A Convexity of composite divergence

To verify convexity of (30), consider two joint probability measures on $W \times \Theta$:

$$\hat{m}_0(w \mid \theta)\tau(w \mid \theta)dv(w)n_0(\theta)d\pi_o(\theta)$$
$$\hat{m}_1(w \mid \theta)\tau(w \mid \theta)dv(w)n_1(\theta)d\pi_o(\theta).$$

A convex combination of these two probability measures is itself a probability measure. Use weights $1-\alpha$ and α to construct a convex combination and then factor it in the following way. First, compute the marginal probability distribution for θ expressed as $n_\alpha(\theta)d\pi_o(\theta)$:

$$n_\alpha(\theta) = (1-\alpha)n_0(\theta) + \alpha n_1(\theta).$$

By the convexity of ϕ_2, it follows that

$$\phi_2[n_\alpha(\theta)] \leqslant (1-\alpha)\phi_2[n_0(\theta)] + \alpha\phi_2[n_1(\theta)]. \tag{35}$$

Next note that

$$\begin{aligned}\hat{m}_\alpha(w \mid \theta) = &\left[\frac{(1-\alpha)n_0(\theta)}{(1-\alpha)n_0(\theta)+\alpha n_1(\theta)}\right]\hat{m}_0(w \mid \theta) \\ &+ \left[\frac{\alpha n_1(\theta)}{(1-\alpha)n_0(\theta)+\alpha n_1(\theta)}\right]\hat{m}_1(w \mid \theta).\end{aligned}$$

By the convexity of ϕ_1

$$\begin{aligned}\phi_1[\hat{m}_\alpha(w \mid \theta)] \leqslant &\left[\frac{(1-\alpha)n_0(\theta)}{(1-\alpha)n_0(\theta)+\alpha n_1(\theta)}\right]\phi_1[\hat{m}_0(w \mid \theta)] \\ &+ \left[\frac{\alpha n_1(\theta)}{(1-\alpha)n_0(\theta)+\alpha n_1(\theta)}\right]\phi_1[\hat{m}_1(w \mid \theta)].\end{aligned}$$

Thus,

$$\phi_1[\hat{m}_\alpha(w \mid \theta)]n_\alpha(\theta) \leqslant (1-\alpha)n_0(\theta)\phi_1[\hat{m}_0(w \mid \theta)] + \alpha n_1(\theta)\phi_1[\hat{m}_1(w \mid \theta)]. \tag{36}$$

Multiply (36) by ξ_1 and (35) by ξ_2, add the resulting two terms, and integrate with respect to $\tau(w \mid \theta)dv(w)d\pi_o(\theta)$ to verify that divergence (30) is indeed convex in probability measures that concern the decision maker.