

THE UNIVERSITY OF CHICAGO

GEOMETRIC ASPECTS OF LIOUVILLE QUANTUM GRAVITY AND DISPERSIVE
EQUATIONS

A DISSERTATION SUBMITTED TO
THE FACULTY OF THE DIVISION OF THE PHYSICAL SCIENCES DIVISION
IN CANDIDACY FOR THE DEGREE OF
DOCTOR OF PHILOSOPHY

DEPARTMENT OF MATHEMATICS

BY
ANDRES A. CONTRERAS HIP

CHICAGO, ILLINOIS

JUNE 2026

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ORCID iD: <https://orcid.org/0000-0002-3429-978X>

To my brothers

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ACKNOWLEDGMENTS

I would like to express my deepest gratitude to my advisors, Carlos Kenig and Ewain Gwynne, for their exceptional guidance and support throughout my PhD. I also wish to thank the members of my committee for their time and for many stimulating discussions. My thanks also go to my collaborators, from whom I have learned a lot. Finally, I want to thank all the other people who have supported me over the years, including my family and friends.

ABSTRACT

In this thesis, we study geometric aspects of exponential metrics associated with log-correlated Gaussian fields, as well as analytic properties of dispersive wave equations. The first part focuses on the structure of exponential metrics in arbitrary dimension, where we establish that Liouville quantum gravity (LQG) metrics can approximate any Riemannian metric equipped with a locally finite measure. In the second part, we investigate geometric features of LQG and related random planar maps, introducing a notion of Gaussian curvature for LQG surfaces and analyzing its discrete counterpart for mated-CRT maps. We obtain asymptotics and scaling limits for curvature observables. The final part focuses on channels of energy estimates for linearized wave equations. This type of estimates is essential for establishing soliton resolution via the channels of energy method of Kenig-Merle. We prove such estimates for the wave maps equation linearized around a ground state, and more generally for a class of equations with analytic nonlinearities in even dimensions greater than or equal to 8.

CHAPTER 1

INTRODUCTION

The goal of this thesis is to study geometric and probabilistic properties of exponential metrics associated with log-correlated Gaussian fields, as well as certain structural properties of dispersive partial differential equations. The first part focuses on random surfaces arising from these metrics, particularly on the flexibility of the associated measure and metric. The second part investigates the discrete Gaussian curvature of a related probabilistic model and its connections to continuum scaling limits. The final part concerns linearized wave-type equations, using the channels of energy method to study the behavior of solutions and the distribution of their energy.

1.1 Geometric properties of exponential metrics associated with log-correlated Gaussian fields

In statistical mechanics, one studies random systems through their large-scale behavior, usually as a physically relevant parameter tends to infinity. Mathematically, one takes scaling limits of discrete models, yielding continuous approximations. Liouville quantum gravity (LQG) is one such limit: a one-parameter family of random fractal surfaces, conjectured to arise as the scaling limit of many discrete models. Introduced by Polyakov and later made rigorous in works such as Duplantier and Sheffield [2011], Rhodes and Vargas [2014], Sheffield [2016b], Ding et al. [2020], and Gwynne and Miller [2021], LQG has been shown to describe limits in only a few cases (notably uniform random planar maps Miller and Sheffield [2015], Miller and Sheffield [2021a], and Miller and Sheffield [2021b]). A natural higher-dimensional analogue of LQG is given by exponential metrics of log-correlated fields. By now, the 2d theory of LQG has been extensively studied. However many questions about higher dimensional extensions remain open: subsequential limits are known to exist due to

Ding et al. [2020], but have not been proven to be unique, and thus at the moment a unique analogue of LQG in higher dimensions has not been properly constructed. Planar maps are often used as canonical discrete models for random geometry. Many are conjectured to converge to an LQG surface (see Figure 1.1 for an illustration of an LQG surface), a canonical model for random surfaces defined as an exponential metric of a Gaussian free field, but few are actually proven to converge. The definition of the LQG surface metric and measure are highly nontrivial: they were established in a series of works Duplantier and Sheffield [2011], Ding et al. [2020], Gwynne and Miller [2021]. A key tool in doing so is the *mating-of-trees theorem* by Duplantier et al. [2021]: it provides a way of constructing certain random planar maps decorated by statistical physics models from two correlated Brownian motions. It complements earlier discrete bijections (Schaeffer [1998], Mullin [1967]) by providing a continuum encoding via correlated Brownian motion.

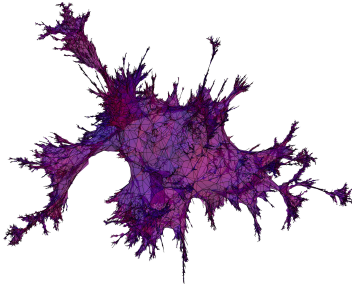


Figure 1.1: Simulation of Liouville quantum gravity.

1.1.1 Support of LQG

Given the rough regularity of LQG surfaces, a natural question is: what metric measure spaces lie in the support of the law of LQG? More precisely, what metric measure spaces can be approximated by the LQG metric and measure in the Gromov-Hausdorff-Prokhorov topology with positive probability? In Contreras Hip and Gwynne [2023], we show that the support includes all Riemannian metric spaces equipped with arbitrary locally finite measures on it. This shows that, despite their low regularity, LQG surfaces can approximate

any smooth metric. The result is valid in all dimensions.

A striking feature of this result is that the metric and measure can be approximated essentially independently, without any compatibility condition between the two, and also to a lesser extent, that the support also includes pearl spaces.

The proof relies on a combination of tools, including the so-called “white noise decomposition” of the GFF in order to separate the rough but small part from the large but smooth component, and the “adding bump functions” technique together with Weyl scaling to control the smooth component. A more precise statement of the result can be found in Chapter 2.

This chapter is based on joint work with Ewain Gwynne.

1.1.2 Gaussian curvature in LQG and random planar maps

LQG surfaces have a metric structure, but at the same time have low-regularity. A natural question to ask is to what extent Riemannian geometry concepts extend to LQG surfaces, and whether they are compatible with analogous concepts in the discrete setting. To this end, we first defined a notion of Gaussian curvature for LQG surfaces, and studied the discrete curvature on a class of mated CRT-maps.

Suppose $\mathcal{G}$ is a planar map. To define the curvature, suppose we set all angles at each corner of a face F of $\mathcal{G}$ to be $\frac{|F|-2}{|F|}\pi$, where $|F|$ is the number of vertices incident to F . Inspired by the Gauss-Bonnet formula, it is natural to define the discrete curvature at a vertex v as

$$K_M(v) := 2\pi - \sum_F \frac{|F|-2}{|F|}\pi$$

where the sum is taken over all possible faces F incident to v , that is, over all faces whose boundary contains v . Note that in the case the graph $\mathcal{G}$ is a triangulation, this is equivalent to

$$K_{\mathcal{G}}(v) = \frac{\pi}{3} (6 - \deg(v)).$$

In particular, we can define the discrete Gaussian curvature for the Poisson mated CRT map $\mathcal{G}_\varepsilon$ (a variant of the usual mated CRT map). We prove two main results: for the first, we look at the discrete curvature summed against a smooth compactly supported test function. We show that with probability going to 1 as $\varepsilon \rightarrow 0$,

$$\sum_{v \in \mathcal{V}\tilde{\mathcal{G}}_\varepsilon} f(\tilde{\eta}(v)) K_{\tilde{\mathcal{G}}_\varepsilon}(v) = \varepsilon^{o(1)}.$$

In contrast, the total curvature on a CRT cell behaves differently, as our other main result shows. We show that with probability going to 1 as $\varepsilon \rightarrow 0$, we have

$$\frac{\sum_{v \in \mathcal{V}\tilde{\mathcal{G}}_\varepsilon \cap \mathcal{C}_0} K_{\tilde{\mathcal{G}}_\varepsilon}(v)}{\varepsilon^{-\frac{1}{4}}} \rightarrow \alpha \mathcal{B},$$

in law, where the law of $\mathcal{B}$ can be described as a Gaussian random variable with variance given by the boundary length of the mated CRT map cell $\mathcal{C}_0$. More precise statements can be found in Chapter 3.

This chapter is based on joint work with Ewain Gwynne.

1.2 Energy channels for linearized wave equations

Some important questions in the theory of Dispersive PDEs are the stability and classification of solutions to nonlinear dispersive equations. These equations include the wave maps, nonlinear Schrödinger, and Yang-Mills equations just to name a few. There has been considerable interest in the study of these equations, modeling wave propagation phenomena arising in multiple areas in physics, such as optics, ferromagnetism, general relativity, and water waves to name a few. The long term behavior of these solutions, the so-called *soliton resolution*, and issues such as blow up, global well-posedness and scattering are of particular interest.

A key tool in understanding the long-time dynamics of dispersive equations is the channels of energy estimate, which quantifies how much energy must escape to infinity for solutions not trapped near a soliton. For the energy-critical focusing nonlinear wave equation in three dimensions, Kenig–Merle introduced this method in Kenig and Merle [2008] and used it to prove soliton resolution for radial solutions. Later, in works by Collot, Duyckaerts, Kenig, and Merle, they proved the soliton resolution for radial solutions to the energy critical focusing wave equation in all odd dimensions, as well as dimensions 4 and 6. They also proved similar results for the wave maps equation and a class of similar equations with appropriate analytic nonlinearities. The only even dimensions they treated were 4 and 6 due to the fact that in even dimensions, the channels of energy estimate is weaker and takes a different form. Later Jendrej-Lawrie proved the soliton resolution conjecture for wave maps in the equivariant case and for energy critical NLW using a different approach bypassing the channels of energy estimate used by Kenig and Merle. Nevertheless, the channels of energy estimate is of independent interest, as it provides key information about the propagation of energy of solutions to the wave maps equation. Proving a channels of energy estimate becomes increasingly difficult as the dimension increases, due to the growing number of degenerate directions of the linear wave operator. In Contreras Hip [2025], we prove such a channels of energy estimate for the wave maps equation linearized around a ground state. More specifically, we establish a quantitative lower bound on the amount of energy that must radiate to spatial infinity for solutions to the linearized equation, after projecting away from a finite-dimensional subspace associated with the symmetries and resonant directions of the problem. This estimate captures the dispersive component of the dynamics in high even dimensions, where the presence of additional degeneracies significantly complicates the analysis. This is a crucial step towards recovering the result of Jendrej and Lawrie [2025] using the channels of energy method. The channels of energy estimate we prove also applies to linearized wave equations corresponding to a class of analytic nonlinearities. A more

precise statment of the result can be found in Chapter 4.

CHAPTER 2

A SUPPORT THEOREM FOR EXPONENTIAL METRICS OF LOG-CORRELATED GAUSSIAN FIELDS IN ARBITRARY DIMENSION

The results in this chapter were developed in collaboration with Ewain Gwynne. The mathematical ideas and proof were developed jointly.

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2.1 Introduction

Liouville quantum gravity (LQG) is a family of random metric measure spaces parametrized by Riemann surfaces, depending on a parameter $\gamma \in (0, 2)$, which describe the scaling limits of random planar maps. LQG was first introduced (non-rigourously) by Polyakov in Polyakov [1981] as a class of canonical models of random surfaces. One can define LQG surfaces with the topology of any desired orientable surface Duplantier et al. [2021], David et al. [2016a,b], Remy [2018], Guillarmou et al. [2019]. However, in this chapter we will only consider LQG surfaces with the topology of the whole plane.

The main results of this chapter describe the closed support of the law of the LQG metric measure space. More precisely, we show that if $\mathfrak{d}$ is any length metric on $\mathbb{R}^2$ and $\mathfrak{m}$ is any probability measure on $\mathbb{R}^2$, then with positive probability (D_h, μ_h) is close to $(\mathfrak{d}, \mathfrak{m})$ with respect to the uniform distance and the Prokhorov distance (Theorems 2.1.4 and Corollary 2.1.6). Our results provide a convenient “black box” for whenever one needs to show that certain events for LQG hold with positive probability.

Several recent works (see, e.g., Ding et al. [2023], Dello Schiavo et al. [2023, 2021], Cerclé [2022]) have studied analogs of LQG in dimension $d \geq 3$. The results of this chapter also

extend to this higher-dimensional setting, conditional on some properties of the higher-dimensional analog of the LQG metric whose proofs have not yet been written down. In this setting, our results raise important questions about what it means for a random Riemannian metric to be “conformally flat”; see Section 2.1.3 for more details.

Roughly speaking, the aforementioned works on higher-dimensional analogs of LQG consider a conformally flat Riemannian metric tensor on $\mathbb{R}^d$, $d \geq 3$, of the form

$$e^{\gamma h}(dx_1^2 + \cdots + dx_d^2), \tag{2.1}$$

where $\gamma \in (0, \sqrt{2d})$ and h is a log-correlated Gaussian field on $\mathbb{R}^d$, or some variant thereof (the works Dello Schiavo et al. [2021], Cerclé [2022] also consider similar Riemannian metric tensors on d -manifolds for even values of d).

As in the two-dimensional case, the Riemannian metric tensor (2.1) does not make literal sense since h is a generalized function and is not pointwise defined. However, one can construct the volume form associated with (2.1) as a random measure on $\mathbb{R}^d$ via the general theory of Gaussian multiplicative chaos Kahane [1985], Rhodes and Vargas [2014], Berestycki [2017], Berestycki and Powell.

The recent paper Ding et al. [2023] proves the tightness of a natural approximation scheme for the Riemannian distance function associated with (2.1). This paper also shows that every subsequential limit is a metric on $\mathbb{R}^d$ which induces the Euclidean topology. It is expected, but not yet proven, that the subsequential limit is unique and is characterized by a similar list of axioms as in the two-dimensional case Gwynne and Miller [2021]; see Definition 2.1.3. Our results for $d \geq 3$ are proven conditional on these axioms.

2.1.1 Measure and metric definitions

We will now discuss the definitions of the measure and metric associated with the log-correlated Gaussian field h .

Definition 2.1.1. The **log-correlated Gaussian field** is the centered Gaussian random generalized function h^0 defined on $\mathbb{R}^d$ up to additive constants, with covariance structure

$$\text{Cov} \left(h^0(z), h^0(w) \right) = G(z, w) := \log \left(\frac{1}{|z - w|} \right). \quad (2.2)$$

That is, if f is a smooth compactly supported test function on $\mathbb{R}^d$ with $\int_{\partial B_1(0)} f d\sigma = 0$, then $\int_{\mathbb{R}^d} f(z) h^0(z) dz$ is a centered Gaussian random variable with variance

$$\int_{\mathbb{R}^d} \int_{\mathbb{R}^d} f(z) f(w) \log \left(\frac{1}{|z - w|} \right) dz dw.$$

We define the field h to be the field h^0 normalized so that its unit sphere average is zero, i.e.,

$$h = h^0 - \int_{\partial B_1(0)} h^0(z) d\sigma(z).$$

In dimension $d = 2$, the field h is the whole-plane Gaussian free field (appropriately normalized) and has the explicit covariance structure given by

$$\text{Cov} (h(z), h(w)) := \log \left(\frac{\max\{|z|, 1\} \max\{|w|, 1\}}{|z - w|} \right). \quad (2.3)$$

For an exposition of the Gaussian free field in $d = 2$, see, e.g., Werner and Powell [2020], Berestycki and Powell, Sheffield [2007]. For an exposition of log-correlated Gaussian fields in general dimension, see, e.g., Lodhia et al. [2016], Duplantier et al. [2017].

Formally, for $\gamma \in (0, \sqrt{2d})$, the γ -**exponential geometry** associated with h is described

by the random Riemannian metric tensor

$$e^{\gamma h} (dx_1^2 + \cdots + dx_d^2), \quad (2.4)$$

where $dx_1^2 + \cdots + dx_d^2$ denotes the Euclidean metric tensor on $\mathbb{R}^d$. This metric tensor does not make literal sense since h is not a random function but instead it is a random generalized function, and hence $e^{\gamma h}$ is not well defined. However, at least in dimension 2, we can still define a random metric and measure associated with (2.4). To do so, we first approximate $\tilde{h}$ with well chosen mollifications, then pass to the limit.

One possible construction in dimension $d = 2$ is as follows. Let p_t denote the heat kernel in dimension 2,

$$p_t(z) := \frac{1}{2\pi t} e^{-\frac{|z|^2}{2t}}. \quad (2.5)$$

Define the mollified log-correlated Gaussian field

$$h_\varepsilon(z) := p_{\varepsilon^2/2} * h(z) = \int_{\mathbb{R}^d} h(w) p_t(z - w) dw,$$

where the integral is interpreted in the distributional sense. We can then define the **γ -LQG measure** as the a.s. weak limit

$$\lim_{\varepsilon \rightarrow 0} \varepsilon^{\gamma^2/2} e^{\gamma h_\varepsilon(z)}. \quad (2.6)$$

This construction is a special case of the theory of **Gaussian multiplicative chaos** Kahane [1985], Rhodes and Vargas [2014], Berestycki [2017], Berestycki and Powell. The measure μ_h is locally finite, non-atomic, assigns positive mass to every open set, and is mutually singular with respect to Lebesgue measure. This construction is also possible in higher dimensions. We also have analogous properties for the Gaussian multiplicative chaos measure μ_h :

1. Almost surely, μ_h is a non-atomic Radon measure.
2. If $U \subset \mathbb{R}^d$ is any deterministic open set, then $\mu_h(U)$ is given by a measurable function

of $h|_U$.

3. Almost surely, we have that for any continuous function $f : \mathbb{R}^d \rightarrow \mathbb{R}$, $e^{\gamma f} \cdot \mu_h = \mu_{h+f}$.
4. Almost surely, we have the following statement. Suppose that ϕ is a composition of dilations, translations, and rotations. Then

$$\mu_{h \circ \phi + Q \log |\phi'|}(A) = \mu_h(\phi(A))$$

for any measurable $A \subset \mathbb{R}^d$.

The γ -LQG metric is defined similarly. Let $d_\gamma > 2$ be the fractal dimension of γ -LQG. Prior to the construction of the LQG metric, this number was shown to arise in various approximations of the LQG metric in Ding et al. [2019], Ding and Gwynne [2018]. After the metric was constructed, it was shown that d_γ is its Hausdorff dimension [Gwynne and Pfeffer, 2019, Corollary 1.7]. We note that d_γ is not known explicitly except that $d_{\sqrt{8/3}} = 4$. Let

$$\xi := \frac{\gamma}{d_\gamma}. \quad (2.7)$$

Similarly to (2.6), we define the approximating metrics

$$D_h^\varepsilon(z, w) := \inf_{P: z \rightarrow w} \int_0^1 e^{\xi h_\varepsilon(P(t))} |P'(t)| dt \quad (2.8)$$

where the infimum is over all piecewise continuously differentiable paths from z to w . It was shown in the series of papers Ding et al. [2020], Gwynne and Miller [2020b], Dubédat et al. [2020], Gwynne and Miller [2020a, 2021] that there are normalizing constants $\{\mathfrak{a}_\varepsilon\}_{\varepsilon>0}$ and a random metric D_h on $\mathbb{R}^d$ such that

$$D_h(z, w) = \lim_{\varepsilon \rightarrow 0} \mathfrak{a}_\varepsilon^{-1} D_h^\varepsilon(z, w) \quad (2.9)$$

in probability with respect to the topology of uniform convergence on compact subsets of $\mathbb{R}^d \times \mathbb{R}^d$. The metric D_h is defined to be the γ -**LQG metric**. The metric D_h induces the same topology on $\mathbb{R}^d$ as the Euclidean metric, but has very different geometric properties. See Ding et al. [2021] for a survey of known results about the LQG metric.

In higher dimensions, the metric measure space D_h associated with the log-correlated Gaussian field h can be defined, but is not yet defined uniquely Ding et al. [2023]. However, it is conjectured that such a metric can be defined uniquely, and that it satisfies the following “strong” axioms. First, we define length metrics.

Definition 2.1.2. Let $(X, \mathfrak{d})$ be a metric space. For a path $P : [a, b] \rightarrow X$, we define the $\mathfrak{d}$ -length of P by

$$\ell_{\mathfrak{d}}(P) := \sup_T \sum_{i=1}^{\#T} \mathfrak{d}(P(t_i), P(t_{i-1}))$$

where the supremum is over all partitions $T : a = t_0 < \dots < t_{\#T} = b$ of $[a, b]$. We say that $\mathfrak{d}$ is a **length metric** if for each $x, y \in X$, the distance $\mathfrak{d}(x, y)$ is equal to the infimum of the $\mathfrak{d}$ -lengths of the $\mathfrak{d}$ -continuous paths from x to y .

Now we will define the Weyl scaling as follows: for any continuous $f : \mathbb{R}^d \rightarrow \mathbb{R}$ and any length metric $\mathfrak{d}$ on $\mathbb{R}^d$ inducing the Euclidean topology, we let

$$(e^f \cdot \mathfrak{d})(x, y) = \inf_P \int_0^{\ell(P)} e^{f(P(t))} dt \quad (2.10)$$

where $\ell(P)$ is the $\mathfrak{d}$ -length of P and the inf is over all paths from x to y parametrized by $\mathfrak{d}$ -length. For any open set $U \subseteq \mathbb{R}^d$ we also define the internal metric $\mathfrak{d}(\cdot, \cdot; U)$ by

$$\mathfrak{d}(x, y; U) = \inf_{P \subseteq U} \ell_{\mathfrak{d}}(P),$$

where the infimum is taken over paths $P : [0, 1] \rightarrow U$ such that $P(0) = x$ and $P(1) = y$. Then we will assume the metric D_h satisfies the following axioms.

Definition 2.1.3. We say a map from random generalized random functions h on $\mathbb{R}^d$ to metric measure spaces $(\mathbb{R}^d, D_h, \mu_h)$ satisfies the strong LQG axioms if there exist constants ξ, γ such that the following hold:

1. Almost surely, D_h is a length metric.
2. Let U be a deterministic open set. Then the internal metric $D_h(\cdot, \cdot; U)$ is a measurable function of $h|_U$.
3. Weyl scaling: Recall the definition of Weyl scaling (2.10). Then, a.s., for every continuous function f , we have $e^{\xi f} \cdot D_h = D_{h+f}$.
4. Coordinate change: let $\lambda > 0$ and $z_0 \in \mathbb{R}^d$. Then almost surely,

$$D_h(\lambda x + z_0, \lambda y + z_0) = D_{h(\lambda \cdot + z_0) + Q \log \lambda}(x, y), \quad \forall x, y \in \mathbb{R}^d$$

where $Q := \frac{\gamma}{2} + \frac{d}{\gamma}$.

Note that the above axioms are satisfied by the LQG metric in dimension $d = 2$ (see Gwynne and Miller [2021] and Dubédat et al. [2020]). It is believed that the above axioms also hold for any subsequential limit obtain in Ding et al. [2023] for $d \geq 3$, but no proof of this has been written yet.

We will also assume that

$$\frac{\gamma}{\xi} > d \tag{2.11}$$

as in the two dimensional case. We expect this to hold, since one expects $\frac{\gamma}{\xi} = \dim(\mathbb{R}^d; D_h)$ and $\dim(\mathbb{R}^d; D_h) > d$ since D_h 's geometry should be strictly "rougher" than Euclidean. We expect this can be proven rigorously with comparison arguments, but we do not do this here, and is the subject of forthcoming work of the first author and Z. Zhuang.

We will additionally assume that

$$\mathbb{E}(\text{diam}_{D_h}([0, 1]^d)^{\bar{p}}) < \infty \quad (2.12)$$

for some $\bar{p} > d$. This also holds for the LQG metric in dimension 2 (see [Dubédat et al., 2020, Proposition 3.9]). The axioms above are all used throughout the paper, while the bounds (2.11) and (2.12) are technical assumption, only used in Subsection 2.3.1 and Subsection 2.3.2, respectively.

2.1.2 Main results

Recall that our goal is to determine the closed support of the law of $(\mathbb{R}^d, D_h, \mu_h)$ with respect to some reasonable topology on the space of metric measure spaces, assuming h is as in Definition 2.1.1.

The LQG metric (and any higher dimensional analogue) is a length metric essentially by construction. The class of length metrics is preserved under various forms of convergence, e.g., uniform convergence [Burago et al., 2001, Exercise 2.4.19] and Gromov-Hausdorff convergence [Burago et al., 2001, Theorem 7.5.1]. Hence, the LQG metric cannot approximate a metric which is not a length metric. Our first main result implies that this is essentially the only constraint.

We say that $(X, \mathfrak{d})$ is **boundedly compact** if each closed bounded subset of X is $\mathfrak{d}$ -compact.¹

Theorem 2.1.4. *Let $\mathfrak{d}$ be a boundedly compact length metric on $\mathbb{R}^d$ which induces the Euclidean topology and let $\mathfrak{m}$ be a locally finite Borel measure on $\mathbb{R}^d$. Suppose that there*

1. The reason for imposing the bounded compactness assumption is to make it so that all of the $\mathfrak{d}$ -geodesics going between points in $[-R, R]^d$ are contained in some ($\mathfrak{d}$ -dependent) compact set. We expect that Theorem 2.1.4 is true without the boundedly compact assumption, and that this can be obtained by approximating the restriction to $[-R, R]^d$ of a length metric which is not boundedly compact by the corresponding restrictions of boundedly compact length metrics. However, for the sake of brevity, we do not carry this out in this chapter.

exists a map $h \mapsto (\mathbb{R}^d, D_h, \mu_h)$ as in Definition 2.1.3 with $\xi < 1$, and suppose h is a log-correlated Gaussian field. Let D_h, μ_h be as Definition 2.1.3. Then D_h and μ_h approximate the metric $\mathfrak{d}$ and the measure $\mathfrak{m}$ respectively with positive probability. More precisely, for any $\varepsilon, R > 0$, we have with positive probability that

$$|D_h(x, y) - \mathfrak{d}(x, y)| < \varepsilon$$

for all $x, y \in [-R, R]^d$ and at the same time for any Borel set $A \subseteq [-R, R]^d$ we have

$$\mathfrak{m}(A) \leq \mu_h(A_\varepsilon) + \varepsilon$$

and

$$\mu_h(A) \leq \mathfrak{m}(A_\varepsilon) + \varepsilon$$

where $A_\varepsilon := \{z \in \mathbb{R}^d : \inf_{u \in A} |z - u| < \varepsilon\}$.

Perhaps surprisingly, we do not need to assume any relationship between $\mathfrak{d}$ and $\mathfrak{m}$ in Theorem 2.1.4. Even though D_h and μ_h are closely related to each other (see, e.g., Gwynne and Sung [2022]), there is a positive chance for them to behave quite differently. A much weaker result in this direction in the case of $d = 2$ appears as [Bou-Rabee and Gwynne, 2022, Proposition 11.9].

In dimension two, Theorem 2.1.4 can be viewed as a general-purpose theorem for making the LQG measure and metric do things with positive probability. In this sense, these results are an LQG analog of “support theorems” for various stochastic processes, which characterize the elements of the state space that they can approximate with positive probability. Examples of such support theorems include the classical fact that d -dimensional Brownian motion can be made to approximate any continuous path in $\mathbb{R}^d$; and analogous statements for Schramm-Loewner evolution (see, e.g., [Miller and Wu, 2017, Section 2]).

We expect that Theorem 2.1.4 and the some of the techniques used in its proof will be useful in future works concerning LQG and its higher dimensional analogs. Indeed, when studying LQG, one often needs to show that D_h and/or μ_h has some prescribed behavior with positive probability. Prior to this work, this was typically done via ad hoc methods based on the fact that adding a smooth bump function to h changes its law in an absolutely continuous way. See [Ding et al., 2021, Section 4.1] for an explanation of this technique. Arguments of this type can sometimes be quite complicated, see, e.g., [Bou-Rabee and Gwynne, 2022, Sections 11.2 and 11.3] or [Gwynne and Miller, 2021, Section 5]. In future work, such arguments could be replaced by applications of Theorem 2.1.4.

In the course of proving Theorem 2.1.4 in dimension $d \geq 3$, we also establish several independently interesting facts about log-correlated Gaussian fields which we expect to be useful elsewhere (see e.g. Lemmas 2.2.4, 2.2.5, and 2.2.6).

2.1.3 Interpretation of results in dimension $d \geq 3$

In dimension $d \geq 3$, Theorem 2.1.4 is of interest for the same reasons as in the two-dimensional case, as discussed just above. It also raises interesting questions about the interpretation of the random Riemannian metric tensor (2.1), which we will discuss in this section.

A Riemannian metric tensor g on a domain $U \subset \mathbb{R}^d$ is **conformally flat** if there is a continuous function $f : U \rightarrow \mathbb{R}$ such that $g = e^f(dx_1^2 + \cdots + dx_d^2)$, where $dx_1^2 + \cdots + dx_d^2$ is the Euclidean Riemmanian metric tensor. More generally, a d -dimensional Riemannian manifold is **locally conformally flat** if it admits an atlas where in each chart, the Riemannian metric is conformally flat. It is a classical theorem that every two dimensional Riemannian manifold is locally conformally flat, i.e., it can locally be expressed in isothermal coordinates. However, this is not true in dimension $d \geq 3$. Rather, a d -dimensional Riemannian manifold is locally conformally flat if and only if its Cotton tensor (for $d = 3$) or its Weyl tensor (for $d \geq 4$) is

identically zero.

Theorem 2.1.4 implies that the random metric and measure associated with (2.1) can approximate *any* length metric and locally finite Borel measure on $\mathbb{R}^d$ with positive probability, even a metric and measure associated with a Riemannian metric tensor which is not conformally flat. This leads one to wonder whether the random Riemannian metric tensor associated with (2.1) should be thought of as a random conformally flat metric on $\mathbb{R}^d$, or as a general random Riemannian metric on $\mathbb{R}^d$. This is closely related to some fundamental questions about analogs of LQG in higher dimensions, e.g., the following.

- Let $n \in \mathbb{N}$ and let M_n be a uniform sample from the set of triangulations of the d -sphere (i.e., simplicial complexes homeomorphic to $\mathbb{S}^d$) with n d -simplices. For $d = 2$, it is known that M_n (equipped with its graph distances, appropriately rescaled) converges in the Gromov-Hausdorff sense to $\sqrt{8/3}$ -LQG Le Gall [2013], Miermont [2013], Miller and Sheffield [2015, 2021a] (see also Holden and Sun [2023] for a stronger topology of convergence). For $d \geq 3$, does M_n converge in the scaling limit (e.g., with respect to the Gromov-Hausdorff distance) to a d -manifold, equipped with a random Riemannian distance function of the form (2.1)? Or is it the case that to get convergence to a random Riemannian distance function of the form (2.1), one needs to instead take a uniform sample from a restricted set of triangulations of $\mathbb{S}^d$ which are required to satisfy some discrete analog of conformal flatness? See Ding et al. [2023], Bou-Rabee and Gwynne [2024] for related discussions.
- Consider the following random Riemannian metric tensor on $\mathbb{R}^d$:

$$\sum_{i,j=1}^d e^{\beta h_{i,j}} dx_i dx_j \tag{2.13}$$

where the $\beta > 0$ is a parameter and $h_{i,j}$ s are log-correlated Gaussian fields on $\mathbb{R}^d$ (coupled together in some manner). Can one make rigorous sense of the distance func-

tion associated with (2.13)? If so, then this distance function superficially seems to be supported on a more general class of metrics than (2.1) since (2.13) is not required to be conformally flat. However, our results show that any length metric on $\mathbb{R}^d$ can be approximated with positive probability by the Riemannian distance function associated with (2.13) (conditional on the conjecture that this Riemannian distance function satisfies the strong axioms). So, in what sense does (2.13) actually give a more general Riemannian metric than (2.1)? Could it be that the limit of a natural approximation scheme for the Riemannian distance function associated with (2.13) is actually just the Riemannian distance function associated with (2.1), where h is some function of $\{h_{i,j} : i, j = 1, \dots, d\}$ and γ is some function of β and d ?

We do not know the answers to the above questions, even conjecturally, but they seem to be worth pursuing further.

2.1.4 Applications of Theorem 2.1.4 in the two-dimensional case

Application to the LQG sphere

As a consequence of Theorem 2.1.4, one can obtain a similar result concerning a special LQG surface called the γ -LQG sphere, which is represented by a certain special variant of the GFF. There are a number of equivalent ways of defining the γ -LQG sphere. The first definitions appeared in Duplantier et al. [2021], David et al. [2016a] and were proven to be equivalent in Aru et al. [2017]. The definition we give here is [Aru et al., 2017, Definition 2.2] with $k = 3$ and $\alpha_1 = \alpha_2 = \alpha_3 = \gamma$.

Definition 2.1.5. Let $\gamma \in (0, 2)$ and let

$$Q := \frac{2}{\gamma} + \frac{\gamma}{2}. \tag{2.14}$$

The **Liouville field** is the random generalized function

$$h = h^{\mathbb{C}} + \gamma G(0, \cdot) + \gamma G(1, \cdot) - (2Q - \gamma) \log |\cdot|_+ \quad (2.15)$$

where $h^{\mathbb{C}}$ is the whole-plane GFF, $G(\cdot, \cdot)$ is its covariance kernel as in (2.2), and $\log |\cdot|_+ := \log(\max\{|z|, 1\})$. The **(triply marked, unit area) γ -LQG sphere** is the LQG surface parametrized by $\mathbb{C}$ represented by the field h , where the law of $\mathbb{P}_h$ of h is equal to the law $\mathbb{P}_{\tilde{h}}$ of

$$\tilde{h} := h - \frac{1}{\gamma} \log \mu_h(\mathbb{C}) \quad (2.16)$$

weighted by a γ -dependent constant times $[\mu_h(\mathbb{C})]^{4/\gamma^2-2}$, i.e., $\frac{d\mathbb{P}_h}{d\mathbb{P}_{\tilde{h}}} = C(\gamma) \times [\mu_h(\mathbb{C})]^{4/\gamma^2-2}$.

We note that if h is as in Definition 2.1.5, then $\mu_h(\mathbb{C}) = 1$ (this is because of the subtraction of $\frac{1}{\gamma} \log \mu_h(\mathbb{C})$ in (2.16)). Furthermore, one can check that the LQG metric associated with the LQG sphere extends continuously to the one-point compactification $\mathbb{C} \cup \{\infty\}$, so it can be viewed as a metric on the sphere (not just on $\mathbb{C}$). One reason why the γ -LQG sphere is special is that it is the LQG surface which arises as the scaling limit of random planar maps with the sphere topology. See e.g. subsection 2.1.4 for more details.

We also have the following result for LQG in dimension 2.

Corollary 2.1.6. *Let $\mathfrak{d}$ be a length metric on the sphere $\mathbb{S}^2 = \mathbb{C} \cup \{\infty\}$ which induces the Euclidean topology and let $\mathfrak{m}$ be a Borel measure on $\mathbb{S}^2$ with $\mathfrak{m}(\mathbb{S}^2) = 1$. Fix $\gamma \in (0, 2)$ and let $D_{\tilde{h}}$ and $\mu_{\tilde{h}}$ denote the metric and measure associated with the γ -LQG quantum sphere, viewed as a metric and measure on $\mathbb{S}^2$. For each $\varepsilon > 0$, it holds with positive probability that*

$$\left| D_{\tilde{h}}(x, y) - \mathfrak{d}(x, y) \right| \leq \varepsilon, \quad \forall x, y \in \mathbb{S}^2 \quad (2.17)$$

and at the same time for any Borel set $A \subset \mathbb{S}^2$,

$$\mathbf{m}(A) \leq \mu_{\tilde{h}}(A_\varepsilon) + \varepsilon \quad (2.18)$$

and

$$\mu_h(A) \leq \mathbf{m}(A_\varepsilon) + \varepsilon \quad (2.19)$$

where $A_\varepsilon := \{z \in \mathbb{C} : \inf_{u \in A} d_S(z, u) < \varepsilon\}$, with d_S being the Euclidean metric on $\mathbb{S}^2$. In particular, with positive probability we have that

$$d_{GHP} \left((\mathbb{S}^2, \mathfrak{d}, \mathbf{m}), (\mathbb{S}^2, D_{\tilde{h}}, \mu_{\tilde{h}}) \right) \leq \varepsilon. \quad (2.20)$$

To obtain Corollary 2.1.6, one can write the quantum sphere field $\tilde{h}$ in terms of the Gaussian free field,

$$\tilde{h} := \bar{h} - \frac{1}{\gamma} \log \mu_{\bar{h}}(\mathbb{C}),$$

where

$$\bar{h} = h + \gamma G(0, \cdot) + \gamma G(1, \cdot) - (2Q - \gamma) \log |\cdot|_+.$$

Since Theorem 2.1.4 lets us “fix” the total measure μ_h , it suffices to consider $\bar{h}$, and show that $(D_{\bar{h}}, \mu_{\bar{h}})$ is close to any conformally flat Riemannian metric $e^f \cdot d_0$ with positive probability. For this, we first approximate the function $\gamma G(0, \cdot) + \gamma G(1, \cdot) - (2Q - \gamma) \log |\cdot|_+$ by a smooth, finite Dirichlet energy function f_ε . This approximation can be chosen so that $f_\varepsilon(z) = \gamma G(0, z) + \gamma G(1, z) - (2Q - \gamma) \log |z|_+$ for $z \in B_{\frac{1}{\varepsilon}}(0) \setminus (B_\varepsilon(0) \cup B_\varepsilon(1))$. By the Cameron Martin property of the GFF, we know that Corollary 2.1.6 holds if we replace $\tilde{h}$ with $h + f_\varepsilon$. Now the final step is to show that both the measure and metric are “unaffected” by the singularities at $0, 1, \infty$. To do this, we note that if P is any geodesic between two points, then we can replace P by a path with similar LQG length avoiding ε neighborhoods of the singularities.

Application to random planar maps

Due to the convergence of uniform random planar maps toward $\sqrt{8/3}$ -LQG, Theorem 2.1.6 has implications for the study of random planar maps. To explain this, let M_n be sampled uniformly from the set of all quadrangulations of the sphere (planar maps whose faces all have degree 4) with $n \in \mathbb{N}$ total faces. Let D_n and μ_n denote the graph distance and the counting measure on vertices of M_n , respectively. We will now recall the Gromov-Hausdorff-Prokhorov distance. Suppose that $\mathbf{m}_1, \mathbf{m}_2$ are Borel measures on a metric space $(X, \mathfrak{d})$. The Prokhorov distance is given by

$$d_P(\mathbf{m}_1, \mathbf{m}_2) := \inf\{\varepsilon > 0 : \mathbf{m}_1(A) \leq \mathbf{m}_2(A_\varepsilon) + \varepsilon, \mathbf{m}_2(A) \leq \mathbf{m}_1(A_\varepsilon) + \varepsilon \text{ for all closed } A\},$$

where

$$A_\varepsilon := \{x \in X : \inf_{y \in A} \mathfrak{d}(x, y) < \varepsilon\}.$$

Also, recalling that the Hausdorff distance between pairs of sets A, B in a metric space $(X, \mathfrak{d})$ is defined by

$$d_H(A, B) := \inf\{r > 0 : A \subseteq \cup_{x \in B} B_{\mathfrak{d}}(x), B \subseteq \cup_{x \in A} B_{\mathfrak{d}}(x)\}.$$

We define the Gromov-Hausdorff distance between two metric spaces $(X_1, \mathfrak{d}_1)$ and $(X_2, \mathfrak{d}_2)$ as the infimum over all $r > 0$ such that there exists a third metric space $(Z, \mathfrak{d})$ and subspaces X'_1, X'_2 that are isometric to X_1 and X_2 respectively, and additionally $d_H(X'_1, X'_2) < r$. Finally, the Gromov-Hausdorff-Prokhorov distance between two metric measure spaces $(X_1, \mathfrak{d}_1, \mu_1)$ and $(X_2, \mathfrak{d}_2, \mu_2)$ is defined as

$$d_{GHP}((X_1, \mathfrak{d}_1, \mu_1), (X_2, \mathfrak{d}_2, \mu_2)) = \inf_{(Z, \mathfrak{d}), \iota_1, \iota_2} d_H(X_1, X_2) + d_P(\iota_1^* \mu_1, \iota_2^* \mu_2),$$

where the infimum is taken over isometric embeddings $\iota_1 : X_1 \rightarrow Z$ and $\iota_2 : X_2 \rightarrow Z$.

It was shown independently by Le Gall [2013] and Miermont [2013], building on many other works, that the metric measure spaces² $(M_n, n^{-1/4}D_n, n^{-1}\mu_n)$ converge in law in the Gromov-Hausdorff-Prokhorov sense to a random metric measure space called the **Brownian map**. See Le Gall [2014] for a survey of this work and Addario-Berry and Albenque [2017], Bettinelli et al. [2014], Addario-Berry and Albenque [2021], Marzouk [2022] for extensions to other types of random planar maps with the sphere topology.

Subsequently, Miller and Sheffield [2015, 2021a,b] constructed a metric associated with $\sqrt{8/3}$ -LQG (using a very different construction from the one described above). They then showed that the $\sqrt{8/3}$ -LQG sphere, equipped with this metric and its LQG area measure, is isometric to the Brownian map. Finally, it was shown in [Gwynne and Miller, 2021, Corollary 1.4] that the Miller-Sheffield metric a.s. coincides with the metric from (2.9) for $\gamma = \sqrt{8/3}$ ($\xi = 1/\sqrt{6}$). Combining these results shows that if h is the random generalized function associated with the $\sqrt{8/3}$ -LQG sphere and M_n is a uniform quadrangulation as above, then

$$(M_n, n^{-1/4}D_n, n^{-1}\mu_n) \rightarrow (\mathbb{S}^2, D_h, \mu_h) \quad (2.21)$$

in law with respect to the Gromov-Hausdorff-Prokhorov distance. The following corollary is immediate from Theorem 2.1.6 and (2.21).

Corollary 2.1.7. *For $n \in \mathbb{N}$, let M_n be a uniform quadrangulation of the sphere with n faces, as above. Let $(X, \mathfrak{d}, \mathfrak{m})$ be a metric space homeomorphic to the sphere, equipped with a length metric and a probability measure. For each $\epsilon > 0$, there exists $p = p(\epsilon, (X, \mathfrak{d}, \mathfrak{m})) > 0$ such that for each sufficiently large $n \in \mathbb{N}$,*

$$\mathbb{P} \left[d_{\text{GHP}} \left((M_n, n^{-1/4}D_n, n^{-1}\mu_n), (X, \mathfrak{d}, \mathfrak{m}) \right) < \epsilon \right] \geq p. \quad (2.22)$$

2. Often this scaling limit result is stated with an additional constant factor in front of $n^{-1/4}D_n$. For convenience we implicitly re-scale the metric on the Brownian map so that this constant is not needed. We do the same for the $\sqrt{8/3}$ -LQG metric. This re-scaling does not affect the statement of Corollary 2.1.7.

The same holds for other classes of random planar maps known to converge to the Brownian map, e.g., uniform k -angulations for $k \geq 3$ Le Gall [2013], Addario-Berry and Albenque [2021] and uniform planar maps with unconstrained face degree Bettinelli et al. [2014].

2.1.5 Outline

In Section 2.3 we prove the following variant of Theorem 2.1.4:

Theorem 2.1.8. *Let $\varepsilon > 0$, and let $f : \mathbb{R}^d \rightarrow \mathbb{R}$ be a smooth compactly supported function. Then with positive probability we have that*

$$\sup_{x,y \in [-R,R]^d} |D_h(x,y) - e^f \cdot d_0(x,y)| \leq \varepsilon$$

and at the same time,

$$\mu_h([-R,R]^d) \leq \varepsilon.$$

To prove this, we use an analogue of the white noise decomposition for the Gaussian free field to decompose h as $h = h_{0,\beta} + h_{\beta,\infty}$, where $\beta > 0$ is small, $h_{0,\beta}$ is a random generalized function, and $h_{\beta,\infty}$ is an independent random smooth function (see (2.35) for a precise statement). The generalized function $h_{0,\beta}$ has very weak long-range correlations and its law is invariant under rotations and translations of $\mathbb{R}^d$. The function $h_{\beta,\infty}$ does not enjoy these properties, but it is smooth. Once we do this, we discretize $\mathbb{R}^d$ and use a first passage percolation argument as in Boivin [1990] followed by scaling arguments for LQG to show that $\beta^{-(\xi Q-1)} D_{h_{0,\beta}}$ is approximately Euclidean and simultaneously $\beta^{-(\gamma Q-d)} \mu_{h_{0,\beta}}$ is approximately the Lebesgue measure with high probability when β is small (Propositions 2.3.2 and 2.3.3). After this, in Lemma 2.3.4 one controls the smooth part corresponding to $h_{\beta,\infty}$ by forcing it to be close to any given function with positive probability. Forcing $h_{\beta,\infty}$ to be approximately $\frac{\xi Q-1}{\xi} \log \beta$ and using Weyl scaling, one obtains that with positive probability, that D_h approximates any conformally flat metric, while $\mu_h([-R,R]^d)$ is bounded

by a multiple of $\theta = O(\beta^{\frac{\gamma}{\xi}-d})$.

In Section 2.4 we show that the measure can be modified in such a way that we do not change the metric very much. In Proposition 2.4.3 we add bump functions to h to show that we can make μ_h approximate an arbitrary finite sum of point masses. Then in Proposition 2.4.4 we show that the metric is not greatly modified, by showing that geodesic can freely go around the added bumps without adding much length. The bump functions we add will be large on the union of a large collection of small squares, and very negative on a small neighborhood of each of these squares. See Section 2.4.1 for a precise definition. This construction is similar to Section 11.2 in Bou-Rabee and Gwynne [2022]. Together with the results in Section 2.3 this gives the proof of Theorem 2.1.6 for the case that our metric is a conformally flat Riemannian one. The last step is to use the following folklore statement.

Theorem 2.1.9. *Let $\bar{d}$ be a length metric on $\mathbb{R}^d$ which induces the same topology as the Euclidean metric d_0 . Let $\varepsilon > 0$ and $R > 0$. Then there exists a bounded continuous function $f : \mathbb{R}^d \rightarrow \mathbb{R}$ such that*

$$\sup_{x,y \in [-R,R]^d} |\bar{d}(x,y) - e^f \cdot d_0(x,y)| \leq \varepsilon.$$

A proof of this is given in Contreras Hip and Gwynne [2024].

Acknowledgements

We thank Dmitri Burago, Tuca Auffinger, and Sahana Vasudevan for helpful discussions. E.G. was partially supported by a Clay research fellowship and by National Science Foundation grant DMS-2245832.

2.2 Preliminaries

Fix $\gamma \in (0, 2)$. Let $\bar{h}$ be a whole-space log-correlated Gaussian field as in (2.2). Let h be a *log-correlated Gaussian field plus a continuous function*, i.e., a random generalized function

on $\mathbb{R}^d$ which can be coupled with h in such a way that $\bar{h} - h$ is a continuous function on $\mathbb{R}^d$. As explained in Section 2.1.1, one can define the γ -LQG metric $D_{\bar{h}}$ and the LQG measure $\mu_{\bar{h}}$ associated with $\bar{h}$.

We will also define the sphere average of a function. Suppose that $g : \mathbb{R}^d \rightarrow \mathbb{R}$ is a function, $r > 0$, and $z_0 \in \mathbb{R}^d$. Then we define

$$g_r(z_0) := \frac{1}{|B_r(z_0)|} \int_{\partial B_r(z_0)} g(z) d\sigma(z),$$

where σ is the uniform measure on the sphere $\partial B_r(z_0)$.

Let h be a log-correlated Gaussian field, as in (2.2). For any $z \in \mathbb{R}^d$ and $r > 0$, we define the sphere average $h_r(z)$ as the average of h over the sphere of radius r centered at z , which is well defined despite the fact that h does not have well-defined pointwise values (see Section 11.1 in Lodhia et al. [2016]).

We note for the future that the law of h up to constant is invariant under complex affine transformations, that is for $a \in \mathbb{R}^d \setminus \{0\}$, $b \in \mathbb{R}^d$, we have

$$h(a \cdot + b) - h_{|a|}(b) \stackrel{d}{=} h. \tag{2.23}$$

We will also use a few additional properties about log-correlated fields, such as an analogue of the white noise decomposition (subsection 2.2.1), and an analogue of the Cameron-Martin property for h (subsection 2.2.2): if $f : \mathbb{R}^d \rightarrow \mathbb{C}$ is a continuously differentiable function whose Dirichlet energy $\int_{\mathbb{R}^d} |\nabla f|^2$ is finite and whose average $(f)_1(0)$ over the unit sphere is zero, then the law of $h + f$ is absolutely continuous with respect to the law of h .

2.2.1 White noise decomposition for log-correlated Gaussian fields

Let W be a space-time white noise on $\mathbb{R}^d \times \mathbb{R}_+$. That is, W is the Gaussian random generalized function such that for each $f \in L^2(\mathbb{R}^d \times \mathbb{R}_+)$, the (formal) integral

$$\int_0^\infty \int_{\mathbb{R}^d} f(x, t) W(dx, dt)$$

is centered Gaussian with variance $\int_0^\infty \int_{\mathbb{R}^2} f(x, t)^2 dx dt$. For $t > 0$, we define the kernel $p_t(x)$ to be the kernel associated with $(-\Delta)^{\frac{d}{2}}$, that is $p_t(x)$ is formally the solution to

$$\begin{cases} \partial_t p_t + (-\Delta)^{\frac{d}{2}} p_t = 0, \\ p_0 = \delta_0. \end{cases} \quad (2.24)$$

Rigorously, $p_t \in L^2(\mathbb{R}^d)$ is the unique kernel such that the (unique) solution to

$$\begin{cases} \partial_t u + (-\Delta)^{\frac{d}{2}} u = 0 \\ u(0, x) = u_0(x) \end{cases}$$

is given by

$$u(t, x) = \int_{\mathbb{R}^d} p_t(x - y) u_0(y) dy.$$

To see that this kernel is well defined, one can take the Fourier transform on both sides of the equation for p_t to obtain

$$\begin{cases} \partial_t \hat{p}_t + (2\pi|\xi|^2)^{\frac{d}{2}} \hat{p}_t = 0, \\ \hat{p}_0(\xi) = 1, \end{cases}$$

which suggests we define

$$p_t = \mathcal{F}^{-1} e^{-(2\pi|\xi|^2)^{\frac{d}{2}} t}.$$

Note that by scaling, we have the identity

$$p_t(x) = \frac{1}{t} p_1\left(\frac{x}{t^{\frac{1}{d}}}\right) \quad (2.25)$$

For $0 \leq \alpha \leq \beta < \infty$, we also define

$$h_{\alpha,\beta}(y) := \int_{\alpha^2}^{\beta^2} \int_{\mathbb{R}^d} p_{t/2}(x-y, t) W(dx, dt). \quad (2.26)$$

We will need the following basic property about the kernel $p_t(x)$.

Lemma 2.2.1. *Let $p_t(x)$ be defined as in (2.24). Then we have the following identity:*

$$\int_{\mathbb{R}^d} p_t(z_1 - x) p_s(x - z_2) dx = p_{t+s}(z_1 - z_2).$$

Proof. By translation invariance, it suffices to show

$$\int_{\mathbb{R}^d} p_t(x) p_s(x - z) dx = p_{s+t}(z).$$

For this, suppose u is the solution to

$$\begin{cases} \partial_t u + (-\Delta)^{\frac{d}{2}} u = 0, \\ u(0, x) = u_0(x) \end{cases}$$

where u_0 is any smooth compactly supported initial condition. Let

$$v(t, x) = \int_{\mathbb{R}^d} \left(\int_{\mathbb{R}^d} p_t(z) p_s(x - y - z) dz \right) u_0(y) dy.$$

Then

$$\begin{aligned}\partial_t v(t, x) &= \partial_t \left(\int_{\mathbb{R}^d} \int_{\mathbb{R}^d} p_t(z) p_s(x - y - z) u_0(y) dy dz \right) \\ &= \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \partial_t(p_t(z)) p_s(x - y - z) u_0(y) dy dz\end{aligned}$$

and so

$$\partial_t v(t, x) = \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \partial_t(p_t(z)) p_s(x - y - z) u_0(y) dy dz. \quad (2.27)$$

On the other hand,

$$\begin{aligned}(-\Delta)^{\frac{d}{2}} v(t, x) &= \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} (-\Delta_x)^{\frac{d}{2}} p_t(x - y - z) (p_s(z)) u_0(y) dy dz \\ &= \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \left((-\Delta_x)^{\frac{d}{2}} p_t \right) (x - y - z) p_s(z) u_0(y) dy dz.\end{aligned}$$

Combining with (2.27) we obtain

$$\left(\partial_t + (-\Delta)^{\frac{d}{2}} \right) v = \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \left(\partial_t + (-\Delta)^{\frac{d}{2}} \right) p_t(z) p_s(x - y - z) u_0(y) dy dz = 0.$$

Finally, note that

$$\begin{aligned}\lim_{t \rightarrow 0} v(t, x) &= \lim_{t \rightarrow 0} \int_{\mathbb{R}^d} p_t(z) \left(\int_{\mathbb{R}^d} p_s(x - y - z) u_0(y) dy \right) dz \\ &= \int_{\mathbb{R}^d} p_s(x - y) u_0(y) dy \\ &= u(s, x).\end{aligned}$$

Therefore $u(s + \cdot, \cdot), v(\cdot, \cdot)$ both solve

$$\begin{cases} \partial_t w + (-\Delta)^{\frac{d}{2}} w = 0, \\ w(0, x) = u(s, x) \end{cases}$$

This implies that $v(t, x) = u(s + t, x)$. Since the initial condition $u_0 \in C_c^\infty$ was arbitrary, This completes the proof. $\square$

For $\alpha > 0$, one can check using the Kolmogorov continuity theorem that $h_{\alpha, \beta}$ is a continuous function. Indeed, for any two points v, w we have

$$\begin{aligned} \text{Var}(h_{\alpha, \beta}(v) - h_{\alpha, \beta}(w)) &= 2(\text{Var}(h_{\alpha, \beta}(v)) - \text{Cov}(h_{\alpha, \beta}(v), h_{\alpha, \beta}(w))) \\ &= 2 \int_{\alpha^2}^{\beta^2} p_s(v, v) - p_s(v, w) ds \lesssim \left(\int_{\alpha^2}^{\beta^2} \|\nabla p_s\|_{L^\infty} ds \right) |v - w| \end{aligned}$$

For $\alpha = 0$, we interpret $h_{0, \beta}$ as a random generalized function.

We collect some basic properties of the family $\{h_{0, \beta}\}_{\beta > 0}$ we will need later on.

Lemma 2.2.2. *The family $\{h_{0, \beta} : \beta > 0\}$ has the following properties:*

- *If $\phi : \mathbb{R}^d \rightarrow \mathbb{R}^d$ is a rotation, reflection, or translation, then $\{h_{0, \beta} \circ \phi : \beta > 0\}$ has the same law as $\{h_{0, \beta} : \beta > 0\}$.*
- *If $\delta \in [0, 1)$ we have that $\{h_{0, \delta\beta}(\delta \cdot) : \beta > 0\}$ and $\{h_{0, \beta} : \beta > 0\}$ agree in law.*
- *If $0 \leq \beta_1 \leq \beta_2 \leq \beta_3 \leq \beta_4 < \infty$ then $h_{0, \beta_2} - h_{0, \beta_1}$ and $h_{0, \beta_4} - h_{0, \beta_3}$ are independent.*

Finally, $h_{0, 1}$ is ergodic with respect to translations of $\mathbb{R}^d$.

The invariance and independence properties are immediate from the definition, see, e.g., Section 3.2 of Ding and Goswami [2019] for the two dimensional case. For the last statement, note that the white noise $W(dx, dt)$ is ergodic with respect to spatial translations, and therefore so is $h_{0, \beta}$. An important consequence is the following scaling property of $D_{h_{0, \beta}}$.

Lemma 2.2.3. *Let $\beta > 0$. Then $\{D_{h_{0, \beta}}(x, y) : x, y \in \mathbb{R}^d\}$ has the same law as*

$$\left\{ \beta^{\xi_Q} D_{h_{0, 1}}(\beta^{-1}x, \beta^{-1}y) : x, y \in \mathbb{R}^d \right\}$$

Proof. This is a direct consequence of the LQG coordinate change formula, Weyl scaling and Lemma 2.2.2. Indeed, by the LQG coordinate change formula, a.s.

$$D_{h_{0,1}}(\beta^{-1}x, \beta^{-1}y) = D_{h_{0,1}(\beta^{-1}\cdot) + Q \log(\beta^{-1})}(x, y), \quad \forall x, y \in \mathbb{R}^d.$$

By Weyl scaling and Lemma 2.2.2, the metric on the right has the same law as $\beta^{-\xi Q} D_{h_{0,\beta}}$. □

Lemma 2.2.4. *For each $\beta > 0$, $h_{\beta,\infty}$ is well-defined as a random function viewed modulo additive constant and agrees in law with $h * p_{\beta^2/2}$, modulo additive constant.*

Proof. Let $f : \mathbb{R}^d \rightarrow \mathbb{R}^d$ be compactly supported and smooth such that $\int_{\mathbb{R}^d} f(z) dz = 0$. Let $R > 0$ be large. Then

$$\text{Var} \left(\int_{\mathbb{R}^d} f(z) h_{\beta,R}(z) dz \right) = \mathbb{E} \left(\left(\int_{\mathbb{R}^d} f(z) h_{\beta,R}(z) dz \right)^2 \right)$$

since

$$\mathbb{E} \left(\int_{\mathbb{R}^d} f(z) h_{\beta,R}(z) dz \right) = 0.$$

Therefore,

$$\begin{aligned}
\text{Var}\left(\int_{\mathbb{R}^d} f(z) h_{\beta,R}(z) dz\right) &= \mathbb{E}\left(\int_{\mathbb{R}^d} \int_{\mathbb{R}^d} f(x) f(y) h_{\beta,R}(x) h_{\beta,R}(y) dx dy\right) \\
&= \mathbb{E}\left(\int_{\mathbb{R}^d} \int_{\mathbb{R}^d} f(x) f(y) \right. \\
&\quad \times \int_{\mathbb{R}^d} \int_{\beta^2}^{R^2} p_{t/2}(x-u) W(du, dt) \\
&\quad \times \left. \int_{\mathbb{R}^d} \int_{\beta^2}^{R^2} p_{s/2}(y-v) W(dv, ds) dx dy\right) \\
&= \mathbb{E}\left(\int_{\mathbb{R}^d} \int_{\mathbb{R}^d} f(x) f(y) \right. \\
&\quad \times \int_{\mathbb{R}^d} \int_{\beta^2}^{R^2} p_{t/2}(x-y-u) W(du, dt) \\
&\quad \times \left. \int_{\mathbb{R}^d} \int_{\beta^2}^{R^2} p_{s/2}(-v) W(dv, ds) dx dy\right) \\
&= \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} f(x) f(y) K_R(x-y) dx dy.
\end{aligned} \tag{2.28}$$

where

$$K_R(x) := \mathbb{E}\left(\int_{\mathbb{R}^d} \int_{\beta^2}^{R^2} \int_{\mathbb{R}^d} \int_{\beta^2}^{R^2} p_{t/2}(x-u) p_{s/2}(-v) W(du, dt) W(dv, ds)\right).$$

Using the definition of the white noise we obtain

$$K_R(x) = \int_{\mathbb{R}^d} \int_{\beta^2}^{R^2} p_{t/2}(x-u) p_{t/2}(u) dt du$$

Using Lemma 2.2.1 we have

$$\int_{\mathbb{R}^d} p_t(z_1 - x) p_t(x - z_2) dx = p_{2t}(z_1 - z_2),$$

and so

$$K_R(x) = \int_{\beta^2}^{R^2} p_t(x) dt. \quad (2.29)$$

Let $\mathcal{F}$ denote the Fourier transform, that is

$$\mathcal{F}(f)(\zeta) = \hat{f}(\zeta) := \int_{\mathbb{R}^d} e^{-2\pi i x \cdot \zeta} f(x) dx.$$

Using Plancherel, (2.29), and the fact that $\mathcal{F}(f * g) = \hat{f} \hat{g}$ we obtain

$$\int_{\mathbb{R}^d} \int_{\mathbb{R}^d} f(x) f(y) K_R(x - y) dx dy = \int_{\mathbb{R}^d} \hat{K}_R(\zeta) |\hat{f}(\zeta)|^2 d\zeta. \quad (2.30)$$

Note that

$$\hat{p}_t(\zeta) = e^{-(2\pi)^d t |\zeta|^d}, \quad (2.31)$$

and so in particular it is integrable in both space and time. Hence

$$\lim_{R \rightarrow \infty} \hat{K}_R(\zeta) = \lim_{R \rightarrow \infty} \int_{\beta^2}^{R^2} \hat{p}_t(\zeta) dt = \int_{\beta^2}^{\infty} \hat{p}_t(\zeta) dt = \int_{\beta^2}^{\infty} e^{-2^d \pi^d t |\zeta|^d} dt = \frac{1}{2^d \pi^d |\zeta|^d} e^{-\frac{2^d \pi^d \beta^2 |\zeta|^d}{2}}. \quad (2.32)$$

By plugging (2.30) into (2.28), then using (2.32), we obtain

$$\lim_{R \rightarrow \infty} \text{Var} \left(\int_{\mathbb{R}^d} f(z) h_{\beta, R}(z) dz \right) = \int_{\mathbb{R}^d} \frac{1}{2^d \pi^d |\zeta|^d} e^{-\frac{2^d \pi^d \beta^2 |\zeta|^d}{2}} |\hat{f}(\zeta)|^2 d\zeta. \quad (2.33)$$

Note that the above integral is well defined since f is smooth and $f(0) = 0$, hence $\frac{|\hat{f}(\zeta)|^2}{|\zeta|^2}$ is integrable at 0. This shows that $f \mapsto \int_{\mathbb{R}^d} f(z) h_{\beta, \infty}(z) dz$ is well-defined as a centered Gaussian process on the set of smooth compactly supported test functions with integral zero. We will now show that this process agrees in law with $f \mapsto \int_{\mathbb{R}^d} f(z) h * p_{\beta^2/2}(z) dz$.

Recalling that $h_1(0) := \int_{\partial B_1(0)} h d\sigma = 0$ and that $\int_{\mathbb{R}^d} f(z) dz = 0$, we obtain

$$\begin{aligned}
& \text{Var} \left(\int_{\mathbb{R}^d} f(z) h * p_{\frac{\beta^2}{2}}(z) dz \right) \\
&= \text{Var} \left(\int_{\mathbb{R}^d} f(z) h^0 * p_{\frac{\beta^2}{2}}(z) dz \right) \\
&= \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} f(x) f(y) \mathbb{E}(h^0 * p_{\frac{\beta^2}{2}}(x) h^0 * p_{\frac{\beta^2}{2}}(y)) dx dy \\
&= \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} f(x) f(y) \left(\int_{\mathbb{R}^d} \int_{\mathbb{R}^d} p_{\frac{\beta^2}{2}}(x-u) p_{\frac{\beta^2}{2}}(y-v) \mathbb{E}(h^0(u) h^0(v)) du dv \right) dx dy \\
&= \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} f(x) f(y) \left(\int_{\mathbb{R}^d} \int_{\mathbb{R}^d} p_{\frac{\beta^2}{2}}(x-u) p_{\frac{\beta^2}{2}}(y-v) G(u, v) du dv \right) dx dy, \tag{2.34}
\end{aligned}$$

where $G(x, y)$ is given by (2.2). Note that by translations invariance we have

$$\begin{aligned}
& \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} f(x) f(y) \left(\int_{\mathbb{R}^d} \int_{\mathbb{R}^d} p_{\beta^2/2}(x-u) p_{\beta^2/2}(y-v) \log(\max\{|u|, 1\}) du dv \right) dx dy \\
&= \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} f(x) f(y) \left(\int_{\mathbb{R}^d} p_{\beta^2/2}(x-u) \log(\max\{|u|, 1\}) du \right) \left(\int_{\mathbb{R}^d} p_{\beta^2/2}(v) dv \right) dx dy \\
&= \left(\int_{\mathbb{R}^d} f(y) dy \right) \int_{\mathbb{R}^d} f(x) \left(\int_{\mathbb{R}^d} p_{\beta^2/2}(x-u) \log(\max\{|u|, 1\}) du \right) \left(\int_{\mathbb{R}^d} p_{\beta^2/2}(v) dv \right) dx \\
&= 0,
\end{aligned}$$

where in the last line we used that $\int_{\mathbb{R}^d} f = 0$. Therefore

$$\begin{aligned}
& \text{Var} \left(\int_{\mathbb{R}^d} f(z) h * p_{\frac{\beta^2}{2}}(z) dz \right) \\
&= \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} f(x) f(y) \left(\int_{\mathbb{R}^d} \int_{\mathbb{R}^d} p_{\frac{\beta^2}{2}}(x-u) p_{\frac{\beta^2}{2}}(y-v) \log \left(\frac{1}{|u-v|} \right) du dv \right) dx dy,
\end{aligned}$$

Let

$$F_x(v) := \int_{\mathbb{R}^d} p_{\beta^2/2}(x-u) \log \left(\frac{1}{|u-v|} \right) du.$$

Note that $F_x = p_{\beta^2/2}(x - \cdot) * \log\left(\frac{1}{|\cdot|}\right)$. Hence

$$\begin{aligned}\hat{F}_x(\zeta) &= \mathcal{F}\left(\log\left(\frac{1}{|\cdot|}\right)\right) \mathcal{F}(p_{\beta^2/2}(x - \cdot))(\zeta) \\ &= \frac{1}{2^d \pi^d |\zeta|^d} \mathcal{F}(p_{\beta^2/2}(x - \cdot))(\zeta) = \frac{1}{2^d \pi^d |\zeta|^d} \hat{p}_{\beta^2/2}(\zeta) e^{-2\pi i x \cdot \zeta}.\end{aligned}$$

Then

$$\begin{aligned}& \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} f(x) f(y) \left(\int_{\mathbb{R}^d} \int_{\mathbb{R}^d} p_{\frac{\beta^2}{2}}(x - u) p_{\frac{\beta^2}{2}}(y - v) \log\left(\frac{1}{|u - v|}\right) du dv \right) dx dy \\ &= \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} f(x) f(y) \left(\int_{\mathbb{R}^d} F_x(v) p_{\beta^2/2}(y - v) dv \right) dx dy \\ &= \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} f(x) f(y) \left(\int_{\mathbb{R}^d} \hat{F}_x(\zeta) e^{2\pi i y \cdot \zeta} \bar{\hat{p}}_{\beta^2/2}(\zeta) d\zeta \right) dx dy \quad (\text{Plancherel}) \\ &= \frac{1}{2^d \pi^d} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} f(x) f(y) \left(\int_{\mathbb{R}^d} \frac{1}{|\zeta|^d} e^{2\pi i (y - x) \cdot \zeta} |\hat{p}_{\beta^2/2}(\zeta)|^2 d\zeta \right) dx dy \\ &= \frac{1}{2^d \pi^d} \int_{\mathbb{R}^d} |\hat{f}(\zeta)|^2 \frac{1}{|\zeta|^d} |\hat{p}_{\beta^2/2}(\zeta)|^2 d\zeta \quad (\text{Defn. of Fourier transform}) \\ &= \int_{\mathbb{R}^d} \frac{\hat{f}(\zeta)^2}{2^d \pi^d |\zeta|^d} e^{-2^d \pi^d \beta^2 |\zeta|^d} d\zeta.\end{aligned}$$

Combining this with (2.34), that

$$\text{Var} \left(\int_{\mathbb{R}^d} f(z) h * p_{\frac{\beta^2}{2}}(z) dz \right) = \int_{\mathbb{R}^d} \frac{\hat{f}(\zeta)^2}{2^d \pi^d |\zeta|^d} e^{-2^d \pi^d \beta^2 |\zeta|^d} d\zeta.$$

Now using (2.33) we obtain

$$\text{Var} \left(\int_{\mathbb{R}^d} f(z) h * p_{\frac{\beta^2}{2}}(z) dz \right) = \lim_{R \rightarrow \infty} \text{Var} \left(\int_{\mathbb{R}^d} f(z) h_{\beta, R}(z) dz \right) = \text{Var} \left(\int_{\mathbb{R}^d} f(z) h_{\beta, \infty}(z) dz \right).$$

Since the variances agree, this completes the proof. $\square$

By sending $\beta \rightarrow 0$ in Lemma 2.2.4, we find that $h_{0, \infty}$ is well-defined as a random generalized function and agrees in law with h , modulo additive constant. We may therefore

couple h with the white noise W in such a way that $\int_{\partial B_1(0)} h_{\beta,\infty} = 0$, and

$$h = h_{0,\beta} + h_{\beta,\infty} - (h_{0,\beta})_1(0), \quad (2.35)$$

where we recall that $g_r(z_0)$ denotes the sphere average $\frac{1}{|B_r(z_0)|} \int_{\partial B_r(z_0)} g(z) d\sigma(z)$. This decomposition will be crucial for our proofs. We prove the following technical lemma, which will be used repeatedly in future calculations.

Lemma 2.2.5. *We have that*

$$(h_{0,\beta})_1(0) \rightarrow 0$$

in probability as $\beta \rightarrow 0$.

Proof. Recall by (2.28) and (2.29) that

$$\text{Var} \left(\int_{\mathbb{R}^d} f(z) h_{\alpha,\beta}(z) dz \right) = \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} f(x) f(y) K_{\alpha,\beta}(x-y) dx dy,$$

where

$$K_{\alpha,\beta} := \int_{\alpha^2}^{\beta^2} p_t(z) dz.$$

Note that the above inequality also holds for f replaced by a measure μ , that is

$$\text{Var} \left(\int_{\mathbb{R}^d} h_{\alpha,\beta}(z) d\mu(z) \right) = \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} K_{\alpha,\beta}(x-y) d\mu(x) d\mu(y).$$

Again using Plancherel as in Lemma 2.2.4, we obtain that

$$\text{Var} \left(\int_{\mathbb{R}^d} f(z) h_{\alpha,\beta}(z) d\mu(z) \right) = \int_{\mathbb{R}^d} \hat{K}_{\alpha,\beta}(\zeta) |\hat{\mu}(\zeta)|^2 d\zeta.$$

Taking μ such that $\int_{\mathbb{R}^d} f(z) d\mu(z) = \int_{\partial B_1(0)} f(z) d\sigma(z)$, we note that μ 's Fourier transform

is well defined. Therefore we have

$$\text{Var}((h_{0,\beta})_1(0)) = \int_{\mathbb{R}^d} \lim_{\alpha \rightarrow 0} \hat{K}_{\alpha,\beta}(\zeta) |\hat{\mu}(\zeta)|^2 d\zeta,$$

and similarly we have

$$\text{Var}((h_{\beta,\infty})_1(0)) = \int_{\mathbb{R}^d} \lim_{R \rightarrow \infty} \hat{K}_{\beta,R}(\zeta) |\hat{\mu}(\zeta)|^2 d\zeta.$$

We compute

$$\lim_{\alpha \rightarrow 0} \hat{K}_{\alpha,\beta}(\zeta) = \lim_{\alpha \rightarrow 0} \pi \int_{\alpha^2}^{\beta^2} \hat{p}_t(\zeta) dt = \int_0^{\beta^2} \pi \hat{p}_t(\zeta) dt = \int_0^{\beta^2} \pi e^{-\frac{4\pi^2 t |\zeta|^2}{2}} dt.$$

Noting that this last expression converges to 0 as $\beta \rightarrow 0$, we see that

$$\lim_{\beta \rightarrow 0} \text{Var}((h_{0,\beta})_1(0)) = 0.$$

This completes the proof. □

2.2.2 Cameron-Martin property for log-correlated Gaussian fields

We will need the following property, analogous to the Cameron-Martin property for the GFF.

Lemma 2.2.6. *Suppose that $f \in C_0^\infty(\mathbb{R}^d)$ is a smooth compactly supported function. Then the law of $h + f$ is absolutely continuous with respect to the law of h up to additive constant.*

Proof. By Proposition 2.3 in Junnila et al. [2018] we have the Karhunen–Loève expansion of h (up to additive constant)

$$h = \sum_{n \geq 0} A_n \sqrt{\lambda_n} \varphi_n(x)$$

where $\{\lambda_n\}_{n \in \mathbb{N}}$ is the set of eigenvalues of the Hilbert-Schmidt operator associated with the kernel p_t , $\{\varphi_n\}_{n \in \mathbb{N}}$ is an orthonormal basis of eigenvectors in L^2 , and $\{A_n\}_{n \geq 0}$ is an

i.i.d. set of standard Gaussian random variables. Since $f \in C_c^\infty(\mathbb{R}^d)$, there exist coefficients $\{c_n\}_{n \in \mathbb{N}} \in \ell^2(\mathbb{N})$ such that

$$f = \sum_{n \in \mathbb{N}} c_n \sqrt{\lambda_n} \varphi_n.$$

Indeed, since $f \in H_0^{\frac{d}{2}}(\mathbb{R}^d)$, we have

$$f = \sum_{n \in \mathbb{N}} d_n \varphi_n$$

for some coefficients d_n , and so

$$\infty > \int_{\mathbb{R}^d} |\Delta^{\frac{d}{4}} f(z)|^2 dz = \int_{\mathbb{R}^d} f(z) \Delta^{\frac{d}{2}} f(z) dz \geq \sum_{n \in \mathbb{N}} |d_n| \frac{|d_n|}{\lambda_n}$$

and so $c_n = \frac{d_n}{\sqrt{\lambda_n}} \in \ell^2$. Then

$$h + f = \sum_{n \in \mathbb{N}} (A_n + c_n) \varphi_n.$$

Therefore $h + f$ is absolutely continuous with respect to h , with Radon-Nikodym derivative given by

$$\prod_{n \in \mathbb{N}} e^{-\frac{c_n^2}{2}} e^{c_n x} = e^{-\frac{1}{2} \sum_{n \in \mathbb{N}} c_n^2 + \sum_{n \in \mathbb{N}} c_n A_n}$$

which converges, since $c_n \in \ell^2$. This completes the proof. $\square$

2.3 Aproximation theorem for Riemannian metrics

In this section we will prove Theorem 2.1.8.

In Section 2.4 we will then generalize this to the case of any measure on $\mathbb{R}^d$, still with the same metric.

We will need the following technical lemma about a moment bound for $\text{diam}_{D_{h_{0,1}}}([0, 1]^d)$.

In this we use the moment assumption (2.12).

Lemma 2.3.1. *There exists a $p > d$ such that*

$$\mathbb{E}(\text{diam}_{D_{h_{0,1}}}(B_1(0))^p) < \infty.$$

Proof. Recall that $h = h_{0,1} + h_{1,\infty}$. Therefore

$$\text{diam}_{D_{h_{0,1}}}([0, 1]^d) \lesssim \text{diam}_{D_h}([0, 1]^d) e^{\xi \sup_{z \in [0, 1]^d} h_{1,\infty}}.$$

By the Borel TIS inequality, we have that $\sup_{z \in [0, 1]^d} h_{1,\infty}$ has an exponential tail, and hence exponential moments of all orders. Therefore for any $d < p < \bar{p}$, we have

$$\mathbb{E}(\text{diam}_{D_{h_{0,1}}}([0, 1]^d)^p) \lesssim \mathbb{E}(\text{diam}_{D_h}([0, 1]^d)^{\bar{p}})^{\frac{p}{\bar{p}}} \mathbb{E} \left(e^{p \left(\frac{\bar{p}}{p} \right)' \xi \sup_{z \in [0, 1]^d} h_{1,\infty}} \right)^{\frac{1}{\left(\frac{\bar{p}}{p} \right)'}} < \infty$$

where we used (2.12) in the last line. This completes the proof. $\square$

Theorem 2.1.8 will be proven by combining the following propositions.

Proposition 2.3.2. *There exists $\nu > 0$ such that the following is true. Let $\varepsilon > 0$. With probability going to 1 as $\beta \rightarrow 0$,*

$$\left| D_{h_{0,\beta}}(x, y) - \beta^{\xi Q - 1} \nu |x - y| \right| \leq \varepsilon \beta^{\xi Q - 1} \quad (2.36)$$

for all $x, y \in [-R, R]^d$.

Proposition 2.3.3. *There exists a constant $C_0 > 0$ such that the following is true. Let $c > 0$. With probability going to 1 as $\beta \rightarrow 0$, we have*

$$\mu_{h_{0,\beta}}([-R, R]^d) \leq C_0 \beta^{\gamma Q - 2 - c}.$$

Let $f \mapsto \hat{f}$ denote the Fourier transform and let $f \mapsto \check{f}$ denote the inverse Fourier transform,

$$\check{f}(x) = \int e^{2\pi i \zeta \cdot x} d\zeta.$$

We will also need the following lemma:

Proposition 2.3.4. *Let $\varepsilon > 0$, and suppose that f is a function such that $\hat{f}$ has compact support and $\int_{\partial B_1(0)} f(z) d\sigma = 0$. For small enough $\beta > 0$, it holds with positive probability that*

$$\|h_{\beta, \infty} - f\|_{L^\infty([-1, 1]^2)} < \varepsilon.$$

We will now assume the results of Propositions 2.3.2, 2.3.3, and 2.3.4, and prove Theorem 2.1.8.

2.3.1 Proof of Theorem 2.1.8

We have the following result, which follows from the same proof as Lemma 7.1 in Dubédat and Falconet [2020].

Lemma 2.3.5. *Let $U \subset \mathbb{R}^d$ be a bounded open set, and let $f : \bar{U} \rightarrow \mathbb{R}$ be a continuous function, and let $\varphi_1, \varphi_2 : (0, \infty) \rightarrow (0, \infty)$ be increasing functions such that $\varphi_1(0^+) = \varphi_2(0^+) = 0$. Suppose $\{d_n\}_{n \geq 0}$ is a sequence of length metrics inducing the Euclidean topology on $\bar{U}$ such that for any $x, y \in \bar{U}$, and any $n \in \mathbb{N}$,*

$$\varphi_1(\|x - y\|) \leq d_n(x, y) \leq \varphi_2(\|x - y\|).$$

Assume additionally that d_n converges uniformly to a metric d_∞ . Then $e^f \cdot d_n$ converges to $e^f \cdot d_\infty$ in the sense that for any $x, y \in \bar{U}$, we have

$$\lim_{n \rightarrow \infty} e^f \cdot d_n(x, y) = e^f \cdot d_\infty(x, y).$$

We will need the following corollary.

Corollary 2.3.6. *Let $U \subset \mathbb{R}^d$ be a bounded open set, and let $f : \bar{U} \rightarrow \mathbb{R}$ be continuous. Let $\varepsilon > 0$, and suppose that $\mathfrak{d}_0$ is a continuous length metric on $\bar{U}$. Then there exists a $\delta > 0$ such that if $\mathfrak{d}$ is another continuous length metric on $\bar{U}$ such that $\sup_{x,y \in \bar{U}} |\mathfrak{d}_0(x,y) - \mathfrak{d}(x,y)| < \delta$, then $|e^f \cdot \mathfrak{d}_0(x,y) - e^f \cdot \mathfrak{d}(x,y)| < \varepsilon$ for all $x, y \in \bar{U}$.*

Proof. If the conclusion does not hold, then there is a sequence of continuous length metrics $\mathfrak{d}^n$ such that

$$\sup_{x,y \in \bar{U}} |\mathfrak{d}_0(x,y) - \mathfrak{d}^n(x,y)| < \frac{1}{n}, \quad (2.37)$$

but

$$\sup_{x,y \in \bar{U}} |e^f \cdot \mathfrak{d}_0(x,y) - e^f \cdot \mathfrak{d}^n(x,y)| > \varepsilon.$$

Therefore $\mathfrak{d}^n$ converges to $\mathfrak{d}_0$ uniformly. We claim that the hypotheses of Lemma 2.3.5 hold, giving us a contradiction. For this we construct the functions φ_1, φ_2 . For any $r > 0$, let

$$\bar{\varphi}_1(r) := \inf_{\substack{x,y:|x-y|=r \\ n \in \mathbb{N}}} \mathfrak{d}^n(x,y),$$

and similarly

$$\bar{\varphi}_2(r) := \sup_{\substack{x,y:|x-y|=r \\ n \in \mathbb{N}}} \mathfrak{d}^n(x,y).$$

Finally, we define

$$\varphi_1(r) := \inf_{s \geq r} \bar{\varphi}_1(s)$$

and

$$\varphi_2(r) := \sup_{s \leq r} \bar{\varphi}_2(s).$$

Note that $\varphi_1(r), \varphi_2(r)$ are both decreasing, and $\bar{\varphi}_1(r) \geq \varphi_1(r), \bar{\varphi}_2(r) \leq \varphi_2(r)$. Therefore by

construction we have that

$$\varphi_1(|x - y|) \leq \mathfrak{d}^n(x, y) \leq \varphi_2(|x - y|).$$

It suffices to check now that $\varphi_1(r) > 0$ if $r > 0$, $\varphi_2(0^+) = 0$, and $\varphi_2(s) < \infty$ for small enough s . To check that $\varphi_1(r) > 0$, suppose by contradiction that this is not the case. Then there exist sequences n_k, x_k, y_k such that $|x_k - y_k| = r$ and

$$\lim_{k \rightarrow \infty} \mathfrak{d}^{n_k}(x_k, y_k) = 0.$$

Note that by passing to subsequence if necessary, we can assume that either $n_k = n$ does not depend on k or $n_k \rightarrow \infty$ as $k \rightarrow \infty$. In the first case, we would have that $\lim_{k \rightarrow \infty} \mathfrak{d}^n(x_k, y_k) = 0$ for some n , which is impossible since $\mathfrak{d}^n$ is positive definite. In the second case, again by passing to a subsequence we can assume that $x_k \rightarrow x$ and $y_k \rightarrow y$. Hence we have by (2.37) that

$$\lim_{k \rightarrow \infty} \mathfrak{d}^{n_k}(x_k, y_k) \geq \liminf_{k \rightarrow \infty} \mathfrak{d}_0(x_k, y_k) - \frac{1}{n_k} = \liminf_{k \rightarrow \infty} \mathfrak{d}_0(x_k, y_k) = \mathfrak{d}_0(x, y) > 0,$$

which yields a contradiction.

A similar contradiction argument shows that $\varphi_2(0^+) = 0$ and $\varphi_2(s) < \infty$. This completes the proof. $\square$

We will need the following lemma about the behavior of metrics of the form $e^F \cdot \mathfrak{d}$ when F is small.

Lemma 2.3.7. *Let $F : \mathbb{R}^d \rightarrow \mathbb{R}^d$ be a continuous function, and suppose that $\mathfrak{d}$ is a length metric. Suppose that $F \in L^\infty(\mathbb{R}^d)$, and that*

$$\text{Diam}_{\mathfrak{d}}([-R, R]^d) := \sup_{x, y \in [-R, R]^d} \mathfrak{d}(x, y) < \infty$$

Then

$$\sup_{x,y \in [-R,R]^d} |e^F \cdot \mathfrak{d}(x,y) - \mathfrak{d}(x,y)| \leq e^{\|F\|_\infty} (e^{\|F\|_\infty} - 1) \text{Diam}_{\mathfrak{d}}([-R,R]^d).$$

Proof. This is an elementary consequence of the definition of $e^F \cdot \mathfrak{d}$. $\square$

Now we will finish the proof of Theorem 2.1.8.

Proof of Theorem 2.1.8. Now assume that we are in the setting of Theorem 2.1.8. Let $f : \mathbb{R}^d \rightarrow \mathbb{R}$ be a continuous function. From now on, we assume that the conclusions of Propositions 2.3.2, 2.3.3,

$$|(h_{0,\beta})_1(0)| \leq \varepsilon \tag{2.38}$$

and 2.3.4 hold simultaneously, which is a positive probability event for small enough $\beta > 0$. This is because the conclusions of Propositions 2.3.2 and 2.3.3 hold with probability going to 1 as $\beta \rightarrow 0$, (2.38) holds with probability going to 1 by Lemma 2.2.5, the conclusion Proposition 2.3.4 holds with positive probability, and also by the independence of $h_{0,\beta}, h_{\beta,\infty}$. We claim that

$$|D_h(x,y) - \nu \beta^{\xi Q-1} e^{\xi f} \cdot d_0(x,y)| \leq \beta^{\xi Q-1} \varepsilon, \quad \forall x,y \in [-R,R]^d. \tag{2.39}$$

This statement might seem odd at first glance, since $D_h(x,y)$ is of constant order while $\nu \beta^{\xi Q-1} e^{\xi f} \cdot d_0(x,y)$ is not. However, it holds because we are conditioning on the event that $\|h_{\beta,\infty} - f\|_\infty < \varepsilon$, which is rare, but holds with positive probability. To prove (2.39) we apply Corollary 2.3.6 to the metrics $\mathfrak{d}_0 = \nu d_0$ and $\mathfrak{d} = \beta^{1-\xi Q} D_{h_{0,\beta}}$ together with the function f , which we know satisfy the hypotheses by Proposition 2.3.2 for small enough β . Thus for small enough $\beta > 0$, we have that for all $x,y \in [-R,R]^d$,

$$|\beta^{1-\xi Q} D_{h_{0,\beta}+f}(x,y) - \nu e^{\xi f} \cdot d_0(x,y)| \leq \varepsilon,$$

or equivalently,

$$|D_{h_{0,\beta}+f}(x, y) - \nu \beta^{\xi Q-1} e^{\xi f} \cdot d_0(x, y)| \leq \beta^{\xi Q-1} \varepsilon. \quad (2.40)$$

We recall the white noise decomposition

$$h = h_{0,\beta} + h_{\beta,\infty} - (h_{0,\beta})_1(0).$$

This implies that

$$\|h - (h_{0,\beta} + f)\|_\infty = \|(h_{\beta,\infty} - f) - (h_{0,\beta})_1(0)\|_\infty \leq \|h_{\beta,\infty} - f\|_\infty + |(h_{0,\beta})_1(0)|.$$

Using Proposition 2.3.4 for the first term and (2.38) for the second one, we see that for small enough β we have

$$\|h - (h_{0,\beta} + f)\|_\infty \leq 2\varepsilon.$$

Therefore by Lemma 2.3.7 we have that for all $x, y \in [-R, R]^d$,

$$|D_h(x, y) - D_{h_{0,\beta}+f}(x, y)| \leq e^{2\xi\varepsilon} (e^{2\xi\varepsilon} - 1) \text{Diam}_{D_{h_{0,\beta}+f}}([-1, 1]^2). \quad (2.41)$$

Applying Lemma 2.3.7 again for the metric $D_{h_{0,\beta}}$ and the perturbation f , we see that

$$\text{Diam}_{D_{h_{0,\beta}+f}}([-R, R]^d) \leq e^{\xi\|f\|_\infty} \text{Diam}_{D_{h_{0,\beta}}}([-R, R]^d).$$

Now using Proposition 2.3.2 we obtain that

$$\text{Diam}_{D_{h_{0,\beta}+f}}([-R, R]^d) \leq \beta^{\xi Q-1} \nu 2\sqrt{2} + \varepsilon \beta^{\xi Q-1}.$$

Plugging this into (2.41) we obtain that if ε is small enough, for all $x, y \in [-R, R]^d$ we have

$$\left| D_h(x, y) - D_{h_{0,\beta}+f}(x, y) \right| \leq \left(e^{2\xi\varepsilon} - 1 \right) \beta^{\xi Q-1} 4\nu.$$

Combining this with (2.40) and using the triangle inequality, we obtain that

$$\begin{aligned} \left| D_h(x, y) - \nu \beta^{\xi Q-1} e^{\xi f} \cdot d_0(x, y) \right| &\leq \left| D_h(x, y) - D_{h_{0,\beta}+f}(x, y) \right| \\ &\quad + |D_{h_{0,\beta}+f}(x, y) - \nu \beta^{\xi Q-1} e^{\xi f} \cdot d_0(x, y)| \\ &\leq \beta^{\xi Q-1} \varepsilon + \left(e^{2\xi\varepsilon} - 1 \right) \beta^{\xi Q-1} 4\nu. \end{aligned}$$

Therefore for small enough $\beta > 0$ and shrinking ε if necessary, we obtain our claim (2.39).

Again recall we are assuming the conclusions of Propositions 2.3.2, 2.3.3 and 2.3.4 hold simultaneously. Using the white noise decomposition $h = h_{p,\beta} + h_{\beta,\infty} - (h_{0,\beta})_1(0)$, we see that by Proposition 2.3.4 and (2.38) we have

$$\|h - h_{0,\beta} - f\|_\infty \leq 2\varepsilon.$$

Hence

$$\mu_h([-R, R]^d) \leq e^{\gamma\|f\|_\infty + 2\gamma\varepsilon} \mu_{h_{0,\beta}}([-R, R]^d) \leq C_0 e^{\gamma\|f\|_\infty + 2\gamma\varepsilon} \beta^{\gamma Q-2-c}. \quad (2.42)$$

This implies that if the conclusions of Propositions 2.3.2, 2.3.3, and 2.3.4 hold, then we have that for any $x, y \in [-R, R]^d$,

$$|D_h(x, y) - \nu \beta^{\xi Q-1} e^{\xi f} \cdot d_0(x, y)| \leq \beta^{\xi Q-1} \varepsilon,$$

and

$$\mu_h([-R, R]^d) \leq C_0 e^{\gamma\|f\|_\infty + 2\gamma\varepsilon} \beta^{\gamma Q-2-c}.$$

Now replacing f by $f + (\frac{1}{\xi} - Q) \log(\beta) - \frac{\log \nu}{\xi}$, we obtain

$$\sup_{x,y \in [-R,R]^d} |D_h(x,y) - e^{\xi f} \cdot d_0(x,y)| \leq \varepsilon$$

and for any Borel set A ,

$$\mu_h(A) \leq \theta \int_{A_\varepsilon} f(z) dz + \varepsilon$$

where

$$\theta = \alpha \nu^{-\frac{\gamma}{\xi}} \beta^{\frac{\gamma}{\xi} - d}.$$

Note that by (2.11), $\theta \rightarrow 0$ as $\beta \rightarrow 0$, and so this completes the proof of Theorem 2.1.8. $\square$

2.3.2 First passage percolation argument

From now on, let Diam_h denote the diameter with respect to the metric D_h and define $\text{Diam}_{h_{0,\beta}}$ analogously. For the proof of this Proposition 2.3.2, we first treat the case where y is fixed. We will look at the asymptotic behaviour of $D_{h_{0,1}}(0, x)$ in a single direction first, and then prove convergence independent of the direction. To this end, we will need the following lemma.

Lemma 2.3.8. *Let $c > 0$ be a sufficiently small fixed constant. Then we have that almost surely,*

$$\sup_{\substack{x,y \in [-2^n, 2^n]^d \\ |x-y| \leq 1}} D_{h_{0,1}}(x,y) \leq 2^{(1-c)n}$$

for all but finitely many integers n .

Proof. We recall that $e(p) := \mathbb{E} \left(\left(\text{Diam}_{D_{h_{0,1}}}([0, 1]^d) \right)^p \right) < \infty$ for some $p > d$ by (2.12). By Markov's inequality,

$$\mathbb{P} \left(\text{Diam}_{D_{h_{0,1}}}([0, 1]^d) > 2^{(1-c)n} \right) \leq \frac{C}{2^{np(1-c)}}.$$

for some absolute constant C only depending on p . Now note that if

$$\sup_{\substack{x, y \in [-2^n, 2^n]^d \\ |x-y| \leq 1}} D_{h_{0,1}}(x, y) > 2^{(1-c)n+d},$$

then there must exist x_0, y_0 such that $|x - y| \leq 1$, but $D_{h_{0,1}}(x, y) > 2^{(1-c)n+2}$. This implies that there must exist a cube $[k_1, k_1 + 1] \times [k_2, k_2 + 1] \times \dots \times [k_d, k_d + 1]$ with $-2^n \leq k_1, k_2, \dots, k_d \leq 2^n - 1$ such that

$$\text{Diam}_{D_{h_{0,1}}}([k_1, k_1 + 1] \times [k_2, k_2 + 1] \times \dots \times [k_d, k_d + 1]) > 2^{(1-c)n}.$$

Since there are 2^{dn} possibilities for the pair (k_1, k_2) , we obtain

$$\mathbb{P} \left(\sup_{\substack{x, y \in [-2^n, 2^n]^d \\ |x-y| \leq 1}} D_{h_{0,1}}(x, y) > 2^{(1-c)n} \right) \leq (2^n)^{d-p(1-c)}.$$

Choosing p to be close enough to 2 such that $d - p(1 - c) < 0$, we can apply Borel-Cantelli to obtain that almost surely, for all but finitely many n we have

$$\sup_{\substack{x, y \in [-2^n, 2^n]^d \\ |x-y| \leq 1}} D_{h_{0,1}}(x, y) \leq 2^{(1-c)n}.$$

This completes the proof. □

Lemma 2.3.9. *Let $x \in \mathbb{R}^d \setminus \{0\}$ be fixed. There is a deterministic constant $\nu > 0$, not depending on x , such that a.s.*

$$\lim_{\lambda \rightarrow \infty} \left| \frac{D_{h_{0,1}}(0, \lambda x)}{\lambda |x|} - \nu \right| = 0.$$

Proof. Recall that $h_{0,1}$ is stationary and ergodic by Lemma 2.2.2 and thus so is $D_{h_{0,1}}(0, x)$, which together with subadditivity (which is the triangle inequality for $D_{h_{0,1}}$) and Kingman's ergodic theorem (see Kingman [1968]) shows that there exists a deterministic function $\nu : \mathbb{S}^1 \rightarrow \mathbb{R}$ such that

$$\lim_{n \rightarrow \infty} \left| \frac{D_{h_{0,1}}(0, nx)}{\lambda|x|} - \nu \left(\frac{x}{|x|} \right) \right| = 0.$$

Combining this with Lemma 2.3.8 we see that

$$\lim_{\lambda \rightarrow \infty} \left| \frac{D_{h_{0,1}}(0, \lambda x)}{\lambda|x|} - \nu \left(\frac{x}{|x|} \right) \right| = 0,$$

where here λ is a real number. Now all that remains is to see why ν is in fact constant. Simply note that the law of $D_{h_{0,1}}(0, x)$ is rotationally invariant with respect to x by Lemma 2.2.2 and hence ν is independent of the direction. $\square$

Lemma 2.3.10. *Almost surely, $\limsup_{|x| \rightarrow \infty, x \in \mathbb{R}^d} \left| \frac{D_{h_{0,1}}(0, x)}{|x|} - \nu \right| = 0$.*

For the proof of this lemma, we follow the argument used in the main theorem in Boivin [1990]. The argument is almost exactly the same, but we reproduce it here for the reader's convenience. First we will outline the proof. The first crucial element is the maximal lemma, Lemma 2.3.11 below, which gives an upper tail bound for the random variable $\sup_{|x| > 1} D_{h_{0,1}}(0, x)/|x|$. The next step is Lemma 2.3.12, which tells us that given any direction θ there is a z' in that direction at a far enough distance from the origin such that $D_{h_{0,1}}(x, z') < \lambda|x|$ for a large constant $\lambda > 0$ which does not depend on x .

Now if the lemma is not true, then we must have a sequence $x_i \rightarrow \infty$ such that

$$\left| \frac{D_{h_{0,1}}(0, x_i)}{|x_i|} - \nu \right| > \bar{\varepsilon}, \quad \forall i \in \mathbb{N}.$$

By extracting a subsequence, we can assume that $\frac{x_i}{|x_i|} \rightarrow y$. We can then compare the distances $D_{h_{0,1}}(0, x_i)$ and $D_{h_{0,1}}(0, n_i M z)$ where $z \in \mathbb{R}^d$ with $|z| = 1$ is an appropriate direc-

tion, M is a large integer, and $n_i \in \mathbb{N}$ is such that $|x_i|, |n_i M z|$ are comparable. Combining everything we can conclude that $\left| \frac{D_{h_{0,1}}(0, x_i)}{|x_i|} - \nu \right| < \bar{\varepsilon}$, yielding a contradiction.

First we claim that the probability of $D_h(0, x)/|x|$ being large is small. More precisely, we need to show the following “maximal lemma”:

Lemma 2.3.11. *For some fixed constant K_1 we have*

$$\mathbb{P} \left\{ \sup_{x \in \mathbb{R}^d, |x| > 1} \frac{D_{h_{0,1}}(0, x)}{|x|} > \lambda \right\} \leq \frac{K_1}{\lambda^d}$$

Proof. Let $\mathcal{S}$ be the set of closed squares of side length 1 with corners in $\mathbb{Z}^d$. For each pair of squares $S, S' \in \mathcal{S}$ which share a side, we define the weight

$$w(S, S') = \text{Diam}_{h_{0,1}}(S) + \text{Diam}_{h_{0,1}}(S').$$

For each $x \in \mathbb{R}^d$, choose $S_x \in \mathcal{S}$ such that $x \in S$ (if x lies on the boundary of a square, we make an arbitrary choice). For $x, y \in \mathbb{R}^d$, let

$$L(x, y) := \inf_P \sum_{i=1}^{|P|} w(P_{i-1}, P_i) = \text{Diam}_{h_{0,1}}(S_x) + \text{Diam}_{h_{0,1}}(S_y) + 2 \inf_P \sum_{i=1}^{|P|-1} \text{Diam}_{h_{0,1}}(P_i)$$

where the infimum runs over all paths $P : P_0, P_1, \dots, P_{|P|}$ of cubes with $P_0 = S_x$ and $P_{|P|} = S_y$. By the triangle inequality,

$$\frac{D_{h_{0,1}}(0, x)}{|x|} \leq \frac{L(0, x)}{|x|}.$$

Hence

$$\mathbb{P} \left\{ \sup_{x \in \mathbb{R}^d, |x| > 1} \frac{D_{h_{0,1}}(0, x)}{|x|} > \lambda \right\} \leq \mathbb{P} \left\{ \sup_{x \in \mathbb{R}^d, |x| > 1} \frac{L(0, x)}{|x|} > \lambda \right\}.$$

Note that $h_{0,1}$ is stationary and ergodic by Lemma 2.2.2, and therefore so is $\text{Diam}_{h_{0,1}} S$. Since

$\text{Diam}_{h_{0,1}} S$ has a finite p -th moment by Lemma 2.3.1, we can apply the maximal lemma in Boivin [1990] (also see [Boivin and Derriennic, 1991, Theorem 6]) to see that

$$\mathbb{P} \left\{ \sup_{x \in \mathbb{R}^d, |x| > 1} \frac{L(0, x)}{|x|} > \lambda \right\} \leq \frac{K_1}{\lambda^d}.$$

□

Let E_λ be the event that

$$\sup_{x \in \mathbb{R}^d, |x| > 1} \frac{D_{h_{0,1}}(y, x)}{|x|} \leq \lambda,$$

and write $E_\lambda = E_\lambda(0)$. By Lemma 2.3.11 we know that

$$\mathbb{P}(E_\lambda) \geq 1 - \frac{K_1}{\lambda^d}.$$

Following the notation in Boivin [1990], for $M \in \mathbb{N}$, let

$$V_M := \left\{ \frac{z}{M} : z \in \mathbb{Z}^d \right\}, \quad V := \bigcup_{M \in \mathbb{N}} V_M, \quad B := \{z \in V : |v| = 1\}$$

and for $k, \rho > 0$, define the partial cone

$$C(k, \rho) := \left\{ x \in \mathbb{R}^d : x_1^2 + \cdots + x_{d-1}^2 \leq x_d^2, 0 \leq x_d \leq \rho \right\}.$$

If $\vec{\theta}$ is a unit vector, let $\vec{\theta} C(k, \rho)$ denote the cone $C(k, \rho)$ rotated so that its axis is $\mathbb{R} \vec{\theta}$.

If f is a function from the set of metrics on $\mathbb{R}^d$ to $\mathbb{R}$, let

$$\text{Av}(\vec{\theta}, k, \rho)(f) := \frac{1}{|\vec{\theta} C(k, \rho)|} \sum_{y \in \vec{\theta} C(k, \rho) \cap \mathbb{Z}^d} f(D_{h_{0,1}}(\cdot + y, \cdot + y)).$$

Then $\text{Av}(\vec{\theta}, k, \rho)(f) \rightarrow \mathbb{E}(f(D_{h_{0,1}}))$ as $\rho \rightarrow \infty$ by the Birkhoff ergodic theorem. Therefore applying this to $f = 1_{E_\lambda}$ we get that almost surely, for each fixed $\lambda, k > 0$ and $\vec{\theta} \in \mathbb{R}$, we have

$$\text{Av}(\vec{\theta}, k, \rho)1_{E_\lambda} \rightarrow \mathbb{P}(E_\lambda)$$

as $\rho \rightarrow \infty$. In particular, there is some $N_1 = N_1(\vec{\theta}, k, \lambda)$ such that for all $\rho \geq N_1$ we have

$$\mathbb{P}(E_\lambda) - \frac{K_1}{\lambda^d} \leq \text{Av}(\vec{\theta}, k, \rho)(1_{E_\lambda}) \leq \mathbb{P}(E_\lambda) + \frac{K_1}{\lambda^d}. \quad (2.43)$$

Lemma 2.3.12. *Suppose that $\vec{\theta} \in \mathbb{R}^d$ is a fixed unit vector. Suppose that $K_1/\lambda^d < 1/4$. Then there exists a random variable N_1 depending on $k, \vec{\theta}, \lambda$ if $\rho > \max\{N_1, 2k\}$ and $\rho' - \rho > \max\{2, K_2\rho\lambda^{-d}\}$ for some large constant K_2 only depending on K_1 , then there is a $z' \in \mathbb{Z}^d \cap (\vec{\theta} C(k, \rho') \setminus \vec{\theta} C(k, \rho))$ such that*

$$\sup_{x \in \mathbb{R}^d, |x - z'| > 1} \frac{D_{h_{0,1}}(z', x)}{|x - z'|} \leq \lambda.$$

Proof. We have

$$1 - \frac{2K_1}{\lambda^d} \leq \mathbb{P}(E_\lambda) - \frac{K_1}{\lambda^d} \leq \text{Av}(\vec{\theta}, k, \rho)(1_{E_\lambda}).$$

Let

$$\Delta := \mathbb{Z}^d \cap \vec{\theta} C(k, \rho), \quad \Delta' := \mathbb{Z}^d \cap (\vec{\theta} C(k, \rho') \setminus \vec{\theta} C(k, \rho))$$

and

$$D = \Delta \cap \{y : E_\lambda(y) \text{ holds}\}, \quad D' = \Delta' \cap \{y : E_\lambda(y) \text{ holds}\}$$

(see Figure 2.1). Then

$$1 - \frac{2K_1}{\lambda^d} \leq \frac{|D| + |D'|}{|\Delta| + |\Delta'|}.$$

Hence

$$|D'| \geq |\Delta'| + (|\Delta| - |D|) - \frac{2K_1}{\lambda^d}(|\Delta| + |\Delta'|) \geq |\Delta'| - \frac{2K_1}{\lambda^d}(|\Delta| + |\Delta'|)$$

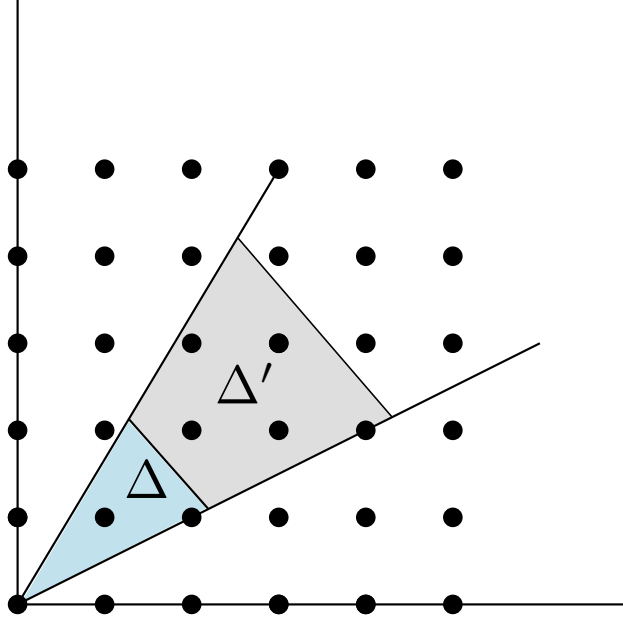


Figure 2.1: The sets Δ and Δ' .

To prove the claim it is enough to have $|D'| \geq 1$. By the above inequality, for this it suffices to have $(1 - \frac{2K_1}{\lambda^d})|\Delta'| \geq \frac{2K_1}{\lambda^d}|\Delta| + 1$. Explicit calculation of $|\Delta|, |\Delta'|$ gives this inequality for large enough ρ and $\rho' - \rho$. $\square$

Now if Lemma 2.3.11 is false, then we would have

$$\mathbb{P} \left\{ \limsup_{x \in \mathbb{R}^d, |x| \rightarrow \infty} \left| \frac{D_{h_{0,1}}(0, x)}{|x|} - \nu \right| > 0 \right\} > 0.$$

If this is the case, then by compactness of $\partial B(0, 1)$, we can find a sequence $\{x_i\}_{i \in \mathbb{N}}$ with $x_i \rightarrow \infty$ such that

$$\frac{x_i}{|x_i|} \rightarrow y \quad \text{and} \quad \left| \frac{D_{h_{0,1}}(0, x_i)}{|x_i|} - \nu \right| > \bar{\varepsilon}, \quad \forall i = 1, 2, \dots \quad (2.44)$$

for some $\bar{\varepsilon} > 0$. Now we make a series of choices. Let λ be large enough so that

$$\lambda^{1-d} < \bar{\varepsilon} / (32\sqrt{d}K_2), \quad (2.45)$$

$$\frac{2K_1}{\lambda^d} < \frac{1}{4} \quad (2.46)$$

Also, take M large enough so that there is a $z \in V_M = \frac{1}{M}\mathbb{Z}^d$ with $|z| = 1$ and

$$|y - z| < \bar{\varepsilon}/10\nu\lambda. \quad (2.47)$$

For $i \in \mathbb{N}$, let n_i be such that

$$n_i M \leq |x_i| < (n_i + 1)M, \quad (2.48)$$

and let N_0 be large enough so that for any $i > N_0$,

$$M/|x_i| < \bar{\varepsilon}/20\nu\lambda, \quad (2.49)$$

$$\left| y - \frac{x_i}{|x_i|} \right| < \bar{\varepsilon}/10\nu\lambda, \quad (2.50)$$

$$|(n_i M)^{-1} D_{h_{0,1}}(0, n_i M z) - \nu| < \bar{\varepsilon}/10 \quad (2.51)$$

(which is possible by the ergodic theorem) and

$$\|n_i M z\| > \max\{2K_2, \lambda^2/K_2, N_1\} \quad (2.52)$$

(see Figure 2.2).

Lemma 2.3.13. *Let $\vec{\theta} \in \partial B_1(0)$ be fixed. Then for large enough k and ρ, ρ' as in Lemma 2.3.12, there exists a $z' \in \mathbb{Z}^d \cap \vec{\theta}(C(k, \rho') \setminus C(k, \rho))$ such that*

$$\left| \frac{D_{h_{0,1}}(0, x_i)}{|x_i|} - \frac{D_{h_{0,1}}(0, n_i M z)}{|x_i|} \right| \leq \lambda \frac{(|x_i - n_i M z| + 2|z' - n_i M z|)}{|x_i|}. \quad (2.53)$$

Proof. By Lemma 2.3.12, there is a $z' \in \mathbb{Z}^d \cap (\vec{\theta} C(k, \rho') \setminus \vec{\theta} C(k, \rho))$ with $\rho = \|n_i M z\|$ such

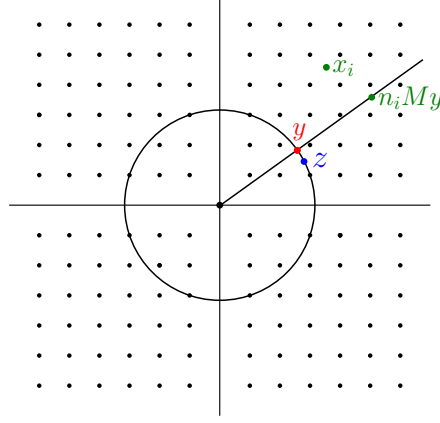


Figure 2.2: The n_i, x_i and the renormalized limit y .

that

$$\sup_{\substack{x \in \mathbb{R}^d \\ |x - z'| \geq 1}} \frac{D_{h_{0,1}}(z', x)}{|z' - x|} \leq \lambda. \quad (2.54)$$

By repeated applications of the triangle inequality,

$$\begin{aligned} \left| \frac{D_{h_{0,1}}(0, x_i)}{|x_i|} - \frac{D_{h_{0,1}}(0, n_i M z)}{|x_i|} \right| &\leq \left| \frac{D_{h_{0,1}}(x_i, n_i M z)}{|x_i|} \right| \\ &\leq \frac{(D_{h_{0,1}}(x_i, z') + D_{h_{0,1}}(z', n_i M z))}{|x_i|} \\ &\leq \lambda \frac{(|x_i - z'| + |z' - n_i M z|)}{|x_i|} \\ &\leq \lambda \frac{(|x_i - n_i M z| + 2|z' - n_i M z|)}{|x_i|}. \end{aligned} \quad (2.55)$$

□

Recall y from (2.44). We claim that

$$\left| \frac{D_{h_{0,1}}(0, x_i)}{|x_i|} - \nu \right| < \bar{\varepsilon} \quad (2.56)$$

which will contradict (2.44). By the triangle inequality,

$$\begin{aligned}
|x_i - n_i M z| &\leq |x_i| \left(\left| \frac{x_i}{|x_i|} - y \right| + \left| y - \frac{n_i M z}{|x_i|} \right| \right) \\
&\leq |x_i| \left(\left| y - \frac{x_i}{|x_i|} \right| + |y - z| + |z| \left(1 - \frac{n_i M}{|x_i|} \right) \right) \\
&= |x_i| \left(\left| y - \frac{x_i}{|x_i|} \right| + |y - z| + \left(1 - \frac{n_i M}{|x_i|} \right) \right). \tag{2.57}
\end{aligned}$$

Let $z' \in \mathbb{Z}^d \cap (\vec{\theta} C(k, \rho') \setminus \vec{\theta} C(k, \rho))$ be as in (2.54) taking $\rho = \|n_i M z\|$, $\rho' - \rho = 2K_2 \rho / \lambda^d$.

Using (2.52) we obtain

$$\begin{aligned}
|z' - n_i M z| &\leq \sqrt{d} \left(\frac{\rho'}{k} + (\rho' - \rho) \right) \\
&\leq \sqrt{d} \left(2K_2 \|n_i M z\| / k \lambda^d + \frac{\|n_i M z\|}{k} + \frac{2K_2 \|n_i M z\|}{\lambda^d} \right) \\
&\leq \sqrt{d} \left(\frac{2K_2}{k \lambda^d} + \frac{2K_2}{\lambda^d} + \frac{1}{k} \right) n_i M \\
&\leq \sqrt{d} \left(\frac{4K_2}{\lambda^d} + \frac{1}{k} \right) n_i M. \tag{2.58}
\end{aligned}$$

In the first line we used the fact that $z', n_i M z \in e^{i\theta}(C(k, \rho') \setminus C(k, \rho))$, and so it is easy to see that

$$|z' - n_i M z|_1 \leq \sqrt{d} |z' - n_i M z| \leq \sqrt{d} \left(\frac{\rho'}{k} + (\rho' - \rho) \right).$$

The first line follows. Plugging (2.57) and (2.58) into (2.53), we obtain

$$\begin{aligned}
&\left| \frac{D_{h_{0,1}}(0, x_i)}{|x_i|} - \frac{D_{h_{0,1}}(0, n_i M z)}{|x_i|} \right| \\
&\leq \lambda \left(\left| y - \frac{x_i}{|x_i|} \right| + |y - z| + \left(1 - \frac{n_i M}{|x_i|} \right) + \sqrt{d} \left(\frac{4K_2}{\lambda^d} + \frac{1}{k} \right) \frac{n_i M}{|x_i|} \right).
\end{aligned}$$

Now using (2.47), (2.49), and (2.50) we obtain that for large enough λ, k we have

$$\left| \frac{D_{h_{0,1}}(0, x_i)}{|x_i|} - \frac{D_{h_{0,1}}(0, n_i M z)}{|x_i|} \right| \leq \lambda \left(\frac{\bar{\varepsilon}}{10\lambda} + \frac{\bar{\varepsilon}}{10\nu\lambda} + \frac{\bar{\varepsilon}}{20\nu\lambda} \right) < \frac{\bar{\varepsilon}}{2}, \quad (2.59)$$

where in the last line we used (2.45). We also have by (2.48) and (2.51) that

$$\begin{aligned} \left| \frac{D_{h_{0,1}}(0, n_i M z)}{|x_i|} - \frac{D_{h_{0,1}}(0, n_i M z)}{n_i M} \right| &\leq \frac{M}{|n_i M x_i|} D_{h_{0,1}}(0, n_i M z) \leq \frac{M}{|x_i|} \left(\nu + \frac{\bar{\varepsilon}}{10} \right) \\ &\leq \frac{\bar{\varepsilon}}{20\nu\lambda} \left(\nu + \frac{\bar{\varepsilon}}{10} \right) \leq \frac{\bar{\varepsilon}}{10} \end{aligned} \quad (2.60)$$

and

$$\left| \frac{D_{h_{0,1}}(0, n_i M z)}{n_i M} - \nu \right| \leq \frac{\bar{\varepsilon}}{10}. \quad (2.61)$$

Finally, combining (2.60), (2.61), and (2.59), we obtain

$$\begin{aligned} \left| \frac{D_{h_{0,1}}(0, x_i)}{|x_i|} - \nu \right| &\leq \left| \frac{D_{h_{0,1}}(0, x_i)}{|x_i|} - \frac{D_{h_{0,1}}(0, n_i M z)}{|x_i|} \right| \\ &\quad + \left| \frac{D_{h_{0,1}}(0, n_i M z)}{|x_i|} - \frac{D_{h_{0,1}}(0, n_i M z)}{n_i M} \right| \\ &\quad + \left| \frac{D_{h_{0,1}}(0, n_i M z)}{n_i M} - \nu \right| \\ &< \bar{\varepsilon} \end{aligned}$$

which proves (2.56).

Now we are ready to prove Proposition 2.3.2.

Proof of Proposition 2.3.2. Let $\Psi : (0, \infty) \rightarrow (0, \infty)$ be such that $\lim_{\beta \rightarrow 0} \Psi(\beta) = 0$ and $\lim_{\beta \rightarrow 0} \Psi(\beta)/\beta = \infty$. Then we claim that for any fixed $\bar{x} \in \delta\mathbb{Z}^d$ we have with probability tending to 1 as $\beta \rightarrow 0$ that

$$\beta^{\xi Q-1} (1-\varepsilon) \nu |x - \bar{x}| \leq D_{h_{0,\beta}}(x, \bar{x}) \leq \beta^{\xi Q-1} (1+\varepsilon) \nu |x - \bar{x}|, \quad \forall x \in \mathbb{R}^d \text{ such that } |x - \bar{x}| > \Psi(\beta).$$

Indeed, in Proposition 2.3.10 0 could be replaced with any other fixed point $\bar{x} \in \mathbb{Z}^d$, that is, almost surely we have that

$$\limsup_{x \rightarrow \infty} \left| \frac{D_{h_{0,1}}(x, \bar{x})}{|x - \bar{x}|} - \nu \right| = 0.$$

By Lemma 2.2.3 and Lemma 2.2.2 we have that

$$\begin{aligned} \mathbb{P}(\beta^{\xi Q-1}(1-\varepsilon)\nu|x-\bar{x}| \leq D_{h_{0,\beta}}(x, \bar{x}) \leq \beta^{\xi Q-1}(1+\varepsilon)\nu|x-\bar{x}|, \forall x \text{ s.t. } |x-\bar{x}| > \Psi(\beta)) \\ = \mathbb{P}((1-\varepsilon)\nu|x| \leq D_{h_{0,1}}(x, 0) \leq (1+\varepsilon)\nu|x|, \forall x \text{ s.t. } |x| > \Psi(\beta)/\beta) \rightarrow 1 \end{aligned} \quad (2.62)$$

as $\beta \rightarrow 0$.

Now fix $\delta > 0$ and assume that β is small enough that $\Psi(\beta) < \delta$. By a union bound, it holds with probability tending to 1 as $\beta \rightarrow 0$ that for each $\bar{x} \in (\delta\mathbb{Z}^d) \cap [-R, R]^d$, and all $x \in \mathbb{R}^d$ such that $|x - \bar{x}| > \delta$, we have

$$(1-\varepsilon)\nu\beta^{\xi Q-1}|x-\bar{x}| \leq D_{h_{0,\beta}}(\bar{x}, x) \leq (1+\varepsilon)\nu\beta^{\xi Q-1}|x-\bar{x}|$$

Now suppose that $x, y \in [-R, R]^d$. Then there exists an $\bar{x} \in \delta\mathbb{Z}^d$ such that

$$\delta < |x - \bar{x}| \leq C\delta$$

for some d dependent constant d and

$$\delta < |\bar{x} - y|.$$

Therefore we obtain that

$$\begin{aligned}
D_{h_{0,\beta}}(x, y) &\leq D_{h_{0,\beta}}(\bar{x}, x) + D_{h_{0,\beta}}(\bar{x}, y) \leq (1 + \varepsilon)\nu\beta^{\xi Q-1}(|x - \bar{x}| + |\bar{x} - y|) \\
&\leq (1 + \varepsilon)\nu\beta^{\xi Q-1}(2|x - \bar{x}| + |x - y|) \\
&\leq (1 + \varepsilon)\nu\beta^{\xi Q-1}|x - y| + 2C\delta(1 + \varepsilon)\nu\beta^{\xi Q-1}.
\end{aligned}$$

Similarly,

$$D_{h_{0,\beta}}(x, y) \geq (1 - \varepsilon)\nu\beta^{\xi Q-1}|x - y| - 4\delta(1 + \varepsilon)\nu\beta^{\xi Q-1}.$$

Noting that $\varepsilon\nu\beta^{\xi Q-1}|x - y| + 4\delta(1 + \varepsilon)\nu\beta^{\xi Q-1}$ can be made smaller than any desired multiple of $\beta^{\xi Q-1}|x - y|$ by making ε and δ small enough, we conclude the proof of Proposition 2.3.2. $\square$

2.3.3 Controlling the total mass

Let α be defined by

$$\alpha = \mathbb{E}(\mu_{h_{0,1}}([0, 1]^d)) < \infty$$

(which is well defined by Theorem 2.11 in Rhodes and Vargas [2014]). Then by translation invariance, we have that for any positive integer n ,

$$\mathbb{E}(\mu_{h_{0,1}}([-2^n, 2^n]^d)) \leq 2^d \alpha 2^{dn}.$$

Therefore, by Markov's inequality we have that for any small $c > 0$,

$$\mathbb{P}(\mu_{h_{0,1}}([-2^n, 2^n]^d)) \leq 2^d \alpha (2^n)^{d+c} \geq 1 - \frac{1}{2^{nc}} \rightarrow 0$$

as $n \rightarrow \infty$. By Lemma 2.2.2, $h_{0,1}$ agrees in law with $h_{0,\beta}(\beta \cdot)$. By the Weyl scaling and LQG coordinate change properties of $\mu_{h_{0,1}}$, we deduce that $\mu_{h_{0,1}}([-2^n, 2^n]^d)$ agrees in law

with $\beta^{-\gamma Q} \mu_{h_{0,\beta}}([-\beta 2^n, \beta 2^n]^d)$. Hence picking n such that $2^n \beta \in [1, 11/10]$, we obtain with probability tending to 1 as $\beta \rightarrow 0$ that

$$\mu_{h_{0,\beta}}([-1, 1]^d) \leq 4\alpha\beta^{\gamma Q-d-c}. \quad (2.63)$$

2.3.4 Making $h_{\beta,\infty}$ approximate with positive probability

We will need a couple of lemmas.

Lemma 2.3.14. *Let $f : \mathbb{R}^d \rightarrow \mathbb{R}$ be a smooth function and let $\beta > 0$. Let $p_{\beta^2/2}$ be as in (2.5). Suppose that f 's Fourier transform $\hat{f}$ is compactly supported. Then there exists a ϕ with $\nabla\phi, \phi \in L^2(\mathbb{R}^d)$ such that*

$$f = p_{\beta^2/2} * \phi.$$

Proof. Since f is compactly supported, its Fourier transform is well defined. Let $\mathcal{F}^{-1}$ denote the inverse Fourier transform. Define

$$\phi := \mathcal{F}^{-1} \left(\frac{\hat{f}}{\hat{p}_{\beta^2/2}} \right).$$

We note that by (2.31), $\hat{p}_{\beta^2/2}$ never vanishes, and hence $\frac{\hat{f}}{\hat{p}_{\beta^2/2}}$ is well defined and by the compact support of $\hat{f}$ it is compactly supported. Therefore $\phi \in H^1(\mathbb{R}^d)$. Also,

$$\mathcal{F}(p_{\beta^2/2} * \phi) = \hat{p}_{\beta^2/2} \frac{\hat{f}}{\hat{p}_{\beta^2/2}} = \hat{f}.$$

This completes the proof. □

The following lemma tells us that we can approximate uniformly by functions with compact Fourier support.

Lemma 2.3.15. *Let $R, \varepsilon > 0$. Then there exists a countable family of smooth functions such*

that for any $f \in \mathcal{F}$, the function $\hat{f}$ is compactly supported, and if g is continuous in $\overline{B_R(0)}$ then there exists an element $f \in \mathcal{F}$ such that $\sup_{z \in B_R(0)} |g(z) - f(z)| < \varepsilon$. In particular, all elements in this net are Schwartz.

Proof. Take $\tilde{\mathcal{F}}$ to be any countable $\varepsilon/2$ -net of smooth functions in $\mathcal{C}(B_R(0))$. Let $\chi : \mathbb{R}^d \rightarrow [0, 1]$ be a smooth function such that $\chi(y) = 1$ for $|y| < 1$ and $\chi(y) = 0$ for $|y| \geq 2$. Then for any $f \in \tilde{\mathcal{F}}$ and $N \in \mathbb{N}$, let $f_N = \mathcal{F}^{-1} \left(\hat{f} \chi \left(\frac{\cdot}{N} \right) \right)$. Then f_N has compactly supported Fourier transform, and $\|f_N - f\|_\infty \leq \|\hat{f}_N - \hat{f}\|_1 \rightarrow 0$ as $N \rightarrow 0$. Now taking $\mathcal{F} = \{f_N : f \in \tilde{\mathcal{F}}, N \in \mathbb{N}\}$ we have our desired net. $\square$

Next, we need to construct functions such that the sphere average at the unit sphere can be arbitrarily large, but when convolved with $p_{\beta^2/2}$ is small.

Lemma 2.3.16. *Let $\beta > 0$, $K > 0$ and $\eta > 0$. Then there is a smooth function $\Psi_{K,\eta} = \Psi_{K,\eta,\beta}$ such that*

$$\int_{\partial B_1(0)} \Psi_{K,\eta} = 2\pi K, \quad (2.64)$$

and

$$\|\Psi_{K,\eta} * p_{\beta^2/2}\|_\infty < \eta. \quad (2.65)$$

Proof. Let $\bar{\eta} > 0$ be a small constant to be chosen later. Let $\Psi_{K,\eta}$ be such that

$$\Psi_{K,\eta}(z) = \begin{cases} K & \text{if } z \in \partial B_1(0), \\ 0 & \text{if } |z| \leq 1 - \bar{\eta} \text{ or } |z| \geq 1 + \bar{\eta}. \end{cases}$$

Suppose additionally that $\Psi_{K,\eta}(z) \in [0, K]$ for all z . Then it is clear that (2.64) holds. For (2.65), note that $p_{\beta^2/2}(z) \lesssim \frac{1}{\beta^d}$ for any $z \in \mathbb{R}^d$. Therefore

$$\begin{aligned} \Psi_{K,\eta} * p_{\beta^2/2}(z_0) &= \int_C \Psi_{K,\eta}(z) p_{\beta^2/2}(z_0 - z) \lesssim \frac{1}{\beta^d} \int_{\mathbb{R}^d} \Psi_{K,\eta}(z) dz \\ &\leq \frac{1}{\beta^d} R \left(\pi(1 + \bar{\eta})^2 - \pi(1 - \bar{\eta})^2 \right) = \frac{4K}{\beta^d} \bar{\eta}. \end{aligned}$$

Therefore if we take $\bar{\eta} \leq \frac{\beta^2 \eta}{4K}$ we obtain (2.65). $\square$

The following lemma tells us that the sphere average of $h * p_{\beta^2/2}$ at the unit sphere is small if β is small enough.

Lemma 2.3.17. *Let $\varepsilon > 0$. It holds with probability converging to 1 as $\beta \rightarrow 0$ that*

$$\left| \left(h * p_{\beta^2/2} \right)_1(0) \right| \leq \varepsilon.$$

Proof. We recall that h is locally in H^{-s} for any $s > 0$ by Proposition 2.7 in Sheffield [2007]. Therefore $h * p_{\beta^2/2} \rightarrow h$ in H^{-s} for any $s > 0$ as $\beta \rightarrow 0$. Therefore $(h * p_{\beta^2/2})_1(0) \rightarrow h_1(0)$ as $\beta \rightarrow 0$. Since $h_1(0) = 0$, this completes the proof. $\square$

We will also need the following lemma.

Lemma 2.3.18. *Let $R > 0$, and $\varepsilon, \delta, \theta > 0$. Suppose that E is an event such that $\mathbb{P}(E) > 0$. Then there is a smooth, compactly supported function ϕ such that $\phi(z) = 0$ if $|z| \geq R + \delta + \theta$, and such that with positive probability,*

$$E \text{ occurs} \quad \text{and} \quad \sup_{x \in B_R(0)} |h_{\beta, \infty}(x) - \phi(x)| \leq \varepsilon.$$

Proof. Let F_R denote a family of functions compactly supported on $B_{R+\delta+\theta}(0)$ that form a countable ε -net in $C(B_{R+\delta}(0))$ (with the L^∞ norm). Then

$$0 < \mathbb{P}(E) \leq \sum_{\phi \in F_R} \mathbb{P} \left(E \text{ holds and } \sup_{x \in B_R(0)} |h_{\beta, \infty}(x) - \phi(x)| \leq \varepsilon \right),$$

so there is some ϕ such that

$$\mathbb{P} \left(E \text{ holds and } \sup_{x \in B_R(0)} |h_{\beta, \infty}(x) - \phi(x)| \leq \varepsilon \right) > 0.$$

This completes the proof. $\square$

Let $\|\cdot\|_\infty$ denote the $L^\infty(B_R(0))$ norm for some large fixed R . Let E denote the event in Lemma 2.3.17. Then by Lemma 2.3.18, there exists a smooth function f_β such that

$$\mathbb{P}\left(E \text{ holds and } \|h_{\beta,\infty} - f_\beta\|_\infty < \frac{\varepsilon}{4}\right) > 0. \quad (2.66)$$

Suppose from now on that E holds. By Lemma 2.3.14, there exists a smooth function g_β such that

$$p_{\beta^2/2} * g_\beta = f - f_\beta. \quad (2.67)$$

By Lemma 2.2.4 we can couple h and $h_{\beta,\infty}$ in such a way that

$$h_{\beta,\infty} = h * p_{\beta^2/2} - (h * p_{\beta^2/2})_1(0)$$

Then we have that

$$\begin{aligned} \|h_{\beta,\infty} - f\|_\infty &\leq \|h_{\beta,\infty} - (f - f_\beta) - f_\beta\|_\infty \\ &\leq \|h_{\beta,\infty} - p_{\beta^2/2} * g_\beta - f_\beta\|_\infty \\ &\leq \|(h - g_\beta) * p_{\beta^2/2} - f_\beta\|_\infty + \|h_{\beta,\infty} - h * p_{\beta^2/2}\|_\infty \\ &= \|(h - g_\beta) * p_{\beta^2/2} - f_\beta\|_\infty + |(h * p_{\beta^2/2})_1(0)|, \end{aligned}$$

where in the last line we used Lemma 2.2.4. Let $K = (g_\beta)_1(0)$, and define $\tilde{g}_\beta = g_\beta - \Psi_{K,\varepsilon}$,

where $\Psi_{K,\varepsilon}$ is given as in Lemma 2.3.16. Then we have by the triangle inequality that

$$\begin{aligned}
\|h_{\beta,\infty} - f\|_\infty &\leq \| (h - g_\beta) * p_{\beta^2/2} - f_\beta \|_\infty + | (h * p_{\beta^2/2})_1(0) | \\
&\leq \| (h - \tilde{g}_\beta) * p_{\beta^2/2} - f_\beta \|_\infty + | (h * p_{\beta^2/2})_1(0) | + \| (g_\beta - \tilde{g}_\beta) * p_{\beta^2/2} \|_\infty \\
&= \| (h - \tilde{g}_\beta) * p_{\beta^2/2} - f_\beta \|_\infty + | (h * p_{\beta^2/2})_1(0) | + \|\Psi_{K,\varepsilon} * p_{\beta^2/2}\|_\infty \\
&\leq \| (h - \tilde{g}_\beta) * p_{\beta^2/2} - f_\beta \|_\infty + | (h * p_{\beta^2/2})_1(0) | + \varepsilon
\end{aligned} \tag{2.68}$$

where in the last line we used Lemma 2.3.16. Now we claim that the event

$$\{ \|h * p_{\beta^2/2} - f_\beta\|_\infty, | (h + \tilde{g}_\beta) * p_{\beta^2/2})_1(0) | \leq \varepsilon \} \tag{2.69}$$

holds with positive probability. Suppose that (2.69) holds for now. Then we have by Lemma 2.2.6 that it holds with positive probability that

$$\{ \| (h - \tilde{g}_\beta) * p_{\beta^2/2} - f_\beta \|_\infty, | (h * p_{\beta^2/2})_1(0) | \leq \varepsilon \}.$$

Combining this with (2.68) we obtain that

$$\|h_{\beta,\infty} - f\|_\infty < 3\varepsilon$$

with positive probability.

It remains to show that (2.69) holds with positive probability. Note that we already have with positive probability that we simultaneously have

$$\|h_{\beta,\infty} - f_\beta\|_\infty < \frac{\varepsilon}{4}$$

and

$$| (h * p_{\beta^2/2})_1(0) | \leq \varepsilon.$$

Then

$$\begin{aligned}
|((h + \tilde{g}_\beta) * p_{\beta^2/2})_1(0)| &\leq \varepsilon + |(\tilde{g}_\beta * p_{\beta^2/2})_1(0)| \\
&\leq \varepsilon + |(g_\beta * p_{\beta^2/2})_1(0)| + |(\Psi_{K,\varepsilon} * p_{\beta^2/2})_1(0)| \\
&= \varepsilon + |(f - f_\beta)_1(0)| + |(\Psi_{K,\varepsilon} * p_{\beta^2/2})_1(0)|,
\end{aligned}$$

where in the last line we used (2.67). Now recall that $f_1(0) = 0$ by hypothesis. Hence by (2.66) we have

$$|(f - f_\beta)_1(0)| = |(f_\beta)_1(0)| \leq |(h_\beta, \infty)_1(0)| + \pi \frac{\varepsilon}{4} \leq \varepsilon.$$

This implies that

$$|((h + \tilde{g}_\beta) * p_{\beta^2/2})_1(0)| \leq 2\varepsilon + |(\Psi_{K,\varepsilon} * p_{\beta^2/2})_1(0)|.$$

Using Lemma 2.3.17 we obtain that (2.69) holds with positive probability.

2.4 Increasing measure without changing metric

Let $f : \mathbb{R}^d \rightarrow \mathbb{R}$ be a bounded continuous function. By Theorem 2.1.8, with positive probability we have

$$e^{\xi f} \cdot d_0(x, y) - \varepsilon < D_h(x, y) < e^{\xi f} \cdot d_0(x, y) + \varepsilon, \quad (2.70)$$

for all $x, y \in [-R, R]^d$ and also

$$\mu_h([-R, R]^d) < \varepsilon. \quad (2.71)$$

Definition 2.4.1. We let E^0 denote the event that both (2.70) and (2.71) hold. Note that this is a positive probability event by Theorem 2.1.8

If we are able to change the measure without changing the metric very much, we can

then use the results from Section 2.3 and prove Theorem 2.1.6 in the case of any Riemannian metric of the form $e^{\xi f} \cdot d_0$ together with any finite measure on $\mathbb{R}^d$. Therefore the set of metric measure spaces that can be approximated by (D_h, μ_h) with positive probability and the closure of Riemannian metrics on $\mathbb{R}^d$ are the same. Now by Contreras Hip and Gwynne [2024], this proves Theorem 2.1.6.

2.4.1 Proof of Theorem 2.1.6

The idea will be to add bump functions to the log-correlated Gaussian field h which will locally add mass at the centers of these bumps, but will not affect distances very much since geodesics can go around these bumps. To this end, suppose that $z_1, \dots, z_M$ is a set of points in $[-R, R]^d$, and suppose that $C_1, \dots, C_M$ are positive weights. Let N, K be large positive integers, and let $r > 0$ be small. For each $1 \leq i \leq M$ we will subdivide the square $z_i + [-r, r]^d$ into 2^d sets $X_{(j_1, \dots, j_d)}^{i, K}$ with $j_1, \dots, j_d \in \{0, 1\}^d$ defined as follows:

$$X_{(j_1, \dots, j_d)}^{i, K} := z_i + \frac{r}{2K}(j_1, \dots, j_d) + \left(\bigcup_{\ell_1=-K}^{K-1} \dots \bigcup_{\ell_d=-K}^{K-1} \left(\frac{r\ell_1}{K}, \dots, \frac{r\ell_d}{K} \right) + \left[0, \frac{r}{2K} \right]^d \right)$$

(see Figure 2.3 for a picture). The idea of considering sets of this form is taken from the proof of Lemma 11.10 in Bou-Rabee and Gwynne [2022].

Let $\zeta > 0$ be a small constant. Since D_h induces the Euclidean topology, the maximum D_h diameter of the cubes $X_{(j_1, \dots, j_d)}^{i, K}$ goes to 0 as $K \rightarrow \infty$. By this and the pigeonhole principle, we can take deterministic $\{(j_1^i, \dots, j_d^i)\}_{1 \leq i \leq M}$ such that with positive probability, E^0 holds, and simultaneously

$$\frac{\max_{-K \leq \ell_1, \dots, \ell_d \leq K} \text{Diam}_{D_h} \left(z_i + \frac{r}{2K}(j_1^i, \dots, j_d^i) + \left(\frac{r\ell_1}{K}, \dots, \frac{r\ell_d}{K} \right) + \left[-\frac{r}{K}, \frac{3r}{2K} \right]^d \right)}{\mu_h(z_i + [-r, r]^d)} \leq \zeta \quad (2.72)$$

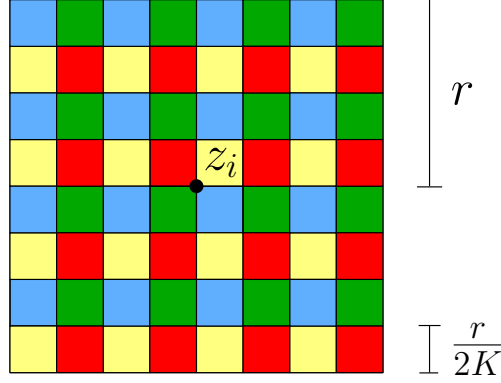


Figure 2.3: The definition of $X_{(j_1, j_2)}^{i, K}$. Different colors correspond to different (j_1, j_2) .

for all $1 \leq i \leq M$, and

$$\mu_h \left(X_{(j_1^i, \dots, j_d^i)}^{i, K} \right) \geq \frac{\mu_h(z_i + [-r, r]^d)}{2^d}. \quad (2.73)$$

To simplify notation, we let

$$X_i^K := X_{(j_1^i, \dots, j_d^i)}^{i, K}. \quad (2.74)$$

Now we will “fix” the μ_h masses around each point z_i . We have the following lemma.

Lemma 2.4.2. *Suppose that $r > 0$ and $\bar{\eta} > 0$. Then there exist integers $n_1, \dots, n_M \in \mathbb{N}$, a deterministic $0 < \rho < \frac{r}{K}$, and a large deterministic constant $\bar{C}$ such that with positive probability, E^0 holds, (2.72) holds, (2.73) holds, we have*

$$\frac{\mu_h \left(\{z : d_0(z, X_i^K) \leq \rho\} \setminus X_i^K \right)}{\mu_h(X_i^K)} \leq \zeta, \quad (2.75)$$

also

$$n_i \bar{\eta} \leq \mu_h(X_i^K) \leq (n_i + 1) \bar{\eta}$$

for any $1 \leq i \leq M$, and we finally have

$$\text{Diam}_{D_h}([-r - \rho, r + \rho]^d \setminus X_i^K) \leq \bar{C}. \quad (2.76)$$

Proof. Note that the event that E^0 , (2.72), (2.73) hold simultaneously is an event of positive probability by construction. The result follows as a direct consequence of the pigeonhole principle. To select ρ , we note that a non-deterministic ρ satisfying (2.75) exists with probability 1, and so there exists a deterministic ρ satisfying (2.75) simultaneously with the conditions E^0 , (2.72), and (2.73). Finally, note that the probability of (2.76) holds with probability going to 1 as $\bar{C} \rightarrow \infty$, so we can also assume this holds simultaneously. This completes the proof. $\square$

Therefore for any $\bar{\eta} > 0$, there exist deterministic constants $\{a_i\}$ such that with positive probability, (2.72), (2.73), (2.75) hold, and simultaneously

$$|\mu_h(X_i^K) - a_i| \leq \bar{\eta}. \quad (2.77)$$

We will define our bump functions now. For any $1 \leq i \leq M$, we let $\varphi_{N,r,\rho}^i$ be defined as follows:

$$\varphi_{N,r,\rho}^i(z) = \begin{cases} \log\left(\frac{C_i}{a_i}\right) & \text{if } z \in X_i^K, \\ -N & \text{if } d_0(x, X_i^K) \geq \rho \text{ and } z \in z_i + [-r - \rho, r + \rho]^d, \\ 0 & \text{if } d_0(z, z_i + [-r, r]^d) \geq 2\rho. \end{cases} \quad (2.78)$$

We additionally impose that $-N \leq \varphi_{N,r,\rho}^i(z) \leq \log\left(\frac{C_i}{a_i}\right)$ if $d_0(x, X_i^K) \leq \rho$, and that $-N \leq \varphi_{N,r,\rho}^i(z) \leq 0$ if $d_0(z, z_i + [-r, r]^d) \leq \rho$, but $z \notin z_i + [-r, r]^d$ (see Figure 2.4 for a picture of $\varphi_{N,r,\rho}^i(z)$ in dimension 2).

Now we define

$$\varphi_N(z) := \sum_{i=1}^M \varphi_{N,r,\rho}^i(z). \quad (2.79)$$

We will prove the following proposition.

Proposition 2.4.3. *Let g be a continuous function on $\mathbb{R}^d$, and let $z_1, \dots, z_M \in \mathbb{R}^d$. Define*

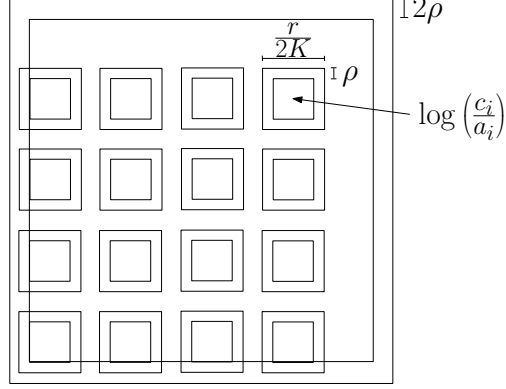


Figure 2.4: Support of $\varphi_{N,\varepsilon,\rho}^i$

φ_N as in (2.79). Suppose that $\int_{\mathbb{R}^d} |g(z)| dz < \infty$, and suppose that the event of Lemma 2.4.2 occurs. Then for any $\zeta > 0$ and small enough ε, r ,

$$\left| \int_{\mathbb{R}^d} g(z) e^{\varphi_N(z)} d\mu_h - \sum_{i=1}^M C_i g(z_i) \right| < \zeta$$

Proof. First we note that we can reduce to the case that $M = 1$. Indeed, let $\{\chi_i\}_{i=1}^M$ be smooth functions such that

$$\chi_i(z) = \begin{cases} 1 & \text{if } z \in z_i + [-r - 2\rho, r + 2\rho]^d, \\ 0 & \text{if } z \notin z_i + [-r - 3\rho, r + 3\rho]^d, \end{cases}$$

and $0 \leq \chi_i(z) \leq 1$. Then by the triangle inequality we have

$$\begin{aligned}
& \left| \int_{\mathbb{R}^d} g(z) e^{\varphi_N(z)} d\mu_h - \sum_{i=1}^M C_i g(z_i) e^{\gamma f(z_i)} \right| \\
& \leq \sum_{i=1}^M \left| \int_{\mathbb{R}^d} \chi_i(z) g(z) e^{\varphi_{N,r,\rho}^i(z)} d\mu_h - C_i \chi_i(z_i) g(z_i) \right| \\
& \quad + \left| \int_{\mathbb{R}^d} g(z) \left(e^{\varphi_N(z)} - \sum_{i=1}^M e^{\varphi_{N,r,\rho}^i(z)} \right) d\mu_h \right| \\
& \leq \sum_{i=1}^M \left| \int_{\mathbb{R}^d} \chi_i(z) g(z) e^{\varphi_{N,r,\rho}^i(z)} d\mu_h - C_i g(z_i) \right| \\
& \quad + \varepsilon \|g\|_\infty \max_{1 \leq i \leq M} |C_i|,
\end{aligned}$$

where in the last line we used the fact that $\mu_h(\mathbb{R}^d) \leq \varepsilon$ by (2.71). Therefore shrinking ζ if necessary, it suffices to prove the proposition for the case when $M = 1$. We then have a single point z_1 with its weight C . Without loss of generality, suppose that $z_1 = 0$. Since we have a single point mass, we will drop the subscript on X_i^K . We split $\mathbb{R}^d$ into three regions,

$$\mathbb{R}^d = X^K \cup \left((X^K + B_\rho(0)) \setminus X^K \right) \cup \left(\mathbb{R}^d \setminus (X^K + B_\rho(0)) \right).$$

Splitting $\mathbb{R}^d$ into these domains we obtain

$$\begin{aligned}
& \left| \int_{\mathbb{R}^d} g(z) e^{\gamma f(z) + \varphi_N(z)} d\mu_h - C g(0) e^{\gamma f(0)} \right| \\
& \leq \left| \int_{X^K} g(z) e^{\varphi_N(z)} d\mu_h - C g(0) \right| \\
& \quad + \left| \int_{((X^K + B_\rho(0)) \setminus X^K)} g(z) e^{\varphi_N(z)} d\mu_h \right| \\
& \quad + \left| \int_{(\mathbb{R}^d \setminus (X^K + B_\rho(0)))} g(z) e^{\varphi_N(z)} d\mu_h \right|.
\end{aligned}$$

Note that

$$\left| \int_{((X^K + B_\rho(0)) \setminus X^K)} g(z) e^{\varphi_N(z)} d\mu_h \right| \leq \mu_h((X^K + B_\rho(0)) \setminus X^K) \|g\|_\infty \frac{C}{a},$$

where we used the fact that $\varphi_N(z) \leq \frac{C}{a}$ if $z \in ((X^K + B_\rho(0)) \setminus X^K)$. Using (2.75), we obtain that

$$\left| \int_{((X^K + B_\rho(0)) \setminus X^K)} g(z) e^{\varphi_N(z)} d\mu_h \right| \leq \|g\|_\infty C\zeta. \quad (2.80)$$

Similarly, we have with probability 1 that

$$\left| \int_{(\mathbb{R}^d \setminus (X^K + B_\rho(0)))} g(z) e^{\varphi_N(z)} d\mu_h \right| \leq \|g\|_\infty \mu_h(\mathbb{R}^d \setminus (X^K + B_\rho(0))) \leq \|g\|_\infty \varepsilon. \quad (2.81)$$

$$\left| \int_{\mathbb{R}^d} g(z) e^{\varphi_N(z)} d\mu_h - Cg(0) \right| \leq \left| \int_{X^K} g(z) \frac{C}{a} - Cg(0) \right| + \|g\|_\infty C\zeta + \|g\|_\infty \varepsilon \quad (2.82)$$

Note that

$$\int_{X^K} g(z) \frac{C}{a} - Cg(0) = o_r(1) \quad (2.83)$$

by (2.77) and the fact that g is continuous. Therefore taking ε to be small enough we obtain

$$\left| \int_{\mathbb{R}^d} g(z) e^{\varphi_N(z)} d\mu_h - Cg(0) \right| \leq \zeta. \quad (2.84)$$

□

Now we claim the metric doesn't change much when adding $\varphi_N(z)$.

Proposition 2.4.4. *Let $\theta > 0$. For large enough N , and small enough r , we have that with positive probability that the conclusion of Lemma 2.4.2 holds, and simultaneously for all*

$x, y \in \mathbb{R}^d$

$$\sup_{x, y \in \mathbb{R}^d} |D_h(x, y) - e^{\varphi_N} \cdot D_h(x, y)| \leq \theta.$$

We will prove Proposition 2.4.4 in Subsection 2.4.2. We will prove now Theorem 2.1.6 assuming Proposition 2.4.4.

Proof of Theorem 2.1.4 assuming Proposition 2.4.4. Suppose that $e^{\xi f} \cdot d_S$ is a Riemannian metric in its isothermal form, and let $\mathbf{m}$ be a probability measure on $\mathbb{R}^d$. We will show Theorem 2.1.6 for this metric and measure. Let $\varphi = \varphi_N$ be as in (2.79). We now introduce a finite set of functions to which we will apply Proposition 2.4.4. Let n be a large integer chosen so that

$$n \geq \frac{2\sqrt{d}R}{\varepsilon}. \quad (2.85)$$

Let $T_{(j_1, \dots, j_d)}$ be defined by

$$T_{(j_1, \dots, j_d)} := \left[\frac{R}{n}(-n + j_1), \frac{R}{n}(-n + j_1 + 1) \right] \times \cdots \times \left[\frac{R}{n}(-n + j_d), \frac{R}{n}(-n + j_d + 1) \right].$$

For each finite subset $S \subseteq [0, 2n - 1]^d$, let g_S be a smooth function such that

$$g_S(z) = \begin{cases} 1 & \text{if } z \in \cup_{(j_1, \dots, j_d) \in S} T_{(j_1, \dots, j_d)}, \\ 0 & \text{if } z \notin \cup_{(j_1, \dots, j_d) \in S} \{x : d_S(x, T_{(j_1, \dots, j_d)}) \leq \frac{\varepsilon}{2}\}. \end{cases}$$

For any $\theta > 0$, there exist points $\{z_i\}_{i=1}^n \subset \mathbb{R}^d$ and weights $\{C_i\}_{i=1}^n \subset (0, \infty)$ such that for any $(j_1, \dots, j_d) \in S$ we have

$$\left| \int_{\mathbb{R}^d} g_S d\mathbf{m} - \sum_{i=1}^M C_i g_S(z_i) \right| < \frac{\theta}{3} \left| \int_{\mathbb{R}^d} g_S d\mathbf{m} \right|. \quad (2.86)$$

Then applying Proposition 2.4.3 to g_S for every $S \subseteq [0, 2n - 1]^d$ taking r to be small enough

we obtain that with positive probability for any $S \subseteq F(\mathcal{T})$, we have

$$\left| \int_{\mathbb{R}^d} g_S(z) e^{\varphi_N(z)} d\mu_h - \sum_{i=1}^M C_i g_S(z_i) \right| < \zeta, \quad (2.87)$$

$$e^{\xi f} \cdot d_0(x, y) - \varepsilon \leq D_h(x, y) \leq e^{\xi f} \cdot d_0(x, y) + \varepsilon, \quad \forall x, y \in \mathbb{C}$$

and simultaneously

$$\mu_h(\mathbb{R}^d) \leq \varepsilon.$$

Therefore using (2.86) we obtain

$$\begin{aligned} \left| \int_{\mathbb{R}^d} g_S(z) e^{\varphi_N(z)} d\mu_h - \int_{\mathbb{R}^d} g_S d\mathbf{m} \right| &\leq \left| \int_{\mathbb{R}^d} g_S(z) d\mathbf{m} - \sum_{i=1}^M C_i g_S(z_i) \right| \\ &+ \left| \int_{\mathbb{R}^d} g_S(z) e^{\varphi_N(z)} d\mu_h - \sum_{i=1}^M C_i g_S(z_i) \right| \\ &\leq \frac{\theta}{3} \left| \int_{\mathbb{R}^d} g_S d\mathbf{m} \right| + \zeta, \end{aligned}$$

where in the last inequality we used (2.86) and (2.87). Again, for large enough N and small enough r , applying Proposition 2.4.4 we obtain that for all $x, y \in \mathbb{R}^d$ we have

$$D_h(x, y) - \frac{\theta}{2} \leq e^{\frac{\xi}{\gamma} \varphi_N} \cdot D_h(x, y) \leq D_h(x, y) + \frac{\theta}{2}. \quad (2.88)$$

Recall that we assumed that $z_i \notin \partial B_1(0)$. Thus using the Cameron-Martin property applied to $\frac{1}{\gamma} \varphi_N$ we obtain that for any $\eta > 0$, we have with positive probability that

$$\sup_{x, y \in \mathbb{R}^d} |D_h(x, y) - e^{\xi f} \cdot d_0(x, y)| \leq \frac{\theta}{2}, \quad (2.89)$$

$$\left| \int_{\mathbb{R}^d} g_S(z) e^{\varphi_N(z)} d\mu_h - \int_{\mathbb{R}^d} g_S d\mathbf{m} \right| \leq \theta \left| \int_{\mathbb{R}^d} g_S d\mathbf{m} \right|, \quad (2.90)$$

and simultaneously

$$\mu_h([-R, R]^d) \leq \varepsilon.$$

To pass to a statement about measures of sets, suppose that A is a measurable set, and let $S \subseteq [0, 2n-1]^d$ such that $\|g_S - 1_A\|_1 \leq \theta$, and $\text{Supp} g_S \subseteq A_\varepsilon$. Then

$$\left| |A| - \int_{\mathbb{R}^d} g_S \right| \leq \frac{\theta}{3}$$

which implies that

$$\left| \mathbf{m}(A) - \int_{\mathbb{R}^d} 1_A(z) e^{\xi f(z) + \varphi(z)} d\mu_h \right| \leq \theta. \quad (2.91)$$

This completes the proof. $\square$

2.4.2 Proof of Proposition 2.4.4

The first ingredient we need is the following lemma.

Lemma 2.4.5. *Assume that the events in the conclusion of Lemma 2.4.2 hold. Let $\bar{\eta} > 0$. Then for large enough N , we have for all $1 \leq i \leq M$, we have that*

$$\text{Diam}_{e^{\frac{\xi}{\gamma}\varphi_N} \cdot D_h} (z_i + [-r - \rho, r + \rho]^d) \leq \bar{\eta}, \quad \text{Diam}_{D_h} (z_i + [-r - \rho, r + \rho]^d) \leq \bar{\eta}.$$

For now we will assume the validity of this lemma and prove Proposition 2.4.4.

Suppose $x, y \in \mathbb{R}^d$, and let P be a $e^{\frac{\xi}{\gamma}\varphi_N} \cdot D_h$ -distance minimizing geodesic between x, y .

We claim that

$$\begin{aligned} e^{\frac{\xi}{\gamma}\varphi_N} \cdot D_h(x, y) &\leq \ell_{e^{\frac{\xi}{\gamma}\varphi_N} \cdot D_h} (P \setminus \bigcup_{i=1}^M (z_i + [-r - 2\rho, r + 2\rho]^d)) \\ &\quad + M \sup_{1 \leq i \leq M} \text{Diam}_{e^{\frac{\xi}{\gamma}\varphi_N} \cdot D_h} (z_i + [-r - 2\rho, r + 2\rho]^d). \end{aligned}$$

Indeed, there exist intervals $[t_i, s_i]_{i=1}^M$ such that $P(t_0) = x$, $P(s_M) = y$, $P_{[t_\ell, s_\ell]} \cap$

$(\bigcup_{i=1}^M X_i^K) = \emptyset$, and additionally, for every $1 \leq i \leq M$, $P(s_i), P(t_{i+1}) \in X_j^K$ for some $1 \leq j \leq M$. Then

$$\begin{aligned}
e^{\frac{\xi}{\gamma}\varphi_N} \cdot D_h(x, y) &\leq \ell_{e^{\frac{\xi}{\gamma}\varphi_N} \cdot D_h} (P \setminus \bigcup_{i=1}^M (z_i + [-r - 2\rho, r + 2\rho]^d)) + \sum_{i=1}^M \ell_{e^{\frac{\xi}{\gamma}\varphi_N} \cdot D_h} (P|_{[s_i, t_{i+1}]}) \\
&\leq \ell_{e^{\frac{\xi}{\gamma}\varphi_N} \cdot D_h} (P \setminus \bigcup_{i=1}^M (z_i + [-r - 2\rho, r + 2\rho]^d)) \\
&\quad + M \sup_{1 \leq i \leq M} \text{Diam}_{e^{\frac{\xi}{\gamma}\varphi_N} \cdot D_h} (z_i + [-r - 2\rho, r + 2\rho]^d)
\end{aligned}$$

This proves the claim. Now note that

$$\begin{aligned}
e^{\frac{\xi}{\gamma}\varphi_N} \cdot D_h(x, y) &\leq \ell_{e^{\frac{\xi}{\gamma}\varphi_N} \cdot D_h} (P \setminus \bigcup_{i=1}^M (z_i + [-r - 2\rho, r + 2\rho]^d)) \\
&\quad + M \sup_{1 \leq i \leq M} \text{Diam}_{e^{\frac{\xi}{\gamma}\varphi_N} \cdot D_h} (z_i + [-r - 2\rho, r + 2\rho]^d) \\
&\leq \ell_{D_h}(P) \\
&\quad + M \sup_{1 \leq i \leq M} \text{Diam}_{e^{\frac{\xi}{\gamma}\varphi_N} \cdot D_h} (z_i + [-r - 2\rho, r + 2\rho]^d).
\end{aligned}$$

Now applying Lemma 2.4.5 we conclude the upper bound for $D_{e^{\frac{\xi}{\gamma}\varphi_N} \cdot D_h}(x, y)$. The lower bound can be obtained in the exact same way interchanging the metrics $D_{e^{\frac{\xi}{\gamma}\varphi_N} \cdot D_h}$ and D_h . This completes the proof of Proposition 2.4.4.

For the proof of Lemma 2.4.5, we will need the following lemma.

Lemma 2.4.6. *Assume that the events in the conclusion of Lemma 2.4.2 hold. Suppose that $x \in X_i^K + [-\rho, \rho]^d$ for some $1 \leq i \leq M$. Let $\bar{\varepsilon} > 0$. Then for large enough K depending on r , we have*

$$e^{\frac{\xi}{\gamma}\varphi_N} \cdot D_h(x, \partial(X_i^K + [-\rho, \rho]^d)) \leq \bar{\varepsilon}.$$

Proof. By (2.73), we have that

$$\left| \frac{C_i}{a_i} \right| \leq \frac{2^d |C_i|}{\mu_h(z_i + [-r, r]^d)}.$$

Therefore if P is any path between x and $\partial X_i^K + [-\rho, \rho]^2$ of Euclidean length at most $\frac{r}{2K}$, we note that

$$e^{\frac{\xi}{\gamma} \varphi_N} \cdot D_h(x, \partial X_i^K + [-\rho, \rho]^d) \leq \ell_{e^{\frac{\xi}{\gamma} \varphi_N} \cdot D_h}(P) \leq \left| \frac{C_i}{a_i} \right| \frac{r}{2K} \leq \frac{2^d |C_i|}{\mu_h(z_i + [-r, r]^d)} \frac{r}{2K}.$$

Now choosing K large enough so that

$$\frac{r}{2K} \leq \frac{\mu_h(z_i + [-r, r]^d)}{2^d |C_i|} \bar{\varepsilon}$$

we conclude. □

Now we will prove Lemma 2.4.5.

Proof of Lemma 2.4.5. Now suppose that $x, y \in z_i + [-r - \rho, r + \rho]^d$. Then by the triangle inequality,

$$D_{e^{\frac{\xi}{\gamma} \varphi_N} \cdot D_h}(x, y) \leq \mathfrak{d}_x + \mathfrak{d}_y + D_{e^{\frac{\xi}{\gamma} \varphi_N} \cdot D_h}(\bar{x}, \bar{y}),$$

where for any point $z \in z_i + [-r - \rho, r + \rho]^d$, we define

$$\mathfrak{d}_z := \begin{cases} e^{\frac{\xi}{\gamma} \varphi_N} \cdot D_h(z, \partial(X_i^K + [-\rho, \rho]^d)) & \text{if } z \in X_i^K + [-\rho, \rho]^d, \\ 0 & \text{otherwise,} \end{cases}$$

and where $\bar{z}$ is defined to be such that $e^{\frac{\xi}{\gamma} \varphi_N} \cdot D_h(z, \partial(X_i^K + [-\rho, \rho]^d)) = e^{\frac{\xi}{\gamma} \varphi_N} \cdot D_h(z, \bar{z})$ if $z \in X_i^K + [-\rho, \rho]^d$, and defined as z otherwise (see Figures 2.5 and 2.6).

By Lemma 2.4.6, for small enough K we have that $\mathfrak{d}_x, \mathfrak{d}_y \leq \bar{\varepsilon}$. Additionally, we note that

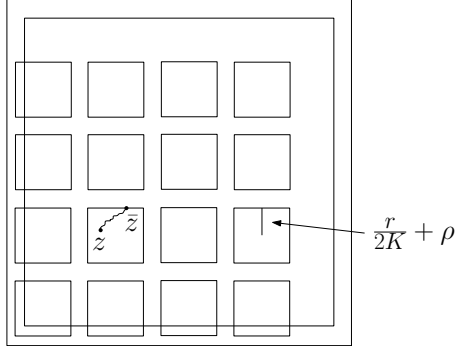


Figure 2.5: Definition of $\bar{z}$

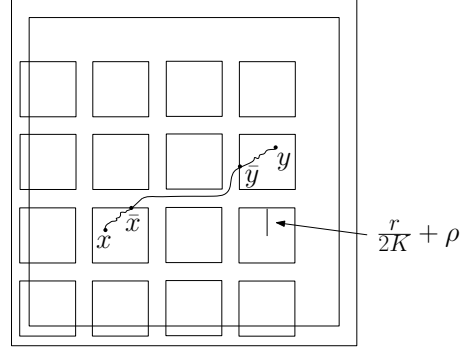


Figure 2.6: Triangle inequality argument

by taking a path P between $\bar{x}, \bar{y}$ not intersecting $X_i^K + [-\rho, \rho]^d$ of D_h -length at most M ,

$$e^{\frac{\xi}{\gamma} \varphi_N} \cdot D_h(\bar{x}, \bar{y}) \leq e^{-\frac{\xi}{\gamma} N} M,$$

and so for a small enough choice of $\bar{\varepsilon}$ and N large enough so that $e^{-\frac{\xi}{\gamma} N} M \leq \bar{\eta}$ we conclude that

$$e^{\frac{\xi}{\gamma} \varphi_N} \cdot D_h(x, y) \leq 2\bar{\varepsilon} + e^{-N} r \leq \bar{\eta}.$$

This completes the proof. □

CHAPTER 3

GAUSSIAN CURVATURE ON RANDOM PLANAR MAPS AND LIOUVILLE QUANTUM GRAVITY

The results in this chapter were developed in collaboration with Ewain Gwynne. The mathematical ideas and proof were developed jointly.

*The results in this chapter have appeared in *Probability theory and Related Fields in Contreras Hip and Gwynne [2025]**

3.1 Introduction

Liouville quantum gravity is a canonical 1-parameter family of models of a random surface first introduced by Polyakov [1981]. LQG is conjectured to be the scaling limit of various random planar map models, which are planar graphs embedded in $\mathbb{C}$ in such a way that no two edges cross, viewed modulo orientation-preserving homomorphisms. There are cases where the convergence to LQG has already been proven: for example, uniform planar maps (uniform triangulations, uniform quadrangulations etc.) have been proven to converge in the Gromov-Hausdorff sense to $\sqrt{8/3}$ -LQG (Le Gall [2013], Miermont [2013], Miller and Sheffield [2015, 2021a], Holden and Sun [2023]). Other random planar map models are expected to converge to LQG such as the spanning tree decorated maps ($\gamma = \sqrt{2}$), maps weighted by critical Fortuin-Kasteleyn partition function Sheffield [2016a] ($\gamma \in (\sqrt{2}, 2)$), and the bipolar-oriented maps Kenyon et al. [2019]. One can use multiple “reasonable” embeddings, such as the Tutte embedding and the circle packing embedding, and still expect convergence. See e.g. Gwynne et al. [2023].

Let $\gamma \in (0, 2)$, and let Q be defined by $Q = \frac{\gamma}{2} + \frac{2}{\gamma}$. Suppose that Φ is a GFF or a variant such as the quantum cone. Formally, the LQG metric tensor associated with Φ is defined as $e^{\gamma\Phi}(dx^2 + dy^2)$. A distance metric D_h and measure μ_h can be defined via approximation

of Φ by appropriate mollifications (see Kahane [1985], Rhodes and Vargas [2014], Berestycki [2017], Duplantier and Sheffield [2011], Ding et al. [2020], Ding and Gwynne [2023]).

There is a vast literature on LQG: see Gwynne [2020], Sheffield [2022], Berestycki and Powell, Guillarmou et al. [2024] for surveys on the main techniques and results in the area. Many geometric properties about LQG surfaces have been studied, such as the confluence property of LQG geodesics established in Gwynne and Miller [2020a], the KPZ formula for Hausdorff dimensions of LQG subsets Gwynne and Pfeffer [2019], Duplantier and Sheffield [2011], Aru [2015], Gwynne et al. [2019a], Gwynne et al. [2020], Berestycki et al. [2016], Gwynne and Miller [2021], Barral et al. [2013], Benjamini and Schramm [2009], Duplantier et al. [2021, 2014], and more general properties about the LQG metric's behavior Ding et al. [2021]. However, given the non smooth nature of LQG surfaces, most of the study of their geometry is restricted to first order information which does not require derivatives of the underlying field. Moreover, LQG does not always behave as its smooth counterpart (e.g. LQG surfaces have Hausdorff dimension greater than 2, and the LQG measure and metric scale differently). Given this, it is natural to ask which geometric properties are shared with smooth 2d Riemannian manifolds, and to what extent notions from Riemannian geometry can be extended for LQG.

A central concept in Riemannian geometry is curvature. In this regard, a first question is whether there is a way of defining Gaussian curvature for LQG surfaces so that it extends the usual definition, and at the same time is compatible with the known LQG theory, in the sense that it arises naturally as a limit of the corresponding discrete curvature for all reasonable random planar map approximations. In this chapter, we define the Gaussian curvature for LQG surfaces (3.1), and we conjecture that it arises as the scaling limit of the discrete curvature defined in (3.2). To this end, the first question to study is whether the discrete curvature converges in a reasonable random planar maps model, and if so, under what scaling. The particular random planar map model we look at in this chapter is the

Poisson mated CRT map defined in Section 3.2. This implies that we need to study the asymptotics of this discrete curvature as the mesh size of the planar map tends to 0, which is the main goal of this chapter and is the purpose of Theorem 3.1.5. After this, we study the total curvature on a “canonical” set, namely space filling SLE segments. In Theorem 3.1.7 we show that the scaling limit of this total curvature converges in law to an explicit random variable. We note that surprisingly the scaling for this quantity is different from that of the discrete Gaussian curvature when summed against a C_c^2 test function.

3.1.1 Curvature for LQG surfaces

Definition 3.1.1. The whole plane Gaussian free field (GFF) Φ is the centered Gaussian random generalized function with covariance structure given by

$$\text{Cov}(\Phi(z), \Phi(w)) = G(v, w) = \log \left(\frac{\max\{|z|, 1\}, \max\{|w|, 1\}}{|z - w|} \right).$$

Note that this correlation function corresponds to a GFF normalized so that the average over the unit circle is 0 [Vargas, 2017, Section 2.1.1].

Definition 3.1.2. An LQG surface is an equivalence class defined as follows. We say pairs $(D_1, \Phi_1), (D_2, \Phi_2)$ consisting of a domain in $\mathbb{C}$ and a generalized function on the domain are equivalent if there exists a conformal map $f : D_1 \rightarrow D_2$ with

$$\Phi_2 = \Phi_1 \circ f^{-1} + Q \log |(f^{-1})'|,$$

where here we recall that $Q = 2/\gamma + \gamma/2$. Then any equivalence class is called γ -LQG surface.

If Φ is a GFF, or more generally if $\Phi = \Phi^0 + f$, where Φ^0 is a GFF and f is a possibly random continuous function, then for $\gamma \in (0, 2)$ we can define the *LQG area measure* μ_Φ as a limit of regularized versions of $e^{\gamma\Phi} dz$, where dz denotes two-dimensional Lebesgue mea-

sure Kahane [1985], Rhodes and Vargas [2014], Berestycki [2017], Duplantier and Sheffield [2011]. This measure is compatible with changes of coordinate, in the sense that if (D_1, Φ_1) and (D_2, Φ_2) are related as in Definition 3.1.2, then the conformal map f pushes forward μ_{Φ_1} to μ_{Φ_2} [Duplantier and Sheffield, 2011, Proposition 2.1].

One can intuitively define the Gaussian curvature associated with LQG as follows. If we consider a general metric in isothermal coordinates $e^\phi(dx^2 + dy^2)$, then the Gaussian curvature is given by

$$\frac{\Delta\phi}{2}e^{-\phi}.$$

Given this, if Φ is the underlying field, after a formal computation one obtains that the Gaussian curvature on an LQG surface K_Φ is given by

$$\int_{\mathbb{C}} K_\Phi(z) f(z) d\mu_\Phi(z) = \int_{\mathbb{C}} \frac{\gamma}{2} \Delta\Phi e^{-\gamma\Phi} f(z) e^{\gamma\Phi} dz = \int_{\mathbb{C}} \frac{\gamma}{2} \Delta\Phi(z) f(z) dz = \int_{\mathbb{C}} \frac{\gamma}{2} \Phi(z) \Delta f(z) dz$$

where f is any C_c^2 test function, and so it is natural to define the Gaussian curvature weakly by

$$\int_{\mathbb{C}} K_\Phi(z) f(z) d\mu_\Phi(z) := \int_{\mathbb{C}} \frac{\gamma}{2} \Phi(z) \Delta f(z) dz. \quad (3.1)$$

Note that this is invariant under LQG coordinate change: indeed, suppose that U is a bounded open set and that f is a C_c^2 test function supported on U . If $\psi : U \rightarrow V$ is a conformal map, then for $\bar{\Phi} = \Phi \circ \psi + Q \log |\psi'|$ we have

$$\int_U K_{\bar{\Phi}}(z) f(z) d\mu_{\bar{\Phi}}(z) = \int_U \Delta f \frac{\gamma}{2} (\Phi \circ \psi) dz + \int_U Q \log |\psi'| \Delta f dz.$$

Since $\log |\psi'|$ is harmonic and the Dirichlet energy is invariant under conformal transforma-

tions, after integrating by parts we obtain

$$\begin{aligned}\int_U K_\Phi(z) f(z) d\mu_\Phi(z) &= \int_U \frac{\gamma}{2} \Phi \circ \psi(z) \Delta f(z) dz = \int_V \frac{\gamma}{2} \Phi(z) (\Delta f) \circ \psi^{-1}(z) |\psi'(z)|^{-2} dz \\ &= \int_V \frac{\gamma}{2} \Phi(z) \Delta(f \circ \psi^{-1})(z) dz = \int_V K_\Phi(z) f \circ \psi^{-1}(z) d\mu_\Phi(z).\end{aligned}$$

3.1.2 Discrete curvature

Suppose that M is a planar map embedded in $\mathbb{C}$. To define the curvature, suppose we set all angles at each corner of a face F to be $\frac{|F|-2}{|F|}\pi$, where $|F|$ is the number of vertices incident to F . Inspired by the Gauss-Bonnet formula, it is natural to define the discrete curvature at a vertex v as

$$K_M(v) := 2\pi - \sum_F \frac{|F| - 2}{|F|} \pi$$

where the sum is taken over all possible faces F incident to v , that is, over all faces whose boundary contains v . Note that in the case the graph $\mathcal{G}$ is a triangulation, this is equivalent to

$$K_M(v) = \frac{\pi}{3} (6 - \deg(v)). \quad (3.2)$$

We note that in some sense the average degree of a vertex is 6, which can be deduced from the Euler characteristic formula, see Corollary A.6 in Berestycki and Gwynne [2022]. See also Angel et al. [2016].

We conjecture that the discrete curvature K_M converges to the curvature K_Φ for any planar map model in the γ -LQG universality class.

Conjecture 3.1.3. *Let $\{M_n\}_{n \geq 1}$ be a sequence of infinite random planar maps believed to be in the universality class of γ -LQG (e.g. uniform infinite triangulations for $\gamma = \sqrt{8/3}$). Suppose we embed the map in the plane via any “reasonable embedding”, such as the circle packing embedding or the Tutte embedding. Then for an appropriate scaling factor c_n , we*

have that for any C_c^2 test function $f : \mathbb{C} \rightarrow \mathbb{R}$,

$$\frac{1}{c_n} \sum_{v \in \mathcal{VM}_n} f(v) K_{M_n}(v) \xrightarrow{d} \int_{\mathbb{C}} f(z) K_{\Phi}(z) d\mu_{\Phi}(z)$$

where Φ is an appropriate variant of the GFF. Moreover, the convergence in distribution holds jointly for any finite collection of C_c^2 test functions.

3.1.3 Mated CRT maps

Mated CRT maps are a family of random planar map models which are in the universality class of γ -LQG for $\gamma \in (0, 2)$. These maps are particularly natural since they are closely connected both to LQG and to other random planar map models. Indeed, as we discuss below, mated CRT maps have a direct connection to SLE and LQG thanks to the results of Duplantier et al. [2021]. On the other hand, mated CRT maps can be seen as "course grained approximations" to other types of random planar maps (e.g., uniform triangulations, spanning tree decorated maps, bipolar oriented maps) due to discrete mating of trees bijections. We refer to Gwynne et al. [2023] for a survey of various results related to mated-CRT maps.

In other literature, the CRT map is defined by discretizing time using $\varepsilon\mathbb{Z}$. However, in this chapter we consider a variant which uses a Poisson point process of intensity ε^{-1} instead. The two resulting graphs are not equivalent; however, they should both be in the same universality class. One can call the variant considered in this chapter "Poisson mated CRT map", but for the sake of brevity we will usually just refer to it as the mated CRT map (there is no risk of ambiguity since we will not consider any other variant). The reason for using this Poisson point process is technical: essentially, the advantage of the Poisson mated CRT map is that the space filling SLE segments depend locally on the underlying field (see Lemma 3.3.7).

Definition 3.1.4. Let $\varepsilon > 0$. We let $\Lambda^\varepsilon = \{y_j\}_{j \in \mathbb{Z}}$ denote a Poisson point process chosen on $\mathbb{R}$ with intensity ε^{-1} defined so that $\{y_j\} \subset \mathbb{R}$ is an increasing sequence. Suppose that $(L, R) : \mathbb{R} \rightarrow \mathbb{R}^2$ is a pair of correlated two sided standard linear Brownian motions, normalized such that $L_0 = R_0 = 0$ and such that $\text{corr}(L_t, R_t) = -\cos\left(\frac{\pi\gamma^2}{4}\right)$ for $t \neq 0$. The mated CRT map is defined to be the random planar map $\tilde{\mathcal{G}}_\varepsilon$ obtained by mating (gluing together) discretized versions of the continuum random trees constructed from L and R , that is, two vertices $y_j, y_k \in \Lambda^\varepsilon$ such that $j < k$ are connected if either

$$\left(\inf_{t \in [y_{j-1}, y_j]} L_t \right) \vee \left(\inf_{t \in [y_{k-1}, y_k]} L_t \right) \leq \left(\inf_{t \in [y_j, y_{k-1}]} L_t \right) \quad (3.3)$$

or the same holds with L replaced by R . If $|j - k| \geq 2$ and (3.3) holds for both L and R , then y_j, y_k are connected by two edges (see Figure 3.1, left).

One can draw the graph $\tilde{\mathcal{G}}_\varepsilon$ in the plane by connecting two vertices $y_j, y_k \in \Lambda^\varepsilon$ by an arc above or below the real line according to whether (3.3) holds for L or R respectively (see Figure 3.1, right). This makes the mated-CRT map into a planar map.

In fact, the mated-CRT map is a triangulation. Indeed, with the above embedding of the planar graph, each face of $\tilde{\mathcal{G}}_\varepsilon$ corresponds to a horizontal strip below the graph of L or above the graph of R , which is bounded by two horizontal segments on the real line between points in Λ^ε and two segments of either the graph of L or of R . Almost surely, neither L nor R attains a local minimum at any point in Λ^ε and neither L nor R has two local minima where it attains the same value. Hence a.s. one of the horizontal segments is a single point. This means that a.s. each face of $\tilde{\mathcal{G}}_\varepsilon$ has exactly three vertices on its boundary, and thus $\tilde{\mathcal{G}}_\varepsilon$ is a triangulation.

Note that the law of a mated CRT map does not depend on the parameter ε , however the parameter ε is relevant when taking scaling limits.

The mated CRT maps constructed above have a well established relation to SLE deco-

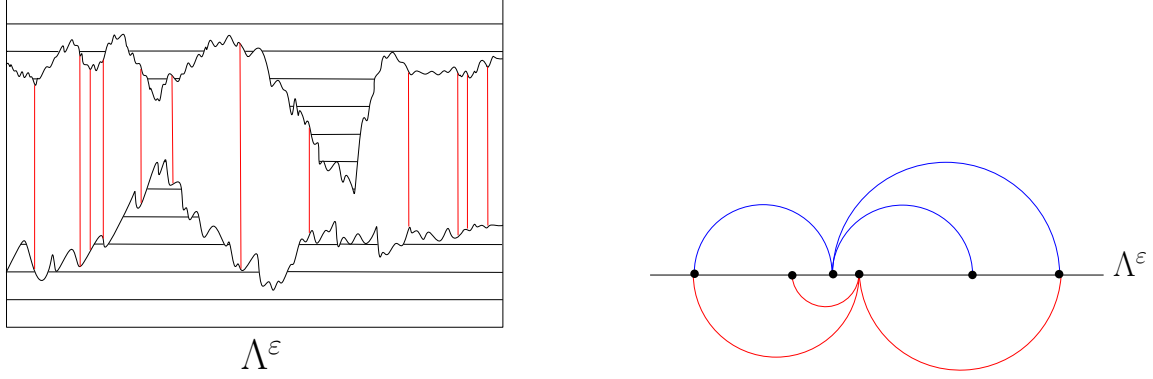


Figure 3.1: **Left:** Poisson mated CRT map. The bottom graph corresponds to the graph of L , while the top graph corresponds to the graph of $C - R$ for some large enough constant C . The red lines are the vertical identifications (hence the “mating”), while the black horizontal lines correspond to the identifications when (3.3) holds. **Right:** A portion of the Poisson mated CRT map. The top blue arcs correspond to when (3.3) holds for R , while the bottom red ones correspond to when (3.3) holds for L . The vertex set is given by the Poisson point process Λ^ε .

rated LQG. Let $\tilde{\Phi}$ be the γ -quantum cone field (a particular variant of the GFF, see Section 3.2.1, Definition 3.2.1), and let $\tilde{\eta}$ be a whole-plane space-filling SLE_κ from ∞ to ∞ sampled independently from $\tilde{\Phi}$, then parameterized by γ -quantum mass with respect to $\tilde{\Phi}$, where $\kappa = 16/\gamma^2 > 4$. That is, $\mu_{\tilde{\Phi}}(\tilde{\eta}[a, b]) = b - a$ for each $a < b \in \mathbb{R}$, and $\tilde{\eta}(0) = 0$.

Then we can define the graph $\tilde{\mathcal{G}}_\varepsilon$ with vertex set Λ^ε where two vertices y_i, y_j are connected if the space filling SLE segments $\tilde{\eta}([y_{i-1}, y_i])$ and $\tilde{\eta}([y_{j-1}, y_j])$ (called *cells*) share a nontrivial boundary arc, with two edges occurring between the two segments if they share nontrivial boundary arcs in both the left and right boundaries. By Duplantier et al. [2021], the resulting graph agrees in law with the mated CRT map defined above. This construction yields a natural embedding of the mated CRT map into $\mathbb{C}$, called the *SLE/LQG* embedding, by assigning each vertex $y_j \in \Lambda^\varepsilon$ to a point in the corresponding mated CRT map cell $\eta([y_{j-1}, y_j])$. It is convenient to take this point to be $\tilde{\eta}(y_j)$. See Section 3.2.2 for more details. We will use this fact throughout the proof of the main Theorems 3.1.5 and 3.1.7 below.

3.1.4 Main results

Henceforth when we talk about the mated CRT map $\tilde{\mathcal{G}}_\varepsilon$, we will always assume that it is coupled with $(L, R, \Lambda^\varepsilon)$ as described at the end of Section 3.1.3. In particular, $\Lambda^\varepsilon \ni v \mapsto \tilde{\eta}(v)$ is the SLE/LQG embedding of the mated CRT map. To show a result such as Conjecture 3.1.3 for $\tilde{\mathcal{G}}_\varepsilon$, one would not only need a definition of the Gaussian curvature that is compatible with the discrete setting, but also precise control over the discrete curvature in order to find the correct scaling factor, and later prove its scaling limit matches our notion of LQG curvature. This is also related to the problem of finding an observable on a random planar map converging to the underlying field: indeed, if f is a C_c^2 test function and we assume that

$$\frac{1}{a_\varepsilon} \sum_{v \in \mathcal{V}\tilde{\mathcal{G}}_\varepsilon} f(\tilde{\eta}(v)) K_{\tilde{\mathcal{G}}_\varepsilon}(v) \rightarrow \int_{\mathbb{C}} f(z) K_{\tilde{\Phi}}(z) d\mu_{\tilde{\Phi}}(z) = \int_{\mathbb{C}} \Delta f(z) \tilde{\Phi}(z) dz,$$

for some appropriate scaling factor a_ε , then we have

$$\frac{1}{a_\varepsilon} \sum_{v \in \mathcal{V}\tilde{\mathcal{G}}_\varepsilon} f(\tilde{\eta}(v)) \Delta_{\tilde{\mathcal{G}}_\varepsilon}^{-1} K_{\tilde{\mathcal{G}}_\varepsilon}(v) \rightarrow \int_{\mathbb{C}} f(z) \tilde{\Phi}(z) dz,$$

and so, at least formally, we should have that $\Delta_{\tilde{\mathcal{G}}_\varepsilon}^{-1} K_{\tilde{\mathcal{G}}_\varepsilon} \rightarrow \tilde{\Phi}$ in some weak sense, where $\Delta_{\tilde{\mathcal{G}}_\varepsilon}^{-1}$ is the inverse of the discrete graph Laplacian on $\tilde{\mathcal{G}}_\varepsilon$.

Our main theorem is a first step towards proving the convergence of the discrete curvature. If we let f be a C_c^2 test function, and we look at the expression $\sum_{v \in \mathcal{V}\tilde{\mathcal{G}}_\varepsilon} f(\tilde{\eta}(v)) K_{\tilde{\mathcal{G}}_\varepsilon}(v)$, at first glance one could expect it to be of order $\varepsilon^{-\frac{1}{2}}$ since it is a sum over $\approx \varepsilon^{-1}$ terms of random signs times values of f at the graph's vertices, and so if we expect square root cancellation one would obtain the order $\varepsilon^{-\frac{1}{2}}$. However, our first main theorem tells us something much stronger: the sum $\sum_{v \in \mathcal{V}\tilde{\mathcal{G}}_\varepsilon} f(\tilde{\eta}(v)) K_{\tilde{\mathcal{G}}_\varepsilon}(v)$ is in fact of a much smaller order.

Now we will state our first main theorem.

Theorem 3.1.5. *Let $\tilde{\mathcal{G}}_\varepsilon$ be the ε mated CRT map defined as in Definition 3.1.4, with $\mathcal{V}\tilde{\mathcal{G}}_\varepsilon =$*

Λ^ε defined to be its vertex set. Let $K_{\tilde{\mathcal{G}}_\varepsilon}$ denote the discrete curvature as defined in (3.2). Suppose that $f \in C_c^2(\mathbb{C})$ is any compactly supported twice differentiable function on $\mathbb{C}$. Then with probability going to 1 as $\varepsilon \rightarrow 0$, we have that

$$\sum_{v \in \mathcal{V}\tilde{\mathcal{G}}_\varepsilon} f(\tilde{\eta}(v)) K_{\tilde{\mathcal{G}}_\varepsilon}(v) = \varepsilon^{o(1)}. \quad (3.4)$$

Note that the $o(1)$ exponent may be negative.

The $\varepsilon^{o(1)}$ asymptotics in the statement of Theorem 3.1.5 are necessary: the next theorem tells us that if an appropriate normalization factor for the curvature summed against a C_c^2 test function exists, then it must be slow varying.

Proposition 3.1.6. *Let $\tilde{\Phi}$ be the embedding of the γ -quantum cone used to define $\tilde{\mathcal{G}}_\varepsilon$. Assume that there exists a twice continuously differentiable, compactly supported function $f : \mathbb{C} \rightarrow \mathbb{R}$, which is not identically zero, and constants $\{a_\varepsilon\}_{\varepsilon>0}$ such that we have the convergence in probability*

$$\frac{1}{a_\varepsilon} \sum_{x \in \mathcal{V}\tilde{\mathcal{G}}_\varepsilon} K_{\tilde{\mathcal{G}}_\varepsilon}(x) f(\tilde{\eta}(x)) \rightarrow \int_{\mathbb{C}} f(z) K_{\tilde{\Phi}}(z) d\mu_{\tilde{\Phi}}(z) \quad (3.5)$$

with respect to the weak topology on generalized functions. Then $\{a_\varepsilon\}_{\varepsilon>0}$ is slowly varying as $\varepsilon \rightarrow 0$, i.e.,

$$\lim_{\varepsilon \rightarrow 0} \frac{a_{C\varepsilon}}{a_\varepsilon} = 1, \quad \forall C > 0. \quad (3.6)$$

The proof is given in Section 3.2.4.

One could ask whether the correct scaling factor should in fact be of constant order. We believe the answer is no. A heuristic argument for this is that if one makes a local change in $\tilde{\mathcal{G}}_\varepsilon$ such as artificially increasing the degree of a vertex by 1, then while the sum $\sum_{v \in \mathcal{V}\tilde{\mathcal{G}}_\varepsilon} K_{\tilde{\mathcal{G}}_\varepsilon}(v) f(\tilde{\eta}(v))$ is affected by a fixed amount for any $\varepsilon > 0$, the scaling limit should not be, hence the scaling factor must make such local changes negligible.

Another interesting quantity is the total discrete curvature in a small region, such as a space filling SLE segment. More precisely, let

$$\mathcal{C}_0 = \tilde{\eta}([0, 1]). \quad (3.7)$$

We define the discrete total curvature on $\mathcal{C}_0$ to be the sum of the discrete curvature on every vertex in $\mathcal{C}_0$,

$$K_{\tilde{\mathcal{G}}_\varepsilon}(\mathcal{C}_0) = \sum_{v \in \mathcal{V}_{\tilde{\mathcal{G}}_\varepsilon} \cap \mathcal{C}_0} K_{\tilde{\mathcal{G}}_\varepsilon}(v). \quad (3.8)$$

This represents the total curvature in a space filling SLE segment.

Our second main theorem is the following.

Theorem 3.1.7. *There exists a deterministic constant $\alpha > 0$ such that*

$$\frac{K_{\tilde{\mathcal{G}}_\varepsilon}(\mathcal{C}_0)}{\varepsilon^{-\frac{1}{4}}} \rightarrow \alpha \mathcal{B},$$

in law, where the law of $\mathcal{B}$ can be described as follows. Sample $\mathcal{L}$ according to the law of the boundary length of the mated CRT map cell $\mathcal{C}_0$. Now let Θ be sampled according to a Gaussian distribution with mean 0 and variance $\mathcal{L}$. Then $\mathcal{B}$ and Θ have the same law.

Note that Theorem 3.1.5 tells us that if there is a scaling limit of the discrete curvature summed against a C_c^2 test function, then the scaling factor used must be of order $\varepsilon^{o(1)}$. However, Theorem 3.1.7 tells us the correct scaling for the total curvature on a mated CRT map cell is $\varepsilon^{\frac{1}{4}}$. This discrepancy is in part because of the fact that Gauss-Bonnet does not apply since LQG is not a proper Riemannian surface. It also suggests that it is impossible to define the Gaussian curvature together with the curvature on rough or lower dimensional sets, which is consistent with the fact that it is not so straightforward to define the total Gaussian curvature on a space-filling SLE segment since $\int_C \Delta \tilde{\Phi}(z) dz$ is not defined due to the poor regularity of $\Delta \tilde{\Phi}$. Moreover, it seems that defining the Gaussian curvature integrated

against C_c^2 test functions and the geodesic curvature on a mated CRT map cell in a consistent manner is impossible. Indeed, if one is to define the geodesic curvature on the boundary of space filling SLE segments, it must be defined in a manner that is compatible with the Gaussian curvature in the sense of the Gauss-Bonnet theorem, such that when “integrated” over the boundary gives us the total curvature over a space-filling SLE segment. Since the total Gaussian curvature uses a different scaling factor than the Gaussian curvature, this suggests that not all notions of curvature (at least on boundaries of space-filling SLE segment) are simultaneously compatible.

One of the main ingredients in the proof of Theorem 3.1.5 is a cancellation when computing $\sum_{v \in \mathcal{V}_{\tilde{\mathcal{G}}_\varepsilon}} f(\tilde{\eta}(v)) K_{\tilde{\mathcal{G}}_\varepsilon}(v)$: one can rewrite this sum as a sum of discrete gradients of f along appropriately oriented edges in $\mathcal{G}_\varepsilon$. This is proved in Section 3.3 via purely combinatorial arguments. After this, we rewrite the sum as an integral of a function which is constant in each mated CRT map cell. To estimate this integral, we estimate its variance and split the double integral into two parts: the diagonal and off diagonal parts (see Propositions 3.4.14 and 3.4.15 respectively). For the diagonal, we use the fact that the measure of the diagonal region is small. For the off diagonal region, we use the long range independence properties of the GFF (see (3.64)). These estimates are proved in subsections 3.4.3 and 3.4.3 respectively. After this, we transfer all our results back to the case of the quantum cone in Section 3.5. Part of our proof is inspired by techniques in Gwynne et al. [2019b], Section 3, however substantial new ideas are needed to get a very exact cancellation. In some sense, our argument pushes the technique in Gwynne et al. [2019b] as far as possible.

For the proof of Theorem 3.1.7, the main ingredients are a discrete analogue of Gauss-Bonnet about sums of degrees on triangulations (see Lemma 3.6.2), together with Donsker’s theorem to ensure the convergence of an average on the boundary of a mated CRT map cell.

Acknowledgements

We thank Rémi Rhodes for helpful discussions. E.G. was partially supported by NSF grant DMS-2245832. We thank an anonymous referee for thankful comments on an earlier version of this article.

3.2 Preliminaries

Throughout the chapter we will need the definitions of discrete gradient and Laplacian. We define them now. Suppose $\mathcal{G}$ is a triangulation of the plane. Suppose that this graph is embedded in $\mathbb{C}$ and suppose that $f : \mathcal{V}\mathcal{G} \rightarrow \mathbb{C}$, where $\mathcal{V}\mathcal{G}$ is the set of vertices in $\mathcal{G}$. We define the discrete Laplacian $\Delta_{\mathcal{G}}f$ by

$$\Delta_{\mathcal{G}}f(x) = \sum_{y \sim x} (f(y) - f(x)). \quad (3.9)$$

Similarly, for any oriented edge $\vec{e}$ with starting vertex v_1 and ending vertex v_2 , we define

$$\mathcal{D}f(\vec{e}) = f(v_2) - f(v_1). \quad (3.10)$$

We will slightly abuse notation, and define $\mathcal{D}f$ the same way for functions $f : \mathbb{C} \rightarrow \mathbb{C}$ by identifying vertices $v \in \mathcal{G}$ with their counterparts $\tilde{\eta}(v) \in \mathbb{C}$, where we recall $\tilde{\eta}$ is a whole-plane space-filling SLE_{κ} from ∞ to ∞ sampled independently from the γ -quantum cone field $\tilde{\Phi}$ as in Definition 3.2.1 with $\alpha = \gamma$, then parameterized by γ -quantum mass with respect to $\tilde{\Phi}$.

3.2.1 Quantum cones

First we start with the definition of the quantum cone, an important field (random generalized function) which we will be using throughout the proof of Theorems 3.1.5 and 3.1.7.

Definition 3.2.1. Let $\alpha < Q$. An α -quantum cone is a doubly marked quantum surface embedded as $(\mathbb{C}, \tilde{\Phi})$ with marked points 0 and ∞ . Its law can be described in the following way. Let A_s be the process defined by

$$A_s = \begin{cases} B_s + \alpha s, & s > 0 \\ \hat{B}_{-s} + \alpha s, & s < 0, \end{cases}$$

where B_s is a Brownian motion with $B_0 = 0$, and $\hat{B}$ is a Brownian motion independent of B conditioned so that $\hat{B}_t + (Q - \alpha)t > 0$ for all $t > 0$. We will define the spaces $\mathcal{H}_1(\mathbb{C}), \mathcal{H}_2(\mathbb{C})$ as follows: we let $\mathcal{H}_1(\mathbb{C})$ denote the set of radially symmetric functions in the Sobolev space $H^1(\mathbb{C})$, and let $\mathcal{H}_2(\mathbb{C})$ denote the set of function in $H^1(\mathbb{C})$ with common mean on all circles centered at the origin. Then we define the α -quantum cone field $\tilde{\Phi}$ as the field with projection onto $\mathcal{H}_1(\mathbb{C})$ given by the function with common value on $\partial B_{e^{-s}}(0)$ equal to A_s for all $s \in \mathbb{R}$, and whose projection onto $\mathcal{H}_2(\mathbb{C})$ is given by the projection onto $\mathcal{H}_2(\mathbb{C})$ of a whole plane GFF as defined in Definition 3.1.1.

Remark 3.2.2. Note that the restriction of an α -quantum cone field to the unit disk has the same law as $(h - \alpha \log |\cdot|)|_{B_1(0)}$ where h is a whole plane GFF normalized so that its average over $\partial B_1(0)$ is zero.

3.2.2 Geometric properties of space-filling SLE segments

The Schramm-Loewner evolution SLE_κ for $\kappa > 0$ is a one parameter family of fractal curves, first defined in Schramm [2000]. These curves exhibit different properties depending on κ . More precisely, SLE_κ is a simple curve for $\kappa \in (0, 4]$, self-touching but not space filling or self crossing for $\kappa \in (4, 8)$, and becomes space-filling for $\kappa \geq 8$ Rohde and Schramm [2005].

Space-filling SLE_κ for $\kappa > 4$ is a variant of SLE_κ introduced in Miller and Sheffield [2017]. We will recall the construction of the version of space-filling SLE_κ from ∞ to ∞ in

$\mathbb{C}$ here. To define it, let

$$\chi^{\text{IG}} := \frac{\sqrt{\kappa}}{2} - \frac{2}{\sqrt{\kappa}}.$$

We define a whole-plane GFF modulo a multiple of $2\pi\chi^{\text{IG}}$ to be an equivalence class of random distributions as follows. First we sample Ψ from the law of a whole-plane GFF normalized so that $\Psi_1(0) = 0$. Suppose that we fix $z \in \mathbb{C}$ and $\theta \in [0, 2\pi]$. Then by [Miller and Sheffield, 2017, Theorem 1.1] we can define the flow line of Ψ at z with angle θ . To define the space filling SLE_κ we will need the flow lines started at points in $z \in \mathbb{Q}^2$ with angles $-\frac{\pi}{2}$ or $\frac{\pi}{2}$, which will be denoted $\tilde{\eta}_z^L, \tilde{\eta}_z^R$ respectively.

For distinct $z, w \in \mathbb{Q}^2$, the flow lines $\tilde{\eta}_z^L, \tilde{\eta}_w^L$ merge a.s. upon intersecting, and similarly for $\tilde{\eta}_z^R, \tilde{\eta}_w^R$. The two flow lines $\tilde{\eta}_z^L, \tilde{\eta}_w^R$ a.s. do not cross, but these flow lines bounce off each other without crossing if and only if $\kappa \in (4, 8)$ [Miller and Sheffield, 2017, Theorem 1.7].

Given this, we can define a total ordering on $\mathbb{Q}^2$ by declaring that z comes before w if and only if w lies in a connected component of $\mathbb{C} \setminus (\tilde{\eta}_z^L \cup \tilde{\eta}_z^R)$ lying to the right of $\tilde{\eta}_z^L$. Using the whole plane analogue of Theorem 4.12 in Miller and Sheffield [2017] we see that there is a well defined curve $\tilde{\eta}$ passing through $\mathbb{Q}^2$ in the order above, such that $\tilde{\eta}^{-1}(\mathbb{Q}^2)$ is a dense set, and is continuous when parametrized by Lebesgue measure, that is in such a way such that $\text{area}(\tilde{\eta}([a, b])) = b - a$ whenever $a < b$. The curve $\tilde{\eta}$ is defined to be the whole-plane space-filling SLE_κ from ∞ to ∞ .

The topology of η changes depending on κ . For $\kappa \geq 8$, η_z^L and η_z^R do not bounce off of each other, and thus $\eta([a, b])$ has the topology of a closed disk. However, if $4 \leq \kappa \leq 8$, η_z^L and η_z^R intersect in an uncountable fractal set, and in this case the interior of $\eta([a, b])$ becomes a countable union of connected components, each with the topology of a closed disk, and $\eta([a, b])$ is the closure of its interior.

We will need a couple of geometric facts about space-filling SLE segments.

Lemma 3.2.3. *[Gwynne et al., 2020, Proposition 3.4] Let $\tilde{\eta}$ be a space filling SLE_κ from ∞ to ∞ in $\mathbb{C}$. For $r \in (0, 1)$, $R > 0$, $\delta_0 > 0$, let $\mathcal{E}_{\delta_0} = \mathcal{E}_{\delta_0}(R, r)$ be the event that the*

following is true. For all $0 < \delta \leq \delta_0$, and each $a < b \in \mathbb{R}$ such that $\tilde{\eta}([a, b]) \subset B_R(0)$ and $\text{diam}(\tilde{\eta}([a, b])) \geq \delta^{1-r}$, the set $\tilde{\eta}([a, b])$ contains a ball of radius at least δ . Then

$$\lim_{\delta_0 \rightarrow 0} \mathbb{P}(\mathcal{E}_{\delta_0}) = 1.$$

In particular, we have the following corollary:

Corollary 3.2.4. *Let $\tilde{\eta}$ be a space filling SLE_κ from ∞ to ∞ in $\mathbb{C}$. Let $r \in (0, 1)$. For $\delta_0 > 0$, let $\tilde{\mathcal{E}}_{\delta_0} = \tilde{\mathcal{E}}_{\delta_0}(r)$ be the event that the following is true. For any $a < b \in \mathbb{R}$ such that $\text{diam}(\tilde{\eta}([a, b])) \leq \delta_0$, we have*

$$\text{area}(\tilde{\eta}([a, b])) \geq \text{diam}(\tilde{\eta}([a, b]))^{\frac{2}{1-r}}.$$

Then

$$\lim_{\delta_0 \rightarrow 0} \mathbb{P}(\tilde{\mathcal{E}}_{\delta_0}) = 1.$$

3.2.3 Mated CRT maps with Gaussian free fields

Recall the SLE/LQG embedding of the mated CRT map $\tilde{\mathcal{G}}_\varepsilon$ as defined at the end of Section 3.1.3. The main results of this chapter concern $\tilde{\mathcal{G}}_\varepsilon$, but it is often convenient to work with a variant of the mated CRT map $\tilde{\mathcal{G}}_\varepsilon$, where the underlying field is a whole-plane GFF Φ instead of the quantum cone $\tilde{\Phi}$. This map, which we call $\mathcal{G}_\varepsilon$ is defined as follows. As in the construction of $\tilde{\mathcal{G}}_\varepsilon$, let $\Lambda^\varepsilon = \{y_j\}_{j \in \mathbb{Z}}$ denote a Poisson point process chosen on $\mathbb{R}$ with intensity ε^{-1} defined so that $\{y_j\} \subset \mathbb{R}$ is an increasing sequence. We also let η be a space-filling SLE_κ curve with $\kappa = 16/\gamma^2$, reparametrized according to unit μ_Φ measure. We let the vertex set of the graph $\mathcal{G}_\varepsilon$ be Λ^ε , and we let two vertices y_i, y_j be connected if the space filling SLE segments $\eta([y_{i-1}, y_i])$ and $\eta([y_{j-1}, y_j])$ share a nontrivial boundary arc, with two edges occurring between the two segments if they share nontrivial boundary arcs in both the left and right boundaries. Moreover, we can naturally embed this graph in $\mathbb{C}$ by

assigning each vertex to a point in the corresponding mated CRT map cell. We note that unlike in the case of $\tilde{\mathcal{G}}_\varepsilon$, the random planar map $\mathcal{G}_\varepsilon$ does not admit a simple description in terms of Brownian motion.

3.2.4 Proof of Proposition 3.1.6

Proof of Proposition 3.1.6. The proof is via a scaling argument where we study what happens when we replace $\tilde{\Phi}$ by $\tilde{\Phi} + \frac{1}{\gamma} \log(1/C)$. Fix $C > 0$. Write $\tilde{\eta}_C$ for the space-filling SLE curve parametrized by $\mu_{\tilde{\Phi} + \frac{1}{\gamma} \log(1/C)}$ -mass instead of $\mu_{\tilde{\Phi}}$ -mass. Let Λ_C^ε be a Poisson point process on $\mathbb{R}$ with intensity $1/\varepsilon$, sampled independently from $\tilde{\Phi}$ and $\tilde{\eta}$. Let $\tilde{\mathcal{G}}_\varepsilon^C$ and its associated curvature $K_{\tilde{\mathcal{G}}_\varepsilon^C}$ be defined as in (3.2), but using the field $\tilde{\Phi} + \frac{1}{\gamma} \log(1/C)$ and Λ_C^ε instead of $\tilde{\Phi}^\varepsilon$ and Λ^ε .

By the definition (3.1) of $K_{\tilde{\Phi}}$, and the fact that the Laplacian of a constant function is zero, we have $K_{\tilde{\Phi} + \frac{1}{\gamma} \log(1/C)} d\mu_{\tilde{\Phi} + \frac{1}{\gamma} \log(1/C)} = K_{\tilde{\Phi}} d\mu_{\tilde{\Phi}}$. By the local absolute continuity of the laws of $\tilde{\Phi}$ and $\tilde{\Phi} + \frac{1}{\gamma} \log(1/C)$ (see, e.g., [Berestycki and Powell, Proposition 1.5.1]), (3.5) implies the convergence in probability

$$\frac{1}{a\varepsilon} \sum_{v \in \mathcal{V}_{\tilde{\mathcal{G}}_\varepsilon^C}} K_{\tilde{\mathcal{G}}_\varepsilon^C}(v) f(\tilde{\eta}_C(v)) \rightarrow \int_{\mathbb{C}} f(z) K_{\tilde{\Phi} + \frac{1}{\gamma} \log(1/C)}(z) d\mu_{\tilde{\Phi} + \frac{1}{\gamma} \log(1/C)} = \int_{\mathbb{C}} f(z) K_{\tilde{\Phi}}(z) d\mu_{\tilde{\Phi}}. \quad (3.11)$$

On the other hand, since LQG area measure satisfies $\mu_{\tilde{\Phi} + \frac{1}{\gamma} \log(1/C)} = C^{-1} \mu_{\tilde{\Phi}}$, we have that $\mu_{\tilde{\Phi}}(\tilde{\eta}_C[a, b]) = C(b - a)$ for each $b > a$. Thus $\tilde{\eta}_C(\cdot) = \tilde{\eta}(C\cdot)$. The point process $C\Lambda_C^\varepsilon$ is a Poisson point process on $\mathbb{R}$ with intensity $1/(C\varepsilon)$. From this, we deduce that $\tilde{\eta}_C(\Lambda_C^\varepsilon) \stackrel{d}{=} \tilde{\eta}(\Lambda^{C\varepsilon})$ and the joint law of $(\tilde{\Phi}, \tilde{\eta}, C\Lambda_C^\varepsilon)$ is the same as the joint law of $(\tilde{\Phi}, \tilde{\eta}, \Lambda^{C\varepsilon})$. Recalling that $\mathcal{V}_{\tilde{\mathcal{G}}_{C\varepsilon}} = \Lambda^{C\varepsilon}$, we obtain the equality of joint laws

$$\left(\sum_{v \in \mathcal{V}_{\tilde{\mathcal{G}}_\varepsilon^C}} K_{\tilde{\mathcal{G}}_\varepsilon^C}^C(v) f(\tilde{\eta}_C(v)) \right) \stackrel{d}{=} \left(\sum_{v \in \mathcal{V}_{\tilde{\mathcal{G}}_{C\varepsilon}}} K_{\tilde{\mathcal{G}}_{C\varepsilon}}(v) f(\tilde{\eta}(v)) \right).$$

Therefore, (3.5) implies the convergence in probability

$$\frac{1}{a_{C\varepsilon}} \sum_{v \in \mathcal{V}\tilde{\mathcal{G}}_\varepsilon^C} K_{\tilde{\mathcal{G}}_\varepsilon^C}^C(v) f(\tilde{\eta}_C(v)) \rightarrow \int_{\mathbb{C}} f(z) K_{\tilde{\Phi}}(z) d\mu_{\tilde{\Phi}}(z) \quad (3.12)$$

By combining (3.11) and (3.12) and using the fact that $\int_{\mathbb{C}} f(z) K_{\tilde{\Phi}}(z) d\mu_{\tilde{\Phi}}(z)$ is a.s. non-zero, we get that $\lim_{\varepsilon \rightarrow 0} a_{C\varepsilon}/a_\varepsilon = 1$, as required. $\square$

3.3 Combinatorial relation for discrete curvature

Let η be a whole-plane space filling SLE with parameter $\kappa = \frac{16}{\gamma^2}$ reparametrized according to unit μ_Φ measure, where Φ is a whole-plane GFF, and let Ψ denote the associated imaginary geometry field. In the rest of the chapter, we will frequently switch between Φ and the quantum cone field, $\tilde{\Phi}$.

Recall that $\mathcal{G}_\varepsilon$ denotes the mated CRT map whose underlying field is the GFF Φ (Section 3.2.3). We let $\mathcal{V}\mathcal{G}_\varepsilon$ and $\mathcal{E}\mathcal{G}_\varepsilon$ denote the vertex set and set of (unoriented) edges of $\mathcal{G}_\varepsilon$ respectively (and we define $\mathcal{V}\tilde{\mathcal{G}}_\varepsilon, \mathcal{E}\tilde{\mathcal{G}}_\varepsilon$ similarly). We also let $\mathcal{F}\mathcal{G}_\varepsilon$ denote the set of faces in $\mathcal{G}_\varepsilon$.

In this section we will only be considering the graph $\mathcal{G}_\varepsilon$ corresponding to the whole plane GFF Φ . We will later transfer all results to the case where the underlying field is a γ -quantum cone (this is done in Section 3.5).

Suppose that $f \in C_c^2(\mathbb{C})$ is a compactly supported function. The main expression we will estimate is

$$\sum_{v \in \mathcal{V}\mathcal{G}_\varepsilon} K_{\mathcal{G}_\varepsilon}(v) f(\eta(v)).$$

The first step will be to rewrite this sum in a more convenient way. To that end, we recall the definition of the discrete gradient: for any oriented edge $e = (v_1, v_2)$ in $\mathcal{G}_\varepsilon$ we let $\mathcal{D}f(e)$ be the discrete gradient, $f(v_2) - f(v_1)$, where we abuse notation and identify v and $\eta(v)$ in

the case that $f : \mathbb{C} \rightarrow \mathbb{C}$.

We will need to split up the set of edges on $\mathcal{G}_\varepsilon$: we set $\mathcal{E}_L\mathcal{G}_\varepsilon, \mathcal{E}_R\mathcal{G}_\varepsilon$ to be the sets of vertices corresponding to pairs y_j, y_k for which (3.3) holds with L and R respectively, and $|j - k| > 1$. We also define $\mathcal{E}_M\mathcal{G}_\varepsilon$ to be the set of edges along the real line, that is edges between y_j, y_k with $|j - k| = 1$. Then

$$\mathcal{E}\mathcal{G}_\varepsilon = \mathcal{E}_M\mathcal{G}_\varepsilon \cup \mathcal{E}_L\mathcal{G}_\varepsilon \cup \mathcal{E}_R\mathcal{G}_\varepsilon.$$

We will need the following notation throughout.

Definition 3.3.1. For any vertex y_j let $H_{y_j}^\varepsilon$ denote the earliest mated CRT map cell containing y_j , that is

$$H_{y_j}^\varepsilon = \eta([y_{j-1}, y_j]).$$

Recall the definition of $K_M(v)$ in (3.2). We have the following identity.

Proposition 3.3.2. *We have two assignments $\mathcal{E}_L\mathcal{G}_\varepsilon \cup \mathcal{E}_M\mathcal{G}_\varepsilon \ni e \mapsto \vec{e}_L^\rightarrow, \mathcal{E}_R\mathcal{G}_\varepsilon \cup \mathcal{E}_M\mathcal{G}_\varepsilon \ni e \mapsto \vec{e}_R^\rightarrow$ assigning to each edge an orientation such that*

$$\sum_{v \in \mathcal{V}\mathcal{G}_\varepsilon} f(\eta(v)) K_{\mathcal{G}_\varepsilon}(v) = \frac{\pi}{3} \sum_{e \in \mathcal{E}_L\mathcal{G}_\varepsilon \cup \mathcal{E}_M\mathcal{G}_\varepsilon} \mathcal{D}f(\vec{e}_L^\rightarrow) + \frac{\pi}{3} \sum_{e \in \mathcal{E}_R\mathcal{G}_\varepsilon \cup \mathcal{E}_M\mathcal{G}_\varepsilon} \mathcal{D}f(\vec{e}_R^\rightarrow). \quad (3.13)$$

Note that each edge in $\mathcal{E}_M\mathcal{G}_\varepsilon$ has two orientations coming from each assignment $e \mapsto \vec{e}_L^\rightarrow, e \mapsto \vec{e}_R^\rightarrow$: these orientations can coincide or could be opposite.

Proof. Let $\mathcal{F}\mathcal{G}_\varepsilon$ denote the set of faces of $\mathcal{G}_\varepsilon$, and let $\mathcal{F}_L\mathcal{G}_\varepsilon, \mathcal{F}_R\mathcal{G}_\varepsilon$ denote the sets of faces whose edges belong to $\mathcal{E}_L\mathcal{G}_\varepsilon \cup \mathcal{E}_M\mathcal{G}_\varepsilon$ and $\mathcal{E}_R\mathcal{G}_\varepsilon \cup \mathcal{E}_M\mathcal{G}_\varepsilon$ respectively. Then for any face $t \in \mathcal{F}\mathcal{G}_\varepsilon$, there are three vertices on its boundary corresponding to three distinct space filling SLE segments. Ordering the cells as $\mathfrak{C}_1, \mathfrak{C}_2, \mathfrak{C}_3$ in chronological order according to η , let $v(t)$ be the vertex corresponding to $\mathfrak{C}_2$. We claim that $v(\cdot)|_{\mathcal{F}_L\mathcal{G}_\varepsilon} : \mathcal{F}_L\mathcal{G}_\varepsilon \rightarrow \mathcal{V}\mathcal{G}_\varepsilon$ and $v(\cdot)|_{\mathcal{F}_R\mathcal{G}_\varepsilon} : \mathcal{F}_R\mathcal{G}_\varepsilon \rightarrow \mathcal{V}\mathcal{G}_\varepsilon$ are both bijective maps. Indeed, if $H_v^\varepsilon = \eta([y_j, y_{j+1}])$, then there

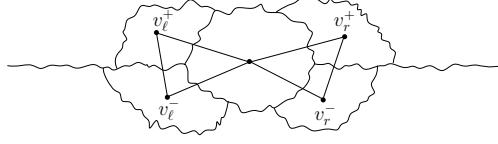


Figure 3.2: The two indicated triangles are mapped to the same vertex.

exist four η segments $H_{w_1}^\varepsilon, H_{w_2}^\varepsilon, H_{w_3}^\varepsilon, H_{w_4}^\varepsilon$ (not necessarily distinct) sharing a nontrivial boundary arc with each of ∂H_v^ε and $\eta((-\infty, y_j]) \cap \eta([y_{j+1}, \infty))$. These four η segments correspond to four vertices, denoted $v_r^\pm$ and $v_\ell^\pm$ ($v_r^\pm$ correspond to the past and future rightmost η cells, while $v_\ell^\pm$ correspond to the past and future leftmost η cells). Then the faces $\{v_r^-, v_r^+, v\}$ and $\{v_\ell^-, v_\ell^+, v\}$ are both such that $v(t) = v$ and the face $\{v_\ell^-, v_\ell^+, v\}$ is in $\mathcal{F}_L \mathcal{G}_\varepsilon$, while the face $\{v_r^-, v_r^+, v\}$ is in $\mathcal{F}_R \mathcal{G}_\varepsilon$ (see Figure 3.2). This shows the surjectivity of $v(\cdot)|_{\mathcal{F}_L \mathcal{G}_\varepsilon}$ and $v(\cdot)|_{\mathcal{F}_R \mathcal{G}_\varepsilon}$. For the injectivity, suppose without loss of generality that $v(t_1) = v(t_2)$ with $t_1, t_2 \in \mathcal{F}_L \mathcal{G}_\varepsilon$. Suppose that the vertices in t_1 are $y_{j_1}, y_{j_2}, y_{j_3} \in \Lambda^\varepsilon$ where $j_1 < j_2 < j_3$. Then $v(t_1) = y_{j_2}$. Since the arc between y_{j_1}, y_{j_3} separates the upper half plane into two connected components, the vertices of t_2 must be between y_{j_1} and y_{j_3} . If the vertices of t are given by $y_{\tilde{j}_1}, y_{\tilde{j}_3}$ and $y_{\tilde{j}_2} = v(t_2)$, then this implies that

$$j_1 < \tilde{j}_1 < j_2 = \tilde{j}_2 < \tilde{j}_3 < j_3.$$

This is impossible, since the arcs between y_{j_1}, y_{j_2} and y_{j_2}, y_{j_3} separate $y_{\tilde{j}_1}, y_{\tilde{j}_3}$ in different connected components of the upper half plane, hence it is impossible for an arc between $y_{\tilde{j}_1}, y_{\tilde{j}_3}$ to exist (see Figure 3.3). Then

$$\sum_{t \in \mathcal{F}_L \mathcal{G}_\varepsilon} f(\eta(v(t))) = \sum_{v \in \mathcal{V} \mathcal{G}_\varepsilon} f(\eta(v)) = \sum_{t \in \mathcal{F}_R \mathcal{G}_\varepsilon} f(\eta(v(t))),$$

while

$$\sum_{v \in \mathcal{V} \mathcal{G}_\varepsilon} f(\eta(v)) \deg(v) = \sum_{t \in \mathcal{F} \mathcal{G}_\varepsilon} \sum_{v \in \partial t} f(\eta(v)),$$

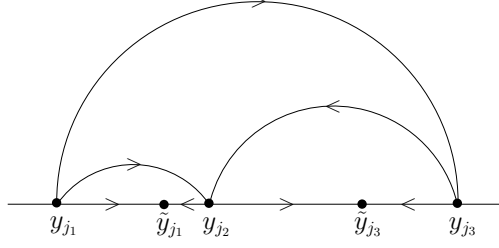


Figure 3.3: It is impossible for two triangles that both belong to $\mathcal{F}_L\mathcal{G}_\varepsilon$ or both belong to $\mathcal{F}_R\mathcal{G}_\varepsilon$ to be mapped to the same vertex under v .

where ∂t denotes the set of vertices on the boundary of the face t . Putting these together we obtain that

$$\begin{aligned} & \sum_{v \in \mathcal{V}\mathcal{G}_\varepsilon} f(\eta(v)) K_{\mathcal{G}_\varepsilon}(v) \\ &= \frac{\pi}{3} \left(\sum_{t \in \mathcal{F}_R\mathcal{G}_\varepsilon} \left(3f(\eta(v(t))) - \sum_{v \in \partial t} f(\eta(v)) \right) + \sum_{t \in \mathcal{F}_L\mathcal{G}_\varepsilon} \left(3f(\eta(v(t))) - \sum_{v \in \partial t} f(\eta(v)) \right) \right). \end{aligned}$$

Finally, noting that

$$3f(\eta(v(t))) - \sum_{v \in \partial t} f(\eta(v)) = \sum_{v \in \partial t \setminus \{v(t)\}} f(\eta(v(t))) - f(\eta(v)).$$

This implies that (3.13) holds. To define the orientation explicitly, let e be any edge with endpoints y_{j_1}, y_{j_2} , with $j_1 < j_2$. If $|j_1 - j_2| > 1$, then the arc between y_{j_1}, y_{j_2} divides the upper half plane into two connected regions. In this case there are two faces incident to e : one whose additional vertex is y_{j_3} with $j_1 < j_3 < j_2$, and the other whose additional vertex is y_{j_4} with $j_4 < j_1$ or $j_2 < j_4$. Let t be the latter face. Then $v(t) \in \{y_{j_1}, y_{j_2}\}$ ($v(t) = y_{j_1}$ if $j_4 < j_1$ and $v(t) = y_{j_2}$ if $j_2 < j_4$). We orient the edge e so that the ending vertex is $v(t)$. In the case that $j_2 = j_1 + 1$, the edge e is incident to a face t , with the vertices $y_{j_1}, y_{j_2}, y_{j_3}$ where $j_3 < j_1$ or $j_3 > j_2$. In the first case, we have $v(t) = y_{j_1}$, while in the second we have $v(t) = y_{j_2}$. In either case, we orient the edge so that $v(t)$ is the ending vertex. This completes

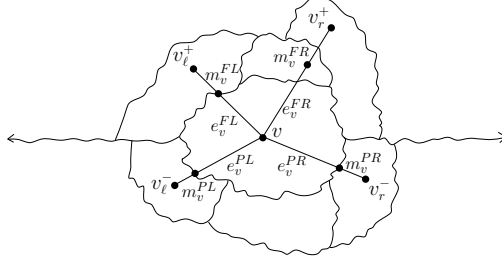


Figure 3.4: The edges $e_v^{PL}, e_v^{PR}, e_v^{FL}, e_v^{FR}$, and the points $m_v^{PL}, m_v^{PR}, m_v^{FL}, m_v^{FR}$.

the proof. $\square$

We will now define convenient mappings between vertices of $\mathcal{G}_\varepsilon$ and oriented edges.

Definition 3.3.3. Let $\tilde{\mathcal{E}}_\varepsilon$ denote the set of oriented edges in the graph $\mathcal{G}_\varepsilon$. We will define four functions $\mathcal{V}\mathcal{G}_\varepsilon \ni v \mapsto \vec{e}_v^{PL}, \vec{e}_v^{PR}, \vec{e}_v^{FL}, \vec{e}_v^{FR} \in \tilde{\mathcal{E}}_\varepsilon$. For each vertex $v \in \mathcal{V}\mathcal{G}_\varepsilon$, we define $\vec{e}_v^{PL}$ to be the leftmost outgoing edge contained in the past of η , $\vec{e}_v^{PR}$ the rightmost outgoing edge contained in the past of η , $\vec{e}_v^{FL}$ the leftmost outgoing edge contained in the future of η , and $\vec{e}_v^{FR}$ the rightmost outgoing edge contained in the future of η (see Figure 3.4).

If $\vec{e}$ is an oriented edge, let e denote the unoriented edge corresponding to $\vec{e}$. Then the maps $v \mapsto e_v^{PL}, v \mapsto e_v^{PR}, v \mapsto e_v^{FL}, v \mapsto e_v^{FR}$ are injective:

Lemma 3.3.4. *The maps $v \mapsto e_v^{PL}, e_v^{PR}, e_v^{FL}, e_v^{FR}$, where $\vec{e}_v^{PL}, \vec{e}_v^{PR}, \vec{e}_v^{FL}, \vec{e}_v^{FR}$ are as defined in Definition 3.3.3, are injective.*

Proof. We will prove this for the map $v \mapsto e_v^{PL}$, the proofs for the other maps being identical. We will also denote e_v^{PL} by e_v for the remainder of the proof to lighten notation. Suppose that $v_1, v_2 \in \mathcal{V}\mathcal{G}_\varepsilon$ are such that $e_{v_1} = e_{v_2}$, but $v_1 \neq v_2$. Since $e_{v_1} = e_{v_2}$ if and only if their endpoints agree, this must mean that e_{v_1} and e_{v_2} are the same edge with different orientations. This implies that v_1, v_2 are the two endpoints of $e_{v_1} = e_{v_2}$, given that $v_1 \neq v_2$. Since $\overrightarrow{v_1 v_2} = e_{v_1}$, this implies that v_2 is in v_1 's past. However, given that $\overrightarrow{v_2 v_1} = e_{v_2}$, this implies that v_1 is in v_2 's past, which yields a contradiction. $\square$

Since these maps are injective, we have the following corollary:

Corollary 3.3.5. *For each vertex v , there exist oriented edges $v \mapsto \vec{e}_v^{PL}$, $v \mapsto \vec{e}_v^{PR}$, $v \mapsto \vec{e}_v^{FL}$, and $v \mapsto \vec{e}_v^{FR}$, with starting point v such that*

$$\sum_{v \in \mathcal{VG}_\varepsilon} f(\eta(v)) K_{\mathcal{G}_\varepsilon}(v) = \frac{\pi}{3} \sum_{v \in \mathcal{VG}_\varepsilon} \left(\mathcal{D}f(e_v^{PL}) + \mathcal{D}f(e_v^{PR}) + \mathcal{D}f(e_v^{FL}) + \mathcal{D}f(e_v^{FR}) \right).$$

Proof. We claim that

$$\{e_v^{FR}\}_{v \in \mathcal{VG}_\varepsilon} \cup \{e_v^{PR}\}_{v \in \mathcal{VG}_\varepsilon} = \mathcal{E}_R \mathcal{G}_\varepsilon \cup \mathcal{E}_M \mathcal{G}_\varepsilon,$$

$$\{e_v^{FL}\}_{v \in \mathcal{VG}_\varepsilon} \cup \{e_v^{PL}\}_{v \in \mathcal{VG}_\varepsilon} = \mathcal{E}_L \mathcal{G}_\varepsilon \cup \mathcal{E}_M \mathcal{G}_\varepsilon,$$

and that $\{e_v^{FR}\}_{v \in \mathcal{VG}_\varepsilon} \cap \{e_v^{PR}\}_{v \in \mathcal{VG}_\varepsilon} = \emptyset$, $\{e_v^{FL}\}_{v \in \mathcal{VG}_\varepsilon} \cap \{e_v^{PL}\}_{v \in \mathcal{VG}_\varepsilon} = \emptyset$. Indeed, to prove the first statement (the second being proved identically), suppose by contradiction that this isn't the case. Note that if e is an edge with endpoints y_{j_1}, y_{j_2} with $|j_1 - j_2| = 1$ (that is, $e \in \mathcal{E}_R \mathcal{G}_\varepsilon$), there exist j_3, j_4 such that $\{y_{j_1}, y_{j_2}, y_{j_3}\}$ and $\{y_{j_1}, y_{j_2}, y_{j_4}\}$ form two faces in $\mathcal{FG}_\varepsilon$ using edges in $\mathcal{E}_R \mathcal{G}_\varepsilon \cup \mathcal{E}_M \mathcal{G}_\varepsilon$. Since the arc connecting y_{j_1}, y_{j_2} separates the upper half plane into two connected components, we can assume without loss of generality that $j_1 < j_4 < j_2$, and that $j_3 \in (-\infty, j_1] \cup [j_2, \infty)$. Then note that if $j_3 < j_1$, then $e_{y_{j_1}}^{FR} = e$ while if $j_3 > j_2$ then $e_{y_{j_2}}^{PR} = e$ (see Figures 3.5 and 3.6). In the case that $j_2 = j_1 + 1$ (that is, $e \in \mathcal{E}_M \mathcal{G}_\varepsilon$), there is a face incident to e with vertices $y_{j_1}, y_{j_2}, y_{j_3}$ for some $j_3 \in (-\infty, j_1 - 1] \cup [j_2 + 1, \infty)$, whose boundary consists of edges in $\mathcal{E}_R \mathcal{G}_\varepsilon \cup \mathcal{E}_M \mathcal{G}_\varepsilon$. Then as in the previous case, $e_{y_{j_1}}^{FR} = e$ if $j_3 < j_1$, while $e_{y_{j_2}}^{PR} = e$ if $j_3 > j_2$. Moreover, if $e = e_{v_1}^{FR} = e_{v_2}^{PR}$, then $v_1 = y_{j_1}$ and $v_2 = y_{j_2}$. Since $e_{y_{j_1}}^{FR} = e$, there exists $j_3 < j_1$ and $j_4 > j_2$ such that $\{y_{j_1}, y_{j_3}, y_{j_2}\}$ and $\{y_{j_1}, y_{j_4}, y_{j_2}\}$ form triangles. Noting that $j_3 = j_4$ by orientation, we obtain a contradiction. The result is now a direct consequence of Proposition 3.3.2 and Lemma 3.3.4. $\square$

We rewrite this sum even further. For any point $z \in \mathbb{C}$, let H_z^ε denote the mated CRT cell

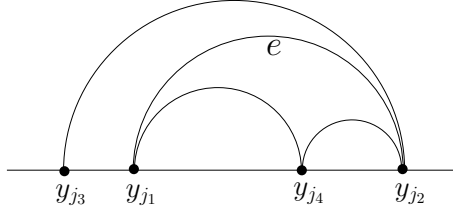


Figure 3.5: The case $j_3 < j_1 < j_4 < j_2$. Here we have $e_{y_{j_1}}^{FR} = e$.

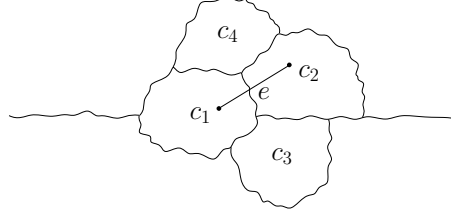


Figure 3.6: The cells C_1, C_2, C_3, C_4 correspond to the vertices $y_{j_1}, y_{j_2}, y_{j_3}, y_{j_4}$ respectively.



Figure 3.7: The definition of the points $m_v \in \{m_v^{PL}, m_v^{PR}, m_v^{FL}, m_v^{FR}\}$

containing z , and let $x_z^\varepsilon \in \mathcal{VG}_\varepsilon^{-1}$ be the corresponding vertex. Also, let $v_z^{PL}, v_z^{PR}, v_z^{FL}, v_z^{FR} \in \mathbb{C}$ denote the endpoints of $e_v^{PL}, e_v^{PR}, e_v^{FL}, e_v^{FR}$ that are not v , where $v = x_z^\varepsilon$. Then there are points $m_z^{PL}, m_z^{PR}, m_z^{FL}, m_z^{FR} \in \mathbb{C}$ (not necessarily vertices in $\mathcal{G}_\varepsilon$) such that

$$|\eta(x_z^\varepsilon) - m_z^{PL}| \leq \text{diam}(H_z^\varepsilon), \quad |m_z^{PL} - v_z^{PL}| \leq \text{diam}(H_{v_z^{PL}}^\varepsilon) \quad (3.14)$$

and the analogous statements for $m_z^{PR}, m_z^{FL}, m_z^{FR}$ (see Figure 3.7). Then we can decompose $\mathcal{D}f(e_z)$ as

$$\mathcal{D}f(e_z) = f(v_z) - f(\eta(x_z^\varepsilon)) = (f(v_z) - f(m_z)) + (f(m_z) - f(\eta(x_z^\varepsilon)))$$

(here $e_z \in \{e_z^{PL}, e_z^{PR}, e_z^{FL}, e_z^{FR}\}$ and $m_z \in \{m_z^{PL}, m_z^{PR}, m_z^{FL}, m_z^{FR}\}$ is the corresponding point). Let

$$G_f^{PL}(z) := f(m_z^{PL}) - f(\eta(x_z^\varepsilon)) + \sum_{y: v_y^{PL}=z} (f(\eta(x_z^\varepsilon)) - f(m_y^{PL}))$$

1. We will sometimes use $z \in \mathbb{C}$ and evaluate functions at x_z^ε , and other times we will use $z \in \mathcal{VG}_\varepsilon$ directly for clarity.

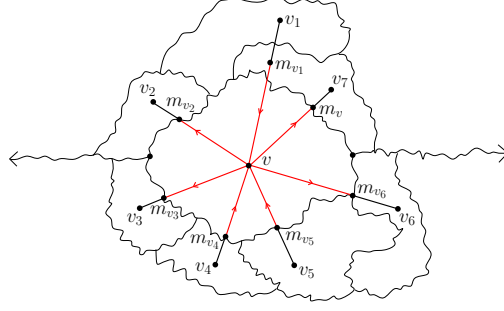


Figure 3.8: G_f is the sum of f 's discrete gradients along the red segments.

and define $G_f^{PR}(z), G_f^{FL}(z), G_f^{FR}(z)$ similarly. Then we let

$$G_f(z) := G_f^{PL}(z) + G_f^{PR}(z) + G_f^{FL}(z) + G_f^{FR}(z), \quad (3.15)$$

that is, G_f is the sum of the discrete gradients over the red segments in Figure 3.8. Note that m_z^q, v_z^q , and G_f are all constant as functions of z on each cell, where $q \in \{FL, FR, PL, PR\}$. We have the following lemma.

Lemma 3.3.6. *Let $G_f(z)$ be defined as in (3.15). Then for any compactly supported function $f \in C_c^2(\mathbb{C})$ we have*

$$\sum_{z \in \mathcal{V}_{\mathcal{G}_\varepsilon}} f(\eta(z)) K_{\mathcal{G}_\varepsilon}(v) = \frac{\pi}{3} \sum_{z \in \mathcal{V}_{\mathcal{G}_\varepsilon}} G_f(z). \quad (3.16)$$

The reason for rewriting the left hand side of (3.16) in terms of G_f is that G_f only depends on a neighborhood of v and therefore satisfies useful independence properties (see Corollary 3.3.8).

Proof. Recall that $v_z^{PL}, v_z^{PR}, v_z^{FL}, v_z^{FR} \in \mathbb{C}$ denote the endpoints of $e_v^{PL}, e_v^{PR}, e_v^{FL}, e_v^{FR}$ that are not v . Then by splitting each gradient $\mathcal{D}f$ into two pieces, and then group each summand according to what cell they correspond to, and then regrouping according to PL or FL, we

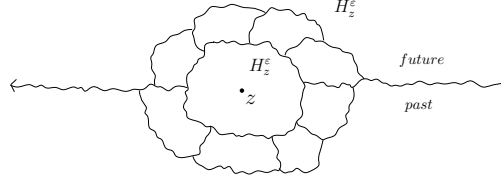


Figure 3.9: The definition of $\widehat{H}_x^\varepsilon$

obtain

$$\begin{aligned}
& \sum_{e \in \mathcal{E}_L \mathcal{G}_\varepsilon \cup \mathcal{E}_M \mathcal{G}_\varepsilon} \mathcal{D}f(\vec{e}_L) \\
&= \sum_{z \in \mathcal{V} \mathcal{G}_\varepsilon} \sum_{e \in \{e_z^{PL}, e_z^{FL}\}} \mathcal{D}f(\vec{e}_L) \\
&= \sum_{v \in \mathcal{V} \mathcal{G}_\varepsilon} \sum_{(m_z, v_z) \in \{(m_z^{PL}, v_z^{PL}), (m_z^{FL}, v_z^{FL})\}} (f(v_z) - f(m_z)) + (f(m_z) - f(\eta(z))) \\
&= \sum_{z \in \mathcal{V} \mathcal{G}_\varepsilon} \sum_{m_z \in \{m_z^{PL}, m_z^{FL}\}} (f(m_z) - f(\eta(z))) \\
&\quad + \left(\sum_{z \in \mathcal{V} \mathcal{G}_\varepsilon} \sum_{y: v_y^{PL} = \eta(z)} f(\eta(z)) - f(m_y^{PL}) \right) + \left(\sum_{z \in \mathcal{V} \mathcal{G}_\varepsilon} \sum_{y: v_y^{FL} = \eta(z)} f(\eta(z)) - f(m_y^{FL}) \right) \\
&= \sum_{z \in \mathcal{V} \mathcal{G}_\varepsilon} G_f^{PL}(z) + G_f^{FL}(z), \tag{3.17}
\end{aligned}$$

and similarly

$$\sum_{e \in \mathcal{E}_R \mathcal{G}_\varepsilon \cup \mathcal{E}_M \mathcal{G}_\varepsilon} \mathcal{D}f(\vec{e}_R) = \sum_{z \in \mathcal{G}_\varepsilon} G_f^{PR}(z) + G_f^{FR}(z). \tag{3.18}$$

Now the proof follows immediately from Corollary 3.3.5 together with (3.17), (3.18). $\square$

Recall the definition of H_z^ε in Definition 3.3.1.

Lemma 3.3.7. *Let U be a deterministic open set. Let K_z^ε be defined so that $K_z^\varepsilon = H_z^\varepsilon$ if $H_z^\varepsilon \subseteq U$, and empty otherwise. Then the collection $\{K_z^\varepsilon\}$ is determined by the set of embedded vertices in U along with the fields Φ, Ψ restricted to U , that is it is determined by $(\eta(\mathcal{V} \mathcal{G}_\varepsilon) \cap U, \Phi|_U, \Psi|_U)$.*

Proof. We know by Lemma 2.4 in Gwynne et al. [2019b] that the space filling SLE segments are determined by the restricted fields $\Phi|_U, \Psi|_U$ together with the ν_Φ - unit length parametrization. This parametrization is in turn locally determined by $\Lambda^\varepsilon, \Phi|_U$ and thus the space filling SLE segments are locally determined by $\eta(\mathcal{VG}_\varepsilon) \cap U, \Phi|_U, \Psi|_U$. $\square$

Let $\widehat{H}_z^\varepsilon$ denote the mated CRT map cell corresponding to z together with all the neighboring cells (see Figure 3.9), that is

$$\widehat{H}_z^\varepsilon = H_z^\varepsilon \cup \bigcup_{y: z \sim y} H_y^\varepsilon. \quad (3.19)$$

We note that the function G_f defined as in (3.15) only depends on the fields Ψ, Φ locally:

Corollary 3.3.8. *Suppose U is an open set and f is a C_c^2 test function. Then for any z , the value of $G_f(z)1_{\widehat{H}_z^\varepsilon \subseteq U}$ is uniquely determined by $(\Phi, \Psi)|_U$ and $\eta(\mathcal{VG}_\varepsilon) \cap U$.*

Proof. This is a direct consequence of Lemma 3.3.7. $\square$

In the following lemma and onwards, we will let $A \lesssim B$ mean that $A \leq CB$ for some absolute constant C . We will prove the following lemma about G_f which will be used later, see (3.63).

Lemma 3.3.9. *Let $f \in C_c^2(\mathbb{C})$ be a C^2 compactly supported function, and define G_f as in (3.15). Then for any vertex $z \in \mathcal{VG}_\varepsilon$,*

$$|G_f(z)| \lesssim (\deg(z) + 1) \|\nabla f\|_\infty \text{diam}(H_z^\varepsilon).$$

Proof. By (3.15) it suffices to show that

$$|G_f^q(z)| \lesssim (\deg(z) + 1) \|\nabla f\|_\infty \text{diam}(H_z^\varepsilon),$$

where $q \in \{PL, PR, FL, FR\}$. We have that

$$\begin{aligned}
|G_f^q(z)| &= \left| f(m_z^q) - f(\eta(z)) + \sum_{y \in \mathcal{V}\mathcal{G}_\varepsilon: v_y^q = z} (f(\eta(z)) - f(m_y^q)) \right| \\
&\leq |f(m_z^q) - f(\eta(z))| + \sum_{y \in \mathcal{V}\mathcal{G}_\varepsilon: v_y^q = z} |f(\eta(z)) - f(m_y^q)| \\
&\leq \|\nabla f\|_\infty |m_z^q - \eta(z)| + \sum_{y \in \mathcal{V}\mathcal{G}_\varepsilon: v_y^q = z} \|\nabla f\|_\infty |\eta(z) - m_y^q|
\end{aligned}$$

By the definition of m_y^q for a general vertex y (3.14), we have that

$$|m_z^q - \eta(z)|, |\eta(z) - m_y^q| \leq \text{diam}(H_z^\varepsilon)$$

if $v_y^q = \eta(z)$. Hence

$$|G_f^q(z)| \leq \|\nabla f\|_\infty |m_z^q - \eta(z)| + \sum_{y \in \mathcal{V}\mathcal{G}_\varepsilon: v_y^q = z} \|\nabla f\|_\infty |\eta(z) - m_y^q| \leq (\deg(z) + 1) \|\nabla f\|_\infty \text{diam}(H_z^\varepsilon).$$

This completes the proof. $\square$

3.4 Control of main sum in the whole-plane GFF case

We recall the definition of $H_{y_j}^\varepsilon$ as in Definition 3.3.1, the definition of m_z in (3.14) and the definition of $G_f(z)$ in (3.15), which will be used throughout the section. We also recall that we fix a smooth compactly supported function f .

First we will control the sum in (3.16).

Proposition 3.4.1. *With probability going to 1 as $\varepsilon \rightarrow 0$, we have*

$$\sum_{z \in \mathcal{V}\mathcal{G}_\varepsilon} f(\eta(z)) K_{\mathcal{G}_\varepsilon}(z) = \varepsilon^{o(1)}. \quad (3.20)$$

where η is a space filling SLE_κ parametrized according LQG area with respect to a whole plane GFF Φ .

Note that this statement differs from Theorem 3.1.5 in that it uses a whole-plane GFF instead of a γ -quantum cone field.

The proof of Theorem 3.1.5 will be done in Section 3.5. We will now focus on proving (3.20).

The first step will be to split up the sum in (3.3.6) into the parts corresponding to $\mathcal{E}_L\mathcal{G}_\varepsilon$ and $\mathcal{E}_R\mathcal{G}_\varepsilon$. We have by Lemma 3.3.6

$$\begin{aligned} \sum_{z \in \mathcal{V}\mathcal{G}_\varepsilon} f(\eta(z)) K_{\mathcal{G}_\varepsilon}(z) &= \frac{\pi}{3} \sum_{v \in \mathcal{V}\mathcal{G}_\varepsilon} G_f(v) \\ &= \frac{\pi}{3} \left(\sum_{v \in \mathcal{V}\mathcal{G}_\varepsilon} G_f^{PL}(v) + G_f^{FL}(v) \right) + \frac{\pi}{3} \left(\sum_{v \in \mathcal{V}\mathcal{G}_\varepsilon} G_f^{PR}(v) + G_f^{FR}(v) \right). \end{aligned}$$

Hence it suffices to show that

$$\sum_{v \in \mathcal{V}\mathcal{G}_\varepsilon} G_f^{PL}(v) + G_f^{FL}(v) = \varepsilon^{o(1)}, \quad (3.21)$$

$$\sum_{v \in \mathcal{V}\mathcal{G}_\varepsilon} G_f^{PR}(v) + G_f^{FR}(v) = \varepsilon^{o(1)}. \quad (3.22)$$

We will show the first identity, the second being proved identically. To this end, we will rewrite this last sum as an integral over $\mathbb{C}$. Let

$$\Gamma_f(z) := G_f^{PL}(z) + G_f^{FL}(z). \quad (3.23)$$

Recalling Definition 3.3.1 and defining let $x_z^\varepsilon \in \mathcal{V}\mathcal{G}_\varepsilon$ be the vertex corresponding to H_z^ε , note that

$$\sum_{v \in \mathcal{V}\mathcal{G}_\varepsilon} \Gamma_f(v) = \int_{\mathbb{C}} \frac{\Gamma_f(x_z^\varepsilon)}{\text{area}(H_z^\varepsilon)} dz. \quad (3.24)$$

Let X_z^ε denote the quotient

$$X_z^\varepsilon = \frac{\Gamma_f(x_z^\varepsilon)}{\text{area}(H_z^\varepsilon)} 1_{z \in \text{supp} f}. \quad (3.25)$$

We will follow the ideas of Section 4 in Gwynne et al. [2019b], particularly Proposition 4.1 therein. However, the techniques used in [Gwynne et al., 2019b, Proposition 4.1] require several refinements to adapt them to our setting. In that proposition, the integrand is of order $\frac{(\text{diam}(H_z^\varepsilon))^2}{\text{area}(H_z^\varepsilon)}$, while in our case the main integrand is roughly of order $\frac{\text{diam}(H_z^\varepsilon)}{\text{area}(H_z^\varepsilon)}$. The first step is to define a sequence of sets Z_j (defined in Definition 3.4.8) which “fix” $\text{diam}(H_z^\varepsilon)$ and $\text{area}(H_z^\varepsilon)$ to conveniently decompose the integral. This is necessary due to the additional factors of $\text{diam}(H_z^\varepsilon)$ that appear when compared to [Gwynne et al., 2019b, Proposition 4.1], which must be controlled separately with the conditions that define Z_j that provide uniform control over cell geometry. This is done in subsection 3.4.1. In subsection 3.4.2 we restrict the integral in (3.24) to a single set Z_j , where the diameter of each corresponding cell is of order ε^{α_j} where α_j is a fixed exponent, along with other regularity properties which are useful when bounding the integrand. Here we also split X_z^ε into its expectation with respect to the circle averages of Φ, Ψ centered at z , and the error $\bar{X}_z^\varepsilon$, which allows us to exploit the long range independence of the fields Φ, Ψ . The main fact used here is that the fields Φ, Ψ become decorrelated *only after normalizing* by the circle averages. In Section 3.4.3 we rewrite the second moment of the integral in (3.21) as a double integral, splitting it into the diagonal and off-diagonal components (see the proof of Proposition 3.4.11), and bound the main contribution for both the diagonal and off diagonal integrals. Here, the decomposition of X_z^ε into $\bar{X}_z^\varepsilon$ and the expectation is needed in order to bound the off-diagonal component, using the independence properties of $\bar{X}_z^\varepsilon$. We also use rotational invariance to bound the expectation of X_z^ε (Lemma 3.4.13), which is used to bound both the diagonal and off-diagonal components. For the diagonal component, a key ingredient is an estimate on the number of cells present in Z_j (see (3.58)), which is crucial in the proof of Proposition 3.4.14. In particular, this estimate is needed in order to cancel a factor of $\varepsilon^{2\alpha_j}$ arising due to the

fact that $\frac{\text{diam}(H_z^\varepsilon)}{\text{area}(H_z^\varepsilon)}$ is expected to be of order $\frac{1}{\text{diam}(H_z^\varepsilon)}$. Finally, in Section 3.4.4 we bound the last term remaining to finish the proof of (3.20), namely the contribution coming from the expectation of X_z^ε .

3.4.1 Defining the truncation

We will need the following technical lemmas on the size for degrees on $\tilde{\mathcal{G}}_\varepsilon$ and $\mathcal{G}_\varepsilon$.

Lemma 3.4.2. *Let $\tilde{\mathcal{G}}_\varepsilon$ be defined as in Definition 3.1.4 (with the γ -quantum cone field $\tilde{\Phi}$), and let K be a deterministic compact set. Let v_0 denote the vertex associated with the mated CRT map cell containing the origin. There are constants $c_0, c_1 > 0$ such that for any $n \in \mathbb{N}$ we have*

$$\mathbb{P}(\deg(v_0) > n^3) \leq c_0 e^{-c_1 n^{\frac{1}{2}}}.$$

Later on we will take $K = \text{supp } f$, however we will show this lemma for general compact sets K for clarity.

Proof. The proof is similar to the proof of Lemma 2.5 in Gwynne et al. [2019b]. Recall that the Poisson point process is denoted $\Lambda^\varepsilon = \{y_j\}_{j \in \mathbb{Z}}$. First, let $j_0 \in \mathbb{Z}$ denote the index such that $0 \in [y_{j_0}, y_{j_0+1})$, and let

$$S := \left\{ j \geq j_0 : \left(\inf_{t \in [y_{j_0}, y_{j_0+1}]} L_t \right) \vee \left(\inf_{t \in [y_j, y_{j+1}]} L_t \right) \leq \inf_{t \in [y_{j_0}, y_j]} L_t \right\}$$

To prove the lemma, by time reversal symmetry, it suffices to show that

$$\mathbb{P}(\#S > n) \leq c_0 e^{-c_1 n^{\frac{1}{2}}}$$

for appropriate constants c_0, c_1 . Note that if $j \in S$, then in particular we have that L_t has

not yet reached $\inf_{t \in [y_{j_0}, y_{j_0+1}]} L_t$ at time y_j , but

$$\inf_{t \in [y_j, y_{j+1}]} L_t < \inf_{t \in [y_{j_0}, y_j]} L_t \leq 0.$$

Therefore if we define $\tilde{\mathcal{N}}_1$ as

$$\tilde{\mathcal{N}}_1 = \#\{j \geq j_0 : \inf_{t \in [y_{j_0}, y_j]} L_t > \inf_{t \in [y_{j_0}, y_{j_0+1}]} L_t, \inf_{t \in [y_j, y_{j+1}]} L_t < 0\},$$

then $\#S \lesssim \tilde{\mathcal{N}}_1$. It suffices to bound $\mathbb{P}(\tilde{\mathcal{N}}_1 > n)$. Now we will define a sequence of stopping times. Let A denote the infimum

$$A = \inf_{t \in [y_{j_0}, y_{j_0+1}]} L_t.$$

We define the times σ_n, τ_n as follows: we let

$$\sigma_n = \tau_n + A^2,$$

and

$$\tau_n = \min\{t : t \geq \sigma_{n-1} : L_t = 0\}.$$

Define

$$\tilde{\mathcal{N}}_2 := \min\{n : \exists t \in [\tau_n, \sigma_n] \text{ s.t. } L_t \leq A\}.$$

Then by definition of $\tilde{\mathcal{N}}_1$ we have that

$$\tilde{\mathcal{N}}_1 \leq \sum_{n=1}^{\tilde{\mathcal{N}}_2} \#(\Lambda^\varepsilon \cap [\tau_n, \sigma_n]). \quad (3.26)$$

First, we will bound $\mathbb{P}(\tilde{\mathcal{N}}_2 > n)$. After this, we will use a union bound to bound the

probability that simultaneously, every interval $[\tau_n, \sigma_n]$ has a bounded number of points in Λ^ε .

By the Markov property of Brownian motion, we have that the expression

$$\mathbb{P}(\exists t \in [\tau_n, \sigma_n] \text{ s.t. } L_t \leq M | \mathcal{F}_{\tau_n})$$

only depends on the constant M , where $\mathcal{F}_{\tau_n}$ is the σ -algebra generated by $L|_{[y_{j_0}, \tau_n]}$.

We will need the following estimate (which is a consequence of Bennett's inequality): if X has a Poisson distribution of intensity λ , then

$$\mathbb{P}(X > x + \lambda) \leq e^{-\frac{x^2}{2(\lambda+x)}}. \quad (3.27)$$

In particular, using the fact that the number of points in Λ^ε contained in a fixed interval follows a Poisson distribution, we have that if $\tilde{\mathcal{N}}_2 \leq n$ then we have that for some constant $c_1 > 0$,

$$\mathbb{P}\left(|\Lambda^\varepsilon \cap [\tau_n, \sigma_n]| \leq A^2 n | A\right) \geq 1 - ne^{-c_1 A^2 n^{\frac{1}{2}}}.$$

Note that A has stretch exponential decay: indeed,

$$\mathbb{P}(A > -M) \leq \int_0^\infty \mathbb{P}\left(\inf_{t \in [0, \ell]} L_t > -M\right) \varepsilon e^{-\varepsilon \ell} d\ell \lesssim \int_0^\infty e^{-C\ell M} \varepsilon e^{-\varepsilon \ell} d\ell \lesssim e^{-CM}.$$

Hence for some $c_2 > 0$, we have

$$\mathbb{P}\left(A^2 > n\right) = \mathbb{P}\left(A > -n^{\frac{1}{2}}\right) \leq e^{-c_2 n^{\frac{1}{2}}}.$$

Thus we obtain that with probability at most $e^{-c_1 A^2 n^{\frac{1}{2}}} n + e^{-c_2 n^{\frac{1}{2}}}$, $\tilde{\mathcal{N}}_2 > n$.

Now, note that after conditioning on the intervals $\{[\tau_n, \sigma_n]\}_n$, the quantities $\#(\Lambda^\varepsilon \cap [\tau_n, \sigma_n])$, $\#(\Lambda^\varepsilon \cap [\tau_m, \sigma_m])$ are independent whenever $n \neq m$. Since the interval $[\tau_n, \sigma_n]$ is

of length A^2 , the law of $\#(\Lambda^\varepsilon \cap [\tau_n, \sigma_n])$ is that of an exponential distribution with mean $\frac{A^2}{\varepsilon}$. Therefore using a union bound and (3.27) we obtain that with probability at least $1 - e^{-c_1 A^2 n^{\frac{1}{2}}} n - e^{-c_2 n^{\frac{1}{2}}} - n e^{-c_3 n^{\frac{1}{2}}}$, we have $\deg(v_0) \leq n^3$. This completes the proof. $\square$

Lemma 3.4.3. *Let $\theta > 0$. Let $\tilde{\mathcal{G}}_\varepsilon$ be defined as in Definition 3.1.4. Then we have*

$$\sup_{z \in B_1(0)} \deg(x_z^\varepsilon) \leq (\log \varepsilon^{-1})^{6+\theta} \quad (3.28)$$

with probability tending to 1 as $\varepsilon \rightarrow 0$.

Proof. By Proposition 6.2 in Holden and Sun [2018] we have that for $M > 0$, $B_1(0) \subset \tilde{\eta}([- \varepsilon^{-M}, \varepsilon^{-M}])$ except on an event of probability at most $C \varepsilon^{-M p^*(\gamma)}$ for some constant C and power $p^*(\gamma)$. Now note that the law of the number of points in the Poisson point process Λ^ε contained in $[- \varepsilon^{-M}, \varepsilon^{-M}]$, is a Poisson distribution with mean $2 \varepsilon^{-M-1}$, hence with probability going to 1 as $\varepsilon \rightarrow 0$, this number is at most ε^{-M-2} . Thus by Lemma 3.4.2 and a union bound, we conclude (3.28) holds with probability tending to 1 as $\varepsilon \rightarrow 0$. $\square$

Now we need to transfer this lemma to the case we have a whole plane GFF instead of the quantum cone field.

Lemma 3.4.4. *Let K be a fixed, deterministic compact set, and let $\theta > 0$. Additionally, let $z \in K$. Then after replacing the quantum cone field $\tilde{\Phi}$ by a whole plane GFF Φ , we have*

$$\sup_{z \in B_1(0)} \deg(x_z^\varepsilon) \leq (\log \varepsilon^{-1})^{6+\theta}$$

with probability tending to 1 as $\varepsilon \rightarrow 0$.

Proof. Note that if we have a $\bar{\theta} > 0$ such that $B_{\bar{\theta}}(0) \cap K = \emptyset$ and $K \subseteq B_1(0)$, then using the absolute continuity of $\Phi|_{\mathbb{C} \setminus B_{\bar{\theta}}(0)}$ with respect to $\tilde{\Phi}|_{\mathbb{C} \setminus B_{\bar{\theta}}(0)}$ and the fact that the law of $\eta(\Lambda^\varepsilon) \cap (\mathbb{C} \setminus B_{\bar{\theta}}(0))$ given Φ only depends on $\Phi|_{\mathbb{C} \setminus B_{\bar{\theta}}(0)}$ we would be done. Hence the conclusion holds as long as K avoids a neighborhood of the origin.

For the general case, we proceed as follows. First, let $r > 1$. We claim that for any two points z, w the laws of $\Phi(\cdot + z)|_{B_r(0)}$ and $\Phi(\cdot + w)|_{B_r(0)}$ are mutually absolutely continuous. Indeed, if we take $r' > r$, then the conditional law of $\Phi(\cdot + z)|_{B_r(0)}$ given $\Phi(\cdot + z)|_{B_{r'}(0)^c}$ is mutually absolutely continuous with respect to the law of a zero boundary GFF restricted to $B_r(z)$ by the Cameron-Martin property for the GFF. Noting that the zero boundary GFF is translation invariant proves the claim. To lift the condition that K avoids the origin, we cover K in finitely many balls of radius r , and use the claim together with a union bound, applying Lemma 3.4.3 to each ball. $\square$

To define the events Z_j we will need the following lemmas.

We show that the μ_Φ measure of space-filling SLE segments is not too large.

Lemma 3.4.5. *Let $K \subset \mathbb{C}$ be a compact set. There exists a $\tilde{\beta}_1 > 0$ such that the following holds. With probability going to 1 as $\varepsilon \rightarrow 0$, for any mated CRT map cell $\mathfrak{C}$ such that $\mathfrak{C} \cap K \neq \emptyset$, we have*

$$\mu_\Phi(\mathfrak{C}) \geq \varepsilon^{\tilde{\beta}_1}$$

Proof. This follows directly from the fact that $y_{i+1} - y_i$ has a Poisson distribution. $\square$

Now, we show that space-filling SLE segments cannot have very large Lebesgue measure.

Lemma 3.4.6. *Let $K \subset \mathbb{C}$ be a compact set. There exists a $\tilde{\beta}_2 > 0$ such that the following holds. With probability going to 1 as $\varepsilon \rightarrow 0$, for any mated CRT map cell $\mathfrak{C}$ such that $\mathfrak{C} \cap K \neq \emptyset$, we have*

$$\text{area}(\mathfrak{C}) \geq \varepsilon^{\tilde{\beta}_2}.$$

Proof. We proceed as in Lemma 3.8 in Ding and Gwynne [2018]. We have the standard moment bound

$$\mathbb{E}(\mu_\Phi(B_\delta(z))^p) \leq \delta^{w(p)+o_\delta(1)}$$

where $p \in \left[0, \frac{4}{\gamma^2}\right)$, $\delta \in (0, 1)$, and $w(p)$ is defined by

$$w(p) := \left(2 + \frac{\gamma^2}{2}\right)p - \frac{\gamma^2}{2}p^2$$

(See Theorem 2.14 in Rhodes and Vargas [2014] and Lemma 5.2 in Gwynne et al. [2020]).

Using Markov's inequality we obtain

$$\mathbb{P}\left(\mu_\Phi(B_\delta(z)) > \delta \tilde{\beta}_2^{-1}\right) \leq \delta^{w(p) - \tilde{\beta}_2^{-1}p + o_\delta(1)}.$$

Now taking $p = \frac{4 + \gamma^2 - 2\tilde{\beta}_2^{-1}}{2\gamma^2}$ for $\tilde{\beta}_2 < \frac{2}{\gamma^2 - 4}$ we obtain

$$\mathbb{P}\left(\mu_\Phi(B_\delta(z)) > \delta \tilde{\beta}_2^{-1}\right) \leq \delta^{\frac{(4\gamma^2 - 2\tilde{\beta}_2^{-1})^2}{8\gamma^2} + o_\delta(1)}.$$

After covering K with a minimal amount of balls of radius δ and applying a union bound over the set of centers, together with Lemma 3.4.5 we conclude. $\square$

We also need the following lemma.

Lemma 3.4.7. *Let $K \subset B_1(0)$ be a compact set. There exists a $\beta > 0$ such that the following holds. With probability going to 1 as $\varepsilon \rightarrow 0$, for any mated CRT map cell $\mathfrak{C}$ such that $\mathfrak{C} \cap K \neq \emptyset$, we have*

$$\text{diam}(\mathfrak{C}) \leq \varepsilon^\beta.$$

Proof. Let $\delta > 0$ be a small positive constant. Note that with probability going to 1 as $\varepsilon \rightarrow 0$, we have that

$$\sup_{i \in I_k} y_{j+1} - y_j \leq \varepsilon^{1-\delta}$$

where $I_K := \{j : K \cap \eta([y_{j-1}, y_j]) \neq \emptyset\}$. Now applying Lemma 2.7 in Gwynne et al. [2019b] for $q = \beta$ and $\varepsilon^{1-\delta}$ instead of ε we conclude. $\square$

Now we are ready to define the sets Z_j . The idea is to split into sets of z for which the associated mated CRT map cell cluster $\widehat{H}_z^\varepsilon$ (as defined in (3.19)) has its diameter at a fixed scale. More precisely, let $\zeta > 0$ be a small positive number, N be a large integer, and fix $\alpha_1 \leq \dots \leq \alpha_N$ such that

$$\sup_{1 \leq j \leq N-1} |\alpha_j - \alpha_{j+1}| < \zeta. \quad (3.29)$$

Note that by choosing N to be large enough, we can force

$$\mathbb{P} \left(\sup_{z: \text{supp } f \cap H_z^\varepsilon \neq \emptyset} \left(\text{diam } \widehat{H}_z^\varepsilon \right) \geq \varepsilon^{\alpha_1} \right), \mathbb{P} \left(\inf_{z: \text{supp } f \cap H_z^\varepsilon \neq \emptyset} \left(\text{diam } \widehat{H}_z^\varepsilon \right) \leq \varepsilon^{\alpha_N} \right)$$

to be arbitrarily small. Indeed, by Lemma 3.4.7 and Lemma 3.4.4 one sees that one can force the first probability to be arbitrarily small, while Lemma 3.4.6 and the fact that $\text{diam}(\widehat{H}_z^\varepsilon) \geq \text{diam}(H_z^\varepsilon) = \sqrt{\text{area}(H_z^\varepsilon)} \left(\frac{(\text{diam}(H_z^\varepsilon))^2}{\text{area}(H_z^\varepsilon)} \right)^{1/2}$ we see that if α_1 is chosen small enough (depending on γ), and α_N is chosen large enough (also depending on γ), then the probability that $\widehat{H}_z^\varepsilon$ has a diameter outside of $[\varepsilon^{\alpha_N}, \varepsilon^{\alpha_1}]$ tends to 0 as $\varepsilon \rightarrow 0$. Let $r \in (0, 1)$ be a small constant as in Corollary 3.2.4, β large as in Lemma 3.4.7, and let $b = \frac{2}{1-r}\beta$.

Definition 3.4.8. Let C be a large constant. We define Z_j to be the set of $z \in \text{supp } f$ such that the following statements hold:

$$\text{diam}(\widehat{H}_z^\varepsilon) \in [\varepsilon^{\alpha_{j+1}}, \varepsilon^{\alpha_j}] \quad (3.30)$$

$$\text{area}(H_z^\varepsilon) \geq \varepsilon^b \quad (3.31)$$

$$\deg x_z^\varepsilon \leq C(\log \varepsilon^{-1})^7 \quad (3.32)$$

$$\frac{\text{diam}(H_z^\varepsilon)^2}{\text{area}(H_z^\varepsilon)} \leq \varepsilon^{-\zeta} \quad (3.33)$$

and if $w \sim z$,

$$\text{area}(H_w^\varepsilon) \geq \varepsilon^b \quad (3.34)$$

$$\deg x_w^\varepsilon \leq C(\log \varepsilon^{-1})^7 \quad (3.35)$$

$$\varepsilon^\zeta \leq \frac{\text{diam}(H_w^\varepsilon)^2}{\text{area}(H_w^\varepsilon)} \leq \varepsilon^{-\zeta} \quad (3.36)$$

The reason for the extra conditions on $\text{area}(H_z^\varepsilon)$, $\deg x_z^\varepsilon$, and $\text{diam}(\widehat{H}_z^\varepsilon)$ is only technical, however we need these conditions for lower bounds in the future for certain expectations of expressions. Additionally, let $\bar{Z}_j$ denote the set of $z \in \text{supp } f$ such that (3.31), (3.32), (3.33) hold, together with

$$\min_{w \in \text{supp } f} |z - w| \leq \varepsilon^{\alpha_j} \quad (3.37)$$

$$\text{diam}(H_z^\varepsilon) \in [\varepsilon^{\alpha_{j+1}}, \varepsilon^{\alpha_j}]. \quad (3.38)$$

We will need the following lemma throughout this section.

Lemma 3.4.9. *Let $\beta, \zeta > 0$ be small fixed positive constants. Let $K \subset \mathbb{C}$ be a compact set. Then with probability tending to 1 as $\varepsilon \rightarrow 0$, we have*

$$K \subseteq \bigcup_{j=1}^N Z_j.$$

Proof. This lemma is a direct consequence of Lemmas 3.4.4, 3.4.6, 3.4.7 and Corollary 3.2.4. □

3.4.2 Restricting to a single set

The integral we are interested in is

$$\int_{\mathbb{C}} X_z^\varepsilon dz$$

where X_z^ε is defined in (3.25). By (3.24) and (3.25), proving (3.21) is equivalent to

$$\mathbb{E} \left(\left(\int_{\mathbb{C}} X_z^\varepsilon dz \right)^2 \right) \leq \varepsilon^{-2\zeta + o(1)}.$$

We claim we can restrict to a single set Z_j .

Proposition 3.4.10. *Let $1 \leq j \leq N-1$. Then we have*

$$\mathbb{E} \left(\left(\int_{\mathbb{C}} 1_{Z_j}(z) X_z^\varepsilon dz \right)^2 \right) \leq \varepsilon^{-15\zeta+o(1)}.$$

Assuming this, we will prove (3.20).

Proof of (3.20). Let F_ε denote the event that $\text{supp } f \subseteq \cup_{j=1}^N Z_j$. By Lemma 3.4.9, we see that $\mathbb{P}(F_\varepsilon) \rightarrow 1$ as $\varepsilon \rightarrow 0$ and thus

$$\mathbb{E} \left(1_{F_\varepsilon} \left(\int_{\mathbb{C}} X_z^\varepsilon dz \right)^2 \right) \rightarrow \mathbb{E} \left(\left(\int_{\mathbb{C}} X_z^\varepsilon dz \right)^2 \right)$$

as $\varepsilon \rightarrow 0$. We also have

$$\begin{aligned} \mathbb{E} \left(1_{F_\varepsilon} \left(\int_{\mathbb{C}} X_z^\varepsilon dz \right)^2 \right) &\lesssim \mathbb{E} \left(\left(\sum_{j=1}^{N-1} \int_{\mathbb{C}} 1_{Z_j}(z) X_z^\varepsilon dz \right)^2 \right) \\ &\leq (N-1) \mathbb{E} \left(\sum_{j=1}^{N-1} \left(\int_{\mathbb{C}} 1_{Z_j}(z) X_z^\varepsilon dz \right)^2 \right) \\ &\leq (N-1)^2 \sup_{j=1, \dots, N-1} \mathbb{E} \left(\left(\int_{\mathbb{C}} 1_{Z_j}(z) X_z^\varepsilon dz \right)^2 \right). \end{aligned}$$

By Proposition 3.4.10, the last expression is bounded above by $\varepsilon^{-15\zeta+o(1)}$. Since $\zeta > 0$ can be made arbitrarily small, this proves (3.21). Combining this with (3.22) (whose proof is identical), we obtain the result. $\square$

From now on we will fix j , and focus on proving Proposition 3.4.10. Let $\Phi_r(z), \Psi_r(z)$ denote the averages of Φ, Ψ on the circle centered at z of radius r , and define $\bar{X}_z^\varepsilon$ by

$$\bar{X}_z^\varepsilon := 1_{Z_j}(z) X_z^\varepsilon - \mathbb{E} \left(1_{Z_j}(z) X_z^\varepsilon \middle| \Phi_{\varepsilon \alpha_j - \zeta}(z), \Psi_{\varepsilon \alpha_j - \zeta}(z) \right). \quad (3.39)$$

We have that

$$\int_{\mathbb{C}} 1_{Z_j}(z) X_z^\varepsilon dz = \int_{\mathbb{C}} \bar{X}_z^\varepsilon dz + \int_{\mathbb{C}} \mathbb{E} \left(1_{Z_j}(z) X_z^\varepsilon \middle| \Phi_{\varepsilon^{\alpha_j - \zeta}}(z), \Psi_{\varepsilon^{\alpha_j - \zeta}}(z) \right) dz$$

and therefore

$$\begin{aligned} & \mathbb{E} \left(\left(\int_{\mathbb{C}} 1_{Z_j}(z) X_z^\varepsilon dz \right)^2 \right) \\ &= \mathbb{E} \left(\left(\int_{\mathbb{C}} 1_{Z_j}(z) \bar{X}_z^\varepsilon dz + \int_{\mathbb{C}} \mathbb{E} \left(1_{Z_j}(z) X_z^\varepsilon \middle| \Phi_{\varepsilon^{\alpha_j - \zeta}}(z), \Psi_{\varepsilon^{\alpha_j - \zeta}}(z) \right) dz \right)^2 \right) \\ &\leq 2\mathbb{E} \left(\left(\int_{\mathbb{C}} 1_{Z_j}(z) \bar{X}_z^\varepsilon dz \right)^2 \right) + 2\mathbb{E} \left(\left(\int_{\mathbb{C}} \mathbb{E} (1_{Z_j}(z) X_z^\varepsilon \middle| \Phi_{\varepsilon^{\alpha_j - \zeta}}(z), \Psi_{\varepsilon^{\alpha_j - \zeta}}(z)) dz \right)^2 \right). \end{aligned} \tag{3.40}$$

We claim each one of these expectations is small.

Proposition 3.4.11. *With probability going to 1 as $\varepsilon \rightarrow 0$, we have*

$$\mathbb{E} \left(\left(\int_{\mathbb{C}} 1_{Z_j}(z) \bar{X}_z^\varepsilon dz \right)^2 \right) \leq \varepsilon^{o(1) - 15\zeta}.$$

Proposition 3.4.12. *With probability going to 1 as $\varepsilon \rightarrow 0$, we have*

$$\mathbb{E} \left(\left(\int_{\mathbb{C}} \mathbb{E} (1_{Z_j}(z) X_z^\varepsilon \middle| \Phi_{\varepsilon^{\alpha_j - \zeta}}(z), \Psi_{\varepsilon^{\alpha_j - \zeta}}(z)) dz \right)^2 \right) \leq \varepsilon^{o(1)}.$$

To prove Proposition 3.4.11 and Proposition 3.4.14 below, we will need the following lemma.

Lemma 3.4.13. *We have that*

$$\begin{aligned} & \mathbb{E} \left(1_{Z_j}(z) X_z^\varepsilon \left| \Phi_{\varepsilon^{\alpha_j - \zeta}}(z), \Psi_{\varepsilon^{\alpha_j - \zeta}}(z) \right. \right) \\ & \leq \|\nabla^2 f\|_\infty \mathbb{E} \left(1_{Z_j \cap \text{supp} f}(z) (\deg(z) + 1) \frac{\text{diam}(H_z^\varepsilon)^2}{\text{area}(H_z^\varepsilon)} \left| \Phi_{\varepsilon^{\alpha_j - \zeta}}(z), \Psi_{\varepsilon^{\alpha_j - \zeta}}(z) \right. \right) + \varepsilon^{o(1)} \end{aligned} \quad (3.41)$$

Proof. Step 1: reduction. We claim that it is enough to show the claim for functions whose Taylor expansion contains no linear term. Indeed, by Taylor expansion, we have that

$$f(w) = f(z) + \nabla f(z) \cdot (w - z) + \mathcal{N}_z f(w - z),$$

where the Taylor expansion error $\mathcal{N}_z f$ satisfies

$$|\mathcal{N}_z f(w - z)| \lesssim \|\nabla^2 f\|_\infty |z - w|^2. \quad (3.42)$$

Let $\tilde{f}$ be the linear function defined by

$$\tilde{f}_z(w) = f(z) + \nabla f(z) \cdot (w - z).$$

The idea is that the contribution in the LHS of (3.41) corresponding to $\tilde{f}$ is small since it is approximately rotationally invariant, while the quadratic term is of smaller order. Then

$$\Gamma_f(w) = \Gamma_{\tilde{f}_z}(w) + \Gamma_{\mathcal{N}_z f(\cdot - z)}(w).$$

Let $a, b \in \mathbb{R}$, and let $\mathbb{P}_{a,b}$ be the law of Φ^0, Ψ^0 , where the fields Φ^0, Ψ^0 are the fields Φ, Ψ normalized so that $\Phi_{\varepsilon^{\alpha_j - \zeta}}^0(z) = a$ and $\Psi_{\varepsilon^{\alpha_j - \zeta}}^0(z) = b$. By the invariance of the fields Φ^0, Ψ^0

under rotations about z , we have that

$$\mathbb{E}_{a,b} \left(\frac{\Gamma_{\tilde{f}_z}(z)}{\text{area}(H_z^\varepsilon)} 1_{Z_j}(z) \right) = 0. \quad (3.43)$$

Let $M_{a,b}$ denote the Radon-Nikodym derivative of the conditional law of $(\Phi, \Psi)_{B_{\varepsilon\alpha_j}(z)}$ given $\{\Phi_{\varepsilon\alpha_j-\zeta}(z) = a, \Psi_{\varepsilon\alpha_j-\zeta}(z) = b\}$ with respect to Φ^0, Ψ^0 . Then

$$\mathbb{E} \left(1_{Z_j}(z) \frac{\Gamma_{\tilde{f}_z}(z)}{\text{area}(H_z^\varepsilon)} \middle| \Phi_{\varepsilon\alpha_j-\zeta}(z) = a, \Psi_{\varepsilon\alpha_j-\zeta}(z) = b \right) = \mathbb{E}_{a,b} \left(\frac{\Gamma_{\tilde{f}_z}}{\text{area}(H_z^\varepsilon)} M_{a,b} \right).$$

Using (3.43), we have that

$$\begin{aligned} & \mathbb{E} \left(1_{Z_j}(z) \frac{\Gamma_{\tilde{f}_z}(z)}{\text{area}(H_z^\varepsilon)} \middle| \Phi_{\varepsilon\alpha_j-\zeta} = a, \Psi_{\varepsilon\alpha_j-\zeta}(z) = b \right) \\ &= \mathbb{E}_{a,b} \left(1_{Z_j}(z) \frac{\Gamma_{\tilde{f}_z}}{\text{area}(H_z^\varepsilon)} (M_{a,b} - 1) \right) + \mathbb{E}_{a,b} \left(\frac{\Gamma_{\tilde{f}_z}}{\text{area}(H_z^\varepsilon)} 1_{Z_j}(z) \right) \\ &= \mathbb{E}_{a,b} \left(1_{Z_j}(z) \frac{\Gamma_{\tilde{f}_z}}{\text{area}(H_z^\varepsilon)} (M_{a,b} - 1) \right) \\ &\leq \left(\mathbb{E}_{a,b} \left(1_{Z_j}(z) \left(\frac{\Gamma_{\tilde{f}_z}}{\text{area}(H_z^\varepsilon)} \right)^2 \right) \right)^{\frac{1}{2}} \left(\mathbb{E}_{a,b} \left((M_{a,b} - 1)^2 \right) \right)^{\frac{1}{2}} \\ &= \left(\mathbb{E}_{a,b} \left(1_{Z_j}(z) \left(\frac{\Gamma_{\tilde{f}_z}}{\text{area}(H_z^\varepsilon)} \right)^2 \right) \right)^{\frac{1}{2}} \left(\mathbb{E}_{a,b} \left(M_{a,b}^2 - 1 \right) \right)^{\frac{1}{2}}, \end{aligned}$$

where in the last line we used that

$$\begin{aligned} \mathbb{E}_{a,b} \left((M_{a,b} - 1)^2 \right) &= \mathbb{E}_{a,b} \left(M_{a,b}^2 \right) + 1 - 2\mathbb{E}_{a,b}(M_{a,b}) \\ &= \mathbb{E}_{a,b} \left(M_{a,b}^2 \right) - 1 \end{aligned}$$

since

$$\mathbb{E}_{a,b}(M_{a,b}) = 1.$$

Step 2: bound for $\mathbb{E}_{a,b} \left(M_{a,b}^2 - 1 \right)$. We claim that

$$\mathbb{E}_{a,b} \left(M_{a,b}^2 - 1 \right) \lesssim \varepsilon^{o(1)}. \quad (3.44)$$

By the proof of Lemma A.1 in Gwynne et al. [2019b] and the fact that Φ and Ψ are independent, we have that $M_{a,b} = M_a M_b$, where M_a, M_b are independent, and M_a agrees in law with

$$\frac{\sqrt{\log \tilde{R}}}{\sqrt{\log \tilde{R} + \log(1 - |\tilde{z}|^2)}} \exp \left(\frac{a^2}{2 \log \tilde{R}} - \frac{\left(a - \Omega_{\varepsilon^{\alpha_j - \zeta}}(z) \right)^2}{2 \left(\log \tilde{R} + \log(1 - |\tilde{z}|^2) \right)} \right),$$

where

$$\tilde{R} = \frac{\left(\varepsilon^{\alpha_j - \zeta} \right)^2 - |z|^2}{\varepsilon^{\alpha_j - \zeta}},$$

$$\tilde{z} = -\frac{z}{\varepsilon^{\alpha_j - \zeta} - |z|^2}$$

and finally $\Omega_{\varepsilon^{\alpha_j - \zeta}}(z)$ is a centered Gaussian with variance $\log \left((1 - |\tilde{z}|^2)^{-1} \right) \lesssim 1$. Therefore

$$\begin{aligned} \mathbb{E}_{a,b}(M_{a,b}^2) - 1 &= \mathbb{E}_{a,b}(M_a^2) \mathbb{E}_{a,b}(M_b^2) - 1 \\ &= \left(\mathbb{E}_{a,b}(M_a^2) - 1 \right) \left(\mathbb{E}_{a,b}(M_b^2) - 1 \right) \\ &\quad + \left(\mathbb{E}_{a,b}(M_a^2) - 1 \right) + \left(\mathbb{E}_{a,b}(M_b^2) - 1 \right). \end{aligned}$$

We have

$$\begin{aligned} &\mathbb{E}_{a,b}(M_a^2) - 1 \\ &= \mathbb{E}_{a,b} \left(\left(\frac{\log \tilde{R}}{\log \tilde{R} + \log(1 - |\tilde{z}|^2)} \right) \exp \left(\frac{a^2}{\log \tilde{R}} - \frac{\left(a - \Omega_{\varepsilon^{\alpha_j - \zeta}}(z) \right)^2}{\left(\log \tilde{R} + \log(1 - |\tilde{z}|^2) \right)} \right) \right) - 1 \\ &= \left(\frac{\log \tilde{R}}{\log \tilde{R} + \log(1 - |\tilde{z}|^2)} \right) \mathbb{E}_{a,b} \left(\exp \left(\frac{a^2}{\log \tilde{R}} - \frac{\left(a - \Omega_{\varepsilon^{\alpha_j - \zeta}}(z) \right)^2}{\left(\log \tilde{R} + \log(1 - |\tilde{z}|^2) \right)} \right) \right) - 1. \end{aligned}$$

Now note that

$$\begin{aligned}
& \mathbb{E}_{a,b} \left(\exp \left(\frac{a^2}{\log \tilde{R}} - \frac{\left(a - \Omega_{\varepsilon}^{\alpha_j - \zeta}(z) \right)^2}{\left(\log \tilde{R} + \log(1 - |\tilde{z}|^2) \right)} \right) \right) \\
&= \int_{\mathbb{R}} \exp \left(\frac{a^2}{\log \tilde{R}} - \frac{(a - \omega)^2}{\left(\log \tilde{R} + \log(1 - |\tilde{z}|^2) \right)} \right) \\
&\quad \cdot \frac{1}{\sqrt{2\pi \log \left((1 - |\tilde{z}|^2)^{-1} \right)}} \exp \left(-\frac{\omega^2}{2 \log \left((1 - |\tilde{z}|^2)^{-1} \right)} \right) d\omega \\
&= \frac{\exp \left(\frac{a^2}{\log \tilde{R}} \right)}{\sqrt{2\pi \log \left((1 - |\tilde{z}|^2)^{-1} \right)}} e^{-\frac{c_1 c_2 a^2}{c_1 + c_2}} \int_{\mathbb{R}} e^{-(c_1 + c_2) \left(\omega - \frac{a c_1}{c_1 + c_2} \right)^2} d\omega \\
&= \frac{\exp \left(\frac{a^2}{\log \tilde{R}} \right)}{\sqrt{2\pi \log \left((1 - |\tilde{z}|^2)^{-1} \right)}} e^{-\frac{c_1 c_2 a^2}{c_1 + c_2}} \left(\frac{\pi}{c_1 + c_2} \right)^{\frac{1}{2}},
\end{aligned}$$

where

$$c_1 = \frac{1}{\log \tilde{R} + \log(1 - |\tilde{z}|^2)}, \quad c_2 = \frac{1}{2 \log \left((1 - |\tilde{z}|^2)^{-1} \right)}.$$

Note that $c_1 \lesssim \frac{1}{\log \tilde{R}} \lesssim \frac{1}{\log \varepsilon} \ll c_2$. This implies that

$$\begin{aligned}
& \mathbb{E}_{a,b} \left(\exp \left(\frac{a^2}{\log \tilde{R}} - \frac{\left(a - \Omega_{\varepsilon}^{\alpha_j - \zeta}(z) \right)^2}{\left(\log \tilde{R} + \log(1 - |\tilde{z}|^2) \right)} \right) \right) \\
&\leq \left(1 + \varepsilon^{o(1)} \right) \frac{\exp \left(\frac{a^2}{\log \tilde{R}} \right)}{\sqrt{2 \log \left((1 - |\tilde{z}|^2)^{-1} \right)}} e^{-\frac{c_1 c_2 a^2}{c_1 + c_2}} \left(\frac{1}{c_2} \right)^{\frac{1}{2}} \\
&\leq \left(1 + \varepsilon^{o(1)} \right) \exp \left(\frac{a^2}{\log \tilde{R}} \right) e^{-\frac{c_1 c_2 a^2}{c_1 + c_2}} \lesssim \varepsilon^{o(1)} \exp \left(\frac{a^2}{\log \tilde{R}} \right) e^{-c_1 a^2} \\
&= 1 + \varepsilon^{o(1)}.
\end{aligned}$$

This implies (3.44).

Step 3: conclusion. We obtain

$$\mathbb{E}_{a,b} \left(\frac{\Gamma_{\tilde{f}_z}(z)}{\text{area}(H_z^\varepsilon)} \middle| \Phi_{\varepsilon^{\alpha_j-\zeta}} = a, \Psi_{\varepsilon^{\alpha_j-\zeta}} = b \right) = \varepsilon^{o(1)}.$$

Let

$$\mathcal{B}_z^\varepsilon := \frac{1}{\text{area}(H_z^\varepsilon)} \left(\mathcal{N}_z f(m_z - z) + \sum_{y:v_y=z} \mathcal{N}_z f(m_y - z) \right).$$

Therefore

$$\begin{aligned} \mathbb{E} \left(1_{Z_j}(z) X_w^\varepsilon \middle| \Phi_{\varepsilon^{\alpha_j-\zeta}}(w), \Psi_{\varepsilon^{\alpha_j-\zeta}}(w) \right) &= \mathbb{E} \left(1_{Z_j \cap \text{supp} f}(z) \frac{\Gamma_f(z)}{\text{area}(H_z^\varepsilon)} \middle| \Phi_{\varepsilon^{\alpha_j-\zeta}}(z), \Psi_{\varepsilon^{\alpha_j-\zeta}}(z) \right) \\ &= \mathbb{E} \left(1_{Z_j \cap \text{supp} f}(z) \mathcal{B}_z^\varepsilon \middle| \Phi_{\varepsilon^{\alpha_j-\zeta}}(z), \Psi_{\varepsilon^{\alpha_j-\zeta}}(z) \right) + \varepsilon^{o(1)}. \end{aligned} \tag{3.45}$$

Now using (3.4.8) we have

$$\begin{aligned} |\mathcal{B}_z^\varepsilon| &\leq \frac{\|\nabla^2 f\|_\infty}{\text{area}(H_z^\varepsilon)} \left(|m_z - z|^2 + \sum_{y:v_y=z} |m_y - z|^2 \right) \\ &\leq \frac{\|\nabla^2 f\|_\infty}{\text{area}(H_z^\varepsilon)} (\deg(x_z^\varepsilon) + 1) (\text{diam}(H_z^\varepsilon))^2. \end{aligned}$$

Combining with (3.45) this completes the proof. $\square$

Assuming Propositions 3.4.11 and 3.4.12 we will now prove Proposition 3.4.10, which will in turn prove (3.20).

Proof of Proposition 3.4.10. We have by (3.40) that

$$\begin{aligned} \mathbb{E} \left(\left(\sum_{z \in \mathcal{V}\mathcal{G}_\varepsilon} f(\eta(z)) K_{\mathcal{G}_\varepsilon}(z) \right)^2 \right) &\leq 2\mathbb{E} \left(\left(\int_{\mathbb{C}} \bar{X}_z^\varepsilon dz \right)^2 \right) \\ &+ 2\mathbb{E} \left(\left(\int_{\mathbb{C}} \mathbb{E}(X_z^\varepsilon | \Phi_{\varepsilon^{\alpha_j - \zeta}}(z), \Psi_{\varepsilon^{\alpha_j - \zeta}}(z)) dz \right)^2 \right). \end{aligned}$$

By Propositions 3.4.11 and 3.4.12 we know that each term is of order $\varepsilon^{o(1)-15\zeta}$. This completes the proof. $\square$

Now we will focus on proving Propositions 3.4.11 and 3.4.12.

3.4.3 Main estimates

Proof of Proposition 3.4.11

To prove Proposition 3.4.11, we write

$$\mathbb{E} \left(\left(\int_{\mathbb{C}} 1_{Z_j}(z) \bar{X}_z^\varepsilon dz \right)^2 \right) = \mathbb{E} \left(\int_{Z_j} \int_{Z_j} \bar{X}_z^\varepsilon \bar{X}_w^\varepsilon dz dw \right).$$

Let $\zeta > 0$. We will decompose this double integral into the set of (z, w) with $|z - w| \leq \varepsilon^{\alpha_j - \zeta}$ and where $|z - w| \geq \varepsilon^{\alpha_j - \zeta}$. More precisely, we have

$$\begin{aligned} \mathbb{E} \left(\left(\int_{\mathbb{C}} 1_{Z_j}(z) \bar{X}_z^\varepsilon dz \right)^2 \right) &= \mathbb{E} \left(\int \int_{|z-w| < \varepsilon^{\alpha_j - \zeta}} 1_{Z_j}(z) 1_{Z_j}(w) \bar{X}_z^\varepsilon \bar{X}_w^\varepsilon dz dw \right) \\ &+ \mathbb{E} \left(\int \int_{|z-w| \geq \varepsilon^{\alpha_j - \zeta}} 1_{Z_j}(z) 1_{Z_j}(w) \bar{X}_z^\varepsilon \bar{X}_w^\varepsilon dz dw \right). \end{aligned} \tag{3.46}$$

We claim each term is of order at most $\varepsilon^{o(1)-15\zeta}$. We will prove this in subsections 3.4.3 and 3.4.3. For the first, we prove the integral is small because of the fact that $\{z, w : |z - w| \leq$

$\varepsilon^{\alpha_j - \zeta}$ has small measure. For the second integral, we use the fact that X_z^ε satisfies a long range independence property (see (3.64)).

Diagonal integral

The main result of this section is the following.

Proposition 3.4.14. *We have*

$$\left| \mathbb{E} \left(\int \int_{|z-w| < \varepsilon^{\alpha_j - \zeta}} \bar{X}_z^\varepsilon \bar{X}_w^\varepsilon dz dw \right) \right| \leq \varepsilon^{o(1) - 15\zeta}.$$

Now we will prove Proposition 3.4.14.

Proof of Proposition 3.4.14. Step 1: Rewriting the integral. We have

$$\begin{aligned} & \int \int_{|z-w| < \varepsilon^{\alpha_j - \zeta}} \bar{X}_z^\varepsilon \bar{X}_w^\varepsilon dz dw \\ &= \int \int_{|z-w| < \varepsilon^{\alpha_j - \zeta}} 1_{Z_j}(z) 1_{Z_j}(w) X_z^\varepsilon X_w^\varepsilon dz dw \\ & - 2 \int \int_{|z-w| < \varepsilon^{\alpha_j - \zeta}} 1_{Z_j}(z) X_z^\varepsilon \mathbb{E} \left(1_{Z_j}(w) X_w^\varepsilon \middle| \Phi_{\varepsilon^{\alpha_j - \zeta}}(w), \Psi_{\varepsilon^{\alpha_j - \zeta}}(w) \right) dz dw \\ & + \int \int_{|z-w| < \varepsilon^{\alpha_j - \zeta}} \mathbb{E} \left(1_{Z_j}(z) X_z^\varepsilon \middle| \Phi_{\varepsilon^{\alpha_j - \zeta}}(z), \Psi_{\varepsilon^{\alpha_j - \zeta}}(z) \right) \\ & \quad \cdot \mathbb{E} \left(1_{Z_j}(w) X_w^\varepsilon \middle| \Phi_{\varepsilon^{\alpha_j - \zeta}}(w), \Psi_{\varepsilon^{\alpha_{j+1}}}(w) \right) dz dw \\ & =: A_1 - 2A_2 + A_3 \end{aligned} \tag{3.47}$$

Now we will bound each term individually. For any $z, w \in \mathbb{C}$, let $E_{z,w}$ denote the event that $|z - w| < \varepsilon^{\alpha_j - \zeta}$, and $z, w \in Z_j$. We have

$$\int \int_{|z-w| < \varepsilon^{\alpha_j - \zeta}} 1_{Z_j}(z) 1_{Z_j}(w) X_z^\varepsilon X_w^\varepsilon dz dw = \int_{\mathbb{C}} \int_{\mathbb{C}} X_z^\varepsilon X_w^\varepsilon 1_{E_{z,w}} dz dw. \tag{3.48}$$

Let V_ε be defined by

$$V_\varepsilon = \{v \in \mathcal{VG}_\varepsilon : H_v^\varepsilon \cap \text{supp } f \neq \emptyset\}.$$

Step 2: Bounding A_1 . Going back to the discrete sum form, we have that

$$|A_1| = \left| \int_{\mathbb{C}} \int_{\mathbb{C}} X_z^\varepsilon X_w^\varepsilon 1_{E_{z,w}} dz dw \right| \leq \sum_{z,w \in V_\varepsilon} |\Gamma_f(z)| |\Gamma_f(w)| 1_{E_{z,w}}. \quad (3.49)$$

Now using Lemma 3.3.9 we obtain that

$$\sum_{z,w \in V_\varepsilon} |\Gamma_f(z)| |\Gamma_f(w)| 1_{E_{z,w}} \leq \|\nabla f\|_\infty^2 \sum_{z,w \in V_\varepsilon} (\deg(z)+1)(\deg(w)+1) \text{diam}(H_z^\varepsilon) \text{diam}(H_w^\varepsilon) 1_{E_{z,w}}. \quad (3.50)$$

Note that if $E_{z,w}$ holds, then

$$\text{diam}(H_z^\varepsilon) \text{diam}(H_w^\varepsilon) \leq \varepsilon^{2\alpha_j}.$$

Hence

$$\begin{aligned} & \sum_{z,w \in V_\varepsilon} (\deg(z)+1)(\deg(w)+1) \text{diam}(H_z^\varepsilon) \text{diam}(H_w^\varepsilon) 1_{E_{z,w}} \\ & \leq \varepsilon^{2\alpha_j} \sum_{z,w \in V_\varepsilon} (\deg(z)+1)(\deg(w)+1) 1_{E_{z,w}}. \end{aligned} \quad (3.51)$$

Then by the definition of Z_j we have

$$\sup_{z \in V_\varepsilon} \deg(z) 1_{Z_j}(z) \leq C(\log \varepsilon^{-1})^3.$$

This implies by (3.30), (3.32), (3.35) that

$$\begin{aligned} & \sum_{z,w \in V_\varepsilon \cap \text{supp } f} (\deg(z) + 1)(\deg(w) + 1) \text{diam}(H_z^\varepsilon) \text{diam}(H_w^\varepsilon) 1_{E_{z,w}} \\ & \leq C^2 \left(\log \varepsilon^{-1} \right)^{14} \varepsilon^{2\alpha_j} \sum_{z,w \in V_\varepsilon} 1_{E_{z,w}}. \end{aligned}$$

Combining this with (3.50) we obtain

$$\sum_{z,w \in V_\varepsilon} |\Gamma_f(z)| |\Gamma_f(w)| 1_{E_{z,w}} \leq \|\nabla f\|_\infty^2 C^2 \left(\log \varepsilon^{-1} \right)^{14} \varepsilon^{2\alpha_j} \sum_{z,w \in V_\varepsilon} 1_{E_{z,w}}. \quad (3.52)$$

Now we will estimate the sum

$$\sum_{z,w \in V_\varepsilon} 1_{E_{z,w}}.$$

We have

$$\sum_{z,w \in V_\varepsilon \cap Z_j} 1_{E_{z,w}} = \sum_{z \in V_\varepsilon \cap Z_j} \#\{w \in V_\varepsilon \cap Z_j : E_{z,w} \text{ holds}\}.$$

We also have that

$$\begin{aligned} & \#\{w \in V_\varepsilon \cap Z_j : E_{z,w} \text{ holds}\} \\ & \leq \#\left\{w \in V_\varepsilon \cap Z_j : H_w^\varepsilon \cap B_{\varepsilon^{\alpha_j - \zeta}}(z) \neq \emptyset, \text{diam}\left(\widehat{H}_w^\varepsilon\right) \in [\varepsilon^{\alpha_j + 1}, \varepsilon^{\alpha_j}]\right\}. \end{aligned} \quad (3.53)$$

By (3.32), (3.35), if $w \in V_\varepsilon$, there are at most $C(\log \varepsilon^{-1})^7$ cells y such that $y \sim w$. Therefore if $\text{diam}(\widehat{H}_w^\varepsilon) \geq \varepsilon^{\alpha_j + 1}$, then there exists a w such that either $y = w$ or $y \sim w$ and

$$\text{diam}(H_y^\varepsilon) \geq \frac{\varepsilon^{\alpha_j + 1}}{C(\log \varepsilon^{-1})^7}.$$

Hence by (3.32), (3.35)

$$\begin{aligned} & \# \left\{ w \in V_\varepsilon \cap Z_j : H_w^\varepsilon \cap B_{\varepsilon^{\alpha_j - \zeta}}(z) \neq \emptyset, \text{diam} \left(\widehat{H}_w^\varepsilon \right) \geq \varepsilon^{\alpha_j + 1} \right\} \\ & \leq C(\log \varepsilon^{-1})^7 \# \left\{ y \in B_{\varepsilon^{\alpha_j - \zeta} + \varepsilon^{\alpha_j}}(z) : \text{diam} \left(H_y^\varepsilon \right) \geq \frac{\varepsilon^{\alpha_j + \zeta}}{C(\log \varepsilon^{-1})^7} \right\}. \end{aligned} \quad (3.54)$$

Now, using (3.33), (3.36) we have that

$$\begin{aligned} & \# \left\{ y \in B_{\varepsilon^{\alpha_j - \zeta} + \varepsilon^{\alpha_j}}(z) : \text{diam} \left(H_y^\varepsilon \right) \geq \frac{\varepsilon^{\alpha_j + \zeta}}{C(\log \varepsilon^{-1})^3} \right\} \\ & \leq \# \left\{ y \in B_{2\varepsilon^{\alpha_j - \zeta}}(z) : \text{area} \left(H_y^\varepsilon \right) \geq \frac{\varepsilon^{(2\alpha_j + 3\zeta)}}{C^2(\log \varepsilon^{-1})^6} \right\} \\ & \leq 4\pi\varepsilon^{2\alpha_j - 2\zeta} \frac{\varepsilon^{-(2\alpha_j + 3\zeta)}}{C^{-2}(\log \varepsilon^{-1})^{-6}} = 4\pi C^2 \varepsilon^{-5\zeta} (\log \varepsilon^{-1})^6 \end{aligned}$$

where in the last line we used the fact that the cells intersect only along their boundaries.

Putting this together with (3.53), (3.54) we obtain

$$\#\{w \in V_\varepsilon \cap Z_j : E_{z,w} \text{ holds}\} \leq 4C^3 \varepsilon^{-5\zeta} (\log \varepsilon^{-1})^9 \quad (3.55)$$

and so

$$\#\{w \in V_\varepsilon \cap Z_j : E_{z,w} \text{ holds}\} \leq \varepsilon^{-5\zeta + o(1)}. \quad (3.56)$$

Hence

$$\sum_{z, w \in V_\varepsilon} 1_{E_{z,w}} \leq \# \left\{ z \in V_\varepsilon \cap Z_j : \text{diam} \left(\widehat{H}_z^\varepsilon \right) \in [\varepsilon^{\alpha_j + 1}, \varepsilon^{\alpha_j}] \right\} \varepsilon^{-5\zeta + o(1)}. \quad (3.57)$$

Using (3.30) we obtain that

$$\begin{aligned}
& \# \left\{ z \in V_\varepsilon \cap Z_j : \text{diam} \left(\widehat{H}_z^\varepsilon \right) \in [\varepsilon^{\alpha_{j+1}}, \varepsilon^{\alpha_j}] \right\} \\
& \leq \# \left\{ z \in V_\varepsilon \cap Z_j : \text{diam} \left(\widehat{H}_z^\varepsilon \right) \geq \varepsilon^{\alpha_{j+1}} \right\} \\
& \leq C \log \varepsilon^{-1} \# \left\{ z \in V_\varepsilon \cap \bar{Z}_j : \text{diam}(H_z^\varepsilon) \geq \frac{\varepsilon^{\alpha_{j+1}}}{C \log \varepsilon^{-1}} \right\}.
\end{aligned}$$

where in the last line we used the fact that there are at most $(\log \varepsilon^{-1})^3 + 1$ cells in $\widehat{H}_z^\varepsilon$, and so if $\text{diam} \left(\widehat{H}_z^\varepsilon \right) \geq \varepsilon^{\alpha_{j+1}}$ then there is a cell in $\widehat{H}_z^\varepsilon$ whose diameter is at least $\frac{\varepsilon^{\alpha_{j+1}}}{C \log \varepsilon^{-1}}$. Now using the same argument as for (3.55) we obtain that

$$\begin{aligned}
& \# \left\{ z \in V_\varepsilon \cap \bar{Z}_j : \text{diam} \left(\widehat{H}_z^\varepsilon \right) \geq \varepsilon^{\alpha_{j+1}} \right\} \\
& \leq C (\log \varepsilon^{-1})^3 \left(\frac{\varepsilon^{\alpha_{j+1}}}{C \log \varepsilon^{-1}} \right)^{-2} \varepsilon^{-2\zeta} (\text{area}(\text{supp} f) + O(\varepsilon^{\alpha_j})) \\
& \lesssim C^3 (\log \varepsilon^{-1})^5 \varepsilon^{-2\alpha_{j+1}-2\zeta}.
\end{aligned}$$

Combining this with (3.57) we obtain

$$\sum_{z,w \in V_\varepsilon \cap Z_j} 1_{E_{z,w}} \leq \varepsilon^{-2\alpha_{j+1}-7\zeta+o(1)}. \quad (3.58)$$

Plugging this into (3.52) and then back into (3.48) and (3.49) we obtain that

$$|A_1| \leq \varepsilon^{o(1)-9\zeta}. \quad (3.59)$$

Step 3: Bounding A_2 . By Lemma 3.4.13 we have

$$\begin{aligned}
& |A_2| \\
& \leq \int \int_{|z-w| < \varepsilon^{\alpha_j - \zeta}} 1_{Z_j}(z) |X_z^\varepsilon| \left| \mathbb{E} \left(1_{Z_j}(z) X_w^\varepsilon \middle| \Phi_{\varepsilon^{\alpha_j - \zeta}}(w), \Psi_{\varepsilon^{\alpha_j - \zeta}}(w) \right) \right| dz dw + \varepsilon^{o(1)} \\
& \leq \|\nabla^2 f\|_\infty \int \int_{|z-w| < \varepsilon^{\alpha_j - \zeta}} 1_{Z_j}(z) |X_z^\varepsilon| \\
& \quad \cdot \mathbb{E} \left((\deg(x_w^\varepsilon) + 1) \frac{\text{diam}(H_w^\varepsilon)^2}{\text{area}(H_w^\varepsilon)} \middle| \Phi_{\varepsilon^{\alpha_j - \zeta}}(w), \Psi_{\varepsilon^{\alpha_j - \zeta}}(w) \right) dz dw + \varepsilon^{o(1)}.
\end{aligned}$$

Note that by Lemma 3.3.9 we have

$$|X_z^\varepsilon| \leq (\deg(x_z^\varepsilon) + 1) \|\nabla f\|_\infty \frac{\text{diam}(H_z^\varepsilon)}{\text{area}(H_z^\varepsilon)}.$$

This implies that

$$\begin{aligned}
& \frac{|A_2|}{\|\nabla f\|_\infty \|\nabla^2 f\|_\infty} \\
& \leq \int \int_{|z-w| < \varepsilon^{\alpha_j - \zeta}} \mathbb{E} \left(1_{E_{z,w}} (\deg(x_z^\varepsilon) + 1) (\deg(x_w^\varepsilon) + 1) \right. \\
& \quad \left. \frac{\text{diam}(H_z^\varepsilon)}{\text{area}(H_z^\varepsilon)} \frac{\text{diam}(H_w^\varepsilon)^2}{\text{area}(H_w^\varepsilon)} \middle| \Phi_{\varepsilon^{\alpha_j - \zeta}}(w), \Psi_{\varepsilon^{\alpha_j - \zeta}}(w) \right) dz dw + \varepsilon^{o(1)}.
\end{aligned}$$

Now by definition of Z_j we have that

$$(\deg(x_z^\varepsilon) + 1) (\deg(x_w^\varepsilon) + 1) \frac{\text{diam}(H_z^\varepsilon)}{\text{area}(H_z^\varepsilon)} \frac{\text{diam}(H_w^\varepsilon)^2}{\text{area}(H_w^\varepsilon)} 1_{E_{z,w}} \lesssim \varepsilon^{-2\zeta + o(1)} \frac{1}{\text{diam}(H_z^\varepsilon)}.$$

This implies by (3.30), (3.33), (3.36) that

$$\begin{aligned}
\frac{|A_2|}{\|\nabla f\|_\infty \|\nabla^2 f\|_\infty} &\leq \varepsilon^{-2\zeta+o(1)} \int \int_{|z-w|<\varepsilon} \frac{1_{E_{z,w}}}{\text{diam}(H_z^\varepsilon)} dz dw + \varepsilon^{o(1)} \\
&\leq \varepsilon^{\alpha_j-3\zeta} \varepsilon^{o(1)} \int \int_{|z-w|<\varepsilon} \frac{1_{E_{z,w}}}{\text{diam}(H_z^\varepsilon)} \frac{\text{diam}(H_z^\varepsilon)^2 \text{diam}(H_w^\varepsilon)^2}{\text{area}(H_z^\varepsilon) \text{area}(H_w^\varepsilon)} dz dw \\
&\quad + \varepsilon^{o(1)} \\
&= \varepsilon^{o(1)} \varepsilon^{\alpha_j-4\zeta} \sum_{z,w \in V_\varepsilon \cap Z_j} \text{diam}(H_z^\varepsilon) \text{diam}(H_w^\varepsilon) 1_{E_{z,w}} + \varepsilon^{o(1)} \\
&\leq \varepsilon^{o(1)} \varepsilon^{\alpha_j-4\zeta} \varepsilon^{2\alpha_j-2\zeta} \sum_{z,w \in V_\varepsilon \cap Z_j} 1_{E_{z,w}} + \varepsilon^{o(1)}.
\end{aligned}$$

By (3.58), we have that

$$\frac{|A_2|}{\|\nabla f\|_\infty \|\nabla^2 f\|_\infty} \lesssim \varepsilon^{-15\zeta+o(1)}$$

hence

$$|A_2| \lesssim \varepsilon^{-15\zeta+o(1)}. \quad (3.60)$$

Step 4: Bounding A_3 and conclusion. We note that by Lemma 3.4.13 together with (3.33), (3.36) we have

$$\begin{aligned}
&\mathbb{E} \left(|1_{Z_j}(z) X_z^\varepsilon| |\Phi_{\varepsilon^{\alpha_j-\zeta}}(z), \Psi_{\varepsilon^{\alpha_j-\zeta}}(z)| \right) \\
&\leq \|\nabla^2 f\|_\infty \mathbb{E} \left(|1_{Z_j}(z) (\deg(x_z^\varepsilon) + 1) \frac{\text{diam}(H_z^\varepsilon)^2}{\text{area}(H_z^\varepsilon)}| |\Phi_{\varepsilon^{\alpha_j-\zeta}}(z), \Psi_{\varepsilon^{\alpha_j-\zeta}}(z)| \right) + \varepsilon^{o(1)} \\
&\lesssim \varepsilon^{o(1)-\zeta},
\end{aligned}$$

thus

$$\begin{aligned}
|A_3| &\leq \varepsilon^{o(1)-2\zeta} \int \int_{|z-w| \leq \varepsilon^{\alpha_j-\zeta}} 1_{\text{supp}(f)}(z) 1_{\text{supp}(f)}(w) 1_{E_{z,w}} dz dw + \varepsilon^{o(1)} \\
&= \varepsilon^{o(1)-4\zeta} \int \int_{|z-w| \leq \varepsilon^{\alpha_j-\zeta}} 1_{\text{supp}(f)}(z) 1_{\text{supp}(f)}(w) 1_{E_{z,w}} \frac{\text{diam}(H_z^\varepsilon)^2}{\text{area}(H_z^\varepsilon)} \frac{\text{diam}(H_w^\varepsilon)^2}{\text{area}(H_w^\varepsilon)} dz dw \\
&\quad + \varepsilon^{o(1)} \\
&\lesssim \varepsilon^{o(1)} \varepsilon^{2\alpha_j-6\zeta} \int \int_{|z-w| \leq \varepsilon^{\alpha_j-\zeta}} 1_{E_{z,w}} \frac{\text{diam}(H_z^\varepsilon)}{\text{area}(H_z^\varepsilon)} \frac{\text{diam}(H_w^\varepsilon)}{\text{area}(H_w^\varepsilon)} dz dw + \varepsilon^{o(1)} \\
&\lesssim \varepsilon^{o(1)} \varepsilon^{2\alpha_j-7\zeta} \sum_{z,w \in V_\varepsilon \cap Z_j} \text{diam}(H_z^\varepsilon) \text{diam}(H_w^\varepsilon) 1_{E_{z,w}} + \varepsilon^{o(1)} \\
&\lesssim \varepsilon^{o(1)} \varepsilon^{4\alpha_j-9\zeta} \sum_{z,w \in V_\varepsilon \cap Z_j} 1_{E_{z,w}} + \varepsilon^{o(1)}.
\end{aligned}$$

Now using (3.58) we obtain that

$$|A_3| \lesssim \varepsilon^{o(1)} \varepsilon^{2\alpha_j-18\zeta} + \varepsilon^{o(1)}$$

Combining this with (3.59), (3.60) and (3.47) yields the result of Proposition 3.4.14. $\square$

Off-diagonal integral

The main result of this section is the following.

Proposition 3.4.15. *We have*

$$\mathbb{E} \left(\int \int_{|z-w| \geq \varepsilon^{\alpha_j-\zeta}} \bar{X}_z^\varepsilon \bar{X}_w^\varepsilon dz dw \right) \lesssim \varepsilon^{-2\zeta+o(1)}.$$

The main ingredient is a long range independence property that $\bar{X}_z^\varepsilon$ satisfies. First we rewrite the integral in Proposition 3.4.15. Recall the definitions of X_z^ε and $\bar{X}_z^\varepsilon$, (3.25) and (3.39). Suppose $z, w \in \mathbb{C}$. We define $\mathcal{F}_{z,w}$ to be the σ -algebra generated by

$(\Phi, \Psi)|_{\mathbb{C} \setminus \left(B_{\frac{|z-w|}{4}}(z) \cup B_{\frac{|z-w|}{4}}(w) \right)}$, that is

$$\mathcal{F}_{z,w} = \sigma \left((\Phi, \Psi)|_{\mathbb{C} \setminus \left(B_{|z-w|/4}(z) \cup B_{|z-w|/4}(w) \right)} \right). \quad (3.61)$$

We will need the following important lemma, which is proven in Gwynne et al. [2019b].

Lemma 3.4.16. *[Gwynne et al., 2019b, Lemma A.3] Let $B_R(w)$ be a ball containing the unit disk and suppose h is a whole-plane GFF normalized so that its circle average over $B_R(w)$ vanishes. Let p, p', s be such that $p' > p > 1$, $s \in (0, 1)$. Then there exist constants $a = a(p, p') > 0$ and $\bar{b} = \bar{b}(p, p', s) > 0$ such that for any $0 < \delta \leq a$ and any random variable X determined almost surely by $h|_{B_\delta(0)}$ and $h_1(0)$, we have*

$$\mathbb{E} \left(\left| \mathbb{E}(X | h_{\mathbb{C} \setminus B_1(0)}) - \mathbb{E}(X | h_1(0)) \right|^p \right) \lesssim \delta^{sp} \mathbb{E}(|X|^p) + e^{-\bar{b}/\delta^{1-s}} \mathbb{E}(|X|^{p'})^{\frac{p}{p'}}.$$

Proof of Proposition 3.4.15. Step 1: We claim that

$$\mathbb{E} \left(\left(\mathbb{E}(\bar{X}_z^\varepsilon | \mathcal{F}_{z,w}) \right)^2 \right) \lesssim \frac{\varepsilon^{2s\alpha_{j+1}+2s\zeta}}{|z-w|^{2s}} \mathbb{E} \left(1_{Z_j}(z) (X_z^\varepsilon)^2 \right). \quad (3.62)$$

To this end, let $\mathcal{F}_{z,w}$ be as in (3.61). Then we have

$$\mathbb{E}(\bar{X}_z^\varepsilon \bar{X}_w^\varepsilon) = \mathbb{E}(\mathbb{E}(\bar{X}_z^\varepsilon \bar{X}_w^\varepsilon | \mathcal{F}_{z,w})). \quad (3.63)$$

By the domain Markov property of the Φ and $\mathcal{F}(Z_j)$, the fields Φ, Ψ when restricted to these two balls are conditionally dependent. By the definition of Z_j , we have that if $z, w \in Z_j$, we have $\hat{H}_z^\varepsilon \subset B_{\frac{|z-w|}{4}}(z)$, $\hat{H}_w^\varepsilon \subset B_{\frac{|z-w|}{4}}(w)$ which are disjoint, and hence $\bar{X}_z^\varepsilon$ and $\bar{X}_w^\varepsilon$ become independent. Hence

$$\mathbb{E}(\bar{X}_z^\varepsilon \bar{X}_w^\varepsilon | \mathcal{F}_{z,w}) = \mathbb{E}(\bar{X}_z^\varepsilon | \mathcal{F}_{z,w}) \mathbb{E}(\bar{X}_w^\varepsilon | \mathcal{F}_{z,w}). \quad (3.64)$$

Using Lemma 3.4.16 for $\delta = \frac{\varepsilon^{\alpha_{j+1}+\zeta}}{|z-w|}$, $p = 2$, $p' = 3$, sending s to 1 and appropriately rescaling we obtain that

$$\mathbb{E} \left(\left(\mathbb{E} (\bar{X}_z^\varepsilon | \mathcal{F}_{z,w}) \right)^2 \right) \leq \frac{\varepsilon^{2s\alpha_{j+1}+2s\zeta}}{|z-w|^{2s}} \mathbb{E} \left((\bar{X}_z^\varepsilon)^2 \right) + e^{-\frac{\bar{b}|z-w|^{1-s}}{\varepsilon^{(1-s)(\alpha_{j+1}+\zeta)}}} \left(\mathbb{E} \left((\bar{X}_z^\varepsilon)^3 \right) \right)^{\frac{2}{3}}. \quad (3.65)$$

Note that

$$\begin{aligned} \mathbb{E} \left((\bar{X}_z^\varepsilon)^3 \right) &= \mathbb{E} \left(\left(1_{Z_j}(z) X_z^\varepsilon - \mathbb{E} \left(1_{Z_j}(z) X_z^\varepsilon \middle| \Phi_{\varepsilon^{\alpha_j-\zeta}}(z), \Psi_{\varepsilon^{\alpha_j-\zeta}}(z) \right) \right)^3 \right) \\ &\leq 4\mathbb{E} \left(1_{Z_j}(z) (X_z^\varepsilon)^3 \right) + 4\mathbb{E} \left(\left(\mathbb{E} \left(1_{Z_j}(z) X_z^\varepsilon \middle| \Phi_{\varepsilon^{\alpha_j-\zeta}}(z), \Psi_{\varepsilon^{\alpha_j-\zeta}}(z) \right) \right)^3 \right) \\ &\leq 4\mathbb{E} \left(1_{Z_j}(z) (X_z^\varepsilon)^3 \right) + 4\mathbb{E} \left(\mathbb{E} \left(1_{Z_j}(z) (X_z^\varepsilon)^3 \middle| \Phi_{\varepsilon^{\alpha_j-\zeta}}(z), \Psi_{\varepsilon^{\alpha_j-\zeta}}(z) \right) \right) \\ &= 8\mathbb{E} \left(1_{Z_j}(z) (X_z^\varepsilon)^3 \right). \end{aligned} \quad (3.66)$$

We also have by Lemma 3.3.9 that

$$\begin{aligned} &\mathbb{E} \left(1_{Z_j}(z) (X_z^\varepsilon)^3 \right) \\ &\leq \|\nabla f\|_\infty^3 \mathbb{E} \left(1_{Z_j}(z) \frac{(\deg(x_z^\varepsilon) + 1)^3 \text{diam}(H_z^\varepsilon)^3}{\text{area}(H_z^\varepsilon)^3} \right) \\ &\leq \|\nabla f\|_\infty^3 \left(\mathbb{E} \left(1_{Z_j}(z) (\deg(x_z^\varepsilon) + 1)^6 \left(\frac{\text{diam}(H_z^\varepsilon)^2}{\text{area}(H_z^\varepsilon)} \right)^6 \right) \right)^{\frac{1}{2}} \\ &\quad \cdot \left(\mathbb{E} \left(1_{Z_j}(z) \frac{1}{\text{diam}(H_z^\varepsilon)^6} \right) \right)^{\frac{1}{2}}. \end{aligned}$$

Recalling the definition of Z_j in Definition 3.4.8 we obtain that we have

$$\mathbb{E} \left(1_{Z_j}(z) (X_z^\varepsilon)^3 \right) \leq \varepsilon^{-3\zeta} \left(\mathbb{E} \left(1_{Z_j}(z) \frac{1}{\text{diam}(H_z^\varepsilon)^6} \right) \right)^{\frac{1}{2}}.$$

Recall that if Z_j holds, then $\text{area}(H_z^\varepsilon) \geq \varepsilon^b$, and hence $\text{diam}(H_z^\varepsilon) \geq (\varepsilon^b/\pi)^{1/2}$. This implies

that

$$\mathbb{E} \left(1_{Z_j}(z) (X_z^\varepsilon)^3 \right) \lesssim \varepsilon^{-\frac{3}{2}\beta - 3\zeta}.$$

This implies the second term in (3.65) is negligible. Plugging this into (3.65) we obtain

$$\begin{aligned} \mathbb{E} \left((\mathbb{E}(\bar{X}_z^\varepsilon | \mathcal{F}_{z,w}))^2 \right) &\leq \frac{\varepsilon^{2s\alpha_{j+1}+2s\zeta}}{|z-w|^{2s}} \mathbb{E} \left((\bar{X}_z^\varepsilon)^2 \right) + e^{-\frac{\bar{b}|z-w|^{1-s}}{\varepsilon^{(1-s)(\alpha_{j+1}+\zeta)}}} \left(\mathbb{E} \left((\bar{X}_z^\varepsilon)^3 \right) \right)^{\frac{2}{3}} \\ &\lesssim \frac{\varepsilon^{2s\alpha_{j+1}+2s\zeta}}{|z-w|^{2s}} \mathbb{E} \left((\bar{X}_z^\varepsilon)^2 \right) \end{aligned}$$

for appropriate constants $s, \bar{b}$ and thus

$$\begin{aligned} &\mathbb{E} \left(\int \int_{|z-w| \geq \varepsilon^{\alpha_j - \zeta}} \bar{X}_z^\varepsilon \bar{X}_w^\varepsilon dz dw \right) \\ &\leq \mathbb{E} \left(\int \int_{|z-w| \geq \varepsilon^{\alpha_j - \zeta}} 1_{z,w \in \text{supp} f} \frac{\varepsilon^{2s\alpha_{j+1}+2s\zeta}}{|z-w|^{2s}} \mathbb{E} \left((\bar{X}_z^\varepsilon)^2 \right) dz dw \right) \end{aligned} \quad (3.67)$$

By computations similar to (3.66) one has

$$\mathbb{E} \left((\bar{X}_z^\varepsilon)^2 \right) \leq 4 \mathbb{E} \left(1_{Z_j}(z) (X_z^\varepsilon)^2 \right).$$

Therefore plugging this into (3.67) we obtain (3.62).

Step 2: conclusion. Plugging into (3.63) and using Cauchy-Schwarz twice we obtain

$$\begin{aligned}
& \mathbb{E} \left(\int \int_{|z-w| \geq \varepsilon^{\alpha_j - \zeta}} \bar{X}_z^\varepsilon \bar{X}_w^\varepsilon dz dw \right) \\
&= \int \int_{|z-w| \geq \varepsilon^{\alpha_j - \zeta}} 1_{z,w \in \text{supp} f} \mathbb{E} \left(\bar{X}_z^\varepsilon \bar{X}_w^\varepsilon \right) dz dw \\
&= \int \int_{|z-w| \geq \varepsilon^{\alpha_j - \zeta}} 1_{z,w \in \text{supp} f} \mathbb{E} \left(\mathbb{E} \left(\bar{X}_z^\varepsilon | \mathcal{F}_{z,w} \right) \mathbb{E} \left(\bar{X}_w^\varepsilon | \mathcal{F}_{z,w} \right) \right) dz dw \\
&\leq \int \int_{|z-w| \geq \varepsilon^{\alpha_j - \zeta}} 1_{z,w \in \text{supp} f} \mathbb{E} \left(\left(\mathbb{E} \left(\bar{X}_z^\varepsilon | \mathcal{F}_{z,w} \right) \right)^2 \right)^{\frac{1}{2}} \mathbb{E} \left(\left(\mathbb{E} \left(\bar{X}_w^\varepsilon | \mathcal{F}_{z,w} \right) \right)^2 \right)^{\frac{1}{2}} dz dw \\
&\leq \left(\int \int_{|z-w| \geq \varepsilon^{\alpha_j - \zeta}} 1_{z,w \in \text{supp} f} \mathbb{E} \left(\left(\mathbb{E} \left(\bar{X}_z^\varepsilon | \mathcal{F}_{z,w} \right) \right)^2 \right) dz dw \right)^{\frac{1}{2}} \\
&\quad \cdot \left(\int \int_{|z-w| \geq \varepsilon^{\alpha_j - \zeta}} 1_{z,w \in \text{supp} f} \mathbb{E} \left(\left(\mathbb{E} \left(\bar{X}_w^\varepsilon | \mathcal{F}_{z,w} \right) \right)^2 \right) dz dw \right)^{\frac{1}{2}} \\
&= \left(\int \int_{|z-w| \geq \varepsilon^{\alpha_j - \zeta}} 1_{z,w \in \text{supp} f} \mathbb{E} \left(\left(\mathbb{E} \left(\bar{X}_z^\varepsilon | \mathcal{F}_{z,w} \right) \right)^2 \right) dz dw \right).
\end{aligned}$$

By (3.62), we have that

$$\begin{aligned}
& \mathbb{E} \left(\int \int_{|z-w| \geq \varepsilon^{\alpha_j - \zeta}} \bar{X}_z^\varepsilon \bar{X}_w^\varepsilon dz dw \right) \\
&\lesssim \int \int_{|z-w| \geq \varepsilon^{\alpha_j - \zeta}} 1_{z,w \in \text{supp} f} \frac{\varepsilon^{2s\alpha_j + 1 + 2s\zeta}}{|z-w|^{2s}} \mathbb{E} \left(1_{Z_j}(z) (X_z^\varepsilon)^2 \right) dz dw. \tag{3.68}
\end{aligned}$$

Now using Lemma 3.3.9 we have that

$$\mathbb{E} \left(1_{Z_j}(z) (X_z^\varepsilon)^2 \right) \lesssim \mathbb{E} \left(1_{Z_j}(z) \frac{(\deg(x_z^\varepsilon) + 1)^2 \text{diam}(H_z^\varepsilon)^2}{\text{area}(H_z^\varepsilon)^2} \right).$$

Recalling the definition of Z_j in Definition 3.4.8 we obtain that

$$\mathbb{E} \left(1_{Z_j}(z) (X_z^\varepsilon)^2 \right) \lesssim \varepsilon^{-\zeta} \mathbb{E} \left(\frac{1_{Z_j}(z)}{\text{diam}(H_z^\varepsilon)^2} \right) \leq \varepsilon^{-2\alpha_j + 1 - \zeta}.$$

Plugging this into (3.68) we have

$$\begin{aligned}
& \mathbb{E} \left(\int \int_{|z-w| \geq \varepsilon} \alpha_j^{-\zeta} \bar{X}_z^\varepsilon \bar{X}_w^\varepsilon dz dw \right) \\
& \lesssim \varepsilon^{2\alpha_{j+1}-2\alpha_j+o(1)} \mathbb{E} \left(\int \int_{|z-w| \geq \varepsilon} \alpha_j^{-\zeta} 1_{z,w \in \text{supp} f} 1_{Z_j}(z) \frac{1}{|z-w|^2} dw dz \right) \\
& \leq |\log(\varepsilon)| \varepsilon^{2\alpha_{j+1}-2\alpha_j-2\zeta+o(1)} \mathbb{E} \left(\int_{\mathbb{C}} 1_{Z_j}(z) dz \right).
\end{aligned}$$

Since $\zeta > 0$ was chosen to be arbitrarily small, this completes the proof. $\square$

End of proof of Proposition 3.4.11

In this subsection we finish the proof of Proposition 3.4.11.

Proof of Proposition 3.4.11. By (3.46) we have that

$$\begin{aligned}
\mathbb{E} \left(\left(\int_{\mathbb{C}} \bar{X}_z^\varepsilon 1_{Z_j}(z) dz \right)^2 \right) &= \mathbb{E} \left(\int \int_{|z-w| \leq \varepsilon} \alpha_j^{-\zeta} 1_{Z_j}(z) 1_{Z_j}(w) \bar{X}_z^\varepsilon \bar{X}_w^\varepsilon dz dw \right) \\
&+ \mathbb{E} \left(\int \int_{|z-w| \geq \varepsilon} \alpha_j^{-\zeta} 1_{Z_j}(z) 1_{Z_j}(w) \bar{X}_z^\varepsilon \bar{X}_w^\varepsilon dz dw \right).
\end{aligned}$$

The first term is $\lesssim \varepsilon^{-15\zeta+o(1)}$ by Proposition 3.4.14, while the second term is $\lesssim \varepsilon^{-2\zeta+o(1)}$ by Proposition 3.4.15. This completes the proof. $\square$

3.4.4 Proof of Proposition 3.4.12

In this subsection we prove Proposition 3.4.12.

Proof of Proposition 3.4.12. By Lemma 3.4.13, we have that

$$\begin{aligned}
& \mathbb{E} \left(\left(\int_{\mathbb{C}} \mathbb{E} \left(1_{Z_j(z)} X_z^\varepsilon \middle| \Phi_{\varepsilon^{\alpha_j - \zeta}}(z), \Psi_{\varepsilon^{\alpha_j - \zeta}}(z) \right) \right)^2 \right) \\
& \lesssim \mathbb{E} \left(\left(\int_{\mathbb{C}} \|\nabla^2 f\|_\infty \mathbb{E}(1_{Z_j \cap \text{supp} f}(z) (\deg(z) + 1) \frac{\text{diam}(H_z^\varepsilon)^2}{\text{area}(H_z^\varepsilon)} \middle| \Phi_{\varepsilon^{\alpha_j - \zeta}}(z), \Psi_{\varepsilon^{\alpha_j - \zeta}}(z)) \right)^2 \right) \\
& \quad + \varepsilon^{o(1)} \\
& \lesssim \mathbb{E} \left(\left(\int_{\text{supp} f} (\log \varepsilon^{-1})^7 \varepsilon^{-\zeta} dz \right)^2 \right) + \varepsilon^{o(1)} \lesssim O(\varepsilon^{-3\zeta}) + \varepsilon^{o(1)}
\end{aligned}$$

where in the last line we used the definition of Z_j , Definition 3.4.8. $\square$

3.5 Transferring to the Quantum cone field

In this section we finish the proof of Theorem 3.1.5, that we pass from statements involving the whole plane GFF to statements involving the γ -quantum cone. The main idea is to use the following fact. Recall that Φ is a whole plane GFF. Suppose we sample a point $\mathcal{P}$ according to the Lebesgue measure on the unit disk. Let

$$\Phi^{(1)} := \Phi - \gamma \log |\cdot - \mathcal{P}| + \gamma \log(\max(|\cdot|, 1)).$$

Then the laws of Φ and $\Phi^{(1)}$ are mutually absolutely continuous (see Lemma A.10 in Duplantier et al. [2021]).

Now we will define three more fields to use as intermediate steps towards the quantum cone. Recall that Φ_r denotes the circle average of Φ centered at the origin of radius r . Let

$$\Phi^{(2)} = \Phi^{(1)} - \gamma \log(\max(|\cdot|, 1))$$

$$\Phi^{(3)} = \Phi^{(2)} - \Phi_{\frac{1}{2}}^2(\mathcal{P})$$

$$\Phi^{(4)} = \Phi^{(3)} \left(\frac{1}{2} \cdot + \mathcal{P} \right)$$

By scaling, it suffices to assume that $\text{supp } f \subseteq B_1(0)$. We will show Theorem 3.1.5 step by step.

Lemma 3.5.1. *We have Theorem 3.1.5 holds if the quantum cone field $\tilde{\Phi}$ is replaced by $\Phi^{(1)}$.*

Proof. Since $\Phi^{(1)}$ and Φ are mutually absolutely continuous, we can use Proposition 3.4.1 directly to conclude. $\square$

Lemma 3.5.2. *We have Theorem 3.1.5 holds if the quantum cone $\tilde{\Phi}$ is replaced by $\Phi^{(2)}$.*

Proof. This comes directly from the fact that $\Phi^{(1)}$ and $\Phi^{(2)}$ coincide in the unit disk together with Lemma 3.5.1. $\square$

Lemma 3.5.3. *We have Theorem 3.1.5 holds if the quantum cone $\tilde{\Phi}$ is replaced by $\Phi^{(3)}$.*

Proof. Let $G_f^{i,\varepsilon}$ denote the analogue of G_f when the underlying field is replaced with $\Phi^{(i)}$. Let $S := e^{\frac{\gamma h_1^2(\mathcal{P})}{2}}$. Then recalling Definition 3.3.1, note that by scaling we have

$$\frac{G_f^{3,S\varepsilon}}{\text{area}(H_z^{S\varepsilon})} = \frac{G_f^{2,\varepsilon}}{\text{area}(H_z^\varepsilon)}$$

Therefore by Lemma 3.5.2 we only need to convert integrals involving $G_f^{3,S\varepsilon}$ to integrals only involving $G_f^{2,\varepsilon}$. Now by Lemma A.2 in Gwynne et al. [2019b], for $\mathbf{a} \in \mathbb{R}$ and $\mathbf{z} \in B_{\varepsilon^\xi}(0)$, the conditional law of $\Phi^{(3)}|_{B_{\frac{1}{2}}(\mathcal{P})}$ given $\{\Phi_{\frac{1}{2}}^2(\mathbf{z}) = \mathbf{a}\} \cap \{\mathcal{P} = \mathbf{z}\}$ is mutually absolutely continuous with respect to the law of $\Phi^{(3)}|_{B_{\frac{1}{2}}(\mathcal{P})}$ given only $\{\mathcal{P} = \mathbf{z}\}$. Following the notation in Gwynne et al. [2019b], if $p > 1$, there is an $r_p = r_p(\rho)$ such that the p -th moment of $\mathcal{M}_{\mathbf{a},\mathbf{z}}$ is bounded above by a constant depending only on p provided $\mathbf{a} \in [-r_p, r_p]$. The rest of the proof follows as in case 3 in the proof of Proposition 4.10 in Gwynne et al. [2019b]. $\square$

Lemma 3.5.4. *We have Theorem 3.1.5 holds if the quantum cone Φ is replaced by $\Phi^{(4)}$.*

Proof. By γ -LQG coordinate change, we have that

$$\frac{G_f^{3,\varepsilon}}{\text{area}(H_z^\varepsilon)}(\cdot) = C \frac{G_f^{4,2^{-\gamma Q_\varepsilon}}}{\text{area}(H_z^{2^{-\gamma Q_\varepsilon}})}(2 \cdot + \mathcal{P})$$

for some deterministic constant C . This constant does not matter in the rest of the proof: one can now follow step 4 in the proof of Proposition 4.10 in Gwynne et al. [2019b]. $\square$

Now the final step is to prove Theorem 3.1.5.

Proof of Theorem 3.1.5. If $\tilde{\Phi}$ is the γ -quantum cone field, then $\tilde{\Phi}$ and $\Phi^{(4)}$ agree in law on the unit disk. Since $\frac{G_f(z)}{\text{area}(H_z^\varepsilon)}$ is locally determined by the underlying fields $\tilde{\Phi}, \Psi|_{B_{\frac{1}{2}}(z)}$, we can replace $\Phi^{(4)}$ by $\tilde{\Phi}$ in Lemma 3.5.4. This completes the proof. $\square$

3.6 Total Curvature on a mated CRT map Cell

3.6.1 Proof of Theorem 3.1.7

We let $\mathcal{C}_0$ be a fixed segment of $\tilde{\eta}$,

$$\mathcal{C}_0 = \tilde{\eta}([0, 1]).$$

We define $\tilde{\mathcal{G}}_\varepsilon(\mathcal{C}_0)$ to be the submap of $\tilde{\mathcal{G}}_\varepsilon$ induced by $\mathcal{V}\tilde{\mathcal{G}}_\varepsilon \cap [0, 1]$. We will look at the total discrete curvature inside $\mathcal{C}_0$. We recall that the discrete total curvature on a segment $\mathcal{C}_0$ of $\tilde{\eta}$ is defined as the sum of the discrete curvature on every vertex in $\mathcal{C}_0$,

$$K_{\tilde{\mathcal{G}}_\varepsilon}(\mathcal{C}_0) = \sum_{v \in \mathcal{V}\tilde{\mathcal{G}}_\varepsilon(\mathcal{C}_0)} K_{\tilde{\mathcal{G}}_\varepsilon}(v).$$

This represents the total curvature in $\mathcal{C}_0$.

Now we will prove Theorem 3.1.7. Our first step will be to rewrite the total curvature in $\mathcal{C}_0$ in a more convenient manner. To this end, we will need the following graph theory

lemma about general triangulations with boundary.

Lemma 3.6.1. *[Berestycki and Gwynne, 2022, Lemma A.7] Let $\mathcal{T}$ be a triangulation with boundary, that is, each face has 3 edges, except for the outer face. We let $\#\mathcal{VT}$, $\#\mathcal{ET}$, $\#\mathcal{FT}$ denote the number of vertices, edges, and faces in T respectively. Let $\text{Perim}(T)$ denote the perimeter of the external face, that is, the number of edges on the boundary of the external face. Then we have that*

$$\#\mathcal{ET} = 3\#\mathcal{VT} + 3 - \text{Perim}(T).$$

We claim the total curvature on $\mathcal{C}_0$ simplifies to an expression only involving degrees on the boundary of $\mathcal{C}_0$. More precisely, let $\deg_{\text{ext}}^{\mathcal{C}_0}$ denote the exterior degree, that is the number of edges between a vertex in $\mathcal{V}_{\tilde{\mathcal{G}}_\varepsilon}(\mathcal{C}_0)$ and $\mathcal{V}_{\tilde{\mathcal{G}}_\varepsilon} \setminus \mathcal{V}_{\tilde{\mathcal{G}}_\varepsilon}(\mathcal{C}_0)$. Then we have the following.

Proposition 3.6.2. *We have*

$$K_{\tilde{\mathcal{G}}_\varepsilon}(\mathcal{C}_0) = \frac{\pi}{3} \left(2\text{Perim}(\mathcal{C}_0) - 6 - \sum_{v \in \mathcal{V}_{\tilde{\mathcal{G}}_\varepsilon}(\mathcal{C}_0)} \deg_{\text{ext}}^{\mathcal{C}_0}(v) \right).$$

Proof. We have that

$$\sum_{v \in \mathcal{V}_{\tilde{\mathcal{G}}_\varepsilon}(\mathcal{C}_0)} K_{\tilde{\mathcal{G}}_\varepsilon}(v) = \frac{\pi}{3} \sum_{v \in \mathcal{V}_{\tilde{\mathcal{G}}_\varepsilon}(\mathcal{C}_0)} (6 - \deg(v)) = 2\pi\#\mathcal{V}_{\tilde{\mathcal{G}}_\varepsilon}(\mathcal{C}_0) - \frac{\pi}{3} \sum_{v \in \mathcal{V}_{\tilde{\mathcal{G}}_\varepsilon}(\mathcal{C}_0)} \deg(v).$$

Recall that $\deg_{\text{ext}}^{\mathcal{C}_0}(v)$ denotes the number of edges connecting v to a point outside of $\mathcal{C}_0$, and similarly let $\deg_{\text{int}}^{\mathcal{C}_0}(v)$ denote the number of edges connecting v with another vertex in $\mathcal{C}_0$. Then note that

$$\sum_{v \in \mathcal{V}_{\tilde{\mathcal{G}}_\varepsilon}(\mathcal{C}_0)} \deg_{\text{int}}^{\mathcal{C}_0}(v) = 2\#\mathcal{EC}_0$$

and thus

$$\sum_{v \in \mathcal{V}_{\tilde{\mathcal{G}}_\varepsilon(\mathcal{C}_0)}} \deg(v) = \sum_{v \in \mathcal{V}_{\tilde{\mathcal{G}}_\varepsilon(\mathcal{C}_0)}} \deg_{\text{ext}}^{\mathcal{C}_0}(v) + \sum_{v \in \mathcal{V}_{\tilde{\mathcal{G}}_\varepsilon(\mathcal{C}_0)}} \deg_{\text{int}}^{\mathcal{C}_0}(v) = \sum_{v \in \mathcal{V}_{\tilde{\mathcal{G}}_\varepsilon(\mathcal{C}_0)}} \deg_{\text{ext}}^{\mathcal{C}_0}(v) + 2\#\mathcal{EC}_0.$$

This implies that

$$\begin{aligned} \sum_{v \in \mathcal{V}_{\tilde{\mathcal{G}}_\varepsilon(\mathcal{C}_0)}} K_{\tilde{\mathcal{G}}_\varepsilon}(v) &= \frac{\pi}{3} \left(6\#\mathcal{V}_{\tilde{\mathcal{G}}_\varepsilon(\mathcal{C}_0)} - \left(\sum_{v \in \mathcal{V}_{\tilde{\mathcal{G}}_\varepsilon(\mathcal{C}_0)}} \deg_{\text{ext}}^{\mathcal{C}_0}(v) + 2\#\mathcal{EC}_0 \right) \right) \\ &= \frac{\pi}{3} \left(6\#\mathcal{V}_{\tilde{\mathcal{G}}_\varepsilon(\mathcal{C}_0)} - 2\#\mathcal{EC}_0 - \sum_{v \in \mathcal{V}_{\tilde{\mathcal{G}}_\varepsilon(\mathcal{C}_0)}} \deg_{\text{ext}}^{\mathcal{C}_0}(v) \right). \end{aligned}$$

Now using Lemma 3.6.1, we obtain

$$\begin{aligned} &\sum_{v \in \mathcal{V}_{\tilde{\mathcal{G}}_\varepsilon(\mathcal{C}_0)}} K_{\tilde{\mathcal{G}}_\varepsilon}(v) \\ &= \frac{\pi}{3} \left(6\#\mathcal{V}_{\tilde{\mathcal{G}}_\varepsilon(\mathcal{C}_0)} - \sum_{v \in \mathcal{V}_{\tilde{\mathcal{G}}_\varepsilon(\mathcal{C}_0)}} \deg_{\text{ext}}^{\mathcal{C}_0}(v) - 2 \left(3\#\mathcal{V}_{\tilde{\mathcal{G}}_\varepsilon(\mathcal{C}_0)} + 3 - \text{Perim}(\mathcal{C}_0) \right) \right) \\ &= \frac{\pi}{3} \left(2\text{Perim}(\mathcal{C}_0) - 6 - \sum_{v \in \mathcal{V}_{\tilde{\mathcal{G}}_\varepsilon(\mathcal{C}_0)}} \deg_{\text{ext}}^{\mathcal{C}_0}(v) \right). \end{aligned}$$

This completes the proof. □

For every vertex v on the boundary of $\mathcal{C}_0$, let

$$K_{\text{ext}}^{\mathcal{C}_0}(v) := \frac{2\pi}{3} - \frac{\pi}{3} \deg_{\text{ext}}^{\mathcal{C}_0}(v).$$

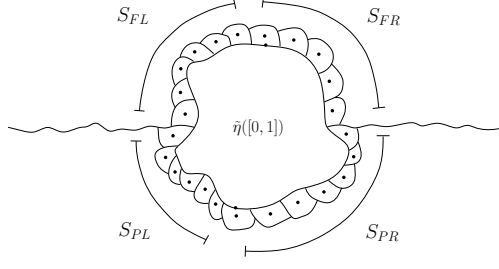


Figure 3.10: Splitting the boundary of a space-filling SLE segment.

Then by Proposition 3.6.2, we have

$$K_{\tilde{\mathcal{G}}_\varepsilon}(\mathcal{C}_0) = \sum_{v \in \partial_{\tilde{\mathcal{G}}_\varepsilon} \mathcal{C}_0} K_{\text{ext}}^{\mathcal{C}_0}(v) - 2\pi \quad (3.69)$$

where $\partial_{\tilde{\mathcal{G}}_\varepsilon} \mathcal{C}_0 := \{v \in \tilde{\mathcal{G}}_\varepsilon(\mathcal{C}_0) : \deg_{\text{ext}}(v) > 0\}$. We can further split this sum. Let

$\mathfrak{C}_{PL}, \mathfrak{C}_{PR}, \mathfrak{C}_{FL}, \mathfrak{C}_{FR}$ denote the past-left, past-right, future-left, future-right components of the boundary of $\mathcal{C}_0$, and let $S^{PL}, S^{PR}, S^{FL}, S^{FR}$ denote the analogous components of $\partial_{\tilde{\mathcal{G}}_\varepsilon} \mathcal{C}_0$ (see Figure 3.10). More precisely, let

$$\mathfrak{C}_{PL} = \eta_0^L \cap \{\text{left boundary of } \tilde{\eta}((-\infty, 0])\},$$

$$\mathfrak{C}_{PR} = \eta_0^R \cap \{\text{right boundary of } \tilde{\eta}((-\infty, 0])\},$$

$$\mathfrak{C}_{FL} = \eta_1^L \cap \{\text{left boundary of } \tilde{\eta}([1, \infty))\},$$

$$\mathfrak{C}_{FR} = \eta_1^R \cap \{\text{right boundary of } \tilde{\eta}([1, \infty))\},$$

and let $S^{PL}, S^{PR}, S^{FL}, S^{FR}$ be the sets of cells sharing a nontrivial arc with the corresponding piece of the boundary of C , with vertices whose corresponding cells share a nontrivial boundary arc with $\mathfrak{C}_{PL}, \mathfrak{C}_{PR}$ (resp. $\mathfrak{C}_{FL}, \mathfrak{C}_{FR}$) are assigned to one of the sets S^{PL}, S^{PR} (resp. S^{FL}, S^{FR}) arbitrarily. Then we claim the following.

Proposition 3.6.3. *There exist independent Brownian motions $B_{FL}, B_{FR}, B_{PL}, B_{PR}$ that*

are also independent of $\tilde{\Phi}, \tilde{\eta}$ and a deterministic constant $\alpha > 0$ such that we have the following convergences in law:

$$\varepsilon^{\frac{1}{4}} \sum_{v \in S_q} K_{\text{ext}}^{\mathcal{C}_0}(v) \rightarrow \alpha B_q(\nu_{\tilde{\Phi}}(\mathfrak{C}_q)) \quad \forall q \in \{PL, PR, FL, FR\} \quad (3.70)$$

where $\nu_{\tilde{\Phi}}$ denotes the LQG length.

It may be surprising that the Brownian motions $B_{PL}, B_{PR}, B_{FL}, B_{FR}$ are independent of $\tilde{\Phi}, \eta$. We will show in the proof that the left hand side of (3.70) depends only on the configuration of the cells in a tiny neighborhood of $\partial\mathcal{C}_0$, and this becomes independent of L, R (and therefore of Φ, η) since L, R can be approximated by processes that are exactly independent of Φ, η . Assuming this proposition we prove Theorem 3.1.7.

Proof of Theorem 3.1.7. Adding up (3.70) for $q \in \{PL, PR, FL, FR\}$ we obtain that

$$\begin{aligned} & \varepsilon^{-\frac{1}{4}} \sum_{v \in \partial_{\tilde{\mathcal{G}}_\varepsilon} \mathcal{C}_0} K_{\text{ext}}^{\mathcal{C}_0}(v) \\ & \rightarrow \alpha \left(B_{PL}(\nu_{\tilde{\Phi}}(\mathfrak{C}_{PL})) + B_{PR}(\nu_{\tilde{\Phi}}(\mathfrak{C}_{PR})) + B_{FL}(\nu_{\tilde{\Phi}}(\mathfrak{C}_{FL})) + B_{FR}(\nu_{\tilde{\Phi}}(\mathfrak{C}_{FR})) \right) \end{aligned}$$

Using (3.69) we obtain that

$$\begin{aligned} & \varepsilon^{\frac{1}{4}} K_{\tilde{\mathcal{G}}_\varepsilon}(\mathcal{C}_0) \\ & \rightarrow \alpha \left(B_{PL}(\nu_{\tilde{\Phi}}(\mathfrak{C}_{PL})) + B_{PR}(\nu_{\tilde{\Phi}}(\mathfrak{C}_{PR})) + B_{FL}(\nu_{\tilde{\Phi}}(\mathfrak{C}_{FL})) + B_{FR}(\nu_{\tilde{\Phi}}(\mathfrak{C}_{FR})) \right). \end{aligned}$$

Now recalling the fact that the laws of $B_{PL}, B_{PR}, B_{FL}, B_{FR}$ are independent Brownian motions by Proposition 3.6.3, we have that

$$B_{PL}(\nu_{\tilde{\Phi}}(\mathfrak{C}_{PL})) + B_{PR}(\nu_{\tilde{\Phi}}(\mathfrak{C}_{PR})) + B_{FL}(\nu_{\tilde{\Phi}}(\mathfrak{C}_{FL})) + B_{FR}(\nu_{\tilde{\Phi}}(\mathfrak{C}_{FR}))$$

and the random variable B in the statement of Theorem 3.1.7 agree in law. This completes the proof. $\square$

3.6.2 Proof of Proposition 3.6.3

For simplicity, suppose $q = FR$. Let $\beta : [0, \infty) \rightarrow \mathbb{C}$ be a parametrization of the right boundary of $\tilde{\eta}([0, \infty))$ according to LQG length, and let S_t be the set of vertices corresponding to cells adjacent to $\beta([0, t])$ contained in $\tilde{\eta}([0, \infty))$. We will now consider the cells intersecting $\beta([0, \infty))$ instead of only the right boundary of $\tilde{\eta}([0, 1])$. The reason for this is that we want to use the central limit theorem, and so it is more convenient to work up to infinity. First, we will need to define a few important quantities. For any positive integer k , let

$$\tau_k^+ := \inf\{t > 0 : R_t \leq -k\},$$

and similarly

$$\tau_k^- := \sup\{t < 0 : R_t \leq -k\}.$$

Let $\{\mathfrak{C}_i^+\}$ be the cells intersecting $\beta([0, \infty))$ contained in $\tilde{\eta}([0, \infty))$ and similarly let $\{\mathfrak{C}_i^-\}$ denote the cells intersecting $\beta([0, \infty))$ that are contained in $\tilde{\eta}((-\infty, 0])$, where $\{\mathfrak{C}_i^\pm\}$ are ordered as they are hit by β . For each cell $\mathfrak{C}_i^\pm$, we define the points $a_i^\pm$ such that $[a_i^\pm, a_{i+1}^\pm]$ is the only non singleton component of $\mathfrak{C}_i^\pm \cap \beta([0, \infty))$ (we remark that when $\kappa \geq 8$ we have $\mathfrak{C}_i^\pm \cap \beta([0, \infty)) = \beta([a_i^\pm, a_{i+1}^\pm])$). Define N_k by

$$N_k^+ := \#\{i : \beta(a_i^+) \in \tilde{\eta}([\tau_{k-1}^+, \tau_k^+])\}, \quad N_k^- := \#\{i : a_i^- \in \tilde{\eta}([\tau_k^-, \tau_{k-1}^-])\}. \quad (3.71)$$

We will prove a few properties about $N_k^\pm$, which will be needed in the proof of Proposition 3.6.3.

Lemma 3.6.4. N_k^+ and N_k^- are determined by

$$\begin{aligned} & \left((R_t - R_{\tau_k^+})|_{[\tau_{k-1}^+, \tau_k^+]}, \Lambda|_{[\tau_{k-1}^+, \tau_k^+]} - \tau_{k-1}^+ \right), \\ & \left((R_t - R_{\tau_k^-})|_{[\tau_k^-, \tau_{k-1}^-]}, \Lambda|_{[\tau_k^-, \tau_{k-1}^-]} - \tau_{k-1}^- \right) \end{aligned}$$

respectively. Moreover, $\{N_k^+\}$ and $\{N_k^-\}$ are each i.i.d. families.

Proof. We will show the first statement only for N_k^+ , the argument for N_k^- being identical. For the first statement, note that N_k^+ is the number of intervals of the form $[y_{j-1}, y_j]$ containing a running minimum of R contained in $[\tau_{k-1}^+, \tau_k^+]$. This in turn only depends on $(R - R_{\tau_k^+})|_{[\tau_{k-1}^+, \tau_k^+]}$, together with $\Lambda|_{[\tau_{k-1}^+, \tau_k^+]}$. The second statement follows from the fact that

$$\begin{aligned} & \left((R_t - R_{\tau_i^+})|_{[\tau_{i-1}^+, \tau_i^+]}, \Lambda|_{[\tau_{i-1}^+, \tau_i^+] \cup [\tau_i^-, \tau_{i-1}^-]} \right), \\ & \left((R_t - R_{\tau_j^+})|_{[\tau_{j-1}^+, \tau_j^+]}, \Lambda|_{[\tau_{j-1}^+, \tau_j^+] \cup [\tau_j^-, \tau_{j-1}^-]} \right) \end{aligned}$$

are independent for distinct i, j , and they determine $N_i^\pm, N_j^\pm$ respectively. $\square$

Lemma 3.6.5. Suppose that $\varepsilon = 1$, and let $N_k^\pm$ be defined as in (3.71). Then

$$\text{Var}(N_k^\pm) < \infty.$$

Proof. First let M denote the numbers of unit length intervals required to cover $\{t \in [0, \tau_1^+] : R_t = \inf_{0 \leq s \leq t} R_s\}$. We claim there exist constants C_1, C_2 such that

$$\mathbb{P}(M > A) \leq C_1 e^{-C_2 A}.$$

We will define a sequence of stopping times for the Brownian motion R_t . Let $r_0 = 0$, and

suppose that r_j, s_j have been defined for $0 \leq j \leq n-1$. We let

$$s_n = \min\{t : t \geq r_{n-1}, \inf_{0 \leq x \leq t} R_x = R_t\},$$

and

$$r_n = s_n + 1.$$

Let $Y_0 = \max\{n : s_n \leq \tau_1^+\}$. Then since the intervals $\{[s_n, r_n]\}_{n=1}^{Y_0}$ cover $\{t \in [0, \tau_1^+] : R_t = \inf_{0 \leq s \leq t} R_s\}$, we have that $M \leq Y_0$. Since $Y_0 \leq \tau_1^+$ and τ_1^+ has an exponential tail, this proves the claim. Note in particular that this implies $\mathbb{E}(M^2) < \infty$. Let $I_1, \dots, I_M$ be the intervals required in the definition of M . Letting $\Lambda := \Lambda^1$, note that conditioning on M , the collection $\{|\Lambda \cap I_j|\}_{j=1}^M$ is i.i.d., where $|\Lambda \cap I_j|$ has a Poisson distribution with parameter 1.

Therefore

$$N_1^+ \leq \sum_{j=1}^M \#\{j : [y_{j-1}, y_j] \cap I_j \neq \emptyset\} \leq \left(\sum_{j=1}^M \#(\Lambda \cap I_j) \right) + M.$$

Hence, for some absolute constant C we have

$$\begin{aligned} \text{Var}(N_1^+) \leq \mathbb{E}((N_1^+)^2) &\leq \mathbb{E} \left(\left(\sum_{j=1}^M |\Lambda \cap I_j| + M \right)^2 \right) \\ &\leq 2\mathbb{E} \left(\mathbb{E} \left(\left(\sum_{j=1}^M |\Lambda \cap I_j| \right)^2 \middle| M \right) \right) + 2\mathbb{E}(M^2) \leq \mathbb{E}(CM^2) < \infty, \end{aligned}$$

where in the second to last inequality we used Cauchy-Schwarz. $\square$

For any $t > 0$ let D_t denote the mated CRT map cell contained in $\tilde{\eta}([0, \infty))$ containing $\beta(t)$, and let $\bar{v}_t$ denote the vertex corresponding to the cell D_t . For any vertex $v \in [0, \infty)$, we also let $\deg_{\text{ext}}^{\tilde{\eta}([0, \infty))}(v)$ be the number of edges between v and a vertex in $(-\infty, 0]$, and define $K_{\text{ext}}^{\tilde{\eta}((0, \infty))}$ by

$$K_{\text{ext}}^{\tilde{\eta}((0, \infty))}(v) = 2 - \deg_{\text{ext}}^{\tilde{\eta}([0, \infty))}(v).$$

We will also need the following lemma.

Lemma 3.6.6. *Recall that $\beta : [0, \infty) \rightarrow \mathbb{C}$ is a unit $\nu_{\tilde{\Phi}}$ length parametrization of the right boundary of $\tilde{\eta}([0, \infty))$, and let S_t be the set of vertices corresponding to cells adjacent to $\beta([0, t])$ contained in $\tilde{\eta}([0, \infty))$. Then*

$$\left| \sum_{k \leq t} (N_k^+ - N_k^-) - \sum_{v \in S_t} K_{\text{ext}}^{\tilde{\eta}([0, \infty))}(v) \right| \leq \deg(\bar{v}_t) + 2.$$

Analogous statements hold if we instead use the left boundary of $\tilde{\eta}([0, \infty))$, the right boundary of $\tilde{\eta}((-\infty, 0])$, or the left boundary of $\tilde{\eta}((-\infty, 0])$.

Proof. We claim that

$$\left| \sum_{v \in S_t} \deg_{\text{ext}}^{\tilde{\eta}([0, \infty))}(v) - \sum_{k \leq t} (N_k^+ + N_k^-) \right| \leq \deg_{\text{ext}}^{\tilde{\eta}([0, \infty))}(\bar{v}_t). \quad (3.72)$$

Indeed,

$$\sum_{v \in S_t} \deg_{\text{ext}}^{\tilde{\eta}([0, \infty))}(v) = \sum_{\mathfrak{C}_1, \mathfrak{C}_2 \text{ are adjacent}} 1_{\mathfrak{C}_1, \mathfrak{C}_2 \text{ are adjacent}},$$

where the sum is taken over cells $\mathfrak{C}_1$ adjacent to $\beta([0, t])$ contained in $\tilde{\eta}([0, \infty))$ and cells $\mathfrak{C}_2$ adjacent to $\beta([0, t])$ contained in $\tilde{\eta}((-\infty, 0])$. Note that each pair of such adjacent cells $\mathfrak{C}_1, \mathfrak{C}_2$ corresponds to their shared boundary arc which is of the form (a, b) , where $a, b \in \{a_i^\pm\}_{i \geq 0} \cap [0, t]$ (see Figure 3.11). However, given that there could be intervals (a, b) of this form not corresponding to a pair of adjacent cells (if there is an i such that $a_{i_0}^+ = \max\{a_i^\pm\} \cap [0, t]$, then $N_k^+ + N_k^-$ counts the edges incident to $\mathfrak{C}_{i_0}^+$) we obtain (3.72).

Now, combining (3.72) with the fact that

$$\left| \sum_{v \in S_t} 2 - 2 \sum_{k \leq t} N_k^+ \right| \leq 2,$$

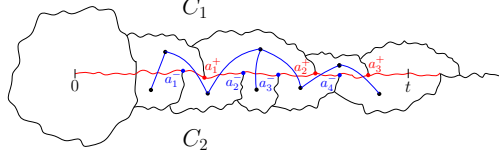


Figure 3.11: Correspondence between pairs of adjacent cells (represented by the edges in blue) and boundary arcs (drawn in red). The points a_i^+ are indicated in red, while the points a_i^- are indicated in blue.

we obtain that

$$\begin{aligned}
 \left| \sum_{v \in S_t} K_{\text{ext}}^{\tilde{\eta}([0, \infty))}(v) - \sum_{k \leq t} (N_k^+ - N_k^-) \right| &\leq \left| \sum_{v \in S_t} \deg_{\text{ext}}^{\tilde{\eta}([0, \infty))}(v) - \sum_{k \leq t} (N_k^+ + N_k^-) \right| \\
 &+ \left| \sum_{v \in S_t} 2 - 2 \sum_{k \leq t} N_k^+ \right| \\
 &\leq \deg_{\text{ext}}^{\tilde{\eta}([0, \infty))}(\bar{v}_t) + 2.
 \end{aligned}$$

□

Before we prove Proposition 3.6.3, we will need one more lemma. Let

$$X_t^\varepsilon = \sum_{v \in S_t} K_{\text{ext}}^{\tilde{\eta}([0, \infty))}(v).$$

Lemma 3.6.7. *There exists an absolute constant α such that $\varepsilon^{\frac{1}{4}} X_t^\varepsilon \xrightarrow{d} \alpha B(t)$ with respect to the local uniform topology on $[0, \infty)$, where B is a Brownian motion.*

Proof. Let

$$Y_t^\varepsilon = \frac{\pi}{3} \sum_{k \leq t} (N_k^+ - N_k^-).$$

We claim that there is an absolute constant α such that

$$\varepsilon^{\frac{1}{4}} Y_t^\varepsilon \rightarrow \alpha B(t) \tag{3.73}$$

in distribution as $\varepsilon \rightarrow 0$, with respect to the local uniform topology. By rescaling, to show this claim it suffices to show that there is an absolute constant α such that if $\varepsilon = 1$, then

$$\frac{Y_{nt}^1}{n^{\frac{1}{2}}} \xrightarrow{d} \alpha B(t) \quad (3.74)$$

as $n \rightarrow \infty$, uniformly over compact sets, where B is a Brownian motion. Indeed, if this convergence holds then taking $n = \varepsilon^{-\frac{1}{2}}$ we obtain

$$\varepsilon^{\frac{1}{4}} Y_{\left(\varepsilon^{-\frac{1}{2}}t\right)}^1 \xrightarrow{d} \alpha B(t),$$

and now since $Y_{\varepsilon^{-\frac{1}{2}}t}^1$ has the same law as Y_t^ε we conclude

$$\varepsilon^{\frac{1}{4}} Y_t^\varepsilon \xrightarrow{d} \alpha B(t).$$

Suppose momentarily that we have proven (3.74) and thus (3.73). Recalling Lemma 3.6.6, note that

$$\left| X_t^\varepsilon - \frac{\pi}{3} \sum_{k \leq t} (N_k^+ - N_k^-) \right| \leq \deg(\bar{v}_t) + 2.$$

Now using Lemma 3.4.3, we have that with probability tending to 1 as $\varepsilon \rightarrow 0$, we have

$$\left| X_t^\varepsilon - \frac{\pi}{3} \sum_{k \leq t} (N_k^+ - N_k^-) \right| \lesssim \left(\log \varepsilon^{-1} \right)^{-6}.$$

Combining this with (3.73) we obtain

$$\varepsilon^{\frac{1}{4}} X_t^\varepsilon = \varepsilon^{\frac{1}{4}} Y_t^\varepsilon + o_\varepsilon(1) \xrightarrow{d} \alpha B_{PL}(t).$$

It remains to prove (3.74). From now on we will assume that $\varepsilon = 1$. Note that by symmetry

we have that

$$\mathbb{E}(N_k^+ - N_k^-) = 0.$$

Therefore using Donsker's theorem together with Lemma 3.6.4 and 3.6.5, we obtain that

$$n^{-\frac{1}{2}} \frac{\pi}{3} \sum_{k \leq nt} (N_k^+ - N_k^-) = \frac{\pi}{3} n^{-\frac{1}{2}} \sum_{k \leq nt} (N_k^+ - N_k^- - \mathbb{E}(N_k^+ - N_k^-)) \rightarrow \alpha B_{PL}(t), \quad (3.75)$$

where

$$\alpha = \frac{\pi}{3} \sqrt{\text{Var}(N_k^+ - N_k^-)}.$$

□

Now we are ready to prove Proposition 3.6.3.

Proof of Proposition 3.6.3. Step 1: Approximations of the Brownian motions L, R . By taking $t = \nu_{\tilde{\Phi}}(\mathfrak{C}_q)$ and using that $S_{\nu_{\tilde{\Phi}}(\mathfrak{C}_q)} = S_q$ in Lemma 3.6.7 we obtain (3.70). The final step is to show that the Brownian motions $B_{PL}, B_{PR}, B_{FL}, B_{FR}$ are all independent, and independent of $\tilde{\Phi}, \tilde{\eta}$ (equivalently, from (L, R)). To show that the limiting Brownian motion B_{PR} is independent from L, R , the basic idea is to approximate each of L, R , and X^ε (as defined just above Lemma 3.6.7) by processes which are exactly independent from each other. This can be done since X^ε depends only on the behavior of L and R in a small neighborhood of the set of times when R attains a record minimum run forward or backward from time 0.

To carry this out precisely, let $\delta > 0$, and let $w_0^+ = 0$. We define w_k^+ by

$$w_k^+ = \inf \left\{ t \geq w_{k-1}^+ + \delta : R_t = \inf_{0 \leq s \leq t} R_s \right\}. \quad (3.76)$$

We recall that $\Lambda^\varepsilon = \{y_k\}$. We let

$$\tilde{S}_t := \{y_k \in S_t : [y_{k-1}, y_k] \subseteq [w_j^+, w_j^+ + \delta] \text{ for some } j\}$$

and we define

$$\tilde{X}_t^\varepsilon := \sum_{v \in \tilde{S}_t} \left(2 - \widetilde{\deg}_{\text{ext}}(v) \right),$$

where $\widetilde{\deg}_{\text{ext}}(v)$ is defined as

$$\begin{aligned} & \widetilde{\deg}_{\text{ext}}(v) \\ &:= \#\{i : \beta([a_i^-, a_{i+1}^-]), H_v^\varepsilon \text{ share a nontrivial boundary arc,} \\ & \quad [a_i^-, a_{i+1}^-] \subseteq [w_j^+, w_j^+ + \delta] \text{ for some } j\}, \end{aligned}$$

that is, $\widetilde{\deg}_{\text{ext}}(v)$ is the number of cells adjacent to both $\beta([0, \infty))$ and the cell H_v^ε corresponding to v , such that its intersection with $\beta([0, \infty))$ is contained in an interval of the form $[w_j^+, w_j^+ + \delta]$.

Step 2: dependence of $\tilde{X}^\varepsilon$. We claim that $\tilde{X}_t^\varepsilon|_{0 \leq t \leq T}$ is determined by

$$\begin{aligned} & \left\{ (R_t - R_{w_k^+})|_{t \in [w_k^+, w_k^+ + \delta]}, (\Lambda_\varepsilon \cap [w_k^+, w_k^+ + \delta]) - w_k^+ \right\}_{k: 0 \leq w_k^+ \leq T} \\ & \left\{ (R_t - R_{w_k^-})|_{t \in [w_k^-, w_k^- + \delta]}, (\Lambda_\varepsilon \cap [w_k^-, w_k^- + \delta]) - w_k^- \right\}_{k: 0 \leq w_k^- \leq T} \end{aligned} \quad (3.77)$$

with probability going to 1 as $\varepsilon \rightarrow 0$. More precisely, there is an event $E_{\varepsilon, \delta}$ such that $X_t^\varepsilon|_{0 \leq t \leq T} 1_{E_{\varepsilon, \delta}}$ is determined by (3.77) and $\mathbb{P}(E_{\varepsilon, \delta}) \rightarrow 1$ as $\varepsilon \rightarrow 0$. Indeed, $\{\tilde{X}_t^\varepsilon\}_{0 \leq t \leq T}$ is determined by $\widetilde{\deg}_{\text{ext}}(v)$ for $v \in \tilde{S}_T$. Recall that $\Lambda^\varepsilon = \{y_k\}_{k \in \mathbb{Z}}$. Note that if for all k such that $0 \leq y_k \leq T$ we have $|y_k - y_{k-1}| \leq \delta$, then

$$\left\{ t : R_t = \inf_{0 \leq s \leq t} R_s \right\} \subseteq \bigcup_{k: 0 \leq w_k^+ \leq T} [w_k^+, w_k^+ + \delta]. \quad (3.78)$$

Let $E_{\varepsilon, \delta}$ be the event that

$$\sup_{k: k \geq 0, y_k \leq T} |y_k - y_{k-1}| \leq \delta.$$

Then $\mathbb{P}(E_{\varepsilon,\delta}) \rightarrow 1$ as $\varepsilon \rightarrow 0$. Moreover, if $E_{\varepsilon,\delta}$ holds, then $\{\widetilde{\deg_{\text{ext}}}(v)\}_{v \in S_T}$ is determined by R 's running minima in $[w_k^+, w_k^+ + \delta] \cup [w_k^-, w_k^- + \delta]$ and $(\Lambda_\varepsilon \cap [w_k^+, w_k^+ + \delta]) - w_k^+$, $(\Lambda_\varepsilon \cap [w_k^-, w_k^- + \delta]) - w_k^-$ and thus by (3.78), $\tilde{X}_t^\varepsilon|_{0 \leq t \leq T}$ is determined by (3.77), if $E_{\varepsilon,\delta}$ holds.

Step 3: $\tilde{X}_\varepsilon$ approximates X_ε . Now let

$$Z_t := (L_t, R_t).$$

Since $\{w_k^+\}$ is a collection of stopping times, if we let $\bar{K}(t)$ be the random variable defined by

$$\bar{K}(t) := \#\{k : w_k^+ \leq t\},$$

then we have

$$\sum_{k:w_k^+ \leq T} (Z_{w_k^+ + \delta} - Z_{w_k^+})$$

is stochastically dominated by $B_{\bar{K}(T)\delta}$, where B is a two dimensional correlated Brownian motion independent of $\bar{K}(T)$. Therefore

$$\sup_{0 \leq t \leq T} \left| \sum_{k:w_k^+ \leq t} (Z_{w_k^+ + \delta} - Z_{w_k^+}) \right| \stackrel{d}{=} \sup_{0 \leq t \leq T} |B_{\bar{K}(t)\delta}|.$$

Note that

$$\begin{aligned} \mathbb{P}(|B_{\bar{K}(t)\delta}| > C\delta^{\frac{1}{4}}) &= \sum_{k \geq 0} \mathbb{P}(|B_{k\delta}| > C\delta^{\frac{1}{4}}) \mathbb{P}(\bar{K}(t) = k) \\ &= \sum_{k \geq 0} \mathbb{P}(|B_1| > C\delta^{-\frac{1}{4}} k^{-\frac{1}{2}}) \mathbb{P}(\bar{K}(t) = k) \\ &\leq \sum_{k \leq C\delta^{-\frac{1}{2}}} \mathbb{P}(|B_1| > C^{\frac{1}{2}}) \mathbb{P}(\bar{K}(t) = k) + \mathbb{P}(\delta^{\frac{1}{2}} \bar{K}(t) > C) = o_\delta(1) \end{aligned}$$

where $C = C(\delta)$ is chosen so that $C(\delta) \rightarrow \infty$ slow enough. Hence if we let

$$\tilde{Z}_t^\delta := \sum_{k=0}^{\bar{K}(t)} \left(Z_{w_{k+1}^+} - Z_{w_k^+ + \delta} \right)$$

then $\tilde{R}_t^\delta \rightarrow R_t$ in law as $\delta \rightarrow 0$, since with probability going to 1 as $\delta \rightarrow 0$, we have

$$\sup_{0 \leq t \leq T} |Z_t - \tilde{Z}_t^\delta| \leq o_\delta(1) \quad (3.79)$$

(this is because of the fact that $Z_{w_k^+ + \delta} - Z_{w_k^+}$ is of order $\delta^{\frac{1}{2}}$ and the fact that the set of running minima of a Brownian motion has Minkowski dimension $\frac{1}{2}$). Finally, note that

$$|\tilde{X}_t^\varepsilon - X_t^\varepsilon| \lesssim \delta^{-\frac{1}{2}} (\log \varepsilon^{-1})^6 \quad (3.80)$$

with probability going to 1 as $\varepsilon \rightarrow 0$. Indeed, since every cell adjacent to $\beta([0, \infty))$ not in $\{H_{w_k^+}^\varepsilon\}$ is contained in an interval of the form $[w_j^+, w_j^+ + \delta]$ (see Figure 3.12), we can write

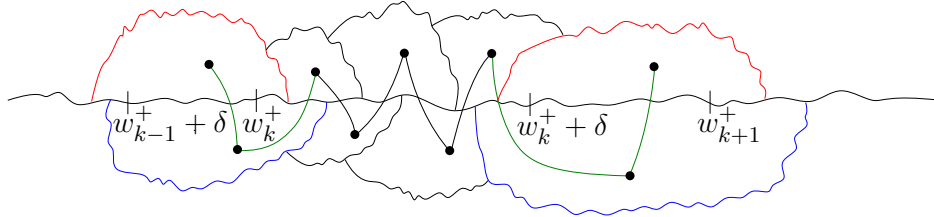


Figure 3.12: The red cells and blue are counted in X_t^ε , but not in $\tilde{X}_t^\varepsilon$. The green edges are those "lost" when computing $\tilde{X}_t^\varepsilon$.

$$\begin{aligned}
|X_t^\varepsilon - \tilde{X}_t^\varepsilon| &\leq \sum_{k=0}^{\bar{K}(t)} \deg_{\text{ext}}(\bar{v}_{w_k^+}) + \sum_{v \in \tilde{S}(t)} |\widetilde{\deg_{\text{ext}}}(v) - \deg_{\text{ext}}(v)| \\
&\lesssim \sum_{v \in \tilde{S}(t)} |\widetilde{\deg_{\text{ext}}}(v) - \deg_{\text{ext}}(v)| + \bar{K}(t)(\log \varepsilon^{-1})^6 \\
&\lesssim \sum_{v \in \tilde{S}(t)} |\widetilde{\deg_{\text{ext}}}(v) - \deg_{\text{ext}}(v)| + \delta^{-\frac{1}{2}}(\log \varepsilon^{-1})^6, \tag{3.81}
\end{aligned}$$

where again we used Lemma 3.4.4 and the fact that $\bar{K}(t) \lesssim \delta^{-\frac{1}{2}}$ due to the fact that the set of running minima of a Brownian motion has Minkowski dimension $\frac{1}{2}$. Now note that

$$\begin{aligned}
&|\widetilde{\deg_{\text{ext}}}(v) - \deg_{\text{ext}}(v)| \\
&\leq \#\{i : \beta([a_i^-, a_{i+1}^-]), H_v^\varepsilon \text{ share a nontrivial boundary arc,} \\
&\quad \nexists j : [a_i^-, a_{i+1}^-] \subseteq [w_j^+, w_j^+ + \delta]\},
\end{aligned}$$

and so if we let $S_0(t)$ denote the set of vertices corresponding to cells adjacent to $\beta([0, t])$ contained in $\tilde{\eta}((-\infty, 0])$, and $\deg_0(v)$ denote the number of cells contained in $\tilde{\eta}([0, \infty))$ adjacent to both $\beta([0, \infty))$ and the cell corresponding to v , proceeding as in (3.81) we have

$$|X_t^\varepsilon - \tilde{X}_t^\varepsilon| \lesssim \sum_{v \in S_0(t)} \deg_0(v) + \delta^{-\frac{1}{2}}(\log \varepsilon^{-1})^6 \lesssim \delta^{-\frac{1}{2}}(\log \varepsilon^{-1})^6.$$

Step 4: Dependence of $\tilde{Z}_t^\delta$. Moreover, note that $\tilde{Z}_t^\delta$ depends only on the collection

$(Z_t - Z_{w_k^+})|_{t \in [w_k^+, w_k^+ + \delta]}$ for $k \geq 0$, and on the other hand that X^ε is independent of these increments. Therefore noting that $(\tilde{Z}^\delta, X^\varepsilon)$ converges to $(\tilde{Z}^\delta, X)$ as $\varepsilon \rightarrow 0$ we see that $\tilde{Z}^\delta$ is independent of X . Now using (3.79) together with Lemma 3.6.7 and taking the limit $\delta \rightarrow 0$ we obtain the result. We have shown that $(L, R, \varepsilon^{1/4} X^\varepsilon)$ converges in distribution as $\varepsilon \rightarrow 0$ to $(L, R, \alpha B_{PR})$, where B_{PR} is a Brownian motion independent from (L, R) , and where

α is as in Lemma 3.6.7. It remains to prove a joint convergence statement for processes associated with four different boundary arcs.

Step 5: Independence of X^ε from Brownian motions. Write $A_{PR}^\varepsilon = X^\varepsilon$. Define A_{PL}^ε analogously to X^ε but with the roles of L and R interchanged. Define A_{FR}^ε and A_{FL}^ε analogously to A_{PR}^ε and A_{PL}^ε but with $\tilde{\eta}((-\infty, 0])$ and $\tilde{\eta}([0, \infty))$ replaced by $\tilde{\eta}([1, \infty))$ and $\tilde{\eta}((-\infty, 1])$, respectively. We claim that

$$(L, R, \varepsilon^{1/4} A_{PR}^\varepsilon, \varepsilon^{1/4} A_{PL}^\varepsilon, \varepsilon^{1/4} A_{FR}^\varepsilon, \varepsilon^{1/4} A_{FL}^\varepsilon) \xrightarrow{d} (L, R, \alpha B_{PR}, \alpha B_{PL}, \alpha B_{FR}, \alpha B_{FL}) \quad (3.82)$$

where the conditional law of $(B_{PR}, B_{PL}, B_{FR}, B_{FL})$ given (L, R) is described as follows.

- The conditional law of (B_{PR}, B_{FR}) is that of a pair of Brownian motions coupled together so that that $B_{PR}|_{[0, \nu_{\tilde{\Phi}}(\mathfrak{C}_{PR})]}$ and $B_{FR}|_{[0, \nu_{\tilde{\Phi}}(\mathfrak{C}_{FR})]}$ are independent and $B_{PR}(t + \nu_{\tilde{\Phi}}(\mathfrak{C}_{PR})) - B_{PR}(\nu_{\tilde{\Phi}}(\mathfrak{C}_{PR})) = B_{PR}(t + \nu_{\tilde{\Phi}}(\mathfrak{C}_{FR})) - B_{FR}(\nu_{\tilde{\Phi}}(\mathfrak{C}_{FR}))$ for each $t \geq 0$.
- The conditional law of (B_{PL}, B_{FL}) is described in the same way but with L in place of R throughout.
- (B_{PR}, B_{FR}) and (B_{PL}, B_{FL}) are conditionally independent.

We will not provide a fully detailed proof of (3.82) since the argument uses very similar ideas to the proof of the convergence $(L, R, X^\varepsilon) \xrightarrow{d} (L, R, \alpha B_{PR})$. The idea of the proof is as follows. Let $\delta > 0$. Let $w_{k, PR}^\pm = w_k^\pm$ for $k \geq 0$ and let $\tilde{A}_{PR}^\varepsilon(t) = \tilde{X}_t^\varepsilon$, defined as above. Define $w_{k, FR}^\pm, w_{k, PL}^\pm, w_{k, FL}^\pm$ and $\tilde{A}_{FR}^\varepsilon, \tilde{A}_{PL}^\varepsilon, \tilde{A}_{FL}^\varepsilon$ in an analogous manner.

We claim that if $\delta > 0$ is small, it holds with high probability that the number of pairs of indices j, k such that the following is true is of order $o(\delta^{-1})$: the intervals $[w_{j, PR}^\pm, w_{j, PR}^\pm + \delta]$ and $[w_{k, PL}^\pm, w_{k, PL}^\pm + \delta]$ have non-empty intersection and the images under $\tilde{\eta}$ of each of these intervals intersects $\tilde{\eta}([0, 1])$. Moreover the same holds with PR, PL replaced by any two distinct elements of $\{PR, FR, PL, FL\}$. Indeed, in the case when $\kappa \geq 8$ ($\gamma \leq \sqrt{2}$), this

follows from the continuity of $\tilde{\eta}$ and the fact that the intersection of the left and right boundaries of $\tilde{\eta}([0, \infty))$ is a.s. the single point $\{0\}$, and similarly for $\tilde{\eta}([1, \infty))$. When $\kappa \in (4, 8)$, the left and right boundaries of $\tilde{\eta}([0, \infty))$ a.s. intersect in an uncountable set. But, the set of times when $\tilde{\eta}$ hits this set has the law of the range of a stable subordinator of index $1 - \kappa/8 < 1/2$ (see [Gwynne and Miller, 2021, Lemma 3.5]), so set of times has Minkowski dimension strictly smaller than $1/2$.

Step 6: Independence of L and R . Using the claim in the previous paragraph and similar arguments as in the proof of the convergence $(L, R, X^\varepsilon) \xrightarrow{d} (L, R, \alpha B_{PR})$, we can define further approximations $\mathring{L}^\delta, \mathring{R}^\delta, \mathring{A}_{PR}^\varepsilon, \mathring{A}_{FR}^\varepsilon, \mathring{A}_{PL}^\varepsilon, \mathring{A}_{FL}^\varepsilon$ of which are close to $L, R, A_{PR}^\varepsilon, A_{FR}^\varepsilon, A_{PL}^\varepsilon, A_{PR}^\varepsilon$, respectively, with high probability when $\varepsilon \rightarrow 0$ and then $\delta \rightarrow 0$ (in similar sense to (3.79), (3.80)) (depending on ε and δ) which exactly satisfy a version of the independence properties stated just after (3.82) (since the only source of dependence is the interfaces between adjacent cells). One can define stopping times $\{\mathring{w}_k\}$ analogous to the times $\{w_k^+\}$ defined in (3.76) and define

$$\mathring{L}_t^\delta := \sum_{k=0}^{\bar{K}(t)} \left(L_{\mathring{w}_{k+1}} - L_{\mathring{w}_k + \delta} \right), \quad \mathring{R}_t^\delta := \sum_{k=0}^{\bar{K}(t)} \left(R_{\mathring{w}_{k+1}} - R_{\mathring{w}_k + \delta} \right)$$

and similarly for $\mathring{A}_q^\varepsilon$ with $q \in \{PR, FR, PL, PR\}$. This completes the proof of (3.82), which in turn implies the proposition statement. $\square$

CHAPTER 4

CHANNELS OF ENERGY FOR THE LINEARIZED ENERGY CRITICAL WAVE EQUATION IN EVEN DIMENSIONS $N \geq 8$

4.1 Introduction

We consider radial solutions to the linearized energy critical wave equation around the ground state in even dimensions N ,

$$\begin{cases} \partial_{tt}u_L - \Delta u_L + V u_L = 0 \\ u_L|_{t=0} = (u_0, u_1) \end{cases} \quad (4.1)$$

with initial conditions $(u_0, u_1) \in \mathcal{H} = \mathcal{H}^1 \times L^2(\mathbb{R}^N)$, where V is the potential given by

$$V = -\frac{N+2}{N-2}W^{\frac{4}{N-2}},$$

with

$$W = \left(1 + \frac{|x|^2}{N(N-2)}\right)^{-\frac{N-2}{2}}.$$

This is the linearized equation for

$$\partial_t^2 u - \Delta u = |u|^{\frac{4}{N-2}}u$$

around the stationary solutions $\pm W$. By standard arguments, (4.1) is well posed in $\dot{H}^1 \times L^2$ and moreover the outer radiated energy

$$E_{\text{out}} := E_{\text{out}}^- + E_{\text{out}}^+, \quad \lim_{t \rightarrow \pm\infty} \int_{|x| > |t|} |\nabla_{t,x} u_L(t, x)|^2 dx$$

is well defined for $t \rightarrow \pm\infty$.

For the free wave equation,

$$\begin{cases} \partial_t^2 u_F - \Delta u_F = 0 \\ (u_F, \partial_t u_F)|_{t=0} = (u_0, u_1) \end{cases}$$

one has the channels of energy estimate

$$\|(u_0, u_1)\|_{\dot{H}^1 \times L^2} \lesssim \sqrt{E_{\text{out}}} \quad (4.2)$$

if N is odd. However, in the case that N is even the full estimate fails (Côte et al. [2014]), instead we have

$$\|u_0\|_{\dot{H}^1} \lesssim \sqrt{E_{\text{out}}}$$

if $N \equiv 0 \pmod{4}$, and

$$\|u_1\|_{L^2} \lesssim \sqrt{E_{\text{out}}}$$

if $N \equiv 2 \pmod{4}$. Extensions to domains of the form $\{|x| \geq R + |t|\}$ (obtained in Duyckaerts et al. [2022], Kenig et al. [2015], Li et al. [2024]) will also be used in the present work. Let $\Lambda = \frac{N-2}{2} + x \cdot \nabla$ denote the scaling generator. For the equation (4.1), there are two natural counterexamples to the analogue of (4.2), namely ΛW , $t\Lambda W$. For odd dimensions N , the analogue of (4.2) is

$$\|\Pi_{\dot{H}^1}^\perp u_0\|_{\dot{H}^1} + \|\Pi_{L^2}^\perp u_0\|_{L^2} \lesssim \sqrt{E_{\text{out}}} \text{ if } N \geq 3 \text{ is odd}$$

where the projections $\Pi_{\dot{H}^1}^\perp, \Pi_{L^2}^\perp$ are defined by

$$\Pi_{\dot{H}^1}^\perp = \Pi_{\dot{H}^1}(\text{Span}(\Lambda W))^\perp, \quad \Pi_{L^2}^\perp = \Pi_{L^2}(\text{Span}(\Lambda W))^\perp. \quad (4.3)$$

Here, if H is any Hilbert space and E any closed subspace, we let $\Pi_H(E)$ denote the orthog-

onal projection onto E in H and we let $\Pi_H(E)^\perp = 1 - \Pi_H(E)$.

In this chapter, we will restrict to even dimensions N . We will need to define logarithmic weakenings of L^2 . Let Z_α be the space defined by the norm

$$\|f\|_{Z_\alpha} := \sup_{R>0} \frac{R^{-\frac{N}{2}-\alpha}}{\langle \log R \rangle} \left(\int_{R<|x|<2R} f^2(x) dx \right)^{\frac{1}{2}}.$$

We will also need the norm

$$\|f\|_{Z_{\alpha,\lambda}} = \sup_{R>0} \frac{R^{-3-\alpha}}{\inf_{1 \leq j \leq J} \left\langle \frac{R}{\lambda_j} \right\rangle} \left(\int_{R \leq |x| \leq 2R} f^2 dx \right)^{\frac{1}{2}},$$

where $\lambda = (\lambda_1, \dots, \lambda_J)$ is such that $\lambda_1 \geq \dots \geq \lambda_J$.

We have the following result.

Theorem 4.1.1. *There exists a constant C such that for any radial solution u_L of (4.1.1) we have*

$$\|\Pi_{\dot{H}^1}^\perp u_0\|_{\dot{H}^1} + \|\Pi_{L^2}^\perp u_1\|_{Z_{-3}} \leq C\sqrt{E_{\text{out}}},$$

if $N \equiv 0 \pmod{4}$,

$$\|\Pi_{L^2}^\perp u_1\|_{L^2} + \|\nabla \Pi_{\dot{H}^1}^\perp u_0\|_{Z_{-3}} \leq C\sqrt{E_{\text{out}}},$$

if $N \equiv 2 \pmod{4}$.

We will also consider the linearized energy critical wave equation around a multisoliton. To define this, let $J \in \mathbb{N}$ and let

$$\Lambda_J := \{\lambda = (\lambda_1, \dots, \lambda_J) \in (0, \infty)^J, \lambda_J < \lambda_{J-1} < \dots < \lambda_1\}.$$

Let

$$V_\lambda = \sum_{j=1}^J V_{\{\lambda_j\}},$$

where $f_{\{\lambda\}}$ denotes the scaling

$$f_{\{\lambda\}}(x) := \frac{1}{\lambda^2} f\left(\frac{x}{\lambda}\right).$$

We define the scalings $f_{(\lambda)}$ as the $\dot{H}^1$ critical scaling

$$f_{(\lambda)}(x) = \frac{1}{\lambda^{\frac{N-2}{2}}} f\left(\frac{x}{\lambda}\right).$$

We define the scalings $f_{[\lambda]}$ as the L^2 critical scaling

$$f_{[\lambda]}(x) = \frac{1}{\lambda^{\frac{N}{2}}} f\left(\frac{x}{\lambda}\right).$$

We will also consider the equation

$$\begin{cases} \partial_{tt} u_L - \Delta u_L + V_\lambda u_L = 0, \\ u_L|_{t=0} = (u_0, u_1), \end{cases} \quad (4.4)$$

which is the energy critical wave equation linearized around a multisoliton.

Let $\gamma(\lambda)$ be defined by

$$\gamma(\lambda) = \sup_{1 \leq j \leq J-1} \frac{\lambda_j^2}{\lambda_{j+1}^2}.$$

We have the following statement for the multisoliton case.

Theorem 4.1.2. *There exists a dimensional constant C such that for any radial solution u_L of (4.4) we have*

$$\|\Pi_{\dot{H}^1}^\perp u_0\|_{L^2} + \|\Pi_{L^2}^\perp u_1\|_{Z_{-3,\lambda}} \leq C \left(\sqrt{E_{\text{out}}} + \gamma(\lambda) \|(u_0, u_1)\|_{\dot{H}^1 \times L^2} \right),$$

if $N \equiv 0 \pmod{4}$,

$$\|\Pi_{L^2}^\perp u_1\|_{L^2} + \|\nabla \Pi_{\dot{H}^1}^\perp u_0\|_{Z_{-3,\lambda}} \leq C \left(\sqrt{E_{\text{out}}} + \gamma(\lambda) \|(u_0, u_1)\|_{\dot{H}^1 \times L^2} \right),$$

if $N \equiv 2 \pmod{4}$.

Theorems 4.1.1 and 4.1.2 were proven in Collot et al. [2023] for the case of dimensions $N = 6, 8$. However, in higher dimensions, substantial additional work is needed. In the one soliton case, to bound $\|\Pi_{\dot{H}^1}^\perp u_0\|_{\dot{H}^1}$ if $N \equiv 0 \pmod{4}$ (respectively $\|\Pi_{L^2}^\perp u_0\|_{L^2}$ if $N \equiv 2 \pmod{4}$) one needs to use a channels of energy estimate for the free wave equation combined with a contradiction argument to show that any asymptotic counterexample to Theorem 4.1.1 is trivial (see Lemma 4.4.5). Additionally, in the multisoliton case (Theorem 4.1.2), one proceeds by induction on the number of solitons. An additional argument is used in the analogous contradiction argument to control the orthogonal projection onto the span of the generalized eigenfunctions of $-\Delta + V$ corresponding to ΛW .

Theorem 4.1.2 will be used in Contreras Hip [2024+] to provide an alternative proof to that of Jendrej and Lawrie in Jendrej and Lawrie [2025] of the soliton resolution for radial equivariant wave maps on $\mathbb{S}^2$ in even dimensions.

Acknowledgements. The author would like to thank Carlos Kenig for suggesting the problem, useful discussions and comments, and for carefully reading earlier versions of this manuscript.

4.2 More general nonlinearities

Our main result also holds for other similar equations. Let us consider the equation

$$\partial_t^2 u - \Delta u + \varphi(r, u) = 0 \tag{4.5}$$

for $r \in \mathbb{R}^+$, $t \in \mathbb{R}$. Suppose that φ is of the form

$$\varphi(r, u) = \sum_{k=2}^{\infty} \varphi_k r^{(k-1)\left(\frac{N}{2}-1\right)-2} u^k, \quad (4.6)$$

where

$$\lim_{k \rightarrow \infty} \tau^k \varphi_k = 0$$

for any $\tau > 0$. We will make some additional assumptions:

(A1) If u is a solution to (4.5), then so is $u_\lambda(r, t) = \lambda^{\frac{N-2}{2}} u(\lambda r, \lambda t)$.

(A2) All static radial solutions $U(r)$ of (4.5) verify

$$\partial_r^\alpha U(r) \sim r^{-(N-2)-\alpha}$$

as $r \rightarrow \infty$ for $\alpha \in \mathbb{N}$.

One can then consider the equation (4.5) linearized around $U(r)$,

$$\partial_t^2 u - \Delta u + \partial_2 \varphi(r, U) u = 0, \quad (4.7)$$

where ∂_2 denotes the derivative with respect to the second variable of φ and where U is any static solution to (4.5).

We have the following result.

Theorem 4.2.1. *Suppose φ is a function satisfying assumptions (A1) and (A2) and is of the form (4.6). Then for any radial solution u of (4.7), we have the channels of energy estimate*

$$\|\Pi_{\dot{H}^1}^\perp u_0\|_{L_2} + \|\Pi_{L^2}^\perp u_1\|_{Z_{-3}} \leq C\sqrt{E_{\text{out}}},$$

if $N \equiv 0 \pmod{4}$,

$$\|\Pi_{L^2}^\perp u_1\|_{L_2} + \|\nabla \Pi_{\dot{H}^1}^\perp u_0\|_{Z_{-3}} \leq C\sqrt{E_{\text{out}}},$$

if $N \equiv 2 \pmod{4}$.

The proof of Theorem 4.2.1 is the same as that of Theorem 4.1.1. The proof of Theorem 4.1.1 only uses the asymptotics of U .

Similarly, we have the following analogue in the case of a multisoliton.

Theorem 4.2.2. *Suppose φ is a function satisfying assumptions (A1) and (A2) and is of the form (4.6). Also, suppose that $\lambda = (\lambda_1, \dots, \lambda_J)$ is such that $\gamma(\lambda) \leq \gamma_0$ for some small enough γ_0 . Then there exists a constant C such that for any radial solution u of*

$$\begin{cases} \partial_t^2 u - \Delta u + \partial_2 (\varphi(r, U_{\lambda_1}) + \dots + \varphi(r, U_{\lambda_J})) u = 0 \\ (u, \partial_t u)|_{t=0} = (u_0, u_1), \end{cases}$$

we have

$$\|\Pi_{\dot{H}^1}^\perp u_0\|_{L^2} + \|\Pi_{L^2}^\perp u_1\|_{Z_{-3,\lambda}} \leq C \left(\sqrt{E_{\text{out}}} + \gamma(\lambda) \|(u_0, u_1)\|_{\dot{H}^1 \times L^2} \right),$$

if $N \equiv 0 \pmod{4}$,

$$\|\Pi_{L^2}^\perp u_1\|_{L^2} + \|\nabla \Pi_{\dot{H}^1}^\perp u_0\|_{Z_{-3,\lambda}} \leq C \left(\sqrt{E_{\text{out}}} + \gamma(\lambda) \|(u_0, u_1)\|_{\dot{H}^1 \times L^2} \right),$$

if $N \equiv 2 \pmod{4}$.

To see that the hypotheses of Theorem 4.2.1 are not vacuous, consider k -equivariant wave maps from Minkowski space into the sphere. Suppose that $\psi(r, t)$ solves

$$\partial_t^2 \psi - \partial_r^2 \psi - \frac{1}{r} \partial_r \psi + k^2 \frac{\sin(2\psi)}{2r^2} = 0.$$

This equation has the static solutions

$$Q_{k,\ell,\lambda}(r) = \ell k + 2 \arctan(\lambda r^k)$$

for all $\ell \in \mathbb{Z}$. If we set u to be defined by

$$u(r, t) = \frac{\psi(r, t)}{r^k},$$

then u solves

$$\partial_t^2 u - \Delta u + \varphi(r, u) = 0,$$

where

$$\varphi = \frac{k^2}{r^{2+k}} \left(r^k u - \frac{\sin(2r^k u)}{2} \right),$$

which satisfies the hypotheses of the theorem.

4.3 Preliminaries

Let $\|u\|_{L_R^p(\mathbb{R}^N)}$ and $\|u\|_{\dot{H}_R^1(\mathbb{R}^N)}$ be the norms defined by

$$\|u\|_{L_R^p(\mathbb{R}^N)} := \left(\int_R^\infty |u(r)|^p r^{N-1} dr \right)^{\frac{1}{p}},$$

$$\|u\|_{\dot{H}_R^1(\mathbb{R}^N)} := \left(\int_R^\infty |\partial_r u(r)|^2 r^{N-1} dr \right)^{\frac{1}{2}}.$$

Similarly, we define the exterior Strichartz norms

$$\|u\|_{L_t^p L_r^q(r > R+|t-\bar{t}|)} := \left(\int_{\mathbb{R}} \left(\int_{R+|t-\bar{t}|}^\infty |u(\bar{t}, r)|^q r^{N-1} dr \right)^{\frac{p}{q}} d\bar{t} \right)^{\frac{1}{p}}$$

In the case that $N \equiv 0 \pmod{4}$, we define the projection π_0 as the orthogonal projection with respect to the $\dot{H}_R^1$ inner product to the subspace

$$\text{Span} \left\{ \frac{1}{|x|^{N-2k}} : 1 \leq k \leq \frac{N}{4} \right\}. \quad (4.8)$$

Similarly, if $N \equiv 2 \pmod{4}$, we define the projection π_1 as the orthogonal projection with respect to the L_R^2 inner product to the subspace

$$\text{Span} \left\{ \frac{1}{|x|^{N-2k}} : 1 \leq k \leq \frac{N-2}{4} \right\}. \quad (4.9)$$

Additionally, we define the alternating norm

$$\|u\|_{\mathcal{A},N,R} = \begin{cases} \|\pi_0^\perp u\|_{\dot{H}_R^1} & \text{if } N \equiv 0 \pmod{4} \\ \|\pi_1^\perp u\|_{L_R^2} & \text{if } N \equiv 2 \pmod{4} \end{cases}$$

where

$$\pi_0^\perp u_0 = u_0 - \pi_0 u_0$$

and

$$\pi_1^\perp u_1 = u_1 - \pi_1 u_1.$$

Let $\mathcal{C}_{t,R}$ be the exterior cone

$$\mathcal{C}_{t,R} := \{(\bar{t}, \bar{r}) \in \mathbb{R} \times (0, \infty) : \bar{r} \geq R + |\bar{t} - t|\}.$$

We also define the exterior energy

$$E_{t,R} = \sup_{\bar{t} \in \mathbb{R}} \left(\|(u, \partial_t u)\|_{\dot{H}_{R+|\bar{t}-t|}^1 \times L_{R+|\bar{t}-t|}^2} \right).$$

We have the following exterior Strichartz estimate.

Lemma 4.3.1. *Suppose that $N > 3$ and that $(u_0, u_1) \in \dot{H}^1(\mathbb{R}^N) \times L^2(\mathbb{R}^N)$ is radial, and u is the solution to*

$$\partial_{tt} u - \Delta u = f$$

with initial conditions $(u, \partial_t u)|_{t=t_0} = (u_0, u_1)$, where $f \in L_t^1 L_r^2$ is radial. Then for any

$R_0 > 0$, $t_0 \in \mathbb{R}$ we have

$$\|(u, \partial_t u)\|_{E_{t_0, R_0}} + \|u\|_{L_t^2 L_r^{\frac{2N}{N-3}}(C_{t_0, R_0})} \lesssim \|(u_0, u_1)\|_{\dot{H}^1 \times L^2} + \|f\|_{L_t^1 L_r^2(C_{t_0, R_0})}.$$

Proof. Without loss of generality, suppose that $t_0 = 0$. Recall that

$$\sup_{t \in \mathbb{R}} \|(u(t), \partial_t u(t))\|_{\dot{H}^1 \times L^2} + \|u\|_{L_t^2 L_r^{\frac{2N}{N-3}}} \lesssim \|(u_0, u_1)\|_{\dot{H}^1 \times L^2} + \|f\|_{L_t^1 L_r^2} \quad (4.10)$$

(see [Keel and Tao, 1998, corollary 1.3]). Let $(\tilde{u}_0, \tilde{u}_1)$ be any extension of $(u_0, u_1)1_{r > R_0}$ such that $(\tilde{u}_0, \tilde{u}_1) \in \dot{H}^1 \times L^2$ with

$$\|(\tilde{u}_0, \tilde{u}_1)\|_{\dot{H}^1 \times L^2} \lesssim \|(u_0, u_1)\|_{\dot{H}_{R_0}^1 \times L_{R_0}^2}.$$

We also let $\tilde{f} = f1_{C_{0, R_0}}$. Note that $\tilde{f} \in L_t^1 L_r^2$. Letting $\tilde{u}$ be the solution to

$$\begin{cases} \partial_t^2 \tilde{u} - \Delta \tilde{u} = \tilde{f}, \\ (u, \partial_t u)|_{t=t_0} = (\tilde{u}_0, \tilde{u}_1) \end{cases}$$

we see that by finite propagation speed we have $u(x, t) = \tilde{u}(x, t)$ for all $(x, t) \in C_{0, R_0}$. Then we have by (4.10) that

$$\begin{aligned} \|(u, \partial_t u)\|_{E_{t_0, R_0}} + \|u\|_{L_t^2 L_r^{\frac{2N}{N-3}}} &\lesssim \sup_{t \in \mathbb{R}} \|(\tilde{u}(t), \partial_t \tilde{u}(t))\|_{\dot{H}^1 \times L^2} + \|\tilde{u}\|_{L_t^2 L_r^{\frac{2N}{N-3}}} \\ &\lesssim \|(\tilde{u}_0, \tilde{u}_1)\|_{\dot{H}^1 \times L^2} + \|\tilde{f}\|_{L_t^1 L_r^2} \\ &\lesssim \|(u_0, u_1)\|_{\dot{H}^1 \times L^2} + \|f\|_{L_t^1 L_r^2(C_{t_0, R_0})}. \end{aligned}$$

This completes the proof. □

We will also need the following exterior energy estimate from Li et al. [2024].

Proposition 4.3.2. *Suppose N is even, and let $u \in \mathbb{R} \times \mathbb{R}^N$ solve*

$$\partial_{tt}u - \Delta u = f,$$

with initial conditions $(u, \partial_t u)|_{t=0} = (u_0, u_1)$ where u_0, u_1 , and f are radial. Then we have

$$\|u\|_{\mathcal{A}, N, R} \lesssim \lim_{t \rightarrow \infty} \|(u(t), \partial_t u(t))\|_{\dot{H}_{R+|t|}^1(\mathbb{R}^N) \times L_{R+|t|}^2(\mathbb{R}^N)} + \|f\|_{L_t^1 L_r^2(\mathcal{C}_{0,R})}. \quad (4.11)$$

4.4 Single soliton case (non resonant component)

We recall that ΛW solves

$$-\Delta(\Lambda W) + V(\Lambda W) = 0.$$

We will say that $f(r) \sim g(r)$ if $r \rightarrow a$, with $a \in \mathbb{R} \cup \{\pm\infty\}$ if $\frac{f(r)}{g(r)} \rightarrow C$ for some constant C as $r \rightarrow a$.

Lemma 4.4.1. *There exists a solution Γ to*

$$-\Delta\Gamma + V\Gamma = 0,$$

where the Wronskian is given by

$$\Gamma(r)\partial_r(\Lambda W)(r) - \Lambda W(r)\partial_r\Gamma(r) = r^{-(N-1)}$$

and Γ has the asymptotics

$$\Gamma(r) \sim c_1 r^{-(N-2)} \quad \text{as } r \rightarrow 0, \quad \Gamma(r) \sim c_2 \quad \text{as } r \rightarrow \infty$$

for some dimensional constants c_1, c_2 .

Proof. We let

$$\Gamma(r) = -\Lambda W(r) \int_1^r s^{-(N-1)} (\Lambda W)^{-2}(s) ds.$$

Then we have

$$-\Delta \Gamma + V\Gamma = 0,$$

and for all $r > 0$,

$$\Gamma(r) \partial_r (\Lambda W)(r) - \Lambda W(r) \partial_r \Gamma(r) = r^{-(N-1)}.$$

Since

$$\Lambda W(r) = \frac{N-2}{2} \left(\frac{1 - \frac{r^2}{N(N-2)}}{1 + \frac{r^2}{N(N-2)}} \right) W(r),$$

we have

$$\Lambda W \sim \frac{N-2}{2} \quad \text{as } r \rightarrow 0, \quad \Lambda W \sim -\frac{N-2}{2} \frac{(N(N-2))^{\frac{N-2}{2}}}{r^{N-2}} \quad \text{as } r \rightarrow \infty \quad (4.12)$$

and so

$$\Gamma(r) \sim c_1 r^{-(N-2)} \quad \text{as } r \rightarrow 0, \quad \Gamma(r) \sim c_2 \quad \text{as } r \rightarrow \infty. \quad (4.13)$$

□

Let $T_0^\infty := \Lambda W$, $T_0^0 := \Gamma$.

Proposition 4.4.2. *Suppose that $N \geq 6$. There exist T_k^∞, T_k^0 for $1 \leq k \leq \frac{N-6}{2}$ such that*

$$-\Delta T_{k+1}^0 + V T_{k+1}^0 = -T_k^0$$

$$-\Delta T_{k+1}^\infty + V T_{k+1}^\infty = -T_k^\infty$$

and we have the asymptotics

$$T_k^0(r) \sim r^{a_k^0} \quad \text{as } r \rightarrow 0, \quad T_k^0(r) \sim r^{b_k^0} \quad \text{as } r \rightarrow \infty, \quad (4.14)$$

$$T_k^\infty(r) \sim r^{a_k^\infty} \quad \text{as } r \rightarrow 0, \quad T_k^\infty(r) \sim r^{b_k^\infty} \quad \text{as } r \rightarrow \infty, \quad (4.15)$$

where $a_k^0, b_k^0, a_k^\infty, b_k^\infty$ are given by

$$a_k^0 = -N + 2 + 2k, b_k^0 = 2k,$$

$$a_k^\infty = -N + 2, b_k^\infty = -N + 2 + 2k$$

if $k > 0$, and if $k = 0$ we have

$$a_0^0 = -N + 2, b_0^0 = 0, a_0^\infty = 0, b_0^\infty = -N + 2.$$

Proof. We prove the result by induction. Suppose by induction that (4.14), (4.15) hold for k . Let $\mathcal{S}_0, \mathcal{S}_\infty$ denote the operators defined by

$$\mathcal{S}_0 f(r) := T_0^0(r) \int_0^r \rho^{N-1} T_0^\infty(\rho) f(\rho) d\rho + T_0^\infty(r) \int_1^r \rho^{N-1} T_0^0(\rho) f(\rho) d\rho.$$

$$\mathcal{S}_\infty f(r) := -T_0^0(r) \int_r^\infty \rho^{N-1} T_0^\infty(\rho) f(\rho) d\rho + T_0^\infty(r) \int_1^r \rho^{N-1} T_0^0(\rho) f(\rho) d\rho.$$

Then let

$$T_k^\infty = -\mathcal{S}_\infty(T_{k-1}^\infty), \quad T_k^0 = -\mathcal{S}_0(T_{k-1}^0).$$

Note that if $f \sim r^\alpha$ as $r \rightarrow 0$, $f \sim r^\beta$ as $r \rightarrow \infty$, then

$$\mathcal{S}_0 f \sim r^{\alpha+2}, \quad \text{as } r \rightarrow 0, \quad \mathcal{S}_0 f \sim C + r^{\beta+2} \quad \text{as } r \rightarrow \infty$$

if $\alpha > -N$, $\alpha + 2 < 0$, $\beta > -2$. Similarly,

$$\mathcal{S}_\infty f \sim r^{\alpha+2} + r^{-(N-2)} + C \quad \text{as } r \rightarrow 0, \quad \mathcal{S}_\infty f \sim \frac{1}{r^{N-2}} + r^{2+\beta} \quad \text{as } r \rightarrow \infty$$

if $\alpha > -N, \beta < -2$.

Therefore

$$T_{k+1}^0 \sim r^{\min\{-(N-2), a_k^0+2\}} \quad \text{as } r \rightarrow 0, \quad T_{k+1}^0 \sim r^{b_k^0+2} \quad \text{as } r \rightarrow \infty$$

and

$$T_{k+1}^\infty \sim r^{\min\{-(N-2), a_k^\infty+2\}} \quad \text{as } r \rightarrow 0, \quad T_{k+1}^\infty \sim r^{b_k^\infty+2} \quad \text{as } r \rightarrow \infty.$$

By using the induction hypothesis this completes the proof, since (4.12), (4.13) prove the case $k = 0$. □

Lemma 4.4.3. *Let T_k^∞, T_k^0 be defined as in Lemma 4.4.2. There exist coefficients e_k^i such that*

$$\tilde{T}_k^0 := T_k^0 - \left(e_0^k T_0^\infty + \cdots + e_{k+1}^k T_{k+1}^\infty \right) \sim r^2 \quad \text{as } r \rightarrow 0.$$

Proof. We will prove this by induction. Note that

$$0 < \int_0^\infty \rho^{N-1} (T_0^\infty(\rho))^2 d\rho < \infty,$$

since $T_0^\infty \sim 1$ as $r \rightarrow 0$ and $T_0^\infty \sim r^{-(N-2)}$ as $r \rightarrow \infty$. Letting

$$e_1^0 := - \left(\int_0^\infty \rho^{N-1} (T_0^\infty(\rho))^2 d\rho \right)^{-1}$$

and

$$e_0^0 = e_1^0 \int_0^1 \rho^{N-1} T_0^0(\rho) T_0^\infty(\rho) d\rho$$

which is well defined since $\rho^{N-1}T_0^0(\rho)T_0^\infty(\rho) \sim \rho$ for $\rho \rightarrow 0$, we see that

$$\begin{aligned} T_0^0 - e_1^0 T_1^\infty &= T_0^0 - e_1^0 \left((e_1^0)^{-1} T_0^0 + T_0^0(r) \int_0^r \rho^{N-1} (T_0^\infty(\rho))^2 d\rho + T_0^\infty(r) \int_1^r \rho^{N-1} T_0^0(\rho) T_0^\infty(\rho) d\rho \right) \\ &= -e_1^0 \left(T_0^0(r) \int_0^r \rho^{N-1} (T_0^\infty(\rho))^2 d\rho + T_0^\infty(r) \int_1^r \rho^{N-1} T_0^0(\rho) T_0^\infty(\rho) d\rho \right) \end{aligned}$$

and thus

$$\begin{aligned} T_0^0 - e_0^0 T_0^\infty - e_1^0 T_1^\infty &= -e_1^0 \left(T_0^0(r) \int_0^r \rho^{N-1} (T_0^\infty(\rho))^2 d\rho + T_0^\infty(r) \int_0^r \rho^{N-1} T_0^0(\rho) T_0^\infty(\rho) d\rho \right) \\ &\sim r^{-(N-2)} \int_0^r \rho^{N-1} d\rho + \int_0^r \rho d\rho \sim r^2. \end{aligned}$$

This proves the base case. For the general case, suppose that e_i^k for $0 \leq i \leq k+1$ have all been defined. Then we have

$$\begin{aligned} T_{k+1}^0 &= -\mathcal{S}_0(T_k^0) \\ &= \mathcal{S}_0 \left(\tilde{T}_k^0 + e_0^k T_0^\infty + \cdots + e_{k+1}^k T_{k+1}^\infty \right) \\ &= \mathcal{S}_0(\tilde{T}_k^0) + \sum_{i=0}^{k+1} e_i^k \mathcal{S}_\infty(T_i^\infty) + \sum_{i=0}^{k+1} e_i^k (\mathcal{S}_0 - \mathcal{S}_\infty) T_i^\infty \\ &= \mathcal{S}_0(\tilde{T}_k^0) - \sum_{i=1}^{k+2} e_{i-1}^k T_i^\infty + \sum_{i=0}^{k+1} e_i^k T_0^0(r) \int_0^\infty \rho^{N-1} T_0^\infty(\rho) T_k^\infty(\rho) d\rho, \end{aligned}$$

where we note that $\mathcal{S}_0(\tilde{T}_k^0)$ is well defined by the induction hypothesis and where

$$\int_0^\infty \rho^{N-1} T_0^\infty(\rho) T_k^\infty(\rho) d\rho$$

is well defined by Lemma 4.4.2. This implies that

$$\begin{aligned} T_{k+1}^0 &= \mathcal{S}_0(\tilde{T}_k^0) + \left(\sum_{i=0}^{k+1} e_i^k \int_0^\infty \rho^{N-1} T_0^\infty(\rho) T_k^\infty(\rho) d\rho \right) \tilde{T}_0^0 \\ &+ \left(\sum_{i=0}^{k+1} e_i^k \int_0^\infty \rho^{N-1} T_0^\infty(\rho) T_k^\infty(\rho) d\rho \right) (e_0^0 T_0^\infty + e_1^0 T_1^\infty) - \sum_{i=1}^{k+2} e_{i-1}^k T_i^\infty. \end{aligned}$$

Noting that the first line in the equation above is of order r^2 as $r \sim 0$, this completes the proof. $\square$

From the solutions obtained from Proposition 4.4.2, we obtain solutions of the homogeneous equation $\partial_{tt}u - \Delta u + Vu = 0$ such that

$$\lim_{t \rightarrow \pm\infty} \int_{|x| > R+|t|} |\nabla_{t,x} u(t, x)|^2 dx = 0$$

for any $R > 0$, namely

$$\begin{aligned} S_k^{\infty,0} &:= \sum_{i=0}^k T_{k-i}^\infty \frac{t^{2i}}{(2i)!}, \\ S_k^{\infty,1} &:= \sum_{i=0}^k T_{k-i}^\infty \frac{t^{2i+1}}{(2i+1)!}. \end{aligned}$$

Note that these are solutions for the systems

$$\begin{cases} \partial_{tt} S_k^{\infty,0} - \Delta S_k^{\infty,0} + V S_k^{\infty,0} = 0 \\ (S_k^{\infty,0}, \partial_t S_k^{\infty,0})|_{t=0} = (T_k^\infty, 0) \end{cases}$$

$$\begin{cases} \partial_{tt} S_k^{\infty,1} - \Delta S_k^{\infty,1} + V S_k^{\infty,1} = 0 \\ (S_k^{\infty,1}, \partial_t S_k^{\infty,1})|_{t=0} = (0, T_k^\infty) \end{cases}$$

Suppose that $N \geq 8$, and let

$$\mathcal{D} := \left\{ S_k^{\infty,0} : 0 \leq k \leq \frac{N-6}{2} \right\} \cup \left\{ S_k^{\infty,1} : 0 \leq k \leq \frac{N-8}{2} \right\}$$

We claim these are the only ones.

Proposition 4.4.4. *Let $R > 0$ and suppose that u is a non-radiative solution to (4.1) with $N \geq 8$. Suppose additionally that*

i) u is an even in time solution in the case that $N \equiv 0 \pmod{4}$;

ii) u is an odd in time solution in the case that $N \equiv 2 \pmod{4}$.

Suppose that $R > 0$. Then:

i) If $N \equiv 0 \pmod{4}$, then there exist coefficients $\{c_k^0\}_{k=0}^{\frac{N-6}{2}}$ such that

$$u = \sum_{k=0}^{\frac{N-6}{2}} c_k^0 S_k^{\infty,0}$$

for all x with $\{|x| > R + |t|\}$.

ii) If $N \equiv 2 \pmod{4}$, then there exist coefficients $\{c_k^1\}_{k=0}^{\frac{N-8}{2}}$ such that

$$u = \sum_{k=0}^{\frac{N-8}{2}} c_k^1 S_k^{\infty,1}$$

for all x with $\{|x| > R + |t|\}$.

If $R = 0$, then there exists a constant c_0 such that $u = c_0 S_0^{\infty,0}$ if $N \equiv 0 \pmod{4}$, and there exists a constant c_1 such that $u = c_1 S_0^{\infty,1}$ if $N \equiv 2 \pmod{4}$,

The analogues of these for $N = 4, 6$ are done in Collot et al. [2023].

Proof. First note that by replacing u with $\frac{1}{2}(u(t, x) - u(-t, x))$ if $N \equiv 0 \pmod{4}$ or $\frac{1}{2}(u(t, x) + u(-t, x))$ if $N \equiv 2 \pmod{4}$, we can assume that $u_0 \equiv 0$ if $N \equiv 0 \pmod{4}$ or $u_1 \equiv 0$ if $N \equiv 2 \pmod{4}$. Fix a large radius $R' > 0$. If $N \equiv 0 \pmod{4}$, let $\{c_k^0\}_{k=0}^{\frac{N-6}{2}}$ be such that

$$\int_{R'}^{\infty} \left(u_1 - \sum_{k=0}^{\frac{N-6}{2}} c_k^0 T_k^{\infty} \right) \frac{1}{r^{N-2j}} r^{N-1} dr = 0,$$

for all $1 \leq j \leq \frac{N}{4}$, while in the case $N \equiv 2 \pmod{4}$ we let $\{c_k^1\}_{k=0}^{\frac{N-8}{2}}$ be such that

$$\int_{R'}^{\infty} \left(u_0 - \sum_{k=0}^{\frac{N-8}{2}} c_k^1 T_k^{\infty} \right) \frac{1}{r^{N-2j}} r^{N-1} dr = 0,$$

for all $0 \leq j \leq \frac{N-2}{4}$. To show the existence of these coefficients, note that their existence is equivalent to the invertibility of the matrices $A = (a_{i,j})_{0 \leq i,j \leq \frac{N-6}{2}}$ (or $0 \leq i,j \leq \frac{N-8}{2}$ if $N \equiv 2 \pmod{4}$) with

$$a_{i,j} = \int_{R'}^{\infty} T_i^{\infty} \frac{1}{r^{N-2j}} r^{N-1} dr.$$

This matrix can be written as

$$A = A_0(R') + A_{\text{error}}(R'),$$

where $A_0(R') = (a_{i,j}^0(R'))_{0 \leq i,j \leq \frac{N-6}{2}}$ (or $(a_{i,j}^0(R'))_{0 \leq i,j \leq \frac{N-8}{2}}$ if $N \equiv 2 \pmod{4}$) with

$$a_{i,j}^0(R') = \int_{R'}^{\infty} \frac{1}{r^{N-2i}} \frac{1}{r^{N-2j}} r^{N-1} dr$$

and $A_{\text{error}}(R')$ is a smaller order term. More precisely, we have by explicit computation

$$\|A_{\text{error}}(R')\|_2 = O(R'^{-1} \|A_0(R')\|_2)$$

where $\|\cdot\|_2$ denotes the L^2 norm for matrices. The matrix $A_0(R')$ is easily seen to be invertible, since if $v = (v_i)_{0 \leq i \leq \frac{N-6}{2}}$ lies in its kernel, then we obtain

$$\int_{R'}^{\infty} \left(\sum_{i=0}^{\frac{N-6}{2}} v_i \frac{1}{r^{N-2i}} r^{N-1} dr \right) v_k = 0$$

for all $1 \leq k \leq \frac{N-6}{2}$, and so $v = 0$. In the case that $N \equiv 2 \pmod{4}$, we set $c_{(N-6)/2}^1 = 0$. Also, let $\sigma := 0$ if $N \equiv 0 \pmod{4}$ and $\sigma := 1$ if $N \equiv 2 \pmod{4}$. Recall that by Proposition 4.3.2 we have

$$\|u - \sum_{k=0}^{\frac{N-6}{2}} c_k^{\sigma} S_k^{\infty, \sigma}\|_{\mathcal{A}, N, R'} \lesssim \|V \left(u - \sum_{k=0}^{\frac{N-6}{2}} c_k^{\sigma} S_k^{\infty, \sigma} \right)\|_{L_t^1 L_r^2(\mathcal{C}_{0, R'})}.$$

Now by Lemma 4.3.1, we have

$$\begin{aligned} & \|u - \sum_{k=0}^{\frac{N-6}{2}} c_k^{\sigma} S_k^{\infty, \sigma}\|_{L^2 L^{\frac{2N}{N-3}}(|x| \geq R' + |t|)} \\ & \lesssim \|V \left(u - \sum_{k=0}^{\frac{N-6}{2}} c_k^{\sigma} S_k^{\infty, \sigma} \right)\|_{L_t^1 L_r^2(\mathcal{C}_{0, R'})} + \|u - \sum_{k=0}^{\frac{N-6}{2}} c_k^{\sigma} S_k^{\infty, \sigma}\|_{\mathcal{A}, N, R'}. \end{aligned}$$

Combining these two estimates and using Hölder, we obtain

$$\begin{aligned} & \|u - \sum_{k=0}^{\frac{N-6}{2}} c_k^{\sigma} S_k^{\infty, \sigma}\|_{L^2 L^{\frac{2N}{N-3}}(|x| \geq R' + |t|)} \\ & \lesssim \|V\|_{L^2 L^{\frac{2N}{3}}(|x| \geq R' + |t|)} \|u - \sum_{k=0}^{\frac{N-6}{2}} c_k^{\sigma} S_k^{\infty, \sigma}\|_{L^2 L^{\frac{2N}{N-3}}(|x| \geq R' + |t|)}. \end{aligned}$$

Since $V \in L_t^2 L_x^{\frac{2N}{3}}$, taking R' large, we obtain the result for $|x| \geq R' + |t|$. Arguing as in [Collot et al., 2023, Lemma 3.2], we obtain it for all $R > 0$. For the case of $R = 0$, note that

$S_k^{\infty,0}, S_k^{\infty,1} \notin E_{0,R}$ if $R = 0, k \geq 1$. This shows that the only choice for u is given by $c_0 S_0^{\infty,0}$ or $c_0 S_0^{\infty,1}$. $\square$

The next step is to show a quantitative version of Proposition 4.4.4. For this, let χ_0 be a cutoff such that

$$\chi_0(r) := \begin{cases} 1 & \text{if } r \leq 10, \\ 0 & \text{if } r \geq 11 \end{cases}$$

with $\chi_0(r) \in [0, 1]$. suppose that $N \geq 8$. We define the projection $\Pi_R^\perp$ by

$$\Pi_R^\perp(u_0, u_1) = \begin{cases} \Pi_{\dot{H}_R^1}(\text{Span}\{\Lambda W\})^\perp(u_0) & \text{for } R = 0 \\ \Pi_{\dot{H}_R^1}(\text{Span}\{T_k^\infty\}_{k=0}^{\frac{N-6}{2}} \cup \{\chi_0 T_k^0\}_{k=0}^{\frac{N-6}{2}})^\perp(u_0) & \text{for } R \in (0, 1) \\ \Pi_{\dot{H}_R^1}(\text{Span}\{T_k^\infty\}_{k=0}^{\frac{N-6}{2}})^\perp(u_0) & \text{for } R \geq 1. \end{cases} \quad (4.16)$$

in the case that $N \equiv 0 \pmod{4}$, and

$$\Pi_R^\perp(u_0, u_1) = \begin{cases} \Pi_{L_R^2}(\text{Span}\{T_k^\infty\}_{k=0}^{\frac{N-8}{2}})^\perp(u_1) & \text{for } R = 0 \text{ or } R \geq 1 \\ \Pi_{L_R^2}(\text{Span}\{T_k^\infty, T_k^0\}_{k=0}^{\frac{N-8}{2}})^\perp(u_1) & \text{for } R \in (0, 1). \end{cases} \quad (4.17)$$

in the case that $N \equiv 2 \pmod{4}$. Let $\vec{T}_k^\infty, \vec{T}_k^0$ be defined as

$$\vec{T}_k^\infty = \begin{cases} (T_k^\infty, 0) & \text{if } N \equiv 0 \pmod{4}, \\ (0, T_k^\infty) & \text{if } N \equiv 2 \pmod{4} \end{cases}$$

and

$$\vec{T}_k^0 = \begin{cases} (T_k^0, 0) & \text{if } N \equiv 0 \pmod{4}, \\ (0, T_k^0) & \text{if } N \equiv 2 \pmod{4}. \end{cases}$$

For convenience we set $T_{\frac{N-6}{2}}^\infty, T_{\frac{N-6}{2}}^0 = 0$ in the case that $N \equiv 2 \pmod{4}$.

Lemma 4.4.5. *There is a constant C such that for all $R > 0$ and any radial solution u_L of*

(4.1) with $u_L \in \mathcal{H}_R$ then

$$\|\Pi_R^\perp(u_0, u_1)\|_{\dot{H}^1}^2 \leq C \sum_{\pm} \lim_{t \rightarrow \pm\infty} \int_{r \geq R+|t|} |\nabla_{t,x} u_L|^2 dx$$

if $N \equiv 0 \pmod{4}$,

$$\|\Pi_R^\perp(u_0, u_1)\|_{L^2} \leq C \sum_{\pm} \lim_{t \rightarrow \pm\infty} \int_{r \geq R+|t|} |\nabla_{t,x} u_L|^2 dx$$

if $N \equiv 2 \pmod{4}$.

Proof. Suppose by contradiction that the result does not hold. Then there exists initial conditions u_n with corresponding solution to (4.1) $\{u_{L,n}\}_n$, where $u_n = (u_{0,n}, 0)$ if $N \equiv 0 \pmod{4}$, and $u_n = (0, u_{1,n})$ if $N \equiv 2 \pmod{4}$, with

$$\|\Pi_R^\perp(u_{0,n}, u_{1,n})\|_{\dot{H}^1} = 1$$

if $N \equiv 0 \pmod{4}$,

$$\|\Pi_R^\perp(u_{0,n}, u_{1,n})\|_{L^2} = 1$$

if $N \equiv 2 \pmod{4}$, and

$$\lim_{n \rightarrow \infty} \left(\lim_{t \rightarrow \infty} \int_{|x| \geq R_n+|t|} |\nabla_{t,x} u_{L,n}(t, x)|^2 dx \right) = 0. \quad (4.18)$$

We can decompose the initial condition u_n in $\{|x| \geq R_n\}$ as

$$u_n = \sum_{k=0}^{\frac{N-6}{2}} c_{k,n} \vec{T}_k^\infty + v_n,$$

where v_n is chosen so that $v_n = (v_{0,n}, 0)$ if $N \equiv 0 \pmod{4}$, $v_n = (0, v_{1,n})$ if $N \equiv 2 \pmod{4}$,

with

$$v_{0,n} \perp_{\dot{H}_{R_n}^1} \overrightarrow{T}_k^\infty \text{ if } N \equiv 0 \pmod{4}, \quad v_{1,n} \perp_{L_{R_n}^2} \overrightarrow{T}_k^\infty \text{ if } N \equiv 2 \pmod{4} \quad (4.19)$$

and where $v_{0,n}, v_{1,n}$ have support in the exterior of $\{|x| \geq R_n\}$. We split into cases. Let $\mathcal{A}_{N,R}$ denote the space $\dot{H}_R^1$ if $N \equiv 0 \pmod{4}$, and the space L_R^2 if $N \equiv 2 \pmod{4}$.

Case 1: For all n , $R_n \geq 1$. By taking a subsequence if necessary, we can additionally assume that $R_n \rightarrow R_\infty$, where $R_\infty \geq 1$. Note that in this case by the definition of the projection $\Pi_R^\perp$,

$$\|v_n\|_{\dot{H}_{R_n}^1 \times L_{R_n}^2} = 1.$$

We claim that

$$v_n \rightharpoonup 0$$

weakly in $\dot{H}^1 \times L^2$. Indeed, first note that if $R_\infty = \infty$ then this is trivial since $v_n(r) = 0$ if $r \leq R_n$. If not, let v^* be any weak subsequential limit. Then by the orthogonality conditions (4.19) we have that

$$v^* \perp_{\dot{H}^1 \times L^2} \overrightarrow{T}_k^\infty$$

by weak convergence and the fact that $\|T_k^\infty\|_{\mathcal{A}_{N,1}} < \infty$. On the other hand, let v_L denote the solution to (4.1) with initial conditions v^* . We also let v_n^R be the solution to (4.1) in $\{|x| \geq R\}$ with initial data $1_{r \geq R} v_n$, where R is chosen so that $R > R_\infty$, and similarly v^R be the solution to (4.1) in $\{|x| \geq R\}$ with initial data $1_{r \geq R} v^*$. Using our contradiction hypothesis and finite propagation speed we have

$$\lim_{n \rightarrow \infty} \lim_{t \rightarrow \infty} \int_{r \geq R+|t|} |\nabla_{t,x} v_{L,n}^R|^2 = 0. \quad (4.20)$$

Since $(v_{L,n}^R(0), \partial_t v_{L,n}^R(0)) \rightharpoonup v^*$ weakly in $\dot{H}^1 \times L^2$, we have that v^* is in the $\dot{H}^1 \times L^2$ closure

of the convex hull of $\{(v_{L,n}^R(0), \partial_t v_{L,n}^R(0))\}_{n \geq N_0}$ for any large N_0 . Hence for any $\varepsilon > 0$, there exist coefficients $\{c_k\}_{k=1}^\infty$ such that all but finitely many are 0, their sum is 1, and such that

$$\|v^* - \sum_{k=0}^\infty c_k (v_{L,k}^R(0), \partial_t v_{L,k}^R(0))\|_{\dot{H}_R^1 \times L_R^2} < \varepsilon.$$

Let w denote the solution to (4.1) with initial condition

$(v^* - \sum_{k=0}^\infty c_k (v_{L,k}^R(0), \partial_t v_{L,k}^R(0)))1_{|x| > R}$. Then by Lemma 2.1 in Duyckaerts et al. [2020] we have that $\sup_{t \in \mathbb{R}} \|(w, \partial_t w)\|_{\dot{H}^1 \times L^2(|x| \geq R+|t|)} \leq C$. This implies that v^R is then in the $L_t^\infty(\dot{H}_R^1 \times L_R^2)$ closure of the convex hull of $\{v_n^R\}_{n \geq N_0}$. Thus combining this with (4.20) we obtain that

$$\lim_{|t| \rightarrow \infty} \int_{r \geq R+|t|} |\nabla_{t,x} v_L^R|^2 = 0. \quad (4.21)$$

Now using Proposition 4.4.4 we see that there exist coefficients c_k such that

$v_L^R = \sum_{k=1}^{\frac{N-6}{2}} c_k(R) \vec{S}_k^\infty$ (or $v_L^R = \sum_{k=1}^{\frac{N-8}{2}} c_k(R) \vec{S}_k^\infty$ if $N \equiv 2 \pmod{4}$) Since $v_L^{R_1}(0), v_L^{R_2}(0)$ coincide for $|x| \geq \max\{R_1, R_2\}$, the coefficient $c_k(R)$ is independent of R , and so we see that $v_L^R = 0$ and therefore $v^* = 0$. Now the claim has been established.

We let $v_{F,n}$ be the free wave $(\partial_{tt} - \Delta)v_{F,n} = 0$ where $v_{F,n}$ has initial conditions v_n . Then by Lemma 3.7 in Duyckaerts et al. [2020] together with the contradiction hypothesis we have

$$\begin{aligned} & \lim_{t \rightarrow \infty} \int_{|x| \geq R_n+|t|} |\nabla_{t,x} v_{F,n}|^2 dx \\ & \geq C \|v_n\|_{\dot{H}^1 \times L^2}^2 - \sum_{j=0}^{\frac{N-4}{4}} \left(\|r^{-(N-2j-2)}\|_{\dot{H}_{R_n}^1} \int_{|x| \geq R_n} \nabla(|x|^{-(N-2j-2)}) \cdot \nabla v_n(x) dx \right)^2. \end{aligned}$$

if $N \equiv 0 \pmod{4}$, and

$$\begin{aligned} & \lim_{t \rightarrow \infty} \int_{|x| \geq R_n + |t|} |\nabla_{t,x} v_{F,n}|^2 dx \\ & \geq C \|v_n\|_{\dot{H}^1 \times L^2}^2 - \sum_{j=0}^{\frac{N-8}{4}} \left(\|r^{-(N-2j-2)}\|_{L^2_{R_n}} \int_{|x| \geq R_n} |x|^{-(N-2j-2)} v_n(x) dx \right)^2. \end{aligned}$$

if $N \equiv 2 \pmod{4}$. Now since $v_n \rightharpoonup v^* = 0$ in $\dot{H}^1 \times L^2$, we have that

$$\lim_{n \rightarrow \infty} \|r^{-(N-2j-2)}\|_{\dot{H}^1_{R_n}} \int_{|x| \geq R_n} \nabla \left(|x|^{-(N-2j-2)} \right) \cdot \nabla v_n(x) dx = 0$$

if $N \equiv 0 \pmod{4}$, and

$$\lim_{n \rightarrow \infty} \|r^{-(N-2j-2)}\|_{\dot{H}^1_{R_n}} \int_{|x| \geq R_n} |x|^{-(N-2j-2)} v_n(x) dx = 0$$

if $N \equiv 2 \pmod{4}$. Combining with (4.21) this yields a contradiction.

Case 2: For all n , $R_n = 0$. This case is similar to the previous one. We let $v_n = \Pi_R^\perp u_n$, and so in particular

$$\|v_n\|_{\dot{H}^1 \times L^2} \leq 1.$$

We again claim that $v_n \rightharpoonup 0$ in $\dot{H}^1 \times L^2$. Let v^* be any subsequential weak limit in $\dot{H}^1 \times L^2$.

Then we have that

$$v^* \perp_{\dot{H}^1 \times L^2} \overrightarrow{T}_0^\infty$$

by weak convergence and the fact that $\|T_k^\infty\|_{\mathcal{A}_N} < \infty$. On the other hand, let v_L denote the solution to (4.1) with initial conditions v^* . We also let v_n^R be the solution to (4.1) in $\{|x| \geq R + |t|\}$ with initial data $1_{r \geq R} v_n$, where R is chosen so that $R > R_\infty$, and similarly v^R be the solution to (4.1) in $\{|x| \geq R\}$ with initial data $1_{r \geq R} v^*$. Using our contradiction

hypothesis and finite propagation speed we have

$$\lim_{n \rightarrow \infty} \lim_{t \rightarrow \infty} \int_{r \geq R+|t|} |\nabla_{t,x} v_{L,n}^R|^2 = 0. \quad (4.22)$$

Since $(v_{L,n}^R(0), \partial_t v_{L,n}^R(0)) \rightharpoonup v^*$ weakly in $\dot{H}^1 \times L^2$, we have that v^* is in the $\dot{H}^1 \times L^2$ closure of the convex hull of $\{(v_{L,n}^R(0), \partial_t v_{L,n}^R(0))\}_{n \geq N_0}$ for any large N_0 . Hence for any $\varepsilon > 0$, there exist coefficients $\{c_k\}_{k=1}^\infty$ such that all but finitely many are 0, their sum is 1, and such that

$$\|v^* - \sum_{k=0}^\infty c_k (v_{L,k}^R(0), \partial_t v_{L,k}^R(0))\|_{\dot{H}_R^1 \times L_R^2} < \varepsilon.$$

Let w denote the solution in $\{r > R\} \times (-\infty, \infty)$ to (4.1) in $\{|x| \geq R\}$ with initial condition $(v^* - \sum_{k=0}^\infty c_k (v_{L,k}^R(0), \partial_t v_{L,k}^R(0)))1_{|x| > R}$. Then by Lemma 2.1 in Duyckaerts et al. [2020] we have that $\sup_{t \in \mathbb{R}} \|(w, \partial_t w)\|_{\dot{H}^1 \times L^2} < C$. This implies that v^R is then in the $L_t^\infty(\dot{H}^1 \times L^2)$ closure of the convex hull of $\{v_n^R\}_{n \geq N_0}$. Thus combining this with (4.22) we obtain that

$$\lim_{|t| \rightarrow \infty} \int_{r \geq R+|t|} |\nabla_{t,x} v_L^R|^2 = 0.$$

Now using Proposition 4.4.4 we see that there exists a coefficient $c = c(R)$ such that $v_L^R = c(R) \vec{S}_0^{\infty, \sigma}$ where $\sigma = 0$ if $N \equiv 0 \pmod{4}$ and $\sigma = 1$ if $N \equiv 2 \pmod{4}$ and where $\vec{S}_0^{\infty, \sigma}$ denotes the σ -th coordinate of $\vec{S}_0^\infty$. Since $v_L^{R_1}(0), v_L^{R_2}(0)$ coincide for $|x| \geq \max\{R_1, R_2\}$, the coefficient $c(R)$ is independent of R and so we see that $v_L^R = 0$ and therefore $v^* = 0$. Now the claim has been established.

We let $v_{F,n}$ be the free wave $(\partial_{tt} - \Delta)v_{F,n} = 0$ where $\vec{v}_{F,n}$ has initial conditions v_n . Then by Lemma 3.7 in Duyckaerts et al. [2020] together with the contradiction hypothesis we have

$$\lim_{n \rightarrow \infty} \lim_{t \rightarrow \infty} \int_{|x| \geq R_n+|t|} |\nabla_{t,x} v_{F,n}|^2 dx \geq C \|v_n\|_{\dot{H}^1 \times L^2}^2.$$

Since $v_n \rightharpoonup v^* = 0$ in $\dot{H}^1 \times L^2$ this yields a contradiction.

Case 3: For all n , $R_n \in (0, 1)$ and $\lim_{n \rightarrow \infty} R_n = R_\infty \in (0, 1]$. Suppose $R_n \in (0, 1)$ and that $\lim_{n \rightarrow \infty} R_n = R_\infty \in (0, 1]$. By (4.18), we have that $\Pi_{R_n}^\perp u_n = \Pi_{R_n}^\perp v_n$, and so $\|v_n\|_{\dot{H}_{R_n}^1 \times L_{R_n}^2} \geq 1$. Let

$$\tilde{u}_n = \frac{u_n}{\|v_n\|_{\dot{H}_{R_n}^1 \times L_{R_n}^2}}, \quad \tilde{v}_n = \frac{v_n}{\|v_n\|_{\dot{H}_{R_n}^1 \times L_{R_n}^2}}$$

and similarly let $\tilde{u}_{L,n}$ be the solution to (4.1) with initial condition $\tilde{u}_n$. Then note that

$$\tilde{u}_n = \sum_{k=0}^{\frac{N-6}{2}} \tilde{c}_{k,n} T_k^\infty + \tilde{v}_n$$

with $\tilde{c}_{k,n} = \frac{c_{k,n}}{\|v_n\|_{\dot{H}_{R_n}^1 \times L_{R_n}^2}}$, $\tilde{v}_n \perp_{\dot{H}^1 \times L^2} \vec{T}_k^\infty$, $\|v_n\|_{\dot{H}_{R_n}^1 \times L_{R_n}^2} = 1$, $R_n \rightarrow R_\infty > 0$, and

$$\lim_{n \rightarrow \infty} \lim_{t \rightarrow \infty} \int_{|x| > R_n + |t|} |\nabla_{t,x} \tilde{u}_{L,n}(t, x)|^2 dx = 0.$$

Now we continue as in case 1.

Case 4: For all n , $R_n \in (0, 1)$ and $\lim_{n \rightarrow \infty} R_n = 0$. The proof is very similar to that of case 3 in the proof of Lemma 3.1 in Collot et al. [2023]. One can bound the asymptotic exterior energy of the solutions to (4.1) with initial conditions given by $(T_k^0, 0)$ and $(0, T_k^0)$. Then one can expand u_n as

$$u_n = \sum_{k=0}^{\frac{N-6}{2}} c_{k,n}^\infty \vec{T}_k^\infty + c_{k,n}^0 \chi_0 \vec{T}_k^0 + \Pi^\perp u_n$$

$$u_n = \sum_{k=0}^{\frac{N-6}{2}} \tilde{c}_{k,n}^\infty \vec{T}_k^\infty + v_n.$$

Let $\tilde{S}_k^0$ denote the solution to (4.1) with initial condition $\chi_0 \vec{T}_k^0$. Let $R > 0$. Since $\chi_0 \vec{T}_k^0 \notin \text{Span}(\{T_k^\infty\}_{k=0}^{\frac{N-6}{2}})$, by Proposition 4.4.4 we have that there exists a constant C such that

$$\frac{1}{C} \leq \lim_{t \rightarrow \pm\infty} \int_{|x| \geq R+|t|} |\nabla \tilde{S}_k^0|^2 dx.$$

On the other hand, letting $\Xi_k := \tilde{S}_k^0 - \chi_0 S_k^{0,\sigma}$ where $\sigma = 0$ if $N \equiv 0 \pmod{4}$ and $\sigma = 1$ if $N \equiv 2 \pmod{4}$, we have

$$\begin{cases} (\partial_t^2 - \Delta + V)\Xi_K = -F_K \\ (\Xi_K, \partial_t \Xi_K)|_{t=0} = (0, 0), \end{cases}$$

where

$$F_K := \chi_0 \left(\sum_{k=1}^K \frac{t^{2(k-1)}}{(2(k-1))!} T_{k-i} \right) + (-\Delta + V) \left(\chi_0 \left(\sum_{k=0}^K \frac{t^{2i}}{(2i)!} T_{k-i} \right) \right)$$

which is 0 if $r \leq 10$ or $r \geq 12$. From this one obtains as in the proof of (4.21), with standard energy estimates that there is a constant C such that

$$\frac{1}{C} \leq \lim_{t \rightarrow \infty} \int_{|x| \geq R+|t|} |\nabla \tilde{S}_K^0|^2 dx \leq C. \quad (4.23)$$

We take the exterior energy on both sides to obtain

$$\begin{aligned} \sum_{k_1, k_2=0}^{\frac{N-6}{2}} a_{k_1 k_2} c_{k_1, n}^0 c_{k_2, n}^0 &= \lim_{t \rightarrow \infty} \int_{|x| > R+|t|} \left| \nabla_{t,x} \sum_{k=0}^{\frac{N-6}{2}} c_{k,n}^0 \tilde{S}_k^0 \right|^2 dx \\ &\lesssim \lim_{t \rightarrow \infty} \int_{|x| > R+|t|} |\nabla_{t,x} u_{L,n}|^2 + |\nabla_{t,x} \bar{w}_{L,n}|^2 \\ &\quad + \left| \nabla_{t,x} \sum_{k=0}^{\frac{N-6}{2}} c_{k,n}^\infty c_{k,n}^\infty S_k^\infty \right|^2 dx \end{aligned}$$

which is uniformly bounded in n , where

$$a_{k_1 k_2} := \lim_{t \rightarrow \pm\infty} \int_{|x| > R+|t|} \nabla_{t,x} \tilde{S}_{k_1}^0 \cdot \nabla_{t,x} \tilde{S}_{k_2}^0 dx.$$

Note that the matrix $A := (a_{k_1 k_2})_{0 \leq k_1, k_2 \leq (N-6)/2}$ is symmetric, positive semi-definite. We claim that it is also invertible, hence strictly positive definite. To see this, suppose that $v = (v_k)_{0 \leq k \leq N-6/2}$ lies in its kernel. Letting $f := \sum_{k=0}^{\frac{N-6}{2}} v_k \tilde{S}_k^0$ we obtain

$$\begin{aligned} 0 &= \sum_{i=0}^{\frac{N-6}{2}} v_i \lim_{t \rightarrow \infty} \int_{|x| > R=|t|} \nabla_{t,x} \tilde{S}_k^0 \cdot \nabla_{t,x} \tilde{S}_i^0 \\ &= \lim_{t \rightarrow \infty} \int_{|x| > R=|t|} \nabla_{t,x} \tilde{S}_k^0 \cdot \nabla_{t,x} f. \end{aligned}$$

Multiplying the above identity by v_k and adding over all k we obtain that

$$\lim_{t \rightarrow \infty} \int_{|x| > R+|t|} |\nabla_{t,x} f|^2 = 0.$$

By Lemma 4.4.4 we have that $f \in \text{Span}\{S_k^\infty\}_{0 \leq k \leq N-6/2}$. Thus $f \in \text{Span}\{S_k^\infty\}_{0 \leq k \leq N-6/2} \cap \text{Span}\{S_k^0\}_{0 \leq k \leq N-6/2}$. This intersection is $\{0\}$ as can be seen by successively taking $(-\Delta + V)$ on an equality of the form $\sum_{k=0}^{\frac{N-6}{2}} d_k T_k^\infty = \sum_{k=0}^{\frac{N-6}{2}} d_k T_k^0$. Therefore A is invertible.

From this we conclude that $c_{k,n}^0$ is bounded. Now we have

$$\sum_{k=0}^{\frac{N-6}{2}} (c_{k,n}^\infty - \tilde{c}_{k,n}^\infty) \vec{T}_k^\infty = \sum_{k=0}^{\frac{N-6}{2}} c_{k,n}^0 (\tilde{S}_k^0(0), \partial_t \tilde{S}_k^0(0)) + \Pi^\perp u_n - v_n.$$

Therefore taking the $\dot{H}^1 \times L^2$ inner product with $\{(\tilde{S}_k^0(0), \partial_t \tilde{S}_k^0(0))\}_{k=0}^{\frac{N-6}{2}}$, we obtain

$$B \vec{c} = \vec{b},$$

where

$$\begin{aligned}\vec{c} &= (c_{k,n}^\infty - \tilde{c}_{k,n}^\infty)_{k=0}^{\frac{N-6}{2}}, \\ \vec{b} &= \left(\sum_{i=0}^{\frac{N-6}{2}} c_{k,n}^0 \langle (\tilde{S}_i^0(0), \partial_t \tilde{S}_i^0(0)), \vec{T}_k^\infty \rangle_{\mathcal{H}_{R_n}} \right)_{k=0}^{\frac{N-6}{2}} + \left(\langle (\Pi^\perp u_n - v_n), \tilde{T}_k^\infty \rangle_{\dot{H}^1 \times L^2} \right)_{k=0}^{\frac{N-6}{2}}\end{aligned}$$

and

$$B = (b_{ij})_{0 \leq i, j \leq \frac{N-6}{2}}$$

with

$$b_{k_1 k_2} = \langle T_{k_1}^\infty, T_{k_2}^\infty \rangle_{\mathcal{H}_{R_n}}.$$

Since B is invertible by the same argument as that for A above, we obtain that $c_{k,n} - \tilde{c}_{k,n}$ is bounded since $\vec{b}$ is bounded. Thus in particular $c_{k,n}^\infty - \tilde{c}_{k,n}^\infty$ is bounded. Now we have

$$\sum_{k=0}^{\frac{N-6}{2}} (c_{k,n}^\infty - \tilde{c}_{k,n}^\infty) \vec{T}_k^\infty = \sum_{k=0}^{\frac{N-6}{2}} c_{k,n}^0 (\tilde{S}_k^0(0), \partial_t \tilde{S}_k^0(0)) + \Pi^\perp u_n - v_n.$$

Therefore taking the $\dot{H}_{R_n}^1 \times L_{R_n}^2$ inner product with $\{(\tilde{S}_k^0(0), \partial_t \tilde{S}_k^0(0))\}_{k=0}^{\frac{N-6}{2}}$, we obtain

$$B \vec{c} = \vec{b},$$

where

$$\begin{aligned}\vec{c} &= (c_{k,n}^\infty - \tilde{c}_{k,n}^\infty)_{k=0}^{\frac{N-6}{2}}, \\ \vec{b} &= \sum_{i=0}^{\frac{N-6}{2}} \left(c_{i,n}^0 \langle (\tilde{S}_i^0(0), \partial_t \tilde{S}_i^0(0)), \vec{T}_k^\infty \rangle_{\mathcal{A}_N} \right)_{k=0}^{\frac{N-6}{2}} + \left(\langle (\Pi^\perp u_n - v_n), \vec{T}_k^\infty \rangle_{\dot{H}^1 \times L^2} \right)_{k=0}^{\frac{N-6}{2}}\end{aligned}$$

and

$$B = (b_{ij})_{0 \leq i, j \leq \frac{N-6}{2}}$$

with

$$b_{k_1 k_2} = \langle T_{k_1}^\infty, T_{k_2}^\infty \rangle_{\mathcal{A}, R_n}.$$

Since B is invertible by the same argument as that for A above, we obtain that $\vec{c} = B^{-1} \vec{b}$.

Noting that b_{0k}, b_{k0} for $0 \leq k \leq \frac{N-6}{2}$ are of order $O(1)$ while all other entries are of order R_n^{-N} if $N \equiv 0 \pmod{4}$, $R_n^{-(N-2)}$ if $N \equiv 2 \pmod{4}$, we see that

$$\det B = \begin{cases} O(R_n^{-N \frac{N-6}{2}}) & \text{if } N \equiv 0 \pmod{4} \\ O(R_n^{-(N-2) \frac{N-6}{2}}) & \text{if } N \equiv 2 \pmod{4} \end{cases}$$

and so if $N \equiv 0 \pmod{4}$ (resp. $N \equiv 2 \pmod{4}$) the coordinates of B^{-1} are of order R_n^N (resp. R_n^{-N+2}), except for b^{00} which is of order $O(1)$. This implies that $\vec{c}$ is bounded. Thus $v_{1,n}$ is bounded in $\dot{H}^1(r^{2N-\frac{5}{2}} \langle r \rangle^{-(N-\frac{3}{2})} dr)$ if $N \equiv 0 \pmod{4}$ or $L^2(r^{2N-\frac{9}{2}} \langle r \rangle^{-(N-\frac{7}{2})} dr)$ if $N \equiv 2 \pmod{4}$. Thus in particular $c_{k,n}^\infty - \tilde{c}_{k,n}^\infty$ is bounded, and thus after proceeding as in case 1 using Lemma 4.4.4 one obtains that $v_n \rightharpoonup 0$ in $\dot{H}_R^1 \times L_R^2$ for any $R > 0$, and so

$$v_n \rightharpoonup 0$$

in $\dot{H}^1(r^{2N-\frac{5}{2}} \langle r \rangle^{-(N-\frac{3}{2})}) \times L^2(r^{2N-\frac{9}{2}} \langle r \rangle^{-(N-\frac{7}{2})})$. Next, we decompose v_n as

$$v_n = \sum_{i=0}^{\frac{N-6}{2}} \tilde{c}_n^i(\chi_0 T_i^0, 0) + w_n$$

if $N \equiv 0 \pmod{4}$,

$$v_n = \sum_{i=0}^{\frac{N-6}{2}} \tilde{c}_n^i(0, \chi_0 T_i^0) + w_n$$

if $N \equiv 2 \pmod{4}$, where $w_n \perp_{\dot{H}^1(r^{2N-\frac{5}{2}} \langle r \rangle^{-(N-\frac{3}{2})}) \times L^2(r^{2N-\frac{9}{2}} \langle r \rangle^{-(N-\frac{7}{2})})} (T_i^0, 0)$. We extend

w_n trivially for $r \leq R_n$. Note that

$$\|w_n\|_{\dot{H}_{R_n}^1 \times L_{R_n}^2}^2 + \sum_{i=0}^{\frac{N-6}{2}} (c_{i,n}^0 - \bar{c}_n^i)^2 \|\chi_0 T_i^0\|_{\dot{H}^1}^2 \lesssim \|\Pi^\perp u_n\|_{\dot{H}_{R_n}^1 \times L_{R_n}^2}^2 + \sum_{k=0}^{\frac{N-6}{2}} (c_{k,n}^\infty - \bar{c}_{k,n}^\infty)^2 \|T_k^\infty\|_{\dot{H}_{R_n}^1}^2$$

if $N \equiv 0 \pmod{4}$,

$$\|w_n\|_{\dot{H}_{R_n}^1 \times L_{R_n}^2}^2 + \sum_{i=0}^{\frac{N-6}{2}} (c_{i,n}^0 - \bar{c}_n^i)^2 \|\chi_0 T_i^0\|_{L_{R_n}^2}^2 \lesssim \|\Pi^\perp u_n\|_{\dot{H}_{R_n}^1 \times L_{R_n}^2}^2 + \sum_{k=0}^{\frac{N-6}{2}} (c_{k,n}^\infty - \bar{c}_{k,n}^\infty)^2 \|T_k^\infty\|_{L_{R_n}^2}^2$$

if $N \equiv 2 \pmod{4}$. Now after proceeding as in step 3 of the proof of Lemma 3.1 in Collet et al. [2023], we obtain that w_n converges to $(0, 0)$ weakly in $\dot{H}^1(r^{2N-\frac{5}{2}}\langle r \rangle^{-(N-\frac{3}{2})}) \times L^2(r^{2N-\frac{9}{2}}\langle r \rangle^{-(N-\frac{7}{2})})$. Now we can proceed as in step 4 of the proof of Lemma 3.1 in Collet et al. [2023]. Letting $w_{F,n}$ be the free wave with initial condition w_n and $w_{L,n}$ the free wave with initial condition w_n , we obtain

$$\lim_{n \rightarrow \infty} \left(\lim_{t \rightarrow \infty} \int_{|x| \geq R_n + |t|} |\nabla_{t,x} w_{F,n}|^2 \right) = 0.$$

Moreover we have

$$\begin{aligned} & \left(\left\| \frac{1}{r^{N-2k}} \right\|_{\dot{H}_{R_n}^1} \right)^{-1} \left| \int_{r \geq R_n} \nabla w_{F,n} \cdot \nabla \frac{1}{r^{N-2k}} r^{N-1} dr \right| \\ &= \left(\left\| \frac{1}{r^{N-2k}} \right\|_{\dot{H}_{R_n}^1} \right)^{-1} \left| \int_{r \geq R_n} \nabla w_{F,n} \cdot \nabla \left(\frac{1}{r^{N-2k}} - \chi_0 T_k^0 \right) r^{N-1} dr \right| \\ &\lesssim \left(\left\| \frac{1}{r^{N-2k}} \right\|_{\dot{H}_{R_n}^1} \right)^{-1} \left\| \frac{1}{r^{N-2k}} - \chi_0 T_k^0 \right\|_{\dot{H}_{R_n}^1} \rightarrow 0 \end{aligned}$$

if $N \equiv 0 \pmod{4}$ and similarly for the case $N \equiv 2 \pmod{4}$. However,

$$\|w_n\|_{\dot{H}_{R_n}^1 \times L_{R_n}^2} \geq \|\Pi^\perp w_n\|_{\dot{H}_{R_n}^1} = \|\Pi^\perp u_n\|_{\dot{H}_{R_n}^1} = 1$$

if $N \equiv 0 \pmod{4}$, and

$$\|w_n\|_{\dot{H}_{R_n}^1 \times L_{R_n}^2} \geq \|\Pi^\perp w_n\|_{L_{R_n}^2} = \|\Pi^\perp u_n\|_{L_{R_n}^2} = 1$$

if $N \equiv 2 \pmod{4}$, which contradicts Lemma 4.3.2. □

4.5 Single soliton case (resonant component)

Let χ_0 be a smooth cutoff such that

$$\chi^0(r) = \begin{cases} 1 & \text{if } r \leq 10 \\ 0 & \text{if } r \geq 11. \end{cases}$$

$$\Pi_{L_R^2}^\perp = \begin{cases} \Pi_{L_R^2}(\text{span}\{T_0^\infty, \dots, T_{\frac{N-6}{2}}^\infty\})^\perp & \text{if } R \geq 1 \\ \Pi_{L_R^2}(\text{span}\{T_0^\infty, \dots, T_{\frac{N-6}{2}}^\infty, \chi^0 T_0^0, \dots, \chi^0 T_{\frac{N-6}{2}}^0\})^\perp & \text{if } 0 < R \leq 1, \end{cases}$$

and similarly

$$\Pi_{\dot{H}_R^1}^\perp = \begin{cases} \Pi_{\dot{H}_R^1}(\text{span}\{T_0^\infty, \dots, T_{\frac{N-6}{2}}^\infty\})^\perp & \text{if } R \geq 1 \\ \Pi_{\dot{H}_R^1}(\text{span}\{T_0^\infty, \dots, T_{\frac{N-6}{2}}^\infty, \chi^0 T_0^0, \dots, \chi^0 T_{\frac{N-6}{2}}^0\})^\perp & \text{if } 0 < R \leq 1. \end{cases}$$

Let

$$\|f\|_{Z_\alpha} = \sup_{R>0} \frac{R^{-\frac{N}{2}-\alpha}}{\langle \log R \rangle} \left(\int_{R<|x|<2R} f^2(x) dx \right)^{\frac{1}{2}},$$

$$\|f\|_{Z_{\alpha,R}} = \sup_{\rho \geq R} \frac{\rho^{-\frac{N}{2}-\alpha}}{\langle \log \frac{\rho}{R} \rangle} \left(\int_{\rho<|x|<2\rho} f^2(x) dx \right)^{\frac{1}{2}},$$

and

$$d_{Z_{\alpha,R}}(u, E) := \inf_{v \in E} \|u - v\|_{Z_{\alpha,R}}.$$

We will prove the following.

Lemma 4.5.1. *Suppose u_L is a radial solution to (4.1). Then*

$$\left. \begin{aligned} \|\Pi_{L^2}^\perp u_1\|_{Z_{-4}}^2 & \quad \text{if } N \equiv 0 \pmod{4} \\ \|\nabla \Pi_{\dot{H}^1}^\perp u_0\|_{Z_{-3}}^2 & \quad \text{if } N \equiv 2 \pmod{4} \end{aligned} \right\} \leq C \sum_{\pm} \lim_{t \rightarrow \pm\infty} \int_{r>|t|} |\nabla_{t,x} u_L(t, x)|^2 dx.$$

We will prove the cases $N \equiv 0 \pmod{4}$ and $N \equiv 2 \pmod{4}$ separately, in subsection 4.5.1 and 4.5.2

4.5.1 The $N \equiv 0 \pmod{4}$ case

Let

$$\|u_L\|_{\tilde{Y}} := \sup_{t \in \mathbb{R}, R > |t|} \|\Pi_{\dot{H}_R^1}^\perp u_L(t)\|_{\dot{H}_R^1}$$

and similarly let

$$\|u_L\|_{\tilde{Y}_R} := \sup_{t \in \mathbb{R}, \rho > |t|+R} \|\Pi_{\dot{H}_\rho^1}^\perp u_L(t)\|_{\dot{H}_\rho^1}.$$

We also let

$$\mathcal{A}f(r) := \int_r^\infty \rho f(\rho) d\rho, \quad \mathcal{A}^{-1} = -\frac{1}{r} \partial_r f(r).$$

We will need the following lemma.

Lemma 4.5.2. *Suppose that u_L is a linear radial solution to (4.1). Let $\Pi_{L^2}^\perp$ be defined as in (4.3). Suppose that u_L is odd in time. Then if $N \equiv 0 \pmod{4}$ and $N \geq 10$, then*

$$\|\mathcal{A} \Pi_{L^2}^\perp \partial_t u_L(0)\|_{Z_{-2}} \lesssim \|u_L\|_{\tilde{Y}}.$$

Note that the above lemma is proven in the case of $N = 8$ in Collot et al. [2023].

Proof. First, note that without loss of generality, we can assume that

$$\Pi_{L^2}^\perp u_L(0) = u_L(0). \quad (4.24)$$

Indeed, suppose we have showed the result for solutions satisfying (4.24). Then if u_L is any solutions satisfying the hypotheses of the lemma, we let $\bar{u}_L$ denote the solution to (4.1) with initial condition given by $(0, \Pi_{L^2}^\perp \partial_t u_L(0))$, and we let $v := u_L - \bar{u}_L$. Then we have

$$\|\mathcal{A}\Pi_{L^2}^\perp \partial_t u_L(0)\|_{Z_{-2}} = \|\mathcal{A}\partial_t \bar{u}_L(0)\|_{Z_{-2}} \lesssim \|\bar{u}_L\|_{\tilde{Y}} \lesssim \|u_L\|_{\tilde{Y}},$$

where the last inequality comes from direct computation, using that by Lemma 4.4.2

$$|T_k^\infty(r)| + \left| r \frac{d}{dr} T_k^\infty(r) \right| \lesssim \langle r \rangle^{-N+2+2k}.$$

Let $R_k := 2^k$. Then note that

$$\begin{aligned} \partial_t u_L(0) &= - \int_0^{R_k} \partial_t \left(\left(1 - \frac{t}{R_k} \right) \partial_t u_L(t) \right) dt \\ &= \int_0^{R_k} \left(1 - \frac{t}{R_k} \right) (-\partial_{tt}) u_L(t) dt + \frac{1}{R_k} \int_0^{R_k} \partial_t u_L(t) dt \\ &= (-\Delta + V) \int_0^{R_k} \left(1 - \frac{t}{R_k} \right) u_L(t) dt + \frac{u_L(R_k)}{R_k} \end{aligned} \quad (4.25)$$

where in the last line we used the fact that u_L is odd in time, hence $u_L(0) = 0$. Let $\tilde{u}_k(t) = \Pi_{\dot{H}_{R_k}^1}^\perp u_L(t)$. Writing T_k^0 in terms of $\tilde{T}_k^0$ by Lemma 4.4.3, there exist coefficients $\alpha_{j,k}^\infty = \alpha_{j,k}^\infty(t), \alpha_{j,k}^0 = \alpha_{j,k}^0(t)$ such that

$$u_L(t) = \sum_{j=0}^{\frac{N-6}{2}} \alpha_{j,k}^\infty T_j^\infty + \alpha_{j,k}^0 \tilde{T}_j^0 + \tilde{u}_k(t)$$

where by convention $\alpha_{j,k}^0 = 0$ if $k \geq 0$ and where we let $\tilde{T}_j^0$ be defined as in Lemma 4.4.3. This implies

$$\begin{aligned} & (-\Delta + V) \left(\int_0^{R_k} \left(1 - \frac{t}{R_k} \right) u_L(t) dt \right) \\ &= \int_0^{R_k} \left(1 - \frac{t}{R_k} \right) \left(\left(\sum_{j=0}^{\frac{N-6}{2}} -\alpha_{j+1,k}^\infty(t) T_j^\infty - \alpha_{j+1,k}^0(t) \tilde{T}_j^0(t) \right) + (-\Delta + V) \tilde{u}_k(t) \right) dt \end{aligned} \quad (4.26)$$

whenever $r \geq R_k$ if $k \geq 0$, or for all $R_k \leq r \leq 10$ if $k \leq -1$. Recall that by Lemma 4.4.3, there exist coefficients $e_{j,i}^\infty, e_{j,i}^0$ such that

$$\tilde{T}_i^0 := T_i^0 - \sum_{j=1}^{i+1} e_{j,i}^\infty T_j^\infty - \sum_{j=1}^{i-1} e_{j,i}^0 \tilde{T}_j^0 \sim r^2$$

for $r \rightarrow 0$. Plugging this and (4.26) into (4.25), we obtain that for some coefficients

$$\{c_{j,k}^\infty\}, \{c_{j,k}^0\}$$

$$\partial_t u(0) = \frac{1}{R_k} \tilde{u}_k(R_k) - \sum_{j=0}^{\frac{N-6}{2}} c_{j,k}^\infty T_j^\infty - \sum_{j=0}^{\frac{N-6}{2}} c_{j,k}^0 \tilde{T}_j^0 + (-\Delta + V) \left(\int_0^{R_k} \left(1 - \frac{t}{R_k} \right) \tilde{u}_k(t) dt \right).$$

Using the above equality for k and $k+1$ we obtain

$$\begin{aligned} & \sum_{j=0}^{\frac{N-6}{2}} (c_{j,k+1}^\infty - c_{j,k}^\infty) T_j^\infty + \sum_{j=0}^{\frac{N-6}{2}} (c_{j,k+1}^0 - c_{j,k}^0) \tilde{T}_j^0 \\ &= \frac{1}{R_{k+1}} \tilde{u}_{k+1}(R_{k+1}) - \frac{1}{R_k} \tilde{u}_k(R_k) \\ &+ (-\Delta + V) \left(\int_0^{R_{k+1}} \left(1 - \frac{t}{R_{k+1}} \right) \tilde{u}_{k+1}(t) dt - \int_0^{R_k} \left(1 - \frac{t}{R_k} \right) \tilde{u}_k(t) dt \right). \end{aligned} \quad (4.27)$$

Now we choose functions $\Phi_k, \tilde{\Phi}_k$ such that $\text{Supp}\Phi_k, \text{Supp}\tilde{\Phi}_k \subseteq [2R_k, 3R_k]$,

$$|\partial_r^j \Phi_k| \lesssim R_k^{-j}, \quad |\partial_r^j \tilde{\Phi}_k| \lesssim R_k^{-j}$$

for $j = 0, \dots, N-6$ and such that

$$\int_{\mathbb{R}^N} \tilde{T}_0^0 \tilde{\Phi}_k = R_k^2, \quad \int_{\mathbb{R}^N} \tilde{T}_j^0 \tilde{\Phi}_k = 0 \text{ if } j \geq 1,$$

$$\int_{\mathbb{R}^N} T_j^\infty \tilde{\Phi}_k = 0 \text{ if } k \geq -1,$$

and

$$\int_{\mathbb{R}^N} T_0^\infty \Phi_k = R_k^2, \quad \int_{\mathbb{R}^N} T_j^\infty \Phi_k = 0 \text{ if } j \geq 1,$$

$$\int_{\mathbb{R}^N} T_j^0 \Phi_k = 0 \text{ if } k \geq -1.$$

To show this exists, we rescale by R_k . Let $\tilde{\Phi}_k(r) := \tilde{\Phi}^k\left(\frac{r}{R_k}\right)$. Then we obtain the conditions

$$\int_0^\infty \frac{\tilde{T}_0^0(R_k r)}{R_k^{-(N-2)}} \tilde{\Phi}^k(r) r^{N-1} dr = R_k^2$$

$$\int_0^\infty F(R_k r) \tilde{\Phi}^k(r) r^{N-1} dr = 0$$

for all $F \in \{\tilde{T}_j^0\}_{j=1}^{\frac{N-6}{2}} \cup \{T_j^\infty\}_{j=0}^{\frac{N-6}{2}}$. Since $R_k^{(N-2)} \tilde{T}_0^0(R_k r) \sim cr^{-(N-2)}$ for some absolute constant c as $r \rightarrow 0$, we can take a function $\Phi^k \perp_{L^2(r^{N-1} dr)} \text{Span}\{F(R_k \cdot) | F \in \{\tilde{T}_j^0\}_{j \geq 1} \cup \{T_j^\infty\}_{j \geq 0}\}$ with

$$\left\langle \Phi^k, \frac{\tilde{T}_0^0(R_k \cdot)}{R_k^{-(N-2)}} \right\rangle = 1$$

and $|\partial_r^j \Phi^k| \lesssim 1$ for all $0 \leq j \leq N-6$. We can then take $\Phi_k(r) := \tilde{\Phi}^k\left(\frac{r}{R_k}\right)$. The existence of $\tilde{\Phi}_k$ is proven similarly.

Fix any $R > 0$, and let k_0 be the unique integer such that $R_{k_0-2} \leq R < R_{k_0-1}$ and let $k_1 = \max\{k_0, 0\}$. Now applying $(-\Delta + V)^j$ for $1 \leq j \leq \frac{N-6}{2}$ on both sides of (4.27) and integrating against Φ_k if $k \geq -1$ and Φ^k if $k \leq -1$ we obtain

$$|c_{j,k+1}^\infty - c_{j,k}^\infty| \leq \|u_L\|_{\tilde{Y}_R} R_k^{-2j}$$

for all $k \geq -1$, and after plugging these estimates back into (4.27) we obtain

$$|c_{j,k+1}^\infty - c_{j,k}^\infty| \lesssim \|u_L\|_{\tilde{Y}_R} R_k^{-1-2j}$$

for all $k \leq -2$. Then if we write $c_{j,k}^\infty = c_{j,k_1}^\infty + \bar{c}_{j,k}^\infty$, $c_{j,k}^0 = c_{j,k_1}^0 + \bar{c}_{j,k}^0$,

$$\bar{c}_{j,k}^\infty = \begin{cases} O(k R_k^{-2j} \|u_L\|_{\tilde{Y}_R}) & \text{if } k \geq 0, \\ O(R_k^{-2j-1} \|u_L\|_{\tilde{Y}_R}) & \text{if } k_0 \leq k \leq -1 \end{cases} \quad (4.28)$$

if $k_0 \leq 0$, and

$$\bar{c}_{j,k}^\infty = O(|k - k_0| \|u_L\|_{\tilde{Y}_R} R_k^{-2j}) \quad (4.29)$$

if $k \geq k_0$. Similarly, we have

$$|c_{j,k+1}^0 - c_{j,k}^0| \lesssim R_k^{-2j} \|u_L\|_{\tilde{Y}_R} \quad (4.30)$$

and so

$$|\bar{c}_{j,k}^0| \lesssim |k| R_k^{-2j} \|u_L\|_{\tilde{Y}_R}. \quad (4.31)$$

Therefore

$$\begin{aligned}
\partial_t u_L(0) &= \frac{1}{R_k} \tilde{u}_k(R_k) + \sum_{j=0}^{\frac{N-6}{2}} c_{j,k}^\infty T_j^\infty + \sum_{j=0}^{\frac{N-6}{2}} c_{j,k}^0 \tilde{T}_j^0 \\
&\quad + (-\Delta + V) \left(\int_0^{R_k} \left(1 - \frac{t}{R_k} \right) \tilde{u}_k(t) dt \right) \\
&= \sum_{j=0}^{\frac{N-6}{2}} c_{j,k_1}^\infty T_j^\infty + \left(\sum_{j=0}^{\frac{N-6}{2}} (\bar{c}_{j,k}^\infty T_j^\infty + c_{j,k}^0 \tilde{T}_j^0) + \frac{1}{R_k} \tilde{u}_k(R_k) \right) \\
&\quad + (-\Delta + V) \left(\int_0^{R_k} \left(1 - \frac{t}{R_k} \right) \tilde{u}_k(t) dt \right).
\end{aligned}$$

Let η be such that $\eta_k := \eta\left(\frac{r}{R_k}\right)$ is a partition of unity, that is

$$\sum_{k=-\infty}^{\infty} \eta\left(\frac{r}{R_k}\right) = 1.$$

Additionally, suppose $\text{Supp} \eta \subseteq [1, 3]$, and let

$$f_k^1 := \left(\sum_{j=0}^{\frac{N-6}{2}} \bar{c}_{j,k}^\infty T_j^\infty + c_{j,k}^0 \tilde{T}_j^0 \right) + \frac{1}{R_k} \tilde{u}_k(R_k).$$

Then if $k \geq k_0$, $R_k \leq r \leq 3R_k$, we have

$$\begin{aligned}
\partial_t u_L(0) &= \sum_{k \geq k_0} \eta_k \partial_t u_L(0) \\
&= \sum_{k \geq k_0} \eta_k \left(\sum_{j=0}^{\frac{N-6}{2}} \bar{c}_{j,k_1}^\infty T_j^\infty + \left(\sum_{j=0}^{\frac{N-6}{2}} (\bar{c}_{j,k}^\infty T_j^\infty + c_{j,k}^0 \tilde{T}_j^0) + \frac{1}{R_k} \tilde{u}_k(R_k) \right) \right. \\
&\quad \left. + (-\Delta + V) \left(\int_0^{R_k} \left(1 - \frac{t}{R_k} \right) \tilde{u}_k(t) dt \right) \right) \\
&= \sum_{j=0}^{\frac{N-6}{2}} c_{j,k_1}^\infty T_j^\infty + \sum_{k \geq k_0} \eta_k \left(\left(\sum_{j=0}^{\frac{N-6}{2}} \bar{c}_{j,k}^\infty T_j^\infty + c_{j,k}^0 \tilde{T}_j^0 \right) + \frac{1}{R_k} \tilde{u}_k(R_k) \right) \\
&\quad + \sum_{k \geq k_0} \eta_k (-\Delta + V) \left(\int_0^{R_k} \left(1 - \frac{t}{R_k} \right) \tilde{u}_k(t) dt \right) \\
&=: \sum_{j=0}^{\frac{N-6}{2}} c_{j,k_1}^\infty T_j^\infty + \sum_{k \geq k_0} \eta_k f_k^1 + \sum_{k \geq k_0} \eta_k (-\Delta + V) \left(\int_0^{R_k} \left(1 - \frac{t}{R_k} \right) \tilde{u}_k(t) dt \right).
\end{aligned}$$

Note that

$$\begin{aligned}
&(-\Delta + V) \left(\eta_k \left(\int_0^{R_k} \left(1 - \frac{t}{R_k} \right) \tilde{u}_k(t) dt \right) \right) - \eta_k (-\Delta + V) \left(\int_0^{R_k} \left(1 - \frac{t}{R_k} \right) \tilde{u}_k(t) dt \right) \\
&= R_k^{-2} (2\eta_k'' - \Delta \eta_k) \left(\frac{r}{R_k} \right) \int_0^{R_k} \left(1 - \frac{t}{R_k} \right) \tilde{u}_k(t) dt \\
&\quad + \partial_r \left(-2R_k^{-1} \eta_k' \left(\frac{r}{R_K} \right) \int_0^{R_k} \left(1 - \frac{t}{R_k} \right) \tilde{u}_k(t) dt \right) \\
&=: f_k^2 + \partial_r \tilde{g}_k.
\end{aligned}$$

This implies that

$$\begin{aligned}\partial_t u_L(0) &= \sum_{j=0}^{\frac{N-6}{2}} c_{j,k_1}^\infty T_j^\infty + \sum_{k \geq k_0} (\eta_k f_k^1 + f_k^2) + \partial_r \left(\sum_{k \geq k_0} \tilde{g}_k \right) \\ &+ (-\Delta + V) \left(\sum_{k \geq k_0} \eta_k \left(\int_0^{R_k} \left(1 - \frac{t}{R_k} \right) \tilde{u}_k(t) dt \right) \right).\end{aligned}$$

We let

$$p_1 := (-\Delta + V) \sum_{k \geq k_0} \eta_k \left(\int_0^{R_k} \left(1 - \frac{t}{R_k} \right) \tilde{u}_k(t) dt \right).$$

Then we have the decomposition

$$\partial_t u_L(0) = p_1 + \sum_{j=1}^{\frac{N-6}{2}} c_{j,k_1}^0 T_j^0 + \tilde{u}_1$$

where

$$-\tilde{u}_1 = f + \partial_r g,$$

where

$$f = \sum_{k \geq k_0} \eta_k f_k^1 + f_k^2, \quad g = \sum_{k \geq k_0} \tilde{g}_k. \quad (4.32)$$

Note that by Hardy's inequality (see e.g. Collot et al. [2023]),

$$\|v\|_{Z_{-2,R}} \lesssim \|\partial_r v\|_{Z_{-3,R}}.$$

Therefore

$$\|\mathcal{A}(\partial_t u_L(0))\|_{Z_{-2,R}} \leq \|\mathcal{A}p_1\|_{Z_{-2,R}} + \sum_{j=1}^{\frac{N-6}{2}} c_{j,k_1}^0 \|\mathcal{A}T_j^0\|_{Z_{-2,R}} + \|\mathcal{A}\tilde{u}_1\|_{Z_{-2,R}}.$$

We will now bound each term: using (4.28), (4.29), (4.30), (4.31),

$$\|\mathcal{A}(\eta_k f_k^1)\|_{Z_{-2,R}} \lesssim \|\partial_r \mathcal{A}(\eta_k f_k^1)\|_{Z_{-3,R}} = \|(r\eta_k f_k^1)\|_{Z_{-3,R}}$$

and

$$\begin{aligned} \|\mathcal{A}(r\eta_k f_k^1)\|_{L^2(R_k \leq r \leq 3R_k)} &= \|r\eta_k \left(\sum_{j=0}^{\frac{N-6}{2}} (\bar{c}_{j,k}^\infty T_j^\infty + c_{j,k}^\infty \tilde{T}_j^0) - \frac{1}{R_k} \tilde{u}_k(R_k) \right)\|_{L^2(R_k \leq r \leq 3R_k)} \\ &\lesssim \left\langle \log \frac{R_k}{\langle R \rangle} \right\rangle \|u_L\|_{\tilde{Y}_R}. \end{aligned}$$

We also have

$$\|\mathcal{A}(f_k^2)\|_{Z_{-2,R}} \lesssim \|\partial_r \mathcal{A}(f_k^2)\|_{Z_{-3,R}} = \|(r f_k^2)\|_{Z_{-3,R}}$$

and

$$\|r f_k^2\|_{L^2(R_k \leq r \leq 3R_k)} \lesssim R_k^{-1} \int_0^{R_k} \|\partial_t u_L(0)\|_{L^2(R_k \leq r \leq 3R_k)} dt \lesssim \|u_L\|_{\tilde{Y}_R}.$$

By definition of the $Z_{-2,R}$ norm,

$$\|f\|_{Z_{-2,R_{k_0}}} \lesssim \|u_L\|_{\tilde{Y}_R}.$$

On the other hand, since

$$\mathcal{A}(\partial_r \tilde{g}_k) = -r \tilde{g}_k(r) - \int_r^\infty \tilde{g}_k(r) dr,$$

we have

$$\|\mathcal{A}(\partial_r \tilde{g}_k)\|_{L^2(R_k \leq r \leq 3R_k)} \lesssim \int_0^{R_k} \|\partial_t u_L(0)\|_{L^2(R_k \leq r \leq 3R_k)} dt \lesssim R_k \|u_L\|_{\tilde{Y}_R}.$$

This implies $\|g\|_{\tilde{Z}_{-2,R_{k_0}}} \lesssim \|u_L\|_{\tilde{Y}_R}$. Now to bound p_1 , note that

$$\begin{aligned}
\mathcal{A}p_1 &= \int_r^\infty \rho p_1(\rho) d\rho \\
&= \sum_{k \geq k_0} \int_r^\infty \rho(-\Delta + V) \left(\eta_k \left(\int_0^{R_k} \left(1 - \frac{t}{R_k}\right) \tilde{u}_k(t) dt \right) \right) d\rho \\
&= \sum_{k \geq k_0} \int_r^\infty \rho V(\rho) \eta_k \left(\int_0^{R_k} \left(1 - \frac{t}{R_k}\right) \tilde{u}_k(t) dt \right) \\
&\quad - r \partial_r \left(\eta_k \int_0^{R_k} \left(1 - \frac{t}{R_k}\right) \tilde{u}_k(t) dt \right) + (N-2) \eta_k \int_0^{R_k} \left(1 - \frac{t}{R_k}\right) \tilde{u}_k(t) dt \\
&=: \sum_{k \geq k_0} B_1^k + B_2^k + B_3^k.
\end{aligned}$$

Since $V(\rho) \lesssim \rho^{-4}$ for $\rho > r$, we have

$$\left\| \sum_{k \geq k_0} B_1^k \right\|_{Z_{-2,R}}, \left\| \sum_{k \geq k_0} B_3^k \right\|_{Z_{-2,R}} \lesssim \|u_L\|_{\tilde{Y}_R}.$$

To bound B_2^k , note that

$$\|B_2^k\|_{L^2(R_k \leq r \leq 3R_k)} \lesssim R_k^{-1} \int_0^{R_k} \|\partial_\rho(\eta_k \tilde{u}_k(t))\|_{L^2(R_k \leq r \leq 3R_k)} dt \lesssim \|u_L\|_{\tilde{Y}_R}$$

and so $\|\mathcal{A}(\partial_t u_L(0))\|_{Z_{-2}} \lesssim \|u_L\|_{\tilde{Y}}$. □

Now we improve the bound with the following lemma.

Lemma 4.5.3. *Suppose that u_L solves (4.1). We have*

$$\|\Pi_{L^2}^\perp \partial_t u_L(0)\|_{Z_{-4}} \lesssim \|\mathcal{A}(\Pi_{L^2}^\perp \partial_t u_L(0))\|_{Z_{-2}} + \|u_L\|_{\tilde{Y}}$$

Proof. First we claim we can suppose without loss of generality that $\partial_t u_L(0) = \Pi_{L^2}^\perp \partial_t u_L(0)$.

Indeed, it suffices to note that if

$$\Pi_{L^2}^\perp \partial_t u_L(0) = \partial_t u_L(0) - \sum_{k=0}^{\frac{N-6}{2}} c_k T_k^\infty,$$

then

$$\|\Pi_{L^2}^\perp \left(\partial_t u_L(0) - \sum_{k=0}^{\frac{N-6}{2}} c_k T_k^\infty \right)\|_{Z_{-3}} \lesssim \|\partial_t u_L(0) - \sum_{k=0}^{\frac{N-6}{2}} c_k T_k^\infty\|_{Z_{-3}}$$

by direct computation, using that by Lemma 4.4.2

$$|T_k^\infty(r)| + \left| r \frac{d}{dr} T_k^\infty(r) \right| \lesssim \langle r \rangle^{-N+2+2k}.$$

Now suppose then that $\partial_t u_L(0) = \Pi_{L^2}^\perp \partial_t u_L(0)$, and let $v = \mathcal{A} \partial_t u$. We also define the operator

$$\Delta_N := \partial_{rr} + \frac{N-1}{r} \partial_r$$

Noting that $\Delta_N = \mathcal{A}^{-1} \Delta_{N-2} \mathcal{A}$ we see that v solves

$$\begin{cases} \partial_t^2 v - \Delta_{N-2} v + \mathcal{A} V \mathcal{A}^{-1} v = 0, \\ \vec{v}(0) = (\mathcal{A} \partial_t u(0), 0). \end{cases}$$

Note that the kernel of the operator $-\Delta_{N-2} + \mathcal{A} V \mathcal{A}^{-1}$ is $\text{Span}\{\mathcal{A} T_0^\infty, \int_1^r \rho T_0^0(\rho) d\rho, 1\}$. Let k be a fixed integer, and let $R_k = 2^k$, $\tau_k = \frac{R_k}{2}$. Let $\tilde{u}_k(t)$ be the solution to (4.1) with initial condition $\Pi_{\dot{H}_{R_k}^1} u_L(t)$. By the proof of Lemma 4.5.2, we can decompose the solution u_L as

$$u_L(t) = \sum_{j=0}^{\frac{N-6}{2}} \alpha_{j,k}^\infty T_j^\infty + \alpha_{j,k}^0 \chi_0 T_j^0 + \tilde{u}_k(t) \quad (4.33)$$

for $|t| \leq R_k$. Also,

$$\begin{aligned}\partial_t v &= -\mathcal{A}(-\Delta_N + V)u(t) = -\int_r^\infty -\rho \partial_{\rho\rho} u - (N-1)\partial_\rho u + V u d\rho = r \partial_r u(r) \\ &\quad + (N-2)u(t) - \mathcal{A}V u(t).\end{aligned}$$

Letting $\bar{T}_{j-1}^0 := (-\Delta_N + V)(\chi_0 T_j^0)$, we see that by applying $\mathcal{A}(-\Delta_N + V)$ to (4.33)

$$\partial_t v = \sum_{j=0}^{\frac{N-6}{2}} \alpha_{j+1,k}^\infty \mathcal{A}T_j^\infty + \alpha_{j+1,k}^0 \mathcal{A}\bar{T}_j^0 - \mathcal{A}(-\Delta_N + V)\tilde{u}_k(t).$$

Let $q := \|\mathcal{A}(\partial_t u(0))\|_{Z_{-2,R}} + \|u\|_{\tilde{Y}_R}$. Recall from Lemma 4.5.2 $R_k := 2^k$, $\tau_k = \frac{R_k}{2}$, and let k_0 such that $R_{k_0-1} \leq 3R \leq R_{k_0}$. We have the following identity for any radial function w and smooth test function ψ :

$$\begin{aligned}&\int_{\mathbb{R}^N} \int_{\mathbb{R}} \psi(\partial_{tt} w - \Delta w + V w) w dt dx \\ &= \int_{\mathbb{R}^N} \int_{\mathbb{R}} (|\nabla w|^2 - (\partial_t w)^2) \psi dt dx + \int_{\mathbb{R}^N} \int_{\mathbb{R}} (V + \frac{1}{2}(\partial_{tt} - \Delta)) \psi w^2 dt dx.\end{aligned}$$

Let $\bar{\chi}$ be a cutoff such that $\bar{\chi}(r) = 1$ for $r \in [3, 4]$ and $\bar{\chi}(r) = 0$ for $r \leq 2$ or $r \geq 10$, and as before let $\bar{\chi}_k(r) = \bar{\chi}\left(\frac{r}{R_k}\right)$, $\tilde{\chi}_k(r) = \tilde{\chi}\left(\frac{r}{R_k}\right)$, $\psi_k(t, r) = \tilde{\chi}_k(t) \bar{\chi}_k(r)$, where $\tilde{\chi}(t) = 1$ if $|t| \leq \frac{1}{2}$, and $\tilde{\chi}(t) = 0$ if $|t| \geq 1$. Let $k \geq k_0$. Note that $\text{supp}(\bar{\chi}_k) \subseteq \{r \geq 6R\}$, $\text{supp}\psi_k \subseteq \{r \geq |t| + R\}$. Let

$$\langle u, v \rangle_{L_{\bar{\chi}_k}^2} = \int_{\mathbb{R}^N} uv \bar{\chi}_k, \quad \|u\|_{L_{\bar{\chi}_k}^2} = \langle u, u \rangle_{L_{\bar{\chi}_k}^2}^{1/2}.$$

Let $w_k = w_k(t)$, $b_{j,k} = b_{j,k}(t)$, $\tilde{b}_{j,k} = \tilde{b}_{j,k}(t)$ be uniquely defined so that

$$u_L(t) = w_k(t) + \sum_{j=0}^{\frac{N-6}{2}} b_{j,k} T_j^\infty + \tilde{b}_{j,k} \tilde{Y}_j^0$$

where $\tilde{Y}_j^0 = \chi_0 T_j^0$, and

$$w_k \perp_{L^2_{\chi_k}} \{T_j^\infty, \tilde{Y}_j^0\}_{j=0}^{\frac{N-6}{2}} \text{ for } k \leq -1, \quad w_k \perp_{L^2_{\chi_k}} \{T_j^\infty\}_{j=0}^{\frac{N-6}{2}} \text{ for } k \geq 0.$$

Then note that

$$\begin{aligned} \int_{\mathbb{R}^N} \int_{\mathbb{R}} |\nabla w_k|^2 \psi_k dt dx &= \int_{\mathbb{R}^N} \int_{\mathbb{R}} (\partial_t w_k)^2 \psi_k dt dx - \int_{\mathbb{R}^N} \int_{\mathbb{R}} \left(V + \frac{1}{2}(\partial_{tt} - \Delta) \right) \psi_k w_k^2 dt dx \\ &+ \int_{\mathbb{R}^N} \int_{\mathbb{R}} \psi_k (\partial_{tt} w_k - \Delta w_k + V w_k) w_k dt dx. \end{aligned}$$

We have

$$\begin{aligned} &\int_{\mathbb{R}^N} \int_{\mathbb{R}} \psi_k (\partial_{tt} w_k - \Delta w_k + V w_k) w_k dt dx \\ &- \int_{\mathbb{R}^N} \int_{\mathbb{R}} \psi_k \left(- \sum_{j=0}^{\frac{N-6}{2}} b_{j,k} (-\Delta + V) T_j^\infty + \tilde{b}_{j,k} (-\Delta + V) \tilde{Y}_j^0 \right. \\ &\quad \left. - \sum_{j=0}^{\frac{N-6}{2}} (\partial_{tt} b_{j,k}) T_j^\infty - (\partial_{tt} \tilde{b}_{j,k}) \tilde{Y}_j^0 \right) w_k dt dx \\ &= 0 \end{aligned}$$

since $w_k \perp_{L^2_{\chi_k}} T_j^\infty, \tilde{Y}_j^0$. Therefore

$$\int_{\mathbb{R}^N} \int_{\mathbb{R}} |\nabla w_k|^2 \psi_k dt dx = \int_{\mathbb{R}^N} \int_{\mathbb{R}} (\partial_t w_k)^2 dt dx - \int_{\mathbb{R}^N} \int_{\mathbb{R}} \left(V + \frac{1}{2}(\partial_{tt} - \Delta) \right) \psi_k w_k^2 dt dx. \quad (4.34)$$

Let

$$g_k = \partial_t v - \partial_t \left(\sum_{j=0}^{\frac{N-6}{2}} b_{j,k} T_j^\infty + \tilde{b}_{j,k} \tilde{Y}_j^0 \right).$$

Now, there exist coefficients $b_{j,k}, \tilde{b}_{j,k}, \beta_{j,k}^\infty, \beta_{j,k}^0$ such that

$$v(0) = w_k(0) + \sum_{j=0}^{\frac{N-6}{2}} b_{j,k}(0) T_j^\infty + \tilde{b}_{j,k}(0) \tilde{Y}_j^0 \quad (4.35)$$

and

$$g_k = \partial_t w_k(t) + \sum_{j=0}^{\frac{N-6}{2}} \beta_{j,k}^\infty(0) T_j^\infty + \beta_{j,k}^0(0) T_j^0, \quad (4.36)$$

where $\beta_{j,k}$ are the unique coefficients such that the above identity holds, and where

$$w_k(0) = \Pi_{L_{R_k}^2} v(0).$$

Using the definition of $Z_{-2,R}$, we obtain

$$\|v_0\|_{L_{\tilde{\chi}_k}^2} \lesssim \|v_0\|_{L^2(2R_k \leq r \leq 10R_k)} \lesssim R_k \langle \log \frac{R_k}{\langle R \rangle} \rangle \|v_0\|_{Z_{-2,R}} \leq R_k \langle \log \frac{R_k}{\langle R \rangle} \rangle q \quad (4.37)$$

and similarly

$$\|g_k\|_{L_{\tilde{\chi}_k}^2} \lesssim \|g_k\|_{L^2(2R_k \leq r \leq 10R_k)} \lesssim \|g_k\|_{L_{2R_k}^2} \leq \|\partial_t v\|_{Y_R} \leq q. \quad (4.38)$$

Now by Pythagoras,

$$\left\| \sum_{j=0}^{\frac{N-6}{2}} b_{j,k} T_j^\infty + \tilde{b}_{j,k} \tilde{Y}_j^0 \right\|_{L_{\tilde{\chi}_k}^2} \leq \|u_0\|_{L_{\tilde{\chi}_k}^2} \lesssim R_k \left\langle \log \frac{R_k}{\langle R \rangle} \right\rangle q,$$

$$\left\| \sum_{j=0}^{\frac{N-6}{2}} \beta_{j,k}^\infty T_j^\infty + \beta_{j,k}^0 \tilde{Y}_j^0 \right\|_{L_{\tilde{\chi}_k}^2} \leq \|g_k(t)\|_{L_{\tilde{\chi}_k}^2} \lesssim q.$$

Note that for general $\gamma_{j,k}^\infty, \gamma_{j,k}^0$, we have

$$\left\| \sum_{j=0}^{\frac{N-6}{2}} \gamma_{j,k}^\infty T_j^\infty + \gamma_{j,k}^0 \tilde{Y}_j^0 \right\|_{L^2(2R_k \leq r \leq 10R_k)} \lesssim \left\| \sum_{j=0}^{\frac{N-6}{2}} \gamma_{j,k}^\infty T_j^\infty + \gamma_{j,k}^0 \tilde{Y}_j^0 \right\|_{L_{\chi_k}^2}.$$

This implies in particular that

$$\left\| \sum_{j=0}^{\frac{N-6}{2}} b_{j,k}^\infty T_j^\infty + b_{j,k}^0 \tilde{Y}_j^0 \right\|_{L^2(2R_k \leq r \leq 10R_k)} \lesssim R_k \left\langle \log \frac{R_k}{\langle R \rangle} \right\rangle q, \quad (4.39)$$

$$\left\| \sum_{j=0}^{\frac{N-6}{2}} \beta_{j,k}^\infty T_j^\infty + \beta_{j,k}^0 \tilde{Y}_j^0 \right\|_{L^2(2R_k \leq r \leq 10R_k)} \lesssim q. \quad (4.40)$$

Putting (4.39), (4.40), (4.37), (4.38) together with (4.35), (4.36) we obtain

$$\|w_k(t)\|_{L^2(2R_k \leq r \leq 10R_k)} \lesssim R_k \left\langle \log \frac{R_k}{\langle R \rangle} \right\rangle q,$$

$$\|\partial_t w_k(t)\|_{L^2(2R_k \leq r \leq 10R_k)} \lesssim q.$$

Combining this with (4.34) we obtain

$$\int_{\mathbb{R}^N} \int_{\mathbb{R}} |\nabla w_k|^2 \psi_k dt dx \lesssim R_k \left\langle \log \frac{R_k}{\langle R \rangle} \right\rangle^2 q^2.$$

By the mean value theorem, there is a t_k with $|t_k| \leq \frac{R_k}{2}$ such that

$$\int_{3R_k \leq r \leq 9R_k} |\nabla w(t_k)|^2 dx \lesssim \left\langle \log \frac{R_k}{\langle R \rangle} \right\rangle^2 q^2.$$

This bounds ∇w_k . To bound $\nabla v(0)$, by finite propagation speed we have for $|t| \leq t_k$,

$4R_k \leq r \leq 8R_k$, that

$$v(t_k) = w_k(t_k) + \sum_{j=0}^{\frac{N-6}{2}} h_{j,k} T_j^\infty + \tilde{h}_{j,k} T_j^0,$$

$$\partial_t v(t_k) = g_k(t_k) + \sum_{j=0}^{\frac{N-6}{2}} a_{j,k} T_j^\infty + \tilde{a}_{j,k} \tilde{Y}_j^0$$

for appropriate coefficients $h_{j,k}, \tilde{h}_{j,k}, a_{j,k}, \tilde{a}_{j,k}$. Hence

$$v(t) = \bar{w}_k(t) + \sum_{j=0}^{\frac{N-6}{2}} (h_{j,k} + (t - t_{j,k})a_{j,k})T_j^\infty + (\tilde{h}_{j,k} + (t - t_{j,k})\tilde{a}_{j,k})\tilde{Y}_j^0$$

where $\bar{w}_k$ solves

$$\begin{cases} \partial_{tt}\bar{w}_k - \Delta\bar{w}_k + V\bar{w}_k = 0, \\ (\bar{w}_k, \partial_t\bar{w}_k)|_{t=t_k} = (w_k(t_k), g_k(t_k)). \end{cases}$$

Letting

$$d_{j,k} = h_{j,k}(t_k) - t_k a_{j,k}(t_k), \quad \tilde{d}_{j,k} = \tilde{h}_{j,k}(t_k) - t_k \tilde{a}_{j,k}(t_k)$$

we see that for all $4R_k \leq r \leq 8R_k$,

$$v_0 = \bar{w}_k(0) + \sum_{j=0}^{\frac{N-6}{2}} d_{j,k} T_j^\infty + \tilde{d}_{j,k} \tilde{Y}_j^0.$$

Let $(\tilde{w}_{k,0}, \tilde{w}_{k,1})$ be such that

$$(\tilde{w}_{k,0}, \tilde{w}_{k,1})(r) = (w_k, g_k)(t, r)$$

for all $r \in [3R_k, 9R_k]$, and

$$\|(\tilde{w}_{k,0}, \tilde{w}_{k,1})\|_{\dot{H}^1 \times L^2} \lesssim \int_{3R_k \leq |x| \leq 9R_k} \left(|\nabla v_k(t_k)|^2 + \frac{|v_k(t_k)|^2}{R_k^2} + |g_k(t_k)|^2 \right) dx.$$

Also let $\tilde{w}_k$ be the solution to

$$\begin{cases} \partial_t^2 \tilde{w}_k - \Delta \tilde{w}_k + V \tilde{w}_k = 0, \\ (\tilde{w}_k(t_k), \partial_t \tilde{w}_k(t_k)) = (\tilde{w}_{k,0}, \tilde{w}_{k,1}). \end{cases}$$

Then again by finite propagation speed $\bar{w}_k(0, r) = \tilde{w}_k(0, r)$ for all $4R_k \leq r \leq 8R_k$. After replacing $\tilde{w}_k$ by $\tilde{w}_k$ multiplied by a cutoff at $r = R_k + |t|$ and standard energy estimates, we obtain

$$\|\tilde{w}_k(0)\|_{\dot{H}_{R_k}^1} \lesssim \|(\tilde{w}_{k,0}, \tilde{w}_{k,1})\|_{\mathcal{H}}.$$

Now applying the previous estimates, we obtain

$$\|\tilde{w}_k(0)\|_{\dot{H}_{R_k}^1} \lesssim \left\langle \log \frac{R_k}{\langle R \rangle} \right\rangle q$$

and thus by Hardy, for all $4R_k \leq r \leq 8R_k$ we have

$$\|\nabla \bar{w}_k(0)\|_{L^2(4R_k \leq r \leq 8R_k)} + R_k^{-1} \|\bar{w}_k(0)\|_{L^2(4R_k \leq r \leq 8R_k)} \lesssim \left\langle \log \frac{R_k}{\langle R \rangle} \right\rangle q.$$

By definition of the Z_{-2} norm, we have

$$\left\| \sum_{j=0}^{\frac{N-6}{2}} d_{j,k} T_j^\infty + \tilde{d}_{j,k} \tilde{Y}_j^0 \right\|_{L^2(4R_k \leq r \leq 8R_k)} \lesssim R_k \left\langle \log \frac{R_k}{\langle R \rangle} \right\rangle q.$$

Therefore

$$\sum_{j=0}^{\frac{N-6}{2}} \|d_{j,k} T_j^\infty\|_{L^2(4R_k \leq r \leq 8R_k)} + \|\tilde{d}_{j,k} \tilde{Y}_j^0\|_{L^2(4R_k \leq r \leq 8R_k)} \lesssim R_k \left\langle \log \frac{R_k}{\langle R \rangle} \right\rangle q$$

and

$$\sum_{j=0}^{\frac{N-6}{2}} \|d_{j,k} T_j^\infty\|_{\dot{H}^1(4R_k \leq r \leq 8R_k)} + \|\tilde{d}_{j,k} \tilde{Y}_j^0\|_{\dot{H}^1(4R_k \leq r \leq 8R_k)} \lesssim q.$$

These imply

$$\|\nabla v_0\|_{L^2(4R_k \leq r \leq 8R_k)} \lesssim \left\langle \log \frac{R_k}{\langle R \rangle} \right\rangle q$$

which implies the result, since $4R_{k_0} \leq 24R$. □

Now Lemma 4.5.1 follows from Lemma 4.5.3, 4.5.2 and Lemma 4.4.5.

4.5.2 The $N \equiv 2 \pmod{4}$ case

This part is similar to the $N \equiv 0$ case. In this case we need an additional lemma about an elliptic estimate:

Lemma 4.5.4. *There exists a constant $C > 0$ such that the following holds. Let $R > 0$, and suppose $u \in Z_{-2,R}(\mathbb{R}^N)$ solves for $r > R$*

$$-\Delta u + Vu = f + \partial_r g$$

where u, f, g are radial and satisfy

$$\|f\|_{Z_{-4,R}}, \|g\|_{Z_{-3,R}} < \infty.$$

Then if $R \geq 1$, then

$$d_{Z_{-2,R}}(u, \text{span}\{T_0^\infty, \dots, T_{\frac{N-6}{2}}^\infty\}) \leq C(\|f\|_{Z_{-4,R}} + \|g\|_{Z_{-3,R}}),$$

while if $0 < R < 1$, we have

$$d_{Z_{-2,R}}(u, \text{span}\{T_0^\infty, \dots, T_{\frac{N-6}{2}}^\infty, T_0^0, \dots, T_{\frac{N-6}{2}}^0\}) \leq C(\|f\|_{Z_{-4,R}} + \|g\|_{Z_{-3,R}}).$$

The quantities $\|\Pi_{L^2}^\perp \partial_t u(0)\|_{Z_{-4}}$ are replaced by $d_{Z_{-2,R}}(u, \text{span}\{T_0^\infty, \dots, T_{\frac{N-6}{2}}^\infty\})$. The proof of Lemma 4.5.3 becomes slightly easier, since the averaging operator $\mathcal{A}$ is not needed to reduce to an $N - 2$ dimensional radial wave equation. The proof follows from the same argument used in the case $N \equiv 0 \pmod{4}$.

4.6 The multisoliton case

In this section we prove Theorem 4.1.2.

First, we note that the problem (4.4) is globally well posed. Indeed, local well posedness follows from the Strichartz estimate Lemma 4.3.1., and global well posedness follows from the fact that (4.4) is linear. Now we have the following lemma.

Lemma 4.6.1. *For all $J \in \mathbb{N}$, there is a $C > 0$ such that for all $\lambda \in \Lambda_J$, if u is a solution in $\mathbb{R}^{1+N}$ of*

$$\begin{cases} \partial_{tt}u - \Delta u + V_\lambda u = f, \\ (u, \partial_t u)|_{t=0} = (u_0, u_1) \in \dot{H}^1 \times L^2 \end{cases}$$

where $f \in L_t^1 L_x^2(\mathbb{R} \times \mathbb{R}^N)$, then

$$\sup_{t \in \mathbb{R}} \|(u, \partial_t u)\|_{\mathcal{H}_{R+|t|}} + \|u\|_{L_t^{\frac{2(N+1)}{N-2}} L_x^{\frac{2(N+1)}{N-2}}(r \geq R+|t|)} \lesssim \|(u_0, u_1)\|_{\mathcal{H}_R} + \|f\|_{L_t^1 L_x^2(r \geq R+|t|)}.$$

Proof. The proof is as in the proof of Lemma 2.8 in Duyckaerts et al. [2023]. □

Next we claim the following inequalities.

Claim 4.6.2. *Suppose that $f \in \dot{H}^1$ is radial. Then $\nabla f \in Z_{-3,\lambda}$ with*

$$\|\nabla f\|_{Z_{-3,\lambda}} \leq \|f\|_{\dot{H}^1} \quad (4.41)$$

and

$$|f(r)| \lesssim \frac{1}{r^2} \left(1 + \left| \min_{1 \leq j \leq J} \log \left(\frac{r}{\lambda_j} \right) \right| \right) \|\nabla f\|_{Z_{-3,\lambda}}. \quad (4.42)$$

Moreover, if χ^0, χ^1 are two smooth cutoffs with $\chi^0(r) = 1$ if $r \leq 1$, $\chi^0(r) = 0$ for $r \geq 2$, χ^1 with compact support outside the origin, and $\chi_R^i(r) := \chi^i\left(\frac{r}{R}\right)$, then

$$\|\chi_R^1 f\|_{\dot{H}^1} \leq C \left(1 + \left| \min_{1 \leq j \leq J} \log \left(\frac{R}{\lambda_j} \right) \right| \right) \|\nabla f\|_{Z_{-3,\lambda}} \quad (4.43)$$

and

$$\|\nabla(\chi_R^0 f)\|_{Z_{-3,\lambda}} \leq C \|\nabla f\|_{Z_{-3,\lambda}}, \quad (4.44)$$

$$\|\nabla((1 - \chi_R^0)f)\|_{Z_{-3,\lambda}} \leq C \|1_{|x| \geq R} \nabla f\|_{Z_{-3,\lambda}}, \quad (4.45)$$

where C is independent of R .

The proof of this claim is the same as that of Claim 4.2 in Collot et al. [2023]. For the moment we will consider a single soliton. We have the following lemma.

Lemma 4.6.3. *There exists $C > 0$ such that the following holds. For any $\lambda > 0$ and any $u_0 \in \dot{H}^1$, the solution u to*

$$\begin{cases} \partial_t^2 u - \Delta u + V_{\{\lambda\}} u = 0 \\ (u(0), \partial_t u(0)) = (u_0, 0) \end{cases}$$

satisfies

$$\|V_{\{\lambda\}} u\|_{L^1 L^2(r > |t|)} \leq C \|\nabla u_0\|_{Z_{-3}}. \quad (4.46)$$

If $u_0(r) = 0$ for all $r \geq R$ for some $R \leq 1$, then

$$\|V_{\{\lambda\}}u\|_{L^1L^2(r \geq |t|)} \leq CR^{\frac{3}{4N}} \|\nabla u_0\|_{Z_{-3}}.$$

If $u_0(0) = 0$ for all $r \leq R$ for some $R \geq 1$, then

$$\|V_{\{\lambda\}}u\|_{L^1L^2(r > |t|)} \leq CR^{-1} \|\nabla u_0\|_{Z_{-3}}.$$

Similarly, if $u_1 \in L^2$ and u is the solution to

$$\begin{cases} \partial_{tt}u - \Delta u + V_\lambda u = f, \\ (u, \partial_t u)|_{t=0} = (0, u_1) \in \dot{H}^1 \times L^2 \end{cases}$$

satisfies

$$\|V_{\{\lambda\}}u\|_{L^1L^2(r > |t|)} \leq C\|u_1\|_{Z_{-3}}. \quad (4.47)$$

If $u_1(r) = 0$ for all $r \geq R$ for some $R \leq 1$, then

$$\|V_{\{\lambda\}}u\|_{L^1L^2(r \geq |t|)} \leq CR^{\frac{3}{4N}} \|u_1\|_{Z_{-3}}.$$

If $u_1(0) = 0$ for all $r \leq R$ for some $R \geq 1$, then

$$\|V_{\{\lambda\}}u\|_{L^1L^2(r > |t|)} \leq CR^{-1} \|u_1\|_{Z_{-3}}.$$

Proof. As in the proof of Lemma 4.3 in Collot et al. [2023], we use a dyadic partition. Let χ be such that $\text{Supp}\chi \subset \left[\frac{1}{2}, 2\right]$ and $1 = \sum_{\ell \in \mathbb{Z}} \chi_{2^\ell}(r)$. We let u_L be the solution to

$$\begin{cases} \partial_t^2 u_\ell - \Delta u_\ell + V_{(\ell)} u_\ell = 0 \\ (u_\ell(0), \partial_t u_\ell(0)) = (u_{\ell,0}, 0) \end{cases}$$

where $u_{\ell,0} = \chi_{2^\ell} u_0$. Note that $u = \sum_{\ell \in \mathbb{Z}} u_\ell$. Now by (4.43), we have

$$\|u_{\ell,0}\|_{\dot{H}^1} \lesssim (1 + |\ell|) \|\nabla u_0\|_{Z_{-3}}.$$

Since $\text{Supp} u_{0,\ell} \subset [2^{\ell-1}, 2^{\ell+1}]$ we have that

$$\text{Supp}(u_{0,\ell}) \subseteq \{\max(0, 2^{\ell-1} - |t|) \leq r \leq 2^{\ell+1} + |t|\}.$$

Now by Hölder and Lemma 4.6.1 we obtain

$$\|V_{\{\lambda\}} u_L\|_{L^1 L^2(r \geq |t|)} \lesssim (1 + |\ell|) \|V\|_{L_t^2 L_x^{\frac{2N}{3}}(S_\ell)} \|\nabla u_0\|_{Z_{-3}}$$

where

$$S_\ell := \{\max(|t|, 2^{\ell-1} - |t|) \leq r \leq 2^{\ell+1} + |t|\}.$$

If $\ell \leq 0$, then

$$\|V\|_{L_t^2 L_x^{\frac{2N}{3}}(S_\ell)} \leq \|V\|_{L_t^2 L_x^{\frac{2N}{3}}(|t| \leq r \leq |t| + 2^{\ell+1})} \lesssim 2^{\frac{3}{2N}\ell}.$$

Similarly, if $\ell \geq 0$, we have

$$\|V\|_{L_t^2 L_x^{\frac{2N}{3}}(S_\ell)} \leq \|V\|_{L_t^2 L_x^{\frac{2N}{3}}(\max(|t|, 2^{\ell-2}) \leq r \leq |t| + 2^{\ell+1})} \lesssim 2^{-2\ell}.$$

This implies that

$$\begin{aligned} \|V_{\{\lambda\}} u\|_{L^1 L^2(r \geq |t|)} &\lesssim \sum_{\ell \in \mathbb{Z}} \|V_{\{\lambda\}} u_\ell\|_{L^1 L^2(r \geq |t|)} \\ &\leq \|\nabla u_0\|_{Z_{-3}} \left(\sum_{\ell \leq 0} (1 + |\ell|) 2^{\frac{3\ell}{2N}} + \sum_{\ell \geq 0} 2^{-2\ell} \right) \lesssim \|\nabla u_0\|_{Z_{-3}}. \end{aligned}$$

Now if we additionally assume that $u_0(r) = 0$ if $r \geq R$ for some $R \geq 1$, we obtain

$$\|V_{\{\lambda\}}u\|_{L^1L^2(r \geq |t|)} \lesssim \|\nabla u_0\|_{Z_{-3}} \sum_{\ell \leq \ell_0} (1 + |\ell|) 2^{\frac{3\ell}{2N}} \lesssim 2^{\frac{3\ell_0}{4N}} \|\nabla u_0\|_{Z_{-3}} \lesssim R^{\frac{3}{4N}} \|\nabla u_0\|_{Z_{-3}}.$$

The last assertion follows similarly. $\square$

We have the following lemma.

Lemma 4.6.4. *Let $J \in \mathbb{N}$. There exists a small enough $\gamma^* > 0$ such that the following holds for any $\lambda \in \Lambda_J$ with $\gamma(\lambda) \leq \gamma^*$. For $1 \leq j \leq J$, let $\vec{\phi}_j$ and $\vec{\psi}_j$ denote the solutions to (4.4) with initial data $(0, (\Lambda W)_{[\lambda_j]})$ and $((\Lambda W)_{(\lambda_j)}, 0)$ respectively. Then for any $1 \leq j \leq J$, we have*

$$\begin{aligned} \sup_{t \in \mathbb{R}} \|\vec{\phi}_{j,k} - (t(\Lambda W)_{[\lambda_j]}, \Lambda W_{[\lambda_j]})\|_{\mathcal{H}_{|t|}} &\lesssim \gamma(\lambda) \\ \sup_{t \in \mathbb{R}} \|\vec{\psi}_{j,k} - (\Lambda W_{(\lambda_j)}, 0)\|_{\mathcal{H}_{|t|}} &\lesssim \gamma(\lambda). \end{aligned}$$

The proof follows from direct computations, which we omit here.

4.6.1 Non resonant component

In this section we will consider solutions satisfying $u(t, r) = (-1)^{\frac{N}{2}} u(-t, r)$. We will show the following lemma.

Proposition 4.6.5. *For any $J \in \mathbb{N}$, there are $\gamma^*, C > 0$ such that for all $\lambda \in \Lambda_J$ with $\gamma(\lambda) \leq \gamma^*$ any radial solution to (4.4) satisfies*

$$\|\Pi_{\dot{H}^1, \lambda}^\perp u_0\|_{\dot{H}^1}^2 \lesssim \sum_{\pm} \lim_{t \rightarrow \pm\infty} \int_{r \geq |t|} |\nabla_{t,x} u|^2 + \gamma(\lambda) \|u_0\|_{\dot{H}^1}^2$$

in the case $N \equiv 0 \pmod{4}$, and

$$\|\Pi_{L^2, \lambda}^\perp u_0\|_{L^2}^2 \lesssim \sum_{\pm} \lim_{t \rightarrow \pm\infty} \int_{r \geq |t|} |\nabla_{t,x} u|^2 + \gamma(\lambda) \|u_0\|_{L^2}^2$$

if $N \equiv 2 \pmod{4}$.

For the proof we will need the following lemma. For $R > 0, \mu > 0$, let $\bar{P}_{\mu,R}^2$ be the orthogonal projection on the orthogonal of $\text{Span}\{(T_k^\infty)_{(\mu)} : 0 \leq k \leq \frac{N-6}{2}\}$ in $\dot{H}_R^1$ if $N \equiv 0 \pmod{4}$ and the orthogonal projection on the orthogonal of $\text{Span}\{(T_k^\infty)_{[\mu]} : 0 \leq k \leq \frac{N-6}{2}\}$ in L_R^2 if $N \equiv 2 \pmod{4}$.

Lemma 4.6.6. *For all $\eta \geq 0$, there is a $C_\eta > 0$ such that for all $\mu > 0$ and a solution u of*

$$\partial_t^2 u - \Delta u + V_{\{\mu\}} u = f, \quad |x| > \eta\mu + |t|$$

with $\vec{u}(0) \in \mathcal{H}, f \in L_t^1 L_x^2$, we have

$$\|\bar{P}_{\mu,\eta\mu} \partial_t u(0)\|_{\dot{H}_{\eta\mu}^1}^2 \leq C_\eta \|1_{|x| > \eta\mu + |t|} f\|_{L_t^1 L_x^2}^2 + C_\eta \sum_{\pm} \lim_{t \rightarrow \pm\infty} \int_{|x| > \eta\mu + |t|} |\nabla_{t,x} u(t, x)|^2 dx$$

if $N \equiv 0 \pmod{4}$, and

$$\|\bar{P}_{\mu,\eta\mu} \partial_t u(0)\|_{L_{\eta\mu}^2}^2 \leq C_\eta \|1_{|x| > \eta\mu + |t|} f\|_{L_t^1 L_x^2}^2 + C_\eta \sum_{\pm} \lim_{t \rightarrow \pm\infty} \int_{|x| > \eta\mu + |t|} |\nabla_{t,x} u(t, x)|^2 dx$$

Proof. By scaling we can assume $\mu = 1$. By time symmetry, we can assume $\partial_t u(0) = 0$ if $N \equiv 0 \pmod{4}$ and $u(0) = 0$ if $N \equiv 2 \pmod{4}$. Now let u_{hom} be the solution to

$$\begin{cases} \partial_t^2 u_{\text{hom}} - \Delta u_{\text{hom}} + V_{\{\mu\}} u_{\text{hom}} = 0 \\ (u_{\text{hom}}(0), \partial_t(u_{\text{hom}}(0))) = (u(0), \partial_t u(0)) \end{cases}$$

and $\bar{u}$ the solution to

$$\begin{cases} \partial_t^2 \bar{u} - \Delta \bar{u} + V_{\{\mu\}} \bar{u} = f, \\ (\bar{u}(0), \partial_t \bar{u}(0)) = (0, 0). \end{cases}$$

Then if $N \equiv 0 \pmod{4}$,

$$\|\Pi_{\mu,\eta\mu}^\perp u(0)\|_{\dot{H}_{\eta\mu}^1}^2 \lesssim \|\Pi_{\mu,\eta\mu}^\perp u_{\text{hom}}\|_{\dot{H}_{\eta\mu}^1}^2 + \|\bar{u}(0)\|_{\dot{H}_{\eta\mu}^1}^2$$

and

$$\|\Pi_{\mu,\eta\mu}^\perp u(0)\|_{L_{\eta\mu}^2(0)}^2 \lesssim \|\Pi_{\mu,\eta\mu}^\perp u_{\text{hom}}\|_{L_{\eta\mu}^2}^2 + \|\bar{u}(0)\|_{L_{\eta\mu}^2}^2$$

if $N \equiv 2 \pmod{4}$. Now let $\tilde{u}$ be the solution to

$$\begin{cases} \partial_t^2 \tilde{u} - \Delta \tilde{u} + V_{\{\mu\}} \tilde{u} = f 1_{|x| > \eta\mu + |t|}, \\ (\tilde{u}(0), \partial_t \tilde{u}(0)) = (0, 0). \end{cases}$$

By finite propagation speed,

$$\|\bar{u}\|_{\dot{H}_{\eta\mu}^1} + \|\partial_t \bar{u}\|_{L_{\eta\mu}^2}^2 \lesssim \|\tilde{u}\|_{\dot{H}_{\eta\mu}^1} + \|\partial_t \tilde{u}\|_{L_{\eta\mu}^2}^2.$$

Now by standard energy estimates,

$$\|\tilde{u}\|_{\dot{H}^1}^2 + \|\partial_t \tilde{u}\|_{L^2}^2 \lesssim \|f\|_{L^1 L^2(|x| > \eta\mu + |t|)}.$$

Therefore it suffices to show the statement for u_{hom} , that is for the case $f \equiv 0$.

Additionally note that if $\eta = 0$, then the result is Lemma 4.4.5 for $R = 0$. We will then assume $\eta > 0$. Using Lemma 4.4.5 we can decompose $u_0 = u(0)$ as

$$u_0 = \sum_{k=0}^{\frac{N-6}{2}} c_k^\infty T_k^\infty + c_k^0 T_k^0 + \Pi^\perp u,$$

with

$$\|\Pi^\perp u\|_{\mathcal{A}_N}^2 \lesssim E_{\text{out}}(\eta) := \lim_{t \rightarrow \infty} \int_{|x| > \eta + |t|} |\nabla_{t,x} u(t, x)|^2 dx.$$

Rewriting this as

$$\sum_{k=0}^{\frac{N-6}{2}} c_k^0 T_k^0 = u_0 - \sum_{k=0}^{\frac{N-6}{2}} c_k^\infty T_k^\infty - \Pi^\perp u$$

and taking the exterior energy as in the proof of Lemma 4.4.5 together with (4.23) we obtain that

$$c_k^0 \lesssim (E_{\text{out}}(\eta))^{\frac{1}{2}}.$$

This implies that

$$\|\Pi^\perp u\|_{\mathcal{A}_N}^2 + \sum_{k=0}^{\frac{N-6}{2}} (c_k^0)^2 \lesssim E_{\text{out}}(\eta).$$

This completes the proof. □

Now we will prove Proposition 4.6.5.

Proof of Proposition 4.6.5. We can proceed as in the proof of Lemma 4.5 in Collot et al. [2023]. First we can assume by time symmetry that

$$(u, \partial_t u)|_{t=0} = \begin{cases} (u_0, 0) & \text{if } N \equiv 0, \pmod{4} \\ (0, u_1) & \text{if } N \equiv 2 \pmod{4}. \end{cases}$$

Moreover, we can decompose

$$u_0 = \Pi_\mu^\perp u + v_0$$

with

$$v_0 \in \text{Span} \left\{ (T_k^\infty)_{(\lambda_j)} : 0 \leq k \leq \frac{N-6}{2}, 1 \leq j \leq J \right\}$$

if $N \equiv 0 \pmod{4}$,

$$u_1 = \Pi_\mu^\perp u + v_1$$

with

$$v_1 \in \text{Span} \left\{ (T_k^\infty)_{[\lambda_j]} : 0 \leq k \leq \frac{N-6}{2}, 1 \leq j \leq J \right\}$$

if $N \equiv 2 \pmod{4}$. Let v be the solution to (4.1) with initial condition $(v_0, 0)$ if $N \equiv 0 \pmod{4}$ and $(0, v_1)$ if $N \equiv 2 \pmod{4}$, we see by Lemma 4.6.4 that

$$\lim_{t \rightarrow \infty} \int_{|x| > |t|} |\nabla_{t,x} v(t, x)|^2 dx \lesssim \gamma(\lambda) \|(v_0, v_1)\|_{\dot{H}^1 \times L^2}^2 \leq \gamma(\lambda) \|(u_0, u_1)\|_{\dot{H}^1 \times L^2}^2.$$

Thus we can reduce to the case that $v = 0$.

We have

$$\partial_t^2 u - \Delta u - V_{\{\lambda_j\}} u = \sum_{k \neq j} V_{\{\lambda_k\}} u.$$

By Lemma 4.6.6, we have for $N \equiv 0 \pmod{4}$

$$\|P_{\lambda_j, \eta_j \lambda_j} u_0\|_{\dot{H}_{\eta_j \lambda_j}^1} \leq C_{\eta_j} \sum_{k \neq j} \|1_{|x| > \eta_j \lambda_j + |t|} V_{\{\lambda_k\}} u\|_{L^1 L^2} + C_{\eta_j} \sqrt{E_{\text{out}}},$$

where $\bar{P}_{\lambda, \rho}^\perp$ is the $\dot{H}_\rho^1$ (respectively L_ρ^2) projection onto the orthogonal of $\text{Span}\{(T_k^\infty)_{(\lambda_j)}\}_{k=0}^{\frac{N-6}{2}}$ for $N \equiv 0 \pmod{4}$ (respectively $N \equiv 2 \pmod{4}$), with a similar statement for $N \equiv 2 \pmod{4}$.

If $k > j$, we have

$$\begin{aligned} \|1_{|x| > \eta_j \lambda_j + |t|} V_{\{\lambda_k\}} u\|_{L_t^1 L_x^2} &\lesssim \|1_{|x| > \eta_j \lambda_j + |t|} V_{\{\lambda_k\}}\|_{L_t^{\frac{2(N+1)}{N+4}} L_x^{\frac{2(N+1)}{3}}} \\ &\quad \cdot \|1_{|x| > \eta_j \lambda_j + |t|} u\|_{L_t^{\frac{2(N+1)}{N-2}} L_x^{\frac{2(N+1)}{N-2}}} \\ &\lesssim \|1_{|x| > \eta_j \lambda_j + |t|} V_{\{\lambda_k\}}\|_{L_t^{\frac{2(N+1)}{N+4}} L_x^{\frac{2(N+1)}{3}}} \|u_0\|_{\dot{H}_{\eta_j \lambda_j}^1} \end{aligned}$$

where in the second to last step we used Lemma 4.6.1. Now let

$$y = \frac{x}{\lambda_k}, \quad s = \frac{t}{\lambda_k}.$$

Then

$$\begin{aligned}
& \|1_{|x|>\eta_j\lambda_j+|t|} V_{\{\lambda_k\}}\|_{L_t^{\frac{2(N+1)}{N+4}} L_x^{\frac{2(N+1)}{3}}} \\
&= \frac{1}{\lambda_k^2} \lambda_k^{\frac{N+4}{2(N+1)} + \frac{3N}{2(N+1)}} \|\chi_{\{|y|\geq \frac{\eta_j\lambda_j}{\lambda_k} + |s|\}} (y)(1+|y|)^{-4}\|_{L_s^{\frac{2(N+1)}{N+4}} L_y^{\frac{2(N+1)}{3}}} \\
&= \|\chi_{\{|y|\geq \frac{\eta_j\lambda_j}{\lambda_k} + |s|\}} (y)(1+|y|)^{-4}\|_{L_s^{\frac{2(N+1)}{N+4}} L_y^{\frac{2(N+1)}{3}}}
\end{aligned}$$

Let $R = \eta_j \frac{\lambda_j}{\lambda_k}$. We will split the integral into $s \in (0, 1)$ and $s \in [1, \infty)$. If $s \geq 1$, then $|y| > 1$ and so $(1 + |y|) = O(|y|)$ which implies that

$$\begin{aligned}
& \left(\int_{|y|>R+s} \frac{dy}{(1+|y|)^{4p}} \right)^{\frac{q}{p}} \\
& \lesssim \left(\int_{|y|>R+|s|} \frac{1}{|y|^{4p}} dy \right)^{\frac{q}{p}} = \left(\int_{R+s}^{\infty} \frac{r^{N-1}}{r^{4p}} \right)^{\frac{q}{p}} \lesssim \left(\frac{1}{(R+s)^{4p-N}} \right)^{\frac{q}{p}} = \frac{1}{(R+s)^{\frac{5N+8}{N+4}}},
\end{aligned}$$

where

$$q = \frac{2(N+1)}{N+4}, \quad p = \frac{2(N+1)}{3}.$$

Now integrating in s we obtain

$$\|1_{|s|\geq 1} 1_{|x|>\eta_j\lambda_j+|t|} V_{\{\lambda_k\}}\|_{L_t^{\frac{2(N+1)}{N+4}} L_x^{\frac{2(N+1)}{3}}} \lesssim \left(\int_{\mathbb{R}} \frac{1}{(R+|s|)^{\frac{5N+8}{N+4}}} ds \right)^{\frac{1}{p}} \lesssim \frac{1}{R^2}.$$

In the case $0 < s < 1$, we split this into the domains $r \geq 1$ and $R < 1$. In the first case, we have that $|y| > R + |s|$ and so we can follow the exact computations as in the case $s \geq 1$. Now if instead $R \leq 1$, we split into the cases that $|y| \geq 2$ and $R + |s| \leq |y| \leq 2$. If $|y| \geq 2$, then again $1 + |y| = O(|y|)$, and so we can proceed as in the case that $|s| \geq 1$. In the remaining

case, we note that $1 + |y| = O(1)$ and so combining these we obtain that

$$\|1_{|s|\leq 1} 1_{|x|>\eta_j\lambda_j+|t|} V_{\{\lambda_k\}}\|_{L_t^{\frac{2(N+1)}{N+4}} L_x^{\frac{2(N+1)}{3}}} \lesssim \frac{1}{R^2} + 1 \lesssim \frac{1}{R^2}.$$

This shows that

$$\|1_{|s|\geq 1} 1_{|x|>\eta_j\lambda_j+|t|} V_{\{\lambda_k\}}\|_{L_t^{\frac{2(N+1)}{N+4}} L_x^{\frac{2(N+1)}{3}}} \lesssim \frac{\lambda_k^2}{\lambda_j^2 \eta_j^2}.$$

After following the same steps, we obtain

$$\|\Pi_{\lambda_j, \eta_j \lambda_j}^\perp u_0\| \leq \frac{\varepsilon_j}{2} \|u_0\|_{\dot{H}^1} + C_{\varepsilon_j, \eta_j, \varepsilon_{j-1}, \eta_{j-1}} \sqrt{E_{\text{out}}}$$

for $N \equiv 0 \pmod{4}$ and similarly for dimensions $N \equiv 2 \pmod{4}$. We can also obtain the bound

$$\left| \int_{|x|\geq \eta_j \lambda_j} \nabla u_0 \cdot \nabla (\Lambda W)_{(\lambda_j)} \right| = o_{\eta_j}(1) \|u_0\|_{\dot{H}^1}$$

for $N \equiv 0 \pmod{4}$ and the analogous bound for $N \equiv 2 \pmod{4}$. Combining everything we obtain

$$\|\bar{P}_{\lambda_j, \eta_j \lambda_j}\|_{\dot{H}_{\eta_j \lambda_j}} \leq C_{\eta_j} \left(\varepsilon_{j-1} + o_{\eta_{j-1}}(1) + \frac{\gamma(\lambda)}{\eta_j^2} \right) \|u_0\|_{\dot{H}^1} + C_{\varepsilon_j, \eta_j, \varepsilon_{j-1}, \eta_{j-1}} \sqrt{E_{\text{out}}}.$$

Hence if $\gamma(\lambda)$ is small enough we obtain

$$\|\bar{P}_{\lambda_j, \eta_j \lambda_j}\|_{\dot{H}_{\eta_j \lambda_j}} \leq \frac{\varepsilon_j}{2} \|u_0\|_{\dot{H}^1} + C_{\varepsilon_j, \eta_j, \varepsilon_{j-1}, \eta_{j-1}} \sqrt{E_{\text{out}}},$$

with an analogous statement for $N \equiv 2 \pmod{4}$. All that remains is to bound the projection onto $\{T_k^\infty\}_{k=1}^{\frac{N-6}{2}}$. We can see by a contradiction argument (as in case 4 of the proof of Lemma

4.4.5) that there exists an absolute constant C such that

$$\|\bar{P}_{\lambda_j, \eta_j \lambda_j} u_0\|_{\dot{H}_{\eta_j \lambda_j}^1} \geq C \|u_0\|_{\dot{H}_{\eta_j \lambda_j}^1}.$$

Now we have

$$\|u_0\|_{\dot{H}_{\eta_j \lambda_j}^1} \leq C^{-1} \|\bar{P}_{\lambda_j, \eta_j \lambda_j} u_0\|_{\dot{H}_{\eta_j \lambda_j}^1} \leq C^{-1} \left(\frac{\varepsilon_j}{2} \|u_0\|_{\dot{H}^1} + C_{\varepsilon_j, \eta_j, \varepsilon_{j-1}, \eta_{j-1}} \sqrt{E_{\text{out}}} \right).$$

This completes the induction. Now to finish the proof, we set $\eta_J = 0$ in the final step. $\square$

4.6.2 Resonant Component

In this section we prove the other half of the energy estimate in Theorem 4.1.2.

Proposition 4.6.7. *For all $J \in \mathbb{N}$, there exist $\gamma^*, C > 0$ such that for all $\lambda \in \Lambda_J$ with $\gamma(\lambda) \leq \gamma^*$ and any radial solution of (4.4) satisfies*

$$\|\Pi_{L^2, \lambda}^\perp u_1\|_{Z_{-3, \lambda}}^2 \leq C \left(\sum_{\pm} \lim_{t \rightarrow \pm\infty} \int_{r \geq |t|} |\nabla_{t,x} u|^2 dx + \gamma(\lambda) \|u_1\|_{Z_{-3, \lambda}}^2 \right)$$

if $N \equiv 0 \pmod{4}$, and similarly

$$\|\nabla \Pi_{\dot{H}^1, \lambda}^\perp u_0\|_{Z_{-3, \lambda}}^2 \leq C \left(\sum_{\pm} \lim_{t \rightarrow \pm\infty} \int_{r \geq |t|} |\nabla_{t,x} u|^2 dx + \gamma(\lambda) \|\nabla u_0\|_{Z_{-3, \lambda}}^2 \right)$$

if $N \equiv 2 \pmod{4}$.

Proof. We will prove it for the case $N \equiv 2 \pmod{4}$, with the $N \equiv 0 \pmod{4}$ case being identical. We claim it suffices to prove that

$$\|\Pi_{L^2, \lambda}^\perp u_1\|_{Z_{-3, \lambda}} \lesssim \sqrt{E_{\text{out}}} \text{ if } u_1 = \Pi_{L^2, \lambda}^\perp u_1 \quad (4.48)$$

Indeed, suppose these hold. Let u be a solution to (4.4). We can assume by time symmetry that

$$(u_0, u_1) = (0, u_1)$$

if $N \equiv 0 \pmod{4}$, and

$$(u_0, u_1) = (u_0, 0)$$

if $N \equiv 2 \pmod{4}$. We write $u_0 = v_0 + w_0$ for $w_0 = \Pi_{\dot{H}^1, \lambda}^\perp u_0$, and similarly $u_1 = v_1 + w_1$ with $w_1 = \Pi_{L^2, \lambda}^\perp u_1$. Then we claim that

$$\lim_{t \rightarrow \infty} \int_{|x| > |t|} |\nabla_{t,x} v(t, x)|^2 dx \lesssim \begin{cases} \gamma(\lambda) \|v_1\|_{Z_{-3, \lambda}}^2 & \text{if } N \equiv 0 \pmod{4}, \\ \gamma(\lambda) \|\nabla v_0\|_{Z_{-3, \lambda}}^2 & \text{if } N \equiv 2 \pmod{4}. \end{cases}$$

If this claim holds then

$$\begin{aligned} \|w_1\|_{Z_{-3, \lambda}}^2 &\lesssim E_{\text{out}} + \lim_{t \rightarrow \infty} \int_{|x| > |t|} |\nabla_{t,x} v(t, x)|^2 dx \\ &\lesssim E_{\text{out}} + \gamma(\lambda) \|v_1\|_{Z_{-3, \lambda}}^2 \lesssim E_{\text{out}} + \gamma(\lambda) \left(\|u_1\|_{Z_{-3, \lambda}}^2 + \|u_1\|_{Z_{-3, \lambda}}^2 \right) \end{aligned}$$

for $N \equiv 0 \pmod{4}$, and similarly for $N \equiv 2 \pmod{4}$. To show (4.48), note that if j_0 is such that $|\alpha_{j_0}| = \sup_{1 \leq j \leq J} |\alpha_j|$ where

$$v_1 = \sum_{j=1}^J \alpha_j (\Lambda W)_{[\lambda_j]}, \quad v_0 = \sum_{j=1}^J \alpha_j (\Lambda W)_{(\lambda_j)}.$$

We obtain

$$\begin{aligned}
\|v_1\|_{Z_{-3,\lambda}} &\gtrsim \frac{1}{\inf_{1 \leq j \leq J} \left\langle \log \frac{\lambda_{j_0}}{\lambda_j} \right\rangle} \left(\int_{\lambda_{j_0} \leq |x| \leq 2\lambda_{j_0}} |v_1|^2 dx \right)^{\frac{1}{2}} \\
&\gtrsim |\alpha_{j_0}| \frac{1}{\inf_{1 \leq j \leq J} \left\langle \log \frac{\lambda_{j_0}}{\lambda_j} \right\rangle} \left(\int_{\lambda_{j_0} \leq |x| \leq 2\lambda_{j_0}} |\Lambda W_{[\lambda_{j_0}]}|^2 dx \right)^{\frac{1}{2}} \\
&\gtrsim |\alpha_{j_0}|
\end{aligned}$$

and similarly $\|\nabla v_0\|_{Z_{-3,\lambda}} \geq |\alpha_{j_0}|$. On the other hand, we have

$$\lim_{t \rightarrow \infty} \int_{|x| \geq |t|} |\nabla_{t,x} v(t, x)|^2 dx \lesssim \lim_{t \rightarrow \infty} \int_{|x| \geq |t|} \left| \nabla_{t,x} \left(\sum_{j=1}^J t \alpha_j (\Lambda W)_{[\lambda_j]} \right) \right|^2 dx + \gamma(\lambda) \sup_{1 \leq j \leq J} \alpha_j^2$$

and similarly for $N \equiv 2 \pmod{4}$. This proves the claim.

Now let $\gamma := \gamma(\lambda)$, and define the radii

$$R_j^+ = \sqrt{\lambda_j \lambda_{j+1}}, \quad R_j^- = \frac{\lambda_{j+1}}{\gamma^{\frac{1}{4}}}.$$

Note that $R_j^- \leq \gamma^{\frac{1}{4}} R_j^+$. For any j , let

$$\|f\|_{Z_{-3,\lambda_j}} := \|f|_{\left[\frac{1}{\lambda_j}\right]}\|_{Z_{-3}}.$$

Then

$$\|f\|_{Z_{-3,\lambda_j}} \lesssim \|f\|_{-3,\lambda},$$

and if $\text{Supp } f \subset [R_j^+, \infty)$, then

$$\|f\|_{Z_{-3,\lambda}} \lesssim \|f\|_{Z_{-3,\lambda_j}}.$$

We will prove the following statement by induction.

Claim 4.6.8. *There is an absolute constant $\alpha > 0$ such that for all $j \in \{1, \dots, J\}$,*

$$\|1_{r \geq R_j^+} u_1\|_{Z_{-3, \lambda_j}} \lesssim \sqrt{E_{\text{out}}} + \gamma^\alpha \|u_1\|_{Z_{-3, \lambda}}$$

if $N \equiv 0 \pmod{4}$, and similarly

$$\|1_{r \geq R_j^+} \nabla u_0\|_{Z_{-3, \lambda_j}} \lesssim \sqrt{E_{\text{out}}} + \gamma^\alpha \|\nabla u_0\|_{Z_{-3, \lambda}}$$

if $N \equiv 2 \pmod{4}$.

We will prove the statement for $N \equiv 2 \pmod{4}$, the other case being identical.

Suppose that $2 \leq J \leq J-1$, and suppose Claim 4.6.8 holds for $j-1$. Let $v_j, \tilde{v}_j$ be the solutions to

$$\begin{cases} \partial_t^2 v_j - \Delta v_j + V_{\{\lambda_j\}} v_j = 0, \\ (v_j(0), \partial_t v_j(0)) = (u, \partial_t u)|_{t=0}, \end{cases}$$

$$\begin{cases} \partial_t^2 \tilde{v}_j - \Delta \tilde{v}_j + V_\lambda \tilde{v}_j = F(v_j) = -\sum_{i \neq j} V_{\{\lambda_i\}} v_j, \\ (\tilde{v}_j(0), \partial_t \tilde{v}_j(0)) = (0, 0). \end{cases}$$

Let χ be a smooth cut off with

$$\chi(r) = \begin{cases} 1 & \text{if } r \leq 1, \\ 0 & \text{if } r \geq 2. \end{cases}$$

Let v_j^m be solutions to

$$\partial_t^2 v_j^m - \Delta v_j^m + V_{\{\lambda_i\}} v_j^m = 0$$

with

$$v_j^1(0) = \chi_{R_{j-1}^+} (1 - \chi_{R_j^-}) u_0, v_j^2(0) = (1 - \chi_{R_{j-1}^+}) u_0, v_j^3(0) = \chi_{\frac{R_j^-}{48}} u_0$$

where $\chi_\rho = \chi\left(\frac{\cdot}{\rho}\right)$. This implies that

$$v_j = v_j^1 + v_j^2 + v_j^3,$$

and so

$$F(v_j) = F(v_j^1) + F(v_j^2) + F(v_j^3).$$

Lemma 4.6.9. *For some absolute constant $\alpha > 0$, we have*

$$\|1_{\left\{r \geq \frac{R_j^-}{24} + |t|\right\}} F(v_j^1)\|_{L_t^1 L_x^2} \lesssim \gamma^\alpha \|\nabla u_0\|_{Z_{-3,\lambda}} \quad (4.49)$$

if $N \equiv 2 \pmod{4}$ and similarly

$$\|1_{\left\{r \geq \frac{R_j^-}{24} + |t|\right\}} F(v_j^1)\|_{L_t^1 L_x^2} \lesssim \gamma^\alpha \|u_1\|_{Z_{-3,\lambda}}$$

in the case that $N \equiv 0 \pmod{4}$,

$$\|1_{\left\{r \geq \frac{R_j^-}{24} + |t|\right\}} F(v_j^2)\|_{L_t^1 L_x^2} \lesssim \sqrt{E_{\text{out}}} + \gamma^\alpha \|\nabla u_0\|_{Z_{-3,\lambda}} \quad (4.50)$$

if $N \equiv 2 \pmod{4}$ with the analogous statement for $N \equiv 0 \pmod{4}$, and

$$\|1_{\left\{r \geq \frac{R_j^-}{24} + |t|\right\}} F(v_j^3)\|_{L_t^1 L_x^2} = 0. \quad (4.51)$$

We will prove this lemma later, and will continue the proof assuming its validity. By Lemma 4.6.1, we have

$$\sup_t \|(\tilde{v}_j(t), \partial_t \tilde{v}_j(t))\|_{\mathcal{H}_{R_j^- + |t|}} \lesssim \sqrt{E_{\text{out}}} + \gamma^\alpha \|\nabla u_0\|_{Z_{-3,\lambda}}.$$

Indeed,

$$\sup_t \|(\tilde{v}_j(t), \partial_t \tilde{v}_j(t))\|_{\mathcal{H}_{R_j^- + |t|}} \lesssim \|F(v_j)\|_{L_t^1 L_x^2(\{r \geq R_j^- + |t|\})} \lesssim \sqrt{E_{\text{out}}} + \gamma^\alpha \|\nabla u_0\|_{Z_{-3,\lambda}}$$

where we used Lemma 4.6.9. Therefore

$$\sum_{\pm} \lim_{t \rightarrow \infty} \int_{|x| \geq \frac{R_j^-}{24} + |t|} |\nabla_{t,x} v_j(t)|^2 dx \lesssim E_{\text{out}} + \gamma^{2\alpha} \|\nabla u_0\|_{Z_{-3,\lambda}}^2.$$

Now by the $N \equiv 2 \pmod{4}$ version of Lemma 4.5.2 there exist coefficients $c_{j,k}^\infty$ such that

$$u_0 = \sum_{k=0}^{\frac{N-6}{2}} c_{j,k}^\infty (T_k^\infty)_{(\lambda_j)} + \bar{u}_{0,j} \quad (4.52)$$

with

$$\|1_{r \geq R_j^-} \nabla \bar{u}_{0,j}\|_{Z_{-3,j}} \lesssim \sqrt{E_{\text{out}}} + \gamma^\alpha \|\nabla u_0\|_{Z_{-3,\lambda}}. \quad (4.53)$$

Taking the $\dot{H}^1(\{R_j^- \leq |x| \leq 2R_j^-\})$ norm, we see that that

$$\begin{aligned} & \int_{R_j^-}^{2R_j^-} \left| \nabla \left(\sum_{k=1}^{\frac{N-6}{2}} c_{j,k}^\infty (T_k^\infty)_{(\lambda_j)} \right) \right|^2 r^{N-1} dr \\ & \lesssim \int_{R_j^-}^{2R_j^-} |\nabla u_0|^2 r^{N-1} dr + \int_{R_j^-}^{2R_j^-} |\nabla \bar{u}_{0,j}|^2 r^{N-1} dr \\ & + (c_{j,0}^\infty)^2 \int_{R_j^-}^{2R_j^-} |\nabla((\Lambda W)_{(\lambda_j)})|^2 r^{N-1} dr. \end{aligned}$$

Note that

$$\begin{aligned}
& \int_{R_j^-}^{2R_j^-} \left| \nabla \left(\sum_{k=1}^{\frac{N-6}{2}} c_{j,k}^\infty (T_k^\infty)_{(\lambda_j)} \right) \right|^2 r^{N-1} dr \\
&= \sum_{k_1, k_2 \in \{1, \dots, \frac{N-6}{2}\}} c_{j,k_1}^\infty c_{j,k_2}^\infty \int_{R_j^-}^{2R_j^-} \nabla (T_{k_1}^\infty)_{(\lambda_j)} \cdot \nabla (T_{k_2}^\infty)_{(\lambda_j)} r^{N-1} dr.
\end{aligned}$$

Let $M = (m_{k_1 k_2})_{1 \leq k_1, k_2 \leq \frac{N-6}{2}}$ where

$$m_{k_1 k_2} := \int_{R_j^-}^{2R_j^-} \nabla (T_{k_1}^\infty)_{(\lambda_j)} \cdot \nabla (T_{k_2}^\infty)_{(\lambda_j)} r^{N-1} dr$$

Using that

$$|(T_k^\infty)_{[\lambda_j]}| \sim c_k \frac{1}{(R_j^-)^{\frac{N}{2}}} \left(\frac{\lambda_j}{\lambda_{j+1}} \right)^{\frac{N-4}{2}} \gamma^{\frac{N-4}{8}}$$

if $r \sim R_j^-$ we see that

$$m_{k_1 k_2} \sim c_{k_1} c_{k_2} \left(\frac{\lambda_j}{\lambda_{j+1}} \right)^{N-4} \gamma^{\frac{N-4}{4}} (R_j^-)^2$$

and so for some $O(1)$ matrix $\tilde{M}$,

$$\begin{aligned}
& \langle \tilde{M}(c_{j,k}^\infty)_{1 \leq k \leq \frac{N-6}{2}}, (c_{j,k}^\infty)_{1 \leq k \leq \frac{N-6}{2}} \rangle \left(\frac{\lambda_j}{\lambda_{j+1}} \right)^{N-2} \gamma^{\frac{N-6}{4}} (R_j^-)^{N-6} \\
& \lesssim \langle \log \gamma \rangle^2 \|\nabla u_0\|_{Z_{-3,\lambda}}^2 + \left\langle \log \frac{\lambda_{j+1}}{\lambda_j \gamma^{\frac{1}{4}}} \right\rangle^2 \|1_{r \geq R_j^-} \nabla \bar{u}_{0,j}\|_{Z_{-3,\lambda_j}}^2 + (c_{j,0}^\infty)^2 \left(\frac{\lambda_{j+1}^{\frac{N+2}{2}}}{\gamma^{\frac{N+2}{8}} \lambda_j^{\frac{N+2}{2}}} \right)^2.
\end{aligned}$$

where we used that

$$|\nabla(\Lambda W)_{(\lambda_i)}| \sim \frac{1}{\lambda_i^{\frac{N}{2}}} \left| (\nabla \Lambda W) \left(\frac{r}{\lambda_i} \right) \right| \sim \left(\frac{\lambda_{j+1}}{\lambda_j} \right)^{\frac{N+2}{2}} \gamma^{-\frac{N+2}{8}} R_j^{-\frac{N}{2}}.$$

This implies that

$$\begin{aligned}
& \left\langle \tilde{M}(c_{j,k}^\infty)_{1 \leq k \leq \frac{N-6}{2}}, (c_{j,k}^\infty)_{1 \leq k \leq \frac{N-6}{2}} \right\rangle \\
& \lesssim \left(\frac{\lambda_{j+1}}{\lambda_j} \right)^{N-6} \gamma^{-\frac{N-6}{4}} (R_j^-)^{-(N-6)} \\
& \quad \cdot \left(\langle \log r \rangle^2 \|\nabla u_0\|_{Z_{-3,\lambda}}^2 + \left\langle \log \frac{\lambda_{j+1}}{\lambda_j \gamma^{\frac{1}{4}}} \right\rangle^2 \|1_{r \geq R_j^-} \nabla \bar{u}_{0,j}\|_{Z_{-3,\lambda}}^2 \right) \\
& \quad + (c_{j,0}^\infty)^2 \left(\frac{\lambda_{j+1}^{\frac{N+2}{2}}}{\gamma^{\frac{N+2}{8}} \lambda_j^{\frac{N+2}{2}}} \right)^2 \\
& \lesssim \frac{1}{\sqrt{\gamma}} \frac{\lambda_{j+1}^2}{\lambda_j^2} \log \left(\frac{1}{\sqrt{\gamma}} \frac{\lambda_{j+1}^2}{\lambda_j^2} \right) \left(\frac{1}{\gamma^{\frac{N+2}{8}}} \left(\frac{\lambda_{j+1}}{\lambda_j} \right)^{\frac{N+2}{2}} |c_{j,0}^\infty| + \|\nabla u_0\|_{Z_{-3,\lambda}} + \sqrt{E_{\text{out}}} \right).
\end{aligned} \tag{4.54}$$

On the other hand, we have $(u_0, u_1) \perp_{\dot{H}^1 \times L^2} ((\Lambda W)_{(\lambda_i)}, (\Lambda W)_{[\lambda_i]})$. Using that

$$|\nabla \Lambda W(r)| \lesssim r, \quad R_j^- \lesssim \gamma(\lambda)^{\frac{3}{4}} \lambda_j,$$

we have

$$\int_{R_j^- \leq |x|} \nabla u_0 \cdot \nabla (\Lambda W)_{(\lambda_j)} = \int_{|x| \leq R_j^-} \nabla u_0 \cdot \nabla (\Lambda W)_{(\lambda_j)} \lesssim O(\gamma^3 |\log \gamma| \|\nabla u_0\|_{Z_{-3,\lambda}}). \tag{4.55}$$

Now using that

$$\int_{|x| \leq \frac{R_j^-}{\lambda_j}} |\nabla \Lambda W|^2 dx \lesssim \int_{|x| \leq \frac{R_j^-}{\lambda_j}} |x|^2 dx \lesssim \left(\frac{R_j^-}{\lambda_j} \right)^{N+2} \lesssim \frac{1}{\gamma^{\frac{N+2}{4}}} \gamma^{N+2} = \gamma^{\frac{3}{4}(N+2)}.$$

We also see that

$$\begin{aligned} \int_{R_j^- \leq |x|} \nabla u_0 \cdot \nabla (\Lambda W)_{(\lambda_j)} &= c_{j,0}^\infty \int_{\mathbb{R}^n} |\nabla \Lambda W|^2 \left(1 + O\left(\gamma^{\frac{3}{4}(N+2)}\right)\right) \\ &+ O\left(\sum_{k=1}^{\frac{N-6}{2}} |c_{j,k}| + \sqrt{E_{\text{out}}} + \gamma^\alpha \|\nabla u_0\|_{Z_{-3,\lambda}}\right). \end{aligned} \quad (4.56)$$

This implies that

$$|c_{j,0}^\infty| \int |\nabla \Lambda W|^2 \left(1 + O\left(\gamma^{\frac{3}{4}(N+2)}\right)\right) \leq \sum_{k=1}^{\frac{N-6}{2}} |c_{j,k}| + \sqrt{E_{\text{out}}} + \gamma^\alpha \|\nabla u_0\|_{Z_{-3,\lambda}} \quad (4.57)$$

where we used (4.55), (4.56). Combining this with (4.54) we obtain that

$$|c_{j,0}^\infty| \lesssim \sqrt{E_{\text{out}}} + \gamma^\alpha \|\nabla u_0\|_{Z_{-3,\lambda}}$$

and using that

$$\left\langle \tilde{M}(c_{j,k}^\infty)_{1 \leq j \leq \frac{N-6}{2}}, (c_{j,k}^\infty)_{1 \leq j \leq \frac{N-6}{2}} \right\rangle \gtrsim \sum_{k=1}^{\infty} |c_{j,k}|$$

we obtain that (since $\frac{N+2}{2} > \frac{N+2}{8}$ and γ is small)

$$\begin{aligned} \sum_{k=1}^{\infty} |c_{j,k}| &\lesssim \frac{1}{\sqrt{\gamma}} \left(\frac{\lambda_{j+1}}{\lambda_j}\right)^2 \log \left(\frac{1}{\sqrt{\gamma}} \frac{\lambda_{j+1}^2}{\lambda_j^2}\right) (\|\nabla u_0\|_{Z_{-3,\lambda}} + \sqrt{E_{\text{out}}}) \\ &\lesssim \frac{1}{\gamma^{\frac{3}{8}}} \frac{\lambda_{j+1}^{\frac{3}{2}}}{\lambda_j^{\frac{3}{2}}} (\|\nabla u_0\|_{Z_{-3,\lambda}} + \sqrt{E_{\text{out}}}). \end{aligned}$$

Since $\|\nabla(\Lambda W)_{(\lambda_i)}\|_{Z_{-3,\lambda_i}} \lesssim 1$, we have

$$\|c_{j,0}^\infty \nabla(\Lambda W)_{(\lambda_i)}\|_{Z_{-3,\lambda_i}} \lesssim |c_{j,0}^\infty| \lesssim \sqrt{E_{\text{out}}} + \gamma^\alpha \|\nabla u_0\|_{Z_{-3,\lambda}}. \quad (4.58)$$

We also have for $r \leq R_j^+$

$$|\nabla(T_k^\infty)_{(\lambda_j)}| \lesssim \frac{1}{\lambda_j^{\frac{N}{2}}} \left(\frac{r}{\lambda_j} \right)^{-(N-1)} = \frac{\lambda_j^{\frac{N}{2}-1}}{r^{N-1}} \lesssim \left(\frac{\lambda_j}{R_j^+} \right)^{\frac{N}{2}-1} \frac{1}{r^{\frac{N}{2}}} = \frac{1}{r^{\frac{N}{2}}} \left(\frac{\lambda_j}{\lambda_{j+1}} \right)^{\frac{N-2}{4}}$$

and so

$$\begin{aligned} \|1_{r \geq R_j^+} c_{j,k}^\infty \nabla(T_k^\infty)_{(\lambda_j)}\|_{Z_{-3,\lambda_j}} &\lesssim |c_{j,k}^\infty| \|1_{\{r \geq R_j^+\}} \frac{1}{r^{\frac{N}{2}}} \left(\frac{\lambda_j}{\lambda_{j+1}} \right)^{\frac{N-2}{4}}\|_{Z_{-3,\lambda_j}} \\ &\lesssim |c_{j,k}^\infty| \left(\frac{\lambda_j}{\lambda_{j+1}} \right)^{\frac{N-2}{4}} \|1_{\{r \geq R_j^+\}} \frac{1}{r^{\frac{N}{2}}}\|_{Z_{-3,\lambda_j}} \\ &\lesssim \frac{1}{\gamma^{\frac{3}{8}}} \left(\frac{\lambda_j}{\lambda_{j+1}} \right)^{-\frac{1}{2}} (\|\nabla u_0\|_{Z_{-3,\lambda}} + \sqrt{E_{\text{out}}}) \\ &\lesssim \gamma^{\frac{1}{8}} (\|\nabla u_0\|_{Z_{-3,\lambda}} + \sqrt{E_{\text{out}}}) \end{aligned} \quad (4.59)$$

Combining (4.58), (4.59), (4.52), (4.53) we obtain

$$\|1_{\{r \geq R_j^+\}} \nabla u_0\|_{Z_{-3,\lambda_j}} \lesssim \sqrt{E_{\text{out}}} + \gamma^\alpha \|\nabla u_0\|_{Z_{-3,\lambda}}$$

By the induction hypothesis we also have

$$\|1_{\{r \geq R_{j-1}^+\}} \nabla u_0\|_{Z_{-3,\lambda_{j-1}}} \lesssim \sqrt{E_{\text{out}}} + \gamma^\alpha \|\nabla u_0\|_{Z_{-3,\lambda}}$$

and so using

$$\|1_{\{r > R_j^+\}} f\|_{Z_{-3,\lambda_j}} \lesssim \|1_{\{r \geq R_{j-1}^+\}} f\|_{Z_{-3,\lambda_{j-1}}} + \|1_{\{r > R_j^+\}} f\|_{Z_{-3,\lambda_j}}$$

we complete the induction.

Now for $j = 1, \dots, J$, there are slight changes. If $j = 1$, we set up $R_{j-1}^- = R_{j-1}^+ = \infty$ and $v_j^2 = 0$.

For $j = J$, we set $c_{j,k}^\infty = 0$ if $k \geq 1$ and $R_j^- = R_J^+ = 0$. This completes the proof of Proposition 4.6.7 assuming Lemma 4.6.9, which we will now prove.

Proof of Lemma 4.6.9. First note that

$$\|1_{\{r \geq \frac{R_j^-}{24} + |t|\}} F(v_j^3)\|_{L_t^1 L_x^2} = 0$$

by finite propagation speed. Also, suppose $1 \leq j \leq J$ with $i \neq j$. Then

$$\|\nabla v_1^j(0)\|_{Z_{-3,\lambda_j}} \lesssim \|\nabla u_0\|_{Z_{-3,\lambda}}.$$

If $i > j$, then note that by definition of v_j^1 ,

$$v_j^1(0, r) = 0 \text{ if } r \leq \frac{\lambda_i}{48\gamma^{\frac{1}{4}}} \leq \frac{R_j^-}{48}.$$

Now by Lemma 4.6.3, we have

$$\|1_{r \geq |t|} V_{(\lambda_i)} v_j^1\|_{L_t^1 L_x^2} \lesssim \gamma^{\frac{1}{4}} \|\nabla v_j^1\|_{Z_{-3,\lambda_i}} \lesssim \gamma^{\frac{1}{4}} \|\nabla u_0\|_{Z_{-3,\lambda}}.$$

If instead $i < j$, then by definition of v_j^1 , $v_j^1(0, r) = 0$ if $r \geq 2\sqrt{\gamma}\lambda_i \geq 2R_{j-1}^+$. Then by Lemma 4.6.3 we obtain

$$\|1_{\{r \geq |t|\}} V_{\{\lambda_i\}} v_j^1\|_{L_t^1 L_x^2} \leq \gamma^{\frac{3}{8N}} \|\nabla u_0\|_{Z_{-3,\lambda}}.$$

These imply (4.49). To show (4.50), suppose $1 \leq i \leq J$, $i \neq j$. Since

$$\|\nabla((1 - \chi_R^0)f)\|_{Z_{-3,\lambda}} \leq C \|1_{|x| \geq R} \nabla f\|_{Z_{-3,\lambda}},$$

we have

$$\|\nabla v_j^2(0)\|_{Z_{-3,\lambda_{j-1}}} \leq C \|1_{\{r \geq R_{j-1}^+\}} \nabla u_0\|_{Z_{-3,\lambda_{j-1}}} \lesssim \sqrt{E_{\text{out}}} + \gamma^\alpha \|\nabla u_0\|_{Z_{-3,\lambda}}.$$

Since $v_j^2(0, r) = 0$ if $r \leq R_{j-1}^+$ we have

$$\|\nabla v_j^2(0)\|_{Z_{-3,\lambda_j}} \lesssim \|\nabla v_j^2(0)\|_{Z_{-3,\lambda_{j-1}}}.$$

Hence

$$\|\nabla v_j^2(0)\|_{Z_{-3,\lambda_i}} \lesssim \sqrt{E_{\text{out}}} + \gamma^\alpha \|\nabla u_0\|_{Z_{-3,\lambda}}.$$

Using Lemma 4.6.3, we obtain

$$\|1_{\{r \geq |t|\}} V_{\{\lambda_i\}} v_j^2\|_{L_t^1 L_x^2} \lesssim \sqrt{E_{\text{out}}} + \gamma^\alpha \|\nabla u_0\|_{Z_{-3,\lambda}}$$

thus proving (4.50). □

□

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