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$\mathcal{D}^{\infty}$-MODULES ON SMOOTH RIGID ANALYTIC VARIETIES AND LOCALLY ANALYTIC REPRESENTATIONS

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To my grandparents, who devoted their lives to education

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ABSTRACT

In this article, we construct the abelian category of coadmissible p-adic $\mathcal{D}^{\infty}$-modules on a smooth rigid analytic variety over a complete discrete valued field. We also consider equivariant $\mathcal{D}^{\infty}$-modules and prove a p-adic analogue of the Beilinson-Bernstein localization theorem for admissible locally analytic representations.

CHAPTER 1

INTRODUCTION

1.1 Notations

Throughout this article, we use the following notations.

Let K be a complete discrete valued field of mixed characteristic, let $\mathcal{O}_K$ be its ring of integers, and let π_K be an uniformizer. Assume p equals the the characteristic of the residue field $k = \mathcal{O}_K/(\pi_K)$, and assume that the p-adic norm $|\cdot|$ on K is normalized so that $|p| = \frac{1}{p}$. Assume $\alpha \in K$ such that $0 < |\alpha| < 1$.

In Chapter 3, let $L \subseteq K$ be a finite extension of $\mathbb{Q}_p$, and $\mathcal{O}_L$ be its ring of integers. We will assume that $\alpha \in L^\times$ such that $0 < |\alpha| < 1$ and $\alpha^{p-1} \in p\mathcal{O}_L$.

If R is a p-adically complete commutative noetherian algebra, then $R\langle x_1, x_2, ..., x_n\rangle$ is the p-adic completion of the polynomial ring $R[x_1, x_2, ..., x_n]$, and $(R \otimes_{\mathbb{Z}} \mathbb{Q})\langle x_1, x_2, ..., x_n\rangle$ is $(R\langle x_1, x_2, ..., x_n\rangle) \otimes_{\mathbb{Z}} \mathbb{Q}$. The symbol $A := B$ means A is defined to be B. The symbol $n >> 0$ (resp. $n << 0$) means there exists $N \in \mathbb{Z}$, such that $n > N$ (resp. $n < N$).

If A is an algebra with an increasing filtration F_nA, then the associated graded algebra is denoted as $\mathrm{gr}^F_{\cdot} A$. If $\mathcal{A}$ is a sheaf of algebra on a topological space X with an increasing filtration $F_n\mathcal{A}$, then $\mathrm{gr}^F_{\cdot} \mathcal{A}$ is the sheaf associated to the presheaf $U \to \bigoplus_n F_{n+1}\mathcal{A}(U)/F_n\mathcal{A}(U)$, where U is an open subset of X. If $(X, \mathcal{O}_X)$ is a locally ringed space and $\mathcal{F}$ is a locally free sheaf of finite rank, then the symmetric algebra of $\mathcal{F}$ over $\mathcal{O}_X$ is $\mathrm{Sym}_{\mathcal{O}_X} \mathcal{F}$. The i-th graded piece of $\mathrm{Sym}_{\mathcal{O}_X} \mathcal{F}$ is $\mathrm{Sym}^i_{\mathcal{O}_X} \mathcal{F}$.

If $\mathbf{X}$ is a scheme over L, then $\mathbf{X}_K := \mathbf{X} \times_{\mathrm{Spec}(L)} \mathrm{Spec}(K)$. If V is a vector space over L, then $V_K := V \otimes_L K$.

1.2 Main results

In this article, we explore the potential p-adic analogues of the beautiful theory of $\mathcal{D}$-modules on smooth complex varieties.

In Chapter 2 we define a sheaf of infinite order twisted differential operators $\mathcal{A}_{\mathcal{T}}^{\infty}$, which carries a natural Fréchet topology, on a smooth rigid analytic variety X over K. We construct a category $M^{\text{coad}}(\mathcal{A}_{\mathcal{T}}^{\infty})$ of modules over $\mathcal{A}_{\mathcal{T}}^{\infty}$ with appropriate finiteness properties. The key result is Proposition 2.3.9, which tells us that a sheaf in $M^{\text{coad}}(\mathcal{A}_{\mathcal{T}}^{\infty})$ satisfies an analogue of Serre's theorem of quasi-coherent sheaves on schemes. Then it follows that the category $M^{\text{coad}}(\mathcal{A}_{\mathcal{T}}^{\infty})$ is abelian. This category is first constructed and studied by Ardakov and Wadsley in [2] and [3]. In comparison, the approach in this article makes more systematic use of formal models and techniques from the paper [1]. We also construct a sheaf of microlocal differential operators $\mathcal{E}$ on the cotangent bundle of X whose restriction to the zero section is the sheaf of differential operators $\mathcal{D}$.

In Chapter 3 we define a category $M_G^{\text{coad}}(\mathcal{A}_{\mathcal{T}}^{\infty})$ of equivariant $\mathcal{A}_{\mathcal{T}}^{\infty}$-module with appropriate finiteness properties and prove that this category is abelian. As an application, we show in Theorem 3.4.4 that under some technical assumptions, when $\mathbf{X}$ is the flag variety of a reductive algebraic group $\mathbf{G}$ over L, and $\mathbf{G}(L)$ is the L-rational points of $\mathbf{G}$ viewed as a p-adic Lie group, the category of admissible locally analytic representations of $\mathbf{G}(L)$ with a fixed infinitesimal central character is equivalent to the category of coadmissible equivariant twisted $\mathcal{D}_X^{\infty}$-modules, where X is the rigid analytification of $\mathbf{X}$. Similar results are also obtained by Huyghe, Patel, Schmidt and Strauch in [17]. In comparison, we do not consider divided power structures because of Lemma 3.2.2, nor do we assume the infinitesimal central character is trivial.

CHAPTER 2

$\mathcal{D}^{\infty}$-MODULES ON SMOOTH RIGID ANALYTIC VARIETIES OVER A COMPLETE DISCRETE VALUED FIELD

2.1 Introduction

To motivate the constructions in this article, let us briefly recall some stories of analytic $\mathcal{D}$-modules on a smooth complex analytic manifold. Suppose X is a smooth complex manifold and $i : Z \hookrightarrow X$ is a closed analytic subset of codimension d with the sheaf of ideals $\mathcal{I}_Z$. Let $\mathcal{D}_X$ and $\mathcal{D}_Z$ be the sheaves of differential operators on X and Z respectively. The classical Kashiwara's equivalence says that the subcategory of coherent $\mathcal{D}_X$-modules which are annihilated by a power of $\mathcal{I}_Z$ is equivalent to the category of coherent $\mathcal{D}_Z$-modules, via the functors

$$\mathcal{N} \to i_*(\mathcal{D}_{X \hookleftarrow Z} \otimes_{\mathcal{D}_Z} \mathcal{N}), \text{ if } \mathcal{N} \text{ is a coherent } \mathcal{D}_Z\text{-module,}$$

and

$$\mathcal{M} \to i^{-1}\mathcal{H}om_{\mathcal{D}_X}(\mathcal{D}_{X \hookleftarrow Z}, \mathcal{M}), \text{ if } \mathcal{M} \text{ is a coherent } \mathcal{D}_X\text{-module such that } \Gamma_{[Z]}(\mathcal{M}) \simeq \mathcal{M},$$

where by definition $\Gamma_{[Z]}(\mathcal{M}) = \varinjlim_k \mathcal{H}om_{\mathcal{O}_X}(\mathcal{O}_X/\mathcal{I}_Z^k, \mathcal{M})$ for any $\mathcal{O}_X$-module $\mathcal{M}$, and $\mathcal{D}_{X \hookleftarrow Z}$ is the transferring $(i^{-1}\mathcal{D}_X, \mathcal{D}_Z)$-bimodule ([21] Theorem 4.30). In particular, under the Kashiwara's equivalence, the algebraic local cohomology

$$\mathcal{B}_{Z/X} = \varinjlim_k \mathcal{E}xt^d_{\mathcal{O}_X}(\mathcal{O}_X/\mathcal{I}_Z^k, \mathcal{O}_X) \simeq i_*(\mathcal{D}_{X \hookleftarrow Z} \otimes_{\mathcal{D}_Z} \mathcal{O}_Z)$$

of Z in X, is a coherent $\mathcal{D}_X$-module. However, if we consider the local cohomology $\mathbf{R}^d\Gamma_Z(\mathcal{O}_X)$, where $\Gamma_Z(\mathcal{M}) = \{s \in \mathcal{M} \mid s|_{X\setminus Z} = 0\}$ for any $\mathcal{O}_X$-module $\mathcal{M}$, and $\mathbf{R}^d\Gamma_Z$ is the d-th derived functor of Γ_Z, this is not coherent over $\mathcal{D}_X$ anymore. It is demonstrated in Mebkhout's paper [24] that there exists a sheaf of infinite order differential operator operator with convergence conditions $\mathcal{D}_X^\infty$, such that $\mathcal{D}_X^\infty \otimes_{\mathcal{D}_X} \mathcal{B}_{Z/X} \simeq \mathbf{R}^d\Gamma_Z(\mathcal{O}_X)$.

The most important feature of a coherent $\mathcal{D}_X$-module is the existence of its characteristic variety, which is a conic closed analytic subset of the cotangent bundle of X, involutive with respect to the canonical symplectic structure on the cotangent bundle. When the associated characteristic variety is Lagrangian, we say the coherent $\mathcal{D}_X$-module is holonomic. The Riemann-Hilbert correspondence states that the category of regular holonomic $\mathcal{D}_X$-modules is equivalent to the category of perverse sheaves via the de Rham functor

$$DR_X : \mathcal{M} \to \mathbf{R}\mathcal{H}om_{\mathcal{D}_X}(\mathcal{O}_X, \mathcal{M}), \text{ if } \mathcal{M} \text{ is a regular holonomic } \mathcal{D}_X\text{-module.}$$

There is no explicit inverse functor to the de Rham functor. However, if we are willing to consider $\mathcal{D}_X^\infty$-modules, for any holonomic $\mathcal{D}_X$-module $\mathcal{M}$, we have (Theorem 3.4.11 in [6]):

$$\mathcal{D}_X^\infty \otimes_{\mathcal{D}_X} \mathcal{M} \simeq \mathbf{R}\mathcal{H}om_{\mathbb{C}_X}(\mathcal{S}ol_X(\mathcal{M}), \mathcal{O}_X),$$

where the solution functor $\mathcal{S}ol_X$ is related to DR_X explicitly:

$$\mathcal{S}ol_X(\mathcal{M}) = \mathbf{R}\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \mathcal{O}_X) \simeq DR_X(\mathbb{D}_X(\mathcal{M})),$$

and

$$\mathbb{D}_X(\mathcal{M}) = \mathbf{R}\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \mathcal{D}_X \otimes_{\mathcal{O}_X} \Omega_X^{-1})[\text{ dimension of } X\ ]$$

is the duality functor. Moreover, there exists a unique regular holonomic $\mathcal{D}_X$-module $\mathcal{M}_{reg}$ such that $\mathcal{D}_X^\infty \otimes_{\mathcal{D}_X} \mathcal{M}_{reg} = \mathcal{D}_X^\infty \otimes_{\mathcal{D}_X} \mathcal{M}$, $\mathcal{M}_{reg}$ contains all the regular holonomic $\mathcal{D}_X$-submodules of $\mathcal{D}_X^\infty \otimes_{\mathcal{D}_X} \mathcal{M}$, and $\mathcal{S}ol_X(\mathcal{M}) = \mathcal{S}ol_X(\mathcal{M}_{reg})$ (Theorem 5.2.22 [6]).

To explore p-adic $\mathcal{D}$-modules, let us first look at a simple case. Let $X = \mathrm{Spa}(\mathbb{Q}_p\langle x\rangle, \mathbb{Z}_p\langle x\rangle)$ be the unit disk, let $Z = \{0\}$ be the origin, and let $j : U \hookrightarrow X$ be the open immersion, where $U = X\backslash\{0\}$. Then the algebraic local cohomology of Z in X is $\mathcal{B}_{Z/X} \simeq \mathcal{D}_X/\mathcal{D}_X x$, and the local cohomology of Z in X is $\mathbf{R}^1\Gamma_{\{0\}}(\mathcal{O}_X) \simeq j_*\mathcal{O}_U/\mathcal{O}_X$. In order to have

$\mathbf{R}^1\Gamma_{\{0\}}(\mathcal{O}_X) \simeq \mathcal{D}_X^{\infty} \otimes_{\mathcal{D}_X} (\mathcal{D}_X/\mathcal{D}_X x)$, we study the convergent condition of $j_*\mathcal{O}_U/\mathcal{O}_X$ at Z and propose the following definition of a presheaf of infinite order differential operators $\mathcal{D}_X^{\infty}$:

$$\mathcal{D}_X^{\infty}(V) = \left\{ \sum_{i\geq 0} a_i \partial^i \;\middle|\; a_i \in \mathcal{O}_X(V), \text{for any } n \in \mathbb{N} \left\| \frac{a_i}{p^{ni}} \right\| \xrightarrow[i\to\infty]{} 0 \right\},$$

where ∂ is the dual to the differential dx, and $V \subseteq X$ is an open affinoid subdomain. We observe that this definition generalizes easily to any smooth rigid analytic variety.

If X is a smooth rigid analytic variety over K, it is proved in Definition-Lemma 2.4.1 that $\mathcal{D}_X^{\infty}$ is a sheaf of Fréchet-Stein algebra (see Definition 2.2.9). A Fréchet-Stein algebra is a inverse limit of noetherian Banach algebra with flat transition maps. In [28], Schneider and Teitelbaum systematically developed a general theory of coadmissible modules over Fréchet-Stein algebras. Here being codamissible is the analogous condition of being coherent. In Lemma 2.4.5 we prove that there exists a good definition of coadmissible modules over sheaves of Fréchet-Stein algebras. For example, when X is the unit disk, $\mathcal{M}$ is a coadmissible $\mathcal{D}_X^{\infty}$-module, if and only if $\Gamma(X, \mathcal{M})$ is a coadmissible module over $\Gamma(X, \mathcal{D}_X^{\infty})$ and $\mathcal{M}(V) \simeq \mathcal{D}_X^{\infty}(V) \widehat{\otimes}_{\Gamma(X, \mathcal{D}_X^{\infty})} \Gamma(X, \mathcal{M})$ is a coadmissible module over $\mathcal{D}_X^{\infty}(V)$, for any $V \subseteq X$ open affinoid subdomain. We also remark that it is proved in [3] that Kashiwara's equivalence holds for coadmissible $\mathcal{D}_X^{\infty}$-modules.

In section 2.5, we give the definitions of p-adic microlocal differential operators $\mathcal{E}_X$ and $\mathcal{E}_X^{\infty}$ on the cotangent bundle T^*X of X.

2.2 Preliminaries

2.2.1 Preliminaries about sheaves on ringed topological spaces

Let $(X, \mathcal{O}_X)$ be a ringed topological space and let $\mathcal{B}_X$ be a set of basis for the topology on X.

A presheaf of $\mathcal{O}_X$-modules $\mathcal{F}$ on $\mathcal{B}_X$ is a rule which assigns to each $U \in \mathcal{B}_X$ a $\mathcal{O}_X(U)$-

module $\mathcal{F}(U)$, and to each inclusion $V \subseteq U$ of elements of $\mathcal{B}_X$ a morphism of $\mathcal{O}_X(U)$-modules $\rho^U_V : \mathcal{F}(U) \to \mathcal{F}(V)$, such that if $W \subseteq V \subseteq U$ in $\mathcal{B}_X$, then $\rho^U_W = \rho^V_W \circ \rho^U_V$. A sheaf $\mathcal{O}_X$-modules $\mathcal{F}$ on $\mathcal{B}_X$ is a presheaf of $\mathcal{O}_X$-modules on $\mathcal{B}_X$ such that for any $U \in \mathcal{B}_X$, any covering $U = \bigcup_{i\in I} U_i$ with $U_i \in \mathcal{B}_X$, and any coverings $U_i \cap U_j = \bigcup_{k\in I_{ij}} U_{ijk}$ with $U_{ijk} \in \mathcal{B}_X$, for any collection of sections $s_i \in \mathcal{F}(U_i)$ such that $s_i|_{U_{ijk}} = s_j|_{U_{ijk}}$, there exists a unique $s \in \mathcal{F}(U)$ such that $s_i = s|_{U_i}$.

There are no distinctions between sheaves of $\mathcal{O}_X$-modules and sheaves of $\mathcal{O}_X$-modules on $\mathcal{B}_X$ because of the following lemma.

Lemma 2.2.1. *The natural restriction functor from the category of sheaves of $\mathcal{O}_X$-modules on X to the category of sheaves of $\mathcal{O}_X$-modules on $\mathcal{B}_X$ is an equivalence.*

Proof. This is [30] Lemma 30.13.

□

Convention-Notation 1. Without further specifying, if X is a scheme, we take $\mathcal{B}_X$ to be the set of open affine subschemes of X. If X is a formal scheme, we take $\mathcal{B}_X$ to be the set of open affine formal subschemes of X. If X is a rigid analytic variety, we take $\mathcal{B}_X$ to be the set of open affinoid subdomains of X. By the theorem of Gerritzen-Grauert, every affinoid subdomain is a finite union of rational subdomains, so equivalently we can take $\mathcal{B}_X$ to be rational subsets.

Convention-Notation 2. If $\underline{U} = \{U_i\}_{i\in I}$ is an open cover of X, fix a well-ordering of the index set I. Let $C^\cdot(\underline{U}, \mathcal{F})$ be the associated Čech complex, where

$$C^p(\underline{U}, \mathcal{F}) := \prod_{\alpha_0<\alpha_1<...<\alpha_p} \mathcal{F}(U_{\alpha_0} \cap U_{\alpha_1} \cap ... \cap U_{\alpha_p})$$

and the coboundry maps $\delta_p : C^p(\underline{U}, \mathcal{F}) \to C^{p+1}(\underline{U}, \mathcal{F})$ are:

$$(\delta\sigma)_{\alpha_0,...,\alpha_{p+1}} := \sum_{j=0}^{p+1} (-1)^j \sigma_{\alpha_0,...,\alpha_{j-1},\alpha_{j+1},...,\alpha_p}|_{U_{\alpha_0}\cap U_{\alpha_1}\cap...\cap U_{\alpha_{p+1}}}.$$

Let $\check{H}^i(\underline{U}, \mathcal{F})$ be the i-th cohomology of the complex $C^{\cdot}(\underline{U}, \mathcal{F})$.

Lemma 2.2.2. *Let $(X, \mathcal{O}_X)$ be a ringed topological space and $\mathcal{B}_X$ is a set of basis for the topology of X such that*

1. *If $U \in \mathcal{B}_X$, then U is quasi-compact.*

2. *If $U, V \in \mathcal{B}_X$, then $U \cap V$ is a finite union of elements in $\mathcal{B}_X$.*

Let $\mathcal{F}^{pre}$ be a presheaf $\mathcal{O}_X$-modules on $\mathcal{B}_X$ equipped with an increasing exhaustive filtration of presheaves of $\mathcal{O}_X$-modules $F.\mathcal{F}$ such that $F_i\mathcal{F} = 0$ for $i << 0$. Assume the presheaf $U \to gr^F_{\cdot}(\mathcal{F}^{pre}(U))$ is a sheaf on $\mathcal{B}_X$. Then

1. *$\mathcal{F}^{pre}$ is a sheaf on $\mathcal{B}_X$.*

2. *If X is a quasi-separated noetherian scheme and $gr^F_{\cdot}\mathcal{F}^{pre}$ is a quasi-coherent $\mathcal{O}_X$-module, then $\mathcal{F}^{pre}$ is a quasi-coherent $\mathcal{O}_X$-module.*

Proof. Let $U \in \mathcal{B}_X$ and let $\underline{U} \subseteq \mathcal{B}_X$ be a finite open cover of U. Suppose $U_i \cap U_j = \bigcup_{k \in I_{ij}} U_{ijk}$ for finitely many $U_{ijk} \in \mathcal{B}_X$.

1. To show $\mathcal{F}^{\text{pre}}$ is a sheaf on $\mathcal{B}_X$, it suffices to check that

$$0 \to \mathcal{F}^{\text{pre}}(U) \xrightarrow{\delta_0} C^0(\underline{U}, \mathcal{F}^{\text{pre}}) \xrightarrow{\delta_1} \prod_{i<j} \prod_{k \in I_{ij}} \mathcal{F}^{\text{pre}}(U_{ijk})$$

is an exact sequence of abelian groups, where $(\delta_1 \sigma)_{ijk} = \sigma_i|_{U_{ijk}} - \sigma_j|_{U_{ijk}}$.

This is reduced to checking that

$$0 \to \text{gr}^F_{\cdot} \mathcal{F}^{\text{pre}}(U) \xrightarrow{\delta_0} C^0(\underline{U}, \text{gr}^F_{\cdot} \mathcal{F}^{\text{pre}}) \xrightarrow{\delta_1} \prod_{i<j} \prod_{k \in I_{ij}} \text{gr}^F_{\cdot} \mathcal{F}^{\text{pre}}(U_{ijk})$$

is exact.

2. Let $V \subseteq U$ in $\mathcal{B}_X$. It suffices to check that the natural morphism of $\mathcal{O}_X(V)$-modules $\mathcal{O}_X(V) \otimes_{\mathcal{O}_X(U)} \mathcal{F}^{\text{pre}}(U) \to \mathcal{F}^{\text{pre}}(V)$ is an isomorphism, which is reduced to checking that $\mathcal{O}_X(V) \otimes_{\mathcal{O}_X(U)} \text{gr}^F_\cdot \mathcal{F}^{\text{pre}}(U) \to \text{gr}^F_\cdot \mathcal{F}^{\text{pre}}(V)$ is an isomorphism.

□

We summarize the following general results proved by Berthelot from [5] section (3.3):

Theorem 2.2.3. *Let $\mathfrak{X}$ be a locally noetherian formal scheme, and let $\mathcal{I} \subseteq \mathcal{O}_{\mathfrak{X}}$ be an ideal of definition. Suppose $\mathcal{D}$ is a sheaf of rings on $\mathfrak{X}$, with a homomorphism $\mathcal{O}_{\mathfrak{X}} \to \mathcal{D}$ such that the image of $\mathcal{I}$ is central in $\mathcal{D}$. Further assume $\mathcal{D}$ satisfies the following conditions:*

1. *$\mathcal{D} \simeq \varprojlim_i \mathcal{D}/\mathcal{I}^i\mathcal{D}$, and $\mathcal{D}/\mathcal{I}^i\mathcal{D}$ is quasi-coherent as a left $\mathcal{O}_{\mathfrak{X}}/\mathcal{I}^i\mathcal{O}_{\mathfrak{X}}$-module.*
2. *If $\mathfrak{U} \subseteq \mathfrak{X}$ is open affine, the ring $\mathcal{D}(\mathfrak{U})$ is left noetherian.*

Let $\mathcal{M}$ be a $\mathcal{D}$-module. The following statements are true:

1. *If $\mathfrak{X}$ is affine, the following statements are equivalent :*
   (a) *For all $i \in \mathbb{Z}^{\geq 1}$, the $\mathcal{D}/\mathcal{I}^i\mathcal{D}$-module $\mathcal{M}/\mathcal{I}^i\mathcal{M}$ is coherent, and $\mathcal{M} \simeq \varprojlim_i \mathcal{M}/\mathcal{I}^i\mathcal{M}$.*
   (b) *There exists an isomorphism $\mathcal{M} \simeq \varprojlim_i \mathcal{M}_i$, where $\{\mathcal{M}_i\}$ is a projective system of coherent $\mathcal{D}/\mathcal{I}^i\mathcal{D}$-modules, and the transition morphisms induce isomorphisms $\mathcal{M}_{i+1}/\mathcal{I}^i\mathcal{M}_{i+1} \simeq \mathcal{M}_i$.*
   (c) *There exists a finite module M over $D := \Gamma(\mathfrak{X}, \mathcal{D})$ and an isomorphism*

   $$\mathcal{M} \simeq \varprojlim_i (\mathcal{D}/\mathcal{I}^i\mathcal{D}) \otimes_{(D/I^iD)} (M/I^iM),$$

   where $I := \Gamma(\mathfrak{X}, \mathcal{I})$ and $\Gamma(\mathfrak{X}, \mathcal{D}/\mathcal{I}^i\mathcal{D}) \simeq D/I^iD$.
   (d) *The D-module $\Gamma(\mathfrak{X}, \mathcal{M})$ is finite, and for any $\mathfrak{U} \subseteq \mathfrak{X}$ open affine, the homomorphism $\mathcal{D}(\mathfrak{U}) \otimes_D \Gamma(\mathfrak{X}, \mathcal{M}) \to \mathcal{M}(\mathfrak{U})$ is an isomorphism.*
   (e) *The $\mathcal{D}$-module $\mathcal{M}$ is coherent over $\mathcal{D}$.*

2. *If $\mathfrak{X}$ is noetherian and $\mathfrak{U}_1 \cup \mathfrak{U}_2$ is an open cover of $\mathfrak{X}$. Let $\mathcal{M}_i$ be a coherent $\mathcal{D}|_{\mathfrak{U}_i}$ without p-torsion, where $i = 1, 2$. Suppose there is an isomorphism $\epsilon : \mathcal{M}_1 \otimes_{\mathbb{Z}} \mathbb{Q} \simeq \mathcal{M}_2 \otimes_{\mathbb{Z}} \mathbb{Q}$ on $\mathfrak{U}_1 \cap \mathfrak{U}_2$. Then there exists a coherent $\mathcal{D}$-module $\mathcal{M}$ without p-torsion extending $\mathcal{M}_1$, together with an isomorphism $\mathcal{M} \otimes_{\mathbb{Z}} \mathbb{Q}|_{\mathfrak{U}_2} \simeq \mathcal{M}_2 \otimes_{\mathbb{Z}} \mathbb{Q}$ extending ϵ.*

We will need the following version of [16] (13.2.4):

Proposition 2.2.4. *Let $\mathcal{C}^{\cdot} = \{C_n^{\cdot}\}_{n \in \mathbb{Z}}$ be a projective system of complexes of K-Fréchet spaces such that the differentials are continuous. Assume $H^{i-1}(C_n^{\cdot}) \to H^{i-1}(C_{n-1}^{\cdot})$ is a continuous morphism between K-Fréchet spaces with dense image. Then*

$$H^i(\varprojlim_n C_n^{\cdot}) \simeq \varprojlim_n H^i(C_n^{\cdot}).$$

2.2.2 Review of the formal models of a rigid analytic variety over K

We refer to [9] section 9.3 for the definition of a rigid analytic space over K. Huber ([18] 1.1.11) constructed a fully faithful functor $\mathbf{r}$ from the category of rigid analytic spaces over $\mathrm{Sp}(K)$ to the category of adic spaces over $\mathrm{Spa}(K, \mathcal{O}_K)$, such that for affinoids $\mathbf{r}(\mathrm{Sp}(A)) = \mathrm{Spa}(A, A^{\circ})$. In this article, a rigid analytic variety X over K is a quasi-separated reduced rigid analytic space locally of topologically finite type over K, viewed as an adic space via the functor $\mathbf{r}$. The notation X^{Tate} will be used if we wish to consider only the classical points in X, and X^{Tate} is equipped with the Grothendieck topology generated by finite unions of rational subdomains. All fibre products are considered within the category of rigid analytic spaces over K.

A formal scheme $\mathfrak{X}$ over $\mathcal{O}_K$ is admissible, if $\mathfrak{X}$ is locally of topologically finite presentation over $\mathcal{O}_K$ and (π_K)-torsion free. The special fiber of $\mathfrak{X}$, which is denoted as $\mathfrak{X}_k$, is a scheme of finite type over k. In [26] Raynaud constructed a functor **rig** from the category of admissible formal schemes over $\mathcal{O}_K$ to the category of rigid analytic spaces over K which commutes with fibre products. A formal scheme $\mathfrak{X}$ over $\mathcal{O}_K$ is an admissible formal model of the rigid analytic variety X over K, if $\mathfrak{X}$ is admissible and $\mathbf{rig}(\mathfrak{X})$ is isomorphic to X.

Recall from [15] section 7.1-7.3 that a prime filter p on X^{Tate} (resp. $\mathfrak{X}_{k,cl}$ the set of closed points of $\mathfrak{X}_k$) is a collection of admissible open subsets (resp. Zariski open subsets), such that

1. $X \in p$ and $\emptyset \in p$.

2. If $U_1, U_2 \in p$ then $U_1 \cap U_2 \in p$.

3. If $V \in p$ and $V \subseteq U$ then $U \in p$.

4. If $U \in p$ and $\{U_i\}_{i \in I}$ is an admissible open covering (resp. open covering) of U, then there exists $i \in I$ such that $U_i \in p$.

Let $\mathcal{P}(X^{Tate})$ be the set of prime filters of X^{Tate}. Then $\mathcal{P}(X^{Tate})$ is a topological space with open sets $\{\mathcal{P}(U^{Tate}) \mid \ U^{Tate}$ is an admissible open in $X^{Tate}\}$. By [**?**] there is an isomorphism between topological spaces $\mathcal{P}(X^{Tate}) \xrightarrow{\simeq} X$, that is compatible with open immersions.

Let $\mathcal{P}(\mathfrak{X}_{k,cl})$ be the set of prime filters of $\mathfrak{X}$. Then $\mathcal{P}(\mathfrak{X}_{k,cl})$ is a topological space with open sets $\{\mathcal{P}(\mathfrak{U}_{k,cl}) \mid \ \mathfrak{U}_k$ is a Zariski open in $\mathfrak{X}_k$ $\}$. There is an isomorphism as topological spaces between $\mathfrak{X}_k \xrightarrow{\simeq} \mathcal{P}(\mathfrak{X}_{k,cl})$, that sends $\mathfrak{p} \in \mathfrak{X}_k$ to the prime filter $\{\mathfrak{U}_k$ Zariski open in $\mathfrak{X}_k \mid$ (the closure of $\mathfrak{p}) \cap \mathfrak{U}_k \neq \emptyset\}$, and is compatible with open immersions.

Proposition 2.2.5. *Let X be a quasi-compact rigid analytic variety over K. Suppose $\{U_i\}_{i \in I}$ is a finite affinoid open cover of X. Then there exists a quasi-compact admissible formal model $\mathfrak{X}$ over $\mathcal{O}_K$ of X with an open cover $\{\mathfrak{U}_i\}_{i \in I}$ such that:*

1. $\mathfrak{U}_i$ is a finite union of open affines.

2. The associated specialization map $sp : X \to \mathfrak{X}_k$ is a continuous surjection between topological spaces.

3. $sp^{-1}(\mathfrak{U}_i) = U_i$, and the inclusions $\mathcal{O}_{\mathfrak{U}_i} \hookrightarrow sp_*\mathcal{O}_{U_i}$ induces the isomorphisms $\mathcal{O}_{\mathfrak{U}_i} \otimes_{\mathbb{Z}} \mathbb{Q} \simeq sp_*\mathcal{O}_{U_i}$ for any $i \in I$.

Proof. By [26] there exists a quasi-compact admissible formal model $\mathfrak{X}'$ of X. By [7] Lemma 4.4, there exists an admissible formal blowing-up $\mathfrak{X} \to \mathfrak{X}'$ with an open cover $\{\mathfrak{U}_i\}_{i\in I}$ of $\mathfrak{X}$ such that $\mathfrak{U}_i$ is a finite union of open affines, and $\mathbf{rig}(\mathfrak{U}_i) \simeq U_i$ for any $i \in I$.

We briefly recall the construction of the specialization map using rig-points in [8] 8.3. If $x \in X$, then there exists an open affine admissible formal subscheme $\mathrm{Spf}(\mathcal{A})$ of $\mathfrak{X}$ such that $\mathbf{rig}(\mathrm{Spf}(\mathcal{A}))$ is an open affinoid neighborhood of x. If $x \in X^{Tate}$, then x is given by a maximal ideal $\mathfrak{m}$ of $\mathcal{A} \otimes_{\mathcal{O}_K} K$. Then $\mathfrak{p} := \mathfrak{m} \cap \mathcal{A}$ is an non-open prime ideal of $\mathcal{A}$ such that $\mathcal{A}/\mathfrak{p}$ is a local integral domain of dimension 1, and $\bar{x} : \mathcal{A} \otimes_{\mathcal{O}_K} k \to (\mathcal{A}/\mathfrak{p}) \otimes_{\mathcal{O}_K} k$ determines a unique closed point in $\mathfrak{X}_k$. Define $sp(x) = \bar{x}$ and this construction is independent of choices of $\mathrm{Spf}(\mathcal{A})$. By [8] page 200 Proposition 8, we see that sp is surjective from X^{Tate} onto $\mathfrak{X}_{k,cl}$.

To extend sp to X, we view a point $x \in U_i$ as a prime filter in $\mathcal{P}(U_i^{Tate})$, and define $sp(x)$ to be the prime filter

$$\{\mathfrak{V} \mid \mathfrak{V} \text{ is an open subscheme of } \mathfrak{U}_i \text{ such that } sp^{-1}(\mathfrak{V}) \in x\}.$$

By the proof of [**?**] Lemma 3.4 and gluing, we get a continous surjection $sp : X \to \mathfrak{X}$, such that if $\mathfrak{U} \subseteq \mathfrak{X}_k$ is open affine, then $sp^{-1}(\mathfrak{U}) \simeq \mathbf{rig}(\mathfrak{U})$.

□

Example 2.2.6. *X is an affinoid rigid analytic variety over K.*

Assume $X \simeq \mathrm{Spa}(A, A^\circ)$, where A is a reduced Banach algebra topologically finite type over K, and A° is the $\mathcal{O}_K$-subalgebra of A consisting of power-bounded elements. Then X is quasi-compact. By [9] p.251 Corollary 6, we see that A° is topologically of finite type over $\mathcal{O}_K$, and in particular is noetherian. Thus $\mathrm{Spf}(A^\circ)$ is an admissible formal model of X, and the specialization map coincides with the canonical reduction in [9] 6.3. If $\mathfrak{X}' \to \mathrm{Spf}(A^\circ)$ and $\mathfrak{X}'' \to \mathfrak{X}'$ are admissible formal blow-ups, then $\mathfrak{X}'' \to \mathrm{Spf}(A^\circ)$ is also an admissible formal blow-up by [8] p.190 Proposition 11. Therefore, given any finite affinoid open cover of X, an admissible formal model $\mathfrak{X}$ of X that satisfies Proposition 1 can be obtained by an admissible formal blow-up of $\mathrm{Spf}(A^\circ)$. In other words, there exists an open coherent ideal sheaf $\mathfrak{a}$ of $\mathrm{Spf}(A^\circ)$, such that $\mathfrak{X}$ is the formal completion of $\widetilde{X}$ along its special fiber, where $\widetilde{X}$ is the blow up of $\mathrm{Spec}(A^\circ)$ along $\Gamma(\mathrm{Spf}(A^\circ), \mathfrak{a})$. Explicitly, assume that $\{g_0, g_1, ..., g_m\} \subseteq A^\circ$ is a set of generators of $\Gamma(\mathrm{Spf}(A^\circ), \mathfrak{a})$. There is an open affine cover $\{\mathfrak{U}_i\}_{i=0,1,...,m}$ of $\mathfrak{X}$, where

$$\mathfrak{U}_i \simeq \mathrm{Spf}(A^\circ\langle y_j; j \neq i\rangle/(g_i y_j - g_j; j \neq i) \text{ modulo } (\pi_K)\text{-torsion }),$$

We observe that the generic fiber $\mathbf{rig}(\mathfrak{U}_i)$ is a rational subset of X. The global section of $\mathcal{O}_{\mathfrak{X}}$ is an $\mathcal{O}_K$-lattice of A, since $A \simeq \Gamma(\mathfrak{X}, \mathcal{O}_{\mathfrak{X}} \otimes_{\mathbb{Z}} \mathbb{Q}) \simeq \Gamma(\mathfrak{X}, \mathcal{O}_{\mathfrak{X}}) \otimes_{\mathbb{Z}} \mathbb{Q}$. The fact that $\widetilde{X}$ is proper over $\mathrm{Spec}(A^\circ)$ implies that $\Gamma(\mathfrak{X}, \mathcal{O}_{\mathfrak{X}})$ is a finite module over A°. Because A° is integrally closed in A, we conclude that $\Gamma(\mathfrak{X}, \mathcal{O}_{\mathfrak{X}}) \simeq A^\circ$.

From the construction, we also see that $\mathcal{L}_{\mathfrak{X}} := \mathfrak{a}\mathcal{O}_{\mathfrak{X}}$ is an invertible sheaf on $\mathfrak{X}$, and $\mathfrak{U}_i$ is the locus in $\mathfrak{X}$ where g_i generates $\mathcal{L}_{\mathfrak{X}}$. If $\mathcal{L}$ is the ample invertible sheaf on the blow-up $\widetilde{X}$, then the formal completion of $\mathcal{L}$ along the special fiber of $\widetilde{X}$ is $\mathcal{L}_{\mathfrak{X}}$. For any $\mathcal{O}_{\mathfrak{X}}$-module $\mathcal{M}$ and $s \in \mathbb{Z}$, let $\mathcal{M}(s)$ be $\mathcal{M} \otimes_{\mathcal{O}_{\mathfrak{X}}} \mathcal{L}_{\mathfrak{X}}^s$, and let $(s)\mathcal{M}$ be $\mathcal{L}_{\mathfrak{X}}^s \otimes_{\mathcal{O}_{\mathfrak{X}}} \mathcal{M}$. Then $\mathcal{M}(s) \simeq (s)\mathcal{M}$ as $\mathcal{O}_{\mathfrak{X}}$-modules. By the theorem of formal functions, we know that $H^i(\mathfrak{X}, \mathcal{L}_{\mathfrak{X}}(s)) = 0$ for $s \gg 0$ and $i \geq 1$. Since $\{\mathcal{L}_{\mathfrak{X}}(s); s \in \mathbb{Z}\}$ generate the category of coherent sheaves of $\mathcal{O}_{\mathfrak{X}}$-modules on $\mathfrak{X}$, we see that $H^i(\mathfrak{X}, \mathcal{M}(s)) = 0$ for a coherent $\mathcal{O}_{\mathfrak{X}}$-module $\mathcal{M}$, for $s \gg 0$ and $i \geq 1$.

Remark 2.2.7. A choice of admissible formal model $\mathfrak{X}$ of X may be viewed as a choice of integral structure on $\mathcal{O}_X$. If $\mathfrak{U} \subseteq \mathfrak{X}$ is an open affine formal subscheme, then a presentation $\mathcal{O}_{\mathfrak{X}}(\mathfrak{U}) \simeq \mathcal{O}_K\langle x_1, x_2, ..., x_N\rangle/I$, where I is a finite generated ideal, gives rise to a presentation of $\mathcal{O}_X(\mathbf{rig}(\mathfrak{U}))$. The gauge norm on $\mathcal{O}_X(\mathbf{rig}(\mathfrak{U}))$ defined by the $\mathcal{O}_K$-lattice $\mathcal{O}_{\mathfrak{X}}(\mathfrak{U})$ is the residue norm associated to the presentation

$$0 \to I \otimes_{\mathcal{O}_K} K \to K\langle x_1, x_2, ..., x_N\rangle \to \mathcal{O}_X(\mathbf{rig}(\mathfrak{U})) \to 0.$$

On the other hand, we have the spectral norm on $\mathcal{O}_X(\mathbf{rig}(\mathfrak{U}))$ with respect to which the unit ball is $\mathcal{O}_X^\circ(\mathbf{rig}(\mathfrak{U}))$. Since the spectral norm is equivalent to all the residue norms by the open mapping theorem [8], we conclude that $\mathcal{O}_X^\circ(\mathbf{rig}(\mathfrak{U}))$ and $\mathcal{O}_{\mathfrak{X}}(\mathfrak{U})$ are commensurable $\mathcal{O}_K$-lattices in $\mathcal{O}_X(\mathbf{rig}(\mathfrak{U}))$.

Lemma 2.2.8. *Let X be an affinoid rigid analytic variety over K. If $\mathcal{M}$ is a presheaf on $\mathcal{B}_X$ such that $sp_*\mathcal{M}$ is a sheaf on $\mathcal{B}_{\mathfrak{X}}$, for any admissible formal model $\mathfrak{X}$ of X obtained by an admissible formal blow-up of* $\mathrm{Spf}(\Gamma(X, \mathcal{O}_X^\circ))$*. Then $\mathcal{M}$ is a sheaf on $\mathcal{B}_X$.*

Proof. Let $U \in \mathcal{B}_X$, and let $\{U_i\}_{i\in I}$ be a finite open cover of affinoid subdomains of U. By Proposition 2.2.5 there exists an admissible formal model $\mathfrak{X}$ of X obtained by an admissible formal blow-up of $\mathrm{Spf}(\Gamma(X, \mathcal{O}_X^\circ))$, such that there is an open cover $\{\mathfrak{U}, \mathfrak{U}_i\}_{i\in I}$ of $\mathfrak{X}$ with $sp^{-1}(\mathfrak{U}_i) = U_i$ and $sp^{-1}(\mathfrak{U}) = U$. Then the conclusion follows from the assumption that $sp_*\mathcal{M}$ is a sheaf on $\mathcal{B}_{\mathfrak{X}}$ for $\mathfrak{X}$.

□

2.2.3 Coadmissible modules over sheaves of Frechet-Stein algebras on admissible formal models

We refer to [28] for the properties of the category of coadmissible modules over Fréchet-Stein algebras.

Definition 2.2.9. Let X be a ringed space such that the structure sheaf $\mathcal{O}_X$ is a K-algebra, and let $\mathcal{B}_X$ be a basis for the topology of X. A sheaf of $\mathcal{O}_X$-algebra $\mathcal{D}$ is a sheaf of Fréchet-Stein algebras on $\mathcal{B}_X$ if there exists a projective system of $\mathcal{O}_X$-algebras $\{\mathcal{D}_n\}_{n\in\mathbb{N}}$, and an isomorphism $\mathcal{D} \simeq \varprojlim_n \mathcal{D}_n$ of $\mathcal{O}_X$-algebras such that

1. $\mathcal{D}_n$ is a coherent sheaf of rings.

2. If $U \in \mathcal{B}_X$, then $\mathcal{D}_n(U)$ is a left and right noetherian K-Banach algebra, and the transition morphisms $\mathcal{D}_n(U) \to \mathcal{D}_{n+1}(U)$ are flat and continuous.

3. If $U, V \in \mathcal{B}_X$ and $V \subseteq U$, then the restriction morphisms $\mathcal{D}_n(U) \to \mathcal{D}_n(V)$ is continuous.

Lemma 2.2.10. *Let $f : D \to B$ be a morphism between Fréchet-Stein algebras, and let M be a coadmissible module over D. Assume the isomorphisms $D \simeq \varprojlim_n D_n$ and $B \simeq \varprojlim_n B_n$ realize the the Fréchet-Stein structures of D and B, such that f factors through $D_n \to B_n$ for each $n \in \mathbb{N}$. Then the isomorphism*

$$B\widehat{\otimes}_D M \simeq \varprojlim_n B_n \otimes_{D_n} M_n,$$

where $M_n = D_n \otimes_D M$, realizes $B\widehat{\otimes}_D M$ as a coadmissible module over B.

Proof. Let ker_n be the kernel of the surjection $B_n\widehat{\otimes}_K M_n \twoheadrightarrow B_n \otimes_{D_n} M_n$, so ker_n is the the closure of the K-linear span of the set $\{ba \otimes m - b \otimes am \mid b \in B_n, a \in D_n, m \in M_n\}$ in $B_n\widehat{\otimes}_K M_n$.

Let ker be the kernel of the surjection $B\widehat{\otimes}_K M \twoheadrightarrow B\widehat{\otimes}_D M$, so ker is the closure of the K-linear span of the set $\{ba \otimes m - b \otimes am \mid b \in B, a \in D, m \in M\}$ in $B\widehat{\otimes}_K M$.

Next, consider the exact sequence

$$0 \to \varprojlim_n ker_n \to \varprojlim_n B_n\widehat{\otimes}_K M_n \to \varprojlim_n B_n \otimes_{D_n} M_n \to R^1 \varprojlim_n ker_n.$$

We know that $B\widehat{\otimes}_K M \simeq \varprojlim_n B_n\widehat{\otimes}_K M_n$. Since $B \to B_n$, $D \to D_n$ and $M \to M_n$ have dense image, we see that $ker \to ker_n$ also has dense image. Therefore $ker \simeq \varprojlim_n ker_n$ and $R^1 \varprojlim_n ker_n = 0$. The desired isomorphism follows.

Since M_n is a finite module over D_n, we see that $B_n \otimes_{D_n} M_n$ is finite module over B_n. Moreover, $B_n \otimes_{D_n} M_n \simeq B_n \otimes_{D_n} (D_n \otimes_{D_{n+1}} M_{n+1}) \simeq B_n \otimes_{B_{n+1}} (B_{n+1} \otimes_{D_{n+1}} M_{n+1})$.

□

Lemma 2.2.11. *Suppose $\mathcal{D} \simeq \varprojlim_n \mathcal{D}_n$ is a sheaf of Fréchet-Stein algebras on $\mathcal{B}_X$, and the isomorphism $D := \Gamma(X, \mathcal{D}) \simeq \varprojlim_n \Gamma(X, \mathcal{D}_n)$ realizes $\Gamma(X, \mathcal{D}_n)$ as a Fréchet-Stein algebra. If M is a coadmissible module over D, then for $U \in \mathcal{B}_X$, Lemma 2.2.10 implies that $\widetilde{M}(U) := \mathcal{D}(U)\widehat{\otimes}_D M$ is a coadmissible module over $\mathcal{D}(U)$. In this way we get a presheaf $\widetilde{M}$ on $\mathcal{B}_X$.*

If the functors $\Gamma(X,-)$ and $\mathcal{D}_n \otimes_{\Gamma(X,\mathcal{D}_n)} -$ induce an equivalence between the category of coherent $\mathcal{D}_n$-modules and the category of coherent $\Gamma(X,\mathcal{D}_n)$-modules, then

1. $\mathcal{M}_n := \mathcal{D}_n \otimes_{\Gamma(X,\mathcal{D})} M$ *is a coherent* $\mathcal{D}_n$*-module.*

2. $\widetilde{M} \simeq \varprojlim_n \mathcal{M}_n$*. In particular* $\widetilde{M}$ *is a sheaf on* $\mathcal{B}_X$*.*

Proof. By [28] Corollary 3.1, we see that $M_n := \Gamma(X,\mathcal{D}_n) \otimes_{\Gamma(X,\mathcal{D})} M$ is a finite generated module over $\Gamma(X,\mathcal{D}_n)$, so $\mathcal{M}_n \simeq \mathcal{D}_n \otimes_{\Gamma(X,\mathcal{D}_n)} M_n$ is a coherent $\mathcal{D}_n$-module.

For any $U \in \mathcal{B}_X$, we have $\mathcal{M}_n(U) \simeq \mathcal{D}_n(U) \otimes_{\Gamma(X,\mathcal{D}_n)} M_n$. Therefore $\widetilde{M}(U) \simeq \varprojlim_n \mathcal{M}_n(U)$. □

Lemma 2.2.12. *Let $\mathfrak{X}$ be an admissible formal model of a rigid analytic variety X over K. Let $\mathcal{D} \simeq \varprojlim_n \mathcal{D}_n$ be a sheaf of Fréchet-Stein algebras on $\mathcal{B}_{\mathfrak{X}}$, such that*

1. $D_n := \Gamma(\mathfrak{X},\mathcal{D}_n)$ *is a left and right noetherian K-Banach algebra for $n \in \mathbb{N}$, and the isomorphism $D := \Gamma(\mathfrak{X},\mathcal{D}) \simeq \varprojlim_n D_n$ realizes D as a Fréchet-Stein algebra.*

2. *If $\mathfrak{U} \in \mathcal{B}_{\mathfrak{X}}$, the restriction morphism $D_n \to \mathcal{D}_n(\mathfrak{U})$ is continuous for $n \in \mathbb{N}$.*

Assume that the functors $\Gamma(\mathfrak{X},-)$ and $\mathcal{D}_n \otimes_{D_n} -$ induce an equivalence between the category of coherent $\mathcal{D}_n$-modules and the category of coherent D_n-modules. If there exists a finite open affine cover $\{\mathfrak{U}_i\}_{i\in I}$ of $\mathfrak{X}$ and coadmissible modules M^i over $\mathcal{D}(\mathfrak{U}_i)$ such that $\mathcal{M}|_{\mathfrak{U}_i} \simeq \widetilde{M^i}$. Then

1. *$\Gamma(\mathfrak{X},\mathcal{M})$ is a coadmissible module over D, and $\mathcal{M} \simeq \widetilde{\Gamma(\mathfrak{X},\mathcal{M})}$ (defined in Lemma 2.2.11).*

2. *$H^i(\mathfrak{X},\mathcal{M}) = 0$ for $i \geq 1$.*

Proof. Denote $\mathcal{M}_n := \mathcal{D}_n \otimes_{\mathcal{D}} \mathcal{M}$ and $M_n := \Gamma(\mathfrak{X},\mathcal{M}_n)$.

1. We know that $M^i \simeq \varprojlim_n M^i_n$ where $M^i_n = \mathcal{D}_n(\mathfrak{U}_i) \otimes_{\mathcal{D}(\mathfrak{U}_i)} M^i$. Since

$$\mathcal{M}_n|_{\mathfrak{U}_i} \simeq \mathcal{D}_n|_{\mathfrak{U}_i} \otimes_{\mathcal{D}|_{\mathfrak{U}_i}} \widetilde{M^i} \simeq \mathcal{D}_n|_{\mathfrak{U}_i} \otimes_{\mathcal{D}_n(\mathfrak{U}_i)} M^i_n,$$

we see that $\mathcal{M}_n$ is a coherent module over $\mathcal{D}_n$. It follows that $M_n := \Gamma(\mathfrak{X}, \mathcal{M}_n)$ is a finite module over $\Gamma(\mathfrak{X}, \mathcal{D}_n)$. Since

$$\begin{aligned}\mathcal{M}_n &\simeq \mathcal{D}_n \otimes_{\mathcal{D}_{n+1}} \mathcal{M}_{n+1} \simeq \mathcal{D}_n \otimes_{D_{n+1}} M_{n+1} \\ &\simeq \mathcal{D}_n \otimes_{D_n} (D_n \otimes_{D_{n+1}} M_{n+1}),\end{aligned}$$

we see that $M_n \simeq D_n \otimes_{D_{n+1}} M_{n+1}$. Since $\mathcal{M}|_{\mathfrak{U}_i} \simeq \varprojlim_n \mathcal{M}_n|_{\mathfrak{U}_i}$, the natural morphism $\mathcal{M} \to \varprojlim_n \mathcal{M}_n$ is an isomorphism. Thus $\Gamma(\mathfrak{X}, \mathcal{M}) \simeq \varprojlim_n M_n$ is a coadmissible module over D, and $\mathcal{M} \simeq \Gamma\widetilde{(\mathfrak{X}, \mathcal{M})}$.

2. Consider the Čech complex of $\mathcal{M}_n$ associated to a finite refinement $\{\mathfrak{V}_j\}_{j\in J}$ of the cover $\{\mathfrak{U}_i\}_{i\in I}$. Then

$$\left\{C^{\cdot}(\{\mathfrak{V}_j\}_{j\in J}, \mathcal{M}_n)\right\}_{n\in\mathbb{N}}$$

is a projective system of K-Fréchet spaces with continuous differentials. By [28] Theorem A, we see that

$$\check{H}^0(\{\mathfrak{V}_j\}_{j\in J}, \mathcal{M}_{n+1}) \to \check{H}^0(\{\mathfrak{V}_j\}_{j\in J}, \mathcal{M}_n)$$

has dense image. Therefore, Proposition 2.2.4 implies the isomorphism

$$\check{H}^1(\{\mathfrak{V}_j\}_{j\in J}, \mathcal{M}) \simeq \varprojlim_n \check{H}^1(\{\mathfrak{V}_j\}_{j\in J}, \mathcal{M}_n) = 0.$$

Hence for $i > 1$ we can apply Proposition 2.2.4 again to conclude that

$$\check{H}^i(\{\mathfrak{V}_j\}_{j\in J}, \mathcal{M}) \simeq \varprojlim_n \check{H}^i(\{\mathfrak{V}_j\}_{j\in J}, \mathcal{M}_n) = 0.$$

Thus $H^i(\mathfrak{X}, \mathcal{M}) = 0$ for $i \geq 1$.

□

2.2.4 Review of smooth and étale morphisms of adic spaces

We first recall from [18] 1.6 and 1.7 some properties of smooth and étale morphisms of rigid analytic varieties.

Proposition 2.2.13. *Let $X \xrightarrow{f} Y$ be a smooth morphism between rigid analytic varieties over K. Then the following statements are true:*

1. *The relative sheaf of differentials $\Omega^1_{X/Y}$ is a locally free coherent $\mathcal{O}_X$-module.*

2. *If $Y \simeq \mathrm{Spa}(B, B^\circ)$ is an affinoid, then for any $x \in X$, there exists an open affinoid neighborhood U of x such that:*

   (a) *$\Omega^1_{X/Y}|_U$ is free.*

   (b) *There exists an étale morphism $U \xrightarrow{g} Spa(B\langle x_1, x_2, ..., x_d\rangle, C^+)$ such that $f|_U = h \circ g$, where C^+ is the integral closure of $B^\circ\langle x_1, x_2, ..., x_d\rangle)$ in $B\langle x_1, x_2, ..., x_d\rangle$, and $\mathrm{Spa}(B\langle x_1, x_2, ..., x_d\rangle, C^+) \xrightarrow{h} \mathrm{Spa}(B, B^\circ)$ is the natural projection.*

3. *If $X \simeq \mathrm{Spa}(A, A^\circ)$ and $Y \simeq \mathrm{Spa}(B, B^\circ)$ are affinoids, then f is étale if and only if there exists a presentation of X as $A \simeq B\langle x_1, x_2, ..., x_N\rangle/(f_1, f_2, ..., f_N)$, such that $f_i \in B\langle x_1, x_2, ..., x_N\rangle$ and the determinant of the Jacobian matrix $(\dfrac{\partial f_i}{\partial x_j})$ is invertible in A.*

Convention-Notation 3. A rigid analytic variety X over K is smooth if the morphism $X \to \mathrm{Spa}(K, K^\circ)$ is smooth. By Proposition 2.2.13 we know that for any $x \in X$, there exists an open affinoid neighborhood U of x, such that $\Omega^1_X|_U$ is free and there exists an étale morphism

$$U \to \mathrm{Spa}(K\langle x_1, x_2, ..., x_d\rangle, K^\circ\langle x_1, x_2, ..., x_d\rangle).$$

We shall call $\{x_1, x_2, ..., x_d\}$ a system of local coordinates around x on X.

Remark 2.2.14. Let $\mathcal{T}_X := \mathcal{H}om_{\mathcal{O}_X}(\Omega^1_X, \mathcal{O}_X)$ be the tangent sheaf of X. If X is smooth over K, a choice of local coordinates for X gives rise to a set of generators $\{\xi_1, \xi_2, ..., \xi_d\} \subseteq \mathcal{T}_X(U)$ of $\mathcal{T}_X|_U$ as a free $\mathcal{O}_X|_U$-module, where ξ_i is dual to dx_i.

If $\mathfrak{U}$ is an admissible formal model of U over $\mathcal{O}_K$, and let $\mathcal{T}_{\mathfrak{U}} := \mathcal{H}om_{\mathcal{O}_{\mathfrak{U}}}(\Omega_{\mathfrak{U}}, \mathcal{O}_{\mathfrak{U}})$ be the tangent sheaf of $\mathfrak{U}$. It follows from Proposition 2.2.5 that the natural morphism $\Omega^1_{\mathfrak{U}} \otimes_{\mathbb{Z}} \mathbb{Q} \xrightarrow{\simeq} sp_*\Omega^1_U$ is an isomorphism, which induces the isomorphism $\mathcal{T}_{\mathfrak{U}} \otimes_{\mathbb{Z}} \mathbb{Q} \simeq sp_*\mathcal{T}_U$. Since $\mathcal{O}_{\mathfrak{U}}$ is (π_K)-torsion free, there is an inclusion $\mathcal{T}_{\mathfrak{U}} \hookrightarrow \mathcal{T}_{\mathfrak{U}} \otimes_{\mathbb{Z}} \mathbb{Q}$. Thus, we get the isomorphisms $\Gamma(\mathfrak{U}, \mathcal{T}_{\mathfrak{U}}) \otimes_{\mathbb{Z}} \mathbb{Q} \simeq \Gamma(\mathfrak{U}, \mathcal{T}_{\mathfrak{U}} \otimes_{\mathbb{Z}} \mathbb{Q}) \simeq \mathcal{T}_X(U)$.

Based on the definitions of a smooth Lie algebroid and the associated twisted differential algebras on schemes given in [4] and [20], we define the following:

Definition 2.2.15. A smooth Lie algebroid $\mathcal{T}$ on a smooth rigid analytic variety X is a locally free coherent $\mathcal{O}_X$-module with a morphism of $\mathcal{O}_X$-modules $\sigma : \mathcal{T} \to \mathcal{T}_X$ and a K-linear pairing $[\cdot, \cdot] : \mathcal{T} \otimes_K \mathcal{T} \to \mathcal{T}$ such that

1. $[\cdot, \cdot]$ is a Lie algebra bracket and σ commutes with the brackets.

2. For $l_1, l_2 \in \mathcal{T}$ and $f \in \mathcal{O}_X$, one has $[l_1, fl_2] = f[l_1, l_2] + \sigma(l_1)(f)l_2$.

Definition 2.2.16. A twisted sheaf of differential algebras $\mathcal{A}_{\mathcal{T}}$ associated to a smooth Lie algebroid $\mathcal{T}$ is a sheaf of algebras equipped with a morphism between sheaves of algebras $i : \mathcal{O}_X \to \mathcal{A}_{\mathcal{T}}$, and an increasing filtration $F.\mathcal{A}_{\mathcal{T}}$ of $\mathcal{O}_X$-modules such that

1. $\mathcal{A}_{\mathcal{T}} = \bigcup_n F_n\mathcal{A}_{\mathcal{T}}$ and $F_n = 0$ if $n < 0$.

2. i induces an isomorphism $\mathcal{O}_X \xrightarrow{\simeq} F_0\mathcal{A}_{\mathcal{T}}$ of $\mathcal{O}_X$-modules.

3. $F_{m_1}\mathcal{A}_{\mathcal{T}} \cdot F_{m_2}\mathcal{A}_{\mathcal{T}} \subseteq F_{m_1+m_2}\mathcal{A}_{\mathcal{T}}$, for $m_1, m_2 \in \mathbb{N}$.

4. $[F_{m_1}\mathcal{A}_{\mathcal{T}}, F_{m_2}\mathcal{A}_{\mathcal{T}}] \subseteq F_{m_1+m_2-1}\mathcal{A}_{\mathcal{T}}$, for $m_1, m_2 \in \mathbb{N}$, where the bracket is the commutator.

5. There is an isomorphism of Lie algebroids: $\mathrm{gr}_1^F \mathcal{A}_{\mathcal{T}} \simeq \mathcal{T}$.

6. The natural morphism $\mathrm{Sym}_{\mathcal{O}_X}(\mathcal{T}) \to \mathrm{gr}_\cdot^F(\mathcal{A}_{\mathcal{T}})$ is an isomorphism of sheaves of $\mathcal{O}_X$-algebras.

A morphism between twisted sheaves of differential algebras $\mathcal{A}_{\mathcal{T}}$ and $\mathcal{A}'_{\mathcal{T}}$ which are associated to a smooth Lie algebroid $\mathcal{T}$ is a morphism $\mathcal{A}_{\mathcal{T}} \to \mathcal{A}'_{\mathcal{T}}$ of sheaves of filtered $\mathcal{O}_X$-algebras.

Lemma 2.2.17. *If $\mathcal{T}$ is a smooth Lie algebroid on X, there exists a twisted sheaf of differential algebras $\mathcal{U}(\mathcal{T})$, which is called the universal enveloping algebra of $\mathcal{T}$, such that*

1. *There is a morphism $\mathcal{T} \xrightarrow{i_{\mathcal{T}}} F_1\mathcal{U}(\mathcal{T})$ that splits the exact sequence of $\mathcal{O}_X$-modules*

$$0 \to \mathcal{O}_X \xrightarrow{i} F_1\mathcal{U}(\mathcal{T}) \to \mathcal{T} \to 0.$$

2. *If $f \in \mathcal{O}_X$ and $\xi, \mu \in \mathcal{T}$, then $[i_{\mathcal{T}}(\xi), i(f)] = i(\sigma(\xi)(f))$ and $[i_{\mathcal{T}}(\xi), i_{\mathcal{T}}(\mu)] = i_{\mathcal{T}}([\xi, \mu])$.*

Proof. We first recall some constructions from [25]. Define $F_0(\mathcal{U}(\mathcal{T})) := \mathcal{O}_X$ and $F_1(\mathcal{U}(\mathcal{T})) := \mathcal{O}_X \oplus \mathcal{T}$, and $i : \mathcal{O}_X \to F_1(\mathcal{U}(\mathcal{T}))$ is the natural inclusion. For $U \in \mathcal{B}_X$, the $\mathcal{O}_X(U)$-module $F_1(\mathcal{U}(\mathcal{T}))(U)$ is a Lie algebra over K with the Lie bracket $[r+\alpha, s+\mu] = \sigma(\alpha)(s) - \sigma(\mu)(r) + [\alpha, \mu]$ for $r, s \in \mathcal{O}_X(U)$ and $\alpha, \mu \in \mathcal{T}(U)$. Let $U^+(\mathcal{O}_X(U), \mathcal{T}(U))$ be subalgebra of the universal enveloping algebra of $F_1(\mathcal{U}(\mathcal{T}))(U)$ generated by the image of $F_1(\mathcal{U}(\mathcal{T}))(U)$. Let $V(\mathcal{O}_X(U), \mathcal{T}(U))$ be the quotient of $U^+(\mathcal{O}_X(U), \mathcal{T}(U))$ by the two-sided ideal generated by

$$\{r \cdot z - \imath(rz) \mid r \in \mathcal{O}_X(U), z \in F_1(\mathcal{U}(\mathcal{T}))(U)\},$$

where $r \cdot z$ denotes the multiplication in $U^+(\mathcal{O}_X(U), \mathcal{T}(U))$ and

$$\imath : F_1(\mathcal{U}(\mathcal{T}))(U) \to U^+(\mathcal{O}_X(U), \mathcal{T}(U))$$

is the canonical inclusion. Let $i_{\mathcal{T}}$ be the composition of morphisms

$$\mathcal{T}(U) \hookrightarrow F_1(\mathcal{U}(\mathcal{T}))(U) \xrightarrow{\imath} U^+(\mathcal{O}_X(U), \mathcal{T}(U)) \twoheadrightarrow V(\mathcal{O}_X(U), \mathcal{T}(U)).$$

For $s \in \mathbb{Z}^{>0}$, let $V_s(\mathcal{O}_X(U), \mathcal{T}(U))$ be the left $\mathcal{O}_X(U)$-submodule of $V(\mathcal{O}_X(U), \mathcal{T}(U))$ generated by $\mathcal{O}_X(U)$ and at most s elements of the image of $\mathcal{T}(U)$ in $V(\mathcal{O}_X(U), \mathcal{T}(U))$, and define $V_0(\mathcal{O}_X(U), \mathcal{T}(U)) := \mathcal{O}_X(U)$. By [25] section 2 and Theorem 3.1, we know that $F_1(\mathcal{U}(\mathcal{T}))(U) \to V(\mathcal{O}_X(U), \mathcal{T}(U))$ induces an isomorphism

$$F_1(\mathcal{U}(\mathcal{T}))(U) \xrightarrow{\simeq} V_1(\mathcal{O}_X(U), \mathcal{T}(U)),$$

and moreover,

$$Sym_{\mathcal{O}_X(U)}(\mathcal{T}(U)) \to \bigoplus_{s\in\mathbb{N}} V_s(\mathcal{O}_X(U), \mathcal{T}(U))/V_{s-1}(\mathcal{O}_X(U), \mathcal{T}(U))$$

is an isomorphism as graded $\mathcal{O}_X(U)$-algebras.

It follows that we can define a presheaf $\mathcal{U}(\mathcal{T})$ such that $\mathcal{U}(\mathcal{T})(U) := V(\mathcal{O}_X(U), \mathcal{T}(U))$ for $U \in \mathcal{B}_X$, and an increasing exhaustive filtration $F_i\mathcal{U}(\mathcal{T})^{\mathrm{pre}}$ such that $F_i\mathcal{U}(\mathcal{T})^{\mathrm{pre}}(U) := V_i(\mathcal{O}_X(U), \mathcal{T}(U))$ for $U \in \mathcal{B}_X$. Since $Sym_{\mathcal{O}_X}\mathcal{T} \simeq gr^F_{\cdot}\mathcal{U}(\mathcal{T})$ as presheaves of $\mathcal{O}_X$-modules, by Lemma 2.2.2 and Lemma 2.2.1, we see that $\mathcal{U}(\mathcal{T})$ is a sheaf on X. Let $F_i\mathcal{U}(\mathcal{T})$ be the sheafification of $F_i\mathcal{U}(\mathcal{T})^{\mathrm{pre}}$, and we see that $\mathcal{U}(\mathcal{T})$ is a twisted sheaf of differential algebras that satisfies (1) and (2).

□

Example 2.2.18.

1. If $X \xrightarrow{f} Y$ is a smooth morphism between smooth rigid analytic varieties over K, then the relative tangent sheaf $\mathcal{T}_{X/Y} := \mathcal{H}om_{\mathcal{O}_X}(\Omega^1_{X/Y}, \mathcal{O}_X)$ is a smooth Lie algebroid on X. If $U \subseteq X$ and $V \subseteq Y$ are open affinoids such that $f(U) \subseteq V$, then $\mathcal{T}_{X/Y}(U)$ is the set of continuous $\mathcal{O}_Y(V)$-derivations from $\mathcal{O}_X(U)$ to $\mathcal{O}_X(U)$, so $\mathcal{T}_{X/Y}$ carries a Lie

bracket. Moreover, the exact sequence of $\mathcal{O}_X$-modules

$$0 \to f^*\Omega^1_Y \to \Omega^1_X \to \Omega^1_{X/Y} \to 0$$

implies that there is a morphism of $\mathcal{O}_X$-modules $\mathcal{T}_{X/Y} \xrightarrow{\sigma} \mathcal{T}_X$ which preserves the Lie brackets.

2. If X is a smooth rigid analytic variety over K, then the tangent sheaf $\mathcal{T}_X$ is a smooth Lie algebroid and $\mathcal{U}(\mathcal{T}_X)$ is the sheaf of finite order differential operators on X, which will be denoted by $\mathcal{D}_X$.

2.3 $\mathcal{D}^\infty$-modules on smooth affinoid rigid analytic varieties over K

In section 2.3, we assume that $X \simeq \mathrm{Spa}(A, A^\circ)$ is an affinoid rigid analytic variety over K and $\mathcal{T}$ is a smooth Lie algebroid on X such that $\mathcal{T}$ is a free $\mathcal{O}_X$-module. Let $\mathfrak{X}$ be an admissible formal model of X obtained by an admissible formal blow-up of $\mathrm{Spf}(A^\circ)$, with $sp : X \to \mathfrak{X}$ defined in Proposition 2.2.5. Let $\mathcal{A}_{\mathcal{T}}$ be a twisted sheaf of differential algebras associated to $\mathcal{T}$.

2.3.1 The Fréchet completion of the twisted sheaf of differential algebra on smooth affinoid rigid analytic varieties over K

In this subsection we define a completion of $\mathcal{A}_{\mathcal{T}}$ which has the structure of a sheaf of Fréchet-Stein algebra on $\mathcal{B}_X$ (see Definition 2.2.9).

There exists $\{\xi_1, \xi_2, ...\xi_d\} \subseteq \Gamma(X, \mathcal{T})$ such that ξ_i generates $\mathcal{T}$ as a free $\mathcal{O}_X$-module. For $n \in \mathbb{N}$, let $\mathfrak{T}_n$ be the free $\mathcal{O}_{\mathfrak{X}}$-submodule of $sp_*\mathcal{T}$ generated by $\{\alpha^n\xi_1, \alpha^n\xi_2, ...\alpha^n\xi_d\}$. By the discussion in Remark 2.2.14, we see that there exists $N > 0$ such that for $n > N$:

1. $[\mathfrak{T}_n, \mathfrak{T}_n] \subseteq \mathfrak{T}_n$.

2. $\sigma(\xi_i) \in \Gamma(\mathfrak{X}, \mathcal{T}_{\mathfrak{X}}) \subseteq \Gamma(X, \mathcal{T}_X)$.

By Definition 2.2.16 we have the exact sequence of $\mathcal{O}_X$-modules:

$$0 \to \mathcal{O}_X \xrightarrow{i_0} F_1(\mathcal{A}_{\mathcal{T}}) \xrightarrow{q_1} \mathcal{T} \to 0$$

Let $\tilde{\xi}_i \in F_1(\mathcal{A}_{\mathcal{T}})$ such that $q_1(\tilde{\xi}_i) = \xi_i$. Let $F_1(\mathcal{A}_{\mathfrak{T}_n,\mathfrak{X}})$ be the sheaf associated to the presheaf $F_1(\mathcal{A}_{\mathfrak{T}_n,\mathfrak{X}})^{\text{pre}}$ on $\mathfrak{X}$, where $F_1(\mathcal{A}_{\mathfrak{T}_n,\mathfrak{X}})^{\text{pre}}(\mathfrak{U})$ is the $\mathcal{O}_{\mathfrak{X}}(\mathfrak{U})$-submodule of $F_1(\mathcal{A}_{\mathcal{T}})(sp^{-1}\mathfrak{U})$ generated by $i_0(\mathcal{O}_{\mathfrak{X}}(\mathfrak{U}))$ and $\alpha^n\tilde{\xi}_i$, for $\mathfrak{U} \in \mathcal{B}_{\mathfrak{X}}$. It follows from the construction that we have the following exact sequences of $\mathcal{O}_{\mathfrak{X}}$-modules:

$$0 \to \mathcal{O}_{\mathfrak{X}} \to F_1(\mathcal{A}_{\mathfrak{T}_n,\mathfrak{X}}) \to \mathfrak{T}_n \to 0.$$

By the structure theorem of coherent $\mathcal{O}_{\mathfrak{X}}$-modules, if $\mathfrak{U}$ is an open affine formal subscheme of $\mathfrak{X}$, then $F_1(\mathcal{A}_{\mathfrak{T}_n,\mathfrak{X}})^{\text{pre}}|_{\mathfrak{U}} \simeq F_1(\mathcal{A}_{\mathfrak{T}_n,\mathfrak{X}})|_{\mathfrak{U}}$ is a sheaf.

Let $\mathcal{A}_{\mathfrak{T}_n,\mathfrak{X}}$ be the sheaf on $\mathfrak{X}$ associated to the presheaf

$$\mathcal{A}^{\text{pre}}_{\mathfrak{T}_n,\mathfrak{X}}(\mathfrak{U}) := \text{the subalgebra of } \mathcal{A}_{\mathcal{T}}(sp^{-1}(\mathfrak{U})) \text{ generated by } F_1(\mathcal{A}_{\mathfrak{T}_n,\mathfrak{X}})(\mathfrak{U}),$$

for $\mathfrak{U} \in \mathcal{B}_{\mathfrak{X}}$. Let us define an increasing exhaustive filtration on $\mathcal{A}_{\mathfrak{T}_n,\mathfrak{X}}$ inductively. Let $F_i(\mathcal{A}_{\mathfrak{T}_n,\mathfrak{X}})$ be sheaf associated to the presheaf

$$F_i(\mathcal{A}_{\mathfrak{T}_n,\mathfrak{X}})^{\text{pre}}(\mathfrak{U}) := F_1(\mathcal{A}_{\mathfrak{T}_n,\mathfrak{X}})^{\text{pre}}(\mathfrak{U}) \cdot F_{i-1}(\mathcal{A}_{\mathfrak{T}_n,\mathfrak{X}})^{\text{pre}}(\mathfrak{U})$$

for $\mathfrak{U} \in \mathcal{B}_{\mathfrak{X}}$ and $i \in \mathbb{Z}^{>0}$, and let $F_0(\mathcal{A}_{\mathfrak{T}_n,\mathfrak{X}}) := \mathcal{O}_{\mathfrak{X}}$. It follows from the construction that $F_i(\mathcal{A}_{\mathfrak{T}_n,\mathfrak{X}}) \cdot F_j(\mathcal{A}_{\mathfrak{T}_n,\mathfrak{X}}) \subseteq F_{i+j}(\mathcal{A}_{\mathfrak{T}_n,\mathfrak{X}})$, and $[F_i(\mathcal{A}_{\mathfrak{T}_n,\mathfrak{X}})), F_j(\mathcal{A}_{\mathfrak{T}_n,\mathfrak{X}})] \subseteq F_{i+j-1}(\mathcal{A}_{\mathfrak{T}_n,\mathfrak{X}})$, so $\text{gr}^F_\cdot(\mathcal{U}_{\mathfrak{X}}(\mathfrak{T}_n)$ is a commutative $\mathcal{O}_{\mathfrak{X}}$-ring. Therefore, in addition to the isomorphism

$\mathrm{gr}_1^F(\mathcal{A}_{\mathfrak{T}_n,\mathfrak{X}}) \xrightarrow{\simeq} \mathfrak{T}_n$, we have a surjective morphism of graded $\mathcal{O}_{\mathfrak{X}}$-rings:

$$\mathrm{Sym}_{\mathcal{O}_{\mathfrak{X}}}(\mathfrak{T}_n) \to \mathrm{gr}_{\cdot}^F(\mathcal{A}_{\mathfrak{T}_n,\mathfrak{X}}).$$

Since $\mathrm{gr}_i^F(\mathcal{A}_{\mathfrak{T}_n,\mathfrak{X}}))$ is flat over $\mathcal{O}_K$, the isomorphism $\mathrm{Sym}^i_{\mathcal{O}_X}(\mathcal{T}) \xrightarrow{\simeq} \mathrm{gr}_i^F(\mathcal{A}_{\mathcal{T}})$ implies that $\mathrm{Sym}^i_{\mathcal{O}_{\mathfrak{X}}}(\mathfrak{T}_n) \to \mathrm{gr}_i^F(\mathcal{A}_{\mathfrak{T}_n,\mathfrak{X}})$ is injective. Therefore $\mathrm{Sym}_{\mathcal{O}_{\mathfrak{X}}}(\mathfrak{T}_n) \to \mathrm{gr}^F(\mathcal{A}_{\mathfrak{T}_n,\mathfrak{X}})$ is an isomorphism.

Lemma 2.3.1. *Let $\widehat{\mathcal{A}}_{\mathfrak{T}_n,\mathfrak{X}}$ be the p-adic completion of $\mathcal{A}_{\mathfrak{T}_n,\mathfrak{X}}$, and $\widehat{\mathcal{A}}_{\mathfrak{T}_n,\mathfrak{X},\mathbb{Q}} := \widehat{\mathcal{A}}_{\mathfrak{T}_n,\mathfrak{X}} \otimes_{\mathbb{Z}} \mathbb{Q}$. Then the following statements are true:*

1. *$\mathcal{A}_{\mathfrak{T}_n,\mathfrak{X}}$ and $\widehat{\mathcal{A}}_{\mathfrak{T}_n,\mathfrak{X}}$ are left (resp. right) coherent sheaves of rings.*

2. *$\widehat{\mathcal{A}}_{\mathfrak{T}_n,\mathfrak{X},\mathbb{Q}}$ is a left (resp. right) coherent sheaf of left (resp. right) noetherian Banach algebra. i.e. $\widehat{\mathcal{A}}_{\mathfrak{T}_n,\mathfrak{X},\mathbb{Q}}$ is a left (resp. right) coherent sheaf of rings, and if $\mathfrak{U} \subseteq \mathfrak{X}$ is open affine, then $\widehat{\mathcal{A}}_{\mathfrak{T}_n,\mathfrak{X},\mathbb{Q}}(\mathfrak{U})$ is a left (resp. right) noetherian Banach algebra.*

3. *$\Gamma(\mathfrak{X}, \widehat{\mathcal{A}}_{\mathfrak{T}_n,\mathfrak{X},\mathbb{Q}})$ is a left (resp. right) noetherian Banach algebra.*

Proof. Since $gr_{\cdot}^F \mathcal{A}_{\mathfrak{T}_n,\mathfrak{X}} \simeq \mathcal{O}_{\mathfrak{X}}[\alpha^n\xi_1, \alpha^n\xi_2, ..., \alpha^n\xi_d]$, by Lemma 2.2.2, we see that $\mathcal{A}_{\mathfrak{T}_n,\mathfrak{X}}$ is a quasi-coherent sheaf of $\mathcal{O}_{\mathfrak{X}}$-module, in the sense that $\mathcal{A}_{\mathfrak{T}_n,\mathfrak{X}}/(\pi_K^i)$ is a quasi-coherent sheaf of $(\mathcal{O}_{\mathfrak{X}}/(\pi_K^i))$-module, for any $i \in \mathbb{Z}^{\geq 1}$. By [28] Proposition 1.2, we also see that $\mathcal{A}_{\mathfrak{T}_n,\mathfrak{X}}(\mathfrak{U})$ is left (resp. right) noetherian. By [5] 3.2.3, the p-adic completion $\widehat{\mathcal{A}}_{\mathfrak{T}_n,\mathfrak{X}}(\mathfrak{U})$ is left (resp. right) noetherian. It follows that the algebra $\widehat{\mathcal{A}}_{\mathfrak{T}_n,\mathfrak{X},\mathbb{Q}}(\mathfrak{U})$ is left (resp. right) noetherian. Also, since $\widehat{\mathcal{A}}_{\mathfrak{T}_n,\mathfrak{X},\mathbb{Q}}(\mathfrak{U}) \simeq \widehat{\mathcal{A}}_{\mathfrak{T}_n,\mathfrak{X}}(\mathfrak{U}) \otimes_{\mathcal{O}_K} K$, we may endow $\widehat{\mathcal{A}}_{\mathfrak{T}_n,\mathfrak{X},\mathbb{Q}}(\mathfrak{U})$ an K-Banach space structure with the gauge norm define by the lattice $\widehat{\mathcal{A}}_{\mathfrak{T}_n,\mathfrak{X}}(\mathfrak{U})$. The continuity of the multiplication follows from the definition.

To show that $\mathcal{A}_{\mathfrak{T}_n,\mathfrak{X}}$, $\widehat{\mathcal{A}}_{\mathfrak{T}_n,\mathfrak{X}}$ and $\widehat{\mathcal{A}}_{\mathfrak{T}_n,\mathfrak{X},\mathbb{Q}}$ are left (resp. right) coherent sheaf of rings, it suffices to show that if $\mathfrak{U}_1 \subseteq \mathfrak{U}_2$ are open affine formal subschemes of $\mathfrak{X}$, the restriction morphisms from $\mathfrak{U}_2$ to $\mathfrak{U}_1$ are flat. First observe that $\mathcal{A}_{\mathfrak{T}_n,\mathfrak{X}}(\mathfrak{U}_2) \to \mathcal{A}_{\mathfrak{T}_n,\mathfrak{X}}(\mathfrak{U}_1)$ is flat, since

$\mathrm{gr}^F_\cdot \mathcal{A}_{\mathfrak{T}_n,\mathfrak{X}}(\mathfrak{U}_2) \to \mathrm{gr}^F_\cdot \mathcal{A}_{\mathfrak{T}_n,\mathfrak{X}}(\mathfrak{U}_1)$ is flat. By [5] 3.2.3, the morphism $\widehat{\mathcal{A}}_{\mathfrak{T}_n,\mathfrak{X}}(\mathfrak{U}_2) \to \widehat{\mathcal{A}}_{\mathfrak{T}_n,\mathfrak{X}}(\mathfrak{U}_1)$ is flat. It follows that $\widehat{\mathcal{A}}_{\mathfrak{T}_n,\mathfrak{X},\mathbb{Q}}(\mathfrak{U}_2) \to \widehat{\mathcal{A}}_{\mathfrak{T}_n,\mathfrak{X},\mathbb{Q}}(\mathfrak{U}_1)$ is flat.

Since $\Gamma(\mathfrak{X}, \mathcal{O}_{\mathfrak{X}}) \simeq A^\circ$ by Example 2.2.6, we see that $\Gamma(\mathfrak{X}, \mathrm{gr}^F_\cdot \mathcal{A}_{\mathfrak{T}_n,\mathfrak{X}}) \simeq A^\circ[\alpha^n\xi_1, \alpha^n\xi_2, ..., \alpha^n\xi_d]$. Therefore $\Gamma(\mathfrak{X}, \mathcal{A}_{\mathfrak{T}_n,\mathfrak{X}})$ is left (resp. right) noetherian, and $\Gamma(\mathfrak{X}, \widehat{\mathcal{A}}_{\mathfrak{T}_n,\mathfrak{X},\mathbb{Q}})$ is a left (resp. right) noetherian Banach algebra.

□

Definition-Lemma 2.3.2. *Consider the projective system of sheaves $\left\{\widehat{\mathcal{A}}_{\mathfrak{T}_n,\mathfrak{X},\mathbb{Q}}\right\}_{n\in\mathbb{N}}$ with transition maps induced by the natural inclusions. Define a sheaf of $\mathcal{O}_{\mathfrak{X}}$-algebra $\mathcal{A}^\infty_{\mathcal{T},\mathfrak{X}}$ on $\mathfrak{X}$ to be $\varprojlim_n \widehat{\mathcal{A}}_{\mathfrak{T}_n,\mathfrak{X},\mathbb{Q}}$. Then*

1. *The definition of $\mathcal{A}^\infty_{\mathcal{T},\mathfrak{X}}$ is independent of the trivializations of $sp_*\mathcal{T}$.*
2. *The admissible blow-up $\mathfrak{X} \to Spf(A^\circ)$ induces an isomorphism*

$$\Gamma(Spf(A^\circ), \mathcal{A}^\infty_{\mathcal{T},Spf(A^\circ)}) \xrightarrow{\simeq} \Gamma(\mathfrak{X}, \mathcal{A}^\infty_{\mathcal{T},\mathfrak{X}}).$$

Proof. If $\mathfrak{T}'$ is another $\mathcal{O}_{\mathfrak{X}}$-submodule of $sp_*\mathcal{T}$ such that $\mathfrak{T}' \otimes_{\mathbb{Z}} \mathbb{Q} \simeq sp_*\mathcal{T}$, then for $n >> 0$, there exists $N \in \mathbb{N}$ such that $\alpha^{-N} F_1\mathcal{A}_{\mathfrak{T}_n,\mathfrak{X}} \subseteq F_1\mathcal{A}_{\mathfrak{T}'_n,\mathfrak{X}} \subseteq \alpha^N F_1\mathcal{A}_{\mathfrak{T}_n,\mathfrak{X}}$, which induces $\mathcal{A}_{\mathfrak{T}_{n-N},\mathfrak{X}} \subseteq \mathcal{A}_{\mathfrak{T}'_n,\mathfrak{X}} \subseteq \mathcal{A}_{\mathfrak{T}_{n+N},\mathfrak{X}}$. Therefore, we get an isomorphism $\varprojlim_n \widehat{\mathcal{A}}_{\mathfrak{T}'_n,\mathfrak{X},\mathbb{Q}} \simeq \varprojlim_n \widehat{\mathcal{A}}_{\mathfrak{T}_n,\mathfrak{X},\mathbb{Q}}$.

(2) follows from Lemma 2.3.1.

□

Definition-Lemma 2.3.3. *If $U \subset X$ is an affinoid, define*

$$\mathcal{A}^\infty_{\mathcal{T},X}(U) := \Gamma(Spf(\mathcal{O}^\circ_X(U)), \mathcal{A}^\infty_{\mathcal{T},Spf(\mathcal{O}^\circ_X(U))}).$$

Then $\mathcal{A}^\infty_{\mathcal{T},X}$ is a presheaf of $\mathcal{O}_X$-algebras.

Proof. If $U_1 \subseteq U_2$ are open affinoids in X, for n sufficiently large, we may assume that $\alpha^n\xi_i \in \Gamma(\mathrm{Spf}(\mathcal{O}^\circ_X(U_j)), \mathcal{T}_{\mathrm{Spf}(\mathcal{O}^\circ_X(U_j))})$, for $i = 1, 2, ..., d$ and $j = 1, 2$. Let $\mathfrak{T}_j$ be the free

$\mathcal{O}_{\mathrm{Spf}(\mathcal{O}_X^\circ(U_j))}$-submodule of $sp_{j*}(\mathcal{T})$ generated by ξ_i for $i = 1, 2, ..., d$, where $sp_j : X \to \mathrm{Spf}(\mathcal{O}_X^\circ(U_j)$ is the specialization map as in Proposition 2.2.5 and $j = 1, 2$. For n sufficiently large, the homomorphism

$$\Gamma(\mathrm{Spf}(\mathcal{O}_X^\circ(U_2)), \widehat{\mathcal{A}}_{\mathfrak{T}_{2,n},\mathrm{Spf}(\mathcal{O}_X^\circ(U_2)),\mathbb{Q}}) \to \Gamma(\mathrm{Spf}(\mathcal{O}_X^\circ(U_1)), \widehat{\mathcal{A}}_{\mathfrak{T}_{1,n},\mathrm{Spf}(\mathcal{O}_X^\circ(U_1)),\mathbb{Q}})$$

induces the restriction morphism

$$\Gamma(\mathrm{Spf}(\mathcal{O}_X^\circ(U_2)), \mathcal{A}^\infty_{\mathcal{T},\mathrm{Spf}(\mathcal{O}_X^\circ(U_2))}) \to \Gamma(\mathrm{Spf}(\mathcal{O}_X^\circ(U_1)), \mathcal{A}^\infty_{\mathcal{T},\mathrm{Spf}(\mathcal{O}_X^\circ(U_1))}).$$

The restriction morphism is independent of the trivializations of $\mathcal{T}$.

□

Lemma 2.3.4.

1. *There is an isomorphism $\mathcal{A}^\infty_{\mathcal{T},\mathfrak{X}} \xrightarrow{\simeq} sp_*\mathcal{A}^\infty_{\mathcal{T},X}$ as presheaves of $\mathcal{O}_{\mathfrak{X}}$-algebras on $\mathcal{B}_{\mathfrak{X}}$, and in particular $\mathcal{A}^\infty_{\mathcal{T}}$ is a sheaf by Lemma 2.2.8.*

2. *There is an inclusion of sheaves of $\mathcal{O}_X$-algebras $\mathcal{A}_{\mathcal{T}} \to \mathcal{A}^\infty_{\mathcal{T}}$ with dense image.*

3. *A morphism $\mathcal{A}_{\mathcal{T}} \to \mathcal{A}'_{\mathcal{T}}$ of twisted sheaf of differential algebras associated to $\mathcal{T}$ extends uniquely to a continuous homomorphism between sheaves of $\mathcal{O}_X$-algebras $\mathcal{A}^\infty_{\mathcal{T}} \to \mathcal{A}'^\infty_{\mathcal{T}}$.*

4. *$\mathcal{A}^\infty_{\mathcal{T}}$ is a sheaf of Fréchet-Stein algebra on $\mathcal{B}_X$.*

Proof. For $U \in \mathcal{B}_X$, let $sp_U : U \to \mathrm{Spf}(\mathcal{O}_X^\circ(U))$ be the specialization map described in Proposition 2.2.5. A choice of trivialization $\{\xi_i\}_{i=1,2,...,d}$ of $\mathcal{T}$ induces an isomorphism $\mathcal{A}^\infty_{\mathcal{T},\mathfrak{X}} \simeq \varprojlim_n \widehat{\mathcal{A}}_{\mathfrak{T}_n,\mathfrak{X},\mathbb{Q}}$. Let $\mathfrak{T}'_n$ be the $\mathcal{O}_{\mathrm{Spf}(\mathcal{O}_X^\circ(U))}$-submodule of $sp_{U*}(\mathcal{T}|_U)$ generated by $\alpha^n\xi_i$ for $i = 1, 2, ..., d$.

1. The natural inclusion of sheaves of algebras $\mathcal{O}_{\mathfrak{X}} \hookrightarrow sp_*\mathcal{O}_X$ factors through the inclusion $sp_*\mathcal{O}_X^\circ \hookrightarrow sp_*\mathcal{O}_X$. For $\mathfrak{U} \in \mathcal{B}_{\mathfrak{X}}$, let $U := sp^{-1}(\mathfrak{U})$. For $n >> 0$, from Lemma 2.3.1 we

see that $\mathcal{A}_{\mathfrak{T}_n,\mathfrak{X}}(\mathfrak{U}) \hookrightarrow \Gamma(\mathrm{Spf}(\mathcal{O}_X^\circ(U)), \mathcal{A}_{\mathfrak{T}'_n,\mathrm{Spf}(\mathcal{O}_X^\circ(U))})$ are commensurable lattices in $\mathcal{A}_{\mathcal{T}}(U)$, and therefore induce an isomorphism $\mathcal{A}^{\infty}_{\mathcal{T},\mathfrak{X}}(\mathfrak{U}) \to \mathcal{A}^{\infty}_{\mathcal{T}}(U)$.

2. For $n >> 0$, we see that $\mathcal{A}_{\mathfrak{T}_n,\mathrm{Spf}(\mathcal{O}_X^\circ(U))} \hookrightarrow sp_{U*}(\mathcal{A}_{\mathcal{T}}|_U)$ induces an isomorphism $\mathcal{A}_{\mathfrak{T}'_n,\mathrm{Spf}(\mathcal{O}_X^\circ(U))}\otimes_{\mathbb{Z}}\mathbb{Q} \xrightarrow{\simeq} sp_{U*}(\mathcal{A}_{\mathcal{T}}|_U)$. Hence we have $sp_{U*}(\mathcal{A}_{\mathcal{T}}|_U) \hookrightarrow \widehat{\mathcal{A}}_{\mathfrak{T}'_n,\mathrm{Spf}(\mathcal{O}_X^\circ(U)),\mathbb{Q}}$, and $sp_{U*}(\mathcal{A}_{\mathcal{T}}|_U) \hookrightarrow \varprojlim_n \widehat{\mathcal{A}}_{\mathfrak{T}'_n,\mathrm{Spf}(\mathcal{O}_X^\circ(U)),\mathbb{Q}}$. Evaluating on U we get $\mathcal{A}_{\mathcal{T}}(U) \hookrightarrow \mathcal{A}^{\infty}_{\mathcal{T}}(U)$ which gives us an inclusion of sheaves of $\mathcal{O}_X$-algebras $\mathcal{A}_{\mathcal{T}} \hookrightarrow \mathcal{A}^{\infty}_{\mathcal{T}}$ with dense image.

3. There exists $N, M \in \mathbb{N}$, such that for $n > N$, we have $\mathcal{A}_{\mathfrak{T}'_{n+M},\mathrm{Spf}(\mathcal{O}_X^\circ(U))} \to \mathcal{A}'_{\mathfrak{T}'_n,\mathrm{Spf}(\mathcal{O}_X^\circ(U))}$, such that the following diagram commutes:

$$\begin{array}{ccc} \mathcal{A}_{\mathfrak{T}'_{n+M},\mathrm{Spf}(\mathcal{O}_X^\circ(U))} \otimes_{\mathbb{Z}} \mathbb{Q} & \longrightarrow & \mathcal{A}'_{\mathfrak{T}'_n,\mathrm{Spf}(\mathcal{O}_X^\circ(U))} \otimes_{\mathbb{Z}} \mathbb{Q} \\ \downarrow \simeq & & \downarrow \simeq \\ sp_{U*}(\mathcal{A}_{\mathcal{T}}|_U) & \longrightarrow & sp_{U*}(\mathcal{A}'_{\mathcal{T}}|_U) \end{array}$$

Therefore we get a continuous morphism of $\mathcal{O}_X(U)$-algebras $\mathcal{A}^{\infty}_{\mathcal{T}}(U) \to \mathcal{A}'^{\infty}_{\mathcal{T}}(U)$ that extends $\mathcal{A}_{\mathcal{T}}(U) \to \mathcal{A}'_{\mathcal{T}}(U)$.

4. We will show that the isomorphism

$$\mathcal{A}^{\infty}_{\mathcal{T}}(U) \simeq \varprojlim_n \Gamma(\mathrm{Spf}(\mathcal{O}_X^\circ(U)), \widehat{\mathcal{A}}_{\mathfrak{T}'_n,\mathrm{Spf}(\mathcal{O}_X^\circ(U)),\mathbb{Q}})$$

realizes $\mathcal{A}^{\infty}_{\mathcal{T}}(U)$-as a Fréchet-Stein algebra. By Lemma 2.3.1 we know

$$\Gamma(\mathrm{Spf}(\mathcal{O}_X^\circ(U)), \widehat{\mathcal{A}}_{\mathfrak{T}'_n,\mathrm{Spf}(\mathcal{O}_X^\circ(U)),\mathbb{Q}})$$

is a noetherian K-Banach algebra. We will apply [14] Proposition 5.3.10 to show that the transition morphism

$$\Gamma(\mathrm{Spf}(\mathcal{O}_X^\circ(U)), \widehat{\mathcal{A}}_{\mathfrak{T}'_{n+1},\mathrm{Spf}(\mathcal{O}_X^\circ(U)),\mathbb{Q}}) \to \Gamma(\mathrm{Spf}(\mathcal{O}_X^\circ(U)), \widehat{\mathcal{A}}_{\mathfrak{T}'_n,\mathrm{Spf}(\mathcal{O}_X^\circ(U)),\mathbb{Q}})$$

is flat. Let $G_{\cdot}$ be the filtration on $\Gamma(\mathrm{Spf}(\mathcal{O}_X^\circ(U)), \mathcal{A}_{\mathfrak{I}'_n, \mathrm{Spf}(\mathcal{O}_X^\circ(U))})$ defined as

$$G_i := \Gamma(\mathrm{Spf}(\mathcal{O}_X^\circ(U)), \mathcal{A}_{\mathfrak{I}'_{n+1}, \mathrm{Spf}(\mathcal{O}_X^\circ(U))}) \cdot \Gamma(\mathrm{Spf}(\mathcal{O}_X^\circ(U)), F_i\mathcal{A}_{\mathfrak{I}'_n, \mathrm{Spf}(\mathcal{O}_X^\circ(U))}).$$

Then $G_0 = \Gamma(\mathrm{Spf}(\mathcal{O}_X^\circ(U)), \mathcal{A}_{\mathfrak{I}'_{n+1}, \mathrm{Spf}(\mathcal{O}_X^\circ(U))})$. To show that $G_i \cdot G_j \subseteq G_{i+j}$, it suffices to check that

$$\Gamma(\mathrm{Spf}(\mathcal{O}_X^\circ(U)), F_i\mathcal{A}_{\mathfrak{I}'_n, \mathrm{Spf}(\mathcal{O}_X^\circ(U))}) \cdot \Gamma(\mathrm{Spf}(\mathcal{O}_X^\circ(U)), \mathcal{A}_{\mathfrak{I}'_{n+1}, \mathrm{Spf}(\mathcal{O}_X^\circ(U))}) \subseteq$$
$$\Gamma(\mathrm{Spf}(\mathcal{O}_X^\circ(U)), \mathcal{A}_{\mathfrak{I}'_{n+1}, \mathrm{Spf}(\mathcal{O}_X^\circ(U))}) \cdot \Gamma(\mathrm{Spf}(\mathcal{O}_X^\circ(U)), F_i\mathcal{A}_{\mathfrak{I}'_n, \mathrm{Spf}(\mathcal{O}_X^\circ(U))}).$$

Fix a multi-index $I := (\nu_1, \nu_2, ..., \nu_d) \in \mathbb{N}^d$ and $|I| := \sum_{i=1}^d \nu_i$. Let $\partial_x^I := \prod_{i=1}^d \tilde{\xi}_i^{\nu_i}$. By induction on i, it suffices to check that if $f = a_I \alpha^{n|I|}\partial_x^I$ where $a_I \in \mathcal{O}_X^\circ(U)$, and $g = a_J \alpha^{(n+1)|J|}\partial_x^J$, then

$$fg \in \Gamma(\mathrm{Spf}(\mathcal{O}_X^\circ(U)), \mathcal{A}_{\mathfrak{I}'_{n+1}, \mathrm{Spf}(\mathcal{O}_X^\circ(U))}) \cdot \Gamma(\mathrm{Spf}(\mathcal{O}_X^\circ(U)), F_{|I|}\mathcal{A}_{\mathfrak{I}'_n, \mathrm{Spf}(\mathcal{O}_X^\circ(U))}).$$

Since

$$fg = a_I a_J \alpha^{(n+1)|J|+n|I|}\partial_x^I \partial_x^J + a_I[\partial_x^J, a_J]\alpha^{(n+1)|J|+n|I|}\partial_x^I,$$

and $[\partial_x^J, a_J]$ has order less than $|J|$, the conclusion follows. Finally, observe that $\mathrm{gr}_{\cdot}^G \Gamma(\mathrm{Spf}(\mathcal{O}_X^\circ(U)), \mathcal{A}_{\mathfrak{I}_n, \mathrm{Spf}(\mathcal{O}_X^\circ(U))})$ is an algebra over $\Gamma(\mathrm{Spf}(\mathcal{O}_X^\circ(U)), \mathcal{A}_{\mathfrak{I}_{n+1}, \mathrm{Spf}(\mathcal{O}_X^\circ(U))})$ generated by the central elements $\left\{\alpha^n \tilde{\xi}_i\right\}_{i=1,2,...,d}$.

□

Remark 2.3.5. The proofs of Lemma 2.3.4 and Lemma 2.3.1 also imply that the isomorphism $\Gamma(X, \mathcal{A}_{\mathcal{T}}^\infty) \simeq \varprojlim_n \Gamma(\mathfrak{X}, \widehat{\mathcal{A}}_{\mathfrak{I}_n, \mathfrak{X}, \mathbb{Q}})$ realizes $\Gamma(X, \Gamma(X, \mathcal{A}_{\mathcal{T}}^\infty))$ as a Fréchet-Stein algebra.

Example 2.3.6. If $\mathcal{T} = \mathcal{T}_X$ and $\mathcal{A}_{\mathcal{T}} = \mathcal{D}_X$, we will use $\mathcal{D}_X^\infty$ to represent $\mathcal{A}_{\mathcal{T}}^\infty$.

Let us choose a system of local coordinates $x := \{x_1, x_2, ..., x_d\}$ on X, which gives rise to a set of sections $\{\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, ..., \frac{\partial}{\partial x_d}\} \subseteq \Gamma(X, \mathcal{T}_X)$ that trivializes $\mathcal{T}_X$ as a free $\mathcal{O}_X$-module. Since the derivations $\frac{\partial}{\partial x_i}$ are continous, replacing $\frac{\partial}{\partial x_i}$ by $\alpha^N \frac{\partial}{\partial x_i}$ for sufficiently large N, we may assume that $\frac{\partial}{\partial x_i}(\mathcal{O}_X^{\circ}(U)) \subseteq \mathcal{O}_X^{\circ}(U)$ for $1 \leq i \leq d$. Recall that for a multi-index $I = (\nu_1, \nu_2, ..., \nu_d) \in \mathbb{N}^d$, we have $|I| = \sum_{i=1}^{d} \nu_i$ and $\partial_x^I = \prod_{i=1}^{d} \frac{\partial^{\nu_i}}{\partial x_i^{\nu_i}}$. As a $\mathcal{O}_X(U)$-module, we have the following description of $\mathcal{D}_X^{\infty}(U)$:

$$\mathcal{D}_X^{\infty}(U) \simeq \left\{ \sum_I a_I \partial_x^I \;\middle|\; a_I \in \mathcal{O}_X(U), \text{for any } n \in \mathbb{N} \; \left\| \frac{a_I}{\alpha^{n|I|}} \right\|_U \xrightarrow[|I| \to \infty]{} 0 \right\},$$

where $\| \; \|_U$ is the spectral norm on U. Note that this identification is compatible with the restriction morphisms, and it is straightforward to check that $\mathcal{D}_X^{\infty}$ is a sheaf by the maximum principle. If we choose another system of local coordinates $z = \{z_1, z_2, ..., z_d\}$ on U. Then $\frac{\partial}{\partial x_i} = \sum_{j=1}^{d} \frac{\partial z_j}{\partial x_i} \frac{\partial}{\partial z_j}$, where the Jacobian matrix $(\frac{\partial z_j}{\partial x_i}) \in M_{d \times d}(\mathcal{O}_X(U))$ is invertible. The convergent condition ensures the definition $\mathcal{D}_X^{\infty}$ is independent of the choice of local coordinates.

2.3.2 Coadmissible modules over $\mathcal{A}_{\mathcal{T}}^{\infty}$ on affinoid rigid analytic varieties over K

In this subsection, we define the category of coadmissible modules over the sheaf of Fréchet-Stein algebra $\mathcal{A}_{\mathcal{T}}^{\infty}$ on $\mathcal{B}_X$, and prove an analogue of Serre's theorem on quasi-coherent sheaves for such coadmissible modules.

Lemma 2.3.7. *We continue to use the notations introduced in section 2.3.1.*

1. *Let $\mathcal{M}_0$ be a coherent left $\widehat{\mathcal{A}}_{\mathfrak{T}_n,\mathfrak{X}}$-module. If $\mathfrak{U} \in \mathcal{B}_{\mathfrak{X}}$, then $\mathcal{M}_0(\mathfrak{U})$ is a p-adically complete coherent left $\widehat{\mathcal{A}}_{\mathfrak{T}_n,\mathfrak{X}}(\mathfrak{U})$-module. Moreover, if $\mathfrak{V} \subseteq \mathfrak{U}$ is open affine, then*

$\mathcal{M}_0(\mathfrak{V})$ is isomorphic to

$$\widehat{\mathcal{A}}_{\mathfrak{T}_n,\mathfrak{X}}(\mathfrak{V}) \otimes_{\widehat{\mathcal{A}}_{\mathfrak{T}_n,\mathfrak{X}}(\mathfrak{U})} \mathcal{M}_0(\mathfrak{U}),$$

which is also isomorphic to

$$\varprojlim_i \mathcal{A}_{\mathfrak{T}_n,\mathfrak{X}}(\mathfrak{V}) \otimes_{\mathcal{A}_{\mathfrak{T}_n,\mathfrak{X}}(\mathfrak{U})} (\mathcal{M}_0(\mathfrak{U})/\pi_K^i \mathcal{M}_0(\mathfrak{U})).$$

In particular, we see that

$$\mathcal{M}_0 \simeq \varprojlim_i \mathcal{M}_0/\pi_K^i \mathcal{M}_0.$$

2. *If $\mathcal{M}$ is a coherent left $\widehat{\mathcal{A}}_{\mathfrak{T}_n,\mathfrak{X},\mathbb{Q}}$-module, then there exists a (π_K)-torsion free coherent left $\widehat{\mathcal{A}}_{\mathfrak{T}_n,\mathfrak{X}}$-module $\mathcal{M}_0$ such that $\mathcal{M} \simeq \mathcal{M}_0 \otimes_{\mathbb{Z}} \mathbb{Q}$.*

Proof.

1. This immediately follows from Theorem 2.2.3.

2. Consider a finite open affine cover $\{\mathfrak{U}_i\}_{i=1,2,...,N}$ of $\mathfrak{X}$. There exists a coherent $\widehat{\mathcal{A}}_{\mathfrak{T}_n,\mathfrak{X}}|_{\mathfrak{U}_i}$-module $\mathcal{M}_i$ such that $\mathcal{M}_i \otimes_{\mathbb{Z}} \mathbb{Q} \simeq \mathcal{M}|_{\mathfrak{U}_i}$, for $i = 1, 2, ..., N$. We may assume $\mathcal{M}_i$ is (π_K)-torsion free since the π_K^j-torsion subsheaf $\mathcal{M}_i\,[\pi_K^j]$ of $\mathcal{M}_i$ is $\widehat{\mathcal{A}}_{\mathfrak{T}_n,\mathfrak{X}}|_{\mathfrak{U}_i}$-coherent. By Theorem 2.2.3 we conclude that there exists a coherent $\widehat{\mathcal{A}}_{\mathfrak{T}_n,\mathfrak{X}}$-module $\mathcal{M}_0$ without (π_K)-torsion, such that $\mathcal{M} \simeq \mathcal{M}_0 \otimes_{\mathbb{Z}} \mathbb{Q}$.

□

Lemma 2.3.8. *$\widehat{\mathcal{A}}_{\mathfrak{T}_n,\mathfrak{X},\mathbb{Q}}$ has vanishing higher cohomology groups.*

Proof. The set $\left\{(\alpha^n\tilde{\xi}_1)^{n_1}(\alpha^n\tilde{\xi}_2)^{n_2}...(\alpha^n\tilde{\xi}_d)^{n_d} \mid (n_1, n_2, ..., n_d) \in \mathbb{N}^d\right\}$ forms a basis of $\mathcal{A}_{\mathfrak{T}_n,\mathfrak{X}}$ as a free $\mathcal{O}_{\mathfrak{X}}$-module. As sheaves of $\mathcal{O}_{\mathfrak{X}} \otimes_{\mathbb{Z}} \mathbb{Q}$-modules, we get an isomorphism

$$\widehat{\mathcal{A}}_{\mathfrak{T}_n,\mathfrak{X},\mathbb{Q}} \simeq (\mathcal{O}_{\mathfrak{X}} \otimes_{\mathbb{Z}} \mathbb{Q})\langle \alpha^n\xi_1, \alpha^n\xi_2, ..., \alpha^n\xi_d\rangle.$$

Let $pr : X \times_K \mathbb{B}(|\alpha^n|)^d \to X$ be the natural projection, where

$$\mathbb{B}(|\alpha^n|)^d := \mathrm{Spa}(K\langle \alpha^n\xi_1, \alpha^n\xi_2, ..., \alpha^n\xi_d\rangle, K^\circ\langle \alpha^n\xi_1, \alpha^n\xi_2, ..., \alpha^n\xi_d\rangle)$$

is the rigid analytic closed ball of radius $\frac{1}{|\alpha^n|}$. Then as $sp_*\mathcal{O}_X$-modules, we have an isomorphism

$$\widehat{\mathcal{A}}_{\mathfrak{T}_n,\mathfrak{X},\mathbb{Q}} \simeq sp_* pr_* \mathcal{O}_{X\times_K \mathbb{B}(|\alpha^n|)^d}.$$

Since $X \times_K \mathbb{B}(|\alpha|)^d \simeq \mathrm{Spa}(A\widehat{\otimes}_K K\langle \alpha^n\xi_1, \alpha^n\xi_2, ..., \alpha^n\xi_d\rangle)$ is an affinoid, we know that the higher cohomology groups of $\mathcal{O}_{X\times_K\mathbb{B}(|\alpha|)^d}$ vanish. Since the Čech complex of $\widehat{\mathcal{A}}_{\mathfrak{T}_n,\mathfrak{X},\mathbb{Q}}$ associated to an open affine cover $\{\mathfrak{U}_i\}$ could be identified with the Čech complex of $\mathcal{O}_{X\times_K\mathbb{B}(|\alpha|)^d}$ associated with the affinoid cover $\{\mathbf{rig}(\mathfrak{U}_i) \times_K \mathbb{B}(|\alpha|)^d\}$, the higher cohomology groups of $\widehat{\mathcal{A}}_{\mathfrak{T}_n,\mathfrak{X},\mathbb{Q}}$ vanish.

□

Proposition 2.3.9. *A coherent $\widehat{\mathcal{A}}_{\mathfrak{T}_n,\mathfrak{X},\mathbb{Q}}$-module $\mathcal{M}$ has trival higher cohomology groups, and is generated by global sections. Therefore, we have $\mathcal{M} \simeq \widehat{\mathcal{A}}_{\mathfrak{T}_n,\mathfrak{X},\mathbb{Q}} \otimes_{\Gamma(\mathfrak{X},\widehat{\mathcal{A}}_{\mathfrak{T}_n,\mathfrak{X},\mathbb{Q}})} \Gamma(\mathfrak{X},\mathcal{M})$, where $\Gamma(\mathfrak{X},\mathcal{M})$ is finitely generated over $\Gamma(\mathfrak{X},\widehat{\mathcal{A}}_{\mathfrak{T}_n,\mathfrak{X},\mathbb{Q}})$.*

Proof. The arguments are inspired by [1], and we use the notations introduced in Example 2.2.6.

Step1. We show that $\left\{\widehat{\mathcal{A}}_{\mathfrak{T}_n,\mathfrak{X}}(s); s \in \mathbb{Z}\right\}$ generate the category of coherent $\widehat{\mathcal{A}}_{\mathfrak{T}_n,\mathfrak{X}}(s)$-modules.

Let $\mathcal{M}_0$ be a (π_K)-torsion free coherent $\widehat{\mathcal{A}}_{\mathfrak{T}_n,\mathfrak{X}}$-module, and let $\mathfrak{m}_k$ be $\mathcal{M}_0/\pi_K^k\mathcal{M}_0$. Thus $\mathfrak{m}_1$ is a coherent module over $\widehat{\mathcal{A}}_{\mathfrak{T}_n,\mathfrak{X}}/\pi_K\widehat{\mathcal{A}}_{\mathfrak{T}_n,\mathfrak{X}} \simeq \mathcal{A}_{\mathfrak{T}_n,\mathfrak{X}}/\pi_K\mathcal{A}_{\mathfrak{T}_n,\mathfrak{X}}$, and is quasi-coherent as a $(\mathcal{O}_\mathfrak{X}/(\pi_K))$-module. Since $\mathfrak{m}_1$ is the direct limit of its coherent $(\mathcal{O}_\mathfrak{X}/(\pi_K))$-submodules, we can find a coherent $(\mathcal{O}_\mathfrak{X}/(\pi_K))$-submodule $\mathcal{F}$ of $\mathfrak{m}_1$, such that there is a surjective morphism $\phi : \mathcal{A}_{\mathfrak{T}_n,\mathfrak{X}} \otimes_{\mathcal{O}_\mathfrak{X}} \mathcal{F} \twoheadrightarrow \mathfrak{m}_1$. The filtration $F_.\mathcal{A}_{\mathfrak{T}_n,\mathfrak{X}}$ induces a filtration on $\mathcal{A}_{\mathfrak{T}_n,\mathfrak{X}}/\pi_K\mathcal{A}_{\mathfrak{T}_n,\mathfrak{X}}$, such that

$$\mathrm{gr}_{F.}(\mathcal{A}_{\mathfrak{T}_n,\mathfrak{X}}/\pi_K\mathcal{A}_{\mathfrak{T}_n,\mathfrak{X}}) \simeq (\mathcal{O}_\mathfrak{X}/(\pi_K))[\alpha^n\xi_1, \alpha^n\xi_2, ..., \alpha^n\xi_d].$$

Define a filtration on $\mathfrak{m}_1$ by letting $F_i(\mathfrak{m}_1)$ be $F_i(\mathcal{A}_{\mathfrak{T}_n,\mathfrak{X}})\cdot\mathcal{F}$. The associated graded sheaf $\mathrm{gr}_{F.}(\mathfrak{m}_1)$ is a coherent $\mathrm{gr}_{F.}(\mathcal{A}_{\mathfrak{T}_n,\mathfrak{X}}/\pi_K\mathcal{A}_{\mathfrak{T}_n,\mathfrak{X}})$-module. Thus $\mathrm{gr}_{F.}(\mathfrak{m}_1)$ can be viewed as a coherent $\mathcal{O}_{\widetilde{X}}\otimes_{A^\circ}A^\circ[\alpha^n\xi_1,\alpha^n\xi_2,...,\alpha^n\xi_d]$-module on the scheme $\widetilde{X}$. In addition, we have that $(s)(\mathrm{gr}_{F.}\mathfrak{m}_1)\simeq\mathcal{L}^s\otimes_{\mathcal{O}_{\widetilde{X}}}\mathrm{gr}_{F.}\mathfrak{m}_1$ as $\mathcal{O}_{\widetilde{X}}$-modules on $\widetilde{X}$. Consider the following digram of schemes over $\mathrm{Spec}(A^\circ)$:

$$\begin{array}{ccc} \widetilde{X}\times_{\mathrm{Spec}(A^\circ)}\mathbb{A}^d & \xrightarrow{p_2} & \mathbb{A}^d \\ {\scriptstyle p_1}\downarrow & & \downarrow \\ \widetilde{X} & \longrightarrow & \mathrm{Spec}(A^\circ), \end{array}$$

where $\mathbb{A}^d:=\mathrm{Spec}(A^\circ[\alpha^n\xi_1,\alpha^n\xi_2,...,\alpha^n\xi_d])$. There exists a coherent $\mathcal{O}_{\widetilde{X}\times\mathbb{A}^d}$-module $\mathfrak{n}_1$, such that $p_{1*}\mathfrak{n}_1\simeq\mathrm{gr}_{F.}\mathfrak{m}_1$. Since ${p_1}^*\mathcal{L}$ is ample relative to p_2, the twist

$$\mathfrak{n}_1(s):=\mathfrak{n}_1\otimes_{\mathcal{O}_{\widetilde{X}\times_{\mathrm{Spec}(A^\circ)}\mathbb{A}^d}}({p_1}^*\mathcal{L})^s$$

is p_{2*}-acyclic for $s\gg 0$. Since $\mathbb{A}^d$ is affine,

$$H^i(\widetilde{X}\times_{\mathrm{Spec}(A^\circ)}\mathbb{A}^d,\mathfrak{n}_1(s))\simeq\Gamma(\mathbb{A}^d,R^ip_{2*}\mathfrak{n}_1(s))=0$$

for $i\geq 1$ and $s\gg 0$. Because p_1 is an affine morphism, we have the isomorphism $H^i(\widetilde{X},p_{1*}(\mathfrak{n}_1(s)))\simeq H^i(\widetilde{X}\times_{\mathrm{Spec}(A^\circ)}\mathbb{A}^d,\mathfrak{n}_1(s))$. By the projection formula $p_{1*}(\mathfrak{n}_1(s))\simeq(\mathrm{gr}_{F.}\mathfrak{m}_1)(s)$, so it follows that $H^i(\mathfrak{X},(\mathrm{gr}_{F.}\mathfrak{m}_1)(s))=0$ for $i\geq 1$ and $s\gg 0$. As a result, each graded component of $(\mathrm{gr}_{F.}\mathfrak{m}_1)(s)$, and therefore each filtered piece $F_i(\mathfrak{m}_1)(s)$, have trivial higher cohomology groups. Since $\mathfrak{m}_1(s)\simeq\bigcup_i F_i(\mathfrak{m}_1)(s)$ as $\mathcal{O}_{\mathfrak{X}}$-modules, and taking cohomology commutes with taking direct limit on noetherian spaces, we conclude that $H^i(\mathfrak{X},\mathfrak{m}_1(s))=0$ for $s\gg 0$ and $i\geq 1$.

Let $\mathcal{A}^{(s)}_{\mathfrak{T}_n,\mathfrak{X}}:=\mathcal{L}^s_{\mathfrak{X}}\otimes_{\mathcal{O}_{\mathfrak{X}}}\mathcal{A}_{\mathfrak{T}_n,\mathfrak{X}}\otimes_{\mathcal{O}_{\mathfrak{X}}}\mathcal{L}^{-s}_{\mathfrak{X}}$, which is a left (resp. right) coherent sheaf of rings locally isomorphic to $\mathcal{A}_{\mathfrak{T}_n,\mathfrak{X}}$. Then $(s)\mathfrak{m}_1$ is naturally a $\mathcal{A}^{(s)}_{\mathfrak{T}_n,\mathfrak{X}}$-module. The surjection ϕ induces a surjection $\mathcal{A}^{(s)}_{\mathfrak{T}_n,\mathfrak{X}}\otimes_{\mathcal{O}_{\mathfrak{X}}}(s)\mathcal{F}\twoheadrightarrow(s)\mathfrak{m}_1$. Since $(s)\mathcal{F}$ is $\mathcal{O}_{\mathfrak{X}}$-coherent, we can find a surjection $\bigoplus_{\text{finite sum}}\mathcal{O}_{\mathfrak{X}}\twoheadrightarrow(s)\mathcal{F}$ for $s\gg 0$. Thus, there exists a surjection $\bigoplus_{\text{finite sum}}\mathcal{A}^{(s)}_{\mathfrak{T}_n,\mathfrak{X}}\twoheadrightarrow$

$(s)\mathfrak{m}_1$ for $s \gg 0$.

Consider the following short exact sequence with the second arrow induced by multiplication by π_K^i:

$$0 \to (s)\mathfrak{m}_1 \to (s)\mathfrak{m}_{i+1} \to (s)\mathfrak{m}_i \to 0.$$

Taking the associated long exact sequence of cohomology groups, we get a surjective map $\Gamma(\mathfrak{X}, (s)\mathfrak{m}_{i+1}) \twoheadrightarrow \Gamma(\mathfrak{X}, (s)\mathfrak{m}_i)$ for $s \gg 0$, because $H^1(\mathfrak{X}, \mathfrak{m}_1(s)) = 0$. Since $\varprojlim_i \mathfrak{m}_i \simeq \mathcal{M}_0$, we are able to lift a finite set of global sections for $(s)\mathfrak{m}_1$ that generate $(s)\mathfrak{m}_1$ as a $\mathcal{A}^{(s)}_{\mathfrak{I}_n,\mathfrak{X}}$-module to global sections of $(s)\mathcal{M}_0$. By Nakayama's lemma we see that there is a surjection $\bigoplus_{\text{finite sum}} \widehat{\mathcal{A}^{(s)}_{\mathfrak{I}_n,\mathfrak{X}}} \twoheadrightarrow (s)\mathcal{M}_0$, where $\widehat{\mathcal{A}^{(s)}_{\mathfrak{I}_n,\mathfrak{X}}}$ is the p-adic completion of $\mathcal{A}^{(s)}_{\mathfrak{I}_n,\mathfrak{X}}$, which is isomorphic to $\mathcal{L}^s_{\mathfrak{X}} \otimes_{\mathcal{O}_{\mathfrak{X}}} \widehat{\mathcal{A}}_{\mathfrak{I}_n,\mathfrak{X}} \otimes_{\mathcal{O}_{\mathfrak{X}}} \mathcal{L}^{-s}_{\mathfrak{X}}$. Thus, we get a surjection $\bigoplus_{\text{finite sum}} \widehat{\mathcal{A}}_{\mathfrak{I}_n,\mathfrak{X}}(-s) \twoheadrightarrow \mathcal{M}_0$.

Step 2. By Lemma 2.3.7, we may choose a coherent (π_K)-torsion free $\widehat{\mathcal{A}}_{\mathfrak{I}_n,\mathfrak{X}}$-module $\mathcal{M}_0$, such that $\mathcal{M}_0 \otimes_{\mathbb{Z}} \mathbb{Q} \simeq \mathcal{M}$. Since $\mathcal{L}_{\mathfrak{X}} \otimes_{\mathbb{Z}} \mathbb{Q} \simeq \mathcal{O}_{\mathfrak{X}} \otimes_{\mathbb{Z}} \mathbb{Q}$, there exists an exact sequence

$$0 \to \mathcal{N}_1 \to \bigoplus_{\text{finite sum}} \widehat{\mathcal{A}}_{\mathfrak{I}_n,\mathfrak{X},\mathbb{Q}} \to \mathcal{M} \to 0,$$

for a coherent $\widehat{\mathcal{A}}_{\mathfrak{I}_n,\mathfrak{X},\mathbb{Q}}$-module $\mathcal{N}_1$. Taking the associated long exact sequence of cohomology groups, we see that $H^i(\mathfrak{X}, \mathcal{M}) \simeq H^{i+1}(\mathfrak{X}, \mathcal{N}_1)$ for $i \geq 1$ by Lemma 2.3.8. We conclude by the dimension shifting argument that $\mathcal{M}$ has no higher cohomology.

Let us choose a finite presentation of $\mathcal{M}$:

$$\bigoplus_{\text{finite sum}} \widehat{\mathcal{A}}_{\mathfrak{I}_n,\mathfrak{X},\mathbb{Q}} \to \bigoplus_{\text{finite sum}} \widehat{\mathcal{A}}_{\mathfrak{I}_n,\mathfrak{X},\mathbb{Q}} \to \mathcal{M} \to 0.$$

Taking the global sections and applying $\widehat{\mathcal{A}}_{\mathfrak{I}_n,\mathfrak{X},\mathbb{Q}} \otimes_{\Gamma(\mathfrak{X}, \widehat{\mathcal{A}}_{\mathfrak{I}_n,\mathfrak{X},\mathbb{Q}})} -$ to the above exact sequence, we get the following commutative diagram, with exact rows and the first two columns being isomorphic:

$$\begin{array}{ccccccc}
\bigoplus_{\text{finite sum}} \widehat{\mathcal{A}}_{\mathfrak{T}_n,\mathfrak{X},\mathbb{Q}} & \longrightarrow & \bigoplus_{\text{finite sum}} \widehat{\mathcal{A}}_{\mathfrak{T}_n,\mathfrak{X},\mathbb{Q}} & \longrightarrow & \widehat{\mathcal{A}}_{\mathfrak{T}_n,\mathfrak{X},\mathbb{Q}} \otimes_{\Gamma(\mathfrak{X},\widehat{\mathcal{A}}_{\mathfrak{T}_n,\mathfrak{X},\mathbb{Q}})} \Gamma(\mathfrak{X},\mathcal{M}) & \longrightarrow & 0 \\
\simeq\downarrow & & \simeq\downarrow & & \downarrow & & \\
\bigoplus_{\text{finite sum}} \widehat{\mathcal{A}}_{\mathfrak{T}_n,\mathfrak{X},\mathbb{Q}} & \longrightarrow & \bigoplus_{\text{finite sum}} \widehat{\mathcal{A}}_{\mathfrak{T}_n,\mathfrak{X},\mathbb{Q}} & \longrightarrow & \mathcal{M} & \longrightarrow & 0
\end{array}$$

By diagram chasing, we see that $\mathcal{M}$ is generated by global sections.

□

If M is a coadmissible module over $\Gamma(X, \mathcal{A}^{\infty}_{\mathcal{T}}) \simeq \varprojlim_n \Gamma(\mathfrak{X}, \widehat{\mathcal{A}}_{\mathfrak{T}_n,\mathfrak{X},\mathbb{Q}})$, then $M \simeq \varprojlim_n M_n$, where M_n is a finitely generated $\Gamma(\mathfrak{X}, \widehat{\mathcal{A}}_{\mathfrak{T}_n,\mathfrak{X},\mathbb{Q}})$-module, and

$$M_n \simeq \Gamma(\mathfrak{X}, \widehat{\mathcal{A}}_{\mathfrak{T}_n,\mathfrak{X},\mathbb{Q}}) \otimes_{\Gamma(\mathfrak{X},\widehat{\mathcal{A}}_{\mathfrak{T}_n,\mathfrak{X},\mathbb{Q}})} M_{n+1}.$$

By Definition-Lemma 2.2.11 we can associate to M a sheaf $M_{\mathfrak{X}}$ of $\mathcal{A}^{\infty}_{\mathcal{T},\mathfrak{X}}$-modules on $\mathfrak{X}$ in the following way:

$$M_{\mathfrak{X}} := \varprojlim_n \widehat{\mathcal{A}}_{\mathfrak{T}_n,\mathfrak{X},\mathbb{Q}} \otimes_{\Gamma(\mathfrak{X},\widehat{\mathcal{A}}_{\mathfrak{T}_n,\mathfrak{X},\mathbb{Q}})} M_n.$$

If $\mathfrak{U} \in \mathcal{B}_{\mathfrak{X}}$, then $M_{\mathfrak{X}}(\mathfrak{U}) \simeq \mathcal{A}^{\infty}_{\mathcal{T},\mathfrak{X}}(\mathfrak{U}) \widehat{\otimes}_{\Gamma(\mathfrak{X},\mathcal{A}^{\infty}_{\mathcal{T}})} M$ is a coadmissible $\mathcal{A}^{\infty}_{\mathcal{T},\mathfrak{X}}(\mathfrak{U})$-module.

Definition-Lemma 2.3.10. *If M is a coadmissible module over $\Gamma(X, \mathcal{A}^{\infty}_{\mathcal{T}})$, let us define a presheaf of $\mathcal{A}^{\infty}_{\mathcal{T}}$-module $\widetilde{M}$ on X in the following way: if $U \in \mathcal{B}_X$, then $\mathcal{M}(U) := \mathcal{A}^{\infty}_{\mathcal{T}}(U) \widehat{\otimes}_{\Gamma(X,\mathcal{A}^{\infty}_{\mathcal{T}})} M$ is a coadmissible module over $\mathcal{A}^{\infty}_{\mathcal{T}}(U)$ by Lemma 2.2.10. Then $\widetilde{M}$ is a sheaf, and we call $\widetilde{M}$ the sheaf of coadmissible $\mathcal{A}^{\infty}_{\mathcal{T}}$-module associated to the coadmissible $\Gamma(X, \mathcal{A}^{\infty}_{\mathcal{T}})$-module M.*

Proof. By Definition-Lemma 2.2.11 and Lemma 2.3.4, we have the isomorphism $M_{\mathfrak{X}} \xrightarrow{\simeq} sp_* \widetilde{M}$ of sheaves of $\mathcal{A}^{\infty}_{\mathcal{T},\mathfrak{X}}$-modules. By Lemma 2.2.8, we see that $\widetilde{M}$ is a sheaf.

□

2.4 $\mathcal{D}^\infty$-modules on smooth rigid analytic variety over K

In section 2.4, let X be a smooth rigid analytic variety over K, let $\mathcal{T}$ be a smooth Lie algebroid on X, and let $\mathcal{A}_\mathcal{T}$ be a twisted differential algebra associated to $\mathcal{T}$. We shall glue the constructions on affinoids in section 2.3 to obtain some global results.

2.4.1 The Fréchet completion of a twisted sheaf of differential algebra on smooth rigid analytic varieties over K

Definition-Lemma 2.4.1. *Let $\{U_i\}_{i\in I}$ be an affinoid open cover of X such that $\mathcal{T}|_{U_i}$ is free, and let $\mathcal{A}^\infty_{U_i,\mathcal{T}}$ be the Fréchet completion of $\mathcal{A}^\infty_\mathcal{T}|_{U_i}$ on U_i as in Definition-Lemma 2.3.3. By Lemma 2.3.4, we have the isomorphisms $\mathcal{A}^\infty_{U_i,\mathcal{T}}|_{U_i\cap U_j} \xrightarrow{t_{ij}} \mathcal{A}^\infty_{U_j,\mathcal{T}}|_{U_j\cap U_i}$, such that $t_{ij} = t_{ji}^{-1}$ and $t_{ik} = t_{jk}\circ t_{ij}$. Therefore we can glue $\mathcal{A}^\infty_{U_i,\mathcal{T}}$ and get a sheaf of $\mathcal{O}_X$-algebras $\mathcal{A}^\infty_{\mathcal{T},\{U_i\}_{i\in I}}$. Then*

1. *There is an inclusion of sheaves of $\mathcal{O}_X$-algebras $\mathcal{A}_\mathcal{T} \hookrightarrow \mathcal{A}^\infty_{\mathcal{T},\{U_i\}_{i\in I}}$.*
2. *If $\{V_j\}_{j\in J}$ is another affinoid open cover of X of finite type such that $\mathcal{T}|_{U_i}$ is free, then there is a unique isomorphism $\mathcal{A}^\infty_{\mathcal{T},\{U_i\}_{i\in I}} \simeq \mathcal{A}^\infty_{\mathcal{T},\{V_j\}_{j\in J}}$ such that the following diagram commutes:*

$$\begin{array}{ccc} \mathcal{A}_\mathcal{T} & \longrightarrow & \mathcal{A}^\infty_{\mathcal{T},\{U_i\}_{i\in I}} \\ \big\downarrow = & & \big\downarrow \simeq \\ \mathcal{A}_\mathcal{T} & \longrightarrow & \mathcal{A}^\infty_{\mathcal{T},\{V_j\}_{j\in J}} \end{array}$$

   Thus we shall write $\mathcal{A}^\infty_{\mathcal{T},\{U_i\}_{i\in I}}$ as $\mathcal{A}^\infty_\mathcal{T}$, which will be called the Fréchet completion of $\mathcal{A}_\mathcal{T}$.

Proof.

1. This follows from Lemma 2.3.4 and gluing.

2. Consider the cover $\{U_i \cap V_j\}_{i\in I, j\in J}$. Then the restriction morphisms

$$\mathcal{A}^{\infty}_{U_i,\mathcal{T}}|_{U_i\cap V_j} \to \mathcal{A}^{\infty}_{U_i\cap V_j,\mathcal{T}}$$

induce an isomorphism $\mathcal{A}^{\infty}_{\mathcal{T},\{U_i\}_{i\in I}} \xrightarrow{\simeq} \mathcal{A}^{\infty}_{\mathcal{T},\{U_i\cap V_j\}_{i\in I,j\in J}}$. Similarly we have an isomorphism

$$\mathcal{A}^{\infty}_{\mathcal{T},\{V_j\}_{j\in J}} \to \mathcal{A}^{\infty}_{\mathcal{T},\{U_i\cap V_j\}_{i\in I,j\in J}}.$$

Thus we get an isomorphism $\mathcal{A}^{\infty}_{\mathcal{T},\{U_i\}_{i\in I}} \xrightarrow{\simeq} \mathcal{A}^{\infty}_{\mathcal{T},\{V_j\}_{j\in J}}$ that is compatible with the inclusion of $\mathcal{A}_{\mathcal{T}}$. The uniqueness follows from Lemma 2.3.4.

□

Proposition 2.4.2. *$\mathcal{A}^{\infty}_{\mathcal{T}}$ is flat over $\mathcal{A}_{\mathcal{T}}$ as a right module. Moreover, if $U \in \mathcal{B}_X$ such that U has a smooth admissible formal model, then $\mathcal{A}^{\infty}_{\mathcal{T}}(U)$ is faithfully flat over $\mathcal{A}_{\mathcal{T}}(U)$.*

Proof. Let $U \subseteq X$ be an open affinoid with such that $\mathcal{T}|_U$ is free. By Lemma 2.3.4 the isomorphism

$$\mathcal{A}^{\infty}_{\mathcal{T}}(U) \simeq \varprojlim_n \Gamma(\mathrm{Spf}(\mathcal{O}^{\circ}_X(U)), \widehat{\mathcal{A}}_{\mathfrak{T}_n,\mathrm{Spf}(\mathcal{O}^{\circ}_X(U)),\mathbb{Q}})$$

realizes $\mathcal{A}^{\infty}_{\mathcal{T}}(U)$ as a Féchet-Stein algebra. It follows from [5] 3.2.3 (iv) that $\Gamma(\mathrm{Spf}(\mathcal{O}^{\circ}_X(U)), \widehat{\mathcal{A}}_{\mathfrak{T}_n,\mathrm{Spf}(\mathcal{O}^{\circ}_X(U))})$ is flat over $\Gamma(\mathrm{Spf}(\mathcal{O}^{\circ}_X(U)), \mathcal{A}_{\mathfrak{T}_n,\mathrm{Spf}(\mathcal{O}^{\circ}_X(U))})$, so $\Gamma(\mathrm{Spf}(\mathcal{O}^{\circ}_X(U)), \widehat{\mathcal{A}}_{\mathfrak{T}_n,\mathrm{Spf}(\mathcal{O}^{\circ}_X(U)),\mathbb{Q}})$ is flat over $\mathcal{A}_{\mathcal{T}}(U)$ as a right module. By [28] Remark 3.2, we see that $\mathcal{A}^{\infty}_X(U)$ is flat over $\Gamma(\mathrm{Spf}(\mathcal{O}^{\circ}_X(U)), \widehat{\mathcal{A}}_{\mathfrak{T}_n,\mathrm{Spf}(\mathcal{O}^{\circ}_X(U)),\mathbb{Q}})$ as a right module.

If U has a smooth integral model, [27] Proposition 3.4 implies the faithful flatness.

□

Remark 2.4.3. It is unclear whether a smooth quasi-compact rigid analytic variety X has a finite open cover of affinoids that have smooth formal models. However, by [10] at every classical point of X, we can find a neighborhood that has a smooth integral model.

2.4.2 Coadmissible modules over $\mathcal{A}_{\mathcal{T}}^{\infty}$ on a smooth rigid variety over K.

Definition 2.4.4. A sheaf of $\mathcal{A}_{\mathcal{T}}^{\infty}$-module $\mathcal{M}$ is coadmissible if there exists an open affinoid cover $\{U_i\}_{i\in I}$ such that

1. $\mathcal{T}|_{U_i}$ is free.

2. $\mathcal{M}|_{U_i} \simeq \widetilde{M_i}$, where $\widetilde{M_i}$ is the sheaf of coadmissible $\mathcal{A}_{\mathcal{T}}^{\infty}|_{U_i}$-module associated to a coadmissible $\mathcal{A}_{\mathcal{T}}^{\infty}(U_i)$-module M_i defined in Definition-Lemma 2.3.10.

Lemma 2.4.5. *Let $\mathcal{M}$ be a sheaf of coadmissible left $\mathcal{A}_{\mathcal{T}}^{\infty}$-module on a smooth rigid analytic variety over K. If U is an affinoid subdomain of X such that $\mathcal{T}|_U$ is free, then*

1. $\mathcal{M}(U)$ is a coadmissible left module over $\mathcal{A}_{\mathcal{T}}^{\infty}(U)$, and $\mathcal{M}|_U \simeq \widetilde{\mathcal{M}(U)}$.

2. The higher cohomology groups of $\mathcal{M}|_U$ vanish.

Proof. Let $\{U_i\}_{i\in I}$ be an open affinoid cover such that $\mathcal{T}|_{U_i}$ is free and $\mathcal{M}|_{U_i}$ is associated to a coadmissible module over $\mathcal{A}_{\mathcal{T}}^{\infty}(U_i)$. Since finitely many $U \cap U_i$ will cover U, by Proposition 2.2.5 there exists an admissible formal model $\mathfrak{U}$ of U obtained by an admissible formal blow-up of Spf$(\mathcal{O}_X^{\circ}(U))$, such that $\{\mathfrak{U}_i\}$ is a finite open cover of $\mathfrak{U}$ and $sp^{-1}(\mathfrak{U}_i) = U \cap U_i$, where $sp : U \to \mathfrak{U}$ is the associated specialization map. Replacing $\{U_i\}_{i\in I}$ by a finite refinement, we may assume that $\{\mathfrak{U}_i\}_{i\in I}$ is a finite open affine cover of $\mathfrak{U}$. By Lemma 2.2.12, we see that $\mathcal{M}(U) \simeq \Gamma(\mathfrak{U}, sp_*\mathcal{M}|_U)$ is a coadmissible module over $\Gamma(\mathfrak{U}, \mathcal{A}_{\mathcal{T},\mathfrak{U}}^{\infty}) \xrightarrow{\simeq} \mathcal{A}_{\mathcal{T}}^{\infty}(U)$, and $H^i(\mathfrak{U}, sp_*(\mathcal{M}|_U)) \simeq \check{H}^i(\{U_i\}_{i\in I}, \mathcal{M}|_U) = 0$ for $i \geq 1$. Therefore, we conclude that $H^i(U, \mathcal{M}|_U) = 0$ for $i \geq 1$.

□

Proposition 2.4.6. *Let $Mod^{coad}(\mathcal{A}_{\mathcal{T}}^{\infty})$ be the full subcategory of the category of sheaves of $\mathcal{A}_{\mathcal{T}}^{\infty}$-modules, whose objects are coadmissible $\mathcal{A}_{\mathcal{T}}^{\infty}$-modules. Then $Mod^{coad}(\mathcal{A}_{\mathcal{T}}^{\infty})$ is an abelian category.*

Proof. This follows from Lemma 2.4.5 and [28] Proposition 2.1.

□

2.5 Microlocal differential operators

In section 2.5, let X be a smooth rigid analytic variety of dimension d, and let $\pi : T^*X \to X$ be the natural projection from of the cotangent bundle. We will construct a sheaf of microlocal differential operators $\mathcal{E}_X$ on T^*X whose restriction to the zero section is $\mathcal{D}_X$. We conjecture that if $\mathcal{M}$ is a coherent $\mathcal{D}_X$-module, then the characteristic variety of $\mathcal{M}$ equals to the support of $\mathcal{E}_X \otimes_{\pi^{-1}\mathcal{D}_X} \pi^{-1}\mathcal{M}$ as sets.

Convention-Notation 4. Let $F_.$ be the order filtration on $\mathcal{D}_X$, so it induces an isomorphism between sheaves of algebras $\mathrm{Sym}_{\mathcal{O}_X}\mathcal{T}_X \xrightarrow{\simeq} \mathrm{gr}_.^F\mathcal{D}_X$, and we view $\mathrm{Sym}_{\mathcal{O}_X}\mathcal{T}_X$ as a subsheaf of rings in $\pi_*\mathcal{O}_{T^*X}$ as in [21] 2.1. The image of an m-th order differential operator P under $\sigma_m : \mathrm{gr}_F^m\mathcal{D}_X \to \mathrm{Sym}^m_{\mathcal{O}_X}\mathcal{T}_X$ is called the principal symbol of P. Let $U \in \mathcal{B}_X$ such that $\Omega^1_X|_U$ is trivial. If $x = (x_1, x_2, ...x_d)$ is a system of local coordinates on U, let ξ_i be the image of $\frac{\partial}{\partial x_i}$ under the inclusion $\mathrm{gr}_{F.}\mathcal{D}_X \hookrightarrow \pi_*\mathcal{O}_{T^*X}$, for $i = 1, 2, ..., d$. We call $(x, \xi) = (x_1, .., x_d, \xi_1, ..., \xi_d)$ a canonical chart on T^*U. For $m \in K$, let $\mathcal{O}_{T^*U}(m)$ be the sheaf that assigns to every $\Omega \in \mathcal{B}_{T^*U}$ the sections

$$\mathcal{O}_{T^*U}(m)(\Omega) = \left\{ f(x,\xi) \in \mathcal{O}_{T^*U}(\Omega) \Big| \sum_{i=1}^{d} \xi_i \frac{\partial}{\partial \xi_i} f(x,\xi) = mf(x,\xi) \right\}.$$

The definition of $\mathcal{O}_{T^*U}(m)$ is independent of choices of canonical charts, since if $(\tilde{x}, \tilde{\xi})$ is another canonical chart on T^*U, then $\sum_{i=1}^d \xi_i \frac{\partial}{\partial \xi_i} = \sum_{i=1}^d \tilde{\xi}_i \frac{\partial}{\partial \tilde{\xi}_i}$. By glueing we get a sheaf $\mathcal{O}_{T^*X}(m)$ on T^*X. Also, note that $\mathrm{gr}_F^{\cdot}\mathcal{D}_X \xrightarrow{\simeq} \pi_*(\bigoplus_{m \in \mathbb{N}} \mathcal{O}_{T^*X}(m))$.

Similar to the case of a complex manifold, there is a canonical 1-form ω on T^*X, characterized by the following condition: for $y \in T^*X^{Tate}$, if we view y as a 1-form on X, then $\langle \omega_p, \mu \rangle = \langle y_{\pi(p)}, d\pi_y(\mu) \rangle$, where $p \in T^*X^{Tate}$, $\mu \in T_y(T^*X)$ the tangent space of T^*X at y, and $d\pi_y : T_y(T^*X) \to T_{\pi(y)}X$ is the derivative of π at y. To properly define ω, we can first define it in a canonical chart $\omega := \sum_{i=1}^{d} \xi_i dx_i$, and then glue. It follows that $\theta_X := d\omega$ is a non-degenerate anti-symmetric bilinear form, which gives rise to a canonical identification

$H : T^*(T^*X) \to T(T^*X)$ such that if $y \in T^*X^{Tate}$, $v \in T_y(T^*X)$ and $\nu \in T_y^*(T^*X)$, then $\langle \theta_X, v \wedge H_y(\nu)) \rangle = \langle \nu, v \rangle$. The Poisson bracket $\{,\}$ on T^*X is defined as $\{f, g\} = (H(df))g$ if $f, g \in \mathcal{O}_{T^*X}$. In a canonical chart $H(d\xi_i) = \frac{\partial}{\partial x_i}$, $H(dx_i) = -\frac{\partial}{\partial \xi_i}$, and

$$\{f(x,\xi), g(x,\xi)\} = \sum_{i=1}^{d} (\frac{\partial f}{\partial \xi_i}\frac{\partial g}{\partial x_i} - \frac{\partial g}{\partial \xi_i}\frac{\partial f}{\partial x_i}).$$

We say a rigid analytic subspace of T^*X is involutive if its defining ideal is closed under the Poisson bracket.

2.5.1 Characteristic varieties of coherent $\mathcal{D}_X$-modules

In this subsection, we check that the construction of the characteristic variety of a coherent $\mathcal{D}$-module on a complex manifold also works in the p-adic case.

Lemma 2.5.1. *$\mathcal{D}_X$ is a noetherian sheaf of rings. (See [21] Appendix A for the definition of a noetherian sheaf of rings.)*

Proof. By [11] Proposition 15.1.1, the local ring of a point in adic space is noetherian. Since $\mathcal{O}_X(U)$ is noetherian for $U \in \mathcal{B}_X$, we see that $\mathcal{O}_X$ is a sheaf of neotherian rings. It follows from [21] Theorem A.29 that $\mathcal{D}_X$ is a noetherian sheaf of rings.

□

Therefore, it follows from [21] Lemma A.26 that a coherent module $\mathcal{M}$ over $\mathcal{D}_X$ locally has a good filtration. Recall that an increasing exhaustive filtration $G_.$ on $\mathcal{M}$ is good if

1. $F_m\mathcal{D}_X \cdot G_l\mathcal{M} \subseteq G_{l+m}\mathcal{M}$ for $m, l \in \mathbb{Z}$.
2. $\bigoplus_{m\in\mathbb{Z}} G_m(\mathcal{M})$ is a locally finitely generated module over the Rees algebra $\bigoplus_{m\in\mathbb{Z}} F_m(\mathcal{D}_X)$.

For instance, if $\mathcal{M}$ is generated by sections $u_1, u_2, ..., u_v$, then $G_n(\mathcal{M}) := \sum_{i=1}^{v} F_n(\mathcal{D}_X)u_i$ is a good filtration. Therefore, if $G_.$ is a good filtration on $\mathcal{M}$, then $\mathrm{gr}_.^G\mathcal{M}$ is coherent over $\mathrm{gr}^F_{\cdot}\mathcal{D}_X$. By [6] Appendix III 3.21, the annihilating ideal $\mathcal{J}_{alg}$ of $\mathrm{gr}^G_{\cdot}\mathcal{M}$ in $\mathrm{gr}^F_{\cdot}\mathcal{D}_X$ is independent of choices of good filtrations. Hence $\mathcal{J} := \mathcal{O}_{T^*X} \otimes_{\mathrm{gr}^F_{\cdot}\mathcal{D}_X} \mathcal{J}_{alg}$ is a coherent sheaf of ideals of $\mathcal{O}_{T^*X}$, and we can define the characteristic variety $\mathrm{Ch}(\mathcal{M})$ of $\mathcal{M}$, to be the rigid analytic subspace of T^*X vanishing on $\mathcal{J}$.

Lemma 2.5.2. *The characteristic variety of a coherent $\mathcal{D}_X$-module $\mathcal{M}$ is a conic involutive rigid analytic subspace of T^*X.*

Proof. It follows from the construction of $\mathrm{Ch}(\mathcal{M})$ that it is preserved by the $\mathbb{G}_m$-action on the fiber of T^*X. By [6] Appendix III Theorem 3.25, we see that under the Poisson bracket $\{\mathcal{J}_{alg,x}, \mathcal{J}_{alg,x}\} \subseteq \mathcal{J}_{alg,x}$ for $x \in X$. By the product rule, we have $\{\mathcal{J}_x, \mathcal{J}_x\} \subseteq \mathcal{J}_x$, and thus $\mathcal{J}$ is closed under the Poisson bracket.

□

2.5.2 Microlocal differential operators on smooth affinoid rigid analytic varieties over K

In this subsection, suppose $X = \mathrm{Spa}(A, A^\circ)$ is a smooth affinoid rigid analytic variety such that Ω^1_X is free. Let (x, ξ) be a canonical chart on T^*X.

Definition 2.5.3. Let $\mathcal{E}_X$ be the presheaf that assigns to every $\Omega \in \mathcal{B}_{T^*X}$ the sections

$$\mathcal{E}_X(\Omega) = \left\{(p_k(x,\xi))_{k\in\mathbb{Z}} \;\middle|\; p_k(x,\xi) \in \mathcal{O}_{T^*X}(k)(\Omega) \text{ satisfying the following conditions 1 and 2.}\right\}$$

1. There exists $N \in \mathbb{Z}$ such that $p_k(x, \xi) = 0$ if $k > N$.

2. There exists $M \geq 0$ such that

$$\lim_{k\to-\infty} \|\frac{p_k(x,\xi)}{p^{kM}}\|_\Omega = 0.$$

By the maximum principle we see that $\mathcal{E}_X$ is a sheaf of K-vector spaces on X.

Definition-Lemma 2.5.4. *If $(p_k(x,\xi))_{k\in\mathbb{Z}}, (q_k(x,\xi))_{k\in\mathbb{Z}} \in \mathcal{E}_X(\Omega)$, define the product $(p_k(x,\xi))_{k\in\mathbb{Z}} \cdot (q_k(x,\xi))_{k\in\mathbb{Z}}$ to be $(r_k(x,\xi))_{k\in\mathbb{Z}}$, where*

$$r_k(x,\xi) = \sum_{\substack{I\in\mathbb{N}^d \\ k=i+j-|I|}} \frac{1}{I!}(\partial_\xi^I p_i(x,\xi))(\partial_x^I q_j(x,\xi)),$$

and if $I = (v_1, ...v_d)$, then $I! = v_1!v_2!...v_d!$ and $0! = 1$.

Then $\mathcal{E}_X(\Omega)$ is an associative ring with a unit.

Proof. To check the product is well-defined, we verify the convergence condition of $r_k(x,\xi)$ when $k \to -\infty$. Since $\frac{\partial}{\partial x_i}$ and $\frac{\partial}{\partial \xi_i}$ are continuous derivations on $\mathcal{O}_{T^*X}(\Omega)$, we assume their operator norms $\|\frac{\partial}{\partial x_i}\| \le |c|$ and $\|\frac{\partial}{\partial \xi_i}\| \le |c|$ for $c \in \bar{K}$ and $i = 1, 2, ...d$. Since $v_p(n!) = \frac{n-\text{sum of } p\text{-adic digits of n}}{p-1}$, we see that $|I!| \ge |p^{|I|}|$. Therefore,

$$\|\frac{r_k(x,\xi)}{p^{kn}}\|_\Omega \le \sum_{\substack{I\in\mathbb{N}^d \\ i\in\mathbb{Z}}} |\frac{1}{p^{|I|}} c^{2|I|} p^{n|I|}| \|\frac{p_i(x,\xi)}{p^{in}}\|_\Omega \|\frac{q_{k+|I|-i}(x,\xi)}{p^{(k+|I|-i)n}}\|_\Omega.$$

Let $M > 0$ such that $\lim_{k\to-\infty} \|\frac{p_k(x,\xi)}{p^{kM}}\|_\Omega = 0$ and $\lim_{k\to-\infty} \|\frac{q_k(x,\xi)}{p^{kM}}\|_\Omega = 0$. Let $n > \max\{M, 1 - 2v_p(c)\}$, so $|\frac{1}{p^{|I|}} c^{2|I|} p^{n|I|}| < 1$. For any $\epsilon > 0$, there exits $N > 0$ such that $p_i(x,\xi) = 0$ for $i > N$ and $q_j(x,\xi) = 0$ for $j > N$, and $\|\frac{q_j(x,\xi)}{p^{jM}}\|_\Omega < \epsilon$ if $j < -N$, and $\|\frac{p_i(x,\xi)}{p^{iM}}\|_\Omega < \epsilon$ if $i < -N$, and $|\frac{1}{p^{|I|}} c^{2|I|} p^{n|I|}| < \epsilon$ if $|I| > N$. Then $\{\|\frac{p_k(x,\xi)}{p^{kn}}\|_\Omega, \|\frac{q_k(x,\xi)}{p^{kn}}\|_\Omega\}_{k\in\mathbb{Z}}$ is bounded by $C > 0$. It follows that

$$\sum_{\substack{I\in\mathbb{N}^d \\ |i|>N}} \|\frac{p_i(x,\xi)}{p^{in}}\|_\Omega \|\frac{q_{k+|I|-i}(x,\xi)}{p^{(k+|I|-i)n}}\|_\Omega < \epsilon \cdot C.$$

If $|i| \le N$, $|I| \le N$ and $k < -3N$, then $k + |I| - i < |I| - 2N \le -N$, so we have

$$\sum_{\substack{|I|\le N \\ |i|\le N}} \|\frac{p_i(x,\xi)}{p^{in}}\|_\Omega \|\frac{q_{k+|I|-i}(x,\xi)}{p^{(k+|I|-i)n}}\|_\Omega < C \cdot \epsilon.$$

If $|i| \leq N$ and $|I| > N$, then

$$\sum_{\substack{|I|>N \\ |i|<N}} |\frac{1}{p^{|I|}} c^{2|I|} p^{n|I|}| \| \frac{p_i(x,\xi)}{p^{in}} \|_\Omega \| \frac{q_{k+|I|-i}(x,\xi)}{p^{(k+|I|-i)n}} \|_\Omega < \epsilon \cdot C^2.$$

We conclude that $\lim_{k \to -\infty} \| \frac{r_k(x,\xi)}{p^{kn}} \|_\Omega = 0$.

The associativity follows from the Leibniz's formula.

□

Lemma 2.5.5. *If $\tilde{x}$ is another choice of local coordinates on X, then $\Phi : \mathcal{E}_X(\Omega) \to \mathcal{E}_X(\Omega)$ defined as*

$$\Phi(p_k(\tilde{x}, \tilde{\xi})) = \sum_{\substack{s \in \mathbb{N} \\ I_1, I_2, \ldots I_s \in \mathbb{N}^d \\ |I_j| \geq 2 (j=1,\ldots,s) \\ l = k + \sum_{j=1}^s (|I_j|-1)}} \frac{\prod_{j=1}^s (\partial_x^{I_j} (\sum_{t=1}^d \tilde{x}_t \tilde{\xi}_t))}{s! I_1! \ldots I_s!} (\partial_\xi^{I_1+\ldots+I_s} p_l(x,\xi)).$$

is an isomorphism between rings, where $\tilde{\xi}_t = \sum_{j=1}^d \frac{\partial x_j}{\partial \tilde{x}_t} \xi_j$. In particular, we see that the definition of $\mathcal{E}_X$ is independent of choices of local coordinates.

Proof. By [21] Lemma 7.4 and 7.5 we are reduced to showing that there exists n such that $\lim_{k \to -\infty} \| \frac{\Phi(p_k(\tilde{x}, \tilde{\xi}))}{p^{kn}} \|_\Omega = 0$. Assume $\| \sum_{t=1}^d \tilde{x}_t \tilde{\xi}_t) \|_\Omega \leq C_1$, and the operator norms $\| \frac{\partial}{\partial x_i} \| \leq |c|$ and $\| \frac{\partial}{\partial \xi_i} \| \leq |c|$ for $c \in \bar{K}$ and $i = 1, 2, \ldots d$. Let $M > 0$ such that $\lim_{k \to -\infty} \| \frac{p_k(x,\xi)}{p^{kM}} \|_\Omega = 0$. Because of the assumption on the summation indexes $\sum_{t=1}^s |I_t| - s \geq s \geq 0$, if $v_p(c) < 1$, then $(2v_p(c) + n - 1)(\sum_{t=1}^s |I_t| - s) + 2s(v_p(c) - 1) \geq \sum_{t=1}^s |I_t| - s$ when $n \geq -4v_p(c) + 4$. Let $n = \max\{M, -2v_p(c) + 2\}$ if $v_p(c) \geq 1$, and let $n = \max\{M, -4v_p(c) + 4\}$ if $v_p(c) < 1$, so $\{\| \frac{p_k(x,\xi)}{p^{kn}} \|_\Omega\}_{k \in \mathbb{Z}}$ is bounded by constant C_2. For any $\epsilon > 0$, there exits $N > 0$ such that $p_i(x,\xi) = 0$ for $i > N$, and $\| \frac{p_i(x,\xi)}{p^{iM}} \|_\Omega < \epsilon$ if $i < -N$, and $|p^{\sum_{t=1}^s |I_t| - s}| < \epsilon$ if

$\sum_{t=1}^{s} |I_t| - s \geq N$. Therefore,

$$\|\frac{\Phi(p_k(\tilde{x},\tilde{\xi}))}{p^{kn}}\|_\Omega \leq \sum_{\substack{s\in\mathbb{N} \\ I_1,I_2,\dots I_s\in\mathbb{N}^d \\ |I_j|\geq 2(j=1,\dots,s)}} |\frac{p^{2v_p(c)(\sum_{t=1}^s |I_t|)+(\sum_{t=1}^s |I_t|-s)n}}{p^{s+\sum_{t=1}^s |I_t|}}|C_1\|\frac{p_{k+\sum_{t=1}^s |I_t|-s}(x,\xi)}{p^{(k+\sum_{t=1}^s |I_t|-s)n}}\|_\Omega$$

$$\leq \sum_{\substack{s\in\mathbb{N} \\ I_1,I_2,\dots I_s\in\mathbb{N}^d \\ |I_j|\geq 2(j=1,\dots,s)}} |p^{\sum_{t=1}^s |I_t|-s}|C_1\|\frac{p_{k+\sum_{t=1}^s |I_t|-s}(x,\xi)}{p^{(k+\sum_{t=1}^s |I_t|-s)n}}\|_\Omega.$$

If $\sum_{t=1}^{s} |I_t| - s \geq N$, then

$$\sum_{\substack{s>N \\ I_1,I_2,\dots I_s\in\mathbb{N}^d \\ |I_j|\geq 2(j=1,\dots,s)}} |p^{\sum_{t=1}^s |I_t|-s}|\|\frac{p_{k+\sum_{t=1}^s |I_t|-s}(x,\xi)}{p^{(k+\sum_{t=1}^s |I_t|-s)n}}\|_\Omega < \epsilon C_2.$$

If $\sum_{t=1}^{s} |I_t| - s < N$, and $k < -2N$, then $k - (\sum_{t=1}^s |I_t| - s) < -N$, so

$$\sum_{\substack{s<N \\ I_1,I_2,\dots I_s\in\mathbb{N}^d \\ |I_j|\geq 2(j=1,\dots,s)}} |p^{\sum_{t=1}^s |I_t|-s}|\|\frac{p_{k+\sum_{t=1}^s |I_t|-s}(x,\xi)}{p^{(k+\sum_{t=1}^s |I_t|-s)n}}\|_\Omega < \epsilon$$

□

Remark 2.5.6. Let $\mathcal{E}_X^\infty$ be the presheaf that assigns to every $\Omega \in \mathcal{B}_{T^*X}$ the sections

$$\mathcal{E}_X^\infty(\Omega) = \{(p_k(x,\xi))_{k\in\mathbb{Z}} \mid p_k(x,\xi) \in \mathcal{O}_{T^*X}(k)(\Omega) \text{ satisfying the following growth conditions.}\}$$

1. For any $n \geq 0$

$$\lim_{k\to+\infty} \|\frac{p_k(x,\xi)}{p^{kn}}\|_\Omega = 0$$

2. There exists $M \geq 0$ such that

$$\lim_{k\to-\infty} \|\frac{p_k(x,\xi)}{p^{kM}}\|_\Omega = 0$$

One can similarly check that $\mathcal{E}_X^\infty$ is a sheaf of K-vector spaces on X. Moreover, the formula in Definition-Lemma 2.5.4 defines a ring structure on $\mathcal{E}_X^\infty$.

2.5.3 Microlocal differential operators on smooth rigid analytic varieties over K

In this subsection, let X be a smooth rigid analytic variety over K.

Definition-Lemma 2.5.7. *Let $\{U_i\}_{i\in I}$ be an affinoid open cover of X such that $\Omega^1_X|_{U_i}$ is trivial. By Lemma 2.5.5 and [21] Lemma 7.5, we can glue $\mathcal{E}_{U_i}$ (see Definition 2.5.3) to get a sheaf of rings $\mathcal{E}_X$ on X. The definition is independent of the choice of covers.*

By the definition of $\mathcal{E}_X$, we see that there is an inclusion of sheaves of rings $\pi^{-1}\mathcal{D}_X \hookrightarrow \mathcal{E}_X$. Moreover, let T^*_XX be the zero section of T^*X, we have $\mathcal{D}_X \simeq \mathcal{E}_X|_{T^*_XX}$.

Remark 2.5.8. Similarly $\mathcal{E}_X^\infty$ is a sheaf on X such that there is an inclusion of sheaves of rings $\pi^{-1}\mathcal{D}_X^\infty \hookrightarrow \mathcal{E}_X^\infty$, and $\mathcal{D}_X^\infty \simeq \mathcal{E}_X^\infty|_{T^*_XX}$. We wonder whether these sheaves could have applications in the study of p-adic differential equations.

CHAPTER 3

EQUIVARIANT $\mathcal{D}^\infty$-MODULES ON SMOOTH RIGID ANALYTIC VARIETIES OVER A COMPLETE DISCRETE VALUED FIELD

3.1 Introduction and notations

Let us briefly recall the relation between the unitary representations of a real semi-simple Lie group $G_{\mathbb{R}}$ with finite center and the geometry of the complex flag variety. Let $\mathfrak{g}$ be the complexification of the Lie algebra of $G_{\mathbb{R}}$, and K be the complexifiction of a maximal compact $K_{\mathbb{R}}$ of $G_{\mathbb{R}}$. Let X be the flag variety of $\mathfrak{g}$, whose analytification is X^{an}. Let $\mathcal{U}(\mathfrak{g})$ be the universal enveloping algebra of $\mathfrak{g}$, and $\mathcal{D}_X$ be the sheaf of differential operators on X. Suppose E is an irreducible unitary representation of $G_{\mathbb{R}}$ over $\mathbb{C}$. On one hand, the subspace $M = E^{K_{\mathbb{R}}-\text{finite}}$ of $K_{\mathbb{R}}$-finite vectors of E is a Harish-Chandra module. For simplicity we assume M has trivial infinitesimal central character. Hence by Beilinson-Bernstein localization theorem, the sheaf $\mathcal{M} = \mathcal{D}_X \otimes_{\mathcal{U}(\mathfrak{g})} E^{K_{\mathbb{R}}-\text{finite}}$ is a K-equivariant coherent $\mathcal{D}_X$-module on X, whose global section is M. Since there are finitely many K-orbits on X, a K-equivariant coherent $\mathcal{D}_X$-module is regular holonomic. Therefore by Riemman-Hilbert correspondence, $DR_X(\mathcal{M}) = \mathbf{R}\mathcal{H}om_{\mathcal{D}_X}(\mathcal{O}_X, \mathcal{M})$ is a K-equivariant perverse sheaves on X. On the other hand, the Matsuki duality generalizes to an equivalence Φ in Theorem 1.42 [22] between the bounded K-equivariant derived category of $\mathbb{C}_X$-sheaves and the $G_{\mathbb{R}}$-equivariant derived category of $\mathbb{C}_X$-sheaves. Let $C^\infty(G_{\mathbb{R}})$ be space of smooth functions on $G_{\mathbb{R}}$. Then the maximal globalization of M is $Hom^{top}_{\mathcal{U}(\mathfrak{g})}(M^*, C^\infty(G_{\mathbb{R}}))$, which is an admissible $G_{\mathbb{R}}$-representations of finite length with nuclear Fréchet topology, whose $K_{\mathbb{R}}$-finite vectors are M. By [22] section 1.7

$$\mathbf{R}Hom^{top}_{\mathcal{U}(\mathfrak{g})}(M^*, C^\infty(G_{\mathbb{R}})) \simeq \mathbf{R}\mathcal{H}om_{\mathbb{C}_X}(\Phi(DR_X(\mathcal{M})), \Omega_{X^{an}}[\text{dimension of } X]).$$

Let $\Gamma_c(G_{\mathbb{R}}, Dist_{G_{\mathbb{R}}})$ be the space of distribution on $G_{\mathbb{R}}$ with compact support. Then the minimal globalization of M is $\Gamma_c(G_{\mathbb{R}}, Dist_{G_{\mathbb{R}}}) \otimes_{\mathcal{U}(\mathfrak{g})} M$, which is an admissible $G_{\mathbb{R}}$-representations of finite length with dual nuclear Fréchet topology, whose $K_{\mathbb{R}}$-finite vectors are M. By [22] section 1.7

$$\Gamma_c(G_{\mathbb{R}}, Dist_{G_{\mathbb{R}}}) \otimes^{\mathbf{L}}_{(\mathfrak{g}, K_{\mathbb{R}})} M \simeq \mathbf{R}\Gamma_c^{top}(X^{an}, \Phi(DR_X(\mathcal{M})) \otimes \mathcal{O}_{X^{an}}).$$

For example, when $\mathbf{G} = GL_2$, $K_{\mathbb{R}} = O(2)$ and $X = \mathbb{P}^1$, there are two K-orbits $\{i, -i\}$ and the complement $\mathbb{P}^1 \backslash \{i, -i\}$. By Matsuki duality, the K-orbit $\{i, -i\}$ corresponds to the $GL_2(\mathbb{R})$-orbit $\mathbb{P}^1 \backslash \mathbb{P}^1(\mathbb{R})$, and the K-orbit $\mathbb{P}^1 \backslash \{i, -i\}$ corresponds to $GL_2(\mathbb{R})$-orbit $\mathbb{P}^1(\mathbb{R})$. By Beilinson-Bernstein localization, the K-orbit $\{i, -i\}$ corresponds to discrete series, and the K-orbit $\mathbb{P}^1 \backslash \{i, -i\}$ corresponds to principal series.

We would like to know whether it is possible to have a parallel story for admissible locally analytic representations of p-adic groups. In section 3.3, if $\mathbb{G}$ is a rigid analytic group acting on X and $\mathcal{A}_{\mathcal{T}_X}$ is a $\mathbb{G}$-equivariant twisted differential operators, we define an abelian category of coadmissible G-equivariant $\mathcal{A}^{\infty}_{\mathcal{T}_X}$-modules on X, where G is the $\mathbb{Q}_p$-rational points of $\mathbb{G}$, viewed as a p-adic Lie group. In Thereom 3.4.4, we show that under certain restrictions, when $\mathbb{G}$ is the rigid analytification of a connected split reductive linear algebraic group $\mathbf{G}$ and X is the rigid analytification of the flag variety of $\mathbf{G}$, the category of admissible locally analytic representations of G with a fixed infinitesimal central character is equivalent to the category of coadmissible G-equivariant twisted $\mathcal{D}^{\infty}$-modules on X.

For example, let $X = \mathbb{P}^1$ and $\mathbb{G} = \mathrm{GL}_2$. If $j : U \hookrightarrow X$ is the natural inclusion of the Drinfeld upper half plane U in $\mathbb{P}^1$ with the complement $\mathbb{P}^1(\mathbb{Q}_p)$, we have the following exact sequence of coadmissible $\mathrm{GL}_2(\mathbb{Q}_p)$-equivariant $\mathcal{D}^{\infty}$-modules on $\mathbb{P}^1$:

$$0 \to \mathcal{O}_X \to j_*\mathcal{O}_U \to \mathcal{H}^1_{\mathbb{P}^1(\mathbb{Q}_p)} \to 0,$$

where $\mathcal{H}^1_{\mathbb{P}^1(\mathbb{Q}_p)}$ is defined to be the quotient $j_*\mathcal{O}_U/\mathcal{O}_X$. The global sections of $\mathcal{H}^1_{\mathbb{P}^1(\mathbb{Q}_p)}$

form a principal series (the Orlik-Strauch induction of weight -2 Verma module). By the computation in section 3.5, we see that in general a principal series corresponds to an equivariant twisted $\mathcal{D}^\infty$-module on $\mathbb{P}^1$ with support in $\mathbb{P}^1(\mathbb{Q}_p)$. By the work of Dospinescu and Le Bras [13], the push-forward of the structure sheaves of Drinfeld covers are also coadmissible $\mathrm{GL}_2(\mathbb{Q}_p)$-equivariant $\mathcal{D}^\infty$-modules on $\mathbb{P}^1$. Motiviated by the classical $GL_2(\mathbb{R})$-picture and the proof of [23] Proposition 7.1.3, we conjecture that if $\mathcal{M}$ is a coadmissible $GL_2(\mathbb{Q}_p)$-equivariant $\mathcal{D}^\infty$-module on $\mathbb{P}^1$, and $u \in \mathbb{P}^1$ is not a $\mathbb{Q}_p$-rational point, then the support of $\mathcal{M}$ cannot be the closure of the $GL_2(\mathbb{Q}_p)$-orbit of u.

In Chapter 3, let $\mathbf{G}$ be a connected linear algebraic group over L of rank q. Let $\mathbb{G}$ be the rigid analytification of $\mathbf{G}_K := \mathbf{G} \times_{\mathrm{Spec}(L)} \mathrm{Spec}(K)$ ([8] section 5.4). Let $m : \mathbb{G} \times_K \mathbb{G} \to \mathbb{G}$ be the group multiplication. Let G be the L-rational points of $\mathbb{G}$, viewed as a locally L-analytic group. Let $\mathfrak{g}$ be the Lie algebra of $\mathbf{G}$ over L. The universal enveloping algebra of $\mathfrak{g}_K := \mathfrak{g} \otimes_L K$ is $\mathcal{U}(\mathfrak{g}_K)$.

3.2 Preliminaries

3.2.1 Review of the Fréchet-Stein structure of the distribution algebra of G

Let us recall some notations and facts from [14] section 5.2. Let $r' \in \bar{K}$ such that $|r'| = r$ and $0 < r \leq 1$. Let $\mathbb{B}(r)^d := \mathrm{Spa}(\hat{\bar{K}}\langle r't_1, r't_2, ..., r't_d\rangle, \mathcal{O}_{\hat{\bar{K}}}\langle r't_1, r't_2, ..., r't_d\rangle)$ be the rigid analytic ball of radius r. If $\mathfrak{h}$ is a sufficiently small $\mathcal{O}_L$-Lie sublattice of $\mathfrak{g}$, then fixing a set of basis of $\mathfrak{h}$, we have an isomorphism between rigid analytic groups $\exp : \mathbb{B}(1)^d \to \mathbb{H}$, where $\mathbb{H}$ is an analytic open subgroup of the rigid analytification of $\mathbf{G}$, and $\mathbb{B}(1)^d$ is equipped with the rigid analytic group structure by the Baker-Campbell-Hausdorff formula. We call such $\mathbb{H}$ a good analytic open subgroup of G obtained by exponentiating $\mathfrak{h}$, and the coordinate $(t_1, t_2, ..., t_d)$ on $\mathbb{B}(1)^d$ is called the canonical coordinates of the second kind of $\mathbb{H}$. Let H be the L-rational points of $\mathbb{H}$. Recall that $\mathbb{H}^\circ := \bigcup_{r<1} \mathbb{B}(r)^d$ is a rigid analytic open subgroup of $\mathbb{H}$. Let $\mathbb{H}_n$ be the good analytic open subgroup of G obtained by exponentiating $\alpha^n\mathfrak{h}$ for

$n \in \mathbb{N}$, and let H_n° be the L-rational points of $\mathbb{H}_n^\circ$.

Remark 3.2.1. As a locally L-analytic group H_n° is isomorphic to the L-rational points of the analytic open subgroup obtained by exponentiating $\pi_L^{e_1 n-1}\mathfrak{h}$, where π_L is an uniformizer of L, and $e_1 \in \mathbb{N}$ such that $|\pi_L^{e_1}| = |\alpha|$. In particular H_n° is a compact open normal subgroup of H of finite index.

Lemma 3.2.2. *Consider the inclusion of algebras* $\mathcal{U}(\mathfrak{g}_K) \to D^{la}(H,K)$ *as in [14] p.94, and let* $\overline{\mathcal{U}(\mathfrak{g}_K)}$ *be the closure of* $\mathcal{U}(\mathfrak{g}_K)$ *in* $D^{la}(H,K)$.

1. *There exists an isomorphism*

$$D^{la}(H^\circ, K) \simeq \varprojlim_n \widehat{\mathcal{U}}(\alpha^n\mathfrak{h})_K \otimes_{K[H_{n+1}^\circ]} K[H^\circ]$$

   that realizes $D^{la}(H^\circ,K)$ *as a Fréchet-Stein algebra.*

2. *Under the isomorphism in 1, we have the isomorphism* $\overline{\mathcal{U}(\mathfrak{g}_K)} \simeq \varprojlim_n \widehat{\mathcal{U}}(\alpha^n\mathfrak{h})_K$ *that realizes* $\overline{\mathcal{U}(\mathfrak{g}_K)}$ *as a Fréchet-Stein algebra, where* $\widehat{\mathcal{U}}(\alpha^n\mathfrak{h})_K$ *is the* p*-adic completion of* $\mathcal{U}(\alpha^n\mathfrak{h})$ *inverting* p*, and* $\mathcal{U}(\alpha^n\mathfrak{h})$ *is the* $\mathcal{O}_K$*-subring of* $\mathcal{U}(\mathfrak{g}_K)$ *generated by* $\alpha^n\mathfrak{h}$*. In particular, the closure* $\overline{\mathcal{U}(\mathfrak{g}_K)}$ *is independent of choice of compact open subgroup* H *of* G.

Proof. Assume that $\{X_j\}_{j=1,2,\ldots,q}$ is a set of $\mathcal{O}_L$-basis of $\mathfrak{h}$. For $m \in \mathbb{N}$, let $\mathcal{U}(\alpha^n\mathfrak{h})^{(m)}$ be the $\mathcal{O}_K$-subring of $\mathcal{U}(\mathfrak{g}_K)$ generated by the elements $\frac{(\alpha^n X_j)^i}{i!}$, where $0 \le i \le p^m$ and $1 \le j \le q$. Let $\widehat{\mathcal{U}}(\alpha^n\mathfrak{h})^{(m)}$ be the p-adic completion of $\mathcal{U}(\alpha^n\mathfrak{h})^{(m)}$, and let $\widehat{\mathcal{U}}(\alpha^n\mathfrak{h})_K^{(m)}$ be $\widehat{\mathcal{U}}(\alpha^n\mathfrak{h})^{(m)} \otimes_{\mathcal{O}_K} K$. Then $D^{an}(\mathbb{H}_n^\circ, K) \simeq \varinjlim_m \widehat{\mathcal{U}}(\alpha^n\mathfrak{h})_K^{(m)}$. In addition, we have the inclusion of algebras $K[H_{n+1}^\circ] \to \widehat{\mathcal{U}}(\alpha^n\mathfrak{h})_K$. Explicitly if $g \in H_{n+1}^\circ$ corresponds to the vector $\mathfrak{x} \in \pi_L^{e_1(n+1)-1}\mathfrak{h}$ under the exponential map, then $g \to \sum_{n=0}^{+\infty} \frac{\mathfrak{x}^n}{n!}$. The conjugation action of H° on $\widehat{\mathcal{U}}(\alpha^n\mathfrak{h})_K$ equips $\widehat{\mathcal{U}}(\alpha^n\mathfrak{h})_K \otimes_{K[H_{n+1}^\circ]} K[H^\circ]$ with an algebra structure, and the natural inclusion $K[H^\circ] \hookrightarrow D^{an}(H^\circ, K)$ factors through

$$K[H^\circ] \to \widehat{\mathcal{U}}(\alpha^n \mathfrak{h})_K \otimes_{K[H^\circ_{n+1}]} K[H^\circ] \to D^{an}(H^\circ, K).$$

1. By [14] 5.3.13, the homomorphism of algebras $D^{an}(\mathbb{H}^\circ_{n+1}, K) \to D^{an}(\mathbb{H}^\circ_n, K)$ factors through $\widehat{\mathcal{U}}(\alpha^n \mathfrak{h})^{(m_n)}_K$, for $m_n \in \mathbb{N}$. Let $r = \max_{0 \le i \le p^{m_n}} \{|\frac{1}{i!}|^{\frac{1}{i}} |\alpha|^n\}$, and choose $r(n) \in \mathbb{N}$ such that $|\alpha^{r(n)+1}| \le r \le |\alpha^{r(n)}|$. Then we have

$$D^{an}(\mathbb{H}^\circ_{n+1}, K) \to \widehat{\mathcal{U}}(\alpha^{r(n)} \mathfrak{h})_K \to D^{an}(\mathbb{H}^\circ_{r(n)}, K).$$

The p-adic ordinal of $i!$ is equal to $\frac{i-s(i)}{p-1}$, where $s(i)$ is the sum of the p-adic digits of i. Therefore $r(n) \to +\infty$ as $n \to +\infty$. Since

$$D^{la}(H^\circ, K) \simeq \varprojlim_n D^{an}(\mathbb{H}^\circ_n, K) \otimes_{K[H^\circ_{n+1}]} K[H^\circ],$$

we see that $D^{la}(H^\circ, K)$ is isomorphic to the projective limit of the system of noetherian Banach algebras $\left\{\widehat{\mathcal{U}}(\alpha^n \mathfrak{h})_K \otimes_{K[H^\circ_{n+1}]} K[H^\circ]\right\}_{n \in \mathbb{N}}$. Consider the following commutative diagram:

$$\begin{array}{ccc} \widehat{\mathcal{U}}(\alpha^{n+1} \mathfrak{h})_K & \xrightarrow{p_n} & \widehat{\mathcal{U}}(\alpha^n \mathfrak{h})_K \\ {\scriptstyle i_{n+1}}\downarrow & & {\scriptstyle i_n}\downarrow \\ \widehat{\mathcal{U}}(\alpha^{n+1} \mathfrak{h})_K \otimes_{K[H^\circ_{n+2}]} K[H^\circ] & \xrightarrow{\tilde{p}_n} & \widehat{\mathcal{U}}(\alpha^n \mathfrak{h})_K \otimes_{K[H^\circ_{n+1}]} K[H^\circ] \end{array}$$

As in [14] Proposition 5.3.18, we find a filtration of normal open subgroups

$$H^\circ_{n+2} = G_0 \trianglelefteq G_1 \trianglelefteq ... \trianglelefteq G_w = H^\circ$$

such that G_{i+1}/G_i is cyclic. Therefore i_{n+1} factor though

$$\widehat{\mathcal{U}}(\alpha^{n+1} \mathfrak{h})_K \to A_1 \to ... \to A_q,$$

where $A_i := \widehat{\mathcal{U}}(\alpha^{n+1}\mathfrak{h})_K \otimes_{K[G_0]} K[G_i]$. Since i_n, i_{n+1} and p_n are flat, by [14] Lemma 5.3.17, we see that $\tilde{p}_n$ is also flat.

2. $\widehat{\mathcal{U}}(\alpha^n\mathfrak{h})_K$ is a closed subspace of $\widehat{\mathcal{U}}(\alpha^n\mathfrak{h})_K \otimes_{K[H^\circ_{n+1}]} K[H^\circ]$. Therefore $\varprojlim_n \widehat{\mathcal{U}}(\alpha^n\mathfrak{h})_K$ is a closed subspace in $D^{la}(H^\circ, K)$. Since $\mathcal{U}(\mathfrak{g}_K)$ is dense in $\varprojlim_n \widehat{\mathcal{U}}(\alpha^n\mathfrak{h})_K$, we conclude that $\overline{\mathcal{U}(\mathfrak{g}_K)} \xrightarrow{\simeq} \varprojlim_n \widehat{\mathcal{U}}(\alpha^n\mathfrak{h})_K$ is an isomorphism.

$\square$

3.2.2 Some equivariant sheaves on smooth rigid analytic varieties over K

Let X be a smooth rigid analytic variety over K. Let the morphisms $p_1 : X \times_K \mathbb{G} \to X$ be the natural projection to the first factor, and $p_{12} : X \times_K \mathbb{G} \times_K \mathbb{G} \to X \times_K \mathbb{G}$ be the natural projection to the first and second factors. Let $e \in G$ be the identity element.

Definition 3.2.3. A rigid analytic variety X over K is a right $\mathbb{G}$-variety if there exists a morphism $a : X \times_K \mathbb{G} \to X$ of rigid analytic varieties over K, such that the following two diagrams commute:

1.

$$\begin{array}{ccc} X \times_K \mathbb{G} \times_K \mathbb{G} & \xrightarrow{a \times id_{\mathbb{G}}} & X \times_K \mathbb{G} \\ \downarrow{\scriptstyle id_X \times m} & & \downarrow{\scriptstyle a} \\ X \times_K \mathbb{G} & \xrightarrow{a} & X \end{array}$$

2.

$$\begin{array}{ccc} X \times_K \mathrm{Spa}(K, K^\circ) & \xrightarrow{id_X \times e} & X \times_K \mathbb{G} \\ \downarrow{\scriptstyle \simeq} & & \downarrow{\scriptstyle a} \\ X & \xrightarrow{id_X} & X \end{array}$$

Definition 3.2.4. Let X be a right $\mathbb{G}$-variety.

1. A sheaf of $\mathcal{O}_X$-module $\mathcal{M}$ is $\mathbb{G}$-equivariant if there exists an isomorphism

$$\theta : p_1^*\mathcal{M} \simeq a^*\mathcal{M}$$

of $\mathcal{O}_{X\times_K\mathbb{G}}$-modules such that the following diagram commutes:

$$\begin{array}{ccc} (a\circ(a\times id_G))^*\mathcal{M} & \xrightarrow{(a\times id_G)^*\theta} & (p_1\circ(a\times id_G))^*\mathcal{M} = (a\circ p_{12})^*\mathcal{M} \\ \big\downarrow = & & \big\downarrow p_{12}^*\theta \\ (a\circ(id_X\times m))^*\mathcal{M} & \xrightarrow{(id_X\times m)^*\theta} & (p_1\circ(id_X\times m))^*\mathcal{M} = (p_1\circ p_{12})^*\mathcal{M} \end{array}$$

2. A sheaf of $\mathcal{O}_X$-module $\mathcal{M}$ is G-equivariant if for all $g\in G$, we have isomorphisms of $\mathcal{O}_X$-modules $\theta_g : g^*\mathcal{M}\to\mathcal{M}$, such that the following diagram commutes:

$$\begin{array}{ccc} (g_1g_2)^*\mathcal{M} & \xrightarrow{g_1^*\theta_{g_2}} & g_1^*\mathcal{M} \\ \big\downarrow \theta_{g_1g_2} & & \theta_{g_1}\big\downarrow \\ \mathcal{M} & \xrightarrow{=} & \mathcal{M} \end{array}$$

where g is viewed as an automorphism of X.

3. A morphism $f:\mathcal{M}\to\mathcal{N}$ is a morphism between sheaves of G-equivariant $\mathcal{O}_X$-modules $\mathcal{M}$ and $\mathcal{N}$, if f is a morphism between sheaves of $\mathcal{O}_X$-modules, such that the following diagram commutes:

$$\begin{array}{ccc} \mathcal{M} & \xrightarrow{f} & \mathcal{N} \\ \big\downarrow \theta_g & & \big\downarrow \theta_g \\ g^*\mathcal{M} & \xrightarrow{g^*f} & g^*\mathcal{N} \end{array}$$

Remark 3.2.5. Let X be a right $\mathbb{G}$-variety.

1. Since $G\hookrightarrow\mathbb{G}$ is a continuous inclusion between topological groups, we see that a induces a jointly continuous action morphism of topological spaces $X\times G\to X$, such that G preserves the classical points of X.

2. The morphism $a^{-1}\mathcal{O}_X\to\mathcal{O}_{X\times_K\mathbb{G}}$ induces an isomorphism

$$a^*\mathcal{O}_X = \mathcal{O}_{X\times_K\mathbb{G}}\otimes_{a^{-1}\mathcal{O}_X} a^{-1}\mathcal{O}_X \xrightarrow{\simeq} \mathcal{O}_{X\times_K\mathbb{G}}.$$

Together with the isomorphism $p_1^*\mathcal{O}_X \xrightarrow{\simeq} \mathcal{O}_{X\times_K\mathbb{G}}$ we see that $\mathcal{O}_X$ is $\mathbb{G}$-equivariant.

3. If $\mathcal{M}$ is a $\mathbb{G}$-equivariant sheaf of $\mathcal{O}_X$-modules on X, and $g \in \mathbb{G}$ is a K-rational point, then restricting the isomorphism $a^*\mathcal{O} \simeq p_1^*\mathcal{M}$ to $\{g\} \times_K X \subseteq \mathbb{G} \times_K X$ gives the isomorphism $g^*\mathcal{M} \simeq \mathcal{M}$. Therefore, $\mathcal{M}$ is G-equivariant. When $\mathcal{M} = \mathcal{O}_X$ and H is a compact open subgroup of G which preserves an affinoid subdomain U of X, then the left action of H on $\mathcal{O}_X(U)$ can be described as $(g.f)(x) = f(x.g)$ for $f \in \mathcal{O}_X$, $x \in U^{Tate}$ and $g \in H$.

   Let $\mathbb{G}^{(1)}$ be the rigid analytification of $\mathrm{Spec}(K \oplus \mathfrak{g}_K^*\epsilon)$, where $\epsilon^2 = 0$. This is the first infinitesimal neighborhood of the identity of $\mathbb{G}$. Restricting the isomorphism $a^*\mathcal{M} \simeq p_1^*\mathcal{M}$ to $\mathbb{G}^{(1)} \times X$ gives us an endomorphism of $\mathcal{M} \otimes_K (K \oplus \mathfrak{g}_K^*\epsilon)$, which gives rise to a morphism $\mathfrak{g}_K \to \mathrm{End}_K(\mathcal{M})$ that preserves Lie brackets. When $\mathcal{M} = \mathcal{O}_X$, we get the morphism between Lie algebras $\mathfrak{g}_K \to \mathcal{T}_X$, which is given by the familar formula: $\mathfrak{x}.f = \lim_{t\in L,|t|\to 0} \dfrac{\exp(t\mathfrak{x}).f - f}{t}$ for $\mathfrak{x} \in \mathfrak{g}$ and $f \in \mathcal{O}_X$. Therefore, we have the morphism between sheaves of algebras $\mathcal{U}(\mathfrak{g}_K) \to \mathcal{D}_X$.

4. $\mathcal{T}_X$ is $\mathbb{G}$-equivariant. If $g \in G$, the action is explicitly given by the formula $(g.\mathfrak{x})f = g(\mathfrak{x}(g^{-1}.f))$, where $\mathfrak{x} \in \mathcal{T}_X$, and $f \in \mathcal{O}_X$ are local sections.

Lemma 3.2.6. *Let X be a right $\mathbb{G}$-variety over K. If $\mathbb{H}$ is a sufficiently small good analytic subgroup of G obtained by exponentiating $\mathfrak{h}$, such that the action of H preserves an affinoid subdomain U of X, then the action of H on $\mathcal{O}_X(U)$ is jointly continuous.*

Proof. Equip $\mathcal{O}_X(U)$ with the spectral norm. By maximum principle H acts through isometry. Scale $\mathfrak{h}$ by a power of α we can assume that the operator norm of $\mathfrak{x} \in \mathfrak{h}$ on $\mathcal{O}_X(U)$ is less than 1. Therefore, if $f \in \mathcal{O}_X(U)$, since $\mathfrak{x}.f = \lim_{t\in L,|t|\to 0} \dfrac{\exp(t\mathfrak{x}).f - f}{t}$, we see that $||\exp(t\mathfrak{x}).f - f||_U \leq ||t\mathfrak{x}.f||_U \leq ||f||_U|t|$. In other words, the orbit map $o_f : H \to \mathcal{O}_X(U)$ is continuous. Therefore, the morphism $H \times \mathcal{O}_X(U) \to \mathcal{O}_X(U)$ is continuous by [14] 3.1.1.

□

Definition 3.2.7. Let X be a smooth right $\mathbb{G}$-variety. A twisted differential algebra $\mathcal{A}_{\mathcal{T}_X}$ associated to $\mathcal{T}_X$ is a $\mathbb{G}$-equivariant twisted differential algebra if $\mathcal{A}_{\mathcal{T}_X}$ is $\mathbb{G}$-equivariant as a $\mathcal{O}_X$-module and there is a morphism of Lie algebras $i_{\mathfrak{g}_K} : \mathfrak{g}_K \to \mathcal{A}_{\mathcal{T}_X}$, such that

1. The $\mathbb{G}$-equivariant structure is compactible with the ring structure of $\mathcal{A}$. i.e. $g \in \mathbb{G}(K)$ acts on $\mathcal{A}_{\mathcal{T}_X}$ as ring homomorphism.

2. $i_{\mathfrak{g}_K}$ is G-equivariant, if $\mathfrak{g}_K$ is equipped with the conjugation action of G.

3. The $\mathfrak{g}_K$-action on $\mathcal{A}_{\mathcal{T}_X}$ that comes from differentiating the $\mathbb{G}$-action (see Remark 3.2.5) is $\mathrm{ad}_{i_{\mathfrak{g}_K}} = [i_{\mathfrak{g}_K}(-), -]$.

Remark 3.2.8. The definition of equivariant twisted differential algebras is modeled based on [4] section 1.8. It implies that $i : \mathcal{O}_X \to \mathcal{A}_{\mathcal{T}_X}$ is G-equivariant.

Example 3.2.9. If X is a smooth right $\mathbb{G}$-variety, then $\mathcal{D}_X$ is a $\mathbb{G}$-equivariant twisted differential algebra. The $\mathbb{G}$-equivariant structure is determined by the equivariant structure on $\mathcal{O}_X$ and $\mathcal{T}_X$. The Lie algebra map $i_{\mathfrak{g}_K} : \mathfrak{g}_K \to \mathcal{D}_X$ is defined in Remark 3.2.5.

3.3 G-equivariant $\mathcal{D}^\infty$-modules on smooth rigid analytic varieties

In section 3.3, we assume that X is a smooth right $\mathbb{G}$-variety over K of dimension d, and $\mathcal{A}_{\mathcal{T}_X}$ is a $\mathbb{G}$-equivariant twisted sheaf of differential algebra associated to $\mathcal{T}_X$. We also assume that there exists a good analytic open subgroup $\mathbb{H}$ of G such and an open affinoid cover $\{U_i\}$ of X such that the action of $\mathbb{H}$ preserves each U_i. We shall define the abelian category of coadmissible G-equivariant $\mathcal{A}^\infty_{\mathcal{T}_X}$-modules.

3.3.1 Coadmissible equivariant $\mathcal{D}^\infty$-Modules on smooth rigid analytic affinoid varieties

The goal of this subsection is to define coadmissible H-equivariant $\mathcal{A}^\infty_{\mathcal{T}_X}|_U$-modules, where $U = \mathrm{Spa}(A, A^\circ)$ is a small affinoid subdomain of X, such that $\mathcal{T}_X|_U$ is trivial, and $\mathbb{H}$ is a good analytic open subgroup of G that acts on U.

Lemma 3.3.1. *Let $\mathfrak{U}$ be an admissible formal model of U obtained by an admissible formal bow-up of* $\mathrm{Spf}(A^\circ)$*. Then there exists a open compact subgroup H' of G such that H' acts on $\mathcal{O}_U(\mathbf{rig}(\mathfrak{V}))$ for $\mathfrak{V} \in \mathcal{B}_{\mathfrak{U}}$. In particular, we see that for any affinoid subdomain V of U, there exists a compact open subgroup of G that acts on $\mathcal{O}_U(V)$.*

Proof. Consider $\mathcal{A}_{\mathcal{T}_X} = \mathcal{D}_X$ and use the notations from section 2.3.1 and 3.2.1. There exists a good analytic open subgroup $\mathbb{H}'$ of H obtained by exponentiating $\mathfrak{h}$ and $N \in \mathbb{N}$ such that the image of $\mathfrak{h}$ under the morphism $\mathfrak{g}_K \to \mathcal{D}_X|_U$ (defined in Example 3.2.9) lies in $\mathcal{A}_{\mathfrak{X}_N,\mathfrak{U}}$. Therefore we have a morphism of sheaves of rings $\widehat{\mathcal{U}}(\mathfrak{h})_K \to \widehat{\mathcal{A}}_{\mathfrak{X}_N,\mathfrak{U},\mathbb{Q}}$. Since $K[H_1^\circ] \hookrightarrow \widehat{\mathcal{U}}(\mathfrak{h})_K$ via exponential map, we see that $H' := H_1^\circ$ acts on $\mathcal{O}_{\mathfrak{U}}(\mathfrak{V}) \otimes_{\mathcal{O}_K} K$.

□

Recall in section 2.3.1, we choose a set sections $\{\xi_1, \xi_2, ..., \xi_d\}$ of $\mathcal{T}_X|_U$ that trivialize $\mathcal{T}_X|_U$ as a free $\mathcal{O}_U$-module, and $\tilde{\xi_i}$ liftings of ξ_i in $F_1(\mathcal{A}_{\mathcal{T}_X}|_U)$. For $n >> 0$, replacing $\mathbb{H}$ by a smaller good analytic open subgroup if necessary, we can assume that

1. The image of $\alpha^n\mathfrak{h}$ under the morphism $\mathfrak{g}_K \xrightarrow{i_{\mathfrak{g}_K}} \mathcal{A}_{\mathcal{T}_X}$ lies in $\mathcal{A}_{\mathfrak{X}_n,\mathfrak{U}}$.
2. $H \cdot \alpha^n\tilde{\xi_i} \subseteq \Gamma(\mathfrak{U}, F_1(\mathcal{A}_{\mathfrak{X}_n,\mathfrak{U}}))$.
3. $H \cdot sp_*(\mathcal{O}_X)(\mathfrak{V}) \subseteq sp_*(\mathcal{O}_X)(\mathfrak{V})$ for $\mathfrak{V} \in \mathcal{B}_{\mathfrak{U}}$. (by Lemma 3.3.1)

Therefore we have a morphism of sheaves of rings $\mathcal{U}(\alpha^n\mathfrak{h}) \to \mathcal{A}_{\mathfrak{X}_n,\mathfrak{U}}$, which extends to a morphism of sheaves of algebras $\widehat{\mathcal{U}}(\alpha^n\mathfrak{h})_K \to \widehat{\mathcal{A}}_{\mathfrak{X}_n,\mathfrak{U},\mathbb{Q}}$. Taking inverse limit over n, we get a morphism between sheaves of Fréchet-Stein algebras $\overline{\mathcal{U}(\mathfrak{g}_K)} \to \mathcal{A}^\infty_{\mathcal{T}_X}|_U$ that extends the morphism $\mathcal{U}(\mathfrak{g}_K) \to \mathcal{A}_{\mathcal{T}_X}|_U$.

Definition-Lemma 3.3.2. *By Lemma 3.2.2 there is an algebra homomorphism $K[H^\circ_{n+1}] \to \widehat{\mathcal{U}}(\alpha^n\mathfrak{h})_K$, so we can define the sheaves of $\mathcal{O}_{\mathfrak{U}}$-modules $\mathcal{A}^{H^\circ}_{\mathfrak{X}_n,\mathfrak{U}} := \widehat{\mathcal{A}}_{\mathfrak{X}_n,\mathfrak{U},\mathbb{Q}} \otimes_{K[H^\circ_{n+1}]} K[H^\circ]$, and $\mathcal{A}^{H^\circ}_{\mathcal{T}_X,\mathfrak{U}} := \varprojlim_n \mathcal{A}^{H^\circ}_{\mathfrak{X}_n,\mathfrak{U}}$. Then*

1. *$\mathcal{A}^{H^\circ}_{\mathfrak{X}_n,\mathfrak{U}}$ is a left (resp. right) coherent sheaf of noetherian Banach algebras. i.e. $\mathcal{A}^{H^\circ}_{\mathfrak{X}_n,\mathfrak{U}}$ is a left (resp. right) coherent sheaf of rings, such that if $\mathfrak{V} \subseteq \mathfrak{U}$ is open affine, then $\mathcal{A}^{H^\circ}_{\mathfrak{X}_n,\mathfrak{U}}(\mathfrak{V})$ is a left (resp. right) noetherian Banach algebra.*

2. *$\mathcal{A}^{H^\circ}_{\mathcal{T}_X,\mathfrak{U}}$ is a sheaf of Fréchet-Stein algebras on $\mathcal{B}_{\mathfrak{U}}$.*

3. *The isomorphism $\Gamma(\mathfrak{U}, \mathcal{A}^{H^\circ}_{\mathcal{T}_X,\mathfrak{U}}) \simeq \varprojlim_n \Gamma(\mathfrak{U}, \mathcal{A}^{H^\circ}_{\mathfrak{X}_n,\mathfrak{U}})$ realizes $\Gamma(\mathfrak{U}, \mathcal{A}^{H^\circ}_{\mathcal{T}_X,\mathfrak{U}})$ as a Féchet-Stein algebra.*

Proof. Let $\mathfrak{V} \in \mathcal{B}_{\mathfrak{U}}$. By the above discussion, we see that H acts on $\Gamma(\mathfrak{U}, F_1(\mathcal{A}_{\mathfrak{X}_n,\mathfrak{U}}))$ and on $F_1(\mathcal{A}_{\mathfrak{X}_n,\mathfrak{U}})(\mathfrak{V})$. Hence H acts on $\mathcal{A}_{\mathfrak{X}_n,\mathfrak{U}}(\mathfrak{V})$ and on $\widehat{\mathcal{A}}_{\mathfrak{X}_n,\mathfrak{U}}(\mathfrak{V})$. It follows that $\widehat{\mathcal{A}}_{\mathfrak{X}_n,\mathfrak{U},\mathbb{Q}}(\mathfrak{V}) \otimes_{K[H^\circ_{n+1}]} K[H^\circ]$ has an algebra structure. Since the morphisms of sheaves of algebras

$$K[H^\circ_{n+1}] \to \widehat{\mathcal{U}}(\alpha^n\mathfrak{h})_K \to \widehat{\mathcal{A}}_{\mathfrak{X}_n,\mathfrak{U},\mathbb{Q}}(\mathfrak{V})$$

are H-equivariant, we get a continuous morphism between algebras

$$\widehat{\mathcal{U}}(\alpha^n\mathfrak{h})_K \otimes_{K[H^\circ_{n+1}]} K[H^\circ] \to \widehat{\mathcal{A}}_{\mathfrak{X}_n,\mathfrak{U},\mathbb{Q}}(\mathfrak{V}) \otimes_{K[H^\circ_{n+1}]} K[H^\circ].$$

1. By Lemma 2.3.1, we see that $\widehat{\mathcal{A}}_{\mathfrak{X}_n,\mathfrak{U},\mathbb{Q}}$ is a coherent sheave of noetherian Banach subalgebras of finite index in $\mathcal{A}^{H^\circ}_{\mathfrak{X}_n,\mathfrak{U}}$. It follows that $\mathcal{A}^{H^\circ}_{\mathfrak{X}_n,\mathfrak{U}}$ is also a coherent sheave of noetherian Banach algebras.

2. The flatness of the transition morphism $\mathcal{A}^{H^\circ}_{\mathfrak{X}_{n+1},\mathfrak{U}} \to \mathcal{A}^{H^\circ}_{\mathfrak{X}_n,\mathfrak{U}}$ can be shown by the same arguments in Lemma 3.2.2.

3. It follows from Lemma 2.3.1.

∎

Lemma 3.3.3. *If $\mathfrak{V} \in \mathcal{B}_{\mathfrak{U}}$, or $\mathfrak{V} = \mathfrak{U}$, there is an isomorphism*

$$\mathcal{A}^{H^\circ}_{\mathcal{T}_X,\mathfrak{U}}(\mathfrak{V}) \simeq \mathcal{A}^{\infty}_{\mathcal{T}_X}(\mathbf{rig}(\mathfrak{V}))\widehat{\otimes}_{\overline{\mathcal{U}(\mathfrak{g}_K)}} D^{la}(H^\circ, K)$$

of $(\mathcal{A}^{\infty}_{\mathcal{T}_X}(\mathbf{rig}(\mathfrak{V})), D^{la}(H^\circ, K))$*-bimodules.*

Proof. The conclusion follows from the following isomorphisms:

$$\mathcal{A}^{\infty}_{\mathcal{T}_X}(\mathbf{rig}(\mathfrak{V}))\widehat{\otimes}_{\overline{\mathcal{U}(\mathfrak{g})}} D^{la}(H^\circ, K)$$

$$\begin{aligned}
&\xrightarrow{\simeq} \varprojlim_n \widehat{\mathcal{A}}_{\mathfrak{X}_n,\mathfrak{U},\mathbb{Q}}(\mathfrak{V}) \otimes_{\widehat{\mathcal{U}}(\alpha^n\mathfrak{h})_K} (\widehat{\mathcal{U}}(\alpha^n\mathfrak{h})_K \otimes_{K[H^\circ_{n+1}]} K[H^\circ]) \\
&\xrightarrow{\simeq} \varprojlim_n \widehat{\mathcal{A}}_{\mathfrak{X}_n,\mathfrak{U},\mathbb{Q}}(\mathfrak{V}) \otimes_{K[H^\circ_{n+1}]} K[H^\circ] \\
&\xrightarrow{\simeq} \mathcal{A}^{H^\circ}_{\mathcal{T}_X,\mathfrak{U}}(\mathfrak{V})
\end{aligned}$$

□

Lemma 3.3.4. *A coherent* $\mathcal{A}^{H^\circ}_{\mathfrak{X}_n,\mathfrak{U}}$*-module has trivial higher cohomology groups and is generated by its global sections.*

Proof. A coherent $\mathcal{A}^{H^\circ}_{\mathfrak{X}_n,\mathfrak{U}}$-module is coherent over $\widehat{\mathcal{A}}_{\mathfrak{X}_n,\mathfrak{U},\mathbb{Q}}$. The result follows from Proposition 2.3.9.

□

If M is a coadmissible $\Gamma(\mathfrak{U}, \mathcal{A}^{H^\circ}_{\mathcal{T}_X,\mathfrak{U}})$-module, similarly as in Definition-Lemma 2.3.10, we could associate to M a sheaf $M_{\mathfrak{U}}$ of $\mathcal{A}^{H^\circ}_{\mathcal{T}_X,\mathfrak{U}}$-module on $\mathfrak{U}$ in the following way:

$$M_{\mathfrak{U}} := \varprojlim_n \mathcal{A}^{H^\circ}_{\mathfrak{X}_n,\mathfrak{U}} \otimes_{\Gamma(\mathfrak{U}, \mathcal{A}^{H^\circ}_{\mathcal{T}_X,\mathfrak{U}})} M,$$

such that if $\mathfrak{V} \in \mathcal{B}_{\mathfrak{U}}$, then we have the isomorphisms

$$\begin{aligned}
M_{\mathfrak{U}}(\mathfrak{V}) &\xrightarrow{\simeq} \varprojlim_n \mathcal{A}^{H^\circ}_{\mathfrak{X}_n,\mathfrak{U}}(\mathfrak{V}) \otimes_{\Gamma(\mathfrak{U}, \mathcal{A}^{H^\circ}_{\mathfrak{X}_n,\mathfrak{U}})} M_n \\
&\xleftarrow{\simeq} \mathcal{A}^{H^\circ}_{\mathcal{T}_X,\mathfrak{U}}(\mathfrak{V})\widehat{\otimes}_{\Gamma(\mathfrak{U}, \mathcal{A}^{H^\circ}_{\mathcal{T}_X,\mathfrak{U}})} M,
\end{aligned}$$

where $M_n := \Gamma(\mathfrak{U}, \mathcal{A}^{H^\circ}_{\mathfrak{X}_n,\mathfrak{U}}) \otimes_{\Gamma(\mathfrak{U}, \mathcal{A}^{H^\circ}_{\mathcal{T}_X,\mathfrak{U}})} M$ is a finite generated module over $\Gamma(\mathfrak{U}, \mathcal{A}^{H^\circ}_{\mathfrak{X}_n,\mathfrak{U}})$.

Lemma 3.3.5. *Let $\mathbb{H}'$ be a good analytic subgroup of G, such that $\mathbb{H}'$ is a normal subgroup of $\mathbb{H}$.*

1. *The natural homomorphism $\mathcal{A}^{H'^\circ}_{\mathcal{T}_X,\mathfrak{U}} \to \mathcal{A}^{H^\circ}_{\mathcal{T}_X,\mathfrak{U}}$ between sheaves of Fréchet-Stein algebras realizes $\mathcal{A}^{H^\circ}_{\mathcal{T}_X,\mathfrak{U}}$ as a coadmissible module over $\mathcal{A}^{H'^\circ}_{\mathcal{T}_X,\mathfrak{U}}$.*

2. *Let M be a coadmissible module over $\Gamma(\mathfrak{U}, \mathcal{A}^{H^\circ}_{\mathcal{T}_X,\mathfrak{U}})$, then by [28] Lemma 3.8, M is also a coadmissible module over $\Gamma(\mathfrak{U}, \mathcal{A}^{H'^\circ}_{\mathcal{T}_X,\mathfrak{U}})$. We have the following isomorphism of sheaves of coadmissible modules over $\mathcal{A}^{H'^\circ}_{\mathcal{T}_X,\mathfrak{U}}$:*

$$\mathcal{A}^{H'^\circ}_{\mathcal{T}_X,\mathfrak{U}} \widehat{\otimes}_{\Gamma(\mathfrak{U}, \mathcal{A}^{H'^\circ}_{\mathcal{T}_X,\mathfrak{U}})} M \xrightarrow{\simeq} \mathcal{A}^{H^\circ}_{\mathcal{T}_X,\mathfrak{U}} \widehat{\otimes}_{\Gamma(\mathfrak{U}, \mathcal{A}^{H^\circ}_{\mathcal{T}_X,\mathfrak{U}})} M$$

   In particular, the definition of $M_{\mathfrak{U}}$ is independent of the choices of $\mathbb{H}$.

Proof.

1. If g is a representative of $\bar{g} \in H^\circ / H'^\circ$ in H°, then

$$D^{la}(H^\circ, K) \simeq \bigoplus_{\bar{g} \in H/H'} D^{la}(H'^\circ g, K) \simeq D^{la}(H'^\circ, K) \otimes_{K[H']^\circ} K[H^\circ].$$

   It follows that $\mathcal{A}^{H^\circ}_{\mathcal{T}_X,\mathfrak{U}} \simeq \mathcal{A}^{H'^\circ}_{\mathcal{T}_X,\mathfrak{U}} \otimes_{K[H'^\circ]} K[H^\circ]$ as a finite $\mathcal{A}^{H'^\circ}_{\mathcal{T}_X,\mathfrak{U}}$-module.

2. If $\mathfrak{V} \in \mathcal{B}_{\mathfrak{U}}$, then $\mathcal{A}^{H^\circ}_{\mathcal{T}_X,\mathfrak{U}}(\mathfrak{V}) \widehat{\otimes}_{\Gamma(\mathfrak{U}, \mathcal{A}^{H^\circ}_{\mathcal{T}_X,\mathfrak{U}})} M$

$$\begin{aligned}
&\xrightarrow{\simeq} \varprojlim_n (\widehat{\mathcal{A}}_{\mathfrak{T}_n,\mathfrak{U},\mathbb{Q}}(\mathfrak{V}) \otimes_{K[H^\circ_{n+1}]} K[H^\circ]) \otimes_{(\Gamma(\mathfrak{U}, \widehat{\mathcal{A}}_{\mathfrak{T}_n,\mathfrak{U},\mathbb{Q}}) \otimes_{K[H^\circ_{n+1}]} K[H^\circ])} M_n \\
&\simeq \varprojlim_n \widehat{\mathcal{A}}_{\mathfrak{T}_n,\mathfrak{U},\mathbb{Q}}(\mathfrak{V}) \otimes_{\Gamma(\mathfrak{U}, \widehat{\mathcal{A}}_{\mathfrak{T}_n,\mathfrak{U},\mathbb{Q}})} M_n \\
&\simeq \mathcal{A}^{\infty}_{\mathcal{T}_X}(\mathbf{rig}(\mathfrak{V})) \widehat{\otimes}_{\mathcal{A}^{\infty}_{\mathcal{T}_X}(U)} M
\end{aligned}$$

□

Definition-Lemma 3.3.6. *If $V \in \mathcal{B}_U$, by Lemma 3.3.3 and Lemma 3.3.1 there exists a good analytic open subgroup $\mathbb{H}$ such that $\mathcal{A}^{\infty}_{\mathcal{T}_X}(V)\widehat{\otimes}_{\overline{\mathcal{U}(\mathfrak{g}_K)}}D^{la}(H^\circ, K)$ has a Fréchet-Stein algebra structure. If M a coadmissible module over $\mathcal{A}^{\infty}_{\mathcal{T}_X}(U)\widehat{\otimes}_{\overline{\mathcal{U}(\mathfrak{g}_K)}}D^{la}(H^\circ, K)$, then*

$$\widetilde{M}(V) := (\mathcal{A}^{\infty}_{\mathcal{T}_X}(V)\widehat{\otimes}_{\overline{\mathcal{U}(\mathfrak{g}_K)}}D^{la}(H^\circ, K))\widehat{\otimes}_{(\mathcal{A}^{\infty}_{\mathcal{T}_X}(U)\widehat{\otimes}_{\overline{\mathcal{U}(\mathfrak{g}_K)}}D^{la}(H^\circ,K))}M$$

is a coadmissible module over $\mathcal{A}^{\infty}_{\mathcal{T}_X}(V)\widehat{\otimes}_{\overline{\mathcal{U}(\mathfrak{g}_K)}}D^{la}(H^\circ, K)$. Lemma 3.3.5 shows that

$$\widetilde{M}(V) \simeq \mathcal{A}^{\infty}_{\mathcal{T}_X}(V)\widehat{\otimes}_{\mathcal{A}^{\infty}_X(U)}M$$

and therefore $\widetilde{M}$ is a presheaf. In fact $\widetilde{M}$ is a sheaf on U, which will be called the sheaf of coadmissible equivariant $\mathcal{A}^{\infty}_{\mathcal{T}_X}|_U$-module associated to M.

Proof. Let $\mathfrak{U}$ be an admissible formal model of U obtained by an admissible formal blow-up of Spf (A°). There is an isomorphism $M_{\mathfrak{U}} \to sp_*(\widetilde{M})$ of $\mathcal{A}^{\infty}_{\mathcal{T}_X,\mathfrak{U}}$-modules. By Lemma 2.2.11, we see that $\widetilde{M}$ is a sheaf.

□

3.3.2 Coadmissible Equivariant $\mathcal{D}^{\infty}$-Modules on smooth rigid analytic varieties over K

Definition 3.3.7. A sheaf of coadmissible G-equivariant $\mathcal{A}^{\infty}_{\mathcal{T}_X}$-module $\mathcal{M}$ is a sheaf of $\mathcal{A}^{\infty}_{\mathcal{T}_X}$-module that satisfies the following conditions:

1. $\mathcal{M}$ is a G-equivariant $\mathcal{O}_X$-module, whose $\mathcal{O}_X$-module structure is compatible with the inclusion $\mathcal{O}_X \to \mathcal{A}^{\infty}_{\mathcal{T}_X}$.
2. There exists an open cover of affinoid subdomains $\{U_i\}_{i\in I}$ such that $\mathcal{T}_X|_{U_i}$ is free, and good analytic open subgroup $\mathbb{H}_i$ of G such that $\mathcal{M}|_{U_i}$ is isomorphic to $\widetilde{M_i}$ (see Definition-Lemma 3.3.6), where M_i is a coadmissible $\mathcal{A}^{\infty}_{\mathcal{T}_X}(U_i)\widehat{\otimes}_{\overline{\mathcal{U}(\mathfrak{g}_K)}}D^{la}(H_i^\circ, K)$-module.

3. The $K[H_i^\circ]$-module structure on $\mathcal{M}(U_i)$ is compatible with the $\mathcal{A}_{\mathcal{T}_X}^\infty(U_i)\widehat{\otimes}_{\overline{\mathcal{U}(\mathfrak{g}_K)}}D^{la}(H_i^\circ,K)$-module structure on M_i.

Lemma 3.3.8. *Let $U \subseteq X$ be an open affinoid with $\mathcal{T}_X|_U$ trivial, and $\mathcal{M}$ is a coadmissible G-equivariant $\mathcal{A}_{\mathcal{T}_X}^\infty$-module on X. Then*

1. *$\mathcal{M}|_U$ is isomorphic to $\widetilde{M}$ (see Definition-Lemma 3.3.6), where M is a coadmissible $\mathcal{A}_{\mathcal{T}_X}^\infty(U)\widehat{\otimes}_{\overline{\mathcal{U}(\mathfrak{g}_K)}}D^{la}(H^\circ,K)$-module, for a sufficiently small good analytic open subgroup $\mathbb{H}$ of G. In particular, we see that the definition of coadmissible G-equivariant modules is independent of the choice of affinoid open covers of X.*

2. *$\mathcal{M}|_U$ has trivial higher cohomology.*

Proof. Let $\{U_i\}_{i\in I}$ be an open affinoid cover of X such that $\mathcal{T}_X|_{U_i}$ is free, and good analytic open subgroups $\mathbb{H}_i$ of G such that $\mathcal{M}|_{U_i\cap U}$ is isomorphic to $\widetilde{M_i}$, where M_i is a coadmissible $\mathcal{A}_{\mathcal{T}_X}^\infty(U_i\cap U)\widehat{\otimes}_{\overline{\mathcal{U}(\mathfrak{g}_K)}}D^{la}(H_i^\circ,K)$-module. Since U is quasi-compact, we can assume that I is finite. By Proposition 2.3.9, let $\mathfrak{U}$ be an admissible formal model of U obtained by an admissible formal blow-up of $\mathrm{Spf}(\mathcal{O}_X^\circ(U))$ with associated specialization map $sp : U \to \mathfrak{U}$, such that $\{\mathfrak{U}_i\}$ is an open cover of $\mathfrak{U}$ with $sp^{-1}(\mathfrak{U}_i) = U_i \cap U$. By passing to a finite refinement, we can assume that $\mathfrak{U}_i$ is affine. Then there exists a good analytic open subgroup $\mathbb{H}$ such that $sp_*(\mathcal{M}|_U)|_{\mathfrak{U}_i}$ is associated to a coadmissible module over $\mathcal{A}_{\mathcal{T}_X,\mathfrak{U}}^{H^\circ}(\mathfrak{U}_i)$. By Lemma 3.3.4 and Lemma 2.2.12, we see that there is a coadmissible module M over $\mathcal{A}_{\mathcal{T}_X}^\infty(U)\widehat{\otimes}_{\overline{\mathcal{U}(\mathfrak{g}_K)}}D^{la}(H^\circ,K)$ such that $M_{\mathfrak{U}} \xrightarrow{\simeq} sp_*(\mathcal{M}|_U)$. It follows that $\widetilde{M} \to \mathcal{M}|_U$ is an isomorphism. Since $H^i(\mathfrak{U}, sp_*(\mathcal{M}|_U)) = 0$ for $i \geq 0$, we conclude that $H^i(U,\mathcal{M}|_U) = 0$ for $i \geq 0$.

□

If $\mathcal{M}$ and $\mathcal{N}$ are sheaves of coadmissible G-equivariant $\mathcal{A}_{\mathcal{T}_X}^\infty$-modules, define $\mathrm{Hom}^G_{\mathcal{A}_{\mathcal{T}_X}^\infty}(\mathcal{M},\mathcal{N})$ to be the set of morphisms of sheaves of $\mathcal{A}_{\mathcal{T}_X}^\infty$-modules that are also G-equivariant. Let $\mathrm{Mod}_G^{\mathrm{coad}}(\mathcal{A}_X^\infty)$ be the category of coadmissible G-equivariant $\mathcal{A}_{\mathcal{T}_X}^\infty$-modules.

Proposition 3.3.9. *$Mod_G^{coad}(\mathcal{A}_X^{\infty})$ is an abelian category.*

Proof. This follows from Lemma 3.3.7 and [28] Proposition 2.1.

□

3.4 Beilinson-Bernstein localization of admissible locally analytic representations of G

In this section we assume that $\mathbf{G}$ is a connected and split reductive linear algebraic group over L. Fix a Borel subgroup $\overline{\mathbf{B}}$ of $\mathbf{G}$ with the unipotent radical $\overline{\mathbf{N}}$. We will use the letter $\mathbf{T}$ to denote the universal Cartan $\overline{\mathbf{B}}/\overline{\mathbf{N}}$ and implicitly use the canonical isomorphism between $\mathbf{T}$ and a maximal torus of $\mathbf{G}$. Let $\mathbf{W}$ be the Weyl group of the root system of $\mathbf{G}$ relative to $\overline{\mathbf{B}}$, and let ρ be the half sum of positive roots. Let $\mathbf{B}$ be the Borel subgroup opposite to $\overline{\mathbf{B}}$. Let $\mathfrak{b}$, $\mathfrak{n}$, $\overline{\mathfrak{n}}$ and $\mathfrak{t}$ be the Lie algebra of $\mathbf{B}$, $\mathbf{N}$, $\overline{\mathbf{N}}$ and $\mathbf{T}$ over L respectively. Let $\mathbf{X} := \mathbf{G}/\mathbf{B}$ be the flag variety, and let $\mathbf{Y} := \mathbf{G}/\mathbf{N}$ be the base affine. The right $\mathbf{G}$-action on $\mathbf{X}$ (resp. $\mathbf{Y}$) is given by $(g'\mathbf{B}).g := g^{-1}g'\mathbf{B}$ (resp. $(g'\mathbf{N}).g := g^{-1}g'\mathbf{N}$) for $g \in \mathbf{G}$ and $g'\mathbf{B} \in \mathbf{X}$ (resp. $g \in \mathbf{G}$ and $g'\mathbf{N} \in \mathbf{Y}$). The $\mathbf{T}$-action on $\mathbf{Y}$ is given by $b\mathbf{N}.g\mathbf{N} := gb\mathbf{N}$ for $b\mathbf{N} \in \mathbf{T}$ and $g\mathbf{N} \in \mathbf{Y}$, and the $\mathbf{T}$ action commuters with the $\mathbf{G}$-action. We know that the natural projection $\xi : \mathbf{Y} \to \mathbf{X}$ is a $\mathbf{T}$-torsor. Let X, Y, $\mathbb{G}$, $\mathbb{T}$ be the rigid analytification of $\mathbf{X}_K$, $\mathbf{Y}_K$, $\mathbf{G}_K$ and $\mathbf{T}_K$ respectively.

We also assume that there exists a connected and split reductive linear algebraic group $\mathfrak{G}$ over $\mathcal{O}_K$ with a Borel subgroup $\mathfrak{B}$ whose unipotent radical is $\mathfrak{N}$ and the universal Cartan $\mathfrak{H}$, such that $\mathfrak{G} \otimes_{\mathcal{O}_K} K \simeq \mathbf{G}_K$, $\mathfrak{B} \otimes_{\mathcal{O}_K} K \simeq \mathbf{B}_K$, $\mathfrak{N} \otimes_{\mathcal{O}_K} K \simeq \mathbf{N}_K$ and $\mathfrak{H} \otimes_{\mathcal{O}_K} K \simeq \mathbf{T}_K$. Let $\mathfrak{g}^\circ$, $\mathfrak{b}^\circ$, $\mathfrak{t}^\circ$ be the Lie algebra of $\mathfrak{G}$, $\mathfrak{B}$, $\mathfrak{H}$ respectively. Then $\mathfrak{g}^\circ$ can be viewed as a $\mathcal{O}_K$-Lie sublattice of $\mathfrak{g}_K$, and $\mathfrak{G}/\mathfrak{B}$ is a smooth projective scheme over $\mathcal{O}_K$ such that $(\mathfrak{G}/\mathfrak{B}) \otimes_{\mathcal{O}_K} K \simeq \mathbf{X}_K$. Let $\mathfrak{X}$ be the formal completion of $\mathfrak{G}/\mathfrak{B}$ along the special fiber, let $\mathfrak{Y}$ be the formal completion of $\mathfrak{G}/\mathfrak{N}$ along the special fiber , and let $\hat{\mathfrak{H}}$ be the formal completion of $\mathfrak{H}$ along the special fiber. Thus $\mathfrak{X}$ (resp. $\mathfrak{Y}$ or $\hat{\mathfrak{H}}$) can be viewed as an admissible formal model of X (resp. Y or $\mathbb{T}$). Let $sp : X \to \mathfrak{X}$ be the specialization map. The Bruhat decomposition implies that there exists a finite open affine cover $\{\mathfrak{U}_i\}$ that trivializes the torsor $\hat{\xi}$, such that $\mathfrak{U}_i \simeq \mathrm{Spf}(\mathcal{O}_K\langle x_1, x_2, ..., x_m\rangle)$.

Let us recall the construction of the construction of $\mathbf{G}_K$-equivariant twisted differential algebras on $\mathbf{X}_K$ [4] section 3.2, 2.5 and 1.8. The Poincaré-Birkhoff-Witt theorem gives a vector space decomposition:

$$\mathcal{U}(\mathfrak{g}_K) \simeq \mathcal{U}(\mathfrak{t}_K) \oplus (\overline{\mathfrak{n}}_K \mathcal{U}(\mathfrak{g}_K) + \mathcal{U}(\mathfrak{g}_K)\mathfrak{n}_K).$$

Let $\mathcal{Z}(\mathfrak{g}_K)$ be the center of $\mathcal{U}(\mathfrak{g}_K)$. The composition of the natural inclusion $\mathcal{Z}(\mathfrak{g}_K) \to \mathcal{U}(\mathfrak{g}_K)$ with the projection $\mathcal{U}(\mathfrak{g}_K) \to \mathcal{U}(\mathfrak{t}_K)$ is an injective algebra homomorphism, which is called the Harish-Chandra homomorphism. We have the following commutative diagram by [1] 4.10:

$$\begin{array}{ccc}
\mathcal{Z}(\mathfrak{g}_K) & \xrightarrow{\text{Harish-Chandra}} & \mathcal{U}(\mathfrak{t}_K) \\
\Big\downarrow \text{inclusion} & & \Big\downarrow \text{differentiate the } \mathbf{T}\text{-action on } \mathbf{Y} \\
\mathcal{U}(\mathfrak{g}_K) & \xrightarrow[\text{differentiate the right } \mathbf{G}\text{-action on } \mathbf{Y}]{} & ((\xi_*\mathcal{D}_{\mathbf{Y}})^{\mathbf{T}}) \otimes_L K
\end{array}$$

Since $\mathcal{U}(\mathfrak{t}_K)$ can be naturally identified with the ring of polynomial functions on $\mathfrak{t}_K^*$, the translation $\mathfrak{t}_K^* \to \mathfrak{t}_K^* : x \to x - \rho$ gives rise to an automorphism of $\mathcal{U}(\mathfrak{t}_K)$. Composing the Harish-Chandra homomorphism with this automorphism of $\mathcal{U}(\mathfrak{t}_K)$, we get the Harish-Chandra isomorphism $\mathcal{Z}(\mathfrak{g}_K) \xrightarrow{\simeq} \mathcal{U}(\mathfrak{t}_K)^{\mathbf{W}}$, and $\mathcal{U}(\mathfrak{t}_K)^{\mathbf{W}}$ is isomorphic to the polynomial ring over K. For $\lambda \in \mathfrak{t}_K^*$, let K_λ be the corresponding 1-dimensional representation of $\mathcal{U}(\mathfrak{t}_K)$, which gives rise to a homomorphism $\chi_\lambda : \mathcal{Z}(\mathfrak{g}_K) \to K$. If $\beta \in \mathfrak{t}_K^*$, then $\chi_\lambda = \chi_\beta$ if and only if $\lambda - \rho$ and $\beta - \rho$ are in the same $\mathbf{W}$-orbit. Let K_θ be the character of $\mathcal{Z}(\mathfrak{g}_K)$ associated to the $K_{\lambda-\rho}$. Define $\mathcal{D}_{\mathbf{X},\lambda} := (\xi_*\mathcal{D}_{\mathbf{Y}})^{\mathbf{T}} \otimes_{\mathcal{U}(\mathfrak{t}_K)} K_{\lambda-\rho}$, and $\mathcal{U}(\mathfrak{g}_K)_\theta := \mathcal{U}(\mathfrak{g}_K) \otimes_{\mathcal{Z}(\mathfrak{g}_K)} K_\theta$. We know by [4] 3.2 that $\mathcal{D}_{\mathbf{X},\lambda}$ is a $\mathbf{G}$-equivariant twisted differential algebra.

Remark 3.4.1. If we further assume that $\mathfrak{G}$ is semi-simple, simply connected and p is a very good prime for $\mathfrak{G}$ as in [1] 6.8, or $\mathfrak{G}$ is GL_n and $n \neq pm$ for $m \in \mathbb{N}$, then:

1. Let $\mathcal{Z}(\mathfrak{g}^\circ)$ be the center of $\mathcal{U}(\mathfrak{g}^\circ)$. By [1] 6.9 we see that $\mathcal{Z}(\mathfrak{g}^\circ)$ is isomorphic to a polynomial ring over $\mathcal{O}_K$. Let $F.\mathcal{Z}(\mathfrak{g}^\circ)$ be the filtration on $\mathcal{Z}(\mathfrak{g}^\circ)$ induced from the PBW filtration on $\mathcal{U}(\mathfrak{g}^\circ)$. Let $\mathcal{Z}(\mathfrak{g}^\circ)_n := \sum \alpha^{in} F_i \mathcal{Z}(\mathfrak{g}^\circ)$, let $\widehat{\mathcal{Z}(\mathfrak{g}^\circ)_n}$ be the p-adic completion of $\mathcal{Z}(\mathfrak{g}^\circ)_n$, and let $(\widehat{\mathcal{Z}(\mathfrak{g}^\circ)_n})_K := \widehat{\mathcal{Z}(\mathfrak{g}^\circ)_n} \otimes_{\mathcal{O}_K} K$. There exists $N \in \mathbb{N}$ such that $\alpha^N(\lambda - \rho)$ maps $\mathfrak{t}^\circ$ to $\mathcal{O}_K$. Let $(\lambda - \rho)_n$ be $\alpha^{Nn}(\lambda - \rho)$, which can be viewed as an element of $\mathrm{Hom}_{\mathcal{O}_K}(\alpha^n \mathfrak{t}^\circ, \mathcal{O}_K)$, and $(\lambda - \rho)_n \otimes_{\mathcal{O}_K} K = \lambda - \rho$. Thus $(\lambda - \rho)_n$ determines a 1-dimensional representation of $\widehat{\mathcal{U}(\alpha^n \mathfrak{t}^\circ)}_K$, which will also be written as $K_{\lambda - \rho}$. If $\overline{\mathcal{Z}(\mathfrak{g}_K)}$ is the center of $\overline{\mathcal{U}(\mathfrak{g}_K)}$, then the isomorphism $\varprojlim_n (\widehat{\mathcal{Z}(\mathfrak{g}^\circ)_n})_K \simeq \overline{\mathcal{Z}(\mathfrak{g}_K)}$ realizes $\overline{\mathcal{Z}(\mathfrak{g}_K)}$ as a Fréchet-Stein algebra. By [1] 6.10, we see that there is a ring homomorphism $\mathcal{Z}(\mathfrak{g}^\circ)_n \to \mathcal{U}(\alpha^n \mathfrak{t}^\circ)$, which agrees with the Harish-Chandra homomorphism after tensoring K over $\mathcal{O}_K$. Thus we get a morphism between Fréchet-Stein algebras $\varprojlim_n (\widehat{\mathcal{Z}(\mathfrak{g}^\circ)_n})_K \to \varprojlim_n \widehat{\mathcal{U}(\alpha^n \mathfrak{t}^\circ)}_K \simeq \overline{\mathcal{U}(\mathfrak{t}_K)}$. We will also use K_θ to represent the 1-dimensional representation of $(\widehat{\mathcal{Z}(\mathfrak{g}^\circ)_n})_K$ determined by $(\lambda - \rho)_n$.

2. By [19] Thereom 2.1.6, the Harish-Chandra isomorphism extends to an isomorphism of Fréchet-Stein algebras $\overline{\mathcal{Z}(\mathfrak{g}_K)} \xrightarrow{\simeq} \overline{\mathcal{U}(\mathfrak{t}_K)}^{\mathbf{W}} \simeq$ rigid analytic functions on $\mathbb{A}^d$. Also [19] tells us that the center of $D^{la}(G, K)$ is isomorphic to $D^{la}(Z, K) \widehat{\otimes}_{\overline{\mathcal{U}(\mathfrak{z}_K)}} \overline{\mathcal{Z}(\mathfrak{g}_K)}$, where Z is center of G and $\mathfrak{z}$ is the Lie algebra of Z.

Definition-Lemma 3.4.2. *Let $\{\mathbf{U}_i\}$ be a finite open affine cover of $\mathbf{X}$ that $\xi|_{\mathbf{U}_i}$ is trivial, and U_i be the rigid analytification of $\mathbf{U}_i$. If $U \subseteq U_i$ is an open affinoid, define $\mathcal{D}_{X,\lambda}(U) := \mathcal{O}_X(U) \otimes_{\mathcal{O}_X(U_i)} \mathcal{D}_{\mathbf{X},\lambda}(\mathbf{U}_i)$.*

1. *$\mathcal{D}_{X,\lambda}$ is a sheaf of $\mathbb{G}$-equivariant twisted differential algebra associated to $\mathcal{T}_X$.*

2. *Let $\mathcal{D}^\infty_{X,\lambda}$ be the Fréchet completion of $\mathcal{D}_{X,\lambda}$ as defined in section 2.4.1. Then $\mathcal{D}^\infty_{X,\lambda}|_{U_i} \simeq \mathcal{D}^\infty_X|_{U_i}$.*

3. *Assume the assumptions in Remark 3.4.1 and define $\overline{\mathcal{U}(\mathfrak{g}_K)}_\theta := \overline{\mathcal{U}(\mathfrak{g}_K)} \widehat{\otimes}_{\overline{\mathcal{Z}(\mathfrak{g}_K)}} K_\theta$. There is a morphism between sheaves of Fréchet-Stein algebras $\overline{\mathcal{U}(\mathfrak{g}_K)}_\theta \to \mathcal{D}^\infty_{X,\lambda}$ that extends the morphism between algebras $\mathcal{U}(\mathfrak{g}_K)_\theta \to \mathcal{D}_{\mathbf{X},\lambda}$.*

Proof.

1. The $\mathbb{G}$-equivariant structure on $\mathcal{D}_{X,\lambda}$ comes from the $\mathbf{G}$-equivariant structure on $((\xi_*\mathcal{D}_{\mathbf{Y}})^{\mathbf{T}}) \otimes_L K$, and the Lie algebra morphism $\mathfrak{g}_K \to \mathcal{D}_{X,\lambda}$ comes from differentiating the right $\mathbf{G}$-action on $\mathbf{Y}$, i.e. the Lie algebra morphism $\mathfrak{g}_K \to ((\xi_*\mathcal{D}_{\mathbf{Y}})^{\mathbf{T}}) \otimes_L K$. It follows that $\mathcal{D}_{X,\lambda}$ is a sheaf of $\mathbb{G}$-equivariant twisted differential algebra associated to $\mathcal{T}_X$.

2. It follows from the isomorphism $\mathcal{D}_{\mathbf{X},\lambda}|_{\mathbf{U}_i} \simeq \mathcal{D}_{\mathbf{X}}|_{\mathbf{U}_i}$.

3. Define $\widehat{\mathcal{U}}(\alpha^n\mathfrak{g}^\circ)_{K,\theta} := \widehat{\mathcal{U}}(\alpha^n\mathfrak{g}^\circ)_K \otimes_{(\widehat{\mathcal{Z}(\mathfrak{g}^\circ)_n})_K} K_\theta$. Then $\widehat{\mathcal{U}}(\alpha^n\mathfrak{g}^\circ)_{K,\theta}$ is isomorphic to

$$\widehat{\mathcal{U}}(\alpha^n\mathfrak{g}^\circ)_K/\{\text{the ideal generated by } z-\theta(z) \text{ for } z \in \mathcal{Z}(\mathfrak{g}_K)\}$$

as topological algebras. Thus the isomorphism $\varprojlim_n \widehat{\mathcal{U}}(\alpha^n\mathfrak{g}^\circ)_{K,\theta} \simeq \overline{\mathcal{U}(\mathfrak{g}_K)}_\theta$ realizes $\overline{\mathcal{U}(\mathfrak{g}_K)}_\theta$ as a Fréchet-Stein algebra. The existence of the morphism $\overline{\mathcal{U}(\mathfrak{g}_K)}_\theta \to \mathcal{D}^\infty_{X,\lambda}$ follows from [1] 4.10.

□

Let us briefly recall the definition of the sheaf of completed twisted differential operators $\widehat{\mathcal{D}}^{\lambda-\rho}_{n,K}$ of deformation parameter n on $\mathfrak{X}$ introduced in [1]. Let $\mathcal{D}'$ (resp. $\mathcal{D}$) be the sheaf of crystalline differential operators on $\mathfrak{G}/\mathfrak{N}$ (resp. $\mathfrak{G}/\mathfrak{B}$). Since $\tilde{\xi} : \mathfrak{G}/\mathfrak{N} \to \mathfrak{G}/\mathfrak{B}$ is a left $\mathfrak{H}$-torsor, the sheaf of algebra $\widetilde{\mathcal{D}} := (\tilde{\xi}_*\mathcal{D}')^{\mathfrak{H}}$ is locally isomorphic to $\mathcal{D} \otimes_{\mathcal{O}_K} \mathcal{U}(\mathfrak{t}^\circ)$. Let $F_\cdot\widetilde{\mathcal{D}}$ be the filtration on $\widetilde{\mathcal{D}}$ induced from the order filtration on $\mathcal{D}'$. Let $\widetilde{\mathcal{D}}_n$ be the sheaf associated to the presheaf $U \to \sum_i \alpha^{in} F_i\widetilde{\mathcal{D}}(U)$. Then $\widetilde{\mathcal{D}}_n$ is locally isomorphic to $\mathcal{D}\otimes_{\mathcal{O}_K}\mathcal{U}(\alpha^n\mathfrak{t}^\circ)$. Let $(\widehat{\widetilde{\mathcal{D}}_n})_K$ be the p-adic completion of $\widetilde{\mathcal{D}}_n$ tensoring K over $\mathcal{O}_K$, and define $\widehat{\mathcal{D}}^{\lambda-\rho}_{n,K} := (\widehat{\widetilde{\mathcal{D}}_n})_K \otimes_{\widehat{\mathcal{U}(\alpha^n\mathfrak{t}^\circ)}_K} K_{\lambda-\rho}$. From the construction, we see that $\widehat{\mathcal{D}}^{\lambda-\rho}_{n,K}|_{\mathfrak{U}_i} \simeq \widehat{\mathcal{A}}_{\mathfrak{T}_n,\mathfrak{U}_i,\mathbb{Q}}$, where $\mathcal{A}_{\mathcal{T}_X} = \mathcal{D}_{X,\lambda}$ and $\mathfrak{T} = \mathcal{T}_{\mathfrak{X}}|_{\mathfrak{U}_i}$, and $\varprojlim_n \widehat{\mathcal{D}}^{\lambda-\rho}_{n,K} \simeq sp_*\mathcal{D}^\infty_{X,\lambda}$ as sheaves of topological algebras.

Let $\mathbb{H}$ be a sufficiently small good analytic open subgroup of G obtained by exponentiating $\mathfrak{h}$, we can assume that $\mathfrak{h} \otimes_{\mathcal{O}_L} \mathcal{O}_K$ is a $\mathcal{O}_K$-Lie sublattic of $\mathfrak{g}^\circ$. Therefore, we have algebra homomorphisms $K[H^\circ_{n+1}] \to \widehat{\mathcal{U}}(\alpha^n \mathfrak{h})_K \to \widehat{\mathcal{U}}(\alpha^n \mathfrak{g}^\circ)_K$. Similar to Definition-Lemma 3.3.2, let us define $\mathcal{D}^{H^\circ}_{\mathfrak{X},n,\lambda} := \widehat{\mathcal{D}}^{\lambda-\rho}_{n,K} \otimes_{K[H^\circ_{n+1}]} K[H^\circ]$, and $\mathcal{D}^{H^\circ}_{\mathfrak{X},\lambda} := \varprojlim_n \mathcal{D}^{H^\circ}_{\mathfrak{X},n,\lambda}$. It follow from the construction that there are isomorphisms of sheaves of topological algebras $\mathcal{D}^{H^\circ}_{\mathfrak{X},n,\lambda}|_{\mathfrak{U}_i} \simeq \mathcal{A}^{H^\circ}_{\mathfrak{U}_i,\mathfrak{T}_n}$, and $\mathcal{D}^{H^\circ}_{\mathfrak{X},\lambda}|_{\mathfrak{U}_i} \simeq \mathcal{A}^{H^\circ}_{\mathcal{T}_X,\mathfrak{U}_i}$.

Lemma 3.4.3. *Let λ be regular and dominant, and assume the assumptions in Remark 3.4.1. The category of finitely generated modules over $\widehat{\mathcal{U}}(\alpha^n \mathfrak{g}^\circ)_{K,\theta} \otimes_{K[H^\circ_{n+1}]} K[H^\circ]$ is equivalent to the category of coherent modules over $\mathcal{D}^{H^\circ}_{\mathfrak{X},n,\lambda}$.*

Proof. By [1] Theorem 6.12, the category of finite generated $\widehat{\mathcal{U}}(\alpha^n \mathfrak{g}^\circ)_{K,\theta}$-modules is equivalent to the category of $\widehat{\mathcal{D}}^{\lambda-\rho}_{n,K}$-modules, via the functors $\widehat{\mathcal{D}}^{\lambda-\rho}_{n,K} \otimes_{\widehat{\mathcal{U}}(\alpha^n \mathfrak{g}^\circ)_{K,\theta}} -$ and $\Gamma(\mathfrak{X}, -)$. It follows that the functors $\mathcal{D}^{H^\circ}_{\mathfrak{X},n,\lambda} \otimes_{(\widehat{\mathcal{U}}(\alpha^n \mathfrak{g}^\circ)_{K,\theta} \otimes_{K[H^\circ_{n+1}]} K[H^\circ])} -$ and $\Gamma(\mathfrak{X}, -)$ induce an equivalence between the category of finitely generated modules over $\widehat{\mathcal{U}}\alpha^n \mathfrak{g}^\circ)_{K,\theta} \otimes_{K[H^\circ_{n+1}]} K[H^\circ]$, and the category of coherent modules over $\mathcal{D}^{H^\circ}_{\mathfrak{X},n,\lambda}$.

□

Theorem 3.4.4. *Let λ be regular dominant, and assume the assumptions in Remark 3.4.1. Define $D^{la}(G,K)_\theta := D^{la}(G,K) \widehat{\otimes}_{\overline{\mathcal{Z}(\mathfrak{g}_K)}} K_\theta$. Then the following statements are true:*

1. *$\overline{\mathcal{U}(\mathfrak{g})}_\theta \xrightarrow{\simeq} \Gamma(X, \mathcal{D}^\infty_{X,\lambda})$ as Fréchet-Stein algebras.*
2. *The category of coadmissible $\overline{\mathcal{U}(\mathfrak{g})}_\theta$-modules is equivalent to the category of sheaves of coadmissible $\mathcal{D}^\infty_{X,\lambda}$-modules on X.*
3. *$D^{la}(H^\circ, K)_\theta \xrightarrow{\simeq} \Gamma(\mathfrak{X}, \mathcal{D}^{H^\circ}_{\mathfrak{X},\lambda})$ as Fréchet-Stein algebras.*
4. *The category of coadmissible $D^{la}(G,K)_\theta$-module is equivalent to the category of sheaves of coadmissible G-equivariant $\mathcal{D}^\infty_{X,\lambda}$-modules on X.*

Proof.

1. This follows from the following isomorphisms:

$$\Gamma(X, \mathcal{D}^{\infty}_{X,\lambda}) \simeq \Gamma(\mathfrak{X}, sp_*\mathcal{D}^{\infty}_{X,\lambda}) \simeq \Gamma(\mathfrak{X}, \varprojlim_n \widehat{\mathcal{D}}^{\lambda-\rho}_{n,K}) \simeq \varprojlim_n \Gamma(\mathfrak{X}, \widehat{\mathcal{D}}^{\lambda-\rho}_{n,K})$$
$$\simeq \varprojlim_n \widehat{\mathcal{U}}(\alpha^n \mathfrak{g}^\circ)_K \otimes_{(\widehat{\mathcal{Z}(\mathfrak{g}^\circ)_n})_K} K_\theta \simeq D^{la}(G,K)_\theta$$

2. If M is a coadmissible $\overline{\mathcal{U}(\mathfrak{g})}_\theta$-module, we can define a presheaf $\Delta(M)^{\text{pre}}$, such that if $U \subseteq X$ is an open affinoid such that $\mathcal{T}_X|_U$ is trivial, then

$$\Delta(M)^{\text{pre}}(U) = \mathcal{D}^{\infty}_{X,\lambda}(U) \widehat{\otimes}_{\overline{\mathcal{U}(\mathfrak{g})}_\theta} M.$$

   Let $\Delta(M)$ be the sheafification of $\Delta(M)^{\text{pre}}$, then $\Delta(M)|_U \simeq \Delta(M)^{\text{pre}}|_U$ is the sheaf associated to the coadmissible module $\mathcal{D}^{\infty}_{X,\lambda}(U) \widehat{\otimes}_{\overline{\mathcal{U}(\mathfrak{g})}_\theta} M$ over $\mathcal{D}^{\infty}_{X,\lambda}(U)$ by Definition-Lemma 2.3.10. In particular, we see that $\Delta(M)$ is a sheaf of coadmissible $\mathcal{D}^{\infty}_{X,\lambda}$-module Definition 2.4.4.

   If $\mathcal{M}$ is a sheaf of coadmissible $\mathcal{D}^{\infty}_{X,\lambda}$-module, then $\mathcal{M}_n := \widehat{\mathcal{D}}^{\lambda-\rho}_{n,K} \otimes_{(sp_*\mathcal{D}^{\infty}_{X,\lambda})} sp_*\mathcal{M}$ is a coherent $\widehat{\mathcal{D}}^{\lambda-\rho}_{n,K}$-module on $\mathfrak{X}$, such that $\mathcal{M}_{n+1} \simeq \widehat{\mathcal{D}}^{\lambda-\rho}_{n+1,K} \otimes_{\widehat{\mathcal{D}}^{\lambda-\rho}_{n,K}} \mathcal{M}_n$, and $sp_*\mathcal{M} \simeq \varprojlim_n \mathcal{M}_n$. By Lemma 2.4.5 and Theorem 6.12 in [1] that $\Gamma(X,\mathcal{M}) \simeq \Gamma(\mathfrak{X}, sp_*\mathcal{M}) \simeq \varprojlim_n \Gamma(\mathfrak{X}, \mathcal{M}_n)$ is a coadmissible $\overline{\mathcal{U}(\mathfrak{g})}_\theta$-module.

   Then we can check directly that $\Delta \circ \Gamma \simeq Id$ and $\Gamma \circ \Delta \simeq Id$.

3. This follows from the isomorphisms:

$$\Gamma(\mathfrak{X}, \mathcal{D}^{H^\circ}_{\mathfrak{X},\lambda}) \simeq \Gamma(\mathfrak{X}, \varprojlim_n \widehat{\mathcal{D}}^{\lambda-\rho}_{n,K} \otimes_{K[H^\circ_{n+1}]} K[H^\circ])$$
$$\simeq \varprojlim_n \Gamma(\mathfrak{X}, \widehat{\mathcal{D}}^{\lambda-\rho}_{n,K}) \otimes_{K[H^\circ_{n+1}]} K[H^\circ]$$
$$\simeq \varprojlim_n (\widehat{\mathcal{U}}(\alpha^n \mathfrak{g}^\circ)_K \otimes_{K[H^\circ_{n+1}]} K[H^\circ]) \otimes_{(\widehat{\mathcal{Z}(\mathfrak{g}^\circ)_n})_K} K_\theta$$
$$\simeq D^{la}(H^\circ, K) \widehat{\otimes}_{\overline{\mathcal{Z}(\mathfrak{g}_K)}} K_\theta$$

4. If M is a coadmissible $D^{la}(G,K)_\theta$-module, and $U \subseteq X$ is an open affinoid such that $\mathcal{T}_X|_U$ is trivial, Let $\mathbb{H}$ be a sufficiently small good analytic open subgroup of G. Then

$$\mathcal{D}^\infty_{X,\lambda}(U)\widehat{\otimes}_{\Gamma(X,\mathcal{D}^\infty_{X,\lambda})}M \simeq (\mathcal{D}^\infty_{X,\lambda}(U)\widehat{\otimes}_{\overline{\mathcal{U}(\mathfrak{g}_K)}}D^{la}(H^\circ,K))\widehat{\otimes}_{D^{la}(H^\circ,K)_\theta}M$$

is a coadmissible module over $\mathcal{D}^\infty_{X,\lambda}(U)\widehat{\otimes}_{\overline{\mathcal{U}(\mathfrak{g}_K)}}D^{la}(H^\circ,K)$. Let us define a presheaf $\Delta^{\mathrm{pre}}(M)$, such that $\Delta^{\mathrm{pre}}(M)(U) = \mathcal{D}^\infty_{X,\lambda}(U)\widehat{\otimes}_{\Gamma(X,\mathcal{D}^\infty_{X,\lambda})}M$. Let $\Delta(M)$ be the sheafification of $\Delta^{\mathrm{pre}}(M)$. Since $\Delta^{\mathrm{pre}}(M)|_U$ is the sheaf associated to the coadmissible module M of $\mathcal{D}^\infty_{X,\lambda}(U)\widehat{\otimes}_{\overline{\mathcal{U}(\mathfrak{g}_K)}}D^{la}(H^\circ,K)$, we see that $\Delta^{\mathrm{pre}}(M)|_U \simeq \Delta(M)|_U$ by Definition-Lemma 3.3.6. Moreover, if $g \in G$, $\mathfrak{x} \in \mathcal{D}^\infty_{X,\lambda}(U)$ and $m \in M$, we have a natural action $g.(\mathfrak{x}\otimes m) := g.\mathfrak{x}\otimes g.m$. Therefore $\Delta(M)$ is a G-equivariant coadmissible $\mathcal{D}^\infty_{X,\lambda}$-module. If $\mathcal{M}$ is a G-equivariant coadmissible $\mathcal{D}^\infty_{X,\lambda}$-module, by Lemma 3.4.3 and Lemma 2.4.5, we see that $\Gamma(\mathfrak{X}, sp_*\mathcal{M})$ is a coadmissible module over $\Gamma(\mathfrak{X},\mathcal{D}^{H^\circ}_{\mathfrak{X},\lambda})$. It follows that $\Gamma(X,\mathcal{M})$ is a coadmissible module over $D^{la}(H^\circ,K)\widehat{\otimes}_{\overline{\mathcal{Z}(\mathfrak{g}_K)}}K_\theta$. Because $\Gamma(X,\mathcal{M})$ is a G-representation, and $D^{la}(G,K) \simeq \bigoplus_{\bar{g}\in G/H^\circ} D^{la}(H^\circ,K) * \delta_g$, where g is a set of representatives of $\bar{g}$ in G, we see that $\Gamma(X,\mathcal{M})$ is a coadmissible $D^{la}(G,K)_\theta$-module. Then we can check directly that $\Delta\circ\Gamma \simeq Id$ and $\Gamma\circ\Delta \simeq Id$.

□

3.5 Examples

3.5.1 $G = \mathbb{Z}_p$

In this subsection, let $X = \mathrm{Spa}(K\langle y\rangle, K^\circ\langle y\rangle)$, and $\mathbb{G} = X$ is the abelian additive group. Let $t := \frac{\partial}{\partial y}$ be the generator of the Lie algebra $\mathfrak{g}$ of G. We have isomorphisms $\exp : p^n\mathbb{Z}_p \xrightarrow{\simeq} G_n$, for $n > 0$. Therefore,

$$D^{la}(G,K) \simeq \varprojlim_n K\langle p^n t\rangle \otimes_{K[G_{n+1}]} K[\mathbb{Z}_p]$$

Let $D_n = \{\sum_{n\geq 0} a_s(p^n t)^s \mid a_n \in K\langle y\rangle, \lim_{s\to+\infty} a_s = 0\}$ whose multiplication structure is given by $ty = yt+1$. Then a coadmissible G-equivariant $\mathcal{D}_X^\infty$-module is a coadmissible module over $\varprojlim_n D_n \otimes_{K[G_{n+1}]} K[\mathbb{Z}_p]$. We see that $D^{la}(G,K) \simeq \varprojlim_n (D_n/D_n y)\otimes_{K[G_{n+1}]} K[\mathbb{Z}_p]$ can be viewed as a G-equivariant $\mathcal{D}_X^\infty$-module. Geometrically, $D^{la}(G,K)$ is a sheaf on X supported on $\mathbb{Z}_p$-rational points.

Remark 3.5.1. By Amice transform $D^{la}(G,K)$ can be identified with the rigid analytic function on the open unit disk:

$$D^{la}(G,K) \simeq \varprojlim_n K\langle p^{-\frac{1}{n}} T\rangle,$$

where $t = \log(1+T)$. The latter ring is usually represented as $\mathcal{R}^+$ in the world of (ϕ,Γ)-modules. Therefore, inverting t in $\mathcal{R}^+$ can be interpreted as inverting differential operators.

3.5.2 $G = GL_2$ and Principal series

Let $U_0 = \mathrm{Spa}(K\langle x\rangle, K^\circ\langle x\rangle)$, and $U_1 = \mathrm{Spa}(K\langle y\rangle, K^\circ\langle y\rangle)$ be the standard cover of $\mathbb{P}^1$. The GL_2-action on $\mathbb{P}^1$ is given by $\begin{pmatrix} a & b \\ c & d\end{pmatrix} x = \frac{ax+b}{cx+d}$, and $\begin{pmatrix} a & b \\ c & d\end{pmatrix} y = \frac{c+dy}{a+by}$. Assume that $y=0$ is the point ∞.

Let $w = \begin{pmatrix} 0 & 1 \\ 1 & 0\end{pmatrix}$, $e = \begin{pmatrix} 0 & 1 \\ 0 & 0\end{pmatrix}$, $f = \begin{pmatrix} 0 & 0 \\ 1 & 0\end{pmatrix}$, $h = \begin{pmatrix} 1 & 0 \\ 0 & -1\end{pmatrix}$. Then the center of $\mathcal{U}(\mathfrak{gl}_2)$ is generated by $z = \begin{pmatrix} 1 & 0 \\ 0 & 1\end{pmatrix}$ and the Casimir $C = \frac{1}{2}h^2 + ef + fe$.

Fix two characters $\delta_1, \delta_2 : \mathbb{Q}_p^* \to K$. Let $\delta = \delta_1\delta_2^{-1}\chi^{-1}$, and $\omega = \delta_1\delta_2\chi^{-1}$. Let $\kappa(\delta_i) = \delta_i'(1)$ be the derivative of δ_i for $i=1,2$. Assume $\kappa(\delta) = \kappa(\delta_1) - \kappa(\delta_2) - 1$ does not equal any nonegative integer. Let $\mathcal{D}_{\mathbb{P}^1,\kappa(\delta)}$ be the twisted differential operator given explicitly by gluing as follows: $i_0 : \mathcal{D}_{\mathbb{P}^1,\kappa(\delta)}|_{U_0} \xrightarrow{\simeq} \mathcal{D}_{U_0}$, $i_1 : \mathcal{D}_{\mathbb{P}^1,\kappa(\delta)}|_{U_1} \xrightarrow{\simeq} \mathcal{D}_{U_1}$, and $i_1 i_0^{-1}|_{U_0\cap U_1}$ is $P \to x^{-\kappa(\delta)} P x^{\kappa(\delta)}$. The homomorphism $\alpha : \mathfrak{gl}_2 \to \Gamma(\mathbb{P}^1, \mathcal{D}_{\mathbb{P}^1,\kappa(\delta)})$ is given by:

$$i_0 \circ \alpha : h \to -2x\partial_x - \kappa(\delta) \qquad\qquad i_1 \circ \alpha : h \to 2y\partial_y + \kappa(\delta)$$

$$e \to -\partial_x \qquad\qquad e \to y^2\partial_x + \kappa(\delta)y$$

$$f \to x^2\partial_x + \kappa(\delta)x \qquad\qquad f \to -\partial_y$$

$$z \to -\kappa(\delta_1) - \kappa(\delta_2) + 1 \qquad\qquad z \to -\kappa(\delta_1) - \kappa(\delta_2) + 1$$

Then C acts as $-\frac{1}{2}((\kappa(\delta_1) - \kappa(\delta_2))^2 - 1)$.

Recall that the locally analytic principal series is: $B^{an}(\delta_1, \delta_2) =$

$$\{f \in C^{la}(\mathbb{Q}_p, K)|\ \delta(x)f(\frac{1}{x}) \text{ can be extend to an analytic function in a neighborhood of } 0\},$$

with $GL_2(\mathbb{Q}_p)$-acts as $\begin{pmatrix} a & b \\ c & d \end{pmatrix} f(x) = \delta_2(ad - bc)\delta(a - cx)f(\frac{dx-b}{a-cx})$. Another description of the locally analytic principal series is:

$$\mathrm{Ind}_{\bar{B}}^{G}(\delta_1\chi^{-1} \otimes \delta_2) = \{\phi \in C^{la}(G, K)|\phi(gb) = (\delta_1\chi^{-1} \otimes \delta_2)(b^{-1})\phi(g) \text{ if } b \in \bar{B}\},$$

with $GL_2(\mathbb{Q}_p)$-action $g\phi(x) = \phi(g^{-1}x)$, where $\bar{B}$ is the lower triangular Borel subgroup. We see that $f(x) = \phi(\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix})$ defines an isomorphism of G-representations

$$\mathrm{Ind}_{\bar{B}}^{G}(\delta_1\chi^{-1} \otimes \delta_2) \xrightarrow{\simeq} B^{an}(\delta_1, \delta_2).$$

By [12] Proposition 4.14 we have the relationship between locally analytic principal series and (ϕ, Γ)-modules:

$$(B^{an}(\delta_1, \delta_2))^* = \mathcal{R}^+(\chi\delta_1^{-1}) \boxtimes_{\omega^{-1}} \mathbb{P}^1.$$

Let $\bar{I}$ be the mod p lower triangular Iwahori subgroup, so the action of $\bar{I}$ on $\mathbb{P}^1$ preserves

U_1. Let $\bar{N}$ be the unipotent subgroup of $\bar{B}$. Then $\bar{I} \cap N \simeq U_1(\mathbb{Z}_p)$ via $\begin{pmatrix} 1 & 0 \\ y & 1 \end{pmatrix} \to y$. Define

$$M_1 := (\mathrm{Ind}_{\bar{I}\cap B}^{\bar{I}}((\delta_2 \otimes \delta_1\chi^{-1}))^*.$$

By Proposition 5.1 in [29], M_1 is a finitely generated $D^{la}(\bar{I}, K)$-module, and as $D^{la}(\bar{I}\cap N, K)$-modules, we have

$$M_1 \simeq D^{la}(\bar{I} \cap N, K) \simeq \mathcal{R}^+.$$

Under the second isomorphism (Amice) we see that $\begin{pmatrix} 1 & 0 \\ 0 & a \end{pmatrix}$ acts $T \to (\chi\delta_1^{-1})(a)(1+T)^a$. Therefore, we can write $M_1 \simeq \mathcal{R}^+(\chi\delta_1^{-1})$.

Using the notations in section 3.5.1, as $\bar{I} \cap \bar{N}$-equivariant $\mathcal{D}^{\infty}_{\mathbb{P}^1,\kappa(\delta)}(U_1)$-modules, we have

$$M_1 \simeq \varprojlim_n (D_n/D_n y) \otimes_{G_{n+1}} K[\bar{I} \cap \bar{N}].$$

Thus M_1 is a finitely generated module over $D^{la}(\bar{I}, K)\widehat{\otimes}_{\overline{\mathcal{U}(\mathfrak{gl}_2)}}\mathcal{D}^{\infty}_{\mathbb{P}^1,\kappa(\delta)}(U_1)$, and $\widetilde{M_1} := \mathcal{D}^{\infty}_{\mathbb{P}^1,\kappa(\delta)}|_{U_1}\widehat{\otimes}_{\mathcal{D}^{\infty}_{\mathbb{P}^1,\kappa(\delta)}(U_1)} M_1$ is therefore an $\bar{I}$-equivariant $\mathcal{D}^{\infty}_{\mathbb{P}^1,\kappa(\delta)}|_{U_1}$-module on U_1. Note that the stalk $\widetilde{M}_{1,\infty} = \bigcap_{n\geq 0} K\langle p^n\partial_y\rangle$ is the delta function supported at ∞, which has an action of the upper triangular Borel B such that $\begin{pmatrix} 1 & 0 \\ 0 & p \end{pmatrix}$ acts as $\partial_y \to (\chi\delta_1^{-1})(p)p\partial_y$.

Similarly, if I is the mod p upper triangular Iwahori subgroup, define

$$M_0 := (\mathrm{Ind}_{I\cap\bar{B}}^{I}(\delta_1\chi^{-1} \otimes \delta_2))^*$$

Then $M_0 \simeq D^{la}(I \cap N, K) \simeq \mathcal{R}^+$ such that $\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix}$ acts as $T \to (\chi\delta_1^{-1})(a)(1+T)^a$. Therefore $\widetilde{M_0} := \mathcal{D}^{\infty}_{\mathbb{P}^1,\kappa(\delta)}|_{U_0}\widehat{\otimes}_{\mathcal{D}^{\infty}_{\mathbb{P}^1,\kappa(\delta)}(U_0)} M_0$ is an I-equivariant $\mathcal{D}^{\infty}_{\mathbb{P}^1,\kappa(\delta)}|_{U_0}$-module on U_0.

It follows that $\widetilde{M_1}(U_0 \cap U_1) \simeq \mathcal{R}^+(\chi\delta_1^{-1}) \boxtimes \mathbb{Z}_p^{\times} \simeq \widetilde{M_0}(U_0 \cap U_1)$. If we glue $\widetilde{M_1}$ and $\widetilde{M_0}$

via $\int_{\mathbb{Z}_p^\times} f(x)\mu_0 = \int_{\mathbb{Z}_p^\times} \delta_2(-1)\delta(-x)f(\frac{1}{x})\mu_1$ for $\mu_i \in M_i$ and i = 1, 2. Then by the definition of Colmez's $\boxtimes$ construction and Čech cohomology, the global section of the glued sheaf is $\mathcal{R}^+(\chi\delta_1^{-1}) \boxtimes_{\omega^{-1}} \mathbb{P}^1$.

Remark 3.5.2. We wonder whether it is possible to give $D_{\mathrm{rig}}^\natural \boxtimes \mathbb{P}^1$ a similar interpretation.

Remark 3.5.3. The locally analytic Steinberg is the global section of Ω_U^1 where $U \hookrightarrow \mathbb{P}^1$ is the Drinfeld upper half plane. However, Ω_U^1 is naturally a G equivariant $\mathcal{D}_{\mathbb{P}^1,2}^\infty$-module.

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