THE UNIVERSITY OF CHICAGO

GEOMETRY AND DYNAMICS IN THE TEICHMÜLLER AND MODULI SPACES OF FLAT SURFACES

A DISSERTATION SUBMITTED TO
THE FACULTY OF THE DIVISION OF THE PHYSICAL SCIENCES
IN CANDIDACY FOR THE DEGREE OF
DOCTOR OF PHILOSOPHY

DEPARTMENT OF MATHEMATICS

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JUNE 2026

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ACKNOWLEDGMENTS

I would like to express my sincere gratitude to everyone who supported me throughout my years in graduate school. I also apologize for any unintentional omissions in the lines below.

First, I would like to thank my advisor, Alex Eskin, whose guidance and support mean more to me than I can fully express. It has been an honor to be his student, and I am eternally grateful for his kindness, encouragement, insight and all I have learnt from him.

I would like to thank Jon Chaika, who suggested the problem that led to the second half of this thesis, for his mentorship, patience and introducing me to many beautiful mathematical ideas.

I would like to thank Howard Masur, Kasra Rafi, Elise Goujard, Anton Zorich, Corinna Ulcigrai, Alex Wright, Simion Filip, Amie Wilkinson and all the other professors who have shared their knowledge, given helpful advice, and taught me during these years.

I would like to thank Daniil Rudenko, Peter May and László Babai, who gave me an opportunity to be part of the UChicago REU summer programs, and I would like to thank everyone who helped with the REU problem sessions during the summer.

I would like to thank my academic sisters, Megan Roda and Rose Elliot Smith, and other graduate students and postdocs I met, Olga Medrano Martin del Campo, Aaron Calderon, Linus Setiabrata, Seraphina Lee, DeVon Ingram, Seung Uk Jang, Aadrita Laha, and many others.

I would like to thank the people from the Uchicago Wushu Club, Angel Zhang, Vikram Ramesh, Bella Wietrzak and Gillian Nielsen with whom we spent such great time training and performing.

I would like to thank my best friend and collaborator Polina Baron, with whom we have been together on this journey for so long.

Finally, I would like to thank my husband Misha and my family.

ABSTRACT

This work consists of two projects studying geometry and dynamics in the Teichmüller and moduli spaces of quadratic and Abelian differentials.

In the first half of this thesis, we prove that the Masur-Veech volume of an Avila-Gouëzel-Yoccoz ball of a sufficiently small fixed radius is uniformly bounded on the principal stratum of quadratic differentials. We also give some estimates on the Avila-Gouëzel-Yoccoz norms of the vectors of a certain relative cohomology basis that captures the geometry of the flat metric on the underlying surface.

In the second half, we show that for almost every choice of the slit the slit double cover of a fixed translation surface with uniquely ergodic vertical flow also has uniquely ergodic vertical flow.

CHAPTER 1

INTRODUCTION

Let $S_{g,n}$ be a closed, connected, oriented topological surface of genus g with n marked points. A point $(X, f) \in \mathcal{T}_{g,n}$ is a pair consisting of a Riemann surface X of genus g with n marked points and a diffeomorphism $f : S_{g,n} \longrightarrow X$ that preserves the marked points (up to isotopy relative to them). The moduli space $\mathcal{M}_{g,n}$ can be defined as the quotient $\mathcal{T}_{g,n}/Mod(S_{g,n})$ of the Teichmüller space by the action of the mapping class group $Mod(S_{g,n})$, the group of relative isotopy classes of orientation-preserving diffeomorphisms of $S_{g,n}$.

$\mathcal{T}_{g,n}$ can be endowed with a number of metrics, one of the best-known being the Teichmüller metric. First introduced by Teichmüller, it has been extensively studied since, and a lot is known about its geodesics and the dynamical properties of its geodesic flow called the Teichmüller geodesic flow (see [Mas09]). The unit cotangent bundle of $\mathcal{T}_{g,n}$ for the Teichmüller metric can be naturally identified with the space $\mathcal{Q}_1\mathcal{T}_{g,n}$ of quadratic differentials of area one with at most simple poles. Thus, the study of $\mathcal{Q}_1\mathcal{T}_{g,n}$ is an inherent part of the study of $\mathcal{T}_{g,n}$.

Each Abelian differential ω and quadratic differential q defines a singular Euclidean metric on X with cone singularities at its zeros (and poles of q in the case of a quadratic differential). X endowed with such a metric is called *a flat surface*. If q has at most simple poles, the corresponding Euclidean area is finite. If this area is equal to one, q is called an area-one quadratic differential. *A saddle connection* on a flat surface X is a segment of a flat-metric geodesic that goes from one singular point to another and has no singular points in its interior.

Another way to look at flat surfaces is to interpret them as collections of polygons in the ordinary Euclidean plane with pairwise identifications of the sides of the polygons by translations in the Abelian differential case and translations with possible addition of flips in the quadratic differential case. Because of these corresponding objects are also called *translation surfaces* and *half-translation surfaces* ([Wri15], [Fil24], [Zor06]).

If we fix the combinatorial data of an Abelian differential, that is set its singularities to be n distinct points $\Sigma = \Sigma(\omega) = \{p_1, \ldots, p_n\}$ with prescribed orders of the zeroes

$$\mathrm{ord}_{p_i}(\omega) = \kappa_i, \qquad \kappa_1 + \cdots + \kappa_n = 2g - 2,$$

we will obtain the corresponding *stratum* of abelian differentials

$$\mathcal{H}(\kappa) = \mathcal{H}(\kappa_1, \ldots, \kappa_n).$$

There are natural local coordinates on each stratum called *period coordinates* given by the periods of the differential ω.

More precisely, one considers the relative homology group $H_1(X, \Sigma; \mathbb{Z})$ If $\gamma_1, \ldots, \gamma_{2g+n-1}$ is a basis of $H_1(X, \Sigma; \mathbb{Z})$, then the *period map* is

$$\Phi(X, \omega) = \left(\int_{\gamma_1} \omega, \ldots, \int_{\gamma_{2g+n-1}} \omega \right) \in \mathbb{C}^{2g+n-1}. \tag{1.0.1}$$

A fundamental fact is that, after fixing the topological type and labeling the zeros, this map gives local holomorphic coordinates on the stratum. Geometrically, the complex number $\int_\gamma \omega$ records the Euclidean displacement vector of the relative cycle γ in the flat geometry defined by ω. In particular, $\dim_{\mathbb{C}} \mathcal{H}(\kappa) = 2g + n - 1$.

Period coordinates also give rise to the natural measure on the stratum, called the *Masur–Veech measure*. The standard Lebesgue measure in the period coordinates defines a measure in each chart. This Lebesgue measure patches together to give a well-defined measure on the stratum.

Because the stratum is conical under the rescaling

$$(X, \omega) \mapsto (X, c\omega), \qquad c \in \mathbb{C}^*,$$

the total Masur–Veech measure of $\mathcal{H}(\kappa)$ is infinite. One therefore often restricts to the area-one locus

$$\mathcal{H}_1(\kappa) = \left\{ (X, \omega) \in \mathcal{H}(\kappa) : \frac{i}{2} \int_X \omega \wedge \overline{\omega} = 1 \right\}.$$

The induced Masur–Veech measure on $\mathcal{H}_1(\kappa)$ is finite, and its total mass is called the *Masur–Veech volume* of the stratum.

As we travel across the stratum, certain geometric features can degenerate. For example, the length of a saddle connection could converge to zero or a cylinder diameter could go to infinity. In particular, the stratum fails to be compact. Thus, to fully study its dynamics, it is crucial to understand the behavior of such degenerating surfaces. In order to do that, various compactifications were invented and studied ([Doz24], [MW17], [CW21], [EMZ03], [BCG+16], [BCG+18], [BCG+19]).

A useful tool in the study of this are Avila-Gouëzel-Yoccoz metric and the Avila-Gouëzel-Yoccoz norm (or the AGY-metric and the AGY-norm).

The AGY-metric was first introduced by Avila, Gouëzel and Yoccoz in [AGY06] as a metric on the space of Abelian differentials and used to prove that the Teichmüller flow is exponentially mixing on connected components of its strata. They proved that it is indeed a norm, that it varies continuously with the surface, and that it gives a Finsler metric on the stratum.

At a point of the stratum of Abelian differentials, the tangent space can be identified with relative cohomology via period coordinates.

Definition 1.0.1. For $v \in H^1(X, \Sigma, \mathbb{C})$ we define its AGY-norm at the point (X, ω) as

$$\|v\|^{AGY}_{(X,\omega)} = \sup_{\gamma} \frac{|v(\gamma)|}{|[\omega](\gamma)|},$$

where γ runs over all saddle connections in (X, ω).

In a subsequent paper ([AG13]) Avila and Gouëzel showed that the AGY-metric has bounded behavior locally and behaves well with respect to the Teichmüller flow. It also behaves in a controlled way close to infinity as observed by Eskin, Mirzakhani and Mohammadi in [EMM21].

The AGY-norm gave a way of measuring tangent vectors and distances that still behaves well when surfaces develop short saddle connections (unlike some other norms, for example, the Hodge norm). More applications of these properties of the AGY-norm can be found in [EM18], [AHC25], [San23], [SZ24].

This work establishes another uniformly bounded behavior result. In particular, we give some estimates on the AGY-norm on certain vectors chosen to capture the flat geometry of the surface, and prove that the Masur-Veech volume of an AGY ball of a sufficiently small fixed radius is uniformly bounded on the principal stratum of quadratic differentials. To be more precise, our main theorem is this.

Theorem. *Let $q_0 \in \mathcal{PQ}_1\mathcal{T}_{g,n}$ be a point in the principal stratum of of the Teichmuller space of quadratic differentials of area one. Let $\mathcal{B}_{q_0}(0, \frac{1}{200})$ be the set of all points $q \in \mathcal{PQ}_1\mathcal{T}_{g,n}$ such that $d_{AGY}(q_0, q) < \frac{1}{200}$. There exists a universal constant $\mathfrak{C}$ (that depends only on the stratum itself) such that the Masur-Veech volume of $\mathcal{B}_{q_0}(0, \frac{1}{200})$ is smaller than $\mathfrak{C}$.*

We prove this theorem in Chapter 2.

Each translation surface comes with a specified vertical direction inherited from the plane. Following the trajectories of the points along the vertical direction produces a flow on X that is called *the vertical flow.* A trajectory of a point is well-defined unless it hits one of the finitely many singularities. This produces a dynamical system on (X, ω).

Note that the dynamical system described above is a dynamical system on a specific fixed flat surface. However, the study of this system can be replaced by the study of a dynamical system on the entire moduli space of Abelian differentials. We can observe that the matrix group $SL_2(\mathbb{R})$ acts on $\mathcal{H}(\alpha)$ by acting on each of the polygons and preserving the gluings of the corresponding sides (note that parallel intervals will be mapped to parallel intervals under this action). The trajectory (X, ω) under the action of the diagonal matrix

$$g_t = \begin{pmatrix} e^t & 0 \\ 0 & e^{-t} \end{pmatrix}$$

is called *the Teichmüller geodesic* of (X, ω), and the action of this matrix on $\mathcal{H}(\alpha)$ gives us *the Teichmüller geodesic flow*, a dynamical system on $\mathcal{H}(\alpha)$. We denote the image of X under the action of g_t by $g_t X$.

A key observation is that the study of dynamics on these spaces often provides insight into problems involving dynamics on individual surfaces belonging to them. The study of $\mathcal{M}_{g,n}$ and $\mathcal{T}_{g,n}$ often combines methods from dynamical systems, ergodic theory, complex analysis and differential geometry and is connected to a number of mathematical fields including geometric group theory, algebraic geometry and topology, hyperbolic geometry and even theoretical physics.

One of the famous examples of how the study of the dynamics on the entire moduli space can give information about the dynamics on an individual surface is Masur's criterion for unique ergodicity, which says that if the vertical flow of a flat surface $(X, \omega) \in \mathcal{H}(\alpha)$ is minimal and not uniquely ergodic, then the trajectory of (X, ω) along its Teichmüller geodesic leaves every compact set of the stratum.

A stronger version of Masur's Criterion was proved by Cheung and Eskin in [CE07a]. They show that a Teichmüller geodesic that leaves every compact set is determined by a surface with a uniquely ergodic vertical foliation provided it has a certain slow rate of divergence. Later, this result was in turn improved by Treviño in [Tre14]. In [KMS86] Kerckhoff, Masur and Smillie used Masur's Criterion to prove that on every translation surface, the straight-line flow is uniquely ergodic in almost every direction. Note that in that setting the direction of the flow is changing while the surface remains the same. Jon Chaika suggested looking at the vertical flow for varying surfaces, thus keeping the direction fixed and the surface changing.

Note that in this setting the direction of the flow is changing while the surface remains the same. Jon Chaika suggested looking at the vertical flow for varying surfaces, thus keeping the direction fixed and the surface changing. We restrict our attention to the surfaces obtained from a fixed flat surface (X, ω) as follows. Let us take two identical copies of (X, ω), make identical slits on each copy and glue the opposite sides of these slits. This is a well-known double-slit construction that allows to assemble higher-genus translation surfaces from low-genus ones (see [Wri15]). The resulting surface $(\widetilde{X}, \widetilde{\omega})$ is a double-cover of (X, ω) ramified at the endpoints of the slit. Note that if we vary our choice of the slit, the resulting double-slit surface will vary as well.

In joint work with Polina Baron, we proved the following result first conjectured by Jon Chaika.

Theorem. *Suppose (X, ω) is a translation surface with uniquely ergodic vertical flow. Then for almost every choice of the slit, the vertical flow on the resulting double-cover $(\widetilde{X}, \widetilde{\omega})$ is uniquely ergodic.*

The proof is presented in Chapter 3.

CHAPTER 2

AGY BALLS IN THE TEICHMÜLLER SPACE OF QUADRATIC DIFFERENTIALS

2.1 Introduction

We now revisit the concepts mentioned in the introduction, and concentrate on our specific setting.

Let $\mathcal{T}_{g,n}$ be the Teichmuller space of the surface $S_{g,n}$. Let $\mathcal{QT}_{g,n}$ be the space of meromorphic quadratic differentials that can only have zeroes or simple poles at the marked points and are holomorphic outside of the marked points. For a tuple of non-negative integers $\kappa = \{\kappa_1, \kappa_2, \cdots, \kappa_{n-k}, k\}$ let $\mathcal{QT}_{g,n}(\kappa)$ be the stratum of quadratic differentials with the orders of zeroes prescribed by $\kappa_1, \kappa_2, \cdots, \kappa_{n-k}$ and k simple poles. Let $\mathcal{Q}_1\mathcal{T}_{g,n}$ be the space of differentials with area one, and let $\mathcal{Q}_1\mathcal{T}_{g,n}(\kappa)$ be the the space of differentials with area one from the stratum given by κ. Let $\mathcal{PQT}_{g,n}$ be the principal stratum – those differentials from $\mathcal{QT}_{g,n}$ that only have simple zeroes. Finally, let $\mathcal{PQ}_1\mathcal{T}_{g,n}$ be the intersection of $\mathcal{PQT}_{g,n}$ and $\mathcal{Q}_1\mathcal{T}_{g,n}$, the principal stratum of half-translation surfaces of area 1 – those differentials that only have simple zeros and no poles.

Let M be the mapping class group of $S_{g,n}$. The spaces $\mathcal{M}_{g,n}$, $\mathcal{QM}_{g,n}$, $\mathcal{QM}_{g,n}(\kappa)$, $\mathcal{Q}_1\mathcal{M}_{g,n}$, $\mathcal{Q}_1\mathcal{M}_{g,n}(\kappa)$, $\mathcal{PQM}_{g,n}$, $\mathcal{PQ}_1\mathcal{M}_{g,n}$ are the corresponding quotients by the action of M.

Period coordinates. Every $(X, q) \in \mathcal{Q}_1\mathcal{T}_{g,n}(\kappa)$ has a neigbourhood $\mathcal{U}$ on which its is possible to define local coordinates called the period coordinates. They are obtained as follows. First, we look at the canonical double-cover of (X, q) $\pi : \tilde{X} \to X$, on which the pullback $\pi^* q = \omega_q^2$ for some abelian differential ω_q on $\tilde{X}$. There is a canonical involution σ on $\tilde{X}$ that switches the sign of ω_q, when ω_q is viewed as an element $[\omega_q]$ of the relative cohomology group $H^1(\tilde{X}, \Sigma, \mathbb{C})$, where Σ is the set of zeroes of ω_q. The relative homology can be decomposed into a sum of two subspaces:

$$H^1(\tilde{X}, \Sigma, \mathbb{C}) = H^1_+(\tilde{X}, \Sigma, \mathbb{C}) \oplus H^1_-(\tilde{X}, \Sigma, \mathbb{C}),$$

where $H^1_+(\tilde{X}, \Sigma, \mathbb{C})$ is the space of those cohomology classes that are invariant under the action of σ, and $H^1_-(\tilde{X}, \Sigma, \mathbb{C})$ is the space of those cohomology classes that switch the sign under the action of σ.

Similarly, the relative homology is decomposed:

$$H_1(\tilde{X}, \Sigma, \mathbb{Z}) = H_1^+(\tilde{X}, \Sigma, \mathbb{Z}) \oplus H_1^-(\tilde{X}, \Sigma, \mathbb{Z}).$$

Clearly, $[\omega_q] \in H^1_-(\tilde{X}, \Sigma, \mathbb{C})$. We can pick a basis $[\alpha_1], \cdots, [\alpha_m]$ of $H_1^-(\tilde{X}, \Sigma, \mathbb{Z})$. The value $[\omega_q]$ takes on $[\alpha_i]$ is determined by the integral of ω_q along a representative of $[\alpha_i]$. There exists such a neigbourhood $\mathcal{U}$ of $(X, q) \in \mathcal{Q}_1\mathcal{T}_{g,n}(\kappa)$ that for any quadratic differential $(X', q') \in \mathcal{U}$ the relative homology group $H_1^-(\tilde{X}', \Sigma', \mathbb{Z})$ obtained by considering the analogous double cover π' and involution σ' for (X', q') can be identified with $H_1^-(\tilde{X}, \Sigma, \mathbb{Z})$. Thus, we can fix the basis $[\alpha_1], \cdots, [\alpha_m]$ in the neighbourhood $\mathcal{U}$. This allows to define the period map Φ on $\mathcal{U}$.

$$\Phi : \mathcal{U} \longrightarrow H_1^-(\tilde{X}, \Sigma, \mathbb{C})$$

$$(X, q) \mapsto [\omega_q]$$

Since the relative cohomology group $H^1_-(\tilde{X}, \Sigma, \mathbb{C})$ can be identified with $\mathbb{C}^d$ (where $d = dim_{\mathbb{C}} H^1_-(\tilde{X}, \Sigma, \mathbb{C})$), the period map provides us with local coordinates on $\mathcal{U}$.

Similarly to the Abelian differential case, the subset of those cohomology classes that take values in $\mathbb{Z} \oplus i\mathbb{Z}$ on $H_1^-(\tilde{X}, \Sigma, \mathbb{Z})$ forms a lattice in $H^1_-(\tilde{X}, \Sigma, \mathbb{C})$. The volume element $d\mu_{MV}$

defined on $\mathcal{PQ}_1\mathcal{T}_g$ is the linear volume element in the vector space $H^1_-(\tilde{X}, \Sigma, \mathbb{C})$ normalized in such a way that the fundamental domain of the lattice defined above has volume 1. The resulting volume is called *the Masur-Veech volume.*

The tangent spaces. Since $\mathcal{U}$ can be identified with $\mathbb{C}^d$, we can identify the tangent space $T_{(X,q)}\mathcal{QT}_{g,n}(\kappa)$ to $\mathcal{QT}_{g,n}(\kappa)$ at some point (X, q) with $\mathbb{C}^d$ as well. This also allows to identify $T_{(X,q)}\mathcal{QT}_{g,n}(\kappa)$ with the tangent space $T_{(X',q')}\mathcal{QT}_{g,n}(\kappa)$ at a different point (X', q'). Denote these identifications

$$i_\Phi : \mathcal{U} \to T_{(X,q)}\mathcal{QT}_{g,n}(\kappa)$$

$$i' : T_{(X,q)}\mathcal{QT}_{g,n}(\kappa) \to T_{(X',q')}\mathcal{QT}_{g,n}(\kappa).$$

The AGY-norm. For a fixed point $(X, q) \in \mathcal{QT}_{g,n}(\kappa)$ we consider the norm $\|\cdot\|^{AGY}_{(X,q)}$ (first introduced by Avila, Gouëzel and Yoccoz in) on the tangent space $T_{(X,q)}\mathcal{QT}_{g,n}(\kappa)$. Since this tangent space can be identified with the cohomology $H^1_-(\tilde{X}, \Sigma, \mathbb{C})$, it suffices to define the norm on the cohomology classes from this space.

Definition 2.1.1. For $v \in H^1(\tilde{X}, \Sigma, \mathbb{C})$ we define its AGY-norm at the point (X, q) as

$$\|v\|^{AGY}_{(X,q)} = \sup_\gamma \frac{|v(\gamma)|}{|[\omega_q](\gamma)|},$$

where γ runs over all saddle connections in $(\tilde{X}, \omega_q)$.

This norm depends continuously on the point $(X, q) \in \mathcal{QT}_{g,n}(\kappa)$. The identification of the tangent space $T_{(X,q)}\mathcal{QT}_{g,n}(\kappa)$ with $H^1_-(\tilde{X}, \Sigma, \mathbb{C})$ allows to define lengths of paths and the distance induced by the AGY-norms.

Definition 2.1.2. For a C^1-path in $\mathcal{QT}_{g,n}(\kappa)$

$$\rho : [0,1] \to \mathcal{QT}_{g,n}(\kappa)$$

we can interpret the tangent vector $\rho'(t)$ as a cohomology class, so its AGY-norm at the point $(X(t), q(t))$ is well-defined. Thus, we define the AGY-length of the path ρ to be

$$l_{AGY}(\rho) = \int_0^1 \left\|\rho'(t)\right\|_{\rho(t)}^{AGY} dt.$$

The AGY-distance $d_{AGY}((X,q),(X',q\prime))$ between two points (X,q) and (X',q') in $\mathcal{QT}_{g,n}(\kappa)$ is then defined to be the infimum of lengths of all C^1-paths from (X,q) to (X',q').

Our main Theorem is this.

Theorem 2.1.3. *Let $q_0 \in \mathcal{PQ}_1\mathcal{T}_{g,n}$ be a point in the principal stratum of of the Teichmuller space of quadratic differentials of area 1. Let $\mathcal{B}_{q_0}(0, \frac{1}{200})$ be the set of all such points $q \in \mathcal{PQ}_1\mathcal{T}_{g,n}$ that $d_{AGY}(q_0, q) < \frac{1}{200}$. There exists a universal constant $\mathfrak{C}$ (that depends only on the stratum itself) such that the Masur-Veech volume of $\mathcal{B}_{q_0}(0, \frac{1}{200})$ is smaller than $\mathfrak{C}$.*

2.2 The Thick-Thin Decomposition and Comparability

2.2.1 Variants of the Thick-Thin Decomposition

The concept of a thick-thin decomposition was originally developed by Margulis for hyperbolic surfaces (see [Mar69]). It is obtained for a sufficiently small number $\epsilon > 0$ by deleting curves whose hyperbolic length is smaller than ϵ from the surface. Later (see [Raf14a], [Raf14b]) Rafi introduced a variation of the thick-thin decomposition that captures the geometry of the flat metric and thus allows to study the connections between the flat metric induced by an Abelian or a quadratic differential and the hyperbolic metric on the underlying surface (which is determined uniquely by a point in the Teichmüller space). The entire surface is decomposed into a finite number of thick components, thin curves serving as their boundaries and, possibly, some thin cylinders. The thick components have bounded geometry. I use a slight modification of Rafi's thick-thin decomposition. While Rafi looks at the hyperbolic metric with cusps at the poles of a quadratic differential only, I look at the metric with cusps at the zeros of it as well.

We will refer to a homotopy class of an essential simple closed curve (that is, a simple closed curve that is not homotopic to a single point, a puncture or a boundary component) simply as a *curve*.

Let X be a compact Riemann surface, and let $\Sigma \subset X$ be the set of zeroes and poles of a differential on X. We consider the punctured surface

$$X^\circ = X \setminus \Sigma. \tag{2.2.1}$$

If

$$\chi(X^\circ) = 2 - 2g - |\Sigma| < 0, \tag{2.2.2}$$

then X° carries a unique complete hyperbolic metric ρ in its conformal class. Every free homotopy class of an essential simple closed curve γ on X° contains a unique *hyperbolic geodesic representative*. We say that *the hyperbolic length of a curve* $L_\rho(\gamma)$ is the hyperbolic length of that representative.

Definition 2.2.1. A *multicurve* on a surface $S_{g,n}$ is a finite collection

$$\Gamma = \{\gamma_1, \ldots, \gamma_k\}$$

of simple closed curves on $S_{g,n}$ such that:

1. the curves $\gamma_1, \ldots, \gamma_k$ are pairwise disjoint;

2. the curves $\gamma_1, \ldots, \gamma_k$ are pairwise non-isotopic;

3. each γ_i is essential, that is, γ_i does not bound a disk containing at most one marked point/puncture.

Two multicurves are considered the same if they are isotopic.

Let Γ be a multicurve on $S_{g,n}$. A point $(X, f, \omega) \in \mathcal{H}_{g,n}(\kappa)$ determines, for each $\gamma \in \Gamma$, a well-defined isotopy class, equivalently free homotopy class $f(\gamma)$, of essential simple closed curves on X, namely the class represented by the image of γ under any representative of the marking f, since f is defined only up to isotopy. Fix some connected component $R \subset S_{g,n} \backslash \Gamma$. For each $\gamma \in \Gamma$ choose the unique hyperbolic geodesic representative from the free homotopy class $f(\gamma)$. If $\Gamma = \{\gamma_1, \ldots \gamma_k\}$, let the corresponding hyperbolic geodesic representatives be $\alpha_1, \ldots, \alpha_k$. Since f is defined up to isotopy, we may replace f by an isotopic representative of the same class such that

$$f(\gamma_i) = \alpha_i. \tag{2.2.3}$$

The connected component R is sent to a unique connected component X_R of $X \backslash \bigcup_{i=1}^{k} \alpha_i$. Given a different point $(X', f', \omega') \in \mathcal{H}_{g,n}(\kappa)$, we can build X'_R in a similar fashion. Since

$$f|_R : R \to X_R \qquad \text{and} \qquad f'|_R : R \to X'_R$$

are homeomorphisms, they induce a homeomorphism

$$X_R \longrightarrow X'_R, \qquad x \longmapsto f'\big((f|_R)^{-1}(x)\big).$$

The first notion of a thick-thin decomposition that we introduce is purely topological, since it records topological data only.

Definition 2.2.2 (Topological Thick-Thin Decomposition)**.** Fix a sufficiently small $\epsilon > 0$. On our Riemann surface X with its hyperbolic metric ρ_X on X°, we can consider all the curves with ρ_X-lengths smaller than ϵ. If $\mathcal{G}$ is the set of such curves, and $\mathcal{Y}$ is the set of all the connected components of $S_{g,n} \setminus f^{-1}(\mathcal{G})$ – the complement of the pullback multicurve of $\mathcal{G}$ to the base surface $S_{g,n}$, then we say that *the topological thick-thin decomposition* of X is the pair $\{f^{-1}(\mathcal{G}), \mathcal{Y}\}$.

The flat thick-thin decomposition was first introduced by Rafi in [Raf14b]. Note that we use a slight variation of that decomposition throughout the paper, since in Rafi's condition there were no punctures at the zeroes of the differential.

A *flat–geodesic representative* of a curve γ is a representative of the free homotopy class of γ with the shortest flat length. This representative is either unique or there is a continuous family of them filling a Euclidean cylinder. For each cylinder there are no singularities in its interior, and the boundaries of each cylinder are concatenations of saddle connections.

Definition 2.2.3 (Flat Thick-Thin Decomposition)**.** Fix a sufficiently small $\epsilon > 0$. Given the corresponding topological thick-thin decomposition $\{\mathcal{G}, \mathcal{Y}\}$, we can consider the flat-geodesic representatives of the free homotopy classes that belong to $\mathcal{G}$. Then for each $Y \in \mathcal{Y}$ there is a unique subsurface of X in the homotopy class of Y whose boundary is flat-geodesic and which is disjoint from the interior of every flat cylinder consisting of geodesic representatives of its boundary curve. *The flat thick-thin decomposition* is obtained by taking these flat representatives of the topological thick components. The cylinder pieces corresponding to the short boundary curves will be called *thin cylinders*.

Another version of the decomposition comes from looking in more detail at the hyperbolic metric.

Definition 2.2.4. Let X° be the punctured surface equipped with its hyperbolic metric ρ_X. For a point $p \in X^\circ$, define the injectivity radius by

$$\mathrm{inj}_{\rho_X}(p) = \sup\{r > 0 : B_{\rho_X}(p, r) \text{ isometrically embeds in } X^\circ\}. \tag{2.2.4}$$

Equivalently,

$$\mathrm{inj}_{\rho_X}(p) = \frac{1}{2} \inf\{\ell_{\rho_X}(\gamma) : \gamma \text{ is a nontrivial closed loop based at } p\}. \tag{2.2.5}$$

For $\epsilon > 0$, define the hyperbolic ϵ-thin and ϵ-thick parts by

$$X^{\circ}_{<\epsilon} = \{p \in X^{\circ} : \mathrm{inj}_{\rho_X}(p) < \epsilon\}, \qquad X^{\circ}_{\geq\epsilon} = \{p \in X^{\circ} : \mathrm{inj}_{\rho_X}(p) \geq \epsilon\}. \tag{2.2.6}$$

For ϵ sufficiently small, each connected component of $X^{\circ}_{<\epsilon}$ is either a cusp neighborhood of a puncture or an annular neighborhood of a simple closed geodesic of hyperbolic length less than 2ϵ. The connected components of $X^{\circ}_{\geq\epsilon}$ are called *the hyperbolic thick components.*

Finally, the last variant of the thick-thin decomposition is this.

Definition 2.2.5 (Extremal Length and Extremal-Length Thick-Thin Decomposition)**.** Let X be a Riemann surface. For a family Γ of curves on X, its *extremal length* is

$$\mathrm{Ext}_X(\Gamma) = \sup_{\rho} \frac{\ell_\rho(\Gamma)^2}{\mathrm{area}_\rho(X)}, \tag{2.2.7}$$

where the supremum is taken over all conformal metrics ρ of finite positive area on X, and

$$\ell_\rho(\Gamma) = \inf_{\gamma\in\Gamma} \mathrm{length}_\rho(\gamma). \tag{2.2.8}$$

For an essential simple closed curve α, we write $\mathrm{Ext}_X(\alpha)$ for the extremal length of its isotopy class.

Fix a sufficiently small $\epsilon > 0$. Let

$$\mathcal{G}^{\mathrm{Ext}}_{\epsilon}(X) = \{\alpha \text{ essential, non-peripheral simple closed curve } : \mathrm{Ext}_X(\alpha) < \epsilon\}\,. \tag{2.2.9}$$

Let $\mathcal{Y}_\epsilon^{\mathrm{Ext}}(X)$ be the set of connected components of

$$S_{g,n} \setminus \bigcup_{\alpha \in \mathcal{G}_\epsilon^{\mathrm{Ext}}(X)} \alpha. \tag{2.2.10}$$

Then the pair

$$\left(\mathcal{G}_\epsilon^{\mathrm{Ext}}(X),\, \mathcal{Y}_\epsilon^{\mathrm{Ext}}(X)\right) \tag{2.2.11}$$

is called *the extremal-length topological thick-thin decomposition* of X.

The following elementary property is immediate from the definitions listed in this section.

Property 2.2.6. Suppose $\varepsilon_1 > \varepsilon_2$. Then any thick component (for any of the four notions defined above) of the ε_1-thick-thin decomposition is contained in some thick component of the ε_2-thick-thin decomposition.

The topological decompositions defined by hyperbolically short curves and extremally short curves are not the same for the same threshold number ϵ. However, they turn into each other after the appropriate change of the threshold. This is due to the following comparison result due to Maskit.

Property 2.2.7 (Maskit Comparison Theorem, [Mas85])**.** Let X be a Riemann surface, and let α be an essential simple closed curve on X. Then the hyperbolic length and the extremal length of α satisfy

$$\frac{1}{\pi} \le \frac{\mathrm{Ext}_X(\alpha)}{\rho_X(\alpha)} \le \frac{1}{2} e^{\rho_X(\alpha)/2}. \tag{2.2.12}$$

In particular, the notions of being hyperbolically short and extremally short are equivalent up to a change of constants.

The inequality implies that if $\rho_X(\alpha) < \epsilon$, then

$$\mathrm{Ext}_X(\alpha) < \frac{\epsilon}{2} e^{\epsilon/2}, \tag{2.2.13}$$

while if $\mathrm{Ext}_X(\alpha) < \epsilon$, then

$$\rho_X(\alpha) < \pi\epsilon. \tag{2.2.14}$$

For the topological and hyperbolic thick-thin decompositions, the appropriate choice of a sufficiently small ε is motivated by the following.

Lemma 2.2.8 (The Margulis Constant, [Mar77], [Thu79]). *There exists a universal constant $\varepsilon_M > 0$ such that every connected component of the ε_M-thin part of the hyperbolic thick-thin decomposition is either a neighborhood of a cusp or an embedded annulus around an essential simple closed geodesic. In particular, all such essential simple closed geodesics are mutually disjoint.*

Thus, whenever we talk about a topological or a hyperbolic thick-thin decomposition, it makes sense to take the threshold ε to be smaller than the Margulis constant. There is a similar notion of being sufficiently small for the extremal-length thick-thin decomposition.

Lemma 2.2.9 ([EMR19], [GM91]). *For any two essential simple closed curves α and β we have*

$$\mathrm{Ext}_X(\alpha) \cdot \mathrm{Ext}_X(\beta) > i(\alpha, \beta)^2, \tag{2.2.15}$$

where $i(\alpha, \beta)$ is the intersection number. In particular, the intersection number is an integer, so two curves with extremal lengths that are smaller than 1 cannot intersect.

For these sufficiently small choices of the threshold, the number of short curves (and thus the number of the components) is finite and uniformly bounded on the entire stratum.

Lemma 2.2.10 ([FM11]). *Any collection of pairwise disjoint, pairwise non-isotopic essential simple closed curves on $S_{g,n}$ has at most $3g - 3 + n$ elements.*

From now on we fix once and for all a small number $\varepsilon_0 > 0$ such that $\varepsilon_M > \pi\varepsilon_0$ and $\frac{\varepsilon_0}{2} e^{\varepsilon_0/2} < 1$. In this way we can switch between the topological, hyperbolic and extremal-length variations of the thick-thin decomposition.

2.2.2 Size of a Thick Component

The thick-thin decomposition introduced by Rafi in [Raf14b] allowed to compare the flat and hyperbolic lengths of the same curves. In general, the thick parts have controlled geometry, so that all the geometric quantities arising from such a component are comparable to a certain number called its size. Various papers ([Raf14a], [EMR19], [BCG+19]) use different, yet comparable, notions of size. We will now describe the four different definitions of a size of a thick component. The main goal of the next subsection is to prove the Comparability Theorem (Theorem 2.2.17) that establishes these sizes are all comparable in our setting. To be more specific:

Definition 2.2.11. We say two functions $f(X)$ and $g(X)$ on $\mathcal{H}_{g,n}(\kappa)$ are comparable if there exist constants $C > c > 0$ that depend only on the stratum such that for any $X \in \mathcal{H}_{g,n}(\kappa)$

$$c < \frac{f(X)}{g(X)} < C. \tag{2.2.16}$$

A Riemannian metric on X defines a norm on the tangent space T_pX at any point $p \in X$. If g_p is the bilinear form given by the metric at the point p, then the norm of a tangent vector $u \in T_pX$ is

$$||u||_p = \sqrt{g_p(u,u)}.$$

A tangent vector u has a dual cotangent vector α_u defined by the pairing

$$\alpha_u : v \mapsto g_p(v,u).$$

If the Riemannian metric is nondegenerate, then for any $\alpha \in T_p^*X$ there is a unique $u \in T_pX$ such that $\alpha = \alpha_u$, and for a pair $\alpha_u, \alpha_v \in T_p^*X$ we can define a bilinear form $\tilde{g_p}(\alpha_u, \alpha_v) =$

$g_p(u, v)$. The norm of a cotanget vector α at the point p is then

$$||\omega||_p = \sqrt{\tilde{g_p}(\alpha, \alpha)}.$$

The Riemannian metric then defines a norm of a 1-form ω the point p by taking the norm

$$|\omega|_p = ||\omega(p)||_p,$$

where $\omega(p)$ is the value of the 1-form ω at the point p.

We will follow [BCG+19] and define the size of a thick component using this norm.

Definition 2.2.12. Suppose Y is a thick component of the hyperbolic ϵ-thick-thin decomposition of X. Then *the size of a thick component* Y is

$$\lambda_\omega(Y) = \sup_{p \in Y} |\omega|_p. \tag{2.2.17}$$

Here are some properties of this size quantity that follow immediately from the definition.

Property 2.2.13. Suppose that the Riemann surface X is fixed. Then for any hyperbolic thick component Y and any meromorphic differential ω the size $\lambda_\omega(Y) > 0$ and is finite.

Proof. For the hyperbolic metric the bilinear form $\tilde{g_p}$ is positive-definite for all p, so the norm is nonzero if $\omega(p)$ is nonzero. So if ω does not have a zero or a pole at the point p, then $|\omega|_p > 0$. Since Y is clearly infinite, and the set of zeroes and poles of a meromorphic differential is finite, for any hyperbolic thick component Y and any meromorphic differential ω the size $\lambda_\omega(Y) > 0$. The norm $|\omega|_p$ depends continuously on the point p. Since for a hyperbolic surface of finite type each thick component is compact, the norm attains a finite supremum on it. □

Now we recall some quantities related to the flat geometry of a thick component that can also serve as a notion of its size.

Definition 2.2.14. For a flat representative of a thick component Y its *flat diameter* $D_\omega(Y)$ is the flat diameter of its flat representative.

For a curve γ contained in Y let its *flat length* $\ell_\omega(\gamma)$ be the flat length of its ω-shortest representative. Let $\ell_\omega(Y)$ be the flat length of the ω-shortest curve contained in Y if Y is not a pair of pants. If Y is a pair of pants, then let $\ell_\omega(Y)$ be the maximum of the flat lengths of its boundary components.

Also, let $L_\omega(Y)$ be the flat length of the shortest saddle connection on the flat representative of Y (among all saddle connections whose interiors are contained in Y).

The following properties are direct consequences of the definitions.

Property 2.2.15. For a nonzero differential ω on a thick component Y all three values $L_\omega(Y)$, $\ell_\omega(Y)$, $D_\omega(Y)$ are well-defined, nonzero and finite.

Property 2.2.16. All the functions introduced above are rescaled accordingly when the differential is:

$$\lambda_{c\omega}(Y) = |c|\lambda_\omega(Y), \qquad D_{c\omega}(Y) = |c|D_\omega(Y),$$
$$\ell_{c\omega}(Y) = |c|\ell_\omega(Y), \qquad L_{c\omega}(Y) = |c|L_\omega(Y) \quad (2.2.18)$$

for any $c \in \mathbb{C}^*$.

2.2.3 The Comparability Theorem

Now we recall that we have fixed a sufficiently small ε_0, and show that all the notions of size are comparable for the corresponding thick components.

Theorem 2.2.17. *For any $(X, f, \omega) \in \mathcal{H}_{g,n}(\kappa)$ and any ε_0-thick component Q of X, each of the three ratios*

$$\frac{L_\omega(Q)}{\lambda_\omega(Q)}, \frac{\ell_\omega(Q)}{\lambda_\omega(Q)}, \frac{D_\omega(Q)}{\lambda_\omega(Q)},$$

is bounded away from zero and bounded above by positive constants that depend only on the stratum.

In this proof we will use the multi-scale compactification of the moduli space of abelian differentials. This space was constructed and explored by Bainbridge, Chen, Gendron, Grushevsky, and Möller in [BCG+19] and its construction relies heavily on their earlier work from [BCG+16] and [BCG+18]. A shorter description of this compactification can also be found in [Doz24]. We will not give a full definition of this space, but we will use some of its properties in the proof below.

We will mainly write the proof for the ratio $\frac{D_\omega(Q)}{\lambda_\omega(Q)}$, since for the other two ratios the proof is analogous.

Before we start the proof, we also recall what convergence in the moduli space means.

Definition 2.2.18. An *exhaustion* of a (possibly open) Riemann surface X is a nested sequence

$$K_1 \subset K_2 \subset ... \subset K_j \subset .... \subset X$$

of compact subsurfaces with boundary such that each K_j is a deformation retract of X and $\cup_{j=1}^{\infty} K_j = X$.

For any exhaustion of X, every compact subset of X is contained in K_m for all sufficiently large m. Indeed, if a compact set $K \subset X$ were not eventually contained in every K_m, one could choose $x_m \in K \setminus K_m$ for a subsequence of m-s with $x_m \to x \in K$. Since $x \in \bigcup_m K_m$, we have to have $x \in K_M$ for some M. Since K_M is closed, all the points in the subsequence eventually have to be in K_M. But $K_M \subset K_m$ for all large m, so we have a contradiction.

Definition 2.2.19. A sequence of points $\{X_j\}$ in the moduli space $\mathcal{M}_{g,n}$ *converges conformally* to a nodal surface $\overline{X}$ on the boundary of the Deligne-Mumford compactification $\overline{\mathcal{M}}_{g,n}$ if there exists an exhaustion $\{K_j\}$ of the complement of the nodes and punctures in $\overline{X}$ such that there is a sequence of conformal diffeomorphisms $G_j : K_j \longrightarrow X_j$.

Recall that a holomorphic differential ω on a Riemann surface X can be written as $\omega = f(z)dz$ in any holomorphic chart $z : U \longrightarrow \mathbb{C}$ for some function $f(z)$ holomorphic on U. A sequence of holomorphic one-forms $\{\omega_j\}$ converges to a one-form ω *uniformly on compact sets* if for any chart U and any compact $K \subset U$ the differentials $\{\omega_j\}$ uniformly converge to ω (i.e. for any $a > 0$ there exists an $N \in \mathbb{N}$ s.t. for any $j > N$ $|f_j(z) - f(z)| < a$ for $\omega_j = f_j(z)dz$ and $\omega = f(z)dz$ on U). This can be used to define convergence of differentials on a nodal component.

Definition 2.2.20. A sequence of abelian differentials $\{(X_j, \omega_j)\}$ in $\Omega\mathcal{M}_{g,n}$ *converges conformally* to a meromorphic differential $(\overline{X}, \overline{\omega})$ on a nodal component $Y \subset \overline{X}$ if the sequence of surfaces $\{X_j\}$ converges conformally to $\overline{X}$ in the sense of Definition 2.2.19, and the sequence of pullbacks $\{G_j^*\omega_j\}$ converges to $\overline{\omega}$ uniformly on compact sets.

Remark 2.2.21. Note that each $G_j^*\omega_j$ is a pullback of a holomorphic differential via a holomorphic map and thus holomorphic itself.

Proof. We will prove the theorem by contradiction. By Lemma 2.2.10 for any $(X, f, \omega) \in \mathcal{H}_{g,n}(\kappa)$ there is a finite number of ε_0-thick components $Q_1, Q_2, ..., Q_{k_\omega}$ in the thick-thin decomposition of (X, f, ω). We define the maximal ratio function

$$\mathrm{Max}_D(X, f, \omega) = \max_{i \in \{1,2,...,k_\omega\}} \left(\frac{D_\omega(Q_i)}{\lambda_\omega(Q_i)} \right).$$

and the minimal ratio function

$$\mathrm{Min}_D(X, f, \omega) = \min_{i \in \{1,2,...,k_\omega\}} \left(\frac{D_\omega(Q_i)}{\lambda_\omega(Q_i)} \right).$$

If the ratio function is not bounded from above (resp., away from zero), there exists such a sequence $\{(X_j, f_j, \omega_j)\}$ of points in $\mathcal{H}_{g,n}(\kappa)$ that the corresponding sequence of values $\{\mathrm{Max}_D(X_j, f_j, \omega_j)\}$ is not bounded from above (resp., $\{\mathrm{Min}_D(X_j, f_j, \omega_j)\}$ is not bounded away from zero). We can pick a subsequence of $\{(X_j, f_j, \omega_j)\}$ for which the sequence of the values of Max converges to infinity (resp., Min converges to zero). We will use the same indexing that we used for the initial sequence and keep calling this subsequence $\{(X_j, f_j, \omega_j)\}$.

Let us project $\{(X_j, f_j, \omega_j)\}$ to a sequence of points $\{(X_j, \omega_j)\}$ in the moduli space $\Omega\mathcal{M}_{g,n}(\kappa)$. The maximal ratio function is well-defined on $\Omega\mathcal{M}_{g,n}(\kappa)$. Indeed, for two marked translation surfaces (X, f, ω) and (X, g, ω) with the same underlying (X, ω) the underlying hyperbolic metric, the thick-thin decomposition and all the flat-geometric quantities arising from it are the same. If two points (X, f, ω) and (Y, g, η) in $\mathcal{H}_{g,n}(\kappa)$ project to the same point in $\Omega\mathcal{M}_{g,n}(\kappa)$, then there exists a biholomorphism $f : X \to Y$ with $f^*\eta = \omega$. So h preserves flat lengths of curves, the hyperbolic metric and the norm λ.

We may regard the sequence $\{(X_j, \omega_j)\}$ as a sequence in the projectivized moduli space of multi-scale differentials

$$\mathbb{P}\Xi\overline{\mathcal{M}}_{g,n}(\kappa) = \Xi\overline{\mathcal{M}}_{g,n}(\kappa)/\mathbb{C}^*.$$

Indeed, if $c \in \mathbb{C}^*$, then replacing ω_j by $c\omega_j$ does not change the underlying Riemann surface X_j, the zero set of the differential, or the punctured surface $X_j \setminus \Sigma_{\omega_j}$. Hence the complete hyperbolic metric on the punctured surface, and therefore the thick-thin decomposition, remain unchanged. On the other hand, the induced flat metric is rescaled by the factor $|c|$, since $|c\omega_j| = |c|\,|\omega_j|$. Thus by Property 2.2.16, the ratio

$$\frac{D_{\omega_j}(Q)}{\lambda_{\omega_j}(Q)}$$

is invariant under the $\mathbb{C}^*$-action, so the maximal ratio function is well-defined on $\mathbb{P}\Xi\overline{\mathcal{M}}_{g,n}(\kappa)$.

We will now apply the compactness of $\mathbb{P}\Xi\overline{\mathcal{M}}_{g,n}(\kappa)$, established in [BCG+19].

Theorem 2.2.22 ([BCG+19], Theorem 7.13)**.** *The multi-scale $\mathbb{P}\Xi\overline{\mathcal{M}}_{g,n}(\kappa)$ is compact and sequentially compact.*

Since $\mathbb{P}\Xi\overline{\mathcal{M}}_{g,n}(\kappa)$ is sequentially compact, the sequence $\{[X_j, \omega_j]\}$ has a convergent (in the standard topology on $\mathbb{P}\Xi\overline{\mathcal{M}}_{g,n}(\kappa)$) subsequence. Following our convention, we will call elements of this subsequence $\{[X_j, \omega_j]\}$ as well.

Let $[\overline{X}, \overline{\omega}]$ be the limit of $\{[X_j, \omega_j]\}$ in $\mathbb{P}\Xi\overline{\mathcal{M}}_{g,n}(\kappa)$. The underlying surface $\overline{X}$ of $[\overline{X}, \overline{\omega}]$ is either nodal or not (depending on whether or not it belongs to the boundary of $\mathbb{P}\Xi\overline{\mathcal{M}}_{g,n}(\kappa)$).

If $\overline{X}$ is not nodal, then $[\overline{X}, \overline{\omega}]$ belongs to $\Omega\mathcal{M}_{g,n}(\kappa)/\mathbb{C}^*$, the interior of $\mathbb{P}\Xi\overline{\mathcal{M}}_{g,n}(\kappa)$. Let us pick some preimage $(\overline{X}, \overline{\omega})$ of $[\overline{X}, \overline{\omega}]$ in the moduli space $\Omega\mathcal{M}_{g,n}(\kappa)$. Since

$$\mathbb{P}\Omega\mathcal{M}_{g,n}(\kappa) = \Omega\mathcal{M}_{g,n}(\kappa)/S^1,$$

after passing to a subsequence, there exist complex numbers $\zeta_j \in S^1$ such that

$$(X_j, \zeta_j\omega_j) \longrightarrow (\overline{X}, \overline{\omega}) \qquad \text{in } \Omega\mathcal{M}_{g,n}(\kappa). \tag{2.2.19}$$

Since $\zeta_j \in S^1$, we have

$$\mathrm{Max}_D(X_j, \zeta_j\omega_j) = \mathrm{Max}_D(X_j, \omega_j). \tag{2.2.20}$$

Let us also consider some preimage $(\overline{X}, \overline{f}, \overline{\omega})$ of $(\overline{X}, \overline{\omega})$ in the Teichmüller space $\mathcal{H}_{g,n}(\kappa)$.

Lemma 2.2.23 (Local Boundedness near a Smooth Point)**.** *Consider our* $(\overline{X}, \overline{\omega}) \in \Omega\mathcal{M}_{g,n}(\kappa)$. *Then there exists an open neighborhood* $U \subset \Omega\mathcal{M}_{g,n}(\kappa)$ *of* $(\overline{X}, \overline{\omega})$ *and constants* $C_U, c_U > 0$ *such that for every* $(X, \omega) \in U$ *and every* ε_0*-thick component* Q *of* X

$$c_U \leq \frac{D_\omega(Q)}{\lambda_\omega(Q)} \leq C_U. \tag{2.2.21}$$

Proof. Let $l_{\min}(\overline{\omega})$ be the length of the shortest saddle connection in the flat metric induced by $\overline{\omega}$ on $\overline{X}$. Let

$$K_c = \{(X, f, \omega) \in \mathcal{H}_{g,n}(\kappa) | l_{\min}(\omega) \geq c\}. \tag{2.2.22}$$

Choose a positive number $s < l_{\min}(\overline{\omega})$, and let

$$K_s^{\text{mod}} = \{(X,\omega) \in \Omega\mathcal{M}_{g,n}(\kappa) \mid l_{\min}(\omega) \geq s\}.$$

By Masur's Compactness Criterion, K_s^{mod} is a compact subset of $\Omega\mathcal{M}_{g,n}(\kappa)$ containing $(\overline{X},\overline{\omega})$.

Since $l_{\min}$ is continuous on $\Omega\mathcal{M}_{g,n}(\kappa)$, there exists some open neighborhood $U^{\text{mod}} \subset K_s^{\text{mod}}$ containing $(\overline{X},\overline{\omega})$.

Since the natural projection

$$\pi : \mathcal{H}_{g,n}(\kappa) \to \Omega\mathcal{M}_{g,n}(\kappa) \tag{2.2.23}$$

is continuous, $\pi^{-1}(U^{\text{mod}})$ is an open neighborhood of $(\overline{X},\overline{f},\overline{\omega})$, and $\pi^{-1}(U^{\text{mod}}) \subset K_s$.

Assume in addition that no essential simple closed curve on $\overline{X}$ has hyperbolic length exactly ε_0. Since there are finitely many curves shorter than that, there exists some $\delta > 0$ such that no essential simple closed curve on $\overline{X}$ has hyperbolic length in $[\varepsilon_0 - \delta, \varepsilon_0 + \delta]$.

Let us pick a small number $r > 0$ such that

$$e^{2r}(\varepsilon_0 - \delta) < \varepsilon_0 \text{ and } e^{-2r}(\varepsilon_0 + \delta) > \varepsilon_0. \tag{2.2.24}$$

If

$$pr : \mathcal{H}_{g,n}(\kappa) \to \mathcal{T}_{g,n}$$

is the natural projection to the Teichmüller space, then consider the set

$$B_{\mathcal{T}}(pr(\overline{X}), r) = \{X \in \mathcal{T}_{g,n} | d_{\mathcal{T}}(X, pr(\overline{X})) < r\}. \tag{2.2.25}$$

The set

$$\widetilde{U} = pr^{-1}(B_{\mathcal{T}}(pr(\overline{X}), r)) \cap \pi^{-1}(U^{\text{mod}}) \tag{2.2.26}$$

is open as an intersection of two open sets. Let $(X, f, \omega) \in \widetilde{U}$. Then

$$d_{\mathcal{T}g,n}(pr(X, f, \omega), pr(\overline{X}, \overline{f}, \overline{\omega})) < r.$$

Recall the following.

Lemma 2.2.24 (Wolpert's Inequality, [Wol10] or Proposition 2.15 in [Ham13])**.** *Let $X_1, X_2 \in \mathcal{T}_{g,n}$. If there exists a K-quasiconformal map $f : X_1 \to X_2$, then for any essential simple closed curve γ we have*

$$\frac{\rho_{X_1}(\gamma)}{K} \leq \rho_{X_2}(\gamma) \leq K\rho_{X_1}(\gamma). \tag{2.2.27}$$

By the definition of the Teichmüller metric, there exists a K-quasiconformal map between the underlying marked surfaces with $K < e^{2r}$. Therefore, by Wolpert's inequality, for every essential simple closed curve γ,

$$e^{-2r}\rho_{\overline{X}}(\gamma) \leq \rho_X(\gamma) \leq e^{2r}\rho_{\overline{X}}(\gamma).$$

Since no essential simple closed curve on $\overline{X}$ has hyperbolic length in $[\varepsilon_0 - \delta, \varepsilon_0 + \delta]$, it follows from the choice of r that $\rho_X(\gamma) < \varepsilon_0$ if and only if $\rho_{\overline{X}}(\gamma) < \varepsilon_0$ for every essential simple closed curve γ.

Let $S_{g,n}$ be the model topological surface and let Γ be the multicurve on $S_{g,n}$ consisting of all essential simple closed curves whose hyperbolic length on $\overline{X}$ is less than ε_0. By the argument above, for every $(X, f, \omega) \in \widetilde{U}$ the set of essential simple closed curves of hyperbolic length less than ε_0 is again exactly Γ. The connected components of $S_{g,n} \setminus \Gamma$ therefore provide a fixed collection of complementary topological subsurfaces. When we say that a thick component of X corresponds to a thick component of $\overline{X}$, we mean that if R is a connected component of $S_{g,n} \setminus \Gamma$, then for each $(X, f, \omega) \in \widetilde{U}$ there is a unique connected component of the thick part of X whose preimage under the marking f is isotopic to R relative to Γ and the punctures. In this sense, the thick-thin decomposition is topologically stable on $\widetilde{U}$. Thus, after fixing a connected component R of $S_{g,n} \setminus \Gamma$, the associated thick component $Y_R(X, \omega)$ is well-defined for every $(X, f, \omega) \in \widetilde{U}$. Therefore the ratio $\frac{D_{\omega_j}(Q)}{\lambda_{\omega_j}(Q)}$ may now be regarded as a function on $\widetilde{U}$. Note that the same is true for $\frac{L_{\omega_j}(Q)}{\lambda_{\omega_j}(Q)}$ and $\frac{\ell_{\omega_j}(Q)}{\lambda_{\omega_j}(Q)}$.

Proposition 2.2.25. *The quantities $\lambda_\omega(Q)$, $D_\omega(Q)$, $L_\omega(Q)$ and $\ell_\omega(Q)$ are all continuous on $\widetilde{U}$.*

Proof. Let $(X_0, f_0, \omega_0) \in \widetilde{U}$. Then for every other $(X, f, \omega) \in \widetilde{U}$, the markings induce a homeomorphism

$$h_{X,f} : Q(X_0) \to Q(X) \tag{2.2.28}$$

between the thick components on X_0 and X corresponding to Q. Define

$$F : \widetilde{U} \times Q(X_0) \to \mathbb{R}$$

by setting

$$F((X, f, \omega), x) := |\omega(h_{X,f}(x))\|_{\rho_X}, \qquad x \in Q(X_0) \tag{2.2.29}$$

Since the marking varies continuously, the hyperbolic metric varies continuously, and ω varies continuously in the stratum, the function F is continuous on $\widetilde{U} \times Q(X_0)$.

Moreover, for every $(X, f, \omega) \in \widetilde{U}$, we have

$$\lambda_\omega(Q(X)) = \sup_{p \in Q(X)} \|\omega(p)\|_{\rho_X} = \sup_{x \in Q(X_0)} F((X, f, \omega), x). \tag{2.2.30}$$

Since $Q(X_0)$ is compact, the supremum is attained, and therefore

$$\lambda_\omega(Q(X)) = \max_{x \in Q(X_0)} F((X, f, \omega), x). \tag{2.2.31}$$

It follows that the function $(X, f, \omega) \longmapsto \lambda_\omega(Q(X))$ is continuous on $\widetilde{U}$.

For $D_\omega(Q)$, the proof is essentially the same. Since on $\widetilde{U}$ the thick-thin decomposition is stable, the markings identify all flat representatives of Q. After making this identification, the flat distance between two points becomes a continuous function. The flat diameter is then obtained by taking the maximum of this distance function over all pairs of points on a fixed compact surface. This means it is continuous.

The proofs for $L_\omega(Q)$ and $\ell_\omega(Q)$ are similar.

□

Since at each specific point, all the four quantities are finite and bounded away from zero, this contradicts the assumption that for the sequence $(X_j, \zeta_j \omega_j)$ their ratio was converging either to infinity or to zero.

If $\overline{X}$ has curves of hyperbolic length exactly ε_0, then there are finitely many possible hyperbolically short multicurves in some neighborhood of $\overline{X}$. Then, we may pass to a subsequence along which this multicurve is stable. For all sufficiently large j, the chosen components $Q(X_j)$ correspond to one fixed topological component $R \subset S_{g,n} \setminus \Gamma$. Now let us cut the limit surface $\overline{X}$ along this stable multicurve Γ, and let $\overline{Y}_R$ be the component corresponding to R. Although $\overline{Y}_R$ need not be an actual ε_0-thick component of $\overline{X}$, it is still a compact subsurface with boundary of the same topological type as each $Q(X_j)$. This means the ratios still converge to the values of the corresponding quantities on $\overline{Y}_R$, which contradicts the assumption that they converge either to infinity or to zero. □

Now suppose that the limit point $[\overline{X}, \overline{\omega}]$ belongs to the boundary of $\mathbb{P}\Xi\overline{\mathcal{M}}_{g,n}(\kappa)$. This means that the underlying sequence of surfaces X_j converges to the nodal surface $\overline{X}$. Since projectivization does not affect the convergence of underlying surfaces, the underlying surfaces of the sequence $(X_j, \zeta_j \omega_j)$ in $\Omega\mathcal{M}_{g,n}(\kappa)$ also converge to $\overline{X}$. Since there are finitely many nodal components in $\overline{X}$, there exists such an $\varepsilon = \varepsilon(\overline{X}) > 0$ that for any nodal component Y of $\overline{X}$ its thick part $Y_\varepsilon(\overline{X})$ is connected. We can also choose our $\varepsilon(\overline{X})$ in such a way that there are no curves of that exact length on $\overline{X}$ (except for the non-essential ones homotopic to the cusps at the nodes or trivial).

Consider a fixed component $Y(\varepsilon(\overline{X}))$ of the $\varepsilon(\overline{X})$-thick-thin decomposition of $\overline{X}$. By cutting a finite number of elements at the beginning of the sequence, we can assume that there is a sequence of components $Y_j(\varepsilon)$ of the $\varepsilon(\overline{X})$-thick-thin decomposition of X_j such that $\{Y_j(\varepsilon)\}$ converges to $Y(\varepsilon)$.

We will use the following result establishing the convergence of a rescaled subsequence of differentials on a single component.

Theorem 2.2.26 ([BCG+19], Theorem 3.10)**.** *Suppose (X_j, ω_j) is a sequence of meromorphic differentials in $\Omega\mathcal{M}_{g,n}(\kappa)$ such that X_j converges to some $\overline{X}$ as a sequence of pointed stable curves. Let $Y \subset \overline{X}$ be a nodal component, and pick $\varepsilon(\overline{X})$ small enough for the hyperbolic $\varepsilon(\overline{X})$-thick part $Y(\varepsilon(\overline{X}))$ to be connected. For large enough j, choose $Y(\varepsilon(\overline{X}))_j$ to be the sequence of thick components of X_j such that $Y(\varepsilon(\overline{X}))_j$ converges to $Y(\varepsilon(\overline{X}))$. Then*

$$\eta_j = \frac{\omega_j}{\lambda_{\omega_j}(Y_j(\varepsilon))} \tag{2.2.32}$$

rescaled by the sizes $\lambda_{\omega_j}(Y_j(\varepsilon))$ converges to a nonzero differential η on the entire nodal component Y.

We also immediately recall:

Theorem 2.2.27 ([BCG+19], Corollary 3.12)**.** *There exists a constant L that depends only on the stratum and $\varepsilon(\overline{X})$), such that for any pointed meromorphic differential $(X, \omega) \in$*

$\mathcal{M}_{g,n}(\kappa)$, *for any points* p *and* q *in the* $\varepsilon(\overline{X})$*-thick part of* X, *we have*

$$|\omega|_p \leq L \cdot |\omega|_q. \tag{2.2.33}$$

We will first prove the following lemma.

Lemma 2.2.28. *For every fixed* $\epsilon > 0$ *sufficiently small for the conditions of Theorem 2.2.26 the corresponding thick component* $Y(\epsilon)_m$ *is contained in the image of the exhaustion map* $g_m(K_m)$ *for all sufficiently large* j*-s.*

Proof. Now let us choose two numbers $\epsilon_- < \epsilon < \epsilon_+$ that are very close to ϵ (in particular, so that the topological thick component remains stable for all of them). Since $Y(\epsilon_-)$ and $Y(\epsilon_+)$ are compact subsets of Y, both are contained in some K_m for all sufficiently large m.

On the compact set $Y(\epsilon_+)$ every point has injectivity radius strictly bigger than ϵ. Since the hyperbolic metrics converge on fixed compact subsets under the maps g_m, for all sufficiently large m every point of $g_m(Y(\epsilon_+))$ still has injectivity radius bigger than ϵ. So

$$g_m(Y(\epsilon_+)) \subset Y(\epsilon)_m. \tag{2.2.34}$$

Also, every point of the boundary $\partial Y(\epsilon_-)$ has injectivity radius exactly ϵ_-, which is strictly less than ϵ. Again by convergence on this fixed compact set, for all sufficiently large m, every point of $g_m(\partial Y(\epsilon_-))$ still has injectivity radius less than ϵ. Hence,

$$g_m(\partial Y(\epsilon_-)) \cap Y(\epsilon)_m = \varnothing. \tag{2.2.35}$$

Now suppose for contradiction that $Y(\epsilon)_m$ is not contained in $g_m(\partial Y(\epsilon_-))$. Then there is a point of $Y(\epsilon)_m$ outside of $g_m(\partial Y(\epsilon_-))$. But $Y(\epsilon)_m$ also contains $g_m(\partial Y(\epsilon_+))$, and $Y(\epsilon)_m$ is connected. So there is a path in $Y(\epsilon)_m$ joining a point of $g_m(\partial Y(\epsilon_+))$ to a point outside $g_m(\partial Y(\epsilon_+))$. Such a path must cross the boundary of $g_m(Y(\epsilon_-))$, namely $g_m(\partial Y(\epsilon_-))$. That is impossible, since $g_m(\partial Y(\epsilon_-))$ contains no ϵ-thick points. Therefore, for all sufficiently large m, we have

$$Y(\epsilon)_m \subset g_m(Y(\epsilon_-)). \tag{2.2.36}$$

Since $Y(\epsilon_-) \subset K_m$ for all sufficiently large m, it follows that

$$Y(\epsilon)_m \subset g_m(K_m) \tag{2.2.37}$$

for all sufficiently large m.

$\square$

Let us choose two thresholds $0 < \epsilon_1 < \epsilon_2$ satisfying the conditions of Theorem 2.2.26 for our limit $(\overline{X}, \overline{\omega})$. Clearly, for a fixed nodal component Y of the limit surface $\overline{X}$ the two hyperbolic thick components are nested: $Y(\epsilon_2) \subset Y(\epsilon_1)$. The same thing is true for the corresponding thick components $Y(\epsilon_2)_j \subset Y(\epsilon_1)_j$.

After passing to a subsequence we may assume that for the sequence $(X_j, \zeta_j\omega_j)$ we have that the rescaled differentials

$$\eta_j = \frac{\zeta_j\omega_j}{\lambda_{\zeta_j\omega_j}(Y(\epsilon_1)_j)} \tag{2.2.38}$$

converges to a nonzero limit differential on Y.

We will now change the notation slightly to keep track of various hyperbolic metrics appearing in the proof below, and for the norm of a differential ω at the point p with respect to a hyperbolic metric ρ we will write $|\omega|_\rho(p)$. If ρ_j is the hyperbolic metric on X_j , then by Corollary 2.2.27, there exists a number $L(\kappa, \epsilon_1)$ such that for any $x, y \in Y(\epsilon_1)_j$ and for every j

$$|\zeta_j\omega_j|_{\rho_j}(x) \leq L(\kappa, \epsilon_1) \cdot |\zeta_j\omega_j|_{\rho_j}(y). \tag{2.2.39}$$

and the same holds for the rescaled differentials

$$|\eta_j|_{\rho_j}(x) \leq L(\kappa, \epsilon_1) \cdot |\eta_j|_{\rho_j}(y). \tag{2.2.40}$$

Now consider two points $p, q \in Y(\epsilon_2)$. Note that by the definition of convergence, and since $Y(\epsilon_2)$ and $Y(\epsilon_1)$ are compact, the points p and q will belong to the exhaustion sets K_j for all sufficiently large j for some exhaustion of Y°, the limiting nodal component with zeroes and poles removed.

Thus, if $g_j : K_j \to X_j$ is the sequence of maps that arise from the differential sequence convergence, then for sufficiently large j the images $g_j(p)$ and $g_j(q)$ are well-defined.

Since both $Y(\epsilon_2)$ and $Y(\epsilon_1)$ are compact, and the sequence of hyperbolic metrics

$$g_j^* \rho_{X_j} \to \rho_Y \tag{2.2.41}$$

uniformly on compact sets, we have convergence of injectivity radii:

$$\mathrm{inj}_{X_j}(g_j(p)) \to \mathrm{inj}_Y(p) \qquad \text{and} \qquad \mathrm{inj}_{X_j}(g_j(q)) \to \mathrm{inj}_Y(q). \tag{2.2.42}$$

If $\mathrm{inj}_Y(p) \geq \epsilon_2$, then for any $\epsilon_1 < \epsilon_2$ the injectivity radius $\mathrm{inj}_{X_j}(g_j(p)) \geq \epsilon_1$ for all sufficiently large j. So for all sufficiently large j, if $p \in Y(\epsilon_2)$, then $g_j(p) \in Y(\epsilon_1)_j$.

So for all sufficiently large j we have $g_j(p), g_j(q) \in Y(\epsilon_1)_j$ and (2.2.40) gives

$$|\eta_j|_{\rho_j}(g_j(p)) \leq L(\kappa, \epsilon_1) \cdot |\eta_j|_{\rho_j}(g_j(q)). \tag{2.2.43}$$

Pulling back the differential η_j, we obtain

$$|g_j^*\eta_j|_{g_j^*\rho_j}(p) \leq L(\kappa, \epsilon_1) \cdot |g_j^*\eta_j|_{g_j^*\rho_j}(q). \tag{2.2.44}$$

Choose an open coordinate neighborhood U_p around the point p. Then the differentails can be written in a local coordinate z as

$$g_j^*\eta_j = h_j(z)\mathrm{dz} \qquad \text{and} \qquad \eta = h(z)\mathrm{dz}, \tag{2.2.45}$$

and the hyperbolic metrics can be written as

$$g_j^*\rho_j = \nu_j(z)|\mathrm{dz}| \qquad \text{and} \qquad \rho_Y = \nu(z)|\mathrm{dz}|, \tag{2.2.46}$$

By the convergence of the differential sequence and the pullback hyperbolic metrics convergence, $h_j \to h$ and $\nu_j \to \nu$ uniformly on compact subsets of U_p. Note that $\nu(p)$ is also nonzero, since ρ_Y is well-defined on Y°.

Since the corresponding norms at z are

$$|g_j^*\eta_j|_{g_j^*\rho_j}(z) = \frac{|h_j(z)|}{\nu_j(z)} \qquad \text{and} \qquad |\eta|_{\rho_Y}(z) = \frac{|h(z)|}{\nu(z)}, \tag{2.2.47}$$

we have

$$|g_j^*\eta_j|_{g_j^*\rho_j}(p) \to |\eta|_{\rho_Y}(p). \tag{2.2.48}$$

Similar convergence occurs at the point q.

This means that

$$|\eta|_{\rho_Y}(p) \leq L(\kappa, \epsilon_1) \cdot |\eta|_{\rho_Y}(q). \tag{2.2.49}$$

Note that this means that the size of the ϵ_2-thick component is well-defined, finite, and nonzero, that is the number

$$\Lambda(\epsilon_2) = \sup_{p \in Y(\epsilon_2)} |\eta|_{\rho_Y}(p) \tag{2.2.50}$$

is finite and nonzero, since the limit differential η is nonzero.

Recall that

$$\lambda_j(\epsilon_2) = \sup_{x \in Y(\epsilon_2)_j} |\eta_j|_{\rho_{X_j}}(x) \tag{2.2.51}$$

.

Proposition 2.2.29. *We have*

$$\lambda_j(\epsilon_2) \to \Lambda(\epsilon_2) \tag{2.2.52}$$

as $j \to \infty$.

Proof. Choose two numbers sufficiently close to ϵ_2 (in particular, they do not disrupt the topological stability of the thick components, and the smaller one is closer to ϵ_2 than ϵ_1):

$$\epsilon_- < \epsilon_2 < \epsilon_+. \tag{2.2.53}$$

Let $Y(\epsilon_-)$, $Y(\epsilon_+)$ be the corresponding thick components of the limit nodal component Y, and let $Y(\epsilon_-)_j$, $Y(\epsilon_+)_j$ be the corresponding thick components of the surfaces X_j. We have the inclusions:

$$Y(\epsilon_+)_j \subset Y(\epsilon_2)_j \subset Y(\epsilon_-)_j \qquad \text{and} \qquad Y(\epsilon_+) \subset Y(\epsilon_2) \subset Y(\epsilon_-). \tag{2.2.54}$$

For all sufficiently large j the exhaustion maps g_j are defined on the entire $Y(\epsilon_+)$:

$$g_j : Y(\epsilon_+) \to X_j. \tag{2.2.55}$$

Since $Y(\epsilon_+)$ is compact, $g_j^*\eta_j \to \eta$ uniformly on $Y(\epsilon_+)$ and $g_j^*\rho_{X_j} \to \rho_Y$ uniformly on $Y(\epsilon_+)$.

It follows that

$$\sup_{p \in Y(\epsilon_+)} \left| |\eta_j|_{\rho_{X_j}}(g_j(p)) - |\eta|_{\rho_Y}(p) \right| \to 0 \tag{2.2.56}$$

as $j \to \infty$.

By the properties of supremums then we have

$$\left| \sup_{p \in Y(\epsilon_+)} |\eta_j|_{\rho_{X_j}}(g_j(p)) - \sup_{p \in Y(\epsilon_+)} |\eta|_{\rho_Y}(p) \right| \leq \sup_{p \in Y(\epsilon_+)} \left| |\eta_j|_{\rho_{X_j}}(g_j(p)) - |\eta|_{\rho_Y}(p) \right|, \quad (2.2.57)$$

so

$$\left| \sup_{p \in Y(\epsilon_+)} |\eta_j|_{\rho_{X_j}}(g_j(p)) - \sup_{p \in Y(\epsilon_+)} |\eta|_{\rho_Y}(p) \right| \to 0 \quad (2.2.58)$$

as $j \to \infty$. By (2.2.54) we have

$$\sup_{p \in Y(\epsilon_+)} |\eta_j|_{\rho_{X_j}}(g_j(p)) \leq \sup_{p \in Y(\epsilon_2)_j} |\eta_j|_{\rho_{X_j}}(g_j(p)) = \lambda_j(\epsilon_2). \quad (2.2.59)$$

By (2.2.58), once we take the limit as $j \to 0$ on each side of (2.2.59), we get

$$\sup_{p \in Y(\epsilon_+)} |\eta|_{\rho_Y}(p) \leq \liminf_{j \to \infty} \lambda_j(\epsilon_2). \quad (2.2.60)$$

Similarly, we can show that

$$\limsup_{j \to \infty} \lambda_j(\epsilon_2) \leq \sup_{p \in Y(\epsilon_-)} |\eta|_{\rho_Y}(p). \quad (2.2.61)$$

Note that the equations above imply that for any ϵ_+, ϵ_- satisfying (2.2.53), we have

$$\sup_{p \in Y(\epsilon_+)} |\eta|_{\rho_Y}(p) \leq \liminf_{j \to \infty} \lambda_j(\epsilon_2). \leq \limsup_{j \to \infty} \lambda_j(\epsilon_2) \leq \sup_{p \in Y(\epsilon_-)} |\eta|_{\rho_Y}(p). \quad (2.2.62)$$

The leftmost expression in this equation converges to $\sup_{p \in Y(\epsilon_2)} |\eta|_{\rho_Y}(p)$ as $\epsilon_+ \to \epsilon_2$, and the rightmost expression converges to $\sup_{p \in Y(\epsilon_2)} |\eta|_{\rho_Y}(p)$ as $\epsilon_- \to \epsilon_2$. So we get

$$\Lambda(\epsilon_2) \leq \liminf_{j\to\infty} \lambda_j(\epsilon_2). \leq \limsup_{j\to\infty} \lambda_j(\epsilon_2) \leq \Lambda(\epsilon_2), \tag{2.2.63}$$

and the proof of Proposition 2.2.29 is finished.

□

It follows immediately that the sequence $\lambda_j(\epsilon_2)$ cannot converge to zero or infinity.

Now suppose that Y is not a pair of pants. Then let us prove that the length of the shortest flat geodesic representative of any non-peripheral essential simple closed curve is bounded from above and away from zero.

Since the thick components are topologically stable along the sequence, a fixed isotopy class of essential non-peripheral simple closed curves in Y_j-s corresponds to some fixed isotopy class of essential non-peripheral simple closed curves in Y. Let β be a representative of that class in $Y(\epsilon_2)$. Let $\beta_j = g_j(\beta)$. Let α_j be the hyperbolic geodesic representative of the same class in X_j. We know that $\ell_{\rho_{X_j}}(\alpha_j) \leq \ell_{\rho_{X_j}}(\beta_j)$.

We can then find the flat length of that representative and get

$$\ell_{\eta_j}(\alpha_j) \;=\; \int_{\alpha_j} |\eta_j| \;=\; \int_{\alpha_j} |\eta_j|_{\rho_{X_j}} \mathrm{ds}_{\rho_{X_j}} \;\leq\; \lambda_j(\epsilon_2)\cdot \ell_{\rho_{X_j}}(\alpha_j) \;\leq\; \lambda_j(\epsilon_2)\cdot \ell_{\rho_{X_j}}(\beta_j). \tag{2.2.64}$$

By the convergence of our sequence, the right-hand side is bounded from above, since

$$\lambda_j(\epsilon_2) \to \Lambda(\epsilon_2) \qquad \text{and} \qquad \ell_{\rho_{X_j}}(\beta_j) \to \ell_{\rho_Y}(\beta). \tag{2.2.65}$$

If this quantity is bounded for one fixed class, then the flat length of the shortest non-peripheral essential simple closed curve on our thick component is also bounded from above, so $\ell_j(Y)$ cannot converge to infinity along our sequence.

If Y is a pair of pants, for each $Y(\epsilon_2)_j$ let us look at their boundary curves $\beta_{1,j}$, $\beta_{2,j}$ and

$\beta_{3,j}$. For each $i \in \{1, 2, 3\}$ we have

$$\ell_{\eta_j}(\beta_{i,j}) = \int_{\beta_{i,j}} |\eta_j| = \int_{\beta_{i,j}} |\eta_j|_{\rho_{X_j}} \mathrm{ds}_{\rho_{X_j}} \leq \lambda_j(\epsilon_2) \cdot \ell_{\rho_{X_j}}(\beta_{i,j}). \tag{2.2.66}$$

But since each $\beta_{i,j}$ is a boundary component of a thick component we have

$$\ell_{\rho_{X_j}}(\beta_{i,j}) < C(\epsilon_2),$$

where $C(\epsilon_2)$ depends only on ϵ_2 so a uniform upper bound still holds for each boundary component, since there are only three boundary components, it holds for all of them.

Now let us prove the lower bound for the case when Y is not a pair of pants. Then let us fix a class of non-peripheral essential simple closed curves, in particular, the hyperbolic length of such a class does not go to zero, and even stays above ϵ_2 along the sequence. On each X_j let α_j be this curve, and let γ_j be its flat representative (we choose the boundary one that excludes the corresponding cylinder if there is one). On Y consider all the boundary curves of $Y(\epsilon_2)$. Eventually all these boundary curves will belong to all exhaustion sets K_m, and will have well-defined images under g_m-s.

Now suppose that

$$\ell_{\eta_j}(\gamma_j) \to 0.$$

As we mentioned above, we have

$$\ell_{\rho_{X_j}}(\alpha_j) > \epsilon_2.$$

We know that $\lambda_j(\epsilon_2)$ is converging to a nonzero number, so we may assume it is greater than some constant λ_0. The flat geodesic representative γ_j can be traced by a nearby essential simple closed curve $\gamma_j(r)$ of the same isotopy class such that the flat distance between that curve and γ_j is no greater than r for an arbitrarily small r. This $\gamma_j(r)$ can be chosen to avoid the singularities of the flat metric and thus the cusps of the hyperbolic one. Naturally, we can choose a family of such curves and have

$$\ell_{\eta_j}(\gamma_j(r)) \to \ell_{\eta_j}(\gamma_j)$$

as $r \to 0$.

Note that $\gamma_j(r)$ cannot be disjoint from $Y(\epsilon_2)_j$. Since α_j is carried by $Y(\epsilon_2)_j$, there is some representative $\beta_j \subset Y(\epsilon_2)_j$ in the same class. Suppose $\gamma_j(r)$ were disjoint from $Y(\epsilon_2)_j$. Then the curves $\gamma_j(r)$ and β_j should bound an annulus, and a boundary component of $Y(\epsilon_2)_j$ should be inside that annulus. That would make the curve α_j peripheral, so this cannot happen. Therefore

$$\gamma_j(r) \cap Y(\epsilon_2)_j \neq \varnothing.$$

Let us choose small flat δ-neighborhoods of the boundary curves of $Y(\epsilon_2)$. Then the images of such neighborhoods under g_j-s will eventually have width no smaller than $\frac{\delta}{2}$. Since $\gamma_j(r)$ has to return to the neighborhood of the cusp, it would have to cross such a strip and accumulate flat length of at least δ. Note that this will be true for a $\gamma_j(r)$ with arbitrarily small r, so the actual flat representative will have to accumulate length at least $\frac{\delta}{2}$, for example. This means that the flat lengths cannot go to zero unless $\gamma_j(r)$ stays in $Y(\epsilon_2)_j$. But this cannot happens since for r small enough $\gamma_j(r)$ has to go near a cusp.

Note that in the case when Y is a pair of pants, by the classification of flat pairs of pants given by Eskin, Kontsevich and Zorich in Section 4 of [EKZ14], the maximal flat representative of a boundary component is either a curve that goes around a saddle connection that connects two distinct singularities or a saddle connection loop that does not stay in the neighborhood of the singularity it connects to itself. So those flat geodesics similarly cannot have their lengths converge to zero by the argument above.

The total flat diameter of the limiting component Y might be infinite, since the limit differential η might be meromorphic. However, the following describes the flat geometry of such a flat surface.

Definition 2.2.30 (Tahar, [Tah18], [Tah20])**.** Let (X, q) be a flat surface with poles of higher order, and let Λ be the set of conical singularities of the induced flat metric. The *core* of X is the convex hull of Σ:

$$\text{core}(X) := \text{Conv}(\Sigma).$$

We write

$$IC(X) := \text{int}(\text{core}(X)), \qquad \partial C(X) := \text{core}(X) \setminus IC(X).$$

The core is said to be *degenerate* if $IC(X) = \varnothing$.

Proposition 2.2.31 (Tahar, [Tah18], [Tah20])**.** *Let (X, q) be a flat surface with p poles of higher order. Then*

$$X \setminus \text{core}(X)$$

has exactly p connected components. Each connected component is a topological disk containing a unique higher-order pole. These connected components are called the domains of poles.

Proposition 2.2.32 (Tahar, [Tah18], [Tah20])**.** *For any flat surface with poles of higher order (X, q), the boundary of the core,*

$$\partial C(X),$$

is a finite union of saddle connections.

Let us consider what the preimages of the node neighborhoods look like in the flat metric. If the node becomes a zero of the differential, then the flat neighborhood around it has finite diameter, and is part of the compact core. If it develops a pole, then it becomes an annular neighborhood, which is in the complement of the compact core, and by 2.2.32, there is some flat geodesic representative of the curve going around such a pole that has finite length. This means that the flat representative of the thick component Y will be cut along such a curve, and will not include the polar neighborhood. Thus the diameter of the flat representative cannot converge to zero or to infinity for our sequence of rescaled differentials.

Since the length of the shortest saddle connection cannot exceed twice the flat diameter of the subsurface, its length is immediately bounded from above. What remains to be proved is now is that the length of the shortest saddle connection along our sequence is bounded away from zero.

First, we will note that a saddle connection whose interior completely belongs to a flat representative of a thick component does note have to be fully contained in the hyperbolic representative of that component. However, it has to intersect the hyperbolic representative either by at least one essential arc or by at least one segment that connects two boundary components: assuming otherwise would contradict the Gauss-Bonnet Theorem. We then recall that on the hyperbolically thick part along our sequence the flat length is comparable to the hyperbolic length. The hyperbolic length of any essential arc or an arc connecting two boundaries is bounded both above and away from zero by constants that depend only on the threshold ϵ_2 and the topology of the subsurface. This is a well-known corollary of Mumford's Compactness criterion.

Theorem 2.2.33 (Mumford's Compactness Criterion, [DL25], [Wri15])**.** *For a fixed $\epsilon > 0$ the moduli space of hyperbolic surfaces with geodesic boundary components whose shortest essential simple closed curve has hyperbolic length no smaller than ϵ and whose boundary lengths are bounded is compact.*

As for the convergence of saddle connections, we note that on the limit surface the length of the shortest saddle connection in the compact core is nonzero and finite. Thus, the shortest saddle connection along a sequence cannot converge to zero or to infinity.

Now let us increase the threshold ϵ_2 to our chosen ε_0. The thick components $Y(\varepsilon_0)$ and $Y(\varepsilon_0)_j$ will be contained in the corresponding components for ϵ_2, and thus the hyperbolic subsurfaces inside them still satisfy Mumford's compactness criterion. Thus, similarly, it would be impossible for any of the three ratios to converge to zero or infinity, so the statement of Theorem 2.2.17 holds. □

Finally, we will note that the sizes of thick components are uniformly bounded on a stratum of unit-area flat surfaces.

Lemma 2.2.34. *Consider a fixed stratum of unit-area flat surfaces and a fixed (sufficiently small) constant ϵ. Then for any surface X from the stratum, and for any hyperbolic thick component Y in the ϵ-thick-thin decomposition of X, the size $\lambda(Y)$ of that component is bounded from above by some constant $\mathfrak{C}$ that depends only on the stratum (and the area upper boundary 1).*

Proof. Let us fix a point $p \in Y$. Then the injectivity radius at p is greater or equal to $\frac{\epsilon}{2}$. Thus the hyperbolic ball $B_{hyp}(p, \frac{\epsilon}{2})$ is embedded in X°. Since X° is a hyperbolic surface, by the Uniformization Theorem, there exists a holomorphic covering map

$$\phi : \mathcal{D} \to X^\circ \tag{2.2.67}$$

from the unit disk $\mathcal{D} = \{z \in \mathbb{C} : |z| < 1\}$ with the Poincaré metric such that $\phi(0) = p$. The preimage $\phi^{-1}B_{hyp}(p, \frac{\epsilon}{2})$ may have many connected components, and we can pick the one that contains 0. That component is the standard hyperbolic disk of radius r again:

$$B_{\mathcal{D}}\left(0, \frac{\epsilon}{2}\right) = \left\{z \in \mathcal{D} : d_{\mathcal{D}}(0, z) < \frac{\epsilon}{2}\right\},$$

where $d_{\mathcal{D}}$ is the Poincaré metric distance.

It is well-known (see, for example, [And05]) that due to the metric formula depending solely on the absolute value of the coordinate and not on the coordinate itself, that Euclidean disks and hyperbolic disks around the origin coincide in $\mathcal{D}$. In particular, a hyperbolic disk of radius no greater than $\frac{\epsilon}{2}$ coincides with a Euclidean disk with radius no greater that $R(\frac{\epsilon}{2})$, where the number $R(\frac{\epsilon}{2})$ depends only on the number ϵ.

Note that on $B_{hyp}(p, \frac{\epsilon}{2})$ the differential ω is holomorphic, since $B_{hyp}(p, \frac{\epsilon}{2})$ is in the hyperbolic thick component.

Using the standard coordinate $z = x + iy$ on $\mathcal{D}$, we can write the pullback of ω simply as

$$\phi^*\omega = f(z)dz \tag{2.2.68}$$

with some holomorphic function $f(z)$.

Note that

$$|\omega|_p = \frac{|f(0)|}{2}. \tag{2.2.69}$$

Indeed, since the cover map is a hyperbolic isometry, we have

$$|\omega|_p = |\phi^*\omega|_0 \tag{2.2.70}$$

Since the hyperbolic metric on $\mathcal{D}$ is

$$\mathrm{ds}_{hyp} = \frac{2|dz|}{1 - |z|^2}, \tag{2.2.71}$$

we have

$$|\phi^*\omega|_z = |f(z)||dz| = |f(z)| \cdot \frac{1 - |z|^2}{2}. \tag{2.2.72}$$

Substituting zero in here gives 2.4.2.

Now note that the area elements satisfy

$$|\phi^*\omega|^2 \mathrm{dA}_{hyp} = |f(z)|^2 \cdot \left(\frac{1-|z|^2}{2}\right)^2 \mathrm{dA}_{hyp} =$$
$$= |f(z)|^2 \cdot \left(\frac{1-|z|^2}{2}\right)^2 \frac{4}{(1-|z|^2)^2} dxdy = |f(z)|^2 dxdy. \quad (2.2.73)$$

By the mean-value property of subharmonic functions (which we are applying simply to a holomorphic $f(z)$ on the Euclidean disk $\mathcal{D}_R$ of radius $R(\frac{\epsilon}{2})$ around the origin on $\mathcal{D}$),

$$|f(0)|^2 \leq \frac{1}{\pi R^2} \int_{\mathcal{D}_R} |f(z)|^2 dxdy. \quad (2.2.74)$$

Using previous computations, we can say that 2.2.74 implies

$$4|\omega|_p^2 \leq \frac{1}{\pi R^2} \int_{\phi^{-1} B_{hyp}(p, \frac{\epsilon}{2})} |\phi^*\omega|^2 \mathrm{dA}_{hyp} = \frac{1}{\pi R^2} \int_{B_{hyp}(p, \frac{\epsilon}{2})} |\omega|^2 \mathrm{dA}_{hyp}. \quad (2.2.75)$$

This becomes

$$|\omega|_p^2 \leq \frac{1}{4\pi R^2} \int_{B_{hyp}(p, \frac{\epsilon}{2})} \mathrm{dA}_{flat}. \quad (2.2.76)$$

On a stratum of unit area flat surfaces,

$$\int_{B_{hyp}(p, \frac{\epsilon}{2})} \mathrm{dA}_{flat} \leq 1,$$

so

$$|\omega|_p \leq \frac{1}{2\sqrt{\pi} R}. \quad (2.2.77)$$

Recall that R depends only on the chosen threshold ϵ, so the proof of Lemma 2.2.34 is complete.

□

2.3 The Regular Triangulation and the Bases of Full and Odd Cohomology

First, we will talk about expanding annuli, whose properties we will need later. We follow [Raf14a] and [EMR19], since in our case the corresponding construction becomes simpler.

We consider the topological ε-thick-thin decomposition on a translation surface (X, ω). Suppose γ is a hyperbolically short curve. Consider its flat-geodesic representative that serves as a boundary curve for a component Q of the flat ε-thick-thin decomposition of X. There exists some small distance $r > 0$, for which the one-sided equidistant r-neighborhood of the fixed flat representative of γ is an embedded annulus with no singularities in its interior. Let us expand this neighborhood in the direction of Q as long as it stays embedded, equidistant, annular and contains no zeroes. If we take the union E of all such neighborhoods, it will be an annulus on the Q-side of the flat representative of γ. We say that E is *the expanding annulus of* γ in the direction of Q.

The flat distance between the boundary components of E is comparable to the flat diameter (and thus the size) of the thick component Q. While in our convention the surface is punctured at a different set of point, the proof presented in [EMR19] only relies on facts that stay intact in our setting and the Comparability Theorem.

Lemma 2.3.1 (Eskin, Mirzakhani, Rafi, [EMR19])**.** *The flat distance between the boundary components of E is comparable to the flat diameter (and thus the size) of the thick component Q.*

2.3.1 The Regular Triangulation

Recall that *a triangulation* of a translation surface (X, ω) is a collection of saddle connections with pairwise non-intersecting interiors such that any connected component of the complement of this collection is a triangle. We will now build a triangulation of our surface that is compatible with the flat thick-thin decomposition.

Definition 2.3.2. We say that a triangulation of (X, ω) is an *(involution-invariant) regular* triangulation if

(a) for each edge Θ of our triangulation the flat length of Θ is comparable to the size of some thick component Q (of size λ_Q):

$$c_{edge}\lambda_Q \leq l_\omega(\Theta) \leq C_{edge}\lambda_Q \tag{2.3.1}$$

for some positive constants c_{edge} and C_{edge} that depend only on the stratum;

(b) if (X, ω) is the orienting double-cover of some quadratic differential q, then for any edge Θ of our triangulation its image $\sigma(\Theta)$ under the canonical involution σ is also an edge of the triangulation.

In order to build this triangulation let us first note that each boundary component of a thick component is a closed chain of saddle connections. We start by adding all the saddle connections that belong to a boundary component to the list of edges of our triangulation. Note that the canonical involution σ is a holomorphic map, so it is an isometry of both the hyperbolic metric corresponding to X and the flat metric induced by ω. Thus it preserves both the hyperbolic and the flat thick-thin decompositions. So for any saddle connection Θ that belongs to a boundary component of the flat thick-thin decomposition its image $\sigma(\Theta)$ is also a saddle connection belonging to some boundary component, and thus is also included in the set of triangulation edges.

The complement of the union of all the boundary components is a disjoint union of thick components and thin cylinders. We will now triangulate each of these separately. In order to triangulate any thick component Q let us do the following. If the component Q is a triangle, we are done. If it is not, let us consider the shortest saddle connection Θ_1 contained in Q that has not yet been chosen to be an edge of our triangulation. Note that $\sigma(\Theta_1)$ belongs to a thick component of the same size. We add both Θ_1 and $\sigma(\Theta_1)$ to the list of edges of our triangulation. Note also that since only a singular point may be mapped to itself by σ, the interiors of Θ_1 and $\sigma(\Theta_1)$ cannot intersect, so they can both be edges in the same triangulation.

The flat length of Θ_1 (and of $\sigma(\Theta_1)$ since their lengths coincide) is comparable to the size λ_Q. Indeed, since Θ_1 was chosen to be the shortest saddle connection in Q its length cannot be greater than the diameter $D_\omega(Q)$, which is comparable to the size λ_Q.

On the next step we look at the components of the complement of the edges chosen so far in Q, and again pick the shortest saddle connection in each connected component and add it to the set of edges along with its image under σ. Note that if we cut Q along Θ_1, then the diameter of any new subcomponent Q' resulting from this cannot be larger than $D_\omega(Q) + 2l_\omega(\Theta_1)$. Since we are always simultaneously cutting along $\sigma(\Theta_1)$, $D_\omega(Q') \leq D_\omega(Q) + 4l_\omega(\Theta_1) \leq 5D_\omega(Q)$. Note that the total number of edges we will eventually pick is bounded by the number of edges in any triangulation of (X, ω), so it depends only on the stratum. Each new edge we add is not longer than the diameter of some component obtained from cuts along previously chosen edges. Thus for any saddle connection Θ that is an edge of our triangulation whose interior is contained in a thick component Q its flat length will be bounded from above by $CD_\omega(Q)$, where C is a constant that depends only on the stratum. Since the flat diameter of each thick component is comparable to its size ($c_{diam}\lambda_Q \leq D_\omega(Q) \leq C_{diam}\lambda_Q$ for some positive constants c_{diam} and C_{diam} that depend only on the stratum), we have that for any edge Θ of our triangulation

$l_\omega(\Theta) \le CC_{diam}(Q)\lambda_Q$. By Theorem 2.2.17 the length of the shortest saddle connection in any thick component Q is comparable to the size, so we now have the lower bound $c_{sc}\lambda_Q \le l_\omega(\Theta)$. Combining everything said above, we get that for any saddle connection Θ that is an edge in our triangulation whose interior is contained in a thick component Q we have

$$c_{sc}\lambda_Q \le l_\omega(\Theta) \le CC_{diam}\lambda_Q. \qquad (2.3.2)$$

Note also that our algorithm guarantees that whenever we add a new pair of edges Θ_{i+1} and $\sigma(\Theta_{i+1})$ they will not intersect any of the previously added edges Θ_1, Θ_2,..., Θ_i and $\sigma(\Theta_1)$, $\sigma(\Theta_2)$,..., $\sigma(\Theta_i)$. For the edge Θ_{i+1} this is automatically true by the choice of it. The edge $\sigma(\Theta_{i+1})$ clearly cannot intersect any of the edges $\sigma(\Theta_1)$, $\sigma(\Theta_2)$,..., $\sigma(\Theta_i)$ (if it did intersect some $\sigma(\Theta_j)$, then Θ_{i+1} would intersect Θ_j, which cannot happen by the choice of Θ_{i+1}). But if $\sigma(\Theta_{i+1})$ intersects any Θ_j, that would mean that $\sigma(\Theta_j)$ intersects Θ_{i+1}, which is not allowed by the choice of Θ_{i+1}.

Let us now build triangulations of cylinders. Suppose that the height of our cylinder is H and its circumference is Π. Consider a parallelogram $ABCD$ from which our cylinder can be glued by identifying the sides AD and BC (see the figure below). Note that the sides AB and CD are boundary curves of our cylinder. Note that both AB and CD are concatenations of saddle connections that are sides of triangles of the regular triangulation of our surface. Suppose that $S_1, S_2, \ldots, S_n$ are the singularities that belong to AB, and $P_1, P_2, \ldots, P_m$ are the singularities that belong to CD. Without loss of generality we can assume that $m \le n$.

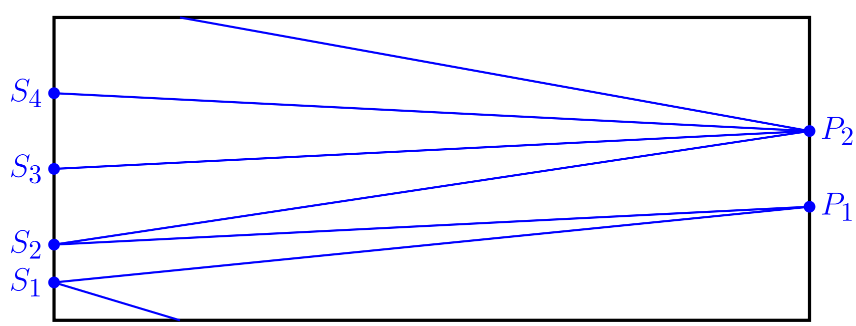

Figure 2.1: An example of a triangulation of a cylinder with $n = 5$ and $m = 2$.

We build our triangulation as follows. First, we connect the points S_1 and P_1 by a saddle connection that is the shortest straight line interval connecting them. Similarly, we build the interval P_1S_2, and then the interval S_2P_2 and so on. This process continues until we build the interval S_mP_m, and then if $m < n$ we proceed to connect all the remaining points $S_{m+1}, S_{m+2}, \ldots, S_n$ to the point P_m. Finally, we build the interval S_1P_m. This gives us a triangulation of the cylinder by saddle connections. Note that if two cylinders are exchanged by σ, we mirror their triangulations accordingly.

2.3.2 The Bases of Full and Odd Cohomology

Each side Θ of our triangulation determines an integral relative homology class $[\Theta]$. We will say that the flat length of a homology class $[\Theta]$ is the flat length of the corresponding triangulation side Θ. Let us now pick a basis of relative homology $H_1(X, \Sigma, \mathbb{Z})$ of homology classes that have the sides of the triangulation as their representatives. First, we pick the class $[\Psi_1]$ with the shortest flat length (if there are several of those, pick any of them). Then we pick the class $[\Psi_2]$ with the shortest length among all the triangle side classes that are linearly independent from $[\Psi_1]$. If $[\sigma(\Psi_1)]$ is independent of $[\Psi_1]$ let us also always pick $[\Psi_2] = [\sigma(\Psi_1)]$. After we have picked the first i classes $[\Psi_1]$, $[\Psi_2]$,...,$[\Psi_i]$ we pick the next one $[\Psi_{i+1}]$ as the triangle side class with the shortest length among all the triangle side classes that are not in the span of $[\Psi_1]$, $[\Psi_2]$,...,$[\Psi_i]$. We also add $[\sigma(\Psi_{i+1})]$ as the next basis vector whenever it is independent on all the vectors added before it ($[\Psi_{i+1}]$ included). Clearly, this will terminate after finitely many steps giving some basis $[\Psi_1]$, $[\Psi_2]$,...,$[\Psi_t]$ of relative homology as a result. Let $[\psi_i]$ be the relative cohomology class dual to $[\Psi_i]$ making $[\psi_1]$, $[\psi_2]$,...,$[\psi_t]$ a basis of relative cohomology $H^1(X, \Sigma, \mathbb{Z})$.

Let $[\Theta_1]$, $[\Theta_2]$, ..., $[\Theta_s]$ be the homology classes of the triangulation sides that are not in the basis chosen above. Let

$$[\Theta_1] = \delta_{11}[\Psi_1] + \delta_{12}[\Psi_2] + ... + \delta_{1t}[\Psi_t];$$

$$[\Theta_2] = \delta_{21}[\Psi_1] + \delta_{22}[\Psi_2] + ... + \delta_{2t}[\Psi_t];$$

$$...$$

$$[\Theta_s] = \delta_{s1}[\Psi_1] + \delta_{s2}[\Psi_2] + ... + \delta_{st}[\Psi_t].$$

be their decompositions into linear sums of the basis vectors.

Lemma 2.3.3. *The possible absolute values $|\delta_{ij}|$ are bounded by a constant that depends only on the stratum.*

Proof. This follows immediately from the fact that the number of triangles and edges in a triangulation is constant on the stratum. □

Now we build the basis of the odd cohomology $H^1(\tilde{X}, \Sigma, \mathbb{C})$. For any $[\Psi] \in H_1(\tilde{X}, \Sigma, \mathbb{C})$ define

$$[\hat{\Psi}] = \begin{Bmatrix} [\Psi] \text{ if } [\sigma(\Psi)] = -[\Psi]; \\ [\Psi] - [\sigma(\Psi)] \text{ otherwise }. \end{Bmatrix} \tag{2.3.3}$$

Lemma 2.3.4 (**Athreya, Eskin, Zorich**, [AEZ14])**.** *For each of the basis edge classes $[\Psi_1]$, $[\Psi_2]$,...,$[\Psi_t]$ consider the corresponding classes $[\hat{\Psi}_1]$, $[\hat{\Psi}_2]$,...,$[\hat{\Psi}_t]$. They span the integral odd homology $H_1(\tilde{X}, \Sigma, \mathbb{Z})$. Consequently, we can pick a subset $[\hat{\Psi}_{s_1}]$, $[\hat{\Psi}_{s_2}]$,...,$[\hat{\Psi}_{s_\nu}]$ that forms a basis of $H_1^-(\tilde{X}, \Sigma, \mathbb{Z})$.*

Proposition 2.3.5. *There exists a basis* $[\Delta_{s_1}]$, $[\Delta_{s_2}]$,...,$[\Delta_{s_\nu}]$ *of* $H^1_-(\tilde{X}, \Sigma, \mathbb{C})$ *such that*

$$[\Delta_{s_i}]([\hat{\Psi_{s_j}}]) = \begin{cases} 1 & \textit{if } i = j; \\ \\ 0 & \textit{otherwise} \end{cases} \tag{2.3.4}$$

and such that for all i *we have* $[\Delta_{s_i}]([\Theta]) = 0$ *for any* $\Theta \in H_1^+(\tilde{X}, \Sigma, \mathbb{C})$.

By the definition of the period map on the principal stratum of quadratic differentials, this is a basis in the local period coordinates on the stratum. Note that the values of these basis vectors on individual saddle connections may be half-integers. In particular

Proposition 2.3.6.

$$[\Delta_{s_i}]([\Psi_{s_j}]) = \begin{cases} \frac{1}{2} & \textit{if } i = j; \\ \\ 0 & \textit{if } i \neq j. \end{cases} \tag{2.3.5}$$

Proof. Rewrite $2[\Psi_{s_i}] = [\Psi_{s_i}] + [\sigma(\Psi_{s_i})] + [\Psi_{s_i}] - [\sigma(\Psi_{s_i})]$. Then since $[\Delta_{s_i}]$ is zero on every even homology class

$$[\Delta_{s_i}](2[\Psi_{s_i}]) = [\Delta_{s_i}]([\Psi_{s_i}] + [\sigma(\Psi_{s_i})]) + [\Delta_{s_i}]([\Psi_{s_i}] - [\sigma(\Psi_{s_i})]) = [\Delta_{s_i}]([\Psi_{s_i}] - [\sigma(\Psi_{s_i})]) = 1.$$

By linearity $[\Delta_{s_i}]([\Psi_{s_i}]) = \frac{1}{2}$. Similarly,

$$[\Delta_{s_i}](2[\Psi_{s_j}]) = [\Delta_{s_i}]([\Psi_{s_j}] + [\sigma(\Psi_{s_j})]) + [\Delta_{s_i}]([\Psi_{s_j}] - [\sigma(\Psi_{s_j})]) = 0.$$

□

2.4 The Upper Estimate on the AGY-norm

2.4.1 Saddle Connection Length Estimates

Lemma 2.4.1. *Suppose that γ is the segment of a saddle connection that intersects two components Q_1 and Q_2 of the thick-thin decomposition. Let λ_1 and λ_2 be the corresponding sizes of these components. Without loss of generality we can assume that $\lambda_1 < \lambda_2$. Then the length of this segment $l(\gamma) > C_a \lambda_2$ for some constant C_a.*

Proof. Among the two directions that we can choose to take while going along the segment γ let us choose the one that visits the smaller component first. Then the segment γ has to cross the expanding annulus in the direction of Q_2. The radius r of this expanding annulus is comparable to the size of Q_2:

$$c_2 \lambda_2 \leq r \leq C_2 \lambda_2$$

for some positive constants c_2 and C_2 that depend only on the stratum. Since $l(\gamma) \geq r$, we have

$$l(\gamma) \geq r \geq c_2 \lambda_2.$$

So we can take $C_a = c_2$. ☐

Theorem 2.4.2. *Suppose that γ is the segment of our saddle connection that is contained in a single component Q of the thick-thin decomposition. Let λ_Q be the size of this component. Then the length $l(\gamma)$ of this segment satisfies*

$$\eta(\gamma)\mathcal{K}_1 - \mathcal{K}_2 < l(\gamma) < \eta(\gamma)\mathcal{K}_1 + \mathcal{K}_2, \tag{2.4.1}$$

where $\mathcal{K}_1$ and $\mathcal{K}_2$ are some positive constants comparable to the size of the component while $\eta(\gamma)$ is is the maximal number of intersections this segment has with any side of the triangles of the regular triangulation of our surface.

Suppose that a segment γ of a saddle connection lies entirely in a single component Q of the thick-thin decomposition. Let T_Q be the number of the triangles in the regular involution-invariant triangulation whose interiors are contained in Q (the number of such triangles is finite and bounded from above by a constant T – the total number of triangles in every triangulation – that depends only on the stratum). Let us consider a path along γ and record which triangles it intersects as we follow it (one and the same triangle might appear multiple times in our records). For each triangle Δ in this recorded sequence we have a subsegment γ_Δ of γ that intersects two of the sides of Δ. Let s_Δ be the side of Δ that γ_Δ does not intersect. Let $l(\gamma_\Delta)$ and $l(s_\Delta)$ be their flat lengths.

For a fixed $\varepsilon > 0$, we say that a triangle Δ from the sequence is ε-*bad* if

$$\frac{l(\gamma_\Delta)}{l(s_\Delta)} < \varepsilon,$$

and we say that Δ is ε-*good* otherwise.

Lemma 2.4.3. *There exist a $\mathfrak{T}_Q \in \mathbb{N}$ and an $\varepsilon_Q > 0$ such that there can be no more than $\mathfrak{T}_Q$ consecutive ε_Q-bad triangles in the resulting sequence of triangles.*

Proof. By the properties of a regular involution-invariant triangulation, any side of a triangle Δ is comparable to the size of Q: there exist two positive constants c_{edge} and C_{edge} that depend only on the stratum such that

$$c_{edge}\lambda_Q \leq l(s) \leq C_{edge}\lambda_Q, \tag{2.4.2}$$

where s is a side of Δ.

There also exists a constant c_{loop} that depends only on the stratum such that for the length of any essential curve α we have

$$c_{loop}\lambda_Q \leq l(\alpha). \tag{2.4.3}$$

Pick N to be the smallest integer satisfying

$$\frac{1}{N-1} < \frac{c_{loop}}{C_{edge}}.$$

Note that since c_{loop} and C_{edge} depend only on the stratum, N also depends only on the stratum, and the difference

$$\mathcal{E} = c_{loop} - \frac{C_{edge}}{N-1}$$

is a positive fixed number that also depends only on the stratum.

Note that there exists a constant M_Q s.t. any saddle connection γ staying in Q can only intersect at most M_Q triangles before intersecting one of the interior edges of Q twice. Note that $1 \leq M_Q \leq 3T$.

If γ intersects some Q-interior edge E of our triangulation N times, then it can intersect at most NM_Q triangles in between the first and the last of the N intersections with E.

The N intersection points subdivide E into $N+1$ subsegments, and there is a pair of points p and q in the interior of E with distance at most $\frac{l(E)}{N-1}$ where $l(E)$ is the length of E. Note that $l(E) \leq C_{edge}\lambda_Q$, so the distance between that pair of points cannot be larger than $\frac{C_{edge}\lambda_Q}{N-1}$.

Suppose that each of the (at most) $(N-1)M_Q$ triangles above is ε-bad. The total length of the segment γ going through all these triangles from p to q cannot be greater than $(N-1)M_Q C_{edge}\lambda_Q\varepsilon$.

Let us now consider a closed curve $\mathcal{A}$ obtained by starting at the point p, following γ until we reach the point q and then going from q to p along the edge E. The total length of $\mathcal{A}$

$$l(\mathcal{A}) \leq (N-1)M_Q C_{edge}\lambda_Q\varepsilon + \frac{C_{edge}\lambda_Q}{N-1}. \tag{2.4.4}$$

Now let us prove that the curve $\mathcal{A}$ belongs to a free homotopy class of some essential simple closed curve.

Suppose this is not true. Then either one of the components of its complement has no singularities, or it contains one singularity, or the curve $\mathcal{A}$ is homotopic to a boundary component of Q.

Let us look at the first two options. Let $Q_{\mathcal{A}}$ be the said complement component. Consider the union $Q_{\mathcal{A}} \cup \mathcal{A}$ of this component with its boundary (our curve). According to the Gauss–Bonnet Theorem:

$$\int_{Q_{\mathcal{A}} \cup \mathcal{A}} \kappa \, dA + \int_{\mathcal{A}} k_g \, ds = 2\pi\chi(Q_{\mathcal{A}} \cup \mathcal{A}),$$

where dA is the area element of $Q_{\mathcal{A}} \cup \mathcal{A}$, ds is the length element along the boundary $\mathcal{A}$, κ is the Gaussian curvature of $Q_{\mathcal{A}} \cup \mathcal{A}$ and k_g is the geodesic curvature of $\mathcal{A}$.

Since the intersection angle between the segment γ and the side S is the same at the points p_1 and p_2, their exterior angles' measures are each other's additive opposites, and thus the integral of the geodesic curvature

$$\int_{\mathcal{A}} k_g \, ds = 0.$$

Thus, the Gauss-Bonnet formula simplifies to

$$\int_{Q_{\mathcal{A}} \cup \mathcal{A}} \kappa \, dA = 2\pi\chi(Q_{\mathcal{A}} \cup \mathcal{A}).$$

In the case when $Q_{\mathcal{A}} \cup \mathcal{A}$ contains no punctures,

$$\int_{Q_{\mathcal{A}} \cup \mathcal{A}} \kappa \, dA = 0$$

and

$$\chi(Q_{\mathcal{A}} \cup \mathcal{A}) = 2.$$

Thus, we arrive at a contradiction.

In the case when there is a single puncture inside $Q_{\mathcal{A}} \cup \mathcal{A}$,

$$\int_{Q_{\mathcal{A}} \cup \mathcal{A}} \kappa \, dA < 0.$$

The Euler characteristic of $Q_{\mathcal{A}} \cup \mathcal{A}$ in this case is equal to

$$\chi(Q_{\mathcal{A}} \cup \mathcal{A}) = 3.$$

So in this case we also obtain a contradiction.

Now if $\mathcal{A}$ is homotopic to some boundary component β_Q of Q, consider the primitive annulus that has the curves $\mathcal{A}$ and β_Q as its boundary components. Note that the geodesic curvature of β_Q with respect to Q is well-defined. Since $\int_{\mathcal{A}} k_g \, ds = 0$, another application of the Gauss-Bonnet Theorem shows that same should be true for β_Q:

$$\int_{\beta_Q} k_g \, ds = 0.$$

Our annulus in this case is actually a flat cylinder, but the thick components were chosen in such a way that they are all disjoint from the interiors of flat cylinders corresponding to their boundary components.

Thus $\mathcal{A}$ represents a free homotopy class of some essential simple closed curve.

Take

$$\varepsilon < \mathcal{E}(N-1) M_Q C_{edge}. (2.4.5)$$

Then

$$\varepsilon \lambda_Q (N-1) M_Q C_{edge} < \left(c_{loop} - \frac{C_{edge}}{N-1} \right) \lambda_Q.$$

And

$$\varepsilon\lambda_Q(N-1)M_Q C_{edge} + \frac{C_{edge}}{N-1}\lambda_Q < c_{loop}\lambda_Q. \tag{2.4.6}$$

The left-hand side of expression (2.4.6) is exactly the upper bound on the length $l(\mathcal{A})$. So $l(\mathcal{A}) < c_{loop}\lambda Q$. Since $\mathcal{A}$ represents a free homotopy class of some essential simple closed curve, this is a contradiction. So there cannot be more than $\mathfrak{T}_Q = N_Q T$ consecutive ε_Q-bad triangles for $\varepsilon_Q = \frac{\mathcal{E}_Q}{N_Q T C_{edge}}$ or any smaller ε_Q. □

Suppose that we have a saddle connection segment γ that crosses M triangles with $M \geq N_Q T + 1$ and is contained entirely in some thick component Q. Then the length

$$l(\gamma) \geq \varepsilon_Q c_{edge}\lambda_Q \left\lfloor \frac{M}{N_Q T + 1} \right\rfloor, \tag{2.4.7}$$

where $\left\lfloor \frac{M}{N_Q T+1} \right\rfloor$ is the integral part of $\frac{M}{N_Q T+1}$. The length of the corresponding path along the triangle edges cannot be greater than

$$l(\Delta_\gamma) \leq 2(N_Q T + 1)MC_{edge}\lambda_Q. \tag{2.4.8}$$

So

$$\frac{l(\Delta_\gamma)}{l(\gamma)} \leq \frac{2(N_Q T + 1)MC_{edge}\lambda_Q}{\varepsilon_Q c_{edge}\lambda_Q \left\lfloor \frac{M}{N_Q T+1} \right\rfloor}. \tag{2.4.9}$$

For $M = N_Q T + 1$ we get

$$\frac{l(\Delta_\gamma)}{l(\gamma)} \leq \frac{2(N_Q T + 1)^2 C_{edge}}{\varepsilon_Q c_{edge}}. \tag{2.4.10}$$

For $M > N_Q T + 1$ we recall that

$$\frac{M}{N_Q T + 1} - 1 \leq \left\lfloor \frac{M}{N_Q T + 1} \right\rfloor. \tag{2.4.11}$$

Combining this with (2.4.9), we get

$$\frac{l(\Delta_\gamma)}{l(\gamma)} \le \frac{2(N_QT+1)MC_{edge}}{\varepsilon_Q c_{edge}\left(\frac{M}{N_QT+1}-1\right)} = \frac{2(N_QT+1)^2MC_{edge}}{\varepsilon_Q c_{edge}(M-N_QT-1)} \le$$
$$\le \frac{2(N_QT+1)^2(N_QT+2)C_{edge}}{\varepsilon_Q c_{edge}}. \quad (2.4.12)$$

In both cases the upper bounds are constants that depend only on the stratum.

If a saddle connection γ crosses $M < N_QT + 1$ triangles and is still contained in a thick component Q, then $l(\gamma) \ge c_{sc}\lambda_Q$. At the same time $l(\Delta_\gamma) \le 2MC_{edge}\lambda_Q$. Their ratio

$$\frac{l(\Delta_\gamma)}{l(\gamma)} \le \frac{2MC_{edge}\lambda_Q}{c_{sc}\lambda_Q} \le \frac{2N_QTC_{edge}}{c_{sc}}. \quad (2.4.13)$$

This upper bound is a constant depending only on the stratum.

Lemma 2.4.4. *Suppose that a segment γ of a saddle connection is crossing an expanding annulus E contained in the thick component Q. Then the ratio $l_{\omega_q}(\Delta_\gamma) \le Cl_{\omega_q}(\gamma)$ where C is some constant that depends only on the stratum.*

Proof. Once a saddle connection segment γ crosses a boundary component of E, it has to cross it, and as shown in Lemma 2.4.1 $l(\gamma) \ge C_a\lambda_2$, where C_a is some constant that depends only on the stratum while λ_2 is the size of the thick component Q_2.

Consider all the triangles of the regular triangulation whose interiors have nonzero intersection with E. The segment γ cannot intersect the any side of any of these triangles twice before it leaves the expanding annulus. Indeed, suppose there is aside that is intersected twice, first at the point p and then at the point q. Then let us build a closed curve by following gamma from p to q and then connecting p and q along the triangulation side they belong to. This creates a curve $\mathcal{A}$ consisting of two saddle connection segments constructed in exactly the same way as in Theorem 2.4.2. Since $\mathcal{A}$ is a simple closed curve contained in the expanding annulus, it has to be either homotopically trivial or homotopic to a boundary component of E. We have shown in the proof of Theorem 2.4.2 that such a curve cannot be homotopically trivial or homotopic to a boundary component of a thick component. So for a segment γ of a saddle connection crossing an expanding annulus, no triangle side whose interior has nonzero intersection with the interior of the annulus can be crossed twice.

Note that for each such triangle, the flat length of each of its sides is bounded by $C_{edge}\lambda_2$. So the length of the path Δ_γ along the triangle sides corresponding to the segment γ cannot be greater than $TC_{edge}\lambda_2$. This means that

$$\frac{l(\Delta_\gamma)}{l(\gamma)} \leq \frac{TC_{edge}\lambda_2}{C_a\lambda_2} = \frac{TC_{edge}}{C_a}$$

$\overline{.(2.4.14)}$

$\square$

Theorem 2.4.5. *Consider a cylinder whose boundary curves are boundary components of some subsurfaces of the thick-thin decomposition of our surface X. Suppose that a saddle connection segment γ is the intersection of some saddle connection with a flat cylinder. Then there exists a path β along the sides of the triangulation of our cylinder constructed above that is homotopic to γ and such that the ratio $\frac{l(\gamma)}{l(\beta)}$ is bounded from below by a constant that only depends on the height and length of the cylinder.*

Proof. Without loss of generality assume that the slope of γ is positive. Suppose that γ intersects AB at the point A_1, then goes through the cylinder and intersects CD at the point A_2. Assume that $A_1 \in S_kS_{k+1}$ and $A_2 \in P_lP_{l+1}$ (the indices here form a cycle, so if $k = n$, then we use 1 instead of $k+1$, and, similarly, when $l = m$, we use 1 instead of $m+1$).

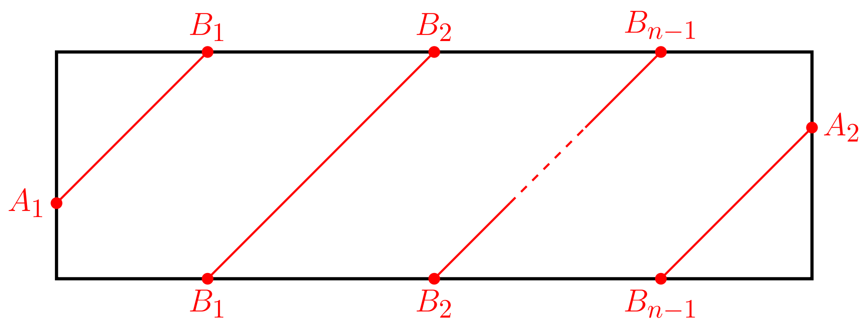

Figure 2.2: A saddle connection intersecting a cylinder.

Let us consider the intersection points B_1, B_2, $\ldots$, B_{n-1} of γ and the interval AD. The length of each of the intervals B_1B_2, B_2B_3 $\ldots$, $B_{n-2}B_{n-1}$ is equal to $\sqrt{\left(\frac{H}{n}\right)^2 + \Pi^2}$. So

$$l(\gamma) \geq (n-2)\sqrt{\left(\frac{H}{n}\right)^2 + \Pi^2}.$$

Now let us consider a path along the sides of the triangles in the triangulation built as follows. We go from the point S_k to the point A (which may not be a singularity, and thus may not be a vertex of the triangulation). Then we go from A back to itself along the boundary of the cylinder $n-2$ times. Then we take the point whose index is the smallest among the points $S_1, \ldots, S_n$ that are connected to P_{l+1} via a side of the triangulation, and go along that side across the cylinder. We now have a path β along the sides of the triangulation.

The length $l(\beta)$ of this path cannot be longer than

$$n\Pi + \sqrt{H^2 + \Pi^2}.$$

If $n > 2$, then the ratio

$$\frac{l(\gamma)}{l(\beta)} \geq \frac{(n-2)\sqrt{\frac{H^2}{n^2}+\Pi^2}}{n\Pi+\sqrt{\Pi^2+H^2}} \geq \frac{(n-2)\sqrt{\frac{H^2}{n^2}+\Pi^2}}{n\Pi+n\sqrt{\Pi^2+H^2}} >$$
$$> \frac{n-2}{n}\cdot\frac{\Pi}{\Pi+\sqrt{\Pi^2+H^2}} > \frac{1}{3}\cdot\frac{\Pi}{\Pi+\sqrt{\Pi^2+H^2}}. \tag{2.4.15}$$

If $n = 1, 2$, then

$$\frac{l(\gamma)}{l(\beta)} \geq \frac{H}{2\Pi+H}.$$

$\square$

Finally, we can combine all the estimates obtained in this section into the following theorem.

Theorem 2.4.6. *For any saddle connection γ there is a triangulation edge path Δ_γ that represents the same relative homology class as γ and satisfies $l_{\omega_q}(\Delta_\gamma) \leq C l_{\omega_q}(\gamma)$ for some constant C that depends only on the stratum.*

2.4.2 The Estimates on Full Homology Classes

Now we will prove our upper estimate on the AGY-norm of the basis relative cohomology classes.

Lemma 2.4.7. *The AGY-norm of the class $[\psi_i]$ is bounded from above:*

$$\|[\psi_i]\|^{AGY}_{qX} \leq \frac{C}{\lambda([\psi_i])}, \tag{2.4.16}$$

where C is a constant that depends only on the stratum and $\lambda([\psi_i])$ is the size of the thick-thin component that Ψ_i belongs to.

Proof. Let γ be an arbitrary saddle connection. Then it represents some relative homology class $[\gamma]$, which can be represented as a linear sum of triangulation classes:

$$[\gamma] = a_1[\psi_1] + a_2[\psi_2] + ... + a_t[\psi_t] + b_1[\theta_1] + b_2[\theta_2] + ... + b_s[\theta_s].$$

Then

$$\frac{|[\psi_i](\gamma)|}{l_{\omega_q}(\gamma)} = \frac{|a_i + b_1\delta_{1i} + b_2\delta_{2i} + ... + b_s\delta_{si}|}{l_{\omega_q}(\gamma)} \leq \frac{|a_i| + \sum_{\{j|\delta_{ji}\neq 0\}} |b_j\delta_{ji}|}{l_{\omega_q}(\gamma)}. \tag{2.4.17}$$

By Theorem 2.4.6 we know that there is a triangulation side path Δ_γ that represents the same homology class as γ and satisfies $l_{\omega_q}(\Delta_\gamma) \leq Cl_{\omega_q}(\gamma)$ for some constant C. So

$$\begin{aligned}\frac{|[\psi_i](\gamma)|}{l_{\omega_q}(\gamma)} &\leq \frac{C(|a_i| + \sum_{\{j|\delta_{ji}\neq 0\}} |b_j\delta_{ji}|)}{|a_i|l_{\omega_q}(\Psi_i) + |b_1|l_{\omega_q}(\Theta_1) + |b_2|l_{\omega_q}(\Theta_2) + ... + |b_s|l_{\omega_q}(\Theta_s)} \leq \\ &\leq \frac{C(|a_i| + \sum_{\{j|\delta_{ji}\neq 0\}} |b_j\delta_{ji}|)}{|a_i|l_{\omega_q}(\Psi_i) + \sum_{\{j|\delta_{ji}\neq 0\}} |b_j|l_{\omega_q}(\Theta_j)} \leq \\ &\leq \frac{C\max(1, |\delta_{1i}|, |\delta_{2i}|, ..., |\delta_{si}|)(|a_i| + \sum_{\{j|\delta_{ji}\neq 0\}} |b_j|)}{|a_i|l_{\omega_q}(\Psi_i) + \sum_{\{j|\delta_{ji}\neq 0\}} |b_j|l_{\omega_q}(\Theta_j)}.\end{aligned} \tag{2.4.18}$$

Note that by the choice of our basis the flat length of each Θ_j with $\delta_{ji} \neq 0$ cannot be smaller than the flat length of Ψ_i. Thus

$$\begin{aligned}\frac{|[\psi_i](\gamma)|}{l_{\omega_q}(\gamma)} &\leq \frac{C\max(1, |\delta_{1i}|, |\delta_{2i}|, ..., |\delta_{si}|)(|a_i| + \sum_{\{j|\delta_{ji}\neq 0\}} |b_j|)}{l_{\omega_q}(\Psi_i)(|a_i| + \sum_{\{j|\delta_{ji}\neq 0\}} |b_j|)} = \\ &= \frac{C\max(1, |\delta_{1i}|, |\delta_{2i}|, ..., |\delta_{si}|)}{l_{\omega_q}(\Psi_i)}.\end{aligned} \tag{2.4.19}$$

By (2.3.1) we know that $l_{\omega_q}(\Psi_i)$ is comparable to the size of the thick-thin component Q that the interior of Ψ_i belongs to, so

$$\frac{|[\psi_i](\gamma)|}{l_{\omega_q}(\gamma)} \leq \frac{C \max(1, |\delta_{1i}|, |\delta_{2i}|, ..., |\delta_{si}|)}{c\lambda([\psi_i])}. \tag{2.4.20}$$

Note that the final upper estimate does not depend on the saddle connection γ, so it holds true for all saddle connections and thus for supremum over all saddle connections. □

2.5 The Masur-Veech Volume Estimate

2.5.1 The Norm Estimates

From now on we will omit the surface X and write the point (X, q) simply as q or q_X. We will also write T_{q_X} for the tangent space $T_{q_X}\mathcal{QT}_{g,n}(\kappa)$.

For a vector $v \in T_{q_X}$ we can look at the path γ starting from q_X which goes along the direction of v ($\gamma'(t) = v$ at all times t). This path is defined for small values of t, but may be undefined for $t = 1$. For those vectors whose path is well-defined on the interval $[0, 1]$, let us define the map Ψ by setting $\Psi(v) = \gamma(1)$.

Let $B_{q_X}(0, r)$ be the ball of vectors in the tangent space T_{q_X} whose AGY-norm at the point q_X does not exceed r. Let $\mathcal{B}_{q_X}(0, r)$ be the ball of all points q_Y from the space such that the AGY-distance from q_X to q_Y does not exceed r.

The following properties, established in [AG13], show that the AGY-norm varies in a controlled way in neighbourhoods of a fixed radius.

Proposition 2.5.1 (Avila–Gouëzel, [AG13], also see [AHC25])**.**

1. *The map Ψ is well-defined on $B_{q_X}(0, \frac{1}{2})$.*

2. For any $v \in B_{q_X}(0, \frac{1}{2})$

$$d_{AGY}(q_X, \Psi(v)) \leq 2 \|v\|_{q_X}^{AGY}.$$

3. For any $v \in B_{q_X}(0, \frac{1}{2})$ *and any* $w \in T_X$

$$\frac{1}{2} \leq \frac{\|w\|_{q_X}^{AGY}}{\|w\|_{\Psi(v)}^{AGY}} \leq 2.$$

4. For any $v \in B_{q_X}(0, \frac{1}{25})$

$$\frac{\|v\|_{q_X}^{AGY}}{2} \leq d_{AGY}(q_X, \Psi(v)).$$

Proposition 2.5.1 allows us to prove the following lemma.

Lemma 2.5.2. *Let* $q_X \in \mathcal{QT}_{g,n}(\kappa)$*, and let* q_Y *and* q_Z *be two points from* $\mathcal{U}$ *whose images under the identification* i_Φ *belong to* $B_{q_X}(0, \frac{1}{100})$*. Then*

$$\frac{1}{4} \|\Phi(q_Z) - \Phi(q_Y)\|_{q_X}^{AGY} \leq d_{AGY}(q_Y, q_Z) \leq 2 \|\Phi(q_Z) - \Phi(q_Y)\|_{q_X}^{AGY}. \tag{2.5.1}$$

Proof. The difference $v_{YZ} = i_\Phi(q_Z) - i_\Phi(q_Y)$ is a vector that belongs to $B_{q_X}(0, \frac{1}{50})$. Consider the path γ from $i_\Phi(q_Y)$ to $i_\Phi(q_Z)$ that follows the vector v_{YZ}

$$\gamma : [0, 1] \to T_{q_X}$$

$$t \mapsto i_\Phi(q_Y) + t(i_\Phi(q_Z) - i_\Phi(q_Y)).$$

Since balls in normed spaces are convex, the entire image of γ (which we will also call γ) also belongs to $B_{q_X}(0, \frac{1}{50})$. Note that if we take the preimage of γ under i_ϕ, we will get a path from q_Y to q_Z in $\mathcal{QT}_{g,n}(\kappa)$. We will call that path γ as well. Note that $\gamma(t)$ viewed as a point in $\mathcal{QT}_{g,n}(\kappa)$ is equal to $\Psi(\gamma(t))$ for $\gamma(t)$ viewed as an element of T_{q_X}. Thus by

Proposition 2.5.1 (c) applied to $w = v_{YZ}$ we get

$$\frac{1}{2} \|v_{YZ}\|_{q_X}^{AGY} \leq \|v_{YZ}\|_{\gamma(t)}^{AGY} \leq 2 \|v_{YZ}\|_{q_X}^{AGY}.$$

The AGY-length of the path γ

$$l_{AGY}(\gamma) = \int_0^1 \|v_{YZ}\|_{\gamma(t)}^{AGY} \, dt$$

is bounded from above and below:

$$\frac{1}{2} \|v_{YZ}\|_{q_X}^{AGY} \leq l_{AGY}(\gamma) \leq 2 \|v_{YZ}\|_{q_X}^{AGY}.$$

This means that

$$d_{AGY}(q_Y, q_Z) \leq 2 \|v_{YZ}\|_{q_X}^{AGY}.$$

So we get the first part of the inequality. To obtain the second part we first apply Proposition 2.5.1 (d) to v_{YZ} at the point q_Y (we can do this since $v_{YZ} \in B_{q_X}(0, \frac{1}{50})$, so another application of (c) gives $v_{YZ} \in B_{q_Y}(0, \frac{1}{25})$):

$$\frac{1}{2} \|v_{YZ}\|_{q_Y}^{AGY} \leq d_{AGY}(q_Y, q_Z).$$

By Proposition 2.5.1 (c)

$$\frac{1}{2} \|v_{YZ}\|_{q_X}^{AGY} \leq \|v_{YZ}\|_{q_Y}^{AGY},$$

and combining the two inequalities above into

$$\frac{1}{4} \|v_{YZ}\|_{q_X}^{AGY} \leq d_{AGY}(q_Y, q_Z).$$

Finally, we note that $\Phi(q_Z) - \Phi(q_Y)$ is the same vector as v_{YZ}, which completes the proof. $\square$

The following corollary follows immediately from Lemma 2.5.2.

Corollary 2.5.3. *For any* $q_X \in \mathcal{QT}_{g,n}(\kappa)$*, if* $q_Y \in \mathcal{B}_{q_X}(0, \frac{1}{100})$*, then the vector* $v_{XY} \in B_{q_X}(0, \frac{1}{25})$*.*

Proof. $q_Y \in \mathcal{B}_{q_X}(0, \frac{1}{100})$ means that $d_{AGY}(q_X, q_Y) < \frac{1}{100}$, so by the first inequality in (2.5.1) we get

$$\|v_{XY}\|_{q_X}^{AGY} \leq \frac{1}{25}.$$

□

We now recall the period box construction build by Eskin, Mirzakhani and Mohammadi in [EMM22].

For a point $X \in \mathcal{Q}_1\mathcal{T}_{g,n}(\kappa)$ choose some period map Φ providing local period coordinates. In those period coordinates, we can consider a box

$$\mathrm{Box}_r(X) = \{\Phi(X) + v : ||v||_{AGY,x} \leq r\}. \tag{2.5.2}$$

Then set

$$B_r(X) = \Phi^{-1}\mathrm{Box}_r(X). \tag{2.5.3}$$

It follows from Corollary 2.5.3, Lemma 2.5.2 and Proposition 2.5.1 that for every such vector the expression $\Phi(X) + v$ actually defines some nearby surface in the same period chart, and that we can treat AGY balls of a sufficiently small fixed radius by means of a single period-coordinate chart uniformly across the stratum.

Lemma 2.5.4. *Let* $v \in B_{\omega_X}(0, \frac{1}{25})$*, and let* $v = c_1[\psi_1] + c_2[\psi_2] + ... + c_t[\psi_t]$ *be its decomposition with respect to the cohomology basis* $[\psi_1]$*,* $[\psi_2]$*,...,*$[\psi_t]$*. Let* Ψ_i *be the side of the regular triangulation that represents the homology class dual to the cohomology class* $[\psi_i]$*. If* Ψ_i *has no intersection with the interior of some thin cylinder, then the for the corresponding coefficient* $|c_i|$ *we have*

$$|c_i| \leq \frac{\mathfrak{M}}{25}, \tag{2.5.4}$$

where $\mathfrak{M}$ is a constant that depends only on the stratum.

Proof. First, let us show that we get the upper bound of this type if $c_1 = 1$. By definition of the AGY-norm we have

$$\|v\|_{\omega_X}^{AGY} = \sup_{\gamma} \frac{|v(\gamma)|}{|[\omega](\gamma)|},$$

where γ runs over all saddle connections in (X, ω). This means that

$$\|v\|_{\omega_X}^{AGY} \geq \frac{|v(\Psi_1)|}{|[\omega](\Psi_1)|},$$

where Ψ_1 is the side of our triangulation that represents the homology class dual to the cohomology class $[\psi_1]$. Note that

$$\frac{|v(\Psi_1)|}{|[\omega](\Psi_1)|} = \frac{1}{l_\omega(\Psi_1)}.$$

If Ψ_1 is a side in our triangulation, whose interior is contained in some thick component Q_1 its flat length is comparable to the size of this thick component:

$$\underline{C}\lambda_\omega(Q_1) \leq l_\omega(\Psi_1) \leq \overline{C}\lambda_\omega(Q_1) \tag{2.5.5}$$

for some constants $\underline{C}$, $\overline{C}$ that depend only on the stratum. The size $\lambda_\omega(Q_1)$ is in turn bounded from above by $\mathfrak{C}$ by Lemma 2.2.34.

If Ψ_1 is a saddle connection that is part of the flat-geodesic representative of a short curve, then

$$l_\omega(\Psi_1) < \mathfrak{E}.$$

In either case we have that

$$\|v\|_{\omega_X}^{AGY} \geq \frac{1}{l_\omega(\Psi_1)} > \frac{1}{\mathfrak{M}}, \tag{2.5.6}$$

where $\mathfrak{M} = \max\{\overline{C}\mathfrak{C}, \mathfrak{E}\}$.

For a vector $v \in B_{\omega_X}(0, \frac{1}{25})$ with $c_1 \neq 0$ we have

$$\frac{1}{25} > \|c_1[\psi_1] + c_2[\psi_2] + ... + c_t[\psi_t]\|_{q_X}^{AGY} = |c_1| \left\| [\psi_1] + \frac{c_2}{c_1}[\psi_2] + ... + \frac{c_t}{c_1}[\psi_t] \right\|_{q_X}^{AGY} \geq \frac{|c_1|}{\mathfrak{M}}.$$

To sum this up,

$$|c_1| \leq \frac{\mathfrak{M}}{25}. \tag{2.5.7}$$

Analogous statements are true for $c_2, \ldots, c_t$. This allows to conclude that the statement of the lemma holds for each $|c_i|$ satisfying its conditions.

$\square$

An analogous proof provides a similar estimate for quadratic differentials.

Lemma 2.5.5. *If a vector $v \in B_{q_X}(0, \frac{1}{25})$, then the absolute value of each of the coefficients c_1, c_2, $...c_\nu$ of the basis vectors $[\Delta_{s_1}]$, $[\Delta_{s_2}]$,...,$[\Delta_{s_\nu}]$ in its decomposition $v = c_1[\Delta_{s_1}] + c_2[\Delta_{s_2}] + ... + c_\nu[\Delta_{s_\nu}]$ is bounded from above by a constant that depends only on the stratum.*

2.5.2 The Volume Estimate

Let $\Phi_1(\omega) = A_1(\omega) + iB_1(\omega), \ldots, \Phi_d(\omega) = A_d(\omega) + iB_d(\omega)$ be the periods of our fixed flat surface (X, ω), and let $\{A_1, \ldots, A_d, B_1, \ldots, B_d\}$ be the real coordinates on the corresponding period chart.

Theorem 2.5.6. *The Masur-Veech volume of $\mathcal{B}(\omega, \frac{1}{100})$ is bounded from above by*

$$Const \cdot \prod_{i=1}^{d} |\Phi_i(\omega)|^2, \tag{2.5.8}$$

where $Const$ depends only on the stratum.

By the estimates found in the previous section, we can conclude that $\mathcal{B}(\omega, \frac{1}{100}) \subset \mathcal{D}$, where $\mathcal{D}$ is the polydisc

$$\mathcal{D} = \left\{ \{A_1(\omega) + A_1, \ldots, A_d(\omega) + A_d, B_1(\omega) + B_1, \ldots, B_d(\omega) + B_d\} : |A_i + iB_i| < |c_i| \right\}. \tag{2.5.9}$$

Clearly, $\mathcal{D} \subset \mathcal{K}$, where $\mathcal{K}$ is the box

$$\mathcal{K} = \left\{ \{A_1(\omega) + A_1, \ldots, A_d(\omega) + A_d, B_1(\omega) + B_1, \ldots, B_d(\omega) + B_d\} : |A_i| < |c_i|, B_i < |c_i| \right\}. \tag{2.5.10}$$

Let $\mathcal{A}(a)$ be the subset of the surfaces in the stratum of (X, ω) with area a. Let $\mathcal{D}(1) = \mathcal{D} \cap \mathcal{A}(1)$ and $\mathcal{K}(1) = \mathcal{K} \cap \mathcal{A}(1)$. Clearly, $\mathcal{D}(1) \subset \mathcal{K}(1)$. We will provide an upper estimate on the Masur-Veech volume of $\mathcal{K}(1)$. Let

$$\mathcal{C}(\delta) = \{\sqrt{a}w : w \in \mathcal{K}(1), a \in [1-\delta, 1+\delta]\}, \tag{2.5.11}$$

where δ is a positive number considerably smaller than 1. Since the flat surfaces we are considering are always orientable, we can choose a consistent orientation to pick sides of triangles with positive cross-products. The area formula thus becomes

$$A = \sum_{i,j=1}^{d} \varkappa_{ij} A_i B_j, \tag{2.5.12}$$

where the coefficients $\varkappa_{ij}$ are integers constant within a fixed period chart and bounded from above by a constant that depends only on the number of triangles in the triangulation. For the surfaces whose area is contained in the interval $[1-\delta, 1+\delta]$ we have

$$\sum_{i,j=1}^{d} \varkappa_{ij} A_i B_j \geq 1 - \delta. \tag{2.5.13}$$

This means that there exists an index $k \in \{1, \cdots, d\}$ such that

$$\sum_{i=1}^{d} \varkappa_{ik} A_i B_k \geq \frac{1-\delta}{d}. \tag{2.5.14}$$

In particular, the partial derivative $\frac{\partial A}{\partial B_k} = \sum_{i=1}^{d} \varkappa_{ik} A_i$ is nonzero, and the standard Euclidean volume form $dV = dA_1 dB_1 \cdots dA_d dB_d$ can be rewritten as

$$dV = \frac{1}{|\sum_{i=1}^{d} \varkappa_{ik} A_i|} dA_1 dB_1 \cdots dA_k dA \cdots dA_d dB_d. \tag{2.5.15}$$

Changing the area variable to the cone radius R we get that this form can be rewritten as

$$dV = \frac{2R}{|\sum_{i=1}^{d} \varkappa_{ik} A_i|} dA_1 dB_1 \cdots dA_k dR \cdots dA_d dB_d. \tag{2.5.16}$$

Then we renormalize the variables according to R by transitioning to a new set of coordinates $\widehat{A_i}$, $\widehat{B_j}$ chosen in such a way that $A_i = R\widehat{A_i}$ for $i \in \{1, \cdots, d\}$ and $B_j = R\widehat{B_j}$ for $j \in \{1, \cdots, d\} \setminus \{k\}$. The Jacobian of this transition is R^{2d-1}, so the standard volume form is rewritten as

$$dV = \frac{2R^{2d-1}}{|\sum_{i=1}^{d} \varkappa_{ik} \widehat{A_i}|} d\widehat{A_1} d\widehat{B_1} \cdots d\widehat{A_k} dR \cdots d\widehat{A_d} d\widehat{B_d}. \tag{2.5.17}$$

For each $k \in \{1, \cdots, d\}$ let $\mathcal{K}_k(1)$ be the subset of $\mathcal{K}(1)$ on which equation (2.5.14) holds for

k. Let $\mathcal{C}_k(\delta)$ be the corresponding cone:

$$\mathcal{C}_k(\delta) = \{\sqrt{a}w : w \in \mathcal{K}_k(1), a \in [1-\delta, 1+\delta]\}. \tag{2.5.18}$$

Lemma 2.5.7. *The Masur-Veech volume of $\mathcal{C}_k(\delta)$ is bounded from above by*

$$\frac{26}{25^{2d}} \cdot \frac{(1+\delta)((1+\delta)^d - (1-\delta)^d)}{1-\delta} \cdot \prod_{i=1}^{d} |\Phi_i(\omega)|^2.$$

Proof. By the definition of $\mathcal{C}_k(\delta)$ the volume form on this set can be rewritten as in equation (2.5.17). Combing this with the restrictions on the coordinates of the points contained in $\mathcal{K}$ we get

$$\frac{1}{|\sum_{i=1}^{d} \varkappa_{ik}\widehat{A_i}|} \leq \frac{d \cdot |B_k|}{1-\delta} \leq \frac{d \cdot (1+\delta)(\mathcal{B}_k(\omega) + |c_k|)}{1-\delta} \tag{2.5.19}$$

and obtain the following estimate on the volume of $\mathcal{C}_k(\delta)$:

$$\int_{\mathcal{C}_k(\delta)} dV \leq \int_{\mathcal{C}_k(\delta)} \frac{2d \cdot (1+\delta)(\mathcal{B}_k(\omega) + |c_k|)R^{2d-1}}{1-\delta} d\widehat{A_1}d\widehat{B_1} \cdots d\widehat{A_k}dR \cdots \widehat{dA_d}d\widehat{B_d}. \tag{2.5.20}$$

Since the R-coordinate of points that belong to $\mathcal{C}_k(\delta)$ varies between $\sqrt{1-\delta}$ and $\sqrt{1+\delta}$, while each of the other coordinates cannot go outside of the bounds of the cube $\mathcal{K}$, the integral in (2.5.20) is in turn bounded:

$$\int_{\mathcal{C}_k(\delta)} \frac{2d\cdot(1+\delta)(\mathcal{B}_k(\omega)+|c_k|)R^{2d-1}}{1-\delta} d\widehat{A_1}d\widehat{B_1}\cdots d\widehat{A_k}dR\cdots d\widehat{A_d}d\widehat{B_d} \leq$$

$$\leq \int_{B_d(\omega)-|c_d|}^{B_d(\omega)+|c_d|}\int_{A_d(\omega)-|c_d|}^{A_d(\omega)+|c_d|}\cdots\int_{\sqrt{1-\delta}}^{\sqrt{1+\delta}}\int_{A_k(\omega)-|c_k|}^{A_k(\omega)+|c_k|}\cdots\int_{B_1(\omega)-|c_1|}^{B_1(\omega)+|c_1|}\int_{A_1(\omega)-|c_1|}^{A_1(\omega)+|c_1|}\frac{2d\cdot(1+\delta)(\mathcal{B}_k(\omega)+|c_k|)R^{2d-1}}{1-\delta}d\widehat{A_1}d\widehat{B_1}\cdots d\widehat{A_k}dR\cdots d\widehat{A_d}d\widehat{B_d} =$$

$$= \frac{(1+\delta)((1+\delta)^d-(1-\delta)^d)}{1-\delta}\cdot|c_1|^2\ldots|c_k|(|\mathcal{B}_k(\omega)|+|c_k|)\ldots|c_d|^2 \leq$$

$$\leq \frac{26}{25^{2d}}\cdot\frac{(1+\delta)((1+\delta)^d-(1-\delta)^d)}{1-\delta}\cdot\prod_{i=1}^{d}|\Phi_i(\omega)|^2. \quad (2.5.21)$$

□

Since $\mathcal{C}(\delta)\subset\bigcup\limits_{i=1}^{d}\mathcal{C}_i(\delta)$ and $\lim\limits_{\delta\to 0}\frac{(1+\delta)((1+\delta)^d-(1-\delta)^d)}{(1-\delta)\delta}=2d$, the Masur-Veech volume

$$V(\mathcal{D}(1)) \leq \frac{52d^2}{25^{2d}}\prod_{i=1}^{d}|\Phi_i(\omega)|^2. \quad (2.5.22)$$

An analogue of Theorem 2.5.6 is true for quadratic differentials.

Theorem 2.5.8. *The Masur-Veech volume of $\mathcal{B}(q_X,\frac{1}{100})$ is bounded from above by*

$$Const\cdot\prod_{i=1}^{d}|\Phi_i(\omega)|^2, \quad (2.5.23)$$

where $Const$ depends only on the stratum.

Proof. Note that in Section 2.3.2 we constructed a basis of $H_1(\tilde{X},\Sigma,\mathbb{Z})$ consisting of such vectors that each either belongs to $H_1^-(\tilde{X},\Sigma,\mathbb{Z})$ or can be paired with its image under the canonical involution σ that is also an element of the basis. In the second case, if a basis vector V has period $A_i + iB_i$, then $\sigma(V)$ has period $-A_i - iB_i$, so the corresponding basis vector in $H_1^-(\tilde{X}, \Sigma, \mathbb{Z})$ has period $2A_i + 2iB_i$. Thus, the area formula similar to (2.5.12) holds in the case of quadratic differentials as well:

$$A = \sum_{i,j=1}^{\dim H_1^-(\tilde{X},\Sigma,\mathbb{Z})} \varkappa_{ij} A_i B_j. \tag{2.5.24}$$

Since the double-cover area is normalized to 2, a full analogue of the proof of Theorem 2.5.6 gives the proof of Theorem 2.5.8. □

Note that the estimate expressions in Theorem 2.5.6 and Theorem 2.5.8 are uniformly bounded from above if the thick-thin decomposition of (X, ω) or $(\tilde{X}, \omega_q)$ has no thin cylinders as its components.

2.6 The Thin Cylinder Case

In this section we will prove that if $(\tilde{X}, \omega_q) \in \mathcal{PQ}_1\mathcal{T}_{g,n}$, then expression (2.5.23) is uniformly bounded even if its thick-thin decomposition does contain thin cylinder components. Note that by construction of our triangulation the only saddle connections that could produce periods of unbounded length are the the ones that connect a pair of singularities that belong to two different boundaries of a thin cylinder (we will say that such saddle connections *cross* the cylinder).

Lemma 2.6.1. *Suppose that S_{CYL} is the shortest saddle connection that crosses a thin cylinder* Cyl. *Suppose that S is a different saddle connection, whose flat length is smaller than the circumference circ of* Cyl. *Let $\Phi(S_{CYL})$ and $\Phi(S)$ be the corresponding periods. Then the product $|\Phi(S_{CYL})| \cdot |\Phi(S)|$ is uniformly bounded from above.*

Proof. Note that $|\Phi(S_{\text{CYL}})| \leq \sqrt{\frac{(circ)^2}{4} + h^2}$, where h is the height of CYL. The product

$$|\Phi(S_{\text{CYL}})| \cdot |\Phi(S)| \leq \sqrt{\frac{(circ)^4}{4} + (circ)^2 h^2}.$$

The first summand inside the square root is uniformly bounded from above since CYL is thin. The second one is uniformly bounded from above by the restriction on the total area of the surface. □

Proposition 2.6.2. *Consider a thin cylinder* CYL *and suppose that* S_{CYL} *is the shortest* CYL*-crossing saddle connection. By construction of the extension of the regular triangulation to thin cylinders,* S_{CYL} *is a triangulation edge. Then the relative homology class of any other* CYL*-crossing triangulation edge is linearly dependent on* $[S_{\text{CYL}}]$ *and the classes of the saddle connections that belong to the boundaries of* CYL.

Proof. This follows immediately from the relations induced by the triangles on the relative homology classes represented by their sides. □

So in full relative homology each thin cylinder CYL can produce at most one independent CYL-crossing basis triangulation edge. We will now prove that for each basis edge of this kind, we will be able to find another basis edge, whose length is smaller than the circumference of CYL, and that the number of such basis edges would be sufficient to provide a separate short edge for each thin cylinder-crossing long one.

Given two saddle connections $\boldsymbol{\gamma}_1$ and $\boldsymbol{\gamma}_2$ on a half-translation surface (X, q), one could ask if they correspond to the same class in $H_1(\tilde{X}, \Sigma, \mathbb{Z})$. Such saddle connections were named $\hat{h}$*omologous* by Masur and Zorich in [MZ08]. The precise definition they introduced is the following.

Definition 2.6.3. Let $\boldsymbol{\gamma}$ be a saddle connection on (X, q). Note that its lifts to the orienting double-cover are naturally swapped by the canonical involution σ. Let us call them $\tilde{\boldsymbol{\gamma}}$ and

$\sigma(\tilde{\boldsymbol{\gamma}})$. Let $[\tilde{\boldsymbol{\gamma}}]$ and $[\sigma(\tilde{\boldsymbol{\gamma}})]$, as usual, be the corresponding classes in $H_1(\tilde{X}, \Sigma, \mathbb{Z})$. Define

$$[\hat{\boldsymbol{\gamma}}] = \begin{cases} [\tilde{\boldsymbol{\gamma}}] \text{ if } [\sigma(\tilde{\boldsymbol{\gamma}})] = -[\tilde{\boldsymbol{\gamma}}]; \\ [\tilde{\boldsymbol{\gamma}}] - [\sigma(\tilde{\boldsymbol{\gamma}})] \text{ otherwise }. \end{cases} \tag{2.6.1}$$

Two saddle connections $\boldsymbol{\gamma}_1$ and $\boldsymbol{\gamma}_2$ on (X, q) are $\hat{h}$omologous if $[\hat{\boldsymbol{\gamma}_1}] = [\hat{\boldsymbol{\gamma}_2}]$ in $H_1(\tilde{X}, \Sigma, \mathbb{Z})$ under an appropriate choice of their orientations.

Masur and Zorich studied maximal collections of pairwise $\hat{\text{h}}$omologous saddle connections on moduli spaces of quadratic differentials. In particular, they establish the following.

Property 2.6.4 (**Masur, Zorich**, [MZ08])**.** For almost every (with respect to the Masur-Veech measure) flat surface in any moduli space stratum $\mathcal{Q}_1\mathcal{M}_{g,n}(\kappa)$, two saddle connections are parallel if and only if they are $\hat{h}$omologous.

If $\mathcal{M}_0$ is the subset of $\mathcal{PQ}_1\mathcal{M}_{g,n}$ for which Proposition 2.6.4 fails, then the preimage $\mathcal{T}_0$ of $\mathcal{M}_0$ in $\mathcal{PQ}_1\mathcal{T}_{g,n}$ also has zero Masur-Veech volume. For any maximal cylinder on a half-translation surface (X, q), all the saddle connections that belong to its boundary are parallel. It follows immediately that they have to be $\hat{\text{h}}$omologous for almost every surface.

Corollary 2.6.5. *Suppose that $(X, q) \in \mathcal{PQ}_1\mathcal{T}_{g,n} \setminus \mathcal{T}_0$. Then for any maximal cylinder on (X, q) any two saddle connections that belong to its boundary are pairwise $\hat{h}$omologous.*

The geometric types of possible collections of $\hat{\text{h}}$omologous saddle connections are called *configurations* ([EMZ03], [MZ08]). They describe the combinatorial geometry of the complements of such collections in (X, q). In particular, the possible configurations that can arise in strata of quadratic differentials with odd-degree singularities only were described by Goujard in [Gou15].

Theorem 2.6.6 (**Goujard**, [Gou15])**.** *Let $(X, q) \in \mathcal{PQ}_1\mathcal{T}_{g,n} \setminus \mathcal{T}_0$. Let $\boldsymbol{\Gamma} = \{\boldsymbol{\gamma_1}, \cdots \boldsymbol{\gamma_s}\}$ be a collection of $\hat{h}$omologous saddle connections on (X, q). Then the complement $X \setminus \boldsymbol{\Gamma}$ contains* *at most one cylinder. Moreover, its boundary saddle connections can be described by one of the*

(i) each boundary component consists of a single saddle connection loop;

(ii) each boundary component consists of a pair of saddle connections of equal length;

(iii) one boundary component consists of a single saddle connection loop while the other consists of a pair of saddle connections of equal length.

Lemma 2.6.7. *Suppose that a* $\mathrm{C{\small YL}}$ *is a cylinder on* (X, q)*. Then if it has a single cylinder* $\widetilde{\mathrm{C{\small YL}}}$ *as its preimage on the orienting double cover, then the core of* $\widetilde{\mathrm{C{\small YL}}}$ *is an odd homology class in* $H_1(\widetilde{X}, \widetilde{\Sigma}, \mathbb{C})$*.*

Proof. Let $\sigma : \widetilde{X} \to \widetilde{X}$ be the canonical involution of the orienting double cover. Since $\widetilde{\mathrm{C{\small YL}}} = \pi^{-1}(\mathrm{C{\small YL}})$ is connected by assumption, σ preserves $\widetilde{\mathrm{C{\small YL}}}$ as a subset. Hence $\sigma(\widetilde{\gamma})$ is again a core curve of the same cylinder $\widetilde{\mathrm{C{\small YL}}}$.

Since $\widetilde{\mathrm{C{\small YL}}}$ is an annulus, any two oriented core curves represent either the same or the opposite homology class. Therefore

$$\sigma_*[\widetilde{\gamma}] = \pm[\widetilde{\gamma}].$$

On the other hand, by definition of the orienting double cover, one has

$$\sigma^*\omega = -\omega.$$

Hence

$$\int_{\sigma(\widetilde{\gamma})} \omega = \int_{\widetilde{\gamma}} \sigma^*\omega = -\int_{\widetilde{\gamma}} \omega.$$

Suppose for contradiction that

$$\sigma_*[\widetilde{\gamma}] = [\widetilde{\gamma}].$$

Then $\sigma(\widetilde{\gamma})$ and $\widetilde{\gamma}$ are homologous, so since ω is closed, their periods must agree:

$$\int_{\sigma(\widetilde{\gamma})} \omega = \int_{\widetilde{\gamma}} \omega.$$

Combining this with the previous identity gives

$$\int_{\widetilde{\gamma}} \omega = -\int_{\widetilde{\gamma}} \omega,$$

so

$$\int_{\widetilde{\gamma}} \omega = 0.$$

But $\widetilde{\gamma}$ is the core curve of a genuine cylinder on the translation surface $(\widetilde{X}, \omega)$, so its ω-period is exactly the holonomy vector of that cylinder, whose norm is the circumference of the cylinder. In particular,

$$\int_{\widetilde{\gamma}} \omega \neq 0,$$

a contradiction. Therefore

$$\sigma_*[\widetilde{\gamma}] = -[\widetilde{\gamma}],$$

so $[\widetilde{\gamma}]$ is odd.

It remains to show that this class is nonzero. If $[\widetilde{\gamma}] = 0$ in $H_1(\widetilde{X}; \mathbb{C})$, then again, since ω is closed,

$$\int_{\widetilde{\gamma}} \omega = 0,$$

contrary to the argument above. Thus $[\widetilde{\gamma}] \neq 0$ in absolute homology.

Finally, since $\widetilde{\Sigma}$ is a finite set, the natural map

$$H_1(\widetilde{X}; \mathbb{C}) \hookrightarrow H_1(\widetilde{X}, \widetilde{\Sigma}; \mathbb{C})$$

is injective. Hence $[\widetilde{\gamma}]$ is also nonzero in relative homology. This proves the lemma. $\square$

Lemma 2.6.8. *Suppose that* $\textsc{Cyl}$ *is a cylinder on* (X, q)*, and suppose that its full preimage in the orienting double cover*

$$\pi : (\widetilde{X}, \omega) \to (X, q)$$

consists of two cylinders $\widetilde{\textsc{Cyl}}_1$ *and* $\widetilde{\textsc{Cyl}}_2$*. Let* $\sigma : \widetilde{X} \to \widetilde{X}$ *be the canonical involution, let* $\widetilde{\gamma}_1$ *be an oriented core curve of* $\widetilde{\textsc{Cyl}}_1$*, and let*

$$\widetilde{\gamma}_2 := \sigma(\widetilde{\gamma}_1),$$

viewed as an oriented core curve of $\widetilde{\textsc{Cyl}}_2$*. Then the class*

$$[\widetilde{\gamma}_1] - [\widetilde{\gamma}_2] \in H_1(\widetilde{X}, \widetilde{\Sigma}; \mathbb{C})$$

is nonzero and odd.

Proof. The involution σ, preserves the full preimage $\pi^{-1}(\textsc{Cyl})$. As the preimage consists of exactly two cylinders, namely $\widetilde{\textsc{Cyl}}_1$ and $\widetilde{\textsc{Cyl}}_2$, the involution exchanges them:

$$\sigma(\widetilde{\textsc{Cyl}}_1) = \widetilde{\textsc{Cyl}}_2, \qquad \sigma(\widetilde{\textsc{Cyl}}_2) = \widetilde{\textsc{Cyl}}_1.$$

Hence $\widetilde{\gamma}_2 = \sigma(\widetilde{\gamma}_1)$ is a core curve of $\widetilde{\textsc{Cyl}}_2$, and

$$\sigma_*[\widetilde{\gamma}_1] = [\widetilde{\gamma}_2], \qquad \sigma_*[\widetilde{\gamma}_2] = [\widetilde{\gamma}_1].$$

Therefore

$$\sigma_*\big([\widetilde{\gamma}_1] - [\widetilde{\gamma}_2]\big) = [\widetilde{\gamma}_2] - [\widetilde{\gamma}_1] = -\big([\widetilde{\gamma}_1] - [\widetilde{\gamma}_2]\big),$$

so the class $[\widetilde{\gamma}_1] - [\widetilde{\gamma}_2]$ is odd.

It remains to show that it is nonzero. By definition of the orienting double cover,

$$\sigma^*\omega = -\omega.$$

Hence

$$\int_{\widetilde{\gamma}_2} \omega = \int_{\sigma(\widetilde{\gamma}_1)} \omega = \int_{\widetilde{\gamma}_1} \sigma^*\omega = -\int_{\widetilde{\gamma}_1} \omega.$$

Suppose for contradiction that

$$[\widetilde{\gamma}_1] - [\widetilde{\gamma}_2] = 0$$

in absolute homology. Then $[\widetilde{\gamma}_1] = [\widetilde{\gamma}_2]$, so since ω is closed, their periods must agree:

$$\int_{\widetilde{\gamma}_1} \omega = \int_{\widetilde{\gamma}_2} \omega.$$

Combining this with the previous identity gives

$$\int_{\widetilde{\gamma}_1} \omega = -\int_{\widetilde{\gamma}_1} \omega,$$

hence

$$\int_{\widetilde{\gamma}_1} \omega = 0.$$

But $\widetilde{\gamma}_1$ is the core curve of a genuine cylinder on the translation surface $(\widetilde{X}, \omega)$, so its ω-period is the holonomy vector of that cylinder, whose norm is the circumference of the cylinder. In particular,

$$\int_{\widetilde{\gamma}_1} \omega \neq 0,$$

so this is a contradiction. Therefore

$$[\widetilde{\gamma}_1] - [\widetilde{\gamma}_2] \neq 0$$

in absolute homology.

Finally, the natural map

$$H_1(\widetilde{X};\mathbb{C}) \hookrightarrow H_1(\widetilde{X},\widetilde{\Sigma};\mathbb{C})$$

is injective, so the same class is nonzero in relative homology. □

Theorem 2.6.9. *Consider a collection of disjoint maximal cylinders on $(X,q) \in \mathcal{PQ}_1\mathcal{T}_{g,n} \setminus \mathcal{T}_0$. Then the collection of relative odd homology representatives of their core curves is linearly independent.*

Proof. Let us cut some cylinder $\textsc{Cyl}$ out of X. Then, then we can glue parts of the boundary of $X \setminus \textsc{Cyl}$ to obtain a new flat surface (with no boundary). How we do this will depend on the configuration of the boundary saddle connections of $\textsc{Cyl}$. We know that the boundary of $\textsc{Cyl}$ is described by the options listed in Theorem 2.6.6. The cut-and-paste operation we perform now to show that the core curve of each cylinder is independent of the others was suggested to us by Anton Zorich.

In case (i) a boundary curve of $\textsc{Cyl}$ consists of regular points that have total cone angle 2π and one singularity R with cone angle 3π. Once $\textsc{Cyl}$ is cut out, the remaining cone angle at each regular point is π, while the remaining cone angle at the singularity is 2π. Let P be the point that splits the saddle connection boundary loop into two parts of equal lengths. Let us glue those halves with a flip. Thus, we will no longer have a boundary, but we will obtain a straight-line segment RP with regular interior points going from a point with total cone angle π to a point with total cone angle 2π. If we do this to both boundary components of $X \setminus \textsc{Cyl}$, we will obtain a new flat surface X', and each straight-line segment obtained as above will be connecting a pole to a regular marked point. Let us call these straight-line segments RP and $R'P'$. We can choose a small number δ such that the δ-neighborhoods $D(RP)$ and $D(R'P')$ of RP and $R'P'$ have no other singularities inside them. Since P is a regular marked point, it can be moved within $D(RP)$ without changing the metric on X'. Similarly, the point P' can be moved inside $D(R'P')$ without changing the metric on X'

either. Let Q and Q' be the points to which P and P' were moved. If we coordinate the way we move P and P' the straight-line segments RQ and $R'Q'$ will remain parallel. We can cut the surface along them, and then glue a modified version of the cylinder $\textsc{Cyl}_{new}$ back in and obtain a new half-translation surface X_{new}. The flat metric on X_{new} coincides with the metric on X outside of the δ-neighborhood of $\textsc{Cyl}$ and the corresponding neighborhood $\textsc{Cyl}_{new}$. Since any maximal cylinder that does not coincide with $\textsc{Cyl}$ has a core curve representative that does not intersect $D(RP)$ or $D(R'P')$, this process of modifying X by cutting and regluing a cylinder can be performed in such a way that the length of the core curve of this cylinder is changed while the lengths of the core curves of all the other maximal cylinders are unchanged.

In case (ii) a boundary curve of $\textsc{Cyl}$ consists of regular points that have total cone angle 2π and two singularities R_1 and R_2 with cone angle 3π. Once $\textsc{Cyl}$ is cut out, the remaining cone angle at each regular point is π, while the remaining cone angle at the singularities is 2π. A boundary component of $X \setminus \textsc{Cyl}$ consists of two saddle connections of equal length, and by gluing them we get a closed loop consisting of two straight-line segments that just connect regular points R_1 and R_2. Choose a δ-neighborhood D of that loop that contains no other singularities again. Then moving R_1 and R_2 does not change the metric on the corresponding glued surface, and thus the straight-line segment R_1R_2 can be modified to become a new straight-line segment Q_1Q_2. This allows to similarly glue back an altered cylinder $\textsc{Cyl}_{new}$, changing the length of the core curve of the original $\textsc{Cyl}$ while keeping the core curves of all other cylinders unchanged.

Finally, in case (iii) we have we can perform the construction from case (i) on one boundary component of $\textsc{Cyl}$ and the construction from case (ii) on the other.

Note that we cannot have two saddle connection loops that meet at the same singularity on the boundary of a cylinder, since such a singularity would have to have four prongs pointing at the same direction, but since it has total angle 3π, it only has three. Note also that the case when the cylinder's boundary components share at least one point is degenerate: they would have to share a saddle connection, so in case (iii) this is impossible, while in case (i), the two boundary components have to coincide completely, so the cylinder is the entire surface. In case (ii), if both saddle connections are glued on each side, then the cylinder is again closed on itself, while if two saddle connections are glued, they have to be on distinct boundary components (otherwise, we will get an angle π at one of the singularities). In the last case, the glued sides can be perturbed independently, and thus so can the core of the cylinder.

It follows immediately that the core curves of any collection of maximal cylinders with disjoint interiors on X cannot be representing relative homology classes that are linearly dependent. Indeed, the length of the core curve in each such cylinder can be infinitesimally altered without altering any of the other core curves. Note also that the corresponding classes are all nonzero. By (2.3.6) that would mean that any linear relation should persist under an infinitesimal change of the holonomy of one class. But that is not possible.

□

Corollary 2.6.10. *Consider a collection of disjoint maximal cylinders on* $(X, q) \in \mathcal{PQ}_1\mathcal{T}_{g,n} \setminus \mathcal{T}_0$*. Then it is possible to pick a collection* $\mathbf{\Gamma}$ *of saddle connections from their respective boundaries in such a way that the collection of relative homology representatives of those saddle connections is linearly independent and the length of each saddle connection in* $\mathbf{\Gamma}$ *can be changed without altering the lengths of any of the other elements from* $\mathbf{\Gamma}$*.*

Proof. Since the cores of the cylinder collection we have all represent independent odd homology classes, there should be at least that many independent vectors in the basis of odd homology that we picked. Note that since we were producing that basis simultaneously with the construction of the full relative homology basis, we have a non-decreasing sequence of corresponding saddle connection lengths. Since by the classification in Theorem 2.6.6, the saddle connections on the boundary of each cylinder lave lengths either equal to the core or equal to half the length of the core, so if the core length is perturbed, so is the saddle connection length. This tells us that their odd homology classes are also independent. So for each such saddle connection we can choose at least the same number of basis classes, such that the lengths of the corresponding saddle connections are pairwise smaller than the lengths of the cylinder boundary saddle connections that we picked. Indeed, let $\gamma_1, \gamma_2, \dots, \gamma_k$ be the saddle connections classes we picked from the cylinder boundaries, ordered in the increasing order of their length. Clearly, the fist basis saddle connection b_1 we picked was no longer than γ_1. Next, γ_1, γ_2 span a 2-dimensional subspace, so they cannot be spun by b_1 alone, which means that one of them could be picked as the basis vector b_2 at this stage, so the length of that vector could not be greater than the length of γ_2. For any positive $m \leq k$, γ_1,...,γ_m cannot be spanned by the first $m-1$ basis vectors, so one of the vectors $m \leq k$, γ_1,...,γ_m can still be chosen, and thus b_m has to be no longer than γ_m. It follows by induction then, that the first basis vectors are pairwise shorter than the cylinder boundary ones we picked. $\square$

Since the value of the Abelian differential ω_q on the odd homology class $[\hat{\boldsymbol{\gamma}}]$ corresponding to a saddle connection $\boldsymbol{\gamma} \subset X$ is twice the length of $\boldsymbol{\gamma}$ on X, the corresponding collection $\hat{\boldsymbol{\Gamma}}$ will consist of homology classes independent in $H_1(\tilde{X}, \Sigma, \mathbb{Z})$. So, we can conclude the following.

Corollary 2.6.11. *Let* $\mathrm{C{\scriptsize YL}}_1, \cdots \mathrm{C{\scriptsize YL}}_k$ *be thin cylinders on* $(\tilde{X}, \omega_q)$ *with* $(X, q) \in \mathcal{PQ}_1\mathcal{T}_{g,n} \setminus \mathcal{T}_0$. *Let the odd projections of the relative homology classes of the* $\mathrm{C{\scriptsize YL}}_i$*-crossing saddle connections*

$$[C_{odd}(1)], \cdots, [C_{odd}(m)]$$

be linearly independent in $H_1(\tilde{X}, \Sigma, \mathbb{Z})$. *The first* m *basis odd homology classes satisfy*

$$|\omega_q[\Delta_i]| \leq circ(\mathrm{C{\scriptsize YL}}_i).$$

As a consequence, product (2.5.23) is bounded for any $(X, q) \in \mathcal{PQ}_1\mathcal{T}_{g,n} \setminus \mathcal{T}_0$.

Proof. This follows immediately from Corollary 2.6.10 and Lemma 2.6.1. The last statement is due to the fact that if two cylinders have core curves exchanged by σ, then their crossing saddle connections that were picked for the basis of relative homology are also exchanged by σ, and thus produce only one basis vector in odd homology. $\square$

This allows us to finish the proof of our main theorem.

Theorem. *Let $q \in \mathcal{PQ}_1\mathcal{T}_{g,n}$. Then there exists a universal constant $\mathfrak{C}$ such that the Masur-Veech volume of $\mathcal{B}_q(0, \frac{1}{200})$ is smaller than $\mathfrak{C}$.*

Proof. If $(X, q) \in \mathcal{PQ}_1\mathcal{T}_{g,n} \setminus \mathcal{T}_0$, then Corollary 2.6.11 holds. If $(X, q) \in \mathcal{T}_0$, then either $\mathcal{B}_q(0, \frac{1}{200})$ has Masur-Veech measure zero or it contains another point $(X_0, q_0) \in \mathcal{PQ}_1\mathcal{T}_{g,n} \setminus \mathcal{T}_0$. Clearly, $\mathcal{B}_q(0, \frac{1}{200}) \subset \mathcal{B}_{q_0}(0, \frac{1}{100})$, and the Masur-Veech volume of $\mathcal{B}_{q_0}(0, \frac{1}{100})$ is again bounded from above by Corollary 2.6.11. $\square$

CHAPTER 3

UNIQUE ERGODICITY OF SLIT DOUBLE COVERS

3.1 The Slit Double Cover Construction

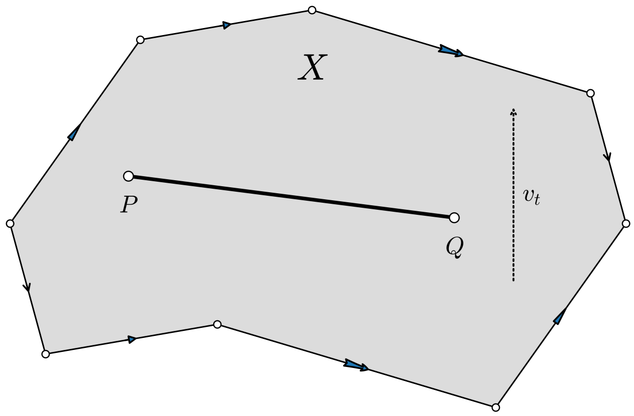

Figure 3.1: An example of a translation surface with a slit.

Let X be a translation surface of finite area, and let Σ be the set of its singularities.

Suppose that the vertical flow v_t on X is uniquely ergodic, and let S be a slit on the surface X with endpoints P and Q (see Figure 3.4). The slit has to be a straight line segment, but can have any slope except for the vertical one and is allowed to be of any length. Take two copies (X_{I}, S_{I}) and (X_{II}, S_{II}) of the surface (X, S) (see Figure 3.2) and glue them crosswise along the slit into a surface $\widetilde{X}$. The vertical flow on $\widetilde{X}$ switches the index i of the copy every time the flow hits either of the slits. In other words, the surface $\widetilde{X}$ is a double cover of the surface X branched at two points P and Q. We will denote by $\widetilde{X}_{\mathrm{I}}$ and $\widetilde{X}_{\mathrm{II}}$ the natural projections of X_{I} and X_{II} respectively onto $\widetilde{X}$. See Figure 3.3 for a visual explanation of the concept.

In the future, we will denote by $\pi : \widetilde{X} \to X$ the natural projection from $\widetilde{X}$ to X (note that $\pi(\widetilde{X}_{\mathrm{I}}) = \pi(\widetilde{X}_{\mathrm{II}}) = X$) and by $\varsigma : \widetilde{X} \to \widetilde{X}$ the involution that leaves intact the X-coordinate of any point on $\widetilde{X}$ but exchanges the copy: $\varsigma(\widetilde{X}_{\mathrm{I}}) = \widetilde{X}_{\mathrm{II}}$. This involution has two fixed points, which are the endpoints $\widetilde{P} = \pi^{-1}P$ and $\widetilde{Q} = \pi^{-1}Q$ of the slit. We will also denote the forward vertical flow on $\widetilde{X}$ by $\widetilde{v}_t$ and the backward vertical flow on $\widetilde{X}$ by $\widetilde{v}_t^- = \widetilde{v}_{-t}$.

Our main result in this chapter is the following.

Theorem 3.1.1. *Let X be a finite-area translation surface whose vertical flow is uniquely ergodic. For almost every pair of points (P, Q) on X and for every slit from P to Q, the resulting branched double cover construction is uniquely ergodic.*

We also note that a completely analogous construction allows to build cyclic n-covers. Chaika's Conjecture, presented in the introduction in the context of double-covers, was originally for all n, and we expect the proof presented here to be extended to arbitrary n with minor adjustments.

First, we discuss in which sense we say that the statement of Theorem 3.1.1 holds for almost every slit.

Lemma 3.1.2. *For a fixed translation surface (X, ω) and fixed points $P, Q \in X$, the double slit construction can produce only finitely many points in the corresponding moduli space of translation surfaces.*

Proof. Let $X^\circ = X \setminus \{P, Q\}$. By the classification theorem for covering spaces, connected double covers of X° up to isomorphism are classified by nontrivial homomorphisms from $\pi_1(X^\circ)$ to $\mathbb{Z}/2\mathbb{Z}$. Since $\pi_1(X^\circ)$ is finitely, generated, there are finitely many such homomorphisms, and thus there are finitely many corresponding double-covers. Given a double cover $\pi : \widetilde{X} \to X$, we can produce a double cover of X° by restricting the map π to the complement of the preimages of P and Q on $\widetilde{X}$. This restriction defines a map from connected branched double covers of X branched at P and Q, up to isomorphism over X, to connected double-covers of X°, up to isomorphism over X°. Two branched covers with isomorphic restrictions to X° are isomorphic over X by uniqueness of extensions, so this map is injective. Therefore, there are only finitely many connected branched double covers of X

branched at P and Q. In particular, the total number of double covers obtained through the slit construction is also finite.

Note that isomorphic double covers of X produce the same point in the corresponding moduli space of translation surfaces. Indeed, if $p_1 : \widetilde{X}_1 \to X$ and $p_2 : \widetilde{X}_2 \to X$ are two branched covers whose restrictions to X° define the same unbranched cover, then, by the uniqueness of extensions to branched covers, there is a biholomorphism $F : \widetilde{X}_1 \to \widetilde{X}_2$ such that $p_2 \circ F = p_1$. The corresponding abelian differentials upstairs are obtained by pullback, so the resulting translation surfaces are $(\widetilde{X}_1, p_1^*\omega)$ and $(\widetilde{X}_2, p_2^*\omega)$. Since $p_2 \circ F = p_1$, we have $F^* p_2^* \omega = p_1^* \omega$, so the two differentials agree, and the two translation surfaces are actually equivalent and determine the same point in the moduli space. $\square$

We will now describe a natural measure on the set of all slit double covers of a translation surface (X, ω). Since (X, ω) is a point in the moduli space, it is defined up to the corresponding equivalence. Let us choose a fixed representative, which we will also call (X, ω), of this equivalence class. Any double slit construction over (X, ω) is determined by an unordered pair of distinct regular points $\{P, Q\} \subset X^\circ$ together with one of the finitely many admissible double-covers arising from this choice of endpoints. Thus, if $\mathcal{M}(\{P, Q\})$ denotes the finite set of translation surfaces in moduli obtained as slit double-covers of (X, ω) with slit endpoints P and Q, we define

$$\mathcal{S}(X, \omega) := \big\{(\{P, Q\}, S) : \{P, Q\} \subset X^\circ,\ P \neq Q,\ S \in \mathcal{M}(\{P, Q\})\big\}. \tag{3.1.1}$$

There is a natural projection

$$\pi : \mathcal{S}(X, \omega) \to \big\{\{P, Q\} : P, Q \in X^\circ,\ P \neq Q\big\}, \qquad \pi(\{P, Q\}, S) = \{P, Q\}, \tag{3.1.2}$$

By construction, the fibers of π are finite. We equip the base with the measure induced from the product flat area measure on $X^\circ \times X^\circ$, and each fiber with counting measure. The resulting natural measure on $\mathcal{S}(X,\omega)$ is thus obtained by integrating the fiber cardinalities over the space of slit endpoints. Let us consider the map

$$F_{(x,\omega)} : \mathcal{S}(X,\omega) \to \mathcal{M},$$

where $\mathcal{M}$ denotes the appropriate moduli space of translation surfaces. In general, the map $F_{(x,\omega)}$ need not be injective. So the natural measure on the family of resulting slit double covers is the pushforward of the measure we defined along the map $F_{(x,\omega)}$. Note that this measure does not depend on the choice of the representative of the initial translation surface. Indeed, suppose that (X_1,ω_1) and (X_2,ω_2) are two representatives of the same moduli space point with the map f being a translation equivalence between them. f is a biholomorphism with $f_*\omega_2 = \omega_1$. In particular, f takes distinct regular points to distinct regular points and sends straight-line segments connecting them to straight-line segments connecting their images. Since such an f in bijective, we obtain a one-to-one correspondence between $\mathcal{S}(X_1,\omega_1)$ and $\mathcal{S}(X_2,\omega_2)$, and since $f_*\omega_2 = \omega_1$, the resulting double covers represent the same point in $\mathcal{M}$.

A translation surface of genus g can have at most g ergodic invariant probability measures [Sat75, Kat73]. Early examples of minimal but non-uniquely ergodic straight-line flows (and IETs) are due to Veech [Vee69], Keane [Kea77], Keynes–Newton [KN76], and Sataev [Sat75]. The example of Veech (1969) is of particular interest to us, as it inspired the constructions used in this work.

We begin with the following simple lemma that describes measures on $\widetilde{X}$ that are ergodic with respect to the vertical flow.

Lemma 3.1.3. *The double-cover surface $\widetilde{X}$ either has only one normalized ergodic measure $\mathcal{L}_{\widetilde{X}}$ (the Lebesgue measure), or it has two normalized measures μ and ν such that*

1. *Both* μ *and* ν *are absolutely continuous with respect to* $\mathcal{L}_{\widetilde{X}}$*;*

2. $\mu(A) = \nu(\varsigma(A))$ *for any Borel subset* $A \subset \widetilde{X}$*;*

3. $\mu + \nu = 2\mathcal{L}_{\widetilde{X}}$*.*

Proof. First, we will show that any ergodic probability measure μ on $\widetilde{X}$ projects to an ergodic probability measure on X, that is, its pushforward $\pi_*\mu$ coincides with the Lebesgue measure $\mathcal{L}_X$ on X. Consider $B \subset X$ invariant under the vertical flow (so we have $v_{-t}B = B$ for all t). Clearly, the preimage $\pi^{-1}B \subset \widetilde{X}$ is $\widetilde{v}_t$-invariant:

$$\widetilde{v}_{-t}(\pi^{-1}B) = \pi^{-1}(v_{-t}B) = \pi^{-1}B.$$

Since μ is ergodic on $\widetilde{X}$, the measure $\mu(\pi^{-1}B)$ is either 0 or 1. By the definition of a pushforward $\pi_*\mu(B) = \mu(\pi^{-1}B)$. So $\pi_*\mu(B)$ is also either 0 or 1, which means exactly that $\pi_*\mu$ is ergodic with respect to v_t on X. There is only one ergodic probability measure on X, so $\pi_*\mu$ has to coincide with $\mathcal{L}_X$.

Now we will prove that μ is absolutely continuous with respect to $\mathcal{L}_{\widetilde{X}}$. If a Borel subset $A \subset \widetilde{X}$ has $\mathcal{L}_{\widetilde{X}}(A) = 0$, then

$$\mu(A) \leq \mu(\pi^{-1}(\pi(A))) = \pi_*\mu(\pi(A)) = \mathcal{L}_X(\pi(A)),$$

which is clearly zero, being a projection of a set of Lebesgue measure zero.

Suppose now for contradiction that there at least three ergodic probability measures μ_1, μ_2 and μ_3 on $\widetilde{X}$. It is well-known that distinct ergodic probability measures have to be mutually singular. In particular, there exists a Borel subset $A_1 \subset \widetilde{X}$ such that $\mu_1(A_1) = 1$ while $\mu_2(A_1) = 0$, and there exists a Borel subset $B_1 \subset \widetilde{X}$ such that $\mu_1(B_1) = 1$ while $\mu_3(B_1) = 0$. The intersection set $E_1 = A_1 \cap B_1$ has $\mu_1(E_1) = 1$ and $\mu_2(E_1) = \mu_3(E_1) = 0$. We can find a similar set for the second measure μ_2: a set G_2 with $\mu_2(G_2) = 1$ and $\mu_1(G_2) =$

$\mu_3(G_2) = 0$. The complement $E_2 = G_2 \setminus E_1$ will satisfy the same equalities and be disjoint from E_1. Finally, the set $E_3 = \widetilde{X} \setminus (E_1 \cup E_2)$ has $\mu_3(E_3) = 1$ and $\mu_1(E_3) = \mu_2(E_3) = 0$. So we have built three pairwise disjoint sets, each of which carries all of one measure, and is invisible to the other two.

Each of the three projections $\pi(E_1)$, $\pi(E_2)$, $\pi(E_3)$ has full Lebesgue measure on X, since we know that all three measures project to the Lebesgue measure on X: $\pi_*\mu_1 = \pi_*\mu_2 = \pi_*\mu_3 = \mathcal{L}_X$:

$$1 = \mu_j(E_j) \leq \mu_j(\pi^{-1}(\pi(E_j))) = \pi_*\mu_j(\pi(E_j)) = \mathcal{L}_X(\pi(E_j)).$$

Therefore

$$\mathcal{L}_X(\pi(E_1) \cap \pi(E_2) \cap \pi(E_3)) = 1.$$

In particular, this intersection is nonempty, so there is some point $x \in \pi(E_1) \cap \pi(E_2) \cap \pi(E_3)$ such that π^{-1} contains at least three distinct points, which is not possible. So there can be at most two invariant ergodic probability measures on $\widetilde{X}$.

Suppose now that there are two distinct measures μ and ν (as above) on $\widetilde{X}$. Since the involution ς commutes with the vertical flow, the pushforward $\varsigma_*\mu$ of an invariant ergodic probability measure μ is again an invariant ergodic probability measure. So either $\varsigma_*\mu = \mu$ or $\varsigma_*\mu = \nu$.

If $\varsigma_*\mu = \mu$, then for every ς-invariant Borel $A \subset \widetilde{X}$ we have $\mu(A) = \mathcal{L}_{\widetilde{X}}(A)$. Such a set coincides with the full preimage of its projection, so

$$\mu(A) = \mu(\pi^{-1}(\pi(A))) = \pi_*\mu(\pi(A)) = \mathcal{L}_X(\pi(A)).$$

Since the normalized Lebesgue measure $\mathcal{L}_{\widetilde{X}}$ also projects to the normalized Lebesgue measure $\mathcal{L}_X$, we get $\mu(A) = \mathcal{L}_{\widetilde{X}}(A)$. So μ agrees with the Lebesgue measure upstairs on all ς-invariant Borel subsets of $\widetilde{X}$. Since both μ and $\mathcal{L}_{\widetilde{X}}$ are ς-invariant, they have to agree on all Borel subsets of $\widetilde{X}$. Indeed, let us consider an arbitrary Borel $B \subset \widetilde{X}$. Both $B \cup \varsigma(B)$ and $B \cap \varsigma(B)$ are ς-invariant, so μ and $\mathcal{L}_{\widetilde{X}}$ coincide on both of these subsets. A simple exclusion-inclusion formula then gives us

$$2\mu(B) = \mu(B \cup \varsigma(B)) + \mu(B \cap \varsigma(B)) = \mathcal{L}_{\widetilde{X}}(B \cup \varsigma(B)) + \mathcal{L}_{\widetilde{X}}(B \cap \varsigma(B)) = 2\mathcal{L}_{\widetilde{X}}(B). \quad (3.1.3)$$

So if $\varsigma_* \mu = \mu$, the measure μ has to coincide with $\mathcal{L}_{\widetilde{X}}$, and so does ν. This contradicts the assumption that μ and ν are distinct. Thus, if $\widetilde{X}$ has two distinct invariant ergodic probability measures, the involution ς has to exchange them.

Finally, we observe that the measure $\frac{\mu+\nu}{2}$ is another ς-invariant probability measure on $\widetilde{X}$ that is also invariant under the vertical flow, so by following the same as above, we conclude that it has to be the same as $\mathcal{L}_{\widetilde{X}}$. $\square$

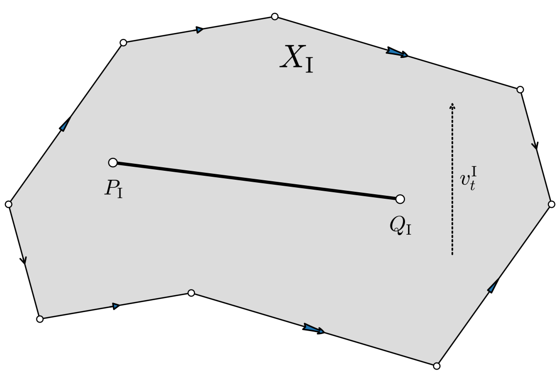

Figure 3.2: Two copies X_{I} and X_{II} of the translation surface X.

This leads to a natural question: **When does $\widetilde{X}$ have one ergodic measure and when does it have two?**

3.2 Measure-generic and Almost Measure-generic Points on $\widetilde{X}$

Lemma 3.2.1. *Consider the inverse vertical flow $\widetilde{v}_t^- = \widetilde{v}_{-t}$ on the translation surface $\widetilde{X}$. Any measure on $\widetilde{X}$ that is ergodic and invariant under the action of the flow $\widetilde{v}_t^-$ is ergodic and invariant under the action of the flow $\widetilde{v}_t$, and vice versa.*

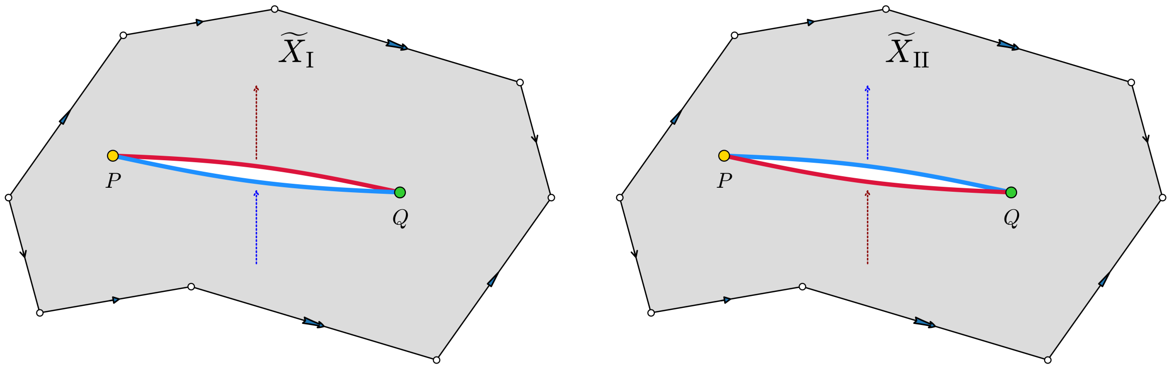

Figure 3.3: Construction of the double-cover surface $\widetilde{X}$ and visualization of its parts $\widetilde{X}_{\mathrm{I}}$ and $\widetilde{X}_{\mathrm{II}}$.

Proof. The argument in both directions is essentially identical, so it suffices to give a proof of only one of them. Suppose that a measure μ is ergodic and invariant on $\widetilde{X}$ under the action of the flow $\widetilde{v}_t$. We will show that it is also invariant and ergodic under the action of the flow $\widetilde{v}_t^-$. Let A be a Borel subset of $\widetilde{X}$, and let us fix a time $t > 0$. To prove invariance, we note that

$$\mu(\widetilde{v}_t^- A) = \mu(\widetilde{v}_t(\widetilde{v}_t^- A)) = \mu(A). \tag{3.2.1}$$

To prove ergodicity, consider a Borel $A \subset \widetilde{X}$ that is invariant under the inverse vertical flow $\widetilde{v}_t^-$. This means that $\widetilde{v}_t^- A = A$, that is, $\widetilde{v}_{-t}(A) = A$. Applying $\widetilde{v}_t$ to both sides, we get $A = \widetilde{v}_t A$, so A is also invariant under the original vertical flow. Since μ is ergodic for $\widetilde{v}_t$, the measure $\mu(A) \in \{0, 1\}$. Hence μ is ergodic for $\widetilde{v}_t^-$ as well. □

3.2.1 Measure-generic Points on $\widetilde{X}$

Let $\widetilde{x}$ be a non-singular point on $\widetilde{X}$, let f be a measurable function on $\widetilde{X}$, and let $t > 0$. We introduce the following notation:

$$I_t^{\uparrow}(f, \widetilde{x}) := \frac{1}{t}\int_0^t f(\widetilde{v}_s \widetilde{x})\,\mathrm{d}s \qquad \text{and} \qquad I_t^{\downarrow}(f, \widetilde{x}) := \frac{1}{t}\int_0^t f(\widetilde{v}_{-s} \widetilde{x})\,\mathrm{d}s. \tag{3.2.2}$$

Let m be an ergodic $\widetilde{v}$-invariant probability measure on $\widetilde{X}$. Suppose that the function f

is m-integrable on $\widetilde{X}$. Denote

$$\mathrm{Av}_m(f) := \int_{\widetilde{X}} f(\widetilde{x})\,\mathrm{d}m(\widetilde{x}). \tag{3.2.3}$$

Let $\mathcal{C}(\widetilde{X})$ be the space of all real-valued continuous functions on $\widetilde{X}$. Since $\widetilde{X}$ is compact, any $f \in \mathcal{C}(\widetilde{X})$ is bounded, hence integrable with respect to any finite regular Borel measure m on $\widetilde{X}$. Thus, any such measure produces a well-defined linear functional

$$\begin{aligned} \Lambda_m : \mathcal{C}(\widetilde{X}) &\longrightarrow \mathbb{R} \\ f &\longmapsto \mathrm{Av}_m(f) \end{aligned} \tag{3.2.4}$$

If $\widetilde{X}$ carries two distinct ergodic measures, then these averages can be used to distinguish between them.

Lemma 3.2.2. *Suppose that the translation surface $\widetilde{X}$ has two distinct ergodic invariant measures μ and ν. Then there exists a 1-Lipschitz function $\mathfrak{h}$ on $\widetilde{X}$ such that $\mathrm{Av}_\mu(\mathfrak{h}) \neq \mathrm{Av}_\nu(\mathfrak{h})$.*

Proof. Note that $\widetilde{X}$ is a compact metric space, so every Lipschitz function on $\widetilde{X}$ is bounded, and, consequently, integrable with respect to both probability measures μ and ν. Moreover, since μ and ν are distinct finite regular Borel measures on $\widetilde{X}$, the Riesz–Markov–Kakutani Representation Theorem implies that they cannot define the same continuous linear functional on $\mathcal{C}(\widetilde{X})$. Therefore, there exists an $\mathfrak{f} \in \mathcal{C}(\widetilde{X})$ such that

$$\mathrm{Av}_\mu(\mathfrak{f}) \neq \mathrm{Av}_\nu(\mathfrak{f}). \tag{3.2.5}$$

Let $\mathrm{Lip}(\widetilde{X})$ and $\mathrm{Lip}_1(\widetilde{X})$ be the spaces of all Lipschitz functions and 1-Lipschitz functions on $\widetilde{X}$, respectively. Clearly, $\mathrm{Lip}_1(\widetilde{X}) \subset \mathrm{Lip}(\widetilde{X}) \subset \mathcal{C}(\widetilde{X})$. We now recall the Stone–Weierstrass Theorem.

Proposition 3.2.3 (The Stone–Weierstrass Theorem, [Con90] – Theorem 8.1, p. 145). *Let $\mathcal{A}$ be a subalgebra of the space of continuous real-valued functions $\mathcal{C}(K)$ on a compact Hausdorff space K. If $\mathcal{A}$ contains all the constant functions and separates points (that is, for every distinct pair $x, y \in K$ there is some $f \in \mathcal{A}$ with $f(x) \neq f(y)$), then $\mathcal{A}$ is dense in $\mathcal{C}(K)$ in the standard supremum norm.*

It follows from the definition of a Lipschitz function that $\mathrm{Lip}(\widetilde{X})$ is closed under addition and scalar multiplication, and since $\widetilde{X}$ is compact, a product of Lipschitz functions is also Lipschitz. Hence, $\mathrm{Lip}(\widetilde{X})$ is a subalgebra in $\mathcal{C}(K)$. Obviously, all constant functions belong to $\mathrm{Lip}(\widetilde{X})$. In addition, $\mathrm{Lip}(\widetilde{X})$ separates points: the function that measures the distance from any point to a fixed point $\widetilde{x}$ is Lipschitz and gives different values on $\widetilde{x}$ itself and any other point $\widetilde{y}$. Therefore, $\mathrm{Lip}(\widetilde{X})$ is dense in $\mathcal{C}(K)$ in the standard supremum norm. Thus, there is a sequence $\{h_j\}_{j\in\mathbb{N}_+}$ with $h_j \in \mathrm{Lip}(\widetilde{X})$ (with Lipschitz constants $\{C_j\}_{j\in\mathbb{N}_+}$) that converges uniformly to $\mathfrak{f}$.

Let

$$\Delta = |\operatorname{Av}_\mu(\mathfrak{f}) - \operatorname{Av}_\nu(\mathfrak{f})| > 0 \tag{3.2.6}$$

Since $h_j \to \mathfrak{f}$ uniformly, there exists some $\mathfrak{j} \in \mathbb{N}_+$ such that for all $j \geq \mathfrak{j}$,

$$||h_j - \mathfrak{f}||_\infty < \frac{\Delta}{4}. \tag{3.2.7}$$

Since μ and ν are probability measures on $\widetilde{X}$, we have

$$\begin{aligned} |\operatorname{Av}_\mu(h_j) - \operatorname{Av}_\mu(\mathfrak{f})| = |\int_{\widetilde{X}} (h_j - \mathfrak{f})\,\mathrm{d}\mu| \leq \\ \leq \int_{\widetilde{X}} |(h_j - \mathfrak{f})|\,\mathrm{d}\mu \leq \\ \leq ||(h_j - \mathfrak{f})||_\infty. \end{aligned} \tag{3.2.8}$$

Similarly,

$$|\operatorname{Av}_\nu(h_j) - \operatorname{Av}_\nu(\mathfrak{f})| \le ||(h_j - \mathfrak{f})||_\infty \tag{3.2.9}$$

By the triangle inequality,

$$\begin{aligned} &|\operatorname{Av}_\mu(h_j) - \operatorname{Av}_\nu(h_j)| \ge \\ \ge |\operatorname{Av}_\mu(\mathfrak{f}) - \operatorname{Av}_\nu(\mathfrak{f})| - |\operatorname{Av}_\mu(h_j) - \operatorname{Av}_\mu(\mathfrak{f})| - &|\operatorname{Av}_\nu(h_j) - \operatorname{Av}_\nu(\mathfrak{f})| \ge \\ &\ge \Delta - 2||(h_j - \mathfrak{f})||_\infty > \frac{\Delta}{2} > 0. \end{aligned} \tag{3.2.10}$$

Thus, for any $j \ge \mathfrak{j}$ we have

$$\operatorname{Av}_\mu(h_j) \ne \operatorname{Av}_\nu(h_j). \tag{3.2.11}$$

Fix any $j \ge \mathfrak{j}$, for example, consider the function $h_{\mathfrak{j}}$. Its Lipschitz constant is some number $C_{\mathfrak{j}}$. Let us rescale $h_{\mathfrak{j}}$ and look at

$$\mathfrak{h} := \frac{h_{\mathfrak{j}}}{\max\{1, C_{\mathfrak{j}}\}}. \tag{3.2.12}$$

The function $\mathfrak{h}$ is 1-Lipschitz, and

$$\operatorname{Av}_\mu(\mathfrak{h}) - \operatorname{Av}_\nu(\mathfrak{h}) = \frac{1}{\max\{1, C_{\mathfrak{j}}\}}(\operatorname{Av}_\mu(h_{\mathfrak{j}}) - \operatorname{Av}_\nu(h_{\mathfrak{j}})) \ne 0. \tag{3.2.13}$$

So, $\mathfrak{h} \in \operatorname{Lip}_1(\widetilde{X})$ and $\operatorname{Av}_\mu(\mathfrak{h}) \ne \operatorname{Av}_\nu(\mathfrak{h})$. $\square$

We now recall one of the fundamental ergodic theorems.

Theorem 3.2.4 (Birkhoff–Khinchin Pointwise Ergodic Theorem for Flows — Corollary 25.9,

p. 517 in [Kal21]). *Let* $(X, \mathcal{B}, m)$ *be a probability space and let* $\{\varphi_t\}_{t\in\mathbb{R}}$ *be a jointly-measurable, m-preserving flow (i.e.* $\varphi_{t+s} = \varphi_t \circ \varphi_s$*,* $\varphi_0 = \mathrm{id}$*, and* $m(\varphi_t^{-1}A) = m(A)$ *for all* $A \in \mathcal{B}$ *and* $t \in \mathbb{R}$*). Then, for m-almost every* $x \in X$*, the integrals*

$$\frac{1}{T}\int_0^T f(\varphi_t(x))\,\mathrm{d}t$$

converge to the conditional expectation of f *with respect to the* σ*-algebra* $\mathcal{I}$ *of* $\{\varphi_t\}$*-invariant sets*

$$\mathbb{E}(f\,|\,\mathcal{I})\,(x),$$

and the convergence also holds in $L^1(m)$*. In particular, if the flow is ergodic, then for every* $f \in L^1(m)$*,*

$$\lim_{T\to\infty}\frac{1}{T}\int_0^T f(\varphi_t(x))\,\mathrm{d}t \;=\; \mathrm{Av}_m(f) \quad \text{for } m\text{-almost every } x.$$

In our notation (if we apply this to the vertical flow),

$$\lim_{T\to\infty} I_T^{\uparrow}(f, \widetilde{x}) = \mathrm{Av}_m(f).$$

As an immediate consequence of Lemma 3.2.1, we obtain the following.

Corollary 3.2.5. *Let* m *be an ergodic invariant measure on* $\widetilde{X}$*. Then the conclusion of Theorem 3.2.4 holds for both the forward and backward vertical flows on* $\widetilde{X}$*.*

We now define the relevant notions of generic behavior of points under the vertical flow.

Definition 3.2.6 (Generic points)**.** Let m be an ergodic invariant measure on $\widetilde{X}$, and let $\widetilde{x} \in \widetilde{X}$ be a non-singular point whose backward and forward vertical trajectories are defined for all times $t \in \mathbb{R}$.

We say that

- $\widetilde{x}$ is *forward m-generic* if for any $f \in \mathcal{C}(\widetilde{X})$

$$\lim_{T\to\infty} I_T^{\uparrow}(f, \widetilde{x}) = \mathrm{Av}_m(f); \tag{3.2.14}$$

- $\widetilde{x}$ is *backward m-generic* if for any $f \in \mathcal{C}(\widetilde{X})$

$$\lim_{T\to\infty} I_T^{\downarrow}(f, \widetilde{x}) = \mathrm{Av}_m(f); \tag{3.2.15}$$

- $\widetilde{x}$ is *forward generic* if $\widetilde{X}$ has multiple ergodic measures and $\widetilde{x}$ is forward m-generic for at least one of them;
- $\widetilde{x}$ is *backward generic* if $\widetilde{X}$ has multiple ergodic measures and $\widetilde{x}$ is backward m-generic for at least one of them;
- $\widetilde{x}$ is *totally m-generic* if it is both forward and backward m-generic for an ergodic invariant measure m on $\widetilde{X}$;
- $\widetilde{x}$ is *totally generic* if $\widetilde{X}$ has multiple ergodic measures and the point is both backward and forward m-generic for at least one of them.

Corollary 3.2.7. *If a non-critical point $\widetilde{x} \in \widetilde{X}$ is forward, backward, or totally μ-generic, then $\varsigma(\widetilde{x})$ is, respectively, forward, backward, or totally ν-generic, and vice versa.*

Proof. Assume that $\widetilde{x}$ is forward μ-generic. Fix $f \in \mathcal{C}(\widetilde{X})$. Since the involution ς commutes with the vertica flow, for every $T > 0$ we have

$$I_T^{\uparrow}(f, \varsigma(\widetilde{x})) = I_T^{\uparrow}(f \circ \varsigma, \widetilde{x}) \tag{3.2.16}$$

Since $f \circ \varsigma \in \mathcal{C}(\widetilde{X})$ (and $\widetilde{x}$ is forward μ-generic),

$$\lim_{T\to\infty} I_T^{\uparrow}(f \circ \varsigma, \widetilde{x}) = \mathrm{Av}_\mu(f \circ \varsigma), \tag{3.2.17}$$

and

$$\lim_{T\to\infty} I_T^{\uparrow}(f,\, \varsigma(\widetilde{x})) = \mathrm{Av}_{\mu}(f\circ\varsigma). \tag{3.2.18}$$

Since, $\varsigma_*\mu = \nu$,

$$\mathrm{Av}_{\mu}(f\circ\varsigma) = \mathrm{Av}_{\nu}(f). \tag{3.2.19}$$

So, $\varsigma(\widetilde{x})$ is forward ν-generic.

The proof for the backward direction is identical, and the conclusion for totally generic points follows immediately. □

Corollary 3.2.8. *The set of points in $\widetilde{X}$ that are either simultaneously forward μ-generic and backward ν-generic, or simultaneously forward ν-generic and backward μ-generic, has Lebesgue measure zero.*

Proof. Let $F_\mu, F_\nu \subset \widetilde{X}$ be the sets of forward μ-generic and forward ν-generic points, respectively. Likewise, let $B_\mu, B_\nu \subset \widetilde{X}$ be the sets of backward μ-generic and backward ν-generic points, respectively. By the Birkhoff–Khinchin Pointwise Ergodic Theorem for flows

$$\mu(F_\mu) = \mu(B_\mu) = 1 \text{ and } \nu(F_\nu) = \nu(B_\nu) = 1. \tag{3.2.20}$$

Then

$$\mu(F_\mu \cap B_\mu) = 1 \text{ and } \nu(F_\nu \cap B_\nu) = 1. \tag{3.2.21}$$

So if

$$G = \{\widetilde{x} \in \widetilde{X} | \widetilde{x} \in F_\mu \cap B_\mu \text{ or } \widetilde{x} \in F_\nu \cap B_\nu\}, \tag{3.2.22}$$

then $\mathcal{L}_{\widetilde{X}}(G) = 1$, and $\mathcal{L}_{\widetilde{X}}(\widetilde{X} \setminus G) = 0$. □

Corollary 3.2.9. *Let m be an ergodic invariant measure on $\widetilde{X}$. The set of totally m-generic points in $\widetilde{X}$ has full m-measure.*

Lemma 3.2.10. *Fix $\tau \in \mathbb{R}$. Let $\widetilde{X}_\tau = g_\tau \widetilde{X}$ be the image of $\widetilde{X}$ under the Teichmüller flow, and let $\widetilde{v}_s^{(\tau)}$ denote the vertical flow on $\widetilde{X}_\tau$. Let m be an ergodic $\widetilde{v}_s$-invariant probability measure on $\widetilde{X}$, and define*

$$m_\tau := (g_\tau)_* m.$$

Then m_τ is an ergodic $\widetilde{v}_s^{(\tau)}$-invariant probability measure on $g_\tau \widetilde{X}$.

Moreover, if $\widetilde{x} \in \widetilde{X}$ is non-critical, then

$$\widetilde{x} \text{ is forward/backward/totally } m\text{-generic on } \widetilde{X}$$

if and only if

$$g_\tau \widetilde{x} \text{ is forward/backward/totally } m_\tau\text{-generic on } \widetilde{X}_\tau.$$

Proof. Since the Teichmüller map

$$g_\tau = \begin{pmatrix} e^\tau & 0 \\ 0 & e^{-\tau} \end{pmatrix}$$

contracts the vertical direction by the factor $e^{-\tau}$, the vertical flows on $\widetilde{X}$ and on $\widetilde{X}_\tau$ are related by

$$\widetilde{v}_s^{(\tau)} \circ g_\tau = g_\tau \circ \widetilde{v}_{e^\tau s} \qquad \text{for all } s \in \mathbb{R}. \tag{3.2.23}$$

In particular, g_τ sends non-critical points to non-critical points.

We first show that m_τ is invariant under $\widetilde{v}_s^{(\tau)}$. Let $A \subset \widetilde{X}_\tau$ be Borel. Then, using (3.2.23),

$$m_\tau\big((\widetilde{v}_s^{(\tau)})^{-1} A\big) = m\big(g_\tau^{-1}((\widetilde{v}_s^{(\tau)})^{-1} A)\big) = m\big((\widetilde{v}_{e^\tau s})^{-1}(g_\tau^{-1} A)\big) = m(g_\tau^{-1} A) = m_\tau(A),$$

because m is $\widetilde{v}_s$-invariant.

Next we show ergodicity. Let $A \subset \widetilde{X}_\tau$ be a Borel set such that

$$(\widetilde{v}_s^{(\tau)})^{-1}(A) = A \qquad \text{for all } s \in \mathbb{R}.$$

Then (3.2.23) implies

$$\widetilde{v}_t^{-1}(g_\tau^{-1}A) = g_\tau^{-1}A \qquad \text{for all } t \in \mathbb{R},$$

since every $t \in \mathbb{R}$ can be written as $t = e^\tau s$. Thus $g_\tau^{-1}A$ is $\widetilde{v}_t$-invariant on $\widetilde{X}$. By ergodicity of m,

$$m(g_\tau^{-1}A) \in \{0, 1\}.$$

Therefore

$$m_\tau(A) = m(g_\tau^{-1}A) \in \{0, 1\},$$

so m_τ is ergodic.

Now suppose that $\widetilde{x}$ is forward m-generic, and let $f \in L^1(m_\tau)$. Then $f \circ g_\tau \in L^1(m)$, and by (3.2.23),

$$\frac{1}{T}\int_0^T f\big(\widetilde{v}_s^{(\tau)}(g_\tau \widetilde{x})\big)\, ds = \frac{1}{T}\int_0^T f\big(g_\tau(\widetilde{v}_{e^\tau s}\widetilde{x})\big)\, ds.$$

Making the change of variables $u = e^\tau s$, so that $ds = e^{-\tau}du$, we get

$$\frac{1}{T}\int_0^T f\big(\widetilde{v}_s^{(\tau)}(g_\tau \widetilde{x})\big)\, ds = \frac{1}{e^\tau T}\int_0^{e^\tau T} (f \circ g_\tau)(\widetilde{v}_u \widetilde{x})\, du.$$

Since $\widetilde{x}$ is forward m-generic, the right-hand side converges, as $T \to \infty$, to

$$\int_{\widetilde{X}} f \circ g_\tau \, dm = \int_{g_\tau \widetilde{X}} f \, d(g_\tau)_* m = \int_{g_\tau \widetilde{X}} f \, dm_\tau.$$

Hence $g_\tau \widetilde{x}$ is forward m_τ-generic.

The backward case is identical:

$$\frac{1}{T}\int_0^T f\big(\widetilde{v}_{-s}^{(\tau)}(g_\tau\widetilde{x})\big)\,ds = \frac{1}{e^\tau T}\int_0^{e^\tau T}(f\circ g_\tau)(\widetilde{v}_{-u}\widetilde{x})\,du,$$

so backward m-genericity of $\widetilde{x}$ implies backward m_τ-genericity of $g_\tau\widetilde{x}$.

Thus total m-genericity of $\widetilde{x}$ implies total m_τ-genericity of $g_\tau\widetilde{x}$.

Finally, the converse implications follow by applying the same argument to the inverse Teichmüller map $g_{-\tau}:\widetilde{X}_\tau\to\widetilde{X}$. $\square$

Measure-generic points are clearly invariant under the action of the vertical flow. This fact is useful in many ways.

Lemma 3.2.11. *For any horizontal interval I on our translation surface $\widetilde{X}$, the set of forward generic points in I has full Lebesgue measure $\mathcal{L}_I$. The same holds for backward generic points and totally generic points.*

Proof. Suppose for contradiction that the set $\widetilde{X}\setminus(F_\mu\cup F_\nu)$ of points that are not forward generic intersects I in a set of positive $\mathcal{L}_I$-measure. Since the set of singularities $\widetilde{\Sigma}$ is finite, we can consider the set $I^\circ = I\setminus\widetilde{\Sigma}$. Clearly, I° has full $\mathcal{L}_I$-measure. Since I° contains no singular point, there exists some number $\varepsilon^\circ>0$ such that the vertical trajectory V° of I° up to time ε° contains no points of $\widetilde{\Sigma}$. Since the set $\widetilde{X}\setminus(F_\mu\cup F_\nu)$ is invariant under the vertical flow,

$$\mathcal{L}_{\widetilde{X}}(V^\circ) = \varepsilon^\circ\cdot\mathcal{L}_I(I^\circ). \tag{3.2.24}$$

and

$$\mathcal{L}_{\widetilde{X}}(V^\circ\cap(\widetilde{X}\setminus(F_\mu\cup F_\nu))) = \varepsilon^\circ\cdot\mathcal{L}_I(I^\circ\cap(\widetilde{X}\setminus(F_\mu\cup F_\nu))). \tag{3.2.25}$$

So if $\mathcal{L}_I(I^\circ\cap(\widetilde{X}\setminus(F_\mu\cup F_\nu)))\neq 0$, then $\mathcal{L}_{\widetilde{X}}(V^\circ\cap(\widetilde{X}\setminus(F_\mu\cup F_\nu)))\neq 0$, contradicting the fact that $\mathcal{L}_{\widetilde{X}}(\widetilde{X}\setminus(F_\mu\cup F_\nu))=0$.

The proof for the backward direction is identical. The set of totally generic points in I is then an intersection of two sets of full $\mathcal{L}_I$-measure, so it has full $\mathcal{L}_I$-measure as well. $\square$

Lemma 3.2.12. *Consider a horizontal interval I on our translation surface $\widetilde{X}$. There exists a pair of points, symmetric with respect to the center of I, that are both generic. The same holds for totally generic points.*

Proof. Suppose not. The reflection $\mathfrak{r}$ across the center of the interval preserves the Lebesgue measure $\mathcal{L}_I$. If G_I is the subset of generic points on I, then $\mathfrak{r}(G) = I \backslash G$ and thus $\mathcal{L}_I(I \backslash G) = \mathcal{L}_I(G)$. This is a contradiction, since $\mathcal{L}_I(I \backslash G) = 0$ and $\mathcal{L}_I(G) = \mathcal{L}_I(I)$ by Lemma 3.2.11. $\square$

3.2.2 Almost Measure-generic Points on $\widetilde{X}$

Let $0 < \epsilon < 1 = \mathcal{L}_{\widetilde{X}}(\widetilde{X})$ and $C > 0$ be some real constants. Define

$$\begin{aligned}
G^{\uparrow}_{\epsilon,C} &= \left\{ \widetilde{x} \in \widetilde{X} \;\middle|\; \begin{array}{r} \exists \text{ erg. inv. meas. } m \text{ on } \widetilde{X} \text{ s.t. } \forall T > C \text{ and } \forall h \in \mathrm{Lip}_1(\widetilde{X}) : \\ \left\| I^{\uparrow}_T(h, \widetilde{x}) - \mathrm{Av}_m(h) \right\| < \epsilon \end{array} \right\}, \\
G^{\downarrow}_{\epsilon,C} &= \left\{ \widetilde{x} \in \widetilde{X} \;\middle|\; \begin{array}{r} \exists \text{ erg. inv. meas. } m \text{ on } \widetilde{X} \text{ s.t. } \forall T > C \text{ and } \forall h \in \mathrm{Lip}_1(\widetilde{X}) : \\ \left\| I^{\downarrow}_T(h, \widetilde{x}) - \mathrm{Av}_m(h) \right\| < \epsilon \end{array} \right\}, \\
G^{\uparrow\downarrow}_{\epsilon,C} &= G^{\uparrow}_{\epsilon,C} \cap G^{\downarrow}_{\epsilon,C}, \\
G_{\epsilon,C} &= \left\{ \widetilde{x} \in \widetilde{X} \;\middle|\; \begin{array}{r} \exists \text{ erg. inv. meas. } m \text{ on } \widetilde{X} \text{ s.t. } \forall T > C \text{ and } \forall h \in \mathrm{Lip}_1(\widetilde{X}) : \\ \left\| I^{\uparrow}_T(h, \widetilde{x}) - \mathrm{Av}_m(h) \right\| < \epsilon \text{ and } \left\| I^{\downarrow}_T(h, \widetilde{x}) - \mathrm{Av}_m(h) \right\| < \epsilon \end{array} \right\}.
\end{aligned} \tag{3.2.26}$$

Theorem 3.2.13. *For any constant $0 < \epsilon < 1 = \mathcal{L}_{\widetilde{X}}(\widetilde{X})$, there exists a positive constant T_ϵ such that*

$$\mathcal{L}_{\widetilde{X}}(G_{\epsilon,T_\epsilon}) > 1 - \epsilon. \tag{3.2.27}$$

Proof. Let

$$G^{\mu}_{\epsilon,C} = \left\{ \widetilde{x} \in \widetilde{X} \;\middle|\; \begin{array}{l} \forall T > C \text{ and } \forall h \in \mathrm{Lip}_1(\widetilde{X}) : \\ \qquad \left\| I^{\uparrow}_{T}(h, \widetilde{x}) - \mathrm{Av}_{\mu}(h) \right\| < \epsilon \text{ and } \left\| I^{\downarrow}_{T}(h, \widetilde{x}) - \mathrm{Av}_{\mu}(h) \right\| < \epsilon \end{array} \right\}, \tag{3.2.28}$$

and let

$$G^{\nu}_{\epsilon,C} = \left\{ \widetilde{x} \in \widetilde{X} \;\middle|\; \begin{array}{l} \forall T > C \text{ and } \forall h \in \mathrm{Lip}_1(\widetilde{X}) : \\ \qquad \left\| I^{\uparrow}_{T}(h, \widetilde{x}) - \mathrm{Av}_{\nu}(h) \right\| < \epsilon \text{ and } \left\| I^{\downarrow}_{T}(h, \widetilde{x}) - \mathrm{Av}_{\nu}(h) \right\| < \epsilon \end{array} \right\}. \tag{3.2.29}$$

Note that

$$G_{\epsilon,C} = G^{\mu}_{\epsilon,C} \cup G^{\nu}_{\epsilon,C}. \tag{3.2.30}$$

First, suppose that we know the estimate μ and ν individually. That is, suppose that the following holds.

Lemma 3.2.14. *For any constant $0 < \epsilon < 1$ there exists a positive constant T^{μ}_{ϵ} and a positive constant T^{ν}_{ϵ} such that*

$$\mu(G^{\mu}_{\epsilon,T^{\mu}_{\epsilon}}) > 1 - \epsilon \textit{ and } \nu(G^{\nu}_{\epsilon,T^{\nu}_{\epsilon}}) > 1 - \epsilon. \tag{3.2.31}$$

Let $T_{\epsilon} = \max\{T^{\mu}_{\epsilon}, T^{\nu}_{\epsilon}\}$. Then

$$G^{\mu}_{\epsilon,T^{\mu}_{\epsilon}} \subseteq G^{\mu}_{\epsilon,T_{\epsilon}} \text{ and } G^{\nu}_{\epsilon,T^{\nu}_{\epsilon}} \subseteq G^{\nu}_{\epsilon,T_{\epsilon}}. \tag{3.2.32}$$

Hence,

$$\mu(G^{\mu}_{\epsilon,T_{\epsilon}}) > 1 - \epsilon \text{ and } \nu(G^{\nu}_{\epsilon,T_{\epsilon}}) > 1 - \epsilon. \tag{3.2.33}$$

By (3.2.30) we have

$$2\mathcal{L}_{\widetilde{X}}(\widetilde{X} \setminus G_{\epsilon, T_\epsilon}) = \mu(\widetilde{X} \setminus G_{\epsilon, T_\epsilon}) + \nu(\widetilde{X} \setminus G_{\epsilon, T_\epsilon}) \leq \mu(\widetilde{X} \setminus G^{\mu}_{\epsilon, T_\epsilon}) + \nu(\widetilde{X} \setminus G^{\nu}_{\epsilon, T_\epsilon}) < 2\epsilon. \quad (3.2.34)$$

So

$$\mathcal{L}_{\widetilde{X}}(G_{\epsilon, T_\epsilon}) > 1 - \epsilon.$$

Now in order to complete the proof of Theorem 3.2.13 we need to prove Lemma 3.2.14. We will prove it for the measure μ, as the proof for ν will be identical.

Once again we can reduce the proof to two twin proofs, by considering the one-sided sets

$$G^{\mu,\uparrow}_{\epsilon, C} = \left\{ \widetilde{x} \in \widetilde{X} \;\middle|\; \forall T > C \text{ and } \forall h \in \mathrm{Lip}_1(\widetilde{X}) : \left\| I^{\uparrow}_{T}(h, \widetilde{x}) - \mathrm{Av}_{\mu}(h) \right\| < \epsilon \right\} \qquad (3.2.35)$$

and

$$G^{\mu,\downarrow}_{\epsilon, C} = \left\{ \widetilde{x} \in \widetilde{X} \;\middle|\; \forall T > C \text{ and } \forall h \in \mathrm{Lip}_1(\widetilde{X}) : \left\| I^{\downarrow}_{T}(h, \widetilde{x}) - \mathrm{Av}_{\mu}(h) \right\| < \epsilon \right\}. \qquad (3.2.36)$$

By definition

$$G^{\mu}_{\epsilon, C} = G^{\mu,\uparrow}_{\epsilon, C} \cap G^{\mu,\downarrow}_{\epsilon, C}. \qquad (3.2.37)$$

Assume that the following holds.

Lemma 3.2.15. *For any $0 < \xi < 1$ there exist constants $C_\uparrow$ and $C_\downarrow$ such that*

$$\mu(G^{\mu,\uparrow}_{\xi, C_\uparrow}) > 1 - \xi \text{ and } \mu(G^{\mu,\downarrow}_{\xi, C_\downarrow}) > 1 - \xi. \qquad (3.2.38)$$

Take $\xi = \frac{\epsilon}{2}$. Let $T^{\mu}_{\epsilon} = \max\{C_{\uparrow}, C_{\downarrow}\}$. Then

$$\mu(G^{\mu,\uparrow}_{\xi, T^{\mu}_{\epsilon}}) > 1 - \frac{\epsilon}{2} \tag{3.2.39}$$

and

$$\mu(G^{\mu,\downarrow}_{\xi, T^{\mu}_{\epsilon}}) > 1 - \frac{\epsilon}{2}. \tag{3.2.40}$$

Hence

$$\mu(G^{\mu,\uparrow}_{\xi, T^{\mu}_{\epsilon}} \cap G^{\mu,\downarrow}_{\xi, T^{\mu}_{\epsilon}}) > 1 - \frac{\epsilon}{2}. \tag{3.2.41}$$

.

Finally, this intersection is contained in $G^{\mu}_{\epsilon, T^{\mu}_{\epsilon}}$, since $\frac{\epsilon}{2} < \epsilon$. We have now reduced the proof of Theorem 3.2.13 to proving Lemma 3.2.15. We will now give a proof for the upward direction, and the downward direction will follow by a similar argument once more.

Proof. Let $\widetilde{x}_0 \in \widetilde{X}$ be some non-singular point on $\widetilde{X}$. Let us consider all 1-Lipschitz functions on $\widetilde{X}$ that vanish at $\widetilde{x}_0$:

$$\mathcal{F}_0 = \left\{ f \in \mathrm{Lip}_1(\widetilde{X}) \;\middle|\; f(\widetilde{x}_0) = 0 \right\}.$$

We will discuss a couple of properties of the set $\mathcal{F}_0$ that will be useful in the proof.

Proposition 3.2.16. *There exists a constant D such that every $f \in \mathcal{F}_0$ is uniformly bounded by D (note that the constant here is the same for all the functions in $\mathcal{F}_0$).*

Proof. For all $f \in \mathcal{F}_0$ and for all $\widetilde{x} \in \widetilde{X}$ we have

$$|f(\widetilde{x})| = |f(\widetilde{x}) - 0| = |f(\widetilde{x}) - f(\widetilde{x}_0)| \leq d(\widetilde{x}, \widetilde{x}_0) \leq \mathrm{diam}(\widetilde{X}) =: D. \tag{3.2.42}$$

□

Proposition 3.2.17. *The set $\mathcal{F}_0 \subset \mathcal{C}(\widetilde{X})$ is compact in the supremum norm $\|\cdot\|_\infty$.*

Proof. First, we will show that $\mathcal{F}_0$ is closed in the supremum norm topology. Suppose that a sequence of functions $\{f_n\}$ from $\mathcal{F}_0$ converges uniformly on $\widetilde{X}$ to some other function f. We will prove that $f \in \mathcal{F}_0$.

Since convergence is uniform, pointwise convergence is present at each point. So

$$f(\widetilde{x}_0) = \lim_{n\to\infty} f_n(\widetilde{x}_0) = \lim_{n\to\infty} 0 = 0, \tag{3.2.43}$$

since every $f_n(\widetilde{x}_0) = 0$.

We now check that f is 1-Lipschitz. Fix any pair of points $\widetilde{x}, \widetilde{y} \in \widetilde{X}$. We have for any fixed f_n

$$|f_n(\widetilde{x}) - f_n(\widetilde{y})| \leq d_{\widetilde{X}}(\widetilde{x}, \widetilde{y}). \tag{3.2.44}$$

Since $f_n(\widetilde{x}) \to f(\widetilde{x})$ and $f_n(\widetilde{y}) \to f(\widetilde{y})$, we have

$$|f(\widetilde{x}) - f(\widetilde{y})| = \lim_{n\to\infty} |f_n(\widetilde{x}) - f_n(\widetilde{y})| \leq d_{\widetilde{X}}(\widetilde{x}, \widetilde{y}). \tag{3.2.45}$$

Now we will show that the closure of $\mathcal{F}_0$ is compact by applying the famous Arzelà–Ascoli Theorem.

Remark 3.2.18 (Arzelà–Ascoli Theorem). Recall that if

- K is a compact metric space;

- $\mathcal{F} \subset \mathcal{C}(K)$ is a subset of continuous functions from K to $\mathbb{R}$.

Then $\mathcal{F}$ is has compact closure in $\mathcal{C}(K)$ with the uniform topology if the two conditions below are met:

1. $\mathcal{F}$ is equicontinuous (that is, for all $x, y \in K$ and for any $\varepsilon > 0$ there exists a $\delta > 0$ such that for any $f \in \mathcal{F}$

$$\text{if } d_K(x,y) < \delta \Rightarrow |f(x) - f(y)| < \varepsilon);$$

2. For every $x \in K$ the set $\{f(x) \mid f \in \mathcal{F}\}$ has compact closure in $\mathbb{R}$.

We do a straightforward check of all the conditions in our case. First, note that $\widetilde{X}$ with its standard flat metric is compact, since it is a finite-area translation surface. Equicontinuity is a consequence of all the functions being 1-Lipschitz for the same constant 1. We can simply take $\delta = \varepsilon$, and then if $d_{\widetilde{X}}(\widetilde{x}, \widetilde{y}) < \delta$ we have by the definition of a 1-Lipschitz function:

$$|f(\widetilde{x}) - f(\widetilde{y})| \leq d_{\widetilde{X}}(\widetilde{x}, \widetilde{y}) < \delta = \varepsilon.$$

Note that the same δ works for all functions in $\mathcal{F}_0$.

By Proposition 3.2.16, for every $\widetilde{x} \in \widetilde{X}$ we have the set

$$\{f(\widetilde{x}) \mid f \in \mathcal{F}_0\}$$

bounded from above by the diameter D of $\widetilde{X}$. It is then a bounded set in $\mathbb{R}$, and bounded sets in $\mathbb{R}$ have compact closures. So $\mathcal{F}_0$ is closed and has compact closure, which means it is compact (in the sup-norm $\|\cdot\|_\infty$). $\square$

After establishing compactness of $\mathcal{F}_0$, we can build a finite subcover. Fix any positive number $\delta > 0$. For every $f \in \mathcal{F}_0$ consider its δ-neighborhood $U_\delta(f)$ in the uniform metric. The collection of all such neighborhoods is an open cover of the set $\mathcal{F}_0$. Since $\mathcal{F}_0$ is compact, it has a finite subcover, so there is a finite subset of functions $\{f_1, f_2, \ldots, f_{M_\delta}\} \subset \mathcal{F}_0$, that are centers of this finite subset of neighborhoods $U_\delta(f_1), U_\delta(f_2), \ldots, U_\delta(f_{M_\delta})$ that cover the entire $\mathcal{F}_0$. In other words, for any $f \in \mathcal{F}_0$ there is at least one $f_j \in \{f_1, f_2, \ldots, f_{M_\delta}\}$ such that

$$||f - f_j||_\infty \le \delta. \tag{3.2.46}$$

By the Birkhoff–Khinchin Pointwise Ergodic Theorem for flows, for μ-almost every point in $\widetilde{X}$ the discrete sequence of the time-average integrals at integer points in time converges to the space average for any fixed f_j :

$$\lim_{N\to\infty} I_N^{\uparrow}(f_j, \widetilde{x}) = \mathrm{Av}_\mu(f_j). \tag{3.2.47}$$

Fix one of the neighborhood-center f_j-s. Then fix N – the integer moment in time. Only the point $\widetilde{x}$ is now varying in the expsession $I_N^{\uparrow}(f_j, \widetilde{x})$, so view that average of f_j at time N as a function of $\widetilde{x}$.

Proposition 3.2.19. *This function $I_N^{\uparrow}(f_j, \cdot)$ is μ-measurable on $\widetilde{X}$.*

Proof. Since the function $(s, \widetilde{x}) \mapsto f_j(\widetilde{v}_s \widetilde{x})$ is continuous on $[0, N] \times \widetilde{X}$, its integral over the compact interval $[0, N]$ is continuous, hence is μ-measurable. □

The space-average $\mathrm{Av}_\mu(f_j)$ does not depend on any point, but can be viewed as a constant function on $\widetilde{X}$ that just takes the value $\mathrm{Av}_\mu(f_j)$ at all points $\widetilde{x} \in \widetilde{X}$ (j is fixed). Being a constant function, it is obviously μ-measurable on $\widetilde{X}$.

We will now prove this.

Proposition 3.2.20. *For any $0 < \zeta < 1$ there exists a μ-measurable set $E(\zeta) \subset \widetilde{X}$ such that $\mu(E(\zeta)) \ge 1 - \zeta$ and such that for any $j \in \{1, \ldots, M_\delta\}$*

$$\lim_{N\to\infty} I_N^{\uparrow}(f_j, \widetilde{x}) = \mathrm{Av}_\mu(f_j) \tag{3.2.48}$$

uniformly on $E(\zeta)$.

Proof. This proof amounts to an application of another classical theorem.

Remark 3.2.21 (Egorov-Severini Theorem). Suppose that (Y, m) is a finite measure space. Let $\psi_n : Y \to \mathbb{R}$ be measurable functions with $\lim_{n\to\infty} \psi_n(y) = \psi(y)$ for m-almost every $y \in Y$. Then for every $\eta > 0$ there exists a measurable set $E \subset Y$ such that $m(Y \setminus E) < \eta$ and $\psi_n \to \psi$ uniformly on E.

Fix one f_j from the finite set $\{f_1, f_2, \ldots, f_{M_\delta}\}$ obtained above. Take $Y = \widetilde{X}$, $m = \mu$ and $\psi_N(\widetilde{x}) = I_N^{\uparrow}(f_j, \widetilde{x})$ (for a fixed f_j). Note that $I_N^{\uparrow}(f_j, \widetilde{x})$ is μ-measurable by Proposition 3.2.19, and

$$\lim_{N\to\infty} I_N^{\uparrow}(f_j, \widetilde{x}) = \mathrm{Av}_\mu(f_j) \tag{3.2.49}$$

for μ-almost every $\widetilde{x} \in \widetilde{X}$ by the Birkhoff–Khinchin Pointwise Ergodic Theorem for flows.

Egorov-Severini Theorem tells us that (in particular) for $\eta = \frac{\zeta}{M_\delta}$ there exists a measurable set $E_j(\frac{\zeta}{M_\delta}) \subset \widetilde{X}$ with $\mu(E_j(\frac{\zeta}{M_\delta})) \geq 1 - \frac{\zeta}{M_\delta}$ and with

$$I_N^{\uparrow}(f_j, \widetilde{x}) \to \mathrm{Av}_\mu(f_j)$$

uniformly on $E_j(\frac{\zeta}{M_\delta})$.

Since this can be applied to each f_j individually, we can consider the intersection

$$E(\zeta) = \cap_{j=1}^{M_\delta} E_j\left(\frac{\zeta}{M_\delta}\right). \tag{3.2.50}$$

Then

$$\mu(\widetilde{X} \setminus E(\zeta)) \leq \sum_{j=1}^{M_\delta} \mu\left(\widetilde{X}_j\left(\frac{\zeta}{M_\delta}\right)\right) < M_\delta \cdot \frac{\zeta}{M_\delta} = \zeta, \tag{3.2.51}$$

and

$$\mu(E(\zeta)) \geq 1 - \zeta. \tag{3.2.52}$$

Since for each j we have $E(\zeta) \subset E_j(\frac{\zeta}{M_\delta})$, we have

$$I_N^{\uparrow}(f_j, \widetilde{x}) \to \mathrm{Av}_\mu(f_j)$$

uniformly on $E(\zeta)$ for every $j \in \{1, ..., M_\delta\}$. So we have proved Proposition 3.2.20. $\square$

We will now combine these j separate uniform convergence statements into one using that there are only finitely many j-s. Expanding the main statement of Proposition 3.2.20, we can write that for every $\rho > 0$ and for each f_j we have some natural $N_j(\delta, \zeta, \rho)$ such that for any $n > N_j(\delta, \zeta, \rho)$ we have

$$\sup_{\widetilde{x} \in E(\zeta)} |I_n^{\uparrow}(f_j, \widetilde{x}) - \mathrm{Av}_\mu(f_j)| < \rho. \tag{3.2.53}$$

Let $N(\delta, \zeta, \rho) = \max_{j \in \{1, ..., M_\delta\}} N_j(\delta, \zeta, \rho)$. This will work for all f_j-s simultaneously. To sum this up, we have shown the following.

Lemma 3.2.22. *For any $0 < \zeta < 1$ there exists a μ-measurable set $E(\zeta) \subset \widetilde{X}$ with $\mu(E(\zeta)) \geq 1 - \zeta$ such that for any $\rho > 0$ there exists a natural number $N(\delta, \zeta, \rho)$ such that for any $n > N(\delta, \zeta, \rho)$, for any $\widetilde{x} \in E(\zeta)$ and for any $j \in \{1, \ldots, M_\delta\}$*

$$|I_n^{\uparrow}(f_j, \widetilde{x}) - \mathrm{Av}_\mu(f_j)| < \rho. \tag{3.2.54}$$

We will now show that the statement of Lemma 3.2.22 holds for all functions in $\mathcal{F}_0$. To be more precise, we will prove the following.

Lemma 3.2.23. *For any $0 < \zeta < 1$ and for any $\rho > 0$ there exists a μ-measurable set $E(\zeta) \subset \widetilde{X}$ with $\mu(E(\zeta)) \geq 1 - \zeta$ such that there exists a natural number $N(\zeta, \rho)$ such that* *for any $n > N(\zeta, \rho)$, for any $\widetilde{x} \in E(\zeta)$ and for any $f \in \mathcal{F}_0$*

$$|I_n^{\uparrow}(f, \widetilde{x}) - \mathrm{Av}_{\mu}(f)| < \rho. \tag{3.2.55}$$

Proof. Let $f \in \mathcal{F}_0$. Take $\delta = \frac{\rho}{3}$ and build the corresponding finite cover and the functions $f_1, \ldots, f_{M_\delta}$. There is some $f_j \in \{f_1, \ldots, f_{M_\delta}\}$ such that

$$||f - f_j||_\infty < \delta. \tag{3.2.56}$$

Let us compare the time and space averages of f and f_j. Using elementary properties of norms we get:

$$|I_n^{\uparrow}(f, \widetilde{x}) - \mathrm{Av}_{\mu}(f)| < |I_n^{\uparrow}(f, \widetilde{x}) - I_n^{\uparrow}(f_j, \widetilde{x})| + |I_n^{\uparrow}(f_j, \widetilde{x}) - \mathrm{Av}_{\mu}(f_j)| + |\mathrm{Av}_{\mu}(f_j) - \mathrm{Av}_{\mu}(f)|.$$

We will now estimate each term in the expression above individually. The first term

$$|I_n^{\uparrow}(f, \widetilde{x}) - I_n^{\uparrow}(f_j, \widetilde{x})| = |\frac{1}{n}\int_0^n (f - f_j)(\widetilde{v}_t\widetilde{x})dt| \leq |\frac{1}{n}\int_0^n ||f - f_j||_\infty dt| \leq \frac{1}{n}\int_0^n \delta dt = \delta$$

For the second term, since $\widetilde{x} \in E(\epsilon)$, and $n > N(\delta, \zeta, \rho)$ we have

$$|I_n^{\uparrow}(f_j, \widetilde{x}) - \mathrm{Av}_{\mu}(f_j)| < \delta.$$

Finally, the third term

$$|\mathrm{Av}_{\mu}(f_j) - \mathrm{Av}_{\mu}(f)| = |\int_{\widetilde{X}} (f_j - f)d\mu| \leq \int_{\widetilde{X}} d\mu \cdot ||f - f_j||_\infty = 1 \cdot ||f - f_j||_\infty < \delta.$$

Combining these we get

$$|I_n^{\uparrow}(f, \widetilde{x}) - \mathrm{Av}_\mu(f)| < 3\delta = \rho. \tag{3.2.57}$$

$\square$

Now that we have proved Lemma 3.2.23, we will upgrade it again and show that a similar statement holds for all moments in time, not only for integer ones.

Lemma 3.2.24. *For any $0 < \zeta < 1$ and for any $\eta > 0$ there exists a μ-measurable set $E(\zeta) \subset \widetilde{X}$ with $\mu(E(\zeta)) \geq 1 - \zeta$ such that there exists a real number $T(\zeta, \eta) > 0$ such that for any $t > T(\zeta, \eta)$, for any $\widetilde{x} \in E(\zeta)$ and for any $f \in \mathcal{F}_0$*

$$|I_t^{\uparrow}(f, \widetilde{x}) - \mathrm{Av}_\mu(f)| < \eta. \tag{3.2.58}$$

Proof. Fix $1 > \zeta > 0$ and $\eta > 0$. Take $\rho = \frac{\eta}{2}$. By Lemma 3.2.23 there is a set $E(\zeta)$ with $\mu(E(\zeta)) \geq 1 - \zeta$ and an $N(\zeta, \rho) > 0$ such that for any $\widetilde{x} \in E(\zeta)$, for every integer $n > N(\zeta, \rho)$ and for any $f \in \mathcal{F}_0$ we have

$$|I_n^{\uparrow}(f, \widetilde{x}) - \mathrm{Av}_\mu(f)| < \rho. \tag{3.2.59}$$

Now fix $t > 0$, $\widetilde{x} \in \widetilde{X}$ and $f \in \mathcal{F}_0$. Let $n := \lfloor t \rfloor$. We want to estimate $|I_t^{\uparrow}(f, \widetilde{x}) - \mathrm{Av}_\mu(f)|$. We can write

$$|I_t^{\uparrow}(f, \widetilde{x}) - \mathrm{Av}_\mu(f)| \leq |I_t^{\uparrow}(f, \widetilde{x}) - I_n^{\uparrow}(f, \widetilde{x})| + |I_n^{\uparrow}(f, \widetilde{x}) - \mathrm{Av}_\mu(f)|. \tag{3.2.60}$$

If $n > N(\zeta, \rho)$, then the second term is smaller than $\rho = \frac{\eta}{2}$.

To estimate the first term, we write:

$$|I_t^{\uparrow}(f,\,\widetilde{x}) - I_n^{\uparrow}(f,\,\widetilde{x})| = \left|\frac{1}{t}\int_0^t f(\widetilde{v}_s(\widetilde{x}))ds - \frac{1}{n}\int_0^n f(\widetilde{v}_s(\widetilde{x}))ds\right| = \left|\frac{1}{t}\int_0^n f(\widetilde{v}_s(\widetilde{x}))ds + \right.$$

$$\left. + \frac{1}{t}\int_n^t f(\widetilde{v}_s(\widetilde{x}))ds - \frac{1}{n}\int_0^n f(\widetilde{v}_s(\widetilde{x}))ds\right| = \left|\left(\frac{1}{t} - \frac{1}{n}\right)\int_0^n f(\widetilde{v}_s(\widetilde{x}))ds + \frac{1}{t}\int_n^t f(\widetilde{v}_s(\widetilde{x}))ds\right| \le$$

$$\le \left|\left(\frac{1}{t} - \frac{1}{n}\right)\right| nD + \frac{1}{t}(t-n)D =$$

$$= \left|\left(\frac{n}{t} - 1\right)\right| D + \frac{1}{t}(t-n)D = 2\frac{t-n}{t}D \quad (3.2.61)$$

by Proposition 3.2.16. Since $t \in [n, n+1]$, we have $0 \le t - n \le 1$. So

$$2\frac{t-n}{t}D \le \frac{2D}{t} \le \frac{2D}{n}$$

So combining all of that we get

$$|I_t^{\uparrow}(f,\,\widetilde{x}) - \mathrm{Av}_{\mu}(f)| \le \frac{2D}{n} + \frac{\eta}{2}$$

Now choose $T(\zeta, \eta)$ large enough so that whenever $t > T(\zeta, \eta)$ and $n = \lfloor t \rfloor$ we have both $n > N(\zeta, \rho)$ and $\frac{2D}{n} < \frac{\eta}{2}$.

Note that this threshold only depends on ζ, η and the constant D. This immediately leads to

$$|I_t^{\uparrow}(f,\,\widetilde{x}) - \mathrm{Av}_{\mu}(f)| \le \frac{\eta}{2} + \frac{\eta}{2} = \eta.$$

This proves Lemma 3.2.24. □

Finally, to prove Lemma 3.2.15, we note that for any constant c

$$I_t^{\uparrow}(f + c,\,\widetilde{x}) = I_t^{\uparrow}(f,\,\widetilde{x}) + c \text{ and } I_t^{\downarrow}(f + c,\,\widetilde{x}) = I_t^{\downarrow}(f,\,\widetilde{x}) + c, \quad (3.2.62)$$

and also

$$\mathrm{Av}_\mu(f+c) = \mathrm{Av}_\mu(f) + c. \tag{3.2.63}$$

This means that

$$I_t^{\uparrow}(f+c,\,\widetilde{x}) - \mathrm{Av}_\mu(f+c) = I_t^{\uparrow}(f,\,\widetilde{x}) - \mathrm{Av}_\mu(f),$$

and

$$I_t^{\downarrow}(f+c,\,\widetilde{x}) - \mathrm{Av}_\mu(f+c) = I_t^{\downarrow}(f,\,\widetilde{x}) - \mathrm{Av}_\mu(f).$$

Thus the statement of Lemma 3.2.15 holds for all $f \in \mathrm{Lip}_1(\widetilde{X})$. This completes the proof of Theorem 3.2.13.

$\square$

$\square$

Definition 3.2.25 (Almost generic points)**.** We call a point $\widetilde{x} \in \widetilde{X}$ *ϵ-almost totally generic* if $\widetilde{x} \in G_{\epsilon, T_\epsilon}$.

Lemma 3.2.26. *Let $0 < \epsilon < 1$ and $C > 0$. If a non-critical point $\widetilde{x} \in \widetilde{X}$ is forward/backward/totally (ϵ, C)-almost μ-generic, then $\varsigma(\widetilde{x})$ is forward/backward/totally (ϵ, C)-almost ν-generic, respectively. The converse statement also holds, with μ and ν interchanged.*

Proof. We prove the forward statement. The backward statement is identical, and the totally (ϵ, C)-generic statement follows by combining the forward and backward cases.

Assume that $\widetilde{x}$ is forward (ϵ, C)-almost μ-generic. We must show that $\varsigma(\widetilde{x})$ is forward (ϵ, C)-almost ν-generic.

Let $h \in \mathrm{Lip}_1(\widetilde{X})$. Since ς is an isometry of $\widetilde{X}$, the function $h \circ \varsigma$ also belongs to $\mathrm{Lip}_1(\widetilde{X})$.

By Lemma 3.1.3, the involution ς pushes μ forward to ν, i.e.

$$\varsigma_* \mu = \nu.$$

Therefore

$$\mathrm{Av}_\mu(h \circ \varsigma) = \int_{\widetilde{X}} h(\varsigma(y))\, d\mu(y) = \int_{\widetilde{X}} h(z)\, d\nu(z) = \mathrm{Av}_\nu(h).$$

Next, by symmetry of the construction, ς commutes with the vertical flow:

$$\widetilde{v}_t \circ \varsigma = \varsigma \circ \widetilde{v}_t \qquad \text{for all } t \in \mathbb{R}.$$

Hence for every $T > 0$,

$$I_T^{\uparrow}(h,\, \varsigma(\widetilde{x})) = \frac{1}{T}\int_0^T h(\widetilde{v}_t(\varsigma(\widetilde{x})))\, dt = \frac{1}{T}\int_0^T h(\varsigma(\widetilde{v}_t(\widetilde{x})))\, dt = \frac{1}{T}\int_0^T (h\circ\varsigma)(\widetilde{v}_t(\widetilde{x}))\, dt = I_T^{\uparrow}(h\circ\varsigma,\, \widetilde{x}).$$

Since $\widetilde{x}$ is forward (ϵ, C)-almost μ-generic, it follows that for every $T > C$,

$$\left| I_T^{\uparrow}(h \circ \varsigma,\, \widetilde{x}) - \mathrm{Av}_\mu(h \circ \varsigma) \right| < \epsilon.$$

Using the two identities above, we obtain

$$\left| I_T^{\uparrow}(h,\, \varsigma(\widetilde{x})) - \mathrm{Av}_\nu(h) \right| < \epsilon \qquad \text{for all } T > C.$$

Since this holds for every $h \in \mathrm{Lip}_1(\widetilde{X})$, the point $\varsigma(\widetilde{x})$ is forward (ϵ, C)-almost ν-generic.

The backward case is proved in exactly the same way, using

$$I_T^{\downarrow}(h,\, \varsigma(\widetilde{x})) = I_T^{\downarrow}(h \circ \varsigma,\, \widetilde{x}),$$

which again follows from $\widetilde{v}_{-t} \circ \varsigma = \varsigma \circ \widetilde{v}_{-t}$.

If $\widetilde{x}$ is totally (ϵ, C)-almost μ-generic, then it is both forward and backward (ϵ, C)-almost μ-generic, so the argument above shows that $\varsigma(\widetilde{x})$ is both forward and backward (ϵ, C)-almost ν-generic. Hence $\varsigma(\widetilde{x})$ is totally (ϵ, C)-almost ν-generic.

The converse statement follows by interchanging μ and ν, since ς is an involution. $\square$

Lemma 3.2.27. *Let I be a horizontal interval on $\widetilde{X}$, and let $0 < \epsilon < 1$. Let $F_I : I \to I$ be the first return map of the vertical flow to I.*

Then there exists a constant $C = C(I, \epsilon) > 0$ such that

$$\mathcal{L}_I\big(G^I_{\epsilon,C}\big) > (1-\epsilon)\mathcal{L}_I(I).$$

Proof. Consider the first return map $F_I : I \to I$ of the vertical flow to I. This is an interval exchange transformation preserving the Lebesgue measure $\mathcal{L}_I$ on I.

Apply the quantitative almost-genericity Theorem 3.2.13 to the measure-preserving system $(I, F_I, \mathcal{L}_I)$. It yields a constant $C = C(I, \epsilon)$ such that the set of ϵ-almost totally generic points for F_I has $\mathcal{L}_I$-measure greater than $(1-\epsilon)\mathcal{L}_I(I)$. □

Lemma 3.2.28. *Let I be a horizontal interval on $\widetilde{X}$, and let $0 < \epsilon < \frac{1}{2}$. Then there exists a constant $C = C(I, \epsilon) > 0$ and a pair of points in I, symmetric with respect to the center of I, that are both ϵ-almost totally generic for the first return map on I.*

Proof. Let $R : I \to I$ be reflection across the center of I. Since R preserves $\mathcal{L}_I$, for every measurable subset $A \subset I$ one has

$$\mathcal{L}_I(R(A)) = \mathcal{L}_I(A).$$

By Lemma 3.2.27, there exists $C = C(I, \epsilon)$ such that

$$\mathcal{L}_I\big(G^I_{\epsilon,C}\big) > (1-\epsilon)\mathcal{L}_I(I).$$

Since $\epsilon < \frac{1}{2}$, this implies

$$\mathcal{L}_I\big(G^I_{\epsilon,C}\big) > \frac{1}{2}\,\mathcal{L}_I(I).$$

Therefore the two sets $G^I_{\epsilon, C}$ and $R(G^I_{\epsilon, C})$ cannot be disjoint, because otherwise

$$\mathcal{L}_I\big(G^I_{\epsilon, C}\big) + \mathcal{L}_I\big(R(G^I_{\epsilon, C})\big) \le \mathcal{L}_I(I),$$

contradicting the fact that each of these two sets has measure $\mathcal{L}_I(G^I_{\epsilon, C}) > \frac{1}{2}\,\mathcal{L}_I(I)$.

Hence there exists $\widetilde{x} \in G^I_{\epsilon, C} \cap R(G^I_{\epsilon, C})$. Then both $\widetilde{x}$ and $R(\widetilde{x})$ belong to $G^I_{\epsilon, C}$, and these two points are symmetric with respect to the center of I. □

3.3 The Circle Argument

As before, let $\pi : \widetilde{X} \to X$ be the slit double-cover with ramification points $\widetilde{P} = \pi^{-1}P$ and $\widetilde{Q} = \pi^{-1}Q$. For an $r > 0$, let $\mathrm{U}(x, r)$ denote the r-neighborhood of the point x in the flat metric. Denote the Teichmüller flow by g_t, and write $X_t = g_t(X)$, and $\widetilde{X}_t = g_t(\widetilde{X})$. Similarly, we set $P_t = g_t P$, $Q_t = g_t Q$.

Our main theorem in this section is the following geometric criterion for unique ergodicity.

Theorem 3.3.1. *Consider one of the slit endpoints on X (without loss of generality, call it Q). Suppose that the following key assumption holds: there exist a number $R > 0$ and a sequence of moments in time $\{t_k\}_{k\in\mathbb{N}_0}$, with $\lim_{k\to\infty} t_k = +\infty$, such that the R-neighborhood $\mathrm{U}_{t_k}(Q_{t_k}, R) \subset X_{t_k}$ of Q_{t_k} is an embedded disk that does not contain P_{t_k}. Then the surface $\widetilde{X}$ is uniquely ergodic.*

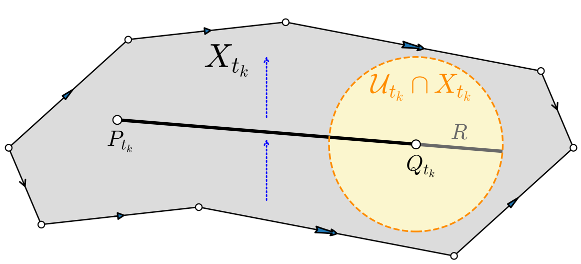

Figure 3.4: Example of the key assumption of Theorem 3.3.1.

Without loss of generality, we may assume that $t_0 = 0$ and choose the local coordinates around the slit endpoint $\widetilde{Q}$ so that it is the origin for the purposes of applying the Teichmüller flow. Suppose that $\widetilde{X}$ is non-uniquely ergodic, and denote the two normalized ergodic measures on it by μ and ν. We will prove the theorem by contradiction.

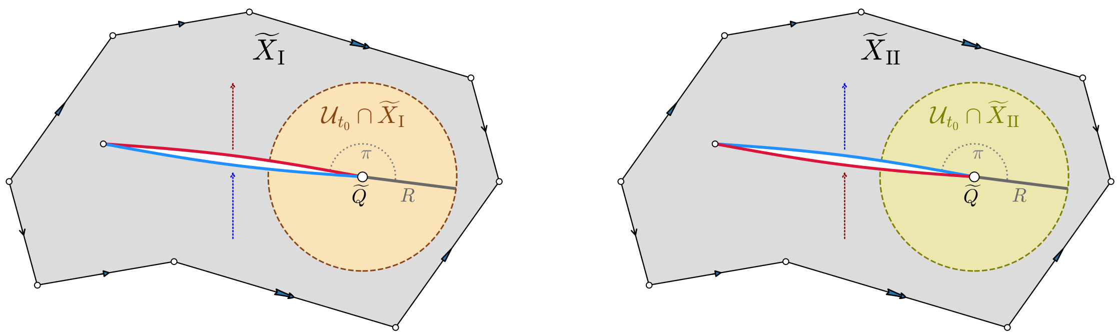

Figure 3.5: Example of the R-neighborhood $\mathcal{U}_{t_0}$ of the endpoint $\widetilde{Q}$ of a slit on the surface $\widetilde{X}_t$.

Proof. Denote by $\mathcal{U}_{t_k}$ the biggest neighborhood of $\widetilde{Q}_{t_k} = g_{t_k}\widetilde{Q}$ such that $\pi(\mathcal{U}_{t_k}) = \mathrm{U}_{t_k}$ (see Figure 3.5 for the visual representation).

I. Passing to a subsequence of times $\{t_{k_i}\}_{i\in\mathbb{N}_0}$.

First, we pick a certain subsequence $\{t_{k_i}\}_{i\in\mathbb{N}_0}$ of the sequence $\{t_{k_i}\}_{i\in\mathbb{N}_0}$. Let us consider the sequence $\{\epsilon_i\}_{i\in\mathbb{N}}$, $\epsilon_i = 1/(i+2)$ of reciprocals of natural numbers. The first element we pick is simply $t_0 = 0$. Starting from $i = 1$, for each ϵ_i, we pick a moment in time t_{k_i} such that

$$\frac{\sqrt{3}}{2} e^{t_{k_i}} R \; > \; \max\{T_{\epsilon_i}, T_{\epsilon_{i+1}}\}, \tag{3.3.1}$$

where T_{ϵ_i} and $T_{\epsilon_{i+1}}$ are the constants from Theorem 3.2.13 corresponding to $\epsilon = \epsilon_i$ and $\epsilon = \epsilon_{i+1}$ respectively (the motivation from this, admittedly somewhat obscure, choice will become apparent in Part V of the proof; more specifically, in Lemma 3.3.3). Moreover, we pick our new elements t_{k_i} in such a way that the subsequence remains strictly increasing.

For the sake of simplicity, we will abuse our notation and denote the resulting subsequence of times t_{k_i} by $\{t_i\}_{i\in\mathbb{N}_0}$. We will also sometimes write $\widetilde{X}_{t_i} = \widetilde{X}_i$.

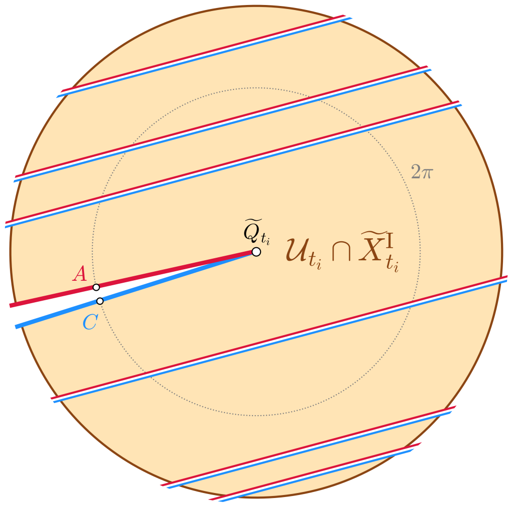

Figure 3.6: Neighborhood of the endpoint O of radius r inside $\widetilde{X}_{t_i}$.

II. The neighborhood $\mathcal{U}_{t_i}$.

Take a point $A \neq \widetilde{Q}_{t_i}$ on the intersection of $\mathcal{U}_{t_i}$ with the slit. Consider the clockwise path around $\widetilde{Q}_{t_i}$ on $\widetilde{X}_{t_i}$ starting from A. It is clear that, after rotating by 2π, we will arrive at $C = \varsigma(A)$, and after rotating by the angle 4π, we will return back to A. In this process, we may cross the slit again some number of times. Every time we do so, we jump between the sheets of $\widetilde{X}_{t_i}$: from $\widetilde{X}^{\mathrm{I}}_i$ to $\widetilde{X}^{\mathrm{II}}_i$, or vice versa. However, since $\mathcal{U}_{t_i} \cap \widetilde{X}^{\mathrm{I}}_i$ and $\mathcal{U}_{t_i} \cap \widetilde{X}^{\mathrm{II}}_i$ are clearly symmetric, inside $\mathcal{U}_{t_i}$, we can reglue the stripes formed by the slit. Therefore, $\mathcal{U}_{t_i}$ is simply a glueing of two radius r_i circles centered at $\widetilde{Q}_{t_i}$ and cut along their parallel radii. We will denote those two circles by $\mathcal{U}^+_{t_i}$ and $\mathcal{U}^-_{t_i}$ (indexing can be chosen, for example, by fixing a side of the slit and stating that $\mathcal{U}^+_{t_i}$ corresponds to counterclockwise rotation by at most 2π and $\mathcal{U}^-_{t_i}$ corresponds to clockwise rotation by at most 2π). See Figure 3.6 for the visual representation.

III. Geometry inside $\mathcal{U}^+_{t_i}$.

In this part, we aim to build a special sequence of pairs of vertical intervals on the initial surface $\widetilde{X}$.

For a fixed $i \in \mathbb{N}_0$, consider the neighborhood $\mathcal{U}_{t_i}$ of the slit endpoint $\widetilde{Q}_{t_i}$. As before, we will represent this neighborhood as two disks, $\mathcal{U}_{t_i}^+$ and $\mathcal{U}_{t_i}^-$, cut and glued along a symmetric radius segment determined by the slit. Let us first describe all our geometric constructions for the forward vertical flow on $\mathcal{U}_{t_i}^+$; from symmetry, geometric constructions on $\mathcal{U}_{t_i}^-$, as well as those for the backward vertical flow, will repeat these. See Figure 3.7 for the visual reference.

Prolong the slit to a diameter of $\mathcal{U}_{t_i}^+$. More specifically, starting from the "top" part of the slit, we rotate it counterclockwise inside $\mathcal{U}_{t_i}^+$ around the endpoint $\widetilde{Q}_{t_i}$ in and consider the separatrix segment that goes in the direction of the slit and is contained in $\mathcal{U}_{t_i}^+$.

By Lemma 3.2.12, Corollary 3.2.7, as well as Theorem 3.2.13 and Lemmas 3.2.27, 3.2.28, we can find points $\widehat{A}_i$ on the top side of the slit, $\widehat{C}_i = \varsigma(\widehat{A}_i)$ on the bottom side of the slit, and $\widehat{B}_i$ on the separatrix such that they are all totally generic (possibly for different measures), ϵ_i-almost totally generic (from Theorem 3.2.13, since the area of the neighborhood $\mathcal{U}_{t_i}$ is strictly bigger than $2\epsilon_i$ by assumption), and satisfy $0 < \widehat{r}_i := |\widetilde{Q}_{t_i}\widehat{A}_i| = |\widetilde{Q}_{t_i}\widehat{B}_i| < R/2$.

Let $\widehat{A}_i'$ and $\widehat{B}_i'$, respectively, be the intersections of the vertical projections on the horizontal diameter of $\mathcal{U}_{t_i}$ (note that $\widehat{A}_i'$ may belong to $\mathcal{U}_{t_i}^-$ dependent on the angle between the slit and the vertical flow). We will denote by $\widehat{H}_i$ the signed vertical distance from $\widehat{B}_i$ to $\widehat{B}_i'$ (i.e. $\widehat{H}_i$ is positive if $\widehat{B}_i' = v_{|\widehat{H}_i|}\widehat{B}_i$ and negative if $\widehat{B}_i = v_{|\widehat{H}_i|}\widehat{B}_i'$). Clearly, the signed vertical distance from $\widehat{A}_i$ to $\widehat{A}_i'$ is $-\widehat{H}_i$.

Let $\widehat{M}_i^{+,\,top}$ and $\widehat{N}_i^{+,\,top}$, respectively, be the points of first intersection with the boundary of $\mathcal{U}_i^+$ of the forward vertical trajectories starting at the points $\widehat{A}_i$ and $\widehat{B}_i$. We will denote by $\widehat{L}_i$ the vertical distance from $\widehat{B}_i'$ to $\widehat{M}_i^{+,\,top}$. Note that $\frac{\sqrt{3}}{2}r_i \leq \widehat{L}_i < r_i$.

Evidently, $\widehat{L}_i = |\widehat{B}_i'\widehat{N}_i^{+,\,top}| = |\widehat{A}_i'\widehat{M}_i^{+,\,top}|$. Moreover, $|\widehat{B}_i\widehat{N}_i^{+,\,top}| = \widehat{L}_i + \widehat{H}_i$ and $|\widehat{A}_i\widehat{M}_i^{+,\,top}| = \widehat{L}_i - \widehat{H}_i$.

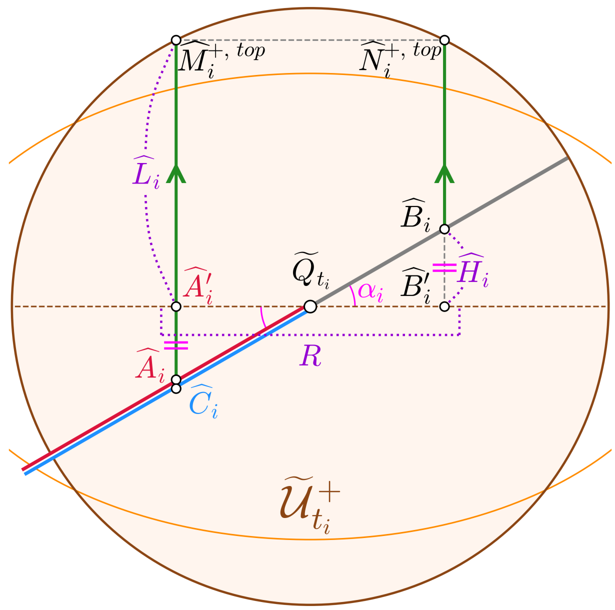

Figure 3.7: Vertical interval sequence construction on $\mathcal{U}^+_{t_i}$.

Next, we describe the preimage of this picture on $\mathcal{U}^+_{t_0} \subset \widetilde{X}$. See Figure 3.9 for visual reference.

Let $A_i = g_{t_i}^{-1}\widehat{A}_i$, $B_i = g_{t_i}^{-1}\widehat{B}_i$, $A'_i = g_{t_i}^{-1}\widehat{A}'_i$, $B'_i = g_{t_i}^{-1}\widehat{B}'_i$, $M_i^{+,\,top} = g_{t_i}^{-1}\widehat{M}_i^{+,\,top}$, and $N_i^{+,\,top} = g_{t_i}^{-1}\widehat{N}_i^{+,\,top}$ be the respective preimages of the points $\widehat{A}_i$, $\widehat{B}_i$, $\widehat{A}'_i$, $\widehat{B}'_i$, $\widehat{M}_i^{+,\,top}$, and $\widehat{N}_i^{+,\,top}$. Note that A'_i and B'_i remain equidistant from $\widetilde{Q}$ and are still totally generic points on $\widetilde{X}$ (as the Teichmüller flow preserves measure-generality of non-critical points, see Lemma 3.2.10). The same is true for A'_i and B'_i, as well as $M_i^{+,\,top}$ and $N_i^{+,\,top}$. By construction, the distance $2R_i = |A_iB_i| = 2e^{-t_i}\widehat{r}_i < e^{-t_i}r_i$. We can find a subsequence $\{t_{i_j}\}_{j\in\mathbb{N}}$ of the sequence of times $\{t_i\}_{i\in\mathbb{N}}$ such that the distances r_{i_j} are strictly decreasing. For simplicity, we will abuse our notation and denote this subsequence of times by $\{t_i\}_{i\in\mathbb{N}}$ too.

For each t_i in the subsequence, consider the parallel intervals $[A'_i,\, M_i^{+,\,top}]$ and $[B'_i,\, N_i^{+,\,top}]$

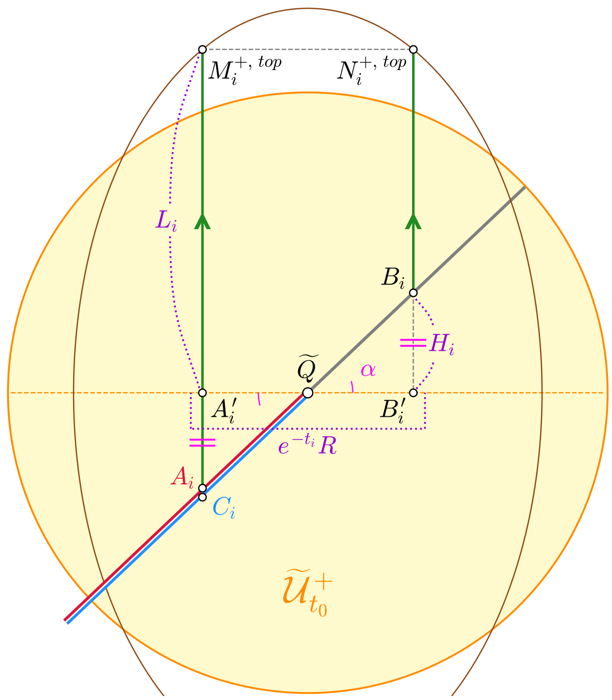

Figure 3.8: Construction of two sequences of vertical intervals, $[A_i,\, M_i^{+,\,top}]$ and $[B_i,\, N_i^{+,\,top}]$, on $\mathcal{U}_{t_0}$.

on the surface $\widetilde{X}$. Clearly, they are both of length $L_i = e^{t_i}\widehat{L}_i$, and

$$T_{\epsilon_i} \leq \frac{\sqrt{3}}{2}e^{t_i}r_i \leq L_i < e^{t_i}r_i, \quad \text{therefore} \quad \lim_{i\to\infty} L_i \geq \lim_{i\to\infty} \frac{\sqrt{3}}{2}e^{t_i}r_i = +\infty. \tag{3.3.2}$$

Recall that the value of T_{ϵ_i} comes from Theorem 3.2.13.

Moreover, the signed distance $H_i = e^t\widehat{H}_i$ from B_i to B_i' clearly tends to 0 as i goes to infinity (thus so does the distance $|\widetilde{Q}B_i'|$).

IV. Geometry inside the entire $\mathcal{U}_{t_i}$.

Since $\mathcal{U}_{t_i}^{+} = \varsigma(\mathcal{U}_{t_i}^{-})$, so the exact same geometric construction can be repeated in $\mathcal{U}_{t_i}^{-}$. Note that $\widehat{A}_i, \widehat{C}_i \in \mathcal{U}_{t_i}^{+} \cap \mathcal{U}_{t_i}^{-}$. The point $\widehat{D}_i = \varsigma(\widehat{B}_i)$ has to be totally generic since $\widehat{B}_i$ is totally generic and not critical. Moreover, $\widehat{D}_i$ has to be ϵ_i-almost totally generic from symmetry (see Lemma 3.2.26). The exact same extension construction as above can be done in the backward direction of the vertical flow on $\widetilde{X}$. The only slight issue is keeping track of

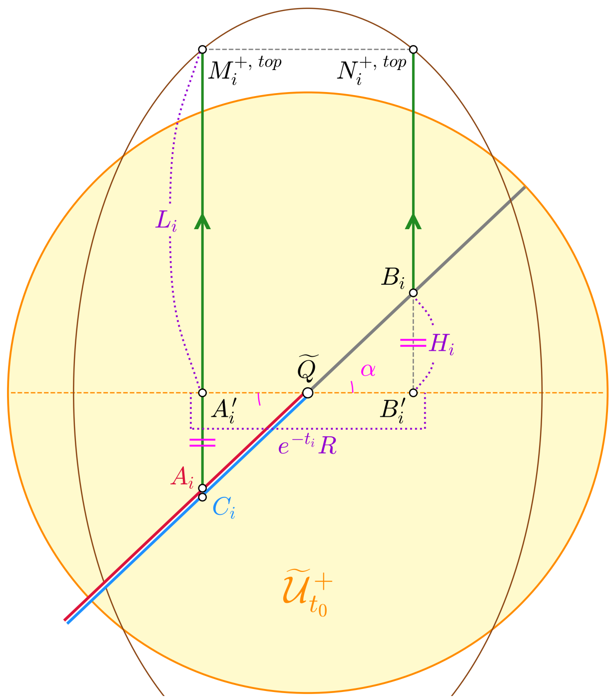

Figure 3.9: Construction of two sequences of vertical intervals, $[A_i,\, M_i^{+,\,top}]$ and $[B_i,\, N_i^{+,\,top}]$, on $\mathcal{U}_{t_0}$.

the relative positions of rays and points, as demonstrated by Figure 3.10.

V. Forward flow limits.

Let f be a function on $\widetilde{X}$, and let $t \geq 0$. For simplicity, slightly change the notation of the integrals from Equation (3.2.2):

$$I^{\uparrow}(f,\, A_i) := I^{\uparrow}_{L_i - H_i}(f,\, A_i) = \frac{1}{L_i - H_i} \int_0^{L_i - H_i} f(v_s A_i)\,\mathrm{d}s, \tag{3.3.3}$$

$$I^{\uparrow}(f,\, A_i') := I^{\uparrow}_{L_i}(f,\, A_i') = \frac{1}{L_i} \int_0^{L_i} f(v_s A_i')\,\mathrm{d}s, \tag{3.3.4}$$

$$I^{\uparrow}(f,\, B_i) := I^{\uparrow}_{L_i + H_i}(f,\, B_i) = \frac{1}{L_i + H_i} \int_0^{L_i + H_i} f(v_s B_i)\,\mathrm{d}s, \tag{3.3.5}$$

$$I^{\uparrow}(f,\, B_i') := I^{\uparrow}_{L_i}(f,\, B_i') = \frac{1}{L_i} \int_0^{L_i} f(v_{-s} B_i')\,\mathrm{d}s. \tag{3.3.6}$$

Lemma 3.3.2. *If* $\lim_{i\to\infty} I^{\uparrow}(f,\, B_i')$ *exists and is finite, then so does* $\lim_{i\to\infty} I^{\uparrow}(f,\, B_i)$*, and*

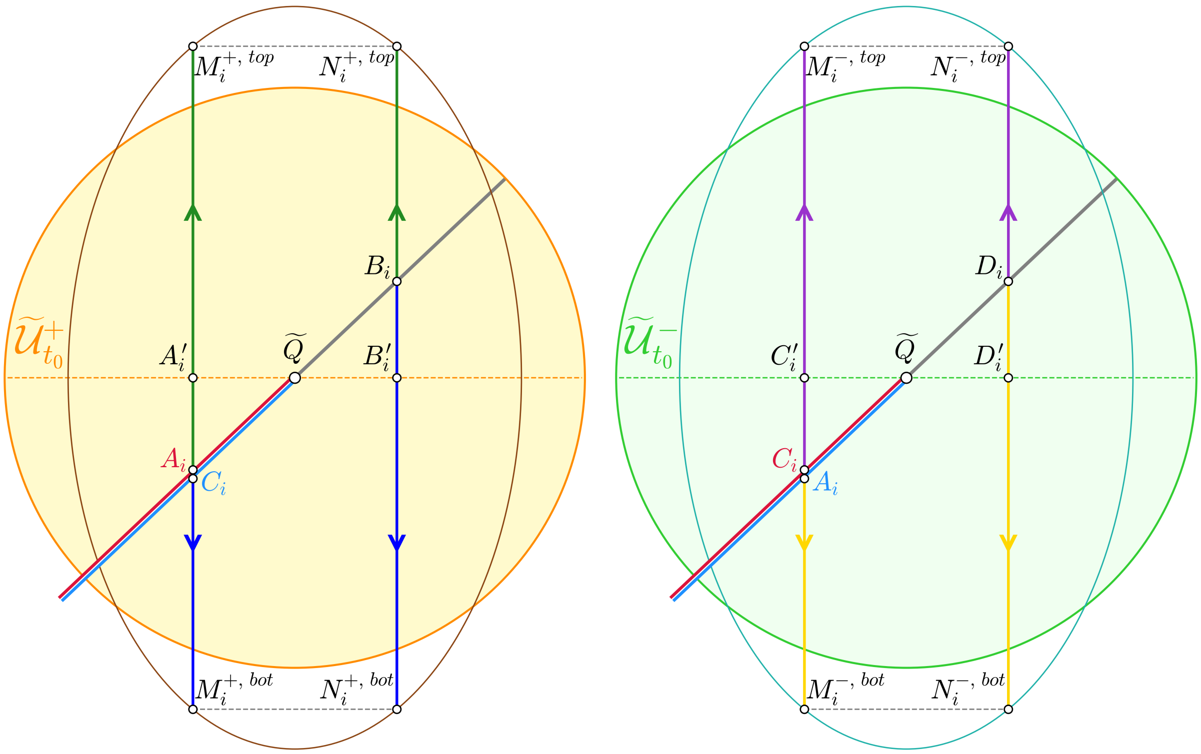

Figure 3.10: The neighborhood $\mathcal{U}_{t_0}$.

vice versa. Moreover, these two limits are equal. The same holds for $\lim_{i\to\infty} I^{\uparrow}(f,\, A_i')$ *and* $\lim_{i\to\infty} I^{\uparrow}(f,\, A_i)$.

Proof. It is enough to prove the statement for B_i' and B_i. There are two cases to consider.

First, assume that B_i' is an image of B_i under the forward vertical flow: $B_i' = g_{|H_i|}B_i$. Then H_i is positive, and

$$\left\| I^{\uparrow}(f,\, B_i) - I^{\uparrow}(f,\, B_i') \right\| = \left\| \frac{1}{L_i + H_i} \int_0^{L_i + H_i} f(v_s B_i)\, \mathrm{d}s - \frac{1}{L_i} \int_0^{L_i} f(v_s B_i')\, \mathrm{d}s \right\| =$$
$$= \left\| \frac{1}{L_i + H_i} \int_0^{H_i} f(v_s B_i)\, \mathrm{d}s + \left(\frac{1}{L_i + H_i} - \frac{1}{L_i} \right) \int_0^{L_i} f(v_s B_i')\, \mathrm{d}s \right\| =$$
$$= \frac{H_i}{L_i + H_i} \left\| I^{\uparrow}_{H_i}(f,\, B_i) - I^{\uparrow}(f,\, B_i') \right\|. \quad (3.3.7)$$

As i goes to infinity, L_i goes to infinity and H_i goes to 0. Therefore $I^{\uparrow}_{H_i}(f,\, B_i)$ goes to 0.

Therefore, if $\lim_{i\to\infty} \|I^{\uparrow}(f,\, B'_i)\|$ exists and is finite, then $\lim_{i\to\infty} \|I^{\uparrow}(f,\, B_i)\|$ exists and is finite, and these two limits are equal. The other direction (from existence of limits for B_i to existence of limits for B'_i) follows analogously from considering:

$$\left\| I^{\uparrow}(f,\, B_i) - I^{\uparrow}(f,\, B'_i) \right\| = \left\| \frac{1}{L_i + H_i} \int_0^{L_i+H_i} f(v_s B_i)\, \mathrm{d}s - \frac{1}{L_i} \int_0^{L_i} f(v_s B'_i)\, \mathrm{d}s \right\| =$$
$$= \left\| \left(\frac{1}{L_i + H_i} - \frac{1}{L_i} \right) \int_0^{L_i+H_i} f(v_s B_i) + \frac{1}{L_i} \int_0^{H_i} f(v_s B_i)\, \mathrm{d}s \right\| = \frac{H_i}{L_i} \left\| I^{\uparrow}_{H_i}(f,\, B_i) - I^{\uparrow}(f,\, B_i). \right\| \tag{3.3.8}$$

Calculations for the case when B'_i is a preimage of B_i under the forward vertical flow ($B_i = g_{|H_i|} B'_i$) are done analogously.

□

Consider an arbitrary 1-Lipschitz function h on $\widetilde{X}$. Since $\widetilde{X}$ is a compact metric space, and since both measures μ and ν are absolutely continuous with respect to the Lebesgue measure $\mathcal{L}_{\widetilde{X}}$ on $\widetilde{X}$ (see Lemma 3.1.3), the function h is going to be both μ- and ν-integrable on $\widetilde{X}$.

Lemma 3.3.3. *The sequence $\{I^{\uparrow}(h,\, A_i)\}_{n\in\mathbb{N}_+}$ converges to a finite limit, and so does the sequence $\{I^{\uparrow}(h,\, B_i)\}_{n\in\mathbb{N}_+}$. Moreover, these two limits are equal to each other and to* $\mathrm{Av}_m(h)$, *where either $m = \mu$ or $m = \nu$.*

Proof. We first prove that the sequence $\{I^{\uparrow}(h,\, A_i)\}_{n\in\mathbb{N}_+} = \{I^{\uparrow}_{L_i - H_i}(h,\, A_i)\}_{n\in\mathbb{N}_+}$ converges (the proof for the sequence $\{I^{\uparrow}(h,\, B_i)\}_{n\in\mathbb{N}_+} = \{I^{\uparrow}_{L_i + H_i}(h,\, B_i)\}_{n\in\mathbb{N}_+}$ is analogous). Then we will prove that the limits are equal.

Step 1: From the previous lemma, we can instead prove convergence of the sequence $\{I^{\uparrow}_{L_i}(h,\, A'_i)\}_{n\in\mathbb{N}_+}$. Recall that each point A_i, $i \geq 1$, is ϵ_i-almost totally generic w.r.t. an ergodic invariant measure m on $\widetilde{X}$, where $m = \mu$ or $m = \nu$. While the same is not necessarily true for points A'_i, $i \geq 1$, we can denote by d_{A_i} the difference $\left\| I^{\uparrow}(f,\, A_i) - I^{\uparrow}(f,\, A'_i) \right\|$, implying that a point A'_i is at most d_{A_i}-away from being ϵ_i-almost totally generic. Keep in mind that d_{A_i} tends to 0 as i tends to ∞.

We want to show that there exists $N \in \mathbb{N}_+$ such that the measure m is the same for all A'_i with $i > N$.

Suppose not. Then, for any $N \in \mathbb{N}_+$, there exist two consecutive points A_i and A_{i+1}, $i \geq N$, such that A_i is ϵ_i-almost totally μ-generic and A_{i+1} is ϵ_{i+1}-almost totally ν-generic. By the construction from the main proof, we have $L_i \geq T_{\epsilon_i}$, $L_i \geq T_{\epsilon_{i+1}}$, and $L_{i+1} \geq T_{\epsilon_{i+1}}$. Therefore

$$\left\| I^{\uparrow}_{L_i}(h,\, A'_i) - \mathrm{Av}_{\mu}(h) \right\| < \epsilon_i + d_{A_i} \quad \text{and} \quad \left\| I^{\uparrow}_{L_{i+1}}(h,\, A'_{i+1}) - \mathrm{Av}_{\nu}(h) \right\| < \epsilon_{i+1} + d_{A_{i+1}}. \tag{3.3.9}$$

Moreover, given Condition (3.3.1) on the sequence of times t_i, we have

$$\left\| I^{\uparrow}_{L_i}(h,\, A'_{i+1}) - \mathrm{Av}_{\nu}(h) \right\| < \epsilon_{i+1} + d_{A_{i+1}}. \tag{3.3.10}$$

Since h is a 1-Lipschitz function, A'_{i+1} is positioned between O and A'_i by construction, and $|OA'_i| \leq \frac{1}{2}R$, we have

$$\begin{aligned} \left\| I^{\uparrow}_{L_i}(h,\, A'_i) - I^{\uparrow}_{L_i}(h,\, A'_{i+1}) \right\| &= \frac{1}{L_i} \left\| \int_0^{L_i} h(v_s A'_i)\, \mathrm{d}s - \int_0^{L_i} h(v_s A'_{i+1})\, \mathrm{d}s \right\| \leq \\ &\leq \frac{1}{L_i} \int_0^{L_i} \left\| h(v_s A'_i) - h(v_s A'_{i+1}) \right\| \, \mathrm{d}s \leq \frac{1}{L_i} \int_0^{L_i} e^{-t_i} \frac{R}{2} \, \mathrm{d}s = e^{-t_i} \frac{R}{2}. \end{aligned} \tag{3.3.11}$$

Combining (3.3.9), (3.3.10) and (3.3.11), we get

$$\begin{aligned} \left\| \mathrm{Av}_{\nu}(h) - \mathrm{Av}_{\mu}(h) \right\| &\leq \left\| I^{\uparrow}_{L_i}(h,\, A'_i) - \mathrm{Av}_{\mu}(h) \right\| + d_{A_i} + \left\| I^{\uparrow}_{L_i}(h,\, A'_{i+1}) - I^{\uparrow}_{L_i}(h,\, A'_i) \right\| + d_{A_i} + d_{A_{i+1}} + \\ &+ \left\| \mathrm{Av}_{\nu}(h) - I^{\uparrow}_{L_i}(h,\, A'_{i+1}) \right\| + d_{A_{i+1}} < \epsilon_i + e^{-t_i} \frac{R}{2} + \epsilon_{i+1} + 2d_{A_i} + 2d_{A_{i+1}}. \end{aligned}$$

Since the sum on the right clearly converges to 0 as i tends to ∞, so, even though the measures are different, their averages $\mathrm{Av}_\mu(h)$ and $\mathrm{Av}_\nu(h)$ are the same. Since we did not specify the function h beyond the fact that it is 1-Lipschitz, we can take the function $\mathfrak{h}$ from Lemma 3.2.2. We have obtained a contradiction.

A completely analogous proof shows that the sequence $\{I^{\uparrow}(h, B_i)\}_{n\in\mathbb{N}_+}$ also converges to $\mathrm{Av}_{\bar{m}}(h)$ for either $\bar{m}=\mu$ or $\bar{m}=\nu$.

Step 2: If $\bar{m}=m$, the proof is complete. It remains to prove that we cannot have $\bar{m}\neq m$.

Recall that h is 1-Lipschitz, $|A_i'B_i'| = |v_sA_i'v_sB_i'|$ for any $s\in[0, L_i]$, and $|A_i'B_i'|\le e^{-t_i}R$ for all $i>0$. Therefore, for any $i>0$,

$$
\begin{aligned}
\left\|I^{\uparrow}(h, B_i') - I^{\uparrow}(h, A_i')\right\| &= \left\|\frac{1}{L_i}\int_0^{L_i} h(v_sB_i')\,\mathrm{d}s - \frac{1}{L_i}\int_0^{L_i} h(v_sA_i')\,\mathrm{d}s\right\| \le \\
&\le \frac{1}{L_i}\int_0^{L_i} \left\|h(v_sB_i') - h(v_sA_i')\right\|\,\mathrm{d}s \le \frac{1}{L_i}\int_0^{L_i} e^{-t_i}R\,\mathrm{d}s = e^{-t_i}R.
\end{aligned}
$$

Therefore $\{I^{\uparrow}(h, A_i)'\}_{n\in\mathbb{N}_+}$ and $\{I^{\uparrow}(h, B_i')\}_{n\in\mathbb{N}_+}$ have to converge to the same limit. Then the same is true for $\{I^{\uparrow}(h, A_i)\}_{n\in\mathbb{N}_+}$ and $\{I^{\uparrow}(h, B_i)\}_{n\in\mathbb{N}_+}$ from Lemma 3.3.2. □

Part VI. Backward flow limits.

Note that the exact same construction as above can be repeated for the backward vertical flow and the sequence of pairs of points $\{(C_i, B_i)\}_{i\in\mathbb{N}_0}$, where $C_i = \varsigma(A_i)$ for all $i\ge 0$. Analogously to Lemma 3.3.3, we can obtain

Lemma 3.3.4. *The sequence $\{I^{\downarrow}(h, C_i)\}_{n\in\mathbb{N}_+}$ converges to a finite limit, and so does the sequence $\{I^{\downarrow}(h, B_i)\}_{n\in\mathbb{N}_+}$. Moreover, these two limits are equal to each other and to $\mathrm{Av}_{\bar{m}}(h)$, where either $\bar{m}=\mu$ or $\bar{m}=\nu$.*

Moreover, since A_i, B_i, and $C_i=\varsigma(A_i)$ are, by construction, both totally generic and ϵ_i-generic for every $i\ge 1$, we obtain

Lemma 3.3.5. *The sequences $\{I^{\uparrow}(h, B_i)\}_{n\in\mathbb{N}_+}$ and $\{I^{\downarrow}(h, B_i)\}_{n\in\mathbb{N}_+}$ converge to the same limit. The same statement holds for the sequences $\{I^{\uparrow}(h, A_i)\}_{n\in\mathbb{N}_+}$ and $\{I^{\downarrow}(h, A_i)\}_{n\in\mathbb{N}_+}$, as well as the sequences $\{I^{\uparrow}(h, C_i)\}_{n\in\mathbb{N}_+}$ and $\{I^{\downarrow}(h, C_i)\}_{n\in\mathbb{N}_+}$.*

From Lemmas 3.3.3, 3.3.4, and 3.3.5, we conclude that, for every $h \in \mathrm{Lip}_1(\widetilde{X})$,

$$\lim_{i\to\infty} I^{\uparrow}(h, A_i) \;=\; \lim_{i\to\infty} I^{\downarrow}(h, A_i) \;=\; \lim_{i\to\infty} I^{\uparrow}(h, \varsigma(A_i)) \;=\; \lim_{i\to\infty} I^{\downarrow}(h, \varsigma(A_i)) \;=\; \mathrm{Av}_m(h), \tag{3.3.12}$$

where either $m = \mu$ or $m = \nu$.

Since A_i, and $\varsigma(A_i)$ are, by construction, totally generic and ϵ_i-generic for every $i \geq 1$, we obtain a contradiction with Corollary 3.2.7. $\square$

3.4 Proof of the Main Theorem

3.4.1 Embedded Disks

Definition 3.4.1. The *embedded radius* of X is

$$r_{\mathrm{emb}}(X) := \sup\Big\{r > 0 : \text{there exists an isometric embedding } B_r(0) \hookrightarrow X\Big\},$$

where $B_r(0) \subset \mathbb{R}^2$ is the open Euclidean disk of radius r.

Theorem 3.4.2. *Let X be a finite-area translation surface whose vertical flow is uniquely ergodic, and fix a marked point $P \in X$. Suppose that there exist $r > 0$ and a sequence $t_k \to \infty$ such that*

$$r_{\mathrm{emb}}(X_{t_k}) \geq r, \qquad X_{t_k} := g_{t_k} X.$$

Then for Lebesgue-almost every point $Q \in X$ there exist a number $R(Q) > 0$ and an infinite subsequence $t_{k_j} \to \infty$ such that, for every j, the surface $X_{t_{k_j}}$ contains an embedded Euclidean disk of radius at least $R(Q)$ centered at $g_{t_{k_j}}(Q)$, and this disk does not contain $g_{t_{k_j}}(P)$.

Proof. Fix R with $0 < R < r$.

For each k, choose an isometric embedding

$$\iota_k : B_R(0) \hookrightarrow X_{t_k},$$

and write

$$D_k^R := \iota_k(B_R(0)).$$

We also consider the concentric disk

$$D_k^{R/2} := \iota_k(B_{R/2}(0)) \subset D_k^R.$$

We now define a measurable subset $A_k \subset D_k^{R/2}$ of uniformly positive area, all of whose points stay a definite Euclidean distance away from $g_{t_k}(P)$ inside the chosen chart.

If $g_{t_k}(P) \notin D_k^R$, set

$$A_k := D_k^{R/2}.$$

If $g_{t_k}(P) \in D_k^R$, let

$$p_k := \iota_k^{-1}(g_{t_k}(P)) \in B_R(0),$$

and set

$$A_k := \iota_k\big(B_{R/2}(0) \setminus B_{R/4}(p_k)\big).$$

In either case, $A_k \subset D_k^{R/2}$ is measurable. Moreover,

$$\mathcal{L}(A_k) \geq \pi(R/2)^2 - \pi(R/4)^2 = \frac{3\pi R^2}{16}.$$

Indeed, if $g_{t_k}(P) \notin D_k^R$, then $\mathcal{L}(A_k) = \pi(R/2)^2$. If $g_{t_k}(P) \in D_k^R$, then removing the set

$$B_{R/2}(0) \cap B_{R/4}(p_k)$$

can decrease the area by at most $\pi(R/4)^2$.

Now define

$$F_k := g_{-t_k}(A_k) \subset X.$$

Since the Teichmüller flow preserves Lebesgue measure,

$$\mathcal{L}_X(F_k) = \mathcal{L}(A_k) \geq \frac{3\pi R^2}{16}.$$

Let

$$U_N := \bigcup_{k \geq N} F_k, \qquad S := \bigcap_{N=1}^{\infty} U_N = \limsup_{k\to\infty} F_k.$$

The sets U_N decrease with N, and $U_N \supset F_N$ for every N. Therefore

$$\mathcal{L}_X(S) = \lim_{N\to\infty} \mathcal{L}_X(U_N) \geq \frac{3\pi R^2}{16} > 0.$$

Now apply the Birkhoff–Khinchin pointwise ergodic Theorem 3.2.4 for the vertical flow to the indicator function $\mathbf{1}_S$. Since the vertical flow is uniquely ergodic, it is ergodic for Lebesgue measure. Hence for Lebesgue-almost every point $Q \in X$,

$$\lim_{T\to\infty} \frac{1}{T} \int_0^T \mathbf{1}_S(v_t Q)\, dt = \mathcal{L}_X(S) > 0.$$

In particular, for Lebesgue-almost every Q, the forward vertical orbit of Q meets S infinitely often. Fix such a point Q, and choose one time $s(Q) \geq 0$ such that

$$v_{s(Q)}Q \in S.$$

Since $v_{s(Q)}Q \in S = \limsup_{k\to\infty} F_k$, there exists an infinite subsequence t_{k_j} such that

$$v_{s(Q)}Q \in F_{k_j}.$$

Equivalently,

$$x_j := g_{t_{k_j}}(v_{s(Q)}Q) \in A_{k_j}.$$

Because $x_j \in D_{k_j}^{R/2}$, its distance to the boundary of $D_{k_j}^R$ is at least $R/2$. Also, if $g_{t_{k_j}}(P) \in D_{k_j}^R$, then by construction of A_{k_j},

$$d_{D_{k_j}^R}\big(x_j,\, g_{t_{k_j}}(P)\big) \geq \frac{R}{4},$$

where $d_{D_{k_j}^R}$ denotes Euclidean distance in the chart $D_{k_j}^R$. If $g_{t_{k_j}}(P) \notin D_{k_j}^R$, then every subset of $D_{k_j}^R$ automatically avoids $g_{t_{k_j}}(P)$.

Now use the relation

$$g_t \circ v_s = v_{e^{-t}s} \circ g_t.$$

Set

$$q_j := g_{t_{k_j}}(Q).$$

Then q_j and x_j lie on the same vertical orbit segment in $X_{t_{k_j}}$, and

$$d(q_j, x_j) = e^{-t_{k_j}}\, s(Q) \longrightarrow 0.$$

After discarding finitely many terms, we may assume that

$$d(q_j, x_j) < \frac{R}{8} \qquad \text{for every } j.$$

We claim that the open metric disk

$$B_{R/8}(q_j)$$

is contained in $D_{k_j}^R$ and does not contain $g_{t_{k_j}}(P)$.

Let $y \in B_{R/8}(q_j)$. Then

$$d(y, x_j) \leq d(y, q_j) + d(q_j, x_j) < \frac{R}{8} + \frac{R}{8} = \frac{R}{4}.$$

Since x_j is at distance at least $R/2$ from $\partial D^R_{k_j}$, it follows that $y \in D^R_{k_j}$. Thus

$$B_{R/8}(q_j) \subset D^R_{k_j}.$$

If $g_{t_{k_j}}(P) \notin D^R_{k_j}$, then this disk certainly avoids $g_{t_{k_j}}(P)$. So suppose instead that

$$g_{t_{k_j}}(P) \in D^R_{k_j}.$$

Since both x_j and y lie in the embedded chart $D^R_{k_j}$, all distances between them are Euclidean chart distances. Therefore

$$d_{D^R_{k_j}}\big(y, g_{t_{k_j}}(P)\big) \geq d_{D^R_{k_j}}\big(x_j, g_{t_{k_j}}(P)\big) - d_{D^R_{k_j}}(y, x_j) > \frac{R}{4} - \frac{R}{4} = 0.$$

Hence $y \neq g_{t_{k_j}}(P)$. Therefore

$$g_{t_{k_j}}(P) \notin B_{R/8}(q_j).$$

We have shown that $B_{R/8}(q_j)$ is contained in the embedded Euclidean disk $D^R_{k_j}$, so it is itself an embedded Euclidean disk in $X_{t_{k_j}}$. This proves the theorem, with

$$R(Q) := \frac{R}{8}.$$

□

Remark 3.4.3. If P is nonsingular and Q lies on the same vertical trajectory as P, then the conclusion generally fails. Indeed, if

$$Q = v_{-a}(P)$$

for some $a \in \mathbb{R}$, then

$$g_t(Q) = v_{-e^{-t}a}(g_t(P)),$$

so

$$d(g_t(Q), g_t(P)) \to 0 \qquad \text{as } t \to \infty.$$

Hence no fixed positive-radius disk centered at $g_t(Q)$ can avoid $g_t(P)$ for all large t. This does not affect the theorem, since a single vertical trajectory has Lebesgue measure zero.

Corollary 3.4.4. *Let X be a finite-area translation surface whose vertical flow is uniquely ergodic, and fix a marked point $P \in X$. Suppose that there exist $r > 0$ and a sequence $t_k \to \infty$ such that*

$$r_{\mathrm{emb}}(X_{t_k}) \geq r, \qquad X_{t_k} := g_{t_k} X.$$

Then for Lebesgue-almost every point $Q \in X$, the double-cover construction determined by the slit from P to Q is uniquely ergodic.

Proof. By Theorem 3.4.2, for Lebesgue-almost every point $Q \in X$ there exist a number $R(Q) > 0$ and an infinite subsequence $t_{k_j} \to \infty$ such that, for every j, the surface $X_{t_{k_j}}$ contains an embedded Euclidean disk of radius at least $R(Q)$ centered at $g_{t_{k_j}}(Q)$, and this disk does not contain $g_{t_{k_j}}(P)$.

Therefore, the conditions of Theorem 3.3.1 are satisfied. □

Note that, apart from finite area and unique ergodicity of the vertical flow, the only additional requirement is the existence of a sequence $t_k \to \infty$ along which the embedded radius of X_{t_k} is bounded below.

3.4.2 Cylinders

Definition 3.4.5 (Flat cylinder)**.** A *flat cylinder* (in direction θ) is an open subset

$$C \subset X \setminus \Sigma$$

isometric, in translation coordinates, to

$$(\mathbb{R}/c\mathbb{Z}) \times (0, h)$$

with the Euclidean metric, for some $c, h > 0$, in such a way that the curves $\{y = \text{const}\}$ map to simple closed geodesics in direction θ on X.

We write:

$$c(C) := \text{circumference}, \qquad h(C) := \text{height}, \qquad a(C) := c(C)h(C), \qquad m(C) := \frac{h(C)}{c(C)}.$$

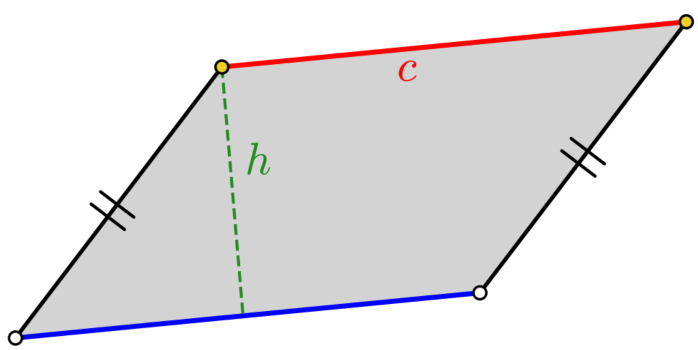

Figure 3.11: A maximal cylinder.

Definition 3.4.6 (Maximal flat cylinder)**.** A flat cylinder C is *maximal* if it is not properly contained in any larger flat cylinder in the same direction. Equivalently, it is a connected component of the set of points whose geodesic in direction θ is periodic. We call two maximal cylinders *disjoint* if they have disjoint interiors.

By [**?** , Lemma 1.6], the closure $\overline{C}$ has two boundary components, each of which is a finite union of saddle connections in direction θ.

Definition 3.4.7. A maximal flat cylinder C is called a *pipe cylinder* if

$$c(C) < h(C),$$

and a *perfect cylinder* if

$$c(C) = h(C).$$

Lemma 3.4.8. *For a fixed translation surface X, any two distinct maximal pipe cylinders are disjoint.*

Proof. Let C_1 and C_2 be two distinct maximal pipe cylinders. If they are parallel, then maximality implies that either they coincide or their interiors are disjoint. So only the non-parallel case needs to be studied.

Assume, for contradiction, that C_1 and C_2 intersect and are not parallel. Then a core curve of C_1 crosses C_2 from one boundary component of C_2 to the other, so its length is at least the height of C_2:

$$c(C_1) \geq h(C_2).$$

Similarly,

$$c(C_2) \geq h(C_1).$$

But since both cylinders are pipes,

$$h(C_1) > c(C_1), \qquad h(C_2) > c(C_2).$$

Combining these inequalities gives

$$c(C_1) \geq h(C_2) > c(C_2) \geq h(C_1) > c(C_1),$$

a contradiction. Hence the interiors of C_1 and C_2 are disjoint. $\square$

Lemma 3.4.9. *For a fixed stratum, the number of maximal pipe cylinders on any surface in that stratum is bounded above by a constant depending only on the stratum.*

Proof. Choose one core curve from each maximal pipe cylinder. By Lemma 3.4.8, these core curves are pairwise disjoint.

Moreover, two such core curves cannot be homotopic in $X \setminus \Sigma$. Indeed, any two simple closed geodesics on a translation surface that avoid the singularities and are freely homotopic lie in the closure of the same maximal cylinder. Hence distinct maximal cylinders give non-homotopic core curves.

Thus the core curves form a collection of pairwise disjoint, pairwise non-homotopic, essential simple closed curves on the punctured surface $X \setminus \Sigma$. The size of such a collection is bounded above by the usual topological constant

$$M = 3g - 3 + n,$$

where g is the genus and $n = |\Sigma|$ is the number of singularities. This bound depends only on the stratum. $\square$

Proposition 3.4.10 (Masur–Smillie, [MS91], Proposition 5.4)**.** *The Delaunay triangulation of an area-*1 *translation surface X consists of edges which either have length at most $2\sqrt{2/\pi}$ or cross a pipe cylinder C. If an edge crosses C, then its length lies in the interval*

$$[h(C),\ \sqrt{h(C)^2 + c(C)^2}].$$

Lemma 3.4.11. *Suppose there exists a sequence $t_k \to \infty$ such that $X_{t_k} = g_{t_k}X$ contains no pipe cylinders. Then there exists a constant $R > 0$ such that every X_{t_k} contains an embedded disk of radius at least R.*

Proof. Fix the stratum of X, and let T be the number of triangles in a Delaunay triangulation of a surface in that stratum. This number depends only on the stratum.

Since the total area is 1, at least one Delaunay triangle Δ has area

$$A_\Delta \geq \frac{1}{T}.$$

If X_{t_k} contains no pipe cylinders, then Proposition 3.4.10 implies that every Delaunay edge has length at most $2\sqrt{2/\pi}$. Therefore the perimeter of Δ satisfies

$$P_\Delta \leq 6\sqrt{2/\pi}.$$

The inradius of a Euclidean triangle is

$$r_\Delta = \frac{2A_\Delta}{P_\Delta},$$

so

$$r_\Delta \geq \frac{2/T}{6\sqrt{2/\pi}} = \frac{\sqrt{\pi}}{3\sqrt{2}\,T}.$$

Since the interior of each Delaunay triangle is isometrically embedded in the surface, the inscribed Euclidean disk in Δ is an embedded disk in X_{t_k}. Thus one may take

$$R := \frac{\sqrt{\pi}}{3\sqrt{2}\,T}.$$

$\square$

Let $\mathfrak{P}_t$ denote the union of the closures of all pipe cylinders on $X_t = g_t X$, and let $A(P_t)$ be its area.

Lemma 3.4.12. *Assume that the condition of Lemma 3.4.11 fails. Suppose there exist a* *constant $K > 0$ and a sequence $t_k \to \infty$ such that*

$$A(\mathfrak{P}_{t_k}) > K \qquad \textit{for all } k.$$

Then there exist a constant $R > 0$ and a sequence $t'_k \to \infty$ such that each $X_{t'_k}$ contains an embedded disk of radius at least R.

Proof. By Lemmas 3.4.8 and 3.4.9, for each k the pipe cylinders on X_{t_k} are pairwise disjoint and there are at most M of them, where M depends only on the stratum. Hence at least one pipe cylinder $C_k \subset X_{t_k}$ has area

$$a(C_k) \geq \frac{K}{M}.$$

Let $c_k(s)$ and $h_k(s)$ be the circumference and height of the image of C_k after flowing forward for time $s \geq 0$. The area is preserved, so

$$c_k(s)\, h_k(s) = a(C_k) \geq \frac{K}{M}.$$

At time $s = 0$, C_k is a pipe cylinder, so $c_k(0) < h_k(0)$. As $s \to +\infty$, the circumference in its periodic direction tends to $+\infty$, so by continuity there exists $s_k \geq 0$ such that

$$c_k(s_k) = h_k(s_k) = \sqrt{a(C_k)} \geq \sqrt{\frac{K}{M}}.$$

Thus, on the later surface

$$X_{t'_k} := X_{t_k + s_k},$$

the image of C_k is a perfect cylinder of height and circumference at least $\sqrt{K/M}$. Any flat cylinder of height L and circumference L contains an embedded Euclidean disk of radius

$L/2$, so $X_{t'_k}$ contains an embedded disk of radius at least

$$R := \frac{1}{2}\sqrt{\frac{K}{M}}.$$

Since $t'_k \geq t_k \to \infty$, we have $t'_k \to \infty$. $\square$

Lemma 3.4.13. *Assume that the conditions of Lemmas 3.4.11 and 3.4.12 both fail. In other words, assume that for all sufficiently large t the surface $X_t = g_t X$ contains at least one pipe cylinder, and that*

$$A(\mathfrak{P}_t) \longrightarrow 0 \qquad \textit{as } t \to \infty.$$

Then there exist $R > 0$ and $\tau > 0$ such that for every $t > \tau$, the surface X_t contains an embedded disk of radius at least R.

Proof. Let $\Sigma_t = \{g_t(s_1), \ldots, g_t(s_n)\}$ be the singular set of X_t, and let

$$\Sigma_t(\delta) := \{x \in X_t : d(x, \Sigma_t) \leq \delta\}.$$

If the cone angle at $g_t(s_i)$ is $2\pi(k_i+1)$, then the Euclidean area of the radius-δ neighborhood of $g_t(s_i)$ is $\pi(k_i + 1)\delta^2$. Hence

$$\mathcal{L}(\Sigma_t(\delta)) \leq \pi\Big(\sum_{i=1}^{n}(k_i + 1)\Big)\delta^2 = \pi(2g - 2 + n)\delta^2.$$

Let

$$C_{\mathrm{str}} := \pi(2g - 2 + n).$$

Choose $\varepsilon > 0$ and $\delta > 0$ so that

$$1 - \varepsilon - C_{\mathrm{str}}\delta^2 > 0.$$

Since $A(\mathfrak{P}_t) \to 0$, there exists $\tau > 0$ such that

$$A(\mathfrak{P}_t) < \varepsilon \qquad \text{for all } t > \tau.$$

For such t, the set

$$F_\delta(t) := X_t \setminus \big(P_t \cup \Sigma_t(\delta)\big)$$

has positive area, hence is nonempty. Fix $x \in F_\delta(t)$.

Because $d(x, \Sigma_t) > \delta$, there is a well-defined local developing map

$$\iota_x : B_\delta(0) \to X_t$$

which is a local isometry and satisfies $\iota_x(0) = x$.

We claim that the restriction of ι_x to $B_{\delta/4}(0)$ is injective. Suppose not. Then there exist distinct points $u, v \in B_{\delta/4}(0)$ such that

$$\iota_x(u) = \iota_x(v).$$

In translation coordinates, the two local branches differ by a translation

$$T(z) = z + w, \qquad w := v - u.$$

Since

$$|w| = |v - u| < \frac{\delta}{2},$$

we have $w \in B_\delta(0)$, and therefore

$$\iota_x(w) = \iota_x(0) = x.$$

So the straight segment from 0 to w projects to a simple closed geodesic γ based at x of length

$$c := |w| < \frac{\delta}{2}.$$

Let C be the maximal cylinder containing γ. Let a and b be the distances from γ to the two boundary components of C, so that

$$h(C) = a + b.$$

Each boundary component has total length c. Therefore on each boundary component one can find a singularity whose tangential displacement from the perpendicular through x is at most $c/2$. It follows that there are singularities at Euclidean distance at most

$$\sqrt{a^2 + (c/2)^2} \qquad \text{and} \qquad \sqrt{b^2 + (c/2)^2}$$

from x. Since $x \notin \Sigma_t(\delta)$, every singularity is at distance bigger than δ from x, hence

$$a > \sqrt{\delta^2 - (c/2)^2}, \qquad b > \sqrt{\delta^2 - (c/2)^2}.$$

Therefore

$$h(C) = a + b > 2\sqrt{\delta^2 - (c/2)^2}.$$

Because $c < \delta/2$, the right-hand side is strictly larger than c, so

$$h(C) > c.$$

Thus C is a pipe cylinder. But $x \in \gamma \subset C \subset \mathfrak{P}_t$, contradicting the choice $x \in X_t \setminus \mathfrak{P}_t$.

This contradiction proves that $\iota_x|_{B_{\delta/4}(0)}$ is injective. Hence X_t contains an embedded disk of radius $\delta/4$ centered at x.

Since $t > \tau$ was arbitrary, the conclusion holds with

$$R = \frac{\delta}{4}.$$

$\square$

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