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RETURNS

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Abstract

Foreign currency sovereign bond spreads tend to be higher than historical sovereign credit losses, and cross-country spread correlations are larger than their macro-economic counterparts. Foreign currency sovereign debt exhibits positive and time-varying risk premia, and standard linear asset pricing models using US-based factors cannot be rejected. The term structure of sovereign credit spreads is upward sloping, and inverts when either (a) the country's fundamentals are bad or (b) measures of US equity or credit market stress are high. I develop a quantitative and tractable continuous-time model of endogenous sovereign default in order to account for these stylized facts. My framework leads to semi-closed form expressions for certain key macro-economic and asset pricing moments of interest, helping disentangle which of the model features influences credit spreads, expected returns and cross-country correlations. Standard pricing kernels used to explain properties of US equity returns can be nested into my quantitative framework in order to test the hypothesis that US-based bond investors are marginal in sovereign debt markets. I show how to leverage my model to study the early 1980's Latin American debt crisis, during which high short term US interest rates and floating rate dollar-denominated debt led to a wave of sovereign defaults.

Chapter 1

Continuous Time Theory of Endogenous Sovereign Defaults

1.1 Introduction

Driven by low real interest rates, high commodity prices and easy credit, Latin American external debt grew significantly in the 1970s. The Volcker shock, combined with debt contracts indexed to US short term rates, contributed to the subsequent debt crisis and the “lost decade” suffered by many Latin American countries in the 1980s. A quarter of a century later, in the fall 2008, the US subprime crisis morphed into a global financial crisis, leading to a shut down of emerging economies’ access to international credit markets and a violent widening of their sovereign spreads. Those two episodes highlight the central importance of the supply of capital for sovereign debt dynamics. However, a large component of the international macroeconomic literature on sovereign credit risk uses economic models where external creditors are risk-neutral, assuming away any possible link between investors’ attributes and government financing and default decisions¹. The modeling hypothesis of this line of research stems from its main focus on macroeconomic quantities (such as the current

1. Two notable exceptions are [9] and [34].

account balance and the debt-to-GDP ratio) as opposed to prices, and from the difficulty of adding one or several dimensions to already complex models of endogenous default. Separately, the fixed income asset pricing literature on sovereign debt takes seriously investors' risk attributes when explaining properties of sovereign credit spreads and returns, but it does so at the expense of modeling the underlying asset cash-flows and their dynamic properties. Indeed, its primary objective is to use bond and credit derivatives' market prices in order to estimate hazard rate of default processes, without having the need to relate them to economic fundamentals.

My paper bridges the gap between these two seemingly disconnected literatures by offering a new model of endogenous sovereign default where the supply of capital takes on a prominent role, as supported by known stylized facts as well as new evidence I document in my empirical work. Thanks to its reduced dimensionality, the proposed framework remains tractable and allows me to obtain semi-closed form expressions for several macroeconomic and asset pricing moments of interest, helping disentangle which features of the model are essential to generate specific moments of the data. In addition, it facilitates the estimation and testing of the model, and an in-depth analysis of the government financing and default policies. It can then be used to answer numerous questions: how much of sovereign governments' financing costs can be attributed to bond investors' risk characteristics, and how much to country-specific macroeconomic risks? Are sovereign debt return co-movements mostly due to correlated fundamentals, or the fact that a common bond buyer base is marginal in sovereign bond markets? Can supply-side shocks to capital markets rationalize the magnitude of current account reversals observed in the context of "sudden-stops" suffered by emerging market economies in Latin America in the early 1980s, or in South East Asia in the late 1990s?

In the empirical section of my paper, I infer market-implied (sometimes called "risk-neutral") default intensities from sovereign credit-default swap ("CDS") premia, and then compute returns on CDS contracts. Leveraging my constructed data-set, I document three

sets of empirical facts that are the counterparts to known properties of foreign currency sovereign bond prices and returns. Those facts will not only guide the construction of my model but also will be used for estimation and testing.

First, I provide evidence that investors in sovereign debt markets do not behave risk-neutrally. To do so, I show that market-implied default intensities are significantly larger than historical default frequencies, and that sovereign CDS' expected excess returns are positive. Together, these empirical properties of sovereign debt spreads and returns illustrate the two sides of the same coin: creditors require compensation for being exposed to a risk (the sovereign default risk) that co-moves with their pricing kernel. While these stylized facts have already been investigated by [10] and [9] in the context of foreign currency sovereign bonds, I contribute to the empirical debate by showing that this property of sovereign credit prices and returns also holds for CDS contracts.

Second, the data supports not only that sovereign debt investors are risk-averse, but also that their pricing of risk is time-varying and relates to measures of US credit and equity market risks. Indeed, the difference between market-implied and historical default intensities is time-varying and cannot be explained by time-varying country-specific macroeconomic risk factors. This stylized fact has been documented previously by [36] and [43], who analyzed local and global factors that explain movements in sovereign CDS premia. Using my constructed CDS return data, I then perform standard linear asset pricing tests, using US equity market returns, and I fail to reject the hypothesis that a linear stochastic discount factor can price my set of excess returns. This exercise lends support to the analysis performed by [9] in the context of sovereign bond returns. Finally, I show that cross-country CDS return correlations are significantly larger than their macroeconomic counterparts, suggesting that a common bond buyer base is marginal in foreign currency sovereign debt markets.

While these facts, taken together, help us understand the required characteristics of a sovereign investors' pricing kernel, they are silent on the type of mechanism leading to sovereign defaults, and how supply side factors may impact a sovereign government's bor-

rowing and default decisions. I speak to this question by illustrating a third set of facts, related to the term structure of market-implied default intensities and returns. First, I show that the term structure of default intensities is upward sloping for most countries, but it flattens and inverts if either (i) a country’s fundamentals deteriorate, or (ii) measures of US credit or equity market stress are high. Second, I show that holding period excess returns are increasing with the maturity of the CDS contract – this latter fact being documented by [10] in the context of foreign currency sovereign bonds. Both properties of the term structure of spreads and returns are consistent with a “first hitting time” model, where a sovereign default is triggered by some – possibly endogenous – mean-reverting fundamental variable exceeding a certain threshold that depends on aggregate financial market conditions.

What might this macroeconomic “fundamental” variable be? In my theoretical setup, it is the debt-to-GDP ratio. I leverage the canonical sovereign default model of [20], further enhanced by [6] and [5], and develop a quantitative continuous time model of sovereign debt issuances and defaults, in which a government uses non-state contingent debt sold to foreign creditors for the purpose of consumption smoothing and consumption tilting². The government’s inability to commit to repay its debt leads to default risk. Following a default, the country suffers an instantaneous discrete drop in output and loses access to capital markets for an exponentially distributed time period. Using a modeling device used in [42], the country then re-enters financial markets with a lower debt burden, the result of an un-modeled renegotiation with its creditors. The sovereign debt-to-GDP ratio naturally arises as the fundamental state variable – a consequence of the homotheticity of the government’s objective function and the linearity of output and debt dynamics. I deviate from the canonical sovereign debt models along several dimensions. Since my focus is on the supply of capital and its impact on sovereign bond prices and returns, I introduce investors, whose preferences and equilibrium consumption lead to a pricing kernel that features regime-

2. As is typically the case in the international macroeconomic literature, the sovereign government will be more impatient than its creditors, providing an incentive to borrow in order to consume early.

dependent risk free rates and risk prices, in the spirit of [13]. Those regimes act as a second – exogenous and discrete – state variable that describes the international capital market environment.

My modeling ingredients lead to sovereign spreads that are greater than model-implied historical credit losses, as I document in the empirical part of the paper. For a panel of emerging market countries, I can then estimate the proportion of the average credit spread that can be attributed to (a) pure default risk and (b) the risk premium charged by international investors. I find that approximately 30% of sovereign governments' financing costs (over and above the risk-free rate) is attributable to required compensation paid to investors for taking on risks that are correlated with their marginal utilities. In the model, spread volatilities stem not only from output shocks, but also from stochastic discount factor ("SDF") shocks, and are thus close to spread volatilities in the data, a moment notoriously difficult to match with standard models ([3]). For the same reason, cross-country sovereign spread correlations are larger than cross-country output correlations. By turning on and off those SDF shocks, I can then infer the proportion of such cross-country spread correlation that relates to correlated fundamentals, and the proportion that relates to pricing by a common stochastic discount factor.

In my model, the sovereign default decision features an optimal debt-to-GDP default boundary that depends on the specific pricing kernel regime. Consistent with the data, this characteristic of my model leads to upward sloping term structures of spreads and default intensities for countries whose economic fundamentals are not too bad and in environments where risk-prices are not too high. Transitions from a "good regime" (where prices of risk are low for example) to a "bad regime" (with higher prices of risk) might cause the sovereign to "jump-to-default". Even if the sovereign government does not jump to default, it adjusts downwards its financing policy, switching from running a current account deficit to a current account surplus, and endogenously creating a sudden stop. For most of my countries of focus, a jump from the most benign capital market environment to the worst environment

leads to current account adjustments of 3% to 5% of GDP, potentially explaining up to half the adjustments observed in the data for the 1980s' Latin American debt crisis or the 1997 Asian tiger crisis. SDF regime transitions are also associated with inversions of the term structure of credit spreads, another feature of the data. When looking across multiple countries, transitions from "good regimes" to "bad regimes" lead to sudden increases in sovereign spreads as well as correlated defaults, arguably a feature of several sovereign debt crisis. The jump-to-default risk induced by SDF shocks also leads to high short term credit spreads, another stylized fact I document in the empirical section of my paper.

The continuous time framework I use has several key advantages over discrete time models that have been the workhorse of the sovereign default literature. First and foremost, it allows me to characterize fully the equilibrium of my model in the particular case where the government is risk-neutral. I provide closed-form solutions for the country's welfare, the debt price, the optimal default boundary of the government, and compute the magnitude of the current account reversal incurred upon an increase in the risk free rate or the price of risk. Outside the knife-edge risk-neutral case, the continuous time framework facilitates the transition from physical probabilities (under which the government optimizes) to risk-neutral probabilities (under which creditors price the debt issued). It allows for semi-closed form expressions of macro and asset pricing moments of interest, providing greater insight into the specific impact of the model assumptions on endogenous quantities of focus. My model features only two state variables – the debt-to-GDP ratio of the country being considered (a continuous variable), and the SDF regime (a discrete variable). This low dimensionality of the state space makes the framework more tractable than alternative models that have been studied in the literature³. It permits an estimation of the key parameters of the model using

3. Other articles focused on sovereign spreads and returns include [9], which feature 4 state variables, and [3], which feature 5 state variables; in order to find an equilibrium in such models, not only does the researcher have to find a global solution to the value function of the government (a function of all the state variables), but he also has to find the bond price schedule, which depends on both (i) the state variables and (ii) the amount of bonds that the government considers issuing. As will be clear in this paper, in continuous time the bond price schedule is no longer a function of the amount of bonds issued "in the next period".

a panel of countries, and gives me the ability to test whether pricing kernels used to explain properties of US equity returns can also explain properties of emerging market sovereign bond returns. In my numerical applications, I test the pricing kernel featured in [33] and show that that the level of risk-prices implied by such SDF is too low to fully account for the expected excess return observed in the data for many emerging market economies.

I finally highlight the flexibility of my framework by testing two new ideas. First, I focus on the contractual structure of sovereign debt and study the spill-over effects of US monetary policy on a government that issues debt whose coupon rate is indexed to US short-term rates. While foreign currency sovereign bonds are nowadays mainly issued in fixed rate form, Latin American countries used floating rate debt in the 1970's and early 80's, since the funding came in the form of loans from US commercial banks. Given that my model features time-varying risk-free rates, I can investigate the impact of US monetary policy on sovereign default risk. In this paper, I show that a simple mechanism may have been at play both (a) in the late 1970s, as Latin American economies took advantage of low US short term rates to significantly increase their external sovereign debt and run current account deficits, and (b) in the early 1980s', as the US monetary authorities increased short term rates to fight domestic inflation, increasing the debt servicing costs for Latin American governments and ultimately triggering the defaults of Mexico and multiple other sovereign issuers after 1982. In my model, in a low US short rate environment, floating rate sovereign issuers run current account deficits. When short term interest rates increase, a combination of lower debt prices and a higher marginal cost of debt issuances make governments adjust their current account balance by up to 15%, consistent in magnitude with what was observed empirically in 1982 in Mexico and other Latin American economies.

In a second application, I no longer assume a small open *endowment* economy but instead introduce a simple “A-K” production technology with investment adjustment costs and capital quality shocks, as in [11]. Sovereign debt is not only useful for consumption smoothing and consumption tilting, but also to build the domestic capital stock via in-

vestments. Thanks to the flexibility of my framework, the state space remains unchanged, with only one additional control variable – investments – added for the small open economy. In this modified environment, I show that two separate sources of debt overhang can lead to under-investments: (a) after a sequence of bad capital quality shocks suffered in the country’s production sector, or (b) after an SDF regime change from a mild capital market environment to one with higher risk-prices. This enhanced model thus leads to a negative correlation between sovereign spreads and investments, as observed in the data by [40] or [53]. It also provides a simple micro-foundation for the output dynamics used in [4], [5] and many other articles in the quantitative sovereign default literature, where log-output growth is a mean-reverting variable. Finally, the debt overhang channel leads to an amplification of the capital quality shocks, and thus to more volatile credit spreads and a wider ergodic debt-to-GDP distribution, getting this class of models closer to the data.

This paper is organized as follows. The first part of the paper focuses on some empirical facts of sovereign CDS premia and returns. I then develop a continuous time version of the canonical model of sovereign borrowing and default, and enhance it by introducing a Markov switching model of the stochastic discount factor used to price sovereign bonds. I estimate the model and perform a variety of exercises to illustrate the tractability of the framework.

1.2 Stylized Facts

In this section, I summarize key stylized facts on foreign currency sovereign credit spreads and returns. Many of these empirical observations have been highlighted in the past in connection with research focused on foreign currency sovereign bonds. In the online appendix, I revisit those facts by looking at a different set of credit instruments: credit default swap contracts referencing emerging market sovereign governments. My empirical analysis supports the existing evidence on sovereign credit spreads and returns, and adds new observations missed by previous studies. I will use my empirical work to guide my model estimation and

validation.

- (1) Hard currency sovereign credit spreads are higher than historical credit losses. This fact is inconsistent with an assumption of investors' risk-neutrality. It also means that holding-period expected excess returns on foreign currency sovereign debt are positive. This aspect of the data is highlighted by multiple studies, including for example [9] and [2]. I add supporting evidence in my online appendix, by showing that (i) hazard rates of default implied by the price of CDS contracts are materially higher than historical default rates, and (ii) CDS expected excess returns are positive.
- (2) The differential between sovereign credit spreads and conditional expected credit losses is time-varying, and is positively correlated with measures of US credit or equity market risk. This fact is highlighted by [2] for example, who regress the level of sovereign bond spreads onto the VIX index. It is also tightly related to a second observation: holding-period excess returns on sovereign bonds are higher for countries with higher US equity market beta. [9] document this fact by looking at returns on sovereign bonds in the EMBI index, and running standard cross-sectional and time-series tests of the CAPM. In the online appendix, I obtain similar results by using CDS returns as opposed to sovereign bond returns. I also emphasize that CDS provide a "cleaner" measure of expected excess returns earned on sovereign credit exposures than bonds – the latter not only being exposed to sovereign credit risk, but also to the term structure of US interest rates⁴. A third observation, made for example by [35], is tightly connected to the other two: there is a strong factor structure in the level of sovereign spreads, as supported by a principal component analysis of the time series of CDS premia for multiple countries. In addition, the first principal component in this decomposition is highly correlated with

4. Most foreign currency sovereign bonds issued by small open economies nowadays are fixed rate bonds denominated in USD. Researchers looking at time-series data on sovereign bonds rely on the EMBI index, compiled by JPMorgan, which provides, on a daily basis, an average price and average spread for a basket of eligible obligations issued by each country included in the index. JPMorgan unfortunately does not provide security-specific prices, making it difficult to extract excess returns attributable purely to sovereign default risk.

US equity market returns. These three observations suggests the presence of US-based marginal investors in foreign currency sovereign credit markets.

- (3) Short term market-implied hazard rates of defaults are non-zero, leading to a rejection of any model under which, at least at short horizons, defaults can be ruled out in some portions of the state space. An example of such model is one where default occurs exclusively when a continuous process hits a barrier – a so-called “first-hitting-time” model. This fact appears to be new in the sovereign default literature, and echos a similar observation made by the corporate finance literature in connection with corporate credit spreads.
- (4) In time series, sovereign credit spreads are (a) negatively related to GDP growth, (b) positively related to debt-to-GDP ratios, and (c) negatively related to measures of US credit or equity market risk. (a) is documented by [40] or [53], who however highlight that the relationship is weak. (b) is highlighted in several studies, including [3]. I provide new empirical evidence supporting (b) and (c) by looking at CDS contracts.
- (5) The term structure of sovereign credit spreads is upward-sloping, except for countries whose credit spreads are high, for which the term structure is either flat or downward sloping. The upward sloping term structure of spreads is highlighted by [43], who focus on CDS contracts referencing Mexico, Turkey and South Korea. The flattening and potential inversion of the term structure of credit spreads is briefly noticed in [10] and [7]. I provide additional supporting evidence for this feature of the data by focusing on CDS for a panel of 27 emerging market economies. This fact is consistent with a “first-hitting-time” model of sovereign default.
- (6) The term structure of sovereign credit spreads “flattens” at times when international investors’ risk prices are high. This feature of the data is different from fact (5), which relates movements of the slope of spreads to the level of spreads for a given country, whereas fact (6) relates movements of the slope of spreads to measures of US equity or

credit market risks for example. This fact appears to be new in the sovereign default literature, and is also consistent with a “first-hitting-time” model, in which the default barrier depends on international financial market conditions. I document it in the online appendix and test whether my model generates this behavior of the term structure of spreads.

- (7) Holding-period expected excess returns on foreign currency sovereign debt increase with the time-to-maturity of the credit instrument; in addition, most of the excess return differential between short term bonds and longer term bonds is earned in “crisis” periods – defined by [10] as a period when the level of credit spreads for the countries of interest are greater than the previous quarterly average *plus* 300bps. I will provide additional evidence supporting this result in the online appendix, by focusing on CDS as opposed to bonds. I will also emphasize that this excess return is actually earned during periods of high risk prices – which can be interpreted as periods during which international debt investors are more risk-averse than usual. This fact is consistent with a sovereign debt “risk exposure” that is increasing with the maturity of the debt instrument.
- (8) Holding-period excess returns on sovereign bonds are higher for countries with worse credit ratings. [9] document this fact by looking at returns on sovereign bonds in the EMBI index, and I provide additional evidence by looking at CDS returns. Whereas [9] argue that they would need a new source of exogenous country heterogeneity in order to account for fact (8), I will argue in the paper that such fact arises because the “risk exposure” of sovereign credit instruments is higher after a country has been hit by a sequence of bad fundamental shocks.

In the next sections I leverage the canonical framework of [20], [6] and [5] in order to build a continuous time model of sovereign defaults where international capital markets take on a prominent role. I will then confront the resulting model to the stylized facts discussed above.

1.3 A Continuous Time Sovereign Default Model

While I focus my empirical and quantitative analyses on the credit risk of different sovereign governments, the theoretical section of this paper only deals with a single government “ n ”. For simplicity, I abstract from interactions that different countries may have (such as cross-border trade flows), except through a common marginal investor in their sovereign debt. I thus abstract from the identity of the government in my notation. Country n is endowed with real output Y_t per unit of time, which evolves according to a Markov modulated geometric Brownian motion:

$$\frac{dY_t}{Y_t} = \mu_{s_t} dt + \boldsymbol{\sigma}_{s_t} \cdot d\mathbf{B}_t \quad (1.1)$$

My notation will use bold letters for vectors. $\{\mathbf{B}_t\}_{t \geq 0}$ is an N_b -dimensional standard Brownian motion on the underlying probability space $(\Omega, \mathcal{F}, \mathbb{P})$. I use multi-dimensional Brownian shocks to be able to discuss how idiosyncratic, regional and global shocks affect spread and return properties of sovereign debt. $\{s_t\}_{t \geq 0}$, taking values in $\{1, \dots, N_s\}$, is a discrete state Markov process with a generator matrix $\Lambda = (\Lambda_{ij})_{1 \leq i, j \leq N_s}$ that is assumed to be conservative (in other words $\sum_{j=1}^{N_s} \Lambda_{ij} = 0$ for all i). I will assume that $\{s_t\}_{t \geq 0}$ is recurrent, thus admitting a unique stationary distribution $\boldsymbol{\pi}$ (an $N_s \times 1$ real-valued positive vector) that solves $\boldsymbol{\pi}' \Lambda = \mathbf{0}$, and whose elements sum to 1. I will note $N_t^{(i,j)}$ the Poisson counting process for transitions from state i to state j . I will refer to \mathbb{P} as the physical probability measure, and note \mathcal{F}_t the σ -algebra generated by the Brownian motion \mathbf{B}_t and the discrete state Markov process s_t .

The Markov state s_t is not essential for modeling the country’s output dynamics – in most of the quantitative applications of this model, I will in fact assume that the expected GDP growth rate and the GDP growth volatility do not depend on the regime s_t . It could also be argued that the length of the GDP growth time series of the countries of interest is too limited to detect such regime shifts in the data⁵. Instead, the discrete regime s_t will be

5. For most countries of interest, I have yearly GDP growth data since 1970 – in other words, approximately 40 data-points. Estimating a Markov-switching output model with 2 Markov states for example would

the key variable describing the state of the creditors' stochastic discount factor, as will be discussed in section 1.3.1. I keep the flexibility to model a country's output dynamics as a Markov modulated geometric Brownian motion for two reasons. First, it allows me to deal with time-varying output growth volatility, a phenomenon empirically relevant for certain countries, as [46] suggests when focusing on Greece, Italy, Spain and Portugal. Second, I argue in section A.1.2 that this stochastic growth model enables me to approximate the output process used by [5] and many other articles in the international macroeconomic literature⁶. Lastly, I show in section 1.8 that a mean-reverting output growth rate can be obtained endogenously by introducing capital accumulation and a simple "AK" production technology. The government objective is to maximize the life-time utility function:

$$J_t = \mathbb{E} \left[\int_t^{+\infty} \varphi(C_s, J_s) ds | \mathcal{F}_t \right] \quad (1.2)$$

The notation \mathbb{E} denotes expectations under the measure \mathbb{P} . The aggregator φ takes the following form:

$$\varphi(C, J) := \delta \frac{1-\gamma}{1-\rho} J \left(\frac{C^{1-\rho}}{((1-\gamma)J)^{\frac{1-\rho}{1-\gamma}}} - 1 \right) \quad (1.3)$$

This preference specification is a generalization of the standard time-separable iso-elastic preferences to a non-time-separable framework, where intertemporal substitution can be decoupled from risk-aversion. δ is the government rate of time preference, $1/\rho$ is the intertemporal elasticity of substitution, while γ is the risk aversion coefficient. The standard iso-elastic time-separable preference specification corresponds to the parameter restriction

require estimating 2 expected growth rates, 2 growth volatilities, and 2 transition probabilities, leading to point estimates likely to have large standard errors.

6. Other articles that use this output process include, amongst others, [9], [3], [4]. In those articles, log output has a unit root, and output growth is a stationary and mean-reverting process. Viewed differently, the martingale decomposition of log output (see [21]) features (a) a time trend, (b) a martingale component with constant volatility and (c) a stationary component that is the sum of two Ornstein-Uhlenbeck processes, one of them fully correlated with the martingale component, while the other is independent.

$\gamma = \rho$. In such case, the life-time utility of the government takes the more familiar form:

$$J_t = \mathbb{E} \left[\int_t^{+\infty} \delta e^{-\delta(s-t)} \frac{C_s^{1-\gamma}}{1-\gamma} ds \mid \mathcal{F}_t \right] \quad (1.4)$$

If the government does not have any financial contracts at its disposal, its life-time utility is equal to:

$$J_{st}(Y_t) = K_{st} Y_t^{1-\gamma} \quad (1.5)$$

Equation 1.5 as well as the N_s constants $\{K_i\}_{i \leq N_s}$ are determined in section A.1.3. In order for equation 1.5 to be well defined, I need to impose a parameter restriction that will be assumed going forward.

Assumption 1. *Let $\{A_i\}_{i \leq N_s}$ be the family of constants defined via:*

$$A_i := \delta + (\rho - 1)(\mu_i - \frac{1}{2}\gamma|\boldsymbol{\sigma}_i|^2) \quad (1.6)$$

Then $(\delta, \rho, \gamma, \{\mu_i\}_{i \leq N_s}, \{\boldsymbol{\sigma}_i\}_{i \leq N_s})$ are such that $A_i > 0$ for all i .

The government does not have a full set of Arrow-Debreu securities at its disposal. Instead, it can only use non-contingent long-term debt contracts, with aggregate face value F_t and coupon rate κ . The incentive for the government to issue debt is two-fold: first, it enables the government to smooth consumption, and to reduce the welfare losses associated with consumption volatility. Second, differences between the government's rate of time preference and sovereign debt investors' discount rates will enable the government to "tilt" consumption into the present.

During each time period $(t, t+dt]$, a constant fraction mdt of the government's total debt amortizes, which the government repays with $mF_t dt$ units of output. This contract structure guarantees a constant debt average life of $1/m$ years, and allows me to carry only one state variable (F_t) as a descriptor of the government's indebtedness, as opposed to the full history of past debt issuances. The long-term debt assumption is also essential in my continuous

time framework in order to insure that an equilibrium with default can be supported: I show in section A.1.1 that the continuous sample paths of my output process preclude short term debt from being supportable in any sovereign default equilibrium. During each time period $(t, t + dt]$, the government can also decide to issue a dollar face amount $I_t dt$ of bonds. This formulation of an admissible issuance policy prevents “lumpy” debt issuances, and results in a government face value process F_t that is absolutely continuous:

$$dF_t = (I_t - mF_t) dt \quad (1.7)$$

Per period flow consumption consists of (a) total per-period output, plus (b) proceeds (in units of consumption goods) raised from capital markets minus (c) debt interest and principal repayments due:

$$C_t = Y_t + I_t D_t - (\kappa + m) F_t \quad (1.8)$$

In the above, D_t is the endogenous debt price per unit of face value, and is determined in equilibrium. My formulation of the debt dynamics as well as the resource constraint for the government lead to a cumulative consumption process that is absolutely continuous; in other words, the government does not consume in “lumpy fashion”, but rather always in “flow” fashion. I can interpret the difference $Y_t - C_t$ as the trade balance. The government cannot commit to repay its debt, which is thus credit risky. In other words, the government will choose a sequence of default times $\{\tau_k\}_{k \geq 1}$ out of the set of sequences of stopping times⁷. Default leads to the following consequences. First, output jumps down, from $Y_{\tau-}$ to $Y_{\tau} = \alpha Y_{\tau-}$, with $\alpha < 1$. Second, the country is locked out of capital markets for a (random)

7. The continuous time setting of this model allows me to abstract from the specific timing assumption of the government bond auction. In discrete time models, [16], [1] and [3] (for example) all assume that the bond auction happens before the default decision is made by the government, while [5], [6] and many other quantitative models of sovereign debt assume that the government makes its default decision before the bond auction takes place. The former timing convention allows, in discrete time, for the existence of potentially multiple equilibria, induced by the creditor’s self-fulfilling belief that the government will default immediately after debt has been issued, leading to a low auction debt price and a rational decision by the government to default. Those considerations are absent from the continuous time environment.

time period τ_e that is exponentially distributed with parameter λ . Once the country emerges from financial autarky, it has an outstanding debt balance that is only a fraction of its pre-default value, according to:

$$F_{\tau+\tau_e} = \theta \frac{Y_{\tau+\tau_e}}{Y_{\tau-}} F_{\tau-} \quad (1.9)$$

One can think of the parameter θ as the outcome of a bargaining game between creditors and the sovereign government, once such government has elected to default. However, for simplicity and since the strategic interactions between the government in default and its creditors are not a focus of this paper, I elect to model the outcome of this renegotiation exogenously⁸.

1.3.1 Creditors

External creditors purchase the debt issued by the government. I model their marginal utility process M_t (which I will also refer to as the stochastic discount factor, or “SDF”) as a random walk with two independent components – a diffusion component, and a jump component. More specifically, M_t evolves according to:

$$\frac{dM_t}{M_{t-}} = -r_{st} dt - \boldsymbol{\nu}_{st} \cdot d\mathbf{B}_t + \sum_{s_t \neq s_{t-}} \left(e^{v(s_{t-}, s_t)} - 1 \right) \left(dN_t^{(s_{t-}, s_t)} - \Lambda_{s_{t-}, s_t} dt \right) \quad (1.10)$$

Conditioned on being in the discrete Markov state i , creditors’ risk free rate r_i and the $N_b \times 1$ risk price vector $\boldsymbol{\nu}_i$ are constant. As section A.1.4 or [13] show, this stochastic discount factor can be obtained for example if creditors have iso-elastic time-separable or recursive preferences and an equilibrium consumption process that follows a Markov modulated geometric Brownian motion. This stochastic discount factor can also be obtained in a general equilibrium environment with a continuum of countries, by re-intrepreting C_t as

8. Note that the adjustment factor $\frac{Y_{\tau+\tau_e}}{Y_{\tau-}}$ in the debt face value post-restructuring is included for tractability purposes, since it will lead me to solve nested ordinary differential equations, as opposed to integro-differential equations. This feature is used in [42].

spending by government n , Y_t as the tax revenues of government n , and introducing a “world investor” who can diversify away all countries’ idiosyncratic risks, as I show in section A.1.5. This latter interpretation has the benefit of tying the world interest rate and the world risk prices to the investor’s preferences and the countries’ endowment growth rates, but would not add any additional insight to the paper. Finally, as explained in section A.1.4, the j^{th} coordinate of ν_i represents the excess return compensation per unit of j^{th} Brownian shock earned by investors – hence why I refer to ν_i as the vector of risk prices in state i . Similarly, $(e^{v(i,j)} - 1)$ is the jump-risk premium earned by investors per unit of jump risk, in connection with shifts from SDF state i to SDF state j .

My formulation of the stochastic discount factor implicitly assumes that government n ’s sovereign debt component of the creditor’s portfolio is negligible, and that government n ’s sovereign debt cash-flows do not alter the equilibrium consumption of creditors. This assumption seems reasonable: according to the World Bank, the aggregate external debt of emerging market countries was approximately \$1tn in 2014; while economically large, this quantity is small compared to the \$19tn market capitalization of stocks traded on the NYSE, the \$7tn market capitalization of stocks traded on the Nasdaq, and the \$35tn size of the US bond market.

Given my assumed investor pricing kernel, any \mathcal{F}_{t+s} -measurable amount A_{t+s} received at time $t + s$ will be valued by investors by weighting such future cash-flow by the investors’ future marginal utility, and taking expectations. One can also use a standard tool of the financial economics literature, and instead discount this future cashflow A_{t+s} at the risk-free rate, while distorting the probability distribution of such future cashflow via the following change in measure:

$$\text{Pricet}(A_{t+s}) = \mathbb{E} \left[\frac{M_{t+s}}{M_t} A_{t+s} | \mathcal{F}_t \right] := \hat{\mathbb{E}} \left[e^{-\int_0^s r_{t+u} du} A_{t+s} | \mathcal{F}_t \right]$$

$\hat{\mathbb{E}}$ is the risk-neutral expectation operator. It implicitly defines the risk-neutral measure \mathbb{Q} ,

under which $\hat{\mathbf{B}}_t := \mathbf{B}_t + \int_0^t \boldsymbol{\nu}_{\mathbf{s}_u} du$ is a standard N_b dimensional Brownian motion, and under which $\{s_t\}_{t \geq 0}$ is a discrete state Markov process with generator matrix $\hat{\Lambda}$, whose (i, j) element is $\hat{\Lambda}_{ij} = e^{v(i,j)} \Lambda_{ij}$, for $i \neq j$ ⁹.

Since most of the elements of the model have been introduced, I conclude this section by introducing two parameter restrictions. The first restriction guarantees that the risk-neutral value of a claim to the government's output be finite.

Assumption 2. $(\{r_i\}_{i \leq N_s}, \{\boldsymbol{\nu}_i\}_{i \leq N_s}, \{\mu_i\}_{i \leq N_s}, \{\boldsymbol{\sigma}_i\}_{i \leq N_s})$ jointly satisfy the following parameter restriction:

$$r_i + \boldsymbol{\nu}_i \cdot \boldsymbol{\sigma}_i - \mu_i > 0 \quad \forall i \in \{1, \dots, N_s\} \quad (1.11)$$

The second restriction insures that the government is impatient enough to front-load consumption in equilibrium. To be specific, when the government has neither debt nor assets outstanding, I need the government's financing policy to be such that it wants to borrow, instead of save. While I do not provide an explicit restriction on the deep model parameters in order to satisfy such condition, I verify ex-post after solving the model that it is the case. Intuitively, this parameter restriction should insure that the rate of time-preference δ of the government is sufficiently greater than the level of interest rates at which the government can finance itself via debt issuances.

1.3.2 Debt Valuation, Government Problem and Equilibrium

In this section, I focus on a Markovian setting and define admissible issuance and default policies of the government. Any admissible issuance and default policy will give rise to controlled Markov processes for the GDP and the debt face value. I then define the sovereign debt price and the life-time utility of the government, discuss the stochastic control prob-

9. $\hat{\Lambda}$ is also assumed to be conservative.

lem of the government, and define a Markov perfect equilibrium. All technical details are relegated to the appendix, in section A.1.6.

The payoff-relevant variables for the sovereign government and creditors are s_t , Y_t and F_t . The state space will be $\{1, \dots, N_s\} \times \mathbb{R}^2$, or a subset thereof. An admissible issuance policy \mathbf{I} will be a set of N_s functions $I_i(Y, F)$ that satisfy a particular integrability condition, and an admissible default policy $\boldsymbol{\tau}$ will be a sequence of increasing stopping times $\{\tau_k\}_{k \geq 1}$ that can be written as first hitting times of a particular subset of the state space. I will also note $\{\tau_{e,k}\}_{k \geq 1}$ the sequence of capital markets' re-entry delays, in other words the sequence of independent exponentially distributed time lengths spent by the country in autarky. I will note \mathcal{I} the set of admissible issuance policies, and \mathcal{T} the set of admissible default policies.

For any given admissible default policy $\boldsymbol{\tau} \in \mathcal{T}$, there is an associated controlled output process $Y^{(\boldsymbol{\tau})}$, which follows (1.1) at all times except when a default occurs, at which point $Y^{(\boldsymbol{\tau})}$ suffers a downward jump. For any given admissible issuance policy $\mathbf{I} \in \mathcal{I}$, and default policy $\boldsymbol{\tau} \in \mathcal{T}$, there is an associated controlled debt face value process $F^{(\mathbf{I}, \boldsymbol{\tau})}$, which follows (1.7) at all times except when a default occurs, at which point the aggregate debt face value stays unchanged, until reset at a lower level according to (1.9).

Creditors price the sovereign debt rationally. If they anticipate that the government will follow admissible policy $(\mathbf{I}, \boldsymbol{\tau}) \in \mathcal{I} \times \mathcal{T}$, they will value one unit of debt of a government currently performing under its contractual obligations as follows:

$$D_i(Y, F; (\mathbf{I}, \boldsymbol{\tau})) := \hat{\mathbb{E}}^{i, Y, F} \left[\int_0^\tau e^{-\int_0^t (r_{su} + m) du} (\kappa + m) dt \right. \\ \left. + e^{-\int_0^\tau (r_{su} + m) du} D_{s_\tau}^d \left(Y_{\tau-}^{(\boldsymbol{\tau})}, F_{\tau-}^{(\mathbf{I}, \boldsymbol{\tau})}; (\mathbf{I}, \boldsymbol{\tau}) \right) \right] \quad (1.12)$$

The stopping time τ in the equation above refers to the first element of the sequence of default times $\boldsymbol{\tau}$. The superscript notation next to the expectation operator denotes the

conditioning on the initial state. $D_i^d(\cdot, \cdot; (\mathbf{I}, \boldsymbol{\tau}))$ is the debt price in default, which satisfies:

$$D_i^d(Y, F; (\mathbf{I}, \boldsymbol{\tau})) := \hat{\mathbb{E}}^{i, Y, F} \left[e^{-\int_0^{\tau_e} r_{su} du} \frac{F_{\tau_e}^{(\mathbf{I}, \boldsymbol{\tau})}}{F} D_{s_{\tau_e}} \left(Y_{\tau_e}^{(\boldsymbol{\tau})}, F_{\tau_e}^{(\mathbf{I}, \boldsymbol{\tau})}; (\mathbf{I}, \boldsymbol{\tau}) \right) \right] \quad (1.13)$$

The stopping time τ_e in (1.13) refers to the first capital markets' re-entry delay of the sequence $\{\tau_{e,k}\}_{k \geq 1}$. I use a notation that makes the dependence of the debt price functions on the anticipated issuance and default policies explicit. Equation 1.12 and 1.13 can be interpreted as follows: creditors receive cash-flows $\kappa + m$ per unit of time on a debt balance that amortizes exponentially at rate m . Following a default, creditors receive no cash-flows for the exponentially distributed random time τ_e , following which their claim face value suffers a haircut. The expectations are taken under the risk-neutral measure \mathbb{Q} .

I then focus on the government life-time utility. Given a debt price schedule $\mathbf{D} := \{D_i(\cdot, \cdot)\}_{i \leq N_s}$ that the government faces, and given admissible issuance and default policies $(\mathbf{I}, \boldsymbol{\tau})$ used by the government (where $(\mathbf{I}, \boldsymbol{\tau})$ might not necessarily be consistent with the debt prices \mathbf{D}), there is a controlled flow consumption process $C_t^{(\mathbf{I}, \boldsymbol{\tau}; \mathbf{D})}$, which satisfies (1.8) when the government is performing, and which is equal to output whenever the government is in default. This leads to the following government life-time utility:

$$J_i(Y, F; (\mathbf{I}, \boldsymbol{\tau}); \mathbf{D}) = \mathbb{E}^{i, Y, F} \left[\int_0^\infty \varphi \left(C_t^{(\mathbf{I}, \boldsymbol{\tau}; \mathbf{D})}, J_{st} \left(Y_t^{(\boldsymbol{\tau})}, F_t^{(\mathbf{I}, \boldsymbol{\tau})}; (\mathbf{I}, \boldsymbol{\tau}); \mathbf{D} \right) \right) dt \right] \quad (1.14)$$

In the time-separable preference case, the life-time utility takes the more familiar form:

$$J_i(Y, F; (\mathbf{I}, \boldsymbol{\tau}); \mathbf{D}) = \mathbb{E}^{i, Y, F} \left[\int_0^\infty \delta e^{-\delta t} \frac{\left(C_t^{(\mathbf{I}, \boldsymbol{\tau}; \mathbf{D})} \right)^{1-\gamma}}{1-\gamma} dt \right] \quad (1.15)$$

In both cases, the expectations are taken under the physical probability measure \mathbb{P} . The government takes as given the family of debt price functions \mathbf{D} and \mathbf{D}^d and chooses its

issuance and default policies in order to solve the following problem:

$$V_i(Y, F; \mathbf{D}) := \sup_{(\mathbf{I}, \boldsymbol{\tau}) \in \mathcal{I} \times \mathcal{T}} J_i(Y, F; (\mathbf{I}, \boldsymbol{\tau}); \mathbf{D}) \quad (1.16)$$

When choosing its issuance policy, the government takes into account the debt price schedule and the impact that such schedule has on flow consumption, via the resource constraint. Consistent with [37], I then define a Markov perfect equilibrium as follows.

Definition 3. *A Markov perfect equilibrium is a set of Markovian issuance and default policies $(\mathbf{I}^*, \boldsymbol{\tau}^*) \in \mathcal{I} \times \mathcal{T}$ such that for any initial state (i, Y, F) ,*

$$(\mathbf{I}^*, \boldsymbol{\tau}^*) = \arg \max_{(\mathbf{I}, \boldsymbol{\tau}) \in \mathcal{I} \times \mathcal{T}} J_i(Y, F; (\mathbf{I}, \boldsymbol{\tau}); \mathbf{D}(\cdot, \cdot; (\mathbf{I}^*, \boldsymbol{\tau}^*)))$$

For a given equilibrium $(\mathbf{I}^*, \boldsymbol{\tau}^*)$, I will note $V_{st}(Y_t, F_t)$ the government's equilibrium value function when performing, and $V_{s_\tau}^d(Y_{\tau-}, F_{\tau-})$ the government's equilibrium value function at default time τ , when the pre-default output is equal to $Y_{\tau-}$ and the pre-default debt face value is equal to $F_{\tau-}$. The following set of lemmas will help narrow down the class of Markov perfect equilibria I will be focusing on.

Lemma 4. *If for each state $i \leq N_s$, the debt price schedule $D_i(\cdot, \cdot)$ is homogeneous of degree zero and decreasing in F , then the life-time utility $V_i(\cdot, \cdot; \mathbf{D})$ is strictly increasing in Y and strictly decreasing in F . In such case, the optimal issuance policy is homogeneous of degree one and the optimal government default policy is a state-dependent barrier policy, in other words there exists a set of positive cutoffs $\{\bar{x}_i\}_{i \leq N_s}$ such that $\tau_{k+1} = \inf\{t \geq \tau_k + \tau_{e,k} : F_t \geq \bar{x}_{s_t} Y_t\}$ (with $\tau_0 = \tau_{e,0} = 0$). Finally, the life-time utilities $V_i(\cdot, \cdot; \mathbf{D})$ are homogeneous of degree $1 - \gamma$.*

The proof of this lemma is detailed in section A.1.7. I then focus on the debt price schedule for specific types of issuance and default policies.

Lemma 5. *If $\mathbf{I} \in \mathcal{I}$ is a homogeneous of degree 1 Markov issuance policy, and if $\boldsymbol{\tau} \in \mathcal{T}$ is a barrier default policy, the debt price functions $D_i(\cdot, \cdot)$ are homogeneous of degree zero and decreasing in F .*

The proof can be found in section A.1.8. As discussed in the next section, by restricting the set of equilibria of focus, lemma 4 and lemma 5 will enable me to reduce the dimensionality of the state space and deal with only one continuous and one discrete state variables.

1.3.3 Equilibrium Debt Value

Using the previous observations, I look for an equilibrium of the model for which $x_t := F_t/Y_t$ (the debt-to-output ratio) and s_t are the unique state variables, and for which the government follows a barrier policy: it defaults when the debt-to-output ratio x_t is at or above a state-dependent threshold \bar{x}_{s_t} . In other words, the sovereign's first time of default is $\tau := \inf\{t \geq 0 : x_t \geq \bar{x}_{s_t}\}$. The government issuance policy can be re-written $I_t = \iota_{s_t}(x_t)Y_t$, where $\iota_{s_t}(x_t)$ represents the rate of debt issuance per unit of output, for a given debt-to-output ratio and when the discrete Markov state is s_t . $\iota > 0$ means that the government is either decumulating net foreign assets (when $x < 0$) or borrowing (when $x > 0$), whereas $\iota < 0$ means that the government is buying back outstanding debt. The dynamic evolution of the controlled stochastic process x_t (under the measure \mathbb{P}) when the government is performing under its debt obligations is as follows:

$$dx_t^{(\iota, \boldsymbol{\tau})} = \left(\iota_{s_t} \left(x_t^{(\iota, \boldsymbol{\tau})} \right) - \left(m + \mu_{s_t} - |\boldsymbol{\sigma}_{s_t}|^2 \right) x_t^{(\iota, \boldsymbol{\tau})} \right) dt - x_t^{(\iota, \boldsymbol{\tau})} \boldsymbol{\sigma}_{s_t} \cdot d\mathbf{B}_t$$

The debt-to-GDP ratio increases with the issuance rate ι_t and with the Itô term $|\boldsymbol{\sigma}_{s_t}|^2 x_t$, and decreases thanks to GDP growth $\mu_{s_t} x_t$ and debt amortizations $m x_t$. Under the risk-neutral measure \mathbb{Q} , following Girsanov's theorem, the drift of x_t must be adjusted upward by $\boldsymbol{\nu}_{s_t} \cdot \boldsymbol{\sigma}_{s_t} x_t$. Creditors take the government issuance policy ι and the government default policy as given when pricing a unit of sovereign debt. Finally, I will postulate (and verify)

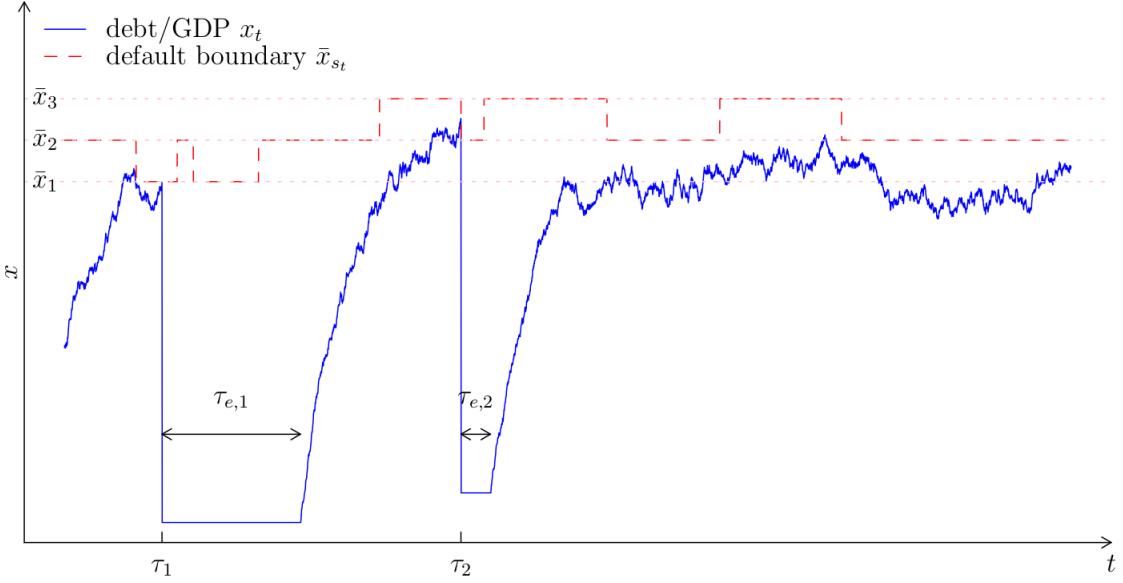


Figure 1.1: Debt-to-GDP ratio x_t

that in equilibrium, $\iota_i(0) > 0$ for all states $i \leq N_s$. This means that when the government has neither financial assets nor financial liabilities, it finds it optimal to borrow and front-load consumption in all states $i \leq N_s$. This also means that once the state x_t enters the interval $(0, \max_i \bar{x}_i)$, it never leaves such interval, since the diffusion term in the stochastic differential equation for x_t vanishes and the drift term is strictly positive. I thus restrict the focus of my analysis to the state space $\{1, \dots, N_s\} \times (0, \max_i \bar{x}_i)$.

A random realization of the state variables x_t and s_t is illustrated in figure 1.1. Defaults occur at times τ_1 and τ_2 . The length of time spent in autarky after the i^{th} default is $\tau_{e,i}$, after which the aggregate debt face amount is reset at a fraction of its pre-default value.

With an abuse of notation, I use $D_i(x; (\boldsymbol{\iota}, \boldsymbol{\tau}))$ (resp. $D_i^d(x; (\boldsymbol{\iota}, \boldsymbol{\tau}))$) to denote the debt value (resp. the debt value in default) per dollar of face-value when the debt-to-output ratio is x and the SDF regime is i . I will also omit the dependence of the debt price function on the government policies $(\boldsymbol{\iota}, \boldsymbol{\tau})$ whenever possible. When s_t is in state i and when $x \in [0, \bar{x}_i]$,

the government is indebted, and the price $D_i(\cdot; (\boldsymbol{\iota}, \boldsymbol{\tau}))$ of defaultable sovereign debt verifies:

$$D_i(x) = \hat{\mathbb{E}}^{i,x} \left[\int_0^\tau e^{-\int_0^t (r_{su} + m) du} (\kappa + m) dt + e^{-\int_0^\tau (r_{su} + m) du} D_{s_\tau}^d(x_{\tau-}) \right] \quad (1.17)$$

Using Feynman-Kac, it is immediate to show that D_i is twice differentiable and satisfies the following HJB equation for $x \in (0, \bar{x}_i)$:

$$(r_i + m) D_i(x) = \kappa + m + \hat{\mathcal{L}}_i^{(\boldsymbol{\iota})} D_i(x) + \sum_{j=1}^{N_s} \hat{\Lambda}_{ij} D_j(x) \quad (1.18)$$

For ease of notation, I have introduced the infinitesimal operator $\hat{\mathcal{L}}_i^{(\boldsymbol{\iota})}$ as follows:

$$\hat{\mathcal{L}}_i^{(\boldsymbol{\iota})} := \left[\iota_i(x) - \left(\mu_i + m - |\boldsymbol{\sigma}_i|^2 - \boldsymbol{\nu}_i \cdot \boldsymbol{\sigma}_i \right) x \right] \frac{\partial}{\partial x} + \frac{1}{2} |\boldsymbol{\sigma}_i|^2 x^2 \frac{\partial^2}{\partial x^2}$$

$2 \times N_s$ boundary conditions are required in order to solve this set of N_s nested second order ordinary differential equations. They are as follows, for $1 \leq i \leq N_s$:

$$D_i(\bar{x}_i) = D_i^d(\bar{x}_i) \quad (1.19)$$

$$(r_i + m) D_i(0) = \kappa + m + \iota_i(0) D_i'(0) + \sum_{j=1}^{N_s} \hat{\Lambda}_{ij} D_j(0) \quad (1.20)$$

For each state i , the first boundary condition is a value matching condition, which says that the debt price at the default boundary $x = \bar{x}_i$ is equal to the price of a claim on the defaulted debt, $D_i^d(\bar{x}_i)$ (which will be calculated later on). The second boundary condition is a Robin boundary condition; it relates the value of the function D_i at the origin to its first derivative at the origin. It can be obtained by simply taking a limit of the HJB equation satisfied by D_i at $x = 0$. I need to compute the debt price in default $D_i^d(x)$, for $x \geq \bar{x}_i$ and $1 \leq i \leq N_s$. Assume that at time of default τ , the state is $s_\tau = i$. When the country exits financial autarky, its debt-to-GDP ratio is equal to $\frac{F_{\tau+\tau_e}}{Y_{\tau+\tau_e}} = \theta \frac{F_{\tau-}}{Y_{\tau-}} = \theta x_{\tau-}$. Note that it is

possible that $x_{\tau-} > \bar{x}_{s_\tau}$ when the sovereign defaults. This happens upon the occurrence of a “jump-to-default”, in other words a situation where the discrete SDF state jumps from $s_{\tau-} = j$ to $s_\tau = i$ and when $\bar{x}_i < x_{\tau-} < \bar{x}_j$. Thus, I have the following for $x \geq \bar{x}_i$:

$$D_i^d(x) = \hat{\mathbb{E}}^i \left[\exp \left(- \int_0^{\tau_e} r_{s_{t+u}} du \right) \frac{F_{t+\tau_e}}{F_{t-}} D_{s_{t+\tau_e}}(\theta x) \right]$$

section A.1.9 establishes the following formula for the defaulted debt price:

$$\mathbf{D}^d(x) = \lambda \theta \alpha \Xi^{-1} \mathbf{D}(\theta x) \quad (1.21)$$

In equation 1.21, $\mathbf{D}^d(x)$ is the $N_s \times 1$ vector with i^{th} element $D_i^d(x)$, and the $N_s \times N_s$ matrix $\Xi := \text{diag}_i(r_i + \boldsymbol{\nu}_i \cdot \boldsymbol{\sigma}_i + \lambda - \mu_i) - \hat{\Lambda}$ is well defined thanks to assumption 2. Finally, note that this equation is valid for each coordinate i for $x \geq \bar{x}_i$.

I end this section by discussing two different aspects of the model. First, the existence of a discrete number of SDF regimes leads to two types of defaults: defaults following a sequence of bad GDP shocks, as well as defaults induced by jumps in the SDF state, from a state of low risk prices to a state of higher risk prices. Both types are illustrated in figure 1.1. In this example, a default occurs at τ_1 , after a sequence of bad GDP shocks that cause the debt-to-GDP ratio to breach the optimal default boundary that the government has set in such SDF regime. In the same figure, a default occurs at τ_2 , triggered by a jump in the SDF state. At such time, an SDF regime shift occurs, from $s_{\tau_2-} = 3$ to $s_{\tau_2} = 2$, and the debt-to-GDP ratio satisfies $\bar{x}_3 > x_{\tau_2-} > \bar{x}_2$. In other words, before the SDF jump, the debt-to-GDP ratio of the sovereign is below the optimal default boundary, but as the SDF regime shifts, the debt-to-GDP ratio is suddenly greater than the new optimal default boundary, causing the sovereign to immediately default. Since the SDF I use will price the sovereign debt of multiple countries, SDF regime shifts induce correlated defaults amongst sovereign governments. Note that jump-to-default risk exists even if the GDP growth rate and GDP growth volatility are not regime-dependent – so long as the SDF exhibits different

risk prices in different regimes.

Second, note that when $x \searrow 0$, the government debt balance is negligible compared to output. However, the price of any infinitesimally small unit of debt is actually not equal to the risk-free debt price, since the debt price needs to factor in the dilution risk of the government, whose optimal issuance policy will dictate to issue debt to front-load consumption. This observation is in stark contrast with what happens in structural corporate credit risk models (see for example [31] or [32]) – in those models, the firm can commit to a financing policy (typically, maintaining the debt face value constant), but cannot commit to a default policy, leading to a debt price that is equal to the risk-free debt price when the level of fundamentals becomes arbitrarily large compared to the debt face value.

1.3.4 Equilibrium Debt Issuance and Default Policies

Now consider the government's problem, as described in section 1.3.2. As a reminder, the government takes the debt price schedule $\mathbf{D}(\cdot)$ as given when solving its optimization problem. Thanks to lemma 4, the government value function in state i can be written as follows:

$$V_i(Y, F) := v_i(x)Y^{1-\gamma} \quad (1.22)$$

In the above, the function v_i will be positive when $\gamma \in (0, 1)$, and negative when $\gamma > 1$. Since V_i is decreasing in F , I also have the sign restriction $v'_i(x) < 0$. An appropriate change-in-measure described in section A.1.10 shows that the HJB equation associated with the government problem, in the continuation region $[0, \bar{x}_i)$, is the following:

$$\begin{aligned} \frac{1-\gamma}{1-\rho} A_i v_i(x) - \sum_{j=1}^{N_s} \Lambda_{ij} v_j(x) = \\ \sup_{\iota_i} \left[\delta \frac{(1 + \iota_i D_i(x) - (\kappa + m)x)^{1-\rho} [(1-\gamma)v_i(x)]^{\frac{\rho-\gamma}{1-\gamma}}}{1-\rho} + \mathcal{L}_i^{(\iota)} v_i(x) \right] \end{aligned} \quad (1.23)$$

In the above, I have used the differential operator $\mathcal{L}_i^{(\iota)}$ defined as follows:

$$\mathcal{L}_i^{(\iota)} := \left[\iota_i - \left(\mu_i + m - \gamma |\boldsymbol{\sigma}_i|^2 \right) x \right] \frac{\partial}{\partial x} + \frac{1}{2} |\boldsymbol{\sigma}_i|^2 x^2 \frac{\partial^2}{\partial x^2}$$

The optimal state-contingent issuance policy ι_i is then given by:

$$\max_{\iota_i} \left[\frac{\delta}{1 - \rho} (1 + \iota_i D_i(x) - (\kappa + m)x)^{1-\rho} [(1 - \gamma)v_i(x)]^{\frac{\rho-\gamma}{1-\gamma}} + \iota_i v'_i(x) \right]$$

This yields the (necessary and sufficient, given the strict concavity of the expression in brackets w.r.t. ι_i) first order condition:

$$D_i(x) \delta c_i(x)^{-\rho} [(1 - \gamma)v_i(x)]^{\frac{\rho-\gamma}{1-\gamma}} = -v'_i(x) \quad (1.24)$$

In the above, I have introduced the consumption-to-GDP ratio $c_i := C/Y$ when the discrete Markov state is i . Focusing on (1.24), I notice that the left-hand side is the product of (a) the marginal utility of consumption $\delta c_i(x)^{-\rho} [(1 - \gamma)v_i(x)]^{\frac{\rho-\gamma}{1-\gamma}}$ and (b) the debt price, while the right-hand side is the marginal cost of taking on one extra unit of debt. The optimal Markov issuance policy function $\iota_i(x)$ is given by:

$$\iota_i(x) = \frac{1}{D_i(x)} \left[\left(\frac{\delta D_i(x) [(1 - \gamma)v_i(x)]^{\frac{\rho-\gamma}{1-\gamma}}}{-v'_i(x)} \right)^{1/\rho} + (\kappa + m)x - 1 \right] \quad (1.25)$$

The expression is well defined since I showed previously that $v'_i(x) < 0$. The dependence of the issuance policy on the model parameters or on the debt price schedule (which the government takes as given) are ambiguous, since those issuance parameters will also have a feedback effect on the felicity function and its derivative. I can however perform a “partial equilibrium” analysis of the debt price schedule in the unit elasticity of substitution case, i.e. when $\rho = 1$. In such case, $\iota_i(x)$ is an increasing function of $D_i(x)$ whenever the sovereign output Y_t is greater than the total debt service owed $(\kappa + m)F_t$, which will always be the

case in equilibrium (in other words in equilibrium, the sovereign will have defaulted before the sovereign output falls low enough that new debt issuances are required to service the existing debt). For the case where the elasticity of substitution is different from 1, I verify numerically that this comparative static result still holds: when the debt price schedule is more beneficial to the sovereign, the latter takes advantage of it through additional issuances.

For a given set of default thresholds $\{\bar{x}_i\}_{i \leq N_s}$, $2 \times N_s$ additional boundary conditions are needed to solve the system of N_s equations 1.23. The first set of conditions relates to value matching at the default boundary \bar{x}_i . Let $V_i^d(Y, F)$ be the government value function in default, if the pre-default output level is Y and the pre-default debt face value is F . I show in section A.1.11 that $V_i^d(Y, F) = v_i^d(\bar{x}_i) (\alpha Y)^{1-\gamma}$, which leads to the following value-matching condition:

$$v_i(\bar{x}_i) = \alpha^{1-\gamma} v_i^d(\bar{x}_i) \quad (1.26)$$

$v_i^d(\bar{x})$ solves a system of non-linear equations discussed in section A.1.11. I also have a set of N_s Robin boundary conditions, linking the value function at the origin to its derivative, via:

$$\frac{1-\gamma}{1-\rho} A_i v_i(0) - \sum_{j=1}^{N_s} \Lambda_{ij} v_j(0) = \delta \frac{(1 + \iota_i(0) D_i(0))^{1-\rho} [(1-\gamma)v_i(0)]^{\frac{\rho-\gamma}{1-\gamma}}}{1-\rho} + \iota_i(0) v_i'(0) \quad (1.27)$$

I finally focus on the optimal default policy. Since it is always an option for the government to default, I must have $V_i(Y, F) - V_i^d(Y, F) \geq 0$ for all states (Y, F) . This leads to a set of N_s smooth-pasting conditions:

$$v_i'(\bar{x}_i) = \alpha^{1-\gamma} (v_i^d)'(\bar{x}_i) \quad (1.28)$$

section A.1.11 establishes more formally this optimality condition and shows how $(v_i^d)'(\bar{x}_i)$ can be expressed as a function of $v_i^d(\bar{x}_i)$ and $v_i(\theta \bar{x}_i)$. I conclude this section by two propositions. First, I establish a standard verification theorem for the government value function.

I then discuss the existence of a Markov perfect equilibrium, subject to the existence of a solution to a set of ordinary differential equations.

Proposition 6. *For any family of decreasing functions $D_i : \mathbb{R}^+ \rightarrow [0, D_i^{rf}]$, assume that there exists a family of functions $v_i(\cdot; \mathbf{D}) \in \mathcal{C}^1(\mathbb{R}^+) \cap \mathcal{C}^2(\mathbb{R}^+ \setminus \{\bar{x}_i\})$, which satisfies for $1 \leq i \leq N_s$:*

$$0 = \max \left[\sup_{\boldsymbol{\iota}} \left[-\frac{1-\gamma}{1-\rho} A_i v_i(x; \mathbf{D}) + \sum_{j=1}^{N_s} \Lambda_{ij} v_j(x; \mathbf{D}) \right. \right. \\ \left. \left. + \delta \frac{(1 + \iota_i D_i(x) - (\kappa + m)x)^{1-\rho} [(1-\gamma)v_i(x; \mathbf{D})]^{\frac{\rho-\gamma}{1-\gamma}}}{1-\rho} + \mathcal{L}^{(\boldsymbol{\iota})} v_i(x; \mathbf{D}) \right] ; \right. \\ \left. \alpha^{1-\gamma} v_i^d(x; \mathbf{D}) - v_i(x; \mathbf{D}) \right],$$

where $\mathbf{v}^d(x; \mathbf{D})$ satisfies (using the $N_s \times N_s$ matrix $\Upsilon := \frac{1-\gamma}{1-\rho} \text{diag}_i(A_i) + \lambda I - \Lambda$):

$$\Upsilon \mathbf{v}^d(x; \mathbf{D}) - \lambda \mathbf{v}(\theta x; \mathbf{D}) = \frac{\delta}{1-\rho} \left[(1-\gamma) \mathbf{v}^d(x; \mathbf{D}) \right]^{\frac{\rho-\gamma}{1-\gamma}},$$

Then for any state $i \leq N_s$ and any $x \in \mathbb{R}^+$, $v_i(x; \mathbf{D}) \geq J_i(1, x; (\boldsymbol{\iota}, \boldsymbol{\tau}); \mathbf{D})$ for any $(\boldsymbol{\iota}, \boldsymbol{\tau}) \in \mathcal{I} \times \mathcal{T}$ that satisfy $\lim_{t \rightarrow +\infty} \inf e^{-\int_0^t \frac{1-\gamma}{1-\rho} A_{su} du} v_{st} \left(x_t^{(\boldsymbol{\iota}, \boldsymbol{\tau})}; \mathbf{D} \right) \leq 0$. Let the family of thresholds $\{\bar{x}_i\}_{1 \leq i \leq N_s} \in (\mathbb{R}^+)^{N_s}$ satisfy:

$$\left(\mathbf{v}^d \right)'(\bar{x}_i) = \lambda \theta \left(\Upsilon + \delta \frac{\gamma - \rho}{1-\rho} \text{diag}_j \left(\left[(1-\gamma) v_j^d(\bar{x}_i) \right]^{-\frac{1-\rho}{1-\gamma}} \right) \right)^{-1} \mathbf{v}'(\theta \bar{x}_i)$$

Let $(\boldsymbol{\iota}^*, \boldsymbol{\tau}^*)$ be defined as follows:

$$\iota_i^*(x; \mathbf{D}) := \frac{1}{D_i(x)} \left[\left(\frac{\delta D_i(x) [(1-\gamma)v_i(x; \mathbf{D})]^{\frac{\rho-\gamma}{1-\gamma}}}{-v_i'(x; \mathbf{D})} \right)^{1/\rho} + (\kappa + m)x - 1 \right]$$

$$\tau^*(\mathbf{D}) := \inf \{t \geq 0 : x_t \geq \bar{x}_{st}\}$$

Then $v_i(x; \mathbf{D}) = J_i(1, x; (\boldsymbol{\iota}^*, \boldsymbol{\tau}^*); \mathbf{D})$ is the value function.

This proposition, proven in section A.1.12, provides for a characterization of the optimal issuance and default policies *given* a decreasing debt price schedule \mathbf{D} . It does not establish the existence of an equilibrium, which is achieved in the next proposition.

Proposition 7. *Assume that there exists a set of functions $\{v_i(\cdot)\}_{i \leq N_s}$, $\{D_i(\cdot)\}_{i \leq N_s}$, and a set of positive thresholds $\{\bar{x}_i\}_{i \leq N_s}$ such that the system of nested ordinary differential equations (1.18), (1.23) subject to value-matching boundary conditions (1.19), (1.20), (1.26) and (1.27) are satisfied, where $\iota_i(\cdot)$ satisfies (1.25) and each threshold \bar{x}_i satisfies the smooth pasting condition (1.28). Then a Markov perfect equilibrium exists.*

Proving the existence of a Markov perfect equilibrium without relying on the (strong) assumptions of proposition 7 is beyond the scope of this paper, and I leave this proof for future research. I provide in section A.1.13 a discussion of the potential route to pursue to establish such result. I also show in section 1.4, for the particular case where $\rho = \gamma = 0$, that a Markov perfect equilibrium exists, and it is unique in the class of “smooth” equilibria (i.e. equilibria in which the debt face value process is restricted to being absolutely continuous).

1.3.5 Asset Pricing Moments

In this section, I discuss the implications of my model for the long term sovereign bond spread, as well as excess returns earned by international investors on such bond. I also show how to compute CDS premia and the excess return on these contracts.

Long Term Sovereign Debt Spreads

The sovereign bond spread $\varsigma_i(x)$ is the constant margin over the risk-free benchmark that is needed to discount the long-term sovereign bond’s cash flow stream assuming away any

default risk. In other words, the credit spread must verify:

$$D_i(x) := \hat{\mathbb{E}}^{i,x} \left[\int_0^\infty e^{-\int_0^t (r_{su} + \varsigma_i(x) + m) du} (\kappa + m) dt \right] \quad (1.29)$$

The credit spread $\varsigma_i(x)$ is the unique positive solution to the following equation:

$$D_i(x) = (m + \kappa) \left[\left(\text{diag}_j (r_j + \varsigma_i(x) + m) - \hat{\Lambda} \right)^{-1} \mathbf{1} \right]_i$$

Using Itô's lemma, credit spread innovations under \mathbb{P} take the following form:

$$\begin{aligned} d\varsigma_t - \mathbb{E}[d\varsigma_t | \mathcal{F}_t] &= -\varsigma'_{s_{t-}}(x_t) x_t \boldsymbol{\sigma}_{s_{t-}} \cdot d\boldsymbol{B}_t \\ &\quad + \sum_{s'} (\varsigma'_{s'}(x_t) - \varsigma'_{s_{t-}}(x_t)) \left(dN^{(s_{t-}, s')} - \Lambda_{s_{t-}, s'} dt \right) \end{aligned} \quad (1.30)$$

What happens upon the occurrence of a GDP shock? section A.1.15 establishes that $\varsigma'_i < 0$ in any state i . Thus, good GDP shocks translate into decreases in sovereign bond spreads. In other words, credit spreads are counter-cyclical in this model – a sequence of good GDP shocks will on average lead to lower spreads, consistent with empirical fact (4).

I can then leverage (1.30) to compute the instantaneous sovereign bond spread volatility:

$$\sigma_t^\varsigma = \sqrt{x_t^2 |\boldsymbol{\sigma}_{s_{t-}}|^2 \varsigma'_{s_{t-}}(x_t)^2 + \sum_{s'} \Lambda_{s_{t-}, s'} (\varsigma'_{s'}(x_t) - \varsigma'_{s_{t-}}(x_t))^2} \quad (1.31)$$

In a model without SDF regime shifts, sovereign spread volatilities are purely driven by the macroeconomic fundamentals of a country (in the context of this model, the debt-to-GDP ratio x). Instead, SDF regime shifts in my model induce an additional component to sovereign spread volatilities. A separate testable implication emerges from (1.31): spread volatilities tend to be higher when the sovereign government is close to its endogenous default boundary. Indeed, I show in the appendix that under mild conditions, the function $x\varsigma'_i(x)$ is increasing, meaning that the component of sovereign spread volatility stemming from

Brownian shocks increases as the sovereign government approaches its default cutoff. Both predictions will be tested as part of my model validation.

Equation 1.30 also illustrates the crucial importance of the different SDF regimes for cross-sectional spread correlations: absent those regime shifts, pairwise local spread correlation between two different sovereign governments would only stem from output correlation, which is at odds with fact (2). If I index by “a” and “b” two countries, the instantaneous spread correlation between those countries takes the following form:

$$\text{corr}_t(\varsigma_{a,t}, \varsigma_{b,t}) = \frac{\varsigma'_{a,s_{t-}} \varsigma'_{b,s_{t-}} x_{a,t} x_{b,t} \boldsymbol{\sigma}_{a,s_{t-}} \cdot \boldsymbol{\sigma}_{b,s_{t-}}}{\sigma_{a,t}^{\varsigma} \sigma_{b,t}^{\varsigma}} + \frac{\sum_{s'} \Lambda_{s_{t-}s'} (\varsigma_{a,s'} - \varsigma_{a,s_{t-}}) (\varsigma_{b,s'} - \varsigma_{b,s_{t-}})}{\sigma_{a,t}^{\varsigma} \sigma_{b,t}^{\varsigma}}$$

In the formula above, for all states $i \leq N_s$, the function $\varsigma'_{a,i}$ is evaluated at $x_{a,t}$ and the function $\varsigma'_{b,i}$ is evaluated at $x_{b,t}$. When the SDF state jumps from a low risk price level s to a high risk price level s' , if both countries’ output processes are positively correlated with the risk price vector in all discrete Markov states, spreads for both country “a” and country “b” jump up, meaning that $(\varsigma_{a,s'} - \varsigma_{a,s_{t-}}) (\varsigma_{b,s'} - \varsigma_{b,s_{t-}}) > 0$. The same reasoning holds upon a jump from a high risk price state to a low risk price state. Thus, the second term in my formula for spread correlations above is positive: spread correlations are induced by SDF regime shifts. This gives my model the potential for being consistent with fact (2) – but only to the extent my countries of interest have output processes whose correlation with the vector of risk prices have the same sign.

Long Term Sovereign Debt Returns

I then compute sovereign debt excess returns. Debt excess returns over the time period $(t, t + dt]$ include capital gains dD_t , coupon payments κdt and principal repayments mdt , while the opportunity cost is $r_{st} dt$ and reinvestment costs are equal mdt . Thus, excess

returns (under the physical measure \mathbb{P}) are equal to:

$$dR_t^e := \frac{dD_t + (\kappa + m)dt}{D_t} - (r_{st} + m)dt$$

Using Itô's lemma and the HJB equation satisfied by the family of debt values $\{D_i(\cdot)\}_{i \leq N_s}$, I obtain expected excess returns (per unit of time) and return volatilities that are equal to:

$$\mathbb{E}[dR_t^e | \mathcal{F}_t] = - \left[\frac{x_t D'_{st}(x_t)}{D_{st}(x_t)} \boldsymbol{\nu}_{st} \cdot \boldsymbol{\sigma}_{st} + \sum_{s'} \Lambda_{sts'} \left(\frac{D_{s'}(x_t)}{D_{st}(x_t)} - 1 \right) \left(e^{v(st, s')} - 1 \right) \right] dt \quad (1.32)$$

$$\text{var}[dR_t^e | \mathcal{F}_t] = \frac{x_t^2 D'_{st}(x_t)^2}{D_{st}(x_t)^2} |\boldsymbol{\sigma}_{st}|^2 dt + \sum_{s'} \Lambda_{sts'} \left(\frac{D_{s'}(x_t)}{D_{st}(x_t)} - 1 \right)^2 dt \quad (1.33)$$

Thus, sovereign bond investors are compensated for taking Brownian risk (the first term on the right hand-side of (1.32)), as well as for taking regime jump risk (the second term on the right hand-side of (1.32)). The expected excess return can be read as (minus) the local covariance between (a) sovereign debt returns and (b) the creditors' pricing kernel. This risk compensation is similar to a standard two-factor asset pricing compensation. Indeed, I can interpret $\frac{-x_t D'_{st}(x_t)}{D_{st}(x_t)}$ as the market beta of sovereign debt w.r.t. the shock \mathbf{B}_t , while $\boldsymbol{\nu}_{st} \cdot \boldsymbol{\sigma}_{st}$ is the sovereign output claim's risk premium earned in connection with such shock. Similarly, the jump compensation (the second term in (1.32)) can be re-written:

$$\sum_{s'} \Lambda_{sts'} \left(e^{v(st, s')} - 1 \right) \left(\frac{P_{s'}}{P_{st}} - 1 \right) \underbrace{\left(\frac{\frac{D_{s'}(x_t)}{D_{st}(x_t)} - 1}{\frac{P_{s'}}{P_{st}} - 1} \right)}_{\text{output claim's premium for jump risk market beta of sovereign debt w.r.t. jump risk}}$$

In the above, P_i is the price of a claim to the output of country i . Using the vector notation, $\mathbf{P} = [\text{diag}_i(r_i + \boldsymbol{\nu}_i \cdot \boldsymbol{\sigma}_i - \mu_i) - \hat{\Lambda}]^{-1} \mathbf{1}$. Alternatively, one can interpret those formulas using the terminology of [24] or [25]: in such case, the expected excess return in (1.32) is the sum-product of (time-varying) risk prices ($\boldsymbol{\nu}_{st}$ for the Brownian shocks and $(e^{v(s, s')} - 1)$

for jump risks) and (time-varying) risk exposures $(\frac{-x_t D'_{s_t}(x_t)}{D_{s_t}(x_t)} \boldsymbol{\sigma}_{s_t})$ for Brownian shocks and $(\frac{D_{s_t}'}{D_{s_t}} - 1)$ for jumps).

Equation 1.32 highlights the crucial role of the local covariance between risk prices and the GDP process for the determination of expected excess returns. When I tie the investor's SDF to US consumption growth and US consumption volatility (as in section A.1.4), risk prices are equal to the product of (i) US investors' risk-aversion times (ii) US consumption growth volatility. But US output and consumption growth exhibit only mild levels of correlation with emerging market economies' output growth, as documented in table "Country-Specific Macro Moments" in the online appendix. One might then ask how this model might explain the high level of expected excess returns earned on emerging market sovereign risks. Even if the risk price vector $\boldsymbol{\nu}_{s_t}$ is not (locally) correlated with the country's output process, expected excess returns can be positive when risk prices are time-varying and co-move with sovereign debt prices. For this latter effect to "bite", the pricing kernel must feature jumps (i.e. some of the $\{v(i, j)\}_{1 \leq i, j \leq N_s}$ must be non-zero); the introduction of different SDF regimes only does not suffice in order to produce large model-implied expected excess returns when $\boldsymbol{\nu}_{s_t} \cdot \boldsymbol{\sigma}_{s_t} \leq 0$ in all states.

Note also that the risk exposure to Brownian shocks (and the corresponding sovereign debt market beta) depends on the elasticity $\frac{-x D'_i(x)}{D_i(x)}$ of the bond price function. It turns out that in all my numerical computations, the debt price function $D_i(\cdot)$ is a concave function, which means that the sovereign debt's risk exposure to Brownian shocks is increasing in x . This leads to another implication of the model: sovereign expected excess returns are increasing in the debt-to-GDP ratio, consistent with fact (8). This implication was also indirectly tested by [9] when sorting sovereign debt portfolios by (a) rating and (b) "market betas", if one interprets the rating as a noisy measure of the debt-to-GDP ratio. But while [9] argue in the model section of their paper that they would need to introduce two sources of heterogeneity in order to recreate their empirical observation, I argue that this is not necessary: not only different countries may have different business cycle correlations with

foreign investors' risk prices, but also countries may have different *risk exposures*.

To conclude this section on long term debt returns, the properties of sovereign bond return volatilities and cross-country correlations should be identical to those of sovereign spread volatilities and cross-country sovereign spread correlations since realized bond returns between t and $t + dt$ are (approximately) proportional to spread changes during that time period.

Credit Default Swap Premia and Returns

To conclude this section, I define $\varsigma_i(x, T)$, the credit default swap premium for a T maturity contract. Conceptually, such premium should, at the time the trade is executed, compensate the writer of protection for expected losses to be suffered on the contract. Mathematically, $\varsigma_i(x, T)$ is defined as follows:

$$\varsigma_i(x, T) := \frac{\hat{\mathbb{E}}^{x,i} \left[1_{\{\tau < T\}} e^{-\int_0^\tau r_{su} du} \max(0, 1 - D_{s_\tau}^d(x_{\tau-})) \right]}{\hat{\mathbb{E}}^{x,i} \left[\int_0^{T \wedge \tau} e^{-\int_0^t r_{su} du} dt \right]} = \frac{L_i(x, T)}{P_i(x, T)}$$

$L_i(x, T)$ is the risk-neutral expected credit loss, while $P_i(x, T)$ is the risk-neutral present-value of CDS premia. Both expected losses and expected CDS premia can be calculated using the Feynman-Kac formula, by solving a set of partial differential equations with boundary conditions discussed in section A.1.16. section A.1.16 also provides formula for computing expected excess returns and conditional return volatilities of CDS contracts of different maturities. I can then test whether the model-implied term structure of spreads is consistent with facts (5) and (6), and whether the term structure of expected excess returns is consistent with fact (8).

My model with multiple SDF regimes (inducing multiple default boundaries, one per regime) is particularly convenient in analyzing short term CDS premia, and confronting them with the data. Indeed, when the CDS contract maturity is arbitrarily small (i.e. when $T \rightarrow 0$), default risk only stems from the risk of regime shifts. Under the assumption that

the discrete SDF states are ordered (i.e. under the assumption that $\bar{x}_1 \leq \dots \leq \bar{x}_{N_s}$), I then have the following lemma, characterizing short term CDS premia.

Lemma 8. *When the contract maturity T becomes arbitrarily small, the sovereign CDS premium converges to the following limit:*

$$\lim_{T \rightarrow 0} \varsigma_i(x, T) = \begin{cases} 0 & \text{if } x \leq \min_j \bar{x}_j \text{ or } i = 1 \\ \sum_{j=1}^{i-1} \hat{\Lambda}_{ij} \left(1 - D_j^d(x)\right) & \text{otherwise} \end{cases}$$

lemma 8 shows that premium compensation for writers of short term sovereign CDS only comes from SDF jump risk, as opposed to output volatility risk. In other words, a model that does not feature multiple discrete SDF regimes would not feature high short-term market-implied hazard rates, and would thus be inconsistent with stylized fact (3).

1.3.6 Macro Moments

One distinguishing feature of emerging market economies is the fact that the ratio of (a) consumption growth volatility over (b) output growth volatility is substantially greater than one (see for example [40] or [4]). In my model, as established in section A.1.17, this ratio takes the following expression:

$$\frac{\text{stdev} \left[\frac{dC_t}{C_t} \mid \mathcal{F}_t \right]}{\text{stdev} \left[\frac{dY_t}{Y_t} \mid \mathcal{F}_t \right]} = \sqrt{\left(1 - \frac{x_t c'_{st}(x_t)}{c_{st}(x_t)}\right)^2 + \frac{1}{|\sigma_{st}|^2} \sum_{s'} \Lambda_{st,s'} \left(\frac{c_{s'}(x_t)}{c_{st}(x_t)} - 1 \right)^2} \quad (1.34)$$

This ratio will thus crucially depend on how the consumption-to-output ratio $c_i(\cdot)$ varies with the debt-to-GDP ratio. As will be seen, as the debt-to-GDP ratio nears the default boundary, the government will adjust its issuance policy downwards, meaning that the consumption-to-output ratio $c_i(\cdot)$ will be a decreasing function of x . This leads to a consumption growth volatility that will be greater than output growth volatility. Note also that such volatility is

exacerbated by SDF regime shifts.

1.4 A Useful Benchmark: Risk-Neutral Government

The model presented above will need to be solved numerically, but there is one particular parameter configuration where closed-form solutions are available: the case where the government is risk-neutral. Leveraging an insight from [17], I develop in section A.1.18 the solution for this problem in the case where $\rho = \gamma = 0$. This benchmark is useful to provide some intuition on the mechanics of the model.

Proposition 9. *Assume that the government is risk-neutral, i.e. that $\gamma = \rho = 0$. Assume that the risk-free rate in all SDF states is strictly less than the government's rate of time preference, i.e. $r_i < \delta$ for all $i \leq N_s$. Assume also that the GDP process is not regime-specific, in other words assume that for all state $i \leq N_s$, $\mu_i = \mu$ and $\sigma_i = \sigma$. In such case, there exists an equilibrium where the life-time government value function $V_i(Y, F)$, the debt price schedule $D_i(Y, F)$ and the optimal default cutoff \bar{x}_i are independent of the SDF state, and have the following expressions:*

$$D(x) = \left(\frac{\kappa + m}{\delta + m} \right) \left[1 - \left(\frac{1 - \frac{\alpha\theta\lambda}{\delta + \lambda - \mu}}{1 - \frac{\alpha\theta\xi\lambda}{\delta + \lambda - \mu}} \right) \left(\frac{x}{\bar{x}} \right)^{\xi-1} \right] \quad (1.35)$$

$$v(x) = \delta \left[\frac{1}{\delta - \mu} \left(1 - \left(\frac{1 - \alpha}{1 - \frac{\alpha\theta\xi\lambda}{\delta + \lambda - \mu}} \right) \left(\frac{x}{\bar{x}} \right)^\xi \right) - xD(x) \right] \quad (1.36)$$

$$\bar{x} = \frac{\xi}{\xi - 1} \left(\frac{\delta + m}{\kappa + m} \right) \left(\frac{\frac{1 - \alpha}{\delta - \mu}}{1 - \frac{\alpha\theta\lambda}{\delta + \lambda - \mu}} \right) \quad (1.37)$$

In the above, $\xi > 1$ is a constant that only depends on the model parameters δ, μ, σ, m , and not on the level of interest rates or the prices of risk. The scaled optimal financing policy

$\iota_i(x)$ is SDF state-dependent and has the following expression:

$$\iota_i(x) = \frac{\delta - r_i}{\xi - 1} \left[\left(\frac{1 - \frac{\alpha\theta\xi\lambda}{\delta+\lambda-\mu}}{1 - \frac{\alpha\theta\lambda}{\delta+\lambda-\mu}} \right) \left(\frac{\bar{x}}{x} \right)^{\xi-1} - 1 \right] x - \boldsymbol{\nu}_i \cdot \boldsymbol{\sigma} x \quad (1.38)$$

The financing policy $\iota_i(\cdot)$ is a strictly decreasing function of x if $\delta + m > |\boldsymbol{\sigma}|^2 - (m + \mu)$, and is otherwise hump-shaped. Conditional expected excess returns on the long-term bond have the following expression:

$$\mathbb{E}[dR_t^e | \mathcal{F}_t] = \frac{\xi - 1}{\left(\frac{1 - \frac{\alpha\theta\xi\lambda}{\delta+\lambda-\mu}}{1 - \frac{\alpha\theta\lambda}{\delta+\lambda-\mu}} \right) \left(\frac{\bar{x}}{x} \right)^{\xi-1} - 1} \boldsymbol{\nu}_i \cdot \boldsymbol{\sigma} \quad (1.39)$$

The equilibrium above is the unique Markov perfect equilibrium featuring an absolutely continuous debt face value process.

The first – and seemingly surprising – result of proposition 9 is that the (output-normalized) welfare value of a government without any debt outstanding (i.e. $v(0)$) is exactly equal to the autarky welfare $\delta/(\delta - \mu)$. In other words, the fact that the risk-neutral government has the option to take on debt financed by creditors with a discount rate that is strictly less than the rate of time preference of the government is not welfare-improving for such government. Similarly, when the government is indebted, the welfare of the government can be expressed as the sum of (a) the welfare of a debt-free government that suffers a downward GDP drop of $(1 - \alpha)$ % each time the state variable x_t hits the boundary \bar{x} , minus (b) the aggregate value of sovereign debt, computed as if creditors were risk-neutral with a discount rate δ . Importantly, neither the risk-free rates $\{r_i\}_{i \leq N_s}$, nor the price of risk $\{\boldsymbol{\nu}_i\}_{i \leq N_s}$ influence the

welfare value $v(x)$:

$$V(Y, F) = \delta \left[\underbrace{\frac{Y}{\delta - \mu} \left(1 - \left(\frac{1 - \alpha}{1 - \frac{\alpha\theta\xi\lambda}{\delta + \lambda - \mu}} \right) \left(\frac{x}{\bar{x}} \right)^\xi \right)}_{\text{value of "credit-risky" endowment}} - \underbrace{FD(Y, F)}_{\text{aggregate debt value}} \right]$$

This result is tightly related to the conjecture made in [15], and formally proven by [47] and [22], who show that a monopolist with constant marginal costs selling a durable good to a continuum of consumers will actually behave competitively, in the continuous-time limit, and not extract any monopoly rent. The argument, in the context of the sovereign default model with a risk-neutral government, works as follows: without commitment, no matter how many bonds the government sold in the past, the government will sell more bonds if there are marginal gains from doing so (in other words if those bonds can be sold at an implied interest rate strictly less than δ , i.e. if $\delta D(Y, F) > -\partial_F V(Y, F)$). But investors perfectly anticipate this behavior and thus price the bonds at an implied interest rate of δ , therefore stripping away any potential welfare gain that the government may extract from facing financiers that discount cash flows at a rate strictly less than δ .

Note that this result is purely due to the continuous-time nature of my model; as highlighted by [47] and as (unreported) computations illustrate, the discrete time counterpart to this model would yield strictly positive welfare gains for the risk-neutral government. Having a non-zero time period during which the government can commit not to issue bonds is crucial in obtaining such result. Similarly, in the case of a risk-averse government, the result above will no longer hold: the concavity in the flow payoff function will be such that the government will extract welfare gains from issuing bonds to investors whose implied interest rate is lower than the government's rate of time preference. The country's welfare, sovereign bond prices and default boundaries will again depend on the level of risk-free rates and prices of risk: since the government dislikes high levels of consumption volatility, its financing policy will not fully adjust to keep sovereign bond prices unchanged; instead, the adjustment will

be partial. This result is analogous to what is showed theoretically in [28] in the context of the durable goods monopoly problem: rents can be extracted by the monopolist if its marginal production costs are increasing.

As a consequence of proposition 9, changes in the supply-side of capital lead the government to adjust its financing policy in such a way that sovereign bond prices and welfare remain unchanged: with higher risk-free rates or higher prices of risk¹⁰, the higher financing costs borne by the government are exactly compensated by a lower pace of debt issuances. This translates into an upward adjustment of the country's current account and trade balances, causing an endogenous sudden stop. In the context of an increase in the price of risk, the magnitude of the current account reversal is high when the debt-to-GDP ratio of the country is high. The financing policy of the government is also interesting to study since it highlights the fact that when bond investors are risk-neutral (i.e. when the price of risk is identically zero), the issuance policy of the government is always positive: it is never efficient for the government to buy back debt. This result echos an insight from [12] who show, in the context of a one-period model of sovereign default with a risk-neutral government and risk-neutral lenders, that it is never welfare-improving for a country to buy back its own debt. This result breaks down in the presence of risk-averse lenders, whose price of risk has a positive correlation with the country's endowment process: in such case, equation 1.38 shows that there are parameter configurations where, for high debt-to-GDP ratios, the country does find it optimal to buy back its own debt. For this to be the case, the parameters of the model need to satisfy the following condition:

$$\frac{\alpha\lambda\theta(1-\theta^{\xi-1})}{\delta+\lambda-\mu-\alpha\theta\lambda} < \frac{\xi-1}{\delta-r_i} \boldsymbol{\sigma} \cdot \boldsymbol{\nu}_i$$

In other words, if either the price of risk $|\boldsymbol{\nu}_i|$ is sufficiently high, or the GDP drop upon default $1-\alpha$ is sufficiently severe, or the risk-free rate r_i is sufficiently close to the government

10. This discussion assumes that the price of risk is positively correlated with the GDP process in all SDF states (i.e. $\boldsymbol{\nu}_i \cdot \boldsymbol{\sigma} > 0$).

discount rate δ , it is sometimes optimal for the government to buy back debt. The key to this result is the fact that the probability measure under which investors discount cash-flows (the “risk-neutral” measure) is different from the probability measure (the “physical” measure) under which the government optimizes. A different interpretation of this result can be put as follows: persistent differences in beliefs about the growth rate of the country’s endowment (where investors would be more “pessimistic” than the government) would also lead the government to buy back debt when the debt-to-GDP ratio of the country is high.

Finally, the expected excess return earned on sovereign bonds is increasing in the debt-to-GDP ratio, confirming the ability of the model to replicate – at least qualitatively – empirical fact (8). In environments with constant risk-free rates, while an increase in the price of risk induces a sudden stop, sovereign bond prices remain unchanged, leading to sovereign credit spreads that are not reacting to these worse capital market conditions. Finally, the closed-form expressions of proposition 9 allow me to derive the following comparative static results.

Corollary 10. *The default boundary \bar{x} is decreasing in the impatience parameter δ , decreasing in the coupon rate κ , increasing in the GDP haircut post default $1 - \alpha$, increasing in the haircut parameter θ , decreasing in the expected autarky time $1/\lambda$.*

One particular comparative static result worth highlighting is the fact that the default boundary (as well as the country’s welfare) is decreasing in the coupon rate κ . This will have its importance when I analyze the Latin American debt crisis of the early 1980s – at that time, most of the sovereign debt contracted by these small open economies was structured with variable coupons indexed to US short term rates. As will be seen in section 1.7, the increase in short term rates creates a current account reversal that is magnified by the contractual structure of sovereign debt at the time.

1.5 Applications

1.5.1 Numerical Illustration and Comparative Static Results

Before estimating my model for a set of countries of interest, I first provide some comparative static analysis, in order to gain some intuition about the role of certain model parameters. To facilitate this investigation, I shut down for now the multiple SDF regimes, and analyze the effect of certain model parameters in an environment where there is only one SDF regime. I solve the model numerically using a Markov chain approximation method, as described in details in section A.3.

Calibration

Parameter	Value	Description
$1/\rho$	0.5	IES
γ	5	Risk aversion
δ	0.2	Rate of time preference
μ	0.035	GDP growth rate
$ \sigma $	0.04	GDP growth volatility
$1 - \alpha$	0.04	GDP % fall at default
$1/\lambda$	5	Capital markets' exclusion (years)
θ	0.50	Debt-to-GDP upon autarky exit
r	0.05	Creditors' risk free rate
$ \nu $	0.625	Creditors' market price of risk
$\text{corr}(\nu, \sigma)$	0.50	Business cycle-risk price correlation
$1/m$	7	Debt average life (years)
κ	0.05	Debt coupon rate

Table 1.1: Calibration Parameters

Table 1.1 highlights the base case parameters I use for this comparative static analysis. I select model parameters that are meant to represent the “average” emerging market economy of the dataset studied in the online appendix. More specifically, the table “Country-Specific Macro Moments” in the online appendix, constructed using data from the World Bank, shows that the average real GDP growth rate for the countries in my dataset is 3.5% p.a., and the average real GDP growth volatility is 4.1% p.a., leading to the parameters μ and σ

in Table 1.1. In the same online appendix, the table “Bond Issuance Average Maturities” shows summary statistics for a dataset of foreign currency sovereign bonds I collected from Bloomberg¹¹; according to such table, the average original maturity date of sovereign bonds issued by my sample of 27 emerging market countries is 13.8 years. Since a country consistently rolling over 13.8-year original maturity bonds has a debt average life of 6.9 years, I pick $1/m = 7$ years. I select jointly the default punishment parameter $1 - \alpha$ and the rate of time preference δ to approximately match two moments of the data: the average debt-to-GDP ratio (equal to 50% for the set of countries of focus, as indicated in the “Country-Specific Macro Moments” table in the online appendix), and the average 5yr CDS premium (equal to 395bps for the set of countries of focus, as indicated in “Country-Specific Debt Price Moments” table in the online appendix). This procedure yields a permanent output drop upon default of $1 - \alpha = 4\%$ ¹², and sovereign rate of time preference $\delta = 20\%$. According to [52], it takes 4.7 years post-default for a country to regain access to capital markets, leading me to pick $1/\lambda = 5$ years. [8] find a mean creditor haircut following a sovereign default of approximately 40%. Since my model assumes that the face value of debt at exit from financial autarky is equal to $\theta Y_{\tau+\tau_e}/Y_{\tau-}$ times the face value of debt pre-default, the model-implied face value haircut is equal to:

$$1 - \theta \alpha \mathbb{E} \left[e^{(\mu - \frac{1}{2}\sigma^2)\tau_e} \right] = 1 - \frac{\alpha \theta \lambda}{\lambda - \left(\mu - \frac{\sigma^2}{2} \right)}$$

I thus pick $\theta = 0.50$, leading to a model-implied average creditor haircut of 42%. I choose an inter-temporal elasticity of substitution equal to $1/\rho = 0.5$ that is consistent with the inter-

11. For all countries in the data-base I construct, I download all bonds listed on Bloomberg and issued by such country. I only keep in my data-base foreign currency bonds denominated in either EUR, GBP, USD, JPY or DEM. I also exclude bonds whose original notional amount is less than USD 100mm, whose original term is less than 1 year or greater than 50 years, or bonds with non-fixed coupon rates. The list of remaining bonds is available upon request.

12. Note that estimates of output drops following a sovereign default vary vastly across the empirical literature: [27] for example calculate the cost of Argentina’s sovereign default to correspond to 9.4% permanent reduction in output; [5] use a (transitory) cost of default of 2% of output in their model, citing evidence from [45], who calculates an 8% decline in international trade following a sovereign default.

national business cycle literature¹³, and a risk-aversion parameter $\gamma = 5$ that is consistent with the asset pricing literature. Risk-free rates are set at 5% p.a., and I pick a sovereign debt coupon rate of 5%, which means that (a) the risk-free value of government debt is 1, and (b) the sovereign debt always trades at a discount to par. The magnitude of the risk price ν is set at the ergodic mean risk price of [33] (the pricing kernel I will use later on when incorporating multiple SDF states). In a model with one risk price state (and therefore no SDF jumps), $|\nu|$ corresponds to the highest Sharpe ratio attainable by any asset priced by international investors (see [23])¹⁴. To start with, I assume a correlation between the creditors' risk price and the government's output process of 50%, and will discuss how this correlation affects equilibrium outcomes in section 11 (of course in this simple model, varying this correlation and keeping risk prices constant is equivalent to keeping this correlation constant and varying risk prices). In all the figures I will be discussing, I plot in dotted lines the ergodic distribution of the state variable x_t in order to focus my attention on the sub-interval of the state space where I expect to see most of my model-implied observations. This ergodic distribution is obtained via integrating a set of Kolmogorov forward equations, as section A.1.14 reveals. In the case of a unique SDF regime, the ergodic distribution admits the following semi-closed form expression:

Lemma 11. *In the absence of multiple SDF states, the ergodic distribution f takes the following form:*

$$f(x) = \begin{cases} \int_{\theta\bar{x}}^{\bar{x}} \exp \left[\int_x^t \frac{2}{|\sigma|^2 s^2} ((m + \mu)s - \iota(s)) ds \right] \frac{2G}{|\sigma|^2 t^2} dt & \text{if } x \in [0, \theta\bar{x}) \\ \int_x^{\bar{x}} \exp \left[\int_x^t \frac{2}{|\sigma|^2 s^2} ((m + \mu)s - \iota(s)) ds \right] \frac{2G}{|\sigma|^2 t^2} dt & \text{if } x \in (\theta\bar{x}, \bar{x}) \end{cases}$$

The constant G is pinned down by the condition $\int_0^{\bar{x}} f(x)dx = 1 - \frac{1}{1+\lambda T(\theta\bar{x})}$, where $T(x)$ is

13. [3], [5], [6] all use an IES of 0.5.

14. Note that this 62.5% maximal Sharpe ratio is slightly higher than the unconditional Sharpe ratio of 40% obtained for US equities – computed using an historical average of US equity market excess returns of 6%, and a yearly volatility of 15%.

the expected default time conditional on the initial debt-to-GDP ratio being equal to x .

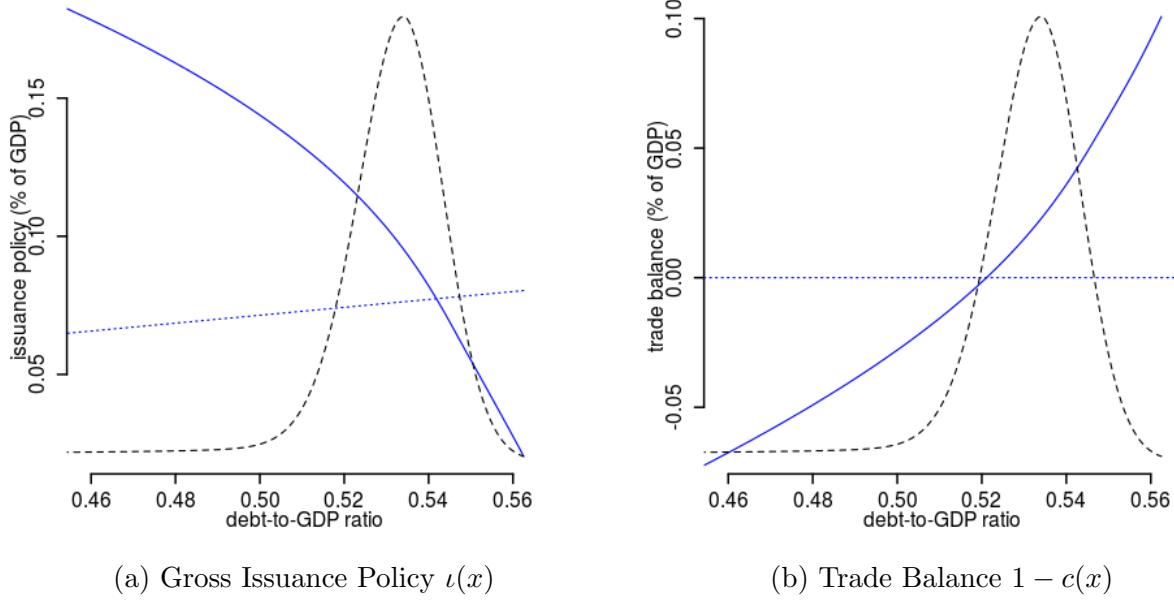
Base Case Calibration Equilibrium

Moment	Notation	Value	Target/ Historical Value
default boundary	\bar{x}	56%	
mean debt-to-GDP	$\mathbb{E}[x_t]$	52%	50%
stdev debt-to-GDP	$\text{stdev}[x_t]$	3%	24%
consumption-output vol ratio	$\text{vol}(C_t) / \text{vol}(Y_t)$	1.94	
default rate	$(1/\lambda + T(\theta\bar{x}))^{-1}$	2.8% p.a.	
bond spread	$\mathbb{E}[\varsigma(x_t)]$	365 bps p.a.	
5y CDS spread	$\mathbb{E}[\varsigma(x_t, 5)]$	414 bps p.a.	407 bps p.a.
5y-1y CDS slope	$\mathbb{E}[\varsigma(x_t, 5) - \varsigma(x_t, 1)]$	60 bps p.a.	72 bps p.a.
bond excess return	$\mathbb{E}[dR_t^e]$	164 bps p.a.	
bond return volatility	$\text{stdev}[dR_t^e]$	525 bps p.a.	
5y CDS excess return	$\mathbb{E}[dR_{t,5}^e]$	230 bps p.a.	508 bps p.a.
5y CDS return volatility	$\text{stdev}[dR_{t,5}^e]$	732 bps p.a.	

Table 1.2: Base Case Calibration Results

In Table 1.2, I display the key model-implied moments of interest, as well as the target and historical values for the “average” of my 27 countries in the online appendix. The base case calibration results in an optimal debt-to-GDP default boundary $\bar{x} = 56\%$, an ergodic mean debt-to-GDP ratio of 52% and an ergodic debt-to-GDP distribution standard deviation of 3.0%. In figure 1.2, I plot the issuance policy ι and the resulting trade balance. The issuance policy is a decreasing function of the debt-to-GDP ratio, positive but reaching levels close to zero at the default boundary. It is important to keep in mind that these represent gross issuances, before taking into account any debt amortization. In figure 1.2a, the dotted blue line represents the locus of points $(\mu + m - |\sigma|^2)x$, i.e. the required value of $\iota(x)$ such that the drift rate of x is zero. It is immediate to notice that the debt-to-GDP ratio is a mean-reverting variable – its drift rate is going to be positive for values of x on the left of the intersection of the solid and dotted blue lines and negative on the right side of such intersection. The resulting trade balance (as a fraction of GDP) is equal to

Figure 1.2: Government Financing Policy and the Trade Balance



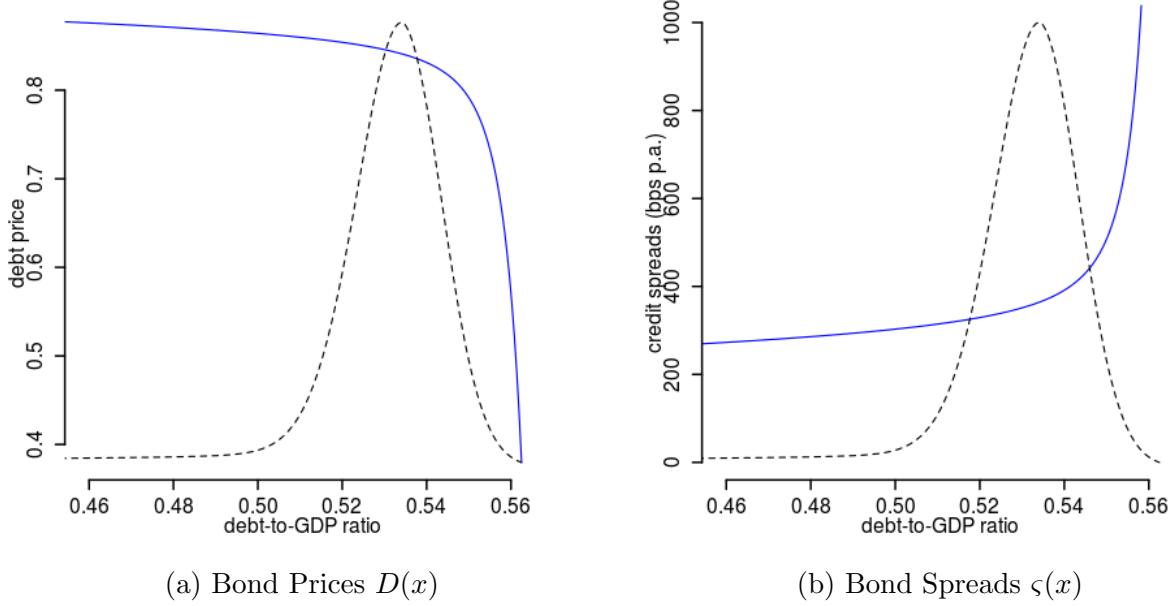
$1 - c(x) = (\kappa + m)x - \iota(x)D(x)$. It is negative for low debt-to-GDP levels and positive otherwise. Thus, consistent with the overwhelming data for emerging market economies, the trade balance in this model is countercyclical.

Since the consumption-to-output ratio $c(x)$ is a decreasing function of the debt-to-GDP ratio, using equation 1.34 it is immediate to see that the model generates a consumption growth volatility that is greater than output growth volatility, an empirical regularity of the data. For the parameters selected, I obtain an ergodic consumption growth vol to output growth vol ratio of 1.94, which is in line with several emerging market economies, as documented in [4]¹⁵. Note also that consumption growth and output growth are perfectly correlated (at least locally) in a model with one SDF regime, which is obviously counterfactual – the model featuring multiple SDF regimes breaks this result.

I plot the sovereign bond price and the sovereign spread in figure 1.3. The ergodic mean credit spread of the exponentially amortizing bond is equal to 365bps p.a., and the ergodic

15. They find a ratio of 1.38 for Argentina, 2.01 for Brazil, 2.39 for Ecuador, 1.70 for Malaysia, 1.24 for Mexico, 0.92 for Peru, 0.62 for the Philippines, 1.61 for South Africa, 1.09 for Thailand and 1.09 for Turkey.

Figure 1.3: Government Bond Prices and Spreads



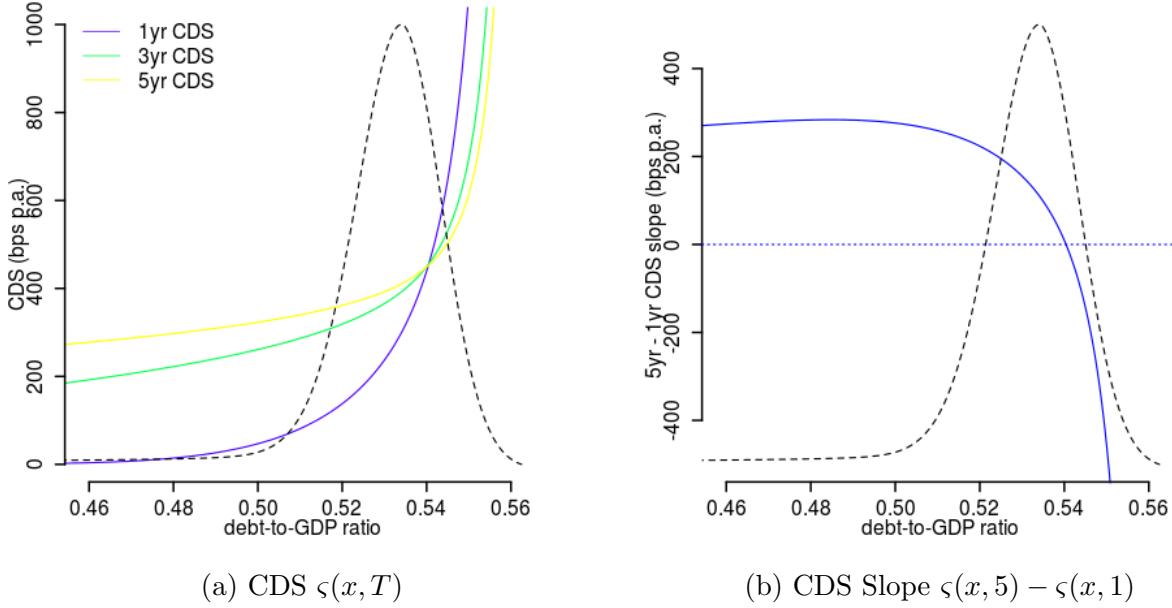
credit spread volatility is equal to 132bps p.a. This ergodic mean credit spread volatility is higher than what is obtained by [3] and the difference stems from the much lower average maturity of sovereign debt (2 years) this latter article assumes, compared to the 7-year average life debt in the data for my countries of interest.

Focusing on figure 1.3, bond prices decrease with the debt-to-GDP ratio, while bond spreads increase. Even at low debt-to-GDP ratios (i.e. lower than the ergodic debt-to-GDP mean), sovereign spreads are far from negligible. When the sovereign has no debt outstanding, sovereign spreads are strictly positive – a simple manipulation of equation 1.20 shows that the sovereign spread at $x = 0$ verifies:

$$\varsigma(0) = -\frac{\iota(0)D'(0)}{D(0)}$$

This equation highlights the role of future debt issuances (and the implicit dilution risk associated with those future debt issuances): at low debt-to-GDP ratios, creditors perfectly anticipate that the government will be issuing large amounts of debt (since the government

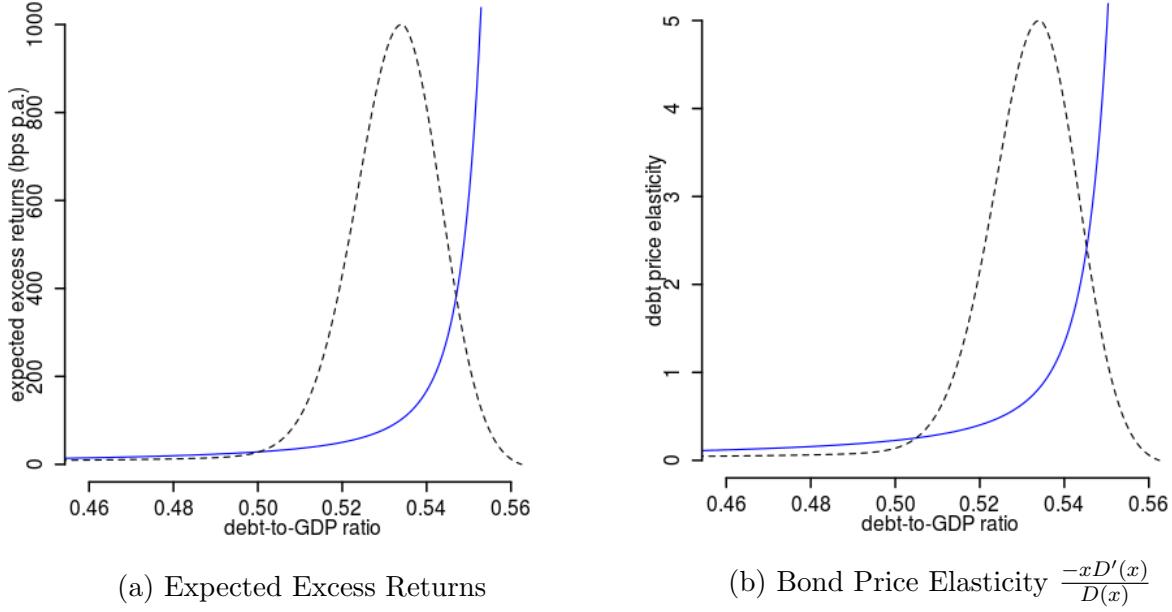
Figure 1.4: Credit Default Swap Premia



is impatient), leading them to price bonds at a discount that reflects such dilution risk.

I then focus on credit default swap premia $\xi(x, T)$. While the government uses 7-year bonds to smooth and front-load consumption, my model allows me to compute the premium $\xi(x, T)$ of CDS contracts at any time-horizon T , as discussed in section 1.3.5. The numerical procedure to solve the relevant PDEs is described in details at the end of section A.3. I plot CDS premia in figure 1.4a for 1-year, 3-year and 5-year contracts, and I plot in figure 1.4b the 5y-1y slope. CDS premia for short-dated contracts are close to zero when the debt-to-GDP ratio is far away from the mean ergodic debt-to-GDP ratio, highlighting the fact that at such low debt-to-GDP levels, the sovereign slowly increases its indebtedness, such that 1-yr credit instruments are almost risk-free. The 5-year CDS premium $\xi(x, 5)$ resembles the credit spread of the exponentially amortizing bond $\xi(x)$ used by the government to finance itself, since the average life of this bond is 7 years. Lastly, the term structure of credit spreads becomes inverted as the sovereign approaches its default boundary. This property of my model ends up being a very general property of default models that are structured

Figure 1.5: Expected Excess Returns and Bond Price Elasticity



as “first-hitting-time” models, where the state is mean-reverting and has continuous sample paths. This feature of the model also fits with stylized fact (5).

I plot in figure 1.5 the bond expected returns and the bond price elasticity. Expected excess returns are non-zero, since I have used a pricing kernel that co-moves with the country’s output process: the unconditional expected excess return is equal to 1.64% p.a., while the unconditional return volatility is equal to 5.25% p.a. Expected excess returns increase with the debt-to-GDP ratio, since the bond price function’s elasticity increases with the state variable x – this is the model counterpart to fact (8) in the data. I plot the bond price elasticity $\frac{-xD'(x)}{D(x)}$ in order to get a better understanding of the magnitude of the sovereign bond’s US equity market beta – this will help me understand whether my model stands a chance at generating the magnitude of expected excess returns observed in the data. In the one-state SDF model, conditional expected excess returns are equal to $\frac{-xD'(x)}{D(x)}\boldsymbol{\sigma} \cdot \boldsymbol{\nu}$. The risk prices will be time-varying in the full model, but will on average be equal to 62.5%. Most of my countries of interest have GDP volatilities of the order of 4%, meaning that at

best, an emerging market output's risk premium is equal to $\sigma \cdot \nu \approx 2.4\%$. In this one-state SDF model, the bond price elasticity has an ergodic average equal to 1.33. Thus, at best (in other words when $\text{corr}(\nu, \sigma) = 1$), the ergodic mean excess return will be equal to 3.3% – high, but not quite sufficient to reach the unconditional expected excess returns in the data for certain countries such as Brazil or Hungary for example.

I end this section by focusing on the historical measure. When the country emerges from financial autarky, it has a debt-to-GDP ratio $\theta\bar{x} = 28.5\%$, and it takes such country on average 24 years to default on its debt, once it has exited from autarky. The sovereign default frequency under the physical measure is thus equal to 2.8% p.a. – slightly higher than the unconditional estimate of 2% cited in multiple studies, but this measure is of course extremely difficult to estimate accurately in the data given the low frequency nature of sovereign defaults¹⁶.

After having investigated some of the key outputs of the model in the base case parametrization, I now study the impact of the model parameters on various endogenous quantities of interest. This comparative static analysis will be used in my estimation in order to provide identification for a subset of parameters of my model.

Comparative Statics

Table A.1 is a summary of the comparative statics with respect to several parameters of the model. These comparative statics are performed starting from the base case parametrization of Table 1.1. In the table, I compute the elasticity of the moment of interest w.r.t. the parameter of interest; blue numbers correspond to positive elasticities, red numbers correspond to negative elasticities. As an example, if one focus on the ergodic debt-to-GDP mean

16. It is also worthwhile noting that this historical default frequency is high due to the assumption that the loss severity suffered by bond investors in connection with a sovereign default is not 100%. In the discrete time literature on sovereign defaults, barring a few exceptions, most papers assume that creditors' loss severity upon a sovereign default is 100%. In the case of creditors' risk-neutrality, this automatically causes the level of credit spreads to be close to the level of historical default intensities; if instead a recovery R is realized by creditors, the historical default intensity would be approximately $1/(1-R)$ times the average sovereign credit spread.

and the variation of such mean w.r.t. the impatience parameter δ , the elasticity is equal to $\frac{\partial \ln \mathbb{E}[x_t]}{\partial \ln \delta} = -0.56$, meaning that a 10% increase in the value of the parameter δ leads to 5.2% decrease in the ergodic mean debt-to-GDP ratio. These elasticities are closely related to the covariance matrix of parameter estimates that will be calculated as part of my model estimation.

The GDP drop upon default $1 - \alpha$ has a significant impact on government behavior – specifically on the debt-to-GDP ratio at default, its ergodic mean and standard deviation. An increase in the default punishment incentivizes the government to support higher levels of debt in equilibrium, a result already known in the literature using discrete time models. The magnitude of the sensitivities are not surprising: my output process being a geometric Brownian motion, any downward GDP drop is a permanent shock that is translated into large welfare losses. Since the default boundary increases with the magnitude of the GDP drop post-default, the government issuance policy adjusts upwards, via an almost-parallel shift.

As the government becomes more impatient (i.e. as δ increases), it tends to front-load consumption when it is not significantly indebted. The impatient government thus has a debt issuance policy with a steeper slope than the patient government, and the optimal default boundary is lower for the former than for the latter. With an impatient government, creditors take into account the dilution risk and price the debt more punitively than in the situation where the government is patient, which leads to higher credit spreads, and a much steeper 5yr-1yr spread slope. The long-run mean spread volatility is higher, which leads to significantly higher risk-premia.

My framework allows me to investigate the separate roles of risk-aversion and inter-temporal elasticity of substitution in the government's decision problem – an analysis that the international macroeconomic literature has not focused on so far. An increase in risk-aversion tends to decrease the debt issuance rate, due to a greater precautionary savings motive. It also increases the incentive to default, since welfare costs of business cycle fluctuations

increase with the level of risk-aversion, and since consumption volatility is greater than output volatility in this class of models. Thus, the equilibrium default boundary is lower with a more risk-averse government.

Increases in GDP volatility decrease the equilibrium debt-to-GDP default boundary. Once again, this is due to the increase in the incentive to default, since welfare costs of business cycle fluctuations increase with the GDP volatility, and since consumption volatility is greater than output volatility. The mean bond spread increases significantly, since if one keeps the government financing policy unchanged, a greater GDP volatility increases the default barrier hitting probabilities (of course there will be an equilibrium response by the government, which will reduce its debt issuances). A greater GDP volatility increases bond spread volatilities, and as expected the sovereign bond risk-premium is materially higher. More surprisingly, a greater GDP volatility does not lead to a meaningfully wider debt-to-GDP distribution, as illustrated by the low sensitivity of $\text{stdev}[x_t]$ to $|\sigma|$. This is disappointing since it suggests that stochastic volatility will not rescue one fundamental weakness of this class of models: the fact that the model-implied ergodic debt-to-GDP distribution is a lot thinner than the one observed in the data – the latter being computed and displayed in table “Country-Specific Macro Moments” in the online appendix.

Breaking Down Governments’ Cost of Financing

In order to better understand the impact that SDF regime shifts will have on equilibrium outcomes, I look at the comparative static w.r.t. ν in more details. Higher risk prices increase sovereign spreads, lower government bond prices, leading the government to adjust its issuance policy downwards. Higher creditors’ risk prices also cause the sovereign government to default at lower debt-to-GDP levels – although the adjustment is relatively small. This mechanism leads, in a multi-SDF-regime version of my model, to jump-to-default risk, induced by risk prices jumping from one level to another (higher) level. When comparing credit spreads for $\nu = \mathbf{0}$ to credit spreads with strictly positive risk prices, one might want to

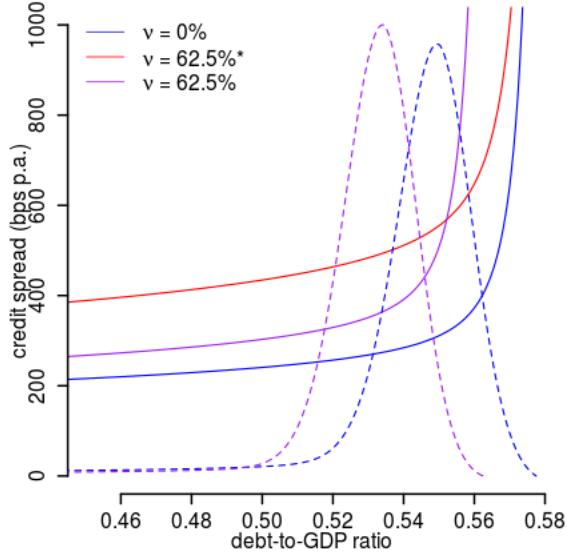


Figure 1.6: Credit Spread Decomposition

interpret the spread differential between those two parameter configurations as the “spread premium” that a sovereign government is paying to its creditors above and beyond what would be actuarially fair. This is not exactly the case since the government endogenously reacts to those higher risk prices by altering its issuance and default policies.

I thus look at the following credit spread decomposition: starting from the equilibrium without risk-pricing, I first adjust risk prices to $|\nu| = 0.625$ (and under the assumption that $\frac{\nu \cdot \sigma}{|\nu| |\sigma|} = 0.50$) and re-compute sovereign debt prices and spreads, keeping the government’s issuance policy constant. I then adjust the government’s issuance policy to reflect such government’s “best response” to this new debt price schedule. I will use this decomposition in section 1.6 in order to compute the cost borne by governments when issuing bonds to risk-averse (US-based) investors. As discussed, this cost cannot be computed by simply evaluating the average excess return paid to creditors for taking on sovereign credit risk, given the fact that sovereign issuance policies depend on investors’ risk attributes; a counterfactual analysis is required to decompose this sovereign financing cost.

The result of such decomposition is illustrated in figure 1.6: the credit spread function

plotted in red is the spread under the intermediate step where risk prices are set to 0.625 but where the government issuance policy has not reacted yet. figure 1.6 shows that positive risk prices alone act as a powerful force to push credit spreads wider. Indeed, when $|\nu|$ is increased from zero to $|\nu| = 0.625$ (and the issuance policy is not adjusted), the drift rate of the state variable x_t increases by $x_t \boldsymbol{\sigma} \cdot \nu \approx 0.5 \times 0.04 \times 0.625 \times 0.5 = 0.625\%$ per annum (using the ergodic debt-to-GDP ratio for x_t and the assumed 50% correlation between risk prices and GDP), and the credit spread is wider by 233bps p.a. on average (this can be seen in the plot by the upward shift from the curve $\nu = 0$ to the curve $\nu = 0.625^*$). But the government responds by adjusting its issuance policy downwards, defaulting at a lower debt-to-GDP burden, and the resulting credit spreads ends up wider by “only” 101bps p.a. on average (using the ergodic distribution with risk-pricing). In other words, in this simple calibration, $101/360 \approx 28\%$ of total financing cost of the government is attributable to investors’ risk aversion. For my panel of countries of interest, as will be seen shortly, this cost will turn out to be very close to 30%.

1.6 Structural Estimation

1.6.1 Procedure

I select a subset of $N = 8$ countries out of the set of 27 countries discussed in the online appendix and for which I have the longest time-series data available. My subset consists of Brazil, Bulgaria, Hungary, Indonesia, Mexico, Philippines, South Africa and Turkey. My model with SDF regime shifts has a large number of parameters to determine. I am going to impose restrictions on those parameters as follows. First, I will use a pricing kernel specification widely used in the asset pricing literature to rationalize properties of US equity market returns: the pricing kernel of [33], originally built to explain (amongst other things) the value premium. This SDF features a constant risk-free rate (2% per annum), and risk prices that follow an AR(1) process. Section A.4 provides a detailed description of the

properties of this SDF, as well as the method I use to transform the original continuous-state specification of [33] into a discrete state continuous-time Markov process $\{\nu_{st}\}_{t \geq 0}$, using a numerical procedure based on matching conditional and unconditional moments of the original model and the approximating model. This gives me the risk free rates $\{r_i\}_{i \leq N_s}$, risk prices $\{\nu_i\}_{i \leq N_s}$, the matrix of intensities Λ as well as the SDF jumps $\{v(i, j)\}_{i, j \leq N_s}$. I choose a number of states $N_s = 5$ in order to be able to solve for a single equilibrium of my model in a few minutes of computing time. My risk prices $\{\nu_i\}_{i \leq N_s}$ are equally spaced between 0% and 150%, with an ergodic mean of 62.5%.

The remaining parameters to estimate are country-specific. For simplicity and due to the small number of GDP data points available, I will assume that for each country of interest, expected consumption growth, consumption growth volatility and the correlation between the country's output process and the SDF risk price do not change with SDF regime shifts, in other words for each country n , $\mu_i^n = \mu_j^n$, $\sigma_i^n = \sigma_j^n$ and $\frac{\sigma_i^n \cdot \nu_i}{|\sigma_i^n| |\nu_i|} = \frac{\sigma_j^n \cdot \nu_j}{|\sigma_j^n| |\nu_j|}$ for any pair of SDF states i, j . I will also assume that the sovereign government has time-separable preferences, in other words I will assume that $\gamma^n = \rho^n$ for each country n . A few other parameters are calibrated using a-priori evidence. I will leverage the average original maturity of bonds issued by each country (as documented in the "Bond Issuance Average Maturities" table in the online appendix) in order to calibrate the debt parameter m^n for country n ¹⁷. The expected time spent in financial autarky $1/\lambda$ and the parameter θ governing the debt-to-GDP post-autarky are kept at their values in table 1.1 and are thus not country-dependent. For each country n , my estimation will then pin down the GDP growth μ^n , GDP volatility $|\sigma^n|$, the preference parameter $\gamma^n = \rho^n$, the correlation between GDP and the risk price vector $\frac{\nu \cdot \sigma^n}{|\nu| |\sigma^n|}$, the GDP drop upon default $1 - \alpha^n$ and the rate of impatience δ^n . The following 6 moments will be used in my estimation. First, the first difference mean and standard deviation of log output will provide information on μ^n and $|\sigma^n|$. Unconditional expected

17. A country consistently rolling over T -year original maturity bonds has a debt average life of $T/2$ years; for such country I thus use $\lambda = 2/T$.

excess returns on 5y CDS contracts will then provide information on the correlation between country n 's output and risk prices. Finally, the level of 5y CDS premia, the mean debt-to-GDP ratio and the term structure slope (i.e. the difference between 5y CDS and 1y CDS spreads) will jointly provide information on $1 - \alpha^n$, δ^n and γ^n .

My simulated method of moment estimation follows closely [30]. I note $H = \frac{1}{T} \sum_{t=1}^T h_t$ the $p \times 1$ vector of target moments in the data, and $\mathcal{H}_k(\Theta) = \frac{1}{T} \sum_{t=1}^T h_t(\{x_t^n\}_{n \leq N}, s_t; \Theta)$ the corresponding $p \times 1$ vector of moments generated by the k^{th} simulation of my model. I use $K = 1000$ simulations, and note $\mathcal{H}(\Theta) = \frac{1}{K} \sum_{k=1}^K \mathcal{H}_k(\Theta)$ the sample average of moments generated by my model across the K simulations. For each simulation, I use an identical seed and generate $2T$ years of data, burn-in the first T years, keeping only the simulated data for the last T years. I minimize the criterion function:

$$\hat{\Theta} = \arg \min_{\Theta} (H - \mathcal{H}(\Theta))' W (H - \mathcal{H}(\Theta))$$

I use the diagonal matrix $W = (\text{diag}_i(H_i^2))^{-1}$ as my weighting matrix, to penalize proportional deviations of the model-implied moments from their data counterparts. I compute the asymptotic covariance matrix of my estimator as follows:

$$\text{covar}(\hat{\Theta}) = \frac{1}{T} \left(1 + \frac{1}{K} \right) \left(\frac{\partial \mathcal{H}'}{\partial \Theta} W \frac{\partial \mathcal{H}}{\partial \Theta} \right)^{-1} \frac{\partial \mathcal{H}'}{\partial \Theta} W' \hat{\Omega} W \frac{\partial \mathcal{H}}{\partial \Theta} \left(\frac{\partial \mathcal{H}'}{\partial \Theta} W \frac{\partial \mathcal{H}}{\partial \Theta} \right)^{-1}$$

In the above, $\hat{\Omega}$ is a consistent estimator of the long run covariance matrix of the moment conditions $\Omega = \sum_{j=-\infty}^{+\infty} \mathbb{E} \left[(h_t - \mathbb{E}[h_t]) (h_{t-j} - \mathbb{E}[h_{t-j}])' \right]$. Since my moments of focus are computed using different data sets, over different time periods and with different frequencies of data available, computing a consistent estimator of Ω (using [41] for example) is difficult. Thus, instead, I compute an estimator of Ω using the null of my model:

$$\hat{\Omega}(\Theta) := \frac{1}{K} \sum_{k=1}^K (\mathcal{H}_k(\Theta) - \mathcal{H}(\Theta)) (\mathcal{H}_k(\Theta) - \mathcal{H}(\Theta))'$$

Under the null hypothesis, $\hat{\Omega}(\Theta)$ is a consistent estimator of Ω .

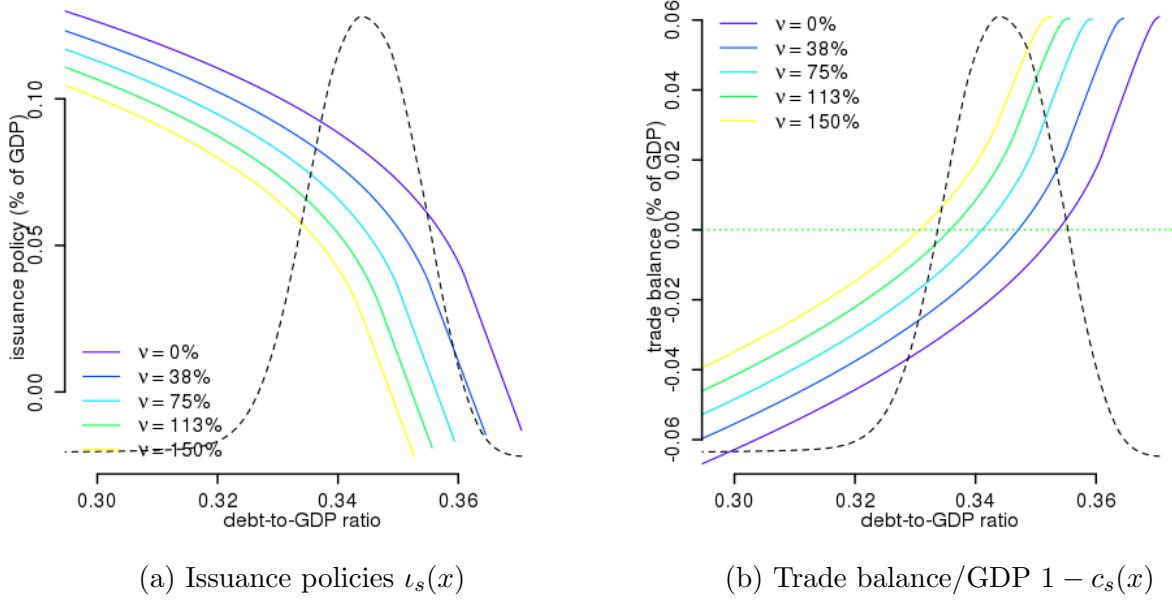
1.6.2 Results – the Case of Brazil

In table A.2 and table A.3, I first display the parameter estimates, then the target and model-implied moments, and finally additional moments that were not targeted in my estimation. I display in figure 1.7, 1.8 and 1.9 the equilibrium outcome for Brazil. In the highest risk price state, Brazil defaults at a debt-to-GDP ratio of 34.8%, whereas in the lowest risk price state, it defaults at a debt-to-GDP ratio of 36.3%, illustrating the relatively small impact of risk prices on optimal default boundaries.

figure 1.7 highlights the endogenous response of Brazil’s financing policy, trade and current account balance to changes in international financial market conditions. Upon an increase in the price of risk, its bonds trade at steeper discounts, inducing Brazil to reduce its debt issuances. Jumps from the lowest risk price state to the highest risk price state lead to a large adjustment to the current account and trade balances of approximately 4% of GDP. Thus, increases in international risk prices constitute an endogenous mechanism for generating sudden stops. Across my model economies, the adjustment in the trade and current account balances varies between 3% and 5%. While this is not sufficient to explain the full adjustments observed in South East Asia post 1997 in connection with the Asian Tiger crisis (as illustrated in figure A.4), those adjustments are nonetheless material. The mechanism used to generate endogenous sudden stops in this paper thus differs from the more standard channel that has been investigated in the past by the international macroeconomics literature – occasionally-binding borrowing constraints in RBC models, such as those featured in [38] or [39], and which depress investment and output when the small open economy suffers a sequence of bad fundamental shocks and bounces against such constraint.

Figure 1.8 then shows that 5yr CDS levels jump up with transitions from low risk price states to high risk price states – at the mean of the debt-to-GDP ratio ergodic distribution for example, 5yr CDS jumps by more than 300bps between the lowest and the highest risk

Figure 1.7: Brazil – Equilibrium Quantities



price states. The term structure of credit spreads may invert for two reasons according to figure 1.8b: either following a sequence of bad fundamental shocks that push Brazil's debt-to-GDP ratio closer to the relevant default boundary, or upon an upward shift in risk prices. This feature of the model is thus consistent with the data, as suggested by facts (5) and (6).

figure 1.9 illustrates aspects of my model that are consistent with facts (7) and (8). Indeed, one can see that CDS expected excess returns increase with the debt-to-GDP ratio, which was expected given that the elasticity of the bond price function is increasing in the debt-to-GDP ratio (this is fact (7)). In addition, longer-term CDS contracts earn higher excess returns than shorter-term CDS contracts, for a given debt-to-GDP level and a given risk price level (this is fact (8)). This effect is due to the risk exposure of CDS contracts, which is increasing in the tenor of such contract.

table A.2 and table A.3 indicate the estimated average bond spread (for the exponentially amortizing long term bond issued by each small open economy), as well as the average bond spread that would prevail if investors were instead risk-neutral (in row titled “Bond spread

Figure 1.8: Brazil – Equilibrium Credit Spreads

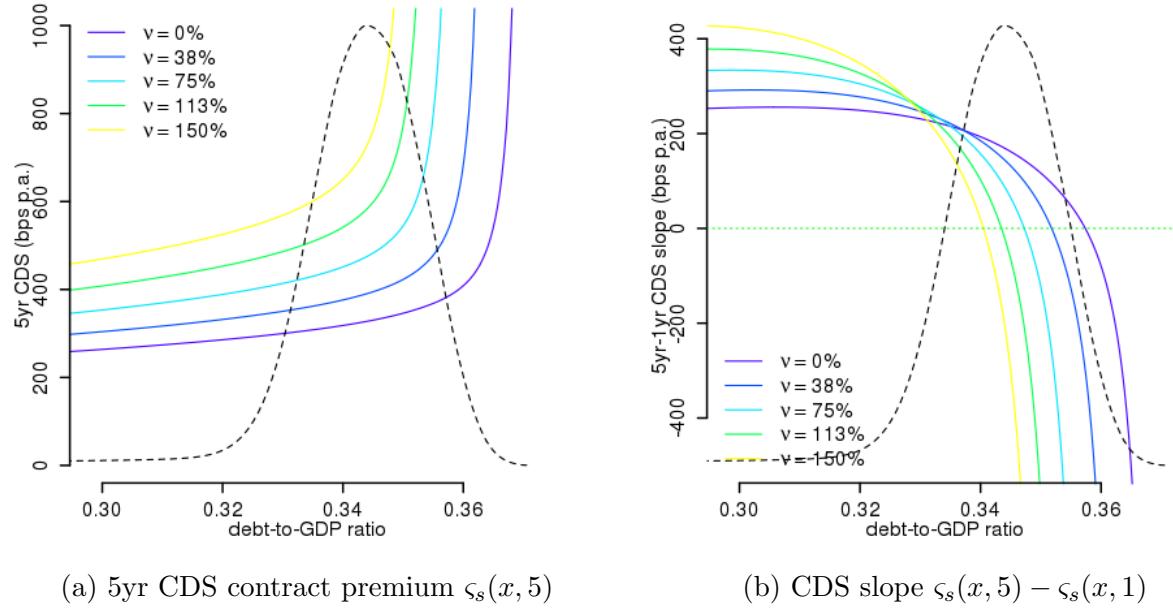
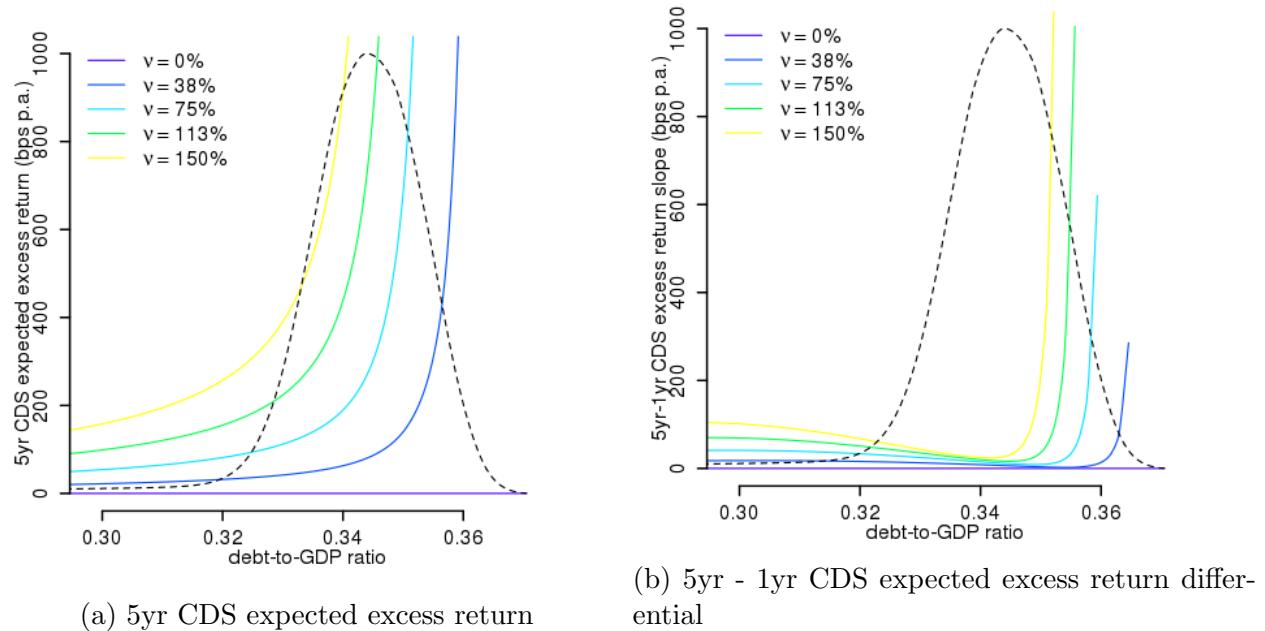


Figure 1.9: Brazil – Equilibrium Credit Returns



– RN model”). The difference between those two spreads can thus be viewed as the excess compensation paid to sovereign debt investors for taking on risks that are correlated with those investors’ marginal utility process. On average, approximately 30% of the credit spread paid by sovereign governments above and beyond the risk free rate is attributable to investors’ risk aversion.

As table A.2 and table A.3 indicate, my estimation leads to target model-implied moments that are relatively close to the data, except for the 5y CDS expected excess return. That moment turns out to be particularly difficult to hit: indeed, for most small open economies in my sample, the estimated correlation between the GDP process and the risk price vector has to be 100%, and model-implied bond and CDS expected excess returns are below those in the data. High levels of correlations between the risk price vector and the GDP process of small open economies also lead to counterfactually high cross-country GDP correlations, another negative consequence. In my model, either risk prices or risk exposures are too low. Whereas risk exposures are functions of the state variables and depend on the deep model parameters, risk prices are taken from the SDF of [33], using the observation that the CAPM is not rejected (meaning that US investors seem to be marginal in sovereign credit markets). One way to address the issue of low model-implied expected excess returns would be to add, as part of the estimation, all the SDF parameters. This turns out to be a numerical challenge, which I plan on tackling in subsequent research.

In addition, the debt-to-GDP ergodic distributions generated by the model have much smaller variances than in the data, even in the presence of time-varying risk prices. The small variance of the model-implied debt-to-GDP ergodic distribution is mostly due to the low volatility (in absolute terms) of GDP shocks suffered by my small open economies of focus¹⁸. One might have conjectured that time-varying risk prices, and thus regime specific default boundaries, will increase the variance of the ergodic debt-to-GDP distribu-

18. While the GDP volatilities of my small open economies of focus are larger than those of developed economies, the absolute level of such volatilities has a first order impact on the variance of the ergodic debt-to-GDP distribution.

tion. Unfortunately, the optimal default boundaries do not strongly react to the presence of regime-specific risk prices, since the government adjusts its issuance policy to reflect different credit market conditions. Making the sovereign output volatility σ_t time varying would help slightly, as suggested by some (unreported) experimentation I ran, but would in no way enable the model-implied distribution variances to match those of the data: as table “Country-Specific Macro Moments” in the online appendix indicates, the standard deviation of the empirical ergodic debt-to-GDP distribution for my countries of focus is approximately 25%, which is an order of magnitude larger than those obtained in my model.

1.7 The 1980’s Latin American Debt Crisis

In this section, I illustrate the flexibility of my framework by studying the 1980’s Latin American debt crisis. For a variety of reasons documented in multiple historical studies (see for example [18] or [14]) Latin American governments borrowed heavily during the 1970s, partly as a consequence of an increase in the supply of loans from US banks recycling petrodollars, partly to take advantage of historically low real interest rates in the US, and partly as a consequence of the need to finance large current account deficits following the two oil price shocks of 1973 and 1979. Latin American sovereign debt increased by an average of 24% per annum between 1970 and 1979, therefore substantially increasing those countries’ debt-to-GDP ratios. The largest sovereign borrowers during that time period were Mexico and Brazil. The World Bank estimates that two third of that debt was in the form of USD-denominated, long-term, syndicated bank loans whose interest rate was indexed to LIBOR, thereby making sovereign governments’ financing costs directly exposed to the US dollar and US monetary policy. In the early 1980s, the Federal Reserve aggressively increased US short term rates to fight domestic inflation, causing the US dollar to appreciate against most currencies, and causing LIBOR rates to skyrocket. In August 1982, as T-bill rates were approaching 16%, Mexico announced that it could no longer meet its debt service payments;

by the end of that year, 40 other nations, including Brazil, Venezuela and Argentina, had defaulted on their sovereign debt.

My model allows me to perform a “lab experiment” on this historical period. Indeed, it is straightforward to change the debt contract structure that the sovereign government enters into from fixed rate contracts to floating rate contracts. Thus, let me now assume that the sovereign issues floating rate debt indexed to the risk free rate r_{st} , which evolves according to a discrete state Markov process with generator matrix Λ . In other words, coupon rates paid by the government are time varying, and equal to $\kappa_{st} = r_{st}$ in state s_t . The government resource constraint (out of financial autarky) becomes:

$$C_t = Y_t + I_t D_t - (\kappa_{st} + m) F_t$$

Debt prices and the life-time utility function for the government satisfy second order ordinary differential equations similar to those presented in section 1.3.4, but appropriately modified to account for the floating rate nature of sovereign debt. Default optimality is still obtained by a set of smooth-pasting conditions. It is worthwhile noting that the price of a risk-free debt instrument that amortizes exponentially and pays a coupon of $r_{st} dt$ for $t \in [t + dt]$ is equal to:

$$\hat{\mathbb{E}} \left[\int_0^\infty e^{-\int_0^t (r_{su} + m) du} (r_{st} + m) dt \mid \mathcal{F}_t \right] = 1$$

In other words, the price of risk-free floating rate debt in this set up is always par, and any credit-risky floating rate instrument (where the coupon paid is equal to the floating rate benchmark) will trade at a discount to par.

In order to compute US short term real risk-free rates, I proceed as follows. I download the US one-month T-bill rate from Ken French website; this will be my time series of US nominal short term risk free rates. I download the consumer price index (CPIAUCSL) from the Saint Louis Fed website, and compute expected inflation by using the one-month ahead

inflation forecast generated from a univariate AR(5) of inflation estimated using the previous 10 years of inflation rates. I then subtract my measure of expected inflation from the nominal rate in order to obtain a measure of real short term interest rates. The US nominal and real short term interest rates obtained are plotted in figure A.1. I then use the time series of US short term real rates in order to estimate the generator matrix Λ , after having discretized the interest rate process into a 4-state Markov process taking values $(0\%, 2\%, 4\%, 6\%)^{19}$. The time series of US real short term interest rates as well as the discrete state Markov process approximation are plotted in figure A.2. On average, s_t spends 35% of the time in the 0% interest rate state, 49% of the time in the 2% interest rate state, 12% of the time in the 4% interest rate state and 4% of the time in the 6% interest rate state.

For this exercise, I use model parameters as in table 1.1; while I assume that creditors' short rates are stochastic, I assume no risk-premia, in other words $\nu_s = \mathbf{0}$ and $v(s, s') = 0$ for all pair of states s, s' . I then solve numerically the model for two separate contractual structures of sovereign debt: first, when the coupon rate on the debt is indexed to US risk free rates (which is the relevant case for Latin American economies in the early 1980's), and second, when the coupon rate on the debt is constant and equal to 2% (i.e. approximately equal to the ergodic risk free rate of 1.7%), in order to understand the importance of the contractual structure of sovereign debt.

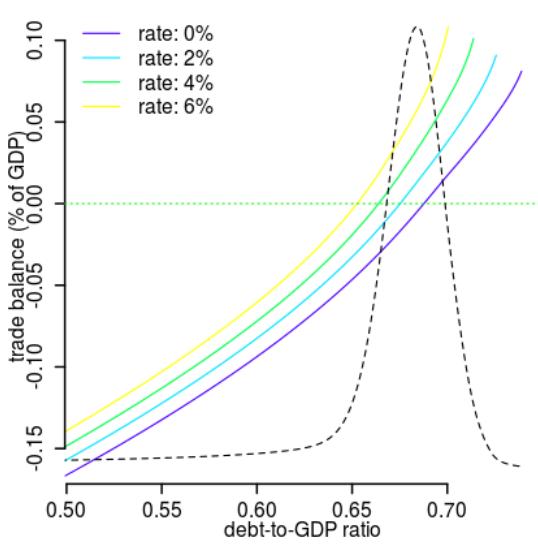
The result of this exercise is striking, as showed in figure 1.10, where I plot the equilibrium trade balance of my model small open economy when the debt issued has floating coupons (figure 1.10a) and fixed coupons (figure 1.10b). With floating rate coupons, the debt-to-GDP default boundaries differ significantly state by state, from 74% in the the lowest US short-rate state to 70% in the highest US short-rate state. Instead, with fixed coupon debt, the debt-to-GDP default boundaries are clustered around 74%. In addition, for both contractual structures, issuance policies depend in a negative way on short rates: for a

19. I restrict the real rate to be weakly positive in my numerical application since the long term bond issued by the sovereign government is linked to such short rate; negative real interest rates in my model would lead the small open economy to *receive* payments from creditors.

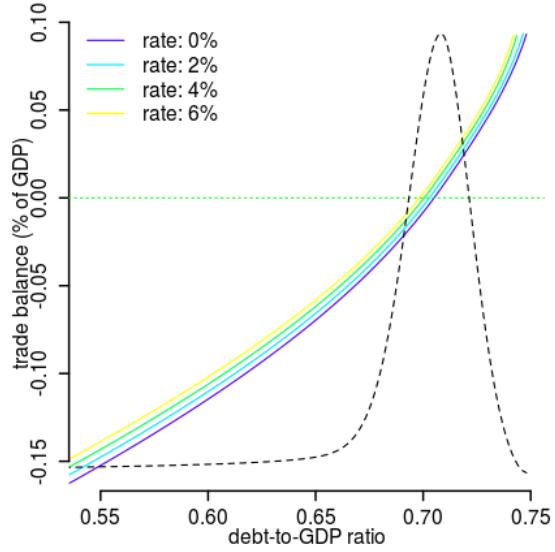
given debt-to-GDP ratio, the lower the interest rate, the higher the issuance rate, leading to current account adjustments when US short term interest rates increase. However, while those current account adjustments are relatively small (of the order of 1%) in the case of fixed coupon debt, they are materially larger when the small open economy issues floating coupon debt – at the ergodic mean of the debt-to-GDP distribution, a jump from 0% US interest rates to 6% US interest rates leads to a current account adjustment of approximately 7%. Of course, if the small open economy’s debt-to-GDP ratio is materially above its ergodic mean at the time of the interest rate increase, the government might be induced to “jump-to-default”. Lastly, a US interest rate increase is experienced by all small open economies at the same time, inducing correlated defaults and correlated current account adjustments for those small open economies that financed themselves using floating coupon debt. The magnitude of the current account adjustment obtained in the model with floating rate debt corresponds approximately to those observed during the large expansion of foreign currency sovereign debt experienced by Latin American economies in the 1970s, as well as following the Volcker shock post-1982, as shown in figure A.3. The figure shows that the current account balance for Argentina, Brazil, Colombia, Mexico and Peru was negative and between -1% and -8% in the late 1970s and early 1980s, but adjusted upwards at the end 1982 by up to 7% for Mexico. Thus, qualitatively, the mechanism highlighted in this section – a large rise of US real rates combined with floating coupon sovereign debt – is a plausible explanation for the behavior of the current account balance of these emerging market economies before and after the Volcker shock.

To understand in more details the mechanism driving the current account behavior discussed above, it is worth looking at the financing policy of the government when bonds that are issued have floating coupons. The optimal Markov issuance policy has the following

Figure 1.10: Trade Balance $1 - c_i(x)$



(a) Floating Rate Debt



(b) Fixed Rate Debt

expression:

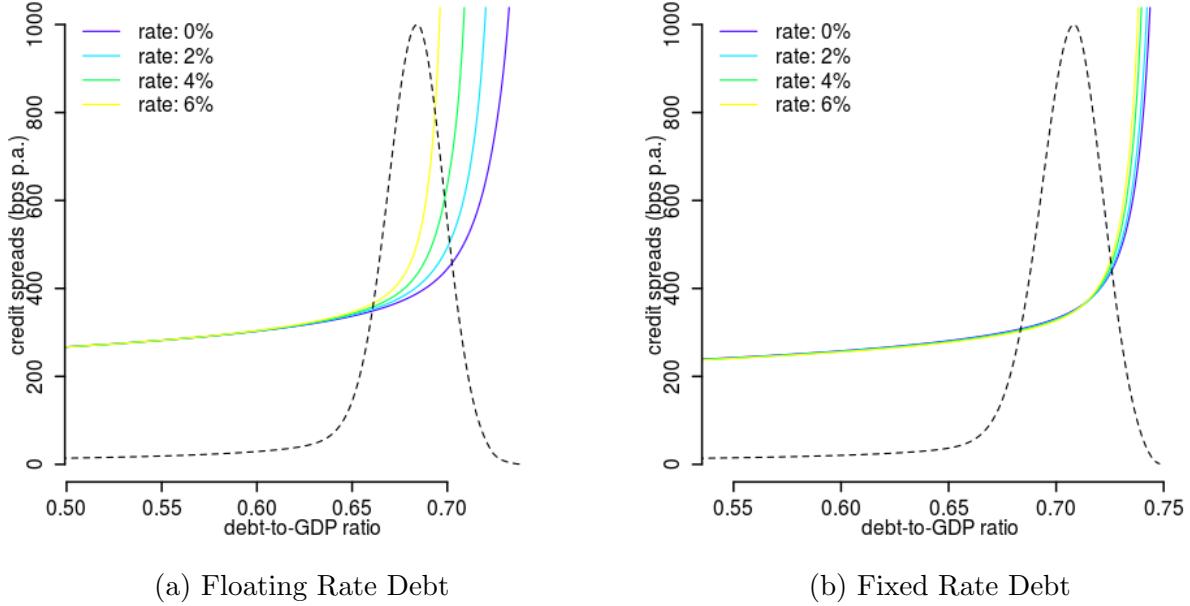
$$\iota_i(x) = \frac{1}{D_i(x)} \left[\left(\frac{\delta D_i(x) [(1 - \gamma)v_i(x)]^{\frac{\rho-\gamma}{1-\gamma}}}{-v_i'(x)} \right)^{1/\rho} + (\kappa_i + m)x - 1 \right]$$

The expression above indicates that for a given debt price schedule \mathbf{D} and life-time utility set of functions \mathbf{v} , the issuance rate increases with the coupon rate paid on the debt – this is a cash-flow effect, due to the fact that in a high interest rate environment, higher debt servicing costs incentivize the government to issue larger amounts of debt to achieve the same level of consumption than in a low rate environment.

So how can we explain the fact that the government issues more debt in a low rate environment, and defaults at a higher debt-to-GDP level? As discussed previously, buyers of risk-free floating rate debt are not exposed to the level of short rates – in other words, the interest rate duration of their investment is zero. On the other side, the sovereign government issuing such floating rate bonds is not indifferent: an increase in short term rates makes the

country strictly worse off, since its financing cost is higher. Thus, everything else equal, one would expect a country to default at a lower debt-to-GDP level when short term interest rates are higher. This means that the price of floating rate sovereign debt should decrease when short term interest rates jump up, and that credit spreads should be higher – this is effectively what happens, as illustrated in figure 1.11. Since the credit cost of issuing debt increases in a high interest rate environment, the sovereign government adjusts its issuance strategy downwards. In addition, the value function for the government (as a function of the debt-to-GDP ratio) is *steeper* in a high interest rate environment: indeed, when the government has no debt, one would expect the value function of the government to not be very sensitive to the level of short term rates; instead, when the government is highly indebted, one would expect the value function to be significantly lower in a high rate environment than in a low rate environment. In other words, $|v'_{r=6\%}(x)| > |v'_{r=0\%}(x)|$, i.e. the marginal cost of issuing debt is higher in a high US interest rate environment, pushing further down the issuance policy of the government, and exacerbating the current account adjustment upon an increase in US short rates.

Figure 1.11: Credit Spreads $\varsigma_i(x)$



1.8 Endogenous Growth

The model studied so far exhibits countries whose output process is specified exogenously. The government's sole motive for taking on debt is a consumption tilting and a smoothing motive. What happens when the government instead borrows from external creditors in order to finance domestic investments and capital accumulation? This section answers this question, and emphasizes two sources of debt overhang channels through which highly indebted sovereign government's capital investment decisions might be distorted. This enhanced model provides a simple micro-foundation to the output process of [4], [5], and many other articles in the sovereign debt literature, in which output growth is a mean-reverting variable.

Assume now that the country n has a production technology $Y_t = aK_t$, with $a > 0$, where K_t is the number of effective units of capital in the small open economy. Assume that

effective capital evolves according to:

$$dK_t = (H_t - \eta K_t)dt + K_t \boldsymbol{\sigma} \cdot d\mathbf{B}_t \quad (1.40)$$

H_t represents effective capital investments and η is the rate of depreciation of capital. The effective capital in this economy is hit by capital quality shocks similar to those in standard continuous time macro models such as [11] or [26]. Capital investments come with adjustment costs equal to $G(H_t, K_t) = g(H_t/K_t)K_t$, with g strictly convex, $g(0) = g'(0) = 0$. For simplicity, I will assume that $g(h) = \frac{\psi}{2}h^2$, where h is the investment rate per unit of effective capital. Government debt dynamics continue to follow equation 1.7. The government's resource constraint can now be written:

$$C_t + H_t = aK_t + I_t D_t - (\kappa + m)F_t - G(H_t, K_t) \quad (1.41)$$

Equation 1.41 simply says that the sum of consumption C_t and investment H_t need to be equal to the sum of output aK_t and capital markets net flows $I_t D_t - (\kappa + m)F_t$, net of investment adjustment costs $G(H_t, K_t)$. If and when the government elects to default, its (efficiency units of) capital stock suffers a discrete drop, falling from $K_{\tau-}$ to $K_{\tau} = \alpha K_{\tau-}$. The government is also excluded from capital markets for an exponentially distributed time period, and exits autarky with a debt-to-capital ratio that is θ times its debt-to-capital ratio pre-default.

The strategy of the government now consists in choosing an issuance policy \mathbf{I} , an investment policy \mathbf{H} , and a default policy $\boldsymbol{\tau}$ in order to maximize its objective function. The debt price follows equations 1.12 and 1.13. I look for an equilibrium where the debt-to-capital ratio $x_t := F_t/K_t$ and the SDF regime s_t are the state variables of this modified environment. The details of the equilibrium calculations are displayed in section A.1.19. When the debt-to-capital ratio of the country is x and when the SDF state is i , the debt price is equal to $D_i(x)$ and the life-time utility function for the government is equal to $v_i(x)K^{1-\gamma}$. The optimal debt

issuance rate and optimal investment rate follow Markov policies noted $I_i(K, F) = K\iota_i(x)$ and $H_i(K, F) = Kh_i(x)$, where ι_i, h_i are now functions of the debt-to-capital ratio x only. The optimal investment rate can be computed as follows:

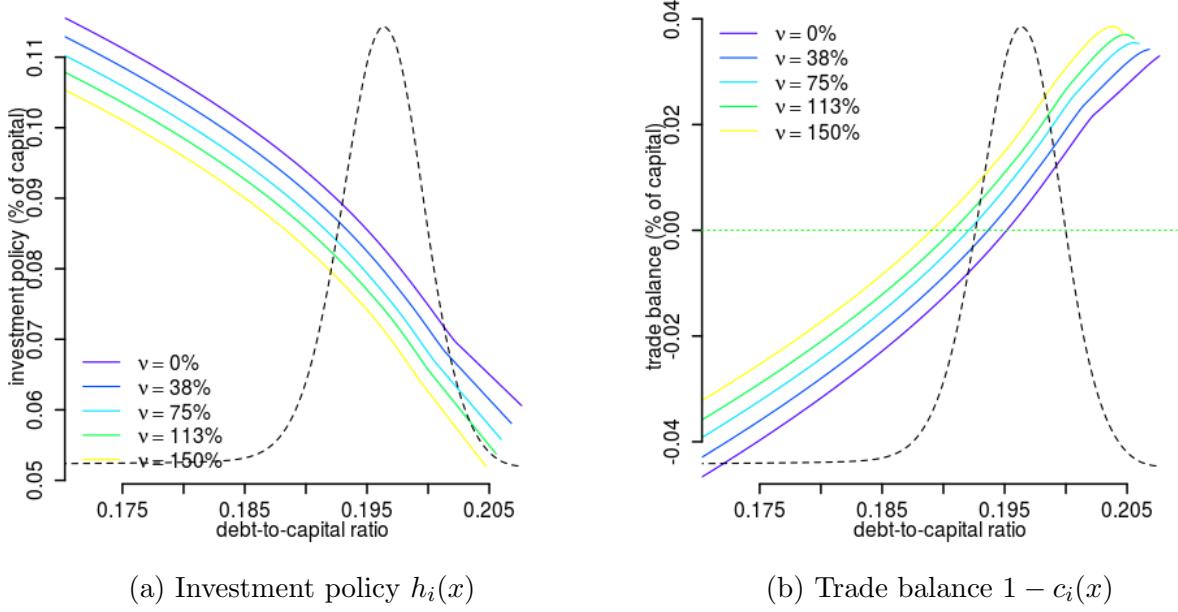
$$h_i(x) = \frac{1}{\psi} \left[xD_i(x) \left(1 - \frac{(1 - \gamma)v_i(x)}{xv'_i(x)} \right) - 1 \right] \quad (1.42)$$

This equation highlights immediately the potential debt overhang issue in this small open economy, as well as the impact of worse international debt market conditions. Indeed, since the parenthesis on the right hand-side of the equation above is always positive, a low debt price $D_i(x)$ depresses investment. This can happen for two reasons: first, if a sequence of bad capital quality shocks move the debt-to-capital ratio close to the default boundary; second, if risk prices in international capital markets are high, depressing sovereign bond prices and thus investment. In such case, expected output growth $h_i(x) - \eta$ is directly impacted.

My numerical illustration uses parameters in table 1.1, and the SDF model of [33]. I assume that the correlation between the country's capital quality shocks and the risk price vector is 100%. I use a depreciation rate $\eta = 7\%$. My productivity parameter a is calibrated in order to obtain a capital-to-output ratio K/Y of 3.3. Finally, the parameter ψ , which governs the investment adjustment costs, is calibrated to match a ratio of investment growth volatility over output growth volatility of 4, consistent with the evidence of [40] for many emerging economies.

Equilibrium investment and trade balance policies are shown in figure 1.12. The investment policy is a decreasing function of the debt-to-capital ratio. Thus, a sequence of bad capital quality shocks leads to lower investments, lower output growth, and higher credit spreads, key features of the data of emerging market economies, as highlighted for example by [53] in the context of several Latin American countries. It also leads to an amplification effect: bad capital quality shocks lead to lower debt prices and lower investments. This not only depresses the debt-to-capital ratio today, but also in the future given the fact that

Figure 1.12: Endogenous Growth – Policies



the investment rate is now lower, and it feeds back into lower debt prices. Spreads in this model should thus be more volatile than in my original endowment economy, and the ergodic debt-to-GDP distribution should be wider.

1.9 Conclusion

In this paper, I develop a new modeling framework to study sovereign debt quantities and prices by leveraging continuous time technology. This tractable approach leads to semi-closed form expressions for key asset pricing moments of interest, which enables me to discuss characteristics of the stochastic discount factor that are needed to explain properties of sovereign spreads, sovereign debt returns and cross-country spread correlations. The approach I propose reduces significantly the number of state variables typically present in discrete time sovereign debt models and opens the door to partial model estimation. My quantitative application shows that this modeling framework has the ability to match many quantities and price moments of the data, but fails on two dimensions: (i) the model-implied

dispersion of the debt-to-GDP ergodic distribution remains an order of magnitude smaller than in the data, and (ii) the magnitude of excess returns on sovereign debt requires very high levels of correlation between countries' output processes and SDFs typically used in the asset pricing literature to explain properties of US equity market returns – if the creditors' SDF is associated with US consumption or output dynamics, the empirical correlation between the US GDP and the GDP of my countries of interest is too low compared to the model estimates required to rationalize this high level of expected excess returns. My framework can be extended in multiple directions without adding any complexity or state variable. I show for example how to layer a linear production technology and study capital accumulation and the impact of debt overhang on investment decisions. One can also analyze an environment with exogenously specified sudden stops, by adding discrete SDF states where the sovereign government is prevented from issuing any debt. Finally, more theoretical work needs to be done, for example by leveraging viscosity theory and existence results for nested ordinary differential equations, in order to establish the existence of a Markov perfect equilibrium of the game between the sovereign government and its creditors.

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Appendix A

Continuous Time Theory of Endogenous Sovereign Defaults

A.1 Proofs and Detailed Calculations

A.1.1 A Useful Discrete Time Limit

In this section, I study a discrete time counterpart to the continuous-time model developed in this paper. Δt will represent a small time interval. I study the limit of a simple sovereign default model as $\Delta t \rightarrow 0$, and show heuristically that no short term debt can be supported in equilibrium at such limit. I focus on a government that has iso-elastic time-separable preferences with rate of time preference δ and risk-aversion γ as follows:

$$\mathbb{E} \left[\sum_{i=0}^{\infty} e^{-\delta i \Delta t} \frac{C_{i \Delta t}^{1-\gamma}}{1-\gamma} \Delta t \mid \mathcal{F}_0 \right]$$

Government output follows the discrete time equivalent of a geometric Brownian motion:

$$\frac{Y_{(i+1)\Delta t}}{Y_{i\Delta t}} = e^{\omega_{(i+1)\Delta t}}$$

In the above, $\omega_{(i+1)\Delta t}$ is an $\mathcal{F}_{(i+1)\Delta t}$ measurable binomial random variable that can take values $+\sigma\sqrt{\Delta t}$ or $-\sigma\sqrt{\Delta t}$ with respective probabilities p_u and p_d :

$$p_u := \frac{1}{2} \left(1 + \frac{\left(\mu - \frac{1}{2}\sigma^2 \right) \sqrt{\Delta t}}{\sigma} \right); \quad p_d := \frac{1}{2} \left(1 - \frac{\left(\mu - \frac{1}{2}\sigma^2 \right) \sqrt{\Delta t}}{\sigma} \right)$$

Thus, $\{\omega_{(i+1)\Delta t}\}_{i \geq 0}$ is a sequence of i.i.d. random variables, and I will use $\omega_{\Delta t}$ for simplicity to denote one of these random variables. When $\Delta t \rightarrow 0$, I have the following limits:

$$\begin{aligned} \mathbb{E}[e^{\omega_{\Delta t}}] &= 1 + \mu\Delta t + o(\Delta t) \\ \text{var}[e^{\omega_{\Delta t}}] &= \sigma^2\Delta t + o(\Delta t) \end{aligned}$$

The government has one-period debt at its disposal. Let $B_{i\Delta t}$ be the stock of debt that the government has to repay at time $i\Delta t$. Let $D(B_{(i+1)\Delta t}, Y_{i\Delta t})$ be the price of one unit of debt if the government plans to issue, at date $i\Delta t$, $B_{(i+1)\Delta t}$ units of debt maturing at period $(i+1)\Delta t$. The government resource constraint at time $i\Delta t$ is as follows:

$$C_{i\Delta t}\Delta t = Y_{i\Delta t}\Delta t - B_{i\Delta t} + D(B_{(i+1)\Delta t}, Y_{i\Delta t})B_{(i+1)\Delta t}$$

Upon default, GDP suffers a permanent shock of size $1 - \alpha$ and the government is in autarky forever after. Thus, if the pre-default output value is $Y_{i\Delta t}$, the value function for the government in default is equal to $V_d(Y_{i\Delta t})$, which satisfies:

$$V_d(Y) = \frac{(\alpha Y)^{1-\gamma}}{1-\gamma} \Delta t + e^{-\delta\Delta t} \mathbb{E}[V_d(e^{\omega_{\Delta t}} Y)]$$

Guessing that $V_d(Y) = v_d Y^{1-\gamma}$, the constant v_d is equal to:

$$v_d = \frac{\frac{\alpha^{1-\gamma}}{1-\gamma} \Delta t}{1 - e^{-\delta\Delta t} \mathbb{E}[e^{(1-\gamma)\omega_{\Delta t}}]} = \frac{\frac{\alpha^{1-\gamma}}{1-\gamma}}{\delta + (\gamma - 1) \left(\mu - \frac{1}{2}\gamma\sigma^2 \right)} + o(1)$$

The latter equality is valid when $\Delta t \rightarrow 0$. The government problem is as follows:

$$V_0(B, Y) = \max(V_c(B, Y), V_d(Y))$$

$$V_d(Y) = v_d Y^{1-\gamma}$$

$$V_c(B, Y) = \max_{B'} \left[\frac{\left(Y + \frac{1}{\Delta t} (D(B', Y) B' - B) \right)^{1-\gamma} \Delta t}{1 - \gamma} + e^{-\delta \Delta t} \mathbb{E} [V_0(B', Y') | Y] \right]$$

The bond price verifies $D(B', Y) = e^{-r \Delta t} \Pr(V_c(B', Y') \geq V_d(Y') | Y)$, where r is the interest rate at which lenders discount risk-free cashflows. This formula assumes that upon default, sovereign creditors recover nothing from their defaulted debt claim. One can show that for any bond price function that is homogeneous of degree zero and decreasing in B' , the value function V_c is homogeneous of degree $1 - \gamma$ and the best “default” response by the government is to follow a linear barrier policy of the form $\tau := \inf\{t : B_t \geq \bar{x} Y_t\}$ for some endogenously determined constant \bar{x} (which depends on the time step Δt)¹. Noting $x := B/Y$, and using the homogeneity property of the value function, the government life-time utility can be written $V_c(B, Y) = v_c(x) Y^{1-\gamma}$, and the government problem can be simplified as follows:

$$v_c(x) = \max_{x'} \left[\frac{\left(1 + \frac{1}{\Delta t} (D(x') x' - x) \right)^{1-\gamma} \Delta t}{1 - \gamma} + e^{-\delta \Delta t} \mathbb{E} \left[e^{(1-\gamma)\omega \Delta t} \max(v_d, v_c(e^{-\omega \Delta t} x')) \right] \right]$$

For simplicity, I assume that the choice set for the debt-to-GDP ratio of the government is discrete. In other words, I discretize the state space into a grid $G_{\Delta t}$ consisting of a countable

1. Of course this statement is assuming the existence of the value function V_c , which is the fixed point of a functional equation. Since my discussion on the lack of equilibria with defaultable debt at the limit of my discrete time economies is only heuristic, I side-step the proof of existence of V_c .

number of points $x_i = e^{i\sigma\sqrt{\Delta t}}$, for $i \in \mathbb{N}$, where I note i_d the default index². In other words, I assume that at $x = x_{i_d}$, the government strictly prefer defaulting over repaying, while for $i < i_d$, the government weakly prefer repaying over defaulting. This model structure guarantees that the debt-to-GDP ratio stays on the grid $G_{\Delta t}$ at all times, irrespective of the sequence of GDP shocks and decisions made by the government. In any cutoff equilibrium with default threshold $\bar{x} = x_{i_d}$, the debt price must satisfy:

$$D(x_i) = \begin{cases} e^{-r\Delta t} & \text{if } i < i_d - 1 \\ \frac{1}{2} \left(1 + \frac{(\mu - \frac{1}{2}\sigma^2)\sqrt{\Delta t}}{\sigma} \right) e^{-r\Delta t} & \text{if } i = i_d - 1 \\ 0 & \text{if } i \geq i_d \end{cases}$$

Now consider the resource constraint for the government if at the beginning of a given period, $x = x_i$, with $i = i_d - 1$. Since $i < i_d$, it is optimal for the government to continue to perform on its debt obligations. If the government selects a debt-to-GDP ratio $x_j < x_i$, the consumption-to-output ratio, using the resource constraint, is equal to:

$$1 + \frac{1}{\Delta t} \left(e^{j\sigma\sqrt{\Delta t} - r\Delta t} - e^{i\sigma\sqrt{\Delta t}} \right) = \frac{(j - i)\sigma}{\sqrt{\Delta t}} + o\left(\frac{1}{\sqrt{\Delta t}}\right) \xrightarrow{\Delta t \rightarrow 0} -\infty$$

If the government selects a debt-to-GDP ratio $x_j = x_i$, the consumption-to-output ratio is equal to:

$$1 + \frac{1}{\Delta t} \left(\frac{1}{2} \left(1 + \frac{(\mu - \frac{1}{2}\sigma^2)\sqrt{\Delta t}}{\sigma} \right) e^{i\sigma\sqrt{\Delta t} - r\Delta t} - e^{i\sigma\sqrt{\Delta t}} \right) = -\frac{1}{2\Delta t} + o\left(\frac{1}{\Delta t}\right) \xrightarrow{\Delta t \rightarrow 0} -\infty$$

Of course if the government selects a debt-to-GDP ratio $x_j = x_{i_d}$, it obtains no proceeds from its debt issuance – this decision cannot be optimal. Thus, as $\Delta t \rightarrow 0$, the only possibility for government consumption to be positive at all points of the state space $x_i < \bar{x}$ is for the

2. This model structure implicitly assumes that the government is prevented from saving. This can also be achieved endogenously by using a sufficiently high rate of time preference δ .

default boundary to converge to zero. Note that this conclusion stems from (A) the shape of the government bond price schedule (at I take $\Delta t \rightarrow 0$, such debt price schedule converges to a step function, equal to the risk-free benchmark for $x < \bar{x}$ and equal to zero at $x \geq \bar{x}$), and (B) the continuous sample path of geometric Brownian motions. Had the output process featured jumps, a cutoff equilibrium with strictly positive debt can be supported. \square

A.1.2 Output Process in this Article vs. [5]

The output process in [5] can be recast in continuous time as follows:

$$\begin{aligned} Y_t &= e^{z_t} \Gamma_t \\ d \ln \Gamma_t &= \ln g_t dt \\ dz_t &= -\kappa_z (z_t - \mu_z) dt + \sigma_z dB_t^z \\ d \ln g_t &= -\kappa_g (\ln g_t - \ln \mu_g) dt + \sigma_g dB_t^g \end{aligned}$$

In the above, B_t^g, B_t^z are standard Brownian motions assumed to be independent, κ_z, κ_g are positive constants that parametrize the speed of mean reversion of the processes z_t and $\ln g_t$ respectively. Two approaches can be used to approximate the output process above by equation 1.1.

In the first approach, conditional and unconditional moments of consumption growth can be matched. One can for example match the s -lagged auto-correlation of log consumption growth $R_{model}(s)$ for both models. Such auto-correlations are equal to:

$$\begin{aligned} R_{AG}(s) &= e^{-\kappa_g s} \\ R_{GBM}(s) &= \frac{\sum_{i=1}^{N_s} \pi_i \left(\mu_i - \frac{1}{2} |\boldsymbol{\sigma}_i|^2 \right) \sum_{j=1}^{N_s} \left[\left(e^{s\Lambda} \right)_{ij} - \pi_j \right] \left(\mu_j - \frac{1}{2} |\boldsymbol{\sigma}_j|^2 \right)}{\sum_{i=1}^{N_s} \pi_i \left(\mu_i - \frac{1}{2} |\boldsymbol{\sigma}_i|^2 \right)^2 - \left(\sum_{i=1}^{N_s} \pi_i \left(\mu_i - \frac{1}{2} |\boldsymbol{\sigma}_i|^2 \right) \right)^2} \end{aligned}$$

One can also match the unconditional log consumption growth volatility Σ_{model} for both models. Such log consumption growth volatilities are equal to:

$$\Sigma_{AG} = \sigma_z$$

$$\Sigma_{GBM} = \sqrt{\sum_{i=1}^{N_s} \pi_i |\boldsymbol{\sigma}_i|^2}$$

In the second approach, one can use a procedure introduced by [21] to extract a martingale component to the logarithm of output for both models, and match separately (a) the volatility of the martingale component of both models, (b) conditional and unconditional moments of the stationary component of both models, and (c) the time trend of both models. For the model of [5], the decomposition of $\ln Y_t$ takes the following form:

$$d \ln Y_t = \frac{\sigma_g}{\kappa_g} dB_t^g - d \left(\frac{1}{\kappa_g} \ln g_t - z_t \right) + \ln \mu_g dt$$

$$\ln Y_t - \ln Y_0 = \underbrace{\int_0^t \frac{\sigma_g}{\kappa_g} dB_s^g}_{\text{martingale component}} + \underbrace{\left[\frac{1}{\kappa_g} \ln g_0 - z_0 \right] - \left[\frac{1}{\kappa_g} \ln g_t - z_t \right]}_{\text{stationary component}} + \underbrace{t \ln \mu_g}_{\text{time trend}}$$

Thus, the permanent component to log output is purely driven by (rescaled) shocks to the state variable $\ln g_t$, while the state variable z_t has a purely transitory role. The long-run time trend in log-output is equal to $\ln \mu_g$. When I focus on the output process driven by equation 1.1 (and assuming, for the purpose of this section, that \mathbf{B}_t is unidimensional), I can compute small increments in logarithms as follows:

$$d \ln Y_t = \left(\mu_{st} - \frac{1}{2} \sigma_{st}^2 \right) dt + \sigma_{st} dB_t$$

In the above, s_t is the discrete state Markov chain with generator matrix Λ and stationary

density $\boldsymbol{\pi}$. In such case, note that:

$$\mathbb{E} [d \ln Y_t | \mathcal{F}_t] = \left(\mu_{s_t} - \frac{1}{2} \sigma_{s_t}^2 \right) dt$$

Let $\kappa_{2,t} := \int_0^t \sigma_{s_u} dB_u$ and note that for $\tau > t$, $\mathbb{E} [\kappa_{2,\tau} | \mathcal{F}_t] = \kappa_{2,t}$, in other words $\kappa_{2,t}$ is a martingale. I then compute the long term average of log output growth:

$$\ln g_\infty := \boldsymbol{\pi} \cdot \left(\boldsymbol{\mu} - \frac{1}{2} \boldsymbol{\sigma}^2 \right)$$

I create the function $f(s) := \mu_s - \frac{1}{2} \sigma_s^2 - \ln g_\infty = \left(\boldsymbol{\mu} - \frac{1}{2} \boldsymbol{\sigma}^2 - \ln g_\infty \mathbf{1} \right) \cdot \mathbf{e}_{s_t}$, \mathbf{e}_s is an $N_s \times 1$ column vector with entry s equal to 1, and all other entries equal to zero, and compute:

$$\int_0^\infty \mathbb{E} [f(s_{t+u}) | \mathcal{F}_t] du = \Lambda^{-1} \left(\boldsymbol{\mu} - \frac{1}{2} \boldsymbol{\sigma}^2 - \ln g_\infty \mathbf{1} \right) \cdot \mathbf{e}_{s_t}$$

In the above, Λ^{-1} is the generalized inverse of the generator matrix Λ (since the rows of the matrix Λ sum up to zero, Λ is not invertible). Finally, I introduce the martingale $\kappa_{1,t}$, defined as follows:

$$\begin{aligned} \kappa_{1,t} := & \int_0^t \sum_{i=1}^{N_s} \Lambda^{-1} \left(\boldsymbol{\mu} - \frac{1}{2} \boldsymbol{\sigma}^2 - \ln g_\infty \mathbf{1} \right) \cdot (\mathbf{e}_i - \mathbf{e}_{s_{u-}}) dN_u^{(s_{u-}, i)} \\ & + \int_0^t \left(\boldsymbol{\mu} - \frac{1}{2} \boldsymbol{\sigma}^2 - \ln g_\infty \mathbf{1} \right) \cdot \mathbf{e}_{s_u} du \end{aligned}$$

I can then decompose increments in log output growth as follows:

$$\begin{aligned} d \ln Y_t &= d(\kappa_{1,t} + \kappa_{2,t}) - \Lambda^{-1} \left(\boldsymbol{\mu} - \frac{1}{2} \boldsymbol{\sigma}^2 - \ln g_\infty \mathbf{1} \right) \cdot (\mathbf{e}_{s_t} - \mathbf{e}_{s_{t-}}) dN_t^{(s_{t-}, s_t)} + \ln g_\infty dt \\ \ln Y_t - \ln Y_0 &= \underbrace{\int_0^t d(\kappa_{1,s} + \kappa_{2,s})}_{\text{martingale component}} + \underbrace{\Lambda^{-1} \left(\boldsymbol{\mu} - \frac{1}{2} \boldsymbol{\sigma}^2 - \ln g_\infty \mathbf{1} \right) \cdot (\mathbf{e}_{s_0} - \mathbf{e}_{s_t})}_{\text{stationary component}} + \underbrace{t \ln g_\infty}_{\text{time trend}} \end{aligned}$$

Thus, in the case of this article, the martingale component of log output is the sum of a

Brownian process $\kappa_{1,t}$ and a jump process $\kappa_{2,t}$, while the stationary component is a pure jump process. By carefully parameterizing $\boldsymbol{\mu}, \boldsymbol{\sigma}, \Lambda$, one can approximate the long run trend, stationary and martingale components in [5] by a stochastic process that follows equation 1.1. \square

A.1.3 No-Debt Life-Time Utility

I note $J_i(Y)$ the life-time utility value when the government cannot issue debt, when the level of output is Y and when the Markov state is $s_t = i$. I postulate that this life-time utility can be written $J_i(Y) = K_i Y^{1-\gamma}$ for some constants $\{K_i\}_{i \leq N_s}$ to be determined. J_i verifies the following HJB equation:

$$0 = \varphi(Y_t, J_{s_t}(Y_t)) + \mathcal{A}J_{s_t}(Y_t)$$

For convenience, I have introduced the differential operator \mathcal{A} , defined for any stochastic process $\{Z_t\}_{t \geq 0}$ (belonging to an appropriate class of stochastic processes) as follows:

$$\mathcal{A}Z_t := \lim_{\epsilon \rightarrow 0} \frac{\mathbb{E}[Z_{t+\epsilon} | \mathcal{F}_t] - Z_t}{\epsilon} \quad (\text{A.1})$$

Injecting my guess function for J_i , I obtain the following equation, for $i \leq N_s$:

$$\begin{aligned} 0 = \delta \frac{1-\gamma}{1-\rho} K_i Y^{1-\gamma} & \left(\frac{1}{[(1-\gamma)K_i]^{\frac{1-\rho}{1-\gamma}}} - 1 \right) \\ & + \mu_i(1-\gamma)K_i Y^{1-\gamma} - \frac{1}{2}\gamma(1-\gamma)|\boldsymbol{\sigma}_i|^2 K_i Y^{1-\gamma} + \sum_{j=1}^{N_s} \Lambda_{ij} K_j Y^{1-\gamma} \end{aligned}$$

Dividing by $K_i Y^{1-\gamma}$ and simplifying, the set of constants $\mathbf{K} := \{K_i\}_{i \leq N_s}$ must satisfy:

$$\left[\text{diag}_i(A_i) - \frac{1-\rho}{1-\gamma} \Lambda \right] [(1-\gamma)\mathbf{K}] = \delta [(1-\gamma)\mathbf{K}]^{\frac{\rho-\gamma}{1-\gamma}}$$

Vector exponentiation in the equation above has to be understood element by element. Under assumption 1, this set of N_s equations in N_s unknown has a unique solution. \square

A.1.4 Creditors' Stochastic Discount Factor

The discussion below is substantially similar to the discussion in [13]. Assume a representative creditor whose equilibrium consumption $C_{c,t}$ follows, under the physical measure \mathbb{P} :

$$\frac{dC_{c,t}}{C_{c,t}} = \mu_{c,s_t} dt + \boldsymbol{\sigma}_{c,s_t} \cdot d\mathbf{B}_t$$

In the above, $\{s_t\}_{t \geq 0}$ follows a discrete state Markov process with generator matrix $\Lambda = (\Lambda_{ij})_{1 \leq i,j \leq N_s}$. N_s is the total number of discrete states. Assume that such creditor ranks consumption streams according to the utility specification:

$$U_t = \mathbb{E} \left[\int_t^{+\infty} \Psi(C_s, U_s) ds | \mathcal{F}_t \right] \quad (\text{A.2})$$

In the above, the aggregator Ψ is assumed to be equal to:

$$\Psi(C, U) := \delta_c \frac{1 - \gamma_c}{1 - \rho_c} U \left(\frac{C^{1 - \rho_c}}{((1 - \gamma_c)U)^{\frac{1 - \rho_c}{1 - \gamma_c}}} - 1 \right)$$

The representative creditor's life-time utility function can be written $U_t = \frac{(h_{s_t} Y_t)^{1 - \gamma}}{1 - \gamma}$, where the constants $\{h_i\}_{1 \leq i \leq N_s}$ satisfy the set of non-linear equations:

$$0 = \delta_c \frac{1 - \gamma_c}{1 - \rho_c} h_i^{\rho_c - 1} + (1 - \gamma_c) \mu_{c,i} - \frac{1}{2} \gamma_c (1 - \gamma_c) |\boldsymbol{\sigma}_{c,i}|^2 - \delta_c \frac{1 - \gamma_c}{1 - \rho_c} + \sum_{j=1}^{N_s} \Lambda_{ij} \left(\frac{h_j}{h_i} \right)^{1 - \gamma_c}$$

As [19] shows, the marginal utility process associated with recursive preferences can be written:

$$M_t = \exp \left[\int_0^t \frac{\partial \Psi}{\partial U} (C_{c,z}, U_{c,z}) dz \right] \frac{\partial \Psi}{\partial C} (C_{c,t}, U_{c,t}) \quad (\text{A.3})$$

Applying Itô's lemma then leads to the following dynamic evolution of M_t :

$$\frac{dM_t}{M_t} = -r_{s_t} dt - \boldsymbol{\nu}_{s_t} \cdot d\mathbf{B}_t + \sum_{s_t \neq s_{t-}} \left(e^{v(s_{t-}, s_t)} - 1 \right) \left(dN_t^{(s_{t-}, s_t)} - \Lambda_{s_{t-}, s_t} dt \right) \quad (\text{A.4})$$

In the above, $N_t^{(s_{t-}, s_t)}$ is a Poisson counting process for transitions from state s_{t-} to state s_t . For each discrete Markov state i , the state dependent risk price vector ν_i is equal to:

$$\boldsymbol{\nu}_i = \gamma_c \boldsymbol{\sigma}_{c,i} \quad (\text{A.5})$$

For each discrete Markov state i , the state dependent risk free rate r_i is equal to:

$$r_i = -\delta_c \frac{1 - \gamma_c}{1 - \rho_c} \left[\left(\frac{\rho_c - \gamma_c}{1 - \gamma_c} \right) h_i^{\rho_c - 1} - 1 \right] + \gamma_c \mu_{c,i} - \frac{1}{2} \gamma_c (1 + \gamma_c) |\boldsymbol{\sigma}_{c,i}|^2 - \sum_{j=1}^{N_s} \Lambda_{ij} e^{v(i,j)} \quad (\text{A.6})$$

Finally, the SDF relative jump sizes encoded via $v(i, j)$ are equal to:

$$v(i, j) = (\rho_c - \gamma_c) \ln \left(\frac{h_j}{h_i} \right) \quad (\text{A.7})$$

Compensations per unit of Brownian risk can be read via the coordinates of the vector $\boldsymbol{\nu}_{s_t}$, while jump compensation is encoded via $e^{v(s_{t-}, s_t)} - 1$. Indeed, take any asset whose valuation V_t follows:

$$\frac{dV_t}{V_t} = \mu_{v,t} dt + \boldsymbol{\sigma}_{v,t} \cdot d\mathbf{B}_t + \sum_{s_t \neq s_{t-}} \left(e^{\xi_v(s_{t-}, s_t)} - 1 \right) \left(dN_t^{(s_{t-}, s_t)} - \Lambda_{s_{t-}, s_t} dt \right)$$

Absence of arbitrage imposes that $M_t V_t$ be a local martingale, in other words $\mathbb{E}[d(M_t V_t)] = 0$. Using Itô's lemma leads to the following pricing restriction:

$$\mu_{v,t} - r_{s_t} = \boldsymbol{\sigma}_{v,t} \cdot \boldsymbol{\nu}_{s_t} - \sum_{s_t \neq s_{t-}} \left(e^{\xi_v(s_{t-}, s_t)} - 1 \right) \left(e^{v(s_{t-}, s_t)} - 1 \right) \Lambda_{s_{t-}, s_t}$$

For example, a claim to the creditor's aggregate consumption $C_{c,t} = Y_{c,t}$ has a price $P_{c,t} = P_{c,s_t}(Y_t)$ that solve the following HJB equation:

$$r_i P_{c,i}(Y) = Y + (\mu_{c,i} - \boldsymbol{\nu}_i \cdot \boldsymbol{\sigma}_{c,i}) Y P'_{c,i}(Y) + \frac{1}{2} |\boldsymbol{\sigma}_{c,i}|^2 Y^2 P''_{c,i}(Y) + \sum_{j=1}^{N_s} \hat{\Lambda}_{ij} P_{c,j}(Y)$$

The solution to the system above is of the form $P_{c,i}(Y) = P_{c,i} Y$, for a set of constants $\{P_{c,i}\}_{i \leq N_s}$ that are equal to:

$$\mathbf{P}_c = \left[\text{diag}_i (r_i + \boldsymbol{\nu}_i \cdot \boldsymbol{\sigma}_{c,i} - \mu_{c,i}) - \hat{\Lambda} \right]^{-1} \mathbf{1} \quad (\text{A.8})$$

Expected excess returns on this claim thus take the following form:

$$\begin{aligned} \mathbb{E} \left[\frac{d(P_{c,s_t} Y_t) + Y_t dt}{P_{c,s_t} Y_t} - r_{s_t} dt \mid \mathcal{F}_t \right] &= \boldsymbol{\nu}_{s_t} \cdot \boldsymbol{\sigma}_{c,s_t} dt \\ &\quad - \sum_{s'} \Lambda_{s_{t-}, s'} \left(\frac{P_{c,s'}}{P_{c,s_{t-}}} - 1 \right) \left(e^{v(s_{t-}, s')} - 1 \right) dt \end{aligned}$$

A.1.5 A General Equilibrium Interpretation

In this section, I describe how to obtain, in a general equilibrium setting, the stochastic discount factor M_t , whose dynamics are given by equation 1.10. For this, I introduce the subscript “ n ” for a country's identity, and assume a continuum of countries of measure 1. Y_t^n is re-interpreted as tax collections of country n – imagine for example that country n has a flow output X_t^n and a tax rate ϵ^n , such that $Y_t^n = \epsilon^n X_t^n$. Suppose country n 's output

evolves as follows:

$$\frac{dX_t^n}{X_t^n} = \mu_{s_t} dt + \boldsymbol{\sigma}_{s_t} \cdot d\mathbf{B}_t + \sigma^n dB_t^n$$

B_t^n is a country-specific Brownian shock that is independent of \mathbf{B}_t , the vector of aggregate Brownian shocks. Note that all countries need to have the same expected growth rate μ_{s_t} in all states s_t , but not necessarily the same output volatility³. The dynamics for aggregate world output is thus:

$$\begin{aligned} dX_t &= \int dX_t^n dn \\ &= \int [\mu_{s_t} X_t^n dt + X_t^n (\boldsymbol{\sigma}_{s_t} \cdot d\mathbf{B}_t + \sigma^n dB_t^n)] dn \\ &= X_t (\mu_{s_t} dt + \boldsymbol{\sigma}_{s_t} \cdot d\mathbf{B}_t) \end{aligned}$$

The latter equality uses the law of large numbers and leverages the fact that X_t^n and dB_t^n are not correlated. Given that tax revenues of country n are equal to $Y_t^n = \epsilon^n X_t^n$, the government revenue process Y_t^n follows the same stochastic differential equation as X_t^n .

Government n resource constraint is the following:

$$C_t^n = Y_t^n + I_t^n D_t^n - (\kappa + m) F_t^n$$

C_t^n represents government spending, I_t^n represents government debt issuances (debt is issued at price D_t^n), and F_t^n represents the face value of government debt. We assume for the moment that ϵ^n is fixed and exogenous, and that the government will only choose its financing and default policies. Due to institutional frictions, government n 's utility function is not exactly equal to the utility function of its citizens; more specifically, the government maximizes

3. Forcing all countries to have the same expected GDP growth rate also guarantees that countries all “survive” as $t \rightarrow +\infty$, in other words no country becomes arbitrarily small asymptotically.

the following objective function (subject to government resource constraint above):

$$J_t^n = \mathbb{E} \left[\int_t^{+\infty} \varphi(C_s^n, J_s^n) ds | \mathcal{F}_t \right]$$

In other words, the government maximizes flow utility over government spending only; moreover, the government will be more impatient than its citizens: δ , the rate of time preference of the government, will verify $\delta > \delta_c$, where δ_c will be the rate of time preference of the citizens/creditors (to be discussed shortly). Government of country n may elect to default on its debt. In such case, the government is shut down from capital markets for some exponentially distributed time (parametrized by λ), and suffers a temporary drop in tax collection efficiency: while in autarky, the government only collects $Y_t^n = \alpha \epsilon^n X_t^n$ from its citizens, with $\alpha < 1$. Upon exit from financial autarky, the government from country n emerges with a lower debt burden, and recovers its pre-default tax efficiency $Y_t^n = \epsilon^n X_t^n$. Note that the drop in tax collections suffered by the defaulting government is only temporary, and lasts the time of the capital market's exclusion.

Government debt is bought by a “representative world investor” – in other words, a citizen whose equilibrium consumption process is proportional to world consumption. Let \mathcal{D}_t be the set of indices of countries in default at time t , and $\mathcal{D}_t^c = [0, 1] \setminus \mathcal{D}_t$ its complement. Remember that world citizens enjoy flow consumption Γ_t equal to:

$$\Gamma_t := \int_{n \in \mathcal{D}_t^c} [(1 - \epsilon^n) X_t^n + (\kappa + m) F_t^n - I_t^n D_t^n] dn + \int_{n \in \mathcal{D}_t} (1 - \alpha \epsilon^n) X_t^n dn = X_t - C_t$$

In the above, C_t represents aggregate government spending. Consumption by the “representative world investor” is simply equal to world output minus taxes paid plus income received on its government debt portfolio minus investments in government debt. Assume this “representative world investor” faces dynamically complete markets, and has preferences over

consumption Γ_t and government spending C_t as follows:

$$U_t = \mathbb{E} \left[\int_t^{+\infty} \Psi(\Gamma_s + C_s, U_s) ds | \mathcal{F}_t \right]$$

In other words, consumption and government spending are perfect substitutes in the representative investor's preferences. Given those assumptions, the investor's equilibrium consumption (via market clearing) is equal to $\Gamma_t + C_t = X_t$, and the investor's marginal utility is given by equation A.3, meaning that the pricing kernel M_t evolves as follows:

$$\frac{dM_t}{M_t} = -r_{st} dt - \boldsymbol{\nu}_{st} \cdot d\mathbf{B}_t + \sum_{s_t \neq s_{t-}} \left(e^{v(s_{t-}, s_t)} - 1 \right) \left(dN_t^{(s_{t-}, s_t)} - \Lambda_{s_{t-}, s_t} dt \right)$$

r_s , $\boldsymbol{\nu}_s$ and the SDF jumps $v(s, s')$ correspond to those described in section A.1.4. \square

A.1.6 Controlled Stochastic Processes $Y_t^{(\tau)}$ and $F_t^{(\mathbf{I}, \tau)}$

An admissible issuance policy $\{I_t\}_{t \geq 0}$ of the government is a progressively measurable process that is a function of the payoff-relevant variables. In other words, $I_t = I_{st}(Y_t, F_t)$, for a set of measurable functions $\mathbf{I} := \{I_i\}_{i \leq N_s}$. I will require the function I_t to satisfy the standard integrability condition, for all $t \geq 0$, almost surely:

$$\mathbb{E} \left[\int_0^t |I_s| ds \right] < +\infty$$

An admissible default policy is an increasing sequence of stopping times (with respect to \mathcal{F}_t) noted $\boldsymbol{\tau} = \{\tau_k\}_{k \geq 1}$, which can be written (for $k \geq 0$), $\tau_{k+1} = \inf\{t \geq \tau_k + \tau_{e,k} : (Y_t, F_t) \in \mathcal{O}_{s_t}\}$, for a finite number of Borel sets $\{\mathcal{O}_i\}_{i \leq N_s}$, where $\{\tau_{e,k}\}_{k \geq 1}$ is a sequence of i.i.d. exponentially distributed times (with parameter λ), and where I have set $\tau_0 = \tau_{e,0} = 0$.

For a given admissible default policy $\boldsymbol{\tau} \in \mathcal{T}$, define $N_{d,t}^{(\boldsymbol{\tau})} := \max\{k \in \mathbb{N} : \tau_k \leq t\}$ (resp. $N_{e,t}^{(\boldsymbol{\tau})} := \max\{k \in \mathbb{N} : \tau_k + \tau_{e,k} \leq t\}$) to be the counting process for default events (resp.

capital markets re-entry events). Let $1_{d,t}^{(\tau)}$ be the default indicator, equal to 1 when the sovereign government is in default, and zero otherwise:

$$1_{d,t}^{(\tau)} := 1_{t \in [\tau_{N_{d,t}}, \tau_{N_{d,t}} + \tau_{e,N_{d,t}})}$$

Using this notation, the dynamic evolution of the controlled stochastic process $Y_t^{(\tau)}$ can be expressed as follows:

$$dY_t^{(\tau)} = Y_{t-}^{(\tau)} \left(\mu_{s_t} dt + \boldsymbol{\sigma}_{s_t} \cdot d\mathbf{B}_t + (\alpha - 1) dN_{d,t}^{(\tau)} \right) \quad (\text{A.9})$$

Similarly, the dynamic evolution of the controlled stochastic process $F^{(\mathbf{I},\tau)}$ can be expressed as follows:

$$\begin{aligned} dF_t^{(\mathbf{I},\tau)} &= \left(I_{t-} - mF_{t-}^{(\mathbf{I},\tau)} \right) \left(1 - 1_{d,t}^{(\tau)} \right) dt \\ &+ \left(\theta \alpha \exp \left[\int_{\tau_{N_{d,t}}}^t \left(\mu_{s_u} - \frac{1}{2} |\boldsymbol{\sigma}_{s_u}|^2 \right) du + \boldsymbol{\sigma}_{s_u} \cdot d\mathbf{B}_u \right] - 1 \right) F_{t-}^{(\mathbf{I},\tau)} dN_{e,t}^{(\tau)} \quad (\text{A.10}) \end{aligned}$$

The drift term in the dynamic evolution of $F_t^{(\mathbf{I},\tau)}$ relates to issuances and debt redemptions when the government is performing under its debt obligations, while the jump term relates to reductions in the debt face value in connection with a restructuring and re-entry into capital markets post-default.

Given a debt price schedule $\mathbf{D} := \{D_i(\cdot, \cdot)\}_{i \leq N_s}$ that the government faces, and given admissible issuance and default policies (\mathbf{I}, τ) used by the government (where (\mathbf{I}, τ) might not necessarily be consistent with the debt prices \mathbf{D}), there is a controlled flow consumption process $C_t^{(\mathbf{I},\tau;\mathbf{D})}$:

$$C_t^{(\mathbf{I},\tau;\mathbf{D})} = \left[Y_t^{(\tau)} + I_t D_{s_t} \left(Y_t^{(\tau)}, F_t^{(\mathbf{I},\tau)} \right) - (\kappa + m) F_t^{(\mathbf{I},\tau)} \right] \left(1 - 1_{d,t}^{(\tau)} \right) + Y_t^{(\tau)} 1_{d,t}^{(\tau)}$$

The indicator functions in this expression highlight the fact that the government can smooth consumption via debt issuances and buy-backs when performing, while it is unable to do so in default.

A.1.7 Monotonicity of V_i

Take a set of debt price functions $\{D_i(\cdot, \cdot)\}_{i \leq N_s}$ that are homogeneous of degree zero and decreasing in F . I focus my attention on two initial levels of output, $Y^{(1)}$ and $Y^{(2)} > Y^{(1)}$, and show that I must have $V_i(Y^{(1)}, F; \mathbf{D}) < V_i(Y^{(2)}, F; \mathbf{D})$. First, take any arbitrary policy $(\mathbf{I}, \boldsymbol{\tau}) \in \mathcal{I} \times \mathcal{T}$ (not necessarily optimal) followed by the government. Following the issuance policy \mathbf{I} and starting in state $(Y^{(2)}, F)$ yields strictly higher flow payoffs at each time t . Indeed, $\{Y_t^{(\boldsymbol{\tau})} | Y_0 = Y^{(2)}\}$ is almost surely greater than $\{Y_t^{(\boldsymbol{\tau})} | Y_0 = Y^{(1)}\}$. In addition, $\{I_t D_{st} \left(Y_t^{(\boldsymbol{\tau})}, F_t^{(\mathbf{I}, \boldsymbol{\tau})} \right) | Y_0 = Y^{(2)}\}$ is almost surely greater than $\{I_t D_{st} \left(Y_t^{(\boldsymbol{\tau})}, F_t^{(\mathbf{I}, \boldsymbol{\tau})} \right) | Y_0 = Y^{(1)}\}$, since the debt price conditioned on starting in state $(Y^{(2)}, F)$ is almost surely greater than the debt price conditioned on starting in state $(Y^{(1)}, F)$ (since I assumed that the debt price is decreasing in the debt face value). Thus, $\{C_t^{(\mathbf{I}, \boldsymbol{\tau}; \mathbf{D})} | Y_0 = Y^{(2)}\}$ is almost surely greater than $\{C_t^{(\mathbf{I}, \boldsymbol{\tau}; \mathbf{D})} | Y_0 = Y^{(1)}\}$, which means that the life-time utility is increasing in Y , for any arbitrary issuance and default policy. Thus, the supremum over all feasible issuance and default policies, $V_i(\cdot, \cdot; \mathbf{D})$, is also increasing in output Y . The proof for the monotonicity of $V_i(\cdot, \cdot; \mathbf{D})$ in F is identical, since consumption $C_t^{(\mathbf{I}, \boldsymbol{\tau}; \mathbf{D})}$ is decreasing in the level of indebtedness F and since $D_i(\cdot, \cdot)$ is decreasing in F .

I then show that the optimal issuance policy is homogeneous of degree 1 and the optimal default policy is barrier type. Take an arbitrary (Y, F) , and the related optimal Markov issuance and default policies $(\mathbf{I}_{\mathbf{Y}, \mathbf{F}}^*, \boldsymbol{\tau}_{\mathbf{Y}, \mathbf{F}}^*) = \arg \max J_i(Y, F; (\mathbf{I}, \boldsymbol{\tau}); \mathbf{D})$. Take $\epsilon > 0$, and focus on starting output and face value levels $(\epsilon Y, \epsilon F)$. Consider the policy $(\mathbf{I}_{\epsilon \mathbf{Y}, \epsilon \mathbf{F}}, \boldsymbol{\tau}_{\epsilon \mathbf{Y}, \epsilon \mathbf{F}})$, such that $\mathbf{I}_{\epsilon \mathbf{Y}, \epsilon \mathbf{F}} = \epsilon \mathbf{I}_{\mathbf{Y}, \mathbf{F}}^*$, and $\boldsymbol{\tau}_{\epsilon \mathbf{Y}, \epsilon \mathbf{F}} = \boldsymbol{\tau}_{\mathbf{Y}, \mathbf{F}}^*$. Since $(\mathbf{I}_{\mathbf{Y}, \mathbf{F}}^*, \boldsymbol{\tau}_{\mathbf{Y}, \mathbf{F}}^*)$ is feasible conditioning on $(Y_0, F_0) = (Y, F)$, since the output dynamics are linear in Y and since the debt face value dynamics are homogeneous of degree 1 in (Y, F) , it must be the case that $(\mathbf{I}_{\epsilon \mathbf{Y}, \epsilon \mathbf{F}}, \boldsymbol{\tau}_{\epsilon \mathbf{Y}, \epsilon \mathbf{F}})$

is feasible conditioning on $(Y_0, F_0) = (\epsilon Y, \epsilon F)$. Thus, I have:

$$J_i(\epsilon Y, \epsilon F; (\mathbf{I}_{\epsilon Y, \epsilon F}, \boldsymbol{\tau}_{\epsilon Y, \epsilon F}); \mathbf{D}) \leq V_i(\epsilon Y, \epsilon F; \mathbf{D})$$

Then assume for a second that the inequality above is strict. If that was the case, then take $(\mathbf{I}_{\epsilon Y, \epsilon F}^*, \boldsymbol{\tau}_{\epsilon Y, \epsilon F}^*) = \arg \max J_i(\epsilon Y, \epsilon F; (\mathbf{I}, \boldsymbol{\tau}); \mathbf{D})$. Consider the policy $(\mathbf{I}_{Y, F}, \boldsymbol{\tau}_{Y, F})$, such that $\mathbf{I}_{Y, F} = \mathbf{I}_{\epsilon Y, \epsilon F}^*/\epsilon$, and $\boldsymbol{\tau}_{Y, F} = \boldsymbol{\tau}_{\epsilon Y, \epsilon F}^*$. Then is it immediate to see that this policy is feasible conditioned on starting at (Y_0, F_0) , and it is also immediate to see that:

$$J_i(Y, F; (\mathbf{I}_{Y, F}, \boldsymbol{\tau}_{Y, F}); \mathbf{D}) > J_i(Y, F; (\mathbf{I}_{Y, F}^*, \boldsymbol{\tau}_{Y, F}^*); \mathbf{D}) = V_i(Y, F; \mathbf{D})$$

This is a contradiction. Thus, the optimal issuance policy is homogeneous of degree 1 in (Y, F) . Since the value function is decreasing in F and increasing in Y , the default policy must be a barrier default policy. \square

A.1.8 Monotonicity of D_i

Let $(\mathbf{I}, \boldsymbol{\tau}) \in \mathcal{I} \times \mathcal{T}$ be admissible issuance and default Markov policies. Assume that \mathbf{I} is homogeneous of degree 1 in (Y, F) and that $\boldsymbol{\tau}$ is barrier. Note $I_i(Y, F) = \iota_i(F/Y)Y$, where $\iota_i(x) := I_i(1, x)$, for each Markov state $i \leq N_s$. Given these assumptions, the default policy can be written:

$$\begin{aligned} \tau &= \inf\{t \geq 0 : F_t \geq Y_t \bar{x}_{s_t}\} \\ &= \inf\{t \geq 0 : x_t \geq \bar{x}_{s_t}\} \end{aligned}$$

Using Itô's lemma, $x_t := F_t/Y_t$ follows the following stochastic differential equation under \mathbb{Q} :

$$\begin{aligned} dx_t^{(\boldsymbol{\iota}, \boldsymbol{\tau})} &= \left(1 - 1_{d,t}^{(\boldsymbol{\tau})}\right) \left[\left(\iota_{st} \left(x_t^{(\boldsymbol{\iota}, \boldsymbol{\tau})}\right)\right. \right. \\ &\quad \left. \left. - \left(m + \mu_{st} - |\boldsymbol{\sigma}_{st}|^2 - \boldsymbol{\nu}_{st} \cdot \boldsymbol{\sigma}_{st}\right) x_t^{(\boldsymbol{\iota}, \boldsymbol{\tau})}\right) dt - x_t^{(\boldsymbol{\iota}, \boldsymbol{\tau})} \boldsymbol{\sigma}_{st} \cdot d\hat{\mathbf{B}}_t \right] \\ &\quad + x_{t-}^{(\boldsymbol{\iota}, \boldsymbol{\tau})} \left(\frac{1}{\alpha} - 1\right) dN_{d,t}^{(\boldsymbol{\tau})} + x_{t-}^{(\boldsymbol{\iota}, \boldsymbol{\tau})} (\theta\alpha - 1) dN_{e,t}^{(\boldsymbol{\tau})} \end{aligned}$$

The debt price is an expected present value of flow payoffs. Since such flow payoffs are homogeneous of degree zero in (Y, F) , and since the default policy is barrier, the debt price function must be homogeneous of degree zero. With an abuse of notation, I will note $D_i(Y, F; (\boldsymbol{\iota}, \boldsymbol{\tau})) = D_i(x; (\boldsymbol{\iota}, \boldsymbol{\tau}))$. Note then that the debt price cannot be greater than the price D_i^{rf} of a risk-free claim to sovereign debt cash-flows, where \mathbf{D}^{rf} verifies:

$$\mathbf{D}^{rf} = (\kappa + m) \left(\text{diag}_i(r_i + m) - \hat{\Lambda} \right)^{-1} \mathbf{1} \quad (\text{A.11})$$

I then use a result that will be proven in section A.1.9: the fact that the defaulted debt price must satisfy:

$$\mathbf{D}^d(x) = \lambda\theta\alpha\Xi^{-1}\mathbf{D}(\theta x)$$

The matrix $N_s \times N_s$ matrix Ξ is equal to $\text{diag}_i(r_i + \boldsymbol{\nu}_i \cdot \boldsymbol{\sigma}_i + \lambda - \mu_i) - \hat{\Lambda}$. I then introduce the operator \mathbb{T} , defined for any $N_s \times 1$ vector \mathbf{f} of continuous decreasing functions whose i^{th} coordinate is $f_i : (0, \max_j \bar{x}_j) \rightarrow [0, D_i^{rf}]$ (D_i^{rf} is the price of risk-free debt in state i , see

equation A.11) as follows. If $x \geq \bar{x}_i$, set $(\mathbb{T}\mathbf{f})_i(x) = \lambda\theta\alpha [\Xi^{-1}\mathbf{f}(\theta x)]_i$. If $x \leq \bar{x}_i$, then:

$$(\mathbb{T}\mathbf{f})_i(x) := \hat{\mathbb{E}}^{i,x} \left[\int_0^\tau e^{-\int_0^t (r_{su} + m) du} (\kappa + m) dt + \lambda\theta\alpha e^{-\int_0^\tau (r_{su} + m) du} [\Xi^{-1}\mathbf{f}(\theta x_\tau)]_{s_\tau} \right] \quad (\text{A.12})$$

$$= D_i^{rf} + \hat{\mathbb{E}}^{i,x} \left[e^{-\int_0^\tau (r_{su} + m) du} \left(\lambda\theta\alpha [\Xi^{-1}\mathbf{f}(\theta x_\tau)]_{s_\tau} - D_{s_\tau}^{rf} \right) \right] \quad (\text{A.13})$$

Given assumption 2, given that $\theta < 1$ and $\alpha < 1$, and given that the function f_i has an image in $[0, D_i^{rf}]$, it must be the case that $\lambda\theta\alpha [\Xi^{-1}\mathbf{f}(\cdot)]_i \leq D_i^{rf}$ for any state i , which means that the term in brackets in equation A.13 is negative. Thus, $(\mathbb{T}\mathbf{f})_i(\cdot)$ is a decreasing function of x , positive and bounded above by D_i^{rf} . The Feynman-Kac theorem also provides for the continuity of the function $(\mathbb{T}\mathbf{f})_i(\cdot)$. Thus, \mathbb{T} maps $N_s \times 1$ vectors of continuous bounded decreasing functions with image in $[0, D_i^{rf}]$ into itself. For any pair of vectors of functions $\mathbf{f}_1, \mathbf{f}_2$ whose components are continuous and decreasing on the interval $[0, \max_i \bar{x}_i]$, I have:

$$\begin{aligned} |(\mathbb{T}\mathbf{f}_2 - \mathbb{T}\mathbf{f}_1)_i|(x) &= \lambda\theta\alpha \hat{\mathbb{E}}^{i,x} \left[e^{-\int_0^\tau (r_{su} + m) du} \left| [\Xi^{-1}(\mathbf{f}_2(\theta x_\tau) - \mathbf{f}_1(\theta x_\tau))]_{s_\tau} \right| \right] \\ &\leq \lambda\theta\alpha \|\Xi^{-1}\| \cdot \|\mathbf{f}_2 - \mathbf{f}_1\|_\infty \end{aligned}$$

Since $\lambda\theta\alpha \|\Xi^{-1}\| < 1$ (given assumption 2), \mathbb{T} is a contraction, and the contraction mapping theorem provides for a unique continuous, bounded and decreasing vector of functions $\mathbf{D}(\cdot; (\boldsymbol{\iota}, \boldsymbol{\tau}))$ whose i^{th} component satisfies the functional equation:

$$D_i(x) = \hat{\mathbb{E}}^{i,x} \left[\int_0^\tau e^{-\int_0^t (r_{su} + m) du} (\kappa + m) dt + \lambda\theta\alpha e^{-\int_0^\tau (r_{su} + m) du} [\Xi^{-1}\mathbf{D}(\theta x_\tau)]_{s_\tau} \right]$$

The function D_i is decreasing as required. \square

A.1.9 Debt Price at Default

I need to compute the debt price in default $D_i^d(x)$, for $x \geq \bar{x}_i$ and $1 \leq i \leq N_s$. Assume that at time of default, the state is $s_\tau = i$. When the country exits financial autarky, its debt-to-GDP ratio is equal to $\frac{F_{\tau+\tau_e}}{Y_{\tau+\tau_e}} = \theta \frac{F_{\tau-}}{Y_{\tau-}} = \theta x_{\tau-}$. It is possible that $x_\tau > \bar{x}_{s_\tau-}$ when the sovereign defaults. This happens upon the occurrence of a “jump-to-default”, in other words a situation where the state jumps from $s_{\tau-} = j$ to $s_\tau = i$ and when $\bar{x}_i < x_{\tau-} < \bar{x}_j$.

Thus, I have the following for $x \geq \bar{x}_i$:

$$\begin{aligned} D_i^d(x) &= \hat{\mathbb{E}} \left[\exp \left(- \int_0^{\tau_e} r_{st+u} du \right) \frac{F_{t+\tau_e}}{F_{t-}} D_{s_{t+\tau_e}}(\theta x) | Y_{t-} = Y, s_{t-} = i \right] \\ &= \theta \alpha \hat{\mathbb{E}} \left[\exp \left(- \int_0^{\tau_e} r_{st+u} du \right) \frac{Y_{t+\tau_e}}{Y_t} D_{s_{t+\tau_e}}(\theta x) | Y_{t-} = Y, s_{t-} = i \right] \end{aligned}$$

In other words, in order to compute $D_i^d(x)$ (for $x \geq \bar{x}_i$), I need to solve a system of N_s equations in N_s unknown $\{D_j^d(x)\}_{1 \leq j \leq N_s}$:

$$r_j D_j^d(x) = (\mu_j - \boldsymbol{\nu}_j \cdot \boldsymbol{\sigma}_j) D_j^d(x) + \lambda \left(\theta \alpha D_j(\theta x) - D_j^d(x) \right) + \sum_{k=1}^{N_s} \hat{\Lambda}_{jk} D_k^d(x)$$

If I introduce the $N_s \times N_s$ matrix $\Xi := \text{diag}_i (r_i + \boldsymbol{\nu}_i \cdot \boldsymbol{\sigma}_i + \lambda - \mu_i) - \hat{\Lambda}$, and if I note $\mathbf{D}^d(x)$ the $N_s \times 1$ vector with i^{th} element $D_i^d(x)$, I obtain:

$$\mathbf{D}^d(x) = \lambda \theta \alpha \Xi^{-1} \mathbf{D}(\theta x)$$

Given assumption 2, the quantity above is well defined and finite. Finally, note that this equation is valid for each coordinate i for $x \geq \bar{x}_i$. Indeed, the model with discrete Markov states for the SDF generates default waves, via “jumps-to-default” created when the state s_t jumps from a state $s_{t-} = i$ (for example a state where risk prices are low) to a state $s_t = j$ (for example a state where risk prices are high), and when $\bar{x}_i > x_t > \bar{x}_j$. When computing the full set of functions $\{D_i(\cdot)\}_{1 \leq i \leq N_s}$, it is thus essential to compute such function on the

interval $[0, \max_i \bar{x}_i]$. \square

A.1.10 Life-Time Utility HJB Equation

Introduce the probability measure $\tilde{\Pr}(A) = \mathbb{E} \left[e^{(1-\gamma) \int_0^t \sigma_{s,u} \cdot d\mathbf{B}_u - \frac{1}{2}(1-\gamma)^2 \int_0^t |\sigma_{s,u}|^2 du} \mathbf{1}_A \right]$, for some arbitrary Borel set $A \subseteq \mathcal{F}_t$. Under such measure, using Girsanov's theorem, the variable x_t evolves as follows:

$$\begin{aligned} dx_t^{(\boldsymbol{\iota}, \boldsymbol{\tau})} &= 1_{d,t}^{(\boldsymbol{\tau})} \left[\left(\iota_{s,t} \left(x_t^{(\boldsymbol{\iota}, \boldsymbol{\tau})} \right) - \left(m + \mu_{s,t} - \gamma |\sigma_{s,t}|^2 \right) x_t^{(\boldsymbol{\iota}, \boldsymbol{\tau})} \right) dt - x_t^{(\boldsymbol{\iota}, \boldsymbol{\tau})} \sigma_{s,t} \cdot d\tilde{\mathbf{B}}_t \right] \\ &\quad + x_{t-}^{(\boldsymbol{\iota}, \boldsymbol{\tau})} \left(\frac{1}{\alpha} - 1 \right) dN_{d,t}^{(\boldsymbol{\tau})} + x_{t-}^{(\boldsymbol{\iota}, \boldsymbol{\tau})} (\alpha\theta - 1) dN_{e,t}^{(\boldsymbol{\tau})} \end{aligned}$$

In the above, $\tilde{\mathbf{B}}_t := \mathbf{B}_t + (\gamma - 1) \int_0^t \sigma_{s,u} du$ is a standard N_b -dimensional Brownian motion under this equivalent probability measure. Let $c_t^{(\boldsymbol{\iota}, \boldsymbol{\tau}; \mathbf{D})}$ be the consumption-to-output ratio when the policy used is $(\boldsymbol{\iota}, \boldsymbol{\tau})$ and when the debt price schedule is \mathbf{D} :

$$c_t^{(\boldsymbol{\iota}, \boldsymbol{\tau}; \mathbf{D})} = \left[1 + \iota_t D_{s,t} \left(x_t^{(\boldsymbol{\iota}, \boldsymbol{\tau})} \right) - (\kappa + m) x_t^{(\boldsymbol{\iota}, \boldsymbol{\tau})} \right] \left(1 - 1_{d,t}^{(\boldsymbol{\tau})} \right) + 1_{d,t}^{(\boldsymbol{\tau})}$$

The value function $V_i(Y, F)$ in state i can be written $V_i(Y, F) = v_i(x)Y^{1-\gamma}$:

$$\begin{aligned} V_i(Y, F) &= \sup_{(\boldsymbol{\iota}, \boldsymbol{\tau}) \in \mathcal{I} \times \mathcal{T}} \mathbb{E}^{i, Y, F} \left[\int_0^\infty \delta \frac{1-\gamma}{1-\rho} v_{s,t}(x_t) Y_t^{1-\gamma} \left(\frac{(c_t^{(\boldsymbol{\iota}, \boldsymbol{\tau}; \mathbf{D})})^{1-\rho}}{((1-\gamma)v_{s,t}(x_t))^{\frac{1-\rho}{1-\gamma}}} - 1 \right) dt \right] \\ &= Y^{1-\gamma} v_i(x) \end{aligned}$$

In the above, I defined $v_i(x)$ as follows:

$$\begin{aligned} v_i(x) &:= \sup_{(\boldsymbol{\iota}, \boldsymbol{\tau}) \in \mathcal{I} \times \mathcal{T}} \tilde{\mathbb{E}}^{i, Y, F} \left[\int_0^\infty \delta \frac{1-\gamma}{1-\rho} v_{s,t}(x_t) \left(\frac{(c_t^{(\boldsymbol{\iota}, \boldsymbol{\tau}; \mathbf{D})})^{1-\rho}}{((1-\gamma)v_{s,t}(x_t))^{\frac{1-\rho}{1-\gamma}}} - 1 \right) \right. \\ &\quad \left. e^{(1-\gamma) \int_0^t (\mu_{s,u} - \frac{1}{2}\gamma |\sigma_{s,u}|^2) du} dt \right] \end{aligned}$$

The corresponding HJB equation for v_i can thus be written, in the continuation region:

$$\begin{aligned} \frac{1-\gamma}{1-\rho} \left(\delta + (\rho-1) \left(\mu_i - \frac{1}{2} \gamma |\boldsymbol{\sigma}_i|^2 \right) \right) v_i(x) - \sum_{j=1}^{N_s} \Lambda_{ij} v_j(x) = \\ \sup_{\iota_i} \left[\delta \frac{(1 + \iota_i D_i(x) - (\kappa + m)x)^{1-\rho} [(1-\gamma)v_i(x)]^{\frac{\rho-\gamma}{1-\gamma}}}{1-\rho} \right. \\ \left. + \left[\iota_i - \left(m + \mu_i - \gamma |\boldsymbol{\sigma}_i|^2 \right) x \right] v_i'(x) + \frac{1}{2} |\boldsymbol{\sigma}_i|^2 x^2 v_i''(x) \right] \end{aligned}$$

The term in brackets on the right-hand side of the equal sign is concave in ι_i , which leads to an optimal issuance policy of the form:

$$\iota_i(x) = \frac{1}{D_i(x)} \left[\left(\frac{\delta D_i(x) [(1-\gamma)v_i(x)]^{\frac{\rho-\gamma}{1-\gamma}}}{-v_i'(x)} \right)^{1/\rho} + (\kappa + m)x - 1 \right]$$

For the particular case $\rho = 1$, the guess value function still takes the form $V_i(Y, F) = v_i(x)Y^{1-\gamma}$, but the HJB equation solved by v_i is now:

$$\begin{aligned} \left(\delta \ln [(1-\gamma)v_i(x)] - (1-\gamma) \left(\mu_i - \frac{1}{2} \gamma |\boldsymbol{\sigma}_i|^2 \right) \right) v_i(x) - \sum_{j=1}^{N_s} \Lambda_{ij} v_j(x) = \\ \sup_{\iota_i} [\delta(1-\gamma)v_i(x) \ln (1 + \iota_i D_i(x) - (\kappa + m)x) \\ + \left[\iota_i - \left(m + \mu_i - \gamma |\boldsymbol{\sigma}_i|^2 \right) x \right] v_i'(x) + \frac{1}{2} |\boldsymbol{\sigma}_i|^2 x^2 v_i''(x)] \end{aligned}$$

Optimality of the issuance policy in such case takes the form:

$$\iota_i(x) = \frac{1}{D_i(x)} \left[\frac{\delta(1-\gamma)v_i(x)D_i(x)}{-v_i'(x)} + (\kappa + m)x - 1 \right]$$

Finally, for the particular case $\gamma = 1$, the guess value function still takes the form $V_i(Y, F) = v_i(x) + \ln Y$, and the HJB equation solved by v_i is now:

$$\begin{aligned} \frac{1}{1-\rho} \left(\delta + (\rho-1) \left(\mu_i - \frac{1}{2} |\boldsymbol{\sigma}_i|^2 \right) \right) - \sum_{j=1}^{N_s} v_j(x) = \\ \sup_{\iota_i} \left[\frac{\delta}{1-\rho} \frac{(1 + \iota_i D_i(x) - (\kappa + m)x)^{1-\rho}}{\exp((1-\rho)v_i(x))} \right. \\ \left. + \left[\iota_i - \left(m + \mu_i - |\boldsymbol{\sigma}_i|^2 \right) x \right] v_i'(x) + \frac{1}{2} |\boldsymbol{\sigma}_i|^2 x^2 v_i''(x) \right] \end{aligned}$$

Optimality of the issuance policy in such case takes the form:

$$\iota_i(x) = \frac{1}{D_i(x)} \left[\left(-\frac{v_i'(x)}{\delta} \right)^{-1/\rho} \exp((1-1/\rho)v_i(x)) + (\kappa + m)x - 1 \right]$$

□

A.1.11 Life-Time Utility at Default

Assume that the state at default time τ is equal to s_τ . The pre-default output is $Y_{\tau-} = Y$, and it falls at the time of default by a factor α . The life-time utility in default consists of the flow value of receiving Y_t until the random time interval τ_e , at which point a lump sum value $V_{\tau+\tau_e}(Y_{\tau+\tau_e}, F_{\tau-})$ should be added. The debt-to-GDP ratio at the time the country is exiting from financial autarky is:

$$\frac{F_{\tau+\tau_e}}{Y_{\tau+\tau_e}} = \theta \frac{Y_{\tau+\tau_e}}{Y_{\tau-}} \frac{F_{\tau-}}{Y_{\tau+\tau_e}} = \theta x_{\tau-}$$

Thus the value function at exit from financial autarky can be expressed as $v_{s_{\tau+\tau_e}}(\theta x_{\tau-}) Y_{\tau+\tau_e}^{1-\gamma}$. I thus look for a function $V_{s_t}^d$ of the form $V_{s_t}^d(Y, F) = v_{s_t}^d(x)(\alpha Y)^{1-\gamma}$, for a set of functions $\{v_i^d(x)\}_{1 \leq i \leq N_s}$ to be determined. For $x \geq \bar{x}_{s_t}$, the function $V_{s_t}^d$ satisfies the recursive

equation:

$$0 = \varphi \left(Y_t, V_{s_t}^d \right) + \mathcal{A}V_{s_t}^d$$

Plugging in my guess function, the HJB equation becomes:

$$0 = \delta \frac{1-\gamma}{1-\rho} v_i^d(x) (\alpha Y)^{1-\gamma} \left(\frac{1}{\left[(1-\gamma) v_i^d(x) \right]^{\frac{1-\rho}{1-\gamma}}} - 1 \right) + \sum_{j=1}^{N_s} \Lambda_{ij} (\alpha Y)^{1-\gamma} \left(v_j^d(x) - v_i^d(x) \right) + \mu_i (1-\gamma) v_i^d(x) (\alpha Y)^{1-\gamma} - \frac{1}{2} \gamma (1-\gamma) |\boldsymbol{\sigma}_i|^2 v_i^d(x) (\alpha Y)^{1-\gamma} + \lambda (\alpha Y)^{1-\gamma} \left(v_i(\theta x) - v_i^d(x) \right)$$

If I introduce the $N_s \times N_s$ matrix $\Upsilon := \frac{1-\gamma}{1-\rho} \text{diag}_i(A_i) + \lambda I - \Lambda$, if I note $\mathbf{v}(x)$ the $N_s \times 1$ vector with i^{th} row $v_i(x)$, and if I note $\mathbf{v}^d(x)$ the $N_s \times 1$ vector with i^{th} row $v_i^d(x)$, I need to solve the non-linear equation:

$$\Upsilon \mathbf{v}^d(x) - \lambda \mathbf{v}(\theta x) = \frac{\delta}{1-\rho} \left[(1-\gamma) \mathbf{v}^d(x) \right]^{\frac{\rho-\gamma}{1-\gamma}} \quad (\text{A.14})$$

In the above $\left[(1-\gamma) \mathbf{v}^d(x) \right]^{\frac{\rho-\gamma}{1-\gamma}}$ is to be understood as an element-by-element power function. Note that $v_i^d(x)$ admits the integral representation (this will be useful in connection with my verification theorem):

$$v_i^d(x) = \mathbb{E}^i \left[\int_0^{\tau_e} e^{-\frac{1-\gamma}{1-\rho} \int_0^t A_{su} du} \frac{\delta}{1-\rho} [(1-\gamma) v_{s_t}(x)]^{\frac{\rho-\gamma}{1-\gamma}} dt + e^{-\frac{1-\gamma}{1-\rho} \int_0^{\tau_e} A_{su} du} v_{s_{\tau_e}}(\theta x) \right] \quad (\text{A.15})$$

Note that for $\gamma = \rho$, I can solve equation A.14 explicitly, obtaining:

$$\mathbf{v}^d(x) = \Upsilon^{-1} \left(\lambda \mathbf{v}(\theta x) + \frac{\delta}{1-\gamma} \mathbf{1} \right)$$

In the above, $\mathbf{1}$ is a $N_s \times 1$ vector of ones. Default optimality can be written $V_i(Y, F) \geq V_i^d(Y, F)$, which, stated in terms of the normalized value functions, can be written:

$$v_{st}(x_t) - \alpha^{1-\gamma} v_{st}^d(x_t) \geq 0 \quad \forall t \geq 0$$

Applying Itô's lemma to $v_{st}(x_t) - \alpha^{1-\gamma} v_{st}^d(x_t)$, the diffusion term is equal to:

$$-x_t \left(v'_{st}(x_t) - \alpha^{1-\gamma} \left(v_{st}^d \right)'(x_t) \right) \boldsymbol{\sigma}_{st} \cdot d\mathbf{B}_t$$

In particular, since $v_{st}(\bar{x}_{st}) = \alpha^{1-\gamma} v_{st}^d(\bar{x}_{st})$, the only way for the inequality $v_{st}(x_t) - \alpha^{1-\gamma} v_{st}^d(x_t) \geq 0$ to be preserved in the presence of Brownian shocks at the default boundary is for the diffusion term above to be identically zero at such boundary. This leads to the smooth-pasting optimality condition:

$$v'_i(\bar{x}_i) = \alpha^{1-\gamma} \left(v_i^d \right)'(\bar{x}_i)$$

Differentiating the implicit equation defining $\mathbf{v}^d(x)$ w.r.t. x , I obtain the following expression for $\left(\mathbf{v}^d \right)'(\bar{x}_i)$:

$$\left(\mathbf{v}^d \right)'(\bar{x}_i) = \lambda \theta \left(\Upsilon + \delta \frac{\gamma - \rho}{1 - \rho} \text{diag}_j \left(\left[(1 - \gamma) v_j^d(\bar{x}_i) \right]^{-\frac{1-\rho}{1-\gamma}} \right) \right)^{-1} \mathbf{v}'(\theta \bar{x}_i)$$

□

A.1.12 Verification Theorem

Let $\{v_i\}_{i \leq N_s}$ be a family of functions such that for each i , $v_i \in \mathcal{C}^1(\mathbb{R}^+) \cap \mathcal{C}^2(\mathbb{R}^+ \setminus \{\bar{x}_i\})$ satisfies the assumptions of the theorem. Let $(\boldsymbol{\iota}, \boldsymbol{\tau}) \in \mathcal{I} \times \mathcal{T}$ be an arbitrary policy, I have

the following Itô formula:

$$\begin{aligned}
e^{-\frac{1-\gamma}{1-\rho} \int_0^t A_{su} du} v_{st}(x_t) &= v_i(x) - \int_0^t e^{-\frac{1-\gamma}{1-\rho} \int_0^z A_{su} du} \left(1 - 1_{d,z}^{(\tau)}\right) x_z v'_{sz}(x_z) \boldsymbol{\sigma}_{sz} \cdot d\tilde{\mathbf{B}}_z \\
&+ \int_0^t e^{-\frac{1-\gamma}{1-\rho} \int_0^z A_{su} du} \left[\sum_{s' \neq s_{z-}} (v_{s'}(x_z) - v_{s_{z-}}(x_z)) \left(dN_z^{s_{z-}, s'} - \Lambda_{s_{z-}, s'} dz\right) \right] \\
&+ \int_0^t e^{-\frac{1-\gamma}{1-\rho} \int_0^z A_{su} du} \left[\left(1 - 1_{d,z}^{(\tau)}\right) \mathcal{L}^{(\iota)} v_{sz}(x_z) + \sum_{s' \neq s_{z-}} \Lambda_{s_{z-}, s'} (v_{s'}(x_z) - v_{s_{z-}}(x_z)) \right. \\
&\quad \left. - \frac{1-\gamma}{1-\rho} A_{sz} v_{sz}(x_z) \right] dz \\
&+ \int_0^t e^{-\frac{1-\gamma}{1-\rho} \int_0^z A_{su} du} (v_{sz}(x_z) - v_{s_{z-}}(x_z)) dN_{d,z}^{(\tau)} \\
&+ \int_0^t e^{-\frac{1-\gamma}{1-\rho} \int_0^z A_{su} du} (v_{sz}(x_z) - v_{s_{z-}}(x_z)) dN_{e,z}^{(\tau)}
\end{aligned}$$

See for example [44]. For the arbitrary control policy (ι, τ) , I note $c_{st}^{(\iota, \tau)}(x_t)$ the resulting consumption-to-output policy. Using Itô's lemma above, using the variational inequality in the assumption of the theorem, and using equation A.15:

$$\begin{aligned}
e^{-\frac{1-\gamma}{1-\rho} \int_0^t A_{su} du} v_{st}(x_t) &\leq v_i(x) - \int_0^t e^{-\frac{1-\gamma}{1-\rho} \int_0^z A_{sz} dz} \delta \frac{c_{sz}^{(\iota, \tau)}(x_z)^{1-\rho} [(1-\gamma)v_{sz}(x_z)]^{\frac{\rho-\gamma}{1-\gamma}}}{1-\rho} dz \\
&+ \int_0^t e^{-\frac{1-\gamma}{1-\rho} \int_0^z A_{su} du} \left[\sum_{s' \neq s_{z-}} (v_{s'}(x_z) - v_{s_{z-}}(x_z)) \left(dN_z^{s_{z-}, s'} - \Lambda_{s_{z-}, s'} dz\right) \right] \\
&\quad - \int_0^t e^{-\frac{1-\gamma}{1-\rho} \int_0^t A_{sz} dz} \left(1 - 1_{d,z}^{(\tau)}\right) x_z v'_{sz}(x_z) \boldsymbol{\sigma}_{sz} \cdot d\tilde{\mathbf{B}}_z
\end{aligned}$$

The terms on the second and third line above are martingales since v_i and v'_i are bounded. Thus, taking expectations on both sides of this equality, I obtain:

$$\begin{aligned}
\tilde{\mathbb{E}}^{i,x} \left[\int_0^t e^{-\frac{1-\gamma}{1-\rho} \int_0^z A_{sz} dz} \delta \frac{c_{sz}^{(\iota, \tau)}(x_z)^{1-\rho} [(1-\gamma)v_{sz}(x_z)]^{\frac{\rho-\gamma}{1-\gamma}}}{1-\rho} dz \right] + \tilde{\mathbb{E}}^{i,x} \left[e^{-\frac{1-\gamma}{1-\rho} \int_0^t A_{sz} dz} v_{st}(x_t) \right] \\
\leq v_i(x)
\end{aligned}$$

Taking $t \rightarrow +\infty$, using the assumption that $\lim_{t \rightarrow +\infty} \inf e^{-\int_0^t \frac{1-\gamma}{1-\rho} A_{su} du} v_{st} \left(x_t^{(\iota, \tau)}; \mathbf{D} \right) \leq 0$, and using the monotone convergence theorem, I then obtain the desired result: $v_i(x; \mathbf{D}) \geq J_i(1, x; (\iota, \tau); \mathbf{D})$ for any admissible control policy. The proof of the second part of the theorem relies on steps identical to those described above, except that inequalities are now replaced by equalities. The uniqueness of $v_i(\cdot; D)$ as a solution to re-scaled sequence problem equation 1.16, shows that $v_i(x; \mathbf{D}) = J_i(1, x; (\iota^*, \tau^*); \mathbf{D})$. \square

A.1.13 Sketch of Equilibrium Existence Proof

I discuss here a possible route to prove that an equilibrium exists in a simpler environment without discrete SDF states, and where the punishment upon default is financial autarky forever. Upon a sovereign default, creditors' recovery value is zero. In this simpler environment, the Markov perfect equilibrium features only the debt-to-GDP ratio as a state variable. Take an arbitrary debt price schedule $D : \mathbb{R}^+ \rightarrow [0, \frac{\kappa+m}{r+m}]$ that is continuous and strictly decreasing on that interval. Given this debt price schedule, construct the sovereign's "best response", in other words construct the value function $v(\cdot; D)$ as well as the optimal issuance and default policies $\iota^*(\cdot; D)$ and $x^*(D)$. This best response exists: given a debt price schedule D , the function $v(\cdot; D)$ is simply the optimal life-time utility in a single-agent optimal control, optimal stopping problem, where the control is the issuance rate ι and the stopping time is the default time τ . Using those issuance and default policies, construct a new debt price schedule $D(\cdot; (\iota^*(\cdot; D), x^*(D)))$.

I have implicitly constructed a functional map \mathbb{T} , which takes a continuous decreasing function $D : \mathbb{R}^+ \rightarrow [0, \frac{\kappa+m}{r+m}]$ and maps it into a continuous decreasing function:

$$D(\cdot; (\iota^*(\cdot; D), x^*(D))) : \mathbb{R}^+ \rightarrow [0, \frac{\kappa+m}{r+m}]$$

In fact, by studying the sovereign's behavior when the debt price is constant and equal to its risk-free value $\frac{\kappa+m}{r+m}$, I can restrict this functional map to functions defined on the

interval $[0, \bar{x}^{rf}]$ (where \bar{x}^{rf} is the optimal sovereign default boundary when the debt is priced at its risk-free value by creditors). Indeed, it is straightforward to show that the default boundary must be decreasing in the debt price schedule – in other words, if for any x , $D_1(x) \geq D_2(x)$, it must be the case that $x^*(D_1) \geq x^*(D_2)$. A Markov perfect equilibrium of my economy is simply a fixed point of the functional map constructed. Schauder’s fixed point theorem (appropriate for infinite dimensional spaces) could then be invoked in order to establish the existence of a fixed point of such functional map. An appropriate space of functions to use is any subset that is closed, bounded and equicontinuous. Indeed, since $[0, \bar{x}^{rf}]$ is compact, Arzela-Ascoli’s theorem guarantees that any such subspace of functions is compact. A good candidate to restrict oneself would be the space of Lipschitz continuous functions that have the same Lipschitz constant. In order to apply Schauder’s fixed point theorem, two theoretical hurdles thus have to be overcome. First, one would need to show that the mapping \mathbb{T} is continuous. Second, one would need to show that the mapping \mathbb{T} preserves Lipschitz continuity. Once those two conditions are established, existence of a Markov perfect equilibrium is straightforward.

A.1.14 Expected Default Times and Ergodic Distribution

I note $T_i(x)$, the risk-natural expected default time conditioned on the debt-to-GDP ratio being equal to x and the state $s_t = i$. Mathematically, the expected default time can be written $T_i(x) := \mathbb{E}^{i,x}[\tau]$. Using Feynman-Kac, it is immediate to show that $T_i(\cdot)$ solves the following HJB equation, for $x \in (0, \bar{x}_i)$:

$$0 = 1 + \left(\iota_i(x) - \left(m + \mu_i - |\boldsymbol{\sigma}_i|^2 \right) x \right) T'_i(x) + \frac{1}{2} |\boldsymbol{\sigma}_i|^2 x^2 T''_i(x) + \sum_{j=1}^{N_s} \Lambda_{ij} T_j(x) \quad (\text{A.16})$$

The $2 \times N_s$ boundary conditions consist in (i) value matching conditions at the default boundary and (ii) Robin boundary conditions at $x = 0$:

$$0 = T_i(x) \quad \forall x \geq \bar{x}_i$$

$$0 = 1 + \iota_i(0)T'_i(0) + \sum_{j=1}^{N_s} \Lambda_{ij}T_j(0)$$

I then focus on the ergodic *measure* f_i of the state variable under the physical measure \mathbb{P} , conditioned on being in state i and conditioned on the government being performing under its debt obligations (I emphasize the word *measure* as opposed to density since f_i does not integrate to 1). For $x \in (0, \bar{x}_i)$ and $x \notin \{\theta\bar{x}_j\}_{1 \leq j \leq N_s}$, f_i solves the following Kolmogorov-forward equation:

$$0 = -\frac{d}{dx} \left[\left(\iota_i(x) - (m + \mu_i - |\boldsymbol{\sigma}_i|^2) x \right) f_i(x) \right] + \frac{1}{2} \frac{d^2}{dx^2} \left[|\boldsymbol{\sigma}_i|^2 x^2 f_i(x) \right] + \sum_{j=1}^{N_s} \Lambda_{jif_j}(x)$$

The equation above is not applicable at the points $\{\theta\bar{x}_j\}_{1 \leq j \leq N_s}$ (the points of re-entry of the sovereign following a “smooth” default, i.e. a default such that $x_{\tau-} = \bar{x}_{s_\tau}$), but the measures f_i are continuous at those points. The following boundary condition holds at $x = \bar{x}_i$:

$$f_i(\bar{x}_i) = 0 \tag{A.17}$$

This equation can be obtained heuristically by approximating the continuous time process $\{x_t\}$ by a discrete state Markov chain, and analyzing transitions in and out of the state $x = \bar{x}_{s_t}$ between time t and time $t + \Delta_t$. It is also a standard condition for absorbing boundaries. I note g_i the fraction of time the sovereign is in autarky in SDF state i – note that such fraction does not depend on the debt-to-GDP ratio at entry into the default state given the memory-less property of the stochastic process s_t , and given that the autarky time length is exponentially distributed and independent of the process $\{s_t\}$. In other words, if

$g := \sum_{i=1}^{N_s} g_i$, then I have:

$$\frac{g_i}{g} = \frac{\mathbb{E}^i \left[\int_0^{\tau_e} 1_{\{s_t=i\}} dt \right]}{\mathbb{E}^i [\tau_e]} = \left(\left(I - \frac{1}{\lambda} \Lambda \right)^{-1} \right)_{ii} \quad (\text{A.18})$$

For any Markov state $s = i$, the integral of the ergodic distribution over the state space $[0, \bar{x}_i]$, in addition to the expected time spent in autarky g_i , need to add up to π_i , the stationary measure of the process s_t :

$$\int_0^{\bar{x}_i} f_i(x) dx + g_i = \pi_i \quad (\text{A.19})$$

Thus, equations A.17 and A.19 gives me $2 \times N_s$ “boundary” conditions, allowing me to solve for the N_s Kolmogorov-forward equations, which are second order ordinary differential equations. However, the constants $\{g_i\}_{1 \leq i \leq N_s}$ in equation A.18 are only determined up to the constant g , which represents the average percentage of time the sovereign spends in autarky post-default. I determine the constant g numerically via a Markov chain approximation method described in section A.3.

Finally, note that in the particular case where there is only one discrete Markov state, I can derive a pseudo-closed form expression for the stationary measure f . Indeed, in such case, the ergodic *measure* f of the state variable under the physical probability measure \mathbb{P} solves the following Kolmogorov-forward equation, valid for $x \in (0, \theta\bar{x}) \cup (\theta\bar{x}, \bar{x})$:

$$0 = -\frac{d}{dx} \left[\left(\iota(x) - \left(m + \mu - |\boldsymbol{\sigma}|^2 \right) x \right) f(x) \right] + \frac{1}{2} \frac{d^2}{dx^2} \left[|\boldsymbol{\sigma}|^2 x^2 f(x) \right] \quad (\text{A.20})$$

f is continuous at $x = \theta\bar{x}$ (the point of re-entry of the sovereign post-autarky). At $x = \bar{x}$, the ergodic distribution must satisfy the absorbing boundary condition:

$$f(\bar{x}) = 0$$

Equation A.20 can be integrated out as follows. For $x \in (\theta\bar{x}, \bar{x})$, I have:

$$0 = G + ((m + \mu)x - \iota(x)) f(x) + \frac{1}{2}|\boldsymbol{\sigma}|^2 x^2 f'(x) \quad (\text{A.21})$$

The constant G is the “flow” of the density in the positive direction. Using $f(\bar{x}) = 0$, I can integrate equation A.21 to obtain, for $x \in (\theta\bar{x}, \bar{x})$:

$$f(x) = \int_x^{\bar{x}} \exp \left[\int_x^t \frac{2}{\sigma^2 s^2} ((m + \mu)s - \iota(s)) ds \right] \frac{2G}{\sigma^2 t^2} dt$$

I also know that the density f is continuous at $\theta\bar{x}$ (even though it is not differentiable at that point). At $x = 0$, the density must be zero. Indeed, note that in a neighborhood of $x = 0$, the stochastic process $\{x_t\}$ behaves similarly to a geometric Brownian motion to which a constant strictly positive drift $\iota(0)$ has been added, and it is straightforward to show that the stationary distribution of a geometric Brownian motion which, at $x = \bar{x}$, is “reset” to $x = \theta\bar{x}$, admits a stationary density with value zero at $x = 0$. Thus, on $[0, \theta\bar{x})$, the density f takes the following form:

$$f(x) = \exp \left[\int_x^{\theta\bar{x}} \frac{2}{\sigma^2 s^2} ((m + \mu)s - \iota(s)) ds \right] f(\theta\bar{x})$$

This integration provides for the continuity of f at $x = \theta\bar{x}$. Finally, the integral of the ergodic measure over the state space $[0, \bar{x}]$, in addition to the expected percentage of time spent in autarky, need to add up to 1:

$$\int_0^{\bar{x}} f(x) dx + \frac{1/\lambda}{1/\lambda + T(\theta\bar{x})} = 1$$

This pins down the unknown constant G . □

A.1.15 Credit Spreads

I leverage the equation that the credit spread $\varsigma_i(x)$ satisfies:

$$D_i(x) = (m + \kappa) \left[\left(\text{diag}_j (r_j + \varsigma_i(x) + m) - \hat{\Lambda} \right)^{-1} \mathbf{1} \right]_i$$

Some algebra can show that for any state i , I have:

$$\varsigma'_i(x) = - \frac{D'_i(x)}{(\kappa + m) \left[\left(\text{diag}_j (r_j + \varsigma_i(x) + m) - \hat{\Lambda} \right)^{-2} \mathbf{1} \right]_i}$$

Since the debt price function D_i is decreasing in the debt-to-GDP ratio x and since the denominator in the expression above is positive, $\varsigma'_i < 0$. Some algebra also shows that the function $x\varsigma'_i(x)$ can be expressed as follows:

$$\frac{d}{dx} (x\varsigma'_i(x)) = \varsigma'_i(x) + \frac{(\kappa + m)x (\varsigma'_i(x))^2 \left[\left(\text{diag}_j (r_j + \varsigma_i(x) + m) - \hat{\Lambda} \right)^{-3} \mathbf{1} \right]_i - x D''_i(x)}{(\kappa + m) \left[\left(\text{diag}_j (r_j + \varsigma_i(x) + m) - \hat{\Lambda} \right)^{-2} \mathbf{1} \right]_i}$$

I have showed previously that $\varsigma'_i > 0$. The second term is positive, and the third term is also positive if the debt price function D_i is concave. \square

A.1.16 Credit Default Swap Premia

As specified in the main text, I define the risk-neutral present value of future credit losses and the risk-neutral present value of future CDS premia as follows:

$$\begin{aligned} L_i(x, T) &:= \hat{\mathbb{E}}^{x,i} \left[1_{\{\tau < T\}} e^{-\int_0^\tau r_{su} du} \max(0, 1 - D_{s_\tau}(x_\tau)) \right] \\ P_i(x, T) &:= \hat{\mathbb{E}}^{x,i} \left[\int_0^{T \wedge \tau} e^{-\int_0^t r_{su} du} dt \right] \end{aligned}$$

The CDS premium is simply the ratio of those two quantities: $\varsigma_i(x, T) = \frac{L_i(x, T)}{P_i(x, T)}$. An application of Feynman-Kac leads to the following partial differential equations satisfied by L_i and P_i , for $1 \leq i \leq N_s$:

$$\begin{aligned} r_i L_i(x, t) &= -\frac{\partial L_i}{\partial t}(x, t) + \hat{\mathcal{L}}_i L_i(x, t) + \sum_{j=1}^{N_s} \hat{\Lambda}_{ij} L_j(x, t) \\ r_i P_i(x, t) &= 1 - \frac{\partial P_i}{\partial t}(x, t) + \hat{\mathcal{L}}_i P_i(x, t) + \sum_{j=1}^{N_s} \hat{\Lambda}_{ij} P_j(x, t) \end{aligned}$$

The boundary conditions are as follows, for $t \in [0, T]$:

$$\begin{aligned} L_i(x, 0) &= 0 & \forall x < \bar{x}_i \\ P_i(x, 0) &= 0 & \forall x < \bar{x}_i \\ L_i(x, t) &= 1 - D_i^d(x) & \forall x \geq \bar{x}_i \\ P_i(x, t) &= 0 & \forall x \geq \bar{x}_i \end{aligned}$$

I can then compute the expected excess return and the return volatility on a T -maturity CDS contract. Imagine that at time t , an investor sells protection on the specific sovereign credit, for \$1 dollar notional amount and using a T -maturity contract. At time t , no cash-flow is exchanged, the value of the CDS contract is zero and the premium agreed upon between the buyer and the seller is equal to $\varsigma_{st}(x_t, T) = L_{st}(x_t, T)/P_{st}(x_t, T)$. At time $t + dt$, the protection seller has accrued $\varsigma_{st}(x_t, T)dt$ of premium income. The value of the “premium leg” of his CDS contract is now equal to $\varsigma_{st}(x_t, T)P_{st+dt}(x_{t+dt}, T - dt)$ while the value of the “default leg” of his CDS contract is now equal to $L_{st+dt}(x_{t+dt}, T - dt)$. In other words, his excess return (computed based on a \$1 notional risky investment) is equal to:

$$dR_{t,T}^e = \varsigma_{st}(x_t, T)dt + \varsigma_{st}(x_t, T)P_{st+dt}(x_{t+dt}, T - dt) - L_{st+dt}(x_{t+dt}, T - dt)$$

This return is viewed as an excess return since the protection seller did not put any money upfront to enter into his contract. To compute CDS expected excess returns and return volatilities, I use Ito's lemma and the relationship $dR_{t,T}^e = \varsigma_t dt + \varsigma_t P_{s_{t+dt}}(x_{t+dt}, T - dt) - L_{s_{t+dt}}(x_{t+dt}, T - dt)$:

$$dR_{t,T}^e = \varsigma_t dt + \varsigma_t \left(P_{s_t} + \mathcal{L}_{s_t} P_{s_t} dt - \frac{\partial P_{s_t}}{\partial t} dt - x_t \frac{\partial P_{s_t}}{\partial x} \boldsymbol{\sigma}_{s_t} \cdot d\mathbf{B}_t + \sum_{s'} (P_{s'} - P_{s_t}) dN_t^{(s_t, s')} \right) - \left(L_{s_t} + \mathcal{L}_{s_t} L_{s_t} - \frac{\partial L_{s_t}}{\partial t} dt - x_t \frac{\partial L_{s_t}}{\partial x} \boldsymbol{\sigma}_{s_t} \cdot d\mathbf{B}_t + \sum_{s'} (L_{s'} - L_{s_t}) dN_t^{(s_t, s')} \right)$$

I then use the relationship between the operators \mathcal{L}_{s_t} and $\hat{\mathcal{L}}_{s_t}$:

$$\hat{\mathcal{L}}_{s_t} = \mathcal{L}_{s_t} + x_t \boldsymbol{\nu}_{s_t} \cdot \boldsymbol{\sigma}_{s_t} \frac{\partial}{\partial x}$$

Using $\varsigma_t = L_{s_t}/P_{s_t}$ and the HJB equation satisfied by L_{s_t} and P_{s_t} , I have:

$$dR_{t,T}^e = -L_{s_t} \sum_{s'} \Lambda_{s_{t-}, s'} \left(e^{v(s_{t-}, s')} - 1 \right) \left(\frac{P_{s'}}{P_{s_{t-}}} - \frac{L_{s'}}{L_{s_{t-}}} \right) dt - L_{s_t} \left(\frac{x_t \frac{\partial P_{s_t}}{\partial x}}{P_{s_t}} - \frac{x_t \frac{\partial L_{s_t}}{\partial x}}{L_{s_t}} \right) \boldsymbol{\nu}_{s_t} \cdot \boldsymbol{\sigma}_{s_t} dt + L_{s_t} \sum_{s'} \left(\frac{P_{s'}}{P_{s_{t-}}} - \frac{L_{s'}}{L_{s_{t-}}} \right) \left(dN_t^{(s_{t-}, s')} - \Lambda_{s_{t-}, s'} dt \right) - L_{s_t} \left(\frac{x_t \frac{\partial P_{s_t}}{\partial x}}{P_{s_t}} - \frac{x_t \frac{\partial L_{s_t}}{\partial x}}{L_{s_t}} \right) \boldsymbol{\sigma}_{s_t} \cdot d\mathbf{B}_t$$

This leads to the following expression for conditional expected excess returns and conditional return volatilities:

$$\mathbb{E} \left[dR_{t,T}^e | \mathcal{F}_t \right] = - \left[\left(\frac{x_t \frac{\partial P_{s_t}}{\partial x}}{P_{s_t}} - \frac{x_t \frac{\partial L_{s_t}}{\partial x}}{L_{s_t}} \right) \boldsymbol{\nu}_{s_t} \cdot \boldsymbol{\sigma}_{s_t} + \sum_{s'} \Lambda_{s_t s'} \left(e^{v(s_t, s')} - 1 \right) \left(\frac{P_{s'}}{P_{s_t}} - \frac{L_{s'}}{L_{s_t}} \right) \right] L_{s_t} dt$$

$$\text{var} \left[dR_{t,T}^e | \mathcal{F}_t \right] = \left[\left(\frac{x_t \frac{\partial P_{s_t}}{\partial x}}{P_{s_t}} - \frac{x_t \frac{\partial L_{s_t}}{\partial x}}{L_{s_t}} \right)^2 |\boldsymbol{\sigma}_{s_t}|^2 + \sum_{s'} \Lambda_{s_t s'} \left(\frac{P_{s'}}{P_{s_t}} - \frac{L_{s'}}{L_{s_t}} \right)^2 \right] L_{s_t}^2 dt$$

□

A.1.17 Consumption Growth vs. Output Growth Volatility

Let me note $\mu_i^x(x_t)$ the drift rate of x_t in SDF regime i , and $\boldsymbol{\sigma}_i^x(x_t)$ its volatility vector:

$$\begin{aligned} \mu_i^x(x_t) &:= \iota_i(x_t) - \left(\mu_i + m - \gamma |\boldsymbol{\sigma}_i|^2 \right) x_t \\ \boldsymbol{\sigma}_i^x(x_t) &:= -x_t \boldsymbol{\sigma}_i \end{aligned}$$

Using Itô's lemma, I can compute consumption growth volatility as follows:

$$\begin{aligned} \frac{dC_t}{C_t} &= \left[\frac{c'_{s_t}(x_t)}{c_{s_t}(x_t)} \mu_{s_t}^x(x_t) + \frac{1}{2} |\boldsymbol{\sigma}_{s_t}^x(x_t)|^2 \frac{c''_{s_t}(x_t)}{c_{s_t}(x_t)} + \mu_{s_t} + \frac{1}{2} |\boldsymbol{\sigma}_{s_t}|^2 + \frac{c'_{s_t}(x_t)}{c_{s_t}(x_t)} \boldsymbol{\sigma}_{s_t}^x(x_t) \cdot \boldsymbol{\sigma}_{s_t} \right] dt \\ &\quad + \sum_{s'} \left(\frac{c'_{s'}(x_t)}{c_{s_t}(x_t)} - 1 \right) dN_t^{(s_t, s')} + \left(\frac{c'_{s_t}(x_t)}{c_{s_t}(x_t)} \boldsymbol{\sigma}_{s_t}^x(x_t) + \boldsymbol{\sigma}_{s_t} \right) \cdot dB_t \end{aligned}$$

In other words, conditioned on being in SDF regime s_t , the ratio of consumption growth volatility to output growth volatility has the following simple expression:

$$\frac{\text{var} \left[\frac{dC_t}{C_t} | \mathcal{F}_t \right]}{\text{var} \left[\frac{dY_t}{Y_t} | \mathcal{F}_t \right]} = \left(1 - \frac{x_t c'_{s_t}(x_t)}{c_{s_t}(x_t)} \right)^2 + \frac{1}{|\boldsymbol{\sigma}_{s_t}|^2} \sum_{s'} \Lambda_{s_t s'} \left(\frac{c'_{s'}(x_t)}{c_{s_t}(x_t)} - 1 \right)^2$$

Thus, the ratio of consumption growth volatility to output growth volatility crucially depends on the elasticity of the consumption function w.r.t. the debt-to-GDP ratio. Moreover, since the consumption function $c_{s_t}(\cdot)$ is decreasing in the debt-to-GDP ratio, it turns out that consumption growth volatility is greater than output growth volatility. Consumption volatility is also enhanced by the SDF shocks. It is also immediate to verify that the presence

of SDF shocks breaks the unit correlation between consumption growth and output growth. Such correlation is equal to:

$$\text{corr} \left[\frac{dC_t}{C_t}, \frac{dY_t}{Y_t} \mid \mathcal{F}_t \right] = \frac{1}{\sqrt{1 + \sum_{s'} \Lambda_{st,s'} \left(\frac{c_{st}(x_t) - c_{s'}(x_t)}{c_{st}(x_t) - x_t c'_{st}(x_t)} \right)^2}} < 1$$

□

A.1.18 Risk-Neutral Government

Assume $\gamma = \rho = 0$, and assume that $\mu_i = \mu$ and $\boldsymbol{\sigma}_i = \boldsymbol{\sigma}$ for all $i \leq N_s$. The government is risk-neutral; its incentive to take on debt is solely due to the fact that it is more impatient than its creditors: $\delta > r_i$, for all state $i \leq N_s$. In any equilibrium with default where the issuance policy ι_i is finite, the HJB equation for the government life-time utility takes the following form:

$$(\delta - \mu) v_i(x) - \sum_{j=1}^{N_s} \Lambda_{ij} v_j(x) = \max_{\iota_i} \left[\delta (1 + \iota_i D_i(x) - (\kappa + m) x) + [\iota_i - (\mu + m) x] v'_i(x) + \frac{1}{2} |\boldsymbol{\sigma}|^2 x^2 v''_i(x) \right]$$

This expression is linear in ι_i , meaning that for an equilibrium to exist with a finite smooth issuance policy, it must be the case that $\delta D_i(x) + v'_i(x) = 0$. Reinjecting this condition into the HJB above leads to:

$$(\delta - \mu) v_i(x) - \sum_{j=1}^{N_s} \Lambda_{ij} v_j(x) = \delta (1 - (\kappa + m) x) - (\mu + m) x v'_i(x) + \frac{1}{2} |\boldsymbol{\sigma}|^2 x^2 v''_i(x)$$

In this HJB equation, the issuance policy and the debt price have disappeared, and neither constants nor boundary conditions are dependent on the SDF state. Thus, one solution to the HJB equation is to have $v_i(x) = v(x)$ for all state $i \leq N_s$. Of course in that case, the default

boundaries are also SDF state-independent, in other words for all $i \leq N_s$, $\bar{x}_i = \bar{x}$. Finally, since $\delta D_i(x) + v'_i(x) = 0$, it is also the case that the debt price is SDF state-independent. v then solves the following:

$$(\delta - \mu) v(x) = \delta (1 - (\kappa + m) x) - (\mu + m) x v'(x) + \frac{1}{2} |\boldsymbol{\sigma}|^2 x^2 v''(x) \quad (\text{A.22})$$

In other words, the government life-time utility is identical to its value if it was allowing its debt to amortize, without ever re-issuing new debt or buying back existing debt. The second order ordinary differential equation admits the following characteristic polynomial:

$$\frac{1}{2} |\boldsymbol{\sigma}|^2 \xi^2 - \left(m + \mu + \frac{1}{2} |\boldsymbol{\sigma}|^2 \right) \xi - (\delta - \mu) = 0$$

Let ξ be the positive root of such polynomial (the other root being strictly negative):

$$\xi := \frac{1}{2} \left(1 + \frac{2(m + \mu)}{|\boldsymbol{\sigma}|^2} \right) \left[1 + \left(1 + \frac{8(\delta - \mu)|\boldsymbol{\sigma}|^2}{(2(m + \mu) + |\boldsymbol{\sigma}|^2)^2} \right)^{1/2} \right] > 1$$

Since the value function must be finite at $x = 0$, it takes the following form:

$$v(x) = \frac{\delta}{\delta - \mu} - \delta \left(\frac{\kappa + m}{\delta + m} \right) x + k_v \left(\frac{x}{\bar{x}} \right)^\xi$$

k_v is a constant of integration that will be found using boundary conditions. At default, $v(\bar{x}) = \alpha v^d(\bar{x})$, where the constant $v^d(\bar{x})$ satisfies:

$$v^d(\bar{x}) = \frac{\delta + \lambda v(\theta \bar{x})}{\delta + \lambda - \mu}$$

Thus the constant k_v solves:

$$\frac{\delta}{\delta - \mu} - \delta \left(\frac{\kappa + m}{\delta + m} \right) \bar{x} + k_v = \frac{\alpha \delta}{\delta + \lambda - \mu} + \frac{\alpha \lambda}{\delta + \lambda - \mu} \left[\frac{\delta}{\delta - \mu} - \delta \left(\frac{\kappa + m}{\delta + m} \right) \theta \bar{x} + k_v \theta^\xi \right]$$

The optimal default boundary \bar{x} then satisfies the smooth pasting condition:

$$v'(\bar{x}) = \alpha(v^d)'(\bar{x}) = \frac{\alpha\theta\lambda v'(\theta\bar{x})}{\delta + \lambda - \mu}$$

This leads to a value of \bar{x} that verifies:

$$\delta \left(\frac{\kappa + m}{\delta + m} \bar{x} \right) \left(1 - \frac{\alpha\theta\lambda}{\delta + \lambda - \mu} \right) = k_v \xi \left(1 - \frac{\alpha\theta\xi\lambda}{\delta + \lambda - \mu} \right)$$

I then deduce the following optimal default boundary \bar{x} and the constant of integration k_v :

$$\begin{aligned} \bar{x} &= \frac{\xi}{\xi - 1} \left(\frac{\delta + m}{\kappa + m} \right) \left(\frac{\frac{1-\alpha}{\delta-\mu}}{1 - \frac{\alpha\theta\lambda}{\delta+\lambda-\mu}} \right) \\ k_v &= \left(\frac{\delta}{\xi - 1} \right) \left(\frac{\frac{1-\alpha}{\delta-\mu}}{1 - \frac{\alpha\theta\xi\lambda}{\delta+\lambda-\mu}} \right) \end{aligned}$$

The debt price is computed using the equality $\delta D(x) + v'(x) = 0$:

$$\begin{aligned} D(x) &= \frac{\kappa + m}{\delta + m} - \frac{\xi k_v}{x^*} \left(\frac{x}{\bar{x}} \right)^{\xi-1} \\ &= \left(\frac{\kappa + m}{\delta + m} \right) \left[1 - \left(\frac{1 - \frac{\alpha\theta\lambda}{\delta+\lambda-\mu}}{1 - \frac{\alpha\theta\xi\lambda}{\delta+\lambda-\mu}} \right) \left(\frac{x}{\bar{x}} \right)^{\xi-1} \right] \end{aligned}$$

D is of course a decreasing function of x since $\xi > 1$. This is the condition that [17] uncover as the necessary and sufficient condition for optimality of a “smooth” financing strategy for the government. It is then easy to show that:

$$v(x) = \delta \left[\frac{1}{\delta - \mu} \left(1 - \left(\frac{1 - \alpha}{1 - \frac{\alpha\theta\xi\lambda}{\delta+\lambda-\mu}} \right) \left(\frac{x}{\bar{x}} \right)^\xi \right) - xD(x) \right]$$

This formula has a natural interpretation: the life-time utility for the government is equal to the present value (from the government’s perspective) of its endowment stream, adjusted for expected welfare losses due to default, minus the aggregate value of sovereign debt. The

debt price function D must satisfy the HJB equation (from Feynman-Kac):

$$(r_i + m) D(x) = \kappa + m + \left[\iota_i(x) - \left(\mu + m - |\boldsymbol{\sigma}|^2 - \boldsymbol{\sigma} \cdot \boldsymbol{\nu}_i \right) x \right] D'(x) + \frac{1}{2} |\boldsymbol{\sigma}|^2 D''(x) \quad (\text{A.23})$$

Take equation A.22, differentiate w.r.t. x , and use $\delta D(x) + v'(x) = 0$ to obtain:

$$(\delta + m) D(x) = \kappa + m - \left(\mu + m - |\boldsymbol{\sigma}|^2 \right) x D'(x) + \frac{1}{2} |\boldsymbol{\sigma}|^2 D''(x) \quad (\text{A.24})$$

I can interpret probabilistically this HJB equation: the price of one unit of debt is equal to the expected discounted net present value of interest and principal repayments on such debt contract, where the discount rate is δ , and where the default time is the first time at which the debt-to-GDP ratio hits the boundary \bar{x} , using a probability measure under which no new debt is ever issued by the government. Subtract equation A.24 from equation A.23, and simplify to obtain:

$$\begin{aligned} \iota_i(x) &= \frac{(\delta - r_i)D(x) + \boldsymbol{\sigma} \cdot \boldsymbol{\nu}_i x D'(x)}{-D'(x)} \\ &= \frac{\delta - r_i}{\xi - 1} \left[\left(\frac{1 - \frac{\alpha\theta\xi\lambda}{\delta+\lambda-\mu}}{1 - \frac{\alpha\theta\lambda}{\delta+\lambda-\mu}} \right) \left(\frac{\bar{x}}{x} \right)^{\xi-1} - 1 \right] x - \boldsymbol{\sigma} \cdot \boldsymbol{\nu}_i x \end{aligned}$$

The time-varying interest rates and prices of risk only impact the financing policy of the government: in periods of high risk-prices or relatively high risk-free rates, the government adjusts its financing policy downwards. Finally, note that the stochastic differential equation for x_t takes the following form (in the continuation region):

$$\begin{aligned} dx_t &= \left(\frac{\delta - r_i}{\xi - 1} \left(\frac{1 - \frac{\alpha\theta\xi\lambda}{\delta+\lambda-\mu}}{1 - \frac{\alpha\theta\lambda}{\delta+\lambda-\mu}} \right) \bar{x}^{\xi-1} x_t^{2-\xi} \right. \\ &\quad \left. - \left(\frac{\delta - r_i}{\xi - 1} + m + \mu - |\boldsymbol{\sigma}|^2 + \boldsymbol{\sigma} \cdot \boldsymbol{\nu}_i \right) x_t \right) dt - x_t \boldsymbol{\sigma} \cdot d\mathbf{B}_t \end{aligned}$$

This equation admits a singularity at $x = 0$, since at that point, the drift rate is unbounded, except when $\xi \in (1, 2]$. \square

A.1.19 Endogeneous Growth

The dynamic equations for the state variables and the resource constraint are as follows:

$$dK_t = (H_t - \eta K_t)dt + K_t \boldsymbol{\sigma} \cdot d\mathbf{B}_t$$

$$dF_t = (I_t - mF_t)dt$$

$$C_t + H_t = aK_t + I_t D_t - (\kappa + m)F_t - G(H_t, K_t)$$

By noting $\iota_t := I_t/K_t$, $h_t := H_t/K_t$, a natural state variable arises: $x_t := F_t/K_t$, which evolves according to:

$$dx_t = \left(\iota_t - \left(m + h_t - \eta - |\boldsymbol{\sigma}|^2 \right) x_t \right) dt - x_t \boldsymbol{\sigma} \cdot d\mathbf{B}_t$$

As usual, the value function, in SDF state i , can be written $V_i(K, F) = v_i(x) K^{1-\gamma}$. The HJB equation satisfied by v_i in the continuation region can be written:

$$\begin{aligned} \frac{1-\gamma}{1-\rho} \left(\delta + (1-\rho) \left(\eta + \frac{1}{2} \gamma |\boldsymbol{\sigma}|^2 \right) \right) v_i(x) - \sum_{j=1}^{N_s} v_j(x) = \\ \max_{\iota_i, h_i} \left[\frac{\delta}{1-\rho} (a + \iota_i D_i(x) - (\kappa + m)x - (h_i + g(h_i)))^{1-\rho} [(1-\gamma)v_i(x)]^{\frac{\rho-\gamma}{1-\gamma}} \right. \\ \left. + h_i(1-\gamma)v_i(x) + \left[\iota_i - \left(m + h_i - \eta - \gamma |\boldsymbol{\sigma}|^2 \right) x \right] v_i'(x) + \frac{1}{2} |\boldsymbol{\sigma}|^2 x^2 v_i''(x) \right] \end{aligned}$$

The (necessary and sufficient) first order conditions for investment and debt issuances are as follows:

$$\delta D_i(x) c_i(x)^{-\rho} [(1 - \gamma)v_i(x)]^{\frac{\rho-\gamma}{1-\gamma}} = -v_i'(x) \quad (\text{A.25})$$

$$\delta (1 + g'(h_i(x))) c_i(x)^{-\rho} [(1 - \gamma)v_i(x)]^{\frac{\rho-\gamma}{1-\gamma}} = (1 - \gamma)v_i(x) - xv_i'(x) \quad (\text{A.26})$$

In the above, I have used the optimal consumption-to-capital ratio $c_i(x)$:

$$c_i(x) := a + \iota_i(x) D_i(x) - (\kappa + m)x - (h_i(x) + g(h_i(x)))$$

The first order condition for optimal investments shows that a marginal unit of output reinvested has two benefits: first, it increases the capital stock (the term $(1 - \gamma)v_i(x)$ on the right-handside of equation A.26), and second, it moves the debt-to-capital ratio away from the default boundary (the term $-xv_i'(x)$ on the right-handside of equation A.26). Taking the ratio of (A.26) over (A.25), I obtain:

$$1 + g'(h_i(x)) = xD_i(x) \left(1 - \frac{(1 - \gamma)v_i(x)}{xv_i'(x)} \right)$$

The equation above can be solved as a function of $h_i(x)$, and then reinjected into the first order condition for the issuance policy $\iota_i(x)$. In the case of quadratic adjustment costs, I have:

$$h_i(x) = \frac{1}{\psi} \left[xD_i(x) \left(1 - \frac{(1 - \gamma)v_i(x)}{xv_i'(x)} \right) - 1 \right]$$

Note that the issuance policy then takes the form:

$$\iota_i(x) = \frac{1}{D_i(x)} \left[\left(\frac{\delta D_i(x) [(1 - \gamma)v_i(x)]^{\frac{\rho-\gamma}{1-\gamma}}}{-v_i'(x)} \right)^{1/\rho} + (\kappa + m)x + h_i(x) + g(h_i(x)) - a \right]$$

Once the government has elected to default, it is in financial autarky for an exponentially distributed time period (parametrized by λ), following which it exits with a resulting debt-to-capital ratio that is θ times its pre-default debt-to-capital ratio. In default, I note $V_i^d(K, F) = v_i^d(x)(\alpha K)^{1-\gamma}$ the government value function. The HJB solved by v_i^d is as follows:

$$\begin{aligned} \left[\frac{1-\gamma}{1-\rho} \left(\delta + (1-\rho) \left(\eta + \frac{1}{2} \gamma |\boldsymbol{\sigma}|^2 \right) \right) + \lambda \right] v_i^d(x) - \sum_{j=1}^{N_s} \Lambda_{ij} v_j^d(x) &= \lambda v_i(\theta x) \\ &+ \max_{h_i} \left[\frac{\delta}{1-\rho} (a - (h_i + g(h_i)))^{1-\rho} \left[(1-\gamma) v_i^d(x) \right]^{\frac{\rho-\gamma}{1-\gamma}} + h_i(1-\gamma) v_i^d(x) \right] \end{aligned}$$

Thus, in default, the investment rate $h_i^d(x)$ (per unit of capital) and the consumption rate $c_i^d(x)$ (per unit of capital) are constant and solve:

$$\begin{aligned} \delta \left(1 + g' \left(h_i^d(x) \right) \right) \left(c_i^d(x) \right)^{-\rho} \left[(1-\gamma) v_i^d(x) \right]^{\frac{\rho-\gamma}{1-\gamma}} &= (1-\gamma) v_i^d(x) \\ c_i^d(x) &= a - \left(h_i^d(x) + g \left(h_i^d(x) \right) \right) \end{aligned}$$

Note $A_i^d(x) := \delta + (\rho-1) \left(h_i^d(x) - \eta - \frac{1}{2} \gamma |\boldsymbol{\sigma}|^2 \right)$. The default value vector $\mathbf{v}^d(x)$ then solves:

$$\left[\frac{1-\gamma}{1-\rho} \text{diag}_i \left(A_i^d(x) \right) + \lambda I - \Lambda \right] \mathbf{v}^d(x) - \lambda \mathbf{v}(\theta x) = \frac{\delta}{1-\rho} \mathbf{c}^d(x)^{1-\rho} \left[(1-\gamma) \mathbf{v}^d(x) \right]^{\frac{\rho-\gamma}{1-\gamma}}$$

□

A.2 Tables and Plots

Figure A.1: US short term nominal and real interest rates

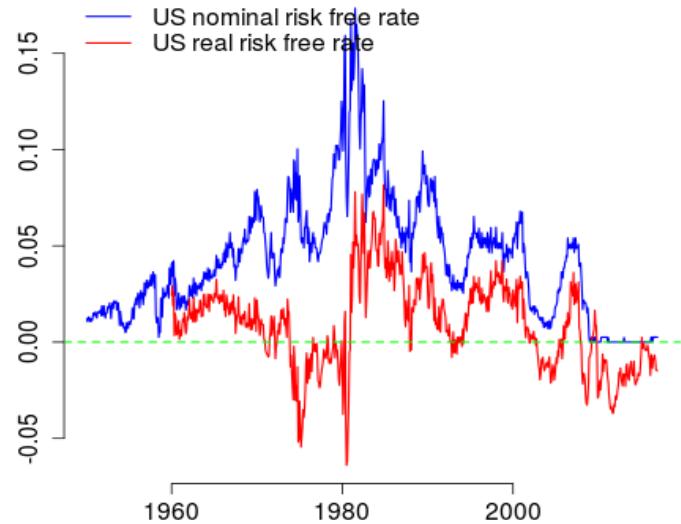


Figure A.2: US short term real rates and discrete state Markov process approximation

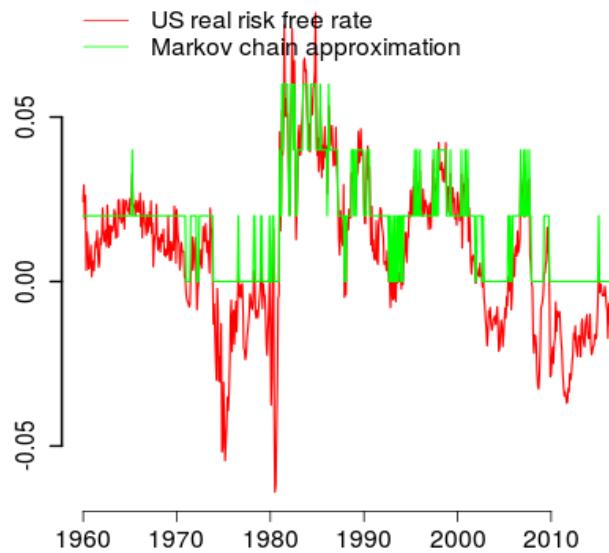


Table A.1: Comparative Statics – Moment Elasticities

Model parameter	1 – α	δ	γ	ρ	μ	$ \sigma $	$ \nu $	r	κ
Parameter description	default GDP drop	degree of patience	risk aversion	inverse IES	GDP growth	GDP growth vol.	risk price	risk free rate	bond coupon rate
Base case	0.04	0.20	5	2	0.035	0.04	0.625	0.05	0.05
default boundary \bar{x}	56%	1.07	-0.57	-0.06	-0.03	0.14	-0.23	-0.03	-0.09
mean debt-to-GDP $\mathbb{E}[x_t]$	52%	1.06	-0.56	-0.06	-0.04	0.14	-0.29	-0.03	-0.09
stdev debt-to-GDP $\text{stdev}[x_t]$	3%	1.01	0.04	-0.13	0.13	0.24	-0.08	-0.22	-0.28
default rate $(1/\lambda + T(\theta\bar{x}))^{-1}$	2.8% p.a.	-0.47	1.57	-0.13	0.03	0.30	0.64	-0.38	-0.35
bond spread $\mathbb{E}[\varsigma(x_t)]$	365 bps p.a.	-0.38	1.55	-0.13	0.9	0.17	0.81	0.14	-0.22
bond spread vol $\mathbb{E}[\sigma^s(x_t)]$	130 bps p.a.	-0.25	1.38	-0.12	0.15	0.10	0.95	-0.06	0.02
5y CDS $\mathbb{E}[\varsigma(x_t, 5)]$	414 bps p.a.	-0.43	1.73	-0.15	0.10	0.23	0.90	0.14	-0.16
5y-1y slope $\mathbb{E}[\varsigma(x_t, 5) - \varsigma(x_t, 1)]$	60 bps p.a.	0.01	1.08	-0.12	0.37	0.11	1.09	0.92	-0.22
bond excess return $\mathbb{E}[dR_t^e]$	164 bps p.a.	-0.19	1.11	-0.10	0.13	0.08	0.80	0.93	-0.16
bond return vol. $\text{stdev}[dR_t^e]$	525 bps p.a.	-0.19	1.11	-0.10	0.13	0.08	0.80	-0.07	-0.16
5y CDS excess return $\mathbb{E}[dR_{t,5}^e]$	230 bps p.a.	-0.23	1.30	-0.11	0.16	0.14	0.83	0.91	-0.13
5y CDS return vol. $\text{stdev}[dR_{t,5}^e]$	732 bps p.a.	-0.23	1.30	-0.11	0.16	0.14	0.83	-0.09	-0.13
									-0.15

Table A.2: Estimation Result (part 1)

Moment/Parameter	Brazil		Bulgaria		Hungary		Colombia	
	Data	Model	Data	Model	Data	Model	Data	Model
δ (% p.a.)		10		10		20		10
$1 - \alpha$ (%)		1.5		2.5		4.5		1.5
corr (σ_t, ν_t) (%)		100		95		100		100
γ		2.5		2.5		3.0		3.0
GDP growth rate (% p.a.)	4.2	4.2	1.7	1.7	1.8	1.8	4.2	4.2
GDP vol (% p.a.)	3.8	3.8	4.7	4.7	2.8	2.8	2.0	2.0
Avg. debt-to-GDP (%)	28.4	34.0	71.1	71.8	93.8	72.1	30.9	35.2
Avg. 5y CDS (bps p.a.)	391	456	177	183	273	327	261	343
Avg. 5y CDS xs return	643	306	291	171	834	223	420	175
Avg. 5y-1y slope	145	81	78	77	130	76	153	51
Stdev. debt-to-GDP (%)	10	2.1	34.0	3.7	43.7	4.5	7.5	2.2
vol($\ln C_t$)/vol($\ln Y_t$)	2.0	1.8		1.8		2.4		1.9
Bond spread (bps p.a.)		374		173		275		297
Bond spread (RN model)		309		113		156		208
Avg. 1y CDS (bps p.a.)	246	375	99	106	143	252	108	292
Avg. 1y CDS xs return		282		113		198		168
5y CDS return volatility	1,247	627	631	350	520	622	875	640

Table A.3: Estimation Result (part 2)

Moment/Parameter	Mexico		Philippines		South Africa		Turkey	
	Data	Model	Data	Model	Data	Model	Data	Model
δ (% p.a.)		10		15		10		10
$1 - \alpha$ (%)		2.0		4.5		1.5		1.5
corr (σ_t, ν_t) (%)		75		100		100		100
γ		1.5		2		2.5		2.5
GDP growth rate (% p.a.)	3.9	3.9	4.1	4.1	3.1	3.1	4.4	4.4
GDP vol (% p.a.)	3.5	3.5	3.0	3.0	2.5	2.5	3.8	3.8
Avg. debt-to-GDP (%)	32.9	56.0	54.9	53.7	23.0	36.6	35.2	32.8
Avg. 5y CDS (bps p.a.)	133	228	241	271	155	195	312	357
Avg. 5y CDS xs return	185	178	391	175	181	128	478	242
Avg. 5y-1y slope	78	54	139	49	84	34	138	67
Stdev. debt-to-GDP (%)	15.1	2.8	20.2	2.9	65.9	1.9	12.0	1.9
vol($\ln C_t$)/vol($\ln Y_t$)		1.7		2.3		2.0		1.8
Bond spread (bps p.a.)		206		236		178		310
Bond spread (RN model)		171		199		133		235
Avg. 1y CDS (bps p.a.)	56	174	102	223	71	161	174	290
Avg. 1y CDS xs return		282		113		198		168
5y CDS return volatility	636	495	759	476	589	295	973	485

Figure A.3: Current Account Reversal during Latin America Debt Crisis

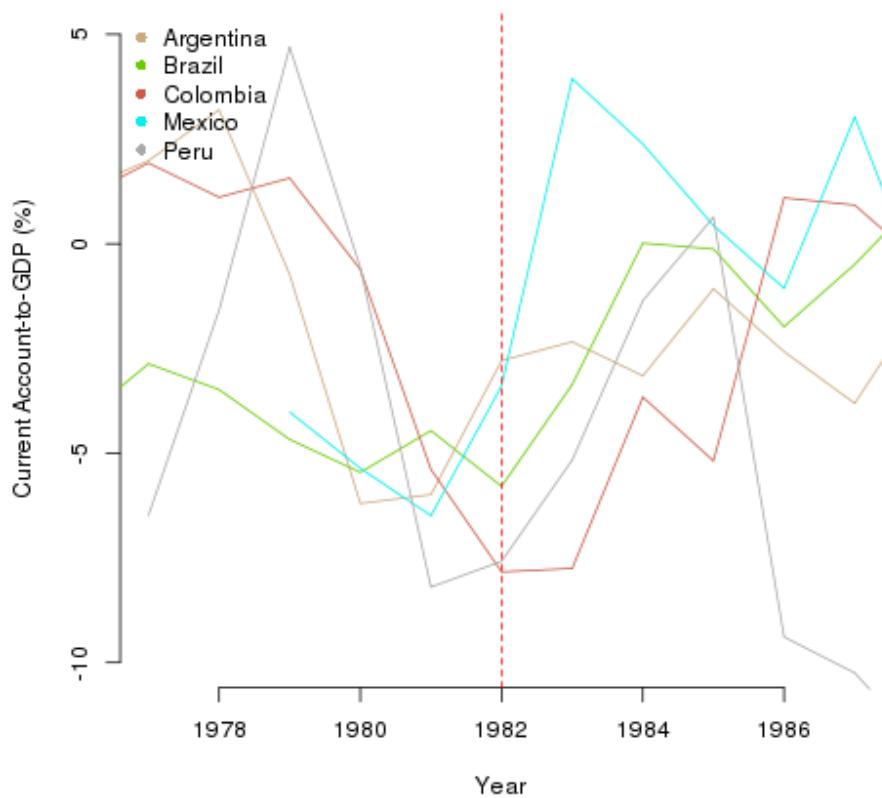
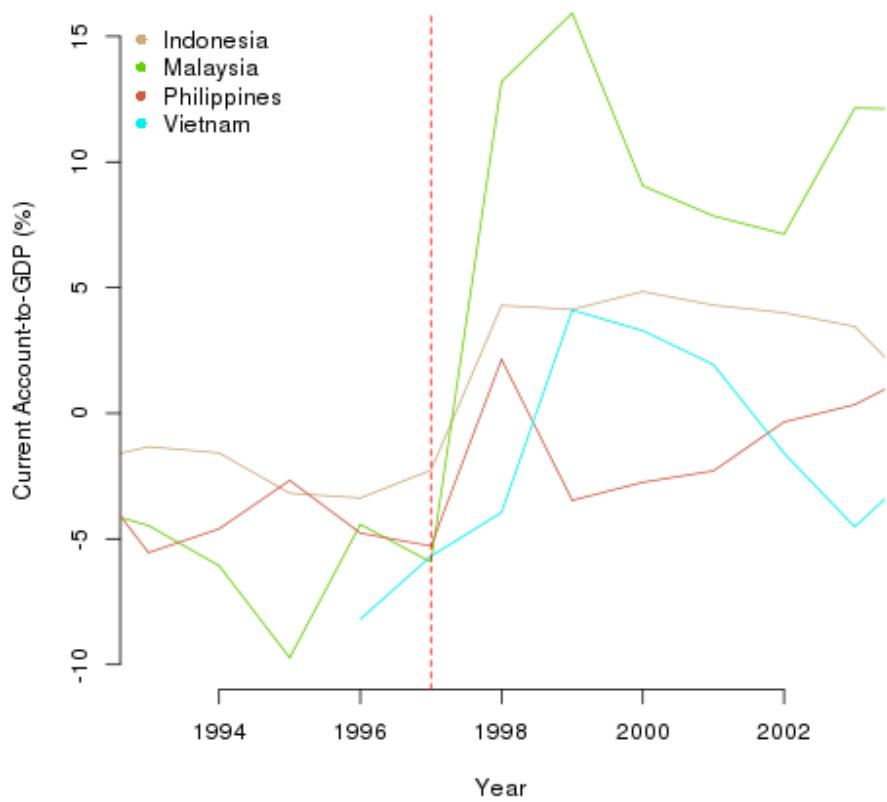


Figure A.4: Current Account Reversal during Asian Tiger Crisis



A.3 Numerical Algorithm – Markov Chain Approximation

I will use a Markov Chain approximation method, as explained in [29]. I select this method over the more traditional finite difference schemes: indeed, to guarantee convergence of both methods to the solution of the differential equations of interest, the latter requires using numerical schemes that are consistent, monotone and stable, and the last two requirements are not always simple to verify⁴, while the former requires the construction of approximating Markov chains whose transition probabilities need to satisfy consistency properties that are extremely simple to verify in practice. I also favor the Markov Chain approximation method over collocation methods: the latter methods require inverting non-sparse matrices, while the former involves the inversion of sparse matrices, making my algorithm significantly faster.

I compute the functions $\{(v_s, D_s)\}_{s \leq N_s}$ numerically over the compact set $[0, \max_s \bar{x}_s]$, by determining their values on an equally-spaced grid G_h , where $h > 0$ is my scalar approximation parameter. I will note $\{x_k = kh\}_{0 \leq k \leq N_h}$ the grid points of G_h . I start with a guess equilibrium cutoff $N_s \times 1$ vector $\bar{\mathbf{x}}^{(1)}$, and a guess issuance policy vector $\boldsymbol{\iota}^{(1,1)} = \{\iota_s^{(1,1)}(x_k)\}_{0 \leq k \leq N_h, 0 \leq s \leq N_s}$. My algorithm has an outer-loop, which updates the equilibrium default cutoff vector $\bar{\mathbf{x}}^{(i)}$, and an inner loop, which, for a given vector of cutoffs $\bar{\mathbf{x}}^{(i)}$, updates the functions $\{v_s^{(i,j)}, D_s^{(i,j)}, \iota_s^{(i,j)}\}$. In the inner loop, I calculate the functions $v_s(\cdot; \bar{\mathbf{x}}^{(i)})$, $D_s(\cdot; \bar{\mathbf{x}}^{(i)})$ and $\iota_s(\cdot; \bar{\mathbf{x}}^{(i)})$ as follows.

I first describe how to compute $\{D_s^{(i,j)}\}_{s \leq N_s}$, given an issuance schedule $\boldsymbol{\iota}^{(i,j)}$ and a default policy $\bar{\mathbf{x}}^{(i)}$. To simplify notation, I omit the superscript (i, j) when possible. Given a set of cutoffs $\{\bar{x}_s\}_{s \leq N_s}$ and issuance policies $\{\iota_s\}_{s \leq N_s}$, the dynamic evolution of the state

4. I note however that upwinding has been used as a classic tool to implement monotone schemes.

variables x_t and s_t under \mathbb{Q} can be written as follows, for $x_t \in [0, \bar{x}_{s_t}]$:

$$\begin{aligned} dx_t &= \left(\iota_{s_t}(x_t) - \left(m + \mu_{s_t} - |\boldsymbol{\sigma}_{s_t}|^2 - \boldsymbol{\nu}_{s_t} \cdot \boldsymbol{\sigma}_{s_t} \right) x_t \right) dt - x_t \boldsymbol{\sigma}_{s_t} \cdot d\hat{\mathbf{B}}_t \\ &= \mu_{\mathbb{Q}}(x_t, s_t) dt - x_t \boldsymbol{\sigma}_{s_t} \cdot d\hat{\mathbf{B}}_t \\ ds_t &= \sum_{s' \neq s_{t-}} (s' - s_{t-}) d\hat{N}_t^{s_{t-}, s'} \end{aligned}$$

In the inner loop, I create a Markov Chain $X_{\mathbb{Q},n}^h := (x_{\mathbb{Q},n}^h, s_{\mathbb{Q},n}^h)$ that approximates the process $\{(x_t, s_t)\}_{t \geq 0}$ under \mathbb{Q} . I introduce $Q_{\mathbb{Q}}^h(x, s)$ and $\Delta t_{\mathbb{Q}}^h(x, s)$ as follows:

$$\begin{aligned} Q_{\mathbb{Q}}^h(x, s) &:= x^2 |\boldsymbol{\sigma}_s|^2 + h |\mu_{\mathbb{Q}}(x, s)| \\ \Delta t_{\mathbb{Q}}^h(x, s) &:= \frac{h^2}{Q_{\mathbb{Q}}^h(x, s)} \end{aligned}$$

Since $\mu_{\mathbb{Q}}(0, s) > 0$ for all discrete SDF state s (in equilibrium the sovereign will be borrowing when it is not indebted, i.e. $\iota_s(0) > 0$), $\inf_{x,s} Q_{\mathbb{Q}}^h(x, s) > 0$, which means that $\Delta t_{\mathbb{Q}}^h(x, s)$ is well defined. For all x and all $s \leq N_s$, I have:

$$\lim_{h \rightarrow 0} \Delta t_{\mathbb{Q}}^h(x, s) = 0$$

I then define the following transition probabilities:

$$\begin{aligned} \Pr \left(X_{\mathbb{Q},n+1}^h = (x+h, s) \mid X_{\mathbb{Q},n}^h = (x, s) \right) &= \frac{e^{\hat{\Lambda}_{ss} \Delta t_{\mathbb{Q}}^h(x, s)}}{Q_{\mathbb{Q}}^h(x, s)} \left(\frac{x^2 |\boldsymbol{\sigma}_s|^2}{2} + h \max(0, \mu_{\mathbb{Q}}(x, s)) \right) \\ \Pr \left(X_{\mathbb{Q},n+1}^h = (x-h, s) \mid X_{\mathbb{Q},n}^h = (x, s) \right) &= \frac{e^{\hat{\Lambda}_{ss} \Delta t_{\mathbb{Q}}^h(x, s)}}{Q_{\mathbb{Q}}^h(x, s)} \left(\frac{x^2 |\boldsymbol{\sigma}_s|^2}{2} + h \max(0, -\mu_{\mathbb{Q}}(x, s)) \right) \\ \Pr \left(X_{\mathbb{Q},n+1}^h = (x, s') \mid X_{\mathbb{Q},n}^h = (x, s) \right) &= \left(\frac{\hat{\Lambda}_{ss'}}{-\hat{\Lambda}_{ss}} \right) \left(1 - e^{\hat{\Lambda}_{ss} \Delta t_{\mathbb{Q}}^h(x, s)} \right) \end{aligned}$$

Notice that these transition probabilities are all greater than zero, less than 1, and they add up to 1. Noting $\Delta x_{\mathbb{Q},n}^h := x_{\mathbb{Q},n+1}^h - x_{\mathbb{Q},n}^h$ and $\Delta s_{\mathbb{Q},n}^h := s_{\mathbb{Q},n+1}^h - s_{\mathbb{Q},n}^h$, the Markov chain

created satisfies the local consistency condition:

$$\mathbb{E}^{x,s} \left[\Delta x_{\mathbb{Q},n}^h \right] = \mu_{\mathbb{Q}}(x, s) \Delta t_{\mathbb{Q}}^h(x, s) \quad (\text{A.27})$$

$$\text{var}^{x,s} \left[\Delta x_{\mathbb{Q},n}^h \right] = x^2 |\boldsymbol{\sigma}_s|^2 \Delta t_{\mathbb{Q}}^h(x, s) + o \left(\Delta t_{\mathbb{Q}}^h(x, s) \right) \quad (\text{A.28})$$

$$\mathbb{E}^{x,s} \left[\Delta s_{\mathbb{Q},n}^h \right] = \sum_{s'} \hat{\Lambda}_{s,s'}(s' - s) \Delta t_{\mathbb{Q}}^h(x, s) \quad (\text{A.29})$$

$$\text{var}^{x,s} \left[\Delta s_{\mathbb{Q},n}^h \right] = \sum_{s'} \hat{\Lambda}_{s,s'}(s' - s)^2 \Delta t_{\mathbb{Q}}^h(x, s) + o \left(\Delta t_{\mathbb{Q}}^h(x) \right) \quad (\text{A.30})$$

$$\text{cov}^{x,s} \left[\Delta x_{\mathbb{Q},n}^h, \Delta s_{\mathbb{Q},n}^h \right] = 0 \quad (\text{A.31})$$

For $\bar{x}_s > x_k \geq 0$, the sovereign government is performing and I compute $D_s^{(i,j)}(x_k)$ as follows:

$$D_s^{(i,j)}(x_k) = (\kappa + m) \Delta t_{\mathbb{Q}}^h(x_k, s) + e^{-(r_s + m) \Delta t_{\mathbb{Q}}^h(x_k, s)} \sum_{x'_{\mathbb{Q}}, s'_{\mathbb{Q}}} \Pr \left(X'_{\mathbb{Q}} | x_k, s \right) D_{s'_{\mathbb{Q}}}^{(i,j)}(x'_{\mathbb{Q}})$$

For $x_k \geq \bar{x}_s$, the sovereign government is in default and I compute $D_s^{(i,j)}(x_k)$ as follows:

$$D_s^{(i,j)}(x_k) = \lambda \theta \alpha \left(\Xi^{-1} \mathbf{D}^{(i,j)}(\theta x_k) \right)_s$$

This is a linear system of $N_s \times (N_h + 1)$ equations in $N_s \times (N_h + 1)$ unknown, which can be solved easily via a simple matrix inversion. Note that the matrix to be inverted is sparse, which greatly reduces computing time.

I then describe how to compute $v_s^{(i,j)}$ in each discrete Markov state s , given an issuance schedule $\boldsymbol{\iota}^{(i,j)}$, a debt price schedule $\mathbf{D}^{(i,j)}$ and a default policy $\bar{\mathbf{x}}^{(i)}$. Once again I omit the superscript (i,j) when possible. Given a vector of cutoffs $\bar{\mathbf{x}}$, an issuance policy $\boldsymbol{\iota}$ and a debt price schedule \mathbf{D} , the dynamic evolution of the state variables x_t and s_t under the probability measure induced via $\tilde{\Pr}(A) = \mathbb{E} \left[e^{(1-\gamma) \int_0^t \boldsymbol{\sigma}_{s,u} \cdot d\mathbf{B}_u - \frac{1}{2}(1-\gamma)^2 \int_0^t |\boldsymbol{\sigma}_{s,u}|^2 du} 1_A \right]$ can be

expressed as follows, for $x_t \in [0, \bar{x}_{s_t}]$:

$$\begin{aligned} dx_t &= \left(\iota_{s_t}(x_t) - \left(m + \mu_{s_t} - \gamma |\boldsymbol{\sigma}_{s_t}|^2 \right) x_t \right) dt - x_t \boldsymbol{\sigma}_{s_t} \cdot d\tilde{\mathbf{B}}_t \\ &= \mu_{\tilde{\mathbb{P}}}(x_t, s_t) dt - x_t \boldsymbol{\sigma}_{s_t} \cdot d\tilde{\mathbf{B}}_t \\ ds_t &= \sum_{s' \neq s_{t-}} (s' - s_{t-}) dN_t^{s_{t-}, s'} \end{aligned}$$

By introducing $Q_{\tilde{\mathbb{P}}}^h(x, s)$ and $\Delta t_{\tilde{\mathbb{P}}}^h(x, s)$, computed in a similar fashion to $Q_{\mathbb{Q}}^h(x, s)$ and $\Delta t_{\mathbb{Q}}^h(x, s)$, I can construct a new Markov Chain $\{X_{\tilde{\mathbb{P}}, n}^h\}_{n \geq 0}$ that approximates the process $\{(x_t, s_t)\}_{t \geq 0}$ under the probability measure $\tilde{\Pr}$. The transition probabilities of this Markov chain will satisfy consistency conditions similar to those of equations A.27, A.28, A.29, A.30 and A.31. For $\bar{x}_s > x_k \geq 0$, the sovereign government is performing and I compute $v_s^{(i,j)}(x_k)$ as follows:

$$\begin{aligned} v_s^{(i,j)}(x_k) &= \frac{\delta}{1 - \rho} (1 + \iota_s(x_k) D_s(x_k) - (\kappa + m) x_k) \left[(1 - \gamma) v_s^{(i,j)}(x_k) \right]^{\frac{\rho - \gamma}{1 - \gamma}} \Delta t_{\tilde{\mathbb{P}}}^h(x_k, s) \\ &\quad + e^{-\frac{1-\gamma}{1-\rho} A_s \Delta t_{\tilde{\mathbb{P}}}^h(x_k, s)} \sum_{x'_{\tilde{\mathbb{P}}}, s'_{\tilde{\mathbb{P}}}} \Pr(X'_{\tilde{\mathbb{P}}} | x_k, s) v_{s'_{\tilde{\mathbb{P}}}}^{(i,j)}(x'_{\tilde{\mathbb{P}}}) \end{aligned}$$

For $x_k \geq \bar{x}_s$, the sovereign government is in default and I compute $v_s^{(i,j)}(x_k)$ as follows:

$$\left(v_s^d \right)^{(i,j)}(x_k) = \left[\Upsilon^{-1} \left(\frac{\delta}{1 - \rho} \left[(1 - \gamma) \left(\mathbf{v}^d \right)^{(i,j)}(x_k) \right]^{\frac{\rho - \gamma}{1 - \gamma}} + \lambda \mathbf{v}^{(i,j)}(\theta x_k) \right) \right]_s$$

Note that the resulting system of $N_s \times (N_h + 1)$ equations in $N_s \times (N_h + 1)$ unknown $\{v_s^{(i,j)}(x_k)\}_{0 \leq k \leq N_h, s \leq N_s}$ is not linear. In order to solve such system, I use a simple procedure:

starting with a guess $\{v^{(i,j,m)}(x_k)\}_{0 \leq k \leq N_h}$, I iterate, for $x_k < \bar{x}_s$, on the following:

$$\begin{aligned} v_s^{(i,j,m+1)}(x_k) &= \frac{\delta}{1-\rho} (1 + \iota_s(x_k) D_s(x_k) - (\kappa + m)x_k) \left[(1-\gamma)v_s^{(i,j,m)}(x_k) \right]^{\frac{\rho-\gamma}{1-\gamma}} \Delta t_{\tilde{\mathbb{P}}}^h(x_k, s) \\ &\quad + e^{-\frac{1-\gamma}{1-\rho} A_s \Delta t_{\tilde{\mathbb{P}}}^h(x_k, s)} \sum_{x'_{\tilde{\mathbb{P}}}, s'_{\tilde{\mathbb{P}}}} \Pr(X'_{\tilde{\mathbb{P}}} | x_k, s) v_{s'_{\tilde{\mathbb{P}}}}^{(i,j,m+1)}(x'_{\tilde{\mathbb{P}}}) \end{aligned}$$

For $x_k \geq \bar{x}_s$, the iteration becomes:

$$(v_s^d)^{(i,j,m+1)}(x_k) = \left[\Upsilon^{-1} \left(\frac{\delta}{1-\rho} \left[(1-\gamma) (v^d)^{(i,j,m+1)}(x_k) \right]^{\frac{\rho-\gamma}{1-\gamma}} + \lambda v^{(i,j,m)}(\theta x_k) \right) \right]_s$$

The iterative procedure is stopped once $\|\mathbf{v}^{(i,j,m+1)} - \mathbf{v}^{(i,j,m)}\|_\infty$ is sufficiently small. Once $\mathbf{v}^{(i,j)}$ and $\mathbf{D}^{(i,j)}$ are computed, I can update the issuance policy as follows, in each state s :

$$\begin{aligned} \iota_s^{(i,j+1)}(x_k) &= \varpi \iota_s^{(i,j)}(x_k) + \frac{1-\varpi}{D_s^{(i,j)}(x_k)} \left[\left(\frac{\delta D_s^{(i,j)}(x_k) \left[(1-\gamma)v_s^{(i,j)}(x_k) \right]^{\frac{\rho-\gamma}{1-\gamma}}}{- (v_s^{(i,j)})'(x_k)} \right)^{1/\rho} \right. \\ &\quad \left. + (\kappa + m)x_k - 1 \right] \end{aligned}$$

In the above $\varpi \in (0, 1)$ is a dampening parameter that “smoothes” the transition from $\iota_s^{(i,j)}$ to $\iota_s^{(i,j+1)}$ and prevents infinite loops between debt price and issuance policy⁵. The derivative $(v_s^{(i,j)})'(x_k)$ is computed by using a centered finite difference approximation. I iterate on the inner loop until $\|\iota^{(i,j+1)} - \iota^{(i,j)}\|_\infty$ is sufficiently small.

At the conclusion of the inner loop, I have obtained $\mathbf{v}^{(i)}, \mathbf{D}^{(i)}, \iota^{(i)}$, all assuming a default policy $\bar{\mathbf{x}}^{(i)}$. I then set $\bar{\mathbf{x}}^{(i+1)}$ by checking the smooth pasting condition at $\bar{x}_s^{(i)}$ for all discrete

5. For most parameter configurations of interest, the issuance policy is increasing in the debt price schedule, and the debt price is a decreasing function of the issuance schedule. Thus, without dampening, the algorithm ends up frequently in an infinite loop: a high debt price at the end of iteration j leads to a high issuance policy in iteration $j+1$; such high issuance policy feeds back into a low debt price at iteration $j+1$, which leads to a low issuance policy in iteration $j+2$, thus creating the infinite loop.

states s :

$$(v^{(i)})'(\bar{x}_s^{(i)}) \gtrsim \alpha^{1-\gamma} \lambda \theta \left[\left(\Upsilon + \delta \frac{\gamma - \rho}{1 - \rho} \text{diag}_j \left(\left[(1 - \gamma) v_j^d(\bar{x}_s) \right]^{-\frac{1-\rho}{1-\gamma}} \right) \right)^{-1} \mathbf{v}'(\theta \bar{x}_s) \right]_s$$

Depending on whether the left handside is greater or less than the right handside, I update $\bar{x}_s^{(i+1)}$ using a binomial search method.

Once the optimal default boundary $\bar{\mathbf{x}}$ and the optimal issuance policy $\boldsymbol{\iota}$ are known, I can compute the expected default time and the ergodic density of x . The computation of the expected default time $\mathbf{T}(x)$ follows the same logic as the computation of the debt price, except that the Markov transition probabilities are adjusted to reflect the stochastic evolution of x_t and s_t under the physical measure \mathbb{P} . Finally, the ergodic density of (x_t, s_t) under \mathbb{P} is constructed as follows. First, I find the unitary eigen-vector $\{p^h(x_k, s)\}_{0 \leq k \leq N_h, 1 \leq s \leq N_s}$ (associated with the eigen-value 1) of the transpose of a Markov matrix whose elements correspond to transition probabilities in and out of performing states (x_k, s) (for $x_k \leq \bar{x}_s$), as well as in and out of default states (x_k, s) (for $x_k \geq \min_s \bar{x}_s$). Once again those transition probabilities are constructed in an identical way to those described previously. For each state (x_k, s) , the ergodic density at such point is approximated by $\pi^h(x_k, s)$, computed as follows:

$$\pi^h(x_k, s) := \frac{p^h(x_k, s) \Delta t_{\mathbb{P}}^h(x_k, s)}{\sum_{x_j, s'} p^h(x_j, s') \Delta t_{\mathbb{P}}^h(x_j, s')}$$

In order to compute CDS prices, I use a slightly modified procedure. As described previously, I need to compute the risk-neutral expected loss $L_s(x, T)$ and the risk-neutral expected present value of CDS premia $P_s(x, T)$. Introduce the constant $\epsilon > 0$. $Q_{\mathbb{Q}}^h(x, s)$ and

$\Delta t_{\mathbb{Q}}^h(x, s)$ are now defined as follows:

$$Q_{\mathbb{Q}}^h(x, s) := x^2 |\boldsymbol{\sigma}_s|^2 + h |\mu_{\mathbb{Q}}(x, s)| + h/\epsilon$$

$$\Delta t_{\mathbb{Q}}^h(x, s) := \frac{h^2}{Q_{\mathbb{Q}}^h(x, s)}$$

I still have $\inf_{x, s} Q_{\mathbb{Q}}^h(x, s) > 0$, which means that $\Delta t_{\mathbb{Q}}^h(x, s)$ is well defined. For all x, s , I have:

$$\lim_{h \rightarrow 0} \Delta t_{\mathbb{Q}}^h(x, s) = 0$$

The state space now includes time-to-maturity T , and the approximating Markov chain is now the three-dimentional process $X_{\mathbb{Q}, n}^h := (x_{\mathbb{Q}, n}^h, s_{\mathbb{Q}, n}^h, T_{\mathbb{Q}, n}^h)$. Given a starting state $X_{\mathbb{Q}, n}^h = (x, s, T)$, I define the following transition probabilities:

$$\Pr(X_{\mathbb{Q}, n+1}^h = (x + h, s, T) | X_{\mathbb{Q}, n}^h) = \frac{e^{\hat{\Lambda}_{ss} \Delta t_{\mathbb{Q}}^h(x, s)}}{Q_{\mathbb{Q}}^h(x, s)} \left(\frac{x^2 |\boldsymbol{\sigma}_s|^2}{2} + h \max(0, \mu_{\mathbb{Q}}(x, s)) \right)$$

$$\Pr(X_{\mathbb{Q}, n+1}^h = (x - h, s, T) | X_{\mathbb{Q}, n}^h) = \frac{e^{\hat{\Lambda}_{ss} \Delta t_{\mathbb{Q}}^h(x, s)}}{Q_{\mathbb{Q}}^h(x, s)} \left(\frac{x^2 |\boldsymbol{\sigma}_s|^2}{2} + h \max(0, -\mu_{\mathbb{Q}}(x, s)) \right)$$

$$\Pr(X_{\mathbb{Q}, n+1}^h = (x, s, T - \epsilon h) | X_{\mathbb{Q}, n}^h) = \frac{e^{\hat{\Lambda}_{ss} \Delta t_{\mathbb{Q}}^h(x, s)} h / \epsilon}{Q_{\mathbb{Q}}^h(x, s)}$$

$$\Pr(X_{\mathbb{Q}, n+1}^h = (x, s', T) | X_{\mathbb{Q}, n}^h) = \left(\frac{\hat{\Lambda}_{ss'}}{-\hat{\Lambda}_{ss}} \right) \left(1 - e^{\hat{\Lambda}_{ss} \Delta t_{\mathbb{Q}}^h(x, s)} \right)$$

Notice that these transition probabilities are all greater than zero, less than 1, and they add up to 1. The Markov chain created satisfies local consistency conditions similar to those in equations A.27, A.28, A.29, A.30 and A.31, in addition to:

$$\mathbb{E}^{s, x} \left[\Delta T_{\mathbb{Q}, n}^h \right] = -\Delta t_{\mathbb{Q}}^h(x, s)$$

$$\text{var}^{s, x} \left[\Delta T_{\mathbb{Q}, n}^h \right] = o \left(\Delta t_{\mathbb{Q}}^h(x, s) \right)$$

The state space grid G_h is three-dimensional, and is of the form:

$$\{(x_i, s, T_j)\}_{1 \leq i \leq N_{x,h}, 1 \leq s \leq N_s, 1 \leq j \leq N_{T,h}}$$

, where the grid points $\{x_i\}$ and the grid points $\{T_j\}$ are equally spaced with distance h and ϵh respectively. For $\bar{x}_s > x_i \geq 0$ and for $T \geq T_i > 0$, I compute $L_s(x_i, T_j)$ and $P_s(x_i, T_j)$ on the grid G_h as follows:

$$L_s(x_i, T_j) = e^{-r_s \Delta t_{\mathbb{Q}}^h(x_i, s)} \sum_{(x'_\mathbb{Q}, s'_\mathbb{Q}, T'_\mathbb{Q})} \Pr(X'_\mathbb{Q} | (x_i, s, T_j)) L_{s'_\mathbb{Q}}(x'_\mathbb{Q}, T'_\mathbb{Q})$$

$$P_s(x_i, T_j) = \Delta t_{\mathbb{Q}}^h(x_i, s) + e^{-r_s \Delta t_{\mathbb{Q}}^h(x_i, s)} \sum_{(x'_\mathbb{Q}, s'_\mathbb{Q}, T'_\mathbb{Q})} \Pr(X'_\mathbb{Q} | (x_i, s, T_j)) P_{s'_\mathbb{Q}}(x'_\mathbb{Q}, T'_\mathbb{Q})$$

I note $x_{N_{x,h,s}}$ the grid point at which the sovereign government defaults optimally in state s , in other words $N_{x,h,s} = \bar{x}_s/h$. The boundary conditions at $T = 0$ are $L_s(x_i, 0) = 0$ for $i < N_{x,h,s}$, $L_s(x_i, 0) = (1 - D(x_i))$ for $i \geq N_{x,h,s}$, and $P_s(x_i, 0) = 0$ for all x_i . The boundary conditions at $T > 0$ are $L_s(x_i, T) = (1 - D(x_i))$ and $P_s(x_i, T) = 0$ for $i \geq N_{x,h,s}$ and for any T . This system of linear equations is solved recursively: starting from $\{L_s(x_i, 0), P_s(x_i, 0)\}_{1 \leq i \leq N_{x,h}, s \leq N_s}$, I can compute $\{L_s(x_i, \epsilon h), P(x_i, \epsilon h)\}_{1 \leq i \leq N_{x,h}, s \leq N_s}$ via the system of linear equations above, and progress backwards.

A.4 Continuous State Markov Process Approximation

Several methods have been implemented over the years to approximate continuous state Markov processes by discrete state Markov chains. Some of these methods are described in [50], [51], and more recently [49] and [48], when dealing with highly persistent processes.

The methods mentioned above work well with a relatively large number of states. Since I use a small number of discrete states in my numerical application, I will use an ad-hoc procedure relying on matching conditional and unconditional moments of the original process and my approximating process. I use the stochastic discount factor of [33] to illustrate my procedure. In their article, the pricing kernel features (a) a constant risk-free rate, and (b) a time-varying price of risk ν_t that follows an AR(1) process – the discrete time equivalent of an Ornstein-Uhlenbeck process:

$$d\nu_t = -\kappa_\nu (\nu_t - \bar{\nu}) dt + \sigma_\nu dZ_t$$

[33] parameterize $\bar{\nu}, \kappa_\nu, \sigma_\nu$ according to table A.4.

Table A.4: Parameters for [33] Model

Parameter	Variable	Value (annualized)
Average risk-price	$\bar{\nu}$	62.5%
Risk-price volatility	σ_ν	24%
Persistence parameter	κ_ν	0.14

In order to approximate the Ornstein-Uhlenbeck process by a discrete state continuous time Markov process, I first choose a discrete number of risk-prices $\{\nu_i\}_{1 \leq i \leq N_s}$, and then minimize the distance between (i) a set of conditional and unconditional moments of the Ornstein-Uhlenbeck process and (ii) the same set of conditional and unconditional moments of my discrete state continuous time Markov process. Table A.5 highlights the set of conditional and unconditional moments I use. This minimization step is usually time consuming if the number of discrete states is large – in such case, the alternative methods described at the beginning of this section are more appropriate, as numerically more efficient. But

since I work with a relatively small number of discrete states, I can afford to go through this minimization procedure.

Table A.5: Target Moments

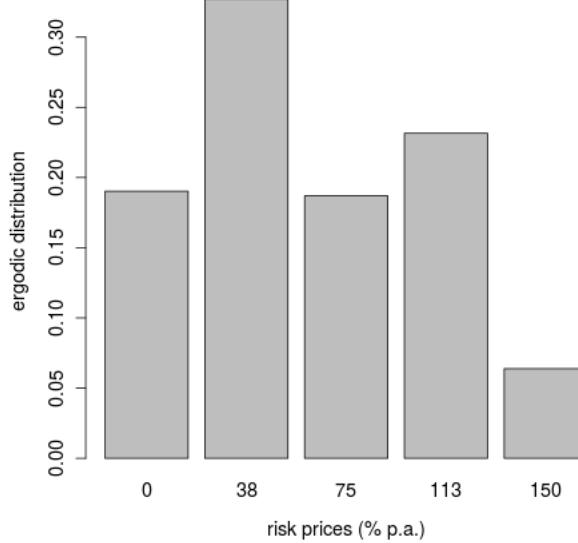
Moment	[33] Model	Discrete State Model
$\mathbb{E}[\nu_t]$	$\bar{\nu}$	$\sum_{i=1}^{N_s} \pi_i \nu_i$
$\text{var}[\nu_t^2]$	$\frac{\sigma_\nu^2}{2\kappa_\nu}$	$\sum_{i=1}^{N_s} \pi_i \nu_i^2 - \left(\sum_{i=1}^{N_s} \pi_i \nu_i\right)^2$
$\mathbb{E}[\nu_{t+s} \mathcal{F}_t]$	$\nu_t e^{-\kappa_\nu s} + \bar{\nu} (1 - e^{-\kappa_\nu s})$	$\sum_{j=1}^{N_s} \left(e^{\Lambda s}\right)_{s_t, j} \nu_j$
$\text{var}[\nu_{t+s} \mathcal{F}_t]$	$\frac{\sigma_\nu^2}{2\kappa_\nu} (1 - e^{-2\kappa_\nu s})$	$\sum_{j=1}^{N_s} \left(e^{\Lambda s}\right)_{s_t, j} \nu_j^2 - \left(\sum_{j=1}^{N_s} \left(e^{\Lambda s}\right)_{s_t, j} \nu_j\right)^2$

My choice of N_s is driven by computational considerations – I pick $N_s = 5$ in order for my numerical algorithm to be able to solve for an equilibrium relatively quickly. This means that I need to compute $N_s \times (N_s - 1) = 20$ transition intensities $(\Lambda_{ij})_{i \neq j}$. I pick an equally spaced grid $\{\nu_i\}_{i \leq N_s} = (0, 0.375, 0.75, 1.125, 1.5)$. In addition to the mean and variance of the ergodic distribution of ν_t , I also match the conditional mean and variance at 2 horizons: 0.25 years, and 2 years. This gives me 22 different moments (20 conditional, and 2 unconditional), for 20 free parameters. My approximation procedure leads to the following generator matrix:

$$\Lambda = \begin{bmatrix} -0.1838 & 0.1637 & 0.0075 & 0.0012 & 0.0113 \\ 0.0703 & -0.1576 & 0.034 & 0.0401 & 0.0132 \\ 0.0417 & 0.0904 & -0.2505 & 0.1154 & 0.003 \\ 0.0239 & 0.002 & 0.1406 & -0.1826 & 0.016 \\ 0.0001 & 0.0001 & 0.0772 & 0.1412 & -0.2186 \end{bmatrix}$$

The resulting ergodic distribution $\boldsymbol{\pi}$ is displayed in figure A.5. The ergodic distribution of my discrete state process has a mean of 62.5% and a standard deviation of 45%, which correspond

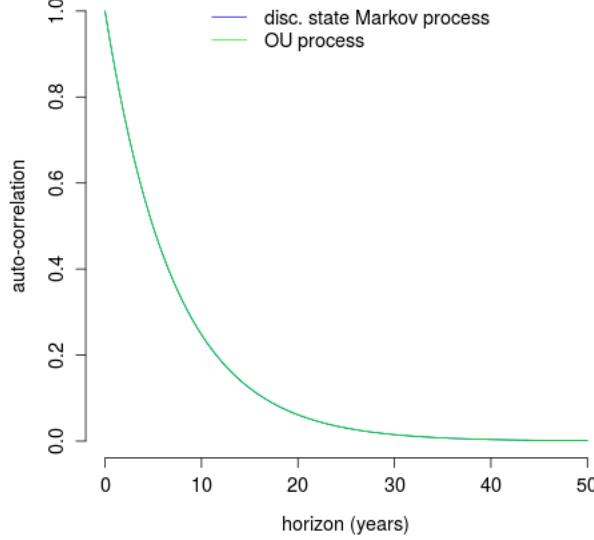
Figure A.5: Stationary Distribution



exactly to the moments of the ergodic distribution of the original Ornstein Uhlenbeck process. The maximum *relative* error in conditional means is equal to 1.7%, while the maximum *relative* error in conditional variances is equal to 7.9%.

Finally, I highlight one particular aspect of the SDF used in this paper: its persistence. Consistent with many models of the stochastic discount factor, whether long-run risk models, habit formation models, or more reduced form specifications such as [33], the persistence of the risk-price process is high – in the particular parametrization of [33], the half-life mean-reversion speed is $\ln 2/\kappa_\nu \approx 5$ years. This feature of asset pricing models has its importance in the context of my sovereign default framework: since the transition intensities in and out of a given state are relatively small, the adjustments to the sovereign financing policy will be larger than if those intensities were higher. This will lead to the relatively large current account adjustments discussed in the core of my paper. Note that the autocorrelation function of my discrete state continuous time Markov process approximation takes the following

Figure A.6: Autocorrelation Function



form:

$$R(s) = \frac{\sum_{i=1}^{N_s} \pi_i \nu_i \sum_{j=1}^{N_s} \left[\left(e^{s\Lambda} \right)_{ij} - \pi_j \right] \nu_j}{\sum_{i=1}^{N_s} \pi_i \nu_i^2 - \left(\sum_{i=1}^{N_s} \pi_i \nu_i \right)^2}$$

One can show that asymptotically, $R(s) \approx e^{-\kappa s}$, where $-\kappa$ is the largest non-zero (and of course negative) eigen-value of the matrix Λ . My numerical procedure leads to $\kappa = 0.139$, which is very close to the persistence parameter $\kappa_\nu = 0.14$ in the original article of [33]. I display $R(s)$ in figure A.6, and on the same graph I plot the autocorrelation function of the original Ornstein-Uhlenbeck process, simply equal to $e^{-\kappa_\nu s}$. As showed in the plot, the two autocorrelation functions are indistinguishable.