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Uniformly expanding random walks on manifolds

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Abstract

In this paper we construct uniformly expanding random walks on smooth manifolds. Potrie showed that given any open set U of $\mathrm{Diff}^\infty_{\mathrm{vol}}(\mathbb{T}^2)$, there exists an uniformly expanding random walk μ supported on a finite subset of U. In this paper we extend those results to closed manifolds of any dimension, building on the work of Potrie and Chung to build a robust class of examples. Adapting to higher dimensions, we work with a new definition of uniform expansion that measures volume growth in subspaces rather than norm growth of single vectors.

Keywords: dynamical systems, uniform expansion, smooth dynamics, dynamics

Mathematics Subject Classification numbers: 37Cxx

1. Introduction

Let M be a closed smooth Riemannian manifold of dimension d, and denote by $\mathrm{Diff}^\infty(M)$ the set of smooth diffeomorphisms of M. We define a random walk on M by a probability measure μ on $\mathrm{Diff}^\infty(M)$; each step of the walk is determined by a random diffeomorphism with distribution μ. In this paper we will discuss a new class of random walks called *uniformly expanding*, and construct a broad set of examples of such walks. Uniform expansion has received quite a bit of attention in recent years, and we will explain some of the motivation for this after stating our main theorem.

Definition 1.1. We say a probability measure μ on $\mathrm{Diff}^{\infty}(M)$ is *uniformly expanding* if there exists a $C > 0, N \in \mathbb{N}$ such that for all $x \in M$, and $v \in T_x^1 M$, we have[1]

$$\int_{\mathrm{Diff}^{\infty}(M)} \log ||D_x f(v)|| \mathrm{d}\mu^{(N)}(f) > C.$$

This property tells us that the dynamical system is, on average, expanding everywhere. Note that uniform expansion is a finitary condition (see lemma 6.4).

We begin with a few natural questions—how common are uniformly expanding random walks, and in what settings can they exist? The answer is not obvious, especially when we restrict to finitely supported random walks. For example, if $\mu = \delta_{f_0}$ for $f_0 \in \mathrm{Diff}^{\infty}_{\mathrm{vol}}(M)$, the random walk with law given by μ is never uniformly expanding, i.e. a purely deterministic conservative system cannot be uniformly expanding. It is then surprising that they prove to be common when $d = 2$; Potrie [1] showed that given any open set U of $\mathrm{Diff}^{\infty}_{\mathrm{vol}}(\mathbb{T}^2)$, there exists an uniformly expanding random walk μ supported on a finite subset of U.

In this paper, we demonstrate the abundance of finitely supported uniformly expanding random walks in smooth dynamical settings of arbitrary dimension, building on the work of Potrie [1]. Additionally, we work with the stronger version of uniform expansion, that is, uniform expansion in dimension k:

Definition 1.2. Fix $k \in \{1, \ldots, d-1\}$. We say μ is *uniformly expanding in dimension k* if there exists a $C > 0, N \in \mathbb{N}$ such that for all $x \in M$, and $v \in \Lambda^k T_x M$ such that $||v|| = 1$ and $v = v_1 \wedge \cdots \wedge v_k$ for $v_1, \ldots, v_k \in T_x M$ independent, we have

$$\int_{\mathrm{Diff}^{\infty}(M)} \log ||D_x f(v)|| \mathrm{d}\mu^{(N)}(f) > C.$$

If μ is uniformly expanding in dimension k for all $k \in \{1, \ldots, d-1\}$, we will call μ *uniformly expanding in all dimensions*. This extension allows us to characterize volume growth in subspaces of TM in addition to the norm growth of single vectors—a natural generalization in higher dimensions.

This leads us to the main theorem of the paper:

Theorem 1.3. *Let M be a closed smooth Riemannian manifold of dimension d. For any open $U \subset \mathit{Diff}^{\infty}_{vol}(M)$, there is a finitely supported measure μ on $\mathit{Diff}^{\infty}_{vol}(M)$ such that $\mathit{supp}(\mu) \subset U$, μ is uniformly expanding in all dimensions in the sense of definition 1.2, and there is no finite μ-invariant subset of M.*

A consequence of uniform expansion is positivity of top Lyapunov exponent. Using proposition 2.1, we obtain:

Corollary 1.4. *For any open $U \subset \mathit{Diff}^{\infty}_{vol}(M)$, there is a finitely supported measure μ on $\mathit{Diff}^{\infty}_{vol}(M)$ such that $\mathit{supp}(\mu) \subset U$, and the top Lyapunov exponent of μ with respect to any stationary measure is positive.*

One motivation for constructing uniformly expanding random walks is the study of stationary and invariant measures. In the case of a single iterate dynamical system, a classic theorem of Krylov and Bogolyubov proves there must be at least one invariant measure on M. Such measures often inform an understanding of the dynamical system, characterizing various properties.

[1] Here we have $\mu^{(N)} := \mu * \mu * \cdots * \mu$ (N times).

In particular, we can construct an invariant measure supported on a subset of the orbit closure of any point $x \in M$, and so understanding invariant measures furthers our understanding of orbits closures as well.

In the random walk setting, however, invariant measures may fail to exist, and measures that are invariant on average—i.e. μ-stationary—take their place. Kukutani showed that μ-stationary measures always exist. Stationary measures can be thought of as a weaker analogue of invariant measures, and they serve many of the same purposes. Thus, it is useful to study when a given stationary measure is genuinely invariant, and when invariant measures exist in a random dynamical system at all.

When all the stationary measures of a system are invariant, we call the system *stiff*. This notion was introduced by Furstenberg in [2], where he studies this property in the homogeneous setting. In smooth dynamics, this question has had very recent developments, some of which are detailed below.

Recent work of Chung [3] in the conservative setting—using the work of Brown–Rodriguez Hertz [4]– showed that all uniformly expanding random walks are stiff when $d = 2$. In combination with our work, we obtain the following corollary:

Corollary 1.5. *Let M be a surface. For μ as in theorem 1.3, volume is the only μ-stationary measure.*

Proof. Let ν be a μ-stationary measure. By proposition 3.1 of [3], ν is either volume, or supported on a finite set of points. By theorem 1.3, there is no finite μ-invariant subset of M, and as ν is μ-stationary, we know $\mathrm{supp}(\nu)$ must be μ-invariant. Thus, we can conclude that ν is volume. □

In higher dimensions, the ongoing work of Brown, Eskin, Filip, and Rodriguez Hertz (*in prep*, [5]) provides an analogous stiffness result. Here, $\mathrm{Diff}^{\infty}_{\mathrm{vol}}(M)$ denotes the set of volume preserving smooth diffeomorphisms of M.

Theorem A ([5]). *Let μ be uniformly expanding in all dimensions, $supp(\mu) \subseteq Diff^{\infty}_{vol}(M)$. Then any μ-stationary measure on M is μ-invariant.*

Thus we see that uniformly expanding random walks form a new class of stiff actions, far removed from the homogeneous setting. An immediate corollary, using the main theorem of this work, is the following:

Corollary 1.6. *Let M be a closed smooth Riemannian manifold of dimension d. For any open $U \subset Diff^{\infty}_{vol}(M)$, there is a finitely supported measure μ on $Diff^{\infty}_{vol}(M)$ such that $supp(\mu) \subset U$ and the action of the random walk defined by μ is stiff.*

Remark. As a bookkeeping comment: only corollary 1.6 depends on the as-of-writing unpublished work of Brown, Eskin, Filip, and Rodriguez Hertz. All proofs below are independent of that work.

2. Setup and rationale

The structure of the paper is as follows: in section 2, we recall some classical results, and formulate a criterion for detecting measures that are not uniformly expanding. In section 3, we will use an invariance principle to sharpen this criterion and describe it in terms of non-random algebraic structures. In sections 4 and 5, we will lay out a concrete construction of a uniformly expanding measure. Finally, in section 6 we will use the tools of section 3 to conclude.

For the remainder of the paper, let M be a closed smooth manifold of dimension d with Riemannian volume form ω. The measure induced by ω will be denoted vol_M. We let μ be a probability measure on $\mathrm{Diff}^\infty_{\mathrm{vol}}(M)$ with bounded support, and consider the random walk on M defined by the law μ. For the time being, this is an unspecified measure—a more concrete construction will be provided later, and we will also denote this measure μ.

One of our primary objects of study will be the stationary measures of μ. We recall the definition here.

Definition 2.1. Let ν be a Borel probability measure on M. We call ν *μ-stationary* if $\nu = \mu * \nu$ in the sense of convolution of measures. A stationary measure ν is ergodic if given any ν measurable set $A \subset M$ such that $\nu(f(A)\Delta A) = 0$ for μ-a.e. f, then $\nu(A) = 0$ or 1.[2]

It is worth noting that μ-stationary measures can also be characterized as follows: let $\sigma : \mathrm{Diff}^\infty(M)^{\mathbb{N}} \to \mathrm{Diff}^\infty(M)^{\mathbb{N}}$ be the left shift map, i.e. $\sigma(\alpha)_i = f_{i+1}$ for $i \in \mathbb{N}$, $\alpha = (f_0, f_1, \ldots) \in \mathrm{Diff}^\infty(M)^{\mathbb{N}}$.[3] Then the μ-stationary measures on M are precisely the measures ν such that $\mu^{\mathbb{N}} \times \nu$ is invariant under the skew product map

$$T\colon \mathrm{Diff}^\infty(M)^{\mathbb{N}} \times M \to \mathrm{Diff}^\infty(M)^{\mathbb{N}} \times M \text{where } T(\alpha, x) = (\sigma(\alpha), f_0(x)).$$

See [6] for details. This is a non-random dynamical system that characterizes μ, ν, and the relationship between them. We will study it often in the remainder of the paper.

Finally, we set out the relevant spaces for our dynamics. In addition to the tangent bundle TM, here we will also study the change of k-dimensional subspaces of the tangent bundle for $k \in \{1, \ldots, d-1\}$. Thus, we extend the action of our dynamics to the kth Grassmanian of T_xM, denoted $\mathbf{Gr}^k(T_xM)$. Under the Plücker Embedding, any element of $\mathbf{Gr}^k(T_xM)$ can be viewed as a decomposable element of $\mathbb{P}(\Lambda^k T_xM)$, the projectivized kth exterior power of TM. Write v for both an element of $\mathbb{P}(\bigwedge^k TM)$ and for an element of $\bigwedge^k TM$ that represents it. Here, decomposable means that it can be written as a single wedge product—such elements span $\Lambda^k T_xM$, and precisely make up the image of $\mathbf{Gr}^k(T_xM)$ under the Plücker Embedding. It is often be useful to move between $\mathbf{Gr}^k(TM)$ and its image in $\mathbb{P}(\bigwedge^k TM)$, and we will use them interchangeably. We denote the full Grassmannian bundle over M by $\mathbf{Gr}^k(TM)$– at each point $x \in M$, the fiber is given by $\mathbf{Gr}^k(T_xM)$.

Similarly, the bundle of conformal structures of TM over M will be denoted by $\mathbf{CS}(TM)$, and at each point $x \in M$ the fiber is denoted by $\mathbf{CS}(T_xM)$, the conformal structures on T_xM, i.e. the set of all possible positive definite inner products up to rotation and scaling. More formally,

$$\mathbf{CS}(T_xM) = \{\text{Riemannian metrics on } T_xM\} / \sim \ ,$$

where two metrics g_1, g_2 are equivalent iff $g_1 = Ag_2$ for a conformal linear transformation A. As M has a Riemannian metric denoted by $\langle \cdot, \cdot \rangle$, each fiber $\mathbf{CS}(T_xM)$ can be identified with the orbits of $\mathcal{S}_d$ under multiplication by $\mathcal{C}_d$, where $\mathcal{S}_d$ is the set of $d \times d$ symmetric positive definite matrices and

$$\mathcal{C}_d = \left\{A \in \mathrm{GL}_d(\mathbb{R}) \mid A^\top A = c^2 I_d, \text{ for some } c > 0\right\}.$$

[2] Here, Δ is symmetric difference.

[3] The ith coordinate of $\alpha \in \mathrm{Diff}^\infty(M)^{\mathbb{N}}$ is denoted by α_i.

Note that the standard norm of a decomposable vector $v = v_1 \wedge \cdots \wedge v_k$ in $\Lambda^k T_x M$ is given by the determinant of the Gram matrix, $[\langle v_i, v_j \rangle]_{ij}$. This is the k-volume of the parallelotope spanned by $v_1, \ldots, v_k$; studying its change is an intuitive measurement of expansion or contraction of our dynamical system. In this setting, given $f \in \mathrm{Diff}^\infty(M)$ and $x \in M$, we extend the action of $D_x f \in \mathrm{Hom}(T_x M, T_{f(x)} M)$ to $D_x f \in \mathrm{Hom}(\Lambda^k T_x M, \Lambda^k T_{f(x)} M)$ by

$$D_x f(v_1 \wedge \cdots \wedge v_k) = D_x f v_1 \wedge \cdots \wedge D_x f v_k.$$

Let $k \in \{1, \ldots, d-1\}$, ν be an ergodic μ-stationary measure on M. Given an $f \in \mathrm{Diff}^\infty(M)$, we define $F_f^k \colon \bigwedge^k TM \to \bigwedge^k TM$ by $F_f^k(x,v) = (f(x), D_x f(v))$. We will suppress the k when it is clear what exterior power we are considering. Consider the cocycle

$$A_k \colon \mathrm{Diff}^\infty(M)^{\mathbb{N}} \times \bigwedge^k TM \to \mathrm{Diff}^\infty(M)^N \times \bigwedge^k TM$$

$$A_k(\alpha, (x,v)) = (\sigma(\alpha), F_{f_0}(x,v)).$$

By Oseledets' theorem [7], $\exists d_k \in \{1, \ldots, \binom{d}{k}\}$ such that for ν-a.e. $x \in M$ and $\mu^{\mathbb{N}}$-a.e. $\alpha \in \mathrm{Diff}^\infty(M)^{\mathbb{N}}$, there is a flag of subspaces $\Lambda^k T_x M = V_1(x,k) \supset \cdots \supset V_{d_k}(x,k)$ and a set of corresponding $\lambda_1(\nu,k) > \cdots > \lambda_{d_k}(\nu,k)$ such that for any $v \in V_i(x,k) \backslash V_{i+1}(x,k)$ with $||v|| = 1$,

$$\lim_{n\to\infty} \frac{1}{n} \log ||D_x f_\alpha^n v|| = \lambda_i(\nu,k).$$

Here ${f^n}_\alpha = f_{n-1} \circ \cdots \circ f_0$. We will call $\lambda_i(\nu,k)$ the ith Lyapunov exponent of A_k with respect to ν, or simply the Lyapunov exponent of ν when the cocycle is clear.

We say two cocycles $B, A_k \colon \mathrm{Diff}^\infty(M)^{\mathbb{N}} \times \bigwedge^k TM \to \mathrm{Diff}^\infty(M)^N \times \bigwedge^k TM$ are Lyapunov cohomologous by a transfer cocycle $C \colon \mathrm{Diff}^\infty(M)^{\mathbb{N}} \times \bigwedge^k TM \to \mathrm{Diff}^\infty(M)^N \times \bigwedge^k TM$ if

$$B_{(\alpha,x)} = C^{-1}_{T(\alpha,x)} (A_k)_{(\alpha,x)} C_{(\alpha,x)}$$

where $B_{(\alpha,x)} \colon \Lambda^k T_x M \to \Lambda^k T_{f0(x)} M]$ is the associated map between fibers of the cocycle B. Note that this conjugation does not lose the information given by Oseledets' Theorem, as we may choose C so that the Lyapunov exponents of B and A_k agree (see [8] proposition 8.2).

Noting that part of proposition 3.17 of [3] can easily be extended to manifolds of any dimension, we can formulate a criterion for verifying that μ is uniformly expanding in all dimensions in terms of μ-stationary measures on $\mathbf{Gr}^k(TM)$. We state the extended theorem and adjusted proof here for clarity. It is worth noting that the core adjustment is the switch from $T^1 M$ to $\mathbf{Gr}^k(TM)$.

Theorem 2.2. *If the measure μ is not uniformly expanding in all dimensions, then for some $k \in \{1, \ldots, d-1\}$, there is an ergodic μ-stationary measure η on $\mathbf{Gr}^k(TM)$ that has non-positive top Lyapunov exponent on $\bigwedge^k TM$.*

Proof. Assume that μ is not uniformly expanding in all dimensions, and fix $\epsilon > 0$. Then $\exists k \in \{1, \ldots, d-1\}$ such that for all $n \in \mathbb{N}$, there exists $(x_n, v_n) \in \mathbb{P}(\bigwedge^k T_{x_n} M)$ such that v_n is decomposable and

$$\int \log ||D_{x_n} f(v_n)|| d\mu^{(n)}(f) < \epsilon.$$

We will construct a limit of finite measures supported on the orbits of the points (x_n, v_n) to build a measure η that has non-positive exponent. For all n, define

$$\eta_n := \frac{1}{n}\sum_{m=0}^{n-1}\mu^{(m)} * \delta_{(x_n,v_n)} = \frac{1}{n}\sum_{m=0}^{n-1}\int \delta_{F_f(x_n,v_n)} d\mu^{(m)}(f).$$

Now, we evaluate the following to estimate how far from μ-stationary η_n is.

$$\begin{aligned}\mu * \eta_n - \eta_n &= \frac{1}{n}\sum_{m=0}^{n-1}\left[\int \delta_{F_f(x_n,v_n)} d\mu^{(m+1)}(f) - \int \delta_{F_f(x_n,v_n)} d\mu^{(m)}(f)\right] \\ &= \frac{1}{n}\left[\int \delta_{F_f(x_n,v_n)} d\mu^{(n)}(f) - \delta_{(x_n,v_n)}\right].\end{aligned}$$

And so we have,

$$\lim_{n\to\infty} \mu * \eta_n - \eta_n = 0.$$

Take η to be a weak-$*$ limit of η_n. By the above we see that η is a μ-stationary measure on $\mathbf{Gr}^k(TM)$, and also induces a μ-stationary measure on $\mathbb{P}(\bigwedge^k TM)$ by the pushforward of η.

Define $\Phi\colon \mathrm{Diff}^\infty(M) \times \mathbf{Gr}^k(TM) \to \mathbb{R}$ by $\Phi(g,(x,v)) = \log||D_x g(v)||$. Note that for $\alpha = (f_0, f_1, \ldots)$ we can write

$$\log||D_x f_\alpha^n(v)|| \leqslant \sum_{m=0}^{n-1}\Phi\left(f_m, F_{f_\alpha^m}(x,v)\right)||.$$

We can separate the variables f_m and f_α^m in integration, and conclude the following for any (x,v):

$$\begin{aligned}\int \log||D_x f(v)||\, d\mu^{(n)}(f) &= \int \log||D_x f_\alpha^n(v)||\, d\mu^{\mathbb{N}}(\alpha) \\ &\leqslant \sum_{m=0}^{n-1}\int \Phi\left(f_m, F_{f_\alpha^m}(x,v)\right)\, d\mu^{\mathbb{N}}(\alpha) \\ &\leqslant \sum_{m=0}^{n-1}\int \Phi(g, F_f(x,v))\, d\mu(g)\, d\mu^{(m)}(f).\end{aligned}$$

Therefore, for any $n \in \mathbb{N}$, we have

$$\begin{aligned}\int\int \log||D_x g(v)||d\mu(g)\, d\eta_n(x,v) &= \int\int \Phi(g,(x,v))\, d\eta_n(x,v)\, d\mu(g) \\ &\leqslant \frac{1}{n}\sum_{m=0}^{n-1}\int\int \Phi(g, F_f(x_n,v_n))\, d\mu^{(m)}(f)\, d\mu(g) \\ &\leqslant \frac{1}{n}\int \log||D_{x_n} f(v_n)||d\mu^{(n)}(f) \\ &\leqslant \frac{\epsilon}{n}.\end{aligned}$$

By weak-$*$ convergence and the continuity of log, we see

$$\int\int \log||D_x g(v)||\, d\mu(g)\, d\eta(x,v) \leqslant 0.$$

Possibly passing to an ergodic component of η that preserves the above, we can assume η is an ergodic μ-stationary measure on $\mathbf{Gr}^k(TM)$ such that $\int\int \log||D_x g(v)||\, d\mu(g)\, d\eta(x,v) \leqslant 0$. Define T: $\mathrm{Diff}^\infty(M)^{\mathbb{N}} \times \mathbf{Gr}^k(TM)$ by $T(\alpha,(x,v)) = (\sigma(\alpha), f_0(x,v))$. As η is ergodic and μ-stationary, we know $\mu^{\mathbb{N}} \times \eta$ is an ergodic T-invariant measure on $\mathrm{Diff}^\infty(M)^{\mathbb{N}} \times \mathbf{Gr}^k(TM)$. Thus, by the Birkhoff Ergodic Theorem, for $\mu^{\mathbb{N}} \times \eta$-a.e. $(\alpha,(x,v))$, we have

$$\begin{aligned}\lim_{n\to\infty} \frac{1}{n}\sum_{m=0}^{n-1} \Phi(f_0, f^n{}_\alpha(x,v)) &= \int\int \Phi(f_0,(x,v))\; d\mu^{\mathbb{N}}(\alpha)\; d\eta(x,v)\\ &= \int\int \log||D_x f(v)||\, d\mu(f)\; d\eta(x,v).\end{aligned}$$

By the construction of η, this equality is non-positive, and so we can conclude

$$\lim_{n\to\infty} \frac{\log||D_x f^n_\alpha(v)||}{n} \leqslant 0.$$

This directly shows that the Lyapunov exponents of η are non-positive, and thus the theorem is proved. □

Definition 2.3. Let ν be a μ-stationary measure on M, $\pi\colon \mathbf{Gr}^k(TM) \to M$ be projection on to M. We say $E \subseteq \mathbf{Gr}^k(TM)$ is a *μ-invariant ν-measurable algebraic structure in* $\mathbf{Gr}^k(TM)$ if:

(1) E_x is an algebraic subset[4] of $\mathbf{Gr}^k(T_xM)$ for ν-a.e $x \in M$, where $E_x = E \cap \pi^{-1}(x)$ is the fiber of E over x, and
(2) for μ-a.e. $f \in \mathrm{Diff}^\infty(M)$, ν-a.e. $x \in M$, $E_{f(x)} = D_x f E_x$.

Similarly, we say C is a *μ-invariant ν-measurable conformal structure on TM* if $C\colon M \to \mathrm{CS}(TM)$ is a ν-measureable map and for ν-a.e. $x \in M$, μ-a.e. $f \in \mathrm{Diff}^\infty(M)$, we have $C_{f(x)} = D_x f C_x$, where $C_x = C(x)$.

Corollary 2.4. *If μ is not uniformly expanding in all dimensions, then there is an ergodic μ-stationary measure ν on M, a $k \in \{1,\ldots,d-1\}$, and a μ-invariant ν-measurable algebraic structure of* $\mathbf{Gr}^k(TM)$*– denoted E– such that the image of E_x in $\bigwedge^k TM$ is contained in the non-positive Lyapunov subspaces of ν on $\bigwedge^k TM$. Further, if $\lambda_1(\nu,k) > 0$, then E is proper.*

Proof. From theorem 2.2, we know there is an ergodic μ-stationary measure η on $\mathbf{Gr}^k(TM)$ that has non-positive top Lyapunov exponent on $\bigwedge^k TM$ for some $k \in \{1,\ldots,d-1\}$. Let $\nu := \pi_*\eta$. Then ν is an ergodic μ-stationary measure on M as π is equivariant and η is ergodic and μ-stationary. This corollary is essentially asking when η has full support on fibers.

Let η_x be the disintegration of η along fibers over $x \in M$, and define $V_x := \mathrm{supp}(\eta_x)$, $V := \cup_{x\in M} V_x$, and $V' := \cup_{x\in M}\mathrm{span}(V_x)$. Note that as V is f-invariant for μ-a.e. $f \in \mathrm{Diff}^\infty(M)$, we see that $V \subseteq \mathbf{Gr}^k(TM)$ defines a μ-a.e. f-invariant closed subset of $\mathbf{Gr}^k(TM)$. V' forms a

[4] By algebraic we mean 'defined by polynomial equations'.

similarly invariant subbundle of $\bigwedge^k TM$. As V' is an invariant subbundle, there is a well-defined restriction of A_k to V', and Lyapunov exponents with respect to ν on V' are non-positive.

Let

$$E := V' \cap \mathbf{Gr}^k (TM).$$

As the fibers of V' are subspaces, they are algebraic. Thus E is a μ-invariant ν-measurable algebraic structure in $\mathbf{Gr}^k(TM)$ with non-positive Lyapunov exponents.

Finally, if ν has positive top Lyapunov exponent, then V is a proper subset of $\mathbf{Gr}^k(TM)$, and so V' must be a proper subbundle of $\bigwedge^k TM$ and E must be a proper subset of $\mathbf{Gr}^k(TM)$, as decomposable elements span $\bigwedge^k TM$. □

Finally, we include a proof of positivity of top Lyapunov exponent. This is nearly identical to the proof of proposition 2.2 in [3], with minor adjustments for higher dimensions. We have included it here for completeness.

Proposition 2.1. *Let μ be uniformly expanding in dimension k with compact support. Then $\exists c > 0$ such that $\lambda_1(\nu,k) > c$ for any ergodic μ-stationary measure ν.*

Proof. For the following proof, we will borrow some notation from probability theory. By assumption, we know that $\exists C > 0, N \in \mathbb{N}$ such that for all $x \in M$, $v \in \mathbb{P}(\Lambda^k T_x M)$,

$$\int_{\mathrm{Diff}^\infty(M)} \log ||D_x f(v)|| d\mu^{(N)}(f) > C.$$

We will let $c = C/N$. Fix $x \in M$, $v \in \mathbb{P}(\Lambda^k T_x M)$. For any $\alpha \in \mathrm{Diff}^\infty(M)^{\mathbb{N}}$, we define $v_n = v_n(\alpha) := D_x f^n{}_\alpha v$ and $x_n = x_n(\alpha) := f^n{}_\alpha(x)$.

For $j \in \mathbb{N}$, we will consider the random variable

$$X_j = X_j(\alpha) := \log ||v_{jN}|| - \log ||v_{(j-1)N}||.$$

Intuitively, this is the change of $||v_n||$ after N steps of the random walk. We want to show this is uniformly bounded from below for $\mu^{\mathbb{N}}$-a.e. α. Let $\mathcal{F}_j$ be the σ-algebra on $\mathrm{Diff}^\infty(M)^{\mathbb{N}}$ that is generated by the cylinder sets on the first jN coordinates. By construction, X_j is $\mathcal{F}_j$ measurable for all j, as they do not depend on the coordinates past jN. Define the random variables $Z_j = Z_j(\alpha) := X_j - \mathbb{E}[X_j \mid \mathcal{F}_{j-1}]$, where this is the conditional expectation of X_j given $\mathcal{F}_{j-1}$. We denote the sum of the Z_j by S_ℓ, i.e. $S_\ell = \sum_{j=1}^{\ell} Z_j$.

Notice that for any $\ell \in \mathbb{N}$,

$$\log ||v_{\ell N}|| = \sum_{j=1}^{\ell} X_j = S_\ell + \sum_{j=1}^{\ell} \mathbb{E}[X_j \mid \mathcal{F}_{j-1}].$$

Note that $\mathbb{E}[X_j \mid \mathcal{F}_{j-1}] = \mathbb{E}[\log ||v_{jN}|| \mid \mathcal{F}_{j-1}] - \log ||v_{(j-1)N}||$

This tells us that to understand the change of the norm after ℓN steps, we can shift to studying S_ℓ and the corresponding sum of conditional expectations. The key observation here is that the conditional expectations are integrals against $\mu^{(N)}$, and are thus bounded from below as μ is uniformly expanding.

We only need to compute $\mathbb{E}[\log ||v_{jN}|| \mid \mathcal{F}_{j-1}]$, as $\mathbb{E}[X_j \mid \mathcal{F}_{j-1}] = \mathbb{E}[\log ||v_{jN}|| \mid \mathcal{F}_{j-1}] - \log ||v_{(j-1)N}||$. Notice that this is an integral against $\mu^{\mathbb{N}}$, but that the random variable does not depend on anything past the jNth coordinate, and the information that is not given by $\mathcal{F}_{j-1}$ is only the $(j-1)N$ thru jNth coordinates. Thus,

$$\mathbb{E}\left[\log||v_{jN}|| \mid \mathcal{F}_{j-1}\right] = \int \log||D_{x_{(j-1)N}}f\left(v_{(j-1)N}\right)||\,\mathrm{d}\mu^{(N)}(f)$$

and so

$$\mathbb{E}\left[X_j \mid \mathcal{F}_{j-1}\right] = \int \log||D_{x_{(j-1)N}}f\left(\frac{v_{(j-1)N}}{||v_{(j-1)N}||}\right)||\,\mathrm{d}\mu^{(N)}(f).$$

Therefore for any j, we have

$$\mathbb{E}\left[X_j \mid \mathcal{F}_{j-1}\right] \geqslant C.$$

Finally, note that S_ℓ is a martingale—

$$\begin{aligned}\mathbb{E}\left[S_{\ell+1} \mid \mathcal{F}_\ell\right] &= \mathbb{E}\left[S_\ell + Z_{\ell+1} \mid \mathcal{F}_\ell\right] = S_\ell + \mathbb{E}\left[X_{\ell+1} - \mathbb{E}\left[X_{\ell+1} \mid \mathcal{F}_\ell\right] \mid \mathcal{F}_\ell\right] \\ &= S_\ell + X_{\ell+1} - \mathbb{E}\left[\left[X_{\ell+1} \mid \mathcal{F}_\ell\right] \mid \mathcal{F}_\ell\right] = S_\ell.\end{aligned}$$

As μ is compactly supported, S_ℓ is square integrable, and so by the Strong Law of Large Numbers for square integrable martingales we have $S_\ell/\ell \to 0$ as $\ell \to \infty$ for $\mu^{\mathbb{N}}$-a.e. choice of α.

Thus, taking $\ell = \lfloor \frac{n}{N} \rfloor$, we see that for $\mu^{\mathbb{N}}$-a.e. α,

$$\liminf_{n\to\infty} \frac{\log||D_x f^n{}_\alpha(v)||}{n} \geqslant \lim_{n\to\infty} \frac{S_\ell}{n} + \lim_{n\to\infty} \frac{1}{n}\sum_{j=1}^{\ell} \mathbb{E}\left[X_j \mid \mathcal{F}_{j-1}\right] \geqslant \frac{C}{N} = c.$$

Given any ergodic μ-stationary measure ν, $\lambda_1(\nu,k) \geqslant \liminf_{n\to\infty} \frac{\log||D_x f^n{}_\alpha(v)||}{n}$, and we may conclude. □

3. Reduction to studying Invariance

In this section, we will discuss how to reduce the problem of establishing uniform expansion to the study of invariant structures on $\mathbf{Gr}^k(TM)$. We start with a lemma.

Lemma 3.1. *Let ν be an ergodic μ-stationary measure, supp$(\mu) \subset$ Diff$^\infty_{vol}(M)$. If $\lambda_1(\nu,k) \leqslant 0$ for some $k \in \{1,\ldots,d-1\}$, then $\lambda_i(\nu,1) = 0$ for any i, i.e. the Lyapunov exponents of A_1 on TM with respect to ν are zero.*

Proof. It suffices to prove the stronger statement that $\lambda_1(\nu,1) \leqslant 0$ if and only if $\lambda_1(\nu,k) \leqslant 0$ for all $k \in \{1,\ldots,d-1\}$. As μ is supported on volume preserving diffeomorphisms, the Lyapunov exponents of A_1 with respect to ν on TM must sum to 0. On the exterior product, the top Lyapunov exponent of A_k is given by the sum of the top k Lyapunov exponents of A_1 on TM. If $\lambda_1(\nu,1) \leqslant 0$, then $\lambda_i(\nu,1) = 0$ for all $i \in \{1,\ldots,d\}$, and so $\lambda_1(\nu,k) = 0$. To prove the other direction, we show the converse: if $\lambda_1(\nu,1) > 0$, then as the Lyapunov exponents are listed in decreasing order $\sum_{i=1}^{d-1} \lambda_i(\nu,1) = -\lambda_d(\nu,1) > 0$. This means $\lambda_1(\nu,k) > 0$ for any $k \in \{1,\ldots,d-1\}$, and so all exterior powers will have positive top exponent. □

We will use a classical invariance principle of Ledrappier [9], extended by Avila-Viana [10], in the following theorem.

Theorem 3.2. *If μ is not uniformly expanding in all dimensions, then there is an ergodic μ-stationary measure ν on M such that one of the following is true:*

(a) for some $k \in \{1,\ldots,d-1\}$, there is a proper μ-a.e. invariant ν-measurable algebraic structure of $\mathbf{Gr}^k(TM)$, or
(b) there is a μ-a.e. invariant ν-measurable conformal structure on TM.

Using this theorem, we will show that all μ-stationary measures have positive top Lyapunov exponent and do not admit a non-random invariant measurable algebraic structure $\mathbf{Gr}^k(TM)$ or a conformal structure on TM. It is with this fact that we will prove our measure is uniformly expanding. If $\lambda_1(\nu,k) > 0$, corollary 2.4 tells us we are in case one. If $\lambda_1(\nu,1) = 0$, we will apply lemma 3.5.

To do this, we will first need to extend our dynamical system T from the one sided shift to the two sided shift. Let $\hat{\sigma}\colon \mathrm{Diff}^\infty(M)^{\mathbb{Z}} \to \mathrm{Diff}^\infty(M)^{\mathbb{Z}}$ be the two sided left shift, i.e. $\hat{\sigma}(\alpha)_i = f_{i+1}$ and $\hat{\sigma}^{-1}(\alpha)_i = f_{i-1}$ for any $\alpha = (\cdots, f_{-1}, f_0, f_1, \ldots)$. We define the map T as before, and its extension $\hat{T}$ by the following:

$$T\colon \mathrm{Diff}^\infty(M)^{\mathbb{N}} \times M \to \mathrm{Diff}^\infty(M)^{\mathbb{N}} \times M \qquad \hat{T}\colon \mathrm{Diff}^\infty(M)^{\mathbb{Z}} \times M \to \mathrm{Diff}^\infty(M)^{\mathbb{Z}} \times M$$
$$T(\alpha,x) = (\sigma(\alpha), f_0(x)) \qquad \hat{T}(\alpha,x) = (\hat{\sigma}(\alpha), f_0(x)).$$

On the tangent bundle, we consider the adjusted cocycle

$$\hat{A}_1\colon \mathrm{Diff}^\infty(M)^{\mathbb{Z}} \times T^1M \to \mathrm{Diff}^\infty(M)^{\mathbb{Z}} \times T^1M, \text{ where}$$
$$\hat{A}_1(\alpha,(x,v)) = \left(\hat{\sigma}(\alpha), \left(f_0(x), \frac{D_xf_0v}{||D_xf_0v||}\right)\right).$$

Lemma 3.3. *Let μ be a probability measure on $\mathrm{Diff}^\infty_{vol}(M)$, γ a μ-stationary probability measure on T^1M. Assume $\lim_{n\to\infty}\frac{1}{n}\log||D_xf^n_\alpha v|| = 0$ for γ-a.e. $(x,v) \in T^1M$, μ-a.e. α. Then $\mu^{\mathbb{Z}} \times \gamma$ is an invariant measure under $\hat{A}_1$.*

Proof. We will work with the natural extension ρ of the measure $\mu^{\mathbb{N}} \times \gamma$ on $\mathrm{Diff}^\infty(M)^{\mathbb{N}} \times M$ to the two sided shift space $\mathrm{Diff}^\infty(M)^{\mathbb{Z}} \times T^1M$. Unlike the one sided case, it is not immediate that $\mu^{\mathbb{Z}} \times \gamma$ is invariant under $\hat{A}_1$ if γ is μ-stationary[5]. Our first step will establish that $\rho = \mu^{\mathbb{Z}} \times \gamma$ under our assumptions.

We know that ρ has a unique characterization given by

$$\rho_\alpha = \lim_{n\to\infty} f_0^{-1} f_{-1}^{-1} \cdots f_{-n+1}^{-1}\gamma.$$

See example 3.13 in [10] or section 1.6 of [6] for details. As before, ρ_α is the disintegration of ρ over $\alpha \in \mathrm{Diff}^\infty(M)^{\mathbb{Z}}$. In particular, ρ is invariant under $\hat{A}_1$ and all Lyapunov exponents of ρ are zero. Note that $\mu^{\mathbb{Z}}$ is a product measure and so has a local product structure, and $\hat{A}_1$ admits s and u holonomies that are constant on the local stable and unstable laminations, respectively. Therefore, by Theorem D of [10], ρ has a disintegration that is su-invariant and varies continuously over the support of $\mu^{\mathbb{Z}}$. As the holonomies are constant on the local stable and unstable laminations, ρ must be independent of the choice of α for $\mu^{\mathbb{Z}}$-a.e. $\alpha \in \mathrm{Diff}^\infty(M)^{\mathbb{Z}}$, and thus, $\rho = \mu^{\mathbb{Z}} \times \gamma$. □

[5] In fact, this is generally not true.

Lemma 3.4. *Let μ be a probability measure on $Diff^\infty_{vol}(M)$, ν a μ-stationary probability measure on M. Assume $\lambda_1(\nu,1)=0$. Then $\mu^{\mathbb{Z}}\times\nu$ is an invariant measure under $\hat{T}$.*

Proof. This proof is identical to the proof of lemma 3.3. □

Lemma 3.5. *Let ν be an ergodic μ-stationary measure on M. If $\lambda_1(\nu,k)\leqslant 0$ for some $k\in\{1,\ldots,d-1\}$, then there is either a proper algebraic subbundle of TM or a conformal structure on TM that is μ-a.e. invariant and ν-measurable.*

Proof. Fix $k\in\{1,\ldots,d-1\}$ such that $\lambda_1(\nu,k)\leqslant 0$. By lemma 3.1, we may assume $k=1$ and all Lyapunov exponents on TM are zero. We are now in the situation to apply the invariance principle, as our extremal Lyapunov exponents agree. We will study μ-stationary measures γ on T^1M that project to ν on M– at least one such measure always exists.

By lemma 3.4, we know $\mu\times\nu$ is $\hat{T}$ invariant. As γ projects to ν on M, we know $\lim_{n\to\infty}\frac{1}{n}\log||D_xf^n_\alpha v||=0$ for γ-a.e. $(x,v)\in T^1M$. Thus by lemma 3.3, $\mu^{\mathbb{Z}}\times\gamma$ is $\hat{A}_1$ invariant. We now consider two cases, based on the structure of our measure and cocycle. We say a cocycle $B\colon \mathrm{Diff}^\infty(M)^{\mathbb{Z}}\times TM\to\mathrm{Diff}^\infty(M)^{\mathbb{Z}}\times TM$ is conformal if $B_{(\alpha,x)}$ is a conformal linear transformation for $\mu^{\mathbb{Z}}\times\nu$-a.e. $(\alpha,x)\in TM$.

Case 1: Assume $\hat{A}_1$ is cohomologous to a conformal cocycle B by a transfer cocycle C. Then we can define a measure β on $\mathrm{Diff}^\infty(M)^{\mathbb{Z}}\times T^1M$ by $\beta_{(\alpha,x)}:=C^{-1}_{(\alpha,x)}m$, where m is Lebesgue measure on $\mathbb{R}^d$, and $\beta=\int\beta_{(\alpha,x)}\,\mathrm{d}\mu^{\mathbb{Z}}\times\nu$. Note that β is invariant under $\hat{A}_1$ by construction, and as we may insist C is $\mu^{\mathbb{Z}}\times\nu$-measurable, β projects to $\mu^{\mathbb{Z}}\times\nu$ on $\mathrm{Diff}^\infty(M)^{\mathbb{Z}}\times M$. By theorem D of [10], the disintegration of β is independent of the choice of $\alpha\in\mathrm{Diff}^\infty(M)^{\mathbb{Z}}$ (but not necessarily independent of the choice of $x\in M$). Thus, for any $(\alpha,x)\in\mathrm{Diff}^\infty(M)^{\mathbb{Z}}\times M$ we have a conformal structure on TM given by $C^{-1}_{(\alpha,x)}g$, where g is the standard euclidean metric, and this conformal structure is μ-invariant and ν-measurable.

Case 2: Assume $\hat{A}_1$ is not cohomologous to a conformal cocycle. Let γ be any stationary measure of μ on T^1M that projects to ν on M. A lemma of Furstenberg (lemma 3.21 of [8]) shows that $\gamma_{(\alpha,x)}$ is supported on the union of two proper subspaces of T_xM for $\mu^{\mathbb{Z}}\times\nu$-a.e. $(\alpha,x)\in\mathrm{Diff}^\infty(M)^{\mathbb{Z}}\times M$. As $\gamma_{(\alpha,x)}$ is not dependent on α, we have that $\mathrm{supp}(\gamma_{f(x)})=D_xf\mathrm{supp}(\gamma_x)$ for μ a.e. $f\in\mathrm{Diff}^\infty(M)$ ν a.e. $x\in M$. This support is proper, and so the structure formed by $\mathrm{supp}(\gamma)$ will be a proper μ-invariant ν-measurable algebraic structure. □

Proof of theorem 3.2. By corollary 2.4 and lemma 3.1, there is an ergodic μ-stationary measure ν on M that has either $\lambda_1(\nu,k)\leqslant 0$ for all $k\in\{1,\ldots,d-1\}$, or admits a proper μ-invariant ν-measurable algebraic structure of $\mathbf{Gr}^k(TM)$ for some $k\in\{1,\ldots,d-1\}$. In the first case, lemma 3.5 shows that there is either a proper μ-a.e. invariant ν measurable algebraic structure or a conformal structure on TM. Thus the theorem is proved. □

With this reduction, we have pivoted from general criteria for uniformly expanding measures, to a specific characterization in terms of invariant structures in the dynamics. It now remains to construct our measure and show that it can have no such invariant objects. Our construction will hinge on picking a specific family of diffeomorphisms of M that are strictly compatible with volume, in the sense that they can only coexist with invariant structures if the μ-stationary measure in question is mutually singular with volume.

4. Studying invariance

In the following section we will construct a set of specific maps $g^a_x\in\mathrm{Diff}^\infty(M)$, for any $x\in M$, that depend smoothly on a. We then demonstrate several key properties of g^a_x, and use the compactness of M to restrict to a finite set of base points $x_1, \ldots, x_j \in M$. The goal of this section is to create $g^a_{x_1}, \ldots, g^a_{x_j}$ so that we may eliminate the invariant structures identified in the previous section.

Let $g^a_x \in \mathrm{Diff}^\infty(M)$ be defined as follows. Fix an $R > 1$, and let $B_h(0)$ denote the ball of radius h in $\mathbb{R}^d$ for $h > 0$. By Moser's Theorem (see 5.1.27 of [11]), for any $x \in M$ there is a neighborhood U'_x of x and a $\phi_x\colon U'_x \to B_R(0)$ such that ϕ_x is a diffeomorphism, $\phi_x(x) = 0$, and ϕ_x takes divergence free vector fields to divergence free vector fields. We then select a more restricted open neighborhood of x, denoted U_x, such that $U_x \subset U'_x$ and $\phi_x(U_x) = B_{1/2}(0)$. For fixed a, we will construct a volume preserving map ψ^a_1 on $B_R(0)$ that is affine on $B_{1/2}(0)$ and smoothly interpolates to the identity on $\partial B_R(0)$. Finally, we will construct g^a_x by conjugating ψ^a_1 on $B_R(0)$ by ϕ_x– g^a_x will be affine on U_x, and the identity outside of U'_x.

To begin, let $d' := d^2 + d - 1$. Set $a = (a_1, \ldots, a_d, a_{d+2}, \cdots a_{d^2+d}) \in \mathbb{R}^d \times \mathbb{R}^{d^2-1} = \mathbb{R}^{d'}$, $b_a := (a_1, \ldots, a_d)$ and consider

$$A'_a := \begin{bmatrix} c_a & a_{d+2} & \cdots & a_{2d} \\ a_{2d+1} & a_{2d+2} & \cdots & a_{3d} \\ \vdots & & & \vdots \\ a_{d^2+1} & \cdots & \cdots & a_{d^2+d} \end{bmatrix}$$

where $c_a = -\sum_{i=2}^d a_{id+i}$. The Lie group G of affine transformations of $\mathbb{R}^d$ is given by $(d+1) \times (d+1)$ matrices of the form $\begin{pmatrix} A & b \\ 0 & 1 \end{pmatrix}$, for $A \in \mathrm{SL}_d(\mathbb{R})$, and its action on $\mathbb{R}^d$ is given by

$$\begin{pmatrix} A & b \\ 0 & 1 \end{pmatrix} \cdot y = Ay + b.$$

We will work in the open neighborhood of the identity in $\mathfrak{g}$, the Lie algebra of G, where the exponential map has inverse given by log. Note that for small a, we may choose b'_a and A_a so that

$$\exp \begin{pmatrix} A'_a & b'_a \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} A_a & b_a \\ 0 & 1 \end{pmatrix}.$$

By the choice of c_a we know $\mathrm{tr}(A'_a) = 0$, and so $\det(A_a) = 1$. Moving forward, we will denote $\begin{pmatrix} A'_a & b'_a \\ 0 & 0 \end{pmatrix} \in \mathfrak{g}$ by X_a.

Take $X_a \in \mathfrak{g}$ for $a \in (-1,1)^{d'}$ small enough, and consider the flow defined on $\mathbb{R}^d$ given by $\phi^a_t(y) = e^{tX_a} \cdot y$. Note that at time $t = 1$, $\phi^a_1(y) = A_a y + b_a$, and as X_a is a trace free vector field, we know ϕ^a_t is a volume preserving diffeomorphism of $\mathbb{R}^d$. Without loss of generality, we choose R to be large so that $A_a(B_1(0)) + b_a \subset B_R(0)$ for any $a \in (-1,1)^{d'}$, that is, the affine transformation $\exp(X_a)$ maps $B_1(0)$ inside $B_R(0)$.

Let $\alpha_a = \iota_{X_a}\omega$. Since α_a is a closed $(d-1)$-form there is a $(d-2)$-form χ_a such that $d\chi_a = \alpha_a$. Let $f\colon B_R(0) \to [0,1]$ be a smooth bump function such that $f(y) = 1$ for $y \in B_1(0)$ and $f(y) = 0$ on a neighborhood of $\partial B_R(0)$. Then $d(f\chi_a) = \iota_{Y_a}\omega$, where Y_a is a trace free vector field that agrees with X_a on $B_1(0)$ and vanishes on a neighborhood of $\partial B_R(0)$. Let $\psi^a_1(y)$ be the time one map of the flow of $d(f\chi_a)$ for small a. This map will agree with ϕ^a_1 on $B_1(0)$, and the identity on $\partial B_R(0)$. Further, it is volume preserving for any given a by the above.

Define $g_x^a(y) = \phi_x^{-1} \circ \psi_1^a \circ \phi_x(y)$ for any $y \in U_x$. By construction, the maps g_x^a are volume preserving diffeomorphisms on M with several key properties that we establish in the following theorem.

Theorem 4.1. *Let $x \in M$, $k \in \{1, \ldots, d-1\}$, $a \in (-1,1)^{d'}$. For fixed $y \in U_x$, $v \in \mathbf{Gr}^k(T_yM)$, the map*

$$a \mapsto (g_x^a(y), D_y g_x^a(v)) \in \mathbf{Gr}^k(TM)$$

is smooth and has surjective derivative at $a = 0$.

Proof. As the question is local, we may conjugate by ϕ_x and assume $M = \mathbb{R}^d$, $U_x = B_{1/2}(0)$, and $g_x^a = \psi_1^a$.

Fix $y \in B_{1/2}(0)$, $y = (y_1, \ldots, y_d)$. On $B_{1/2}(0)$, $\psi_1^a(y) = \exp(X_a) \cdot y$. As $a \mapsto X_a$ is invertible, we restrict to a neighborhood $\mathfrak{g}(\epsilon)$ of the identity of $\mathfrak{g}$. Thus we reduce to the study of the derivative of $F\colon \mathfrak{g}(\epsilon) \to \mathbb{R}^d \times \mathbf{Gr}^k(\mathbb{R}^d)$ where

$$F(X_a) = (\exp(X_a) \cdot y, D_y \exp(X_a) \cdot y(v)) = (A_a y + b_a, D_y(A_a y)(v)).$$

Note that the derivative of the exponential map at zero is the identity, and so the derivative of F at zero is $D_0F\colon \mathfrak{g} \to \mathbb{R}^d \times T_v\mathbf{Gr}^k(\mathbb{R}^d)$,[6] given by

$$D_0F(X_a) = (X_a \cdot (y_1, \ldots, y_d, 1), X_a \cdot (1, \ldots, 1, 0)(v)) = (A_a' y + b_a', A_a' v).$$

It remains to argue why this is surjective, but this is clear—all but one of the entries of A_a' can be changed independently between $(-1,1)$, and so A_a' acts transitively on subspaces of dimension k. Further, translation by b_a' is transitive, and so the entire map is surjective. $\square$

These facts give us a local transitive property on the manifold and every Grassmanian, as they imply the maps $a \mapsto g_x^a(y), a \mapsto D_y g_x^a(v)$ are submersions. Finally, note that as M is compact, we may choose a subset $g_{x_1}^a, \ldots, g_{x_j}^a$ such that the corresponding sets $U_{x_1}, \ldots, U_{x_j}$ cover M. We are now equipped to begin constructing our measure in earnest.

5. Constructing the measure

Take $f_0 \in U \subset \mathrm{Diff}^\infty_{\mathrm{vol}}(M)$, for U an open set. Let $0 < \epsilon < 1/2$ be such that $g_{x_i}^a \circ f_0 \in U$ for all $a \in (-\epsilon, \epsilon)^{d'}, i \in \{1, \ldots, j\}$. Choose $p_i > 0$ so $\sum_{i=0}^j p_i = 1$. Define

$$\mu = Cp_0 \delta_{f_0} + C \sum_{i=1}^{j} p_i \int_{[-\epsilon,\epsilon]^{d^2+d}} \delta_{g_x^a \circ f_0} da,$$

where $C > 0$ is a normalizing constant that makes μ a probability measure. Going forward, we will absorb the C into p_i. Observe that μ can be split into an outer integral over $a_{d+2}, \ldots, a_{d^2+d}$ and an inner integral over $a_1, \ldots, a_d$. This inner integral corresponds to the pure translation portion of $g_{x_i}^a$, and will be used to force additional regularity with respect to volume.

[6] Under the identification $T_0\mathfrak{g}(\epsilon) \simeq \mathfrak{g}$.

Theorem 5.1. *Let μ be as defined above, and ν be a μ-stationary probability measure. Then $\nu = \rho + \nu'$, where ρ, ν' are positive measures such that $\rho \ll vol_M$ and $\rho(M) > 0$.*

Proof. Let ν be a μ-stationary probability measure on M. We will show that convolution by μ produces a non-zero absolutely continuous part of ν.

For fixed $z \in M$, we know that $f_0(z) \in U_{x_k}$ for at least one k. Fix such a k. For $y \in U_{x_k}$, we have $g^a_{x_k}(y) = \phi^{-1}_{x_k} \circ \phi_1 \circ \phi_{x_k}(y)$. Let $a_{d+2}, \ldots, a_{d'}$ be fixed, A_a be as above, and let $b_a = (a_1, \ldots, a_d)$ be variable. By construction, $A_a\phi_{x_k}(f_0(z)) \in B_{1/2}(0)$, and so $A_a\phi_{x_k}(f_0(z)) + b_a \in B_1(0)$. Where $\phi_{x_i}(f_0(z))$ is well defined, let

$$B_i(z) := \phi^{-1}_{x_i}\left(A_a\phi_{x_i}(f_0(z)) + [-\epsilon, \epsilon]^d\right).$$

We will bound the inner integral from below. As ω is a Riemannian volume form, there is a smooth positive $g : U'_{x_k} \to \mathbb{R}$ such that $\omega = g dx_1 \wedge \cdots dx_d$ on U'_{x_k}. Fix $c_k > 0$ such that $c_k^{-1} < g < c_k$ on U_{x_k}. Note that for any such A_a, these constants and the compactness of M imply that $vol_M(B_i(z))$ will be uniformly bounded from below and above by a constant not depending on z, A_a. Fix A_a. Then for $E \subseteq M$ measurable, we have,

$$\begin{aligned}
\sum_{i=1}^{j} p_i\left(\int_{[-\epsilon,\epsilon]^d} \delta_{g^a_{x_i}\circ f_0(z)}(E)\, da_1\cdots da_d\right) &= \sum_{i=1}^{j} p_i\left(\int_{[-\epsilon,\epsilon]^d} 1_E\left(\phi^{-1}_{x_i}\circ\psi^a_1\circ\phi_{x_i}\circ f_0(z)\right) da_1\cdots da_d\right)\\
&\geqslant p_k\int_{[-\epsilon,\epsilon]^d} 1_E\left(\phi^{-1}_{x_k}\left(A_a\phi_{x_k}(f_0(z)) + b_a\right)\right) db_a\\
&\geqslant p_k\int_{B_k(z)} \frac{1_E(y)}{g(y)} d\mathrm{vol}_M(y)\\
&\geqslant \frac{p_k}{c_k}\mathrm{vol}_M(E\cap B_k(z)).
\end{aligned}$$

We now consider $\mu * \nu$. Note that

$$\mu = p_0\delta_{f_0} + \int_{[-\epsilon,\epsilon]^{d^2-1}} \sum_{i=1}^{j} p_i\left(\int_{[-\epsilon,\epsilon]^d} \delta_{g^a_{x_i}\circ f_0} da_1\cdots da_d\right) da_{d+2}\cdots da_{d^2+d}.$$

For any ν we may write $\nu(E) = \int_M \delta_z(E) d\nu(z)$. By Fubini's Theorem and the fact $\mu * \nu = \nu$, we have,

$$\begin{aligned}
\int_M \delta_z(E)\, d\nu(z) = \nu(E) = \mu * \nu(E) &= \int_U\int_M \delta_{h(z)}(E)\, d\nu(z)\, d\mu(h)\\
&= \int_M\int_U \delta_{h(z)}(E)\, d\mu(h)\, d\nu(z).
\end{aligned}$$

Since this holds for any E measurable, it follows that for any E we have $\nu_z(E) = \int_U \delta_{h(z)}(E) d\mu(h)$, where $\nu = \int \nu_z\ d\nu(z)$. It then suffices to show ν_z has an absolutely continuous part for any given z. But

$$
\begin{aligned}
\nu_z(E) &= \int_U \delta_{h(z)}(E)\, d\mu(h) \\
&= p_0 \delta_{f_0(z)}(E) + \sum_{i=1}^{j} p_i \int_{[-\epsilon,\epsilon]^{d'}} \delta_{g_x^a \circ f_0(z)}(E)\, da \\
&\geqslant \int_{[-\epsilon,\epsilon]^{d^2-1}} \sum_{i=1}^{j} p_i \int_{B_i(z)} \frac{\mathbf{1}_E(y)}{g(y)} \cdot \mathbf{1}_{U_i}(f_0(z))\, d\mathrm{vol}_M(y)\, dA_a \\
&\geqslant \int_{[-\epsilon,\epsilon]^{d^2-1}} \max_i \left\{ \frac{p_i}{c_i} \mathrm{vol}_M(E \cap B_i(z)) \right\} dA_a .
\end{aligned}
$$

Let $\rho_z(E) := \int_{[-\epsilon,\epsilon]^{d^2-1}} \max_i\{\frac{p_i}{c_i}\mathrm{vol}_M(E\cap B_i(z))\}dA_a$. Then ρ_z is absolutely continuous with respect to volume. If $\mathrm{vol}_M(E) > 0$, then there is a positive volume set of z such that $\rho_z(E)$ is bounded from below by a uniform constant not depending on z. Thus, we can conclude. □

6. Proving uniform expansion

Theorem 6.1. *Let ν be a μ-stationary probability measure on M such that $\nu = \nu_0 + \nu_\perp$, where $\nu_0, \nu_\perp$ are positive measures and $\nu_0 \ll vol_M$. Then there are no ν-measurable proper algebraic μ-invariant structures of $\mathbf{Gr}^k(TM)$.*

Proof. We will proceed by contradiction, assuming there is a ν-measurable μ-invariant algebraic structure E and showing $\nu_0 = 0$. Given an algebraic subset E_x of $\mathbf{Gr}^k(TM)$, there is a corresponding subspace $V_x = \langle E_x \rangle$ generated by its image in $\bigwedge^k TM$. As $D_x f E_x = E_{f(x)}$ for ν-a.e. $x \in M$, μ-a.e. $f \in \mathrm{Diff}^\infty(M)$, definition 2.3 implies $h = h(x) := \dim V_x$ is constant ν-almost everywhere. Thus, any μ-invariant ν-measurable algebraic structure E in $\mathbf{Gr}^k(TM)$ can be written as a ν-measurable μ-invariant map $E\colon M \to \mathbf{Gr}^h(\bigwedge^k TM)$, for some fixed $h \in \mathbb{N}$. By the construction, μ-invariance implies E is invariant under the action of both $g_{x_i}^a \circ f_0$ and f_0 for any i, m-a.e. $a \in (-\epsilon,\epsilon)^{d'}$. This implies E is invariant under $g_{x_i}^a$ for any i, m-a.e. $a \in (-\epsilon,\epsilon)^{d'}$.

Let $\epsilon > 0$. By Lusin's Theorem, there is a compact set $K_\epsilon \subset M$ such that $\nu_0(K_\epsilon) \geqslant (1-\epsilon)\nu_0(M)$ and E is continuous when restricted to K_ε. As ν_0 is absolutely continuous with respect to volume, we may assume K_ϵ has no isolated points. As M is compact, there is some U_{x_k} such that $\nu_0(K_\epsilon \cap U_{x_k}) > 0$. For any $y \in K_\epsilon \cap U_{x_k}$, we may choose $a' \in (-\epsilon,\epsilon)^{d'}$ such that $g_{x_k}^{a'}(y) = y$ and $D_y g_{x_k}^{a'} E(y) \neq E(y)$. This is due to theorem 4.1 and the fact $E(y)$ corresponds to a proper closed subset of $\mathbf{Gr}^k(T_yM)$, so surjectivity on a neighborhood of $\mathbf{Gr}^k(T_yM)$ lets us perturb proper subspaces.

Let

$$
L := \left\{ a \in (-\epsilon,\epsilon)^{d'} \mid E\left(g_{x_k}^a(y)\right) = D_y g_{x_k}^a E(y) \right\}
$$

and define $L' := \{z \in U_{x_k} \mid z = g_{x_k}^a(y), a \in L\}$. We know $m(L) = 1$, so by theorem 4.1, we have $\mathrm{vol}_M(L' \cap K_\epsilon) = \mathrm{vol}_M(K_\epsilon \cap U_{x_k})$. As ν_0 is absolutely continuous with respect to vol_M, $\nu_0(L') > 0$. Thus, there is a sequence $a_n \to a'$ such that $a_n \in L$ and $g_{x_k}^{a_n}(y) \in K_\epsilon$. As E is continuous on K_ϵ and $a \to g_{x_k}^a$ is continuous, we have

$$
D_y g_{x_k}^{a_n} E(y) = E\left(g_{x_k}^{a_n}(y)\right) \to E\left(g_{x_k}^{a'}(y)\right) = E(y)
$$

but

$$D_y g_{x_k}^{a_n} E(y) \to D_y g_{x_k}^{a'} E(y) \neq E(y),$$

a contradiction. Thus, $\nu_0 = 0$. □

Theorem 6.2. *Let ν be a μ-stationary probability measure on M such that $\nu = \nu_0 + \nu_\perp$, where $\nu_0, \nu_\perp$ are positive measures and $\nu_0 \ll vol_M$. Then there are no ν-measurable μ-invariant conformal structures on TM.*

Proof. For the sake of contradiction, assume there is a map $C\colon M \to \mathbf{CS}(TM)$ that is μ-invariant and ν-measurable. We will show $\nu_0 = 0$. Let $\epsilon > 0$. By Lusin's Theorem, there is a compact set $K_\epsilon \subset M$ such that $\nu_0(K_\epsilon) \geqslant (1-\epsilon)\nu_0(M)$ and C is continuous when restricted to K_ϵ. As ν_0 is absolutely continuous with respect to volume, we may assume K_ϵ has no isolated points. As M is compact, there is some U_{x_k} such that $\nu_0(K_\epsilon \cap U_{x_k}) > 0$.

Note that the image of $a \to A_a$ is a neighborhood of the identity in $\mathrm{SL}_d(\mathbb{R})$, so for some choice of a, the corresponding matrix A_a is not conformal. Thus $D_y g_{x_i}^a$ will not preserve a conformal structure on T_yM for any $y \in U_{x_k}$. Arguing as in the proof of theorem 6.1, we obtain that for any $y \in K_\epsilon$ there is an $a' \in (-\epsilon,\epsilon)^{d'}$ such that $g_{x_k}^{a'}(y) = y$ and $D_y g_{x_k}^{a'} C(y) \neq C(y)$. The remainder of the proof is identical to the proof of 6.1. □

With these two theorems we now have all the machinery in place to show that μ is uniformly expanding.

Theorem 6.3. *The measure μ is uniformly expanding in all dimensions.*

Proof. Assume for the sake of contradiction that μ is not uniformly expanding in some dimension k. By theorem 3.2, there is a μ-stationary measure ν such that there is either a μ-invariant ν-measurable algebraic structure on $\mathbf{Gr}^k(TM)$ or a μ-invariant ν-measurable conformal structure on TM. By theorem 5.1, ν must have an absolutely continuous part with respect to volume. But by theorems 6.1 and 6.2, absolute continuity prevents the existence of invariant structures. Thus, the theorem is proved. □

Finally, let us show that uniform expansion in all dimensions is an open property—thus, we can discretize μ to construct a measure with finite support that is also uniformly expanding.

Lemma 6.4. *Let μ_0 be a probability measure on $\mathrm{Diff}^\infty(M)$ that is uniformly expanding in dimension $k \in \{1,\ldots,d-1\}$. Assume that $\mathrm{supp}(\mu_0)$ is contained in some compact set K. Then there is an open neighborhood V of μ_0 in the weak-$*$ topology of $P(K)$ so that any $\rho \in V$ is also uniformly expanding in dimension k. Here, P(K) is the set of probability measures on K.*

Proof. We know $\exists C > 0, N > 0$ such that for any $(x,v) \in \mathbf{Gr}^k(TM)$, we have

$$\int_K \log ||D_x f v|| \mathrm{d}\mu_0^{(N)}(f) > C.$$

As K is compact, $\phi_{(x,v)}(f) := \log ||D_x f v||$ is continuous and so uniformly bounded for $f \in K$. Define

$$\phi\colon K \times \mathbf{Gr}^k(TM) \to \mathbb{R} \text{by } \phi(f,(v,v)) := \phi_{(x,v)}(f).$$

This is a continuous function in all variables, and the domain is compact, so the family of continuous functions from $\mathbf{Gr}^k(TM) \to \mathbb{R}$ given by $\{\phi \circ f \,|\, f \in K\}$ is uniformly equicontinuous. By compactness of K and M, there exists a finite cover $B_1,\ldots,B_\ell$ of $\mathbf{Gr}^k(TM)$ by open balls of fixed radius $\delta > 0$ centered at $(x_1, v_1), \ldots (x_\ell, v_\ell)$ in $\mathbf{Gr}^k(TM)$ such that for all i, for any $(x, v) \in B_i$:

$$|\phi_{(x_i,v_i)}(f) - \phi_{(x,v)}(f)| < \frac{C}{4} \text{for all} f \in K, \tag{$\star$}$$

and

$$\left| \int \phi_{(x_i,v_i)}(f)\, d\rho^{(N)} - \int \phi_{(x,v)}(f)\, d\rho^{(N)} \right| < \frac{C}{4} \text{for all } \rho \in P(\mathrm{Diff}^\infty(M)). \tag{$\star\star$}$$

We now define the sets

$$V_i := \left\{ \rho^{(N)} \in P(K) \;\middle|\; \left| \int_K \phi_{(x_i,v_i)}(f)\, \rho^{(N)}(f) - C \right| < \frac{C}{4} \right\}.$$

These are open non-empty sets in the weak-$*$ topology by definition, and comprise of all measures that meet the criteria for uniform expansion on the point (x_i, v_i). By ($\star$) and ($\star\star$), for any $(x, v) \in B_i$, $\rho^{(N)}$ is uniformly expanding at the point (x, v) by constant $\frac{C}{2}$. Thus, $\rho \in V := \cap_{i=1}^{\ell} V_i$ is uniformly expanding at all points in M. As V is the finite intersection of non-empty open sets and $\mu_0 \in V_i$ for all i, it too is a non-empty open set, and the claim is shown. □

Finally, we may prove the main theorem of the paper:

Proof of theorem 1.3. Let μ be the measure constructed in section 5. We will build a sequence of finitely supported measures μ_n such that $\mu_n \to \mu$ in the weak-$*$ topology, and there are no finite μ_n-invariant subsets of M.

We will first eliminate finite invariant sets. Choose $a_0, a_1 \in [-\epsilon, \epsilon]^{d'}$ so that A_{a_0} and A_{a_1} are distinct irrational rotations of $\mathbb{R}^d$ that do not share a real eigenvector, and $b_{a_0} = b_{a_1} = 0$. Note that $\{0\}$ is the only finite set in $\mathbb{R}^d$ invariant under the action $y \mapsto A_{a_0} y + b_{a_0}$ and $y \mapsto A_{a_1} y + b_{a_1}$. For any $x_i \in \{x_1, \ldots, x_j\}$, let $h_i^0 = g_{x_i}^{a_0}$ and $h_i^1 = g_{x_i}^{a_1}$. As $b_{a_0} = b_{a_1} = 0$, for any $y \in U_{x_i}$ we have $h_i^0 = \phi_{x_i}^{-1}(A_{a_0} \phi_{x_i}(f_0(y)))$ and $h_i^1 = \phi_{x_i}^{-1}(A_{a_1} \phi_{x_i}(f_0(y)))$. This implies that the only proper subset of U_{x_i} preserved by both h_i^0 and h_i^1 is $\{x_i\}$.

For each $i \in \{1, \ldots, j\}$, define $K_i = \{g_{x_i}^a \circ f_0 \mid a \in [-\epsilon, \epsilon]^{d'}\}$. As M is compact and K_i is a compact subset of a finite parameter family of diffeomorphisms, K_i is a complete metric space under the uniform metric and thus a Baire space. Define $D_i^j := \{g \in K_i \mid g^j(x_i) \neq x_i\}$. This is an open and dense set in K_i, and so $D_i = \cap_{j \geqslant 1} D_i^j$ is also open and dense in K_i by the Baire Category Theorem. The set D_i is all diffeomorphisms in K_i that do not have x_i as a periodic point. For each i, fix $h_i \in D_i$. By construction, there is no finite subset of U_{x_i} invariant under the action of $\{h_i^1, h_i^0, h_i\}$.

Fix $n \in \mathbb{N}$. We subdivide $[-\epsilon, \epsilon]^{d'}$ into $n^{d'}$ cubical cells of side length $2\epsilon/n$. Let $z_1, \ldots, z_N$ be the points in $[-\epsilon, \epsilon]^{d'}$ corresponding to the vertices of this subdivision, where $N = (n+1)^{d'}$. We will use these points to approximate the Lebesgue integral. Here we denote $g_{x_i}^{z_\ell}$ by $g_{i\ell}$.

Define

$$\mu_n = p_0 \delta_{f_0} + \sum_{i=1}^{j} \frac{p_i}{3+N} \left[\delta_{h_i^1} + \delta_{h_i^0} + \delta_{h_i} + \sum_{\ell=1}^{N} \delta_{g_{i\ell} \circ f_0} \right].$$

Note that the weak-$*$ limit of μ_n is μ. For any n, μ_n is a finitely supported probability measure and there is no finite μ_n-invariant subset of M. Further, the support of both μ and μ_n is contained in the compact set $K := \cup_{i=1}^{j} K_i$. By applying theorem 6.4 for each dimension k

in $\{1,\ldots,d-1\}$, we see there is an open neighborhood V of μ such that every measure in V is uniformly expanding in all dimensions. As $\mu_n \xrightarrow{*} \mu$, the sequence $\{\mu_n\}$ must eventually enter the set V. Thus there is some n such that μ_n is uniformly expanding. □

Data availability statement

No new data were created or analysed in this study.

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