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Walrasian equilibrium in large, quasilinear markets

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In an economy with indivisible goods, a continuum of agents, and quasilinear utility, we show that equilibrium exists regardless of the nature of agents' preferences over bundles. This contrasts with results for economies with a finite number of agents, which require restrictions on preferences (such as substitutability) to guarantee existence. When the distribution of preferences has full support, equilibrium prices are unique.

Keywords. Walrasian equilibrium, quasilinear utility, indivisible goods, continuum economies, complementary preferences.

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1. Introduction

An important finding of the literature on markets with a finite number of agents, indivisible goods and preferences that are quasilinear in a divisible numeraire is that Walrasian equilibrium is guaranteed to exist only under restrictive conditions on preferences. Most such results rely on consumers viewing goods as substitutes in a strong sense.[1] However, in some such markets, complex patterns of complementarity seem inevitable.

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[1]Examples of papers that impose such requirements of substitutes are Kelso and Crawford (1982), Gul and Stacchetti (1999), Milgrom and Strulovici (2009), and Hatfield et al. (2011). These papers argue that strongly substitute preferences are "necessary" for equilibrium to exist in the sense that they constitute a "maximal domain": if any individual in an economy has a preference outside of this class, then all other individuals in the economy having preferences in this class does not guarantee the existence of equilibrium. However, other maximal domains exist and in particular Sun and Yang (2006), Hatfield and Kominers (2011), and Baldwin and Klemperer (2013) consider domains with complementarity where equilibrium is also guaranteed to exist under other restrictions on demand patterns. By contrast, in a large market, we show that no restrictions on preferences are needed.

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For instance, used car part 1 could be useless, to a given consumer, without part 2, while someone else might require parts 2, 3, and either 10 or 11. In this paper, we show that such preferences pose no threat to existence, provided that the number of consumers is sufficiently large. Specifically, we consider a model analogous to that of Gul and Stacchetti, but with a continuum of consumers. A competitive equilibrium exists even if agents have arbitrary preferences over bundles of items and it is unique if the distribution of preferences has full support.

At the core of our approach is Farrell's (1959) classic insight that the impact of nonconvexities in any individual consumer's demand diminishes as the number of consumers increases. As a result, even though individual demand is not convex, when there is an infinite number of consumers, aggregate demand exhibits the convexity necessary to prove the existence of an equilibrium price vector using a standard fixed-point argument. This is true independently of the preference distribution, which, for instance, can have atoms. To see the intuition for this, note that when prices are such that a positive mass of agents are indifferent between two bundles, one can allocate them to the two bundles in the proper proportion so as to exactly exhaust supply. By contrast, in the finite agent model, an indifferent consumer must go "all in" to one bundle or the other, creating nonconvexities. Thus considering large economies not only helps justify price-taking behavior but also offers a complementary alternative means of ensuring equilibrium existence to Gul and Stacchetti's limits on the range of consumer preferences.

Similarly, we show that incorporating rich heterogeneity smooths the economy further: if the distribution of preferences has full support, then equilibrium prices are unique, utilitarian social welfare is differentiable with respect to aggregate supply, and the marginal contribution of each good equals its equilibrium price. Thus, we restore the uniqueness typically characteristic of quasilinear economies.

Following Mas-Colell (1975), our model differs from Aumann's (1966) formalization of Farrell's argument in that we consider indivisible goods, whereas he considers a continuous, convex consumption set with continuous preferences. In an example, Mas-Colell (1977) shows that indivisibility may cause a failure of upper hemicontinuity of the aggregate demand correspondence. The problem is that an infinitesimal change in prices may make an indivisible commodity affordable to a positive mass of consumers, creating discontinuous income effects. To avoid this, Mas-Colell assumes a smooth distribution of endowments so that each such large income effect impacts at most a measure zero of agents.

Our contribution is to connect the more recent literature on quasilinear indivisible good economies to this older literature on economies with continua of agents. In particular, we note that by ruling out all income effects, including the extreme ones Mas-Colell considers, quasilinearity restores upper hemicontinuity without any distributional assumptions. Schmeidler (1973) uses a similar observation in the context of game theory (bounds on reactions to aggregate strategic variables rather than prices) to ensure the existence of pure strategy Nash equilibria in large, anonymous, finite action games.[2]

[2]Because Rath (1992) employs a distributional approach similar to the one we adopt, the link of our paper to his is clearer than to Schmeidler's.

However, as far as we know, no one has previously used this approach directly in general equilibrium theory. Quasilinearity also ensures the essential uniqueness of equilibrium allocations, as they must maximize total welfare. Thus equilibrium prices are unique under full support because any change in price implies a change in quantity.

One field that frequently uses a continuum representation of consumers is industrial organization and our results may prove useful in this area. Most directly, any application of the Cournot (Nash-in-quantities) solution concept requires prices to be well defined functions of quantities. Our results ensure the existence of such functions in discrete choice demand systems with complementarities. Moreover, in empirical industrial organization, invertibility of demand systems is an area of active research (Berry et al. forthcoming). Our results extend those of Berry (1994) on the invertibility of the "market-share equation" under unit demand to the case where consumers demand arbitrary bundles.

The main paper opens, in Section 2, with a simple example that illustrates our results. We then develop the continuum model and state the results in Section 3. Section 4 concludes and the Appendix contains all the proofs.

2. Illustrative example

We begin by illustrating the results with a simple example, adapted from Hatfield et al. (2011). An economy has two consumers, Charlie and Sonia, and two indivisible goods, as well as money. Consumers' utility is quasilinear (viz. additively separable and linear) in money, and utilities from discrete bundles may thus be measured in dollar equivalents. Charlie views the goods as perfect complements, deriving \$1 of value from consuming both goods and 0 otherwise. Sonia views the goods as perfect substitutes, valuing each one at \$0.75 in the absence of the other, but receiving no additional benefit from consuming both.

The unique efficient allocation assigns both goods to Charlie, as the most utility Sonia can earn is \$0.75 and allowing this would require forfeiting Charlie's \$1. However, there exists no price vector that supports such an allocation and, therefore no competitive equilibrium. To see this, note that for Charlie to demand both goods, it must be the case that $p_1 + p_2 \leq 1$, where p_g denotes the price of good $g = 1, 2$. If that condition is satisfied, however, at least one of the two prices must be strictly less than \$0.75, implying that Sonia will demand exactly one good.

Now consider an alternative economy in which there are lots of consumers like Charlie and Sonia. In particular, consider the *continuum replication* (Budish 2011) of the above economy, where half of all consumers are like Sonia, half are like Charlie, and there is enough of each commodity for half of all consumers.

The efficient allocation in this economy is simple. One-half of the Sonias receive good 1 and half receive good 2; however, only half of all Charlies receive any good and those who do, receive both of the two goods. This is efficient because Sonias value each good at \$0.75 per unit, while Charlies effectively value each at \$0.50 and thus all Sonias should be satisfied before any Charlies are. Moreover this efficient allocation is supported by a Walrasian price vector: each good costs \$0.50. At these prices, Sonias strictly prefer to consume some good, but are indifferent as to which one, while Charlies are indifferent between consuming the bundle and consuming nothing. Thus, as efficiency prescribes, Charlies are collectively happy to mix between demanding both goods and demanding neither, while Sonias are collectively content to mix between demanding one good and the other. Our main result shows the intuition from this example to be robust: Theorem 1 guarantees that an equilibrium always exists in an economy with a continuum of consumers.

Why does a simple competitive equilibrium exist in the continuum replication, but not in the finite economy? When prices equal \$0.50, in the continuum economy, the Charlies may, in aggregate, demand bundle $(0,0)$, bundle $(1,1)$, or any convex combination of these. Likewise, the Sonias demand $(1,0)$, $(0,1)$, or anything in between. This means that aggregate demand may be any point in the convex hull of $(0,\frac{1}{2})$, $(\frac{1}{2},0)$, $(\frac{1}{2},1)$, or $(1,\frac{1}{2})$. In particular, it can be equal to aggregate supply $(\frac{1}{2},\frac{1}{2})$, which is in the center of this polytope. However, with a finite number of consumers, indivisibilities make it so that convex combinations that yield demand exactly equal to aggregate supply are not possible.

3. Model and results

There is a set G of goods $g = 1, 2, \dots, G$ in addition to a numeraire commodity. The numeraire may be consumed in a continuous amount, agents are endowed with a large quantity of it, and utility is quasilinear. The other goods can only be consumed at a fixed quantity, 1, or not at all, 0.[3] A bundle of goods $x_0 \in 2^G =: X$ denotes whether an agent consumes each good. Agents have utility functions over bundles $u : X \to \Re$, measured in units of the numeraire. Therefore, a utility function is simply a 2^G-dimensional vector. We denote the set of all possible utility functions, or agent *types*, by $\mathcal{U} = \mathbb{R}^{2^G-1}$, a high-dimensional Euclidean space on which we consider the Lebesgue σ-algebra.

Following Hart et al. (1974), we define an economy by an endowment vector $q \in (0,1)^G$ of quantities of each good and a probability distribution η over the set of agent types $\mathcal{U}$. We do not specify which agents initially own the endowment because, with quasilinear preferences and financially unconstrained agents, the initial allocation is moot.

Let ΔX be the set of probability distributions over the set of bundles X.[4] An allocation is a measurable map $\mathbf{x} : \mathcal{U} \to \Delta X$ that assigns a distribution of bundles to each agent type. The variable x denotes a typical distribution over bundles and $\tilde{x}$ denotes the G-dimensional vector of the measure of each good consumed in a distribution of bundles x. That is, the gth coordinate of $\tilde{x}$ is the measure that x assigns to bundles that contain good g.

[3] This restriction is immaterial for our results and was made for notational simplicity. All proofs carry over to the case where consumption of each indivisible good g may take on integer values $0, 1, 2, \dots, M_g$.

[4] Since X is finite, ΔX is a finite-dimensional simplex.

To clarify this notation, consider the following example. There are three goods, $G = \{1, 2, 3\}$. Take the distribution over bundles $x = \mathbf{x}(u)$ that assigns bundle $\{1\}$ to $\frac{1}{2}$ the consumers and bundle $\{1, 2\}$ to the other $\frac{1}{2}$. Then the measure of good 1 demanded is 1, of good 2 is $\frac{1}{2}$, and of good 3 is 0. Therefore, the vector $\tilde{x} = (1, \frac{1}{2}, 0)$.

An allocation is feasible if

$$\int \tilde{\mathbf{x}}\, d\eta = q.$$

To simplify notation, we extend u to distributions over bundles linearly. We may then define social welfare given an allocation $\mathbf{x}(u)$ as $W(\mathbf{x}) = \int u(\mathbf{x}(u))\, d\eta$. A feasible allocation is efficient if no other feasible allocation generates higher welfare. The supremum of the welfare attainable by any allocation that is feasible given q is denoted by $W^*(q)$.

A price vector $p \in \Re^G$ of prices specifies a price for each good. Given a price vector p, define type u's demand as the set of all distributions with support on the set of the best bundles at those prices, that is,

$$D(p, u) = \arg\max_{x \in \Delta X} u(x) - p \cdot \tilde{x}.$$

A competitive equilibrium is a price–allocation pair $(p, \mathbf{x})$ such that markets clear and all agents demand bundles that are optimal at prices p. Formally, this requires that $\mathbf{x}$ is feasible and for every u, we have $\mathbf{x}(u) \in D(p, u)$.

We now state our main result.

Theorem 1. *An equilibrium exists and all equilibria are efficient. If η has full support, then the equilibrium price vector is unique.*

The proof of the existence statement is a standard fixed-point argument. The role of the continuum of consumers is to convexify the aggregate demand correspondence, permitting the use of Kakutani's fixed-point theorem. Uniqueness follows from observing that, with full support, there are always buyers and nonbuyers of each good who are arbitrarily close to indifferent. This implies that the derivative of social welfare with respect to each good's quantity is well defined and must equal the price of the good. Therefore, the equilibrium price vector must be unique.

In the case of full support, denote by $P^*(q)$ the unique equilibrium price vector given q. We then have the following proposition.

Proposition 1. *If η has full support, $W^*(q)$ is concave and C^1, and $\nabla W^*(q) = P^*(q)$. Moreover, $P^*(q)$ is continuous and $P^*_g(q)$ is decreasing in q_g.*

Like the uniqueness part of Theorem 1, the proposition follows from the fact that, with full support, there are always agents who are close to indifferent between buying a good or not. Proofs of the results are given in the Appendix.

4. Conclusion

This paper shows that adding a continuum of consumers to a model with quasilinear utility and indivisible goods eliminates the existence problems created by complementarities. In a supplementary file on the journal website, http://econtheory.org/supp/1060/supplement.pdf, we show that, in large finite replicas, equilibrium prices of the continuum replication market, which always exist by Theorem 1, approximately clear the market and clear it exactly for infinitely many such replicas. Furthermore, by construction, under these prices aggregate supply is in the convex hull of aggregate demand and thus any pressure on prices is not systematic, satisfying much of the motivation for Walrasian equilibrium. Thus payments corresponding to the continuum replication equilibrium may be reasonable substitutes for recently popular Walrasian payments in package auctions.

Appendix: Proofs

To prove the existence of an equilibrium, we define an aggregate demand correspondence and a tâtonnement process whose fixed points correspond to equilibria. We then show that this tâtonnement maps a large cube into itself, and we use Kakutani's theorem to show that a fixed point exists.

Given p, we define (aggregate) demand $D:\Re^G \rightrightarrows \Re^G$ as

$$D(p)=\left\{\int \tilde{\mathbf{x}}\,d\eta:\mathbf{x}(u)\in D(p,u)\text{ for all }u\right\}.$$

That is, D is a correspondence that maps a price vector p into all possible vectors of aggregate quantities demanded by the agents at these prices. The reason why it is a correspondence is that some positive mass of agents may be indifferent between different bundles. Had we ruled this out, D would be a function. Note that a price vector p is part of an equilibrium if and only if $D(p)\ni q$. The following technical lemma is useful to show the existence of an equilibrium.

Lemma 1. *For all p, $D(p)$ is nonempty and convex. Moreover, $D(\cdot)$ has a closed graph.*

Proof. The set of bundles $D(p)$ is nonempty because, given u, there is always at least one optimal bundle, as the set of bundles is finite; therefore, there exists at least one allocation $\mathbf{x}$ satisfying the conditions in the definition of $D(p)$.

To see that $D(p)$ is convex, consider two points d, d' in $D(p)$. By the definition of demand, there exist allocations $\mathbf{x}$, $\mathbf{x}'$ with $\mathbf{x}(u)\in D(p,u)$ for all u, $\int \mathbf{x}(u)\,d\eta=d$, and likewise for $\mathbf{x}'$, d'. For any $\alpha\in(0,1)$, since $D(p,u)$ is convex, we have that $\alpha\mathbf{x}(u)+(1-\alpha)\mathbf{x}'(u)\in D(p,u)$ for all u. Therefore, $\alpha d+(1-\alpha)d'=\int \alpha\mathbf{x}+(1-\alpha)\mathbf{x}'$ is in $D(p)$.

To show that $D(\cdot)$ has a closed graph, consider a sequence (p^n,d^n), with $d^n\in D(p^n)$ converging to some (p,d). We must show that $d\in D(p)$. To reach a contradiction, assume that this is not the case. Since $d^n\in D(p^n)$, there must exist $\mathbf{x}^n$ such that, for all u, $\mathbf{x}^n(u)\in D(p,u)$ and $\int \tilde{\mathbf{x}}^n\,d\eta=d^n$. Note that the set of allocations is compact according to the L^1 norm. Therefore, it is without loss of generality to assume that $\mathbf{x}^n$ converges in the L^1 norm. Let $\mathbf{x}$ be this limit. Then

$$\int \tilde{\mathbf{x}}\, d\eta = \lim_{n\to\infty} \int \tilde{\mathbf{x}}^n\, d\eta = \lim_{n\to\infty} d^n = d.$$

Since we assumed that $d \notin D(p)$, there must exist $\mathbf{x}'$ such that

$$\int u(\mathbf{x}') - p\tilde{\mathbf{x}}'\, d\eta > \int u(\mathbf{x}) - p\tilde{\mathbf{x}}\, d\eta.$$

This implies that for high enough n,

$$\int u(\mathbf{x}') - p^n\tilde{\mathbf{x}}'\, d\eta > \int u(\mathbf{x}^n) - p^n\tilde{\mathbf{x}}^n\, d\eta.$$

Therefore, there exists u such that $u(\mathbf{x}'(u)) - p^n\tilde{\mathbf{x}}'(u) > u(\mathbf{x}^n(u)) - p^n\tilde{\mathbf{x}}^n(u)$, which contradicts $\mathbf{x}^n(u) \in D(p^n, u)$, completing the proof. □

We now define the tâtonnement correspondence $T : \Re^G \rightrightarrows \Re^G$:

$$Tp = p + [D(p) - q].$$

That is, T takes a price vector and increases the prices of all goods that are in excess demand and decreases it for all goods that are in excess supply. By definition, p is an equilibrium price vector if and only if p is a fixed point of T. Note that, by Lemma 1, T takes on nonempty convex values and has a closed graph. It also has the following property.

Lemma 2. *There exists K large enough such that p takes the cube $[-K, K]^G$ into itself.*

Proof. Consider a good g. If type u demands g at prices p, it must be the case that there exists a bundle x_0 such that

$$u(x_0 \cup \{g\}) - u(x_0) \geq p_g.$$

Therefore, the sup norm

$$\|u\|_\infty \geq p_g/2.$$

Now take K such that

$$\eta\left(\|u\|_\infty \geq \frac{K-1}{2}\right) < q_g.$$

For any price such that $K-1 \leq p_g \leq K$, then $(D(p))_g \leq q_g$ and, therefore, $(Tp)_g \leq K$. For any price with $p_g \leq K-1$, since demand is bounded by 1, we have $Tp \leq p+1 \leq K$. Therefore, for any price with $0 \leq p_g \leq K$, we have $(Tp)_g \subseteq [-1, K] \subseteq [-K, K]$.

An analogous argument for prices with $-K \leq p_g \leq 0$ yields that we may take K such that for all prices with $p_g \in [-K, K]$, we have $(Tp)_g \in [-K, K]$. Since there is only a finite number of goods, we may take K uniform for all goods, completing the proof. □

Lemmas 1 and 2 imply, by Kakutani's fixed-point theorem, that T has a fixed point. Therefore, an equilibrium exists.

To see that all equilibria are efficient, consider an equilibrium $(p, \mathbf{x})$. Let $\mathbf{x}'$ be another feasible allocation. For $\mathbf{x}(u)$ to be optimal for agents of type u given p, we must have

$$u(\mathbf{x}(u)) - p \cdot \tilde{\mathbf{x}}(u) \geq u(\mathbf{x}'(u)) - p \cdot \tilde{\mathbf{x}}'(u).$$

Integrating this over all u, we have

$$W(\mathbf{x}) - pq \geq W(\mathbf{x}') - pq \quad \text{or}$$
$$W(\mathbf{x}) \geq W(\mathbf{x}').$$

Assume henceforth that η has full support. To show that equilibrium is unique, fix a quantity q and a good g. Denote by q' a vector with the same quantity as q of each good, plus an extra $dq > 0$ of good g. Let $(p, \mathbf{x})$ be an equilibrium with respect to quantities q and let $(p', \mathbf{x}')$ be an equilibrium with respect to q'. Since $(p, \mathbf{x})$ is an equilibrium, we must have that for all u,

$$u(\mathbf{x}(u)) - p\tilde{\mathbf{x}} \geq u(\mathbf{x}'(u)) - p\tilde{\mathbf{x}}'.$$

Integrating over all u and taking into account the fact that equilibria are efficient, we have

$$W^*(q') - W^*(q) \leq p_g \, dq. \tag{1}$$

Take $\epsilon > 0$. Consider now the set of agents who are marginal nonbuyers of good g. That is, agents who are not buying the good at prices p, but who would benefit at least $p_g - \epsilon$ from adding it to their current bundles. Formally,

$$\mathrm{NB}(\epsilon) = \big\{u : \exists x_0 \in \text{support}(\mathbf{x}(u)) : p - \epsilon \leq u(x_0 \cup \{g\}) - u(x_0) < p_g\big\}.$$

By the full support assumption, there must exist a positive measure of agents in $\mathrm{NB}(\epsilon)$. Therefore, for small enough dq, starting from allocation $\mathbf{x}$, it is possible to give the extra dq of good g to agents in $\mathrm{NB}(\epsilon)$; therefore, the welfare $W^*(q')$ has to be bounded below by $W^*(q) + (p_g - \epsilon)\, dq$. Combining this with inequality (1), we have

$$p_g - \epsilon \leq \frac{W^*(q') - W^*(q)}{dq} \leq p_g.$$

Since for any ϵ, this holds for small enough dq, then $W^*(q)$ has a right derivative equal to p_g. An analogous argument for the left derivative shows that it must also equal p_g, that W^* has a directional derivative, and that the derivative equals p_g. Since this directional derivative is uniquely determined, the equilibrium price vector is unique.

We now prove Proposition 1. Remember that we denote by $P^*(q)$ the unique equilibrium price at a quantity q. Applying the argument used to derive inequality (1), and switching q and q', we obtain

$$P^*(q') \leq \frac{W^*(q') - W^*(q)}{dq} \leq P^*(q).$$

Therefore, $P^*(q)_g$ is decreasing in q_g. This implies that W^* is concave in q_g and, therefore, it is C^1 in q_g. Because we can repeat this argument for each good, W^* is C^1 in all variables in an open set, which implies W^* is C^1, and, therefore, P^* is continuous. Moreover, due to inequality (1), we have that W^* is concave.

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