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RANDOM WALKS ON CARTESIAN PRODUCTS OF CERTAIN NONAMENABLE GROUPS AND INTEGER LATTICES

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Non quia difficilia sunt non audemus, sed quia non audemus difficilia sunt.

– Seneca, Epistulae Morales, 104.26

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ABSTRACT

A random walk on a discrete group satisfies a local limit theorem with power law exponent α if the return probabilities follow the asymptotic law

$$P\{\text{ return to starting point after } n \text{ steps }\} \sim C\varrho^n n^{-\alpha}.$$

A group has a *universal local limit theorem* if all random walks on the group with finitely supported step distributions obey a local limit theorem with the same power law exponent. Given two groups that obey universal local limit theorems, it is not known whether their cartesian product also has a universal local limit theorem. We settle the question affirmatively in one case, by considering a random walk on the cartesian product of a nonamenable group whose Cayley graph is a tree, and the integer lattice. As corollaries, we derive large deviations estimates and a central limit theorem.

CHAPTER 1

INTRODUCTION

Let Γ be a finitely generated group and μ a probability measure, or step distribution, on Γ. A *(right) random walk* on Γ with step distribution μ is a sequence $\{S_n\}_{n\geq 0}$ of $\Gamma-$valued random variables such that the *steps* $\xi_n := S_n^{-1}S_{n+1}$ are independent and identically distributed with common distribution μ. The $n-$step transition probabilities $P^n(x,y)$ of the random walk are the conditional probabilities

$$P^n(x,y) = P(S_n = y \,|\, S_0 = x) = \mu^{*n}(x^{-1}y),$$

where μ^{*n} denotes the $n-$fold convolution of μ by itself. This dissertation concerns the asymptotic behavior of the n-step return probabilities $P^n(e,e)$. Sharp asymptotic estimates for these are known as *local limit theorems* (LLT's).

We begin with a key early result of H. Kesten [Kesten(1959a)], [Kesten(1959b)]. Let Γ be a finitely generated group, with a finite, symmetric generating set S, and let μ be a symmetric measure on Γ, that is, $\mu(g) = \mu(g^{-1})$ for all $g \in \Gamma$. Define the *spectral radius* of the random walk with step distribution μ to be

$$\varrho = \limsup_{n\to\infty} P^n(e,e)^{1/n}. \tag{1.0.1}$$

This limsup coincides with the (usual) spectral radius of the convolution operator on $\ell^2(\Gamma)$ induced by μ (cf. [Kesten(1959a)]), hence the term. If $\mu(e) > 0$ then the limsup is actually a limit; this follows because if $\mu(e) > 0$ then the sequence $a_n := -\log P^n(e,e)$ is subadditive. Kesten's theorem [Kesten(1959b)] asserts that if μ is symmetric and $\mu(x) > 0$ for all $x \in S$, then $\rho < 1$ if and only if Γ is nonamenable. Thus, for nonamenable groups, the transition probabilities $P^n(e,e)$ decay at an exponential rate. In general, the spectral radius will depend on the step distribution μ. In Subsection 3.1 we compute ϱ for the simple nearest neighbor random walk on F_2, the free group on two letters. The computation is straightforward in this case, but in most other cases explicit computation of the spectral radius is impossible. The spectral radius of a surface group, the fundamental group of a genus 2 or greater surface, was estimated from above by [Nagnibeda(1997)] and [Żuk(1997)], and from below by [Gouezel(2015)].

The most basic example of a random walk on a discrete group is the symmetric, nearest-neighbor random walk on the integer lattice $\Gamma = \mathbb{Z}^d$, where μ is supported by the natural generators $\pm e_i$, and $\mu(e_i) = \mu(-e_i)$. If $\mu(e_i) > 0$ for each of the generators e_i then elementary Fourier analysis shows that[1]

$$P^{2n}(e,e) \sim C_d (4\pi n)^{-d/2}, \tag{1.0.2}$$

where $C_d > 0$ (see Proposition 2.7.6 of [Spitzer(1976)]). In particular, since $\mathbb{Z}^d$ is amenable and $\rho = 1$, we confirm Kesten's theorem.

From Kesten's theorem we know that if Γ is nonamenable, then the probability of return to the identity decays exponentially. We would like sharper estimates on the asymptotic rate of return, in particular, estimates for which the relative error goes to zero. For the random walks we shall study, the probability of return will be of the form

$$C\frac{\varrho^n}{n^\gamma} = CR^{-n}n^{-\gamma}$$

where we refer to the $n^{-\gamma}$ exponent as the *exponent power law,* and $R := 1/\varrho$. For random walk on $\mathbb{Z}^d$, Fourier methods allow us to determine that the exponent power law is $n^{-\frac{d}{2}}$. However, for nonamenable groups Fourier-style techniques are generally inapplicable, because the representation theory for discrete groups is insufficiently well-understood. Local limit

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[1] Here and throughout the dissertation the symbol $\sim$ means that the ratio of the two sides converges to 1; thus, the *relative* error in approximating the probability on the left by the quantity on the right converges to 0.

results are instead proven by analyzing the singularities of certain generating functions.

We next review some of the existing results concerning return probabilities of random walks on nonamenable groups. An early theorem due to [Bougerol(1981)] concerns random walks on semisimple Lie groups whose step distributions are absolutely continuous probability measures, or more generally probability measures with an absolutely continuous component. Bougerol proved that the probability of returning to a fixed neighborhood of the identity is approximately $CR^{-n}n^{-\delta}$, where the $R > 1$ depends on the step distribution and δ depends on the geometry of the group. The representation theory of semisimple Lie groups is essential to Bougerol's proof. It has been conjectured that the exponent power law $n^{-\delta}$ persists for cocompact lattices of semisimple Lie groups, but currently this is known to be true only for rank-1 groups.

In an early paper, Gerl and Woess [Gerl and Woess(1986)] proved a local limit theorem for a nearest-neighbor random walk on a finitely generated free group. (*Nearest neighbor* means that the individual steps ξ_n of the random walk are elements of the natural generating set.) They proved that the return probabilities obey an $n^{-3/2}$ power law:

$$P^{2n}(e,e) \sim \frac{C}{R^n n^{3/2}}, \tag{1.0.3}$$

where $C > 0$. By similar methods, Cartwright and Soardi [Cartwright and Soardi(1986)] and Woess [Woess(1986)] proved a local limit theorem for nearest neighbor random walks on free products $*_{j \in J} G_j$ of a family $(G_j)_{j \in J}$ of finite or infinite cyclic groups, where the step distribution is a convex combination $\sum \alpha_j p_j$ of probabilities p_j on the groups G_j. Several years later, a local limit theorem for a non-nearest-neighbor random walk with finite support on a finitely generated free group was proved by Lalley [Lalley(1993)] (and independently by T. Steger, in an unpublished manuscript). In Lalley's theorem, the measure is not required to be symmetric. Lalley's method extends to other free products whose Cayley graphs are trees, such as $\mathbb{Z}_2 * \mathbb{Z}_2 * \mathbb{Z}_2$. Once again, the return probability is of the form (1.0.3).

The results of Gerl, Woess, and Lalley are further generalized in [Lalley(2001)], where random walks on regular languages (defined in Section 2.4 below) are considered. These include not only random walks on free products of finite groups, but also on virtually free groups, such as $PSL(2, \mathbb{Z})$ and its finite-index subgroups. In [Lalley(2001)], it was shown that the n-step return probabilities of a random walk on a regular language must obey one of three types of power laws, namely, R^{-n} times one of $n^{-1/2}, n^{-1}, n^{-3/2}$. In particular, in the case of virtually free, nonamenable groups, the power law is known to be $R^{-n}n^{-3/2}$.

The results of [Lalley(2001)] show that for a wide class of lattices of $PSL(2, \mathbb{R})$ there is a universal local limit theorem, and that the exponent power law is the same as for the ambient Lie group. By entirely different methods, Gouezel and Lalley [Gouëzel and Lalley(2013)] proved that the local limit theorem, with exponent power law $n^{-3/2}$, extends to every co-compact Fuchsian group. Subsequently, in [Gouëzel(2014)] Gouezel demonstrated that equation (1.0.3) holds for random walks on all nonelementary hyperbolic groups.

In all these examples, when there is a local limit theorem for random walk on a group Γ, it is *universal* in form, that is, the same exponent power law $n^{-\delta}$ holds for all step distributions (as long as the step distribution has finite support that generates G as a semigroup). Indeed, [Gerl(1981)] conjectured that for any two step distributions, μ_1, μ_2 on a group satisfying local limit laws of the form $CR^{-n}n^{-\delta_1}, CR^{-n}n^{-\delta_2}$, necessarily $\delta_1 = \delta_2$. This conjecture has been shown to be false: there are discrete groups Γ that support distinct finitely-supported probability measures subject to local limit theorems with different exponent power laws. In particular, as Cartwright [Cartwright(1988)] has shown, there are random walks on $\mathbb{Z}^d * \mathbb{Z}^d$ for $d \geq 5$ that obey at least two different power laws, $n^{-d/2}$ and $n^{-3/2}$, depending on the step distribution. Moreover, Candellero and Gilch [Candellero and Gilch(2012)] have proven that the range of different asymptotic powers is wider: for $\mathbb{Z}^{d_1} * \cdots * \mathbb{Z}^{d_m}$ and all d_i different, exactly $m + 1$ different asymptotic behaviors may occur.

Let G_1, G_2 be two groups with universal exponent power laws. In general, we cannot expect that the exponential decay rate of a random walk on $G_1 \times G_2$ is the product of the exponential decay rates of the projected random walks, as we show below in 3.2. However, it is conceivable that universality of exponent power laws for the groups G_1, G_2 could imply the universality of the exponent power law in the local limit theorem on the product group $G_1 \times G_2$. For a *product measure* $\mu = \mu_1 \times \mu_2$ on $G_1 \times G_2$ it is trivial that

$$(\mu_1 \times \mu_2)^{*n} = \mu_1^{*n} \times \mu_2^{*n},$$

and so the return probabilities for a random walk on $G_1 \times G_2$ with step distribution $\mu = \mu_1 \times \mu_2$ are the product of the return probabilities for each factor. Thus, the random walk must obey a local limit theorem if the component random walks on G_1 and G_2 do. [Cartwright and Soardi(1987)] proved that in certain simple cases the local limit theorem extends to probability measures on $G_1 \times G_2$ that are *not* product measures. In particular, if the measure on $G_1 \times G_2$ is given by $\alpha\mu_1 + (1-\alpha)\mu_2$, and the power laws for the random walks on G_1, G_2 are $n^{-\delta_1}, n^{-\delta_2}$, respectively, then the local limit theorem holds with exponent power law $n^{-\delta_1-\delta_2}$. It is unknown whether such behavior extends to all finitely supported step distributions on product groups. Our primary goal in this dissertation is to show that for products of the form $\Gamma = \Gamma_1 \times \mathbb{Z}^d$, it does.

Theorem 1.0.4. *Assume now that the ambient group is*

$$\Gamma = \Gamma_1 \times \mathbb{Z}^d,$$

where Γ_1 is assumed to be a group whose Cayley graph (with respect to some finite generating set S) is a tree[2]*. Let μ be a step distribution on Γ, subject to the assumptions given in Section*

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[2] Thus, for instance, Γ_1 could be a finitely generated free group, or more generally a finite free product whose factors are copies of $\mathbb{Z}$ or $\mathbb{Z}_2$.

4.1. Then:

$$P^n((e,0),(e,0)) \sim C\varrho^n n^{-(3+d)/2},$$

where $C > 0$ is a constant that depends on the step distribution μ, and ϱ is the spectral radius of the random walk.

In particular, the question of whether the Cartesian product preserves universal power laws is settled affirmatively in the case of Γ.

For ease of exposition, we let $d = 1$ for the remainder of the paper. The Fourier analysis easily extends to higher d. The two main sources of inspiration for arguments in the proof are Section III.19.A in [Woess(2000)] (for sections 4.7, 4.8) and [Lalley(2001)] (for the applications of implicit function, regular language, and Perron-Frobenius theory throughout the paper).

CHAPTER 2

PRELIMINARIES

2.1 Green's function of a random walk

Let S_n be a random walk on a discrete group Γ. The *Green's function* of the random walk is the function defined by the power series

$$G(z) = \sum_{n=0}^{\infty} P\{S_n = \text{identity}\} z^n. \tag{2.1.1}$$

Because the coefficients are probabilities, the radius of convergence of this series is at least 1. The spectral radius of the random walk ϱ is equal to reciprocal of the radius of convergence R of the Green's function. Furthermore, if S_n is an irreducible random walk on a nonamenable group Γ, Kesten's theorem implies that the radius of convergence is strictly greater than 1. The Green's function can also be interpreted as a sum over paths: if P is the set of all paths in Γ that begin and end at the group identity, then

$$G(z) = \sum_{\gamma \in \mathcal{P}} p(\gamma) z^{|\gamma|} \tag{2.1.2}$$

where $|z|$ represents the length of the path and $p(\gamma)$ its probability, that is, $p(\gamma) = \prod \mu(\xi_i)$ where ξ_i are the individual steps of the path.

The Green's function encapsulates all information concerning the return probabilities of the random walk. The idea, in brief, is that the probabilities of interest can be recovered from the Green's function by contour integration:

$$P\{S_n = \text{identity}\} = \frac{1}{2\pi i} \oint_C G(z) \frac{dz}{z^{n+1}} \tag{2.1.3}$$

where C is any contour surrounding $z = 0$ that contains no singularities of G. Because the contour can be deformed in any manner that does not take it through a singularity, the singularities of G – in particular, the singularity of minimum absolute value – will play a key role in determining the asymptotic behavior of the coefficients.

In section 4.2 below we shall introduce a related function of two arguments z, w better suited to studying the return probabilities of a random walk on a product group $\Gamma_1 \times \mathbb{Z}$. We shall also refer to this as a Green's function.

2.2 Algebraic systems of functional equations

Recall that a complex-valued function $G(z)$ defined in a domain D is said to be *algebraic* (or more precisely, a branch of an algebraic function) if there is an irreducible polynomial $P(z, w)$ such that

$$P(z, G(z)) = 0$$

for z in D. The asymptotic behavior of the coefficients in the power series expansion of an algebraic function is determined by the singularities of the function. One key result from analytic function theory (as demonstrated, for instance, in [Hille(1962)]) is this.

Proposition 2.2.1. *If $G(z)$ is an algebraic function, then it has only finitely many singularities in the (extended) complex plane. Each of these singularities is either a pole or a branch point. Near each of the branch points, $G(z)$ may be expanded as a Puiseux series, i.e. a function of the form $\sum\limits_{k=0}^{\infty} c_k z^{k/n}$ for n a positive integer.*

If a function can be expanded as a Puiseux series, then near each singularity ξ_i of order α_i, it can be written in the form $A_i(z)(1 - z/\xi_i)^{\alpha_i} + B_i(z)$, where A_i, B_i are analytic near ξ_i. Then we may apply Darboux's theorem (see p. 277 of [Comtet(1974)], for instance) to obtain asymptotics on the coefficients:

Theorem 2.2.2 (Darboux's theorem)**.** *Let $G(z) = \sum_{n=0}^{\infty} a_n z^n$ be a power series with finite radius of convergence. Suppose $G(z)$ has finitely many singularities $\xi_1, \xi_2, \ldots, \xi_m$ on the* *circle of convergence, near each of which it has the form*

$$G(z) = A_i(z)(1 - z/\xi_i)^{\alpha_i} + B_i(z)$$

with $A_i(z), B_i(z)$ *analytic near* $z = \xi_i$ *and* $\alpha_i \in \mathbb{R}\backslash\{0, -1, -2, \ldots\}$*. Then as* $n \to \infty$*,*

$$a_n \sim \sum_{i=1}^{m} \frac{A_i(\xi_i)}{\Gamma(-\alpha_i) n^{1+\alpha_i} \xi_i^n} \tag{2.2.3}$$

provided the right side is nonzero.

We now present a strategy for showing that the Green's function $G(z)$ of a random walk on a free group is algebraic. First, we will define a finite set of auxiliary generating functions $F_i(z)$, which we will write in vector form as $\vec{F}(z)$. We will then show that $G(z)$ is a rational function of z and the functions F_i, and that the auxiliary generating functions F_i are themselves algebraically interrelated, by deriving an equation

$$\vec{F}(z) = z\vec{Q}(\vec{F}(z))$$

where Q is a vector of quadratic polynomials. Elimination theory – the process of solving a system of polynomial equations by eliminating variables – will then imply that the function $G(z)$, and each of the auxiliary functions $F_i(z)$, is algebraic in z. It will then follow, by Proposition 2.2.1, that the singularities of G are poles or branch points. Determination of the orders of poles and branch points must generally be done by other means (cf. sec. 4.10).

Example 2.2.4. Let X_n be a nearest neighbor random walk on $\Gamma = \mathbb{Z}_2 * \mathbb{Z}_2 * \mathbb{Z}_2$. Then Γ has a group presentation $\langle a, b, c | a^2 = b^2 = c^2 = e \rangle$, and the Cayley graph is a homogenous tree T_3 with three edges connected to each vertex. Let $S = \{a, b, c\}$. Since the random walk is nearest neighbor, the step distribution is supported by the natural generators $\{a, b, c\}$, and we let $p(i)$ denote the probability allotted to generator i for each $i \in S$. Thus, $p(i)$ is the probability of moving from e to i in one step. We define generating functions:

$$G(z) = \sum_{n=0}^{\infty} P^n(e,e)z^n \quad \text{and}$$
$$F_i(z) = \sum_{n=0}^{\infty} P(X_0 = e, X_n = i, T(i) = n)z^n, \quad i \in S,$$

where $T(i)$ is the first time the random walk reaches i. Since the coefficients are probabilities, each series converges in the unit disk.

By conditioning on the first step, we obtain the following system of functional equations:

$$G(z) = 1 + z[\sum_{i\in S} p(i)F_i(z)G(z)], \tag{2.2.5}$$
$$F_\ell(z) = z[p(\ell) + \sum_{w\neq\ell\in S} p(w)F_w(z)F_\ell(z)] \qquad i \in S. \tag{2.2.6}$$

These equations hold on $|z| < 1$, since on the unit disk the power series for the generating functions converge. We now explain how these equations are derived. First, observe that the Green's function $G(z)$ can be interpreted as an expectation of "discounted rewards" along the random walk path, where a reward of 1 is earned at every visit to the group identity e, with discount factor z^n, where n is the time of a visit. The initial time $n = 0$ always gives a reward, since the random walk starts at e. After $n = 0$, no further rewards are earned until the first time $T \geq 1$ that the random walk returns to e (if ever); at this time, a new, independent random walk is begun at e, but future rewards are discounted by the factor z^T; thus,

$$G(z) = 1 + (\mathbb{E}z^T)G(z).$$

where $\mathbb{E}$ denotes expectation. Now the conditional distribution of T, given that the very first step of the random walk is to i, is the same as the distribution of the first time $T(i)$

that the random walk visits i (by symmetry). Consequently,

$$\mathbb{E}z^T = \sum_{i=a,b,c} p(i) z F_i(z),$$

and so equation (2.2.5) follows.

A similar argument can be given to justify the equations (2.2.6). Equation (2.2.6) holds because either the random walk moves to the goal ℓ on the first step, or it moves to one of the two other branches of the tree; in the latter case it must first return to the root e (for the first time) before moving to ℓ (for the first time). More formally, at $n = 1$, $P(X_0 = e, X_n = \ell, T(\ell) = n) = p(\ell)$ and for $n \geq 2$,

$$\begin{aligned} P(X_0 = e, X_n = \ell, T(\ell) = n) = \sum_{\substack{w \neq \ell \in \{a,b,c\} \\ 0 \leq j < n}} & p(w) P(X_0 = w, X_{n-j-1} = e, \\ & X_k \neq e \text{ for } k < n-j-1) P(X_0 = e, \\ & X_j = \ell, X_k \neq e \text{ for } k < j). \end{aligned}$$

By symmetry,

$$\begin{aligned} & P(X_0 = e, X_{n-j-1} = w, X_k \neq w \text{ for } k < n-j-1) \\ & \qquad = P(X_0 = w, X_{n-j-1} = e, X_k \neq e \text{ for } k < n-j-1). \end{aligned}$$

For even n, we see that $P(X_0 = e, X_n = \ell, T(\ell) = n) = 0$.

Given the equations (2.2.5) and (2.2.6), it follows that the functions $G(z)$ and $F_i(z)$ are algebraic. Moreover, since equations (2.2.6) involve all of the functions $F_i(z)$, and since the coefficients of the polynomials in (2.2.6) are positive, all of the functions $F_i(z)$ must have a common radius of convergence $R \geq 1$, and since the power series for $F_i(z)$ has nonnegative coefficients, Pringsheim's theorem (p.133 of [Hille(1962)]) implies that $z = R$ is a singularity of each $F_i(z)$. Equation (2.2.5), which can be rewritten as

$$G(z) = \left\{ z \sum_{i \in S} p(i) F_i(z) \right\}^{-1},$$

implies that either G has a pole in $|z| < R$ or G has a singularity at $z = R$. A theorem of Guivarch (see[Woess(2000)], Th. 7.8) implies that $G < \infty$ at its radius of convergence, and so G cannot have a pole as its leading singularity; hence the radius of convergence of the power series for G is also R, and by Pringsheim's theorem, this must be a singularity. Since G is algebraic, it follows that $z = R$ is in fact a branch point. Finally, Kesten's theorem implies that $R > 1$.

In Example 2.2.4, where the algebraic system has only three quadratic polynomials, it is possible to solve for F_ℓ explicitly. However, in general, when the algebraic system of equations is large, elimination theory implies that solving for an F_ℓ produces a polynomial equation in F_ℓ of large degree, for which solution by quadrature is impossible . In these cases we need alternative methods to find the exponents α_i in Darboux's theorem above. For this we will use Perron-Frobenius theory, as in [Lalley(2001)].

2.3 Laplace's method and saddlepoint technique

When generating functions are multivariable, Darboux's theorem does not apply. Another approach to studying coefficient asymptotics may be useful in such cases. We describe the *saddlepoint technique*, which we first motivate by recalling *Laplace's method*.

Proposition 2.3.1 (Laplace's method). *Let $g \in C^2[a, b]$ be such that for some $x_* \in (a, b)$,*

$$-g''(x_*) = 1/\sigma^2 > 0 \quad \textit{and} \quad g(x) < g(x_*) \quad \textit{for all} \ \ x \neq x_*.$$

Consider the integral

$$J_n := \int_a^b e^{ng(x)} dx$$

Then as $n \to \infty$,

$$J_n \sim e^{ng(x_*)} \sqrt{\frac{2\pi}{n\sigma}}.$$

The more conventional statement is the case where $g(x_*) = 0$, in which case $J_n \sim \sqrt{\frac{2\pi}{n\sigma}}$; we have stated the result in the form more useful to us. If x_* has the property that $g'(x_*) = 0$ and $g(x) < g(x_*)$ for x near x_*, we will call x_* a *saddlepoint* of g. If x_* is a saddlepoint, with large n, values of $e^{ng(x)}$ are concentrated near x_*, so that the value of the integral is determined by the value of the integrand near x_*. The proof of Laplace's method is a straightforward application of Taylor's theorem and the Gaussian integral.

Following Chapter VIII of [Flajolet and Sedgewick(2009)], we consider the *saddlepoint method* to be the complex counterpart of Laplace's method. Whereas in Laplace's method, one integrates over a real interval, the saddlepoint method involves a choice of complex contour. Both scenarios are characterized by a saddlepoint x_* of a function f and the availability of a local expansion of f near x_*.

The saddlepoint method for the integral $\int_A^B e^{nf(z)} dz$ has a few essential steps. First, we choose a simple closed contour C that intersects (or is near) the saddlepoint of f. We divide the contour C into two parts C_1, C_2, where C_1 is a neighborhood of the saddlepoint, and C_2 is its complement. Since x_* is a saddlepoint of f, the integral along C_2 contributes negligibly to the overall integral and can be ignored. Along C_1, we approximate $f(z)$ by a quadratic function, and obtain a Gaussian integral $e^{nf(x_*)} \oint_{C_1} e^{\frac{nf''(x_*)}{2}(w-x_*)^2} dw$.

2.4 Random walks on regular languages

We introduce the concept of random walks on regular languages, studied in detail in [Lalley(2001)]. First we provide some definitions; for further detail, see for instance [Yu(1997)].

An *alphabet* is a finite nonempty set of symbols, and a *word* over an alphabet A is a finite (possibly empty) sequence of symbols from A. A language L over A is a set of words over A.

Definition 2.4.1. A *deterministic finite automaton* is a quintuple

$$(Q, A, \delta, s, F),$$

where

- Q is the finite set of states,,
- A is the input alphabet,
- $\delta : Q \times A \to Q$ is the state transition function,
- $s \in Q$ is the starting state,
- $F \subset Q$ is the the set of accept states.

A *regular language* is defined as a language recognized by a finite automaton. That is, each word in the language determines a path in the automaton starting from s and ending in a state in F.

Example 2.4.2. The group $\mathbb{Z}_2 * \mathbb{Z}_2 * \mathbb{Z}_2$ with group presentation $\langle a, b, c | a^2 = b^2 = c^2 \rangle$ may be expressed as a regular language. Let s denote a starting state, and s_1, s_2, s_3 be accept states. Figure 2.4.2 is a diagram of a finite state automaton that encodes $\mathbb{Z}_2 * \mathbb{Z}_2 * \mathbb{Z}_2$ as a regular language. Following convention, the double circles denote accept states, which are elements of the set $F = \{s_1, s_2, s_3\}$. A word in the language on $\langle a, b, c \rangle$ is in $\mathbb{Z}_2 * \mathbb{Z}_2 * \mathbb{Z}_2$ if and only if it is reduced. Thus as Figure 2.4.2 shows, all finite concatenations of elements from $\{a, b, c\}$ are accepted unless a letter is followed by its inverse.

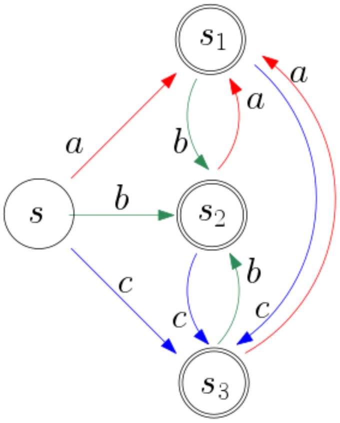


Figure 2.1: Automaton representing $\mathbb{Z}_2 * \mathbb{Z}_2 * \mathbb{Z}_2$

For more on the subject of automata and groups, including examples of groups that *cannot* be expressed as regular languages, see [Epstein et al.(1992)Epstein, Cannon, Holt, Levy, Paterson, and Thurston]. Following [Lalley(2001)], we next define a random walk on a regular language, where we use a weaker definition of a regular language that will suit our purposes.

Definition 2.4.3. Let A be a finite alphabet. A *nearest neighbor random walk* on a regular language with alphabet A is a Markov chain X_n on the set $\cup_{n\geq 0} A^n$ of all finite words whose transitions obey the rules:

- Only the last two letters of the current word may be modified.
- At most one letter may be adjoined or deleted.
- Adjunction or deletion may only be done at the end of a word.
- Probabilities of modification, deletion, and/or adjunction depend only on the last two letters of the current word.

These features of random walks on regular languages are formalized in [Lalley(2001)]. The definition may be weakened in various ways. For instance, instead of assuming only the last two letters may be modified, we may assume that the last K letters may be modified. For any such process, by a *blocking* trick explained below, an equivalent random walk obeying the restrictions of Definition 2.4.3. We now provide two examples of random walks on regular languages.

Example 2.4.4. Let Γ be a virtually free group, that is, a group with free subgroup H of finite index n. Let S be the set of free generators of H and their inverses, and assume S is finite. We claim Γ can be expressed as a regular language.

Let $a_1, \ldots, a_n \in \Gamma$ be coset representatives of the set of right cosets Γ/H. Then any element $x \in \Gamma$ may be written in the form

$$x = h \cdot a_i,$$

where $h = h_1 \ldots h_k$ is a reduced product of elements of S, and the expression $h \cdot a_i$ is possibly unreduced. Since there are only finitely many a_i, there are only finitely many types of reductions of expressions of the form $h \cdot a_i$. Thus there exists a finite set B such all $x \in \Gamma$ may be written in the form $x = h \cdot b$ for $h \in S, b \in B$ and x is reduced. Thus we may write Γ as a regular language, with alphabet $S \cup B$.

Example 2.4.5. Regular languages languages allow us to consider a wider class of cases than groups. Let $\Gamma = F_2$ and consider $T = N \times \Gamma$, where $N = \{1, 2, \ldots, n\}$, so that T is a disjoint union of n copies of Γ. If $\{a, b\}$ is the generating set of Γ, let $S = \{a, b, a^{-1}, b^{-1}\}$. We may define a random walk according to a law

$$\alpha_h(i, j) := P((i, x), (j, xh)),$$

where $h \in S$ and probabilities are translation invariant in the second coordinate. Then α_h defines a step distribution for a random walk on the regular language T with alphabet $N \times S$.

2.5 Perron-Frobenius theory

In the proof of the main result, we rely heavily on several results related to irreducible, nonnegative matrices. For completeness, we state those theorems here; see XIII.2 in [Gantmacher(1959)] for proofs of these results.

Proposition 2.5.1 (Perron-Frobenius). *Let A be an irreducible non-negative $n \times n$ matrix. Then it has an eigenvalue $\lambda > 0$ such that:*

1. *λ is a simple root of the characteristic polynomial.*
2. *λ has a strictly positive left eigenvector v^T and right eigenvector w, which may be normalized so that $v^T w = 1$.*
3. *Any eigenvalue μ of A is such that $|\mu| \leq \lambda$, with strict inequality if A is aperiodic.*
4. *If A is aperiodic, we may write $A = \lambda w v^T + \tilde{A}$ where $\varrho(\tilde{A}) < \lambda$, and $Aw = 0 = v^T A$.*

Definition 2.5.2. For a matrix C let $|C|$ denote the matrix obtained by taking the modulus of each entry.

Proposition 2.5.3 (Wielandt). *Let A, C be two $n \times n$ matrices, such that A is irreducible, and $|C| \leq A$. Then for every eigenvalue γ of C, and for the maximal eigenvalue of r of A (which exists by the Perron-Frobenius theorem),*

$$|\gamma| \leq r.$$

Proposition 2.5.4. *Given two aperiodic, irreducible $n \times n$ Perron-Frobenius matrices A, B such that $A \leq B$, if there exists an i, j such that $(A)_{ij}$ (the ij^{th} entry of A) is strictly less than $(B)_{ij}$, we find*

$$\varrho(A) < \varrho(B). \tag{2.5.5}$$

Sketch. Since A, B are irreducible, if there exists an index ij such that $(A)_{ij} < (B)_{ij}$, there exists an m_0 such that for $m \geq m_0$, $(A^m)_{ij} < (B^m)_{ij}$ for all indices ij. (This is easily seen from the interpretation of A, B as transition matrices of an graph with n nodes.) Thus, there exists an $\epsilon > 0$ such that $A \leq (1-\epsilon)B$, so that $A^m \leq (1-\epsilon)^m B^m$, and then by Wielandt's theorem,

$$\varrho(A)^m = \varrho(A^m) \leq (1-\epsilon)^m \varrho(B^m) = (1-\epsilon)^m \varrho(B)^m$$

so that $\varrho(A) \leq (1-\epsilon)\varrho(B)$ and therefore $\varrho(A) < \varrho(B)$. □

CHAPTER 3

SPECTRAL RADII OF RANDOM WALKS ON CARTESIAN PRODUCTS: SOME EXAMPLES

We take a detour from the main proof to address a claim made in the introduction. Let the *period* of a random walk on a group be defined by $d = \gcd\{n \in \mathbb{N} : P^n(e,e) > 0\}$. For a random walk of period $d = 2$, the the spectral radius may be defined by

$$\varrho = \limsup_{n\to\infty} P^{2n}(e,e)^{1/2n}.$$

We stated that the spectral radius of a random walk on a cartesian product of groups is not in general the same as the product of the spectral radii obtained from projection in each coordinate. We first present an instructive computation that shows one case where the spectral radius of the walk on the cartesian product *is* the product of the spectral radii of the projected walks. We then present a counterexample that verifies our claim.

First, we give some preliminary notation. Recall that $f(n) \sim g(n)$ if $f(n)/g(n) \to 1$ as $n \to \infty$. We will let

$$f(n) \approx g(n) \text{ if } \frac{\log f(n)}{n} \sim \frac{\log g(n)}{n}.$$

In other words, $f(n) \approx g(n)$ if $f(n) \sim g(n)$ up to subexponential terms. We will need the following form of Stirling's approximation:

$$n! \approx (n/e)^n.$$

Also, in the following we will repeatedly use the fact stated in the introduction that the n^{th} step return probability of a random walk on a free group F_k is

$$\sim CR^{-n}n^{-3/2},$$

where $R = 1/\varrho$ is the radius of convergence of the Green's function of the random walk. Finally, given a random walk $S_n = (X_n, Y_n)$ on $G_1 \times G_2$, the *projected random walk* onto the first, second coordinate is the given by X_n, Y_n respectively.

3.1 An example from Cartwright and Soardi

Let $(G_1, \mu_1), (G_2, \mu_2)$, be two finitely generated groups and step distributions such that the return probabilities have the form $\dfrac{C_i}{n^{a_i} R_i^n}$, for $C_i > 0$. Then a theorem due to [Cartwright and Soardi(1987)] states that a random walk on $G_1 \times G_2$ with step distribution $P = \alpha P_1 \otimes I_2 + (1-\alpha) I_1 \otimes P_2$ (for $\alpha \in [0, 1]$) has return probabilities given by

$$\frac{C}{n^{a_1+a_2}(\alpha R_1 + (1-\alpha)R_2)^n}.$$

Here the $R_i = 1/\varrho_i$, where ϱ_i is the spectral radius of the (original, unprojected) random walk on (G_i, μ_i). Thus the weighted product of the random walks corresponds to weighted addition of their spectral radii.

Taking cue from Cartwright and Soardi's theorem, we consider a random walk which walks either horizontally (with probability 3/4) or vertically (with probability 1/4) in the Cartesian product $\Gamma \times \mathbb{Z}$. In particular, we consider $(\Gamma, \mu), (\mathbb{Z}, \eta)$ where $\Gamma = F_2$, the free group on two letters, and where measures μ, η are defined on $\Gamma, \mathbb{Z}$ respectively as:

$$\mu = \frac{1}{4}[\delta_a + \delta_{a^{-1}} + \delta_b + \delta_{b^{-1}}],$$
$$\eta = \frac{1}{2}[\delta_{-1} + \delta_1].$$

The measures of the projected random walks on $\Gamma, \mathbb{Z}$ respectively are given by

$$\nu = \frac{3}{4}\mu + \frac{1}{4}\delta_e,$$

$$\tilde{\nu} = \frac{1}{4}\eta + \frac{3}{4}\delta_e.$$

We let the spectral radius of this random walk on $\Gamma \times \mathbb{Z}$ be denoted $\varrho_{\Gamma\times\mathbb{Z}}$, and we let the spectral radius of the projected random walks on $\Gamma, \mathbb{Z}$ be denoted by $\varrho_{\Gamma,\nu}, \varrho_{\mathbb{Z},\tilde{\nu}}$ respectively. We we would like to show that

$$\varrho_{\Gamma\times\mathbb{Z}} = \varrho_{\Gamma,\nu}\varrho_{\mathbb{Z},\tilde{\nu}} = \varrho_{\Gamma,\nu}$$

The latter equality comes from the fact that the spectral radius of the projected random walk on $\mathbb{Z}$ is 1, which is seen from Kesten's theorem. Cartwright and Soardi's result implies

$$\varrho_{\Gamma\times\mathbb{Z}} = \frac{3}{4}\varrho_{\Gamma,\mu} + \frac{1}{4}\varrho_{\mathbb{Z},\eta} = \frac{3}{4}\varrho_{\Gamma,\mu} + \frac{1}{4}.$$

We let

$$\varrho_\mu := \varrho_{\Gamma,\mu}, \quad \varrho_\nu := \varrho_{\Gamma,\nu}, \quad R_\mu = 1/\varrho_\mu.$$

Thus the goal is now to show that

$$\frac{1}{4} + \frac{3}{4}\varrho_\mu = \varrho_\nu. \tag{3.1.1}$$

To do so, we express ϱ_ν in terms of ϱ_μ. First, let N_μ be the number of steps in the Γ direction. We will let $k = \lfloor nt \rfloor$, with $t \in [0,1]$. A preliminary computation, using Stirling's approximation:

$$\begin{aligned}\binom{2n}{2k} &\approx \frac{n^{2n}}{(n-k)^{2(n-k)}k^{2k}}.\\ &\sim \frac{1}{(1-t)^{2n(1-t)}t^{2nt}}.\end{aligned}$$

Thus,

$$\begin{aligned}
P^{\nu}(X_{2n} = e) &= \sum_{k=0}^{n} P(X_{2n} = e \cap N_{\mu} = 2k) \\
&= \sum_{k=0}^{n} P(N_{\mu} = 2k) P(X_{2n} = e | N_{\mu} = 2k) \\
&= \sum_{k=0}^{n} \binom{2n}{2k} (3/4)^{2k} (1/4)^{2(n-k)} P^{\mu}(X_{2k} = e) \\
&\approx \int_0^1 \frac{(3/4)^{2nt}(1/4)^{2n(1-t)}}{R^{-2nt} t^{2nt} (1-t)^{2n(1-t)}} dt \\
&= \int_0^1 e^{2n\phi(t)} dt, \text{ where} \\
\phi(t) &= t\log(3/4) + (1-t)\log(1/4) - t\log(R_{\mu}) - t\log(t) - (1-t)\log(1-t) \\
&= t\log(3/4) + (1-t)\log(1/4) - t\log(R_{\mu}) + H(t)
\end{aligned}$$

where $H(t)$ is Shannon's entropy. We solve the equation $\phi'(t_*) = 0$, and obtain

$$t_* = \frac{3}{R_{\mu} + 3}.$$

Next, we observe that $\phi''(t_*) = H''(t_*) < 0$. Therefore after approximating $\phi(t)$ by its quadratic Taylor approximation, using the saddlepoint method and the fact that $f(n) \approx G(n)f(n)$ for G subexponential, we have

$$P^{\nu}(X_{2n} = e) \approx e^{2n\phi(t_*)}.$$

We set $s(t) = t\log(3/4) + (1-t)\log(1/4) + H(t)$. Then

$$\begin{aligned}
P^{\nu}(X_{2n} = e) &\approx e^{2n\phi(t_*)}, \\
&= e^{-2nt_* \log(R_{\mu})} e^{2ns(t_*)}
\end{aligned}$$

$$= \varrho_\mu^{2nt_*} e^{2ns(t_*)}.$$

Then taking the $2n^{th}$ root of both sides,

$$P^\nu(X_{2n} = e)^{\frac{1}{2n}} \approx \varrho_\mu^{t_*} e^{s(t_*)}$$

so

$$\varrho_\nu = \varrho_\mu^{t_*} e^{s(t_*)}.$$

Plugging this into equation 3.1.1, we need to show

$$\frac{1}{4} + \frac{3}{4}\varrho_\mu = \varrho_\mu^{t_*} e^{s(t_*)}. \tag{3.1.2}$$

We may make the computation directly. To do so, we find the value of ϱ_μ. Let X_{2n} be the $2n^{th}$ step of the random walk on Γ. With probability 3/4, the random walk moves 1 step further from the origin, and with probability 1/4 it moves 1 step closer to the origin. The return probability of X_{2n} is the same as that for the random walk $|X_{2n}|$ that measures distance from the origin. Then $|X_{2n}|$ is approximately (ignoring steps starting at 0) a random walk on $\mathbb{Z}_+$ that moves one step to the right with probability 3/4, and to the left with probability 1/4. Thus the probability of return is:

$$\begin{aligned} P(X_{2n} = e) &\sim \binom{2n}{n}(1/4)^n(3/4)^n \\ &\approx 2^{2n}(1/4)^n(3/4)^n \\ &= (3/4)^n. \text{ Thus,} \\ \varrho_\mu &= \frac{\sqrt{3}}{2}. \end{aligned}$$

Substituting this value of ϱ_μ into equation 3.1.2, we find that now we need to show:

$$\frac{2+3\sqrt{3}}{8} = \left(\frac{\sqrt{3}}{2}\right)^{t_*} e^{s(t_*)},$$

where

$$t_* = \frac{3}{R_\mu + 3} = \frac{3\sqrt{3}}{2+3\sqrt{3}},$$
$$1 - t_* = \frac{2}{2+3\sqrt{3}},$$

and

$$\begin{aligned} s(t_*) &= t_* \log(3/4) + (1-t_*)\log(1/4) + H(t_*) \\ &= \log\left(\frac{(3/4)^{t_*}(1/4)^{1-t_*}}{(1-t_*)^{1-t_*} t_*^{t_*}}\right). \end{aligned}$$

In sum, we need to show that:

$$\frac{2+3\sqrt{3}}{8} = \left(\frac{\sqrt{3}}{2}\right)^{\frac{3\sqrt{3}}{2+3\sqrt{3}}} \left(\frac{(3/4)^{\frac{3\sqrt{3}}{2+3\sqrt{3}}}(1/4)^{\frac{2}{2+3\sqrt{3}}}}{(\frac{2}{2+3\sqrt{3}})^{\frac{2}{2+3\sqrt{3}}} \frac{3\sqrt{3}}{2+3\sqrt{3}}^{\frac{3\sqrt{3}}{2+3\sqrt{3}}}}\right)$$

We confirm the validity of the expression with MATHEMATICA.

3.2 The counterexample: a random walk on $\Gamma \times \Gamma$

Next, we turn to an example where the spectral radius of the random walk on a Cartesian product is *not* the product of the spectral radii of the projected random walks. We consider $\Gamma \times \Gamma$, where again $\Gamma = F_2$. Suppose we flip a fair coin so that with probability 1/2, the random walk moves 1 step in the first coordinate and 2 steps in the second, and with probability 1/2 the random walk moves 2 steps in the first coordinate and 1 step in the second. The step distribution associated with this random walk is not a product measure, because steps in the second coordinate depend on the first. We also assume the random walk is isotropic in each coordinate; in other words, in each coordinate the walker has equal probability moving in any direction of the same length. We let ϱ be the spectral radius associated with this random walk on $\Gamma \times \Gamma$. Thanks to symmetry considerations, the spectral radii of the projected random walks in either coordinate are equal and will be denoted ϱ_ν. Finally let ϱ_μ be the spectral radius of the simple random walk on Γ, which we found in the previous subsection to be $\sqrt{3}/2$; again we let $R_\mu := 1/\varrho_\mu$. We will show that

$$\varrho \neq \varrho_\nu^2;$$

thus we confirm that the spectral radius of a random walk on a Cartesian product is *not* in general the same as the product of the spectral radii of the projected random walks in each coordinate.

Let $N_{(1,2)}$ denote the number of times the random walk moves 1 step in the first coordinate and 2 steps in the second. Then

$$\begin{aligned}
P((X_{2n}, Y_{2n}) = (e,e)) &= \sum_{k=0}^{n} P(X_{2n} = e \cap Y_{2n} = e \cap N_{(1,2)} = 2k) \\
&= \sum_{k=0}^{n} P(N_{(1,2)} = 2k) P(X_{2n} = e \cap Y_{2n} = e | N_{(1,2)} = 2k) \\
&= \sum_{k=0}^{n} P(N_{(1,2)} = 2k) P(X_{2n} = e | N_{(1,2)} = 2k) \cdot \\
&\cdot P(Y_{2n} = e | N_{(1,2)} = 2k) \\
&\approx \sum_{k=0}^{n} \binom{2n}{2k} 2^{-2k} 2^{-2(n-k)} R_\mu^{-2k} R_\mu^{-2(2n-2k)} R_\mu^{-4k} R_\mu^{-(2n-2k)} \\
&= \sum_{k=0}^{n} \binom{2n}{2k} 2^{-2n} R_\mu^{-6n}
\end{aligned}$$

$$\approx \int_0^1 (1-t)^{-2n(1-t)} t^{-2nt} 2^{-2n} R_\mu^{-6n} dt. \text{ Then}$$

$$P(X_{2n}, Y_{2n}) = (e,e)) \approx \int_0^1 e^{-2n\phi(t)} dt,$$

where $\phi(t) = -H(t) + \log 2 + 3\log R_\mu$. Then if $\phi'(t_*) = 0$, $t_* = \frac{1}{2}$, and

$$\begin{aligned} P((X_{2n}, Y_{2n}) = (e,e)) &\approx e^{-2n\phi(1/2)} \\ &= (R_\mu^3)^{-2n}. \text{ Thus,} \\ \varrho = \varrho_\mu^3 &= \frac{3\sqrt{3}}{8} \end{aligned}$$

We now compute the spectral radius of the projected random walk.

$$\begin{aligned} P(X_{2n} = e) &= \sum_{k=0}^{n} P(X_{2n} = e \cap N_{(1,2)} = 2k) \\ &= \sum_{k=0}^{n} P(N_{(1,2)} = 2k) P(X_{2n} = e | N_{(1,2)} = 2k) \\ &\approx \sum_{k=0}^{n} \binom{2n}{2k} (1/2)^{2n} R_\mu^{-(2k+2(2n-2k))} \\ &\approx \int_0^1 (1-t)^{-2n(1-t)} t^{-2nt} 2^{-2n} R_\mu^{-2n(2-t)} dt. \\ &\approx \int_0^1 e^{-2n\psi(t)} dt, \end{aligned}$$

where $\psi(t) = -H(t) + (2-t)\log R_\mu + \log 2$. Then solving $\psi'(\tilde{t}) = 0$, we find $\tilde{t} = \frac{R_\mu}{1+R_\mu}$. Then

$$\begin{aligned} P(X_{2n} = e) &\approx e^{-2n\psi(\frac{R_\mu}{1+R_\mu})}. \text{ We have,} \\ \psi\left(\frac{R_\mu}{1+R_\mu}\right) &= \frac{R_\mu}{1+R_\mu} \log \frac{R_\mu}{1+R_\mu} + \frac{1}{1+R_\mu} \log \frac{1}{1+R_\mu} + \frac{2+R_\mu}{1+R_\mu} \log R_\mu + \log 2 \\ &= \log \frac{2R_\mu^2}{1+R_\mu}, \text{ so} \end{aligned}$$

$$\begin{aligned}
P(X_{2n} = e) &\approx \left(\frac{2R_\mu^2}{1 + R_\mu} \right)^{-2n} \\
&= \left(\frac{\varrho_\mu^2 + \varrho_\mu}{2} \right)^{2n} = \left(\frac{3 + 2\sqrt{3}}{8} \right)^{2n}. \text{ Thus,} \\
\varrho_\nu &= \frac{3 + 2\sqrt{3}}{8}. \text{ Then,} \\
\frac{3\sqrt{3}}{8} &\neq \left(\frac{3 + 2\sqrt{3}}{8} \right)^2,
\end{aligned}$$

so in this case, the spectral radius ϱ of the random walk on the product $\Gamma \times \Gamma$ is not the same as ϱ_ν^2.

CHAPTER 4

MAIN RESULT

This chapter is devoted to the proof of our main result, Theorem 1.0.4.

4.1 Definitions and hypotheses

As in the Introduction, let Γ_1 be a discrete group which has a (symmetric) generating set S relative to which the Cayley graph is a tree. Let μ be a probability distribution on $\Gamma = \Gamma_1 \times \mathbb{Z}$, which is not necessarily a product measure. Denote the support of μ by

$$U := \{y \in \Gamma : \mu(y) > 0\},$$

and let

$$A = \{\pi_1(y) : y \in U\},$$

where $\pi_1 : \Gamma \to \Gamma_1$ is the projection onto the first coordinate. A random walk on Γ with step distribution μ is a Markov chain $S_n = (X_n, L_n)$ on Γ, defined by

$$\begin{aligned} S_{n+1} &= S_n * (\xi_{n+1}, \zeta_{n+1}) \\ &:= (X_n \xi_{n+1}, L_n + \zeta_{n+1}), \end{aligned} \tag{4.1.1}$$

where $\{(\xi_i, \zeta_i)\}_{i \in \mathbb{N}}$ are i.i.d. random variables with common distribution μ. Unless otherwise specified, we assume that the initial state S_0 is the group identity, i.e., $S_0 = (e, 0)$. To denote the law of a random walk started at initial point (x, k), we will use a superscript on the underlying probability measure: thus, under $P^{x,k}$, the sequence S_n is a random walk with step distribution μ (cf. equation (4.1.1)) and initial point $S_0 = (x, k)$. When $(x, k) = (e, 0)$ is the group identity, we will (when there is no danger of ambiguity) drop the superscript $(e, 0)$ and simply write P.

We make the following hypotheses on the step distribution μ.

Assumption 1 (Finite support). μ has finite support; that is, $|U| < \infty$.

Assumption 2 (Irreducibility). For any two elements $x, y \in \Gamma$ there is a path from x to y of positive probability.

Assumption 3 (Positive and negative support). For each $j \in A$, there is $k \in \mathbb{N}, m \in -\mathbb{N}$ so that $(j, k), (j, m)$ have positive probability.

Assumption 4 (Aperiodicity). $(e, 0) \in U$. In particular, we have no parity issues in the statement of the main theorem.

For the next assumption, we first introduce some notation. Let S be a set of free generators of Γ_1 and their inverses, such that the Cayley graph of Γ_1 with respect to S is a tree. Recall that A is the support of the projection of the distribution on the first coordinate. Let $|\cdot|$ denote the usual word norm with respect to S, and let

$$K = \max_{w \in A} \{|w|\}.$$

Assumption 5. A is equal to the set of all elements of word length $\leq K$. In other words, $A = \bigcup_{i=0}^{K} S^i$, where $S^0 := \{e\}$.

Our strategy will be to derive a set of algebraic relationships among a suitable collection of generating functions, similar to those introduced in section 2.2, from which the return probabilities can be extracted by residue calculus. Assumptions 1 – 5 will be of crucial importance in the derivation. To obtain the algebraic relations, we shall exhibit the (projected) random walk $X_n = \pi_1(S_n)$ on Γ_1 as a random walk on a regular language. Treating the identity element e (which is represented by the empty word in the symbols S) as a special letter, we view A as an alphabet. To translate words in the alphabet S to words in the new alphabet A, we group words in the elements of S into K-sized blocks starting from the left.

For example, if $K = 2$, $s_i \in S$:

$$s_1 s_2 s_3 \to (s_1 s_2) s_3.$$

Thus, every reduced word in the letters S is translated to a word in the alphabet A in which all but the final symbol are elements of A^K. When this translation is applied to the random walk X_n, the resulting process (which we will continue to denote by X_n) is a random walk on a regular language over the alphabet A, because by Assumption 5, right multiplication by elements of A modifies at most the last two letters; cf. Definition 2.4.3. We provide an example that illustrates the blocking trick.

Example 4.1.2. Let F_2 be the free group on two letters, and let the generating set and the probability support of a random walk on F_2 be

$$S = \{a, b, a^{-1}, b^{-1}\}, A = \bigcup_{i=0}^{2} S^i,$$

respectively, so that $K = \max_{w \in A}\{|w|\} = 2$. The Cayley graph of F_2 with respect to S is a homogeneous tree of valence 4. Each reduced word x in S, written uniquely as

$$x = g_1 \dots g_n$$

for $g_i \in S$, can be rewritten uniquely via blocking as

$$x = b_1 \dots b_{k-1} b_k,$$

such that $b_i \in A$, $|b_i| = 2$ for $i < k$, and $|b_k| \leq 2$. If $k \geq 2$, concatenating x by an element of A modifies at most the last two letters $b_{k-1}b_k$, and if $k = 1$ concatenation modifies at most the letter b_1. Thus the random walk on the Cayley graph of F_2 defined by the distribution support A corresponds to a random walk on a regular language $\mathcal{L}$ over the alphabet A.

Translation by blocking changes word-length: given a reduced word $x = x_1 \dots x_k$ of length $k \geq 1$ in the letters S, the corresponding word $x = b_1 \dots b_{k'-1} b_{k'}$ in the regular language over A has length k' which satisfies $k = K(k'-1) + L$, where $1 \leq L \leq K$. Henceforth we shall refer to k' as the *block length*, written $|x|_B = k'$, to distinguish it from the (usual) *word length* $|x| = k$. The empty word is assigned block length 0.

Next, we define a system of generating functions for the random walk on Γ. These will be functions of two variables z, w (unlike the generating functions of section 2.2); the second variable w will be used to keep track of the level L_n of the random walk in the $\mathbb{Z}$ coordinate. For $s \in \Gamma_1$ such that $|s|_B = 1$, and for $(z, w) \in \mathbb{C} \times \mathbb{C}$, define

$$G(z,w) := \sum_{m \in \mathbb{Z}} \sum_{n \in \mathbb{N}} P^{(e,0)}(S_n = (e,m)) z^n w^m,$$

$$F_{s,e}(z,w) := \sum_{m \in \mathbb{Z}} \sum_{n \in \mathbb{N}} P^{(s,0)}(T = n \text{ and } S_n = (b,m)) z^n w^m.$$

Here,

$$T = \min\{n \geq 1 : X_n = e\},$$

or $T = \infty$ on the event that there is no such n. For $a, b, c \in \Gamma_1$ such that $|bc|_B = 2$, $|a|_B = 1$ we define

$$F_{bc,a}(z,w) := \sum_{m \in \mathbb{Z}} \sum_{n \in \mathbb{N}} P^{(bc,0)}(T' = n \text{ and } S_n = (a,m)) z^n w^m.$$

Here, $T' := \min\{n \geq 1 : |X_n|_B = 1\}$, that is, T is the first time the projected random walk X_n returns to the ball of radius K centered at e in the Cayley graph.

4.2 Algebraic relations among the generating functions

Let $p_{s,k} = \mu((s,k))$ be the probability of moving from $(e,0)$ to (s,k) in one step, and set

$$p_s(w) := \sum_{k\in\mathbb{Z}} p_{s,k} w^k.$$

To obtain relations among the various generating functions introduced above, we condition on the first step of the random walk X_n. In its first step, a random walk starting at e either stays at e or moves to $a \in A$, where $|a|_B = 1$. Thus,

$$G(z,w) = 1 + zp_e(w)G(z,w) + \sum_{\substack{a\in A \\ a\neq e}} zp_a(w)F_{a,e}(z,w)G(z,w). \tag{4.2.1}$$

Furthermore, since a random walk that is at a word a of block length 1 can move to a word of block length $2, 1$, or 0, we have

$$\begin{aligned} F_{s,e}(z,w) = zp_{s^{-1}}(w) + z\sum_a p_{s^{-1}a}(w)F_{a,e}(z,w) + \\ z\sum_{bc}\sum_a p_{s^{-1}bc}(w)F_{bc,a}(z,w)F_{a,e}(z,w). \end{aligned} \tag{4.2.2}$$

Here $\sum_{bc}$ denotes a sum over elements of block length 2, and $\sum_a$ is a sum over elements of block length 1.

Thus, each $F_{s,e}$ can be expressed in a quadratic relationship with other first passage functions of the form $F_{a,e}, F_{ab,c}$. Similar considerations lead to quadratic relationships among the functions $F_{ab,c}$. From the start point ab, the random walk X_n can jump to a word of block length $1, 2$, or 3. On the event that the first step is to a word of block length 3, it must then visit a word of block length 2 before it can reach the set of words of block length

1. Thus,

$$\begin{aligned} F_{ab,c}(z,w) = & z p_{(ab)^{-1}c}(w) + z p_e(w) F_{ab,c}(z,w) + \\ & z \sum_{dg} p_{(ab)^{-1}dg}(w) F_{dg,c}(z,w) + \\ & + z \sum_{\ell hf} p_{(ab)^{-1}\ell hf}(w) \sum_{g} F_{hf,g}(z,w) F_{\ell g,c}(z,w), \end{aligned} \tag{4.2.3}$$

where $\sum_{\ell hf}$ is a sum over elements of block length 3.

We now introduce a vector notation we will use throughout the proof. We let $\vec{F}_1$ be the vector of first passage functions of the form $F_{s,e}$, let $\vec{F}_2 := \vec{F}$ be the vector of first passage functions of the form $F_{ab,c}$, and let $\vec{F}' = (\vec{F}_1^T, \vec{F}_2^T)^T$ (the superscript T denotes matrix transpose) be the vector of both kinds of first passage functions. With these conventions, the system of algebraic relations (4.2.3) can be written in vector form as

$$\vec{F} = z\vec{Q}(w, \vec{F})$$

where $\vec{Q}$ is a vector of polynomials in the variables $\vec{F}$ and w.

Lemma 4.2.4. *For each first passage function F and for each w, the function $F(z,w)$ is an algebraic function of z.*

Proof. Recall that two polynomials $f(x), g(x)$ have a common solution if and only if their resultant, which is a polynomial in the coefficients of $f(x), g(x)$, equals 0 (see Theorem IX.4.1 in [Lang(2002)] for details). This result applies to polynomials with coefficients in any field.

Recall we have let $\vec{F}'$ be the vector of first passage functions of the form $F_{s,e}, F_{ab,c}$. We combine the finite systems of polynomial equations (4.2.2), (4.2.3), writing:

$$0 = S_1(\vec{F}', z, w; \vec{p}),$$

$$\vdots$$

$$0 = S_k(\vec{F'}, z, w; \vec{p}).$$

Here $\vec{p}$ is the vector of coefficients of the form $p_\ell(w)$; for the moment $\vec{p}$ is treated as a constant vector.

These equations are simultaneously equal to 0 if and only if for each of the component functions F there is a polynomial P, obtained by repeatedly applying resultants, such that $P(F, z, w; \vec{p}) = 0$. For completeness, we briefly explain the elimination process. By definition, the resultant of two polynomials $f(x), g(x)$ is a polynomial in the coefficients of $f(x), g(x)$. Thus we may choose, for instance, the first two polynomial equations in the list above, S_1, S_2, choose a variable F_* in common, and write S_1, S_2 as polynomials in the variable F_* with coefficients in the remaining variables. We then eliminate F_* by replacing the S_1, S_2 polynomial equations with their resultant, which is a polynomial in the remaining variables. We continue the elimination process in this manner until the original system has been reduced to a single polynomial equation involving F, z, w, and the transition probabilities $\vec{p}$ (which depend on w). The existence of such a polynomial relation implies the result. □

Lemma 4.2.5. *Assumption 5 implies Irreducibility Assumptions 2.3 and 2.4 in [Lalley(2001)]. In particular, with $\mathcal{L}$ a regular language over the alphabet $B = \bigcup_{i=1}^{K} S^i$, we have the following:*

1. *Let*

   $$w_1 = x_1 x_2 \dots x_m a \in \mathcal{L}$$

   and

   $$w_2 = x_1 x_2 \dots x_m \in \mathcal{L}$$

   such that $|x_i| = K$ for $i < m, |x_m| = L \leq K$ and $|a| \leq K$. There exists a path of positive probability from w_1 to w_2 through words of length at least $(m-1)K + L$.

2. Given

$$w_1 = a_1 \dots a_k \in \mathcal{L}$$

and

$$w_2 = b_1 \dots b_{k'} y_1 \dots y_m \in \mathcal{L}$$

such that $|a_k|, |y_m| \leq K$ *and all other* $a_i, b_i's, y_i's$ *and* w *have length* K*, there exists a word* $\in \mathcal{L}$ *of the form*

$$w_3 = b_1 \dots b_{k'} w a_1 \dots a_k \in \mathcal{L},$$

and there is a positive probability path from w_2 *to* w_3 *through words of length* $\geq k'$*.*

Proof. Since by Assumption 5 the support A contains all elements of length 1 and their inverses, this is clear, since we have full freedom to reduce words by applying appropriate inverses of length 1 and build words by applying generators of length 1. □

Since $\vec{F} = z\vec{Q}(w, \vec{F})$, $\vec{F}$ is a fixed point in x of the equation $x = z\vec{Q}(x)$. For each component $F_{ab,c}$, we may resubstitute the right hand side of the ab, c^{th} component of $z\vec{Q}(w, \vec{F}(z, w))$ back into each $F_{ab,c}$ term. After doing this m times for each component, we obtain the equation $\vec{F} = z^m \vec{Q}^{\circ m}(\vec{F})$, where $\vec{Q}^{\circ m}$ denotes m-times functional composition.

Lemma 4.2.6. *There exists an integer* $m \geq 1$ *such that for any two first passage functions of the form* $F_{ab,c}$*,*$F_{a'b',c'}$*, the variable* $F_{a'b',c'}$ *appears as a term with a positive coefficient in the* ab, c^{th} *index of*

$$\vec{F} = z^m \vec{Q}^{\circ m}(\vec{F}). \tag{4.2.7}$$

Proof. It suffices to show that for any first passage functions $F_{a'b',c'}$, $F_{a'b',c'}$, there exists an m such that $F_{a'b',c'}$ appears as a term in the ab, c^{th} index of the right hand side of equation (4.2.7). This is because the right hand side of (4.2.3) comprises terms that z times a probability, then times a $F_{ab,c}$ function, and all coefficients are positive. Thus once a function appears as a term at some m it does so in all subsequent iterations (with a higher power of z).

The ab, c^{th} term of the right hand side of equation (4.2.7) corresponds to a path from ab to c through words of block length > 1 once m intermediary conditions are taken into account. The presence of $F_{a'b',c'}$ in the ab, c^{th} term corresponds existence of a subpath from $a'b'$ to c' through words of block length > 1. Lemma 4.3.13 guarantees the existence of such paths. $\square$

4.3 Definition of the branch point $R(w)$

For each $w > 0$, let $R(w)$ be the radius of convergence of $G(\cdot, w)$. In this section we will show that $R(w)$ is also the radius of convergence of each of the first passage functions $F_{a,e}(\cdot, w)$ and $F_{ab,c}(\cdot, w)$, and that $R(w)$ is a branch point for all of them. We will then extend $R(w)$ off of the positive real axis.

Proposition 4.3.1. *For each* $w > 0$ *we have* $0 < R(w) < \infty$.

Proof. First we show that $R(w)$ is bounded above by a positive number. By hypothesis, the random walk S_n is irreducible, and so there is a positive-probability path of some finite length $m \geq 1$ that begins and ends at $(e, 0)$. Hence,

$$P\{S_m = (e, 0)\} := q > 0.$$

But it then follows that for any $n > 1$,

$$P\{S_{nm} = (e, 0)\} \geq q^n,$$

and so $\limsup P\{S_n = (e, 0)\}^{1/n} \geq q^{1/m} > 0$, which implies that the radius of convergence of the series $G(\cdot, w)$ is bounded above by $1/q^{1/m} < \infty$.

Next we show that the radius of convergence of $G(\cdot, w)$ is positive. It suffices to show that the coefficients

$$c_n := \sum_{k\in\mathbb{Z}} P\{X_n = e;\, L_n = k\} w^k$$

grow no faster than exponentially in n, i.e., that there is some $r < \infty$ such that $c_n \leq r^n$ for all n.

Let K be the maximum value of $|k|$ for which $p_{x,k} > 0$ for some $x \in \Gamma_1$. Then in each step of the random walk, the $L-$coordinate changes by at most $\pm K$, and so for all $n = 1, 2, \dots,$

$$|L_n| \leq nK.$$

Consequently, for any $w > 0$,

$$\begin{aligned} c_n &= \sum_{k\in\mathbb{Z}} P\{X_n = e;\, L_n = k\} w^k \\ &\leq (w + w^{-1})^{nK} \sum_{k\in\mathbb{Z}} P\{X_n = e;\, L_n = k\} \\ &\leq (w + w^{-1})^{nK} \end{aligned}$$

□

Proposition 4.3.2. *For each $w > 0$,*

$$G(R(w), w) < \infty. \tag{4.3.3}$$

The proof of Proposition 4.3.2 will rely on the following corollary of a result of Y. Guivarc'h, cf. Theorem 7.8 in [Woess(2000)].

Proposition 4.3.4. *Let Y_n be an irreducible random walk on a nonamenable group $\tilde{\Gamma}$, and* *let* $\tilde{G}(z)$ *be its Green's function, that is,*

$$\tilde{G}(z) = \sum_{n=0}^{\infty} P\{Y_n = e\} z^n.$$

If $r < \infty$ *is the radius of convergence of the power series for* $\tilde{G}$*, then*

$$\tilde{G}(r) < \infty.$$

Proof of Proposition 4.3.2. Fix $w > 0$ and define a probability distribution $(q_x)_{x \in \Gamma_1}$ on the group Γ_1 by

$$q_x := \frac{\sum_{k \in \mathbb{Z}} p_{x,k} w^k}{\varphi(w)} \quad \text{where} \quad \varphi(w) = \sum_{x \in \Gamma_1} \sum_{k \in \mathbb{Z}} p_{x,k} w^k.$$

Let Y_n be the random walk on Γ_1 with step distribution $(q_x)_{x \in \Gamma_1}$; then a routine induction on n shows that for any $n \geq 0$ and $y \in \Gamma_1$,

$$P\{Y_n = y\} = \sum_{k \in \mathbb{Z}} P\{X_n = y;\, L_n = k\} w^k / \varphi(w)^n.$$

Consequently, the Green's function $G(z, w)$ of the random walk (X_n, L_n) satisfies

$$G(z, w) = \sum_{n=0}^{\infty} P\{Y_n = e\} (z\varphi(w))^n = \tilde{G}(z\varphi(w))$$

where $\tilde{G}(z)$ denotes the Green's function of the random walk Y_n. Hence, the radius of convergence of $G(\cdot, w)$ is $\tilde{r}/\varphi(w)$, where $\tilde{r}$ is the radius of convergence of $\tilde{G}$. Guivarc'h's theorem implies that $\tilde{G}(\tilde{r}) < \infty$, and so the inequality (4.3.3) follows. $\square$

Corollary 4.3.5. *G is a holomorphic function of w and the first passage functions $F_{a,e}$. Moreover, for fixed $w > 0$, none of the first passage functions $F_{a,e}(z, w)$ can have a singularity in the disk $|z| < R(w)$, and so each must have radius of convergence at least $R(w)$.*

Proof. As before, let $\vec{F_1}$ be the vector of first passage functions of the form $F_{a,e}$. By solving equation (4.2.1) for G,

$$G = \frac{1}{1 - zp_e - z\sum_{a\neq e} p_a F_{a,e}} := \Theta(z, \vec{F_1}) \tag{4.3.6}$$

The right hand side of (4.3.6) can be understood as a rational function Θ of the variables $z, \vec{F_1}$; this function is holomorphic at any point where the denominator in its definition (4.3.6) is nonzero. Furthermore, since the denominator is in fact *linear* in the variables $\vec{F_1}$, none of the entries of $\vec{F_1}$ can have a singularity in the disk $|z| < R(w)$. Finally, In Proposition 4.3.2 we showed that $G(R(w), w) < \infty$ for $w > 0$; therefore, at this point the denominator is nonzero, and so G is holomorphic in $\vec{F_1}$. $\square$

We recall that $\vec{F} := \vec{F_2}$ is the vector of first passage functions of the form $F_{cd,f}$. Fixing $w > 0$ and suppressing it from notation, equation (4.2.2) may be written in vector form

$$\begin{aligned} &\vec{F_1} = zM(\vec{F}(z))\vec{F_1} + z\vec{p} \text{ for } \vec{p} > 0 && \Longrightarrow \\ &\vec{F_1}(I - zM(\vec{F}(z))) = z\vec{p} \end{aligned} \tag{4.3.7}$$

where $M(\vec{F}(z))$ is a square matrix whose entries are linear functions of the entries $F_{ab,c}$ of $\vec{F}$, all of whose coefficients are nonnegative (see (4.2.2)). Thus for z sufficiently small,

$$\vec{F_1} = (I - zM(\vec{F}(z)))^{-1} z\vec{p} \tag{4.3.8}$$

Therefore, $\vec{F_1}$ has a singularity at $z = R(w)$ if and only if one of the following hold: first, if $M(\vec{F}(z))$ has a singularity as a function of z at $z = R(w)$, or second, if the matrix $I - zM(\vec{F}(z))$ is noninvertible. We show the latter is not possible:

Lemma 4.3.9. $\det(I - zM(\vec{F}(z)) \neq 0$ *at* $z = R(w)$.

Proof. For each fixed $w > 0$ the generating functions $G, F_{a,e}$, and $F_{ab,c}$ are all positive and nondecreasing in z for $z > 0$. Furthermore, the functions $F_{a,e}$ and $F_{ab,c}$ converge to 0 as $z \to 0$. Consequently, for all sufficiently small $z > 0$, the matrix $I - zM(\vec{F}(z)$ is invertible. Let z_* be the smallest value of z for which $I - zM(\vec{F}(z)$ is *non*-invertible; then $0 < z_* \leq \infty$.

For each $z > 0$ the matrix $zM(\vec{F}(z))$ is a Perron-Frobenius matrix whose entries are monotone and vary continuously with z. (That $zM(\vec{F}(z)$ is irreducible and aperiodic follows by considerations similar to Lemma 4.3.13). Thus, the Perron-Frobenius theorem applies.The lead eigenvalue $\lambda_1(z)$ is increasing and continuous in z, and so z_* must be the value of z at which $\lambda_1(z) = 1$. Let $h_1(z), v_1(z)$ be the left and right Perron-Frobenius eigenvectors, normalized so that $v_1(z)h_1(z)^T = 1$. Then equation (4.3.8) can be rewritten as

$$\begin{aligned}
\vec{F}_1 &= (I - zM(\vec{F}(z)))^{-1} z\vec{p} \\
&= \left[\sum_{n=0}^{\infty} (zM)^n\right] z\vec{p} \\
&= \left[h_1(z)v_1(z)^T \sum_{n=0}^{\infty} \lambda_1(z)^n + \text{ error of size } (1-\epsilon)^n \lambda_1(z)^n)\right] z\vec{p}.
\end{aligned}$$

At $z = z_*$, the series diverges, and thus some component of $\vec{F}_1$ must have a pole at z_*. It now follows that $R(w) < z_*$, because in Corollary 4.3.5 we established that G is holomorphic in $\vec{F}_1$, and thus up to higher order terms is linear in the components of $\vec{F}_1$. If $\vec{F}_1$ had a component which has a pole at $z = R(w)$, this would contradict Proposition 4.3.2. We conclude that $I - zM(\vec{F}(z))$ is invertible at $z = R(w)$. ☐

Proposition 4.3.10. *For each $w > 0$, the radius of convergence $R(w)$ of $G(z, w)$ is also the radius of convergence of each passage function $F_{s,e}, F_{ab,c}$. Furthermore, $z = R(w)$ is a branch point of $G(z, w)$ and the first passage functions, and for each of these functions the branch point has the same degree.*

Proof. Throughout this proof, $w > 0$ will be fixed, and all generating functions will be viewed as functions of z. By Pringsheim's theorem, $G(z, w)$ has a singularity at its radius of convergence, and by Proposition 4.3.2, $G(R(w), w) < \infty$. Consequently, by equation (4.3.6), all of the functions $F_{a,e}(z, w)$ must be finite at $z = R(w)$, and at least one of them must have a singularity, since otherwise G would be analytic at $z = R(w)$.

Recall that $\vec{F}_1 = (I - zM(\vec{F}(z)))^{-1}z\vec{p}$, by equation (4.3.8). From Lemma 4.3.9, $\vec{F}_1$ has a singularity at $z = R(w)$ if and only if $zM(\vec{F}(z))$ has a singularity as a function of z at $z = R(w)$. Moreover, since $(I - zM(\vec{F}))^{-1}$ is a holomorphic function of $\vec{F}$ (by Cramer's rule), $\vec{F}_1$ is a holomorphic function of $\vec{F}$. It now follows that none of the functions $F_{ab,c}(z, w)$ can have a singularity in $|z| < R(w)$ (since this would force one of the functions $F_{a,e}$ to also have a singularity in the disk, contradicting Corollary 4.3.5), but at least one of them must have a singularity at $z = R(w)$.

We now show that $R(w)$ is the radius of convergence of each of the first passage functions, and that the singularity type is the same for each first passage function and G. We have seen that there is a component F of $\vec{F}_1$ whose radius of convergence of is $R(w)$. Hence, by equation (4.3.8), there is a first passage function in $\vec{F}$ whose radius of convergence is $R(w)$. Lemma 4.2.6 implies that all functions in $\vec{F}$ have the same radius of convergence $R(w)$ and singularity type at $R(w)$. Since all functions in $\vec{F}$ have the same radius of convergence and singularity type, equation (4.3.8) implies all the functions in $\vec{F}_1$ do as well. Finally, equation (4.3.6) implies $R(w)$ has the same singularity type for $G(w)$ as for $\vec{F}_1$. □

As before, let $\vec{F}$ be the vector of first passage functions of the form $F_{ab,c}$. Recall that $\vec{F}$ satisfies a system $\vec{F} = z\vec{Q}(w, \vec{F})$ of quadratic polynomials. Define

$$M(z, w) := \frac{\partial \vec{Q}(w, \vec{F}(z, w))}{\partial \vec{F}}\Big|_{\vec{F}=\vec{F}(z,w)} \tag{4.3.11}$$

to be the Jacobian matrix of this system. Differentiation of the equation $\vec{F} = z\vec{Q}(w, \vec{F})$ gives

$$d\vec{F} = (dz)\vec{Q}(w, \vec{F}(z, w)) + zM(z, w)d\vec{F},$$

which implies

$$\left(I - zM(z,w)d\vec{F}\right) = (dz)\vec{Q}(w, \vec{F}(z,w)). \tag{4.3.12}$$

By the implicit function theorem, the vector-valued function $z \mapsto \vec{F}(z,w)$ has a singularity at z (i.e., some component $F_{ab,c}$ has a singularity) if $d\vec{F}$ is not solvable in terms of dz, or equivalently, if $I - zM(z,w)$ is not invertible.

Lemma 4.3.13. *For each $z > 0$ and $w > 0$, the matrix $M(z,w)$ is an irreducible, aperiodic Perron-Frobenius matrix.*

Proof. It suffices to show that some (positive integer) power $M(z,w)^m$ of $M(z,w)$ has strictly positive entries. This follows directly from Lemma 4.2.6, because

$$M(z,w)^m = \left(\frac{\partial \vec{Q}(\vec{F})}{\partial \vec{F}}\right)^m = \frac{\partial(\vec{Q}^{\circ m}(\vec{F}))}{\partial \vec{F}}, \tag{4.3.14}$$

□

Therefore, the Perron-Frobenius theorem implies that for each pair $w > 0, z > 0$ there is a positive eigenvalue $\lambda(z,w)$ such that

$$\lambda(z,w) = \varrho(M(z,w)),$$

where ϱ denotes spectral radius.This eigenvalue has multiplicity one, and all other eigenvalues of $M(z,w)$ have norms strictly less than $\lambda(z,w)$. Moreover, since the entries of the matrix $M(z,w)$ vary continuously with z and w, so does $\lambda(z,w)$; and since for each $w > 0$ the entries of $M(z,w)$ are monotone in $z > 0$, so is $\lambda(z,w)$.

Proposition 4.3.15. *For each $w > 0$, $R(w)$ is the smallest positive z-solution of*

$$z\lambda(z,w) = 1 \tag{4.3.16}$$

Proof. First, suppose that $z\lambda(z, w) > 1$ at $z = R(w)$. Then by the intermediate value theorem there is some $0 < z_* < R(w)$ where $z_*\lambda(z_*, w) = 1$. But then one of the first passage functions $F_{ab,c}$ must have a singularity at z_*, by the implicit function theorem. This contradicts the fact that $R(w)$ is the least positive singularity in norm of $F_{ab,c}$.

On the other hand, suppose that $z\lambda(z, w) < 1$ at $R(w)$. Then the spectrum of $zM(z, w)$ is contained in a disk of radius < 1, so the matrix $I - zM(z, w)$ is invertible. But this implies that $d\vec{F}$ is solvable in terms of dz, so by the implicit function theorem none of the components of $\vec{F}$ have a singularity at $R(w)$. This contradicts Pringsheim's theorem, which states that a power series with positive coefficients has its least positive singularity at its radius of convergence. $\square$

Corollary 4.3.17. *$R(w)$ extends to a branch of an algebraic function of w.*

Proof. By Proposition 4.3.15, for each $w > 0$ the value $R(w)$ is the smallest positive root of the equation (4.3.16), and hence is a root of the determinental equation

$$\det\left(I - z\frac{\partial\vec{Q}}{\partial\vec{F}}\right) = 0.$$

Here the Jacobian matrix is evaluated at $\vec{F}(z, w)$; thus, the determinant is a polynomial in the variables $z, w, \vec{F}$. But for any $w > 0$, the function(s) $\vec{F}(z, w)$ solve the algebraic system $\vec{F} = z\vec{Q}(w, \vec{F})$; consequently, by elimination, $R(w)$ satisfies a (minimal) polynomial equation $H(w, R(w)) = 0$ for $w > 0$. It follows that $R(w)$ can be analytically continued to a branch of an algebraic function. $\square$

Proposition 4.3.18. *The function $w \mapsto R(w)$ is analytic at every $w > 0$, that is, it has no branch points on the positive axis.*

This result will be of central importance, as it will justify the contour deformation in the proof of the main result below.

Proof. If the function $R(w)$ were to have a branch point at some $w_* > 0$, then $R'(w) \to \infty$ as $w \to w_*$ from below. We will show that this cannot be the case.

Recall that for $w > 0$ the value $R(w)$ is the radius of convergence of the Green's function

$$G(z, w) = \sum_{n=0}^{\infty} \sum_{k \in \mathbb{Z}} P\{X_n = e; L_n = k\} w^k z^n.$$

By Assumption 1, there is an integer $M < \infty$ such that the step distribution μ has support contained in $\Gamma_1 \times [-M, M]$; thus, the projection L_n of the random walk cannot jump by more than $\pm M$ in one step. It follows that $|L_n| \leq nM$, and so the inner sum in the series for the Green's function contains only terms with $-nM \leq k \leq nM$. Now for each $k \in [-M, M]$ the monomial w^k has finite derivative at $w = w_*$, and so because there are only finitely many possibilities for k there exists $C < \infty$ such that for all $|\delta| > 0$ sufficiently small,

$$\begin{aligned} &(w_* + \delta)^k \leq (1 + C\delta) w_*^k \quad \text{and} \\ &(w_* + \delta)^k \geq (1 + C\delta)^{-1} w_*^k. \end{aligned}$$

This implies that for every $n \geq 0$,

$$\begin{aligned} &\sum_{k \in \mathbb{Z}} P\{X_n = e;\, L_n = k\}(w_* + \delta)^k \leq (1 + C\delta)^n \sum_{k \in \mathbb{Z}} P\{X_n = e;\, L_n = k\} w_*^k \quad \text{and} \\ &\sum_{k \in \mathbb{Z}} P\{X_n = e;\, L_n = k\}(w_* + \delta)^k \geq (1 + C\delta)^{-n} \sum_{k \in \mathbb{Z}} P\{X_n = e;\, L_n = k\} w_*^k. \end{aligned}$$

Therefore, the ratio $R(w_* + \delta)/R(w_*)$ is bounded above and below by $(1 + C\delta)^{\pm 1}$ for all δ in a neighborhood of 0. This proves that the derivative of $\log R$ cannot be infinite at any $w_* > 0$, and it follows, since $0 < R(w) < \infty$ on the positive axis, that the derivative of R cannot be infinite at any $w_* > 0$. $\square$

Proposition 4.3.19. *The function* $R(w)$ *has a saddle point at some* $w = \rho > 0$*, that is,*

$$\begin{aligned} R'(\rho) &= 0 \quad \textit{and} \\ R''(\rho) &< 0. \end{aligned} \tag{4.3.20}$$

The proof is long and rather technical, and therefore we postpone it to Section 4.10. A notational remark: we use $\rho > 0$ to denote this saddlepoint; the symbol ϱ denotes spectral radius.

Definition 4.3.21. Let ρ be the saddle point (4.3.20). Let $\alpha > 0$ be such that $R(\rho e^{i\theta})$ is defined for $|\theta| \leq \alpha$. For brevity, we shall in the remainder of the chapter denote

$$R_\theta := R(\rho e^{i\theta})$$

Finally, for $|\theta| \leq \alpha$, write

$$G_\theta(z) := G(z, \rho e^{i\theta}).$$

Proposition 4.3.22. *For* $w > 0$ *near* $\rho, R(w)$ *is a square-root singularity of* $z \mapsto G(z, w)$*, that is, the singularity of* $G(z, w)$ *at* $z = R(w)$ *is a branch point of order* 2*.*

Remark 4.3.23. In the special case where the random walk is symmetric, the existence of a saddle point is straightfoward, and the saddlepoint occurs at $w = 1$. This is because $G(z, e^{i\theta})$ is a function with real coefficients, such that there is local maximum of $w = e^{i\theta}$ at $\theta = 0$, and thus correspondingly a local minimum of $R(w)$ at $R(1)$. Moreover, the symmetry of R in θ implies that $R''(e^{i\theta}) < 0$.

Sketch. The proof is similar to Proposition 7.17 of [Lalley(2001)], which we summarize here. We fix $w > 0$ and often suppress it from notation. First, we recall that $R(w)$ is a singularity of the same type for G as for first passage functions in $\vec{F}$, so it suffices to show that for w near $\rho, R(w)$ is a square-root singularity of some $z \mapsto F(z,w)$. $R(w)$ is defined for $w > 0$ near ρ. We would like to show that $F(R(w)) - F(z) \sim C_F(R(w) - z)^{1/2}$ for $C_F > 0$.

Writing $\Delta\vec{F} := \vec{F}(R(w)) - \vec{F}(z)$ and $\Delta z := R(w) - z$, from the system $\vec{F} = z\vec{Q}(\vec{F}(z))$ we obtain an equation

$$\begin{aligned}(I - R(w)M(R(w)))\Delta\vec{F} =& (\Delta z)\vec{Q}(\vec{F}(R(w))) - R(w)(\text{quadratic terms in } \Delta\vec{F})\\ &- (\Delta z)(\text{linear and quadratic terms in } \Delta\vec{F}).\end{aligned}$$

Now let v be the left eigenvector of $M(R(w))$ and $\lambda(R(w))$ the Perron-eigenvalue as before. Since $R(w)\lambda(R(w)) = 1$, we have:

$$\begin{aligned}(\Delta z)(v^T\vec{Q}(\vec{F}(R(w))) =& R(w)(\text{quadratic form in } \Delta\vec{F})\\ &- (\Delta z)(\text{linear and quadratic terms in } \Delta\vec{F}).\end{aligned}$$

Since v has strictly positive entries, $v^T\vec{Q}(\vec{F}(R(w))$ is positive; also, the quadratic form on the right hand side is nonnegative definite and nondegenerate by irreducibility assumptions on the random walk. ☐

Corollary 4.3.24. *For w near each $w_* > 0$, $R(w)$ is a square root singularity.*

Proof. Fix $w_* > 0$. Let C be a small oriented circle centered at $R(w_*)$, so that there are no other singularities of $G(z,w)$ in or on the circle. As z winds around C, the argument of $G(z,w_*) - G(R(w_*),w_*)$ is π, by Proposition 4.3.22. By the continuity of $R(w)$ in w, for w near w_*, $R(w)$ is a small distance from $R(w_*)$, and we may assume that $R(w_*)$ is inside C. Since the argument of $G(z,w) - G(R(w),w)$ is continuous in w, as z winds around C, the argument is still π. ☐

4.4 Outline of proof

Assuming Proposition 4.3.19, the main steps of the proof of Theorem 1.0.4 are:

(i) For each $|\theta| < \pi$, we can find a simple closed curve C_θ in $\mathbb{C}$ which has 0 in its interior and all singularities of G_θ in its exterior. Applying Cauchy's integral theorem, we write the return probability as:

$$P(X_n = e, L_n = 0) = \frac{1}{4\pi^2 i} \int_{-\pi}^{\pi} \oint_{C_\theta} \frac{G_\theta(z)}{z^{n+1}} dz d\theta.$$

(ii) We show that

$$P(X_n = e, L_n = 0) \sim \frac{1}{4\pi^2 i} \int_{|\theta|<\alpha} \oint_{C_\theta} \frac{G_\theta(z)}{z^{n+1}} dz d\theta.$$

(iii) We deform the contour C_θ. We show we may integrate just over a line segment that intersects the point R_θ.

(iv) After changing variables and writing $G_\theta(z)$ as a Puiseux series near R_θ, we show that for some $k_1 > 0$,

$$P(X_n = e, L_n = 0) \sim \frac{k_1}{n^{3/2}} \int_{-\alpha}^{\alpha} \frac{1}{R_\theta^{n-3/2}} d\theta.$$

(v) Finally, we show that that for some $k_2 > 0$,

$$\int_{-\alpha}^{\alpha} R_\theta^{-(n-1)} d\theta \sim \frac{k_2}{n^{1/2} R_0^{n-1}}.$$

4.5 Integral concentration

Lemma 4.5.1. *For all $\epsilon > 0$ sufficiently small, there exists a $\delta > 0$ such that for all $\theta \in [-\pi, \pi] \backslash [-\delta, \delta]$, $G(\cdot, \rho e^{i\theta})$ has no singularities in the disk $\{z : |z| \leq R_0\}$.*

Proof. The lemma statement is equivalent to the statement that for all $\epsilon > 0$ sufficiently small, there exists a $\delta > 0$ such that for all $\theta \in [-\pi, \pi]\backslash[-\delta, \delta]$ and $|z| \leq R_0$,

$$\varrho(zM(z, \rho e^{i\theta})) < 1 - \epsilon,$$

where ϱ denotes the spectral radius. For then at such z the matrix $I - zM(z, w)$ is nonsingular, $d\vec{F}$ is solvable in terms of dz via equation (4.3.12), and therefore z is not a branch point. It suffices to show that for all $\theta \neq 0 \in [-\pi, \pi]$ and $|z| \leq R_0$,

$$\varrho(zM(z, \rho e^{i\theta})) < 1.$$

That this statement is sufficient follows from the continuity of the elements of the spectrum in θ, and the fact $D_1(0)$ is open.

We now outline the remaining argument. In order to show that $\varrho(zM(z, \rho e^{i\theta})) < 1$ for all $\theta \neq 0 \in [-\pi, \pi]$ and $|z| \leq R_0$, we will use the two theorems to compare $zM(z, \rho e^{i\theta})$ to $R_0 M(R_0, \rho)$, as follows. First, from Wielandt's theorem (Proposition 2.5.3),

$$\varrho(zM(z, \rho e^{i\theta})) \leq \varrho(|zM(z, \rho e^{i\theta})|).$$

Now suppose we can find a matrix B such that

$$|zM(z, \rho e^{i\theta})| \leq B.$$

We may reapply Wielandt's theorem to find that

$$\varrho(|zM(z, \rho e^{i\theta})|) \leq \varrho(B).$$

From Proposition 2.5.4, if we can show that there exists an entry index ij such that

$$(|B|)_{ij} < (R_0 M(R_0, \rho))_{ij},$$

and if we can show these matrices are aperiodic and irreducible, we conclude the proof by applying (2.5.5) and recalling that

$$\varrho(R_0 M(R_0, \rho)) = 1.$$

We obtain this bounding matrix B by applying the triangle inequality in each entry of $|zM(z, \rho e^{i\theta})|$ so that the entries are sums of absolute values of z, p_ℓ and F. We exhibit some of the entries:

$$\begin{gathered}|z|(|p_e(\rho e^{i\theta})| + \sum_{bc} |p_{s^{-1}b}(\rho e^{i\theta})||F_{bc,s}(z, \rho e^{i\theta})|), \\ |z||p_{s^{-1}de}(\rho e^{i\theta})||F_{k,e}(z, \rho e^{i\theta})|, \\ |z|(|p_{s^{-1}k}(\rho e^{i\theta})| + \sum_{bc} |p_{s^{-1}bc}(\rho e^{i\theta})||F_{bc,k}(z, \rho e^{i\theta})|).\end{gathered} \tag{4.5.2}$$

All of the entries of B have a property exhibited in the formulae (4.5.2), namely they are $|z|$ times a sum of terms of the form

$$|p_\ell(\rho e^{i\theta})| \quad \text{or} \quad |p_\ell(\rho e^{i\theta}) F(z, \rho e^{i\theta})|,$$

or both. By comparison, all the entries of $R_0 M(R_0, \rho)$ are R_0 times a sum of terms of the form

$$p_\ell(\rho), p_\ell(\rho) F(R_0, \rho).$$

Thus it suffices to show that $|p_\ell(\rho e^{i\theta})| < p_\ell(\rho)$ and $|F(z, \rho e^{i\theta})| \leq F(R_0, \rho)$ for all first passage functions F.

We first recall a classical result. Let $a_n \geq 0$, where a_n is strictly positive for at least two

n. If $\gcd\{n : a_n > 0\} = 1$, then

$$|\sum_{n\in\mathbb{Z}} a_n e^{in\theta}| < \sum_{n\in\mathbb{Z}} a_n$$

for $0 < |\theta| < 2\pi$. This is because if m, n are coprime, the pair $e^{in\theta}, e^{im\theta}$ are linearly independent as vectors in $\mathbb{R}^2$ for all $0 < |\theta| < 2\pi$, but they must be dependent for a maximum to be reached. From this result and the aperiodicity assumption we conclude that indeed $|p_\ell(\rho e^{i\theta})| < p_\ell(\rho)$.

For the first passage functions $F(z, \rho e^{i\theta})$, let $p_{m,n} > 0$ denote the coefficients. Then:

$$\begin{aligned}
|F(z, \rho e^{i\theta})| &= |\sum_m \sum_n p_{m,n} z^n (\rho e^{i\theta})^m| \\
&\leq \sum_m \sum_n |p_{m,n} z^n (\rho e^{i\theta})^m| \\
&= \sum_m \sum_n p_{m,n} |z|^n \rho^m \\
&\leq \sum_m \sum_n p_{m,n} (R_0)^n \rho^m = F(R_0, \rho)
\end{aligned}$$

□

Corollary 4.5.3. *For each $|\theta| < \pi$, we can find a simple closed curve C_θ in $\mathbb{C}$ which has 0 in its interior and all singularities of G_θ in its exterior.*

Proof. First, for all θ in a sufficiently small neighborhood of 0, $|R_\theta|$ is the radius of convergence of $G_\theta(z)$. R_0 is the radius of convergence and the smallest singularity of $G_0(z)$ in norm, and since the singularities of $F_{ab,c}(z, \rho e^{i\theta})$ perturb continuously with θ, for small enough perturbations, R_θ remains the smallest singularity in norm of G_θ. Moreover, an analytic function has at least one singularity on its circle of convergence, so R_θ must be on the circle of convergence. Thus, for small θ, we may take C_θ to be a circle of radius $0 < r < |R_\theta|$. Second, for θ larger than 0, the previous lemma 4.5.1 implies that we may take C_θ to be the circle of radius R_0. ☐

As above, let $\alpha > 0$ be such that $R(\rho e^{i\theta})$ is defined for $|\theta| \leq \alpha$. We may conclude that the $d\theta$ integral is concentrated in an α-neighborhood of $\theta = 0$:

Lemma 4.5.4.

$$P(X_n = e, L_n = 0) \sim \frac{1}{4\pi^2 i} \int\limits_{|\theta|<\alpha} \oint\limits_{C_\theta} \frac{G_\theta(z)}{z^{n+1}} dz d\theta.$$

Proof. Fix $\epsilon > 0$. Using Lemma 4.5.1, there exists an $\delta > 0$ such that $G_\theta(z)$ has no singularities in the disk $\{z : |z| < R_0 + \epsilon\}$ for $\delta < |\theta| < \pi$. Then redefine α to be be the minimum of α above and this δ. We may write

$$\begin{aligned} P(X_n = e, L_n = 0) = &\frac{1}{4\pi^2 i} \int\limits_{|\theta|<\alpha} \oint\limits_{C_\theta} \frac{G_\theta(z)}{z^{n+1}} dz d\theta \\ &+ \frac{1}{4\pi^2 i} \int\limits_{\alpha<|\theta|<\pi} \oint\limits_{\{z:|z|=R_0+\epsilon\}} \frac{G_\theta(z)}{z^{n+1}} dz d\theta. \end{aligned}$$

It suffices to observe that since $G_\theta(z)$ is bounded in θ and z, there exists a $C > 0$ such that

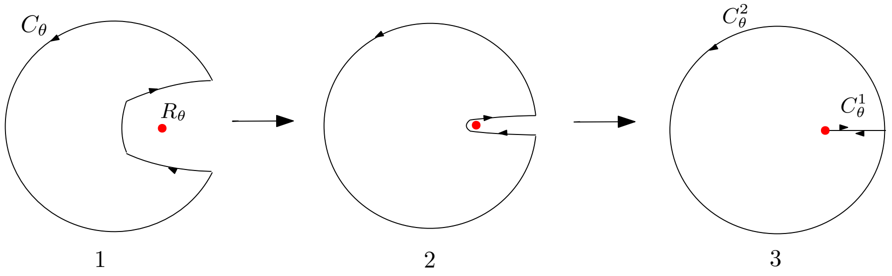
$$\Big| \frac{1}{4\pi^2 i} \int\limits_{\alpha<|\theta|<\pi} \oint\limits_{\{z:|z|=R_0+\epsilon\}} \frac{G_\theta(z)}{z^{n+1}} dz d\theta \Big| < \frac{C}{(R_0 + \epsilon)^n}.$$

Then this term is asymptotically negligible by comparison with the term

$$\int\limits_{|\theta|<\alpha} \oint\limits_{C_\theta} \frac{G_\theta(z)}{z^{n+1}} dz d\theta,$$

provided we can show that the latter is $\approx \dfrac{C'}{(R_0)^n}$ for $C' > 0$, which we will do below. ☐

Figure 4.1: Contour deformation to $C_\theta^1 \cup C_\theta^2$; contours $1, 2, 3$



4.6 Contour deformation

By choosing α sufficiently small, for each $|\theta| < \alpha$ we may assume that R_θ is the smallest singularity in norm of G_θ. Now let C_θ be a positively oriented simple closed curve in $\mathbb{C}$ which has 0 in its interior and R_θ in its exterior. The return probability may then be written

$$P(X_n = e, L_n = 0) \sim \frac{1}{4\pi^2 i} \int\limits_{|\theta|<\alpha} \oint\limits_{C_\theta} \frac{G_\theta(z)}{z^{n+1}} dz d\theta.$$

Following Figure 4.6, for each $\theta < \alpha$ we may deform C_θ to the slit contour $C_\theta^1 \cup C_\theta^2$. Here C_θ^2 is the circle of radius $R_\theta + \beta$ for some $\beta > 0$ such that all singularities of G_θ except R_θ are outside the circle, and C_θ^1 is the horizontal line segment from R_θ to C_θ^2. We observe that the integral $\oint\limits_{C_\theta} \frac{G_\theta(z)}{z^{n+1}} dz$ over C_θ (contour 1 in the Figure) is the same as the integral over contour 2, by Cauchy's integral theorem. That contour 2 is then equal to the integral over the slit contour $C_\theta^1 \cup C_\theta^2$ follows from the dominated convergence theorem. Then we may write,

$$P(X_n = e, L_n = 0) \sim \frac{1}{4\pi^2 i} \int\limits_{|\theta|<\alpha} \oint\limits_{C_\theta} \frac{G_\theta(z)}{z^{n+1}} dz d\theta$$

$$= \frac{1}{4\pi^2 i} \int\limits_{|\theta|<\alpha} \left[\oint\limits_{C_\theta^1} \frac{G_\theta(z)}{z^{n+1}} dz + \oint\limits_{C_\theta^2} \frac{G_\theta(z)}{z^{n+1}} dz \right] d\theta$$

$$\sim \frac{1}{4\pi^2 i} \int\limits_{|\theta|<\alpha} \oint\limits_{C_\theta^1} \frac{G_\theta(z)}{z^{n+1}} dz d\theta.$$

The approximation is justified by an argument similar to that of Lemma 4.5.4. In particular, since the radius of C_θ^2 is equal to $R_\theta + \beta$ for some $\beta > 0$, we have $\Big| \oint\limits_{C_\theta^2} \frac{G_\theta(z)}{z^{n+1}} \Big| < \frac{C}{(R_\theta + \delta)^{n+1}}$ for some $C > 0$, so that this term contributes negligibly compared to the integral over C_θ^1.

4.7 Puiseux series and change of variables

Recall that for $|\theta| < \alpha$, $G_\theta(z)$ has a singularity R_θ of order 1/2. Therefore for such θ we can write $G_\theta(z)$ as a Puiseux series around R_θ with exponent 1/2. In turn, we can write:

$$\frac{G_\theta(z)}{z} = H_\theta(z) + \sqrt{R_\theta - z} K_\theta(z), \tag{4.7.1}$$

where $H_\theta(z)$ and $K_\theta(z)$ are functions that are analytic in z (for each fixed θ) within the radius of convergence of $G_\theta(z)$. In particular, after factoring out terms from the Puiseux expansion:

$$\frac{1}{z}[G_\theta(z) - G_\theta(R_\theta)] = \frac{1}{z} \sum_{n\geq 1} \frac{G_\theta^{2n}(R_\theta)}{(2n)!} (z - R_\theta)^n + \sqrt{z - R_\theta} \cdot \tag{4.7.2}$$

$$\cdot \frac{1}{z} \sum_{n\geq 0} \frac{G_\theta^{(2n+1)}}{(2n+1)!} (R_\theta)(z - R_\theta)^n, \tag{4.7.3}$$

so that

$$H_\theta(z) = \frac{1}{z} \sum_{n\geq 1} \frac{G_\theta^{2n}(R_\theta)}{(2n)!} (z - R_\theta)^n$$

$$K_\theta(z) = \frac{1}{z}\sum_{n\geq 0}\frac{G_\theta^{(2n+1)}}{(2n+1)!}(R_\theta)(z-R_\theta)^n.$$

$H_\theta(z), K_\theta(z)$ are analytic in z near R_θ. Since $H_\theta(z)$ is analytic, its contour integral over t^2+R_θ from $-\sqrt{s_\theta}$ to $\sqrt{s_\theta}$ is equal to 0.

Next, changing variables from z to t removes the singularity at R_θ, since if

$$z = R_\theta + t^2$$

then

$$it = \sqrt{R_\theta - z}$$

We can perform a further change of variables, setting $\beta = t\sqrt{n}$, which will allow us to remove terms involving n from inside the integral.

Thus we will integrate over

$$z(\beta) := \frac{\beta^2}{n} + R_\theta,$$

so that $dz = \frac{2|\beta|}{n}d\beta$, and integration occurs over $[-\sqrt{s_\theta n}, \sqrt{s_\theta n}]$. Furthermore we have that $\sqrt{R_\theta - z} = i\frac{|\beta|}{\sqrt{n}}$.

4.8 Obtaining the $n^{-3/2}$ term

Continuing the analysis above, we get, supposing n is large:

$$P(S_n = (e,0)) \sim \frac{1}{4\pi^2 i}\int\limits_{|\theta|<\alpha}\oint\limits_{C_\theta}\frac{G_\theta(z)}{z^{n+1}}dzd\theta \tag{4.8.1}$$

$$\sim \frac{1}{4\pi^2 i}\int\limits_{|\theta|<\alpha}\oint\limits_{C_\theta^1}\frac{G_\theta(z)}{z^{n+1}}dzd\theta \tag{4.8.2}$$

$$= \frac{1}{4\pi^2 i}\int_{-\alpha}^{\alpha}\int_{-\sqrt{s_\theta n}}^{\sqrt{s_\theta n}} \frac{G_\theta(z(\beta))}{z(\beta)} e^{-n\log z(\beta)} \frac{2}{n}|\beta| \, d\beta d\theta$$

$$= \frac{1}{2\pi^2}\int_{-\alpha}^{\alpha}\int_{-\sqrt{s_\theta n}}^{\sqrt{s_\theta n}} -i\frac{i}{\sqrt{n}}\frac{K_\theta(z(\beta))}{n} e^{-n\log z(\beta)} \beta^2 d\beta d\theta$$

$$\sim \int_{-\alpha}^{\alpha} \frac{K_\theta(R_\theta)}{2\pi^2 n^{\frac{3}{2}}} \int_{-\sqrt{s_\theta n}}^{\sqrt{s_\theta n}} \beta^2 e^{-n\log(R_\theta+\beta^2/n)} d\beta d\theta \tag{4.8.3}$$

$$\sim \int_{-\alpha}^{\alpha} \frac{K_\theta(R_\theta)}{2\pi^2 n^{\frac{3}{2}}} \int_{-\sqrt{s_\theta n}}^{\sqrt{s_\theta n}} \beta^2 e^{-n[\log(R_\theta)+\frac{\beta^2}{nR_\theta}]} d\beta d\theta \tag{4.8.4}$$

$$= \int_{-\alpha}^{\alpha} \frac{K_\theta(R_\theta)}{2\pi^2 (R_\theta)^n n^{\frac{3}{2}}} \int_{-\sqrt{s_\theta n}}^{\sqrt{s_\theta n}} \beta^2 e^{-\frac{\beta^2}{R_\theta}} d\beta d\theta$$

$$\sim \int_{-\alpha}^{\alpha} \frac{K_0(R_0)}{2\pi^2 (R_\theta)^n n^{\frac{3}{2}}} \int_{-\sqrt{s_\theta n}}^{\sqrt{s_\theta n}} \beta^2 e^{-\frac{\beta^2}{R_\theta}} d\beta d\theta \tag{4.8.5}$$

$$\sim \int_{-\alpha}^{\alpha} \frac{K_0(R_0)}{2\pi^2 (R_\theta)^n n^{\frac{3}{2}}} \int_{-\infty}^{\infty} \beta^2 e^{-\frac{\beta^2}{R_\theta}} d\beta d\theta \tag{4.8.6}$$

$$= \frac{K_0(R_0)}{4n^{\frac{3}{2}}\pi^{\frac{3}{2}}} \int_{-\alpha}^{\alpha} \frac{1}{(R_\theta)^{n-3/2}} d\theta$$

The approximation (4.8.5) follows from the fact that the term $(R_\theta)^n = (R(\rho e^{i\theta}))^n$ spikes as n grows large in a neighborhood around $\theta = 0$, and so as $n \to \infty$ the denominator term R_θ^n dominates the $K_\theta(R_\theta)$ term in the numerator.

4.9 Obtaining the $n^{-1/2}$ term

Now considering our aim, we would like to show that the term

$$\int_{-\alpha}^{\alpha} (R_\theta)^{\frac{3}{2}-n} d\theta$$

gives us an extra $n^{-1/2}$. According to Proposition 4.3.19 for $w > 0$,

$$R(w) = R(\rho) + \frac{R''(\rho)}{2}(w-\rho)^2 + \sum_{k=3}^{\infty} \frac{R^{(k)}}{k!}(w-\rho)^k$$

where $R(\rho) > 0, R''(\rho) < 0$. Since $R(w)$ according to Proposition 4.3.19 is locally analytic, the Taylor expansion extends to a complex neighborhood of ρ. In particular, along the imaginary line $w = \rho + it$, the coefficient of the second term in the expansion changes sign from its value along the real axis:

$$\frac{R''(\rho)}{2}(w-\rho)^2 = \frac{R''(\rho)}{2}(it)^2 = -\frac{R''(\rho)}{2}t^2.$$

Thus the second derivative at ρ evaluated from the vertical direction is positive. We are interested in the second derivative of R at ρ in the direction $\rho e^{i\theta}, \theta \to 0$. Since the second derivative is continuous, this is approximately the second derivative in the vertical direction. Recall $R_0 = R(\rho)$. We may then conclude:

$$\begin{aligned}
\frac{1}{2\pi^2}\int_{-\alpha}^{\alpha} R_\theta^{-(n-3/2)} d\theta &\sim \int_{-\alpha}^{\alpha} e^{-(n-3/2)\log(R_0 + R''(\rho)\theta^2)} d\theta \\
&\sim \int_{-\alpha}^{\alpha} e^{-(n-3/2)[\log(R_0) + \frac{2R''(\rho)}{R_0}\theta^2]} d\theta \\
&= \frac{1}{\sqrt{n}} R_0^{-(n-3/2)} \int_{-\alpha\sqrt{n}}^{\alpha\sqrt{n}} e^{-(n-1)\frac{2R''(\rho)\beta^2}{nR_0}} d\beta \\
&\sim \frac{1}{\sqrt{n}} R_0^{-(n-3/2)} \int_{-\alpha\sqrt{n}}^{\alpha\sqrt{n}} e^{-\frac{2R''(\rho)\beta^2}{R_0}} d\beta
\end{aligned}$$

$$\begin{aligned}&\to \frac{1}{\sqrt{n}} R_0^{-(n-3/2)} \int_{-\infty}^{\infty} e^{-\frac{2R''(\rho)\beta^2}{R_0}} d\beta \\ &= \frac{1}{\sqrt{n}} R_0^{-(n-3/2)} \sqrt{\frac{R_0 \pi}{2R''(\rho)}}\end{aligned}$$

Therefore, altogether we find

$$P(X_n = e, L_n = 0) = \frac{C}{n^2 (R_0)^n}$$

where

$$\begin{aligned}C &= \frac{(R_0)^2 K_0(R_0)\pi}{4\sqrt{2} R''(\rho)} \\ &> 0.\end{aligned}$$

4.10 Existence of a saddle point

In this section we prove Proposition 4.3.19. In particular, we show that:

(i) the maximum of $R(w)$ for $w \in (0, \infty)$ is achieved at some $\rho > 0$.

(ii) At this ρ, R is analytic, and $R'(\rho) = 0$.

(iii) $R''(\rho) < 0$.

Lemma 4.10.1. *$R(w)$ has a maximum at some $\rho > 0$.*

Proof. It suffices to show that as $w \to 0$, and $w \to \infty$, $\lambda(w, R(w)) \to \infty$. For we may then conclude that as $w \to 0$, $w \to \infty$, $R(w) \to 0$. The continuity of $R(w)$ in w (which follows from its algebraicity in Lemma 4.2.4) will then imply that $R(w)$ has a maximum along the positive real numbers. We choose a maximum and call it ρ.

For each $z, w > 0$, $F_{s,e}(z,w)$ and $F_{ab,c}(z,w)$, the two kinds of first passage functions present in $\vec{F}$, are $\geq zp_e(w)$ (by functional equations (4.2.2) and (4.2.3)). Moreover, we have assumed that for each $j \in A$, there is $k \in \mathbb{N}, m \in -\mathbb{N}$ so that $(j,k),(j,m)$ have positive probability. Therefore, each p_j has a positive coefficient in front of a term with a positive exponent, and one in front of a negative exponent, so as $w \to 0$, $w \to \infty$, $p_j(w) \to \infty$.

The diagonal entries of $M(z,w)$ have the form

$$p_e(w) + \sum_{\ell} p_{(ab)^{-1}\ell ab}(w) F_{\ell c,c}(z,w) + \sum_{hf} p_{b^{-1}hf}(w) F_{hdf,b}(z,w),$$

which is $\geq p_e$. Let

$$D(w) = p_e(w)I$$

where I is the identity matrix. Let $\lambda_{D(w)} = p_e(w)$ denote the spectral radius of $D(w)$. Then as $w \to 0, w \to \infty, \lambda_{D(w)} \to \infty$. By Wielandt's theorem (Proposition 2.5.3),

$$\begin{aligned} \lambda(z,w) &\geq \lambda_{D(w)}, \text{ so} \\ z\lambda(z,w) &\geq z\lambda_{D(w)}, \text{ so} \\ 1 = R(w)\lambda(R(w),w) &\geq R(w)\lambda_{D(w)}, \text{ and} \\ R(w) &\leq \frac{1}{\lambda_{D(w)}} \end{aligned} \tag{4.10.2}$$

Thus as $w \to 0$, $w \to \infty$, $R(w) \to 0$. □

Proposition 4.10.3. *For all* $\theta \in \mathbb{R}$, $R''(e^\theta) < 0$.

The rest of this section will be devoted to the proof. First, we introduce some notation. Fix $\theta \in \mathbb{R}$. Centering the second variable of G at e^θ, we let

$$G(z, e^{\theta+t}) = \sum_n z^n \mathbb{E}[\exp((\theta+t)L_n)\mathbb{1}_{X_n=e}],$$

so that

$$\frac{1}{R(e^{\theta+t})} = \limsup_n [\mathbb{E}[\exp((\theta+t)L_n)\mathbb{1}_{X_n=e}]]^{1/n}.$$

Recall A is probability support of the random walk projected on the first coordinate. We consider A^n to be the collection of n tuples $(y_1, \ldots, y_n)$ which represent paths γ in Γ_1 (as a group element $y_1 \ldots y_n$ is unreduced). With earlier notation, the position of the random walk in the first coordinate at time n is $X_n = y_1 \ldots y_n$, after reduction. Each path $\gamma = (y_1, \ldots, y_n)$ has probability

$$\begin{aligned} p(\gamma) &= P\{(X_m)_{0\le m\le n} = \gamma\} \\ &= \prod_{i=1}^{n} \sum_{\ell} p_{y_i,\ell}. \end{aligned}$$

Furthermore,

$$\mathbb{E}[\exp((\theta+t)L_n)|\gamma)] = \prod_{i=1}^{n} \mathbb{E}[\exp((\theta+t)\xi_i)|y_i].$$

Setting

$$\phi_{y_i}(t) = \mathbb{E}[e^{(\theta+t)\xi_i}|y_i],$$

we observe that ϕ_{y_i} is the conditional moment generating function of ξ_i given y_i (and centered at θ.) We define the conditional cumulant generating function as:

$$\psi_{y_i}(t) = \log \phi_{y_i}(t).$$

Since the conditional random variables $\xi_i|y_i$ are nonconstant by assumptions on the support, for each y_i, $\psi''_{y_i}(0) > 0$. Using the notation of ψ, we rewrite $G(z, e^{\theta+t})$:

$$G(z, e^{\theta+t}) = \sum_n z^n \sum_{\substack{\gamma\in A^n \\ \gamma=(y_1,\dots,y_n) \\ y_1\cdots y_n=e}} p(\gamma) \exp\left(n \sum_{i=1}^{n} \frac{\psi_{y_i}}{n}\right). \tag{4.10.4}$$

Suppose $|A| = d > 1$. Each $\gamma \in A^n$ determines a probability distribution on A. In particular, the frequency $f^{\gamma}_{a_j}$ of each letter $a_j \in A$ in $\gamma = (y_1, \dots, y_n)$ determines a weight $\frac{f^{\gamma}_{a_j}}{n}$ on A. In turn, probability distributions on A are naturally points in a $d-1$-dimensional simplex $\mathcal{S}_{d-1}$, where vertices v_i are associated with letters $a_i \in A$. Overall, we obtain a function $\pi : A^n \to \mathcal{S}_{d-1}$, defined by the formula

$$\pi(\gamma) := \left(\frac{f^{\gamma}_{a_1}}{n}, \dots, \frac{f^{\gamma}_{a_d}}{n}\right).$$

Next, we define a collection of probability measures measures β_n on $\mathcal{S}_{d-1}$ by

$$\beta_n(B) = \frac{\sum\limits_{\substack{\gamma\in A^n \\ \pi(\gamma)\in B}} p(\gamma)}{\sum\limits_{\gamma\in A^n} p(\gamma)} = \frac{\sum\limits_{\substack{\gamma\in A^n \\ \pi(\gamma)\in B}} p(\gamma)}{P\{X_n = e\}},$$

and a point of $\mathcal{S}_{d-1}$ is called an *empirical letter frequency.* The support of $\beta_n(\pi)$ is finite since by assumption A is finite. Let π_i be the projection of $\pi \in \mathcal{S}_{d-1}$ on the i^{th} coordinate, which is seen as the weight of letter a_i. We may rewrite (4.10.4) as

$$G(z, e^{\theta+t}) = \sum_n z^n P(X_n = e) \sum_{\pi\in\mathcal{S}_{d-1}} \beta_n(\pi) \exp\left(n \sum_i \pi_i \psi_{a_i}(\theta)\right), \tag{4.10.5}$$

and therefore

$$\frac{1}{R(e^{\theta+t})} = \limsup_{n} \left[P(X_n = e) \sum_{\pi \in \mathcal{S}_{d-1}} \beta_n(\pi) \exp\left(n \sum_i \pi_i \psi_{a_i}(t) \right) \right]^{1/n}. \qquad (4.10.6)$$

Definition 4.10.7. We define the (upper) large deviation rate function

$$J(\pi) = -\log \inf_{\epsilon > 0} \limsup_{n \to \infty} \beta_n(B_\epsilon(\pi))^{1/n},$$

where $B_\epsilon(\pi)$ is the ϵ-ball centered at π in the simplex S. For convenience we may take the metric to be the box (or ℓ^∞) metric on $\mathbb{R}^d$. We observe that J is upper semi-continuous on $\mathcal{S}_{d-1}$.

From upper semi-continuity, we observe that:

Lemma 4.10.8. *There exists a* $\pi^\theta \in S$ *such that*

$$\sup_{\pi \in \mathcal{S}_{d-1}} \{-J(\pi) + \sum_i \pi_i \psi_i(\theta)\} = -J(\pi^\theta) + \sum_i \pi_i^\theta \psi_i(\theta). \qquad (4.10.9)$$

We characterize the function $R(\theta)$ in terms of the large deviation rate function J.

Proposition 4.10.10.

$$\limsup_{n \to \infty} \left\{ \mathbb{E} e^{\theta L_n} \mathbf{1}\{X_n = e\} \right\}^{1/n} = \frac{1}{R(0)} \sup_{\pi \in \mathcal{S}_{d-1}} \exp \left\{ -J(\pi) + \sum_i \pi_i \psi_i(\theta) \right\}. \qquad (4.10.11)$$

Proof of $\geq$*.* The local limit theorem for random walks on free products, as mentioned in the Introduction, implies that

$$\lim_{n \to \infty} P\{X_{2n} = e\}^{1/2n} = R(0)^{-1},$$

so it suffices to show that

$$\limsup_{n\to\infty}\left\{\int_{\mathcal{S}_{d-1}}\exp\{n\sum_i \pi_i\psi_i(\theta)\}\,d\beta_n(\pi)\right\}^{1/n}$$
$$\geq \sup_{\pi\in\mathcal{S}_{d-1}}\exp\left\{-J(\pi)+\sum_i \pi_i\psi_i(\theta)\right\}.$$

Then it suffices to show that the limsup on the left is at least

$$\exp\left\{-J(\pi^\theta)+\sum_i \pi_i^\theta\psi_i(\theta)\right\}.$$

For any $\varepsilon>0$ there exists $\delta>0$ such that for all n in some subsequence of $\mathbb{N}$,

$$\beta_n(B_\delta(\pi^\theta))\geq \exp\{-nJ(\pi^\theta)-n\varepsilon\};$$

consequently, for all n in this subsequence,

$$\begin{aligned}\int_{\mathcal{S}}\exp\{n\sum_i \pi_i\psi_i(\theta)\}\,d\beta_n(\pi) &\geq \int_{B_\delta(\pi^\theta)}\exp\{n\sum_i \pi_i\psi_i(\theta)\}\,d\beta_n(\pi)\\ &\geq \exp\{n\sum_i(\pi_i^\theta-\delta)\psi_i(\theta)\}\beta_n(B_\delta(\pi^\theta))\\ &\geq \exp\{n\sum_i(\pi_i^\theta-\delta)\psi_i(\theta)-nJ(\pi^\theta)-n\varepsilon\}.\end{aligned}$$

(Here we have used the assumption that the underlying metric on $\mathcal{S}$ is the ℓ^∞ metric.) Since $\varepsilon>0$ and $\delta>0$ can be taken arbitrarily small, the inequality follows. □

Proof of $\leq$. By the same reasoning as in the proof of the lower bound, it suffices to prove that

$$\limsup_{n\to\infty}\left\{\int_{\mathcal{S}}\exp\{n\sum_i \pi_i\psi_i(\theta)\}\,d\beta_n(\pi)\right\}^{1/n}\leq$$

$$\sup_{\pi \in \mathcal{S}} \exp\left\{ -J(\pi) + \sum_i \pi_i \psi_i(\theta) \right\}.$$

By definition of the large deviation rate function J, for every $\varepsilon > 0$ and every point $\pi \in \mathcal{S}_{d-1}$ there exists $\delta = \delta(\pi) > 0$ such that

$$\limsup_n \beta_n(B_\delta(\pi))^{1/n} \leq \exp\{-J(\pi) + \varepsilon\}. \tag{4.10.12}$$

Since the simplex $\mathcal{S}_{d-1}$ is compact, it follows that for any $\varepsilon > 0$ there exist a finite set $\{\pi^k\}_{k \leq K} \subset \mathcal{S}_{d-1}$ and a real number $\delta > 0$ such that the balls $B_\delta(\pi^k)$ cover $\mathcal{S}_{d-1}$ and such that the relation (4.10.12) holds for each $\pi = \pi^k$. Thus, for all sufficiently large n,

$$\begin{aligned}
\int_{\mathcal{S}} \exp\{n \sum_i \pi_i \psi_i(\theta)\}\, d\beta_n(\pi) &\leq \sum_{k=1}^{K} \exp\{n \sum_i (\pi_i^k + \delta)\psi_i(\theta)\} \beta_n(B_\delta(\pi^k)) \\
&\leq \sum_{k=1}^{K} \exp\{n \sum_i (\pi_i^k + \delta)\psi_i(\theta)\} \\
&\cdot \exp\{-nJ(\pi) + 2n\varepsilon\} \\
&\leq K \sup_{\pi \in \mathcal{S}} \exp\{-nJ(\pi) + n \sum_i \pi_i \psi_i(\theta)\} \\
&\cdot \exp\{2n\varepsilon + n\delta \sum_i \psi_i(\theta)\}.
\end{aligned}$$

Finally, since $\varepsilon > 0$ and $\delta > 0$ can be made arbitrarily small, the result follows by taking nth roots and letting $n \to \infty$. □

Proof of Proposition 4.10.3. It suffices to prove that the function

$$\varrho(\theta) := \limsup_{n \to \infty} \left\{ \int_{\mathcal{S}} \exp\left\{ n \sum_i \pi_i \psi_i(\theta)\, d\beta_n(\pi) \right\} \right\}^{1/n}$$

satisfies $\varrho''(\theta) > 0$. Proposition 4.10.10 implies that for every $\theta \in \mathbb{R}$,

$$\varrho(\theta) = \sup_{\pi \in \mathcal{S}_{d-1}} \exp\left\{-J(\pi) + \sum_i \pi_i \psi_i(\theta)\right\}.$$

Since $\varrho(\theta)$ is a smooth function of θ, to prove that $\varrho''(\theta) > 0$ it suffices to show that there are real numbers a and $b > 0$ such that for all t in some neighborhood $(-\varepsilon, \varepsilon)$ of 0,

$$\varrho(\theta + t) \geq \varrho(\theta) + at + bt^2. \tag{4.10.13}$$

Fix θ, and let $\pi^\theta \in \mathcal{S}_{d-1}$ be a point for which relation (4.10.9) is valid. Now for any $t \in \mathbb{R}$,

$$\begin{aligned}\varrho(\theta + t) &= \sup_{\pi \in \mathcal{S}} \exp\left\{-J(\pi) + \sum_i \pi_i \psi_i(\theta + t)\right\} \qquad (4.10.14)\\ &\geq \exp\left\{-J(\pi^\theta) + \sum_i \pi_i^\theta \psi_i(\theta + t)\right\}.\end{aligned}$$

Each function ψ_i, as a cumulant generating function of a nonconstant random variable, has a nonnegative second derivative on $\mathbb{R}$. By hypothesis, for each i there are at least two distinct integers k, k' such that $q_{i,k} q_{i,k'} > 0$, and so the cumulant generating function ψ_i must in fact have *strictly positive* second derivative everywhere on $\mathbb{R}$. Therefore, there are constants a_* and $b_* > 0$ such that

$$\sum_i \pi_i^\theta \psi_i(\theta + t) \geq \sum_i \pi_i^\theta \psi_i(\theta) + a_* t + b_* t^2$$

for all t in a neighborhood $(-\varepsilon, \varepsilon)$ of 0. This together with the inequality (4.10.14) implies (4.10.13). $\square$

Corollary 4.10.15. *There is exactly one saddlepoint of $R(w), w > 0$ so we may refer to ρ*

as the *saddlepoint of R*.

CHAPTER 5

COROLLARIES

We first prove some consequences for the random walk $S_n = (X_n, L_n)$ on $\Gamma \times \mathbb{Z}$.

First, recall that by Assumption 1, there is an integer $M < \infty$ such that the step distribution μ has support contained in $\Gamma_1 \times [-M, M]$.

5.1 Large deviations

Proposition 5.1.1. *There exist* $C, w_* > 0$, *and* $k_1, \rho > 0$ *as above, such that*

$$P(X_n = e, L_n = \lfloor \gamma n \rfloor) \sim \frac{Ck_1}{R(w_*\rho)^n w_*^{\gamma n} n^2}$$

for $|\gamma| < M$.

Proof. First,

$$P(X_n = e, L_n = \lfloor \gamma n \rfloor) = \frac{1}{4\pi^2 i} \int\limits_{\mathbb{T}} \oint\limits_{C_\theta} \frac{G_\theta(z)}{z^{n+1}} e^{-i\theta \lfloor n\gamma \rfloor} dz d\theta.$$

Then we claim,

$$P(X_n = e, L_n = \lfloor \gamma n \rfloor) \sim \frac{1}{4\pi^2 i} \int\limits_{|\theta|<\alpha} \oint\limits_{C_\theta} \frac{G_\theta(z)}{z^{n+1}} e^{-i\theta \lfloor n\gamma \rfloor} dz d\theta.$$

The argument is similar to the argument for Lemma 4.5.4. The only modification in the argument is due to the term $e^{-i\theta\lfloor n\gamma \rfloor}$. The numerator becomes $G_\theta(z) e^{-i\theta\lfloor n\gamma \rfloor}$, which is still bounded over the z, θ domain.

Following the argument in section 4.8, we have

$$P(X_n = e, L_n = \lfloor \gamma n \rfloor) \sim \frac{k_1}{n^{3/2}} \int_{-\alpha}^{\alpha} \frac{1}{R_\theta^{n-3/2}} e^{-i\theta\lfloor n\gamma \rfloor} d\theta$$

$$= \frac{k_1 R_0^{3/2}}{n^{3/2}} \int_{-\alpha}^{\alpha} \frac{1}{(R(\rho e^{i\theta})e^{i\theta\gamma})^n} \frac{R_\theta^{3/2}}{R_0^{3/2}} e^{-i\theta\{n\gamma\}} d\theta$$

$$\to \frac{k_1 R_0^{3/2}}{n^{3/2}} \int_{-\alpha}^{\alpha} \frac{1}{(R(\rho e^{i\theta})e^{i\theta\gamma})^n} d\theta,$$

where $0 \leq \{n\gamma\} < 1$ is the fractional part of $n\gamma$, and the last expression holds as $n \to \infty$ if we can show that $\frac{1}{(R(\rho e^{i\theta})e^{i\theta\gamma})^n}$ has a saddlepoint at $\theta = 0$.

Since $\rho > 0$ we may assume without loss of generality that $\rho = 1$. We claim that the function $\phi(w) = R(w)w^\gamma$ has a saddlepoint $w_* > 0$, that is, $\phi'(w_*) = 0$ and $\phi''(w_*) < 0$.

First, we show that if $w \to 0, w \to \infty$, then $\phi(w) \to 0$. We already know that as $w \to \infty, w \to 0$, $R(w) \to 0$. Suppose first that γ is positive and $< M$. We need to show that as $w \to \infty$, $R(w) \to 0$ faster than $w^{-\gamma} \to 0$. Then equation (4.10.2)

$$R(w) \leq \frac{1}{\sum\limits_{k \in \mathbb{Z}} p(e,k)w^k}.$$

This implies that $R(w) \in O(w^{-M})$, so since $\gamma < M$, $w^{-\gamma} \in o(R(w))$. The argument is similar for the case that $-M < \gamma < 0$.

Recall that $R(w)$ is analytic for all $w > 0$. Then $\phi(w)$ is infinitely differentiable and thus $\phi'(w_*) = 0$. Now let us change variables to $w = e^\theta$. Let $w_* = e^{\theta'}$; then

$$R'(e^{\theta'})e^{\theta'} = -R(e^{\theta'})\gamma e^{\theta'}.$$

Thus,

$$\phi''(e^{\theta'}) = e^{\theta'}[R''(e^{\theta'}) - \gamma^2 R(e^{\theta'})],$$

and since $R''(e^{\theta'}) < 0$, we have $\phi''(e^{\theta'}) < 0$. Thus a saddlepoint w_* of ϕ exists.

Next, we change contours to pass through $w_* e^{i\theta}$ instead of $\rho e^{i\theta}$, remove the tail end of the contour as before, and set

$$f(\theta) = R(\rho w_* e^{i\theta})(w_* e^{i\theta})^{\gamma}.$$

Then

$$\begin{aligned}
P(X_n = e, L_n = \lfloor \gamma n \rfloor) &\sim \frac{k_1 R_0^{3/2}}{n^{3/2}} \int_{-\alpha}^{\alpha} f(\theta)^{-n} d\theta \\
&= \frac{k_1 R_0^{3/2}}{n^{3/2}} \int_{-\alpha}^{\alpha} e^{-n \log f(\theta)} d\theta \\
&\to \frac{k_1 R_0^{3/2} f(0)^n}{n^2} \int_{-\infty}^{\infty} e^{f''(0)/f(0)} d\beta \\
&= \frac{k_1 R_0^{3/2} f(0)^n}{n^2} \sqrt{\frac{f(0)\pi}{-f''(0)}},
\end{aligned}$$

which concludes the proof. ☐

5.2 Central limit theorem

Proposition 5.2.1 (Central limit theorem for $\Gamma_1 \times \mathbb{Z}$). *Let $S_n = (X_n, L_n)$ denote the random walk on $\Gamma_1 \times \mathbb{Z}$ as before. Suppose the walk is symmetric in the second coordinate, that is, for each $a \in A$, $p(a, m) = p(a, -m)$. Then $\frac{L_n}{\sqrt{n}}$ conditioned on $X_n = e$ converges in distribution to a normal random variable with variance σ^2, that is,*

$$P\left(-a \leq \frac{L_n}{\sqrt{n}} \leq a \middle| X_n = e\right) \to \Phi(a) - \Phi(-a).$$

In particular, $\sigma^2 = -R_0''/R_0$.

Proof. Recall that if the random walk is symmetric, $\rho = 1$, and $R'(\rho) = 0, R''(\rho) < 0$.

Recall from the Introduction that R is the radius of convergence of the random walk on Γ_1, and is the exponential growth constant in its asymptotic rate of return. Then $R = R_0$, since by plugging in $\theta = 0$ into $G(z, \theta)$ we obtain the Green's function for the random walk on Γ_1, which has radius of convergence R.

To show that $\frac{L_n}{\sqrt{n}}|(X_n = e)$ converges in distribution to a normal, it suffices to show that the induced measure of $\frac{L_n}{\sqrt{n}}|(X_n = e)$ with variance $\sigma^2 > 0$ converges to the induced measure of a normal distribution with mean $\mu = 0$ and variance σ^2. Equivalently, for all θ in a neighborhood around 0

$$\mathbb{E}(e^{i\theta \frac{L_n}{\sqrt{n}}}|X_n = e) \to e^{\frac{-\sigma^2\theta^2}{2}},$$

uniformly in θ. Equivalently, for θ in a neighborhood of 0

$$\begin{aligned}\mathbb{E}(e^{i\theta \frac{L_n}{\sqrt{n}}}\mathbb{1}_{\{X_n=e\}}) &\to e^{\frac{-\sigma^2\theta^2}{2}}P(X_n = e)\\ &\sim \frac{C}{e^{\frac{\sigma^2\theta^2}{2}}R^n n^{3/2}}\end{aligned}$$

for $C_\theta > 0$ uniformly in θ. Since

$$\begin{aligned}G(z, \theta) &= \sum_{n\in\mathbb{N}}\sum_{m\in\mathbb{Z}} P^{(e,0)}(X_n = e, L_n = m)e^{i\theta m}z^n\\ &= \sum_{n\in\mathbb{N}} \mathbb{E}(e^{i\theta L_n}\mathbb{1}_{\{X_n=e\}})z^n,\end{aligned}$$

it suffices to show that

$$\frac{1}{2\pi i}\oint \frac{G(z, \frac{\theta}{\sqrt{n}})}{z^{n+1}} \to \frac{C}{e^{\frac{\sigma^2\theta^2}{2}}R^n n^{3/2}}$$

uniformly in θ. Following section 4.8, we find that

$$\frac{1}{2\pi i}\oint \frac{G(z, \frac{\theta}{\sqrt{n}})}{z^{n+1}} \to \frac{C_{\frac{\theta}{\sqrt{n}}}}{(R_{\frac{\theta}{\sqrt{n}}})^n n^{\frac{3}{2}}} \tag{5.2.2}$$

$$\sim \frac{C_0}{(R_{\frac{\theta}{\sqrt{n}}})^n n^{\frac{3}{2}}}$$

where

$$C_{\frac{\theta}{\sqrt{n}}} = \frac{K_0(R)R_{\frac{\theta}{\sqrt{n}}}^{3/2}}{4\pi^{\frac{3}{2}}}.$$

The convergence in (5.2.2) is uniform in θ. From Section 4.8, we need to check uniformity in θ for approximations (4.8.1), (4.8.2), (4.8.3), (4.8.4), and (4.8.6). In all cases, it is sufficient that R_θ and s_θ are continuous in θ, and therefore over $[-\alpha, \alpha]$ have compact images. In particular, all of the approximations may be bounded above and below by functions depending continuously on R_θ (and sometimes s_θ). Then thanks to the continuity of R_θ and s_θ in θ, those bounding functions are themselves are bounded by constants. For example, in equation (4.8.4), we need to check that $\sup_\theta$ of the error term

$$\int_{-\infty}^{\sqrt{s_\theta n}} e^{-\beta^2/R_\theta} \beta^2 d\beta$$

tends to 0. Using integration by parts and changing variables, we find that this expression is

$$\leq e^{-\frac{s_\theta n}{R_\theta}} \left(\frac{\sqrt{s_\theta n}R_\theta}{2} + \frac{(R_\theta)^2}{4\sqrt{s_\theta n}}\right). \tag{5.2.3}$$

The expression (5.2.3) can be bounded above by constants in θ using the continuity of (5.2.3) in θ. Thanks to the term $e^{-\frac{s_\theta n}{R_\theta}}$ the overall expression tends to 0 as $n \to \infty$.

Since

$$\left(R_{\frac{\theta}{\sqrt{n}}}\right)^n \sim R^n \left(1 + \frac{R_0''}{nR}\frac{\theta^2}{2}\right)^n \sim R^n e^{\frac{\theta^2 R_0''}{2R}}$$

we obtain the conclusion with

$$\sigma^2 = \frac{R_0''}{R}.$$

□

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