

THE UNIVERSITY OF CHICAGO

THE GRAVITATIONAL WAVE MEMORY EFFECT AND CLASSICAL SCATTERING
PROBLEMS

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This work is dedicated to my family,
to my parents Theodore and Cheryl, my brother Douglas, and my sister Madeline,
in thanks for their constant encouragement and support.

Perhaps it was completed at the University of Chicago in 2017,
but it was begun with them at the Adler Planetarium in 1995.

ἔτι τοίνυν καὶ τὰ κατὰ τὴν ψυχὴν σκεψώμεθα. σωφροσύνην τι καλεῖς
καὶ δικαιοσύνην . . . καὶ μνήμην καὶ πάντα τὰ τοιαῦτα;

Now, let us examine the virtues of the soul. These are the sorts of things you call
prudence, a sense of justice . . . and memory, are they not?

—Plato, *Meno* 88a-b

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ABSTRACT

The gravitational wave memory effect is a permanent change in the relative separations of a configuration of inertial test particles, initially at rest with respect to each other, which make up a gravitational wave detector, after a pulse of gravitational radiation passes. It is a phenomenon of both observational and theoretical interest, and there is a growing body of research on its various aspects.

Below we describe some of the questions regarding memory which we have raised and explored over the past five years, namely: (I) is it possible to provide a consistent and meaningful definition of the memory effect in spacetimes that are not asymptotically flat and thus lack a notion of null infinity (particularly, expanding cosmological spacetimes)? (II) Why is the memory effect unique to four spacetime dimensions? (III) Can the nonlinear memory of Christodoulou be understood as a tidal effect as stress-energy passes a detector at null infinity? (IV) Does the ordinary memory of Bieri and Garfinkle “imitate” null memory for ultrarelativistic matter?

We answer these questions by explicitly calculating, within linearized gravity, the memory accompanying classical particle-scattering sources. We find that: (I) memory can be defined using derivative-of-delta-function features in the Riemann curvature tensor radiating away from the scattering event, and we use this to study memory in a cosmological spacetime. (II) These delta-derivative features appear only at sub-leading order in spacetimes of dimension greater than four and so are physically negligible. (III) Nonlinear memory is not a tidal effect, but a purely radiative phenomenon without Newtonian analog. (IV) Ordinary memory smoothly extrapolates to null memory, and null sources are not “double counted” in both ordinary and null memory.

CHAPTER 1

INTRODUCTION

It is well known that the passage of a gravitational wave past a detector composed of freely falling test particles can give rise to relative motion of the particles. Indeed, this is the physical mechanism behind the Laser Interferometer Gravitational-Wave Observatory (LIGO) collaboration’s recent direct detection of gravitational radiation [1]. Likewise, a simplified schematic of a ring of test particles “breathing” as a plane wave passes can be found in almost any introductory textbook on general relativity:

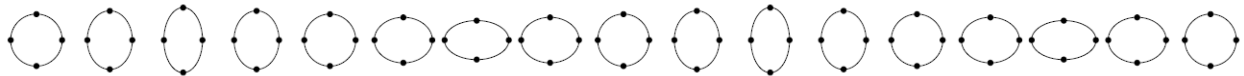


Figure 1.1: A simplified gravitational wave detector. As a gravitational wave passes into the plane of the paper, the proper distances amongst test particles vary along with the metric. The particles appear to move with respect to one another, and the ring “breathes.”

A gravitational wave can be thought of as self-propagating variations in the metric; as the metric changes, so to do the proper distances amongst the particles. A sinusoidal wave will give rise to sinusoidal relative motion. Astrophysically realistic radiation, however, takes the form of finite pulses rather than endlessly repeating plane waves. Zel’dovich and Polnarev [2] showed that such a pulse, created by the gravitational interactions of stars and black holes in a galactic nucleus, can give rise to a finite, permanent change in the separations of detector particles, which return to rest with respect to each other after the wave has passed. This phenomenon is known as the *gravitational wave memory effect*. In terms of the schematic in figure 1.1, a ring may begin as a perfect circle, breath for a finite amount of time, and then stop; it will no longer be a circle.

The memory effect remains a topic of active research, on both experimental and theoretical fronts. We have recently entered the era of gravitational-wave astronomy and memory, defined as it is in terms of permanent changes in a detector, makes a tempting gravitational wave phenomenon to measure. Although memory has remained a theoretical prediction



Figure 1.2: A visual representation of the memory effect. A gravitational wave pulse causes detector test particles to move; when the pulse has passed, the particles stop moving, but they are not in their initial configuration.

unverified by observation since the foundational work of [2], there is widespread optimism that we are drawing ever closer to directly detecting the memory effect. There is a general consensus that we will be unable to observe the memory of a wave using LIGO, primarily due to seismic noise [3]. Measurements using the next generation of gravitational wave detection experiments, such as the Evolved Laser Interferometer Space Antenna (eLISA) and pulsar timing arrays, are much more realistic [4]-[10].

One interesting area of current theoretical work is the relationship between memory and the asymptotic symmetries of null infinity. As will be discussed in more detail below, Strominger and Zhiboedov [11] have shown that the transformation relating the metric at null infinity before and after a gravitational wave passes is a combination of a Lorentz boost and another form of diffeomorphism on null infinity called a *supertranslation*. The supertranslation portion corresponds to a permanent change in the metric that can be measured by a detector near null infinity as the memory effect.

Hawking, Perry, and Strominger [12],[13] believe that this understanding of the memory effect may help resolve one of the most famous puzzles regarding the intersection of general relativity and quantum physics: they claim that something analogous to memory might be able to explain the information paradox. They say that a flux of matter, across not future null infinity but a black hole’s horizon, might in a similar fashion leave “black hole memory” in the form of deviations of the horizon’s generators (rather than deviations in detector particle’s worldlines, as is the case for memory at infinity), providing a form of “soft hair” on the black hole which saves information about the influx. As the black hole evaporates, this information can be accessed through the details of the Hawking radiation.

1.1 Mathematical Properties of the Memory Effect

In this section, we discuss in more detail some interesting and important properties of the memory effect. Almost all the work on memory has made use, either explicitly or implicitly, of null infinity, and for good reason: it allows us to isolate radiation from other forms of gravity via the peeling theorem. The peeling theorem allows us to identify different “portions” of curvature (according to the Petrov classification). It is natural to assume that our gravitational wave detector is very far away from the source, so that, if the spacetime asymptotically flat, we can put the source at a point p in the bulk of spacetime but place the detector near a point q on null infinity. A null geodesic $\gamma(\lambda)$ with affine parameterization $\lambda \in (0, \infty)$ joins $p = \gamma(0)$ and $q = \lim_{\lambda \rightarrow \infty} \gamma(\lambda)$. Near infinity, the physical curvature is fully described by the Weyl tensor C_{abcd} . Near null infinity and on γ , we can expand C_{abcd} in inverse powers of λ , and the coefficient to $1/\lambda$ is of Petrov class IV/N—*i.e.*, transverse gravitational radiation. Furthermore, near null infinity the usual radial coordinate r becomes a valid affine parameter for an outgoing null geodesic, so the peeling theorem allows us to invariantly identify gravitational radiation as the $1/r$ portion of the Weyl curvature. So, in studying the memory effect in asymptotically flat spacetimes, we can expand gravitational fields in powers of $1/r$ and neglect $\mathcal{O}(1/r^2)$ terms.

One important consequence of this definition is that it allows us to analytically distinguish between dominant effects that take place over a short timescale (which presumably can be measured in a realistic experiment) and sub-dominant effects that take place over a long timescale. For example, it is well known that observers in an asymptotically Schwarzschild spacetime will feel tidal forces. These tidal forces act at order $1/r^3$ but persist for all time. Therefore, if we run a gravitational wave detector in a spacetime that is asymptotically Schwarzschild but also has outgoing radiation bursts, and we allow the detector to run for an arbitrarily long time and naively measure the change in particle separation over the course of the experiment, we can find that tidal forces contribute as much to “memory” as the radiation did. A similar problem will occur if our detector consists of particles at rest in the cosmic

fluid of a Friedmann-Lemaître-Robertson-Walker (FLRW) spacetime: the particles will drift away from (or towards) each other as the universe expands (or contracts), and given enough time, this drift can match or even exceed the actual gravitational wave memory effect. In fact, even if there are no “background” forces competing with the radiation, this expansion is important: as we shall see below, gravitational waves can also impart “velocity kicks” (permanent changes in the relative velocity of test particles) at sub-leading orders of $1/r$. If we allow the detector to run on a timescale of order r/c (where c is the speed of light), the separation due to this residual motion will be comparable to the memory left by the wave’s immediate passage. None of these “secular drifts” will be observable in a realistic experiment, so we naturally wish to exclude them from our theoretical work on memory.

The original work of Zel’dovich and Polnarev was concerned with weak gravitational waves. They investigated solutions to the linearized Einstein equation for interacting particles and made use of the expansion in powers of $1/r$ mentioned above. If d^a is the spatial separation between two detector particles, Δd^a is the change in the separation (*i.e.*, the memory) and r is the distance from the gravitational wave source to the detector, then it can be shown that memory takes the form

$$\Delta d^a = \frac{1}{r} \Delta^a_b d^b . \quad (1.1)$$

Here, Δ_{ab} is a *memory tensor* that contains all the interesting details of the memory—angular dependence, energy and mass scales, etc. Braginsky and Thorne [14] considered the memory arising from a particle scattering source, made up of an in-state and an out-state both consisting of particles of mass $m^{(i)}$ and three-velocity $\mathbf{v}_{(i)}$. If $\hat{\mathbf{r}}$ points to the location of the detector on the sphere at null infinity, the memory tensor is

$$\Delta_{ab} = 2\Delta \sum_{(i)} \left[\frac{m_{(i)}}{\sqrt{1-v_{(i)}^2}} \frac{(v_{(i)})_a (v_{(i)})_b}{1 - \hat{\mathbf{r}} \cdot \mathbf{v}_{(i)}} \right]^{\text{TT}} . \quad (1.2)$$

Here the Δ in front of the sum refers to the difference between the out-state sum and the in-state sum, and

$$[X_{ab}]^{\text{TT}} = q_a^c q_b^d X_{cd} - \frac{1}{2} q^{cd} X_{cd} q_{ab} , \quad (1.3)$$

where q_{ab} is the projection of the Minkowski metric onto a unit sphere, is the usual transverse-traceless projection operator

However, as a bi-product of his work with Klainerman on the global nonlinear stability of Minkowski spacetime [15], Christodoulou showed that there can also be significant nonlinear contributions to memory associated with the Bondi flux of the gravitational radiation’s effective stress-energy at null infinity [16]:

$$\Delta_{ab} = 2 \oint \left[F(\hat{\mathbf{r}}') \frac{r'_a r'_b}{1 - \hat{\mathbf{r}} \cdot \hat{\mathbf{r}}'} \right]^{\text{TT}} d^2\Omega' . \quad (1.4)$$

$\hat{\mathbf{r}}'$ and $\hat{\mathbf{r}}$ are the unit vectors pointing, respectively, to the element of integration and to the detector on the sphere at null infinity. $F(\hat{\mathbf{r}}')$ is the total gravitational energy radiated to infinity, per unit solid angle in the direction $\hat{\mathbf{r}}'$. This soon became known as the “nonlinear memory effect,” in contrast with the “linear memory” of Zel’dovich and Polnarev.

Thorne [17] and Wiseman and Will [18] were quick to offer an intuitive understanding of the nonlinear memory effect. They pointed out that Christodoulou’s formula takes essentially the same form as Thorne and Braginsky’s formula for linear memory, but instead of summing the energies of particles, we now integrate the energy flux density over the sphere at null infinity. Likewise, velocities are replaced by the directional vectors $\hat{\mathbf{r}}'$, of magnitude $1 = c$, at which elements of energy reach null infinity. They therefore interpreted nonlinear memory as the linear memory of the creation of the “gravitons” comprising the primary gravitational radiation.

Bieri, Chen, and Yau [19],[20], Bieri and Garfinkle [21],[22] and Bieri, Garfinkle, and Yau [23] provide us with another way of thinking about nonlinear memory. They found that when *any* form stress-energy escapes to null infinity—*e.g.*, electromagnetic radiation and

(hypothetical) massless neutrinos—it leaves memory according to Christodoulou’s nonlinear memory formula. In general, the memory tensor can be separated into ordinary (1) and null (2) pieces:

$$\Delta_{ab} = \Delta_{ab}^{(1)} + \Delta_{ab}^{(2)} \quad (1.5)$$

$$\Delta_{ab}^{(1)} = -2 \oint \left[\Delta P(\hat{\mathbf{r}}') \frac{r'_a r'_b}{1 - \hat{\mathbf{r}} \cdot \hat{\mathbf{r}}'} \right]^{\text{TT}} d^2\Omega' \quad (1.6)$$

$$\Delta_{ab}^{(2)} = 2 \oint \left[F_{\text{null}}(\hat{\mathbf{r}}') \frac{r'_a r'_b}{1 - \hat{\mathbf{r}} \cdot \hat{\mathbf{r}}'} \right]^{\text{TT}} d^2\Omega' . \quad (1.7)$$

Here $\Delta P(\hat{\mathbf{r}}')$ refers to the change in the quantity

$$P(\hat{\mathbf{r}}') = \lim_{r \rightarrow \infty} r^3 C_{abcd}(r\hat{\mathbf{r}}') r'^a t^b r'^c t^d \quad (1.8)$$

between early and late times, while $F_{\text{null}}(\hat{\mathbf{r}}')$ represents all energy escaping to null infinity per unit solid angle in the direction $\hat{\mathbf{r}}'$, not just that of gravitational radiation. Christodoulou’s nonlinear memory is simply one form of a broader class of “null memory,” which is *all* memory that can be associated with (effective) stress-energy reaching null infinity. Instead of distinguishing between nonlinear and linear memory, it is more natural to speak of null and ordinary memory (where ordinary memory is the “leftover” memory effect, associated with stress-energy that does not escape to null infinity).

Furthermore, using asymptotic flatness and null infinity to define memory allows us to use the formalism of Strominger *et al.*¹ mentioned above. Asymptotic flatness is defined in terms of the fall-off rate at which the metric approaches the flat Minkowski metric at large distances from some central region, such as the mass in a Schwarzschild spacetime. There is no *a priori* method for specifying a single, universally accepted asymptotic condition in any given spacetime, but there are generally accepted boundaries on what makes a condition too strict

1. These authors use the language and imagery of quantum field theory and particle physics—soft particles, degenerate vacua, scattering amplitudes, etc. These quantum interpretations do not concern us here.

or too lax. A spacetime’s asymptotic symmetry group is the collection of diffeomorphisms that preserve these conditions; weaker conditions imply less structure at infinity and an enlarged asymptotic symmetry group.

In particular, these conditions and their symmetry groups must be considered near both spatial and null infinity. Near spatial infinity, it is possible to specify satisfactory conditions strict enough that the asymptotic symmetry group here is reduced to the symmetry group of Minkowski spacetime itself—*i.e.*, the Poincare group, made up of translations, rotations, and Lorentz boosts. There is no compelling reason to consider weaker conditions or larger symmetry groups [24]. This is not necessarily the case near null infinity; indeed, Bondi, van der Burg, and Metzner [25] and Sachs [26] showed that, for a generic asymptotically flat four-dimensional spacetime, the full symmetry group at null infinity (which has come to be known as the Bondi-van der Burg-Metzner-Sachs (BMS) group) will be an enlargement of the Poincare group, including as well the aforementioned *supertranslations*, which are angle-dependent translations at null infinity.

Strominger and Zhiboedov [11] have shown that, when stress-energy (or effective stress-energy, in the case of gravitational radiation) escapes to future null infinity, the transformation relating the metric at null infinity after the flux to the metric at null infinity before the flux is a combination of a Lorentz boost and a supertranslation. What’s more, the portion of the change in the metric that is described by the supertranslation is precisely the portion that can be measured as memory in a gravitational wave detector. They also note that this change in the metric is equal to the graviton field of Weinberg’s [27] “soft gravitons,” produced in the same interaction whence came the radiation reaching null infinity. They propose that all three things—supertranslations, memory, and soft gravitons—are the three vertices of an “infrared triangle,” and that they are different aspects of the same underlying physical effect.

Hollands, Ishibashi, and Wald [28] have shown that, in an asymptotically flat spacetime of four dimensions, if a gravitational wave burst results in a change in the Bondi four-

momentum (E, \mathbf{P}) of the spacetime, then the resulting supertranslation leaves memory Δ_{ab} whose nonzero components can be expressed in the form

$$\Delta_{AB} = - \left(D_A D_B - \frac{1}{2} q_{AB} D^C D_C \right) T(\hat{\mathbf{r}}) , \quad (1.9)$$

where

$$T(\hat{\mathbf{r}}) = 2\Delta (E - \hat{\mathbf{r}} \cdot \mathbf{P}) \ln (E - \hat{\mathbf{r}} \cdot \mathbf{P}) . \quad (1.10)$$

Capital Latin indices are coordinate indices on the sphere. They are raised and lowered with q_{AB} , the intrinsic metric of the unit sphere, which has covariant derivative D_A .

However, they found that in asymptotically flat spacetimes of dimension n where $n > 4$ is even,² all metric transformations at null infinity accompanying fluxes are boosts; we have no need for supertranslations. They conclude that the asymptotic symmetry group at null infinity can be reduced to the Poincare group in higher dimensions, and that there is no memory effect in higher-dimensional spacetimes.

1.2 Open Questions Regarding Memory

Despite all this progress, there remains much about the memory effect that is not well understood. Here we discuss a few questions, puzzles, and apparent paradoxes.

1.2.1 *How Can We Define Memory in a Spacetime That Is Not Asymptotically Flat?*

We have seen above that when a spacetime is asymptotically flat and has a well-defined notion of null infinity, we have a natural way to distinguish between radiative and non-radiative gravity, and thereby determine what detector particle motions are legitimate memory and

2. Null infinity cannot be defined in the usual way, as the conformal boundary of spacetime, for odd-dimensional spacetimes [29]. Without null infinity and supertranslations, the entire BMS formalism for memory falls apart.

what are not. It is not immediately clear how we can accomplish this in a spacetime that is not asymptotically flat. On an intuitive level, we naturally expect that a finite pulse of gravitational radiation *should* leave memory, regardless of spacetime’s asymptotic structure, and it would be useful to have a precise definition of what that memory is. More pragmatically, we are already conducting cosmological gravitational wave astronomy.³ As the precision of gravitational wave detection continues to improve (both in the sense that we will be able to measure memory and that we will be able to observe waves originating from ever more distant sources) it will become imperative that we understand the nature of gravitational wave memory in an expanding universe, which is most certainly *not* asymptotically flat.

There is a growing body of research done on memory in non-asymptotically flat backgrounds (particularly expanding FLRW spacetimes), but in each case the methods used to evade the above-mentioned difficulties were tailored to specific backgrounds and cosmological models, and so limit their general applicability.

For example, Bieri, Garfinkle, and Yau [23] make use of FLRW spacetimes’ conformal flatness and the Weyl curvature’s conformal invariance by considering linearized perturbations to the Weyl tensor. However, the full Riemann curvature, which governs geodesic deviations and thus memory, depends on both Weyl and Ricci curvature. Ricci curvature represents gravity associated with the matter content of spacetime, so their analysis is greatly simplified in vacuum spacetimes—specifically, in vacuum de Sitter spacetime, with expansion driven by a cosmological constant rather than some dark energy fluid. In this case, they find that, for locally similar gravitational wave sources and detectors equal luminosity distances away in Minkowski and de Sitter spacetimes, the null memory in de Sitter is enhanced by a factor of $1 + z$ (where z is the cosmological redshift between the source and the detector).

Meanwhile, Kehagias and Riotto [30] confine themselves to a decelerating universe without cosmological constant. Although not asymptotically flat, such a spacetime does have a

3. The source of GW150914 is believed to be a black hole merger at cosmological redshift somewhere around $z = 0.09$ [1].

notion of future null infinity. They can therefore use an adapted form of Strominger *et al.*'s BMS formalism. Their results can be interpreted the same as [23]; namely, we find a $1 + z$ enhancement of null memory in de Sitter over Minkowski for equal luminosity distances. They do not discuss ordinary memory.

Chu [31]-[33] neither limits himself to a single cosmological model, nor does he distinguish between null and ordinary memory. He directly subtracts out the background metric and defines memory to be the permanent change in the metric perturbation over all time. However, due to his emphasis on gravitational wave tails (gravity propagating within light cones) as well as the velocity kicks noted above, his calculations fail to differentiate between secular drift and what we regard as true observable memory.

Instead of relying on *ad hoc* methods, we would like a consistent definition of memory and a procedure for computing it that is valid in any non-asymptotically flat spacetime (albeit with an emphasis on cosmological spacetimes). We offer one possible definition. Instead of limiting ourselves to a particular background spacetime, we limit ourselves to a particular kind of gravitational wave source: classical particle scattering. In this case, memory can be characterized invariantly as the detector particle motion associated with derivative-of-a-delta-function-like features in the curvature tensor. We use this definition to study the memory effect in an FLRW spacetime of arbitrary matter content.

1.2.2 Why Is There No Memory Effect in Higher-Dimensional Spacetimes?

Hollands, Ishibashi and Wald [28] have shown that in n -dimensional asymptotically flat spacetimes where $n > 4$ is even, we do not need supertranslations to describe the transformation of the metric between the regions of null infinity before and after radiation bursts. Consequently, there is no memory in higher (even) dimensional spacetimes. Despite the generality of this result, the abstract nature of the analysis of asymptotic symmetry groups impedes an intuitive understanding of *why* it is so. What is special about four dimensions?

We show that, although gravitational waves in higher dimensions do still cause detector

particles to move, they always return the particles to their initial positions, leaving no permanent memory.

1.2.3 *Is Nonlinear/Null Memory a Form of Tidal Effect?*

At first glance, the Christodoulou/Bieri and Garfinkle [16],[19]-[23] and the Thorne/Wiseman and Will [17],[18] characterizations of nonlinear or null memory seem inconsistent. Both place the gravitational wave detector and energy fluxes on (or near) future null infinity, but while [17],[18] require some sort of burst event in the bulk of spacetime to “create” the energy that escapes to null infinity and make a change between in- and out-states meaningful, thereby making memory an explicitly radiative phenomenon, the calculations of [16],[19]-[23] appear to depend *only* on quantities on null infinity. Do we actually need a burst, or can (effective) stress-energy traveling from past null infinity to future null infinity leave memory as well? If so, would that make memory a tidal rather than a radiative effect, possibly with a Newtonian analogue?⁴

We address this question by considering a source in which a single particle reaches null infinity. This provides us with a straightforward way to isolate the issue of whether or not the radiation is created in a burst. We find that a null particle traveling from past null infinity to future null infinity leaves not memory, but rather a relative velocity kick. The magnitude and direction of this kick is equal (up to a factor of 2) to that caused by the tidal force of a massive particle (with mass equal to the null particle’s energy) traveling at the speed of light in Newtonian gravity. We conclude that memory itself is *not* a tidal phenomenon.

4. Nowhere do any the authors of [16],[19]-[23] claim or even suggest this interpretation; in fact, their assumption of their spacetimes’ asymptotic flatness excludes the possibility of stress-energy simply following a worldline from past null infinity to future null infinity, and so implicitly requires a creation event somewhere within the bulk.

1.2.4 *What Is the Status of Ordinary and Null Memory for Ultrarelativistic Matter?*

While it is reasonable to distinguish between Bieri and Garfinkle’s null and ordinary memory, we do expect that they should “match up” for sources that are “nearly null.” For example, we can imagine a particle-scattering source in which exactly one of the outgoing particles has no mass. The entire scattering process will leave a certain amount of memory, a finite portion of it being null memory due entirely to this particle. A second scattering source is almost identical—except the massless particle is replaced by a particle with, say, the mass of a neutrino, but the same energy. (The masses and momenta of the other particles are also very slightly changed in order to conserve momentum.) We expect the total memory of this process to be almost exactly the same as that of the first, and yet null memory is exactly zero.

This is not obvious from the Bieri-Garfinkle equations for ordinary and null memory (1.7),(1.6). The null memory naturally depends on the energy flux density on the sphere at future null infinity. The ordinary memory, on the other hand, depends on the change in a certain component of the Weyl tensor near null infinity between early and late times. Energy density and curvature are certainly coupled by the Einstein equations, but the Weyl tensor famously represents gravity’s vacuum degrees of freedom. It is surprising that, as matter becomes ultrarelativistic, change in certain components of curvature should so closely mimic energy flux—and that this change of curvature should suddenly vanish for sources moving at the speed of light, so that its memory is not “double counted” as both null and ordinary memory. Yet we find, by explicit calculation, that this is so.

These are the questions that we shall address in this thesis. In chapter 2 we define what we mean by a “classical point-particle scattering” interaction and provide a template for calculating the memory accompanying such a source. This discussion is based off of material first presented in [34] by Garfinkle, Hollands, Ishibashi, Tolish, and Wald. In chapter 3 we use these results to motivate one useful definition for memory in non-asymptotically flat

spacetimes. We follow by using this definition to find a number of results regarding memory in an expanding universe. This section is based on the results first presented in [35] by Tolish and Wald. In chapter 4 we use similar arguments to present a physically intuitive explanation for the claim in [28] that there is no memory in higher even-dimensional spacetimes. These findings were first obtained in [34]. In chapter 5 we explore whether null memory depends on the creation of (effective) stress-energy, or whether it is a tidal-like effect that only depends on the passage of energy at infinity, based on the work first presented in [36] by Tolish and Wald. Finally, in chapter 6, we study the behavior of ordinary memory for ultra-relativistic matter and investigate how it converges to null memory. This section presents the results first obtained in [37] by Tolish, Bieri, Garfinkle, and Wald. Where relevant, we also explore analogous scalar and electromagnetic wave memory effects.

1.3 Notation and Conventions

We largely follow the notation and conventions of *General Relativity* by Wald [38]. In particular, we use the “East Coast” metric signature $(-, +, +, +)$, geometrized units ($G = c = 1$), and the abstract index notation for tensors. That is, Latin indices from the beginning of the alphabet (X_{ab}) simply denote tensor type and Greek indices ($X_{\mu\nu}$) refer to generic tensor components. In certain circumstances where there is a preferred local inertial coordinate system (LICS) a “0” index and Latin indices from the middle of the alphabet (X_{00}, X_{0i}, X_{ij}) refer to time and space components, respectively, and capital Latin indices (X_{AB}) are indices on the unit hypersphere S^{n-2} . Symmetric and antisymmetric parts of a tensor are denoted by

$$X_{(ab)} = \frac{1}{2}(X_{ab} + X_{ba}) , \tag{1.11}$$

$$X_{[ab]} = \frac{1}{2}(X_{ab} - X_{ba}) . \tag{1.12}$$

When symmetrizing and antisymmetrizing, we can pass over indices by enclosing them in $|\dots|$; for example,

$$X_{[a|bc|d]} = \frac{1}{2} (X_{abcd} - X_{dbca}) . \quad (1.13)$$

A Latin index in parentheses ($X_{ab}^{(i)}$) is not a spacetime index, but refers to one of the particles in the source.

The n^{th} derivative of a one-dimensional Dirac delta function is denoted $\delta^{(n)}$, and the m -dimensional coordinate Delta function is denoted δ_m , *i.e.*, $\int d^m x \delta_m(x) = 1$. The Heaviside step function is denoted Θ .

Except in sections 3 and 4, we work with linearized gravitational perturbations on a four-dimensional Minkowski background, using the usual coordinate systems—cartesian (t, x, y, z) , cylindrical (t, ρ, ϕ, z) , and spherical (t, r, θ, ϕ) . In section 4 we consider higher-dimensional Minkowski spacetimes, in which case we use the usual hyperspherical coordinates $(t, r, \{z^A\})$. We frequently use these coordinates to construct a unit vector basis, where the vectors point in the direction of increasing coordinate (*e.g.*, $t^a = -\nabla^a t$ is future-pointing; $r^a = \nabla^a r$ is outward-pointing). We also make use of the retarded times $U = t - r$ and $u = t - z$, with future-pointing null vectors

$$K^a = -\nabla^a U = t^a + r^a , \quad (1.14)$$

$$k^a = -\nabla^a u = t^a + z^a . \quad (1.15)$$

Where it is necessary to label an event in spacetime rather than a particular coordinate, we will do so using lower-case script Latin letters (χ, p, q, \dots).

The intrinsic metric of the unit hypersphere S^{n-2} is q_{AB} , with covariant derivative D_A , while the projection of the n -dimensional Minkowski metric onto S^{n-2} is denoted q_{ab} . We can use this to define an operator which projects the transverse-traceless part of a generic

symmetric tensor X_{ab} : in n dimensions it is

$$[X_{ab}]^{\text{TT}} = q_a^c q_b^d X_{cd} - \frac{1}{n-2} q^{cd} X_{cd} q_{ab} . \quad (1.16)$$

It reduces to the usual TT-projector for $n = 4$. The subscript $X_{[1]}$ represents the sum of the $\ell = 0$ and $\ell = 1$ spherical harmonic modes of the scalar field X on the unit sphere.

CHAPTER 2

THE MEMORY OF SCATTERING PARTICLES IN MINKOWSKI SPACETIME

Before we directly address the issue of memory in non-asymptotically flat spacetimes, we consider the retarded solutions to wave equations on a four-dimensional Minkowski spacetime. For simplicity we begin with the scalar wave equation, and then consider Maxwell's equation and the linearized Einstein equation. We assume classical point-particle scattering sources and explicitly calculate the retarded fields and their associated forces on a detector made of test particles near future null infinity. This relatively simple problem provides useful intuition and context for memory and will motivate our methods in the rest of this thesis.

2.1 Scalar Fields

We wish to find the retarded solution to the scalar wave equation

$$\nabla^a \nabla_a \varphi = -4\pi S, \tag{2.1}$$

where φ is a scalar field and S is a scalar charge distribution. Specifically, we take S to represent a system of charged point-particles following inertial trajectories except at a single “interaction vertex” p , where they may interact, be created, or be destroyed. A spacetime diagram of such a source is shown in figure 2.1.

Let (t, \mathbf{x}) be a globally inertial coordinate system (GICS). Without loss of generality, we can choose our GICS so that p is at the origin ($t = 0, \mathbf{x} = \mathbf{0}$). Then S takes the form

$$S(\chi) = \sum_{(i) \text{ in}} q_{(i)} \frac{d\tau_{(i)}}{dt} \delta_3(\mathbf{x} - \mathbf{y}_{(i)}(t)) \Theta(-t) + \sum_{(j) \text{ out}} q_{(j)} \frac{d\tau_{(j)}}{dt} \delta_3(\mathbf{x} - \mathbf{y}_{(j)}(t)) \Theta(t), \tag{2.2}$$

where $q_{(i)}$ are the scalar charges of the particles as measured in their rest frame, and

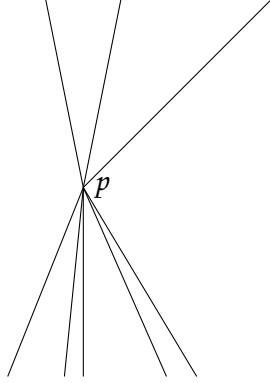


Figure 2.1: A spacetime diagram of the sort of radiation source we will consider. Here 5 incoming point particles travel to a single interaction vertex p , and 3 emerge. The worldlines of particles may be timelike or null.

$(t, \mathbf{y}_{(i)}(t))$ (with $\mathbf{y}_{(i)}(0) = \mathbf{0}$) are the particle worldlines parametrized with the GICS time coordinate. Intuitively, each term represents one charged particle, and the particles do not interact except at one point; here, incoming charges are “destroyed” by the $\Theta(-t)$ factor, while outgoing charges are “created” by the $\Theta(t)$.

We can find the retarded solution to eq. (2.1) by convolving the charge density S with the retarded Green’s function G of eq. (2.1):

$$\varphi(\chi) = 4\pi \int d^4x' G(\chi, \chi') S(\chi'). \quad (2.3)$$

This retarded Green’s function is given by

$$G(\chi, \chi') = \frac{1}{2\pi} \delta(\sigma^2(\chi, \chi')) \Theta(t - t'), \quad (2.4)$$

where $\sigma^2(\chi, \chi') = -(t - t')^2 + |\mathbf{x} - \mathbf{x}'|^2$ is the squared geodesic distance between field point χ and source point χ' . Holding χ fixed, the Green’s function’s support in χ' coincides with χ ’s past light cone.

The convolution of the retarded Green’s function with our distributional sources is well defined, as one can see by a standard wave-front-set argument. Firstly, viewed as a bi-distribution, the retarded propagator is known to have on any globally hyperbolic spacetime

M the wave front set [39] $\text{WF}(G)$ consisting of those $(\chi, k_a, \chi', -k'_a)$ in the cotangent bundle $T^*(M \times M) \setminus 0$ minus the zero section for which there exists a future directed null-geodesic γ connecting χ and χ' such that k_a and k'_a are co-tangent to- and parallel transported along γ . Secondly, for a single particle, the wave front set $\text{WF}(S)$ consists of the co-normal bundle of the particle trajectory (i.e. the set of all non-zero covectors annihilating the tangent vector of the particle). Since the particle trajectory is timelike, it follows that there cannot exist a $(\chi, 0, \chi', -k'_a) \in \text{WF}(G)$ and a $(\chi', p'_a) \in \text{WF}(S)$ such that $p'_a - k'_a = 0$. Consequently, by the wave front set calculus [39], the distributional product $G(\chi, \chi')S(x')$ is well defined on any globally hyperbolic spacetime. Our calculations show furthermore that integration over all of χ' is admissible in Minkowski spacetime, i.e. there are no infra-red divergences. The case of more particles is treated in the same way, since the source is just the sum of contributions from the individual particles.

To obtain the retarded solution, we consider a source $S_{\text{in}, \mathbf{v}=0}$ corresponding to a single massive particle “at rest,” which is destroyed at p : *i.e.*,

$$S_{\text{in}, \mathbf{v}=0} = q\delta_3(\mathbf{x})\Theta(-t) . \quad (2.5)$$

The retarded field is then

$$\varphi_{\text{in}, \mathbf{v}=0}(\chi) = 2q \int d^4x' \delta \left[-(t-t')^2 + |\mathbf{x} - \mathbf{x}'|^2 \right] \Theta(t-t') \delta_3(\mathbf{x}') \Theta(-t') . \quad (2.6)$$

Carrying out the spatial integral first, we obtain

$$\begin{aligned} \varphi_{\text{in}, \mathbf{v}=0}(\chi) &= 2q \int dt' \delta \left[-(t-t')^2 + r^2 \right] \Theta(t-t') \theta(-t') \\ &= 2q \int dt' \frac{1}{2|t-t'|} \delta \left[t' - (t-r) \right] \Theta(t-t') \Theta(-t') \\ &= \frac{q}{r} \Theta(-U) + \mathcal{O} \left(\frac{1}{r^2} \right) . \end{aligned} \quad (2.7)$$

Thus, the leading order behavior of $\varphi_{\text{in}, \mathbf{v}=0}$ is $1/r$. Since we are interested in detectors in the radiation zone, we shall ignore sub-leading terms.

The field of a particle created with velocity \mathbf{v} can be found by boosting eq. (2.7). For a particle following the worldline $(t, \mathbf{y}(t))$ with coordinate-velocity $\mathbf{v} = d\mathbf{y}/dt$, we obtain to leading order in $1/r$

$$\varphi_{\text{in}, \mathbf{v}}(\chi) = \frac{q}{r} \frac{d\tau}{dt} \frac{1}{1 - \hat{\mathbf{r}} \cdot \mathbf{v}} \Theta(-U) . \quad (2.8)$$

These calculations can be repeated for a particle that is “created” at p

$$S_{\text{out}, \mathbf{v}}(\chi) = q\delta_3(\mathbf{x} - \mathbf{y}(t))\Theta(t) , \quad (2.9)$$

in which case we find that at large distances the field of the boosted particle behaves like

$$\varphi_{\text{out}, \mathbf{v}}(\chi) = \frac{q}{r} \frac{d\tau}{dt} \frac{1}{1 - \hat{\mathbf{r}} \cdot \mathbf{v}} \Theta(U) . \quad (2.10)$$

A general source of the form eq. (2.2) can be written as a linear superposition of such created and destroyed particles, so its field can be written as a superposition of individual fields like eqs. (2.8) and (2.10). Thus, we find that the retarded solution with source (2.2) is given by

$$\varphi = \frac{1}{r} (\Theta(U)\alpha(\hat{\mathbf{r}}) + \Theta(-U)\beta(\hat{\mathbf{r}})) \quad (2.11)$$

to leading order in $1/r$, where

$$\alpha = \sum_{(i), \text{out}} \frac{d\tau_{(i)}}{dt} \frac{q_{(i)}}{1 - \hat{\mathbf{r}} \cdot \mathbf{v}_{(i)}} , \quad \beta = \sum_{(j), \text{in}} \frac{d\tau_{(j)}}{dt} \frac{q_{(j)}}{1 - \hat{\mathbf{r}} \cdot \mathbf{v}_{(j)}} \quad (2.12)$$

For $U < 0$ (*i.e.*, $t < r$), an observer will simply observe a collection of charges moving at various constant velocities, and so measure a superposition of boosted Coulomb-like fields. For $U > 0$ ($t > r$), he will observe a different collection of charges with different constant velocities. In between these two regions we find a “scalar wave” propagating with a Heaviside

step wavefront on the future light cone of the interaction point p .

What effect will this have on a “scalar wave detector” made of a massive test charge initially at rest in the GICS near future null infinity? The scalar force on a test particle of mass M_0 and charge Q is given by

$$f^a = Q \nabla^a \varphi . \quad (2.13)$$

The leading order force at large distances for the field (2.11) is

$$f^a(U, \mathbf{x}) = -\frac{Q}{r} (\alpha - \beta) \delta(U) K^a , \quad (2.14)$$

where

$$K^a = -\nabla^a U . \quad (2.15)$$

If the test particle is initially at rest, then the change in its momentum can be found by integrating (2.14) with respect to time:

$$\Delta P^a(U) = \int_{-\infty}^U dU' f^a(U', \mathbf{x}) = -\frac{Q}{r} (\alpha - \beta) \Theta(U) K^a . \quad (2.16)$$

The change in momentum goes like $r^{-1} \Theta(U) K^a$. Thus a test particle will get a “momentum kick” as a result of the scalar radiation emitted by the interactions of the particles. Note that, generally, a test particle exposed to such a kick will experience a change in mass:¹

$$M_1^2 = -\eta_{ab} (P_0^a + \Delta P^a) (P_0^b + \Delta P^b) = M_0^2 - 2P_0^a \Delta P_a = M_0^2 - 2Q(\alpha - \beta) \frac{M_0}{r} , \quad (2.17)$$

up to terms of order $1/r^2$. However, the detector particle will not exhibit memory in the form of a permanent, finite change in position as a result of its exposure to the scalar radiation

1. Changes in particle mass associated with relativistic scalar fields have been calculated before: see, see, *e.g.*, [40] and [41].

emitted by the decay.

2.2 Electromagnetic Fields

Maxwell's equation for the four-potential A^a in Lorenz gauge ($\nabla_a A^a = 0$) also takes the form of a wave equation very similar to eq. (2.1):

$$\nabla^b \nabla_b A^a = -4\pi J^a , \quad (2.18)$$

where J^a is the electromagnetic current density, which satisfies the charge-conservation law $\nabla_a J^a = 0$. We can find the retarded electromagnetic field for a given current density by using the scalar retarded integral (2.3) on each GICS component of (2.18). We again consider ingoing and outgoing point charges that scatter at an event p , taken to be at the origin of our GICS. Each of the massive incoming particles has a charge-current of the form

$$J_{(i)}^a = q_{(i)} \frac{d\tau_{(i)}}{dt} u_{(i)}^a \delta_3 \left(\mathbf{x} - \mathbf{y}_{(i)}(t) \right) \Theta(-t) , \quad (2.19)$$

where $u_{(i)}^a$ is the normalized tangent vector, $\tau_{(i)}$ is the proper times along the worldline $(t, \mathbf{y}_{(i)})$ and $q_{(i)}$ is the electromagnetic charge. Massless incoming charges, on the other hand, have charge-current

$$J_{(j)}^a = q_{(j)} w_{(j)}^a \delta_3 \left(\mathbf{x} - \mathbf{y}_{(j)}(t) \right) \Theta(-t) , \quad (2.20)$$

where $q_{(j)}$ is again the charge and $w_{(j)}^a$ is the tangent vector to the particle's null worldline, with normalization such that an observer with four-velocity t^a will measure $w_{(j)}^a t_a = -1$. The outgoing particles have similar current densities, with destruction factors $\Theta(-t)$ replaced

by creation factors $\Theta(t)$. The complete current density is

$$J^a = \sum_{(i) \text{ in, massive}} J_{(i)}^a + \sum_{(j) \text{ in, null}} J_{(j)}^a + \sum_{(k) \text{ out, massive}} J_{(k)}^a + \sum_{(l) \text{ out, null}} J_{(l)}^a. \quad (2.21)$$

Conservation of J^a implies conservation of charge at the interaction vertex p , *i.e.*,

$$\sum_{(i) \text{ out}} q_{(i)} = \sum_{(j) \text{ in}} q_{(j)}. \quad (2.22)$$

Below, we limit ourselves to considering massive charges, for reasons of notational simplicity. Nevertheless, our results can easily be generalized to include massless charged particles. Using (2.11) on each GICS component of Maxwell's equation (2.18), we find that the retarded solution for the electromagnetic potential A_a is to leading order in $1/r$

$$A^a = \frac{1}{r} (\Theta(U)\alpha^a + \Theta(-U)\beta^a) \quad (2.23)$$

where

$$\alpha^a(\hat{\mathbf{r}}) = \sum_{(i) \text{ out}} \frac{d\tau_{(i)}}{dt} \frac{q_{(i)} u_{(i)}^a}{1 - \hat{\mathbf{r}} \cdot \mathbf{v}_{(i)}}, \quad (2.24)$$

$$\beta^a(\hat{\mathbf{r}}) = \sum_{(j) \text{ in}} \frac{d\tau_{(j)}}{dt} \frac{q_{(j)} u_{(j)}^a}{1 - \hat{\mathbf{r}} \cdot \mathbf{v}_{(j)}}. \quad (2.25)$$

The field tensor $F_{ab} = 2\nabla_{[a}A_{b]}$ is thus given to leading order in $1/r$ by

$$F^{ab} = \frac{2}{r} K^{[a}(\alpha^{b]} - \beta^{b]})\delta(U). \quad (2.26)$$

Using conservation of charge (2.22), it can be seen that

$$K^{[a}(\alpha^{b]} - \beta^{b]}) = \sum_{(i) \text{ in, out}} \frac{d\tau_{(i)}}{dt} \frac{\eta_{(i)} q_{(i)}}{1 - \hat{\mathbf{r}} \cdot \mathbf{v}_{(i)}} K^{[a} q^{b]} c_{u_{(i)}c}, \quad (2.27)$$

where the factor $\eta_{(i)}$ equals +1 if particle (i) is outgoing and -1 if it is ingoing. In particular, we have $F_{ab}K^b = 0$.

The force acting on a test particle with charge Q and four-velocity V^a is

$$f^a = QF^{ab}V_b. \quad (2.28)$$

We assume that the test particle is initially at rest in our GICS, $V^a = t^a$. Then,

$$f^a(U, \mathbf{x}) = \frac{Q}{r} \left[\sum_{(i) \text{ in, out}} \frac{\eta_{(i)}q_{(i)}}{1 - \hat{\mathbf{r}} \cdot \mathbf{v}_{(i)}} \frac{d\tau_{(i)}}{dt} q^{ab} u_{(i)b} \right] \delta(U). \quad (2.29)$$

Its change in momentum is

$$\Delta P^a(U) = \int_{-\infty}^U dU' f^a(U', \mathbf{x}) \quad (2.30)$$

$$= \frac{Q}{r} \left[\sum_{(i) \text{ in, out}} \frac{\eta_{(i)}q_{(i)}}{1 - \hat{\mathbf{r}} \cdot \mathbf{v}_{(i)}} \frac{d\tau_{(i)}}{dt} q^{ab} u_{(i)b} \right] \Theta(U). \quad (2.31)$$

Since $f_a V^a = 0$, the electromagnetic force cannot produce a change in mass. Otherwise we obtain similar results to in the scalar case: we find a velocity kick in a direction tangent to the sphere centered at p , as previously found in [21].

2.3 Gravitational Fields

Finally we consider the case of linearized gravity. It is well-known that if we express the metric as a perturbation off of Minkowski background $g_{ab} = \eta_{ab} + h_{ab}$, introduce the trace-reversed perturbation $\bar{h}_{ab} = h_{ab} - \frac{1}{2}\eta_{ab}h$ (where $h = \eta^{ab}h_{ab}$), and impose the harmonic gauge condition $\nabla^a \bar{h}_{ab} = 0$, then the linearized Einstein equation takes the form

$$\nabla^c \nabla_c \bar{h}_{ab} = -16\pi T_{ab}, \quad (2.32)$$

where T_{ab} is the stress-energy tensor, which obeys the conservation law $\nabla^a T_{ab} = 0$. Once again our source consists of point particles² interacting at a single point p , as in figure 2.1. The stress-energy of the i^{th} massive incoming particle (with rest mass $m^{(i)}$ and following geodesic $(t, \mathbf{y}^{(i)}(t))$ with four-velocity $u^{(i)}$) takes the form

$$T_{ab}^{(i)} = m^{(i)} u_a^{(i)} u_b^{(i)} \delta_3(\mathbf{x} - \mathbf{y}^{(i)}(t)) \frac{d\tau^{(i)}}{dt} \Theta(-t); \quad (2.33)$$

that of the j^{th} massless incoming particle, on geodesic $(t, \mathbf{y}^{(j)}(t))$ and with energy E as measured by an observer at rest in the GICS, is

$$T_{ab}^{(j)} = E w_a^{(j)} w_b^{(j)} \delta_3(\mathbf{x} - \mathbf{y}^{(j)}(t)) \frac{d\lambda^{(j)}}{dt} \Theta(-t), \quad (2.34)$$

where $w_a^{(j)}$ is again normalized so that $t^a w_a^{(j)} = -1$. The stress-energy of each of the outgoing massive and massless particles takes the form of (2.33) and (2.34), respectively, except that $\Theta(-t)$ is replaced by $\Theta(t)$. The total stress-energy of the particle sources we consider takes the form

$$T_{ab} = \sum_{(i) \text{ massive, in}} T_{ab}^{(i)} + \sum_{(j) \text{ null, in}} T_{ab}^{(j)} + \sum_{(k) \text{ massive, out}} T_{ab}^{(k)} + \sum_{(l) \text{ null, out}} T_{ab}^{(l)}. \quad (2.35)$$

Conservation of stress-energy, $\nabla^a T_{ab} = 0$, holds in the distributional sense away from p by virtue of the fact that each particle moves on a geodesic [42]. Conservation of stress-energy will hold at p if and only if we have at p

$$\sum_{(i) \text{ massive, in}} m^{(i)} u_a^{(i)} + \sum_{(j) \text{ null, in}} w_a^{(j)} = \sum_{(k) \text{ massive, out}} m^{(k)} u_a^{(k)} + \sum_{(l) \text{ null, out}} w_a^{(l)}. \quad (2.36)$$

2. Although point particles are not consistent with the full nonlinear theory of relativity [43], they are allowed in the context of linearized gravitational perturbations off of fixed background spacetimes.

As in the electromagnetic case, we consider only massive particles. The retarded solution to the linearized Einstein equation (2.32) is then

$$\bar{h}_{ab}(\chi) = 16\pi \int d^4x' \left(G \cdot I_{ab}{}^{c'd'} \right) (\chi, \chi') T_{c'd'}(\chi'), \quad (2.37)$$

with G denoting the retarded Green's function of the scalar wave equation, and $I_{ab}{}^{c'd'}$ the bi-tensor of parallel transport, which is of course trivial in a GICS. The retarded metric perturbation is

$$h_{ab} = \frac{4}{r} (\Theta(U)\alpha_{ab} + \Theta(-U)\beta_{ab}) , \quad (2.38)$$

where

$$\alpha_{ab}(\hat{\mathbf{r}}) = \sum_{(i) \text{ out}} \frac{m^{(i)}}{1 - \hat{\mathbf{r}} \cdot \mathbf{v}^{(i)}} \frac{d\tau^{(i)}}{dt} \left(u_a^{(i)} u_b^{(i)} + \frac{1}{2} \eta_{ab} \right) \quad (2.39)$$

and

$$\beta_{ab}(\hat{\mathbf{r}}) = \sum_{(j) \text{ in}} \frac{m^{(j)}}{1 - \hat{\mathbf{r}} \cdot \mathbf{v}^{(j)}} \frac{d\tau^{(j)}}{dt} \left(u_a^{(j)} u_b^{(j)} + \frac{1}{2} \eta_{ab} \right) . \quad (2.40)$$

The linearized Riemann tensor R_{abcd} computed from a metric perturbation h_{ab} is

$$R_{abcd} = 2\nabla_{[a} \nabla_{|d} h_{c|b]} ; \quad (2.41)$$

for the perturbation (2.38) we find

$$R_{abcd} = \frac{4}{r} K_{[a} \Delta_{b][c} K_{d]} \delta'(U) , \quad (2.42)$$

where

$$\Delta_{ab} = 2 \sum_{(i) \text{ in, out}} \frac{\eta_{(i)} m_{(i)}}{1 - \hat{\mathbf{r}} \cdot \mathbf{v}^{(i)}} \frac{d\tau^{(i)}}{dt} \left(q_{ac} u_{(i)}^c q_{bd} u_{(i)}^d + \frac{1}{2} q_{ab} \right) . \quad (2.43)$$

Again, $\eta_{(i)}$ is +1 for outgoing and -1 for incoming particles. Within the basis induced by spherical coordinates, the only components of (2.43) which do not vanish are those associated

with the coordinates on the sphere:

$$\Delta_{AB} = \left(D_A D_B - \frac{1}{2} q_{AB} D^C D_C \right) T, \quad (2.44)$$

where D_A is the covariant derivative of the intrinsic metric of the unit sphere q_{AB} and

$$T(\hat{\mathbf{r}}) = 2 \sum_{(i) \text{ in, out}} \eta_{(i)} \left(E_{(i)} - \hat{\mathbf{r}} \cdot \mathbf{p}_{(i)} \right) \ln \left(E_{(i)} - \hat{\mathbf{r}} \cdot \mathbf{p}_{(i)} \right). \quad (2.45)$$

Here

$$(E, \mathbf{p}) = m \frac{dt}{d\tau} (1, \mathbf{v}) \quad (2.46)$$

is a particle's four-momentum.

Note that while the scalar force and the electromagnetic field depended on one derivative of the scalar field and potential, the curvature is the second derivative of the metric perturbation. Therefore while the leading order terms of the scalar force and electromagnetic field are proportional to a delta function in retarded time, the curvature is proportional to a derivative of a delta function in retarded time.

The curvature is relevant to the memory effect via the geodesic deviation equation. If two freely falling test particles are initially at rest with respect to each other with spatial separation d^a and common four-velocity V^a , then their relative separation evolves according to the equation

$$V^e \nabla_e V^f \nabla_f d^a = -R_{bcd}{}^a V^b V^d d^c. \quad (2.47)$$

If the particles are part of a gravitational wave detector at rest in the GICS (so that $V^a = t^a$), the coordinate version of the geodesic deviation equation becomes

$$\frac{d^2 d^i}{dt^2} = R_{j00}{}^i d^j. \quad (2.48)$$

The memory of the detector can be found by integrating the electric components of the

Riemann tensor (R_{i00j}) twice:

$$\Delta d_i(U) = \int_{-\infty}^U dU' \int_{-\infty}^{U'} dU'' \frac{d^2 d_i}{dU''^2} = \frac{1}{r} \Delta_{ij} d^j \Theta(U) . \quad (2.49)$$

$$(2.50)$$

The detector particles undergo a permanent, finite change in their separation when they come into causal contact with the scattering event, given by the memory tensor (2.43). This, at last, is true memory. It is consistent with the previous findings of [14]. Furthermore, as discussed in [28], the quantity T appearing in (2.45) describes the supertranslation associated with the memory effect.

CHAPTER 3

MEMORY IN NON-ASYMPTOTICALLY FLAT SPACETIMES

We are now in a position to consider the memory effect in more general circumstances—particularly when our spacetime is not asymptotically flat and we cannot depend on placing our detector near null infinity to isolate the relevant portion of the gravitational field. In the previous section, we found that the retarded solution to the linearized Einstein equation with a particle-scattering source has the property that the radiative portion of the curvature has the form of a derivative-of-a-delta-function of retarded time with respect to the decay event. Integration of the geodesic deviation equation then shows that the $\mathcal{O}(1/r)$ effect on test particles is to produce a sharp step function in their relative separation. Thus, for this kind of idealized source, the memory effect can be characterized by the presence of a derivative of a delta-function in the linearized curvature and a corresponding step function behavior in the relative separation of test particles. Of course, if we were to consider a less idealized source with a smoothed out stress-energy tensor, then the Riemann tensor also will be smoothed out, and the relative separation of the test particles will not undergo a sharp, sudden change in separation; rather, separation of the particles that results in a memory effect would occur continuously on the same timescale as that of the event itself.

For our present purposes, the main advantage of considering sources consisting of point-particles undergoing instantaneous interactions is that the characterization of the memory effect in terms of derivative-of-a-delta-function behavior in the curvature holds at all distances from the interaction event—*i.e.*, one does not need to employ the peeling theorem or go to null infinity for any other reason to extract the memory effect. This characterization may therefore be imported straightforwardly to other spacetimes.

Thus we shall again restrict consideration to linearized gravity off of now smooth but otherwise arbitrary background spacetimes. The linearized perturbations are once more sourced by (massive or massless) point particles. The particles behave as they did in section 2: interactions of the particles will be modeled by having their worldlines intersect (and,

possibly, begin or end) at a single event, p , in spacetime as illustrated in figure 2.1. The stress-energy of the particles largely take the same forms (2.33),(2.34) as before. For some LICs of p such that p lies at $t = 0$, an incoming massive particle contributes

$$T_{ab}^{(i)} = m^{(i)} u_a^{(i)} u_b^{(i)} \delta_3(\mathbf{x} - \mathbf{z}(t)) \frac{1}{\sqrt{-g}} \frac{d\tau^{(i)}}{dt} \Theta(-t), \quad (3.1)$$

and an incoming massless particle contributes

$$T_{ab}^{(j)} = w_a w_b \delta_3(\mathbf{x} - \mathbf{y}(t)) \frac{1}{\sqrt{-g}} \frac{d\lambda}{dt} \Theta(-t). \quad (3.2)$$

The only difference is the $1/\sqrt{-g}$ factor, which is the inverse of the background metric’s determinant. Again, the stress-energy of each of the outgoing particles also take these forms, except that $\Theta(-t)$ is replaced by $\Theta(t)$. Conservation of stress-energy then requires that (i) the particle worldlines are geodesics of the background spacetime [42] away from p , and (ii) total four-momentum is conserved at p . Within the LICs of p , conservation of four-momentum takes its familiar special-relativistic form, eq. (2.36). *“Memory” will then be characterized by the presence of a derivative of a delta-function in the curvature of the retarded solution with this source. This characterization does not require that the detector be placed near “infinity.”*

Another advantage to considering the above idealized particle sources is that—since all spacetimes are “locally flat” on sufficiently small scales—there is a well defined notion of having similar sources in different spacetimes. In particular, two sources are similar if we can introduce LICs in a neighborhood of both such that each particle has exactly one counterpart, in the sense that their four-momenta, written as linear combinations of the LICs basis vectors, are the same. Similarly, there is a well defined notion of having the “same detector”—i.e., inertial test particles initially at rest and with small separation—in different spacetimes. Thus, we can compare the memory effect in two different spacetimes provided only that we specify the location and “rest frame” of both the source and the

detector in the two spacetimes.

Below, we shall use this definition to study the memory of particle-scattering interactions in perhaps the most interesting and relevant non-asymptotically flat spacetimes: spatially flat FLRW cosmologies. In section 3.1, we analyze linearized perturbations off of spatially flat FLRW spacetimes with such sources in terms of the scalar, vector, and tensor sectors proposed by Bardeen and others. In section 3.2, we consider the tensor mode contribution to memory. We show that only the “light cone portion” of the retarded Green’s function will contribute to memory. Furthermore, we note that even though the scalar modes *do* support acoustic waves in the cosmic fluid, they do not contribute to memory. Since any spatially flat FLRW spacetime is conformal to Minkowski spacetime, it is particularly useful to state our results by comparing the memory effect in FLRW spacetime to that in Minkowski spacetime, which we shall do at the end of the section.

3.1 Cosmological Perturbation Theory

We wish to consider linearized perturbations off of a spatially flat FLRW background,

$$ds^2 = -d\tau^2 + a^2(\tau)(dx^2 + dy^2 + dz^2). \quad (3.3)$$

As usual, it is convenient to introduce conformal time $d\eta = d\tau/a$ so that the background FLRW metric takes the manifestly conformally flat form

$$ds^2 = a^2(\eta) \left(-d\eta^2 + dx^2 + dy^2 + dz^2 \right). \quad (3.4)$$

Throughout the rest of the section, “0” and “*i*” (i.e., spatial) indices will denote components of tensors with respect to these coordinates, and an overdot will denote a derivative with respect to η . We will write $\partial^i = \delta^{ij}\partial_j$ and $\nabla^2 = \partial^i\partial_i = \delta^{ij}\partial_i\partial_j$, i.e., ∇^2 is the Laplacian with respect to the spatial metric δ_{ij} given by $dx^2 + dy^2 + dz^2$.

We assume that Einstein’s equation holds (possibly with a cosmological constant Λ) and that the matter stress-energy—apart from the particle matter that we will add as a perturbation—is that of a perfect fluid

$$T_{ab}^{(F)} = (\rho + p)u_a u_b + p g_{ab}, \quad (3.5)$$

with 4-velocity u^a , density ρ and pressure p . The fluid is assumed to be described by a one-parameter (“barotropic”) equation of state $p = p(\rho)$. The density and pressure are perturbations away from homogeneous background values $\bar{\rho}$ and \bar{p} which satisfy the Friedmann equations

$$\left(\frac{1}{a} \frac{da}{d\tau}\right)^2 = \frac{1}{3} (8\pi\bar{\rho} + \Lambda), \quad (3.6)$$

$$\frac{1}{a} \frac{d^2 a}{d\tau^2} = \frac{1}{3} (-4\pi(\bar{\rho} + 3\bar{p}) + \Lambda). \quad (3.7)$$

We write the perturbed metric as

$$g_{ab} = \bar{g}_{ab} + a^2 h_{ab} \quad (3.8)$$

where \bar{g}_{ab} denotes the background FLRW metric. The perturbed fluid is described by δu^μ , $\delta\rho$, and $\delta p = c_s^2 \delta\rho$, where

$$c_s^2 = \frac{dp}{d\rho}. \quad (3.9)$$

We wish to consider the metric perturbation resulting from the presence of a particle stress-energy of the form (2.35). The particle sources are assumed to have no direct interaction with the fluid present in the FLRW background; the particle stress-energy is separately conserved. However, since the particles affect the perturbed metric, they automatically affect the fluid (even at the linearized level), so the fluid perturbations cannot be ignored.

Analysis of the perturbations is most easily done using the gauge-invariant methods of

Bardeen [44] with modifications by Durrer [45],[46] allowing for additional forms of matter perturbations.¹ These methods rely on decomposing the metric, fluid stress-energy, and particle stress-energy perturbations into scalar, vector, and tensor parts, and working with gauge invariant quantities in each sector. We can decompose a general symmetric tensor field X_{ab} on spacetime into its scalar, vector, and tensor parts by writing its coordinate components as

$$X_{\mu\nu} = \left(\begin{array}{c|c} \varphi & \partial_i\chi + \xi_i \\ \hline \partial_i\chi & \psi\delta_{ij} + (\partial_i\partial_j - \frac{1}{3}\delta_{ij}\nabla^2)\omega \\ + & \\ \xi_i & +\partial_{(i}\zeta_{j)} + \mathcal{X}_{ij} \end{array} \right), \quad (3.10)$$

where the scalar parts are given by²

$$\begin{aligned} \varphi &= X_{00} \\ \nabla^2\chi &= \partial^i X_{0i} \\ \psi &= \frac{1}{3}\delta^{ij}X_{ij} \\ \nabla^2\nabla^2\omega &= \frac{3}{2}\left(\partial^i\partial^j - \frac{1}{3}\delta^{ij}\nabla^2\right)X_{ij}, \end{aligned} \quad (3.11)$$

1. Both Bardeen and Durrer allow for general stress-energies with non-fluid properties like anisotropic pressures. However, as we have discussed above, we want our perturbed fluid and particles to interact only gravitationally, which means that the stress-energies of the fluid and the particles must be conserved independently. Bardeen does not discuss this scenario, but it corresponds to Durrer's notion of cosmological seeds.

2. If the spatial slices have topology \mathbb{R}^3 , we need to impose boundary conditions at infinity in order to get a unique solution to the Poisson equations for χ and ω (and ζ_i below), which, in turn, may put restrictions on the asymptotic behavior of X_{ab} . However, as we are ultimately interested in singular behavior of the perturbations, it does not matter what solutions of the Poisson equations we choose. For convenience, we shall assume that the spatial slices have the topology of three-tori, with the dimensions of the tori being much larger than the dimensions of the physical problem. The solutions are then unique up to the addition of constants, which do not affect the decomposition.

the vector parts are given by

$$\begin{aligned}\xi_i &= X_{0i} - \partial_i \chi \\ \nabla^2 \zeta_i &= 2 \left(\partial^j X_{ij} - \partial_i \psi - \frac{2}{3} \nabla^2 \partial_i \omega \right),\end{aligned}\tag{3.12}$$

and the tensor part is

$$\mathcal{X}_{ij} = X_{ij} - \psi \delta_{ij} - \left(\partial_i \partial_j - \frac{1}{3} \delta_{ij} \nabla^2 \right) \omega - \partial_{(i} \zeta_{j)}.\tag{3.13}$$

If the metric perturbation is written in this way,

$$h_{\mu\nu} = \left(\begin{array}{c|c} \varphi^{(h)} & \partial_i \chi^{(h)} + \xi_i^{(h)} \\ \hline \partial_i \chi^{(h)} & \psi^{(h)} \delta_{ij} + (\partial_i \partial_j - \frac{1}{3} \delta_{ij} \nabla^2) \omega^{(h)} \\ + & + \partial_{(i} \zeta_{j)}^{(h)} + \mathfrak{h}_{ij} \\ \xi_i^{(h)} & \end{array} \right),\tag{3.14}$$

then

$$\Phi = \varphi^{(h)} + 2\dot{\chi}^{(h)} + 2\frac{\dot{a}}{a}\chi^{(h)} + \ddot{\omega}^{(h)} + \frac{\dot{a}}{a}\omega^{(h)}\tag{3.15}$$

$$\Psi = \psi^{(h)} + 2\frac{\dot{a}}{a}\chi^{(h)} - \frac{1}{3}\nabla^2 \omega^{(h)} - \frac{\dot{a}}{a}\dot{\omega}^{(h)}\tag{3.16}$$

are gauge-invariant scalar quantities, whereas

$$\Xi_i = \xi_i^{(h)} - \dot{\zeta}_i^{(h)}\tag{3.17}$$

is a gauge-invariant vector quantity, and \mathfrak{h}_{ij} is a gauge-invariant tensor quantity. The above two scalar fields, Φ and Ψ , one transverse three-vector field, Ξ_i , and one transverse-traceless three-tensor field, \mathfrak{h}_{ij} , contain all of the physical (non-gauge) information concerning the metric perturbation.

The stress-energy tensor of the particles (2.35) can also be decomposed in this way:

$$T_{\mu\nu}^{(P)} = \left(\begin{array}{c|c} \varphi^{(P)} & \partial_i \chi^{(P)} + \xi_i^{(P)} \\ \hline \partial_i \chi^{(P)} & \psi^{(P)} \delta_{ij} + (\partial_i \partial_j - \frac{1}{3} \delta_{ij} \nabla^2) \omega^{(P)} \\ + & + \partial_{(i} \zeta_{j)}^{(P)} + \mathcal{F}_{ij} \\ \xi_i^{(P)} & \end{array} \right). \quad (3.18)$$

Because there is no “background” particle stress-energy, each of the individual component fields $\varphi^{(P)}, \chi^{(P)}$, etc. are already gauge invariant to first order. These quantities are related to $T_{\mu\nu}^{(P)}$ by eqs.(3.11)-(3.13). Since $T_{\mu\nu}^{(P)}$ is distributional, these quantities will also be distributional.

We can also find gauge-invariant combinations of the perturbed stress-energy $\delta T_{\mu\nu}^{(F)}$ of the fluid, (3.5). We define

$$\delta_\rho = \frac{\rho - \bar{\rho}}{\bar{\rho}}. \quad (3.19)$$

We decompose the perturbed 4-velocity as

$$\delta u^\mu = \frac{1}{a} \begin{pmatrix} \delta u^0 \\ \partial^i v + v^i \end{pmatrix} \quad (3.20)$$

with $\partial_i v^i = 0$, and we remind the reader that $\partial^i v = \delta^{ij} \partial_j v$. Note that the quantity δu^0 is not independent, since it is fixed by the normalization condition $g_{ab} u^a u^b = -1$. In terms of these quantities and the perturbed metric, we can obtain the following gauge-invariant fluid variables:

$$V = v + \frac{1}{2} \dot{\omega}^{(h)} \quad (3.21)$$

$$A = \delta_\rho + 3 \left(1 + \frac{\bar{p}}{\bar{\rho}} \right) \left(\frac{1}{2} \left(\psi^{(h)} - \frac{1}{3} \nabla^2 \omega^{(h)} \right) - \frac{\dot{a}}{a} V - \Phi \right) \quad (3.22)$$

$$W_i = \delta_{ij} v^j + \frac{1}{2} \dot{\xi}_i^{(h)}. \quad (3.23)$$

The fields V , A , and W_i thus provide us, respectively, with gauge-invariant measures of fluid's peculiar velocity with respect to the Hubble flow, its perturbed density, and its vorticity.

The linearized Einstein equation decomposes into decoupled sets of equations involving the scalar, vector, and tensor parts of the perturbations. These equations can be written entirely in terms of the gauge-invariant quantities introduced above. The scalar equations are

$$\nabla^2 \Psi - 3 \frac{\dot{a}}{a} \left(\dot{\Psi} + \frac{\dot{a}}{a} \Phi \right) = -8\pi \left(a^2 \bar{\rho} A - 3a\dot{a}(\bar{\rho} + \bar{p})V + \varphi^{(P)} \right) \quad (3.24)$$

$$\partial_i \left(\dot{\Psi} + \frac{\dot{a}}{a} \Phi \right) = -8\pi \left(a^2 (\bar{\rho} + \bar{p}) \partial_i V - \partial_i \chi^{(P)} \right) \quad (3.25)$$

$$\partial_i \partial_j (\Psi - \Phi) = -16\pi \partial_i \partial_j \omega^{(P)} \quad (3.26)$$

$$\begin{aligned} \ddot{\Psi} + 2 \frac{\dot{a}}{a} \dot{\Psi} + \frac{\dot{a}}{a} \dot{\Phi} + \left(2 \frac{\ddot{a}}{a} - \left(\frac{\dot{a}}{a} \right)^2 \right) \Phi + \frac{2}{3} \nabla^2 (\Psi - \Phi) \\ = -8\pi \left(a^2 c_s^2 \bar{\rho} A - 3(\bar{\rho} + \bar{p}) \frac{\dot{a}}{a} V + \psi^{(P)} \right). \end{aligned} \quad (3.27)$$

The vector equations are

$$\nabla^2 \Xi_i = -8\pi \left(2a^2 (\bar{\rho} + \bar{p}) W_i - \xi_i^{(P)} \right) \quad (3.28)$$

$$\partial_{(i} \dot{\Xi}_{j)} + 2 \frac{\dot{a}}{a} \partial_{(i} \Xi_{j)} = -8\pi \partial_{(i} \zeta_{j)}^{(P)}. \quad (3.29)$$

Finally, the tensor equation is

$$-\ddot{h}_{ij} - 2 \frac{\dot{a}}{a} \dot{h}_{ij} + \nabla^2 h_{ij} = -16\pi \mathcal{T}_{ij}. \quad (3.30)$$

If the various perturbation fields do not grow in an unbounded fashion at large distances,

the unique solutions to (3.25) and (3.26) are

$$\dot{\Psi} + \frac{\dot{a}}{a}\Phi = -8\pi \left(a^2(\bar{\rho} + \bar{p})V - \chi^{(P)} \right) \quad (3.31)$$

$$\Psi - \Phi = -16\pi\omega^{(P)}. \quad (3.32)$$

Equations (3.24) and (3.27) then simplify to

$$\nabla^2\Psi = -8\pi \left(a^2\bar{\rho}A + \varphi^{(P)} + 3\chi^{(P)} \right) \quad (3.33)$$

$$\begin{aligned} \ddot{\Psi} + 2\frac{\dot{a}}{a}\dot{\Psi} + \frac{\dot{a}}{a}\dot{\Phi} + \left(2\frac{\ddot{a}}{a} - \left(\frac{\dot{a}}{a} \right)^2 \right) \Phi \\ = -8\pi \left(a^2c_s^2 \left(\bar{\rho}A - 3(\bar{\rho} + \bar{p})\frac{\dot{a}}{a}V \right) + \psi^{(P)} - \frac{2}{3}\nabla^2\omega^{(P)} \right). \end{aligned} \quad (3.34)$$

Similarly, (3.29) implies that

$$\dot{\Xi}_i + 2\frac{\dot{a}}{a}\Xi_i = -8\pi\zeta_i^{(P)}. \quad (3.35)$$

The full set of Einstein's equation thus reduces to (3.31)-(3.35) together with (3.28) and (3.30).

The perturbed conservation of stress energy for the fluid, $\delta[\nabla^\mu T_{\mu\nu}^{(F)}] = 0$, yields the scalar equations

$$\dot{V} + \frac{\dot{a}}{a}V = -\frac{c_s^2\bar{\rho}A}{\bar{\rho} + \bar{p}} - \frac{1}{2}\Phi \quad (3.36)$$

$$\dot{A} - 3\frac{\bar{p}}{\bar{\rho}}\frac{\dot{a}}{a}A = -\left(1 + \frac{\bar{p}}{\bar{\rho}} \right) \left(\nabla^2V - 3\chi^{(P)} \right) \quad (3.37)$$

as well as the vector equation

$$\dot{W}_i - 3\frac{\dot{a}}{a}c_s^2W_i = 0. \quad (3.38)$$

Similarly, conservation of stress-energy for the particles can also be expressed in terms of the

fields (3.18) as

$$\dot{\varphi}^{(P)} + \frac{\dot{a}}{a}\varphi^{(P)} - \nabla^2\chi^{(P)} + 3\frac{\dot{a}}{a}\psi^{(P)} = 0 \quad (3.39)$$

$$\dot{\chi}^{(P)} + 2\frac{\dot{a}}{a}\chi^{(P)} - \psi^{(P)} - \frac{2}{3}\nabla^2\omega^{(P)} = 0 \quad (3.40)$$

$$\dot{\xi}_i^{(P)} + 2\frac{\dot{a}}{a}\xi_i^{(P)} - \frac{1}{2}\nabla^2\zeta_i^{(P)} = 0. \quad (3.41)$$

A very useful equation for A can be derived as follows [45]: We differentiate (3.37) with respect to η , and substitute from (3.36) to eliminate \dot{V} . Then we use (3.37) to eliminate $\nabla^2 V$, and we use (3.32) and (3.33) to eliminate $\nabla^2\Phi$. Finally, we use (3.40) to eliminate $\dot{\chi}^{(P)}$. We thereby obtain a wave equation for A with particle sources,

$$\begin{aligned} -\ddot{A} - \frac{\dot{a}}{a}\left(1 + 3c_s^2 - 6\frac{\bar{p}}{\bar{\rho}}\right)\dot{A} + c_s^2\nabla^2 A + 3\left(\frac{\ddot{a}\bar{p}}{a\bar{\rho}} - 3\left(\frac{\dot{a}}{a}\right)^2\left(c_s^2 - \frac{\bar{p}}{\bar{\rho}}\right) + a^2\left(1 + \frac{\bar{p}}{\bar{\rho}}\right)\frac{4\pi}{3}\bar{\rho}\right)A \\ = -4\pi a^2\left(1 + \frac{\bar{p}}{\bar{\rho}}\right)\left(\varphi^{(P)} + 3\psi^{(P)}\right). \end{aligned} \quad (3.42)$$

Physically, this equation describes the propagation of sound waves in the fluid. Although there is no direct coupling between the particles and the fluid, there are “particle source terms” in (3.42) resulting from the gravitational interactions between the particles and the fluid.

As previously explained, the memory effect will be identified with the presence of a derivative of a delta function in the curvature. The curvature is given by an expression involving at most 2 derivatives of the metric variables. In particular, in the Newtonian gauge, the perturbation to the electric components of the Riemann tensor is [47]

$$\begin{aligned} \delta R_{i00}{}^j = -\frac{1}{2}\left(\left(\partial_i\partial_k - \frac{1}{3}\delta_{ik}\nabla^2\right)\Phi + \left(\ddot{\Psi} + \frac{\dot{a}}{a}(\dot{\Psi} - \dot{\Phi})\right)\delta_{ik} \right. \\ \left. - \partial_{(i}\left(\dot{\Xi}_{k)} + \frac{\dot{a}}{a}\Xi_{k)}\right) + \left(\ddot{h}_{ik} + \frac{\dot{a}}{a}\dot{h}_{ik}\right)\right)\delta^{jk}. \end{aligned} \quad (3.43)$$

Thus, a derivative of a delta function in the curvature requires a step function (or worse) discontinuity in the gauge invariant metric variables.

We are now in a position to analyze how discontinuities could arise. First, it is important to note that the equations (3.11)-(3.13) giving the scalar, vector, and tensor parts of a tensor $X_{\mu\nu}$ involve solving elliptic and/or algebraic equations, with “source” given by components of $X_{\mu\nu}$ and their derivatives. It follows immediately that the scalar, vector, and tensor parts of $X_{\mu\nu}$ are smooth wherever $X_{\mu\nu}$ itself is smooth. In particular, the scalar, vector, and tensor parts of the particle stress-energy (3.18) are smooth away from the worldlines of the particles.

Consider the scalar perturbations. Eqs. (3.33) and (3.31) are elliptic in Φ and Ψ , so these quantities—which fully characterize the scalar part of the metric perturbation—can be singular only where the source terms in these equations are singular. These source terms involve the scalar part of the particle source and the quantity A . The scalar part of the particle source is smooth away from the particle world lines. The quantity A satisfies the hyperbolic equation (3.42), which, in turn is sourced by the scalar parts of the particle stress-energy. Although A can be discontinuous along the sound cone of the source event p , there cannot be any discontinuities in Φ or Ψ off of the particles’ worldlines (we shall demonstrate this later, using results found investigating tensor-mode memory in the section below). Thus, no memory effect can occur in the scalar sector.

Consider, now, the vector perturbations. The quantity Ξ_i satisfies the elliptic equation (3.28) and thus is smooth wherever the sources are smooth. However, the particle source term $\xi_i^{(P)}$ is smooth away from the worldlines of the particles and the fluid source term W_i satisfies the source free evolution equation (3.38) and is thus nonsingular everywhere. Thus, Ξ_i is smooth away from the particle worldlines and no memory effect can occur in the vector sector.

In the next section, we calculate the memory effect occurring in the tensor sector.

3.2 The Memory Effect in an Expanding Universe

The tensor perturbations are described by the quantity h_{ij} , which satisfies (see (3.30))

$$-\ddot{h}_{ij} - 2\frac{\dot{a}}{a}\dot{h}_{ij} + \nabla^2 h_{ij} = -16\pi \mathcal{T}_{ij}, \quad (3.44)$$

where \mathcal{T}_{ij} is the tensor part of the particle stress energy. Thus, each component of h_{ij} in the coordinates (3.4) satisfies a decoupled scalar wave equation and it suffices to analyze the behavior of solutions to the scalar wave equation.

We are interested in the contribution to the retarded integral

$$h_{ij}(\chi) = 16\pi \int \sqrt{-g(\chi')} d^4x' G(\chi, \chi') \mathcal{T}_{ij}(\chi') \quad (3.45)$$

arising from a small neighborhood of the source event p , where $G(\chi, \chi')$ denotes the retarded Green's function for the scalar wave equation

$$-\ddot{\phi} - 2\frac{\dot{a}}{a}\dot{\phi} + \nabla^2 \phi = -16\pi T. \quad (3.46)$$

Specifically, we seek to determine whether a discontinuity can arise in h_{ij} and, if so, to determine its magnitude. Such discontinuities will give rise to derivative of delta function contributions to the curvature, which, in turn, will produce a memory effect.

To proceed, we need to know the form of $G(\chi, \chi')$. Consider an equation of the general form

$$L[\phi] = g^{\mu\nu} \partial_\mu \partial_\nu \phi + b^\mu \partial_\mu \phi + c\phi = -16\pi T, \quad (3.47)$$

where $g_{\mu\nu}$ is a metric of Lorentz signature. It is well known [48], [49] that, in 4 spacetime dimensions, the retarded Green's function for this equation takes the form

$$G(\chi, \chi') = \left[U(\chi, \chi') \delta(\sigma^2) + V(\chi, \chi') \Theta(-\sigma^2) \right] \Theta(t - t'), \quad (3.48)$$

where σ^2 denotes the squared geodesic distance between χ and χ' in the metric $g_{\mu\nu}$ and t is a global time function. The quantities U (the Van Vleck-Morette determinant) and V are smooth functions; we refer to $U(\chi, \chi')\delta(\sigma^2)$ as the “direct part” and $V(\chi, \chi')\Theta(-\sigma^2)$ as the “tail part” of $G(\chi, \chi')$. In general, the form (3.48) for $G(\chi, \chi')$ will hold only locally in a convex normal neighborhood, but in the case of (3.46), the spacetime metric corresponding to (3.47) is flat, and this form of $G(\chi, \chi')$ holds globally, with

$$\sigma^2(\chi, \chi') = -(\eta - \eta')^2 + (x - x')^2 + (y - y')^2 + (z - z')^2. \quad (3.49)$$

The Van Vleck-Morette U is determined by integrating an ODE along a geodesic connecting χ and χ' [48], [49]. The quantities appearing in this ODE depend on $g^{\mu\nu}$ and b^μ but do not depend on c . We could integrate this ODE directly, but we can greatly simplify the calculation of U by working with the rescaled variable $\tilde{\phi} = a\phi$ (where ϕ denotes a component of h_{ij}), which satisfies the equation

$$-\ddot{\tilde{\phi}} + \nabla^2 \tilde{\phi} + \frac{\ddot{a}}{a} \tilde{\phi} = -16\pi a T. \quad (3.50)$$

This change of variables eliminates the term involving b^μ in (3.47), so \tilde{U} is determined by exactly the same equation for the wave equation in flat spacetime. Thus we obtain the unique solution $\tilde{U}(\chi, \chi') = (4\pi)^{-1}$. However, the retarded Green’s function for ϕ is related to the retarded Green’s function for $\tilde{\phi}$ by [40]-[41]

$$G(\chi, \chi') = \frac{a(\eta')}{a(\eta)} \tilde{G}^{\text{ret}}(\chi, \chi'). \quad (3.51)$$

Thus, we obtain

$$U(\chi, \chi') = \frac{1}{4\pi} \frac{a(\eta')}{a(\eta)}. \quad (3.52)$$

This holds for any spatially flat FLRW universe, i.e., we have not assumed any particular equation of state $\bar{p} = \bar{p}(\bar{\rho})$ (and, thus, we have not assumed any particular expansion law)

in the background spacetime. By contrast, $V(\chi, \chi')$ will depend on the expansion history of the FLRW universe between η' and η . Note, however, that by spatial Euclidean invariance, V depends on (χ, χ') only via η, η' , and $|\mathbf{x} - \mathbf{x}'|$.

As previously stated, we are interested in the possible discontinuities in h_{ij} resulting from the source behavior near p , where we take p to have coordinates $\mathbf{x} = t = 0$. To analyze this, let us first introduce a new intermediate mathematical problem, wherein we consider retarded solutions to the equation

$$-\ddot{H}_{ij} - 2\frac{\dot{a}}{a}\dot{H}_{ij} + \nabla^2 H_{ij} = -16\pi T_{ij}^{(P)}. \quad (3.53)$$

Eq. (3.53) differs from the equation of interest (3.44) in that we have not taken the tensor part of the particle source or perturbation and, correspondingly, we do not require H_{ij} or $T_{ij}^{(P)}$ to be transverse or traceless. It would be extremely cumbersome to compute \mathcal{T}_{ij} using the operations (3.11)-(3.13) and then perform a direct analysis of the behavior of the retarded Green's function integral involving \mathcal{T}_{ij} . Fortunately, we can bypass this by noting that the operation of “taking the tensor part” commutes with the wave operator appearing in (3.44) and (3.53). It follows that the desired quantity, h_{ij} , given by (3.45), is related to H_{ij} by

$$h_{ij} = [H_{ij}]^T, \quad (3.54)$$

where “[X_{ij}] T ” denotes the operation of taking the “tensor part” of X_{ij} , as given by (3.13), not the transverse-traceless projection. Thus, our analysis reduces to extracting information about how the operation of taking the tensor part of a quantity affects the nature of its singularities. To analyze this, we note first that since “taking the tensor part” consists of algebraic operations involving differentiations and inversions of Laplacians (see (3.10)-(3.13)), the tensor part, \mathcal{X}_{ij} , of a distribution X_{ij} must be smooth wherever X_{ij} is smooth. It then follows that the singular behavior of the tensor part of X_{ij} at χ is the same as that of the tensor part of ψX_{ij} , where ψ is any smooth function of compact support with $\psi = 1$

in a neighborhood of χ .³ However, the singular behavior of ψX_{ij} is characterized by the decay (or lack thereof) of its Fourier transform at large k_μ . The key point is that in the operation of “taking the tensor part,” there are exactly as many total “inverse derivatives” from Laplacian inversions in (3.13) as there are differentiations. It follows that the Fourier transform of the tensor part of ψX_{ij} is related to the Fourier transform of ψX_{ij} by a function that is everywhere bounded in k_μ . In particular, the decay of the Fourier transform of the tensor part of ψX_{ij} at large k_μ cannot be slower than that of ψX_{ij} . Therefore the intermediate problem involving H_{ij} can help us learn the singular structure of h_{ij} .

By (2.35), $T_{ij}^{(P)}$ consists of a sum of terms, each one of which has the form $\delta_3(\mathbf{x} - \mathbf{z}(t)) \Theta(\pm t)$, where $\mathbf{z}(t)$ describes a timelike or null geodesic. The contribution of the tail part, $V(\chi, \chi') \Theta(-\sigma)$, of $G(\chi, \chi')$ to the retarded integral is thus a sum of terms of the form

$$H_{ij}^{\text{tail}}(\chi) = \int d^4x' f_{ij}(\chi') V(\chi, \chi') \Theta(-\sigma(\chi, \chi')) \delta_3(\mathbf{x}' - \mathbf{z}(t')) \Theta(\pm t'), \quad (3.55)$$

where f_{ij} is smooth. It is not difficult to see that H_{ij}^{tail} is smooth whenever χ does not lie on the future light cone of p . On the other hand, if $\mathbf{z}(t)$ is a null geodesic and if χ lies on (the continuation of) this null geodesic—i.e., if one of the incoming or outgoing null particles is “aimed” directly at an observer at χ —then the singularities of $\delta_3(\mathbf{x}' - \mathbf{z}(t'))$ and $\Theta(-\sigma(\chi, \chi'))$ will coincide and H_{ij}^{tail} will, in general, be “highly singular” at χ in the sense that, in general, it will be defined only distributionally in a neighborhood of χ . We exclude such special points from consideration. The case of main interest is one where χ lies near the future light cone of p but does not lie on the special direction defined by $\mathbf{z}(t)$ (if null). Then the $\delta_3(\mathbf{x}' - \mathbf{z}(t'))$ singularity in the integral will be transverse to the step function singularity of $\Theta(-\sigma(\chi, \chi'))$ as well as to that of $\Theta(\pm t)$. One can then integrate over \mathbf{x}' , leaving one with an integral only over t' . The integrand will be proportional to $V(\chi; \mathbf{z}(t'), t') \Theta(U - t') \Theta(\pm t')$, where $U = t - |\mathbf{x}|$ denotes the retarded time of χ . The integral over t' thus yields a result of

3. Proof: $X_{ij} - \psi X_{ij}$ vanishes in a neighborhood of χ and hence is smooth there, so the tensor part of this difference is smooth in a neighborhood of χ .

the form $F_{ij}(\chi)U\Theta(U)$, where F_{ij} is smooth. Thus, we see that H_{ij}^{tail} is continuous (although not continuously differentiable) at χ .

The analysis of the contribution, H_{ij}^{dir} , of the “direct part,” $U(\chi, \chi')\delta(\sigma)$, of $G(\chi, \chi')$ to H_{ij} is similar, with $U(\chi, \chi')\delta(\sigma)$ replacing $V(\chi, \chi')\Theta(-\sigma)$ in (3.55). Again, H_{ij}^{dir} is smooth whenever x does not lie on the future light cone of q , and is highly singular if $\mathbf{z}(t)$ is a null geodesic and if x lies on (the continuation of) this null geodesic. If we exclude such special points, then the integral over \mathbf{x}' can again be done, and we are again left with an integral over t' . However, now the integrand is proportional to $\delta(U - t')\Theta(\pm t')$, where U is the retarded time of χ . Consequently, H_{ij}^{dir} has the form $\tilde{F}_{ij}(x)\Theta(U)$ for some smooth \tilde{F}_{ij} , and thus it has a discontinuity along the future light cone of q .

The above argument regarding the relationship between the physical problem and the intermediate mathematical problem can be applied to the present case as follows to get the key conclusion that we need. The Fourier transform of the tensor part of ψH_{ij}^{dir} differs from the Fourier transform of ψH_{ij}^{dir} by a bounded function of k_μ . Therefore, since H_{ij}^{dir} has a Heaviside discontinuity in U , then so does its tensor part h_{ij}^{dir} ; by (3.43), this portion of the metric perturbation contributes a derivative-of-a-delta-function in the Riemann curvature. *The direct contribution to h_{ij} does give rise to memory.*

On the other hand, it is easily seen from the explicit behavior of H_{ij}^{tail} found above that the Fourier transform of $\psi H_{ij}^{\text{tail}}$ lies in L^1 for any smooth function ψ of compact support. Therefore, the Fourier transform of the tensor part of $\psi H_{ij}^{\text{tail}}$ —which differs from the Fourier transform of $\psi H_{ij}^{\text{tail}}$ by a bounded function of k_μ —also lies in L^1 . But that implies that the tensor part of $\psi H_{ij}^{\text{tail}}$ is continuous for all ψ , which implies that the tensor part of H_{ij}^{tail} is continuous. *Thus, we have shown that the tail contribution to h_{ij} is continuous and thus cannot contribute to the memory effect.*

We can also use these arguments to justify our claim above that there is no scalar contribution to the memory effect. Recall that the gauge-invariant density perturbation A satisfies the sound-wave equation (3.42). Equation (3.42) is a hyperbolic wave equation of the general

form (3.47), with the Lorentz metric $g_{\mu\nu}$ now being the “acoustic metric,”

$$ds^2 = -d\eta^2 + \frac{1}{c_s^2}[dx^2 + dy^2 + dz^2], \quad (3.56)$$

and with the source term T proportional to the scalar particle fields $\varphi^{(P)}$ and $\psi^{(P)}$. Thus, the retarded Green’s function for (3.42) takes the general Hadamard form (3.48), with σ replaced by the squared geodesic distance, σ_s , in the acoustic metric (3.56), i.e., we have

$$G_s(\chi, \chi') = [U_s(\chi, \chi')\delta(\sigma_s) + V_s(\chi, \chi')\Theta(-\sigma_s)] \Theta(\eta - \eta'), \quad (3.57)$$

where U_s and V_s are again smooth functions in both χ and χ' . Furthermore, it can be seen from (3.18) that both $\varphi^{(P)}$ and $\psi^{(P)}$ are obtained from $T_{\mu\nu}^{(P)}$ by algebraic operations (i.e., no differentiations or Laplace inversions). It follows immediately that the source term appearing in (3.42) takes the same form (namely, proportional to $\delta_3(\mathbf{x} - \mathbf{z}(\eta)) \Theta(\pm\eta)$), as considered above. We may therefore repeat the above analysis regarding H_{ij} to draw the following conclusion: suppose that all of the particles in $T_{\mu\nu}^{(P)}$ are moving with velocity smaller than the speed of sound.⁴ Then A is smooth except on the future sound cone of p . Furthermore, on the sound cone, A will, in general, be discontinuous, but it cannot have “worse” singular behavior.

The metric perturbation variables Φ and Ψ satisfy elliptic equations, with source terms given by A and the scalar parts of the particle sources. It follows that Φ and Ψ must be smooth everywhere apart from the worldlines of the particles and the points at which A fails to be smooth, i.e., the future sound cone of q . We are not interested in the singularities at the particle worldlines. However, on the future sound cone of p , $\nabla^2\Phi$ and $\nabla^2\Psi$ are at worst discontinuous, so Φ and Ψ themselves are at least C^1 . *Thus, the scalar-sector perturbations*

4. If any of the particles are moving with velocity greater than the speed of sound, there will be additional “Cherenkov radiation” singularities occurring at points χ where the past sound cone of χ intersects a particle world line orthogonally (in the sound metric)—i.e., a “sonic boom.” These additional singularities are not of interest for the memory effect.

cannot contribute a derivative of delta-function to the Riemann curvature (3.43), and do not contribute to any memory effect.

The above conclusions are all that is needed to derive our results on the memory effect, because, as we have seen above, the direct contribution to the retarded solution is universal, and does not depend on the expansion history. Furthermore, Minkowski spacetime lies within the class of $k = 0$ FLRW spacetimes to which our analysis applies. Thus, we can relate the memory effect in an arbitrary spatially flat FLRW spacetime to that in Minkowski spacetime as follows. Consider a source event at p in the FLRW spacetime that is observed near event q . Let η_s and η_o denote the conformal times of the events p and q respectively. For convenience, rescale the coordinates, if necessary, so that $a(\eta_s) = 1$. This corresponds to choosing the comoving coordinates to correspond to proper distances at $\eta = \eta_s$. Now, identify the FLRW spacetime with Minkowski spacetime by identifying the coordinates (3.4) of the FLRW spacetime with global inertial coordinates of Minkowski spacetime. Place a source and observer at the events \bar{p} and \bar{q} of Minkowski spacetime that are identified in this manner with events p and q in the FLRW spacetime. Since $a(\eta_s) = 1$, the Minkowski source will physically correspond to the FLRW source provided that the masses and 4-velocities of each of the particles agree (under this identification) at p . It follows immediately from (3.52) that the direct part, $\bar{h}_{ij}^{\text{dir}}$, of $h_{ij}(\chi)$ near q will be a factor of $1/a(\eta_o)$ times the same function of χ as it is in the Minkowski case, i.e., near q ,

$$\bar{h}_{ij}^{\text{dir}}(\chi) = \frac{1}{a(\eta_o)} \bar{h}_{ij}^{\text{dir}}(\chi), \quad (3.58)$$

which agrees with the results of a WKB analysis of Damour and Vilenkin [50] regarding the propagation of high-frequency gravitational wave modes on FLRW backgrounds. It then follows immediately from (3.43) that the direct parts of the linearized Riemann curvature

tensor are similarly related, i.e., near q

$$\delta R_{i00}^{\text{dir}^j}(\chi) = \frac{1}{a(\eta_o)} \delta \bar{R}_{i00}^{\text{dir}^j}(\chi). \quad (3.59)$$

Suppose, now, that we place a gravitational wave detector near q , composed of two nearby particles initially at rest in the cosmic reference frame. By the geodesic deviation equation, the deviation vector, d^a , describing the displacement of the particles will satisfy

$$V^e \nabla_e V^f \nabla_f d^a = -R_{bcd}{}^a V^b V^d d^c, \quad (3.60)$$

where V^a is the unit tangent to the particles' geodesic. Since $V^\mu \approx 1/a(\eta_o)(\partial/\partial\eta)^\mu$ and the Hubble expansion is negligible over the relevant timescale, we can rewrite this equation as

$$\frac{d^2}{d\eta^2} d^i = R_{j00}{}^i d^j. \quad (3.61)$$

Let Δ_{ij} denote the coordinate components of the memory tensor, obtained by integrating (3.61) twice with respect to η and stripping away the trivial (comoving) distance and d^i dependencies. In view of (3.59) and the fact, proven above, that the “direct part” of the Riemann tensor contains the full memory effect, we see that

$$\Delta_{ij} = \frac{1}{a(\eta_o)} \bar{\Delta}_{ij}, \quad (3.62)$$

where $\bar{\Delta}_{ij}$ denotes the corresponding memory tensor in Minkowski spacetime, assuming that the initial displacement was $\bar{d}^i = d^i$. Thus, the relationship between Δd^i and d^i in an arbitrary FLRW spacetime differs from the corresponding Minkowski result by a factor of $1/a(\eta_o)$.

Thus, we have shown that *if we identify the FLRW spacetime with Minkowski spacetime via the coordinates (3.4) in such a way that $a(\eta_s) = 1$, and we place the same physical source*

at q and the same physical detector at p in both spacetimes, then the memory effect in the FLRW spacetime will be a factor of $1/a(\eta_o) = 1/(1+z)$ smaller than the corresponding memory effect in Minkowski spacetime. Note that placing the source at the same proper distance at the time of emission corresponds to placing the source at the same angular diameter distance in both spacetimes.

The above result compares the memory effect in FLRW and Minkowski spacetime when the source and detector are at the same proper distance at the source emission time, *i.e.*, when they are at the same location with $a(\eta_s) = 1$. Since the memory effect in Minkowski spacetime falls off as $1/r$, this result may be reformulated in numerous equivalent ways. In particular, we have

- *If the source and detector are placed so that they are at the same proper distance at the time of detection (rather than emission), then the memory effect in the FLRW spacetime is identical to the corresponding memory effect in Minkowski spacetime.*
- *If the source and detector are placed so that the source is at the same luminosity distance in both cases, then the memory effect in the FLRW spacetime is larger by a factor of $(1+z)$ as compared with the corresponding memory effect in Minkowski spacetime.*

The above result is in agreement with the results of [23], [30], [31], and [32] in the cases where the results of those references apply. Although our analysis is restricted to the context of linear perturbation theory with idealized particle sources, we expect that our main results should be valid completely generally for any sources whose spatial and time variation scales are small compared with the Hubble scale. Indeed, the main difficulty in generalizing our results to non-particle-like sources and to the nonlinear regime would be to give a precise definition of “memory” outside of the context we consider. Thus, if one wishes to compute the memory effect resulting from, say, the coalescence of two black holes in a distant galaxy in a spatially flat FLRW spacetime, it should suffice to compute the memory effect arising from a similar coalescence in an asymptotically flat spacetime and

then use the above correspondence. However, we shall not attempt to formulate or prove such a generalization here.

CHAPTER 4

MEMORY IN HIGHER DIMENSIONS

An explicit calculation of memory for scattering sources can also be used to shed light on Hollands, Ishibashi and Wald’s [28] finding that there is no memory effect in higher even dimensional spacetimes. We have seen that, for particle scattering in four dimensions, the term in the curvature tensor which is leading-order in $1/r$ is also proportional to a derivative of a delta function in retarded time (2.42). Integration of the geodesic deviation equation (2.49) tells us that the change in detector particles’ separation goes like a step function in retarded time—*i.e.*, the memory is finite and permanent. We show below that in an n -dimensional spacetime, where n is even, the leading-order term of the Riemann tensor is of the form $\delta^{(n/2-1)}(U)$, and the change in separation of detector particles goes like $d^{(n/2-2)}\Theta(U)/dU$. If $n > 4$, the particles may move with respect to each other in response to the gravitational radiation, but they will return to their original separation once the wave has passed. They therefore show no measurable memory.

As was previously mentioned, the authors of [28] did not make any claims regarding memory in odd-dimensional spacetimes, as we cannot construct null infinity as the conformal boundary of such spacetimes. Without null infinity and the accompanying asymptotic symmetry group, they could not use supertranslations to characterize memory. We limit ourselves to even dimensions for a different reason. The retarded Green’s function of the wave equation takes a fundamentally different form in odd dimensions: in particular, it has support within, rather than on, the past light cone.

We begin in section 4.1 by reviewing the form of the retarded Green’s function for the wave equation in an even-dimensional Minkowski spacetime, and obtaining the form of the retarded solution for scalar point particle interactions of the kind discussed in section 2. We then obtain the higher-dimensional scalar and electromagnetic field analogs of the memory effect in section 4.2. We calculate the gravitational field and show the dimensional dependence of memory in section 4.3. Finally, in section 4.4, we obtain formulas for memory in the

limit of slow motion of the sources, where it can be readily understood why the gravitational memory effect vanishes for $n > 4$. As in section 2, we consider only massive particles for notational clarity.

4.1 The n -Dimensional Wave Equation and Its Retarded Solution

We consider once more the inhomogeneous scalar wave equation (2.1), now on an n -dimensional Minkowski spacetime. Our scalar charge density again takes the form given in eq. (2.2), with the three-dimensional spatial delta functions are replaced with $(n - 1)$ -dimensional deltas. Finding the retarded field will again require convolving the source with the wave equation's retarded Green's function; for even n , the retarded Green's function is given by

$$G(\chi, \chi') = \frac{1}{(2\pi)^{\frac{n}{2}-1}} \delta^{(\frac{n}{2}-2)}(\sigma^2(\chi, \chi')) \Theta(t - t') \quad (4.1)$$

(see, for example, Sec. 6.1 of [49]), where

$$\sigma^2(\chi, \chi') = -(t - t')^2 + |\mathbf{x} - \mathbf{x}'|^2 . \quad (4.2)$$

This can be rewritten as

$$G(\chi, \chi') = \frac{1}{2} \frac{1}{(2\pi)^{\frac{n}{2}-1}} \Theta(t - t') \left(-\frac{1}{\Xi} \frac{\partial}{\partial \Xi} \right)^{n/2-2} \left(\frac{\delta((t - t') - \Xi)}{\Xi} \right) , \quad (4.3)$$

where, after taking the derivative, we substitute $\Xi = |\mathbf{x} - \mathbf{x}'|$. Our argument in section 2 that the convolution of such a Green's function with such sources, made using wavefront-set theory, remains valid. To obtain the retarded solution, we begin, again, with a source S corresponding to a single massive particle "at rest," which is destroyed at p : *i.e.*,

$$S_{\text{out}, \mathbf{v}=0} = q \delta_{n-1}(\mathbf{x}) \Theta(-t) . \quad (4.4)$$

The retarded field of such a source is

$$\varphi_{\text{in}, \mathbf{v}=0} = 2\pi q \frac{1}{(2\pi r)^{n/2-1}} \frac{d^{n/2-2}}{dU^{n/2-2}} \Theta(-U) + \mathcal{O}\left(\frac{1}{r^{n/2}}\right), \quad (4.5)$$

The leading order behavior of $\varphi_{\text{in}, \mathbf{v}=0}$ is now $(1/r)^{n/2-1}$. As before, we shall ignore sub-leading terms. Also note the derivative acting on the Heaviside discontinuity: as dimension increases, the wavefront becomes more singular. The field of a particle with velocity \mathbf{v} which is destroyed at p can be found by boosting eq. (4.5). For a particle following the worldline $(t, \mathbf{y}(t))$ with coordinate-velocity $\mathbf{v} = d\mathbf{y}/dt$, we obtain to leading order in $1/r$

$$\varphi_{\text{in}, \mathbf{v}}(\chi) = 2\pi q \frac{1}{(2\pi r)^{n/2-1}} \frac{d\tau}{dt} \frac{1}{1 - \hat{\mathbf{r}} \cdot \mathbf{v}} \frac{d^{n/2-2}}{dU^{n/2-2}} \Theta(-U). \quad (4.6)$$

The field of a particle with velocity \mathbf{v} which is created at p will have the same form as (4.6), except with U inside of the step function instead of $-U$. As in the $n = 4$ -dimensional case, a general source of the form eq. (2.2) (with 3-dimensional delta functions generalized to $n - 1$ -dimensional delta functions) can be written as a linear superposition of boosted created and destroyed particles, so its field can be written as a superposition of individual fields:

$$\varphi = \frac{2\pi}{(2\pi r)^{n/2-1}} \frac{\partial^{n/2-2}}{\partial U^{n/2-2}} (\Theta(U)\alpha(\hat{\mathbf{r}}) + \Theta(-U)\beta(\hat{\mathbf{r}})). \quad (4.7)$$

to leading order in $1/r$, where, as before

$$\alpha = \sum_{(i), \text{out}} \frac{d\tau_{(i)}}{dt} \frac{q_{(i)}}{1 - \hat{\mathbf{r}} \cdot \mathbf{v}_{(i)}}, \quad \beta = \sum_{(j), \text{in}} \frac{d\tau_{(j)}}{dt} \frac{q_{(j)}}{1 - \hat{\mathbf{r}} \cdot \mathbf{v}_{(j)}}. \quad (4.8)$$

4.2 Scalar and Electromagnetic Memory

We now consider the effects of the scalar field (4.7) on test particles in an n -dimensional spacetime. The scalar force on a test particle of mass M_0 and charge Q is still given by

$$f^a = Q\nabla^a\varphi \quad (4.9)$$

The leading order force at large distances for the field (2.11) is

$$f^a(U, \mathbf{x}) = -2\pi Q \frac{\alpha - \beta}{(2\pi r)^{n/2-1}} \frac{d^{n/2-2}}{dU^{n/2-2}} \delta(U) K^a, \quad (4.10)$$

where $K^a = -\nabla^a U$. If the test particle is initially at rest, then the change in its momentum can be found by integrating (4.10) with respect to time:

$$\Delta P^a(U) = \int_{-\infty}^U dU' f^a(U', \mathbf{x}) = -2\pi Q \frac{\alpha - \beta}{(2\pi r)^{n/2-1}} \frac{d^{n/2-2}}{dU^{n/2-2}} \Theta(U) K^a. \quad (4.11)$$

We have already seen that in $n = 4$ dimensions the detector particle receives a momentum kick, including a change in mass. In $n = 6$ dimensions, the leading order momentum change of the test particle goes like $r^{-2}\delta(U)K^a$. Thus, to leading order, the test particle's momentum returns to its initial value after the scalar wave passes; there is no velocity kick or change of mass. For the idealized case considered here—where the radiation is instantaneous—there is no time for the test particle to move, so there is no change of position either. However, for a smoothed-out source—where the radiation acts over a finite time—the test particle would undergo a finite radial displacement. In this sense, we obtain a scalar wave memory effect for the position of a test particle when $n = 6$.

In $n > 6$ dimensions, both the momentum and position of the test particle return to their initial values after the passage of the wavefront. Even for a smoothed out source, the momentum change of the test particle averages to zero during the passage of the wave, so there is no permanent displacement. Thus, there is no scalar memory effect for $n > 6$.

We can repeat this procedure in classical electromagnetism. Using (4.7) on each GICS component of Maxwell's equation (2.18) generalized to n -dimensional spacetimes, we find that the retarded solution for the electromagnetic potential A_a is to leading order in $1/r$

$$A^a = \frac{2\pi}{(2\pi r)^{n/2-1}} \frac{\partial^{n/2-2}}{\partial U^{n/2-2}} (\Theta(U)\alpha^a + \Theta(-U)\beta^a) \quad (4.12)$$

where

$$\alpha^a(\hat{\mathbf{r}}) = \sum_{(i) \text{ out}} \frac{d\tau_{(i)}}{dt} \frac{q_{(i)}u_{(i)}^a}{1 - \hat{\mathbf{r}} \cdot \mathbf{v}_{(i)}} , \quad (4.13)$$

$$\beta^a(\hat{\mathbf{r}}) = \sum_{(j) \text{ in}} \frac{d\tau_{(j)}}{dt} \frac{q_{(j)}u_{(j)}^a}{1 - \hat{\mathbf{r}} \cdot \mathbf{v}_{(j)}} . \quad (4.14)$$

The field tensor $F_{ab} = 2\nabla_{[a}A_{b]}$ is thus given to leading order in $1/r$ by

$$F^{ab} = \frac{4\pi}{(2\pi r)^{n/2-1}} K^{[a}(\alpha^{b]} - \beta^{b]) \frac{d^{n/2-2}}{dU^{n/2-2}} \delta(U) . \quad (4.15)$$

The n -dimensional form of conservation of charge (2.22) gives us

$$K^{[a}(\alpha^{b]} - \beta^{b]) = \sum_{(i) \text{ in, out}} \frac{d\tau_{(i)}}{dt} \frac{\eta_{(i)}q_{(i)}}{1 - \hat{\mathbf{r}} \cdot \mathbf{v}_{(i)}} K^{[a}q^{b]c}u_{(i)c} , \quad (4.16)$$

where, as before, the factor $\eta_{(i)}$ equals $+1$ if particle (i) is outgoing and -1 if it is ingoing. In particular, we still have $F_{ab}K^b = 0$.

The force acting on a test particle with charge Q and n -velocity V^a is

$$f^a = QF^{ab}V_b . \quad (4.17)$$

We assume that the test particle is initially at rest in our GICS, $V^a = t^a$. Then,

$$f^a(U, \mathbf{x}) = \frac{2\pi Q}{(2\pi r)^{n/2-1}} \left[\sum_{(i) \text{ in, out}} \frac{\eta_{(i)} q_{(i)}}{1 - \hat{\mathbf{r}} \cdot \mathbf{v}_{(i)}} \frac{d\tau_{(i)}}{dt} q^{ab} u_{(i)b} \right] \frac{d^{n/2-2}}{dU^{n/2-2}} \delta(U) . \quad (4.18)$$

Its change in momentum is

$$\Delta P^a(U) = \int_{-\infty}^U dU' f^a(U', \mathbf{x}) \quad (4.19)$$

$$= \frac{2\pi Q}{(2\pi r)^{n/2-1}} \left[\sum_{(i) \text{ in, out}} \frac{\eta_{(i)} q_{(i)}}{1 - \hat{\mathbf{r}} \cdot \mathbf{v}_{(i)}} \frac{d\tau_{(i)}}{dt} q^{ab} u_{(i)b} \right] \frac{d^{n/2-2}}{dU^{n/2-2}} \Theta(U) . \quad (4.20)$$

Since $f_a V^a = 0$, the electromagnetic force still cannot produce a change in mass. For $n = 6$, for a smoothed-out source, we can obtain a finite displacement of the test particle's position tangent to the sphere. For $n > 6$, there is no electromagnetic memory effect.

4.3 Gravitational Fields and Memory

We now turn our attention to gravitational memory arising from linearized gravitational perturbations off of an even dimensional Minkowski background. The retarded solution to the n -dimensional generalization of the linearized Einstein equation (2.32) is

$$\bar{h}_{ab}(\chi) = 16\pi \int d^n x' (G \cdot I_{ab}{}^{c'd'}) (\chi, \chi') T_{c'd'}(\chi') \quad (4.21)$$

with G denoting the retarded Green's function of the n -dimensional scalar wave equation (4.3), and the bi-tensor of parallel transport $I_{ab}{}^{c'd'}$ remaining trivial. In fact, it follows that near null infinity we have

$$\bar{h}_{\mu\nu}(r, U, \hat{\mathbf{r}}) = \frac{8\pi}{(2\pi r)^{n/2-1}} \frac{\partial^{n/2-2}}{\partial U^{n/2-2}} \int d^{n-1} \mathbf{y} T_{\mu\nu}(U + \hat{\mathbf{r}} \cdot \mathbf{y}, \mathbf{y}) . \quad (4.22)$$

Again, we consider a stress-energy tensor of the form (2.35) (with three-dimensional delta functions replaced by $n - 1$ -dimensional delta functions). The retarded metric perturbation is now

$$h_{ab} = \frac{8\pi}{(2\pi r)^{n/2-1}} \frac{\partial^{n/2-2}}{\partial U^{n/2-2}} (\Theta(U)\alpha_{ab} + \Theta(-U)\beta_{ab}) , \quad (4.23)$$

where

$$\alpha_{ab}(\hat{\mathbf{r}}) = \sum_{(i) \text{ out}} \frac{m^{(i)}}{1 - \hat{\mathbf{r}} \cdot \mathbf{v}^{(i)}} \frac{d\tau^{(i)}}{dt} \left(u_a^{(i)} u_b^{(i)} + \frac{1}{n-2} \eta_{ab} \right) \quad (4.24)$$

and

$$\beta_{ab}(\hat{\mathbf{r}}) = \sum_{(j) \text{ in}} \frac{m^{(j)}}{1 - \hat{\mathbf{r}} \cdot \mathbf{v}^{(j)}} \frac{d\tau^{(j)}}{dt} \left(u_a^{(j)} u_b^{(j)} + \frac{1}{n-2} \eta_{ab} \right) . \quad (4.25)$$

The relationship between the linearized Riemann tensor and the metric perturbation (2.41) is valid for perturbations off of Minkowski backgrounds of any dimension, so the linearized Riemann tensor computed from (4.23) is

$$R_{abcd} = \frac{8\pi}{(2\pi r)^{n/2-1}} K_{[a} \Delta_{b][c} K_{d]} \frac{d^{n/2-1}}{dU^{n/2-1}} \delta(U) , \quad (4.26)$$

where

$$\Delta_{ab} = 2 \sum_{(i) \text{ in, out}} \frac{\eta_{(i)} m_{(i)}}{1 - \hat{\mathbf{r}} \cdot \mathbf{v}^{(i)}} \frac{d\tau^{(i)}}{dt} \left\{ q_{ac} u_{(i)}^c q_{bd} u_{(i)}^d + \frac{1}{n-2} q_{ab} \right\} . \quad (4.27)$$

Again, $\eta_{(i)}$ is +1 for outgoing and -1 for incoming particles.

Now integrating the geodesic deviation equation twice with respect to time we find that the change in displacement of the two test particles is

$$\Delta d^a(U) = \frac{2\pi}{(2\pi r)^{n/2-1}} \frac{d^{n/2-2}}{dU^{n/2-2}} \Theta(U) \Delta^a_b d^b , \quad (4.28)$$

For $n > 4$, the change in separation goes like $\delta(U)$ or derivatives of $\delta(U)$. The test particles return to their original relative displacement, and there is no memory effect.

4.4 The Slow Motion Limit

Further insight into the absence of gravitational memory for $n > 4$ can be seen from consideration of radiation in the slow-motion limit of the source. To analyze this, there is no need to restrict consideration to particle sources, and we shall not make this restriction below except where stated. To leading order in the velocity of the source, we may neglect the variation of the retarded time over the source. For the scalar field (2.1) with source S (both generalized to n dimensions), to lowest order in source velocity and leading order in $1/r$, we thereby obtain

$$\begin{aligned}\varphi(U, \mathbf{x}) &= \frac{2\pi}{(2\pi r)^{n/2-1}} \frac{d^{n/2-2}}{dU^{n/2-2}} \int d^{n-1}\mathbf{x}' S(U, \mathbf{x}') \\ &= \frac{2\pi}{(2\pi r)^{n/2-1}} \frac{d^{n/2-2}\Omega}{dU^{n/2-2}},\end{aligned}\tag{4.29}$$

where $\Omega = \Omega(U)$ denotes the monopole moment of the source at retarded time U . Thus, the leading order contribution to scalar radiation comes from variation of the monopole moment. For electromagnetic radiation, we similarly obtain for the spatial components, A_i , of the vector potential,

$$A_i(U, \mathbf{x}) = \frac{2\pi}{(2\pi r)^{n/2-1}} \frac{d^{n/2-2}}{dU^{n/2-2}} \int d^{n-1}\mathbf{x}' J_i(U, \mathbf{x}').\tag{4.30}$$

However, using conservation of J^a , we have

$$\int d^{n-1}\mathbf{x}' J^i(U, \mathbf{x}') = \int d^{n-1}\mathbf{x}' J^j(U, \mathbf{x}') \partial_j x'^i = \frac{d}{dU} \int d^{n-1}\mathbf{x}' J^0(U, \mathbf{x}') x'^i = \frac{dp^i}{dU}\tag{4.31}$$

where $p^i = p^i(U)$ is the electric dipole moment of the source. Thus, we obtain

$$A_i(U) = \frac{2\pi}{(2\pi r)^{n/2-1}} \frac{d^{n/2-1} p_i}{dU^{n/2-1}},\tag{4.32}$$

and the dominant form of electromagnetic radiation in the slow motion limit is electric dipole radiation.

Similarly, in the gravitational case, we have

$$\int d^{n-1}\mathbf{x}' T_{ij}(U, \mathbf{x}') = \frac{1}{2} \frac{d^2}{dU^2} \int d^{n-1}\mathbf{x}' T_{00}(U, \mathbf{x}') x'_i x'_j \equiv \frac{1}{2} \frac{d^2 I_{ij}}{dU^2} , \quad (4.33)$$

where $I_{ij} = I_{ij}(U)$ is the inertia tensor. The dominant contribution to the spatial components of the metric perturbation is thus

$$\bar{h}_{ij}(U) = \frac{4\pi}{(2\pi r)^{n/2-1}} \frac{d^{n/2} I_{ij}}{dU^{n/2}} . \quad (4.34)$$

Note however that gravitational radiation depends only on the projected tensor $[\bar{h}_{ij}]^{\text{TT}}$. The inertia tensor is decomposed as

$$I_{ij} = Q_{ij} + \frac{1}{n-1} I^k{}_k \delta_{ij} , \quad (4.35)$$

where Q_{ij} is trace-free and is called the quadrupole tensor. Since $[\delta_{ij}]^{\text{TT}} = 0$, it follows that gravitational radiation depends only on the quadrupole tensor. Thus the dominant form of gravitational radiation in the slow motion limit is (polar parity) quadrupole radiation.

The gravitational memory effect is determined by the change in \bar{h}_{ij} between asymptotically early and late times. In $n = 4$ spacetime dimensions, this will be given by the change in $d^2 I_{ij}/dU^2$. If the matter is moving in from infinity in an inertial manner at early times and moving out to infinity in an inertial manner at late times, it is not difficult to produce a change in this quantity. In particular, for particle sources, we have

$$I_{ij}(t) = \sum_{(i)} m^{(i)} x_i^{(i)} x_j^{(i)} , \quad (4.36)$$

and if the particles are in inertial motion, we have

$$\frac{d^2 I_{ij}}{dU^2} = 2 \sum_{(i)} m^{(i)} v_i^{(i)} v_j^{(i)} . \quad (4.37)$$

In general, this quantity is nonvanishing, and its value for incoming particles need not equal its value for outgoing particles.

By contrast, for $n > 4$, the memory effect in the slow motion limit is given by the change in derivatives of I_{ij} of higher order than second. However, for matter in inertial motion, we have

$$\frac{d^3 I_{ij}}{dU^3} = 0 . \quad (4.38)$$

Thus, one can thereby see that there can be no memory effect in the slow motion limit when $n > 4$.

CHAPTER 5

IS MEMORY A TIDAL EFFECT?

We now inquire whether or not memory is a tidal phenomenon. Recall that Christodoulou’s (1.4) and Bieri and Garfinkle’s (1.7) formulae for nonlinear and null memory depend only upon the passage of (effective) stress-energy past the detector near null infinity—they make no mention of where it “came from;” their formulae appear to allow gravitational waves traveling from past null infinity, for example. Thorne’s and Wiseman and Will’s interpretation of nonlinear memory, on the other hand, explicitly requires that the gravitational wave come from a burst-type source in the interior of spacetime.

We attempt to resolve this apparent paradox by calculating the null memory—if there is memory—left by stress-energy passing from past to future null infinity. In particular, we consider a single null particle. It is straightforward to generalize the calculation of section 2 to find the memory left when such a particle is created in a burst event; below, we also find the gravitational field of a particle which travels from past null infinity to future null infinity by sending the interaction point p back to past null infinity. As before, we begin by studying the analogous scalar problem.

5.1 Scalar Sources from Past Null Infinity

Consider a massless scalar charge, traveling at the speed of light in the \hat{z} -direction, created *ex nihilo* at some interaction point p_0 which occurs at $t = z = t_0, x = y = 0$:

$$S_0 = q\delta(x)\delta(y)\delta(z - t)\Theta(t - t_0) . \tag{5.1}$$

The retarded scalar field is

$$\begin{aligned}
\varphi_0 &= 2q \int d^4\delta \left[-(t-t')^2 + |\mathbf{x} - \mathbf{x}'|^2 \right] \Theta(t-t') \delta(x') \delta(y') \delta(z'-t') \Theta(t'-t_0) \\
&= 2q \int dt' \delta \left[-(t-t')^2 + x^2 + y^2 + (z-t')^2 \right] \Theta(t-t') \Theta(t'-t_0) \\
&= 2q \int dt' \delta \left[2(t-z)t' - t^2 + x^2 + y^2 + z^2 \right] \Theta(t-t') \Theta(t'-t_0) \\
&= \frac{q}{t-z} \Theta \left(t - \frac{t^2 - x^2 - y^2 - z^2}{2(t-z)} \right) \Theta \left(\frac{t^2 - x^2 - y^2 - z^2}{2(t-z)} - t_0 \right). \tag{5.2}
\end{aligned}$$

The two step functions can be combined into a single step function to produce our final result

$$\varphi_0 = \frac{q}{t-z} \Theta(U_0). \tag{5.3}$$

where $U_0 = (t-t_0) - \sqrt{x^2 + y^2 + (z-t_0)^2}$ is a null coordinate such that $U_0 = 0$ corresponds to the future light cone of p_0 . Note that although φ_0 is unbounded (since it diverges as $t \downarrow z$ at $x = y = 0$) it is locally in L^1 and thus is well defined as a distribution. The resulting four-force on a test particle of charge Q is

$$f_0^a = Q \nabla^a \varphi_0 = \frac{qQ}{u^2} k^a \Theta(U_0) - \frac{qQ}{u} \left(t^a + \frac{rr^a - t_0 z^a}{\sqrt{x^2 + y^2 + (z-t_0)^2}} \right) \delta(U_0), \tag{5.4}$$

where $u = t - z$ and $k^a = -\nabla^a u$. To leading order in $1/r$, our expression (5.4) becomes

$$f_0^a = -\frac{qQ}{r} \frac{K^a}{1 - \cos \theta} \delta(U). \tag{5.5}$$

This force can be seen as a limiting case of (2.14), where we have only one outgoing particle with speed $|\mathbf{v}| \rightarrow 1$. As we have seen in sections 2 and 4, this δ -function contribution to f^a will give rise to an instantaneous “kick” in the four-momentum of the test particle. If the test particle is initially “at rest” and its motion remains non-relativistic, then the change in

4-momentum due to this instantaneous kick is given by

$$\Delta P_0^a = -\frac{qQ}{r} \frac{K^a}{1 - \cos \theta}. \quad (5.6)$$

Note that this expression for the net kick is independent of t_0 , i.e., a change in t_0 affects the kick only to higher order in $1/r$ (although, of course, a change of t_0 affects the time at which the kick is felt). Since the kick arises from the $\delta(U_0)$ term in the force, the kick can be understood as being produced by a burst of radiation emitted when the source was created. Note that the kick diverges as $\theta \rightarrow 0$. Let us now take the limit as $t_0 \rightarrow -\infty$. Naively taking the limit of (5.3), we obtain

$$\varphi = \lim_{t_0 \rightarrow -\infty} \varphi_0 = \frac{q}{t - z} \Theta(t - z). \quad (5.7)$$

However, the right side of this equation is not locally in L^1 and does not make sense as a distribution. Indeed, it is easy to see that for any fixed, non-negative test function f with $f \neq 0$ at some point at which $t = z$ we have

$$\lim_{t_0 \rightarrow -\infty} \int \varphi_0 f = \infty, \quad (5.8)$$

so the weak distributional limit of φ_0 does not exist as $t_0 \rightarrow -\infty$. We conclude, therefore, that for the scalar wave equation, it does not make sense to talk about the retarded field of a charged particle¹ source that moves forever on a null geodesic. As our derivation has indicated, the problem with obtaining a distributional solution arises from the “forever” (i.e., non-compactness) character of the source rather than its “null” character.

Nevertheless, although $\lim_{t_0 \rightarrow -\infty} \varphi_0$ does not exist as a distribution, some aspects of this limit do exist. Specifically, let $k^a = t^a + z^a$ be the vector field on Minkowski spacetime

1. Since, in Minkowski spacetime, averaging over the observation point is equivalent to averaging over the source, the failure to obtain a distributional solution for a particle source moving forever on a null geodesic implies the failure to have any retarded solution at all for a smooth, null fluid source with everywhere parallel 4-velocity.

that is everywhere parallel to the tangent to the null geodesic source (5.1). Then we claim that the weak distributional limit $\lim_{t_0 \rightarrow -\infty} k^{[a\nabla b]}\varphi_0$ does exist. To see this, let α^{ab} be a smooth, antisymmetric tensor field of compact support. We wish to evaluate

$$\lim_{t_0 \rightarrow -\infty} - \int \varphi_0 k_a \nabla_b \alpha^{ab} = \lim_{t_0 \rightarrow -\infty} - \int_{U_0 > 0} d^4x \frac{1}{u} k_a \nabla_b \alpha^{ab}. \quad (5.9)$$

Integrating by parts, we obtain

$$- \int_{U_0 > 0} d^4x \frac{1}{u} k_a \nabla_b \alpha^{ab} = - \int_{U_0 > 0} d^4x \frac{1}{u^2} k_a \nabla_b u \alpha^{ab} - \int_{U_0 = 0} \frac{1}{u} k_a n_b \alpha^{ab} \quad (5.10)$$

where n^a is the normal to the $U_0 = 0$ surface,

$$n^a = t^a + \frac{\rho \rho^a + (z - t_0) z^a}{\sqrt{\rho^2 + (z - t_0)^2}}. \quad (5.11)$$

(Here $\rho = (x^2 + y^2)^{1/2}$ and $\rho^a = \nabla^a \rho$.) The bulk integral vanishes because α^{ab} is antisymmetric and $k_{[a} \nabla_{b]} u = 0$. The surface term is

$$\begin{aligned} & - \int_{U_0 = 0} \frac{1}{u} k_a n_b \alpha^{ab} \\ &= - \int_{U_0 = 0} \rho d\rho d\phi dz \frac{\sqrt{\rho^2 + (z - t_0)^2} k_{[a} t_{b]} + \rho k_{[a} \rho_{b]} + (z - t_0) k_{[a} z_{b]}}{\left(\sqrt{\rho^2 + (z - t_0)^2} - (z - t_0)\right) \sqrt{\rho^2 + (z - t_0)^2}} \alpha^{ab}. \end{aligned} \quad (5.12)$$

As $t_0 \rightarrow -\infty$, the numerator in this expression converges uniformly on compact sets to $\rho k_{[a} \rho_{b]}$, whereas the denominator converges uniformly on compact sets to $\rho^2/2$. Furthermore, as $t_0 \rightarrow -\infty$, we have $U_0 \rightarrow u$. From this it can be seen that the (weak) limit of $k^{[a\nabla b]}\varphi_0$ as $t_0 \rightarrow -\infty$ exists and is given by

$$\lim_{t_0 \rightarrow -\infty} k^{[a\nabla b]}\varphi_0 = 2q \frac{1}{\rho} \rho^{[a} k^{b]} \delta(u). \quad (5.13)$$

Thus, in the limit $t_0 \rightarrow -\infty$, the force exerted on a test particle is well defined modulo

addition of multiples of k^a . Since this force also has a δ -function character, it gives rise to a 4-momentum kick of the form

$$\Delta P_\infty^a = -2qQ \frac{1}{\rho} \rho^a \quad (5.14)$$

modulo multiples of k^a . This 4-momentum kick is very different in form from the kick (5.6) produced by the burst of radiation arising from a “creation event.”

5.2 Electromagnetic Sources from Past Null Infinity

In this section, we wish to obtain the retarded solution to Maxwell’s equations with a charged particle source moving on a null geodesic. As in the case of the scalar wave equation, in order to have a well defined solution, we would like to “create” the source at a finite time t_0 and then consider the limit $t_0 \rightarrow -\infty$. However, unlike the scalar case, we cannot “create” a charge at a finite time because Maxwell’s equations require conservation of charge. Therefore, we consider, instead, a situation where a charge sits “at rest” until time $t = t_0$ and thereafter moves on a null geodesic, i.e., we take the current density to be

$$J_0^a = q\delta(x)\delta(y) [\delta(z - t_0)\Theta(t_0 - t)t^a + \delta(z - t)\Theta(t - t_0)k^a] , \quad (5.15)$$

where $k^a = t^a + z^a$ is tangent to the null geodesic $x = y = 0, t = z$. We can immediately write down the retarded solution in Lorenz gauge using the well known Coulomb solution for the source for $t < t_0$ and using (5.3) for $t \geq t_0$. We obtain

$$A_0^a = \frac{q}{\sqrt{x^2 + y^2 + (z - t_0)^2}} \Theta(-U_0)t^a + \frac{q}{t - z} \Theta(U_0)k^a \quad (5.16)$$

The electromagnetic field tensor is given in terms of the vector potential by $F^{ab} = 2\nabla^{[a}A^{b]}$. From (5.16), we obtain

$$(F_0)^{ab} = -2q \frac{1}{r} \frac{\sin \theta}{1 - \cos \theta} \theta^{[a} K^{b]} \delta(U_0) + O(1/r^2) . \quad (5.17)$$

The force on a test particle of charge Q and 4-velocity u^a is $f_a = QF_{ab}u^b$. As in the scalar case, the leading order in $1/r$ contribution to f_a is a δ -function term, which will give the particle an instantaneous momentum kick. In the case of electromagnetism, f^a is automatically orthogonal to u^a and, hence, does not change the rest mass of the test particle, i.e., the particle gets only a “velocity kick.” For a test particle that is initially “at rest” and whose motion remains non-relativistic, the instantaneous kick in 4-momentum is given by

$$\Delta P^a = qQ \frac{1}{r} \frac{\sin \theta}{1 - \cos \theta} \theta^a. \quad (5.18)$$

This agrees with the velocity kick obtained by Bieri and Garfinkle [21]. It is the limit of (2.31) for one incoming charge at rest and one outgoing charge with velocity $v\hat{\mathbf{z}}$ as $v \rightarrow 1$.

We now attempt to take the limit $t_0 \rightarrow -\infty$. The contribution of the first (Coulomb) term in (5.16) clearly goes to zero in this limit. However, apart from the factor of k^a , the contribution of the second term in (5.16) is identical to the scalar case, and hence it does *not* have a distributional limit. We conclude that the retarded solution for the vector potential of a charged particle that moves forever on a null geodesic does not exist in Lorenz gauge. Nevertheless, since the Coulomb contribution vanishes in the limit, we see that that

$$F_{ab} \equiv \lim_{t_0 \rightarrow -\infty} (F_0)_{ab} = -2 \lim_{t_0 \rightarrow -\infty} k_{[a} \nabla_{b]} \varphi_0 \quad (5.19)$$

with φ_0 given by (5.3). As we showed in the previous section, the limit on the right side of this equation *does* exist as a distribution, and we obtain

$$F_{ab} = -4q \frac{1}{\rho} \rho_{[a} k_{b]} \delta(u). \quad (5.20)$$

Equation (5.20) may thus be interpreted as providing the retarded field² of a charged particle

2. Note that although we showed above that the retarded vector potential in Lorenz gauge does not exist for this solution, one can find other gauges in which a distributional vector potential for the field (5.20) can be found; see [54] and [55].

that moves on a null geodesic for all time, in agreement with Jackiw, Kabat and Oritz [54] (see also problem 11.18 of the third edition of Jackson [55]).

The field (5.20) produces an instantaneous momentum kick on a test particle of charge Q (assumed to be initially at rest) given by

$$\Delta P_\infty^a = 2qQ \frac{1}{\rho} \rho^a. \quad (5.21)$$

Again, this differs in form from the momentum kick (5.18) produced by the burst of radiation associated with the instantaneous change of motion of the source at time $t_0 = 0$.

5.3 Gravitational Sources from Past Null Infinity

We now turn to the case of linearized gravity, with a source T_{ab} corresponding to a particle moving on a null geodesic. As in the scalar and electromagnetic cases, we would like to “create” this particle at time t_0 and then take the limit $t_0 \rightarrow -\infty$. However, the linearized Einstein equation requires conservation of stress-energy, which, for particle sources, requires conservation of 4-momentum. Thus, the simplest case to consider would be a particle of mass m which is at rest until time t_0 at which time it emits a null particle of energy, E , and then loses mass and recoils so as to conserve four-momentum. Thus, we consider a stress-energy source of the form

$$\begin{aligned} T_{ab} &= T_{ab}^{(\text{I})} + T_{ab}^{(\text{II})} + T_{ab}^{(\text{III})} \\ T_{ab}^{(\text{I})} &= m\delta_3(\mathbf{x})t_a t_b \Theta(-t) \\ T_{ab}^{(\text{II})} &= \tilde{m}\delta_2(x, y)\delta(\tilde{z})\tilde{t}_a \tilde{t}_b \Theta(t) \\ T_{ab}^{(\text{III})} &= E\delta_2(x, y)\delta(z - t)k_a k_b \Theta(t). \end{aligned} \quad (5.22)$$

The recoil coordinates $(\tilde{t}, x, y, \tilde{z})$ are the GICS of recoiling particle’s rest frame, and $\tilde{t}^a = -\nabla^a \tilde{t}$. The recoil coordinates and recoil mass \tilde{m} are chosen so that four-momentum is

conserved at p . The metric perturbation is

$$\begin{aligned}
(h_0)_{ab} &= (h_0^{(\text{I})})_{ab} + (h_0^{(\text{II})})_{ab} + (h_0^{(\text{III})})_{ab} \\
(h_0^{(\text{I})})_{ab} &= \frac{2m}{\sqrt{x^2 + y^2 + (z - t_0)^2}} \left(2t^a t^b + \eta_{ab} \right) \Theta(-U_0) \\
(h_0^{(\text{II})})_{ab} &= \frac{2\tilde{m}}{\sqrt{x^2 + y^2 + (\tilde{z} - t_0)^2}} \left(2\tilde{t}^a \tilde{t}^b + \eta_{ab} \right) \Theta(U_0) \\
(h_0^{(\text{III})})_{ab} &= \frac{4E}{t - z} k_a k_b \Theta(U_0) .
\end{aligned} \tag{5.23}$$

Portions (I) and (II) are the linearized Schwarzschild metric in the massive particles' rest frames. Portion (III) is the gravitational analog of the null particle field (5.3). Assuming $E \ll m$ and keeping only the leading order term in E/m , we find the linearized Riemann tensor to be

$$\begin{aligned}
R_{abcd} &= 4E \frac{1}{r} \left[\frac{2}{1 - \cos \theta} k_{[a} K_{b]} K_{[c} k_{d]} \right. \\
&\quad \left. - \left(2K_{[a} (t_b] z_{[c} + z_b] t_{[c} K_{d]} \right) + (1 + \cos \theta) \left(2K_{[a} t_b] t_{[c} K_{d]} + K_{[a} \eta_b]_{[c} K_{d]} \right) \right] \delta'(U) .
\end{aligned} \tag{5.24}$$

Although we have chosen a particular decay/recoil process in order to do these calculations, the details of the process are irrelevant at $\mathcal{O}(E/m)$ provided that all of the particles apart from the null particle are non-relativistic, i.e., the details of the decay process would affect (5.24) only at higher orders in E/m . Upon taking the electric components of (5.24) and integrating twice, we find the memory tensor to be

$$\Delta_{ab} = E(1 + \cos \theta)(\theta_a \theta_b + \phi_a \phi_b) . \tag{5.25}$$

It can be shown that (5.25) is a limiting case of (2.43) for a massive particle emitting an ultrarelativistic massive particle of energy E and recoiling.

Let us now take the limit as $t_0 \rightarrow -\infty$. It is clear that

$$\lim_{t_0 \rightarrow -\infty} (h_0^{\text{I}})_{ab} = \lim_{t_0 \rightarrow -\infty} (h_0^{\text{II}})_{ab} = 0. \quad (5.26)$$

On the other hand, $(h_0^{\text{III}})_{ab} = 4\varphi_0 k_a k_b$ (with q replaced by E), so the limit as $t_0 \rightarrow -\infty$ of $(h_0^{\text{III}})_{ab}$ does not exist. We conclude that, as was true for the electromagnetic four-potential, the retarded solution for the metric perturbation of a particle that moves forever on a null geodesic does not exist in the Lorenz gauge. On the other hand, the contribution of $(h_0^{\text{III}})_{ab}$ to the linearized Riemann tensor is

$$(R_0^{\text{III}})_{abcd} = 8\nabla_{[a}\nabla_{|[d}\varphi_0 k_{c]}k_{b]} \quad (5.27)$$

and it follows that the limit as $t_0 \rightarrow -\infty$ of $(R_0^{\text{III}})_{abcd}$ *does* exist. In fact, we obtain

$$R_{abcd} = \lim_{t_0 \rightarrow -\infty} (R_0^{\text{III}})_{abcd} = 4k_{[a}\nabla_{b]}F_{cd} \quad (5.28)$$

where F_{ab} is given by (5.20) (with q replaced by E) and the derivative is taken in the distributional sense. To calculate this distributional derivative more explicitly, let β^{abcd} be smooth and of compact support and have the tensor symmetries of the linearized Riemann tensor. We wish to evaluate

$$-16 \int \nabla_b \beta^{abcd} k_a \left(-\frac{1}{\rho} \rho_c k_d \right) \rho d\rho d\phi dz. \quad (5.29)$$

To do so, we exclude a disc of radius ϵ about $\rho = 0$, integrate by parts with respect to ρ , and then let $\epsilon \rightarrow 0$. We thereby obtain

$$R_{abcd} = 16E \frac{1}{\rho^2} k_{[a}(\rho_b)\rho_{[c} - \phi_b]\phi_{[c]}k_{d]}\delta(u) - 16\pi E k_{[a}p_{b][c}k_{d]}\delta(x)\delta(y)\delta(u), \quad (5.30)$$

where p_{ab} is the projection of the metric into the “ x - y ” plane (*i.e.*, $p_{ab} = x_a x_b + y_a y_b$). Equa-

tion (5.30) agrees with the Riemann curvature tensor of the Aichelburg-Sexl [56] solution—apart from several sign discrepancies, which are undoubtedly misprints in eq. (3.12) of their paper.³ Equation (5.30) may be interpreted as the linearized curvature⁴ of the retarded field of particle of energy E that moves on a null geodesic forever. Although, as we have seen, the retarded solution for the perturbed metric in the Lorenz gauge does not exist as a distribution, it should be possible to find other gauges in which a distributional metric perturbation giving rise to (5.30) does exist.

Unlike (5.24), the Riemann tensor (5.30) does not have a derivative of a δ -function term. Furthermore, its effects fall off at large distances like $1/r^2$ rather than $1/r$. Consequently, *we conclude there is no memory effect associated with the retarded field of a particle that moves on a null geodesic forever.* However, the δ -function in (5.30) will produce an instantaneous “relative velocity kick” to a system of test particles moving on geodesics. Integrating (5.30), we find that if the particles have initially separation D^a , the relative velocity kick will be

$$\Delta v_a = 4E \frac{1}{\rho^2} (\rho_a \rho_b - \phi_a \phi_b) d^b. \quad (5.31)$$

This velocity kick can be given a simple interpretation in terms of Newtonian tidal effects. Consider, in Newtonian gravity, a particle of mass E traveling with velocity c along the z -axis. The Newtonian potential produced by such a particle at time t is

$$\chi = -\frac{E}{\sqrt{x^2 + y^2 + (z - ct)^2}}. \quad (5.32)$$

3. In particular, the Ricci component R_{00} is easily computed by adding together the first two lines of their eq. (3.12) and does not agree with the (correct) expression they give below eq. (3.12); their eq. (3.12) also fails to be rotationally invariant in the plane orthogonal to the direction of the particle.

4. As Aichelburg and Sexl have argued, this solution may be interpreted as a solution to the full, nonlinear Einstein equation, not merely the linearized Einstein equation. Indeed, Aichelburg and Sexl obtained their solution by taking an infinite boost limit of the exact Schwarzschild solution.

The tidal tensor associated with this potential is

$$\Phi_{ab} = \frac{E}{r^3} (3r'_a r'_b - \delta_{ab}), \quad (5.33)$$

where $r' = \sqrt{x^2 + y^2 + (z - ct)^2}$ and $r'_a = \nabla_a r'$. We can integrate the tidal tensor once to get the net relative velocity change of two neighboring test particles over all time. For test particles initially separated by the displacement d^j , we obtain

$$\begin{aligned} \Delta v_i &= \int_{-\infty}^{\infty} dt' \Phi_{ij}(t', x, y, z) d^j \\ &= 2E \frac{1}{\rho^2} (\rho_i \rho_j - \phi_i \phi_j) d^j. \end{aligned} \quad (5.34)$$

Thus, apart from a factor of 2, the net relative velocity change in the Newtonian case produced by a particle of mass E that moves forever along the z -axis at velocity c agrees with the relative velocity kick in linearized gravity produced by a particle of energy E that moves forever on a corresponding null geodesic. The only difference is that in Newtonian gravity, these tidal effects occur over all time, whereas in linearized gravity, the tidal effects are “compressed” into a null plane traveling along with the source. Thus, the Newtonian tidal acceleration is gradual and continuous, whereas in linearized gravity, one obtains an instantaneous velocity kick.

We have investigated the retarded solution for a scalar field, an electromagnetic field, and a linearized gravitational field associated with the creation of a null particle at time t_0 in Minkowski spacetime. In the scalar case, we can simply create a charged null particle; in the electromagnetic and linearized gravitational cases, other sources must also be present in order to conserve, respectively, charge and four-momentum. There are then two distinct limits of this retarded solution that we can take. The first is to fix t_0 and extract the leading order in $1/r$ behavior of the solution. In all three cases, there are effects produced on distant test particles at order $1/r$ caused by the creation of the null particle.

In the scalar and electromagnetic cases, they give rise to an instantaneous “kick” to the four-momentum of a test particle. In the linearized gravitational case, the $\mathcal{O}(1/r)$ effect is to produce an instantaneous relative displacement of test particles—the memory effect.

The alternative limit is to fix the observation point and let $t_0 \rightarrow -\infty$. This limit can be thought of as providing the retarded field of a null particle that moves on a null geodesic forever. In the scalar case, we found that this limit does not exist as a distribution. However, in the electromagnetic and linearized gravitational cases, although the limits of the Lorenz gauge vector potential and Lorenz gauge metric perturbation similarly do not exist, the limits of the electromagnetic field tensor and linearized Riemann tensor do exist. In the electromagnetic case, the limiting electromagnetic field tensor gives rise to a velocity kick on distant test particles at order $1/r$, but the form of this velocity kick is very different from the $\mathcal{O}(1/r)$ velocity kick produced by the creation of a null charge at finite time t_0 . In the linearized gravitational case, the limiting linearized Riemann tensor yields the Aichelburg-Sexl solution. It falls off as $1/r^2$ and thus produces no effects of any kind at order $1/r$. In particular, there is no memory effect. The leading order ($1/r^2$) effect of this linearized Riemann tensor is to produce an instantaneous relative velocity kick on test particles, of exactly the same form as the integrated Newtonian tidal force would produce.

We conclude that in linearized gravity, the “radiation field” (retarded solution) produced by a particle moving on a null geodesic forever is the Aichelburg-Sexl solution, which is a pure “tidal field” that produces no associated memory effect. Thus, the memory effect should not be interpreted as being caused merely by the passage of (effective) stress-energy to null infinity. However, the Aichelburg-Sexl solution is not physically acceptable if we wish to consider null and ordinary memory: it fails to be asymptotically flat at spatial infinity (even if we “smooth out” the source, as we can in linearized gravity), since the Riemann tensor vanishes in all non-equatorial directions and falls off too slowly (as $1/r^2$) in equatorial directions near spatial infinity. The Aichelburg-Sexl solution therefore has no notion of null infinity. One way of producing a physically acceptable solution is to

create the null particle at a finite time t_0 via an “emission event,” as we have considered. In that case, there will be a burst of radiation associated with the emission event that produces a nontrivial memory effect, in agreement with previous results. More generally, the requirement of asymptotic flatness at spatial infinity implies either the finite time creation of the null particle or the presence of additional “incoming radiation” from past null infinity that is not directly associated with the null particle. We believe that the memory effect is most naturally interpreted as being caused by either the emission event or by the additional incoming radiation from past null infinity, rather than by the passage of the particle to future null infinity.

CHAPTER 6

NULL AND ORDINARY MEMORY FOR ULTRARELATIVISTIC MATTER

While Bieri and Garfinkle’s classification of memory into null and ordinary portions is useful and interesting, it is also in some ways puzzling. The formulae for ordinary (1.6) and null (1.7) memory tell us that they come from two very different looking sources: null memory is proportional to the total energy reaching future null infinity (and does not appear to depend at all on early times), while ordinary memory depends explicitly on a change in certain components of the Weyl curvature between early and late times. Nevertheless, we expect that in the ultrarelativistic limit, the ordinary memory of a massive matter should smoothly extrapolate to the null memory of massless sources of the same energy. Indeed, we have seen this to be the case for particle stress-energy in section 5. Yet it is not obvious from (1.6) and (1.7) that this is so.

In this section, we seek a better understanding of the relationship between null and ordinary memory by re-examining the simple decay problem of section 5.3 in the context of the Bieri-Garfinkle formalism. In particular, we will investigate why the total memory found in section 5.3 is entirely null memory despite there being massive particles as well by finding the memory due to the decay process without making any assumption of smallness of E/m . We will also generalize the calculation of 5.3 by requiring that the emitted particle will travel at a speed $\beta < 1$ rather than at the speed of light. Here the memory will only be ordinary memory, since the emitted particle is traveling at a speed slower than light. We will see to what extent and in what sense our expectation that ordinary memory “imitates” null memory in the ultrarelativistic limit is correct.

6.1 High-Energy Decay Products

We again specialize to the particular case treated in section 5.3: the decay, at time $t = 0$, of a particle of mass m at rest into a null particle of energy E that travels in the \hat{z} direction and a recoiling particle of mass \tilde{m} that travels in the $-\hat{z}$ direction. Due to the axisymmetry of the problem, it follows that to order $1/r$ the electric portion of the Riemann curvature—that is, the portion of the curvature that leaves memory through the geodesic deviation equation—is

$$R_{abcd}t^a t^c = W(\theta_b \theta_d - \phi_b \phi_d) , \quad (6.1)$$

for some scalar W . From this equation it follows that we only need to calculate one component of $R_{abcd}t^a t^c$ to find all components. In particular, it follows from the standard expressions for spherical coordinates that

$$W = \frac{R_{0x0y}}{(1 + \cos^2 \theta) \cos \phi \sin \phi} . \quad (6.2)$$

The reason for choosing R_{0x0y} is that for the metric of section 5.3, this component is particularly simple to calculate. Since h_{xy} , h_{0x} and h_{0y} all vanish, it follows from the formula relating the linearized Riemann tensor to the metric perturbation (2.41) that

$$R_{0x0y} = -\frac{1}{2} \partial_x \partial_y h_{00} , \quad (6.3)$$

so the only component of the metric that we need is h_{00} . To leading order in $1/r$, we can manipulate (5.23) into the form

$$h_{00} = \frac{k}{r} + \frac{1}{r} \left[-2m + \frac{2\tilde{m}\gamma(1 + \beta^2)}{1 + \beta \cos \theta} + \frac{4E}{1 - \cos \theta} \right] \Theta(U) ; . \quad (6.4)$$

Here k is a constant, and β and $\gamma = 1/\sqrt{1 - \beta^2}$ are respectively the speed and gamma factor of the recoiling particle. Note however that \tilde{m} and β are not independent quantities. Rather

they are related to m and E by the conservation of four-momentum:

$$\tilde{m}\gamma = m - E , \quad (6.5)$$

$$\tilde{m}\gamma\beta = E . \quad (6.6)$$

Using these identities and some straightforward but tedious algebra, we can re-write eq. (6.4) as

$$h_{00} = \frac{k}{r} + \frac{2}{r} \frac{E(1 + \cos^2 \theta)}{(1 - \cos \theta)(1 - (E/m)(1 - \cos \theta))} \Theta(U) , \quad (6.7)$$

so

$$R_{0x0y} = \frac{-1}{r} \frac{E(1 + \cos^2 \theta) \sin^2 \theta \cos \phi \sin \phi}{(1 - \cos \theta)(1 - (E/m)(1 - \cos \theta))} \delta'(U) . \quad (6.8)$$

Combining eqs. (6.2) and (6.8), we find

$$W = -\frac{1}{r} \frac{E(1 + \cos \theta)}{1 - (E/m)(1 - \cos \theta)} \delta'(U) ; \quad (6.9)$$

inserting this back into the electric portion of the Riemann tensor (6.1) and integrating the geodesic deviation equation, we find the total memory tensor to be

$$\Delta_{ab} = \frac{E(1 + \cos \theta)}{1 - (E/m)(1 - \cos \theta)} (\theta_a \theta_b - \phi_a \phi_b) . \quad (6.10)$$

We now compare the total memory tensor to the individual ordinary and null memory tensors. The total tensor is the solution to the local form of eqs. (1.6) and (1.7) on the sphere at infinity

$$D_A D^A \Upsilon = \Delta P - F_{\text{null}} , \quad (6.11)$$

$$D^B \Delta_{AB} = D_A \Upsilon . \quad (6.12)$$

As before, D_A is the covariant derivative of the metric on S^2 , F_{null} is the total energy

radiated to a point on S^2 in future null infinity, and

$$\Delta P(\hat{\mathbf{r}}) = \lim_{U \rightarrow \infty} \lim_{r \rightarrow \infty} r^3 C_{abcd}(U, r\hat{\mathbf{r}}) t^a r^b t^c r^d - \lim_{U \rightarrow -\infty} \lim_{r \rightarrow \infty} r^3 C_{abcd}(U, r\hat{\mathbf{r}}) t^a r^b t^c r^d, \quad (6.13)$$

where C_{abcd} is the Weyl tensor. Due to the axisymmetry of the source, there must be functions of the polar angle alone, $A(\theta)$ and $B(\theta)$, such that

$$\Delta P - F_{\text{null}} = A(\theta), \quad (6.14)$$

$$\Upsilon = B(\theta). \quad (6.15)$$

The consistency of eqs. (6.11) and (6.12) requires that A have vanishing $\ell = 0$ and $\ell = 1$ harmonic modes.

Using the ansatz of eqs. (6.14), (6.15) and

$$\Delta_{AB} = C(\theta)(\theta_A \theta_B - \phi_A \phi_B), \quad (6.16)$$

we find that eqs. (6.11) and (6.12) become

$$\frac{d}{d\theta} \left(\sin \theta \frac{dB}{d\theta} \right) = \sin \theta A, \quad (6.17)$$

$$\frac{d}{d\theta} \left(\sin^2 \theta C \right) = \sin^2 \theta \frac{dB}{d\theta}. \quad (6.18)$$

The memory can be divided into ordinary and null parts as follows: $\Upsilon = \Upsilon^{(1)} + \Upsilon^{(2)}$ which satisfy

$$D_A D^A \Upsilon^{(1)} = \Delta P - (\Delta P)_{[1]}, \quad (6.19)$$

$$D_A D^A \Upsilon^{(2)} = - \left(F_{\text{null}} - (F_{\text{null}})_{[1]} \right), \quad (6.20)$$

where the subscript [1] denotes the sum of the $\ell = 0$ and $\ell = 1$ harmonic modes. Then

$\Delta_{AB} = \Delta_{AB}^{(1)} + \Delta_{AB}^{(2)}$, where

$$D^B \Delta_{AB}^{(1)} = D_A \Upsilon^{(1)}, \quad (6.21)$$

$$D^B \Delta_{AB}^{(2)} = D_A \Upsilon^{(2)}. \quad (6.22)$$

Here $\Delta_{AB}^{(1)}$, the memory due to ΔP , is the ordinary memory, while $\Delta_{AB}^{(2)}$, the memory due to F_{null} , is the null memory. In each case, eqs. (6.14) and (6.15) hold separately for each kind of memory, with $A^{(1)}(\theta) = \Delta P - (\Delta P)_{[1]}$ in the case of ordinary memory and $A^{(2)}(\theta) = -(F_{\text{null}} - (F_{\text{null}})_{[1]})$ in the case of null memory.

We now work out the null memory for the case of the decay of a particle of mass m emitting a null particle of energy E . Since the particle is emitted in the \hat{z} direction, it follows that $F_{\text{null}} = E\delta$, where δ is the delta function which vanishes everywhere except $\theta = 0$ and whose integral over the unit two sphere is 1. We then find

$$8\pi \left(F_{\text{null}} - (F_{\text{null}})_{[1]} \right) = -2E(-4\pi\delta + (1 + 3\cos\theta)). \quad (6.23)$$

Thus to find the null memory, we must solve eqns. (6.17),(6.18) with A given by the right hand side of eqn. (6.23). Note that since δ vanishes for $\theta > 0$, what we need to do is to solve eqns. (6.17),(6.18) for $\theta > 0$ with $A = 2E(1 + 3\cos\theta)$. Given a solution for $\theta > 0$ we can then verify that eqns. (6.11),(6.12) are satisfied in a distributional sense.

For $\theta > 0$, eq. (6.17) becomes

$$\frac{d}{d\theta} \left(\sin\theta \frac{dB}{d\theta} \right) = E(2\sin\theta + 6\sin\theta\cos\theta), \quad (6.24)$$

from which we find

$$\sin\theta \frac{dB}{d\theta} = E(-2\cos\theta + 3\sin^2\theta + c_0), \quad (6.25)$$

for some constant c_0 . Since the left-hand side of this equation vanishes at $\theta = \pi$, we must

have $c_0 = -2$, and therefore

$$\sin \theta \frac{dB}{d\theta} = E(1 - 2 \cos \theta - 2 \cos^2 \theta) . \quad (6.26)$$

Now from eq. (6.18) we obtain

$$\frac{d}{d\theta}(\sin^2 \theta C) = E \sin \theta (1 - 2 \cos \theta - 3 \cos^2 \theta) , \quad (6.27)$$

for which the solution is

$$\sin^2 \theta C = E(-\cos \theta - \sin^2 \theta + \cos^3 \theta + c_1) \quad (6.28)$$

for some constant c_1 . Since the left-hand side vanishes at $\theta = \pi$, it follows that $c_1 = 0$ and thus

$$C = -E(1 + \cos \theta) . \quad (6.29)$$

We now verify that this is actually a distributional solution. For $\Upsilon^{(2)}$ to be a distributional solution to eq. (6.20) means that for any smooth function f on S^2 we have

$$\oint d\Omega \left[\Upsilon^{(2)} D_A D^A f + 8\pi \left(F_{\text{null}} - (F_{\text{null}})_{[1]} \right) f \right] = 0 , \quad (6.30)$$

where the integral is over S^2 with $d\Omega$ the usual two-sphere volume element. Thus, we must evaluate the left-hand side of eq. (6.30) with $\Upsilon^{(2)}$ equal to the B specified in eq. (6.26) and $F_{\text{null}} - (F_{\text{null}})_{[1]}$ given in eq. (6.23). If the result is zero, then the solution is a distributional

solution. We have

$$\begin{aligned}
& \oint d\Omega [BD_A D^a f + 2E(4\pi\delta - (1 + 3\cos\theta))f] \\
&= 8\pi E f|_{\theta=0} + \lim_{\epsilon \rightarrow 0} \int_{\theta < \epsilon} d\Omega [BD_A D^A f - 2E(1 + 3\cos\theta)f] \\
&= 8\pi E f|_{\theta=0} + \lim_{\epsilon \rightarrow 0} \int_{\theta < \epsilon} d\Omega D_A (BD^A f - f D^A B) \\
&\quad + \lim_{\epsilon \rightarrow 0} \int d\Omega f [D_A D^A B - 2E(1 + 3\cos\theta)] \\
&= 8\pi E f|_{\theta=0} + \lim_{\epsilon \rightarrow 0} \left[(-2\pi \sin\theta) \left(B \frac{\partial f}{\partial \theta} - f \frac{\partial B}{\partial \theta} \right) \right]_{\theta=\epsilon} \\
&\quad + \lim_{\epsilon \rightarrow 0} \int_{\theta < \epsilon} d\Omega f \left[\frac{1}{\sin\theta} \frac{d}{d\theta} \left(\sin\theta \frac{dB}{d\theta} \right) - 2E(1 + 3\cos\theta) \right] \\
&= (2\pi f|_{\theta=0}) \left[4E + \lim_{\theta \rightarrow 0} \sin\theta \frac{dB}{d\theta} \right] \\
&\quad + \lim_{\epsilon \rightarrow 0} \int_{\theta < \epsilon} d\Omega f \left[\frac{1}{\sin\theta} \frac{d}{d\theta} \left(E(1 - 2\cos\theta - 3\cos^2\theta) \right) - 2E(1 + 3\cos\theta) \right] \\
&= (2\pi f|_{\theta=0}) \left[4E + \lim_{\theta \rightarrow 0} E(1 - 2\cos\theta - 3\cos^2\theta) \right] = 0. \tag{6.31}
\end{aligned}$$

Therefore the B specified by eq. (6.26) is a distributional solution of eq. (6.20) with the $F_{\text{null}} - (F_{\text{null}})_{[1]}$ given in eq. (6.23).

For eq. (6.20) to be satisfied in a distributional sense means that for any smooth vector field ω^A on S^2 we have

$$\oint d\Omega \left[\Delta_{AB}^{(2)} D^B \omega^B - \Upsilon^{(2)} D_A \omega^A \right] = 0. \tag{6.32}$$

thus we must evaluate the left-hand side of eq. (6.32) with $\Delta_{AB}^{(2)}$ given by the expression in eq. (6.16) with C given in eq. (6.29) and with $\Upsilon^{(2)}$ equal to the B specified in eq. (6.26). If

the result is zero, then eq. (6.20) is satisfied in a distributional sense. We have

$$\begin{aligned}
& \oint d\Omega \left[C(\theta_A \theta_B - \phi_A \phi_B) D^B \omega^A - B D_A \omega^A \right] \\
&= \lim_{\theta < \epsilon} \int d\Omega \left[C(\theta_A \theta_B - \phi_A \phi_B) D^B \omega^A - B D_A \omega^A \right] \\
&= \lim_{\epsilon \rightarrow 0} \int_{\theta < \epsilon} d\Omega D^B \left[C(\theta_A \theta_B - \phi_A \phi_B) \omega^A - B \omega_B \right] \\
&\quad + \lim_{\epsilon \rightarrow 0} \int_{\theta < \epsilon} d\Omega \left[-\omega^A D^B (C(\theta_A \theta_B - \phi_A \phi_B)) + \omega^B D_B B \right] \\
&= \lim_{\theta \rightarrow 0} 2\pi \sin \theta \omega^\theta (B - C) + \lim_{\epsilon \rightarrow 0} \int_{\theta < \epsilon} d\Omega \omega^\theta \left[\frac{dB}{d\theta} - \frac{dC}{d\theta} - 2 \cos \theta C \right] \\
&= \lim_{\epsilon \rightarrow 0} \int_{\theta < \epsilon} d\Omega \frac{\omega^\theta}{\sin \theta} \left[\sin \theta \frac{dB}{d\theta} - \sin \theta \frac{dC}{d\theta} - 2 \cos \theta C \right] \\
&= \lim_{\epsilon \rightarrow 0} \int_{\theta < \epsilon} d\Omega \frac{\omega^\theta}{\sin \theta} \left[E(1 - 2 \cos \theta - 3 \cos^2 \theta) - E \sin^2 \theta + 2E \cos \theta (1 + \cos \theta) \right] = 0 .
\end{aligned} \tag{6.33}$$

Therefore the B and C given respectively by eqs. (6.26) and (6.29) provide a distributional solution for eq. (6.22). The null memory of this source is

$$\Delta_{AB}^{(2)} = E(1 + \cos \theta)(\theta_A \theta_B - \phi_A \phi_B) , \tag{6.34}$$

which is the $\mathcal{O}(E/m)$ term of the entire memory found in section 2.

We now calculate the ordinary memory. For this we must calculate ΔP . Note that before the particle decays, the metric perturbation is just that of a Schwarzschild metric of mass m . Therefore $P(-\infty)$ is just the P of Schwarzschild. After the decay, and after the null particle has hit null infinity, the metric perturbation is again, that of a Schwarzschild metric, but now with mass \tilde{m} and boosted with velocity β in the \hat{z} direction. We thus need to calculate the P of both boosted and unboosted Schwarzschild.

Associated with the usual spherical coordinates (t, r, θ, ϕ) there is the usual orthonormal

tetrad $(t^a, r^a, \theta^a, \phi^a)$. Introduce as well the null tetrad $(\ell^a, n^a, m^a, \bar{m}^a)$ given by

$$\ell^a = \frac{1}{\sqrt{2}}(t^a + r^a), \quad (6.35)$$

$$n^a = \frac{1}{\sqrt{2}}(t^a - r^a), \quad (6.36)$$

$$m^a = \frac{1}{\sqrt{2}}(\theta^a + i\phi^a), \quad (6.37)$$

$$\bar{m}^a = \frac{1}{\sqrt{2}}(\theta^a - i\phi^a). \quad (6.38)$$

The Schwarzschild metric of mass m has (to first order in perturbation of a flat background metric) has Weyl tensor [57]

$$C_{abcd} = -\frac{m}{r^3} \left(\eta_{ac}\eta_{bd} - \eta_{ad}\eta_{bc} + 12\ell_{[a}n_b]\ell_{[c}n_{d]} + 12m_{[a}\bar{m}_b]m_{[c}\bar{m}_{d]} \right). \quad (6.39)$$

Now consider a mass m moving with velocity $\beta\hat{z}$. then the mass is at rest in the coordinate system (t', x', y', z') , where

$$t' = \gamma(t - \beta z), \quad (6.40)$$

$$z' = \gamma(z - \beta t), \quad (6.41)$$

where $\gamma = 1/\sqrt{1 - \beta^2}$ and the x and y coordinates are unchanged. The Weyl tensor then takes the form

$$C_{abcd} = -\frac{m}{r^3} \left(\eta_{ac}\eta_{bd} - \eta_{ad}\eta_{bc} + 12\ell'_{[a}n'_{b]}\ell'_{[c}n'_{d]} + 12m'_{[a}\bar{m}'_{b]}m'_{[c}\bar{m}'_{d]} \right). \quad (6.42)$$

We would like to express the Weyl tensor of the moving mass in terms of the coordinates and null tetrad of the stationary observer. Since we are interested in quantities at null infinity,

we will work only to leading order in $1/r$. From eqs. (6.40) and (6.41) it follows that

$$r' = r\gamma(1 - \beta \cos \theta) , \quad (6.43)$$

$$u' = \frac{u}{\gamma(1 - \beta \cos \theta)} . \quad (6.44)$$

From eq. (6.44) we obtain

$$\ell'_a = \frac{1}{\gamma(1 - \beta \cos \theta)} \ell_a , \quad (6.45)$$

and from eqs. (6.40) and (6.45) we find

$$n'_a = \gamma \left[\frac{\beta^2 \sin^2 \theta}{1 - \beta \cos \theta} \ell_a + (1 - \beta \cos \theta) n_a - \beta \sin \theta (m_a + \bar{m}_a) \right] . \quad (6.46)$$

Finally, using the fact that ϕ and $r \sin \theta$ are unchanged by the Lorentz transformation, we obtain

$$m'_a = m_a - \frac{\beta \sin \theta}{1 - \beta \cos \theta} \ell_a . \quad (6.47)$$

(The complex conjugate of this equation gives the transformation of \bar{m}_a .) We then find the quantity P is given by

$$\begin{aligned} P &= r^3 C_{0r0r} \\ &= r^3 \ell^a n^b \ell^c n^d C_{abcd} \\ &= -m \left(\frac{r}{r'} \right)^3 (-1 + 3(\ell^a n'_a)^2 (n^b \ell'_b)^2) \end{aligned} \quad (6.48)$$

$$= \frac{-2m}{\gamma^3 (1 - \beta \cos \theta)^3} . \quad (6.49)$$

However, $m\gamma$ is the energy \mathcal{E} of the particle, and $\gamma^{-4} = (1 - \beta^2)^2$. We therefore obtain

$$P = \frac{-2\mathcal{E}(1 - \beta^2)^2}{(1 - \beta \cos \theta)^3} . \quad (6.50)$$

Before the decay, we have a particle of mass m and zero velocity, so it follows that $P(-\infty) =$

$-2m$. After the decay, it follows from eqs. (6.5) and (6.6) that the recoiling particle has energy $m - E$ and velocity $\beta = -E/(m - E)$, so

$$P(\infty) = \frac{-2m(1 - \frac{2E}{m})^2}{(1 - \frac{E}{m}(1 - \cos \theta))^3}, \quad (6.51)$$

and therefore

$$\Delta P = 2m \left[1 - \frac{(1 - \frac{2E}{m})^2}{1 - \frac{E}{m}(1 - \cos \theta)} \right]. \quad (6.52)$$

Subtracting off the $\ell = 0, 1$ harmonic modes of eq. (6.52), we find

$$\Delta P - (\Delta P)_{[1]} = 2m \left[1 - \frac{(1 - \frac{2E}{m})^2}{(1 - \frac{E}{m}(1 - \cos \theta))^3} - \frac{E}{m}(1 + 3 \cos \theta) \right]. \quad (6.53)$$

We are now in a position to explain the agreement of the calculation of section 2 with the null memory. Since that calculation is first order in E/m and agrees with the null memory, it follows that to first order in E/m the ordinary memory must vanish. However, the ordinary memory is due to the recoiling particle, and we would certainly expect that ΔP of the recoiling particle contains terms that are first order in E/m . Indeed, it follows from eq. (6.52) that to first order in E/m we have $\Delta P = 2E(1 + 3 \cos \theta)$. Thus, though to first order ΔP does not vanish; rather, it consists purely of $\ell = 0$ and $\ell = 1$ parts. Since those parts do not contribute to the memory, it follows that to first order in E/m the ordinary memory for this process vanishes. Now to find the ordinary memory, we must solve eqs. (6.17) and (6.18) with A given by the right hand side of eq. (6.53). Define the quantities s and X by $s = E/m$ and

$$X = 1 - s(1 - \cos \theta). \quad (6.54)$$

Then eq. (6.17) becomes

$$\frac{d}{d\theta} \left(\sin \theta \frac{dB}{d\theta} \right) = m \sin \theta \left[8(1 - s) - 6X - 2(1 - 2s^2)X^{-3} \right]; \quad (6.55)$$

integrating this equation we find

$$\sin \theta \frac{dB}{d\theta} = -\frac{m}{s} \left[8(1-s)X - 3X^2 + (1-2s)^2 X^{-2} + c_0 \right], \quad (6.56)$$

where c_0 is a constant. This constant must be chosen so that the right-hand side of eq. (6.56) vanishes at $\theta = 0$, which corresponds to $X = 1$. It then follows that

$$c_0 = -8(1-s) + 3 - (1-2s)^2. \quad (6.57)$$

Equation (6.18) then becomes

$$\frac{d}{d\theta}(\sin^2 \theta C) = -\frac{m \sin \theta}{s} \left[8(1-s)X - 3X^2 + (1-2s)^2 X^{-2} + c_0 \right], \quad (6.58)$$

where c_1 is a constant. The right-hand side of eq. (6.58) must vanish at $\theta = 0$, which yields

$$c_1 = -4(1-2) + 1 + (1-2s)^2 - c_0. \quad (6.59)$$

Using these expressions for c_0 and c_1 , some straightforward algebra yields

$$\sin^2 \theta C = -\frac{m}{s^2} X^{-1} (X-1)^2 (X-[1-2s])^2. \quad (6.60)$$

Then using (6.54) we find the ordinary memory to be

$$\Delta_{AB}^{(1)} = \frac{E^2}{m} \frac{\sin^2 \theta}{1 - \frac{\epsilon}{m}(1 - \cos \theta)} (\theta_A \theta_B - \phi_A \phi_B). \quad (6.61)$$

Adding the null (6.34) and ordinary (6.61) memory tensors we recover the complete tensor (6.10).

6.2 Massive Decay Products

We now consider the memory due to the decay of a particle of mass m where both particles produced in the decay are timelike. The particle moving in the \hat{z} direction will have energy E and velocity $\beta\hat{z}$ where $0 < \beta < 1$. The recoil particle will have energy \tilde{E} and velocity $\tilde{\beta}\hat{z}$ where $-1 < \tilde{\beta} < 0$. Note that \tilde{E} and $\tilde{\beta}$ are not independent quantities: the conservation of energy and momentum in the decay requires

$$m = E + \tilde{E} , \quad (6.62)$$

$$0 = E\beta + \tilde{E}\tilde{\beta} , \quad (6.63)$$

which yields

$$\tilde{E} = m - E , \quad (6.64)$$

$$\tilde{\beta} = \frac{-\beta E}{m - E} . \quad (6.65)$$

As in the section above, the axisymmetry of the problem means that the electric part of the Weyl tensor is of the form in eq. (6.1), with W given by eq. (6.2) and R_{0x0y} given by eq. (6.3). Thus, we only need to calculate the perturbed metric component h_{00} . Note that the situation is very similar to that of section 5.3, with the same metric before the decay, and after the decay the null particle and recoiling particle replaced by two timelike particles. It then follows that to leading order in $1/r$ we have

$$h_{00} = \frac{k}{r} + \frac{1}{r} \left[-2m + \frac{2E(1 + \beta^2)}{1 - \beta \cos \theta} + \frac{2\tilde{E}(1 + \tilde{\beta}^2)}{1 - \tilde{\beta} \cos \theta} \right] \Theta(U) \quad (6.66)$$

$$= \frac{k}{r} + \frac{1}{r} \frac{2E\beta(\beta - \tilde{\beta})(1 + \cos^2 \theta)}{(1 - \beta \cos \theta)(1 - \tilde{\beta} \cos \theta)} \Theta(U) . \quad (6.67)$$

Therefore

$$R_{0x0y} = -\frac{1}{r} \frac{E\beta(\beta - \tilde{\beta})(1 + \cos^2 \theta)}{(1 - \beta \cos \theta)(1 - \tilde{\beta} \cos \theta)} \sin^2 \theta \cos \phi \sin \phi \delta'(U) \quad (6.68)$$

and

$$W = -\frac{1}{r} \frac{E\beta(\beta - \tilde{\beta}) \sin^2 \theta}{(1 - \beta \cos \theta)(1 - \tilde{\beta} \cos \theta)} \delta'(U) . \quad (6.69)$$

Finally, using this expression for W in (6.1), we can integrate the geodesic deviation equation to find the total memory

$$\Delta_{ab} = \frac{E\beta^2 \sin^2 \theta}{(1 - \beta \cos \theta)(1 - (E/m)(1 - \beta \cos \theta))} (\theta_a \theta_b - \phi_a \phi_b) . \quad (6.70)$$

In order to get further insight into the relation between ordinary memory and null memory, we calculate the memory of the timelike decay again, but this time using the method of ordinary and null memory—with special emphasis on the ultrarelativistic ($\beta \rightarrow 1$) limit. It follows from eq. (6.52) that for this decay process we have

$$\Delta P = 2m - \frac{2E(1 - \beta^2)^2}{(1 - \beta \cos \theta)^3} - \frac{2\tilde{E}(1 - \tilde{\beta}^2)^2}{(1 - \tilde{\beta} \cos \theta)^3} . \quad (6.71)$$

Because in this case the entire memory is ordinary memory, it follows that the $\ell = 0$ and $\ell = 1$ parts of ΔP vanish, so there is no need to perform a subtraction of these parts. To find the memory, we need to solve eqs. (6.17)-(6.18) with A given by the right hand side of eq. (6.71). Integrating eq. (6.17) we obtain

$$\sin \theta \frac{dB}{d\theta} = -2m \cos \theta + \frac{E(1 - \beta^2)^2}{\beta(1 - \beta \cos \theta)^2} + \frac{\tilde{E}(1 - \tilde{\beta}^2)^2}{\tilde{\beta}(1 - \tilde{\beta} \cos \theta)^2} + c_0 . \quad (6.72)$$

The constant of integration c_0 is fixed by demanding that the right-hand side of the equation vanish at $\theta = 0$, which yields

$$c_0 = - \left(\frac{E}{\beta} + \frac{\tilde{E}}{\tilde{\beta}} \right) . \quad (6.73)$$

Equation (6.18) then becomes

$$\frac{d}{d\theta}(\sin^2 \theta C) = \sin \theta \left[-2m \cos \theta + \frac{E(1 - \beta^2)^2}{\beta(1 - \beta \cos \theta)^2} + \frac{\tilde{E}(1 - \tilde{\beta}^2)^2}{\tilde{\beta}(1 - \tilde{\beta} \cos \theta)^2} + c_0 \right], \quad (6.74)$$

from which we obtain

$$\sin^2 \theta C = m \cos^2 \theta - \frac{E(1 - \beta^2)^2}{\beta(1 - \beta \cos \theta)^2} - \frac{\tilde{E}(1 - \tilde{\beta}^2)^2}{\tilde{\beta}(1 - \tilde{\beta} \cos \theta)^2} - c_0 \cos \theta + c_1. \quad (6.75)$$

The constant of integration is fixed by demanding that the right-hand side vanish at $\theta = 0$, which yields

$$c_1 = \frac{E}{\beta^2} + \frac{\tilde{E}}{\tilde{\beta}^2} - 2m. \quad (6.76)$$

Finally, solving eq. (6.75) for C and simplifying, we find the ordinary memory to be

$$\Delta_{AB}^{(1)} = \frac{E\beta^2 \sin^2 \theta}{(1 - \beta \cos \theta)(1 - (E/m)(1 - \beta \cos \theta))} (\theta_A \theta_B - \phi_A \phi_B), \quad (6.77)$$

which is precisely the entire memory found above (6.70).

We now consider the null limit of the timelike decay, *i.e.*, we consider at fixed E the limit as $\beta \rightarrow 1$. First note that in the limit as $\beta \rightarrow 1$ eq. (6.70) goes to eq. (6.10). That is, as the timelike particle approaches the speed of light the memory produced by the timelike decay approaches the memory produced by the null decay. Though this is certainly what we would intuitively expect, we now consider how to reconcile this limit with Bieri and Garfinkle's picture of the two types of gravitational wave memory. The null decay has both ordinary memory sourced by ΔP and null memory sourced by F_{null} . The timelike decay has only ordinary memory. Thus, since the memory of the timelike decay approaches that of the null decay in the limit as $\beta \rightarrow 1$, it follows that some piece of ΔP must “mimic” the $-F_{\text{null}}$ of the null particle. In particular, define ΔP_E to be the middle term on the right hand side of

eq. (6.71):

$$\Delta P_E = -\frac{2E(1-\beta^2)^2}{(1-\beta\cos\theta)^3}. \quad (6.78)$$

In physical terms, one can think of ΔP_E as the contribution of the particle of energy E to the source of the memory. It follows that for $\theta \neq 0$ we have

$$\lim_{\beta \rightarrow 1} \Delta P_E = 0 \quad (6.79)$$

and that for all $\beta < 1$ we have

$$\oint \Delta P_E d\Omega = -E. \quad (6.80)$$

It then follows that in a distributional sense we have

$$\lim_{\beta \rightarrow 1} \Delta P_E = -E\delta. \quad (6.81)$$

Thus, in the limit as the timelike particle becomes null the ΔP of the timelike particle becomes the $-F_{\text{null}}$ of the null particle.

CHAPTER 7

CONCLUSIONS

We have investigated the gravitational wave memory effect, along with its scalar and electromagnetic analogs, for radiation emitted from classical particle-scattering interactions. In particular, we have found the retarded solution of the inhomogeneous wave equation for sources which are confined by three-dimensional Dirac delta functions to one-dimensional worldlines which may begin or end at a single interaction vertex p . The dominant feature of the field at large distances from such sources is a Heaviside step discontinuity in retarded time on the future light cone of p . In the context of linearized gravity on a Minkowski background in Lorenz gauge, this means there will be a step in the metric perturbation. The dominant feature in the curvature tensor at large distances will therefore be a derivative-of-a-delta-function singularity on the future light cone of p . The geodesic deviation equation expresses the relative acceleration of two nearby test particles (such as those that make up an interferometric gravitational wave detector) in terms of the curvature tensor; integrating the geodesic deviation equation twice, we see that this delta-derivative gives rise to a permanent, finite step in the test particles' separation—*i.e.*, it leaves memory. We have found this characterization of memory useful in answering questions and clarifying puzzling issues related to the memory effect.

It has given us one way to define memory in non-asymptotically flat spacetimes. Most of the previous work done on memory has been done on asymptotically flat backgrounds, where we can isolate the short-timescale effects of radiation from other long-timescale gravitational effects by putting our detector near future null infinity. Other *ad hoc* definitions of memory have been offered for specific non-asymptotically flat spacetimes. We define memory as the detector particle motion associated with a derivative of a delta function in the Riemann curvature of the retarded solution arising from a scattering interaction. This definition does not depend on any limits to null infinity, so it is valid for linearized perturbations of arbitrary background spacetimes. It also provides us a way to compare memory for “similar

sources” and “similar detectors” in different spacetimes.

We have made such a comparison between memory in FLRW and Minkowski spacetimes. If we identify the two manifolds via the same coordinate system (η, \mathbf{x}) (which is a GICS in Minkowski and a comoving coordinate system in FLRW) such that the FLRW scale factor $a(\eta_p) = 1$ at the source event p , and place the same physical source at p and the same physical detector at q in both spacetimes, then the memory effect in the FLRW spacetime will be a factor of $1/a(\eta_q) = 1/(1+z)$ (where $(1+z)$ is the redshift of the source as seen by the detector) smaller than the corresponding memory effect in Minkowski spacetime. Note that placing the source at the same proper distance at the time of emission corresponds to placing the source at the same angular diameter distance in both spacetimes. If, instead, the source is the same luminosity distance from the detector in both cases, then the FLRW memory is enhanced over the Minkowski memory by a factor of $(1+z)$; if the sources are the same proper distance from the detectors at the time of detection, then the memories are identical.

We have also used this characterization to help us understand why the memory effect disappears in higher-(even)-dimensional spacetimes. In n dimensions, the wave equation’s retarded Green’s function is dominated by the $\delta^{(n/2-2)}(\sigma^2)$ (*i.e.*, the $(n/2-2)^{\text{th}}$ derivative of a delta function of the squared geodesic distance between the source and field points)—as dimension increases, the Green’s function becomes more singular. These singularities are passed on to the radiation field; for $n > 4$, at large distances from p the metric perturbation’s most prominent feature is proportional to $\delta^{(n/2-3)}(U)$ rather than a $\Theta(U)$. Using the geodesic deviation equation, we see that the separation of test particles will also go like $\delta^{(n/2-3)}(U)$; to leading order in $1/r$, the particles will return to their original separation and will not exhibit memory of the wave.

We have used particle sources to explore under what conditions gravitational wave can leave memory. We have shown that stress-energy traveling from past null infinity to a detector near future null infinity does *not* leave memory; rather, it provides a “relative

velocity kick” (*i.e.*, the detector particles will begin moving, but will not return to rest with respect to each other) at $\mathcal{O}(1/r^2)$. This velocity kick is related to the tidal motion we see in Newtonian gravity. Stress energy escaping to null infinity will only leave memory if it is created in a burst-type event at some finite time in the past. We therefore conclude that memory is a purely radiative phenomenon, totally distinct from any Newtonian-like tidal effects.

Finally, we have used an explicit scattering source, in which a massive particle emits a (possibly massless) daughter particle and recoils, to elucidate Bieri and Garfinkle’s distinction between null and ordinary memory. We have shown that, if the daughter is massless and its energy is far less than the parent’s mass, then almost all of the total energy will be nonlinear/null rather than linear/ordinary. We have also confirmed that, if the daughter is massive, then in the ultrarelativistic limit its ordinary memory will smoothly extrapolate to the null memory of a massless particle of the same energy.

These results raise a number of new questions. From a theoretical perspective, it would be interesting to explore memory in odd-dimensional spacetimes. In odd dimensions, the Green’s function of the wave equation has support *within* an observation point’s past light cone rather than *on* it. Therefore even in odd-dimensional Minkowski spacetimes gravitational waves will possess large tails. This defies our identification of memory with a single derivative of a delta function term in curvature in section 3; nevertheless, the relative simplicity of point-particle calculations make them a natural starting point.

It would also be helpful to expand the analysis of section 6 by examining the limiting case of *incoming* ultrarelativistic matter. This might help shed light on the curious fact that ordinary memory depends on a change between early and late times, but null memory appears to depend only on energy fluxes at late times.

Furthermore, even though the classical scattering problem we have been studying can model a wide variety of gravitational wave sources, it does have some significant limitations. For example, in requiring that the particles interact only at a single point but otherwise

strictly follow geodesics of the background, we exclude some physically very relevant sources, such as binary in-spirals and mergers. It would be useful to find a new environment of approximations and assumptions in which we could relax some of our conditions but still make use of point particles and the delta-derivative characterization of memory. Post-Newtonian methods provide a possible way forward here.

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