THE UNIVERSITY OF CHICAGO

# DISPERSIVE EQUATIONS WITH MULTIPLE POTENTIALS

A DISSERTATION SUBMITTED TO
THE FACULTY OF THE DIVISION OF THE PHYSICAL SCIENCES
IN CANDIDACY FOR THE DEGREE OF
DOCTOR OF PHILOSOPHY

DEPARTMENT OF MATHEMATICS

BY
GONG CHEN

CHICAGO, ILLINOIS
JUNE 2017

To my family, my loved ones.

# CONTENTS

# ACKNOWLEDGMENTS

First and foremost, I would like to thank my advisor, Professor Wilhelm Schlag. No words can express my thanks to his kind encouragement, countless hours that he has spent teaching and guiding me. The way Professor Schlag taught me and explained problems influenced me a lot. He made me understand that even when I was completely miserable about the progress of my work but things would work out as long as I kept working. Professor Schlag did not simply give me problems to work on but also created a path for me to follow for the rest of my career. He is always an idol in mathematics for me. He not only taught me to be a mathematician but also a mature person. I am honored and regard it as a privilege to be his student.

Secondly, I want to thank my "academic brothers" – Marius Beceanu and Andrew Lawrie. Their played crucial roles during my time at Chicago. My thesis works were influenced by Marius' works in many aspects. The conversations with Marius provided a very valuable perspective on dispersive decays. I am very grateful to Andy for a lot of support and encouragement. Sometimes, Andy is just like my second advisor.

Next, I would like to thank Professor Carlos Kenig, Professor Madhav Nori, Professor Luis Silvestre and Professor Takis Sougandis. I have learned so much from their classes. I appreciate the time and patience that them spent taking care of me. They had a profound impact on my mathematical education. My works on parabolic equations [16, 17, 18] benefited a lot from many conversations with Professor Silvestre.

I am also extremely grateful to our analysis postdocs – Jacek Jendrej, Hao Jia, Tianling Jin, Wenjia Jing, Baoping Liu, Guixiang Xu and Jinxin Xue. Thank them for many fruitful discussions on mathematics and teaching me so much. In particular, thanks Baoping and Jinxin for listening to me during my hard times and giving me a lot of suggestions to overcome many difficulties.

I want thank my mentors at University of Minnesota – Professor Dennis Hejhal and

Professor Peter Olver. Even after I left Minnesota, I got tremendous encouragement and support from them. In particular, Professor Hejhal's two visits and phone calls supported me to overcome my hard periods. I am also grateful to Professor Mikhail Safonov and Professor Vladimir Sverak who influenced me to become an analyst.

I would never have made myself survive through my time here at Chicago, in particular the first-year, without my close friends and classmates, Daniel Campos, Wei Chai, Yang Ge, Jon Rubin, Ben Seeger, Yiwen Zhou, Foling Zou and all my first-year fellows.

Finally, tremendous thanks go to my family. Nothing could come close to touching what them mean to me. They helped, supported, and made me very happy over the years. Thank you for putting up with me, and for everything. I am sorry being so annoying and for making you worry.

# ABSTRACT

In this thesis, we study dispersive equations with several moving potentials, a.k.a. charge transfer Hamiltonians. We mainly focus on two models: the Schödinger equation and the wave equation. We prove linear estimates and analyze nonlinear models based on them.

In Chapter 1, we briefly survey the historical backgrounds and motivation for the main results in this thesis.

Then, in Chapter 2, we prove Strichartz estimates for scattering states of the scalar charge transfer models in $\mathbb{R}^3$. More precisely, we study the time-dependent charge transfer Hamiltonian

$$H(t) = -\frac{1}{2}\Delta + \sum_{j=1}^{m} V_j(x - \vec{v}_j t) \tag{0.0.1}$$

with rapidly decaying smooth potentials $V_j(x)$, say, exponentially decaying and a set of mutually non-parallel constant velocities $\vec{v}_j$. We prove Strichartz estimates for the evolution

$$\frac{1}{i}\partial_t \psi + H(t)\psi = 0. \tag{0.0.2}$$

Based on the idea of the proof of Strichartz estimates which follows [22, 51], we also show the energy of the whole evolution is bounded independent of time without using the phase space method, for example, in [27]. One can easily generalize our argument to $\mathbb{R}^n$ for $n \geq 3$. We also discuss the extension of above results to matrix charge transfer models in $\mathbb{R}^3$.

Next, in Chapter 3, we prove Strichartz estimates (both regular and reversed) for a scattering state to the wave equation with a charge transfer Hamiltonian in $\mathbb{R}^3$:

$$\partial_{tt} u - \Delta u + \sum_{j=1}^{m} V_j \left(x - \vec{v}_j t\right) u = 0 \tag{0.0.3}$$

where $V_j(x)$'s are rapidly decaying smooth potentials and $\left\{\vec{v}_j\right\}$ is a set of distinct constant

velocities such that

$$|\vec{v}_i| < 1, \ 1 \leq i \leq m. \tag{0.0.4}$$

The energy estimate and the local energy decay of a scattering state are also established. In order to study nonlinear multisoltion systems, we will present the inhomogeneous generalizations of Strichartz estimates. As an application of our results, we show that scattering states indeed scatter to solutions to the free wave equation.

In Chapter 4, we study the endpoint reversed Strichartz estimates along general time-like trajectories for wave equations in $\mathbb{R}^3$. We also discuss some applications of the reversed Strichartz estimates and the structure formula of wave operators to the wave equation with one potential. These techniques are useful to analyze the stability problem of traveling solitons.

In Chapter 5, lastly, we construct multisoliton solutions to the defocusing energy critical wave equation with potentials in $\mathbb{R}^3$

$$\partial_{tt} u - \Delta u + \sum_{j=1}^{m} V_j \left( x - \vec{v}_j t \right) u + u^5 = 0 \tag{0.0.5}$$

based on regular and reversed Strichartz estimates developed in Chapter 3 for wave equations with charge transfer Hamiltonians. We also show the asymptotic stability of multisoliton solutions. The multisoliton structures with both stable and unstable solitons are covered. Since each soliton decays slowly with rate $\frac{1}{\langle x \rangle}$, the interactions among the solitons are strong. Some reversed Strichartz estimates and local decay estimates for the charge transfer model are established to handle strong interactions.

# CHAPTER 1

# INTRODUCTION

## 1.1 Background

This thesis consists of various results on dispersive equations with multiple moving potentials. Roughly speaking, dispersion means that when no boundary is present, waves of different wavelengths travel at different phase speeds: long wavelength components propagate faster than short ones. This is the reason why over time dispersive waves spread out in space as they evolve in time, while conserving some form of energy. The main focus will be Schödinger equations and wave equations. In this section, to set up and motivate later results in this thesis. we discuss three subjects: dispersive estimates, Strichartz estimates and smoothing effects.

### *1.1.1 Free linear Schrödinger equations*

First of all, we consider the free Schrödinger equation in $\mathbb{R}^n$:

$$i\partial_t u + \Delta u = 0 \tag{1.1.1}$$

with initial data

$$u(0) = f.$$

The dispersive estimate for the free linear Schrödinger equation is the following estimate from $L^1$ to $L^\infty$:

$$\|u(t)\|_{L^\infty_x} \leq C|t|^{-\frac{n}{2}} \|f\|_{L^1_x} \tag{1.1.2}$$

With the trivial conservation of $L^2$ norm of the solution and interpolation, we get

$$\|u(t)\|_{L_x^q(\mathbb{R}^n)} \leq C\,|t|^{(-n/2)(1/q'-1/q)}\,\|f\|_{L_x^{q'}(\mathbb{R}^n)} \tag{1.1.3}$$

for $1 \leq q' \leq 2$.

The dispersive estimates can be understood as the consequence of the conservation of $L^2$ norm of the solution and the dispersion effects. Heuristically, the constant $L^2$ mass spreads on a region of volume comparable with $t^n$. This statement can be made more precise by calculating the Schrödinger evolution of a wave packet which is localized near position $x_0$ with frequency is localized near $\xi_0$. The dispersive estimate for the linear free Schrödinger can be directly derived from the explicit formula for the solution. Or one might conclude the dispersive estimate from rescaling argument used to derive the fundamental solution. But in order to be consist with the later sections, we will derive the dispersive estimate from the view of oscillatory integrals.

### *1.1.2 Stationary and non-stationary phase.*

Following [46], we consider oscillatory integrals of the form

$$I(\lambda) = \int_{\mathbb{R}^n} e^{i\lambda\phi(\xi)} a(\xi)\, d\xi, \tag{1.1.4}$$

with a compactly supported function $a \in C_0^\infty$ and smooth real-valued phase $\phi \in C^\infty$. In order to estimate the decay of oscillatory integrals, we list the results of stationary and non-stationary phase.

**Lemma 1.1.1** (Non-stationary phase)**.** *If* $\nabla\phi \neq 0$ *on supp*$(a)$ *then the integral* (1.1.4) *decays as*

$$\left|\int_{\mathbb{R}^n} e^{i\lambda\phi(\xi)} a(\xi)\, d\xi\right| \leq C(N,\, a,\, \phi)\lambda^{-N}, \quad \lambda \to \infty,$$

*for arbitrary $N \geq 1$.*

**Lemma 1.1.2** (Stationary phase)**.** *If $\nabla\phi(\xi_0) = 0$ for some $\xi_0 \in supp(a)$, $\nabla\phi \neq 0$ away from $\xi_0$, and the Hessian of $\phi$ at the stationary point $\xi_0$ is nondegenerate, i.e., $\det D^2\phi(\xi_0) \neq 0$, then for all $\lambda \geq 1$*

$$\left| \int_{\mathbb{R}^n} e^{i\lambda\phi(\xi)} a(\xi)\, d\xi \right| \leq C(N,\, a,\, \phi)\lambda^{-n/2}.$$

To connect to our dispersive estimates, we apply the above lemmas to concrete examples. We investigate the Fourier transform of $d\mu = \phi d\sigma$ where $\sigma$ is the surface measure of a hypersurface $\mathcal{M}$ in $\mathbb{R}^n$ and $\phi$ is compactly supported and smooth. Denote the unit vector normal to $\mathcal{M}$ at $x \in \mathcal{M}$ by $N(x)$.

**Corollary 1.1.3.** *Let $\xi = \lambda\nu$ where $\nu \in S^{n-1}$ and $\lambda \geq 1$. If $\nu$ is not parallel to $N(x)$ for any $x \in supp(\phi)$ the $\hat{\mu} = \mathcal{O}(\lambda^{-k})$ as $\lambda \to \infty$ for any positive integer $k$. Assume that $\mathcal{M}$ has nonvanishing Gaussian curvature on $supp(\phi)$. Then*

$$\hat{\mu}(\xi) \leq C(\mu, \mathcal{M})|\lambda|^{-(n-1)/2}$$

*for all $\lambda \geq 1$ and all $\nu$. This is optimal if $\nu$ belongs to the normal bundle of $\mathcal{M}$ on $supp(\phi)$.*

Another corollary which is useful to analyze the decay of the linear wave equation is the following:

**Corollary 1.1.4.** *One has the representation*

$$\widehat{\sigma_{S^{n-1}}}(x) = e^{i|x|}\omega_+(|x|) + e^{-i|x|}\omega_-(|x|), \quad |x| \geq 1,$$

*where $\omega_\pm$ are smooth and satisfy*

$$\left|\partial_r^k \omega_\pm(r)\right| \leq C_k r^{-(n-1)/2-k},\ \forall r \geq 1$$

*and all* $k \geq 1$.

We apply the above Corollary 1.1.3 to the linear Schrödinger equation. Consider the linear Schrödinger equation

$$\partial_t u + i\frac{1}{2\pi}\Delta_x u = 0, \;\; u|_{t=0} = f. \tag{1.1.5}$$

Then

$$u(x,t) = \int_{\mathbb{R}^n} e^{2\pi i(x\cdot\xi + t|\xi|^2)}\hat{f}(\xi)d\xi = \left(\hat{f}\mu\right)^{\vee}(x,t), \tag{1.1.6}$$

where $\mu$ is the measure in $\mathbb{R}^{n+1}$ defined by the integral

$$\int_{\mathbb{R}^{n+1}} h(\xi,\tau)\mu(d(\xi,\tau)) = \int_{\mathbb{R}^n} h(\xi,|\xi|^2)d\xi, \tag{1.1.7}$$

for all $h \in C^0(\mathbb{R}^{n+1})$. Now if we assume $supp(\hat{f}) \subset B(0,1)$, by the above corollary, we have

$$\|u(t)\|_{L^\infty_x} \leq C|t|^{-n/2}\,\|f\|_{L^1_x}.$$

The general case can be obtained by a simple rescaling argument. One can find details on this point in next section on Strichartz estimates.

### *1.1.3 Free linear wave equations.*

We briefly discuss the free linear wave equation

$$\Box u = \partial_{tt}u - \Delta u = h, \;\; u|_{t=0} = f, \;\; \partial_t u|_{t=0} = g.$$

For the linear wave equation, with Littlewood-Paley decomposition, it is crucial to derive the dispersive estimate for the evolution $e^{\pm it\sqrt{-\Delta}}$.

**Lemma 1.1.5.** *For $f \in \mathcal{S}$ satisfying supp$(\hat{f}) \subset \{1 \leq |\xi| \leq 2\}$, we have*

$$\left\| e^{\pm it\sqrt{-\Delta}} \right\|_{L^\infty_x} \leq C|t|^{-(n-1)/2} \|f\|_{L^1_x}, \; |t| \geq 1.$$

There is also some sort of physical interpretation of this phenomenon. The conservation of $L^2$ norm and the solution concentrates in a shell of constant thickness around the light cone in odd dimension. For even dimension, the solution tends to concentrate along the light cone. Thus, the energy spreads over a volume of size $t^{n-1}$.

### *1.1.4 Linear Schrödinger operator with stationary potentials.*

To analyze the dispersive estimate of linear dispersive equations with potentials, we consider the dispersive estimates the flows

$$e^{i\sqrt{H}t}P_c, \; e^{itH}P_c, \quad \text{where} \;, \; H = -\Delta + V.$$

on $\mathbb{R}^n$, where $P_c$ is the projection onto the continuous spectrum of $H$. $V$ is a real-valued potential that is assumed to satisfy some decay condition at infinity. This decay is typically expressed in terms of the point-wise decay $|V(x)| \leq C < x >^{-\beta}$, for all $x \in \mathbb{R}^n$ and for some $\beta > 0$. We use the notation $< x >= (1 + |x|^2)^{\frac{1}{2}}$. Occasionally, we will use an integrability condition $V \in L^p(\mathbb{R}^n)$ (or a weighted variant of it) instead of a point-wise condition. These decay conditions will also be such that $H$ is asymptotically complete:

$$L^2(\mathbb{R}^n) = L^2_{p.p.}(\mathbb{R}^n) \oplus L^2_{a.c.}(\mathbb{R}^n)$$

where the spaces on the right-hand side refer to the span of all eigenfunctions, and the absolutely continuous subspace, respectively.

For the results and historical progress of this topic, one can find details and reference from [53].

### *1.1.5 Strichartz estimates*

We mainly focus the Schödinger equation. The idea for the wave equations is similar.

Roughly speaking, Strichartz estimates can be regarded as smoothing effects in $L^q_x$ spaces. For example, when we consider the Schrödinger equation, compared with trivial conservation of $L^2$ norm of the solution, in Strichartz estimates one gains space integrability from $q = 2$ to $q > 2$, but one looses time integrability from $p = \infty$ to $p < \infty$. It is very easy to see, that if we take a function $g \in L^2$ but $g \notin L^q_x$ for $q > 2$. Then we take the initial data $f = e^{it_0\Delta} g$ as the initial data for the free linear Schrödinger equation, then we can see that at $t = t_0$, $e^{-it_0\Delta} f \notin L^q_x$. So without integration/ average on time, there is no hope to get $L^q_x$ estimate for all the time for general $L^2$ initial data. From the discussion here, we know that under the dispersion phenomenon, the solution tends to spread out in the space, as least in time-average sense, therefore we obtain higher integrability.

Mainly following [46], we can present a argument based the $TT^*$ argument due to Ginibre and Velo in a series of papers. First we present general Strichartz estimates for the linear Schrödinger equation. We still consider the linear Schrödinger equation

$$\partial_t u + i\frac{1}{2\pi}\Delta_x u = h, \;\; u|_{t=0} = f. \tag{1.1.8}$$

**Definition.** We call a pair $(p, q)$ Schrödinger-admissible if and only if

$$\frac{2}{p} + \frac{n}{q} = \frac{n}{2} \tag{1.1.9}$$

and if $2 \leq q \leq \infty$ with $(p, q) \neq (2, \infty)$.

**Theorem 1.1.6.** *Let $h$ be a space-time Schwartz function in $n + 1$ dimensions and $f$ a* *spacial Schwartz function, let $u(x,t)$ solve equation* (1.1.8)*. Then*

$$\|u\|_{L_t^p L_x^q(\mathbb{R}^{n+1})} \leq C\left(\|f\|_{L_x^2} + \|h\|_{L_t^{a'} L_x^{b'}(\mathbb{R}^{n+1})}\right), \tag{1.1.10}$$

*where $(p,q)$ and $(a,b)$ are Schrödinger-admissible with $a > 2$ and $p > 2$. Finally, these estimates localize it time: if on the left-hand side $u$ is restricted to some time interval $I \ni 0$ then on the right-hand side $h$ can be restricted to $I$.*

*Remark.* The estimate (1.1.10) is scaling invariant, so the pairs $(p,q)$ and $(a,b)$ must be Schrödinger-admissible.

*Remark.* The requirement $q > 2$ is technical. The results of endpoint of Strichartz estimates, $q = 2$ and $n \geq 3$, can not be obtained by $T^*T$ argument, they are verified by finer interpolations with results from non-endpoint cases. One can find details in [37] .

Under stronger conditions on $h$ and $f$, it is easy to solve equation (1.1.8) by means of Duhamel's formula,

$$u(t) = e^{-i\Delta t/2\pi} f + \int_0^t e^{-i\Delta(t-s)/2\pi} h(s)ds. \tag{1.1.11}$$

Now we are ready to proceed the proof of Theorem 1.1.6.

*Proof.* (Sketch) First of all, we assume $h = 0$ and let $U(t)$ denote the propagator which means $U(t)f$ is the solution of equation (1.1.8) as (1.1.11). By the dispersive estimate, we have

$$\|U(t)f\|_{L^q(\mathbb{R}^n)} \leq C\,|t|^{(-n/2)(1/q'-1/q)}\,\|f\|_{L^{q'}(\mathbb{R}^n)} \tag{1.1.12}$$

for $1 \leq q' \leq 2$. By the $TT^*$ argument, we have the following three estimates are equivalent:

$$\|Uf\|_{L_t^p L_x^q(\mathbb{R}^{n+1})} \leq C\,\|f\|_{L_x^2},$$

$$\|U^*F\|_{L_x^2} \leq C\,\|F\|_{L_t^{p'} L_x^{q'}(\mathbb{R}^{n+1})},$$

$$\|U \circ U^* F\|_{L_t^p L_x^q(\mathbb{R}^{n+1})} \leq C^2 \|F\|_{L_t^{p'} L_x^{q'}(\mathbb{R}^{n+1})}.$$

We know

$$U^* F = \int_{-\infty}^{\infty} U(-s)F(s)ds,$$

and hence we also get

$$(U \circ U^*)F(t) = \int_{-\infty}^{\infty} U(t-s)F(s)ds.$$

Then applying the norm and the estimate (1.1.12), we have

$$\|(U \circ U^*)F(t)\|_{L_x^q} \leq C \int_{-\infty}^{\infty} |t|^{(-n/2)(1/q'-1/q)} \|F(s)\|_{L_x^{q'}} ds.$$

Then by the Schrödinger-admissible relation,

$$1 + \frac{1}{p} = \frac{1}{p'} + \frac{n}{2}\left(\frac{1}{q'} - \frac{1}{q}\right), \ p > 2$$

so

$$0 < \frac{n}{2}\left(\frac{1}{q'} - \frac{1}{q}\right) < 1.$$

Next applying fractional integration over time, one has

$$\|U \circ U^* F\|_{L_t^p L_x^q(\mathbb{R}^{n+1})} \leq C^2 \|F\|_{L_t^{p'} L_x^{q'}(\mathbb{R}^{n+1})}.$$

We are done with $h = 0$. Now suppose $f = 0$, then we can write

$$u(t) = \int_0^t U(t-s)F(s)ds = \int_{-\infty}^{\infty} \chi_{[0<s<t]} U(t-s)F(s)ds.$$

By similar steps to the above, we get

$$\|u\|_{L_t^p L_x^q} \leq C \|F\|_{L_t^{p'} L_x^{q'}}.$$

Clearly,

$$\|u(t)\|_{L_x^q} \le \int_{-\infty}^{\infty} \|U(t-s)F(s)ds\|_{L_x^q} \, ds,$$

then

$$\begin{aligned} \|u\|_{L_t^p L_x^q} &\le C \int_{-\infty}^{\infty} \|U(t-s)F(s)ds\|_{L_t^p L_x^q} \, ds \\ &\le C \int_{-\infty}^{\infty} \|F(s)\|_{L_x^2} \, ds = \|F\|_{L_t^1 L_x^2} . \end{aligned}$$

Finally, by duality and interpolation,

$$\|u\|_{L_t^p L_x^q} \le C \, \|F\|_{L_t^{a'} L_x^{b'}}$$

for any Schrödinger-admissible pairs $(a, b)$ and $(p, q)$ with $a > 2$ and $q > 2$. The result about time localization follows from the solution formula (1.1.11). □

Similarly, for wave equations, with dispersive estimates above, $TT^*$ argument and the Christ-Kiselev lemma, we have the general Strichartz estimates for linear wave equations. We just state the results here.

**Definition.** A pair $(p, q)$ is called wave-admissible if $p$, $q \ge 2$ and

$$\frac{1}{q} + \frac{n-1}{2p} \le \frac{n-1}{4}.$$

**Theorem 1.1.7.** *Let $u$ be a solution to the wave equation*

$$\partial_{tt} u - \Delta u = h, \;\; u|_{t=0} = (f, g),$$

*where $(f,\, g) \in \dot{H}^{\gamma} \times \dot{H}^{\gamma-1}$ and $h \in L_t^{a'} L_x^{b'}$. Then we have the following a-priori estimate:*

$$\|u\|_{L_t^p L_x^q} \le C \left( \|u(0)\|_{\dot{H}^\gamma \times \dot{H}^{\gamma-1}} + \|h\|_{L_t^{a'} L_x^{b'}(\mathbb{R}^{n+1})} \right)$$

*provided* $(p,q)$ *and* $(a.b)$ *are wave-admissible and with* $\gamma$, *they satisfy the scaling condition*

$$\frac{1}{q} + \frac{n}{p} = \frac{1}{a'} + \frac{n}{b'} - 2 = \frac{n}{2} - \gamma.$$

*Remark.* Due the singularity of the vertex of the double light cone, as above, when we do Strichartz estimates for wave equations, we need to apply the Littlewood-Paley decompostion. We do estimate for each piece and then with certain rules to sum up them.

### *1.1.6 Smoothing effects*

In this subsection, we introduce the notation of the smoothness of an operator in Kato's sense. And with the help of Kato's smoothness, we derive Strichartz estimates for the solutions of Schrödinger equations with potentials decaying with the rate $|x|^{-2-\epsilon}$ at spacial infinity. From a more abstract view, we can apply Kato's smoothing to the multiplication operator $|V|^{\frac{1}{2}}$ relative to $H$ and $H_0$ to relate $L_t^q L_x^p$ estimates for the semigroup $e^{-itH_0}$ to the corresponding estimates for $e^{-itH}$ with $H = H_0 + V$.

We first introduce some basic definitions.

**Definition.** An unbounded (linear) operator $A$ between Banach spaces $B$ and $C$ is a (linear) map $A : \mathcal{D}(A) \to C$ where $\mathcal{D}(A) \subset B$. And $A$ is called closed if its graph is closed and densely defined if $\mathcal{D}(A)$ is dense in B.

We will consider closed and densely defined operators. We recall that with a abstract Hilbert space $\mathcal{H}$( we will focus on $L^2$ later on), for a self-adjoint operator $H$ , we can define the resolvent of it as $R_H(\mu) = (H - \mu)^{-1}$. A closed operator $A$ is called $H$-smooth in sense of Kato if and only if for each $\phi \in \mathcal{H}$ and $\epsilon \neq 0$, $R_H(\lambda + i\epsilon)\phi \in \mathcal{D}(A)$ for almost all $\lambda \in \mathbb{R}$

and moreover, we have

$$\|A\|_H^2 := \sup_{\|\phi\|=1, \epsilon>0} \frac{1}{4\pi^2}\int_{-\infty}^{\infty} \left(\|AR_H(\lambda+i\epsilon)\phi\|^2 + \|AR_H(\lambda-i\epsilon)\phi\|^2\right) d\lambda < \infty. \quad (1.1.13)$$

*Remark.* Since $A$ is closed, it suffices to check formula (1.1.13) for a dense subset of $\phi$.

*Remark.* By the uniform boundedness principle, it is enough to check for each $\phi$,

$$\frac{1}{4\pi^2}\int_{-\infty}^{\infty} \left(\|AR_H(\lambda+i\epsilon)\phi\|^2 + \|AR_H(\lambda-i\epsilon)\phi\|^2\right) d\lambda$$

is bounded by a constant independent of $\epsilon$.

In order the apply the notation of $H$-smoothness, we have several equivalent conditions to (1.1.13). Some of them are easier to check and apply in practice.

**Theorem 1.1.8.** *Let $A$ be a closed operator and let $H$ be a self-adjoint operator. Then the following are equivalent:*

*(0) $A$ is $H$-smooth.*

*(1) For all $\phi \in \mathcal{H}$, $e^{-itH}\phi \in \mathcal{D}(A)$ for almost all $t$ and*

$$c_1 = \sup_{\|\phi\|=1} \frac{1}{2\pi}\int_{-\infty}^{\infty} \left\|Ae^{-iHt}\phi\right\|^2 dt < \infty.$$

*(2) $\mathcal{D}(H) \subset \mathcal{D}(A)$ and with $E_\Omega$ as the spectral family for $H$, we have*

$$c_2 = \sup_{\|\phi\|=1, -\infty<a<b<\infty} \frac{\|AE(a,b]\phi\|^2}{|b-a|} < \infty,$$

*(3)*

$$c_3 = \sup_{\|\phi\|=1, \phi\in D(A^*), -\infty<a<b<\infty} \frac{\|E(a,b]A^*\phi\|^2}{|b-a|} < \infty.$$

*(4)*

$$c_4 = \frac{1}{2\pi} \sup_{\|\phi\|=1, \phi\in D(A^*), \mu\notin\mathbb{R}} |(A^*\phi, [R_H(\mu) - R_H(\bar{\mu})]\, A^*\phi)| < \infty.$$

*(5)*

$$c_5 = \frac{1}{\pi} \sup_{\|\phi\|=1, \phi\in D(A^*), \mu\notin\mathbb{R}} \|R_H(\mu)A^*\phi\|^2\, |\Im\mu| < \infty.$$

*(6)* $\mathcal{D}(H) \subset \mathcal{D}(A)$ *and*

$$c_6 = \frac{1}{\pi} \sup_{\|\phi\|=1, \mu\notin\mathbb{R}} \|AR_H(\mu)\phi\|^2\, |\Im\mu| < \infty.$$

*(7)* $\mathcal{D}(H) \subset \mathcal{D}(A)$ *and*

$$c_7 = \frac{1}{\pi} \sup_{\|\phi\|=1, \mu\notin\mathbb{R}} \left(\|AR_H(\mu)\phi\|^2 + \|AR_H(\bar{\mu})\phi\|^2\right) |\Im\mu| < \infty.$$

*Moreover, if any of* $c_i$ *is finite, then all* $c_i$*'s are finite and they all equal* $\|A\|_H^2$.

Now we are ready to apply Kato's smoothing in our problem. Following [50], we consider a self-adjoint operator $H_0$ on $L^2(\mathbb{R}^n)$ with domain $\mathcal{D}(H_0)$. Let $e^{-itH_0}$ be the associated unitary semigroup, which is a solution operator to the Schrödinger equation

$$i\partial_t u - H_0 u = 0, \quad u|_{t=0} = u_0.$$

In our concrete example, $H_0 = -\Delta$, $H = -\Delta + V$. We denote by $R_0(z)$ the resolvent of $H_0$. Let $A$ and $B$ be a pair of bounded operators on $L^2(\mathbb{R}^n)$ and consider a self-adjoint operator $H = H_0 + B^*A$ with domain $\mathcal{D}(H_0)$. The corresponding semigroup is $e^{-itH}$ and the resolvent is $R(z)$. Clearly, we have for $\Im z \neq 0$, we have

$$R(z) = R_0(z) - R_0(z)B^*AR(z). \tag{1.1.14}$$

Also $e^{-itH}$ and $e^{-itH_0}$ are related by Duhamel's formula

$$e^{-itH}u_0 = e^{-itH_0}u_0 + i\int_0^t e^{-i(t-s)H_0}B^*Ae^{-isH}u_0 ds \tag{1.1.15}$$

which holds for any $u_0 \in L^2(\mathbb{R}^n)$. From the Theorem 1.1.8 above, we know for a self-adjoint operator $\tilde{H}$, an operator $\Gamma$ is called $\tilde{H}$-smooth if for any $f \in \mathcal{D}(\tilde{H})$, we have

$$\left\|\Gamma e^{-it\tilde{H}}f\right\|_{L_t^2 L_x^3} \le C_\Gamma(\tilde{H}) \left\|f\right\|_{L_x^2}, \tag{1.1.16}$$

or using another formula, we have

$$\sup_{\epsilon>0}\left\|\Gamma R_{\tilde{H}}(\lambda \pm i\epsilon)f\right\|_{L_\lambda^2 L_x^2} \le C_\Gamma(\tilde{H}) \left\|f\right\|_{L_x^2}. \tag{1.1.17}$$

We call $C_\Gamma(\tilde{H})$ the smoothing bound of $\Gamma$ with respect to $\tilde{H}$. Let $\Omega \subset \mathbb{R}$ and let $E_\Omega$ be a spectral projection of $\tilde{H}$ associated to a set $\Omega$. We say $\Gamma$ is $\tilde{H}$-smooth on $\Omega$ if $\Gamma E_\Omega$ is $\tilde{H}$-smooth. If we denote the corresponding smoothing bound to be $C_\Gamma(\tilde{H}, \Omega)$, we have

$$\sup_{\epsilon>0, \lambda\in\Omega}\left\|\Gamma R_{\tilde{H}}(\lambda \pm i\epsilon)f\right\|_{L_\lambda^2 L_x^2} \le C_\Gamma(\tilde{H}, \Omega) \left\|f\right\|_{L_x^2} \tag{1.1.18}$$

Now with above notations, we have the main result of this section.

**Theorem 1.1.9** ([50])**.** *Let $H_0$ and $H = H_0 + B^*A$ as above. We assume that $B$ is $H_0$-smooth with a smoothing bound $C_B(H_0)$ and that for some $\Omega \subset \mathbb{R}$, the operator $A$ is $H$-smooth on $\Omega$ with the smoothing bound $C_A(H, \Omega)$. We also assume that the unitary semigroup $e^{-itH_0}$ satisfies the estimate*

$$\left\|e^{-itH_0}u_0\right\|_{L_t^q L_x^r} \le C_{H_0} \left\|u_0\right\|_{L_x^2} \tag{1.1.19}$$

*for some $q \in (2, \infty]$ and $r \in [1, \infty]$. Then the semigroup $e^{-itH}$ associated with $H = H_0 +$*

*$B^*A$, restricted to the spectral set $\Omega$, also verifies the estimate* (1.1.19) ,

$$\left\| e^{-itH} E_\Omega u_0 \right\|_{L_t^q L_x^r} \leq C_{H_0} C_B(H_0) C_A(H,\Omega) \|u_0\|_{L_x^2} \tag{1.1.20}$$

*Proof.* (Sketch) With the help of the Duhamel formula (1.1.15), we have

$$e^{-itH} u_0 = e^{-itH_0} u_0 + i \int_0^t e^{-i(t-s)H_0} B^* A e^{-isH} u_0 ds.$$

Plug in $L_t^q L_x^r$ norm and the projection, we have

$$\begin{aligned} \left\| e^{-itH} E_\Omega u_0 \right\|_{L_t^q L_x^r} &\leq \left\| e^{-itH_0} E_\Omega u_0 \right\|_{L_t^q L_x^r} + \left\| \int_0^t e^{-i(t-s)H_0} B^* A e^{-isH} E_\Omega u_0 ds \right\|_{L_t^q L_x^r} \\ &\leq C_{H_0} \|u_0\|_{L_x^2} + \left\| \int_0^t e^{-i(t-s)H_0} B^* A e^{-isH} E_\Omega u_0 ds \right\|_{L_t^q L_x^r} \end{aligned} \tag{1.1.21}$$

If we write the Duhamel term as

$$D := \tilde{K} \left( A e^{-isH} E u_0 \right) = \int_0^t e^{-i(t-s)H_0} B^* A e^{-isH} E_\Omega u_0 ds, \tag{1.1.22}$$

with the help of the Chirst-Kiselev lemma, we have

$$\|D\|_{L_t^q L_x^r} \lesssim \|K\|_{L^2([0,\infty):L_x^2) \to L^q([0,\infty);L_x^r)} \left\| A e^{-isH} E_\Omega u_0 \right\|_{L_t^2 L_x^2}. \tag{1.1.23}$$

So next, we estimate the norm of $K$,

$$\begin{aligned} \|KF\|_{L_t^q L_x^r} &= \left\| \int_0^\infty e^{-i(t-s)H_0} B^* F(s) ds \right\|_{L_t^q L_x^r} \\ &= \left\| e^{-itH_0} \int_0^\infty e^{isH_0} B^* F(s) ds \right\|_{L_t^q L_x^r} \\ &\leq C_{H_0} \left\| \int_0^\infty e^{isH_0} B^* F(s) ds \right\|_{L_x^2}. \end{aligned} \tag{1.1.24}$$

To estimate $\left\|\int_0^\infty e^{isH_0}B^*F(s)ds\right\|_{L_x^2}$, we use duality

$$\begin{aligned}\left\|\int_0^\infty e^{isH_0}B^*F(s)ds\right\|_{L_x^2} &= \sup_{\|\phi\|_{L_x^2}=1}\left\langle\int_0^\infty e^{isH_0}B^*F(s)ds,\phi\right\rangle \\ &= \sup_{\|\phi\|_{L_x^2}=1}\int_0^\infty ds\left\langle F(s),Be^{-isH_0}\phi\right\rangle \\ &\leq \|F\|_{L_t^2L_x^2}\sup_{\|\phi\|_{L_x^2}=1}\left\|Be^{-isH_0}\phi\right\|_{L_t^2L_x^2} \\ &\leq C_B(H_0)\|F\|_{L_t^2L_x^2}\|\phi\|_{L_x^2},\end{aligned}$$

by the smoothness of $B$ with respect to $H_0$. Thus the operator $K(t,s)=e^{-i(t-s)H_0}B^*$ is bounded from $L^2([0,\infty):L_x^2)\to L^q([0,\infty);L_x^r)$. Finally, we need to estimate $\left\|Ae^{-isH}E_\Omega u_0\right\|_{L_t^2L_x^2}$. Note that $A$ is $H$-smooth on $\Omega$, so we have

$$\left\|Ae^{-isH}.E_\Omega u_0\right\|_{L_t^2L_x^2}\leq C_A(H,\Omega)\|u_0\|_{L_x^2}. \tag{1.1.25}$$

With all estimates above, we finally have

$$\left\|e^{-itH}E_\Omega u_0\right\|_{L_t^qL_x^r}\leq C_{H_0}C_B(H_0)C_A(H,\Omega)\|u_0\|_{L_x^2}.$$

We are done. □

In our concrete example, we apply the above theorem with $H_0=-\Delta$ and $H=H_0+V(x)$. We assume the the potential $V$ satisfies

$$|V(x)|\leq C_V(1+|x|^2)^{-1-\epsilon} \tag{1.1.26}$$

with some constants $C_V$ and $\epsilon>0$.

*Remark.* The decay assumption (1.1.26) helps us rule out the positive singular continuous spectrum and positive eigenvalues. Actually, it also guarantees the absence of the singular continuous spectrum. See [50] for details.

*Remark.* By Weyl's criterion, the essential spectrum of $H$ is $[0,\infty)$. It might be possible, we have negative eigenvalues for $H$. So it is natural to project all our data orthogonal to the eigenfunctions associated with negative eigenvalues. Therefore, we use spectral projection $P$ of $H$ corresponding to $[0,\infty)$.

We know in our concrete example,

$$\left\| e^{-itH_0}u_0 \right\|_{L_t^q L_x^r} \leq C \left\| u_0 \right\|_{L_x^2}, \quad \forall (q,r,n) \neq (2, \frac{2n}{n-2}, n), \ \frac{2}{q} = n(\frac{1}{2} - \frac{1}{r}),$$

for all $u_0 \in L^2(\mathbb{R}^n)$. We factor $V$ as

$$V = B^* A, \ \ B = |V|^{\frac{1}{2}}, \quad A = |V|^{\frac{1}{2}} sgn(V).$$

Apply our Theorem 1.1.9, then we have the following theorem.

**Theorem 1.1.10** ([50])**.** *Let $V$ be a potential as above. In addition, we impose the condition that the point $\lambda = 0$ in the spectrum of the operator $H = -\Delta + V$ is neither an eigenvalue nor a resonance. Then if $P$ is the spectral projection of $H$ corresponding to the interval $[0,\infty)$, we have*

$$\left\| e^{-itH} P u_0 \right\|_{L_t^q L_x^r} \leq C \left\| u_0 \right\|_{L_x^2}, \ \forall (q,r,n), \ n \geq 3, \ \frac{2}{q} = n(\frac{1}{2} - \frac{1}{r}),$$

All the details on verifying the conditions in Theorem 1.1.9 can be found in [50].

*Remark.* Notice that the above theorem can not give Keel-Tao endpoint results since we use the Christ-Kiselev lemma which requires $q > 2$.

### *1.1.7 Time-dependent potentials*

There are extra difficulties when dealing with time-dependent potentials. For example, given a general time-dependent potential $V(x,t)$, it is not clear how to introduce an analog of bound states and a spectral projection. The evolution might not satisfy group properties any more, so $TT^*$ argument always fails. The time-dependent potential might also result in the growth of certain norms of the solutions, see the book by Bourgain [9]. In this thesis, we focus on the charge transfer Hamiltonian in $\mathbb{R}^3$:

$$H(t) = -\Delta + \sum_{j=1}^{m} V_j \left( x - \vec{v}_j t \right), \tag{1.1.27}$$

which appears naturally in the study of nonlinear multisoliton system, see Rodnianski-Schlag-Soffer [52] for the Schrödinger model. For the wave model, due relativistic effects, we consider

$$\partial_{tt} u - \Delta u + \sum_{j=1}^{m} V_j \left( x - \vec{v}_j t \right) u = 0, \quad |\vec{v}_i| < 1,\ 1 \leq i \leq m. \tag{1.1.28}$$

These models are of crucial importance to analyze the stabilty of multi-soliton states. In this thesis, we will also present the multisoliton system based on the wave model.

## 1.2 Outline of the Thesis

In this thesis we study the local dispersive decays, Strichartz estimates and their applications. The main results have all appeared in research articles [11, 12, 13, 14, 15].

In Chapter 2, we prove Strichartz estimates for scattering states of the scalar charge transfer models in $\mathbb{R}^3$:

$$\frac{1}{i} \partial_t \psi + H(t)\psi = 0 \tag{1.2.1}$$

where

$$H(t) = -\frac{1}{2}\Delta + \sum_{j=1}^{m} V_j(x - \vec{v}_j t) \tag{1.2.2}$$

with rapidly decaying smooth potentials $V_j(x)$, say, exponentially decaying and a set of mutually non-parallel constant velocities $\vec{v}_j$. Based on the idea of the proof of Strichartz estimates which follows [22, 51], we also show the energy of the whole evolution is bounded independent of time without using the phase space method, for example, in [27]. One can easily generalize our argument to $\mathbb{R}^n$ for $n \geq 3$. Finally, in the last section, we discuss the extension of above results to matrix charge transfer models in $\mathbb{R}^3$. Restricting to the case $m = 2$, we formulate the main result in the this chapter.

**Definition 1.2.1.** Let $U(t,0)\psi_0 = \psi(t,x)$ the solution of equation (3.1.3). We say that $\psi_0$ or $\psi(x,t)$ is asymptotically orthogonal to the bound states of $H_1$ and $H_2$ if

$$\|P_b(H_1)\, U(t,0)\psi_0\|_{L^2} + \|P_b(H_2,t)\, U(t,0)\psi_0\|_{L^2} \to 0, \ \ t \to \infty. \tag{1.2.3}$$

Here

$$P_b(H_2,t) := \mathfrak{g}_{\vec{e_1}}(t)^{-1} P_b(H_2)\, \mathfrak{g}_{\vec{e_1}}(t), \quad \forall t,$$

and $\mathfrak{g}_{\vec{e_1}}$ is the Gallilei transformation to reduce the moving potential to the stationary case. We also call $\psi$ a scattering state.

**Theorem 1.2.2.** *Sppose $\psi$ is a scattering state. Then for a Schrödinger admission pair $(p,q)$ in $\mathbb{R}^3$, i.e.,*

$$\frac{2}{p} + \frac{3}{q} = \frac{3}{2} \tag{1.2.4}$$

*and if $2 \leq q \leq \infty$ with $(p,q) \neq (2,\infty)$, we have*

$$\|\psi\|_{L^p_t L^q_x} \leq C\, \|\psi_0\|_{L^2_x}. \tag{1.2.5}$$

**Theorem 1.2.3.** *For $\psi_0 \in H^1$, we have*

$$\|U(t,0)\psi_0\|_{H^1} \leq C \|\psi_0\|_{H^1}. \tag{1.2.6}$$

This chapter is based on the work in [11]. The proof is based on the idea in Rodnianski-Schlag discussed above that the local decay estimates imply Strichartz estimates. In order to handle the moving potential, we need some weighted decay estimates with time-dependent weight.

In Chapter 3, Chapter 4 and Chapter 5, we turn our attention to wave equations.

In Chapter 3, we prove Strichartz estimates (both regular and reversed) for a scattering state to the wave equation with a charge transfer Hamiltonian in $\mathbb{R}^3$:

$$\partial_{tt}u - \Delta u + \sum_{j=1}^{m} V_j\left(x - \vec{v}_j t\right) u = 0.$$

The energy estimate and the local energy decay of a scattering state are also established. In order to study nonlinear multisoltion systems, we will present the inhomogeneous generalizations of Strichartz estimates. As an application of our results, we show that scattering states indeed scatter to solutions to the free wave equation.

The study of Schrödinger equations with a charge transfer Hamiltonian can be found in Rodnianski-Schlag-Soffer [51], Cai [10], Chen [11] and Deng-Soffer-Yao [23]. For the Schrödinger model, there is no need to require $|\vec{v}_i| < 1$. In Rodnianski-Schlag-Soffer [51], the authors proved the dispersive estimates for both the scalar and matrix Schrödinger charge transfer models. They introduced Galilei transformations to interchange stationary frames with respect to different potentials. Basically, they applied a bootstrap argument via a semi-classical propagation lemma for low frequencies and Kato's smoothing estimate for high frequencies. With careful analysis of wave operators, the authors also obtain the results on the asymptotic completeness. Their works inspired the subsequent development in Cai

[10] where the $L^1 \to L^\infty$ dispersive estimate is proved. Later on, by Chen [11], Strichartz estimates for both the scalar and matrix Schrödinger charge transfer models were presented based on a time-dependent local decay estimate and the endpoint Strichartz estimate for the free equations. Alternatively, Strichartz estimates can be obtained by analysis of wave operators, see Deng-Soffer-Yao [23].

Compared with Schrödinger equations, wave equations have some natural difficulties, for example the evolution of bound states of wave equations leads to exponential growth as we pointed out above, meanwhile the evolution of bound states of Schrödinger equations are merely multiplied by oscillating factors. The structure of wave operators in the wave equation setting is not clear either. Moreover, the endpoint Strichartz estimate for free equations, an important tool used in the paper [11], also fails for free wave equations in $\mathbb{R}^3$. Last but not least, Lorentz transformations are space-time rotations, therefore one can not hope to succeed by the approach used with Schrödinger equations based on Galilei transformations. Galilei transformations are bounded in any $L^p$ space, but it is not clear under Lorentz transformations whether the energy with respect to the new frame stays comparable to the energy in the original frame.

In Chapters 3, 4, 5, to resolve the above issues, we develop the endpoint reversed Strichartz estimates along time-like trajectories and the energy comparison with respect different Lorentz frames.

Here are the main results from [14].

**Definition 1.2.4.** Let

$$\partial_{tt}u - \Delta u + V_1(x)u + V_2(x - \vec{v}t)u = 0, \tag{1.2.7}$$

with initial data

$$u(x,0) = g(x),\ u_t(x,0) = f(x). \tag{1.2.8}$$

If $u$ also satisfies

$$\|P_b\,(H_1)\,u(t)\|_{L^2_x} \to 0,\ \left\|P_b\,(H_2)\,u_L(t')\right\|_{L^2_{x'}} \to 0\ \ t,t' \to \infty, \tag{1.2.9}$$

we call it a scattering state.

**Theorem 1.2.5.** *Suppose $u$ is a scattering state. For $p > 2$ and $(p,q)$ satisfying*

$$\frac{1}{2} = \frac{1}{p} + \frac{3}{q}, \tag{1.2.10}$$

*we have*

$$\|u\|_{L^p_t([0,\infty),\,L^q_x)} \lesssim \|g\|_{L^2} + \|f\|_{\dot{H}^1}. \tag{1.2.11}$$

Secondly, one has the energy estimate:

**Theorem 1.2.6.** *Suppose $u$ is a scattering state. Then we have*

$$\sup_{t\in\,\mathbb{R}} \left(\|\nabla u(t)\|_{L^2} + \|u_t(t)\|_{L^2}\right) \lesssim \|g\|_{L^2} + \|f\|_{\dot{H}^1}. \tag{1.2.12}$$

Even more importantly, we obtain the endpoint reversed Strichartz estimates for $u$.

**Theorem 1.2.7.** *Suppose $u$ is a scattering state. Then*

$$\sup_{x\in\mathbb{R}^3} \int_0^\infty |u(x,t)|^2\,dt \lesssim \left(\|g\|_{L^2} + \|f\|_{\dot{H}^1}\right)^2, \tag{1.2.13}$$

*and*

$$\sup_{x\in\mathbb{R}^3} \int_0^\infty |u(x+\vec{v}t,t)|^2\,dt \lesssim \left(\|g\|_{L^2} + \|f\|_{\dot{H}^1}\right)^2. \tag{1.2.14}$$

In Chapter 4, we develop reversed Strichartz estimates along time-like trajectories in more details. We also discuss the energy comparison with respect to different Lorentz frames in more details, in particular, we show Agmon's estimates based on wave equations. We also develop a unified way to pass estimates for the free equation to the perturbed case based on the structure formula of wave operator developed by Beceanu and Schlag. Here we formulate main results from [12].

**Definition 1.2.8** (Admissible trajectories)**.** A trajectory $\vec{Y}(t) \in \mathbb{R}^3$ is said to be admissible if $\vec{Y}(t)$ is $C^1$ and there exists $0 \leq \ell < 1$ such $\left|\vec{Y}'(t)\right| < \ell < 1$ for $t \in \mathbb{R}$.

Consider the solution to the free wave equation ($H_0 = -\Delta$),

$$u(x,t) = \frac{\sin\left(t\sqrt{-\Delta}\right)}{\sqrt{-\Delta}} f + \cos\left(t\sqrt{-\Delta}\right) g + \int_0^t \frac{\sin\left((t-s)\sqrt{-\Delta}\right)}{\sqrt{-\Delta}} F(s)\, ds \tag{1.2.15}$$

and let $\vec{Y}(t) \in \mathbb{R}^3$ be an admissible trajectory. Setting

$$u^S(x,t) := u\left(x + \vec{Y}(t), t\right), \tag{1.2.16}$$

we estimate

$$\sup_{x\in\mathbb{R}^3} \int \left|u^S(x,t)\right|^2 dt \tag{1.2.17}$$

in terms of the initial energy and various norms of $F$. The idea behind these estimates is that the fundamental solution of the free wave equation is supported on the light cone. Along a time-like curve, the propagation will only meet the light cone once.

**Theorem 1.2.9.** *Let $\vec{Y}(t)$ be an admissible trajectory. We have*

$$\left\|u^S(x,t)\right\|_{L_x^\infty L_t^2} \lesssim \|f\|_{L^2} + \|g\|_{\dot{H}^1} + \|F\|_{\dot{W}_x^{1,1} L_t^2}. \tag{1.2.18}$$

*If $\vec{Y}(t)$ does not change the direction, then*

$$\left\|u^S(x,t)\right\|_{L_x^\infty L_t^2} \lesssim \|f\|_{L^2} + \|g\|_{\dot{H}^1} + \|F\|_{L_d^1 L_{\widehat{d}}^{2,1} L_t^2}, \tag{1.2.19}$$

*where $d$ is the direction of $\vec{Y}(t)$.*

*Let $\vec{Z}(t)$ be another admissible trajectory, we have the same estimates above with $F$ replaced by*

$$F^{S'}(x,t) := F\left(x + \vec{Z}(t), t\right). \tag{1.2.20}$$

**Remark 1.2.10.** *If $\vec{Y}(t) = \vec{Z}(t)$, one can obtain*

$$\left\| u^S(x,t) \right\|_{L_x^\infty L_t^2} \lesssim \|f\|_{L^2} + \|g\|_{\dot{H}^1} + \left\| F^S \right\|_{L_x^{\frac{3}{2},1} L_t^2}. \tag{1.2.21}$$

*The other extreme exponents are $L^\infty$ for $t$ and $L^6$ for $x$. To be more precise, we have the following endpoint estimates.*

**Theorem 1.2.11.** *Let $\vec{Y}(t)$ be an admissible trajectory. First of all, for the standard case, one has*

$$\|u\|_{L_x^{6,2} L_t^\infty} \lesssim \|f\|_{L^2} + \|g\|_{\dot{H}^1} + \|F\|_{L_x^{\frac{6}{5},2} L_t^\infty}. \tag{1.2.22}$$

*Consider the estimates along the trajectory $\vec{Y}(t)$, one has*

$$\left\| u^S(x,t) \right\|_{L_x^{6,2} L_t^\infty} \lesssim \|f\|_{L^2} + \|g\|_{\dot{H}^1} + \|F\|_{\dot{W}_x^{1,\frac{6}{5}} L_t^1}. \tag{1.2.23}$$

*Let $\vec{Z}(t)$ be another admissible trajectory, we have the same estimates above with $F$ replaced by*

$$F^{S'}(x,t) := F\left(x + \vec{Y}(t), t\right). \tag{1.2.24}$$

We can extend the above estimates to wave equations with perturbed Hamiltonian,

$$H = -\Delta + V \tag{1.2.25}$$

by the structure formulas for wave operators developed in [5, 7], for the potential $V$ such that

$$H = -\Delta + V \tag{1.2.26}$$

by the structure formulas for wave operators developed in [5, 7], for the potential $V$ such that

$$V \in B^{1+} \cap L^2, \tag{1.2.27}$$

where

$$B^{\beta} = \left\{ V \mid \sum_{k \in \mathbb{Z}} 2^{\beta k} \left\| \chi_{\{|x| \in [2^k, 2^{k+1}]\}}(x) V(x) \right\|_{L^2} < \infty \right\}. \tag{1.2.28}$$

and $H$ admits neither eigenfunctions nor resonances at 0. Recall that $\psi$ is a resonance at 0 if it is a distributional solution of the equation $H\psi = 0$ which belongs to the space $L^2\left(\langle x \rangle^{-\sigma} dx\right) := \left\{ f : \langle x \rangle^{-\sigma} f \in L^2 \right\}$ for any $\sigma > \frac{1}{2}$, but not for $\sigma = \frac{1}{2}$.

**Theorem 1.2.12.** *Let $\vec{Y}(t)$ be an admissible trajectory. Suppose*

$$H = -\Delta + V \tag{1.2.29}$$

*admits neither eigenfunctions nor resonances at* 0*. Set*

$$u(x,t) = \frac{\sin\left(t\sqrt{H}\right)}{\sqrt{H}} P_c f + \cos\left(t\sqrt{-H}\right) P_c g + \int_0^t \frac{\sin\left((t-s)\sqrt{H}\right)}{\sqrt{H}} P_c F(s)\, ds \tag{1.2.30}$$

*and*

$$u^S(x,t) := u\left(x + \vec{Y}(t), t\right), \tag{1.2.31}$$

*where $P_c$ is the projection onto the continuous spectrum of $H$.*

*Then*

$$\left\| u^S(x,t) \right\|_{L_x^\infty L_t^2} \lesssim \|f\|_{L^2} + \|g\|_{\dot{H}^1} + \|F\|_{\dot{W}_x^{1,1} L_t^2}. \tag{1.2.32}$$

*If $\vec{Y}(t)$ does not change the direction, then*

$$\left\| u^S(x,t) \right\|_{L_x^\infty L_t^2} \lesssim \|f\|_{L^2} + \|g\|_{\dot{H}^1} + \|F\|_{L_d^1 L_{\widehat{d}}^{2,1} L_t^2}, \tag{1.2.33}$$

*where $d$ is the direction of $\vec{Y}(t)$.*

*Let $\vec{Z}(t)$ be another admissible trajectory, we have the same estimates above with $F$ replaced by*

$$F^{S'}(x,t) := F\left(x + \vec{Z}(t), t\right). \tag{1.2.34}$$

We also have the perturbed version of the second endpoint reversed space-time estimates.

**Theorem 1.2.13.** *Let $\vec{Y}(t)$ be an admissible trajectory. Suppose*

$$H = -\Delta + V \tag{1.2.35}$$

*admits neither eigenfunctions nor resonances at 0. Set*

$$u(x,t) = \frac{\sin\left(t\sqrt{H}\right)}{\sqrt{H}} P_c f + \cos\left(t\sqrt{-H}\right) P_c g + \int_0^t \frac{\sin\left((t-s)\sqrt{H}\right)}{\sqrt{H}} P_c F(s)\, ds \tag{1.2.36}$$

*and*

$$u^S(x,t) := u\left(x + \vec{Y}(t), t\right), \tag{1.2.37}$$

*where $P_c$ is the projection onto the continuous spectrum of $H$. First of all, for the standard case, one has*

$$\|u\|_{L_x^{6,2} L_t^\infty} \lesssim \|f\|_{L^2} + \|g\|_{\dot{H}^1} + \|F\|_{L_x^{\frac{6}{5},2} L_t^\infty}. \tag{1.2.38}$$

*Consider the estimates along the trajectory $\vec{Y}(t)$, one has*

$$\left\|u^S(x,t)\right\|_{L_x^{6,2} L_t^\infty} \lesssim \|f\|_{L^2} + \|g\|_{\dot{H}^1} + \|F\|_{\dot{W}_x^{1,\frac{6}{5}} L_t^1}. \tag{1.2.39}$$

*Let $\vec{Z}(t)$ be another admissible trajectory, we have the same estimates above with $F$ replaced by*

$$F^{S'}(x,t) := F\left(x + \vec{Z}(t), t\right). \tag{1.2.40}$$

In Chapter 5, we apply the linear estimates from the previous chapters to our nonlinear model: the multisoliton structure to the defocusing energy critical wave equation with potentials in $\mathbb{R}^3$:

$$\partial_{tt}u - \Delta u + \sum_{j=1}^{m} V_j\left(x - \vec{v}_j t\right)u + u^5 = 0. \tag{1.2.41}$$

To the author's knowledge, this model is the first one to produce multisoliton structures for wave equations in $\mathbb{R}^3$. Unlike Klein-Gordon equations and wave equations in higher dimensions, see Côte-Muñoz [22], Côte-Martel [19], Jendrej [30, 31], Martel-Merle [44], in our case, the static solutions to the associated elliptic equations decay slowly like $\langle x\rangle^{-1}$. It is of crucial importance to understand the multisoliton structure in order to establish the soliton resolution. In fact, if we remove the potentials and replace the positive sign in front of the nonlinearity by the negative sign, the equation becomes the well-known focusing energy critical wave equation. Duyckaerts, Jia, Kenig and Merle establish the soliton resolution (along a well-chosen time sequence) in [24, 25]. But to construct the multisoliton in this case is open. For higher dimensions cases, Martel and Merle construct the multisoliton in dimension higher than 5 by the energy method in [44]. They point out that the slow decay of the ground state is the obstruction to obtain a multisoliton in $\mathbb{R}^3$. Although the structure of our model is different from the pure-power nonlinear equation, the construction in this paper illustrates that we can overcome the slow decay. But the zero eigenfunctions and resonances for the linearized operator from the pure-power nonlinear equation near each soliton will be the challenge for the linear theory. Another interesting point is that unlike the constructions in Côte-Muñoz [22], Côte-Martel [19], Jendrej [30, 31], Martel-Merle [44] which choose the initial data based on the Brouwer's fixed point theorem, in this paper, we construct the initial data for the unstable soliton case based on the Banach's fixed point theorem.

Here we formulate the main results from [15].

Let $W_j^{v_j}$ be the stable static state to

$$-\Delta W_j^{v_j} + V_j^{v_j}(x) W_j^{v_j} + \left(W_j^{v_j}\right)^5 = 0 \tag{1.2.42}$$

By a stable state, we mean that the linearized operator

$$-\Delta + V_j^{v_j}(x) + 5\left(W_j^{v_j}\right)^4 \tag{1.2.43}$$

has no eigenvalues nor zero resonance.

Set

$$W_j(x) = W_j^{v_j}\left(m_{v_j}^{-1}\rho_{v_j}(x)\right). \tag{1.2.44}$$

We also need the Hamiltonian structure of wave equations to discuss scattering. In general, we can write a general wave equation as

$$\partial_{tt} u - \Delta u = F(u,t) \tag{1.2.45}$$

with initial data

$$u(x,0) = f(x),\ u_t(x,0) = g(x). \tag{1.2.46}$$

Also consider the homogeneous free wave equation,

$$\partial_{tt} u_0 - \Delta u_0 = 0 \tag{1.2.47}$$

with initial data

$$u_0(x,0) = f_0(x),\ (u_0)_t(x,0) = g_0(x). \tag{1.2.48}$$

We reformulate the wave equation as a Hamiltonian system,

$$\partial_t \begin{pmatrix} u \\ \partial_t u \end{pmatrix} - \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} -\Delta & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} u \\ \partial_t u \end{pmatrix} = \begin{pmatrix} 0 \\ F(u) \end{pmatrix}. \tag{1.2.49}$$

Setting

$$U := \begin{pmatrix} u \\ \partial_t u \end{pmatrix}, \; J := \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \; H_F := \begin{pmatrix} -\Delta & 0 \\ 0 & 1 \end{pmatrix} \text{ and } F(U) := \begin{pmatrix} 0 \\ F(u,t) \end{pmatrix}, \tag{1.2.50}$$

we can rewrite the free wave equation as

$$\dot{U}_0 - JH_F U_0 = 0, \tag{1.2.51}$$

$$U_0[0] = \begin{pmatrix} f_0 \\ g_0 \end{pmatrix} \tag{1.2.52}$$

and the nonlinear wave equation as

$$\dot{U} - JH_F U = F(U), \tag{1.2.53}$$

$$U[0] = \begin{pmatrix} f \\ g \end{pmatrix}. \tag{1.2.54}$$

The solution of the free wave equation is given by

$$U_0 = e^{tJH_F} U_0[0]. \tag{1.2.55}$$

In the following, we write

$$U[t] = (u, u_t)^t \, , \, W[t] = \left( \sum_{j=1}^{m} W_j \left( x - \vec{v}_j t \right) , \, \partial_t \sum_{j=1}^{m} W_j \left( x - \vec{v}_j t \right) \right)^t . \tag{1.2.56}$$

With the preparations and notations above, we can formulate our main theorems:

**Theorem 1.2.14.** *There exists a solution $u$ to*

$$\partial_{tt} u - \Delta u + \sum_{j=1}^{m} V_j \left( x - \vec{v}_j t \right) u + u^5 = 0, \tag{1.2.57}$$

*such that*

$$\lim_{t \to \infty} \| U[t] - W[t] \|_{\dot{H}^1 \times L^2} \to 0. \tag{1.2.58}$$

*Moreover, we have the decay rate*

$$\| U[t] - W[t] \|_{\dot{H}^1 \times L^2} \lesssim \frac{1}{\sqrt{t}}. \tag{1.2.59}$$

as $t \to \infty$.

Next we have the asymptotic stability of the multisoliton structure.

**Theorem 1.2.15.** *Suppose that $0 < \epsilon \ll 1$ is small enough and $1 \ll t_0$ is large enough. Let $u$ solve*

$$\partial_{tt} u - \Delta u + \sum_{j=1}^{m} V_j \left( x - \vec{v}_j t \right) u + u^5 = 0. \tag{1.2.60}$$

*Suppose at $t = t_0$,*

$$\| U[t_0] - W[t_0] \|_{\dot{H}^1 \times L^2} \le \epsilon. \tag{1.2.61}$$

*Then there exist free data*

$$U_0[0] = (f_0, g_0)^t \in \dot{H}^1 \times L^2 \tag{1.2.62}$$

*such that*

$$\left\| U[t] - W[t] - e^{tJH_F} U_0[0] \right\|_{\dot{H}^1 \times L^2} \to 0.$$

*In other words, the error* $u(t) - \sum_{j=1}^{m} W_j \left( x - \vec{v}_j t \right)$ *scatters to the free wave.*

### *1.2.1 Notation*

"$A := B$" or "$B =: A$" is the definition of $A$ by means of the expression $B$. We use the notation $\langle x \rangle = \left(1 + |x|^2\right)^{\frac{1}{2}}$. The bracket $\langle \cdot, \cdot \rangle$ denotes the distributional pairing and the scalar product in the spaces $L^2$, $L^2 \times L^2$ . For positive quantities $a$ and $b$, we write $a \lesssim b$ for $a \leq Cb$ where $C$ is some prescribed constant. Also $a \simeq b$ for $a \lesssim b$ and $b \lesssim a$. We denote $B_R(x)$ the open ball of centered at $x$ with radius $R$ in $\mathbb{R}^3$. We also denote by $\chi$ a standard $C^\infty$ cut-off function, that is $\chi(x) = 1$ for $|x| \leq 1$, $\chi(x) = 0$ for $|x| > 2$ and $0 \leq \chi(x) \leq 1$ for $1 \leq |x| \leq 2$.

# CHAPTER 2

# STRICHARTZ ESTIMATES FOR CHARGE TRANSFER MODELS

## 2.1 Introduction

In this chapter, following the work of [51, 10], charge transfer models for Schrödinger equations in $\mathbb{R}^3$ will be considered. We study the time-dependent charge transfer Hamiltonian

$$H(t) = -\frac{1}{2}\Delta + \sum_{j=1}^{m} V_j(x - \vec{v}_j t) \tag{2.1.1}$$

with rapidly decaying smooth potentials $V_j(x)$, say, exponentially decaying and a set of mutually non-parallel constant velocities $\vec{v}_j$. Strichartz estimates for the evolution

$$\frac{1}{i}\partial_t \psi + H(t)\psi = 0 \tag{2.1.2}$$

associated with a charge transfer Hamiltonian $H(t)$ will be proved.

The starting point is the well-known $L^p$ estimates for the free Schrödinger equation ($H_0 = -\frac{1}{2}\Delta$) on $\mathbb{R}^n$ :

$$\left\| e^{iH_0 t} f \right\|_{L^p} \leq C_p \left| t \right|^{-n(\frac{1}{2}-\frac{1}{p})} \left\| f \right\|_{L^{p'}}, \tag{2.1.3}$$

where $2 \leq p \leq \infty$, $\frac{1}{p} + \frac{1}{p'} = 1$.

To analyze the dispersive estimate of linear Schrödinger equations with potentials, we consider the dispersive estimates of the Schrödinger flow

$$e^{itH} P_c, \quad H = -\frac{1}{2}\Delta + V \tag{2.1.4}$$

on $\mathbb{R}^n$, where $P_c$ is the projection onto the continuous spectrum of $H$. For Schrödinger equations with potentials, there may be bound states, i.e., $L^2$ eigenfunctions of $H$. Under the evolution $e^{itH}$, such bound states are merely multiplied by oscillating factors and thus do not disperse. So we need to project away any bound state. $V$ is a real-valued potential that is assumed to satisfy some decay condition at infinity. This decay is typically expressed in terms of the point-wise decay $|V(x)| \leq C \langle x \rangle^{-\beta}$, for all $x \in \mathbb{R}^n$ and for some $\beta > 0$. We use the notation $\langle x \rangle = \left(1 + |x|^2\right)^{\frac{1}{2}}$. Occasionally, we will use an integrability condition $V \in L^p(\mathbb{R}^n)$ (or a weighted variant of it) instead of a point-wise condition. These decay conditions will also be such that $H$ is asymptotically complete:

$$L^2(\mathbb{R}^n) = L^2_{p.p.}(\mathbb{R}^n) \oplus L^2_{a.c.}(\mathbb{R}^n)$$

where the spaces on the right-hand side refer to the span of all eigenfunctions, and the absolutely continuous subspace, respectively.

The dispersive estimate for the linear Schrödinger equations with potentials, which we will be most concerned with is of the form

$$\sup_{t \neq 0} |t|^{\frac{n}{2}} \left\| e^{itH} P_c f \right\|_{L^\infty_x} \leq C \|f\|_{L^1_x}, \quad \forall f \in L^1_x(\mathbb{R}^n) \cap L^2_x(\mathbb{R}^n). \tag{2.1.5}$$

Interpolating with the $L^2$ bound $\left\| e^{itH} P_c f \right\|_{L^2_x} \leq C \|f\|_{L^2_x}$, we get

$$\sup_{t \neq 0} |t|^{n(\frac{1}{2} - \frac{1}{p})} \left\| e^{itH} P_c f \right\|_{L^{p'}_x} \leq C \|f\|_{L^p_x} \quad \forall f \in L^1_x(\mathbb{R}^n) \cap L^2_x(\mathbb{R}^n), \tag{2.1.6}$$

where $1 \leq p \leq 2$.

It is well-known that via a $T^*T$ argument the dispersive estimate (2.1.5) gives rise to the class of Strichartz estimates

$$\left\| e^{itH} P_c f \right\|_{L_t^q L_x^p} \lesssim \|f\|_{L^2} \tag{2.1.7}$$

for all $\frac{2}{q} + \frac{n}{p} = \frac{n}{2}$. The endpoint $q = 2$ holds for $n \geq 3$ but it is not captured by this approach, see [37].

Roughly speaking, Strichartz estimates can be regarded as smoothing effects in $L_x^p$ spaces. For example, when we consider the free Schrödinger equation, compared with the trivial conservation of the $L^2$ norm of the solution, in Strichartz estimates one gains space integrability from $p = 2$ to $p > 2$, if one chooses to loses time integrability from $q = \infty$ to $q < \infty$. In other words, we gain space integrability in the integral sense. To be more precise, we can take a function $g \in L^2$ with $g \notin L_x^p$ for $p > 2$. If we take $f = e^{it_0\Delta} g$ as the initial data for the free linear Schrödinger equation, then we can see that at $t = t_0$, $e^{-it_0\Delta} f \notin L_x^p$. So without integration or averaging in time, there is no hope to get $L_x^p$ estimates for all times for general $L^2$ initial data. Strichartz estimates are crucial for the study of long-time behavior of associated nonlinear models.

For the results and historical progress of dispersive estimates and smoothing effects of Schrödinger operators, one can find further details and references in [53].

There are extra difficulties for Schrödinger equations with time-dependent potentials. For example, given a general time-dependent potential $V(x,t)$, it is not clear how to introduce an analog of bound states and the spectral projection. And the evolution of equation might not satisfy group properties any more. In this chapter, we focus on a particular case of time-dependent potentials, i.e. the charge transfer models in $\mathbb{R}^3$.

Firstly, we consider the scalar model in the following sense:

**Definition 2.1.1.** By a charge transfer model we mean a Schrödinger equation

$$\frac{1}{i}\partial_t \psi - \frac{1}{2}\Delta\psi + \sum_{j=1}^{m} V_j(x - \vec{v}_j t)\psi = 0, \tag{2.1.8}$$

$$\psi|_{t=0} = \psi_0,\ x \in \mathbb{R}^3.$$

where $\vec{v}_j$'s are distinct vectors in $\mathbb{R}^3$ , and the real potentials $V_k$ are such that for every $1 \le k \le m$

1) $V_k$ is time-independent and decays exponentially (or has compact support)

2) 0 is neither an eigenvalue nor a resonance of the operators

$$H_k = -\frac{1}{2}\Delta + V_k(x). \tag{2.1.9}$$

Recall that $\psi$ is a resonance at 0 if it is a distributional solution of the equation $H_k\psi = 0$ which belongs to the space $\left\{f : \langle x\rangle^{-\sigma} f \in L^2\right\}$ for any $\sigma > \frac{1}{2}$, but not for $\sigma = 0$.

To simplify our argument, we discuss when $m = 2$ case with $V_1$ is stationary and $V_2$ moves along $\overrightarrow{e_1}$ with the unit speed. It is easy to see our arguments work for general cases.

*Remark.* The assumptions are always assumed when we want to prove dispersive estimate and Strichartz estimates, e.g, [35, 53, 55, 51, 10]. The decay required of the potentials is not optimal but merely for convenience.

An indispensable tool in the study of charge transfer models are the Galilei transformations

$$\mathfrak{g}_{\vec{v},y}(t) = e^{i\frac{|\vec{v}|^2}{2}t}e^{ix\cdot\vec{v}}e^{-i(y+\vec{v}t)\cdot\vec{p}}, \tag{2.1.10}$$

cf. [27, 10, 51], where $\vec{p} = -i\vec{\nabla}$. They are the quantum analogues of the classical Galilei transforms

$$x \mapsto x - t\vec{v} - y, \quad \vec{p} \mapsto \vec{p} - \vec{v}. \tag{2.1.11}$$

To see this, we take a Schwartz function $f$ such that $f$ and $\hat{f}$ are centered around the origin, then $\mathfrak{g}_{\vec{v},y}(t)f$ is centered around $t\vec{v} + y$, and $\widehat{\mathfrak{g}_{\vec{v},y}(t)f}$ is centered around $\vec{v}$. The Galilei transformations have a very important conjugacy property:

$$\mathfrak{g}_{\vec{v},y}(t)e^{it\frac{\Delta}{2}} = e^{it\frac{\Delta}{2}}\mathfrak{g}_{\vec{v},y}(0). \tag{2.1.12}$$

Moreover, notice that with $H = -\frac{1}{2}\Delta + V$, then

$$\psi(t) := \mathfrak{g}_{\vec{v},y}(t)^{-1}e^{-itH}\mathfrak{g}_{\vec{v},y}(0)\psi_0, \;\; \mathfrak{g}_{\vec{v},y}(t)^{-1} = e^{-iy\vec{v}}\mathfrak{g}_{-\vec{v},-y}(t), \tag{2.1.13}$$

solves

$$\frac{1}{i}\partial_t\psi - \frac{1}{2}\Delta\psi + V\left(\cdot - t\vec{v} - y\right)\psi = 0, \;\; \psi|_{t=0} = \psi_0. \tag{2.1.14}$$

Another important property of the Galilei transformations is that $\mathfrak{g}_{\vec{v},y}(t)$ are isometries in all $L^p$ spaces. Finally, in our case, as discussed above, we always assume $y = 0$. To simplify our notations, we write $\mathfrak{g}_{\vec{v}}(t) := \mathfrak{g}_{\vec{v},0}(t)$ and notice that $\mathfrak{g}_{\vec{e_1}}(t)^{-1} = \mathfrak{g}_{-\vec{e_1}}(t)$.

We recall some consequences from [51, 10]. Again, we consider

$$\frac{1}{i}\partial_t\psi - \frac{1}{2}\Delta\psi + V_1\psi + V_2\left(\cdot - t\vec{e_1}\right)\psi = 0, \;\; \psi|_{t=0} = \psi_0, \tag{2.1.15}$$

with $V_1$ and $V_2$ decaying rapidly. Let $w_1, \ldots, w_m$ and $u_1, \ldots, u_\ell$ be the normalized bound states of $H_1$ and $H_2$ associated to the negative eigenvalues $\lambda_1, \ldots, \lambda_m$ and $\mu_1, \ldots, \mu_\ell$ respectively (notice that by our assumptions, 0 is not an eigenvalue).

Following the notations in [51], we denote by $P_b\left(H_1\right)$ and $P_b\left(H_2\right)$ the projections onto the the bound states of $H_1$ and $H_2$, respectively, and let $P_c\left(H_i\right) = Id - P_b\left(H_i\right), \, i = 1, 2$. To be more explicit, we have

$$P_b\left(H_1\right) = \sum_{i=1}^{m}\left\langle\cdot, w_i\right\rangle w_i, \quad P_b\left(H_2\right) = \sum_{j=1}^{\ell}\left\langle\cdot, u_j\right\rangle u_j. \tag{2.1.16}$$

It is well-known, from the standard case with stationary potentials that we need to project away from bound states as we discussed at the very beginning. Here following [51], we recall the analogous condition in our case.

**Definition 2.1.2.** Let $U(t,0)\psi_0 = \psi(t,x)$ be the solution of equation (2.1.15). We say that $\psi_0$ or $\psi(x,t)$ is asymptotically orthogonal to the bound states of $H_1$ and $H_2$ if

$$\|P_b(H_1)\, U(t,0)\psi_0\|_{L^2} + \|P_b(H_2,t)\, U(t,0)\psi_0\|_{L^2} \to 0, \quad t \to \pm\infty. \tag{2.1.17}$$

Here

$$P_b(H_2,t) := \mathfrak{g}_{\vec{e_1}}(t)^{-1} P_b(H_2)\, \mathfrak{g}_{\vec{e_1}}(t), \quad \forall t.$$

It is clear that all $\psi_0$ that satisfy (2.1.17) form a closed subspace of $L^2(\mathbb{R}^n)$. We call elements in this subspace scattering states at $t = 0$ and denote the subspace by $H_s(0)$. We name $H_s(0)$ as scattering space at $t = 0$. With $H_s(0)$, we define $P_s(0)$ to be the projection onto $H_s(0)$.

*Remark.* The subspace above coincides with the space of scattering states for the charge transfer problem which appears in Graf's asymptotic completeness result [27]. We will see more details in Section 2.2.

We now formulate our main results.

**Theorem 2.1.3** (Strichartz estimates)**.** *Consider the charge transfer model as in Definition 2.1.1 with two potentials in $\mathbb{R}^3$ as above. Suppose the initial data $\psi_0 \in L^2(\mathbb{R}^3)$ is asymptotically orthogonal to the bound states of $H_1$ and $H_2$ in the sense of Definition 2.1.2. Then for $\psi(t,x) = U(t,0)\psi_0$ and a Schrödinger admissible pair $(p,q)$ in $\mathbb{R}^3$, i.e.,*

$$\frac{2}{p} + \frac{3}{q} = \frac{3}{2} \tag{2.1.18}$$

*with $2 \le q \le \infty$, $p \ge 2$, we have*

$$\|\psi\|_{L^p_t([0,\infty),\, L^q_x)} \le C\, \|\psi_0\|_{L^2_x}. \tag{2.1.19}$$

We also have the boundedness of the energy.

**Theorem 2.1.4.** *Let $\psi_0 \in H^1(\mathbb{R}^3)$ and $\psi(t,x) = U(t,0)\psi_0$ be a solution to* (2.1.15) *with the initial data $\psi_0$. Then*

$$\sup_{t\in\mathbb{R}} \|U(t,0)\psi_0\|_{H^1} \leq C\|\psi_0\|_{H^1}. \tag{2.1.20}$$

The chapter is organized as follows: In Section 2.2, we will recall some results from [51, 10]. Then in Section 2.3, we establish Strichartz estimates for the evolution that is not associated to the bound states of $H_j$ for the scalar charge transfer model. In Section 2.4, we will show the energy of the whole evolution is bounded independently of time. Finally, we will generalize our arguments to non-selfadjoint matrix cases in Section 2.5.

## 2.2 Preliminaries

In this section, we formulate the important results from [51, 10] which are crucial for later sections.

First of all, if the evolution is asymptotically orthogonal to the bound states of $H_1$ and $H_2$ , we can actually get a decay rate for

$$\|P_b(H_1)U(t,0)\psi_0\|_{L^2} + \|P_b(H_2,t)U(t,0)\psi_0\|_{L^2} \to 0.$$

**2.2.1** ([51], Proposition 3.1)**.** *Let $\psi(t,x)$ be a solution to* (2.1.15) *which is asymptotically orthogonal to the bound states of $H_1$ and $H_2$ in the sense of Definition 2.1.2. Then we have the decay rate that*

$$\|P_b(H_1)U(t,0)\psi_0\|_{L^2} + \|P_b(H_2,t)U(t,0)\psi_0\|_{L^2} \lesssim e^{-\alpha|t|}\|\psi_0\|_{L^2} \tag{2.2.1}$$

*for some $\alpha > 0$.*

As pointed out above, $\forall \psi_0 \in L^2$ such that the asymptotically orthogonal condition 2.1.17 are satisfied, they form a subspace $H_s(0) \subset L^2$. We can do a more general time-dependent construction. Denote the evolution starting from $\tau$ to $t$ by $U(t, \tau)$. Similar as our original construction there is a subspace $H_s(\tau) \subset L^2$ such that for $\psi \in H_s(\tau)$,

$$\|P_b\,(H_1)\,U(t,\tau)\psi\|_{L^2} + \|P_b\,(H_2,t)\,U(t,\tau)\psi\|_{L^2} \to 0.$$

Similarly as above, we can also obtain a decay rate that for some $\alpha > 0$,

$$\|P_b\,(H_1)\,U(t,\tau)\psi\|_{L^2} + \|P_b\,(H_2,t)\,U(t,\tau)\psi\|_{L^2} \lesssim e^{-\alpha(t-\tau)}\,\|\psi\|_{L^2}\,,\ \psi \in H_s(s).$$

It is crucial to notice an important property of $H_s(\tau)$.

**Lemma 2.2.2.** *Denote $P_s(\tau)$ to be the projection onto $H_s(\tau)$. Then far arbitrary $s$, $\tau \in \mathbb{R}$,*

$$P_s(s)U(s,\tau) = U(s,\tau)P_s(\tau). \tag{2.2.2}$$

*Proof.* Notice that for $\psi \in H_s(\tau)$, then $U(s,\tau)\psi \in H_s(s)$. Since

$$\begin{aligned}&\|P_b\,(H_1)\,U(t,s)U(s,\tau)\psi\|_{L^2} + \|P_b\,(H_2,t)\,U(t,s)U(s,\tau)\psi\|_{L^2}\\ &\quad= \|P_b\,(H_1)\,U(t,\tau)\psi\|_{L^2} + \|P_b\,(H_2,t)\,U(t,\tau)\psi\|_{L^2} \to 0\end{aligned}$$

as $t \to \infty$ by the definition of $H_s(\tau)$. Then again by the definition of $H_s(s)$, it is clear $U(s,\tau)\psi \in H_s(s)$. Conversely, by symmetry, for $\psi \in H_s(s)$, then $U(\tau,s)\psi \in H_s(\tau)$. Therefore, we have the scattering spaces are invariant under the flow $U(s,\tau)$,

$$H_s(s) = U(s,\tau)H_s(\tau). \tag{2.2.3}$$

Let $\phi \in L^2$, then $U(s,\tau)P_s(\tau)\phi \in H_s(s)$ by construction. Then

$$\begin{aligned} U(s,\tau)P_s(\tau)\phi &= (1-P_s(s))\,U(s,\tau)P_s(\tau)\phi + P_s(s)U(s,\tau)P_s(\tau)\phi \\ &= P_s(s)U(s,\tau)P_s(\tau)\phi. \end{aligned}$$

Similarly,

$$P_s(s)U(s,\tau)\phi = P_s(s)U(s,\tau)P_s(\tau)\phi.$$

Hence

$$P_s(s)U(s,\tau) = U(s,\tau)P_s(\tau),$$

as claimed. □

If the evolution is asymptotically orthogonal to the bound states, one also have the usual $L^1 \to L^\infty$ dispersive estimate.

**Theorem 2.2.3** ([51],[10]). *Consider the charge transfer model as in Definition 2.1.1 with two potentials as above. Assume $\widehat{V_1}, \widehat{V_2} \in L^1(\mathbb{R}^n)$. Then for any initial data $\psi_0 \in L^1(\mathbb{R}^n)$, which is asymptotically orthogonal to the bound states of $H_1$ and $H_2$ in the sense of Definition 2.1.2, one has the decay estimate*

$$\|U(t,0)\psi_0\|_{L^\infty} \lesssim |t|^{-\frac{n}{2}} \|\psi_0\|_{L^1}. \tag{2.2.4}$$

*A similar estimate holds for any number of potentials.*

Note that since the potentials depend on time, Strichartz estimates do not follow from the dispersive estimate and $TT^*$ argument.

With the decay estimate (2.2.4), we obtain the asymptotic completeness of the charge transfer Hamiltonian:

**Theorem 2.2.4** ([51, 10]). *Let $w_1, \ldots, w_m$ and $u_1, \ldots, u_\ell$ be the normalized bound states of $H_1$ and $H_2$ associated to the negative eigenvalues $\lambda_1, \ldots, \lambda_m$ and $\mu_1, \ldots, \mu_\ell$. Then for any initial data $\psi_0 \in L^2(\mathbb{R}^n)$, the solution $\psi(t,x) = U(t,0)\psi_0$ of the charge transfer model, equation* (2.1.15)*, can be written in the form*

$$\psi(t,x) = U(t,0)\psi_0 = \sum_{r=1}^{m} A_r e^{-i\lambda_r t} w_r + \sum_{s=1}^{\ell} B_s e^{-i\mu_s t} \mathfrak{g}_{-\vec{e_1}}(t) u_s + e^{-it\frac{\Delta}{2}} \phi_0 + R(t) \quad (2.2.5)$$

*with some choice of the constants $A_r$, $B_s$ and the function $\phi_0$. The remainder term $R(t)$ satisfies the estimate,*

$$\|R(t)\|_{L^2} \to 0, \ t \to \pm\infty. \quad (2.2.6)$$

With the asymptotic completeness of the charge transfer Hamiltonian, we can construct a time-dependent decomposition of $L^2$ with scattering states and analogous of bound states associated with $H_1$ and $H_2$. The construction should be similar as [27, 56]. Following the notations in [51, 27], with the proof of Theorem 2.2.4, we know the existence of the following wave operators in $L^2$: for $s \in \mathbb{R}$,

$$\Omega_0^-(s) = \lim_{t\to\infty} U(s,t) e^{-iH_0(t-s)} \quad (2.2.7)$$

$$\Omega_1^-(s) = \lim_{t\to\infty} U(s,t) e^{-iH_1(t-s)} P_b(H_1) \quad (2.2.8)$$

$$\Omega_2^-(s) = \lim_{t\to\infty} U(s,t) \mathfrak{g}_{\vec{e_1}}(t)^{-1} e^{-iH_2(t-s)} P_b(H_2)\, \mathfrak{g}_{\vec{e_1}}(t) \quad (2.2.9)$$

where limits are taken as strong operator topology. From [51], the ranges of the above operators has the following relation:

$$L^2 = \mathrm{Ran}\Omega_0^-(s) \oplus \mathrm{Ran}\Omega_1^-(s) \oplus \mathrm{Ran}\Omega_2^-(s). \quad (2.2.10)$$

Naturally the above constructions will introduce a time-dependent decomposition of $L^2$ and one can observe that

$$\mathrm{Ran}\Omega_0^-(\tau) = H_s(\tau).$$

We introduce projections $P_i(\tau)$ to the projection onto the range of $\Omega_i^-(\tau)$, $i=1,2$. Clearly. $\mathrm{Ran}\Omega_1^-(\tau)$ and $\mathrm{Ran}\Omega_2^-(\tau)$ are analogous as the spans bound states associated with $H_1$ and $H_2$ respectively. Notice that by construction, one can find a basis for $\Omega_i^-(\tau)$, $i=1,2$. With our notations above, $w_1, \ldots, w_m$ and $u_1, \ldots, u_\ell$be the normalized bound states of $H_1$ and $H_2$ associated to the negative eigenvalues $\lambda_1, \ldots, \lambda_m$ and $\mu_1, \ldots, \mu_\ell$ respectively. Then $\{w_{i,\tau} = \Omega_1^-(\tau) w_i\}_{i=1}^m$ is a basis for $\mathrm{Ran}\Omega_1^-(\tau)$. Similarly, $\{u_{j,\tau} = \Omega_2^-(\tau)\mathfrak{g}_{\vec{e_1}}(\tau) u_j\}_{i=1}^m$ is a basis for $\mathrm{Ran}\Omega_2^-(\tau)$. By asymptotic completeness and intuition, as $\tau \to \infty$, $\Omega_1^-(\tau) w_i \to w_i$ and $\Omega_2^-(\tau)\mathfrak{g}(\tau) u_j \to \mathfrak{g}_{\vec{e_1}}(\tau)^{-1} u_j$. Actually, following [51], one can actually extract a convergent rate. We focus on$\Omega_1^-(\tau) w_i \to w_i$ and for the other case, we just need to apply the same argument after applying a Galilei transformation.

**2.2.5.** *For some $\alpha > 0$ ,*

$$\left\| \Omega_1^-(\tau) w_i - w_i \right\|_{L^2} \lesssim e^{-\alpha\tau}. \tag{2.2.11}$$

*Proof.* For $1 \leq i \leq m$, with Duhamel's formula

$$U(s,t) e^{-iH_1(t-s)} w_i = w_i + i \int_s^t U(s,t) V_2\left(\cdot - \vec{e_1}\tau\right) e^{-i\lambda_i(\tau - s)} w_i \, d\tau.$$

It suffices to estimate the $L^2$ norm of

$$\int_s^\infty U(s,t) V_2\left(\cdot - \vec{e_1}\tau\right) e^{-i\lambda_i(\tau - s)} w_i \, d\tau. \tag{2.2.12}$$

By Agmon's estimate,

$$\begin{aligned} \left\| \int_s^\infty U(s,t) V_2\left(\cdot - \vec{e_1}\tau\right) e^{-i\lambda_i(\tau - s)} w_i \, d\tau \right\|_{L^2} &\lesssim \left\| \int_s^\infty V_2\left(\cdot - \vec{e_1}\tau\right) w_i \, d\tau \right\|_{L^2} \\ &\lesssim e^{-\alpha s}. \end{aligned} \tag{2.2.13}$$

Therefore

$$\left\|\Omega_1^-(\tau)w - w_i\right\|_{L^2} \lesssim e^{-\alpha\tau},$$

as claimed. □

## 2.3 Strichartz Estimates

In this section, we prove Strichartz estimates for charge transfer models. The ideas will be based on methods in [22, 51]. Certainly, we need to project away from the bound states of $H_1$ and the moving bound states associated to $H_2(t)$. We will show certain weighted estimates for the evolution of states in the scattering space defined in [51] and in the sense of Definition 2.1.2.

Now we formulate the following two estimates when our initial state is in the scattering space. The first one is:

**Lemma 2.3.1.** *For $\sigma > \frac{3}{2}$ and $t \geq t_0$*

$$\left\|\langle x - x_0\rangle^{-\sigma} U(t,t_0) P_s(t_0) \langle x - x_1\rangle^{-\sigma}\right\|_{2\to 2} \leq C \frac{1}{\langle t - t_0\rangle^{\frac{3}{2}}} \tag{2.3.1}$$

*for all $x_0$ and $x_1$.*

Here $2 \to 2$ means the norm as an operator from $L^2$ to $L^2$ and $P_s$ defined as the projection onto the scattering space as above in sense of Definition 2.1.2. Also as usual, $\langle x\rangle = \left(|x|^2 + 1\right)^{\frac{1}{2}}$. The second weighted estimate we want to show is the following:

**Lemma 2.3.2.** *For $\sigma > \frac{3}{2}$*

$$\int_0^\infty \left\|\langle x - x(t)\rangle^{-\sigma} U(t,t_0) P_s(t_0) u\right\|_{L_x^2}^2 dt \leq C \left\|u\right\|_{L^2}^2 \tag{2.3.2}$$

*for all $x(t) \in \mathcal{C}\left([0,\infty), \mathbb{R}^3\right)$.*

Heuristically, we can see the above two estimates hold for the evolution of a free Schrödinger equation since a free particle moves towards infinity. The weights just play roles like indicator functions of certain finite regions. Then surely, as time evolves, the particle will leave any of those regions. So we have the decay of the wave function. In our case, the state in the scattering space will just move asymptotically like a free particle, so we should expect the above result. The second estimate is a variant of the above heuristics adjusted to our model since we have moving potentials.

Before we prove Lemma 2.3.1 and Lemma 2.3.2, we show how to derive Strichartz estimates for the charge transfer model based on them.

*Proof of Theorem 2.1.3.* Let $\psi(t) = U(t,0)\psi_0$ and by our assumption we have $P_s(0)\psi_0 = \psi_0$. Rewrite the charge transfer model as

$$i\psi_t + \frac{1}{2}\Delta\psi = V_1\psi + V_2(\cdot - t\vec{e_1})\psi.$$

Now we apply the endpoint Strichartz estimate [37] for the free Schrödinger equation, we get for a Schrödinger admissible pair $(p,q)$ in $\mathbb{R}^3$, one has

$$\|\psi\|_{L^p_t([0,\infty),\, L^q_x)} \leq C\left\|V_1\psi + V_2(\cdot - t\vec{e_1})\psi\right\|_{L^2_t\left([0,\infty),\, L_x^{\frac{6}{5}}\right)} + \|\psi_0\|_{L^2_x}$$

Since our potentials decay fast, we can pick $m$ large (in particular $m > \frac{3}{2}$) such that by Hölder's inequality we have,

$$\begin{aligned}
\left\|V_1\psi + V_2(\cdot - t\vec{e_1})\psi\right\|_{L^2_t\left([0,\infty),\, L_x^{\frac{6}{5}}\right)} &\leq \|V_1\psi\|_{L^2_t\left([0,\infty),\, L_x^{\frac{6}{5}}\right)} + \left\|V_2(\cdot - t\vec{e_1})\psi\right\|_{L^2_t\left([0,\infty),\, L_x^{\frac{6}{5}}\right)} \\
\|V_1\psi\|_{L^2_t\left([0,\infty),\, L_x^{\frac{6}{5}}\right)} &\leq C_V\left\|\langle x\rangle^{-m}\psi\right\|_{L^2_t([0,\infty),\, L^2_x)} \\
\left\|V_2(\cdot - t\vec{e_1})\psi\right\|_{L^2_t\left([0,\infty),\, L_x^{\frac{6}{5}}\right)} &\leq C_V\left\|\langle x - t\vec{e_1}\rangle^{-m}\psi\right\|_{L^2_t([0,\infty),\, L^2_x)}
\end{aligned}$$

Now by our above two claimed estimates Lemma 2.3.1 and Lemma 2.3.2, we have

$$\left\| \langle x \rangle^{-m} \psi \right\|_{L^2_t([0,\infty), L^2_x)} \leq C \left\| \psi_0 \right\|_{L^2},$$

$$\left\| \langle x - t\vec{e_1} \rangle^{-m} \psi \right\|_{L^2_t([0,\infty), L^2_x)} \leq C \left\| \psi_0 \right\|_{L^2}.$$

Then combine all estimates above, we get

$$\left\| \psi \right\|_{L^p_t([0,\infty), L^q_x)} \leq C \left\| \langle x \rangle^{-m} \psi \right\|_{L^2_t([0,\infty), L^2_x)} + C \left\| \langle x - t\vec{e_1} \rangle^{-m} \psi \right\|_{L^2_t([0,\infty), L^2_x)} \leq C \left\| \psi_0 \right\|_{L^2}.$$

Therefore, we have the desired Strichartz estimate

$$\left\| \psi \right\|_{L^p_t([0,\infty), L^q_x)} \leq C \left\| \psi_0 \right\|_{L^2}.$$

as claimed. □

In the next section, we will show as a byproduct of Strichartz estimates, (2.1.19), we can get the energy boundedness of the whole evolution of the charge transfer model.

### *2.3.1 Proof of lemmas 2.3.1 and 2.3.2*

To rigorously show Lemmas 2.3.1 and 2.3.2 are consistent with our heuristics, we consider the free evolution first. We claim that the first estimate (2.3.1) holds for the free Schrödinger equation.

**Lemma 2.3.3.** *For $\sigma > \frac{3}{2}$*

$$\left\| \langle x - x_0 \rangle^{-\sigma} e^{i\frac{\Delta}{2}(t-t_0)} \langle x - x_1 \rangle^{-\sigma} \right\|_{2\to 2} \leq C \frac{1}{\langle t - t_0 \rangle^{\frac{3}{2}}} \tag{2.3.3}$$

*for all $x_0$ and $x_1$ in $\mathbb{R}^3$.*

*Proof.* Let $s = t - t_0$, then if $|s| \leq 1$, clearly by $\left\| e^{i\frac{\Delta}{2}s} \right\|_{2\to 2} \leq 1$ and $\sigma > 0$, we can get the desired result.

If $|s| \geq 1$, we apply the dispersive estimate for the free evolution. Then by Young's inequality we get

$$\left\| \langle x - x_0 \rangle^{-\sigma} e^{i\frac{\Delta}{2}(t-t_0)} \langle x - x_1 \rangle^{-\sigma} \right\|_{2\to 2} \lesssim \left\| \langle x \rangle^{-\sigma} \right\|_{L^2}^2 \left\| e^{is\frac{\Delta}{2}} \right\|_{1\to\infty} \lesssim |s|^{-\frac{3}{2}}.$$

So the desired estimate holds. □

Also the second estimate (2.3.2) holds for the free Schrödinger evolution by the endpoint Strichartz estimate, estimate (2.1.7).

**Lemma 2.3.4.** *For $\sigma > \frac{3}{2}$*

$$\int_0^\infty \left\| \langle x - x(t) \rangle^{-\sigma} e^{it\frac{\Delta}{2}} u \right\|_{L_x^2}^2 dt \leq C \left\| u \right\|_{L^2}^2. \tag{2.3.4}$$

*Proof.* By Hölder's inequality, we have

$$\left\| \langle x - x(t) \rangle^{-\sigma} e^{it\frac{\Delta}{2}} u \right\|_{L_x^2} \lesssim \left\| e^{it\frac{\Delta}{2}} u \right\|_{L_x^6}.$$

Then by the endpoint Strichartz estimate in $\mathbb{R}^3$, [37], we have

$$\left\| e^{it\frac{\Delta}{2}} u \right\|_{L_t^2 L_x^6} \leq C \left\| u \right\|_{L^2}.$$

Therefore, we can conclude

$$\int_0^\infty \left\| \langle x - x(t) \rangle^{-\sigma} e^{it\frac{\Delta}{2}} u \right\|_{L_x^2}^2 dt \leq C \left\| u \right\|_{L^2}^2.$$

The Lemma is proved. □

Now we show Lemmas 2.3.1 and 2.3.2 by a bootstrap argument similar to the one in [51]. As usual, the constant $C$ varies from line to line.

First of all, we note the following simple facts: Since $P_s(t_0)u$ satisfies following estimates, for $p \geq 2$ and $t \geq t_0$

$$\|P_b(H_1)\, U(t,t_0)P_s(t_0)\|_{L^2\to L^p} \lesssim e^{-\alpha(p)|t|},$$

and

$$\|P_b(H_2,t)\, U(t,t_0)P_s(t_0)\|_{L^2\to L^p} \lesssim e^{-\beta(p)|t|}.$$

Then surely,

$$\left\|\langle x-x_0\rangle^{-\sigma} P_b(H_1)\, U(t,t_0)P_s(t_0)\langle x-x_1\rangle^{-\sigma}\right\|_{2\to 2} \leq C\frac{1}{\langle t-t_0\rangle^{\frac{3}{2}}},$$

$$\left\|\langle x-x_0\rangle^{-\sigma} P_b(H_2,t)\, U(t,t_0)P_s(t_0)\langle x-x_1\rangle^{-\sigma}\right\|_{2\to 2} \leq C\frac{1}{\langle t-t_0\rangle^{\frac{3}{2}}}.$$

For the second weighted estimate, with some $p \geq 2$,

$$\left\|\langle x-x(t)\rangle^{-\sigma} P_b(H_1)\, U(t,t_0)P_s(t_0)u\right\|_{L^2_x} \lesssim e^{-\alpha(p)|t|}\,\|u\|_{L^2_x},$$

$$\left\|\langle x-x(t)\rangle^{-\sigma} P_b(H_2,t)\, U(t,t_0)P_s(t_0)u\right\|_{L^2_x} \lesssim e^{-\beta(p)|t|}\,\|u\|_{L^2_x}.$$

So

$$\int_0^\infty \left\|\langle x-x(t)\rangle^{-\sigma} P_b(H_1)\, U(t,t_0)P_s(t_0)u\right\|_{L^2_x} dt \lesssim \|u\|^2_{L^2_x},$$

and

$$\int_0^\infty \left\|\langle x-x(t)\rangle^{-\sigma} P_b(H_2,t)\, U(t,t_0)P_s(t_0)u\right\|_{L^2_x} dt \lesssim \|u\|^2_{L^2_x}.$$

By the Duhamel formula, we write

$$U(t,t_0)P_s(t_0) = e^{i\frac{1}{2}\Delta(t-t_0)}P_s(t_0) + i\int_{t_0}^{t} e^{i\frac{1}{2}\Delta(t-s)}V_1U(s,t_0)P_s(t_0)\,ds + i\int_{t_0}^{t} e^{i\frac{1}{2}\Delta(t-s)}V_2(\cdot - s\vec{e_1})U(s,t_0)P_s(t_0)\,ds, \tag{2.3.5}$$

and let

$$U(t,t_0) = F + iL + iG \tag{2.3.6}$$

Surely, there is no problem with the free piece $F$ as we discussed above by Lemmas 2.3.3 and 2.3.4.

Now fix $T$ large enough and apply Gronwall's equality. Then we can find a large constant $C(T)$ such that

$$\left\|\langle x-x_0\rangle^{-\sigma}U(t,t_0)P_s(t_0)\langle x-x_1\rangle^{-\sigma}\right\|_{2\to 2} \le C(T)\frac{1}{\langle t-t_0\rangle^{\frac{3}{2}}} \tag{2.3.7}$$

$$\int_0^T \left\|\langle x-x(t)\rangle^{-\sigma}\,U(t,t_0)P_s(t_0)u\right\|_{L_x^2}^2\,dt \le C^2(T)\,\|u\|_{L^2}^2 \tag{2.3.8}$$

hold for $t_0 \le t \le T$.

Next we imitate the bootstrap process in [51] and [22]. Fix a large constant $A$ to be determined later. We also assume $T - t_0 \gg A$. As in [51], for $t \le T$ we consider the decomposition of interval $[t_0,t] = [t_0, t_0+A] \cup [t_0+A, t-A] \cup [t-A, t]$. Set

$$L_1 = \int_{t_0}^{t_0+A} e^{i\frac{1}{2}\Delta(t-s)}V_1U(s,t_0)P_s(t_0)\,ds$$

$$L_2 = \int_{t_0+A}^{t-A} e^{i\frac{1}{2}\Delta(t-s)}V_1U(s,t_0)P_s(t_0)\,ds$$

$$L_3 = \int_{t-A}^{t} e^{i\frac{1}{2}\Delta(t-s)}V_1U(s,t_0)P_s(t_0)\,ds$$

$$G_1 = \int_{t_0}^{t_0+A} e^{i\frac{1}{2}\Delta(t-s)} V_2(\cdot - s\vec{e_1}) U(s,t_0) P_s(t_0)\, ds$$

$$G_2 = \int_{t_0+A}^{t-A} e^{i\frac{1}{2}\Delta(t-s)} V_2(\cdot - s\vec{e_1}) U(s,t_0) P_s(t_0)\, ds$$

$$G_3 = \int_{t-A}^{t} e^{i\frac{1}{2}\Delta(t-s)} V_2(\cdot - s\vec{e_1}) U(s,t_0) P_s(t_0)\, ds.$$

First, we bound $L_1$. With Lemma 2.3.3, we have

$$\begin{aligned}
&\left\| \langle x - x_0 \rangle^{-\sigma} L_1 \langle x - x_1 \rangle^{-\sigma} \right\|_{2\to 2} \\
&\leq C \int_{t_0}^{t_0+A} \frac{1}{\langle t-s \rangle^{\frac{3}{2}}} \left\| \langle x \rangle^{-\sigma} U(s,t_0) P_s(t_0) \langle x - x_1 \rangle^{-\sigma} \right\|_{2\to 2} ds \\
&\qquad \leq C(A) \int_{t_0}^{t_0+A} \frac{1}{\langle t-s \rangle^{\frac{3}{2}}} \frac{1}{\langle s-t_0 \rangle^{\frac{3}{2}}}\, ds \\
&\qquad\qquad \leq C(A) \frac{1}{\langle t-t_0 \rangle^{\frac{3}{2}}}.
\end{aligned}$$

Here we just emphasize that the constant in above estimate does not depend on $T$. Notice that $G_1$ can be bounded similarly as above.

For the second part, with Lemma 2.3.4,

$$\begin{aligned}
&\int_0^T \left\| \langle x - x(t) \rangle^{-\sigma} L_1 u \right\|_{L_x^2}^2 dt \\
&\leq C \int_0^T \left( \int_{t_0}^{t_0+A} \langle t-s \rangle^{-\frac{3}{2}} \left\| \langle x \rangle^{-\sigma} U(s,t_0) P_s(t_0) u \right\|_{L_x^2} ds \right)^2 dt \\
&\qquad \leq C \left\| \langle x \rangle^{-\sigma} U(s,t_0) P_s(t_0) u \right\|_{L_t^2((t_0,t_0+A),\, L_x^2)}^2 \\
&\qquad\qquad \leq C(A) \left\| u \right\|_{L^2}^2.
\end{aligned}$$

Also $G_1$ can be bounded similarly.

Next, we analyze $L_2$. With Lemma 2.3.3 and the bootstrap assumption (3.4.10),

$$
\begin{aligned}
&\left\| \langle x - x_0\rangle^{-\sigma} L_2 \langle x - x_1\rangle^{-\sigma} \right\|_{2\to 2} \\
&\quad \leq C \int_{t_0+A}^{t-A} \frac{1}{\langle t-s\rangle^{\frac{3}{2}}} \left\| \langle x\rangle^{-\sigma} U(s,t_0) P_s(t_0) \langle x - x_1\rangle^{-\sigma} \right\|_{2\to 2} ds \\
&\quad \leq C(T) \int_{t_0+A}^{t-A} \frac{1}{\langle t-s\rangle^{\frac{3}{2}}} \frac{1}{\langle s-t_0\rangle^{\frac{3}{2}}} ds \\
&\quad \leq CC(T) A^{-\frac{1}{2}} \langle t-t_0\rangle^{-\frac{3}{2}}
\end{aligned}
$$

for an absolute constant $C$.

For the other estimate, with Lemma 2.3.4 and bootstrap assumption (3.4.11), we conclude that

$$
\begin{aligned}
\int_0^T \left\| \langle x - x(t)\rangle^{-\sigma} L_2 u \right\|_{L_x^2}^2 dt &\leq C \int_0^T \left( \int_{t_0+A}^{t-A} \langle t-s\rangle^{-\frac{3}{2}} \left\| \langle x\rangle^{-\sigma} U(s,t_0) P_s(t_0) u \right\|_{L_x^2} ds \right)^2 dt \\
&\leq h(A) \left\| \langle x\rangle^{-\sigma} U(s,t_0) P_s(t_0) u \right\|_{L_t^2((0,T),\, L_x^2)}^2 \\
&\leq h(A) C^2(T) \left\| u \right\|_{L^2}^2
\end{aligned}
$$

where

$$h(A) \lesssim A^{-1}$$

by Young's inequality applied to the convolution

$$\int_{t_0+A}^{t-A} \langle t-s\rangle^{-\frac{3}{2}} \left\| \langle x\rangle^{-\sigma} U(s,t_0) P_s(t_0) u \right\|_{L_x^2} ds.$$

So when $A$ is large, we recapture our bootstrap argument conditions, i.e., $h(A)C^2(T)$ will be a small portion of $C(T)$ provided $A$ is large enough. Similar estimates hold for $G_2$.

It remains to analyze $L_3$ and $G_3$. We will expand $U$ again. And the following two versions of weighted estimates for Schrödinger equations with rapidly decaying potentials will be used.

**Lemma 2.3.5.** *For $\sigma > \frac{3}{2}$, and $H_j = -\frac{1}{2}\Delta + V_j$, where $V_j$ satisfies the decay assumption for our charge transfer Hamiltonian, then we have*

$$\left\| \langle x - x_0 \rangle^{-\sigma} e^{iH_j(t-t_0)} P_c \left(H_j\right) \langle x - x_1 \rangle^{-\sigma} \right\|_{2\to 2} \le C \frac{1}{\langle t - t_0 \rangle^{\frac{3}{2}}} \tag{2.3.9}$$

*and*

$$\int_0^\infty \left\| \langle x - x(t) \rangle^{-\sigma} e^{itH_j} P_c \left(H_j\right) u \right\|_{L_x^2}^2 dt \le C \left\| u \right\|_{L^2}^2, \tag{2.3.10}$$

*where $P_c \left(H_j\right)$ is the projection onto the continuous spectrum of $H_j$.*

*Proof.* These two estimates follow from the boundedness of wave operators [55] and Lemma 2.3.3 and Lemma 2.3.4. Or one can apply the dispersive estimate and Strichartz estimates for perturbed Schrödinger equations. □

Now we analyze

$$L_3 = \int_{t-A}^{t} e^{i\frac{1}{2}\Delta(t-s)} V_1 U(s,t_0) P_s(t_0)\, ds.$$

Splitting $L_3$ with respect to the spectrum of $H_1$, one has

$$\begin{aligned} L_3 &= \int_{t-A}^{t} e^{i\frac{1}{2}\Delta(t-s)} V_1 P_c\left(H_1\right) U(s,t_0) P_s(t_0)\, ds \\ &\quad + \int_{t-A}^{t} e^{i\frac{1}{2}\Delta(t-s)} V_1 P_b\left(H_1\right) U(s,t_0) P_s(t_0)\, ds \\ &= L_{3,c} + L_{3,b}. \end{aligned}$$

Surely, there is no problem with $L_{3,b}$ by the discussion at the very beginning of this section$P_b\left(H_1\right) U(s,t_0) P_s$ decays exponentially.

For $L_{3,c}$, we use the ideas from [51] to decompose our evolution into low velocity and high velocity pieces. For the low velocity piece, we directly use a commutator argument, non-stationary phase and the fact the supports of $V_1$ and $V_2$ become almost disjoint. For the high velocity part, we use a version of the Kato smoothing estimate.

Expanding $U$ with respect to $H_1$, we can write

$$U(t,t_0) = e^{-iH_1(t-t_0)} + i\int_{t_0}^{t} e^{-iH_1(t-s)} V_2(\cdot - s\vec{e_1})U(s,t_0)\, ds.$$

Then we can write

$$\begin{aligned} L_{3,c} &= \int_{t-A}^{t} e^{i\frac{1}{2}\Delta(t-s)} V_1 P_c(H_1)\, U(s,t_0) P_s(t_0)\, ds \\ &= \int_{t-A}^{t} e^{i\frac{1}{2}\Delta(t-s)} V_1 P_c(H_1)\, e^{-iH_1(s-t_0)} P_s(t_0)\, ds \\ &\quad + i\int_{t-A}^{t}\int_{t_0}^{s} V_1 P_c(H_1)\, e^{-iH_1(s-\tau)} V_2(\cdot - \tau\vec{e_1}) U(\tau,t_0) P_s(t_0)\, d\tau ds. \end{aligned}$$

Consider the decomposition

$$L_{3,c} = I + iK,$$

$$I = \int_{t-A}^{t} e^{i\frac{1}{2}\Delta(t-s)} V_1 P_c(H_1)\, e^{-iH_1(s-t_0)} P_s(t_0)\, ds,$$

$$K = \int_{t-A}^{t} e^{i\frac{1}{2}\Delta(t-s)} V_1 \int_{t_0}^{s} P_c(H_1)\, e^{-iH_1(s-\tau)} V_2(\cdot - \tau\vec{e_1}) U(\tau,t_0) P_s(t_0)\, d\tau ds.$$

There is no problem with $I$ by similar arguments for the free case with Lemma 2.3.5.

Next, we decompose $K$ further as follows:

$$K = J + S + Z,$$

$$S = \int_{t-A}^{t} e^{i\frac{1}{2}\Delta(t-s)} V_1 \int_{t_0}^{t_0+B} P_c(H_1)\, e^{-iH_1(s-\tau)} V_2(\cdot - \tau\vec{e_1}) U(\tau,t_0) P_s(t_0)\, d\tau ds,$$

$$Z = \int_{t-A}^{t} e^{i\frac{1}{2}\Delta(t-s)} V_1 \int_{t_0+B}^{s-B} P_c(H_1)\, e^{-iH_1(s-\tau)} V_2(\cdot - \tau\vec{e_1}) U(\tau,t_0) P_s(t_0)\, d\tau ds,$$

$$J = \int_{t-A}^{t}\int_{s-B}^{s} e^{i\frac{1}{2}\Delta(t-s)} V_1 P_c(H_1)\, e^{-iH_1(s-\tau)} V_2(\cdot - \tau\vec{e_1}) U(\tau,t_0) P_s(t_0)\, d\tau ds.$$

For $S$, a similar argument as for $L_1$ implies

$$
\begin{aligned}
&\left\|\langle x-x_0\rangle^{-\sigma} S \langle x-x_1\rangle^{-\sigma}\right\|_{2\to 2}\\
&\lesssim C\int_{t-A}^{t}\frac{1}{\langle t-s\rangle^{\frac{3}{2}}}\int_{t_0}^{t_0+B}\frac{1}{\langle s-\tau\rangle^{\frac{3}{2}}}\left\|\langle x\rangle^{-\sigma}U(s,t_0)P_s(t_0)\langle x-x_1\rangle^{-\sigma}\right\|_{2\to 2}\,d\tau ds\\
&\qquad\lesssim C(B)\int_{t-A}^{t}\int_{t_0}^{t_0+B}\frac{1}{\langle t-s\rangle^{\frac{3}{2}}}\frac{1}{\langle s-\tau\rangle^{\frac{3}{2}}}\frac{1}{\langle s-t_0\rangle^{\frac{3}{2}}}dsd\tau\\
&\qquad\le C(A,B)\frac{1}{\langle t-t_0\rangle^{\frac{3}{2}}}.
\end{aligned}
$$

As usual, the constant $C$ does not depend on $T$.

For the second piece, we also have

$$
\begin{aligned}
&\int_0^T \left\|\langle x-x(t)\rangle^{-\sigma} Su\right\|_{L_x^2}^2 dt\\
&\le C\int_0^T\left(\int_{t-A}^{t}\langle t-s\rangle^{-\frac{3}{2}}\int_{t_0}^{t_0+B}\langle s-\tau\rangle^{-\frac{3}{2}}\left\|\langle x\rangle^{-\sigma}U(s,t_0)P_s(t_0)u\right\|_{L_x^2}d\tau ds\right)^2 dt\\
&\qquad\lesssim C(B)A\left\|\langle x\rangle^{-\sigma}U(s,t_0)P_s u\right\|_{L_t^2((t_0,t_0+B),\,L_x^2)}^2\\
&\qquad\lesssim C(A,B)\left\|u\right\|_{L^2}^2.
\end{aligned}
$$

Next, for $Z$, following a similar argument to $L_2$, we obtain

$$
\begin{aligned}
&\left\|\langle x-x_0\rangle^{-\sigma} Z \langle x-x_1\rangle^{-\sigma}\right\|_{2\to 2}\\
&\lesssim \int_{t-A}^{t}\frac{1}{\langle t-s\rangle^{\frac{3}{2}}}ds\int_{t_0+B}^{s-B}\frac{1}{\langle s-\tau\rangle^{\frac{3}{2}}}\left\|\langle x\rangle^{-\sigma}U(\tau,t_0)P_s(t_0)\langle x-x_1\rangle^{-\sigma}\right\|_{2\to 2}d\tau\\
&\qquad\lesssim C(T)\int_{t-A}^{t}\int_{t_0+B}^{s-B}\frac{1}{\langle t-s\rangle^{\frac{3}{2}}}\frac{1}{\langle s-\tau\rangle^{\frac{3}{2}}}\frac{1}{\langle \tau-t_0\rangle^{\frac{3}{2}}}\,d\tau ds\\
&\qquad\lesssim C(T)AB^{-\frac{1}{2}}\langle t-t_0\rangle^{-\frac{3}{2}}.
\end{aligned}
$$

For the second estimate,

$$\begin{aligned}
&\int_0^T \left\| \langle x - x(t)\rangle^{-\sigma} Zu \right\|_{L_x^2}^2 dt \\
&\quad \lesssim \int_0^T \left( \int_{t-A}^t \langle t-s\rangle^{-\frac{3}{2}} \int_{t_0+B}^{s-B} \langle s-\tau\rangle^{-\frac{3}{2}} \left\| \langle x\rangle^{-\sigma} U(\tau,t_0) P_s(t_0) u \right\|_{L_x^2} d\tau ds \right)^2 dt \\
&\qquad\qquad \lesssim h(B) A \left\| \langle x\rangle^{-\sigma} U(\tau,t_0) P_s(t_0) u \right\|_{L_t^2((0,T),\, L_x^2)}^2 \\
&\qquad\qquad \lesssim h(B) A C^2(T) \left\| u \right\|_{L^2}^2
\end{aligned}$$

where as before,

$$h(B) \lesssim B^{-1}.$$

Therefore, when we pick $B$ large enough, we have satisfied all the conditions for the bootstrap argument.

Finally, we analyze $J$:

$$J = \int_{t-A}^t \int_{s-B}^s e^{i\frac{1}{2}\Delta(t-s)} V_1 P_c(H_1) e^{-iH_1(s-\tau)} V_2(\cdot - \tau\vec{e_1}) U(\tau,t_0) P_s(t_0)\, d\tau ds.$$

We decompose the integral into low and high frequency parts:

$$J_L = \int_{t-A}^t \int_{s-B}^s e^{i\frac{1}{2}\Delta(t-s)} V_1 F(|\vec{p}| \leq M) e^{-iH_1(s-\tau)} P_c(H_1) V_2(\cdot - \tau\vec{e_1}) U(\tau,t_0) P_s(t_0)\, d\tau ds,$$

$$J_H = \int_{t-A}^t \int_{s-B}^s e^{i\frac{1}{2}\Delta(t-s)} V_1 F(|\vec{p}| \geq M) e^{-iH_1(s-\tau)} P_c(H_1) V_2(\cdot - \tau\vec{e_1}) U(\tau,t_0) P_s(t_0)\, d\tau ds,$$

where $F(|\vec{p}| \leq M)$ and $F(|\vec{p}| \geq M)$ denote smooth projections onto frequencies $|\vec{p}| \leq M$ and $|\vec{p}| \geq M$ respectively.

To analyze the low frequency part, we observe that for arbitrary $\epsilon > 0$,

$$\int_{s-B}^s \langle -\tau\vec{e_1}\rangle^{-\frac{3}{2}} d\tau \leq \epsilon$$

provided $s$ is large enough.

Set $V_2^\sigma(x) = V_2(x)\langle x\rangle^\sigma$, then we look at the following quantity,

$$\left\| V_1 F\left(|\vec{p}| \le M\right) e^{-i\frac{1}{2}(s-\tau)\Delta} V_2^\sigma\left(\cdot - \tau\vec{e}_1\right) u \right\|_{L^2} = \left\| \int_{\mathbb{R}^3} K(x,\eta)\hat{u}(\eta)\, d\eta \right\|_{L^2}$$

where

$$K(x,\eta) = V_1(x) \int_{\mathbb{R}^3} e^{-i\frac{1}{2}(s-\tau)\xi^2 + i\xi(x+\tau\vec{e}_1)} \chi\left(\frac{\xi}{M}\right) \widehat{V_2^\sigma}(\xi-\eta) e^{-i\eta\tau\vec{e}_1}\, d\xi.$$

Observe that

$$|K(x,\eta)| \le C_M \langle x\rangle^{-N} \langle \tau\vec{e}_1\rangle^{-N} \langle \eta\rangle^{-N}.$$

This decay result follows from the following two facts:

Integration by parts with

$$e^{iI_2\xi\tau\vec{e}_1} = \left(\frac{\tau\vec{e}_1\nabla_\xi}{|\tau\vec{e}_1|^2}\right)^N e^{I_2\xi\tau\vec{e}_1};$$

and the decay estimate:

$$\left| D^\beta \widehat{V_2^\sigma}(\xi-\eta) \right| \lesssim \langle \eta\rangle^{-N},\ |\xi| \lesssim M.$$

So we can conclude that for any $N > 0$,

$$\left\| V_1 F\left(|\vec{p}| \le M\right) e^{-i\frac{1}{2}(s-\tau)\Delta} V_2^\sigma\left(\cdot - \tau\vec{e}_1\right) \right\|_{2\to 2} \le C_{N,M} \langle \tau\vec{e}_1\rangle^{-N}.$$

By some similar calculations in [27], we conclude

$$\left\| V_1 F\left(|\vec{p}| \le M\right) e^{-i(s-\tau)H_1} P_c\left(H_1\right) V_2^\sigma\left(\cdot - \tau\vec{e}_1\right) \right\|_{2\to 2} \le C_{N,M} \langle \tau\vec{e}_1\rangle^{-N}.$$

But in our particular situation, one can do easy calculations based on Duhamel formula,

$$e^{-i(s-\tau)H_1} = e^{-i(s-\tau)H_0} - i\int_\tau^s e^{-i(s-\tau)H_0} V_1 e^{-i(r-\tau)H_1}\, dr$$

$$\begin{aligned} & F\left(|\vec{p}| \leq M\right) e^{-i(s-\tau)H_1} P_c\left(H_1\right) V_2^\sigma\left(\cdot - \tau\vec{e}_1\right) \\ & = F\left(|\vec{p}| \leq M\right) e^{-i(s-\tau)H_0} V_2^\sigma\left(\cdot - \tau\vec{e}_1\right) \\ -i\int_\tau^s & F\left(|\vec{p}| \leq M\right) e^{-i(s-\tau)H_0} V_1 e^{-i(r-\tau)H_1} V_2^\sigma\left(\cdot - \tau\vec{e}_1\right)\, dr. \end{aligned}$$

$$\begin{aligned} & \int_\tau^s F\left(|\vec{p}| \leq M\right) e^{-i(s-\tau)H_0} V_1 e^{-i(r-\tau)H_1} V_2^\sigma\left(\cdot - \tau\vec{e}_1\right)\, dr \\ = & \int_\tau^s e^{-i(s-\tau)H_0} V_1 F\left(|\vec{p}| \leq M\right) e^{-i(r-\tau)H_1} V_2^\sigma\left(\cdot - \tau\vec{e}_1\right)\, dr \\ + & \int_\tau^s e^{-i(s-\tau)H_0}\left[V_1, F\left(|\vec{p}| \leq M\right)\right] e^{-i(r-\tau)H_1} V_2^\sigma\left(\cdot - \tau\vec{e}_1\right)\, dr. \end{aligned}$$

$$\begin{aligned} & \left\| \int_\tau^s F\left(|\vec{p}| \leq M\right) e^{-i(s-\tau)H_0} V_1 e^{-i(r-\tau)H_1} V_2^\sigma\left(\cdot - \tau\vec{e}_1\right)\, dr \right\|_{L^2} \\ & \leq \int_\tau^s \left\| V_1 F\left(|\vec{p}| \leq M\right) e^{-i(r-\tau)H_1} V_2^\sigma\left(\cdot - \tau\vec{e}_1\right) \right\|_{L^2} dr \\ & \left\| \int_\tau^s e^{-i(s-\tau)H_0}\left[V_1, F\left(|\vec{p}| \leq M\right)\right] e^{-i(r-\tau)H_1} V_2^\sigma\left(\cdot - \tau\vec{e}_1\right)\, dr \right\|_{L^2}. \end{aligned}$$

Notice from construction, $0 \leq s-\tau \leq B$,

$$\left\| \int_\tau^s e^{-i(s-\tau)H_0}\left[V_1, F\left(|\vec{p}| \leq M\right)\right] e^{-i(r-\tau)H_1} V_2^\sigma\left(\cdot - \tau\vec{e}_1\right)\, dr \right\|_{L^2} \lesssim \frac{B}{M} \left\| V_2^\sigma \right\|_{L^2}$$

By Gronwall's inequality, with the fact $0 \leq s-\tau \leq B$, one has

$$\begin{aligned}\int_{s-B}^{s} &\left\| V_1 F\left(|\vec{p}| \leq M\right) e^{-i(s-\tau)H_1} P_c\left(H_1\right) V_2^{\sigma}\left(\cdot - \tau\vec{e}_1\right)\right\|_{2\to 2} \\ &\lesssim e^B \int_{s-B}^{s} \left( C_{N,M} \left\langle \tau\vec{e}_1\right\rangle^{-N} + \frac{B}{M}\left\|V_2^{\sigma}\right\|_{L^2}\right) d\tau \\ &\lesssim \epsilon + \frac{B^2 e^B}{M} \\ &\lesssim \epsilon\end{aligned} \tag{2.3.11}$$

provided $M$ is large enough.

Therefore, for $J_L$,

$$\begin{aligned}\left\|\langle x-x_0\rangle^{-\sigma} J_L \langle x-x_1\rangle^{-\sigma}\right\|_{2\to 2} &\\ \leq CC(T)\int_{t-A}^{t}\int_{s-B}^{s} &\langle t-s\rangle^{-\frac{3}{2}} \langle s-\tau\rangle^{-\frac{3}{2}} \langle \tau\vec{e}_1\rangle^{-\frac{3}{2}} \, ds d\tau \\ \leq CC(T)\langle t-t_0\rangle^{-\frac{3}{2}} \int_{t-A}^{t}\int_{s-B}^{s} &\langle t-s\rangle^{-\frac{3}{2}} \langle \tau\vec{e}_1\rangle^{-\frac{3}{2}} \, ds d\tau \\ \leq \epsilon CC(T)&\langle t-t_0\rangle^{-\frac{3}{2}}\end{aligned}.$$

So when $A$, $B$ is large, we conclude that the coefficient satisfies the bootstrap conditions.

For the second part,

$$\begin{aligned}&\left\|\langle x-x(t)\rangle^{-\sigma} J_L u_0\right\|_{L^2((0,T),L_x^2)} \\ &\leq C\left\|\int_{t-A}^{t} ds \int_{s-B}^{B} \langle t-s\rangle^{-\frac{3}{2}} \langle -\tau\vec{e}_1\rangle^{-\frac{3}{2}} \left\|\langle x-\tau\vec{e_1}\rangle^{-\sigma} U(\tau,t_0) P_s(t_0) u_0\right\|\right\| \\ &\leq CC(T)\sqrt{A}\left\|\langle -\tau\vec{e}_1\rangle^{-\frac{3}{2}}\right\|_{L^2(s-B,s)} \left\|U(\tau,t_0)P_s(t_0)u_0\right\|_{L^\infty((0,T),L_x^2)} \\ &\leq CC(T)\sqrt{A\epsilon}\left\|u_0\right\|_{L^2}.\end{aligned}$$

Again, we know when $\epsilon$ is small, we recapture the bootstrap conditions.

Finally, we need to check $J_{c,H}$. We will use the following version of the Kato smoothing estimate, or we can apply a variant of Kato's smoothing estimate from [51].

**Lemma 2.3.6** ([41]). *For $\sigma > \frac{1}{2}$, we have*

$$\int\limits_{\mathbb{R}} \left\| \langle x\rangle^{-\sigma} \langle \vec{p}\rangle^{-\frac{1}{2}} \nabla e^{-i\frac{1}{2}(s-\tau)\Delta} \right\|_{2\to 2}^{2} d\tau \leq C_{\tau},$$

*we also have*

$$\int\limits_{\mathbb{R}} \left\| \langle x\rangle^{-\sigma} \langle \vec{p}\rangle^{-\frac{1}{2}} \nabla e^{-i(s-\tau)H_1} P_c\left(H_1\right) \right\|_{2\to 2}^{2} d\tau \leq C_{\tau,V_1}.$$

We will use Lemma 2.3.6, but for the sake of completeness, we formulate the result from [51].

**Lemma** ([51]). *Let $H = -\frac{1}{2}\Delta + V$, $\|V\|_{\infty} < \infty$, set $\psi(t) = e^{-itH}\psi_0$, then for all $T > 1$ and $\alpha > 0$, we have*

$$\sup_{x_0\in\mathbb{R}^n} \int_0^T \int_{\mathbb{R}^n} \frac{\left|\nabla \langle \nabla \rangle^{-\frac{1}{2}} \psi(x,t)\right|^2}{\left(1+|x-x_0|^{\alpha}\right)^{\frac{1}{\alpha}+1}} dxdt \leq C_{\alpha,n} T \left(1+\|V\|_{\infty}\right) \|\psi_0\|_{L^2}.$$

Consider

$$\begin{aligned}\int_{s-B}^{s} &\left\| V_1 F\left(|\vec{p}| \geq M\right) e^{-iH_1(s-\tau)} P_c\left(H_1\right) V_2(\cdot - \tau\vec{e_1}) \right\|_{2\to 2} d\tau \\ &\leq \int_{s-B}^{s} \left\| \left[V_1, F\left(|\vec{p}| \geq M\right)\right] e^{-iH_1(s-\tau)} P_c\left(H_1\right) V_2(\cdot - \tau\vec{e_1}) \right\|_{2\to 2} d\tau \\ +B^{\frac{1}{2}} M^{-\frac{1}{2}} &\left( \int_{s-B}^{s} \left\| \langle \vec{p}\rangle^{-\frac{1}{2}} \nabla V_1 e^{-iH_1(s-\tau)} P_c\left(H_1\right) V_2(\cdot - \tau\vec{e_1}) \right\|_{2\to 2} d\tau \right)^{\frac{1}{2}}.\end{aligned}$$

By Young's inequality [51, 10],

$$\left\| \left[V_1, F\left(|\vec{p}| \geq M\right)\right] \right\|_{2\to 2} \lesssim \frac{1}{M}.$$

Also note that

$$\langle \vec{p} \rangle^{-\frac{1}{2}} \nabla V_1 = V_1 \langle \vec{p} \rangle^{-\frac{1}{2}} \nabla + \left[ \langle \vec{p} \rangle^{-\frac{1}{2}} \nabla, V_1 \right],$$

and

$$\left\| \left[ \langle \vec{p} \rangle^{-\frac{1}{2}} \nabla, V_1 \right] \right\|_{2\to 2} \lesssim 1.$$

So for the first estimate, with bootstrap assumption (3.4.10), we get

$$\begin{aligned}
&\left\| \langle x - x_0 \rangle^{-\sigma} J_H \langle x - x_1 \rangle^{-\sigma} \right\|_{2\to 2} \\
&\lesssim C(T) \langle t - t_0 \rangle^{-\frac{3}{2}} \int_{t-A}^{t} \langle t - s \rangle^{-\frac{3}{2}} \\
&\quad \times \int_{s-B}^{s} \left\| V_1 F\left(|\vec{p}| \geq M\right) e^{-iH_1(s-\tau)} P_c\left(H_1\right) V_2(\cdot - \tau \vec{e_1}) \right\|_{2\to 2} d\tau ds \\
&\lesssim C(T) \langle t - t_0 \rangle^{-\frac{3}{2}} \int_{t-A}^{t} \langle t - s \rangle^{-\frac{3}{2}} \left( M^{-1} B + M^{-\frac{1}{2}} \sqrt{B} (1 + \sqrt{B}) \right) \\
&\qquad \lesssim C(T) A B M^{-\frac{1}{2}} \langle t - t_0 \rangle^{-\frac{3}{2}}.
\end{aligned}$$

For the other estimate,

$$\begin{aligned}
&\left\| \langle x - x(t) \rangle^{-\sigma} J_H u_0 \right\|_{L^2((0,T), L_x^2)} \\
&\lesssim \left\| \int_{t-A}^{t} \langle t - s \rangle^{-\frac{3}{2}} \int_{s-B}^{s} \left( M^{-1} + M^{-\frac{1}{2}} (B + \sqrt{B}) \right) \left\| \langle x - \tau \vec{e_1} \rangle^{-\sigma} U(\tau, t_0) P_s(t_0) u_0 \right\| \right\|_{L^2(0,T)} \\
&\qquad \lesssim C(T) B M^{-\frac{1}{2}} \left\| u_0 \right\|_{L^2}.
\end{aligned}$$

So we can pick $M$ large, then the coefficient satisfies the bootstrap condition again.

To sum up, when we pick $A$, $B$ and $M$ large enough independent of $T$, if we have for $t \in [t_0, T]$

$$\left\| \langle x - x_0 \rangle^{-\sigma} U(t, t_0) P_s \langle x - x_1 \rangle^{-\sigma} \right\|_{2\to 2} \leq C(T) \frac{1}{\langle t - t_0 \rangle^{\frac{3}{2}}},$$

we can improve it to

$$\left\|\langle x-x_0\rangle^{-\sigma}U(t,t_0)P_s\langle x-x_1\rangle^{-\sigma}\right\|_{2\to 2}\le \frac{1}{2}C(T)\frac{1}{\langle t-t_0\rangle^{\frac{3}{2}}}+C\frac{1}{\langle t-t_0\rangle^{\frac{3}{2}}}.$$

Therefore, we can make for $t\in[t_0,T]$,

$$\left\|\langle x-x_0\rangle^{-\sigma}U(t,t_0)P_s\langle x-x_1\rangle^{-\sigma}\right\|_{2\to 2}\le C,$$

for some constant independent of $T$. So we conclude

$$\left\|\langle x-x_0\rangle^{-\sigma}U(t,t_0)P_s\langle x-x_1\rangle^{-\sigma}\right\|_{2\to 2}\le C$$

holds for arbitrary $t$ which shows Lemma 2.3.1.

For the second part we proceed analogously. Indeed, if we suppose

$$\int_0^T \left\|\langle x-x(t)\rangle^{-\sigma}U(t,t_0)P_s u\right\|_{L_x^2}^2 dt \;\le\; C^2(T)\left\|u\right\|_{L^2}^2,$$

then we can improve the estimate to

$$\int_0^T \left\|\langle x-x(t)\rangle^{-\sigma}U(t,t_0)P_s u\right\|_{L_x^2}^2 dt\le C\left\|u\right\|_{L^2}^2+\frac{1}{2}C^2(T)\left\|u\right\|_{L^2}^2.$$

So we can obtain a bound for

$$\int_0^T \left\|\langle x-x(t)\rangle^{-\sigma}U(t,t_0)P_s u\right\|_{L_x^2}^2 dt\le C\left\|u\right\|_{L^2}^2$$

which is independent of $T$. Therefore, we can send $T$ to $\infty$ above. Finally, we obtain

$$\int_0^\infty \left\|\langle x-x(t)\rangle^{-\sigma}U(t,t_0)P_s u\right\|_{L_x^2}^2 dt\le C\left\|u\right\|_{L^2}^2$$

which establishes Lemma 2.3.2.

**Remark 2.3.7.** *With Theorem 4.7.1, one can show Lemma 2.3.1 easily as the free case. Set $s = t - t_0$, first, if $|s| \leq 1$, clearly by $\|U(t,t_0)\|_{2\to 2} \leq 1$ and the integrability condition in $\mathbb{R}^3$, i.e. $\sigma > \frac{3}{2}$, we can get the desired result. If $|s| \geq 1$, we apply the dispersive estimate for the free motion, by Young's inequality we get*

$$\left\| \langle x - x_0 \rangle^{-\sigma} U(t,t_0) P_s(t_0) \langle x - x_1 \rangle^{-\sigma} \right\|_{2\to 2} \lesssim \left\| \langle x \rangle^{-\sigma} \right\|_{L^2}^2 \|U(t,t_0) P_s(t_0)\|_{1\to\infty},$$

*and from Theorem 4.7.1,*

$$\|U(t,t_0) P_s(t_0)\|_{1\to\infty} \lesssim |t - t_0|^{-\frac{3}{2}}.$$

*But we proved Lemma 2.3.1 together with Lemma 2.3.2, since the dispersive estimate might not be available in other contexts.*

## 2.4 Boundedness of the Energy

In this section, we use Strichartz estimates to show that the energy of the whole evolution of the charge transfer model is bounded independently of time. The asymptotic completeness of the Hamiltonian shown in [51] will be used. We will still consider the model with two potentials as in the previous section.

*Proof of Theorem 2.1.4.* From Theorem 2.2.4, we can write the evolution as: for some $\phi_0 \in L^2(\mathbb{R}^3)$,

$$U(t,0)\psi_0 = \sum_{r=1}^{m} A_r e^{-i\lambda_r t} u_r + \sum_{s=1}^{\ell} B_s e^{-i\mu_s t} \mathfrak{g}_{-\overrightarrow{e_1}}(t) w_s + e^{-it\frac{\Delta}{2}} \phi_0 + R(t) \tag{2.4.1}$$

where $\mathfrak{g}$ is the Galilei transformation. It is trivial to see the part associated with bound states and moving bound states,

$$\sum_{r=1}^{m} A_r e^{-i\lambda_r t} w_r + \sum_{s=1}^{\ell} B_s e^{-i\mu_s t} \mathfrak{g}_{-\vec{e_1}}(t) u_s \tag{2.4.2}$$

has bounded energy. Indeed, to be more precise, we have

$$\left\| \sum_{r=1}^{m} A_r e^{-i\lambda_r t} w_r + \sum_{s=1}^{\ell} B_s e^{-i\mu_s t} \mathfrak{g}_{-\vec{e_1}}(t) u_s \right\|_{H^1} \lesssim \sum_{r=1}^{m} \|w_r\|_{H^1} \|\psi_0\|_{L^2} + \sum_{s=1}^{\ell} \|u_s\|_{H^1} \|\psi_0\|_{L^2}.$$

So it suffices to consider

$$\psi(t) := U(t,0)\psi_0 = e^{-it\frac{\Delta}{2}}\phi_0 + R(t) \tag{2.4.3}$$

where

$$\|R(t)\|_{L^2} \to 0, \ t \to \pm\infty.$$

In other words, we might assume

$$P_s(t)\psi(t) = \psi(t).$$

Rewrite the equation,

$$i\psi_t + \frac{\Delta}{2}\psi = V_1\psi + V_2(\cdot - t\vec{e_1})\psi. \tag{2.4.4}$$

We can differentiate the equation (2.4.4) and set

$$v = \partial_{x_1}\psi =: \partial_1\psi,$$

then $v$ satisfies

$$iv_t + \Delta v - V_1 v - V_2(\cdot - t\vec{e_1})v = \partial_1 V_1 \psi + \partial_1 V_2(\cdot - t\vec{e_1})\psi. \tag{2.4.5}$$

Again, it suffices to consider $\psi$ is in the scattering space. Since other components are easily to be bounded. To see this, we look at

$$\langle v, w_r\rangle_{L^2} = -\left\langle e^{-it\frac{\Delta}{2}}\phi_0 + R(t), \partial_{x_1} w_r\right\rangle_{L^2}.$$

By the asymptotic completeness result, we know

$$\|R(t)\|_{L^2} \to 0,\ t \to \pm\infty.$$

In particular, we know

$$\|R(t)\|_{L^2} \lesssim C,$$

so

$$\left|\langle R(t), \partial_{x_1} w_r\rangle_{L^2}\right| \lesssim \|R(0)\|_{L^2} \lesssim \|\psi_0\|_{L^2}$$

since from Agmon's estimate, $\partial_{x_1} w_r$ is still exponentially decaying.

Notice that

$$\left|\left\langle e^{-it\frac{\Delta}{2}}\phi_0, \partial_{x_1} w_r\right\rangle_{L^2}\right| \to 0,\ t \to \infty,$$

since we can approximate $\phi_0$ by $\phi_n \in L^2 \cap L^1$ in $L^2$ and then by the dispersive estimate for the free equation

$$\left\|e^{-it\frac{\Delta}{2}}\phi_n\right\|_{L^\infty} \lesssim \frac{1}{|t|^{\frac{3}{2}}}\|\phi_n\|_{L^1}.$$

A similar discussion holds for $u_s$, we can conclude that

$$\|P_b(H_1)\, v(t)\|_{L^2} + \|P_b(H_2, t)\, v(t)\|_{L^2} \to 0,\ \ t \to \pm\infty,$$

and

$$\|P_b(H_1)\, v(t)\|_{L^2} + \|P_b(H_2, t)\, v(t)\|_{L^2} \lesssim \|\psi_0\|_{L^2}.$$

By the above argument, we can actually conclude that $v$ is asymptotically orthogonal to the bound states of $H_1$ and moving bound states associated to $H_2(t)$. We can in fact obtain an explicit rate of decay for the term

$$\|P_b\left(H_1\right)v(t)\|_{L^2} + \|P_b\left(H_2,t\right)v(t)\|_{L^2}$$

goes to 0, but it is enough for our purposes to know that it is just bounded by $\|\psi_0\|_{H^1}$.

Then by Proposition 2.2.5,

$$\|(1 - P_s(t)v(t)\|_{L^2} \lesssim \|\psi_0\|_{H^1}.$$

Therefore, it is sufficient to estimate

$$\|P_s(t)v(t)\|_{L^2}$$

and hence, without loss of generality, we assume

$$P_s(t)v(t).$$

We do a similar argument as the proof for Strichartz estimates, Theorem 2.1.3.

Setting $F(x,t) = \partial_1 V_1\psi + \partial_1 V_2(\cdot - t\vec{e_1})\psi$, we can write (2.4.5) in the form

$$iv_t + \frac{\Delta}{2}v = V_1 v + V_2(\cdot - t\vec{e_1})v + F(x,t).$$

By the endpoint Strichartz estimate for the free Schrdinger equation, we obtain

$$\|v\|_{L_t^p L_x^q} \leq C\,\|V_1 v + V_2(\cdot - t\vec{e_1})v + F\|_{L_t^2 L_x^{\frac{6}{5}}}$$

$$\begin{aligned}
\|V_1 v + V_2(\cdot - t\vec{e_1})v + F\|_{L_t^2 L_x^{\frac{6}{5}}} &\leq \|V_1 v\|_{L_t^2 L_x^{\frac{6}{5}}} + \|V_2(\cdot - t\vec{e_1})v\|_{L_t^2 L_x^{\frac{6}{5}}} + \|F\|_{L_t^2 L_x^{\frac{6}{5}}}, \\
\|V_1 v\|_{L_t^2 L_x^{\frac{6}{5}}} &\leq C_V \left\| \langle x \rangle^{-m} v \right\|_{L_t^2 L_x^2}, \\
\|V_2(\cdot - t\vec{e_1})v\|_{L_t^2 L_x^{\frac{6}{5}}} &\leq C_V \left\| \langle x - t\vec{e_1} \rangle^{-m} v \right\|_{L_t^2 L_x^2}.
\end{aligned}$$

So it suffices to estimate

$$\left\| \langle x - x(t) \rangle^{-m} v \right\|_{L_t^2 L_x^2}$$

for $x(t)$ a smooth curve in $\mathbb{R}^3$.

By Duhamel's formula,

$$v(t) = e^{i\frac{1}{2}\Delta t} v_0 + i \int_0^t e^{i\frac{1}{2}\Delta(t-s)} \left(V_1 + V_2(\cdot - s\vec{e_1})\right) v(s)\, ds + i \int_0^t e^{i\frac{1}{2}\Delta(t-s)} F(x,s)\, ds.$$

We write

$$v(t) = U_1 + iU_2 + iU_3.$$

Certainly, it is easy to bound $U_1$ as Lemmas 2.3.3 and 2.3.4.

Next we bound $U_3$. We again apply Hölder's inequality and the endpoint Strichartz estimate,

$$\begin{aligned}
\left\| \langle x - x(t) \rangle^{-m} \int_{t_0}^t e^{i\frac{1}{2}\Delta(t-s)} F(x,s)\, ds \right\|_{L_t^2 L_x^2} &\leq \left\| \int_{t_0}^t e^{i\frac{1}{2}\Delta(t-s)} F(x,s)\, ds \right\|_{L_t^2 L_x^6} \\
&\leq \|F\|_{L_t^2 L_x^{\frac{6}{5}}}.
\end{aligned}$$

It remains to bound

$$\left\| \int_0^t e^{i\frac{1}{2}\Delta(t-s)} \left(V_1 + V_2(\cdot - s\vec{e_1})\right) v(s)\, ds \right\|_{L_t^2 L_x^2}.$$

Rewrite

$$v(s) = U(s,0)v_0 + i \int_0^s U(s,\tau) F(x,\tau)\, d\tau$$

By our assumption:

$$v(s) = P_s(s)U(s,0)v_0 + i\int_0^s P_s(s)U(s,\tau)F(x,\tau)\,d\tau.$$

By Lemma 2.3.1, Lemma 2.3.2, we have

$$\begin{aligned}
&\left\|\langle x - x(t)\rangle^{-m}\int_0^t e^{i\frac{1}{2}\Delta(t-s)}\left(V_1 + V_2(\cdot - s\vec{e_1})\right)P_s(s)U(s,0)v_0\,ds\right\|_{L_t^2L_x^2}\\
&\qquad\lesssim \left\|\langle x\rangle^{-\beta}P_s(s)U(t,0)v_0\right\|_{L_t^2L_x^2} + \left\|\langle x - t\vec{e_1}\rangle^{-\beta}P_s(s)U(t,0)v_0\right\|_{L_t^2L_x^2}\\
&\qquad\lesssim \|v_0\|_{L^2}.
\end{aligned}$$

Also, we can get

$$\begin{aligned}
&\left\|\langle x - x(t)\rangle^{-m}\int_0^t e^{i\frac{1}{2}\Delta(t-s)}\left(V_1 + V_2(\cdot - s\vec{e_1})\right)\int_0^s P_s(s)U(s,\tau)F(x,\tau)\,d\tau ds\right\|_{L_t^2L_x^2}\\
&\qquad\lesssim \left\|\int_0^t \langle t-s\rangle^{-\frac{3}{2}}\int_0^s \langle s-\tau\rangle^{-\frac{3}{2}}\left\|\langle x\rangle^{\alpha}F(\tau)\right\|_{L_x^2}d\tau ds\right\|_{L_t^2}\\
&\qquad\lesssim \left\|\int_0^t \langle t-\tau\rangle^{-\frac{3}{2}}\left\|\langle x\rangle^{\alpha}F(\tau)\right\|_{L_x^2}d\tau\right\|\\
&\qquad\leq \left\|\langle x\rangle^{\alpha}F(\tau)\right\|_{L_t^2L_x^2}.
\end{aligned}$$

So we have shown that

$$\|v\|_{L_t^pL_x^q} \leq C\left\|V_1v + V_2(\cdot - t\vec{e_1})v + F\right\|_{L_t^2L_x^{\frac{6}{5}}} \lesssim \|\psi_0\|_{L^2} + \left\|\langle x\rangle^{\alpha}F\right\|_{L_t^2L_x^2} + \|F\|_{L_t^2L_x^{\frac{6}{5}}}$$

for any Schrödinger admissible pair $(p,q)$.

Plugging in $F = \partial_1V_1\psi + \partial_1V_2(\cdot - t\vec{e_1})\psi$, it is easy to estimate

$$\left\|\langle x\rangle^{\alpha}F\right\|_{L_t^2L_x^2} + \|F\|_{L_t^2L_x^{\frac{6}{5}}} \lesssim \|\psi_0\|_{L^2}.$$

For the second piece, we use

$$\begin{aligned}
\|\partial_1 V_1\psi+\partial_1 V_2(\cdot-t\vec{e_1})\psi\|_{L_t^2L_x^{\frac{6}{5}}} &\leq \|\partial_1 V_1\psi\|_{L_t^2L_x^{\frac{6}{5}}}+\|\partial_1 V_2(\cdot-t\vec{e_1})\psi\|_{L_t^2L_x^{\frac{6}{5}}}\\
\|\partial_1 V_1\psi\|_{L_t^2L_x^{\frac{6}{5}}} &\leq C_V\left\|\langle x\rangle^{-m}\psi\right\|_{L_t^2L_x^2}\lesssim\|\psi_0\|_{L^2}\\
\|\partial_1 V_2(\cdot-t\vec{e_1})\psi\|_{L_t^2L_x^{\frac{6}{5}}} &\leq C_V\left\|\langle x-t\vec{e_1}\rangle^{-m}\psi\right\|_{L_t^2L_x^2}\lesssim\|\psi_0\|_{L^2}.
\end{aligned}$$

For the first piece, by Hölder's inequality, we have

$$\|\langle x\rangle^{\alpha}\left(\partial_1 V_1\psi+\partial_1 V_2(\cdot-t\vec{e_1})\psi\right)\|_{L_x^2}\lesssim\|\psi\|_{L_x^6}.$$

Then applying the endpoint Strichart estimate to $\psi$ by Theorem 2.1.3, we get

$$\|\langle x\rangle^{\alpha}\left(\partial_1 V_1\psi+\partial_1 V_2(\cdot-t\vec{e_1})\psi\right)\|_{L_t^2L_x^2}\lesssim\|\psi\|_{L_t^2L_x^6}\lesssim\|\psi_0\|_{L_x^2}.$$

So in particular, we infer that

$$\|v\|_{L_t^\infty L_x^2}\lesssim\|v_0\|_{L^2}+\|\psi_0\|_{L^2}=\|\psi_0\|_{H^1}. \tag{2.4.6}$$

The same argument applies to all other partial derivatives of $\psi$. So we can conclude that

$$\sup_{t\in\mathbb{R}}\|\psi(t)\|_{H^1}\lesssim\|\psi_0\|_{H^1}. \tag{2.4.7}$$

The theorem is proved □

By a simple inductive argument, we obtain the following corollary:

**Corollary 2.4.1.** *For $\psi_0\in H^k\left(\mathbb{R}^3\right)$ where $k$ is a non-negative integer, then*

$$\sup_{t\in\mathbb{R}}\|U(t,0)\psi_0\|_{H^k}\leq C\|\psi_0\|_{H^k}. \tag{2.4.8}$$

*Remark.* As a concluding remark, we notice that we proved the boundedness of the energy based on Strichartz estimates and the asymptotic completeness of the Hamiltonian. In [27], Graf proved the asymptotic completeness based on the boundedness of the energy. So, we can see, modulo some technical assumptions on the spectrum of the Schrdinger operator, the boundedness of the energy is equivalent to the asymptotic completeness of the Hamiltonian. Also note that the asymptotic completeness can be also proved by the dispersive estimate as in [51].

## 2.5 Matrix Charge Transfer Models

In this section, we extend our above results to matrix charge transfer models in $\mathbb{R}^3$ similarly as the work in [51]. For the sake of completeness, we start from the basic definitions following [51].

**Definition 2.5.1.** By a matrix charge transfer model we mean a system

$$\frac{1}{i}\partial_t\vec{\psi} + \begin{pmatrix} -\frac{1}{2}\Delta & 0 \\ 0 & \frac{1}{2}\Delta \end{pmatrix}\vec{\psi} + \sum_{j=1}^{v} V_j\left(\cdot - \vec{v_j}t\right)\vec{\psi} = 0, \;\; \vec{\psi}|_{t=0} = \vec{\psi}_0$$

where $\vec{v_j}$ are distinct vectors in $\mathbb{R}^3$, and $V_j$ are matrix potentials of the form

$$V_j(t,x) = \begin{pmatrix} U_j(x) & -e^{i\theta_j(t,x)}W_j(x) \\ e^{-i\theta_j(t,x)}W_j(x) & -U_j(x) \end{pmatrix},$$

where $\theta_j(t,x) = \left(\left|\vec{v_j}\right|^2 + \alpha_j^2\right)t + 2x\cdot\vec{v_j} + \gamma_j$ with $\alpha_j, \gamma_j \in \mathbb{R}$ and $\alpha_j \neq 0$. Furthermore, we require that each

$$H_j = \begin{pmatrix} -\frac{1}{2}\Delta + \frac{1}{2}\alpha_j^2 + U_j & -W_j \\ W_j & \frac{1}{2}\Delta - \frac{1}{2}\alpha_j^2 - U_j \end{pmatrix}$$

satisfies the admissible conditions (Definition2.5.2) and stability condition (Definition 2.5.3) defined below.

Here we give the definitions of stability condition and admissible conditions for a matrix Hamiltonian $A = B + V$ where

$$B = \begin{pmatrix} -\frac{1}{2}\Delta + \mu & 0 \\ 0 & \frac{1}{2}\Delta - \mu \end{pmatrix}, \quad V = \begin{pmatrix} U & -W \\ W & -U \end{pmatrix}$$

with $\mu > 0$ and $U$, $W$ are of real-valued.

**Definition 2.5.2.** Let $A = B + V$ as above with $V$ exponentially decaying. We call the operator $A$ on $\mathcal{H} := L^2(\mathbb{R}^3) \times L^2(\mathbb{R}^3)$ admissible provided the following hold:

1. $spec(A) \subset \mathbb{R}$ and $spec(A) \cap (-\mu, \mu) = \{\omega_\ell : 0 \le \ell \le M\}$, where $\omega_0 = 0$ and all $\omega_j$ are distinct eigenvalues. There are no eigenvalues in $spec_{ess}(A)$.

2. For $1 \le \ell \le M$, $L_\ell := ker\,(A - \omega_\ell)^2 = ker\,(A - \omega_\ell)$ and $ker\,(A) \subsetneq ker\,(A^2) = ker\,(A^3) =: L_0$. Moreover, these spaces are finite-dimensional.

3. The ranges $Ran\,(A - \omega_\ell)$, for $1 \le \ell \le M$ and $Ran\,(A^2)$ are closed.

4. The spaces $L_\ell$ are spanned by exponentially decreasing functions in $\mathcal{H}$ (say, with bound $e^{-\epsilon_0 |x|}$).

5. All these assumptions hold as well for the adjoint $A^*$. We denote the corresponding (generalized) eigenspaces by $L_\ell^*$.

6. The points $\pm\mu$ are not resonances of $A$.

*Remark.* For detailed definition of resonance here, one can find it in [51] Remark 7.10.

Following the above admissible conditions for $A$, we have can define analogous projections onto continuous spectrum and point spectrum following [51] Lemma 7.3.

**Lemma** ([51], Lemma 7.3)**.** *There a direct sum decomposition*

$$\mathcal{H} = \sum_{j=1}^{M} L_j + \left( \sum_{j=1}^{M} L_j^* \right)^{\perp}.$$

*The decomposition is invariant under $A$. Let $P_c$ denote the projection onto $\left(\sum_{j=1}^{M} L_j^*\right)^{\perp}$ and set $P_b = Id - P_c$. Notice that here $P_c$ is not an orthogonal projection. It is easy to see $AP_c = P_c A$, and there exist numbers $c_{ij}$ such that*

$$P_b = \sum_{i,j} \phi_j c_{ij} \langle f, \psi_i \rangle \,, \ \forall f \in \mathcal{H}$$

*where $\phi_j$ and $\psi_i$ are exponentially decreasing functions.*

**Definition 2.5.3.** For $A$ satisfying the admissible conditions, we say $A$ satisfies the stability condition if

$$\sup_{t\in\mathbb{R}} \left\| e^{itA} P_c \right\|_{\mathcal{H}\to\mathcal{H}} < \infty.$$

In order the study the matrix charge transfer model, we need the vector-valued Galilei transformation similarly as in the scalar case:

$$\mathcal{G}_{\vec{v},y}(t) \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} := \begin{pmatrix} \mathfrak{g}_{\vec{v},y}(t)\psi_1 \\ \overline{\mathfrak{g}_{\vec{v},y}(t)\overline{\psi_2}} \end{pmatrix},$$

where $\mathfrak{g}_{\vec{v},y}(t)$ is the scalar version Galilei transformation. In contrast to the scalar case, the conjugated transformation now involves a modulation $\mathcal{M}(t)$. We cite Lemma 8.2 in [51].

**Lemma** ([51], Lemma 8.2)**.** *Let $\alpha \in \mathbb{R}$ and let*

$$A := \begin{pmatrix} -\frac{1}{2}\Delta + \frac{1}{2}\alpha^2 + U & -W \\ W & \frac{1}{2}\Delta - \frac{1}{2}\alpha^2 - U \end{pmatrix}$$

*with real-valued $U$ and $W$. Moreover, let $\vec{v} \in \mathbb{R}^3$, $\theta(t,x) = \left(|\vec{v}|^2 + \alpha^2\right) t + 2x \cdot \vec{v} + \gamma\,, \gamma \in \mathbb{R}$, and define*

$$H(t) := \begin{pmatrix} -\frac{1}{2}\Delta + U\left(\cdot - \vec{v}t\right) & -e^{i\theta(t,\cdot-\vec{v}t)} W\left(\cdot - \vec{v}t\right) \\ e^{-i\theta(t,\cdot-\vec{v}t)} W\left(\cdot - \vec{v}t\right) & \frac{1}{2}\Delta - \frac{1}{2}\alpha^2 - U\left(\cdot - \vec{v}t\right) \end{pmatrix}.$$

*Let $S(0) = Id$, $S(t)$ denote the propagator of the system*

$$\frac{1}{i}\partial_t S(t) + H(t)S(t) = 0.$$

*Finally, let*

$$\mathcal{M}(t) = \mathcal{M}_{\alpha,\gamma}(t) = \begin{pmatrix} e^{-\frac{i\omega(t)}{2}} & 0 \\ 0 & e^{\frac{i\omega(t)}{2}} \end{pmatrix}$$

*where $\omega(t) = \alpha^2 t + \gamma$. Then we have the following relation*

$$S(t) = \mathcal{G}_{\vec{v}}(t)^{-1}\mathcal{M}(t)^{-1}e^{-itA}\mathcal{M}(0)\mathcal{G}_{\vec{v}}(0).$$

For matrix charge transfer models, the analysis should be similar to the scalar case except that we have to modify the asymptotic orthogonality condition. Recall that as we remarked above, it is not necessary to use the asymptotic completeness results. In the scalar case, the asymptotic orthogonality condition is sufficient for us. In the matrix case, the asymptotic orthogonality condition is replaced by the definition of "scattering states" in Definition 8.3 in [51] which is similar to the scattering space in the sense of Definition 2.1.2 for the scalar case.

**Definition 2.5.4.** Let $U(t)\vec{\psi_0} = \vec{\psi}(t,\cdot)$, we call that $\vec{\psi_0}$ a scattering state relative to $H_j$ if

$$\left\|P_b\left(H_j,t\right)U(t)\vec{\psi_0}\right\|_{L^2} \to 0, \;\; t \to \infty.$$

Here

$$P_b\left(H_j,t\right) := \mathcal{G}_{\vec{v}_j}(t)^{-1}\mathcal{M}_j(t)^{-1}P_b\left(H_j\right)\mathcal{M}_j(t)\mathcal{G}_{\vec{v}_j}(t)$$

with $\mathcal{M}_j(t) = \mathcal{M}_{\alpha_j,\gamma_j}(t)$.

By the discussion in Section 8.3 in [51], if $\vec{\psi_0}$ a scattering state relative to each $H_j$, we have the rate of convergence similar to the scalar case,

$$\left\| P_b\left(H_1,t\right)U(t)\vec{\psi}_0\right\|_{L^2}+\left\| P_b\left(H_2,t\right)U(t)\vec{\psi}_0\right\|_{L^2}\lesssim e^{-\alpha t}\left\|\vec{\psi}_0\right\|_{L^2}$$

for some $\alpha>0$.

With all the preparations above, we now can formulate our Strichartz estimates for matrix charge transfer models.

**Theorem 2.5.5.** *Consider the matrix charge transfer model as in Definition 2.5.1. We denote $\vec{\psi}(t)=U(t,0)\vec{\psi}_0$ and assume $\vec{\psi}_0$ is a scattering state relative to each $H_j$ in sense of Definition 2.5.4. Then for a Schrödinger admissible pair $(p,q)$ in $\mathbb{R}^3$, i.e.,*

$$\frac{2}{p}+\frac{3}{q}=\frac{3}{2} \tag{2.5.1}$$

*with $2\le q\le\infty$, $p\ge 2$, we have*

$$\left\|\vec{\psi}\right\|_{L_t^p([0,\infty),\,L_x^q)}\le C\left\|\vec{\psi}_0\right\|_{L_x^2}. \tag{2.5.2}$$

*for some finite constant $C$.*

As in the scalar case, the proof Theorem 2.5.5 is based on certain weighted estimates which rely on a bootstrap argument. Since the proof is basically identical as with the scalar case, we do not carry out the details. We only discuss it briefly. Recall that in our proof, there are several important ingredients: dispersive estimates for stationary potentials, the boundedness of wave operators, the Kato smoothing estimate. All of them hold for the matrix case. For the dispersive estimates for stationary potentials, one can find details in [21, 51, 26]; for the boundedness of wave operators, the results are discussed in [21]; the Kato smoothing estimates can be obtained as for the scalar case in [51]. Hence with the remark at the beginning of the second section, and all the proofs above, we can conclude that Strichartz estimates hold for the matrix case.

*Remark.* With the dispersive estimate for matrix transfer models and the results on scattering states, we can follow the proof in [51] to prove the asymptotic completeness for matrix charge transfer Hamiltonians.

Similar to the scalar case, we also have the energy estimate.

**Theorem 2.5.6.** *For $\vec{\psi}_0 \in \mathcal{H}^1 := H^1\left(\mathbb{R}^3\right) \times H^1\left(\mathbb{R}^3\right)$, we have*

$$\sup_{t\in\mathbb{R}} \left\| U(t,0)\vec{\psi}_0 \right\|_{\mathcal{H}^1} \leq C \left\| \psi_0 \right\|_{\mathcal{H}^1}. \tag{2.5.3}$$

**Corollary 2.5.7.** *For $\vec{\psi}_0 \in \mathcal{H}^k := H^k\left(\mathbb{R}^3\right) \times H^k\left(\mathbb{R}^3\right)$ where $k$ is a non-negative integer, then we have*

$$\sup_{t\in\mathbb{R}} \left\| U(t,0)\psi_0 \right\|_{\mathcal{H}^k} \leq C \left\| \psi_0 \right\|_{\mathcal{H}^k}. \tag{2.5.4}$$

# CHAPTER 3

# STRICHARTZ ESTIMATES FOR WAVE EQUATIONS WITH CHARGE TRANSFER HAMILTONIANS

## 3.1 Introduction

In this chapter, we study wave equations with charge transfer Hamiltonians in $\mathbb{R}^3$. To be more precise, consider the wave equation with the time-dependent charge transfer Hamiltonian

$$H(t) = -\Delta + \sum_{j=1}^{m} V_j\left(x - \vec{v}_j t\right) \tag{3.1.1}$$

where $V_j(x)$'s are rapidly decaying smooth potentials and $\left\{\vec{v}_j\right\}$ is a set of distinct constant velocities such that

$$|\vec{v}_i| < 1,\ 1 \leq i \leq m. \tag{3.1.2}$$

Due to the nature of our problem, we focus on initial data in the energy space. We will prove Strichartz estimates, energy estimates, the local energy decay and the boundedness of the total energy for a scattering state to the wave equation

$$\partial_{tt}\psi + H(t)\psi = 0 \tag{3.1.3}$$

associated with a charge transfer Hamiltonian $H(t)$. Throughout, we use $\partial_{tt}u := \frac{\partial^2}{\partial t \partial t}$, $u_t := \frac{\partial}{\partial_t}u$, $\Delta := \sum_{i=1}^{n} \frac{\partial^2}{\partial x_i \partial x_i}$ and occasionally, $\Box := -\partial_{tt} + \Delta$.

### *3.1.1 Historical background*

In this subsection, we briefly discuss some background of Strichartz estimates, reversed Strichartz estimates.

Our starting point is the free wave equation ($H_0 = -\Delta$) on $\mathbb{R}^3$

$$\partial_{tt}u - \Delta u = 0 \tag{3.1.4}$$

with initial data

$$u(x,0) = g(x),\ u_t(x,0) = f(x). \tag{3.1.5}$$

We can write down $u$ explicitly,

$$u = \frac{\sin\left(t\sqrt{-\Delta}\right)}{\sqrt{-\Delta}} f + \cos\left(t\sqrt{-\Delta}\right) g. \tag{3.1.6}$$

Then for $p > \frac{2}{s}$ and $(p,q)$ satisfying

$$\frac{3}{2} - s = \frac{1}{p} + \frac{3}{q}, \tag{3.1.7}$$

one has

$$\|u\|_{L_t^p L_x^q} \lesssim \|g\|_{\dot{H}^s} + \|f\|_{\dot{H}^{s-1}}. \tag{3.1.8}$$

Strichartz estimates (4.1.7), which are stated precisely in Theorem 4.2.1, are estimates of solutions in terms of space-time integrability properties. The non-endpoint estimates for the wave equations can be found in Ginibre-Velo [28]. Keel–Tao [37] also obtained sharp Strichartz estimates for the free wave equation in $\mathbb{R}^n$, $n \geq 4$ and everything except the endpoint in $\mathbb{R}^3$. See Keel-Tao [37] and Tao's book [54] for more details on the subject's background and the history.

In $\mathbb{R}^3$, there is no hope to obtain such an estimate with the $L_t^2 L_x^\infty$ norm, the so-called endpoint Strichartz estimate for free wave equations, cf. Klainerman-Machedon [38] and Machihara-Nakamura-Nakanishi-Ozawa [42]. But if we reverse the order of space-time integration, one can obtain a version of reversed Strichartz estimates from the Morawetz estimate, cf. Theorem 4.2.3:

$$\left\| \frac{\sin\left(t\sqrt{-\Delta}\right)}{\sqrt{-\Delta}} f \right\|_{L^\infty_x L^2_t} \lesssim \|f\|_{L^2(\mathbb{R}^3)}, \tag{3.1.9}$$

$$\left\| \cos\left(t\sqrt{-\Delta}\right) g \right\|_{L^\infty_x L^2_t} \lesssim \|g\|_{\dot{H}^1(\mathbb{R}^3)}. \tag{3.1.10}$$

These types of estimates are extended to inhomogeneous cases and perturbed Hamiltonians in Beceanu-Goldberg [3]. In Section 4.2 and Section 3.3, we will rely crucially on these estimates and their generalizations.

Consider a linear wave equation with a real-valued stationary potential in $\mathbb{R}^3$,

$$H = -\Delta + V, \tag{3.1.11}$$

$$\partial_{tt} u + Hu = \partial_{tt} u - \Delta u + Vu = 0, \tag{3.1.12}$$

$$u(x,0) = g(x),\ u_t(x,0) = f(x). \tag{3.1.13}$$

One substantial difference between the perturbed Hamiltonian $H = -\Delta + V$ and the free Laplacian $-\Delta$ is the possible existence of eigenvalues and bound states, i.e., $L^2$ eigenfunctions of $H$. For the class of short-range potentials we consider in this chapter, the essential spectrum of $H$ is $[0, \infty)$ and the point spectrum may include a countable number of eigenvalues in a bounded subset of the real axis that is discrete away from zero. We further assume that zero is a regular point of the spectrum of $H$. Under our hypotheses $H$ only has pure absolutely continuous spectrum on $[0, \infty)$ and a finite number of negative eigenvalues. It is very crucial to notice that if $E < 0$ is a negative eigenvalue, the associated eigenfunction responds to the wave equation propagators by scalar multiplication by $\cos\left(t\sqrt{E}\right)$ or $\frac{\sin\left(t\sqrt{E}\right)}{E^{\frac{1}{2}}}$, both of which will grow exponentially since $\sqrt{E}$ is purely imaginary. Thus, dispersive estimates and Strichartz estimates for $H$ must include a projection $P_c$ onto the continuous spectrum in order to get away from this situation.

The problem of dispersive decay and Strichartz estimates for the wave equation with a potential has received much attention in recent years, see the papers by Beceanu-Goldberg [3], Krieger-Schlag [39] and the survey by Schlag [53] for further details and references.

The Strichartz estimates in this case are in the form:

$$\left\| \frac{\sin\left(t\sqrt{H}\right)}{\sqrt{H}} P_c f + \cos\left(t\sqrt{H}\right) P_c g \right\|_{L_t^p L_x^q} \lesssim \|g\|_{\dot{H}^1} + \|f\|_{L^2}, \tag{3.1.14}$$

with $2 < p$, $\frac{1}{2} = \frac{1}{p} + \frac{3}{q}$. One also has the endpoint reversed Strichartz estimates:

$$\left\| \frac{\sin\left(t\sqrt{H}\right)}{\sqrt{H}} P_c f + \cos\left(t\sqrt{H}\right) P_c g \right\|_{L_x^\infty L_t^2} \lesssim \|f\|_{L^2} + \|g\|_{\dot{H}^1}, \tag{3.1.15}$$

see Theorem 4.2.4.

There are extra difficulties when dealing with time-dependent potentials. For example, given a general time-dependent potential $V(x,t)$, it is not clear how to introduce an analog of bound states and a spectral projection. The evolution might not satisfy group properties any more. It might also result in the growth of certain norms of the solutions, see the book by Bourgain [9]. In this chapter, we focus on the charge transfer Hamiltonian (3.1.1) in $\mathbb{R}^3$:

$$H(t) = -\Delta + \sum_{j=1}^{m} V_j\left(x - \vec{v}_j t\right), \tag{3.1.16}$$

which appears naturally in the study of nonlinear multisoliton system, see Rodnianski-Schlag-Soffer [52] for the Schrödinger model. For the wave model,

$$\partial_{tt} u - \Delta u + \sum_{j=1}^{m} V_j\left(x - \vec{v}_j t\right) u = 0, \quad |\vec{v}_i| < 1,\ 1 \le i \le m, \tag{3.1.17}$$

in this chapter, we prove Strichartz estimates, energy estimates, the local energy decay which are essential to analyze the stability of multi-soliton states. In Chen [15], relying on this linear model, we construct a multisoliton structure to the defocusing energy critical wave equation with potentials in $\mathbb{R}^3$:

$$\partial_{tt}u - \Delta u + \sum_{j=1}^{m} V_j\left(x - \vec{v}_j t\right) u + u^5 = 0. \tag{3.1.18}$$

We also analyze the asymptotic stability of the multisoliton. Since each bubble in the multi-solion structure decays slowly like $\frac{1}{\langle x \rangle}$, the interactions among each bubble are very strong. Our linear theory and reversed Strichartz estimaes play a pivotal rule in the construction. And it turns out that this model is the first multisoliton structure for wave equations in $\mathbb{R}^3$.

The study of Schrödinger equations with a charge transfer Hamiltonian can be found in Rodnianski-Schlag-Soffer [51], Cai [10], Chen [11] and Deng-Soffer-Yao [23]. For the Schrödinger model, there is no need to require $|\vec{v}_i| < 1$. In Rodnianski-Schlag-Soffer [51], the authors proved the dispersive estimates for both the scalar and matrix Schrödinger charge transfer models. They introduced Galilei transformations to interchange stationary frames with respect to different potentials. Basically, they applied a bootstrap argument via a semi-classical propagation lemma for low frequencies and Kato's smoothing estimate for high frequencies. With careful analysis of wave operators, the authors also obtain the results on the asymptotic completeness. Their works inspired the subsequent development in Cai [10] where the $L^1 \to L^\infty$ dispersive estimate is proved. Later on, by Chen [11], Strichartz estimates for both the scalar and matrix Schrödinger charge transfer models were presented based on a time-dependent local decay estimate and the endpoint Strichartz estimate for the free equations. Alternatively, Strichartz estimates can be obtained by analysis of wave operators, see Deng-Soffer-Yao [23].

Compared with Schrödinger equations, wave equations have some natural difficulties, for example the evolution of bound states of wave equations leads to exponential growth as we pointed out above, meanwhile the evolution of bound states of Schrödinger equations are merely multiplied by oscillating factors. The structure of wave operators in the wave equation setting is not clear either. Moreover, the endpoint Strichartz estimate for free equations, an important tool used in the paper [11], also fails for free wave equations in $\mathbb{R}^3$. Last but not least, Lorentz transformations are space-time rotations, therefore one can not hope to succeed by the approach used with Schrödinger equations based on Galilei transformations. Galilei transformations are bounded in any $L^p$ space, but it is not clear under Lorentz transformations whether the energy with respect to the new frame stays comparable to the energy in the original frame. To the author's knowledge, for wave equations with even just one potential moving along a space-like line, Strichartz estimates, scattering, and the asymptotic decomposition of the evolution are new. We refer to [12] for more information on wave equations with one moving potential.

### *3.1.2 Charge transfer model and main results*

Before we give the precise definition of our model, it is necessary to introduce Lorentz transformations. Given a vector $\vec{\mu} \in \mathbb{R}^3$, there is a Lorentz transformation $L(\vec{\mu})$ acting on $(x,t) \in \mathbb{R}^{3+1}$ such that it makes the moving frame $(x-\vec{\mu}t, t)$ stationary. We can use a $4\times 4$ matrix $B(\vec{\mu})$ to represent the transformation $L(\vec{\mu})$. Moreover, for the given vector $\vec{\mu} = (\mu_1, \mu_2, \mu_3) \in \mathbb{R}^3$, there is a $3\times 4$ matrix $M(\vec{\mu})$ such that

$$(x-\vec{\mu}t)^T = M(\vec{\mu})(x,t)^T, \tag{3.1.19}$$

where the superscript $T$ denotes the transpose of a vector.

With the preparations above, we can set up our model. We consider the scalar charge transfer model for wave equations in the following sense:

**Definition 3.1.1.** By a wave equation with a charge transfer Hamiltonian we mean a wave equation

$$\partial_{tt}u - \Delta u + \sum_{j=1}^{m} V_j\left(x - \vec{v}_j t\right) u = 0, \tag{3.1.20}$$

$$u|_{t=0} = g,\ \partial_t u|_{t=0} = f,\ x \in \mathbb{R}^3,$$

where $\vec{v}_j$'s are distinct vectors in $\mathbb{R}^3$ with

$$|\vec{v}_i| < 1,\ 1 \leq i \leq m. \tag{3.1.21}$$

and the real potentials $V_j$ are such that $\forall 1 \leq j \leq m$

1) $V_j$ is time-independent and decays with rate $\langle x\rangle^{-\alpha}$ with $\alpha > 3$

2) 0 is neither an eigenvalue nor a resonance of the operators

$$H_j = -\Delta + V_j\left(S\left(\vec{v}_j\right)x\right), \tag{3.1.22}$$

where $S\left(\vec{v_j}\right)x = M\left(\vec{v}_j\right)B^{-1}\left(\vec{v}_j\right)(x,0)^T$.

Recall that $\psi$ is a resonance at 0 if it is a distributional solution of the equation $H_k\psi = 0$ which belongs to the space $L^2\left(\langle x\rangle^{-\sigma}dx\right) := \left\{f : \langle x\rangle^{-\sigma} f \in L^2\right\}$ for any $\sigma > \frac{1}{2}$, but not for $\sigma = \frac{1}{2}$.

*Remark.* The construction of $S\left(\vec{v}_j\right)$ is clear from the change between different frames under Lorentz transformations. In our concrete problem below (5.3.7), $S\left(\vec{v}_j\right)$ can be written down explicitly.

To simplify our argument, throughout this chapter, we discuss the wave equation with a charge transfer Hamiltonian in the sense of Definition 5.3.1 with $m = 2$, a stationary $V_1$ and a $V_2$ moving along $\overrightarrow{e_1}$ with speed $|v| < 1$, i.e., the velocity is

$$\vec{v} = (v, 0, 0). \tag{3.1.23}$$

Under this setting, by Definition 5.3.1,

$$H_1 = -\Delta + V_1(x), \tag{3.1.24}$$

and

$$H_2 = -\Delta + V_2\left(\sqrt{1-|v|^2}x_1, x_2, x_3\right). \tag{3.1.25}$$

It is easy to see that our arguments work for $m > 2$.

An indispensable tool we need to study the charge transfer model is the Lorentz transformation. Throughout this chapter, we apply Lorentz transformations $L$ with respect to a moving frame with speed $|v| < 1$ along the $x_1$ direction. After we apply the Lorentz transformation $L$, under the new coordinates, $V_2$ is stationary meanwhile $V_1$ will be moving.

Writing down the Lorentz transformation $L$ explicitly, we have

$$\begin{cases} t' = \gamma\left(t - vx_1\right) \\ x_1' = \gamma\left(x_1 - vt\right) \\ x_2' = x_2 \\ x_3' = x_3 \end{cases} \tag{3.1.26}$$

with

$$\gamma = \frac{1}{\sqrt{1-|v|^2}}. \tag{3.1.27}$$

We can also write down the inverse transformation of the above:

$$\begin{cases} t = \gamma\left(t' + vx_1'\right) \\ x_1 = \gamma\left(x_1' + vt'\right) \\ x_2 = x_2' \\ x_3 = x_3' \end{cases}. \tag{3.1.28}$$

Under the Lorentz transformation $L$, if we use the subscript $L$ to denote a function with respect to the new coordinate $\left(x', t'\right)$, we have

$$u_L\left(x_1', x_2', x_3', t'\right) = u\left(\gamma\left(x_1' + vt'\right), x_2', x_3', \gamma\left(t' + vx_1'\right)\right) \tag{3.1.29}$$

and

$$u(x,t) = u_L\left(\gamma\left(x_1 - vt\right), x_2, x_3, \gamma\left(t - vx_1\right)\right). \tag{3.1.30}$$

Let $w_1, \ldots, w_m$ and $m_1, \ldots, m_\ell$ be the normalized bound states of $H_1$ and $H_2$ associated with the negative eigenvalues $-\lambda_1^2, \ldots, -\lambda_m^2$ and $-\mu_1^2, \ldots, -\mu_\ell^2$ respectively (notice that by our assumptions, 0 is not an eigenvalue). In other words, we assume

$$H_1 w_i = -\lambda_i^2 w_i, \ \ w_i \in L^2, \ \lambda_i > 0. \tag{3.1.31}$$

$$H_2 m_i = -\mu_i^2 m_i, \ \ m_i \in L^2, \ \mu_i > 0. \tag{3.1.32}$$

We denote by $P_b\left(H_1\right)$ and $P_b\left(H_2\right)$ the projections on the the bound states of $H_1$ and $H_2$, respectively, and let $P_c\left(H_i\right) = Id - P_b\left(H_i\right), \ i = 1, 2$. To be more explicit, we have

$$P_b\left(H_1\right) = \sum_{i=1}^{m} \langle \cdot, w_i\rangle\, w_i, \quad P_b\left(H_2\right) = \sum_{j=1}^{\ell} \langle \cdot, m_j\rangle\, m_j. \tag{3.1.33}$$

In order to study the equation with time-dependent potentials, we need to introduce a suitable projection. Again, with Lorentz transformations $L$ associated with the moving potential $V_2(x-vt)$, we use the subscript $L$ to denote a function under the new frame $(x', t')$.

**Definition 3.1.2** (Scattering states)**.** Let

$$\partial_{tt}u - \Delta u + V_1(x)u + V_2(x-vt)u = 0, \tag{3.1.34}$$

with initial data

$$u(x,0) = g(x),\ u_t(x,0) = f(x). \tag{3.1.35}$$

If $u$ also satisfies

$$\|P_b(H_1)\,u(t)\|_{L^2_x} \to 0,\ \left\|P_b(H_2)\,u_L(t')\right\|_{L^2_{x'}} \to 0\ \ t, t' \to \infty, \tag{3.1.36}$$

we call it a scattering state.

**Remark 3.1.3.** *Clearly, the set of* $(g,\, f) \in H^1\left(\mathbb{R}^3\right) \times L^2\left(\mathbb{R}^3\right)$ *which produces a scattering state forms a subspace of* $H^1\left(\mathbb{R}^3\right) \times L^2\left(\mathbb{R}^3\right)$*. We will see a detailed discussion on this subspace later on in Section 3.6.*

**Remark 3.1.4.** *Notice that in order to perform Lorentz transformations, one needs the existence of global solutions. The existence and the uniqueness of global solutions to wave equatioans with more general time-dependent potentials are presented by contraction arguments in [12].*

With the above preparations, we state our main results. First of all, we have Strichartz estimates:

**Theorem 3.1.5** (Strichartz estimates)**.** *Suppose $u$ is a scattering state in the sense of Definition 5.3.2 which solves*

$$\partial_{tt}u - \Delta u + V_1(x)u + V_2(x - vt)u = 0 \tag{3.1.37}$$

*with initial data*

$$u(x, 0) = g(x),\ u_t(x, 0) = f(x). \tag{3.1.38}$$

*Then for $p > 2$ and $(p, q)$ satisfying*

$$\frac{1}{2} = \frac{1}{p} + \frac{3}{q}, \tag{3.1.39}$$

*we have*

$$\|u\|_{L_t^p([0,\infty),\, L_x^q)} \lesssim \|f\|_{L^2} + \|g\|_{\dot{H}^1}. \tag{3.1.40}$$

The above theorem is extended to the inhomogeneous case in Section 3.6.

As in [42], if we all the norm to be inhomogeneous with respect to the radial and angular variables, one can recover the endpoint Strichartz estimate:

**Theorem 3.1.6** (Endpoint Strichartz estimate)**.** *Let $|v| < 1$. Suppose $u$ is a scattering state in the sense of Definition 5.3.2 which solves*

$$\partial_{tt}u - \Delta u + V_1(x)u + V_2(x - vt)u = 0 \tag{3.1.41}$$

*with initial data*

$$u(x, 0) = g(x),\ u_t(x, 0) = f(x). \tag{3.1.42}$$

*Then for $1 \leq p < \infty$,*

$$\|u\|_{L_t^2([0,\infty),\, L_r^\infty L_\omega^p)} \lesssim \|f\|_{L^2} + \|g\|_{\dot{H}^1} \tag{3.1.43}$$

Next, one has the energy estimate:

**Theorem 3.1.7** (Energy estimate)**.** *Suppose $u$ is a scattering state in the sense of Definition 5.3.2 which solves*

$$\partial_{tt}u - \Delta u + V_1(x)u + V_2(x - vt)u = 0 \tag{3.1.44}$$

*with initial data*

$$u(x, 0) = g(x),\ u_t(x, 0) = f(x). \tag{3.1.45}$$

*Then we have*

$$\sup_{t\geq 0} \left(\|\nabla u(t)\|_{L^2} + \|u_t(t)\|_{L^2}\right) \lesssim \|f\|_{L^2} + \|g\|_{\dot{H}^1}. \tag{3.1.46}$$

Associated with the energy estimate, we also have the local energy decay:

**Theorem 3.1.8** (Local energy decay)**.** *Suppose $u$ is a scattering state in the sense of Definition 5.3.2 which solves*

$$\partial_{tt}u - \Delta u + V_1(x)u + V_2(x - vt)u = 0 \tag{3.1.47}$$

*with initial data*

$$u(x, 0) = g(x),\ u_t(x, 0) = f(x). \tag{3.1.48}$$

*Then for $\forall\epsilon > 0$, $|\mu| < 1$, we have*

$$\left\|(1 + |x - \mu t|)^{-\frac{1}{2}-\epsilon} (|\nabla u| + |u_t|)\right\|_{L^2_{t,x}} \lesssim_{\mu,\epsilon} \|f\|_{L^2} + \|g\|_{\dot{H}^1}.$$

Even more importantly, we obtain the endpoint reversed Strichartz estimates for $u$.

**Theorem 3.1.9** (Endpoint reversed Strichartz estimate)**.** *Suppose $u$ is a scattering state in the sense of Definition 5.3.2 which solves*

$$\partial_{tt}u - \Delta u + V_1(x)u + V_2(x - vt)u = 0 \tag{3.1.49}$$

*with initial data*

$$u(x,0) = g(x),\; u_t(x,0) = f(x). \tag{3.1.50}$$

*Then*

$$\sup_{x\in\mathbb{R}^3} \int_0^\infty |u(x,t)|^2\, dt \lesssim \left(\|f\|_{L^2} + \|g\|_{\dot{H}^1}\right)^2, \tag{3.1.51}$$

*and*

$$\sup_{x\in\mathbb{R}^3} \int_0^\infty |u(x+vt,t)|^2\, dt \lesssim \left(\|f\|_{L^2} + \|g\|_{\dot{H}^1}\right)^2. \tag{3.1.52}$$

With the endpoint estimate along $(x+vt,t)$, one can derive the boundedness of the total energy. We denote the total energy of the system as

$$E(t) = \int |\nabla_x u|^2 + |\partial_t u|^2 + V_1\, |u|^2 + V_2(x-vt)\, |u|^2\, dx. \tag{3.1.53}$$

**Corollary 3.1.10** (Boundedness of the total energy)**.** *Suppose $u$ is a scattering state in the sense of Definition 5.3.2 which solves*

$$\partial_{tt} u - \Delta u + V_1(x)u + V_2(x-vt)u = 0 \tag{3.1.54}$$

*with initial data*

$$u(x,0) = g(x),\; u_t(x,0) = f(x). \tag{3.1.55}$$

*Assume*

$$\|\nabla V_2\|_{L^1} < \infty, \tag{3.1.56}$$

*then $E(t)$ is bounded by the initial energy independently of $t$,*

$$\sup_{t\geq 0} E(t) \lesssim \|(g,f)\|^2_{\dot{H}^1\times L^2}. \tag{3.1.57}$$

### 3.1.3 Main ideas

Here we briefly discuss the main ideas in our analysis and sketch our proofs. We follow the philosophy from Rodnianski-Schlag [50] that *local decay estimates* imply Strichartz estimates. The main stream of ideas is that *the endpoint Strichartz estimate* implies weighted estimates, based on which we can derive Strichartz estimates, energy estimates, the local energy decay, and the boundedness of the total energy.

An essential step to approach wave equations with moving potentials is to understand the change of energy under Lorentz transformations. In subsection 4.3, we show that the energy along a space-like slanted line stays comparable to the energy of the initial data. This in particular implies that under Lorentz transformations, the initial energy with respect to the new frame is comparable to the initial energy of the original frame. As a byproduct, we can also obtain Agmon's estimates for the decay of eigenfunctions. The arguments hold for all dimensions and even for wave equations with time-dependent potentials, cf. [12].

In order to handle time-dependent potentials, we need a time-dependent weight in the local decay estimate, see Chen [11]. More precisely, we will show that for $|v| < 1$,

$$\int_{\mathbb{R}} \int_{\mathbb{R}^3} \frac{1}{\langle x \rangle^{\alpha}} u^2(x,t)\, dxdt \lesssim \|(g,f)\|^2_{\dot{H}^1 \times L^2} \tag{3.1.58}$$

and

$$\int_{\mathbb{R}} \int_{\mathbb{R}^3} \frac{1}{\langle x - vt \rangle^{\alpha}} u^2(x,t)\, dxdt \lesssim \|(g,f)\|^2_{\dot{H}^1 \times L^2}. \tag{3.1.59}$$

We notice that for $\alpha > 3$,

$$\int_{\mathbb{R}} \int_{\mathbb{R}^3} \frac{1}{\langle x \rangle^{\alpha}} u^2(x,t)\, dxdt \lesssim \sup_{x \in \mathbb{R}^3} \int_{\mathbb{R}} u^2(x,t)\, dt \tag{3.1.60}$$

and

$$\int_{\mathbb{R}} \int_{\mathbb{R}^3} \frac{1}{\langle x - vt \rangle^{\alpha}} u^2(x,t)\, dxdt \lesssim \sup_{x \in \mathbb{R}^3} \int_{\mathbb{R}} u^2(x+vt,t)\, dt \tag{3.1.61}$$

which are in the form of *endpoint reversed Strichartz estimates*. But we also need to *integrate over a time-like slanted line*. These are carefully analyzed in Section 3.3. Intuitively, the reversed Strichartz estimates are based on the fact that the fundamental solutions of the wave equation in $\mathbb{R}^3$ is supported on the light cone. For fixed $x$, the propagation will only meet the light cone once. Meanwhile, away from the light cone, the solution decays fast. We note that a time-like slanted line will also only intersect the light cone only once, hence we should have the same endpoint estimate along it. Our analysis crucially relies on these types of estimates. Many estimates in Section 3.3 also hold for more general trajectories provided that their speeds are strictly less than 1.

After performing the Lorentz transformation $L$, we have

$$u_L\left(x_1', x_2', x_3', t'\right) = u\left(\gamma\left(x_1' + vt'\right), x_2', x_3', \gamma\left(t' + vx_1'\right)\right) \tag{3.1.62}$$

and

$$u(x,t) = u_L\left(\gamma\left(x_1 - vt\right), x_2, x_3, \gamma\left(t - vx_1\right)\right). \tag{3.1.63}$$

It is crucial to notice that from the expressions above, the standard endpoint Strichartz estimate for $u$ is equivalent to the endpoint Strichartz estimate along a slanted line for $u_L$ and vice versa. It is important to note that with the above fact, we can always apply Lorentz transformations to exchange different frames if we consider several endpoint Strichartz estimates together.

Based on the observations above, we apply a bootstrap procedure for the case with two potentials. Let

$$u^S\left(x,t\right) = u(x+vt,t).$$

For a scattering state in the sense of Definition 5.3.2, we show that the bootstrap assumptions with big constants $C_1(T)$ and $C_2(T)$,

$$\sup_{x\in\mathbb{R}^3}\int_0^T |u(x,t)|^2\,dt \leq C_1(T)\left(\|f\|_{L^2}+\|g\|_{\dot{H}^1}\right)^2 \tag{3.1.64}$$

and

$$\sup_{x\in\mathbb{R}^3}\int_0^T \left|u^S(x,t)\right|^2 dt \leq C_2(T)\left(\|f\|_{L^2}+\|g\|_{\dot{H}^1}\right)^2, \tag{3.1.65}$$

imply

$$\sup_{x\in\mathbb{R}^3}\int_0^T |u(x,t)|^2\,dt \leq \left(\widetilde{C}_1+\frac{1}{2}C_1(T)\right)\left(\|f\|_{L^2}+\|g\|_{\dot{H}^1}\right)^2 \tag{3.1.66}$$

and

$$\sup_{x\in\mathbb{R}^3}\int_0^T \left|u^S(x,t)\right|^2 dt \leq \left(\widetilde{C}_2+\frac{1}{2}C_2(T)\right)\left(\|f\|_{L^2}+\|g\|_{\dot{H}^1}\right)^2. \tag{3.1.67}$$

Then we can conclude

$$\sup_{x\in\mathbb{R}^3}\int_0^T |u(x,t)|^2\,dt \leq C_1\left(\|f\|_{L^2}+\|g\|_{\dot{H}^1}\right)^2 \tag{3.1.68}$$

and

$$\sup_{x\in\mathbb{R}^3}\int_0^T \left|u^S(x,t)\right|^2 dt \leq C_2\left(\|f\|_{L^2}+\|g\|_{\dot{H}^1}\right)^2, \tag{3.1.69}$$

for some constants $C_1$ and $C_2$ independent of $T$ by the bootstrap argument. Therefore, as we pointed out above, we obtain two local decay estimates (3.1.58) and (3.1.59). To run the bootstrap argument, we use the fact that the distance between $V_1$ and $V_2$ becomes larger and larger and both potentials are of short-range. Therefore, for different regions in $\mathbb{R}^3$, the evolution will be dominated by different Hamiltonians. To make this intuition precise, in Section 3.4, we apply a partition of unity to carry out the decomposition into channels. For each channel, we use Duhamel's formula to compare the evolution to the associated dominating Hamiltonian. For every dominating Hamiltonian, both of the endpoint estimates hold. In each Duhamel expansion, based on the fact that $V_1$ and $V_2$ move far away from each other, it suffices to consider the endpoint estimates of the following integrals,

$$k_A(x,t) := \int_0^{t-A} \frac{\sin\left((t-s)\sqrt{H}\right)}{\sqrt{H}} P_c F\, ds. \tag{3.1.70}$$

and

$$k_A^S(x,t) := k_A(x+vt,t).$$

From Section 3.3, we have

$$\begin{aligned} \|k_A\|_{L_x^\infty L_t^2[A,T]} &= \left\| \int_0^{t-A} \frac{\sin\left((t-s)\sqrt{H}\right)}{\sqrt{H}} P_c F\, ds \right\|_{L_x^\infty L_t^2[A,T]} \\ &\lesssim \frac{1}{A}\left( \|F\|_{L_x^1 L_t^2} + \|F\|_{L_x^{\frac{3}{2},1} L_t^2} \right) \end{aligned} \tag{3.1.71}$$

and

$$\left\| k_A^S(x,t) \right\|_{L_x^\infty L_t^2[A,T]} \lesssim \frac{1}{A}\left( \|F\|_{L_x^{\frac{3}{2},1} L_t^2} + \|F\|_{L_x^1 L_t^2} \right). \tag{3.1.72}$$

Therefore for $A > 0$ large but independent of $T$, this term can be absorbed to the left-hand side to improve our bootstrap assumptions.

From (3.1.58) and (3.1.59), Strichartz estimates follow from the general scheme introduced in Rodnianski-Schlag [50, 40].

## *Organization*

The chapter is organized as follows: In Section 4.2, we discuss some preliminary results for the free wave equation and the wave equation with a stationary potential. We will also analyze the change of the energy under Lorentz transformations. In Section 3.3, estimates of homogeneous and inhomogeneous forms of wave equations along time-like slanted lines will be discussed. In Section 3.4 and Section 3.5, we show Strichartz estimates, energy estimates, the local energy decay and the boundedness of the total energy for a scattering state to the wave equation with a charge transfer Hamiltonian. In order to consider nonlinear applications, in Section 3.6 we discuss inhomogeneous Strichartz estimates and local decay estimates. Finally, in Section 4.6, we confirm that a scattering state indeed scatters to a solution to the free wave equation.

## 3.2 Preliminaries

In this section, we present some preliminary results on wave equations to prepare further discussions in later sections. Throughout, we will only consider equations in $\mathbb{R}^3$.

### *3.2.1 Strichartz estimates and local energy decay*

We start with the regular Strichartz estimates for free wave equations.

Consider the free wave equation,

$$\partial_{tt}u - \Delta u = F \tag{3.2.1}$$

with initial data

$$u(x,0) = g(x),\ u_t(x,0) = f(x). \tag{3.2.2}$$

We can write down the solution using the Fourier transform:

$$u = \frac{\sin\left(t\sqrt{-\Delta}\right)}{\sqrt{-\Delta}} f + \cos\left(t\sqrt{-\Delta}\right) g + \int_0^t \frac{\sin\left((t-s)\sqrt{-\Delta}\right)}{\sqrt{-\Delta}} F(s)\, ds. \tag{3.2.3}$$

It obeys the energy inequality,

$$E_F(t) = \int_{\mathbb{R}^3} |\partial_t u(t)|^2 + |\nabla u(t)|^2\ dx \lesssim \int_{\mathbb{R}^3} |f|^2 + |\nabla g|^2\ dx + \int_0^t \int_{\mathbb{R}^3} |F(s)|^2\ dxds. \tag{3.2.4}$$

We also have the well-known dispersive estimates for the free wave equation ($H_0 = -\Delta$) on

$\mathbb{R}^3$:

$$\left\| \frac{\sin\left(t\sqrt{-\Delta}\right)}{\sqrt{-\Delta}} f \right\|_{L^\infty(\mathbb{R}^3)} \lesssim \frac{1}{|t|} \left\| \nabla f \right\|_{L^1(\mathbb{R}^3)}, \tag{3.2.5}$$

$$\left\| \cos\left(t\sqrt{-\Delta}\right) g \right\|_{L^\infty(\mathbb{R}^3)} \lesssim \frac{1}{|t|} \left\| \Delta g \right\|_{L^1(\mathbb{R}^3)}. \tag{3.2.6}$$

Notice that the estimate (4.1.6) is slightly different from the estimates commonly in the literature. For example, in Krieger-Schlag [39], one needs the $L^1$ norm of $D^2 g$ instead of $\Delta g$. One can find a detailed proof in [12] based on an idea similar to the endpoint reversed Strichartz estimate.

Strichartz estimates can be derived abstractly from these dispersive inequalities and the energy inequality. The following theorem is standard. One can find a detailed proof in, for example, Keel-Tao [37].

**Theorem 3.2.1** (Strichartz estimate)**.** *Suppose*

$$\partial_{tt} u - \Delta u = F \tag{3.2.7}$$

*with initial data*

$$u(x,0) = g(x),\ u_t(x,0) = f(x). \tag{3.2.8}$$

*Then for* $p,\ a > \frac{2}{s}$, $(p,q)$, $(a,b)$ *satisfying*

$$\frac{3}{2} - s = \frac{1}{p} + \frac{3}{q} \tag{3.2.9}$$

$$\frac{3}{2} - s = \frac{1}{a} + \frac{3}{b} \tag{3.2.10}$$

*we have*

$$\|u\|_{L_t^p L_x^q} \lesssim \|g\|_{\dot{H}^s} + \|f\|_{\dot{H}^{s-1}} + \|F\|_{L_t^{a'} L_x^{b'}} \tag{3.2.11}$$

*where* $\frac{1}{a} + \frac{1}{a'} = 1$, $\frac{1}{b} + \frac{1}{b'} = 1$.

The endpoint $(p, q) = (2, \infty)$ can be recovered for radial functions in Klainerman-Machedon [38] for the homogeneous case and Jia-Liu-Schlag-Xu [34] for the inhomogeneous case. The endpoint estimate can also be obtained when a small amount of smoothing (either in the Sobolev sense, or in relaxing the integrability) is applied to the angular variable, by Machihara-Nakamura-Nakanishi-Ozawa [42].

**Theorem 3.2.2** ([42])**.** *For any $1 \leq p < \infty$, suppose $u$ solves the free wave equation*

$$\partial_{tt}u - \Delta u = 0 \tag{3.2.12}$$

*with initial data*

$$u(x, 0) = g(x),\ u_t(x, 0) = f(x). \tag{3.2.13}$$

*Then*

$$\|u\|_{L_t^2 L_r^\infty L_\omega^p} \leq C(p) \left( \|f\|_{L^2} + \|g\|_{\dot{H}^1} \right). \tag{3.2.14}$$

The regular Strichartz estimates fail at the endpoint. But if one switches the order of space-time integration, it is possible to estimate the solution using the fact that the solution decays quickly away from the light cone. Therefore, we introduce reversed Strichartz estimates. Since we will only use the endpoint reversed Stricharz estimate, we will restrict our focus to that case. The detailed proof for free equations is presented for the sake of completeness.

**Theorem 3.2.3** (Endpoint reversed Strichartz estimate)**.** *Suppose*

$$\partial_{tt}u - \Delta u = F \tag{3.2.15}$$

*with initial data*

$$u(x, 0) = g(x),\ u_t(x, 0) = f(x). \tag{3.2.16}$$

*Then*

$$\|u\|_{L_x^\infty L_t^2} \lesssim \|f\|_{L^2} + \|g\|_{\dot{H}^1} + \|F\|_{L_x^{\frac{3}{2},1} L_t^2}, \tag{3.2.17}$$

*and for* $T > 0$*,*

$$\|u\|_{L_x^\infty L_t^2[0,T]} \lesssim \|f\|_{L^2} + \|g\|_{\dot{H}^1} + \|F\|_{L_x^{\frac{3}{2},1} L_t^2[0,T]}. \tag{3.2.18}$$

*Proof.* Writing down $u$ explicitly, we have

$$u = \frac{\sin\left(t\sqrt{-\Delta}\right)}{\sqrt{-\Delta}} f + \cos\left(t\sqrt{-\Delta}\right) g + \int_0^t \frac{\sin\left((t-s)\sqrt{-\Delta}\right)}{\sqrt{-\Delta}} F(s)\, ds. \tag{3.2.19}$$

We will analyze each term separately. By symmetry, we may assume that $t \geq 0$.

For the first term,

$$\frac{\sin\left(t\sqrt{-\Delta}\right)}{\sqrt{-\Delta}} f = \frac{1}{4\pi t} \int_{|x-y|=t} f(y)\, \sigma\left(dy\right). \tag{3.2.20}$$

So in polar coordinates,

$$\begin{aligned}
\left\| \frac{\sin\left(t\sqrt{-\Delta}\right)}{\sqrt{-\Delta}} f \right\|_{L_t^2}^2 &\lesssim \int_0^\infty \left( \int_{\mathbb{S}} f(x + r\omega) r\, d\omega \right)^2 dr \\
&\lesssim \left( \int_0^\infty \int_{\mathbb{S}} f(x + r\omega)^2 r^2\, d\omega dr \right) \left( \int_{\mathbb{S}^2} d\omega \right) \\
&\lesssim \|f\|_{L^2}^2 .
\end{aligned}$$

Therefore,

$$\left\| \frac{\sin\left(t\sqrt{-\Delta}\right)}{\sqrt{-\Delta}} f \right\|_{L_x^\infty L_t^2} \lesssim \|f\|_{L^2} . \tag{3.2.21}$$

For the second term,

$$
\begin{aligned}
\left\|\cos\left(t\sqrt{-\Delta}\right)g\right\|_{L_t^2}^2 &= \int_0^\infty \left(\int_{\mathbb{S}^2} g\left(x+r\omega\right)d\omega + r\partial_r g\left(x+r\omega\right)\, d\omega\right)^2 dr \\
&\lesssim \int_0^\infty \int_{\mathbb{S}^2} \left(g\left(x+r\omega\right)d\omega\right)^2 dr + \int_0^\infty \int_{\mathbb{S}^2} \left(\partial_r g\left(x+r\omega\right)d\omega\right)^2 r^2\, dr \\
&\lesssim \left(\int_0^\infty \int_{\mathbb{S}^2} g\left(x+r\omega\right)^2 d\omega dr\right)\left(\int_{\mathbb{S}^2} d\omega\right) \\
&\quad + \left(\int_0^\infty \int_{\mathbb{S}^2} \partial_r g\left(x+r\omega\right)^2 d\omega r^2 dr\right)\left(\int_{\mathbb{S}^2} d\omega\right) \\
&\lesssim \left\|\nabla g\right\|_{L^2}^2,
\end{aligned}
$$

where for the last inequality, we applied Hardy's inequality

$$\left\| |x|^{-1} g \right\|_{L^2} \lesssim \|g\|_{\dot{H}^1}. \tag{3.2.22}$$

Therefore,

$$\left\|\cos\left(t\sqrt{-\Delta}\right)g\right\|_{L_x^\infty L_t^2} \lesssim \|\nabla g\|_{L^2}. \tag{3.2.23}$$

For the third term,

$$
\begin{aligned}
\left\|\int_0^t \frac{\sin\left((t-s)\sqrt{-\Delta}\right)}{\sqrt{-\Delta}} F(s)\,ds\right\|_{L_t^2} &= \left\|\int_0^t \int_{|x-y|=t-s} \frac{1}{|x-y|} F(y,s)\,\sigma\left(dy\right) ds\right\|_{L_t^2} \\
&= \left\|\int_{|x-y|\le t} \frac{1}{|x-y|} F\left(y, t-|x-y|\right)\, dy\right\|_{L_t^2} \\
&\lesssim \int \frac{1}{|x-y|} \left\|F\left(y,t-|x-y|\right)\right\|_{L_t^2} dy \\
&\lesssim \sup_{x\in\mathbb{R}^3} \int \frac{1}{|x-y|} \left\|F\left(y,t\right)\right\|_{L_t^2} dy \\
&\lesssim \|F\|_{L_x^{\frac{3}{2},1} L_t^2},
\end{aligned}
$$

where we applied Minkowski's inequality in the third line. Here $L^{\frac{3}{2},1}$ is the Lorentz space.

In the last inequality, we apply the following fact:

$$\sup_{y\in\mathbb{R}^3}\int_{\mathbb{R}^3}\frac{|h(x)|}{|x-y|}\,dx=\sup_{y\in\mathbb{R}^3}\int_{\mathbb{R}^3}\frac{|h(x-y)|}{|x|}\,dx. \tag{3.2.24}$$

Then for fixed $y$, we apply Hölder's inequality for Lorentz spaces, see Theorem 3.5 in O'Neil [48],

$$\int_{\mathbb{R}^3}\frac{|h(x-y)|}{|x|}\,dx=\left\|\frac{|h(x-y)|}{|x|}\right\|_{L^{1,1}}\lesssim\left\|\frac{1}{|x|}\right\|_{L^{3,\infty}}\|h(x-y)\|_{L_x^{\frac{3}{2},1}}. \tag{3.2.25}$$

Notice that

$$\|h(x-y)\|_{L_x^{\frac{3}{2},1}}=\|h(x)\|_{L_x^{\frac{3}{2},1}}. \tag{3.2.26}$$

Therefore,

$$\sup_{y\in\mathbb{R}^3}\int_{\mathbb{R}^3}\frac{|h(x)|}{|x-y|}\,dx\lesssim\|h\|_{L_x^{\frac{3}{2},1}}. \tag{3.2.27}$$

Hence

$$\left\|\int_0^t\frac{\sin\left((t-s)\sqrt{-\Delta}\right)}{\sqrt{-\Delta}}F(s)\,ds\right\|_{L_x^\infty L_t^2}\lesssim\|F\|_{L_x^{\frac{3}{2},1}L_t^2}. \tag{3.2.28}$$

We also notice that for $T>0$,

$$\left\|\int_0^t\frac{\sin\left((t-s)\sqrt{-\Delta}\right)}{\sqrt{-\Delta}}F(s)\,ds\right\|_{L_t^2[0,T]}\lesssim\left\|\int_{|x-y|\leq t}\frac{1}{|x-y|}F\left(y,t-|x-y|\right)\,dy\right\|_{L_t^2[0,T]}$$

with

$$0\leq t-|x-y|\leq T,$$

whence

$$\begin{aligned}\left\|\int_0^t\frac{\sin\left((t-s)\sqrt{-\Delta}\right)}{\sqrt{-\Delta}}F(s)\,ds\right\|_{L_t^2[0,T]} &\lesssim \left\|\int_{|x-y|\leq t}\frac{1}{|x-y|}F\left(y,t-|x-y|\right)\,dy\right\|_{L_t^2[0,T]}\\ &\lesssim \int\frac{1}{|x-y|}\|F\left(y,t\right)\|_{L_t^2[0,T]}\,dy\\ &\lesssim \|F\|_{L_x^{\frac{3}{2},1}L_t^2[0,T]}.\end{aligned}$$

Therefore,

$$\left\| \int_0^t \frac{\sin\left((t-s)\sqrt{-\Delta}\right)}{\sqrt{-\Delta}} F(s)\, ds \right\|_{L_t^2[0,T]} \lesssim \|F\|_{L_x^{\frac{3}{2},1} L_t^2[0,T]}. \tag{3.2.29}$$

The theorem is proved. □

The above results from Theorem 4.2.1 and Theorem 4.2.3 can be generalized to wave equations with real stationary potentials.

Denote

$$H = -\Delta + V, \tag{3.2.30}$$

where the potential $V$ satisfies the assumption in Definition 5.3.1.

Consider the wave equation with potential in $\mathbb{R}^3$:

$$\partial_{tt} u - \Delta u + V u = 0 \tag{3.2.31}$$

with initial data

$$u(x,0) = g(x),\ u_t(x,0) = f(x). \tag{3.2.32}$$

One can write down the solution to it explicitly:

$$u = \frac{\sin\left(t\sqrt{H}\right)}{\sqrt{H}} f + \cos\left(t\sqrt{H}\right) g. \tag{3.2.33}$$

Let $P_b$ be the projection onto the point spectrum of $H$, $P_c = I - P_b$ be the projection onto the continuous spectrum of $H$.

With the above setting, we formulate the results from [3].

**Theorem 3.2.4** (Strichartz and reversed Strichartz estimates)**.** *Suppose $H$ has neither eigenvalues nor resonances at zero. Then for all $0 \le s \le 1$, $p > \frac{2}{s}$, and $(p,q)$ satisfying*

$$\frac{3}{2} - s = \frac{1}{p} + \frac{3}{q} \tag{3.2.34}$$

*we have*

$$\left\| \frac{\sin\left(t\sqrt{H}\right)}{\sqrt{H}} P_c f + \cos\left(t\sqrt{H}\right) P_c g \right\|_{L_t^p L_x^q} \lesssim \|g\|_{\dot{H}^s} + \|f\|_{\dot{H}^{s-1}}. \tag{3.2.35}$$

*For the endpoint reversed Strichartz estimate, we have*

$$\left\| \frac{\sin\left(t\sqrt{H}\right)}{\sqrt{H}} P_c f + \cos\left(t\sqrt{H}\right) P_c g \right\|_{L_x^\infty L_t^2} \lesssim \|f\|_{L^2} + \|g\|_{\dot{H}^1}, \tag{3.2.36}$$

$$\left\| \int_0^t \frac{\sin\left((t-s)\sqrt{H}\right)}{\sqrt{H}} P_c F(s)\, ds \right\|_{L_x^\infty L_t^2} \lesssim \|F\|_{L_x^{\frac{3}{2},1} L_t^2}, \tag{3.2.37}$$

*and for $T > 0$,*

$$\left\| \int_0^t \frac{\sin\left((t-s)\sqrt{H}\right)}{\sqrt{H}} P_c F(s)\, ds \right\|_{L_x^\infty L_t^2[0,T]} \lesssim \|F\|_{L_x^{\frac{3}{2},1} L_t^2[0,T]}. \tag{3.2.38}$$

One can find detailed arguments and more estimates in [3]. The above theorem can also be established by passing the estimates for free wave equations in Theorem 4.2.1 and Theorem 4.2.3 to the perturbed case via the structure of wave operators. This general strategy is discussed in detail in [12].

**Remark 3.2.5.** *In [3], the above theorem is shown for potentials $V$ with a finite global Kato norm. The Kato space $K$ is the Banach space of measures with the property that*

$$\|V\|_K = \sup_{y\in\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{|V(x)|}{|x-y|}\, dx. \tag{3.2.39}$$

*They consider the space of potentials $V$ which are taken in the Kato norm closure of the set of bounded, compactly supported functions, which is denoted by $K_0$. Note that from estimate*

(3.2.27), $L_x^{\frac{3}{2},1} \subset K$ .

Next, we formulate one fundamental mechanism of wave equations: local energy decay. It suffices to consider the half-wave operator.

**Theorem 3.2.6** (Local energy decay)**.** *$\forall \epsilon > 0$, one has*

$$\left\| (1+|x|)^{-\frac{1}{2}-\epsilon} e^{it\sqrt{-\Delta}} f \right\|_{L^2_{t,x}} \lesssim_\epsilon \|f\|_{L^2_x}. \tag{3.2.40}$$

See Corollary 3.2.11 for a more general formulation with time-dependent weight. A detailed proof can be found in the appendices in [12].

The following Christ-Kiselev Lemma is important in our derivation of Strichartz estimates.

**Lemma 3.2.7** (Christ-Kiselev)**.** *Let $X$, $Y$ be two Banach spaces and let $T$ be a bounded linear operator from $L^\beta\left(\mathbb{R}^+; X\right)$ to $L^\gamma\left(\mathbb{R}^+; Y\right)$, such that*

$$Tf(t) = \int_0^\infty K(t,s) f(s)\, ds. \tag{3.2.41}$$

*Then the operator*

$$\widetilde{T} f = \int_0^t K(t,s) f(s)\, ds \tag{3.2.42}$$

*is bounded from $L^\beta\left(\mathbb{R}^+; X\right)$ to $L^\gamma\left(\mathbb{R}^+; Y\right)$ provided $\beta < \gamma$, and the*

$$\left\| \widetilde{T} \right\| \leq C(\beta,\gamma) \|T\| \tag{3.2.43}$$

*with*

$$C(\beta,\gamma) = \left(1 - 2^{\frac{1}{\gamma}-\frac{1}{\beta}}\right)^{-1}. \tag{3.2.44}$$

### *3.2.2 Lorentz Transformations and Energy*

In this chapter, Lorentz transformations will be important for us to reduce some estimates to stationary cases. In order to approach our problem from the viewpoint of Lorentz transformations, the first natural step is to understand the change of energy under Lorentz transformations. In this subsection, we show that under Lorentz transformations, the energy stays comparable to that of the initial data. Recall that after we apply the Lorentz transformation, for function $u$, under the new coordinates, we denote

$$u_L\left(x_1', x_2', x_3', t'\right) = u\left(\gamma\left(x_1' + vt'\right), x_2', x_3', \gamma\left(t' + vx_1'\right)\right). \tag{3.2.45}$$

Now let $u$ be a solution to some wave equation and set $t' = 0$. We notice that in order to show under Lorentz transformations, the energy stays comparable to that of the initial data up to an absolute constant, it suffices to prove

$$\begin{aligned} &\int \left|\nabla_x u\left(x_1, x_2, x_3, vx_1\right)\right|^2 + \left|\partial_t u\left(x_1, x_2, x_3, vx_1\right)\right|^2 dx \\ &\quad \simeq \int \left|\nabla_x u\left(x_1, x_2, x_3, 0\right)\right|^2 + \left|\partial_t u\left(x_1, x_2, x_3, 0\right)\right|^2 dx. \end{aligned} \tag{3.2.46}$$

provided $|v| < 1$.

Throughout this subsection, we will assume all functions are smooth and decay fast. We will obtain estimates independent of the additional smoothness assumption. It is easy to pass the estimates to general cases with a density argument.

**Remark 3.2.8.** *One can observe that all discussions in this section hold for $\mathbb{R}^n$. We choose $n = 3$ since we will only consider the charge transfer model in $\mathbb{R}^3$ in later parts of this chapter.*

In this subsection, a more general situation is analyzed. We consider wave equations with time-dependent potentials

$$\partial_{tt}u - \Delta u + V(x,t)u = 0 \tag{3.2.47}$$

under some uniform decay conditions

$$|V(x, \mu x_1)| \lesssim \frac{1}{\langle x \rangle^3} \tag{3.2.48}$$

uniformly for $0 \leq |\mu| \leq 1$. These in particular apply to wave equations with moving potentials with speed strictly less than 1. For example,

$$V(x,t) = V(x - vt) \tag{3.2.49}$$

with

$$|V(x)| \lesssim \frac{1}{\langle x \rangle^3}. \tag{3.2.50}$$

**Theorem 3.2.9.** *Let $|v| < 1$. Suppose*

$$\partial_{tt}u - \Delta u + V(x,t)u = 0 \tag{3.2.51}$$

*and*

$$|V(x, \mu x_1)| \lesssim \frac{1}{\langle x \rangle^3} \tag{3.2.52}$$

*uniformly with respect to $0 \leq |\mu| < 1$. Then*

$$\begin{aligned} &\int |\nabla_x u(x_1, x_2, x_3, vx_1)|^2 + |\partial_t u(x_1, x_2, x_3, vx_1)|^2 \, dx \\ &\simeq \int |\nabla_x u(x_1, x_2, x_3, 0)|^2 + |\partial_t u(x_1, x_2, x_3, 0)|^2 \, dx, \end{aligned} \tag{3.2.53}$$

*where the implicit constant depends on $v$ and $V$.*

*Proof.* Up to performing a Lorentz transformation or a change of variable, it suffices to show

$$\begin{aligned}\int & \left|\nabla_x u\left(x_1, x_2, x_3, v x_1\right)\right|^2+\left|\partial_t u\left(x_1, x_2, x_3, v x_1\right)\right|^2 d x \\ & \lesssim \int\left|\nabla_x u\left(x_1, x_2, x_3, 0\right)\right|^2+\left|\partial_t u\left(x_1, x_2, x_3, 0\right)\right|^2 d x .\end{aligned} \tag{3.2.54}$$

Set

$$E_1(\mu)=\int\left|\nabla_x u\left(x_1, x_2, x_3, \mu x_1\right)\right|^2 d x, \tag{3.2.55}$$

$$E_2(\mu)=\int\left|\partial_t u\left(x_1, x_2, x_3, \mu x_1\right)\right|^2 d x . \tag{3.2.56}$$

In the following computations, for a function $f(x, t)$, we use the short-hand notation

$$\int f\, d x=\int_{\mathbb{R}^3} f\left(x_1, x_2, x_3, \mu x_1\right) d x .$$

Then

$$\begin{aligned}\frac{d E_1}{d \mu} & =2 \int x_1 \nabla_x u\left(x_1, x_2, x_3, \mu x_1\right) \nabla_x u_t\left(x_1, x_2, x_3, \mu x_1\right)\, d x \\ & =2 \int x_1 \nabla_x u \nabla_x u_t\, d x\end{aligned} \tag{3.2.57}$$

and

$$\begin{aligned}\frac{d E_2}{d \mu} & =2 \int x_1 \partial_t u\left(x_1, x_2, x_3, \mu x_1\right) \partial_{t t} u\left(x_1, x_2, x_3, \mu x_1\right)\, d x \\ & =2 \int x_1 \partial_t u \partial_{t t} u\, d x .\end{aligned} \tag{3.2.58}$$

Integration by parts in (3.2.57) gives

$$\frac{d E_1}{d \mu}=-2 \int \partial_{x_1} u \cdot u_t\, d x-2 \int x_1 \Delta u \cdot u_t\, d x-2 \mu \int x_1 \partial_{x_1} u_t \cdot u_t\, d x . \tag{3.2.59}$$

And using the fact that $u$ solves the wave equation implies

$$\frac{dE_2}{d\mu} = 2\int x_1\partial_t u \cdot \Delta u \, dx - 2\int x_1\partial_t u \cdot V u \, dx. \tag{3.2.60}$$

Consider the following integral appearing as the last term in (3.2.59),

$$\int x_1\partial_{x_1} u_t \cdot u_t \, dx. \tag{3.2.61}$$

Integration by parts in $x$, one has

$$\int x_1\partial_{x_1} u_t \cdot u_t \, dx = -\int |u_t|^2 \, dx - \int x_1\partial_{x_1} u_t \cdot u_t \, dx - \mu \int x_1 u_t \cdot u_{tt} \, dx.$$

Therefore,

$$\int x_1\partial_{x_1} u_t \cdot u_t \, dx = -\frac{1}{2}\int |u_t|^2 \, dx - \frac{\mu}{4}\frac{dE_2}{d\mu}. \tag{3.2.62}$$

Combining identities (3.2.59), (3.2.60) and (3.2.62) together, we have

$$E_1'(\mu) + \left(1 - \frac{\mu^2}{2}\right) E_2'(\mu) = H(\mu),$$

where

$$H(\mu) = -2\int \partial_{x_1} u \cdot u_t \, dx - 2\int x_1\partial_t u \cdot V u \, dx + \mu \int |u_t|^2 \, dx. \tag{3.2.63}$$

By Cauchy-Schwarz and Hardy's inequality,

$$|H(\mu)| \lesssim E_1(\mu) + E_2(\mu),$$

and hence

$$\left| E_1'(\mu) + \left(1 - \frac{\mu^2}{2}\right) E_2'(\mu) \right| \lesssim E_1(\mu) + E_2(\mu). \tag{3.2.64}$$

Setting

$$E_3(\mu) = E_1(\mu) + \left(1 - \frac{\mu^2}{2}\right) E_2(\mu),$$

one has

$$E_3'(\mu) = E_1'(\mu) + \left(1 - \frac{\mu^2}{2}\right) E_2'(\mu) - \mu E_2(\mu)$$

and

$$\left|E_3'(\mu)\right| \lesssim E_1(\mu) + E_2(\mu) + \mu E_2(\mu) \lesssim E_1(\mu) + E_2(\mu)$$

by (3.2.64).

Since $0 \leq \mu < 1$,

$$E_1(\mu) + E_2(\mu) \lesssim E_1(\mu) + \left(1 - \frac{\mu^2}{2}\right) E_2(\mu) = E_3(\mu), \tag{3.2.65}$$

so

$$\left|E_3'(\mu)\right| \lesssim E_3(\mu). \tag{3.2.66}$$

Applying Grönwall's inequality,

$$E_1(\mu) + E_2(\mu) \lesssim E_3(\mu) \lesssim e^{\mu} E_3(0) \lesssim E_1(0) + E_2(0). \tag{3.2.67}$$

Therefore, by the definitions of $E_1(\mu)$ and $E(\mu)$, we have

$$\begin{aligned} &\int |\nabla_x u(x_1, x_2, x_3, vx_1)|^2 + |\partial_t u(x_1, x_2, x_3, vx_1)|^2 \, dx \\ &\lesssim \int |\nabla_x u(x_1, x_2, x_3, 0)|^2 + |\partial_t u(x_1, x_2, x_3, 0)|^2 \, dx. \end{aligned}$$

The theorem is proved. □

**Remark 3.2.10.** *The above theorem can be also obtained by local energy conservation and the control of the energy flux. And this approach will only require the potential to decay with* *rate* $\langle x\rangle^{-2}$*. See [12] for more details.*

Applying Theorem 5.3.6 in the setting of Theorem 4.8.2, we obtain a more general formulation of the local energy decay estimate.

**Corollary 3.2.11.** $\forall\epsilon > 0$ $|\vec{\mu}| < 1$*, one has*

$$\left\|(1+|x-\vec{\mu}t|)^{-\frac{1}{2}-\epsilon}\, e^{it\sqrt{-\Delta}}f\right\|_{L^2_{t,x}} \lesssim_\epsilon \|f\|_{L^2_x}. \tag{3.2.68}$$

As a by product of Theorem 5.3.6, we obtain Agmon's estimates [1] for the decay of eigenfunctions associated with negative eigenvalues. One can find a detailed proof in [12].

## 3.3 Estimates along Slanted Lines

In order to obtain reversed Strichartz estimates for wave equations with moving potentials, we need to understand the analogous estimates along slanted lines. With the results from subsection 4.3, we first consider the estimates along slanted lines for free wave equations. For the free evolution, the results can be obtained by explicit calculations with the Kirchhoff formula or the Fourier transforms, for example see the calculations in [12]. In this section, we will approach those estimates with a viewpoint of Lorentz transformations. The reason is that this approach will be more consistent with our construction later on.

### *3.3.1 Free wave equations*

First of all, we will consider

$$\partial_{tt}u - \Delta u = 0, \tag{3.3.1}$$

with initial data

$$u(x,0) = g,\ u_t(x,0) = f(x). \tag{3.3.2}$$

We can write

$$u(x,t) = \frac{\sin\left(t\sqrt{-\Delta}\right)}{\sqrt{-\Delta}} f + \cos\left(t\sqrt{-\Delta}\right) g. \tag{3.3.3}$$

By our preliminary discussions in Theorem 4.2.3, we know

$$\|u\|_{L_x^\infty L_t^2} \lesssim \|f\|_{L_x^2} + \|\nabla g\|_{L_x^2}. \tag{3.3.4}$$

We consider an analogous estimate to (3.3.4) along slanted lines. To be more concrete, we integrate $u^2$ along slanted lines

$$(x+vt,t) = (x_1+vt,\, x_2,\, x_3,\, t) \tag{3.3.5}$$

Denote

$$u^S(x,t) := u(x+vt,t), \tag{3.3.6}$$

we estimate

$$\left\|u^S\right\|_{L_x^\infty L_t^2}. \tag{3.3.7}$$

**Lemma 3.3.1.** *Let $|v| < 1$ and suppose $u$ solves*

$$\partial_{tt}u - \Delta u = 0 \tag{3.3.8}$$

*with initial data*

$$u(0) = g,\, u_t(0) = f. \tag{3.3.9}$$

*Then*

$$\left\|u^S\right\|_{L_x^\infty L_t^2} \lesssim \|f\|_{L^2} + \|g\|_{\dot{H}^1}. \tag{3.3.10}$$

*Proof.* Recall that performing the Lorentz transformation with respect to $v$, in the new frame, one has

$$u_L\left(x_1', x_2', x_3', t'\right) := u\left(\gamma\left(x_1' + vt'\right), x_2', x_3', \gamma\left(t' + vx_1'\right)\right) \tag{3.3.11}$$

and

$$\partial_{t't'} u_L - \Delta_{x'} u_L = 0. \tag{3.3.12}$$

Notice that from (3.3.11), to estimate the $L_x^\infty L_t^2$ norm of

$$u^S = u(x + vt, t), \tag{3.3.13}$$

is equivalent to integrating of $u_L$ along $t'$ up to a multiplication of an absolute constant only depending on $v$ and $\gamma$.

Therefore, by the endpoint reversed Strichartz estimate for $u_L$, we have

$$\left\|u^S\right\|_{L_x^\infty L_t^2} \lesssim \|u_L\|_{L_x^\infty L_t^2} \lesssim \|\partial_t u_L(0)\|_{L^2} + \|u_L(0)\|_{\dot{H}^1} \lesssim \|f\|_{L^2} + \|g\|_{\dot{H}^1}, \tag{3.3.14}$$

where in the last inequality, we apply Theorem 5.3.6 with $V \equiv 0$. □

### 3.3.2 Wave equations with stationary potentials

In this subsection, we consider the perturbed Hamiltonian,

$$H = -\Delta + V, \tag{3.3.15}$$

and the wave equation with potential,

$$\partial_{tt} u + Hu = 0 \tag{3.3.16}$$

with initial data

$$u(x,0)=g,\, u_t(x,0)=f.$$

The results in this section can always be obtained by the related estimates for the free case via the structure formula of wave operators, cf. [12]. But in order to make our exposition self-contained, we will prove all estimates independent of the structure formula.

For simplicity, from now on till the end of this section, we will assume $g=0$. For the other case, the analysis is similar with $L^2$ norm replaced by $\dot{H}^1$ norm.

**Theorem 3.3.2.** *Let $|v|<1$ and set*

$$u(x,t)=\frac{\sin\left(t\sqrt{H}\right)}{\sqrt{H}}P_cf. \tag{3.3.17}$$

*Denote*

$$u^S(x,t):=u(x+vt,t) \tag{3.3.18}$$

*then*

$$\left\|u^S\right\|_{L^\infty_x L^2_t}\lesssim \|P_cf\|_{L^2}\lesssim \|f\|_{L^2}\,. \tag{3.3.19}$$

*Proof.* By Duhamel's formula, we write

$$\begin{aligned} u(x,t) &= \frac{\sin\left(t\sqrt{-\Delta}\right)}{\sqrt{-\Delta}}P_cf-\int_0^t\frac{\sin\left((t-s)\sqrt{-\Delta}\right)}{\sqrt{-\Delta}}V\frac{\sin\left(s\sqrt{H}\right)}{\sqrt{H}}P_cf\,ds,\\ &=: A+B \end{aligned} \tag{3.3.20}$$

Now consider the estimate along slanted lines. The estimate for $A$ is known from the free evolution, Lemma 3.3.1. For the second term, we use the explicit representation of the free evolution $\frac{\sin(t\sqrt{-\Delta})}{\sqrt{-\Delta}}$.

Set

$$D(\cdot,t)=\int_0^t\frac{\sin\left((t-s)\sqrt{-\Delta}\right)}{\sqrt{-\Delta}}F(s)\,ds \tag{3.3.21}$$

along slanted lines. First of all, by our preliminary results, Theorem 4.2.3,

$$\|D\|_{L^\infty_x L^2_t} \lesssim \|F\|_{L^{\frac{3}{2},1}_x L^2_t}. \tag{3.3.22}$$

For the estimate along slanted lines, by Kirchhoff's formula, we know

$$D^S(x,t) := D(x+vt,t) = \int_0^t \int_{|x+vt-y|=t-s} \frac{F(y,s)}{|x+vt-y|}\,\sigma(dy)\,ds \tag{3.3.23}$$

and

$$\begin{aligned} \left\|D^S(x,\cdot)\right\|_{L^2_t} &= \left\|\int_0^t \int_{|x+vt-y|=t-s} \frac{F(y,s)}{|x+vt-y|}\,\sigma(dy)\,ds\right\|_{L^2_t} \\ &= \left\|\int_{|y|\le t} \frac{F(x+vt-y,t-|y|)}{|y|}\,dy\right\|_{L^2_t} \\ &\le \left\|\int_{\mathbb{R}^3} \frac{|F(x-y,t-|y+vt|)|}{|y+vt|}\,dy\right\|_{L^2_t} \\ &\le \left\|\int_{\mathbb{R}^3} \frac{|F(x-y,t-|y+vt|)|}{\sqrt{y_2^2+y_3^2}}\,dy\right\|_{L^2_t}, \end{aligned} \tag{3.3.24}$$

where in the third line, we use a change of variable and for the last inequality and reduce the norm of $y$ to the norm of the component of $y$ orthogonal to the direction of the motion.

Finally,

$$\left\|\int_{\mathbb{R}^3} \frac{F(x-y,t-|y+vt|)}{\sqrt{y_2^2+y_3^2}}\,dy\right\|_{L^2_t} \le \int_{\mathbb{R}^3} \frac{\|F(x-y,t-|y+vt|)\|_{L^2_t}}{\sqrt{y_2^2+y_3^2}}\,dy \tag{3.3.25}$$

For fixed $y$, if we apply a change of variable of $t$ here, the Jacobian is bounded by $1-|v|$

and $1+|v|$, so

$$\int_{\mathbb{R}^3} \frac{\|F(x-y,t-|y+vt|)\|_{L_t^2}}{\sqrt{y_2^2+y_3^2}}\,dy \lesssim \int_{\mathbb{R}^3} \frac{\|F(x-y,\cdot)\|_{L_t^2}}{\sqrt{y_2^2+y_3^2}}\,dy \lesssim \|F\|_{L^1_{x_1} L^{2,1}_{\widehat{x_1}} L^2_t} \tag{3.3.26}$$

where $\widehat{x_1}$ denotes the subspace orthogonal to $x_1$ (more generally, the subspace orthogonal to the direction of the motion). Here $L^{2,1}$ is the Lorentz norm and the last inequality follows from Hölder's inequality of Lorentz spaces. Therefore,

$$\left\|D^S\right\|_{L^\infty_x L^2_t} \lesssim \|F\|_{L^1_{x_1} L^{2,1}_{\widehat{x_1}} L^2_t}. \tag{3.3.27}$$

By a similar discussion as the estimate (3.2.29), we also have for $T>0$,

$$\left\|D^S\right\|_{L^\infty_x L^2_t[0,T]} \lesssim \|F\|_{L^1_{x_1} L^{2,1}_{\widehat{x_1}} L^2_t[0,T]}. \tag{3.3.28}$$

With estimate (3.3.27), we know for $u^S(x,t) := u(x+vt,t)$,

$$\begin{aligned}
\left\|u^S\right\|_{L^\infty_x L^2_t} &\lesssim \|P_c f\|_{L^2} + \left\| V \frac{\sin\left(t\sqrt{H}\right)}{\sqrt{H}} P_c f \right\|_{L^1_{x_1} L^{2,1}_{\widehat{x_1}} L^2_t} \\
&\lesssim \|P_c f\|_{L^2} + \|V\|_{L^1_{x_1} L^{2,1}_{\widehat{x_1}}} \left\| \frac{\sin\left(t\sqrt{H}\right)}{\sqrt{H}} P_c f \right\|_{L^\infty_x L^2_t} \\
&\lesssim \|P_c f\|_{L^2} + \|V\|_{L^1_{x_1} L^{2,1}_{\widehat{x_1}}} \|P_c f\|_{L^2_x} \\
&\lesssim \|f\|_{L^2}.
\end{aligned} \tag{3.3.29}$$

where in the third line, we use the endpoint reversed Strichartz estimate of the wave equation with potentials as Theorem 4.2.4.

Therefore,

$$\|u\|_{L_x^\infty L_t^2} \lesssim \|P_c f\|_{L_x^2} \lesssim \|f\|_{L_x^2} \tag{3.3.30}$$

$$\left\|u^S\right\|_{L_x^\infty L_t^2} \lesssim \|P_c f\|_{L_x^2} \lesssim \|f\|_{L_x^2} \tag{3.3.31}$$

as claimed. □

As a byproduct, we have the following inhomogeneous estimates from (3.3.27) and (3.3.28).

**Corollary 3.3.3.** *For $|v| < 1$ we have*

$$\left\|\int_0^t \int_{|x+vt-y|=t-s} \frac{F(y,s)}{|x+vt-y|}\,\sigma(dy)ds\right\|_{L_x^\infty L_t^2} \lesssim \|F\|_{L_{x_1}^1 L_{\widehat{x_1}}^{2,1} L_t^2}, \tag{3.3.32}$$

*and for $T > 0$,*

$$\left\|\int_0^t \int_{|x+vt-y|=t-s} \frac{F(y,s)}{|x+vt-y|}\,\sigma(dy)ds\right\|_{L_x^\infty L_t^2[0,T]} \lesssim \|F\|_{L_{x_1}^1 L_{\widehat{x_1}}^{2,1} L_t^2[0,T]}. \tag{3.3.33}$$

From the discussion above, we can also obtain the following truncated versions of inhomogeneous estimates which are crucial in our later bootstrap arguments.

**Corollary 3.3.4.** *Suppose $A > 0$ and $|v| < 1$, then*

$$\sup_x \left\|\int_0^{t-A} \frac{\sin\left((t-s)\sqrt{-\Delta}\right)}{\sqrt{-\Delta}}\,F ds\right\|_{L_t^2[A,\infty)} \lesssim \frac{1}{A}\|F\|_{L_x^1 L_t^2}. \tag{3.3.34}$$

*and for $T > 0$,*

$$\sup_x \left\|\int_0^{t-A} \frac{\sin\left((t-s)\sqrt{-\Delta}\right)}{\sqrt{-\Delta}}\,F ds\right\|_{L_t^2[A,T]} \lesssim \frac{1}{A}\|F\|_{L_x^1 L_t^2[0,T]}. \tag{3.3.35}$$

*Similarly,*

$$\sup_x \left\| \int_0^{t-A} \int_{|x+vt-y|=t-s} \frac{F(y,s)}{|x+vt-y|}\,\sigma(dy)ds \right\|_{L_t^2[A,\infty)} \lesssim \frac{1}{A}\,\|F\|_{L_x^1 L_t^2}\,, \tag{3.3.36}$$

and for $T > 0$

$$\sup_x \left\| \int_0^{t-A} \int_{|x+vt-y|=t-s} \frac{F(y,s)}{|x+vt-y|}\,\sigma(dy)ds \right\|_{L_t^2[A,T]} \lesssim \frac{1}{A}\,\|F\|_{L_x^1 L_t^2[0,T]}\,. \tag{3.3.37}$$

*Proof.* By a similar discussion above with Kirchhoff 's formula,

$$\begin{aligned} \left\| \int_0^{t-A} \frac{\sin\left((t-s)\sqrt{-\Delta}\right)}{\sqrt{-\Delta}} F(s)\,ds \right\|_{L_t^2[A,\infty)} &= \left\| \int_{A\le|y|\le t} \frac{F(x-y,t-|y|)}{|y|}\,dy \right\|_{L_t^2[A,\infty)} \\ &\lesssim \int_{A\le|y|} \frac{\|F(x-y,t-|y|)\|_{L_t^2}}{|y|}\,dy \\ &\lesssim \frac{1}{A}\,\|F\|_{L_x^1 L_t^2}\,. \end{aligned} \tag{3.3.38}$$

Therefore,

$$\left\| \int_0^{t-A} \frac{\sin\left((t-s)\sqrt{-\Delta}\right)}{\sqrt{-\Delta}} F(s)\,ds \right\|_{L_x^\infty L_t^2[A,\infty)} \lesssim \frac{1}{A}\,\|F\|_{L_x^1 L_t^2}\,. \tag{3.3.39}$$

With the same argument as (3.3.28), we also have

$$\left\| \int_0^{t-A} \frac{\sin\left((t-s)\sqrt{-\Delta}\right)}{\sqrt{-\Delta}}\, F(s)\,ds \right\|_{L_t^2[A,T]} \lesssim \frac{1}{A}\,\|F\|_{L_x^1 L_t^2[0,T]}\,. \tag{3.3.40}$$

Similarly to the way we derive estimates (3.3.32) and (3.3.33), one obtains

$$\left\| \int_0^{t-A} \int_{|x+vt-y|=t-s} \frac{F(y,s)}{|x+vt-y|}\,\sigma(dy)ds \right\|_{L_x^\infty L_t^2[A,\infty)} \lesssim \frac{1}{A}\,\|F\|_{L_x^1 L_t^2}\,, \tag{3.3.41}$$

$$\left\| \int_0^{t-A} \int_{|x+vt-y|=t-s} \frac{F(y,s)}{|x+vt-y|}\, \sigma(dy)ds \right\|_{L_x^\infty L_t^2[A,T]} \lesssim \frac{1}{A} \|F\|_{L_x^1 L_t^2[0,T]}\,. \tag{3.3.42}$$

We are done. □

Next, we consider estimates in inhomogeneous forms for the perturbed evolution along slanted lines. In the following proofs, essentially, we pass the effects caused by the integration along slanted lines to the free evolution by Duhamel expansion and use the standard case for the perturbed evolution.

Define

$$k(\cdot,t) := \int_0^t \frac{\sin\left((t-s)\sqrt{H}\right)}{\sqrt{H}} P_c F(s)\, ds. \tag{3.3.43}$$

Then from the endpoint reversed Strichartz estimate, Theorem 4.2.4, we have

$$\|k\|_{L_x^\infty L_t^2} \lesssim \|F\|_{L_x^{\frac{3}{2},1} L_t^2}\,. \tag{3.3.44}$$

**Theorem 3.3.5.** *Let $|v|<1$ and suppose $H=-\Delta+V$ has neither resonances nor eigenfunctions at $0$. Define*

$$k^S(x,t) = k(x+vt,t). \tag{3.3.45}$$

*Then we have*

$$\left\|k^S\right\|_{L_x^\infty L_t^2} \lesssim \|F\|_{L_{x_1}^1 L_{\widehat{x_1}}^{2,1} L_t^2} + \|F\|_{L_x^{\frac{3}{2},1} L_t^2}\,, \tag{3.3.46}$$

*and for $T>0$,*

$$\left\|k^S\right\|_{L_x^\infty L_t^2[0,T]} \lesssim \|F\|_{L_{x_1}^1 L_{\widehat{x_1}}^{2,1} L_t^2[0,T]} + \|F\|_{L_x^{\frac{3}{2},1} L_t^2[0,T]}\,, \tag{3.3.47}$$

*where $\widehat{x_1}$ is the subspace orthogonal to to $\vec{e}_1$.*

*Proof.* By Duhamel's formula, we write

$$\frac{\sin\left((t-s)\sqrt{H}\right)}{\sqrt{H}}P_cF(s) = \frac{\sin\left((t-s)\sqrt{-\Delta}\right)}{\sqrt{-\Delta}}P_cF(s) - \int_s^t \frac{\sin\left((t-m)\sqrt{-\Delta}\right)}{\sqrt{-\Delta}}V\frac{\sin\left((m-s)\sqrt{H}\right)}{\sqrt{H}}P_cF(s)\,dm. \quad (3.3.48)$$

Therefore,

$$\int_0^t \frac{\sin\left((t-s)\sqrt{H}\right)}{\sqrt{H}}P_cF\,ds = \int_0^t \frac{\sin\left((t-s)\sqrt{-\Delta}\right)}{\sqrt{-\Delta}}F(s)\,ds - \int_0^t\int_s^t \frac{\sin\left((t-m)\sqrt{-\Delta}\right)}{\sqrt{-\Delta}}V\frac{\sin\left((m-s)\sqrt{H}\right)}{\sqrt{H}}P_cF(s)\,dmds.$$

Denote

$$R(x,t) := \int_0^t\int_s^t \frac{\sin\left((t-m)\sqrt{-\Delta}\right)}{\sqrt{-\Delta}}V\frac{\sin\left((m-s)\sqrt{H}\right)}{\sqrt{H}}P_cF(s)\,dmds \quad (3.3.49)$$

and

$$R^S(x,t) := R(x+vt,t). \quad (3.3.50)$$

Then

$$\left\|k^S\right\|_{L^\infty_x L^2_t} \lesssim \left\|D^S\right\|_{L^\infty_x L^2_t} + \left\|R^S\right\|_{L^\infty_x L^2_t}, \quad (3.3.51)$$

where

$$D^S(x,t) = D(x+vt,t) = \int_0^t\int_{|x+vt-y|=t-s}\frac{F(y,s)}{|x+vt-y|}\,dyds. \quad (3.3.52)$$

From Corollary 3.3.3, we know

$$\left\|D^S\right\|_{L^\infty_x L^2_t} \lesssim \|F\|_{L^1_{x_1}L^{2,1}_{\widehat{x_1}}L^2_t}. \quad (3.3.53)$$

To estimate

$$\left\| \int_0^t \int_s^t \frac{\sin\left((t-k)\sqrt{-\Delta}\right)}{\sqrt{-\Delta}} V \frac{\sin\left((k-s)\sqrt{H}\right)}{\sqrt{H}} P_c F(s)\, dkds \right\|_{L_t^2}, \tag{3.3.54}$$

we notice that with an exchange of the order of integration,

$$\begin{aligned} R(x,t) &= \int_0^t \int_s^t \frac{\sin\left((t-k)\sqrt{-\Delta}\right)}{\sqrt{-\Delta}} V \frac{\sin\left((k-s)\sqrt{H}\right)}{\sqrt{H}} P_c F(s)\, dkds \\ &= \int_0^t \frac{\sin\left((t-k)\sqrt{-\Delta}\right)}{\sqrt{-\Delta}} \left( \int_0^k V \frac{\sin\left((k-s)\sqrt{H}\right)}{\sqrt{H}} P_c F(s)\, ds \right) dk. \end{aligned} \tag{3.3.55}$$

Then applying our estimate for the free evolution estimate in the inhomogeneous case, Corollary 3.3.3, we have

$$\begin{aligned} \left\| R^S(x,t) \right\|_{L_x^\infty L_t^2} &\lesssim \left\| \int_0^t V \frac{\sin\left((t-s)\sqrt{H}\right)}{\sqrt{H}} P_c F(s)\, ds \right\|_{L_{x_1}^1 L_{\widehat{x_1}}^{2,1} L_t^2} \\ &\lesssim \|V\|_{L_{x_1}^1 L_{\widehat{x_1}}^{2,1}} \left\| \int_0^t \frac{\sin\left((t-s)\sqrt{H}\right)}{\sqrt{H}} P_c F(s)\, ds \right\|_{L_x^\infty L_t^2} \\ &\lesssim \|F\|_{L_x^{\frac{3}{2},1} L_t^2} \end{aligned} \tag{3.3.56}$$

where in the third inequality, we use the endpoint reversed Strichartz estimate (3.3.44).

Therefore, we conclude that

$$\begin{aligned} \left\| k^S \right\|_{L_x^\infty L_t^2} &\lesssim \left\| D^S \right\|_{L_x^\infty L_t^2} + \left\| R^S \right\|_{L_x^\infty L_T^2} \\ &\lesssim \|F\|_{L_{x_1}^1 L_{\widehat{x_1}}^{2,1} L_t^2} + \|F\|_{L_x^{\frac{3}{2},1} L_t^2}. \end{aligned} \tag{3.3.57}$$

When we restrict to $[0,T]$, as above, we can obtain

$$\left\|k^S\right\|_{L_x^\infty L_t^2[0,T]} \lesssim \|F\|_{L_{x_1}^1 L_{\widehat{x_1}}^{2,1} L_t^2[0,T]} + \|F\|_{L_x^{\frac{3}{2},1} L_t^2[0,T]}. \tag{3.3.58}$$

The lemma is proved. □

To prepare our bootstrap arguments in the later section, similarly to the free case, we also consider the truncated versions of the above estimates.

By the same method we used to estimate

$$\int_0^t \frac{\sin\left((t-s)\sqrt{H}\right)}{\sqrt{H}} P_c F(s)\, ds \tag{3.3.59}$$

along slanted lines, we obtain the following:

**Corollary 3.3.6.** *For $|v|<1$ and $A>0$, suppose $H=-\Delta+V$ has neither resonances nor eigenfunctions at 0. Let*

$$k_A(\cdot,t) := \int_0^{t-A} \frac{\sin\left((t-s)\sqrt{H}\right)}{\sqrt{H}} P_c F(s)\, ds. \tag{3.3.60}$$

*Then*

$$\|k_A\|_{L_x^\infty L_t^2[A,\infty)} \lesssim \frac{1}{A}\left(\|F\|_{L_x^1 L_t^2} + \|F\|_{L_x^{\frac{3}{2},1} L_t^2}\right), \tag{3.3.61}$$

*and for $T>0$,*

$$\|k_A\|_{L_x^\infty L_t^2[A,T]} \lesssim \frac{1}{A}\left(\|F\|_{L_x^1 L_t^2[0,T]} + \|F\|_{L_x^{\frac{3}{2},1} L_t^2[0,T]}\right). \tag{3.3.62}$$

*Define*

$$k_A^S(x,t) := k_A(x+vt,t). \tag{3.3.63}$$

*then*

$$\left\| k_A^S(x,t) \right\|_{L_x^\infty L_t^2[A,\infty)} \lesssim \frac{1}{A} \left( \|F\|_{L_x^{\frac{3}{2},1} L_t^2} + \|F\|_{L_x^1 L_t^2} \right). \tag{3.3.64}$$

*and for* $T > 0$*,*

$$\left\| k_A^S(x,t) \right\|_{L_x^\infty L_t^2[A,T]} \lesssim \frac{1}{A} \left( \|F\|_{L_x^{\frac{3}{2},1} L_t^2[0,T]} + \|F\|_{L_x^1 L_t^2[0,T]} \right). \tag{3.3.65}$$

Finally, in order to handle moving potentials, we consider some estimates with inhomogeneous terms along slanted lines:

Setting

$$F^S(x,t) = F\left(x + vt, t\right) \tag{3.3.66}$$

we have the following results.

**Lemma 3.3.7.** *Let* $A > 0$ *and* $|\vec{\mu}| < 1$*,* $|v| < 1$*. Suppose*

$$D_A(x,t) := \int_0^{t-A} \frac{\sin\left((t-s)\sqrt{-\Delta}\right)}{\sqrt{-\Delta}} F(s)\, ds, \tag{3.3.67}$$

$$D_A^S(x,t) = g_A(x + \vec{\mu}t, t). \tag{3.3.68}$$

*We have*

$$\|D_A(x,t)\|_{L_x^\infty L_t^2[A,\infty)} \lesssim \frac{1}{A} \left\| F^S \right\|_{L_x^1 L_t^2}, \tag{3.3.69}$$

$$\left\| D_A^S(x,t) \right\|_{L_x^\infty L_t^2[A,\infty)} \lesssim \frac{1}{A} \left\| F^S \right\|_{L_x^1 L_t^2}, \tag{3.3.70}$$

*and for* $T > 0$*,*

$$\|D_A(x,t)\|_{L_x^\infty L_t^2[A,T]} \lesssim \frac{1}{A} \left\| F^S \right\|_{L_x^1 L_t^2[0,T]}, \tag{3.3.71}$$

$$\left\| D_A^S(x,t) \right\|_{L_x^\infty L_t^2[A,T]} \lesssim \frac{1}{A} \left\| F^S \right\|_{L_x^1 L_t^2[0,T]}. \tag{3.3.72}$$

*Proof.* We know explicitly,

$$D_A(x,t) = \int_0^{t-A} \int_{|x-y|=t-s} \frac{F(y,s)}{|x-y|}\, dyds. \tag{3.3.73}$$

Taking $z = y - sv$, we have

$$\begin{aligned}
|D_A(x,t)| &= \left| \int_0^{t-A} \int_{|x-y|=t-s} \frac{F(y,s)}{|x-y|}\, dyds \right| \\
&= \left| \int_0^{t-A} \int_{|x-z-vs|=t-s} \frac{F^S(z,s)}{|x-z-vs|}\, dzds \right| \\
&\lesssim \int_0^{t-A} \int_{|m|=t-s} \frac{\left|F^S(x-vs-m,s)\right|}{|m|}\, dmds \qquad (3.3.74) \\
&\lesssim \int_0^{t-A} \int_{|m|=t-s} \frac{\left|F^S(x-v\,(t-|m|)-m, t-|m|)\right|}{|m|}\, dmds \\
&\lesssim \frac{1}{A} \int \left| F^S(x-v\,(t-|m|)-m, t-|m|) \right| dm
\end{aligned}$$

In the third line above, we apply a change of variable $m = x - z - vs$ and in the fifth line, we again apply a change of variable $v\,|m| + m = h$ with bounded Jabobian.

Therefore, if we set $q = v\,(t - |m|) + m$, we have

$$\begin{aligned}
\|D(x,\cdot)\|_{L_t^2[A,\infty)} &\lesssim \frac{1}{A} \int \left\| F^S(x-q,\cdot) \right\|_{L_t^2} dq \\
&\lesssim \frac{1}{A} \left\| F^S \right\|_{L_x^1 L_t^2} \qquad (3.3.75)
\end{aligned}$$

The estimate for $\left\| D_A^S(x,t) \right\|_{L_x^\infty L_t^2}$ is the same as we did for Corollary 3.3.4. Hence we obtain

$$\left\| D_A^S(x,t) \right\|_{L_x^\infty L_t^2[A,\infty)} \lesssim \frac{1}{A} \left\| F^S \right\|_{L_x^1 L_t^2}. \tag{3.3.76}$$

The the same as above, when we restrict to $[0,T]$, one has

$$\|D_A(x,t)\|_{L_x^\infty L_t^2[A,T]} \lesssim \frac{1}{A}\left\|F^S\right\|_{L_x^1 L_t^2[0,T]}, \tag{3.3.77}$$

$$\left\|D_A^S(x,t)\right\|_{L_x^\infty L_t^2[A,T]} \lesssim \frac{1}{A}\left\|F^S\right\|_{L_x^1 L_t^2[0,T]}. \tag{3.3.78}$$

The lemma is proved. □

The above lemma can also be established by a duality argument. For the sake of completeness, we sketch the argument here. We only focus on

$$\|D_A(x,t)\|_{L_x^\infty L_t^2[A,T]} \lesssim \frac{1}{A}\left\|F^S\right\|_{L_x^1 L_t^2}.$$

Testing a function $H(x,t) \in L_x^1 L_t^2$, one has

$$\begin{aligned}
\int_{\mathbb{R}^3}\int_A^T H(x,t)D_A(x,t)\,dtdx &= \int_{\mathbb{R}^3}\int_A^T H(x,t)\int_0^{t-A}\int_{|x-y|=t-s}\frac{F(y,s)}{|x-y|}\,\sigma(dy)\,dsdtdx \\
&= \int_{\mathbb{R}^3}\int_0^{T-A} F(y,s)\int_{s+A}^T\int_{|x-y|=t-s}\frac{H(y,t)}{|x-z-vs|}\sigma(dx)\,dtdsdy \\
&= \int_{\mathbb{R}^3}\int_0^{T-A} F^S(z,s)\int_{s+A}^T\int_{|x-y|=t-s}\frac{H(y,t)}{|x-z-vs|}\sigma(dx)\,dtdsdy.
\end{aligned}$$

Then it suffices to show

$$\left\|\int_{s+A}^T\int_{|x-y|=t-s}\frac{H(y,t)}{|x-z-vs|}\sigma(dx)\,dt\right\|_{L_z^\infty L_s^2[0,T-A]} \lesssim \frac{1}{A}\|H\|_{L_x^1 L_t^2[0,T]}. \tag{3.3.79}$$

But with an almost identical argument as Corollary 3.3.4, the estimate (3.3.79) indeed holds, and therefore, our desired estimate holds too.

By [3] or applying the structure formula of wave operators, with the calculations in the proof of Lemma 3.3.7, Corollary 3.3.6 and Theorems 3.3.2, 3.3.5, we have the perturbed version of the estimates (3.3.69) and (3.3.70). We omit the details here since the calculations are more or less identical.

**Theorem 3.3.8.** *For $|\vec{\mu}| < 1, |v| < 1$ and $A > 0$, suppose $H = -\Delta + V$ has neither resonances nor eigenfunctions at 0. Define*

$$k_A(x,t) := \int_0^{t-A} \frac{\sin\left((t-s)\sqrt{H}\right)}{\sqrt{H}} P_c F(s)\, ds \tag{3.3.80}$$

$$k_A^S(x,t) = k_A\left(x + \vec{\mu}t, t\right). \tag{3.3.81}$$

*We have*

$$\|k_A(x,t)\|_{L_x^\infty L_t^2[A,\infty)} \lesssim \frac{1}{A}\left(\left\|F^S\right\|_{L_x^1 L_t^2} + \left\|F^S\right\|_{L_x^{\frac{3}{2},1} L_t^2}\right), \tag{3.3.82}$$

$$\left\|k_A^S(x,t)\right\|_{L_x^\infty L_t^2[A,\infty)} \lesssim \frac{1}{A}\left(\left\|F^S\right\|_{L_x^1 L_t^2} + \left\|F^S\right\|_{L_x^{\frac{3}{2},1} L_t^2}\right), \tag{3.3.83}$$

*and for $T > 0$,*

$$\|k_A(x,t)\|_{L_x^\infty L_t^2[A,T]} \lesssim \frac{1}{A}\left(\left\|F^S\right\|_{L_x^1 L_t^2[0,T]} + \left\|F^S\right\|_{L_x^{\frac{3}{2},1} L_t^2[0,T]}\right), \tag{3.3.84}$$

$$\left\|k_A^S(x,t)\right\|_{L_x^\infty L_t^2[A,T]} \lesssim \frac{1}{A}\left(\left\|F^S\right\|_{L_x^1 L_t^2[0,T]} + \left\|F^S\right\|_{L_x^{\frac{3}{2},1} L_t^2[0,T]}\right). \tag{3.3.85}$$

By careful analysis and more complicated computations, one can extend all the results above to the linear Klein-Gordon equation, cf. [13].

## 3.4 Endpoint Reversed Strichartz Estimates

In this section, we show the endpoint reversed Strichartz estimates for the wave equation with charge transfer Hamiltonian. More precisely, we consider

$$\partial_{tt}u - \Delta u + V_1(x)u + V_2(x - vt)u = 0 \tag{3.4.1}$$

with initial data

$$u(x,0) = g(x),\ u_t(x,0) = f(x).$$

Throughout this section, for simplicity, we furthermore assume $V_i$ is compactly supported. With a little bit more careful analysis, one can easily obtain the same results for general case, see Remark 3.4.6.

Recall that after we apply the associated Lorentz transformation $L$, under the new coordinate, we denote

$$u_L\left(x_1', x_2', x_3', t'\right) = u\left(\gamma\left(x_1' + vt'\right), x_2', x_3', \gamma\left(t' + vx_1'\right)\right), \tag{3.4.2}$$

and with the inverse transformation $L^{-1}$

$$u(x,t) = u_L\left(\gamma\left(x - vt\right), x_2, x_3, \gamma\left(t - vx_1\right)\right). \tag{3.4.3}$$

Under the above setting, we state the main result of this section.

**Theorem 3.4.1.** *Let $|v| < 1$. Suppose $u$ is a scattering state in the sense of Definition 5.3.2 and solves*

$$\partial_{tt}u - \Delta u + V_1(x)u + V_2(x - vt)u = 0 \tag{3.4.4}$$

*with initial data*

$$u(x,0) = g(x),\ u_t(x,0) = f(x). \tag{3.4.5}$$

*Then*

$$\sup_{x\in\mathbb{R}^3} \int_0^\infty |u(x,t)|^2\, dt \lesssim \left(\|f\|_{L^2} + \|g\|_{\dot{H}^1}\right)^2. \tag{3.4.6}$$

*Furthermore, if we denote*

$$u^S(x,t) := u(x + vt, t), \tag{3.4.7}$$

*then*

$$\sup_{x\in\mathbb{R}^3}\int_0^\infty \left|u^S(x,t)\right|^2 dt \lesssim \left(\|f\|_{L^2}+\|g\|_{\dot{H}^1}\right)^2. \tag{3.4.8}$$

To show Theorem 3.4.1, we will apply a bootstrap process and decomposition into channels in the spirit of [51]. If there are no bound states, the bootstrap arguments simply work for the entire evolution. But in the presence of bound states, a more careful analysis is necessary. We will construct a truncated evolution and show that the estimates we obtain are independent of the truncation. Finally, we pass our estimates to the entire evolution.

## *3.4.1 Bootstrap argument*

We set up the bootstrap argument and prove the initial assumptions for the bootstrap argument hold for big $T$ with some positive constants.

By Duhamel's formula,

$$u(x,t)=\frac{\sin\left(t\sqrt{-\Delta}\right)}{\sqrt{-\Delta}}f+\cos\left(t\sqrt{-\Delta}\right)g-\int_0^t\frac{\sin\left((t-s)\sqrt{-\Delta}\right)}{\sqrt{-\Delta}}\left(V_1+V_2(\cdot-vs)\right)u(s)\,ds. \tag{3.4.9}$$

By Grönwall's inequality, the endpoint reversed Strichartz estimates and the estimate along slanted lines for the free evolution, we have the following estimates as bootstrap assumptions.

**Lemma 3.4.2.** *For $T>0$ large, there exist constants $C_1(T)$ and $C_2(T)$ such that*

$$\sup_{x\in\mathbb{R}^3}\int_0^T |u(x,t)|^2 dt \le C_1(T)\left(\|f\|_{L^2}+\|g\|_{\dot{H}^1}\right)^2 \tag{3.4.10}$$

*and if we denote*

$$u^S(x,t)=u(x+vt,t),$$

*then*

$$\sup_{x\in\mathbb{R}^3}\int_0^T \left|u^S(x,t)\right|^2 dt \le C_2(T)\left(\|f\|_{L^2}+\|g\|_{\dot{H}^1}\right)^2. \tag{3.4.11}$$

*Proof.* To establish the bootstrap assumptions, we first notice that by the expression (3.4.9) and Grönwall inequality, we have

$$\int_{\mathbb{R}^3} |\nabla u(x,t)|^2 + |\partial_t u(x,t)|^2 \, dx \lesssim e^{C|t|} \left( \|f\|_{L^2} + \|g\|_{\dot{H}^1} \right)^2. \tag{3.4.12}$$

Clearly, estimates (3.4.10) and (3.4.11) hold for $T = 0$. Next, we note that for arbitrary $T_0 > 0$, from Theorem 4.2.3,

$$\sup_{x\in\mathbb{R}^3} \int_0^{T_0} |u(x,t)|^2 \, dt \lesssim \left( \|f\|_{L^2} + \|g\|_{\dot{H}^1} \right)^2 + C(T_0) \left( \sup_{x\in\mathbb{R}^3} \int_0^{T_0} |u(x,t)|^2 \, dt + \sup_{x\in\mathbb{R}^3} \int_0^{T_0} \left| u^S(x,t) \right|^2 dt \right) \tag{3.4.13}$$

where $C(T_0)$ can be computed explicitly, see Theorem 4.2.3 and duality argument as Lemma 3.3.7:

$$C(T_0) = \sup_{x\in\mathbb{R}^3} \int_{|x-y|\le T_0} \frac{1}{|x-y|} |V_1| \, dy + \int_{|\hat{y}_1 - \hat{x}_1| \le T_0} \int |V_2| \, dy_1 d\hat{y}_1.$$

We can perform a similar estimate for $\sup_{x\in\mathbb{R}^3} \int_0^{T_0} \left| u^S(x,t) \right|^2 dt$.

Therefore, for $T_0$ small enough,

$$\sup_{x\in\mathbb{R}^3} \int_0^{T_0} |u(x,t)|^2 \, dt \le C(T_0) \left( \|f\|_{L^2} + \|g\|_{\dot{H}^1} \right)^2. \tag{3.4.14}$$

$$\sup_{x\in\mathbb{R}^3} \int_0^{T_0} \left| u^S(x,t) \right|^2 dt \le C(T_0) \left( \|f\|_{L^2} + \|g\|_{\dot{H}^1} \right)^2. \tag{3.4.15}$$

Iterating the above construction with the energy growth estimate (3.4.12), we can obtain that for $T > 0$ large, there exists constant $C_1(T)$, $C_2(T)$ such that

$$\sup_{x\in\mathbb{R}^3} \int_0^{T} |u(x,t)|^2 \, dt \le C_1(T) \left( \|f\|_{L^2} + \|g\|_{\dot{H}^1} \right)^2, \tag{3.4.16}$$

$$\sup_{x\in\mathbb{R}^3}\int_0^T \left|u^S(x,t)\right|^2 dt \le C_2(T)\left(\|f\|_{L^2}+\|g\|_{\dot{H}^1}\right)^2, \tag{3.4.17}$$

as claimed. □

Based on estimates (3.4.10), (3.4.11), we will run a bootstrap argument to improve these two estimate and reduce to estimates with constants independent of $T$.

We also have a perturbed version of Lemma 3.4.2 with the same constants in estimates (3.4.10) and (3.4.11) up to multiplication of a constant only depending on the potentials. Let

$$H_i = -\Delta + V_i,\ i = 1, 2$$

and $P_c(H_i)$ to be the projection onto the continuous spectrum of $H_i$.

**Lemma 3.4.3.** *For $T > 0$ large, there exist constants $C_1(T)$ and $C_2(T)$ such that*

$$\sup_{x\in\mathbb{R}^3}\int_0^T |P_c(H_1)u(x,t)|^2 dt \lesssim_{V_1} C_1(T)\left(\|f\|_{L^2}+\|g\|_{\dot{H}^1}\right)^2, \tag{3.4.18}$$

$$\sup_{x\in\mathbb{R}^3}\int_0^T |P_c(H_2)u_L(x,t)|^2 dt \lesssim_{V_2} C_1(T)\left(\|f\|_{L^2}+\|g\|_{\dot{H}^1}\right)^2 \tag{3.4.19}$$

**Remark 3.4.4.** *By symmetry, with $C_1(T)$ and $C_2(T)$, we also have with $T > 0$,*

$$\sup_{x\in\mathbb{R}^3}\int_{-T}^0 |u(x,t)|^2 dt \le C_1(T)\left(\|f\|_{L^2}+\|g\|_{\dot{H}^1}\right)^2 \tag{3.4.20}$$

*and*

$$\sup_{x\in\mathbb{R}^3}\int_{-T}^0 \left|u^S(x,t)\right|^2 dt \le C_2(T)\left(\|f\|_{L^2}+\|g\|_{\dot{H}^1}\right)^2. \tag{3.4.21}$$

### *3.4.2 Bound states*

Before we start the bootstrap analysis, it is necessary to understand the evolution of bound states.

In the following, for simplicity, we assume $H_i - \Delta + V_i$, $i = 1, 2$ has only one negative eigenvalue. With $\lambda > 0,\ \mu > 0$,

$$H_1 w = -\lambda^2 w,\ H_2 m = -\mu^2 m. \tag{3.4.22}$$

$w$ and $m$ decay exponentially by Agmon's estimate. The analysis can be easily adapted to the most general situation.

Set $U(t,s)$ as evolution from $s$ to $t$ associated to the initial velocity and formally, we use $\dot{U}(t,s)$ to denote the evolution associated the other initial data.

Suppose $u(x,t)$ is a scattering state. We decompose the evolution as following,

$$u(x,t) = U(t,0)f + \dot{U}(t,0)g = a(t)w(x) + b\left(\gamma(t - vx_1)\right) m_v\left(x,t\right) + r(x,t) \tag{3.4.23}$$

where

$$m_v(x,t) = m\left(\gamma\left(x_1 - vt\right), x_2, x_3\right).$$

With our decomposition, we know

$$P_c\left(H_1\right) r = r \tag{3.4.24}$$

and

$$P_c\left(H_2\right) r_L = r_L \tag{3.4.25}$$

where the Lorentz transformation $L$ makes $V_2$ stationary.

Surely, since $u(x,t)$ is asymptotically orthogonal to the bound states of $H_1$ and $H_2$, it forces $a(t)$ to go 0 and $b(t)$ go to 0. Following the above construction, we do some preliminary calculations.

Plugging the evolution (5.2.135) into the equation (5.3.13) and taking inner product with

$w$, we get

$$\ddot{a}(t) - \lambda^2 a(t) + a(t) \langle V_2 (x - vt) w, w \rangle$$
$$+ \langle V_2 (x - vt) (b (\gamma(t - vx_1)) m_v (x, t) + r(x, t)), w \rangle = 0. \tag{3.4.26}$$

One can write

$$\ddot{a}(t) - \lambda^2 a(t) + a(t)c(t) + h(t) = 0, \tag{3.4.27}$$

where

$$c(t) := \langle V_2 (x - vt) w, w \rangle \tag{3.4.28}$$

and

$$h(t) := \langle V_2 (x - vt) (b (\gamma(t - vx_1)) m_v (x, t) + r(x, t)), w \rangle . \tag{3.4.29}$$

Since $w$ is exponentially localized by Agmon's estimate, we know

$$|c(t)| \lesssim e^{-\alpha|t|}. \tag{3.4.30}$$

The existence of the solution to the ODE (4.5.17) is clear. We study the long-time behavior of the solution. Write the equation as

$$\ddot{a}(t) - \lambda^2 a(t) = - [a(t)c(t) + h(t)], \tag{3.4.31}$$

and denote

$$N(t) := - [a(t)c(t) + h(t)]. \tag{3.4.32}$$

Then

$$a(t) = \frac{e^{\lambda t}}{2} \left[ a(0) + \frac{1}{\lambda}\dot{a}(0) + \frac{1}{\lambda} \int_0^t e^{-\lambda s} N(s)\, ds \right] + R(t) \tag{3.4.33}$$

where

$$|R(t)| \lesssim e^{-\beta t}, \tag{3.4.34}$$

for some positive constant $\beta > 0$. Therefore, the stability condition forces

$$a(0) + \frac{1}{\lambda}\dot{a}(0) + \frac{1}{\lambda}\int_0^\infty e^{-\lambda s} N(s)\, ds = 0. \tag{3.4.35}$$

Then under the stability condition (5.2.145),

$$a(t) = e^{-\lambda t}\left[a(0) + \frac{1}{2\lambda}\int_0^\infty e^{-\lambda s} N(s) ds\right] + \frac{1}{2\lambda}\int_0^\infty e^{-\lambda|t-s|} N(s)\, ds. \tag{3.4.36}$$

We notice that in order to estimate $a(t)$ and $b(t)$, we need a non-local term

$$\int_0^\infty e^{-\lambda s} N(s)\, ds, \tag{3.4.37}$$

and in all estimates, a global estimate for

$$\left\| b\left(\gamma(t - v x_1)\right) m_v\left(x, t\right) + r(x,t) \right\|_{L_x^\infty L_t^2[0,\infty)} \tag{3.4.38}$$

is involved. But for the general charge transfer model, a-priori, we do not have any global estimates. Therefore, we will consider a truncated version of the above construction restricted to interval $t \in [0, T]$ for large positive $T$. Then one can run the bootstrap procedure for our truncated evolution.

For $t \in [0, T]$, we construct the following truncated version of the evolution:

$$u_T(x,t) = U(t,0)f + \dot{U}(t,0)g = a_T(t)w(x) + b_T\left(\gamma(t - v x_1)\right) m_v\left(x, t\right) + r_T(x,t). \tag{3.4.39}$$

For $a_T(t)$, we analyze the same ODE for $a(t)$ again but restricted to $[0, T]$ and instead of the stability condition

$$a(0)+\frac{1}{\lambda}\dot{a}(0)+\frac{1}{\lambda}\int_0^\infty e^{-\lambda s}N(s)\,ds=0 \tag{3.4.40}$$

we impose the condition that

$$a_T(0)+\frac{1}{\lambda}\dot{a}_T(0)+\frac{1}{\lambda}\int_0^T e^{-\lambda s}N(s)\,ds=0. \tag{3.4.41}$$

The same construction can be applied to $b_T$.

**Lemma 3.4.5.** *From the construction above, we have the following estimates: for* $0 \ll A \ll T$*,*

$$\|a_T\|_{L^\infty[0,T]} \lesssim \left(C(A,\lambda)+\frac{1}{\lambda A}C_1(T)\right)\left(\|f\|_{L^2}+\|g\|_{\dot{H}^1}\right), \tag{3.4.42}$$

$$\|a_T\|_{L^1[0,T]} \lesssim \left(C(A,\lambda)+\frac{1}{\lambda A}C_1(T)\right)\left(\|f\|_{L^2}+\|g\|_{\dot{H}^1}\right), \tag{3.4.43}$$

$$\|b_T\|_{L^\infty[0,T]} \lesssim \left(C(A,\mu)+\frac{1}{\mu A}C_1(T)\right)\left(\|f\|_{L^2}+\|g\|_{\dot{H}^1}\right), \tag{3.4.44}$$

*and*

$$\|b_T\|_{L^1[0,T]} \lesssim \left(C(A,\mu)+\frac{1}{\mu A}C_1(T)\right)\left(\|f\|_{L^2}+\|g\|_{\dot{H}^1}\right). \tag{3.4.45}$$

*Proof.* First of all, by the bootstrap assumption (3.4.18),

$$\left\|b_T\left(\gamma(t-vx_1)\right)m_v\left(x,t\right)+r_T(x,t)\right\|_{L^\infty_x L^2_t[0,T]} \leq C_1(T)\left(\|f\|_{L^2}+\|g\|_{\dot{H}^1}\right). \tag{3.4.46}$$

For $a_T(t)$, we know that

$$\begin{aligned}\ddot{a}_T(t)-\lambda^2 a_T(t)+a_T(t)\left\langle V_2\left(x-vt\right)w,w\right\rangle\\ +\left\langle V_2\left(x-vt\right)\left(b_T\left(\gamma(t-vx_1)\right)m_v\left(x,t\right)+r_T(x,t)\right),w\right\rangle=0.\end{aligned} \tag{3.4.47}$$

We obtain

$$a_T(t) = \frac{e^{\lambda t}}{2}\left[a_T(0) + \frac{1}{\lambda}\dot{a}_T(0) + \frac{1}{\lambda}\int_0^t e^{-\lambda s}N(s)ds\right] + R(t) \tag{3.4.48}$$

where

$$|R(t)| \lesssim e^{-\beta t}, \tag{3.4.49}$$

With notations introduced above, we consider the truncated version of the stability condition,

$$a_T(0) + \frac{1}{\lambda}\dot{a}_T(0) + \frac{1}{\lambda}\int_0^T e^{-\lambda s}N(s)\,ds = 0. \tag{3.4.50}$$

So

$$a_T(t) = e^{-\lambda t}\left[a_T(0) + \frac{1}{2\lambda}\int_0^T e^{-\lambda s}N(s)ds\right] + \frac{1}{2\lambda}\int_0^T e^{-\lambda|t-s|}N(s)\,ds. \tag{3.4.51}$$

where

$$N(t) = -\left[a_T(t)c(t) + h(t)\right] \tag{3.4.52}$$

with

$$|c(t)| \lesssim e^{-\alpha|t|} \tag{3.4.53}$$

$$h(t) := \left\langle V_2\left(x - vt\right)\left[b_T(t - vx_1)m_v\left(x,t\right) + r_T(x,t)\right], w\right\rangle. \tag{3.4.54}$$

For $0 \ll A \ll T$ fixed, we can always bound the $L^\infty$ norm of $a_T$ on the interval $[0, A]$ by Grönwall's inequality. Therefore, it suffices to estimate the $L^\infty$ norm of $a_T$ from $A$ to $T$. Note that $|c(t)| \lesssim e^{-\alpha|t|}$, for $A$ large, one can always absorb the effects from $\int_A^T a_T(t)c(t)\,dt$ into the left-hand side. Hence it reduces to estimate the $L_t^1$ norm of $h(t)$ restricted to $[A, T]$.

Consider the integral

$$\int_A^T |h(t)|\,dt = \int_A^T \left|\left\langle V_2\left(x - vt\right)\left[b_T\left(\gamma(t - vx_1)\right)m_v\left(x,t\right) + r_T(x,t)\right], w\right\rangle\right|\,dt.$$

Clearly,

$$\int_A^T |V_2(x-vt)[b_T(\gamma(t-vx_1))m_v(x,t)+r_T(x,t)]|\,dt \lesssim$$
$$\left(\int_A^T |(b_T(\gamma(t-vx_1))m_v(x,t)+r_T(x,t))|^2\,dt\right)^{\frac{1}{2}}\left(\int_A^T |V_2(x-vt)|^2\,dt\right)^{\frac{1}{2}}.$$

Note that

$$\left|\left\langle\left(\int_A^T |V_2(\cdot-vt)|^2\,dt\right)^{\frac{1}{2}}, w\right\rangle\right| \lesssim \frac{1}{A}. \tag{3.4.55}$$

By the preliminary calculations above, we can estimate the $L^\infty$ norm of $a_T(t)$,

$$\begin{aligned}
\|a_T\|_{L^\infty[0,T]} &\lesssim C(A,\lambda)\left(\|f\|_{L^2}+\|g\|_{\dot{H}^1}\right)+\frac{1}{\lambda}\int_A^T |h(t)|\,dt \\
&\lesssim C(A,\lambda)\left(\|f\|_{L^2}+\|g\|_{\dot{H}^1}\right) \\
&\quad +\frac{1}{\lambda A}\left(\int_A^T |(b_T(\gamma(t-vx_1))m_v(x,t)+r_T(x,t))|^2\,dt\right)^{\frac{1}{2}} \qquad (3.4.56)\\
&\lesssim \left(C(A,\lambda)+\frac{1}{\lambda A}C_1(T)\right)\left(\|f\|_{L^2}+\|g\|_{\dot{H}^1}\right).
\end{aligned}$$

Similarly, for the $L^1$ norm of $a_T(t)$,

$$\|a_T\|_{L^1[0,T]} \lesssim \left(C(A,\lambda)+\frac{1}{\lambda A}C_1(T)\right)\left(\|f\|_{L^2}+\|g\|_{\dot{H}^1}\right). \tag{3.4.57}$$

After applying a Lorentz transformation, we have analogous estimates for $b_T(t)$:

$$\|b_T\|_{L^\infty[0,T]} \lesssim \left(C(A,\mu)+\frac{1}{\mu A}C_1(T)\right)\left(\|f\|_{L^2}+\|g\|_{\dot{H}^1}\right), \tag{3.4.58}$$

$$\|b_T\|_{L^1[0,T]} \lesssim \left(C(A,\mu)+\frac{1}{\mu A}C_1(T)\right)\left(\|f\|_{L^2}+\|g\|_{\dot{H}^1}\right). \tag{3.4.59}$$

The lemma is proved. □

In the following subsections, we will show estimates with constants independent of $T$,

$$\sup_{x\in\mathbb{R}^3}\int_0^T |u_T(x,t)|^2\,dt \le C_1\left(\|f\|_{L^2}+\|g\|_{\dot{H}^1}\right)^2 \tag{3.4.60}$$

$$\sup_{x\in\mathbb{R}^3}\int_0^T \left|u_T^S(x,t)\right|^2 dt \le C_2\left(\|f\|_{L^2}+\|g\|_{\dot{H}^1}\right)^2. \tag{3.4.61}$$

Then we know our construction of $u_T$ has estimates independent of $T$. As $T\to\infty$, the stability condition (5.2.145) will be recovered from

$$a_T(0)+\frac{1}{\lambda}\dot{a}_T(0)+\frac{1}{\lambda}\int_0^T e^{-\lambda s}N(s)\,ds=0. \tag{3.4.62}$$

and

$$a_T(t)=e^{-\lambda t}\left[a_T(0)+\frac{1}{2\lambda}\int_0^T e^{-\lambda s}N(s)ds\right]+\frac{1}{2\lambda}\int_0^T e^{-\lambda|t-s|}N(s)\,ds. \tag{3.4.63}$$

Therefore, from the estimates for $u_T$, we can obtain the desired estimates for a scattering state $u(x,t)$,

$$\sup_{x\in\mathbb{R}^3}\int_0^\infty |u(x,t)|^2\,dt \le C_1\left(\|f\|_{L^2}+\|g\|_{\dot{H}^1}\right)^2 \tag{3.4.64}$$

$$\sup_{x\in\mathbb{R}^3}\int_0^\infty \left|u^S(x,t)\right|^2 dt \le C_2\left(\|f\|_{L^2}+\|g\|_{\dot{H}^1}\right)^2. \tag{3.4.65}$$

Therefore in the remaining part of this section, we will analyze the bootstrap process for $u_T(x,t)$ carefully.

### *3.4.3 Decomposition into channels*

Following the notations above, for $t\in[0,T]$, consider

$$u_T(x,t)=U(t,0)f+\dot{U}(t,0)g=a_T(t)w(x)+b_T\left(\gamma(t-vx_1)\right)m_v\left(x,t\right)+r_T(x,t).$$

There exist constants $C_1(T)$ and $C_2(T)$ such that

$$\sup_{x\in\mathbb{R}^3}\int_0^T |u_T(x,t)|^2\,dt \le C_1(T)\left(\|f\|_{L^2}+\|g\|_{\dot{H}^1}\right)^2 \tag{3.4.66}$$

and

$$\sup_{x\in\mathbb{R}^3}\int_0^T \left|u_T^S(x,t)\right|^2 dt \le C_2(T)\left(\|f\|_{L^2}+\|g\|_{\dot{H}^1}\right)^2. \tag{3.4.67}$$

We run our bootstrap argument for $u_T(x,t)$. Notice that since $V_i$, $i=1,2$ is a short-range potential and $V_2(x-vt)$ moves away from $V_1$, intuitively, $u(x,t)$ will have different dominant behaviors in different regions in $\mathbb{R}^3$. To make this heuristic rigorous, we perform a decomposition of channels based on it. For some fixed small $\delta>0$, we introduce a partition of unity associated with the sets

$$B_{\delta t}(0)=\{x:\,|x|\le\delta t\}, \qquad B_{\delta t}(tv)=\{x:\,|x-(tv,0,0)|\le\delta t\} \tag{3.4.68}$$

and

$$\mathbb{R}^3\backslash\left(B_{\delta t}(0)\cup B_{\delta t}(tv)\right). \tag{3.4.69}$$

To be more precisely, let $\chi_1(x,t)$ be a smooth cutoff function such that

$$\chi_1(x,t)=1,\ \forall x\in B_{\delta t}(0), \qquad \chi_1(x,t)=0,\ \forall x\in\mathbb{R}^3\backslash B_{2\delta t}(0). \tag{3.4.70}$$

One might assume $t\ge t_0$ for some large $t_0$. We also define

$$\chi_2(x,t)=\chi_1(x-vt,t), \qquad \chi_3=1-\chi_1-\chi_2. \tag{3.4.71}$$

Note that we only consider the estimates for large $t$, so one might also assume the support of $\chi_1(x,t)$ contains the support of $V_1(x)$ and support of $\chi_2(x,t)$ contains the support of $V_2(\cdot-vt)$.

With the partition above, we rewrite the evolution as

$$u_T(x,t) = \chi_1(x,t)u_T(x,t) + \chi_2(x,t)u_T(x,t) + \chi_3(x,t)u_T(x,t). \tag{3.4.72}$$

We will discuss $\chi_i(x,t)u_T(x,t)$, $i = 1,2,3$, separately.

Based on Duhamel's formula, we will compare $u$ to different evolution groups on different "channels".

For

$$\chi_1(x,t)u_T(x,t), \tag{3.4.73}$$

we will compare it to

$$W_1(t)f + \dot{W}_1(t)g \tag{3.4.74}$$

where

$$W_1(t) := \frac{\sin\left(t\sqrt{H_1}\right)}{\sqrt{H_1}}. \tag{3.4.75}$$

As to

$$\chi_2(x,t)u_T(x,t), \tag{3.4.76}$$

it will be compared to

$$W_2(t)f + \dot{W}_2(t)g \tag{3.4.77}$$

where $W_2\,(t,s)$ denotes the evolution associated with the Hamiltonian $-\Delta + V_2(x - vt)$ and initial velocity $f$. starting from $s$ to $t$. And formally, $\dot{W}_2(t,s)$ is used to denote the evolution associated with $g$ from $s$ to $t$. Here the dot in $\dot{W}_2$ is not the time derivative but simply a notation. These evolution can be obtained from the entries of the solution map if we write the wave equation $\partial_{tt}u - \Delta u + V_2(x - vt)u = 0$ using the Hamiltonian structure. We also use the short-hand notation $W_2(t)$ and $\dot{W}_2(t)$ to denote the evolution starting at $s = 0$.

Finally

$$\chi_3(x,t)u_T(x,t) \tag{3.4.78}$$

is compared with

$$W_0(t)f + \dot{W}_0(t)g \tag{3.4.79}$$

where

$$W_0(t) := \frac{\sin\left(t\sqrt{-\Delta}\right)}{\sqrt{-\Delta}}. \tag{3.4.80}$$

To be more explicit, we write

$$\begin{aligned} \chi_1(x,t)u_T(x,t) &= \chi_1(x,t)W_1(t)f + \chi_1(x,t)\dot{W}_1(t)g \\ &\quad -\chi_1(x,t)\int_0^t W_1(t-s)V_2(\cdot - sv)u_T(s)\,ds, \end{aligned} \tag{3.4.81}$$

$$\begin{aligned} \chi_2(x,t)u_T(x,t) &= \chi_2(x,t)W_2(t)f + \chi_2(x,t)\dot{W}_2(t)g \\ &\quad -\chi_2(x,t)\int_0^t W_2(t,s)V_1u_T(s)\,ds \end{aligned} \tag{3.4.82}$$

and

$$\begin{aligned} \chi_3(x,t)u_T(x,t) &= \chi_3(x,t)W_0(t)f + \chi_3(x,t)\dot{W}_0(t)g \\ &\quad -\chi_3(x,t)\int_0^t W_0(t-s)\left(V_1 + V_2(\cdot - vs)\right)u_T(s)\,ds. \end{aligned} \tag{3.4.83}$$

### *3.4.4 Analysis of the three channels*

We will use the notations

$$\begin{aligned} u_T(x,t) &= a_T(t)w(x) + b_T(\gamma(t-vx_1))\, m_v(x,t) + r_T(x,t) \\ &=: a_T(t)w(x) + u_{T,1}(x,t) \\ &=: b_T(\gamma(t-vx_1))\, m_v(x,t) + u_{T,2}(x,t). \end{aligned} \tag{3.4.84}$$

Note that

$$P_c(H_1)\left(u_{T,1}\right) = u_{T,1} \tag{3.4.85}$$

and

$$P_c(H_2)\left(u_{T,2}\right)_L = \left(u_{T,2}\right)_L. \tag{3.4.86}$$

The free channel and the channel associated with $H_1$ are easy to analyze with the endpoint reversed Strichartz estimate and results for estimates along slanted lines, Theorems 4.2.3, 4.2.4, 3.3.2 and Lemma 3.3.1.

## Analysis of $\chi_1(x,t)u_T(x,t)$:

We consider

$$\begin{aligned} \chi_1(x,t)u_T(x,t) &= \chi_1(x,t)W_1(t)f + \chi_1(x,t)\dot{W}_1(t)g \\ &\quad -\chi_1(x,t)\int_0^t W_1(t-s)V_2(\cdot - sv)u_T(s)\,ds. \end{aligned} \tag{3.4.87}$$

Given $B$ fixed and $0 \ll B \ll T$, one can always bound the integrals restricted to $[0,B]$,

$$\int_0^B |\chi_1(x,t)u_T(x,t)|^2\,dt, \quad \int_0^B \left|\chi_1(x,t)u_T^S(x,t)\right|^2 dt$$

by a prescribed constant by Grönwall's inequality as Lemma 3.4.2. Therefore, it suffices to consider the integrals over $[B, T]$. If we fixed $0 \ll A \ll T$ large, one can always find a big constant $B$ such that $A \ll \frac{(v-2\delta)}{1+v} B$. Then when we consider the integrals from $B$ to $T$, by the finite speed of propagation and the fact that $V_2$ is compactly supported, we can further reduce

$$\begin{aligned} \chi_1(x,t)u_T(x,t) &= \chi_1(x,t)W_1(t)f + \chi_1(x,t)\dot{W}_1(t)g \\ &\quad -\chi_1(x,t)\int_0^{t-A} W_1(t-s)V_2(\cdot - sv)u_T(s)\,ds. \end{aligned} \tag{3.4.88}$$

For $s > t - A$, the center of $V_2$ is of distance at least $|(t-A)\,v|$ away from the center of the support of $\chi_1$. Meanwhile, $t-s$ is at most $A$. So the effects caused by $W_1(t-s)V_2(\cdot - sv)u_T(s)$ will not influence the points in the support of $\chi_1$.

First, we consider the endpoint reversed Strichartz estimate (3.4.10),

$$\begin{aligned} \int_0^T \left|\chi_1(x,t)u_{T,1}(x,t)\right|^2 dt &\lesssim \int_0^T \left|\chi_1(x,t)W_1(t)P_c(H_1)f + \chi_1(x,t)\dot{W}_1(t)P_c(H_1)g\right|^2 dt \\ &\quad + \int_0^T \left|\chi_1(x,t)\int_0^{t-A} W_1(t-s)P_c(H_1)V_2(\cdot - sv)u_T(s)\,ds\right|^2 dt \\ &\lesssim \left(\|f\|_{L^2} + \|g\|_{\dot{H}^1}\right)^2 \\ &\quad + \int_0^B \left|\chi_1(x,t)\int_0^t W_1(t-s)P_c(H_1)V_2(\cdot - sv)u_T(s)\,ds\right|^2 dt \\ &\quad + \int_B^T \left|\chi_1(x,t)\int_0^{t-A} W_1(t-s)P_c(H_1)V_2(\cdot - sv)u_T(s)\,ds\right|^2 \\ &\lesssim \left(\|f\|_{L^2} + \|g\|_{\dot{H}^1}\right)^2 + C(B)\left(\|f\|_{L^2} + \|g\|_{\dot{H}^1}\right)^2 \\ &\quad + \int_B^T \left|\chi_1(x,t)\int_0^{t-A} W_1(t-s)P_c(H_1)V_2(\cdot - sv)u_T(s)\,ds\right|^2 dt \\ &\lesssim \left(\|f\|_{L^2} + \|g\|_{\dot{H}^1}\right)^2 + C(B)\left(\|f\|_{L^2} + \|g\|_{\dot{H}^1}\right)^2 \\ &\quad + \frac{1}{A}C_2(T)\left(\|f\|_{L^2} + \|g\|_{\dot{H}^1}\right)^2. \end{aligned}$$

In the above calculations, for the second inequality, we applied the endpoint Strichartz estimate for perturbed wave equations, cf. Theorem 4.2.4:

$$\int_0^T \left|\chi_1(x,t)W_1(t)P_c\left(H_1\right)f+\chi_1(x,t)\dot{W}_1(t)P_c\left(H_1\right)g\right|^2 dt \lesssim \left(\|f\|_{L^2}+\|g\|_{\dot{H}^1}\right)^2.$$

For the third inequality, we used the fact that $B$ is a fixed big constant, one can always find $C(B)$ independent of $T$ to ensure the inequality holds as we did in Lemma 3.4.3:

$$\int_0^B \left|\chi_1(x,t)\int_0^t W_1(t-s)P_c\left(H_1\right)V_2(\cdot-sv)u_T(s)\,ds\right|^2 dt \lesssim C(B)\left(\|f\|_{L^2}+\|g\|_{\dot{H}^1}\right)^2.$$

For the last inequality, we used the bootstrap assumption (3.4.11) and the results from the section on estimates along slanted lines, Theorem 3.3.8 and Corollary 3.3.6. By Theorem 3.3.8,

$$\begin{aligned}\int_B^T \left|\chi_1(x,t)\int_0^{t-A} W_1(t-s)P_c\left(H_1\right)V_2(\cdot-sv)u_T(s)\,ds\right|^2 dt &\lesssim \frac{1}{A^2}\left\|V_2\right\|^2_{L^1_x}\sup_x\int_0^T\left|u_T(t)\right|^2 dt\\ &\lesssim \frac{1}{A}C_2(T)\left(\|f\|_{L^2}+\|g\|_{\dot{H}^1}\right).^2\end{aligned}$$

Therefore,

$$\int_0^T \left|\chi_1(x,t)u_{T,1}(x,t)\right|^2 dt \lesssim \left(C_0+C(A)+\frac{1}{A}C_2(T)\right)\left(\|f\|_{L^2}+\|g\|_{\dot{H}^1}\right)^2. \tag{3.4.89}$$

For the remaining piece, by estimates (3.4.42), (3.4.43) and Agmon's estimate,

$$\int_0^T \left|\chi_1(x,t)a_T(t)w(x)\right|^2 dt \lesssim \left(C(A,\lambda)+\frac{1}{\lambda A}C_1(T)\right)\left(\|f\|_{L^2}+\|g\|_{\dot{H}^1}\right)^2. \tag{3.4.90}$$

Therefore, with estimates (3.4.89) and (3.4.90), for the endpoint reversed estimate, we obtain

$$C_1(T) \lesssim C_0 + C(A,B) + \frac{1}{A} C_2(T) \tag{3.4.91}$$

in the first channel. So for $A$ large, in this channel, we have the condition for the bootstrap argument.

Next we consider the estimate along the slanted line $(x+vt, t)$.

Denoting

$$u^S_{T,1}(x,t) = \chi_1(x+vt,t) u_{T,1}(x+vt,t), \tag{3.4.92}$$

we want to estimate

$$\int_0^T \left| \chi_1(x+vt,t) u_{T,1}(x+vt,t) \right|^2 dt = \int_0^T \left| u^S_{T,1}(x,t) \right|^2 dt. \tag{3.4.93}$$

Furthermore, we introduce

$$D^S_1(x,t) := D_1\left(x+vt,t\right) \tag{3.4.94}$$

where

$$D_1(x,t) := \chi_1(x,t) W_1(t) P_c\left(H_1\right) f + \chi_1(x,t) \dot{W}_1(t) P_c\left(H_1\right) g; \tag{3.4.95}$$

$$k^S_1(x,t) := k_1\left(x+vt,t\right) \tag{3.4.96}$$

where

$$k_1(x,t) := \chi_1(x,t) \int_0^t W_1(t-s) P_c\left(H_1\right) V_2(\cdot - sv) u_T(s)\, ds; \tag{3.4.97}$$

$$E^S_1(x,t) := E_1\left(x+vt,t\right) \tag{3.4.98}$$

where

$$E_1(x,t) := \chi_1(x,t)\int_0^{t-A} W_1(t-s)P_c(H_1)\,V_2(\cdot - sv)u_T(s)\,ds. \tag{3.4.99}$$

Then we can conclude

$$\begin{aligned}\int_0^T \left|u_{T,1}^S\right|^2 dt &\lesssim \int_0^T \left|D_1^S\right|^2 dt + \int_0^B \left|k_1^S\right|^2 dt + \int_B^T \left|E_1^S\right|^2 dt \\ &\lesssim \left(\|f\|_{L^2} + \|g\|_{\dot{H}^1}\right)^2 + C(B)\left(\|f\|_{L^2} + \|g\|_{\dot{H}^1}\right)^2 \\ &\quad + \frac{1}{A}C_2(T)\left(\|f\|_{L^2} + \|g\|_{\dot{H}^1}\right)^2 \end{aligned} \tag{3.4.100}$$

similar to the analysis of estimate (3.4.89) via Theorems 3.3.2, 3.3.8 and Corollary 3.3.6.

For the piece with bound states, by estimate (3.4.43) and Agmon's estimate,

$$\begin{aligned}&\int_0^T |\chi_1(x+vt,t)a_T(t)w(x+vt)|^2\,dt \\ \lesssim &\left(C(A,\lambda) + \frac{1}{\lambda A}C_1(T)\right)\left(\|f\|_{L^2} + \|g\|_{\dot{H}^1}\right)^2.\end{aligned} \tag{3.4.101}$$

Therefore, with estimates (3.4.100) and (3.4.101), we obtain

$$C_2(T) \lesssim C_0 + C(A,B) + \frac{1}{A}C_2(T) \tag{3.4.102}$$

in the first channel. So for $A$ large, in this channel, we obtain the desired reduction for the bootstrap argument.

Analysis of $\chi_2(x,t)u_T(x,t)$:

Now we consider the most delicate channel which is the channel associated to the moving potential.

$$\begin{aligned}\chi_2(x,t)u_T(x,t) &= \chi_2(x,t)W_2(t)f + \chi_2(x,t)\dot{W}_2(t)g \\ &\quad -\chi_2(x,t)\int_0^t W_2(t,s)V_1u_T(s)\,ds.\end{aligned} \tag{3.4.103}$$

Again, by the finite speed of propagation, it suffices to consider

$$\begin{aligned}\chi_2(x,t)u_T(x,t) &= \chi_2(x,t)W_2(t)f + \chi_2(x,t)\dot{W}_2(t)g \\ &\quad -\chi_2(x,t)\int_0^{t-A} W_2(t,s)V_1u_T(s)\,ds.\end{aligned} \tag{3.4.104}$$

Note that with the Lorentz transformation associated with $V_2(x-vt)$, we have

$$(u_T)_L\left(x_1',x_2',x_3',t'\right) = u_T\left(\gamma\left(x_1'+vt'\right),x_2',x_3',\gamma\left(t'+vx_1'\right)\right) \tag{3.4.105}$$

and

$$u_T(x,t) = (u_T)_L\left(\gamma\left(x_1-vt\right),x_2,x_3,\gamma\left(t-vx_1\right)\right). \tag{3.4.106}$$

The endpoint reversed Strichartz estimate (3.4.10) for this channel is equivalent to the estimate along the slanted line $(x-vt,t)$ under the new frame. Meanwhile, the estimate along the slanted line $(x+vt,t)$, see (3.4.11), for this channel is equivalent to the endpoint reversed Strichartz estimate with respect to the new frame.

Denote $\widetilde{g}$ and $\widetilde{f}$ to denote the initial data with respect to this new frame under which $V_2$ is stationary and $V_1$ is moving. We use $W_2^L(t)$ and $\dot{W}_2^L(t)$ to denote the evolutions associated to $\tilde{f}$ and $\widetilde{g}$ respectively in the new frame. By construction, in the new frame, $W_2^L(t)$ is the sine evolution with respect to $H_2$. By Theorem 5.3.6, we know

$$\left(\|\widetilde{f}\|_{L^2}+\|\widetilde{g}\|_{\dot{H}^1}\right)\simeq\left(\|f\|_{L^2}+\|g\|_{\dot{H}^1}\right). \tag{3.4.107}$$

Denote

$$D_2^S(x,t):=D_2\left(x-vt,t\right) \tag{3.4.108}$$

where

$$D_2(x,t):=W_2^L(t)P_c\left(H_2\right)\widetilde{f}+\dot{W}_2^L(t)P_c\left(H_2\right)\widetilde{g}; \tag{3.4.109}$$

$$k_2^S(x,t):=k_2\left(x-vt,t\right) \tag{3.4.110}$$

where

$$k_2(x,t):=\int_0^t W_2^L(t-s)P_c\left(H_2\right)V_1(s)u_T(s)\,ds; \tag{3.4.111}$$

$$E_2^S(x,t):=E_2\left(x-vt,t\right) \tag{3.4.112}$$

where

$$E_2(x,t)=\int_0^{t-A} W_2^L(t-s)P_c\left(H_2\right)V_1(s)u_T(s)\,ds. \tag{3.4.113}$$

With the estimates along the slanted line $(x-vt,t)$ for $W_2^L(t)$, Theorem 3.3.2, we know

$$\begin{aligned}
\int_0^T \left|\chi_2(x,t)u_{T,2}(x,t)\right|^2 dt &= \int_0^T \left|\left(u_{T,2}\right)_L\left(\gamma\left(x_1-vt\right),x_2,x_3,\gamma\left(t-vx_1\right)\right)\right|^2 dt \\
&\lesssim \int_0^T \left|D_2^S\right|^2 dt + \int_0^B \left|k_2^S\right|^2 dt + \int_B^T \left|E_2^S\right|^2 dt \\
&\lesssim \left(\|f\|_{L^2}+\|g\|_{\dot{H}^1}\right)^2 + C(B)\left(\|f\|_{L^2}+\|g\|_{\dot{H}^1}\right)^2 \\
&\quad + \frac{1}{A}\left(\|V_1u_T\|_{L_x^1L_t^2[0,T]}\right)^2 \qquad (3.4.114) \\
&\lesssim \left(\|f\|_{L^2}+\|g\|_{\dot{H}^1}\right)^2 + C(B)\left(\|f\|_{L^2}+\|g\|_{\dot{H}^1}\right)^2 \\
&\quad + \frac{1}{A}\left(\|u_T\|_{L_x^\infty L_t^2[0,T]}\right)^2 \\
&\lesssim \left(\|f\|_{L^2}+\|g\|_{\dot{H}^1}\right)^2 + C(B)\left(\|f\|_{L^2}+\|g\|_{\dot{H}^1}\right)^2 \\
&\quad + \frac{1}{A}C_1(T)\left(\|f\|_{L^2}+\|g\|_{\dot{H}^1}\right)^2
\end{aligned}$$

by the bootstrap assumption (3.4.10), Theorem 3.3.8 and Corollary 3.3.6. For the third inequality, we also use the fact $A$ is a fixed big constant, we can always find $C(A)$ independent of $T$ to ensure the inequality holds.

For the piece with bound states, by estimate (3.4.44) and Agmon's estimate, one has

$$\begin{aligned}
&\int_0^T \left|\chi_2(x,t)b_T\left(\gamma(t-vx_1)\right)m_v\left(x,t\right)\right|^2 dt \qquad (3.4.115) \\
\lesssim &\left(C(A,\mu)+\frac{1}{\mu A}C_1(T)\right)\left(\|f\|_{L^2}+\|g\|_{\dot{H}^1}\right)^2.
\end{aligned}$$

Hence in this channel, with estimates (3.4.114) and (3.4.115),

$$C_1(T) \lesssim C_0 + C(A,B) + \frac{1}{A}C_1(T). \qquad (3.4.116)$$

So for $A$ large, in this channel, we achieve the condition for the bootstrap argument.

Now we analyze the estimate along $(x+vt,t)$. The argument here is similar to the analysis for the first channel.

Denote

$$u^S_{T,2}(x,t) := \chi_2(x+vt,t)u_{T,2}(x+vt,t). \tag{3.4.117}$$

Then

$$\begin{aligned}
\int_0^T \left|u^S_{T,2}(x,t)\right|^2 dt &\lesssim \int_0^T \left|\left(u_{T,2}\right)_L(x,t)\right|^2 dt \\
&\lesssim \int_0^T |D_2(x,t)|^2\,dt + \int_0^T |k_2(x,t)|^2\,dt + \int_0^T |E_2(x,t)|^2\,dt \\
&\lesssim \left(\|f\|_{L^2} + \|g\|_{\dot{H}^1}\right)^2 + C(B)\left(\|f\|_{L^2} + \|g\|_{\dot{H}^1}\right)^2 \\
&\quad + \frac{1}{A}\left(\|V_1 u_T\|_{L^1_x L^2_t(0,T)}\right)^2 \qquad (3.4.118)\\
&\lesssim \left(\|f\|_{L^2} + \|g\|_{\dot{H}^1}\right)^2 + C(B)\left(\|f\|_{L^2} + \|g\|_{\dot{H}^1}\right)^2 \\
&\quad + \frac{1}{A}C_1(T)\left(\|f\|_{L^2} + \|g\|_{\dot{H}^1}\right)^2
\end{aligned}$$

with the bootstrap assumption (3.4.10) and Corollary 3.3.6.

For the remaining piece with bound states, by a similar argument to estimate (3.4.90), we have

$$\begin{aligned}
&\int_0^T \left|\chi_2(x+vt,t)\left(b_T\left(\gamma\left(\left(1-v^2\right)t - vx_1\right)\right) m_v\left(x+vt,t\right)\right)\right|^2 dt \\
&\qquad \lesssim \left(C(A,\mu) + \frac{1}{\mu A}C_1(T)\right)\left(\|f\|_{L^2} + \|g\|_{\dot{H}^1}\right)^2. \qquad (3.4.119)
\end{aligned}$$

Therefore, in this channel, we obtain

$$C_2(T) \lesssim C_0 + C(A,B) + \frac{1}{A}C_1(T). \tag{3.4.120}$$

For $A$ large, in this channel, we recapture the condition for the bootstrap argument.

## Analysis of $\chi_3(x,t)u_T(x,t)$:

Finally, we consider the free channel $\chi_3(x,t)u_T(x,t)$. In this channel, we can estimate all pieces together since the dominant evolution is the free ones.

We know

$$\begin{aligned}\chi_3(x,t)u_T(x,t) &= \chi_3(x,t)W_0(t)f+\chi_3(x,t)\dot{W}_0(t)g \\ &\quad -\chi_3(x,t)\int_0^t W_0(t-s)\left(V_1+V_2(\cdot-vs)\right)u_T(s)\,ds. \end{aligned} \tag{3.4.121}$$

By the finite speed of propagation as above, it suffices to consider

$$\begin{aligned}\chi_3(x,t)u_T(x,t) &= \chi_3(x,t)W_0(t)f+\chi_3(x,t)\dot{W}_0(t)g \\ &\quad -\chi_3(x,t)\int_0^{t-A} W_0(t-s)\left(V_1+V_2(\cdot-vs)\right)u_T(s)\,ds. \end{aligned} \tag{3.4.122}$$

Consider the endpoint reversed Strichartz estimate,

$$\begin{aligned}\int_0^T |\chi_3(x,t)u_T(x,t)|^2\,dt &\lesssim \int_0^T \left|\chi_3(x,t)W_0(t)f+\chi_3(x,t)\dot{W}_0(t)g\right|^2 dt \\ &\quad +\int_0^T \left|\chi_3(x,t)\int_0^{t-A} W_0(t-s)V_1u_T(s)\,ds\right|^2 dt \qquad (3.4.123) \\ &\quad +\int_0^T \left|\chi_3(x,t)\int_0^{t-A} W_0(t-s)V_2(\cdot-sv)u_T(s)\,ds\right|^2 dt \\ &\lesssim \left(\|f\|_{L^2}+\|g\|_{\dot{H}^1}\right)^2+C(B)\left(\|f\|_{L^2}+\|g\|_{\dot{H}^1}\right)^2 \\ &\quad +\frac{1}{A}\left(C_1(T)+C_2(T)\right)\left(\|f\|_{L^2}+\|g\|_{\dot{H}^1}\right)^2\end{aligned}$$

For the last inequality, we apply the bootstrap assumptions (3.4.10) and (3.4.11).

$$C_1(T)\lesssim C_0+C(A,B)+\frac{1}{A}\left(C_1(T)+C_2(T)\right). \tag{3.4.124}$$

So for $A$ large, in this channel, we obtain the condition for bootstrap argument.

Next we consider the estimate along slanted line $(x+vt,t)$.

Denote

$$u_T^S(x,t) = \chi_3(x+vt,t)u_T(x+vt,t), \tag{3.4.125}$$

$$u_{T,3}^S(x,t) := u_{T,3}\left(x+vt,t\right) \tag{3.4.126}$$

where

$$u_{T,3}(x,t) := \chi_3(x,t)W_0(t)f + \chi_3(x,t)\dot{W}_0(t)g; \tag{3.4.127}$$

$$k_3^S(x,t) := k_3\left(x+vt,t\right) \tag{3.4.128}$$

where

$$k_3(x,t) := \chi_3(x,t)\int_0^t W_0(t-s)\left(V_1+V_2(\cdot - sv)\right)u_T(s)\,ds; \tag{3.4.129}$$

$$E_3^S(x,t) := E_3\left(x+vt,t\right) \tag{3.4.130}$$

where

$$E_3(x,t) := \chi_3(x,t)\int_0^{t-A} W_0(t-s)\left(V_1+V_2(\cdot - sv)\right)u_T(s)\,ds. \tag{3.4.131}$$

Then

$$\begin{aligned}\int_0^T \left|u_T^S(x,t)\right|^2 dt &\lesssim \int_0^T \left|u_{T,3}^S(x,t)\right|^2 dt + \int_0^B \left|k_3^S(x,t)\right|^2 dt + \int_B^T \left|E_3^S(x,t)\right|^2 dt \\ &\lesssim \left(\|f\|_{L^2}+\|g\|_{\dot{H}^1}\right)^2 + C(B)\left(\|f\|_{L^2}+\|g\|_{\dot{H}^1}\right)^2 \\ &\quad + \frac{1}{A}\left(C_1(T)+C_2(T)\right)\left(\|f\|_{L^2}+\|g\|_{\dot{H}^1}\right)^2 \end{aligned} \tag{3.4.132}$$

Therefore, we obtain

$$C_2(T) \lesssim C_0 + C(A,B) + \frac{1}{A}\left(C_1(T) + C_2(T)\right) \tag{3.4.133}$$

along the free channel. So for $A$ large, we recapture the condition for the bootstrap argument.

### *3.4.5 Conclusion*

Finally, by the results from the analysis of three channels above, we conclude

$$C_1(T) \lesssim C_0 + \frac{1}{A}C_1(T) + \frac{1}{A}C_2(T) + C(A,B) \tag{3.4.134}$$

$$C_2(T) \lesssim C_0 + \frac{1}{A}C_1(T) + \frac{1}{A}C_2(T) + C(A,B) \tag{3.4.135}$$

where $C_1(T)$ is the constant appearing for the bootstrap assumption (3.4.10) for the endpoint reversed Strichartz estimate and $C_2(T)$ is the constant for the bootstrap assumption (3.4.11) for the estimate along $(x+vt,t)$.

We apply the bootstrap argument for these two estimates simultaneously. We conclude that $C_1(T)$ and $C_2(T)$ are independent of $T$. In other words, one has

$$\sup_{x\in\mathbb{R}^3}\int_0^T |u_T(x,t)|^2\,dt \le C_1\left(\|f\|_{L^2} + \|g\|_{\dot{H}^1}\right)^2 \tag{3.4.136}$$

$$\sup_{x\in\mathbb{R}^3}\int_0^T \left|u_T^S(x,t)\right|^2\,dt \le C_2\left(\|f\|_{L^2} + \|g\|_{\dot{H}^1}\right)^2. \tag{3.4.137}$$

Finally, as we discussed above, passing $T$ to $\infty$, we will recover those two estimates for a scattering state $u(x,t)$:

$$\sup_{x\in\mathbb{R}^3}\int_0^\infty |u(x,t)|^2\,dt \lesssim \left(\|f\|_{L^2} + \|g\|_{\dot{H}^1}\right)^2 \tag{3.4.138}$$

and

$$\sup_{x\in\mathbb{R}^3}\int_0^\infty \left|u^S(x,t)\right|^2 dt \lesssim \left(\|f\|_{L^2}+\|g\|_{\dot{H}^1}\right)^2. \tag{3.4.139}$$

**Remark 3.4.6.** *In the above analysis, we assumed $V_1$ and $V_2$ are compactly supported. With more careful calculations, it is easy to extend the above results to $V_1$ and $V_2$ decay as we assume in the Definition 5.3.1. In this case, instead of vanishing, the smallness conditions of our bootstrap procedure are from the smallness of tails of $V_1$ and $V_2$ in $L^1_x$ and $L^{\frac{3}{2},1}_x$ in the estimates for the following terms*

$$\chi_1\int_{t-A}^t W_1(t-s)V_2(\cdot - sv)u_T(s)\,ds, \tag{3.4.140}$$

$$\chi_2\int_{t-A}^t W_2^L(t-s)V_1(s)u_T(s)\,ds \tag{3.4.141}$$

*and*

$$\chi_3\int_{t-A}^t W_0(t-s)\left(V_1+V_2(\cdot - vs)\right)u_T(s)\,ds. \tag{3.4.142}$$

*To demonstrate, we compute a concrete example below.*

$$\begin{aligned}
&\left\|\int_{t-A}^t W_0(t-s)V_2(\cdot - sv)u_T(s)\,ds\right\|_{L^2_t[A,T]}|_{x=0}\\
&=\left\|\int_{|y|\leq A}\frac{1}{|y|}V_2(y-v\left(t-|y|\right))u_T((t-|y|))\;dy\right\|_{L^2_t[B,T]}\\
&\lesssim\left(\frac{A^2}{\langle A\rangle^\alpha}\right)\sup_x\|u_T\|_{L^2_t[0,T]}\\
&\lesssim\frac{1}{A}\sup_x\|u_T\|_{L^2_t[0,T]}.
\end{aligned} \tag{3.4.143}$$

*All other terms can be estimated by a similar way.*

## 3.5 Strichartz Estimates and Energy Bound

We know from the introduction that weighted estimates play important roles in building Strichartz estimates. In this section, we establish weighted estimates for a scattering state to the wave equation with charge transfer Hamiltonian. Just for the sake of convenience, we will restate our main theorems in this section.

Throughout this subsection, we will use the short-hand notation

$$L_t^p L_x^q := L_t^p \left([0,\infty),\, L_x^q\right). \tag{3.5.1}$$

**Corollary 3.5.1.** *Let $|v| < 1$. Suppose $u$ is a scattering state in the sense of Definition 5.3.2 and that it solves*

$$\partial_{tt} u - \Delta u + V_1(x)u + V_2(x - vt)u = 0 \tag{3.5.2}$$

*with initial data*

$$u(x,0) = g(x),\, u_t(x,0) = f(x) \tag{3.5.3}$$

*Then for $\alpha > 3$,*

$$\int_{\mathbb{R}^+} \int_{\mathbb{R}^3} \frac{1}{\langle x\rangle^{\alpha}} u^2(x,t)\, dxdt \lesssim \left(\|f\|_{L^2} + \|g\|_{\dot{H}^1}\right)^2 \tag{3.5.4}$$

*and*

$$\int_{\mathbb{R}^+} \int_{\mathbb{R}^3} \frac{1}{\langle x - vt\rangle^{\alpha}} u^2(x,t)\, dxdt \lesssim \left(\|f\|_{L^2} + \|g\|_{\dot{H}^1}\right)^2 \tag{3.5.5}$$

*Proof.* The two weighted estimates above follow easily from Theorem 3.4.1.

For the first one,

$$\begin{aligned}
\int_{\mathbb{R}^+}\int_{\mathbb{R}^3}\frac{1}{\langle x\rangle^{\alpha}}u^2(x,t)\,dxdt &\lesssim \left(\int_{\mathbb{R}^3}\frac{1}{\langle x\rangle^{\alpha}}dx\right)\sup_x\int_{\mathbb{R}^+}u^2(x,t)\,dt\\
&\lesssim \left(\|f\|_{L^2}+\|g\|_{\dot{H}^1}\right)^2 \qquad (3.5.6)
\end{aligned}$$

by the endpoint reversed Strichartz estimate (3.4.6) for $u$.

For the second one, one has

$$\begin{aligned}
\int_{\mathbb{R}^+}\int_{\mathbb{R}^3}\frac{1}{\langle x-vt\rangle^{\alpha}}u^2(x,t)\,dxdt &\lesssim \int_{\mathbb{R}}\int_{\mathbb{R}^3}\frac{1}{\langle y\rangle^{\alpha}}u^2(y+vt,t)\,dydt\\
&\lesssim \left(\int_{\mathbb{R}^3}\frac{1}{\langle y\rangle^{\alpha}}\,dy\right)\sup_x\int_{\mathbb{R}^+}\left|u^S(x,t)\right|^2dt\\
&\lesssim \left(\|f\|_{L^2}+\|g\|_{\dot{H}^1}\right)^2 \qquad (3.5.7)
\end{aligned}$$

by our estimate (3.4.8) along the slanted line $(x+vt,t)$.

We are done. □

**Theorem 3.5.2.** *Let $|v|<1$. Suppose $u$ is a scattering state in the sense of Definition 5.3.2 which solves*

$$\partial_{tt}u-\Delta u+V_1(x)u+V_2(x-vt)u=0 \qquad (3.5.8)$$

*with initial data*

$$u(x,0)=g(x),\ u_t(x,0)=f(x). \qquad (3.5.9)$$

*Then for $p>2$, and $(p,q)$ satisfying*

$$\frac{1}{2}=\frac{1}{p}+\frac{3}{q} \qquad (3.5.10)$$

*we have*

$$\|u\|_{L^p_t([0,\infty),\,L^q_x)}\lesssim\|f\|_{L^2}+\|g\|_{\dot{H}^1} \qquad (3.5.11)$$

*Proof.* Following [40], we set $A = \sqrt{-\Delta}$ and notice that

$$\|Af\|_{L^2} \simeq \|f\|_{\dot{H}^1}, \ \forall f \in C^{\infty}\left(\mathbb{R}^3\right). \tag{3.5.12}$$

For real-valued $u = (u_1, u_2) \in \mathcal{H} = \dot{H}^1\left(\mathbb{R}^3\right) \times L^2\left(\mathbb{R}^3\right)$, we write

$$U := Au_1 + iu_2. \tag{3.5.13}$$

From (3.5.12), we know

$$\|U\|_{L^2} \simeq \|(u_1, u_2)\|_{\mathcal{H}}. \tag{3.5.14}$$

We also notice that $u$ solves (3.5.8) if and only if

$$U := Au + i\partial_t u \tag{3.5.15}$$

satisfies

$$i\partial_t U = AU + V_1 u + V_2 (x - vt) u, \tag{3.5.16}$$

$$U(0) = Ag + if \in L^2\left(\mathbb{R}^3\right). \tag{3.5.17}$$

By Duhamel's formula,

$$U(t) = e^{itA}U(0) - i \int_0^t e^{-i(t-s)A} \left(V_1 u(s) + V_2 (\cdot - vs) u(s)\right) ds. \tag{3.5.18}$$

Let $P := A^{-1}\Re$, then from Strichartz estimates for the free evolution,

$$\left\|Pe^{itA}U(0)\right\|_{L_t^p L_x^q} \lesssim \|U(0)\|_{L^2}. \tag{3.5.19}$$

Writing $V_1 = V_3 V_4$, $V_2 = V_5 V_6$ , since $V_1$ and $V_2$ decay like $\langle x\rangle^{-\alpha}$ with $\alpha > 3$, we can make $V_3$ and $V_5$ satisfy the weight condition in Theorem 4.8.2. Also $V_4^2$, $V_6^2$ decay with rate

$\langle x\rangle^{-\alpha}$. By the Christ-Kiselev lemma, cf. Lemma 4.2.7, it suffices to bound

$$\left\| P\int_0^\infty e^{-i(t-s)A} V_3 V_4 u(s)\, ds \right\|_{L_t^p L_x^q}, \tag{3.5.20}$$

and

$$\left\| P\int_0^\infty e^{-i(t-s)A} V_5 V_6(\cdot - vs) u(s)\, ds \right\|_{L_t^p L_x^q}. \tag{3.5.21}$$

It is clear that

$$\left\| P\int_0^\infty e^{-i(t-s)A} V_3 V_4 u(s)\, ds \right\|_{L_t^p L_x^q} \le \|K\|_{L^2_{t,x}\to L_t^p L_x^q} \|V_4 u\|_{L^2_{t,x}}, \tag{3.5.22}$$

where

$$(KF)(t) := P\int_0^\infty e^{-i(t-s)A} V_3 F(s)\, ds. \tag{3.5.23}$$

Similarly,

$$\left\| P\int_0^\infty e^{-i(t-s)A} V_5 V_6(\cdot - vs) u(s)\, ds \right\|_{L_t^p L_x^q} \le \left\|\widetilde{K}\right\|_{L^2_{t,x}\to L_t^p L_x^q} \|V_6(x-vt)u\|_{L^2_{t,x}}, \tag{3.5.24}$$

where

$$\left(\widetilde{K}F\right)(t) := P\int_0^\infty e^{-i(t-s)A} V_5(\cdot - vs) F(s)\, ds. \tag{3.5.25}$$

We need to estimate

$$\|K\|_{L^2_{t,x}\to L_t^p L_x^q}, \ \left\|\widetilde{K}\right\|_{L^2_{t,x}\to L_t^p L_x^q}. \tag{3.5.26}$$

Testing against $F\in L^2_{t,x}$, clearly,

$$\|KF\|_{L_t^p L_x^q} \le \left\|Pe^{-itA}\right\|_{L^2\to L_t^p L_x^q} \left\| \int_0^\infty e^{isA} V_3 F(s)\, ds \right\|_{L^2}. \tag{3.5.27}$$

$$\left\|\widetilde{K}F\right\|_{L_t^p L_x^q} \le \left\|Pe^{-itA}\right\|_{L^2\to L_t^p L_x^q} \left\| \int_0^\infty e^{isA} V_5(\cdot - vs) F(s)\, ds \right\|_{L^2}. \tag{3.5.28}$$

The first factors on the right-hand side of (3.5.27) and (3.5.28) is bounded by Strichartz estimates for the free evolution. Consider the second factors, by duality, it suffices to show

$$\left\| V_3 e^{-itA}\phi \right\|_{L^2_{t,x}} \lesssim \|\phi\|_{L^2}, \forall \phi \in L^2\left(\mathbb{R}^3\right) \tag{3.5.29}$$

$$\left\| V_5(x-vt) e^{-itA}\phi \right\|_{L^2_{t,x}} \lesssim \|\phi\|_{L^2}, \forall \phi \in L^2\left(\mathbb{R}^3\right). \tag{3.5.30}$$

which holds by Theorem 4.8.2 and Corollary 3.2.11.

Hence

$$\left\| \int_0^\infty e^{isA} V_3 F(s)\, ds \right\|_{L^2} \lesssim \|F\|_{L^2_{t,x}}, \tag{3.5.31}$$

$$\left\| \int_0^\infty e^{isA} V_5(\cdot - vs) F(s)\, ds \right\|_{L^2} \lesssim \|F\|_{L^2_{t,x}}. \tag{3.5.32}$$

Therefore, indeed, we have

$$\|K\|_{L^2_{t,x}\to L^p_t L^q_x} \le C, \left\| \widetilde{K} \right\|_{L^2_{t,x}\to L^p_t L^q_x} \le C \tag{3.5.33}$$

and from (3.5.22), it follows that

$$\left\| P\int_0^\infty e^{-i(t-s)A} V_3 V_4 u(s)\, ds \right\|_{L^p_t L^q_x} \lesssim \|V_4 u\|_{L^2_{t,x}}, \tag{3.5.34}$$

$$\left\| P\int_0^\infty e^{-i(t-s)A} V_5 V_6(\cdot - vs) u(s)\, ds \right\|_{L^p_t L^q_x} \lesssim \|V_6(x-vt) u\|_{L^2_{t,x}}. \tag{3.5.35}$$

By estimates (3.5.4) and (3.5.5) from Corollary 3.5.1,

$$\|V_4 u\|_{L^2_{t,x}} \lesssim \left( \int_{\mathbb{R}^+}\int_{\mathbb{R}^3} \frac{1}{\langle x\rangle^\alpha} |u(x,t)|^2\, dxdt \right)^{\frac{1}{2}} \lesssim \|f\|_{L^2} + \|g\|_{\dot{H}^1}, \tag{3.5.36}$$

$$\|V_6(x-vt) u\|_{L^2_{t,x}} \lesssim \left( \int_{\mathbb{R}^+}\int_{\mathbb{R}^3} \frac{1}{\langle x-vt\rangle^\alpha} |u(x,t)|^2\, dxdt \right)^{\frac{1}{2}} \lesssim \|f\|_{L^2} + \|g\|_{\dot{H}^1}. \tag{3.5.37}$$

They follows that

$$\left\| P \int_0^\infty e^{-i(t-s)A} V_3 V_4 u(s)\, ds \right\|_{L_t^p L_x^q} \lesssim \|f\|_{L^2} + \|g\|_{\dot{H}^1}. \tag{3.5.38}$$

$$\left\| P \int_0^\infty e^{-i(t-s)A} V_5 V_6(\cdot - vs) u(s)\, ds \right\|_{L_t^p L_x^q} \lesssim \|f\|_{L^2} + \|g\|_{\dot{H}^1}. \tag{3.5.39}$$

Therefore, by estimates (3.5.19), (3.5.38) and (3.5.39), for $p > 2$, and

$$\frac{1}{2} = \frac{1}{p} + \frac{3}{q}$$

we have

$$\|u\|_{L_t^p([0,\infty),\, L_x^q)} \lesssim \|f\|_{L^2} + \|g\|_{\dot{H}^1}. \tag{3.5.40}$$

as claimed. □

Taking the case $p = q$ in the regular Strichartz estimate (3.5.11) and interpolating it with the endpoint reversed Strichartz estimate (3.4.6), we obtain more reversed Strichartz estimates.

**Corollary 3.5.3.** *Let $|v| < 1$. Suppose $u$ is a scattering state in the sense of Definition 5.3.2 which solves*

$$\partial_{tt} u - \Delta u + V_1(x) u + V_2(x - vt) u = 0 \tag{3.5.41}$$

*with initial data*

$$u(x,0) = g(x),\ u_t(x,0) = f(x). \tag{3.5.42}$$

*Then for $(p,q)$ satisfying*

$$\frac{1}{2} = \frac{1}{p} + \frac{3}{q} \tag{3.5.43}$$

*with*

$$2 \le p \le 8, \tag{3.5.44}$$

*we have*

$$\|u\|_{L_x^q(\mathbb{R}^3, L_t^p[0,\infty))} \lesssim \|f\|_{L^2} + \|g\|_{\dot{H}^1}. \tag{3.5.45}$$

**Theorem 3.5.4.** *Let $|v| < 1$. Suppose $u$ is a scattering state in the sense of Definition 5.3.2 which solves*

$$\partial_{tt}u - \Delta u + V_1(x)u + V_2(x - vt)u = 0 \tag{3.5.46}$$

*with initial data*

$$u(x, 0) = g(x),\ u_t(x, 0) = f(x). \tag{3.5.47}$$

*Then we have*

$$\sup_{t\geq 0} \left(\|\nabla u(t)\|_{L^2} + \|u_t(t)\|_{L^2}\right) \lesssim \|f\|_{L^2} + \|g\|_{\dot{H}^1}. \tag{3.5.48}$$

*Proof.* The proof is similar to Theorem 3.5.2. We still use the notations from the above proof of Theorem 3.5.2.

Set

$$U := Au + i\partial_t u, \tag{3.5.49}$$

then by Duhamel's formula,

$$U(t) = e^{itA}U(0) - i \int_0^t e^{-i(t-s)A} \left(V_1 u(s) + V_2 \left(\cdot - vs\right) u(s)\right)\, ds. \tag{3.5.50}$$

It suffices to estimate the $L^2$ norm of $U(t)$.

From the energy estimate for the free evolution,

$$\sup_{t\geq 0} \left\|e^{itA}U(0)\right\|_{L_x^2} \lesssim \|U(0)\|_{L^2}. \tag{3.5.51}$$

Writing $V_1 = V_3V_4$, $V_2 = V_5V_6$ as above, it suffices to bound

$$\sup_{t\geq 0} \left\|\int_0^\infty e^{-i(t-s)A} V_3V_4u(s)\, ds\right\|_{L_x^2}, \tag{3.5.52}$$

and

$$\sup_{t\geq 0}\left\|\int_0^{\infty} e^{-i(t-s)A}V_5V_6(\cdot - vs)u(s)\,ds\right\|_{L_x^2}. \tag{3.5.53}$$

It is clear that

$$\sup_{t\geq 0}\left\|\int_0^{\infty} e^{-i(t-s)A}V_3V_4u(s)\,ds\right\|_{L_x^2} \leq \|K\|_{L_t^2L_x^2\to L_t^\infty L_x^2}\,\|V_4u\|_{L_{t,x}^2}, \tag{3.5.54}$$

where

$$(KF)(t) := \int_0^{\infty} e^{-i(t-s)A}V_3F(s)\,ds. \tag{3.5.55}$$

Similarly,

$$\left\|\int_0^{\infty} e^{-i(t-s)A}V_5V_6(\cdot - vs)u(s)\,ds\right\|_{L_t^\infty L_x^2} \leq \left\|\widetilde{K}\right\|_{L_t^2L_x^2\to L_t^\infty L_x^2}\|V_6(x-vt)u\|_{L_t^2L_x^2}, \tag{3.5.56}$$

where

$$\left(\widetilde{K}F\right)(t) := \int_0^{\infty} e^{-i(t-s)A}V_5(\cdot - vs)F(s)\,ds. \tag{3.5.57}$$

We need to estimate

$$\|K\|_{L_t^2L_x^2\to L_t^\infty L_x^2},\ \left\|\widetilde{K}\right\|_{L_t^2L_x^2\to L_t^\infty L_x^2}. \tag{3.5.58}$$

Testing against $F\in L_t^2\left([0,\infty),\ L_x^2\right)$, clearly,

$$\|KF\|_{L_t^\infty L_x^2} \leq \left\|e^{-itA}\right\|_{L^2\to L_t^\infty L_x^2}\left\|\int_0^{\infty} e^{isA}V_3F(s)\,ds\right\|_{L^2}. \tag{3.5.59}$$

$$\left\|\widetilde{K}F\right\|_{L_t^\infty L_x^2} \leq \left\|e^{-itA}\right\|_{L^2\to L_t^\infty L_x^2}\left\|\int_0^{\infty} e^{isA}V_5(\cdot - vs)F(s)\,ds\right\|_{L^2}. \tag{3.5.60}$$

The first factors on the right-hand side of (3.5.59) and (3.5.60) is bounded by the energy estimates for the free evolution. The second factors are estimated in the same manner as for (3.5.27) and (3.5.28).

Therefore, we have

$$\|K\|_{L_t^2L_x^2\to L_t^\infty L_x^2}\le C,\ \left\|\widetilde{K}\right\|_{L_t^2L_x^2\to L_t^\infty L_x^2}\le C \tag{3.5.61}$$

and from (3.5.54),

$$\sup_{t\ge0}\left\|\int_0^\infty e^{-i(t-s)A}V_3V_4u(s)\,ds\right\|_{L_x^2}\lesssim\|V_4u\|_{L_{t,x}^2}, \tag{3.5.62}$$

$$\sup_{t\ge0}\left\|\int_0^\infty e^{-i(t-s)A}V_5V_6(\cdot-vs)u(s)\,ds\right\|_{L_x^2}\lesssim\|V_6(x-vt)u\|_{L_{t,x}^2}. \tag{3.5.63}$$

From Corollary 3.5.1,

$$\|V_4u\|_{L_t^2L_x^2}\lesssim\left(\int_{\mathbb{R}^+}\int_{\mathbb{R}^3}\frac{1}{\langle x\rangle^\alpha}|u(x,t)|^2\,dxdt\right)^{\frac12}\lesssim\|f\|_{L^2}+\|g\|_{\dot H^1}, \tag{3.5.64}$$

$$\|V_6(x-vt)u\|_{L_t^2L_x^2}\lesssim\left(\int_{\mathbb{R}^+}\int_{\mathbb{R}^3}\frac{1}{\langle x-vt\rangle^\alpha}|u(x,t)|^2\,dxdt\right)^{\frac12}\lesssim\|f\|_{L^2}+\|g\|_{\dot H^1}. \tag{3.5.65}$$

They imply

$$\sup_{t\ge0}\left\|\int_0^\infty e^{-i(t-s)A}V_3V_4u(s)\,ds\right\|_{L_x^2}\lesssim\|f\|_{L^2}+\|g\|_{\dot H^1}. \tag{3.5.66}$$

$$\sup_{t\ge0}\left\|\int_0^\infty e^{-i(t-s)A}V_5V_6(\cdot-vs)u(s)\,ds\right\|_{L_x^2}\lesssim\|f\|_{L^2}+\|g\|_{\dot H^1}. \tag{3.5.67}$$

Therefore, with estimates (3.5.51), (3.5.66) and (3.5.67), we have

$$\sup_{t\ge0}\left(\|\nabla u(t)\|_{L^2}+\|u_t(t)\|_{L^2}\right)\lesssim\|f\|_{L^2}+\|g\|_{\dot H^1}. \tag{3.5.68}$$

as claimed. □

Similarly, one can also obtain the local energy decay estimate:

**Theorem 3.5.5.** *Let $|v|<1$. Suppose $u$ is a scattering state in the sense of Definition 5.3.2* *which solves*

$$\partial_{tt}u - \Delta u + V_1(x)u + V_2(x - vt)u = 0 \tag{3.5.69}$$

*with initial data*

$$u(x, 0) = g(x),\ u_t(x, 0) = f(x). \tag{3.5.70}$$

*Then for* $\forall \epsilon > 0$, $|\mu| < 1$, *we have*

$$\left\|(1 + |x - \mu t|)^{-\frac{1}{2}-\epsilon}\left(|\nabla u| + |u_t|\right)\right\|_{L^2([0,\infty),\, L^2_x)} \lesssim_{\mu,\epsilon} \|f\|_{L^2} + \|g\|_{\dot{H}^1}. \tag{3.5.71}$$

*Proof.* The proof is the same as above with the energy estimate for the free wave equation replaced by the local energy decay estimate for the free wave equation.

$$\left\|(1 + |x - \mu t|)^{-\frac{1}{2}-\epsilon}\, e^{it\sqrt{-\Delta}} f\right\|_{L^2([0,\infty),\, L^2_x)} \lesssim_{\mu,\epsilon} \|f\|_{L^2_x}. \tag{3.5.72}$$

The claim follows easily. □

Finally, we consider the boundedness of the total energy. We denote the total energy by

$$E(t) = \int |\nabla_x u|^2 + |\partial_t u|^2 + V_1 |u|^2 + V_2(x - vt) |u|^2 \, dx. \tag{3.5.73}$$

**Corollary 3.5.6.** *Let* $|v| < 1$ *and suppose* $u$ *is a scattering state in the sense of Definition 5.3.2 and solves*

$$\partial_{tt}u - \Delta u + V_1(x)u + V_2(x - vt)u = 0 \tag{3.5.74}$$

*with initial data*

$$u(x, 0) = g(x),\ u_t(x, 0) = f(x). \tag{3.5.75}$$

*Assume*

$$\|\nabla V_2\|_{L^1} < \infty, \tag{3.5.76}$$

*then $E(t)$ is bounded by the initial energy independently of $t$,*

$$\sup_{t\geq 0} E(t) \lesssim \|(g,f)\|^2_{\dot{H}^1\times L^2}.$$

*Proof.* We might assume $u$ is smooth. Taking time derivative of $E(t)$, with the fact $u$ solves

$$\partial_{tt}u - \Delta u + V_1(x)u + V_2(x-vt)u = 0, \tag{3.5.77}$$

one obtains

$$\partial_t E(t) = \int_{\mathbb{R}^3} \partial_t V_2(x-vt)\,|u|^2\,(x,t)dx = -v\int_{\mathbb{R}^3} \partial_x V_2(x)\left|u^S(x,t)\right|^2 dx \tag{3.5.78}$$

by a simple change of variable.

Note that

$$\begin{aligned}\int_0^\infty |\partial_t E(t)|\,dt &\lesssim \int_0^\infty \int_{\mathbb{R}^3} |\partial_x V_2(x)|\left|u^S(x,t)\right|^2 dxdt \\ &= \|\partial_x V_2\|_{L^1_x}\left\|u^S\right\|^2_{L^\infty_x L^2_t[0,\infty)} \\ &\lesssim \|(g,f)\|^2_{\dot{H}^1\times L^2}.\end{aligned} \tag{3.5.79}$$

For arbitrary $t\in\mathbb{R}$, we have

$$E(t) - E(0) \leq \int_{\mathbb{R}^+} |\partial_t E(t)|\,dt \lesssim \|(g,f)\|^2_{\dot{H}^1\times L^2} \tag{3.5.80}$$

which implies

$$\sup_{t\geq 0} E(t) \lesssim \|(g,f)\|^2_{\dot{H}^1\times L^2} \tag{3.5.81}$$

as claimed.

To finish this section, we prove a version of endpoint Strichartz estimate which is inhomogeneous with respect to angular and radial variables. □

**Theorem 3.5.7.** *Let $|v| < 1$. Suppose $u$ is a scattering state in the sense of Definition 5.3.2 which solves*

$$\partial_{tt}u - \Delta u + V_1(x)u + V_2(x - vt)u = 0 \tag{3.5.82}$$

*with initial data*

$$u(x, 0) = g(x),\ u_t(x, 0) = f(x). \tag{3.5.83}$$

*Then for $1 \leq p < \infty$,*

$$\|u\|_{L_t^2([0,\infty),\, L_r^\infty L_\omega^p)} \lesssim \|f\|_{L^2} + \|g\|_{\dot{H}^1} \tag{3.5.84}$$

*Proof.* First of all, we consider a auxiliary function given by

$$\begin{aligned} v(x,t) = &\frac{\sin\left(t\sqrt{-\Delta}\right)}{\sqrt{-\Delta}} f + \cos\left(t\sqrt{-\Delta}\right) \\ &+ \int_0^\infty \frac{\sin\left((t-s)\sqrt{-\Delta}\right)}{\sqrt{-\Delta}} \left(|V_1 u(s)| + |V_2\left(\cdot - vs\right) u(s)|\right) ds. \end{aligned} \tag{3.5.85}$$

Since in $\mathbb{R}^3$ $\frac{\sin(t\sqrt{-\Delta})}{\sqrt{-\Delta}}$ is a positive operator, we know

$$\begin{aligned} &\left| \int_0^t \frac{\sin\left((t-s)\sqrt{-\Delta}\right)}{\sqrt{-\Delta}} \left(V_1 u(s) + V_2\left(\cdot - vs\right) u(s)\right) ds \right| \\ \lesssim &\int_0^\infty \frac{\sin\left((t-s)\sqrt{-\Delta}\right)}{\sqrt{-\Delta}} \left(|V_1 u(s)| + |V_2\left(\cdot - vs\right) u(s)|\right) ds \end{aligned} \tag{3.5.86}$$

and it follows

$$u(x,t) \leq v(x,t). \tag{3.5.87}$$

We need to estimate the Strichartz norm of

$$\int_0^\infty \frac{\sin\left((t-s)\sqrt{-\Delta}\right)}{\sqrt{-\Delta}} \left(|V_1 u(s)| + |V_2\left(\cdot - vs\right) u(s)|\right) ds. \tag{3.5.88}$$

Clearly,

$$\int_0^\infty \frac{\sin\left((t-s)\sqrt{-\Delta}\right)}{\sqrt{-\Delta}} \left(|V_1 u(s)| + |V_2(\cdot - vs)\, u(s)|\right) ds \tag{3.5.89}$$
$$= P \int_0^\infty e^{-i(t-s)A} \left(|V_1 u(s)| + |V_2(\cdot - vs)\, u(s)|\right) ds.$$

So now we can follow the same scheme as before to consider the following two estimates

$$\left\| P \int_0^\infty e^{-i(t-s)A} |V_1 u(s)|\ ds \right\|_{L_t^2 L_r^\infty L_\omega^p}, \tag{3.5.90}$$

$$\left\| P \int_0^\infty e^{-i(t-s)A} |V_2(\cdot - vs)\, u(s)|\ ds \right\|_{L_t^2 L_r^\infty L_\omega^p}. \tag{3.5.91}$$

As above, we write $V_1 = V_3 V_4$, $V_2 = V_5 V_6$ . It is clear that

$$\left\| P \int_0^\infty e^{-i(t-s)A} |V_1 u(s)|\ ds \right\|_{L_t^2 L_r^\infty L_\omega^p} \leq \|K\|_{L_{t,x}^2 \to L_t^2 L_r^\infty L_\omega^p} \|V_4 u\|_{L_{t,x}^2}, \tag{3.5.92}$$

where

$$(KF)(t) := P \int_0^\infty e^{-i(t-s)A} |V_3| F(s)\, ds. \tag{3.5.93}$$

Similarly,

$$\left\| P \int_0^\infty e^{-i(t-s)A} |V_2(\cdot - vs)\, u(s)|\ ds \right\|_{L_t^2 L_r^\infty L_\omega^p} . \leq \left\| \widetilde{K} \right\|_{L_{t,x}^2 \to L_t^2 L_r^\infty L_\omega^p} \|V_6(x - vt) u\|_{L_{t,x}^2}, \tag{3.5.94}$$

where

$$\left(\widetilde{K} F\right)(t) := P \int_0^\infty e^{-i(t-s)A} V_5(\cdot - vs) F(s)\, ds. \tag{3.5.95}$$

We need to estimate

$$\|K\|_{L_{t,x}^2 \to L_t^2 L_r^\infty L_\omega^p}, \ \left\| \widetilde{K} \right\|_{L_{t,x}^2 \to L_t^2 L_r^\infty L_\omega^p}. \tag{3.5.96}$$

Testing against $F \in L^2_{t,x}$, clearly,

$$\|KF\|_{L^p_t L^q_x} \leq \left\|Pe^{-itA}\right\|_{L^2 \to L^2_t L^\infty_r L^p_\omega} \left\|\int_0^\infty e^{isA} V_3 F(s)\, ds\right\|_{L^2}. \tag{3.5.97}$$

$$\left\|\widetilde{K}F\right\|_{L^p_t L^q_x} \leq \left\|Pe^{-itA}\right\|_{L^2 \to L^2_t L^\infty_r L^p_\omega} \left\|\int_0^\infty e^{isA} V_5(\cdot - vs) F(s)\, ds\right\|_{L^2}. \tag{3.5.98}$$

The first factors on the right-hand side of (3.5.97) and (3.5.98) is bounded by the endpoint Strichartz estimates for the free evolution. For the second factors, we can bound them as previous proofs.

Therefore, indeed, we have

$$\|K\|_{L^2_{t,x} \to L^2_t L^\infty_r L^p_\omega} \leq C, \; \left\|\widetilde{K}\right\|_{L^2_{t,x} \to L^2_t L^\infty_r L^p_\omega} \leq C \tag{3.5.99}$$

and from (3.5.92), it follows that

$$\left\|P \int_0^\infty e^{-i(t-s)A} \left|V_1 u(s)\right| \, ds\right\|_{L^2_t L^\infty_r L^p_\omega} \lesssim \|V_4 u\|_{L^2_{t,x}}, \tag{3.5.100}$$

$$\left\|P \int_0^\infty e^{-i(t-s)A} \left|V_2\left(\cdot - vs\right) u(s)\right| \, ds\right\|_{L^2_t L^\infty_r L^p_\omega} \lesssim \|V_6(x - vt) u\|_{L^2_{t,x}}. \tag{3.5.101}$$

They follows that

$$\left\|P \int_0^\infty e^{-i(t-s)A} \left|V_1 u(s)\right| \, ds\right\|_{L^2_t L^\infty_r L^p_\omega} \lesssim \|f\|_{L^2} + \|g\|_{\dot{H}^1}. \tag{3.5.102}$$

$$\left\|P \int_0^\infty e^{-i(t-s)A} \left|V_2\left(\cdot - vs\right) u(s)\right| \, ds\right\|_{L^2_t L^\infty_r L^p_\omega} \lesssim \|f\|_{L^2} + \|g\|_{\dot{H}^1}. \tag{3.5.103}$$

Therefore,

$$\begin{aligned}&\left\|\int_0^t \frac{\sin\left((t-s)\sqrt{-\Delta}\right)}{\sqrt{-\Delta}}\left(V_1u(s)+V_2\left(\cdot-vs\right)u(s)\right)ds\right\|_{L_t^2L_r^\infty L_\omega^p} \\ \lesssim &\left\|\int_0^\infty \frac{\sin\left((t-s)\sqrt{-\Delta}\right)}{\sqrt{-\Delta}}\left(|V_1u(s)|+|V_2\left(\cdot-vs\right)u(s)|\right)ds\right\|_{L_t^2L_r^\infty L_\omega^p} \\ &\lesssim \|f\|_{L^2}+\|g\|_{\dot{H}^1}.\end{aligned} \tag{3.5.104}$$

And hence

$$\|u\|_{L_t^2([0,\infty),\,L_r^\infty L_\omega^p)} \lesssim \|f\|_{L^2}+\|g\|_{\dot{H}^1} \tag{3.5.105}$$

as claimed. □

## 3.6 Inhomogeneous Estimates

When we consider nonlinear applications, it is useful to have estimates for inhomogeneous equations. Again, for simplicity we consider the case of two potentials.

### *3.6.1 Scattering states*

We start with revisiting scattering states.

Recall that if $u$ solves

$$\partial_{tt}u-\Delta u+V_1(x)u+V_2(x-vt)u=0 \tag{3.6.1}$$

and $u$ satisfies

$$\|P_b\left(H_1\right)u(t)\|_{L_x^2}\to 0,\ \left\|P_b\left(H_2\right)u_L(t')\right\|_{L_{x'}^2}\to 0\ \ t,t'\to\infty, \tag{3.6.2}$$

then we call it a scattering state.

Clearly, the set of $(g,\ f) \in H^1(\mathbb{R}^3) \times L^2(\mathbb{R}^3)$ which produce a scattering state forms a subspace of $H^1(\mathbb{R}^3) \times L^2(\mathbb{R}^3)$. In order to study this more precisely, we reformulate the wave equation as a Hamiltonian system,

$$\partial_t \begin{pmatrix} u \\ \partial_t u \end{pmatrix} - \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} -\Delta + V_1(x) + V_2(x-vt) & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} u \\ \partial_t u \end{pmatrix} = 0. \tag{3.6.3}$$

Setting

$$U := \begin{pmatrix} u \\ \partial_t u \end{pmatrix},\ J := \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \tag{3.6.4}$$

$$H(t) := \begin{pmatrix} -\Delta + V_1(x) + V_2(x-vt) & 0 \\ 0 & 1 \end{pmatrix}, \tag{3.6.5}$$

and defining

$$P_1(U) := u, \tag{3.6.6}$$

we can rewrite the wave equation with charge transfer Hamiltonian as

$$\dot{U} - JH(t)U = 0, \tag{3.6.7}$$

$$U(0) = \begin{pmatrix} g \\ f \end{pmatrix}. \tag{3.6.8}$$

With the above notations, we define the solution operator starting from $\tau$ to $t$ as $S(t,\tau)$. In particular, one can write

$$U(t) = S(t,0)U(0). \tag{3.6.9}$$

As pointed out above, the set of $(g,\ f) \in H^1(\mathbb{R}^3) \times L^2(\mathbb{R}^3)$ which produce a scattering state in the sense of Definition 5.3.2 forms a subspace

$$\mathcal{H}_s(0)\subset H^1\left(\mathbb{R}^3\right)\times L^2\left(\mathbb{R}^3\right).$$

We can do a more general time-dependent construction. One considers the evolution from $\tau$ to $t$, i.e., $S(t,\tau)$. Similar as our original construction there is a subspace

$$\mathcal{H}_s(\tau)\subset H^1\left(\mathbb{R}^3\right)\times L^2\left(\mathbb{R}^3\right)$$

such that for $\Phi\in\mathcal{H}_s(\tau)$,

$$\left\|P_b\left(H_1\right)S(t,\tau)\Phi\right\|_{L^2_x}\to 0,\ \left\|P_b\left(H_2\right)\left(S(\cdot,\tau)\Phi\right)_{L_\tau}\left(t'\right)\right\|_{L^2_{x'}}\to 0\ \ t,t'\to\infty. \tag{3.6.10}$$

It is important to notice a fundamental property of $\mathcal{H}_s(\tau)$.

**Lemma 3.6.1.** *Denote $P_s(\tau)$ as the projection onto $\mathcal{H}_s(\tau)$. Then $\forall s,\ \tau\in\mathbb{R}$,*

$$P_s(s)S(s,\tau)=S(s,\tau)P_s(\tau). \tag{3.6.11}$$

*Proof.* Notice that for $\Phi\in\mathcal{H}_s(\tau)$, then $S(s,\tau)\Phi\in\mathcal{H}_s(s)$. Since

$$\left\|P_b\left(H_1\right)S(t,s)S(s,\tau)\Phi\right\|_{L^2}=\left\|P_b\left(H_1\right)S(t,\tau)\Phi\right\|_{L^2}\to 0, \tag{3.6.12}$$

$$\left\|P_b\left(H_2\right)\left(S(\cdot,s)S(s,\tau)\Phi\right)_{L_s}\left(t'\right)\right\|_{L^2_{x'}}=\left\|P_b\left(H_2\right)\left(S(\cdot,\tau)\Phi\right)_{L_\tau}\left(t'\right)\right\|_{L^2_{x'}}\to 0 \tag{3.6.13}$$

as $t,\ t'\to\infty$ by the definition of $\mathcal{H}_s(\tau)$. Then again by the definition of $\mathcal{H}_s(s)$, it is clear $S(s,\tau)\Phi\in\mathcal{H}_s(s)$. Conversely, by symmetry, for $\Phi\in\mathcal{H}_s(s)$, then $S(\tau,s)\Phi\in\mathcal{H}_s(\tau)$. Therefore, we have that the scattering spaces are invariant under the flow $S(s,\tau)$,

$$\mathcal{H}_s(s)=S(s,\tau)\mathcal{H}_s(\tau). \tag{3.6.14}$$

Let $\Phi \in H^1\left(\mathbb{R}^3\right) \times L^2\left(\mathbb{R}^3\right)$, then $S(s,\tau)P_s(\tau)\Phi \in \mathcal{H}_s(s)$ by construction. So

$$\begin{aligned} S(s,\tau)P_s(\tau)\Phi &= \left(1 - P_s(s)\right) S(s,\tau)P_s(\tau)\Phi + P_s(s)S(s,\tau)P_s(\tau)\Phi \\ &= P_s(s)S(s,\tau)P_s(\tau)\Phi. \end{aligned} \tag{3.6.15}$$

Similarly,

$$P_s(s)S(s,\tau)\Phi = P_s(s)S(s,\tau)P_s(\tau)\Phi. \tag{3.6.16}$$

Hence

$$P_s(s)S(s,\tau) = S(s,\tau)P_s(\tau), \tag{3.6.17}$$

as claimed. □

For wave equations, it is always necessary to exchange the scalar formulation and the Hamiltonian formulation. Here we introduce some notations which are useful in our later analysis. We define $P_s(\tau)$ via the Hamiltonian formulation above. Now consider a scalar function $v(x,t) \in C\left(\mathbb{R}, H^1\left(\mathbb{R}^3\right)\right) \cap C\left(\mathbb{R}, L^2\left(\mathbb{R}^3\right)\right)$, it can give the data $(v, v_t)$ for the charge transfer model. We define

$$P_s^{\mathbf{S}}(\tau)\, v := P_1 P_s(\tau)\,(v,\, v_t)\,, \tag{3.6.18}$$

where $P_1$ is the projection onto the first component as in (3.6.6). For a vector-valued function $V = \begin{pmatrix} v \\ v_t \end{pmatrix} \in H^1\left(\mathbb{R}^3\right) \times L^2\left(\mathbb{R}^3\right)$,

$$P_s^{\mathbf{V}}(\tau)\, V := P_1 P_s(\tau)\, V, \tag{3.6.19}$$

Given data $(g,\, f) \in H^1\left(\mathbb{R}^3\right) \times L^2\left(\mathbb{R}^3\right)$, formally, we can define the evolution from $\tau$ to

$t$ associated with $f$ as

$$U(t,\tau)f \tag{3.6.20}$$

and the evolution associated with $g$ as

$$\dot{U}(t,\tau)g. \tag{3.6.21}$$

Here $\dot{U}$ is just a formal notation.

Finally, we consider two special cases.

Setting $g = 0$, then the set of $f \in L^2(\mathbb{R}^3)$ such that $(0, f) \in \mathcal{H}_s(\tau)$ forms a subspace of $L^2(\mathbb{R}^3)$. We use $L^2_s(\tau)$ to denote this subspace and let $P^L_s(\tau)$ to be the associated projection.

Setting $f = 0$, then the set of $g \in H^1(\mathbb{R}^3)$ such that $(g, 0) \in \mathcal{H}_s(\tau)$ forms a subspace of $H^1(\mathbb{R}^3)$. We use $H^1_s(\tau)$ to denote this subspace and let $P^H_s(\tau)$ to be the associated projection.

### *3.6.2 Inhomogeneous local decay estimate and Strichartz estimates*

Throughout this subsection, we will use the short-hand notation

$$L^p_t L^q_x := L^p_t\left([0,\infty),\, L^q_x\right). \tag{3.6.22}$$

Let $u$ solve

$$\partial_{tt}u - \Delta u + V_1(x)u + V_2(x - vt)u = 0 \tag{3.6.23}$$

with initial data

$$u(x,0) = g(x),\, u_t(x,0) = f(x). \tag{3.6.24}$$

Denote the evolution as

$$u(x,t) = U(t,0)f + \dot{U}(t,0)g. \tag{3.6.25}$$

From the endpoint reversed Strichartz estimate, Theorem 3.4.1, with the notations introduced above, we know

$$\sup_x \int_0^\infty \left| P_s^{\mathbf{S}}(t)\, u(x,t) \right|^2 dt \lesssim \left( \|f\|_{L^2} + \|g\|_{\dot{H}^1} \right)^2 \tag{3.6.26}$$

and

$$\sup_x \int_0^\infty \left| \left( P_s^{\mathbf{S}}(t)\, u(x,t) \right)^S \right|^2 dt \lesssim \left( \|f\|_{L^2} + \|g\|_{\dot{H}^1} \right)^2. \tag{3.6.27}$$

Furthermore, we have the following corollary as particular situations:

**Corollary 3.6.2.** *For the evolution $U(t,\tau)$ and the projections $P_s^{\mathbf{S}}(t)$, $P_s^L(\tau)$ defined above, one has*

$$\sup_x \int_0^\infty \left| P_s^{\mathbf{S}}(t)\, U(t,\tau) f \right|^2 dt = \sup_x \int_0^\infty \left| U(t,\tau) P_s^L(\tau) f \right|^2 dt \lesssim \|f\|_{L^2}^2, \tag{3.6.28}$$

$$\sup_x \int_0^\infty \left| \left( P_s^{\mathbf{S}}(t)\, U(t,\tau) f \right)^S \right|^2 dt = \sup_x \int_0^\infty \left| U^S(t,\tau) P_s^L(\tau) f \right|^2 dt \lesssim \|f\|_{L^2}^2, \tag{3.6.29}$$

*where $U^S$ denotes the integration along the slanted line $(x+vt,t)$.*

*Proof.* This is just the particular cases of what we have discussed above. □

By Corollary 3.6.2, we have the weighted estimates for the inhomogeneous evolution.

**Lemma 3.6.3.** *For $\alpha > 3$, with $U(t,\tau)$ and projections $P_s^{\mathcal{S}}(t)$, $P_s^L(\tau)$ defined above, we have*

$$\left\| \langle x \rangle^{-\frac{\alpha}{2}} \int_0^t U(t,\tau) P_s^L(\tau) H(\tau)\, d\tau \right\|_{L_t^2 L_x^2} = \left\| \langle x \rangle^{-\frac{\alpha}{2}} \int_0^t P_s^{\mathbf{S}}(t)\, U(t,\tau) H(\tau)\, d\tau \right\|_{L_t^2 L_x^2}, \tag{3.6.30}$$

$$\left\| \langle x\rangle^{-\frac{\alpha}{2}} \int_0^t U(t,\tau) P_s^L(\tau) H(\tau)\, d\tau \right\|_{L_t^2 L_x^2} \lesssim \|H(t)\|_{L_t^1 L_x^2}\,, \tag{3.6.31}$$

$$\left\| \langle x\rangle^{-\frac{\alpha}{2}} \int_0^t U^S(t,\tau) P_s^L(\tau) H(\tau)\, d\tau \right\|_{L_t^2 L_x^2} = \left\| \langle x\rangle^{-\frac{\alpha}{2}} \int_0^t \left( P_s^{\mathbf{S}}(t)\,(t)\, U(t,\tau)\right)^S H(\tau)\, d\tau \right\|_{L_t^2 L_x^2}, \tag{3.6.32}$$

$$\left\| \langle x - vt\rangle^{-\frac{\alpha}{2}} \int_0^t U(t,\tau) P_s^L(\tau) H(\tau)\, d\tau \right\|_{L_t^2 L_x^2} \lesssim \|H(t)\|_{L_t^1 L_x^2}\,. \tag{3.6.33}$$

*Proof.* By the definition of projections, we have

$$\left\| \langle x\rangle^{-\frac{\alpha}{2}} \int_0^t U(t,\tau) P_s^L(\tau) H(\tau)\, d\tau \right\|_{L_t^2 L_x^2} = \left\| \langle x\rangle^{-\frac{\alpha}{2}} \int_0^t P_s^{\mathbf{S}}(t)\, U(t,\tau) H(\tau)\, d\tau \right\|_{L_t^2 L_x^2},$$

$$\left\| \langle x\rangle^{-\frac{\alpha}{2}} \int_0^t U^S(t,\tau) P_s^L(\tau) H(\tau)\, d\tau \right\|_{L_t^2 L_x^2} = \left\| \langle x\rangle^{-\frac{\alpha}{2}} \int_0^t \left( P_s^{\mathbf{S}}(t)\, U(t,\tau)\right)^S H(\tau)\, d\tau \right\|_{L_t^2 L_x^2}.$$

Applying Minkowski's inequality and Corollary 3.6.2, we have

$$\begin{aligned}
\left\| \langle x\rangle^{-\alpha} \int_0^t U(t,\tau) P_s^L(\tau) H(\tau)\, d\tau \right\|_{L_t^2 L_x^2} &\lesssim \left\| \langle x\rangle^{-\alpha} \int_0^t \left| U(t,\tau) P_s^L(\tau) H(\tau)\right|\, d\tau \right\|_{L_t^2 L_x^2} \\
&\lesssim \left\| \langle x\rangle^{-\alpha} \int_0^\infty \left| U(t,\tau) P_s^L(\tau) H(\tau)\right|\, d\tau \right\|_{L_t^2 L_x^2} \\
&\lesssim \int_0^\infty \left\| U(t,\tau) P_s^L(\tau) H(\tau)\right\|_{L_x^\infty L_t^2} d\tau \\
&\lesssim \|H(t)\|_{L_t^1 L_x^2}
\end{aligned}$$

and similarly,

$$\begin{aligned}
\left\| \langle x - vt\rangle^{-\alpha} \int_0^t U(t,\tau) P_s^L(\tau) H(\tau)\, d\tau \right\|_{L_t^2 L_x^2} &\lesssim \int_0^\infty \left\| U^S(t,\tau) P_s^L(\tau) H(\tau)\right\|_{L_x^\infty L_t^2} d\tau \\
&\lesssim \|H(t)\|_{L_t^1 L_x^2}\,.
\end{aligned}$$

The lemma is proved. $\square$

With the preparations above, we are ready to proceed to the analysis of inhomogeneous Strichartz estimates. As one can observe from previous sections on the homogeneous Strichartz estimates that it suffices to establish certain local decay estimates.

Now we set

$$\partial_{tt}u - \Delta u + V_1(x)u + V_2(x - vt)u = F \tag{3.6.34}$$

with initial data

$$u(x, 0) = g(x),\ u_t(x, 0) = f(x) \tag{3.6.35}$$

**Lemma 3.6.4.** *Suppose $u$ solves*

$$\partial_{tt}u - \Delta u + V_1(x)u + V_2(x - vt)u = F \tag{3.6.36}$$

*with initial data*

$$u(x, 0) = g(x),\ u_t(x, 0) = f(x). \tag{3.6.37}$$

*Then for $\alpha > 3$ $|v| < 1$, we have*

$$\left\| \langle x\rangle^{-\frac{\alpha}{2}}\, P_s^{\mathbf{S}}\,(t)\, u \right\|_{L_t^2([0,\infty),\, L_x^2)} \lesssim \|\nabla g\|_{L_x^2} + \|f\|_{L_x^2} + \|F\|_{L_t^1([0,\infty),\, L_x^2)}\,, \tag{3.6.38}$$

*and*

$$\left\| \langle x - vt\rangle^{-\frac{\alpha}{2}}\, P_s^{\mathbf{S}}\,(t)\, u \right\|_{L_t^2([0,\infty),\, L_x^2)} \lesssim \|\nabla g\|_{L_x^2} + \|f\|_{L_x^2} + \|F\|_{L_t^1([0,\infty),\, L_x^2)}\,. \tag{3.6.39}$$

*Proof.* By Duhamel's formula, we write

$$u(x, t) = U(t, 0)f + \dot{U}(t, 0)g + \int_0^t U(t, s)F(s)\, ds. \tag{3.6.40}$$

$$
\begin{aligned}
P_s^{\mathbf{S}}(t)\, u(x,t) &= P_s^{\mathbf{S}}(t)\left(U(t,0)f + \dot{U}(t,0)g\right) + \int_0^t P_s^{\mathbf{S}}(t)\, U(t,s)F(s)\, ds \\
&= P_s^{\mathbf{S}}(t)\left(U(t,0)f + \dot{U}(t,0)g\right) + \int_0^t U(t,s)P_s^{L}(s)F(s)\, ds. \quad (3.6.41)
\end{aligned}
$$

Applying the weighted norms, for the homogeneous part, we know

$$
\left\| \langle x \rangle^{-\alpha} P_s^{\mathbf{S}}(t)\left(U(t,0)f + \dot{U}(t,0)g\right)\right\|_{L_t^2 L_x^2} \lesssim \|\nabla g\|_{L_x^2} + \|f\|_{L_x^2} \quad (3.6.42)
$$

and

$$
\left\| \langle x - vt \rangle^{-\alpha} P_s^{\mathbf{S}}(t)\left(U(t,0)f + \dot{U}(t,0)g\right)\right\|_{L_t^2 L_x^2} \lesssim \|\nabla g\|_{L_x^2} + \|f\|_{L_x^2}. \quad (3.6.43)
$$

For the inhomogeneous part, by our discussion above, one has

$$
\left\| \langle x \rangle^{-\alpha} \int_0^t U(t,s)P_s^{L}(s)F(s)\, ds \right\|_{L_t^2 L_x^2} \lesssim \|F\|_{L_t^1 L_x^2}, \quad (3.6.44)
$$

and

$$
\left\| \langle x - vt \rangle^{-\alpha} \int_0^t U(t,s)P_s^{L}(s)F(s)\, ds \right\|_{L_t^2 L_x^2} \lesssim \|F\|_{L_t^1 L_x^2}. \quad (3.6.45)
$$

Therefore, one can conclude that

$$
\left\| \langle x \rangle^{-\frac{\alpha}{2}} P_s^{\mathbf{S}}(t)\, u \right\|_{L_t^2([0,\infty),\, L_x^2)} \lesssim \|\nabla g\|_{L_x^2} + \|f\|_{L_x^2} + \|F\|_{L_t^1([0,\infty),\, L_x^2)}, \quad (3.6.46)
$$

$$
\left\| \langle x - vt \rangle^{-\frac{\alpha}{2}} P_s^{\mathbf{S}}(t)\, u \right\|_{L_t^2([0,\infty),\, L_x^2)} \lesssim \|\nabla g\|_{L_x^2} + \|f\|_{L_x^2} + \|F\|_{L_t^1([0,\infty),\, L_x^2)}. \quad (3.6.47)
$$

The lemma is proved. □

With the decay estimate Lemma 3.6.4, we can establish Strichartz estimates using almost identical procedures as for the homogeneous Strichartz estimates.

**Theorem 3.6.5.** *Let $|v|<1$ and suppose $u$ solves*

$$\partial_{tt}u-\Delta u+V_1(x)u+V_2(x-vt)u=F \tag{3.6.48}$$

*with initial data*

$$u(x,0)=g(x),\ u_t(x,0)=f(x). \tag{3.6.49}$$

*Then for $p$, $\tilde{p}>2$, and*

$$\frac{1}{2}=\frac{1}{p}+\frac{3}{q},\ \frac{1}{2}=\frac{1}{\tilde{p}}+\frac{3}{\tilde{q}} \tag{3.6.50}$$

*we have*

$$\left\|P_s^{\mathbf{S}}(t)\,u\right\|_{L_t^p([0,\infty),\,L_x^q)}\lesssim\|f\|_{L^2}+\|g\|_{\dot{H}^1}+\|F\|_{L_t^{\tilde{p}'}\left([0,\infty),\,L_x^{\tilde{q}'}\right)\cap L_t^1([0,\infty),\,L_x^2)} \tag{3.6.51}$$

*where $\tilde{p}'$, $\tilde{q}'$ are Hölder conjugate of $\tilde{p}$, $\tilde{q}$.*

*Proof.* The proof is almost identical to the one for Theorem 3.5.2. But we need some preliminary calculations. By Lemma 3.6.1, we know

$$P_s(s)S(s,\tau)=S(s,\tau)P_s(\tau). \tag{3.6.52}$$

Differentiating (3.6.52) with respect to $s$ and then setting both $\tau=s=t$, we have

$$\dot{P}_s(t)=-JH(t)P_s(t)+P_s(t)JH(t). \tag{3.6.53}$$

Just as we discussed about projections, we write

$$\partial_{tt}u-\Delta u+V_1(x)u+V_2(x-vt)u=F \tag{3.6.54}$$

as a system:

$$\partial_t \begin{pmatrix} u \\ \partial_t u \end{pmatrix} - \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} -\Delta + V_1(x) + V_2(x-vt) & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} u \\ \partial_t u \end{pmatrix} = \begin{pmatrix} 0 \\ F(t) \end{pmatrix}. \tag{3.6.55}$$

Then

$$P_s(t)\partial_t \begin{pmatrix} u \\ \partial_t u \end{pmatrix} - P_s(t) \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} -\Delta + V_1(x) + V_2(x-vt) & 0 \\ 0 & 1 \end{pmatrix} = P_s(t) \begin{pmatrix} 0 \\ F(t) \end{pmatrix} \tag{3.6.56}$$

which is

$$P_s(t)\dot{U}(t) - P_s(t)JH(t) = P_s(t)F(t). \tag{3.6.57}$$

By equations (3.6.52) and (3.6.57), one has

$$\frac{d}{dt}\left(P_s(t)U(t)\right) - JH(t)P_s(t)U(t) = P_s(t)F(t). \tag{3.6.58}$$

Hence returning to our scalar setting, we have

$$\partial_{tt}\left(P_s^{\mathbf{S}}(t)\,u\right) + \left(-\Delta + V_1(x) + V_2(x-vt)\right) P_s^{\mathbf{S}}(t)\,u = P_s^{\mathbf{S}}(t)\,F(t). \tag{3.6.59}$$

Now we are ready to proceed to the Strichartz estimates argument similar to the case in Theorem 3.5.2.

Again, following [40], setting $A = \sqrt{-\Delta}$ and taking

$$U(t) = AP_s^{\mathbf{S}}(t)\,u(t) + i\partial_t\left(P_s^{\mathbf{S}}(t)\,u(t)\right),$$

then $U$ satisfies

$$i\partial_t U = AU + V_1 P_s^{\mathbf{S}}(t)\, u(t) + V_2(x - vt)\, P_s^{\mathbf{S}}(t)\, u(t) + P_s^{\mathbf{S}}(t)\, F, \tag{3.6.60}$$

By Duhamel's formula,

$$U(t) = e^{itA}U(0) - i\int_0^t e^{-i(t-s)A}\left(V_1 P_s^{\mathbf{S}}(s)\, u(s) + V_2(\cdot - vs)\, P_s^{\mathbf{S}}(s)\, u(s) + P_s^{\mathbf{S}}(s)\, F(s)\right) ds. \tag{3.6.61}$$

Let $P := A^{-1}\Re$, then from Strichartz estimates for the free evolution,

$$\left\| P e^{itA} U(0) \right\| \lesssim \|U(0)\|_{L^2}, \tag{3.6.62}$$

and

$$\left\| \int_0^t e^{-i(t-s)A} P_s^{\mathbf{S}}(s)\, F(s)\, ds \right\|_{L_t^p L_x^q} \lesssim \|F\|_{L_t^{\tilde{p}'} L_x^{\tilde{q}'}}. \tag{3.6.63}$$

As in the proof of Theorem 3.5.2, writing $V_1 = V_3 V_4$, $V_2 = V_5 V_6$ , it suffices to bound

$$\left\| P \int_0^\infty e^{-i(t-s)A} V_3 V_4 P_s^{\mathbf{S}}(s)\, u(s)\, ds \right\|_{L_t^p L_x^q}, \tag{3.6.64}$$

and

$$\left\| P \int_0^\infty e^{-i(t-s)A} V_5 V_6(\cdot - vs) P_s^{\mathbf{S}}(s)\, u(s)\, ds \right\|_{L_t^p L_x^q}. \tag{3.6.65}$$

In the same manner as we did in the proof of Theorem 3.5.2, one has

$$\left\| P \int_0^\infty e^{-i(t-s)A} V_3 V_4 P_s^{\mathbf{S}}(s)\, u(s)\, ds \right\|_{L_t^p L_x^q} \lesssim \left\| V_4 P_s^{\mathbf{S}}(t)\, u \right\|_{L_t^2 L_x^2}, \tag{3.6.66}$$

$$\left\| P \int_0^\infty e^{-i(t-s)A} V_5 V_6(\cdot - vs) P_s^{\mathbf{S}}(s)\, u(s)\, ds \right\|_{L_t^p L_x^q} \lesssim \left\| V_6(x - vt) P_s^{\mathbf{S}}(t)\, u \right\|_{L_t^2 L_x^2}. \tag{3.6.67}$$

By estimates (3.6.38) and (3.6.39) from Lemma 3.6.4,

$$\left\| V_4 P_s^{\mathbf{S}}(t)\, u \right\|_{L_t^2 L_x^2} \lesssim \|\nabla g\|_{L_x^2} + \|f\|_{L_x^2} + \|F\|_{L_t^1 L_x^2}, \tag{3.6.68}$$

$$\left\| V_6(x-vt) P_s^{\mathbf{S}}(t)\, u \right\|_{L_t^2 L_x^2} \lesssim \|\nabla g\|_{L_x^2} + \|f\|_{L_x^2} + \|F\|_{L_t^1 L_x^2}. \tag{3.6.69}$$

Therefore, by the same argument as for the homogeneous Strichartz estimates, we have

$$\|P_s^{\mathbf{S}}(t)\, u\|_{L_t^p([0,\infty),\, L_x^q)} \lesssim \|f\|_{L^2} + \|g\|_{\dot{H}^1} + \|F\|_{L_t^{\tilde{p}'}\left([0,\infty),\, L_x^{\tilde{q}'}\right) \cap L_t^1([0,\infty),\, L_x^2)}. \tag{3.6.70}$$

as claimed. □

From the discussions above, we can also conclude the endpoint reversed Strichartz estimate.

**Theorem 3.6.6.** *Let $|v| < 1$ and suppose $u$ solves*

$$\partial_{tt} u - \Delta u + V_1(x) u + V_2(x - vt) u = F \tag{3.6.71}$$

*with initial data*

$$u(x,0) = g(x),\ u_t(x,0) = f(x). \tag{3.6.72}$$

*Then we have*

$$\sup_x \int_0^\infty \left| P_s^{\mathbf{S}}(t)\, u \right|^2 dt \lesssim \left( \|f\|_{L^2} + \|g\|_{\dot{H}^1} + \|F\|_{L_t^1([0,\infty),\, L_x^2)}. \right)^2 \tag{3.6.73}$$

Taking the case $p = q$ in the regular Strichartz estimate and interpolating it with the endpoint reversed Strichartz estimate (3.6.73), we obtain more reversed Strichartz estimates.

**Corollary 3.6.7.** *Let $|v| < 1$ and suppose $u$ solves*

$$\partial_{tt}u - \Delta u + V_1(x)u + V_2(x - vt)u = F \tag{3.6.74}$$

*with initial data*

$$u(x,0) = g(x),\ u_t(x,0) = f(x).$$

*Then for*

$$2 \leq p,\ \tilde{p} \leq 8 \tag{3.6.75}$$

*and*

$$\frac{1}{2} = \frac{1}{p} + \frac{3}{q},\ \frac{1}{2} = \frac{1}{\tilde{p}} + \frac{3}{\tilde{q}} \tag{3.6.76}$$

*we have*

$$\left\| P_s^{\mathbf{S}}(t)\, u \right\|_{L_x^q\left(\mathbb{R}^3,\, L_t^p[0,\infty)\right)} \lesssim \|f\|_{L^2} + \|g\|_{\dot{H}^1} + \|F\|_{L_x^{\tilde{q}'}\left(\mathbb{R}^3,\, L_t^{\tilde{p}'}[0,\infty)\right)\cap L_t^1([0,\infty),\, L_x^2)}. \tag{3.6.77}$$

*where $\tilde{p}'$, $\tilde{q}'$ are Hölder conjugate of $\tilde{p}$, $\tilde{q}$.*

We also have the endpoint Strichartz estimate with norm inhomogeneous with respect to radial and angular variables

**Theorem 3.6.8.** *Let $|v| < 1$ and suppose $u$ solves*

$$\partial_{tt}u - \Delta u + V_1(x)u + V_2(x - vt)u = F \tag{3.6.78}$$

*with initial data*

$$u(x,0) = g(x),\ u_t(x,0) = f(x). \tag{3.6.79}$$

*Then for $1 \leq p < \infty$, we have*

$$\|P_s^{\mathbf{S}}(t)\, u\|_{L_t^p([0,\infty),\, L_r^\infty L_\omega^p)} \lesssim \|f\|_{L^2} + \|g\|_{\dot{H}^1} + \|F\|_{L_t^1([0,\infty),\, L_x^2)}\,. \tag{3.6.80}$$

### 3.6.3 Reversed endpoint Strichartz estimates with inhomogeneous terms in revered norms

In some nonlinear applications, the interactions among potentials and solitons are strong which cause the inhomogeneous terms is not in $L_t^1 L_x^2$. So to finish this section, we discuss the reversed endpoint Strichartz estimates with inhomogeneous terms in revered norms. We need a slightly different formulation. As we did in the homogeneous, we recall the definition of scattering states in inhomogeneous setting.

**Definition 3.6.9.** (scattering states) Let

$$\partial_{tt} u - \Delta u + V_1(x) u + V_2(x - vt) u = F, \tag{3.6.81}$$

with initial data

$$u(x,0) = g(x),\ u_t(x,0) = f(x). \tag{3.6.82}$$

If $u$ also satisfies

$$\|P_b\,(H_1)\, u(t)\|_{L_x^2} \to 0,\ \left\|P_b\,(H_2)\, u_L(t')\right\|_{L_{x'}^2} \to 0\ \ t, t' \to \infty, \tag{3.6.83}$$

we call it a scattering state.

Set the space $I$

$$I = \left\{ G(x,t) \in L_x^{\frac{3}{2},1} L_t^2 \bigcap L_{x_1}^1 L_{\widehat{x_1}}^{2,1} L_t^2 \bigcap L_{t,x}^2 \right\} \tag{3.6.84}$$

for the strong interactions terms.

**Theorem 3.6.10.** *Let $|v| < 1$. Suppose $u$ is a scattering state in the sense of Definition 3.6.9 which solves*

$$\partial_{tt}u - \Delta u + V_1(x)u + V_2(x - vt)u = F \tag{3.6.85}$$

*with initial data*

$$u(x, 0) = g(x),\ u_t(x, 0) = f(x). \tag{3.6.86}$$

*Then*

$$\left(\sup_{x\in\mathbb{R}^3} \int_0^\infty |u(x, t)|^2\, dt\right)^{\frac{1}{2}} \lesssim \|f\|_{L^2} + \|g\|_{\dot{H}^1} + \|F\|_I \tag{3.6.87}$$

*and*

$$\left(\sup_{x\in\mathbb{R}^3} \int_0^\infty |u(x + vt, t)|^2\, dt\right)^{\frac{1}{2}} \lesssim \|f\|_{L^2} + \|g\|_{\dot{H}^1} + \|F\|_I\,. \tag{3.6.88}$$

*One can replace $F$ in the above estimates by $F^S$.*

These estimates can be proved by the same ideas as the homogeneous case. Here we briefly sketch the arguments since many steps are identical as the homogeneous case.

First of all, we need the energy comparison. By similar arguments as we did as the homogeneous case, one has the following comparison results.

**Theorem 3.6.11.** *Let $|v| < 1$. Suppose*

$$\partial_{tt}u - \Delta u + V(x, t)u = F(x, t) \tag{3.6.89}$$

*and*

$$|V(x, \mu x_1)| \lesssim \frac{1}{\langle x\rangle^2} \tag{3.6.90}$$

*for* $0 \leq |\mu| < 1$. *Then*

$$\begin{aligned}\int |\nabla_x u(x_1,x_2,x_3,vx_1)|^2 + |\partial_t u(x_1,x_2,x_3,vx_1)|^2\,dx \\ \lesssim \int |\nabla_x u(x_1,x_2,x_3,0)|^2 + |\partial_t u(x_1,x_2,x_3,0)|^2\,dx \\ + \int_{\mathbb{R}}\int_{\mathbb{R}^3} |F(x,t)|^2\,dxdt\end{aligned} \tag{3.6.91}$$

*and*

$$\begin{aligned}\int |\nabla_x u(x_1,x_2,x_3,0)|^2 + |\partial_t u(x_1,x_2,x_3,0)|^2\,dx \\ \lesssim \int |\nabla_x u(x_1,x_2,x_3,vx_1)|^2 + |\partial_t u(x_1,x_2,x_3,vx_1)|^2\,dx \\ + \int_{\mathbb{R}}\int_{\mathbb{R}^3} |F(x,t)|^2\,dxdt\end{aligned} \tag{3.6.92}$$

*where the implicit constant depends on* $v$ *and* $V$.

From the theorem above, we know initial energy with respect to different frames stays comparable up to $\|F\|_{L^2_{t,x}}$. For a detailed proof, please see [12].

Next, from Section 3.3, we have all necessary reversed type estimates. So one has all the basic tools to run the bootstrap arguments as in the homogeneous case. We just need to understand the evolution of bound states more carefully. Nothing changes substantially but with one more inhomogeneous term in the ODE.

Let $u(x,t)$ be a scattering state. Following the notations from 3.4.2, we decompose the evolution as following,

$$u(x,t) = a(t)w(x) + b\left(\gamma(t - vx_1)\right)m_v(x,t) + r(x,t) \tag{3.6.93}$$

where

$$m_v(x,t) = m\left(\gamma\left(x_1 - vt\right), x_2, x_3\right).$$

With our decomposition, we know

$$P_c\left(H_1\right) r = r \tag{3.6.94}$$

and

$$P_c\left(H_2\right) r_L = r_L \tag{3.6.95}$$

where the Lorentz transformation $L$ makes $V_2$ stationary.

Plugging the evolution (3.6.93) into the equation (5.3.13) and taking inner product with $w$, we get

$$\begin{aligned} \ddot{a}(t) - \lambda^2 a(t) + a(t)\left\langle V_2\left(x - vt\right) w, w\right\rangle \\ + \left\langle V_2\left(x - vt\right)\left(b\left(\gamma(t - vx_1)\right) m_v\left(x,t\right) + r(x,t)\right), w\right\rangle = \left\langle F, w\right\rangle. \end{aligned} \tag{3.6.96}$$

One can write

$$\ddot{a}(t) - \lambda^2 a(t) + a(t)c(t) + h(t) + h_1(t) = 0, \tag{3.6.97}$$

where

$$c(t) := \left\langle V_2\left(x - vt\right) w, w\right\rangle, \tag{3.6.98}$$

$$h_1\left(t\right) = \left\langle F, w\right\rangle \tag{3.6.99}$$

and

$$h(t) := \left\langle V_2\left(x - vt\right)\left(b\left(\gamma(t - vx_1)\right) m_v\left(x,t\right) + r(x,t)\right), w\right\rangle. \tag{3.6.100}$$

The existence of the solution to the ODE (3.6.97) is clear. We study the long-time behavior of the solution. Write the equation as

$$\ddot{a}(t) - \lambda^2 a(t) = -\left[a(t)c(t) + h(t) + h_1(t)\right], \tag{3.6.101}$$

and denote

$$N(t) := -\left[a(t)c(t) + h(t) + h_1(t)\right]. \tag{3.6.102}$$

Then

$$a(t) = \frac{e^{\lambda t}}{2}\left[a(0) + \frac{1}{\lambda}\dot{a}(0) + \frac{1}{\lambda}\int_0^t e^{-\lambda s} N(s)\, ds\right] + R(t) \tag{3.6.103}$$

where

$$|R(t)| \lesssim e^{-\beta t}, \tag{3.6.104}$$

for some positive constant $\beta > 0$. Therefore, the stability condition forces

$$a(0) + \frac{1}{\lambda}\dot{a}(0) + \frac{1}{\lambda}\int_0^\infty e^{-\lambda s} N(s)\, ds = 0. \tag{3.6.105}$$

Then under the stability condition (3.6.105),

$$a(t) = e^{-\lambda t}\left[a(0) + \frac{1}{2\lambda}\int_0^\infty e^{-\lambda s} N(s) ds\right] + \frac{1}{2\lambda}\int_0^\infty e^{-\lambda|t-s|} N(s)\, ds. \tag{3.6.106}$$

As in the homogeneous case, we just need to estimate the non-local term,

$$\int_0^\infty e^{-\lambda s} N(s)\, ds. \tag{3.6.107}$$

The same idea as the homogeneous case, for $t \in [0, T]$, we construct the following truncated version of the evolution:

$$u_T(x,t) == a_T(t)w(x) + b_T\left(\gamma(t - vx_1)\right) m_v\left(x,t\right) + r_T(x,t). \tag{3.6.108}$$

For $a_T(t)$, we analyze the same ODE for $a(t)$ again but restricted to $[0, T]$ and instead of the stability condition

$$a(0)+\frac{1}{\lambda}\dot{a}(0)+\frac{1}{\lambda}\int_0^{\infty} e^{-\lambda s}N(s)\,ds=0 \tag{3.6.109}$$

we impose the condition that

$$a_T(0)+\frac{1}{\lambda}\dot{a}_T(0)+\frac{1}{\lambda}\int_0^{T} e^{-\lambda s}N(s)\,ds=0. \tag{3.6.110}$$

The same construction can be applied to $b_T$.

In current setting, we only estimate the $L^2$ norms of $a_T$ and $b_T$.

**Lemma 3.6.12.** *From the construction above, we have the following estimates: for* $0 \ll A \ll T$*,*

$$\|a_T\|_{L^\infty[0,T]} \lesssim \left(C(A,\lambda)+\frac{1}{\lambda A}C_1(T)\right)\left(\|f\|_{L^2}+\|g\|_{\dot{H}^1}+\|F\|_I\right), \tag{3.6.111}$$

$$\|a_T\|_{L^2[0,T]} \lesssim \left(C(A,\lambda)+\frac{1}{\lambda A}C_1(T)\right)\left(\|f\|_{L^2}+\|g\|_{\dot{H}^1}+\|F\|_I\right), \tag{3.6.112}$$

$$\|b_T\|_{L^\infty[0,T]} \lesssim \left(C(A,\mu)+\frac{1}{\mu A}C_1(T)\right)\left(\|f\|_{L^2}+\|g\|_{\dot{H}^1}+\|F\|_I\right), \tag{3.6.113}$$

*and*

$$\|b_T\|_{L^2[0,T]} \lesssim \left(C(A,\mu)+\frac{1}{\mu A}C_1(T)\right)\left(\|f\|_{L^2}+\|g\|_{\dot{H}^1}+\|F\|_I\right). \tag{3.6.114}$$

*Proof.* First of all, as in the homogeneous case, by the bootstrap assumption,

$$\left\|b_T\left(\gamma(t-vx_1)\right)m_v\left(x,t\right)+r_T(x,t)\right\|_{L_x^\infty L_t^2[0,T]} \leq C_1(T)\left(\|f\|_{L^2}+\|g\|_{\dot{H}^1}+\|F\|_I\right).$$

For $a_T(t)$, we know that

$$\ddot{a}_T(t) - \lambda^2 a_T(t) + a_T(t)\left\langle V_2\left(x - vt\right) w, w\right\rangle$$
$$+\left\langle V_2\left(x-vt\right)\left(b_T\left(\gamma(t - vx_1)\right) m_v\left(x,t\right) + r_T(x,t)\right), w\right\rangle = \langle w, F\rangle . \quad (3.6.115)$$

We obtain

$$a_T(t) = \frac{e^{\lambda t}}{2}\left[a_T(0) + \frac{1}{\lambda}\dot{a}_T(0) + \frac{1}{\lambda}\int_0^t e^{-\lambda s} N(s) ds\right] + R(t) \quad (3.6.116)$$

where

$$|R(t)| \lesssim e^{-\beta t}, \quad (3.6.117)$$

With notations introduced above, we consider the truncated version of the stability condition,

$$a_T(0) + \frac{1}{\lambda}\dot{a}_T(0) + \frac{1}{\lambda}\int_0^T e^{-\lambda s} N(s)\, ds = 0. \quad (3.6.118)$$

So

$$a_T(t) = e^{-\lambda t}\left[a_T(0) + \frac{1}{2\lambda}\int_0^T e^{-\lambda s} N(s) ds\right] + \frac{1}{2\lambda}\int_0^T e^{-\lambda|t-s|} N(s)\, ds. \quad (3.6.119)$$

where

$$N(t) = -\left[a_T(t)c(t) + h(t) + h_1(t)\right] \quad (3.6.120)$$

with

$$|c(t)| \lesssim e^{-\alpha|t|},\ h_1(t) = \langle w, F\rangle \quad (3.6.121)$$

$$h(t) := \left\langle V_2\left(x - vt\right)\left[b_T(t - vx_1) m_v\left(x,t\right) + r_T(x,t)\right], w\right\rangle. \quad (3.6.122)$$

For $0 \ll A \ll T$ fixed, we can always bound the $L^\infty$ norm of $a_T$ on the interval $[0, A]$ by Grönwall's inequality. Therefore, it suffices to estimate the $L^\infty$ norm of $a_T$ from $A$ to $T$.

Note that $|c(t)| \lesssim e^{-\alpha|t|}$, for $A$ large, one can always absorb the effects from $\int_A^T a_T(t)c(t)\,dt$ into the left-hand side. Hence it reduces to estimate the $L_t^1$ norm of $h(t)$ restricted to $[A,T]$ and the $L_t^2$ norm of $h_1(t)$.

From the computations in 3.4.2, we have

$$\int_A^T |h(t)|\, dt \lesssim C(A,\lambda)\left(\|f\|_{L^2} + \|g\|_{\dot{H}^1} + \|F\|_I\right) + \frac{1}{\lambda A}\left(\int_A^T |(b_T(\gamma(t-vx_1))\, m_v(x,t) + r_T(x,t))|^2\, dt\right)^{\frac{1}{2}}. \tag{3.6.123}$$

Clearly,

$$\int_0^T |h_1(t)|^2 \lesssim \int_0^T \|F(t)\|_{L_x^2}\, dt. \tag{3.6.124}$$

We can estimate the $L^\infty$ norm of $a_T(t)$,

$$\begin{aligned} \|a_T\|_{L^\infty[0,T]} &\lesssim C(A,\lambda)\left(\|f\|_{L^2} + \|g\|_{\dot{H}^1} + \|F\|_I\right) \\ &\quad + + \frac{1}{\lambda}\int_A^T |h(t)|\,dt + \frac{1}{\lambda}\left(\int_0^T |h_1(t)|^2\right)^{\frac{1}{2}} \\ &\lesssim C(A,\lambda)\left(\|f\|_{L^2} + \|g\|_{\dot{H}^1} + \|F\|_I\right) \\ &\quad + \frac{1}{\lambda A}\left(\int_A^T |(b_T(\gamma(t-vx_1))\, m_v(x,t) + r_T(x,t))|^2\, dt\right)^{\frac{1}{2}} \\ &\lesssim \left(C(A,\lambda) + \frac{1}{\lambda A}C_1(T)\right)\left(\|f\|_{L^2} + \|g\|_{\dot{H}^1} + \|F\|_I\right). \end{aligned} \tag{3.6.125}$$

Similarly, for the $L^2$ norm of $a_T(t)$,

$$\|a_T\|_{L^2[0,T]} \lesssim \left(C(A,\lambda) + \frac{1}{\lambda A}C_1(T)\right)\left(\|f\|_{L^2} + \|g\|_{\dot{H}^1} + \|F\|_I\right). \tag{3.6.126}$$

After applying a Lorentz transformation, we have analogous estimates for $b_T(t)$:

$$\|b_T\|_{L^\infty[0,T]} \lesssim \left(C(A,\mu) + \frac{1}{\mu A}C_1(T)\right)\left(\|f\|_{L^2} + \|g\|_{\dot{H}^1} + \|F\|_I\right), \tag{3.6.127}$$

$$\|b_T\|_{L^2[0,T]} \lesssim \left(C(A,\mu) + \frac{1}{\mu A}C_1(T)\right)\left(\|f\|_{L^2} + \|g\|_{\dot{H}^1} + \|F\|_I\right). \tag{3.6.128}$$

The lemma is proved. □

With the preparations above, we can run the bootstrap argument and channel decomposition as in the homogeneous case.

*Proof of Theorem 3.6.10.* As in the homogeneous case, let $\chi_1(x,t)$ be a smooth cutoff function such that

$$\chi_1(x,t) = 1,\ \forall x \in B_{\delta t}(0), \qquad \chi_1(x,t) = 0,\ \forall x \in \mathbb{R}^3 \backslash B_{2\delta t}(0). \tag{3.6.129}$$

One might assume $t \geq t_0$ for some large $t_0$. We also define

$$\chi_2(x,t) = \chi_1(x - vt, t), \qquad \chi_3 = 1 - \chi_1 - \chi_2. \tag{3.6.130}$$

Note that we only consider the estimates for large $t$, so one might also assume the support of $\chi_1(x,t)$ contains the support of $V_1(x)$ and support of $\chi_2(x,t)$ contains the support of $V_2(\cdot - vt)$.

With the partition above, we rewrite the evolution as

$$u_T(x,t) = \chi_1(x,t)u_T(x,t) + \chi_2(x,t)u_T(x,t) + \chi_3(x,t)u_T(x,t). \tag{3.6.131}$$

We will discuss $\chi_i(x,t)u_T(x,t)$, $i = 1,2,3$, separately. We only analyze $\chi_1(x,t)u_T(x,t)$ here since other pieces are can be done in the same manner in the homogeneous case. T

$$\begin{aligned} \chi_1(x,t)u_T(x,t) &= \chi_1(x,t)W_1(t)f + \chi_1(x,t)\dot{W}_1(t)g \\ &-\chi_1(x,t)\int_0^t W_1(t-s)V_2(\cdot - sv)u_T(s)\,ds \\ &+\chi_1(x,t)\int_0^t W_1(t-s)F(s)\,ds. \end{aligned} \tag{3.6.132}$$

We will use the notations

$$\begin{aligned} u_T(x,t) &= a_T(t)w(x) + b_T\left(\gamma(t-vx_1)\right)m_v\left(x,t\right) + r_T(x,t) \\ &=: a_T(t)w(x) + u_{T,1}\left(x,t\right) \\ &=: b_T\left(\gamma(t-vx_1)\right)m_v\left(x,t\right) + u_{T,2}(x,t). \end{aligned} \tag{3.6.133}$$

Note that

$$P_c\left(H_1\right)\left(u_{T,1}\right) = u_{T,1} \tag{3.6.134}$$

and

$$P_c\left(H_2\right)\left(u_{T,2}\right)_L = \left(u_{T,2}\right)_L. \tag{3.6.135}$$

As in the homogeneous case, we can further reduce to

$$\begin{aligned} \chi_1(x,t)u_T(x,t) &= \chi_1(x,t)W_1(t)f + \chi_1(x,t)\dot{W}_1(t)g \\ &-\chi_1(x,t)\int_0^{t-A} W_1(t-s)V_2(\cdot - sv)u_T(s)\,ds \\ &+\chi_1(x,t)\int_0^t W_1(t-s)F(s)\,ds. \end{aligned} \tag{3.6.136}$$

First, we consider the endpoint reversed Strichartz estimate,

$$
\begin{aligned}
\int_0^T \left|\chi_1(x,t)u_{T,1}(x,t)\right|^2 dt \;\lesssim\;& \int_0^T \left|\chi_1(x,t)W_1(t)P_c\left(H_1\right)f + \chi_1(x,t)\dot{W}_1(t)P_c\left(H_1\right)g\right|^2 dt \\
&+ \int_0^T \left|\chi_1(x,t)\int_0^{t-A} W_1(t-s)P_c\left(H_1\right)V_2(\cdot - sv)u_T(s)\,ds\right|^2 dt \\
&\int_0^T \left|\chi_1(x,t)\chi_1(x,t)\int_0^t W_1(t-s)P_c\left(H_1\right)F(s)\,ds\right|^2 dt \\
\lesssim\;& \left(\|f\|_{L^2} + \|g\|_{\dot{H}^1} + \|F\|_I\right)^2 \\
&+ \int_0^B \left|\chi_1(x,t)\int_0^t W_1(t-s)P_c\left(H_1\right)V_2(\cdot - sv)u_T(s)\,ds\right|^2 dt \\
&+ \int_B^T \left|\chi_1(x,t)\int_0^{t-A} W_1(t-s)P_c\left(H_1\right)V_2(\cdot - sv)u_T(s)\,ds\right|^2 \\
\lesssim\;& \left(\|f\|_{L^2} + \|g\|_{\dot{H}^1} + \|F\|_I\right)^2 + C(B)\left(\|f\|_{L^2} + \|g\|_{\dot{H}^1} + \|F\|_I\right)^2 \\
&+ \int_B^T \left|\chi_1(x,t)\int_0^{t-A} W_1(t-s)P_c\left(H_1\right)V_2(\cdot - sv)u_T(s)\,ds\right|^2 dt \\
\lesssim\;& \left(\|f\|_{L^2} + \|g\|_{\dot{H}^1} + \|F\|_I\right)^2 + C(B)\left(\|f\|_{L^2} + \|g\|_{\dot{H}^1} + \|F\|_I\right)^2 \\
&+ \frac{1}{A}C_2(T)\left(\|f\|_{L^2} + \|g\|_{\dot{H}^1} + \|F\|_I\right)^2.
\end{aligned}
$$

Therefore,

$$
\int_0^T \left|\chi_1(x,t)u_{T,1}(x,t)\right|^2 dt \lesssim \left(C_0 + C(A) + \frac{1}{A}C_2(T)\right)\left(\|f\|_{L^2} + \|g\|_{\dot{H}^1} + \|F\|_I\right)^2. \tag{3.6.137}
$$

For the remaining piece, by Lemma 3.6.12

$$
\int_0^T \left|\chi_1(x,t)a_T(t)w(x)\right|^2 dt \lesssim \left(C(A,\lambda) + \frac{1}{\lambda A}C_1(T)\right)\left(\|f\|_{L^2} + \|g\|_{\dot{H}^1} + \|F\|_I\right)^2. \tag{3.6.138}
$$

Therefore, with estimates (3.6.137) and (3.6.138), for the endpoint reversed estimate, we obtain

$$C_1(T) \lesssim C_0 + C(A,B) + \frac{1}{A} C_2(T) \tag{3.6.139}$$

in the first channel. So for $A$ large, in this channel, we have the condition for the bootstrap argument.

Next we consider the estimate along the slanted line $(x+vt,t)$.

Denoting

$$u^S_{T,1}(x,t) = \chi_1(x+vt,t)u_{T,1}(x+vt,t), \tag{3.6.140}$$

we want to estimate

$$\int_0^T \left|\chi_1(x+vt,t)u_{T,1}(x+vt,t)\right|^2 dt = \int_0^T \left|u^S_{T,1}(x,t)\right|^2 dt. \tag{3.6.141}$$

Furthermore, we introduce

$$D^S_1(x,t) := D_1\left(x+vt,t\right) \tag{3.6.142}$$

where

$$D_1(x,t) := \chi_1(x,t)W_1(t)P_c\left(H_1\right)f + \chi_1(x,t)\dot{W}_1(t)P_c\left(H_1\right)g; \tag{3.6.143}$$

$$k^S_1(x,t) := k_1\left(x+vt,t\right) \tag{3.6.144}$$

where

$$k_1(x,t) := \chi_1(x,t)\int_0^t W_1(t-s)P_c\left(H_1\right)V_2(\cdot - sv)u_T(s)\,ds; \tag{3.6.145}$$

$$E^S_1(x,t) := E_1\left(x+vt,t\right) \tag{3.6.146}$$

where

$$E_1(x,t) := \chi_1(x,t)\int_0^{t-A} W_1(t-s)P_c\left(H_1\right)V_2(\cdot - sv)u_T(s)\,ds. \tag{3.6.147}$$

$$E_2(x,t) := \chi_1(x,t) \int_0^t W_1(t-s) P_c(H_1) F(s) u_T(s)\, ds.$$

Then we can conclude

$$\begin{aligned} \int_0^T \left| u_{T,1}^S \right|^2 dt &\lesssim \int_0^T \left| D_1^S \right|^2 dt + \int_0^B \left| k_1^S \right|^2 dt + \int_B^T \left| E_1^S \right|^2 dt + \int_0^T \left| E_2^S \right|^2 dt \\ &\lesssim \left( \|f\|_{L^2} + \|g\|_{\dot{H}^1} + \|F\|_I \right)^2 + C(B) \left( \|f\|_{L^2} + \|g\|_{\dot{H}^1} + \|F\|_I \right)^2 \\ &\quad + \frac{1}{A} C_2(T) \left( \|f\|_{L^2} + \|g\|_{\dot{H}^1} + \|F\|_I \right)^2 . \end{aligned} \tag{3.6.148}$$

For the piece with bound states, by Lemma 3.6.12 and Agmon's estimate,

$$\begin{aligned} &\int_0^T \left| \chi_1(x+vt,t) a_T(t) w(x+vt) \right|^2 dt \\ \lesssim &\left( C(A,\lambda) + \frac{1}{\lambda A} C_1(T) \right) \left( \|f\|_{L^2} + \|g\|_{\dot{H}^1} + \|F\|_I \right)^2 . \end{aligned} \tag{3.6.149}$$

Therefore, with estimates (3.6.148) and (3.6.149), we obtain

$$C_2(T) \lesssim C_0 + C(A,B) + \frac{1}{A} C_2(T) \tag{3.6.150}$$

in the first channel. So for $A$ large, in this channel, we obtain the desired reduction for the bootstrap argument.

The other two channels can be analyzed by the same steps as above. Therefore, after passing $T$ to $\infty$, we can conclude that

$$\left( \sup_{x\in\mathbb{R}^3} \int_0^\infty |u(x,t)|^2 dt \right)^{\frac{1}{2}} \lesssim \|f\|_{L^2} + \|g\|_{\dot{H}^1} + \|F\|_I \tag{3.6.151}$$

and

$$\left(\sup_{x\in\mathbb{R}^3}\int_0^\infty |u(x+vt,t)|^2\,dt\right)^{\frac{1}{2}} \lesssim \|f\|_{L^2} + \|g\|_{\dot{H}^1} + \|F\|_I\,. \tag{3.6.152}$$

The same arguments work for $F$ replaced by $F^S$.

We are done. □

### *3.6.4 Reversed type local decay estimates*

To handle the strong interactions of solitons and potentials, in this subsection we establish some reversed type local decay estimates. These estimates are also important to handle multisoliton structures as in [15].

**Theorem 3.6.13.** *Let $|v| < 1$. Suppose $u$ is a scattering state in the sense of Definition 3.6.9 which solves*

$$\partial_{tt}u - \Delta u + V_1(x)u + V_2(x-vt)u = F \tag{3.6.153}$$

*with initial data*

$$u(x,0) = g(x),\ u_t(x,0) = f(x). \tag{3.6.154}$$

*Then*

$$\left\|\langle x\rangle^{-3}\,u(x,t)\right\|_{L_x^{\frac{3}{2},1}L_t^\infty \bigcap L^1_{x_1}L^{2,1}_{\widehat{x_1}}L^\infty_t} \lesssim \|f\|_{L^2} + \|g\|_{\dot{H}^1} + \|F\|_D \tag{3.6.155}$$

$$\left\|\langle x\rangle^{-3}\,u(x+vt,t)\right\|_{L_x^{\frac{3}{2},1}L_t^\infty \bigcap L^1_{x_1}L^{2,1}_{\widehat{x_1}}L^\infty_t} \lesssim \|f\|_{L^2} + \|g\|_{\dot{H}^1} + \|F\|_D\,. \tag{3.6.156}$$

*Here the space $D$ is*

$$D := \left\{ G(x,t) \in L_x^{\frac{3}{2},1} L_t^\infty \bigcap L_{x_1}^1 L_{\widehat{x_1}}^{2,1} L_t^\infty \bigcap L_t^2 L_x^2 \right\}. \tag{3.6.157}$$

*Again, one can replace $F$ by $F^S$ in the above estimates. We can also replace the space $D$ by*

$$D_1 := \left\{ G(x,t) \in L_x^{\frac{3}{2},1} L_t^1 \bigcap L_{x_1}^1 L_{\widehat{x_1}}^{2,1} L_t^1 \bigcap L_t^2 L_x^2 \right\}. \tag{3.6.158}$$

This theorem can be proved by the same way as Theorem 3.6.10 provided we have the energy comparison and the related reversed type estimates for the free wave equations and perturbed equations.

We start with the free equation again. We set

$$u_F(x,t) = \frac{\sin\left(t\sqrt{-\Delta}\right)}{\sqrt{-\Delta}} f + \cos\left(t\sqrt{-\Delta}\right) g, \tag{3.6.159}$$

and

$$D(\cdot,t) = \int_0^t \frac{\sin\left((t-s)\sqrt{-\Delta}\right)}{\sqrt{-\Delta}} F(s)\, ds. \tag{3.6.160}$$

**Theorem 3.6.14.** *Let $|v| < 1$. Then first of all, for the standard case, one has*

$$\|u_F\|_{L_x^{6,2} L_t^\infty} \lesssim \|f\|_{L^2} + \|g\|_{\dot{H}^1}, \tag{3.6.161}$$

*in particular,*

$$\left\| \langle x \rangle^{-3} u_F \right\|_{L_x^{\frac{3}{2},1} L_t^\infty \bigcap L_{x_1}^1 L_{\widehat{x_1}}^{2,1} L_t^\infty} \lesssim \|f\|_{L^2} + \|g\|_{\dot{H}^1}. \tag{3.6.162}$$

*Also for the inhomogeneous term,*

$$\left\| \langle x \rangle^{-3} D \right\|_{L_x^{\frac{3}{2},1} L_t^\infty \bigcap L_{x_1}^1 L_{\widehat{x_1}}^{2,1} L_t^\infty} \lesssim \|F\|_{L_x^{\frac{3}{2},1} L_t^\infty}. \tag{3.6.163}$$

*We can also estimate these pieces along slanted lines and obtain*

$$\left\| u_F^S \right\|_{L_x^{6,2} L_t^\infty} \lesssim \|f\|_{L^2} + \|g\|_{\dot{H}^1}, \tag{3.6.164}$$

$$\left\| \langle x \rangle^{-3} u_F^S \right\|_{L_x^{\frac{3}{2},1} L_t^\infty \bigcap L^1_{x_1} L^{2,1}_{\widehat{x_1}} L_t^\infty} \lesssim \|f\|_{L^2} + \|g\|_{\dot{H}^1}, \tag{3.6.165}$$

*and*

$$\left\| \langle x \rangle^{-3} D^S \right\|_{L_x^{\frac{3}{2},1} L_t^\infty \bigcap L^1_{x_1} L^{2,1}_{\widehat{x_1}} L_t^\infty} \lesssim \|F\|_{L^1_{x_1} L^{2,1}_{\widehat{x_1}} L_t^\infty}. \tag{3.6.166}$$

*We can replace the $L_t^\infty$ norm of $F$ by the $L_t^1$ norm. Also one can replace $F$ in the above estimates by $F^S$.*

*Proof.* We will only prove (4.4.35). (4.4.67) is a consequence of estimate (4.4.35) after applying Hø"lder's inequality. (3.6.165) and (3.6.164) follow from estimate (4.4.35) after performing a Lorentz transformation and energy comparison as in Section 3.3. For the inhomogeneous estimates, we do the same arguments as in Section 3.3 with $L_t^2$ replaced by $L_t^\infty$. For example,

$$\begin{aligned}
\left\| \int_0^t \frac{\sin\left((t-s)\sqrt{-\Delta}\right)}{\sqrt{-\Delta}} F(s)\, ds \right\|_{L_t^\infty} &= \left\| \int_0^t \int_{|x-y|=t-s} \frac{1}{|x-y|} F(y,s)\, \sigma(dy)\, ds \right\|_{L_t^\infty} \\
&= \left\| \int_{|x-y|\le t} \frac{1}{|x-y|} F\left(y, t-|x-y|\right)\, dy \right\|_{L_t^\infty} \\
&\lesssim \int \frac{1}{|x-y|} \left\| F\left(y, t-|x-y|\right) \right\|_{L_t^2} dy \\
&\lesssim \sup_{x\in\mathbb{R}^3} \int \frac{1}{|x-y|} \left\| F(y,t) \right\|_{L_t^\infty} dy \qquad (3.6.167) \\
&\lesssim \|F\|_{L_x^{\frac{3}{2},1} L_t^\infty}.
\end{aligned}$$

Therefore,

$$\|D\|_{L_x^\infty L_t^\infty} \lesssim \|F\|_{L_x^{\frac{3}{2},1} L_t^\infty}, \tag{3.6.168}$$

and (4.4.63) follows after applying Hölder's inequality. On the other hand,

$$\begin{aligned}
\left\| \int_0^t \frac{\sin\left((t-s)\sqrt{-\Delta}\right)}{\sqrt{-\Delta}} F(s)\, ds \right\|_{L_t^\infty} &= \left\| \int_0^t \int_{|x-y|=t-s} \frac{1}{|x-y|} F(y,s)\, \sigma(dy)\, ds \right\|_{L_t^\infty} \\
&\lesssim \int \frac{1}{|x-y|} \left\| F(y,\cdot) \right\|_{L_t^1} dy \\
&\lesssim \|F\|_{L_x^{\frac{3}{2},1} L_t^1}. \qquad (3.6.169)
\end{aligned}$$

Hence,

$$\|D\|_{L_x^\infty L_t^\infty} \lesssim \|F\|_{L_x^{\frac{3}{2},1} L_t^1}. \qquad (3.6.170)$$

So as claimed in the statement, one can replace the $L_t^\infty$ norm for $F$ by the $L_t^1$ norm.

Now we prove (4.4.35). Consider $t \geq 0$ and define

$$Tf = \frac{\sin\left(t\sqrt{-\Delta}\right)}{\sqrt{-\Delta}} f \qquad (3.6.171)$$

then

$$T^*F = \int_0^\infty \frac{\sin\left(t\sqrt{-\Delta}\right)}{\sqrt{-\Delta}} F(t)\, dt, \qquad (3.6.172)$$

and

$$\begin{aligned}
TT^*F &= \int_0^\infty \frac{\sin\left(t\sqrt{-\Delta}\right)}{\sqrt{-\Delta}} \frac{\sin\left(s\sqrt{-\Delta}\right)}{\sqrt{-\Delta}} F(s)\, ds \qquad (3.6.173) \\
&= \frac{1}{2} \int_0^\infty \left( \frac{\cos\left((t-s)\sqrt{-\Delta}\right)}{-\Delta} - \frac{\cos\left((t+s)\sqrt{-\Delta}\right)}{-\Delta} \right) F(s)\, ds.
\end{aligned}$$

We compute the kernel of

$$\frac{\cos\left(h\sqrt{-\Delta}\right)}{-\Delta} F = \int_{\mathbb{R}^3} K(x,y,h) F(y)\, dy. \qquad (3.6.174)$$

By straightforward computations, one has

$$\frac{\cos\left(h\sqrt{-\Delta}\right)}{-\Delta}=\frac{1}{-\Delta}-\int_0^h\frac{\sin\left(s\sqrt{-\Delta}\right)}{\sqrt{-\Delta}}\,ds=\int_h^\infty\frac{\sin\left(s\sqrt{-\Delta}\right)}{\sqrt{-\Delta}}\,ds. \tag{3.6.175}$$

By the explicit kernel of $\frac{\sin(s\sqrt{-\Delta})}{\sqrt{-\Delta}}$, we know that

$$K(x,y,h)=\begin{cases}\frac{1}{|x-y|} & |x-y|\geq h\\ 0 & |x-y|<h\end{cases}. \tag{3.6.176}$$

Notice that in $\mathbb{R}^3$, $\frac{1}{|x|}\in L^{3,\infty}$, so

$$\left\|\int_0^\infty\left(\frac{\cos\left((t-s)\sqrt{-\Delta}\right)}{-\Delta}\right)F(s)\,ds\right\|_{L_x^{6,2}L_t^\infty}\lesssim\|F\|_{L_x^{\frac{6}{5},2}L_t^1} \tag{3.6.177}$$

by Young's inequality for convolution. It follows that

$$\|Tf\|_{L_x^{6,2}L_t^\infty}=\left\|\frac{\sin\left(t\sqrt{-\Delta}\right)}{\sqrt{-\Delta}}f\right\|_{L_x^{6,2}L_t^\infty}\lesssim\|f\|_{L^2}. \tag{3.6.178}$$

We are done. □

By the same arguments in Section 3.3, we can extend all the above estimates to perturbed cases. Define

$$u_H(x,t)=\frac{\sin\left(t\sqrt{H}\right)}{\sqrt{H}}P_cf+\cos\left(t\sqrt{H}\right)P_cg, \tag{3.6.179}$$

and

$$k(\cdot,t):=\int_0^t\frac{\sin\left((t-s)\sqrt{H}\right)}{\sqrt{H}}P_cF(s)\,ds. \tag{3.6.180}$$

**Theorem 3.6.15.** *Let* $|v|<1$ *and suppose* $H=-\Delta+V$ *has neither resonances nor* *eigenfunctions at* 0*. Then first of all, for the standard case, one has*

$$\|u_H\|_{L_x^{6,2}L_t^\infty} \lesssim \|f\|_{L^2} + \|g\|_{\dot{H}^1}, \tag{3.6.181}$$

*in particular,*

$$\left\|\langle x\rangle^{-3} u_H\right\|_{L_x^{\frac{3}{2},1}L_t^\infty \bigcap L_{x_1}^1 L_{\widehat{x_1}}^{2,1} L_t^\infty} \lesssim \|f\|_{L^2} + \|g\|_{\dot{H}^1}. \tag{3.6.182}$$

*Also for the inhomogeneous term,*

$$\left\|\langle x\rangle^{-3} k\right\|_{L_x^{\frac{3}{2},1}L_t^\infty \bigcap L_{x_1}^1 L_{\widehat{x_1}}^{2,1} L_t^\infty} \lesssim \|F\|_{L_x^{\frac{3}{2},1}L_t^\infty}. \tag{3.6.183}$$

*We can also estimate these pieces along slanted lines and obtain*

$$\left\|u_H^S\right\|_{L_x^{6,2}L_t^\infty} \lesssim \|f\|_{L^2} + \|g\|_{\dot{H}^1}, \tag{3.6.184}$$

$$\left\|\langle x\rangle^{-3} u_H^S\right\|_{L_x^{\frac{3}{2},1}L_t^\infty \bigcap L_{x_1}^1 L_{\widehat{x_1}}^{2,1} L_t^\infty} \lesssim \|f\|_{L^2} + \|g\|_{\dot{H}^1}, \tag{3.6.185}$$

*and*

$$\left\|\langle x\rangle^{-3} k^S\right\|_{L_x^{\frac{3}{2},1}L_t^\infty \bigcap L_{x_1}^1 L_{\widehat{x_1}}^{2,1} L_t^\infty} \lesssim \|F\|_{L_{x_1}^1 L_{\widehat{x_1}}^{2,1} L_t^\infty}. \tag{3.6.186}$$

*We can replace the $L_t^\infty$ norm of $F$ by the $L_t^1$ norm. Also one can replace $F$ in the above estimates by $F^S$.*

By identical discussions to Section 3.3 with $L_t^2$ replaced by $L_t^\infty$, we have all the estimates for $D_A$ and $k_A$, the truncated version of $D$ and $k$, with factor $\frac{1}{A}$. So we have all the necessary ingredients for our bootstrap process. With the decomposition of channels and all the estimates above, we can conclude Theorem 3.6.13. We omit the details since they are identical as the proof for Theorem 3.6.10.

## 3.7 Scattering

In this section, we show some applications of the results in this chapter. We will study the long-time behaviors for a scattering state in the sense of Definition 5.3.2.

Following the notations from above section, we will still use the short-hand notation

$$L_t^p L_x^q := L_t^p \left([0, \infty),\, L_x^q\right). \tag{3.7.1}$$

In general, we can write a general wave equation as

$$\partial_{tt} u - \Delta u = F(u, t) \tag{3.7.2}$$

with initial data

$$u(x, 0) = g(x),\, u_t(x, 0) = f(x). \tag{3.7.3}$$

Also consider the homogeneous free wave equation,

$$\partial_{tt} u_0 - \Delta u_0 = 0 \tag{3.7.4}$$

with initial data

$$u_0(x, 0) = g_0(x),\, (u_0)_t\,(x, 0) = f_0(x). \tag{3.7.5}$$

For scattering states, we consider the following question: given data $(g, f) \in \dot{H}^1 \times L^2$ and a corresponding solution $u \in \dot{H}^1 \times L^2$ to the perturbed problem $\Box u = F(u, t)$ with initial data $(g, f) \in \dot{H}^1 \times L^2$, can we find data $(g_0, f_0) \in \dot{H}^1 \times L^2$ such that the solution $u_0 \in \dot{H}^1 \times L^2$ to the corresponding homogeneous problem $\partial_{tt} u_0 - \Delta u_0 = 0$, $(g_0, f_0) \in \dot{H}^1 \times L^2$ is such that

$$\|u(t) - u_0(t)\|_{\dot{H}^1 \times L^2} \to 0, \quad t \to \infty \tag{3.7.6}$$

To do this, as we discuss about projections, we reformulate the wave equation as a Hamiltonian system,

$$\partial_t \begin{pmatrix} u \\ \partial_t u \end{pmatrix} - \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} -\Delta & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} u \\ \partial_t u \end{pmatrix} = \begin{pmatrix} 0 \\ F(u) \end{pmatrix}. \tag{3.7.7}$$

Setting

$$U := \begin{pmatrix} u \\ \partial_t u \end{pmatrix}, \ J := \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \ H_F := \begin{pmatrix} -\Delta & 0 \\ 0 & 1 \end{pmatrix} \text{ and } F(U) := \begin{pmatrix} 0 \\ F(u,t) \end{pmatrix}, \tag{3.7.8}$$

we can rewrite the free wave equation as

$$\dot{U}_0 - JH_F U_0 = 0, \tag{3.7.9}$$

$$U_0[0] = \begin{pmatrix} g_0 \\ f_0 \end{pmatrix} \tag{3.7.10}$$

and the perturbed wave equation as

$$\dot{U} - JH_F U = F(U), \tag{3.7.11}$$

$$U[0] = \begin{pmatrix} g \\ f \end{pmatrix}. \tag{3.7.12}$$

The solution of the free wave equation is given by

$$U_0 = e^{tJH_F} U_0[0], \tag{3.7.13}$$

on the other hand, by Duhamel's formula, the solution to the perturbed wave equation is given by

$$U[t] = e^{tJH_F}U[0] + \int_0^t e^{(t-s)JH_F} F\left(U(s)\right)\, ds. \tag{3.7.14}$$

We consider the charge transfer model,

$$\partial_{tt}u - \Delta u + V_1(x)u + V_2(x-vt)u = 0 \tag{3.7.15}$$

for which

$$F(u,t) = -\left(V_1(x)u + V_2(x-vt)u\right) \tag{3.7.16}$$

**Theorem 3.7.1.** *Suppose $u$ is a scattering state in the sense of Definition 5.3.2 which solves*

$$\partial_{tt}u - \Delta u + V_1(x)u + V_2(x-vt)u = 0. \tag{3.7.17}$$

*Write*

$$U = (u, u_t)^t \in C^0\left([0,\infty);\, \dot{H}^1\right) \times C^0\left([0,\infty);\, L^2\right), \tag{3.7.18}$$

*with initial data $U[0] = (g,f)^t \in \dot{H}^1 \times L^2$.* Then there exist free data

$$U_0[0] = (g_0, f_0)^t \in \dot{H}^1 \times L^2$$

such that

$$\left\| U[t] - e^{tJH_F}U_0[0] \right\|_{\dot{H}^1 \times L^2} \to 0 \tag{3.7.19}$$

as $t \to \infty$.

*Proof.* We will still use the formulation in Theorem 3.5.2. We set $A = \sqrt{-\Delta}$ and notice that

$$\|Af\|_{L^2} \simeq \|f\|_{\dot{H}^1}, \ \forall f \in C^\infty\left(\mathbb{R}^3\right). \tag{3.7.20}$$

For real-valued $u = (u_1, u_2) \in \mathcal{H} = \dot{H}^1\left(\mathbb{R}^3\right) \times L^2\left(\mathbb{R}^3\right)$, we write

$$U := Au_1 + iu_2. \tag{3.7.21}$$

We know

$$\|U\|_{L^2} \simeq \|(u_1, u_2)\|_{\mathcal{H}}. \tag{3.7.22}$$

We also notice that $u$ solves (5.2.93) if and only if

$$U := Au + i\partial_t u \tag{3.7.23}$$

satisfies

$$i\partial_t U = AU + V_1 u + V_2\left(x - vt\right)u, \tag{3.7.24}$$

$$U(0) = Ag + if \in L^2\left(\mathbb{R}^3\right). \tag{3.7.25}$$

By Duhamel's formula, for fixed $T$

$$U(T) = e^{iTA}U(0) - i\int_0^T e^{-i(T-s)A}\left(V_1 u(s) + V_2\left(\cdot - vs\right)u(s)\right)\,ds. \tag{3.7.26}$$

Applying the free evolution backwards, we obtain

$$e^{-iTA}U(T) = U(0) - i\int_0^T e^{isA}\left(V_1 u(s) + V_2\left(\cdot - vs\right)u(s)\right)\,ds. \tag{3.7.27}$$

Letting $T$ go to $\infty$, we define

$$U_0(0) := U(0) - i\int_0^\infty e^{isA}\left(V_1 u(s) + V_2\left(\cdot - vs\right)u(s)\right)\,ds \tag{3.7.28}$$

By construction, we just need to show $U_0[0]$ is well-defined in $L^2$, then automatically,

$$\left\| U(t) - e^{itA}U_0(0) \right\|_{L^2} \to 0. \tag{3.7.29}$$

It suffices to show

$$\int_0^\infty e^{isA} \left( V_1 u(s) + V_2 \left( \cdot - vs \right) u(s) \right) \, ds \in L^2. \tag{3.7.30}$$

Then following the argument as in the proof of Theorem 3.5.2, we write $V_1 = V_3V_4$, $V_2 = V_5V_6$.

We consider

$$\left\| \int_0^\infty e^{isA} V_3 V_4 u(s) \, ds \right\|_{L^2_x} \leq \|K_1\|_{L^2_{t,x} \to L^2_x} \|V_4 u\|_{L^2_{t,x}}, \tag{3.7.31}$$

where

$$\left( K_1 F \right)(t) := \int_0^\infty e^{isA} V_3 F(s) \, ds. \tag{3.7.32}$$

Similarly,

$$\left\| \int_0^\infty e^{isA} V_5 V_6(\cdot - vs) u(s) \, ds \right\|_{L^2_x} \leq \left\| \widetilde{K}_1 \right\|_{L^2_{t,x} \to L^2_x} \|V_6(x - vt) u\|_{L^2_{t,x}}, \tag{3.7.33}$$

where

$$\left( \widetilde{K}_1 F \right)(t) := \int_0^\infty e^{isA} V_3(\cdot - vs) F(s) \, ds. \tag{3.7.34}$$

By the same argument in the proof of Theorem 5.3.4, one has

$$\|K_1\|_{L^2_{t,x} \to L^2_x} \leq C_1, \; \left\| \widetilde{K}_1 \right\|_{L^2_{t,x} \to L^2_x} \leq C_2. \tag{3.7.35}$$

Therefore

$$\left\| \int_0^\infty e^{isA} V_3 V_4 u(s)\, ds \right\|_{L^2_x} \lesssim \|V_4 u\|_{L^2_{t,x}}, \tag{3.7.36}$$

$$\left\| \int_0^\infty e^{isA} V_5 V_6(\cdot - vs) u(s)\, ds \right\|_{L^2_x} \lesssim \|V_6(x - vt) u\|_{L^2_{t,x}}. \tag{3.7.37}$$

By estimates (3.5.4) and (3.5.5) from Corollary 3.5.1,

$$\|V_4 u\|_{L^2_{t,x}} \lesssim \left( \int_{\mathbb{R}^+} \int_{\mathbb{R}^3} \frac{1}{\langle x \rangle^\alpha} |u(x,t)|^2 \, dx dt \right)^{\frac{1}{2}} \lesssim \|f\|_{L^2} + \|g\|_{\dot{H}^1}, \tag{3.7.38}$$

$$\|V_6(x - vt) u\|_{L^2_{t,x}} \lesssim \left( \int_{\mathbb{R}^+} \int_{\mathbb{R}^3} \frac{1}{\langle x - vt \rangle^\alpha} |u(x,t)|^2 \, dx dt \right)^{\frac{1}{2}} \lesssim \|f\|_{L^2} + \|g\|_{\dot{H}^1}. \tag{3.7.39}$$

We conclude

$$\int_0^\infty e^{isA} \left( V_1 u(s) + V_2 \left( \cdot - vs \right) u(s) \right) \, ds \in L^2$$

with

$$\left\| \int_0^\infty e^{isA} \left( V_1 u(s) + V_2 \left( \cdot - vs \right) u(s) \right) \, ds \right\|_{L^2} \lesssim \|f\|_{L^2} + \|g\|_{\dot{H}^1}. \tag{3.7.40}$$

So

$$U_0(0) := U(0) - i \int_0^\infty e^{isA} \left( V_1 u(s) + V_2 \left( \cdot - vs \right) u(s) \right) \, ds \tag{3.7.41}$$

is well-defined in $L^2$ and

$$\left\| U(t) - e^{itA} U_0(0) \right\|_{L^2} \to 0. \tag{3.7.42}$$

Define

$$(g_0, f_0) := \left( A^{-1} \Re U_0(0), \, \Im U_0(0) \right). \tag{3.7.43}$$

By construction, notice that

$$U[t] = \left( A^{-1} \Re U(t), \, \Im U(t) \right) \tag{3.7.44}$$

and

$$\left\| U[t] - e^{tJH_F} U_0[0] \right\|_{\dot{H}^1 \times L^2} \to 0. \tag{3.7.45}$$

We are done. □

The above theorem confirms that scattering states indeed scatter to free waves.

# CHAPTER 4

# WAVE EQUATIONS WITH MOVING POTENTIALS

## 4.1 Introduction

Our starting point is the free wave equation ($H_0 = -\Delta$) on $\mathbb{R}^3$

$$\partial_{tt}u - \Delta u = 0 \tag{4.1.1}$$

with initial data

$$u(x,0) = g(x),\ u_t(x,0) = f(x). \tag{4.1.2}$$

We can write down $u$ explicitly,

$$u = \frac{\sin\left(t\sqrt{-\Delta}\right)}{\sqrt{-\Delta}} f + \cos\left(t\sqrt{-\Delta}\right) g. \tag{4.1.3}$$

It obeys the energy inequality,

$$E_F(t) = \int_{\mathbb{R}^3} |\partial_t u(t)|^2 + |\nabla u(t)|^2\, dx \lesssim \int_{\mathbb{R}^3} |f|^2 + |\nabla g|^2\, dx. \tag{4.1.4}$$

We also have the well-known dispersive estimates for the free wave equation ($H_0 = -\Delta$) on $\mathbb{R}^3$:

$$\left\| \frac{\sin\left(t\sqrt{-\Delta}\right)}{\sqrt{-\Delta}} f \right\|_{L^\infty(\mathbb{R}^3)} \lesssim \frac{1}{|t|} \|\nabla f\|_{L^1(\mathbb{R}^3)}, \tag{4.1.5}$$

$$\left\| \cos\left(t\sqrt{-\Delta}\right) g \right\|_{L^\infty(\mathbb{R}^3)} \lesssim \frac{1}{|t|} \|\Delta g\|_{L^1(\mathbb{R}^3)}. \tag{4.1.6}$$

For the sake of completeness, the proofs of estimates (4.1.5) and (4.1.6) are provided in details in Appendix A. (Notice that the estimate (4.1.6) is slightly different from the estimates commonly used in the literature, such as Krieger-Schlag [39] where one needs the $L^1$ norm of $D^2 g$ instead of $\Delta g$).

Strichartz estimates can be derived abstractly from these dispersive inequalities and the energy inequality. With some appropriate $(p, q, s)$, one has

$$\|u\|_{L_t^p L_x^q} \lesssim \|g\|_{\dot{H}^s} + \|f\|_{\dot{H}^{s-1}} \tag{4.1.7}$$

The non-endpoint estimates for the wave equations can be found in Ginibre-Velo [28]. Keel–Tao [37] also obtained sharp Strichartz estimates for the free wave equation in $\mathbb{R}^n$, $n \geq 4$ and everything except the endpoint in $\mathbb{R}^3$. See Keel-Tao [37] and Tao's book [54] for more details on the subject's background and the history.

In $\mathbb{R}^3$, there is no hope to obtain such an estimate with the $L_t^2 L_x^\infty$ norm, the so-called endpoint Strichartz estimate for free wave equations, cf. Klainerman-Machedon [38] and Machihara-Nakamura-Nakanishi-Ozawa [42]. But if we reverse the order of space-time integration, one can obtain a version of reversed Strichartz estimates from the Morawetz estimate, cf. Theorem 4.2.3:

$$\left\| \frac{\sin\left(t\sqrt{-\Delta}\right)}{\sqrt{-\Delta}} f \right\|_{L_x^\infty L_t^2} \lesssim \|f\|_{L^2(\mathbb{R}^3)}\,, \; \left\| \cos\left(t\sqrt{-\Delta}\right) g \right\|_{L_x^\infty L_t^2} \lesssim \|g\|_{\dot{H}^1(\mathbb{R}^3)}\,. \tag{4.1.8}$$

These types of estimates are extended to inhomogeneous cases and perturbed Hamiltonian in Beceanu-Goldberg [3]. In Section 4.4, we will study these estimates and their generalizations intensively. We will also study the other extreme case with the norm $L_x^6 L_t^\infty$ as in Beceanu-Goldberg [3]:

$$\left\| \frac{\sin\left(t\sqrt{-\Delta}\right)}{\sqrt{-\Delta}} f \right\|_{L_x^6 L_t^\infty} \lesssim \|f\|_{L^2(\mathbb{R}^3)}\,, \; \left\| \cos\left(t\sqrt{-\Delta}\right) g \right\|_{L_x^6 L_t^\infty} \lesssim \|g\|_{\dot{H}^1(\mathbb{R}^3)}\,. \tag{4.1.9}$$

These two estimates can be combined together to remedy the failure of the regular endpoint Strichartz estimate.

Next, we consider a linear wave equations with a real-valued stationary potential,

$$H = -\Delta + V, \tag{4.1.10}$$

$$\partial_{tt} u + Hu = \partial_{tt} u - \Delta u + Vu = 0, \tag{4.1.11}$$

$$u(x,0) = g(x),\ u_t(x,0) = f(x). \tag{4.1.12}$$

Explicitly, we have

$$u = \frac{\sin\left(t\sqrt{H}\right)}{\sqrt{H}} f + \cos\left(t\sqrt{H}\right) g. \tag{4.1.13}$$

For the class of short-range potentials we consider in this chapter, under our hypotheses $H$ only has pure absolutely continuous spectrum on $[0,\infty)$ and a finite number of negative eigenvalues. It is very crucial to notice that if there is a negative eigenvalue $E < 0$, the associated eigenfunction responds to the wave equation propagators with a scalar factor by $\cos\left(t\sqrt{E}\right)$ or $\frac{\sin\left(t\sqrt{E}\right)}{E^{\frac{1}{2}}}$, both of which will grow exponentially since $\sqrt{E}$ is purely imaginary. Thus, Strichartz estimates for $H$ must include a projection $P_c$ onto the continuous spectrum in order to get away from this situation.

The problem of dispersive decay and Strichartz estimates for the wave equation with a potential has received much attention in recent years, see the papers by Beceanu-Goldberg [3], Krieger-Schlag [39] and the survey by Schlag [53] for further details and references.

The Strichartz estimates in this case are in the form:

$$\left\| \frac{\sin\left(t\sqrt{H}\right)}{\sqrt{H}} P_c f + \cos\left(t\sqrt{H}\right) P_c g \right\|_{L_t^p L_x^q} \lesssim \|g\|_{\dot{H}^1} + \|f\|_{L^2} \tag{4.1.14}$$

with $2 < p$, $\frac{1}{2} = \frac{1}{p} + \frac{3}{q}$. One also has the endpoint reversed Strichartz estimates:

$$\left\| \frac{\sin\left(t\sqrt{H}\right)}{\sqrt{H}} P_c f + \cos\left(t\sqrt{H}\right) P_c g \right\|_{L_x^\infty L_t^2} \lesssim \|f\|_{L^2} + \|g\|_{\dot{H}^1}, \tag{4.1.15}$$

see Theorem 4.2.4. For the other extreme case, we have

$$\left\| \frac{\sin\left(t\sqrt{H}\right)}{\sqrt{H}} P_c f + \cos\left(t\sqrt{H}\right) P_c g \right\|_{L_x^6 L_t^\infty} \lesssim \|f\|_{L^2} + \|g\|_{\dot{H}^1}. \tag{4.1.16}$$

In Section 4.2 and Section 4.4, we will systematically pass the estimates for free equations to the perturbed case via the structure formula of wave operators. This strategy also works in many other contexts provided that the free solution operators commute with certain symmetries.

For wave equations in $\mathbb{R}^3$, there are several difficulties. For example, the failure of the $L_t^2 L_x^\infty$ estimate and the weakness of decay power $\frac{1}{t}$ in dispersive estimates. The reversed Strichartz estimates might circumvent these difficulties. Reversed Strichartz estimates along time-like trajectories play an important role in the analysis of wave equations of moving potentials. For example, in [14], we used some preliminary versions of these estimates to show Strichartz estimates for wave equations with charge transfer Hamiltonian.

There are extra difficulties when dealing with time-dependent potentials. For example, given a general time-dependent potential $V(x,t)$, it is not clear how to introduce an analog of bound states and a spectral projection. The evolution might not satisfy group properties any more. It might also result in the growth of certain norms of the solutions, see Bourgain's book [9].

The second part of this chapter, we apply the endpoint reversed Strichartz estimates along trajectories to study the wave equation with one moving potential:

$$\partial_{tt}u - \Delta u + V\left(x - \vec{Y}(t)\right)u = 0 \tag{4.1.17}$$

which appears naturally in the study of stability problems of traveling solitons. We impose that the trajectories are asymptotic to straight lines as in [27].

For Schrödinger equations with moving potentials, one can find references and progress, for example in Beceanu-Soffer [8], Rodnianski-Schlag-Soffer [51]. Compared with Schrödinger equations, wave equations have some natural difficulties, for example the evolution of bound states of wave equations leads to exponential growth meanwhile the evolution of bound states of Schrödinger equations are merely multiplied by oscillating factors. We also notice that Lorentz transformations are space-time rotations, therefore one can not hope to succeed by the approach used with Schrödinger equations based on Galilei transformations. The geometry becomes much more complicated in the wave equation context. A crucial step to study wave equations with moving potentials is to understand the change of the energy under Lorentz transformations. In Chen [14], we obtained that the energy stays comparable under Lorentz transformations. In this chapter, we study this by a different approach based on local energy conservation which requires less decay of the potential. As a byproduct, we also obtain Agmon's estimates for the decay of eigenfunctions associated to negatives eigenvalues of $H$.

### *4.1.1 Main results*

**Definition 4.1.1** (Admissible trajectories)**.** A trajectory $\vec{Y}(t) \in \mathbb{R}^3$ is said to be admissible if $\vec{Y}(t)$ is $C^1$ and there exists $0 \leq \ell < 1$ such $\left|\vec{Y}'(t)\right| < \ell < 1$ for $t \in \mathbb{R}$.

Consider the solution to the free wave equation ($H_0 = -\Delta$),

$$u(x,t) = \frac{\sin\left(t\sqrt{-\Delta}\right)}{\sqrt{-\Delta}} f + \cos\left(t\sqrt{-\Delta}\right) g + \int_0^t \frac{\sin\left((t-s)\sqrt{-\Delta}\right)}{\sqrt{-\Delta}} F(s)\, ds \tag{4.1.18}$$

and let $\vec{Y}(t) \in \mathbb{R}^3$ be an admissible trajectory. Setting

$$u^S(x,t) := u\left(x + \vec{Y}(t), t\right), \tag{4.1.19}$$

we estimate

$$\sup_{x\in\mathbb{R}^3} \int \left| u^S(x,t) \right|^2 dt \tag{4.1.20}$$

in terms of the initial energy and various norms of $F$. The idea behind these estimates is that the fundamental solution of the free wave equation is supported on the light cone. Along a time-like curve, the propagation will only meet the light cone once.

**Theorem 4.1.2.** *Let $\vec{Y}(t)$ be an admissible trajectory. We have*

$$\left\| u^S(x,t) \right\|_{L_x^\infty L_t^2} \lesssim \|f\|_{L^2} + \|g\|_{\dot{H}^1} + \|F\|_{\dot{W}_x^{1,1} L_t^2}. \tag{4.1.21}$$

*If $\vec{Y}(t)$ does not change the direction, then*

$$\left\| u^S(x,t) \right\|_{L_x^\infty L_t^2} \lesssim \|f\|_{L^2} + \|g\|_{\dot{H}^1} + \|F\|_{L_d^1 L_{\hat{d}}^{2,1} L_t^2}, \tag{4.1.22}$$

*where $d$ is the direction of $\vec{Y}(t)$.*

*Let $\vec{Z}(t)$ be another admissible trajectory, we have the same estimates above with $F$ replaced by*

$$F^{S'}(x,t) := F\left(x + \vec{Z}(t), t\right). \tag{4.1.23}$$

**Remark 4.1.3.** *If $\vec{Y}(t) = \vec{Z}(t)$, one can obtain*

$$\left\| u^S(x,t) \right\|_{L_x^\infty L_t^2} \lesssim \|f\|_{L^2} + \|g\|_{\dot{H}^1} + \left\| F^S \right\|_{L_x^{\frac{3}{2},1} L_t^2}. \tag{4.1.24}$$

*The other extreme exponents are $L^\infty$ for $t$ and $L^6$ for $x$. To be more precise, we have the following endpoint estimates.*

**Theorem 4.1.4.** *Let $\vec{Y}(t)$ be an admissible trajectory. First of all, for the standard case, one has*

$$\|u\|_{L_x^{6,2} L_t^\infty} \lesssim \|f\|_{L^2} + \|g\|_{\dot{H}^1} + \|F\|_{L_x^{\frac{6}{5},2} L_t^\infty}. \tag{4.1.25}$$

*Consider the estimates along the trajectory $\vec{Y}(t)$, one has*

$$\left\| u^S(x,t) \right\|_{L_x^{6,2} L_t^\infty} \lesssim \|f\|_{L^2} + \|g\|_{\dot{H}^1} + \|F\|_{\dot{W}_x^{1,\frac{6}{5}} L_t^1}. \tag{4.1.26}$$

*Let $\vec{Z}(t)$ be another admissible trajectory, we have the same estimates above with $F$ replaced by*

$$F^{S'}(x,t) := F\left(x + \vec{Y}(t), t\right). \tag{4.1.27}$$

We can extend the above estimates to wave equations with perturbed Hamiltonian,

$$H = -\Delta + V \tag{4.1.28}$$

by the structure formulas for wave operators developed in [5, 7], for the potential $V$ such that

$$H = -\Delta + V \tag{4.1.29}$$

by the structure formulas for wave operators developed in [5, 7], for the potential $V$ such that

$$V \in B^{1+} \cap L^2, \tag{4.1.30}$$

where

$$B^{\beta}=\left\{V\mid\sum_{k\in\mathbb{Z}}2^{\beta k}\left\|\chi_{\{|x|\in[2^k,2^{k+1}]\}}(x)V(x)\right\|_{L^2}<\infty\right\}. \tag{4.1.31}$$

and $H$ admits neither eigenfunctions nor resonances at 0. Recall that $\psi$ is a resonance at 0 if it is a distributional solution of the equation $H\psi=0$ which belongs to the space $L^2\left(\langle x\rangle^{-\sigma}dx\right):=\left\{f:\langle x\rangle^{-\sigma}f\in L^2\right\}$ for any $\sigma>\frac{1}{2}$, but not for $\sigma=\frac{1}{2}$.

**Theorem 4.1.5.** *Let $\vec{Y}(t)$ be an admissible trajectory. Suppose*

$$H=-\Delta+V \tag{4.1.32}$$

*admits neither eigenfunctions nor resonances at* $0$. *Set*

$$u(x,t)=\frac{\sin\left(t\sqrt{H}\right)}{\sqrt{H}}P_cf+\cos\left(t\sqrt{-H}\right)P_cg+\int_0^t\frac{\sin\left((t-s)\sqrt{H}\right)}{\sqrt{H}}P_cF(s)\,ds \tag{4.1.33}$$

*and*

$$u^S(x,t):=u\left(x+\vec{Y}(t),t\right), \tag{4.1.34}$$

*where $P_c$ is the projection onto the continuous spectrum of $H$.*

*Then*

$$\left\|u^S(x,t)\right\|_{L^\infty_xL^2_t}\lesssim\|f\|_{L^2}+\|g\|_{\dot{H}^1}+\|F\|_{\dot{W}^{1,1}_xL^2_t}. \tag{4.1.35}$$

*If $\vec{Y}(t)$ does not change the direction, then*

$$\left\|u^S(x,t)\right\|_{L^\infty_xL^2_t}\lesssim\|f\|_{L^2}+\|g\|_{\dot{H}^1}+\|F\|_{L^1_dL^{2,1}_{\hat{d}}L^2_t}, \tag{4.1.36}$$

*where $d$ is the direction of $\vec{Y}(t)$.*

*Let $\vec{Z}(t)$ be another admissible trajectory, we have the same estimates above with $F$*

*replaced by*

$$F^{S'}(x,t) := F\left(x+\vec{Z}(t),t\right). \tag{4.1.37}$$

We also have the perturbed version of the second endpoint reversed space-time estimates.

**Theorem 4.1.6.** *Let $\vec{Y}(t)$ be an admissible trajectory. Suppose*

$$H = -\Delta + V \tag{4.1.38}$$

*admits neither eigenfunctions nor resonances at* $0$. *Set*

$$u(x,t) = \frac{\sin\left(t\sqrt{H}\right)}{\sqrt{H}} P_c f + \cos\left(t\sqrt{-H}\right) P_c g + \int_0^t \frac{\sin\left((t-s)\sqrt{H}\right)}{\sqrt{H}} P_c F(s)\, ds \tag{4.1.39}$$

*and*

$$u^S(x,t) := u\left(x+\vec{Y}(t),t\right), \tag{4.1.40}$$

*where $P_c$ is the projection onto the continuous spectrum of $H$. First of all, for the standard case, one has*

$$\|u\|_{L_x^{6,2}L_t^\infty} \lesssim \|f\|_{L^2} + \|g\|_{\dot{H}^1} + \|F\|_{L_x^{\frac{6}{5},2}L_t^\infty}. \tag{4.1.41}$$

*Consider the estimates along the trajectory $\vec{Y}(t)$, one has*

$$\left\|u^S(x,t)\right\|_{L_x^{6,2}L_t^\infty} \lesssim \|f\|_{L^2} + \|g\|_{\dot{H}^1} + \|F\|_{\dot{W}_x^{1,\frac{6}{5}}L_t^1}. \tag{4.1.42}$$

*Let $\vec{Z}(t)$ be another admissible trajectory, we have the same estimates above with $F$ replaced by*

$$F^{S'}(x,t) := F\left(x+\vec{Z}(t),t\right). \tag{4.1.43}$$

We will rely on the structure formula of the wave operators by Beceanu-Schlalg [7]. Although one can obtain similar results without using the structure formula, see [14], the goal of our exposition is the illustrate a general strategy that one can pass the estimates for the free evolution to the perturbed one via the structure formula provided there are some symmetries of the free solution operators.

As applications of the above estimates, we study both regular and reversed Strichartz estimates for scattering states to a wave equation with a moving potential with the trajectory asymptotically like a straight line. Suppose $\vec{Y}(t) \in \mathbb{R}^3$ is a trajectory such that there exist $\vec{\mu} \in \mathbb{R}^3$

$$\left|\vec{Y}(t) - \vec{\mu}t\right| \lesssim \langle t\rangle^{-\beta}, \ \beta > 1.$$

Consider

$$\partial_{tt}u - \Delta u + V\left(x - \vec{Y}(t)\right) u = 0 \tag{4.1.44}$$

with initial data

$$u(x, 0) = g(x),\ u_t(x, 0) = f(x). \tag{4.1.45}$$

An indispensable tool we need to study wave equations with moving potentials is the Lorentz transformations. Without loss of generality, we assume $\vec{\mu}$ is along $\vec{e_1}$. We apply the Lorentz transformation $L$ with respect to a moving frame with speed $|\mu| < 1$ along the $x_1$ direction. Writing down the Lorentz transformation explicitly, we have

$$\begin{cases} t' = \gamma\left(t - \mu x_1\right) \\ x_1' = \gamma\left(x_1 - \mu t\right) \\ x_2' = x_2 \\ x_3' = x_3 \end{cases} \tag{4.1.46}$$

with

$$\gamma = \frac{1}{\sqrt{1 - |\mu|^2}}. \tag{4.1.47}$$

We can also write down the inverse transformation of the above:

$$\begin{cases} t = \gamma \left(t' + v x_1'\right) \\ x_1 = \gamma \left(x_1' + \mu t'\right) \\ x_2 = x_2' \\ x_3 = x_3' \end{cases} . \tag{4.1.48}$$

Under the Lorentz transformation $L$, if we use the subscript $L$ to denote a function with respect to the new coordinate $\left(x', t'\right)$, we have

$$u_L \left(x_1', x_2', x_3', t'\right) = u \left(\gamma \left(x_1' + \mu t'\right), x_2', x_3', \gamma \left(t' + \mu x_1'\right)\right) \tag{4.1.49}$$

and

$$u(x,t) = u_L \left(\gamma \left(x_1 - \mu t\right), x_2, x_3, \gamma \left(t - v\mu x\right)\right). \tag{4.1.50}$$

In order to study the equation with time-dependent potentials, we need to introduce a suitable projection. Let

$$H = -\Delta + V\left(\sqrt{1-|\mu|^2} x_1, x_2, x_3\right).$$

Let $m_1, \ldots, m_w$ be the normalized bound states of $H$ associated to the negative eigenvalues $-\lambda_1^2, \ldots, -\lambda_w^2$ respectively (notice that by our assumptions, 0 is not an eigenvalue). In other words, we assume

$$Hm_i = -\lambda_i^2 m_i, \;\; m_i \in L^2, \; \lambda_i > 0. \tag{4.1.51}$$

We denote by $P_b$ the projections on the the bound states of $H$ and let $P_c = Id - P_b$. To be more explicit, we have

$$P_b = \sum_{j=1}^{\ell} \langle \cdot, m_j \rangle m_j. \tag{4.1.52}$$

With Lorentz transformations $L$ associated to the moving frame $(x - \vec{\mu}t, t)$, we use the subscript $L$ to denote a function under the new frame $\left(x', t'\right)$.

**Definition 4.1.7.** Let

$$\partial_{tt} u - \Delta u + V\left(x - \vec{Y}(t)\right) u = 0 \tag{4.1.53}$$

with initial data

$$u(x,0) = g(x),\ u_t(x,0) = f(x). \tag{4.1.54}$$

If $u$ also satisfies

$$\left\| P_b u_L(t') \right\|_{L^2_{x'}} \to 0 \ \ t, t' \to \infty, \tag{4.1.55}$$

we call it a scattering state.

**Theorem 4.1.8** (Strichartz estimates)**.** *Suppose $u$ is a scattering state in the sense of Definition 5.3.2 which solves the equation* (4.1.44)*. Then for $p > 2$ and $(p,q)$ satisfying*

$$\frac{1}{2} = \frac{1}{p} + \frac{3}{q}, \tag{4.1.56}$$

*we have*

$$\|u\|_{L^p_t([0,\infty),\, L^q_x)} \lesssim \|f\|_{L^2} + \|g\|_{\dot{H}^1}. \tag{4.1.57}$$

The above theorem can be extended to the inhomogeneous case, see for example [14].

Secondly, one has the energy estimate:

**Theorem 4.1.9** (Energy estimate)**.** *Suppose $u$ is a scattering state in the sense of Definition 5.3.2 which solves the equation* (4.1.44)*. Then we have*

$$\sup_{t \geq 0} \left(\|\nabla u(t)\|_{L^2} + \|u_t(t)\|_{L^2}\right) \lesssim \|f\|_{L^2} + \|g\|_{\dot{H}^1}. \tag{4.1.58}$$

We also obtain the endpoint reversed Strichartz estimates for $u$.

**Theorem 4.1.10.** *Let $\vec{h}(t)$ be an admissible trajectory. Suppose $u$ is a scattering state in the sense of Definition 5.3.2 which solves the equation* (4.1.44)*. Then*

$$\sup_{x\in\mathbb{R}^3}\int_0^\infty |u(x,t)|^2\,dt \lesssim \left(\|f\|_{L^2}+\|g\|_{\dot{H}^1}\right)^2, \tag{4.1.59}$$

*and*

$$\sup_{x\in\mathbb{R}^3}\int_0^\infty \left|u(x+\vec{h}(t),t)\right|^2 dt \lesssim \left(\|f\|_{L^2}+\|g\|_{\dot{H}^1}\right)^2. \tag{4.1.60}$$

With the endpoint estimate along $\left(x+\vec{Y}(t),t\right)$, one can derive the boundedness of the total energy. We denote the total energy of the system as

$$E_V(t)=\int |\nabla_x u|^2+|\partial_t u|^2+V\left(x-\vec{Y}(t)\right)|u|^2\,dx. \tag{4.1.61}$$

**Corollary 4.1.11.** *Suppose $u$ is a scattering state in the sense of Definition 5.3.2 which solves the equation* (4.1.44)*. Assume*

$$\|\nabla V\|_{L^1}<\infty, \tag{4.1.62}$$

*then $E_V(t)$ is bounded by the initial energy independently of $t$,*

$$\sup_{t\ge 0} E_V(t)\lesssim \|(g,f)\|^2_{\dot{H}^1\times L^2}. \tag{4.1.63}$$

## *Organization*

The chapter is organized as follows: In Section 4.2, we discuss some preliminary results for the free wave equation and the wave equation with a stationary potential. In Section 4.3, we will analyze the change of the energy under Lorentz transformations. Agmon's estimates are also presented as a consequence of our comparison results. In Section 4.4, the endpoint reversed Strichartz estimates of homogeneous and inhomogeneous forms are derived along admissible trajectories. In Section 4.5, we show Strichartz estimates, energy estimates, the local energy decay and the boundedness of the total energy for a scattering state to the wave equation with a moving potential. Finally, in Section 4.6, we confirm that a scattering state indeed scatters to a solution to the free wave equation and also obtain a version of the asymptotic completeness description of the wave equations with one moving potential. In appendices, for the sake of completeness, we show the dispersive estimates for wave equations in $\mathbb{R}^3$ based on the idea of reversed Strichartz estimates, the local energy decay of free wave equations and the global existence of solutions to the wave equation with a time-dependent potential. A Fourier analytic proof of the endpoint reversed Strichartz estimates is also presented.

## 4.2 Preliminaries

### *4.2.1 Strichartz estimates and the endpoint reversed Strichartz estimates*

We start with Strichartz estimates for free wave equations. Strichartz estimates can be derived abstractly from these dispersive inequalities and the energy inequality. The following theorem is standard. One can find a detailed proof in, for example, Keel-Tao [37].

**Theorem 4.2.1** (Strichartz estimates)**.** *Suppose*

$$\partial_{tt}u - \Delta u = F \tag{4.2.1}$$

*with initial data*

$$u(x,0) = g(x),\ u_t(x,0) = f(x). \tag{4.2.2}$$

*Then for* $p$, $a > \frac{2}{s}$, $(p,q)$, $(a,b)$ *satisfying*

$$\frac{3}{2} - s = \frac{1}{p} + \frac{3}{q} \tag{4.2.3}$$

$$\frac{3}{2} - s = \frac{1}{a} + \frac{3}{b} \tag{4.2.4}$$

*we have*

$$\|u\|_{L_t^p L_x^q} \lesssim \|g\|_{\dot{H}^s} + \|f\|_{\dot{H}^{s-1}} + \|F\|_{L_t^{a'} L_x^{b'}} \tag{4.2.5}$$

*where* $\frac{1}{a} + \frac{1}{a'} = 1$, $\frac{1}{b} + \frac{1}{b'} = 1$.

The endpoint $(p,q) = (2,\infty)$ can be recovered for radial functions in Klainerman-Machedon [38] for the homogeneous case and Jia-Liu-Schlag-Xu [34] for the inhomogeneous case. The endpoint estimate can also be obtained when a small amount of smoothing (either in the Sobolev sense, or in relaxing the integrability) is applied to the angular variable, see Machihara-Nakamura-Nakanishi-Ozawa [42].

**Theorem 4.2.2** ([42])**.** *For any* $1 \leq p < \infty$, *suppose* $u$ *solves the free wave equation*

$$\partial_{tt} u - \Delta u = 0 \tag{4.2.6}$$

*with initial data*

$$u(x,0) = g(x),\ u_t(x,0) = f(x). \tag{4.2.7}$$

*Then*

$$\|u\|_{L_t^2 L_r^\infty L_\omega^p} \leq C(p) \left( \|f\|_{L^2} + \|g\|_{\dot{H}^1} \right). \tag{4.2.8}$$

The regular Strichartz estimates fail at the endpoint. But if one switches the order of space-time integration, it is possible to estimate the solution using the fact that the solution decays quickly away from the light cone. Therefore, we introduce reversed Strichartz estimates. Since we will only use the endpoint reversed Stricharz estimate, we will restrict our focus to that case.

**Theorem 4.2.3** (Endpoint reversed Strichartz estimates)**.** *Suppose*

$$\partial_{tt}u - \Delta u = F \tag{4.2.9}$$

*with initial data*

$$u(x,0) = g(x),\ u_t(x,0) = f(x). \tag{4.2.10}$$

*Then*

$$\|u\|_{L_x^\infty L_t^2} \lesssim \|f\|_{L^2} + \|g\|_{\dot{H}^1} + \|F\|_{L_x^{\frac{3}{2},1} L_t^2} \tag{4.2.11}$$

*and*

$$\|u\|_{L_x^{6,2} L_t^\infty} \lesssim \|f\|_{L^2} + \|g\|_{\dot{H}^1} + \|F\|_{L_x^{\frac{6}{5},2} L_t^\infty}. \tag{4.2.12}$$

See Section 4.4 for the detailed proof. For (4.2.11), one can find an alternative proof based on the Fourier transform in Appendix D.

The above results from Theorem 4.2.1 and Theorem 4.2.3 can be generalized to the wave equation with a real stationary potentials.

For the perturbed Hamiltonian,

$$H = -\Delta + V, \tag{4.2.13}$$

with $V \lesssim \langle x \rangle^{-\alpha}$ for $\alpha > 3$, consider the wave equation with potential in $\mathbb{R}^3$:

$$\partial_{tt}u - \Delta u + Vu = 0 \tag{4.2.14}$$

with initial data

$$u(x,0) = g(x),\ u_t(x,0) = f(x). \tag{4.2.15}$$

One can write down the solution to it explicitly:

$$u = \frac{\sin\left(t\sqrt{H}\right)}{\sqrt{H}} f + \cos\left(t\sqrt{H}\right) g. \tag{4.2.16}$$

Let $P_b$ be the projection onto the point spectrum of $H$, $P_c = I - P_b$ be the projection onto the continuous spectrum of $H$.

With the above setting, we formulate the results from [3].

**Theorem 4.2.4** (Strichartz and reversed Strichartz estimates)**.** *Consider the perturbed Hamiltonian $H = -\Delta + V$ in $\mathbb{R}^3$. Suppose $H$ has neither eigenvalues nor resonance at zero. Then for all $0 \leq s \leq 1$, $p > \frac{2}{s}$, and $(p, q)$ satisfying*

$$\frac{3}{2} - s = \frac{1}{p} + \frac{3}{q} \tag{4.2.17}$$

*we have*

$$\left\| \frac{\sin\left(t\sqrt{H}\right)}{\sqrt{H}} P_c f + \cos\left(t\sqrt{H}\right) P_c g \right\|_{L_t^p L_x^q} \lesssim \|g\|_{\dot{H}^s} + \|f\|_{\dot{H}^{s-1}}. \tag{4.2.18}$$

*For the endpoint of reversed Strichartz estimates, we have*

$$\left\| \frac{\sin\left(t\sqrt{H}\right)}{\sqrt{H}} P_c f + \cos\left(t\sqrt{H}\right) P_c g \right\|_{L_x^\infty L_t^2} \lesssim \|f\|_{L^2} + \|g\|_{\dot{H}^1}, \tag{4.2.19}$$

$$\left\| \int_0^t \frac{\sin\left((t-s)\sqrt{H}\right)}{\sqrt{H}} P_c F(s)\, ds \right\|_{L_x^\infty L_t^2} \lesssim \|F\|_{L_x^{\frac{3}{2},1} L_t^2}. \tag{4.2.20}$$

*One also has*

$$\left\| \frac{\sin\left(t\sqrt{H}\right)}{\sqrt{H}} P_c f + \cos\left(t\sqrt{H}\right) P_c g \right\|_{L_x^{6,2} L_t^\infty} \lesssim \|f\|_{L^2} + \|g\|_{\dot{H}^1}, \tag{4.2.21}$$

$$\left\| \int_0^t \frac{\sin\left((t-s)\sqrt{H}\right)}{\sqrt{H}} P_c F(s)\, ds \right\|_{L_x^{6,2} L_t^2} \lesssim \|F\|_{L_x^{\frac{6}{5},2} L_t^\infty}. \tag{4.2.22}$$

One can find detailed arguments and more estimates in [3]. We will apply the structure of the wave operators to show the above theorem in Section 4.4.

### *4.2.2 Structure formulas of wave operators and their applications*

Next, we discuss structure formulas of wave operators. Again consider

$$H = -\Delta + V. \tag{4.2.23}$$

For wave operators, we define

$$W^+ = s - \lim_{t\to\infty} e^{itH} e^{it\Delta}. \tag{4.2.24}$$

We know

$$W^+(-\Delta) = HW^+ \tag{4.2.25}$$

and

$$\left(W^+\right)^* = s - \lim_{t\to\infty} e^{itH_0} e^{-itH} P_c. \tag{4.2.26}$$

By Beceanu and Schlag [7], suppose that

$$V \in B^{1+} \cap L^2 \tag{4.2.27}$$

where

$$B^{\beta} = \left\{ V \mid \sum_{k\in\mathbb{Z}} 2^{\beta k} \left\| \chi_{\{|x|\in[2^k,2^{k+1}]\}}(x)V(x) \right\|_{L^2} < \infty \right\}, \tag{4.2.28}$$

we have a structure formula for $W^+$ and $\left(W^+\right)^*$.

**Theorem 4.2.5** ([7])**.** *Assume $H = -\Delta + V$ admits neither eigenfunction nor resonances at 0. Then for both $W^+$ and $\left(W^+\right)^*$, we have for $f \in L^2$,*

$$Wf(x) = f(x) + \int_{\mathbb{S}^2}\int_{\mathbb{R}^3} g(x,y,\omega) f\left(S_\omega x + y\right)\, dy d\omega, \tag{4.2.29}$$

*for some $g(x,y,\omega)$ such that*

$$\int_{\mathbb{S}^2}\int_{\mathbb{R}^3} \|g(x,y,\omega)\|_{L^\infty_x}\, dy d\omega < \infty \tag{4.2.30}$$

*and where*

$$S_\omega x = x - (2x\cdot\omega)\omega. \tag{4.2.31}$$

*is the reflection by the plane orthogonal to $\omega$. Here $W$ is either of $W^+$ or $\left(W^+\right)^*$.*

The structure formula (4.2.29) in Theorem 4.2.5 is very powerful. One can easily pass many estimates from the free case to the perturbed case provided the solution operators of the free problem commute with certain symmetries. Here we illustrate this idea by a concrete computation based on Theorem 4.2.2.

**Theorem 4.2.6.** *Assume $H = -\Delta + V$ admits neither eigenfunction nor resonances at 0. Setting*

$$u^H = \frac{\sin\left(t\sqrt{H}\right)}{\sqrt{H}} P_c f + \cos\left(t\sqrt{H}\right) P_c g, \tag{4.2.32}$$

*then for any $1 \le p < \infty$, one has*

$$\left\| u^H \right\|_{L^2_t L^\infty_r L^p_\omega} \le C(p)\left( \|f\|_{L^2} + \|g\|_{\dot{H}^1} \right) \tag{4.2.33}$$

*Proof.* It suffices to consider

$$u^H = \frac{\sin\left(t\sqrt{H}\right)}{\sqrt{H}} P_c f. \tag{4.2.34}$$

By construction,

$$\frac{\sin\left(t\sqrt{H}\right)}{\sqrt{H}} P_c = W^+ \frac{\sin\left(t\sqrt{-\Delta}\right)}{\sqrt{-\Delta}} \left(W^+\right)^*. \tag{4.2.35}$$

$$\frac{\sin\left(t\sqrt{H}\right)}{\sqrt{H}} P_c f = W^+ \frac{\sin\left(t\sqrt{-\Delta}\right)}{\sqrt{-\Delta}} \left(W^+\right)^* P_c f. \tag{4.2.36}$$

Denoting

$$h = \left(W^+\right)^* P_c f, \tag{4.2.37}$$

we have

$$\left\|P_c f\right\|_{L^2} \simeq \left\|h\right\|_{L^2}. \tag{4.2.38}$$

Setting

$$G = \frac{\sin\left(t\sqrt{-\Delta}\right)}{\sqrt{-\Delta}} h, \tag{4.2.39}$$

by Theorem 4.2.5, it is sufficient to consider the boundedness of

$$G + \int_{\mathbb{S}^2} \int_{\mathbb{R}^3} g(x, y, \tau) G\left(S_\tau x + y\right)\, dy d\tau. \tag{4.2.40}$$

Clearly, by Theorem 4.2.2,

$$\left\|G\right\|_{L_t^2 L_r^\infty L_\omega^p} \lesssim \left\|h\right\|_{L^2} \simeq \left\|P_c f\right\|_{L^2}. \tag{4.2.41}$$

Next, by Minkowski's inequality,

$$\begin{aligned} &\left\|\int_{\mathbb{S}^2} \int_{\mathbb{R}^3} g(x, y, \tau) G\left(S_\tau x + y\right)\, dy d\tau\right\|_{L_t^2 L_r^\infty L_\omega^p} \\ \lesssim &\int_{\mathbb{S}^2} \int_{\mathbb{R}^3} \left\|g(x, y, \tau) G\left(S_\tau x + y\right)\right\|_{L_t^2 L_r^\infty L_\omega^p} dy d\tau \end{aligned} \tag{4.2.42}$$

$$\|g(x,y,\tau)G(S_\tau x+y)\|_{L_t^2L_r^\infty L_\omega^p} \lesssim \|g(x,y,\tau)\|_{L_x^\infty} \|G(S_\tau x+y)\|_{L_t^2L_r^\infty L_\omega^p}. \tag{4.2.43}$$

Since reflections with respect to a fixed plane and translations commute with the solution of a free wave equation, we obtain

$$G(S_\tau x+y) = \frac{\sin\left(t\sqrt{-\Delta}\right)}{\sqrt{-\Delta}} h(S_\tau x+y). \tag{4.2.44}$$

Therefore,

$$\|G(S_\tau x+y)\|_{L_t^2L_r^\infty L_\omega^p} \lesssim \|h(S_\tau x+y)\|_{L^2} \lesssim \|h\|_{L^2} \simeq \|P_c f\|_{L^2}. \tag{4.2.45}$$

It follows

$$\begin{aligned} &\left\|G + \int_{\mathbb{S}^2}\int_{\mathbb{R}^3} g(x,y,\tau)G(S_\tau x+y)\, dyd\tau\right\|_{L_t^2L_r^\infty L_\omega^p} \\ \lesssim &\left(1+\int_{\mathbb{S}^2}\int_{\mathbb{R}^3} \|g(x,y,\tau)\|_{L_x^\infty}\, dyd\tau\right)\|P_c f\|_{L^2} \lesssim \|f\|_{L^2}. \end{aligned} \tag{4.2.46}$$

Then we conclude

$$\left\|u^H\right\|_{L_t^2L_r^\infty L_\omega^p} \le C(p,V)\left(\|f\|_{L^2} + \|g\|_{\dot{H}^1}\right), \tag{4.2.47}$$

as claimed. □

One can do similar arguments to obtain many other estimates for the perturbed wave equations, for example the local energy decay estimate, the energy estimate and many weighted estimates.

The following Christ-Kiselev Lemma is important in our derivation of Strichartz estimates.

**Lemma 4.2.7** (Christ-Kiselev)**.** *Let $X$, $Y$ be two Banach spaces and let $T$ be a bounded* *linear operator from* $L^\beta\left(\mathbb{R}^+;X\right)$ *to* $L^\gamma\left(\mathbb{R}^+;Y\right)$*, such that*

$$Tf(t)=\int_0^\infty K(t,s)f(s)\,ds. \tag{4.2.48}$$

*Then the operator*

$$\widetilde{T}f=\int_0^t K(t,s)f(s)\,ds \tag{4.2.49}$$

*is bounded from* $L^\beta\left(\mathbb{R}^+;X\right)$ *to* $L^\gamma\left(\mathbb{R}^+;Y\right)$ *provided* $\beta<\gamma$*, and the*

$$\left\|\widetilde{T}\right\|\leq C(\beta,\gamma)\left\|T\right\| \tag{4.2.50}$$

*with*

$$C(\beta,\gamma)=\left(1-2^{\frac{1}{\gamma}-\frac{1}{\beta}}\right)^{-1}. \tag{4.2.51}$$

## 4.3 Lorentz Transformations and Energy

When we consider wave equations with moving potentials, Lorentz transformations will be important for us to reduce some estimates to stationary cases. In order to approach our problem from the viewpoint of Lorentz transformations as in [14], the first natural step is to understand the change of energy under Lorentz transformations.

Indeed, in [14], we shown that under Lorentz transformations, the energy stays comparable to that of the initial data. The method in [14] is based on integration by parts. Here we present an alternative approach based on the local energy conservation which is more natural and requires less decay of the potential. We notice that the method in [14] can be viewed as the differential version of the argument here.

Throughout this section, we perform Lorentz transformations with respect to a moving frame with speed $|v| < 1$, say, along the $x_1$ direction, i.e., the velocity is

$$\vec{Y} = (v, 0, 0)\,. \tag{4.3.1}$$

Recall that after we apply the Lorentz transformation, for function $u$, under the new coordinates, we denote

$$u_L\left(x_1', x_2', x_3', t'\right) = u\left(\gamma\left(x_1' + vt'\right), x_2', x_3', \gamma\left(t' + vx_1'\right)\right). \tag{4.3.2}$$

Now let $u$ be a solution to some wave equation and set $t' = 0$. We notice that in order to show under Lorentz transformations, the energy stays comparable to that of the initial data, up to an absolute constant it suffices to prove

$$\begin{aligned}&\int \left|\nabla_x u\left(x_1, x_2, x_3, vx_1\right)\right|^2 + \left|\partial_t u\left(x_1, x_2, x_3, vx_1\right)\right|^2 dx \\ &\quad \simeq \int \left|\nabla_x u\left(x_1, x_2, x_3, 0\right)\right|^2 + \left|\partial_t u\left(x_1, x_2, x_3, 0\right)\right|^2 dx.\end{aligned} \tag{4.3.3}$$

Throughout this section, we will assume all functions are smooth and decay fast. We will obtain estimates independent of the additional smoothness assumption. It is easy to pass the estimates to general cases with a density argument.

**Remark 4.3.1.** *One can observe that all discussions in this section hold for* $\mathbb{R}^n$.

### *4.3.1 Energy comparison*

In this section, a more general situation is analyzed. We consider wave equations with time-dependent potentials

$$\partial_{tt} u - \Delta u + V(x,t)u = 0 \tag{4.3.4}$$

with

$$|V(x, \mu x_1)| \lesssim \frac{1}{\langle x \rangle^2} \tag{4.3.5}$$

uniformly for $0 \leq |\mu| \leq 1$. These in particular apply to wave equations with moving potentials with speed strictly less than 1. For example, if the potential is of the from

$$V(x,t) = V\left(x - \nu t\right) \tag{4.3.6}$$

with

$$|V(x)| \lesssim \frac{1}{\langle x \rangle^2} \tag{4.3.7}$$

then it is transparent that

$$|V(x, \mu x_1)| = |V(x - \nu \mu x_1)| \lesssim \frac{1}{\langle x \rangle^2}. \tag{4.3.8}$$

## Homogeneous comparison

Suppose

$$\partial_{tt} u - \Delta u + V(x,t)u = 0, \tag{4.3.9}$$

then it is clear that

$$\begin{aligned} 0 &= u_t \left( \Box u - V(t)u \right) \\ &= -\partial_t \left( \frac{|u_t|^2}{2} + \frac{|u_x|^2}{2} \right) + \operatorname{div}\left(\nabla u u_t\right) - V(x,t) u u_t. \end{aligned} \tag{4.3.10}$$

**Lemma 4.3.2.** *Let* $|v| < 1$. *Suppose*

$$\partial_{tt} u - \Delta u + V(x,t)u = 0 \tag{4.3.11}$$

*and*

$$|V(x,\mu x_1)| \lesssim \frac{1}{\langle x\rangle^2} \tag{4.3.12}$$

*uniformly for* $0 \le |\mu| < 1$*. Then for arbitrary* $R > 0$*, there exists some constant* $M(v) > 1$ *depending on* $v$ *such that*

$$\begin{aligned}\int_{|x|>M(v)R} & |\nabla_x u(x_1,x_2,x_3,vx_1)|^2 + |\partial_t u(x_1,x_2,x_3,vx_1)|^2 \, dx \\ & \lesssim \int_{|x|>R} |\nabla_x u(x_1,x_2,x_3,0)|^2 + |\partial_t u(x_1,x_2,x_3,0)|^2 \, dx,\end{aligned} \tag{4.3.13}$$

*where the implicit constant depends on* $v$*,* $V$*.*

*Proof.* Denote

$$L_+^U(u,\mu,R) = \int_{|x|>M(\mu)R,\, x_1>0} |V(x,\mu x_1)|\, |u(x_1,x_2,x_3,\mu x_1)|^2 \, dx. \tag{4.3.14}$$

$$T_+^U(u,\mu,R) = \int_{|x|>M(\mu)R,\, x_1>0} |V(x,\mu x_1)|\, |u_t(x_1,x_2,x_3,\mu x_1)|^2 \, dx. \tag{4.3.15}$$

$$E_+^U(u,\mu,R) = \int_{|x|>M(\mu)R,\, x_1>0} |\nabla_x u(x_1,x_2,x_3,\mu x_1)|^2 + |\partial_t u(x_1,x_2,x_3,\mu x_1)|^2 \, dx, \tag{4.3.16}$$

where

$$M(\mu) = \frac{1}{1-|\mu|}. \tag{4.3.17}$$

One observes that if

$$|V(x,\mu x_1)| \lesssim \frac{1}{\langle x\rangle^2} \tag{4.3.18}$$

uniformly with respect to $0 \le |\mu| < 1$, by Hardy's inequality,

$$L_+^U(u,\mu,R) + T_+^U(u,\mu,R) \lesssim E_+^U(u,\mu,R). \tag{4.3.19}$$

With these notations, we need to show

$$E_{+}^{U}(u,v,R) \lesssim E_{+}^{U}(u,0,R). \tag{4.3.20}$$

For fixed $R>0$, we construct two regions as follows:

$L_R^+$ with equation

$$x_1^2+x_2^2+x_3^3 \leq (R+t)^2, \ \ 0\leq t<\infty,\, x_1\geq 0 \tag{4.3.21}$$

and $L_R^-$ with equation

$$x_1^2+x_2^2+x_3^3 \leq (R-t)^2, \ \ -\infty<t\leq 0,\, x_1\leq 0. \tag{4.3.22}$$

Denote the region bounded by $\left([0,\infty)\times\mathbb{R}^2\times[0,\infty)\right)\backslash L_R^+$ and the plane $(x_1,x_2,x_3,vx_1)$ by $Y^+$ and use $Y^-$ to denote the region bounded by $\left((-\infty,0]\times\mathbb{R}^2\times(-\infty,0]\right)\backslash L_R^-$ and the plane $(x_1,x_2,x_3,vx_1)$.

By symmetry, it suffices to analyze $Y^+$. We apply the space-time divergence theorem to

$$\left(\nabla u u_t, -\left(\frac{|u_t|^2}{2}+\frac{|u_x|^2}{2}\right)\right) \tag{4.3.23}$$

in region $Y^+$. We denote the top of $Y^+$ by $Y_T^+$, the bottom as $Y_B^+$ and the lateral boundary as $Y_L^+$. We should notice $Y_B^+$ actually is $\{x_1>0\}\cap\left(\mathbb{R}^3\backslash B_R(0)\right)$.

The unit outward-pointing normal vector on the plane $(x_1,x_2,x_3,vx_1)$ is

$$\frac{1}{\sqrt{v^2+1}}(-v,0,0,1). \tag{4.3.24}$$

The outward-pointing normal vector on the bottom of $Y^+$ is

$$(0,0,0,-1). \tag{4.3.25}$$

From (5.3.42), one obtains

$$\begin{aligned}\frac{1}{\sqrt{v^2+1}}\int_{Y_T^+}\left[v\partial_{x_1}uu_t-\left(\frac{|u_t|^2}{2}+\frac{|u_x|^2}{2}\right)\right]dx+\int_{Y_B^+}\left(\frac{|u_t|^2}{2}+\frac{|u_x|^2}{2}\right)dx\\ =\frac{1}{2\sqrt{2}}\int_{Y_L^+}|\nabla u-n_L(x)u_t|^2\,d\sigma+\int_{Y^+}V(x,t)uu_t\,dxdt,\end{aligned} \tag{4.3.26}$$

where $n_L(x)$ is a vector of norm 1.

Note that

$$\int_{Y^+}|V(x,t)uu_t|\,dxdt\lesssim\int_{Y^+}|V(x,t)|\left(|u|^2+|u_t|^2\right)dxdt. \tag{4.3.27}$$

Hence we can conclude

$$\begin{aligned}\frac{1}{\sqrt{v^2+1}}\int_{Y_T^+}\left[-v\partial_{x_1}uu_t+\left(\frac{|u_t|^2}{2}+\frac{|u_x|^2}{2}\right)\right]dx\\ \lesssim\int_{Y_B^+}\left(\frac{|u_t|^2}{2}+\frac{|u_x|^2}{2}\right)dx+\int_{Y^+}|V(x,t)|\left(|u|^2+|u_t|^2\right)dxdt.\end{aligned} \tag{4.3.28}$$

Consider the integral

$$\int_{Y^+}|V(x,t)|\left(|u|^2+|u_t|^2\right)dxdt, \tag{4.3.29}$$

by a change of variable and Fubini's Theorem, it follows

$$\begin{aligned}\int_{Y^+}|V(x,t)|\left(|u|^2+|u_t|^2\right)dxdt &= \int_0^v\left(L_+^U(u,\mu,R)+T_+^U(u,\mu,R)\right)d\mu\\ &\lesssim \int_0^v E_+^U(u,\mu,R)\,d\mu.\end{aligned} \tag{4.3.30}$$

Note that with $|v| < 1$ and the AM–GM inequality, we obtain

$$(1 - |v|) \left( \frac{|u_t|^2}{2} + \frac{|u_x|^2}{2} \right) \leq -v\partial_{x_1} u u_t + \left( \frac{|u_t|^2}{2} + \frac{|u_x|^2}{2} \right), \tag{4.3.31}$$

which implies

$$\begin{aligned} \frac{(1 - |v|)}{\sqrt{v^2 + 1}} \int_{Y_T^+} \left( \frac{|u_t|^2}{2} + \frac{|u_x|^2}{2} \right) dx \ \ \lesssim \ \ & \int_{Y_B^+} \left( \frac{|u_t|^2}{2} + \frac{|u_x|^2}{2} \right) dx \\ & + \int_0^v E_+^U (u, \mu, R) \ d\mu \end{aligned} \tag{4.3.32}$$

With our notations, we have

$$E_+^U (u, v, R) \lesssim \frac{\sqrt{v^2 + 1}}{|1 - |v||} \left( E_+^U (u, 0, R) + \int_0^v E_+^U (u, \mu, R) \ d\mu \right). \tag{4.3.33}$$

By Grönwall's inequality with respect to $v$, one obtains

$$E_+^U (u, v, R) \lesssim E_+^U (u, 0, R) \tag{4.3.34}$$

provided $|v| < 1$.

By construction, we have

$$\begin{aligned} & \int_{|x| > M(v)R, \, x_1 > 0} |\nabla_x u (x_1, x_2, x_3, vx_1)|^2 + |\partial_t u (x_1, x_2, x_3, vx_1)|^2 \, dx \\ & \qquad \lesssim \int_{|x| > R, \, x_1 > 0} |\nabla_x u (x_1, x_2, x_3, 0)|^2 + |\partial_t u (x_1, x_2, x_3, 0)|^2 \, dx \end{aligned} \tag{4.3.35}$$

A similar argument for $Y^-$ gives

$$\begin{aligned} & \int_{|x| > M(v)R, \, x_1 \leq 0} |\nabla_x u (x_1, x_2, x_3, vx_1)|^2 + |\partial_t u (x_1, x_2, x_3, vx_1)|^2 \, dx \\ & \qquad \lesssim \int_{|x| > R, \, x_1 \leq 0} |\nabla_x u (x_1, x_2, x_3, 0)|^2 + |\partial_t u (x_1, x_2, x_3, 0)|^2 \, dx. \end{aligned} \tag{4.3.36}$$

Hence, we get

$$\int_{|x|>M(v)R} |\nabla_x u(x_1,x_2,x_3,vx_1)|^2 + |\partial_t u(x_1,x_2,x_3,vx_1)|^2 \, dx$$
$$\lesssim \int_{|x|>R} |\nabla_x u(x_1,x_2,x_3,0)|^2 + |\partial_t u(x_1,x_2,x_3,0)|^2 \, dx \tag{4.3.37}$$

as claimed. □

**Lemma 4.3.3.** *Let* $|v|<1$. *Suppose*

$$\partial_{tt} u - \Delta u + V(x,t)u = 0 \tag{4.3.38}$$

*and*

$$|V(x,\mu x_1)| \lesssim \frac{1}{\langle x\rangle^2} \tag{4.3.39}$$

*uniformly for* $0 \le |\mu| < 1$. *Then for arbitrary* $R>0$, *there exists some constant* $M_1(v) < 1$ *depending* $v$ *such that*

$$\int_{|x|>R} |\nabla_x u(x_1,x_2,x_3,0)|^2 + |\partial_t u(x_1,x_2,x_3,0)|^2 \, dx \tag{4.3.40}$$
$$\lesssim \int_{|x|>M_1(v)R} |\nabla_x u(x_1,x_2,x_3,vx_1)|^2 + |\partial_t u(x_1,x_2,x_3,vx_1)|^2 \, dx,$$

*where the implicit constant depends on* $v$, $V$.

*Proof.* Denote

$$L_+^L(u,\mu,R) = \int_{|x|>M_1(\mu)R,\, x_1>0} |V(x,\mu x_1)| \, |u(x_1,x_2,x_3,\mu x_1)|^2 \, dx. \tag{4.3.41}$$

$$T_+^L(u,\mu,R) = \int_{|x|>M_1(\mu)R,\, x_1>0} |V(x,\mu x_1)| \, |u_t(x_1,x_2,x_3,\mu x_1)|^2 \, dx. \tag{4.3.42}$$

$$E_{+}^{L}(u,\mu,R)=\int_{|x|>M_1(\mu)R,\, x_1>0}|\nabla_x u(x_1,x_2,x_3,\mu x_1)|^2+|\partial_t u(x_1,x_2,x_3,\mu x_1)|^2\,dx, \tag{4.3.43}$$

where

$$M_1(\mu)=\frac{1}{1+|\mu|}. \tag{4.3.44}$$

One observes that if

$$|V(x,\mu x_1)|\lesssim\frac{1}{\langle x\rangle^2} \tag{4.3.45}$$

uniformly with respect to $0\le|\mu|<1$, by Hardy's inequality,

$$L_{+}^{L}(u,\mu,R)+T_{+}^{L}(u,\mu,R)\lesssim E_{+}^{L}(u,\mu,R). \tag{4.3.46}$$

With these notations, we need to show

$$E_{+}^{L}(u,0,R)\lesssim E_{+}^{L}(u,v,R). \tag{4.3.47}$$

For fixed $R>0$, we construct two regions as follows:

$C_R^+$ with equation

$$x_1^2+x_2^2+x_3^3\le(R-t)^2,\ \ 0\le t\le R,\ x_1\ge 0 \tag{4.3.48}$$

and $C_R^-$ with equation

$$x_1^2+x_2^2+x_3^3\le(R+t)^2,\ \ -R\le t\le 0,\ x_1\le 0. \tag{4.3.49}$$

Denote the region bounded by $\left([0,\infty)\times\mathbb{R}^2\times[0,\infty)\right)\backslash C_R^+$ and the plane $(x_1,x_2,x_3,vx_1)$ by $K^+$ and use $K^-$ to denote the region $\left((-\infty,0]\times\mathbb{R}^2\times(-\infty,0]\right)\backslash C_R^-$ and the plane

$(x_1, x_2, x_3, vx_1)$.

By symmetry, it suffices to analyze $K^+$. We again apply the space-time divergence theorem to

$$\left(\nabla u u_t, -\left(\frac{|u_t|^2}{2} + \frac{|u_x|^2}{2}\right)\right) \tag{4.3.50}$$

in region $K^+$ as above. We denote the top of $K^+$ by $K_T^+$, the bottom as $K_B^+$ and the lateral boundary as $K_L^+$. The unit outward-pointing normal vector on the plane $(x_1, x_2, x_3, vx_1)$ is

$$\frac{1}{\sqrt{v^2+1}}\left(-v, 0, 0, 1\right). \tag{4.3.51}$$

The outward-pointing normal vector on the bottom of $K^+$ is

$$(0, 0, 0, -1). \tag{4.3.52}$$

One obtains from (5.3.42),

$$\begin{aligned}\frac{1}{\sqrt{v^2+1}}\int_{K_T^+}\left[v\partial_{x_1} u u_t - \left(\frac{|u_t|^2}{2} + \frac{|u_x|^2}{2}\right)\right] dx + \int_{K_B^+}\left(\frac{|u_t|^2}{2} + \frac{|u_x|^2}{2}\right) dx \\ \leq -\frac{1}{2\sqrt{2}}\int_{K_L^+}|\nabla u - n(x)u_t|^2 \, d\sigma + \int_{K^+}|V(x,t)|\left(|u|^2 + |u_t|^2\right) dxdt,\end{aligned} \tag{4.3.53}$$

where $n(x)$ is a vector of norm 1.

Note that

$$\int_{K^+}|V(t)uu_t|\, dxdt \lesssim \int_{K^+}|V(x,t)|\left(|u|^2 + |u_t|^2\right) dxdt. \tag{4.3.54}$$

Hence we can conclude

$$\begin{aligned}\int_{K_B^+}\left(\frac{|u_t|^2}{2} + \frac{|u_x|^2}{2}\right) dx \;\lesssim\; & \frac{1}{\sqrt{v^2+1}}\int_{K_T^+}\left[-v\partial_{x_1} u u_t + \left(\frac{|u_t|^2}{2} + \frac{|u_x|^2}{2}\right)\right] dx \\ & + \int_{K^+}|V(x,t)|\left(|u|^2 + |u_t|^2\right) dxdt.\end{aligned} \tag{4.3.55}$$

Again, consider the integral

$$\int_{K^+} |V(x,t)| \left( |u|^2 + |u_t|^2 \right) dxdt, \tag{4.3.56}$$

by a change of variable and Fubini's Theorem, it follows

$$\begin{aligned} \int_{K^+} |V(x,t)| \left( |u|^2 + |u_t|^2 \right) dxdt &= \int_0^v \left( L_+^L(u,\mu,R) + T_+^L(u,\mu,R) \right) d\mu \\ &\lesssim \int_0^v E_+^L(u,\mu,R)\ d\mu. \end{aligned} \tag{4.3.57}$$

Hence we can conclude

$$\begin{aligned} \int_{K_B^+} \left( \frac{|u_t|^2}{2} + \frac{|u_x|^2}{2} \right) dx &\lesssim \frac{1+|v|}{\sqrt{v^2+1}} \int_{K_T^+} \left( \frac{|u_t|^2}{2} + \frac{|u_x|^2}{2} \right) dx \\ &\quad + \int_0^v E_+^L(u,\mu,R)\ d\mu \end{aligned} \tag{4.3.58}$$

In other words,

$$E_+^L(u,0,R) \lesssim \frac{1+|v|}{\sqrt{v^2+1}} \left( E_+^L(u,v,R) + \int_0^v E_+^L(u,\mu,R)\ d\mu \right). \tag{4.3.59}$$

With Grönwall's inequality again, it implies

$$E_+^L(u,0,R) \lesssim E_+^L(u,v,R) \tag{4.3.60}$$

Therefore,

$$\int_{K_B^+} \left( \frac{|u_t|^2}{2} + \frac{|u_x|^2}{2} \right) dx \lesssim \int_{K_T^+} \left( \frac{|u_t|^2}{2} + \frac{|u_x|^2}{2} \right) dx. \tag{4.3.61}$$

By construction and a similar argument applied to $K^-$, we obtain precisely

$$\int_{|x|>R} |\nabla_x u(x_1,x_2,x_3,0)|^2 + |\partial_t u(x_1,x_2,x_3,0)|^2\,dx \lesssim \int_{|x|>M_1(v)R} |\nabla_x u(x_1,x_2,x_3,vx_1)|^2 + |\partial_t u(x_1,x_2,x_3,vx_1)|^2\,dx, \tag{4.3.62}$$

as claimed. □

**Theorem 4.3.4.** *Let* $|v|<1$. *Suppose*

$$\partial_{tt}u - \Delta u + V(x,t)u = 0 \tag{4.3.63}$$

*and*

$$|V(x,\mu x_1)| \lesssim \frac{1}{\langle x\rangle^2} \tag{4.3.64}$$

*for* $0 \le |\mu| < 1$. *Then*

$$\int |\nabla_x u(x_1,x_2,x_3,vx_1)|^2 + |\partial_t u(x_1,x_2,x_3,vx_1)|^2\,dx \simeq \int |\nabla_x u(x_1,x_2,x_3,0)|^2 + |\partial_t u(x_1,x_2,x_3,0)|^2\,dx, \tag{4.3.65}$$

*where the implicit constant depends on* $v$ *and* $V$.

*Proof.* We first show

$$\int |\nabla_x u(x_1,x_2,x_3,vx_1)|^2 + |\partial_t u(x_1,x_2,x_3,vx_1)|^2\,dx \lesssim \int |\nabla_x u(x_1,x_2,x_3,0)|^2 + |\partial_t u(x_1,x_2,x_3,0)|^2\,dx. \tag{4.3.66}$$

This follows from Lemma 4.3.2 which implies for $R > 0$,

$$\begin{aligned}\int_{|x|>M(v)R} &|\nabla_x u\,(x_1, x_2, x_3, vx_1)|^2 + |\partial_t u\,(x_1, x_2, x_3, vx_1)|^2\, dx \\ &\lesssim \int_{|x|>R} |\nabla_x u\,(x_1, x_2, x_3, 0)|^2 + |\partial_t u\,(x_1, x_2, x_3, 0)|^2\, dx \qquad (4.3.67) \\ &\lesssim \int |\nabla_x u\,(x_1, x_2, x_3, 0)|^2 + |\partial_t u\,(x_1, x_2, x_3, 0)|^2\, dx\end{aligned}$$

with an implicit constant independent of $R$. By the monotone convergence theorem, letting $R \to 0$, we get

$$\begin{aligned}\int &|\nabla_x u\,(x_1, x_2, x_3, vx_1)|^2 + |\partial_t u\,(x_1, x_2, x_3, vx_1)|^2\, dx \\ &\lesssim \int |\nabla_x u\,(x_1, x_2, x_3, 0)|^2 + |\partial_t u\,(x_1, x_2, x_3, 0)|^2\, dx. \qquad (4.3.68)\end{aligned}$$

Next, we establish the converse inequality. By Lemma 4.3.3, for $R > 0$,

$$\begin{aligned}&\int_{|x|>R} |\nabla_x u\,(x_1, x_2, x_3, 0)|^2 + |\partial_t u\,(x_1, x_2, x_3, 0)|^2\, dx \\ &\lesssim \int_{|x|>M_1(v)R} |\nabla_x u\,(x_1, x_2, x_3, vx_1)|^2 + |\partial_t u\,(x_1, x_2, x_3, vx_1)|^2\, dx \qquad (4.3.69) \\ &\lesssim \int |\nabla_x u\,(x_1, x_2, x_3, vx_1)|^2 + |\partial_t u\,(x_1, x_2, x_3, vx_1)|^2\, dx. \qquad (4.3.70)\end{aligned}$$

with an implicit constant independent of $R$. Letting $R \to 0$, from the dominated convergence theorem, it follows that

$$\begin{aligned}\int &|\nabla_x u\,(x_1, x_2, x_3, 0)|^2 + |\partial_t u\,(x_1, x_2, x_3, 0)|^2\, dx \qquad (4.3.71) \\ &\lesssim \int |\nabla_x u\,(\gamma x_1, x_2, x_3, \gamma v x_1)|^2 + |\partial_t u\,(\gamma x_1, x_2, x_3, \gamma v x_1)|^2\, dx.\end{aligned}$$

Hence, we conclude

$$\int |\nabla_x u (\gamma x_1, x_2, x_3, \gamma v x_1)|^2 + |\partial_t u (\gamma x_1, x_2, x_3, \gamma v x_1)|^2 \, dx$$
$$\simeq \int |\nabla_x u (x_1, x_2, x_3, 0)|^2 + |\partial_t u (x_1, x_2, x_3, 0)|^2 \, dx. \tag{4.3.72}$$

The theorem is proved. □

## Inhomogeneous comparison

In nonlinear applications, we also need to handle inhomogeneous equations. So here we briefly discuss the energy comparison.

Suppose

$$\partial_{tt} u - \Delta u + V(x,t)u = F, \tag{4.3.73}$$

then it is clear that

$$\begin{aligned} F u_t &= u_t (\Box u - V(t)u) \\ &= -\partial_t \left( \frac{|u_t|^2}{2} + \frac{|u_x|^2}{2} \right) + \operatorname{div} (\nabla u u_t) - V(x,t) u u_t. \end{aligned} \tag{4.3.74}$$

We apply the space-time divergence theorem to

$$\left( \nabla u u_t, - \left( \frac{|u_t|^2}{2} + \frac{|u_x|^2}{2} \right) \right). \tag{4.3.75}$$

then one has the following comparison as in the homogeneous case.

**Lemma 4.3.5.** *Let $|v| < 1$. Suppose*

$$\partial_{tt} u - \Delta u + V(x,t)u = F(x,t) \tag{4.3.76}$$

*and*

$$|V(x,\mu x_1)| \lesssim \frac{1}{\langle x\rangle^2} \tag{4.3.77}$$

*uniformly for* $0 \leq |\mu| < 1$. *Then for arbitrary* $R > 0$, *there exists some constant* $M(v) > 1$ *depending on* $v$ *such that*

$$\begin{aligned}\int_{|x|>M(v)R} |\nabla_x u(x_1,x_2,x_3,vx_1)|^2 + |\partial_t u(x_1,x_2,x_3,vx_1)|^2\,dx\\ \lesssim \int_{|x|>R} |\nabla_x u(x_1,x_2,x_3,0)|^2 + |\partial_t u(x_1,x_2,x_3,0)|^2\,dx\\ + \int_{\mathbb{R}}\int_{\mathbb{R}^3} |F(x,t)|^2\,dxdt,\end{aligned} \tag{4.3.78}$$

*where the implicit constant depends on* $v$, $V$.

*Proof.* By the identical arguments as Lemma 4.3.2, denote

$$L_+^U(u,\mu,R) = \int_{|x|>M(\mu)R,\,x_1>0} |V(x,\mu x_1)|\,|u(x_1,x_2,x_3,\mu x_1)|^2\,dx. \tag{4.3.79}$$

$$T_+^U(u,\mu,R) = \int_{|x|>M(\mu)R,\,x_1>0} |V(x,\mu x_1)|\,|u_t(x_1,x_2,x_3,\mu x_1)|^2\,dx. \tag{4.3.80}$$

$$E_+^U(u,\mu,R) = \int_{|x|>M(\mu)R,\,x_1>0} |\nabla_x u(x_1,x_2,x_3,\mu x_1)|^2 + |\partial_t u(x_1,x_2,x_3,\mu x_1)|^2\,dx, \tag{4.3.81}$$

where

$$M(\mu) = \frac{1}{1-|\mu|}. \tag{4.3.82}$$

By Hardy's inequality,

$$L_+^U(u,\mu,R) + T_+^U(u,\mu,R) \lesssim E_+^U(u,\mu,R). \tag{4.3.83}$$

With these notations, we need to show

$$E_{+}^{U}\left(u,v,R\right)\lesssim E_{+}^{U}\left(u,0,R\right). \tag{4.3.84}$$

Using the same notations as the proof of Lemma 4.3.2, we apply the space-time divergence theorem to

$$\left(\nabla u u_t,-\left(\frac{|u_t|^2}{2}+\frac{|u_x|^2}{2}\right)\right) \tag{4.3.85}$$

in region $Y^{+}$. From (4.3.74), one obtains

$$\begin{aligned}&\frac{1}{\sqrt{v^2+1}}\int_{Y_T^+}\left[v\partial_{x_1}uu_t-\left(\frac{|u_t|^2}{2}+\frac{|u_x|^2}{2}\right)\right]dx+\int_{Y_B^+}\left(\frac{|u_t|^2}{2}+\frac{|u_x|^2}{2}\right)dx\\&\quad=\frac{1}{2\sqrt{2}}\int_{Y_L^+}\left|\nabla u-n_L(x)u_t\right|^2d\sigma+\int_{Y^+}V(x,t)uu_t\,dxdt+\int_{Y^+}Fu_t\,dxdt,\end{aligned} \tag{4.3.86}$$

where $n_L(x)$ is a vector of norm 1.

Note that

$$\int_{Y^+}|V(x,t)uu_t|\,dxdt\lesssim\int_{Y^+}|V(x,t)|\left(|u|^2+|u_t|^2\right)dxdt. \tag{4.3.87}$$

Hence we can conclude

$$\frac{1}{\sqrt{v^2+1}}\int_{Y_T^+}\left[-v\partial_{x_1}uu_t+\left(\frac{|u_t|^2}{2}+\frac{|u_x|^2}{2}\right)\right]dx$$

$$\lesssim\int_{Y_B^+}\left(\frac{|u_t|^2}{2}+\frac{|u_x|^2}{2}\right)dx+\int_{Y^+}|V(x,t)|\left(|u|^2+|u_t|^2\right)dxdt \tag{4.3.88}$$

$$+\int_{Y^+}\left(|F(x,t)|^2+|u_t|^2\right)dxdt \tag{4.3.89}$$

We know that

$$\int_{Y^+} |V(x,t)| \left(|u|^2 + |u_t|^2\right) dxdt = \int_0^v \left(L_+^U(u,\mu,R) + T_+^U(u,\mu,R)\right) d\mu \lesssim \int_0^v E_+^U(u,\mu,R)\, d\mu. \tag{4.3.90}$$

So as we did in the proof of Lemma 4.3.2, one has

$$\frac{(1-|v|)}{\sqrt{v^2+1}} \int_{Y_T^+} \left(\frac{|u_t|^2}{2} + \frac{|u_x|^2}{2}\right) dx \lesssim \int_{Y_B^+} \left(\frac{|u_t|^2}{2} + \frac{|u_x|^2}{2}\right) dx + \int_0^v E_+^U(u,\mu,R)\, d\mu + \int_{Y^+} |F(x,t)|^2\, dxdt \tag{4.3.91}$$

With our notations, we have

$$E_+^U(u,v,R) \lesssim \frac{\sqrt{v^2+1}}{|1-|v||} \left(E_+^U(u,0,R) + \int_{\mathbb{R}}\int_{\mathbb{R}^3} |F(x,t)|^2\, dxdt + \int_0^v E_+^U(u,\mu,R)\, d\mu\right). \tag{4.3.92}$$

By Grönwall's inequality with respect to $v$, one obtains

$$E_+^U(u,v,R) \lesssim E_+^U(u,0,R) + \int_{\mathbb{R}}\int_{\mathbb{R}^3} |F(x,t)|^2\, dxdt \tag{4.3.93}$$

provided $|v| < 1$.

By construction, we have

$$\int_{|x|>M(v)R,\, x_1>0} |\nabla_x u(x_1,x_2,x_3,vx_1)|^2 + |\partial_t u(x_1,x_2,x_3,vx_1)|^2\, dx \lesssim \int_{|x|>R,\, x_1>0} |\nabla_x u(x_1,x_2,x_3,0)|^2 + |\partial_t u(x_1,x_2,x_3,0)|^2\, dx + \int_{\mathbb{R}}\int_{\mathbb{R}^3} |F(x,t)|^2\, dxdt. \tag{4.3.94}$$

So by the same steps as in Lemma 4.3.2, we get

$$\begin{aligned}\int_{|x|>M(v)R} |\nabla_x u\,(x_1,x_2,x_3,vx_1)|^2 + |\partial_t u\,(x_1,x_2,x_3,vx_1)|^2\,dx \\ \lesssim \int_{|x|>R} |\nabla_x u\,(x_1,x_2,x_3,0)|^2 + |\partial_t u\,(x_1,x_2,x_3,0)|^2\,dx \\ + \int_{\mathbb{R}}\int_{\mathbb{R}^3} |F(x,t)|^2\,dxdt\end{aligned} \tag{4.3.95}$$

as claimed. □

In the same manner, one has the estimate from the other side. We omit the proof.

**Lemma 4.3.6.** *Let* $|v|<1$. *Suppose*

$$\partial_{tt}u - \Delta u + V(x,t)u = F(x,t) \tag{4.3.96}$$

*and*

$$|V(x,\mu x_1)| \lesssim \frac{1}{\langle x\rangle^2} \tag{4.3.97}$$

*uniformly for* $0\le|\mu|<1$. *Then for arbitrary* $R>0$, *there exists some constant* $M_1(v)<1$ *depending* $v$ *such that*

$$\begin{aligned}\int_{|x|>R} |\nabla_x u\,(x_1,x_2,x_3,0)|^2 + |\partial_t u\,(x_1,x_2,x_3,0)|^2\,dx \\ \lesssim \int_{|x|>M_1(v)R} |\nabla_x u\,(x_1,x_2,x_3,vx_1)|^2 + |\partial_t u\,(x_1,x_2,x_3,vx_1)|^2\,dx \\ + \int_{\mathbb{R}}\int_{\mathbb{R}^3} |F(x,t)|^2\,dxdt,\end{aligned} \tag{4.3.98}$$

*where the implicit constant depends on* $v$, $V$.

By a limiting argument similar to the homogeneous case, one has the following comparison with the inhomogeneous term.

**Theorem 4.3.7.** *Let $|v| < 1$. Suppose*

$$\partial_{tt} u - \Delta u + V(x,t)u = F(x,t) \tag{4.3.99}$$

*and*

$$|V(x, \mu x_1)| \lesssim \frac{1}{\langle x \rangle^2} \tag{4.3.100}$$

*for $0 \le |\mu| < 1$. Then*

$$\begin{aligned} \int |\nabla_x u\,(x_1, x_2, x_3, v x_1)|^2 + |\partial_t u\,(x_1, x_2, x_3, v x_1)|^2\,dx \\ \lesssim \int |\nabla_x u\,(x_1, x_2, x_3, 0)|^2 + |\partial_t u\,(x_1, x_2, x_3, 0)|^2\,dx \\ + \int_{\mathbb{R}} \int_{\mathbb{R}^3} |F(x,t)|^2\,dxdt \end{aligned} \tag{4.3.101}$$

*and*

$$\begin{aligned} \int |\nabla_x u\,(x_1, x_2, x_3, 0)|^2 + |\partial_t u\,(x_1, x_2, x_3, 0)|^2\,dx \\ \lesssim \int |\nabla_x u\,(x_1, x_2, x_3, v x_1)|^2 + |\partial_t u\,(x_1, x_2, x_3, v x_1)|^2\,dx \\ + \int_{\mathbb{R}} \int_{\mathbb{R}^3} |F(x,t)|^2\,dxdt \end{aligned} \tag{4.3.102}$$

*where the implicit constant depends on $v$ and $V$.*

From the theorem above, we know initial energy with respect to different frames stays comparable up to $\|F\|_{L^2_{t,x}}$.

### *4.3.2 Agmon's estimates via wave equations*

As a by product of Theorem 5.3.6, we show Agmon's estimates [1] for the decay of eigenfunctions associated with negative eigenvalues of

$$H = -\Delta + V. \tag{4.3.103}$$

Again, we restrict our attention to the class of potentials satisfying the assumption

$$|V(x)| \leq C_V \left(1 + x^2\right)^{-1}, \ \forall x \in \mathbb{R}^3. \tag{4.3.104}$$

As in Remark 4.3.1, all arguments and discussions are valid for $x \in \mathbb{R}^n$.

**Theorem 4.3.8** (Agmon)**.** *Let $V$ satisfy the assumption* (5.0.3)*. Suppose $\phi \in W^{2,2}$*

$$-\Delta\phi + V\phi = E\phi, \ E < 0. \tag{4.3.105}$$

*Then $\forall \alpha \in [0, 2\sqrt{-E})$*

$$\int_{\mathbb{R}^3} e^{\alpha|x|} |\phi(x)|^2 \, dx \simeq \int_{\mathbb{R}^3} |\phi(x)|^2 \, dx, \tag{4.3.106}$$

*with implicit constants depending on $\alpha$, $V$.*

*Furthermore, if $V \in W^k(\mathbb{R}^3)$ where $k > \frac{3}{2}$ and for $0 \leq i \leq k$,*

$$\left|\nabla^i V(x)\right| \leq C_{V,i} \left(1 + x^2\right)^{-1} \tag{4.3.107}$$

*then*

$$|\phi(x)| \lesssim e^{-\frac{\alpha}{2}|x|}. \tag{4.3.108}$$

*Proof.* It suffices to show $\forall \alpha \in [0, 2\sqrt{-E})$

$$\int_{\mathbb{R}^3} e^{\alpha |x_j|} |\phi(x)|^2 \, dx \simeq \int_{\mathbb{R}^3} |\phi(x)|^2 \, dx, \ \forall j = 1, 2, 3.$$

Without loss of generality, we pick $j = 1$.

With Theorem 5.3.6, we know if $u_{tt} + Hu = 0$, then with $|v| < 1$

$$\begin{aligned} &\int |\nabla_x u(x_1, x_2, x_3, vx_1)|^2 + |\partial_t u(\gamma x_1, x_2, x_3, vx_1)|^2 \, dx \\ &\simeq \int |\nabla_x u(x_1, x_2, x_3, 0)|^2 + |\partial_t u(x_1, x_2, x_3, 0)|^2 \, dx. \end{aligned} \tag{4.3.109}$$

We can rewrite the above result using half-wave operator $e^{it\sqrt{H}}$, for $f \in L^2$ then

$$\int \left( \left| e^{ivt\sqrt{H}} f \right|_{t=x_1}^2 \right) dx \simeq \int |f|^2 \, dx. \tag{4.3.110}$$

We pick $f = \phi$ satisfying

$$-\Delta \phi + V\phi = E\phi, \ E < 0, \tag{4.3.111}$$

then

$$\int e^{-vx_1 2\sqrt{-E}} |\phi|^2 \, dx = \int \left| e^{-vx_1\sqrt{-E}} \phi \right|^2 dx \simeq \int |\phi|^2 \, dx. \tag{4.3.112}$$

With $v$ replaced by $-v$, we obtain

$$\int e^{vx_1 2\sqrt{-E}} |\phi|^2 \, dx = \int \left| e^{-vx_1\sqrt{-E}} \phi \right|^2 dx \simeq \int |\phi|^2 \, dx. \tag{4.3.113}$$

Therefore,

$$\int e^{|2v\sqrt{-E}||x_1|} |\phi|^2 \, dx \simeq \int |\phi|^2 \, dx. \tag{4.3.114}$$

Fixed an $\alpha \in [0, 2\sqrt{-E})$, we can find $|v| \in [0, 1)$ such that $\alpha = \left|2v\sqrt{-E}\right|$, then it follows that

$$\int e^{\alpha|x_1|} |\phi|^2 \, dx \simeq \int |\phi|^2 \, dx. \tag{4.3.115}$$

Therefore the estimate (4.3.106) is proved.

Next we move to (4.3.108). Since

$$-\Delta\phi + V\phi = E\phi, \; E < 0, \tag{4.3.116}$$

then

$$\int |\nabla\phi|^2 \, dx + \int V |\phi|^2 \, dx = E \int |\phi|^2 \, dx, \tag{4.3.117}$$

$$\int |\nabla\phi|^2 \, dx \leq \|V\|_{L^\infty} \int |\phi|^2 \, dx. \tag{4.3.118}$$

Differentiating the equation, for any multi-index $\beta$

$$-\Delta\left(\partial^\beta \phi\right) + \partial^\beta (V\phi) = E\partial^\beta \phi \tag{4.3.119}$$

we can conclude

$$\int \left|\nabla\left(\partial^\beta \phi\right)\right|^2 dx \leq \int \partial^\beta (V\phi)\, \partial^\beta \phi \, dx. \tag{4.3.120}$$

By induction, we obtain

$$\int \left|\nabla\left(\partial^\beta \phi\right)\right|^2 dx \leq \|V\|_{W^{|\beta|,\infty}} \int |\phi|^2 \, dx. \tag{4.3.121}$$

Let $\psi$ be a smooth bump-cutoff function such that $\psi = 1$ in $B_1(0)$ and $\psi = 0$ in $\mathbb{R}^3 \backslash B_2(0)$. We localize our estimate,

$$\int (-\Delta\phi(x) + V\phi(x))\, \bar{\phi}(x)\psi^2(x-y)\, dx = E \int |\phi(x)|^2 \, \psi^2(x-y)\, dx. \tag{4.3.122}$$

Integrating by parts, we know

$$
\begin{aligned}
\int (-\Delta\phi(x) + V\phi(x))\,\bar{\phi}(x)\psi^2(x-y)\,dx \\
= \int V\,|\phi(x)|^2\,\psi^2(x-y)\,dx \\
+ \int |\nabla\phi(x)|^2\,\psi^2(x-y)\,dx \\
+2\int \nabla\phi(x)\bar{\phi}(x)\psi(x-y)\nabla\psi(x-y)\,dx.
\end{aligned}
\tag{4.3.123}
$$

Therefore, by the Cauchy-Schwarz inequality,

$$
\begin{aligned}
\int |\nabla\phi(x)|^2\,\psi^2(x-y)\,dx \;\lesssim\; & E\int |\phi(x)|^2\,\psi^2(x-y)\,dx \\
& + \int V\,|\phi(x)|^2\,\psi^2(x-y)\,dx \\
& +2\int |\phi(x)\nabla\psi(x-y)|^2\,dx.
\end{aligned}
\tag{4.3.124}
$$

It follows,

$$
\sup_{y\in\mathbb{R}^3}\int_{|x-y|\le 1} |\nabla\phi(x)|^2\,dx \lesssim (\|V\|_{L^\infty} + 1 + |E|)\int_{|x-y|\le 2} |\phi(x)|^2\,dx. \tag{4.3.125}
$$

Inductively as above, we have

$$
\sup_{y\in\mathbb{R}^3}\int_{|x-y|\le 1} \left|\nabla\left(\partial^\beta\phi\right)\right|^2\,dx \lesssim \left(\|V\|_{W^{|\beta|,\infty}} + 1 + |E|\right)\int_{|x-y|\le 2} |\phi(x)|^2\,dx. \tag{4.3.126}
$$

Finally by Sobolev's embedding theorem,

$$
\begin{aligned}
\sup_{y\in\mathbb{R}^3}\sup_{|x-y|\le 1}|\phi(x)|^2 &\lesssim \sum_{\beta\le k}\sup_{y\in\mathbb{R}^3}\int_{|x-y|\le 1}\left|\left(\partial^\beta\phi\right)\right|^2\,dx \\
&\lesssim \int_{|x-y|\le 2}|\phi(x)|^2\,dx \qquad (4.3.127)\\
&\lesssim e^{-\alpha(|y|-2)}\int_{|x-y|\le 2}e^{\alpha|x|}|\phi(x)|^2\,dx \\
&\lesssim e^{-\alpha|y|}\int|\phi(x)|^2\,dx \\
&\lesssim e^{-\alpha|y|}.
\end{aligned}
$$

Hence,

$$\sup_{y\in\mathbb{R}^3}|\phi(y)|\lesssim e^{-\frac{\alpha}{2}|y|} \qquad (4.3.128)$$

as claimed. □

## 4.4 Endpoint Reversed Strichartz Estimates

In [14], we analyzed the endpoint reversed Strichartz estimates along slanted lines for both homogeneous and inhomogeneous cases. Intuitively, the reversed Strichartz estimates along slanted lines are based on the fact that the fundamental solutions of the wave equation in $\mathbb{R}^3$ is supported on the light cone. For fixed $x$, the propagation will only meet the light cone once. Here we further note that for a general smooth trajectory with velocity strictly less than 1, it will also only intersect the light cone once. In this section, we will study the reversed Strichartz estimates along general trajectories in several different settings.

Recall that a trajectory $\vec{Y}(t)\in\mathbb{R}^3$ is called an admissible trajectory if $\vec{Y}(t)$ is $C^1$ and there exists $0\le\ell<1$ such $\left|\vec{Y}(t)\right|<\ell<1$ for $t\in\mathbb{R}$.

### 4.4.1 Free wave equations

In this subsection, we set

$$u(x,t)=\frac{\sin\left(t\sqrt{-\Delta}\right)}{\sqrt{-\Delta}}f+\cos\left(t\sqrt{-\Delta}\right)g+\int_0^t\frac{\sin\left((t-s)\sqrt{-\Delta}\right)}{\sqrt{-\Delta}}F(s)\,ds \tag{4.4.1}$$

and

$$u^S(x,t):=u\left(x+\vec{Y}(t),t\right). \tag{4.4.2}$$

**Theorem 4.4.1.** *Let $\vec{Y}(t)$ be an admissible trajectory. First of all, for the standard case, one has*

$$\|u\|_{L_x^\infty L_t^2}\lesssim\|f\|_{L^2}+\|g\|_{\dot{H}^1}+\|F\|_{L_x^{\frac{3}{2},1}L_t^2}. \tag{4.4.3}$$

*Consider the estimates along the trajectory $\vec{Y}(t)$, one has*

$$\left\|u^S(x,t)\right\|_{L_x^\infty L_t^2}\lesssim\|f\|_{L^2}+\|g\|_{\dot{H}^1}+\|F\|_{\dot{W}_x^{1,1}L_t^2}. \tag{4.4.4}$$

*If $\vec{Y}(t)$ does not change the direction, then*

$$\left\|u^S(x,t)\right\|_{L_x^\infty L_t^2}\lesssim\|f\|_{L^2}+\|g\|_{\dot{H}^1}+\|F\|_{L_d^1L_{\widehat{d}}^{2,1}L_t^2}, \tag{4.4.5}$$

*where $d$ is the direction of $\vec{Y}(t)$.*

*Let $\vec{Z}(t)$ be another admissible trajectory, we have the same estimates above with $F$ replaced by*

$$F^{S'}(x,t):=F\left(x+\vec{Z}(t),t\right). \tag{4.4.6}$$

*Proof.* For the first term,

$$u_1(x,t)=\frac{\sin\left(t\sqrt{-\Delta}\right)}{\sqrt{-\Delta}}f=\frac{1}{4\pi t}\int_{|x-y|=t}f(y)\,\sigma\left(dy\right). \tag{4.4.7}$$

So in polar coordinates,

$$\left\|u_1^S(x,t)\right\|_{L_t^2[0,\infty)}^2 \lesssim \int_0^\infty \left(\int_{\mathbb{S}} f(x+\vec{Y}(r)+r\omega)r\,d\omega\right)^2 dr. \tag{4.4.8}$$

Up to translation, it suffices to estimate when $x=0$, so we consider

$$\int_0^\infty \left(\int_{\mathbb{S}} f(\vec{Y}(r)+r\omega)r\,d\omega\right)^2 dr.$$

By Cauchy-Schwarz, one has

$$\int_0^\infty \left(\int_{\mathbb{S}} f(\vec{Y}(r)+r\omega)r\,d\omega\right)^2 dr \lesssim \left(\int_0^\infty \int_{\mathbb{S}} f(\vec{Y}(r)+r\omega))^2 r^2\,d\omega dr\right)\left(\int_{\mathcal{S}^2} d\omega\right). \tag{4.4.9}$$

Performing the change of variable that

$$(r,\omega) \to x' = \vec{Y}(r)+r\omega \tag{4.4.10}$$

we compare the Jacobian of this change variable with the Jacobian of the regular polar coordinate:

$$(r,\omega) \to x = r\omega. \tag{4.4.11}$$

It is equivalent to show the change of variable

$$x \to x' = \vec{Y}(|x|)+x \tag{4.4.12}$$

has a Jacobian which is bounded from above and below.

Letting $\vec{Y}(|x|) = (Y_1(|x|), Y_2(|x|), Y_3(|x|))$, we compute the Jocobian and obtain

$$\frac{\partial x'}{\partial x} = I + \left\{ \begin{array}{ccc} Y_1' (|x|) \frac{x_1}{|x|} & Y_1' (|x|) \frac{x_2}{|x|} & Y_1' (|x|) \frac{x_3}{|x|} \\ Y_2' (|x|) \frac{x_1}{|x|} & Y_2' (|x|) \frac{x_2}{|x|} & Y_2' (|x|) \frac{x_3}{|x|} \\ Y_3' (|x|) \frac{x_1}{|x|} & Y_3' (|x|) \frac{x_2}{|x|} & Y_3' (|x|) \frac{x_3}{|x|} \end{array} \right\} =: I + R. \tag{4.4.13}$$

Then it reduced to show that $R$ has an operator norm less than 1 uniformly with respect to $x$.

Setting $\vec{d}(x) = \left(\frac{x_1}{|x|}, \frac{x_3}{|x|}, \frac{x_3}{|x|}\right)$, we notice that $\forall \xi \in \mathbb{R}^3$,

$$\begin{aligned} \left|\xi R \xi^T\right| &= \xi \vec{Y}' (|x|) \otimes \vec{d}(x) \xi^T \\ &= \left\langle \vec{Y}' (|x|), \xi \right\rangle \left\langle \vec{d}(x), \xi \right\rangle \\ &\lesssim \left|\vec{Y}' (|x|)\right| \left|\vec{d}(x)\right| |\xi|^2 \\ &\lesssim \left|\vec{Y}' (|x|)\right| |\xi|^2 . \end{aligned} \tag{4.4.14}$$

Since $\left|\vec{Y}' (|x|)\right| < \ell < 1$, $R(x)$ has an operator norm less than $\ell$. Hence the Jacobian $\frac{\partial x'}{\partial x}$ is bounded from above and below uniformly. Therefore the Jacobian of the change of variable

$$(r, \omega) \to x' = \vec{Y}(r) + r\omega \tag{4.4.15}$$

is comparable with the Jacobian of

$$(r, \omega) \to x = r\omega \tag{4.4.16}$$

which is $r^2$ uniformly.

So we can conclude that

$$\left(\int_0^\infty \int_{\mathbb{S}} f(\vec{Y}(r)+r\omega))^2 r^2 \, d\omega dr\right) \lesssim \int \left|f\left(x'\right)\right|^2 \, dx' \\ \lesssim \|f\|_{L^2}^2 . \tag{4.4.17}$$

A similar argument holds for

$$u_2(x,t) = \cos\left(t\sqrt{-\Delta}\right) g. \tag{4.4.18}$$

Therefore

$$\left\|u_1^S\right\|_{L_x^\infty L_t^2} + \left\|u_2^S\right\|_{L_x^\infty L_t^2} \lesssim \|f\|_{L^2} + \|g\|_{\dot{H}^1}. \tag{4.4.19}$$

In particular,

$$\|u_1\|_{L_x^\infty L_t^2} + \|u_2\|_{L_x^\infty L_t^2} \lesssim \|f\|_{L^2} + \|g\|_{\dot{H}^1}. \tag{4.4.20}$$

as claimed.

Next, we consider the inhomogenous case,

$$D(x,t) = \int_0^t \frac{\sin\left((t-s)\sqrt{-\Delta}\right)}{\sqrt{-\Delta}} F(s)\, ds. \tag{4.4.21}$$

For the standard case, we consider

$$\begin{aligned}
\left\| \int_0^t \frac{\sin\left((t-s)\sqrt{-\Delta}\right)}{\sqrt{-\Delta}} F(s)\,ds \right\|_{L_t^2} &= \left\| \int_0^t \int_{|x-y|=t-s} \frac{1}{|x-y|} F(y,s)\,\sigma(dy)\,ds \right\|_{L_t^2} \\
&= \left\| \int_{|x-y|\le t} \frac{1}{|x-y|} F\left(y, t-|x-y|\right)\,dy \right\|_{L_t^2} \\
&\lesssim \int \frac{1}{|x-y|} \left\| F\left(y, t-|x-y|\right)\right\|_{L_t^2} dy \\
&\lesssim \sup_{x\in\mathbb{R}^3} \int \frac{1}{|x-y|} \left\| F(y,t)\right\|_{L_t^2} dy \\
&\lesssim \|F\|_{L_x^{\frac{3}{2},1} L_t^2}.
\end{aligned}$$

Therefore, indeed,

$$\|D\|_{L_x^\infty L_t^2} \lesssim \|F\|_{L_x^{\frac{3}{2},1} L_t^2}. \tag{4.4.22}$$

Actually, we have

$$\|D\|_{L_x^\infty L_t^p} \lesssim \|F\|_{L_x^{\frac{3}{2},1} L_t^p},\ 1 \le p \le \infty.$$

Now we consider the estimate along an admissible trajectory $\vec{Y}(t) \in \mathbb{R}^3$.

We first notice that from the above discussion or the argument in Appendix D,

$$T := \frac{e^{it\sqrt{-\Delta}}}{\sqrt{-\Delta}} \tag{4.4.23}$$

is a bounded operator from $L_x^2$ to $L_x^\infty L_t^2$. Also the operator $T^S$:

$$T^S f := (Tf)^S = \left(\frac{e^{it\sqrt{-\Delta}}}{\sqrt{-\Delta}} f\right)^S \tag{4.4.24}$$

is a bounded operator from $L_x^2$ to $L_x^\infty L_t^2$.

Writing down the inhomogeneous evolution explicitly, one has

$$\begin{aligned}\left\|\int_0^t \frac{\sin\left((t-s)\sqrt{-\Delta}\right)}{\sqrt{-\Delta}} F(s)\,ds\right\|_{L_t^2} &= \left\|\int_0^t \int_{|x-y|=t-s} \frac{1}{|x-y|} F(y,s)\,\sigma\,(dy)\,ds\right\|_{L_t^2} \\ &\lesssim \left\|\int_{|x-y|\leq t} \frac{1}{|x-y|} F\left(y, t-|x-y|\right)\,dy\right\|_{L_t^2} \end{aligned} \tag{4.4.25}$$

Therefore,

$$\begin{aligned}\sup_{x\in\mathbb{R}^3}\left\|\int_{|x-y|\leq t} \frac{1}{|x-y|} F\left(y, t-|x-y|\right)\,dy\right\|_{L_t^2} &\lesssim \sup_{x\in\mathbb{R}^3}\left\|\int \frac{1}{|x-y|}\left|F\left(y, t-|x-y|\right)\right|dy\right\|_{L_t^2} \\ &\lesssim \sup_{x\in\mathbb{R}^3}\left\|\int_0^\infty \frac{\sin\left((t-s)\sqrt{-\Delta}\right)}{\sqrt{-\Delta}}\left|F(s)\right|\,ds\right\|_{L_t^2} \\ &\lesssim \sup_{x\in\mathbb{R}^3}\left\|\Re\left(TT^*\sqrt{-\Delta}\,|F|\right)\right\|_{L_t^2}. \end{aligned} \tag{4.4.26}$$

Hence we know

$$\begin{aligned}\sup_{x\in\mathbb{R}^3}\left\|D^S\left(x+v(t),t\right)\right\|_{L_t^2} &\lesssim \sup_{x\in\mathbb{R}^3}\left\|\Re\left(T^S T^*\sqrt{-\Delta}\,|F|\right)\right\|_{L_t^2} \\ &\lesssim \left\|\sqrt{-\Delta}\,|F|\,(x,t)\right\|_{L_x^1 L_t^2} \\ &\|F\|_{\dot{W}_x^{1,1} L_t^2}. \end{aligned} \tag{4.4.27}$$

If the trajectory does not change the direction, we can obtain an estimate which does not require $\sqrt{-\Delta}F$ by a similar argument to the estimates along slanted lines in [14]. Without loss of generality, we assume the direction of the trajectory is along $x_1$. Then

$$D^S(x,t) = \int_0^t \int_{\left|x+\vec{Y}(t)-y\right|=t-s} \frac{F(y,s)}{\left|x+\vec{Y}(t)-y\right|}\,\sigma\,(dy)\,ds \tag{4.4.28}$$

and

$$\begin{aligned}
\left\|D^S(x,\cdot)\right\|_{L^2_t} &= \left\|\int_0^t \int_{\left|x+\vec{Y}(t)-y\right|=t-s} \frac{F(y,s)}{\left|x+\vec{Y}(t)-y\right|}\,\sigma\,(dy)\,ds\right\|_{L^2_t} \\
&= \left\|\int_{|y|\le t} \frac{F(x+\vec{Y}(t)-y,t-|y|)}{|y|}\,dy\right\|_{L^2_t} \qquad (4.4.29)\\
&\le \left\|\int_{\mathbb{R}^3} \frac{\left|F(x-y,t-\left|y+\vec{Y}(t)\right|)\right|}{\left|y+\vec{Y}(t)\right|}\,dy\right\|_{L^2_t} \\
&\le \left\|\int_{\mathbb{R}^3} \frac{\left|F(x-y,t-\left|y+\vec{Y}(t)\right|)\right|}{\sqrt{y_2^2+y_3^2}}\,dy\right\|_{L^2_t},
\end{aligned}$$

where in the third line, we used a change of variable and for the last inequality and reduce the norm of $y$ to the norm of the component of $y$ orthogonal to the direction of the motion.

Finally,

$$\left\|\int_{\mathbb{R}^3} \frac{F(x-y,t-\left|y+\vec{Y}(t)\right|)}{\sqrt{y_2^2+y_3^2}}\,dy\right\|_{L^2_t} \le \int_{\mathbb{R}^3} \frac{\left\|F(x-y,t-\left|y+\vec{Y}(t)\right|)\right\|_{L^2_t}}{\sqrt{y_2^2+y_3^2}}\,dy \qquad (4.4.30)$$

For fixed $y$, if we apply a change of variable of $t$ here, the Jacobian is bounded by $1-|v'|$ and $1+|v'|$, so

$$\begin{aligned}
\int_{\mathbb{R}^3} \frac{\|F(x-y,t-|y+vt|)\|_{L^2_t}}{\sqrt{y_2^2+y_3^2}}\,dy &\lesssim \int_{\mathbb{R}^3} \frac{\|F(x-y,\cdot)\|_{L^2_t}}{\sqrt{y_2^2+y_3^2}}dy \\
&\lesssim \|F\|_{L^1_{x_1}L^{2,1}_{\widehat{x_1}}L^2_t} \qquad (4.4.31)
\end{aligned}$$

where $\widehat{x_1}$ denotes the subspace orthogonal to $x_1$ (more generally, the subspace orthogonal to the direction of the motion). Here $L^{2,1}$ is the Lorentz norm and the last inequality follows from Hölder's inequality of Lorentz spaces. Therefore,

$$\left\| D^S \right\|_{L_x^\infty L_t^2} \lesssim \|F\|_{L_{x_1}^1 L_{\widehat{x_1}}^{2,1} L_t^2}. \tag{4.4.32}$$

as claimed.

Finally, we consider the estimate with the source term $F$ along an admissible trajectory. This follows from a duality or the same argument as in [14]. So we conclude that

$$\left\| D^S \right\|_{L_x^\infty L_t^2} \lesssim \left\| F^{S'} \right\|_{\dot{W}_x^{1,1} L_t^2}, \tag{4.4.33}$$

and

$$\left\| D^S \right\|_{L_x^\infty L_t^2} \lesssim \left\| F^{S'} \right\|_{L_{x_1}^1 L_{\widehat{x_1}}^{2,1} L_t^2} \tag{4.4.34}$$

provided $\vec{Y}(t)$ moves along $x_1$.

The theorem is proved. □

**Remark 4.4.2.** *We notice that from Sobolev's embedding,*

$$\dot{W}_x^{1,1} \hookrightarrow L^{\frac{3}{2},1}.$$

*Therefore indeed, the estimates along general curves requires slightly more regularity than the standard cases.*

We have the other endpoint version of reversed space-time estimates.

**Theorem 4.4.3.** *Let $\vec{Y}(t)$ be an admissible trajectory. First of all, for the standard case, one has*

$$\|u\|_{L_x^{6,2} L_t^\infty} \lesssim \|f\|_{L^2} + \|g\|_{\dot{H}^1} + \|F\|_{L_x^{\frac{6}{5},2} L_t^\infty}. \tag{4.4.35}$$

*Consider the estimates along the trajectory $\vec{Y}(t)$, one has*

$$\left\| u^S(x,t) \right\|_{L_x^{6,2} L_t^\infty} \lesssim \|f\|_{L^2} + \|g\|_{\dot{H}^1} + \|F\|_{\dot{W}_x^{1,\frac{6}{5}} L_t^1}. \tag{4.4.36}$$

*Let $\vec{Z}(t)$ be another admissible trajectory, we have the same estimates above with $F$ replaced by*

$$F^{S'}(x,t) := F\left(x + \vec{Z}(t), t\right). \tag{4.4.37}$$

**Remark 4.4.4.** *From the embedding of Lorentz spaces, from*

$$\|u\|_{L_x^{6,2} L_t^\infty} \lesssim \|f\|_{L^2} + \|g\|_{\dot{H}^1} + \|F\|_{L_x^{\frac{6}{5},2} L_t^\infty} \tag{4.4.38}$$

*one has*

$$\|u\|_{L_x^6 L_t^\infty} \lesssim \|f\|_{L^2} + \|g\|_{\dot{H}^1} + \|F\|_{L_x^{\frac{6}{5}} L_t^\infty}, \tag{4.4.39}$$

*similarly,*

$$\left\| u^S(x,t) \right\|_{L_x^6 L_t^\infty} \lesssim \|f\|_{L^2} + \|g\|_{\dot{H}^1} + \|F\|_{\dot{W}_x^{\frac{6}{5},1} L_t^1}. \tag{4.4.40}$$

*Proof.* Consider $t \geq 0$ and define

$$Tf = \frac{\sin\left(t\sqrt{-\Delta}\right)}{\sqrt{-\Delta}} f \tag{4.4.41}$$

then

$$T^*F = \int_0^\infty \frac{\sin\left(t\sqrt{-\Delta}\right)}{\sqrt{-\Delta}} F(t)\, dt, \tag{4.4.42}$$

and

$$\begin{aligned} TT^*F &= \int_0^\infty \frac{\sin\left(t\sqrt{-\Delta}\right)}{\sqrt{-\Delta}} \frac{\sin\left(s\sqrt{-\Delta}\right)}{\sqrt{-\Delta}} F(s)\, ds \\ &= \frac{1}{2} \int_0^\infty \left( \frac{\cos\left((t-s)\sqrt{-\Delta}\right)}{-\Delta} - \frac{\cos\left((t+s)\sqrt{-\Delta}\right)}{-\Delta} \right) F(s)\, ds. \end{aligned} \tag{4.4.43}$$

We compute the kernel of

$$\frac{\cos\left(h\sqrt{-\Delta}\right)}{-\Delta}F = \int_{\mathbb{R}^3} K(x,y,h)F(y)\,dy. \tag{4.4.44}$$

By straightforward computations, one has

$$\frac{\cos\left(h\sqrt{-\Delta}\right)}{-\Delta} = \frac{1}{-\Delta} - \int_0^h \frac{\sin\left(s\sqrt{-\Delta}\right)}{\sqrt{-\Delta}}\,ds = \int_h^\infty \frac{\sin\left(s\sqrt{-\Delta}\right)}{\sqrt{-\Delta}}\,ds. \tag{4.4.45}$$

By the explicit kernel of $\frac{\sin(s\sqrt{-\Delta})}{\sqrt{-\Delta}}$, we know that

$$K(x,y,h) = \begin{cases} \frac{1}{|x-y|} & |x-y| \geq h \\ 0 & |x-y| < h \end{cases}. \tag{4.4.46}$$

Notice that in $\mathbb{R}^3$, $\frac{1}{|x|} \in L^{3,\infty}$, so

$$\left\| \int_0^\infty \left( \frac{\cos\left((t-s)\sqrt{-\Delta}\right)}{-\Delta} \right) F(s)\,ds \right\|_{L_x^{6,2}L_t^\infty} \lesssim \|F\|_{L_x^{\frac{6}{5},2}L_t^1} \tag{4.4.47}$$

by Young's inequality for convolution. It follows that

$$\|Tf\|_{L_x^{6,2}L_t^\infty} = \left\| \frac{\sin\left(t\sqrt{-\Delta}\right)}{\sqrt{-\Delta}} f \right\|_{L_x^{6,2}L_t^\infty} \lesssim \|f\|_{L^2}. \tag{4.4.48}$$

Now we consider the shifted version:

$$T^S f = (Tf)^S = \left( \frac{\sin\left(t\sqrt{-\Delta}\right)}{\sqrt{-\Delta}} f \right)^S \tag{4.4.49}$$

From the computations above, the kernel of

$$T^S \left(T^S\right)^* \tag{4.4.50}$$

can be written as two parts

$$K\left(x+\vec{Y}(t), y+\vec{Y}(s), t-s\right)+K\left(x+\vec{Y}(t), y+\vec{Y}(s), t+s\right). \tag{4.4.51}$$

By (4.4.46), we have

$$K\left(x+\vec{Y}(t), y+\vec{Y}(s), t-s\right)=\begin{cases}\frac{1}{\left|x+\vec{Y}(t)-\left(y+\vec{Y}(s)\right)\right|} & \left|x+\vec{Y}(t)-\left(y+\vec{Y}(s)\right)\right| \geq |t-s| \\ 0 & \left|x+\vec{Y}(t)-\left(y+\vec{Y}(s)\right)\right|<|t-s|\end{cases} \tag{4.4.52}$$

For $\left|x+\vec{Y}(t)-\left(y+\vec{Y}(s)\right)\right| \geq t-s$,

$$\begin{aligned}\left|x+\vec{Y}(t)-\left(y+\vec{Y}(s)\right)\right| &\geq |x-y|-\left|\vec{Y}(t)-\vec{Y}(s)\right| \\ &\geq |x-y|-\ell\,|t-s| \\ &\geq |x-y|-\ell\left|x+\vec{Y}(t)-\left(y+\vec{Y}(s)\right)\right|\end{aligned} \tag{4.4.53}$$

Therefore,

$$|x-y| \lesssim\left|x-\vec{Y}(t)-\left(y-\vec{Y}(s)\right)\right| \tag{4.4.54}$$

provided

$$\left|x+\vec{Y}(t)-\left(y+\vec{Y}(s)\right)\right| \geq t-s. \tag{4.4.55}$$

Hence

$$\left|K\left(x+\vec{Y}(t), y+\vec{Y}(s), t-s\right)\right|_{L_s^{\infty}} \lesssim \frac{1}{|x-y|}. \tag{4.4.56}$$

For the second kernel, by similar computations, one has

$$K\left(x+\vec{Y}(t),y+\vec{Y}(s),t+s\right)=\begin{cases}\frac{1}{\left|x+\vec{Y}(t)-\left(y+\vec{Y}(s)\right)\right|} & \left|x+\vec{Y}(t)-\left(y+\vec{Y}(s)\right)\right|\geq t+s\\ 0 & \left|x+\vec{Y}(t)-\left(y+\vec{Y}(s)\right)\right|<t+s\end{cases}. \tag{4.4.57}$$

If $\left|x+\vec{Y}(t)-\left(y+\vec{Y}(s)\right)\right|\geq t+s$,

$$\begin{aligned}\left|x+\vec{Y}(t)-\left(y+\vec{Y}(s)\right)\right| &\geq |x-y|-\left|\vec{Y}(t)-\vec{Y}(s)\right| \\ &\geq |x-y|-\ell\,|t-s| \\ &\geq |x-y|-\ell\,|t+s| \\ &|\geq x-y|-\ell\left|x+\vec{Y}(t)-\left(y+\vec{Y}(s)\right)\right|.\end{aligned} \tag{4.4.58}$$

Hence

$$|x-y|\lesssim\left|x+\vec{Y}(t)-\left(y+\vec{Y}(s)\right)\right| \tag{4.4.59}$$

provided

$$\left|x+\vec{Y}(t)-\left(y+\vec{Y}(s)\right)\right|\geq t+s. \tag{4.4.60}$$

Therefore,

$$\left|K\left(x+\vec{Y}(t),y+\vec{Y}(s),t+s\right)\right|_{L_s^\infty}\lesssim\frac{1}{|x-y|}. \tag{4.4.61}$$

By estimates (4.4.56) and (4.4.61), we conclude that

$$\left\|\left|T^S\left(T^S\right)^*F\right|(x,\cdot)\right\|_{L_t^\infty}\lesssim\int\frac{1}{|x-y|}\left\|F(y,\cdot)\right\|_{L_t^1}\,dy \tag{4.4.62}$$

and

$$\left\|T^S\left(T^S\right)^*F\right\|_{L_x^{6,2}L_t^\infty}\lesssim\left\|F(y,\cdot)\right\|_{L_x^{\frac{6}{5},2}L_t^1}. \tag{4.4.63}$$

Therefore,

$$u_1(x,t) := Tf = \frac{\sin\left(t\sqrt{-\Delta}\right)}{\sqrt{-\Delta}} f \tag{4.4.64}$$

satisfies

$$\left\| u_1^S(x,t) \right\|_{L_x^{6,2} L_t^\infty} = \left\| u_1^S(x+\vec{Y}(t),t) \right\|_{L_x^{6,2} L_t^\infty} \lesssim \|f\|_{L^2}. \tag{4.4.65}$$

By a similar argument, we have that

$$u_2(x,t) = \cos\left(t\sqrt{-\Delta}\right) g \tag{4.4.66}$$

satisfies

$$\left\| u_2^S(x,t) \right\|_{L_x^{6,2} L_t^\infty} = \left\| u_2^S(x+\vec{Y}(t),t) \right\|_{L_x^{6,2} L_t^\infty} \lesssim \|g\|_{\dot{H}^1}. \tag{4.4.67}$$

For the inhomogeneous case, we again consider

$$D(x,t) = \int_0^t \frac{\sin\left((t-s)\sqrt{-\Delta}\right)}{\sqrt{-\Delta}} F(s)\, ds. \tag{4.4.68}$$

For the standard case, as above,

$$\left\| \int_0^t \frac{\sin\left((t-s)\sqrt{-\Delta}\right)}{\sqrt{-\Delta}} F(s)\, ds \right\|_{L_t^\infty} \lesssim \int \frac{1}{|x-y|} \left\| F(y,\cdot) \right\|_{L_t^2} dy, \tag{4.4.69}$$

so

$$\left\| \int_0^t \frac{\sin\left((t-s)\sqrt{-\Delta}\right)}{\sqrt{-\Delta}} F(s)\, ds \right\|_{L_x^{6,2} L_t^\infty} \lesssim \|F\|_{L_x^{\frac{6}{5},2} L_t^\infty}. \tag{4.4.70}$$

Two sum three pieces up, we conclude that

$$\|u\|_{L_x^{6,2} L_t^\infty} \lesssim \|f\|_{L^2} + \|g\|_{\dot{H}^1} + \|F\|_{L_x^{\frac{6}{5},2} L_t^\infty}. \tag{4.4.71}$$

For (4.4.36), it follows from estimates (4.4.65), (4.4.67) and (4.4.63) with the same argument for (4.4.4). Therefore,

$$\left\| u^S(x,t) \right\|_{L_x^{6,2} L_t^\infty} \lesssim \|f\|_{L^2} + \|g\|_{\dot{H}^1} + \|F\|_{\dot{W}_x^{1,\frac{6}{5}} L_t^1}$$

as claimed. □

## 4.4.2 *Perturbed wave equations.*

Finally, we extend all of our estimates to the perturbed Hamiltonian. In [14], we relied on Duhamel expansion of the perturbed evolution, the estimates along trajectories for free ones and the standard estimates for the perturbed ones. Here we present an alternative approach based on the structure formula of the wave operators as in Section 4.2. We only present the standard cases and other estimates can be obtained similarly.

In this section, we suppose

$$H = -\Delta + V \tag{4.4.72}$$

such that

$$V \in B^{1+} \cap L^2,$$

where

$$B^\beta = \left\{ V \mid \sum_{k \in \mathbb{Z}} 2^{\beta k} \left\| \chi_{\{|x| \in [2^k, 2^{k+1}]\}}(x) V(x) \right\|_{L^2} < \infty \right\}.$$

and $H$ admits neither eigenfunctions nor resonances at 0. Set

$$u(x,t) = \frac{\sin\left(t\sqrt{H}\right)}{\sqrt{H}} P_c f + \cos\left(t\sqrt{-H}\right) P_c g + \int_0^t \frac{\sin\left((t-s)\sqrt{H}\right)}{\sqrt{H}} P_c F(s)\, ds \tag{4.4.73}$$

and

$$u^S(x,t) := u\left(x + \vec{Y}(t), t\right), \tag{4.4.74}$$

where $P_c$ is the projection onto the continuous spectrum of $H$.

**Theorem 4.4.5.** *Let $\vec{Y}(t)$ be an admissible trajectory. First of all, for the standard endpoint reversed Strichartz estimates, we have*

$$\|u(x,t)\|_{L_x^\infty L_t^2} \lesssim \|f\|_{L^2} + \|g\|_{\dot{H}^1} + \|F\|_{L_x^{\frac{3}{2},1} L_t^2}. \tag{4.4.75}$$

*Consider the estimates along the trajectory $\vec{Y}(t)$, one has*

$$\left\|u^S(x,t)\right\|_{L_x^\infty L_t^2} \lesssim \|f\|_{L^2} + \|g\|_{\dot{H}^1} + \|F\|_{\dot{W}_x^{1,1} L_t^2}. \tag{4.4.76}$$

*If $\vec{Y}(t)$ does not change the direction, then*

$$\left\|u^S(x,t)\right\|_{L_x^\infty L_t^2} \lesssim \|f\|_{L^2} + \|g\|_{\dot{H}^1} + \|F\|_{L_d^1 L_{\widehat{d}}^{2,1} L_t^2}, \tag{4.4.77}$$

*where $d$ is the direction of $\vec{Y}(t)$.*

*Let $\vec{Z}(t)$ be another admissible trajectory, we have the same estimates above with $F$ replaced by*

$$F^{S'}(x,t) := F\left(x + \vec{Z}(t), t\right).$$

*Proof.* It suffices to consider

$$\frac{\sin\left(t\sqrt{H}\right)}{\sqrt{H}} P_c f. \tag{4.4.78}$$

By construction,

$$\frac{\sin\left(t\sqrt{H}\right)}{\sqrt{H}} P_c = W^+ \frac{\sin\left(t\sqrt{-\Delta}\right)}{\sqrt{-\Delta}} \left(W^+\right)^*. \tag{4.4.79}$$

$$\frac{\sin\left(t\sqrt{H}\right)}{\sqrt{H}} P_c f = W^+ \frac{\sin\left(t\sqrt{-\Delta}\right)}{\sqrt{-\Delta}} \left(W^+\right)^* P_c f. \tag{4.4.80}$$

Denoting

$$h = \left(W^+\right)^* P_c f, \tag{4.4.81}$$

we have

$$\|P_c f\|_{L^2} \simeq \|h\|_{L^2}. \tag{4.4.82}$$

Setting

$$G = \frac{\sin\left(t\sqrt{-\Delta}\right)}{\sqrt{-\Delta}} h, \tag{4.4.83}$$

by Theorem 4.2.5, it is sufficient to consider the boundedness of

$$G + \int_{\mathbb{S}^2}\int_{\mathbb{R}^3} g(x,y,\tau) G\left(S_\tau x + y\right)\, dy d\tau. \tag{4.4.84}$$

Clearly, by the endpoint reversed Strichartz estimate for the free case,

$$\|G\|_{L^\infty_x L^2_t} \lesssim \|h\|_{L^2} \simeq \|P_c f\|_{L^2}. \tag{4.4.85}$$

Next, by Minkowski's inequality,

$$\begin{aligned} &\left\| \int_{\mathbb{S}^2}\int_{\mathbb{R}^3} g(x,y,\tau) G\left(S_\tau x + y\right)\, dy d\tau \right\|_{L^\infty_x L^2_t} \\ &\lesssim \int_{\mathbb{S}^2}\int_{\mathbb{R}^3} \left\| g(x,y,\tau) G\left(S_\tau x + y\right)\right\|_{L^\infty_x L^2_t} dy d\tau \end{aligned} \tag{4.4.86}$$

$$\left\| g(x,y,\tau) G\left(S_\tau x + y\right)\right\|_{L^\infty_x L^2_t} \lesssim \|g(x,y,\tau)\|_{L^\infty_x} \left\| G\left(S_\tau x + y\right)\right\|_{L^\infty_x L^2_t}. \tag{4.4.87}$$

Since reflections with respect to a fixed plane and translations commute with the solution of a free wave equation, we obtain

$$G\left(S_\tau x + y\right) = \frac{\sin\left(t\sqrt{-\Delta}\right)}{\sqrt{-\Delta}} h\left(S_\tau x + y\right). \tag{4.4.88}$$

Therefore,

$$\left\| G\left(S_\tau x + y\right)\right\|_{L^\infty_x L^2_t} \lesssim \left\|h\left(S_\tau x + y\right)\right\|_{L^2} \lesssim \|h\|_{L^2} \simeq \|P_c f\|_{L^2}. \tag{4.4.89}$$

It follows

$$\left\| G + \int_{\mathbb{S}^2} \int_{\mathbb{R}^3} g(x,y,\tau) G\left(S_\tau x + y\right)\, dy d\tau \right\|_{L_x^\infty L_t^2} \tag{4.4.90}$$
$$\lesssim \left(1 + \int_{\mathbb{S}^2} \int_{\mathbb{R}^3} \|g(x,y,\tau)\|_{L_x^\infty}\, dy d\tau\right) \|P_c f\|_{L^2} \lesssim \|f\|_{L^2}.$$

Then we conclude

$$\left\| \frac{\sin\left(t\sqrt{H}\right)}{\sqrt{H}} P_c f \right\|_{L_x^\infty L_t^2} \lesssim \|f\|_{L^2}, \tag{4.4.91}$$

as claimed. □

For the other endpoint revered type estimate, we have

**Theorem 4.4.6.** *Let $\vec{Y}(t)$ be an admissible trajectory. First of all, for the standard case, one has*

$$\|u\|_{L_x^{6,2} L_t^\infty} \lesssim \|f\|_{L^2} + \|g\|_{\dot{H}^1} + \|F\|_{L_x^{\frac{6}{5},2} L_t^\infty}. \tag{4.4.92}$$

*Consider the estimates along the trajectory $\vec{Y}(t)$, one has*

$$\left\| u^S(x,t) \right\|_{L_x^{6,2} L_t^\infty} \lesssim \|f\|_{L^2} + \|g\|_{\dot{H}^1} + \|F\|_{\dot{W}_x^{1,\frac{6}{5}} L_t^1}. \tag{4.4.93}$$

*Let $\vec{Z}(t)$ be another admissible trajectory, we have the same estimates above with $F$ replaced by*

$$F^{S'}(x,t) := F\left(x + \vec{Z}(t), t\right). \tag{4.4.94}$$

### *4.4.3 Wave equations with moving potentials*

Finally in this section, we consider the wave equation

$$\partial_{tt} u - \Delta u + V\left(x - \vec{\mu} t\right) u = 0 \tag{4.4.95}$$

$$u(x,0)=g(x),\ u_t(x,0)=f(x). \tag{4.4.96}$$

Again without of loss of generality, we assume $\vec{\mu}$ is along $\vec{e}_1$ and $\vec{\mu}<1$. Recall that associated to this model, we define

$$H=-\Delta+V\left(\sqrt{1-|\vec{\mu}|^2}x_1,x_2,x_3\right). \tag{4.4.97}$$

Let $m_1,\ \ldots,\ m_w$ be the normalized bound states of $H$ associated to the negative eigenvalues $-\lambda_1^2,\ \ldots,\ -\lambda_w^2$ respectively (notice that by our assumptions, 0 is not an eigenvalue). We denote by $P_b$ the projections on the the bound states of $H$ , respectively, and let $P_c=Id-P_b$.

Performing a Lorentz transformation $L$ with respect to the moving frame $(x-\vec{\mu}t,t)$, we have

$$\partial_{t't'}u_L+Hu_L=0, \tag{4.4.98}$$

$$u_L(x',0)=\tilde{g}(x'),\ (u_L)_t\,(x',0)=\tilde{f}(x') \tag{4.4.99}$$

and

$$\|f\|_{L^2}+\|g\|_{\dot{H}^1}\simeq\|\tilde{f}\|_{L^2}+\|\tilde{g}\|_{\dot{H}^1}. \tag{4.4.100}$$

We can write

$$u_L\left(x',t'\right)=\sum_{i=1}^{w}a_i(t')m_i(x')+r_L\left(x',t'\right), \tag{4.4.101}$$

such that

$$P_cr_L=r_L. \tag{4.4.102}$$

Return to our original coordinate, we have a decomposition for $u$ that

$$u(x,t)=\sum_{i=1}^{w}a_i\left(\gamma(t-vx_1)\right)(m_i)_\mu\,(x,t)+r\,(x,t) \tag{4.4.103}$$

where

$$(m_i)_\mu (x,t) = m_i \left(\gamma \left(x_1 - \mu t\right), x_2, x_3\right). \tag{4.4.104}$$

**Theorem 4.4.7.** *Let $\vec{Y}(t) \in \mathbb{R}^3$ be an admissible trajectory. With the notations from above, we have*

$$\left\| r^S \right\|_{L_x^\infty L_t^2} \lesssim \|f\|_{L^2} + \|g\|_{\dot{H}^1}, \tag{4.4.105}$$

*in particular,*

$$\int_0^\infty \int_{\mathbb{R}^3} \frac{1}{\left\langle x - \vec{Y}(t) \right\rangle^\alpha} r^2(x,t)\, dxdt \lesssim \|f\|_{L^2} + \|g\|_{\dot{H}^1}. \tag{4.4.106}$$

*Proof.* Notice that if $\vec{Y}(t)$ is an admissible trajectory in our original frame $(x,t)$, then if we perform a Lorentz transformation $L(\vec{\mu})$, in the new frame, the trajectory $\vec{Y}(t)$ can be written as $\vec{Z}(t')$ with $\left|\vec{Z}'(t')\right| < \phi\left(\lambda, \vec{\ell}\right) < 1$. In other words, in the new coordinate, the trajectory is still admissible. Then for fixed $x \in \mathbb{R}^3$,

$$\int \left| r^S(x,t) \right|^2 dt \lesssim \sup_{x' \in \mathbb{R}^3} \int \left| r_L^{S'}(x',t') \right|^2 dt', \tag{4.4.107}$$

where

$$r_L^{S'}\left(x',t'\right) = r_L\left(x' + \vec{Z}(t'), t'\right). \tag{4.4.108}$$

By construction and Theorem 5.3.6,

$$\sup_{x' \in \mathbb{R}^3} \int \left| r_L^{S'}(x',t') \right|^2 dt' \lesssim \left( \|\tilde{f}\|_{L^2} + \|\tilde{g}\|_{\dot{H}^1} \right)^2 \simeq \left( \|f\|_{L^2} + \|g\|_{\dot{H}^1} \right)^2 \tag{4.4.109}$$

and hence

$$\left\| r^S \right\|_{L_x^\infty L_t^2} \lesssim \|f\|_{L^2} + \|g\|_{\dot{H}^1}. \tag{4.4.110}$$

The theorem is proved. □

## 4.5 Strichartz Estimates and Energy Estimates

In this section, we establish Strichartz estimates and energy estimates for scattering states to the wave equation

$$\partial_{tt}u - \Delta u + V\left(x - \vec{Y}(t)\right)u = 0, \tag{4.5.1}$$

$$u(x,0) = g(x),\ u_t(x,0) = f(x)$$

with

$$\left|\vec{Y}(t) - \vec{\mu}t\right| \lesssim \langle t\rangle^{-\beta},\ \beta > 1,\ |\vec{\mu}| < 1. \tag{4.5.2}$$

To simplify the problem, we assume

$$H = -\Delta + V\left(\sqrt{1-|\vec{\mu}|^2}x_1, x_2, x_3\right) \tag{4.5.3}$$

only has one bound state $m$ such that

$$Hm = -\lambda^2 m,\ \lambda > 0. \tag{4.5.4}$$

One can observe that our arguments work for the general case.

We start with reversed Strichartz estimates.

**Theorem 4.5.1.** *Let $\vec{h}(t)$ be an admissible trajectory and $u$ be a scattering in the sense of Definition 5.3.2. Then for*

$$u^S(x,t) = u\left(x + \vec{h}(t), t\right) \tag{4.5.5}$$

*one has*

$$\left\|u^S(x,t)\right\|_{L_x^\infty L_t^2[0,\infty)} \lesssim \|f\|_{L^2} + \|g\|_{\dot{H}^1}. \tag{4.5.6}$$

*In particular, it implies for* $\alpha > 3$,

$$\int_0^\infty \int_{\mathbb{R}^3} \frac{1}{\left\langle x - \vec{h}(t) \right\rangle^\alpha} u^2(x,t)\, dxdt \lesssim \|f\|_{L^2} + \|g\|_{\dot{H}^1}. \tag{4.5.7}$$

*Proof.* First of all, we need to understand the evolution of bound states. Writing the equation as

$$\partial_{tt} u - \Delta u + V\left(x - \vec{\mu} t\right) u = \left[ V\left(x - \vec{\mu} t\right) - V\left(x + \vec{Y}(t)\right)\right] u. \tag{4.5.8}$$

Recall that we assume $\vec{\mu}$ is along $x_1$. Suppose $u(x,t)$ is a scattering state. As in (4.4.103), we decompose the evolution as following,

$$u(x,t) = a\left(\gamma(t - \mu x_1)\right) m_\mu\left(x,t\right) + r(x,t) \tag{4.5.9}$$

where

$$m_\mu(x,t) = m\left(\gamma\left(x_1 - \mu t\right), x_2, x_3\right) \tag{4.5.10}$$

and

$$P_c\left(H\right) r_L = r_L. \tag{4.5.11}$$

Performing the Lorentz transformation $L$ with respect to the moving frame $(x - \vec{\mu} t, t)$, we have

$$u_L(x', t') = a\left(t'\right) m\left(x'\right) + r_L(x', t'), \tag{4.5.12}$$

and

$$\partial_{t't'} u_L + H u_L = -M(x', t') u_L \tag{4.5.13}$$

where

$$M(x', t') = -\left[ V\left(x - \vec{\mu} t\right) - V\left(x + \vec{Y}(t)\right)\right]_L. \tag{4.5.14}$$

When $u$ is a scattering state in the sense Definition 5.3.2, the scattering condition forces $a(t)$

to go 0.

Plugging the evolution (4.5.12) into the equation (4.5.13) and taking inner product with $m$, we get

$$\ddot{a}(t') - \lambda^2 a(t') + a(t') \langle Mm, m\rangle + \langle Mr_L, m\rangle = 0 \tag{4.5.15}$$

Notice that

$$\left|M(x', t')\right| \lesssim \frac{1}{\langle \gamma \left(t' + \mu x_1'\right)\rangle^{\beta}}. \tag{4.5.16}$$

One can write

$$\ddot{a}(t') - \lambda^2 a(t') + a(t')c(t') + h(t') = 0, \tag{4.5.17}$$

$$c(t') := \langle Mm, m\rangle \tag{4.5.18}$$

and

$$h(t') := \langle Mr_L, m\rangle . \tag{4.5.19}$$

Since $w$ is exponentially localized by Agmon's estimate, we know

$$\left|c(t')\right| \lesssim e^{-b|t'|},\ b > 0. \tag{4.5.20}$$

The existence of the solution to the ODE (4.5.17) is clear. We study the long-time behavior of the solution. Write the equation as

$$\ddot{a}(t') - \lambda^2 a(t') = -\left[a(t')c(t') + h(t')\right], \tag{4.5.21}$$

and denote

$$N(t') := -\left[a(t')c(t') + h(t')\right]. \tag{4.5.22}$$

Then

$$a(t') = \frac{e^{\lambda t'}}{2}\left[a(0) + \frac{1}{\lambda}\dot{a}(0) + \frac{1}{\lambda}\int_0^{t'} e^{-\lambda s} N(s)\, ds\right] + R(t') \tag{4.5.23}$$

where

$$\left|R(t')\right| \lesssim e^{-ct'}, \tag{4.5.24}$$

for some positive constant $c > 0$. Therefore, the stability condition forces

$$a(0) + \frac{1}{\lambda}\dot{a}(0) + \frac{1}{\lambda}\int_0^\infty e^{-\lambda s} N(s)\, ds = 0. \tag{4.5.25}$$

Then under the stability condition (5.2.145),

$$a(t') = e^{-\lambda t'}\left[a(0) + \frac{1}{2\lambda}\int_0^\infty e^{-\lambda s} N(s) ds\right] + \frac{1}{2\lambda}\int_0^\infty e^{-\lambda|t-s|} N(s)\, ds. \tag{4.5.26}$$

By Young's inequality, to estimate all $L^p$ norms of $a(t')$, it suffices to estimate the $L^1$ norm of $h(t')$, see [14].

By Cauchy-Schwarz and Theorem 4.4.7 ,

$$\int_0^\infty |\langle Mr_L, m\rangle|\, dt \lesssim \|r_L\|_{L^\infty_{x'} L^2_{t'}} \lesssim \|f\|_{L^2} + \|g\|_{\dot{H}^1}.$$

Therefore,

$$\|a(t)\|_{L^p[0,\infty)} \lesssim_p \|f\|_{L^2} + \|g\|_{\dot{H}^1}. \tag{4.5.27}$$

Given $\vec{h}(t)$ an admissible trajectory, set

$$B(x,t) = a\left(\gamma(t - \mu x_1)\right) m_\mu\left(x, t\right), \tag{4.5.28}$$

$$B^S(x,t) = B\left(x + \vec{h}(t), t\right). \tag{4.5.29}$$

By Agmon's estimate, see Theorem 4.3.8, and the $L^1$ norm estimate for $h(t')$, we have

$$\left\|B^S(x,t)\right\|_{L^\infty_x L^2_t[0,\infty)} \lesssim \|f\|_{L^2} + \|g\|_{\dot{H}^1}. \tag{4.5.30}$$

By Theorem 4.4.7, we also know

$$\left\| r^S(x,t) \right\|_{L_x^\infty L_t^2[0,\infty)} \lesssim \|f\|_{L^2} + \|g\|_{\dot{H}^1} \tag{4.5.31}$$

Therefore, one has

$$\left\| u^S(x,t) \right\|_{L_x^\infty L_t^2[0,\infty)} \lesssim \|f\|_{L^2} + \|g\|_{\dot{H}^1}. \tag{4.5.32}$$

We notice that this in particular implies

$$\int_0^\infty \int_{\mathbb{R}^3} \frac{1}{\left\langle x - \vec{h}(t) \right\rangle^\alpha} u^2(x,t)\, dxdt \lesssim \|f\|_{L^2} + \|g\|_{\dot{H}^1}. \tag{4.5.33}$$

The theorem is proved. □

Next, we show Strichartz estimates following [50, 40, 14]. In the following, we use the short-hand notation

$$L_t^p L_x^q := L_t^p \left([0,\infty),\, L_x^q\right). \tag{4.5.34}$$

**Theorem 4.5.2.** *Suppose $u$ is a scattering state in the sense of Definition of 5.3.2 which solves*

$$\partial_{tt} u - \Delta u + V\left(x - \vec{Y}(t)\right) u = 0 \tag{4.5.35}$$

*with initial data*

$$u(x,0) = g(x),\, u_t(x,0) = f(x). \tag{4.5.36}$$

*Then for $p > 2$, and $(p,q)$ satisfying*

$$\frac{1}{2} = \frac{1}{p} + \frac{3}{q} \tag{4.5.37}$$

*we have*

$$\|u\|_{L_t^p L_x^q} \lesssim \|f\|_{L^2} + \|g\|_{\dot{H}^1}. \tag{4.5.38}$$

*Proof.* Following [40], we set $A = \sqrt{-\Delta}$ and notice that

$$\|Af\|_{L^2} \simeq \|f\|_{\dot{H}^1}, \ \forall f \in C^\infty\left(\mathbb{R}^3\right). \tag{4.5.39}$$

For real-valued $u = (u_1, u_2) \in \mathcal{H} = \dot{H}^1\left(\mathbb{R}^3\right) \times L^2\left(\mathbb{R}^3\right)$, we write

$$U := Au_1 + iu_2. \tag{4.5.40}$$

From (4.5.39), we know

$$\|U\|_{L^2} \simeq \|(u_1, u_2)\|_{\mathcal{H}}. \tag{4.5.41}$$

We also notice that $u$ solves (4.5.35) if and only if

$$U := Au + i\partial_t u \tag{4.5.42}$$

satisfies

$$i\partial_t U = AU + V\left(x - \vec{Y}(t)\right)u, \tag{4.5.43}$$

$$U(0) = Ag + if \in L^2\left(\mathbb{R}^3\right). \tag{4.5.44}$$

By Duhamel's formula,

$$U(t) = e^{itA}U(0) - i\int_0^t e^{-i(t-s)A}V\left(\cdot - \vec{Y}(s)\right)u(s)\,ds. \tag{4.5.45}$$

Let $P := A^{-1}\Re$, then from Strichartz estimates for the free evolution,

$$\left\|Pe^{itA}U(0)\right\|_{L_t^p L_x^q} \lesssim \|U(0)\|_{L^2}. \tag{4.5.46}$$

Writing $V = V_1 V_2$ and with the Christ-Kiselev lemma, Lemma 4.2.7, it suffices to bound

$$\left\| P \int_0^\infty e^{-i(t-s)A} V_1 V_2 \left( \cdot - \vec{Y}(s) \right) u(s)\, ds \right\|_{L_t^p L_x^q}. \tag{4.5.47}$$

We only need to analyze

$$\left\| P \int_0^\infty e^{-i(t-s)A} V_1 V_2 \left( \cdot - \vec{Y}(s) \right) u(s)\, ds \right\|_{L_t^p L_x^q} \leq \left\| \widetilde{K} \right\|_{L^2_{t,x} \to L_t^p L_x^q} \left\| V_2 \left( x - \vec{Y}(s) \right) u \right\|_{L^2_{t,x}}$$

where

$$\left( \widetilde{K} F \right)(t) := P \int_0^\infty e^{-i(t-s)A} V_1 \left( \cdot - \vec{Y}(s) \right) F(s)\, ds. \tag{4.5.48}$$

To show $\left\| \widetilde{K} \right\|_{L^2_{t,x} \to L_t^p L_x^q}$ is bounded, we test it against $F \in L^2_{t,x}$, clearly,

$$\left\| \widetilde{K} F \right\|_{L_t^p L_x^q} \leq \left\| P e^{-itA} \right\|_{L^2 \to L_t^p L_x^q} \left\| \int_0^\infty e^{isA} V_1 \left( \cdot - \vec{Y}(s) \right) F(s)\, ds \right\|_{L^2}. \tag{4.5.49}$$

The first factor on the right-hand side of (4.5.49) is bounded by Strichartz estimates for the free evolution. Consider the second factor, by duality, it is sufficient to show

$$\left\| V_1 \left( \cdot - \vec{Y}(t) \right) e^{-itA} \phi \right\|_{L^2_{t,x}} \lesssim \|\phi\|_{L^2}, \; \forall \phi \in L^2 \left( \mathbb{R}^3 \right). \tag{4.5.50}$$

By our assumption,

$$\left| \vec{Y}(t) - \vec{\mu} t \right| \lesssim \langle t \rangle^{-\beta}, \; \beta > 1, \; |\vec{\mu}| < 1.$$

Therefore, it reduces to show

$$\left\| (1 + |x - \vec{\mu} t|)^{-\frac{1}{2} - \epsilon} e^{-itA} \phi \right\|_{L^2_{t,x}} \lesssim \|\phi\|_{L^2}, \; \forall \phi \in L^2 \left( \mathbb{R}^3 \right). \tag{4.5.51}$$

Notice that this is a consequence of that the energy of the free wave equation stays comparable under Lorentz transformations, Theorem 5.3.6. To show estimate (4.5.51), one can apply the Lorentz transformation $L$. In the new frame $\left(x', t'\right)$, then we can use the standard local energy decay for free wave equations, estimate (4.8.12) in Appendix B. Finally after applying an inverse transformation back to the original frame, we obtain (4.5.51).

From estimate (4.5.51), one does have

$$\left\| V_1 \left( \cdot - \vec{Y}(t) \right) e^{-itA} \phi \right\|_{L^2_{t,x}} \lesssim \|\phi\|_{L^2} . \tag{4.5.52}$$

Therefore, indeed,

$$\left\| \widetilde{K} \right\|_{L^2_{t,x} \to L^p_t L^q_x} \leq C. \tag{4.5.53}$$

Hence

$$\left\| P \int_0^\infty e^{-i(t-s)A} V_1 V_2 \left( \cdot - \vec{Y}(s) \right) u(s)\, ds \right\|_{L^p_t L^q_x} \lesssim \left\| V_2 \left( x - \vec{Y}(s) \right) u \right\|_{L^2_{t,x}} . \tag{4.5.54}$$

By our estimate (4.5.7),

$$\left\| V_2 \left( x - \vec{Y}(s) \right) u \right\|_{L^2_{t,x}} \lesssim \left( \int_{\mathbb{R}^+} \int_{\mathbb{R}^3} \frac{1}{\left\langle x - \vec{Y}(t) \right\rangle^\alpha} |u(x,t)|^2 \, dx dt \right)^{\frac{1}{2}} \lesssim \|f\|_{L^2} + \|g\|_{\dot{H}^1}. \tag{4.5.55}$$

Therefore,

$$\left\| P \int_0^\infty e^{-i(t-s)A} V_1 V_2 \left( \cdot - \vec{Y}(s) \right) u(s)\, ds \right\|_{L^p_t L^q_x} \lesssim \|f\|_{L^2} + \|g\|_{\dot{H}^1}. \tag{4.5.56}$$

Hence one can conclude

$$\|u\|_{L^p_t L^q_x} \lesssim \|f\|_{L^2} + \|g\|_{\dot{H}^1}, \tag{4.5.57}$$

as we claimed. $\square$

The energy estimates can be established in a similar manner.

**Theorem 4.5.3.** *Suppose $u$ is a scattering state in the sense of Definition of 5.3.2 which solves*

$$\partial_{tt}u - \Delta u + V\left(x - \vec{Y}(t)\right)u = 0 \tag{4.5.58}$$

*with initial data*

$$u(x,0) = g(x),\ u_t(x,0) = f(x). \tag{4.5.59}$$

*Then we have*

$$\sup_{t\in\mathbb{R}}\left(\|\nabla u(t)\|_{L^2} + \|u_t(t)\|_{L^2}\right) \lesssim \|f\|_{L^2} + \|g\|_{\dot{H}^1}. \tag{4.5.60}$$

*Proof.* Again, we set $A = \sqrt{-\Delta}$ and notice that

$$\|Af\|_{L^2} \simeq \|f\|_{\dot{H}^1},\ \forall f \in C^\infty\left(\mathbb{R}^3\right). \tag{4.5.61}$$

For real-valued $u = (u_1, u_2) \in \mathcal{H} = \dot{H}^1\left(\mathbb{R}^3\right) \times L^2\left(\mathbb{R}^3\right)$, we write

$$U := Au_1 + iu_2. \tag{4.5.62}$$

and we know

$$\|U\|_{L^2} \simeq \|(u_1, u_2)\|_{\mathcal{H}}. \tag{4.5.63}$$

We also notice that $u$ solves the original equation if and only if

$$U := Au + i\partial_t u \tag{4.5.64}$$

satisfies

$$i\partial_t U = AU + V\left(x - \vec{Y}(t)\right)u, \tag{4.5.65}$$

$$U(0) = Ag + if \in L^2\left(\mathbb{R}^3\right). \tag{4.5.66}$$

By Duhamel's formula,

$$U(t) = e^{itA}U(0) - i\int_0^t e^{-i(t-s)A}\left(V\left(\cdot - \vec{Y}(s)\right)u(s)\right)\,ds. \tag{4.5.67}$$

From the energy estimate for the free evolution,

$$\sup_{t\in\mathbb{R}}\left\|e^{itA}U(0)\right\|_{L^2_x} \lesssim \|U(0)\|_{L^2}. \tag{4.5.68}$$

Writing $V = V_1V_2$, it suffices to bound

$$\sup_{t\in\mathbb{R}}\left\|\int_0^\infty e^{-i(t-s)A}V_1V_2\left(\cdot - \vec{Y}(s)\right)u(s)\,ds\right\|_{L^2_x}. \tag{4.5.69}$$

This is can be bounded in a same manner as Theorem 4.5.2.

It is clear that

$$\left\|\int_0^\infty e^{-i(t-s)A}V_1V_2\left(\cdot - \vec{Y}(s)\right)u(s)\,ds\right\|_{L^\infty_tL^2_x} \leq \left\|\widetilde{K}\right\|_{L^2_tL^2_x\to L^\infty_tL^2_x}\left\|V_2\left(x - \vec{Y}(t)\right)u\right\|_{L^2_tL^2_x}, \tag{4.5.70}$$

where

$$\left(\widetilde{K}F\right)(t) := \int_0^\infty e^{-i(t-s)A}V_1\left(\cdot - \vec{Y}(s)\right)F(s)\,ds. \tag{4.5.71}$$

We need to estimate

$$\left\|\widetilde{K}\right\|_{L^2_tL^2_x\to L^\infty_tL^2_x}. \tag{4.5.72}$$

Testing against $F \in L^2_tL^2_x$, clearly,

$$\left\|\widetilde{K}F\right\|_{L^\infty_tL^2_x} \leq \left\|e^{-itA}\right\|_{L^2\to L^\infty_tL^2_x}\left\|\int_0^\infty e^{isA}V_1\left(\cdot - \vec{Y}(s)\right)F(s)\,ds\right\|_{L^2}. \tag{4.5.73}$$

The first factors on the right-hand side of (5.2.19) is bounded by the energy estimates for the free evolution. Consider the second factor, by duality, it suffices to show

$$\left\| V_1 \left( x - \vec{Y}(t) \right) e^{-itA} \phi \right\|_{L_t^2 L_x^2} \lesssim \|\phi\|_{L^2}, \forall \phi \in L^2 \left( \mathbb{R}^3 \right). \tag{4.5.74}$$

From our discussions Theorem 4.5.2, we know

$$\left\| V_1 \left( x - \vec{Y}(t) \right) e^{-itA} \phi \right\|_{L_t^2 L_x^2} \lesssim \|\phi\|_{L^2}. \tag{4.5.75}$$

Hence

$$\left\| \int_0^\infty e^{isA} V_1 \left( \cdot - \vec{Y}(s) \right) F(s) \, ds \right\|_{L^2} \lesssim \|F\|_{L_t^2 L_x^2}. \tag{4.5.76}$$

Therefore, indeed, we have

$$\left\| \widetilde{K} \right\|_{L_t^2 L_x^2 \to L_t^\infty L_x^2} \leq C \tag{4.5.77}$$

and

$$\sup_{t \in \mathbb{R}} \left\| \int_0^\infty e^{-i(t-s)A} V_1 V_2 \left( \cdot - \vec{Y}(s) \right) u(s) \, ds \right\|_{L_x^2} \lesssim \left\| V_2 \left( \cdot - \vec{Y}(s) \right) u \right\|_{L_{t,x}^2}. \tag{4.5.78}$$

By the weighted estimate (4.5.7),

$$\left\| V_2 \left( \cdot - \vec{Y}(s) \right) u \right\|_{L_t^2 L_x^2} \lesssim \left( \int_0^\infty \int_{\mathbb{R}^3} \frac{1}{\left\langle x - \vec{Y}(t) \right\rangle^\alpha} |u(x,t)|^2 \, dx dt \right)^{\frac{1}{2}} \lesssim \|f\|_{L^2} + \|g\|_{\dot{H}^1}. \tag{4.5.79}$$

It implies

$$\sup_{t \in \mathbb{R}} \left\| \int_0^\infty e^{-i(t-s)A} V_1 V_2 \left( \cdot - \vec{Y}(s) \right) u(s) \, ds \right\|_{L_x^2} \lesssim \|f\|_{L^2} + \|g\|_{\dot{H}^1}. \tag{4.5.80}$$

Therefore, by estimates (4.5.68) and (4.5.80), we have

$$\sup_{t\geq 0}\left(\|\nabla u(t)\|_{L^2}+\|u_t(t)\|_{L^2}\right)\lesssim \|f\|_{L^2}+\|g\|_{\dot{H}^1} \tag{4.5.81}$$

as claimed. □

To finish this section, we show one important application of Theorem 4.5.1.

We denote

$$E_V(t)=\int_{\mathbb{R}^3}|\nabla_x u|^2+|\partial_t u|^2+V\left(x-\vec{Y}(t)\right)|u|^2\,dx. \tag{4.5.82}$$

**Corollary 4.5.4.** *Suppose $u$ is a scattering state in the sense of Definition of 5.3.2 which solves*

$$\partial_{tt}u-\Delta u+V(x-\vec{Y}(t))u=0 \tag{4.5.83}$$

*with initial data*

$$u(x,0)=g(x),\, u_t(x,0)=f(x). \tag{4.5.84}$$

*Assume*

$$\|\nabla V\|_{L^1}<\infty, \tag{4.5.85}$$

*then $E_V(t)$ is bounded by the initial energy independently of $t$,*

$$\sup_t E_V(t)\lesssim \|(g,f)\|^2_{\dot{H}^1\times L^2}. \tag{4.5.86}$$

*Proof.* We might assume $u$ is smooth. Taking the time derivative of $E_V(t)$ and by the fact that $u$ solves equation, we obtain

$$\partial_t E_V(t)=\int_{\mathbb{R}^3}\partial_t V(x-\vec{Y}(t))\,|u(x,t)|^2\,dx=-\vec{Y}'(t)\int_{\mathbb{R}^3}\partial_y V(y)\left|u^S(y,t)\right|^2 dy. \tag{4.5.87}$$

by a simple change of variable.

Note that

$$\begin{aligned} \int_0^\infty \left|\partial_t E_V(t)\right| dt &\lesssim \int_0^\infty \int_{\mathbb{R}^3} \left|\partial_y V(y)\right| \left|u^S(y)\right|^2 dydt, \\ &\lesssim \left\|\partial_x V\right\|_{L^1_x} \left\|u^S\right\|^2_{L^\infty_x L^2_t} \\ &\lesssim \left\|(g,f)\right\|^2_{\dot{H}^1\times L^2} \end{aligned} \tag{4.5.88}$$

where in the last inequality, we applied Theorem 4.5.1.

Therefore, for arbitrary $t \in \mathbb{R}^+$, we have

$$E_V(t) - E_V(0) \leq \int_0^\infty \left|\partial_t E_V(t)\right| dt \lesssim \left\|(g,f)\right\|^2_{\dot{H}^1\times L^2} \tag{4.5.89}$$

which implies

$$\sup_t E_V(t) \lesssim \left\|(g,f)\right\|^2_{\dot{H}^1\times L^2}. \tag{4.5.90}$$

We are done. □

With endpoint Strichartz estimates along smooth trajectories, we can also derive inhomogenenous Strichartz estimates. One can find a detailed argument in [14].

## 4.6 Scattering and Asymptotic Completeness

In this section, we show some applications of the results in this chapter. We will study the long-time behaviors for a scattering state in the sense of Definition 5.3.2.

Following the notations from above section, we will still use the short-hand notation

$$L^p_t L^q_x := L^p_t\left([0,\infty),\, L^q_x\right). \tag{4.6.1}$$

We reformulate the wave equation as a Hamiltonian system,

$$U' = JE'(U) \tag{4.6.2}$$

where $J$ is a skew symmetric matrix and $E'(U)$ is the Frechet derivative of the conserved quantity. Setting

$$U := \begin{pmatrix} u \\ \partial_t u \end{pmatrix}, \tag{4.6.3}$$

$$J := \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \tag{4.6.4}$$

$$H_F := \begin{pmatrix} -\Delta & 0 \\ 0 & 1 \end{pmatrix}, \tag{4.6.5}$$

we can rewrite the free wave equation as

$$\dot{U}_0 - JH_F U_0 = 0, \tag{4.6.6}$$

$$U_0[0] = \begin{pmatrix} g_0 \\ f_0 \end{pmatrix} \tag{4.6.7}$$

The solution of the free wave equation is given by

$$U_0 = e^{tJH_F} U_0[0]. \tag{4.6.8}$$

**Theorem 4.6.1.** *Suppose $u$ is a scattering state in the sense of Definition of 5.3.2 which solves*

$$\partial_{tt} u - \Delta u + V(x - \vec{Y}(t))u = 0 \tag{4.6.9}$$

*with initial data*

$$u(x,0) = g(x),\ u_t(x,0) = f(x). \tag{4.6.10}$$

*Write*

$$U = (u, u_t)^t \in C^0\left([0,\infty);\ \dot{H}^1\right) \times C^0\left([0,\infty);\ L^2\right), \tag{4.6.11}$$

*with initial data* $U[0] = (g, f)^t \in \dot{H}^1 \times L^2$. Then there exist free data

$$U_0[0] = (g_0, f_0)^t \in \dot{H}^1 \times L^2$$

such that

$$\left\| U[t] - e^{tJH_F} U_0[0] \right\|_{\dot{H}^1 \times L^2} \to 0 \tag{4.6.12}$$

as $t \to \infty$.

*Proof.* We will still use the formulation in Theorem (4.5.38). We set $A = \sqrt{-\Delta}$ and notice that

$$\|Af\|_{L^2} \simeq \|f\|_{\dot{H}^1},\ \forall f \in C^\infty\left(\mathbb{R}^3\right). \tag{4.6.13}$$

For real-valued $u = (u_1, u_2) \in \mathcal{H} = \dot{H}^1\left(\mathbb{R}^3\right) \times L^2\left(\mathbb{R}^3\right)$, we write

$$U := Au_1 + iu_2. \tag{4.6.14}$$

We know

$$\|U\|_{L^2} \simeq \|(u_1, u_2)\|_{\mathcal{H}}. \tag{4.6.15}$$

We also notice that $u$ solves (4.6.9) if and only if

$$U := Au + i\partial_t u \tag{4.6.16}$$

satisfies

$$i\partial_t U = AU + V\left(x - \vec{Y}(t)\right)u, \tag{4.6.17}$$

$$U(0) = Ag + if \in L^2\left(\mathbb{R}^3\right). \tag{4.6.18}$$

By Duhamel's formula, for fixed $T$

$$U(T) = e^{iTA}U(0) - i\int_0^T e^{-i(T-s)A}\left(V\left(\cdot - \vec{Y}(s)\right)u(s)\right)\,ds. \tag{4.6.19}$$

Applying the free evolution backwards, we obtain

$$e^{-iTA}U(T) = U(0) - i\int_0^T e^{isA}\left(V\left(\cdot - \vec{Y}(s)\right)u(s)\right)\,ds. \tag{4.6.20}$$

Letting $T$ go to $\infty$, we define

$$U_0(0) := U(0) - i\int_0^\infty e^{isA}\left(V\left(\cdot - \vec{Y}(s)\right)u(s)\right)\,ds \tag{4.6.21}$$

By construction, we just need to show $U_0[0]$ is well-defined in $L^2$, then automatically,

$$\left\|U(t) - e^{itA}U_0(0)\right\|_{L^2} \to 0. \tag{4.6.22}$$

It suffices to show

$$\int_0^\infty e^{isA}\left(V\left(\cdot - \vec{Y}(s)\right)u(s)\right)\,ds \in L^2. \tag{4.6.23}$$

Then following the argument as in the proof of Theorem 4.5.2, we write $V = V_1V_2$.

We consider

$$\left\|\int_0^\infty e^{isA}V_1V_2\left(\cdot - \vec{Y}(s)\right)u(s)\,ds\right\|_{L^2_x} \leq \|K\|_{L^2_{t,x}\to L^2_x}\left\|V_2\left(\cdot - \vec{Y}(s)\right)u\right\|_{L^2_{t,x}}, \tag{4.6.24}$$

where

$$(KF)(t) := \int_0^\infty e^{isA} V_1 \left(\cdot - \vec{Y}(s)\right) F(s)\, ds. \tag{4.6.25}$$

By the same argument in the proof of Theorem 4.5.3, one has

$$\|K\|_{L^2_{t,x} \to L^2_x} \leq C. \tag{4.6.26}$$

Therefore

$$\left\| \int_0^\infty e^{isA} V_1 V_2 \left(\cdot - \vec{Y}(s)\right) u(s)\, ds \right\|_{L^2_x} \lesssim \left\| V_2 \left(x - \vec{Y}(t)\right) u \right\|_{L^2_{t,x}}. \tag{4.6.27}$$

By estimate (4.5.7),

$$\left\| V_2 \left(x - \vec{Y}(t)\right) u \right\|_{L^2_{t,x}} \lesssim \left( \int_{\mathbb{R}^+} \int_{\mathbb{R}^3} \frac{1}{\left\langle x - \vec{Y}(t) \right\rangle^{\alpha}} |u(x,t)|^2\, dx dt \right)^{\frac{1}{2}} \lesssim \|f\|_{L^2} + \|g\|_{\dot{H}^1}. \tag{4.6.28}$$

We conclude

$$\int_0^\infty e^{isA} \left( V \left(\cdot - \vec{Y}(s)\right) u(s) \right)\, ds \in L^2$$

with

$$\left\| \int_0^\infty e^{isA} \left( V \left(\cdot - \vec{Y}(s)\right) u(s) \right)\, ds \right\|_{L^2} \lesssim \|f\|_{L^2} + \|g\|_{\dot{H}^1}. \tag{4.6.29}$$

So

$$U_0(0) := U(0) - i \int_0^\infty e^{isA} \left( V \left(\cdot - \vec{Y}(s)\right) u(s) \right)\, ds \tag{4.6.30}$$

is well-defined in $L^2$ and

$$\left\| U(t) - e^{itA} U_0(0) \right\|_{L^2} \to 0. \tag{4.6.31}$$

Define

$$(g_0, f_0) := \left( A^{-1} \Re U_0(0),\, \Im U_0(0) \right). \tag{4.6.32}$$

By construction, notice that

$$U[t] = \left(A^{-1}\Re U(t),\, \Im U(t)\right) \tag{4.6.33}$$

and

$$\left\|U[t] - e^{tJH_F}U_0[0]\right\|_{\dot{H}^1\times L^2} \to 0. \tag{4.6.34}$$

We are done. □

To finish this section, we show the asymptotic completeness for the wave equation with the potential moving along a straight line.

Consider the wave equation

$$\partial_{tt}u - \Delta u + V(x - \vec{\mu}t)u = 0 \tag{4.6.35}$$

with initial data

$$u(x,0) = g(x),\, u_t(x,0) = f(x). \tag{4.6.36}$$

Without loss of generality, we still that assume $\vec{u}$ is along $\vec{e}_1$.

Let $m_1$, …, $m_w$ be the normalized bound states of

$$H = -\Delta + V\left(\sqrt{1-|\mu|}x_1, x_2, x_3\right) \tag{4.6.37}$$

associated with eigenvalues $-\lambda_1^2$, …, $-\lambda_w^2$ respectively with $\lambda_i > 0$, $i = 1, \ldots, w$. Setting

$$A_H = \begin{pmatrix} 0 & 1 \\ -H & 0 \end{pmatrix}, \tag{4.6.38}$$

then the point spectrum of $A_H$ is

$$\sigma_p = \bigcup_{i=1}^{w} \{\pm\lambda_i\} \tag{4.6.39}$$

and the continuous spectrum is

$$\sigma_c = i\,(-\infty, \infty)\,. \tag{4.6.40}$$

Setting

$$E_i^{\pm} = \begin{pmatrix} m_i \\ \pm\lambda_i m_i \end{pmatrix},\ i = 1, \ldots, w, \tag{4.6.41}$$

we know $E_i^{\pm}$ are eigenvectors of $A_H$ with (4.6.4) with eigenvalues $\pm\lambda_i$. One can define the associated Riesz projection

$$P_{i,\pm}(H) := \left\langle \cdot, JE_i^{\mp} \right\rangle E_i^{\pm} \tag{4.6.42}$$

onto $E_i^{\pm}$. One can check

$$P_{i,\pm}(H) \begin{pmatrix} u \\ \partial_t u \end{pmatrix} = \langle \pm\lambda_i u(t) + \partial_t u(t),\, m_i \rangle\,. \tag{4.6.43}$$

From the standard asymptotic completeness results, if we write

$$\dot{U} = A_H U \tag{4.6.44}$$

where as (5.0.19),

$$U = \begin{pmatrix} u \\ \partial_t u \end{pmatrix}, \tag{4.6.45}$$

and

$$U[0] = \begin{pmatrix} g \\ f \end{pmatrix} \tag{4.6.46}$$

then one can decompose the evolution as

$$U(t) = \sum_{i=1}^{w} \left\langle U[0], JE_{i,\mp} \right\rangle e^{\pm\lambda_i t} E_i^{\pm} + e^{tH_F} U_0[0] + R(t) \tag{4.6.47}$$

where $e^{tH_F} U_0[0]$ is the free evolution with initial data $U_0[0]$ and

$$\|R(t)\|_{\dot{H}^1 \times L^2} \to 0, \ t \to \infty. \tag{4.6.48}$$

With notations above, we can obtain a similar decomposition as (4.6.47) when the potential is moving.

**Corollary 4.6.2.** *Suppose $H$ admits no eigenfunction nor resonances at zero. Let $u$ solve*

$$\partial_{tt} u - \Delta u + V(x - \vec{\mu} t) u = 0. \tag{4.6.49}$$

*Write*

$$U = (u, u_t)^t \in C^0\left([0,\infty); \, \dot{H}^1\right) \times C^0\left([0,\infty); \, L^2\right), \tag{4.6.50}$$

*with initial data $U[0] = (g, f)^t \in \dot{H}^1 \times L^2$.* Then there exist free data

$$U_0[0] = (g_0, f_0)^t \in \dot{H}^1 \times L^2$$

such that with $\gamma = \frac{1}{\sqrt{1-|\mu|^2}}$

$$U(t) = \sum_{i=1}^{w} a_{i,\pm} e^{\pm\lambda_i \gamma (t - \mu x_1)} E_{i,\mu}^{\pm}(x,t) + e^{tH_F} U_0[0] + R(t) \tag{4.6.51}$$

where

$$E_{i,\mu}^{\pm}(x,t) = E_i^{\pm}\left(\gamma\left(x_1 - \mu t\right), x_2, x_3\right)$$

and

$$\|R(t)\|_{\dot{H}^1\times L^2} \to 0,\ t\to\infty. \tag{4.6.52}$$

*Proof.* Applying a Lorentz transformation such that under the new frame $(x', t')$, $V$ is stationary, by the standard asymptotic completeness decomposition, one can write

$$U_L(x', t') = \sum_{i=1}^{w} \langle U_L[0], JE_{i,\mp}\rangle e^{\pm\lambda_i t'} E_i^{\pm}(x') + \mathcal{R}_L(x', t'), \tag{4.6.53}$$

where again, we used subscript $L$ to denote the function under the new frame.

Clearly, by the above decomposition (4.6.53),

$$P_b(H)\mathcal{R}_L(x', t') = 0. \tag{4.6.54}$$

Then in the original frame,

$$U(t) = \sum_{i=1}^{w} a_{i,\pm} e^{\pm\lambda_i\gamma(t-\mu x_1)} E_{i,\mu}^{\pm}(x,t) + \mathcal{R}(x,t). \tag{4.6.55}$$

where

$$a_{i,\pm} = \langle U_L[0], JE_{i,\mp}\rangle. \tag{4.6.56}$$

By construction, $\mathcal{R}(x,t)$ satisfies the conditions in Theorem 5.3.7. Hence

$$\mathcal{R}(x,t) = e^{tH_F}U_0[0] + R(t) \tag{4.6.57}$$

where $e^{tH_F}U_0[0]$ is the free evolution with initial data $U_0[0]$ and

$$\|R(t)\|_{\dot{H}^1\times L^2} \to 0,\ t\to\infty. \tag{4.6.58}$$

Therefore, finally, we can write

$$U(t) = \sum_{i=1}^{w} a_{i,\pm} e^{\pm\lambda_i \gamma(t-\mu x_1)} E_{i,\mu}^{\pm}(x,t) + e^{tH_F} U_0[0] + R(t) \tag{4.6.59}$$

with

$$E_{i,\mu}^{\pm}(x,t) = E_i^{\pm}(\gamma(x_1 - \mu t), x_2, x_3)$$

and

$$\|R(t)\|_{\dot{H}^1 \times L^2} \to 0, \ t \to \infty. \tag{4.6.60}$$

The theorem is proved. □

**Remark 4.6.3.** *As a final remark, we point out that there is no hope to establish an elegant asymptotic completeness if the potential is not moving along a straight line. If there is a perturbation from that case, the interaction among bound states becomes complicated. Basically, the mechanism is that if the evolution of one bound state is activated, say the bound state with the highest energy, then it will not only cause exponential growth with highest rate for itself but also make the evolution of other bound states grow exponentially. Meanwhile, if we have a scattering state, the evolution of bound states is controllable.*

## 4.7 Appendix A

For the sake of completeness, in this appendix, we provide the proof of dispersive estimates for the free wave equation in $\mathbb{R}^3$ based on the idea of reversed Strichartz estimates.

Consider

$$u_{tt} - \Delta u = 0 = \Box u \tag{4.7.1}$$

with initial data

$$u(0) = g, \ u_t(0) = f. \tag{4.7.2}$$

One can write down $u$ explicitly,

$$u = \frac{\sin\left(t\sqrt{-\Delta}\right)}{\sqrt{-\Delta}} f + \cos\left(t\sqrt{-\Delta}\right) g. \tag{4.7.3}$$

**Theorem 4.7.1.** *In $\mathbb{R}^3$, suppose $f \in L^2$, $\nabla f \in L^1$ and $g \in L^2$, $\Delta g \in L^1$. Then one has the following estimates:*

$$\left\| \frac{\sin\left(t\sqrt{-\Delta}\right)}{\sqrt{-\Delta}} f \right\|_{L_x^\infty} \lesssim \frac{1}{|t|} \left\|\nabla f\right\|_{L_x^1}, \tag{4.7.4}$$

$$\left\| \cos\left(t\sqrt{-\Delta}\right) g \right\|_{L_x^\infty} \lesssim \frac{1}{|t|} \left\|\Delta g\right\|_{L_x^1}. \tag{4.7.5}$$

*Remark.* Note that the second estimate is slightly different from the estimates commonly used in the literature. For example, in Krieger-Schlag [39] one needs the $L^1$ norm of $D^2 g$ instead of $\Delta g$.

*Proof.* First of all, we consider

$$\frac{\sin\left(t\sqrt{-\Delta}\right)}{\sqrt{-\Delta}} f. \tag{4.7.6}$$

In $\mathbb{R}^3$, one has

$$\frac{\sin\left(t\sqrt{-\Delta}\right)}{\sqrt{-\Delta}} f = \frac{1}{4\pi t} \int_{|x-y|=t} f(y)\, dy. \tag{4.7.7}$$

Without loss of generality, we assume $t \geq 0$.

Multiplying $t$ and integrating, we obtain

$$\begin{aligned} \int_0^\infty \left| t \frac{\sin\left(t\sqrt{-\Delta}\right)}{\sqrt{-\Delta}} f \right| dt &\lesssim \int_0^\infty \int_{\mathbb{S}^2} |f(x+r\omega)|\, r^2\, d\omega dr \\ &\lesssim \|f\|_{L_x^1}. \end{aligned} \tag{4.7.8}$$

Therefore,

$$\left\| t \frac{\sin\left(t\sqrt{-\Delta}\right)}{\sqrt{-\Delta}} f \right\|_{L_x^\infty L_t^1} \lesssim \|f\|_{L_x^1}. \tag{4.7.9}$$

Notice that, from the above estimate, we also have

$$\left\| \int_t^\infty \frac{\sin\left(s\sqrt{-\Delta}\right)}{\sqrt{-\Delta}} f\, ds \right\|_{L_x^\infty} \lesssim \frac{1}{|t|} \|f\|_{L_x^1}. \tag{4.7.10}$$

Replacing $f$ with $\Delta f$, it implies that

$$\left\| \int_t^\infty \sqrt{-\Delta} \sin\left(s\sqrt{-\Delta}\right) f\, ds \right\|_{L_x^\infty} \lesssim \frac{1}{|t|} \|\Delta f\|_{L_x^1}. \tag{4.7.11}$$

On the other hand,

$$\begin{aligned} \int_0^\infty \left| t \cos\left(t\sqrt{-\Delta}\right) f \right| dt &\lesssim \int_0^\infty \int_{\mathbb{S}^2} \left| r f\left(x + r\omega\right) d\omega + r^2 \partial_r f\left(x + r\omega\right) \right| d\omega dr \\ &\lesssim \|\nabla f\|_{L_x^1} \end{aligned} \tag{4.7.12}$$

where in the last inequality, we applied integration by parts in $r$ in the first term of the RHS of the first line.

Therefore,

$$\left\| t \cos\left(t\sqrt{-\Delta}\right) f \right\|_{L_x^\infty L_t^1} \lesssim \|\nabla f\|_{L_x^1}. \tag{4.7.13}$$

Hence

$$\left\| \int_t^\infty \cos\left(s\sqrt{-\Delta}\right) f\, ds \right\|_{L_x^\infty} \lesssim \frac{1}{|t|} \|\nabla f\|_{L_x^1}. \tag{4.7.14}$$

Finally, we check

$$\frac{\sin\left(t\sqrt{-\Delta}\right)}{\sqrt{-\Delta}} f = \int_t^\infty \cos\left(s\sqrt{-\Delta}\right) f\, ds, \tag{4.7.15}$$

and

$$\cos\left(t\sqrt{-\Delta}\right) g = \int_t^\infty \sqrt{-\Delta} \sin\left(s\sqrt{-\Delta}\right) g\, ds. \tag{4.7.16}$$

Let $f$, $g$, $h$ be any test functions. Define

$$Ag = \cos\left(t\sqrt{-\Delta}\right) g - \int_t^\infty \sqrt{-\Delta}\sin\left(s\sqrt{-\Delta}\right) g\, ds \tag{4.7.17}$$

and

$$Bf = \frac{\sin\left(t\sqrt{-\Delta}\right)}{\sqrt{-\Delta}} f - \int_t^\infty \cos\left(s\sqrt{-\Delta}\right) f\, ds. \tag{4.7.18}$$

It is easy to check that $A$, $B$ are independent of $t$.

For $A$, one observes that

$$\left\langle \cos\left(t\sqrt{-\Delta}\right) g,\, h \right\rangle \to 0 \tag{4.7.19}$$

and

$$\left\| \int_t^\infty \frac{\sin\left(s\sqrt{-\Delta}\right)}{\sqrt{-\Delta}} f\, ds \right\|_{L_x^\infty} \lesssim \frac{1}{|t|} \left\| f \right\|_{L_x^1}. \tag{4.7.20}$$

Therefore,

$$\langle Ag,\, h\rangle \to 0,\, t \to \infty. \tag{4.7.21}$$

Since $A$ is independent of $t$, one concludes that

$$\langle Ag,\, h\rangle = 0 \tag{4.7.22}$$

for any pair of test functions and hence

$$A = 0. \tag{4.7.23}$$

Similarly, we get

$$B = 0. \tag{4.7.24}$$

Therefore by our calculations above, we can obtain the dispersive estimates for the free wave equation,

$$\begin{aligned}\left\|\frac{\sin\left(t\sqrt{-\Delta}\right)}{\sqrt{-\Delta}}f\right\|_{L_x^\infty} &= \left\|\int_t^\infty \cos\left(s\sqrt{-\Delta}\right) f\,ds\right\|_{L_x^\infty} \\ &\lesssim \frac{1}{|t|}\left\|\nabla f\right\|_{L_x^1}, \end{aligned} \tag{4.7.25}$$

and

$$\begin{aligned}\left\|\cos\left(t\sqrt{-\Delta}\right) g\right\|_{L_x^\infty} &= \left\|\int_t^\infty \sqrt{-\Delta}\sin\left(s\sqrt{-\Delta}\right) g\,ds\right\|_{L_x^\infty} \\ &\lesssim \frac{1}{|t|}\left\|\Delta g\right\|_{L_x^1}. \end{aligned} \tag{4.7.26}$$

The theorem is proved. □

## 4.8 Appendix B

We derive the local energy decay estimate for the free wave equation by the Fourier method.

Recall the coarea formula: for a a real-valued Lipschitz function $u$ and a $L^1$ function $g$ then

$$\int_{\mathbb{R}^n} g(x)\left|\nabla u(x)\right| dx = \int_{\mathbb{R}}\int_{\{u(x)=t\}} g(x)\,d\sigma(x)dt, \tag{4.8.1}$$

where $\sigma$ is the surface measure.

**Lemma 4.8.1.** *For $F \in C_0^\infty$, $\phi$ smooth and non-degenerate,i.e. $|\nabla\phi(x)| \neq 0$, one has*

$$\int_{\mathbb{R}}\int_{\mathbb{R}^n} e^{i\lambda\phi(x)}F(x)\,dxd\lambda = (2\pi)^n\int_{\{\phi=0\}}\frac{F(x)}{|\nabla\phi(x)|}\,d\sigma(x). \tag{4.8.2}$$

*Proof.* From (4.8.1),

$$\int_{\mathbb{R}}\int_{\mathbb{R}^n} e^{i\lambda\phi(x)}F(x)\,dxd\lambda = \int_{\mathbb{R}}\int_{\mathbb{R}} e^{i\lambda y}\int_{\{\phi=y\}}\frac{F(x)}{|\nabla\phi(x)|}\,d\sigma(x)dyd\lambda. \tag{4.8.3}$$

Denote $\int_{\{\phi=y\}} \frac{F(x)}{|\nabla\phi(x)|}\, d\sigma(x) = g(y)$, then

$$\begin{aligned}\int_{\mathbb{R}}\int_{\mathbb{R}^n} e^{i\lambda\phi(x)}F(x)\,dxd\lambda &= \int_{\mathbb{R}}\int_{\mathbb{R}} e^{i\lambda y}g(y)\,dyd\lambda \\ &= (2\pi)^{\frac{n}{2}}\int_{\mathbb{R}}\hat{g}(\lambda)\,d\lambda \\ &= (2\pi)^n\, g(0) \\ &= \int_{\{\phi=0\}}\frac{F(x)}{|\nabla\phi(x)|}\,d\sigma(x). \end{aligned} \tag{4.8.4}$$

We are done. □

It suffices to consider the half wave evolution,

$$e^{it\sqrt{-\Delta}}f. \tag{4.8.5}$$

**Theorem 4.8.2** (Local energy decay)**.** *Let $\chi \geq 0$ be a smooth cut-off function such that $\hat{\chi}$ has compact support. Then*

$$\left\|\chi(x)e^{it\sqrt{-\Delta}}f\right\|_{L^2_{t,x}} \lesssim \|f\|_{L^2_x}. \tag{4.8.6}$$

*Proof.* Consider

$$\begin{aligned}\int_{\mathbb{R}}\int_{\mathbb{R}^n}\left|e^{it\sqrt{-\Delta}}f\right|^2(x)\chi(x)\,dxdt &= \int_{\mathbb{R}}\left\langle e^{it\sqrt{-\Delta}}f, \chi(x)e^{it\sqrt{-\Delta}}f\right\rangle_{L^2} dt \\ &= \int_{\mathbb{R}}\left\langle e^{it|\xi|}\hat{f}(\xi), \left[e^{it|\xi|}\hat{f}(\xi)\right] * \hat{\chi}(\xi)\right\rangle_{L^2} dt \\ &= \int_{\mathbb{R}}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} e^{it(|\xi|-|\eta|)}\hat{\chi}(\xi-\eta)\,\hat{f}(\xi)\,\hat{f}(\eta)\; d\eta d\xi dt.\end{aligned} \tag{4.8.7}$$

Applying Lemma 4.8.1 with $\phi(\xi,\eta) = |\xi| - |\eta|$, the surface $\{\phi = 0\}$ becomes $\{|\xi| = |\eta|\}$ and

$|\nabla\phi| = \sqrt{2}$. It follows that

$$\begin{aligned}\int_{\mathbb{R}}\int_{\mathbb{R}^n}\left|e^{it\sqrt{-\Delta}}f\right|^2(x)\chi(x)\,dxdt &\simeq \int_{|\xi|=|\eta|}\hat{\chi}(\xi-\eta)\,\hat{f}(\xi)\,\hat{f}(\eta)\;d\sigma \\ &\lesssim \int_{|\xi|=|\eta|}|\hat{\chi}(\xi-\eta)|\left[\left|\hat{f}(\xi)\right|^2+\left|\hat{f}(\eta)\right|^2\right]\,d\sigma \\ &\lesssim \int_{\mathbb{R}^n}\left|\hat{f}(\xi)\right|^2\int_{|\xi|=|\eta|}|\hat{\chi}(\xi-\eta)|\;d\sigma d\xi \\ &\lesssim \sup_{\xi}|K(\xi)|\int_{\mathbb{R}^n}\left|\hat{f}(\xi)\right|^2 d\xi \qquad (4.8.8) \\ &\lesssim \int_{\mathbb{R}^n}|f(x)|^2\,dx.\end{aligned}$$

It reduces to show that

$$K(\xi) = \int_{|\xi|=|\eta|}|\hat{\chi}(\xi-\eta)|\;d\sigma \qquad (4.8.9)$$

is bounded uniformly in $\xi$. Since $\hat{\chi}(\xi)$ decays fast, we have

$$|\hat{\chi}(\xi)| \lesssim \langle\xi\rangle^{-N}$$

and

$$|\hat{\chi}(\xi)| \lesssim |\xi|^{1-\epsilon-n},$$

where as usual, $\langle\xi\rangle = \left(1+|\xi|^2\right)^{\frac{1}{2}}$.

Note that

$$
\begin{aligned}
K(\xi) &= \int_{|\xi|=|\eta|} |\hat{\chi}(\xi-\eta)| \, d\sigma \\
&= \int_{|\zeta-\xi|=|\xi|} |\hat{\chi}(\zeta)| \, d\sigma \\
&\lesssim \int_{|\zeta-\xi|=|\xi|, |\zeta|<1} |\hat{\chi}(\zeta)| \, d\sigma \\
&\quad + \int_{|\zeta-\xi|=|\xi|, |\zeta|>1} |\hat{\chi}(\zeta)| \, d\sigma \\
&\lesssim C(n)
\end{aligned}
\tag{4.8.10}
$$

which is uniformly bounded in $\xi$ and only depends on $n$.

Therefore, we can conclude

$$
\left\| \chi(x) e^{it\sqrt{-\Delta}} f \right\|_{L^2_{t,x}} \lesssim \|f\|_{L^2_x}. \tag{4.8.11}
$$

We are done. □

With dyadic decomposition and weights, one has a global version of the above result:

**Corollary 4.8.3.** *$\forall \epsilon > 0$, one has*

$$
\left\| (1+|x|)^{-\frac{1}{2}-\epsilon} e^{it\sqrt{-\Delta}} f \right\|_{L^2_{t,x}} \lesssim_\epsilon \|f\|_{L^2_x}. \tag{4.8.12}
$$

*Proof.* Let $\chi(x)$ from Theorem 4.8.2 be a smooth version of $1_{B_1(0)}$, the indicator function of the unit ball. It follows that

$$\begin{aligned}
&\left\|\chi\left(2^{-j}x\right)e^{it\sqrt{-\Delta}}f\right\|_{L^2_{t,x}} \\
&= 2^{\frac{jn}{2}}2^{\frac{j}{2}}\left\|\chi\left(x\right)\left(e^{it\sqrt{-\Delta}}f\right)\left(2^jt,2^jx\right)\right\|_{L^2_{t,x}} \\
&= 2^{\frac{jn}{2}}2^{\frac{j}{2}}\left\|\chi\left(x\right)\left(e^{it\sqrt{-\Delta}}f\left(2^j\cdot\right)\right)\right\|_{L^2_{t,x}} \\
&\lesssim 2^{\frac{jn}{2}}2^{\frac{j}{2}}\left\|f\left(2^j\cdot\right)\right\|_{L^2_x} \\
&\lesssim 2^{\frac{j}{2}}\left\|f\right\|_{L^2_x}. \qquad (4.8.13)
\end{aligned}$$

Notice that

$$\left(1+|x|\right)^{-\frac{1}{2}-\epsilon}\simeq\sum_{j\geq 0}2^{-j\left(\frac{1}{2}+\epsilon\right)}\chi\left(2^{-j}x\right) \qquad (4.8.14)$$

then with our computations above, we can conclude that

$$\left\|\sum_{j\geq 0}2^{-j\left(\frac{1}{2}+\epsilon\right)}\chi\left(2^{-j}x\right)e^{it\sqrt{-\Delta}}f\right\|_{L^2_{t,x}}\lesssim_\epsilon \left\|f\right\|_{L^2_x}, \qquad (4.8.15)$$

and hence

$$\left\|\left(1+|x|\right)^{-\frac{1}{2}-\epsilon}e^{it\sqrt{-\Delta}}f\right\|_{L^2_{t,x}}\lesssim_\epsilon \left\|f\right\|_{L^2_x}. \qquad (4.8.16)$$

The corollary is proved. □

## 4.9 Appendix C

In this appendix, we discuss the global existence of solutions to the wave equation with time-dependent potentials. Lorentz transformations are important tools in our analysis. Lorentz transformations are rotations of space-time, therefore, a priori, one needs to show the global existence of solutions to wave equations with time-dependent potentials.

**Theorem 4.9.1.** *Assume $V(x,t)\in L^\infty_{t,x}$. Then for each $(g,\ f)\in H^1\left(\mathbb{R}^3\right)\times L^2\left(\mathbb{R}^3\right)$, there* *is a unique solution* $(u,\, u_t) \in C\left(\mathbb{R},\, H^1\left(\mathbb{R}^3\right)\right) \times C\left(\mathbb{R},\, L^2\left(\mathbb{R}^3\right)\right)$ *to*

$$\partial_{tt}u - \Delta u + V(x,t)u = 0 \tag{4.9.1}$$

*with initial data*

$$u(x,0) = g,\, \partial_t u(x,0) = f. \tag{4.9.2}$$

*Proof.* By Duhamel's formula, we might write the solution as

$$u = \frac{\sin\left(t\sqrt{-\Delta}\right)}{\sqrt{-\Delta}} f + \cos\left(t\sqrt{-\Delta}\right) g + \int_0^t \frac{\sin\left((t-s)\sqrt{-\Delta}\right)}{\sqrt{-\Delta}} V(\cdot,s)u(s)\, ds. \tag{4.9.3}$$

Starting from the local existence, we try to construct the solution in

$$X = C\left([0,\, T],\, H^1\left(\mathbb{R}^3\right)\right) \times C\left([0,\, T),\, L^2\left(\mathbb{R}^3\right)\right) \tag{4.9.4}$$

with $T \leq 1$. One can view $u$ as the fixed-point of the map

$$S(h)(t) = \frac{\sin\left(t\sqrt{-\Delta}\right)}{\sqrt{-\Delta}} f + \cos\left(t\sqrt{-\Delta}\right) g + \int_0^t \frac{\sin\left((t-s)\sqrt{-\Delta}\right)}{\sqrt{-\Delta}} V(\cdot,s)h(s)\, ds. \tag{4.9.5}$$

Let

$$R = 2\left\| \frac{\sin\left(t\sqrt{-\Delta}\right)}{\sqrt{-\Delta}} f + \cos\left(t\sqrt{-\Delta}\right) g \right\|_X. \tag{4.9.6}$$

We will show when $T$ is small enough, $S$ will be a contraction map in $B_X(0,R)$.

Clearly,

$$\|S(h)(t)\|_X \leq \left\| \frac{\sin\left(t\sqrt{-\Delta}\right)}{\sqrt{-\Delta}} f + \cos\left(t\sqrt{-\Delta}\right) g \right\|_X + \left\| \int_0^t \frac{\sin\left((t-s)\sqrt{-\Delta}\right)}{\sqrt{-\Delta}} V(\cdot,s)h(s)\, ds \right\|_X. \tag{4.9.7}$$

By direct calculations,

$$\left\| \int_0^t \frac{\sin\left((t-s)\sqrt{-\Delta}\right)}{\sqrt{-\Delta}} V(\cdot,s)h(s)\,ds \right\|_{L_x^2} \leq T^2 \left\| V(\cdot,t)\,h(t) \right\|_{L^2}, \tag{4.9.8}$$

$$\left\| \int_0^t \frac{\sin\left((t-s)\sqrt{-\Delta}\right)}{\sqrt{-\Delta}} V(\cdot,s)h(s)\,ds \right\|_{\dot{H}_x^1} \leq T \left\| V(\cdot,t)\,h(t) \right\|_{L^2}, \tag{4.9.9}$$

and

$$\left\| \partial_t \left( \int_0^t \frac{\sin\left((t-s)\sqrt{-\Delta}\right)}{\sqrt{-\Delta}} V(\cdot,s)h(s)\,ds \right) \right\|_{L_x^2} \leq T \left\| V(\cdot,t)\,h(t) \right\|_{L^2}. \tag{4.9.10}$$

Therefore, we can pick $T \|V\|_{L_{t,x}^\infty} < \frac{1}{10}$, we have

$$\|S(h)(t)\|_X \leq \left\| \frac{\sin\left(t\sqrt{-\Delta}\right)}{\sqrt{-\Delta}} f + \cos\left(t\sqrt{-\Delta}\right) g \right\|_X + \frac{1}{2} \|h\|_X. \tag{4.9.11}$$

Hence, $S$ maps $B_X(0,R)$ into itself.

Next we show $S$ is a contraction. The calculations are straightforward.

$$\|S(h_1-h_2)(t)\|_X \leq \left\| \int_0^t \frac{\sin\left((t-s)\sqrt{-\Delta}\right)}{\sqrt{-\Delta}} V(\cdot,s)\left(h_1(s)-h_2(s)\right)\,ds \right\|_X. \tag{4.9.12}$$

The the same arguments as above give

$$\|S(h_1-h_2)(t)\|_X \leq \frac{1}{2} \|(h_1-h_2)(t)\|_X. \tag{4.9.13}$$

Therefore, by fixed point theorem, there is $u \in X$ such that

$$u = S(u), \tag{4.9.14}$$

in other words, there exist $u \in C\left([0,\, T],\, H^1\left(\mathbb{R}^3\right)\right) \times C\left([0,\, T),\, L^2\left(\mathbb{R}^3\right)\right)$ such that

$$u = \frac{\sin\left(t\sqrt{-\Delta}\right)}{\sqrt{-\Delta}} f + \cos\left(t\sqrt{-\Delta}\right) g + \int_0^t \frac{\sin\left((t-s)\sqrt{-\Delta}\right)}{\sqrt{-\Delta}} V(\cdot, s) u(s)\, ds. \tag{4.9.15}$$

We notice that the choice of $T$ is independent of the size of the initial data. Then we can repeat the above argument with $(u(T), \partial_t u(T))$ as initial condition to construct the solution from $T$ to $2T$. Iterating this process, one can easily construct the solution $(u,\, u_t) \in C\left(\mathbb{R},\, H^1\left(\mathbb{R}^3\right)\right) \times C\left(\mathbb{R},\, L^2\left(\mathbb{R}^3\right)\right)$.

Finally, we notice the uniqueness of the solution follows from Grönwall's inequality. Suppose one has two solutions $u_1$ and $u_2$ to our equation with the same data, then

$$\|u_1 - u_2\|_{H^1 \times L^2}(t) \leq \int_0^t (t-s) \|u_1 - u_2\|(s)\, ds. \tag{4.9.16}$$

Applying Grönwall's inequality over $[0, T]$, we obtain

$$\|u_1 - u_2\|_X = 0, \tag{4.9.17}$$

which means $u_1 \equiv u_2$ on $[0, T]$. Then by the same iteration argument as above, we can conclude that in $C\left(\mathbb{R},\, H^1\left(\mathbb{R}^3\right)\right) \times C\left(\mathbb{R},\, L^2\left(\mathbb{R}^3\right)\right)$

$$u_1 \equiv u_2. \tag{4.9.18}$$

Therefore, one obtains the uniqueness.

The theorem is proved. □

In our setting, $V(x, t) = V\left(x - \vec{Y}(t)\right)$ satisfies the assumption of Theorem 4.9.1, therefore we have the global existence and uniqueness.

**Corollary 4.9.2.** *For each* $(g,\, f) \in H^1\left(\mathbb{R}^3\right) \times L^2\left(\mathbb{R}^3\right)$, *there is a unique global solution*

$(u,\, u_t) \in C\left(\mathbb{R},\, H^1\left(\mathbb{R}^3\right)\right) \times C\left(\mathbb{R},\, L^2\left(\mathbb{R}^3\right)\right)$ *to the wave equation*

$$\partial_{tt}u - \Delta u + V\left(x - \vec{Y}(t)\right)u = 0 \tag{4.9.19}$$

*with initial data*

$$u(x,0) = g,\ \partial_t u(x,0) = f. \tag{4.9.20}$$

*Remark.* The above theorem also applies to the charge transfer model in [14]:

$$\partial_{tt}u - \Delta u + \sum_{i=1}^{m}\sum_{j=1}^{m} V_{v_j}\left(x - \vec{v}_j t\right)u = 0.$$

## 4.10 Appendix D

In this appendix, we present an alternative approach to the homogeneous endpoint reversed Strichartz estimates based on the Fourier transformation.

We only consider $\frac{\sin\left(t\sqrt{-\Delta}\right)}{\sqrt{-\Delta}}f = \frac{1}{2}\frac{e^{it\sqrt{-\Delta}}}{\sqrt{-\Delta}}f - \frac{1}{2}\frac{e^{-it\sqrt{-\Delta}}}{\sqrt{-\Delta}}f$. We can further reduce to consider

$$\frac{e^{it\sqrt{-\Delta}}}{\sqrt{-\Delta}}f \tag{4.10.1}$$

With Fourier transform and polar coordinates $\xi = \lambda\omega$, we have

$$\begin{aligned}
\frac{e^{it\sqrt{-\Delta}}}{\sqrt{-\Delta}}f &= \int_0^\infty \int_{\mathbb{S}^2} \frac{e^{2\pi it\lambda}}{\lambda} e^{2\pi i\lambda(\omega\cdot x)}\lambda^2 \hat{f}\left(\lambda\omega\right) d\omega d\lambda \\
&= \int_{\mathbb{R}} e^{2\pi it\lambda}\left(\chi_{[0,\infty)}(\lambda)\int_{\mathbb{S}^2} e^{2\pi i\lambda(\omega\cdot x)}\lambda\hat{f}\left(\lambda\omega\right) d\omega\right) d\lambda \\
&= \int_{\mathbb{R}} e^{2\pi it\lambda} G(x,\lambda) d\lambda
\end{aligned} \tag{4.10.2}$$

where

$$G(x,\lambda) = \chi_{[0,\infty)}(\lambda)\int_{\mathbb{S}^2} e^{2\pi i\lambda(\omega\cdot x)}\lambda\hat{f}\left(\lambda\omega\right) d\omega. \tag{4.10.3}$$

By Plancherel's Theorem, we know for fixed $x$,

$$\left\| \frac{e^{it\sqrt{-\Delta}}}{\sqrt{-\Delta}} f \right\|_{L^2_t} = \|G(x,\lambda)\|_{L^2_\lambda}. \tag{4.10.4}$$

$$\begin{aligned} G^2(x,\lambda) &= \left( \chi_{[0,\infty)}(\lambda) \int_{\mathbb{S}^2} e^{2\pi i \lambda(\omega\cdot x)} \lambda \hat{f}(\lambda\omega)\, d\omega \right)^2 \\ &\lesssim \chi_{[0,\infty)}(\lambda) \int_{\mathbb{S}^2} \lambda^2 \left| \hat{f}(\lambda\omega) \right|^2 d\omega \end{aligned} \tag{4.10.5}$$

$$\begin{aligned} \left\| \frac{e^{it\sqrt{-\Delta}}}{\sqrt{-\Delta}} f \right\|^2_{L^2_t} &\lesssim \int_0^\infty \int_{\mathbb{S}^2} \lambda^2 \left| \hat{f}(\lambda\omega) \right|^2 d\omega d\lambda \\ &\lesssim \int \left| \hat{f}(\xi) \right|^2 d\xi \\ &= \int |f(x)|^2 dx. \end{aligned} \tag{4.10.6}$$

Therefore,

$$\left\| \frac{\sin\left(t\sqrt{-\Delta}\right)}{\sqrt{-\Delta}} f \right\|_{L^\infty_x L^2_t} \lesssim \|f\|_{L^2} \tag{4.10.7}$$

as desired.

*Remark.* The two dimension version was obtained in [49] and is mentioned in [6]:

$$\left\| e^{it\sqrt{-\Delta}} f \right\|_{L^\infty_x L^2_t} \lesssim \|f\|_{\dot{B}^{1/2}_{2,1}}. \tag{4.10.8}$$

# CHAPTER 5

# MULTISOLITONS FOR THE DEFOCUSING ENERGY CRITICAL WAVE EQUATION WITH POTENTIALS

In this chapter, we consider multisoliton structures to the defocusing energy critical wave equation with potentials in $\mathbb{R}^3$:

$$\partial_{tt}u - \Delta u + \sum_{j=1}^{m} V_j\left(x - \vec{v}_j t\right)u + u^5 = 0, \tag{5.0.1}$$

where $V_j(x)$'s are rapidly decaying smooth potentials and $\left\{\vec{v}_j\right\}$ is a set of distinct constant velocities such that

$$|\vec{v}_i| < 1,\ 1 \leq i \leq m. \tag{5.0.2}$$

Based on both regular and reversed Strichartz estimates developed in Chen [13] for wave equations with charge transfer Hamiltonians, we construct purely multi-soliton solutions and establish the asymptotic stability of the multisoliton solutions.

To the author's knowledge, this model is the first one to produce multisoliton structures for wave equations in $\mathbb{R}^3$. Unlike Klein-Gordon equations and wave equations in higher dimensions, see Côte-Muñoz [20], Côte-Martel [19], Jendrej [30, 31], Martel-Merle [44], in our case, the static solutions to the associated elliptic equations decay slowly like $\langle x\rangle^{-1}$. It is of crucial importance to understand the multisoliton structure in order to establish the soliton resolution. In fact, if we remove the potentials and replace the positive sign in front of the nonlinearity by the negative sign, the equation becomes the well-known focusing energy critical wave equation. Duyckaerts, Jia, Kenig and Merle establish the soliton resolution (along a well-chosen time sequence) in [24, 25]. But to construct the multisoliton in this case is open. For higher dimensions cases, Martel and Merle construct the multisoliton in dimension higher than 5 by the energy method in [44]. They point out that the slow decay of the ground state is the obstruction to obtain a multisoliton in $\mathbb{R}^3$. Although the structure of our model is different from the pure-power nonlinear equation, the construction in this chapter illustrates that we can overcome the slow decay. But the zero eigenfunctions and resonances for the linearized operator from the pure-power nonlinear equation near each soliton will be the challenge for the linear theory. Another interesting point is that unlike the constructions in Côte-Muñoz [20], Côte-Martel [19], Jendrej [30, 31], Martel-Merle [44] which choose the initial data based on the Brouwer's fixed point theorem, in this chapter, we construct the initial data for the unstable soliton case based on the Banach's fixed point theorem.

Returning to our model, the intuition is that for each potential, it will trap some profile provided that $V_j$ has large negative part. With the defocusing structure, the potentials and the nonlinearity will produce stable solitons. They can also form excited solitons that is the excited states to associated elliptic equations. In this chapter, we will construct the multisoliton structures with stable solitons and unstable solitons. Notice that one needs more delicate analysis in order to handle the unstable solitons in Section 5.2.

Throughout the chapter, we assume that

$$\left|V_j(x)\right| \lesssim \frac{1}{\langle 1+|x|\rangle^{\beta}},\ \beta > 3. \tag{5.0.3}$$

Before we formulate the main theorems, we recall Lorentz transformations along the $x_1$ axis since one can deform a rotation to reduce the general cases to this specific one. More precisely, for the moving frame, $(x-\vec{v}t)$, there is a unique rotation $\rho_v$ so that after rotating, in the new frame $(z_1, z_2, z_3)$, the vector $\vec{v}$ is along $z_1$, i.e. the moving frame becomes $(z_1 - |\vec{v}|\,\vec{e}_1 t, z_2, z_3)$.

Then we apply the Lorentz transformation along $\vec{e}_1$ with velocity $v$. Define

$$\Lambda_v := \begin{pmatrix} \gamma & -|\vec{v}|\gamma & 0 & 0 \\ -|\vec{v}|\gamma & |\vec{v}|\gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{5.0.4}$$

where

$$\gamma = \frac{1}{\sqrt{1-|\vec{v}|^2}}. \tag{5.0.5}$$

Then we consider the the following change of variables:

$$\begin{pmatrix} t' \\ x_1' \\ x_2' \\ x_3' \end{pmatrix} = \Lambda_v \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & & & \\ 0 & & \rho_v & \\ 0 & & & \end{pmatrix} \begin{pmatrix} t \\ x_1 \\ x_2 \\ x_3 \end{pmatrix}. \tag{5.0.6}$$

In our model, applying the above transformation, in the new frame $(x', t')$, the moving potential $V_j(x - \vec{v}_j t)$ becomes

$$V_j\left(\sqrt{1-|\vec{v}_j|^2}x_1', x_2', x_3'\right) =: V_j\left(m_{v_j}x'\right) \tag{5.0.7}$$

Setting

$$V_j^{v_j}(x') = V_j\left(m_{v_j}x'\right), \tag{5.0.8}$$

then we consider the Schrödinger operator

$$-\Delta + V_j^{v_j}(x) \tag{5.0.9}$$

Let $W_j^{v_j}$ be the stable static state to

$$-\Delta W_j^{v_j} + V_j^{v_j}(x)\, W_j^{v_j} + \left(W_j^{v_j}\right)^5 = 0. \tag{5.0.10}$$

Note that

$$W_j^{v_j}(x') = W_j^{v_j}\left(m_{v_j}^{-1}\rho_{v_j}(x - v_j t)\right).$$

By a stable state, we mean that the linearized operator

$$-\Delta + V_j^{v_j}(x) + 5\left(W_j^{v_j}\right)^4 \tag{5.0.11}$$

has no eigenvalues nor zero resonance. For detailed definitions, see Section 5.1 and the Appendix on the linear theory.

Set

$$W_j(x) = W_j^{v_j}\left(m_{v_j}^{-1}\rho_{v_j}(x)\right). \tag{5.0.12}$$

It is crucial to notice that

$$\left|W_j(x)\right| \simeq \frac{1}{\langle x\rangle} \tag{5.0.13}$$

which causes the interactions among different solitons in our construction are very strong. For more detailed discussions on the existence and decay estimates, see Section 5.1.

We also need the Hamiltonian structure of wave equations to discuss scattering. In general, we can write a general wave equation as

$$\partial_{tt}u - \Delta u = F(u,t) \tag{5.0.14}$$

with initial data

$$u(x,0) = f(x),\ u_t(x,0) = g(x). \tag{5.0.15}$$

Also consider the homogeneous free wave equation,

$$\partial_{tt} u_0 - \Delta u_0 = 0 \tag{5.0.16}$$

with initial data

$$u_0(x,0) = f_0(x),\ (u_0)_t\,(x,0) = g_0(x). \tag{5.0.17}$$

We reformulate the wave equation as a Hamiltonian system,

$$\partial_t \begin{pmatrix} u \\ \partial_t u \end{pmatrix} - \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} -\Delta & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} u \\ \partial_t u \end{pmatrix} = \begin{pmatrix} 0 \\ F(u) \end{pmatrix}. \tag{5.0.18}$$

Setting

$$U := \begin{pmatrix} u \\ \partial_t u \end{pmatrix},\ J := \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},\ H_F := \begin{pmatrix} -\Delta & 0 \\ 0 & 1 \end{pmatrix} \text{ and } F(U) := \begin{pmatrix} 0 \\ F(u,t) \end{pmatrix}, \tag{5.0.19}$$

we can rewrite the free wave equation as

$$\dot{U}_0 - JH_F U_0 = 0, \tag{5.0.20}$$

$$U_0[0] = \begin{pmatrix} f_0 \\ g_0 \end{pmatrix} \tag{5.0.21}$$

and the nonlinear wave equation as

$$\dot{U} - JH_F U = F(U), \tag{5.0.22}$$

$$U[0] = \begin{pmatrix} f \\ g \end{pmatrix}. \tag{5.0.23}$$

The solution of the free wave equation is given by

$$U_0 = e^{tJH_F} U_0[0]. \tag{5.0.24}$$

In the following, we write

$$U[t] = (u, u_t)^t\,,\; W[t] = \left( \sum_{j=1}^{m} W_j\left(x - \vec{v}_j t\right),\, \partial_t \sum_{j=1}^{m} W_j\left(x - \vec{v}_j t\right) \right)^t. \tag{5.0.25}$$

With the preparations and notations above, we can formulate our main theorems with stable solitons:

**Theorem 5.0.1** (Existence of purely multi-soliton solutions)**.** *In $\mathbb{R}^3$, there exists a solution $u$ to*

$$\partial_{tt} u - \Delta u + \sum_{j=1}^{m} V_j\left(x - \vec{v}_j t\right) u + u^5 = 0, \tag{5.0.26}$$

*such that*

$$\lim_{t\to\infty} \|U[t] - W[t]\|_{\dot{H}^1 \times L^2} = 0. \tag{5.0.27}$$

*Moreover, we have the decay rate*

$$\|U[t] - W[t]\|_{\dot{H}^1 \times L^2} \lesssim \frac{1}{\sqrt{t}}. \tag{5.0.28}$$

as $t \to \infty$.

Next we have the asymptotic stability of the multisoliton structure.

**Theorem 5.0.2** (Asymptotic stability of the multisoliton)**.** *Suppose that $0 < \epsilon \ll 1$ is small enough and $1 \ll t_0$ is large enough. Let $u$ solve*

$$\partial_{tt} u - \Delta u + \sum_{j=1}^{m} V_j\left(x - \vec{v}_j t\right) u + u^5 = 0,\;\; x \in \mathbb{R}^3. \tag{5.0.29}$$

*Suppose at* $t = t_0$,

$$\|U[t_0] - W[t_0]\|_{\dot{H}^1 \times L^2} \leq \epsilon. \tag{5.0.30}$$

*Then there exists free data*

$$U_0[0] = (f_0, g_0)^t \in \dot{H}^1 \times L^2 \tag{5.0.31}$$

*such that*

$$\left\| U[t] - W[t] - e^{tJH_F} U_0[0] \right\|_{\dot{H}^1 \times L^2} \to 0.\ t \to \infty. \tag{5.0.32}$$

*In other words, the error* $u(t) - \sum_{j=1}^{m} W_j \left(x - \vec{v}_j t\right)$ *scatters to a free wave.*

Here we briefly discuss strong interactions among these solitons. For simplicity, we consider the case when $m = 2$ as in Section 5.2. Around near two solitons, we define

$$h(t) := u(t) - W_1(x) - W_2(x - \vec{v}t). \tag{5.0.33}$$

We consider the equation for $h$. Plugging everything in the equation, we have

$$\begin{aligned} \partial_{tt} h - \Delta h + h^5 + \left(V_1(x) + 5W_1^4(x)\right) h + \left(V_2(x - \vec{v}t) + 5W_2^4(x - \vec{v}t)\right) h + a(x,t)h \\ = F_1(x,t) + F_2(x,t) + F(x,t) + N(h,x,t) \end{aligned} \tag{5.0.34}$$

with

$$a(x,t) = 20W_1^3(x)W_2(x - \vec{v}t) + 30W_1^2(x)W_2^2(x - \vec{v}t) + 20W_1(x)W_2^3(x - \vec{v}t) \tag{5.0.35}$$

$$F_1(x,t) = 5W_1^4(x)W_2(x - \vec{v}t) + V_1(x)W_2(x - \vec{v}t) \tag{5.0.36}$$

$$F_2(x,t) = 5W_1(x)W_2^4(x - \vec{v}t) + W_1(x)V_2(x - \vec{v}t) \tag{5.0.37}$$

$$F(x,t) = 10W_1^3(x)W_2^2(x - \vec{v}t) + 10W_1^2(x)W_2^3(x - \vec{v}t) \tag{5.0.38}$$

and $N(h, x, t)$ is quadratic or higher in $h$. For more details, see Section 5.2. We notice that $F(x, t)$ is easy to handle but $F_1$, $F_2 \notin L^1_t L^2_x$ . One can not simply apply the energy estimate and Strichartz estimates directly. $F_1$ and $F_2$ precisely show that due to the slow decay rate of the solitons, see Section 5.1, some terms in the nonlinear interactions decay slowly. To overcome these terms, we need the local energy decay and reversed Strichartz estimates with inhomogeneous terms in the reversed norm. Moreover, due to the failure of the endpoint Strichartz estimate in $\mathbb{R}^3$, to handle the quadratic term of $h$ in the nonlinearity, one also needs the endpoint reversed Strichartz estimate and revered type local decay estimates. It is also novel in the nonradial setting.

All the above results can be extended to the multisoliton construction with unstable excited solitons, see Section 5.2. The linear model still plays a pivotal role. It is interesting to compare our method which based on linear estimates with the constructions of multisolitons by nonlinear techniques developed in for example, in Martel [43], Merle [45], Côte-Muñoz [20], Côte-Martel [19], Jendrej [30, 31], Martel-Merle [44]. First of all, our linear model can be used to analyze the stability of the multisoliton structure. Secondly the scattering state we construct in this chapter is based on the Banach's fixed point theorem other than the Brouwer's fixed-point theorem. On the other hand, when we need to deal with the purely-soliton solution with unstable solitons, we also need the weak convergence technique which is commonly used in the nonlinear method.

## *Organization*

The chapter is organized as follows: In Section 5.1, we list some existence and decay results on the solutions to elliptic equations. In Section 5.2, we establish the main theorems in this chapter. The constructions with unstable solitons will be shown. Finally, in the Appendix, we briefly recall the linear theory that we need in this chapter based on results from Chen [14, 12]. We also discuss the scattering behavior of the nonlinear equation.

## 5.1 Preliminaries: Static States

In order to construct the multisoliton solution to the equation

$$\partial_{tt}u - \Delta u + \sum_{j=1}^{m} V_j\left(x - \vec{v}_j t\right)u + u^5 = 0, \tag{5.1.1}$$

we first have to understand the soliton trapped by each potential separately.

Performing a Lorentz transformation, it suffices to understand the model elliptic equation:

$$-\Delta u + Vu + u^5 = 0. \tag{5.1.2}$$

**Definition 5.1.1** (Stability of a static solution)**.** A solution $W$ to the equation (5.1.2) is called a stable solution if the linearized opearator

$$L_W = -\Delta + V + 5W^4 \tag{5.1.3}$$

has no eigenvalues nor zero resonance.

We follow Jia-Liu-Xu [33] and Jia-Liu-Schlag-Xu [34]. Define the energy functional

$$J(u) := \int_{R^3} \frac{|\nabla u|^2}{2} + \frac{Vu^2}{2} + \frac{u^6}{6}(x,t)\,dx. \tag{5.1.4}$$

In general when the negative part $V$ of the potential is large, one can expect that there is a unique positive ground state, which is the global minimizer of energy functional and has negative energy. In addition, there can be a number of "excited states" with higher energies (see Appendix A of [33] for more details). It is well known the ground state is asymptotically stable at least when $V$ decays fast. However the dynamics around the excited states can be very complicated even in perturbative regime (even with radial data), involving stable and unstable manifolds. It arises some difficulties to take these unstable excited states as the solitons in our construction.

Here we list some important results regarding the elliptic equation from [33, 34].

**Lemma 5.1.2.** *Consider $J$ as a functional defined in $\dot{H}^1(\mathbb{R}^3)$. If the operator $-\Delta - V$ has negative eigenvalues then there exists a global minimizer $Q > 0$ with $J(Q) < 0$. If $-\Delta - V$ has no negative eigenvalues, then the only steady state solution $u \in \dot{H}^1(\mathbb{R}^3)$ to equation (5.1.2) is $u \equiv 0$.*

**Theorem 5.1.3.** *Fix $\beta > 2$. Define*

$$Y := \{V \in C(\mathbb{R}^3) : V \text{ is radial and } \sup_x (1 + |x|)^\beta |V(x)| < \infty\}. \tag{5.1.5}$$

*For $V$ in a dense open set $\Omega \subset Y$, there are only finitely many radial steady states to equation (5.1.2).*

**Theorem 5.1.4.** *Let $V \in Y$. For any $c \in \mathbb{R}$, there exists a unique radial solution $u_c \in \dot{H}^1(B_r^c)$ for any $r > 0$ to*

$$-\Delta u + V(x)u + u^5 = 0, \text{ in } R^3\backslash\{0\}, \tag{5.1.6}$$

*with*

$$\left| u(x) - \frac{c}{|x|} \right| = o(\frac{1}{|x|}), \text{ as } |x| \to \infty. \tag{5.1.7}$$

*If $u_c \in \dot{H}^1(\mathbb{R}^3)$, then $u_c \in C^1(\mathbb{R}^3)$ and*

$$-\Delta u_c + V u_c + u_c^5 = 0 \text{ in } R^3. \tag{5.1.8}$$

If we take the ground states as the solitons, we notice that the optimal decay rate is $\frac{1}{\langle x \rangle}$. Even if one can assume that $V(x)$ decays very fast, there is no hope to improve the decay rate for the ground state.

**Lemma 5.1.5.** *Let $R$, $\beta_1$ be sufficiently large. There exist $\epsilon$, $\delta > 0$, such that if*

$$\sup_{x \in B_R^c} |x|^{\beta_1} |b(x)| < \delta, \tag{5.1.9}$$

*then any solution $u$ to*

$$-\Delta u + b(x)u + u^5 = 0, \ x \in B_R^c \tag{5.1.10}$$

*with $u|_{\partial B_R} \geq 1$ satisfies*

$$|u(x)| \geq \frac{\epsilon}{|x|}, \text{ for } x \in B_R^c. \tag{5.1.11}$$

Naively one might expect excited states to be unstable, since they change sign. However in general this may not be the case, as seen from the following theorem.

**Theorem 5.1.6.** *There exists an open set $\mathcal{O} \in Y$ such that for any $V \in \mathcal{O}$, there exists an excited state $\phi$ to equation* (5.1.2) *which is stable.*

## 5.2 The Construction and Stability of Multisolitons

In this section, we prove the main results of this chapter. For simplicity, we discuss the case when $m = 2$:

$$\partial_{tt} u - \Delta u + V_1(x)u + V_2\left(x - \vec{v}t\right)u + u^5 = 0 \tag{5.2.1}$$

and $\vec{v}$ is along the $x_1$ direction.

We start with an energy estimate based on the local energy decay for the free wave equation, c.f. the Appendices of [12]. This lemma is particularly useful to handle strong interaction terms, see Remark 5.2.2.

**Lemma 5.2.1.** *Consider*

$$\partial_{tt} u - \Delta u = H \tag{5.2.2}$$

*with initial data*

$$u(0) = f,\ u_t(0) = g. \tag{5.2.3}$$

*Then for any* $\epsilon > 0$ *and* $|\vec{v}| < 1$, *one has*

$$\sup_{t\geq 0} \left(\|\nabla u(t)\|_{L^2} + \|u_t(t)\|_{L^2}\right) \lesssim_\epsilon \|g\|_{L^2} + \|f\|_{\dot{H}^1} + \left\| \langle x\rangle^{\frac{1}{2}+\epsilon} H(t)\right\|_{L^2_{t,x}} \tag{5.2.4}$$

*and*

$$\sup_{t\geq 0} \left(\|\nabla u(t)\|_{L^2} + \|u_t(t)\|_{L^2}\right) \lesssim_\epsilon \|g\|_{L^2} + \|f\|_{\dot{H}^1} + \left\| \langle x - \vec{v}t\rangle^{\frac{1}{2}+\epsilon} H(t)\right\|_{L^2_{t,x}} \tag{5.2.5}$$

*Proof.* we set $A = \sqrt{-\Delta}$ and notice that

$$\|Af\|_{L^2} \simeq \|f\|_{\dot{H}^1}, \ \forall f \in C^\infty\left(\mathbb{R}^3\right). \tag{5.2.6}$$

For real-valued $u = (u_1, u_2) \in \mathcal{H} = \dot{H}^1\left(\mathbb{R}^3\right) \times L^2\left(\mathbb{R}^3\right)$, we write

$$U := Au_1 + iu_2. \tag{5.2.7}$$

and then

$$\|U\|_{L^2} \simeq \|(u_1, u_2)\|_{\mathcal{H}}. \tag{5.2.8}$$

We also notice that $u$ solves the original equation if and only if

$$U := Au + i\partial_t u \tag{5.2.9}$$

satisfies

$$i\partial_t U = AU + H, \tag{5.2.10}$$

$$U(0) = Af + ig \in L^2\left(\mathbb{R}^3\right). \tag{5.2.11}$$

By Duhamel's formula,

$$U(t) = e^{itA}U(0) - i\int_0^t e^{-i(t-s)A}H(s)\,ds. \tag{5.2.12}$$

We will only prove the first estimate (5.2.4). The second one (5.2.5) follows the same way with the the standard local energy decay replaced by the local energy decay developed in [14, 12].

From the energy estimate for the free evolution,

$$\sup_{t\in\mathbb{R}} \left\| e^{itA}U(0) \right\|_{L^2_x} \lesssim \|U(0)\|_{L^2}. \tag{5.2.13}$$

It suffices to bound

$$\sup_{t\in\mathbb{R}} \left\| \int_0^\infty e^{-i(t-s)A} \langle x\rangle^{-\frac{1}{2}-\epsilon} \left( \langle x\rangle^{\frac{1}{2}+\epsilon} H(s) \right) ds \right\|_{L^2_x}. \tag{5.2.14}$$

Denote

$$\langle x\rangle^{\frac{1}{2}+\epsilon} H(s) = N. \tag{5.2.15}$$

It is clear that

$$\left\| \int_0^\infty e^{-i(t-s)A} \langle x\rangle^{-\frac{1}{2}-\epsilon} N(s)\,ds \right\|_{L^\infty_t L^2_x} \leq \left\| \widetilde{K} \right\|_{L^2_t L^2_x \to L^\infty_t L^2_x} \|N\|_{L^2_t L^2_x}, \tag{5.2.16}$$

where

$$\left(\widetilde{K}F\right)(t) := \int_0^\infty e^{-i(t-s)A} \langle x\rangle^{-\frac{1}{2}-\epsilon} F(s)\,ds. \tag{5.2.17}$$

We need to estimate

$$\left\| \widetilde{K} \right\|_{L^2_t L^2_x \to L^\infty_t L^2_x}. \tag{5.2.18}$$

Testing against $F \in L_t^2 L_x^2$, clearly,

$$\left\| \widetilde{K} F \right\|_{L_t^\infty L_x^2} \leq \left\| e^{-itA} \right\|_{L^2 \to L_t^\infty L_x^2} \left\| \int_0^\infty e^{isA} \langle x \rangle^{-\frac{1}{2}-\epsilon} F(s)\, ds \right\|_{L^2}. \tag{5.2.19}$$

The first factors on the right-hand side of (5.2.19) is bounded by the energy estimate for the free evolution. Consider the second factor, by duality, it suffices to show

$$\left\| \langle x \rangle^{-\frac{1}{2}-\epsilon} e^{-itA} \phi \right\|_{L_t^2 L_x^2} \lesssim \|\phi\|_{L^2}, \forall \phi \in L^2 \left( \mathbb{R}^3 \right), \tag{5.2.20}$$

which is local energy decay.

For estimate (5.2.5), we apply

$$\left\| \langle x - \vec{v} t \rangle^{-\frac{1}{2}-\epsilon} e^{-itA} \phi \right\|_{L_t^2 L_x^2} \lesssim \|\phi\|_{L^2}, \forall \phi \in L^2 \left( \mathbb{R}^3 \right) \tag{5.2.21}$$

in the appendices in [12] or Corollary 2.10 in [14].

Hence

$$\left\| \int_0^\infty e^{isA} \langle x \rangle^{-\frac{1}{2}-\epsilon} F(s)\, ds \right\|_{L^2} \lesssim \|F\|_{L_t^2 L_x^2}. \tag{5.2.22}$$

Therefore, indeed, we have

$$\left\| \widetilde{K} \right\|_{L_t^2 L_x^2 \to L_t^\infty L_x^2} \leq C \tag{5.2.23}$$

and

$$\sup_{t \in \mathbb{R}} \left\| \int_0^\infty e^{-i(t-s)A} \langle x \rangle^{-\frac{1}{2}-\epsilon} N(s)\, ds \right\|_{L_x^2} \lesssim \|N\|_{L_{t,x}^2}. \tag{5.2.24}$$

Therefore, we have

$$\sup_{t \geq 0} \left( \|\nabla u(t)\|_{L^2} + \|u_t(t)\|_{L^2} \right) \lesssim \|g\|_{L^2} + \|f\|_{\dot{H}^1} + \left\| \langle x \rangle^{\frac{1}{2}+\epsilon} H(t) \right\|_{L_{t,x}^2} \tag{5.2.25}$$

and

$$\sup_{t\geq 0}\left(\|\nabla u(t)\|_{L^2}+\|u_t(t)\|_{L^2}\right)\lesssim \|g\|_{L^2}+\|f\|_{\dot{H}^1}+\left\|\langle x-\vec{v}t\rangle^{\frac{1}{2}+\epsilon}H(t)\right\|_{L^2_{t,x}} \tag{5.2.26}$$

as claimed □

**Remark 5.2.2.** *As a concrete example, we set*

$$H(t)=\frac{1}{\langle x\rangle^4}\frac{1}{\langle x-\vec{v}t\rangle}. \tag{5.2.27}$$

*which is one of the interaction terms in the nonlinear model.*

*We want to solve*

$$\partial_{tt}u-\Delta u=H \tag{5.2.28}$$

*Taking $1<\eta\ll|\vec{v}|$, consider*

$$\int_{\mathbb{R}^3}\frac{\langle x\rangle^{1+2\epsilon}}{\langle x\rangle^8}\frac{1}{\langle x-\vec{v}t\rangle^2}\,dx. \tag{5.2.29}$$

*Splitting integral into three pieces:*

$$\int_{|x|\leq\eta t}\frac{\langle x\rangle^{1+2\epsilon}}{\langle x\rangle^8}\frac{1}{\langle x-\vec{v}t\rangle^2}\,dx\lesssim\frac{1}{\langle t\rangle^2} \tag{5.2.30}$$

$$\int_{|x-\vec{v}t|\leq\eta t}\frac{\langle x\rangle^{1+2\epsilon}}{\langle x\rangle^8}\frac{1}{\langle x-\vec{v}t\rangle^2}\,dx\lesssim\frac{1}{\langle t\rangle^{6-2-2\epsilon-1}}\lesssim\frac{1}{\langle t\rangle^2} \tag{5.2.31}$$

$$\int_{|x-\vec{v}t|\geq\eta t,\,|x|\geq\eta t}\frac{\langle x\rangle^{1+2\epsilon}}{\langle x\rangle^8}\frac{1}{\langle x-\vec{v}t\rangle^2}\,dx\lesssim\frac{1}{\langle t\rangle^2}. \tag{5.2.32}$$

*Therefore*

$$\int_{\mathbb{R}^3}\frac{\langle x\rangle^{2+2\epsilon}}{\langle x\rangle^8}\frac{1}{\langle x-\vec{v}t\rangle^2}\,dx\lesssim\frac{1}{\langle t\rangle^2}. \tag{5.2.33}$$

*Clearly*

$$\left\| \langle x\rangle^{\frac{1}{2}+\epsilon} H(t) \right\|_{L^2_{t,x}} \lesssim \left( \int_0^\infty \frac{1}{\langle t\rangle^2} \right)^{\frac{1}{2}} < \infty. \tag{5.2.34}$$

*If we consider the case that*

$$\|u(t)\|_{\dot{H}^1\times L^2} \to 0, \ t\to\infty, \tag{5.2.35}$$

*then*

$$\|u(t_1)\|_{\dot{H}^1\times L^2} = \left( \int_{t_1}^\infty \left( \int_{\mathbb{R}^3} \frac{\langle x\rangle^{2+2\epsilon}}{\langle x\rangle^8} \frac{1}{\langle x-\vec{v}t\rangle^2}\, dx \right) dt \right)^{\frac{1}{2}} \lesssim \frac{1}{\sqrt{t_1}}. \tag{5.2.36}$$

*We point out that for this $H(t)$ as the inhomogeneous term, the trivial energy estimate fails since $H(t) \notin L^1_t L^2_x$.*

### *5.2.1 Stable solitons*

Later on, throughout this section, we will use the short-hand notation:

$$L^p_t L^q_x := L^p_t \left( [t_0, \infty),\, L^q_x \right). \tag{5.2.37}$$

where $t_0$ is the large time which only depends on prescribed constants from Theorem 5.0.2.

We first prove Theorem 5.0.2. Setting

$$h(t) := u(t) - W_1(x) - W_2(x-\vec{v}t), \tag{5.2.38}$$

it is well-known that we just need to show that $h$ is bounded in Strichartz norms, see Chen [14] or Theorem 5.3.7 in the Appendix.

*Proof of Theorem 5.0.2.* By construction, we have

$$\partial_{tt}h - \Delta h + h^5 + \left(V_1(x) + 5W_1^4(x)\right)h + \left(V_2\left(x - \vec{v}t\right) + 5W_2^4\left(x - \vec{v}t\right)\right)h + a(x,t)h$$
$$= F_1(x,t) + F_2(x,t) + F(x,t) + N(h,x,t) \tag{5.2.39}$$

with

$$a(x,t) := 20W_1^3(x)W_2\left(x - \vec{v}t\right) + 30W_1^2(x)W_2^2\left(x - \vec{v}t\right) + 20W_1(x)W_2^3\left(x - \vec{v}t\right) \tag{5.2.40}$$

$$F_1(x,t) := 5W_1^4(x)W_2\left(x - \vec{v}t\right) + V_1(x)W_2(x - \vec{v}t) \tag{5.2.41}$$

$$F_2(x,t) := 5W_1(x)W_2^4\left(x - \vec{v}t\right) + W_1(x)V_2(x - \vec{v}t) \tag{5.2.42}$$

$$F(x,t) := 10W_1^3(x)W_2^2\left(x - \vec{v}t\right) + 10W_1^2(x)W_2^3\left(x - \vec{v}t\right) \tag{5.2.43}$$

$$\begin{aligned} N(h,x,t) := &\left(10W_1^3(x) + 30W_1^2(x)W_2\left(x - \vec{v}t\right) + 30W_1(x)W_2^2\left(x - \vec{v}t\right) + 10W_2^3(x - \vec{v}t)\right)h^2 \\ &+ \left(10W_1^2(x) + 3W_1(x)W_2\left(x - \vec{v}t\right) + 10W_2^2\left(x - \vec{v}t\right)\right)h^3 \\ &+ \left(5W_1(x) + 5W_2\left(x - \vec{v}t\right)\right)h^4 \end{aligned} \tag{5.2.44}$$

Furthermore, we denote

$$M_1(h,x,t) := M_{1,1}(h,x,t) + M_{1,2}(h,x,t) + M_{1,3}(h,x,t) + M_{1,4}(h,x,t) \tag{5.2.45}$$

where

$$M_{1,1}(h,x,t) := 10W_1^3(x)h^2,\ M_{1,2}(h,x,t) := 30W_1^2(x)W_2\left(x - \vec{v}t\right)h^2 \tag{5.2.46}$$

and

$$M_{1,3}(h,x,t) := 30W_1(x)W_2^2\left(x-\vec{v}t\right)h^2,\ M_{1,4}(h,x,t) := 10W_2^3(x-\vec{v}t)h^2. \tag{5.2.47}$$

Also we use the notation:

$$M_2(h,x,t) := \Big(10W_1^2(x) + 3W_1(x)W_2\left(x-\vec{v}t\right) + 10W_2^2\left(x-\vec{v}t\right)\Big)h^3, \tag{5.2.48}$$

$$M_3(h,x,t) := \left(5W_1(x) + 5W_2\left(x-\vec{v}t\right)\right)h^4 \tag{5.2.49}$$

Consider the iteration scheme

$$\begin{aligned}\partial_{tt}h_{i+1} - \Delta h_{i+1} + \Big(V_1(x) + 5W_1^4(x)\Big)h_{i+1} + \Big(V_2\left(x-\vec{v}t\right) + 5W_2^4\left(x-\vec{v}t\right)\Big)h_{i+1}\\ = F_1(x,t) + F_2(x,t) + F(x,t) + N(h_i,x,t) - a(x,t)h_i - h_i^5.\end{aligned} \tag{5.2.50}$$

Define the Strichartz norm of $h$ as

$$\|h\|_{Stri} := \sup_{\frac{1}{2}=\frac{1}{p}+\frac{3}{q},\, p=3,4,5} \|h\|_{L_t^p([t_0,\infty),\, L_x^q)}. \tag{5.2.51}$$

For a function $G(x,t)$, we use the notation:

$$G^S(x,t) := G(x+\vec{v}t,t). \tag{5.2.52}$$

As in the Appendix, for fixed $\epsilon_0 > 0$ small, define the strong interactions spaces $I$ as

$$I := \left\{G(x,t) \in L_x^{\frac{3}{2},1}L_t^2 \bigcap L_{x_1}^1 L_{\widehat{x_1}}^{2,1} L_t^2, \text{ such that } \left\|\langle x\rangle^{\frac{1}{2}+\epsilon_0} G\right\|_{L_{t,x}^2} < \infty\right\}, \tag{5.2.53}$$

and local decay space $D$ as

$$D := \left\{ \langle x \rangle^{-3} G(x,t) \in L_x^{\frac{3}{2},1} L_t^\infty \bigcap L_{x_1}^1 L_{\widehat{x_1}}^{2,1} L_t^\infty \right\}. \tag{5.2.54}$$

Define

$$\|G\|_I := \max \left\{ \left\| \langle x \rangle^{\frac{1}{2}+\epsilon_0} G \right\|_{L_{t,x}^2}, \|G\|_{L_{x_1}^1 L_{\widehat{x_1}}^{2,1} L_t^2}, \|G\|_{L_x^{\frac{3}{2},1} L_t^2} \right\}, \tag{5.2.55}$$

and

$$\|G\|_D := \max \left\{ \left\| \langle x \rangle^{-3} G(x,t) \right\|_{L_{x_1}^1 L_{\widehat{x_1}}^{2,1} L_t^\infty}, \left\| \langle x \rangle^{-3} G \right\|_{L_x^{\frac{3}{2},1} L_t^\infty} \right\}. \tag{5.2.56}$$

Using the notations before, by estimate (5.3.28) in Theorem 5.3.5 from the linear theory in the Appendix, we have

$$\begin{aligned} \sup_{x \in \mathbb{R}^3} \left( \int_{t_0}^\infty |h_{i+1}(x,t)|^2 \, dt \right)^{\frac{1}{2}} \lesssim\ & \|f\|_{L^2} + \|g\|_{\dot{H}^1} + \|F\|_{L_t^1 L_x^2} \\ & + \|a(x,t) h_i\|_{L_t^1 L_x^2} + \left\| h_i^5 \right\|_{L_t^1 L_x^2} \\ & + \|M_2(h_i,x,t)\|_{L_t^1 L_x^2} + \|M_3(h_i,x,t)\|_{L_t^1 L_x^2} \\ & + \left\|M_{1,2}(h_i,x,t)\right\|_{L_t^1 L_x^2} + \left\|M_{1,3}(h_i,x,t)\right\|_{L_t^1 L_x^2} \\ & + \left\|M_{1,1}(h_i,x,t)\right\|_I + \left\| M_{1,4}^S(h_i,x,t) \right\|_I \\ & + \|F_1\|_I + \left\| F_2^S \right\|_I . \end{aligned} \tag{5.2.57}$$

Applying Hölder's inequality and Strichartz estimates, we can estimate

$$\|a(x,t) h_i\|_{L_t^1 L_x^2} \lesssim \|a\|_{L_t^{\frac{5}{4}} L_x^{\frac{5}{2}}} \|h_i\|_{L_t^5 L_x^{10}}, \tag{5.2.58}$$

$$\|M_2(h_i,x,t)\|_{L_t^1 L_x^2} \lesssim \|h_i\|_{L_t^3 L_x^{18}}, \tag{5.2.59}$$

$$\|M_3(h_i,x,t)\|_{L_t^1 L_x^2} \lesssim \|h_i\|_{L_t^4 L_x^{12}}, \tag{5.2.60}$$

and

$$\left\|M_{1,2}(h_i,x,t)\right\|_{L_t^1 L_x^2}, \left\|M_{1,3}(h_i,x,t)\right\|_{L_t^1 L_x^2} \lesssim \|h_i\|_{L_t^4 L_x^{12}}. \tag{5.2.61}$$

For the strong-interaction terms, we notice that

$$\begin{aligned}\left\|M_{1,1}(h_i,x,t)\right\|_{L_x^{\frac{3}{2},1} L_t^2} &\lesssim \left\|W_1^3(x) h_i\right\|_{L_x^{\frac{3}{2},1} L_t^\infty} \|h_i\|_{L_x^\infty L_t^2} \\ &\lesssim \|h_i\|_D \|h_i\|_{L_x^\infty L_t^2}. \end{aligned} \tag{5.2.62}$$

In the same manner, we can estimate all other norms in the definition of $I$ and conclude that

$$\left\|M_{1,1}(h_i,x,t)\right\|_I \lesssim \|h_i\|_D \|h_i\|_{L_x^\infty L_t^2}. \tag{5.2.63}$$

Similarly,

$$\left\|M_{1,4}^S(h_i,x,t)\right\|_I \lesssim \left\|h_i^S\right\|_D \left\|h_i^S\right\|_{L_x^\infty L_t^2}. \tag{5.2.64}$$

Therefore, we know that

$$\begin{aligned}\sup_{x\in\mathbb{R}^3}\left(\int_{t_0}^\infty |h_{i+1}(x,t)|^2\,dt\right)^{\frac{1}{2}} \lesssim &\|f\|_{L^2} + \|g\|_{\dot{H}^1} + \|F\|_{L_t^1 L_x^2} \\ &+ \|h_i\|_D \|h_i\|_{L_x^\infty L_t^2} + \left\|h_i^S\right\|_D \left\|h_i^S\right\|_{L_x^\infty L_t^2} \\ &+ \|h_i\|_{Stri} + \|h_i\|_{Stri}^5 + \|F_1\|_I + \left\|F_2^S\right\|_I. \end{aligned} \tag{5.2.65}$$

Similarly, by estimate (5.3.29) from in Theorem 5.3.5, one has

$$\sup_{x\in\mathbb{R}^3}\left(\int_{t_0}^{\infty}|h_{i+1}(x+\vec{v}t,t)|^2\,dt\right)^{\frac{1}{2}} \lesssim \|f\|_{L^2}+\|g\|_{\dot{H}^1}+\|F\|_{L^1_tL^2_x} + \|h_i\|_D\|h_i\|_{L^\infty_xL^2_t}+\left\|h_i^S\right\|_D\left\|h_i^S\right\|_{L^\infty_xL^2_t} + \|h_i\|_{Stri}+\|h_i\|^5_{Stri}+\|F_1\|_I+\left\|F_2^S\right\|_I. \quad (5.2.66)$$

For the local decay, by the estimates (5.3.30) and (5.3.31) from Theorem 5.3.5, we conclude that

$$\|h_{i+1}(x,t)\|_D \lesssim \|g\|_{L^2}+\|f\|_{\dot{H}^1}+\|h_i\|_{Stri}+\|h_i\|^5_{Stri} + \|h_i\|_D\|h_i\|_{L^\infty_xL^2_t}+\left\|h_i^S\right\|_D\left\|h_i^S\right\|_{L^\infty_xL^2_t} + \|F_1\|_I+\left\|F_2^S\right\|_I+\|F\|_{L^1_tL^2_x}. \quad (5.2.67)$$

$$\|h_{i+1}(x+\vec{v}t,t)\|_D \lesssim \|g\|_{L^2}+\|f\|_{\dot{H}^1}+\|h_i\|_{Stri}+\|h_i\|^5_{Stri} + \|h_i\|_D\|h_i\|_{L^\infty_xL^2_t}+\left\|h_i^S\right\|_D\left\|h_i^S\right\|_{L^\infty_xL^2_t} + \|F_1\|_I+\left\|F_2^S\right\|_I+\|F\|_{L^1_tL^2_x}. \quad (5.2.68)$$

Following the argument in Section 5 from [14] and in the Appendix, using the reversed Strichartz estimates to derive regular Strichartz estimates, one has

$$\|h_{i+1}\|_{Stri}+\sup_{t\geq t_0}\|h_{i+1}\|_{\dot{H}^1\times L^2} \lesssim \|g\|_{L^2}+\|f\|_{\dot{H}^1}+\|h_i\|_{Stri}+\|h_i\|^5_{Stri} + \|h_i\|_D\|h_i\|_{L^\infty_xL^2_t}+\left\|h_i^S\right\|_D\left\|h_i^S\right\|_{L^\infty_xL^2_t} + \|F_1\|_I+\left\|F_2^S\right\|_I+\|F\|_{L^1_tL^2_x}. \quad (5.2.69)$$

By the computations in Remark 5.2.2, we can choose $t_0$ large enough, so that

$$\|F\|_{L_t^1([t_0,\infty),\,L_x^2)} + \left\|\langle x\rangle^{\frac{1}{2}+\epsilon_0}\left(F_2^S + F_1\right)\right\|_{L_t^2([t_0,\infty),\,L_x^2)} \ll \epsilon \tag{5.2.70}$$

where $\epsilon$ is the small constant appearing in the contraction.

First we show that $h_{i+1}$ is bounded in all Strichartz norms and the energy norm.

Define the space $S$ as

$$S = \left\{\|u\|_{Stri}\,,\ \|u\|_{\dot{H}^1\times L^2}\,,\ \|u\|_{L_x^\infty L_t^2}\,,\ \left\|u^S\right\|_{L_x^\infty L_t^2}\,,\ \|u\|_D\,,\ \left\|u^S\right\|_D < \infty\right\} \tag{5.2.71}$$

By the iteration scheme:

$$\partial_{tt}h_{i+1} - \Delta h_{i+1} + \left(V_1(x) + 5W_1^4(x)\right)h_{i+1} + \left(V_2\left(x-\vec{v}t\right) + 5W_2^4\left(x-\vec{v}t\right)\right)h_{i+1}$$
$$= F_1(x,t) + F_2(x,t) + F(x,t) + N(h_i,x,t) - a(x,t)h_i - h_i^5. \tag{5.2.72}$$

with data

$$(h_i(t_0), \partial_t h_i(t_0)) = (f,g) \tag{5.2.73}$$

Then

$$\left\|h_{i+1}-h_{j+1}\right\|_{S} \lesssim \eta \left\|h_{i}-h_{j}\right\|_{L_t^5 L_x^{10}} \tag{5.2.74}$$
$$+ \left(\left\|h_{i}\right\|_{D}+\left\|h_{j}\right\|_{D}\right)\left\|h_{i}-h_{j}\right\|_{L_x^{\infty} L_t^2}$$
$$+ \left(\left\|h_{i}^{S}\right\|_{D}+\left\|h_{j}^{S}\right\|_{D}\right)\left\|h_{i}^{S}-h_{j}^{S}\right\|_{L_x^{\infty} L_t^2}$$
$$+ \left(\left\|h_{i}^{2}\right\|_{L_t^{\frac{3}{2}} L_x^{9}}+\left\|h_{j}^{2}\right\|_{L_t^{\frac{3}{2}} L_x^{9}}\right)\left\|h_{i}-h_{j}\right\|_{L_t^3 L_x^{18}}$$
$$+ \left(\left\|h_{i}^{3}\right\|_{L_t^{\frac{4}{3}} L_x^{4}}+\left\|h_{j}^{3}\right\|_{L_t^{\frac{4}{3}} L_x^{4}}\right)\left\|h_{i}-h_{j}\right\|_{L_t^4 L_x^{12}}$$
$$+ \left(\left\|h_{i}^{4}\right\|_{L_t^{\frac{5}{4}} L_x^{\frac{5}{2}}}+\left\|h_{j}^{4}\right\|_{L_t^{\frac{5}{4}} L_x^{\frac{5}{2}}}\right)\left\|h_{i}-h_{j}\right\|_{L_t^5 L_x^{10}}$$

Let $h_{-1}=0$. We have

$$\partial_{tt} h_0-\Delta h_0+\left(V_1(x)+5 W_1^4(x)\right) h_0+\left(V_2(x-\vec{v} t)+5 W_2^4(x-\vec{v} t)\right) h_0$$
$$=F_1(x, t)+F_2(x, t)+F(x, t). \tag{5.2.75}$$

with

$$\left(h_0(t_0), \partial_t h_0(t_0)\right)=(f, g) \tag{5.2.76}$$

such that

$$\|(f, g)\|_{\dot{H}^1 \times L^2} \ll \epsilon. \tag{5.2.77}$$

Then by our Strichartz estimates from Theorem 5.3.3, Theorem 5.3.5 and Theorem 5.3.4, one has

$$\left\|h_0\right\|_{S} \leq \epsilon \ll 1. \tag{5.2.78}$$

By induction, suppose that

$$\left\|h_j\right\|_{S} \leq 2 \epsilon \ll 1. \tag{5.2.79}$$

By similar computations to the above, we can conclude that

$$\begin{aligned}
\left\|h_{j+1}-h_0\right\|_S \lesssim & \,\eta \left\|h_j\right\|_{L_t^5 L_x^{10}} \\
& + \left(\left\|h_j\right\|_D \left\|h_j\right\|_{L_x^\infty L_t^2} + \left\|h_j^S\right\|_D \left\|h_j^S\right\|_{L_x^\infty L_t^2}\right) \\
& + \left(\left\|h_j^2\right\|_{L_t^{\frac{3}{2}} L_x^9} \left\|h_j\right\|_{L_t^3 L_x^{18}}\right) \\
& + \left(\left\|h_j^3\right\|_{L_t^{\frac{4}{3}} L_x^4} \left\|h_j\right\|_{L_t^4 L_x^{12}}\right) \\
& + \left(\left\|h_j^4\right\|_{L_t^{\frac{5}{4}} L_x^{\frac{5}{2}}} \left\|h_j\right\|_{L_t^5 L_x^{10}}\right) \\
\lesssim & \,\eta \left\|h_j\right\|_S + \left\|h_j\right\|_S^2 + \left\|h_j\right\|_S^3 + \left\|h_j\right\|_S^4 + \left\|h_j\right\|_S^5 \\
\leq & \, C\left(2\eta\epsilon + 4\epsilon^2 + 8\epsilon^3 + 16\epsilon^4 + 32\epsilon^5\right) \\
\leq & \, C\left(2\eta + 4\epsilon + 8\epsilon^2 + 16\epsilon^3 + 32\epsilon^4\right)\epsilon.
\end{aligned} \tag{5.2.80}$$

One just needs to pick $\epsilon$ small and $\eta$ small such that

$$C\left(\eta + 4\epsilon + 8\epsilon^2 + 16\epsilon^3 + 32\epsilon^4\right) < \frac{1}{2}. \tag{5.2.81}$$

$$\eta = \|a(x,t)\|_{L_t^{\frac{5}{4}}\left([t_0,\infty), L_x^{\frac{5}{2}}\right)}. \tag{5.2.82}$$

Note that

$$\left\|20W_1^3(x)W_2\left(x-\vec{v}t\right) + 30W_1^2(x)W_2^2\left(x-\vec{v}t\right) + 20W_1(x)W_2^3\left(x-\vec{v}t\right)\right\|_{L_t^{\frac{5}{4}}\left([t_0,\infty), L_x^{\frac{5}{2}}\right)} \tag{5.2.83}$$

can be made sufficiently small provided $t_0$ is large enough. Therefore

$$\|a(x,t)\|_{L_t^{\frac{5}{4}}\left([t_0,\infty), L_x^{\frac{5}{2}}\right)} \tag{5.2.84}$$

can be made arbitrarily small provided $t_0$ is large.

Therefore, by induction, we have

$$\left\|h_{j+1}\right\|_S \leq 2\epsilon. \tag{5.2.85}$$

Next we show that the above construction gives a contraction. By almost the same computations as above,

$$\begin{aligned}
\|h_{i+1}-h_i\|_S \lesssim\ & \eta\,\|h_i-h_{i-1}\|_{L_t^5 L_x^{10}} \\
& + \left(\|h_i\|_D+\|h_{i-1}\|_D\right)\|h_i-h_{i-1}\|_{L_x^\infty L_t^2} \\
& + \left(\left\|h_i^S\right\|_D+\left\|h_{i-1}^S\right\|_D\right)\left\|h_i^S-h_{i-1}^S\right\|_{L_x^\infty L_t^2} \\
& + \left(\left\|h_i^2\right\|_{L_t^{\frac{3}{2}} L_x^9}+\left\|h_{i-1}^2\right\|_{L_t^{\frac{3}{2}} L_x^9}\right)\|h_i-h_{i-1}\|_{L_t^3 L_x^{18}} \\
& + \left(\left\|h_i^3\right\|_{L_t^{\frac{4}{3}} L_x^4}+\left\|h_{i-1}^3\right\|_{L_t^{\frac{4}{3}} L_x^4}\right)\|h_i-h_{i-1}\|_{L_t^4 L_x^{12}} \\
& + \left(\left\|h_i^4\right\|_{L_t^{\frac{5}{4}} L_x^{\frac{5}{2}}}+\left\|h_{i-1}^4\right\|_{L_t^{\frac{5}{4}} L_x^{\frac{5}{2}}}\right)\|h_i-h_{i-1}\|_{L_t^5 L_x^{10}} \\
\leq\ & \frac{1}{2}\|h_i-h_{i-1}\|_S\,.
\end{aligned} \tag{5.2.86}$$

Therefore by the Banach fixed-point theorem, there exists a unique solution to

$$\begin{aligned}
\partial_{tt}h-\Delta h+h^5+\left(V_1(x)+5W_1^4(x)\right)h+\left(V_2\left(x-\vec{v}t\right)+5W_2^4\left(x-\vec{v}t\right)\right)h+a(x,t)h \\
=F_1(x,t)+F_2(x,t)+F(x,t)+N(h,x,t)
\end{aligned} \tag{5.2.87}$$

such that

$$\|h\|_S \leq 2\epsilon. \tag{5.2.88}$$

Hence by Theorem 5.3.7,

$$u(t) - W_1(x) - W_2(x - \vec{v}t) \tag{5.2.89}$$

scatters to free wave. □

**Remark 5.2.3.** *The quadratic term can also be handled by estimate* (5.3.26) *from Theorem 5.3.3.*

Secondly, we show the existence of the purely multi-soliton solution. We solve the equation for $h$ backwards from infinity.

*Proof of Theorem 5.0.1.* Again, we consider

$$h(t) := u(t) - W_1(x) - W_2(x - \vec{v}t), \tag{5.2.90}$$

then

$$\begin{aligned}\partial_{tt}h - \Delta h + h^5 + \left(V_1(x) + 5W_1^4(x)\right)h + \left(V_2(x - \vec{v}t) + 5W_2^4(x - \vec{v}t)\right)h& \\ =: F_1(x,t) + F_2(x,t) + F(x,t) + N(h,x,t) - a(x,t)h& \qquad (5.2.91)\\ =: \mathrm{F}(h,x,t)& \qquad (5.2.92)\end{aligned}$$

and

$$\partial_{tt}h - \Delta h =: G(h,x,t), \tag{5.2.93}$$

$$\|h(t)\|_{\dot{H}^1\times L^2} \to 0,\ t \to \infty.$$

As the beginning of this section, we set $A = \sqrt{-\Delta}$ and notice that $h$ solves (5.2.93) if and only if

$$H := Ah + i\partial_t h \tag{5.2.94}$$

satisfies

$$i\partial_t H = AH + G(h, x, t), \tag{5.2.95}$$

$$H(t) \to 0,\ t \to \infty \tag{5.2.96}$$

in the sense of $L^2$ norm.

By Duhamel's formula, for fixed $T$

$$H(t) = e^{i(t-T)A} H(T) - i \int_T^t e^{-i(t-s)A} G(h, \cdot, s)\, ds. \tag{5.2.97}$$

Letting $T$ go to $\infty$, we know $H(T) \to 0$, so

$$H(t) := i \int_t^\infty e^{i(t-s)A} G\left(h, \cdot, s\right)\, ds. \tag{5.2.98}$$

By construction, we just need to show $H(t)$ is well-defined in $L^2$, then automatically,

$$\left\| u(t) - W_1\left(x\right) - W_2\left(x - \vec{v}t\right) \right\|_{\dot{H}^1 \times L^2} \to 0. \tag{5.2.99}$$

It suffices to show

$$H(t) = i \int_t^\infty e^{i(t-s)A} G\left(h, \cdot, s\right)\, ds \in L^2, \tag{5.2.100}$$

is well-defined. We show the existence of such a solution for $t \geq t_0$ provided $t_0$ is large enough. This can be done by a similar contraction argument to the previous proof.

Indeed, we consider

$$\begin{aligned} \partial_{tt} h_{i+1} - \Delta h_{i+1} + \left(V_1(x) + 5W_1^4(x)\right) h_{i+1} + \left(V_2\left(x - \vec{v}t\right) + 5W_2^4\left(x - \vec{v}t\right)\right) h_{i+1} \\ = F_1(x,t) + F_2(x,t) + F(x,t) + N(h_i, x, t) - a(x,t)h_i - h_i^5. \end{aligned} \tag{5.2.101}$$

Again setting $h_{-1} = 0$, then by Duhamel's formula and the equation (5.2.100), we have

$$
\begin{aligned}
h_0(x,t) &= \int_t^\infty U(t,s)\left(F_1(\cdot,t) + F_2(\cdot,s) + F(\cdot,s)\right)\,ds \\
&= \int_t^\infty U(t,s)\mathrm{F}(h_{-1},\cdot,s)\,ds. \qquad (5.2.102)
\end{aligned}
$$

Then by estimates (5.3.25) and (5.3.29) from Theorem 5.3.5, one has

$$
\begin{aligned}
\sup_{x\in\mathbb{R}^3}\left(\int_{t_1}^\infty |h_0(x,t)|^2\,dt\right)^{\frac{1}{2}} &\lesssim \|F\|_{L_t^1[t_1,\infty)L_x^2} + \|F_1\|_{I_{t_1}} + \left\|F_2^S\right\|_{I_{t_1}} \\
&\lesssim \frac{1}{\sqrt{t_1}} \qquad (5.2.103)
\end{aligned}
$$

and

$$
\begin{aligned}
\sup_{x\in\mathbb{R}^3}\left(\int_{t_1}^\infty |h_0(x+\vec{v}t,t)|^2\,dt\right)^{\frac{1}{2}} &\lesssim \|F\|_{L_t^1[t_1,\infty)L_x^2} + \|F_1\|_{I_{t_1}} + \left\|F_2^S\right\|_{I_{t_1}} \\
&\lesssim \frac{1}{\sqrt{t_1}}, \qquad (5.2.104)
\end{aligned}
$$

where $I_{t_1}$ is the space obtained by restricting space $I$ given by (5.2.53) onto $[t_1,\infty)$.

Then by the argument in Lemma 5.2.1 and Remark 5.2.2, we write

$$
\begin{aligned}
\partial_{tt}h_0 - \Delta h_0 = &-\left(V_1(x) + 5W_1^4(x)\right)h_0 + \left(V_2\left(x-\vec{v}t\right) + 5W_2^4\left(x-\vec{v}t\right)\right)h_0 \\
&+ F_1(x,t) + F_2(x,t) + F(x,t) \\
&=: \mathrm{D}\left(h_{-1},x,t\right) \qquad (5.2.105)
\end{aligned}
$$

and then conclude that

$$
\|h_0(t_1)\|_{\dot{H}^1\times L^2} \lesssim \frac{1}{\sqrt{t_1}}. \qquad (5.2.106)
$$

It follows that

$$\|h_0\|_{S_{t_1}} \lesssim \frac{1}{\sqrt{t_1}} \tag{5.2.107}$$

where $S_{t_1}$ is the space given by (5.2.71) restricted onto onto $[t_1, \infty)$.

Next, we can run the contraction argument as the proof Theorem 5.0.2. We consider the iteration give by the following formula.

$$h_{i+1}(x,t) = \int_t^{\infty} U(t,s)\mathrm{F}(h_i, \cdot, s)\, ds. \tag{5.2.108}$$

Then by the same computations as (5.2.74) restricted onto $[t_1, \infty)$, one has

$$\begin{aligned}
\left\|h_{i+1} - h_{j+1}\right\|_{S_{t_1}} \lesssim\, & \eta \left\|h_i - h_j\right\|_{L_t^5[t_1,\infty)L_x^{10}} \\
& + \left(\|h_i\|_{D_{t_1}} + \left\|h_j\right\|_{D_{t_1}}\right)\left\|h_i - h_j\right\|_{L_x^{\infty}L_t^2[t_1,\infty)} \\
& + \left(\left\|h_i^S\right\|_{D_{t_1}} + \left\|h_j^S\right\|_{D_{t_1}}\right)\left\|h_i^S - h_j^S\right\|_{L_x^{\infty}L_t^2[t_1,\infty)} \\
& + \left(\left\|h_i^2\right\|_{L_t^{\frac{3}{2}}[t_1,\infty)L_x^9} + \left\|h_j^2\right\|_{L_t^{\frac{3}{2}}[t_1,\infty)L_x^9}\right)\left\|h_i - h_j\right\|_{L_t^3[t_1,\infty)L_x^{18}} \\
& + \left(\left\|h_i^3\right\|_{L_t^{\frac{4}{3}}[t_1,\infty)L_x^4} + \left\|h_j^3\right\|_{L_t^{\frac{4}{3}}[t_1,\infty)L_x^4}\right)\left\|h_i - h_j\right\|_{L_t^4[t_1,\infty)L_x^{12}} \\
& + \left(\left\|h_i^4\right\|_{L_t^{\frac{5}{4}}[t_1,\infty)L_x^{\frac{5}{2}}} + \left\|h_j^4\right\|_{L_t^{\frac{5}{4}}[t_1,\infty)L_x^{\frac{5}{2}}}\right)\left\|h_i - h_j\right\|_{L_t^5[t_1,\infty)L_x^{10}}.
\end{aligned} \tag{5.2.109}$$

For all $t_1$ such that

$$t_0 \ll t_1,\ \frac{1}{\sqrt{t_1}} \ll \epsilon. \tag{5.2.110}$$

where $t_0$ and $\epsilon$ are constants depend on prescribed constants as in the proof of Theorem 5.0.2.

We can conclude that

$$\|h_i - h_0\|_{S_{t_1}} \lesssim \frac{1}{\sqrt{t_1}} \tag{5.2.111}$$

$$\|h_{i+1} - h_i\|_{S_{t_1}} \leq \frac{1}{2} \|h_i - h_{i-1}\|_{S_{t_1}} \tag{5.2.112}$$

Therefore by the Banach fixed-point theorem, there exists a unique solution to

$$h(x,t) = \int_t^{\infty} U(t,s)\mathrm{F}(h,\cdot,s)\, ds \tag{5.2.113}$$

such that

$$\|h(t)\|_{\dot{H}^1 \times L^2} \lesssim \frac{1}{\sqrt{t}}. \tag{5.2.114}$$

Therefore, we conclude that if we write

$$U[t] = (u, u_t)^t\,, \; W[t] = \left(W_1(x) - W_2(x - \vec{v}t)\,, \, \partial_t \left(W_1(x) - W_2(x - \vec{v}t)\right)\right)^t, \tag{5.2.115}$$

there exists a solution $u$ to

$$\partial_{tt} u - \Delta u + V_1(x)\, u + V_2(x - \vec{v}t)\, u + u^5 = 0, \tag{5.2.116}$$

such that

$$\lim_{t \to \infty} \|U[t] - W[t]\|_{\dot{H}^1 \times L^2} = 0. \tag{5.2.117}$$

Moreover, we have the decay rate

$$\|U[t] - W[t]\|_{\dot{H}^1 \times L^2} \lesssim \frac{1}{\sqrt{t}}. \tag{5.2.118}$$

as $t \to \infty$. We are done. □

### *5.2.2 Unstable solitons*

To finish this section, we discuss the case that we have some unstable solitons. From the discussion above, the linear model still plays a pivotal role. But in this case, the analysis is much more involved due to the unstable structure. Consider

$$\partial_{tt}u - \Delta u + V_1(x)u + V_2\left(x - \vec{v}t\right)u + u^5 = 0 \tag{5.2.119}$$

and

$$h = u(t) - Q_1\left(x\right) - Q_2\left(x - \vec{v}t\right). \tag{5.2.120}$$

where both $Q_1$ and $Q_2$ are unstable. For simplicity, suppose that

$$L_{Q_1} = -\Delta + V_1 + 5Q_1^4 \tag{5.2.121}$$

has one negative eigenvalue and zero is neither an eigenvalue nor resonance. Also suppose

$$L_{Q_2} = -\Delta + V_2^v + 5\left(Q_2^v\right)^4 \tag{5.2.122}$$

has one negative eigenvalue and zero is neither an eigenvalue nor resonance.

With $\lambda > 0,\ \mu > 0$,

$$L_{Q_1}w = -\lambda^2 w,\ L_{Q_2}m = -\mu^2 m. \tag{5.2.123}$$

$w$ and $m$ decay exponentially by Agmon's estimate, see [1, 12]. The analysis can be easily adapted to the most general situation.

Set

$$Q[t] = \left(Q_1\left(x\right) - Q_2\left(x - \vec{v}t\right), \partial_t\left(Q_1\left(x\right) - Q_2\left(x - \vec{v}t\right)\right)\right)^t. \tag{5.2.124}$$

**Theorem 5.2.4.** *Suppose that $0 < \epsilon \ll 1$ is small enough and $1 \ll t_0$ is large enough. There is a codimension $1+1$ smooth manifold $\mathcal{M} \in \dot{H}\left(\mathbb{R}^3\right) \times L^2\left(\mathbb{R}^3\right)$ around the $\epsilon$ neighborhood around $Q[t_0]$ such that if $u \in \mathcal{M}$ solve*

$$\partial_{tt}u - \Delta u + V_1(x)u + V_2\left(x - \vec{v}t\right)u + u^5 = 0, \tag{5.2.125}$$

*and*

$$\|U[t_0] - Q[t_0]\|_{\dot{H}^1 \times L^2} \leq \epsilon. \tag{5.2.126}$$

*Then there exists free data*

$$U_0[0] = (f_0, g_0)^t \in \dot{H}^1 \times L^2 \tag{5.2.127}$$

*such that*

$$\left\| U[t] - Q[t] - e^{tJH_F} U_0[0] \right\|_{\dot{H}^1 \times L^2} \to 0, \ t \to \infty. \tag{5.2.128}$$

*In other words, the error* $u(t) - Q_1(x) - Q_2(x - \vec{v}t)$ *scatters to the free wave.*

*Proof.* As in the stable case, by construction, we have

$$\begin{aligned} \partial_{tt} h - \Delta h + h^5 + \Big(V_1(x) + 5Q_1^4(x)\Big) h + \Big(V_2(x - \vec{v}t) + 5Q_2^4(x - \vec{v}t)\Big) h + a(x,t)h \\ = F_1(x,t) + F_2(x,t) + F(x,t) + N(h,x,t) \end{aligned} \tag{5.2.129}$$

Again, we consider the evolution starting from $t_0$ where $t_0$ is large enough and only depends on prescribed constants. When dealing with unstable solitons, we need to make sure the evolution under the iteration is a scattering in each iterated step. So we need to modify the data after each iteration.

As in the stable case, we consider the iteration:

$$\begin{aligned} \partial_{tt} h_{i+1} - \Delta h_{i+1} + \Big(V_1(x) + 5Q_1^4(x)\Big) h_{i+1} + \Big(V_2(x - \vec{v}t) + 5Q_2^4(x - \vec{v}t)\Big) h_{i+1} \\ = F_1(x,t) + F_2(x,t) + F(x,t) + N(h_i, x, t) - a(x,t)h_i - h_i^5 \end{aligned} \tag{5.2.130}$$

and

$$h_{-1} \equiv 0. \tag{5.2.131}$$

Denote

$$\mathrm{V}_{\mathbf{i}} = V_i(x) + 5Q_i^4(x), \tag{5.2.132}$$

$$\mathrm{D}(h_i, x, t) = N(h_i, x, t) - a(x,t)h_i - h_i^5 \tag{5.2.133}$$

and

$$\mathrm{F}(h_i, x, t) = F_1(x,t) + F_2(x,t) + F(x,t) + \mathrm{D}(h_i, x, t). \tag{5.2.134}$$

Decompose $h_i$ into three pieces:

$$h_i(x,t) = a_i(t)w(x) + b_i\left(\gamma(t - vx_1)\right) m_v\left(x,t\right) + r_i(x,t) \tag{5.2.135}$$

where

$$m_v(x,t) = m\left(\gamma\left(x_1 - \vec{v}t\right), x_2, x_3\right). \tag{5.2.136}$$

We notice that

$$P_c\left(H_1\right) r_i = r_i \tag{5.2.137}$$

and

$$P_c\left(H_2\right)\left(r_i\right)_L = \left(r_i\right)_L \tag{5.2.138}$$

where the Lorentz transformation $L$ makes $\mathrm{V}_2$ stationary.

Under the iteration, for the initial data, we impose that for $i \geq 1$.

$$a(t_0) = a_i(t_0) = a_0(t_0), \tag{5.2.139}$$

$$b\left(\sqrt{1-|v|^2}t_0\right) = b_i\left(\sqrt{1-|v|^2}t_0\right) = b_0\left(\sqrt{1-|v|^2}t_0\right) \tag{5.2.140}$$

and

$$r(x,t_0) = r_i\left(x,t_0\right) = r_0\left(x,t_0\right). \tag{5.2.141}$$

We first analyze the behavior of the bound states as in [14]. Plugging the evolution (5.2.135) into the equation (5.2.130) and taking inner product with $w$, we get

$$\ddot{a}_i(t) - \lambda^2 a_i(t) + \langle \mathrm{V}_2\left(x - \vec{v}t\right) h_i, w\rangle = \langle \mathrm{F}(h_i, x, t), w\rangle$$

Denote

$$N_i(t) := \langle \mathrm{F}(h_i, x, t), w\rangle - \langle \mathrm{V}_2\left(x - \vec{v}t\right) h_i, w\rangle . \tag{5.2.142}$$

Then

$$a_i(t) = \frac{e^{\lambda t}}{2}\left[a_i(0) + \frac{1}{\lambda}\dot{a}_i(0) + \frac{1}{\lambda}\int_{t_0}^{t} e^{-\lambda s} N_i(s)\, ds\right] + R(t) \tag{5.2.143}$$

where

$$|R(t)| \lesssim e^{-\beta t}, \tag{5.2.144}$$

for some positive constant $\beta > 0$. Therefore, the stability condition from scattering conditions in the sense of Definition 5.3.2 forces

$$a_i(t_0) + \frac{1}{\lambda}\dot{a}_i(t_0) + \frac{1}{\lambda}\int_{t_0}^{\infty} e^{-\lambda s} N_i(s)\, ds = 0. \tag{5.2.145}$$

So as the discussion in [14], given $a(t_0)$, there is a unique $\dot{a}(t_0)$ such that the stability condition (5.2.145) is satisfied. Similar results hold for $b_0\left(t\right)$ up to performing a Lorentz transformation. These stability conditions will ensure that $h_i$ is a scattering state. Therefore, we can employ the estimates from Theorem 5.3.3, Theorem 5.3.4 and Theorem 5.3.5 as in the proof of Theorem 5.0.2.

Consider the iteration for $\dot{a}_i(t_0)$,

$$\dot{a}_{i+1}(t_0) - \dot{a}_{j+1}(t_0) = -\int_{t_0}^{\infty} e^{-\lambda s}\left(N_{i+1}(s) - N_{j+1}(s)\right)\, ds. \tag{5.2.146}$$

Note that

$$\left|N_{i+1}(s)-N_{j+1}(s)\right| \lesssim \left|\left\langle \mathrm{V}_2\left(x-\vec{v}t\right)\left(h_{i+1}-h_{j+1}\right), w\right\rangle\right| \\ + \left|\left\langle \mathrm{D}(h_i,x,t)-\mathrm{D}(h_j,x,t), w\right\rangle\right|. \tag{5.2.147}$$

Then for $1 \leq p \leq 2$, by Minkowski's inequality and Hölder's inequality,

$$\left\|\left|\left\langle \mathrm{V}_2\left(x-\vec{v}t\right)\left(h_{i+1}-h_{j+1}\right), w\right\rangle\right|\right\|_{L_t^p[t_0,\infty)} \\ \lesssim \left|\left\langle \left\|\left|\mathrm{V}_2\left(x-\vec{v}t\right)\left(h_{i+1}-h_{j+1}\right)\right|\right\|_{L_t^p[t_0,\infty)}, w\right\rangle\right| \\ \lesssim \frac{1}{\langle t_0\rangle}\left\|h_{i+1}-h_{j+1}\right\|_{L_x^\infty L_t^2[t_0,\infty)}. \tag{5.2.148}$$

To estimate the difference between $\mathrm{D}(h_i,x,t)$ and $\mathrm{D}(h_j,x,t)$, we do the same computations as in the stable solitons case, see (5.2.74),

$$\left|\left\langle \mathrm{D}(h_i,x,t)-\mathrm{D}(h_j,x,t), w\right\rangle\right|_{L_t^p[t_0,\infty)} \lesssim \left\|h_i-h_j\right\|_S. \tag{5.2.149}$$

Then combine (5.2.148) and (5.2.149) together, one has

$$\left|\dot{a}_{i+1}(t_0)-\dot{a}_{j+1}(t_0)\right| \lesssim e^{-\lambda t_0}\left(\int_{t_0}^\infty \left|N_{i+1}(s)-N_{j+1}(s)\right|^2 ds\right)^{\frac{1}{2}} \\ \lesssim e^{-\lambda t_0}\left(\frac{1}{\langle t_0\rangle}\left\|h_{i+1}-h_{j+1}\right\|_S+\left\|h_i-h_j\right\|_S\right) \tag{5.2.150}$$

Similarly,

$$\left|\dot{b}_{i+1}\left(\sqrt{1-|v|^2}t_0\right)-\dot{b}_{j+1}\left(\sqrt{1-|v|^2}t_0\right)\right| \\ \lesssim e^{-\sqrt{1-|v|^2}\mu t_0}\left(\frac{1}{\langle t_0\rangle}\left\|h_{i+1}-h_{j+1}\right\|_S+\left\|h_i-h_j\right\|_S\right). \tag{5.2.151}$$

with

$$\|h_0(t_0)\|_{\dot{H}^1\times L^2} \ll \epsilon \tag{5.2.152}$$

Then by our Strichartz estimates

$$\|h_0\|_S \lesssim \epsilon \ll 1. \tag{5.2.153}$$

Next we consider the estimate in our iteration scheme as (5.2.74), (5.2.80) and (5.2.86). It suffices to estimate:

$$\begin{aligned}\left\|h_{i+1} - h_{j+1}\right\|_S \le & \left|\dot{a}_{i+1}(t_0) - \dot{a}_{j+1}(t_0)\right| \\ & + \left|\dot{b}_{j+1}\left(\sqrt{1-|v|^2}t_0\right) - \dot{b}_{i+1}\left(\sqrt{1-|v|^2}t_0\right)\right| \\ & + \frac{1}{4}\left\|h_i - h_j\right\|_S \end{aligned} \tag{5.2.154}$$

Therefore as the stable case, (5.2.80) and (5.2.86), we haves

$$\|h_{i+1} - h_i\|_S \le \frac{1}{2}\|h_i - h_{i-1}\|_S \tag{5.2.155}$$

and

$$\left|\dot{a}_{i+1}(t_0) - \dot{a}_{j+1}(t_0)\right| + \left|\dot{b}_{i+1}\left(\sqrt{1-|v|^2}t_0\right) - \dot{b}_{j+1}\left(\sqrt{1-|v|^2}t_0\right)\right| \le \frac{1}{8}\left\|h_i - h_j\right\|_S \tag{5.2.156}$$

Therefore by the Banach fixed-point theorem, there exist $h$, $\dot{a}(t_0)$ and $\dot{b}\left(\sqrt{1-|v|^2}t_0\right)$ such that

$$\|h_i - h\|_S \to 0, \tag{5.2.157}$$

$$|\dot{a}_i(t_0) - \dot{a}(t_0)| \to 0, \tag{5.2.158}$$

and

$$\left|\dot{b}_i\left(\sqrt{1-|v|^2}t_0\right) - \dot{b}\left(\sqrt{1-|v|^2}t_0\right)\right| \to 0 \tag{5.2.159}$$

as $i \to \infty$.

Moreover,

$$a(t_0) + \frac{1}{\lambda}\dot{a}(t_0) + \frac{1}{\lambda}\int_{t_0}^{\infty} e^{-\lambda s} N(s)\, ds = 0 \tag{5.2.160}$$

where

$$N(t) := \langle \mathrm{F}(h,x,t), w \rangle - \langle \mathrm{V}_2\,(x - \vec{v}t)\, h, w \rangle\,, \tag{5.2.161}$$

the same condition holds for $b(t)$.

It follows that $h$ is scattering state and satisfies

$$\begin{aligned} \partial_{tt} h - \Delta h + \Big(V_1(x) + 5Q_1^4(x)\Big)\, h + \Big(V_2\,(x - \vec{v}t) + 5Q_2^4\,(x - \vec{v}t)\Big)\, h \\ = F_1(x,t) + F_2(x,t) + F(x,t) + N(h,x,t) - a(x,t)h - h^5, \end{aligned} \tag{5.2.162}$$

and

$$\|h\|_S \lesssim \epsilon. \tag{5.2.163}$$

Hence

$$u(t) - Q_1\,(x) - Q_2\,(x - \vec{v}t) \tag{5.2.164}$$

scatters to free wave. □

**Remark 5.2.5.** *We can also consider the most general case. Suppose that*

$$L_{Q_1} = -\Delta + V_1 + 5Q_1^4 \tag{5.2.165}$$

*has $k_1$ negative eigenvalues and zero is neither an eigenvalue nor resonance. Also suppose*

$$L_{Q_2} = -\Delta + V_2^v + 5\,(Q_2^v)^4 \tag{5.2.166}$$

*has $k_2$ negative eigenvalues and zero is neither an eigenvalue nor resonance.*

*Then by similar arguments as above, we can obtain the general conditional stability results: Suppose that* $0 < \epsilon \ll 1$ *is small enough and* $1 \ll t_0$ *is large enough. There is a codimension* $k_1 + k_2$ *smooth manifold* $\mathcal{M} \in \dot{H}\left(\mathbb{R}^3\right) \times L^2\left(\mathbb{R}^3\right)$ *around the* $\epsilon$ *neighborhood around* $Q[t_0]$ *such that if* $u \in \mathcal{M}$ *solve*

$$\partial_{tt}u - \Delta u + V_1(x)u + V_2\left(x - \vec{v}t\right)u + u^5 = 0, \tag{5.2.167}$$

*and*

$$\|U[t_0] - Q[t_0]\|_{\dot{H}^1 \times L^2} \leq \epsilon. \tag{5.2.168}$$

*Then there exists free data*

$$U_0[0] = (f_0, g_0)^t \in \dot{H}^1 \times L^2 \tag{5.2.169}$$

*such that*

$$\left\|U[t] - Q[t] - e^{tJH_F}U_0[0]\right\|_{\dot{H}^1 \times L^2} \to 0,\ t \to \infty. \tag{5.2.170}$$

*In other words, the error* $u(t) - Q_1(x) - Q_2(x - \vec{v}t)$ *scatters to the free wave.*

We also have the existence of the purely-soliton solution with unstable excited states.

**Theorem 5.2.6.** *In* $\mathbb{R}^3$*, there exists a solution* $u$ *to*

$$\partial_{tt}u - \Delta u + V_1(x)u + V_2\left(x - \vec{v}t\right)u + u^5 = 0 \tag{5.2.171}$$

*such that*

$$\lim_{t \to \infty} \|U[t] - Q[t]\|_{\dot{H}^1 \times L^2} = 0.$$

In order to deal with bound states, here we need more complicated arguments. We will follow an idea based on the weak convergence from Merle [45] and Martel [43] which are also used in many other constructions of multisoltions, for example in [44, 30, 31, 20, 19].

*Proof.* We still take $t_0$ large enough as before. Taking a sequence $t_n \to \infty$. Consider the equation for $h$:

$$\partial_{tt}h - \Delta h + h^5 + \left(V_1(x) + 5Q_1^4(x)\right) h + \left(V_2\left(x - \vec{v}t\right) + 5Q_2^4\left(x - \vec{v}t\right)\right) h$$
$$= F_1(x,t) + F_2(x,t) + F(x,t) + N(h,x,t) + a(x,t)h \qquad (5.2.172)$$

We can construct a scattering state $h_n$ to equation (5.2.172) as in the proof of Theorem 5.2.4 such that

$$\left\|h_n\left(t_n\right)\right\|_{\dot{H}^1\times L^2} \lesssim \frac{1}{\sqrt{t_n}}. \qquad (5.2.173)$$

Moreover, by the estimates (5.3.28) and (5.3.29), we have

$$\sup_{x\in\mathbb{R}^3}\left(\int_t^{t_n} |h_n(x,t)|^2\,dt\right)^{\frac{1}{2}} \lesssim \frac{1}{\sqrt{t_n}} + \frac{1}{\sqrt{t}}, \qquad (5.2.174)$$

and

$$\sup_{x\in\mathbb{R}^3}\left(\int_t^{t_n} |h_n(x+\vec{v}t,t)|^2\,dt\right)^{\frac{1}{2}} \lesssim \frac{1}{\sqrt{t_n}} + \frac{1}{\sqrt{t}}. \qquad (5.2.175)$$

Furthermore, by a similar argument to the proof of Theorem 5.0.1, Lemma 5.2.1 and Remark 5.2.2, we can conclude that

$$\|h_n(t)\|_{\dot{H}^1\times L^2} \lesssim \frac{1}{\sqrt{t_n}} + \frac{1}{\sqrt{t}}. \qquad (5.2.176)$$

Notice that over $[t_0, t_n]$,

$$\|h_n(t)\|_{\dot{H}^1\times L^2} \lesssim \frac{1}{\sqrt{t_n}} + \frac{1}{\sqrt{t}} \lesssim \frac{1}{\sqrt{t}} \qquad (5.2.177)$$

with a constant independent of $n$.

Then up to passing to a subsequence

$$h_n(t_0) \rightharpoonup h_0 \in \dot{H}^1 \times L^2 \qquad (5.2.178)$$

weakly. Let $h$ be a solution of the equation (5.2.172) with $h_0$ as the initial data at $t = t_0$. By the weak continuity of the flow, for example in [2, 30, 33], one can obtain that $h$ exists on the time interval from $[t_0, \infty)$ and for $t \in [t_0, \infty)$,

$$h_n(t) \rightharpoonup h(t) \in \dot{H}^1 \times L^2. \tag{5.2.179}$$

Then passing to the weak limit in (5.2.177), one has

$$\|h(t)\|_{\dot{H}^1 \times L^2} \lesssim \frac{1}{\sqrt{t}}. \tag{5.2.180}$$

Therefore, we conclude that if we write

$$U[t] = (u, u_t)^t \,, \, Q[t] = \left(Q_1 (x) - Q_2 (x - \vec{v}t) \,, \, \partial_t \left(Q_1 (x) - Q_2 (x - \vec{v}t)\right)\right)^t, \tag{5.2.181}$$

there exists a solution $u$ to

$$\partial_{tt} u - \Delta u + V_1 (x)\, u + V_2 (x - \vec{v}t)\, u + u^5 = 0, \tag{5.2.182}$$

such that

$$\lim_{t \to \infty} \|U[t] - Q[t]\|_{\dot{H}^1 \times L^2} \to 0. \tag{5.2.183}$$

Moreover, we have the decay rate

$$\|U[t] - Q[t]\|_{\dot{H}^1 \times L^2} \lesssim \frac{1}{\sqrt{t}}. \tag{5.2.184}$$

We are done. □

**Remark 5.2.7.** *As in Remark 5.2.5, the above construction holds for the general case.*

## 5.3 Appendix: Linear Theory

In this Appendix, we recall the results from Chen [14] on wave equations with a charge transfer Hamiltonian in $\mathbb{R}^3$. In order to handle the strong interaction of solitons in our nonlinear application, we also need some refined version of inhomogeneous reversed Strichartz estimates.

### 5.3.1 *Charge transfer model*

Before we give the precise definition of our model, it is necessary to introduce Lorentz transformations. Given a vector $\vec{\mu} \in \mathbb{R}^3$, there is a Lorentz transformation $L(\vec{\mu})$ acting on $(x,t) \in \mathbb{R}^{3+1}$ such that it makes the moving frame $(x - \vec{\mu}t, t)$ stationary. We can use a $4 \times 4$ matrix $B(\vec{\mu})$ to represent the transformation $L(\vec{\mu})$. Moreover, for the given vector $\vec{\mu} = (\mu_1, \mu_2, \mu_3) \in \mathbb{R}^3$, there is a $3 \times 4$ matrix $M(\vec{\mu})$ such that

$$(x - \vec{\mu}t)^T = M(\vec{\mu})(x,t)^T, \tag{5.3.1}$$

where the superscript $T$ denotes the transpose of a vector.

With the preparations above, we can set up our model. We consider the scalar charge transfer model for wave equations in the following sense:

**Definition 5.3.1.** By a wave equation with a charge transfer Hamiltonian we mean a wave equation

$$\partial_{tt}u - \Delta u + \sum_{j=1}^{m} \mathrm{V}_j\left(x - \vec{v}_j t\right) u = 0, \tag{5.3.2}$$

$$u|_{t=0} = f,\ \partial_t u|_{t=0} = g,\ x \in \mathbb{R}^3,$$

where $\vec{v}_j$'s are distinct vectors in $\mathbb{R}^3$ with

$$|\vec{v}_i| < 1,\ 1 \le i \le m. \tag{5.3.3}$$

and the real potentials $\mathrm{V}_j$ are such that $\forall 1 \leq j \leq m$

1) $\mathrm{V}_j$ is time-independent and decays with rate $\langle x\rangle^{-\alpha}$ with $\alpha > 3$

2) 0 is neither an eigenvalue nor a resonance of the operators

$$H_j = -\Delta + \mathrm{V}_j\left(S\left(\vec{v_j}\right)x\right), \tag{5.3.4}$$

where $S\left(\vec{v_j}\right)x = M\left(\vec{v_j}\right)B^{-1}\left(\vec{v_j}\right)(x,0)^T$.

Recall that $\psi$ is a resonance at 0 if it is a distributional solution of the equation $H_k\psi = 0$ which belongs to the space $L^2\left(\langle x\rangle^{-\sigma}dx\right) := \left\{f : \langle x\rangle^{-\sigma} f \in L^2\right\}$ for any $\sigma > \frac{1}{2}$, but not for $\sigma = \frac{1}{2}$.

*Remark.* The construction of $S\left(\vec{v_j}\right)$ is clear from the change between different frames under Lorentz transformations. In our concrete problem below (5.3.7), $S\left(\vec{v_j}\right)$ can be written down explicitly.

To be consistent with our nonlinear application, throughout this section, we discuss the wave equation with a charge transfer Hamiltonian in the sense of Definition 5.3.1 with $m = 2$, a stationary $\mathrm{V}_1$ and a $\mathrm{V}_2$ moving along $\overrightarrow{e_1}$ with speed $|v| < 1$, i.e., the velocity is

$$\vec{v} = (v, 0, 0). \tag{5.3.5}$$

Under this setting, by Definition 5.3.1,

$$H_1 = -\Delta + \mathrm{V}_1(x), \tag{5.3.6}$$

and

$$H_2 = -\Delta + \mathrm{V}_2\left(\sqrt{1-|v|^2}x_1, x_2, x_3\right). \tag{5.3.7}$$

An indispensable tool we need to study the charge transfer model is the Lorentz transformation. Again, we apply Lorentz transformations $L$ with respect to a moving frame with speed $|v| < 1$ along the $x_1$ direction. After we apply the Lorentz transformation $L$, under the new coordinates, $\mathrm{V}_2$ is stationary meanwhile $\mathrm{V}_1$ will be moving.

Writing down the Lorentz transformation $L$ explicitly, we have

$$\begin{cases} t' = \gamma\left(t - vx_1\right) \\ x_1' = \gamma\left(x_1 - vt\right) \\ x_2' = x_2 \\ x_3' = x_3 \end{cases} \tag{5.3.8}$$

with

$$\gamma = \frac{1}{\sqrt{1 - |v|^2}}. \tag{5.3.9}$$

We can also write down the inverse transformation of the above:

$$\begin{cases} t = \gamma\left(t' + vx_1'\right) \\ x_1 = \gamma\left(x_1' + vt'\right) \\ x_2 = x_2' \\ x_3 = x_3' \end{cases}. \tag{5.3.10}$$

Under the Lorentz transformation $L$, if we use the subscript $L$ to denote a function with respect to the new coordinate $\left(x', t'\right)$, we have

$$u_L\left(x_1', x_2', x_3', t'\right) = u\left(\gamma\left(x_1' + vt'\right), x_2', x_3', \gamma\left(t' + vx_1'\right)\right) \tag{5.3.11}$$

and

$$u(x,t) = u_L\left(\gamma\left(x_1 - vt\right), x_2, x_3, \gamma\left(t - vx_1\right)\right). \tag{5.3.12}$$

### 5.3.2 Strichartz estimates

With the above preparations, we recall some important results from Chen [14]. Adapting the linear model to our nonlinear setting, we consider the following problem.

Suppose $u$ solves

$$\partial_{tt}u - \Delta u + \mathrm{V}_1(x)u + \mathrm{V}_2(x - \vec{v}t)u = F + F_1 + F_2 \tag{5.3.13}$$

with initial data

$$u(x, 0) = f(x),\ u_t(x, 0) = g(x). \tag{5.3.14}$$

Let $w_1, \dots, w_m$ and $m_1, \dots, m_\ell$ be the normalized bound states of $H_1$ and $H_2$ associated with the negative eigenvalues $-\lambda_1^2, \dots, -\lambda_m^2$ and $-\mu_1^2, \dots, -\mu_\ell^2$ respectively (notice that by our assumptions, 0 is not an eigenvalue). In other words, we assume

$$H_1 w_i = -\lambda_i^2 w_i,\ \ w_i \in L^2,\ \lambda_i > 0. \tag{5.3.15}$$

$$H_2 m_i = -\mu_i^2 m_i,\ \ m_i \in L^2,\ \mu_i > 0. \tag{5.3.16}$$

We denote by $P_b\left(H_1\right)$ and $P_b\left(H_2\right)$ the projections on the the bound states of $H_1$ and $H_2$, respectively, and let $P_c\left(H_i\right) = Id - P_b\left(H_i\right),\ i = 1, 2$. To be more explicit, we have

$$P_b\left(H_1\right) = \sum_{i=1}^{m} \langle \cdot, w_i \rangle\, w_i, \quad P_b\left(H_2\right) = \sum_{j=1}^{\ell} \left\langle \cdot, m_j \right\rangle m_j. \tag{5.3.17}$$

In order to study the equation with time-dependent potentials, we need to introduce a suitable projection. Again, with Lorentz transformations $L$ associated with the moving potential $\mathrm{V}_2(x - \vec{v}t)$, we use the subscript $L$ to denote a function under the new frame $\left(x', t'\right)$.

**Definition 5.3.2** (Scattering states)**.** Let

$$\partial_{tt}u - \Delta u + \mathrm{V}_1(x)u + \mathrm{V}_2(x - \vec{v}t)u = F + F_1 + F_2, \tag{5.3.18}$$

with initial data

$$u(x,0) = f(x),\ u_t(x,0) = g(x). \tag{5.3.19}$$

If $u$ also satisfies

$$\left\|P_b\left(H_1\right)u(t)\right\|_{L^2_x} \to 0,\ \left\|P_b\left(H_2\right)u_L(t')\right\|_{L^2_{x'}} \to 0\ \ t, t' \to \infty, \tag{5.3.20}$$

we call it a scattering state.

Define the space $I$ as

$$I = \left\{ G(x,t) \in L_x^{\frac{3}{2},1} L_t^2 \bigcap L^1_{x_1} L^{2,1}_{\widehat{x_1}} L^2_t \text{ such that } \left\| \langle x \rangle^{\frac{1}{2}+\epsilon} G \right\|_{L^2_{t,x}} < \infty \right\} \tag{5.3.21}$$

for the strong interactions terms where $\widehat{x_1}$ is the subspace orthogonal to the $x_1$ direction. Define

$$\|G\|_I = \max\left\{ \left\| \langle x \rangle^{\frac{1}{2}+\epsilon} G \right\|_{L^2_{t,x}},\ \|G\|_{L^1_{x_1} L^{2,1}_{\widehat{x_1}} L^2_t},\ \|G\|_{L_x^{\frac{3}{2},1} L_t^2} \right\}. \tag{5.3.22}$$

Also recall that for a function $G(x,t)$, we use the notation:

$$G^S(x,t) := G(x + \vec{v}t, t). \tag{5.3.23}$$

First of all, we have Strichartz estimates:

**Theorem 5.3.3.** *Suppose $u$ is a scattering state in the sense of Definition 5.3.2. Then for $p > 2$ and $(p,q)$ satisfying*

$$\frac{1}{2} = \frac{1}{p} + \frac{3}{q}, \tag{5.3.24}$$

*we have*

$$\|u\|_{L_t^p([0,\infty),\,L_x^q)} \lesssim \|g\|_{L^2} + \|f\|_{\dot{H}^1} + \|F\|_{L_t^1 L_x^2} \\ + \|F_1\|_I + \left\|F_2^S\right\|_I . \tag{5.3.25}$$

*and*

$$\|u\|_{L_t^2([0,\infty),\,L_r^\infty L_\omega^2)} \lesssim \|g\|_{L^2} + \|f\|_{\dot{H}^1} + \|F\|_{L_t^1 L_x^2} \\ + \|F_1\|_I + \left\|F_2^S\right\|_I . \tag{5.3.26}$$

Secondly, one has the energy estimate:

**Theorem 5.3.4.** *Suppose $u$ is a scattering state in the sense of Definition 5.3.2, then we have*

$$\sup_{t\in \mathbb{R}} \left(\|\nabla u(t)\|_{L^2} + \|u_t(t)\|_{L^2}\right) \lesssim \|g\|_{L^2} + \|f\|_{\dot{H}^1} + \|F\|_{L_t^1 L_x^2} \\ + \|F_1\|_I + \left\|F_2^S\right\|_I . \tag{5.3.27}$$

Even more importantly, we obtain the endpoint reversed Strichartz estimates for $u$.

**Theorem 5.3.5.** *Suppose $u$ is a scattering state in the sense of Definition 5.3.2, then*

$$\sup_{x\in\mathbb{R}^3} \left(\int_0^\infty |u(x,t)|^2\, dt\right)^{\frac{1}{2}} \lesssim \|g\|_{L^2} + \|f\|_{\dot{H}^1} + \|F\|_{L_t^1 L_x^2} \\ + \|F_1\|_I + \left\|F_2^S\right\|_I , \tag{5.3.28}$$

*and*

$$\sup_{x\in\mathbb{R}^3}\left(\int_0^\infty |u(x+\vec{v}t,t)|^2\,dt\right)^{\frac{1}{2}} \lesssim \|g\|_{L^2}+\|f\|_{\dot{H}^1}+\|F\|_{L^1_tL^2_x} + \|F_1\|_I + \left\|F_2^S\right\|_I. \tag{5.3.29}$$

*Moreover, one has*

$$\left\|\langle x\rangle^{-3}u(x,t)\right\|_{L_x^{\frac{3}{2},1}L_t^\infty\bigcap L^1_{x_1}L^{2,1}_{\widehat{x_1}}L^\infty_t} \lesssim \|g\|_{L^2}+\|f\|_{\dot{H}^1}+\|F\|_{L^1_tL^2_x} + \|F_1\|_{D_1} + \left\|F_2^S\right\|_{D_1} \tag{5.3.30}$$

$$\left\|\langle x\rangle^{-3}u(x+\vec{v}t,t)\right\|_{L_x^{\frac{3}{2},1}L_t^\infty\bigcap L^1_{x_1}L^{2,1}_{\widehat{x_1}}L^\infty_t} \lesssim \|g\|_{L^2}+\|f\|_{\dot{H}^1}+\|F\|_{L^1_tL^2_x} + \|F_1\|_{D_1} + \left\|F_2^S\right\|_{D_1}, \tag{5.3.31}$$

*where*

$$D_1 := \left\{G(x,t)\in L_x^{\frac{3}{2},1}L^1_t\bigcap L^1_{x_1}L^{2,1}_{\widehat{x_1}}L^1_t\bigcap L^2_tL^2_x\right\} \tag{5.3.32}$$

*and*

$$\|G\|_{D_1} := \max\left\{\|G\|_{L_x^{\frac{3}{2},1}L^1_t},\ \|G\|_{L^1_{x_1}L^{2,1}_{\widehat{x_1}}L^1_t},\ \|G\|_{L^2_tL^2_x}\right\}. \tag{5.3.33}$$

*One can replace* $D_1$ *by*

$$D_2 := \left\{G(x,t)\in L_x^{\frac{3}{2},1}L^\infty_t\bigcap L^1_{x_1}L^{2,1}_{\widehat{x_1}}L^\infty_t\bigcap L^2_tL^2_x\right\} \tag{5.3.34}$$

*and*

$$\|G\|_{D_2} := \max\left\{\|G\|_{L_x^{\frac{3}{2},1}L^\infty_t},\ \|G\|_{L^1_{x_1}L^{2,1}_{\widehat{x_1}}L^\infty_t},\ \|G\|_{L^2_tL^2_x}\right\}. \tag{5.3.35}$$

### 5.3.3 Energy comparison

Next, we recall the energy comparison for wave equations with respect to different Lorentz frames.

Following Chen [12, 14], we consider wave equations with time-dependent potentials

$$\partial_{tt}u - \Delta u + V(x,t)u = F \tag{5.3.36}$$

with

$$|V(x, \mu x_1)| \lesssim \frac{1}{\langle x \rangle^2} \tag{5.3.37}$$

uniformly for $0 \leq |\mu| \leq 1$. These in particular apply to wave equations with moving potentials with speed strictly less than 1. For example, if the potential is of the form

$$V(x,t) = V\left(x - \vec{v}t\right) \tag{5.3.38}$$

with

$$|V(x)| \lesssim \frac{1}{\langle x \rangle^2} \tag{5.3.39}$$

then it is transparent that

$$|V(x, \mu x_1)| = |V(x - \vec{v}\mu x_1)| \lesssim \frac{1}{\langle x \rangle^2}. \tag{5.3.40}$$

We sketch the argument in [12], suppose

$$\partial_{tt}u - \Delta u + V(x,t)u = F, \tag{5.3.41}$$

then it is clear that

$$\begin{aligned} Fu_t &= u_t\left(\Box u - V(t)u\right) \\ &= -\partial_t\left(\frac{|u_t|^2}{2} + \frac{|u_x|^2}{2}\right) + \operatorname{div}\left(\nabla u u_t\right) - V(x,t)uu_t. \end{aligned} \tag{5.3.42}$$

We apply the space-time divergence theorem to

$$\left(\nabla u u_t, -\left(\frac{|u_t|^2}{2} + \frac{|u_x|^2}{2}\right)\right) \tag{5.3.43}$$

then one has the following comparison with the inhomogeneous term, see Chen [12].

**Theorem 5.3.6.** *Let $|v| < 1$. Suppose*

$$\partial_{tt}u - \Delta u + V(x,t)u = F(x,t) \tag{5.3.44}$$

*and*

$$|V(x,\mu x_1)| \lesssim \frac{1}{\langle x\rangle^2} \tag{5.3.45}$$

*for $0 \le |\mu| < 1$. Then*

$$\begin{aligned} &\int |\nabla_x u\left(x_1,x_2,x_3,vx_1\right)|^2 + |\partial_t u\left(x_1,x_2,x_3,vx_1\right)|^2\,dx \\ &\quad\lesssim \int |\nabla_x u\left(x_1,x_2,x_3,0\right)|^2 + |\partial_t u\left(x_1,x_2,x_3,0\right)|^2\,dx \\ &\qquad + \int_{\mathbb{R}}\int_{\mathbb{R}^3} |F(x,t)|^2\,dxdt \end{aligned} \tag{5.3.46}$$

*and*

$$\begin{aligned}&\int |\nabla_x u(x_1,x_2,x_3,0)|^2 + |\partial_t u(x_1,x_2,x_3,0)|^2\,dx\\ \lesssim &\int |\nabla_x u(x_1,x_2,x_3,vx_1)|^2 + |\partial_t u(x_1,x_2,x_3,vx_1)|^2\,dx \\ &+\int_{\mathbb{R}}\int_{\mathbb{R}^3}|F(x,t)|^2\,dxdt\end{aligned} \tag{5.3.47}$$

*where the implicit constant depends on $v$ and $V$.*

From the theorem above, we know that the initial energy with respect to different frames stays comparable up to $\|F\|_{L^2_{t,x}}$.

### *5.3.4 Scattering*

In this subsection, we discuss the scattering behavior of the solution to the nonlinear equation for $h$:

$$\begin{aligned}\partial_{tt}h - \Delta h + h^5 + \Big(V_1(x) + 5W_1^4(x)\Big)h + \Big(V_2(x-\vec{v}t) + 5W_2^4(x-\vec{v}t)\Big)h \\ = F_1(x,t) + F_2(x,t) + F(x,t) + N(h,x,t) + a(x,t)h.\end{aligned} \tag{5.3.48}$$

We show that if $h$ is bounded in the $S$ norm where

$$S = \left\{\|u\|_{Stri}\,,\ \|u\|_{\dot{H}^1\times L^2}\,,\ \|u\|_{L^\infty_x L^2_t}\,,\ \left\|u^S\right\|_{L^\infty_x L^2_t}\,,\ \|u\|_D\,,\ \left\|u^S\right\|_D < \infty\right\} \tag{5.3.49}$$

and

$$D := \left\{\langle x\rangle^{-3} G(x,t) \in L^{\frac{3}{2},1}_x L^\infty_t \bigcap L^1_{x_1} L^{2,1}_{\widehat{x_1}} L^\infty_t\right\}, \tag{5.3.50}$$

then $h$ scatters to a free wave.

We will use the notations from the introduction.

**Theorem 5.3.7.** *Suppose that $h$ is a solution to* (5.3.48) *such that*

$$\|h\|_S < \infty, \left\| \langle x \rangle^{\frac{1}{2}+\epsilon} F_1 \right\|_{L^2_{t,x}} < \infty, \left\| \langle x \rangle^{\frac{1}{2}+\epsilon} F_2^S \right\|_{L^2_{t,x}} < \infty \text{ and } \|F\|_{L^1_t L^2_x} < \infty. \tag{5.3.51}$$

*Write*

$$H[t] = (h, h_t)^t \in C^0\left([0,\infty);\, \dot{H}^1\right) \times C^0\left([0,\infty);\, L^2\right), \tag{5.3.52}$$

*with initial data* $H[0] = (f, g)^t \in \dot{H}^1 \times L^2$. Then there exists free data

$$H_0[0] = (f_0, g_0)^t \in \dot{H}^1 \times L^2$$

such that

$$\left\| H[t] - e^{tJH_F} H_0[0] \right\|_{\dot{H}^1 \times L^2} \to 0 \tag{5.3.53}$$

as $t \to \infty$.

*Proof.* We set $A = \sqrt{-\Delta}$ and notice that

$$\|Af\|_{L^2} \simeq \|f\|_{\dot{H}^1}, \ \forall f \in C^\infty\left(\mathbb{R}^3\right). \tag{5.3.54}$$

For real-valued $u = (u_1, u_2) \in \mathcal{H} = \dot{H}^1\left(\mathbb{R}^3\right) \times L^2\left(\mathbb{R}^3\right)$, we write

$$U := Au_1 + iu_2. \tag{5.3.55}$$

We know

$$\|U\|_{L^2} \simeq \|(u_1, u_2)\|_{\mathcal{H}}. \tag{5.3.56}$$

We also notice that $h$ solves (5.3.48) if and only if

$$H := Ah + i\partial_t h \tag{5.3.57}$$

satisfies

$$
\begin{aligned}
i\partial_t H &= AH - h^5 - \Big(V_1(x) + 5W_1^4(x)\Big)\, h - \Big(V_2\left(x - \vec{v}t\right) + 5W_2^4\left(x - \vec{v}t\right)\Big)\, h \\
&\quad + F_1(x,t) + F_2(x,t) + F(x,t) + N(h,x,t) - a(x,t)h \\
&=: AH + D(h,x,t). \qquad (5.3.58)
\end{aligned}
$$

and

$$
H[0] = Af + ig \in L^2\left(\mathbb{R}^3\right). \qquad (5.3.59)
$$

By Duhamel's formula, for fixed $T$

$$
H[T] = e^{iTA}H[0] - i\int_0^T e^{-i(T-s)A}\left(D(h,\cdot,s)\right)\, ds. \qquad (5.3.60)
$$

Applying the free evolution backwards, we obtain

$$
e^{-iTA}H[T] = H[0] - i\int_0^T e^{isA}\left(D(h,\cdot,s)\right)\, ds. \qquad (5.3.61)
$$

Letting $T$ go to $\infty$, we define

$$
H_0[0] := H[0] - i\int_0^\infty e^{isA}\left(D(h,\cdot,s)\right)\, ds \qquad (5.3.62)
$$

By construction, we just need to show $H_0[0]$ is well-defined in $L^2$, then automatically,

$$
\left\| H[t] - e^{tJH_F}H_0[0]\right\|_{L^2} \to 0. \qquad (5.3.63)
$$

It suffices to show

$$
\int_0^\infty e^{isA}\left(D(h,\cdot,s)\right)\, ds \in L^2 \qquad (5.3.64)
$$

as $t \to \infty$.

Recall that

$$D(h,\cdot,s) = -\left(V_1(x) + 5W_1^4(x)\right)h - \left(V_2\left(x - \vec{v}s\right) + 5W_2^4\left(x - \vec{v}s\right)\right)h$$
$$+ F_1(x,s) + F_2(x,s) + F(x,s) + N(h,x,s) - a(x,s)h. \tag{5.3.65}$$

We also recall the precise expression of $N$:

$$\begin{aligned} N(h,x,t) := & \left(10W_1^3(x) + 30W_1^2(x)W_2\left(x - \vec{v}t\right) + 30W_1(x)W_2^2\left(x - \vec{v}t\right) + 10W_2^3(x - \vec{v}t)\right)h^2 \\ & + \left(10W_1^2(x) + 3W_1(x)W_2\left(x - \vec{v}t\right) + 10W_2^2\left(x - \vec{v}t\right)\right)h^3 \\ & + \left(5W_1(x) + 5W_2\left(x - \vec{v}t\right)\right)h^4. \end{aligned} \tag{5.3.66}$$

Furthermore, we have

$$M_1(h,x,t) := \left(10W_1^3(x) + 30W_1^2(x)W_2\left(x - \vec{v}t\right) + 30W_1(x)W_2^2\left(x - \vec{v}t\right) + 10W_2^3(x - \vec{v}t)\right)h^2 \tag{5.3.67}$$

and

$$M_{1,1}(h,x,t) = 10W_1^3(x)h^2,\ M_{1,2}(h,x,t) = 30W_1^2(x)W_2\left(x - \vec{v}t\right)h^2 \tag{5.3.68}$$

$$M_{1,3}(h,x,t) = 30W_1(x)W_2^2\left(x - \vec{v}t\right)h^2,\ M_{1,4}(h,x,t) = 10W_2^3(x - \vec{v}t)h^2. \tag{5.3.69}$$

Also we use the notation:

$$M_2(h,x,t) := \left(10W_1^2(x) + 3W_1(x)W_2\left(x - \vec{v}t\right) + 10W_2^2\left(x - \vec{v}t\right)\right)h^3, \tag{5.3.70}$$

$$M_3(h,x,t) := \left(5W_1(x) + 5W_2\left(x - \vec{v}t\right)\right)h^4 \tag{5.3.71}$$

We estimate each piece separately. By the identical argument as Lemma 5.2.1, we have

$$\left\|\int_0^\infty e^{isA}\left(\left(V_1(x) + 5W_1^4(x)\right)h\right)ds\right\|_{L^2} \lesssim \|h\|_{L_x^\infty L_t^2} \lesssim \|h\|_S. \tag{5.3.72}$$

$$\left\| \int_0^\infty e^{isA} \left( \left( V_2 \left( x - \vec{v}s \right) + 5W_2^4 \left( x - \vec{v}s \right) \right) h \right) \, ds \right\|_{L^2} \lesssim \left\| h^S \right\|_{L_x^\infty L_t^2} \lesssim \|h\|_S . \tag{5.3.73}$$

$$\left\| \int_0^\infty e^{isA} \left( F_1(x,s) \right) \, ds \right\|_{L^2} \lesssim \left\| \langle x \rangle^{\frac{1}{2}+\epsilon} F_1 \right\|_{L^2_{t,x}} \tag{5.3.74}$$

$$\left\| \int_0^\infty e^{isA} \left( F_2(x,s) \right) \, ds \right\|_{L^2} \lesssim \left\| \langle x \rangle^{\frac{1}{2}+\epsilon} F_2^S \right\|_{L^2_{t,x}} \tag{5.3.75}$$

$$\left\| \int_0^\infty e^{isA} \left( M_{1,1}(h,x,t) \right) \, ds \right\|_{L^2} \lesssim \|h\|_{L_x^\infty L_t^2} \|h\|_D \lesssim \|h\|_S^2 \tag{5.3.76}$$

$$\left\| \int_0^\infty e^{isA} \left( M_{1,4}(h,x,t) \right) \, ds \right\|_{L^2} \lesssim \left\| h^S \right\|_{L_x^\infty L_t^2} \left\| h^S \right\|_D \lesssim \|h\|_S^2 \tag{5.3.77}$$

And by trivial energy estimate for the free evolution:

$$\left\| \int_0^\infty e^{isA} \left( F(x,s) \right) \, ds \right\|_{L^2} \lesssim \|F\|_{L_t^1 L_x^2} \tag{5.3.78}$$

$$\left\| \int_0^\infty e^{isA} \left( h^5 \right) \, ds \right\|_{L^2} \lesssim \|h\|^5_{L_t^5 L_x^{10}} \lesssim \|h\|_S^5 \tag{5.3.79}$$

$$\left\| \int_0^\infty e^{isA} \left( a(x,s)h \right) \, ds \right\|_{L^2} \lesssim \|a(x,t)h\|_{L_t^1 L_x^2} \tag{5.3.80}$$

$$\left\| \int_0^\infty e^{isA} \left( M_2(h,x,t) \right) \, ds \right\|_{L^2} \lesssim \|M_2(h,x,t)\|_{L_t^1 L_x^2} \tag{5.3.81}$$

$$\left\| \int_0^\infty e^{isA} \left( M_3(h,x,t) \right) \, ds \right\|_{L^2} \lesssim \|M_3(h,x,t)\|_{L_t^1 L_x^2} \tag{5.3.82}$$

$$\left\| \int_0^\infty e^{isA} \left( M_{1,2}(h,x,t) \right) \, ds \right\|_{L^2} \lesssim \left\| M_{1,2}(h,x,t) \right\|_{L_t^1 L_x^2} \tag{5.3.83}$$

$$\left\| \int_0^\infty e^{isA} \left( M_{1,3}(h,x,t) \right) \, ds \right\|_{L^2} \lesssim \left\| M_{1,3}(h,x,t) \right\|_{L_t^1 L_x^2} \tag{5.3.84}$$

Applying Hölder's inequality and Strichartz estimates, we can estimate

$$\|a(x,t)h\|_{L_t^1 L_x^2} \lesssim \|a\|_{L_t^{\frac{5}{4}} L_x^{\frac{5}{2}}} \|h\|_{L_t^5 L_x^{10}} \lesssim \|h\|_S , \tag{5.3.85}$$

$$\|M_2(h,x,t)\|_{L_t^1 L_x^2} \lesssim \|h\|_{L_t^3 L_x^{18}} \lesssim \|h\|_S \,, \tag{5.3.86}$$

$$\|M_3(h,x,t)\|_{L_t^1 L_x^2} \lesssim \|h\|_{L_t^4 L_x^{12}} \lesssim \|h\|_S \,, \tag{5.3.87}$$

and

$$\left\|M_{1,2}(h,x,t)\right\|_{L_t^1 L_x^2}, \left\|M_{1,3}(h,x,t)\right\|_{L_t^1 L_x^2} \lesssim \|h\|_{L_t^4 L_x^{12}} \lesssim \|h\|_S \,. \tag{5.3.88}$$

Therefore,

$$\begin{aligned}\left\|\int_0^\infty e^{isA}\left(D(h,\cdot,s)\right)\,ds\right\|_{L^2} \lesssim & \left\|\langle x\rangle^{\frac{1}{2}+\epsilon} F_1\right\|_{L^2_{t,x}} + \left\|\langle x\rangle^{\frac{1}{2}+\epsilon} F_2^S\right\|_{L^2_{t,x}} \\ & + \|F\|_{L_t^1 L_x^2} + \|h\|_S + \|h\|_S^5 \,. \end{aligned} \tag{5.3.89}$$

And hence

$$H_0[0] := H[0] - i\int_0^\infty e^{isA}\left(D(h,\cdot,s)\right)\,ds \in L^2. \tag{5.3.90}$$

We are done. □

# BIBLIOGRAPHY


[1] Agmon S. Spectral properties of Schrdinger operators and scattering theory. *Ann. Scuola Norm. Sup. Pisa Cl. Sci.* (4) 2 (1975), no. 2, 151–218.

[2] Bahouri, H. Grard, P. High frequency approximation of solutions to critical nonlinear wave equations. *Amer. J. Math.* 121 (1999), no. 1, 131–175.

[3] Beceanu, M. and Goldberg, M. Strichartz estimates and maximal operators for the wave equation in $\mathbb{R}^3$. *J. Funct. Anal.* 266 (2014), no. 3, 1476–1510.

[4] Beceanu, M. Personal communication.

[5] Beceanu, M. Structure of wave operators for a scaling-critical class of potentials. *Amer. J. Math.* 136 (2014), no. 2, 255–308.

[6] Beceanu, M. Decay estimates for the wave equation in two dimensions. *J. Differential Equations* 260 (2016), no. 6, 5378–5420.

[7] Beceanu, M. and Schlag, W. Structure formulas for wave operators. Preprint (2016), arXiv:1612.07304.

[8] Beceanu, M. and Soffer, A. The Schrdinger equation with a potential in rough motion. *Comm. Partial Differential Equations* 37 (2012), no. 6, 969–1000.

[9] Bourgain, J. *Global solutions of nonlinear Schrdinger equations. American Mathematical Society Colloquium Publications,* 46. American Mathematical Society, Providence, RI, 1999. viii+182 pp.

[10] Cai, K. Fine properties of charge transfer models.Preprint (2003), arXiv:math-ph/0311048v1.

[11] Chen, G. Strichartz estimates for charge transfer models. *Discrete Contin. Dyn. Syst.* 37 (2017), no. 3, 1201-1226.

[12] Chen, G. Wave equations with moving potentials. Preprint (2016), arXiv:1610.09586.

[13] Chen, G. Klein-Gordon equation with charge transfer Hamiltoian and multi-solitons.

[14] Chen, G. Strichartz estimates for wave equtions with charge transfer Hamiltonian. Preprint (2016), arXiv:1610.05226.

[15] Chen, G. Multisolitons for the defocusing energy critical wave equation with potentials. Preprint (2017).

[16] Chen, G. Non-divergence parabolic equations of second order with critical drift in Lebesgue spaces. *J. Differential Equations.* 262 (2017), no. 3, 24142448.

[17] Chen, G. Non-divergence parabolic equations of second order with critical drift in Morrey spaces. Preprint(2014), arXiv:1602.00819.

[18] Chen, G. and Safonov, M.V. On second order elliptic and parabolic equations of mixed type. *J. Functional Analysis* 272 (2017), vol. 8, 32163237.

[19] Côte, R. and Martel, Y. Multi-travelling waves for the nonlinear klein-gordon equation. Preprint (2016), arXiv:1612.02625.

[20] Côte, R. and Muñoz, C. Multi-solitons for nonlinear Klein-Gordon equations. *Forum Math. Sigma* 2 (2014), e15, 38 pp.

[21] Cuccagna, S. Stabilization of solutions to nonlinear Schrdinger equations. *Comm. Pure Appl. Math.* 54 (2001), no. 9, 1110–1145.

[22] Cuccagna, S. and Maeda, M. On weak interaction between a ground state and a non–trapping potential, *J. Differential Equations* 256 (2014), no. 4, 1395–1466.

[23] Deng, Q., Soffer, A. and Yao X. Endpoint Strichartz estimates for charge transfer Hamiltonians. preprint (2015), arXiv:1507.03870.

[24] Duyckaerts, T., Kenig, C. and Merle, F.. Classification of radial solutions of the focusing, energy-critical wave equation. *Camb. J. Math.* 1 (2013), no. 1, 75–144.

[25] Duyckaerts, T., Jia, H., Kenig, C. and Merle, F. Soliton resolution along a sequence of times for the focusing energy critical wave equation. Preprint (2016), arXiv:1601.01871.

[26] Erdogan, M. B. and Schlag, W. Dispersive estimates for Schrödinger operators in the presence of a resonance and/or an eigenvalue at zero energy in dimension three: II. *J. Anal. Math.* 99 (2006), 199–248.

[27] Graf, J. M. Phase space analysis of the charge transfer model. *Helv. Physica Acta* 63 (1990), 107–138.

[28] Ginibre, J. and Velo, G. Generalized Strichartz inequalities for the wave equation. *J. Funct. Anal.* 133 (1995), no. 1, 50–68.

[29] Goldberg, M. and Schlag, W. Dispersive estimates for Schrdinger operators in dimensions one and three, *Comm. Math. Phys.* vol. 251, no. 1 (2004), 157-178.

[30] Jendrej, J. Construction of two-bubble solutions for energy-critical wave equations. To appear in *Amer. J. Math.*

[31] Jendrej, J. Construction of two-bubble solutions for the energy-critical NLS. Preprint (2016), arXiv:1610.01093.

[32] Jensen, A. and Kato, T. Spectral properties of Schrdinger operators and time-decay of the wave functions. *Duke Math. J.* 46 (1979), no. 3, 583–611.

[33] Jia, H., Liu, B.P and Xu, G.X. Long time dynamics of defocusing energy critical 3+1 dimensional wave equation with potential in the radial case. *Comm. Math. Phys.* 339 (2015), no. 2, 353–384.

[34] Jia, H., Liu, B.P., Schlag, W. and Xu, G.X. Generic and non-generic behavior of solutions to the defocusing energy critical wave equation with potential in the radial case. arXiv preprint (2015) arXiv:1506.04763.

[35] Journe, J.-L., Soffer, A. and Sogge, C. D. Decay estimates for Schrödinger operators. *Comm. Pure Appl. Mat.* 44 (1991), 573–604.

[36] Kato, T and Yajima, K. Some examples of smooth operators and the associated smoothing effect. *Rev. Math. Phys.* 1 (1989), no. 4, 481–496.

[37] Keel, M. and Tao, T. Endpoint Strichartz estimates. *Amer. J. Math.* 120 (1998), no. 5, 955–980.

[38] Klainerman, S. and Machedon, M. Space-time estimates for null forms and the local existence theorem. *Comm. Pure Appl. Math.* 46 (1993), no. 9, 1221–1268.

[39] Krieger J. and Schlag W. On the focusing critical semi-linear wave equation. *Amer. J. Math.* 129 (2007), no. 3, 843–913.

[40] Lawrie, A. and Schlag, W. Scattering for wave maps exterior to a ball. *Adv. Math.* 232 (2013), 57–97.

[41] Linares, L. and Ponce, G. *Introduction to nonlinear dispersive equations* Universitext. Springer, New York, 2009.

[42] Machihara, S., Nakamura, M., Nakanishi, K. and Ozawa, T. Endpoint Strichartz estimates and global solutions for the nonlinear Dirac equation. *J. Funct. Anal.* 219 (2005), no. 1, 1–20.

[43] Martel, Y. Asymptotic N-soliton-like solutions of the subcritical and critical generalized Korteweg-de Vries equations. *Amer. J. Math.* 127 (2005), no. 5, 1103–1140.

[44] Martel, Y. and Merle, F. Construction of multi-solitons for the energy-critical wave equation in dimension 5. *Arch. Ration. Mech. Anal.* 222 (2016), no. 3, 1113–1160.

[45] Merle, F. Construction of solutions with exactly k blow-up points for the Schrdinger equation with critical nonlinearity. *Comm. Math. Phys.* 129 (1990), no. 2, 223–240.

[46] Muscalu, C. and Schlag, W. *Classical and multilinear harmonic analysis. Vol. I. Cambridge Studies in Advanced Mathematics*, 138. Cambridge University Press*, Cambridge,* 2013. xvi+324 pp.

[47] Nakanishi, K. and Schlag, W. *Invariant manifolds and dispersive Hamiltonian evolution equations. Zurich Lectures in Advanced Mathematics.* European Mathematical Society (EMS), Zrich, 2011. vi+253 pp.

[48] O'Neil, R. Convolution operators and L(p,q) spaces. *Duke Math. J.* 30 (1963) 129–142.

[49] Oh, S-J. A reversed Strichartz Estimate in $\mathbb{R}^{1+2}$.

[50] Rodnianski, I. and Schlag, W. Time decay for solutions of Schrdinger equations with rough and time-dependent potentials. *Invent. Math.* 155 (2004), no. 3, 451–513.

[51] Rodnianski, I., Schlag, W. and Soffer, A. Dispersive analysis of charge transfer models. *Comm. Pure Appl. Math.* 58 (2005), no. 2, 149–216.

[52] Rodnianski, I., Schlag, W. and Soffer, A. Asymptotic stability of N-soliton states of NLS. preprint (2003), arXiv preprint math/0309114.

[53] Schlag, W. Dispersive estimates for Schrdinger operators: a survey. *Mathematical aspects of nonlinear dispersive equations,* 255–285, Ann. of Math. Stud., 163, Princeton Univ. Press, Princeton, NJ, 2007.

[54] Tao, T. *Nonlinear dispersive equations. Local and global analysis. CBMS Regional Conference Series in Mathematics*, 106. Published for the Conference Board of the Mathematical Sciences, Washington, DC; by the American Mathematical Society, Providence, RI, 2006. xvi+373 pp.

[55] Yajima, K. The $W^{k,p}$ continuity of wave operators for Schrdinger operators. *J. Math. Soc. Japan* 47 (1995), no.$\tilde{3}$, 551–581.

[56] Yajima, K. A multichannel scattering theory for some time dependent Hamiltonians, charge transfer problem. *Comm. Math. Phys.* 75 (1980), no. 2, 153–178