 \mapsto \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$ and $\sigma_2 \mapsto \left(\begin{smallmatrix} 1 & 0 \\ -1 & 1 \end{smallmatrix}\right)$. (The connection between $\bar{\rho}_3$ and ρ_3 is explained by Proposition 3.2.3.) We give a partial answer to Margalit's problem on $B_n[m]$ for $n = 3, 4$ and $m \geq 6$ [Mar19, Problem 3.5].

Theorem C. *Let $m \geq 1$, and $R \subseteq B_3[m]$ be such that the normal closure of $\bar{\rho}_3(R)$ in $\mathrm{SL}_2(\mathbb{Z})$ contains $\mathrm{SL}_2(\mathbb{Z})[2m]$. Then, for $n = 3, 4$, the level m congruence subgroup $B_n[m]$ is*

normally generated in B_n by R, σ_1^m, and $(\sigma_1\sigma_2)^6$.

One way to construct R satisfying the conditions of Theorem C is to find a normal generating set of $\mathrm{SL}_2(\mathbb{Z})[m]$. This can be extracted from a finite presentation of $\mathrm{SL}_2(\mathbb{Z}/m\mathbb{Z})$, such as the one explicitly written down for all $m \geq 1$ by Hsu [Hsu96, Lemma 3.5].

It follows from work of Miller [Mil02], see also [New72, Chapter VII, Section 13], that when $1 \leq m \leq 5$ the image of $\bar{\rho}_3(\sigma_1^m)$ normally generates $\mathrm{SL}_2(\mathbb{Z})[m]$ in $\mathrm{SL}_2(\mathbb{Z})$. Thus Theorem C implies Theorem 4.1.1 in the special cases $n = 3, 4$ and $1 \leq m \leq 5$.

Related work

For the cases $n = 3$ and $1 \leq m \leq 5$, Theorem 4.1.1 follows from Miller [Mil02]. Theorem 4.1.1 was also implicitly known in the special cases where $B_n[m] = B_n^m$, however in general this equality does not hold (see Remark 3.1.2). A notable case where this equality does hold is $B_n[2] = B_n^2$, which follows from Arnol'd's identification of $B_n[2]$ with the pure braid group PB_n [Arn68], and Moore's presentation of the symmetric group S_n [Moo97].

For all $n \geq 2$, a normal generating set of $B_n[3]$ was obtained by Assion [Ass78a], which was extended to a normal generating set of $B_n[p]$ for odd primes p by Wajnryb [Waj91]. Stylianakis has also found a normal generating set of $B_n[2p]$, and outlines how to construct normal generating sets for $B_n[2m]$ for square-free m [Sty18, Corollary 5.4].

Brendle and Margalit showed for all $n \geq 2$ that $B_n[4]$ is generated by all squares of Dehn twists [BM18]; our Theorem 4.1.1 implies for all $n \geq 2$ that $B_n[4]$ is generated by just the squares of Dehn twists about all curves surrounding 2, 3, or 5 points. Nakamura has also produced a different normal generating set of $B_n[4]$ [Nak22, Theorem 1.3].

The groups $B_n[m]$ are all finitely generated, but the problem of determining explicit finite generating sets of these groups remains open in general. This has been done for the pure braid group $B_n[2]$ by Artin [Art25], and for $B_3[p]$ where p is an odd prime by Bellingeri, Damiani, Ocampo, and Stylianakis [Bel+24, Theorem 3.9]. In fact, these papers give finite

presentations on the generators they provide.

We would also like to draw the reader's attention to related work of Bloomquist, Patzt, and Scherich, who completed the determination of the images of B_n and $B_n[m]$ under the integral Burau representation [BPS23].

Remark 3.1.1. When $n = 3, 4$ and $m \geq 6$ the subgroup $\langle \mathcal{BI}_n, B_n^m \rangle$ is not equal to $B_n[m]$; in fact it has infinite index in B_n. Indeed, for $n = 3, 4$, the braid Torelli group $\mathcal{BI}_n$ is normally generated in B_n by $(\sigma_1\sigma_2)^6$ by [BMP15, Theorem C]. Therefore, the special homomorphism $B_4 \to B_3$ defined by $\sigma_1, \sigma_3 \mapsto \sigma_1$ and $\sigma_2 \mapsto \sigma_2$ descends to a surjective homomorphism

$$B_4/\langle \mathcal{BI}_4, B_4^m \rangle \to B_3/\langle \mathcal{BI}_3, B_3^m \rangle.$$

Since $Z(B_3) = \langle (\sigma_1\sigma_2)^3 \rangle$, the quotient $B_3/\langle \mathcal{BI}_3, B_3^m \rangle$ surjects onto $B_3/\langle Z(B_3), B_3^m \rangle$. On the generators $x = (\sigma_1\sigma_2)^{-1}$ and $y = \sigma_1\sigma_2\sigma_1$ of B_3, the relation $\sigma_1\sigma_2\sigma_1 = \sigma_2\sigma_1\sigma_2$ is equivalent to $x^{-3} = y^2$. We also have the identity $\sigma_1 = xy$. Hence

$$B_3/\langle Z(B_3), B_3^m \rangle \cong \langle x, y \mid x^3 = y^2 = (xy)^m = 1 \rangle.$$

This group, known as the $(2, 3, m)$-von Dyck group, is infinite for $m \geq 6$ by a result of Miller [Mil02], see also [CM57, Section 5.3]. Therefore, when $n = 3, 4$ and $m \geq 6$ the subgroup $\langle \mathcal{BI}_n, B_n^m \rangle$ has infinite index in B_n.

Remark 3.1.2. In Proposition 3.2.2 we observe that $B_n^m \leq B_n[m]$ for all $m \geq 1$ and $n \geq 2$. Thus, $B_n^m = B_n[m]$ if and only if B_n/B_n^m and $B_n/B_n[m]$ have the same order. The orders of the quotients B_n/B_n^m and $B_n/B_n[m]$ are all known. The determination of the orders of B_n/B_n^m was completed by Coxeter [Cox59], and the same task for $B_n/B_n[m]$ was completed by Bloomquist, Patzt and Scherich [BPS23].

Coxeter showed that B_n/B_n^m is finite if and only if $1/m + 1/n > 1/2$ [Cox59], see also [Ass78b]. By comparing the orders of the quotients B_n/B_n^m and $B_n/B_n[m]$ for the remaining

values of $m \geq 1$ and $n \geq 2$ for which B_n/B_n^m is finite, we find that

- $B_n[m] = B_n^m$ if and only if $m = 1, 2$, or $n = 2$, or $m = 3$ and $n = 3, 4$,
- $B_3[4]/B_3^4 \cong \mathbb{Z}/2\mathbb{Z}$, $B_3[5]/B_3^5 \cong \mathbb{Z}/5\mathbb{Z}$, and $B_5[3]/B_5^3 \cong \mathbb{Z}/3\mathbb{Z}$,
- $B_n[m]/B_n^m$ is infinite otherwise.

Outline of the paper

In §§3.2 and 3.3 we give background on the Burau representation and the integral symplectic group $\mathrm{Sp}_{2g}(\mathbb{Z})$. In §3.4 we prove an intermediate result: $B_n[m] = \langle B_n[2m], B_n^m \rangle$. In §3.5 we use this to reduce Theorem 4.1.1 and Theorem C to Theorem 3.5.1, which asserts that $\rho_n(B_n^m)$ contains all $2m$th powers of certain elements of $\mathrm{GL}_n(\mathbb{Z})$. Finally, we prove Theorem 3.5.1 in §3.6.

3.2 The integral Burau representation as a symplectic representation

There is a sense in which the integral Burau representation $\rho_n \colon B_n \to \mathrm{GL}_n(\mathbb{Z})$ is a symplectic representation of the braid group, which we explain in Proposition 3.2.3. To state this proposition, we first establish some notation that we use throughout the paper. We write

$$\Gamma_n \coloneqq \rho_n(B_n), \quad \Gamma_n^m \coloneqq \rho_n(B_n^m), \quad \Gamma_n[m] \coloneqq \rho_n(B_n[m]) = \Gamma_n \cap \mathrm{GL}_n(\mathbb{Z})[m].$$

Note that we have the short exact sequence

$$1 \to \mathcal{BI}_n \to B_n[m] \to \Gamma_n[m] \to 1,$$

and thus $B_n[m] = \langle \mathcal{BI}_n, B_n^m \rangle$ is equivalent to $\Gamma_n[m] = \Gamma_n^m$.

Definition (**Transvections**)**.** Given a $\mathbb{Z}$-module V equipped with an alternating $\mathbb{Z}$-bilinear form $\langle -, - \rangle$, for each $x \in V$ we define the *transvection* $T_x \in \mathrm{Aut}(V, \langle -, - \rangle)$ by the formula

$$T_x(v) \coloneqq v - \langle v, x \rangle x \quad \forall v \in V.$$

Note that

$$AT_xA^{-1} = T_{Ax} \quad \text{for all } x \in V \text{ and } A \in \mathrm{Aut}(V, \langle -, - \rangle).$$

Now let $e_1, \ldots, e_n$ denote the standard basis of $\mathbb{Z}^n$, and equip $\mathbb{Z}^n$ with the alternating $\mathbb{Z}$-bilinear form defined by

$$\langle e_i, e_j \rangle = \begin{cases} 1 & i < j \\ 0 & i = j \\ -1 & i > j. \end{cases}$$

Define $c_i \coloneqq e_i - e_{i+1}$ for $1 \leq i < n$. Then we have the following equivalent definition of the integral Burau representation that is easily checked.

Proposition 3.2.1. *For $n \geq 2$, the integral Burau representation ρ_n satisfies*

$$\rho_n(\sigma_i) = T_{c_i} \quad 1 \leq i < n.$$

This provides a quick way to verify the following proposition.

Proposition 3.2.2. *If $m \geq 1$ and $n \geq 2$ then $B_n^m \leq B_n[m]$.*

Proof. If T is a transvection then T^m is the identity mod m, hence $B_n^m \leq B_n[m]$. □

We also define the following sublattices of $\mathbb{Z}^n$.

Definition (**The sublattices W_n and V_n**)**.** Let $n \geq 2$, and let W_n be the sublattice of $\mathbb{Z}^n$

given by

$$W_n \coloneqq \mathbb{Z}c_1 + \cdots + \mathbb{Z}c_{n-1} = \{\lambda_1 e_1 + \cdots + \lambda_n e_n \in \mathbb{Z}^n : \lambda_1 + \cdots + \lambda_n = 0\}.$$

Let V_n be the sublattice of $\mathbb{Z}^n$ given by

$$V_n \coloneqq \begin{cases} \mathbb{Z}^n & n \text{ even} \\ W_n & n \text{ odd.} \end{cases}$$

Note that $\operatorname{rank}(V_n) = 2\lfloor n/2 \rfloor$ is even for all $n \geq 2$.

The following proposition is well-known, but is typically expressed with different notation.

Proposition 3.2.3. *Let $n \geq 2$.*

(a) The sublattices W_n and V_n are Γ_n-invariant.

(b) The bilinear form $\langle -, - \rangle$ on $\mathbb{Z}^n$ restricts to a symplectic form on V_n.

(c) The vector

$$w \coloneqq e_1 - e_2 + \cdots + (-1)^{n-1} e_n \in \mathbb{Z}^n$$

is fixed by Γ_n.

(d) If n is odd, then $\mathbb{Z}^n$ decomposes as a sum $V_n + \mathbb{Z}w$.

(e) If n is even, then

$$w = e_1 - e_2 + \cdots - e_n = c_1 + c_3 + \cdots + c_{n-1} \in W_n,$$

and the orthogonal complement $w^\perp$ of w in $\mathbb{Z}^n$ is W_n.

(f) The natural map

$$\Gamma_n \to \operatorname{Sp}(V_n) \cong \operatorname{Sp}_{2\lfloor n/2 \rfloor}(\mathbb{Z})$$

is injective, and when Γ_n *is viewed as a subgroup of* $\mathrm{Sp}(V_n)$ *we have*

$$\Gamma_n[m] = \Gamma_n \cap \mathrm{Sp}(V_n)[m].$$

Proof. Parts (a), (c), (d), and (e) are easy to verify. To prove part (b) it suffices to find a symplectic basis of V_n, see e.g., [BM18, Section 2.1] or the proof of [BPS23, Lemma 2.1]. Part (f) is immediate when n is even, and when n is odd it follows from parts (c) and (d). □

With respect to the $\mathbb{Z}$-basis (c_1, c_2) of V_3, the images $\rho_3(\sigma_1)$ and $\rho_3(\sigma_2)$ act by the matrices $\left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$ and $\left(\begin{smallmatrix} 1 & 0 \\ -1 & 1 \end{smallmatrix}\right)$, which generate $\mathrm{SL}_2(\mathbb{Z}) \cong \mathrm{Sp}(V_3)$. Thus, by Proposition 3.2.3 we may identify ρ_3 with the surjection $\bar{\rho}_3 \colon B_3 \to \mathrm{SL}_2(\mathbb{Z})$ mentioned in the statement of Theorem C.

3.3 Background on the integral symplectic group

The purpose of this section is to establish some useful facts about the integral symplectic group and its stabilizer subgroups. If a group G acts on a set X and $x \in X$, denote the stabilizer of x in G by G_x.

Let $g \geq 2$ and equip $\mathbb{Z}^{2g}$ with a symplectic form $\langle -, - \rangle$. If $w \in \mathbb{Z}^{2g}$ is a primitive vector, then $\langle -, - \rangle$ descends to a symplectic form on $w^\perp / w \cong \mathbb{Z}^{2g-2}$.

If $T \in \mathrm{Sp}_{2g}(\mathbb{Z})_w$, then T preserves the sublattice $w^\perp$ of $\mathbb{Z}^{2g}$, and thus descends to an element of $\mathrm{Sp}(w^\perp/w) \cong \mathrm{Sp}_{2g-2}(\mathbb{Z})$. This gives rise to a natural map $\mathrm{Sp}_{2g}(\mathbb{Z})_w \to \mathrm{Sp}(w^\perp/w)$, which restricts to a homomorphism $\mathrm{Sp}_{2g}(\mathbb{Z})[m]_w \to \mathrm{Sp}(w^\perp/w)[m]$ between the level m congruence subgroups for all $m \geq 1$.

Proposition 3.3.1. *If* $g \geq 2$, $m \geq 1$ *and* $w \in \mathbb{Z}^{2g}$ *is primitive then the group*

$$G \coloneqq \langle T_u^m : u \in \mathbb{Z}^{2g}, u \perp w \rangle$$

contains the kernel of the natural map

$$K \coloneqq \ker(\mathrm{Sp}_{2g}(\mathbb{Z})[m]_w \to \mathrm{Sp}(w^\perp/w)[m]).$$

Proof. Fix $v \in \mathbb{Z}^{2g}$ with $\langle v, w\rangle = 1$. We first prove that $T \mapsto Tv - v$ defines a set-theoretic bijection $K \to mw^\perp$. Let $T \in K$. Since $Tw = w$, we have $\langle v, w\rangle = \langle Tv, Tw\rangle = \langle Tv, w\rangle$, and hence $\langle Tv - v, w\rangle = 0$. Also, $Tv - v$ is a multiple of m since T is the identity modulo m. Hence $Tv - v \in mw^\perp$.

To prove the map $K \to mw^\perp$ is surjective, note that for each $x \in w^\perp$ we may define $S_x \in K$ with $S_x v - v = mx$ as follows:

$$S_x v \coloneqq v + mx, \qquad S_x u \coloneqq u + \langle u, x\rangle mw \quad \forall u \in w^\perp.$$

For injectivity, suppose that $T \in K$ with $Tv - v = mx$, where $x \in w^\perp$. Let $T' \coloneqq S_x^{-1}T \in K$. Suppose that $u \in w^\perp$. We have $T'v = v$, hence $T'u - u \in v^\perp$. Furthermore, $T'u - u \in \mathbb{Z}w$, since T' acts trivially on $w^\perp/w$. Therefore $T'u - u = 0$, and so T' is the identity. Thus $T = S_x$, which proves injectivity.

Next, note that $T_w^m \in G \cap K$, and also if $T \in K$ and $k \in \mathbb{Z}$, then

$$T_w^k Tv - v = T_w^k(Tv - v) + (T_w^k v - v) = (Tv - v) - kw.$$

Hence, it suffices to prove that for all $x \in w^\perp$, there exists $T \in G \cap K$ such that $Tv - v \equiv mx$

(mod mw). Let $x \in w^\perp$ and define $T \coloneqq T_x^m T_{x+w}^{-m} \in G \cap K$. We have

$$\begin{aligned} Tv - v &= T_x^m(v + m\langle v, x+w\rangle(x+w)) - v \\ &= T_x^m(v + m(\langle v, x\rangle + 1)(x+w)) - v \\ &= m(\langle v, x\rangle + 1)(x+w) - m\langle v, x\rangle x \\ &\equiv mx \pmod{mw}. \end{aligned}$$

□

Remark 3.3.2. A small addition to the above proof also shows that the kernel K in Proposition 3.3.1 is a 2-step nilpotent group, with center $Z(K) = \langle T_w^m \rangle$ and the mapping $T \mapsto \frac{1}{m}(Tv - v)$ inducing an isomorphism $K/Z(K) \cong w^\perp/w$.

Proposition 3.3.3.

(a) If $g \geq 2$ and $m \geq 1$ then

$$\mathrm{Sp}_{2g}(\mathbb{Z})[w] = \langle T_u^m : u \in \mathbb{Z}^{2g} \rangle.$$

(b) If $g \geq 3$, $m \geq 1$ and $w \in \mathbb{Z}^{2g}$ is primitive then

$$\mathrm{Sp}_{2g}(\mathbb{Z})[m]_w = \langle T_u^m : u \in \mathbb{Z}^{2g},\ u \perp w \rangle.$$

Proof. Part (a) is due to Mennicke [Men65, Section 10]. For part (b), note that part (a) implies that $\mathrm{Sp}(w^\perp/w)[m]$ is generated by mth powers of transvections. Hence Proposition 3.3.1 implies the result. □

The following proposition is due to Newman and Smart [NS64].

Proposition 3.3.4 ([NS64, Theorem 7])**.** *Let $g \geq 1$ and $m, \ell \geq 2$ with $m \mid \ell$. There are*

isomorphisms of abelian groups

$$\mathrm{Sp}_{2g}(\mathbb{Z})[\ell]/\,\mathrm{Sp}_{2g}(\mathbb{Z})[m\ell] \cong \mathfrak{sp}_{2g}(\mathbb{Z}/m\mathbb{Z}) \cong (\mathbb{Z}/m\mathbb{Z})^{2g^2+g}.$$

While we do not include Newman and Smart's complete proof of the proposition, we do outline the construction of the first isomorphism. Each $A \in \mathrm{Sp}_{2g}(\mathbb{Z})[\ell]$ has the form $A = I + \ell B$. It can be shown that $B \bmod m \in \mathfrak{sp}_{2g}(\mathbb{Z}/m\mathbb{Z})$. Furthermore, the mapping $A = I + \ell B \mapsto B \bmod m$ defines a surjective homomorphism $\mathrm{Sp}_{2g}(\mathbb{Z})[\ell] \to \mathfrak{sp}_{2g}(\mathbb{Z}/m\mathbb{Z})$ whose kernel is $\mathrm{Sp}_{2g}(\mathbb{Z})[m\ell]$.

In particular, if $T \in \mathrm{Sp}_{2g}(\mathbb{Z})$ is a transvection, then $T^\ell \in \mathrm{Sp}_{2g}(\mathbb{Z})[\ell]$ is carried to $T - I \bmod m \in \mathfrak{sp}_{2g}(\mathbb{Z}/m\mathbb{Z})$ by this mapping.

3.4 Level m from level $2m$ and mth powers

As a stepping stone towards our ultimate goal of proving that $\Gamma_n[m] = \Gamma_n^m$ for the values of m and n specified in Theorem 4.1.1, in this section we prove the following equality.

Theorem 3.4.1. *If $m \geq 1$ and $n \geq 2$ then*

$$\langle \Gamma_n^m, \Gamma_n[2m] \rangle = \Gamma_n[m].$$

With this theorem in hand, the equality $\Gamma_n[m] = \Gamma_n^m$ is clearly equivalent to the inclusion $\Gamma_n[2m] \leq \Gamma_n^m$, which we consider in later sections.

Theorem 3.4.1 also immediately implies Theorem C in the case of $n = 3$, since the isomorphism $\Gamma_3 \cong \mathrm{SL}_2(\mathbb{Z})$ given by Proposition 3.2.3 takes $\Gamma_3[2m]$ to $\mathrm{SL}_2(\mathbb{Z})[2m]$. To handle the $n = 4$ case we will need results from subsequent sections.

3.4.1 Proof when m is a power of 2

We being by proving the following special case of Theorem 3.4.1.

Lemma 3.4.2. *If $r \geq 0$ and $n \geq 2$, then*

$$\langle \Gamma_n^{2^r}, \Gamma_n[2^{r+1}] \rangle = \Gamma_n[2^r].$$

Proof. When $r = 0$ this follows from the equalities $\Gamma_n^1 = \Gamma_n = \Gamma_n[1]$. Assume now that $r \geq 1$. Let $\Gamma \coloneqq \langle \Gamma_n^{2^r}, \Gamma_n[2^{r+1}] \rangle \leq \Gamma_n$. It suffices to show that $\Gamma/\Gamma_n[2^{r+1}] = \Gamma_n[2^r]/\Gamma_n[2^{r+1}]$.

By Proposition 3.2.3 we may identify Γ_n with its image in $\mathrm{Sp}(V_n)$, and hence $\Gamma_n[2^r]/\Gamma_n[2^{r+1}]$ with a subgroup of the quotient group $\mathrm{Sp}(V_n)[2^r]/\mathrm{Sp}(V_n)[2^{r+1}]$. The latter group is abelian and isomorphic to $\mathfrak{sp}(V_n/2V_n)$ by Proposition 3.3.4. Furthermore, the construction of the isomorphism is such that if $T \in \mathrm{Sp}(V_n)$ is a transvection, then the image of T^{2^r} in $\mathrm{Sp}(V_n)[2^r]/\mathrm{Sp}(V_n)[2^{r+1}]$ is carried to the element $T - I \bmod 2$ of $\mathfrak{sp}(V_n/2V_n)$.

It follows that for each $n \geq 2$, the correctness of Lemma 3.4.2 does not depend on the value of $r \geq 1$. Hence, it suffices to prove it when $r = 1$. This special case follows from the equality $B_n^2 = B_n[2]$, which holds by combining work of Moore [Moo97] and Arnol'd [Arn68] as mentioned in the introduction. □

3.4.2 Proof of the general case

Proof of Theorem 3.4.1. Let $m \geq 1$ and $n \geq 2$. Write $m = 2^r s$ where s is odd. Let $\Gamma \coloneqq \langle \Gamma_n^m, \Gamma_n[2m] \rangle$. It suffices to show that $\Gamma/\Gamma_n[2m] = \Gamma_n[m]/\Gamma_n[2m]$.

By the Chinese remainder theorem, the natural reduction maps define an isomorphism

$$\Gamma_n/\Gamma_n[2m] \cong \Gamma_n/\Gamma_n[2^{r+1}] \times \Gamma_n/\Gamma_n[s].$$

Recall that $\Gamma_n^m \leq \Gamma_n[m]$ by Proposition 3.2.2. The image of $\Gamma_n[m]$ in $\Gamma_n/\Gamma_n[s]$ is trivial since

$s \mid m$. Hence, it suffices to show that Γ_n^m and $\Gamma_n[m]$ have the same image in $\Gamma_n/\Gamma_n[2^{r+1}]$.

If $T \in \Gamma_n$ is a transvection, then T^m and T^{2^r} agree modulo 2^{r+1}. Hence Γ_n^m and $\Gamma_n^{2^r}$ have the same image in $\Gamma_n/\Gamma_n[2^{r+1}]$. By Lemma 3.4.2 this implies Γ_n^m and $\Gamma_n[2^r]$ have the same image in $\Gamma_n/\Gamma_n[2^{r+1}]$. Since $\Gamma_n^m \leq \Gamma_n[m] \leq \Gamma_n[2^r]$, we find that Γ_n^m and $\Gamma_n[m]$ also have the same image in $\Gamma_n/\Gamma_n[2^{r+1}]$, as required. □

3.5 Reduction of the main theorems to a result on transvections

In this section we reduce Theorem 4.1.1 and Theorem C to the following theorem.

Theorem 3.5.1. *If $m \geq 1$ and $n \geq 2$, then $T_x^{2m} \in \Gamma_n^m$ for all $x \in W_n$.*

We prove Theorem 3.5.1 in §3.6.

3.5.1 Proof of Theorem C from Theorem 3.5.1

Proof. As mentioned in §3.4, the case $n = 3$ immediately follows from Theorem 3.4.1, so it remains to handle $n = 4$.

Let $m \geq 1$ and let $R \subseteq B_3[m]$ be such that the normal closure of $\bar{\rho}_3(R)$ in $\mathrm{SL}_2(\mathbb{Z})$ contains $\mathrm{SL}_2(\mathbb{Z})[2m]$. Let Γ be the normal closure of $\rho_4(R)$ in Γ_4.

Since $\mathcal{BI}_4$ is normally generated by $(\sigma_1\sigma_2)^6$ by [BMP15, Theorem C], it suffices to show that $\langle \Gamma, \Gamma_4^m \rangle = \Gamma_4[m]$. Hence, by Theorem 3.4.1 it suffices to show that $\Gamma_4[2m] \leq \langle \Gamma, \Gamma_4^m \rangle$.

Recall that we may view Γ_4 as a subgroup of $\mathrm{Sp}(V_4)_w$ by Proposition 3.2.3. As discussed in §3.3, there is a natural homomorphism $\mathrm{Sp}(V_4)[2m]_w \to \mathrm{Sp}(w^\perp/w)[2m]$. By Proposition 3.3.1 and Theorem 3.5.1, Γ_4^m contains the kernel of this homomorphism. So it suffices to show that the image of Γ in $\mathrm{Sp}(w^\perp/w)$ contains $\mathrm{Sp}(w^\perp/w)[2m]$.

To prove this, we note that the following diagram commutes.

$$\begin{array}{ccccc} B_4 & \xrightarrow{\rho_4} & \Gamma_4 & \hookrightarrow & \mathrm{Sp}(V_4)_w \\ \varphi\downarrow & & & & \downarrow \\ B_3 & \xrightarrow[\bar{\rho}_3]{} & \mathrm{SL}_2(\mathbb{Z}) \cong \mathrm{Sp}(w^\perp/w) & & \end{array}$$

Here $\varphi\colon B_4 \to B_3$ denotes the special homomorphism given by $\sigma_1, \sigma_3 \mapsto \sigma_1$ and $\sigma_2 \mapsto \sigma_2$, and the isomorphism $\mathrm{Sp}(w^\perp/w) \cong \mathrm{SL}_2(\mathbb{Z})$ arises from using the images of c_1 and c_2 in $w^\perp/w$ as a $\mathbb{Z}$-basis to identify $w^\perp/w$ with $\mathbb{Z}^2$.

Since the standard inclusion $B_3 \to B_4$ is a section of φ, the hypothesis that the normal closure of $\rho_3(R)$ in $\mathrm{SL}_2(\mathbb{Z})$ contains $\mathrm{SL}_2(\mathbb{Z})[2m]$ implies that the image of Γ in $\mathrm{Sp}(w^\perp/w)$ contains $\mathrm{Sp}(w^\perp/w)[2m]$, as required. □

3.5.2 Proof of Theorem 4.1.1 from Theorem 3.5.1

Proof. When $n = 2$ we have $B_2[m] = B_2^m$ for all $m \geq 1$ (see Remark 3.1.2). As noted in the introduction, when combined with Miller's result [Mil02], Theorem C implies Theorem 4.1.1 in the cases $n = 3, 4$.

Assume now that $n \geq 5$. Recall that by Proposition 3.2.3 we may view Γ_n as a subgroup of $\mathrm{Sp}(V_n)$, and if n is even then Γ_n lies in the stabilizer subgroup $\mathrm{Sp}(V_n)_w$. By Propositions 3.2.3 and 3.3.3 we have

$$\langle T_x^{2m} : x \in W_n \rangle = \begin{cases} \mathrm{Sp}(V_n)[2m] & n \text{ odd} \\ \mathrm{Sp}(V_n)[2m]_w & n \text{ even.} \end{cases}$$

Hence, Theorem 3.5.1 implies that $\Gamma_n[2m] = \langle T_x^{2m} : x \in W_n \rangle \leq \Gamma_n^m$. Therefore, by Theorem 3.4.1 we have $\Gamma_n[m] = \Gamma_n^m$, and hence $B_n[m] = \langle \mathcal{BI}_n, B_n^m \rangle$. □

3.6 Proof of the result on transvections

It remains to prove Theorem 3.5.1, namely that for all $m \geq 1$ and $n \geq 2$

$$T_x^{2m} \in \Gamma_n^m \qquad \forall x \in W_n = \mathbb{Z}c_1 + \cdots + \mathbb{Z}c_{n-1}.$$

Many of the ingredients in the proof of Theorem 3.5.1 are adapted from arguments that appear in Section 3 of Janssen's thesis [Jan85]. Janssen only considers the case of $m = 1$, but works in a more general setting than the integral Burau representation. Our main contributions are noticing that some proofs there can be adapted to work for $m > 1$, as well as simplifying the arguments for the special case of the integral Burau representation.

Proof of Theorem 3.5.1. We proceed by induction on n. For $n = 2$ we have $W_2 = \mathbb{Z}c_1$, hence the following stronger statement holds: $T_x^m \in \Gamma_2^m$ for all $x \in W_2$. Assume now that $n \geq 3$.

Let $x \in W_n$ and write $x = \lambda_1 c_1 + \cdots + \lambda_{n-1} c_{n-1}$, where $\lambda_1, \ldots, \lambda_{n-1} \in \mathbb{Z}$.

3.6.1 Step one: it suffices to assume $\lambda_{n-2} = 0$.

To reduce to the case where $\lambda_{n-2} = 0$, we will construct x' in the Γ_n-orbit of x such that

$$x' \in \mathbb{Z}c_1 + \cdots + \mathbb{Z}c_{n-3} + \mathbb{Z}c_{n-1}.$$

Since T_x and $T_{x'}$ will be Γ_n-conjugate, and Γ_n^m is a normal subgroup of Γ_n, this will show that $T_x^{2m} \in \Gamma_n^m \iff T_{x'}^{2m} \in \Gamma_n^m$.

Consider the $\mathbb{Z}$-linear map $A\colon W_n \to \mathbb{Z}^2$ defined as follows. Let

$$Ac_{n-2} \coloneqq \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad Ac_{n-1} \coloneqq \begin{pmatrix} -1 \\ 0 \end{pmatrix}, \qquad \text{and if } n \geq 4\text{:} \quad Ac_{n-3} \coloneqq \begin{pmatrix} 1 \\ 0 \end{pmatrix},$$

and $Ac_1 \coloneqq \cdots \coloneqq Ac_{n-4} \coloneqq 0$. A straightforward calculation shows that

$$A \circ T_{c_{n-2}}|_{W_n} = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix} \circ A, \qquad A \circ T_{c_{n-1}}|_{W_n} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \circ A.$$

Let d be the gcd of the two entries of Ax. By the Euclidean algorithm, there exists an element $B \in \langle T_{c_{n-2}}, T_{c_{n-1}} \rangle \leq \Gamma_n$ such that $ABx = \left(\begin{smallmatrix} d \\ 0 \end{smallmatrix}\right)$. Hence $x' \coloneqq Bx$ lies in

$$A^{-1}(\mathbb{Z}\left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right)) = \mathbb{Z}c_1 + \cdots + \mathbb{Z}c_{n-3} + \mathbb{Z}c_{n-1}.$$

3.6.2 Step two: L_+ and L_-

By the previous section, we may assume $\lambda_{n-2} = 0$. We will define self maps L_+ and L_- on W_n such that $T_x^{2m} \in \Gamma_n^m \iff T_{L_\pm x}^{2m} \in \Gamma_n^m$, assuming $\lambda_{n-2} = 0$.

We will make use of the following proposition due to Wajnryb [Waj80].

Proposition 3.6.1 ([Waj80, Proposition 1])**.** *Let $n \geq 3$. If $y, y' \in W_n$ are such that*

$$\langle y, W_n \rangle = \langle y', W_n \rangle = \mathbb{Z},$$

and $y - y' \in 2W_n$, then y and y' are in the same Γ_n-orbit.

Now, let

$$y_\pm = (\pm 1 + 2(\lambda_{n-3} - \lambda_{n-1}))c_{n-1},$$

and define $L_\pm x$ to be the quantity $x + y_\pm$. Note that $T_{y_\pm}^m \in \Gamma_n^m$, since $y_\pm \in \mathbb{Z}c_{n-1}$. Furthermore, $\langle 2x + y_\pm, c_{n-2} \rangle = \mp 1$ and $2x + y_\pm - c_{n-1} \in 2W_n$. By Proposition 3.6.1, $2x + y_\pm$ is in the Γ_n-orbit of c_{n-1}, and hence $T_{2x+y_\pm}^m \in \Gamma_n^m$.

Since $\lambda_{n-2} = 0$, we have $\langle x, y_\pm \rangle = 0$, and hence the following identity of commuting

transvections holds:

$$T_x^2 T_{x+y_\pm}^2 = T_{2x+y_\pm} T_{y_\pm} \implies T_x^{2m} T_{L_\pm x}^{2m} = T_{2x+y_\pm}^m T_{y_\pm}^m \in \Gamma_n^m.$$

In particular, we have $T_x^{2m} \in \Gamma_n^m \iff T_{L_\pm x}^{2m} \in \Gamma_n^m$.

3.6.3 Step three: it suffices to also assume $\lambda_{n-1} = 0$

Let L_+ and L_- be the operations defined in the previous subsection. We note two easily checkable facts:

(i) $L_+ x - x$ is an odd multiple of c_{n-1},

(ii) $L_\pm L_\mp x = x \pm 2c_{n-1}$.

It follows from (i) that it suffices to assume that λ_{n-1} is even, and then it follows from (ii) that it suffices to assume that $\lambda_{n-1} = 0$.

3.6.4 Step four: use the inductive hypothesis

By the previous steps, it suffices to prove for all $n \geq 3$ that

$$T_x^{2m} \in \Gamma_n^m \qquad \forall x \in \mathbb{Z}c_1 + \cdots + \mathbb{Z}c_{n-3}.$$

When $n = 3$ the sum $\mathbb{Z}c_1 + \cdots + \mathbb{Z}c_{n-3}$ is the empty sum, so this is immediate. Assume now that $n \geq 4$. If $\sigma \in B_{n-2}$, then $\rho_n(\sigma) = \rho_{n-2}(\sigma) \oplus \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$. Furthermore, if $x \in \mathbb{Z}c_1 + \cdots + \mathbb{Z}c_{n-3}$ then the transvection T_x preserves $\mathbb{Z}e_1 + \cdots + \mathbb{Z}e_{n-2}$, and fixes both e_{n-1} and e_n.

By induction, for all $x \in \mathbb{Z}c_1 + \cdots + \mathbb{Z}c_{n-3}$ we have

$$T_x^{2m} \in \left\{ \begin{pmatrix} A & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} : A \in \Gamma_{n-2}^m \right\} \leq \Gamma_n^m \cap \rho_n(B_{n-2}). \qquad \square$$

CHAPTER 4

NONINJECTIVITY OF THE MONODROMY OF CERTAIN EQUICRITICAL STRATA

This chapter is joint work with Nick Salter [HS24], and is a preprint on the arXiv: Peter Huxford and Nick Salter. *Noninjectivity of the monodromy of certain equicritical strata.* 2024. arXiv: `2411.02222v1 [math.GT]`. Both authors contributed to the ideas, writing, and figures.

4.1 Introduction

Let $\mathrm{Poly}_n(\mathbb{C})$ denote the space of monic squarefree complex polynomials of degree n. Identifying monic polynomials with their roots induces an isomorphism with the space $\mathrm{UConf}_n(\mathbb{C})$ of unordered n-tuples in $\mathbb{C}$. From the polynomial perspective, it is natural to consider the stratification on $\mathrm{Poly}_n(\mathbb{C})$ induced by the multiplicities of the critical points of $f(z) \in \mathrm{Poly}_n(\mathbb{C})$, or equivalently the multiplicities of the roots of $f'(z)$. For a partition $\kappa = \{k_1, \dots, k_r\}$ of $n-1$, define the *equicritical stratum* $\mathrm{Poly}_n(\mathbb{C})[\kappa]$ as the locus in $\mathrm{Poly}_n(\mathbb{C})$ consisting of those f whose critical points have multiplicities specified by κ.

Each $\mathrm{Poly}_n(\mathbb{C})[\kappa]$ admits a natural map into the space $\mathrm{UConf}_{n+r}(\mathbb{C})$ by associating to f the $(n+r)$-tuple of roots and critical points (note that a root coincides with a critical point if and only if f is not squarefree). Moreover, after lifting to a finite cover of $\mathrm{UConf}_{n+r}(\mathbb{C})$ that distinguishes roots and critical points, this map is an embedding. Passing to the fundamental group, this induces a *monodromy homomorphism*

$$\rho : \mathscr{B}_n[\kappa] \to B_{n+r},$$

where we have defined $\mathscr{B}_n[\kappa] := \pi_1(\mathrm{Poly}_n(\mathbb{C})[\kappa])$ and $B_{n+r} = \pi_1(\mathrm{UConf}_{n+r}(\mathbb{C}))$. The latter is the usual braid group (here on $n+r$ strands), while we call the former a *stratified braid*

group.

Equicritical strata are simple examples of families of smooth varieties embedded in some fixed ambient space, subject to certain geometric constraints. Generally in such problems, it is of interest to understand how much of the topology of such a family is captured by various flavors of monodromy representation. There are at least two variants of monodromy one can study: the *intrinsic* monodromy, valued in some automorphism group associated to the variety itself (in our setting, this is just the induced permutation on the roots and critical points), as well as the finer *embedded* monodromy, which tracks the automorphism of the *pair* of the ambient space and the subvariety.

The importance of the embedded monodromy has been recognized by many authors in many problems, going at least as far back as the work of Dolgachev–Libgober [DL81] in the context of families of smooth degree-d projective hypersurfaces. In spite of this, there have been only sporadic results. The purpose of this paper is to show that for a certain class of equicritical strata, namely those with exactly two critical points, the embedded monodromy ρ is, somewhat to the authors' surprise, non-injective.

Theorem 4.1.1. *For all $n \geq 3$ and all $p, q \geq 1$ such that $p + q = n - 1$, the monodromy map*

$$\rho \colon \mathscr{B}_n[p, q] \to B_{n+2}$$

is not injective.

Context: equicritical strata. The equicritical strata were introduced by the second-named author in [Sal23] (though are implicit in the work [DM22; DM24] of Dougherty–McCammond). Subsequently in [Sal24], the image of ρ in B_{n+r} was completely determined. In the nontrivial regime $r > 1$, the image turns out to be a "framed braid group" — a certain infinite-index subgroup of B_{n+r} consisting of braids that preserve a certain "winding number function" assigning winding numbers to arcs on a punctured disk. Ultimately this constraint

arises from the geometric interpretation of the logarithmic derivative of f (a meromorphic differential on $\mathbb{CP}^1$) as a translation surface. In general the stratified braid groups $\mathscr{B}_n[\kappa]$ themselves are rather mysterious, but can be understood as an extension of their image in B_{n+r} by $\ker(\rho)$. Following the work of [Sal24] which determines $\mathrm{im}(\rho)$, attention shifts to the problem of understanding $\ker(\rho)$. Note, however, that in the case $r = 2$ studied here, the algebraic structure of $\mathscr{B}_n[\kappa]$ is well-understood — Proposition 4.2.1 shows that $\mathscr{B}_n[p, q] \cong F_2 \times \mathbb{Z}$ (in the case $p = q$, only after passing to a subgroup of index 2).

What about higher r? It is not difficult to see that the explicit kernel elements constructed in §4.4 generalize to kernel elements for general equicritical strata. The issue is that some new mechanism for certifying their nontriviality in $\mathscr{B}_n[\kappa]$ must be developed. In the case $r = 2$, the derivative map D studied in §4.2 realizes $\mathrm{Poly}_n(\mathbb{C})[p, q]$ as the total space of a fiber bundle with fiber having free fundamental group. As observed in Remark 4.2.3, this no longer holds as soon as $r \geq 3$. It is not clear to the authors whether ρ should be injective for $r \geq 3$.

Outline of proof and organization of paper. In §4.2, we show that, after possibly passing to a two-sheeted cover, $\mathrm{Poly}_n(\mathbb{C})[p, q]$ can be identified with a *product* $\mathrm{PConf}_2(\mathbb{C}) \times (\mathbb{C} \setminus \{0, 1\})$ (here and throughout, $\mathrm{PConf}_n(\mathbb{C})$ denotes the configuration space of *ordered* n-tuples in $\mathbb{C}$), and describe a pair of explicit free generators x, y for the fundamental group of the fiber. The monodromy image of x, y is determined in §4.3. Then in §4.4, we exhibit an explicit nontrivial word in x, y and prove that it lies in the kernel of ρ.

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4.2 A global description of $\mathrm{Poly}_n(\mathbb{C})[p, q]$

We now fix $p, q \geq 1$, and let $n = p + q + 1$.

4.2.1 Bundle structure

Definition. Let $\mathrm{Poly}_n(\mathbb{C})[[p,q]]$ be the (one- or two-sheeted) connected cover of $\mathrm{Poly}_n(\mathbb{C})[p,q]$ that equips a polynomial with an ordering of its two critical points.

Note that this cover is two-sheeted if and only if $p = q$.

Proposition 4.2.1. *There is an isomorphism of varieties*

$$\mathrm{Poly}_n(\mathbb{C})[[p,q]] \cong \mathrm{PConf}_2(\mathbb{C}) \times (\mathbb{C} \setminus \{0,1\}).$$

Proof. An element of $\mathrm{Poly}_n(\mathbb{C})[[p,q]]$ is determined by

- the ordered pair of its critical points,
- the constant term of the underlying polynomial.

This allows us to view $\mathrm{Poly}_n(\mathbb{C})[[p,q]]$ as a Zariski-open subset of $\mathbb{C}^3$. To more easily describe what this subset actually is, we choose the following coordinates (a,b,c) on $\mathbb{C}^3$, where:

- a is the first critical point (of multiplicity p),
- b is the second critical point (of multiplicity q),
- c is the value of the underlying polynomial at a.

The polynomial associated to the triple (a,b,c) is

$$f_{(a,b,c)}(z) = n\int_a^z (u-a)^p(u-b)^q\,du + c.$$

The critical points of $f_{(a,b,c)}$ are a and b. Thus the critical values are $f_{(a,b,c)}(a) = c$ and

$$f_{(a,b,c)}(b) = n\int_a^b (u-a)^p(u-b)^q\,du + c = n(b-a)^n\int_0^1 v^p(v-1)^q\,dv + c = \mu(b-a)^n + c,$$

where $\mu \neq 0$ is the constant $n \int_0^1 v^p (v-1)^q \, dv$. Therefore

$$\mathrm{Poly}_n(\mathbb{C})[[p,q]] = \mathbb{C}^3 - V(b-a) - V(c) - V(\mu(b-a)^n + c).$$

Making the change of coordinates $d = \frac{-c}{\mu(b-a)^n}$ gives the identification

$$\mathrm{Poly}_n(\mathbb{C})[[p,q]] = \mathbb{C}^3 - V(b-a) - V(d) - V(d-1),$$

from which the identification

$$\mathrm{Poly}_n(\mathbb{C})[[p,q]] \cong \mathrm{PConf}_2(\mathbb{C}) \times (\mathbb{C} \setminus \{0,1\})$$

follows. □

Remark 4.2.2. The projection map $D : \mathrm{Poly}_n(\mathbb{C})[[p,q]] \to \mathrm{PConf}_2(\mathbb{C})$ is a disguised form of the ordinary derivative, assigning as it does to $f \in \mathrm{Poly}_n(\mathbb{C})[[p,q]]$ the ordered tuple of its critical points. The proof of Proposition 4.2.1 shows that D realizes $\mathrm{Poly}_n(\mathbb{C})[[p,q]]$ as a globally trivial fiber bundle over $\mathrm{PConf}_2(\mathbb{C})$ with fiber the twice-punctured plane. When $p = q$, D descends to realize $\mathrm{Poly}_n(\mathbb{C})[p,p]$ as a *nontrivial* bundle over $\mathrm{UConf}_2(\mathbb{C})$ with fiber the twice-punctured plane.

Remark 4.2.3. For stratified configurations spaces for at least 3 critical points, the map D recording the critical points is no longer a locally trivial fibration. For example, for the map $\mathrm{Poly}_4[1,1,1](\mathbb{C}) \to \mathrm{UConf}_3(\mathbb{C})$ that records a polynomial's critical points, the fiber over a configuration in $\mathrm{UConf}_3(\mathbb{C})$ is generically a three-times punctured complex plane, but it is a twice punctured complex plane if and only if the configuration has the form $\{u, v, (u+v)/2\}$ where $u \neq v$.

4.2.2 Explicit free generators

Definition. We define two elements $x, y \in \mathscr{B}_n[p,q]$ as follows. Let $f(z)$ be the polynomial

$$f(z) = n \int_0^z u^p (u-1)^q \, du. \tag{4.2.4}$$

Note that $f(z) - c \in \mathrm{Poly}_n(\mathbb{C})[p,q]$ if and only if $c \in \mathbb{C} - \{0, \mu\}$, where $\mu \neq 0$ is the constant $n \int_0^1 u^p (u-1)^q \, du$. We consider elements of $\mathscr{B}_n[p,q]$ that can be represented as loops of the form $f(z) - \gamma(t)$, where $\gamma(t)$ is a path in $\mathbb{C} - \{0, \mu\}$.

Let $x, y \in \mathscr{B}_n[p,q]$ be the elements corresponding to $\gamma(t)$ being a counter-clockwise circular loop around 0, and around μ, respectively, both based at $\mu/2$.

Implicit in this definition we are taking $f - \mu/2$ for our base point of $\mathrm{Poly}_n(\mathbb{C})[p,q]$. That is, we are identifying $\mathscr{B}_n[p,q]$ with the fundamental group $\pi_1(\mathrm{Poly}_n(\mathbb{C})[p,q], f - \mu/2)$.

Lemma 4.2.5. *The elements $x, y \in \mathscr{B}_n[p,q]$ generate a free group F_2 of rank 2.*

Proof. Since x and y can be represented by loops of polynomials whose critical points $\{0, 1\}$ remain fixed, they lift to loops in (the one- or two-sheeted cover) $\mathrm{Poly}_n(\mathbb{C})[[p,q]]$. The isomorphism $\mathrm{Poly}_n(\mathbb{C})[[p,q]] \cong \mathrm{PConf}_2(\mathbb{C}) \times (\mathbb{C} - \{0,1\})$ from Proposition 4.2.1 gives an isomorphism

$$\pi_1(\mathrm{Poly}_n(\mathbb{C})[[p,q]], f - \mu/2) \cong \pi_1(\mathrm{PConf}_2(\mathbb{C}), (0,1)) \times \pi_1(\mathbb{C} - \{0,1\}, 1/2),$$

which is isomorphic to $\mathbb{Z} \times F_2$. The generators x and y lie entirely inside the $\pi_1(\mathbb{C} - \{0,1\}, 1/2) \cong F_2$ factor, and are represented by counter-clockwise loops around 0 and 1, respectively. In particular they generate this free group of rank 2. □

4.3 Explicit description of the monodromy homomorphism

Let $x, y \in \mathscr{B}_n[p,q]$ be as in the previous section.

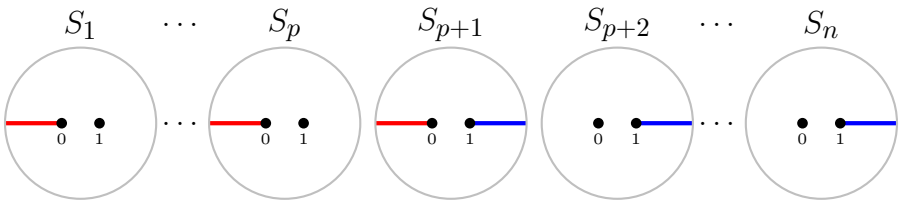

Figure 4.1: Patches of the surfaces $S_1, \ldots, S_n$

Proposition 4.3.1. *The monodromy homomorphism* $\rho\colon \mathscr{B}_n[p,q] \to B_{n+2}$ *satisfies*

$$\rho(x) = \sigma_1\sigma_2\cdots\sigma_{p+1}\sigma_1, \quad \rho(y) = \sigma_{p+2}\sigma_{p+3}\cdots\sigma_{p+q+2}\sigma_{p+2}$$

Technically, we have only defined the monodromy homomorphism ρ up to conjugation in B_{n+2}. To define $x, y \in \mathscr{B}_n[p,q]$, we chose to view $\mathscr{B}_n[p,q]$ as the fundamental group based at $f - \mu/2$, i.e. $\pi_1(\mathrm{Poly}_n(\mathbb{C})[p,q], f - \mu/2)$. Since $f - \mu/2$ has critical points at 0 and 1, to make sense of the monodromy homomorphism $\mathscr{B}_n[p,q] \to B_{n+2}$, we must choose an identification of B_{n+2} with the fundamental group $\pi_1(\mathrm{UConf}_{n+2}(\mathbb{C}), f^{-1}(\mu/2) \cup \{0,1\})$. We describe how to obtain the required identification in §4.3.2. In order to do this, and compute the monodromies $\rho(x)$ and $\rho(y)$, we first give a description of f as a branched cover in §4.3.1.

4.3.1 Description of $f\colon \mathbb{CP}^1 \to \mathbb{CP}^1$ *as a branched cover*

Construct a surface S as follows. Begin with n copies $S_1, \ldots, S_n$ of $\mathbb{CP}^1 = \mathbb{C} \cup \{\infty\}$. In the first p copies $S_1, \ldots, S_p$ cut a "red" slit between 0 and ∞ along the interval $(-\infty, 0) \subset \mathbb{C}$. In the last q copies $S_{p+2}, \ldots, S_{p+q+1} = S_n$ cut a "blue" slit between 1 and ∞ along the interval $(1, \infty) \subset \mathbb{C}$. In the final remaining copy S_{p+1} we cut both a red slit along $(-\infty, 0)$ and a blue slit along $(1, \infty)$. See Figure 4.1.

We now glue the surfaces S_i as follows. For $1 \leq i \leq p$, glue the bottom of the red slit of S_i to the top of the red slit of S_{i+1}, and glue the bottom of the red slit of S_{p+1} to the top

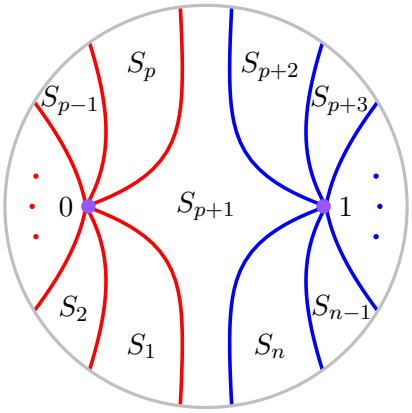

Figure 4.2: A patch of the glued surface $S \cong \mathbb{CP}^1$

of the red slit of S_1. For $p+1 \leq i < n$, glue the top of the blue slit of S_i to the bottom of the blue slit of S_{i+1}, and glue the top of the blue slit of S_n to the bottom of the blue slit of S_{p+1}. Identify all copies of ∞ among $S_1, \ldots, S_n$, and separately identify all copies of 0 among $S_1, \ldots, S_{p+1}$, and all copies of 1 among $S_{p+1}, \ldots, S_n$.

After doing all the gluing and identifications, we obtain a topological surface S that is homeomorphic to the 2-sphere. The identifications of each S_i with $\mathbb{CP}^1$ descend to a well-defined, degree n branched covering map $S \to \mathbb{CP}^1$. This branched covering map induces a complex structure on S by Riemann's existence theorem, and the resulting Riemann surface S is isomorphic to $\mathbb{CP}^1$ by uniformization. We identify S with $\mathbb{CP}^1$ via the unique isomorphism that sends the identified copy of ∞ from $S_1, \ldots, S_n$ to $\infty \in \mathbb{CP}^1$, the identified copy of 0 from $S_1, \ldots, S_{p+1}$ to $0 \in \mathbb{CP}^1$, and the identified copy of 1 from $S_{p+1}, \ldots, S_n$ to $1 \in \mathbb{CP}^1$. We now view the original surfaces $S_1, \ldots, S_n$ as subsets of $\mathbb{CP}^1$. See Figure 4.2 for a depiction of this away from ∞.

This identification of S with $\mathbb{CP}^1$ identifies the degree n branched cover $S \to \mathbb{CP}^1$ with a rational map $g\colon \mathbb{CP}^1 \to \mathbb{CP}^1$. Since $g^{-1}(\infty) = \{\infty\}$, it follows that g is a polynomial. The derivative $g'(z)$ of this polynomial is a scalar multiple of $f'(z)$, and $g(0) = 0 = f(0)$, while $g(1) = 1$ and $f(1) = \mu$. Hence $f = \mu g$.

4.3.2 Identification with the braid group

Let $m \geq 2$ and $X \in \mathrm{UConf}_m(\mathbb{C})$. Consider an arc δ in $\mathbb{C}$, whose endpoints are two distinct elements of X, and whose interior does not meet X. The *half twist* along δ is the element of $\pi_1(\mathrm{UConf}_m(\mathbb{C}), X)$ represented by a loop that fixes all elements of S except the endpoints of δ, which are swapped in a counterclockwise fashion, each endpoint traveling halfway around the boundary of a small thickening of δ. Note that the definition of a half twist does not depend on any orientation δ may have, and only depends on δ up to homotopies that fix its endpoints.

To specify an isomorphism $B_m \cong \pi_1(\mathrm{UConf}_m(\mathbb{C}), X)$, it suffices to specify a collection of arcs $\delta_1, \ldots, \delta_{m-1}$ that connect pairs of points $\{x_1, x_2\}, \{x_2, x_3\}, \ldots, \{x_{m-1}, x_m\}$, where $X = \{x_1, \ldots, x_m\}$. The interiors of these arcs must also avoid the points in the configuration X. Then the standard generators $\sigma_1, \ldots, \sigma_{m-1}$ of B_m will correspond to the half twists about $\delta_1, \ldots, \delta_{m-1}$.

In our setting, we have $m = n + 2$, and $X = f^{-1}(\mu/2) \cup \{0, 1\}$. For each $1 \leq i \leq n$, let z_i be the unique point in S_i such that $f(z_i) = \mu/2$, so that $X = \{z_1, \ldots, z_n, 0, 1\}$.

To specify arcs of the sort mentioned above, we will use the branched cover $f \colon \mathbb{C} \to \mathbb{C}$ to lift paths in $\mathbb{C}$ whose endpoints lie in $\{0, \mu/2, \mu\}$, to paths in $\mathbb{C}$ whose endpoints lie in $\{z_1, \ldots, z_n, 0, 1\}$. Choosing arcs of this form will allow us to easily read off the values of $\rho(x)$ and $\rho(y)$ in the braid group B_{n+2}.

Definition. Define arcs $\alpha_1, \ldots, \alpha_n$, $\beta_1, \ldots, \beta_n$, $\ell_1^0, \ldots, \ell_n^0$, and $\ell_1^1, \ldots, \ell_n^1$ in $\mathbb{C}$ as follows. For each $1 \leq i \leq n$, the arcs $\alpha_i, \beta_i, \ell_i^0, \ell_i^1$ all end at $z_i \in S_i$. Furthermore, they are all obtained by using f to lift paths as follows:

- α_i is a lift of the counterclockwise circular loop centered at 0, based at $\mu/2$.
- β_i is a lift of the counterclockwise circular loop centered at μ, based at $\mu/2$.
- ℓ_i^0 is a lift of the line segment connecting 0 and $\mu/2$.

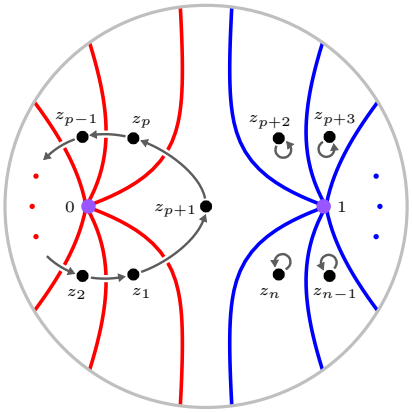

Figure 4.3: The monodromy $\rho(x)$

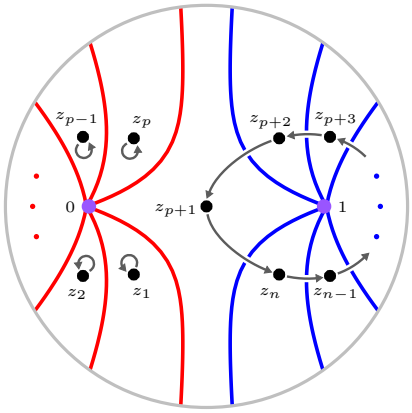

Figure 4.4: The monodromy $\rho(y)$

- ℓ_i^1 is a lift of the line segment connecting 1 and $\mu/2$.

By definition of x and y, the braids $\rho(x)$ and $\rho(y)$ are represented by the loops $\{\alpha_1, \ldots, \alpha_n, 0, 1\}$ and $\{\beta_1, \ldots, \beta_n, 0, 1\}$ in $\mathrm{UConf}_{n+2}(\mathbb{C})$.

From the description of $f\colon \mathbb{C} \to \mathbb{C}$ as a branched cover, the following topological facts are immediate:

- The arcs $\alpha_1, \ldots, \alpha_{p+1}$ form a single loop connecting the points $z_1, \ldots, z_{p+1}$, and each of these arcs only meet two of the S_j. For $p+2 \leq i \leq n$, the arc α_i is a loop based at z_i completely contained in S_i. See Figure 4.3.

- The arcs $\beta_{p+1}, \ldots, \beta_n$ form a single loop connecting the points $z_{p+1}, \ldots, z_n$, and each of these arcs only meet two of the S_j. For $1 \leq i \leq p$, the arc β_i is a loop based at z_i that is completely contained in S_i. See Figure 4.4.

- For $1 \leq i \leq p+1$, the arc ℓ_i^0 connects z_i and 0, and the interior is completely contained in S_i.

- For $p+1 \leq i \leq n$, the arc ℓ_i^1 connects z_i and 1, and the interior is completely contained in S_i.

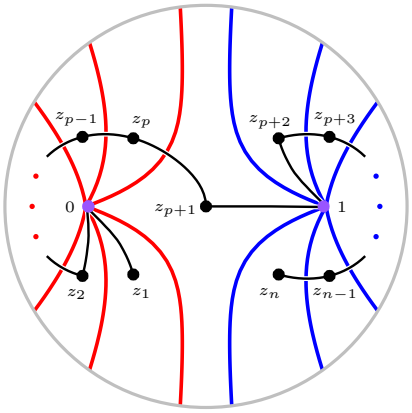

Figure 4.5: The sequence of arcs $\ell_1^0, \ell_2^0, \alpha_2, \ldots, \alpha_p, \ell_{p+1}^1, \ell_{p+2}^1, \beta_{p+2}, \ldots, \beta_{n-1}$.

In particular, we can represent $\rho(x)$ by the path $\{\alpha_1, \ldots, \alpha_{p+1}, z_{p+2}, \ldots, z_n, 0, 1\}$, and $\rho(y)$ by the path $\{z_1, \ldots, z_p, \beta_{p+1}, \ldots, \beta_n, 0, 1\}$.

The sequence of arcs that we used to identify $\pi_1(\mathrm{UConf}_{n+2}(\mathbb{C}), \{z_1, \ldots, z_n, 0, 1\})$ with B_{n+2} is as follows (see Figure 4.5):

$$\ell_1^0, \ell_2^0, \alpha_2, \ldots, \alpha_p, \ell_{p+1}^1, \ell_{p+2}^1, \beta_{p+2}, \ldots, \beta_{n-1}.$$

Implicit in this sequence of arcs is the ordering $z_1, 0, z_2, \ldots, z_{p+1}, 1, z_{p+2}, \ldots, z_n$ of the elements of $\{z_1, \ldots, z_n, 0, 1\}$.

From our description of $\rho(x)$ and $\rho(y)$, as well as our observations about ℓ_i^0 and ℓ_i^1, it follows that with respect to our identification of $\pi_1(\mathrm{UConf}_{n+2}(\mathbb{C}), \{z_1, \ldots, z_n, 0, 1\})$ with B_{n+2}, we do indeed have

$$\rho(x) = \sigma_1 \cdots \sigma_{p+1} \sigma_1, \qquad \rho(y) = \sigma_{p+2} \cdots \sigma_{n+1} \sigma_{p+2},$$

as required. See Figures 4.6 and 4.7. This completes the proof of Proposition 4.3.1. □

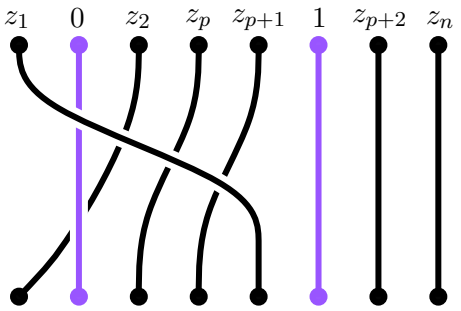

Figure 4.6:
$\rho(x) \in B_{p+q+3}$ for $p = 3$ and $q = 2$.

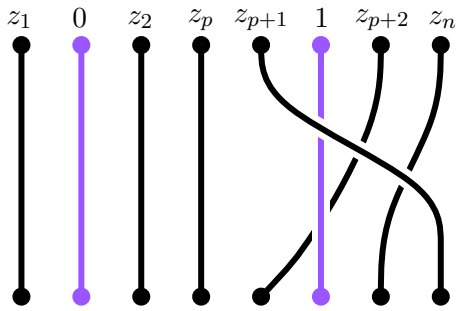

Figure 4.7:
$\rho(y) \in B_{p+q+3}$ for $p = 3$ and $q = 2$.

4.4 Construction of non-trivial elements in the kernel

The following theorem establishes Theorem 4.1.1 by explicitly constructing a non-trivial element of $\mathscr{B}_n[p, q]$ with trivial monodromy.

Theorem 4.4.1. *The word*

$$[x^{p+1}, yx^p y^{-1} xy] \in \mathscr{B}_n[p, q]$$

is a non-trivial element of $\ker(\rho\colon \mathscr{B}_n[p, q] \to B_{n+2})$.

The non-triviality of this word follows immediately from Lemma 4.2.5: the centralizer of x^{p+1} in the free group on x and y is $\langle x \rangle$, and $yx^p y^{-1} xy$ is not a power of x. The main content of Theorem 4.4.1 is that the image of this word is trivial in B_{n+2}. To prove Theorem 4.4.1 we will write $\rho(yx^p y^{-1} xy)$ as a product of braids that all commute with $\rho(x^{p+1})$. To facilitate writing this product, we begin by defining some special elements of the braid group.

Figure 4.8: The braids $\sigma_{[2,4]}$ (left) and $\bar{\sigma}_{[2,4]}$ (right) in B_7.

4.4.1 Products of consecutive standard generators

Let $m \geq 2$. For $1 \leq i, j < m$ we define the following elements of the braid group B_m.

$$\sigma_{[i,j]} \coloneqq \begin{cases} \sigma_i \cdots \sigma_j & i \leq j \\ 1 & i > j, \end{cases} \qquad \bar{\sigma}_{[i,j]} \coloneqq \begin{cases} \sigma_j \cdots \sigma_i & i \leq j \\ 1 & i > j. \end{cases}$$

See Figure 4.8 for an example depicting these braids. The commuting relations in the braid group imply that for $1 \leq i, j, i', j' < m$

$$\sigma_{[i,j]}\sigma_{[i',j']} = \sigma_{[i',j']}\sigma_{[i,j]}, \quad \text{if } |i' - j| \geq 2 \text{ or } |i - j'| \geq 2.$$

For $1 \leq i < m - 1$, the braid relation $\sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1}$ can be expressed by the identity

$$\sigma_{[i,i+1]}\sigma_i = \sigma_{i+1}\sigma_{[i,i+1]}.$$

In fact, if $1 \leq i < j < m$, then we have

$$\sigma_{[i,j]}\sigma_i = \sigma_{[i,i+1]}\sigma_{[i+2,j]}\sigma_i = \sigma_{[i,i+1]}\sigma_i\sigma_{[i+2,j]} = \sigma_{i+1}\sigma_{[i,i+1]}\sigma_{[i+2,j]} = \sigma_{i+1}\sigma_{[i,j]}.$$

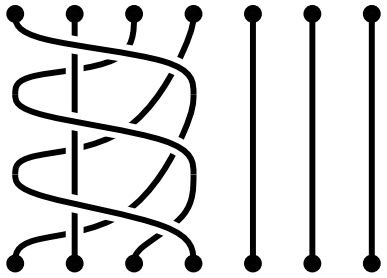

Figure 4.9: Illustration of twist_4 in B_7.

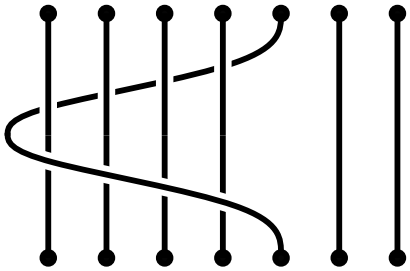

Figure 4.10: Illustration of push_5 in B_7.

4.4.2 Dehn twists

Let $2 \leq k \leq m$ and define the following element of PB_m:

$$\text{twist}_k \coloneqq (\sigma_{[1,k-1]}\sigma_1)^{k-1}.$$

See Figure 4.9 for an example of this braid. If we view B_m as a mapping class group of an m-times punctured closed disk D_m, then twist_k can be represented by a Dehn twist about the boundary of a disk D_k containing the first k punctures. This disk D_k is such that the inclusion $D_k \subseteq D_m$ induces the standard inclusion homomorphism $B_k \to B_m$.

Therefore, twist_k commutes with each of $\sigma_1, \ldots, \sigma_{k-1}$ and $\sigma_{k+1}, \ldots, \sigma_{m-1}$.

4.4.3 Point pushes

Let $2 \leq k \leq m$, and define the following element of PB_m:

$$\text{push}_k \coloneqq \bar{\sigma}_{[1,k-1]}\sigma_{[1,k-1]} = (\sigma_{k-1} \cdots \sigma_1)(\sigma_1 \cdots \sigma_{k-1}).$$

See Figure 4.10 for an example of this braid. Once again view B_m as a mapping class group of an m-times punctured closed disk D_m, and let $D_{k-1} \subset D_k \subseteq D_m$ be inclusions that induce the standard inclusion homomorphisms $B_{k-1} \to B_k \to B_m$. Define the closed annulus $A_k \coloneqq D_k - \text{int}(D_{k-1})$. Then push_k is the image of a counterclockwise generator of

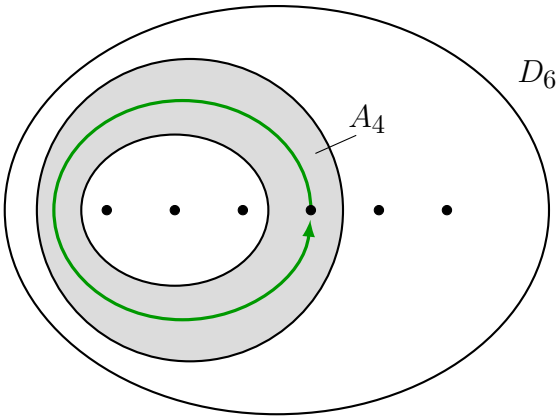

Figure 4.11: A path in $A_4 \subset D_4 \subset D_6$ representing the generator of $\pi_1(A_4, p_4)$ corresponding to push_4 in B_6

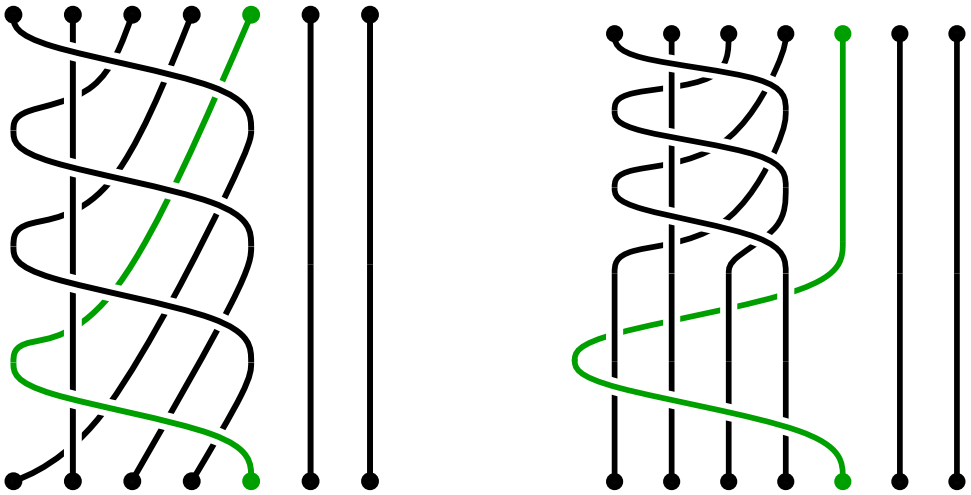

Figure 4.12: The braids twist_5 (left) and $\text{twist}_4 \cdot \text{push}_5$ (right) in B_7. The fifth strand is colored green only for visual clarity.

$\pi_1(A_k, p_k)$ under the point-pushing homomorphism $\pi_1(A_k, p_k) \to PB_m$. See Figure 4.11.

Therefore, push_k commutes with each of $\sigma_1, \ldots, \sigma_{k-2}$ and $\sigma_{k+1}, \ldots, \sigma_{m-1}$. We also have the following identity in PB_m:

$$\text{twist}_{k+1} = \text{twist}_k \cdot \text{push}_{k+1}\,.$$

See Figure 4.12 for a depiction of this identity. To justify this identity, note that each term is supported on the once-marked annulus A_k. The identity asserts that the outer Dehn twist

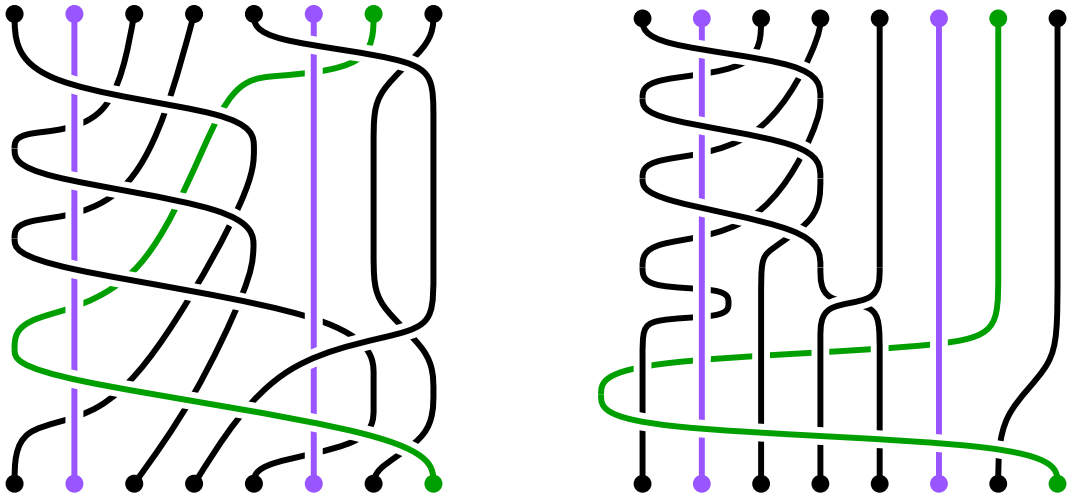

Figure 4.13: The left and right sides of the identity from Lemma 4.4.2 when $p = 3$ and $q = 2$. The second and sixth strands corresponding to critical points are colored purple, whereas the seventh strand is colored green only for visual clarity.

twist_{k+1} is equal to the inner Dehn twist twist_k composed with the counter-clockwise point push push_k. It is not hard to see that $\text{twist}_k \cdot \text{push}_k \cdot \text{twist}_{k+1}^{-1}$ may be represented by a full rotation of the annulus, i.e. the identity. Alternatively, the identity is easy to verify directly in the braid group when $k = 2$, and the above description implies that its truth does not depend on the value of k.

4.4.4 Rewriting $\rho(yx^py^{-1}xy)$

We are now in a position to express $\rho(yx^py^{-1}xy)$ as a product of braids that all commute with $\rho(x^{p+1})$.

Lemma 4.4.2. *The following identity holds:*

$$\rho(yx^py^{-1}xy) = \text{twist}_{p+1}\, \sigma_1^{-1}\sigma_{p+1}^{-1}\sigma_1\sigma_{p+3}\, \text{push}_{p+3}\, \sigma_{[p+3,n+1]}.$$

See Figure 4.13 for an example of this identity. Note that the factor $\sigma_1^{-1}\sigma_{p+1}^{-1}\sigma_1$ simplifies to σ_{p+1}^{-1} if and only if $p \geq 2$.

Proof. We will compute $\rho(yx^px^{-1}y^{-1})$ in chunks, and combine everything at the end. By Proposition 4.3.1 we have

$$\rho(x) = \sigma_{[1,p+1]}\sigma_1 = \sigma_2\sigma_{[1,p+1]},$$
$$\rho(y) = \sigma_{[p+2,n+1]}\sigma_{p+2} = \sigma_{p+3}\sigma_{[p+2,n+1]}.$$

Hence,

$$\begin{aligned}\rho(yx^py^{-1}) &= (\sigma_{p+3}\sigma_{[p+2,n+1]})\rho(x^p)(\sigma_{p+3}\sigma_{[p+2,n+1]})^{-1}\\ &= (\sigma_{p+3}\sigma_{p+2})\rho(x^p)(\sigma_{p+3}\sigma_{p+2})^{-1},\end{aligned}$$

where the last equality holds because $\rho(x) \in B_{p+2}$ commutes with $\sigma_{[p+3,n+1]}$. We also have

$$\begin{aligned}(\sigma_{p+3}\sigma_{p+2})^{-1}\rho(x) &= \sigma_{p+2}^{-1}\sigma_{p+3}^{-1}\sigma_{[1,p+1]}\sigma_1\\ &= \sigma_{[1,p]}\sigma_{p+2}^{-1}\sigma_{p+1}\sigma_1\sigma_{p+3}^{-1}\\ &= \sigma_{[1,p]}(\sigma_{p+1}\sigma_{p+2}\sigma_{p+1}^{-1}\sigma_{p+2}^{-1})\sigma_1\sigma_{p+3}^{-1}\\ &= \sigma_{[1,p+1]}\sigma_{p+2}\sigma_{p+1}^{-1}\sigma_1\sigma_{p+2}^{-1}\sigma_{p+3}^{-1}\\ &= \rho(x)\sigma_1^{-1}(\sigma_{p+2}\sigma_{p+1}^{-1}\sigma_1\sigma_{p+2}^{-1}\sigma_{p+3}^{-1})\\ &= \rho(x)\sigma_{p+2}(\sigma_1^{-1}\sigma_{p+1}^{-1}\sigma_1)(\sigma_{p+2}^{-1}\sigma_{p+3}^{-1}).\end{aligned}$$

Furthermore,

$$(\sigma_{p+2}^{-1}\sigma_{p+3}^{-1})\rho(y) = \sigma_{p+2}^{-1}\sigma_{p+3}^{-1}\sigma_{p+3}\sigma_{[p+2,n+1]} = \sigma_{[p+3,n+1]}.$$

We will also use the fact that

$$\rho(x^{p+1}) = \mathrm{twist}_{p+2}\,.$$

Therefore,

$$\begin{aligned}
\rho(yx^py^{-1}xy) &= (\sigma_{p+3}\sigma_{p+2})\rho(x^p)(\sigma_{p+3}\sigma_{p+2})^{-1}\rho(x)\rho(y) \\
&= (\sigma_{p+3}\sigma_{p+2})\rho(x^{p+1})\sigma_{p+2}(\sigma_1^{-1}\sigma_{p+1}^{-1}\sigma_1)(\sigma_{p+2}^{-1}\sigma_{p+3}^{-1})\rho(y) \\
&= (\sigma_{p+3}\sigma_{p+2})\rho(x^{p+1})\sigma_{p+2}(\sigma_1^{-1}\sigma_{p+1}^{-1}\sigma_1)\sigma_{[p+3,n+1]} \\
&= (\sigma_{p+3}\sigma_{p+2})\,\mathrm{twist}_{p+2}\,\sigma_{p+2}(\sigma_1^{-1}\sigma_{p+1}^{-1}\sigma_1)\sigma_{[p+3,n+1]} \\
&= (\sigma_{p+3}\sigma_{p+2})\,\mathrm{twist}_{p+1}\,\mathrm{push}_{p+2}\,\sigma_{p+2}(\sigma_1^{-1}\sigma_{p+1}^{-1}\sigma_1)\sigma_{[p+3,n+1]} \\
&= \sigma_{p+3}\,\mathrm{twist}_{p+1}\,\mathrm{push}_{p+3}(\sigma_1^{-1}\sigma_{p+1}^{-1}\sigma_1)\sigma_{[p+3,n+1]} \\
&= \mathrm{twist}_{p+1}\,\sigma_1^{-1}\sigma_{p+1}^{-1}\sigma_1\sigma_{p+3}\,\mathrm{push}_{p+3}\,\sigma_{[p+3,n+1]}. \qquad \square
\end{aligned}$$

4.4.5 Proof that $\rho(x^{p+1})$ and $\rho(yx^py^{-1}xy)$ commute

Proof. It was observed earlier that twist_{p+2} commutes with $\sigma_1, \ldots, \sigma_{p+1}$ and $\sigma_{p+3}, \ldots, \sigma_{n+1}$. We also observed that push_{p+3} commutes with everything in B_{p+2}, including twist_{p+2}. Hence $\rho(x^{p+1}) = \mathrm{twist}_{p+2}$ commutes with $\rho(yx^py^{-1}xy)$ by Lemma 4.4.2. $\square$

This completes the proof of Theorem 4.4.1, and hence also Theorem 4.1.1.

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