THE UNIVERSITY OF CHICAGO

DIRAC COMPOSITE FERMION THEORY OF JAIN'S SEQUENCES

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ABSTRACT

An interacting two-dimensional electron gas in a strong magnetic field can form nontrivial topologically ordered gapped states, the fractional quantum Hall (FQH) states. The composite fermion construction was introduced by Jain to understand strongly correlated electron systems such as in FQH [38, 39]. While the connection between original electrons and composite fermions is unclear, composite fermion has been employed to study fractional quantum Hall (FQH) with tremendous success. The composite fermion construction was used by Halperin, Lee and Read (HLR)[33] to explore the half filling state $\nu = 1/2$, which can be thought of as the large n limit of Jain's sequences $\nu = \frac{n}{2n+1}, \frac{n+1}{2n+1}$. The HLR theory predicts precisely the appearance of a Fermi-liquid behavior of quasi-particle near half filling state.However, there are still some issues related to the composite fermion model, one of them is the particle-hole symmetry of the lowest Landau level (LLL). The idea of Dirac composite fermion, proposed recently by Son [93], can be the resolution of this question. In this thesis work, I will investigate Dirac composite fermion model to calculate physical quantities of incompressible states that belong to Jain's sequences $\nu = \frac{n}{2n+1}$ and $\nu = \frac{n+1}{2n+1}$ in the large n limit. The particle-hole symmetry is satisfied explicitly in closed form, the universal topological coefficients of Jain's states are reproduced exactly in the Dirac composite fermion model. The magnetoroton dispersion relation is derived and confirmed by experiment results [41, 52].

CHAPTER 1

INTRODUCTION

In this chapter, we will review the basic concept of fractional quantum Hall effect and the composite fermion models. Since the discovery of fractional quantum Hall effect (FQHE) [95], a vast amount of work had been done both theoretically and experimentally to explain this fascinating phenomenon. However, since no satisfying answer has been proposed yet, FQHE is still the most non-trivial problem of condensed matter physics. The groundbreaking idea allowing to solve the problem is the composite fermion picture, which also provides the framework for many other corners of theoretical physics. The theory of the composite fermion begins by Jain [38, 39]. The microscopic theory can be explained in term of the flux attachment approach [102]. Under mean field approximation assumption, we arrive at the effective field theory of FQH, which is known as Chern-Simon (CS) fermionic theory. The CS was first used by Lopez and Fradkin to describe Jain's sequence of incompressible fractionally quantized Hall states [60].

The CS fermionic field theory is later used by Halperin, Lee, and Read (HLR) [33] to describe the FQH states with the even denominator. The most interesting result of this theory is the appearance of a compressible FQH state, the composite fermion forms a Fermi-liquid. The Fermi-liquid-like behavior of composite fermions near half-filling has been confirmed experimentally [90],[42],[23]. The experiments give the firm evidence that the composite fermion is a physical object which is a quasi-particle near half-filling.

However, the more recent experiments [40, 65] raise the question of particle-hole symmetry of the composite fermion picture. In the CS fermionic field theory approach, the composite fermion density is equated with the electron density, but in the experiments [40, 65], this relation seems to be broken. The composite fermion density needs to be less than the electron density if $\nu > 1/2$. Furthermore, by theoretical argument, Kivelson, Lee, Krotov and Gan [46, 56] showed that HLR theory results disagreed with the particle-hole symmetric result for the conductivity tensor σ_{ij}. The CS composite fermion theory needs to be replaced

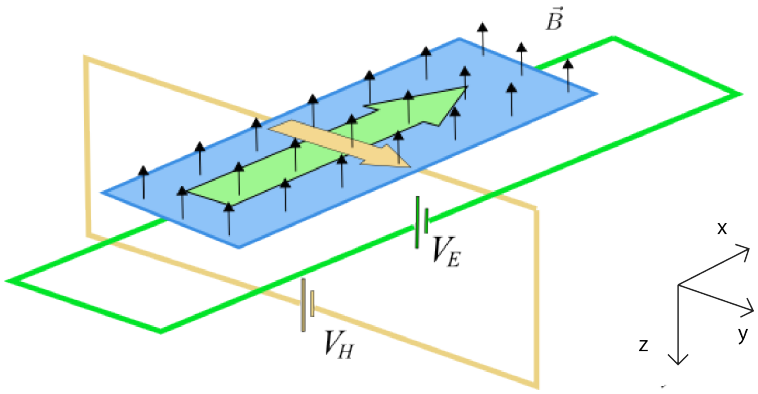

Figure 1.1: 2 dimensional electron gas (2EDG) in applied mangetic field

by an effective theory that preserves PH symmetry and match the experiment results. One of the approaches is the Dirac composite fermion model, which is proposed by Son recently. Section 1.1 will be devoted to reviewing the FQH and the lowest Landau level projection. In section 5.4.3, I will give a detail "derivation" of HLR theory from the microscopic point of view. The PH symmetry issue of HLR theory is discussed in section 1.3. The motivation and the details of Dirac composite fermion idea are given in the section 1.4. Last but not least, in section 1.5, I will briefly summarize the content of my thesis and my contribution to the development of Dirac composite fermion model.

1.1 Fractional quantum Hall effect (FQHE) and lowest Landau level limit

1.1.1 Hamiltonian

The quantum Hall problem has very simple starting point; it begins with Hamiltonian of two-dimensional electron gas (2DEG) in a constant background magnetic field (Figure 1.1.1), where the electrons interact with each other through a two-body interaction which only depends on the distance between the two electrons

$$H = \sum_{a=1}^{N} \frac{(\mathbf{p}_a - \frac{e}{c}\mathbf{A}(\mathbf{x}_a))^2}{2m} + \sum_{<a,b>} V(|\mathbf{x}_a - \mathbf{x}_a|). \tag{1.1}$$

Here $\mathbf{A}$ is the gauge vector potential corresponding to the constant magnetic field $B = \nabla \times \mathbf{A}$, e is the electron's charge and c is the velocity of light. The interaction is normally taken to be the Coulomb potential

$$V(r) = \frac{e^2}{r}. \tag{1.2}$$

It is very surprising that so many phenomena are generated from this very simple Hamiltonian. Let's recover the interesting physics of this system.

1.1.2 Integer quantum Hall (IQH) effect: a free electrons problem

If we ignore the interaction between electrons, the system can be described by independent Hamiltonian of single particle in mangetic field. In Landau gauge, we can rewrite the Hamiltonians as

$$H = \frac{1}{2m}(\vec{p} - e\vec{A}/c)^2, \quad p = -i\hbar\partial, \quad A_y = Bx. \tag{1.3}$$

The eigen-value problem can be solved exactly to get the eigen wavefunction

$$\psi_{n,k}(x,y) = e^{iky} \exp\left[-\frac{1}{2}\left(\frac{x}{\ell_B} - \ell_B k\right)^2\right] H_n\left[\frac{x}{\ell_B} - \ell_B k\right], \tag{1.4}$$

with eigen energy

$$E_{n,k} = \left(n + \frac{1}{2}\right)\hbar\omega_c, \tag{1.5}$$

where $H_n(x)$ is the Hermite polynomial. We also introduced the definition of cyclotron frequency

$$\omega_c = \frac{Be}{mc}, \tag{1.6}$$

and magnetic length

$$\ell_B = \sqrt{\frac{\hbar c}{eB}} = 25\text{nm}\sqrt{1/B[\text{Tesla}]}. \tag{1.7}$$

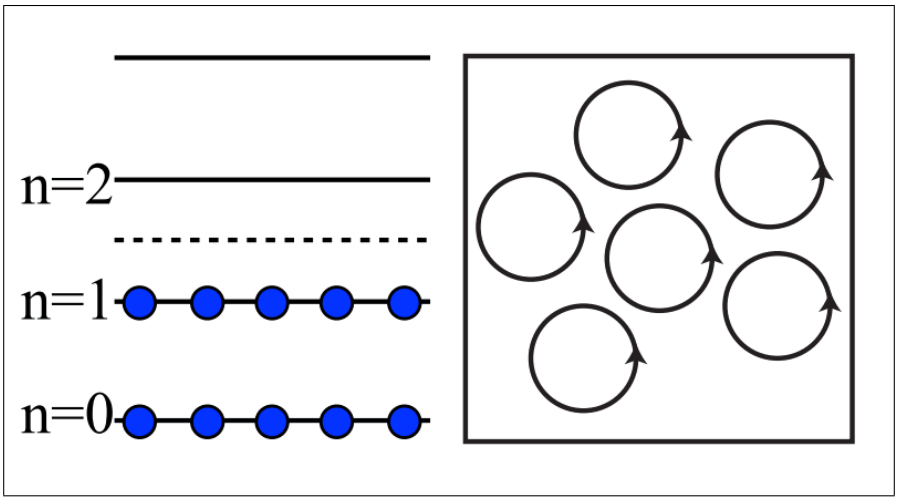

Figure 1.2: Electrons distribute in Landau levels

For each value of n, k can take B/ϕ_0 difference value per unit area of the system, where the quantum flux is defined as

$$\phi_0 = \frac{2\pi\hbar c}{e}. \tag{1.8}$$

As a consequence, the spectrum is highly degenerate and break down to Landau bands whose degeneracy equals to the number of quantum flux in the sample

$$N_\phi = \frac{B \times A}{\phi_0}, \tag{1.9}$$

in which A is the area of the sample. N_ϕ is the number of available orbital in each Landau band. The filling fraction is defined as the ratio of electron and the quantum flux number

$$\nu = \frac{N_e}{N_\phi} = \frac{\rho_e \phi_0}{B} = 2\pi\rho_e \ell_B^2, \tag{1.10}$$

where ρ_e denotes the electron density, and N_e is the total number of electrons in the system. ν is integer and gives the number of Landau levels completely filled. When there are integer numbers of filled Landau levels, the chemical potential is discontinuous, there is a energy

gap in the between ground state and first excited state

$$E_g = \hbar\omega_c = \frac{\hbar eB}{mc}. \tag{1.11}$$

Employing Galilean invariant, we can show that in a perfect clean system, the IQH system has zero longitudinal DC resistivity

$$\rho_{xx} = 0, \tag{1.12}$$

and quantized DC Hall resistivity

$$\rho_{xy} = \frac{1}{\nu}\frac{h}{e^2}. \tag{1.13}$$

The argument is following, we consider the applied electric field $\mathbf{E}$, in the reference frame moving with velocity $\mathbf{v}$, the electric field is

$$\mathbf{E}' = \mathbf{E} + \frac{\mathbf{v}\times\mathbf{B}}{c}. \tag{1.14}$$

If we chose

$$\mathbf{v} = c\frac{(\mathbf{E}\times\mathbf{B})}{B^2}, \tag{1.15}$$

we see that

$$\mathbf{E}' = 0. \tag{1.16}$$

So that in the new reference frame, we simply have the system of electron in constant magnetic field, there is no net current (moving electron). Mapping back to lab reference frame, the current is nothing but the boost velocity times the charge density, and we find that

$$\mathbf{j} = \hat{\mathbf{z}}\times\mathbf{E}\frac{\rho_e ec}{B}, \tag{1.17}$$

or

$$\rho_{xx} = 0, \qquad \rho_{xy} = \frac{\rho_e ec}{B} = \frac{1}{\nu}\frac{h}{e^2}. \tag{1.18}$$

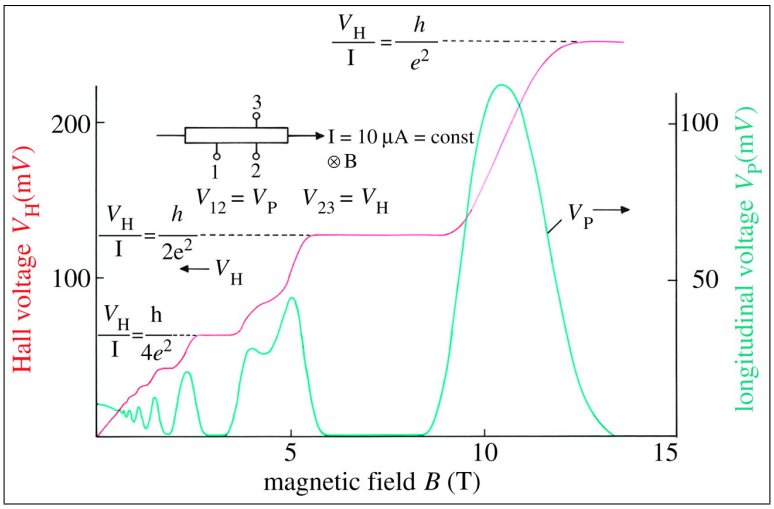

Figure 1.3: Inter quantum Hall system measurement

The more general theorem states that, with incompressible ground state with filling fraction ν, in the appearance of small disorder, the DC resistivity matrix is given by

$$\hat{\rho} = \frac{h}{e^2} \begin{pmatrix} 0 & 1/\nu \\ -1/\nu & 0 \end{pmatrix}. \tag{1.19}$$

The quantization of Hall resistivity was discovered experimentally by Von Klitzing [51] (Figure 1.3). It was proposed that this effect could be used for precision measurements of the resistance quantum h/e^2.

1.1.3 Fractional quantum Hall effect: an interacting electrons problem

If a Landau level is partially filled, there is a huge degeneracy. There is an enormous number of ways to distribute the electrons. If the filling fraction is $\nu < 1$, then the degeneracy is $\frac{(N_e/\nu)!}{N_e!(N_e\frac{1-\nu}{\nu})!}$. If we ignore the interaction, the ground state is gapless. However, when the interaction is turned on, there is a preferred distribution of electron that minimizes the energy (Figure 1.4).

The degeneracy is lifted, and we have gapped ground states at some particular filling

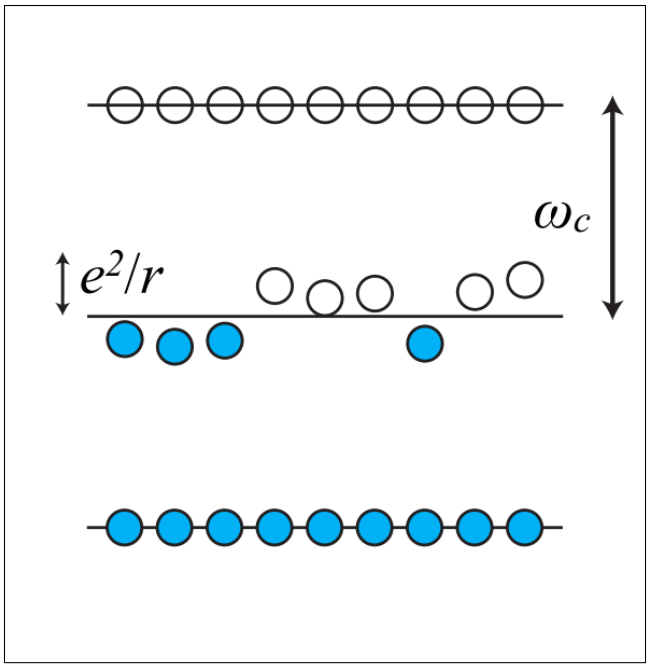

Figure 1.4: Interaction splits the degeneracy

fractions. The incompressible quantum Hall states can form at many fractional filling fractions [95]. The appearance of incompressible fractional quantum Hall states tells us that we need to include the interaction to explain this strange behavior. In the experiment, we also see the quantization of Hall resistivity at incompressible fractional Hall state (Figure 1.5).The DC Hall conductivity is fractionally quantized

$$\sigma_{xy} = \nu \frac{e^2}{2\pi\hbar} \tag{1.20}$$

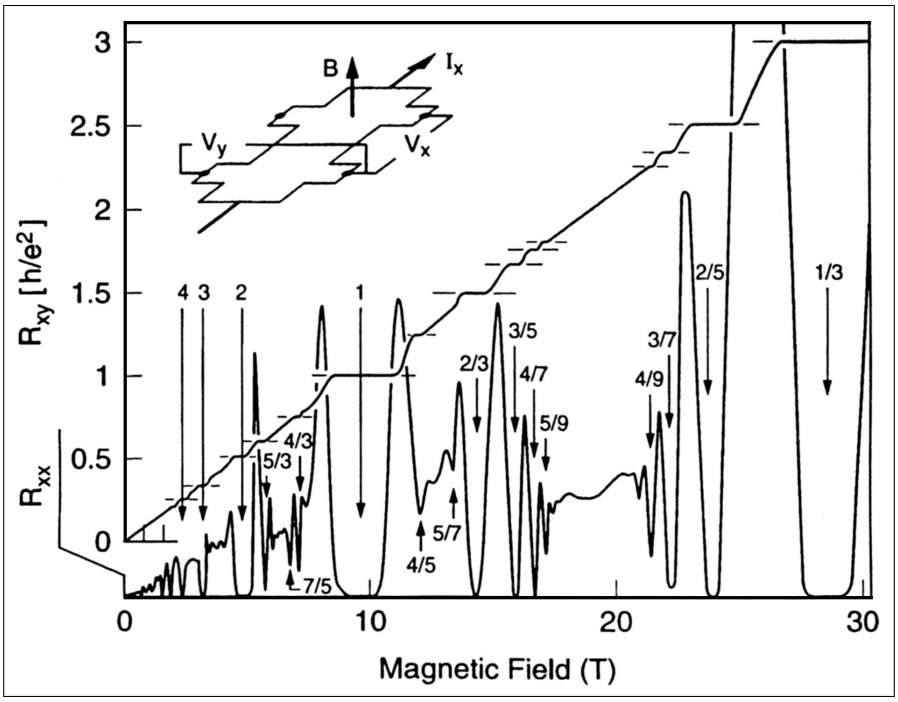

Figure 1.5: Fractional quantum Hall system measurement

1.1.4 Lowest Landau level projection: a strongly interacting electrons problem

There are obviously two energy scales of fractional quantum Hall problem; the first is cyclotron gap between Landau levels (inter-Landau level) energy

$$E_g = \hbar\omega_c = \frac{\hbar eB}{mc}, \tag{1.21}$$

and the Coulomb energy gap which is induced by interaction between the electron in fractionally filled Landau level (intra-Landau level)

$$\Delta_C \approx \frac{e^2}{\ell_B} = \frac{e^2}{\sqrt{\frac{\hbar c}{eB}}} \tag{1.22}$$

The ratio between those energy scales

$$\frac{E_g}{\Delta_C} = \frac{\sqrt{B}}{m}\sqrt{\frac{\hbar^3}{e^3 c}}. \tag{1.23}$$

If we take the natural unit convention, $(e = \hbar = c = 1)$, we have

$$\frac{E_g}{\Delta_C} = \frac{\sqrt{B}}{m}. \tag{1.24}$$

when the ratio goes to infinity $\frac{E_g}{\Delta_C} \to \infty$, we can ignore the Landau level mixing and zoom into a single Landau level to study the low energy effective theory of FQH. The field theory of single Landau level limit is called lowest Landau level (LLL) projection. In a single Landau level, because of the degeneracy of kinetic energy, all the electron have the same kinetic contribution to the energy, effectively electron has no kinetic energy. The only contribution in the LLL projected Hamiltonian is the interaction term. The Hamiltonian in the LLL limit can be reformulated as

$$H_{LLL} = \mathcal{P}_{LLL} \sum_{<a,b>} V(|\mathbf{x}_a - \mathbf{x}_a|), \tag{1.25}$$

where $\mathcal{P}_{LLL}$ is the projection to the LLL. We arrive at the strongly interacting many-body problem; there is no way to study FQH using perturbation theory of electron due to lack of a small parameter. From equation (1.24), we can arrive the LLL limit by either going to high field limit $B \to \infty$ or massless limit $m \to 0$. In the remaining part of the thesis, we will consider the second approach, whenever I consider the LLL limit, I mean $m \to 0$.

1.2 Flux attachment and HLR theory

In this section, I will review the flux attachment approach and derive the HLR theory using mean field approximation assumption. The flux attachment or Chern-Simon transformation is the exact unitary transformation of non-relativistic theory, however the mean field approximation is ambiguous and problematic.

1.2.1 Flux attachment: Chern-Simon transformation

We begin with the Hamiltonian

$$H = \sum_a \frac{(\mathbf{p}_a + \frac{e}{c}\mathbf{A}(\mathbf{x}_a))^2}{2m} - \sum_a eA_0(\mathbf{x}_a) + \sum_{<a,b>} V(|\mathbf{x}_a - \mathbf{x}_a|). \tag{1.26}$$

The Lagrangian density corresponding to this Hamiltonian in the second quantization language is

$$\mathcal{L} = i\Psi^\dagger(\partial_t - iA_0)\Psi - \frac{1}{2m}|(\partial_i - iA_i)\Psi|^2 - \frac{1}{2}\int d^2\mathbf{x}d^2\mathbf{y}\delta\rho(\mathbf{x})V(|\mathbf{x}-\mathbf{x}|)\delta\rho(\mathbf{y}). \tag{1.27}$$

Where the electron field operator $\Psi(x)$ satisfies the anti-commutation relation

$$\{\Psi(x), \Psi^\dagger(y)\} = \delta(x-y). \tag{1.28}$$

We define the density operator in field theory language

$$\rho(x) = \Psi^\dagger(x)\Psi(x), \qquad \delta\rho(x) = \rho(x) - \bar{\rho}, \tag{1.29}$$

where $\bar{\rho}$ is the average density. The vector potential can be written in symmetric gauge

$$A_i = \frac{1}{2}B\epsilon_{ij}x^j, \tag{1.30}$$

and A_0 is the scalar potential of external electric field

$$E_i = \partial_i A_0 \tag{1.31}$$

Since the electron are completely polarized, the wave function $\Psi(\mathbf{x}_1, .., \mathbf{x}_N)$ must be totally antisymmetric, due to Pauli principle. The Schrodinger equation

$$H\Psi(\mathbf{x}_1, .., \mathbf{x}_N) = E\Psi(\mathbf{x}_1, .., \mathbf{x}_N) \tag{1.32}$$

together with total antisymmetry requirement defines the quantum eigenvalue problem. We can perform a singular gauge transformation to map this problem to a different one, the transformation of wave function can be written as

$$\Phi(\mathbf{x}_1, .., \mathbf{x}_N) = \left[\prod_{<a,b>} e^{-i\tilde{\phi}\theta(\mathbf{x}_a - \mathbf{x}_b)}\right] \Psi(\mathbf{x}_1, .., \mathbf{x}_N), \tag{1.33}$$

where $\theta(\mathbf{x}_a - \mathbf{x}_b)$ is the angle between $\mathbf{x}_a - \mathbf{x}_b$ and $\hat{\mathbf{x}}$ axis. $\tilde{\phi}$ is an integer number which helps us map the problem to bosonic one or fermionic one (the wave function $\Phi(\mathbf{x}_1, .., \mathbf{x}_N)$ is symmetric or antisymmetric correspondingly). If $\tilde{\phi}$ is an even number

$$\tilde{\phi} = 2n, \tag{1.34}$$

$\Phi(\mathbf{x}_1, .., \mathbf{x}_N)$ is antisymmetric, and we arrive at composite fermion picture. Note that the angle function $\theta(\mathbf{x}_a - \mathbf{x}_b)$ is defined up to 2π ambiguity, but it is not the problem since $\theta(\mathbf{x}_a - \mathbf{x}_b)$ only appears in the exponential. The transformation (1.33) is unitary transformation $\Phi = U\Psi$, with

$$U = e^{-i\sum_{<a,b>}\tilde{\phi}\theta(\mathbf{x}_a - \mathbf{x}_b)}. \tag{1.35}$$

Since $H\Psi(\mathbf{x}_1, .., \mathbf{x}_N) = E\Psi(\mathbf{x}_1, .., \mathbf{x}_N)$, we have

$$H'\Phi(\mathbf{x}_1, .., \mathbf{x}_N) = E\Phi(\mathbf{x}_1, .., \mathbf{x}_N), \tag{1.36}$$

where

$$H' = UHU^{\dagger}. \tag{1.37}$$

We can immediately derive the transformed Hamiltonian

$$H = \sum_a \frac{(\mathbf{p}_a + \frac{e}{c}\mathbf{A}(\mathbf{x}_a) - \frac{e}{c}\mathbf{a}(\mathbf{x}_a))^2}{2m} - \sum_a eA_0(\mathbf{x}_a) + \sum_{<a,b>} V(|\mathbf{x}_a - \mathbf{x}_a|), \tag{1.38}$$

where

$$\mathbf{a}(\mathbf{x}) = \frac{\tilde{\phi}\phi_0}{2\pi}\vec{\nabla}\left[\sum_a \theta(\mathbf{x} - \mathbf{x}_a)\right] = \frac{\tilde{\phi}\phi_0}{2\pi}\sum_a \frac{\hat{\mathbf{z}} \times (\mathbf{x} - \mathbf{x}_a)}{|\mathbf{x} - \mathbf{x}_a|^2}. \tag{1.39}$$

Due to non-singular value of $\theta(\mathbf{x} - \mathbf{x}_b)$, we have

$$\vec{\nabla} \times \mathbf{a}(\mathbf{x}) = \tilde{\phi}\phi_0 \sum_a \delta(\mathbf{x} - \mathbf{x}_a). \tag{1.40}$$

the CS magnetic field associated with vector potential $\mathbf{a}$ can be written as

$$b(\mathbf{x}) = \vec{\nabla} \times \mathbf{a}(\mathbf{x}) = \phi_0\tilde{\phi}\hat{\rho}(\mathbf{x}) \tag{1.41}$$

where $\rho(\mathbf{x})$ is the density operator the first quantization language

$$\hat{\rho}(\mathbf{x}) = \sum_a \delta(\mathbf{x} - \mathbf{x}_a). \tag{1.42}$$

We can interpret the physical meaning of equation (1.41) as there is $\tilde{\phi}$ quanta flux attached to each electron, this is the original of the name "flux attachment". We can quantize the fermionic Hamiltonian (1.38) by introducing fermion field operators $\Phi(x)$ which is satisfied the anti-commutation relation

$$\{\Phi(x), \Phi^\dagger(y)\} = \delta(x - y). \tag{1.43}$$

In the second quantization language, we can write the Hamiltonian as

$$H = \int d^2\mathbf{x}\Phi^\dagger(x)\left[\frac{(-i\hbar\vec{\partial} + \frac{e}{c}\mathbf{A}(\mathbf{x}) - \frac{e}{c}\mathbf{a}(\mathbf{x}))^2}{2m} - eA_0(x)\right]\Phi(x) + \frac{1}{2}\int d^2\mathbf{x}d^2\mathbf{y}\delta\rho(\mathbf{x})V(|\mathbf{x}-\mathbf{x}|)\delta\rho(\mathbf{y}). \tag{1.44}$$

We defined the density operator in field theory language

$$\rho(x) = \Phi^\dagger(x)\Phi(x) = \Psi^\dagger(x)\Psi(x), \qquad \delta\rho(x) = \rho(x) - \bar{\rho}, \tag{1.45}$$

in which $\bar{\rho}$ is the average electron density. The constraint equation (1.40) becomes

$$\vec{\nabla} \times \mathbf{a}(\mathbf{x}) = \phi_0\tilde{\phi}\rho(\mathbf{x}). \tag{1.46}$$

We can use continuity equation $\partial_t\rho(x) + \partial_i J^i(x,t) = 0$ to obtain

$$\epsilon^{ij}\partial_i\partial_t a_j = -\phi_0\tilde{\phi}\partial_i J^i, \tag{1.47}$$

which gives the constraint [1]

$$\epsilon^{ij}\partial_t a_j = -\phi_0\tilde{\phi}J^i, \tag{1.48}$$

or in term of field operator $\Phi(x)$

$$\epsilon^{ij}\partial_t a_j = -\phi_0\tilde{\phi}\frac{1}{m}\Phi^\dagger(\overleftrightarrow{\partial}_i - iA_i + ia_i)\Phi, \tag{1.49}$$

in which we define

$$\Phi^\dagger\overleftrightarrow{\partial}_i\Phi = \frac{1}{2}(\Phi^\dagger\partial_i\Phi - \partial_i\Phi^\dagger\Phi), \tag{1.50}$$

1. Up to a constant.

since the current operator in second quantization language is

$$J^i = \frac{1}{m}\Phi^\dagger(\overleftrightarrow{\partial}_i - iA_i + ia_i)\Phi = \frac{1}{m}\Psi^\dagger(\overleftrightarrow{\partial}_i - iA_i)\Psi \tag{1.51}$$

We can write down the Lagrangian density corresponding to Hamiltonian (1.44) as [2]

$$\mathcal{L} = i\Phi^\dagger(\partial_t - iA_0)\Phi - \frac{1}{2m}|(\partial_i - iA_i + ia_i)\Phi|^2 - \frac{1}{2}\int d^2\mathbf{x}d^2\mathbf{y}\delta\rho(\mathbf{x})V(|\mathbf{x}-\mathbf{x}|)\delta\rho(\mathbf{y}). \tag{1.52}$$

The constraint (1.46) and (1.49) can be imposed by introducing the Lagrange multiplier a_0 and the CS term into the Lagrangian density [3]

$$\begin{aligned}\mathcal{L} =& i\Phi^\dagger(\partial_t - iA_0 + ia_0)\Phi - \frac{1}{2m}|(\partial_i - iA_i + ia_i)\Phi|^2 + \frac{1}{\tilde{\phi}}\frac{1}{4\pi}\epsilon^{\mu\nu\lambda}a_\mu\partial_\nu a_\lambda \\ &- \frac{1}{2}\int d^2\mathbf{x}d^2\mathbf{y}\delta\rho(\mathbf{x})V(|\mathbf{x}-\mathbf{x}|)\delta\rho(\mathbf{y}).\end{aligned} \tag{1.53}$$

the field equation of a_μ gives us the constraint

$$\vec{\nabla}\times\mathbf{a} = \tilde{\phi}2\pi\rho(\mathbf{x}), \tag{1.54}$$

and

$$\epsilon^{ij}(\partial_j a_0 - \partial_0 a_j) = 2\pi\tilde{\phi}J^i, \tag{1.55}$$

we recover the equation (1.46) and (1.49), after we use the gauge $a_0 = 0$, since the Lagrangian (1.53) is invariant [4] under the gauge transformation $\delta a_\mu = \partial_\mu\Lambda(x)$.

2. We go to natural unit where $e = c = \hbar = 1$

3. In the natural unit, $\phi_0 = 2\pi$

4. Up to total derivative terms

1.2.2 HLR theory and mean field approximation

(1.53) is the Lagrangian density of composite fermion field Φ after doing flux attachment (CS transformation). We need to emphasize that the Lagrangian (1.53) is equivalent to the Lagrangian (1.27), since the CS transformation is unitary and exact. By the "Conservation of difficulty", we can't solve the Lagrangian (1.53) exactly since we can't solve the Lagrangian (1.27). If we take the value $\tilde{\phi} = 2$ and rename the fermionic field Φ by Ψ_{CF}, we end up with the HLR Lagrangian

$$\begin{aligned}\mathcal{L} =& i\Psi^{\dagger}_{CF}(\partial_t - iA_0 + ia_0)\Psi_{CF} - \frac{1}{2m}|(\partial_i - iA_i + ia_i)\Psi_{CF}|^2 + \frac{1}{2}\frac{1}{4\pi}\epsilon^{\mu\nu\lambda}a_\mu\partial_\nu a_\lambda \\ &- \frac{1}{2}\int d^2\mathbf{x}d^2\mathbf{y}\delta\rho(\mathbf{x})V(|\mathbf{x}-\mathbf{x}|)\delta\rho(\mathbf{y}).\end{aligned} \tag{1.56}$$

There are two features of the Lagrangian (1.56) that are different from the Dirac composite fermion model which will be introduced latter.

- The number of the composite fermion is the same as the number of the electron. It can be observed from equations (1.53) or (1.45) , the composite fermions is the results of flux attaching to electrons.
- The appearance of CS term of a_μ, which is also the key ingredient of flux attachment.

Since there is no way to go forward with the Lagrangian (1.56) without imposing an approximation from physical intuition. The simplest approach to analyzing this transformed Chern-Simons fermion system is to make the mean field approximation in which we replace the dynamical gauge field a_μ by its average value assuming that the density is constant. The flux quanta attached to the CF is smeared out to have constant magnetic field

$$< b >= \tilde{\phi}\phi_0\bar{\rho} = 4\pi\bar{\rho} \tag{1.57}$$

The effective magnetic field that is seen by Ψ_{CF} is the mean residue magnetic field

$$B_{\text{eff}} = B - < b >= B - 4\pi\bar{\rho} \tag{1.58}$$

The mean-field Hamiltonian becomes

$$H_{MF} = \int d^2\mathbf{x}\Psi^{\dagger}_{CF}(x)\left[\frac{(-i\hbar\vec{\partial} + \frac{e}{c}\mathbf{A}_{\text{eff}}(\mathbf{x}))^2}{2m} - eA_0(x)\right]\Psi_{CF}(x) + \frac{1}{2}\int d^2\mathbf{x}d^2\mathbf{y}\delta\rho(\mathbf{x})V(|\mathbf{x}-\mathbf{x}|)\delta\rho(\mathbf{y}), \tag{1.59}$$

where $\vec{\nabla} \times \mathbf{A}_{\text{eff}} = B_{\text{eff}}$. This is the Hamiltonian of a quantum Hall problem with effective filling fraction

$$\nu_{CF} = \frac{\bar{\rho}}{B_{\text{eff}}/2\pi} = \frac{\nu}{1-2\nu}. \tag{1.60}$$

In particular, when we consider the Jain's sequence $\nu = \frac{n}{2n+1}$, we reach the IQH problem of CF with filling fraction $\nu_{CF} = n$. In this way, we map the FQH problem of electron to IQH problem of CF. The explanation of energy gap for FQH appears as the cyclotron energy gap of CF.

Other than the explanation of the energy gap, HLR theory achieved many successes, especially the prediction of Fermi-liquid state of CF at filling fraction $\nu = 1/2$. In this particular case, the residue magnetic field vanishes identically,

$$B_{\text{eff}} = 0, \tag{1.61}$$

the CF with finite density should form gapless Fermi-surface. There were experiments that confirmed this prediction [90],[42],[23]. It should be emphasized that it is an extremely surprising result that one can add a huge magnetic field to a system and end up with an effective system that behaves in some ways as if it were in zero magnetic fields.

If we are at a magnetic field such that we are close to (but not exactly at) $\nu = \frac{1}{2}$, then the applied magnetic field and the Chern-Simons flux do not quite exactly cancel and B_{eff} is nonzero. Thus, we have a Fermi sea in a small magnetic field, in which case the elementary quasiparticle excitations above the Fermi sea travel in large cyclotron orbits of radius

$$R_C^* = \frac{\hbar k_F}{eB_{\text{eff}}}, \tag{1.62}$$

This motion of CF in residue magnetic field is also confirmed experimentally [90], [40], [65].

Despite its success, the HLR theory (and all the fluxed attachment approach of FQH) has been criticized on some fundamental issues. Firstly, there is no information about LLL projection that appears in the mean-field Hamiltonian (1.59). The energy gap of the problem $\Delta_{\text{eff}} = \frac{B_{\text{eff}}}{m}$ has no reflection of Coulomb energy scale Δ_C, which is the natural energy scale of the problem. In another word, there is no natural way to achieve the LLL limit by taking $m \to 0$. To go around this issue, one may assume that the energy gap is determined by an effective mass m^*, which is the renormalized value of mass in the low energy effective theory. Since the interaction of the problem is Coulomb, the effective mass should know about the Coulomb energy scale under the RG flow. However, this is a hand-waving argument, since it is very nontrivial assumption about the low energy effective theory of strong interaction problem. There is no sharp solution to the derivation of m^* from microscopic theory, even though some attempts were made [82, 89]. One may argue that we can waive this issue and study the theory with effective mass m^* given as an input parameter of the theory. As a consequence, the effective field theory will not contain all the information of the microscopic Hamiltonian (1.1). Also, we require the new effective theory, with effective mass m^*, to obey all the fundamental symmetry of electrons in the massless limit.

Secondly, the HLR theory lacks particle-hole symmetry, which is the exact symmetry of the original electron problem. The more detail of this issues will be stressed out in the next sections of this chapter.

1.3 Particle-hole symmetry

1.3.1 Particle-hole symmetry of LLL electrons

The system of interacting non-relativistic electrons admits two discrete symmetries:

- Parity ($\mathcal{P}$)

$$x \to x, \qquad y \to -y, \tag{1.63}$$

- Time reversal ($\mathcal{T}$)

$$t \to -t. \tag{1.64}$$

In the constant background magnetic field, both $\mathcal{P}$ and $\mathcal{T}$ are broken spontaneously. The magnetic field prefers the specific direction of space and time. However the product $\mathcal{PT}$ is preserved, $\mathcal{PT}$ transforms the wave function in the following manner

$$\mathcal{PT}\Psi(x_i, y_i, t) \to \Psi^*(x_i, -y_i, -t). \tag{1.65}$$

Indeed, all the proposed wavefunctions, including Laughlin and Moore-Read wavefuntions, are invariant under $\mathcal{PT}$.

In the lowest Landau level limit, we have one extra symmetry, which is particle-hole symmetry. To define PH symmetry, we need to pick up a complete basis of the single-particle wavefunction in the LLL, $\psi_k(x)$. In the symmetric gauge, we can choose

$$\psi_k(z) = \frac{z^k}{\sqrt{2\pi 2^k k!}} e^{-\frac{1}{4\ell_B^2}|z|^2}, \tag{1.66}$$

where $z = x + iy$. The basis define the field operator in the lowest Landau level in term of creation and annihilation operators $c_k^\dagger, c_k$

$$\hat{\Psi}_L(z) = \sum_{k=1}^{N_\phi} \psi_k(z) c_k, \qquad \hat{\Psi}_L^\dagger(z) = \sum_{k=1}^{N_\phi} \psi_k^*(z) c_k^\dagger, \tag{1.67}$$

in which the creation and annihilation operators satisfy the anti-commutation relation

$$\{c_k^\dagger, c_{k'}\} = \delta_{k,k'} \tag{1.68}$$

The many body Fock state is obtained by acting products of creation operators on the empty lowest Landau level $|\text{empty}\rangle$. The particle hole conjugation is defined as an ati-unitary operator, which map the empty Landau level to the full one

$$\Theta : |\text{empty}\rangle \to |\text{full}\rangle = \prod_{k=1}^{N_\phi} c_k^\dagger |\text{empty}\rangle. \tag{1.69}$$

$$\Theta i \Theta^{-1} = -i \tag{1.70}$$

It also maps a creation operator to an annihilation operator and vice versa

$$\Theta c_k^\dagger \Theta^{-1} = c_k. \tag{1.71}$$

In term of LLL field operator, one can rewrite the Hamiltonian (1.25) in second quantization language as

$$H_{LLL} = \int d^2\mathbf{x} d^2\mathbf{y} : \hat{\Psi}_L^\dagger(\mathbf{x}) \hat{\Psi}_L(\mathbf{x}) V(|\mathbf{x}-\mathbf{y}|) \hat{\Psi}_L^\dagger(\mathbf{y}) \hat{\Psi}_L(\mathbf{y}) :, \tag{1.72}$$

where $: \cdots :$ denotes the normal ordering. From the explicit form of LLL field operator, one can show that [58] the projected Hamiltonian map to it self, up to an additional of chemical potential

$$\Theta H_{LLL} \Theta^{-1} = H_{LLL} - \Delta\mu \sum_{k=1}^{N_\phi} c_k^\dagger c_k \tag{1.73}$$

where $\Delta\mu$ depends on the explicit form of two-body interacting potential $V(|\mathbf{x}-\mathbf{y}|)$. If we chose the chemical potential $\mu = \frac{\Delta\mu}{2}$, the Hamiltonian

$$H_{LLL} - \mu \hat{N} \tag{1.74}$$

maps to itself under PH conjugation. We can conclude that, at this chemical potential, the Hamiltonian is particle-hole symmetric. Under the particle hole conjugation, the filling fractional transforms as

$$\nu \to 1 - \nu. \tag{1.75}$$

In particular $\nu = 1/2$ maps to itself under PH conjugation. Moreover, two sides of Jain's sequences also map to each other

$$\nu = \frac{n}{2n+1} \leftrightarrow \nu = \frac{n+1}{2n+1}, \tag{1.76}$$

under PH conjugation.

1.3.2 Particle hole symmetry and HLR theory

The HLR theory (1.56) only has $\mathcal{PT}$ symmetry. The Chern-Simon term does not have any discrete symmetry which can be associated with particle-hole conjugation. This reflects on the asymmetry treatment of Jain's sequences. The state with filling fraction $\nu = \frac{n}{2n+1}$ maps to the CF with filling fraction $\nu_{CF} = n$, while the PH dual state with filling fraction $\nu = \frac{n+1}{2n+1}$ maps to the CF with filling fraction $\nu_{CF} = n + 1$.

Moreover, the particle-hole conjugation of Fermi-liquid state $\nu = 1/2$ is suspicious. Naively, one may guess the PH transformation of Fermi-disk with $k < k_F = \sqrt{B}$ to a hollow with $k > k_F$. It is obviously unacceptable; the underline reason is the lack of PH symmetry of the flux attachment. In the Chern-Simon transformation, which is described in the previous section (5.4.3), one can only attach fluxes to electrons, not to holes. Moreover, HLR theory has no information about the LLL projection, since there is no natural way to take the electron massless limit.

The lacking of PH symmetry of HLR theory was recorgnized early on by Kivelson, Lee, Krotov and Gan [46, 56]. When disorder is PH symmetric, particle hole symmetry implies

that

$$\sigma_{xy} = \frac{1}{2}\frac{e^2}{h}. \tag{1.77}$$

In the random phase approximation of HLR theory, the resistivity of electron can be derived as

$$\rho_{xy} = \rho_{xy}^{CF} + \frac{2h}{e^2}. \tag{1.78}$$

Since at $\nu = 1/2$, CF moves in zero effective mangetic field $B_{\text{eff}} = 0$, we can set $\rho_{xy}^{CF} = 0$, which implies $\rho_{xy} = \frac{2h}{e^2}$. Whenever we have nonzero longitudial resistivity due to disorder $\rho_{xx} \neq 0$, we have $\sigma_{xy} \neq \frac{1}{2}\frac{e^2}{h}$.

One consequence of PH symmetry is that the susceptibility of particle-hole dual states are equivalent, which also implies that the dispersion relation of collective modes is identical. From the straightforward calculation, HLR theory fails to reproduce these results. With more careful analyzing, one [97] can show that the magnetoroton minima of Jain's dual states $\nu = \frac{n}{2n+1}$ and $\nu = \frac{n+1}{2n+1}$ coincide. However, in MMRPA calculations of HLR theory, the susceptibility of Jain's states differ from each other by a factor of $\frac{(n+1)^2}{n^2}$ at the leading oder in gradient expansion ($\mathcal{O}(k^4)$).

1.3.3 PH symmetry and the effective field theories

Despite the lack of explicit PH symmetry of HLR theory, it may have the emergent PH symmetry in the low-energy limit. In the old work by Kivelson et al. [46], a careful diagram calculation indicates that there is no clear mechanism, which helps us to recover PH symmetry as a hidden symmetry of HLR theory. Additional analysis has been done to resolve this issue [97], however, there is no convincing solution yet.

Although the electron theory has PH symmetry, the symmetry can be broken spontaneously at low energy limit, in this case, we should have two distinct $\nu = 1/2$ states. This idea was investigated by Barkeshli et al. [5]. However, there is no experiment and numerical results that support the spontaneously broken PH symmetry. The recent numerical result

[20] indicates that the state $\nu = 1/2$ is PH symmetric, it is the PH dual of itself.

In the next section, we will study one version of the effective field theory of FQH, which differs from HLR theory and has PH symmetry explicitly. This effective theory is the Dirac composite fermion model, which is proposed by Son [93].

1.4 Dirac composite fermion

1.4.1 Motivation

In the demand of an effective theory, which has PH symmetry explicitly and preserves all successful phenomenological predictions of HLR theory, Son proposes Dirac composite fermion model [93]. In the Dirac composite fermion, the composite particles don't transform to composite holes under particle-hole conjugation. Only the momentum of the composite fermion flips sign under the conjugation

$$\Theta : \mathbf{k} \to -\mathbf{k} \tag{1.79}$$

There are some argument to convince that the composite fermion is a massless Dirac particle. One hint comes from the mapping of Jain's sequences to IQH of composite fermion. In HLR theory, Jain's state $\nu = \frac{n}{2n+1}$ and its PH dual state $\nu = \frac{n+1}{2n+1}$ map to CF with filling fraction n and $n+1$ respectively. If we require the PH symmetry, one may naively guess that the filling fraction of CF should be $n + \frac{1}{2}$. How can we have the half-integer filling fraction in an IQH system? This filling fraction can be obtained in IQH system of massless Dirac fermion.

The other observation support the Dirac nature of composite fermion comes form the property of PH symmetry itself. Consider a generic state in the LLL

$$|\psi\rangle = \prod_{i=1}^{N_e} c^\dagger_{k_i} |\text{empty}\rangle. \tag{1.80}$$

Under PH conjugation, $|\psi\rangle$ transforms as

$$\Theta : |\psi\rangle \to \prod_{i=1}^{N_e} c_{k_i} |\text{full}\rangle = \prod_{i=1}^{N_e} c_{k_i} \prod_{k=1}^{N_\phi} c_k^\dagger |\text{empty}\rangle. \tag{1.81}$$

We can derive the transformation of $|\psi\rangle$ after applying Θ twice

$$\Theta^2 : |\psi\rangle \to \prod_{i=1}^{N_e} c_{k_i}^\dagger \prod_{k=1}^{N_\phi} c_k^\dagger \prod_{k'=1}^{N_\phi} c_{k'}^\dagger |\text{empty}\rangle = (-1)^{N_\phi(N_\phi-1)/2} |\psi\rangle. \tag{1.82}$$

If $N_\phi = 2N_{CF}$, we have

$$\Theta^2 : |\psi\rangle = (-1)^{N_{CF}} |\psi\rangle \tag{1.83}$$

The factor $(-1)^{N_{CF}}$ can be understood as there is N_{CF} number of Dirac composite fermion in the state $|\psi\rangle$, and each Dirac composite fermions is associated with (-1) under Θ^2. Furthermore, the number of composite fermion is pinned with the number of total fluxes.

$$N_{CF} = \frac{N_\phi}{2}, \tag{1.84}$$

and doesn't depend on the number of electrons. It implies that in the Dirac composite fermion model, we treat electrons and holes equally.

1.4.2 The LLL duality of non-relativistic electrons and Dirac electrons

In this subsection, we will derive the duality between Dirac electron system and non-relativistic electron system in the LLL limit. We begin with the theory of massless Dirac fermion in 2+1 dimensions couples to $U(1)$ gauge (electromagnetic) field in 3+1 dimensions

$$S_e^D = \int d^3x i\bar{\Psi}\gamma^\mu \left(\partial_\mu - iA_\mu\right)\Psi - \frac{1}{4e^2}\int d^4x F_{\mu\nu}^2, \tag{1.85}$$

where Ψ is a 2 components spinor, $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$. We choose the γ-matrices as follows

$$\gamma^0 = \sigma^3, \qquad \gamma^1 = i\sigma^2, \qquad \gamma^2 = -i\sigma^1, \tag{1.86}$$

where σ^i are the Pauli's matrices. Under a constant back ground magnetic field in $\hat{\mathbf{z}}$ direction $\bar{B} = \partial_1 A_2 - \partial_2 A_1$, the Hamiltonian of free Dirac fermion has the form

$$H_0 = \begin{pmatrix} -\mu & -i\sqrt{\bar{B}}(D_i - iD_2) \\ -i\sqrt{\bar{B}}(D_i + iD_2) & -\mu \end{pmatrix}, \tag{1.87}$$

where μ is the chemical potential, and we define covariant derivative as

$$D_\mu = \partial_\mu - iA_\mu \tag{1.88}$$

Dirac fermions form Landau levels (Figure 1.6(a)) with energy level

$$E_n^D = -\mu + \mathrm{sgn}(n)\sqrt{2|n|\bar{B}}, \tag{1.89}$$

with corresponding eigen wavefunction

$$\Psi_n^D = \begin{cases} \frac{1}{\sqrt{2}} \begin{pmatrix} \Psi_n^{NR} \\ -\mathrm{sgn}(n)\Psi_{n-1}^{NR} \end{pmatrix} & , |n| > 0 \\ \begin{pmatrix} \Psi_L^{NR} \\ 0 \end{pmatrix} & , n = 0 \end{cases}. \tag{1.90}$$

Where Ψ_n^D is the wavefunction of Dirac fermion on Landau level n, which can be written in term of wavefunctions of non-relativistic fermion on Landau level n (Ψ_n^{NR}) and on Landau level $n-1$ (Ψ_{n-1}^{NR}). The wavefunction of Dirac fermion on the 0^{th} Landau level has one component, which is the LLL wavefunction of non-relativistic fermion Ψ_L^{NR}. The cyclotron

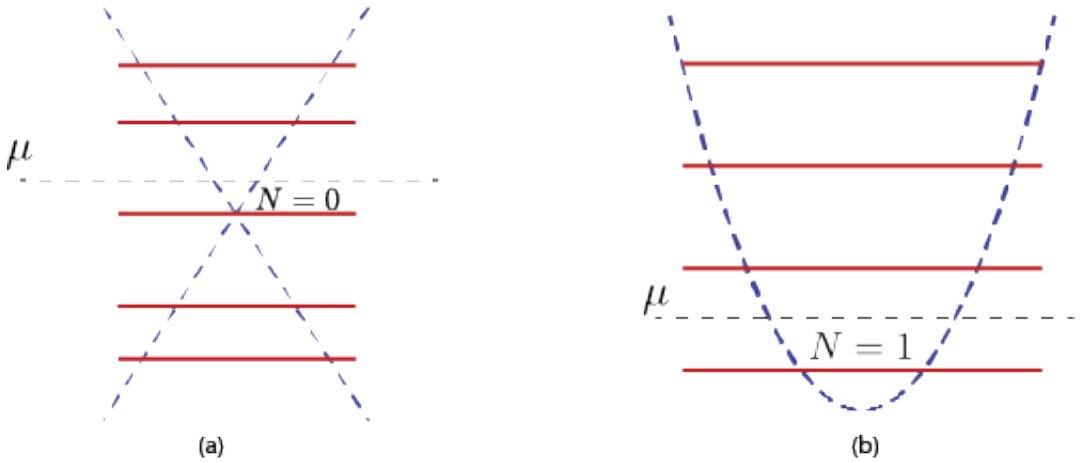

Figure 1.6: Under constant magnetic field, both massless Dirac electrons (a) and non-relativistic electron (b) form Landau level spectrum.

energy gap of Dirac fermion is

$$\omega_c^D \sim \sqrt{B}, \tag{1.91}$$

and energy scale of Coulomb interaction is

$$\Delta_C^D \sim \frac{e^2}{\ell_B} \sim e^2\sqrt{B}. \tag{1.92}$$

In the limit

$$\frac{\Delta_C^D}{\omega_c^D} = e^2 \ll 1, \tag{1.93}$$

we can ignore Landau level mixing and study the physics of single Landau level. We consider the case when the chemical potential μ is fine tuned such that the landau level $N = 0$ is filled and the Landau level $N = 1$ is empty (Figure 1.6(a)). We can write down the component of Dirac field as

$$\Psi^D = \begin{pmatrix} \psi \\ \chi \end{pmatrix}, \tag{1.94}$$

The Lagrangian is rewritten as

$$\mathcal{L}_e^D = i\psi^\dagger D_t\psi + i\chi^\dagger D_t\chi + i\psi^\dagger(D_1 - iD_y)\chi + i\chi^\dagger(D_1 + iD_2)\psi + \mathcal{L}_{int}, \tag{1.95}$$

where $\mathcal{L}_{int}$ represents interaction terms. Without Landau level mixing, we see that $\chi = 0$ from equation (1.90). With small Landau level mixing, we have

$$\chi \sim e^2. \tag{1.96}$$

Furthermore, in the $N = 0$ Landau level, the energy of Dirac fermion only comes from interaction, thus

$$D_t \to i\omega \sim e^2. \tag{1.97}$$

Therefore we can ignore the term $i\chi^\dagger D_t \chi$ in the Lagrangian since it is order e^6. We write down the remaining part of the Lagrangian in this regime

$$\mathcal{L}_e^D = i\psi^\dagger D_t \psi + i\psi^\dagger (D_1 - iD_y)\chi + i\chi^\dagger (D_1 + iD_2)\psi + \mathcal{L}_{int}. \tag{1.98}$$

Next, we consider the action of non-relativistic electron in magnetic field

$$\mathcal{L}_e^{NR} = i\psi^\dagger D_t \psi - \frac{\delta^{ij}}{2m} D_i \psi^\dagger D_j \psi + \mathcal{L}_{int}, \tag{1.99}$$

As derived before, we can approach the LLL limit of non-relativistic electron by taking the massless limit $m \to 0$. In order to take $m \to 0$ smoothly, we introduce the coupling of electron spin and external B field

$$\mathcal{L}_e^{NR} = i\psi^\dagger D_t \psi - \frac{\delta^{ij}}{2m} D_i \psi^\dagger D_j \psi + \frac{g}{4m} B\psi^\dagger \psi + \mathcal{L}_{int}. \tag{1.100}$$

At $g = 2$, we can rewrite the Lagrangian density as

$$\mathcal{L}_e^{NR} = i\psi^\dagger D_t \psi - \frac{1}{2m} (D_1 - iD_2)\psi^\dagger (D_1 + iD_2)\psi + \mathcal{L}_{int}, \tag{1.101}$$

which can be written to in final form with the help of Hubbard Stratonovich transformation

$$\mathcal{L}_e^{NR} = i\psi^\dagger D_t\psi + i\psi^\dagger(D_1 - iD_y)\chi + i\chi^\dagger(D_1 + iD_2)\psi + 2m\chi^\dagger\chi + \mathcal{L}_{int}. \tag{1.102}$$

The massless limit can be taken by removing $2m\chi^\dagger\chi$, which gives us

$$\mathcal{L}_e^{NR} = i\psi^\dagger D_t\psi + i\psi^\dagger(D_1 - iD_y)\chi + i\chi^\dagger(D_1 + iD_2)\psi + \mathcal{L}_{int}. \tag{1.103}$$

We see that the Lagrangian density (1.98) has the same form as (1.103), which gives us the evidence of duality between massless Dirac electron and non-relativistic electron in the Lowest landau level limit. χ and $\chi^\dagger$ play the role of Lagrange multipliers that provide the constraint on LLL field

$$(D_1 + iD_2)\psi = 0, \qquad (D_1 - iD_2)\psi^\dagger = 0 \tag{1.104}$$

However, to study the dynamic of the LLL duality, we need to analyze the LLL Dirac electron side more carefully. On the non-relativistic electron side, there is only one Landau level partially occupied by electrons. On the other hand, on the Dirac electron side, not only the $N = 0$ Landau level is partially filled, all the negative energy bands are fully occupied as well (Figure 1.7). Since each fully filled band contributes a finite amount to the particle density

$$\rho_1 = \frac{B}{2\pi}, \tag{1.105}$$

thus the extra density from negative bands is

$$\Delta\rho = \sum_{N=-1}^{\infty} \frac{B}{2\pi} = \frac{B}{2\pi}\sum_{N=-1}^{\infty} 1 = \frac{B}{2\pi}\zeta(0) = -\frac{B}{4\pi}, \tag{1.106}$$

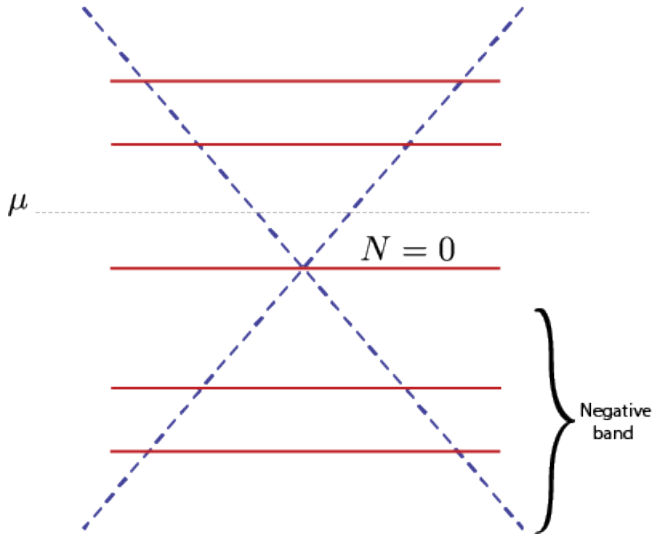

Figure 1.7: In the LLL limit of Dirac electrons, not only the $N = 0$ band is partially filled, all the negative energy bands are fully occupied as well

where $\zeta(x)$ is the Riemann zeta function. Furthermore, each fully filled band also adds a fixed contribution to the Hall conductivity

$$\sigma_1^H = \frac{1}{2\pi}. \tag{1.107}$$

thus the extra Hall conductivity from negative bands is

$$\Delta\sigma^H = \sum_{N=-1}^{\infty} \frac{1}{2\pi} = \frac{1}{2\pi}\zeta(0) = -\frac{1}{4\pi}. \tag{1.108}$$

Thus, we have the map of density

$$\rho_D = \rho_{NR} - \frac{B}{4\pi} \tag{1.109}$$

We define the filling fraction of Dirac eletron as

$$\nu_D = \frac{\rho_D}{B/2\pi} = \nu_{NR} - \frac{1}{2}, \tag{1.110}$$

which explains the IQH system of Dirac fermion has half integer filling fraction. Similarly, we can derive, at limit $k \to 0$,

$$\sigma_{xx}^{D}(\omega) = \sigma_{xx}^{NR}(\omega), \tag{1.111}$$

$$\sigma_{xy}^{D}(\omega) = \sigma_{xy}^{NR}(\omega) - \frac{1}{2}. \tag{1.112}$$

From equations (1.109),(1.111) and (1.112), we need to add an extra term to the LLL Dirac electron side to get the exact map to the LLL non-relativistic electron

$$\mathcal{L}_e^D(\Psi^D, A_\mu) + \frac{1}{8\pi} AdA \overset{LLL}{\longleftrightarrow} \mathcal{L}_e^{NR}(\psi, A_\mu), \tag{1.113}$$

where we use abbreviation $AdA = \epsilon^{\mu\nu\lambda} A_\mu \partial_\nu A_\lambda$.

1.4.3 Dirac composite fermion theory: A heuristic derivation

The duality of Dirac electron and Dirac composite fermion

Following the conjecture of dualities between Dirac electron and Dirac composite fermion [86],[43], we quote an element of duality web

$$\begin{aligned} S_e^D &= \int d^3x i\bar{\Psi}\gamma^\mu \left(\partial_\mu - iA_\mu\right)\Psi - \frac{1}{4e^2}\int d^4x F_{\mu\nu}^2 + \mathcal{L}_{int} \overset{\text{dual}}{\longleftrightarrow} \\ S_{CF} &= \int d^3x i\bar{\Psi}_{CF}\gamma^\mu \left(\partial_\mu - ia_\mu\right)\Psi_{CF} - \frac{1}{4\pi} Ada - \frac{1}{4e^2}\int d^4x F_{\mu\nu}^2 + \mathcal{L}_{int} + \cdots , \end{aligned} \tag{1.114}$$

$\cdots$ includes higher gradient expansion term, Maxwell term of a_μ in $2+1$ dimensions. Ψ_{CF} is Dirac field that describes quasi-partilce, a_μ is an emergent gauge field. This duality is a fermionic version of particle-vortex duality, which acquires attention after Son's proposal [93]. It claims that the duality between two theories (1.114) implies the composite fermion theory for FQH near half filled Landau level. The argument is following : a finite mangetic field in the free Dirac electron side corresponding to finite density of composite fermion

density as

$$\rho_{CF} = \langle \bar{\Psi}_{CF} \gamma^0 \Psi_{CF} \rangle = \frac{B}{4\pi}, \tag{1.115}$$

which is nothing but the equation of motion of a_0. The composite fermions form Fermi-liquid, which is implied from experiments and HLR theory also. Combining the two dualities (1.113) and (1.114), we impose the composite fermion theory that duals to LLL non-relativistic electrons near half filled Landau level as

$$\mathcal{L}_{CF} = i\Psi^\dagger_{CF}(\partial_t - ia_0 - v_F\sigma^i(\partial_i - ia_i))\Psi_{CF} - \frac{1}{4\pi}Ada + \frac{1}{8\pi}AdA + \mathcal{L}_{int} + \cdots \tag{1.116}$$

The apperance of Fermi-velocity v_F is the input of Dirac composite fermion theory, which is equivalent with the imposing of the effective mass m^*. At this point, the derivation of v_F (or m^*) from microscopic electron theory is still an open question, it is the energy scale problem that we mentioned in the section (5.4.3). We can think of v_F as the renormalized value of composite fermion velocity under RG flow. Ψ_{CF} is neutral under external gauge field A_μ, the DCF system couples with external gaugel field through a_μ and the BF coupling $-\frac{1}{4\pi}Ada$. The composite fermion has finite density and sees residue magnetic field

$$b = \epsilon^{ij}\partial_i a_j. \tag{1.117}$$

Dirac composite fermion theory: A Fermi-liquid with $\pm\pi$ Berry phase

We can derive the electron density from the Lagrangian density (1.116) as

$$\rho_e = \frac{\delta S}{\delta A_0} = \frac{B - b}{4\pi}. \tag{1.118}$$

The density of composite fermion is given by

$$\rho_{CF} = \langle \bar{\Psi}_{CF} \gamma^0 \Psi_{CF} \rangle = \frac{B}{4\pi}, \tag{1.119}$$

so that the CF density is set by external magnetic field. The role of magnetic field and the density switches after one goes from original electrons to composite fermions. This is the evidence of particle-vortex duality. Furthermore, from equation (1.119), we observe

$$N_{CF} = \frac{N_\phi}{2}, \tag{1.120}$$

which satisfies the consequence of particle hole conjugation of LLL electrons states. If we define the filling fraction of composite fermion by

$$\nu_{CF} = \frac{2\pi\rho_{CF}}{b}, \tag{1.121}$$

we have the map of filling fraction

$$\nu_{CF} = -\frac{1}{4(\nu - 1/2)}. \tag{1.122}$$

In particular the Jain's state $\nu = \frac{n}{2n+1}$ maps to $\nu_{CF} = n + 1/2$, which is IQH state of Dirac composite fermion. In addition the PH dual state $\nu = \frac{n+1}{2n+1}$ maps to $\nu_{CF} = -(n + 1/2)$, which is the same filling fraction up to a reverse sign of residue magnetic field $b \to -b$.

Even though composite fermion is Dirac, we will never reach the Dirac cone since the Dirac CF has finite density. We can study the low energy excitation within the Dirac composite fermion model, which means we only have access to the modes exist near Fermi-surface (Fig 1.8). The Dirac nature of CF only expresses as the quasiparticles Berry phase $\pm\pi$ around the Fermi surface.

PH transformation of Dirac composite fermion theory

We recall the discrete symmetry of Dirac electron theory

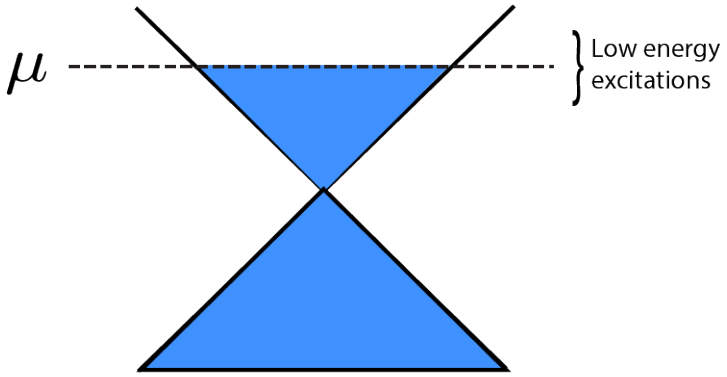

Figure 1.8: Dirac composite fermions form Dirac Fermi-liquid with $\pm\pi$ Berry phase

(i) Charge conjugation,

$$\mathcal{C} A_\mu \mathcal{C}^{-1} = -A_\mu \,, \tag{1.123a}$$

$$\mathcal{C} \Psi \mathcal{C}^{-1} = \sigma^1 \Psi^* \tag{1.123b}$$

(ii) Spatial parity $\mathbf{x} = (x, y) \to \mathbf{x}' = (x, -y)$,

$$\mathcal{P} A_0(t, \mathbf{x}) \mathcal{P}^{-1} = \quad A_0(t, \mathbf{x}'), \tag{1.124a}$$

$$\mathcal{P} A_1(t, \mathbf{x}) \mathcal{P}^{-1} = \quad A_1(t, \mathbf{x}'), \tag{1.124b}$$

$$\mathcal{P} A_2(t, \mathbf{x}) \mathcal{P}^{-1} = -A_2(t, \mathbf{x}'), \tag{1.124c}$$

$$\mathcal{P} \Psi(t, \mathbf{x}) \mathcal{P}^{-1} = \sigma^1 \Psi(t, \mathbf{x}'). \tag{1.124d}$$

(iii) Time reversal $t \to -t$

$$\mathcal{T} A_0(t, \mathbf{x}) \mathcal{T}^{-1} = \quad A_0(-t, \mathbf{x}), \tag{1.125a}$$

$$\mathcal{T} A_i(t, \mathbf{x}) \mathcal{T}^{-1} = -A_i(-t, \mathbf{x}), \tag{1.125b}$$

$$\mathcal{T} \Psi(t, \mathbf{x}) \mathcal{T}^{-1} = -i\sigma^2 \Psi(-t, \mathbf{x}). \tag{1.125c}$$

Each of $\mathcal{C}, \mathcal{P}$ and $\mathcal{T}$ symmetries is broken by magnetic field B. However $\mathcal{CP}$, $\mathcal{CT}$ and $\mathcal{PT}$ leave the magnetic field unchanged and are the symmetries of Dirac electron theory. We have $(\mathcal{CP})(\mathcal{CT})(\mathcal{PT}) \sim 1$, we only have two idenpendent symmetries. Since we require $\mathcal{PT}$ is the symmetry of LLL electron theory in addition to PH symmetry, it implies that the remaining independent symmetry $\mathcal{CT}$ is chosen to be the PH symmetry.

From the duality (1.114), we can easily derive the transformation of Dirac composite fermion under discrete symmetries of Dirac electron theory (and non-relativistic electron theory)

(i) charge conjugation,

$$\mathcal{C} a_\mu(t,\mathbf{x})\mathcal{C}^{-1} = -a_\mu(t,\mathbf{x}), \tag{1.126a}$$

$$\mathcal{C}\Psi_{CF}(t,\mathbf{x})\mathcal{C}^{-1} = \sigma^1\Psi^*_{CF}(t,\mathbf{x}); \tag{1.126b}$$

(ii) spatial parity,

$$\mathcal{P} a_0(t,\mathbf{x})P^{-1} = -a_0(t,\mathbf{x}'), \tag{1.127a}$$

$$\mathcal{P} a_1(t,\mathbf{x})P^{-1} = -a_1(t,\mathbf{x}'), \tag{1.127b}$$

$$\mathcal{P} a_2(t,\mathbf{x})P^{-1} = \quad a_2(t,\mathbf{x}'), \tag{1.127c}$$

$$\mathcal{P}\Psi_{CF}(t,\mathbf{x})P^{-1} = \Psi^*_{CF}(t,\mathbf{x}'); \tag{1.127d}$$

(iii) time reversal,

$$\mathcal{T} a_0(t,\mathbf{x})T^{-1} = -a_0(-t,\mathbf{x}), \tag{1.128a}$$

$$\mathcal{T} a_i(t,\mathbf{x})T^{-1} = \quad a_i(-t,\mathbf{x}), \tag{1.128b}$$

$$\mathcal{T}\Psi_{CF}(t,\mathbf{x})T^{-1} = \sigma^3\Psi^*_{CF}(-t,\mathbf{x}). \tag{1.128c}$$

Under PH transformation, $\Theta = (\mathcal{CT})$, we have

$$(\mathcal{CT})a_0(t\mathbf{x})(\mathcal{CT})^{-1} = a_0(-t,\mathbf{x}) \tag{1.129}$$

$$(\mathcal{CT})a_i(t\mathbf{x})(\mathcal{CT})^{-1} = -a_i(-t,\mathbf{x}) \tag{1.130}$$

$$(\mathcal{CT})\Psi_{CF}(t,\mathbf{x})(\mathcal{CT})^{-1} = -i\sigma^2\Psi_{CF}(-t,\mathbf{x}), \tag{1.131}$$

Thus, PH symmetry maps composite fermion to itself Ψ_{CF}, and doesn't map to its complex conjugate Ψ^*_{CF}. The Dirac composite fermion density remains unchanged under PH conjugation. The residue magnetic field b reverses the sign, since a_i maps to $-a_i$. Therefore, PH conjugation map the CF fillfing fraction as

$$\nu_{CF} \to -\nu_{CF}, \tag{1.132}$$

In particular for the DCF theory of Jain's sequence,

$$\nu_{CF} = (n+1/2) \to \nu_{CF} = -(n+1/2). \tag{1.133}$$

Moreover, the Dirac Fermi-liquid (Fig 1.8), that corresponds to $\nu = 1/2$ state, is PH symmetric and maps to itself under PH conjugation.

Electric dipole term

Since we are working on an effective field theory of the LLL electrons system, we bypass the energy scale problem by introducing the Fermi velocity v_F as the input of our theory. The PH symmetry is also satisfied explicitly by the construction of Dirac composite fermion model. We need to make sure that the Galilean invariant of the LLL electrons system is also satisfied by the effective field theory. The Galilean invariant and Kohn theorem determine the long wavelength limit behavior of susceptibility, the leading order of susceptibility should begin at q^4 [22]. On the other hand, the Dirac action (1.116) is not invariant under Galilean

transformation

$$\Psi_{CF}(t,\mathbf{x}) \to \Psi'_{CF}(t,\mathbf{x}) = \Psi_{CF}(t,\mathbf{x}-\Delta\mathbf{v}t), \tag{1.134}$$

where $\Delta\mathbf{v}$ is the parameter of Galilean transformation[5]. We need to replace

$$\frac{i}{2}\Psi^{\dagger}_{CF}\overleftrightarrow{\partial}_t\Psi_{CF} \to \frac{i}{2}\Psi^{\dagger}_{CF}\overleftrightarrow{\partial}_t\Psi_{CF} + \frac{iv^i}{2}\left(\Psi^{\dagger}_{CF}D_i\Psi_{CF} - D_i\Psi^{\dagger}_{CF}\Psi_{CF}\right), \tag{1.135}$$

where v^i is local velocity, which transform as

$$v^i \to v^i + \Delta v^i, \tag{1.136}$$

under Galilean transformation. One candidate for the local velocity field is

$$v^i = \frac{\epsilon^{ij}E_j}{B}, \tag{1.137}$$

since under Galilean transformation,

$$E_i \to E_i - \epsilon^{ij}\Delta v_j B. \tag{1.138}$$

We can replace (1.135) as

$$\frac{i}{2}\Psi^{\dagger}_{CF}\overleftrightarrow{\partial}_t\Psi_{CF} + \frac{i}{2}\frac{\epsilon^{ij}E_j}{B}\left(\Psi^{\dagger}_{CF}D_i\Psi_{CF} - D_i\Psi^{\dagger}_{CF}\Psi_{CF}\right), \tag{1.139}$$

meaning of the extra term is the electric dipole of composite fermion with dipole moment perpendicular to the momentum

$$\mathbf{d} = -\ell_B^2\mathbf{p}\times\hat{\mathbf{z}}, \tag{1.140}$$

which implies the idea of Read [81] and Zhang [101].

5. The new reference frame has relative velocity $-\Delta\mathbf{v}$ in comparing with the old reference frame

1.4.4 Dirac composite fermion vs HLR

In this section, I will compare the consequences of Dirac composite fermion theory and HLR theory. We will see that there are distinct results of Dirac composite fermion theory that are supported by experiments and numberical calculations.

Paricle hole symmetry

Since Dirac composite fermion is PH symmtric by construction, it is not surprising DCF theory gives PH symmetric results. In particular, the susceptibility of Jain's states $\nu = \frac{n}{2n+1}$ and $\nu = \frac{n+1}{2n+1}$ are identical [71]. HLR theory fails to reproduce this results. Further calculation for Jain's states shows that the results of Hall conductivity σ^H and static structure factor (SSF)

$$\bar{s}_\nu(k) = \frac{1}{N_e}\langle \rho^L_\nu(k)\rho^L_\nu(-k)\rangle_c, \tag{1.141}$$

where ρ^L_ν is the LLL density operator, agree with topological calculation. The DCF theory reproduces all the universal topological coefficients that are calculated independently from a different method [66]. In particular, the coefficient s_4 of $(k\ell_B)^4$ term of SSF saturates the Haldane's bound for both $\nu = \frac{n}{2n+1}$ and $\nu = \frac{n+1}{2n+1}$ states

$$s_4 \geq \frac{|\mathcal{S}_\nu - 1|}{8}, \tag{1.142}$$

where $\mathcal{S}$ is the Wen-Zee shift is defined through Wen-Zee shift is defined through the formula of total electron number on a sphere

$$N_e = \nu(N_\phi + \mathcal{S}_\nu). \tag{1.143}$$

A calculation using MMRPA method shows that the static structure factor for state $\nu = \frac{n}{2n+1}$ of HLR theory violates the Haldane's bound

$$s_4 < \frac{|\mathcal{S}_\nu - 1|}{8} \tag{1.144}$$

Composite fermion density

Near half filling, both HLR theory and Dirac composite fermion theory suggest the formation of Fermi-liquid. However the density of composite fermion (or the Fermi momentum) are different in two theories. From the previous section (5.4.3), we see that the density of composite is equal to the density of electron. When we go closely (but not exactly) to $\nu = \frac{1}{2}$. The composite fermions form Fermi-liquid with Fermi momentum

$$k_F^{\text{HLR}} = \sqrt{4\pi\rho_{CF}} = \sqrt{4\pi\rho_e}, \tag{1.145}$$

in the case of HLR theory. With in the Dirac composite fermion theory, the Fermi-momentum is determined by

$$k_F^{\text{Dirac}} = \sqrt{4\pi\rho_{CF}} = \sqrt{B} \tag{1.146}$$

The composite fermions see the effective magnetic field

$$B_{\text{eff}} = B - 4\pi\rho_e. \tag{1.147}$$

Consequently, the composite fermions establish the cyclotron motion with radius

$$R_C^* = \frac{k_F}{B_{\text{eff}}}. \tag{1.148}$$

The cyclotron radius can be measured by experiments [42], [40] using resistivity resonance. The result shows that the density of composite fermions can not be the same as the density

of electrons. One can analyze the carefully and show that the experiment reusults [40] fit nicely with Dirac composite fermion theory [15].

Absence of backscattering

In the non-relativistic Fermi-liquid theory, which is the prediction of HLR theory for $\nu = 1/2$ FQH, there would be a non-analytical two-point correlation function at zero frequency and exchange momentum $q = 2k_F$. It dues to the backscattering process of a particle-hole pair at opposite points on the Fermi-surface. The well-known phenomenon is the Friedel oscillations in Fermi-liquid theory literature, in which derivatives of Lindhards dielectric function show singularities at $q = 2k_F$.

The Dirac composite fermion also predicts the formation of Fermi-surface for $\nu = 1/2$ FQH. However, due to the non-trivial Berry phase $\pm\pi$ of Dirac composite fermion, the backscattering process is forbidden at leading order in perturbation theory, whenever the disorder is PH symmetric. As a consequence, the correlation function of PH symmetric operator shows no singularity at $q = 2k_F$ at leading order perturbation theory. This conclusion is confirmed recently by a careful numerical calculation [20]. This result strongly confirms the Dirac nature of composite fermion.

1.5 Plan of the thesis

The thesis is planned as follows: in chapter 2, I will derive the electromagnetic response of IQH system, both for non-relativistic and Dirac fermion. We will study the electromagnetic response of Landau level electrons in great detail. I will explain how to calculate the linear response to the arbitrary order in the expansion in momentum and frequency and give a compact expression for the polarization tensor in both non-relativistic and Dirac cases. Moreover, I show the equivalence semi-classical approximation and field theory random phase approximation in the large filling fraction. The duality of non-relativistic electrons theory

and Dirac electrons theory in the lowest Landau level limit when the mixing between Landau levels is neglected. In chapter 3, I will derive the particle-hole duality relations using wave function approach. Furthermore, I will show the calculation of universal topological coefficients of Jain's states $\nu = \frac{n}{2n+1}$ and $\nu = \frac{n+1}{2n+1}$. I also explain how to calculate the topological coefficients of state with filling fraction $1 - \nu$ with the given coefficients of its PH dual state with filling fraction ν. In chapter 4, I will introduce the higher spin theory of magnetoroton. Using the semi-classical approximation of Dirac composite fermion model, I show that the positions of magneto-roton minima are universal. The theoretical prediction is compared with experiments results. In chapter 5, I will give detail calculation of electron magnetic response functions of Jain's sequences using Dirac composite fermion model. I will also provide the comparison with HLR theory.

CHAPTER 2

ELECTRO-MAGNETIC RESPONSE OF INTEGER QUANTUM HALL

2.1 Introduction

2.1.1 Electrons in magnetic field

Two-dimensional non-interacting electrons subject to the strong external magnetic field organize into N highly degenerate Landau levels. Such many-body states are gapped when the chemical potential lies anywhere between the Landau levels and exhibit the same qualitative behavior when electrons have either Dirac or non-relativistic nature. While qualitative features such as quantized Hall conductance and absence of the ground state degeneracy on a torus are identical, there is a quantitative difference in the local linear response functions. Detailed investigation of these fine distinctions as well as certain universalities in the behavior of both lowest Landau level and large N limit of the linear response functions is the objective of the present paper. Additionally, our results should be useful in the analysis of interacting FQH states using the composite fermion [60] and boson[102] approaches.

We will study the electromagnetic response of Landau level electrons in great detail. To start we present a straightforward method that allows to calculate linear response functions in the form of the generating functional for both relativistic and non-relativistic electrons filling an arbitrary number of Landau levels. In the non-relativistic case some of the results are available[14, 80, 60], however we present a simpler method of derivation as well as provide a number of new results. This method was first used by one of us in [1], but only a few results were presented. We will explain in detail how to calculate the linear response to arbitrary order in the expansion in momentum and frequency and give a compact expression for the polarization tensor in both non-relativistic and Dirac cases. In addition to the general expressions we present the leading order corrections in momentum and frequency expansion

for all linear response functions in explicit form.

With the exact expressions at hand we will investigate the linear response of the lowest Landau level (LLL) in the limit when the mixing between the Landau levels is neglected. It turns out that linear response of the Dirac electrons can be extracted from the linear response of the non-relativistic electrons via simple relation (2.108), which is valid to *all* orders in the gradient expansion. We check this relation via an explicit computation as well as using the well-known relation between momentum-dependent Hall conductivity and the static structure factor.

Next, we will meticulously investigate the validity of the semiclassical approximation in the large N limit. Our results on the large N limit are summarized in Fig 2.1. In this limit the electrons form a Fermi sphere and experience a weak magnetic field. The linear response can be calculated either using Landau's Fermi liquid (FL) theory or by directly taking the large N limit of the exact expressions. We will explain how to include a finite g-factor g into the Fermi liquid theory and evaluate the polarization tensor exactly. We will find that in the non-relativistic case the Fermi liquid and direct large N limit agree in the leading and sub-leading order in N to *all* orders in the gradient expansion and for *arbitrary* value of the g-factor (provided the latter was correctly accounted for in the FL theory, which we explain how to do). The large N limit of polarization operator of Dirac electrons agrees in leading and sub-leading order in N with the FL theory and non-relativistic results after the Berry phase of the Dirac cone is taken into account for the value of the g-factor $\mathrm{g} = 0$ (this may come as a surprise since Dirac electrons in vacuum correspond to $\mathrm{g} = 2$). The FL computation is done using the novel approach of [24] where the Boltzmann equation is phrased in terms of the (bosonic) fluctuations of the shape of the Fermi surface. This formulation allows to effortlessly obtain the large N polarization tensor to *all* orders in momentum and frequency in a closed form. We also explain how to include the effects of the short range interactions.

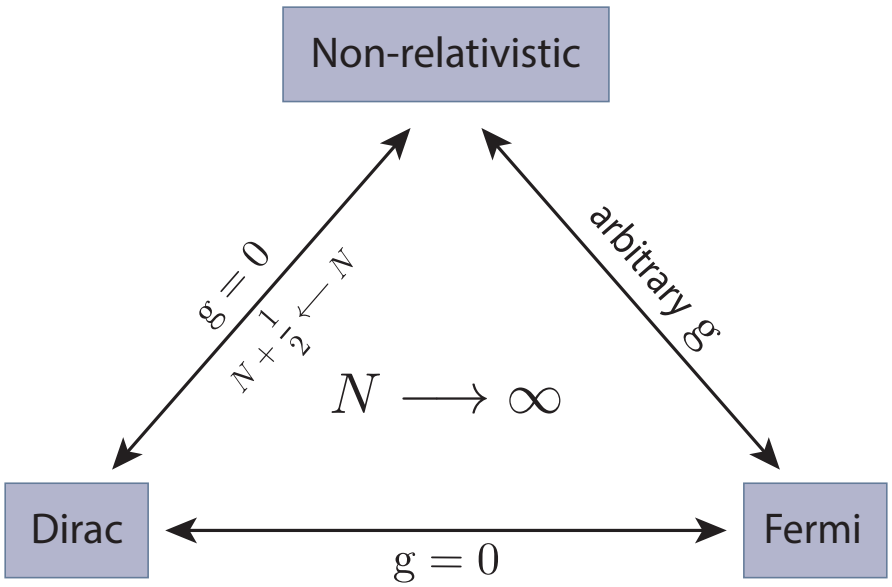

Figure 2.1: In the large N limit electromagnetic linear response of non-relativistic electrons agrees with the response of the Fermi liquid to all orders in gradient expansion and arbitrary g-factor. Electromagnetic response of Dirac electrons (in the large N limit) can be extracted from either non-relativistic or Fermi liquid result upon setting $\mathrm{g} = 0$ and replacing $N \longrightarrow N + \frac{1}{2}$. The replacement is needed to account for the contribution of the π Berry phase of the Dirac cone.

2.1.2 Generalities

Now we will introduce the main objects of interest, mainly to fix the notations. Given an action $S[\psi, \psi^\dagger; A_\mu]$ describing the (relativistic or non-relativistic, bosonic or fermionic) charged matter fields ψ, coupled to external electromagnetic field $A_\mu = \bar{A}_\mu + \delta A_\mu$, we define the generating functional as follows

$$W[\delta A_\mu] = -i \ln \int \mathscr{D}\psi \mathscr{D}\psi^\dagger e^{iS\left[\psi, \psi^\dagger; \bar{A}_\mu + \delta A_\mu\right]} \tag{2.1}$$

where $\bar{A}_\mu$ is the background value of the vector potential chosen to fix the chemical potential $\bar{A}_0 = \mu$ and background magnetic field $\epsilon^{ij}\partial_i \bar{A}_j = \bar{B} = \ell^{-2}$, where we have chosen the natural units $\hbar = c = e = 1$. These units will be used throughout the paper.

Generating functional is a compact way to encode the multipoint correlation functions

via

$$\left\langle \prod_{i=1}^{n} J^{\mu}(x_i) \right\rangle = \prod_{i=1}^{n} \frac{\delta}{\delta\, \delta A_{\mu}(x_i)} W[\delta A_{\mu}]\,. \tag{2.2}$$

The correlation functions obtained this way are always time-ordered.

In the present paper we will be interested in the *linear* response functions, *i.e.* the two-point functions with perturbations of the external fields turned off. For example, the polarization tensor encodes linear response of electric current to the electric field and is given by

$$\Pi^{\mu\nu}(x_1, x_2) = \left(\frac{\delta}{\delta\, \delta A_{\mu}(x_1)} \frac{\delta}{\delta\, \delta A_{\nu}(x_2)} W[\delta A_{\mu}] \right) \Big|_{\delta A_{\mu}=0}\,. \tag{2.3}$$

Assuming that $S[\psi, \psi^{\dagger}; \bar{A}_{\mu}]$ describes a physical system with a spectral gap and the perturbations δA_{μ} are weak and slowly varying on the spatial scale of magnetic length ℓ and the time scale set by the gap, we can expand the generating functional $W[\delta A_{\mu}]$ in powers of external fields and in the gradients of external fields δA_{μ}. If we also assume translational invariance then the gradient expansion can be converted into the expansion in momentum $\mathbf{k}$ and frequency Ω. To study the linear response functions we need to keep only the terms quadratic in δA_{μ}, but to arbitrary order in momentum and frequency. The most general expansion of this form is

$$W[\delta A_{\mu}] \quad = \int \frac{d\Omega d^2\mathbf{k}}{(2\pi)^3} \left[\bar{\rho} \delta A_0 + \frac{1}{2} \delta A_{\mu}(\mathbf{k}, \Omega) \Pi^{\mu\nu}(\mathbf{k}, \Omega) \delta A_{\nu}(-\mathbf{k}, -\Omega) \right]\,,$$

where the matrix $\Pi^{\mu\nu}(\mathbf{k}, \Omega)$ is known as the polarization operator or polarization tensor. Each entry of this 3×3 matrix is an infinite double expansion in momentum and frequency. We have also implicitly assumed in Eq.(2.4) that the expectation value of the electric current vanishes in the unperturbed groundstate. Gauge invariance implies a Ward identity

$$\Omega \Pi^{0\mu}(\mathbf{k}, \Omega) + k_i \Pi^{i\mu}(\mathbf{k}, \Omega) = 0\,. \tag{2.4}$$

It is easy to see that conductivity tensor is expressed in terms of the polarization tensor as

$$\sigma^{ij}(\mathbf{k},\Omega) = \frac{1}{i\Omega}\Pi^{ij}(\mathbf{k},\Omega)\,. \tag{2.5}$$

The plan of the paper is as follows. We will calculate the polarization tensor $\Pi^{\mu\nu}(\mathbf{k},\Omega)$ for non-relativistic electrons filling N Landau levels in Section II. The main result of the Section II is the exact expression for the polarization tensor (2.63). In the Section III we will calculate the polarization tensor for massless Dirac electrons filling N Landau levels and compare it with the non-relativistic one in the large N limit. In the Section IV we investigate the electromagnetic response of the lowest Landau level and find an exact relation between the linear response functions for non-relativistic and Dirac electrons. In the Section V we will obtain a closed form expression for the large N polarization tensor for non-relativistic electrons using the exact result (2.63) and using the FL theory. We find that both approaches agree exactly and differ from the Dirac electrons by the contribution of the Berry phase of the Dirac cone. For reader's convenience in every Section we present an explicit form of the polarization tensor in the leading and sub-leading orders in momentum and frequency that can be understood and used without reading the rest of the paper. Various Appendices are devoted to (often tedious) technical details.

2.2 Non-relativistic electrons

In this Section we will explain the method for calculation of the polarization operator for the non-relativistic electrons filling N Landau levels.

2.2.1 Model

Our starting point is the system of two-dimensional non-interacting non-relativistic fermions in external electromagnetic field described by a $U(1)$ vector potential A_μ. The action has a

form

$$S_{nr} = \int d^2xdt \Big[i\psi^\dagger D_0 \psi - \frac{1}{2m} |D_i \psi|^2 \Big] . \tag{2.6}$$

We assume that the fermions are spin polarized and, consequently, $\psi(x,t)$ is a complex Grassmann scalar. The covariant derivative

$$D_\mu = \partial_\mu - i(\bar{A}_\mu + \delta A_\mu) \tag{2.7}$$

includes both background vector potential and a weak perturbation. We will omit the chemical potential from the equations, but it will be implicitly assumed that the first N Landau levels are completely filled in the ground state and the chemical potential lies anywhere in the gap.

2.2.2 Computation of the Generating Functional

We will compute the generating functional as a gradient expansions in the external fields. Throughout the computation we will only keep the terms quadratic in the external fields, but to arbitrary order in the gradients. This expansion is well-defined because there is a cyclotron gap in the energy spectrum. The gradient expansion can be viewed as the expansion in the inverse gap and magnetic length ℓ which is small compared to any other spatial scale in the problem.

We start with rewriting the action as a differential operator sandwiched between the fermionic fields

$$S_{nr} = \int d^2xdt \ \psi^\dagger G^{-1} \psi \, , \tag{2.8}$$

where G^{-1} is the differential operator obtained by integrating by parts the derivatives acting

on $\psi^\dagger$. Since we assume that the perturbations of external fields are small we can write

$$G^{-1} = G_0^{-1} + V\,, \tag{2.9}$$

where G_0^{-1} is the "bare" Green's function given by

$$G_0^{-1} = i\partial_0 - \frac{1}{2m}|\bar{D}_i|^2\,, \tag{2.10}$$

where $\bar{D}_\mu = \partial_\mu - i\bar{A}_\mu$ and V encodes the terms at least linear in the perturbations of the external fields.

$$V = \delta A_0 + \frac{1}{2m}\{\delta A_i, \partial_i\} - \frac{1}{2m}\delta A_i \delta A_i\,. \tag{2.11}$$

Since the functional integral is quadratic in the external fields it can be formally written as a determinant of the perturbed (differential) operator G^{-1}. The generating functional of (connected) correlations functions is

$$\begin{aligned} W_{\rm nr}[\delta A_\mu] &= \frac{1}{i}\ln\int \mathscr{D}\bar{\psi}\mathscr{D}\psi\, e^{iS_{nr}[\psi,\delta A_\mu]} = \frac{1}{i}\ln\det[G^{-1}] \\ &= -\frac{1}{i}\ln G_0 + \frac{1}{i}{\rm Tr}\,(G_0 V) - \frac{1}{2}\frac{1}{i}{\rm Tr}\,(G_0 V G_0 V) + \dots\,, \end{aligned} \tag{2.12}$$

where in the last line we kept only the terms that contribute to the linear response. We can also disregard the (diverging) first term in the last line since it will not contribute to the linear response because it does not depend on the perturbations of the external fields by construction. To summarize, the object we are interested in is given by

$$W_{\rm nr} = W_{\rm nr}^{(1)} + W_{\rm nr}^{(2)} + W_{c,{\rm nr}}^{(2)} + \dots\,, \tag{2.13}$$

where $W_{\rm nr}^{(1)}$ and $W_{\rm nr}^{(2)}$ are the terms linear and quadratic in external fields correspondingly, while $W_{c,{\rm nr}}^{(2)}$ contains the so-called contact terms (See Fig. 2.2).

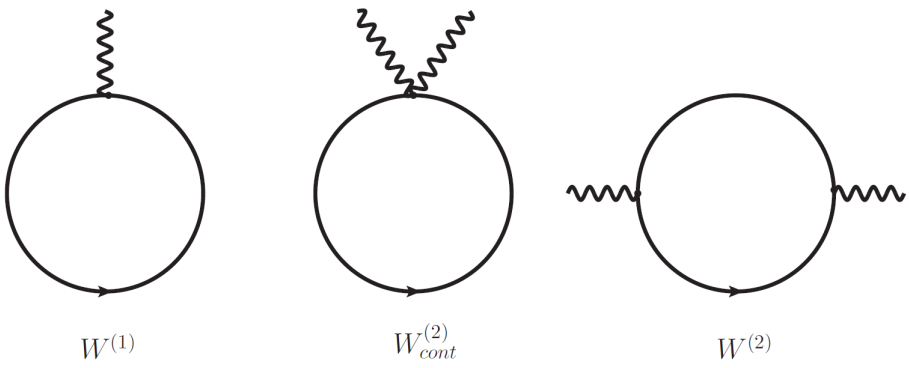

Figure 2.2: The generating functional to quadratic order in external fields is given by the sum of three diagrams. The first diagram, $W^{(1)}$, is linear in the perturbations of electromagnetic field and describes the constant background density of electrons. The second diagram, $W^{(2)}$, contains the main contribution to the generating functional, including the Chern-Simons term. Finally, the third diagram, $W^{(2)}_c$, contains the contact terms. Note, that the last diagram vanishes for the Dirac electrons.

2.2.3 Fock Representation

The Hilbert space of a particle in magnetic field can be mapped to the Hilbert space of two decoupled harmonic oscillators. To make this manifest we will use Fock representation for the basis states instead of the coordinate representation. The advantage of this approach is that we do not need to fix the gauge, thus our results will be manifestly gauge invariant. We will work in complex coordinates $z = x + iy$.

Define the creation and annihilation operators

$$a = \frac{i}{\sqrt{2}}\ell\bar{D}_{\bar{z}} = \frac{i}{\sqrt{2}}\ell(\bar{D}_1 + i\bar{D}_2)\,, \tag{2.14}$$

$$a^\dagger = \frac{i}{\sqrt{2}}\ell\bar{D}_z = \frac{i}{\sqrt{2}}\ell(\bar{D}_1 - i\bar{D}_2)\,. \tag{2.15}$$

The inverse relations are

$$\bar{D}_z = -i\frac{1}{\sqrt{2}\ell}a^\dagger\,, \quad \bar{D}_{\bar{z}} = -i\frac{1}{\sqrt{2}\ell}a\,. \tag{2.16}$$

It can be easily verified that

$$[a, a^\dagger] = 1\,. \tag{2.17}$$

In terms of these operators the inverse Green's function takes form

$$\begin{aligned} G_0^{-1} &= i\hbar\partial_0 - \omega_c\left(a^\dagger a + \frac{1}{2}\right) \\ &= i\partial_0 - H_0\,, \end{aligned} \tag{2.18}$$

where $H_0 = \omega_c\left(a^\dagger a + \frac{1}{2}\right)$ is the Hamiltonian for the particle in magnetic field.

We also define one more oscillator via

$$b^\dagger = -a + \frac{i}{\sqrt{2}\ell}z\,, \quad b = -a^\dagger - \frac{i}{\sqrt{2}\ell}\bar{z}\,. \tag{2.19}$$

It can be verified that $[b, b^\dagger] = 1$ and all a's commute with all b's.

Operators $a^\dagger, b^\dagger$ generate the entire Hilbert space of the single particle problem. From this point of view the coordinates themselves must be understood as operators acting on the Hilbert space according to

$$z = -\sqrt{2}\ell i(b^\dagger + a)\,, \quad \bar{z} = \sqrt{2}\ell i(a^\dagger + b)\,. \tag{2.20}$$

The basis in the Hilbert space is given by

$$|nm\rangle = |n\rangle \otimes |m\rangle = \frac{(a^\dagger)^n}{\sqrt{n!}}\frac{(b^\dagger)^m}{\sqrt{m!}}|0\rangle \otimes |0\rangle\,. \tag{2.21}$$

The a-operators induce the transitions between the Landau levels, whereas b-operators generate the states of different angular momentum within each Landau level since

$$[H_0, b] = [H_0, b^\dagger] = 0\,. \tag{2.22}$$

The bare Green's function is then given by

$$G_0 = \int \frac{d\Omega}{2\pi} \sum_{nm} e^{-i\Omega t} \frac{|nm\rangle\langle nm|}{\Omega - E_n}\,, \tag{2.23}$$

where

$$E_n = \left(n + \frac{1}{2}\right)\omega_c \tag{2.24}$$

is the spectrum of the unperturbed Hamiltonian H_0.

It is easy to check that

$$G_0^{-1} G_0 = \Big[i\partial_0 - H_0\Big] G_0 = \delta(t) \cdot \sum_{m,n} |nm\rangle\langle nm| = \mathbf{1}. \tag{2.25}$$

The trace of a local operator $\mathcal{O}$ over the Hilbert space and time is defined as follows

$$\mathrm{Tr}\,(\mathcal{O}) \equiv \sum_{n,l,t} \langle nlt|\mathcal{O}|nlt\rangle = \int dt \sum_{n,l} \langle nl|\mathcal{O}(t)|nl\rangle\,. \tag{2.26}$$

2.2.4 Setting up the "Feynman rules"

In this Section we will derive the differential operators that will appear in the vertices of the diagrams in Fig. C.1.

First, we expand the classical action to the second order in *external* electromagnetic field

$$S_{nr} = S_{nr}^{(0)} + S_{nr}^{(1)} + S_{nr}^{(2)}\,. \tag{2.27}$$

The unperturbed action is given by

$$\begin{aligned} S_{nr}^{(0)} \quad &= \int d^2x dt\ \psi^\dagger \left[i\partial_0 - \omega_c \left(a^\dagger a + \frac{1}{2}\right)\right] \psi \\ &= \int d^2x dt\ \Psi^\dagger G_0^{-1} \Psi\,. \end{aligned} \tag{2.28}$$

The part of the action linear in external fields is given by

$$\begin{aligned} S_{nr}^{(1)} &= \int d^2x dt\, \psi^\dagger \Big[\delta A_0 - \frac{1}{2\sqrt{2}m\ell}(\{a^\dagger, \delta A_{\bar{z}}\} + \{a, \delta A_z\})\Big]\psi \\ &= \int d^2x dt\ \psi^\dagger V^{(1)} \psi\,, \end{aligned} \tag{2.29}$$

where $\{a, \delta A_z\}$ is the anticommutator (recall that a is a differential operator that we understand as acting to the right).

The part of the action quadratic in external fields is given by

$$S_{nr}^{(2)} = -\int d^2x dt\ \psi^\dagger \left[\frac{1}{2m}|\delta A|^2\right] \psi = \int d^2x dt\ \psi^\dagger V^{(2)} \psi\,. \tag{2.30}$$

The full "vertex operator" consists of the terms linear and quadratic in external fields

$$V(x,t) = V^{(1)}(x,t) + V^{(2)}(x,t)\,. \tag{2.31}$$

Using (2.20) we interpret V as an operator on the Fock space.

We re-write all the vertices in Fourier space and introduce a vector $\mathcal{V}_\mu^{(1)}(\mathbf{k}, \Omega)$ according to

$$V^{(1)} = \mathcal{V}_\mu^{(1)}(\mathbf{k}, \Omega)\delta A_\mu(\mathbf{k}, \Omega)\,, \tag{2.32}$$

which is always possible because $V^{(1)}$ is linear in the external fields by definition. Consider the terms in V linear in, say, $\delta A_{\bar{z}}$

$$V(x,t)\Big|_{\delta A_{\bar{z}}} = \frac{1}{2\sqrt{2}m\ell}\{a^\dagger, \delta A_{\bar{z}}\}\,. \tag{2.33}$$

In momentum space this takes form

$$V(\mathbf{k}, \Omega)\Big|_{\delta A_{\bar{z}}} = e^{-i\Omega t}\frac{1}{2\sqrt{2}m\ell}\{a^\dagger, e^{i\mathbf{k}\cdot\mathbf{x}}\}\delta A_{\bar{z}}(\mathbf{k}, \Omega)\,. \tag{2.34}$$

Then, using (2.20)

$$e^{i\vec{k}\vec{x}} = e^{\frac{i}{2}k\bar{z}} e^{\frac{i}{2}\bar{k}z} = e^{-\frac{k\ell}{\sqrt{2}}a^\dagger} e^{\frac{\bar{k}\ell}{\sqrt{2}}a} e^{-\frac{k\ell}{\sqrt{2}}b} e^{\frac{\bar{k}\ell}{\sqrt{2}}b^\dagger}\,, \tag{2.35}$$

where we introduced the complex momentum $k = k_1 + ik_2$. Finally, using that a's and b's commute with each other we get an expression for

$$\mathcal{V}_z^{(1)}(\mathbf{k},\Omega) = \frac{1}{2\sqrt{2}m\ell} e^{-i\Omega t} e^{-\frac{k\ell}{\sqrt{2}}b} e^{\frac{\bar{k}\ell}{\sqrt{2}}b^\dagger} \left\{ a^\dagger, e^{-\frac{k\ell}{\sqrt{2}}a^\dagger} e^{\frac{\bar{k}\ell}{\sqrt{2}}a} \right\}. \tag{2.36}$$

Expressions for the other vertices can be derived in the same way

$$\mathcal{V}_0^{(1)} = e^{-i\Omega t} e^{-\frac{k\ell}{\sqrt{2}}b} e^{\frac{\bar{k}\ell}{\sqrt{2}}b^\dagger} e^{-\frac{k\ell}{\sqrt{2}}a^\dagger} e^{\frac{\bar{k}\ell}{\sqrt{2}}a}\,, \tag{2.37}$$

$$\mathcal{V}_{\bar{z}}^{(1)} = \frac{-1}{2\sqrt{2}m\ell} e^{-i\Omega t} e^{-\frac{k\ell}{\sqrt{2}}b} e^{\frac{\bar{k}\ell}{\sqrt{2}}b^\dagger} \{ a^\dagger, e^{-\frac{k\ell}{\sqrt{2}}a^\dagger} e^{\frac{\bar{k}\ell}{\sqrt{2}}a} \}\,. \tag{2.38}$$

Notice that part of the vertices that depends on both time and b's has completely factorized and *is the same for all vertices*. We will be able to use this fact to integrate over time and to trace over the Fock space generated by $b^\dagger$ before tracing over the Fock space generated by $a^\dagger$. It is the trace over a where all of the complexity is concentrated. For this reason it will be convenient to introduce a separate notation for the part of the "vertex operators" that acts only in the Fock space generated by $a^\dagger$. Thus we define

$$\mathcal{V}_\mu^{(1)} = e^{-i\Omega t} e^{-\frac{k\ell}{\sqrt{2}}b} e^{\frac{\bar{k}\ell}{\sqrt{2}}b^\dagger} \tilde{V}_\mu\,, \tag{2.39}$$

where

$$\tilde{V}_0 = e^{-\frac{k\ell}{\sqrt{2}}a^\dagger} e^{\frac{\bar{k}\ell}{\sqrt{2}}a}\,, \tag{2.40}$$

$$\tilde{V}_{\bar{z}} = -\frac{1}{2\sqrt{2}m\ell} \{ a^\dagger, e^{-\frac{k\ell}{\sqrt{2}}a^\dagger} e^{\frac{\bar{k}\ell}{\sqrt{2}}a} \}\,, \tag{2.41}$$

$$\tilde{V}_z = -\frac{1}{2\sqrt{2}m\ell} \{ a, e^{-\frac{k\ell}{\sqrt{2}}a^\dagger} e^{\frac{\bar{k}\ell}{\sqrt{2}}a} \}\,. \tag{2.42}$$

2.2.5 Generating Functional to the Second Order

In this Section we will perform an *exact* computation of the entire quadratic generating functional to *all* orders in the gradient expansion. Before diving into details we briefly pause to mention a few relations that will be heavily used in the sequel

$$[b, f(b^\dagger)] = f'(b^\dagger)\,, \tag{2.43}$$

$$e^{Qb} f(b^\dagger) = f(b^\dagger + Q) e^{Qb}\,. \tag{2.44}$$

Using these relations and elementary properties of the oscillator algebra we can evaluate the following expectation values

$$\langle n| e^{-\frac{k\ell}{\sqrt{2}} a^\dagger} e^{\frac{\bar{k}\ell}{\sqrt{2}} a} |m\rangle = \sqrt{\frac{n!}{m!}} \left(\frac{\bar{k}\ell}{\sqrt{2}}\right)^{m-n} L_n^{m-n}\left(\frac{|k\ell|^2}{2}\right) \tag{2.45}$$

for $m \geq n$, and

$$\langle n| e^{-\frac{k\ell}{\sqrt{2}} a^\dagger} e^{\frac{\bar{k}\ell}{\sqrt{2}} a} |m\rangle = \sqrt{\frac{m!}{n!}} \left(\frac{-k\ell}{\sqrt{2}}\right)^{n-m} L_m^{n-m}\left(\frac{|k\ell|^2}{2}\right). \tag{2.46}$$

for $m \leq n$. Similar equations can be found in [74].

There will be two major contributions to the generating functional in quadratic order. One contribution comes from the contact terms. These are obtained by plugging $V^{(2)}$ into

$$-i \mathrm{Tr}\, G_0 V\,. \tag{2.47}$$

These contributions are always evaluated at zero momentum and zero frequency. In fact, the contact terms can be restored simply via analyzing the Ward identities for electric charge conservation. We will denote the contribution of the contact terms to the polarization operator via $\Pi^{\mu\nu}_{c,\mathrm{nr}}$.

The main contribution comes from

$$\frac{i}{2}\mathrm{Tr}\, G_0 V^{(1)} G_0 V^{(1)}\,. \tag{2.48}$$

First, we will trace over the Fock space generated by $b, b^\dagger$, then over frequency and in the end we will be left with an irreducible expression for the trace over the Fock space generated by $a, a^\dagger$. The details of the steps outlined above can be found in the Appendix A.2.

We find

$$\mathrm{Tr}\, G_0 V^{(1)} G_0 V^{(1)} = \sum_{n,l,t} \langle tnl| G_0 V^{(1)} G_0 V^{(1)} |tnl\rangle \tag{2.49}$$

$$= \frac{1}{2\pi}\int \frac{d^2k d\Omega}{(2\pi)^3} e^{-\frac{|k\ell|^2}{2}} \sum_{n'<N, n\geq N} \frac{\langle n|\tilde{V}^{(1)}_\mu(k)|n'\rangle\langle n'|\tilde{V}^{(1)}_\nu(-k)|n\rangle + \langle n'|\tilde{V}^{(1)}_\nu(k)|n\rangle\langle n|\tilde{V}^{(1)}_\mu(-k)|n'\rangle}{E_n - E_{n'} - \Omega} \delta A_\mu(k)\delta A_\nu(-k). \tag{2.50}$$

In the remainder of the Section we will simplify this expression.

We introduce the following notation

$$\Gamma^\mu_{nn'}(k,\Omega) = \langle n|\tilde{V}^{(1)}_\mu(k)|n'\rangle\,, \tag{2.51}$$

then (using the dimensionless frequency $\omega = \Omega/\omega_c$)

$$\Pi^{\mu\nu}_{\mathrm{nr}}(k,\omega) = \frac{m}{2\pi} e^{-\frac{|k\ell|^2}{2}} \sum_{n'<N, n\geq N} \frac{\Gamma^\mu_{nn'}(k)\Gamma^\nu_{n'n}(-k) + \Gamma^\nu_{nn'}(k)\Gamma^\mu_{n'n}(-k)}{n - n' - \omega} + \Pi^{\mu\nu}_{c,\mathrm{nr}}\,. \tag{2.52}$$

This is the main result of the Section. In the following we will show that all of the components of the polarization tensor can be reconstructed from a *single* generating function.

2.2.6 The Generating Function

While (2.52) is indeed the final expression that cannot be reduced further, it is not convenient to work with since one has to use complicated expressions for $\Gamma^{\mu}_{nn'}$. We will introduce a trick that will allow to express *all* of the components of the polarization operator in terms of derivatives of a *single* function.

We define the generating function $\mathcal{G}(k,k';N)$

$$\begin{aligned}\mathcal{G}(k,k';N) &= \sum_{n\geq N,n'<N}\left(\frac{\Gamma^0_{nn'}(k)\Gamma^0_{n'n}(k')}{n-n'-\omega}+\frac{\Gamma^0_{nn'}(k')\Gamma^0_{n'n}(k)}{n-n'+\omega}\right)\\ &= \sum_{n\geq N,n'<N}\left(-\frac{\ell^2}{2}\right)^{n-n'}\frac{n'!}{n!}\left(\frac{(k\bar{k}')^{n-n'}}{n-n'-\omega}+\frac{(\bar{k}k')^{n-n'}}{n-n'+\omega}\right)\\ &\quad L^{n-n'}_{n'}\left(\frac{|k\ell|^2}{2}\right)L^{n-n'}_{n'}\left(\frac{|k'\ell|^2}{2}\right)\end{aligned} \tag{2.53}$$

First, we notice (with the help of (2.45)) that

$$\Pi^{00}_{nr} = \frac{m}{2\pi}e^{-\frac{|k\ell|^2}{2}}\mathcal{G}(k,-k;N)\,. \tag{2.54}$$

other components of $\Pi^{\mu\nu}_{\rm nr}$ can be expressed as derivatives of $\mathcal{G}(k,k';N)$ with respect to momenta. To see this we use the identities

$$e^{-ka^\dagger}e^{\bar{k}a}a^\dagger = (-\partial_k+\bar{k})e^{-ka^\dagger}e^{\bar{k}a}\,, \tag{2.55}$$

$$ae^{-ka^\dagger}e^{\bar{k}a} = (\partial_{\bar{k}}-k)e^{-ka^\dagger}e^{\bar{k}a}\,. \tag{2.56}$$

These identities allow us to re-write the vertex insertions (2.40)-(2.42) in terms of derivatives

with respect to momentum as follows

$$\tilde{V}_0(k) = e^{-\frac{k\ell}{\sqrt{2}}a^\dagger} e^{\frac{\bar{k}\ell}{\sqrt{2}}a}, \tag{2.57}$$

$$\tilde{V}_{\bar{z}}(k) = -\frac{1}{2\sqrt{2}\ell m}\left(-\frac{2\sqrt{2}}{\ell}\partial_k + \frac{\ell}{\sqrt{2}}\bar{k}\right) e^{-\frac{k\ell}{\sqrt{2}}a^\dagger} e^{\frac{\bar{k}\ell}{\sqrt{2}}a}, \tag{2.58}$$

$$\tilde{V}_z(k) = -\frac{1}{2\sqrt{2}\ell m}\left(\frac{2\sqrt{2}}{\ell}\partial_{\bar{k}} - \frac{\ell}{\sqrt{2}}k\right) e^{-\frac{k\ell}{\sqrt{2}}a^\dagger} e^{\frac{\bar{k}\ell}{\sqrt{2}}a}. \tag{2.59}$$

Next we introduce a separate notation for the differential operators acting on $e^{-\frac{k\ell}{\sqrt{2}}a^\dagger} e^{\frac{\bar{k}\ell}{\sqrt{2}}a}$ in Eqs.(2.57)-(2.59) as follows

$$\hat{\mathcal{P}}^0_{\rm nr}(k) = 1\,, \tag{2.60}$$

$$\hat{\mathcal{P}}^z_{\rm nr}(k) = -\frac{1}{2\sqrt{2}\ell m}\left(-\frac{2\sqrt{2}}{\ell}\partial_k + \frac{\ell}{\sqrt{2}}\bar{k}\right), \tag{2.61}$$

$$\hat{\mathcal{P}}^{\bar{z}}_{\rm nr}(k) = -\frac{1}{2\sqrt{2}\ell m}\left(\frac{2\sqrt{2}}{\ell}\partial_{\bar{k}} - \frac{\ell}{\sqrt{2}}k\right). \tag{2.62}$$

Then an arbitrary element of the polarization operator is given by

$$\Pi^{\mu\nu}_{nr}(\omega,k) = \frac{m}{4\pi}e^{-\frac{|k\ell|^2}{2}} \lim_{k'\to -k} \hat{\mathcal{P}}^\mu_{\rm nr}(k)\hat{\mathcal{P}}^\nu_{\rm nr}(k')\mathcal{G}(k,k';N) + \Pi^{\mu\nu}_{c,\rm nr}\,. \tag{2.63}$$

This expression is the one we will use for practical computations and is the first main result of the present manuscript.

The contact terms are obtained from $\mathrm{Tr}\, G_0 V^{(2)}$. The only contact term is the well-known diamagnetic term given by

$$W^{(2)}_{c,\rm nr} = -i\mathrm{Tr}\, G_0 V^{(2)} = N\frac{\omega_c}{4\pi}\int \frac{d^2k}{(2\pi)^2}\delta A_z(k)\delta A_{\bar{z}}(-k)\,.$$

This term is evaluated at zero frequency $\omega = 0$. The contribution to momentum space

polarization tensor is

$$\Pi^{z\bar{z}}_{c,\mathrm{nr}} = N\frac{\omega_c}{16\pi}\delta_{z\bar{z}}\,. \tag{2.64}$$

It can be checked explicitly that this term restores the Ward identity (2.4). Direct application of Eqs. (2.63)-(2.64) to N filled Landau levels allows us to write the generating functional in the leading orders in the gradient expansion

$$W[A_\mu] = \frac{N}{4\pi}\int d^2x dt\Big[AdA + \frac{1}{\omega_c}|\delta\vec{E}|^2 - \frac{N}{m}\delta B^2 - \frac{3N}{2}\ell^2\delta B(\partial_i\delta E_i) + \dots\Big]\,, \tag{2.65}$$

where δE_i and δB are made from the perturbations of the electromagnetic field δA_μ. We have also absorbed the linear term $\bar{\rho}\delta A_0$ into the Chern-Simons term by including the background $\bar{A}_\mu$. Higher order terms can also be easily obtained from (2.63). Finally, note that $|\delta\vec{E}|^2$ and B^2 terms do *not* combine into $\delta F^{\mu\nu}\delta F_{\mu\nu}$ due to apparent absence of Lorentz invariance. We have also checked that Ward Identities of the Galilean symmetry studied in Ref. [[36]] are satisfied.

2.2.7 Including the g*-factor*

For the future applications we also need to include the effects of finite g-factor of the electron, g, by adding an extra "Zeeman" term to the matter action

$$\delta S_{nr}[\psi,\psi^\dagger] = \frac{\mathrm{g}}{4m}\int B\psi^\dagger\psi\,. \tag{2.66}$$

This results in the redefinition of the number current

$$J^i(g) = J^i(0) + \frac{\mathrm{g}}{4m}\epsilon^{ij}\partial_j\rho \tag{2.67}$$

as well as the vertices

$$\hat{\mathcal{P}}^{z}_{\rm nr}(k;g) = -\frac{1}{2\sqrt{2}\ell m}\left(-\frac{2\sqrt{2}}{\ell}\partial_k + \left(1-\frac{\rm g}{2}\right)\frac{\ell}{\sqrt{2}}\bar{k}\right), \tag{2.68}$$

$$\hat{\mathcal{P}}^{\bar{z}}_{\rm nr}(k;g) = -\frac{1}{2\sqrt{2}\ell m}\left(\frac{2\sqrt{2}}{\ell}\partial_{\bar{k}} - \left(1-\frac{\rm g}{2}\right)\frac{\ell}{\sqrt{2}}k\right). \tag{2.69}$$

Note that the generating functional for finite g can be expressed in terms of the generating functional at $\rm g = 0$ as follows

$$W_{\rm nr}\left[\delta A_0, \delta A_i; {\rm g}\right] = W_{\rm nr}\left[\delta A_0 + \frac{\rm g}{4m}\delta B, \delta A_i; {\rm g}=0\right]. \tag{2.70}$$

In particular, Eq.(2.70) implies the following relations between the components of the polarization tensor

$$\Pi^{00}_{\rm nr}(\mathbf{k},\Omega;{\rm g}) = \Pi^{00}_{\rm nr}(\mathbf{k},\Omega;0) \tag{2.71}$$

$$\Pi^{0i}_{\rm nr}(\mathbf{k},\Omega;{\rm g}) = \Pi^{0i}_{\rm nr} + i\frac{\rm g}{4m}\epsilon^{ij}k_j\Pi^{00}_{\rm nr}(\mathbf{k},\Omega;0) \tag{2.72}$$

$$\begin{aligned}\Pi^{ij}_{\rm nr}(\mathbf{k},\Omega;{\rm g}) &= \Pi^{ij}_{\rm nr}(\mathbf{k},\Omega;0) + i\frac{\rm g}{4m}k_l\Big(\epsilon^{lj}\Pi^{0i}_{\rm nr}(\mathbf{k},\Omega;0) - \epsilon^{li}\Pi^{j0}_{\rm nr}(\mathbf{k},\Omega;0)\Big)\\ &\quad + \frac{{\rm g}^2}{16m^2}\left(|k|^2\delta^{ij} - k^ik^j\right)\Pi^{00}_{\rm nr}(\mathbf{k},\Omega;0).\end{aligned} \tag{2.73}$$

The g-factor will be used in Section IV to take the LLL projection and as an extra control parameter in Section V where we will compare the large N limit of the non-relativistic polarization operator with a semiclassical computation.

2.3 Dirac electrons

2.3.1 Model and the "Feynman rules"

In this Section we will calculate the polarization tensor for Dirac[1] fermions in strong magnetic field, filling N Landau levels. The action is given by

$$S_D = \int d^3x i\bar{\Psi}\not{D}\Psi\,, \tag{2.74}$$

where Ψ is a two-component spinor, $\bar{\Psi} = \Psi^\dagger\gamma^0$ and $\not{D} = \gamma^0 D_0 + v_F\gamma^i D_i$, where the covariant derivative D_μ is given by (2.7) and includes both constant magnetic field and its perturbations, v_F is the Fermi velocity. To emphasize Lorentz invariance we will use the notation $d^3x = dtd^2x$.

The Hamiltonian is

$$H = -iv_F\gamma^0\gamma^i D_i - A_0\,, \tag{2.75}$$

where v_F is the Fermi velocity. We choose the γ-matrices as follows

$$\gamma^0 = \sigma^3, \qquad \gamma^1 = i\sigma^2, \qquad \gamma^2 = -i\sigma^1, \tag{2.76}$$

where σ^i are the Pauli's matrices.

As in the non-relativistic case the Hilbert space maps on two copies of the Fock space generated by $a^\dagger$ and $b^\dagger$ defined in (2.14)-(2.20). The unperturbed Hamiltonian can be explicitly written as

$$H_0 = \begin{pmatrix} -\mu & -v_F\sqrt{2\bar{B}}a^\dagger \\ -v_F\sqrt{2\bar{B}}a & -\mu \end{pmatrix}, \tag{2.77}$$

where μ is the chemical potential. There are three types of eigenstates of H_0

1. The Dirac nature of the fermions does not have to come from spacetime symmetries. It can be also internal $SU(2)$ symmetry as it happens in graphene.

(i) zero energy, $\left|\Psi_{0,m}\right\rangle = \begin{pmatrix} |0,m\rangle \\ 0 \end{pmatrix}$, $E_0^D + \mu = 0$,

(ii) positive energy, $\left|\Psi^+_{n,m}\right\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} |n,m\rangle \\ -|n-1,m\rangle \end{pmatrix}$,

$E_n^{D+} + \mu = +v_F\sqrt{2\bar{B}n}$.

(iii) negative energy, $\left|\Psi^-_{n,m}\right\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} |n,m\rangle \\ +|n-1,m\rangle \end{pmatrix}$,

$E_n^{D-} + \mu = -v_F\sqrt{2\bar{B}n}$.

We introduce a uniform notation for all of the eigenstates as follows

$$\begin{aligned} \left|\Psi_{n,m}\right\rangle &= \mathrm{norm}(n) \begin{pmatrix} ||n|,m\rangle \\ -\mathrm{sgn}(n)\,|(|n|-1),m\rangle \end{pmatrix} = |\Psi_n\rangle \otimes |m\rangle \ , \\ E_n^D &= \mathrm{sgn}(n) v_F \sqrt{2\bar{B}|n|} - \mu, \end{aligned} \tag{2.78}$$

where it is understood that $|-1,m\rangle = 0$ and $\mathrm{norm}(n) = 1/\sqrt{2}$ for $|n| > 0$ and $\mathrm{norm}(0) = 1$. With this notation at hand $n \in \mathbb{Z}$, $m \in \mathbb{Z}_+$. The unperturbed Green's function is

$$G_0(t) = \int \frac{d\Omega}{2\pi} e^{-i\Omega t} \sum_{n,m} \frac{\left|\Psi_{n,m}\right\rangle \left\langle\Psi_{n,m}\right|}{\Omega - E_n^D + i\epsilon\ \mathrm{sgn}(E_n^D)} \,. \tag{2.79}$$

The massless Dirac action is easily decomposed into the terms free of and linear in external electromagnetic field perturbations (about the constant magnetic field and chemical potential),

$$S_D = S_D^{(0)} + S_D^{(1)} \,, \tag{2.80}$$

where

$$S_D^{(0)} = \int d^3x \Psi^\dagger \begin{pmatrix} i\partial_0 + \mu & v_F\sqrt{2\bar{B}}a^\dagger \\ v_F\sqrt{2\bar{B}}a & i\partial_0 + \mu \end{pmatrix} \Psi,$$
$$S_D^{(1)} = \int d^3x \Psi^\dagger \begin{pmatrix} \delta A_0 & v_F \delta A_z \\ v_F \delta A_{\bar{z}} & \delta A_0 \end{pmatrix} \Psi. \tag{2.81}$$

The first term gives the bare propagator,

$$S_D^{(0)} = \int d^3x \Psi^\dagger G_0^{-1} \Psi, \tag{2.82}$$

that satisfies

$$G_0^{-1}|\Psi_{n,m}\rangle = (\Omega + \mu - E_n^D)|\Psi_{n,m}\rangle\,, \tag{2.83}$$

and

$$G_0^{-1}G_0 = \delta(t)\sum_i |\Psi_i\rangle\langle\Psi_i| = \mathbb{1}\,. \tag{2.84}$$

The second term gives the vertex in position space

$$V(t,x) = \begin{pmatrix} \delta A_0(t,x) & v_F\delta A_z(t,x) \\ v_F\delta A_{\bar{z}}(t,x) & \delta A_0(t,x) \end{pmatrix}. \tag{2.85}$$

Note that the vertices have no explicit coordinate dependence or derivatives, and it simply remains to Fourier transform them, using (2.20) and (2.35).

Following the Section 2.2, we wish to evaluate $\mathrm{Tr}\, G_0 V G_0 V$ to derive the generating functional. In this case there are no contact terms because Dirac Hamiltonian (and the action) is linear in external fields. Using the results of Appendices A.2 and A.3 we can take

the trace over the Fock space generated by $b^\dagger$ and over time

$$\begin{aligned}
\mathrm{Tr}\, G_0 V G_0 V &= \int dt \sum_{m,n} \left\langle \Psi_{n,m}(t) \right| G_0 V G_0 V \left| \Psi_{n,m}(t) \right\rangle \\
&= \frac{1}{2\pi\ell^2} \int \frac{d\Omega'}{2\pi} \frac{d\Omega}{2\pi} \frac{d^2\mathbf{k}}{(2\pi)^2} e^{-\frac{|k\ell|^2}{2}} \\
&\times \sum_{nn'} \frac{\hat{V}_{nn'}(\mathbf{k},\Omega)}{\Omega' + \Omega - E_n^D + i\epsilon \mathrm{sgn}(E_n^D)} \frac{\hat{V}_{n'n}(-\mathbf{k},-\Omega)}{\Omega' - E_{n'}^D + i\epsilon \mathrm{sgn}(E_{n'}^D)},
\end{aligned} \tag{2.86}$$

in which the vertex operator in the momentum space is defined as

$$\begin{aligned}
\hat{V}_{nn'}(\mathbf{k},\Omega) = \langle \Psi_n | \begin{pmatrix} \delta A_0(\mathbf{k},\Omega) & v_F \delta A_z(\mathbf{k},\Omega) \\ v_F \delta A_{\bar{z}}(\mathbf{k},\Omega) & \delta A_0(\mathbf{k},\Omega) \end{pmatrix} \\
\times e^{-\frac{k\ell a^\dagger}{\sqrt{2}}} e^{\frac{\bar{k}\ell a}{\sqrt{2}}} \left| \Psi_{n'} \right\rangle .
\end{aligned} \tag{2.87}$$

The $i\epsilon$ prescription is crucial in evaluating the frequency integral

$$\begin{aligned}
&\int \frac{d\Omega'}{2\pi} \frac{1}{\Omega' + \Omega - E_n^D + i\epsilon\ \mathrm{sgn}(E_n^D)} \frac{1}{\Omega' - E_{n'}^D + i\epsilon\ \mathrm{sgn}(E_{n'}^D)} \\
&= \begin{cases} \frac{i}{E_n^D - E_{n'}^D - \Omega} & , \quad E_n^D < 0,\ E_{n'}^D > 0 \\ \frac{i}{E_{n'}^D - E_n^D + \Omega} & , \quad E_n^D > 0,\ E_{n'}^D < 0 \\ 0 & , \quad \text{else} \end{cases} .
\end{aligned} \tag{2.88}$$

The polarization tensor $\Pi_D^{\mu\nu}(\Omega,\mathbf{k})$ is given by

$$\Pi_D^{\mu\nu}(\Omega,\mathbf{k}) = \frac{e^{-\frac{|k\ell|^2}{2}}}{2\pi\ell^2} \sum_{n'\le N, n>N} \left(\frac{\Gamma^\mu_{Dnn'}(\mathbf{k}) \Gamma^\nu_{Dn'n}(-\mathbf{k})}{E_n^D - E_{n'}^D - \Omega} + \frac{\Gamma^\nu_{Dnn'}(-\mathbf{k}) \Gamma^\mu_{Dn'n}(\mathbf{k})}{E_n^D - E_{n'}^D + \Omega} \right) \tag{2.89}$$

where

$$\Gamma^\mu_{Dnn'}(\vec{k}) = \langle \Psi_n | \mathcal{P}_D^\mu e^{-\frac{k\ell a^\dagger}{\sqrt{2}}} e^{-\frac{\bar{k}\ell a}{\sqrt{2}}} \left| \Psi_{n'} \right\rangle \tag{2.90}$$

and

$$\mathcal{P}_D^0 = \mathbb{1}, \quad \mathcal{P}_D^1 = v_F\sigma^1, \quad \mathcal{P}_D^2 = v_F\sigma^2. \tag{2.91}$$

Using (2.45) we evaluate the vertices $\Gamma^{\mu}_{Dnn'}(\mathbf{k})$. The expressions turn out to be quite complicated and so we list them in the Appendix D.

2.3.2 Dirac Polarization Tensor

In this Section, we write out explicit expressions for the Landau level polarization tensor for Dirac fermion in the leading order in momentum and frequency. While Eq.(2.89) looks similar to the corresponding Eq.(2.52) for the non-relativistic fermions we want to emphasize that there is a difficulty in evaluating the summation, even when we limit ourselves to some finite order in momentum and frequency. The reason is that every component of the polarization tensor, contains the sum over n' (*i.e.* the sum over the Dirac sea) from $-\infty$ to N, where N is the number of filled Landau levels. We remind the reader that in the non-relativistic case this sum consisted of a *finite* number of terms (because the parabolic dispersion relation has a bottom, see Fig. 2.4) in every order in momentum and frequency. In the present case the the sum has infinite number of terms, however, it is *convergent* and does *not* need to be regularized. To simplify the expressions we fix a coordinate frame in which $\mathbf{k} = (k_1, 0)$. Leaving the details to the Appendix A.5 we present the leading order terms below

$$\Pi_D^{12}(\Omega,\mathbf{k}) = -i\Omega\frac{N+1/2}{2\pi} + i\Omega(k_1\ell)^2\frac{6N^2+6N+1}{16\pi} - i\Omega^3\frac{\ell^2}{v_F^2}\frac{8N^2+8N+1}{8\pi} + \cdots \tag{2.92}$$

$$\Pi_D^{00}(\Omega,\mathbf{k}) = -k_1^2\frac{3\ell}{2\sqrt{2}\pi v_F}\zeta\left(-\frac{1}{2}, N+1\right) + \cdots \tag{2.93}$$

$$\Pi_D^{11}(\Omega,\mathbf{k}) = -\Omega^2\frac{3\ell}{2\sqrt{2}\pi v_F}\zeta\left(-\frac{1}{2}, N+1\right) + \cdots \tag{2.94}$$

$$\Pi_D^{22}(\Omega,\mathbf{k}) = -\Omega^2\frac{3\ell}{2\sqrt{2}\pi v_F}\zeta\left(-\frac{1}{2}, N+1\right) + k_1^2\frac{3\ell v_F}{4\sqrt{2}\pi}\zeta\left(-\frac{1}{2}, N+1\right) + \cdots, \tag{2.95}$$

where $\zeta(s,n)$ is the Hurwitz ζ-function [2]. We stress that each component of the polarization tensor is *finite* without any need for regularization. The use of ζ-function is a convenient choice that allows to evaluate the sums analytically.

In the coordinate space the generating functional is given by

$$W_{\rm D} = \int d^3x \qquad \left[\frac{N+\frac{1}{2}}{4\pi} AdA - \frac{3\ell}{4\sqrt{2}\pi v_F}\zeta\left(-\frac{1}{2}, N+1\right)|\delta\vec{E}|^2 \right.$$
$$\left. + \frac{3\ell v_F}{8\sqrt{2}\pi}\zeta\left(-\frac{1}{2}, N+1\right)\delta B^2 - \ell^2\frac{6(N+\frac{1}{2})^2-\frac{1}{2}}{8\pi}\delta B(\partial_i\delta E_i) + \ldots\right]. \quad (2.96)$$

Eq.(2.96) is valid in arbitrary coordinate frame. Eq. (2.96) is the main result of the present Section. Note that despite the Lorentz invariance of the action (2.74) the generating functional is not Lorentz invariant. This happens because the the Lorentz invariance is broken by the background magnetic field, which is held at a fixed value.

We can subject the above results to several checks. First, we extract the Hall conductivity via[3]

$$\sigma^H(\Omega,\mathbf{k}) = \frac{\delta}{\delta E_2(\mathbf{k},\Omega)}\Big\langle J_1(\mathbf{k},\Omega)\Big\rangle \quad (2.97)$$

Second, define the finite frequency and momentum corrections to the Hall conductivity via

$$\sigma^H(\Omega,\mathbf{k}) = \sigma^H(0) + \sigma^H_{k^2}|\mathbf{k}\ell|^2 + \sigma^H_{\Omega^2}\Omega^2 + \ldots \quad (2.98)$$

According to the Ref. [25] there is a relations between Ω^2 and $|\mathbf{k}\ell|^2$ coefficients of finite frequency and momentum Hall conductivity given by

$$2\sigma^H_{k^2} + v_F^2\sigma^H_{\Omega^2} = \frac{\mathcal{S}\ell^2}{4\pi}\,, \quad (2.99)$$

2. See the appendix A.5 for more detail

3. We choose to define the Hall conductivity $\sigma^H(\mathbf{k},\Omega)$ as the variation of the current over the electric field, because the naively defined off-diagonal component of the conductivity tensor, $\sigma^{12}(\mathbf{k},\Omega) = \frac{\Pi^{12}(\mathbf{k},\Omega)}{i\Omega}$, is divergent in the zero-frequency limit.

where $\mathcal{S} = N(N+1)$ is the relativistic version of the Shift [99] of the integer quantum Hall state of Dirac electrons at filling fraction $\nu = N$. We have checked explicitly that Eq. (2.99) holds.

Next, we compare the polarization tensors for the Dirac and non-relativistic electrons in the large N limit. Using the results of Section 2.2 we have in the non-relativistic case

$$\Pi^{12}_{\rm nr}(\Omega, \mathbf{k}) = -i\Omega \frac{N}{2\pi} + i\Omega k_1^2 \ell^2 \frac{3N^2}{8\pi} - i\Omega^3 \ell^2 \frac{N^2}{\pi v_F^2} + \cdots , \tag{2.100}$$

$$\Pi^{00}_{\rm nr}(\Omega, \mathbf{k}) = k_1^2 \frac{\ell N^{3/2}}{v_F \sqrt{2}\pi} + \cdots , \tag{2.101}$$

$$\Pi^{11}_{\rm nr}(\Omega, \mathbf{k}) = \Omega^2 \frac{\ell N^{3/2}}{v_F \sqrt{2}\pi} + \cdots , \tag{2.102}$$

$$\Pi^{22}_{\rm nr}(\Omega, \mathbf{k}) = \Omega^2 \frac{\ell N^{3/2}}{v_F \sqrt{2}\pi} - k_1^2 \frac{\ell v_F N^{3/2}}{2\sqrt{2}\pi} \cdots , \tag{2.103}$$

where ω_c can be written in terms of Fermi velocity v_F and Fermi momentum k_F as follows

$$\omega_c = \frac{\bar{B}}{m} = \frac{\bar{B} v_F}{k_F} = \frac{v_F \sqrt{\bar{B}}}{\sqrt{2N}} , \tag{2.104}$$

where we used the relation between filling fraction and Fermi momentum

$$N = \frac{\bar{\rho}}{\bar{B}/2\pi} = \frac{k_F^2}{2\bar{B}}, \tag{2.105}$$

where $\bar{\rho}$ is the non-relativistic electron density.

Using the asymptotic formula for the ζ-function at large N

$$\zeta\left(-\frac{1}{2}, N+1\right) \approx -\frac{2}{3}\left(N + \frac{1}{2}\right)^{3/2} \tag{2.106}$$

we find that the non-relativistic and Dirac polarization tensors agree in leading and sub-leading order in N, provided we replace $N \longrightarrow N + \frac{1}{2}$. The latter replacement comes up due to the contribution of the Berry phase in the Dirac's case. We stress that the equivalence

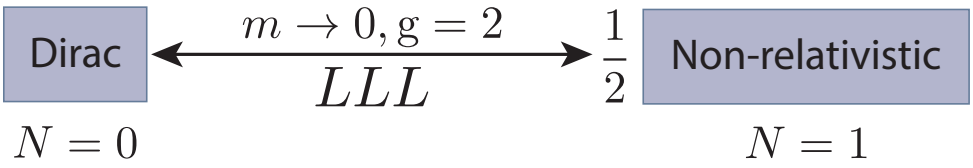

Figure 2.3: The linear response of the LLL is universal. The generating functionals of non-relativistic and Dirac (upon discarding divergent terms) electrons differ from each other by a factor of $\frac{1}{2}$ to *all* orders in the gradient expansion.

holds when the g-factor of the non-relativistic electrons vanishes and does *not* equal to 2 as one may naively expect.

The terms that are sub-sub-leading order in N do not agree, which can be shown by an explicit calculation. The agreement of the leading and sub-leading orders is not surprising, since in large N limit, which is the case of high density and small applied magnetic field, the semiclassical approximation applies equally well to both systems, however Dirac theory has an extra Berry phase contribution. We will study the large N limit in more detail in the Section V.

2.4 Universality of the projected lowest Landau level

In this Section we will show that the exact electromagnetic linear response functions of the lowest Landau level of non-relativistic and Dirac electrons agree to all orders in the gradient expansion, in the limit, where the transitions to higher Landau levels are neglected. The main result of this Section is summarized in Fig 2.3. In the non-relativistic case this limit is accomplished by taking $m \to 0$, keeping magnetic field fixed. Generally, this limit is not well-defined since the exact degeneracy of the LLL is split in inhomogeneous magnetic field, however when $\mathrm{g} = 2$ the LLL is exactly degenerate for any smooth, inhomogeneous background of magnetic field. In the relativistic case this limit is taken via sending $v_F \to \infty$. Both limits send the spectral gap to infinity supressing the contributions of the higher Landau levels. Note that LLL means $N = 1$ for the non-relativistic case and $N = 0$ for Dirac.

It was demonstrated in Ref.[[93]] that the non-relativistic action (2.6) reduces to (2.74)

in the limit $m \to 0$, $\mathrm{g} = 2$ and provided that transitions across the gap are neglected. This argument was used to deduce the following relationship between the generating functionals

$$W_{\mathrm{D}}[\delta A_\mu] = W_{\mathrm{nr}}[\delta A_\mu] - \frac{1}{2}\frac{1}{4\pi}\int A dA\,. \tag{2.107}$$

This relation holds only in the leading order in the gradient expansion. We will show that there is an exact version of this relation which reads

$$W_{\mathrm{D}}[\delta A_\mu] = W_{\mathrm{nr}}[\delta A_\mu] - \frac{1}{2}W_{\mathrm{nr}}[\delta A_\mu] = \frac{1}{2}W_{\mathrm{nr}}[\delta A_\mu]\,. \tag{2.108}$$

Eq. (2.108) can be understood as follows. Completely filled 0-th Landau level contributes $W_{nr}[\delta A_\mu]$ to the linear response, however the filled negative energy bands contribute total of $-\frac{1}{2}W_{nr}[\delta A_\mu]$, which leads to exact relation (2.113). To prove (2.108) we first turn to the non-relativistic generating functional. In the leading order we have

$$\begin{aligned} W_{\mathrm{nr}}[\delta A_\mu] = \frac{1}{4\pi}\int d^2x dt \Big[& A dA \\ & + \frac{1}{\omega_c}|\vec{\delta}E|^2 - \frac{2-\mathrm{g}}{2m}\delta B^2 - \frac{3-\mathrm{g}}{2}\ell^2 \delta B(\partial_i \delta E_i) + \dots \Big]\,. \end{aligned} \tag{2.109}$$

In the limit $m \to 0, \mathrm{g} = 2$ we find

$$W_{\mathrm{nr}}^{m\to 0}[\delta A_\mu] = \frac{1}{4\pi}\int A dA - \frac{1}{2}\ell^2 \delta B(\partial_i \delta E_i) + \dots\,. \tag{2.110}$$

In fact, by dimensional analysis and the regularity of the massless limit only the terms linear in the electric field survive[4]. These terms contribute to the (momentum dependent) Hall conductivity, which can be calculated exactly at zero frequency [10]

$$\sigma_H^{\mathrm{nr}}(k) = \frac{1}{2\pi}\frac{1}{|k\ell|^2}s(k)\,, \tag{2.111}$$

4. This also can be seen, with some work, from the Eq.(2.63)

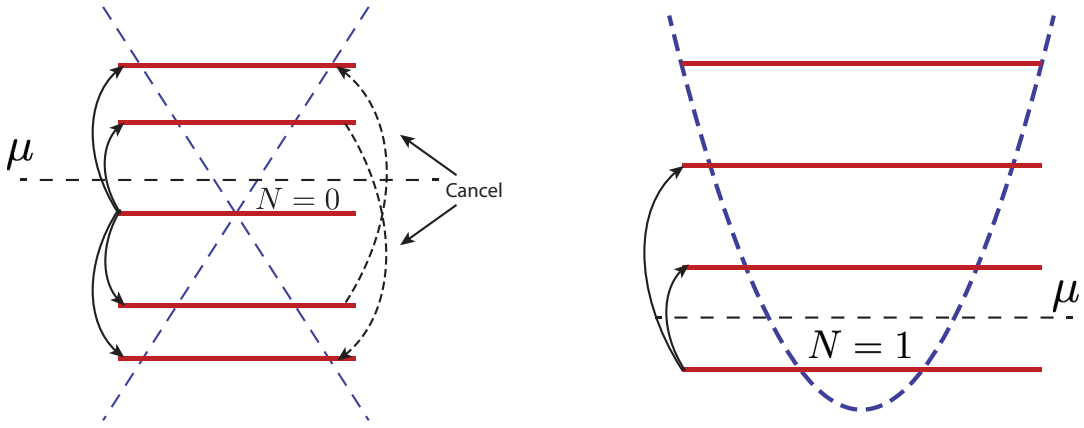

Figure 2.4: **Left,** Spectrum of Dirac operator in magnetic field. Dashed lines illustrate the transitions "across the Fermi sea", while solid lines illustrate the transitions between LLL and the excited levels. Total contribution to the linear response of the LLL of Dirac electrons from the transitions "across the Fermi sea" adds up precisely to 0, leading to an exact relation (2.108). **Right,** Spectrum of non-relativistic electrons in magnetic field. Transitions that contribute to the electromagnetic response of the LLL are qualitatively the same is in the Dirac case. The presence of the filled Dirac sea results in the overall factor of 1/2 in the Hall conductivity of Dirac electrons.

where $s(k) = 1 - e^{-\frac{|k\ell|^2}{2}}$ is the static structure factor[22]. Eq.(2.111) agrees with the results produced from (2.63) upon setting the frequency $\omega = 0$.

The Dirac electrons are more tricky since in addition to the contribution of the LLL there are also, in principle, contributions from the transitions across the Dirac sea as illustrated on Fig. 2.4. We have checked that these transitions sum up to zero in $|k\ell|^0, |k\ell|^2$, $|k\ell|^4$ and $|k\ell|^6$ orders of the momentum expansion. If we assume that these transitions do not contribute to *all* orders in the gradient expansion then the Hall conductivity is given by (using (2.89)-(2.90))

$$\sigma_H^D(k) = \frac{e^{-\frac{|k\ell|^2}{2}}}{2\pi} \sum_{n=1}^{\infty} \frac{2^{-n-1}}{n!} |k\ell|^{2n-2} = \frac{\left(1 - e^{-\frac{|k\ell|^2}{2}}\right)}{4\pi |k\ell|^2} \tag{2.112}$$

which leads to the exact relationship

$$\sigma_H^D(k) = \frac{1}{2} \sigma_H^{\rm nr}(k)\,, \tag{2.113}$$

which is equivalent to (2.108), provided we neglect the LL mixing (that is take g $= 2, m \to 0$ in the r.h.s. and $v_F \to \infty$ in the l.h.s.).

There is, however, a subtlety as we have not explained how to take the infinite gap limit of the Dirac generating functional (2.96). Since the energy levels are given by $E_n = \pm v_F\sqrt{2\bar{B}n}$ we should take the limit $v_F \to \infty$ which removes all of the energy levels except E_0. In this limit the term quadratic in the electric field indeed vanishes and the term linear in the electric field survives, however the term quadratic in magnetic field diverges linearly with v_F. In the non-relativistic case this term was removed by an appropriate choice of the g-factor g $= 2$, but in the Dirac case there is no such mechanism. Thus, in order to ensure the regularity of the $v_F \to \infty$ limit we must subtract this term "by hand". The regular part of the generating functional for Dirac electrons then satisfies (2.108).

2.5 Universality of the Large N limit

In this Section, we calculate the polarization tensor in the large N limit and then re-derive it using the semiclassical approximation. The result of the semiclassical calculation agrees with previous work [91], but we will present a simpler method of calculation. Furthermore, we show that in the large N limit, the result of the RPA calculation agrees with the Fermi liquid theory[76, 72, 53].

2.5.1 Large N Limit of RPA

Working at large filling factors N means that we consider a regime in which the density of electrons is much bigger than the external magnetic field

$$N = \frac{\bar{\rho}}{\bar{B}/2\pi} \gg 1. \tag{2.114}$$

Non-interacting electrons in a weak magnetic field form a Fermi sphere. Furthermore, large filling also implies $k_F\ell \gg 1$. Therefore the gradient expansion in k is valid in the range of

momenta that satisfy $k\ell\sqrt{N} \sim 1$, which is the right regime for Landau's Fermi liquid theory.

First, we will explicitly take the large N limit of the RPA result (2.63). We will use the asymptotic form of Laguerre polynomial (valid in the leading order in N)

$$\lim_{N\to\infty} L_N^{\alpha}(x) \approx \frac{N^{\alpha/2}}{x^{\alpha/2}} e^{\frac{x}{2}} J_\alpha(2\sqrt{Nx})\,. \tag{2.115}$$

In the expression (2.63) only the terms for $n \approx n'$ contribute to the final result. This remains true for any N. Thus in the following we will use approximation

$$\frac{n'!}{n!} \approx \frac{1}{N^{n-n'}}\,, \tag{2.116}$$

that is valid for $n \approx n' \approx N$.

The generating function $\mathcal{G}^{as}(k,k',N)$ in the large N limit takes form

$$\mathcal{G}^{\rm as}(k,k',N) = \sum_{n=1}^{\infty} n\,(-1)^n \frac{e^{\frac{\ell^2(|k'|^2+|k|^2)}{4}} \left(\frac{(k\bar{k}')^n}{n-\omega} + \frac{(\bar{k}k')^n}{n+\omega}\right) J_n\left(2\sqrt{N\frac{k\bar{k}\ell^2}{2}}\right) J_n\left(2\sqrt{N\frac{k'\bar{k}'\ell^2}{2}}\right)}{(k\bar{k})^{n/2}(k'\bar{k}')^{n/2}}, \tag{2.117}$$

where ω is the dimensionless frequency.

The asymptotic form of generating function $\mathcal{G}^{as}(k,k',N)$ agrees with the exact generating function $\mathcal{G}(k,k',N)$ up to sub-leading order in N, which can be checked order by order in the momentum expansion. We choose a frame where $\mathbf{k} = (k_1, 0)$, in this case $k = \bar{k} = k_1$. It will be convenient to use the rescaled momentum $q = k_1\ell\sqrt{2N} = k_1 k_F \ell^2$.

Finally, using Eq.(2.117) together with (2.63), we obtain the polarization tensor for any

g-factor

$$\Pi^{11}(q,\omega) = -\frac{\bar{\rho}}{m} + \sum_{n=1}^{\infty} -\frac{n^4\omega_c[J_n(q)]^2}{2\pi N(\omega^2-n^2)}, \tag{2.118}$$

$$\Pi^{22}(q,\omega) = -\frac{\bar{\rho}}{m} + \sum_{n=1}^{\infty} -\frac{Nn^2\omega_c[J_{n-1}(q)-J_{n+1}(q)]^2}{2\pi(\omega^2-n^2)}$$
$$-\mathrm{g}\sum_{n=1}^{\infty}\frac{n^2\omega_c q[J_{n-1}(q)-J_{n+1}(q)]J_n(q)}{4\pi(\omega^2-n^2)} - \mathrm{g}^2\sum_{n=1}^{\infty}\frac{q^2n^2\omega_c[J_n(q)]^2}{32N\pi(\omega^2-n^2)}, \tag{2.119}$$

$$\Pi^{12}(q,\omega) = \sum_{n=1}^{\infty}\frac{iNn^2\omega\omega_c J_n(q)q[J_{n-1}(q)-J_{n+1}(q)]}{\pi(\omega^2-n^2)} + \mathrm{g}\sum_{n=1}^{\infty}\frac{i\omega n^2\omega_c[J_n(q)]^2}{4\pi(\omega^2-n^2)}. \tag{2.120}$$

Note that using Eqs.(2.71)-(2.73) one can restore all of the components of the polarization tensor at vanishing g-factor.

Remarkably, the infinite sums for each component of the non-relativistic polarization tensor can be evaluated in a closed form. The details of the calculation are presented in the Appendix A.6. The results are written for the conductivity tensor (2.5)

$$\sigma^{11}(q,\omega) = \frac{iN}{\pi}\left(-\frac{\omega}{q^2} + \frac{\pi\omega^2 J_\omega(q)J_{-\omega}(q)}{q^2\sin(\pi\omega)}\right), \tag{2.121}$$

$$\sigma^{22}(q,\omega) = \frac{iN}{\pi}\left(-\frac{\omega}{q^2} + \frac{\pi\omega^2 J_\omega(q)J_{-\omega}(q)}{q^2\sin(\pi\omega)} + \frac{\pi J_{1+\omega}(q)J_{1-\omega}(q)}{\sin(\pi\omega)}\right) + \frac{igq}{8\sin(\pi\omega)}\frac{\partial}{\partial q}\left[J_\omega(q)J_{-\omega}(q)\right]$$
$$-\frac{ig^2q^2}{64\pi N\omega}\left(1 - \frac{\pi\omega}{\sin(\pi\omega)}J_\omega(q)J_{-\omega}(q)\right) \tag{2.122}$$

$$\sigma^{12}(q,\omega) = -N\frac{\omega}{2q}\frac{1}{\sin(\pi\omega)}\frac{\partial}{\partial q}\left(J_\omega(q)J_{-\omega}(q)\right) + \frac{\mathrm{g}}{8\pi}\left(1 - \frac{\pi\omega}{\sin(\pi\omega)}J_\omega(q)J_{-\omega}(q)\right).$$

Eqs.(2.121) - (2.123) are the main result of this Section. Next we will compare these results to a semiclassical computation.

2.5.2 Semiclassical Computation

Review of the Fermi liquid theory with a g-factor

In this Section, we review the derivation of Boltzmann's equation mostly to fix the notation. The derivation follows closely to the bosonization of Fermi liquid [12, 31, 78, 87]. We assume a system of two dimensional non-interacting spinless fermions with Fermi momentum k_F

and mass m in magnetic field $B(x,t) = \bar{B} + b(x,t)$ and electric field $\vec{E}(x,t)$. We will assume that $b(x,t)$ and $\vec{E}(x,t)$ are weak and slowly varying. We denote the distribution function as $f(\mathbf{K},\mathbf{x},t)$. The collective modes are described by the perturbation of distribution function

$$f(\mathbf{K},\mathbf{x},t) = f^0(\mathbf{K}) + \delta f(\mathbf{K},\mathbf{x},t). \tag{2.123}$$

The perturbations of the distribution function caused by weak fields are also assumed to be weak. Where the unperturbed distribution function is

$$f^0(\mathbf{K}) = \Theta(k_F - |\mathbf{K}|), \tag{2.124}$$

where $\Theta(x)$ is the step function. Employing the collisionless limit of the Boltzmann equation[100, 77, 44], we obtain the time evolution equation for the distribution function $f(\mathbf{K},\mathbf{x},t)$.

$$\begin{aligned}&\partial_t f(\mathbf{K},\mathbf{x},t) + \vec{v}(\mathbf{K})\cdot\vec{\nabla}_{\mathbf{x}} f(\mathbf{K},\mathbf{x},t)\\ &+\big(\vec{E} + \frac{\mathrm{g}}{4m}\vec{\nabla}_{\mathbf{x}} B(\mathbf{x},t) + \vec{v}(\mathbf{K})\times\vec{B}(\mathbf{x},t)\big)\cdot\vec{\nabla}_{\mathbf{K}} f(\mathbf{K},\mathbf{x},t) = 0,\end{aligned}$$

where $\vec{v}(\mathbf{K}) = \vec{\nabla}_{\mathbf{K}}\epsilon_{\mathbf{K}}$ is the group velocity and $\epsilon_{\mathbf{K}}$ is the non-relativistic dispersion relation. We also introduce a vector, normal to the Fermi surface via $\vec{v}(\mathbf{K}) = v_F\vec{n}_\theta$. Note that we included the term $\frac{\mathrm{g}}{4m}\vec{\nabla}_{\mathbf{x}} B(\mathbf{x},t)\vec{\nabla}_{\mathbf{K}} f(\mathbf{K},\mathbf{x},t)$ which is necessary to account for the finite g-factor.

In the low energy limit we take the momentum to be $|\mathbf{K}| = k_f + u(x,\theta,t)$ (see Fig. 2.5). Then the perturbations of the distribution function occur only close the the Fermi surface

$$\delta f(\mathbf{K},\mathbf{x},t) = u(\theta,\mathbf{x},t)\delta(k_F - |\mathbf{K}|), \tag{2.125}$$

where θ is the direction of $\mathbf{K}$ on the Fermi surface.

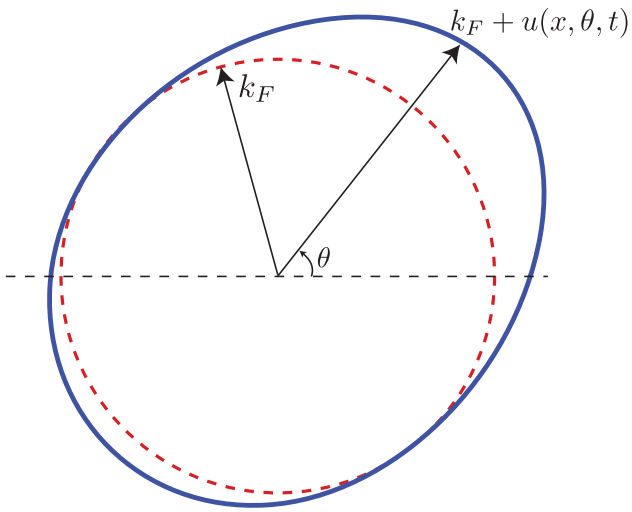

Figure 2.5: Fluctuating Fermi surface. The function $u(x,\theta,t)$ describes the fluctuations of the surface in space and time. The harmonics $u_n(x,t)$ describe the dipolar, quadruploar, etc. deformations of the Fermi surface.

Then Boltzmann equation takes form[24, 68]

$$\begin{aligned}&\partial_t u(\theta,\mathbf{x},t) + v_F \vec{n}_\theta \cdot \vec{\nabla}_{\mathbf{x}} u(\theta,\mathbf{x},t) - \omega_c \partial_\theta u(\theta,\mathbf{x},t) \\ &-\vec{n}_\theta \cdot \left(\vec{E}(\mathbf{x},t) + \frac{g}{4m}\vec{\nabla}_{\mathbf{x}} B(\mathbf{x},t)\right) = 0,\end{aligned} \tag{2.126}$$

where $v_F = \frac{k_F}{m}$ is the Fermi velocity, and $\vec{n}_\theta$ is the normal vector to the Fermi surface. We ignore terms that are second order in $\vec{E}(\mathbf{x},t)$, $\vec{b}(\mathbf{x},t)$ and $\delta f(\mathbf{K},\mathbf{x},t)$. The charge density of the electrons can be written in terms of $u(\theta,\mathbf{x},t)$ as follows

$$\rho(\mathbf{x},t) = \int \frac{d^2\mathbf{K}}{(2\pi)^2} f(\mathbf{K},\mathbf{x},t) = \bar{\rho} + \int d\theta \frac{k_F}{(2\pi)^2} u(\theta,\mathbf{x},t), \tag{2.127}$$

where the background charge density is given by

$$\bar{\rho} = \int \frac{d^2\mathbf{k}}{(2\pi)^2} f^0(\mathbf{K}) = \frac{k_F^2}{4\pi}\,. \tag{2.128}$$

At nonzero g-factor the current density is defined as (cf. Eq. (2.67))

$$J^i(\mathbf{x},t) = \int \frac{d^2\mathbf{K}}{(2\pi)^2} f(\mathbf{K},\mathbf{x},t) v^i(\mathbf{K}) + \frac{\mathrm{g}}{4m}\epsilon^{ij}\partial_j \rho\,.$$

Which in terms of $u(\theta,\mathbf{x},t)$ is given by

$$\begin{aligned} J^i(\mathbf{x},t) &= \frac{k_F v_F}{2\pi}\int \frac{d\theta}{2\pi} n^i_\theta u(\theta,\mathbf{x},t) \\ &\quad + \frac{\mathrm{g}}{4m}\frac{k_F}{2\pi}\epsilon^{ij}\partial_j \int \frac{d\theta}{2\pi} u(\theta,\mathbf{x},t). \end{aligned} \tag{2.129}$$

The equation (2.126),(2.127) and (2.129) are the key ingredients for the semiclassical calculations.

Semiclassical calculation for the non-relativistic polarization tensor

We will work in the temporal gauge $A_0 = 0$. In this gauge the electric field is given by $E_i(\vec{q},\omega) = i\omega A_i(\vec{q},\omega)$. We decompose $u(\theta,\mathbf{x},t)$ into Fourier modes

$$u(\theta,\mathbf{x},t) = \int \frac{d\Omega d^2\mathbf{k}}{(2\pi)^3} \sum_{n=-\infty}^{\infty} u_n(\mathbf{k},\Omega) e^{in\theta} e^{i\mathbf{k}\cdot\mathbf{x}} e^{-i\Omega t}\,. \tag{2.130}$$

Next, we will fix the frame where $\mathbf{k} = (k_1, 0)$ and introduce the notation

$$q = k_1 \ell\sqrt{2N} = k_1 k_F \ell^2\,. \tag{2.131}$$

Then Boltzmann equation (2.126) takes form

$$\begin{aligned} (\omega+n)u_n(q,\omega) \quad & -\tfrac{q}{2}\left(u_{n+1}(q,\omega) + u_{n-1}(q,\omega)\right) + \omega\left[\delta_{n,1}\left(A_z + \mathrm{g}\tfrac{q^2}{32N\omega}(A_z - A_{\bar{z}})\right) + \right. \\ & \left. \delta_{n,-1}\left(A_{\bar{z}} + \mathrm{g}\tfrac{q^2}{32N\omega}(A_z - A_{\bar{z}})\right)\right] = 0. \end{aligned} \tag{2.132}$$

The solution of the above equation of motion for $u_n(q,\omega)$ with $n>0$ and $n<0$ is

$$u_n(q,\omega) = F(q,\omega)J_{n+\omega}(q) \qquad (n>0) \tag{2.133}$$

$$u_n(q,\omega) = (-1)^n G(q,\omega)J_{-n-\omega}(q) \qquad (n<0) \tag{2.134}$$

where $J_\nu(x)$ is the Bessel function of the first kind. The functions $F(q,\omega)$ and $G(q,\omega)$ depend on the external field and are not fixed by the equations for $|n|>1$. We will fix these functions using the equations of motion for u_{-1}, u_0, u_1.

The equation of motion for $u_0(q,\omega)$ gives us

$$\omega u_0(q,\omega) = \frac{q}{2}\left(F(q,\omega)J_{1+\omega}(q) - G(q,\omega)J_{1-\omega}(q)\right). \tag{2.135}$$

Using (2.133),(2.134) and (2.135) in the equation of motion for $u_1(q,\omega)$ and $u_{-1}(q,\omega)$ we find

$$\begin{aligned} F(q,\omega) = \frac{\pi\omega}{\sin(\pi\omega)}\Bigg(&J_{-1-\omega}(q)\left(A_z + \mathrm{g}\frac{q^2}{32N\omega}(A_z - A_{\bar{z}})\right) \\ &+J_{1-\omega}(q)\left(A_{\bar{z}} + \mathrm{g}\frac{q^2}{32N\omega}(A_z - A_{\bar{z}})\right)\Bigg), \end{aligned} \tag{2.136}$$

$$\begin{aligned} G(q,\omega) = -\frac{\pi\omega}{\sin(\pi\omega)}\Bigg(&J_{-1+\omega}(q)\left(A_z + \mathrm{g}\frac{q^2}{32N\omega}(A_z - A_{\bar{z}})\right) \\ &+J_{1+\omega}(q)\left(A_{\bar{z}} + \mathrm{g}\frac{q^2}{32N\omega}(A_z - A_{\bar{z}})\right)\Bigg), \end{aligned} \tag{2.137}$$

where we used the following Bessel function identity

$$J_{1-\omega}(q)J_{1+\omega}(q) - J_{-1-\omega}(q)J_{-1+\omega}(q) = \frac{4\omega\sin(\pi\omega)}{\pi q^2}. \tag{2.138}$$

Functions $u_1(q,\omega)$ and $u_{-1}(q,\omega)$ are then given by

$$u_1(q,\omega) = F(q,\omega)J_{1+\omega}(q), \quad u_{-1}(q,\omega) = -G(q,\omega)J_{1-\omega}(q). \tag{2.139}$$

To calculate the response functions in terms of the applied electric field, we write equation (2.129) in terms of the Fourier modes

$$J^1 = \frac{N\omega_c}{2\pi}(u_1 + u_{-1}), \quad J^2 = \frac{iN\omega_c}{2\pi}(u_1 - u_{-1}) - \frac{igq\omega_c}{8\pi}u_0, \tag{2.140}$$

where $N = \frac{1}{2}k_F^2\ell^2$ is the number of filled Landau levels.

Using Eqs. (2.139) we can derive the current density in terms of vector potential in the usual form

$$J^i(q,\omega) = \Pi^{ij}(-q,-\omega)A_j(q,\omega)\,, \tag{2.141}$$

from where we can extract the polarization tensor which is again given exactly by (2.121)-(2.122) combined with Eqs. (2.71)-(2.73). Reducing Eq.(2.141) to the form of Eqs. (2.121)-(2.122) involves non-trivial manipulations with the Bessel functions. We leave these details to the Appendix A.6. We conclude that the RPA approximation in the large N limit is equivalent to the semiclassical approximation for *any* value of the g-factor.

2.6 Conclusion

We have calculated the electromagnetic response of the IQH states of non-relativistic and massless Dirac electrons to *all* orders in the gradient expansion. In the non-relativistic case we obtained a simple closed form expression (2.63) wich agrees with the one loop calculation from the previous work [60] for non-relativistic electrons as well as general (non-linear) structure of the generating functional [47, 50]. The method we used is extended naturally to the massless Dirac theory in magnetic field. We explicitly check that the polarization tensors of non-relativistic and Dirac electrons match in the large N limit up to substitution $N \to N + 1/2$. The extra $1/2$ is due to the Berry phase of the Dirac cone. Furthermore, in the Dirac case, we checked that the Ω^2 and k^2 corrections to the Hall conductivity satisfy the relation (2.99) imposed by the Lorentz invariance[25].

We have used the semiclassical approximation to calculate the electromagnetic response

function of the Fermi liquid in weak constant background magnetic field, the polarization tensor can be written in a closed form, given in terms of Bessel's functions, and agrees with the previous work [91], however we have used a simpler method of calculation. Our computation can be easily modified to include the effect of short range interactions via introducing the Landau parameters [68]. The results, which include short range interaction, can still be derived in a closed form. Next, we showed explicitly, that the large N limit of RPA calculation in the non-relativistic case matches the semi-classical approximation at the leading and sub-leading order in N, without including the short range interactions. The agreement implies the equivalence of Fermi liquid theory in a weak background magnetic field and large N limit of RPA calculation. Finally, in view of the previous result we see that the Fermi liquid theory must be modified by the Berry phase effects in order to work for the Dirac fermions. This effect can be easily incorporated via the substitution $N \to N + 1/2$.

We expect that our computations will find many applications to quantum Hall physics. The explicit expression for the polarization tensor is necessary in composite fermion [60] and boson [102] approaches to fractional quantum Hall (FQH) states. These results can also serve as a starting point to accounting for lattice, quenched disorder and weak interactions corrections 9to the linear response theory. Moreover, some of the gradient corrections to the transport coefficients, under certain symmetry assumptions, cary universal information about the quantum Hall states[36, 6, 26, 27, 10], thus the knowledge of these corrections as well as general method of their computation is of its own interest. The large N results should be useful in the recently proposed theory of composite fermions[93], where the latter are viewed as neutral Dirac fermions interacting with an internal gauge field. Finally, all of the exact results are useful in testing the recently discovered set of dualities in $2 + 1D$ [62, 86, 43].

CHAPTER 3

PARTICLE HOLE DUALITY: A WAVE FUNCTION APPROACH

3.1 Introduction.

Particle-hole (PH) transformation for fractional quantum Hall (FQH) states was introduced by Girvin [21]. This transformation relates a FQH state at filling fraction ν to a FQH state at filling fraction $1-\nu$. In the absence of Landau level mixing the projected lowest Landau level (LLL) Hamiltonian is PH-symmetric and, therefore, two states related by a PH transformation have the same energy (up to a shift in the chemical potential). Despite the physical clarity of PH-symmetry, the PH-transformed wave functions look quite complicated and are difficult to work with. PH-transformed states contain a different number of particles, have different transport properties and different topological order. In this Letter we will explain that all of the information about PH-transformed state is encoded in the original state, so that both states are a different representation for essentially the same physics. For this reason we feel it is more appropriate to refer to the PH-transformation as a *particle-hole duality* (PHD).

Recent years have also brought the rise of interest in the role of PHD in the problem of the half-filled Landau level. To resolve the issue of the apparent absence of the PH-invariance in the Halperin-Lee-Read [33] theory, Son has proposed a manifestly PH-invariant effective theory of composite fermions with π Berry phase around the composite Fermi surface [93]. This theory can successfully be used to describe Jain states at fillings close to $\nu = 1/2$ and a PH-invariant (or self-dual) version of the Pfaffian state [13, 93], which is a viable candidate for the observed $\nu = 5/2$ plateau [103].

PH-transformation, as defined by Girvin [21], works in flat space and homogeneous magnetic field. It was recently appreciated that placing a FQH state in inhomogeneous background magnetic field and curved geometry allows one to extract considerable information

about the flat space properties of the state [99, 57, 32, 49, 1, 27, 28, 30, 29, 7, 8, 17, 48, 10, 11, 54, 9, 50, 16, 37]. For example, the projected static structure factor (SSF) [22] in leading and sub-leading order in momentum, and long-wave corrections to Hall conductivity and Hall viscosity can be calculated from the properties that become apparent in curved space [10, 6, 36, 27, 92]. Integer quantum Hall states in curved geometry are available in (synthetic) photonic systems [85].

In this Letter, we will use the approach of [49, 10] to extend Girvin's construction to inhomogeneous magnetic field and curved geometry. Next, we will derive several exact relations between Hall conductivity, Hall viscosity, Berry phases, and the SSF of the holomorphic, chiral FQH states and their PH-duals. These relations establish the PHD quantitatively and show that properties of the PH-dual state are completely determined by the original state. The duality is non-trivial since the calculations can be easily done before the PH-transformation, but are difficult to do after.

Under certain assumptions, the long-wave corrections to Hall conductivity, Hall viscosity and the SSF are determined by topological quantum numbers [36, 6, 27, 10, 11]: filling fraction ν, shift $\mathcal{S} = 2\bar{s}$ [99], chiral central charge c_- [45] , and the orbital spin variance $\mathrm{var}(s)$ [29, 8]. We will explain how the topological quantum numbers transform under the PHD and prove that the aforementioned long-wave corrections are still determined by the (transformed) topological quantum numbers, albeit via different relations. We will check the derived relations against the explicit computation of the corresponding quantities for Jain states done in Son's theory of composite fermions and find complete agreement.

3.2 FQH states in inhomogeneous background.

We start with a brief review of the construction [49, 10, 11] of a LLL FQH state in inhomogeneous magnetic field and on a surface with nonuniform spatial curvature. Consider a holomorphic FQH state $\Psi_\nu(\{\xi\})$, where $\{\xi\} = \xi_1, \ldots, \xi_N$ denotes the collection of particle coordinates. The complex coordinate $\xi = x + iy$ will be used to label the particle position in

the plane. We will assume that the magnetic field B is inhomogeneous and the background geometry is curved. Then the unnormalized wavefunction $\Psi_\nu(\{\xi\})$ takes the following form [49, 10]

$$\Psi_\nu(\{\xi\}) = f_\nu(\{\xi\}) e^{\frac{1}{2}\sum_{i=1}^{N} \mathcal{Q}(\xi_i,\bar{\xi}_i)}\,, \tag{3.1}$$

where $f_\nu(\{\xi\})$ is holomorphic in complex coordinates ξ_i, and $\mathcal{Q}$ is the magnetic potential [1] defined by

$$\Delta_g \mathcal{Q} = -2B\,, \tag{3.2}$$

where Δ_g is the Laplace operator for the metric g_{ij}. Throughout the Letter we will fix the coordinates so that $g_{ij} = \sqrt{g}\delta_{ij}$. In these coordinates (also known as the "conformal gauge") the Laplacian is given by $\Delta_g = \frac{4}{\sqrt{g}}\partial_z\partial_{\bar{z}},$. When the magnetic field is homogeneous, but the space is curved, the magnetic potential is given by

$$\mathcal{Q} = -\frac{\mathcal{K}}{2\ell^2}\,, \tag{3.3}$$

where $\mathcal{K}$ is the Kähler potential satisfying $\partial_z\partial_{\bar{z}}\mathcal{K} = \sqrt{g}$, and $\ell = B^{-1/2}$ is the magnetic length. It is of crucial importance that $f_\nu(\{\xi\})$ *does not depend on* $\mathcal{Q}$ or the metric $\sqrt{g}$. This will not be the case for PH-dual states. Such states comprise a proper subset of chiral FQH states, i.e. FQH states with fully chiral effective edge theories [2].

The central object of interest is the logarithm of the normalization factor

$$\mathcal{Z}_\nu[W] = \int [d\xi] |f_\nu(\{\xi\})|^2\, e^{\sum_i W(\xi_i,\bar{\xi}_i)}\,, \tag{3.4}$$

where $[d\xi] = d^2\xi_1 \cdots d^2\xi_N$ with $d^2\xi = dxdy$, and $W = \mathcal{Q} + \log\sqrt{g}$. We assume that for constant magnetic field and flat space, when $W = -|z|^2/2\ell^2$, the state is normalized and $\mathcal{Z}_\nu = 1$. It is not hard to see that $\log \mathcal{Z}_\nu[W]$ is the generating functional of the density

1. We have chosen to use the natural units $e = c = \hbar = 1$

2. More precisely, we call a FQH wave function chiral if it describes a bulk FQH state which possesses a chiral effective edge theory.

correlation functions [10]

$$\langle\rho_\nu(\zeta)\rangle \equiv \langle\Psi_\nu|\rho_\nu(\zeta)|\Psi_\nu\rangle = \frac{1}{\sqrt{g}}\frac{\delta \log \mathcal{Z}_\nu[W]}{\delta W(\zeta)}\,, \tag{3.5}$$

where $\rho_\nu(\zeta) = \frac{1}{\sqrt{g}}\sum_{i=1}^{N_\nu}\delta(\zeta-\xi_i)$ is the density operator, and N_ν is the number of particles in the state Ψ_ν. In writing $\langle\rho_\nu\rangle$ we will always implicitly assume that the expectation value is taken in the state with the filling factor ν.

The second variation produces the connected two-point function [10]

$$\langle\rho_\nu(\zeta)\rho_\nu(\zeta')\rangle_c = \frac{1}{\sqrt{g(\zeta')}}\frac{\delta}{\delta W(\zeta')}\langle\rho_\nu(\zeta)\rangle\,, \tag{3.6}$$

where $\langle\rho_\nu(\zeta)\rho_\nu(\zeta')\rangle_c = \langle\rho_\nu(\zeta)\rho_\nu(\zeta')\rangle - \langle\rho_\nu(\zeta)\rangle\langle\rho_\nu(\zeta')\rangle$. The static structure factor (SSF) is defined as the Fourier transform (q is the dimensionless momentum)

$$s_\nu(q) = \frac{1}{\bar\rho_\nu}\langle\rho_\nu(q)\rho_\nu(-q)\rangle_c\,, \tag{3.7}$$

where $\bar\rho_\nu = \nu/(2\pi\ell^2)$ is the mean electron density in the homogeneous limit (in the bulk of the FQH droplet).

It follows from (3.2) that in flat space, derivatives w.r.t. W and B can be traded with each other. Going to momentum space we recover [10]

$$s_\nu(q) = \frac{q^2}{2}\frac{2\pi}{\nu}\frac{\delta\langle\rho_\nu(q)\rangle}{\delta B(-q)} = \frac{q^2}{2}\frac{\sigma_\nu^H(q)}{\sigma_\nu^H(0)}\,, \tag{3.8}$$

where we used the Stereda formula $\delta\langle\rho_\nu\rangle/\delta B = \sigma_\nu^H$ [94], and the DC Hall conductance $\sigma_\nu^H(0) = \nu/2\pi$.

We will also need to know how the electron density depends on the spatial curvature. This dependence is captured by the function $\eta_\nu(q) = \frac{2\pi}{\nu}\delta\langle\rho_\nu\rangle/\delta R$ [10]. In general, $\eta_\nu(q)$ has

the following momentum expansion

$$\eta_\nu(q) = \frac{\mathcal{S}}{4} - \frac{b}{4\nu} q^2 + O(q^4)\,, \tag{3.9}$$

where the constant b is an *a priori* non-universal parameter. However in the LLL it is determined by the topological quantum numbers [27]

$$b = \nu \bar{s}\,(1-\bar{s}) + \frac{\tilde{c}}{12}\,, \tag{3.10}$$

where $\tilde{c} = c_- - 12\nu\,\mathrm{var}(s)$. It is also known to control the Berry curvature on the moduli space of higher genus surfaces [48]. Note that the kinematic Hall viscosity [3] follows from the zero momentum limit of the curvature response $\eta_\nu^H/\bar{\rho}_\nu = \eta_\nu(0)$ [11]. At the same time using the expression for the scalar curvature $R = -\Delta_g \log\sqrt{g}$, and the general relation, valid for any metric-independent operator $\mathcal{O}$ [10]

$$-\frac{\ell^2}{2}\Delta_g \frac{\delta\langle\mathcal{O}\rangle}{\delta\sqrt{g}} = \left(1 - \frac{\ell^2}{2}\Delta_g\right)\langle\mathcal{O}\rho_\nu\rangle_c \tag{3.11}$$

it is possible to show that [10]

$$s_\nu(q) = \frac{q^2/2}{1+q^2/2}\left(q^2\,\eta_\nu(q) + 1\right)\,. \tag{3.12}$$

These relations imply

$$s_\nu(q) = \frac{1}{2}q^2 + \frac{\mathcal{S}-2}{8}q^4 - \left[\frac{b}{8\nu} + \frac{(\mathcal{S}-2)}{16}\right]q^6 + \ldots\,. \tag{3.13}$$

This relation, together with Eq.(3.10), was derived for the Laughlin states in [10], and conjectured to hold for states of the form (3.1) with $\mathrm{var}(s) = 0$ in [11]. We conjecture that it holds generally for chiral FQH states.

Finally, combining (3.8) and (3.12) we establish an exact relation between Hall conduc-

tivity and $\eta_\nu(q)$

$$\frac{\sigma_\nu^H(q)}{\sigma_\nu^H(0)} = \frac{1}{1+q^2/2}\left(q^2\eta_\nu(q)+1\right) . \tag{3.14}$$

Relations (3.8), (3.12) and (3.14) hold to all orders in q, under the assumption of the absence of Landau level mixing and long-range interactions. Together with (3.10) these relations imply that first 3 terms in the momentum expansion of $\sigma_\nu^H(q)$ and $s_\nu(q)$ are completely determined by the topological quantum numbers.

3.3 Particle-hole transformation in inhomogeneous background.

Following Girvin [21], we define $\Psi_{1-\nu}(\{z\})$ as a state of *holes* at filling ν, which, when viewed as a state of electrons, has filling $1-\nu$. Let $z_1,\ldots,z_M$ be the coordinates of electrons and $\xi_1,\ldots,\xi_N$ be the coordinates of holes. Then the PH dual state is defined as

$$\Psi_{1-\nu}(\{z\}) = \sqrt{\frac{(N+M)!}{N!M!}}\int [d\xi]\Psi_1(\{z\},\{\xi\})\Psi_\nu^*(\{\xi\}) , \tag{3.15}$$

where $\Psi_\nu(\{\xi\})$ is given by (3.1) and Ψ_1 is the $\nu=1$ state. The overall factor is required to ensure that the PH-dual state is normalized to 1 in constant magnetic field and flat space. A defining property of the PH transformation is that it is an involution

$$\Psi_{1-(1-\nu)}(\{\xi\}) = \pm\Psi_\nu(\{\xi\}) . \tag{3.16}$$

Property (3.16) is ensured by the following identity. First, we define an n-particle reduced density matrix [61]

$$\mathcal{P}_\eta^{(n)}(\xi_1,\cdots,\xi_n;\xi_1',\cdots,\xi_n') = \frac{N_\eta!}{n!(N_\eta-n)!}\frac{1}{\mathcal{Z}_\eta[W]}\int [d\hat{\xi}]\Psi_\eta(\hat{\xi}_{n+1},\cdots,\hat{\xi}_{N_\eta},\xi_1,\cdots,\xi_n)$$
$$\times\Psi_\eta^*(\hat{\xi}_{n+1},\cdots,\hat{\xi}_{N_\eta},\xi_1',\cdots,\xi_n') . \tag{3.17}$$

For $\eta = 1$, this density matrix is a projector to the LLL satisfying

$$\Psi_\nu(\{\xi\}) = \int [d\xi'] \mathcal{P}_1^{(N_\nu)}(\{\xi\}, \{\xi'\}) \Psi_\nu(\{\xi'\}) \,. \tag{3.18}$$

Eq. (3.16) follows by applying PH conjugation to $\Psi_{1-\nu}$ and expressing the resulting convolution integral for $\Psi_{1-(1-\nu)}$ in terms of (3.18). We make extensive use of the following formula relating the 2-particle reduced density matrices between PH-dual states [3]

$$\begin{aligned}\mathcal{P}^{(2)}_{1-\nu}(\xi_1,\xi_2;\xi_1,\xi_2) = \quad & \mathcal{P}^{(2)}_1(\xi_1,\xi_2;\xi_1,\xi_2) + \mathcal{P}^{(2)}_\nu(\xi_1,\xi_2;\xi_1,\xi_2) + \frac{1}{2}\mathcal{P}^{(1)}_1(\xi_1;\xi_2)\mathcal{P}^{(1)}_\nu(\xi_2;\xi_1) \\ & + \frac{1}{2}\mathcal{P}^{(1)}_1(\xi_2;\xi_1)\mathcal{P}^{(1)}_\nu(\xi_1;\xi_2) - \frac{1}{2}\langle\rho_1(\xi_1)\rangle\langle\rho_\nu(\xi_2)\rangle - \frac{1}{2}\langle\rho_1(\xi_2)\rangle\langle\rho_\nu(\xi_1)\rangle\end{aligned} \tag{3.19}$$

Integrating over position ξ_2 reduces this to a simple formula relating the electron density (in inhomogeneous background)

$$\langle\rho_\nu\rangle + \langle\rho_{1-\nu}\rangle = \langle\rho_1\rangle \,. \tag{3.20}$$

Equations (3.19) and (3.20) reveal the PHD, and are the central results of the present Letter. Next we will discuss the physical consequences of the duality.

3.4 Particle-hole duality.

The Hall conductivity and curvature response in the PH-dual state can be found using (3.20). Taking a derivative w.r.t. the magnetic field $B(q)$, and applying the Streda formula we obtain an exact relation between the Hall conductivities

$$\sigma^H_\nu(q) + \sigma^H_{1-\nu}(q) = \sigma^H_1(q) \,. \tag{3.21}$$

3. For a proof of this formula, and further discussion of the reduced density matrix, see the Supplemental Material.

Similarly we find

$$\nu\,\eta_\nu(q) + (1-\nu)\eta_{1-\nu}(q) = \eta_1(q)\,. \tag{3.22}$$

These are exact relations connecting the linear response functions of PH conjugate pair states, and hold for *all* LLL wave functions connected by (3.15). They are new results of the present Letter.

Next, we turn to the normalization factor. It follows directly from the definition of the reduced density matrix and the reproducing formula (3.18), as well as the definition of the generating functional (3.4) that

$$\frac{\mathcal{Z}_{1-\nu}}{\mathcal{Z}_\nu} = \mathcal{Z}_1 \quad \Rightarrow \quad \log\mathcal{Z}_{1-\nu} - \log\mathcal{Z}_\nu = \log\mathcal{Z}_1\ , \tag{3.23}$$

where we have dropped the argument of $\mathcal{Z}_\nu$ for brevity. Eq.(3.23) is an *exact* relation between the generating functionals for a pair of PH-dual states. Eq.(3.23) clearly illustrates the duality, and is a new result of the present Letter.

Assuming now that Ψ_ν has the form (3.1), variation of $\log\mathcal{Z}_{1-\nu}$ over $W(\zeta,\bar\zeta)$ is given by

$$\frac{\delta}{\delta W(\zeta)}\log\mathcal{Z}_{1-\nu} = \langle\rho_1\rangle + \langle\rho_\nu\rangle = \langle\rho_{1-\nu}\rangle + 2\langle\rho_\nu\rangle\,. \tag{3.24}$$

We emphasize that since (3.24) does not have the same form as (3.5), the wavefunction $\Psi_{1-\nu}$ does not have the form (3.1). More precisely, we see that $f_{1-\nu}$ *has to* depend on W. In other words, the dual states couple differently to inhomogeneous magnetic field. It appears that the condition for f_ν to be independent of W has to do with the chirality of a state [4]. For instance, all conformal block trial states share this property. This complication indicates that identity (3.23) is not sufficient to extract all of the observables in a PH-dual state in terms of the observables in the original state, because the relationship between the observables and variations of $\log\mathcal{Z}_{1-\nu}$ is more complicated for $\Psi_{1-\nu}$ states.

4. We speculate that this property of f_ν is a sufficient condition for a state to be chiral, but it is not necessary (Jain states being a notable exception)

Now we will derive an analogue of (3.8) for the dual states. We utilize the fact that, by definition, the two-point density correlation function is related to the 2-particle density matrix via

$$\langle \rho_\nu(\zeta)\rho_\nu(\zeta')\rangle = \langle \rho_\nu(\zeta)\rangle \delta(\zeta-\zeta') + 2\mathcal{P}_\nu^{(2)}(\zeta,\zeta';\zeta,\zeta')\,.$$

This allows us to gain insight using the exact formula (3.19). Then the one-particle density matrix for the ν state is known in the translation-invariant limit to be [61]

$$\mathcal{P}_\nu^{(1)}(\zeta;\zeta') = \bar{\rho}_\nu e^{\zeta\bar{\zeta}'/2l^2 - |\zeta'|^2/2\ell^2}. \tag{3.25}$$

Using this in (3.19), we find

$$\langle \rho_{1-\nu}(\zeta)\rho_{1-\nu}(\zeta')\rangle_c = \langle \rho_\nu(\zeta)\rho_\nu(\zeta')\rangle_c + \frac{\bar{\rho}_1 - 2\bar{\rho}_\nu}{\bar{\rho}_1}\langle \rho_1(\zeta)\rho_1(\zeta')\rangle_c\,. \tag{3.26}$$

Taking the Fourier transform, we find a beautiful exact relation between the projected static structure factors $\bar{s}_\nu = s_\nu - s_1$ for a pair of PH-dual states [5]

$$\bar{\rho}_\nu \bar{s}_\nu(q) = \bar{\rho}_{1-\nu}\bar{s}_{1-\nu}(q)\,. \tag{3.27}$$

This relation is another novel result of the Letter. Eq. (3.27) also follows from the relation between the pair-correlation functions [4] for the dual states [6]. We stress here that (3.27) does not require any assumptions such as (3.1) on the wave function other than holomorphicity.

Now we are in position to relate the Hall conductivity to the SSF of the PH-dual state

$$\sigma^H_{1-\nu}(q) = \sigma^H_{1-\nu}(0)\frac{2}{q^2}\Big(s_1(q) - \bar{s}_{1-\nu}(q)\Big)\,. \tag{3.28}$$

The simplest way to obtain (3.28) is to use (3.21) and (3.27), and assume (3.8).

5. Levin and Son have obtained the same relation using a different set of arguments [58]

6. We thank A.C. Balram for bringing [4] to our attention

Next we will derive an analogue of (3.12) for PH-dual states. Using (3.21), (3.22) and (3.14) we find

$$\bar{s}_{1-\nu}(q) = s_1(q) - \frac{q^2/2}{1+q^2/2}(q^2\eta_{1-\nu}(q)+1)\,. \tag{3.29}$$

Excluding $s_{1-\nu}(q)$ from (3.28)-(3.29) we come to a surprising conclusion – the relation between $\sigma^H_{1-\nu}(q)$ and $\eta_{1-\nu}(q)$ is precisely the same as before the PH-transformation (3.14), up to replacing ν by $1-\nu$. In other words, Eq. (3.14) is invariant under PH transformation.

3.5 Berry curvature.

Next we turn to the dependence of the PH-dual states on parameters such as adiabatically varying fluxes of magnetic field or the modular parameter of a torus τ. Denote any of these parameters in complex coordinates as x and $\bar{x}$. Berry curvature can be computed under the assumption that the state Ψ_ν is holomorphic in the coordinates on the parameter space, except for the real-analytic normalization factor. The *normalized* states have the form [83, 8, 48]

$$\psi_\nu\left(\{\xi,\bar{\xi}\};x,\bar{x}\right) = \frac{1}{\sqrt{\mathcal{Z}_\nu[x,\bar{x}]}}\Psi_\nu\left(\{\xi,\bar{\xi}\};x\right)\,. \tag{3.30}$$

Then the holomorphic component of the Berry connection is determined entirely by the normalization factor

$$A_x \equiv i\langle\psi_\nu|\partial_x|\psi_\nu\rangle = \frac{i}{2}\partial_x \log \mathcal{Z}_\nu, \tag{3.31}$$

which follows by using the identity $\partial_x\langle\psi_\nu|\psi_\nu\rangle = 0$ to trade derivatives of Ψ_ν for derivatives of $\mathcal{Z}_\nu$. Thus, for such holomorphic states the Berry curvature is a Kähler form with the Kähler potential $\mathcal{U}_\nu = \log\mathcal{Z}_\nu$, and is given by

$$\Omega_\nu = \frac{i}{2}\left(\partial_x\partial_{\bar{x}}\mathcal{U}_\nu\right)\,dx\wedge d\bar{x}\,. \tag{3.32}$$

This structure is nearly preserved for the PH-dual state. A straightforward calculation shows that the Kähler potential is $\mathcal{U}_{1-\nu} = \log(\mathcal{Z}_1/\mathcal{Z}_\nu)$, which is *not* the logarithm of the normalization as before. Thus, in contrast to the formula (3.23), the Berry curvature obeys

$$\Omega_\nu + \Omega_{1-\nu} = \Omega_1. \tag{3.33}$$

3.6 PHD and Chern-Simons terms.

The first few terms in the long wave expansion of $\sigma^H_\nu(q)$, $s_\nu(q)$ and $\eta_\nu(q)$ are determined by the topological quantum numbers, which appear as the coefficients of the Chern-Simons terms in the effective action [6, 10, 27].

The Chern-Simons part of the effective action is given by [29, 8]

$$\mathcal{W}^\nu_{\rm CS} = \frac{\nu}{4\pi}\int \Big(A + \bar{s}\omega\Big) d\Big(A + \bar{s}\omega\Big) - \frac{\tilde{c}}{48\pi}\int \omega d\omega\,, \tag{3.34}$$

where $\tilde{c} = c_- - 12\nu\,\mathrm{var}(s)$ and other coefficients are the topological quantum numbers discussed in the introduction. We have also introduced ω_μ – a spatial part of the spin connection satisfying $\partial_1\omega_2 - \partial_2\omega_1 = \frac{1}{2}\sqrt{g}R$. This effective action encodes the linear response functions. Notably, the Hall conductance, shift, and the Hall viscosity, averaged over the sample, are given by

$$\sigma^H_\nu = \frac{\nu}{2\pi}\,, \quad \mathcal{S} = \frac{\nu^{-1}N - N_\phi}{\chi/2}\,, \quad \eta^H_\nu = \frac{\bar{s}}{2}\bar{\rho}_\nu + \frac{\tilde{c}}{24}\frac{\chi}{A}\,, \tag{3.35}$$

where A and χ are respectively the area and the Euler characteristic of the sample, N is the number of electrons and N_ϕ is the total magnetic flux in units of the flux quantum. When a FQH state is constructed as a single conformal block in a conformal field theory [63, 8] $\tilde{c} = c_-$. However, in general (and notably for Jain states) $\mathrm{var}(s)$ does not vanish [29].

The action of PH-transformation on the Chern-Simons part of the effective action is

$$\mathcal{W}^{\nu}_{\rm CS} + \mathcal{W}^{1-\nu}_{\rm CS} = \mathcal{W}^{1}_{\rm CS}\,. \tag{3.36}$$

This can be seen as a consequence of the formula for the Berry curvature (3.33) following the arguments of [8]. In addition to $\nu^{PH} = 1-\nu$ it implies

$$\mathcal{S}^{PH} = \frac{1-\nu\mathcal{S}}{1-\nu}\,, \quad \mathrm{var}(s)^{PH} = \frac{\nu}{\nu-1}\left(\frac{(1-\mathcal{S})^2}{4(1-\nu)} + \mathrm{var}(s)\right)\,. \tag{3.37}$$

PHD also transforms the chiral central charge according to $c_-^{PH} = 1 - c_-$, and $\tilde{c}^{PH} = c_-^{PH} - 12\nu^{PH}\mathrm{var}(s)^{PH}$. Curiously, if the initial state had $\mathrm{var}(s) = 0$, then $\mathrm{var}(s)^{PH} \neq 0$, unless $\mathcal{S} = 1$.

These relations are in agreement with the results derived above. For example, using the transformation laws (3.37), combined with (3.10) we can check that relations (3.21) and (3.22) hold at the order q^4 and q^2 correspondingly.

As another example we provide an explicit formula for first two terms in the long-wave expansion of the projected SSF of a PH-dual state

$$\bar{s}_{1-\nu}(q) = \frac{\nu(\mathcal{S}-1)}{8(1-\nu)}q^4 + \frac{(-6b+5\nu-3\nu\mathcal{S})}{48(1-\nu)}q^6 + \dots\,, \tag{3.38}$$

where b is given by (3.10) and all of the topological quantum numbers are known for a large variety of states [29]. Eq. (3.38) follows from using (3.13) in (3.27), and is another new result of the present Letter. For example, it can be readily used to determine the projected SSF for the $\nu = \frac{q-1}{q}$ states, related to the Laughlin states by the PHD. Note that all of the quantum numbers are taken from the state at filling ν. We are not aware of this type of general result in the literature.

Finally, we can ask about the values of the topological quantum numbers for a *self-dual* state which satisfies $\Omega_\nu = \Omega_{1-\nu}$. This state has been previously discussed by Son [93] and

is known as the PH-Pfaffian. We find that it must have $\nu = 1/2$, $\mathcal{S} = 1$, $c_- = 1/2$ and $\mathrm{var}(s) = 0$.

3.7 Jain states.

We apply our relations to the PH-duals of $\nu = \frac{N}{2N+1}$ Jain states with the relevant topological quantum numbers given by

$$\mathcal{S} = N + 2\,, \quad c_- = N\,, \quad \nu\,\mathrm{var}(s) = \frac{N(N^2-1)}{12}\,. \tag{3.39}$$

Then the projected SSF of the PH-dual state is given by

$$\bar{s}_{\frac{N+1}{2N+1}}(q) = \frac{N}{8}q^4 + \frac{N^4+2N^3-2N^2-2N}{48(N+1)}q^6 + \dots\,. \tag{3.40}$$

To the best of our knowledge (3.40) is a new result.

Hall conductivity of Jain's state at the filling factor $\nu = \frac{N}{2N+1}$ and its PH-dual state at the filling factor $(1-\nu) = \frac{N+1}{2N+1}$ can be calculated exactly, in the large N limit using the Dirac composite fermion theory and the results of [69]. In the regime $z = q(2N+1) \sim 1$ [71] the result is

$$\sigma^H_\nu(q) = \frac{((4N+2)^2 - z^2)\left(8N + \frac{2zJ_2(z)}{J_1(z)}\right)}{64\pi(2N+1)^3}\,, \tag{3.41}$$

$$\sigma^H_{1-\nu}(q) = \frac{((4N+2)^2 - z^2)\left(8N + 8 - \frac{2zJ_2(z)}{J_1(z)}\right)}{64\pi(2N+1)^3}\,. \tag{3.42}$$

where $J_\alpha(z)$ is the Bessel function. The correction to this is order $O(N^{-4})$. We have also assumed absence of long-range interactions.

The projected SSF can also be derived exactly using Dirac composite fermion theory [93]

$$\bar{s}_\nu(q) = \frac{z^3((4N+2)^2 - z^2)J_2(z)}{32N(2N+1)^4 J_1(z)}\,, \tag{3.43}$$

$$\bar{s}_{1-\nu}(q) = \frac{z^3((4N+2)^2 - z^2)J_2(z)}{32(N+1)(2N+1)^4 J_1(z)}\,. \tag{3.44}$$

With these expressions at hand we can check that (3.21) holds up to order N^{-2}. It also follows from (3.41)-(3.44) that (3.8) and (3.28) hold in the large N limit at leading and sub-leading orders in N. We emphasize that these are quite non-trivial checks that probe the relations we derived in *all* orders in the momentum expansion.

Long-range interactions can be included in the computation of [71], which leads to the breakdown of relations (3.12) and (3.29) at $O(q^7)$ order and of (3.14) at $O(q^3)$ order.

3.8 Conclusion.

We have presented arguments for the particle-hole duality in the lowest Landau level. This duality implies several exact, non-perturbative relations between the observables in the pair of PH-dual states such as static structure factor, Hall conductivity and response of the electron density to curvature. Our results for the coefficients of the small momentum expansion of these response functions likely do not hold for PH-dual states which are both non-chiral (as defined in [70], see also [32]) – a notable example of such a state is the PH-Pfaffian. However, Eq. (3.27) should be applicable to such states as well. We leave the investigation of general non-chiral states (and in particular the PH-Pfaffian) to future work.

CHAPTER 4

A HIGHER-SPIN THEORY OF THE MAGNETO-ROTONS

4.1 Introduction

Interacting electrons moving in two dimensions in a strong magnetic field can form nontrivial topological states: the fractional quantum Hall liquids [95, 55]. When the lowest Landau level is filled at certain rational filling fractions, including $\nu = N/(2N+1)$ and $\nu = (N+1)/(2N+1)$ (the Jain's sequences), the quantum Hall liquid is gapped, and the lowest energy mode is a neutral mode. Girvin, MacDonald, and Platzman [22] proposed, based on a variational ansatz that the neutral excitation has a broad minimum at $q\ell_B \sim 1$ at the Laughlin plateau $\nu = 1/3$. Several years later, the existence of a neutral mode was confirmed experimentally [75]. Later experiments reveal surprising richness in the structure of the spectrum of neutral excitations. Unexpectedly, the $\nu = 1/3$ state may have more than one branches of excitations [34]. Furthermore, higher in the Jain sequence, i.e., for $\nu = 2/5$, $3/7$, etc., the lowest excitation has been found to have a dispersion with more than one minima [41, 52]. Various theoretical approaches have been brought to the problem of the magneto-roton [**?** 91, 84**?** **?**]. Currently, the most common viewpoint is based on the composite fermion picture of the fractional quantum Hall effect, in which the neutral modes are bound states of a composite fermion and a composite hole.

The notion of the composite fermion is tightly connected to the Halperin-Lee-Read (HLR) field theory [33], proposed as the low-energy description of the half-filled Landau level. Recently, an analysis of the particle-hole symmetry of the lowest Landau level has lead to a revision of the HLR proposal: the low-energy degrees of freedom is now a Dirac composite fermion coupled to a gauge field [93]. Magneto-rotons provide a rare window into the dynamics of a Fermi surface coupled to a gauge field, a long-standing problem of condensed matter physics [2, 79].

None of the previous analytical approaches to the magneto-roton can deal with the non-

Fermi liquid at $\nu = 1/2$, or even with a composite Fermi liquid with general nonzero values of the Landau parameters. In this paper, we develop a theory of neutral excitations in the quantum Hall liquid, reliable in the limit $N \to \infty$ in the Jain's series $\nu = N/(2N+1)$, where quantum Hall plateaux have been found to up to at least $N = 10$ [73]. In this theory, the neutral excitations are viewed as quantized shape fluctuations of the Fermi surface. This interpretation is quite different from what have been suggested so far, and is one with a predictive power. In particular, one can relate the whole dispersion curves of the neutral excitations to the excitation energies at zero momentum. We find that the dispersion curves have deep magneto-roton minima at large N. Remarkably, the momenta at the magneto-roton minima are independent of all microscopic dynamics, and are in quantitative agreement with existing experimental data even for small N.

4.2 Quantizing the shape of the Fermi surface

To find the magneto-rotons we will first bosonize the Fermi surface. This procedure was studied previously [31, 35, 12, 64]. Our approach relies on a commutation algebra of fluctuations of the shape of the Fermi surface, first derived by Haldane [31]. Here we provide a simple semiclassical derivation of this algebra.

We assume that the $\nu = 1/2$ state is gapless and has a Fermi surface with the Fermi momentum p_F, related to the external magnetic field B by $p_F^2 = B$. The Fermi liquid is characterized by the Fermi velocity v_F and the Landau's parameters F_n. The effective mass is defined as $m_* = p_F/v_F$; the Fermi energy scale as $\epsilon_F = v_F p_F$.

In the fractional quantum Hall $\nu = N/(2N+1)$ state, the composite fermions live in a magnetic field $b = B/(2N+1)$, effectively forming an integer quantum Hall state. We are interested in the regime of frequency and momentum of order N^{-1} compared to the Fermi energy and momentum. We now propose that all low-energy excitations can be viewed as deformations of the Fermi surface from the circular shape, which we parameterize by a function $p_F(t, \mathbf{x}, \theta)$ that depends on time and space, and also on the direction in momentum

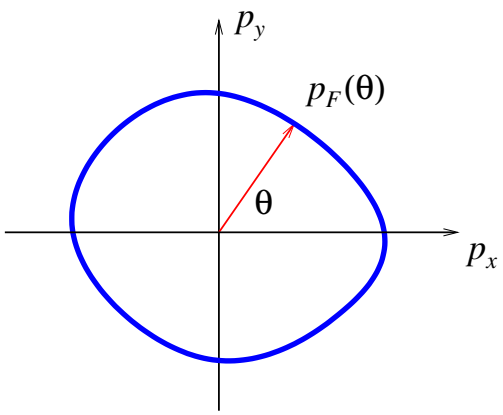

Figure 4.1: A deformed Fermi surface.

space θ ($p_y/p_x = \tan\theta$) (see Fig. 4.1). Furthermore, we decompose the perturbation into different angular momentum channels,

$$p_F(t, \mathbf{x}, \theta) = p_F^0 + u(t, \mathbf{x}, \theta) = p_F^0 + \sum_{n=-\infty}^{\infty} u_n(t, \mathbf{x})\, e^{-in\theta}. \tag{4.1}$$

In the language of Landau's Fermi liquid theory, the state parameterized by $p_F(t, \mathbf{x}, \theta)$ corresponds to a distribution function $n_{\mathbf{p}}(t, \mathbf{x})$ which is one inside the Fermi line and zero outside the line.

We now derive the commutation relation between the u_ns with the following prescription. If we define an operator F (and similarly G) as

$$F = \int \frac{d\mathbf{x}\, d\mathbf{p}}{(2\pi)^2}\, F(\mathbf{x}, \mathbf{p}) n_{\mathbf{p}}(\mathbf{x}), \tag{4.2}$$

where $n_{\mathbf{p}}(\mathbf{x})$ is the quasiparticle distribution function, then we need to impose the condition on the commutation relation so that

$$[F,\, G] = -i \int \frac{d\mathbf{x}\, d\mathbf{p}}{(2\pi)^2}\, \{F,\, G\}(\mathbf{x}, \mathbf{p}) n_{\mathbf{p}}(\mathbf{x}), \tag{4.3}$$

where the $\{F,\, G\}$ is the classical Poisson bracket between F and G,

$$\{F,\, G\} = \frac{\partial F}{\partial p_i}\frac{\partial G}{\partial x_i} - \frac{\partial G}{\partial p_i}\frac{\partial F}{\partial x_i} - b\epsilon^{ij}\frac{\partial F}{\partial p_i}\frac{\partial G}{\partial p_j}, \tag{4.4}$$

where we have allowed the composite fermions to be in an external magnetic field b. For Jain's sequences $b = \pm B/(2N+1)$. Restricting $n_{\mathbf{p}}$ to be of the form of the step function (1 inside the Fermi line, 0 outside), F, G, and the right-hand side of Eq. (4.3) become functionals of the shape of the Fermi surface, and one can easily derive the commutator of the small perturbations u:

$$[u(\mathbf{x},\theta),\, u(\mathbf{x}',\theta')] \;=\; \frac{i(2\pi)^2}{p_F}\left[-n_i(\theta)\frac{\partial}{\partial x_i} \,+\, \frac{b}{p_F}\frac{\partial}{\partial\theta}\right][\delta(\mathbf{x}\,-\,\mathbf{x}')\delta(\theta\,-\,\theta')] \,+\, O(u), \tag{4.5}$$

where $\mathbf{n}(\theta) = (\cos\theta,\, \sin\theta)$. In terms of u_n, the formula reads

$$[u_m(\mathbf{q}),\, u_n(\mathbf{q}')] \;=\; \frac{\pi}{p_F}\left[\frac{2bm}{p_F}\delta_{m+n,0} + \delta_{m+n,1}q_+ + \delta_{m+n,-1}q_-\right](2\pi)^2\delta(\mathbf{q}+\mathbf{q}') + O(u), \tag{4.6}$$

where $q_\pm = q_x \pm iq_y$. This commutation relation has been previously derived in Ref. [31] by extending Tomonaga's bosonization method to higher dimensions. Note that the algebra depends only on the size of the Fermi surface p_F, but not on any dynamic properties (Fermi velocity, Landau's parameters etc.).

4.3 Gauging the Fermi surface

The composite fermion is coupled to a dynamical gauge field. Fermi surface coupled to a gauge field is a long-standing theoretical problem, and the bosonized language allows us to partly address it.

In the bosonic description, the temporal component of the gauge field a_0 is coupled to u_0 and the spatial components are coupled to $u_{\pm 1}$. In the Dirac composite fermion theory,

the leading term in action for a_μ is the Maxwell term. If the dynamical gauge field is at infinitely strong coupling, then the coupling to external field simply imposes the constraints $u_0 = u_{\pm 1} = 0$ as the result of the equations of motion $\delta S/\delta a_\mu = 0$. The assumption of strong gauge coupling should become better and better in the limit $N \to \infty$. This is due to two reasons. Firstly, the coupling of the composite fermions to the gauge field is set at the Fermi energy ϵ_F and momentum p_F, while the scales of interest for our problem are ϵ_F/N and p_F/N. This gauge coupling is relevant for contact and marginal for Coulomb interactions. Secondly, at these low energies the Fermi surface is effectively $O(N)$ fermionic species (corresponding to $O(N)$ patches on the Fermi surface in the renormalization group treatment [78, 87]), boosting the 't Hooft coupling by an additional factor of N. (The argument is more complicated in the case of the HLR theory with a Chern-Simons term in the action for a_μ, but the conclusion is the same).

Hamiltonian and equation of motion.—Assuming the composite fermions form a Fermi liquid with Landau's parameters F_n, the Hamiltonian of the system is

$$H = \frac{v_F p_F}{4\pi} \int d\mathbf{x} \sum_{n=-\infty}^{\infty} (1 + F_n) u_n(\mathbf{x}) u_{-n}(\mathbf{x}), \tag{4.7}$$

where F_n are the Landau parameters. In the case of a marginal Fermi liquid, we may understand by F_n the Landau parameters evaluated at the scale of the energy gap. The Hamiltonian (4.7) and the commutation relations (4.6) form our theory of the neutral excitations in the fractional quantum Hall fluid. This theory involves an infinite number of fields u_n, reminiscent of higher-spin relativistic field theories [19, 18].

Let us first consider zero wavenumber. Then according to Eq. (4.6) the operators u can be divided into pairs of creation and annihilation operators (u_{-2}, u_2), (u_{-3}, u_3) etc, with u_n for $n > 0$ being the annihilation and with $n < 0$, creation operators. The frequency of the oscillators are

$$\omega_n^{(0)} = n(1 + F_n)\omega_c, \quad \omega_c = \frac{b}{m_*}\,. \tag{4.8}$$

The index n can be interpreted as the spin of the excitation. For example, the contribution of spin-n mode to the spectral density of the density operator is expected to be q^{2n} at small n, so the leading contribution to the spectral weight comes from the $n = 2$ mode. The ordering in energy of the modes depend on F_n; in the simplest scenario $n = 2$ is the lowest mode. Since $\omega_c \sim N^{-1}$, and the cutoff of our theory is $O(N^0)$, one should expect $O(N)$ of these modes (provided that F_n does not increase as a power of n).

If one puts $F_n = 0$ in Eq. (4.8) one would find $\omega_n^{(0)} = n\omega_c$. This can be interpreted as the energy of creating a pair of a quasiparticle and a quasihole, separated by n Landau-level steps. Note that the naïve lowest mode with $n = 1$ disappears due to the coupling to the dynamical gauge field [**?**]. As far as we know, Eq. (4.8) does not have a simple interpretation when the Landau parameters are nonzero.

To find the dispersion relation at finite wavenumber q one needs to solve the linearized equation of motion, which can be obtained by taking the the commutator with the Hamiltonian (4.7). In momentum space, choosing $\mathbf{q}$ to point along the x axis, the equation is

$$[\omega - n(1 + F_n)\omega_c]u_n = \frac{v_F q}{2}[(1 + F_{n-1})u_{n-1} + (1 + F_{n+1})u_{n+1}] \quad (4.9)$$

for $n \geq 2$ and $n \leq -2$ and where by construction $u_{\pm 1} = 0$. The task of finding the spectrum of excitations thus reduces to finding the eigenvalues of a certain tridiagonal matrix. Using Eq. (4.8) this equation can be rewritten as

$$(\omega - \omega_n^{(0)})u_n = \frac{2N+1}{2} q\ell_B \left[\frac{\omega_{n-1}^{(0)}}{n-1} u_{n-1} + \frac{\omega_{n+1}^{(0)}}{n+1} u_{n+1}\right]. \quad (4.10)$$

Remarkably, Eq. (4.10) determines completely the dispersion curves from their starting points at $q = 0$. Thus we speculate that Eq. (4.10) is valid even when the $\nu = 1/2$ state is a non-Fermi liquid. For small q the equation can be solved perturbatively over q. For

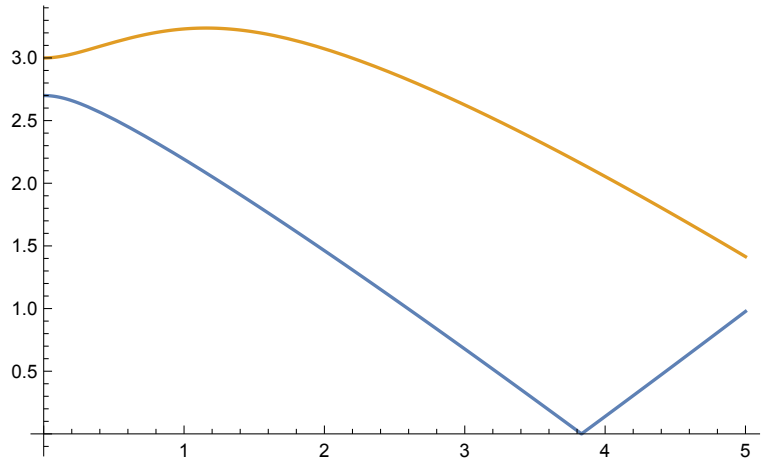

Figure 4.2: The dispersion curves for the lowest two modes for $F_2 = 0.35$, $F_n = 0$ with $n \geq 3$. The horizontal axis is $(2N+1)q\ell_B$ and the vertical axis is the energy in units of ω_c. The cusp at zero energy is an artifact of the infinite N limit.

example, for the $n = 2$ mode we find

$$\frac{\omega_2(q)}{\omega_2^{(0)}} = 1 - \frac{(2N+1)^2}{24\big(1-\omega_2^{(0)}/\omega_3^{(0)}\big)}(q\ell_B)^2 + O(q^4). \tag{4.11}$$

If the spin-2 mode is the lightest one, then its dispersion curve bends down when we go to finite q. Equation (4.11) relates the curvature at $q = 0$ of the lowest mode and the ratio of the energies of the spin-3 and spin-2 modes, and is one prediction of the theory.

It is intriguing that Ref. [34] found two modes at $\nu = 1/3$. While it is tempting to identified them with spin-2 and spin-3 excitations, it is unclear if such identification can be made at such a low value of N, $N = 1$.

4.4 The magneto-roton minima

For $Nq\ell_B \sim 1$ one has to solve the full system of equations, Eq. (4.9) or (4.10), to find the dispersion curves. In Fig. 4.2 we plot a typical result. We note that the energy of the lowest mode goes to zero at a finite momentum. We now show analytically that this always happens at infinitely strong gauge coupling. We need to solve Eq. (4.10) with $\omega = 0$ and the boundary conditions $u_1 = 0$ and $u_n \to 0$ when $n \to \infty$. The solution to this recursion

relation, which satisfies the boundary condition $u_n \to 0$ when $n \to \infty$, is

$$u_n = \frac{(-1)^n}{1+F_n} J_n\Big(\frac{p_F q}{b}\Big). \tag{4.12}$$

The boundary condition $u_1 = 0$ requires $J_1(p_F q/b) = 0$. The latter occurs at $q = z_i b/p_F$ where z_i are the zeros of the Bessel function J_1. One can write this as

$$q\ell_B = z_i \frac{b}{p_F^2} = z_i \frac{b}{B} = \frac{z_i}{2N+1} \tag{4.13}$$

for the filling fractions $\nu = N/(2N+1)$ and $\nu = (N+1)/(2N+1)$.

The fact that the energy of an excitation is exactly zero is an artifact of the strong gauge coupling approximation, which we have argued to occur at infinite N; when the hard constraints on $u_0 = u_{\pm 1} = 0$ are relaxed, these zeros of the dispersion relation should become minima. The values of the energy at the minima are smaller by a power of N compared to the energy scale of the excitations at $q = 0$ ($\omega_n^{(0)}$), but are nevertheless nonzero [1]. This is confirmed in a more detailed treatment of the composite fermions, taking into account the density-density Coulomb interaction [68]. On the other hand, the strict $N = \infty$ limit of infinitely strong gauge coupling allows us to determine analytically the locations of the minima of the dispersion curves. Here we find a surprising result that the positions of the minima on the momentum axis do not at all depend on the parameters appearing in the Hamiltonian [2].

We now show that the robustness of the locations of the magneto-roton minima is due to them being determined by the commutator algebra (4.6), but not by the Hamiltonian. In fact, at the values of q set by Eq. (4.13), there exist a pair of operators, $\hat{O}$ and $\hat{O}^\dagger$, which

1. This qualitative feature has been seen previously in an improved RPA calculations within the HLR theory [91]

2. In Ref. [33] it was been noticed that the minima in the longitudinal conductivity σ_{xx} occur when $(2N+1)q\ell_B$ is near the zeros of the Bessel function J_1.

commutes with all u_n (and consequently with the Hamiltonian) to leading order in u,

$$\hat{O} = \sum_{n=2}^{\infty} (-1)^n J_n\left(\frac{p_F}{b} q\right) u_n. \tag{4.14}$$

In other words, if one defines the commutator matrix C_{mn} as

$$[u_m(\mathbf{q}),\, u_{-n}(\mathbf{q}')] = C_{mn}(2\pi)^2 \delta(\mathbf{q} + \mathbf{q}') \tag{4.15}$$

for $m, n > 0$, where

$$C_{mn} = \frac{2\pi b}{p_F^2} \begin{pmatrix} 2 & z & 0 & 0 & \dots \\ z & 3 & z & 0 & \dots \\ 0 & z & 4 & z & \dots \\ 0 & 0 & z & 5 & \dots \\ \dots & \dots & \dots & \dots & \dots \end{pmatrix}, \quad z = \frac{2N+1}{2} q\ell_B, \tag{4.16}$$

then at the momenta (4.13) the matrix C has a zero eigenvalue. Across these momenta, a role of creation and annihilation operators is exchanged for one pair of operators. It is not difficult to show that any Hamiltonian quadratic in u's needs to have a zero eigenvalue when such an exchange occurs.

The positions of the magneto-roton minima (4.13) and their complete independence of the details of the Hamiltonian are the central result of this paper. In the past, model calculations have shown that the positions of the magneto-roton minima depend very weakly on the interactions (see, e.g., Ref. [**?**]), but the fundamental reason behind this fact was not understood.

It is worth remembering, however, that our derivation requires $q\ell_B \ll 1$, which means that z_i in Eq. (4.13) should be one of the first $o(N)$ roots of J_1. However, the values found in Eq. (4.13) seem to fit existing data quite well even for relatively large $q\ell_B$. Limiting ourselves to the range explored in Ref. [52], $q\ell_B \lesssim 1.2$, our prediction for the locations of the

magneto-roton minima is summarized in the following table (experimental values extracted from Ref. [52] in bracket)

	$n=1$	$n=2$	$n=3$
$\nu=2/5$	0.77 (0.86)		
$\nu=3/7$	0.55 (0.52)	1.00 (1.06)	
$\nu=4/9$	0.43 (0.40)	0.78 (0.85)	1.13 (1.25)

All these values are surprisingly close (within 15% or less) to existing experimental [52] and in numerical [84] results, despite the smallness of N and the large values of the $q\ell_B$ under discussion. Even for $N=1$, the calculated position of the magneto-roton $q\ell_B = 1.28$ is in good agreement with the original estimate of Ref. [22]. We interpret the agreement as confirming the validity of the interpetation of the low-lying neutral excitations as shape fluctuations of the Fermi surface.

Since the locations of the magneto-roton minima depend only on the commutator algebra, which originates from the kinematics of the Fermi surface, rather than from Hamiltonian, we expect the minima would survive even in the non-Fermi-liquid regime of short-ranged electron-electron interactions.

4.5 Conclusion

To summarize, the universal momenta at the magneto-roton minima (4.13), along with the existence of multiple branches of neutral excitations, each with a distinct value of the spin at $q=0$, are the main predictions of this paper. These predictions should be valid in any system described by a Fermi surface coupled to a dynamical gauge field in a small background magnetic field.

CHAPTER 5

LARGE-N APPROACH TO THE QUANTUM HALL EFFECT NEAR HALF FILLING

5.1 Introduction

An interacting two-dimensional electron gas in a strong magnetic field can form nontrivial topologically ordered gapped states, the factional quantum Hall (FQH) states. The composite fermion construction was introduced by Jain to understand strongly correlated electron systems such as in FQH [38, 39]. While the connection between original electrons and composite fermions is unclear, composite fermion has been employed to study fractional quantum Hall (FQH) with tremendous success. The composite fermion construction was used by Halperin, Lee and Read (HLR)[33] to explore the half filling state $\nu = 1/2$, which can be thought of as the large n limit of Jain's sequences $\nu = \frac{n}{2n+1}, \frac{n+1}{2n+1}$. The HLR theory predicts precisely the appearance of the Fermi-liquid behavior of quasi-particle near half filling states.

However, there are still some issues related to the composite fermion model, one of them is the particle-hole (PH) invariant of the lowest Landau level (LLL). The PH symmetry issue was recognized theoretically [46, 56] and experimentally [40, 65]. The idea of Dirac composite fermion proposed recently [93], can be the resolution of this question. In this model, the PH symmetry is satisfied explicitly by the effective theory of composite fermion with π Berry phase around the composite Fermi surface. The Dirac natural of the composite fermion is strongly supported by the numerical work[20].

In work [26], we propose a semi-classical approximation to study the low energy dynamics of Dirac composite fermion model near half filling; we predict fairly accurate the positions of magneto-roton minima. Motivated by the success of this approach, in this paper we expand it to calculate the electromagnetic response of Jain's sequences at long wavelength regime $q\ell_B \sim \frac{1}{2n+1}$. We find that the DC Hall conductivity $\sigma^H(\mathbf{q})$ and projected static structure

factor (SSF) $\bar{s}(q)$ have universal closed forms in the absence of long-range interaction. The results also provide precisely the topological coefficients, which can be compared with a previous work [66]. Moreover, the results show that both states $\nu = \frac{n}{2n+1}$ and $\nu = \frac{n+1}{2n+1}$ saturate the Haldane bound [32]

$$s_4 \equiv \lim_{q\to 0} \frac{\bar{s}(q)}{(q\ell_B)^4} = \frac{|\mathcal{S}-1|}{8}, \tag{5.1}$$

where ℓ_B is the magnetic length and $\mathcal{S}$ is the Wen-Zee shift [99, 83]. This result is derived earlier for a large number of trial states [67]. However, Jain's sequences do not belong to those states. The PH dual relations [66] can be checked explicitly. We also compare the final results with recent work in HLR theory [97].

The plan of the paper is as follows. We will review the Dirac composite fermion model in section 5.2. In this section, we provide the effective field theory action of FQH near half filling. Using Galilean invariant, we argue the requirement of electric dipole term. In the section (5.3), we will derive the Fermi liquid version of Dirac composite fermion. The section (5.4) will be devoted for electromagnetic response calculation of Jain's sequences. We will comment on the results and compare them with HLR theory in the section 5.4.3. Appendices will be devoted to technical details.

5.2 Dirac composite fermion model

5.2.1 Diffeomorphism invariance of LLL electron theory

In flat space, ignoring the Coulomb interaction, the action can be written as

$$S = \int d^3x \frac{i}{2}\left(\psi^\dagger \overleftrightarrow{D}_t \psi + v^i \psi^\dagger \overleftrightarrow{D}_i \psi + v_F \psi^\dagger \sigma^i \overleftrightarrow{D}_i \psi\right) + \int d^3x \left(-\frac{1}{4\pi}\tilde{A}da + \frac{1}{8\pi}\tilde{A}d\tilde{A}\right) \tag{5.2}$$

5.2.2 Electric dipole moment of composite fermion

The field equation of a_0 gives

$$\rho_{CF} = \psi^\dagger \psi = \frac{B}{4\pi}. \tag{5.3}$$

From which we can derive the average composite fermion density and Fermi momentum as

$$\bar{\rho}_{CF} = \frac{\bar{B}}{4\pi} = \frac{k_F^2}{4\pi}, \tag{5.4}$$

$$k_F = \sqrt{\bar{B}} = \frac{1}{\ell_B}, \tag{5.5}$$

where $\bar{B}$ is the average background magnetic field. One can show that, in the long wavelength regime $q\ell_B \sim \frac{1}{2n+1}$ and the regime of random phase approximation (RPA), one can show that

$$\int d^3x \frac{i}{2}\left(\psi^\dagger \overleftrightarrow{D}_t \psi + v^i \psi^\dagger \overleftrightarrow{D}_i \psi + v_F \psi^\dagger \sigma^i \overleftrightarrow{D}_i \psi\right) \approx \int d^3x \frac{i}{2}\left(\psi^\dagger \overleftrightarrow{\tilde{D}}_t \psi + v_F \psi^\dagger \sigma^i \overleftrightarrow{\tilde{D}}_i \psi\right) \tag{5.6}$$

where $\approx$ means equivalent to subleading order in $1/n$ expansion, and

$$\tilde{D}_\mu = \partial_\mu - i\tilde{a}_\mu, \quad \tilde{a}_0 = a_0 + \frac{k_F}{2v_F} v_i v^i, \quad \tilde{a}_i = a_i - \frac{k_F}{v_F} v_i. \tag{5.7}$$

The derivation of equation (5.6) is left for Appendix (C). With the help of (5.6), we can rewrite the action (5.2) as

$$\begin{aligned} S = &\int d^3x \frac{i}{2}\left(\psi^\dagger \overleftrightarrow{\tilde{D}}_t \psi + v_F \psi^\dagger \sigma^i \overleftrightarrow{\tilde{D}}_i \psi\right) \\ &+ \int d^3x \frac{1}{4\pi}\left(-\tilde{A} d\tilde{a} + \frac{1}{2}\tilde{A} d\tilde{A} + \frac{k_F}{2v_F} v_i v^i B - \frac{k_F}{v_F} v_i \epsilon^{ij} \tilde{E}_j\right), \end{aligned} \tag{5.8}$$

where we define

$$\tilde{E}_i = \partial_i \tilde{A}_0 - \partial_0 A_i \tag{5.9}$$

We further define the composite fermion action as

$$S_{CF} = \int d^3x \mathcal{L}_{CF} = \int d^3x \frac{i}{2} \left(\psi^\dagger \overleftrightarrow{\tilde{D}_t} \psi + v_F \psi^\dagger \sigma^i \overleftrightarrow{\tilde{D}_i} \psi \right), \tag{5.10}$$

and the composite fermion current is defined as

$$J^\mu_{CF} = \frac{\delta S_{CF}}{\delta \tilde{a}_\mu}. \tag{5.11}$$

From the action (5.8) and the definition of composite fermion current, we receive the constraint equations J^i_{CF},

$$J^i_{CF} = v_F \psi^\dagger \sigma^i \psi = \frac{\epsilon^{ij} \tilde{E}_j}{4\pi} \tag{5.12}$$

as the field equation of $\tilde{a}_i$. Combining equation (5.3) and (5.12), we see that composite fermion current and density are determined totally by applied electromagnetic field. The continuity equation $\partial_t \rho_{CF} + \partial_i J^i_{CF} = 0$ is fulfilled by the Bianchi identity of $\tilde{A}_\mu$.

The action (5.8) suggests the saddle point value of local velocity $v^i = \frac{\epsilon^{ij} \tilde{E}_j}{B}$ as we expected. This is the velocity of a moving frame, in which electric field $\tilde{E}_i$ vanishes entirely. Substituting the saddle point value of v^i to the action (5.8), we obtain

$$S = \int d^3x \frac{i}{2} \left(\psi^\dagger \overleftrightarrow{\tilde{D}_t} \psi + v_F \psi^\dagger \sigma^i \overleftrightarrow{\tilde{D}_i} \psi \right) + \int d^3x \frac{1}{4\pi} \left(-\tilde{A} d\tilde{a} + \frac{1}{2} \tilde{A} d\tilde{A} - \frac{k_F}{2 v_F B} \tilde{E}_i \tilde{E}^i \right). \tag{5.13}$$

The electromagnetic current can be determined as the variation of the action with respect to external potential

$$J^\mu = \frac{\delta S}{\delta A_\mu}, \tag{5.14}$$

which provides

$$\begin{aligned}\rho =&(1+\frac{1}{4}\ell_B^2\partial_i\partial^i)\left(\frac{B-b}{4\pi}-\partial_k d_k\right),\\ J^i =&\frac{\epsilon^{ij}(\tilde{E}_j-e_j)}{4\pi}+\partial_0 d_i-\frac{\ell_B^2}{4}\partial_0\partial_i\left(\frac{B-b}{4\pi}-\partial_k d_k\right),\end{aligned} \tag{5.15}$$

where we define $b=\epsilon^{ij}\partial_i\tilde{a}_j$, $e_i=\partial_i\tilde{a}_0-\partial_0\tilde{a}_i$, and the definition of electric dipole moment is given as

$$d_i=\frac{k_F}{v_F B}\epsilon_{ij}J^j_{CF}=-\frac{k_F}{4\pi v_F B}\tilde{E}_i. \tag{5.16}$$

The continuity equation $\partial_t\rho+\partial_i J^i=0$ is satisfied automatically by the explicit definition of d_i and Bianchi identities of $U(1)$ gauge fields $\tilde{A}_\mu$ and $\tilde{a}_\mu$. In the low energy limit, composite fermions can be considered as quasi-particle will effective mass $m_*=\frac{k_F}{v_F}$, thus the dipole moment can be rewritten as $\mathbf{d}=\ell_B^2\mathbf{p}\times\hat{\mathbf{z}}$, in which $\mathbf{p}=m_*\mathbf{J}_{CF}$ is the momentum density of composite fermion. This relation can be realized as the electric dipole moment is perpendicular to the momentum, which implies the idea of Read [81] and Zhang [101]. The picture of the electric dipole moment of the composite fermion is employed recently by Wang and Senthil [98] to derive the π Berry phase of composite fermion around the composite Fermi surface. However, they obtain the local Berry phase in contrast with global Berry phase of Dirac composite fermion model. The dipole term is suggested to be the crucial ingredient of the LLL physics near half-filling [96, 88]. In Dirac composite fermion model, the dipole term appears as the requirement of Galilean invariant.

5.3 Semi-classical approximation

In this section, we will derive the semi-classical formalism to calculate electromangetic response of Dirac composite fermion model in long wavelength approximation.

5.3.1 Boltzmann equation for Dirac composite fermion

To make analyze the equation of motion along with the boundary conditions we look at an alternative derivation via the quantum Boltzmann equation. We consider composite fermion has finite densiry $\bar{\rho}_{CF} = \frac{k_F^2}{4\pi} = \frac{\bar{B}}{4\pi}$ under backround magnetic field $\bar{b}$. The Dirac composite fermion of Jain's sequences $\nu = \frac{n}{2n+1}$ and $\nu = \frac{n+1}{2n+1}$ correspond to $\bar{b} = \frac{\bar{B}}{2n+1}$ and $\bar{b} = -\frac{\bar{B}}{2n+1}$ respectively. In the large n limit, $\bar{b} \ll k_F^2$, and long wavelength regime, $q \ll k_F$, we are able to employ the Fermi-liquid formalism [69]. Under those conditions, the composite fermions form two-dimensional Fermi-surface with radius k_F. The Fermi-liquid feels the magnetic field $b(x,t)$ with average value $\bar{b}$ and perturbation $\delta b = \epsilon^{ij}\partial_i \delta\tilde{a}_j$, in additional with with electric field $e_i = \partial_i \delta\tilde{a}_0 - \partial_0 \delta\tilde{a}_i$. The derivation follows closely to the bosonization of Fermi liquid [12, 31, 78, 87]. We will assume that $\delta b(x,t)$ and $\vec{e}(x,t)$ are weak and slowly varying. The low energy physics is described by distribution function

$$n_{\mathbf{k}}(t,\mathbf{x}) = n^0(\mathbf{k}) + \delta n_{\mathbf{k}}(t,\mathbf{x}) \tag{5.17}$$

where the fermionic distribution function

$$n^0(\mathbf{k}) = \Theta(k_F - k) \tag{5.18}$$

Using quantum Boltzmann's equation (QBE) in collisionless limit, we have the evolution equation for $\delta n_{\mathbf{k}}(t,\mathbf{x})$

$$\begin{aligned} &\frac{\partial \delta n_{\mathbf{k}}(t,\mathbf{x})}{\partial t} + \vec{v}(\mathbf{k}).\vec{\nabla}_{\mathbf{x}} \delta n_{\mathbf{k}}(t,\mathbf{x}) + \vec{e}(\mathbf{x}).\vec{\nabla}_{\mathbf{k}} \delta n_{\mathbf{k}}(t,\mathbf{x}) \\ &+(\vec{v}(\mathbf{k}) \times \vec{b}(\mathbf{x})).\vec{\nabla}_{\mathbf{k}} \delta n_{\mathbf{k}}(t,\mathbf{x}) + \vec{v}(\mathbf{k}).\vec{e}(\mathbf{x}) \left(\frac{\partial n^0(\mathbf{k})}{\partial \epsilon_{\mathbf{k}}}\right) = 0 \end{aligned} \tag{5.19}$$

where $\vec{v}(\mathbf{k}) = \vec{\nabla}_{\mathbf{k}} \epsilon_{\mathbf{k}}$ is the group velocity and $\epsilon_{\mathbf{k}}$ is the dispersion relation. We also consider the perturbation of distribution function is closed to the Fermi surface (low energy modes)

by substitution

$$\delta n_{\mathbf{k}}(t,\mathbf{x}) = u(\theta,\mathbf{x},t)\delta(k_F - k) \tag{5.20}$$

in which θ is the direction of $\mathbf{k}$ on the Fermi surface. We can rewrite the Boltzmann equation in term of $u(\theta,\mathbf{x},t)$

$$\frac{\partial u(\theta,\mathbf{x},t)}{\partial t} + v_F\vec{n}_\theta.\vec{\nabla}_{\mathbf{x}}u(\theta,\mathbf{x},t) - \frac{v_F}{k_F}\bar{b}\frac{\partial u(\theta,\mathbf{x},t)}{\partial\theta} - \vec{n}_\theta.\vec{e}(\mathbf{x}) = 0 \tag{5.21}$$

where we introduced a unit vector, normal to the Fermi surface via $\vec{v}(\mathbf{k}) = v_F\vec{n}_\theta$, we have

$$\frac{\partial u(\theta,\mathbf{x},t)}{\partial t} + v_F\vec{n}_\theta.\vec{\nabla}_{\mathbf{x}}u(\theta,\mathbf{x},t) - \mathrm{sgn}(\bar{b})\omega_c\frac{\partial u(\theta,\mathbf{x},t)}{\partial\theta} - \vec{n}_\theta.\vec{e}(\mathbf{x}) = 0 \tag{5.22}$$

where the cyclotron frequency is defied $\omega_c = \frac{|\bar{b}|}{k_F/v_F}$ and we have ignored the term $\vec{e}.\vec{\nabla}_{\mathbf{k}}\delta n_{\mathbf{k}}(t,\mathbf{x})$ and $(\vec{v}(\mathbf{k}) \times \delta\vec{b}(\mathbf{x})).\vec{\nabla}_{\mathbf{k}}\delta n_{\mathbf{k}}(t,\mathbf{x})$ as the quantities $\vec{e}(\mathbf{x})$ and $\delta\vec{b}(\mathbf{x})$ are small.We also have the relation between $J^\mu_{CF}(\mathbf{x},t)$ and $n_{\mathbf{k}}(t,\mathbf{x})$

$$\rho_{CF}(\mathbf{x},t) = \int \frac{d^2\mathbf{k}}{(2\pi)^2}n_{\mathbf{k}}(t,\mathbf{x}) \tag{5.23}$$

$$J^i_{CF}(\mathbf{x},t) = \int \frac{d^2\mathbf{k}}{(2\pi)^2}n_{\mathbf{k}}(t,\mathbf{x})v^i(\mathbf{k}) = \int \frac{d^2\mathbf{k}}{(2\pi)^2}n_{\mathbf{k}}(t,\mathbf{x})\frac{k^i}{m^*} \tag{5.24}$$

We also use the temporal gauge so that

$$e_i = -\partial_0\delta\tilde{a}_i \qquad b = \bar{b} + \partial_1\delta\tilde{a}_2 - \partial_2\delta\tilde{a}_1 \tag{5.25}$$

We use Fourier transform

$$u(\theta,\mathbf{x},t) = \int \frac{d^3q}{(2\pi)^3}u(\theta,\mathbf{q},\omega)e^{iq_\mu x^\mu} \tag{5.26}$$

And also

$$u(\theta, \mathbf{q}, \omega) = \sum_{n=-\infty}^{\infty} u_l(\mathbf{q}, \omega) e^{il\theta} \tag{5.27}$$

We have the Fourier transformation of (5.22)

$$\begin{aligned} -i\omega u(\theta, \mathbf{q}, \omega) + iv_F \vec{n}_\theta . \vec{q} u(\theta, \mathbf{q}, \omega) \\ -\mathrm{sgn}(\bar{b})\omega_c \frac{\partial u(\theta, \mathbf{q}, \omega)}{\partial \theta} - \vec{n}_\theta . \vec{e}(\mathbf{q}, \omega) = 0 \end{aligned} \tag{5.28}$$

From above equations, we can write down the recursion equation

$$(\omega + \mathrm{sgn}(\bar{b}) l\omega_c) u_l(\tilde{q}, \omega) - \frac{\bar{B}}{|\bar{b}|}\omega_c(\tilde{q}_{\bar{z}} u_{l+1}(\tilde{q}, \omega) + \tilde{q}_z u_{l-1}(\tilde{q}, \omega)) + \omega(\delta_{l,1}\delta\tilde{a}_z + \delta_{l,-1}\delta\tilde{a}_{\bar{z}}) = 0 \tag{5.29}$$

where we defined

$$\delta\tilde{a}_z = \tfrac{1}{2}(\delta\tilde{a}_1 - i\delta\tilde{a}_2) \tag{5.30}$$

$$\delta\tilde{a}_{\bar{z}} = \tfrac{1}{2}(\delta\tilde{a}_1 + i\delta\tilde{a}_2) \tag{5.31}$$

$$\tilde{q}_i = q_i l_B \tag{5.32}$$

We can now include the short range interaction simply by turning the Landau's parameters on [26] :

$$\begin{aligned} (\omega + \mathrm{sgn}(\bar{b}) l(1 + F_l)\omega_c) u_l(\tilde{q}, \omega) - \frac{\bar{B}}{|\bar{b}|}\omega_c(\tilde{q}_{\bar{z}}(1 + F_{l+1}) u_{l+1}(\tilde{q}, \omega) + \tilde{q}_z(1 + F_{l-1}) u_{l-1}(\tilde{q}, \omega)) \\ +\omega(\delta_{l,1}\delta\tilde{a}_z + \delta_{l,-1}\delta\tilde{a}_{\bar{z}}) = 0, \end{aligned} \tag{5.33}$$

with the construction, $F_l = F_{-l}$. The composite fermion current J^i_{CF} can be rewritten as [1]

$$J^1_{CF} = \frac{k_F v_F}{4\pi}(u_1 + u_{-1}), \qquad J^2_{CF} = \frac{ik_F v_F}{4\pi}(u_1 - u_{-1}), \tag{5.34}$$

since under the rotational symmetry of Fermi surface, the contribution from $n^0(\mathbf{k})$ to J^i_{CF} vanishes identically. The composite fermion density as

$$\rho_{CF} = \bar{\rho}_{CF} + \frac{k_F}{2\pi}u_0 \tag{5.35}$$

The continuity equation is nothing but the equation of motion for u_0

$$\omega u_0 - q_{\bar{z}} v_F u_1 - q_z v_F u_{-1} = 0 \tag{5.36}$$

5.3.2 Semi-classical calculation of the electromagnetic response

The process to calculate the electromagnetic response is straight forward as follows. Combining (5.35) and (5.34) with (5.3) and (5.12), we are able to translate the constraint equations for J^μ_{CF} to the ones for u_l

$$\frac{k_F}{2\pi}u_0 = \frac{\delta B}{4\pi} = \frac{\epsilon^{ij}\partial_i \delta A_j}{4\pi}, \tag{5.37}$$

$$\frac{1}{2\pi}\frac{\bar{B}}{|\bar{b}|}\omega_c u_1 = i\frac{\tilde{E}_z}{2\pi}, \tag{5.38}$$

$$\frac{1}{2\pi}\frac{\bar{B}}{|\bar{b}|}\omega_c u_{-1} = -i\frac{\tilde{E}_{\bar{z}}}{2\pi}, \tag{5.39}$$

1. Note that we set $F_1 = 0$ by the default construction, parameter F_1 can be always absorbed to the definition of v_F.

where we define

$$\tilde{E}_z = \frac{1}{2}(\tilde{E}_1 - i\tilde{E}_2), \tag{5.40}$$

$$\tilde{E}_{\bar{z}} = \frac{1}{2}(\tilde{E}_1 + i\tilde{E}_2). \tag{5.41}$$

The equation of motion for u_0 (5.36) is satisfied automatically as the consequence of Bianchi identity of $\tilde{A}_\mu$. One can solve the equation of motion (5.33) associated with constraint equations (5.37),(5.38) and (5.39) to attain a_i and u_n in term of δA_μ. Consequently, we can derive J^μ using (5.15) and (5.16). From the outcome of J^μ, one can read off the electromagnetic response polarization tensor from the definition

$$J^\mu = \Pi^{\mu\nu}\delta A_\nu, \tag{5.42}$$

in the linear response formalism.

5.4 Electromagnetic response of Jain's sequences

In this section, we will calculate the electromagnetic response functions of Jain's sequences, in particularly the DC Hall conductivity and susceptibility as well as the SSF. From the equation (5.33), one can argue that the electromagnetic response depends on Landau parameters F_l which encode the detail of the low energy effective theory phenomenologically. However, we will show that the zero frequency limit, $\omega = 0$, the DC Hall conductivity can be derived in a closed universal form which is independent on F_l. With more detail analysis, one can show that the SSF is also invariant under the changing of F_l. One can read off the topological coefficients ([66]) of the result of DC Hall conductivity and SSF. In the appearance of Coulomb interaction [68], those conclusions need to be modified. However, one can still show that [68] the topological coefficients are unchanged. Employing rotational symmetry, we can take $\mathbf{q} = (q, 0)$, thus $q_z = q_{\bar{z}} = \frac{q}{2}$. To simplified the formulae, we introduce the

notation

$$\Omega = \frac{\omega}{\omega_c}, \qquad z = q\ell_B \frac{\bar{B}}{|\bar{b}|} = q\ell_B(2n+1), \tag{5.43}$$

equation (5.33) transforms to

$$(\Omega + \mathrm{sgn}(\bar{b})l(1+F_l))u_l(z,\Omega) - \frac{z}{2}((1+F_{l+1})u_{l+1}(z,\Omega) + (1+F_{l-1})u_{l-1}(z,\Omega)) \\ +\Omega(\delta_{l,1}\delta\tilde{a}_z + \delta_{l,-1}\delta\tilde{a}_{\bar{z}}) = 0 \tag{5.44}$$

5.4.1 Universal DC Hall conductivity

In the DC regime, $\Omega \to 0$, we substitute

$$\tilde{u}_l(z,\Omega) = u_l(z,\Omega)(1+F_l) \tag{5.45}$$

to the equation (5.44) and obtain

$$\mathrm{sgn}(\bar{b})n\tilde{u}_n(z,0) - \frac{z}{2}(\tilde{u}_{n+1}(z,0)+ \\ \tilde{u}_{n-1}(z,0)) + \frac{\omega_c}{i}(\delta_{n,1}e_z + \delta_{n,-1}e_{\bar{z}}) = 0. \tag{5.46}$$

Since all F_l are absorbed in the definition of $\tilde{u}_l$ and To derive DC Hall conductivity for Jain's sequence $\nu_p = \frac{n}{2n+1}$, we substitute $\mathrm{sgn}(\bar{b}) = 1$, equations (5.46) for $|l| > 1$ are satisfied by the ansartz

$$u_l(z,0) = F(z)J_l(z) \qquad (l > 0), \tag{5.47}$$

$$u_l(z,0) = (-1)^l G(z)J_{-l}(z) \qquad (l < 0), \tag{5.48}$$

where $J_l(z)$ is the Bessel function. The equations (5.46) for $l = \pm 1$ provide

$$e_z = -i\frac{zF(z)J_0(z)\omega_c}{2}, \qquad e_{\bar{z}} = -i\frac{zG(z)J_0(z)\omega_c}{2} \tag{5.49}$$

We apply DC electric field $E_1(z,0)$ by turn on inhomogeneous $\delta A_0(z,0)$, in this case $\delta B = 0$ and

$$\tilde{E}_1 = E_1\left(1 - \frac{z^2}{4(2n+1)^2}\right), \qquad \tilde{E}_2 = 0. \tag{5.50}$$

Using (5.38) and (5.39), we obtain

$$F(z) = \frac{i\tilde{E}_1}{2J_1(z)(2n+1)\omega_c}, \quad G(z) = \frac{i\tilde{E}_1}{2J_1(z)(2n+1)\omega_c}. \tag{5.51}$$

The y component of electromagnetic current $J^2(z,0)$ is determined from (5.15) as[2]

$$J^2(z,0) = -\frac{\tilde{E}_1 - e_1}{4\pi} = \sigma^{21}(z,0)E_1 = -\sigma^H_{\nu_p}(q)E_1, \tag{5.52}$$

where we can derive DC Hall conductivity easily from (5.49), (5.50) and (5.51)

$$\sigma^H_{\nu_p}(q) = \frac{((4n+2)^2 - z^2)\left(8n + \frac{2zJ_2(z)}{J_1(z)}\right)}{64\pi(2n+1)^3}. \tag{5.53}$$

To derive DC Hall conductivity for Jain's sequence $\nu_h = \frac{n+1}{2n+1}$, we substitute $\mathrm{sgn}(\bar{b}) = -1$, then perform similar calculation to obtain

$$\sigma^H_{\nu_h}(q) = \frac{((4n+2)^2 - z^2)\left(8n + 8 - \frac{2zJ_2(z)}{J_1(z)}\right)}{64\pi(2n+1)^3}. \tag{5.54}$$

Providing the results (5.53) and (5.54), we derive the gradient expansion of DC Hall con-

2. $q_2 = 0$ and $\partial_0 d_2 = 0$ in the DC regime

ductivity

$$\sigma^H_{\nu_p}(q) = \frac{1}{2\pi}\frac{n}{2n+1}\left(1+\left(\frac{n}{4}+\frac{1}{16n}\right)(q\ell_B)^2\right)+\mathcal{O}(q^3), \tag{5.55}$$

$$\sigma^H_{\nu_h}(q) = \frac{1}{2\pi}\frac{n+1}{2n+1}\left(1-\left(\frac{n+1}{4}+\frac{1}{16(n+1)}\right)(q\ell_B)^2\right)+\mathcal{O}(q^3). \tag{5.56}$$

Both results (5.55) and (5.56) satisfy the relation

$$\sigma^H_{\nu}(q) = \frac{\nu}{2\pi}\left(1+\frac{\mathcal{S}_\nu-2}{4}(q\ell_B)^2\right)+\mathcal{O}(q^3), \tag{5.57}$$

up to subleading order in $1/n$ expansion. $\mathcal{S}_\nu$ is the Wen-Zee shift for state ν, which is determined previously[66] for both Jain's sequences ν_p and ν_h

$$\mathcal{S}_{\nu_p} = n+2, \qquad \mathcal{S}_{\nu_h} = -n+1. \tag{5.58}$$

The relation (5.57) can be extracted from previous work [92] for FQH using Galilean invariant and the g-factor of non-relativistic electrons takes the value $g = 2$. In other words, the relation (5.57) gives us the evidence of Galilean invariant of Dirac composite fermion model. Using results in the work [66], with some assumptions on LLL wavefunction, one can derive the relation (5.57) easily. However, the particular wavefunctions of Jain's sequences are unknown. With the appearance of Coulomb interaction, one can show that [68] the DC Hall conductivity doesn't have the universal form and depends on both Coulomb interaction coupling and Landau parameter F_l. However, the contribution of Coulomb interaction to DC Hall conductivity appears at $\mathcal{O}(q^3)$. Thus the relation (5.57) is still satisfied.

5.4.2 Static structure factor

Jain's sequence $\nu_p = \frac{n}{2n+1}$. To calculate the susceptibility and static structure factor, we turn on inhomogeneous scalar potential $\delta A_0(z,\omega)$. In this case $\delta B = 0$, and up to

subleading order in $1/n$ expansion

$$\tilde{E}_1 = i\frac{z}{\ell_B(2n+1)}\delta A_0, \qquad \tilde{E}_2 = 0. \tag{5.59}$$

We can derive

$$u_1 = -\frac{q\ell_B^2 k_F}{2v_F}\delta A_0 = -\frac{z}{2\ell_B(2n+1)^2\omega_c}\delta A_0 \tag{5.60}$$

$$u_{-1} = \frac{q\ell_B^2 k_F}{2v_F}\delta A_0 = \frac{z}{2\ell_B(2n+1)^2\omega_c}\delta A_0 \tag{5.61}$$

Using $\delta B = 0$, and (5.37), we derive the recursion relation (5.33) for $l = \pm 1$

$$(\omega+\omega_c)u_1(z,\omega) - \frac{z\omega_c}{2}(1+F_2)u_2(z,\omega) + \omega\delta\tilde{a}_z = 0 \tag{5.62}$$

$$(\omega-\omega_c)u_{-1}(z,\omega) - \frac{z\omega_c}{2}(1+F_2)u_{-2}(z,\omega) + \omega\delta\tilde{a}_{\bar{z}} = 0 \tag{5.63}$$

From the constraint equations (5.38), (5.39) and the recursion equations (5.33), (5.62) and (5.63) , we can formally derive δa_z and $\delta a_{\bar{z}}$ as

$$\delta a_z(\omega,q) = \frac{1}{\omega}\mathcal{F}(\{F_l\},z,\omega)\delta A_0 \tag{5.64}$$

$$\delta a_{\bar{z}}(\omega,q) = \frac{1}{\omega}\mathcal{G}(\{F_l\},z,\omega)\delta A_0 \tag{5.65}$$

The susceptibility can be derived from (5.15) as the variation of density with respect to δA_0 up to subleading order in $1/n$ expansion

$$\chi_{\nu_p}(q,\omega) = -\frac{1}{4\pi}\left[q\left(\frac{\mathcal{F}(\{F_l\},z,\omega)}{\omega} - \frac{\mathcal{G}(\{F_l\},z,\omega)}{\omega}\right) + \frac{k_F}{v_F}(q\ell_B)^2\right] \tag{5.66}$$

We can evaluate static structure factor directly from susceptibility as

$$\bar{s}_{\nu_p}(q) = -\frac{i}{\bar{\rho}_{\nu_p}} \int \frac{d\omega}{2\pi} \chi_{\nu_p}(\omega + i\varepsilon \mathrm{sgn}(\omega), q), \tag{5.67}$$

with

$$\bar{\rho}_{\nu_p} = \nu_p \frac{\bar{B}}{2\pi} = \frac{n}{2n+1} \frac{1}{2\pi \ell_B^2} \tag{5.68}$$

where ε is an infinitesimal positive number. The integral can be done by evaluated the residue of integrand at all positive pole of ω,

$$\bar{s}_{\nu_p} = -\frac{1}{\bar{\rho}} \sum_{\omega_i > 0} Res(\chi_{\nu_p}(\omega, q), \omega_i) \tag{5.69}$$

By the Kohn's theorem and the explicit calculation[68], the leading order in momentum expansion of susceptibility is q^4, therefore the last term in (5.66) will be canceled by the explicit form of the first and the second term. Also from the constraint (5.37),(5.38) and (5.39), we do not expect the pole of $\Pi^{\mu,\nu}(\omega, q)$ at $\omega = 0$ and $\omega = \pm\omega_c$, in fact the dispersion relation and the explicit form of response functions [68, 24] confirm this. Therefore the $\frac{1}{\omega}$ coefficient of the first and the second term cancel each other. From the recursion equation (5.33) for Jain's sequence $\nu_p = \frac{n}{2n+1}$, with assumption $1 + F_l > 0$, the positive ω poles appear in $\mathcal{G}(\{F_l\}, \omega, z)$ and negative ω poles appear in $\mathcal{F}(\{F_l\}, \omega, z)$. Therefore, we can write down $\mathcal{F}(\{F_l\}, \omega, z)$ and $\mathcal{G}(\{F_l\}, \omega, z)$ in the form

$$\mathcal{F}(\{F_l\},\omega,z) = F^{(0)}(\{F_l\},z)\omega + F^{(1)}(\{F_l\},z) + \sum_i \frac{F_i^{(2)}(\{F_l\},z)}{(\omega+\omega_i)} + \sum_{ij} \frac{F_{ij}^{(3)}(\{F_l\},z)}{(\omega+\omega_i)(\omega+\omega_j)} + \cdots , \tag{5.70}$$

$$\mathcal{G}(\{F_l\},\omega,z) = G^{(0)}(\{F_l\},z)\omega + G^{(1)}(\{F_l\},z) + \sum_i \frac{G_i^{(2)}(\{F_l\},z)}{(\omega-\omega_i)} + \sum_{ij} \frac{G_{ij}^{(3)}(\{F_l\},z)}{(\omega-\omega_i)(\omega-\omega_j)} + \cdots , \tag{5.71}$$

with $\omega_i = i(1+F_i)\omega_c$. From the above arguments, we arrive at the constraints

$$q\left(F^{(0)}(\{F_l\},z) - G^{(0)}(\{F_l\},z)\right) = -\frac{k_F}{v_F}(q\ell_B)^2 \tag{5.72}$$

$$F^{(1)}(\{F_l\},z) - G^{(1)}(\{F_l\},z) = 0 \tag{5.73}$$

From recursion relations (5.62), (5.63) and $F_1 = 0$, we see that $F^{(0)}(\{F_l\},z), F^{(1)}(\{F_l\},z)$ and $G^{(0)}(\{F_l\},z), G^{(1)}(\{F_l\},z)$ can be read off directly from u_1 and u_{-1}, all of them are independent of F_l

$$F^{(0)}(z) = -G^{(0)}(z) = -\frac{q\ell_B^2 k_F}{2v_F} \tag{5.74}$$

$$F^{(1)}(z) = G^{(1)}(z) = \frac{z}{2\ell_B(2n+1)^2} \tag{5.75}$$

From analytical property of function $\frac{1}{\omega(\omega-\omega_i)\cdots(\omega-\omega_j)}$, we see that

$$Res(\frac{\mathcal{G}(\{F_l\},z,\omega)}{\omega},0) + \sum_{\omega_i>0} Res(\frac{\mathcal{G}(\{F_l\},z,\omega)}{\omega},\omega_i) = G^{(1)}(z) \tag{5.76}$$

Thus

$$\sum_{\omega_i>0} Res(\frac{\mathcal{G}(\{F_l\},z,\omega)}{\omega},\omega_i) = G^{(1)}(z) - Res(\frac{\mathcal{G}(\{F_l\},z,\omega)}{\omega},0)$$
$$= G^{(1)}(z) - \mathcal{G}(\{F_l\},z,0) \tag{5.77}$$

We see that $\mathcal{G}(\{F_l\},z,0)$ and $\mathcal{F}(\{F_l\},z,0)$ are independent of F_l and can be calculated in closed form similarly as the previous section. Up to subleading order in $1/n$ expansion, we obtain

$$\mathcal{F}(z,0) = \mathcal{G}(z,0) = \frac{z^2}{\ell_B(4(2n+1)^2)}\frac{J_0(z)}{J_1(z)} \tag{5.78}$$

Therefore,

$$\bar{s}_{\nu_p}(q) = -\frac{1}{\bar{\rho}_{\nu_p}}\sum_{\omega_i>0} Res(\chi_{\nu_p}(\omega,q),\omega_i) = \frac{z^3}{8n(2n+1)^2}\frac{J_2(z)}{J_1(z)} \tag{5.79}$$

Jain's sequence $\nu_h = \frac{n+1}{2n+1}$. We can show that the susceptibility of Jain's sequence $\nu_h = \frac{n+1}{2n+1}$ is identical to the one of Jain's sequence $\nu_p = \frac{n}{2n+1}$ as follows. The recursion equation (5.33) with $\mathrm{sgn}(\bar{b}) = -1$ and constraint equations (5.38), (5.39) can be solved from the solution of he recursion equation (5.33) with $\mathrm{sgn}(\bar{b}) = 1$ by the substitutions

$$u_l(q,\omega) \to -u_{-l}(q,\omega), \tag{5.80}$$

$$\delta a_z(q,\omega) \to -\delta a_{\bar{z}}(q,\omega), \quad \delta a_{\bar{z}}(q,\omega) \to -\delta a_z(q,\omega). \tag{5.81}$$

With the help of (5.15), one can show that the electron density is unchanged, since $\delta b(q,\omega) = q(\delta a_{\bar{z}}(q,\omega) - \delta a_z(q,\omega))$ is invariant under substitutions (5.81). We can conclude that

$$\chi_{\nu_p}(q,\omega) = \chi_{\nu_h}(q,\omega) \tag{5.82}$$

which is the signature of PH invariant. The equivalent between $\chi_{\nu_p}(q,\omega)$ and $\chi_{\nu_h}(q,\omega)$ can be extended to the case with Coulomb interaction [68]. The relation (5.82) is obtained

previously using wavefunctional approach [66]. In the work [59], the authors prove this PH invariant relation in more general cases. With no extra work, we can derive the static structure factor of Jain's sequence $\nu_h = \frac{n+1}{2n+1}$

$$\bar{s}_{\nu_h}(q) = \frac{z^3}{8(n+1)(2n+1)^2}\frac{J_2(z)}{J_1(z)}, \tag{5.83}$$

where we used (5.82) and $\bar{\rho}_{\nu_p}/\bar{\rho}_{\nu_h} = n/(n+1)$. We consider the gradient expansion of SSF up to subleading order in $1/n$ expansion

$$\bar{s}_{\nu_p}(q) \approx \frac{(n+1)}{8}(q\ell_B)^4 + \left(\frac{n^3}{48} + \frac{n^2}{24}\right)(q\ell_B)^6 + \mathcal{O}(q^7), \tag{5.84}$$

$$\bar{s}_{\nu_h}(q) \approx \frac{n}{8}(q\ell_B)^4 + \left(\frac{n^3}{48} + \frac{n^2}{48}\right)(q\ell_B)^6 + \mathcal{O}(q^7). \tag{5.85}$$

where $\approx$ is replaced for equivalent up to subleading in $1/n$ expansion.

A comment is in order. There is a well-known bound on the coefficient $\bar{s}_4$ defined as $\bar{s}(q) = \bar{s}_4(q\ell_B)^4 + \ldots$ that is valid for any LLL state [32]

$$\bar{s}_4 \geq \frac{|\mathcal{S} - 1|}{8}, \tag{5.86}$$

where $\mathcal{S}$ is the Wen-Zee shift. The above results of SSF suggest that both Jain's sequences saturate the bound[3]

$$\bar{s}_4 = \frac{|\mathcal{S} - 1|}{8}, \tag{5.87}$$

In which the shift for Jain's sequences are provided in the equation (5.58). Moreover, the q^4 and q^6 coefficients of (5.84) and (5.85) satisfy the independent topological evaluation [66]. In the presence of Coulomb potential, one can still calculate the static structure factor[68], the contribution of Coulomb interaction appears at $\mathcal{O}(q^7)$. Thus, the coefficients of q^4 and q^6 are still universal; they are independent of both Coulomb interaction and Landau parameter

3. Which is called Haldane's bound in the literature.

F_l of Fermi-liquid theory of composite fermion.

5.4.3 Compare with HLR theory

We recall the HLR theory given by the Lagrangian Ref[33, 97] with an extra dipole term

$$\mathcal{L} = i\psi^\dagger D_t\psi - \frac{1}{2m}|D_i\psi|^2 + \frac{v^i}{2}\psi^\dagger \overleftrightarrow{D}_i\psi + \frac{1}{2}\frac{1}{4\pi}ada, \tag{5.88}$$

where we define $D_\mu = \partial_\mu - iA_\mu + ia_\mu$. ψ in HLR theory is non-relativistic fermion. One can do the substitution

$$a'_\mu = -a_\mu + A_\mu, \tag{5.89}$$

to derive the new Lagrangian

$$\begin{aligned}\mathcal{L} =& i\psi^\dagger D'_t\psi - \frac{1}{2m}|D'_i\psi|^2 + \frac{v^i}{2}\psi^\dagger\overleftrightarrow{D}'_i\psi \\ &+ \frac{1}{2}\frac{1}{4\pi}a'da' - \frac{1}{4\pi}Ada' + \frac{1}{8\pi}AdA,\end{aligned} \tag{5.90}$$

where $D'_\mu = \partial_\mu - ia'_\mu$. We can shift again

$$\tilde{a}'_0 = a'_0 - \frac{1}{2}mv_iv^i, \tilde{a}'_i = a'_i + mv_i \tag{5.91}$$

to obtain

$$\begin{aligned}\mathcal{L} =& i\psi^\dagger\tilde{D}'_t\psi - \frac{1}{2m}|\tilde{D}'_i\psi|^2 + \frac{1}{2}\frac{1}{4\pi}\tilde{a}'d\tilde{a}' - \frac{1}{4\pi}Ad\tilde{a}' + \frac{1}{8\pi}AdA \\ &+ \frac{1}{4\pi}\left(\frac{1}{2}mv_iv^i(B-b) - mv_i\epsilon^{ij}(E_j - e_j) - m^2v_1\partial_tv_2\right),\end{aligned} \tag{5.92}$$

where $b = \epsilon^{ij}\partial_i\tilde{a}'_j$, $e_i = \partial_i\tilde{a}'_0 - \partial_t\tilde{a}'_i$ in which the local velocity takes value

$$v^i = \frac{\epsilon^{ij}E_j}{B}, \tag{5.93}$$

gives us

$$\mathcal{L} = i\psi^\dagger \tilde{D}'_t \psi - \frac{1}{2m}|\tilde{D}'_i\psi|^2 + \frac{1}{2}\frac{1}{4\pi}\tilde{a}' d\tilde{a}' - \frac{1}{4\pi}Ad\tilde{a}' + \frac{1}{8\pi}AdA$$
$$\frac{1}{4\pi}\left(-\frac{1}{2}m\frac{E_iE^i(B+b)}{B^2} + \frac{m}{B}E_ie^i + \frac{m^2}{B^2}E_1\partial_0 E_2\right), \tag{5.94}$$

Jain's sequence $\nu_p = \frac{n}{2n+1}$. The composite fermions have average density $\bar{\rho}_{cf} = \frac{n}{2n+1}\frac{B}{2\pi}$, sense residue magnetic field $\bar{b} = \frac{B}{2n+1}$. Thus the composite fermions fill $\nu_{cf} = n$ Landau level. We can integrate out non-relativistic composite fermion, within RPA, in the same method as in the Ref [69]. Then, we integrate out $\tilde{a}'_\mu$ to obtain the generating function of external gauge field $\mathcal{W}[\delta A_\mu]$. The DC Hall conductivity and SSF can be obtained straight forward from the generating function. We report here the results up to sub leading in large n limit

$$\sigma^H_{\nu_p} = \frac{1}{2\pi}\frac{n}{2n+1}\left(1 + \frac{n}{4}(q\ell_B)^2\right) + \mathcal{O}(q^3) \tag{5.95}$$

$$\bar{s}_{\nu_p}(q) \approx \frac{n}{8}(q\ell_B)^4 + \left(\frac{n^3}{48} + \frac{n^2}{24}\right)(q\ell_B)^6 + \mathcal{O}(q^7), \tag{5.96}$$

Jain's sequence $\nu_p = \frac{n+1}{2n+1}$. We can do the similar calculation with one modification, $\nu_{cf} = -(n+1)$ instead of n, and obtain

$$\sigma^H_{\nu_h}(q) = \frac{1}{2\pi}\frac{n+1}{2n+1}\left(1 - \frac{n+1}{4}(q\ell_B)^2\right) + \mathcal{O}(q^3). \tag{5.97}$$

$$\bar{s}_{\nu_h}(q) \approx \frac{n+1}{8}(q\ell_B)^4 + \left(\frac{n^3}{48} + \frac{n^2}{48}\right)(q\ell_B)^6 + \mathcal{O}(q^7). \tag{5.98}$$

Comparing with the results of Dirac composite fermion approach, the Hall conductivities agree up to q^2 in gradient expansion. The static structure factors and therefore susceptibilities differ at q^4. The results of HLR calculation explicitly show that PH invariant relation

(5.82) is violated. The coefficient $\bar{s}_4$ of state ν_p disobeys the Haldane bound since

$$\bar{s}_4 = \frac{n}{8} < \frac{|\mathcal{S} - 1|}{8} = \frac{n+1}{8}. \tag{5.99}$$

5.5 Conclusion

We develop a the semi-classical approximation approach for Jain's sequences near half filling. We calculate DC Hall conductivity and SSF in universal closed form. The results are independent on phenomenological parameters, in the absence of Coulomb interaction . We successfully reproduce the topological coefficients of Jain's sequences in the large n limit. The results satisfy PH symmetry in closed form. The SSF outcomes suggest that Jain's sequences saturate Haldane's bound. The comparision with HLR theory is provided.

APPENDIX A

GREEN'S FUNCTION FORMALISM OF INTEGER QUANTUM HALL

A.1 Generating functional summary

For the reader's convenience we list together all of the final expressions derived in the Section II in terms of dimensionless momentum $q = \frac{kl}{\sqrt{2}}$ and for arbitrary g-factor.

$$\delta S_{\rm g} = \frac{\rm g}{4m} \int dt d^2x B \psi^\dagger \psi \,. \tag{A.1}$$

The generating function is given by

$$\mathcal{G}(q, q'; N) = \sum_{n \geq N, n' < N} (-1)^{n-n'} \frac{n'!}{n!} \left(\frac{(q\bar{q}')^{n-n'}}{n - n' - \omega} + \frac{(\bar{q}q')^{n-n'}}{n - n' + \omega} \right) L_{n'}^{n-n'} \left(\frac{|q|^2}{2} \right) L_{n'}^{n-n'} \left(\frac{|q'|^2}{2} \right) , \tag{A.2}$$

The vertices are given by the following relations

$$\hat{\mathcal{P}}^0_{\rm nr}(q) = 1 \,, \tag{A.3}$$

$$\hat{\mathcal{P}}^{\bar{z}}_{\rm nr}(q) = -\frac{1}{2\sqrt{2}m\ell} \left(2\partial_{\bar{q}} - \left(1 - \frac{\rm g}{2} \right) q \right) , \tag{A.4}$$

$$\tag{A.5}$$

where we have also added the dependence on the g-factor that describes the non-minimal coupling of the electrons to the magnetic field due to the intrinsic magnetic moment. The

polarization operator is given by

$$\Pi_{\mu\nu} = \frac{m}{4\pi} e^{-|q|^2} \lim_{q \to -q'} \hat{\mathcal{P}}_\mu(q) \hat{\mathcal{P}}_\nu(q') G(q, q'; N) + \Pi_c^{\mu\nu} \tag{A.6}$$

and

$$W_c^{(2)} = \frac{N\omega_c}{4\pi} \int \frac{d^2\vec{q}}{(2\pi)^2} |\delta A(\vec{q}, 0)|^2 \tag{A.7}$$

A.2 Derivation of (2.50)

Summation over b subspace

The first step in evaluation of (2.50) is to perform the summation over the Fock space generated by $b, b^\dagger$ operators. This can be done easily because the $b, b^\dagger$ operators completely factorize from the expression for the vertices (2.37)-(2.36), because the perturbed action does not depend on b and $(a, a^\dagger)$ commute with $(b, b^\dagger)$. We compute the trace over the Fock spaces (suppressing the frequency integration)

$$\begin{aligned} \mathrm{Tr}_{a,b} G_0 V^{(1)} G_0 V^{(1)} &= \sum_{n,n',m,m'} \langle nm|G_0|nm\rangle \langle nm|V^{(1)}|n'm'\rangle \langle n'm'|G_0|n'm'\rangle \langle n'm'|V^{(1)}|nm\rangle \\ &= \sum_{n,n',m,m'} \frac{1}{\omega - E_n} \frac{1}{\omega' - E_{n'}} \langle nm|V^{(1)}|n'm'\rangle \langle n'm'|V^{(1)}|nm\rangle \end{aligned} \tag{A.8}$$

The matrix elements $\langle n'm'|V^{(1)}|nm\rangle$ factorize as

$$\langle n'm'|V_\mu^{(1)}|nm\rangle = \langle m'|e^{-\frac{k\ell}{\sqrt{2}}b} e^{\frac{\bar{k}\ell}{\sqrt{2}}b^\dagger}|m\rangle\Big|_b \cdot \langle n'|\mathcal{V}_\mu^{(1)}|n\rangle\Big|_a \delta A_\mu \tag{A.9}$$

because a commutes with b. In Eq. (A.9) $\langle m|X|m'\rangle\Big|_b$ means that the average value of operator X is computed in the Fock space generated by the $b^\dagger$. Then

$$\begin{aligned} Tr_{a,b}G_0V^{(1)}G_0V^{(1)} = & \sum_{n,n',m,m',k,q} \frac{1}{\omega - E_n}\frac{1}{\omega' - E_{n'}}\langle m|e^{-\frac{k\ell}{\sqrt{2}}b}e^{\frac{\bar{k}\ell}{\sqrt{2}}b^\dagger}|m'\rangle\Big|_b \\ & \times \langle m'|e^{-\frac{q\ell}{\sqrt{2}}b}e^{\frac{\bar{q}\ell}{\sqrt{2}}b^\dagger}|m\rangle\Big|_b\langle n'|\mathcal{V}_\mu^{(1)}|n\rangle\Big|_a\langle n'|\mathcal{V}_\nu^{(1)}|n\rangle\Big|_a\delta A_\mu(k)\delta A_\nu(q) \\ = & \sum_{n,n',m,k,q} \frac{1}{\omega - E_n}\frac{1}{\omega' - E_{n'}}\langle m|e^{-\frac{k\ell}{\sqrt{2}}b}e^{\frac{\bar{k}\ell}{\sqrt{2}}b^\dagger}e^{-\frac{q\ell}{\sqrt{2}}b}e^{\frac{\bar{q}\ell}{\sqrt{2}}b^\dagger}|m\rangle\Big|_b \\ & \times \langle n'|\mathcal{V}_\mu^{(1)}|n\rangle\Big|_a\langle n'|\mathcal{V}_\nu^{(1)}|n\rangle\Big|_a\delta A_\mu(k)\delta A_\nu(q)\,, \end{aligned} \tag{A.10}$$

where in the last line we have used that $|m\rangle$ form a complete basis in the Fock space generated by b.

$$\sum_{m'}|m'\rangle\langle m'| = \mathbf{1}_b\,, \tag{A.11}$$

where $\mathbf{1}_b$ is an identity operator in the Fock space spanned by b operators. We have, thus, established that in all of the components of the generalized polarization operator the summation over m can be done explicitly and amounts to the computation of the sum

$$\begin{aligned} \sum_m\langle m|e^{-\frac{k\ell}{\sqrt{2}}b}e^{\frac{\bar{k}\ell}{\sqrt{2}}b^\dagger}e^{-\frac{q\ell}{\sqrt{2}}b}e^{\frac{\bar{q}\ell}{\sqrt{2}}b^\dagger}|m\rangle &= \tfrac{1}{\pi}\int d\alpha e^{|\alpha|^2}\langle 0|e^{\alpha b}e^{-\frac{k\ell}{\sqrt{2}}b}e^{\frac{\bar{k}\ell}{\sqrt{2}}b^\dagger}e^{-\frac{q\ell}{\sqrt{2}}b}e^{\frac{\bar{q}\ell}{\sqrt{2}}b^\dagger}e^{\bar{\alpha}b^\dagger}|0\rangle \\ &= \tfrac{2\pi}{\ell^2}e^{-\frac{|k\ell|^2}{2}}\delta^{(2)}(k+q)\,. \end{aligned} \tag{A.12}$$

In the first line we replaced the summation in m with integration over the coherent states (we explain how to do it in the Appendix A.3). In resume: for any component of the polarization tensor summation over m can be replaced by $\frac{2\pi}{\ell^2}e^{-\frac{|k\ell|^2}{2}}\delta^{(2)}(k+q)$. This delta function is the manifestation of the momentum conservation - after b-summation the fully filled Landau level looks translationally invariant.

Frequency integral

Next we perform the trace over time and frequency

$$
\begin{aligned}
&\mathrm{Tr}_t G_0 V^{(1)} G_0 V^{(1)} = \sum_t \langle t|G_0 V^{(1)} G_0 V^{(1)}|t\rangle \\
= &\sum_{t,\omega}\sum_{t',\omega'} \langle t|\omega\rangle\langle\omega|G_0|\omega\rangle\langle\omega|t'\rangle\langle t'|V^{(1)}_{nn'}|t'\rangle\langle t'|\omega'\rangle\langle\omega'|G_0|\omega'\rangle\langle\omega'|t\rangle\langle t|V^{(1)}_{n'n}|t\rangle \\
= &\sum_{n,n'}\sum_{t,\omega}\sum_{t',\omega'} e^{it(\omega-\omega')}e^{-it'(\omega-\omega')}\frac{1}{\omega-E_n}V^{(1)}_{nn'}(t)\frac{1}{\omega'-E_{n'}}V^{(1)}_{n'n}(t') \\
= &\sum_{n,n'}\sum_{t,\omega,\Omega}\sum_{t',\omega',\Omega} e^{it(\omega-\omega'-\Omega)}e^{-it'(\omega-\omega'-\Omega')}\frac{1}{\omega-E_n}V^{(1)}_{nn'}(\Omega)\frac{1}{\omega'-E_{n'}}V^{(1)}_{n'n}(\Omega') \\
= &\sum_{n,n'}\sum_{\omega,\Omega}\sum_{\omega',\Omega} \delta(\omega-\omega'-\Omega)\delta(\omega-\omega'-\Omega')\frac{1}{\omega-E_n}V^{(1)}_{nn'}(\Omega)\frac{1}{\omega'-E_{n'}}V^{(1)}_{n'n}(\Omega') \\
= &\sum_{n,n'}\sum_{\omega,\Omega} \frac{1}{(\omega+\Omega)-E_n}V^{(1)}_{nn'}(\Omega)\frac{1}{\omega-E_{n'}}V^{(1)}_{n'n}(-\Omega) \\
= &\sum_{n,n'}\int \frac{d\Omega}{2\pi}\frac{d\omega}{2\pi}\frac{1}{(\omega+\Omega)-E_n}\frac{1}{\omega-E_{n'}}V^{(1)}_{nn'}(\Omega)V^{(1)}_{n'n}(-\Omega)\,,
\end{aligned}
$$

where we have introduced a shorthand $V_{nn'}$ for matrix elements $\langle n|V|n'\rangle$. To perform the frequency integration we rewrite the fraction as a sum

$$
\frac{1}{(\omega+\Omega)-E_n}\frac{1}{\omega-E_{n'}} = \left(\frac{1}{(\omega+\Omega)-E_n}-\frac{1}{\omega-E_{n'}}\right)\frac{-1}{\Omega-(E_n-E'_n)}\,. \tag{A.13}
$$

and take only first N poles in the integral over ω. This integration will project onto Hilbert space of the first N Landau levels. When this is done we have

$$
\begin{aligned}
Tr_t G_0 V^{(1)} G_0 V^{(1)} = &\int \tfrac{d\Omega}{2\pi}\left(\textstyle\sum_{n,n'}\frac{\theta(N-n)V^{(1)}_{n'n}(\Omega)V^{(1)}_{nn'}(-\Omega)}{E_{n'}-E_n-\Omega}\right. \\
&\left.-\frac{\theta(N-n')V^{(1)}_{n'n}(-\Omega)V^{(1)}_{nn'}(\Omega)}{E_{n'}-E_n-\Omega}\right) \\
= &\int \tfrac{d\Omega}{2\pi}\textstyle\sum_{n\le N,n'>N}\left(\frac{V^{(1)}_{nn'}(\Omega)V^{(1)}_{n'n}(-\Omega)}{E_{n'}-E_n-\Omega}+\frac{V^{(1)}_{n'n}(\Omega)V^{(1)}_{nn'}(-\Omega)}{E_n-E_{n'}+\Omega}\right).
\end{aligned} \tag{A.14}
$$

This is the final outcome of the computation (we have suppressed the dependence on the momentum). This computation yields (2.50).

A.3 Coherent states

In this Appendix we will describe the coherent states that will be useful for multiple calculations. Here we follow Perelomov [74], but customize the notations to agree with the main text.

A.3.1 Heisenberg-Weyl group

We define Heisenberg-Weyl algebra via relations

$$[a, a^\dagger] = 1\,, \qquad [a, 1] = [a^\dagger, 1] = 0\,, \tag{A.15}$$

an arbitrary element of the algebra is given by a linear combination

$$W = is \cdot \mathbf{1} + qa^\dagger - \bar{q}a, \tag{A.16}$$

where s is real and q is complex.

We want to exponentiate the algebra into the group. Arbitrary Heisenbrg-Weyl group element is given by

$$\begin{aligned} e^W \quad &= e^{is} \cdot e^{qa^\dagger - \bar{q}a} = e^{is} e^{qa^\dagger} e^{-\bar{q}a} e^{-\frac{1}{2}[qa^\dagger, -\bar{q}a]} \\ &= e^{is} e^{-\frac{|q|^2}{2}} e^{qa^\dagger} e^{-\bar{q}a}, \end{aligned} \tag{A.17}$$

where we have used $e^{A+B} = e^{-\frac{1}{2}[A,B]} e^A e^B$, which is true for linear combinations of creation/annihilation operators. We also denote

$$D(q) = e^{qa^\dagger - \bar{q}a}\,. \tag{A.18}$$

These operators form a representation of the Heisenberg-Weyl group. Representations for different values of s are inequivalent. For fixed value of s all representations are unitary equivalent. So from now on we fix s and drop e^{is} factor.

We can freely switch between $D(q)$ and $e^{qa^\dagger}e^{-\bar{q}a}$ at the cost of an exponent, that is

$$D(q) = e^{-\frac{|q|^2}{2}} e^{qa^\dagger} e^{-\bar{q}a} \,. \tag{A.19}$$

Operators $D(q)$ have the following multiplication rule

$$D(q)D(k) = e^{i\mathrm{Im}(q\bar{k})} D(k+q) \,. \tag{A.20}$$

This can be checked using the following simple identities

$$e^{ca} f(a^\dagger) = f(a^\dagger + c) e^{ca} \,, \tag{A.21}$$

$$e^{ca^\dagger} f(a) = f(a - c) e^{ca^\dagger} \,. \tag{A.22}$$

These relations can be used to prove the multiplication law. The latter can be obviously generalized as follows

$$\prod_{i=M}^{i=1} D(q_i) = e^{i\sum_{i<j}\mathrm{Im}(q_j\bar{q}_i)} D\left(\sum_{i=1}^{M} q_i\right) \,. \tag{A.23}$$

The multiplication law implies the permutation relation

$$D(q)D(k) = e^{2i\mathrm{Im}(q\bar{k})} D(k)D(q) \,. \tag{A.24}$$

A.3.2 Generalized coherent states

Operators a, $a^\dagger$ naturally generate a Fock space $\mathcal{H}$ with an orthnormal basis

$$|n\rangle = \frac{a^\dagger}{\sqrt{n!}}|0\rangle\,, \tag{A.25}$$

where $|0\rangle$ is defined via $a|0\rangle = 0$. Consider an arbitrary state $|\Psi_0\rangle \in \mathcal{H}$. States of the form

$$D(q)|\Psi_0\rangle = |q\rangle \tag{A.26}$$

are generalized coherent states. One gets usual coherent states choosing $|\Psi_0\rangle = |0\rangle$. Most of relations for coherent sates hold for any $|\Psi_0\rangle$. The overlap of the coherent sates is

$$\langle q|k\rangle = e^{i\mathrm{Im}(k\bar{q})}\langle\Psi_0|D(k-q)|\Psi_0\rangle \qquad |\langle q|k\rangle|^2 \equiv \rho(k-q) \tag{A.27}$$

Also we have

$$D(k)|q\rangle = e^{i\mathrm{Im}(k\bar{q})}|k+q\rangle\,. \tag{A.28}$$

Since the Fock space $\mathcal{H}$ is projected $D(k)$ acts on the q-plane by translations. Therefore an invariant (under the action of Heisenberg-Weyl group) measure is

$$d\mu(k) = Cdk_1dk_2\,, \qquad \text{with} \qquad k = k_1 + ik_2\,, \tag{A.29}$$

where C is arbitrary constant to be fixed momentarily. Consider an operator

$$A = \int d\mu(k)|k\rangle\langle k|\,. \tag{A.30}$$

We find that for any k we have $[D(k), A] = 0$, thus $A = \lambda\hat{1}$ due to Schur's lemma. We also can always choose C to set $\lambda = 1$. We take $C = \frac{1}{\pi}$ then resolution of identity takes form

(this particular value of C will be explained shortly)

$$\int \frac{dk_1 dk_2}{\pi} |k\rangle\langle k| = \hat{1}\,. \tag{A.31}$$

We now present some relations that are valid *only* for $|\Psi_0\rangle = |0\rangle$.

$$D^\dagger(q) a D(q) = a + q\,. \tag{A.32}$$

and

$$a|k\rangle = k|k\rangle\,. \tag{A.33}$$

Similarly we have

$$|k\rangle = D(k)|0\rangle = e^{-\frac{|k|^2}{2}} e^{ka^\dagger}|0\rangle = \sum_{n=0}^{\infty} \frac{k^n}{\sqrt{n!}}|n\rangle \tag{A.34}$$

Using the last relation we find

$$|\langle k|0\rangle|^2 = \rho(k) = e^{-|k|^2} \qquad |\langle k|q\rangle|^2 = \rho(q-k) = e^{-|k-q|^2} \tag{A.35}$$

Noticing that (A.31) is equivalent to $\int d\mu(k)\rho(k) = 1$ we find that $C = \frac{1}{\pi}$ as advertised.

We want to be able to evaluate traces in $\mathcal{H}$. In the Fock basis we have

$$\begin{aligned} \mathrm{Tr}\, O &= \sum_n \langle n|O|n\rangle = \sum_n \int d\mu(k) d\mu(q) \langle q|n\rangle\langle n|k\rangle\langle k|O|q\rangle \\ &= \int d\mu(k) d\mu(q) \langle q|k\rangle\langle k|O|q\rangle \end{aligned} \tag{A.36}$$

Using resolution of identity

$$\mathrm{Tr}\, O = \int d\mu(q) \langle q|O|q\rangle = \int d\mu(q) e^{-|q|^2} \langle 0|e^{\bar{q}a} O e^{qa^\dagger}|0\rangle \tag{A.37}$$

So we have derived

$$\mathrm{Tr}_a \hat{O} = \frac{1}{\pi} \int dq_1 dq_2 \left(e^{-|q|^2} \langle 0 | e^{\bar{q}a} \hat{O} e^{qa^\dagger} | 0 \rangle \right) \tag{A.38}$$

Consider a matrix element of $D(k)$

$$G(\bar{k}, q; p) \equiv e^{\frac{1}{2}(|k|^2+|q|^2)} \langle k | D(p) | q \rangle = e^{-\frac{|p|^2}{2}} e^{\bar{k}q + \bar{k}p - \bar{q}p} \tag{A.39}$$

Inserting resolution of unity in terms $|n\rangle$ we find

$$G(\bar{k}, q; p) = \sum_{m,n} \bar{u}_m(k) u_n(q) D_{mn}(p) \,, \tag{A.40}$$

where

$$u_n(k) \equiv \langle n | k \rangle = \frac{k^m}{\sqrt{m!}} \tag{A.41}$$

G is a generating function of the matrix elements of $D_{mn}(p)$. The latter are obtain expanding (A.39) in series in k and q.

$$D_{nm}(p) = \sqrt{\frac{n!}{m!}} e^{-\frac{|p|^2}{2}} p^{m-n} L_n^{m-n}(|p|^2) \,, \; m \geq n \tag{A.42}$$

$$D_{nm}(p) = \sqrt{\frac{m!}{n!}} e^{-\frac{|p|^2}{2}} (-\bar{p})^{n-m} L_m^{n-m}(|p|^2) \, n \geq m \tag{A.43}$$

We also find simple relations for the traces

$$\mathrm{Tr}\, D(p) = \pi \delta^{(2)}(p) \tag{A.44}$$

$$\mathrm{Tr} \left[D(p) D^{-1}(q) \right] = \pi \delta^{(2)}(p - q) \,. \tag{A.45}$$

A.3.3 Application: Trace over b-subspace

We want to evaluate trace of a product of local operators

$$\mathrm{Tr}_b\left[\prod_{i=1}^{M} O_i(x_i)\right] = \int [dk] \prod_{i=1}^{M} \left[\mathrm{Tr}_b\left[e^{i\sum_i \mathbf{k}_i\cdot\mathbf{x}_i}\right] O_i(k_i)\right], \tag{A.46}$$

where we have introduce a shorthand notation $[dk] = \prod_{i=1}^{M} \frac{d^2\mathbf{k}_i}{(2\pi)^{2M}}$ and $O_i(k_i)$ is understood as a Fourier transform of $O_i(x_i)$. Now we re-write the exponent in terms of a and b.

$$e^{i\mathbf{k}\cdot\mathbf{x}} = e^{\frac{\bar{k}\ell}{\sqrt{2}}a - \frac{k\ell}{\sqrt{2}}a^\dagger} e^{-\frac{\bar{k}\ell}{\sqrt{2}}b^\dagger + \frac{k\ell}{\sqrt{2}}b} = e^{\bar{q}a - qa^\dagger} e^{-\bar{q}b^\dagger + qb}, \tag{A.47}$$

where we introduced $q = \frac{k\ell}{\sqrt{2}}$, so that $[dk] = \left(\frac{2}{\ell^2}\right)^M [dq]$. We have for the exponent

$$e^{i\mathbf{k}\cdot\mathbf{x}} = D_a(-q) e^{-\frac{|q|^2}{2}} e^{-\bar{q}b^\dagger} e^{qb}. \tag{A.48}$$

Now, we plug this back into the trace

$$\left(\frac{2}{l^2}\right)^M \prod_{i=1}^{M} \int [dq] D_a(-q_i) \left[\mathrm{Tr}_b\left[e^{-\frac{|q_i|^2}{2}} e^{-\bar{q}_i b^\dagger} e^{q_i b}\right] O_i(q_i)\right]. \tag{A.49}$$

To proceed we use (A.38)

$$\begin{aligned}
&\mathrm{Tr}_b\left[\prod_{i=1}^{M} e^{-\frac{|q_i|^2}{2}} e^{-\bar{q}_i b^\dagger} e^{q_i b}\right] \\
&= \frac{1}{\pi} e^{-\sum_i \frac{|q_i|^2}{2}} \int d^2p \left[e^{-|p|^2} \langle 0| e^{\bar{p}b} \prod_{i=1}^{M} e^{-\bar{q}_i b^\dagger} e^{q_i b} e^{pb^\dagger} |0\rangle\right].
\end{aligned}$$

We want to normal order the product. In order to do this we use permutation relations

$$e^{\bar{p}b}\prod_{i=1}^{M}e^{-\bar{q}_i b^\dagger}e^{q_i b}e^{pb^\dagger} = \\ :e^{\bar{p}b}\prod_{i=1}^{M}e^{-\bar{q}_i b^\dagger}e^{q_i b}e^{pb^\dagger}: e^{|p|^2}e^{\sum_{i>j}-\bar{q}_i q_j}e^{-\bar{p}\sum_i \bar{q}_i}e^{p\sum_i q_i}.$$

Denoting $\sum_i q_i = Q$ and using

$$\langle 0|:e^{\bar{p}b}\prod_{i=1}^{M}e^{-\bar{q}_i b^\dagger}e^{q_i b}e^{pb^\dagger}:|0\rangle = 1 \tag{A.50}$$

we have

$$\mathrm{Tr}_b\left[\prod_{i=1}^{M}e^{-\frac{|q_i|^2}{2}}e^{-\bar{q}_i b^\dagger}e^{q_i b}\right] \\ = \frac{1}{\pi}e^{-\sum_i\frac{|q_i|^2}{2}}e^{\sum_{i>j}-\bar{q}_i q_j}\int d^2p e^{-\bar{p}Q}e^{p\bar{Q}}. \tag{A.51}$$

The latter integral is a δ-function

$$\frac{1}{\pi}\int dp_1 dp_2 e^{-\bar{p}Q}e^{p\bar{Q}} = \pi\delta^{(2)}(\mathbf{Q}) = \pi\int\frac{d^2\lambda}{(2\pi)^2}e^{i\lambda\cdot\mathbf{Q}}. \tag{A.52}$$

We also use $\bar{q}_i q_j = \mathbf{q_i}\cdot\mathbf{q_j} + i\mathbf{q_i}\wedge\mathbf{q_j}$, where $\mathbf{a}\wedge\mathbf{b} = a_1 b_2 - a_2 b_1$. As well as

$$\frac{1}{2}\sum_i|q_i|^2 + \sum_{i<j}\mathbf{q_i}\cdot\mathbf{q_j} = \frac{1}{2}\mathbf{Q}^2 \tag{A.53}$$

then

$$\mathrm{Tr}_b\left[\prod_{i=1}^{M}e^{-\frac{|q_i|^2}{2}}e^{-\bar{q}_i b^\dagger}e^{q_i b}\right] = \pi e^{-i\sum_{i>j}\mathbf{q_i}\wedge\mathbf{q_j}}\int\frac{d^2\lambda}{(2\pi)^2}e^{i\lambda\cdot\mathbf{Q}} \tag{A.54}$$

We have proven that

$$\mathrm{Tr}_b\left[\prod_{i=1}^{M} O_i(x_i)\right] = \pi\left(\frac{2}{l^2}\right)^M \int \frac{d^2\lambda}{(2\pi)^2}\prod_{i=1}^{M}\int [dq]\left[D_a(-q_i)e^{i\lambda\cdot\mathbf{q_i}}O_i(q_i)\right] e^{-i\sum_{i>j}\mathbf{q_i}\wedge\mathbf{q_j}}\,. \tag{A.55}$$

Finally using

$$D_a(-q_1)\cdot\ldots\cdot D_a(-q_M) = e^{i\sum_{i>j} q_i\wedge q_j} D_a(-q_1-\ldots-q_M) = e^{i\sum_{i>j} q_i\wedge q_j} D_a(-Q) \tag{A.56}$$

we arrive at the trace formula

$$\mathrm{Tr}_b\left[\prod_{i=1}^{M} O_i(x_i)\right] = \pi\left(\frac{2}{l^2}\right)^M \int \frac{d^2\lambda}{(2\pi)^2}\prod_{i=1}^{M}\int [dq]\left[e^{i\lambda\cdot\mathbf{q_i}}\tilde{O}_\ell i(q_i)\right]\,. \tag{A.57}$$

This is the generalization of the b-summation formula that was used in Appendix B to arbitrary number of external legs. This formula is useful if one is aiming to evaluate the generating functional to arbitrary order in external fields.

A.4 Vertices for the Dirac polarization tensor

In this Appendix we explicitly write out the vertices for the Dirac polarization tensor. These are obtained by straightforwardly combining (2.78) with (2.90) and using (2.45), for $|n'| \geq |n| > 0$

$$\begin{aligned}\Gamma^0_{Dnn'}(\vec{k}) = \quad & \tfrac{1}{2}\left[\sqrt{\tfrac{|n|!}{|n'|!}}\left(\tfrac{\bar{k}\ell}{\sqrt{2}}\right)^{|n'|-|n|} L_{|n|}^{|n'|-|n|}\left(\tfrac{|k\ell|^2}{2}\right)\right. \\ & \left.+\mathrm{sgn}(n)\mathrm{sgn}(n')\sqrt{\tfrac{(|n|-1)!}{(|n'|-1)!}}\left(\tfrac{\bar{k}\ell}{\sqrt{2}}\right)^{|n|'-|n|} L_{|n|-1}^{|n'|-|n|}\left(\tfrac{|k\ell|^2}{2}\right)\right]\end{aligned} \tag{A.58}$$

$$\Gamma^1_{Dnn'}(\vec{k}) = -\frac{v_F}{2}\Big[\mathrm{sgn}(n') \quad \sqrt{\frac{|n|!}{(|n'|-1)!}}\left(\frac{\bar{k}\ell}{\sqrt{2}}\right)^{|n'|-|n|-1} L_{|n|}^{|n'|-|n|-1}\left(\frac{|k\ell|^2}{2}\right)$$
$$+\mathrm{sgn}(n)\sqrt{\frac{(|n|-1)!}{(|n'|)!}}\left(\frac{\bar{k}\ell}{\sqrt{2}}\right)^{|n|'-|n|+1} L_{|n|-1}^{|n'|-|n|+1}\left(\frac{|k\ell|^2}{2}\right)\Big]$$
, (A.59)

$$\Gamma^2_{Dnn'}(\vec{k}) = \frac{iv_F}{2}\left[\mathrm{sgn}(n')\sqrt{\frac{|n|!}{(|n'|-1)!}}\left(\frac{\bar{k}\ell}{\sqrt{2}}\right)^{|n'|-|n|-1} L_{|n|}^{|n'|-|n|-1}\left(\frac{|k\ell|^2}{2}\right)\right.$$
$$\left.-\mathrm{sgn}(n)\sqrt{\frac{(|n|-1)!}{(|n'|)!}}\left(\frac{\bar{k}\ell}{\sqrt{2}}\right)^{|n|'-|n|+1} L_{|n|-1}^{|n'|-|n|+1}\left(\frac{|k\ell|^2}{2}\right)\right]. \quad \text{(A.60)}$$

For the case $|n| \geq |n'| > 0$, the expressions for the vertices are obtained by using (2.46) instead of (2.45).

For the case $n = 0$, $|n'| > 0$

$$\Gamma^0_{D0n'}(\vec{k}) = \frac{1}{\sqrt{2}}\left(\sqrt{\frac{1}{|n'|!}}\left(\frac{\bar{k}\ell}{\sqrt{2}}\right)^{|n'|} L_0^{|n'|}\left(\frac{|k\ell|^2}{2}\right)\right), \quad \text{(A.61)}$$

$$\Gamma^1_{D0n'}(\vec{k}) = -\frac{v_F}{\sqrt{2}}\left(\mathrm{sgn}(n')\sqrt{\frac{1}{(|n'|-1)!}}\left(\frac{\bar{k}\ell}{\sqrt{2}}\right)^{|n'|-1}\right.$$
$$\left.\times\, L_0^{|n'|-1}\left(\frac{|k\ell|^2}{2}\right)\right), \quad \text{(A.62)}$$

$$\Gamma^2_{D0n'}(\vec{k}) = \frac{iv_F}{\sqrt{2}}\left(\mathrm{sgn}(n')\sqrt{\frac{1}{(|n'|-1)!}}\left(\frac{\bar{k}\ell}{\sqrt{2}}\right)^{|n'|-1}\right.$$
$$\left.\times\, L_0^{|n'|-1}\left(\frac{|k\ell|^2}{2}\right)\right), \quad \text{(A.63)}$$

For the case $|n| > 0$, $n' = 0$

$$\Gamma^0_{Dn0}(\vec{k}) = \frac{1}{\sqrt{2}} \left(\sqrt{\frac{1}{|n|!}} \left(\frac{-k\ell}{\sqrt{2}} \right)^{|n|} L_0^{|n|} \left(\frac{|k\ell|^2}{2} \right) \right), \tag{A.64}$$

$$\Gamma^1_{Dn0}(\vec{k}) = -\frac{v_F}{\sqrt{2}} \left(\mathrm{sgn}(n) \sqrt{\frac{1}{(|n|-1)!}} \left(\frac{-k\ell}{\sqrt{2}} \right)^{|n|-1} \times L_0^{|n|-1} \left(\frac{|k\ell|^2}{2} \right) \right), \tag{A.65}$$

$$\Gamma^2_{Dn0}(\vec{k}) = -\frac{iv_F}{\sqrt{2}} \left(\mathrm{sgn}(n) \sqrt{\frac{1}{(|n|-1)!}} \left(\frac{-k\ell}{\sqrt{2}} \right)^{|n|-1} \times L_0^{|n|-1} \left(\frac{|k\ell|^2}{2} \right) \right), \tag{A.66}$$

we can write down the explicit form of equation (2.89) as the summation of product of Laguerre polynomials for each pair of indices μ and ν.

A.5 Evaluation of the infinite sums

In this Appendix, we again use $\vec{k} = (k_1, 0)$. The components of polarization tensors can be obtained from equation (2.89) and the explicit form of vertex operator $\Gamma^{\mu}_{Dnn'}(\vec{k})$

$$\Pi^{12}_D(\Omega, \vec{k}) = i\Omega\Pi^{12}_1 + i\Omega k_1^2 \Pi^{12}_2 + i\Omega^3 \Pi^{12}_3 + \cdots, \tag{A.67}$$

where Π^{12}_1,Π^{12}_2 and Π^{12}_3 are the result of Taylor expansion of (2.89) at specific order of ω and p, $\cdots$ represents the higher order of frequency and momentum. The explicit form of Π^{12}_1 is

$$\Pi^{12}_1 = \frac{i}{8\pi} \left(\sum_{n=N+1}^{\infty} \left[(\sqrt{n} - \sqrt{n-1})^2 - (\sqrt{n+1} - \sqrt{n})^2 \right] + (\sqrt{N} + \sqrt{N+1})^2 \right), \tag{A.68}$$

the first two terms come from the summation with $n' < 0$ and $n > 0$, the last term is from $n, n' > 0$. Similarly,

$$\begin{aligned}\Pi_2^{12} = \frac{i\ell^2}{64\pi}\Bigg(\sum_{n=N+1}^{\infty}\Big[&4(2n+1)(\sqrt{n+1}-\sqrt{n})^2 - 4(2n-1)(\sqrt{n}-\sqrt{n-1})^2 \\ &+(n-1)(\sqrt{n}-\sqrt{n-2})^2 - (n+1)(\sqrt{n+2}-\sqrt{n})^2\Big] \\ &- 4(2N-1)(\sqrt{N+1}+\sqrt{N})^2 + N(\sqrt{N+1}+\sqrt{N-1})^2 \\ &+(N+1)(\sqrt{N+2}+\sqrt{N})^2\Bigg)\,, \end{aligned} \tag{A.69}$$

$$\Pi_3^{12} = \frac{i\ell^2}{16\pi v_F^2}\left(\sum_{n=N+1}^{\infty}\Big[(\sqrt{n}-\sqrt{n-1})^4 - (\sqrt{n+1}-\sqrt{n})^4\Big] + (\sqrt{N}+\sqrt{N+1})^4\right). \tag{A.70}$$

The summations are convergent and can evaluated Π_1^{12}, Π_2^{12} and Π_3^{12} to obtain

$$\Pi^{12}(\Omega,\vec{k}) = i\Omega\frac{N+1/2}{2\pi} - i\Omega k_1^2\ell^2\frac{6N^2+6N+1}{16\pi} + i\Omega^3\frac{\ell^2}{v_F^2}\frac{8N^2+8N+1}{8\pi} + \cdots\,. \tag{A.71}$$

We derive similarly

$$\Pi_D^{00}(\Omega,\vec{k}) = k_1^2\Pi_1^{00} + \cdots\,, \tag{A.72}$$

$$\Pi_D^{11}(\Omega,\vec{k}) = \Omega^2\Pi_1^{11} + \cdots\,, \tag{A.73}$$

$$\Pi_D^{22}(\Omega,\vec{k}) = \Omega^2\Pi_1^{22} + k_1^2\Pi_2^{22} + \cdots\,. \tag{A.74}$$

There is no k_1^2 term in $\Pi^{11}(\Omega,\vec{k})$ and no Ω^2 term in $\Pi^{00}(\Omega,\vec{k})$, we can calculate the coefficients

$$\Pi_1^{00}=\Pi_1^{11}=\Pi_1^{22}=\frac{\ell}{8\sqrt{2}\pi v_F}\left(\sum_{n=N+1}^{\infty}\left[(\sqrt{n+1}-\sqrt{n})^3+(\sqrt{n}-\sqrt{n-1})^3\right]\right.$$
$$\left.+(\sqrt{N}+\sqrt{N+1})^3\right)\,. \tag{A.75}$$

The summation is convergent and is given by

$$\sum_{n=N+1}^{\infty}\left[(\sqrt{n+1}-\sqrt{n})^3+(\sqrt{n}-\sqrt{n-1})^3\right]=-(\sqrt{N}+\sqrt{N+1})^3-12\zeta(-\frac{1}{2},N+1), \tag{A.76}$$

where $\zeta(s,n)$ is the Hurwitz ζ-function, which is defined as

$$\zeta(s,q)=\sum_{n=0}^{\infty}\frac{1}{(n+q)^s}. \tag{A.77}$$

As the result, we have

$$\Pi_1^{00}=\Pi_1^{11}=\Pi_1^{22}=-\frac{3\ell}{2\sqrt{2}\pi v_F}\zeta(-\frac{1}{2},N+1)\,. \tag{A.78}$$

The coefficient Π_2^{22} can be also calculated similarly

$$\Pi_2^{22}=\frac{3\ell v_F}{4\sqrt{2}\pi}\zeta(-\frac{1}{2},N+1). \tag{A.79}$$

We summarize these results in the Section 2.3.

A.6 Evaluate the summations of non-relativistic polarization tensor at large N limit

In this Appendix, we will show the explicit calculation of polarization tensors in the Section 2.5.1. The following Bessel function identities will come in handy

$$J_{1-\omega}(q)J_{1+\omega}(q) - J_{-1-\omega}(q)J_{-1+\omega}(q) = \frac{4\omega\sin(\pi\omega)}{\pi q^2}. \tag{A.80}$$

$$\omega J_{\omega}(q) = \frac{q}{2}(J_{1+\omega}(q) + J_{-1+\omega}(q)) \tag{A.81}$$

$$\omega J_{-\omega}(q) = -\frac{q}{2}(J_{1-\omega}(q) + J_{-1-\omega}(q)) \tag{A.82}$$

The summations (2.118),(2.119) and (2.120) can be recast as

$$\Pi^{11}(q,\omega) = -\frac{N\omega_c}{2\pi} + \sum_{n=1}^{\infty} -\frac{2Nn^4\omega_c\left[J_n(q)\right]^2}{\pi q^2(\omega^2 - n^2)}. \tag{A.83}$$

$$
\begin{aligned}
\Pi^{22}(q,\omega) =& -\frac{N\omega_c}{2\pi} + \sum_{n=1}^{\infty} -\frac{Nn^2\omega_c \left[J_{n-1}(q) - J_{n+1}(q)\right]^2}{2\pi(\omega^2 - n^2)} \\
& - \mathrm{g}\sum_{n=1}^{\infty} \frac{n^2\omega_c q \left[J_{n-1}(q) - J_{n+1}(q)\right] J_n(q)}{4\pi(\omega^2 - n^2)} \\
& - \mathrm{g}^2 \sum_{n=1}^{\infty} \frac{q^2 n^2 \omega_c \left[J_n(q)\right]^2}{32N\pi(\omega^2 - n^2)} \\
=& -\frac{N\omega_c}{2\pi} - \frac{N\omega_c}{2\pi}\sum_{n=1}^{\infty} \frac{n^2\left[J_{n-1}(q) - J_{n+1}(q)\right]^2}{(\omega^2 - n^2)} - \mathrm{g}\omega_c q \frac{\partial}{\partial q}\sum_{n=1}^{\infty} \frac{n^2\left[J_n(q)\right]^2}{4\pi(\omega^2 - n^2)} \\
& - \mathrm{g}^2 \sum_{n=1}^{\infty} \frac{q^2 n^2 \omega_c \left[J_n(q)\right]^2}{32N\pi(\omega^2 - n^2)} \\
=& -\frac{N\omega_c}{2\pi} - \frac{N\omega_c}{2\pi}\sum_{n=1}^{\infty} \left[\frac{-4n^2 J_{n-1}(q) J_{n+1}(q)}{(\omega^2 - n^2)} + \frac{4n^4 J_n(q) J_n(q)}{q^2(\omega^2 - n^2)}\right] \\
& - \mathrm{g}\omega_c q \frac{\partial}{\partial q}\sum_{n=1}^{\infty} \frac{n^2\left[J_n(q)\right]^2}{4\pi(\omega^2 - n^2)} - \mathrm{g}^2 \sum_{n=1}^{\infty} \frac{q^2 n^2 \omega_c \left[J_n(q)\right]^2}{32N\pi(\omega^2 - n^2)} \\
=& \Pi^{11}(q,\omega) + \frac{2N\omega_c}{\pi}\sum_{n=1}^{\infty} \frac{n^2 J_{n-1}(q) J_{n+1}(q)}{(\omega^2 - n^2)} - \mathrm{g}\omega_c q \frac{\partial}{\partial q}\sum_{n=1}^{\infty} \frac{n^2\left[J_n(q)\right]^2}{4\pi(\omega^2 - n^2)} \\
& - \mathrm{g}^2 \sum_{n=1}^{\infty} \frac{q^2 n^2 \omega_c \left[J_n(q)\right]^2}{32N\pi(\omega^2 - n^2)}
\end{aligned}
\quad , \tag{A.84}
$$

$$
\begin{aligned}
\Pi^{12}(q,\omega) =& \sum_{n=1}^{\infty} -\frac{iNn^2\omega\omega_c J_n(q)\left[J_{n-1}(q) - J_{n+1}(q)\right]}{\pi q(\omega^2 - n^2)} - \mathrm{g}\sum_{n=1}^{\infty} \frac{i\omega n^2 \omega_c \left[J_n(q)\right]^2}{4\pi(\omega^2 - n^2)} \\
=& -\frac{iN\omega\omega_c}{\pi q}\frac{\partial}{\partial q}\sum_{n=1}^{\infty} \frac{n^2 J_n(q) J_n(q)}{(\omega^2 - n^2)} - \mathrm{g}\sum_{n=1}^{\infty} \frac{i\omega n^2 \omega_c \left[J_n(q)\right]^2}{4\pi(\omega^2 - n^2)}
\end{aligned}
\tag{A.85}
$$

where we used

$$
\frac{\partial}{\partial x} J_n(x) = \frac{1}{2}(J_{n-1}(x) - J_{n+1}(x)). \tag{A.86}
$$

Using the identity

$$\sum_{n=1}^{\infty} n^2 \left[J_n(x)\right]^2 = \frac{x^2}{4} \tag{A.87}$$

we can rewrite $\Pi^{11}(q,\omega)$ as

$$\Pi^{11}(q,\omega) = -\frac{2N\omega^2\omega_c}{\pi q^2} \sum_{n=1}^{\infty} \frac{n^2 \left[J_n(q)\right]^2}{(\omega^2 - n^2)}. \tag{A.88}$$

Next we need to evaluate

$$\sum_{n=1}^{\infty} \frac{n^2 \left[J_n(q)\right]^2}{(\omega^2 - n^2)} = \omega^2 \sum_{n=1}^{\infty} \frac{\left[J_n(q)\right]^2}{(\omega^2 - n^2)} - \sum_{n=1}^{\infty} \left[J_n(q)\right]^2, \tag{A.89}$$

$$\sum_{n=1}^{\infty} \frac{n^2 J_{n-1}(q) J_{n+1}(q)}{(\omega^2 - n^2)} = \omega^2 \sum_{n=1}^{\infty} \frac{J_{n-1}(q) J_{n+1}(q)}{(\omega^2 - n^2)} - \sum_{n=1}^{\infty} J_{n-1}(q) J_{n+1}(q) \tag{A.90}$$

to derive the closed form of $\Pi^{ij}(p,\omega)$. Both of the above summations can be evaluated using the tricks in reference [91], which gives us

$$\sum_{n=1}^{\infty} \frac{n^2 \left[J_n(q)\right]^2}{(\omega^2 - n^2)} = -\frac{1}{2} + \frac{\pi\omega}{2\sin(\pi\omega)} J_{\omega}(q) J_{-\omega}(q) \tag{A.91}$$

$$\sum_{n=1}^{\infty} \frac{n^2 J_{n-1}(q) J_{n+1}(q)}{(\omega^2 - n^2)} = -\frac{\pi\omega}{2\sin(\pi\omega)} J_{1-\omega}(q) J_{1+\omega}(q) \tag{A.92}$$

We therefore can derive the closed form of polarization tensor

$$\Pi^{11}(q,\omega) = \frac{N\omega^2\omega_c}{\pi q^2}\left(1-\frac{\pi\omega}{\sin(\pi\omega)}J_\omega(q)J_{-\omega}(q)\right), \tag{A.93}$$

$$\Pi^{12}(q,\omega) = -\frac{iN\omega\omega_c}{2\pi q}\frac{\pi\omega}{\sin(\pi\omega)}\frac{\partial}{\partial q}\left[J_\omega(q)J_{-\omega}(q)\right]+i\frac{g\omega\omega_c}{8\pi}\left(1-\frac{\pi\omega}{\sin(\pi\omega)}J_\omega(q)J_{-\omega}(q)\right), \tag{A.94}$$

$$\begin{aligned}\Pi^{22}(q,\omega) = {} & \frac{N\omega^2\omega_c}{\pi q^2}\left(1-\frac{\pi\omega}{\sin(\pi\omega)}J_\omega(q)J_{-\omega}(q)\right)-\frac{N\omega_c\omega}{\sin(\pi\omega)}J_{1-\omega}(q)J_{1+\omega}(q)\\ & -\frac{gq\omega\omega c}{8\sin(\pi\omega)}\frac{\partial}{\partial q}\left[J_\omega(q)J_{-\omega}(q)\right]+\frac{g^2q^2\omega\mathrm{c}}{64\pi\mathrm{N}}\left(1-\frac{\pi\omega}{\sin(\pi\omega)}J_\omega(q)J_{-\omega}(q)\right).\end{aligned} \tag{A.95}$$

Since the closed form of polarization tensor is obtained, we can derive the large N approximation of conductivity and compare with Fermi liquid calculation.

APPENDIX B

LOWEST LANDAU LEVEL BASIS

B.1 Lowest Landau level states

The orthonormal complete basis set of single particle wave functions in the LLL is

$$\phi_m(z) = \frac{z^m}{\sqrt{2\pi 2^m m!}} e^{-\frac{1}{4\ell^2}|z|^2} \,. \tag{B.1}$$

We will take $\ell = 1$ to simplify the formulas. We have the projected annihilation field operator

$$\psi_L(z) = \sum_{m=0}^{\infty} \hat{a}_m \phi_m(z) = \sum_{m=0}^{\infty} \hat{a}_m \frac{z^m}{\sqrt{2\pi 2^m m!}} e^{-\frac{1}{4}|z|^2} \,, \tag{B.2}$$

where $\hat{a}_m$ and $\hat{a}_m^\dagger$ annihilate and create normalized LLL states and obey the Fermi canonical relation

$$\{\hat{a}_n, \hat{a}_m^\dagger\} = \delta_{mn} \,, \qquad \{\hat{a}_n^\dagger, \hat{a}_m^\dagger\} = \{\hat{a}_n, \hat{a}_m\} = 0 \,. \tag{B.3}$$

Thus equal time anti-commutators of projected electron field operators are

$$\{\psi_L^\dagger(z_1), \psi_L(z_2)\} = \sum_{m=0}^{\infty} \phi_m^*(z_1)\phi_m(z_2) = \frac{1}{2\pi} e^{-\frac{1}{4}|z_1 - z_2|^2} e^{\frac{1}{4}(\bar{z}_1 z_2 - \bar{z}_2 z_1)} \equiv \{z_1|z_2\}. \tag{B.4}$$

Since the LLL single-particle states are not complete in the *full* Hilbert space $\{z_1|z_2\} \neq \delta^2(z_1 - z_2)$. In fact, $\{z_1|z_2\}$ acts as the δ-function in the LLL

$$\int d^2 z_1 F(z_1)\{z_1|z_2\} = F(z_2) \,, \tag{B.5}$$

where $F(z)$ is any function of the form $F(z) = f(z)e^{-\frac{1}{4}|z|^2}$ and $f(z)$ is holomorphic. Further more,

$$\int d^2 z_2 \{z_1|z_2\} G(\bar{z}_2) = G(\bar{z}_1) \,, \tag{B.6}$$

where $G(\bar{z})$ is any function of the form $G(\bar{z}) = g(\bar{z})e^{-\frac{1}{4}|z|^2}$ and $g(\bar{z})$ is anti-holomorphic.

Finally, $\{z_1|z_2\}$ satisfies the composition rule

$$\int d^2 z_2 \{z_1|z_2\}\{z_2|z_3\} = \{z_1|z_3\}\,, \qquad \{z_1|z_2\}^* = \{z_2|z_1\}\,. \tag{B.7}$$

B.2 LLL at finite number of particles

Next we assume that only N orbitals are available in the LLL. The finite N version of the LLL δ-function is

$$K_N(z_1; z_2) = \sum_{m=0}^{N-1} \phi_m(z_1)\phi_m^*(z_2)\,, \tag{B.8}$$

which is known as the reproducing kernel for the Hilbert space of states spanning the LLL (also known as the Bergman kernel), which satisfies the composition rule

$$K_N(z; z') = \int d^2\xi K_N(z;\xi)K_N(\xi; z'), \tag{B.9}$$

and the reproducing formula

$$\int d^2 z_2 K_N(z_1; z_2)F_N(z_2) = F_N(z_1)\,, \tag{B.10}$$

where $F_N(z)$ is any linear combination of $\phi_m(z)$ with m running from 0 to $N-1$. Similarly,

$$\int d^2 z_1 G_N(\bar{z}_1)K_N(z_1; z_2) = G_N(\bar{z}_2)\,, \tag{B.11}$$

where $G_N(\bar{z})$ is any linear combination of $\phi_m^*(z)$ with m running from 0 to $N-1$.

The normalized wave function of the $\nu = 1$ integer quantum Hall state is

$$\Psi_1(z_1, \cdots, z_N) = \frac{1}{\sqrt{N!}} \begin{vmatrix} \phi_0(z_1) & \phi_0(z_2) & \cdots & \phi_0(z_N) \\ \phi_1(z_1) & \phi_1(z_2) & \cdots & \phi_1(z_N) \\ \cdots & \cdots & \cdots & \cdots \\ \phi_{N-1}(z_1) & \phi_{N-1}(z_2) & \cdots & \phi_{N-1}(z_N) \end{vmatrix} . \tag{B.12}$$

We define the $n-$particle density matrix for this state by

$$\mathcal{P}_1^{(n)}(\{\xi\}, \{\xi'\}) = \frac{N!}{(N-n)!n!} \int [dz] \Psi_1(\{z\}, \{\xi\}) \Psi_1^*(\{z\}, \{\xi'\}) . \tag{B.13}$$

where we used the shorthand notation

$$\{z\} = z_1, \cdots, z_M \qquad \{\xi\} = z_{M+1}, \cdots, z_N \qquad \{\xi'\} = z'_{M+1}, \cdots, z'_N. \tag{B.14}$$

with $n = N - M$. It can be shown that

$$\mathcal{P}_1^{(n)}(\xi_1, ..., \xi_n; \xi'_1, ..., \xi'_n) = \frac{1}{n!} \det \left[K_N(\xi_i; \xi'_j) \right]_{1 \le i,j \le n} . \tag{B.15}$$

For example, $\mathcal{P}_1^{(1)}(\xi; \xi') = K_N(\xi; \xi')$, which at equal points coincides with the mean density of the integer state $\langle \rho_1(\xi) \rangle = K_N(\xi; \xi)$. Furthermore, it's clear that

$$\mathcal{P}_1^{(2)}(\xi_1, \xi_2; \xi'_1, \xi'_2) = \frac{1}{2}(K_N(\xi_1; \xi'_1) K_N(\xi_2; \xi'_2) - K_N(\xi_1; \xi'_2) K_N(\xi_2; \xi'_1)) . \tag{B.16}$$

On the diagonal, $\mathcal{P}^{(2)}(\xi_1, \xi_2; \xi_1, \xi_2)$ is related to the two-particle distribution function $\mathcal{P}^{(2)}(\xi_1, \xi_2; \xi_1, \xi_2) = \frac{1}{2} n^{(2)}(\xi_1, \xi_2)$, which can be expressed in terms of the pair distribution function $n^{(2)}(\xi_1, \xi_2) = \langle \rho(\xi_1) \rangle \langle \rho(\xi_2) \rangle g(\xi_1, \xi_2)$. Similar to the reproducing kernel, the reduced

density matrix has the property that

$$\int [d\xi'] \mathcal{P}_1^{(n)}(\{\xi\};\{\xi'\}) F_N(\{\xi'\}) = F_N(\{\xi\})\,. \tag{B.17}$$

for any *antisymmetric* function $F_N(\{\xi\}) = F_N(\xi_1, ..., \xi_n)$ built by linear superposition of the N LLL orbitals. In particular, for a fermionic $M-$ particle wave function Ψ_ν, it clearly follows that

$$\int [d\xi'] \mathcal{P}_1^{(M)}(\{\xi\};\{\xi'\}) \Psi_\nu(\{\xi'\}) = \Psi_\nu(\{\xi\})\,. \tag{B.18}$$

Then defining PH conjugation by

$$\Psi_{1-\nu}(\{z\}) = \sqrt{\frac{N!}{(N-M)!M!}} \int [d\xi] \Psi_1(\{z\},\{\xi\}) \Psi_\nu^*(\{\xi\})\,. \tag{B.19}$$

we find

$$\Psi_{1-\nu}^{PH} = \Psi_{1-(1-\nu)} = (-1)^{M(N-M)} \Psi_\nu. \tag{B.20}$$

Results of this Section remain true in inhomogeneous magnetic field and curvature.

B.3 PHD for the density matrix

Here we obtain a relation between the density matrices computed for different filling fractions. Specifically

$$\mathcal{P}_\nu^{(n)}(\{\xi\};\{\xi'\}) = \frac{N_\nu!}{(N_\nu - n)!n!} \int [dz] \Psi_\nu(\{z\},\{\xi\}) \Psi_\nu^*(\{z\},\{\xi'\}), \tag{B.21}$$

where $N_\nu = M$ is the total number of particles in the state Ψ_ν. It is similarly defined for the PH-conjugate state $\Psi_{1-\nu}$, with $N_{1-\nu} = N - M$. We start with the one-particle density

matrix

$$\mathcal{P}^{(1)}_{1-\nu}(z;z') \equiv \frac{(M+1)!}{M!}\int [d\xi][d\xi']\mathcal{P}^{(M+1)}_1(z,\{\xi\};z',\{\xi'\})\Psi_\nu(\{\xi'\})\Psi^*_\nu(\{\xi\}), \tag{B.22}$$

$$= \mathcal{P}^{(1)}_1(z;z') - \mathcal{P}^{(1)}_\nu(z;z')\,, \tag{B.23}$$

where the second equality follows directly from the following expansion of the determinant in the definition of the density matrix

$$\mathcal{P}^{(M+1)}_1(z,\{\xi\};z',\{\xi'\}) = \frac{1}{M+1}\Big[K_N(z;z')\mathcal{P}^{(M)}_1(\{\xi\},\{\xi'\}) - \sum_{k=1}^{M} K_N(z;\xi'_k)\mathcal{P}^{(M)}_1(\xi_k,\{\xi\}_k;z',\{\xi'\}_k)\Big]\,, \tag{B.24}$$

where $\{\xi\}_k = \{\xi_1,...,\xi_{k-1},\xi_{k+1},...,\xi_M\}$ is the ordered array of coordinates with ξ_k excluded. On the diagonal, $\mathcal{P}^{(1)} = \langle\rho\rangle$, and (B.23) implies the duality relation for the mean density $\langle\rho_{1-\nu}\rangle = \langle\rho_1\rangle - \langle\rho_\nu\rangle$.

Next, we consider the two-particle reduced density matrix, which can be written as a convolution involving the $(M+2)$-particle density matrix

$$\mathcal{P}^{(2)}_{1-\nu}(z_1,z_2;z_1,z_2) = \frac{(M+2)!}{2(M!)}\int [d\xi][d\xi']\mathcal{P}^{(M+2)}_1(z_1,z_2,\{\xi\};z_1,z_2,\{\xi'\})\Psi_\nu(\{\xi'\})\Psi^*_\nu(\{\xi\}) \tag{B.25}$$

We next utilize the expansion

$$\mathcal{P}^{(M+2)}_1(z_1,z_2,\{\xi\};z_1,z_2,\{\xi'\}) = \frac{1}{M+2}\Big[K_N(z_1;z_1)\mathcal{P}^{(M+1)}_1(z_2,\{\xi\};z_2,\{\xi'\}) - K_N(z_1;z_2)\mathcal{P}^{(M+1)}_1(z_2,\{\xi\};z_1,\{\xi'\}) - \sum_{k=1}^{M}(-1)^k K_N(z_1;\xi'_k)\mathcal{P}^{(M+1)}_1(z_2,\{\xi\};z_1,z_2,\{\xi'\}_k)\Big] \tag{B.26}$$

Inserting this into the definition and using (B.22), we get

$$\begin{aligned}\mathcal{P}^{(2)}_{1-\nu}(z_1,z_2;z_1,z_2) = \frac{1}{2}\Big[&K_N(z_1;z_1)\mathcal{P}^{(1)}_{1-\nu}(z_2;z_2) - K_N(z_1;z_2)\mathcal{P}^{(1)}_{1-\nu}(z_2;z_1)\\ &-\frac{1}{2}(M+1)\sum_k(-1)^k\int[d\xi'][d\xi]K_N(z_1;\xi'_k)\\ &\mathcal{P}^{(M+1)}_1(z_2,\{\xi\};z_1,z_2,\{\xi'\}_k)\Psi_\nu(\xi'_k,\{\xi'\})\Psi^*_\nu(\xi_k,\{\xi\}_k)\Big]. \end{aligned} \tag{B.27}$$

In order to evaluate the sum in the second line, we use the expansion

$$\begin{aligned}\mathcal{P}^{(M+1)}_1(z_2,\{\xi\};z_1,z_2,\{\xi'\}_k) =&\frac{1}{(M+1)}\Big[K_N(z_2;z_1)\mathcal{P}^{(M)}_1(\{\xi\};z_2,\{\xi'\}_k)\\ &- K_N(z_2,z_2)\mathcal{P}^{(M)}_1(\{\xi\};z_1,\{\xi'\}_k)\\ &+\sum_{j\neq k=1}^{M}(-1)^{j+1}K_N(z_2,\xi'_j)\mathcal{P}^{(M)}_1(\{\xi\};z_1,z_2,\{\xi'\}_{j,k})\Big].\end{aligned} \tag{B.28}$$

Plugging this in and using (B.17) gives

$$\begin{aligned}\mathcal{P}^{(2)}_{1-\nu}(z_1,z_2;z_1,z_2) = \frac{1}{2}\Big[&K_N(z_1;z_1)\mathcal{P}^{(1)}_{1-\nu}(z_2;z_2) - K_N(z_1;z_2)\mathcal{P}^{(1)}_{1-\nu}(z_2;z_1)\\ &-\frac{1}{2}\sum_k(-1)^k\int[d\xi']K_N(z_1;\xi'_k)K_N(z_2;z_1)\Psi_\nu(\{\xi'\})\Psi^*_\nu(z_2,\{\xi'\}_k)\\ &+\frac{1}{2}\sum_k(-1)^k\int[d\xi']K_N(z_1;\xi'_k)K_N(z_2;z_2)\Psi_\nu(\{\xi'\})\Psi^*_\nu(z_1,\{\xi'\}_k)\\ &-\frac{1}{2}\sum_{k\neq j}(-1)^{k+j+1}\int[d\xi']K_N(z_1;\xi'_k)K_N(z_2;\xi'_j)\Psi_\nu(\{\xi'\})\Psi^*_\nu(z_1,z_2,\{\xi'\}_{j,k})\Big].\end{aligned} \tag{B.29}$$

which can be expressed in terms of the reduced density matrix of the Ψ_ν state as

$$\begin{aligned}\mathcal{P}^{(2)}_{1-\nu}(z_1,z_2;z_1,z_2) = \frac{1}{2}\Big[&K_N(z_1;z_1)\mathcal{P}^{(1)}_{1-\nu}(z_2;z_2) - K_N(z_1;z_2)\mathcal{P}^{(1)}_{1-\nu}(z_2;z_1)\\ &+\frac{1}{2}\int d^2\xi_1' K_N(z_1;\xi_1')K_N(z_2;z_1)\mathcal{P}^{(1)}_{\nu}(\xi_1';z_2)\\ &-\frac{1}{2}\int d^2\xi_1' K_N(z_1;\xi_1')K_N(z_2;z_2)\mathcal{P}^{(1)}_{\nu}(\xi_1';z_1)\\ &+\int d^2\xi_1' d^2\xi_2' K_N(z_1;\xi_1')K_N(z_2;\xi_2')\mathcal{P}^{(2)}_{\nu}(\xi_1',\xi_2';z_1,z_2)\Big] \end{aligned} \tag{B.30}$$

using the reproducing property of the kernel, and the determinant expression for the 2-particle density matrix $\mathcal{P}^{(2)}_1$, this can be brought into the form presented in the paper.

B.4 Two-point functions of density from the generating functional

Next we explain how to obtain connected two-point functions of density $\rho_{1-\nu}$ from the generating functional $\log \mathcal{Z}_{1-\nu}$. Indeed, differentiating (3.23) w.r.t. $W(\zeta')$

$$\frac{\delta}{\delta W(\zeta')}\langle\rho_{1-\nu}(\zeta)\rangle = \langle\rho_{1-\nu}(\zeta')\rho_{1-\nu}(\zeta)\rangle_c + [\rho_\nu(\zeta');\rho_{1-\nu}(\zeta)]\,, \tag{B.31}$$

where we have introduced a "mixed" correlator

$$\begin{aligned}[\rho_\nu(\zeta');\rho_{1-\nu}(\zeta)] = 2\frac{(N+M)!}{N!M!}&\int [d\xi][d\xi']\Psi_\nu(\{\xi\})\Psi^*_\nu(\{\xi'\})\rho_\nu(\zeta')\\ &\int [dz]\Psi_1(\{z\},\{\xi'\})\Psi^*_1(\{z\},\{\xi\})\rho_{1-\nu}(\zeta)\,, \end{aligned} \tag{B.32}$$

which signals the failure of the formula (3.6) for PH-dual states. To evaluate this term we

combine (B.31) together with (3.20) and (3.24) to obtain

$$\begin{aligned}\frac{\delta^2 \ln \mathcal{Z}_{1-\nu}}{\delta W(\zeta')\delta W(\zeta)} &= \langle \rho_{1-\nu}(\zeta)\rho_{1-\nu}(\zeta')\rangle_c + 2\langle \rho_\nu(\zeta)\rho_\nu(\zeta')\rangle_c + [\rho_\nu(\zeta');\rho_{1-\nu}(\zeta)] \\ &= 2\langle \rho_1(\zeta)\rho_1(\zeta')\rangle_c - \langle \rho_{1-\nu}(\zeta)\rho_{1-\nu}(\zeta')\rangle_c - [\rho_\nu(\zeta');\rho_{1-\nu}(\zeta)]\,. \end{aligned} \tag{B.33}$$

Which implies

$$[\rho_\nu(\zeta');\rho_{1-\nu}(\zeta)] = \langle \rho_1(\zeta)\rho_1(\zeta')\rangle_c - \langle \rho_{1-\nu}(\zeta)\rho_{1-\nu}(\zeta')_c - \langle \rho_\nu(\zeta)\rho_\nu(\zeta')\rangle_c\,. \tag{B.34}$$

In this form, one can see that if the connected two-point functions were related in a manner similar to the one-point functions (3.20) of the main text, then the mixed correlator would vanish. In the translation-invariant state, we can use (3.26) to evaluate this in momentum space, and combined with (3.27) we find a simple expression for the mixed correlator

$$[\rho_\nu;\rho_{1-\nu}](q) = -2\bar{\rho}_\nu \bar{s}_\nu = -2\bar{\rho}_{1-\nu}\bar{s}_{1-\nu}\,. \tag{B.35}$$

A comment is in order. There is a well-known bound on the coefficient $\bar{s}_4$ defined as $\bar{s}_{1-\nu}(q) = \bar{s}_4 q^4 + \ldots$ that is valid for any LLL state [22, 32]

$$\bar{s}_4 \geq \left|\frac{\mathcal{S}-1}{8}\right|. \tag{B.36}$$

This inequality is saturated by a class of states annihilated by a certain short-range potential [70]. It is easy to see that PH-dual states also saturate this inequality.

$$\sigma^H_\nu(q) \approx \frac{2\bar{\rho}_\nu}{q^2} s_\nu(q)\,, \tag{B.37}$$

and

$$\sigma^H_{1-\nu}(q) \approx \frac{2\bar{\rho}_{1-\nu}}{q^2}(2s_1(q) - s_{1-\nu}(q))\,, \tag{B.38}$$

where "≈" means equality up to sub-leading in the 1/N expansion.

APPENDIX C

DERIVATION OF EQUATION (5.6)

In this appendix, we will derive the equation (5.6) in detail. We consider the Jain's sequences $\nu = \frac{n}{2n+1}, \frac{n+1}{2n+1}$. In this case the residue magnetic field felled by composite fermion is

$$\bar{b} = \pm \frac{\bar{B}}{2n+1}, \tag{C.1}$$

The Dirac composite fermion with density $\bar{\rho}_{CF} = \frac{\bar{B}}{4\pi}$ will from IQH with filling fraction $\nu_{CF} = \pm(n + 1/2)$. Consider the action

$$S = \int d^3x \frac{i}{2} \left(\psi^\dagger \overleftrightarrow{D}_t \psi + v^i \psi^\dagger \overleftrightarrow{D}_i \psi + v_F \psi^\dagger \sigma^i \overleftrightarrow{D}_i \psi \right) \tag{C.2}$$

We can integrate out fermion field to derive the generating function

$$\mathcal{W}[\delta a_\mu, v_i] = \frac{1}{i} \ln \int \mathscr{D}\bar{\psi} \mathscr{D}\psi \, e^{iS[\psi, \delta a_\mu, v_i]} \tag{C.3}$$

We separate the contributions to the generating function as in Figure C.1

$$\mathcal{W}[\delta a_\mu, v_i] = W^{(1)} + W^{(2)} + W_c^{(2)} \tag{C.4}$$

We can derive the first $W^{(1)}$ and the third $W_c^{(2)}$ contribution easily, they are related to the Dirac composite fermion density

$$W^{(1)} = \int d^3x \bar{\rho}_{CF} \delta a_0 \tag{C.5}$$

$$W_c^{(2)} = \int d^3x \bar{\rho}_{CF} \delta^{ij} \delta a_i v_j \tag{C.6}$$

In order calculate $W^{(2)}$, we need to evaluate the loop integral of IQH system of massless Dirac fermion. In the Ref [69], the authors develope an algebraic method to translate the

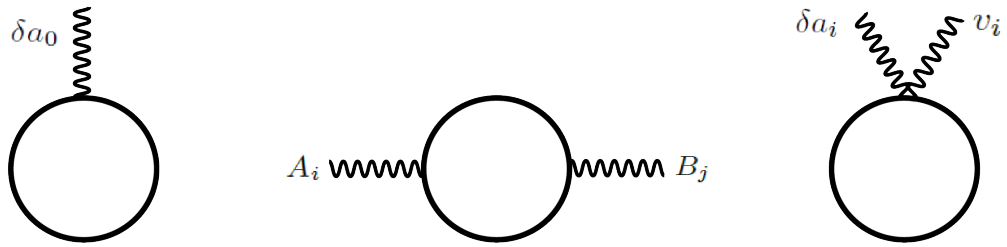

Figure C.1: The generating functional to quadratic order in external fields is given by the sum of three diagrams. The first diagram, $W^{(1)}$, is linear in the perturbations of the electromagnetic field and describes the constant background density of composite fermion. The second diagram, $W^{(2)}$, contains the main contribution to the generating functional, A_i and B_j can be either a_i or v_j. Finally, the third diagram, $W_c^{(2)}$, contains the contact terms

loop integral to the sums including Laguerre polynomials. The explicit calculation of $W^{(2)}$ follows similarly as in the Ref [69] with the identical unperturbed Green's function. We separate $W^{(2)}$ as

$$W^{(2)} = W_1^{(2)}[\delta a_\mu] + W_2^{(2)}[v_i] + W_3^{(2)}[\delta a_\mu, v_i], \tag{C.7}$$

Where $W_1^{(2)}[\delta a_\mu]$ is the Dirac polarization tensor of IQH system

$$W_1^{(2)}[\delta a_\mu] = \frac{1}{2}\int \frac{d^2\mathbf{k}}{4\pi^2}\frac{d\omega}{2\pi}\delta a_i(\omega,\mathbf{k})\Pi_D^{ij}(\omega,\mathbf{k})\delta a_j(-\omega,-\mathbf{k}) \tag{C.8}$$

$W_2^{(2)}[v_i]$ is the result of the loop diagram with the external lines are v_i

$$W_2^{(2)}[v_i] = \frac{1}{2}\int \frac{d^2\mathbf{k}}{4\pi^2}\frac{d\omega}{2\pi}v_i(\omega,\mathbf{k})\mathcal{V}^{ij}(\omega,\mathbf{k})v_j(-\omega,-\mathbf{k}) \tag{C.9}$$

$W_3^{(2)}[\delta a_i, v_i]$ is the result of the loop diagram with one external is δa_μ and the other one is v_i

$$W_3^{(2)}[\delta a_i, v_i] = \int \frac{d^2\mathbf{k}}{4\pi^2}\frac{d\omega}{2\pi}\delta a_\mu(\omega,\mathbf{k})\Sigma^{\mu i}(\omega,\mathbf{k})v_i(-\omega,-\mathbf{k}) \tag{C.10}$$

To simplify the calculation, we chose the reference frame in which $\mathbf{k} = (k_1, 0)$. The algebra is tedious but straightforward, in this appendix we only quote the final results with the terms contribute to sub-leading in large n regime. The space components of Dirac electromagnetic

polarization tensor are

$$\Pi_D^{12}(\omega,\mathbf{k}) = -i\omega\frac{n+1/2}{2\pi} + i\omega(k_1\ell)^2\frac{6n^2+6n+1}{16\pi} - i\omega^3\frac{\ell^2}{v_F^2}\frac{8n^2+8n+1}{8\pi} + \cdots, \quad \text{(C.11)}$$

$$\Pi_D^{11}(\omega,\mathbf{k}) = -\omega^2\frac{3\ell}{2\sqrt{2}\pi v_F}\zeta\left(-\frac{1}{2}, n+1\right) + k_1^2\omega^2\frac{15\ell^3}{16\sqrt{2}\pi v_F}\zeta(-\frac{3}{2}, n+1) + \cdots, \quad \text{(C.12)}$$

$$\Pi_D^{22}(\omega,\mathbf{k}) = -\omega^2\frac{3\ell}{2\sqrt{2}\pi v_F}\zeta\left(-\frac{1}{2}, n+1\right) + k_1^2\frac{3\ell v_F}{4\sqrt{2}\pi}\zeta\left(-\frac{1}{2}, n+1\right) + \cdots, \quad \text{(C.13)}$$

where we define the residue magnetic length $\ell = 1/\sqrt{|\bar{b}|}$. $\zeta(s,q)$ is the Hurwitz $\zeta-$function

$$\zeta(s,q) = \sum_{n=0}^{\infty}\frac{1}{(n+q)^s} \quad \text{(C.14)}$$

For $x \neq -1$, we can check that, at large n limit

$$\zeta(-x, n+1) \approx -\frac{1}{x+1}(n+1/2)^{x+1}, \quad \text{(C.15)}$$

where $\approx$ means equal upto subleading order in $1/n$ expansion

$$\mathcal{V}^{12}(\omega,\mathbf{k}) = -i\omega\frac{8n^2+8n+1}{8\pi\ell^2 v_F^2} + i\omega k_1^2\frac{(2n+1)(6n^2+6n+1/2)}{16\pi v_F^2} + \cdots \quad \text{(C.16)}$$

$$\mathcal{V}^{11}(\omega,\mathbf{k}) = -\frac{3}{2\sqrt{2}\pi\ell^3 v_F}\zeta(-\frac{1}{2}, n+1) - \omega^2\frac{5}{\sqrt{2}\pi\ell v_F^3}\zeta(-\frac{3}{2}, n+1) + k_1^2\omega^2\frac{21\ell}{8\sqrt{2}\pi v_F^3}\zeta(-\frac{5}{2}, n+1) + \cdots \quad \text{(C.17)}$$

$$\mathcal{V}^{22}(\omega,\mathbf{k}) = -\frac{3}{2\sqrt{2}\pi\ell^3 v_F}\zeta(-\frac{1}{2},n+1) - \omega^2\frac{5}{\sqrt{2}\pi\ell v_F^3}\zeta(-\frac{3}{2},n+1) + k_1^2\frac{5}{2\sqrt{2}\pi\ell v_F}\zeta(-\frac{3}{2},n+1)+\cdots \tag{C.18}$$

We can define the regular term and the contact term as

$$\mathcal{V}^{ij}(\omega,\mathbf{k}) = \mathcal{V}^{ij}_{ct} + \mathcal{V}^{ij}_{reg}(\omega,\mathbf{k}) \tag{C.19}$$

where

$$\mathcal{V}^{ij}_{ct} = -\delta^{ij}\frac{3}{2\sqrt{2}\pi\ell^3 v_F}\zeta(-\frac{1}{2},N+1) \tag{C.20}$$

At large n limit, applying (C.15), we obtain

$$\frac{1}{2}\mathcal{V}^{ij}_{ct} \approx \delta^{ij}\frac{1}{2\sqrt{2}\pi\ell^3}(n+\frac{1}{2})^{3/2} = \frac{1}{2}\frac{k_F}{v_F}\bar{\rho}_{CF} \tag{C.21}$$

where we used

$$\bar{\rho}_{CF} = \frac{\bar{B}}{4\pi} = \frac{(2n+1)\ell^{-2}}{4\pi} \tag{C.22}$$

$$k_F = \sqrt{4\pi\rho_{CF}} = \ell^{-1}\sqrt{2n+1} \tag{C.23}$$

Using equation (C.15), we also can show that (up to the order of gradient expanssion that we calculated)

$$\Pi^{ij}_D(\Omega,\mathbf{k}) \approx \frac{v_F^2}{k_F^2}\mathcal{V}^{ij}_{reg}(\omega,\mathbf{k}) = v_F^2\ell^2\frac{\mathcal{V}^{ij}_{norm}(\Omega,\mathbf{k})}{2n+1} \tag{C.24}$$

The result of $\Sigma^{ij}(\omega,\mathbf{k})$ is summarized as follow

$$\Sigma^{12}(\omega,\mathbf{k}) = -i\omega\frac{3}{2\sqrt{2}\pi\ell v_F}\zeta(-\frac{1}{2},n+1) + i\omega k_1^2\ell\frac{3}{10\sqrt{2}\pi v_F}\zeta(-\frac{3}{2},n+1)+\cdots \tag{C.25}$$

$$\Sigma^{11}(\omega,\mathbf{k}) = -\frac{n+1/2}{2\pi\ell^2} - \omega^2\frac{(n+1/2)^2}{\pi v_F^2} + k_1^2\omega^2\frac{3\ell^2(n+1/2)^3}{8\pi v_F^2} + \cdots \tag{C.26}$$

$$\Sigma^{22}(\omega,\mathbf{k}) = -\frac{n+1/2}{2\pi\ell^2} - \omega^2\frac{(n+1/2)^2}{\pi v_F^2} + k_1^2\frac{(n+1/2)}{2\pi} + \cdots \tag{C.27}$$

We also separate $\Sigma^{ij}(\omega,\mathbf{k})$ to the contact part and the regular part

$$\Sigma^{ij}(\omega,\mathbf{k}) = \Sigma^{ij}_{ct} + \Sigma^{ij}_{reg}(\omega,\mathbf{k}) \tag{C.28}$$

where

$$\Sigma^{ij}_{ct} = -\delta^{ij}\frac{2n+1}{4\pi\ell^2} = -\delta^{ij}\bar{\rho}_{CF} \tag{C.29}$$

This contact part will be canceled exactly by the contact term diagram which is

$$\begin{aligned} W^{(2)}_{ct}[\delta a_i, v_i] &= \int d^3x\delta^{ij}\bar{\rho}_{CF}\delta a_i v^j \\ &= \int \frac{d^2\mathbf{k}}{4\pi^2}\frac{d\omega}{2\pi}\delta^{ij}\bar{\rho}_{CF}\delta a_i(\omega,\mathbf{k})v_j(-\omega,-\mathbf{k}) \end{aligned} \tag{C.30}$$

With the help of (C.15), up to leading and sub-leading order in $1/n$ expansion, we obtain

$$\Pi^{ij}_D(\omega,\mathbf{k}) \approx -\frac{k_F}{v_F}\Sigma^{ij}_{reg}(\omega,\mathbf{k}) = -\frac{\ell}{v_F\sqrt{2n+1}}\Sigma^{ij}_{norm}(\omega,\mathbf{k}). \tag{C.31}$$

The relation

$$\Pi^{0j}_D(\omega,\mathbf{k}) \approx -\frac{k_F}{v_F}\Sigma^{0j}_{reg}(\omega,\mathbf{k}), \tag{C.32}$$

follows obviously from gauge invariant. Combining $W^{(1)}$, $W^{(2)}$ and $W^{(2)}_{ct}$ we have the result of Dirac fermion loop integral, up to quadratic order in perturbation

$$\mathcal{W}(\delta a_i, v_i) \approx \int dtd^2\mathbf{x}\bar{\rho}_{CF}\delta\tilde{a}_0 + \frac{1}{2}\int\frac{d^2\mathbf{k}}{4\pi^2}\frac{d\Omega}{2\pi}\delta\tilde{a}_\mu(\omega,\mathbf{k})\Pi^{\mu\nu}_D(\omega,\mathbf{k})\tilde{a}_\nu(-\omega,-\mathbf{k}) \tag{C.33}$$

where

$$\delta\tilde{a}_0 = \delta a_0 + \frac{1}{2}\frac{k_F}{v_F}v_i v^i, \quad \delta\tilde{a}_i = \delta a_i - \frac{k_F}{v_F}v_i. \tag{C.34}$$

Therefore we can conclude that, in RPA calculation and large n limit

$$\int d^3x \frac{i}{2}\left(\psi^\dagger \overleftrightarrow{D}_t \psi + v^i \psi^\dagger \overleftrightarrow{D}_i \psi + v_F \psi^\dagger \sigma^i \overleftrightarrow{D}_i \psi\right) \approx \int d^3x \frac{i}{2}\left(\psi^\dagger \overleftrightarrow{\tilde{D}}_t \psi + v_F \psi^\dagger \sigma^i \overleftrightarrow{\tilde{D}}_i \psi\right) \tag{C.35}$$

Since we obtain the same generating function $\mathcal{W}(\delta a_i, v_i)$ as the result of integrating out the Dirac composite fermion.

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