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Dedicated to Mark Mahowald

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ABSTRACT

My thesis focuses on computations of stable homotopy groups of spheres, with applications and connections to differential geometry and motivic homotopy theory. The Adams spectral sequences and Toda brackets play a major role in my work. We have introduced two methods to compute Adams differentials and solve extension problems: one is very technical but inductive, using the algebraic Kahn-Priddy theorem; the other one is more systematic, using a new connection between motivic homotopy theory and chromatic homotopy theory. Combining both methods, we have computed stable stems into a larger range. As a consequence, I solved the strong Kervaire invariant problem in dimension 62 and showed that the 61-sphere has a unique smooth structure, which is the last odd dimensional case.

CHAPTER 1

INTRODUCTION

Understanding the stable homotopy groups of spheres is one of the most fundamental and important problems in algebraic topology. It has many connections to other areas in mathematics: geometric topology, differential geometry, algebraic geometry, number theory and algebraic K-theory. Despite its simple definition, which was available sixty years ago, it is notoriously hard to compute: all known methods at any time only give a complete answer through a range, and then reach an obstacle until a new method is introduced. More specifically, to compute stable stems, one has Adams type spectral sequences converging from algebra to homotopy; to identify the E_2-pages, one needs algebraic spectral sequences converging from simpler algebra to more complicated algebra. For any spectral sequences, there are difficulties computing differentials, and in solving all extension problems. Differences in methods lead to trade-offs: one method may compute some types of differentials and extension problems more efficiently, but will definitely leave other types open, which perhaps are unsolvable by that technique. To obtain complete calculations, one must be eclectic, applying and combining different methodologies. Even so, combining all known methods, there are eventually some problems that cannot be solved. Because of this, experts in the field tend to believe that computing the stable homotopy groups of spheres *completely* is a mission that will never succeed: This is also known as **Mahowald's uncertainty principle**.

Nonetheless, I devote myself to pushing forward the computations and to finding new connections to other mathematics. I have discovered two new methods, with complementary advantages, which allow me to successfully extend known computations into a much larger range. A major goal in the near future, is to apply these new methods to solve the last open case of the Kervaire invariant problem, the one in dimension 126, which is arguably the most famous unsolved problem in stable homotopy theory. Besides the methods and computations, I found a new connection between chromatic homotopy theory and motivic homotopy theory.

Although complicated computations are hard to summarize, here are some theoretical outcomes of my work:

Theorem 1.0.1 ((88))**.** *The strong Kervaire invariant problem in dimension 62 is true: the Kervaire invariant element* $\theta_5 \in \pi_{62}$ *has order 2.*

Here π_{62} is the 62nd 2-localized stable homotopy groups of spheres. If one can further show that $\theta_5^2 = 0$, then it solves the only open case of the Kervaire invariant problem in dimension 126. Another approach, which requires a much *lower* dimensional computation in dimension 93, is summarized in the following unpublished technical theorem. I will sketch the proof later.

Theorem 1.0.2. *If the Toda bracket* $\langle \theta_5, 2, \theta_4 \rangle$ *in* π_{93} *contains an element of the form* $2\phi_1 + \nu\phi_2$ *for some* $\phi_1 \in \pi_{93}$ *and* $\phi_2 \in \pi_{90}$*, then* θ_6 *exists.*

In particular, if the Toda bracket contains zero, then θ_6 exists.

The following theorem is joint work with Guozhen Wang.

Theorem 1.0.3 ((86))**.** *The sphere* S^{61} *has a unique smooth structure. Moreover, this is the last odd dimensional case: only the spheres* S^1*,* S^3*,* S^5 *and* S^{61} *have a unique smooth structure.*

The statements of Theorem 1.0.3 rely on Kervaire-Milnor's work (45) on exotic smooth structures and Hill-Hopkins-Ravenel's work (30) on the Kervaire invariant problem.

The following theorem was conjectured by Isaksen and proved joint with Bogdan Gheorghe and Guozhen Wang.

Theorem 1.0.4 ((27))**.** *There is an equivalence of two triangulated categories with t-structures at each prime p*

$$\mathcal{D}^b(BP_*BP - comod) \simeq Ho^b(C\tau - mod),$$

between the bounded derived category of p-completed BP_*BP-comodules which are concentrated in even degrees, and the p-completed motivic homotopy category of cellular module spectra over the E_∞-ring spectrum $C\tau$, whose $BPGL$-homology has bounded Chow degree.

We actually proved a stronger version of this theorem: an equivalence of stable ∞-categories in the sense of Jacob Lurie (48). Here $C\tau$ is the cofiber of the element that corresponds to Tate twist in the motivic stable homotopy groups of the p-completed spheres over *Spec* $\mathbb{C}$, BP is the Brown-Peterson spectrum, $BPGL$ is its motivic analogue, and BP_*BP is the Hopf algebroid of BP-cooperations. The derived category of BP_*BP-comodules is also known as the derived category of quasi-coherent sheaves on the moduli stack of formal groups. This connection is foundational to chromatic homotopy theory, and is due to Quillen (70) and Goerss-Hopkins (28; 32). Our theorem further connects these to motivic homotopy theory.

As an application, the following theorem helps compute the stable homotopy groups of spheres:

Theorem 1.0.5 ((27))**.** *There is an isomorphism of tri-graded spectral sequences at each prime p: the algebraic Novikov spectral sequence that converges to the Adams-Novikov E_2-page, and the motivic Adams spectral sequence of $C\tau$ that converges to the motivic homotopy groups of $C\tau$.*

Theorem 1.0.5 originated from an observation of Isaksen (35) that the motivic homotopy groups of $C\tau$ are essentially algebraic. This isomorphism of spectral sequences therefore gives a way of producing ***topological*** differentials (in the motivic Adams spectral sequence) from purely ***algebraic*** data! An application of Theorem 1.0.5 is that, as I will explain in detail in Subsection 1.2.2, it reproves and generalizes Miller's theorem (54) on the correspondence of classical Adams d_r differentials and algebraic Novikov d_r differentials from Miller's case $r = 2$ to $r \geq 3$!

More details regarding the Kervaire invariant problem in dimension 62 and 126 are discussed in Section 1.1. In Section 1.2, I will give a summary of the two new methods of computations of the stable homotopy groups of spheres and present the new connection between chromatic and motivic homotopy theory. The organization of the remaining chapters of my thesis is discussed in Section 1.3.

1.1 the Kervaire invariant elements θ_5 and θ_6

The Kervaire invariant is the most fundamental invariant of a $(4n+2)$-dimensional framed differentiable manifold (or more generally PL-manifold). It is defined as the Arf invariant of the quadratic form determined by the framing on the middle-dimensional $\mathbb{F}_2$-coefficient homology group. Using his invariant in dimension 10, Kervaire (41) constructed a 10-dimensional PL manifold with no differential structure. It also plays a key role in Kervaire and Milnor's work (45) on the computations of exotic smooth structures on spheres.

The Kervaire invariant problem asks in which dimensions do there exist framed manifolds with nonzero Kervaire invariant. Browder (14) reduced it to a computation of stable homotopy groups of spheres at the prime 2: it is only possible for dimensions $2^{j+1}-2$ and in that case only when the element h_j^2 in the E_2-page of the Adams spectral sequence survives. In these cases, the relevant homotopy classes in $\pi_{2^{j+1}-2}$ are called θ_j. It was known by 1984 that θ_j exists for $j \leq 5$. Recently, Hill, Hopkins and Ravenel (30) showed that θ_j does not exist for $k \geq 7$, leaving the existence of θ_6 as the only open case.

Before Hill-Hopkins-Ravenel's result, Barratt-Jones-Mahowald (8) had an inductive approach, trying to show that all θ_j exist: If θ_j exists, $2\theta_j = 0$, and $\theta_j^2 = 0$, then θ_{j+1} exists, and $2\theta_{j+1} = 0$. This inductive approach motivates the strong Kervaire invariant problem, which asserts that θ_j exists and has order 2. To solve the open case that θ_6 exists using this approach, one first needs to know if θ_5 has order 2, which motivates Theorem 1.0.1.

In 1984, Barratt-Jones-Mahowald (7) constructed θ_5 using obstruction theory and the theory of cell diagrams. In fact, they constructed a 9-cell stable complex X, with two stable maps

$$f : S^{62} \longrightarrow X, \ g : X \longrightarrow S^0,$$

and showed that the composition $g \circ f$ is represented by θ_5. Although θ_5 is in π_{62}, the proof only involves computations of certain Toda brackets through dimension 47. Doing further computations, I reduced the 9-cell complex of Barratt-Jones-Mahowald to a 4-cell complex (88) and later to a 3-cell complex (86).

However, they were not be able to show that this θ_5 has order 2, and therefore could not make use of the inductive approach to deduce the existence of θ_6. The full statement of Theorem 1.0.1 reproves that θ_5 exists.

Theorem 1.1.1 (Xu (88))**.** *The class* $\theta_4^2 = 0 \in \pi_{60}$*, and therefore* $\theta_5 \in \pi_{62}$ *exists and has order 2.*

If one could further show that $\theta_5^2 = 0$, then we would know that θ_6 exists, has order 2, and $\theta_6^2 \neq 0$ since θ_7 does not exist.

The idea of the proof of Theorem 1.1.1 is to decompose θ_4 as a Toda bracket of classes in the stable homotopy groups of spheres in *lower* dimensions. The Adams spectral sequence is essential to the computations of the following Toda brackets, which the proof of Theorem 1.1.1 is based on.

Theorem 1.1.2 (Xu (88))**.**

$$\theta_4 = \langle 2, \sigma^2 + \kappa, 2\sigma, \sigma \rangle \quad in \quad \pi_{30},$$

$$0 \in \langle \theta_4, 2, \sigma^2 + \kappa \rangle \quad in \quad \pi_{45}.$$

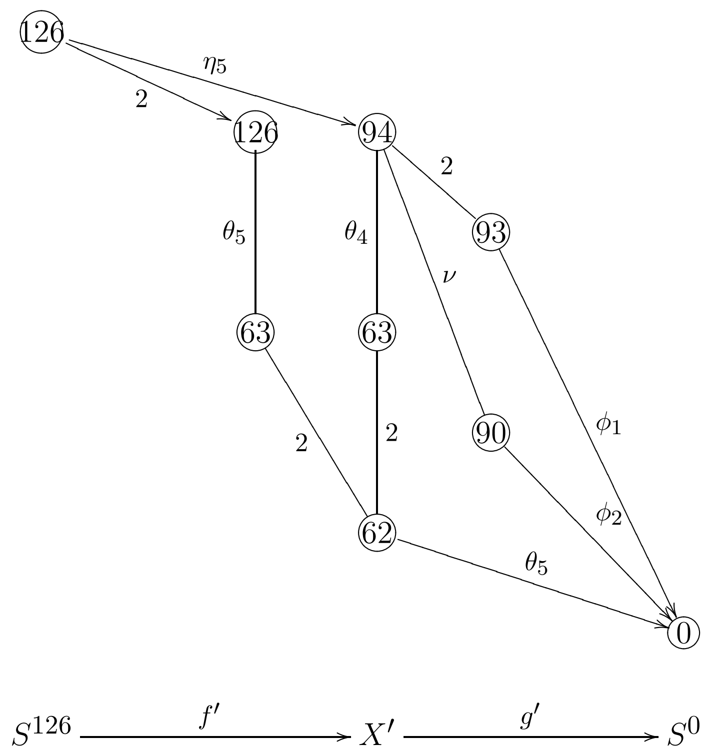
To prove that $\theta_5^2 = 0$, one needs to know a Toda bracket description of θ_5. The only known one is the following:

Theorem 1.1.3 (Wang - Xu (86))**.**

$$\theta_5 \in \langle 2, \theta_4, \theta_4, 2 \rangle \quad in \quad \pi_{62}.$$

This leads us to consider the Toda bracket $\langle \theta_4, 2, \theta_5 \rangle$ in π_{93}. This bracket is also crucial if we use the obstruction theory approach, summarized in Theorem 1.0.2.

Figure 1.1: My proposed construction of θ_6



The idea of the proof is to construct a 7-cell stable complex X', and two stable maps

$$f' : S^{126} \longrightarrow X', \; g' : X' \longrightarrow S^0,$$

and to use obstruction theory to show that the composition $g' \circ f'$ is represented by θ_6. See Figure 1.1 above.

Of course, to see if θ_6 exists in π_{126}, one can always try to compute the stable homotopy groups of spheres up to dimension 126. If some condition similar to Theorem 1.0.2 is satisfied, then we have a shortcut to see the existence of θ_6 with *much less* computation, as Barratt-Jones-Mahowald did for θ_5. This project won't be interesting at all, if one cannot provide methods to extend the known range of computations. This is what I will describe in the next section.

1.2 Two methods of computing stable stems

The stemwise computations of stable homotopy groups of spheres have been pursued by many people, by introducing different methods: Serre, Toda, Adams, May, Barratt, Mahowald, Tangora, Bruner, Nakamura, Aubry, Ravenel, Kochman, Isaksen, others and myself. I will focus the discussion at the prime 2, where the Kervaire invariant problem lies. Among all spectral sequences which converge to the stable homotopy groups of spheres, the classical Adams spectral sequence is the most effective one. My collaborators and I introduced two new methods of computing differentials and solving extension problems in the Adams spectral sequence, with complementary advantages.

The first method, referred to as the RP^{∞}-method, is designed to prove a single differential or extension left by other methods. It translates the problem in the sphere spectrum to a sequence of problems in ***lower*** stems of truncated projective spectra ***inductively***. If the problem at hand is the last stemwise unsolved problem, then this method has a good chance of solving it, using only information about lower stems. Using this method, Guozhen Wang and I (86) proved a notoriously hard differential. As a consequence, we showed $\pi_{61} = 0$, and therefore the 61-sphere has a unique smooth structure. The shortcoming is also obvious: it took us more than 40 pages to set up the theory and do the minimal computations to prove one differential; this method is ***not*** for systematic computations.

The second method, referred to as the motivic $C\tau$-method, is on the contrary very systematic: it produces a huge amount of differentials and extensions from purely algebraic computations, some of which are hard to prove by other methods. Using this method, Dan Isaksen, Guozhen Wang and I have re-computed the known range without much effort, with only a few problems left open. The shortcoming is the universal one: the leftover problems need to be solved by other methods.

Combining the two new methods with classical ones, we have extended stable stems from

59 to 75, and haven't seen the limit of our computations. It is quite plausible that we can achieve 20 more stems and have a chance to compute the Toda bracket in π_{93}.

From our computations, we discovered some new phenomena in the E_2-page of the Adams-Novikov spectral sequence. It is proved by Andrews-Miller (5) that, removing the alpha family elements, there is a vanishing line of slope 1/5.

Observation/Theorem 1.2.1. In between the 1/5-vanishing line and a slope 1/11 line, there is an Adams type periodicity with bidegree $(t-s, s) = (20, 4)$.

Classically, this observation is just algebraic, however, motivically, since motivic Novikov differentials leave τ-torsion behind, this observation actually gives new nontrivial periodic families of homotopy classes, some of which were earlier discovered by Andrews (4).

1.2.1 the RP^∞-method

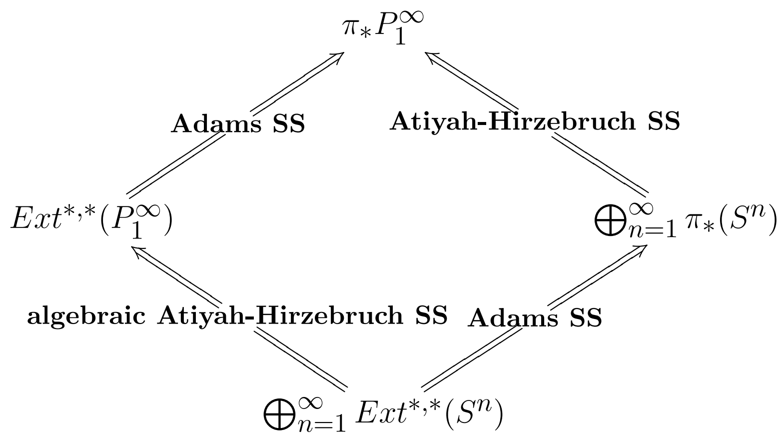
I will first explain the idea of this method, then sketch the proof of the hard differential which took us 40 pages to prove.

The starting point is the algebraic Kahn-Priddy theorem: let S^0 be the sphere spectrum, P_1^∞ be the suspension spectrum of RP^∞, and $Ext(Y) = Ext_{\mathcal{A}}(H^*(Y), \mathbb{F}_2)$ be the abbreviation for the Adams E_2-page for any spectrum Y. Then the homomorphism

$$Ext(P_1^\infty) \longrightarrow Ext(S^0)$$

induced by the transfer map is an epimorphism in positive stems. Due to the surjectivity, we can pull back the elements which are involved in a differential or an extension from $Ext(S^0)$ to $Ext(P_1^\infty)$.

We have the following Mahowald's square to compute Adams differentials of P_1^∞ from differentials in the other three spectral sequences.



Mahowald's square

Since the action of the Steenrod algebra on $H^*(P_1^\infty)$ is well understood, the differentials in the algebraic Atiyah-Hirzebruch spectral sequence are very computable through Massey products. Since the action of the Steenrod algebra on the cohomology of the cofiber of the transfer map is also well understood, we can compute the map $Ext(P_1^\infty) \to Ext(S^0)$ and trace the elements to the E_1-page of the algebraic Atiyah-Hirzebruch spectral sequence, i.e., the cells of P_1^∞ that they come from. These computations are carried out by Wang and myself (85) in the range $t < 72$.

The next step is the crucial step: we use a "zigzag" method to further trace the elements through certain subquotients of P_1^∞, so the differential or extension that we are trying to prove is in a finite cell complex with as few cells as possible. The "zigzag" process is based on the combination of algebraic Atiyah-Hirzebruch differentials and the Adams differentials in ***lower*** stems. The subquotients coming out of the "zigzag" process are highly dependent on the elements we started with. The "zigzag" process is mainly for intuition: it suggests which subquotients to consider, and which differentials or extensions to prove. However, when presenting the proofs, we only use naturality of the spectral sequences.

Figure 1.2 below is the rough outline of how we use this method to prove a very hard differential

$$d_3(D_3) = B_3$$

in the 61-stem, which leads to our proof that $\pi_{61} = 0$, and therefore the uniqueness of smooth structure on the sphere S^{61}.

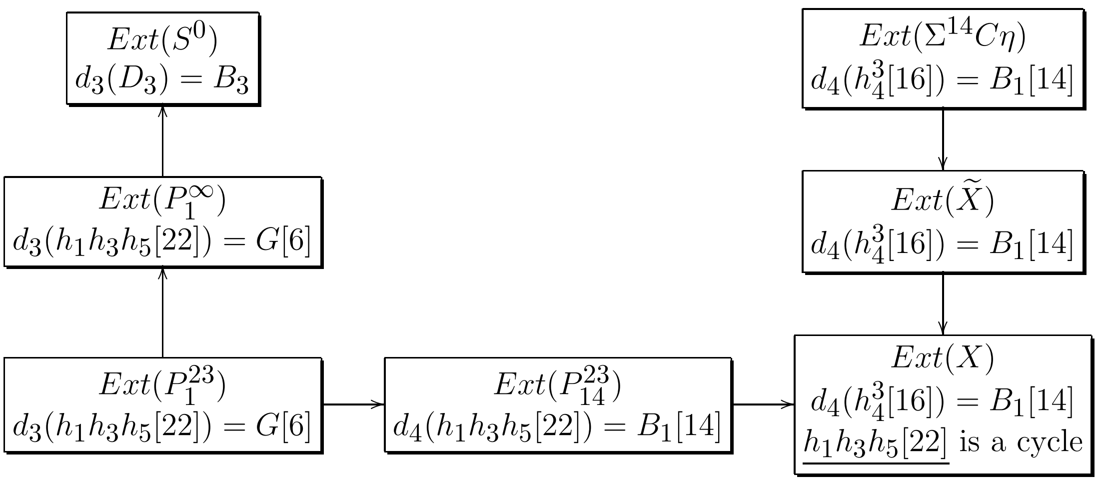


Figure 1.2: The "road map" for the Adams differential $d_3(D_3) = B_3$

Here P_1^{23}, X, $\widetilde{X}$ and $\Sigma^{14}C\eta$ are certain subquotients of P_1^∞. The last subquotient $\Sigma^{14}C\eta$ is a two cell complex: the 14-suspension of the cofiber of $\eta \in \pi_1$. The differential $d_4(h_4^3[16]) = B_1[14]$ in the Adams spectral sequence of $\Sigma^{14}C\eta$ follows from an η-extension from h_4^3 to B_1 in the 45-stem of the sphere spectrum. We first establish this differential, then prove the rest one by one using naturality.

Using this method, Wang and I (87) also solved a few extensions problems in the 46 and 51-stems, which were left open by Isaksen's previous computations (35): These are the last unsolved 2, η and ν-extensions up to the 58-stem.

1.2.2 the motivic $C\tau$-method

Let $H\mathbb{M}_p$ be the motivic Eilenberg-Mac Lane spectrum that represents the mod p motivic cohomology over $Spec\ \mathbb{C}$. Let τ be the element that corresponds to Tate twist in the $H\mathbb{M}_p$-

completed motivic homotopy groups of spheres. Let $C\tau$ be the cofiber of τ.

$$\tau \in \pi_{0,-1}(S^{0,0}{}^{\wedge}_{H\mathbb{M}_p})$$

This method starts with Isaksen's observation that the motivic homotopy groups of $C\tau$ are ***algebraic***: More precisely, Isaksen (35) proved that the motivic Adams-Novikov E_2-page of $C\tau$ is isomorphic to the classical Adams-Novikov E_2-page of the sphere spectrum. Since the motivic Adams-Novikov spectral sequence of $C\tau$ collapses for filtration reasons, there is an isomorphism:

$$\pi_{t-s,\frac{t}{2}}(C\tau) \cong Ext^{s,t}_{BP_*BP}(BP_*, BP_*).$$

between the $H\mathbb{M}_p$-completed motivic homotopy groups of $C\tau$, and the classical Adams-Novikov E_2-page of the sphere spectrum.

For both sides of the isomorphism above, there are spectral sequences converging to them: the motivic Adams spectral sequence of $C\tau$, and the algebraic Novikov spectral sequence. Isaksen (35) computed the first one up to around the 60th stem. After comparing with the computations of the second one in the same range, which was done by a computer program made by Wang, we noticed an isomorphism of spectral sequences: isomorphic tri-graded E_2-pages, and bijections between the differentials and extensions! This is a very surprising discovery: the algebraic Novikov spectral sequence is purely ***algebraic*** - the differentials can be computed by imbedding them in the cobar complex, and therefore systematically produced by a computer program; the motivic Adams differentials are ***topological*** - they are hard to compute in general!

During discussion, Isaksen conjectured Theorem 1.0.4, giving the theoretical reason why this isomorphism of spectral sequences is true. It is now proved by my collaborators and myself (27).

For the proof, we constructed a t-structure on the homotopy category of $C\tau$-modules, such that the heart associated to the t-structure consists of $C\tau$-modules whose $BPGL$-homology is concentrated in Chow degree 0. We then identify this heart with the category of BP_*BP-comodules by taking $BPGL$-homology and forgetting its motivic weight. The version we proved is actually in a stronger form: the equivalence is between two stable ∞-categories in the sense of Lurie (48). The stronger form relies on a theorem of Lurie (48).

As an application, we proved, as stated in Theorem 1.0.5, that there is an isomorphism of spectral sequences between the motivic Adams spectral sequence of $C\tau$, and the algebraic Novikov spectral sequence. The idea of the proof is the following: after re-grading, the algebraic Novikov filtration gives a sequence of subcomplexes in the cobar resolution of BP_*. Using the functor produced by Theorem 1.0.4, we identify the subquotients as motivic Eilenberg-Mac Lane spectra smashed with $C\tau$, compute the maps between them, and prove they form a motivic Adams tower for $C\tau$.

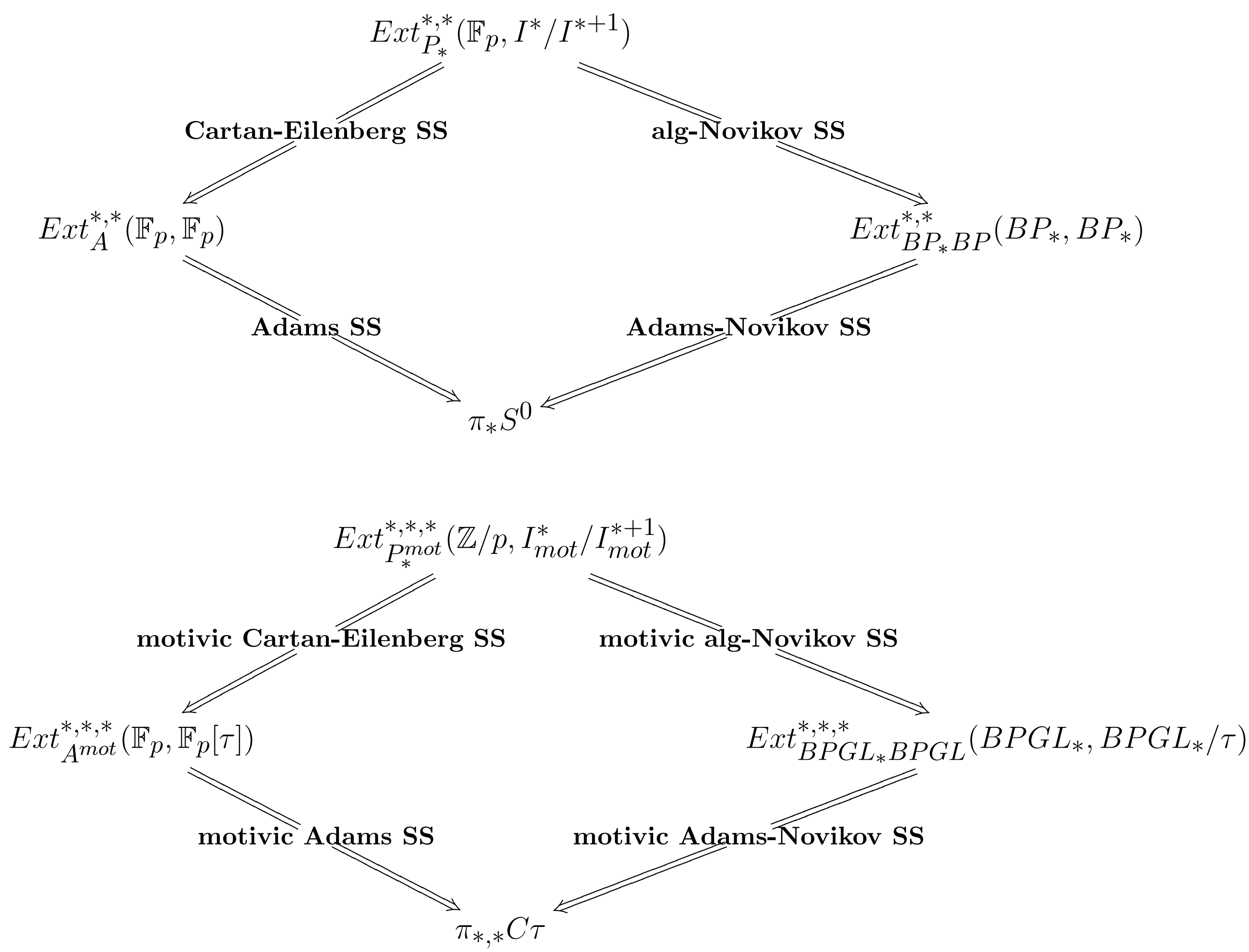
$$\begin{array}{ccc} Ext_{P_*}^{s,2w}(\mathbb{F}_p, I^{a-s}/I^{a-s+1}) & \xrightarrow{\cong} & Ext_{\mathcal{A}^{mot}}^{a,2w-s+a,w}(\mathbb{F}_p, \mathbb{F}_p[\tau]) \\ \Downarrow & & \Downarrow \\ Ext_{BP_*BP}^{s,2w}(BP_*, BP_*) & \xrightarrow{\cong} & \pi_{2w-s,w}(C\tau) \end{array}$$

Here P_* is a sub-Hopf algebra of the dual Steenrod algebra: $\mathbb{F}_p[\xi_1^2, \xi_2^2, \cdots]$ if $p = 2$; $\mathbb{F}_p[\xi_1, \xi_2, \cdots]$ if $p > 2$, I is the ideal $(p, v_1, v_2, \cdots) \subseteq BP_*$, and $\mathcal{A}^{mot}$ is the dual motivic Steenrod algebra over $Spec\ \mathbb{C}$.

Theorem 1.0.5 allows a generalization of Miller's theorem (54) on the correspondence between classical Adams d_2 differentials and algebraic Novikov d_2 differentials to the cases d_r for $r \geq 3$: Given any nontrivial algebraic Novikov d_r differential, it corresponds to a nontrivial motivic Adams d_r differential for $C\tau$, which can be further pulled back to a motivic $d_{r'}$ $(r' \leq r)$ Adams differential of the bottom cell of $C\tau$. After applying the realization functor, one get a classical Adams $d_{r'}$ differential for the sphere spectrum.

Another way to think of Theorem 1.0.5 is through the motivic analogue of Miller's square:

Figure 1.3: Miller's square and its motivic analogue



since both the motivic Cartan-Eilenberg and the motivic Adams-Novikov spectral sequences collapse for filtration reasons, the other two spectral sequences are isomorphic.

1.3 Organization of my thesis

The rest of my thesis is organized as follows. In Chapter 2, I will discuss the proof of the strong Kervaire invariant problem in dimension 62, namely Theorem 1.0.1. Chapter 2 is published as (88). In Chapter 3, I will present the RP^∞-method together with the proof of Theorem 1.0.3. Chapter 3 is joint with Guozhen Wang and is submitted (86). In

Chapter 4, I will discuss a few extension examples that are solved by the RP^∞-method. In Chapter 5, I will discuss the computations of the algebraic Atiyah-Hirzebruch spectral sequence through the Curtis algorithm. The output of the computer data are included as tables in the appendix. Both Chapter 4 and 5 are joint work with Guozhen Wang. In Chapter 6, I will discuss the proof of Theorem 1.0.5 using Theorem 1.0.4. Chapter 6 is joint with Bogdan Gheorghe and Guozhen Wang.

CHAPTER 2

THE STRONG KERVAIRE INVARIANT PROBLEM IN DIMENSION 62

This chapter is published as (88).

2.1 Introduction and main results

The Kervaire invariant problem is one of the most interesting problems that relates geometric topology and stable homotopy theory. One way of formulating it, due to Browder (14), is in terms of the classical Adams spectral sequence (ASS) at the prime 2:

For each n, the element $h_n^2 \in Ext^{2,2^{n+1}-2}$ survives in the ASS.

If h_n^2 survives, we denote the corresponding detecting elements in homotopy by $\theta_n \in \pi_{2^{n+1}-2}S^0$ and we say that θ_n exists. The strong Kervaire invariant problem for n is the following.

θ_n exists, and there exists a θ_n such that $2\theta_n = 0$.

It is well-known that the first three Kervaire invariant elements θ_1, θ_2 and θ_3 can be chosen to be η^2, ν^2 and σ^2. And they all have order 2. Mahowald and Tangora (50) showed that θ_4 exists and $2\theta_4 = 0$ by an ASS computation. In (7), Barratt, Jones and Mahowald showed that θ_5 exists by constructing a 9-cell complex and using the Peterson-Stein formula. Recently, using equivariant homotopy technology, Hill, Hopkins and Ravenel (30) in their marvelous paper showed that θ_n does not exist for all $n \geq 7$, which left the existence of θ_6 as the only open case.

In (8), Barratt, Jones and Mahowald gave the following inductive approach to the strong Kervaire invariant problem:

Theorem 2.1.1. *Suppose that there exists an element θ_n such that $2\theta_n = 0$ and $\theta_n^2 = 0$. Then there exists an element θ_{n+1} with $2\theta_{n+1} = 0$.*

In this chapter, we prove the following:

Theorem 2.1.2. $\theta_4^2 = 0$.

Since θ_4 is unique and $2\theta_4 = 0$, we have the following corollary:

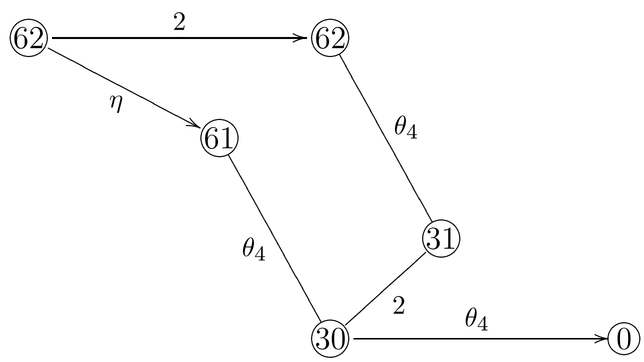
Corollary 2.1.3. *θ_5 exists and there exists a θ_5 such that $2\theta_5 = 0$.*

Remark 2.1.4. In (53), R. J. Milgram claims to show that under the same condition as in Theorem 1.1, one has θ_{n+2} exists. If this were true, then we would have that θ_6 exists. However, Milgram's argument fails because of a computational mistake (17).

Remark 2.1.5. Note that if one can further prove that the same θ_5 has the property $\theta_5^2 = 0$, then Theorem 1.1 will imply the open case θ_6 exists and that there exists a θ_6 such that $2\theta_6 = 0$.

For the case θ_5, Lin (46) shows that there exists a θ_5 such that $2\theta_5 = 0$ based on a computation of the Toda bracket $\langle \theta_4, 2, \sigma^2 \rangle$. Based on the same Toda bracket but a different computational result, Kochman (43) also shows that $\theta_4^2 = 0$ and hence that there exists a θ_5 such that $2\theta_5 = 0$. Recently, Isaksen (36) computed this Toda bracket using more straightforward arguments. His result contradicts the results of both Lin and Kochman. For more details about where Lin and Kochman's arguments fails, see Remark 3.4. Our proof uses Isaksen's computation. Since Isaksen's computation of $\langle \theta_4, 2, \sigma^2 \rangle$ gives a more complicated answer than the earlier claims, we must study several other Toda brackets to prove $\theta_4^2 = 0$.

Knowing $\theta_4^2 = 0$, we give a second proof of the existence of θ_5. In (7), Barratt, Jones and Mahowald constructed a 9-cell complex X', and maps $f' : S^{62} \to X'$, $g' : X' \to S^0$, such that the composite $g' \circ f' : S^{62} \to S^0$ realizes a θ_5. We simplify this 9-cell complex X' into a 4-cell complex X, and construct maps $f : S^{62} \to X$, $g : X \to S^0$ as indicated in the following cell diagram. We follow Barratt, Jones and Mahowald's notation of cell diagrams.



Here each circle represents a cell. The number in each circle represents the dimension of that cell. The middle 4 cells represent the cell structure of X, where the three lines without arrow heads represent attaching maps of X. The map g is an extension of θ_4, and the map f is a co-extension of $\eta \vee 2$. In other words, if we restrict the map g on the bottom cell of X: $g|_{S^{30}} : S^{30} \to S^0$, we have θ_4. If we pinch down the 31-skeleton of X: $p : X \to S^{61} \vee S^{62}$, then the composite $p \circ f : S^{62} \to S^{61} \vee S^{62}$ is $\eta \vee 2$. For more details about cell diagrams, see (7).

Theorem 2.1.6. *The composite of maps $g \circ f : S^{62} \to S^0$ realizes a θ_5.*

Proof. We first show that we can form this cell diagram. For primary obstructions, we have $2\theta_4 = 0$ and $\theta_4^2 = 0$. For secondary obstructions, we have $\eta\theta_4 \in \langle 2, \theta_4, 2\rangle$ and $0 \in \langle \theta_4, 2, \theta_4\rangle$. The latter is shown in (7). It is straightforward to check that the following two facts are true: for $i \leq 4$ the functional cohomology operations

$$Sq_g^{2^i} : H^0 S^0 \longrightarrow H^{2^i - 1} X$$

are all zero, while $Sq^{32}_g : H^0 S^0 \to H^{31} X$ is nonzero; the functional cohomology operation Sq^{32}_f is nonzero on $Sq^{32}_g H^0 S^0 = H^{31} X$. Note that all cohomology is understood to have mod 2 coefficients. As used in (7), it follows from the Peterson-Stein formula ((59),(66)) that the composite $g \circ f$ is detected by the secondary cohomology operation $\phi_{5,5}$. Therefore $g \circ f$ realizes a θ_5. □

We present the proof of Theorem 2.1.2 in Section 2.2. The proof uses several theorems and lemmas whose proofs we postpone. We include Isaksen's computation of $\langle \theta_4, 2, \sigma^2 \rangle$ in Section 2.3 for completeness. In Section 2.4, we discuss two more Toda brackets in the 45-stem, namely $\langle \theta_4, 2, \kappa \rangle$ and $\langle \theta_4, 2, \sigma^2 + \kappa \rangle$. The proof of the main theorem depends on the computation of the latter bracket. We give a modified 4-fold Toda bracket for θ_4 in Section 2.5. We complete our proof of the main theorem by proving several lemmas in Section 2.6.

2.2 The proof of the main theorem

We will use the following Toda brackets to prove Theorem 2.1.2.

Theorem 2.2.1. $\langle \theta_4, 2, \sigma^2 + \kappa \rangle$ *contains* 0 *with indeterminacy* $\{0, \rho_{15}\theta_4\}$.

Theorem 2.2.2. $\theta_4 = \langle 2, \sigma^2 + \kappa, 2\sigma, \sigma \rangle$ *with zero indeterminacy.*

Lemma 2.2.3. $\sigma\pi_{53} = 0$.

Lemma 2.2.4. $\langle \rho_{15}\theta_4, 2\sigma, \sigma \rangle = 0$ *with zero indeterminacy.*

We postpone the proof of Theorem 2.2.1 to Section 2.4, the proof of Theorem 2.2.2 to Section 2.5 and the proofs of Lemma 2.2.3 and 2.2.4 to Section 2.6. Now we present the proof of Theorem 2.1.2.

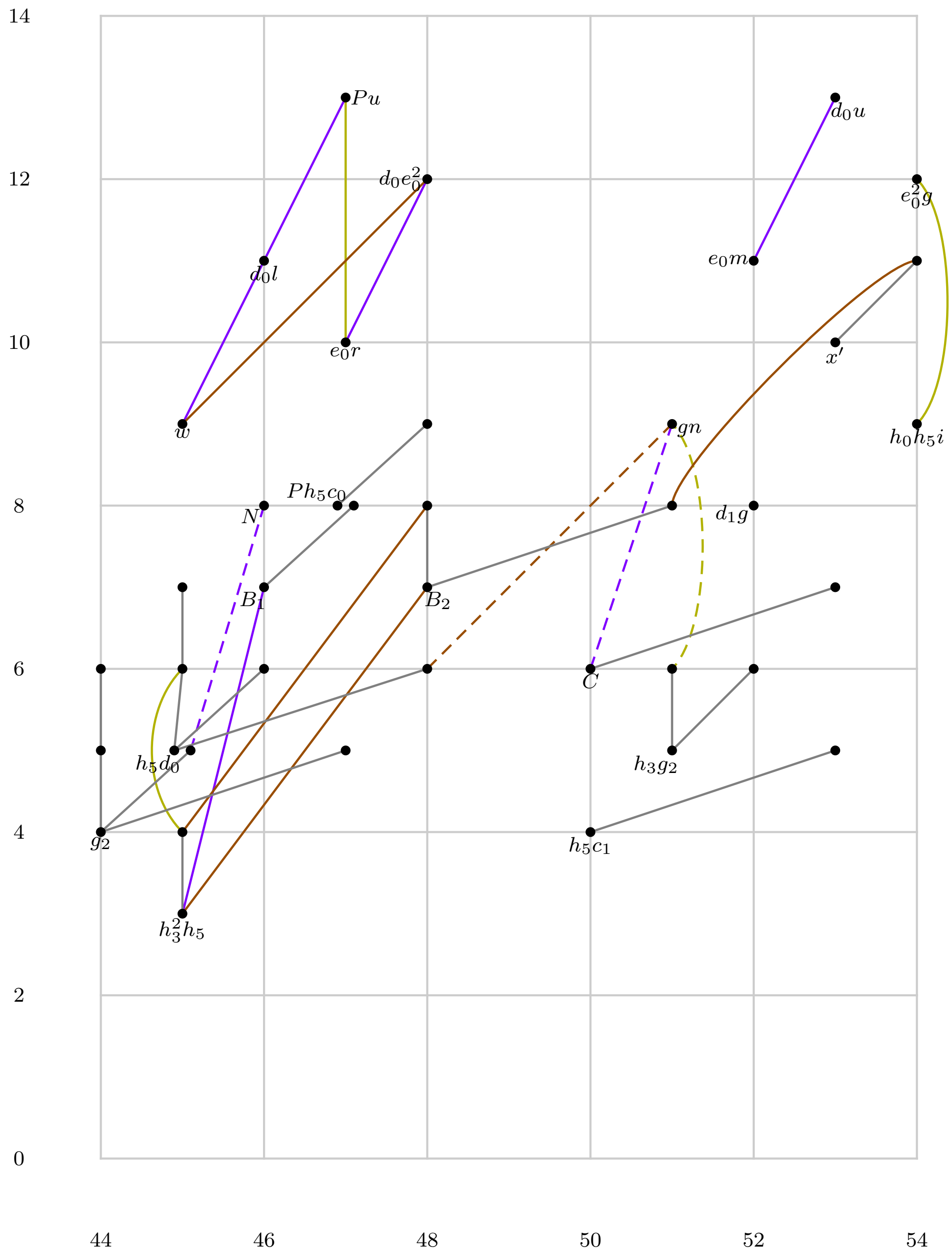
Proof. Following Theorems 2.2.1 and 2.2.2, we have

$$
\begin{aligned}
\theta_4^2 &= \theta_4\langle 2, \sigma^2+\kappa, 2\sigma, \sigma\rangle \\
&\subseteq \langle\langle\theta_4, 2, \sigma^2+\kappa\rangle, 2\sigma, \sigma\rangle \\
&= \text{the union of } \ \langle 0, 2\sigma, \sigma\rangle \text{ and } \langle\rho_{15}\theta_4, 2\sigma, \sigma\rangle
\end{aligned}
$$

By Lemma 2.2.3 and Lemma 2.2.4 above, both brackets contain a single element zero. Therefore, we have that $\theta_4^2 = 0$. □

If a is a surviving cycle in ASS, we use $\{a\}$ to denote the set of elements in the homotopy group that are detected by a. For elements in the E_∞-page of the ASS, we include part of Isaksen's chart (36).

Figure 2.1: The Adams E_∞-page of stems 44 to 54



We do not include elements in filtration higher than 14. Those elements are detected by the $K(1)$-local sphere, and are not relevant to our proof. Here we use colored lines to denote nontrivial extensions. For example, the line between Pu and e_0r indicates that $2\{e_0r\}$ is nontrivial and is detected by Pu. The 2, η and ν-extensions are completely known in this range with a few exceptions that are indicated by dashed lines.[1] But these extensions are irrelevant to our purpose.

2.3 A Toda bracket $\langle\theta_4, 2, \sigma^2\rangle$

The following theorem is due to Isaksen (35). For completeness, we include the proof.

Theorem 2.3.1. *$\langle\theta_4, 2, \sigma^2\rangle$ contains an element of order 2 that can be detected by $h_0h_4^3$.*

Remark 2.3.2. Before presenting the proof, we mention that the indeterminacy of this Toda bracket is well-known. Namely, it is the set $\{0, \rho_{15}\theta_4\}$, where ρ_{15} is the generator of ImJ in π_{15}, and is detected by $h_0^3h_4$. Furthermore, $\rho_{15}\theta_4 \neq 0$ is detected by $h_0^2h_5d_0$. This is shown by Tangora in (78).

Proof. In the Adams E_3-page, we have $\langle h_4^2, h_0, h_3^2\rangle = h_4^2h_4 + h_5h_3^2 = 0$ in the Adams filtration 3. Therefore, by the Moss Theorem (60), there is an element in $\langle\theta_4, 2, \sigma^2\rangle$ that is detected by some element of filtration at least 4. Since the nontrivial element in the indeterminacy has filtration 7, any element in $\langle\theta_4, 2, \sigma^2\rangle$ has filtration at least 4. We have

$$2\langle\theta_4, 2, \sigma^2\rangle = \langle 2, \theta_4, 2\rangle\sigma^2 = \eta\theta_4\sigma^2 = 0.$$

Note that the indeterminacy of $\langle 2, \theta_4, 2\rangle\sigma^2$ is $2\sigma^2\pi_{31} = 0$. Therefore, any element in $\langle\theta_4, 2, \sigma^2\rangle$ has order 2.

Now consider the product $\nu_4\theta_4$.

$$\nu_4\theta_4 = \langle\sigma, \nu, \sigma\rangle\theta_4 \subseteq \langle\sigma, \nu, \sigma\theta_4\rangle \subseteq \langle\sigma, \nu, \{x\}\rangle.$$

1. Added in proof. It is now known that all of these nontrivial extensions do in fact exist.

Here, since $2\theta_4 = 0$, we can ignore the difference between ν_4, which is by definition $\langle \nu, \sigma, 2\sigma \rangle$, and $\langle \sigma, \nu, \sigma \rangle = 7\nu_4$. In the Adams E_2-page, we have $h_2h_5d_0 = \langle h_3, h_2, x \rangle$ with zero indeterminacy. In fact, this follows from

$$h_2\langle h_3, h_2, x\rangle = \langle h_2, h_3, h_2\rangle x = h_3^2 x = h_2^2 h_5 d_0.$$

Therefore, $\nu_4\theta_4$ is contained in $\langle \sigma, \nu, \{x\}\rangle \subseteq \{h_2h_5d_0\}$.

On the other side, $\nu_4\theta_4$ is contained in $\theta_4\langle 2, \sigma^2, \nu\rangle = \langle \theta_4, 2, \sigma^2\rangle\nu$. For the indeterminacy, note that $\rho_{15}\theta_4\nu = 0$. Therefore, we actually have

$$\nu_4\theta_4 = \langle \theta_4, 2, \sigma^2\rangle\nu.$$

Combining this with the fact that $\nu_4\theta_4$ is also contained in $\{h_2h_5d_0\}$, we deduce that there exists an element in $\langle \theta_4, 2, \sigma^2\rangle$ such that ν times it is detected by $h_2h_5d_0$, which has filtration 6. Therefore, $\langle \theta_4, 2, \sigma^2\rangle$ contains an element with filtration at most 5. Furthermore, it cannot be detected by h_1g_2, which has filtration 5, since otherwise the ν multiple won't be detected by $h_2h_5d_0$. Therefore, the statement of the theorem is the only possibility left. □

Remark 2.3.3. Another way to describe the statement of this theorem is the following:

$$\langle \theta_4, 2, \sigma^2\rangle \text{ contains an order 2 element of the form } 2\alpha + \beta,$$

where α is detected by $h_3^2h_5$ and β is detected by h_5d_0. Note that the nontrivial 2-extension in the 45-stem means that there exist elements α and γ, which are detected by $h_3^2h_5$ and h_5d_0 respectively, such that $4\alpha = 2\gamma$. Since γ has order 8, one can choose β to be $-\gamma = 7\gamma$, so that $2\alpha + \beta$ has order 2.

Remark 2.3.4. In (47), Lin showed that this bracket contains 0. The step that rules out the element Isaksen got is invalid. In (43), Kochman showed that this bracket contains $\eta\{g_2\}$ or 0. His argument failed because essentially of the inconsistency of the $\nu-$extension on

$\{h_2h_5d_0\}$ and the $\sigma-$extension on $\{h_0^2g_2\}$, which allowed him to eliminate the right element. The inconsistency is discussed in (35).

2.4 More about the 45-stem

We first consider the Toda bracket $\langle\theta_4, 2, \kappa\rangle$ in π_{45}.

Lemma 2.4.1. *$\langle\theta_4, 2, \kappa\rangle$ contains an element of order 2 that can be detected by $h_0h_4^3$.*

Proof. The Adams differential $d_3(h_0h_4) = h_0d_0$ implies that in the Adams E_4-page, $\langle h_4^2, h_0, d_0\rangle = h_0h_4^3$ in the Adams filtration 4. Then by the Moss convergence theorem (60), there is an element in $\langle\theta_4, 2, \kappa\rangle$ that is detected by $h_0h_4^3$. From

$$2\langle\theta_4, 2, \kappa\rangle = \langle 2, \theta_4, 2\rangle\kappa = \eta\theta_4\kappa = 0,$$

we know that any element in $\langle\theta_4, 2, \kappa\rangle$ has order 2. The indeterminacy of $\langle 2, \theta_4, 2\rangle\kappa$ is $2\kappa\pi_{31} = 0$. Here we also used that $\kappa\theta_4 = 0$, which is known for filtration reasons. In fact, since $d_0h_4^2 = 0$ in Ext^6, $\kappa\theta_4$ must be detected by an element of filtration at least 7. However, in the 44-stem of the E_∞-page, there are no elements of filtration 7 or higher. Therefore $\langle\theta_4, 2, \kappa\rangle$ contains an element of order 2 that can be detected by $h_0h_4^3$. □

Remark 2.4.2. The indeterminacy of this bracket is the same as that of $\langle\theta_4, 2, \sigma^2\rangle$, i.e., $\{0, \rho_{15}\theta_4\}$. In fact, π_{31} is generated by $\eta\theta_4$, $\{n\}$ and ρ_{31}, where ρ_{31} is the generator of ImJ in π_{31}, and is detected by $h_0^{10}h_5$. Since $\kappa\theta_4 = 0$, $\eta\kappa\theta_4 = 0$. Again for filtration reasons, $\kappa\{n\} = 0$ and $\kappa\rho_{31} = 0$. Therefore $\kappa\pi_{31} = 0$. This shows that the indeterminacy of $\langle\theta_4, 2, \kappa\rangle$ is $\{0, \rho_{15}\theta_4\}$.

Although both $\langle\theta_4, 2, \kappa\rangle$ and $\langle\theta_4, 2, \sigma^2\rangle$ contain an element of order 2 that can be detected by $h_0h_4^3$, we do not necessarily know if they have an element in common. The following theorem confirms that they do.

Now we restate Theorem 2.2.1.

Theorem 2.4.3. $\langle \theta_4, 2, \sigma^2 + \kappa \rangle$ *contains* 0 *with indeterminacy* $\{0, \rho_{15}\theta_4\}$.

We need the following lemma to prove the theorem.

Lemma 2.4.4. $\sigma^2 \pi_{33} = 0$.

Proof. We know that π_{33} is generated by $\eta\eta_5$, $\nu\theta_4$, $\eta\{q\}$, $\eta^2\rho_{31}$ and $\{P^4h_1\}$. Since $\eta\sigma^2 = 0$ and $\nu\sigma^2 = 0$, we only need to show that $\{P^4h_1\}\sigma^2$=0. In fact, we have

$$\{P^4h_1\}\sigma^2 = \eta\rho_{39}\sigma = 0$$

for filtration reasons. Here ρ_{39} is the generator of ImJ in π_{39}, and is detected by $P^2h_0^2i$. Therefore, $\sigma^2\pi_{33} = 0$. □

Now we present the proof of Theorem 2.4.3.

Proof. The indeterminacy is straightforward, as in Remark 2.4.2.

Since all elements in $\langle \theta_4, 2, \kappa \rangle$ and $\langle \theta_4, 2, \sigma^2 \rangle$ have order 2 and can be detected by $h_0h_4^3$ in the Adams filtration 4, elements in $\langle \theta_4, 2, \sigma^2 + \kappa \rangle$ must be detected by elements of filtration at least 5 and have order 2. To prove the theorem, we need to rule out both $\{w\}$ and $\eta\{g_2\}$.

For $\{w\}$, by Lemma 2.4.4, we have that

$$\eta^2\langle \theta_4, 2, \sigma^2 \rangle = \langle \eta^2, \theta_4, 2 \rangle \sigma^2 \in \pi_{33}\sigma^2 = 0.$$

Next we have that

$$\eta^2\langle \theta_4, 2, \kappa \rangle = \theta_4\langle 2, \kappa, \eta^2 \rangle.$$

In the Adams E_4-page, we have that $\langle h_0, d_0, h_1^2 \rangle = h_0h_4h_1^2 = 0$ in the Adams filtration 4. Then the Moss Theorem tells us that $\langle 2, \kappa, \eta^2 \rangle$ might contain a nontrivial element of higher filtration, namely a combination of $\nu\kappa, \eta^2\rho_{15}$ and $\{P^2h_1\}$. Note that we have that $\nu\kappa\theta_4 = 0$ and by Lemma 2.6.1 we have that $\eta^2\rho_{15}\theta_4 = 0$. To show that $\{P^2h_1\}\theta_4 = 0$, we first show that $\{Ph_1\}\theta_4 = 0$.

In fact, $\{Ph_1\}\theta_4 \in \langle \eta, 8\sigma, 2\rangle\theta_4 = \eta\langle 8\sigma, 2, \theta_4\rangle$, which contains 0. This holds since $\eta\langle 8\sigma, 2, \theta_4\rangle$ intersects $\eta\{h_0^3 h_3 h_5\}$, which contains a single element zero. The indeterminacy is $\eta\pi_8\theta_4 = 0$. This gives that $\{Ph_1\}\theta_4 = 0$. Then we have

$$\{P^2h_1\}\theta_4 \in \theta_4\langle\{Ph_1\}, 2, 8\sigma\rangle = \langle\theta_4, \{Ph_1\}, 2\rangle 8\sigma \subseteq \pi_{40}8\sigma = 0.$$

Therefore, no matter what $\langle 2, \kappa, \eta^2\rangle$ equals, we always have that

$$\eta^2\langle\theta_4, 2, \kappa\rangle = \langle 2, \kappa, \eta^2\rangle\theta_4 \text{ contains } 0.$$

The indeterminacy of $\eta^2\langle\theta_4, 2, \kappa\rangle$ is zero since $\eta^2\theta_4 = 0$ and $\eta^2\kappa = 0$. Then

$$\eta^2\langle\theta_4, 2, \kappa\rangle = 0.$$

Therefore,

$$\eta^2\langle\theta_4, 2, \sigma^2 + \kappa\rangle = 0.$$

Then the fact that $\eta^2\{w\} \neq 0$ rules out $\{w\}$, since otherwise we would have that $\eta^2\langle\theta_4, 2, \sigma^2 + \kappa\rangle = \eta^2\{w\} \neq 0$.

For $\eta\{g_2\}$, first note that $\sigma\eta\{g_2\} \neq 0$ is detected by $h_1h_3g_2$. We have that

$$\langle\theta_4, 2, \kappa\rangle\sigma = \theta_4\langle 2, \kappa, \sigma\rangle \subseteq \theta_4\pi_{22} = 0.$$

In fact, π_{22} is generated by $\nu\overline{\sigma}$ and $\eta^2\overline{\kappa}$. We have that $\eta^2\overline{\kappa}\theta_4 = 0$ and $\nu\overline{\sigma}\theta_4 = 0$ for filtration reasons. As a remark, we can actually prove that $\langle 2, \kappa, \sigma\rangle = \nu\overline{\sigma}$ by studying the cofiber of 2, but we don't need this fact here.

On the other side, as explained in Remark 2.3.3, $\langle\theta_4, 2, \sigma^2\rangle$ contains $2\alpha+\beta$. Therefore,

$$\langle\theta_4, 2, \sigma^2\rangle\sigma \text{ contains } 2\alpha\sigma+\beta\sigma.$$

We have that $2\alpha\sigma \in 2\pi_{52} = 0$. In the Adams E_3-page, we compute directly that $\langle h_0, h_4^2, d_0\rangle = h_5 d_0$. Then Moss's Theorem shows that $\langle 2, \theta_4, \kappa\rangle$ contains an element that equals to β plus possibly higher filtration terms. Note that $\sigma\{w\} = 0$ by using tmf. In fact, if $\sigma\{w\} \neq 0$, the only possibility is that $\sigma\{w\}$ is detected by $\{e_0 m\}$. This implies that $\eta\sigma\{w\} = \kappa\{u\}$ because of the two nontrivial η-extensions. Since both $\eta\{w\}$ and $\kappa\{u\}$ are detected by tmf and $\sigma = 0$ in $\pi_* tmf$, mapping this relation into tmf gives a contradiction. Besides, from tmf, we know that $\{d_0 l\}$ detects $\kappa\{q\}$, then the contradiction also follows from $\kappa\sigma = 0$. See (10),(29) for example.

Then we have that

$$\beta\sigma \in \langle 2, \theta_4, \kappa\rangle\sigma = 2\langle\theta_4, \kappa, \sigma\rangle \subseteq 2\pi_{52} = 0.$$

Therefore, $\langle\theta_4, 2, \sigma^2\rangle\sigma$ contains $2\alpha\sigma+\beta\sigma = 0$. Note that $\rho_{15}\theta_4\sigma \in \theta_4\pi_{22} = 0$, the indeterminacy is hence zero. Then we have that

$$\langle\theta_4, 2, \sigma^2\rangle\sigma = 0.$$

Therefore,

$$\langle\theta_4, 2, \sigma^2+\kappa\rangle\sigma = 0.$$

Combined with the fact that $\eta\{g_2\}\sigma \neq 0$, this rules out $\eta\{g_2\}$.

This completes the proof.

□

Remark 2.4.5. $\sigma^2+\kappa$ is another element in π_{14} that deserves to be called θ_3.

Remark 2.4.6. We can actually show that the bracket $\langle 2, \theta_4, \eta^2 \rangle$ contains $\eta\eta_5 + \nu\theta_4$ with indeterminacy $\{0, \eta^2\rho_{31}\}$.

2.5 A modified 4-fold Toda bracket for θ_4

We have the following well-known 4-fold Toda brackets for θ_4. See (9),(43),(44) for example.

$$\begin{aligned} \theta_4 &= \langle 2, \sigma^2, 2, \sigma^2 \rangle \\ &= \langle 2, \sigma^2, \sigma^2, 2 \rangle \\ &= \langle 2\sigma, \sigma, 2\sigma, \sigma \rangle \\ &= \langle 2, \sigma^2, 2\sigma, \sigma \rangle \end{aligned}$$

All of them have zero indeterminacy. This is partially discussed in (9),(43),(44). For completeness, we include a proof here.

Lemma 2.5.1. *All four Toda brackets above have zero indeterminacy.*

Proof. In general, suppose a 4-fold Toda bracket $\langle \alpha_1, \alpha_2, \alpha_3, \alpha_4 \rangle$ is defined, where $\alpha_i \in \pi_{n_i}$. Then its indeterminacy is contained in the union of three types of 3-fold Toda brackets:

$$\langle \alpha_1, \alpha_2, \pi_{n_3+n_4+1} \rangle, \langle \alpha_1, \pi_{n_2+n_3+1}, \alpha_4 \rangle \text{ and } \langle \pi_{n_1+n_2+1}, \alpha_3, \alpha_4 \rangle.$$

In our case, the indeterminacy for all of them is contained in the union of the following eight brackets:

$$\langle \pi_{15}, 2, \sigma^2 \rangle, \langle 2, \pi_{15}, \sigma^2 \rangle, \langle 2, \sigma^2, \pi_{15} \rangle, \langle 2, \pi_{29}, 2 \rangle,$$

$$\langle \pi_{15}, 2\sigma, \sigma \rangle, \langle 2\sigma, \pi_{15}, \sigma \rangle, \langle 2\sigma, \sigma, \pi_{15} \rangle, \langle 2, \pi_{22}, \sigma \rangle.$$

We will show that they are all zero. Note that $\pi_{30} \cong \mathbb{Z}/2$ and is generated by θ_4, which is indecomposable. So for each of them, we only need to show that it does not contain θ_4. They all follow for filtration reasons.

For $\langle \pi_{15}, 2, \sigma^2 \rangle, \langle 2, \sigma^2, \pi_{15} \rangle, \langle \pi_{15}, 2\sigma, \sigma \rangle$ and $\langle 2\sigma, \sigma, \pi_{15} \rangle$, the corresponding Massey products are all well-defined on the Adams E_3-page. Since π_{15} is generated by elements of filtration at least 4, the Massey products all take values in filtration at least 5. Therefore, by the Moss Theorem, all of them are all zero.

For $\langle 2, \pi_{15}, \sigma^2 \rangle$ and $\langle 2\sigma, \pi_{15}, \sigma \rangle$, the corresponding Massey products are all well-defined on the Adams E_2-page. Since π_{15} is generated by elements of filtration at least 4, the Massey products all take values in filtration at least 6. Therefore, by the Moss Theorem, all of them are all zero.

For $\langle 2, \pi_{22}, \sigma \rangle$, there are essentially two Toda brackets to check: $\langle 2, \nu\overline{\sigma}, \sigma \rangle$ and $\langle 2, \eta^2\overline{\kappa}, \sigma \rangle$, where $\nu\overline{\sigma}$ is detected by h_2c_1. Both brackets have zero indeterminacy. We have that

$$\langle 2, \nu\overline{\sigma}, \sigma \rangle = \langle 2, \overline{\sigma}, \nu\sigma \rangle = \langle 2, \overline{\sigma}, 0 \rangle = 0,$$

and that

$$\langle 2, \eta^2\overline{\kappa}, \sigma \rangle = \langle 2, \eta^2, \overline{\kappa}\sigma \rangle = \langle 2, \eta^2, 0 \rangle = 0.$$

Here we used the fact that $2\overline{\sigma} = 0$ and $\overline{\kappa}\sigma = 0$.

At last, $\langle 2, \pi_{29}, 2 \rangle = 0$, since $\pi_{29} = 0$. This completes the proof. □

Now we prove a modified 4-fold Toda bracket based on the last one. Again, note that $\pi_{30} \cong \mathbb{Z}/2$ and is generated by θ_4.

Theorem 2.5.2. $\theta_4 = \langle 2, \sigma^2 + \kappa, 2\sigma, \sigma \rangle$ *with zero indeterminacy.*

Proof. We have $\langle \sigma^2 + \kappa, 2\sigma, \sigma \rangle \subseteq \pi_{29} = 0$. And

$$\langle 2, \sigma^2 + \kappa, 2\sigma \rangle \supseteq \langle 2, \sigma^2 + \kappa, 2 \rangle \sigma \ni \eta(\sigma^2 + \kappa)\sigma = 0.$$

The indeterminacy of the bracket $\langle 2, \sigma^2 + \kappa, 2\sigma \rangle$ is $2\pi_{22} + 2\sigma\pi_{15} = 0$, and we have $\langle 2, \sigma^2 +$

$\kappa, 2\sigma\rangle = 0$. Therefore, this 4-fold Toda bracket is strictly defined, and the indeterminacy is

$$\langle 2, \sigma^2 + \kappa, \pi_{15}\rangle + \langle 2, \pi_{22}, \sigma\rangle + \langle \pi_{15}, 2\sigma, \sigma\rangle.$$

Note that $\langle 2, \sigma^2 + \kappa, \pi_{15}\rangle = 0$ for filtration reasons as in the proof of Lemma 5.1. The other two parts of the indeterminacy follow from the indeterminacy of $\langle 2, \sigma^2, 2\sigma, \sigma\rangle$, which we know is zero. Then the theorem follows from the next lemma and the fact that $\theta_4 = \langle 2, \sigma^2, 2\sigma, \sigma\rangle$. □

Lemma 2.5.3. $\langle 2, \kappa, 2\sigma, \sigma\rangle = 0$ *with zero indeterminacy.*

Proof. Again, $\langle \kappa, 2\sigma, \sigma\rangle \subseteq \pi_{29} = 0$. And

$$\langle 2, \kappa, 2\sigma\rangle \supseteq \langle 2, \kappa, 2\rangle\sigma \ni \eta\kappa\sigma = 0.$$

The indeterminacy of $\langle 2, \kappa, 2\sigma\rangle$ is zero. Therefore, this 4-fold Toda bracket is strictly defined. Again, $\langle 2, \kappa, \pi_{15}\rangle = 0$ for filtration reasons. And the other two parts of the indeterminacy are zero, which follows from the indeterminacy of $\langle 2, \sigma^2, 2\sigma, \sigma\rangle$.

To see this bracket contains zero, we multiply by ν.

$$\langle 2, \kappa, 2\sigma, \sigma\rangle\nu \subseteq \langle 2, \kappa, \langle 2\sigma, \sigma, \nu\rangle\rangle = \langle 2, \kappa, \nu_4\rangle.$$

Since in the Adams E_4-page $\langle h_0, d_0, h_2h_4\rangle = 0$ in the Adams filtration 4, there is an element in $\langle 2, \kappa, \nu_4\rangle$ that is detected by an element in filtration strictly higher than 4. The indeterminacy of this bracket is $2\pi_{33} + \nu_4\pi_{15} = \nu_4\pi_{15}$, which also contains elements in filtration strictly higher than 4. On the other side, $\nu\theta_4$ is detected by p in Ext^4. Therefore $\langle 2, \kappa, \nu_4\rangle$ does not contain $\nu\theta_4$. Then the lemma follows from the fact that $\pi_{30} \cong \mathbb{Z}/2$ and is generated by θ_4. □

Remark 2.5.4. We can show directly that $\langle 2, \kappa, \nu_4\rangle = 0$ with zero indeterminacy.

2.6 A few proofs

We first prove Lemma 2.2.3 which states that $\sigma\pi_{53} = 0$.

Proof. As shown in (35), $\pi_{53} \cong \mathbb{Z}/2 \oplus \mathbb{Z}/2 \oplus \mathbb{Z}/2 \oplus \mathbb{Z}/2$. One set of generators can be chosen to be elements in $\nu\{h_5c_1\}, \nu\{C\}, \epsilon\{h_3^2h_5\}$ and $\kappa\{u\}$ respectively. Note that x' detects $\epsilon\{h_3^2h_5\}$. Then the lemma follows from $\nu\sigma = 0, \epsilon\sigma = 0$ and $\kappa\sigma = 0$. ☐

The following lemma is shown by Tangora in (78). We first sketch his proof, then give a more direct proof.

Lemma 2.6.1. $\rho_{15}\theta_4 = 2\sigma\{h_0^2h_3h_5\}$.

Proof. Tangora first showed that $\rho_{15}\theta_4 \neq 0$ and is detected by $h_0^2h_5d_0$. We have

$$\rho_{15}\theta_4 = \rho_{15}\langle\sigma, 2\sigma, \sigma, 2\sigma\rangle = \langle\rho_{15}, \sigma, 2\sigma, \sigma\rangle 2\sigma.$$

Then the only possibility is that $\langle\rho_{15}, \sigma, 2\sigma, \sigma\rangle$ is detected by $h_0^2h_3h_5$.

We present another proof. In the Adams E_3-page, we have $\langle h_3, h_0h_3, h_0^3\rangle = h_0^3h_4$. Therefore, ρ_{15} is contained in $\langle\sigma, 2\sigma, 8\rangle$. Then we have

$$\begin{aligned}\rho_{15}\theta_4 &= \langle\sigma, 2\sigma, 8\rangle\theta_4 \\ &= \sigma\langle 2\sigma, 8, \theta_4\rangle \\ &= \sigma\langle 8\sigma, 2, \theta_4\rangle \\ &= \sigma\{h_0^3h_3h_5\} \\ &= 2\sigma\{h_0^2h_3h_5\}.\end{aligned}$$

For the first equation, $\langle\sigma, 2\sigma, 8\rangle\theta_4$ has no indeterminacy, hence the equality. For the last equation, the difference between $\{h_0^3h_3h_5\}$ and $2\{h_0^2h_3h_5\}$ contains elements of higher filtration, namely $\eta\sigma\theta_4$ in this case. The equality holds since $\eta\sigma^2\theta_4 = 0$. ☐

Now we prove Lemma 2.2.4 which states that $\langle \rho_{15}\theta_4, 2\sigma, \sigma\rangle = 0$ with zero indeterminacy.

Proof. The indeterminacy is $\rho_{15}\theta_4\pi_{15} + \sigma\pi_{53} = \rho_{15}\theta_4\pi_{15}$. π_{15} is generated by $\eta\kappa$ and ρ_{15}. We have $\rho_{15}^2 = 0$ and $\kappa\theta_4 = 0$ both for filtration reasons. Therefore the indeterminacy is equal to $\rho_{15}\theta_4\pi_{15} = 0$.

By Lemma 2.6.1, $\langle \rho_{15}\theta_4, 2\sigma, \sigma\rangle = \langle 2\sigma\{h_0^2h_3h_5\}, 2\sigma, \sigma\rangle$ contains $\sigma\langle 2\{h_0^2h_3h_5\}, 2\sigma, \sigma\rangle$. Note that $\langle 2\{h_0^2h_3h_5\}, 2\sigma, \sigma\rangle \subseteq \pi_{53}$ and $\sigma\pi_{53} = 0$. This completes the proof. $\square$

CHAPTER 3

THE RP^∞-METHOD AND THE 61-STEM

This chapter is joint with Guozhen Wang (86).

3.1 Introduction

In 1904, Poincaré proposed the following famous conjecture:

Conjecture 3.1.1. *Let M be a closed 3-manifold. If M is simply connected, then M is homeomorphic to the 3-sphere.*

This is the celebrated Poincaré conjecture. It was proved by Perelman (65) in 2002, using geometric analytic methods. Note that a closed 3-manifold is simply connected if and only if it is homotopy equivalence to the 3-sphere.

This conjecture can be generalized to higher dimensions as the following question.

Question 3.1.2. Let M be a closed n-manifold. Suppose M is homotopy equivalent to S^n. Is M homeomorphic to S^n?

The answer turns out to be yes for all dimensions. For $n = 4$, it was proved by Freedman (24) in 1982. For $n \geq 5$, it was proved by Smale (75) in 1962, using the theory of h-cobordisms, and by Newman (63) in 1966 and by Connell (22) in 1967. The statement Smale proved assumes further that the n-manifold M admits a smooth structure, while the statement Newman and Connell proved does not require such a condition.

In summary, we have the following theorem:

Theorem 3.1.3. *((75; 63; 22; 24; 65)) Any closed n-manifold that is homotopy equivalent to S^n is homeomorphic to S^n.*

We can also generalize this question into the smooth category.

Question 3.1.4. Let M be a closed n-manifold. Suppose M is homeomorphic to S^n. Is M diffeomorphic to S^n?

For $n = 3$, the answer is yes. It is due to Moise (58) that every closed 3-manifold has a unique smooth structure. In particular, the 3-sphere has a unique smooth structure. For $n = 4$, this question is wildly open.

For higher dimensions, Milnor (56) constructed an exotic smooth structure on S^7. Furthermore, Kervaire and Milnor (42) showed that the answer is not true in general for $n \geq 5$.

Since the answer to Question 3.1.4 is not true in general, there come two natural questions:

Question 3.1.5. How many exotic structures are there on S^n?

Question 3.1.6. For which n's does there exist a unique smooth structure on S^n?

Kervaire and Milnor reduced Question 3.1.5 to a computation of the stable homotopy groups of spheres. In fact, Kervaire and Milnor constructed a group Θ_n, which is the group of h-cobordism classes of homotopy n-spheres. The group Θ_n classifies the differential structures on S^n for $n \geq 5$. This group Θ_n has a subgroup Θ_n^{bp}, which consists of homotopy spheres that bound parallelizable manifolds. The relation between Θ_n and π_n (the n-th stable homotopy group of the spheres) can be summarized by the following theorem.

Theorem 3.1.7. *(Kervaire-Milnor (42)) Suppose that $n \geq 5$.*

1. *The subgroup Θ_n^{bp} is cyclic, and has the following order:*

$$|\Theta_n^{bp}| = \begin{cases} 1, & \textit{if } n \textit{ is even}, \\ 1 \textit{ or } 2, & \textit{if } n = 4k+1, \\ 2^{2k-2}(2^{2k-1}-1)B(k), & \textit{if } n = 4k-1. \end{cases}$$

Here $B(k)$ is the numerator of $4B_{2k}/k$ and B_{2k} is the Bernoulli number.

2. *For $n \not\equiv 2$ (mod 4), there is an exact sequence*

$$0 \longrightarrow \Theta_n^{bp} \longrightarrow \Theta_n \longrightarrow \pi_n/J \longrightarrow 0.$$

Here π_n/J is the cokernel of the J-homomorphism.

3. *For $n \equiv 2$ (mod 4), there is an exact sequence*

$$0 \longrightarrow \Theta_n^{bp} \longrightarrow \Theta_n \longrightarrow \pi_n/J \xrightarrow{\Phi} \mathbb{Z}/2 \longrightarrow \Theta_{n-1}^{bp} \longrightarrow 0.$$

Here the map Φ is the Kervaire invariant.

Remark 3.1.8. In the first part of Theorem 3.1.7, the case $n \equiv 3$ (*mod* 4) depends on the computation of the order of the image of the J-homomorphism. The case $n \equiv 1$ (*mod* 4) depends on the Kervaire invariant in dimension $n+1$. The computation of the image of the J-homomorphism at $4k-1$ stems is a special case of the Adams conjecture. The proof was completed by Mahowald (49), and the full Adams conjecture was proved by Quillen (69), Sullivan (76), and by Becker-Gottlieb (11).

For Question 3.1.6, it is clear from Theorem 3.1.7 that, for $n = 4k+3$ with $k \geq 1$, the smooth structure on the n-sphere is never unique. For $n = 4k+1$ with $k \geq 1$, the answer depends on the existence of the Kervaire invariant elements. In 2009, Hill, Hopkins and Ravenel (30) showed that the only dimensions in which the Kervaire invariant elements exist are 2, 6, 14, 30, 62 and possibly 126. That is, in other dimensions, the Kervaire invariant map

$$\pi_n/J \xrightarrow{\Phi} \mathbb{Z}/2$$

in part (3) of Theorem 3.1.7 is always zero and the group Θ_{n-1}^{bp} is $\mathbb{Z}/2$. Therefore, the only odd dimensional spheres that could have a unique smooth structure are $S^1, S^3, S^5, S^{13}, S^{29}, S^{61}$

and S^{125}. Further, the cases S^{13} and S^{29} can be ruled out by May's (52) 3-primary computation of the stable homotopy groups of spheres.

For dimension 61, we have the main theorem of this chapter.

Theorem 3.1.9. *The 2-primary $\pi_{61} = 0$, and therefore the sphere S^{61} has a unique smooth structure.*

We postpone the proof of the first claim of Theorem 3.1.9 to Section 2, and present the proof of the second claim now.

Proof. In (7), Barratt, Jones and Mahowald showed that the Kervaire invariant element θ_5 exists. The second author gave a new proof in (88). By Theorem 3.1.7, this implies that $\Theta_{61}^{bp} = 0$.

At an odd prime p, the first nontrivial element in the cokernel of J is β_1, which lies in the stem $2p^2 - 2p - 2$. (This is proved in Section 4 of (71).) This value is 82 if $p = 7$. For $p = 3$ and $p = 5$, the table in Appendix A3 of Ravenel's green book (71) shows that the cokernel of J in dimension 61 vanishes. Therefore, the cokernel of J in dimension 61 vanishes at all odd primes.

Combining the first claim of Theorem 3.1.9 with Theorem 3.1.7, this proves the second claim of Theorem 3.1.9. □

There is an important corollary of our theorem, regarding the Kervaire invariant element $\theta_5 \in \pi_{62}$.

Corollary 3.1.10. *The Kervaire invariant class $\theta_5 \in \pi_{62}$ is contained in the strictly defined 4-fold Toda bracket $\langle 2, \theta_4, \theta_4, 2\rangle$.*

Proof. We first check this 4-fold Toda bracket is strictly defined. In (88), the second author showed that $\theta_4^2 = 0$. Note that the 3-fold Toda bracket $\langle 2, \theta_4, \theta_4\rangle$ is contained in $\pi_{61} = 0$.

Therefore, this 4-fold Toda bracket is strictly defined. In the Adams E_3 page, we have a Massey product

$$\langle h_0, h_4^2, h_4^2, h_0 \rangle = h_5^2,$$

because of the Adams differential $d_2(h_5) = h_0 h_4^2$. Then the theorem follows from Moss's Theorem (60, Theorem 1.2). □

Remark 3.1.11. When computing stable stems, it is crucial to understand Toda brackets decompositions of multiplicatively indecomposable classes. A theorem of Joel Cohen (21) says that any classes in the stable homotopy groups of spheres can be decomposed as a (matric) Toda bracket starting only from the classes that correspond to the Hopf maps. However, in practice, it is usually hard to find such a description. For the Kervaire invariant class θ_5, our Corollary 3.1.10 gives the first known Toda bracket of it. Note that θ_4 was known to have multiple Toda bracket decompositions using the Hopf maps.

By a theorem of Barratt, Jones and Mahowald (8), if θ_5 has order 2 and $\theta_5^2 = 0$, then θ_6 exists and has order 2. It is proved by the second author (88) that θ_5 has order 2. Our Toda bracket of θ_5 in Corollary 3.1.10 therefore leads us to consider the Toda bracket $\langle \theta_5, 2, \theta_4 \rangle$ in π_{93}, which is in a much lower stem than θ_6 itself. Using obstruction theory as Barratt-Jones-Mahowald did in (7), one can show that if the Toda bracket $\langle \theta_5, 2, \theta_4 \rangle$ contains zero, then θ_6 exists. The Toda bracket of θ_5 in Corollary 3.1.10 has also been very helpful in ongoing work of Isaksen and the authors of extending computations of stable stems.

For dimension 125, we have the following proposition.

Proposition 3.1.12. *The sphere S^{125} does not have a unique smooth structure.*

Proof. This proof uses the Hurewicz image of tmf (the spectrum of topological modular forms). See (10; 29) for computations of the homotopy groups of tmf.

Let $\{w\} \in \pi_{45}$ be the unique homotopy class detected by w in Adams filtration 9. It is known that both $\overline{\kappa} \in \pi_{20}$ and $\{w\}$ are detected by tmf, that is, they map nontrivially under the following map:

$$\pi_* S^0 \longrightarrow \pi_* tmf.$$

We have that $\overline{\kappa}^4\{w\} \neq 0$ in $\pi_{125} tmf$. Therefore, $\overline{\kappa}^4\{w\} \neq 0$ in $\pi_{125} S^0$ and it lies in the cokernel of J. This shows that S^{125} does not have a unique smooth structure. □

Therefore, we have the following corollary.

Corollary 3.1.13. *The only odd dimensional spheres with a unique smooth structure are* S^1, S^3, S^5 *and* S^{61} .

For even dimensions, since the subgroup Θ_n^{bp} is always zero, we need to understand the cokernel of the J-homomorphism.

In (57), Milnor states that up to dimension 64, the only dimensions where the n-sphere has a unique smooth structure are $n = 1, 2, 3, 5, 6, 12, 61$ and possibly $n = 4$. This observation is based on the computation of 2-primary stable homotopy groups of spheres up to the 64 stem by Kochman and Mahowald (45) from 1995. Recently, Isaksen (35) discovered several errors in Kochman and Mahowald's computations, and he was able to give rigorous proofs of computations through the 59 stem. One major correction is that, instead of having order 4, π_{56} is of order 2 and is generated by a class in the image of J. Consequently, we have the following theorem by Isaksen:

Theorem 3.1.14. *The sphere* S^{56} *has a unique smooth structure.*

Proof. It is clear from Theorem 3.1.7 that $\Theta_{56}^{bp} = 0$. Ravenel's computation (71) shows that the cokernel of J in dimension 56 vanishes at odd primes. Recent computation of Isaksen (35) shows that the cokernel of J in dimension 56 vanishes at the prime 2. Then this theorem follows from part (2) of Theorem 3.1.7. □

The technique used by Kochman and Mahowald (45) is quite different from the classical technique used by Barratt, Bruner, Mahowald, May and Tangora (52; 50; 9; 77; 78; 79; 15) through dimension 45, and the motivic technique used by Isaksen and the second author (35; 38) through dimension 59. For more details of known techniques, see Section 2.

Based on Isaksen's computation, we give rigorous proofs regarding π_{60} and π_{61}. Besides the classical technique of Toda brackets, one of our proofs relies heavily on the transfer map from the infinite real projective spectrum to the sphere spectrum. The success of this technique suggests a theoretical way to improve our understanding through a bigger range.

Combining our computations with the previous knowledge of π_*, we have another corollary of the main theorem.

Corollary 3.1.15. *For $5 \leq n \leq 61$, the only dimensions that S^n has a unique smooth structure are $n = 5, 6, 12, 56$ and 61.*

Proof. The range for $n < 19$ was known to Kervaire and Milnor. For even dimensions between 20 and 60, it is straightforward to check that at $p = 2$, the only dimension in which the cokernel of J vanishes is 56. Note that the Kervaire invariant θ_4 exists in dimension 30. In fact, Barratt, Mahowald and Tangora (9) showed that π_{30} is $\mathbb{Z}/2$, generated by θ_4. Therefore, we need to consider odd primary computations in this dimension. May (52) showed that at the prime 3, the cokernel of J in dimension 30 is $\mathbb{Z}/3$, which implies that S^{30} does not have a unique smooth structure. Combining with Theorems 3.1.7 and 3.1.9 and Corollary 3.1.13, this completes the proof. □

Remark 3.1.16. Recent work of Behrens, Hill, Hopkins and Mahowald (12) shows that the next sphere with a unique smooth structure, if exists, is in dimension at least 126.

Based on our current knowledge on π_*, we have the following conjecture.

Conjecture 3.1.17. *For dimensions greater than 4, the only spheres with a unique smooth structure are S^5, S^6, S^{12}, S^{56}, and S^{61}.*

The rest of this chapter is organized as follows.

In Section 3.2, we give a brief review of the stem-wise computation of π_* with a focus on the prime 2. We compare the known techniques. We reduce $\pi_{61} = 0$ to three Adams differentials.

From Section 3.3 to Section 3.10, we present the proof of the hardest differential $d_3(D_3) = B_3$. In Section 3.3, we summarize the strategy of our technique and explain how we organize the details of the proof in Sections 4 through 10. The intuition behind part of this proof is included in Appendix II, which is Section 3.14.

We present the proof of the other two differentials in Sections 3.11 and 3.12. The targets of these two differentials detect certain homotopy classes. We use the theory of Toda brackets to show that these homotopy classes must vanish.

3.2 The stable homotopy groups of spheres

The computation of the stable homotopy groups of spheres is a long standing and very challenging problem in algebraic topology. We will first give a brief review of the history from the stem-wise point of view, and then talk about some recent progress.

After the geometric computation of the first three stems (31; 25; 84; 67; 72), Serre (74) did the computation of π_n for $n < 9$ with the aid of the Serre spectral sequence and the Eilenberg-Maclane spectra. Serre also showed that these stable groups are finite in positive stems, so we can compute them one prime at a time. Afterwards, at each prime, Adams (1) constructed the Adams spectral sequence whose E_2-term encodes the information that we could obtain via primary cohomology operations. The Adams spectral sequence gives an upper bound on π_n and therefore determining the Adams differentials becomes a major method in computing the stable homotopy groups. Generalizing Adams's idea, Novikov constructed the Adams-Novikov spectral sequence using the complex cobordism spectrum.

There is another method using the EHP sequence, which computes the unstable homotopy groups inductively. Using this method, together with the Toda bracket operations, Toda (81) succeeded to do the computation of π_n for $n \leq 19$.

It turns out that the Adams-Novikov spectral sequence is more successful at odd primes than at the prime 2. In the 1980's, using the Adams-Novikov spectral sequence, Ravenel (71) computed up to the 108-stem at the prime 3, and the 999-stem at the prime 5. Previously, the computation was due independently to Nakamura (62) and Tangora (80) up to the 103-stem at the prime 3, and to Aubry (6) up to the 760-stem at the prime 5.

At the prime 2, the Adams spectral sequence is still the most efficient way. In (52), May constructed the May spectral sequence, which converges to the E_2-page of the Adams spectral sequence. This works at all primes. In particular, May computed π_n for $n \leq 28$ at the prime 2. In the 1960's, using the Adams spectral sequence, and with the aid of the technique of Toda brackets, Barratt, Mahowald and Tangora (9) determined the differentials in the Adams spectral sequence up to the 45-stem. About one and a half decades later, Bruner (15) discovered a gap in (9), and proved a new Adams differential in the 38-stem. Bruner's differential therefore corrected the result of π_{37} and π_{38}, and along with that corrected some relations in the stable homotopy ring.

In 1990, based on the Atiyah-Hirzebruch spectral sequence of the Brown-Peterson spectrum, Kochman (43) made an algorithm and implemented it into computer programmes. In this way, he produced a table of π_n up to the 64-stem. However, his method is not completely reviewed by others due to its complexity, and his result is not fully accepted by the experts. In 1995, Kochman and Mahowald (45) made a few corrections to (43), in the range from 52 to 64. A tentative chart of the Adams spectral sequence is included in the appendix of (43) and (45) without proofs. Note that the Adams differentials in this chart are deduced from the stable homotopy groups, not the other way around.

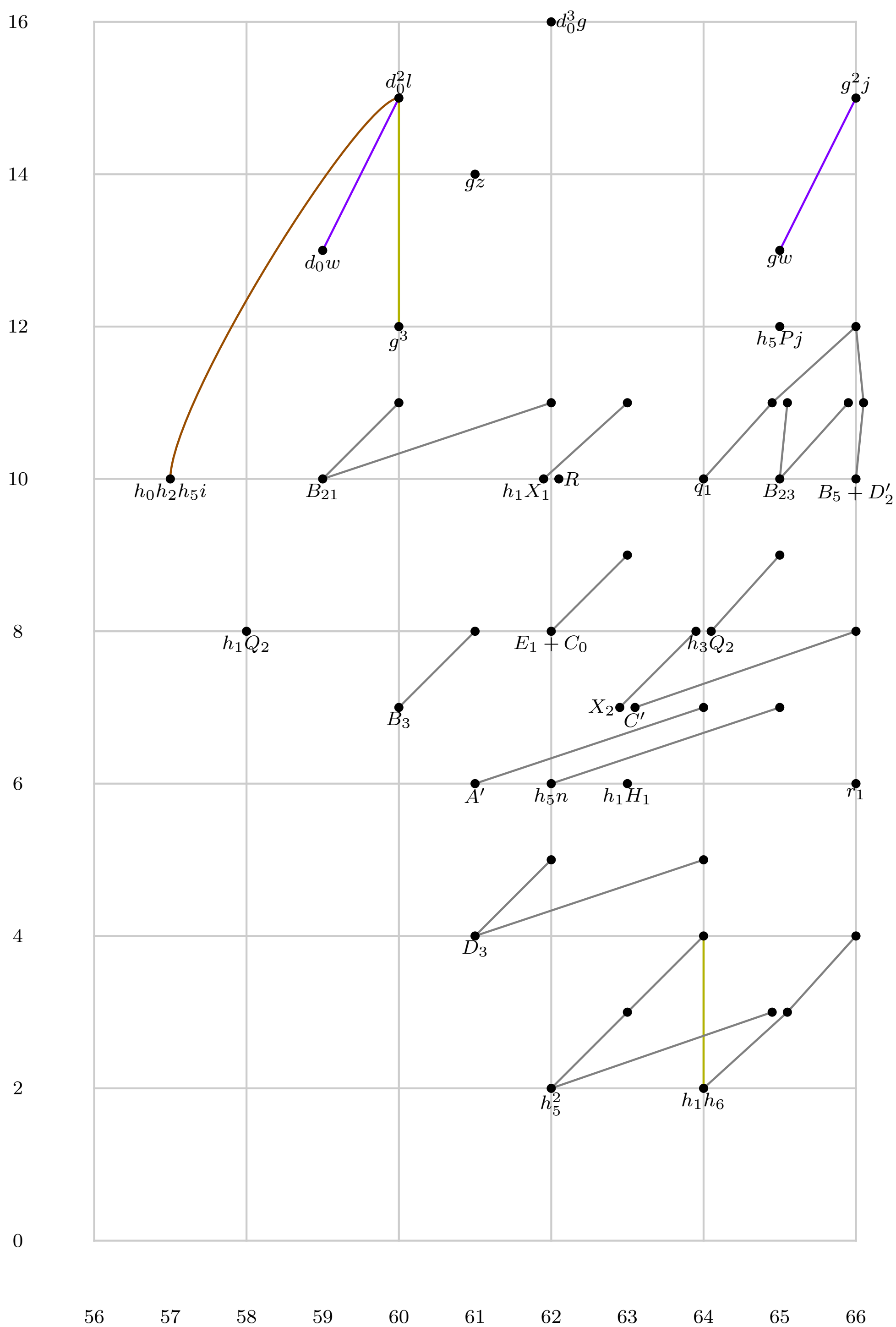
For about two decades, much of our knowledge regarding π_n, in the range from 45 to 64, relied on (45). Recently, by comparing the motivic Adams spectral sequence and the classical Adams spectral sequence, Isaksen (35) gave rigorous proofs to all but one Adams differentials up to the 59 stem. The exception was later proved by the second author (38) based on Isaksen's motivic computation. Along with a few corrections to some relations in the stable homotopy ring, Isaksen proved a new Adams differential in the 57-stem, which was not included in (45). This also corrects π_{56} and π_{57} as we used in the proof of Theorem 3.1.12.

In the range beyond the 59-stem, Isaksen (35) also proved a few differentials. The part which Isaksen did not fully understand can be summarized in his Adams E_∞ chart (36), which we include in the following page.

Note that we do not include elements in filtration higher than 16. Those elements are detected by the $K(1)$-local sphere, and are not relevant to our proof. Here we use colored lines to denote nontrivial extensions: yellow lines correspond to hidden 2-extensions, blue lines correspond to hidden η-extensions, and brown lines correspond to hidden ν-extensions. Note that because of differentials unknown to Isaksen, the actual E_∞-page beyond the 59-stem is a subquotient of what is shown in this chart.

Now we reduce the first claim of Theorem 3.1.9, i.e., $\pi_{61} = 0$, to three Adams differentials.

Figure 3.1: Isaksen's picture



Proof. It is proven in Theorem 3.3.1 (and this is the crux of this chapter) that

$$d_3(D_3) = B_3$$

and therefore

$$d_3(h_1 D_3) = h_1 B_3.$$

It is proven in Theorem 3.12.1 that

$$d_5(A') = h_1 B_{21}.$$

It is proven in Theorem 3.11.1 that the element gz must be killed by some Adams differential.

There are no elements left in the E_∞-page of the 61-stem. ☐

It is clear that these differentials also settle π_{60}.

Corollary 3.2.1. *The 2-primary π_{60} is $\mathbb{Z}/4$, generated by $\overline{\kappa}^3$.*

Proof. The elements g^3 and $d_0^2 l$ are the only elements left, and there is a hidden 2-extension between them. The element g detects $\overline{\kappa} \in \pi_{20}$. Therefore, the 2-primary group π_{60} is $\mathbb{Z}/4$, generated by $\overline{\kappa}^3$. ☐

3.3 Intuition and the proof of the differential $d_3(D_3) = B_3$

We have developed a general method to prove a differential in the Adams spectral sequence of the sphere spectrum. The strategy can be summarized in three parts:

1. Using the algebraic Kahn-Priddy theorem, we pullback a differential in the Adams spectral sequence of the sphere spectrum to one in the Adams spectral sequence of the suspension spectrum of RP^∞.

2. Using our knowledge of the cell structure of RP^{∞} and the algebraic Atiyah-Hirzebruch spectral sequence, we deduce the Adams differential in RP^{∞} from one in a certain $H\mathbb{F}_2$-subquotient of RP^{∞}.

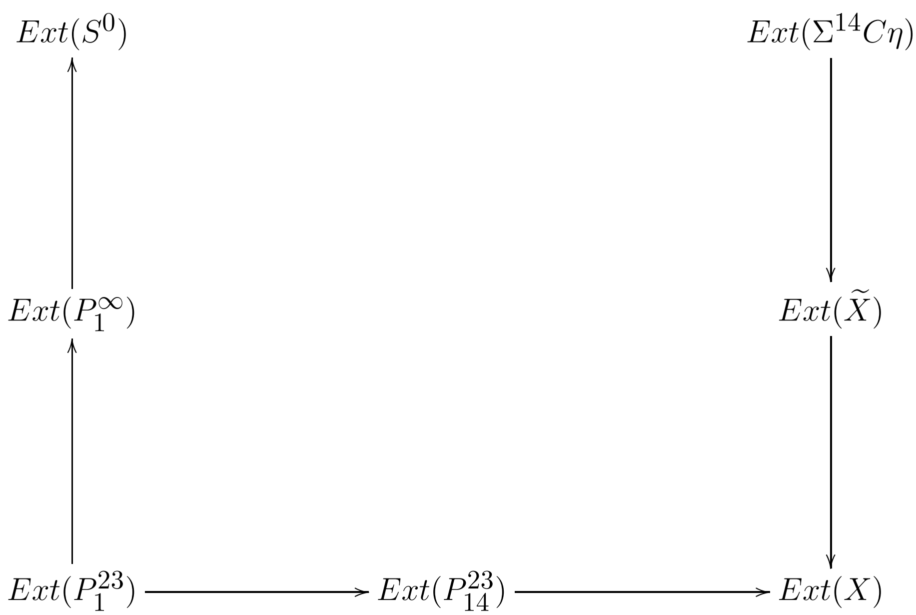
3. Using our knowledge of the Adams spectral sequence of the sphere spectrum, and the cell structure of this $H\mathbb{F}_2$-subquotient, we reduce the computation of the Adams differential in this $H\mathbb{F}_2$-subquotient to that of a product (or more generally a Toda bracket) in a *lower* stem of the stable homotopy groups of spheres.

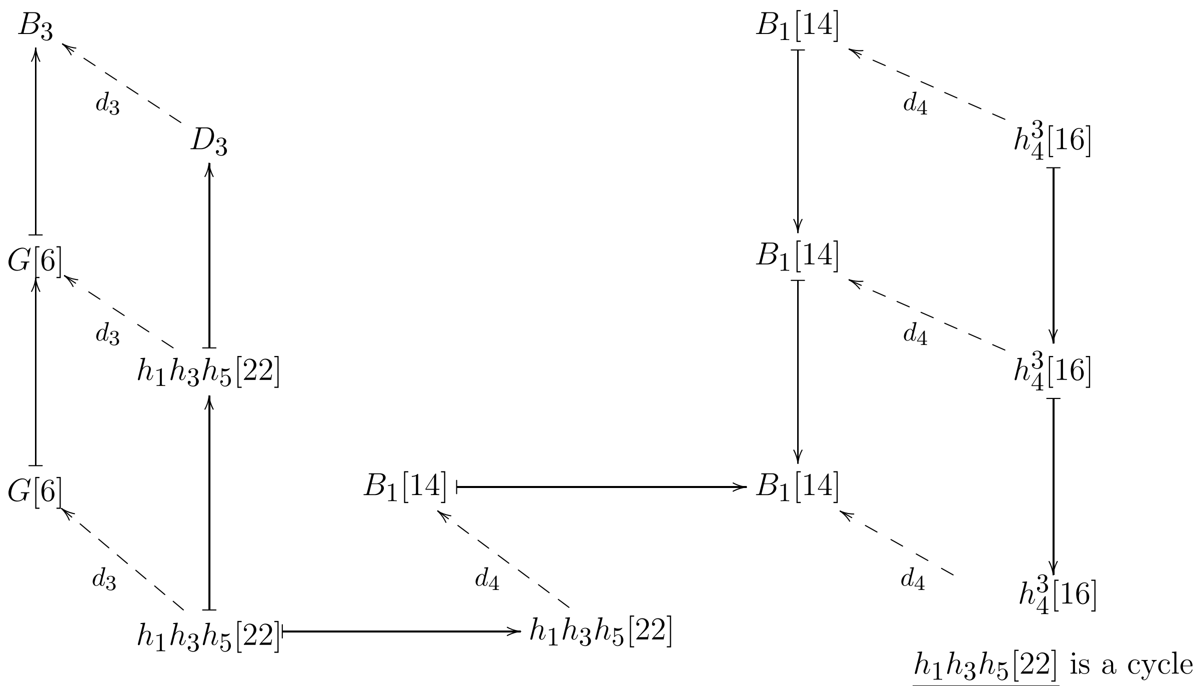
Intuitively, an $H\mathbb{F}_2$-subquotient of a CW complex is a subquotient to the eyes of mod 2 homology, in a sense that will be made precise in Definition 3.4.1.

The technical heart of this chapter, explained in Sections 3.3 - 3.10, is to apply this method to prove the following theorem.

Theorem 3.3.1. *We have the Adams differential:* $d_3(D_3) = B_3$.

With notations to be explained, here is a "road map" of the proof.





The first part of this "road map" describes seven Adams spectral sequences and maps among them; the second part describes certain Adams d_3 or d_4 differentials in the 61-stem of each of the spectral sequences and maps in the Adams E_2-page among the sources and targets of these differentials.

Notation 3.3.2. All spectra are localized at the prime 2. Suppose Z is a spectrum. Let $Ext(Z)$ denote its Adams E_2-page.

For spectra, let S^0 be the sphere spectrum, and P_1^∞ be the suspension spectrum of RP^∞. In general, we use P_n^{n+k} to denote the suspension spectrum of RP^{n+k}/RP^{n-1}. Recall that we have the James periodicity for the stunted projective spectra:

$$\Sigma^{\phi(k)} P_n^{n+k} \simeq P_{n+\phi(k)}^{n+k+\phi(k)},$$

where $\phi(k) = 2^{\psi(k)}$, and

$$\psi(k) = \lfloor \frac{k}{2} \rfloor + \begin{cases} -1, & k \equiv 0 \; mod \; 8 \\ 0, & k \equiv 1 \\ 0, & k \equiv 2 \\ 1, & k \equiv 3 \\ 0, & k \equiv 4 \\ 1, & k \equiv 5 \\ 0, & k \equiv 6 \\ 0, & k \equiv 7. \end{cases}$$

For example, $\phi(7) = 2^{\psi(7)} = 8$, hence we have $P_{16}^{23} \simeq \Sigma^8 P_8^{15} \simeq \Sigma^{16} P_0^7$.

The spectrum X is a quotient spectrum of P_{14}^{23} and $\widetilde{X}$ is a subspectrum of X. The spectrum $C\eta$ is the cofiber of $\eta \in \pi_1$, and $\Sigma^{14} C\eta$ turns out to be a subspectrum of $\widetilde{X}$. The precise definitions of the spectra X and $\widetilde{X}$ can be found in Definition 5.1.

For sources and targets of these differentials, we use the following way to denote the elements in the Adams E_2-page of P_1^∞ and its $H\mathbb{F}_2$-subquotients. One way to compute $Ext(P_1^\infty)$ is to use the algebraic Atiyah-Hirzebruch spectral sequence.

$$E_1 = \bigoplus_{n=1}^{\infty} Ext(S^n) \Longrightarrow Ext(P_1^\infty)$$

Notation 3.3.3. We denote any element in $Ext(S^n)$ to be $a[n]$, where $a \in Ext(S^0)$, and n suggests that it comes from $Ext(S^n)$. We will abuse notation and write the same symbol $a[n]$ for an element of $Ext(P_1^\infty)$ detected by the element $a[n]$ of the Atiyah-Hirzebruch E_∞ page. Thus, there is indeterminacy in the notation $a[n]$ that is detected by Atiyah-Hirzebruch E_∞ elements in lower filtration. When $a[n]$ is the element of lowest Atiyah-Hirzebruch filtration in the Atiyah-Hirzebruch E_∞ page in a given bidegree (s, t), then $a[n]$ also is a well-defined element of $Ext(P_1^\infty)$. Sometimes we will need to be precise about a particular element of $Ext(P_1^\infty)$ detected by $a[n]$. We will use the notation $\underline{a[n]}$ to denote a particular choice, and we must provide a definition that specifies $\underline{a[n]}$ in this case. We use this same notation for all $H\mathbb{F}_2$-subquotients of P_1^∞. There won't be any confusion on the index n since any $H\mathbb{F}_2$-subquotient contains at most one cell in each dimension.

Remark 3.3.4. In (85), we computed the Adams E_2-page of P_1^∞ in the range of $t < 72$ by the Lambda algebra. This Lambda algebra computation gives us a lot of information on the algebraic Atiyah-Hirzebruch spectral sequence. In particular, there is a one-to-one correspondence between the differentials in the Lambda algebra computation and differentials in the algebraic Atiyah-Hirzebruch spectral sequence.

Remark 3.3.5. Despite the indeterminacy in Notation 3.3, there is a huge advantage of it. Suppose $f : Q \to Q'$ is a map between two $H\mathbb{F}_2$-subquotients of P_1^∞, which is a composite of inclusion and quotient maps. Suppose further that there exists an element $a[n]$ which is a generater of both $Ext^{s,t}(Q)$ and $Ext^{s,t}(Q')$ for some bidegree (s, t) (this implies both Q and Q' have a cell in dimension n). We therefore must have that, with the right choices, $a[n]$ in $Ext^{s,t}(Q)$ maps to $a[n]$ in $Ext^{s,t}(Q')$. This property follows from the naturality of the algebraic Atiyah-Hirzebruch spectral sequence.

$$\begin{array}{ccc}
\bigoplus_{i\in I} Ext(S^i) & \longrightarrow & \bigoplus_{i\in I'} Ext(S^i) \\
\Downarrow & & \Downarrow \\
Ext(Q) & \longrightarrow & Ext(Q') \\
 & & \\
a[n] & \longmapsto & a[n]
\end{array}$$

Example 3.3.6. As an example, the group $Ext^{3,64}(X) = \mathbb{Z}/2 \oplus \mathbb{Z}/2 \oplus \mathbb{Z}/2$, is generated by $h_4^3[16]$, $h_1h_3h_5[22]$ and $h_0h_3h_5[23]$, as explained in Table 6 in Section 9. The element $h_4^3[16]$ is uniquely determined by our notation, since it has the lowest Atiyah-Hirzebruch filtration.

In fact, the 16-skeleton of X is $\Sigma^{14}C\eta$. The inclusion map specifies the element $h_4^3[16]$ in $Ext^{3,64}(X)$ as the image of the element $h_4^3[16]$ in $Ext^{3,64}(\Sigma^{14}C\eta)$.

$$\begin{array}{ccc} Ext(\Sigma^{14}C\eta) & \longrightarrow & Ext(X) \\ h_4^3[16] & \longmapsto & h_4^3[16] \end{array}$$

As a comparison, the element $h_1h_3h_5[22]$ in our notation does not specify a unique element in $Ext^{3,64}(X)$. In fact, suppose A and B are elements in $Ext^{3,64}(X)$, which are detected by $h_4^3[16]$ and $h_1h_3h_5[22]$ in the algebraic Atiyah-Hirzebruch spectral sequence of X. The element $A+B$ is therefore also detected by $h_1h_3h_5[22]$. Our notation $h_1h_3h_5[22]$ in $Ext^{3,64}(X)$ does *not* distinguish the elements B and $A+B$.

It turns out making a choice for $h_1h_3h_5[22]$ is essential to our proof. In fact, we use a 4 cell complex X^{22} (see Definition 5.6) to specify such a choice. The complex X^{22} is an $H\mathbb{F}_2$-subcomplex of X, and contains a cell in dimension 22, but not in dimension 16. The group $Ext^{3,64}(X^{22}) = \mathbb{Z}/2$, generated by $h_1h_3h_5[22]$, as explained in Table 3.4 in Section 3.8. We denote the image of $h_1h_3h_5[22]$ in $Ext^{3,64}(X^{22})$ to be $\underline{h_1h_3h_5[22]}$ in $Ext^{3,64}(X)$.

$$\begin{array}{ccc} Ext(X^{22}) & \longrightarrow & Ext(X) \\ h_1h_3h_5[22] & \longmapsto & \underline{h_1h_3h_5[22]} \end{array}$$

Now, we explain the main steps of the proof for the Adams differential $d_3(D_3) = B_3$.

1. **Step 1**: We establish a d_4 differential in the Adams spectral sequence of $\Sigma^{14}C\eta$:

$$d_4(h_4^3[16]) = B_1[14].$$

   This is stated as Theorem 3.7.1 and proved in Section 3.7.

2. **Step 2**: Using the inclusion map $\Sigma^{14}C\eta \to \widetilde{X}$, we push forward the Adams d_4 differ-

ential in Step 1 to an Adams d_4 differential in $\widetilde{X}$:

$$d_4(h_4^3[16]) = B_1[14].$$

This is stated as Theorem 3.8.1 and proved in Section 3.8.

3. **Step 3**: Using the inclusion map $\widetilde{X} \to X$, we push forward the Adams d_4 differential in Step 2 to an Adams d_4 differential in X:

$$d_4(h_4^3[16]) = B_1[14].$$

This is stated as Theorem 3.9.1 and proved in Section 3.9.

4. **Step 4**: We show that the chosen element $\underline{h_1 h_3 h_5[22]}$ (as explained in Example 3.3.6) is a permanent cycle in the Adams spectral sequence of X. This is stated as Theorem 3.9.2 and proved in Section 3.9.

   Combining with Step 3, we have an immediate Adams d_4 differential in X:

$$d_4(\underline{h_1 h_3 h_5[22]} + h_4^3[16]) = B_1[14].$$

   This is stated as Corollary 3.9.3.

5. **Step 5**: Using the quotient map $P_1^{23} \to X$, we pull back the Adams d_4 differential in Step 4 to an Adams d_3 differential in P_1^{23}:

$$d_3(h_1 h_3 h_5[22]) = G[6].$$

   This is stated as Theorem 3.10.1 and proved in Section 3.10.

6. **Step 6**: Using the inclusion map $P_1^{23} \to P_1^{\infty}$ and the transfer map $P_1^{\infty} \to S^0$, we

push forward the Adams d_3 differential in Step 4 to an Adams d_3 differential in S^0:

$$d_3(D_3) = B_3.$$

This is our main theorem and is proved in this section.

We have several comments before we dive into the details of the proofs.

Remark 3.3.7. Step 1 is the origin of all our differentials. It follows essentially from a relation in the stable homotopy groups of spheres: there is a nontrivial η-extension from h_4^3 to B_1.

Remark 3.3.8. Intuitively, the most mysterious step is Step 3.5. The intuition behind such an argument is explained in detail in Section 3.14, which is Appendix II. But note that the intuition is irrelevant to our proofs. For the proof, when we pull back a d_4 differential, the preimage of the source must support a d_2, d_3 or d_4 differential. To get the d_3 differential as claimed in Step 5, we rule out all other possibilities.

Remark 3.3.9. Logically, the most complicated step is Step 2. The intuition seems straightforward: we push forward a d_4 differential to get a d_4 differential. But note that we need to show that the image of the target survives to the E_4 page, i.e., it is not killed by a d_2 or d_3 differential. It turns out in the corresponding bidegrees, there are 10 elements which have the potential to support a d_2 or d_3 differential. To rule out these possibilities, we will show in Section 3.8 that 3.9 elements out of the 10 are permanent cycles, and the other one supports a d_2 differential which is irrelevant. Our way to show these elements are permanent cycles is by showing they are permanent cycles in some $H\mathbb{F}_2$-subcomplexes of X. For this purpose, in Section 3.5, we study the cell structure of X, as well as its several $H\mathbb{F}_2$-subcomplexes.

Remark 3.3.10. The intuitive reason why this method works is due to the geometric and algebraic Kahn-Priddy theorems. It is because of Step 6 that we can reduce the computation of an Adams differential in S^0 to one in P_1^∞, and further to one in a *lower* stem of S^0.

In the rest of this section, we prove Step 6.

Recall that we have the Kahn-Priddy Theorem (40), stated as follows.

Theorem 3.3.11. *The transfer map $P_1^\infty \to S^0$ induces a surjection on homotopy groups in positive stems.*

We also have the algebraic Kahn-Priddy Theorem due to Lin (46).

Theorem 3.3.12. *The transfer map also induces a surjection:*

$$Ext^{s,t}(P_1^\infty) \to Ext^{s+1,t+1}(S^0)$$

for $t - s > 0$.

Now we prove Step 6.

Proof. For the purpose of the differential $d_3(D_3) = B_3$, we check the two tables in the appendix of (85). See (85) for more details of the Lambda algebra notation we used here. We rewrite $Ext^{(s,t)}$ as $Ext^{(s,s+(t-s))}$ to indicate that it is in stem $t - s$.

The element D_3 is in $Ext^{4,61+4}(S^0) = \mathbb{Z}/2$. Checking the table for P_1^∞, we have that

$$\begin{aligned} Ext^{3,61+3}(P_1^\infty) = \mathbb{Z}/2, &\quad \text{generated by} \quad (22)\ 21\ 11\ 7, \\ Ext^{3,61+3}(P_1^{23}) = (\mathbb{Z}/2)^2, &\quad \text{generated by} \quad (22)\ 21\ 11\ 7, \\ &\quad \phantom{\text{generated by}} \quad (23)\ 22\ 13\ 3. \end{aligned}$$

The element 21 11 7 lies in

$$Ext^{3,39+3}(S^0) = \mathbb{Z}/2, \quad \text{generated by} \quad h_1 h_3 h_5.$$

Therefore, the element $h_1h_3h_5[22]$ maps to D_3.

The element B_3 is in $Ext^{7,60+7}(S^0) = \mathbb{Z}/2$. Checking the table for P_1^∞, we have that

$$Ext^{6,60+6}(P_1^\infty) = (\mathbb{Z}/2)^2, \quad \text{generated by } (6)\ 2\ 4\ 7\ 11\ 15\ 15,\ (20)\ 5\ 5\ 9\ 7\ 7\ 7,$$
$$Ext^{6,60+6}(P_1^{23}) = (\mathbb{Z}/2)^4, \quad \text{generated by } (6)\ 2\ 4\ 7\ 11\ 15\ 15,\ (20)\ 5\ 5\ 9\ 7\ 7\ 7,$$
$$(22)\ 3\ 5\ 9\ 7\ 7\ 7,\ (23)\ 13\ 2\ 3\ 5\ 7\ 7.$$

In the table for the transfer, we have that the element (20) 5 5 9 7 7 7 (with certain choice) maps to 0. Due to the algebraic Kahn-Priddy Theorem, we must have the element (6) 2 4 7 11 15 15 maps to B_3. The element 2 4 7 11 15 15 lies in

$$Ext^{6,54+6}(S^0) = \mathbb{Z}/2, \quad \text{generated by } G.$$

Therefore, the element $G[6]$ maps to B_3.

$$Ext^{3,61+3}(P_1^{23}) \longrightarrow Ext^{3,61+3}(P_1^\infty) \longrightarrow Ext^{4,61+4}(S^0)$$

$$h_1h_3h_5[22] \longmapsto h_1h_3h_5[22] \longmapsto D_3$$

$$Ext^{6,60+6}(P_1^{23}) \longrightarrow Ext^{6,60+6}(P_1^\infty) \longrightarrow Ext^{7,60+7}(S^0)$$

$$G[6] \longmapsto G[6] \longmapsto B_3$$

Note that in both $Ext(P_1^\infty)$ and $Ext(P_1^{23})$, the elements $h_1h_3h_5[22]$ and $G[6]$ are uniquely determined by our notation, since they have the lowest Atiyah-Hirzebruch filtrations in their bidegrees.

In the Adams spectral sequence for S^0, the element B_3 survives to the E_3-page: there is no element that could kill B_3 by a d_2 differential. Therefore, the Adams d_3 differential in P_1^{23}:

$$d_3(h_1h_3h_5[22]) = G[6]$$

in Step 5 (Theorem 10.1) implies the Adams d_3 differential in S^0:

$$d_3(D_3) = B_3.$$

□

3.4 $H\mathbb{F}_2$-subquotients for CW spectra

In this section, we introduce the definitions of $H\mathbb{F}_2$-subcomplexes and $H\mathbb{F}_2$-quotient complexes for CW spectra. We also discuss an important $H\mathbb{F}_2$-subcomplex of P_1^6 in Theorem 3.4.7.

Definition 3.4.1. Let A, B, C and D be CW spectra, i and q be maps

$$A \overset{i}{\hookrightarrow} B, \qquad B \overset{q}{\twoheadrightarrow} C$$

We say that (A, i) is an $H\mathbb{F}_2$-subcomplex of B, if the map i induces an injection on mod 2 homology. We denote an $H\mathbb{F}_2$-subcomplex by an hooked arrow as above.

We say that (C, q) is an $H\mathbb{F}_2$-quotient complex of B, if the map q induces a surjection on mod 2 homology. We denote an $H\mathbb{F}_2$-quotient complex by a double headed arrow above.

When the maps involved are clear in the context, we also say A is an $H\mathbb{F}_2$-subcomplex of B, and C is an $H\mathbb{F}_2$-quotient complex of B.

Furthermore, we say D is an $H\mathbb{F}_2$-subquotient of B, if D is an $H\mathbb{F}_2$-subcomplex of an $H\mathbb{F}_2$-quotient complex of B, or an $H\mathbb{F}_2$-quotient complex of an $H\mathbb{F}_2$-subcomplex of B.

Remark 3.4.2. Note that our definitions of $H\mathbb{F}_2$-subcomplexes and $H\mathbb{F}_2$-quotient complexes are *not* necessarily subcomplexes and quotient complexes on the point set level. Our definitions should be thought as in the homological or homotopical sense. Here is a motivating example of why we use these definitions. The top cell of the spectrum P_1^3 splits off, therefore there is a map from S^3 to P_1^3 that induces an injection on mod 2 homology. This is an $H\mathbb{F}_2$-subcomplex in our sense. However, on the point set level, the image of the attaching map is not a point, therefore S^3 is not a subcomplex of P_1^3 in the classical sense.

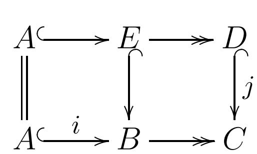
Remark 3.4.3. It follows directly from Definition 4.1 that if (A, i) is an $H\mathbb{F}_2$-subcomplex of B, then the cofiber of i is an $H\mathbb{F}_2$-quotient complex of B, which we sometimes denote as B/A. Dually, if (C, q) is an $H\mathbb{F}_2$-quotient complex of B, then the fiber of q is an $H\mathbb{F}_2$-subcomplex of B.

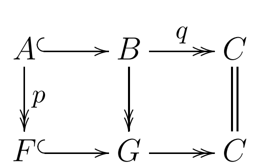
The following lemma is useful in constructing $H\mathbb{F}_2$-subquotients.

Lemma 3.4.4. *Suppose (A, i) is an $H\mathbb{F}_2$-subcomplex of B. Let C be the cofiber of i. Let (D, j) be an $H\mathbb{F}_2$-subcomplex of C. Define E to be the homotopy pullback of D along $B \to C$. We have that E is an $H\mathbb{F}_2$-subcomplex of B. Moreover, A is an $H\mathbb{F}_2$-subcomplex of E with quotient D.*

Dually, suppose (C, q) is an $H\mathbb{F}_2$-quotient complex of B. Let A be the fiber of q. let (F, p) be an $H\mathbb{F}_2$-quotient complex of A. Define G to be the homotopy pushout of F along $A \to B$. We have that G is an $H\mathbb{F}_2$-quotient complex of B. Moreover, C is an $H\mathbb{F}_2$-quotient complex of G with fiber F.

Proof. This follows from the short exact sequences of homology induced by the following commutative diagrams of cofiber sequences and diagram chasing.





□

We first study the spectrum P_1^6. For attaching maps, we abuse notation and refer to a homotopy class by its detecting element in the E_1-page of the Atiyah-Hirzebruch spectral sequence. We use similar notation as in the algebraic case in Notation 3.3.3. The readers who are familiar with the notation of cell diagrams from (7) should compare with the cell diagrams in Remark 3.4.8 for the intuition of the following Lemmas 3.4.5, 3.4.6 and Theorem 3.4.7.

Lemma 3.4.5. *There is an $H\mathbb{F}_2$-subcomplex of P_1^5 with a 3-cell and a 5-cell that forms $\Sigma^3 C\eta$.*

Proof. Firstly, by the solution of the Hopf invariant one problem, the top cell of P_1^3 splits off. It follows that S^3 is an $H\mathbb{F}_2$-subcomplex of P_1^3, and therefore an $H\mathbb{F}_2$-subcomplex of P_1^5.

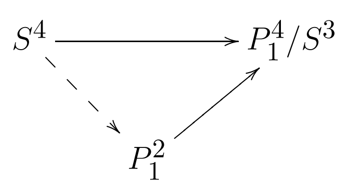
Secondly, we consider the $H\mathbb{F}_2$-quotient complex P_1^5/S^3. We claim the top cell of P_1^5/S^3 splits off. We prove this claim by showing the attaching map is homotopic to zero. In fact, the following composition is trivial:

$$S^4 \to P_1^4/S^3 \to S^4,$$

where the second map is the quotient map. Otherwise, we would have a nontrivial

$$Sq^1 : H^4(P_1^5/S^3) \to H^5(P_1^5/S^3),$$

which we don't. This shows that the attaching map factors through P_1^2.



The group $\pi_4(P_1^2)$ is generated by $\eta^2[2]$ and $\nu[1]$. However, the element $\eta^2[2]$ is killed by $\eta[4]$

in the Atiyah-Hirzebruch spectral sequence of P_1^4/S^3. The element $\nu[1]$ does not detect the attaching map either, since otherwise we would have a nontrivial

$$Sq^4 : H^1(P_1^5/S^3) \to H^5(P_1^5/S^3),$$

which we don't. Therefore, the attaching map $S^4 \to P_1^4/S^3$ is trivial, and S^5 is an $H\mathbb{F}_2$-subcomplex of P_1^5/S^3.

Now we pull back S^5 along the quotient map $P_1^5 \to P_1^5/S^3$. We claim that we have $\Sigma^3 C\eta$ as an $H\mathbb{F}_2$-subcomplex of P_1^5.

$$\begin{array}{ccccc} S^3 & \hookrightarrow & \Sigma^3 C\eta & \twoheadrightarrow & S^5 \\ \| & & \downarrow & & \downarrow \\ S^3 & \hookrightarrow & P_1^5 & \twoheadrightarrow & P_1^5/S^3 \end{array}$$

In fact, by Lemma 3.4.4, we have an $H\mathbb{F}_2$-subcomplex of P_1^5 with nontrivial H^3 and H^5. Since there is a nontrivial

$$Sq^2 : H^3(P_1^5) \to H^5(P_1^5),$$

we must have $\Sigma^3 C\eta$ as the $H\mathbb{F}_2$-subcomplex. □

Lemma 3.4.6. *If we quotient out the $H\mathbb{F}_2$-subcomplex $\Sigma^3 C\eta$ in P_1^6, then the 6-cell splits off. Therefore, S^6 is an $H\mathbb{F}_2$-subcomplex of $P_1^6/\Sigma^3 C\eta$.*

Proof. We claim that the attaching map $S^5 \to P_1^4/S^3$ is trivial.

In fact, the group $\pi_5(P_1^4/S^3) \cong \mathbb{Z}/2$, generated by $\eta[4]$. To compute it, note that the E_1-page of the Atiyah-Hirzebruch spectral sequence of P_1^4/S^3 is $\pi_5(S^1) \oplus \pi_5(S^2) \oplus \pi_5(S^4) =$

$\mathbb{Z}/8 \oplus \mathbb{Z}/2$, generated by $\nu[2]$ and $\eta[4]$. We have the following Atiyah-Hirzebruch differentials:

$$\begin{aligned} \nu[2] &\to 2\nu[1] \\ 2\nu[2] &\to 4\nu[1] \\ \eta^2[4] &\to 4\nu[2] = \eta^3[2] \end{aligned}$$

Therefore, the element $\eta[4]$ is the only one left in the E_∞-page.

Since we have

$$Sq^2 = 0 : H^4(P_1^6) \to H^6(P_1^6),$$

we must have

$$Sq^2 = 0 : H^4(P_1^6/\Sigma^3 C\eta) \to H^6(P_1^6/\Sigma^3 C\eta).$$

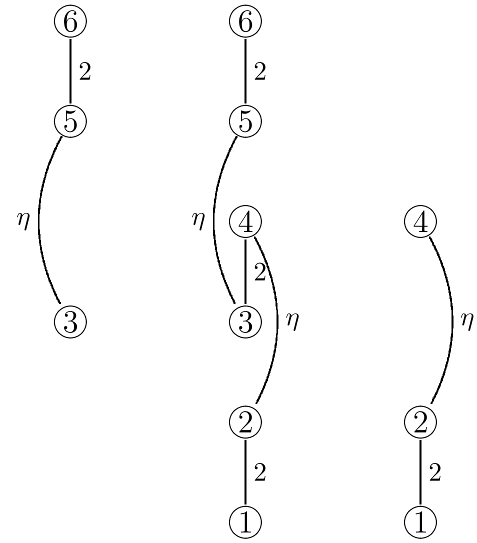
Therefore, the attaching map is not detected by $\eta[4]$, and is trivial. This proves that S^6 is an $H\mathbb{F}_2$-subcomplex of $P_1^6/\Sigma^3 C\eta$. □

Theorem 3.4.7. *There is an $H\mathbb{F}_2$-subcomplex Y of P_1^6 consisting of the 3-cell, 5-cell and the 6-cell, which is the pullback of S^6 along the quotient map $P_1^6 \to P_1^6/\Sigma^3 C\eta$.*

$$\begin{array}{ccccc} \Sigma^3 C\eta & \hookrightarrow & Y & \twoheadrightarrow & S^6 \\ \| & & \downarrow & & \downarrow \\ \Sigma^3 C\eta & \hookrightarrow & P_1^5 & \twoheadrightarrow & P_1^5/\Sigma^3 C\eta \end{array}$$

Proof. This follows directly from Lemmas 3.4.3 and 3.4.6. □

Remark 3.4.8. The cell diagrams of the cofiber sequences in Theorem 3.4.7 are the following:



3.5 Some $H\mathbb{F}_2$-subquotients of P_1^∞

In this section, we discuss the cell structures of certain $H\mathbb{F}_2$-subquotients of P_1^∞. All of them turn out to be $H\mathbb{F}_2$-subcomplexes of a 9 cell complex X. The existence of these $H\mathbb{F}_2$-subquotients is used extensively in the proofs in Sections 3.8, 3.9 and 3.10. For illustration purpose, we include the cell diagrams of these $H\mathbb{F}_2$-subquotients. The definition of cell diagrams is reviewed in Section 3.13, which is Appendix I.

We define the 9 cell complex X.

Definition 3.5.1. Recall that the 15-skeleton of P_{14}^{23} is $P_{14}^{15} = S^{14} \vee S^{15}$. The complex X is defined to be the cofiber of the inclusion map $S^{15} \hookrightarrow P_{14}^{23}$, i.e., X fits into the cofiber sequence

$$S^{15} \hookrightarrow P_{14}^{23} \twoheadrightarrow X.$$

We also define the 22-skeleton of X to be $\widetilde{X}$. In other words, $\widetilde{X}$ fits into the cofiber sequence

$$S^{15} \hookrightarrow P_{14}^{22} \twoheadrightarrow \widetilde{X}.$$

Now we establish the following lemmas on the cell structure of X.

Lemma 3.5.2. *There is a quotient map $X \twoheadrightarrow S^{16}$.*

Proof. There is a quotient map $P_0^7 \twoheadrightarrow S^0$, since the bottom cell splits off. By James periodicity, this gives a quotient map $P_{16}^{23} \twoheadrightarrow S^{16}$. Since the 14-skeleton of X is S^{14}, we have a quotient map to its cofiber P_{16}^{23}.

$$S^{14} \hookrightarrow X \twoheadrightarrow P_{16}^{23}.$$

Pre-composing the quotient map $P_{16}^{23} \twoheadrightarrow S^{16}$ with the quotient map $X \twoheadrightarrow P_{16}^{23}$, we get the desired quotient map $X \twoheadrightarrow S^{16}$. □

Lemma 3.5.3. *We have S^{17} as an $H\mathbb{F}_2$-subcomplex of $\widetilde{X}$ and of X.*

Proof. We claim that the top cell of the 17-skeleton of $\widetilde{X}$ splits off, and therefore S^{17} is an $H\mathbb{F}_2$-subcomplex of $\widetilde{X}$ and X.

The 16-skeleton of $\widetilde{X}$ is $\Sigma^{14}C\eta$ because of the nontrivial Sq^2. The group $\pi_{16}(\Sigma^{14}C\eta)$ is generated by $2[16]$. Note that in the Atiyah-Hirzebruch spectral sequence, the element $\eta^2[14]$ is killed by $\eta[16]$. Therefore, it follows from James periodicity that the attaching map is trivial. □

Now we define some $H\mathbb{F}_2$-subcomplexes of X. The relationships among the $H\mathbb{F}_2$-subcomplexes are summarized in Remark 5.12. The reader should compare with the cell diagrams in Remark 5.13 for the intuition of the following definitions.

Definition 3.5.4. We define $\widehat{X^{20}}$ to be the 20-skeleton of X, and X^{20} to be the fiber of the following composition:

$$\widehat{X^{20}} \hookrightarrow \widetilde{X} \twoheadrightarrow S^{16}.$$

Note that the composition is a quotient map, and therefore X^{20} is an $H\mathbb{F}_2$-subcomplex of $\widehat{X^{20}}$.

Definition 3.5.5. Quotienting out the 16-skeleton of $\widetilde{X}$, we have the $H\mathbb{F}_2$-quotient complex P^{22}_{17}. We define $\widehat{X^{22}}$ to be the pullback of $\Sigma^{16}Y$ along the quotient map $\widetilde{X} \to P^{22}_{17}$. Note that by Theorem 3.4.7 and James periodicity, $\Sigma^{16}Y$ is an $H\mathbb{F}_2$-subcomplex of P^{22}_{17}.

$$\begin{array}{ccccc}
\Sigma^{14}C\eta & \hookrightarrow & \widehat{X^{22}} & \twoheadrightarrow & \Sigma^{16}Y \\
\| & & \downarrow & & \downarrow \\
\Sigma^{14}C\eta & \hookrightarrow & \widetilde{X} & \twoheadrightarrow & P^{22}_{17} = \Sigma^{16}P^{6}_{1}
\end{array}$$

Definition 3.5.6. We define X^{22} to be the fiber of the following composition:

$$\widehat{X^{22}} \hookrightarrow \widetilde{X} \twoheadrightarrow S^{16}.$$

Note that the composition is a quotient map, and therefore X^{22} is an $H\mathbb{F}_2$-subcomplex of $\widehat{X^{22}}$.

Definition 3.5.7. We define $\widehat{X^{21}}$ to be the 21-skeleton of $\widehat{X^{22}}$, and X^{21} to be the 21-skeleton of X^{22}.

Remark 3.5.8. Note that S^{19} is an $H\mathbb{F}_2$-subcomplex of X^{21}. In fact, the 19-skeleton of X^{21} is $S^{19} \vee S^{14}$. The attaching map $S^{18} \to S^{14}$ is trivial since $\pi_4 = 0$.

Definition 3.5.9. The top cell of P^7_1 splits off due to the solution of the Hopf invariant one problem. By James periodicity, this implies that the top cell of P^{23}_{17} splits off. Therefore, S^{23} is an $H\mathbb{F}_2$-subcomplex of P^{23}_{17}.

We define $\widehat{X^{23}}$ to be the pullback of S^{23} along the quotient map $X \to P^{23}_{16}$.

$$\begin{array}{ccccc}
\Sigma^{14}C\eta & \hookrightarrow & \widehat{X^{23}} & \twoheadrightarrow & S^{23} \\
\| & & \downarrow & & \downarrow \\
\Sigma^{14}C\eta & \hookrightarrow & X & \twoheadrightarrow & P^{23}_{17} = \Sigma^{16}P^{7}_{1}
\end{array}$$

Definition 3.5.10. We define X^{23} to be the fiber of the following composition:

$$\widehat{X^{23}} \hookrightarrow X \twoheadrightarrow S^{16}.$$

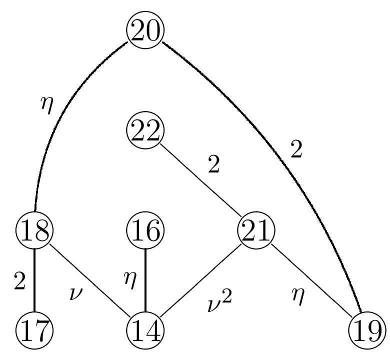
Note that the composition is a quotient map, and therefore X^{23} is an $H\mathbb{F}_2$-subcomplex of $\widehat{X^{23}}$.

Remark 3.5.11. We do not know if the top cell of X^{23} splits off. If not, then the attaching map is detected by a nontrivial homotopy class in π_8. Since homotopy classes in π_8 have Adams filtration at least 2, $Ext(X^{23})$ splits as a direct sum of $Ext(S^{14})$ and $Ext(S^{23})$ in either case.

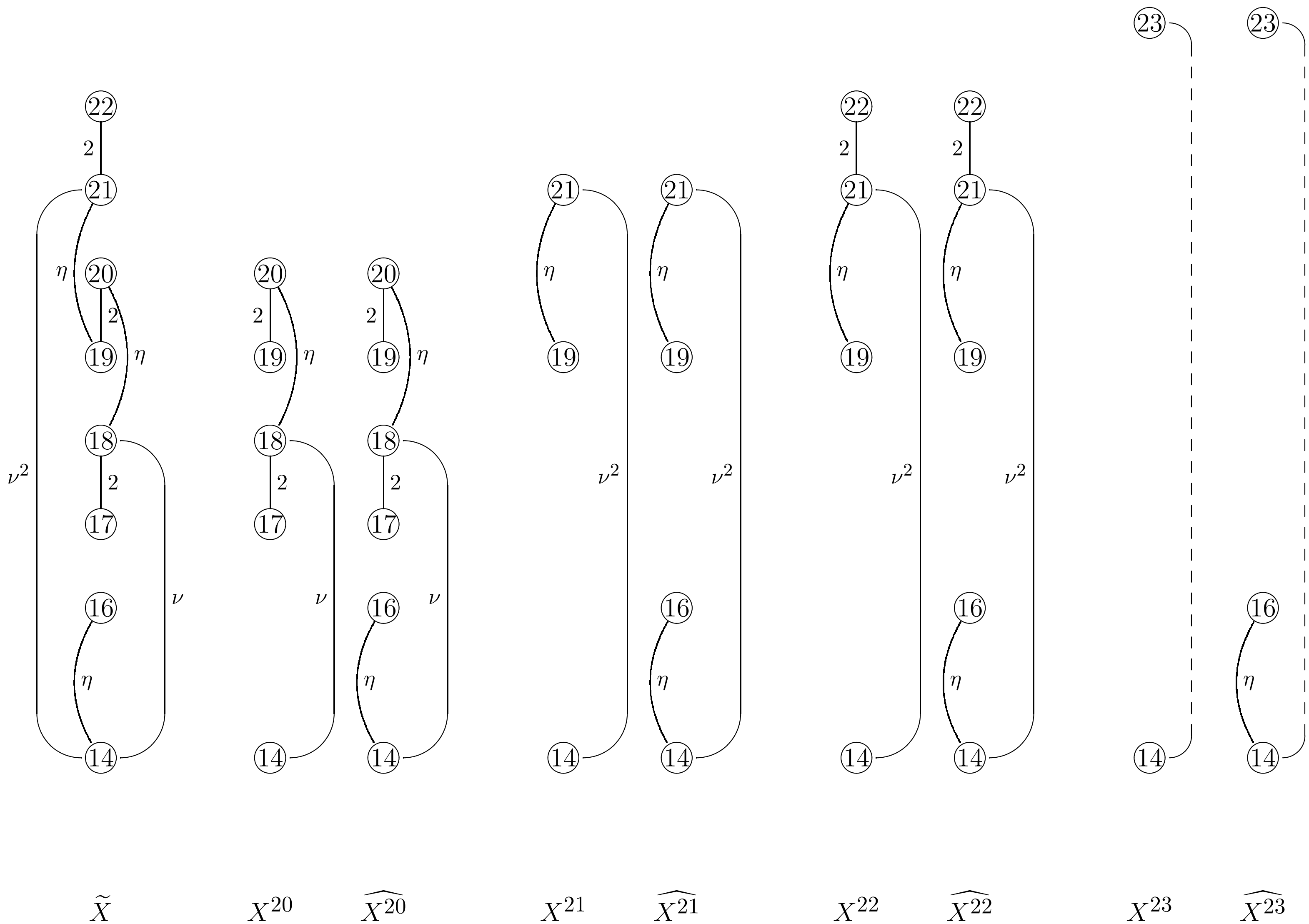
Remark 3.5.12. We summarize in the following diagram the relationships among the $H\mathbb{F}_2$-subcomplexes defined in Definitions 3.5.4, 3.5.5, 3.5.6, 3.5.7, 3.5.9 and 3.5.10. For the name convention, we have been using the notation X^n, not to be confused with the $n-$skeleton of X, to indicate a kind of "$n-$skeleton" to the eyes of mod 2 homology, and the notation $\widehat{X^n}$ to indicate "adding" the 16-cell to X^n. The cases for $n = 23$ do not necessarily follow this convention, since we do not know if the top cell of X^{23} splits off.

$$\begin{array}{ccccccccc}
 & & & & & & & & P^{23}_{14} \\
 & & & & & & & & \downarrow \\
 & & X^{23} & \hookrightarrow & \widehat{X^{23}} & \hookrightarrow & \widetilde{X} & \hookrightarrow & X \\
 & & & & & & \| & & \\
 & & X^{22} & \hookrightarrow & \widehat{X^{22}} & \hookrightarrow & \widetilde{X} & & \\
 & & \uparrow & & \uparrow & & \| & & \\
S^{19} & \hookrightarrow & X^{21} & \hookrightarrow & \widehat{X^{21}} & \hookrightarrow & \widetilde{X} & & \\
 & & & & & & \| & & \\
 & & X^{20} & \hookrightarrow & \widehat{X^{20}} & \hookrightarrow & \widetilde{X} & &
\end{array}$$

In Section 3.8, we need to show certain elements in $Ext(X)$ are permanent cycles. We will show these elements are permanent cycles in the corresponding $H\mathbb{F}_2$-subcomplexes, and use the naturality of Adams spectral sequences and the algebraic Atiyah-Hirzebruch spectral sequences to show they are permanent cycles in X. The intuition of finding these $H\mathbb{F}_2$-subcomplexes is due to the rearrangement of the cell diagram of $\widetilde{X}$. Following the cell diagram, one could reconstruct $\widetilde{X}$ layer by layer. Firstly, consider the cells in the bottom layer: $S^{14} \vee S^{17} \vee S^{19}$. Secondly, attach the cells in the next layer: the ones in dimension 16, 18 and 21. Lastly, attach the cells in dimension 20 and 22. Any $H\mathbb{F}_2$-subcomplex consists of a collection of cells, such that for each cell contained in this collection, any cells in lower layers that this cell is attached to are also contained in this collection. The reader should compare this with the cell diagrams in Remark 3.5.13.



Remark 3.5.13. For readers who are familiar with the notation of cell diagrams from (7), we include the cell diagrams as illustrations of the $H\mathbb{F}_2$-subcomplexes we defined. The definition and some examples of cell diagrams are explained in Appendix I.


22
21
20
19
18
17
16
14
23
2
η
ν
ν²
X̃
X^20
X̂^20
X^21
X̂^21
X^22
X̂^22
X^23
X̂^23

Here the dashed lines in X^{23} and $\widehat{X^{23}}$ mean some possible attaching maps as explained in Remark 3.5.11.

For the cell diagram of $\widetilde{X}$, note that we have a nonzero Sq^8 on $H^{14}(\widetilde{X})$. However, $\Sigma^{14}C\sigma$ is not an $H\mathbb{F}_2$-subquotient of $\widetilde{X}$, we therefore do not draw the attaching map σ. The non-existence of the $H\mathbb{F}_2$-subquotient is due to the existence of the attaching map ν^2, which is proved in the following Theorem 3.5.14.

By Remark 3.5.8, we have S^{19} as an $H\mathbb{F}_2$-subcomplex of X^{21}. The cofiber X^{21}/S^{19} is therefore a 2 cell complex with cells in dimension 14 and 21. We have the following theorem.

Theorem 3.5.14. *The complex X^{21}/S^{19} is $\Sigma^{14}C\nu^2$, where $C\nu^2$ is the cofiber of ν^2.*

This theorem implies the following corollary.

Corollary 3.5.15. *The complex $\Sigma^{14}C\nu^2$ is an $H\mathbb{F}_2$-subquotient of X^{21}, $\widehat{X^{21}}$, X^{22} and $\widehat{X^{22}}$.*

In the rest of this section, we prove Theorem 3.5.14. Note that since $\pi_6 = \mathbb{Z}/2$ is generated by ν^2, the complex X^{21}/S^{19} is either $\Sigma^{14}C\nu^2$ or $S^{14} \vee S^{21}$. Theorem 3.5.14 and Corollary 3.5.15 will be used in several proofs in Section 3.6. However, the proofs in Section 6 do not depend on these results. In fact, if the complex X^{21}/S^{19} were $S^{14} \vee S^{21}$, the proofs in Section 3.6 would be strictly much easier. The reader should feel free to skip the proof of Theorem 3.5.14: knowing either case could be true is good enough for the proofs in Section 8. Since this theorem may be of other interest, we include the proof of Theorem 3.5.14 for completeness.

To prove Theorem 3.5.14, we first consider the spectrum $\mathbb{C}P_1^3$, which is the suspension spectrum of $\mathbb{C}P^3$. As we will explain in Example 3.13.5, the top cell does not split off and is attached to $\mathbb{C}P_1^2$ via $2\nu[2]$. We have a standard quotient map $P_1^7 \to \mathbb{C}P_1^3$, which is induced by the quotient map on the space level. Then pre-composing it with the inclusion map, we have a map

$$q : P_1^6 \twoheadrightarrow \mathbb{C}P_1^3.$$

Recall that in Theorem 3.4.7, we showed that there exists a 3 cell complex Y, which is an $H\mathbb{F}_2$-subcomplex of P_1^6.

Theorem 3.5.16. *The composition*

$$S^3 \hookrightarrow Y \hookrightarrow P_1^6 \overset{q}{\twoheadrightarrow} \mathbb{C}P_1^3$$

is trivial, therefore the composition

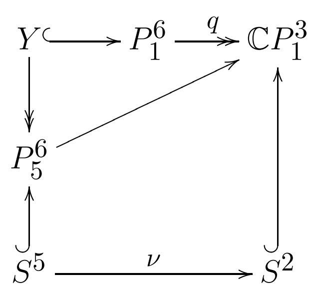
$$Y \hookrightarrow P_1^6 \overset{q}{\twoheadrightarrow} \mathbb{C}P_1^3$$

maps through P_5^6. Furthermore, the composition

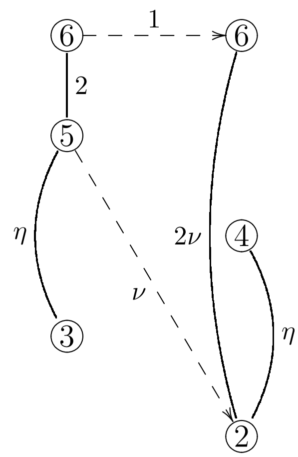
$$S^5 \hookrightarrow P_5^6 \longrightarrow \mathbb{C}P_1^3$$

is nontrivial, and detected by $\nu[2]$ in the Atiyah-Hirzebruch spectral sequence of $\mathbb{C}P_1^3$.

Remark 3.5.17. We have the following commutative diagram:



In other words, the cell diagrams of the composition $Y \to \mathbb{C}P_1^3$ can be described as follows:



Proof. The first claim of Theorem 3.5.16 follows from the fact that $\pi_3(\mathbb{C}P_1^3) = 0$. In fact, in the E_1-page of the Atiyah-Hirzebruch spectral sequence of $\mathbb{C}P_1^3$, there is only one candidate that lies in the degree that converges to π_3: $\eta[2]$. However, because of the attaching map in $\mathbb{C}P_1^2$, we have an Atiyah-Hirzebruch differential

$$1[4] \to \eta[2].$$

Therefore, $\pi_3(\mathbb{C}P_1^3) = 0$.

For the second claim, we first show that the composition

$$S^5 \hookrightarrow P_5^6 \longrightarrow \mathbb{C}P_1^3$$

maps through S^2. This follows from the fact that $\pi_5(\mathbb{C}P_1^3) = \mathbb{Z}/2$, generated by $\nu[2]$. In fact, because of the attaching maps in $\mathbb{C}P_1^3$, we have the Atiyah-Hirzebruch differentials

$$1[6] \to 2\nu[2]$$

$$2[6] \to 4\nu[2]$$

$$\eta[4] \to \eta^2[2],$$

which leave $\nu[2]$ as the only nontrivial element in the Atiyah-Hirzebruch E_∞-page that converges to $\pi_5(\mathbb{C}P_1^3)$.

Next, we consider the following commutative diagram of cofiber sequences

$$\begin{array}{ccccccc}
S^5 & \xrightarrow{2\nu[2]} & \mathbb{C}P_1^2 & \hookrightarrow & \mathbb{C}P_1^3 & \twoheadrightarrow & S^6 \\
\uparrow & & \uparrow & & \uparrow & & \uparrow \\
S^5 & \xrightarrow{2} & S^5 & \hookrightarrow & P_5^6 & \twoheadrightarrow & S^6
\end{array}$$

Since the composition

$$S^5 \hookrightarrow P_5^6 \longrightarrow \mathbb{C}P_1^3 \twoheadrightarrow S^6$$

is trivial, it maps through the quotient $P_5^6/S^5 = S^6$. Since the map $P_5^6 \to \mathbb{C}P_1^3$ induces an isomorphism on H^6, so does $S^6 \dashrightarrow S^6$. Therefore, we can choose it to be the identity map. To make the left square commute, we must identify the map $S^5 \to \mathbb{C}P_1^2$ as $\nu[2]$ modulo the indeterminacy $2\nu[2]$. Therefore, the composition

$$S^5 \hookrightarrow \mathbb{C}P_1^2 \longrightarrow \mathbb{C}P_1^3$$

is nontrivial, and detected by $\nu[2]$ in the Atiyah-Hirzebruch spectral sequence of $\mathbb{C}P_1^3$. $\square$

Proof of Theorem 3.5.14. We show that there is an attaching map ν^2 in X^{21}.

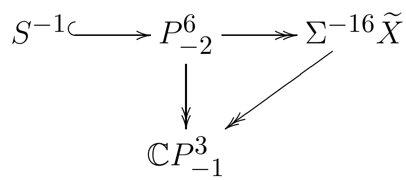
Firstly, we have a quotient map

$$P_{-2}^6 \to \mathbb{C}P_{-1}^3,$$

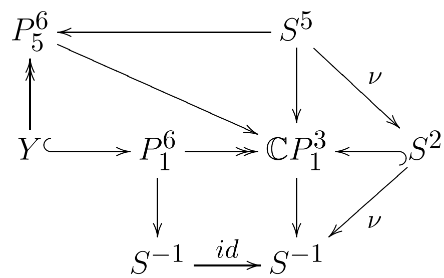
which is induced by the quotient map $\mathbb{R}P_{14}^{22} \to \mathbb{C}P_7^{11}$ on the space level and James periodicity. It maps through $\Sigma^{-16}\widetilde{X}$, since $\pi_{-1}(\mathbb{C}P_{-1}^3) = 0$. In fact, in the Atiyah-Hirzebruch spectral sequence of $\mathbb{C}P_{-1}^3$, we have a differential

$$1[0] \to \eta[-2],$$

which kills the only nontrivial element $\eta[-2]$ in the E_1-page.



Secondly, by Theorem 3.5.16, we have the following commutative diagram



where the map $\nu : S^2 \to S^{-1}$ is due to the nontrivial Sq^4 on $H^{-2}(\mathbb{C}P^3_{-1})$.

Therefore, in the cofiber of the composition

$$Y \hookrightarrow P_1^6 \longrightarrow S^{-1},$$

we have ν^2 as an attaching map. Since this cofiber is $\Sigma^{-15}X^{22}$, this proves the attaching map ν^2 in X^{21}. □

3.6 Two lemmas on Atiyah-Hirzebruch differentials

In this section, we establish two general lemmas regarding the relationship of 3-fold Toda brackets and differentials in the Atiyah-Hirzebruch spectral sequences of certain 3 and 4 cell complexes. As examples, we use these lemmas to prove Proposition 3.6.3 and 3.6.4, whose statements will be used in Section 3.8.

We recall some facts from the construction of the Atiyah-Hirzebruch spectral sequence. Let X be a complex with at most one cell in each dimension. Let X^n denote its n-skeleton.

Not to be confused with the notation we use in the rest of this chapter, the n-skeleton notation only applies in the next four pages.

We have the following facts about the Atiyah-Hirzebruch spectral sequence of X:

1. The E_1-page is
$$E_1^{s,t} = \pi_t(X^s/X^{s-1}).$$
As used in the previous two sections, we denote any element in the E_1-page to be $\alpha[s]$, where α is an element in the stable homotopy groups of spheres, and s suggests its Atiyah-Hirzebruch filtration. We will abuse the notation and write the same symbol $\alpha[s]$ for an element in $\pi_*(X)$.

2. The E_r-page is
$$E_r^{s,t} = \frac{Im(\pi_t(X^s/X^{s-r}) \to \pi_t(X^s/X^{s-1}))}{Im(\pi_{t+1}(X^{s+r-1}/X^s) \to \pi_t(X^s/X^{s-1}))},$$
where the top map is induced by the quotient map
$$X^s/X^{s-r} \twoheadrightarrow X^s/X^{s-1},$$
and the bottom map is induced by the attaching map in the cofiber sequence
$$X^s/X^{s-1} \hookrightarrow X^{s+r-1}/X^{s-1} \twoheadrightarrow X^{s+r-1}/X^s \longrightarrow \Sigma X^s/X^{s-1}.$$

3. The differential
$$d_r : E_r^{s,t} \to E_r^{s-r,t-1}$$
is defined as the following. Let $\widetilde{\alpha}$ be a class in $\pi_t(X^s/X^{s-r})$, such that it maps to $\alpha[s] \in E_r^{s,t}$ under the projection to the top cell: $X^s/X^{s-r} \twoheadrightarrow X^s/X^{s-1}$. We define $d_r(\alpha[s])$

to be the composition of $\widetilde{\alpha}$ with the attaching map $X^s/X^{s-r} \to \Sigma X^{s-r}/X^{s-r-1}$.

$$S^t \xrightarrow{\widetilde{\alpha}} X^s/X^{s-r} \longrightarrow \Sigma X^{s-r}/X^{s-r-1}.$$

One can check that this is well-defined.

4. Suppose we have a nontrivial differential in the Atiyah-Hirzebruch spectral sequence of X:

$$d_{s_1-s_2}(\alpha[s_1]) = \beta[s_2],$$

where $\alpha \in \pi_*(X^{s_1}/X^{s_1-1})$ and $\beta \in \pi_*(X^{s_2}/X^{s_2-1})$. This implies that, in the Atiyah-Hirzebruch spectral sequence of X^{s_1-1}, the element $\beta[s_2]$ is a permanent cycle. Furthermore, under the attaching map $S^{s_1-1} \to X^{s_1-1}$, the image of $\alpha[s_1]$ is detected by $\beta[s_2]$.

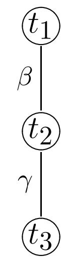
We have the following lemma to compute differentials in the Atiyah-Hirzebruch spectral sequence of 3 cell complexes:

Lemma 3.6.1. *Let T be a three cell complex with cells in dimensions t_1, t_2, t_3, where $t_3 < t_2 < t_1$. Suppose we have cofiber sequences*

$$\Sigma^{t_3} C\gamma \xhookrightarrow{i_1} T \xrightarrow{q_1} S^{t_1} \xrightarrow{a_1} \Sigma^{t_3+1} C\gamma$$

$$S^{t_3} \xhookrightarrow{i_2} T \xrightarrow{q_2} \Sigma^{t_2} C\beta \xrightarrow{a_2} \Sigma S^{t_3},$$

where $C\beta$ is the cofiber of $\beta \in \pi_{t_1-t_2-1}$, $C\gamma$ is the cofiber of $\gamma \in \pi_{t_2-t_3-1}$ and β, γ are *nontrivial classes such that $\beta \cdot \gamma = 0$. In other words, the cell diagram of T is the following:*



Suppose the class $\alpha \in \pi_{t_0}$ satisfies the condition: $\alpha \cdot \beta = 0$ in $\pi_{t_0+t_1-t_2-1}$. Then we have an Atiyah-Hirzebruch differential:

$$d_{t_1-t_3}(\alpha[t_1]) \subseteq \langle \alpha, \beta, \gamma \rangle [t_3].$$

If moreover $\alpha \cdot \pi_{t_1-t_3-1} \subseteq \gamma \cdot \pi_{t_0+t_1-t_2}$ in $\pi_{t_0+t_1-t_3-1}$, then we have an Atiyah-Hirzebruch differential:

$$d_{t_1-t_3}(\alpha[t_1]) = \langle \alpha, \beta, \gamma \rangle [t_3].$$

Here the indeterminacy of $\langle \alpha, \beta, \gamma \rangle [t_3]$ is zero in the $E_{t_1-t_3}$-page.

Furthermore, in the latter case, if $0 \in \langle \alpha, \beta, \gamma \rangle$, then $\alpha[t_1]$ is a permanent cycle in the Atiyah-Hirzebruch spectral sequence of T.

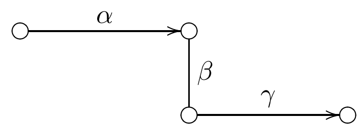
Proof. Following the condition $\alpha \cdot \beta = 0$, $\alpha[t_1]$ survives in the Atiyah-Hirzebruch spectral sequence of $\Sigma^{t_2} C\beta$. In fact, this follows from the long exact sequence of homotopy groups associated to the cofiber sequence

$$S^{t_2} \hookrightarrow \Sigma^{t_2} C\beta \twoheadrightarrow S^{t_1}.$$

By naturality of the Atiyah-Hirzebruch spectral sequence induced by the quotient map $T \twoheadrightarrow \Sigma^{t_2} C\beta$, we have the differential in the Atiyah-Hirzebruch spectral sequence of T:

$$d_{t_1-t_2}(\alpha[t_1]) = 0.$$

Now consider any class in $\pi_{t_0+t_1}(\Sigma^{t_2}C\beta)$ which is detected by $\alpha[t_1]$. We abuse the notation to denote such a class by $\alpha[t_1]$. By the definition of the Toda bracket $\langle\alpha,\beta,\gamma\rangle$, the class $a_{2*}(\alpha[t_1])$ is an element in $\langle\alpha,\beta,\gamma\rangle[t_3]$.



$$S^{t_0+t_1} \longrightarrow \Sigma^{t_2}C\beta \xrightarrow{a_2} \Sigma S^{t_3}$$

The indeterminacy of this Toda bracket is $\alpha\cdot\pi_{t_1-t_3-1}+\gamma\cdot\pi_{t_0+t_1-t_2}$. From the construction of the Atiyah-Hirzebruch spectral sequence, $a_{2*}(\alpha[t_1])$ is also a representative for $d_{t_1-t_3}(\alpha[t_1])$. The indeterminacy of the target of this differential is the image of

$$d_{t_2-t_3} : \pi_{t_0+t_1-t_2+t_3+1}(S^{t_2}) \to \pi_{t_0+t_1}(\Sigma S^{t_3}),$$

which is $\gamma\cdot\pi_{t_0+t_1-t_2}$, since it is induced by multiplication by γ map. Hence the first claim.

If $\alpha\cdot\pi_{t_1-t_3-1}\subseteq\gamma\cdot\pi_{t_0+t_1-t_2}$ in $\pi_{t_0+t_1-t_3-1}$, then $d_{t_1-t_3}(\alpha[t_1])$ and $\langle\alpha,\beta,\gamma\rangle[t_3]$ have a common element with the same indeterminacy. Hence the second statement.

The third statement follows directly from the second one, since the $E_{t_1-t_3+1}$-page is the E_∞-page for the Atiyah-Hirzebruch spectral sequence of T. $\square$

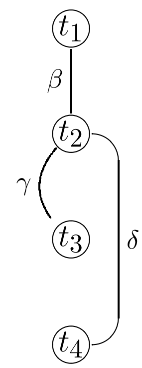
Lemma 3.6.2. *Let U be a four cell complex with cells in dimensions t_1, t_2, t_3, t_4, where $t_4 < t_3 < t_2 < t_1$. Suppose we have cofiber sequences*

$$S^{t_3}\vee S^{t_4} \xhookrightarrow{i_3} U \xrightarrow{q_3} \Sigma^{t_2}C\beta \xrightarrow{a_3} \Sigma S^{t_3}\vee\Sigma S^{t_4}$$

$$V \xhookrightarrow{i_4} U \xrightarrow{q_4} S^{t_1} \xrightarrow{a_4} \Sigma V$$

$$S^{t_3}\vee S^{t_4} \xhookrightarrow{i_5} V \xrightarrow{q_5} S^{t_2} \xrightarrow{a_5} \Sigma S^{t_3}\vee\Sigma S^{t_4}$$

where $C\beta$ is the cofiber of $\beta \in \pi_{t_1-t_2-1}$, the map $a_5 : S^{t_2} \to \Sigma S^{t_3} \vee \Sigma S^{t_4}$ is defined component-wise by multiplication by $\gamma \in \pi_{t_2-t_3-1}$ and $\delta \in \pi_{t_2-t_4-1}$ map, and β, γ, δ are nontrivial classes such that $\beta \cdot \gamma = 0$, $\beta \cdot \delta = 0$. In other words, the cell diagram of U is the following:



Suppose the class $\alpha \in \pi_{t_0}$ satisfies the following conditions:

1. $\alpha \cdot \beta = 0$ *in* $\pi_{t_0+t_1-t_2-1}$,
2. $\alpha \cdot \pi_{t_1-t_3-1} \subseteq \gamma \cdot \pi_{t_0+t_1-t_2}$ *in* $\pi_{t_0+t_1-t_3-1}$,
3. $0 \in \langle \alpha, \beta, \gamma \rangle$ *in* $\pi_{t_0+t_1-t_3-1}$.

We then have an Atiyah-Hirzebruch differential

$$d_{t_1-t_4}(\alpha[t_1]) \subseteq \langle \alpha, \beta, \delta \rangle [t_4].$$

If furthermore the following two conditions are satisfied:

4. $\alpha \cdot \pi_{t_1-t_4-1} = 0$ *in* $\pi_{t_0+t_1-t_4-1}$,
5. $\delta \cdot \pi_{t_0+t_1-t_2} = 0$ *in* $\pi_{t_0+t_1-t_4-1}$,

then we have an Atiyah-Hirzebruch differential

$$d_{t_1-t_4}(\alpha[t_1]) = \langle \alpha, \beta, \delta \rangle [t_4].$$

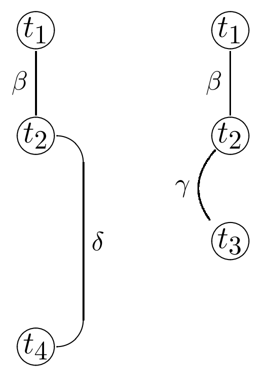
Moreover, in the latter case, if $0 \in \langle \alpha, \beta, \delta \rangle$*, then* $\alpha[t_1]$ *is a permanent cycle in the Atiyah-Hirzebruch spectral sequence of* U*.*

Proof. We consider the following two cofiber sequences:

$$S^{t_3} \hookrightarrow U \xrightarrow{p_3} T'$$

$$S^{t_4} \hookrightarrow U \xrightarrow{p_4} T''.$$

Both 3 cell complexes T' and T'' (with the following cell diagrams) satisfy the assumptions in Lemma 6.1.



T' $\quad$ T''

By Lemma 3.6.1, in the Atiyah-Hirzebruch spectral sequence of T'', we have a differential

$$d_{t_1 - t_3}(\alpha[t_1]) = \langle \alpha, \beta, \gamma \rangle [t_3] = 0.$$

The last equality follows from condition (3). Using the naturality for the quotient map $p'' : U \twoheadrightarrow T''$, we pull back a differential in the Atiyah-Hirzebruch spectral sequence of U:

$$d_{t_1 - t_3}(\alpha[t_1]) = 0.$$

By Lemma 3.6.1, in the Atiyah-Hirzebruch spectral sequence of T', we have a differential

$$d_{t_1-t_4}(\alpha[t_1]) \subseteq \langle \alpha, \beta, \delta \rangle [t_4].$$

Using the naturality of the quotient map $p_3 : U \twoheadrightarrow T'$, we pull it back to get a differential in the Atiyah-Hirzebruch spectral sequence of U:

$$d_{t_1-t_4}(\alpha[t_1]) \subseteq \langle \alpha, \beta, \delta \rangle [t_4].$$

The second and third statements follow directly from the first one, since the Toda bracket $\langle \alpha, \beta, \delta \rangle$ has zero indeterminacy under conditions (4) and (5), and the $E_{t_1-t_4+1}$-page is the E_∞-page for the Atiyah-Hirzebruch spectral sequence of U. □

Now we apply Lemma 3.6.2 to the complex X^{22}.

In π_{39}, consider the three homotopy classes $\alpha = \sigma\eta_5$, $\alpha' \in \{h_5 c_0\}$ such that $2 \cdot \alpha' = 0$, $\sigma \cdot \alpha' = 0$, and $\alpha'' = \sigma\{d_1\}$. Here we use the notation $\{a\}$ to denote the set of homotopy classes that are detected by a, where a is a surviving element in the E_∞-page of the Adams spectral sequence. One can choose $\alpha' = \langle \theta_4, 2, \epsilon \rangle$. Moss's theorem tells us $\alpha' \in \{h_5 c_0\}$. We have

$$2 \cdot \alpha' = 2\langle \theta_4, 2, \epsilon \rangle = \langle 2, \theta_4, 2 \rangle \epsilon = \eta\theta_4\epsilon = 0.$$

The last equation follows from filtration reasons. From the proof of Lemma 6.5, we also have $\sigma \cdot \alpha' = 0$. Note also that there are indeterminacies in the notation $\{d_1\}$ and η_5, but for our purpose, any choices work. The reader should compare with Isaksen's computations in (35; 36).

Proposition 3.6.3. *In the Atiyah-Hirzebruch spectral sequence of X^{22}, we have the following*

d_8 differentials:

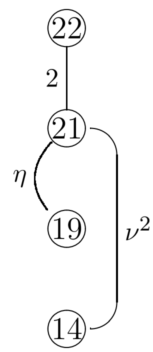
$$d_8(\alpha[22]) = 0,$$

$$d_8(\alpha'[22]) = \eta\phi[14],$$

$$d_8(\alpha''[22]) \subseteq \eta^2\pi_{44}[14],$$

where $\phi \in \pi_{45}$ is detected by h_5d_0, such that $\eta \cdot \phi \in \langle \alpha', 2, \nu^2 \rangle$.

Proof. The complex X^{22} satisfies the conditions in Lemma 3.6.2, with $\beta = 2 \in \pi_0$, $\gamma = \eta \in \pi_1$ and $\delta = \nu^2 \in \pi_6$.



We verify that the classes α and α' satisfy conditions (1) through (5), and α'' satisfy conditions (1) through (3) in Lemma 3.6.2:

1. $\alpha \cdot 2 = 0$ in π_{39}. This follows from $2 \cdot \eta_5 = 0$.

   $\alpha' \cdot 2 = 0$ in π_{39}. This follows from our definition of α'.

   $\alpha'' \cdot 2 = 0$ in π_{39}. This follows from $2 \cdot \{d_1\} = 0$.

2. $\alpha \cdot \pi_2 \subseteq \eta \cdot \pi_{40}$ in π_{41}.

   $\alpha' \cdot \pi_2 \subseteq \eta \cdot \pi_{40}$ in π_{41}.

   $\alpha'' \cdot \pi_2 \subseteq \eta \cdot \pi_{40}$ in π_{41}.

   These follow from the fact that π_2 is generated by η^2.

3. $0 \in \langle \alpha, 2, \eta \rangle$ in π_{41}.

   $0 \in \langle \alpha', 2, \eta \rangle$ in π_{41}.

   $0 \in \langle \alpha'', 2, \eta \rangle$ in π_{41}.

These follow from the fact that the Cokernel of J in π_{41} is contained in the image of $\eta : \pi_{40} \to \pi_{41}$. In fact, suppose for example $\langle \alpha, 2, \eta \rangle$ does not contain 0. It therefore must contain an element in the image of J. Therefore, mapping this Toda bracket to the $K(1)$-local sphere gives a contradiction, since the class α maps to 0. The cases α' and α'' work the same way.

4. $\alpha \cdot \pi_7 = 0$ in π_{46}.

   $\alpha' \cdot \pi_7 = 0$ in π_{46}.

   These follow from the fact that π_7 is generated by σ and the proof of Lemma 3.6.5.

5. $\nu^2 \cdot \pi_{40} = 0$ in π_{46}. This follows from $\nu \cdot \pi_{43} = 0$ for filtration reasons.

For the targets of these differentials, we apply Lemma 3.6.2 by computing the following Toda brackets

$$\langle \alpha, 2, \nu^2 \rangle,\ \langle \alpha', 2, \nu^2 \rangle,\ \langle \alpha'', 2, \nu^2 \rangle.$$

For the element $\alpha = \sigma\eta_5$, we have

$$\langle \sigma \cdot \eta_5, 2, \nu^2 \rangle \supseteq \sigma \langle \eta_5, 2, \nu^2 \rangle = \eta_5 \langle 2, \nu^2, \sigma \rangle = \eta_5 \{0, \sigma^2\} = 0.$$

Note that the last equation holds because in the proof of Lemma 3.6.5 we have $\sigma^2\eta_5 = 0$. Therefore, by Lemma 3.6.2, we have the Atiyah-Hirzebruch differential

$$d_8(\alpha[22]) = 0.$$

For the element $\alpha' \in \{h_5 c_0\}$, we have

$$\begin{aligned} \langle \alpha', 2, \nu^2 \rangle &= \langle \alpha', 2, \langle \eta, \nu, \eta \rangle \rangle \\ &\supseteq \langle \alpha', 2, \eta, \nu \rangle \cdot \eta \\ &\subseteq \{h_5 d_0\} \cdot \eta, \end{aligned}$$

where the last inequality follows from the following Massey product in Ext, and Moss's theorem (60, Theorem 1.2).

$$\langle h_5c_0, h_0, h_1, h_2\rangle = h_5\langle c_0, h_0, h_1, h_2\rangle = h_5d_0.$$

That is, there exists a class ϕ in $\{h_5d_0\}$ in π_{45} such that $\eta \cdot \phi \in \langle \alpha', 2, \nu^2\rangle$. Therefore, by Lemma 3.6.2, we have the Atiyah-Hirzebruch differential

$$d_8(\alpha'[22]) = \eta\phi[14].$$

For the element $\alpha'' = \sigma\{d_1\}$, we have

$$\langle \sigma \cdot \{d_1\}, 2, \nu^2\rangle \supseteq \sigma\langle \{d_1\}, 2, \nu^2\rangle \subseteq \sigma \cdot \pi_{39} \subseteq \eta^2\pi_{44}.$$

The indeterminacy of the Toda bracket $\langle \sigma \cdot \{d_1\}, 2, \nu^2\rangle$ is

$$\sigma\{d_1\} \cdot \pi_7 + \nu^2 \cdot \pi_{40} = \sigma\{d_1\} \cdot \pi_7 \subseteq \sigma \cdot \pi_{39} \subseteq \eta^2\pi_{44}.$$

Therefore, we have

$$\langle \sigma \cdot \{d_1\}, 2, \nu^2\rangle \subseteq \eta^2\pi_{44}.$$

By Lemma 3.6.2, we have the Atiyah-Hirzebruch differential

$$d_8(\alpha''[22]) \subseteq \eta^2\pi_{44}[14].$$

□

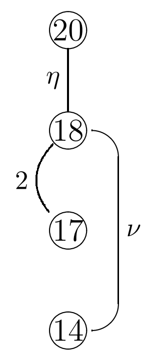
We also apply Lemma 3.6.2 to the complex X^{20}/S^{19}. Note that by Lemma 3.4.4 and Remark 3.5.8, we have S^{19} as an $H\mathbb{F}_2$-subcomplex of X^{20}.

In π_{41}, we consider the homotopy class $\alpha''' = \sigma\{h_0h_2h_5\}$. Note that the notation

$\{h_0h_2h_5\}$ has indeterminacy. Since $h_0h_2h_5$ does not support any hidden η-extension in the E_∞-page of the Adams spectral sequence of S^0, we choose a class in $\{h_0h_2h_5\}$ such that its η-multiple is zero. The class $\alpha''' = \sigma\{h_0h_2h_5\}$ is therefore unique.

Proposition 3.6.4. *In the Atiyah-Hirzebruch spectral sequence of X^{20}/S^{19}, the element $\alpha'''[20]$ is a permanent cycle.*

Proof. The complex X^{20}/S^{19} satisfies the conditions in Lemma 3.6.2, with $\beta' = \eta \in \pi_1$, $\gamma' = 2 \in \pi_0$ and $\delta' = \nu \in \pi_3$.



We verify that $\alpha''' = \sigma\{h_0h_2h_5\} \in \pi_{41}$ satisfies the conditions (1) through (5) in Lemma 3.6.2:

1. $\sigma\{h_0h_2h_5\} \cdot 2 = 0$ in π_{41}. This follows from $2 \cdot \pi_{41} = 0$.

2. $\sigma\{h_0h_2h_5\} \cdot \pi_2 \subseteq 2 \cdot \pi_{43}$ in π_{43}. This follows from the fact that π_2 is generated by η^2, and that
$$\eta^2 \cdot \pi_{41} = \{0, 4\{P^5h_2\}\} \subseteq 2 \cdot \pi_{43}.$$

3. $0 \in \langle \sigma\{h_0h_2h_5\}, \eta, 2\rangle$ in π_{43}. This follows from $\sigma \cdot \pi_{36} = 0$ in π_{43}. In fact, since we chose the element in $\{h_0h_2h_5\}$ such that its η-multiple is zero, we have
$$\langle \sigma \cdot \{h_0h_2h_5\}, \eta, 2\rangle \supseteq \sigma\langle \{h_0h_2h_5\}, \eta, 2\rangle \subseteq \sigma \cdot \pi_{36} = 0.$$

4. $\sigma\{h_0h_2h_5\} \cdot \pi_5 = 0$ in π_{46}. This follows from $\pi_5 = 0$.

5. $\nu \cdot \pi_{43} = 0$ in π_{46}.

We further verify that $0 \in \langle \sigma\{h_0h_2h_5\}, \eta, \nu\rangle$ in π_{46}. Since we chose the element in $\{h_0h_2h_5\}$ such that its η-multiple is zero, we have

$$\langle \sigma \cdot \{h_0h_2h_5\}, \eta, \nu\rangle \supseteq \sigma\langle\{h_0h_2h_5\}, \eta, \nu\rangle = \{h_0h_2h_5\} \cdot \langle \eta, \nu, \sigma\rangle \subseteq \{h_0h_2h_5\} \cdot \pi_{12} = 0.$$

The last equation follows from the fact that $\pi_{12} = 0$. Therefore, by Lemma 3.6.2, the element $\alpha'''[20] = \sigma\{h_0h_2h_5\}[20]$ is a permanent cycle in the Atiyah-Hirzebruch spectral sequence of X^{20}/S^{19}. □

In the rest of this section, we prove the following relation in the stable homotopy groups of spheres, which was used in Propositions 3.6.3 and 3.6.4.

Lemma 3.6.5.

$$\sigma \cdot \pi_{39} \subseteq \eta^2\pi_{44} = \{0, \eta^2\{g_2\}\}.$$

Moreover, there is at most one nontrivial σ-extension from π_{39} to π_{46}, namely

$$\sigma^2\{d_1\} = \eta^2\{g_2\}.$$

Proof. The group π_{39} is generated by classes that are detected by $P^2h_0^2i$, u, h_2t, h_3d_1, h_5c_0 and $h_1h_3h_5$ in the Adams E_∞-page. To prove this lemma, we check that for each element in the Adams E_∞-page, σ annihilates one class it detects, with the possible exception of h_3d_1. For the element h_3d_1, we show that there is a possible σ-extension from h_3d_1 to N, and it is equivalent to an η-extension from h_1g_2 to N. It is now known that this nontrivial σ-extension does in fact exist, but it is irrelevant to the proofs in this chapter.

1. For $P^2h_0^2i$, we have $\sigma \cdot \{P^2h_0^2i\} = 0$ for filtration reasons.

2. For u, suppose $\sigma \cdot \{u\} \neq 0$. The only possibility is $\sigma \cdot \{u\} = \{d_0l\}$ for filtration reasons. However, this cannot happen, since both $\{u\}$ and $\{d_0l\}$ are detected by tmf, and

$\sigma = 0$ in $\pi_* tmf$: mapping this relation to $\pi_* tmf$ gives a contradiction. Therefore, $\sigma \cdot \{u\} = 0$.

3. For $h_2 t$, one class that it detects is $\nu\{t\}$. It follows from $\nu \cdot \sigma = 0$ that $\sigma \cdot \{h_2 t\} = 0$.

4. For $h_3 d_1$, note that there is a relation in Ext: $h_3 d_1 = h_1 e_1$. Following Bruner's differential (15, Theorem 4.1)

$$d_3(e_1) = h_1 t = h_2^2 n,$$

we have a Massey product in the Adams E_4-page

$$\langle h_2 n, h_2, h_1 \rangle = h_1 e_1.$$

By Moss's theorem (60, Theorem 1.2), we have that the Toda bracket $\langle \nu\{n\}, \nu, \eta \rangle$ is detected by $h_1 e_1 = h_3 d_1$. Therefore,

$$\sigma \cdot \langle \nu\{n\}, \nu, \eta \rangle = \langle \sigma, \nu\{n\}, \nu \rangle \cdot \eta.$$

By Bruner's differential and Moss's theorem, we have that the Toda bracket $\langle \sigma, \nu\{n\}, \nu \rangle$ is detected by

$$h_1 g_2 = h_3 e_1 = \langle h_3, h_2 n, h_2 \rangle.$$

Since the only element with higher filtration than $h_1 g_2$ that supports an η-extension is w, to show that

$$\sigma \cdot \{h_3 d_1\} = \eta^2 \{g_2\},$$

we only need to show that

$$\sigma \cdot \langle \nu\{n\}, \nu, \eta \rangle \neq \{w\} \cdot \eta.$$

Suppose the opposite is true. Multiplying the equation by η gives a contradiction, since h_3d_1 does not support hidden η-extension while d_0l does. Therefore, we have

$$\sigma \cdot \{h_3d_1\} = \eta^2\{g_2\}.$$

5. For h_5c_0, by Moss's theorem, $\alpha' = \langle \theta_4, 2, \epsilon \rangle$ is detected by h_5c_0. We have

$$\langle \theta_4, 2, \epsilon \rangle \cdot \sigma = \theta_4 \cdot \langle 2, \epsilon, \sigma \rangle = \theta_4 \cdot 0 = 0.$$

   Therefore, we have the class $\alpha' = \langle \theta_4, 2, \epsilon \rangle$ in $\{h_5c_0\}$ such that $\sigma \cdot \alpha' = 0$.

6. For $h_1h_3h_5$, it detects $\alpha = \sigma\eta_5$. Since $\nu \cdot \eta_5 = 0$, we have

$$\sigma \cdot \sigma\eta_5 = \langle \nu, \sigma, \nu \rangle \eta_5 = \nu \langle \sigma, \nu, \eta_5 \rangle \subseteq \nu \cdot \pi_{43} = 0.$$

   Therefore, we have the class $\alpha = \sigma\eta_5$ in $\{h_1h_3h_5\}$ such that $\sigma \cdot \alpha = 0$.

In sum, we have $\sigma \cdot \pi_{39} \subseteq \eta^2\pi_{44} = \{0, \eta^2\{g_2\}\}$. □

3.7 The cofiber of η

In this section, we establish Step 1 by proving the following theorem.

Theorem 3.7.1. *In the Adams spectral sequence of $\Sigma^{14}C\eta$, we have a d_4 differential in the 61-stem:*

$$d_4(h_4^3[16]) = B_1[14].$$

Proof. The cofiber sequence

$$S^{15} \xrightarrow{\eta} S^{14} \xrightarrow{i} \Sigma^{14}C\eta \xrightarrow{p} S^{16}$$

gives us a short exact sequence on cohomology

$$0 \longrightarrow H^*(S^{16}) \xrightarrow{p^*} H^*(\Sigma^{14}C\eta) \xrightarrow{i^*} H^*(S^{14}) \longrightarrow 0$$

and therefore a long exact sequence of *Ext* groups

$$Ext^{s-1,t-1}(S^{15}) \xrightarrow{h_1} Ext^{s,t}(S^{14}) \xrightarrow{i_\sharp} Ext^{s,t}(\Sigma^{14}C\eta) \xrightarrow{p_\sharp} Ext^{s,t}(S^{16}).$$

From this long exact sequence, we have in Table 3.1 the Adams E_2 page of $\Sigma^{14}C\eta$ in the 60 and 61 stems for $s \leq 7$.

Table 3.1: The Adams E_2 page of $\Sigma^{14}C\eta$ in the 60 and 61 stems for $s \leq 7$

$s\backslash t-s$	60	61
7	$B_1[14]$	$h_0^2h_5d_0[16]$
6	$h_0^2g_2[16]$	$h_0h_2g_2[14]$
		$h_0h_5d_0[16]$
5	$h_0g_2[16]$	$h_2g_2[14]$
		$h_1g_2[16]$
4		$h_0h_4^3[16]$
3		$h_4^3[16]$

Firstly, since there is an η-extension from h_4^3 to B_1 in S^0, The class $B_1[14]$ in $Ext(\Sigma^{14}C\eta)$ detects zero in $\pi_{60}(\Sigma^{14}C\eta)$, and therefore must be killed by some element. There are four candidates: $h_4^3[16]$ in filtration 3, $h_0h_4^3[16]$ in filtration 4, and $h_2g_2[14]$, $h_1g_2[16]$ in filtration 5.

Secondly, the element $h_4^3[16]$ in $Ext(\Sigma^{14}C\eta)$ cannot survive. Suppose it did. We would then have $q_\sharp(h_4^3[16]) = h_4^3[16]$, where the image survives in $Ext(S^{16})$. However, the homotopy class detected by $h_4^3[16]$ in $Ext(S^{16})$ maps nontrivially to a class in $\pi_{60}(\Sigma S^{14})$ because of the same η-extension. This contradicts the exactness of the long exact sequence of homotopy groups.

Thirdly, the element $h_2g_2[14]$ is a permanent cycle and therefore cannot kill $B_1[14]$. In fact, the element $h_2g_2[14]$ is a permanent cycle in $Ext(S^{14})$. The image $i_\sharp(h_2g_2[14]) =$

$h_2g_2[14]$ must also be a permanent cycle.

At last, the kernel of the map

$$\eta : \pi_{45} \longrightarrow \pi_{46}$$

is $\mathbb{Z}/8 \oplus \mathbb{Z}/2$, generated by an order 8 element detected by $h_0h_4^3$ and $\eta\{g_2\}$. Since $h_0h_4^3$ and h_1g_2 have filtration 4 and 5, we must have two more surviving cycles in $\pi_{61}(\Sigma^{14}C\eta)$ with filtration strictly smaller than 6 besides $h_2g_2[14]$. The only possibility is $h_0h_4^3[16]$ and $h_1g_2[16]$, since we know $h_4^3[16]$ cannot survive.

Therefore, the only possibility to kill $B_1[14]$ is $h_4^3[16]$. ☐

Corollary 3.7.2. *The elements $h_0h_4^3[16]$, $h_2g_2[14]$ and $h_1g_2[16]$ survive in the Adams spectral sequence of $\Sigma^{14}C\eta$.*

Proof. This follows directly from the proof of Theorem 7.1 and filtration reasons. ☐

3.8 The Adams spectral sequence of $\widetilde{X}$

In this section, based on Theorem 3.7.1, we prove the following Theorem 3.8.1 in Step 2.

Theorem 3.8.1. *In the Adams spectral sequence of $\widetilde{X}$, we have the differential*

$$d_4(h_4^3[16]) = B_1[14].$$

The proof of Theorem 3.8.1 is summarized as in the following Table 2.

Here the element $\underline{h_1h_3h_5[22]}$ is defined to be the image of $h_1h_3h_5[22]$ in $Ext(X^{22})$. In fact, the group $Ext^{3,64}(X^{22}) = \mathbb{Z}/2$ is generated by $h_1h_3h_5[22]$ as we will show in Lemma 3.8.8. Each $\bullet$ represents a nontrivial element in its bidegree. But these elements are irrelevant to our purpose.

Proof. Firstly, as we will show in Lemma 3.8.2, the Adams E_2-page of $\widetilde{X}$ in the 60 and 61 stems for $s \leq 7$ is as claimed in Table 3.2. In particular, there are 10 elements in Adams

Table 3.2: The Adams E_2 page of $\widetilde{X}$ in the 60 and 61 stems for $s \leq 7$

$s\backslash t-s$	60	61	status	proof	$H\mathbb{F}_2$-subquotients used
7	$B_1[14]$	$\bullet$			
	$\bullet$	$\bullet$			
6	$h_0^2 f_1[20]$	$\bullet$			
	$\bullet$	$\bullet$			
	$\bullet$	$\bullet$			
5	$\bullet$	$h_2 g_2[14]$	permanent cycle	Lemma 8.3	$\Sigma^{14}C\eta$
	$\bullet$	$h_1 g_2[16]$	permanent cycle	Lemma 8.3	$\Sigma^{14}C\eta$
		$h_1 f_1[20]$	permanent cycle	Lemma 8.10	X^{20}
		$h_1 h_5 c_0[21]$	permanent cycle	Lemma 8.7	X^{21}
		$h_3 d_1[22]$	permanent cycle	Lemma 8.8	X^{22} and $\widehat{X^{22}}$
4	$\bullet$	$h_0 h_4^3[16]$			
		$g_2[17]$	permanent cycle	Lemma 8.4	S^{17}
		$f_1[21]$	$d_2(f_1[21]) = h_0^2 f_1[20]$	Lemma 8.5	P_{19}^{21}
		$h_1^2 h_3 h_5[21]$	permanent cycle	Lemma 8.7	X^{21}
		$h_5 c_0[22]$	permanent cycle	Lemma 8.8	X^{22} and $\widehat{X^{22}}$
3	$\bullet$	$h_4^3[16]$	$d_4(h_4^3[16]) = B_1[14]$		
		$h_1 h_3 h_5[22]$	permanent cycle	Lemma 8.8	X^{22} and $\widehat{X^{22}}$

filtration 4 and 5. Secondly, by the Lemmas 3.8.3, 3.8.4, 3.8.5, 3.8.7, 3.8.8 and 3.8.10 in later part of this section, the element $B_1[14]$ in Adams filtration 7 cannot be killed by any d_2 or d_3 differentials from these 10 elements. In fact, one of these 10 elements in Adams filtration 4 supports a d_2 differential, and the rest are permanent cycles. Therefore, the element $B_1[14]$ survives to the E_4-page of the Adams spectral sequence of $\widetilde{X}$. Theorem 3.8.1 follows from naturality of the Adams spectral sequences and Theorem 3.7.1. □

Lemma 3.8.2. *The Adams E_2 page of $\widetilde{X}$ in the 60 and 61 stem for $s \leq 7$ is as claimed in Table 3.2.*

Proof. Because of the cell structure of $\widetilde{X}$, there exists a cofiber sequence

$$S^{14} \xrightarrow{i} \widetilde{X} \xrightarrow{q} P_{16}^{22} \xrightarrow{a} \Sigma S^{14}$$

This cofiber sequence gives us a short exact sequence on cohomology

$$0 \longrightarrow H^*(P_{16}^{22}) \xrightarrow{q^*} H^*(\widetilde{X}) \xrightarrow{i^*} H^*(S^{14}) \longrightarrow 0$$

and therefore a long exact sequence on Ext groups

$$Ext^{s,t}(S^{14}) \xrightarrow{i_\sharp} Ext^{s,t}(\widetilde{X}) \xrightarrow{q_\sharp} Ext^{s,t}(P_{16}^{22}) \xrightarrow{\delta} Ext^{s+1,t+1}(\Sigma S^{14}).$$

Note that the Adams filtration of the attaching map $a : P_{16}^{22} \to \Sigma S^{14}$ is 1. In fact, in its cofiber $\widetilde{X}$, the 16-cell is attached to the 14-cell by η, which has the Adams filtration 1. Therefore, the boundary map in the long exact sequence on Ext groups raises the Adams filtration by 1.

In Section 6 of (85), we explained how to obtain the Adams E_2-page of P_n^{n+k} from our Curtis table of P_1^∞. In particular, we have the Adams E_2 page of P_{16}^{22} in the 60 and 61 stem for $s \leq 7$.

To compute $Ext(\widetilde{X})$ from the long exact sequence on Ext groups, we also need to compute the boundary homomorphism $\delta : Ext^{s,t}(P_{16}^{22}) \to Ext^{s+1,t+1}(\Sigma S^{14})$. In fact, in the 61 stem for $s \leq 5$, there is only one element $h_5 d_0[16]$ (with the right choices of other elements) which maps nontrivially: $\delta(h_5 d_0[16]) = h_1 h_5 d_0[14]$. This follows from the naturality of the boundary homomorphism induced by the inclusion map $\Sigma^{14} C\eta \to \widetilde{X}$, and the fact that

$$Ext^{s,s+46}(S^0) = 0 \ \text{ for } \ s \leq 5$$
$$Ext^{6,6+46}(S^0) = \mathbb{Z}/2, \text{generated by } \ h_1 h_5 d_0$$

$$\begin{array}{ccccccc}
Ext^{s,t}(S^{14}) & \xrightarrow{i_\sharp} & Ext^{s,t}(\Sigma^{14} C\eta) & \xrightarrow{q_\sharp} & Ext^{s,t}(S^{16}) & \longrightarrow & Ext^{s+1,t+1}(\Sigma S^{14}) \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
Ext^{s,t}(S^{14}) & \xrightarrow{i_\sharp} & Ext^{s,t}(\widetilde{X}) & \xrightarrow{q_\sharp} & Ext^{s,t}(P_{16}^{22}) & \xrightarrow{\delta} & Ext^{s+1,t+1}(\Sigma S^{14})
\end{array}$$

Note that the boundary homomorphism δ corresponds to differentials in the algebraic

Atiyah-Hirzebruch spectral sequence of $\widetilde{X}$. One can check, using the naturality of the algebraic Atiyah-Hirzebruch spectral sequence for the quotient map $P_{14}^{22} \twoheadrightarrow \widetilde{X}$, the other elements (with the right choices) maps to zero under the boundary homomorphism δ.

This completes the proof. ☐

The following lemma is a consequence of Corollary 3.7.2 and naturality of the Adams spectral sequence.

Lemma 3.8.3. *In the Adams spectral sequence of $\widetilde{X}$, the elements $h_2g_2[14]$, $h_1g_2[16]$ and $h_0h_4^3[16]$ are permanent cycles.*

Proof. By Corollary 3.7.2, the elements $h_2g_2[14]$, $h_1g_2[16]$ and $h_0h_4^3[16]$ are surviving cycles in the Adams spectral sequence of $\Sigma^{14}C\eta$. In particular, they are permanent cycles. Since $\Sigma^{14}C\eta$ is the 16-skeleton of $\widetilde{X}$, by naturality for the map

$$\Sigma^{14}C\eta \hookrightarrow \widetilde{X},$$

these elements are also permanent cycles in the Adams spectral sequence of $\widetilde{X}$. ☐

Lemma 3.8.4. *In the Adams spectral sequence of $\widetilde{X}$, the element $g_2[17]$ is a permanent cycle.*

Proof. By Lemma 3.5.3, S^{17} is an $H\mathbb{F}_2$-subcomplex of $\widetilde{X}$. Since g_2 is a permanent cycle in the Adams spectral sequence of S^0, by the naturality for the inclusion map, it is also a permanent cycle in the Adams spectral sequence of $\widetilde{X}$. ☐

Lemma 3.8.5. *In the Adams spectral sequence of $\widetilde{X}$, we have a d_2 differential*

$$d_2(f_1[21]) = h_0^2 f_1[20].$$

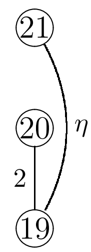
To prove Lemma 3.8.5, we need to prove the following lemma.

Lemma 3.8.6. *We have a quotient map* $q : P_{19}^{21} \twoheadrightarrow S^{20}$*. Moreover, we have* $q_\sharp(f_1[21]) = h_0c_2[20]$*, where* $q_\sharp : Ext(P_{19}^{21}) \to Ext(S^{20})$ *is the induced map on the Adams* E_2*-page.*

Proof. By James periodicity, the quotient map $q : P_{19}^{21} \twoheadrightarrow S^{20}$ maps through P_{20}^{21}.

$$P_{19}^{21} \xrightarrow{q_1} P_{20}^{21} \xrightarrow{q_2} S^{20}$$

The cell diagram of P_{19}^{21} is the following:



In $Ext(P_{20}^{21})$, we define the element $\underline{f_1[21]}$ to be the image of $f_1[21]$ in $Ext(S^{21})$ under the inclusion map $i : S^{21} \hookrightarrow P_{20}^{21}$, i.e., $\underline{f_1[21]} = i_\sharp(f_1[21])$.

$$\begin{array}{ccccc} & & Ext(S^{21}) & & \\ & & \downarrow i_\sharp & & \\ Ext(P_{19}^{21}) & \xrightarrow{q_{1\sharp}} & Ext(P_{20}^{21}) & \xrightarrow{q_{2\sharp}} & Ext(S^{20}) \end{array}$$

$$f_1[21] \longmapsto \begin{array}{c}\underline{f_1[21]} \\ +h_0c_2[20]\end{array} \longmapsto h_0c_2[20]$$

By naturality of the algebraic Atiyah-Hirzebruch spectral sequence, we have $q_{2\sharp}(\underline{f_1[21]}) = 0$. Therefore, in $Ext(P_{20}^{21})$, the element $\underline{f_1[21]} + h_0c_2[20]$ maps to $h_0c_2[20]$ in $Ext(S^{20})$, i.e.,

$$q_{2\sharp}(\underline{f_1[21]} + h_0c_2[20]) = h_0c_2[20].$$

Now we consider the cofiber sequence associated to the map q_1.

$$S^{19} \hookrightarrow P_{19}^{21} \xrightarrow{q_1} P_{20}^{21} = S^{21} \vee S^{20} \longrightarrow \Sigma S^{19}$$

Both elements $\underline{f_1[21]}$ and $h_0c_2[20]$ map to $h_1f_1[19]$ in $Ext(\Sigma S^{19})$. In fact, it follows from the fact that the 21-cell is attached to the 19-cell by η, and the 20-cell is attached to the 19-cell by 2. Note also that there is a relation $h_0^2c_2 = h_1f_1$ in Ext. Therefore, the sum $\underline{f_1[21]} + h_0c_2[20]$ maps to 0 in $Ext(\Sigma S^{19})$, and must comes from $Ext(P_{19}^{21})$ by exactness. By naturality of the algebraic Atiyah-Hirzebruch spectral sequence, it must come from $f_1[21]$, i.e.,

$$q_{1\sharp}(f_1[21]) = \underline{f_1[21]} + h_0c_2[20].$$

Combining with

$$q_{2\sharp}(\underline{f_1[21]} + h_0c_2[20]) = h_0c_2[20],$$

we have

$$q_\sharp(f_1[21]) = h_0c_2[20].$$

□

Now we present the proof of Lemma 3.8.5.

Proof. In the Adams spectral sequence of S^0, we have a differential

$$d_2(h_0c_2) = h_0^2f_1.$$

Now consider the following commutative diagram:

$$\begin{array}{ccccc} \widetilde{X} & \xrightarrow{q_3} & P_{19}^{22} & \xrightarrow{q_4} & S^{20} \\ & & {\scriptstyle i}\uparrow & & \| \\ & & P_{19}^{21} & \xrightarrow{q} & S^{20} \end{array}$$

where q_3 is obtained from $\widetilde{X}$ by quotienting out its 18-skeleton, q_4 is a quotient map that follows essentially from Theorem 3.4.7 and James periodicity, and i is an inclusion map. By Lemma 3.8.6, the d_2 differential in S^{20}:

$$d_2(h_0c_2[20]) = h_0^2f_1[20]$$

can be pulled back to get a d_2 differential in P_{19}^{21}:

$$d_2(f_1[21]) = h_0^2f_1[20].$$

This differential can be further pushed forward by i, and then pulled back by q_3 to get the d_2 differential in $\widetilde{X}$:

$$d_2(f_1[21]) = h_0^2f_1[20].$$

Note that in $Ext(\widetilde{X})$, elements of lower Atiyah-Hirzebruch filtrations, i.e., $h_2g_2[14]$ and $h_1g_2[16]$, have already been shown to survive by Lemma 3.8.3. □

Lemma 3.8.7. *The elements $h_1h_5c_0[21]$ and $h_1^2h_3h_5[21]$ are permanent cycles in $Ext(\widetilde{X})$.*

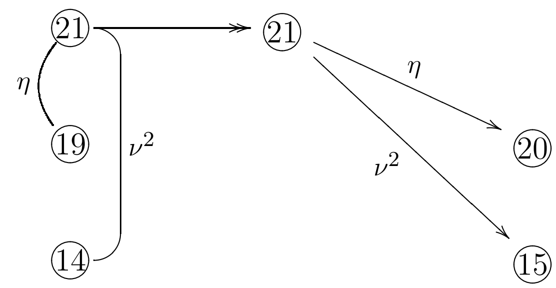
Proof. We consider the $H\mathbb{F}_2$-subcomplex X^{21}. Since there are only three cells in X^{21}, the computation of the Adams E_2 page of X^{21} in the 61 stem for $s \leq 5$ is straightforward by using the algebraic Atiyah-Hirzebruch spectral sequence.

Table 3.3: The Adams E_2 page of X^{21} and S^{21} in the 61 stem for $s \leq 5$

$s\backslash 61-$stem of	X^{21}	S^{21}
5	$h_1h_5c_0[21]$	$h_1h_5c_0[21]$
	$h_2g_2[14]$	$\bullet$
4	$h_1^2h_3h_5[21]$	$h_1^2h_3h_5[21]$
		$f_1[21]$

By Theorem 3.5.14, the $H\mathbb{F}_2$-subcomplex X^{21} fits into a cofiber sequence

$$X^{21} \xrightarrow{q_{21}} S^{21} \xrightarrow{(\eta,\nu^2)} S^{20} \vee S^{15}$$



Here q_{21} is the quotient map. We therefore have a long exact sequence of homotopy groups. Suppose $\alpha \in \pi_{61}(S^{21})$, and α lies in the kernel of the map

$$(\eta, \nu^2) : \pi_{61}(S^{21}) \longrightarrow \pi_{61}(S^{20}) \oplus \pi_{61}(S^{15}).$$

Then α must satisfy the following conditions:

$$\eta \cdot \alpha = 0, \ \nu^2 \cdot \alpha = 0.$$

We verify that the elements $h_1h_5c_0[21]$ and $h_1^2h_3h_5[21]$ each detect a class that satisfies the above condition. In fact, we have that

$$0 \in \eta \cdot \{h_1^2h_3h_5\}, \ 0 \in \eta \cdot \{h_1h_5c_0\}, \ \text{ and } \ \nu \cdot \pi_{40}(S^0) = 0.$$

Therefore, by exactness of homotopy groups, in $\pi_{61}(X^{21})$, there exist classes that map nontrivially to $\pi_{61}(S^{21})$. Furthermore, these classes are in Adams filtration at most 5. By naturality of the algebraic Atiyah-Hirzebruch spectral sequence, the classes detected by $h_2g_2[14]$ map trivially to $\pi_{61}(S^{21})$. It follows that $h_1h_5c_0[21]$ and $h_1^2h_3h_5[21]$ survive in the Adams spectral sequence of X^{21}. In particular, they are permanent cycles. Since X^{21} is an $H\mathbb{F}_2$-subcomplex of $\widetilde{X}$, both elements are permanent cycles in the Adams spectral sequence of $\widetilde{X}$. □

Lemma 3.8.8. *The elements* $h_3d_1[22]$*,* $h_5c_0[22]$ *and* $\underline{h_1h_3h_5[22]}$ *are permanent cycles in the Adams spectral sequence of* $\widetilde{X}$*.*

Proof. For the element $\underline{h_1h_3h_5[22]}$, we consider the $H\mathbb{F}_2$-subcomplex X^{22}, since it is defined by the image of $h_1h_3h_5[22]$ in $Ext(X^{22})$. For the elements $h_3d_1[22]$ and $h_5c_0[22]$, we use both of the $H\mathbb{F}_2$-subcomplexes X^{22} and $\widehat{X^{22}}$. The reason we use different $H\mathbb{F}_2$-subcomplexes here is explained in Remark 3.8.9.

Using the algebraic Atiyah-Hirzebruch spectral sequences, and their naturality for the maps

$$X^{22} \hookrightarrow \widehat{X^{22}} \hookrightarrow \widetilde{X},$$

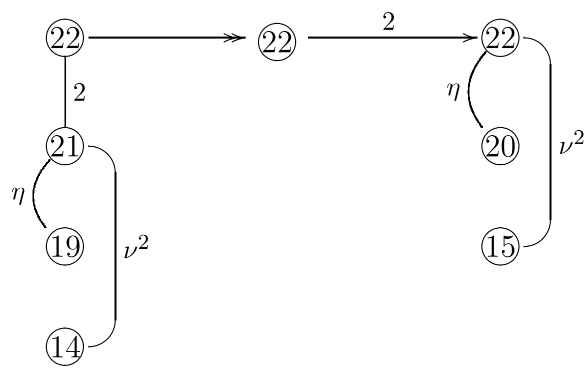
we compute the Adams E_2-page of X^{22} and $\widehat{X^{22}}$ in the 61 stem for $s \leq 5$.

Table 3.4: The Adams E_2 page of $X^{22}, \widehat{X^{22}}$ and S^{22} in the 61 stem for $s \leq 5$

$s\backslash 61-$ stem of	X^{22}	$\widehat{X^{22}}$	S^{22}
5	$h_3d_1[22]$	$h_3d_1[22]$	$h_3d_1[22]$
	$h_1h_5c_0[21]$	$h_1h_5c_0[21]$	
	$h_2g_2[14]$	$h_2g_2[14]$	
		$h_1g_2[16]$	
4	$h_5c_0[22]$	$h_5c_0[22]$	$h_5c_0[22]$
	$h_1^2h_3h_5[21]$	$h_1^2h_3h_5[21]$	
		$h_0h_4^3[16]$	
3	$h_1h_3h_5[22]$	$h_1h_3h_5[22]$	$h_1h_3h_5[22]$
		$h_4^3[16]$	

By Definition 5.6, the complex X^{22} fits into a cofiber sequence

$$X^{22} \xrightarrow{q} S^{22} \xrightarrow{a} \Sigma X^{21}$$



Here q is the quotient map, and a is the suspension of the attaching map of the 22-cell in X^{22}. We have a long exact sequence of homotopy groups associated to this cofiber sequence. Suppose $\alpha[22]$ is an element in $\pi_{61}(S^{22})$. Suppose further that $\alpha[22]$ supports a differential in the Atiyah-Hirzebruch spectral sequence of X^{22}. By the construction of the Atiyah-Hirzebruch spectral sequence, the target of the differential that $\alpha[22]$ supports detects $\Delta(\alpha[22])$ in the homotopy groups of lower skeleton, where the map

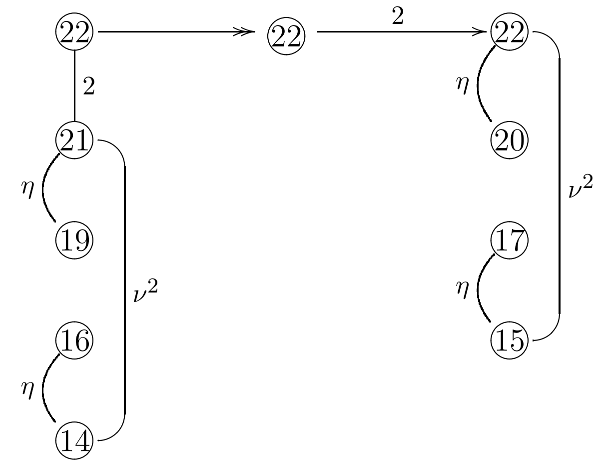
$$\Delta : \pi_{61}(S^{22}) \longrightarrow \pi_{61}(\Sigma X^{21})$$

is the boundary homomorphism in the long exact sequence of homotopy groups.

For the element $\underline{h_1h_3h_5[22]}$, we consider the homotopy class $\alpha = \sigma\eta_5 \in \pi_{39}$, which is detected by $h_1h_3h_5$ in the E_∞-page of the Adams spectral sequence of S^0. By Proposition 3.6.3, the element $\alpha[22]$ is a permanent cycle in the Atiyah-Hirzebruch spectral sequence of X^{22}. Therefore, by exactness of the long exact sequence of homotopy groups, there exists a homotopy class in $\pi_{61}(X^{22})$, which has Adams filtration at most 3. This implies the element $h_1h_3h_5[22]$ survives in $Ext(X^{22})$, since it is the only element with Adams filtration at most 3. In particular, it is a permanent cycle. Therefore, its image in $Ext(\widetilde{X})$, i.e., $\underline{h_1h_3h_5[22]}$, is also a permanent cycle.

By Definition 3.5.5, the complex $\widehat{X^{22}}$ fits into a cofiber sequence

$$\widehat{X^{22}} \xrightarrow{\ q'\ } S^{22} \xrightarrow{\ a'\ } \Sigma\widehat{X^{21}}$$



Here q' is the quotient map, and a' is the suspension of the attaching map of the 22-cell in $\widehat{X^{22}}$. We have a long exact sequence of homotopy groups associated to this cofiber sequence. Suppose $\alpha'[22]$ is an element in $\pi_{61}(S^{22})$.

Suppose further that $\alpha'[22]$ supports a differential in the Atiyah-Hirzebruch spectral sequence of $\widehat{X^{22}}$. By the construction of the Atiyah-Hirzebruch spectral sequence, the target of the differential that $\alpha'[22]$ supports detects $\Delta'(\alpha'[22])$ in the homotopy groups of lower skeleton, where the map

$$\Delta' : \pi_{61}(S^{22}) \longrightarrow \pi_{61}(\Sigma\widehat{X^{21}}).$$

is the boundary homomorphism in the long exact sequence of homotopy groups.

For the element $h_5c_0[22]$, we consider a homotopy class α' in $\{h_5c_0\} \in \pi_{39}$, such that $2 \cdot \alpha' = 0$. Such a class exists, since there is no 2-extension from h_5c_0 in the E_∞-page of the Adams spectral sequence of S^0. By Proposition 3.6.3, we have a differential in the Atiyah-Hirzebruch spectral sequence of X^{22}:

$$d_8(\alpha'[22]) = \eta\phi[14],$$

where $\phi \in \pi_{45}$ is detected by $h_5 d_0$, such that $\eta \cdot \phi \in \langle \alpha', 2, \nu^2 \rangle$.

We map this differential to the Atiyah-Hirzebruch spectral sequence of $\widehat{X^{22}}$. Since the 16-skeleton of $\widehat{X^{22}}$ is $\Sigma^{14} C\eta$, we have a differential in the Atiyah-Hirzebruch spectral sequence of $\widehat{X^{22}}$:

$$d_2(\phi[16]) = \eta\phi[14].$$

This implies the following differential

$$d_8(\alpha'[22]) = 0.$$

That is, $\alpha'[22]$ is a permanent cycle in the Atiyah-Hirzebruch spectral sequence of $\widehat{X^{22}}$. Therefore, by exactness of the long exact sequence of homotopy groups, there exists a homotopy class in $\pi_{61}(\widehat{X^{22}})$, which has Adams filtration at most 4. By naturality of the Adams spectral sequence for the quotient map $\widehat{X^{22}} \twoheadrightarrow S^{22}$, the class that detects $\alpha'[22]$ in $Ext(\widehat{X^{22}})$ must map nontrivially to $Ext(S^{22})$.

$$\begin{array}{ccc} Ext(\widehat{X^{22}}) & \longrightarrow & Ext(S^{22}) \\ \Downarrow & & \Downarrow \\ \pi_*(\widehat{X^{22}}) & \longrightarrow & \pi_*(S^{22}) \end{array}$$

Since the element $\underline{h_1 h_3 h_5[22]}$ is already accounted for, by filtration arguments, the only possibility is that $h_5 c_0[22]$ detects $\alpha'[22]$. In particular, the element $h_5 c_0[22]$ is a permanent cycle in the Adams spectral sequence of $\widehat{X^{22}}$. Therefore, its image in $Ext(\widetilde{X})$ is also a permanent cycle.

For the element $h_3 d_1[22]$, we consider the homotopy class $\alpha'' = \sigma\{d_1\} \in \pi_{39}$, which is detected by $h_3 d_1$ in the E_∞-page of the Adams spectral sequence of S^0. Note that the notation $\{d_1\}$ has indeterminacy, but for our purpose, any class in the set $\{d_1\}$ works. By

Proposition 3.6.3, we have a differential in the Atiyah-Hirzebruch spectral sequence of X^{22}:

$$d_8(\alpha''[22]) \subseteq \eta^2\pi_{44}[14].$$

We map this differential to the Atiyah-Hirzebruch spectral sequence of $\widehat{X^{22}}$. Since the 16-skeleton of $\widehat{X^{22}}$ is $\Sigma^{14}C\eta$, we have some d_2 differentials in the Atiyah-Hirzebruch spectral sequence of $\widehat{X^{22}}$ that kill $\eta^2\pi_{44}[14]$. This implies the following differential

$$d_8(\alpha''[22]) = 0.$$

That is, $\alpha''[22]$ is a permanent cycle in the Atiyah-Hirzebruch spectral sequence of $\widehat{X^{22}}$. Therefore, by exactness of the long exact sequence of homotopy groups, there exists a homotopy class in $\pi_{61}(\widehat{X^{22}})$, which has Adams filtration at most 5. By naturality of the Adams spectral sequence for the quotient map $\widehat{X^{22}} \twoheadrightarrow S^{22}$, the class that detects $\sigma\{d_1\}[22]$ in $Ext(\widehat{X^{22}})$ must map nontrivially to $Ext(S^{22})$. Since the elements $\underline{h_1h_3h_5[22]}$ and $h_5c_0[22]$ are already accounted for, by filtration arguments, the only possibility is $h_3d_1[22]$. In particular, the element $h_3d_1[22]$ is a permanent cycle in the Adams spectral sequence of $\widehat{X^{22}}$. Therefore, its image in $Ext(\widetilde{X})$ is also a permanent cycle. □

Remark 3.8.9. For the element $h_5c_0[22]$, if we use the $H\mathbb{F}_2$-subcomplex X^{22} instead of $\widehat{X^{22}}$, it would support an Adams d_2 differential that kills $h_1h_5d_0[14]$. With the 16-cell, $h_1h_5d_0[14]$ is killed by $h_5d_0[16]$ in the Curtis table, therefore isn't present in the Adams E_2 page of $\widehat{X^{22}}$.

Lemma 3.8.10. *The element $h_1f_1[20]$ is a permanent cycle in the Adams spectral sequence of $\widetilde{X}$.*

Proof. We consider the $H\mathbb{F}_2$-subcomplex X^{20}. Using the algebraic Atiyah-Hirzebruch spectral sequence, we compute the Adams E_2 page of X^{20} in the 61 stem for $s \leq 5$. This computation is straightforward: all differentials in this range follow by the multiplication by

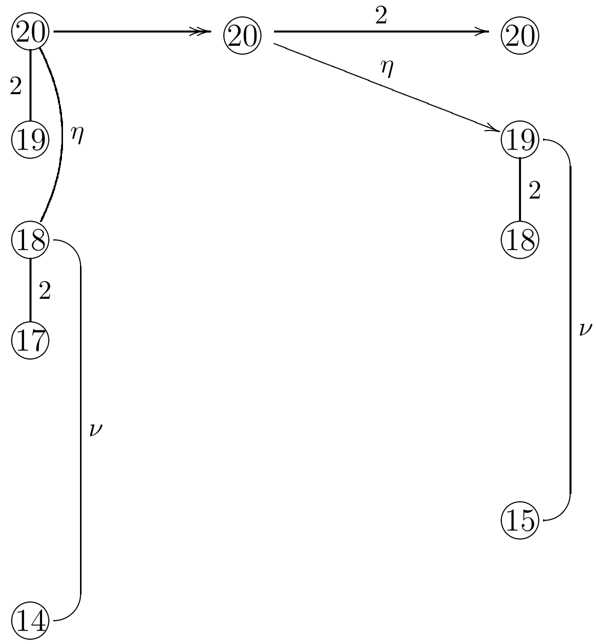
2 attaching maps.

Table 3.5: The Adams E_2 page of X^{20} and S^{20} in the 61 stem for $s \leq 5$

$s \backslash 61-$ stem of	X^{20}	S^{20}
5	$h_1 f_1[20]$	$h_1 f_1[20]$
	$h_2 g_2[14]$	
4	$g_2[17]$	$\bullet$
3		$\bullet$

The complex X^{20} fits into a cofiber sequence

$$X^{20} \xrightarrow{q'''} S^{20} \xrightarrow{a'''} \Sigma(S^{19} \vee X^{18})$$

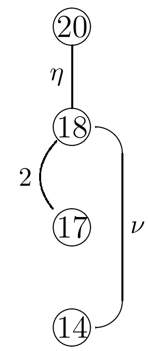


Here q''' is the quotient map, X^{18} is the 18-skeleton of X^{20}, a''' is suspension of the attaching map of the 20-cell in X^{20}. We have a long exact sequence of homotopy groups associated to this cofiber sequence. Suppose $\alpha'''[20]$ is an element in $\pi_{61}(S^{20})$. Suppose further that $\alpha'''[20]$ supports a differential in the Atiyah-Hirzebruch spectral sequence of X^{20}. By the construction of the Atiyah-Hirzebruch spectral sequence, the target of the differential that $\alpha'''[20]$ supports detects $\Delta'''(\alpha'''[20])$ in the homotopy groups of lower skeleton, where the map

$$\Delta''' : \pi_{61}(S^{22}) \longrightarrow \pi_{61}(\Sigma X^{21})$$

is the boundary homomorphism in the long exact sequence of homotopy groups.

By Lemma 3.4.4 and Remark 3.5.8, we have S^{19} as an $H\mathbb{F}_2$-subcomplex of X^{20}. We consider its cofiber X^{20}/S^{19}.



X^{20}/S^{19}

For the element $h_1 f_1[20]$, we consider the homotopy class $\alpha''' = \sigma\{h_0 h_2 h_5\} \in \pi_{41}$. Because of Lemma 3.11.4, $h_1 f_1$ detects $\sigma\{h_0 h_2 h_5\}$ in the Adams E_∞-page of S^0. By Proposition 6.4, the element $\alpha'''[20]$ is a permanent cycle in the Atiyah-Hirzebruch spectral sequence of X^{20}/S^{19}.

In the Atiyah-Hirzebruch spectral sequence of X^{20}, we have the differential

$$d_1(\alpha'''[20]) = 0$$

since the attaching map from the 20-cell to the 19-cell is multiplication by 2 and

$$2 \cdot \alpha''' \in 2 \cdot \pi_{41} = 0.$$

Using the fact that the 19-cell of the 19-skeleton of X^{20} splits off, and the naturality of the Atiyah-Hirzebruch spectral sequences for the quotient map $X^{20} \twoheadrightarrow X^{20}/S^{19}$, the element $\alpha'''[20]$ survives in the Atiyah-Hirzebruch spectral sequence of X^{20}.

Therefore, by exactness of the long exact sequence of homotopy groups, there exists a homotopy class in $\pi_{61}(X^{20})$, which has Adams filtration at most 5. By naturality of the Adams spectral sequence for the quotient map $X^{20} \twoheadrightarrow S^{20}$, the class that detects $\alpha'''[20]$ in $Ext(X^{20})$ must map nontrivially to $Ext(S^{20})$.

$$\begin{array}{ccc} Ext(X^{20}) & \longrightarrow & Ext(S^{20}) \\ \Downarrow & & \Downarrow \\ \pi_*(X^{20}) & \longrightarrow & \pi_*(S^{20}) \end{array}$$

By filtration arguments, the only possibility is $h_1 f_1[20]$. In particular, the element $h_1 f_1[20]$ is a permanent cycle in the Adams spectral sequence of X^{20}. Therefore, its image in $Ext(\widetilde{X})$ is also a permanent cycle. □

3.9 The Adams spectral sequence of X

In this section, we establish Step 3 and Step 4 by proving Theorems 3.9.1 and 3.9.2. Combining them together, we have Corollary 3.9.3.

Theorem 3.9.1. *In the Adams spectral sequence of X, we have the differential*

$$d_4(h_4^3[16]) = B_1[14].$$

The following Theorem 3.9.2 is a consequence of Lemma 3.8.8.

Theorem 3.9.2. *In the Adams spectral sequence of X, the chosen element $\underline{h_1 h_3 h_5[22]}$ is a permanent cycle. Here $\underline{h_1 h_3 h_5[22]}$ is defined to be the image of $h_1 h_3 h_5[22]$ in $Ext(X^{22})$.*

Proof. Since the map $X^{22} \hookrightarrow X$ maps through $\widetilde{X}$, we have $\underline{h_1 h_3 h_5[22]}$ in $Ext(\widetilde{X})$ maps to

$\underline{h_1h_3h_5[22]}$ in $Ext(X)$.

$$\begin{array}{ccccc} Ext^{3,61+3}(X^{22}) & \longrightarrow & Ext^{3,61+3}(\widetilde{X}) & \longrightarrow & Ext^{3,61+3}(X) \\ h_1h_3h_5[22] & \longmapsto & \underline{h_1h_3h_5[22]} & \longmapsto & \underline{h_1h_3h_5[22]} \end{array}$$

By Lemma 3.8.8, $\underline{h_1h_3h_5[22]}$ is a permanent cycle in $Ext(\widetilde{X})$. Therefore, by naturality of the Adams spectral sequences, $\underline{h_1h_3h_5[22]}$ is also a permanent cycle in $Ext(X)$. □

From Theorem 3.9.1 and 3.9.2, we have the following corollary.

Corollary 3.9.3. *In the Adams spectral sequence of X, we have the differential*

$$d_4(\underline{h_1h_3h_5[22]} + h_4^3[16]) = B_1[14].$$

In the rest of this section, we prove Theorem3. 9.1. The idea is to push the d_4 differential in the Adams spectral sequence of $\widetilde{X}$ into that of X, and check the element $B_1[14]$ is not killed by an Adams d_2 or d_3 differential.

Proof. Recall from Remark 3.5.11 that the Adams E_2 page of X splits as follows:

$$Ext(X) = Ext(\widetilde{X}) \oplus Ext(S^{23}).$$

Therefore, by Lemma 3.8.2, we have the Adams E_2 page of X in the 60 and 61 stems for $s \leq 7$ in the following Table 3.6.

Note that by naturality of the Adams spectral sequences for the inclusion map $\widetilde{X} \hookrightarrow X$, and the proof of the Theorem 3.8.1, no $\bullet$'s in Adams filtration 4 and 5 can kill $B_1[14]$. Therefore, to prove Theorem 3.9.1, we only need to show that

$$d_2(h_0^3h_3h_5[23]) \neq B_1[14],$$

Table 3.6: The Adams E_2 page of X in the 60 and 61 stems for $s \leq 7$

$s \backslash t-s$	60	61
7	$B_1[14]$	$\bullet$
	$\bullet$	$\bullet$
	$h_1t[23]$	$\bullet$
	$h_0^2x[23]$	
6	$\bullet$	$\bullet$
	$\bullet$	$\bullet$
	$\bullet$	$\bullet$
	$h_0x[23]$	$\bullet$
		$\bullet$
5	$\bullet$	$\bullet$
	$\bullet$	$\bullet$
	$x[23]$	$\bullet$
		$\bullet$
		$\bullet$
		$h_0^3h_3h_5[23]$
4	$\bullet$	$\bullet$
		$\bullet$
		$\bullet$
		$\bullet$
		$\bullet$
		$e_1[23]$
		$h_0^2h_3h_5[23]$
3	$\bullet$	$h_4^3[16]$
	$\bullet$	$h_1h_3h_5[22]$
		$h_0h_3h_5[23]$
2		$h_3h_5[23]$

$$d_3(h_0^2h_3h_5[23]) \neq B_1[14],$$

$$d_3(e_1[23]) \neq B_1[14].$$

For the elements $h_0^2h_3h_5[23]$ and $h_0^3h_3h_5[23]$, we will show that

$$d_2(h_3h_5[23]) = 0,$$

$$d_3(h_3h_5[23]) = 0,$$

which by Leibniz's rule implies that

$$d_2(h_0^3 h_3 h_5[23]) = h_0^3 \cdot d_2(h_3 h_5[23]) = 0,$$

$$d_3(h_0^2 h_3 h_5[23]) = h_0^2 \cdot d_3(h_3 h_5[23]) = 0.$$

We consider the $H\mathbb{F}_2$-subcomplex X^{23} in Definition 3.5.10. Recall that X^{23} consists of two cells in dimension 14 and 23. Since there is no primary Steenrod operation connecting them, we have

$$Ext(X^{23}) = Ext(S^{14}) \oplus Ext(S^{23}).$$

Therefore, we have the Adams spectral sequence of X^{23} in the 60 and 61 stems for $s \leq 5$ in the following Table 3.7. In the Adams spectral sequence of X^{23}, we have $d_2(h_3h_5[23]) = 0$, since

Table 3.7: The Adams E_2 page of X^{23} in the 60 and 61 stems for $s \leq 5$

$s \backslash t-s$	60	61
5	$x[23]$	$h_0^3 h_3 h_5[23]$ $\bullet$
4		$e_1[23]$ $h_0^2 h_3 h_5[23]$
3	$\bullet$	$h_0 h_3 h_5[23]$
2		$h_3 h_5[23]$

the target lies in the zero group. If $d_3(h_3h_5[23]) \neq 0$, then we must have that $d_3(h_3h_5[23]) = x[23]$, since that is the only possibility. By mapping through the quotient map $X^{23} \twoheadrightarrow S^{23}$, this differential would imply that $d_3(h_3h_5[23]) = x[23]$ in the Adams spectral sequence of S^{23}. However, in S^0, we have that $d_3(h_3h_5) = 0$. Contradiction! Therefore, we must have the differential $d_3(h_3h_5[23]) = 0$ in the Adams spectral sequence of X^{23}, and therefore also in that of X.

For the element $e_1[23]$, suppose we have $d_3(e_1[23]) = B_1[14]$ in the Adams spectral sequence of X. By naturality for the quotient map $X \twoheadrightarrow S^{23}$, we have $d_3(e_1[23]) = 0$ in the Adams spectral sequence of S^{23}, since the target $B_1[14]$ maps to zero in the E_2-page by naturality of the algebraic Atiyah-Hirzebruch spectral sequences. However, this contradicts Bruner's differential (15, Theorem 4.1) in S^0:

$$d_3(e_1) = h_1 t.$$

Therefore, we must have $d_3(e_1[23]) \neq B_1[14]$, which completes the proof. $\square$

3.10 The pull back

In this section, we prove Step 5: based on Corollary 3.9.3, we prove the following Theorem 3.10.1.

Theorem 3.10.1. *In the Adams spectral sequence of P_1^{23}, we have a d_3 differential:*

$$d_3(h_1h_3h_5[22]) = G[6].$$

Proof. We have the Adams E_2-page of P_1^{23} from the Curtis table.

We will show in Lemma 3.10.3 that

$$f_\sharp(h_1h_3h_5[22]) = \underline{h_1h_3h_5[22]} + h_4^3[16],$$

where $f_\sharp : Ext(P_1^{23}) \to Ext(X)$ is induced by the composition of the two quotient maps $f_1 : P_1^{23} \twoheadrightarrow P_{14}^{23}$, $f_2 : P_{14}^{23} \twoheadrightarrow X$. By Corollary 3.9.3 that in the Adams spectral sequence of X, we have the differential

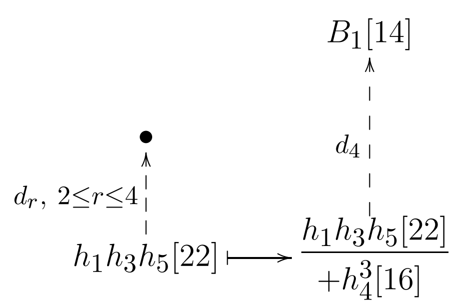
$$d_4(\underline{h_1h_3h_5[22]} + h_4^3[16]) = B_1[14],$$

and the naturality of the Adams spectral sequence, the element $h_1h_3h_5[22]$ in $Ext(P_1^{23})$ must support a nontrivial d_2, d_3 or d_4 differential.

Table 3.8: The Adams E_2-page of P_1^{23} in the 60 and 61 stems for $s \leq 7$

$s\backslash t-s$	60	61
7	$\bullet[3]$	$\bullet$
	$\bullet[5]$	$\bullet$
	$\bullet[21]$	$\bullet$
	$\bullet[23]$	$\bullet$
	$\bullet[23]$	$\bullet$
		$\bullet$
6	$G[6]$	$\bullet$
	$\bullet[20]$	$\bullet$
	$\bullet[22]$	$\bullet$
	$\bullet[23]$	$\bullet$
		$\bullet$
5		$\bullet$
		$\bullet$
		$\bullet$
		$\bullet$
		$\bullet$
		$\bullet$
4	$\bullet$	$\bullet$
		$\bullet$
		$\bullet$
3	$\bullet$	$\bullet$
		$h_1h_3h_5[22]$

$$Ext(P_1^{23}) \xrightarrow{f_\sharp} Ext(X)$$



From the table of the Adams E_2-page of P_1^{23}, we have the following three possibilities.

1. It supports a nontrivial d_3 or d_4 differential that kills one of the elements $\bullet[i]$ with $20 \leq i \leq 23$.

2. It supports a nontrivial d_4 differential that kills one of the elements $\bullet[i]$ with $i = 3, 5$.

3. It supports a nontrivial d_3 differential that kills $G[6]$.

For (1), since these target elements map nontrivially to $Ext(X)$, this would contradict Theorem 3.9.1. For (2), from the Curtis table, these two elements exist in $Ext(P_1^n)$ for all $n \geq 5$. In particular, they exist in $Ext(P_1^{13})$, and map trivially to $Ext(P_{14}^{23})$ in the following long exact sequence

$$\cdots \longrightarrow Ext(P_1^{13}) \longrightarrow Ext(P_1^{23}) \longrightarrow Ext(P_{14}^{23}) \longrightarrow \cdots,$$

and hence trivially to $Ext(X)$. Since they have the same filtration as $B_1[14]$, this would contradict Theorem 3.9.1.

Therefore, (3) is the only possibility. □

Remark 3.10.2. The reason we use P_1^{23} instead of P_1^{22} is that, in the bidegree $(s, t-s) = (5, 60)$ of the Curtis table, the element $h_5 f_0[11]$ is killed by a $\bullet[23]$. Therefore, in $Ext(P_1^{22})$, the element $h_5 f_0[11]$ is present, and leaves a possibility of a nontrivial Adams d_2 differential. We add the 23-cell to make this go away.

We now prove Lemma 3.10.3.

Lemma 3.10.3. *We have*

$$f_\sharp(h_1 h_3 h_5[22]) = \underline{h_1 h_3 h_5[22]} + h_4^3[16],$$

where $f_\sharp : Ext(P_1^{23}) \to Ext(X)$ *is the homomorphism induced by the composition of the two quotient maps*

$$f_1 : P_1^{23} \twoheadrightarrow P_{14}^{23}, \ f_2 : P_{14}^{23} \twoheadrightarrow X.$$

Proof. By naturality of the algebraic Atiyah-Hirzebruch spectral sequences, we have

$$f_{1\sharp}(h_1h_3h_5[22]) = h_1h_3h_5[22].$$

We only need to show that

$$f_{2\sharp}(h_1h_3h_5[22]) = \underline{h_1h_3h_5[22]} + h_4^3[16].$$

$$\begin{array}{ccccc} & & & & Ext(X^{22}) \\ & & & & \downarrow i_\sharp \\ Ext(P_1^\infty) & \xrightarrow{f_{1\sharp}} & Ext(P_{14}^{23}) & \xrightarrow{f_{2\sharp}} & Ext(X) \end{array}$$

$$h_1h_3h_5[22] \longmapsto h_1h_3h_5[22] \longmapsto \begin{array}{c}\underline{h_1h_3h_5[22]} \\ +h_4^3[16]\end{array}$$

Consider the cofiber sequence that defines X

$$S^{15} \hookrightarrow P_{14}^{23} \xrightarrow{f_2} X \longrightarrow \Sigma S^{15}.$$

This gives a long exact sequence of Ext groups:

$$\cdots \longrightarrow Ext(S^{15}) \longrightarrow Ext(P_{14}^{23}) \xrightarrow{f_{2\sharp}} Ext(X) \xrightarrow{\Delta_2} Ext(\Sigma S^{15}) \longrightarrow \cdots.$$

We only need to show that the boundary map Δ_2 satisfies

$$\Delta_2(\underline{h_1h_3h_5[22]} + h_4^3[16]) = 0.$$

In fact, by exactness, the element $\underline{h_1h_3h_5[22]} + h_4^3[16]$ must come from $Ext(P_{14}^{23})$. By naturality of the algebraic Atiyah-Hirzebruch spectral sequence, it must comes from $h_1h_3h_5[22]$, i.e., we must have

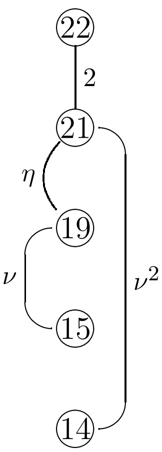
$$f_{2\sharp}(h_1h_3h_5[22]) = \underline{h_1h_3h_5[22]} + h_4^3[16],$$

which completes the proof.

To show $\Delta_2(\underline{h_1h_3h_5[22]} + h_4^3[16]) = 0$, we consider an $H\mathbb{F}_2$-subcomplex W of X. Since X^{22} is an $H\mathbb{F}_2$-subcomplex of X, we define W to be the homotopy pull back of X^{22} along the quotient map $f_2 : P_{14}^{23} \twoheadrightarrow X$. By Lemma 3.4.4, we have (W, j) as an $H\mathbb{F}_2$-subcomplex of P_{14}^{23} in the following commutative diagram of cofiber sequences:

$$\begin{array}{ccccccc}
S^{15} & \hookrightarrow & W & \twoheadrightarrow & X^{22} & \xrightarrow{a_1} & \Sigma S^{15} \\
\| & & \downarrow{\scriptstyle j} & & \downarrow{\scriptstyle i} & & \| \\
S^{15} & \hookrightarrow & P_{14}^{23} & \xrightarrow{f_2} & X & \xrightarrow{a_2} & \Sigma S^{15}
\end{array}$$

As an illustration, the cell diagram of W is the following:



We will show in the following Lemma 3.10.4 that

$$\Delta_1(h_1h_3h_5[22]) = h_0h_4^3[15],$$

where Δ_1 is the boundary map of Ext groups associated to the cofiber sequence defining W. Therefore, following the commutative diagram of cofiber sequences and the definition of the element $\underline{h_1h_3h_5[22]}$, we have

$$\Delta_2(\underline{h_1h_3h_5[22]}) = h_0h_4^3[15].$$

The fact that the 16-cell in P_{14}^{23} is attached to the 15-cell by 2 gives us

$$\Delta_2(h_4^3[16]) = h_0 h_4^3[15].$$

Therefore, we have

$$\Delta_2(\underline{h_1 h_3 h_5[22]} + h_4^3[16]) = 0,$$

as claimed. $\square$

Lemma 3.10.4. $\Delta_1(h_1 h_3 h_5[22]) = h_0 h_4^3[15]$.

Proof. We use the Lambda complex (see Section 7.1 of (68)) to compute the E_2-page of the Adams spectral sequence in a functorial way. Recall from (68) that, for any spectrum Y, we can construct a differential graded module $H_*(Y) \otimes \Lambda^{*,*}$ over the Lambda algebra $\Lambda^{*,*}$. Differentials in this complex are generated by

$$d(x) = \Sigma_{i \geq 1} Sq_*^i(x) \otimes \lambda_{i-1}$$

for $x \in H_*(Y)$, where Sq_*^i is the transpose of Sq^i.

In our case, we abuse notation to denote the unique generator of $H_i(Y)$ by e_i, for any $H\mathbb{F}_2$-subquotient of X.

By naturality of the Steenrod operations, we have nontrivial Sq^4 and Sq^8 in the cohomology of W.

$$\begin{array}{ccc} H^{15}(W) & \xrightarrow{Sq^4 \neq 0} & H^{19}(W) \\ {\scriptstyle j^*}\uparrow{\scriptstyle \cong} & & {\scriptstyle j^*}\uparrow{\scriptstyle \cong} \\ H^{15}(P_{14}^{23}) & \xrightarrow{Sq^4 \neq 0} & H^{19}(P_{14}^{23}) \end{array} \qquad \begin{array}{ccc} H^{14}(W) & \xrightarrow{Sq^8 \neq 0} & H^{22}(W) \\ {\scriptstyle j^*}\uparrow{\scriptstyle \cong} & & {\scriptstyle j^*}\uparrow{\scriptstyle \cong} \\ H^{14}(P_{14}^{23}) & \xrightarrow{Sq^8 \neq 0} & H^{22}(P_{14}^{23}) \end{array}$$

Moreover, in the cohomology of W, we have $Sq^1 Sq^2 Sq^4 \neq 0$ on H^{15}. Dually, we have the following nontrivial operations:

$$Sq^1_*(e_{22}) = e_{21},$$

$$Sq^3_*(e_{22}) = e_{19},$$

$$Sq^7_*(e_{22}) = e_{15},$$

$$Sq^8_*(e_{22}) = e_{14}.$$

By naturality, we have the following nontrivial operations in $H_*(X^{22})$:

$$Sq^1_*(e_{22}) = e_{21},$$

$$Sq^3_*(e_{22}) = e_{19},$$

$$Sq^8_*(e_{22}) = e_{14}.$$

We claim that in $H_*(X^{22}) \otimes \Lambda^{*,*}$ the cycle

$$x = e_{22} \otimes \lambda_1\lambda_7\lambda_{31} + e_{14} \otimes \lambda_{13}\lambda_{19}\lambda_{15}$$

represents the class $h_1h_3h_5[22]$ in $Ext(X^{22})$.

In fact, we can check directly that x is a cycle:

$$\begin{aligned}
d(e_{22} \otimes \lambda_1\lambda_7\lambda_{31}) &= e_{21} \otimes \lambda_0\lambda_1\lambda_7\lambda_{31} + e_{19} \otimes \lambda_2\lambda_1\lambda_7\lambda_{31} + e_{14} \otimes \lambda_7\lambda_1\lambda_7\lambda_{31} \\
&= e_{14} \otimes \lambda_7\lambda_1\lambda_7\lambda_{31} \\
&= e_{14} \otimes (\lambda_{13}\lambda_{15}\lambda_{11}\lambda_7 + \lambda_{11}\lambda_{17}\lambda_{11}\lambda_7 + \lambda_7\lambda_{13}\lambda_{11}\lambda_{15}), \\
d(e_{14} \otimes \lambda_{13}\lambda_{19}\lambda_{15}) &= e_{14} \otimes d(\lambda_{13}\lambda_{19}\lambda_{15}) \\
&= e_{14} \otimes (\lambda_{13}\lambda_{15}\lambda_{11}\lambda_7 + \lambda_{11}\lambda_{17}\lambda_{11}\lambda_7 + \lambda_7\lambda_{13}\lambda_{11}\lambda_{15}).
\end{aligned}$$

We compute

$$\lambda_1\lambda_7\lambda_{31} = \lambda_{21}\lambda_{11}\lambda_7 + \lambda_{13}\lambda_{11}\lambda_{15},$$

and check the Curtis table in (77) that $Ext^{3,3+39} = \mathbb{Z}/2$, generated by an element with the leading term $\lambda_{21}\lambda_{11}\lambda_7$. Since $Ext^{3,3+39} = \mathbb{Z}/2$ is generated by $h_1h_3h_5$, we conclude that x represents the class $h_1h_3h_5[22]$ in $Ext(X^{22})$.

However, in $H_*(W) \otimes \Lambda^{*,*}$ the element

$$x = e_{22} \otimes \lambda_1\lambda_7\lambda_{31} + e_{14} \otimes \lambda_{13}\lambda_{19}\lambda_{15}$$

is not a cycle anymore: there is one more term in $d(x)$ due to the extra nontrivial operation

$$Sq_*^7(e_{22}) = e_{15}.$$

In fact, we have that

$$\begin{aligned}
d(x) &= d(e_{22} \otimes \lambda_1\lambda_7\lambda_{31} + e_{14} \otimes \lambda_{13}\lambda_{19}\lambda_{15}) \\
&= e_{21} \otimes \lambda_0\lambda_1\lambda_7\lambda_{31} + e_{19} \otimes \lambda_2\lambda_1\lambda_7\lambda_{31} + e_{15} \otimes \lambda_6\lambda_1\lambda_7\lambda_{31} \\
&\quad + e_{14} \otimes \lambda_7\lambda_1\lambda_7\lambda_{31} + e_{14} \otimes d(\lambda_{13}\lambda_{19}\lambda_{15}) \\
&= e_{15} \otimes \lambda_6\lambda_1\lambda_7\lambda_{31} \\
&= e_{15} \otimes \lambda_{14}\lambda_{13}\lambda_{11}\lambda_7.
\end{aligned}$$

Therefore, by the definition of the boundary homomorphism $\Delta_1 : Ext(X^{22}) \to Ext(\Sigma S^{15})$, we have

$$\Delta_1(x) = e_{15} \otimes \lambda_{14}\lambda_{13}\lambda_{11}\lambda_7.$$

$$\begin{array}{ccccc}
H_*(S^{15}) \otimes \Lambda^{*,*} & \longrightarrow & H_*(W) \otimes \Lambda^{*,*} & \longrightarrow & H_*(X^{22}) \otimes \Lambda^{*,*} \\
 & & x & \longmapsto & x \\
 & & \downarrow d & & \\
e_{15} \otimes \lambda_{14}\lambda_{13}\lambda_{11}\lambda_7 & \longmapsto & e_{15} \otimes \lambda_{14}\lambda_{13}\lambda_{11}\lambda_7 & &
\end{array}$$

We check the Curtis table in (77) that $Ext^{4,4+45} = \mathbb{Z}/2$, generated by an element with the leading term $\lambda_{14}\lambda_{13}\lambda_{11}\lambda_7$. Since $Ext^{4,4+45} = \mathbb{Z}/2$ is generated by $h_0h_4^3$, we conclude that $e_{15} \otimes \lambda_{14}\lambda_{13}\lambda_{11}\lambda_7$ represents the class $h_0h_4^3[15]$ in $Ext(\Sigma S^{15})$. □

Remark 3.10.5. One can think of the boundary homomorphism in Lemma 3.10.4 as an algebraic attaching map, and therefore its computation corresponds to a 4-fold Massey product. In $Ext(S^0)$, we have the strictly defined 4-fold Massey product

$$h_0h_4^3 = \langle h_2, h_1, h_0, h_1h_3h_5 \rangle$$

with zero indeterminacy. It is straightforward to check this by a Lambda algebra computation:

$$\begin{array}{ccccccc} \langle h_2 & , & h_1 & , & h_0 & , & h_1h_3 \rangle \\ \lambda_3 & & \lambda_1 & & \lambda_0 & & \lambda_5\lambda_3 \\ & \lambda_5 & & \lambda_2 & & * & \\ & & \lambda_6 & & * & & \end{array}$$

Here $*$ means the products are zero in the Lambda algebra. Note that the leading term of $h_0h_3^2$ is $\lambda_6\lambda_5\lambda_3$ from the Curtis table for S^0. Therefore,

$$h_0h_3^2 = \langle h_2, h_1, h_0, h_1h_3 \rangle.$$

Then it follows from a relation in Ext: $h_0h_4^3 = h_0h_3^2h_5$.

3.11 A homotopy relation

In this section, we prove a relation in the homotopy groups of spheres. This relation will lead to an Adams differential that kills the element gz in the 61-stem. We will explain in Remark 11.2 which element supports the differential that kills gz. But to prove $\pi_{61} = 0$, all we need is that gz is gone. We will use certain relations in Ext in the proofs, see (16) for these relations.

Theorem 3.11.1. *We have the homotopy relation $\eta\overline{\kappa}^3 = 0$ in π_{61}. Therefore the element gz must be killed by some Adams differential.*

Using several lemmas that will be proved later in this section, we present the proof of Theorem 3.11.1.

Proof. We first prove the second claim. By (9, Corollary 3.4.2), the permanent cycle z in the 41-stem detects the homotopy class $\eta\overline{\kappa}^2$. It follows that the element gz detects $\eta\overline{\kappa}^3$, since g detects $\overline{\kappa}$. Therefore, if $\eta\overline{\kappa}^3 = 0$, we must have gz killed by some Adams differential.

Now we prove the relation $\eta\overline{\kappa}^3 = 0$.

We have a 4-fold Toda bracket for $\overline{\kappa}$ (61, page 43-44):

$$\overline{\kappa} \in \langle \kappa, 2, \eta, \nu \rangle \text{ with indeterminacy even multiples of } \overline{\kappa}.$$

The indeterminacy will be killed after multiplying by η. We will prove in Lemma 3.11.3 that

$$\langle \eta\overline{\kappa}^2, \kappa, 2 \rangle = 0 \text{ in } \pi_{56}.$$

Therefore

$$\begin{aligned}
\eta\overline{\kappa}^3 &= \eta\overline{\kappa}^2 \langle \kappa, 2, \eta, \nu \rangle \\
&\subseteq \langle \langle \eta\overline{\kappa}^2, \kappa, 2 \rangle, \eta, \nu \rangle \\
&= \langle 0, \eta, \nu \rangle \\
&= \nu \cdot \pi_{58} \\
&= 0
\end{aligned}$$

The last equation is stated as Lemma 3.11.7 that we will prove later in this section. Therefore, we have the homotopy relation

$$\eta\overline{\kappa}^3 = 0 \text{ in } \pi_{61}.$$

□

Remark 3.11.2. Alternatively, we can show that h_1X_1 must support an Adams differential, and

$$d_4(h_1X_1) = gz$$

is the only possibility. The idea is to consider the Massey product $\langle g^2, d_0^2, h_1\rangle = h_1W_1 + g^2r$ in the Adams E_4-page, and to conclude that h_1W_1 must support a nontrivial differential as g^2r does (See Lemma 3.3.49 of (35)), since the sum is a permanent cycle by Moss's Theorem. Suppose that h_1X_1 is a permanent cycle. We have that

$$\begin{aligned} h_1W_1 &= Ph_1X_1 \\ &= X_1\langle h_1, h_0^3h_3, h_0\rangle \\ &= \langle h_1X_1, h_0^3h_3, h_0\rangle \end{aligned}$$

is also a permanent cycle by Moss's Theorem. We therefore have a contradiction.

We first prove Lemma 3.11.3.

Lemma 3.11.3. *We have a Toda bracket* $\langle \eta\overline{\kappa}^2, \kappa, 2\rangle = 0$ *in* π_{56}.

Proof. By (35; 36),

$$\pi_{55} \cong \mathbb{Z}/16 \ \text{ and is generated by an element } \ \rho_{55} \ \text{ in } \ ImJ.$$

Therefore, we have the relation

$$\eta\overline{\kappa}^2\kappa = 0 \ \text{ in } \ \pi_{55}.$$

This follows from the fact that both κ and $\overline{\kappa}$ map trivially to the $K(1)$-local sphere. In fact, suppose that $\eta\overline{\kappa}^2\kappa$ is some multiple of ρ_{55}. Then mapping the relation to the $K(1)$-local sphere tells us the multiple must be zero. Therefore, this Toda bracket is defined.

By (35; 36),

$$\pi_{56} \cong \mathbb{Z}/2 \text{ and is generated by } \eta\rho_{55} \text{ in } ImJ.$$

Therefore, we have the relation

$$\langle \eta\overline{\kappa}^2, \kappa, 2\rangle = 0.$$

This follows similarly by mapping the Toda bracket to the $K(1)$-local sphere.

□

To prove Lemma 3.11.7, we need the following three lemmas.

Lemma 3.11.4. *The product $\sigma \cdot \{h_0h_2h_5\}$ is nontrivial in π_{41}, and is detected by h_1f_1.*

Proof. By (50), we have the following two Adams differentials

$$d_3(h_2h_5) = h_1d_1, \text{ and } d_2(h_0c_2) = h_1h_3d_1.$$

Note that we have a relation $h_3d_1 = h_1e_1$ in Ext. Therefore, we have a Massey product in the Adams E_4-page

$$\langle d_1, h_1, h_0\rangle = h_0h_2h_5$$

and a Massey product in the Adams E_3-page

$$\langle h_3d_1, h_1, h_0\rangle = h_0^2c_2 = h_1f_1.$$

Note that the second equation is a relation in Ext. Then by Moss's Theorem (60, Theorem 1.2), we have the following Toda brackets

$$\langle \{d_1\}, \eta, 2\rangle \text{ contains an element that is detected by } h_0h_2h_5,$$

$$\langle \sigma\{d_1\}, \eta, 2\rangle \text{ contains an element that is detected by } h_1f_1.$$

Since

$$\sigma\langle\{d_1\}, \eta, 2\rangle \subseteq \langle\sigma\{d_1\}, \eta, 2\rangle,$$

the product $\sigma \cdot \{h_0h_2h_5\}$ is nontrivial, and is detected by h_1f_1. □

Lemma 3.11.5. *We have the relation* $\langle\{t\}, \eta, \nu\rangle \subseteq \sigma\{h_0h_2h_5\}$ *in* π_{41}.

Proof. By (15, Theorem 4.1) we have Bruner's differential

$$d_3(e_1) = h_1t.$$

Therefore, we have a Massey product in the Adams E_4-page

$$\langle t, h_1, h_2\rangle = h_2e_1 = h_1f_1.$$

The second equation is a relation in *Ext*. Therefore, by Moss's Theorem (60), we have the following Toda bracket:

$$\langle\{t\}, \eta, \nu\rangle \text{ is detected by } h_1f_1.$$

Note that the Toda bracket $\langle\{t\}, \eta, \nu\rangle$ has no indeterminacy.

Combining with Lemma 3.11.4, both $\sigma\{h_0h_2h_5\}$ and $\langle\{t\}, \eta, \nu\rangle$ are detected by h_1f_1. But in the same column of the E_∞ page of the Adams spectral sequence, there are several elements with higher filtration than h_1f_1. Therefore, to prove this lemma, we need to show that their difference is actually zero. We prove this by multiplying by η. First note that

$$\eta \cdot \sigma\{h_0h_2h_5\} = 0.$$

In fact, $\eta\{h_0h_2h_5\}$ contains non-zero classes $\eta\kappa\overline{\kappa} = \nu\{q\}$ and $\eta^2\{P^4h_1\}$. Both classes are annihilated by σ. Next note that

$$\langle \{t\}, \eta, \nu \rangle \eta = \{t\} \langle \eta, \nu, \eta \rangle = \{t\} \nu^2 = 0.$$

For the last equation, by filtration arguments, the only other possibility is that $\{t\}\nu^2 = \kappa^3$. (For reader's convenience, note that $\kappa^3 = \eta^2 \overline{\kappa}^2$.) However, mapping this relation to $\pi_*(tmf)$ gives a contradiction.

Since all elements of higher filtration than $h_1 f_1$ in the cokernel of J support non-zero η-extensions, this proves the lemma. □

Lemma 3.11.6. *We have a Toda bracket* $\langle \overline{\kappa}, \{t\}, \eta \rangle = \{h_1 Q_2\}$ *in* π_{58}.

Proof. By (35, Table 20), (36), we have Isaksen's differential

$$d_3(Q_2) = gt.$$

Therefore, combining with Bruner's differential (15, Theorem 4.1) $d_3(e_1) = h_1 t$, we have a Massey product in the Adams E_4-page

$$\langle g, t, h_1 \rangle = h_1 Q_2.$$

Note that $ge_1 = 0$ in Ext. Therefore, the lemma follows from Moss's Theorem (60, Theorem 1.2). Both sides of $\langle \overline{\kappa}, \{t\}, \eta \rangle = \{h_1 Q_2\}$ have the same indeterminacy that lies in the image of J. □

Now we prove Lemma 3.11.7.

Lemma 3.11.7. $\nu \cdot \pi_{58} = 0$.

Proof. By (35; 36),

$$\pi_{58} \text{ is } \mathbb{Z}/2 \oplus \mathbb{Z}/2, \text{ and generated by } \{h_1 Q_2\} \text{ and } \eta\{P^7 h_1\}.$$

By Lemma 3.11.5 that

$$\langle \{t\}, \eta, \nu \rangle \subseteq \sigma\{h_0 h_2 h_5\} \text{ in } \pi_{41},$$

and Lemma 3.11.6 that

$$\langle \overline{\kappa}, \{t\}, \eta \rangle = \{h_1 Q_2\} \text{ in } \pi_{58},$$

we have that

$$\begin{aligned} \nu \cdot \{h_1 Q_2\} &= \langle \overline{\kappa}, \{t\}, \eta \rangle \nu \\ &= \overline{\kappa} \langle \{t\}, \eta, \nu \rangle \\ &\subseteq \overline{\kappa} \sigma \{h_0 h_2 h_5\} = 0. \end{aligned}$$

The last equation follows from the relation that $\overline{\kappa}\sigma = 0$. Therefore, we have that

$$\nu \cdot \pi_{58} = 0.$$

□

3.12 Another homotopy relation and the Adams differential $d_5(A') = h_1 B_{21}$

In this section, we prove another relation in the homotopy groups of spheres. This relation will lead to an Adams differential, which is the only possibility to kill the element $h_1 B_{21}$ in the 60-stem.

Theorem 3.12.1. *We have the relation $\eta\kappa\theta_{4.5} = 0$ in π_{60}. Here $\theta_{4.5}$ is a homotopy class in π_{45} defined by Isaksen in Section 1.7 of (35), with an extra condition that it maps to zero in $\pi_{45}(tmf)$. This implies the Adams differential*

$$d_5(A') = h_1 B_{21}.$$

In Isaksen's definition, $\theta_{4.5}$ is a homotopy class detected by h_4^3 in the 45-stem, with indeterminacy containing even multiples of itself and the element $\{w\}$. Our definition of $\theta_{4.5}$ is a refinement of Isaksen's. Since $\{w\}$ has a strictly higher Adams filtration than $\theta_{4.5}$, and is detected by tmf, the indeterminacy of our $\theta_{4.5}$ does not contain the element $\{w\}$.

Using several lemmas that will be proved later in this section, we present the proof of Theorem 3.12.1.

Proof. We first prove the second claim. By (7, Theorem 3.1(i)), the permanent cycle B_1 detects the homotopy class $\eta\theta_{4.5}$. We have the following relation in *Ext*:

$$h_1 B_{21} = d_0 B_1.$$

Since d_0 detects κ, the permanent cycle $h_1 B_{21} = d_0 B_1$ detects the homotopy class $\eta\kappa\theta_{4.5}$. Therefore, if $\eta\kappa\theta_{4.5} = 0$, we must have $h_1 B_{21}$ killed by some Adams differential. By Theorem 3.1, we have that

$$d_3(D_3) = B_3,\ d_3(h_1 D_3) = h_1 B_3.$$

This leaves the element A' to be the only possibility to kill $h_1 B_{21}$ as the source. Therefore, we have the Adams d_5 differential $d_5(A') = h_1 B_{21}$.

Now we prove the relation $\eta\kappa\theta_{4.5} = 0$.

Recall that there is a strictly defined 4-fold Toda bracket for $\kappa \in \pi_{14}$ with zero indeterminacy:

$$\kappa = \langle \epsilon, \nu, \eta, 2 \rangle.$$

It follows that

$$\eta\kappa = \eta\langle \epsilon, \nu, \eta, 2 \rangle \in \langle \eta\epsilon, \nu, \eta, 2 \rangle,$$

and that

$$\eta\kappa\theta_{4.5} \in \theta_{4.5}\langle \eta\epsilon, \nu, \eta, 2 \rangle.$$

We will show in Lemma 3.12.6 that there is a strictly defined 4-fold Toda bracket in π_{15}:

$$\rho_{15} \in \langle \{Ph_1\}, \nu, \eta, 2\rangle \text{ with indeterminacy even multiples of } \rho_{15}.$$

We will show in Lemma 3.12.7 that

$$\rho_{15}\theta_{4.5} = 0 \text{ in } \pi_{60}.$$

Thus

$$0 = \rho_{15}\theta_{4.5} = \theta_{4.5}\langle \{Ph_1\}, \nu, \eta, 2\rangle.$$

We will show in Lemma 3.12.5 that

$$\theta_{4.5}(\eta\epsilon + \{Ph_1\}) = 0,$$

and in Lemma 3.12.9 that

$$\langle \theta_{4.5}, \{Ph_1\} + \eta\epsilon, \nu\rangle = 0 \text{ with zero indeterminacy in } \pi_{58}.$$

Therefore

$$\begin{aligned}
\eta\kappa\theta_{4.5} &= \eta\kappa\theta_{4.5} + \rho_{15}\theta_{4.5} \\
&\in \theta_{4.5}\langle \eta\epsilon, \nu, \eta, 2\rangle + \theta_{4.5}\langle \{Ph_1\}, \nu, \eta, 2\rangle \\
&= \theta_{4.5}\langle \{Ph_1\} + \eta\epsilon, \nu, \eta, 2\rangle \\
&\subseteq \langle \langle \theta_{4.5}, \{Ph_1\} + \eta\epsilon, \nu\rangle, \eta, 2\rangle \\
&= \langle 0, \eta, 2\rangle \\
&= 2 \cdot \pi_{60} = \{0, 2\overline{\kappa}^3\}.
\end{aligned}$$

Note that the following three Toda brackets

$$\langle \eta\epsilon, \nu, \eta, 2\rangle,\ \langle \{Ph_1\}, \nu, \eta, 2\rangle,\ \langle \{Ph_1\} + \eta\epsilon, \nu, \eta, 2\rangle$$

have the same indeterminacy: $2 \cdot \pi_{15}$ = even multiples of ρ_{15}, which is annihilated by $\theta_{4.5}$.

To prove that $\eta\kappa\theta_{4.5} = 0$, we only need to show that

$$\eta\kappa\theta_{4.5} \neq 2\overline{\kappa}^3.$$

Note that $2\overline{\kappa}^3$ is detected by tmf, while $\theta_{4.5}$ is chosen not to be detected by tmf. Suppose we have the relation

$$\eta\kappa\theta_{4.5} = 2\overline{\kappa}^3.$$

Then mapping this relation into tmf gives us $2\overline{\kappa}^3 = 0$, which contradicts the fact that $2\overline{\kappa}^3$ is detected in $\pi_*(tmf)$. Therefore, we must have that

$$\eta\kappa\theta_{4.5} = 0.$$

□

Now we present the proofs of Lemmas 3.12.5, 3.12.6, 3.12.7, 3.12.9, and a few other lemmas that will be needed for the proofs.

Lemma 3.12.2. *In the Adams E_2 page, we have a Massey product*

$$h_1 x' = \langle h_0^2 g_2, h_0, Ph_1\rangle.$$

Proof. In Proposition 4.19 of (77), Tangora showed that we have a May d_6 differential

$$d_6(Y) = h_0^3 g_2.$$

Here we follow Isaksen's notation (35) for names of the elements in the May spectral sequence. Then combining with the fact that $h_1 x' = YPh_1$ in the May E_6 page, this lemma follows from May's convergence theorem (51). □

Lemma 3.12.3. *We have the relation*

$$\{Ph_1\} \cdot \{h_5 d_0\} = 0 \quad \text{in} \quad \pi_{54}.$$

Proof. First note that the Toda bracket

$$\langle 2, \theta_4, \kappa \rangle \text{ is detected by } h_5 d_0.$$

This follows from the Adams d_2 differential $d_2(h_5) = h_0 h_4^2$ and Moss's theorem. Note that to apply the Moss's theorem here, we need to use the fact that $\theta_4 \kappa = 0$, which is obtained by filtration reasons.

We compute the product $\{Ph_1\}\langle 2, \theta_4, \kappa \rangle$ next.

$$\begin{aligned}
\{Ph_1\}\langle 2, \theta_4, \kappa \rangle &= \langle \{Ph_1\}, 2, \theta_4 \rangle \kappa \\
&\subseteq \langle \kappa \{Ph_1\}, 2, \theta_4 \rangle \\
&= \langle \eta^3 \overline{\kappa}, 2, \theta_4 \rangle \\
&\supseteq \eta^2 \overline{\kappa} \langle \eta, 2, \theta_4 \rangle \\
&= \eta^3 \langle 2, \theta_4, \overline{\kappa} \rangle \subseteq \eta^3 \pi_{51} = 0.
\end{aligned}$$

In other words, both $\{Ph_1\}\langle 2, \theta_4, \kappa \rangle$ and 0 are contained in the same Toda bracket

$$\langle \eta^3 \overline{\kappa}, 2, \theta_4 \rangle.$$

Therefore, their difference must be contained in the indeterminacy of this Toda bracket, which is

$$\eta^3\overline{\kappa} \cdot \pi_{31} + \pi_{24} \cdot \theta_4.$$

It is clear that $\eta^3\overline{\kappa} \cdot \pi_{31} \subseteq \eta^3\pi_{51} = 0$. Recall that

$$\pi_{24} \cong \mathbb{Z}/2 \oplus \mathbb{Z}/2 \ \text{ and is generated by } \ \eta\sigma\eta_4 \ \text{ and } \ \eta\rho_{23} \ \text{ in the } \ ImJ.$$

Multiplying by θ_4, both products are zero. This is due to the fact that $\eta\eta_4\theta_4 = 0$ (See Lemma 4.1 in (7)) and filtration reasons. Therefore, we have achieved that

$$\{Ph_1\}\langle 2, \theta_4, \kappa\rangle = 0.$$

Then, from the fact that $2\{Ph_1\} = 0$ and filtration reasons, the product of $\{Ph_1\}$ and all elements in the E_∞ page of higher filtration than h_5d_0 are zero. Therefore, combining with the fact that the Toda bracket

$$\langle 2, \theta_4, \kappa\rangle \ \text{ is detected by } \ h_5d_0,$$

we have the homotopy relation that

$$\{Ph_1\} \cdot \{h_5d_0\} = 0 \ \text{ in } \ \pi_{54}.$$

□

Lemma 3.12.4. *The permanent cycle h_1x' in the 54-stem detects the homotopy class $\theta_{4.5}\{Ph_1\}$.*

Proof. By Lemma 3.12.2 and Moss's theorem, we have that

$$h_1x' \ \text{ detects an element in the Toda bracket } \ \langle \sigma^2\theta_4, 2, \{Ph_1\}\rangle.$$

Recall that Barratt, Mahowald and Tangora (9) showed that

$$h_0^2 g_2 \quad \text{detects} \;\; \sigma^2\theta_4.$$

We have the relation that

$$\theta_4\langle\sigma^2, 2, \{Ph_1\}\rangle \subseteq \langle\sigma^2\theta_4, 2, \{Ph_1\}\rangle.$$

Since also

$$\theta_4\langle\sigma^2, 2, \{Ph_1\}\rangle \subseteq \theta_4 \cdot \pi_{24} = 0,$$

which we showed in the proof of Lemma 12.3, we have that

$$0 \in \langle\sigma^2\theta_4, 2, \{Ph_1\}\rangle.$$

Note that one can also show directly that $\langle\sigma^2, 2, \{Ph_1\}\rangle = 0$.

Recall that Isaksen (35) showed that $h_1 x'$ is a surviving permanent cycle, and it detects both $\nu^3\theta_{4.5}$ and equally $\eta\epsilon\theta_{4.5}$. Therefore, $h_1 x'$ must detect a nontrivial homotopy class in the indeterminacy of the Toda bracket

$$\langle\sigma^2\theta_4, 2, \{Ph_1\}\rangle.$$

The indeterminacy of this Toda bracket is

$$\sigma^2\theta_4 \cdot \pi_{10} + \pi_{45} \cdot \{Ph_1\}.$$

First note that

$$\pi_{10} \cong \mathbb{Z}/2 \;\; \text{and is generated by} \;\; \eta\{Ph_1\}.$$

Since $\eta\sigma^2 = 0$, we must have that

$$\sigma^2\theta_4 \cdot \pi_{10} = 0.$$

Next note that $2\{Ph_1\} = 0$, and the generators of π_{45} can be chosen to be the following

$$\theta_{4.5} \in \{h_4^3\},\ \eta\{g_2\},\ \{h_5 d_0\},\ \{w\}.$$

We have that

$$\{w\} \cdot \{Ph_1\} = 0 \ \text{ for filtration reasons.}$$

We also have that

$$\begin{aligned} \{Ph_1\} \cdot \eta\{g_2\} &\subseteq \langle \eta, 2, 8\sigma \rangle \eta\{g_2\} \\ &= \eta\langle 2, 8\sigma, \{g_2\}\rangle \eta \\ &= \eta^2 \langle 2, 8\sigma, \{g_2\}\rangle \\ &\subseteq \eta^2 \pi_{52} = 0. \end{aligned}$$

Note here we use the fact that $8\sigma\{g_2\} = 0$. Then combining with Lemma 3.12.3 that

$$\{Ph_1\} \cdot \{h_5 d_0\} = 0,$$

the only possibility is that

$$h_1 x' \ \text{ detects the homotopy class } \ \theta_{4.5}\{Ph_1\}.$$

□

Lemma 3.12.5. *In π_{54}, we have a relation $\theta_{4.5}(\eta\epsilon + \{Ph_1\}) = 0$.*

Proof. The element $d_0 g^2$ is the only element in the 54-stem of the E_∞ page with higher filtration than $h_1 x'$. It detects the homotopy class $\kappa\overline{\kappa}^2$, which is also detected in the Hurewicz image of tmf. Since $\theta_{4.5}$ is chosen not to be detected in the Hurewicz image of tmf, and h_1x' detects both $\eta\epsilon\theta_{4.5}$ and $\{Ph_1\}\theta_{4.5}$, we must have a relation

$$\theta_{4.5}(\eta\epsilon + \{Ph_1\}) = 0.$$

□

Lemma 3.12.6. *We have a strictly defined 4-fold Toda bracket*

$$\rho_{15} \in \langle\{Ph_1\}, \nu, \eta, 2\rangle \quad in \quad \pi_{15},$$

with indeterminacy $2\pi_{15}$ given by even multiples of ρ_{15}, where ρ_{15} is a generator of the ImJ in π_{15}.

Proof. We first check that this 4-fold Toda bracket is strictly defined. It is clear that

$$\langle\nu, \eta, 2\rangle \subseteq \pi_5 = 0.$$

In the Adams E_2 page, we have that

$$\langle Ph_1, h_2, h_1\rangle = Ph_2^2 = h_0^2 d_0.$$

The element $h_0^2d_0$ is killed by the Adams d_3 differential

$$d_3(h_0^2h_4) = h_0^2d_0.$$

Therefore,

$$0 \in \langle\{Ph_1\}, \nu, \eta\rangle.$$

It is straightforward to check the indeterminacy of this 3-fold Toda bracket is zero. Therefore, this 4-fold Toda bracket is strictly defined.

We next check the indeterminacy of this 4-fold Toda bracket. The indeterminacy is contained in the union of the following

$$\langle \{Ph_1\}, \nu, \pi_2 \rangle, \langle \{Ph_1\}, \pi_5, 2 \rangle, \langle \pi_{13}, \eta, 2 \rangle.$$

Note that $\pi_5 = 0, \pi_{12} = 0, \pi_{13} = 0$, π_2 is generated by η^2 and π_6 is generated by ν^2. We have

$$\langle \{Ph_1\}, \nu, \eta^2 \rangle \supseteq \langle \{Ph_1\}, \nu, \eta \rangle \eta = 0.$$

$$\{Ph_1\} \cdot \nu^2 \in \nu \cdot \pi_{12} = 0.$$

Therefore, the indeterminacy is $2\pi_{15}$.

Now we multiply this 4-fold Toda bracket by η^2:

$$\langle \{Ph_1\}, \nu, \eta, 2 \rangle \eta^2 = \{Ph_1\} \langle \nu, \eta, 2, \eta^2 \rangle = \{Ph_1\} \epsilon.$$

The 4-fold Toda bracket $\epsilon = \langle \nu, \eta, 2, \eta^2 \rangle$ is strictly defined with zero indeterminacy. The homotopy class $\{Ph_1\}\epsilon$ is detected by the surviving cycle Ph_1c_0. We have a nontrivial extension:

$$\eta^2 \rho_{15} \in \{Ph_1c_0\}.$$

Therefore, we must have that the 4-fold Toda bracket

$$\langle \{Ph_1\}, \nu, \eta, 2 \rangle \ \text{ contains } \ \rho_{15} \ \text{ or } \ \rho_{15} + \eta\kappa.$$

To eliminate the second possibility, we multiply this 4-fold Toda bracket by $\overline{\kappa}$. Note that

$$\overline{\kappa}\{Ph_1\} \subseteq \pi_{29} = 0,$$

$$\langle \overline{\kappa}, \{Ph_1\}, \nu \rangle = 0 \;\text{ with indeterminacy }\; \{0, \nu\theta_4\} \;\text{ in }\; \pi_{33}.$$

In fact, in the Adams E_2 page, we have the Massey product

$$\langle g, Ph_1, h_2 \rangle = 0 \;\text{ in Adams filtration 9.}$$

The homotopy classes that survive in π_{33} with filtration higher than 9 are detected by the $K(1)$-local sphere. Since the class $\overline{\kappa}$ maps trivially to the $K(1)$-local sphere, we must have that

$$\langle \overline{\kappa}, \{Ph_1\}, \nu \rangle \;\text{ contains 0.}$$

Then it is straightforward to check the indeterminacy is

$$\overline{\kappa} \cdot \pi_{13} + \pi_{30} \cdot \nu = \{0, \nu\theta_4\}.$$

Now we have that

$$\begin{aligned} \overline{\kappa}\langle \{Ph_1\}, \nu, \eta, 2 \rangle &\subseteq \langle \langle \overline{\kappa}, \{Ph_1\}, \nu \rangle, \eta, 2 \rangle \\ &= \langle \{0, \nu\theta_4\}, \eta, 2 \rangle \\ &= \;\text{ the union of }\; \langle 0, \eta, 2 \rangle \;\text{ and }\; \langle \nu\theta_4, \eta, 2 \rangle \\ &= 2 \cdot \pi_{35}. \end{aligned}$$

Note that $2 \cdot \pi_{35}$ is detected in the $K(1)$-local sphere. Since the class $\overline{\kappa}$ maps trivially to the $K(1)$-local sphere, we have that

$$\overline{\kappa}\langle \{Ph_1\}, \nu, \eta, 2 \rangle = 0.$$

On the other hand, it is clear that

$$\eta\kappa\overline{\kappa} \neq 0 \;\text{ and is detected by }\; h_1 d_0 g,$$

and that

$$\rho_{15}\overline{\kappa} \in \langle 8, 2\sigma, \sigma\rangle\overline{\kappa} = 8\langle 2\sigma, \sigma, \overline{\kappa}\rangle \subseteq 8\pi_{35} = 0.$$

Here by Moss's theorem, the relation

$$\rho_{15} \in \langle 8, 2\sigma, \sigma\rangle$$

follows from the Adams differential $d_2(h_4) = h_0h_3^2$ and the Massey product in the E_3 page

$$\langle h_0^3, h_0h_3, h_3\rangle = h_0^3h_4 \ \text{ with zero indeterminacy.}$$

Therefore, the 4-fold Toda bracket

$$\langle\{Ph_1\}, \nu, \eta, 2\rangle \ \text{ contains } \ \rho_{15}.$$

□

Lemma 3.12.7. *We have the relation* $\rho_{15}\theta_{4.5} = 0$ *in* π_{60}.

Proof. We first claim that

$$\rho_{15}\theta_4 = 8\theta_{4.5}.$$

In fact, they are both detected by the surviving cycle $h_0^2h_5d_0$ (See Tangora (78)). However, there is one more element w in higher filtration in the E_∞ page, so the two classes might differ by that. Since

$$\eta^2\theta_4 = 0, \ \text{ and } \ \eta^2\{w\} \neq 0,$$

their difference is not $\{w\}$, and hence must be zero. Note that one can also show this by mapping the relation into tmf.

Then we have that

$$\begin{aligned}\rho_{15}\theta_{4.5} &\subseteq \langle 8, 2\sigma, \sigma\rangle\theta_{4.5} \\ &\subseteq \langle 8\theta_{4.5}, 2\sigma, \sigma\rangle \\ &= \langle \rho_{15}\theta_4, 2\sigma, \sigma\rangle \\ &= 0 \;\; \text{with zero indeterminacy.}\end{aligned}$$

The last equation is proved by the second author as Lemma 2.4 in (88). ☐

Lemma 3.12.8. *We have a Toda bracket in* π_{20}*:*

$$\langle\{Ph_1\} + \eta\epsilon, \nu, \sigma\rangle = 0 \;\; \textit{with zero indeterminacy.}$$

Proof. We consider the two brackets $\langle\{Ph_1\}, \nu, \sigma\rangle$ and $\langle\eta\epsilon, \nu, \sigma\rangle$ one by one.

For the first bracket, in the Adams E_2 page we have the Massey product

$$\langle Ph_1, h_2, h_3\rangle = 0$$

with zero indeterminacy in Adams filtration 6. Since there is no surviving class in Adams filtration 7 or higher, it contains zero. For filtration reasons and the fact that $\pi_{13} = 0$, the indeterminacy of the first bracket is

$$\{Ph_1\} \cdot \pi_{11} + \pi_{13} \cdot \sigma = 0.$$

Therefore,

$$\langle\{Ph_1\}, \nu, \sigma\rangle = 0 \;\; \text{with zero indeterminacy.}$$

For the second bracket, we have that

$$\langle \eta\epsilon, \nu, \sigma \rangle \supseteq \epsilon\langle \eta, \nu, \sigma \rangle \subseteq \epsilon \cdot \pi_{12} = 0.$$

Therefore, it contains 0. Again, by filtration reasons and the fact that $\pi_{13} = 0$, the indeterminacy of the second bracket is

$$\eta\epsilon \cdot \pi_{11} + \pi_{13} \cdot \sigma = 0.$$

Therefore,

$$\langle \eta\epsilon, \nu, \sigma \rangle = 0 \ \text{ with zero indeterminacy.}$$

Summing these two relations, we have that

$$\langle \{Ph_1\} + \eta\epsilon, \nu, \sigma \rangle = 0 \ \text{ with zero indeterminacy.}$$

□

Lemma 3.12.9. *We have a Toda bracket in π_{58}:*

$$\langle \theta_{4.5}, \{Ph_1\} + \eta\epsilon, \nu \rangle = 0 \ \textit{ with zero indeterminacy.}$$

Proof. First, by Lemma 3.12.5, we have the relation

$$\theta_{4.5} \cdot (\{Ph_1\} + \eta\epsilon) = 0.$$

Therefore, this Toda bracket is defined.

Recall that

$$\text{the cokernel of J in } \pi_{58} \text{ is } \mathbb{Z}/2, \text{ and generated by } \{h_1Q_2\}.$$

The indeterminacy equals

$$\theta_{4.5} \cdot \pi_{13} + \pi_{55} \cdot \nu = 0.$$

The relation $\pi_{55} \cdot \nu = 0$ follows from filtration reasons. As a side remark, one can actually prove that

$$\{h_1Q_2\} \text{ is indecomposable.}$$

This can be shown by the Adams-Novikov filtration of this element. See Isaksen (35) for details.

In (35), Isaksen showed that the permanent cycle $h_1h_3Q_2$ cannot be killed by r_1. The only other candidate to kill $h_1h_3Q_2$ is $h_1^3h_6$, which is obviously a permanent cycle: it detects $\eta^2\eta_6$. Therefore,

$$h_1h_3Q_2 \text{ is a surviving cycle, and detects } \sigma\{h_1Q_2\}.$$

By Lemma 3.12.8, we have that

$$\langle\theta_{4.5}, \{Ph_1\} + \eta\epsilon, \nu\rangle\sigma = \theta_{4.5}\langle\{Ph_1\} + \eta\epsilon, \nu, \sigma\rangle = 0.$$

Therefore,

$$\langle\theta_{4.5}, \{Ph_1\} + \eta\epsilon, \nu\rangle \text{ does not contain } \{h_1Q_2\},$$

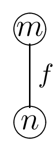
and hence is 0 with zero indeterminacy. □

3.13 Appendix I

The theory of cell diagrams is very helpful when thinking of finite CW spectra. We use them as illustration purpose in Section 3.5. In this section, we recall the definition of cell diagrams from (7). We also include several examples.

Definition 3.13.1. Let Z be a finite CW spectrum. Then a cell diagram for Z consists of nodes and edges. The nodes are in 1-1 correspondence with a chosen basis of the mod 2 homology of Z, and may be labeled with symbols to indicate the dimension. When two nodes are joined by an edge, then it is possible to form an $H\mathbb{F}_2$-subquotient

$$Z'/Z'' = S^n \smile_f e^m,$$



which is the cofiber of f with certain suspension. Here f, the attaching map, is an element in the stable homotopy groups of spheres. For simplicity, we do not draw an edge if the corresponding f is null.

Suppose we have two nodes labeled n and m with $n < m$, and there is no edge joining them. Then there are two possibilities.

The first one is that there is an integer k, and a sequence of nodes labeled $n_i, 0 \leq i \leq k$, with $n = n_0 < n_1 < \cdots < n_k = m$, and edges joining the nodes n_i to the nodes n_{i+1}. In this case we do not assert that there is an $H\mathbb{F}_2$-subquotient of the form above; this does not imply that there is no such $H\mathbb{F}_2$-subquotient.

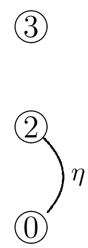
The second one is that there is no such sequence as in the first case. In this case, there exists an $H\mathbb{F}_2$-subquotient which a wedge of spheres $S^n \vee S^m$.

Remark 3.13.2. In (7)'s original definition, they use subquotients instead of $H\mathbb{F}_2$-subquotients.

Example 3.13.3. Let f be the composite of the following two maps:

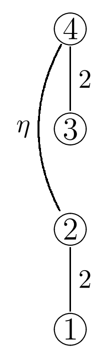
$$S^2 \xrightarrow{\eta^2} S^0 \xrightarrow{i} C\eta,$$

where the second map i is the inclusion of the bottom cell. Consider the cofiber of f: Cf, which is a 3 cell complex with the following cell diagram:

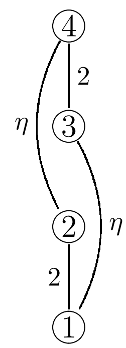


It is clear that the top cell of Cf splits off, since η^2 can be divided by η. So we do not have to draw any attaching map from the cell in dimension 3 to the one in dimension 0. Note that the cofiber of η^2 is in fact an $H\mathbb{F}_2$-subcomplex of Cf. One could think this as the indeterminacy of cell diagrams associated to a given CW spectrum.

Example 3.13.4. Let $X_1 = P_1^4$. The cell diagram of X_1 is the following:



As a comparison, let $X_2 = C2 \wedge C\eta$, where $C2$ and $C\eta$ are the cofibers of 2 and η. Then the cell diagram of X_2 is the following:

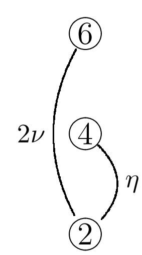


We give a more interesting example.

Example 3.13.5. Consider the suspension spectrum of $\mathbb{C}P^3$. It consists of three cells: one each in dimensions 2, 4 and 6. It is shown in (2) by Adams that, the secondary cohomology operation Ψ, which is associated to the relation

$$Sq^4Sq^1 + Sq^2Sq^1Sq^2 + Sq^1Sq^4 = 0,$$

is nonzero on this spectrum. In other words, there exists an attaching map between the cells in dimension 2 and 6, which is detected by h_0h_2 in the 3-stem of the Adams E_∞ page. Note that h_0h_2 detects two homotopy classes: $2\nu,\ 6\nu$. Their difference is $4\nu = \eta^3$, which is divisible by η. Therefore, we have its cell diagram as the following:



3.14 Appendix II

This section is about intuition.

We summarize and explain the major ideas of how we think of the "road map" of the proof of the differential $d_3(D_3) = B_3$, especially of Step 4. The "zigzag" part of the explanation is crucial if one wants to generalize this method to other Adams differentials.

We try to prove an Adams d_3 differential in P_1^∞:

$$d_3(h_1h_3h_5[22]) = G[6].$$

The element G supports a differential (35; 36) in the Adams spectral sequence of S^0:

$$d_3(G) = Ph_5d_0.$$

From the computation of the transfer map, we have that

$$Ph_5d_0[6] \quad \text{maps to} \quad B_{21}$$

It is shown in (35) that $d_3(B_3) \neq B_{21}$. Therefore, the only possibility is that

$$G[6] \quad \text{supports a} \quad d_2 \quad \text{differential in} \quad P_1^6.$$

Checking the bidegree gives us the only element there: $h_5i[5]$. This argument can be summarized in the following diagram:

$$\begin{array}{ccccccc}
Ext(S^6) & \longleftarrow & Ext(P_1^6) & \longrightarrow & Ext(P_1^\infty) & \longrightarrow & Ext(S^0) \\
Ph_5d_0[6] & \longleftarrow\!\!\dashv & Ph_5d_0[6] & \longmapsto & Ph_5d_0[6] & \longmapsto & B_{21} \\
\uparrow d_3 & & h_5i[5] & & & & \\
 & & \uparrow d_2 & & & & \\
G[6] & \longleftarrow\!\!\dashv & G[6] & \longmapsto & G[6] & \longmapsto & B_3
\end{array}$$

Remark 3.14.1. The above argument implies that in the Adams spectral sequence of P_1^2, we have a differential

$$d_2(G[2]) = h_5 i[1].$$

This differential in the mod 2 Moore spectrum is not obtained by a zigzag.

The Curtis table shows that

$$h_5 i[5] \quad \text{is killed by} \quad B_1[14].$$

Note that the element B_1 in $Ext(S^0)$ is a surviving cycle.

This zigzag suggests that, if the element $G[6]$ were going to survive in the Adams spectral sequence of P_1^{23}, then it would jump the Adams filtration by 1 to the element $B_1[14]$ in the Adams spectral sequence of P_{14}^{23}. This is the first half of the intuition of Step 4: we reduce the Adams d_3 differential in P_1^{23} to an Adams d_4 differential in P_{14}^{23}.

The second half of the intuition is related to the source element $h_1h_3h_5[22]$. The Massey product $h_0h_4^3 = \langle h_2, h_1, h_0, h_1h_3h_5\rangle$ and the nonzero Steenrod operation $Sq^1Sq^2Sq^4$ on the 15 dimensional class in $H^*(P_{14}^{23})$ suggest that we should have a differential

$$h_1h_3h_5[22] \ \text{ kills } \ h_0h_4^3[15]$$

in the Curtis table of P_1^{∞}. However, the element $h_0h_4^3[15]$ is killed by $h_4^3[16]$ in the Curtis table because P_{15}^{16} is a suspension of the mod 2 Moore spectrum. Therefore, if we remove the 15-cell in P_{14}^{23}, we can "separate" the two elements $h_1h_3h_5[22]$ and $h_4^3[16]$. To do this, we take the cofiber of the inclusion of the 15-cell to get the spectrum X, and reduce the Adams d_4 differential in P_{14}^{23} to an Adams d_4 differential in X.

It is therefore clear that the η-extension from h_4^3 to B_1 gives us the d_4 differential in X, since the 16-cell is attached to the 14-cell by η.

CHAPTER 4

SOME EXTENSION PROBLEM BY THE RP^∞-METHOD

4.1 Introduction

The computation of the stable homotopy groups of spheres is both a fundamental and a difficult problem in homotopy theory. Recently, using Massey products and Toda brackets, Isaksen (35) extended the 2-primary Adams spectral sequence computations to the 59-stem, with a few $2, \eta, \nu$-extensions unsettled.

Based on the algebraic Kahn-Priddy theorem, the authors (86) introduced a new method to compute differentials in the Adams spectral sequence, and proved that $\pi_{61} = 0$. The 61-stem result has the geometric consequence that the 61-sphere has a unique smooth structure, and it is the last odd dimensional case. In the article (86), it took us more than 40 pages to introduce the method and prove one Adams differential $d_3(D_3) = B_3$. Here B_3 and D_3 are certain elements in the 60 and 61-stem. Our notation will be consistent with (35) and (86).

In this chapter, we show that our method can also be used to solve extension problems in the Adams spectral sequence. We establish a nontrivial 2-extension in the 51-stem, together with a few other extensions left unsolved by Isaksen (35). As a result, we have the following proposition.

Proposition 4.1.1. *There is a nontrivial 2-extension from $h_0h_3g_2$ to gn in the 51-stem.*

We'd like to point out that this is also a nontrivial 2-extension in the Adams-Novikov spectral sequence.

Combining with Theorem 1.1 of (38), which describes the group structure of π_{51} up to this 2-extension, we have the following corollary.

Corollary 4.1.2. *The 2-primary π_{51} is $\mathbb{Z}/8 \oplus \mathbb{Z}/8 \oplus \mathbb{Z}/2$, generated by elements that are detected by h_3g_2, P^6h_2 and h_2B_2.*

Using a Toda bracket argument, Proposition 4.1.1 is deduced from the following σ-extension in the 46-stem.

Proposition 4.1.3.

1. *There is a nontrivial σ-extension from h_3d_1 to N in the 46-stem.*

2. *There is a nontrivial η-extension from h_1g_2 to N in the 46-stem.*

As a corollary, we prove a few more extensions.

Corollary 4.1.4.

1. *There is a nontrivial η-extension from C to gn in the 51-stem.*

2. *There is a nontrivial ν-extension from $h_2h_5d_0$ to gn in the 51-stem.*

3. *There is a nontrivial σ-extension from $h_0^2g_2$ to gn in the 51-stem. In particular, the element gn detects $\sigma^3\theta_4$.*

Remark 4.1.5. In (35), Isaksen had an argument that implies the nonexistence of the two η-extensions on h_1g_2 and C, which is contrary to our results in Proposition 4.1.3 and Corollary 4.1.4. Isaksen's argument fails because of neglect of the indeterminacy of a certain Massey product in a subtle way. For more details, see Remark 4.2.3.

The proof of the σ-extension in Proposition 4.1.3 is the major part of this article: we prove it by the RP^∞ technique as a demonstration of the effectiveness of our method.

The rest of this chapter is organized as the following.

In Section 4.2, we deduce Proposition 4.1.1 and Corollary 4.1.4 from Proposition 4.1.3. We also show the two statements in Proposition 4.1.3 are equivalent. In Section 4.3, we recall a few notations from (86). We also give a brief review of how to use the RP^∞ technique to prove differentials and to solve extension problems. In Section 4.4, we present the proof of Proposition 4.1.3. In the Appendix, we prove a lemma which is used in Section 4.4. The lemma gives a general connection that relates Toda brackets and extension problems in 2-cell complexes.

4.2 the 51-stem and some extensions

We first establish the following lemma.

Lemma 4.2.1. *In the Adams E_2 page, we have the following Massey products in the 46-stem:*

$$gn = \langle N, h_1, h_2 \rangle = \langle N, h_2, h_1 \rangle$$

Proof. By Bruner's computation (16), there is a relation in bidegree $(t-s, s) = (81, 15)$:

$$gnr = mN.$$

We have $Ext^{15,81+15} = \mathbb{Z}/2 \oplus \mathbb{Z}/2$, generated by gnr and $h_1 x_{14,42}$. Moreover, the element gnr is not divisible by h_1, and neither of the generators is divisible by h_2.

By Tangora's computation (77), we have a Massey product in the Adams E_2 page,

$$m = \langle r, h_1, h_2 \rangle.$$

Therefore,

$$gn \cdot r = m \cdot N = N \cdot \langle r, h_1, h_2 \rangle = \langle N \cdot r, h_1, h_2 \rangle = r \cdot \langle N, h_1, h_2 \rangle$$

with zero indeterminacy. This implies

$$gn = \langle N, h_1, h_2 \rangle.$$

Because of the relation $h_2 \cdot N = 0$ in $Ext^{9,49+9} = 0$, we also have

$$gn \cdot r = m \cdot N = \langle r, h_1, h_2 \rangle \cdot N = r \cdot \langle h_1, h_2, N \rangle.$$

This implies

$$gn = \langle N, h_2, h_1 \rangle.$$

□

Based on Proposition 4.1.3, we prove part (1) of Corollary 4.1.4.

Proof. By Proposition 4.1.3, N detects the homotopy class $\sigma^2\{d_1\}$. Then the Massey product

$$gn = \langle N, h_2, h_1 \rangle$$

and Moss's theorem (60) imply that gn detects a homotopy class that is contained in the Toda bracket

$$\langle \sigma^2\{d_1\}, \nu, \eta \rangle.$$

The indeterminacy of this Toda bracket is

$$\eta \cdot \pi_{50} + \sigma^2\{d_1\} \cdot \pi_5 = \eta \cdot \pi_{50},$$

since $\pi_5 = 0$. Shuffling this bracket, we have

$$\langle \sigma^2\{d_1\}, \nu, \eta \rangle \supseteq \sigma\{d_1\} \cdot \langle \sigma, \nu, \eta \rangle = 0,$$

since $\langle \sigma, \nu, \eta \rangle \subseteq \pi_{12} = 0$.

Therefore, gn detects a homotopy class that lies in the indeterminacy, and hence is divisible by η.

For filtration reasons, the only other possibility is h_5c_1. However, Lemma 4.2.51 of (35) states that there is no η-extension from h_5c_1 to gn. Therefore, we must have an η-extension from C to gn. □

Based on Proposition 4.1.3, we prove part (2) of Corollary 4.1.4.

Proof. By Proposition 4.1.3, N detects the homotopy class $\sigma^2\{d_1\}$. Then the Massey product

$$gn = \langle N, h_1, h_2 \rangle$$

and Moss's theorem (60) imply that gn detects a homotopy class that is contained in the Toda bracket

$$\langle \sigma^2\{d_1\}, \eta, \nu \rangle.$$

The indeterminacy of this Toda bracket is

$$\nu \cdot \pi_{48} + \sigma^2\{d_1\} \cdot \pi_5 = \nu \cdot \pi_{48},$$

since $\pi_5 = 0$. Shuffling this bracket, we have

$$\langle \sigma^2\{d_1\}, \eta, \nu \rangle \supseteq \sigma \cdot \langle \sigma\{d_1\}, \eta, \nu \rangle = \sigma\{d_1\} \cdot \langle \eta, \nu, \sigma \rangle = 0,$$

since $\langle \eta, \nu, \sigma \rangle \subseteq \pi_{12} = 0$.

Therefore, gn detects a homotopy class that lies in the indeterminacy, and hence is divisible by ν.

For filtration reasons, the only possibility is $h_2h_5d_0$, which completes the proof. □

Now we prove part (3) of Corollary 4.1.4, and Proposition 4.1.1.

Proof. Lemma 4.2.31 from Isaksen's computation (35) states that the 2-extension from $h_0h_3g_2$ to gn is equivalent to the ν-extension from $h_2h_5d_0$ to gn. This proves Proposition 4.1.1.

It is clear that Proposition 4.1.1 is equivalent to part (3) of Corollary 4.1.4, since σ is detected by h_3, and $\sigma^2\theta_4$ is detected by $h_0^2g_2$. (See (9; 35) for the second fact.) □

In the following Lemma 4.2.2, we show that the two statements in Proposition 4.1.3 are equivalent.

Lemma 4.2.2. *There is a σ-extension from h_3d_1 to N if and only if there is an η-extension from h_1g_2 to N.*

Proof. First note that there are relations in Ext:

$$h_3d_1 = h_1e_1, h_3e_1 = h_1g_2.$$

By Bruner's differential (15, Theorem 4.1)

$$d_3(e_1) = h_1t = h_2^2n,$$

we have Massey products in the Adams E_4-page

$$h_3d_1 = h_1e_1 = \langle h_2n, h_2, h_1\rangle,\ h_1g_2 = h_3e_1 = \langle h_3, h_2n, h_2\rangle.$$

Then Moss's theorem implies that they converge to Toda brackets

$$\langle \nu\{n\}, \nu, \eta\rangle,\ \langle \sigma, \nu\{n\}, \nu\rangle.$$

Therefore, the lemma follows from the shuffling

$$\sigma \cdot \langle \nu\{n\}, \nu, \eta\rangle = \langle \sigma, \nu\{n\}, \nu\rangle \cdot \eta.$$

□

We give a remark on the two η-extensions we proved.

Remark 4.2.3. In Lemma 4.2.47 and Lemma 4.2.52 of (35), Isaksen showed that there are no η-extensions from h_1g_2 to N or from C to gn. Both arguments are based the statement of Lemma 3.3.45 of (35), whose proof implicitly studied the following Massey product

$$\langle h_1^2, Ph_1h_5c_0, c_0\rangle \ni Ph_1^3h_5e_0$$

in the 59-stem of the Adams E_3-page, which therefore converges to a Toda bracket. However, in the Adams E_3-page, the element $Ph_1^3h_5e_0$ is in the indeterminacy of this Massey product, since $Ph_1h_5e_0$ is present in the E_3-page (it supports a d_3 differential). Therefore, we have

$$\langle h_1^2, Ph_1h_5c_0, c_0\rangle = \{Ph_1^3h_5e_0, 0\}$$

instead. The statement of Moss's theorem gives us the convergence of *only one* permanent cycle in the Massey product, therefore, in this case, it is inconclusive.

4.3 The method and notations

In this section, we recall a few notations from (86) and set up terminology that will be used in Section 4.4.

Notation 4.3.1. All spectra are localized at the prime 2. Suppose Z is a spectrum. Let $Ext(Z)$ denote its Adams E_2-page.

For spectra, let S^0 be the sphere spectrum, and P_1^∞ be the suspension spectrum of RP^∞. In general, we use P_n^{n+k} to denote the suspension spectrum of RP^{n+k}/RP^{n-1}.

Let α be a class in the stable homotopy groups of spheres. We use $C\alpha$ to denote the cofiber of α.

Definition 4.3.2. Let A, B, C and D be CW spectra, i and q be maps

$$A \overset{i}{\hookrightarrow} B, \qquad B \overset{q}{\twoheadrightarrow} C$$

We say that (A, i) is an $H\mathbb{F}_2$-subcomplex of B, if the map i induces an injection on mod 2 homology. We denote an $H\mathbb{F}_2$-subcomplex by a hooked arrow as above.

We say that (C, q) is an $H\mathbb{F}_2$-quotient complex of B, if the map q induces a surjection on mod 2 homology. We denote an $H\mathbb{F}_2$-quotient complex by a double headed arrow as above.

When the maps involved are clear in the context, we also say A is an $H\mathbb{F}_2$-subcomplex of B, and C is an $H\mathbb{F}_2$-quotient complex of B.

Furthermore, we say D is an $H\mathbb{F}_2$-subquotient of B, if D is an $H\mathbb{F}_2$-subcomplex of an $H\mathbb{F}_2$-quotient complex of B, or an $H\mathbb{F}_2$-quotient complex of an $H\mathbb{F}_2$-subcomplex of B.

One example is that S^1, S^3, S^7 are $H\mathbb{F}_2$-subcomplexes of P_1^∞, due to the solution of the Hopf invariant one problem. Another example is that $\Sigma^7 C\eta$ is an $H\mathbb{F}_2$-subcomplex of P_7^9.

We use the following way to denote the elements in the Adams E_2-page of P_1^∞ and its $H\mathbb{F}_2$-subquotients. One way to compute $Ext(P_1^\infty)$ is to use the algebraic Atiyah-Hirzebruch spectral sequence.

$$E_1 = \bigoplus_{n=1}^{\infty} Ext(S^n) \Longrightarrow Ext(P_1^\infty)$$

Notation 4.3.3. We denote any element in $Ext(S^n)$ by $a[n]$, where $a \in Ext(S^0)$, and n suggests that it comes from $Ext(S^n)$. We will abuse notation and write the same symbol $a[n]$ for an element of $Ext(P_1^\infty)$ detected by the element $a[n]$ of the Atiyah-Hirzebruch E_∞ page. Thus, there is indeterminacy in the notation $a[n]$ that is detected by Atiyah-Hirzebruch E_∞ elements in lower filtration. When $a[n]$ is the element of lowest Atiyah-Hirzebruch filtration in the Atiyah-Hirzebruch E_∞ page in a given bidegree (s, t), then $a[n]$ also is a well-defined element of $Ext(P_1^\infty)$.

We use similar notations for homotopy classes.

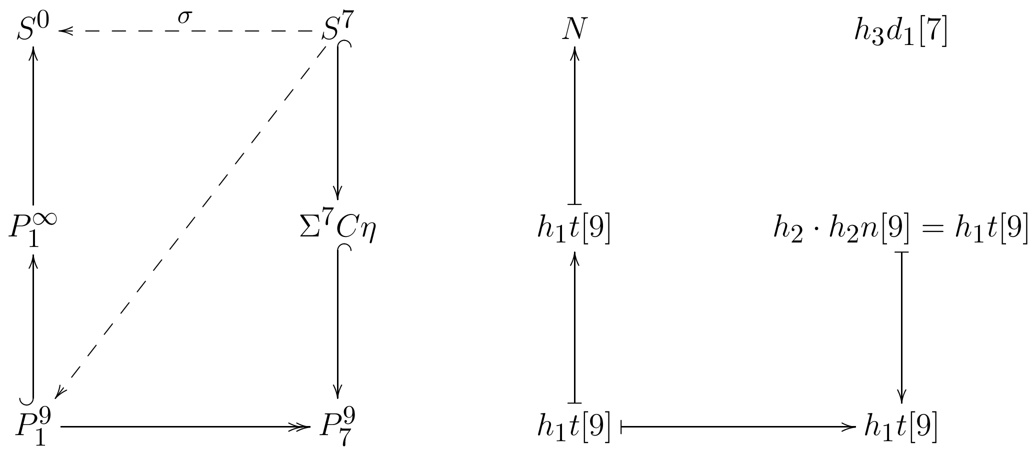
Remark 4.3.4. In (85), we computed the Adams E_2-page of P_1^∞ in the range of $t < 72$ by the Lambda algebra. This Lambda algebra computation gives us a lot of information on the algebraic Atiyah-Hirzebruch spectral sequence. In particular, there is a one-to-one correspondence between the differentials in the Lambda algebra computation and the differentials in the algebraic Atiyah-Hirzebruch spectral sequence. From the Lambda algebra computation, one can also read off information about $Ext(P_n^{n+k})$. For details, see (85).

Remark 4.3.5. Despite the indeterminacy in Notation 4.3.3, there is a huge advantage of it. Suppose $f : Q \to Q'$ is a map between two $H\mathbb{F}_2$-subquotients of P_1^{∞}, and there exists an element $a[n]$ which is a generater of both $Ext^{s,t}(Q)$ and $Ext^{s,t}(Q')$ for some bidegree (s, t) (this implies both Q and Q' have a cell in dimension n). We must have that, with the right choices, $a[n]$ in $Ext^{s,t}(Q)$ maps to $a[n]$ in $Ext^{s,t}(Q')$. This property follows from the naturality of the algebraic Atiyah-Hirzebruch spectral sequence.

$$\begin{array}{ccc} \bigoplus_{i\in I} Ext(S^i) & \longrightarrow & \bigoplus_{i\in I'} Ext(S^i) \\ \Downarrow & & \Downarrow \\ Ext(Q) & \longrightarrow & Ext(Q') \\ & & \\ a[n] & \longmapsto & a[n] \end{array}$$

4.4 the σ-extension on h_3d_1

In this section, we prove part (1) of Proposition 4.1.3. The proof can be summarized in the following “road map” with 4 main steps:



Here the elements in the right side of the "road map" are elements in the 46-stem of the E_∞ page of the Adams spectral sequences of the spectra in the corresponding positions.

1. **Step 1**: We show that the element $h_1 t[9]$ is a permanent cycle in the Adams spectral sequence of $\Sigma^7 C\eta$, and hence a permanent cycle in the Adams spectral sequence of P_7^9. This is stated as Proposition 4.3.

2. **Step 2**: Under the inclusion map $S^7 \hookrightarrow \Sigma^7 C\eta$, we show that the element $h_1 t[9]$ detects the image of $\sigma\{d_1\}[7]$ in $\pi_{46}(\Sigma^7 C\eta)$. By naturality, the same statement is true, after we further map it to $\pi_{46}(P_7^9)$. This is stated as Proposition 4.6.

3. **Step 3**: Under the inclusion map $S^7 \hookrightarrow P_1^9$, we show that the element $h_1 t[9]$ in $Ext(P_1^9)$ detects the image of $\sigma\{d_1\}[7]$ in $\pi_{46}(P_1^9)$. This is stated as Proposition 4.7.

4. **Step 4**: Using the inclusion map $P_1^9 \to P_1^\infty$ and the transfer map $P_1^\infty \to S^0$, we push forward the element $h_1 t[9]$ in the E_∞ page of P_1^9 to the element N in the E_∞ page of S^0. Since the composition

$$S^7 \hookrightarrow P_1^9 \longrightarrow P_1^\infty \longrightarrow S^0$$

is just σ, we have the desired σ-extension from h_3d_1 to N in the Adams spectral sequence for S^0.

Remark 4.4.1. Step 2 is the essential step. Intuitively, it comes from the zigzag of the following two differentials:

$$d_3(e_1) = h_1t$$

in the Adams spectral sequence of S^0, and

$$d_2(e_1[9]) = h_1e_1[7] = h_3d_1[7]$$

in the algebraic Atiyah-Hirzebruch spectral sequence of $\Sigma^7 C\eta$. Here

$$h_1e_1 = h_3d_1$$

is a relation in *Ext*. This zigzag suggested that we consider the fact that $h_1t[9]$ detects $\sigma\{d_1\}[7]$ in $\pi_{46}(\Sigma^7 C\eta)$.

We start with Step 1. Proposition 4.4.3 is a consequence of the following lemma.

Lemma 4.4.2. *The element $h_2n[9]$ is a permanent cycle in $\Sigma^7 C\eta$, which detects a homotopy class that maps to $\nu\{n\}[9]$ under the quotient map*

$$\Sigma^7 C\eta \twoheadrightarrow S^9.$$

Proof. The cofiber sequence

$$S^7 \xhookrightarrow{i} \Sigma^7 C\eta \xrightarrow{p} S^9 \xrightarrow{\eta} S^8$$

gives us a long exact sequence of homotopy groups

$$\pi_{43}(S^7) \xrightarrow{i_*} \pi_{43}(\Sigma^7 C\eta) \xrightarrow{p_*} \pi_{43}(S^9) \xrightarrow{\eta} \pi_{43}(S^8).$$

Since h_2n detects $\nu\{n\}$, and

$$\eta \cdot \nu\{n\} = 0,$$

there is an element α in $\pi_{43}(\Sigma^7 C\eta)$ such that $p_*\alpha = \nu\{n\}[9]$.

The element $h_2n[9]$ in $Ext(S^9)$ has Adams filtration 6, therefore by naturality, if it were not detected by $h_2n[9]$ in $Ext(\Sigma^7 C\eta)$, it would be detected by an element with Adams filtration at most 5.

From the same cofiber sequence, we have a short exact sequence on cohomology

$$0 \longrightarrow H^*(S^9) \xrightarrow{p^*} H^*(\Sigma^7 C\eta) \xrightarrow{i^*} H^*(S^7) \longrightarrow 0$$

and therefore a long exact sequence of Ext groups

$$Ext^{s-1,t-1}(S^8) \xrightarrow{h_1} Ext^{s,t}(S^7) \xrightarrow{i_\sharp} Ext^{s,t}(\Sigma^7 C\eta) \xrightarrow{p_\sharp} Ext^{s,t}(S^9).$$

This gives us the Adams E_2 page of $\Sigma^7 C\eta$ in the 42 and 43 stems for $s \leq 6$ in Table 4.1.

Table 4.1: The Adams E_2 page of $\Sigma^7 C\eta$ in the 42 and 43 stems for $s \leq 6$

$s \backslash t-s$	42	43
6		$h_2n[9]$
		$t[7]$
5	$h_0p[9]$	
	$h_2d_1[7]$	
4	$p[9]$	$h_0^2h_2h_5[9]$
3		$h_0h_2h_5[9]$
2		$h_2h_5[9]$

The element $h_2h_5[9]$ must support a nontrivial differential, since its image $p_\sharp(h_2h_5[9])$ supports a d_3 differential that kills $h_0p[9]$ in the Adams spectral sequence of S^9.

The elements $h_0h_2h_5[9]$ and $h_0^2h_2h_5[9]$ survive and detect homotopy classes that map to $\{h_0h_2h_5\}[9]$ and $\{h_0^2h_2h_5\}[9]$ in $\pi_{43}(S^9)$. In fact, since there is no η-extension on $h_0h_2h_5$ and $h_0^2h_2h_5$, we can choose homotopy classes in $\pi_{43}(S^9)$, which are detected by $h_0h_2h_5[9]$ and

$h_0^2 h_2 h_5[9]$ and are zero after multiplying by η. Therefore, they have nontrivial pre-images under the map p_* in the long exact sequence of homotopy groups. For filtration reasons, their pre-images must be detected by $h_0 h_2 h_5[9]$ and $h_0^2 h_2 h_5[9]$ in the Adams spectral sequence of $\Sigma^7 C\eta$.

Therefore, the only possibility left is $h_2 n[9]$, which completes the proof. □

We prove Proposition 4.4.3 in Step 1.

Proposition 4.4.3. *The elements $h_2 n[9]$ and $h_1 t[9]$ are permanent cycles in the Adams spectral sequence of $\Sigma^7 C\eta$, and hence also in that of P_7^9.*

Proof. We have a relation in Ext:

$$h_2 \cdot h_2 n = h_1 t.$$

Therefore, $h_1 t[9]$ is product of permanent cycles. The second claim follows from the naturality of the Adams spectral sequences. □

For Step 2, we first show the following lemma.

Lemma 4.4.4. *The element $h_1 t[9]$ is not a boundary in the Adams spectral sequences of $\Sigma^7 C\eta$ and P_7^9.*

Proof. The element $h_1 t[9]$ is hit by a d_3 differential on $e_1[9]$

In the Adams spectral sequence of S^9, we have the Bruner differential

$$d_3(e_1[9]) = h_1 t[9].$$

However, the element $e_1[9]$ is not present in either $Ext(\Sigma^7 C\eta)$ or $Ext(P_7^9)$.

Therefore, by naturality, the element $h_1 t[9]$ cannot be hit by any d_r differential for $r \leq 3$ in the Adams spectral sequence of $\Sigma^7 C\eta$ and P_7^9.

Table 4.2: The Adams E_2 page of $\Sigma^7 C\eta$ and P_7^9 in the 46 and 47 stems for $s \leq 7$

	$Ext(\Sigma^7 C\eta)$		$Ext(P_7^9)$	
$s \backslash t-s$	46	47	46	47
7	$h_1 t[9]$	$\bullet$	$h_1 t[9]$	$\bullet$
	$h_0^2 x[9]$		$h_0^2 x[9]$	$\bullet$
6	$h_0 x[9]$	$\bullet$	$h_0 x[9]$	$\bullet$
		$\bullet$	$h_1 x[8]$	$\bullet$
5		$\bullet$	$\bullet$	$\bullet$
		$\bullet$		$\bullet$
4	$\bullet$	$\bullet$	$\bullet$	$\bullet$
		$\bullet$	$\bullet$	$\bullet$
				$\bullet$
3	$\bullet$	$h_0 h_3 h_5[9]$	$\bullet$	$h_0 h_3 h_5[9]$
				$h_1 h_3 h_5[8]$

We have the Adams E_2 page of $\Sigma^7 C\eta$ and P_7^9 in the 46 and 47 stems for $s \leq 7$ in Table 4.2.

We need to rule out two candidates: $h_0 h_3 h_5[9]$ and $h_1 h_3 h_5[8]$.

In the Adams spectral sequence of S^9, we have a d_4 differential:

$$d_4(h_0 h_3 h_5[9]) = h_0^2 x[9].$$

By naturality of the quotient map to S^9, the element $h_0 h_3 h_5[9]$ cannot support a d_4 differential that kills $h_1 t[9]$.

For the element $h_1 h_3 h_5[8]$, it is straightforward to check it is a permanent cycle in the Adams spectral sequence of P_7^8, and hence a permanent cycle in that of P_7^9. This rules out the candidate $h_1 h_3 h_5[8]$ and completes the proof. □

Remark 4.4.5. In $Ext^{6,6+46}(P_7^9)$, the element $h_1 x[8]$ is clearly a surviving cycle. There are two possibilities for the other element $h_0 x[9]$: it is either killed by a d_3 differential from $h_0 h_3 h_5[9]$, or it survives and detects $\{h_1 h_3 h_5\}[7]$. We will leave the reader to figure out which way it goes.

We prove Proposition 4.4.6 in Step 2.

Proposition 4.4.6. *Under the inclusion map $S^7 \hookrightarrow \Sigma^7 C\eta$, the element $h_1 t[9]$ detects the image of $\sigma\{d_1\}[7]$ in $\pi_{46}(\Sigma^7 C\eta)$. By naturality, the same statement is true after we further map it to $\pi_{46}(P_7^9)$.*

Proof. By Lemma 4.4.2 and Proposition 4.4.3, the element $h_2 n[9]$ survives in the Adams spectral sequence of $\Sigma^7 C\eta$, and detects a homotopy class that maps to $\nu\{n\}[9]$ under the quotient map

$$\Sigma^7 C\eta \twoheadrightarrow S^9.$$

By Lemma 4.4.4, the element $h_1 t[9] = h_2 \cdot h_2 n[9]$ survives and detects the homotopy class $\nu \cdot \nu\{n\}[9]$. As showed in the proof of Lemma 4.2.2, the element $h_3 d_1 = h_1 e_1$ detects an element in the Toda bracket

$$\langle \nu, \nu\{n\}, \eta \rangle.$$

Therefore, by Lemma 4.5.2, we have

$$\nu \cdot \nu\{n\}[9] = \langle \nu, \nu\{n\}, \eta \rangle[7] = \sigma\{d_1\}[7]$$

in $\pi_{46}(\Sigma^7 C\eta)$. □

Now we prove Step 3.

Proposition 4.4.7. *Under the inclusion map $S^7 \hookrightarrow P_1^9$, the element $h_1 t[9]$ in $Ext(P_1^9)$ detects the image of $\sigma\{d_1\}[7]$ in $\pi_{46}(P_1^9)$.*

The idea of the proof of Proposition 4.4.6 is to make use of naturality of the Adams filtrations.

$$S^7 \hookrightarrow P_1^9 \twoheadrightarrow P_7^9$$

$$h_3 d_1[7] \qquad\qquad h_1 t[9]$$

$$AF = 5 \qquad\qquad AF = 7$$

The homotopy class $\sigma\{d_1\}[7]$ is detected by $h_3 d_1[7]$ in S^7, which has Adams filtration 5, while its image in $\pi_{46}(P_7^9)$ is detected by $h_1 t[9]$ by Proposition 4.5, which has Adams filtration 7. Therefore, to prove Proposition 4.6, we only need to rule out surviving cycles in the Adams filtration 6, which also lie in the kernel of the map

$$P_1^9 \twoheadrightarrow P_7^9$$

in the Adams E_∞ page. Note that the element $h_3 d_1[7]$ is not present in $Ext(P_1^9)$.

Proof. We have the Adams E_2 page of P_1^9 and P_1^∞ in the 46 and 47 stems for $s \leq 8$ in Table 4.3.

There are 4 elements in $Ext^{6,6+46}(P_1^9)$:

$$Ph_1h_5[6],\ h_0^2 g_2[2],\ h_1 x[8],\ h_0 x[9].$$

Remark 4.5 rules out the last two candidates, since they do not lie in the kernel of the map

$$P_1^9 \twoheadrightarrow P_7^9$$

in the Adams E_∞ page.

In the table for the transfer map in (85), we have that the element $h_0^2 g_2[2]$ maps to B_1. If the image of the homotopy class $\sigma\{d_1\}[7]$ were detected by $h_0^2 g_2[2]$, then we would have a σ-extension from $h_3 d_1$ to B_1 in $\pi_{46} S^0$, which by Lemma 4.2.2 is equivalent to an η-extension

Table 4.3: The Adams E_2 page of P_1^9 and P_1^∞ in the 46 and 47 stems for $s \leq 8$

	$Ext(P_1^9)$		$Ext(P_1^\infty)$	
$s \backslash t-s$	46	47	46	47
8	$Ph_1^3h_5[4]$	•	$Ph_1^3h_5[4]$	•
	•	•		
	•	•		
7	$Ph_1^2h_5[5]$	•	$Ph_1^2h_5[5]$	•
	$h_1t[9]$	•	$h_1t[9]$	•
	$h_0^2x[9]$	•		•
		•		
6	$Ph_1h_5[6]$	•	$Ph_1h_5[6]$	$h_1h_5d_0[1]$
	$h_0^2g_2[2]$	•	$h_0^2g_2[2]$	$h_1x[9]$
	$h_1x[8]$	•	$h_1x[8]$	
	$h_0x[9]$			
5	$h_0^3h_3h_5[8]$	•	$h_0^3h_3h_5[8]$	$h_1g_2[2]$
	•	•	•	$h_1f_1[6]$
	•	•	•	
			•	
4	•	•	$h_1^3h_5[12]$	$h_0h_4^3[2]$
		•		$g_2[3]$
		•		$f_1[7]$
		•		
3	•	•	$h_1^2h_5[13]$	
	•		•	
			•	
2			$h_1h_5[14]$	$h_2h_5[13]$
1			$h_5[15]$	

from h_1g_2 to B_1 in $\pi_{46}S^0$. However, the proof of Lemma 4.2.47 of (35) shows the latter is not true.

The only candidate left is $Ph_1h_5[6]$. To rule it out, we notice there is a long h_0 tower in the 46 stem of P_1^∞: from $h_5[15]$ to $Ph_1^3h_5[4]$. In particular, we have

$$h_0 \cdot Ph_1h_5[6] = Ph_1^2h_5[5],\ h_0 \cdot Ph_1^2h_5[5] = Ph_1^3h_5[4].$$

Since

$$2 \cdot \sigma\{d_1\} = 0,$$

the image of the homotopy class $\sigma\{d_1\}[7]$ must have order 2. Therefore, we only need to show the element $Ph_1^2h_5[5]$ is not a boundary. In the following Lemma 4.8, we show that the elements in Adams filtration 4 to 6 of $Ext(P_1^\infty)$ are all permanent cycles. This only leaves the possibility that $h_2h_5[13]$ kills $Ph_1^3h_5[4]$, but not $Ph_1^2h_5[5]$, and hence completes the proof. □

Lemma 4.4.8. *The elements in Adams filtration 4 to 6 of $Ext(P_1^\infty)$ are all permanent cycles.*

Proof. There are 7 elements:

$$h_1h_5d_0[1],\ h_1x[9],\ h_1g_2[2],\ h_1f_1[6],\ h_0h_4^3[2],\ g_2[3],\ f_1[7].$$

The spheres S^1, S^3, S^7 are $H\mathbb{F}_2$-subcomplexes of P_1^∞ by the solution of the Hopf invariant one problem. Since the elements $h_1h_5d_0,\ g_2,\ f_1$ are permanent cycles in the Adams spectral sequence for S^0, The elements $h_1h_5d_0[1],\ h_1g_2[2],\ f_1[7]$ are permanent cycles.

The element $h_1f_1[6] = h_0 \cdot f_1[7]$ is therefore also a permanent cycle.

It is straightforward to show that the elements $h_1g_2[2]$ and $h_0h_4^3[2]$ are permanent cycles in the Adams spectral sequence of P_1^2. By naturality they are permanent cycles in that of P_1^∞.

For the element $h_1x[9]$, one can use the $H\mathbb{F}_2$-subcomplex that contains cells in dimensions 3, 5, 7, 9 to show it is a permanent cycle. We will leave the details to the reader. Alternatively, since $h_1x[9]$ is divisible by h_1, while the potential target $Ph_1^3h_5[4]$ is not divisible by h_1, we have

$$d_2(h_1x[9]) = 0.$$

This will also do the job. □

Now we prove Step 4.

Lemma 4.4.9. *The element $h_1t[9]$ maps to N under the transfer map.*

Proof. We check the two tables in the appendix of (85). See (85) for more details of the Lambda algebra notation we use here. The element N is in $Ext^{8,8+46}(S^0) = \mathbb{Z}/2$. Checking the table for P_1^∞, we have that

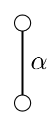
$$Ext^{7,7+46}(P_1^\infty) = (\mathbb{Z}/2)^2, \quad \text{generated by} \quad (5)\ 11\ 12\ 4\ 5\ 3\ 3\ 3,\ (9)\ 3\ 5\ 7\ 3\ 5\ 7\ 7,$$

which means $Ext^{7,7+46}(P_1^\infty)$ is generated by $Ph_1^2h_5[5]$ and $h_1t[9]$. Since $Ph_1^2h_5[5]$ is divisible by h_0 in $Ext(P_1^\infty)$, while N is not divisible by h_0 in $Ext(S^0)$, $Ph_1^2h_5[5]$ cannot map to N under the transfer map. By the algebraic Kahn-Priddy theorem, the other generator $h_1t[9]$ has to map to N. □

4.5 Appendix

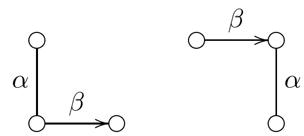
We use cell diagrams for the statements of the lemmas in this appendix. For the definition of cell diagrams, see (7).

In particular, let α, β be classes in the stable homotopy groups of spheres such that $\alpha \cdot \beta = 0$. We denote the cofiber of α by



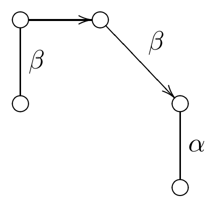
We denote the inclusion of the bottom cell and the quotient to the top cell maps by

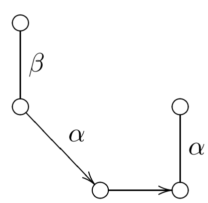
and the extension and co-extension maps of β by



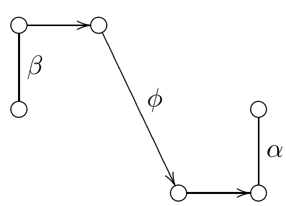
Note that while inclusion and quotient maps are uniquely determined by their cell diagrams, there are sets of extension and co-extension maps that correspond to the same cell diagrams. When taking compositions, the next lemma says the set of co-extensions precomposing with quotient maps equal to the set of extensions post-composing with inclusion maps.

Lemma 4.5.1. *Suppose that $\alpha \cdot \beta = 0$. Then the following two sets of maps equal:*





Proof. It is clear that the indeterminacy of both sets of maps is given by the following compositions:



where $\phi \in \pi_{|\alpha|+|\beta|+1}S^0$ could be any class. Then we need to show that they contain one common element. We have the following diagram:

$$\begin{array}{ccccccccc}
 & & S^{|\alpha|} & \xrightarrow{\alpha} & S^0 & \xrightarrow{i_1} & C(\alpha) & \xrightarrow{p_1} & S^{|\alpha|+1} & \xrightarrow{\alpha} & S^1 \\
 & & \| & & \uparrow f & & \uparrow g & & \| & & \\
S^{|\alpha|+|\beta|} & \xrightarrow{\beta} & S^{|\alpha|} & \xrightarrow{i_2} & \Sigma^{|\alpha|}C(\beta) & \xrightarrow{p_2} & S^{|\alpha|+|\beta|+1} & \xrightarrow{\beta} & S^{|\alpha|+1} & &
\end{array}$$

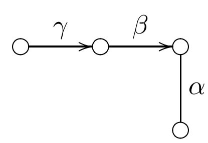
where i_1, i_2 are inclusions, p_1, p_2 are quotient maps. Since $\alpha\beta = 0$, we have an extension $f : \Sigma^{|\alpha|}C(\beta) \to S^0$. Since both lines are cofiber sequences, there exists a co-extension $g : S^{|\alpha|+|\beta|+1} \to C(\alpha)$ such that the diagram commutes. Therefore, we have that

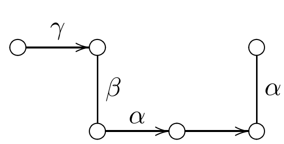
$$i_1 \circ f = g \circ p_2 : \Sigma^{|\alpha|}C(\beta) \to C(\alpha).$$

□

Now we prove the main lemma in this Appendix that gives a general connection that relates Toda brackets and extension problems in 2-cell complexes.

Lemma 4.5.2. *Suppose that $\alpha \cdot \beta = 0$, $\beta \cdot \gamma = 0$. The following two sets of maps equal:*

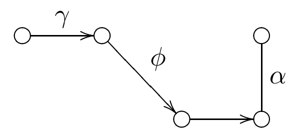




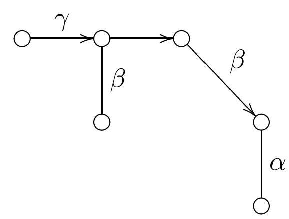
In the Atiyah-Hirzebruch notation, we have a γ-extension in the cofiber of α:

$$\gamma \cdot \beta[|\alpha| + 1] = \langle \gamma, \beta, \alpha \rangle [0]$$

Proof. It is clear that the indeterminacy of both sets of maps is given by the following compositions:



where $\phi \in \pi_{|\alpha|+|\beta|+1}S^0$ could be any class. Then we need to show that they contain one common element. Since $\beta \cdot \gamma = 0$, the map γ can be mapped through the cofiber of β. Then the first map can be decomposed as the following composition:



Then the previous lemma completes the proof. □

CHAPTER 5

THE ALGEBRAIC ATIYAH-HIRZEBRUCH SPECTRAL SEQUENCE OF RP^∞

5.1 Notations

We work at the prime 2 in this chapter. All cohomology groups are taken with coefficients $\mathbb{Z}/2$.

Let $\mathcal{A}$ be the Steenrod algebra. For any $\mathcal{A}$-module M, we will abbreviate $Ext_{\mathcal{A}}(M, \mathbb{Z}/2)$ by $Ext(M)$.

Let V be a vector space with $\{v_j\}$ an ordered basis. We say that an element $v = \sum a_i v_i$ has leading term $a_k v_k$ if k is the largest number for which $a_k \neq 0$.

For spectra, let S^0 be the sphere spectrum, and P_1^∞ be the suspension spectrum of $\mathbb{R}P^\infty$. In general, we use P_n^{n+k} to denote the suspension spectrum of $\mathbb{R}P^{n+k}/\mathbb{R}P^{n-1}$.

5.2 The Curtis table

We recall the notion of Curtis table in a general setting in this section.

Let $X_0 \to X_1 \to \dots$ be a complex of vector spaces (over $\mathbb{F}_2$). For each X_i, let $\{x_{i,j}\}$ be an ordered basis.

Definition 5.2.1. A Curtis table for X_* associated with the basis $\{x_{i,j}\}$ consists of a list L_i for each i.

The items on the list L_i are either an element $x_{i,j}$ for some j, or a tag of the form $x_{i,j} \leftarrow x_{i-1,k}$ for some j, k.

These lists satisfy the following:

1. Each element $x_{i,j}$ appears in these lists exactly once.

2. For any i, j, an item of the form $x_{i,j}$ or a tag of the form $x_{i,j} \leftarrow x_{i-1,k}$ appears in the list L_i if an only if there is a cycle in X_i with leading term $x_{i,j}$.

3. If a tag of the form $x_{i,j} \leftarrow x_{i-1,k}$ appears in the list L_i, then there is an element in X_{i-1} with leading term $x_{i-1,k}$ whose boundary has leading term $x_{i,j}$.

Remark 5.2.2. By Theorem 5.3.3 and Corollary 5.3.4, the Curtis table exists and is unique for a finite dimensional complex with ordered basis.

The Curtis algorithm constructs a Curtis table from a basis, and can output the full cycle from the input of a leading term.

For example, the Curtis table in the usual sense is for the lambda algebra with the basis of admissible monomials in lexicographic order. In (79) Tangora computed the Curtis table for the lambda algebra up to stem 51.

Another example is the minimal resolution for the sphere spectrum. This case is indeed trivial in the sense that there are no tags in the Curtis table.

5.3 The Curtis algorithm

The Curtis algorithm produces a Curtis table from an ordered basis. It can be described as follows:

Algorithm 5.3.1. (Curtis)

1. For each i, construct a list L_i which contains every $x_{i,j}$ such that the items are ordered with j ascending.

2. For $i = 0, 1, 2, \ldots$ do the following:

   (a) Construct a pointer p with initial value pointing to the beginning of L_i.

   (b) If p points to the end of L_i (i.e. after the last element), stop and proceed to the next i.

   (c) If the item pointed by p is tagged, move p to the next item and go to Step 2b.

   (d) Construct a vector $c \in X_i$. Give c the initial value of the item pointed by p.

(e) Compute the boundary $b \in X_{i+1}$ of c.

(f) If $b = 0$, move p to the next item and go to Step 2b.

(g) Search the leading term y of b in L_{i+1}.

(h) If y is untagged, tag y with the leading term of c. Remove the item pointed by p and move p to the next item. Go to Step 2b.

(i) If y is tagged by z, add z to c. Go to Step 2e.

Example 5.3.2. As an example, we compute the Curtis table for the lambda algebra for $t = 3$. We start with

$$L_1 = \{\lambda_2\}$$
$$L_2 = \{\lambda_1\lambda_0\}$$
$$L_3 = \{\lambda_0^3\}$$

We next compute the boundary of λ_2:

$$d(\lambda_2) = \lambda_1\lambda_0.$$

We therefore remove it from L_1 and tag $\lambda_1\lambda_0$ with λ_2. The output gives us the following:

$$L_1 = \emptyset$$
$$L_2 = \{\lambda_1\lambda_0 \leftarrow \lambda_2\}$$
$$L_3 = \{\lambda_0^3\}$$

Theorem 5.3.3. *(Curtis) The Curtis algorithm ends after finitely many steps when X_* is finite dimensional. Moreover, let Y_* be the graded vector space generated by those untagged items on the L_i's. Denote by C_* the subspace of cycles in X_*. There is an algorithm which constructs a map $Y_i \to C_i$ and a map $C_i \to Y_i$ which induce an isomorphism between Y_* and the homology of X_*.*

Proof. See (79). □

Corollary 5.3.4. *The Curtis table is unique for a finite dimensional complex X_* with ordered basis. In fact, it is specified in the following way:*

Let $l(x)$ denote the leading term of x.

If there is a tag $a \leftarrow b$, then a is the minimal element of the set $\{l(d(x))|l(x) = b\}$.

If an item a is untagged, then it is the leading term of an element with lowest leading term in a homology class.

Proof. See (79). □

5.4 Curtis table and spectral sequences

Now suppose V is a filtered vector space with $\cdots \subset F_iV \subset F_{i+1}V \cdots \subset V$. We call an ordered set of basis $\{v_k\}$ compatible if for any i there is a k_i such that F_iV is spanned by $\{v_k : k \leq k_i\}$.

Let $X_0 \to X_1 \to \ldots$ be a complex of filtered vector spaces such that the differentials preserve the filtration. Then there is a spectral sequence converging to the homology of X with the E_1-term $F_kX_i/F_{k-1}X_i$. Suppose we have compatible bases $\{x_{i,j}\}$ of X_i.

Theorem 5.4.1. *The Curtis table of X_* consists of the following:*

1. *The tags of the Curtis table for (E_r, d_r) of the spectral sequence, for all $r \geq 1$.*

2. *The untagged items from the E_∞-term.*

Here we label the basis of E_r as the following. In the E_1-page, we use the image of the $x_{i,j}$'s as the basis, and label them by the same name. Inductively, we use Theorem 5.3.3 to label a basis of E_r by the untagged items in the Curtis table of E_{r-1}.

Proof. We check the conditions of Definition 5.2.1. They follow directly from the definition of the spectral sequence, the conditions for the Curtis tables of the E_r's, and Theorem 5.3.3. □

Consequently, we can identify the Curtis table with the table for the differentials and permanent cycles of the spectral sequence. For example, in the lambda algebra, we have a filtration by the first number of an admissible sequence. The induced spectral sequence is the algebraic EHP sequence. So the usual Curtis table can be identified with the algebraic EHP sequence. See (18) for more details.

In practice, the Curtis table for the E_1 terms is often known before hand. Then we could skip those part of the Curtis algorithm dealing with the tags coming from the E_1 term. And we often omit this part in the output of Curtis table.

5.5 The algebraic Atiyah-Hirzebruch spectral sequence

Let X be a spectrum. There is a filtration on $H^*(X)$ by the degrees. For any n there is a short exact sequence $0 \to H^{\geq n+1}(X) \to H^{\geq n}(X) \to H^n(X) \to 0$. This induces a long exact sequence

$$\cdots \to Ext(\mathbb{Z}/2) \otimes H^n(X) \to Ext(H^{\geq n}(X)) \to Ext(H^{\geq n+1}(X)) \to \dots$$

Combining the long exact sequences for all n we get the algebraic Atiyah-Hirzebruch spectral sequence

$$\oplus_n Ext(\mathbb{Z}/2) \otimes H^n(X) \Rightarrow Ext(H^*(X))$$

There is another way to look at the algebraic Atiyah-Hirzebruch spectral sequence.

Let us fix a free resolution $\cdots \to F_1 \to F_0 \to \mathbb{F}_2$ of $\mathbb{F}_2$ as $\mathcal{A}$-modules. For example, we can take F_* to be the Koszul resolution, which gives the lambda algebra constructed in (13). We can also take F_* to the the minimal resolution.

Then for X a finite CW spectrum, we can identify $R\mathcal{H}om_{\mathcal{A}}(H^*(X), \mathbb{Z}/2)$ with the complex $C^*(H^*(X)) = Hom_{\mathcal{A}}(H^*(X) \otimes_{\mathbb{F}_2} F_*, \mathbb{F}_2)$ where we take the diagonal action of the Steenrod algebra on $H^*(X) \otimes_{\mathbb{F}_2} F_*$ using the Cartan formula.

The cell filtration on $H_*(X)$ induces a filtration on $H^*(X) \otimes_{\mathbb{F}_2} F_*$, and we can identify the algebraic Atiyah-Hirzebruch spectral sequence with the spectral sequence generated by this filtration. In fact, the map $H^*(X) \otimes_{\mathbb{F}_2} F_* \to H^*(X)$ preserves these filtrations and induces a quasi-isomorphism on each layer. So they define equivalent sequences in the derived category, hence generate the same spectral sequence.

5.6 The Curtis algorithm in computing the algebraic Atiyah-Hirzebruch spectral sequence

Let X be a finite CW spectrum.

Let $r_{i,j}^* \in F_i$ be a set of $\mathcal{A}$-basis for the free $\mathcal{A}$-module F_i. Let $r_{i,j} \in Hom_{\mathcal{A}}(F_i, \mathbb{Z}/2)$ be the dual basis.

We choose an ordered $\mathbb{F}_2$-basis e_k^* of $H^*(X)$ such that elements with lower degrees come first. Let $e_k \in H_*(X)$ be the dual basis. Then the set $\{e_k^* \otimes r_{i,j}^*\}$ is a set of $\mathcal{A}$-basis for $H^*(X) \otimes_{\mathbb{F}_2} F_*$. Let $e_k \otimes r_{i,j} \in Hom_{\mathcal{A}}(H^*(X) \otimes_{\mathbb{F}_2} F_*, \mathbb{F}_2)$ be the dual basis with the lexicographic order.

The following is a corollary of Theorem 5.4.1.

Theorem 5.6.1. *The Curtis table for $C^*(H^*(X)) = Hom_{\mathcal{A}}(H^*(X) \otimes_{\mathbb{F}_2} F_*, \mathbb{F}_2)$ satisfies*

1. *If there is a tag $a \leftarrow b$ in the Curtis table of $Hom_{\mathcal{A}}(F_i, \mathbb{Z}/2)$, there are tags of the form $e_k \otimes a \leftarrow e_k \otimes b$.*

2. *The table of all tags which are not contained in Case 1 is the same as the table for the algebraic Atiyah-Hirzebruch differentials of X.*

3. *The items not contained in the previous cases are untagged items. They correspond to the permanent cycles in the algebraic Atiyah-Hirzebruch spectral sequence.*

Consequently, we can read off the E_2-term of the Adams spectral sequence of any truncation of X.

Theorem 5.6.2. *Let X_m^n be the truncation of X which consists of all cells of X in dimensions between (and including) m and n. Therefore in the Curtis table of X_m^n, all the tags are those tags in the Curtis table of X lying within the corresponding range. (Note there could be more untagged items, which are just those not appearing in any tags.)*

Proof. This follows from the previous theorem because the Atiyah-Hirzebruch spectral sequence is truncated this way. □

We present two examples. The latter one is used in our computation in (86) that the 2-primary $\pi_{61} = 0$. For notation, in the Lambda algebra, we will abbreviate an element $\lambda_{i_1} \dots \lambda_{i_n}$ by $i_1 \dots i_n$. In the Lambda complex of P_1^∞, we will abbreviate an element $e_k \otimes \lambda_{i_1} \dots \lambda_{i_n}$ by $(k)i_1 \dots i_n$. The Curtis table is separated into lists labeled by $(t-s, t)$ on the top, in which those untagged items give a basis for $Ext^{s-1,t-1}(H^*(P_1^\infty))$.

Example 5.6.3. As a relatively easy example, we compute $Ext^{2,2+9}(H^*(P_2^8))$ using the Curtis table of P_1^∞ in the Appendix.

There are only two boxes that are used in this computation: the ones labeled with $(9,3)$ and $(8,4)$. The box labeled with $(9,3)$ is the following:

$$(1)\ 5\ 3$$

$$(3)\ 3\ 3$$

$$(7)\ 1\ 1 \leftarrow (9)\ 1$$

The spectrum P_2^8 only has cells in dimensions 2 through 8. We remove the item (1) 5 3, since it comes from the cell in dimension 1. We also remove the tag (9) 1, since it comes from the cell in dimension 9. Therefore, the only items remaining in this box are (3) 3 3 and (7) 1 1.

The box labeled with $(8,4)$ is the following:

$$(1)\ 5\ 1\ 1 \leftarrow (2)\ 6\ 1$$

$$(5)\ 1\ 1\ 1 \leftarrow (6)\ 2\ 1$$

After removing the element (1) 5 1 1, which comes from the cell in dimension 1, the element (2) 6 1 tags nothing. We move the element (2) 6 1 from the box labeled with $(8,4)$ to the one labeled with $(9,3)$. Therefore, we have the conclusion that the group $Ext^{2,2+9}(H^*(P_2^8))$ has dimension 3, generated by

$$(3)\ 3\ 3,\ \ (7)\ 1\ 1,\quad \text{and}\quad (2)\ 6\ 1.$$

One can even recover the names of these generators in the algebraic Atiyah-Hirzebruch spectral sequence. See Notation 3.3 in (86) for the notation. In $Ext(\mathbb{Z}/2)$, the elements 3 3, 1 1 and 6 1 all lie in the bidegrees which contain only one nontrivial element. Therefore, we can identify their Adams E_2-page names as h_2^2, h_1^1 and h_0h_3. This gives us the algebraic Atiyah-Hirzebruch E_1-page names of these generators:

$$h_2^2[3],\ \ h_1^1[7],\quad \text{and}\quad h_0h_3[2].$$

Example 5.6.4. We present the computation of the Adams E_2 page of P_{16}^{22} in the 60 and 61 stem for $s \leq 7$, which is used in the proof of Lemma 8.2 in (86). The boxes that are used in this computation have the following labels:

$$(59, s)\ \text{ for }\ s \leq 7,\quad \text{and}\quad (60, s'), (61, s')\ \text{ for }\ s \leq 8.$$

The spectrum P_{16}^{22} consists of cells in dimensions 16 through 22.

We start with the 60 stem.

We have $Ext^{1,1+60}(P_{16}^{22}) = Ext^{2,2+60}(P_{16}^{22}) = 0$, since the boxes labeled with $(60,2)$, $(59,3)$ and $(60,3)$, $(59,4)$ becomes empty.

We have $Ext^{3,3+60}(P_{16}^{22}) = \mathbb{Z}/2$, generated by (19) 11 15 15 from the box labeled with $(59,5)$. The box labeled with $(60,4)$ becomes empty. Since 11 15 15 $\in Ext^{3,3+41} = \mathbb{Z}/2$, generated by c_2, we identify (19) 11 15 15 with its Atiyah-Hirzebruch name $c_2[19]$.

We have $Ext^{4,4+60}(P_{16}^{22}) = \mathbb{Z}/2 \oplus \mathbb{Z}/2$, generated by (16) 13 13 11 7 from the box labeled with $(59,6)$, and by (20) 19 7 7 7 from the box labeled with $(60,5)$. We find their Atiyah-Hirzebruch names $g_2[16]$ and $f_1[20]$.

We have $Ext^{5,5+60}(P_{16}^{22}) = \mathbb{Z}/2 \oplus \mathbb{Z}/2$, generated by (16) 11 14 5 7 7 and (21) 7 13 5 7 7 from the box labeled with $(59,7)$. The box labeled with $(60,6)$ becomes empty. We find their Atiyah-Hirzebruch names $h_0g_2[16]$ and $h_1e_1[21]$.

We have $Ext^{6,6+60}(P_{16}^{22}) = \mathbb{Z}/2 \oplus \mathbb{Z}/2 \oplus \mathbb{Z}/2$, generated by (16) 7 14 4 5 7 7 from the box labeled with $(59,8)$, and by (20) 5 5 9 7 7 7 and (22) 3 5 9 7 7 7 from the box labeled with $(60,7)$. We find their Atiyah-Hirzebruch names $h_0^2g_2[16]$, $h_0^2f_1[20]$ and $h_1x[22]$.

We have $Ext^{7,7+60}(P_{16}^{22}) = \mathbb{Z}/2$, generated by (21) 3 5 9 3 5 7 7 from the box labeled with $(60,8)$. The box labeled with $(59,9)$ becomes empty. We find its Atiyah-Hirzebruch name $h_1y[21]$.

Similarly, one can compute the 61 stem. The computation is summarized in the following Table 5.1.

Table 5.1: The Adams E_2 page of P_{16}^{22} in the 60 and 61 stems for $s \leq 7$

$s\backslash t-s$	60	61
7	$h_1y[21]$	$h_0^2h_5d_0[16]$ $h_1y[22]$
6	$h_0^2g_2[16]$ $h_0^2f_1[20]$ $h_1x[22]$	$h_0h_5d_0[16]$ $Ph_2h_5[19]$
5	$h_0g_2[16]$ $h_1e_1[21]$	$h_1g_2[16]$ $h_5d_0[16]$ $h_1f_1[20]$ $h_1h_5c_0[21]$ $h_3d_1[22]$
4	$g_2[16]$ $f_1[20]$	$h_0h_4^3[16]$ $g_2[17]$ $f_1[21]$ $h_1^2h_3h_5[21]$ $h_5c_0[22]$
3	$c_2[19]$	$h_4^3[16]$ $h_1h_3h_5[22]$

5.7 The homomorphism induced by a map

Let $f : X \to Y$ be a map which induces the zero map on homology. Let Z be the cofiber of f. Then the homology of Z can be identified with the direct sum of $H_*(X)$ and $H_*(Y)$ as a vector space. If $x_1, \ldots, x_k$ is an ordered basis of $H_*(X)$ and $y_1, \ldots, y_l$ is an ordered basis of $H_*(Y)$, then $y_1, \ldots, y_l, x_1, \ldots, x_k$ is an ordered basis of $H_*(Z)$ with certain degree shifts. Note we do not make elements with lower degree go first here. Instead elements y_i always go before elements x_j regardless of degree.

Note that in this case, there is a map of Adams spectral sequence of X and Y which raises the Adams filtration by one, and on the E_2 page it is the boundary homomorphism for the Ext group for the exact sequence $0 \to H^{*+1}(X) \to H^*(Z) \to H^*(Y) \to 0$. We call this the map induced by f.

Theorem 5.7.1. *The Curtis table for $C^*(H^*(Z)) = Hom_{\mathcal{A}}(H^*(Z) \otimes_{\mathbb{F}_2} F_*, \mathbb{F}_2)$ from Section 5 with this ordered basis satisfies*

1. *All of the tags in the Curtis table for $C^*(H^*(X))$ and for $C^*(H^*(Y))$ also appear in the Curtis table for $C^*(H^*(Z))$.*

2. *The remaining tags give the table for the homomorphism on the Adams E_2-page induced by f.*

3. *The untagged items give basis for the kernel and cokernel of the homomorphism induced by f.*

Proof. This follows from Theorem 5.4.1 by using the filtration $Y \subset Z$, and identifying the d_2-differential with the attaching map $X \to Y$. $\square$

So we can use the Curtis algorithm to compute the homomorphism induced by a map.

CHAPTER 6

BP_*BP-COMODULES AND MOTIVIC $C\tau-$MODULES

In this Chapter, we use Theorem 1.0.4 to prove Theorem 1.0.5. We refer the read the proof of Theorem 1.0.4 to (27). This is joint work with Bogdan Gheorghe and Guozhen Wang (27).

6.1 Introduction

Motivic homotopy theory, introduced by Voevodsky and others, is a success of applying abstract homotopy theory to solve problems in number theory and algebraic geometry. Recently, Isaksen and his collaborators start showing applications in the other way around: using motivic Adams spectral sequence to prove nontrivial differentials and extensions in the classical Adams spectral sequence, and in return compute classical stable homotopy groups of spheres.

In this chapter, we prove two results in the stable motivic homotopy category over *Spec* $\mathbb{C}$ in the second direction, which give a new method of computing stable homotopy groups of spheres. The first result builds a connection between stable motivic homotopy category over *Spec* $\mathbb{C}$ and the derived category of BP_*BP comodules. The second result gives a systematic way of obtaining nontrivial differentials and extensions in the Adams spectral sequence. Using the second result, Isaksen and the second and third authors not only significantly simplify proofs of known Adams differentials, but have also been able to extend computations of stable homotopy groups of spheres into a larger range.

We work over *Spec* $\mathbb{C}$ in this chapter. Fix a prime p. Let H be the mod p motivic Eilenberg-Mac Lane spectrum, which represents the mod p motivic cohomology. Let $S^{0,0}$ be the motivic sphere spectrum, and $\widehat{S}^{0,0}$ be its completion with respect to H. Let $\tau \in$

$\pi_{0,-1}(\widehat{S}^{0,0})$ be the homotopy class which corresponds to the Tate twist. Recall that

$$\pi_{*,*}H = \mathbb{F}_p[\tau],$$

generated by the Hurewicz image of τ. Let $C\tau$ be the two cell complex - the cofiber of τ. It is a theorem of the first author (26) that $C\tau$ is an E_∞ ring object in the category of motivic spectra. In particular, it is an associative algebra object in the stable ∞-category of motivic spectra in the sense of Lurie (48). We denote by $C\tau - Mod_{cell}$ the stable ∞-category of cellular left $C\tau$ modules. Let $BPGL$ be the motivic Brown-Peterson spectrum completed with respect to H. We denote $C\tau - Mod_{cell}^b$ be the full subcategory whose objects are $BPGL$-completed and have bounded Chow degrees in their $BPGL$-homology.

Let $BP_*BP - Comod$ be the Abelian category of p-completed locally bounded below BP_*BP comodules which are concentrated in even degrees. Let $\mathcal{D}^b(BP_*BP - Comod)$ be the bounded derived category of $BP_*BP - Comod$ as a stable ∞-category (See Higher Algebra).

Our first result is the following theorem.

Theorem 6.1.1. *There is an equivalence of stable ∞-categories with t-structures at each prime p*

$$\mathcal{D}^b(BP_*BP - Comod) \simeq C\tau - Mod_{cell}^b.$$

Let $h\mathcal{D}^b(BP_*BP - Comod)$ and $hC\tau - Mod_{cell}^b$ be their homotopy categories. As a corollary, we prove a conjecture of Isaksen.

Corollary 6.1.2. *There is an equivalence of triangulated categories with t-structures at each prime p*

$$h\mathcal{D}^b(BP_*BP - Comod) \simeq hC\tau - Mod_{cell}^b.$$

The second result is suggested by a theorem of Isaksen that $\pi_{*,*}(C\tau)$ is isomorphic to $Ext_{BP_*BP}(BP_*, BP_*)$, the classical Adams-Novikov E_2-page for the sphere spectrum, as bigraded Abelian groups. For $\pi_{*,*}(C\tau)$, we have the motivic Adams spectral spectral for $C\tau$ converging to it. For $Ext_{BP_*BP}(BP_*, BP_*)$, we have the algebraic Novikov spectral sequence for sphere converging to it. Our following theorem says that they are isomorphic.

Theorem 6.1.3. *There is an isomorphism of spectral sequences at each prime p, between the the motivic Adams spectral sequence for $C\tau$ and the algebraic Novikov spectral sequence for the sphere spectrum. Moreover, this isomorphism preserves the multiplicative structures which are induced by composition.*

Using Theorem 6.1.3, we can produce classical Adams differentials from purely algebraic computations, which can be computed effectively. More specifically, start with an algebraic Novikov d_r differential for the sphere spectrum for any r. It is identical to a motivic Adams d_r differential for $C\tau$ by Theorem 6.1.3. Since $C\tau$ is a two cell complex, with a bottom cell in topological dimension 0, and a top cell in topological dimension 1, one can pull back the motivic Adams d_r differential for $C\tau$ to a motivic Adams $d_{r'}$ differential with $r' \leq r$ for $S^{0,0}$. Taking the realization functor from the category of motivic spectra to the category of classical spectra, one gets a classical Adams $d_{r'}$ differential for the sphere spectrum.

Remark 6.1.4. In (54), Miller proved that there is a strong correspondence between classical Adams d_2 differentials and algebraic Novikov d_2 differentials. The differentials in the Adams spectral sequences are *topological*, while the differentials in the algebraic Novikov spectral sequences are *algebraic* - in theory, the algebraic ones can be computed explicitly by imbedding them into the cobar complex. This result of Miller is striking, since one can get nontrivial topological differentials from purely algebraic data. Using this technique, Miller proved the Telescope Conjecture at chromatic level 1 at odd primes. In our method, if we let $r = 2$, then we must have $r' = 2$, since the motivic Adams spectral sequences start with the E_2-page. This recovers Miller's theorem.

Let $\mathcal{A}^{mot}$ be the mod p motivic dual Steenrod algebra. Recall that the motivic Adams

E_2 page of X is

$$Ext(X) = Ext^{*,*,*}_{\mathcal{A}^{mot}}(\mathbb{F}_p[\tau], H_{*,*}X).$$

We have the following two maps

$$Ext(S^{0,0}) \longrightarrow Ext(S^{0,0})/\tau \longrightarrow Ext(C\tau).$$

Because τ acts trivially on $Ext(C\tau)$, the $\mathbb{F}_p[\tau]$-module generators of $Ext(S^{0,0})$ maps non-trivially under the composition of the two maps above. This fact implies that the motivic Adams differentials for $S^{0,0}$, whose sources and targets are not divisible by τ, can be detected by our method.

Remark 6.1.5. Based on Theorem 6.1.3, our method produces a huge amount of differentials and extensions from purely algebraic computations, some of which are hard to prove (and to understand) by other methods. For example, the Adams d_3 differential in the 15-stem

$$d_3(h_0h_4) = h_0d_0$$

is known due to Toda's unstable computations (81). As another example, the Adams d_4 differential in the 38-stem

$$d_4(h_3h_5) = h_0x$$

is proved by Mahowald-Tangora (50) by an ad-hoc method using a certain finite CW spectrum. Moreover, in (86), the Adams d_3 differential in the 61-stem

$$d_3(D_3) = B_3$$

is proved by the second and third authors using the RP^∞-method, which took more than 40 pages to present all the details of the proof. All three differentials can now be proved using our new method directly from algebraic computation, which can be obtained from an automated program.

Remark 6.1.6. Based on Theorem 6.1.3, ongoing work of Isaksen and the second and the third authors also extends the known computations into a much larger range. For some of the differentials, our method gives the only proof. For example, we have an Adams d_3 differential in the 68-stem

$$d_3(d_2) = h_0^2 Q_3,$$

which implies the non-existence of the homotopy class κ_2 in π_{68}.

In Section 6.2, we use Theorem 6.1.1 to prove the isomorphism of the two spectral sequences (Theorem 6.1.3). We postpone the proof of a technical lemma to Section 6.3.

6.2 Two isomorphic spectral sequences

The goal of this section is to prove the following theorem.

Theorem 6.2.1. *There is an isomorphism of tri-graded spectral sequences: the motivic Adams spectral sequence for $C\tau$, which converges to the motivic homotopy groups of $C\tau$, and the re-graded algebraic Novikov spectral sequence, which converges to the Adams-Novikov E_2-page for sphere. More precisely, the indexes are indicated in the following diagram:*

$$\begin{array}{ccc}
Ext_{P_*}^{s,2w}(\mathbb{F}_p, I^{a-s}/I^{a-s+1}) & \xrightarrow{\cong} & Ext_{\mathcal{A}^{mot}}^{a,2w-s+a,w}(\mathbb{F}_p, \mathbb{F}_p[\tau]) \\
\Downarrow \;\textit{Algebraic Novikov SS} & & \Downarrow \;\textit{Motivic Adams SS} \\
Ext_{BP_*BP}^{s,2w}(BP_*, BP_*) & \xrightarrow{\cong} & \pi_{2w-s,w}(C\tau)
\end{array}$$

We first want to point out that, it was known to Isaksen (35) that the abutments of the two spectral sequences are isomorphic.

Theorem 6.2.2. *(Isaksen (36, Prop 6.2.5)) The motivic Adams-Novikov spectral sequence* *for $C\tau$ collapses for filtration reasons, and therefore we have an isomorphism*

$$\pi_{t-s,\frac{t}{2}}(C\tau) \cong Ext^{s,t}_{BP_*BP}(BP_*, BP_*).$$

When t is odd, we always have

$$Ext^{s,t}_{BP_*BP}(BP_*, BP_*) = 0,$$

therefore the weight of classes in $\pi_{t-s,\frac{t}{2}}(C\tau)$ are always integers: $\frac{t}{2}$.

To describe the two spectral sequences, we start with the E_1-page of the algebraic Novikov spectral sequence, which was introduced and discussed by Novikov (64) and Miller (55). Another good reference for this material is Ravenel's green book (71).

The algebraic Novikov spectral sequence comes from the I-filtration on BP_* and BP_*BP. We can therefore form the associated graded E_0BP_* and E_0BP_*BP.

Theorem 6.2.3. *(Novikov (64), Miller (55)) There is a tri-graded spectral sequence with*

$$E_1^{s,i,t} = Ext^{s,t}_{E_0BP_*BP}(E_0BP_*, E_0BP_*),$$

where i is the I-filtration, and

$$d_r : E_r^{s,i,t} \longrightarrow E_r^{s+1,i+r,t},$$

converging to

$$Ext^{s,t}_{BP_*BP}(BP_*, BP_*).$$

We can identify the E_1-page of the algebraic Novikov spectral sequence as

$$E_1^{s,i,t} = Ext^{s,t}_{P_*}(\mathbb{Z}/p, I^i/I^{i+1}).$$

In fact, recall that the dual Steenrod algebra A_* is

$$A_* = \begin{cases} \mathbb{F}_p(\xi_1, \xi_2, \cdots) & |\xi_i| = 2^i - 1 \quad \text{for } p = 2 \\ \Lambda_{\mathbb{F}_p}(\tau_0, \tau_1, \cdots) & |\tau_i| = 2p^i - 1 \\ \otimes \mathbb{F}_p(\xi_1, \xi_2, \cdots) & |\xi_i| = 2p^i - 2 \quad \text{for } p > 2. \end{cases}$$

Let P_* be the sub-Hopf algebra of A_* that

$$P_* = \begin{cases} \mathbb{F}_p(\xi_1^2, \xi_2^2, \cdots) & \text{for } p = 2 \\ \mathbb{F}_p(\xi_1, \xi_2, \cdots) & \text{for } p > 2. \end{cases}$$

and

$$E_* = A_* \otimes_{P_*} \mathbb{Z}/p = \begin{cases} \Lambda_{\mathbb{F}_p}(\xi_1, \xi_2, \cdots) & \text{for } p = 2 \\ \Lambda_{\mathbb{F}_p}(\tau_0, \tau_1, \cdots) & \text{for } p > 2. \end{cases}$$

Since ξ_i^2's correspond to t_i's, we have an isomorphism

$$BP_*BP/I \cong E_0BP_*BP \otimes_{E_0BP_*} \mathbb{Z}/p \cong P_*.$$

Therefore, by the change-of-rings isomorphism, we have the E_1-page of the algebraic Novikov spectral sequence as

$$E_1^{s,i,t} = Ext_{P_*}^{s,t}(\mathbb{Z}/p, I^i/I^{i+1}).$$

For the statement of the main theorem of this section, we re-grade the algebraic Novikov spectral sequence.

Definition 6.2.4. We define the a-filtration for the algebraic Novikov spectral sequence as

$$a = i + s.$$

Remark 6.2.5. We explain the 4 filtrations in the cobar resolution for BP_* as follows.

The t-filtration is the internal degree, which is preserved by differentials. The homological s-filtration is the number of bar's in the cobar resolution. The i-filtration is the number of v's in the cobar resolution. And finally our new a-filtration is the sum of numbers of bar's and v's in the cobar resolution.

After the re-grading, the d_r differentials, which are used to raise the i-filtration by r, now raise the a-filtration by $r+1$. This is because of they also raise the s-filtration by 1. We therefore re-name the d_r's as d_{r+1}'s. The re-graded algebraic Novikov spectral sequence will therefore start with the E_2-page, instead of the E_1-page.

In summary, we have the following re-graded algebraic Novikov spectral sequence.

Corollary 6.2.6. *There is a tri-graded spectral sequence with*

$$E_2^{s,a,t} = Ext_{P_*}^{s,t}(\mathbb{Z}/p, I^{a-s}/I^{a-s+1}),$$

and

$$d_r : E_r^{s,a,t} \longrightarrow E_r^{s+1,a+r,t},$$

converging to

$$Ext_{BP_*BP}^{s,t}(BP_*, BP_*).$$

Since we an extension of Hopf algebras

$$P_* \longrightarrow A_* \longrightarrow E_*,$$

there is a Cartan-Eilenberg spectral sequence with

$$E_2^{s,i,t} = Ext_{P_*}^{s}(\mathbb{Z}/p, Ext_{E_*}^{i,t}(\mathbb{Z}/p, \mathbb{Z}/p)),$$

and

$$d_r : E_r^{s,i,t} \longrightarrow E_r^{s+r,i-r+1,t},$$

converging to

$$Ext_A^{s+i,t}(\mathbb{Z}/p, \mathbb{Z}/p),$$

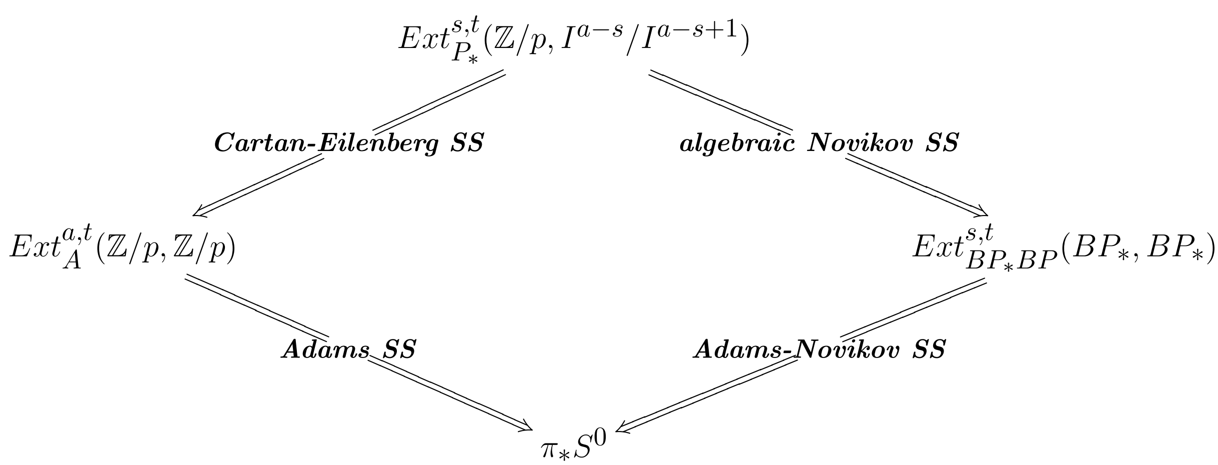
where A is the mod p Steenrod algebra. Since E_* is an exterior algebra, we can identify

$$Ext_{E_*}(\mathbb{Z}/p, \mathbb{Z}/p))$$

as a polynomial algebra which is isomorphic to BP_*, and the i-filtration corresponds the I-filtration of BP_*. Therefore, we can identify the E_2-page of the Cartan-Eilenberg spectral sequence as the E_2-page of the re-graded algebraic Novikov spectral sequence with the re-grading

$$a = s + i.$$

In fact, the 4 spectral sequences we discussed fit into the Miller's square (54):



Remark 6.2.7. This square is not "commutative": the $*$ is $t - a$ if converging from the Adams spectral sequence, and $t - s$ if converging from the Adams-Novikov spectral sequence. In general, an element in the stable homotopy groups of sphere does not necessarily have the same Adams and Adams-Novikov filtration, therefore not the same t-filtration. For example, the multiplication by p map has Adams filtration 1 and Adams-Novikov filtration 0.

Next, we discuss how to convert the re-graded algebraic Novikov spectral sequence to the motivic Adams spectral sequence of $C\tau$, using the equivalence of triangulated categories proved in Section 2. We show that the re-regraded algebraic Novikov tower, coming from the a-filtration in Definition 6.2.4, corresponds to a motivic Adams tower in the sense of Dugger-Isaksen (23).

In fact, let C_0^* be the cobar resolution of BP_* over BP_*BP, with

$$C_0^s = BP_*BP \otimes_{BP_*} \overline{BP_*BP}^{\otimes s} \otimes_{BP_*} BP_*,$$

where $\overline{BP_*BP}$ is the kernel of the counit map $BP_*BP \to BP_*$.

Let $\{C_m^*\}_{m \geq 1}$ be the subcomplexes of C_0^* with respect to the a-filtration. Namely, at cohomological degree s, we have

$$C_m^s = BP_*BP \otimes_{BP_*} \overline{BP_*BP}^{\otimes s} \otimes_{BP_*} I^{m-s}$$

It is understood that $I^r = BP_*$ for $r \leq 0$. Therefore, for $s \geq m$, we have $C_m^s = C_0^s$.

Let Q_m^* be the quotient complex of the inclusion map

$$C_{m+1}^* \xrightarrow{i_m} C_m^*,$$

for $m \geq 0$, and therefore has the form

$$Q_m^s = BP_*BP \otimes_{BP_*} \overline{BP_*BP}^{\otimes s} \otimes_{BP_*} I^{m-s}/I^{m-s+1}.$$

Then we have a tower of chain complexes, or in other words, a tower in the derived category of BP_*BP-comodules:

$$\begin{array}{ccccccccc} BP_* & = & C_0 & \xleftarrow{i_0} & C_1 & \xleftarrow{i_1} & C_2 & \xleftarrow{i_2} & \cdots \\ & & \downarrow{\scriptstyle q_0} & & \downarrow{\scriptstyle q_1} & & \downarrow{\scriptstyle q_2} & & \\ & & Q_0 & & Q_1 & & Q_2 & & \end{array}$$

Note that each C_m^*, and therefore Q_m^*, has bounded cohomological degree. It is standard homological algebra to show that this tower in $D^b(BP_*BP - comod)$ gives rise to the re-graded algebraic Novikov spectral sequence that computes $Ext_{BP_*BP}(BP_*, BP_*)$.

Now we map this tower in $D^b(BP_*BP - comod)$ to get a tower in $Ho(C\tau - mod)$:

$$\begin{array}{ccccccccc} C\tau & = & X_0 & \xleftarrow{g_0} & X_1 & \xleftarrow{g_1} & X_2 & \xleftarrow{g_2} & \cdots \\ & & \downarrow{\scriptstyle f_0} & & \downarrow{\scriptstyle f_1} & & \downarrow{\scriptstyle f_2} & & \\ & & K_0 & & K_1 & & K_2 & & \end{array}$$

and we show that it is indeed a motivic Adams tower for $C\tau$ in the sense of Dugger-Isaksen (23).

To fix notation, let MH_p be the motivic spectrum that represents the mod p motivic cohomology theory. It is shown by Hu-Kriz-Ormsby (34) and Hoyois (33) that the motivic Eilenberg-Mac Lanc spectrum MH_p is cellular.

There are two things that need to be checked:

1. Each motivic spectrum K_m is $C\tau$ smashed a wedge of suspensions of the motivic Eilenberg-Mac Lane spectrum.

2. Each map $f_m : X_m \to K_m$ induces a surjection on mod p motivic cohomology. Or equivalently, each map $g_m : X_{m+1} \to X_m$ induces zero map on mod p motivic cohomology.

We will prove (1) and (2) by the following lemmas.

Lemma 6.2.8. *The BP_*BP comodule $BP_*BP \otimes_{BP_*} \mathbb{Z}/p$ corresponds to $MH_p \wedge C\tau$.*

Proof. Since $MH_{p_{*,*}} = \mathbb{Z}/p[\tau] = BPGL_{*,*}/I$, we have MH_p obtained from $BPGL$ by killing the v_i's inductively. Therefore,

$$BPGL_{*,*}MH_p = BPGL_{*,*}BPGL/I = \mathbb{Z}/p[\tau][t_i|i \geq 1].$$

Since all the t_i's are in Chow degree 0, the $BPGL$-homology of $MH_p \wedge C\tau$ is concentrated in Chow degree 0.

$$\begin{aligned} BPGL_{*,*}(MH_p \wedge C\tau) &= \pi_{*,*}(BPGL \wedge MH_p \wedge C\tau) \\ &= \pi_{*,*}(BPGL \wedge MH_p)/\tau \\ &= \pi_* BP \wedge H\mathbb{Z}/p \quad \text{(forgetting the motivic weight)} \\ &= BP_*(H\mathbb{Z}/p) \\ &= BP_*BP/I \\ &= BP_*BP \otimes_{BP_*} \mathbb{Z}/p \end{aligned}$$

The third equation follows from the fact that the $BPGL$-homology of MH_p is τ-free. □

Lemma 6.2.9. *Let N be a locally finitely generated BP_*-module that is concentrated in even degrees, and is annihilated by I. Then any comodule of the form $BP_*BP \otimes_{BP_*} N$ corresponds to $C\tau$ smashed a wedge of suspensions of MH_p.*

Proof. Since $BP_*/I = \mathbb{Z}/p$, any finitely generated BP_*-module which is annihilated by I is a direct sum of copies of $\mathbb{Z}/p$ in different degrees. Therefore, N is isomorphic to a direct sum of copies of $\mathbb{Z}/p$ in even degrees.

By Lemma 3.8, the comodule $BP_*BP \otimes_{BP_*} \mathbb{Z}/p$ corresponds to $MH_p \wedge C\tau$. Therefore, we have the comodule $BP_*BP \otimes_{BP_*} \Sigma^{2n}\mathbb{Z}/p$ corresponds to

$$MH_p \wedge C\tau \wedge S^{2n,n}$$

Therefore, the comodule $BP_*BP \otimes_{BP_*} N$ corresponds to $C\tau$ smashed a wedge of suspensions of MH_p. ∎

Now we prove (1).

Proposition 6.2.10. *The differentials in the subquotients Q_m^* are all zero, and therefore Q_m^* splits as a direct sum of subcomplexes which are concentrated in one cohomological degree. Furthermore, each split subcomplex of Q_m^* corresponds to $C\tau$ smashed a wedge of suspensions of MH_p, and hence also Q_m^*. Therefore, each K_i is $C\tau$ smashed a wedge of suspensions of MH_p.*

Proof. In Q_m^*, the differentials $Q_m^s \to Q_m^{s+1}$ has the form

$$BP_*BP \otimes_{BP_*} \overline{BP_*BP}^{\otimes s} \otimes_{BP_*} I^{m-s}/I^{m-s+1}$$

$$\longrightarrow BP_*BP \otimes_{BP_*} \overline{BP_*BP}^{\otimes s} \otimes_{BP_*} I^{m-s-1}/I^{m-s},$$

and therefore are all zero due to filtration reasons. Since each I^{m-s}/I^{m-s+1} splits as a direct sum of even suspensions of $\mathbb{Z}/p$, and each $\overline{BP_*BP}^{\otimes s}$ splits as a direct sum of even suspensions of BP_*, the complex splits into a direct sum of complexes of the form

$$\cdots \longrightarrow 0 \longrightarrow \Sigma^{2l} BP_*BP \otimes_{BP_*} \mathbb{Z}/p \longrightarrow 0 \longrightarrow \cdots$$

in cohomological degree k, and therefore corresponds to

$$MH_p \wedge C\tau \wedge S^{2l-k,l-k}.$$

∎

To prove (2), we first prove the following lemma.

Lemma 6.2.11. *Let X be a $C\tau$-module whose $BPGL$-homology has bounded Chow degree. Let $C_{BP_*BP}(X)$ be the corresponding complex of BP_*BP-comodules. Let $C_{BP_*}(X)$ be its* *underlying complex as BP_*-modules. Then the mod p motivic cohomology of X can be computed as*

$$\mathbf{R}Hom_{BP_*}(C_{BP_*}(X), \mathbb{Z}/p).$$

Proof. Since the mod p motivic cohomology is represented as the functor $[-, MH_p]$, we have

$$\begin{aligned}
[X, MH_p] &= [X, MH_p \wedge C\tau]_{C\tau} \\
&= \mathbf{R}Hom_{BP_*BP}(C_{BP_*BP}(X), BP_*BP \otimes_{BP_*} \mathbb{Z}/p) \\
&= \mathbf{R}Hom_{BP_*}(C_{BP_*}(X), \mathbb{Z}/p)
\end{aligned}$$

The last equation comes from the adjunction of the derived functor of $BP_*BP \otimes_{BP_*} -$ and the forgetful functor. □

We also need the following lemma, whose proof is technical, and is postponed to the appendix.

Lemma 6.2.12. *The following homomorphism induced by the inclusion $I^{m+1} \to I^m$ is zero:*

$$Ext_{BP_*}(I^{m+1}, \mathbb{Z}/p) \longrightarrow Ext_{BP_*}(I^m, \mathbb{Z}/p).$$

Now we prove (2).

Proposition 6.2.13. *Each map $g_m : X_{m+1} \to X_m$ induces zero map on mod p motivic cohomology.*

Proof. Consider the cobar resolution C_0^*. Its underlying complex over BP_* splits as a direct sum of complexes over BP_*: $C_0^* = \bigoplus_j D_{0,j}^*$, where

$$D_{0,0}^* : \qquad BP_* \longrightarrow 0 \longrightarrow 0 \longrightarrow 0 \longrightarrow \cdots$$

$$D_{0,1}^* : \qquad \overline{BP_*BP} \otimes_{BP_*} BP_* \xrightarrow{id} \overline{BP_*BP} \otimes_{BP_*} BP_* \longrightarrow 0 \longrightarrow 0 \longrightarrow \cdots$$

$$D^*_{0,2}: \qquad 0 \longrightarrow \overline{BP_*BP}^{\otimes 2} \otimes_{BP_*} BP_* \xrightarrow{id} \overline{BP_*BP}^{\otimes 2} \otimes_{BP_*} BP_* \longrightarrow 0 \longrightarrow \cdots$$

and so forth. In particular, each $D^s_{0,j}$ is a free BP_*-module, and $H^*(D^*_{0,j}) = 0$ for $j \geq 1$. The a-filtration gives a splitting of $C^*_m = \bigoplus_j D^*_{m,j}$, where

$$D^s_{m,j} = D^s_{0,j} \otimes_{BP_*} I^{m-s}.$$

For example, the complex $D^*_{4,2}$ is

$$0 \longrightarrow \overline{BP_*BP}^{\otimes 2} \otimes_{BP_*} I^3 \xrightarrow{id} \overline{BP_*BP}^{\otimes 2} \otimes_{BP_*} I^2 \longrightarrow 0 \longrightarrow \cdots$$

Therefore, we have

$$H^*(D^*_{m,0}) = I^m$$

concentrated in cohomological degree 0, and

$$H^*(D^*_{m,j}) = \overline{BP_*BP}^{\otimes j} \otimes_{BP_*} I^{m-2}/I^{m-3}$$

concentrated in cohomological degree j for $j \geq 1$.

Now we consider the maps

$$\mathbf{R}Hom_{BP_*}(D^*_{m,j}, \mathbb{Z}/p) \longrightarrow \mathbf{R}Hom_{BP_*}(D^*_{m-1,j}, \mathbb{Z}/p)$$

induced by the inclusions $D^*_{m,j} \to D^*_{m-1,j}$. For $j \geq 1$, these maps are zero, since they can be identified as

$$\mathbf{R}Hom_{BP_*}(H^*(D^*_{m,j}), \mathbb{Z}/p) \longrightarrow \mathbf{R}Hom_{BP_*}(H^*(D^*_{m-1,j}), \mathbb{Z}/p),$$

and the maps $H^*(D^*_{m,j}) \to H^*(D^*_{m-1,j})$ are zero.

For $j = 0$, the maps can be rewritten as

$$Ext_{BP_*}(I^m, \mathbb{Z}/p) \longrightarrow Ext_{BP_*}(I^{m+1}, \mathbb{Z}/p).$$

By Lemma 6.2.12, they are also zero. Therefore, the maps

$$\mathbf{R}Hom_{BP_*}(C_m^*, \mathbb{Z}/p) \longrightarrow \mathbf{R}Hom_{BP_*}(C_{m-1}^*, \mathbb{Z}/p)$$

are all zero, since they are zero on each summand of the direct sum.

Since all C_m^* are concentrated in bounded cohomological degrees, all X_m have $BPGL$-homology in bounded Chow degrees. Therefore, by Lemma 6.2.11, each map $g_m : X_{m+1} \to X_m$ induces zero map on mod p motivic cohomology. □

Combining Proposition 6.2.10 and 6.2.13, we have shown the resulting tower is a motivic Adams tower for $C\tau$, and hence proved Theorem 6.2.1.

6.3 Proof of Lemma 6.2.12

We prove Lemma 6.2.12 in this section. The following Noetherian version is well known to algebraists (see (73) for example). In fact, let

$$R = \mathbb{Z}_p[x_1, x_2, \cdots]$$

$$m = (p, x_1, x_2, \cdots)$$

$$R_t = \mathbb{Z}_p[x_1, x_2, \cdots, x_t]$$

$$m_t = (p, x_1, x_2, \cdots, x_t)$$

Then we have both the following maps

$$Tor^{R_t}(m_t^{n+1}, \mathbb{F}_p) \longrightarrow Tor^{R_t}(m_t^n, \mathbb{F}_p)$$

$$Ext_{R_t}(m_t^{n+1}, \mathbb{F}_p) \longrightarrow Ext_{R_t}(m_t^n, \mathbb{F}_p)$$

are zero for all $n \geq 0$ and $t \geq 1$, which are induced by the inclusion map $m_t^{n+1} \to m_t^n$. Note that the second statement follows from the first one, since

$$Ext_{R'}(M, k) = Hom_{R'}(Tor^{R'}(M, k), k)$$

is true for any polynomial k-algebra R' and any module M over R'.

Therefore, to prove Lemma 6.2.12, we only need to prove the dual

$$Tor^{R}(m^{n+1}, \mathbb{F}_p) \longrightarrow Tor^{R}(m^n, \mathbb{F}_p)$$

is zero.

Take a free resolution $F_\bullet$ of $\mathbb{F}_p$ over R. Since R is free over R_t, we can also regard $F_\bullet$ as a free resolution of $\mathbb{F}_p$ over R_t. Since

$$R = \varinjlim R_t, \; m^n = \varinjlim m_t^n,$$

we have

$$\varinjlim F_\bullet \otimes_{R_t} m_t^n = F_\bullet \otimes_R m^n.$$

Therefore, the following diagram commutes, and hence the proof of Lemma 6.2.12.

$$\begin{array}{ccccccc}
Tor^{R_t}(m_t^n, \mathbb{F}_p) & = & H^*(F_\bullet \otimes_{R_t} m_t^n) & \longrightarrow & H^*(F_\bullet \otimes_R m^n) & = & Tor^{R}(m^n, \mathbb{F}_p) \\
 & & {\scriptstyle 0}\uparrow & & \uparrow & & \\
Tor^{R_t}(m_t^{n+1}, \mathbb{F}_p) & = & H^*(F_\bullet \otimes_{R_t} m_t^{n+1}) & \longrightarrow & H^*(F_\bullet \otimes_R m^{n+1}) & = & Tor^{R}(m^{n+1}, \mathbb{F}_p)
\end{array}$$

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APPENDIX A

THE ALGEBRAIC ATIYAH-HIRZEBRUCH SPECTRAL SEQUENCE OF THE REAL PROJECTIVE SPECTRA

We use the Curtis algorithm to compute the algebraic Atiyah-Hirzebruch spectral sequence for the real projective spectra. We take the lambda algebra for the resolution of $\mathbb{Z}/2$ and use the usual Curtis table for the sphere spectrum as input. We have carried out the computation through stems with $t < 72$. As a usual convention to out put the Curtis table, we abbreviate the sequence 2 4 1 1 by $*$; when there are multiple 2's consecutively, we replace them by the same amount of dots.

Together with the algebraic Kahn-Priddy theorem (46) and known information of $Ext(\mathbb{Z}/2)$, this gives the Adams E_2-page of P_1^∞ up to $t - s \leq 61$.

We also compute the transfer map. Recall that the fiber of the transfer map has one more cell than P_1^∞ in dimension -1, and all the Sq^i acts nontrivially on the class in dimension -1. We will use Theorem 5.7.1 to identify the table for transfer with a portion of the Curtis table for this complex.

For notation, in the Lambda algebra, we will abbreviate an element $\lambda_{i_1} \dots \lambda_{i_n}$ by $i_1 \dots i_n$. We will abbreviate an element $e_k \otimes \lambda_{i_1} \dots \lambda_{i_n}$ by $(k)i_1 \dots i_n$ in the Lambda complex of P_1^∞. The symbol o means zero. The Curtis table is separated into lists labeled by $(t-s, t)$ on the top, in which the untagged items give a basis for $Ext^{s-1,t-1}(H^*(P_1^\infty))$.

The table for the transfer is the output of the algorithm: (We put the table for the transfer map first since it is shorter)

In this table we only list the nontrivial items. Others either map to something with the same name, or to the only choice comparable with the algebraic Kahn-Priddy theorem. For example, (1) maps to 1, i.e. the inclusion of the bottom cell maps to η. As another example, (5) 3 maps to 5 3, which can be proved independently by the Massey product

$$\langle h_2, h_1, h_2 \rangle = h_1 h_3.$$

We do not include such items in the transfer table.

A.1 The transfer table

Table A.1: The table for the transfer map

(1) 7	5 3
(1) 5 3	3 3 3
(1) 6 2 3 3	2 4 3 3 3
(1) 15	13 3
(1) 13 3	11 3 3
(3) 15	11 7
(2) 13 3	10 5 3
(1) 11 3 3	9 3 3 3
(4) 6 5 3	5 7 3 3
(1) 8 3 3 3	4 5 3 3 3
(3) 8 3 3 3	4 7 3 3 3
(3) 11 7	7 7 7
(1) 6 6 5 3	3 5 7 3 3
(5) 11 3 3	3 5 7 7
(2) 7 7 7	4 5 7 7
(1) 6 2 3 4 4 1 1 1	2 4 1 1 2 4 3 3 3
(1) 4 5 7 7	2 3 5 7 7
(3) 13 1 2 4 1 1 1	4 2 2 4 5 3 3 3
(1) 8 1 1 2 4 3 3 3	2 2 2 2 4 5 3 3 3
(5) 13 1 2 4 1 1 1	6 2 2 4 5 3 3 3
(3) 8 1 1 2 4 3 3 3	4 2 2 2 4 5 3 3 3

(1) 6 2 2 4 5 3 3 3	2 2 2 2 3 5 7 3 3
(8) 3 5 7 7	12 9 3 3 3
(1) 15 15	13 11 7
(12) 5 7 7	o
(1) 6 2 3 4 4 1 1 2 4 1 1 1	2 4 1 1 2 4 1 1 2 4 3 3 3
(1) 31	29 3
(2) 12 9 3 3 3	9 3 6 6 5 3
(1) 5 6 2 4 5 3 3 3	2 2 2 3 3 6 6 5 3
(1) 29 3	27 3 3
(1) 13 5 7 7	11 3 5 7 7
(1) 9 3 6 6 5 3	5 5 3 6 6 5 3
(3) 12 4 5 3 3 3	5 5 3 6 6 5 3
(3) 31	27 7
(2) 29 3	26 5 3
(1) 27 3 3	25 3 3 3
(3) 9 3 5 7 7	5 7 3 5 7 7
(4) 12 9 3 3 3	5 7 3 5 7 7
(1) 8 1 1 2 4 1 1 2 4 3 3 3	2 2 2 2 2 2 2 2 4 5 3 3 3
(3) 13 5 7 7	5 9 7 7 7
(5) 12 4 5 3 3 3	6 5 2 3 5 7 7
(1) 5 6 2 3 5 7 3 3	2 4 3 3 3 6 6 5 3
(3) 8 1 1 2 4 1 1 2 4 3 3 3	4 2 2 2 2 2 2 2 4 5 3 3 3
(3) 27 7	23 7 7
(1) 5 9 3 5 7 7	3 5 7 3 5 7 7
(4) 5 5 3 6 6 5 3	4 7 3 3 6 6 5 3
(7) 31	23 15

(6) 29 3	22 13 3
(5) 27 3 3	21 11 3 3
(4) 25 3 3 3	20 9 3 3 3
(1) 14 4 5 7 7	3 5 9 7 7 7
(1) 23 15	21 11 7
(5) 27 7	21 11 7
(2) 23 7 7	20 5 7 7
(1) 17 7 7 7	7 13 5 7 7
(1) 8 12 9 3 3 3	3 5 9 3 5 7 7
(1) 21 11 7	19 7 7 7
(3) 23 7 7	19 7 7 7
(9) 13 11 7	11 15 7 7
(1) 20 5 7 7	18 3 5 7 7
(2) 17 7 7 7	7 14 5 7 7
(1) 7 13 5 7 7	5 5 9 7 7 7
(2) 20 9 3 3 3	17 3 6 6 5 3
(1) 11 15 7 7	9 11 7 7 7
(3) 17 7 7 7	9 11 7 7 7
(1) 17 3 6 6 5 3	11 12 4 5 3 3 3
(4) 20 9 3 3 3	12 12 9 3 3 3
(2) 17 3 6 6 5 3	10 9 3 6 6 5 3
(1) 11 12 4 5 3 3 3	9 5 5 3 6 6 5 3
(2) 5 6 5 2 3 5 7 7	4 5 5 5 3 6 6 5 3
(7) 23 15	15 15 15
(6) 21 11 7	14 13 11 7
(1) 13 13 11 7	9 15 7 7 7

(6) 20 5 7 7	13 13 5 7 7 +9 15 7 7 7
(7) 17 7 7 7	9 15 7 7 7
(5) 18 3 5 7 7	12 11 3 5 7 7
(3) 12 12 9 3 3 3	7 8 12 9 3 3 3
(1) 13 13 5 7 7	11 5 9 7 7 7
(8) 20 9 3 3 3	11 5 9 7 7 7
(2) 7 14 4 5 7 7	8 3 5 9 7 7 7
(6) 17 3 6 6 5 3	8 3 5 9 7 7 7
(3) 13 13 11 7	7 11 15 7 7
(2) 9 15 7 7 7	6 9 11 7 7 7
(1) 11 5 9 7 7 7	9 3 5 9 7 7 7
(1) 8 3 5 9 7 7 7	4 5 3 5 9 7 7 7
(2) 7 8 12 9 3 3 3	8 3 5 9 3 5 7 7
(11) 23 7 7	7 11 15 15
(8) 19 7 7 7	10 17 7 7 7
(3) 13 13 5 7 7	9 7 13 5 7 7
(2) 11 5 9 7 7 7	8 5 5 9 7 7 7
(4) 7 14 4 5 7 7	5 10 11 3 5 7 7
(2) 8 3 5 9 7 7 7	4 6 3 5 9 7 7 7
(1) 4 5 3 5 9 7 7 7	2 3 5 3 5 9 7 7 7
(1) 8 3 5 9 3 5 7 7	4 5 3 5 9 3 5 7 7
(4) 14 13 11 7	9 11 15 7 7
(1) 10 17 7 7 7	8 9 11 7 7 7
(5) 15 15 15	9 11 15 15
(6) 14 13 11 7	7 13 13 11 7
(3) 10 17 7 7 7	6 9 15 7 7 7

(2) 8 9 11 7 7 7	4 6 9 11 7 7 7
(4) 9 3 5 9 7 7 7	3 5 10 11 3 5 7 7
(1) 7 13 13 11 7	5 7 11 15 7 7
(3) 9 11 15 7 7	5 7 11 15 7 7
(3) 9 11 15 15	5 7 11 15 15
(1) 8 4 5 3 5 9 3 5 7 7	4 5 4 5 3 5 9 3 5 7 7
(3) 16 2 3 5 5 3 6 6 5 3	4 5 4 5 3 5 9 3 5 7 7
(1) 27 12 4 5 3 3 3	25 5 5 3 6 6 5 3
(20) 5 5 9 7 7 7	o
(3) 27 12 4 5 3 3 3	5 1 2 4 7 11 15 15
(2) 25 5 5 3 6 6 5 3	3 6 4 6 9 11 7 7 7
(1) 6 2 3 5 10 11 3 5 7 7	2 2 4 5 9 3 5 9 7 7 7
(4) 5 5 9 3 5 7 3 5 7 7	o
(7) 18 2 4 3 3 3 6 6 5 3	8 4 2 3 5 3 5 9 7 7 7 +2 2 4 5 9 3 5 9 7 7 7
(8) 5 7 11 15 15	o
(9) 4 7 11 15 15	28 11 3 5 7 7
(3) 28 12 9 3 3 3	23 8 12 9 3 3 3
(1) 6 2 4 7 11 15 15	2 4 3 4 7 11 15 15

The Curtis table for the Adams E_2-page of P_1^∞ in the range of $t < 72$ is the following:

$* = 2\ 4\ 1\ 1$

$. = 2$

A.2 The Curtis table

Table A.2: The Curtis table

(1, 1)

(1)

(2, 2)

(1) 1

(3, 1)

(3)

(3, 2)

(2) 1

(3, 3)

(1) 1 1

(4, 2)

(1) 3

(3) 1 $\leftarrow$ (5)

(4, 3)

(1) 2 1 $\leftarrow$ (2) 3

(2) 1 1 $\leftarrow$ (4) 1

(4, 4)

(1) 1 1 1 $\leftarrow$ (2) 2 1

(5, 3)

(3) 1 1 $\leftarrow$ (5) 1

(5, 4)

(2) 1 1 1 $\leftarrow$ (4) 1 1

(6, 2)

(3) 3

(6, 3)

(3) 2 1 $\leftarrow$ (4) 3

(6, 4)

(3) 1 1 1 $\leftarrow$ (4) 2 1

(7, 1)

(7)

(7, 2)

(6) 1

(7, 3)

(1) 3 3

(5) 1 1

(7, 4)

(4) 1 1 1

(8, 2)

(1) 7

(5) 3

(7) 1 $\leftarrow$ (9)

(8, 3)

(1) 6 1 $\leftarrow$ (2) 7

(2) 3 3

(5) 2 1 $\leftarrow$ (6) 3

(6) 1 1 $\leftarrow$ (8) 1

$(8, 4)$

(1) 5 1 1 ← (2) 6 1

(5) 1 1 1 ← (6) 2 1

$(8, 5)$

(1) 4 1 1 1 ← (2) 5 1 1

$(9, 3)$

(1) 5 3

(3) 3 3

(7) 1 1 ← (9) 1

$(9, 4)$

(1) 2 3 3

(6) 1 1 1 ← (8) 1 1

$(9, 5)$

(2) 4 1 1 1

$(10, 2)$

(3) 7

(7) 3 ← (11)

$(10, 3)$

(2) 5 3

(3) 6 1 ← (4) 7

(4) 3 3 ← (10) 1

(7) 2 1 ← (8) 3

$(10, 4)$

(1) 3 3 3

(2) 2 3 3 ← (9) 1 1

(3) 5 1 1 ← (4) 6 1

(7) 1 1 1 ← (8) 2 1

$(10, 5)$

(1) 1 2 3 3 ← (8) 1 1 1

(3) 4 1 1 1 ← (4) 5 1 1

$(10, 6)$

(1) * 1

$(11, 3)$

(3) 5 3 ← (5) 7

(5) 3 3 ← (9) 3

$(11, 4)$

(2) 3 3 3 ← (4) 5 3

(3) 2 3 3 ← (6) 3 3

$(11, 5)$

(2) 1 2 3 3 ← (4) 2 3 3

(4) 4 1 1 1

$(11, 6)$

(2) * 1

$(11, 7)$

(1) 1 * 1

(12, 2)

(11) 1 ← (13)

(12, 3)

(5) 6 1 ← (6) 7

(9) 2 1 ← (10) 3

(10) 1 1 ← (12) 1

(12, 4)

(3) 3 3 3 ← (5) 5 3

(5) 5 1 1 ← (6) 6 1

(9) 1 1 1 ← (10) 2 1

(12, 5)

(3) 1 2 3 3 ← (5) 2 3 3

(5) 4 1 1 1 ← (6) 5 1 1

(12, 6)

(1) 4 4 1 1 1 ← (4) 1 2 3 3

(3) * 1 ← (6) 4 1 1 1

(12, 7)

(1) 2 * 1 ← (2) 4 4 1 1 1

(2) 1 * 1 ← (4) * 1

(12, 8)

(1) 1 1 * 1 ← (2) 2 * 1

(13, 3)

(7) 3 3 ← (11) 3

(11) 1 1 ← (13) 1

(13, 4)

(4) 3 3 3 ← (8) 3 3

(10) 1 1 1 ← (12) 1 1

(13, 7)

(3) 1 * 1 ← (5) * 1

(13, 8)

(2) 1 1 * 1 ← (4) 1 * 1

(14, 2)

(7) 7

(14, 3)

(6) 5 3

(7) 6 1 ← (8) 7

(11) 2 1 ← (12) 3

(14, 4)

(5) 3 3 3 ← (9) 3 3

(6) 2 3 3

(7) 5 1 1 ← (8) 6 1

(11) 1 1 1 ← (12) 2 1

(14, 5)

(5) 1 2 3 3

(7) 4 1 1 1 ← (8) 5 1 1

(14, 6)

(3) 4 4 1 1 1

(14, 7)

(3) 2 * 1 ← (4) 4 4 1 1 1

(14, 8)

(3) 1 1 * 1 ← (4) 2 * 1

(15, 1)

(15)

(15, 2)

(14) 1

(15, 3)

(1) 7 7

(7) 5 3 ← (9) 7

(13) 1 1

(15, 4)

(1) 6 5 3 ← (2) 7 7

(6) 3 3 3 ← (8) 5 3

(7) 2 3 3 ← (10) 3 3

(12) 1 1 1

(15, 5)

(1) 6 2 3 3

(6) 1 2 3 3 ← (8) 2 3 3

(8) 4 1 1 1

(15, 6)

(1) 5 1 2 3 3 ← (2) 6 2 3 3

(6) * 1

(15, 7)

(1) 3 4 4 1 1 1 ← (2) 5 1 2 3 3

(5) 1 * 1

(15, 8)

(4) 1 1 * 1

(16, 2)

(1) 15

(13) 3

(15) 1 ← (17)

(16, 3)

(1) 14 1 ← (2) 15

(9) 6 1 ← (10) 7

(13) 2 1 ← (14) 3

(14) 1 1 ← (16) 1

(16, 4)

(1) 13 1 1 ← (2) 14 1

(2) 6 5 3

(7) 3 3 3 ← (9) 5 3

(9) 5 1 1 ← (10) 6 1

(13) 1 1 1 ← (14) 2 1

(16, 5)

(1) 12 1 1 1 ← (2) 13 1 1

(7) 1 2 3 3 ← (9) 2 3 3

(9) 4 1 1 1 ← (10) 5 1 1

$(16,6)$

(1) 2 4 3 3 3

(1) 8 4 1 1 1 ← (2) 12 1 1 1

(5) 4 4 1 1 1 ← (8) 1 2 3 3

(7) * 1 ← (10) 4 1 1 1

$(16,7)$

(1) 6 * 1 ← (2) 8 4 1 1 1

(2) 3 4 4 1 1 1

(5) 2 * 1 ← (6) 4 4 1 1 1

(6) 1 * 1 ← (8) * 1

$(16,8)$

(1) 5 1 * 1 ← (2) 6 * 1

(5) 1 1 * 1 ← (6) 2 * 1

$(16,9)$

(1) 4 1 1 * 1 ← (2) 5 1 * 1

$(17,3)$

(1) 13 3

(3) 7 7

(11) 3 3

(15) 1 1 ← (17) 1

$(17,4)$

(3) 6 5 3 ← (4) 7 7

(8) 3 3 3

(14) 1 1 1 ← (16) 1 1

$(17,5)$

(3) 6 2 3 3

$(17,6)$

(2) 2 4 3 3 3

(3) 5 1 2 3 3 ← (4) 6 2 3 3

$(17,7)$

(1) 1 2 4 3 3 3

(3) 3 4 4 1 1 1 ← (4) 5 1 2 3 3

(7) 1 * 1 ← (9) * 1

$(17,8)$

(1) 2 3 4 4 1 1 1

(6) 1 1 * 1 ← (8) 1 * 1

$(17,9)$

(2) 4 1 1 * 1

$(18,2)$

(3) 15

(11) 7

(15) 3 ← (19)

(18, 3)

(2) 13 3

(3) 14 1 ← (4) 15

(10) 5 3

(11) 6 1 ← (12) 7

(12) 3 3 ← (18) 1

(15) 2 1 ← (16) 3

(18, 4)

(1) 11 3 3

(3) 13 1 1 ← (4) 14 1

(4) 6 5 3

(9) 3 3 3

(10) 2 3 3 ← (17) 1 1

(11) 5 1 1 ← (12) 6 1

(15) 1 1 1 ← (16) 2 1

(18, 5)

(1) 8 3 3 3

(3) 12 1 1 1 ← (4) 13 1 1

(9) 1 2 3 3 ← (16) 1 1 1

(11) 4 1 1 1 ← (12) 5 1 1

(18, 6)

(1) 3 6 2 3 3 ← (2) 8 3 3 3

(3) 2 4 3 3 3 ← (5) 6 2 3 3

(3) 8 4 1 1 1 ← (4) 12 1 1 1

(7) 4 4 1 1 1 ← (12) 4 1 1 1

(18, 7)

(1) ..4 3 3 3 ← (2) 3 6 2 3 3

(2) 1 2 4 3 3 3 ← (4) 2 4 3 3 3

(3) 6 * 1 ← (4) 8 4 1 1 1

(4) 3 4 4 1 1 1 ← (10) * 1

(7) 2 * 1 ← (8) 4 4 1 1 1

(18, 8)

(1) 1 1 2 4 3 3 3 ← (2) ..4 3 3 3

(2) 2 3 4 4 1 1 1 ← (9) 1 * 1

(3) 5 1 * 1 ← (4) 6 * 1

(7) 1 1 * 1 ← (8) 2 * 1

(18, 9)

(1) 1 2 3 4 4 1 1 1 ← (8) 1 1 * 1

(3) 4 1 1 * 1 ← (4) 5 1 * 1

(18, 10)

(1) * * 1

(19, 3)

(1) 11 7

(3) 13 3 ← (5) 15

(5) 7 7

(11) 5 3 ← (13) 7

(13) 3 3 ← (17) 3

$(19, 4)$

(1) 10 5 3 ← (2) 11 7

(2) 11 3 3 ← (4) 13 3

(5) 6 5 3 ← (6) 7 7

(10) 3 3 3 ← (12) 5 3

(11) 2 3 3 ← (14) 3 3

$(19, 5)$

(1) 5 7 3 3

(1) 9 3 3 3 ← (2) 10 5 3

(10) 1 2 3 3 ← (12) 2 3 3

$(19, 6)$

(1) 4 5 3 3 3 ← (2) 5 7 3 3

(5) 5 1 2 3 3 ← (6) 6 2 3 3

$(19, 7)$

(3) 1 2 4 3 3 3 ← (5) 2 4 3 3 3

(5) 3 4 4 1 1 1 ← (6) 5 1 2 3 3

$(19, 8)$

(2) 1 1 2 4 3 3 3 ← (4) 1 2 4 3 3 3

(3) 2 3 4 4 1 1 1 ← (6) 3 4 4 1 1 1

$(19, 9)$

(2) 1 2 3 4 4 1 1 1 ← (4) 2 3 4 4 1 1 1

(4) 4 1 1 * 1

$(19, 10)$

(2) * * 1

$(19, 11)$

(1) 1 * * 1

$(20, 2)$

(19) 1 ← (21)

$(20, 3)$

(5) 14 1 ← (6) 15

(13) 6 1 ← (14) 7

(17) 2 1 ← (18) 3

(18) 1 1 ← (20) 1

$(20, 4)$

(1) 5 7 7

(3) 11 3 3 ← (5) 13 3

(5) 13 1 1 ← (6) 14 1

(6) 6 5 3

(11) 3 3 3 ← (13) 5 3

(13) 5 1 1 ← (14) 6 1

(17) 1 1 1 ← (18) 2 1

$(20, 5)$

(2) 9 3 3 3 ← (4) 11 3 3

(3) 8 3 3 3

(5) 12 1 1 1 ← (6) 13 1 1

(11) 1 2 3 3 ← (13) 2 3 3

(13) 4 1 1 1 ← (14) 5 1 1

(20, 6)

(2) 4 5 3 3 3

(3) 3 6 2 3 3 ← (4) 8 3 3 3

(5) 8 4 1 1 1 ← (6) 12 1 1 1

(9) 4 4 1 1 1 ← (12) 1 2 3 3

(11) * 1 ← (14) 4 1 1 1

(20, 7)

(3) ..4 3 3 3 ← (4) 3 6 2 3 3

(5) 6 * 1 ← (6) 8 4 1 1 1

(9) 2 * 1 ← (10) 4 4 1 1 1

(10) 1 * 1 ← (12) * 1

(20, 8)

(3) 1 1 2 4 3 3 3 ← (4) ..4 3 3 3

(5) 5 1 * 1 ← (6) 6 * 1

(9) 1 1 * 1 ← (10) 2 * 1

(20, 9)

(3) 1 2 3 4 4 1 1 1 ← (5) 2 3 4 4 1 1 1

(5) 4 1 1 * 1 ← (6) 5 1 * 1

(20, 10)

(1) 4 4 1 1 * 1 ← (4) 1 2 3 4 4 1 1 1

(3) * * 1 ← (6) 4 1 1 * 1

(20, 11)

(1) 2 * * 1 ← (2) 4 4 1 1 * 1

(2) 1 * * 1 ← (4) * * 1

(20, 12)

(1) 1 1 * * 1 ← (2) 2 * * 1

(21, 3)

(3) 11 7

(7) 7 7

(15) 3 3 ← (19) 3

(19) 1 1 ← (21) 1

(21, 4)

(2) 5 7 7

(3) 10 5 3 ← (4) 11 7

(7) 6 5 3 ← (8) 7 7

(12) 3 3 3 ← (16) 3 3

(18) 1 1 1 ← (20) 1 1

(21, 5)

(1) 6 6 5 3

(3) 5 7 3 3

(3) 9 3 3 3 ← (4) 10 5 3

(7) 6 2 3 3 ← (14) 2 3 3

(21, 6)

(1) 4 7 3 3 3 ← (2) 6 6 5 3

(3) 4 5 3 3 3 ← (4) 5 7 3 3

(6) 2 4 3 3 3 ← (13) 1 2 3 3

(7) 5 1 2 3 3 ← (8) 6 2 3 3

(21, 7)

(1) 2 4 5 3 3 3 ← (2) 4 7 3 3 3

(5) 1 2 4 3 3 3 ← (11) 4 4 1 1 1

(7) 3 4 4 1 1 1 ← (8) 5 1 2 3 3

(11) 1 * 1 ← (13) * 1

(21, 8)

(4) 1 1 2 4 3 3 3 ← (8) 3 4 4 1 1 1

(10) 1 1 * 1 ← (12) 1 * 1

(21, 11)

(3) 1 * * 1 ← (5) * * 1

(21, 12)

(2) 1 1 * * 1 ← (4) 1 * * 1

(22, 2)

(7) 15

(15) 7 ← (23)

(22, 3)

(6) 13 3

(7) 14 1 ← (8) 15

(14) 5 3 ← (22) 1

(15) 6 1 ← (16) 7

(19) 2 1 ← (20) 3

(22, 4)

(1) 7 7 7

(3) 5 7 7

(5) 11 3 3

(7) 13 1 1 ← (8) 14 1

(8) 6 5 3 ← (21) 1 1

(13) 3 3 3 ← (17) 3 3

(15) 5 1 1 ← (16) 6 1

(19) 1 1 1 ← (20) 2 1

(22, 5)

(4) 9 3 3 3

(5) 8 3 3 3 ← (20) 1 1 1

(7) 12 1 1 1 ← (8) 13 1 1

(15) 4 1 1 1 ← (16) 5 1 1

(22, 6)

(1) 3 5 7 3 3

(4) 4 5 3 3 3 ← (16) 4 1 1 1

(5) 3 6 2 3 3 ← (6) 8 3 3 3

(7) 2 4 3 3 3 ← (9) 6 2 3 3

(7) 8 4 1 1 1 ← (8) 12 1 1 1

(22, 7)

(2) 2 4 5 3 3 3 ← (14) * 1

(5) ..4 3 3 3 ← (6) 3 6 2 3 3

(6) 1 2 4 3 3 3 ← (8) 2 4 3 3 3

(7) 6 * 1 ← (8) 8 4 1 1 1

(11) 2 * 1 ← (12) 4 4 1 1 1

(22, 8)

(5) 1 1 2 4 3 3 3 ← (6) ..4 3 3 3

(6) 2 3 4 4 1 1 1

(7) 5 1 * 1 ← (8) 6 * 1

(11) 1 1 * 1 ← (12) 2 * 1

(22, 9)

(5) 1 2 3 4 4 1 1 1

(7) 4 1 1 * 1 ← (8) 5 1 * 1

$(22, 10)$

(3) 4 4 1 1 * 1

$(22, 11)$

(3) 2 * * 1 ← (4) 4 4 1 1 * 1

$(22, 12)$

(3) 1 1 * * 1 ← (4) 2 * * 1

$(23, 3)$

(5) 11 7

(7) 13 3 ← (9) 15

(9) 7 7 ← (21) 3

(15) 5 3 ← (17) 7

$(23, 4)$

(2) 7 7 7

(4) 5 7 7

(5) 10 5 3 ← (6) 11 7

(6) 11 3 3 ← (8) 13 3

(9) 6 5 3 ← (10) 7 7

(14) 3 3 3 ← (16) 5 3

(15) 2 3 3 ← (18) 3 3

$(23, 5)$

(1) 3 5 7 7

(3) 6 6 5 3

(5) 5 7 3 3 ← (10) 6 5 3

(5) 9 3 3 3 ← (6) 10 5 3

(14) 1 2 3 3 ← (16) 2 3 3

$(23, 6)$

(2) 3 5 7 3 3

(3) 4 7 3 3 3 ← (4) 6 6 5 3

(5) 4 5 3 3 3 ← (6) 5 7 3 3

(9) 5 1 2 3 3 ← (10) 6 2 3 3

$(23, 7)$

(3) 2 4 5 3 3 3 ← (4) 4 7 3 3 3

(7) 1 2 4 3 3 3 ← (9) 2 4 3 3 3

(9) 3 4 4 1 1 1 ← (10) 5 1 2 3 3

(13) 1 * 1

$(23, 8)$

(6) 1 1 2 4 3 3 3 ← (8) 1 2 4 3 3 3

(7) 2 3 4 4 1 1 1 ← (10) 3 4 4 1 1 1

(12) 1 1 * 1

$(23, 9)$

(1) 6 2 3 4 4 1 1 1

(6) 1 2 3 4 4 1 1 1 ← (8) 2 3 4 4 1 1 1

(8) 4 1 1 * 1

$(23, 10)$

(1) 5 1 2 3 4 4 1 1 1 ← (2) 6 2 3 4 4 1 1 1

(6) * * 1

$(23, 11)$

(1) 3 4 4 1 1 * 1 ← (2) 5 1 2 3 4 4 1 1 1

(5) 1 * * 1

$(23, 12)$

(4) 1 1 * * 1

$(24, 2)$

(23) 1 ← (25)

$(24, 3)$

(9) 14 1 ← (10) 15

(17) 6 1 ← (18) 7

(21) 2 1 ← (22) 3

(22) 1 1 ← (24) 1

$(24, 4)$

(3) 7 7 7

(5) 5 7 7 ← (19) 3 3

(7) 11 3 3 ← (9) 13 3

(9) 13 1 1 ← (10) 14 1

(15) 3 3 3 ← (17) 5 3

(17) 5 1 1 ← (18) 6 1

(21) 1 1 1 ← (22) 2 1

$(24, 5)$

(1) 4 5 7 7

(2) 3 5 7 7

(6) 9 3 3 3 ← (8) 11 3 3

(7) 8 3 3 3 ← (16) 3 3 3

(9) 12 1 1 1 ← (10) 13 1 1

(15) 1 2 3 3 ← (17) 2 3 3

(17) 4 1 1 1 ← (18) 5 1 1

$(24, 6)$

(1) 3 6 6 5 3

(3) 3 5 7 3 3 ← (5) 6 6 5 3

(6) 4 5 3 3 3 ← (11) 6 2 3 3

(7) 3 6 2 3 3 ← (8) 8 3 3 3

(9) 8 4 1 1 1 ← (10) 12 1 1 1

(13) 4 4 1 1 1 ← (16) 1 2 3 3

(15) * 1 ← (18) 4 1 1 1

$(24, 7)$

(1) 2 3 5 7 3 3 ← (2) 3 6 6 5 3

(4) 2 4 5 3 3 3 ← (10) 2 4 3 3 3

(7) ..4 3 3 3 ← (8) 3 6 2 3 3

(9) 6 * 1 ← (10) 8 4 1 1 1

(13) 2 * 1 ← (14) 4 4 1 1 1

(14) 1 * 1 ← (16) * 1

$(24, 8)$

(1) 13 1 * 1 ← (9) 1 2 4 3 3 3

(7) 1 1 2 4 3 3 3 ← (8) ..4 3 3 3

(9) 5 1 * 1 ← (10) 6 * 1

(13) 1 1 * 1 ← (14) 2 * 1

$(24, 9)$

(1) 12 1 1 * 1 ← (2) 13 1 * 1

(7) 1 2 3 4 4 1 1 1 ← (9) 2 3 4 4 1 1 1

(9) 4 1 1 * 1 ← (10) 5 1 * 1

$(24, 10)$

(1) * 2 4 3 3 3

(1) 8 4 1 1 * 1 ← (2) 12 1 1 * 1

(5) 4 4 1 1 * 1 ← (8) 1 2 3 4 4 1 1 1

(7) * * 1 ← (10) 4 1 1 * 1

(24, 11)

(1) 6 * * 1 ← (2) 8 4 1 1 * 1

(2) 3 4 4 1 1 * 1

(5) 2 * * 1 ← (6) 4 4 1 1 * 1

(6) 1 * * 1 ← (8) * * 1

(24, 12)

(1) 5 1 * * 1 ← (2) 6 * * 1

(5) 1 1 * * 1 ← (6) 2 * * 1

(24, 13)

(1) 4 1 1 * * 1 ← (2) 5 1 * * 1

(25, 3)

(7) 11 7 ← (11) 15

(11) 7 7 ← (19) 7

(23) 1 1 ← (25) 1

(25, 4)

(4) 7 7 7 ← (10) 13 3

(6) 5 7 7 ← (18) 5 3

(7) 10 5 3 ← (8) 11 7

(11) 6 5 3 ← (12) 7 7

(22) 1 1 1 ← (24) 1 1

(25, 5)

(2) 4 5 7 7 ← (9) 11 3 3

(3) 3 5 7 7 ← (17) 3 3 3

(7) 5 7 3 3 ← (12) 6 5 3

(7) 9 3 3 3 ← (8) 10 5 3

(25, 6)

(1) 2 3 5 7 7 ← (8) 9 3 3 3

(4) 3 5 7 3 3 ← (9) 8 3 3 3

(5) 4 7 3 3 3 ← (6) 6 6 5 3

(7) 4 5 3 3 3 ← (8) 5 7 3 3

(11) 5 1 2 3 3 ← (12) 6 2 3 3

(25, 7)

(2) 2 3 5 7 3 3 ← (8) 4 5 3 3 3

(5) 2 4 5 3 3 3 ← (6) 4 7 3 3 3

(11) 3 4 4 1 1 1 ← (12) 5 1 2 3 3

(15) 1 * 1 ← (17) * 1

(25, 8)

(8) 1 1 2 4 3 3 3

(14) 1 1 * 1 ← (16) 1 * 1

(25, 9)

(3) 6 2 3 4 4 1 1 1

(25, 10)

(2) * 2 4 3 3 3

(3) 5 1 2 3 4 4 1 1 1 ← (4) 6 2 3 4 4 1 1 1

(25, 11)

(1) 1 * 2 4 3 3 3

(3) 3 4 4 1 1 * 1 ← (4) 5 1 2 3 4 4 1 1 1

(7) 1 * * 1 ← (9) * * 1

(25, 12)

(1) 2 3 4 4 1 1 * 1

(6) 1 1 * * 1 ← (8) 1 * * 1

(25, 13)

(2) 4 1 1 * * 1

$(26, 2)$

(23) 3 ← (27)

$(26, 3)$

(11) 14 1 ← (12) 15

(19) 6 1 ← (20) 7

(20) 3 3 ← (26) 1

(23) 2 1 ← (24) 3

$(26, 4)$

(5) 7 7 7 ← (9) 11 7

(7) 5 7 7 ← (13) 7 7

(11) 13 1 1 ← (12) 14 1

(18) 2 3 3 ← (25) 1 1

(19) 5 1 1 ← (20) 6 1

(23) 1 1 1 ← (24) 2 1

$(26, 5)$

(3) 4 5 7 7 ← (6) 7 7 7

(4) 3 5 7 7 ← (8) 5 7 7

(11) 12 1 1 1 ← (12) 13 1 1

(17) 1 2 3 3 ← (24) 1 1 1

(19) 4 1 1 1 ← (20) 5 1 1

$(26, 6)$

(2) 2 3 5 7 7 ← (4) 4 5 7 7

(3) 3 6 6 5 3

(5) 3 5 7 3 3 ← (9) 5 7 3 3

(9) 3 6 2 3 3 ← (10) 8 3 3 3

(11) 2 4 3 3 3 ← (13) 6 2 3 3

(11) 8 4 1 1 1 ← (12) 12 1 1 1

(15) 4 4 1 1 1 ← (20) 4 1 1 1

$(26, 7)$

(3) 2 3 5 7 3 3 ← (4) 3 6 6 5 3

(6) 2 4 5 3 3 3

(9) ..4 3 3 3 ← (10) 3 6 2 3 3

(10) 1 2 4 3 3 3 ← (12) 2 4 3 3 3

(11) 6 * 1 ← (12) 8 4 1 1 1

(12) 3 4 4 1 1 1 ← (18) * 1

(15) 2 * 1 ← (16) 4 4 1 1 1

$(26, 8)$

(3) 13 1 * 1

(9) 1 1 2 4 3 3 3 ← (10) ..4 3 3 3

(10) 2 3 4 4 1 1 1 ← (17) 1 * 1

(11) 5 1 * 1 ← (12) 6 * 1

(15) 1 1 * 1 ← (16) 2 * 1

$(26, 9)$

(1) 8 1 1 2 4 3 3 3

(3) 12 1 1 * 1 ← (4) 13 1 * 1

(9) 1 2 3 4 4 1 1 1 ← (16) 1 1 * 1

(11) 4 1 1 * 1 ← (12) 5 1 * 1

$(26, 10)$

(1) 3 6 2 3 4 4 1 1 1 ← (2) 8 1 1 2 4 3 3 3

(3) * 2 4 3 3 3 ← (5) 6 2 3 4 4 1 1 1

(3) 8 4 1 1 * 1 ← (4) 12 1 1 * 1

(7) 4 4 1 1 * 1 ← (12) 4 1 1 * 1

(26, 11)

(1) 2 * 2 4 3 3 3 ← (2) 3 6 2 3 4 4 1 1 1

(2) 1 * 2 4 3 3 3 ← (4) * 2 4 3 3 3

(3) 6 * * 1 ← (4) 8 4 1 1 * 1

(4) 3 4 4 1 1 * 1 ← (10) * * 1

(7) 2 * * 1 ← (8) 4 4 1 1 * 1

(26, 12)

(1) 1 1 * 2 4 3 3 3 ← (2) 2 * 2 4 3 3 3

(2) 2 3 4 4 1 1 * 1 ← (9) 1 * * 1

(3) 5 1 * * 1 ← (4) 6 * * 1

(7) 1 1 * * 1 ← (8) 2 * * 1

(26, 13)

(1) 1 2 3 4 4 1 1 * 1 ← (8) 1 1 * * 1

(3) 4 1 1 * * 1 ← (4) 5 1 * * 1

(26, 14)

(1) * * * 1

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(10) 11 3 3 ← (12) 13 3

(13) 6 5 3 ← (14) 7 7

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(19) 2 3 3 ← (22) 3 3

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(7) 6 6 5 3 ← (14) 6 5 3

(9) 9 3 3 3 ← (10) 10 5 3

(18) 1 2 3 3 ← (20) 2 3 3

(27, 6)

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(7) 4 7 3 3 3 ← (8) 6 6 5 3

(9) 4 5 3 3 3 ← (10) 5 7 3 3

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(13) 13 1 1 ← (14) 14 1

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(28, 5)

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(10) 9 3 3 3 ← (12) 11 3 3

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(7) 3 5 7 3 3 ← (9) 6 6 5 3

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(19) * 1 ← (22) 4 1 1 1

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(17) 2 * 1 ← (18) 4 4 1 1 1

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(28, 8)

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(17) 1 1 * 1 ← (18) 2 * 1

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(15) 6 5 3 ← (16) 7 7

(20) 3 3 3 ← (24) 3 3

(26) 1 1 1 ← (28) 1 1

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(7) 3 5 7 7 ← (11) 5 7 7

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(29, 6)

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(15) 5 1 2 3 3 ← (16) 6 2 3 3

(29, 7)

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(30, 8)

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(15) 5 1 * 1 ← (16) 6 * 1

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(2) 4 5 3 3 3

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(11) 2 * * 1 ← (12) 4 4 1 1 * 1

(30, 12)

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(13) 11 7

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(23) 5 3 ← (25) 7

(29) 1 1

(31, 4)

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(12) 5 7 7

(13) 10 5 3 ← (14) 11 7

(14) 11 3 3 ← (16) 13 3

(17) 6 5 3 ← (18) 7 7

(22) 3 3 3 ← (24) 5 3

(23) 2 3 3 ← (26) 3 3

(28) 1 1 1

(31, 5)

(1) 13 11 3 3 ← (2) 14 13 3

(9) 3 5 7 7

(13) 5 7 3 3 ← (18) 6 5 3

(13) 9 3 3 3 ← (14) 10 5 3

(22) 1 2 3 3 ← (24) 2 3 3

(24) 4 1 1 1

(31, 6)

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(7) 2 3 5 7 7 ← (9) 4 5 7 7

(11) 4 7 3 3 3 ← (12) 6 6 5 3

(13) 4 5 3 3 3 ← (14) 5 7 3 3

(17) 5 1 2 3 3 ← (18) 6 2 3 3

(22) * 1

(31, 7)

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(5) 3 3 6 6 5 3 ← (8) 2 3 5 7 7

(11) 2 4 5 3 3 3 ← (12) 4 7 3 3 3

(15) 1 2 4 3 3 3 ← (17) 2 4 3 3 3

(17) 3 4 4 1 1 1 ← (18) 5 1 2 3 3

(21) 1 * 1

(31, 8)

(1) 10 2 4 5 3 3 3 ← (2) 12 4 5 3 3 3

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(14) 1 1 2 4 3 3 3 ← (16) 1 2 4 3 3 3

(15) 2 3 4 4 1 1 1 ← (18) 3 4 4 1 1 1

(20) 1 1 * 1

(31, 9)

(1) 7 13 1 * 1 ← (2) 10 2 4 5 3 3 3

(3) 6 ..4 5 3 3 3

(5) 4 ..4 5 3 3 3 ← (6) 6 2 4 5 3 3 3

(14) 1 2 3 4 4 1 1 1 ← (16) 2 3 4 4 1 1 1

(16) 4 1 1 * 1

$(31, 10)$

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(2) 3 5 7 3 3

(3) 4 ...4 5 3 3 3 ← (4) 6 ..4 5 3 3 3

(5) ....4 5 3 3 3 ← (6) 4 ..4 5 3 3 3

(9) 5 1 2 3 4 4 1 1 1 ← (10) 6 2 3 4 4 1 1 1

(14) * * 1

$(31, 11)$

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(3) .....4 5 3 3 3 ← (4) 4 ...4 5 3 3 3

(7) 1 * 2 4 3 3 3 ← (9) * 2 4 3 3 3

(9) 3 4 4 1 1 * 1 ← (10) 5 1 2 3 4 4 1 1 1

(13) 1 * * 1

$(31, 12)$

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(6) 1 1 * 2 4 3 3 3 ← (8) 1 * 2 4 3 3 3

(7) 2 3 4 4 1 1 * 1 ← (10) 3 4 4 1 1 * 1

(12) 1 1 * * 1

$(31, 13)$

(1) 6 2 3 4 4 1 1 * 1

(6) 1 2 3 4 4 1 1 * 1 ← (8) 2 3 4 4 1 1 * 1

(8) 4 1 1 * * 1

$(31, 14)$

(1) 5 1 2 3 4 4 1 1 * 1 ← (2) 6 2 3 4 4 1 1 * 1

(6) * * * 1

$(31, 15)$

(1) 3 4 4 1 1 * * 1 ← (2) 5 1 2 3 4 4 1 1 * 1

(5) 1 * * * 1

$(31, 16)$

(4) 1 1 * * * 1

$(32, 2)$

(1) 31

(29) 3

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$(32, 3)$

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(30) 1 1 ← (32) 1

$(32, 4)$

(1) 13 11 7

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(13) 5 7 7

(15) 11 3 3 ← (17) 13 3

(17) 13 1 1 ← (18) 14 1

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(25) 5 1 1 ← (26) 6 1

(29) 1 1 1 ← (30) 2 1

$(32, 5)$

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(14) 9 3 3 3 ← (16) 11 3 3

(15) 8 3 3 3 ← (24) 3 3 3

(17) 12 1 1 1 ← (18) 13 1 1

(23) 1 2 3 3 ← (25) 2 3 3

(25) 4 1 1 1 ← (26) 5 1 1

(32, 6)

(1) 9 3 5 7 7

(1) 24 4 1 1 1 ← (2) 28 1 1 1

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(9) 3 6 6 5 3

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(17) 8 4 1 1 1 ← (18) 12 1 1 1

(21) 4 4 1 1 1 ← (24) 1 2 3 3

(23) * 1 ← (26) 4 1 1 1

(32, 7)

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(9) 2 3 5 7 3 3 ← (10) 3 6 6 5 3

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(32, 8)

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(7) 8 1 1 2 4 3 3 3 ← (16) 1 1 2 4 3 3 3

(9) 12 1 1 * 1 ← (10) 13 1 * 1

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(32, 10)

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(9) 8 4 1 1 * 1 ← (10) 12 1 1 * 1

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(15) * * 1 ← (18) 4 1 1 * 1

(32, 11)

(1) .....3 5 7 3 3 ← (2) 3 6 ..4 5 3 3 3

(1) 14 * * 1 ← (2) 16 4 1 1 * 1

(4) .....4 5 3 3 3 ← (10) * 2 4 3 3 3

(7) 2 * 2 4 3 3 3 ← (8) 3 6 2 3 4 4 1 1 1

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(14) 1 * * 1 ← (16) * * 1

(32, 12)

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(2) ......4 5 3 3 3 ← (9) 1 * 2 4 3 3 3

(7) 1 1 * 2 4 3 3 3 ← (8) 2 * 2 4 3 3 3

(9) 5 1 * * 1 ← (10) 6 * * 1

(13) 1 1 * * 1 ← (14) 2 * * 1

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(9) 4 1 1 * * 1 ← (10) 5 1 * * 1

(32, 14)

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(7) * * * 1 ← (10) 4 1 1 * * 1

(32, 15)

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(32, 17)

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(33, 3)

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(3) 15 15

(15) 11 7 ← (19) 15

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(33, 7)

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(33, 10)

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(7) ....4 5 3 3 3 ← (8) 4 ..4 5 3 3 3

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(33, 11)

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(3) 4 ....4 5 3 3 3 ← (4) 5 8 1 1 2 4 3 3 3

(5) .....4 5 3 3 3 ← (6) 4 ...4 5 3 3 3

(11) 3 4 4 1 1 * 1 ← (12) 5 1 2 3 4 4 1 1 1

(15) 1 * * 1 ← (17) * * 1

(33, 12)

(3) ......4 5 3 3 3 ← (4) 4 ....4 5 3 3 3

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(14) 1 1 * * 1 ← (16) 1 * * 1

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(33, 14)

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(33, 16)

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(2) 4 1 1 * * * 1

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(34, 3)

(2) 29 3

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(1) 4 7 3 3 6 6 5 3

(1) 5 6 3 3 6 6 5 3 ← (2) 7 12 4 5 3 3 3

(3) 3 6 3 3 6 6 5 3 ← (4) 6 5 2 3 5 7 7

(7) 5 6 2 4 5 3 3 3 ← (13) 6 2 4 5 3 3 3

(7) 20 1 1 * 1 ← (8) 21 1 * 1

(9) 3 6 2 4 5 3 3 3 ← (10) 6 2 3 5 7 3 3

(13) 8 1 1 2 4 3 3 3 ← (28) 1 1 * 1

(15) 12 1 1 * 1 ← (16) 13 1 * 1

(23) 4 1 1 * 1 ← (24) 5 1 * 1

(38, 10)

(1) 3 5 6 2 3 5 7 3 3 ← (2) 5 6 3 3 6 6 5 3

(3) 2 4 3 3 3 6 6 5 3 ← (4) 3 6 3 3 6 6 5 3

(6) ...3 3 6 6 5 3 ← (11) 6 ..4 5 3 3 3

(7) 3 6 ..4 5 3 3 3 ← (8) 5 6 2 4 5 3 3 3

(7) 16 4 1 1 * 1 ← (8) 20 1 1 * 1

(9) ....3 5 7 3 3 ← (10) 3 6 2 4 5 3 3 3

(12) ....4 5 3 3 3 ← (24) 4 1 1 * 1

(13) 3 6 2 3 4 4 1 1 1 ← (14) 8 1 1 2 4 3 3 3

(15) * 2 4 3 3 3 ← (17) 6 2 3 4 4 1 1 1

(15) 8 4 1 1 * 1 ← (16) 12 1 1 * 1

(38, 11)

(1) ..4 3 3 3 6 6 5 3 ← (2) 3 5 6 2 3 5 7 3 3

(4) ....3 3 6 6 5 3 ← (10) 3 5 7 3 3

(7) .....3 5 7 3 3 ← (8) 3 6 ..4 5 3 3 3

(7) 14 * * 1 ← (8) 16 4 1 1 * 1

(10) .....4 5 3 3 3 ← (22) * * 1

(13) 2 * 2 4 3 3 3 ← (14) 3 6 2 3 4 4 1 1 1

(14) 1 * 2 4 3 3 3 ← (16) * 2 4 3 3 3

(15) 6 * * 1 ← (16) 8 4 1 1 * 1

(19) 2 * * 1 ← (20) 4 4 1 1 * 1

(38, 12)

(1) 6 .....3 5 7 3 3 ← (8) 3 5 7 3 3

(7) 13 1 * * 1 ← (8) 14 * * 1

(8) ......4 5 3 3 3 ← (21) 1 * * 1

(13) 1 1 * 2 4 3 3 3 ← (14) 2 * 2 4 3 3 3

(15) 5 1 * * 1 ← (16) 6 * * 1

(19) 1 1 * * 1 ← (20) 2 * * 1

(38, 13)

(1) 3 6 .....4 5 3 3 3 ← (2) 6 .....3 5 7 3 3

(5) 8 1 1 * 2 4 3 3 3 ← (20) 1 1 * * 1

(7) 12 1 1 * * 1 ← (8) 13 1 * * 1

(15) 4 1 1 * * 1 ← (16) 5 1 * * 1

(38, 14)

(1) ....3 5 7 3 3 ← (2) 3 6 .....4 5 3 3 3

(4) ....4 5 3 3 3 ← (16) 4 1 1 * * 1

(5) 3 6 2 3 4 4 1 1 * 1 ← (6) 8 1 1 * 2 4 3 3 3

(7) * * 2 4 3 3 3 ← (9) 6 2 3 4 4 1 1 * 1

(7) 8 4 1 1 * * 1 ← (8) 12 1 1 * * 1

(38, 15)

(2) ....4 5 3 3 3 ← (14) * * * 1

(5) 2 * * 2 4 3 3 3 ← (6) 3 6 2 3 4 4 1 1 * 1

(6) 1 * * 2 4 3 3 3 ← (8) * * 2 4 3 3 3

(7) 6 * * * 1 ← (8) 8 4 1 1 * * 1

(11) 2 * * * 1 ← (12) 4 4 1 1 * * 1

(38, 16)

(5) 1 1 * * 2 4 3 3 3 ← (6) 2 * * 2 4 3 3 3

(6) 2 3 4 4 1 1 * * 1

(7) 5 1 * * * 1 ← (8) 6 * * * 1

(11) 1 1 * * * 1 ← (12) 2 * * * 1

(38, 17)

(5) 1 2 3 4 4 1 1 * * 1

(7) 4 1 1 * * * 1 ← (8) 5 1 * * * 1

(38, 18)

(3) 4 4 1 1 * * * 1

(38, 19)

(3) 2 * * * * 1 ← (4) 4 4 1 1 * * * 1

(38, 20)

(3) 1 1 * * * * 1 ← (4) 2 * * * * 1

(39, 3)

(1) 23 15

(5) 27 7

(7) 29 3 ← (9) 31

(21) 11 7

(23) 13 3 ← (25) 15

(25) 7 7 ← (37) 3

(31) 5 3 ← (33) 7

(39, 4)

(1) 22 13 3 ← (2) 23 15

(2) 23 7 7

(5) 26 5 3 ← (6) 27 7

(6) 27 3 3 ← (8) 29 3

(9) 14 13 3 ← (10) 15 15

(20) 5 7 7

(21) 10 5 3 ← (22) 11 7

(22) 11 3 3 ← (24) 13 3

(25) 6 5 3 ← (26) 7 7

(30) 3 3 3 ← (32) 5 3

(31) 2 3 3 ← (34) 3 3

(39, 5)

(1) 17 7 7 7

(1) 21 11 3 3 ← (2) 22 13 3

(5) 25 3 3 3 ← (6) 26 5 3

(7) 13 5 7 7

(9) 13 11 3 3 ← (10) 14 13 3

(21) 5 7 3 3 ← (26) 6 5 3

(21) 9 3 3 3 ← (22) 10 5 3

(30) 1 2 3 3 ← (32) 2 3 3

(39, 6)

(1) 20 9 3 3 3 ← (2) 21 11 3 3

(4) 5 9 7 7 7

(6) 11 3 5 7 7 ← (8) 13 5 7 7

(9) 12 9 3 3 3 ← (10) 13 11 3 3

(15) 2 3 5 7 7 ← (17) 4 5 7 7

(19) 4 7 3 3 3 ← (20) 6 6 5 3

(21) 4 5 3 3 3 ← (22) 5 7 3 3

(25) 5 1 2 3 3 ← (26) 6 2 3 3

(39, 7)

(1) 3 5 9 7 7 7

(1) 8 12 9 3 3 3

(3) 5 9 3 5 7 7

(5) 5 7 3 5 7 7 ← (9) 9 3 5 7 7

(7) 9 3 6 6 5 3 ← (10) 12 9 3 3 3

(13) 3 3 6 6 5 3 ← (16) 2 3 5 7 7

(19) 2 4 5 3 3 3 ← (20) 4 7 3 3 3

(23) 1 2 4 3 3 3 ← (25) 2 4 3 3 3

(25) 3 4 4 1 1 1 ← (26) 5 1 2 3 3

(39, 8)

(1) 6 9 3 6 6 5 3 ← (2) 8 12 9 3 3 3

(2) 3 5 7 3 5 7 7 ← (4) 5 9 3 5 7 7

(6) 5 5 3 6 6 5 3 ← (8) 9 3 6 6 5 3

(7) 6 3 3 6 6 5 3 ← (14) 3 3 6 6 5 3

(9) 10 2 4 5 3 3 3 ← (10) 12 4 5 3 3 3

(22) 1 1 2 4 3 3 3 ← (24) 1 2 4 3 3 3

(23) 2 3 4 4 1 1 1 ← (26) 3 4 4 1 1 1

(39, 9)

(1) 3 6 5 2 3 5 7 7 ← (2) 6 9 3 6 6 5 3

(2) 4 7 3 3 6 6 5 3

(5) 5 6 2 3 5 7 3 3 ← (11) 6 2 3 5 7 3 3

(7) 3 6 2 3 5 7 3 3 ← (8) 6 3 3 6 6 5 3

(9) 7 13 1 * 1 ← (10) 10 2 4 5 3 3 3

(13) 4 ..4 5 3 3 3 ← (14) 6 2 4 5 3 3 3

(22) 1 2 3 4 4 1 1 1 ← (24) 2 3 4 4 1 1 1

(39, 10)

(1) 2 3 5 5 3 6 6 5 3 ← (2) 3 6 5 2 3 5 7 7

(4) 2 4 3 3 3 6 6 5 3 ← (9) 5 6 2 4 5 3 3 3

(5) 3 5 6 2 4 5 3 3 3 ← (6) 5 6 2 3 5 7 3 3

(7) ...3 3 6 6 5 3 ← (8) 3 6 2 3 5 7 3 3

(9) 5 8 1 1 2 4 3 3 3 ← (10) 7 13 1 * 1

(11) 4 ...4 5 3 3 3 ← (12) 6 ..4 5 3 3 3

(13) ....4 5 3 3 3 ← (14) 4 ..4 5 3 3 3

(17) 5 1 2 3 4 4 1 1 1 ← (18) 6 2 3 4 4 1 1 1

(39, 11)

(2) ..4 3 3 3 6 6 5 3 ← (8) ...3 3 6 6 5 3

(5) ....3 3 6 6 5 3 ← (6) 3 5 6 2 4 5 3 3 3

(9) 4 ....4 5 3 3 3 ← (10) 5 8 1 1 2 4 3 3 3

(11) .....4 5 3 3 3 ← (12) 4 ...4 5 3 3 3

(15) 1 * 2 4 3 3 3 ← (17) * 2 4 3 3 3

(17) 3 4 4 1 1 * 1 ← (18) 5 1 2 3 4 4 1 1 1

(39, 12)

(5) 6 .....4 5 3 3 3

(9) ......4 5 3 3 3 ← (10) 4 ....4 5 3 3 3

(14) 1 1 * 2 4 3 3 3 ← (16) 1 * 2 4 3 3 3

(15) 2 3 4 4 1 1 * 1 ← (18) 3 4 4 1 1 * 1

(39, 13)

(3) 6 ......4 5 3 3 3

(5) 4 ......4 5 3 3 3 ← (6) 6 .....4 5 3 3 3

(14) 1 2 3 4 4 1 1 * 1 ← (16) 2 3 4 4 1 1 * 1

(39, 14)

(2) 3 5 7 3 3

(3) 4 ....4 5 3 3 3 ← (4) 6 ......4 5 3 3 3

(5) ....4 5 3 3 3 ← (6) 4 ......4 5 3 3 3

(9) 5 1 2 3 4 4 1 1 * 1 ← (10) 6 2 3 4 4 1 1 * 1

(39, 15)

(3) ....4 5 3 3 3 ← (4) 4 ....4 5 3 3 3

(7) 1 * * 2 4 3 3 3 ← (9) * * 2 4 3 3 3

(9) 3 4 4 1 1 * * 1 ← (10) 5 1 2 3 4 4 1 1 * 1

(13) 1 * * * 1

(39, 16)

(6) 1 1 * * 2 4 3 3 3 ← (8) 1 * * 2 4 3 3 3

(7) 2 3 4 4 1 1 * * 1 ← (10) 3 4 4 1 1 * * 1

(12) 1 1 * * * 1

(39, 17)

(1) 6 2 3 4 4 1 1 * * 1

(6) 1 2 3 4 4 1 1 * * 1 ← (8) 2 3 4 4 1 1 * * 1

(8) 4 1 1 * * * 1

(39, 18)

(1) 5 1 2 3 4 4 1 1 * * 1 ← (2) 6 2 3 4 4 1 1 * * 1

(6) * * * * 1

(39, 19)

(1) 3 4 4 1 1 * * * 1 ← (2) 5 1 2 3 4 4 1 1 * * 1

(5) 1 * * * * 1

(39, 20)

(4) 1 1 * * * * 1

(40, 2)

(39) 1 ← (41)

(40, 3)

(9) 30 1 ← (10) 31

(25) 14 1 ← (26) 15

(33) 6 1 ← (34) 7

(37) 2 1 ← (38) 3

(38) 1 1 ← (40) 1

(40, 4)

(1) 21 11 7

(3) 23 7 7

(7) 27 3 3 ← (9) 29 3

(9) 13 11 7

(9) 29 1 1 ← (10) 30 1

(19) 7 7 7

(21) 5 7 7 ← (35) 3 3

(23) 11 3 3 ← (25) 13 3

(25) 13 1 1 ← (26) 14 1

(31) 3 3 3 ← (33) 5 3

(33) 5 1 1 ← (34) 6 1

(37) 1 1 1 ← (38) 2 1

(40, 5)

(1) 20 5 7 7
(2) 17 7 7 7
(6) 25 3 3 3 ← (8) 27 3 3
(7) 14 5 7 7
(9) 28 1 1 1 ← (10) 29 1 1
(18) 3 5 7 7
(22) 9 3 3 3 ← (24) 11 3 3
(23) 8 3 3 3 ← (32) 3 3 3
(25) 12 1 1 1 ← (26) 13 1 1
(31) 1 2 3 3 ← (33) 2 3 3
(33) 4 1 1 1 ← (34) 5 1 1

(40, 6)

(1) 7 13 5 7 7
(2) 20 9 3 3 3
(3) 14 4 5 7 7
(5) 5 9 7 7 7 ← (9) 13 5 7 7
(7) 11 3 5 7 7 ← (8) 14 5 7 7
(9) 24 4 1 1 1 ← (10) 28 1 1 1
(17) 3 6 6 5 3
(19) 3 5 7 3 3 ← (21) 6 6 5 3
(22) 4 5 3 3 3 ← (27) 6 2 3 3
(23) 3 6 2 3 3 ← (24) 8 3 3 3
(25) 8 4 1 1 1 ← (26) 12 1 1 1
(29) 4 4 1 1 1 ← (32) 1 2 3 3
(31) * 1 ← (34) 4 1 1 1

(40, 7)

(2) 3 5 9 7 7 7 ← (8) 11 3 5 7 7
(3) 13 2 3 5 7 7 ← (4) 14 4 5 7 7
(6) 5 7 3 5 7 7 ← (10) 9 3 5 7 7
(9) 22 * 1 ← (10) 24 4 1 1 1
(17) 2 3 5 7 3 3 ← (18) 3 6 6 5 3
(20) 2 4 5 3 3 3 ← (26) 2 4 3 3 3
(23) ..4 3 3 3 ← (24) 3 6 2 3 3
(25) 6 * 1 ← (26) 8 4 1 1 1
(29) 2 * 1 ← (30) 4 4 1 1 1
(30) 1 * 1 ← (32) * 1

(40, 8)

(1) 3 5 9 3 5 7 7
(3) 3 5 7 3 5 7 7 ← (5) 5 9 3 5 7 7
(3) 7 12 4 5 3 3 3 ← (4) 13 2 3 5 7 7
(5) 6 5 2 3 5 7 7
(7) 5 5 3 6 6 5 3 ← (9) 9 3 6 6 5 3
(9) 21 1 * 1 ← (10) 22 * 1
(17) 13 1 * 1 ← (25) 1 2 4 3 3 3
(23) 1 1 2 4 3 3 3 ← (24) ..4 3 3 3
(25) 5 1 * 1 ← (26) 6 * 1
(29) 1 1 * 1 ← (30) 2 * 1

(40, 9)

(3) 4 7 3 3 6 6 5 3 ← (8) 5 5 3 6 6 5 3
(3) 5 6 3 3 6 6 5 3 ← (4) 7 12 4 5 3 3 3
(5) 3 6 3 3 6 6 5 3 ← (6) 6 5 2 3 5 7 7
(9) 20 1 1 * 1 ← (10) 21 1 * 1
(11) 3 6 2 4 5 3 3 3 ← (12) 6 2 3 5 7 3 3
(15) 8 1 1 2 4 3 3 3 ← (24) 1 1 2 4 3 3 3
(17) 12 1 1 * 1 ← (18) 13 1 * 1
(23) 1 2 3 4 4 1 1 1 ← (25) 2 3 4 4 1 1 1
(25) 4 1 1 * 1 ← (26) 5 1 * 1

(40, 10)

(1) 2 4 7 3 3 6 6 5 3
(2) 2 3 5 5 3 6 6 5 3 ← (4) 4 7 3 3 6 6 5 3
(3) 3 5 6 2 3 5 7 3 3 ← (4) 5 6 3 3 6 6 5 3
(5) 2 4 3 3 3 6 6 5 3 ← (6) 3 6 3 3 6 6 5 3
(9) 3 6 ..4 5 3 3 3 ← (10) 5 6 2 4 5 3 3 3
(9) 16 4 1 1 * 1 ← (10) 20 1 1 * 1
(11) ....3 5 7 3 3 ← (12) 3 6 2 4 5 3 3 3
(14) ....4 5 3 3 3 ← (19) 6 2 3 4 4 1 1 1
(15) 3 6 2 3 4 4 1 1 1 ← (16) 8 1 1 2 4 3 3 3
(17) 8 4 1 1 * 1 ← (18) 12 1 1 * 1
(21) 4 4 1 1 * 1 ← (24) 1 2 3 4 4 1 1 1
(23) * * 1 ← (26) 4 1 1 * 1

(40, 11)

(3) ..4 3 3 3 6 6 5 3 ← (4) 3 5 6 2 3 5 7 3 3
(6) 3 3 6 6 5 3
(9) .....3 5 7 3 3 ← (10) 3 6 ..4 5 3 3 3
(9) 14 * * 1 ← (10) 16 4 1 1 * 1
(12) .....4 5 3 3 3 ← (18) * 2 4 3 3 3
(15) 2 * 2 4 3 3 3 ← (16) 3 6 2 3 4 4 1 1 1
(17) 6 * * 1 ← (18) 8 4 1 1 * 1
(21) 2 * * 1 ← (22) 4 4 1 1 * 1
(22) 1 * * 1 ← (24) * * 1

(40, 12)

(3) 6 .....3 5 7 3 3
(9) 13 1 * * 1 ← (10) 14 * * 1
(10) ......4 5 3 3 3 ← (17) 1 * 2 4 3 3 3
(15) 1 1 * 2 4 3 3 3 ← (16) 2 * 2 4 3 3 3
(17) 5 1 * * 1 ← (18) 6 * * 1
(21) 1 1 * * 1 ← (22) 2 * * 1

(40, 13)

(1) 5 6 .....4 5 3 3 3
(3) 3 6 .....4 5 3 3 3 ← (4) 6 .....3 5 7 3 3
(7) 8 1 1 * 2 4 3 3 3 ← (16) 1 1 * 2 4 3 3 3
(9) 12 1 1 * * 1 ← (10) 13 1 * * 1
(15) 1 2 3 4 4 1 1 * 1 ← (17) 2 3 4 4 1 1 * 1
(17) 4 1 1 * * 1 ← (18) 5 1 * * 1

(40, 14)

(1) 3 6 ......4 5 3 3 3 ← (2) 5 6 .....4 5 3 3 3
(3) ....3 5 7 3 3 ← (4) 3 6 .....4 5 3 3 3
(6) ....4 5 3 3 3 ← (11) 6 2 3 4 4 1 1 * 1
(7) 3 6 2 3 4 4 1 1 * 1 ← (8) 8 1 1 * 2 4 3 3 3
(9) 8 4 1 1 * * 1 ← (10) 12 1 1 * * 1
(13) 4 4 1 1 * * 1 ← (16) 1 2 3 4 4 1 1 * 1
(15) * * * 1 ← (18) 4 1 1 * * 1

(40, 15)

(1) ....3 5 7 3 3 ← (2) 3 6 ......4 5 3 3 3
(4) ....4 5 3 3 3 ← (10) * * 2 4 3 3 3
(7) 2 * * 2 4 3 3 3 ← (8) 3 6 2 3 4 4 1 1 * 1
(9) 6 * * * 1 ← (10) 8 4 1 1 * * 1
(13) 2 * * * 1 ← (14) 4 4 1 1 * * 1
(14) 1 * * * 1 ← (16) * * * 1

(40, 16)

(1) 13 1 * * * 1 ← (9) 1 * * 2 4 3 3 3
(7) 1 1 * * 2 4 3 3 3 ← (8) 2 * * 2 4 3 3 3
(9) 5 1 * * * 1 ← (10) 6 * * * 1
(13) 1 1 * * * 1 ← (14) 2 * * * 1

(40, 17)

(1) 12 1 1 * * * 1 ← (2) 13 1 * * * 1
(7) 1 2 3 4 4 1 1 * * 1 ← (9) 2 3 4 4 1 1 * * 1
(9) 4 1 1 * * * 1 ← (10) 5 1 * * * 1

(40, 18)

(1) * * * 2 4 3 3 3
(1) 8 4 1 1 * * * 1 ← (2) 12 1 1 * * * 1
(5) 4 4 1 1 * * * 1 ← (8) 1 2 3 4 4 1 1 * * 1
(7) * * * * 1 ← (10) 4 1 1 * * * 1

(40, 19)

(1) 6 * * * * 1 ← (2) 8 4 1 1 * * * 1
(2) 3 4 4 1 1 * * * 1
(5) 2 * * * * 1 ← (6) 4 4 1 1 * * * 1
(6) 1 * * * * 1 ← (8) * * * * 1

(40, 20)

(1) 5 1 * * * * 1 ← (2) 6 * * * * 1
(5) 1 1 * * * * 1 ← (6) 2 * * * * 1

(40, 21)

(1) 4 1 1 * * * * 1 ← (2) 5 1 * * * * 1

(41, 3)

(3) 23 15
(7) 27 7 ← (11) 31
(11) 15 15
(23) 11 7 ← (27) 15
(27) 7 7 ← (35) 7
(39) 1 1 ← (41) 1

(41, 4)

(2) 21 11 7
(3) 22 13 3 ← (4) 23 15
(4) 23 7 7 ← (10) 29 3
(7) 26 5 3 ← (8) 27 7
(10) 13 11 7
(11) 14 13 3 ← (12) 15 15
(20) 7 7 7 ← (26) 13 3
(22) 5 7 7 ← (34) 5 3
(23) 10 5 3 ← (24) 11 7
(27) 6 5 3 ← (28) 7 7
(38) 1 1 1 ← (40) 1 1

(41, 5)

(1) 11 15 7 7
(1) 19 7 7 7
(2) 20 5 7 7 ← (9) 27 3 3
(3) 17 7 7 7
(3) 21 11 3 3 ← (4) 22 13 3
(7) 25 3 3 3 ← (8) 26 5 3
(11) 13 11 3 3 ← (12) 14 13 3
(18) 4 5 7 7 ← (25) 11 3 3
(19) 3 5 7 7 ← (33) 3 3 3
(23) 5 7 3 3 ← (28) 6 5 3
(23) 9 3 3 3 ← (24) 10 5 3

(41, 6)

(1) 7 14 5 7 7 ← (2) 11 15 7 7
(1) 18 3 5 7 7 ← (8) 25 3 3 3
(2) 7 13 5 7 7 ← (4) 17 7 7 7
(3) 20 9 3 3 3 ← (4) 21 11 3 3
(6) 5 9 7 7 7 ← (10) 13 5 7 7
(11) 12 9 3 3 3 ← (12) 13 11 3 3
(17) 2 3 5 7 7 ← (24) 9 3 3 3
(20) 3 5 7 3 3 ← (25) 8 3 3 3
(21) 4 7 3 3 3 ← (22) 6 6 5 3
(23) 4 5 3 3 3 ← (24) 5 7 3 3
(27) 5 1 2 3 3 ← (28) 6 2 3 3

(41, 7)

(1) 5 5 9 7 7 7 ← (2) 7 14 5 7 7
(1) 17 3 6 6 5 3
(3) 3 5 9 7 7 7 ← (5) 14 4 5 7 7
(3) 8 12 9 3 3 3
(7) 5 7 3 5 7 7 ← (11) 9 3 5 7 7
(11) 12 4 5 3 3 3
(15) 3 3 6 6 5 3 ← (19) 3 6 6 5 3
(18) 2 3 5 7 3 3 ← (24) 4 5 3 3 3
(21) 2 4 5 3 3 3 ← (22) 4 7 3 3 3
(27) 3 4 4 1 1 1 ← (28) 5 1 2 3 3
(31) 1 * 1 ← (33) * 1

(41, 8)

(2) 3 5 9 3 5 7 7
(3) 6 9 3 6 6 5 3 ← (4) 8 12 9 3 3 3
(4) 3 5 7 3 5 7 7 ← (8) 5 7 3 5 7 7
(9) 6 3 3 6 6 5 3 ← (16) 3 3 6 6 5 3
(11) 10 2 4 5 3 3 3 ← (12) 12 4 5 3 3 3
(15) 6 2 4 5 3 3 3 ← (22) 2 4 5 3 3 3
(30) 1 1 * 1 ← (32) 1 * 1

(41, 9)

(1) 5 6 5 2 3 5 7 7
(3) 3 6 5 2 3 5 7 7 ← (4) 6 9 3 6 6 5 3
(7) 5 6 2 3 5 7 3 3 ← (13) 6 2 3 5 7 3 3
(9) 3 6 2 3 5 7 3 3 ← (10) 6 3 3 6 6 5 3
(11) 7 13 1 * 1 ← (12) 10 2 4 5 3 3 3
(13) 6 ..4 5 3 3 3 ← (19) 13 1 * 1
(15) 4 ..4 5 3 3 3 ← (16) 6 2 4 5 3 3 3

(41, 10)

(2) 2 4 7 3 3 6 6 5 3
(3) 2 3 5 5 3 6 6 5 3 ← (4) 3 6 5 2 3 5 7 7
(6) 2 4 3 3 3 6 6 5 3 ← (11) 5 6 2 4 5 3 3 3
(7) 3 5 6 2 4 5 3 3 3 ← (8) 5 6 2 3 5 7 3 3
(9) ...3 3 6 6 5 3 ← (10) 3 6 2 3 5 7 3 3
(11) 5 8 1 1 2 4 3 3 3 ← (12) 7 13 1 * 1
(12) ....3 5 7 3 3 ← (17) 8 1 1 2 4 3 3 3
(13) 4 ...4 5 3 3 3 ← (14) 6 ..4 5 3 3 3
(15) ....4 5 3 3 3 ← (16) 4 ..4 5 3 3 3
(19) 5 1 2 3 4 4 1 1 1 ← (20) 6 2 3 4 4 1 1 1

(41, 11)

(1) 1 2 4 7 3 3 6 6 5 3
(4) ..4 3 3 3 6 6 5 3 ← (10) ...3 3 6 6 5 3
(7) ....3 3 6 6 5 3 ← (8) 3 5 6 2 4 5 3 3 3
(10) .....3 5 7 3 3 ← (16) 4 5 3 3 3
(11) 4 ....4 5 3 3 3 ← (12) 5 8 1 1 2 4 3 3 3
(13) .....4 5 3 3 3 ← (14) 4 ...4 5 3 3 3
(19) 3 4 4 1 1 * 1 ← (20) 5 1 2 3 4 4 1 1 1
(23) 1 * * 1 ← (25) * * 1

(41, 12)

(1) 6 ....3 3 6 6 5 3 ← (8) 3 3 6 6 5 3
(7) 6 .....4 5 3 3 3 ← (14) 4 5 3 3 3
(11) ......4 5 3 3 3 ← (12) 4 ....4 5 3 3 3
(22) 1 1 * * 1 ← (24) 1 * * 1

(41, 13)

(1) 3 6 .....3 5 7 3 3 ← (2) 6 ....3 3 6 6 5 3
(5) 6 ......4 5 3 3 3 ← (12) 4 5 3 3 3
(7) 4 ......4 5 3 3 3 ← (8) 6 .....4 5 3 3 3

(41, 14)

(1) ....3 3 6 6 5 3 ← (2) 3 6 .....3 5 7 3 3
(4) ....3 5 7 3 3 ← (9) 8 1 1 * 2 4 3 3 3
(5) 4 ....4 5 3 3 3 ← (6) 6 ......4 5 3 3 3
(7) ....4 5 3 3 3 ← (8) 4 ......4 5 3 3 3
(11) 5 1 2 3 4 4 1 1 * 1 ← (12) 6 2 3 4 4 1 1 * 1

(41, 15)

(2) ....3 5 7 3 3 ← (8) 4 5 3 3 3
(5) ....4 5 3 3 3 ← (6) 4 ....4 5 3 3 3
(11) 3 4 4 1 1 * * 1 ← (12) 5 1 2 3 4 4 1 1 * 1
(15) 1 * * * 1 ← (17) * * * 1

(41, 16)

(8) 1 1 * * 2 4 3 3 3
(14) 1 1 * * * 1 ← (16) 1 * * * 1

(41, 17)

(3) 6 2 3 4 4 1 1 * * 1

(41, 18)

(2) * * * 2 4 3 3 3
(3) 5 1 2 3 4 4 1 1 * * 1 ← (4) 6 2 3 4 4 1 1 * * 1

(41, 19)

(1) 1 * * * 2 4 3 3 3
(3) 3 4 4 1 1 * * * 1 ← (4) 5 1 2 3 4 4 1 1 * * 1
(7) 1 * * * * 1 ← (9) * * * * 1

(41, 20)

(1) 2 3 4 4 1 1 * * * 1
(6) 1 1 * * * * 1 ← (8) 1 * * * * 1

(41, 21)

(2) 4 1 1 * * * * 1

(42, 2)

(39) 3 ← (43)

(42, 3)

(11) 30 1 ← (12) 31
(27) 14 1 ← (28) 15
(35) 6 1 ← (36) 7
(36) 3 3 ← (42) 1
(39) 2 1 ← (40) 3

(42, 4)

(1) 11 15 15
(3) 21 11 7 ← (5) 23 15
(5) 23 7 7 ← (9) 27 7
(11) 13 11 7 ← (13) 15 15
(11) 29 1 1 ← (12) 30 1
(21) 7 7 7 ← (25) 11 7
(23) 5 7 7 ← (29) 7 7
(27) 13 1 1 ← (28) 14 1
(34) 2 3 3 ← (41) 1 1
(35) 5 1 1 ← (36) 6 1
(39) 1 1 1 ← (40) 2 1

(42, 5)

(1) 10 13 11 7 ← (2) 11 15 15
(2) 19 7 7 7 ← (4) 21 11 7
(3) 20 5 7 7 ← (6) 23 7 7
(9) 14 5 7 7 ← (12) 13 11 7
(11) 28 1 1 1 ← (12) 29 1 1
(19) 4 5 7 7 ← (22) 7 7 7
(20) 3 5 7 7 ← (24) 5 7 7
(27) 12 1 1 1 ← (28) 13 1 1
(33) 1 2 3 3 ← (40) 1 1 1
(35) 4 1 1 1 ← (36) 5 1 1

(42, 6)

(1) 9 11 7 7 7 ← (2) 10 13 11 7
(2) 18 3 5 7 7 ← (4) 20 5 7 7
(3) 7 13 5 7 7 ← (5) 17 7 7 7
(4) 20 9 3 3 3
(7) 5 9 7 7 7 ← (11) 13 5 7 7
(9) 11 3 5 7 7 ← (10) 14 5 7 7
(11) 24 4 1 1 1 ← (12) 28 1 1 1
(12) 12 9 3 3 3
(18) 2 3 5 7 7 ← (20) 4 5 7 7
(21) 3 5 7 3 3 ← (25) 5 7 3 3
(25) 3 6 2 3 3 ← (26) 8 3 3 3
(27) 2 4 3 3 3 ← (29) 6 2 3 3
(27) 8 4 1 1 1 ← (28) 12 1 1 1
(31) 4 4 1 1 1 ← (36) 4 1 1 1

(42, 7)

(2) 5 5 9 7 7 7 ← (4) 7 13 5 7 7
(2) 17 3 6 6 5 3
(4) 3 5 9 7 7 7 ← (8) 5 9 7 7 7
(5) 13 2 3 5 7 7 ← (6) 14 4 5 7 7
(6) 5 9 3 5 7 7 ← (12) 9 3 5 7 7
(10) 9 3 6 6 5 3
(11) 22 * 1 ← (12) 24 4 1 1 1
(19) 2 3 5 7 3 3 ← (20) 3 6 6 5 3
(25) ..4 3 3 3 ← (26) 3 6 2 3 3
(26) 1 2 4 3 3 3 ← (28) 2 4 3 3 3
(27) 6 * 1 ← (28) 8 4 1 1 1
(28) 3 4 4 1 1 1 ← (34) * 1
(31) 2 * 1 ← (32) 4 4 1 1 1

(42, 8)

(1) 11 12 4 5 3 3 3
(3) 3 5 9 3 5 7 7 ← (5) 8 12 9 3 3 3
(5) 3 5 7 3 5 7 7 ← (9) 5 7 3 5 7 7
(5) 7 12 4 5 3 3 3 ← (6) 13 2 3 5 7 7
(7) 6 5 2 3 5 7 7 ← (13) 12 4 5 3 3 3
(9) 5 5 3 6 6 5 3
(11) 21 1 * 1 ← (12) 22 * 1
(25) 1 1 2 4 3 3 3 ← (26) ..4 3 3 3
(26) 2 3 4 4 1 1 1 ← (33) 1 * 1
(27) 5 1 * 1 ← (28) 6 * 1
(31) 1 1 * 1 ← (32) 2 * 1

(42, 9)

(2) 5 6 5 2 3 5 7 7
(5) 4 7 3 3 6 6 5 3 ← (11) 6 3 3 6 6 5 3
(5) 5 6 3 3 6 6 5 3 ← (6) 7 12 4 5 3 3 3
(7) 3 6 3 3 6 6 5 3 ← (8) 6 5 2 3 5 7 7
(11) 20 1 1 * 1 ← (12) 21 1 * 1
(13) 3 6 2 4 5 3 3 3 ← (14) 6 2 3 5 7 3 3
(19) 12 1 1 * 1 ← (20) 13 1 * 1
(25) 1 2 3 4 4 1 1 1 ← (32) 1 1 * 1
(27) 4 1 1 * 1 ← (28) 5 1 * 1

(42, 10)

(3) 2 4 7 3 3 6 6 5 3 ← (6) 4 7 3 3 6 6 5 3
(4) 2 3 5 5 3 6 6 5 3 ← (9) 5 6 2 3 5 7 3 3
(5) 3 5 6 2 3 5 7 3 3 ← (6) 5 6 3 3 6 6 5 3
(7) 2 4 3 3 3 6 6 5 3 ← (8) 3 6 3 3 6 6 5 3
(11) 3 6 ..4 5 3 3 3 ← (12) 5 6 2 4 5 3 3 3
(11) 16 4 1 1 * 1 ← (12) 20 1 1 * 1
(13) ....3 5 7 3 3 ← (14) 3 6 2 4 5 3 3 3
(17) 3 6 2 3 4 4 1 1 1 ← (18) 8 1 1 2 4 3 3 3
(19) * 2 4 3 3 3 ← (21) 6 2 3 4 4 1 1 1
(19) 8 4 1 1 * 1 ← (20) 12 1 1 * 1
(23) 4 4 1 1 * 1 ← (28) 4 1 1 * 1

(42, 11)

(1) ..4 7 3 3 6 6 5 3 ← (8) 2 4 3 3 3 6 6 5 3
(2) 1 2 4 7 3 3 6 6 5 3 ← (4) 2 4 7 3 3 6 6 5 3
(5) ..4 3 3 3 6 6 5 3 ← (6) 3 5 6 2 3 5 7 3 3
(11) .....3 5 7 3 3 ← (12) 3 6 ..4 5 3 3 3
(11) 14 * * 1 ← (12) 16 4 1 1 * 1
(17) 2 * 2 4 3 3 3 ← (18) 3 6 2 3 4 4 1 1 1
(18) 1 * 2 4 3 3 3 ← (20) * 2 4 3 3 3
(19) 6 * * 1 ← (20) 8 4 1 1 * 1
(20) 3 4 4 1 1 * 1 ← (26) * * 1
(23) 2 * * 1 ← (24) 4 4 1 1 * 1

(42, 12)

(1) 1 1 2 4 7 3 3 6 6 5 3 ← (2) ..4 7 3 3 6 6 5 3
(5) 6 .....3 5 7 3 3
(11) 13 1 * * 1 ← (12) 14 * * 1
(17) 1 1 * 2 4 3 3 3 ← (18) 2 * 2 4 3 3 3
(18) 2 3 4 4 1 1 * 1 ← (25) 1 * * 1
(19) 5 1 * * 1 ← (20) 6 * * 1
(23) 1 1 * * 1 ← (24) 2 * * 1

(42, 13)

(3) 5 6 .....4 5 3 3 3
(5) 3 6 .....4 5 3 3 3 ← (6) 6 .....3 5 7 3 3
(11) 12 1 1 * * 1 ← (12) 13 1 * * 1
(17) 1 2 3 4 4 1 1 * 1 ← (24) 1 1 * * 1
(19) 4 1 1 * * 1 ← (20) 5 1 * * 1

(42, 14)

(2) 3 3 6 6 5 3
(3) 3 6 ......4 5 3 3 3 ← (4) 5 6 .....4 5 3 3 3
(5) ....3 5 7 3 3 ← (6) 3 6 .....4 5 3 3 3
(9) 3 6 2 3 4 4 1 1 * 1 ← (10) 8 1 1 * 2 4 3 3 3
(11) * * 2 4 3 3 3 ← (13) 6 2 3 4 4 1 1 * 1
(11) 8 4 1 1 * * 1 ← (12) 12 1 1 * * 1
(15) 4 4 1 1 * * 1 ← (20) 4 1 1 * * 1

(42, 15)

(3) ....3 5 7 3 3 ← (4) 3 6 ......4 5 3 3 3
(6) 4 5 3 3 3
(9) 2 * * 2 4 3 3 3 ← (10) 3 6 2 3 4 4 1 1 * 1
(10) 1 * * 2 4 3 3 3 ← (12) * * 2 4 3 3 3
(11) 6 * * * 1 ← (12) 8 4 1 1 * * 1
(12) 3 4 4 1 1 * * 1 ← (18) * * * 1
(15) 2 * * * 1 ← (16) 4 4 1 1 * * 1

(42, 16)

(3) 13 1 * * * 1
(9) 1 1 * * 2 4 3 3 3 ← (10) 2 * * 2 4 3 3 3
(10) 2 3 4 4 1 1 * * 1 ← (17) 1 * * * 1
(11) 5 1 * * * 1 ← (12) 6 * * * 1
(15) 1 1 * * * 1 ← (16) 2 * * * 1

(42, 17)

(1) 8 1 1 * * 2 4 3 3 3
(3) 12 1 1 * * * 1 ← (4) 13 1 * * * 1
(9) 1 2 3 4 4 1 1 * * 1 ← (16) 1 1 * * * 1
(11) 4 1 1 * * * 1 ← (12) 5 1 * * * 1

(42, 18)

(1) 3 6 2 3 4 4 1 1 * * 1 ← (2) 8 1 1 * * 2 4 3 3 3
(3) * * * 2 4 3 3 3 ← (5) 6 2 3 4 4 1 1 * * 1
(3) 8 4 1 1 * * * 1 ← (4) 12 1 1 * * * 1
(7) 4 4 1 1 * * * 1 ← (12) 4 1 1 * * * 1

(42, 19)

(1) 2 * * * 2 4 3 3 3 ← (2) 3 6 2 3 4 4 1 1 * * 1
(2) 1 * * * 2 4 3 3 3 ← (4) * * * 2 4 3 3 3
(3) 6 * * * * 1 ← (4) 8 4 1 1 * * * 1
(4) 3 4 4 1 1 * * * 1 ← (10) * * * * 1
(7) 2 * * * * 1 ← (8) 4 4 1 1 * * * 1

(42, 20)

(1) 1 1 * * * 2 4 3 3 3 ← (2) 2 * * * 2 4 3 3 3
(2) 2 3 4 4 1 1 * * * 1 ← (9) 1 * * * * 1
(3) 5 1 * * * * 1 ← (4) 6 * * * * 1
(7) 1 1 * * * * 1 ← (8) 2 * * * * 1

(42, 21)

(1) 1 2 3 4 4 1 1 * * * 1 ← (8) 1 1 * * * * 1
(3) 4 1 1 * * * * 1 ← (4) 5 1 * * * * 1

(42, 22)

(1) * * * * * 1

(43, 3)

(11) 29 3 ← (13) 31
(27) 13 3 ← (29) 15
(35) 5 3 ← (37) 7
(37) 3 3 ← (41) 3

(43, 4)

(5) 22 13 3 ← (6) 23 15
(9) 26 5 3 ← (10) 27 7
(10) 27 3 3 ← (12) 29 3
(13) 14 13 3 ← (14) 15 15
(25) 10 5 3 ← (26) 11 7
(26) 11 3 3 ← (28) 13 3
(29) 6 5 3 ← (30) 7 7
(34) 3 3 3 ← (36) 5 3
(35) 2 3 3 ← (38) 3 3

(43, 5)

(3) 11 15 7 7
(3) 19 7 7 7 ← (5) 21 11 7
(5) 21 11 3 3 ← (6) 22 13 3
(9) 25 3 3 3 ← (10) 26 5 3
(13) 13 11 3 3 ← (14) 14 13 3
(21) 3 5 7 7 ← (25) 5 7 7
(23) 6 6 5 3 ← (30) 6 5 3
(25) 9 3 3 3 ← (26) 10 5 3
(34) 1 2 3 3 ← (36) 2 3 3

(43, 6)

(2) 9 11 7 7 7 ← (6) 17 7 7 7
(3) 7 14 5 7 7 ← (4) 11 15 7 7
(3) 18 3 5 7 7 ← (5) 20 5 7 7
(5) 20 9 3 3 3 ← (6) 21 11 3 3
(10) 11 3 5 7 7 ← (12) 13 5 7 7
(13) 12 9 3 3 3 ← (14) 13 11 3 3
(19) 2 3 5 7 7 ← (21) 4 5 7 7
(22) 3 5 7 3 3 ← (27) 8 3 3 3
(23) 4 7 3 3 3 ← (24) 6 6 5 3
(25) 4 5 3 3 3 ← (26) 5 7 3 3
(29) 5 1 2 3 3 ← (30) 6 2 3 3

(43, 7)

(1) 12 12 9 3 3 3 ← (4) 18 3 5 7 7
(3) 5 5 9 7 7 7 ← (4) 7 14 5 7 7
(3) 17 3 6 6 5 3 ← (6) 20 9 3 3 3
(5) 3 5 9 7 7 7 ← (9) 5 9 7 7 7
(7) 5 9 3 5 7 7 ← (13) 9 3 5 7 7
(11) 9 3 6 6 5 3 ← (14) 12 9 3 3 3
(17) 3 3 6 6 5 3 ← (20) 2 3 5 7 7
(20) 2 3 5 7 3 3 ← (26) 4 5 3 3 3
(23) 2 4 5 3 3 3 ← (24) 4 7 3 3 3
(27) 1 2 4 3 3 3 ← (29) 2 4 3 3 3
(29) 3 4 4 1 1 1 ← (30) 5 1 2 3 3

(43, 8)

(1) 10 9 3 6 6 5 3 ← (2) 12 12 9 3 3 3
(2) 11 12 4 5 3 3 3 ← (4) 17 3 6 6 5 3
(4) 3 5 9 3 5 7 7 ← (10) 5 7 3 5 7 7
(5) 6 9 3 6 6 5 3 ← (6) 8 12 9 3 3 3
(6) 3 5 7 3 5 7 7 ← (8) 5 9 3 5 7 7
(10) 5 5 3 6 6 5 3 ← (12) 9 3 6 6 5 3
(13) 10 2 4 5 3 3 3 ← (14) 12 4 5 3 3 3
(17) 6 2 4 5 3 3 3 ← (24) 2 4 5 3 3 3
(26) 1 1 2 4 3 3 3 ← (28) 1 2 4 3 3 3
(27) 2 3 4 4 1 1 1 ← (30) 3 4 4 1 1 1

(43, 9)

(1) 9 5 5 3 6 6 5 3 ← (2) 10 9 3 6 6 5 3
(3) 5 6 5 2 3 5 7 7 ← (9) 6 5 2 3 5 7 7
(5) 3 6 5 2 3 5 7 7 ← (6) 6 9 3 6 6 5 3
(11) 3 6 2 3 5 7 3 3 ← (12) 6 3 3 6 6 5 3
(13) 7 13 1 * 1 ← (14) 10 2 4 5 3 3 3
(15) 6 ..4 5 3 3 3 ← (21) 13 1 * 1
(17) 4 ..4 5 3 3 3 ← (18) 6 2 4 5 3 3 3
(26) 1 2 3 4 4 1 1 1 ← (28) 2 3 4 4 1 1 1

(43, 10)

(1) 4 5 5 5 3 6 6 5 3 ← (4) 5 6 5 2 3 5 7 7
(5) 2 3 5 5 3 6 6 5 3 ← (6) 3 6 5 2 3 5 7 7
(9) 3 5 6 2 4 5 3 3 3 ← (10) 5 6 2 3 5 7 3 3
(11) ...3 3 6 6 5 3 ← (12) 3 6 2 3 5 7 3 3
(13) 5 8 1 1 2 4 3 3 3 ← (14) 7 13 1 * 1
(14) ....3 5 7 3 3 ← (19) 8 1 1 2 4 3 3 3
(15) 4 ...4 5 3 3 3 ← (16) 6 ..4 5 3 3 3
(17) ....4 5 3 3 3 ← (18) 4 ..4 5 3 3 3
(21) 5 1 2 3 4 4 1 1 1 ← (22) 6 2 3 4 4 1 1 1

(43, 11)

(3) 1 2 4 7 3 3 6 6 5 3 ← (5) 2 4 7 3 3 6 6 5 3
(6) ..4 3 3 3 6 6 5 3
(9) ....3 3 6 6 5 3 ← (10) 3 5 6 2 4 5 3 3 3
(12) .....3 5 7 3 3 ← (18) 4 5 3 3 3
(13) 4 ....4 5 3 3 3 ← (14) 5 8 1 1 2 4 3 3 3
(15) .....4 5 3 3 3 ← (16) 4 ...4 5 3 3 3
(19) 1 * 2 4 3 3 3 ← (21) * 2 4 3 3 3
(21) 3 4 4 1 1 * 1 ← (22) 5 1 2 3 4 4 1 1 1

(43, 12)

(2) 1 1 2 4 7 3 3 6 6 5 3 ← (4) 1 2 4 7 3 3 6 6 5 3
(3) 6 ....3 3 6 6 5 3
(9) 6 .....4 5 3 3 3 ← (16) 4 5 3 3 3
(13) ......4 5 3 3 3 ← (14) 4 ....4 5 3 3 3
(18) 1 1 * 2 4 3 3 3 ← (20) 1 * 2 4 3 3 3
(19) 2 3 4 4 1 1 * 1 ← (22) 3 4 4 1 1 * 1

(43, 13)

(1) 5 6 .....3 5 7 3 3
(3) 3 6 .....3 5 7 3 3 ← (4) 6 ....3 3 6 6 5 3
(7) 6 ......4 5 3 3 3 ← (14) 4 5 3 3 3
(9) 4 ......4 5 3 3 3 ← (10) 6 .....4 5 3 3 3
(18) 1 2 3 4 4 1 1 * 1 ← (20) 2 3 4 4 1 1 * 1

(43, 14)

(1) 3 5 6 .....4 5 3 3 3 ← (2) 5 6 .....3 5 7 3 3
(3) ....3 3 6 6 5 3 ← (4) 3 6 .....3 5 7 3 3
(6) ....3 5 7 3 3 ← (11) 8 1 1 * 2 4 3 3 3
(7) 4 ....4 5 3 3 3 ← (8) 6 ......4 5 3 3 3
(9) ....4 5 3 3 3 ← (10) 4 ......4 5 3 3 3
(13) 5 1 2 3 4 4 1 1 * 1 ← (14) 6 2 3 4 4 1 1 * 1

(43, 15)

(1) ....3 3 6 6 5 3 ← (2) 3 5 6 .....4 5 3 3 3
(4) ....3 5 7 3 3 ← (10) 4 5 3 3 3
(7) ....4 5 3 3 3 ← (8) 4 ....4 5 3 3 3
(11) 1 * * 2 4 3 3 3 ← (13) * * 2 4 3 3 3
(13) 3 4 4 1 1 * * 1 ← (14) 5 1 2 3 4 4 1 1 * 1

(43, 16)

(1) 6 ....4 5 3 3 3 ← (8) 4 5 3 3 3
(10) 1 1 * * 2 4 3 3 3 ← (12) 1 * * 2 4 3 3 3
(11) 2 3 4 4 1 1 * * 1 ← (14) 3 4 4 1 1 * * 1

(43, 17)

(1) 4 ....4 5 3 3 3 ← (2) 6 ....4 5 3 3 3
(10) 1 2 3 4 4 1 1 * * 1 ← (12) 2 3 4 4 1 1 * * 1

(43, 18)

(1) ....4 5 3 3 3 ← (2) 4 ....4 5 3 3 3
(5) 5 1 2 3 4 4 1 1 * * 1 ← (6) 6 2 3 4 4 1 1 * * 1

(43, 19)

(3) 1 * * * 2 4 3 3 3 ← (5) * * * 2 4 3 3 3
(5) 3 4 4 1 1 * * * 1 ← (6) 5 1 2 3 4 4 1 1 * * 1

(43, 20)

(2) 1 1 * * * 2 4 3 3 3 ← (4) 1 * * * 2 4 3 3 3
(3) 2 3 4 4 1 1 * * * 1 ← (6) 3 4 4 1 1 * * * 1

(43, 21)

(2) 1 2 3 4 4 1 1 * * * 1 ← (4) 2 3 4 4 1 1 * * * 1
(4) 4 1 1 * * * * 1

(43, 22)

(2) * * * * * 1

(43, 23)

(1) 1 * * * * * 1

(44, 2)

(43) 1 ← (45)

(44, 3)

(13) 30 1 ← (14) 31
(29) 14 1 ← (30) 15
(37) 6 1 ← (38) 7
(41) 2 1 ← (42) 3
(42) 1 1 ← (44) 1

(44, 4)

(3) 11 15 15
(7) 23 7 7 ← (11) 27 7
(11) 27 3 3 ← (13) 29 3
(13) 13 11 7
(13) 29 1 1 ← (14) 30 1
(23) 7 7 7 ← (27) 11 7
(27) 11 3 3 ← (29) 13 3
(29) 13 1 1 ← (30) 14 1
(35) 3 3 3 ← (37) 5 3
(37) 5 1 1 ← (38) 6 1
(41) 1 1 1 ← (42) 2 1

(44, 5)

(3) 10 13 11 7 ← (4) 11 15 15
(4) 19 7 7 7 ← (8) 23 7 7
(10) 25 3 3 3 ← (12) 27 3 3
(11) 14 5 7 7
(13) 28 1 1 1 ← (14) 29 1 1
(22) 3 5 7 7 ← (26) 5 7 7
(26) 9 3 3 3 ← (28) 11 3 3
(29) 12 1 1 1 ← (30) 13 1 1
(35) 1 2 3 3 ← (37) 2 3 3
(37) 4 1 1 1 ← (38) 5 1 1

(44, 6)

(3) 9 11 7 7 7 ← (4) 10 13 11 7
(5) 7 13 5 7 7
(7) 14 4 5 7 7
(11) 11 3 5 7 7 ← (12) 14 5 7 7
(13) 24 4 1 1 1 ← (14) 28 1 1 1
(21) 3 6 6 5 3 ← (27) 5 7 3 3
(23) 3 5 7 3 3 ← (25) 6 6 5 3
(27) 3 6 2 3 3 ← (28) 8 3 3 3
(29) 8 4 1 1 1 ← (30) 12 1 1 1
(33) 4 4 1 1 1 ← (36) 1 2 3 3
(35) * 1 ← (38) 4 1 1 1

(44, 7)

(4) 5 5 9 7 7 7
(6) 3 5 9 7 7 7 ← (10) 5 9 7 7 7
(7) 13 2 3 5 7 7 ← (8) 14 4 5 7 7
(13) 22 * 1 ← (14) 24 4 1 1 1
(18) 3 3 6 6 5 3 ← (24) 3 5 7 3 3
(21) 2 3 5 7 3 3 ← (22) 3 6 6 5 3
(27) ..4 3 3 3 ← (28) 3 6 2 3 3
(29) 6 * 1 ← (30) 8 4 1 1 1
(33) 2 * 1 ← (34) 4 4 1 1 1
(34) 1 * 1 ← (36) * 1

(44, 8)

(3) 11 12 4 5 3 3 3 ← (5) 17 3 6 6 5 3
(5) 3 5 9 3 5 7 7 ← (11) 5 7 3 5 7 7
(7) 3 5 7 3 5 7 7 ← (9) 5 9 3 5 7 7
(7) 7 12 4 5 3 3 3 ← (8) 13 2 3 5 7 7
(11) 5 5 3 6 6 5 3 ← (13) 9 3 6 6 5 3
(13) 21 1 * 1 ← (14) 22 * 1
(15) 6 2 3 5 7 3 3 ← (22) 2 3 5 7 3 3
(27) 1 1 2 4 3 3 3 ← (28) ..4 3 3 3
(29) 5 1 * 1 ← (30) 6 * 1
(33) 1 1 * 1 ← (34) 2 * 1

(44, 9)

(2) 9 5 5 3 6 6 5 3 ← (4) 11 12 4 5 3 3 3
(7) 4 7 3 3 6 6 5 3 ← (12) 5 5 3 6 6 5 3
(7) 5 6 3 3 6 6 5 3 ← (8) 7 12 4 5 3 3 3
(9) 3 6 3 3 6 6 5 3 ← (10) 6 5 2 3 5 7 7
(13) 5 6 2 4 5 3 3 3 ← (19) 6 2 4 5 3 3 3
(13) 20 1 1 * 1 ← (14) 21 1 * 1
(15) 3 6 2 4 5 3 3 3 ← (16) 6 2 3 5 7 3 3
(21) 12 1 1 * 1 ← (22) 13 1 * 1
(27) 1 2 3 4 4 1 1 1 ← (29) 2 3 4 4 1 1 1
(29) 4 1 1 * 1 ← (30) 5 1 * 1

(44, 10)

(2) 4 5 5 5 3 6 6 5 3
(6) 2 3 5 5 3 6 6 5 3 ← (8) 4 7 3 3 6 6 5 3
(7) 3 5 6 2 3 5 7 3 3 ← (8) 5 6 3 3 6 6 5 3
(9) 2 4 3 3 3 6 6 5 3 ← (10) 3 6 3 3 6 6 5 3
(12) ...3 3 6 6 5 3 ← (17) 6 ..4 5 3 3 3
(13) 3 6 ..4 5 3 3 3 ← (14) 5 6 2 4 5 3 3 3
(13) 16 4 1 1 * 1 ← (14) 20 1 1 * 1
(15) ....3 5 7 3 3 ← (16) 3 6 2 4 5 3 3 3
(19) 3 6 2 3 4 4 1 1 1 ← (20) 8 1 1 2 4 3 3 3
(21) 8 4 1 1 * 1 ← (22) 12 1 1 * 1
(25) 4 4 1 1 * 1 ← (28) 1 2 3 4 4 1 1 1
(27) * * 1 ← (30) 4 1 1 * 1

(44, 11)

(3) ..4 7 3 3 6 6 5 3 ← (6) 2 4 7 3 3 6 6 5 3
(7) ..4 3 3 3 6 6 5 3 ← (8) 3 5 6 2 3 5 7 3 3
(10) ....3 3 6 6 5 3 ← (16) 3 5 7 3 3
(13) .....3 5 7 3 3 ← (14) 3 6 ..4 5 3 3 3
(13) 14 * * 1 ← (14) 16 4 1 1 * 1
(19) 2 * 2 4 3 3 3 ← (20) 3 6 2 3 4 4 1 1 1
(21) 6 * * 1 ← (22) 8 4 1 1 * 1
(25) 2 * * 1 ← (26) 4 4 1 1 * 1
(26) 1 * * 1 ← (28) * * 1

(44, 12)

(1) 6 ..4 3 3 3 6 6 5 3 ← (5) 1 2 4 7 3 3 6 6 5 3
(3) 1 1 2 4 7 3 3 6 6 5 3 ← (4) ..4 7 3 3 6 6 5 3
(7) 6 .....3 5 7 3 3 ← (14) 3 5 7 3 3
(13) 13 1 * * 1 ← (14) 14 * * 1
(19) 1 1 * 2 4 3 3 3 ← (20) 2 * 2 4 3 3 3
(21) 5 1 * * 1 ← (22) 6 * * 1
(25) 1 1 * * 1 ← (26) 2 * * 1

(44, 13)

(1) 3 6 ....3 3 6 6 5 3 ← (2) 6 ..4 3 3 3 6 6 5 3
(5) 5 6 .....4 5 3 3 3 ← (11) 6 .....4 5 3 3 3
(7) 3 6 .....4 5 3 3 3 ← (8) 6 .....3 5 7 3 3
(13) 12 1 1 * * 1 ← (14) 13 1 * * 1
(19) 1 2 3 4 4 1 1 * 1 ← (21) 2 3 4 4 1 1 * 1
(21) 4 1 1 * * 1 ← (22) 5 1 * * 1

(44, 14)

(1) .....4 3 3 3 6 6 5 3 ← (2) 3 6 ....3 3 6 6 5 3
(4) ....3 3 6 6 5 3 ← (9) 6 ......4 5 3 3 3
(5) 3 6 ......4 5 3 3 3 ← (6) 5 6 .....4 5 3 3 3
(7) ....3 5 7 3 3 ← (8) 3 6 .....4 5 3 3 3
(11) 3 6 2 3 4 4 1 1 * 1 ← (12) 8 1 1 * 2 4 3 3 3
(13) 8 4 1 1 * * 1 ← (14) 12 1 1 * * 1
(17) 4 4 1 1 * * 1 ← (20) 1 2 3 4 4 1 1 * 1
(19) * * * 1 ← (22) 4 1 1 * * 1

(44, 15)

(2) ....3 3 6 6 5 3 ← (8) 3 5 7 3 3
(5) ....3 5 7 3 3 ← (6) 3 6 ......4 5 3 3 3
(11) 2 * * 2 4 3 3 3 ← (12) 3 6 2 3 4 4 1 1 * 1
(13) 6 * * * 1 ← (14) 8 4 1 1 * * 1
(17) 2 * * * 1 ← (18) 4 4 1 1 * * 1
(18) 1 * * * 1 ← (20) * * * 1

(44, 16)

(5) 13 1 * * * 1
(11) 1 1 * * 2 4 3 3 3 ← (12) 2 * * 2 4 3 3 3
(13) 5 1 * * * 1 ← (14) 6 * * * 1
(17) 1 1 * * * 1 ← (18) 2 * * * 1

(44, 17)

(3) 8 1 1 * * 2 4 3 3 3
(5) 12 1 1 * * * 1 ← (6) 13 1 * * * 1
(11) 1 2 3 4 4 1 1 * * 1 ← (13) 2 3 4 4 1 1 * * 1
(13) 4 1 1 * * * 1 ← (14) 5 1 * * * 1

(44, 18)

(2) 4 5 3 3 3
(3) 3 6 2 3 4 4 1 1 * * 1 ← (4) 8 1 1 * * 2 4 3 3 3
(5) 8 4 1 1 * * * 1 ← (6) 12 1 1 * * * 1
(9) 4 4 1 1 * * * 1 ← (12) 1 2 3 4 4 1 1 * * 1
(11) * * * * 1 ← (14) 4 1 1 * * * 1

(44, 19)

(3) 2 * * * 2 4 3 3 3 ← (4) 3 6 2 3 4 4 1 1 * * 1
(5) 6 * * * * 1 ← (6) 8 4 1 1 * * * 1
(9) 2 * * * * 1 ← (10) 4 4 1 1 * * * 1
(10) 1 * * * * 1 ← (12) * * * * 1

(44, 20)

(3) 1 1 * * * 2 4 3 3 3 ← (4) 2 * * * 2 4 3 3 3
(5) 5 1 * * * * 1 ← (6) 6 * * * * 1
(9) 1 1 * * * * 1 ← (10) 2 * * * * 1

(44, 21)

(3) 1 2 3 4 4 1 1 * * * 1 ← (5) 2 3 4 4 1 1 * * * 1
(5) 4 1 1 * * * * 1 ← (6) 5 1 * * * * 1

(44, 22)

(1) 4 4 1 1 * * * * 1 ← (4) 1 2 3 4 4 1 1 * * * 1
(3) * * * * * 1 ← (6) 4 1 1 * * * * 1

(44, 23)

(1) 2 * * * * * 1 ← (2) 4 4 1 1 * * * * 1
(2) 1 * * * * * 1 ← (4) * * * * * 1

(44, 24)

(1) 1 1 * * * * * 1 ← (2) 2 * * * * * 1

(45, 3)

(7) 23 15
(15) 15 15
(31) 7 7 ← (39) 7
(39) 3 3 ← (43) 3
(43) 1 1 ← (45) 1

(45, 4)

(6) 21 11 7
(7) 22 13 3 ← (8) 23 15
(11) 26 5 3 ← (12) 27 7
(14) 13 11 7
(15) 14 13 3 ← (16) 15 15
(24) 7 7 7 ← (38) 5 3
(27) 10 5 3 ← (28) 11 7
(31) 6 5 3 ← (32) 7 7
(36) 3 3 3 ← (40) 3 3
(42) 1 1 1 ← (44) 1 1

(45, 5)

(1) 13 13 11 7
(5) 11 15 7 7
(5) 19 7 7 7 ← (9) 23 7 7
(6) 20 5 7 7
(7) 17 7 7 7
(7) 21 11 3 3 ← (8) 22 13 3
(11) 25 3 3 3 ← (12) 26 5 3
(13) 13 5 7 7
(15) 13 11 3 3 ← (16) 14 13 3
(22) 4 5 7 7 ← (32) 6 5 3
(23) 3 5 7 7 ← (27) 5 7 7
(27) 9 3 3 3 ← (28) 10 5 3
(31) 6 2 3 3 ← (38) 2 3 3

(45, 6)

(1) 11 14 5 7 7 ← (2) 13 13 11 7
(4) 9 11 7 7 7
(5) 7 14 5 7 7 ← (6) 11 15 7 7
(5) 18 3 5 7 7
(6) 7 13 5 7 7 ← (8) 17 7 7 7
(7) 20 9 3 3 3 ← (8) 21 11 3 3
(12) 11 3 5 7 7
(14) 9 3 5 7 7 ← (24) 3 5 7 7
(15) 12 9 3 3 3 ← (16) 13 11 3 3
(21) 2 3 5 7 7 ← (29) 8 3 3 3
(25) 4 7 3 3 3 ← (26) 6 6 5 3
(27) 4 5 3 3 3 ← (28) 5 7 3 3
(30) 2 4 3 3 3 ← (37) 1 2 3 3
(31) 5 1 2 3 3 ← (32) 6 2 3 3

(45, 7)

(1) 7 14 4 5 7 7 ← (2) 11 14 5 7 7
(3) 12 12 9 3 3 3
(5) 5 5 9 7 7 7 ← (6) 7 14 5 7 7
(7) 3 5 9 7 7 7 ← (9) 14 4 5 7 7
(7) 8 12 9 3 3 3
(15) 12 4 5 3 3 3 ← (28) 4 5 3 3 3
(19) 3 3 6 6 5 3 ← (25) 3 5 7 3 3
(25) 2 4 5 3 3 3 ← (26) 4 7 3 3 3
(29) 1 2 4 3 3 3 ← (35) 4 4 1 1 1
(31) 3 4 4 1 1 1 ← (32) 5 1 2 3 3
(35) 1 * 1 ← (37) * 1

(45, 8)

(3) 10 9 3 6 6 5 3 ← (4) 12 12 9 3 3 3
(6) 3 5 9 3 5 7 7 ← (10) 5 9 3 5 7 7
(7) 6 9 3 6 6 5 3 ← (8) 8 12 9 3 3 3
(8) 3 5 7 3 5 7 7
(13) 6 3 3 6 6 5 3 ← (26) 2 4 5 3 3 3
(15) 10 2 4 5 3 3 3 ← (16) 12 4 5 3 3 3
(28) 1 1 2 4 3 3 3 ← (32) 3 4 4 1 1 1
(34) 1 1 * 1 ← (36) 1 * 1

(45, 9)

(3) 9 5 5 3 6 6 5 3 ← (4) 10 9 3 6 6 5 3
(5) 5 6 5 2 3 5 7 7
(7) 3 6 5 2 3 5 7 7 ← (8) 6 9 3 6 6 5 3
(11) 5 6 2 3 5 7 3 3 ← (23) 13 1 * 1
(13) 3 6 2 3 5 7 3 3 ← (14) 6 3 3 6 6 5 3
(15) 7 13 1 * 1 ← (16) 10 2 4 5 3 3 3
(19) 4 ..4 5 3 3 3 ← (20) 6 2 4 5 3 3 3
(23) 6 2 3 4 4 1 1 1 ← (30) 2 3 4 4 1 1 1

(45, 10)

(3) 4 5 5 5 3 6 6 5 3 ← (6) 5 6 5 2 3 5 7 7
(7) 2 3 5 5 3 6 6 5 3 ← (8) 3 6 5 2 3 5 7 7
(10) 2 4 3 3 3 6 6 5 3 ← (21) 8 1 1 2 4 3 3 3
(11) 3 5 6 2 4 5 3 3 3 ← (12) 5 6 2 3 5 7 3 3
(13) ...3 3 6 6 5 3 ← (14) 3 6 2 3 5 7 3 3
(15) 5 8 1 1 2 4 3 3 3 ← (16) 7 13 1 * 1
(17) 4 ...4 5 3 3 3 ← (18) 6 ..4 5 3 3 3
(19) ....4 5 3 3 3 ← (20) 4 ..4 5 3 3 3
(22) * 2 4 3 3 3 ← (29) 1 2 3 4 4 1 1 1
(23) 5 1 2 3 4 4 1 1 1 ← (24) 6 2 3 4 4 1 1 1

(45, 11)

(1) 2 4 5 5 5 3 6 6 5 3 ← (4) 4 5 5 5 3 6 6 5 3
(8) ..4 3 3 3 6 6 5 3 ← (20) 4 5 3 3 3
(11) ....3 3 6 6 5 3 ← (12) 3 5 6 2 4 5 3 3 3
(15) 4 ....4 5 3 3 3 ← (16) 5 8 1 1 2 4 3 3 3
(17) .....4 5 3 3 3 ← (18) 4 ...4 5 3 3 3
(21) 1 * 2 4 3 3 3 ← (27) 4 4 1 1 * 1
(23) 3 4 4 1 1 * 1 ← (24) 5 1 2 3 4 4 1 1 1
(27) 1 * * 1 ← (29) * * 1

(45, 12)

(4) 1 1 2 4 7 3 3 6 6 5 3
(5) 6 ....3 3 6 6 5 3 ← (18) 4 5 3 3 3
(15) ......4 5 3 3 3 ← (16) 4 ....4 5 3 3 3
(20) 1 1 * 2 4 3 3 3 ← (24) 3 4 4 1 1 * 1
(26) 1 1 * * 1 ← (28) 1 * * 1

(45, 13)

(3) 5 6 .....3 5 7 3 3 ← (16) 4 5 3 3 3
(5) 3 6 .....3 5 7 3 3 ← (6) 6 ....3 3 6 6 5 3
(11) 4 ......4 5 3 3 3 ← (12) 6 .....4 5 3 3 3
(15) 6 2 3 4 4 1 1 * 1 ← (22) 2 3 4 4 1 1 * 1

(45, 14)

(2) .....4 3 3 3 6 6 5 3 ← (13) 8 1 1 * 2 4 3 3 3
(3) 3 5 6 .....4 5 3 3 3 ← (4) 5 6 .....3 5 7 3 3
(5) ....3 3 6 6 5 3 ← (6) 3 6 .....3 5 7 3 3
(9) 4 ....4 5 3 3 3 ← (10) 6 ......4 5 3 3 3
(11) ....4 5 3 3 3 ← (12) 4 ......4 5 3 3 3
(14) * * 2 4 3 3 3 ← (21) 1 2 3 4 4 1 1 * 1
(15) 5 1 2 3 4 4 1 1 * 1 ← (16) 6 2 3 4 4 1 1 * 1

(45, 15)

(3) ....3 3 6 6 5 3 ← (4) 3 5 6 .....4 5 3 3 3
(6) 3 5 7 3 3
(9) ....4 5 3 3 3 ← (10) 4 ....4 5 3 3 3
(13) 1 * * 2 4 3 3 3 ← (19) 4 4 1 1 * * 1
(15) 3 4 4 1 1 * * 1 ← (16) 5 1 2 3 4 4 1 1 * 1
(19) 1 * * * 1 ← (21) * * * 1

(45, 16)

(3) 6 ....4 5 3 3 3
(12) 1 1 * * 2 4 3 3 3 ← (16) 3 4 4 1 1 * * 1
(18) 1 1 * * * 1 ← (20) 1 * * * 1

(45, 17)

(1) 6 ....4 5 3 3 3
(3) 4 ....4 5 3 3 3 ← (4) 6 ....4 5 3 3 3
(7) 6 2 3 4 4 1 1 * * 1 ← (14) 2 3 4 4 1 1 * * 1

(45, 18)

(1) 4 ....4 5 3 3 3 ← (2) 6 ....4 5 3 3 3
(3) ....4 5 3 3 3 ← (4) 4 ....4 5 3 3 3
(6) * * * 2 4 3 3 3 ← (13) 1 2 3 4 4 1 1 * * 1
(7) 5 1 2 3 4 4 1 1 * * 1 ← (8) 6 2 3 4 4 1 1 * * 1

(45, 19)

(1) ....4 5 3 3 3 ← (2) 4 ....4 5 3 3 3
(5) 1 * * * 2 4 3 3 3 ← (11) 4 4 1 1 * * * 1
(7) 3 4 4 1 1 * * * 1 ← (8) 5 1 2 3 4 4 1 1 * * 1
(11) 1 * * * * 1 ← (13) * * * * 1

(45, 20)

(4) 1 1 * * * 2 4 3 3 3 ← (8) 3 4 4 1 1 * * * 1
(10) 1 1 * * * * 1 ← (12) 1 * * * * 1

(45, 23)

(3) 1 * * * * * 1 ← (5) * * * * * 1

(45, 24)

(2) 1 1 * * * * * 1 ← (4) 1 * * * * * 1

(46, 2)

(15) 31
(31) 15 ← (47)

(46, 3)

(14) 29 3
(15) 30 1 ← (16) 31
(30) 13 3 ← (46) 1
(31) 14 1 ← (32) 15
(39) 6 1 ← (40) 7
(43) 2 1 ← (44) 3

(46, 4)

(1) 15 15 15
(5) 11 15 15
(7) 21 11 7 ← (9) 23 15
(13) 27 3 3
(15) 13 11 7 ← (17) 15 15
(15) 29 1 1 ← (16) 30 1
(25) 7 7 7 ← (33) 7 7
(29) 11 3 3 ← (45) 1 1
(31) 13 1 1 ← (32) 14 1
(37) 3 3 3 ← (41) 3 3
(39) 5 1 1 ← (40) 6 1
(43) 1 1 1 ← (44) 2 1

(46, 5)

(1) 14 13 11 7 ← (2) 15 15 15
(5) 10 13 11 7 ← (6) 11 15 15
(6) 19 7 7 7 ← (8) 21 11 7
(7) 20 5 7 7 ← (10) 23 7 7
(12) 25 3 3 3
(13) 14 5 7 7 ← (16) 13 11 7
(14) 13 5 7 7 ← (28) 5 7 7
(15) 28 1 1 1 ← (16) 29 1 1
(23) 4 5 7 7 ← (26) 7 7 7
(28) 9 3 3 3 ← (44) 1 1 1
(31) 12 1 1 1 ← (32) 13 1 1
(39) 4 1 1 1 ← (40) 5 1 1

(46, 6)

(1) 9 15 7 7 7

(1) 13 13 5 7 7

(5) 9 11 7 7 7 ← (6) 10 13 11 7

(6) 18 3 5 7 7 ← (8) 20 5 7 7

(7) 7 13 5 7 7 ← (9) 17 7 7 7

(8) 20 9 3 3 3

(11) 5 9 7 7 7

(13) 11 3 5 7 7 ← (14) 14 5 7 7

(15) 9 3 5 7 7 ← (25) 3 5 7 7

(15) 24 4 1 1 1 ← (16) 28 1 1 1

(16) 12 9 3 3 3 ← (40) 4 1 1 1

(22) 2 3 5 7 7 ← (24) 4 5 7 7

(23) 3 6 6 5 3 ← (27) 6 6 5 3

(29) 3 6 2 3 3 ← (30) 8 3 3 3

(31) 2 4 3 3 3 ← (33) 6 2 3 3

(31) 8 4 1 1 1 ← (32) 12 1 1 1

(46, 7)

(1) 12 11 3 5 7 7 ← (2) 13 13 5 7 7

(2) 7 14 4 5 7 7

(6) 5 5 9 7 7 7 ← (8) 7 13 5 7 7

(6) 17 3 6 6 5 3

(8) 3 5 9 7 7 7

(9) 13 2 3 5 7 7 ← (10) 14 4 5 7 7

(12) 5 7 3 5 7 7 ← (16) 9 3 5 7 7

(14) 9 3 6 6 5 3 ← (38) * 1

(15) 22 * 1 ← (16) 24 4 1 1 1

(20) 3 3 6 6 5 3 ← (26) 3 5 7 3 3

(23) 2 3 5 7 3 3 ← (24) 3 6 6 5 3

(29) ..4 3 3 3 ← (30) 3 6 2 3 3

(30) 1 2 4 3 3 3 ← (32) 2 4 3 3 3

(31) 6 * 1 ← (32) 8 4 1 1 1

(35) 2 * 1 ← (36) 4 4 1 1 1

(46, 8)

(1) 7 8 12 9 3 3 3 ← (2) 12 11 3 5 7 7

(5) 11 12 4 5 3 3 3

(7) 3 5 9 3 5 7 7 ← (9) 8 12 9 3 3 3

(9) 3 5 7 3 5 7 7

(9) 7 12 4 5 3 3 3 ← (10) 13 2 3 5 7 7

(11) 6 5 2 3 5 7 7 ← (37) 1 * 1

(13) 5 5 3 6 6 5 3 ← (17) 12 4 5 3 3 3

(15) 21 1 * 1 ← (16) 22 * 1

(17) 6 2 3 5 7 3 3 ← (24) 2 3 5 7 3 3

(29) 1 1 2 4 3 3 3 ← (30) ..4 3 3 3

(31) 5 1 * 1 ← (32) 6 * 1

(35) 1 1 * 1 ← (36) 2 * 1

(46, 9)

(4) 9 5 5 3 6 6 5 3

(9) 4 7 3 3 6 6 5 3 ← (36) 1 1 * 1

(9) 5 6 3 3 6 6 5 3 ← (10) 7 12 4 5 3 3 3

(11) 3 6 3 3 6 6 5 3 ← (12) 6 5 2 3 5 7 7

(15) 5 6 2 4 5 3 3 3 ← (21) 6 2 4 5 3 3 3

(15) 20 1 1 * 1 ← (16) 21 1 * 1

(17) 3 6 2 4 5 3 3 3 ← (18) 6 2 3 5 7 3 3

(23) 12 1 1 * 1 ← (24) 13 1 * 1

(31) 4 1 1 * 1 ← (32) 5 1 * 1

(46, 10)

(1) 5 5 6 5 2 3 5 7 7

(7) 2 4 7 3 3 6 6 5 3 ← (10) 4 7 3 3 6 6 5 3

(8) 2 3 5 5 3 6 6 5 3 ← (32) 4 1 1 * 1

(9) 3 5 6 2 3 5 7 3 3 ← (10) 5 6 3 3 6 6 5 3

(11) 2 4 3 3 3 6 6 5 3 ← (12) 3 6 3 3 6 6 5 3

(14) ...3 3 6 6 5 3 ← (19) 6 ..4 5 3 3 3

(15) 3 6 ..4 5 3 3 3 ← (16) 5 6 2 4 5 3 3 3

(15) 16 4 1 1 * 1 ← (16) 20 1 1 * 1

(17) ....3 5 7 3 3 ← (18) 3 6 2 4 5 3 3 3

(21) 3 6 2 3 4 4 1 1 1 ← (22) 8 1 1 2 4 3 3 3

(23) * 2 4 3 3 3 ← (25) 6 2 3 4 4 1 1 1

(23) 8 4 1 1 * 1 ← (24) 12 1 1 * 1

(46, 11)

(2) 2 4 5 5 5 3 6 6 5 3

(5) ..4 7 3 3 6 6 5 3 ← (30) * * 1

(6) 1 2 4 7 3 3 6 6 5 3 ← (8) 2 4 7 3 3 6 6 5 3

(9) ..4 3 3 3 6 6 5 3 ← (10) 3 5 6 2 3 5 7 3 3

(12) ....3 3 6 6 5 3 ← (18) 3 5 7 3 3

(15) .....3 5 7 3 3 ← (16) 3 6 ..4 5 3 3 3

(15) 14 * * 1 ← (16) 16 4 1 1 * 1

(21) 2 * 2 4 3 3 3 ← (22) 3 6 2 3 4 4 1 1 1

(22) 1 * 2 4 3 3 3 ← (24) * 2 4 3 3 3

(23) 6 * * 1 ← (24) 8 4 1 1 * 1

(27) 2 * * 1 ← (28) 4 4 1 1 * 1

(46, 12)

(3) 6 ..4 3 3 3 6 6 5 3 ← (29) 1 * * 1

(5) 1 1 2 4 7 3 3 6 6 5 3 ← (6) ..4 7 3 3 6 6 5 3

(9) 6 .....3 5 7 3 3 ← (16) 3 5 7 3 3

(15) 13 1 * * 1 ← (16) 14 * * 1

(21) 1 1 * 2 4 3 3 3 ← (22) 2 * 2 4 3 3 3

(23) 5 1 * * 1 ← (24) 6 * * 1

(27) 1 1 * * 1 ← (28) 2 * * 1

(46, 13)

(1) 4 1 1 2 4 7 3 3 6 6 5 3 ← (28) 1 1 * * 1
(3) 3 6 ....3 3 6 6 5 3 ← (4) 6 ..4 3 3 3 6 6 5 3
(7) 5 6 .....4 5 3 3 3 ← (13) 6 .....4 5 3 3 3
(9) 3 6 .....4 5 3 3 3 ← (10) 6 .....3 5 7 3 3
(15) 12 1 1 * * 1 ← (16) 13 1 * * 1
(23) 4 1 1 * * 1 ← (24) 5 1 * * 1

(46, 14)

(3) .....4 3 3 3 6 6 5 3 ← (4) 3 6 ....3 3 6 6 5 3
(6) ....3 3 6 6 5 3 ← (11) 6 ......4 5 3 3 3
(7) 3 6 ......4 5 3 3 3 ← (8) 5 6 .....4 5 3 3 3
(9) ....3 5 7 3 3 ← (10) 3 6 .....4 5 3 3 3
(12) 4 5 3 3 3
(13) 3 6 2 3 4 4 1 1 * 1 ← (14) 8 1 1 * 2 4 3 3 3
(15) * * 2 4 3 3 3 ← (17) 6 2 3 4 4 1 1 * 1
(15) 8 4 1 1 * * 1 ← (16) 12 1 1 * * 1

(46, 15)

(4) ....3 3 6 6 5 3 ← (10) 3 5 7 3 3
(7) ....3 5 7 3 3 ← (8) 3 6 ......4 5 3 3 3
(10) 4 5 3 3 3
(13) 2 * * 2 4 3 3 3 ← (14) 3 6 2 3 4 4 1 1 * 1
(14) 1 * * 2 4 3 3 3 ← (16) * * 2 4 3 3 3
(15) 6 * * * 1 ← (16) 8 4 1 1 * * 1
(19) 2 * * * 1 ← (20) 4 4 1 1 * * 1

(46, 16)

(1) 6 ....3 5 7 3 3 ← (8) 3 5 7 3 3
(7) 13 1 * * * 1
(13) 1 1 * * 2 4 3 3 3 ← (14) 2 * * 2 4 3 3 3
(15) 5 1 * * * 1 ← (16) 6 * * * 1
(19) 1 1 * * * 1 ← (20) 2 * * * 1

(46, 17)

(1) 3 6 ....4 5 3 3 3 ← (2) 6 ....3 5 7 3 3
(5) 8 1 1 * * 2 4 3 3 3
(7) 12 1 1 * * * 1 ← (8) 13 1 * * * 1
(15) 4 1 1 * * * 1 ← (16) 5 1 * * * 1

(46, 18)

(1) ....3 5 7 3 3 ← (2) 3 6 ....4 5 3 3 3
(4) 4 5 3 3 3
(5) 3 6 2 3 4 4 1 1 * * 1 ← (6) 8 1 1 * * 2 4 3 3 3
(7) * * * 2 4 3 3 3 ← (9) 6 2 3 4 4 1 1 * * 1
(7) 8 4 1 1 * * * 1 ← (8) 12 1 1 * * * 1

(46, 19)

(2) 4 5 3 3 3
(5) 2 * * * 2 4 3 3 3 ← (6) 3 6 2 3 4 4 1 1 * * 1
(6) 1 * * * 2 4 3 3 3 ← (8) * * * 2 4 3 3 3
(7) 6 * * * * 1 ← (8) 8 4 1 1 * * * 1
(11) 2 * * * * 1 ← (12) 4 4 1 1 * * * 1

(46, 20)

(5) 1 1 * * * 2 4 3 3 3 ← (6) 2 * * * 2 4 3 3 3
(6) 2 3 4 4 1 1 * * * 1
(7) 5 1 * * * * 1 ← (8) 6 * * * * 1
(11) 1 1 * * * * 1 ← (12) 2 * * * * 1

(46, 21)

(5) 1 2 3 4 4 1 1 * * * 1
(7) 4 1 1 * * * * 1 ← (8) 5 1 * * * * 1

(46, 22)

(3) 4 4 1 1 * * * * 1

(46, 23)

(3) 2 * * * * * 1 ← (4) 4 4 1 1 * * * * 1

(46, 24)

(3) 1 1 * * * * * 1 ← (4) 2 * * * * * 1

(47, 3)

(13) 27 7
(15) 29 3 ← (17) 31
(29) 11 7 ← (45) 3
(31) 13 3 ← (33) 15
(39) 5 3 ← (41) 7

(47, 4)

(9) 22 13 3 ← (10) 23 15
(13) 26 5 3 ← (14) 27 7
(14) 27 3 3 ← (16) 29 3
(17) 14 13 3 ← (18) 15 15
(29) 10 5 3 ← (30) 11 7
(30) 11 3 3 ← (32) 13 3
(33) 6 5 3 ← (34) 7 7
(38) 3 3 3 ← (40) 5 3
(39) 2 3 3 ← (42) 3 3

(47, 5)

(2) 14 13 11 7

(3) 13 13 11 7

(7) 11 15 7 7

(7) 19 7 7 7 ← (9) 21 11 7

(9) 21 11 3 3 ← (10) 22 13 3

(13) 25 3 3 3 ← (14) 26 5 3

(15) 13 5 7 7 ← (29) 5 7 7

(17) 13 11 3 3 ← (18) 14 13 3

(29) 5 7 3 3 ← (34) 6 5 3

(29) 9 3 3 3 ← (30) 10 5 3

(38) 1 2 3 3 ← (40) 2 3 3

(47, 6)

(2) 9 15 7 7 7

(3) 11 14 5 7 7 ← (4) 13 13 11 7

(6) 9 11 7 7 7

(7) 7 14 5 7 7 ← (8) 11 15 7 7

(7) 18 3 5 7 7 ← (9) 20 5 7 7

(9) 20 9 3 3 3 ← (10) 21 11 3 3

(12) 5 9 7 7 7 ← (26) 3 5 7 7

(14) 11 3 5 7 7 ← (16) 13 5 7 7

(17) 12 9 3 3 3 ← (18) 13 11 3 3

(23) 2 3 5 7 7 ← (25) 4 5 7 7

(27) 4 7 3 3 3 ← (28) 6 6 5 3

(29) 4 5 3 3 3 ← (30) 5 7 3 3

(33) 5 1 2 3 3 ← (34) 6 2 3 3

(47, 7)

(1) 11 5 9 7 7 7

(3) 7 14 4 5 7 7 ← (4) 11 14 5 7 7

(5) 12 12 9 3 3 3 ← (8) 18 3 5 7 7

(7) 5 5 9 7 7 7 ← (8) 7 14 5 7 7

(7) 17 3 6 6 5 3 ← (10) 20 9 3 3 3

(9) 3 5 9 7 7 7

(11) 5 9 3 5 7 7 ← (25) 3 6 6 5 3

(13) 5 7 3 5 7 7 ← (17) 9 3 5 7 7

(15) 9 3 6 6 5 3 ← (18) 12 9 3 3 3

(21) 3 3 6 6 5 3 ← (24) 2 3 5 7 7

(27) 2 4 5 3 3 3 ← (28) 4 7 3 3 3

(31) 1 2 4 3 3 3 ← (33) 2 4 3 3 3

(33) 3 4 4 1 1 1 ← (34) 5 1 2 3 3

(47, 8)

(1) 8 3 5 9 7 7 7

(2) 7 8 12 9 3 3 3

(5) 10 9 3 6 6 5 3 ← (6) 12 12 9 3 3 3

(6) 11 12 4 5 3 3 3 ← (8) 17 3 6 6 5 3

(8) 3 5 9 3 5 7 7

(9) 6 9 3 6 6 5 3 ← (10) 8 12 9 3 3 3

(10) 3 5 7 3 5 7 7 ← (12) 5 9 3 5 7 7

(14) 5 5 3 6 6 5 3 ← (16) 9 3 6 6 5 3

(15) 6 3 3 6 6 5 3 ← (22) 3 3 6 6 5 3

(17) 10 2 4 5 3 3 3 ← (18) 12 4 5 3 3 3

(30) 1 1 2 4 3 3 3 ← (32) 1 2 4 3 3 3

(31) 2 3 4 4 1 1 1 ← (34) 3 4 4 1 1 1

(47, 9)

(1) 9 3 5 7 3 5 7 7

(5) 9 5 5 3 6 6 5 3 ← (6) 10 9 3 6 6 5 3

(7) 5 6 5 2 3 5 7 7 ← (13) 6 5 2 3 5 7 7

(9) 3 6 5 2 3 5 7 7 ← (10) 6 9 3 6 6 5 3

(13) 5 6 2 3 5 7 3 3 ← (19) 6 2 3 5 7 3 3

(15) 3 6 2 3 5 7 3 3 ← (16) 6 3 3 6 6 5 3

(17) 7 13 1 * 1 ← (18) 10 2 4 5 3 3 3

(21) 4 ..4 5 3 3 3 ← (22) 6 2 4 5 3 3 3

(30) 1 2 3 4 4 1 1 1 ← (32) 2 3 4 4 1 1 1

(47, 10)

(2) 5 5 6 5 2 3 5 7 7

(5) 4 5 5 5 3 6 6 5 3 ← (8) 5 6 5 2 3 5 7 7

(9) 2 3 5 5 3 6 6 5 3 ← (10) 3 6 5 2 3 5 7 7

(12) 2 4 3 3 3 6 6 5 3 ← (17) 5 6 2 4 5 3 3 3

(13) 3 5 6 2 4 5 3 3 3 ← (14) 5 6 2 3 5 7 3 3

(15) ...3 3 6 6 5 3 ← (16) 3 6 2 3 5 7 3 3

(17) 5 8 1 1 2 4 3 3 3 ← (18) 7 13 1 * 1

(19) 4 ...4 5 3 3 3 ← (20) 6 ..4 5 3 3 3

(21) ....4 5 3 3 3 ← (22) 4 ..4 5 3 3 3

(25) 5 1 2 3 4 4 1 1 1 ← (26) 6 2 3 4 4 1 1 1

(47, 11)

(3) 2 4 5 5 5 3 6 6 5 3 ← (6) 4 5 5 5 3 6 6 5 3

(7) 1 2 4 7 3 3 6 6 5 3 ← (9) 2 4 7 3 3 6 6 5 3

(10) ..4 3 3 3 6 6 5 3 ← (16) ...3 3 6 6 5 3

(13) ....3 3 6 6 5 3 ← (14) 3 5 6 2 4 5 3 3 3

(17) 4 ....4 5 3 3 3 ← (18) 5 8 1 1 2 4 3 3 3

(19) .....4 5 3 3 3 ← (20) 4 ...4 5 3 3 3

(23) 1 * 2 4 3 3 3 ← (25) * 2 4 3 3 3

(25) 3 4 4 1 1 * 1 ← (26) 5 1 2 3 4 4 1 1 1

(47, 12)

(1) ..4 5 5 5 3 6 6 5 3 ← (4) 2 4 5 5 5 3 6 6 5 3
(6) 1 1 2 4 7 3 3 6 6 5 3 ← (8) 1 2 4 7 3 3 6 6 5 3
(7) 6 ....3 3 6 6 5 3 ← (14) 3 3 6 6 5 3
(17) ......4 5 3 3 3 ← (18) 4 ....4 5 3 3 3
(22) 1 1 * 2 4 3 3 3 ← (24) 1 * 2 4 3 3 3
(23) 2 3 4 4 1 1 * 1 ← (26) 3 4 4 1 1 * 1

(47, 13)

(2) 4 1 1 2 4 7 3 3 6 6 5 3
(5) 5 6 .....3 5 7 3 3 ← (11) 6 .....3 5 7 3 3
(7) 3 6 .....3 5 7 3 3 ← (8) 6 ....3 3 6 6 5 3
(13) 4 ......4 5 3 3 3 ← (14) 6 .....4 5 3 3 3
(22) 1 2 3 4 4 1 1 * 1 ← (24) 2 3 4 4 1 1 * 1
(24) 4 1 1 * * 1

(47, 14)

(4) .....4 3 3 3 6 6 5 3 ← (9) 5 6 .....4 5 3 3 3
(5) 3 5 6 .....4 5 3 3 3 ← (6) 5 6 .....3 5 7 3 3
(7) ....3 3 6 6 5 3 ← (8) 3 6 .....3 5 7 3 3
(11) 4 ....4 5 3 3 3 ← (12) 6 ......4 5 3 3 3
(13) ....4 5 3 3 3 ← (14) 4 ......4 5 3 3 3
(17) 5 1 2 3 4 4 1 1 * 1 ← (18) 6 2 3 4 4 1 1 * 1
(22) * * * 1

(47, 15)

(1) 1.........4 5 3 3 3 ← (8) 3 3 6 6 5 3
(5) ....3 3 6 6 5 3 ← (6) 3 5 6 .....4 5 3 3 3
(11) ....4 5 3 3 3 ← (12) 4 ....4 5 3 3 3
(15) 1 * * 2 4 3 3 3 ← (17) * * 2 4 3 3 3
(17) 3 4 4 1 1 * * 1 ← (18) 5 1 2 3 4 4 1 1 * 1
(21) 1 * * * 1

(47, 16)

(1) 10 ....4 5 3 3 3 ← (2) 1.........4 5 3 3 3
(5) 6 ....4 5 3 3 3
(14) 1 1 * * 2 4 3 3 3 ← (16) 1 * * 2 4 3 3 3
(15) 2 3 4 4 1 1 * * 1 ← (18) 3 4 4 1 1 * * 1
(20) 1 1 * * * 1

(47, 17)

(1) 7 13 1 * * * 1 ← (2) 10 ....4 5 3 3 3
(3) 6 ....4 5 3 3 3
(5) 4 ....4 5 3 3 3 ← (6) 6 ....4 5 3 3 3
(14) 1 2 3 4 4 1 1 * * 1 ← (16) 2 3 4 4 1 1 * * 1
(16) 4 1 1 * * * 1

(47, 18)

(1) 5 8 1 1 * * 2 4 3 3 3 ← (2) 7 13 1 * * * 1
(2) 3 5 7 3 3
(3) 4 ....4 5 3 3 3 ← (4) 6 ....4 5 3 3 3
(5) ....4 5 3 3 3 ← (6) 4 ....4 5 3 3 3
(9) 5 1 2 3 4 4 1 1 * * 1 ← (10) 6 2 3 4 4 1 1 * * 1
(14) * * * * 1

(47, 19)

(1) 4 ....4 5 3 3 3 ← (2) 5 8 1 1 * * 2 4 3 3 3
(3) ....4 5 3 3 3 ← (4) 4 ....4 5 3 3 3
(7) 1 * * * 2 4 3 3 3 ← (9) * * * 2 4 3 3 3
(9) 3 4 4 1 1 * * * 1 ← (10) 5 1 2 3 4 4 1 1 * * 1
(13) 1 * * * * 1

(47, 20)

(1) ....4 5 3 3 3 ← (2) 4 ....4 5 3 3 3
(6) 1 1 * * * 2 4 3 3 3 ← (8) 1 * * * 2 4 3 3 3
(7) 2 3 4 4 1 1 * * * 1 ← (10) 3 4 4 1 1 * * * 1
(12) 1 1 * * * * 1

(47, 21)

(1) 6 2 3 4 4 1 1 * * * 1
(6) 1 2 3 4 4 1 1 * * * 1 ← (8) 2 3 4 4 1 1 * * * 1
(8) 4 1 1 * * * * 1

(47, 22)

(1) 5 1 2 3 4 4 1 1 * * * 1 ← (2) 6 2 3 4 4 1 1 * * * 1
(6) * * * * * 1

(47, 23)

(1) 3 4 4 1 1 * * * * 1 ← (2) 5 1 2 3 4 4 1 1 * * * 1
(5) 1 * * * * * 1

(47, 24)

(4) 1 1 * * * * * 1

(48, 2)

(47) 1 ← (49)

(48, 3)

(17) 30 1 ← (18) 31
(33) 14 1 ← (34) 15
(41) 6 1 ← (42) 7
(45) 2 1 ← (46) 3
(46) 1 1 ← (48) 1

(48, 4)

(3) 15 15 15
(7) 11 15 15
(11) 23 7 7
(15) 27 3 3 ← (17) 29 3
(17) 13 11 7 ← (43) 3 3
(17) 29 1 1 ← (18) 30 1
(27) 7 7 7 ← (35) 7 7
(31) 11 3 3 ← (33) 13 3
(33) 13 1 1 ← (34) 14 1
(39) 3 3 3 ← (41) 5 3
(41) 5 1 1 ← (42) 6 1
(45) 1 1 1 ← (46) 2 1

(48, 5)

(3) 14 13 11 7 ← (4) 15 15 15
(7) 10 13 11 7 ← (8) 11 15 15
(8) 19 7 7 7
(10) 17 7 7 7
(14) 25 3 3 3 ← (16) 27 3 3
(15) 14 5 7 7 ← (30) 5 7 7
(17) 28 1 1 1 ← (18) 29 1 1
(30) 9 3 3 3 ← (32) 11 3 3
(31) 8 3 3 3 ← (40) 3 3 3
(33) 12 1 1 1 ← (34) 13 1 1
(39) 1 2 3 3 ← (41) 2 3 3
(41) 4 1 1 1 ← (42) 5 1 1

(48, 6)

(1) 7 11 15 7 7
(3) 9 15 7 7 7 ← (5) 13 13 11 7
(3) 13 13 5 7 7
(7) 9 11 7 7 7 ← (8) 10 13 11 7
(9) 7 13 5 7 7
(11) 14 4 5 7 7 ← (27) 3 5 7 7
(13) 5 9 7 7 7 ← (17) 13 5 7 7
(15) 11 3 5 7 7 ← (16) 14 5 7 7
(17) 24 4 1 1 1 ← (18) 28 1 1 1
(27) 3 5 7 3 3 ← (29) 6 6 5 3
(30) 4 5 3 3 3 ← (35) 6 2 3 3
(31) 3 6 2 3 3 ← (32) 8 3 3 3
(33) 8 4 1 1 1 ← (34) 12 1 1 1
(37) 4 4 1 1 1 ← (40) 1 2 3 3
(39) * 1 ← (42) 4 1 1 1

(48, 7)

(1) 6 9 11 7 7 7 ← (2) 7 11 15 7 7
(2) 11 5 9 7 7 7
(3) 12 11 3 5 7 7 ← (4) 13 13 5 7 7
(4) 7 14 4 5 7 7
(8) 5 5 9 7 7 7
(10) 3 5 9 7 7 7 ← (16) 11 3 5 7 7
(11) 13 2 3 5 7 7 ← (12) 14 4 5 7 7
(14) 5 7 3 5 7 7 ← (18) 9 3 5 7 7
(17) 22 * 1 ← (18) 24 4 1 1 1
(25) 2 3 5 7 3 3 ← (26) 3 6 6 5 3
(28) 2 4 5 3 3 3 ← (34) 2 4 3 3 3
(31) ..4 3 3 3 ← (32) 3 6 2 3 3
(33) 6 * 1 ← (34) 8 4 1 1 1
(37) 2 * 1 ← (38) 4 4 1 1 1
(38) 1 * 1 ← (40) * 1

(48, 8)

(1) 9 3 5 9 7 7 7
(2) 8 3 5 9 7 7 7
(3) 7 8 12 9 3 3 3 ← (4) 12 11 3 5 7 7
(7) 11 12 4 5 3 3 3 ← (9) 17 3 6 6 5 3
(9) 3 5 9 3 5 7 7 ← (19) 12 4 5 3 3 3
(11) 3 5 7 3 5 7 7 ← (13) 5 9 3 5 7 7
(11) 7 12 4 5 3 3 3 ← (12) 13 2 3 5 7 7
(15) 5 5 3 6 6 5 3 ← (17) 9 3 6 6 5 3
(17) 21 1 * 1 ← (18) 22 * 1
(25) 13 1 * 1 ← (33) 1 2 4 3 3 3
(31) 1 1 2 4 3 3 3 ← (32) ..4 3 3 3
(33) 5 1 * 1 ← (34) 6 * 1
(37) 1 1 * 1 ← (38) 2 * 1

(48, 9)

(1) 4 5 3 5 9 7 7 7
(1) 8 3 5 9 3 5 7 7
(2) 9 3 5 7 3 5 7 7
(6) 9 5 5 3 6 6 5 3 ← (8) 11 12 4 5 3 3 3
(11) 4 7 3 3 6 6 5 3 ← (16) 5 5 3 6 6 5 3
(11) 5 6 3 3 6 6 5 3 ← (12) 7 12 4 5 3 3 3
(13) 3 6 3 3 6 6 5 3 ← (14) 6 5 2 3 5 7 7
(17) 20 1 1 * 1 ← (18) 21 1 * 1
(19) 3 6 2 4 5 3 3 3 ← (20) 6 2 3 5 7 3 3
(23) 8 1 1 2 4 3 3 3 ← (32) 1 1 2 4 3 3 3
(25) 12 1 1 * 1 ← (26) 13 1 * 1
(31) 1 2 3 4 4 1 1 1 ← (33) 2 3 4 4 1 1 1
(33) 4 1 1 * 1 ← (34) 5 1 * 1

(48, 10)

(3) 5 5 6 5 2 3 5 7 7 ← (9) 5 6 5 2 3 5 7 7
(10) 2 3 5 5 3 6 6 5 3 ← (12) 4 7 3 3 6 6 5 3
(11) 3 5 6 2 3 5 7 3 3 ← (12) 5 6 3 3 6 6 5 3
(13) 2 4 3 3 3 6 6 5 3 ← (14) 3 6 3 3 6 6 5 3
(17) 3 6 ..4 5 3 3 3 ← (18) 5 6 2 4 5 3 3 3
(17) 16 4 1 1 * 1 ← (18) 20 1 1 * 1
(19) ....3 5 7 3 3 ← (20) 3 6 2 4 5 3 3 3
(22) ....4 5 3 3 3 ← (27) 6 2 3 4 4 1 1 1
(23) 3 6 2 3 4 4 1 1 1 ← (24) 8 1 1 2 4 3 3 3
(25) 8 4 1 1 * 1 ← (26) 12 1 1 * 1
(29) 4 4 1 1 * 1 ← (32) 1 2 3 4 4 1 1 1
(31) * * 1 ← (34) 4 1 1 * 1

(48, 11)

(1) 4 3 5 6 5 2 3 5 7 7 ← (4) 5 5 6 5 2 3 5 7 7
(7) ..4 7 3 3 6 6 5 3 ← (10) 2 4 7 3 3 6 6 5 3
(11) ..4 3 3 3 6 6 5 3 ← (12) 3 5 6 2 3 5 7 3 3
(17) .....3 5 7 3 3 ← (18) 3 6 ..4 5 3 3 3
(17) 14 * * 1 ← (18) 16 4 1 1 * 1
(20) .....4 5 3 3 3 ← (26) * 2 4 3 3 3
(23) 2 * 2 4 3 3 3 ← (24) 3 6 2 3 4 4 1 1 1
(25) 6 * * 1 ← (26) 8 4 1 1 * 1
(29) 2 * * 1 ← (30) 4 4 1 1 * 1
(30) 1 * * 1 ← (32) * * 1

(48, 12)

(2) ..4 5 5 5 3 6 6 5 3
(5) 6 ..4 3 3 3 6 6 5 3 ← (9) 1 2 4 7 3 3 6 6 5 3
(7) 1 1 2 4 7 3 3 6 6 5 3 ← (8) ..4 7 3 3 6 6 5 3
(17) 13 1 * * 1 ← (18) 14 * * 1
(18) ......4 5 3 3 3 ← (25) 1 * 2 4 3 3 3
(23) 1 1 * 2 4 3 3 3 ← (24) 2 * 2 4 3 3 3
(25) 5 1 * * 1 ← (26) 6 * * 1
(29) 1 1 * * 1 ← (30) 2 * * 1

(48, 13)

(3) 4 1 1 2 4 7 3 3 6 6 5 3 ← (8) 1 1 2 4 7 3 3 6 6 5 3
(5) 3 6 ....3 3 6 6 5 3 ← (6) 6 ..4 3 3 3 6 6 5 3
(11) 3 6 .....4 5 3 3 3 ← (12) 6 .....3 5 7 3 3
(15) 8 1 1 * 2 4 3 3 3 ← (24) 1 1 * 2 4 3 3 3
(17) 12 1 1 * * 1 ← (18) 13 1 * * 1
(23) 1 2 3 4 4 1 1 * 1 ← (25) 2 3 4 4 1 1 * 1
(25) 4 1 1 * * 1 ← (26) 5 1 * * 1

(48, 14)

(1) * 2 4 7 3 3 6 6 5 3
(1) 24 4 1 1 * * 1 ← (4) 4 1 1 2 4 7 3 3 6 6 5 3
(5) .....4 3 3 3 6 6 5 3 ← (6) 3 6 ....3 3 6 6 5 3
(9) 3 6 ......4 5 3 3 3 ← (10) 5 6 .....4 5 3 3 3
(11) ....3 5 7 3 3 ← (12) 3 6 .....4 5 3 3 3
(14) ....4 5 3 3 3 ← (19) 6 2 3 4 4 1 1 * 1
(15) 3 6 2 3 4 4 1 1 * 1 ← (16) 8 1 1 * 2 4 3 3 3
(17) 8 4 1 1 * * 1 ← (18) 12 1 1 * * 1
(21) 4 4 1 1 * * 1 ← (24) 1 2 3 4 4 1 1 * 1
(23) * * * 1 ← (26) 4 1 1 * * 1

(48, 15)

(1) 22 * * * 1 ← (2) 24 4 1 1 * * 1
(6) 3 3 6 6 5 3
(9) ....3 5 7 3 3 ← (10) 3 6 ......4 5 3 3 3
(12) ....4 5 3 3 3 ← (18) * * 2 4 3 3 3
(15) 2 * * 2 4 3 3 3 ← (16) 3 6 2 3 4 4 1 1 * 1
(17) 6 * * * 1 ← (18) 8 4 1 1 * * 1
(21) 2 * * * 1 ← (22) 4 4 1 1 * * 1
(22) 1 * * * 1 ← (24) * * * 1

(48, 16)

(1) 21 1 * * * 1 ← (2) 22 * * * 1
(3) 6 ....3 5 7 3 3
(9) 13 1 * * * 1 ← (17) 1 * * 2 4 3 3 3
(15) 1 1 * * 2 4 3 3 3 ← (16) 2 * * 2 4 3 3 3
(17) 5 1 * * * 1 ← (18) 6 * * * 1
(21) 1 1 * * * 1 ← (22) 2 * * * 1

(48, 17)

(1) 5 6 ....4 5 3 3 3
(1) 20 1 1 * * * 1 ← (2) 21 1 * * * 1
(3) 3 6 ....4 5 3 3 3 ← (4) 6 ....3 5 7 3 3
(7) 8 1 1 * * 2 4 3 3 3 ← (16) 1 1 * * 2 4 3 3 3
(9) 12 1 1 * * * 1 ← (10) 13 1 * * * 1
(15) 1 2 3 4 4 1 1 * * 1 ← (17) 2 3 4 4 1 1 * * 1
(17) 4 1 1 * * * 1 ← (18) 5 1 * * * 1

(48, 18)

(1) 3 6 ....4 5 3 3 3 ← (2) 5 6 ....4 5 3 3 3
(1) 16 4 1 1 * * * 1 ← (2) 20 1 1 * * * 1
(3) ....3 5 7 3 3 ← (4) 3 6 ....4 5 3 3 3
(6) ....4 5 3 3 3 ← (11) 6 2 3 4 4 1 1 * * 1
(7) 3 6 2 3 4 4 1 1 * * 1 ← (8) 8 1 1 * * 2 4 3 3 3
(9) 8 4 1 1 * * * 1 ← (10) 12 1 1 * * * 1
(13) 4 4 1 1 * * * 1 ← (16) 1 2 3 4 4 1 1 * * 1
(15) * * * * 1 ← (18) 4 1 1 * * * 1

(48, 19)

(1) ....3 5 7 3 3 ← (2) 3 6 ....4 5 3 3 3
(1) 14 * * * * 1 ← (2) 16 4 1 1 * * * 1
(4) ....4 5 3 3 3 ← (10) * * * 2 4 3 3 3
(7) 2 * * * 2 4 3 3 3 ← (8) 3 6 2 3 4 4 1 1 * * 1
(9) 6 * * * * 1 ← (10) 8 4 1 1 * * * 1
(13) 2 * * * * 1 ← (14) 4 4 1 1 * * * 1
(14) 1 * * * * 1 ← (16) * * * * 1

(48, 20)

(1) 13 1 * * * * 1 ← (2) 14 * * * * 1
(2) ....4 5 3 3 3 ← (9) 1 * * * 2 4 3 3 3
(7) 1 1 * * * 2 4 3 3 3 ← (8) 2 * * * 2 4 3 3 3
(9) 5 1 * * * * 1 ← (10) 6 * * * * 1
(13) 1 1 * * * * 1 ← (14) 2 * * * * 1

(48, 21)

(1) 12 1 1 * * * * 1 ← (2) 13 1 * * * * 1
(7) 1 2 3 4 4 1 1 * * * 1 ← (9) 2 3 4 4 1 1 * * * 1
(9) 4 1 1 * * * * 1 ← (10) 5 1 * * * * 1

(48, 22)

(1) * * * * 2 4 3 3 3
(1) 8 4 1 1 * * * * 1 ← (2) 12 1 1 * * * * 1
(5) 4 4 1 1 * * * * 1 ← (8) 1 2 3 4 4 1 1 * * * 1
(7) * * * * * 1 ← (10) 4 1 1 * * * * 1

(48, 23)

(1) 6 * * * * * 1 ← (2) 8 4 1 1 * * * * 1
(2) 3 4 4 1 1 * * * * 1
(5) 2 * * * * * 1 ← (6) 4 4 1 1 * * * * 1
(6) 1 * * * * * 1 ← (8) * * * * * 1

(48, 24)

(1) 5 1 * * * * * 1 ← (2) 6 * * * * * 1
(5) 1 1 * * * * * 1 ← (6) 2 * * * * * 1

(48, 25)

(1) 4 1 1 * * * * * 1 ← (2) 5 1 * * * * * 1

(49, 3)

(11) 23 15
(15) 27 7 ← (19) 31
(19) 15 15 ← (43) 7
(31) 11 7 ← (35) 15
(47) 1 1 ← (49) 1

(49, 4)

(10) 21 11 7
(11) 22 13 3 ← (12) 23 15
(12) 23 7 7 ← (18) 29 3
(15) 26 5 3 ← (16) 27 7
(18) 13 11 7 ← (42) 5 3
(19) 14 13 3 ← (20) 15 15
(28) 7 7 7 ← (34) 13 3
(31) 10 5 3 ← (32) 11 7
(35) 6 5 3 ← (36) 7 7
(46) 1 1 1 ← (48) 1 1

(49, 5)

(1) 7 11 15 15
(4) 14 13 11 7
(9) 11 15 7 7
(9) 19 7 7 7
(10) 20 5 7 7 ← (17) 27 3 3
(11) 17 7 7 7 ← (41) 3 3 3
(11) 21 11 3 3 ← (12) 22 13 3
(15) 25 3 3 3 ← (16) 26 5 3
(19) 13 11 3 3 ← (20) 14 13 3
(26) 4 5 7 7 ← (33) 11 3 3
(31) 5 7 3 3 ← (36) 6 5 3
(31) 9 3 3 3 ← (32) 10 5 3

(49, 6)

(1) 10 17 7 7 7
(4) 9 15 7 7 7
(5) 11 14 5 7 7 ← (6) 13 13 11 7
(8) 9 11 7 7 7
(9) 7 14 5 7 7 ← (10) 11 15 7 7
(9) 18 3 5 7 7 ← (16) 25 3 3 3
(10) 7 13 5 7 7 ← (12) 17 7 7 7
(11) 20 9 3 3 3 ← (12) 21 11 3 3
(14) 5 9 7 7 7 ← (18) 13 5 7 7
(19) 12 9 3 3 3 ← (20) 13 11 3 3
(25) 2 3 5 7 7 ← (32) 9 3 3 3
(28) 3 5 7 3 3 ← (33) 8 3 3 3
(29) 4 7 3 3 3 ← (30) 6 6 5 3
(31) 4 5 3 3 3 ← (32) 5 7 3 3
(35) 5 1 2 3 3 ← (36) 6 2 3 3

(49, 7)

(1) 9 7 13 5 7 7 ← (2) 10 17 7 7 7
(2) 6 9 11 7 7 7
(3) 11 5 9 7 7 7 ← (5) 13 13 5 7 7
(5) 7 14 4 5 7 7 ← (6) 11 14 5 7 7
(7) 12 12 9 3 3 3 ← (12) 20 9 3 3 3
(9) 5 5 9 7 7 7 ← (10) 7 14 5 7 7
(11) 3 5 9 7 7 7 ← (13) 14 4 5 7 7
(11) 8 12 9 3 3 3 ← (20) 12 9 3 3 3
(15) 5 7 3 5 7 7 ← (19) 9 3 5 7 7
(23) 3 3 6 6 5 3 ← (27) 3 6 6 5 3
(26) 2 3 5 7 3 3 ← (32) 4 5 3 3 3
(29) 2 4 5 3 3 3 ← (30) 4 7 3 3 3
(35) 3 4 4 1 1 1 ← (36) 5 1 2 3 3
(39) 1 * 1 ← (41) * 1

(49, 8)

(1) 5 10 11 3 5 7 7
(1) 8 5 5 9 7 7 7 ← (2) 9 7 13 5 7 7
(2) 9 3 5 9 7 7 7 ← (4) 11 5 9 7 7 7
(3) 8 3 5 9 7 7 7 ← (6) 7 14 4 5 7 7
(4) 7 8 12 9 3 3 3 ← (10) 17 3 6 6 5 3
(7) 10 9 3 6 6 5 3 ← (8) 12 12 9 3 3 3
(10) 3 5 9 3 5 7 7 ← (18) 9 3 6 6 5 3
(11) 6 9 3 6 6 5 3 ← (12) 8 12 9 3 3 3
(12) 3 5 7 3 5 7 7 ← (16) 5 7 3 5 7 7
(17) 6 3 3 6 6 5 3 ← (24) 3 3 6 6 5 3
(19) 10 2 4 5 3 3 3 ← (20) 12 4 5 3 3 3
(23) 6 2 4 5 3 3 3 ← (30) 2 4 5 3 3 3
(38) 1 1 * 1 ← (40) 1 * 1

(49, 9)

(1) 4 6 3 5 9 7 7 7 ← (2) 5 10 11 3 5 7 7
(2) 4 5 3 5 9 7 7 7 ← (4) 8 3 5 9 7 7 7
(2) 8 3 5 9 3 5 7 7 ← (9) 11 12 4 5 3 3 3
(3) 9 3 5 7 3 5 7 7 ← (17) 5 5 3 6 6 5 3
(7) 9 5 5 3 6 6 5 3 ← (8) 10 9 3 6 6 5 3
(11) 3 6 5 2 3 5 7 7 ← (12) 6 9 3 6 6 5 3
(15) 5 6 2 3 5 7 3 3 ← (21) 6 2 3 5 7 3 3
(17) 3 6 2 3 5 7 3 3 ← (18) 6 3 3 6 6 5 3
(19) 7 13 1 * 1 ← (20) 10 2 4 5 3 3 3
(21) 6 ..4 5 3 3 3 ← (27) 13 1 * 1
(23) 4 ..4 5 3 3 3 ← (24) 6 2 4 5 3 3 3

(49, 10)

(1) 2 3 5 3 5 9 7 7 7 ← (2) 4 6 3 5 9 7 7 7
(1) 4 5 3 5 9 3 5 7 7 ← (8) 9 5 5 3 6 6 5 3
(7) 4 5 5 5 3 6 6 5 3 ← (10) 5 6 5 2 3 5 7 7
(11) 2 3 5 5 3 6 6 5 3 ← (12) 3 6 5 2 3 5 7 7
(14) 2 4 3 3 3 6 6 5 3 ← (19) 5 6 2 4 5 3 3 3
(15) 3 5 6 2 4 5 3 3 3 ← (16) 5 6 2 3 5 7 3 3
(17) ...3 3 6 6 5 3 ← (18) 3 6 2 3 5 7 3 3
(19) 5 8 1 1 2 4 3 3 3 ← (20) 7 13 1 * 1
(20) ....3 5 7 3 3 ← (25) 8 1 1 2 4 3 3 3
(21) 4 ...4 5 3 3 3 ← (22) 6 ..4 5 3 3 3
(23) ....4 5 3 3 3 ← (24) 4 ..4 5 3 3 3
(27) 5 1 2 3 4 4 1 1 1 ← (28) 6 2 3 4 4 1 1 1

(49, 11)

(2) 4 3 5 6 5 2 3 5 7 7
(5) 2 4 5 5 5 3 6 6 5 3 ← (8) 4 5 5 5 3 6 6 5 3
(12) ..4 3 3 3 6 6 5 3 ← (18) ...3 3 6 6 5 3
(15) ....3 3 6 6 5 3 ← (16) 3 5 6 2 4 5 3 3 3
(18) .....3 5 7 3 3 ← (24) 4 5 3 3 3
(19) 4 ....4 5 3 3 3 ← (20) 5 8 1 1 2 4 3 3 3
(21) .....4 5 3 3 3 ← (22) 4 ...4 5 3 3 3
(27) 3 4 4 1 1 * 1 ← (28) 5 1 2 3 4 4 1 1 1
(31) 1 * * 1 ← (33) * * 1

(49, 12)

(3) ..4 5 5 5 3 6 6 5 3 ← (6) 2 4 5 5 5 3 6 6 5 3
(9) 6 ....3 3 6 6 5 3 ← (16) 3 3 6 6 5 3
(15) 6 .....4 5 3 3 3 ← (22) 4 5 3 3 3
(19) ......4 5 3 3 3 ← (20) 4 ....4 5 3 3 3
(30) 1 1 * * 1 ← (32) 1 * * 1

(49, 13)

(1) ...4 5 5 5 3 6 6 5 3 ← (4) ..4 5 5 5 3 6 6 5 3
(7) 5 6 .....3 5 7 3 3 ← (13) 6 .....3 5 7 3 3
(9) 3 6 .....3 5 7 3 3 ← (10) 6 ....3 3 6 6 5 3
(13) 6 ......4 5 3 3 3 ← (20) 4 5 3 3 3
(15) 4 ......4 5 3 3 3 ← (16) 6 .....4 5 3 3 3

(49, 14)

(2) * 2 4 7 3 3 6 6 5 3
(6) .....4 3 3 3 6 6 5 3 ← (11) 5 6 .....4 5 3 3 3
(7) 3 5 6 .....4 5 3 3 3 ← (8) 5 6 .....3 5 7 3 3
(9) ....3 3 6 6 5 3 ← (10) 3 6 .....3 5 7 3 3
(12) ....3 5 7 3 3 ← (17) 8 1 1 * 2 4 3 3 3
(13) 4 ....4 5 3 3 3 ← (14) 6 ......4 5 3 3 3
(15) ....4 5 3 3 3 ← (16) 4 ......4 5 3 3 3
(19) 5 1 2 3 4 4 1 1 * 1 ← (20) 6 2 3 4 4 1 1 * 1

(49, 15)

(1) 1 * 2 4 7 3 3 6 6 5 3
(3) 1.........4 5 3 3 3 ← (10) 3 3 6 6 5 3
(7) ....3 3 6 6 5 3 ← (8) 3 5 6 .....4 5 3 3 3
(10) ....3 5 7 3 3 ← (16) 4 5 3 3 3
(13) ....4 5 3 3 3 ← (14) 4 ....4 5 3 3 3
(19) 3 4 4 1 1 * * 1 ← (20) 5 1 2 3 4 4 1 1 * 1
(23) 1 * * * 1 ← (25) * * * 1

(49, 16)

(1) 6 ....3 3 6 6 5 3 ← (8) 3 3 6 6 5 3
(3) 10 ....4 5 3 3 3 ← (4) 1.........4 5 3 3 3
(7) 6 ....4 5 3 3 3 ← (14) 4 5 3 3 3
(22) 1 1 * * * 1 ← (24) 1 * * * 1

(49, 17)

(1) 3 6 ....3 5 7 3 3 ← (2) 6 ....3 3 6 6 5 3
(3) 7 13 1 * * * 1 ← (4) 10 ....4 5 3 3 3
(5) 6 ....4 5 3 3 3 ← (11) 13 1 * * * 1
(7) 4 ....4 5 3 3 3 ← (8) 6 ....4 5 3 3 3

(49, 18)

(1) ....3 3 6 6 5 3 ← (2) 3 6 ....3 5 7 3 3
(3) 5 8 1 1 * * 2 4 3 3 3 ← (4) 7 13 1 * * * 1
(4) ....3 5 7 3 3 ← (9) 8 1 1 * * 2 4 3 3 3
(5) 4 ....4 5 3 3 3 ← (6) 6 ....4 5 3 3 3
(7) ....4 5 3 3 3 ← (8) 4 ....4 5 3 3 3
(11) 5 1 2 3 4 4 1 1 * * 1 ← (12) 6 2 3 4 4 1 1 * * 1

(49, 19)

(2) ....3 5 7 3 3 ← (8) 4 5 3 3 3
(3) 4 ....4 5 3 3 3 ← (4) 5 8 1 1 * * 2 4 3 3 3
(5) ....4 5 3 3 3 ← (6) 4 ....4 5 3 3 3
(11) 3 4 4 1 1 * * * 1 ← (12) 5 1 2 3 4 4 1 1 * * 1
(15) 1 * * * * 1 ← (17) * * * * 1

(49, 20)

(3) ....4 5 3 3 3 ← (4) 4 ....4 5 3 3 3
(8) 1 1 * * * 2 4 3 3 3
(14) 1 1 * * * * 1 ← (16) 1 * * * * 1

(49, 21)

(3) 6 2 3 4 4 1 1 * * * 1

(49, 22)

(2) * * * * 2 4 3 3 3
(3) 5 1 2 3 4 4 1 1 * * * 1 ← (4) 6 2 3 4 4 1 1 * * * 1

(49, 23)

(1) 1 * * * * 2 4 3 3 3
(3) 3 4 4 1 1 * * * * 1 ← (4) 5 1 2 3 4 4 1 1 * * * 1
(7) 1 * * * * * 1 ← (9) * * * * * 1

(49, 24)

(1) 2 3 4 4 1 1 * * * * 1
(6) 1 1 * * * * * 1 ← (8) 1 * * * * * 1

(49, 25)

(2) 4 1 1 * * * * * 1

(50, 2)

(47) 3 ← (51)

(50, 3)

(19) 30 1 ← (20) 31
(35) 14 1 ← (36) 15
(43) 6 1 ← (44) 7
(44) 3 3 ← (50) 1
(47) 2 1 ← (48) 3

(50, 4)

(5) 15 15 15
(9) 11 15 15
(11) 21 11 7 ← (13) 23 15
(13) 23 7 7 ← (17) 27 7
(19) 13 11 7 ← (21) 15 15
(19) 29 1 1 ← (20) 30 1
(29) 7 7 7 ← (33) 11 7
(31) 5 7 7 ← (37) 7 7
(35) 13 1 1 ← (36) 14 1
(42) 2 3 3 ← (49) 1 1
(43) 5 1 1 ← (44) 6 1
(47) 1 1 1 ← (48) 2 1

(50, 5)

(2) 7 11 15 15
(5) 14 13 11 7 ← (6) 15 15 15
(9) 10 13 11 7 ← (10) 11 15 15
(10) 19 7 7 7 ← (12) 21 11 7
(11) 20 5 7 7 ← (14) 23 7 7
(17) 14 5 7 7 ← (20) 13 11 7
(19) 28 1 1 1 ← (20) 29 1 1
(27) 4 5 7 7 ← (30) 7 7 7
(28) 3 5 7 7 ← (32) 5 7 7
(35) 12 1 1 1 ← (36) 13 1 1
(41) 1 2 3 3 ← (48) 1 1 1
(43) 4 1 1 1 ← (44) 5 1 1

(50, 6)

(1) 9 11 15 7 7
(3) 7 11 15 7 7
(5) 9 15 7 7 7
(9) 9 11 7 7 7 ← (10) 10 13 11 7
(10) 18 3 5 7 7 ← (12) 20 5 7 7
(11) 7 13 5 7 7 ← (13) 17 7 7 7
(15) 5 9 7 7 7 ← (19) 13 5 7 7
(17) 11 3 5 7 7 ← (18) 14 5 7 7
(19) 24 4 1 1 1 ← (20) 28 1 1 1
(26) 2 3 5 7 7 ← (28) 4 5 7 7
(29) 3 5 7 3 3 ← (33) 5 7 3 3
(33) 3 6 2 3 3 ← (34) 8 3 3 3
(35) 2 4 3 3 3 ← (37) 6 2 3 3
(35) 8 4 1 1 1 ← (36) 12 1 1 1
(39) 4 4 1 1 1 ← (44) 4 1 1 1

(50, 7)

(1) 8 9 11 7 7 7 ← (2) 9 11 15 7 7
(3) 6 9 11 7 7 7 ← (4) 7 11 15 7 7
(5) 12 11 3 5 7 7 ← (6) 13 13 5 7 7
(10) 5 5 9 7 7 7 ← (12) 7 13 5 7 7
(12) 3 5 9 7 7 7 ← (16) 5 9 7 7 7
(13) 13 2 3 5 7 7 ← (14) 14 4 5 7 7
(14) 5 9 3 5 7 7 ← (20) 9 3 5 7 7
(19) 22 * 1 ← (20) 24 4 1 1 1
(27) 2 3 5 7 3 3 ← (28) 3 6 6 5 3
(33) ..4 3 3 3 ← (34) 3 6 2 3 3
(34) 1 2 4 3 3 3 ← (36) 2 4 3 3 3
(35) 6 * 1 ← (36) 8 4 1 1 1
(36) 3 4 4 1 1 1 ← (42) * 1
(39) 2 * 1 ← (40) 4 4 1 1 1

(50, 8)

(2) 8 5 5 9 7 7 7
(3) 9 3 5 9 7 7 7 ← (5) 11 5 9 7 7 7
(5) 7 8 12 9 3 3 3 ← (6) 12 11 3 5 7 7
(11) 3 5 9 3 5 7 7 ← (13) 8 12 9 3 3 3
(13) 3 5 7 3 5 7 7 ← (17) 5 7 3 5 7 7
(13) 7 12 4 5 3 3 3 ← (14) 13 2 3 5 7 7
(15) 6 5 2 3 5 7 7 ← (21) 12 4 5 3 3 3
(19) 21 1 * 1 ← (20) 22 * 1
(33) 1 1 2 4 3 3 3 ← (34) ..4 3 3 3
(34) 2 3 4 4 1 1 1 ← (41) 1 * 1
(35) 5 1 * 1 ← (36) 6 * 1
(39) 1 1 * 1 ← (40) 2 * 1

(50, 9)

(3) 4 5 3 5 9 7 7 7 ← (5) 8 3 5 9 7 7 7
(3) 8 3 5 9 3 5 7 7 ← (6) 7 8 12 9 3 3 3
(4) 9 3 5 7 3 5 7 7
(13) 4 7 3 3 6 6 5 3 ← (19) 6 3 3 6 6 5 3
(13) 5 6 3 3 6 6 5 3 ← (14) 7 12 4 5 3 3 3
(15) 3 6 3 3 6 6 5 3 ← (16) 6 5 2 3 5 7 7
(19) 20 1 1 * 1 ← (20) 21 1 * 1
(21) 3 6 2 4 5 3 3 3 ← (22) 6 2 3 5 7 3 3
(27) 12 1 1 * 1 ← (28) 13 1 * 1
(33) 1 2 3 4 4 1 1 1 ← (40) 1 1 * 1
(35) 4 1 1 * 1 ← (36) 5 1 * 1

(50, 10)

(2) 2 3 5 3 5 9 7 7 7 ← (4) 4 5 3 5 9 7 7 7
(2) 4 5 3 5 9 3 5 7 7 ← (4) 8 3 5 9 3 5 7 7
(5) 5 5 6 5 2 3 5 7 7
(11) 2 4 7 3 3 6 6 5 3 ← (14) 4 7 3 3 6 6 5 3
(12) 2 3 5 5 3 6 6 5 3 ← (17) 5 6 2 3 5 7 3 3
(13) 3 5 6 2 3 5 7 3 3 ← (14) 5 6 3 3 6 6 5 3
(15) 2 4 3 3 3 6 6 5 3 ← (16) 3 6 3 3 6 6 5 3
(19) 3 6 ..4 5 3 3 3 ← (20) 5 6 2 4 5 3 3 3
(19) 16 4 1 1 * 1 ← (20) 20 1 1 * 1
(21) ....3 5 7 3 3 ← (22) 3 6 2 4 5 3 3 3
(25) 3 6 2 3 4 4 1 1 1 ← (26) 8 1 1 2 4 3 3 3
(27) * 2 4 3 3 3 ← (29) 6 2 3 4 4 1 1 1
(27) 8 4 1 1 * 1 ← (28) 12 1 1 * 1
(31) 4 4 1 1 * 1 ← (36) 4 1 1 * 1

(50, 11)

(3) 4 3 5 6 5 2 3 5 7 7 ← (6) 5 5 6 5 2 3 5 7 7
(9) ..4 7 3 3 6 6 5 3 ← (16) 2 4 3 3 3 6 6 5 3
(10) 1 2 4 7 3 3 6 6 5 3 ← (12) 2 4 7 3 3 6 6 5 3
(13) ..4 3 3 3 6 6 5 3 ← (14) 3 5 6 2 3 5 7 3 3
(19) .....3 5 7 3 3 ← (20) 3 6 ..4 5 3 3 3
(19) 14 * * 1 ← (20) 16 4 1 1 * 1
(25) 2 * 2 4 3 3 3 ← (26) 3 6 2 3 4 4 1 1 1
(26) 1 * 2 4 3 3 3 ← (28) * 2 4 3 3 3
(27) 6 * * 1 ← (28) 8 4 1 1 * 1
(28) 3 4 4 1 1 * 1 ← (34) * * 1
(31) 2 * * 1 ← (32) 4 4 1 1 * 1

(50, 12)

(1) 2 4 3 5 6 5 2 3 5 7 7 ← (4) 4 3 5 6 5 2 3 5 7 7
(7) 6 ..4 3 3 3 6 6 5 3 ← (14) ..4 3 3 3 6 6 5 3
(9) 1 1 2 4 7 3 3 6 6 5 3 ← (10) ..4 7 3 3 6 6 5 3
(19) 13 1 * * 1 ← (20) 14 * * 1
(25) 1 1 * 2 4 3 3 3 ← (26) 2 * 2 4 3 3 3
(26) 2 3 4 4 1 1 * 1 ← (33) 1 * * 1
(27) 5 1 * * 1 ← (28) 6 * * 1
(31) 1 1 * * 1 ← (32) 2 * * 1

(50, 13)

(2) ...4 5 5 5 3 6 6 5 3

(5) 4 1 1 2 4 7 3 3 6 6 5 3 ← (11) 6 ....3 3 6 6 5 3

(7) 3 6 ....3 3 6 6 5 3 ← (8) 6 ..4 3 3 3 6 6 5 3

(13) 3 6 .....4 5 3 3 3 ← (14) 6 .....3 5 7 3 3

(19) 12 1 1 * * 1 ← (20) 13 1 * * 1

(25) 1 2 3 4 4 1 1 * 1 ← (32) 1 1 * * 1

(27) 4 1 1 * * 1 ← (28) 5 1 * * 1

(50, 14)

(3) * 2 4 7 3 3 6 6 5 3 ← (6) 4 1 1 2 4 7 3 3 6 6 5 3

(3) 24 4 1 1 * * 1 ← (9) 5 6 .....3 5 7 3 3

(7) .....4 3 3 3 6 6 5 3 ← (8) 3 6 ....3 3 6 6 5 3

(11) 3 6 ......4 5 3 3 3 ← (12) 5 6 .....4 5 3 3 3

(13) ....3 5 7 3 3 ← (14) 3 6 .....4 5 3 3 3

(17) 3 6 2 3 4 4 1 1 * 1 ← (18) 8 1 1 * 2 4 3 3 3

(19) * * 2 4 3 3 3 ← (21) 6 2 3 4 4 1 1 * 1

(19) 8 4 1 1 * * 1 ← (20) 12 1 1 * * 1

(23) 4 4 1 1 * * 1 ← (28) 4 1 1 * * 1

(50, 15)

(1) 2 * 2 4 7 3 3 6 6 5 3 ← (8) 4 3 3 3 6 6 5 3

(2) 1 * 2 4 7 3 3 6 6 5 3 ← (4) * 2 4 7 3 3 6 6 5 3

(3) 22 * * * 1 ← (4) 24 4 1 1 * * 1

(11) ....3 5 7 3 3 ← (12) 3 6 ......4 5 3 3 3

(17) 2 * * 2 4 3 3 3 ← (18) 3 6 2 3 4 4 1 1 * 1

(18) 1 * * 2 4 3 3 3 ← (20) * * 2 4 3 3 3

(19) 6 * * * 1 ← (20) 8 4 1 1 * * 1

(20) 3 4 4 1 1 * * 1 ← (26) * * * 1

(23) 2 * * * 1 ← (24) 4 4 1 1 * * 1

(50, 16)

(1) 1 1 * 2 4 7 3 3 6 6 5 3 ← (2) 2 * 2 4 7 3 3 6 6 5 3

(3) 21 1 * * * 1 ← (4) 22 * * * 1

(5) 6 ....3 5 7 3 3

(17) 1 1 * * 2 4 3 3 3 ← (18) 2 * * 2 4 3 3 3

(18) 2 3 4 4 1 1 * * 1 ← (25) 1 * * * 1

(19) 5 1 * * * 1 ← (20) 6 * * * 1

(23) 1 1 * * * 1 ← (24) 2 * * * 1

(50, 17)

(3) 5 6 ....4 5 3 3 3

(3) 20 1 1 * * * 1 ← (4) 21 1 * * * 1

(5) 3 6 ....4 5 3 3 3 ← (6) 6 ....3 5 7 3 3

(11) 12 1 1 * * * 1 ← (12) 13 1 * * * 1

(17) 1 2 3 4 4 1 1 * * 1 ← (24) 1 1 * * * 1

(19) 4 1 1 * * * 1 ← (20) 5 1 * * * 1

(50, 18)

(2) 3 3 6 6 5 3

(3) 3 6 ....4 5 3 3 3 ← (4) 5 6 ....4 5 3 3 3

(3) 16 4 1 1 * * * 1 ← (4) 20 1 1 * * * 1

(5) ....3 5 7 3 3 ← (6) 3 6 ....4 5 3 3 3

(9) 3 6 2 3 4 4 1 1 * * 1 ← (10) 8 1 1 * * 2 4 3 3 3

(11) * * * 2 4 3 3 3 ← (13) 6 2 3 4 4 1 1 * * 1

(11) 8 4 1 1 * * * 1 ← (12) 12 1 1 * * * 1

(15) 4 4 1 1 * * * 1 ← (20) 4 1 1 * * * 1

(50, 19)

(3) ....3 5 7 3 3 ← (4) 3 6 ....4 5 3 3 3

(3) 14 * * * * 1 ← (4) 16 4 1 1 * * * 1

(6) 4 5 3 3 3

(9) 2 * * * 2 4 3 3 3 ← (10) 3 6 2 3 4 4 1 1 * * 1

(10) 1 * * * 2 4 3 3 3 ← (12) * * * 2 4 3 3 3

(11) 6 * * * * 1 ← (12) 8 4 1 1 * * * 1

(12) 3 4 4 1 1 * * * 1 ← (18) * * * * 1

(15) 2 * * * * 1 ← (16) 4 4 1 1 * * * 1

(50, 20)

(3) 13 1 * * * * 1 ← (4) 14 * * * * 1

(4) 4 5 3 3 3

(9) 1 1 * * * 2 4 3 3 3 ← (10) 2 * * * 2 4 3 3 3

(10) 2 3 4 4 1 1 * * * 1 ← (17) 1 * * * * 1

(11) 5 1 * * * * 1 ← (12) 6 * * * * 1

(15) 1 1 * * * * 1 ← (16) 2 * * * * 1

(50, 21)

(1) 8 1 1 * * * 2 4 3 3 3

(3) 12 1 1 * * * * 1 ← (4) 13 1 * * * * 1

(9) 1 2 3 4 4 1 1 * * * 1 ← (16) 1 1 * * * * 1

(11) 4 1 1 * * * * 1 ← (12) 5 1 * * * * 1

(50, 22)

(1) 3 6 2 3 4 4 1 1 * * * 1 ← (2) 8 1 1 * * * 2 4 3 3 3

(3) * * * * 2 4 3 3 3 ← (5) 6 2 3 4 4 1 1 * * * 1

(3) 8 4 1 1 * * * * 1 ← (4) 12 1 1 * * * * 1

(7) 4 4 1 1 * * * * 1 ← (12) 4 1 1 * * * * 1

(50, 23)

(1) 2 * * * * 2 4 3 3 3 ← (2) 3 6 2 3 4 4 1 1 * * * 1

(2) 1 * * * * 2 4 3 3 3 ← (4) * * * * 2 4 3 3 3

(3) 6 * * * * * 1 ← (4) 8 4 1 1 * * * * 1

(4) 3 4 4 1 1 * * * * 1 ← (10) * * * * * 1

(7) 2 * * * * * 1 ← (8) 4 4 1 1 * * * * 1

(50, 24)

(1) 1 1 * * * * 2 4 3 3 3 ← (2) 2 * * * * 2 4 3 3 3
(2) 2 3 4 4 1 1 * * * * 1 ← (9) 1 * * * * * 1
(3) 5 1 * * * * * 1 ← (4) 6 * * * * * 1
(7) 1 1 * * * * * 1 ← (8) 2 * * * * * 1

(51, 3)

(19) 29 3 ← (21) 31
(35) 13 3 ← (37) 15
(43) 5 3 ← (45) 7
(45) 3 3 ← (49) 3

(51, 4)

(13) 22 13 3 ← (14) 23 15
(17) 26 5 3 ← (18) 27 7
(18) 27 3 3 ← (20) 29 3
(21) 14 13 3 ← (22) 15 15
(33) 10 5 3 ← (34) 11 7
(34) 11 3 3 ← (36) 13 3
(37) 6 5 3 ← (38) 7 7
(42) 3 3 3 ← (44) 5 3
(43) 2 3 3 ← (46) 3 3

(51, 5)

(1) 9 11 15 15
(3) 7 11 15 15
(6) 14 13 11 7
(7) 13 13 11 7
(11) 11 15 7 7 ← (21) 13 11 7
(11) 19 7 7 7 ← (13) 21 11 7
(13) 21 11 3 3 ← (14) 22 13 3
(17) 25 3 3 3 ← (18) 26 5 3
(21) 13 11 3 3 ← (22) 14 13 3
(29) 3 5 7 7 ← (33) 5 7 7
(31) 6 6 5 3 ← (38) 6 5 3
(33) 9 3 3 3 ← (34) 10 5 3
(42) 1 2 3 3 ← (44) 2 3 3

(51, 6)

(3) 10 17 7 7 7
(6) 9 15 7 7 7
(7) 11 14 5 7 7 ← (8) 13 13 11 7
(10) 9 11 7 7 7 ← (14) 17 7 7 7
(11) 7 14 5 7 7 ← (12) 11 15 7 7
(11) 18 3 5 7 7 ← (13) 20 5 7 7
(13) 20 9 3 3 3 ← (14) 21 11 3 3
(18) 11 3 5 7 7 ← (20) 13 5 7 7
(21) 12 9 3 3 3 ← (22) 13 11 3 3
(27) 2 3 5 7 7 ← (29) 4 5 7 7
(30) 3 5 7 3 3 ← (35) 8 3 3 3
(31) 4 7 3 3 3 ← (32) 6 6 5 3
(33) 4 5 3 3 3 ← (34) 5 7 3 3
(37) 5 1 2 3 3 ← (38) 6 2 3 3

(51, 7)

(1) 5 9 15 7 7 7
(2) 8 9 11 7 7 7
(3) 9 7 13 5 7 7 ← (4) 10 17 7 7 7
(4) 6 9 11 7 7 7
(7) 7 14 4 5 7 7 ← (8) 11 14 5 7 7
(9) 12 12 9 3 3 3 ← (12) 18 3 5 7 7
(11) 5 5 9 7 7 7 ← (12) 7 14 5 7 7
(11) 17 3 6 6 5 3 ← (14) 20 9 3 3 3
(13) 3 5 9 7 7 7 ← (17) 5 9 7 7 7
(15) 5 9 3 5 7 7 ← (21) 9 3 5 7 7
(19) 9 3 6 6 5 3 ← (22) 12 9 3 3 3
(25) 3 3 6 6 5 3 ← (28) 2 3 5 7 7
(28) 2 3 5 7 3 3 ← (34) 4 5 3 3 3
(31) 2 4 5 3 3 3 ← (32) 4 7 3 3 3
(35) 1 2 4 3 3 3 ← (37) 2 4 3 3 3
(37) 3 4 4 1 1 1 ← (38) 5 1 2 3 3

(51, 8)

(3) 5 10 11 3 5 7 7
(3) 8 5 5 9 7 7 7 ← (4) 9 7 13 5 7 7
(4) 9 3 5 9 7 7 7
(9) 10 9 3 6 6 5 3 ← (10) 12 12 9 3 3 3
(10) 11 12 4 5 3 3 3 ← (12) 17 3 6 6 5 3
(12) 3 5 9 3 5 7 7 ← (18) 5 7 3 5 7 7
(13) 6 9 3 6 6 5 3 ← (14) 8 12 9 3 3 3
(14) 3 5 7 3 5 7 7 ← (16) 5 9 3 5 7 7
(18) 5 5 3 6 6 5 3 ← (20) 9 3 6 6 5 3
(21) 10 2 4 5 3 3 3 ← (22) 12 4 5 3 3 3
(25) 6 2 4 5 3 3 3 ← (32) 2 4 5 3 3 3
(34) 1 1 2 4 3 3 3 ← (36) 1 2 4 3 3 3
(35) 2 3 4 4 1 1 1 ← (38) 3 4 4 1 1 1

(51, 9)

(3) 4 6 3 5 9 7 7 7 ← (4) 5 10 11 3 5 7 7
(5) 9 3 5 7 3 5 7 7
(9) 9 5 5 3 6 6 5 3 ← (10) 10 9 3 6 6 5 3
(11) 5 6 5 2 3 5 7 7 ← (17) 6 5 2 3 5 7 7
(13) 3 6 5 2 3 5 7 7 ← (14) 6 9 3 6 6 5 3
(19) 3 6 2 3 5 7 3 3 ← (20) 6 3 3 6 6 5 3
(21) 7 13 1 * 1 ← (22) 10 2 4 5 3 3 3
(23) 6 ..4 5 3 3 3 ← (29) 13 1 * 1
(25) 4 ..4 5 3 3 3 ← (26) 6 2 4 5 3 3 3
(34) 1 2 3 4 4 1 1 1 ← (36) 2 3 4 4 1 1 1

(51, 10)

(3) 2 3 5 3 5 9 7 7 7 ← (4) 4 6 3 5 9 7 7 7
(3) 4 5 3 5 9 3 5 7 7 ← (5) 8 3 5 9 3 5 7 7
(9) 4 5 5 5 3 6 6 5 3 ← (12) 5 6 5 2 3 5 7 7
(13) 2 3 5 5 3 6 6 5 3 ← (14) 3 6 5 2 3 5 7 7
(17) 3 5 6 2 4 5 3 3 3 ← (18) 5 6 2 3 5 7 3 3
(19) ...3 3 6 6 5 3 ← (20) 3 6 2 3 5 7 3 3
(21) 5 8 1 1 2 4 3 3 3 ← (22) 7 13 1 * 1
(22) ....3 5 7 3 3 ← (27) 8 1 1 2 4 3 3 3
(23) 4 ...4 5 3 3 3 ← (24) 6 ..4 5 3 3 3
(25) ....4 5 3 3 3 ← (26) 4 ..4 5 3 3 3
(29) 5 1 2 3 4 4 1 1 1 ← (30) 6 2 3 4 4 1 1 1

(51, 11)

(1) 5 5 5 6 5 2 3 5 7 7
(7) 2 4 5 5 5 3 6 6 5 3 ← (10) 4 5 5 5 3 6 6 5 3
(11) 1 2 4 7 3 3 6 6 5 3 ← (13) 2 4 7 3 3 6 6 5 3
(17) ....3 3 6 6 5 3 ← (18) 3 5 6 2 4 5 3 3 3
(20) .....3 5 7 3 3 ← (26) 4 5 3 3 3
(21) 4 ....4 5 3 3 3 ← (22) 5 8 1 1 2 4 3 3 3
(23) .....4 5 3 3 3 ← (24) 4 ...4 5 3 3 3
(27) 1 * 2 4 3 3 3 ← (29) * 2 4 3 3 3
(29) 3 4 4 1 1 * 1 ← (30) 5 1 2 3 4 4 1 1 1

(51, 12)

(2) 2 4 3 5 6 5 2 3 5 7 7
(5) ..4 5 5 5 3 6 6 5 3 ← (8) 2 4 5 5 5 3 6 6 5 3
(10) 1 1 2 4 7 3 3 6 6 5 3 ← (12) 1 2 4 7 3 3 6 6 5 3
(17) 6 .....4 5 3 3 3 ← (24) 4 5 3 3 3
(21) ......4 5 3 3 3 ← (22) 4 ....4 5 3 3 3
(26) 1 1 * 2 4 3 3 3 ← (28) 1 * 2 4 3 3 3
(27) 2 3 4 4 1 1 * 1 ← (30) 3 4 4 1 1 * 1

(51, 13)

(3) ...4 5 5 5 3 6 6 5 3 ← (6) ..4 5 5 5 3 6 6 5 3
(11) 3 6 .....3 5 7 3 3 ← (12) 6 ....3 3 6 6 5 3
(15) 6 ......4 5 3 3 3 ← (22) 4 5 3 3 3
(17) 4 ......4 5 3 3 3 ← (18) 6 .....4 5 3 3 3
(26) 1 2 3 4 4 1 1 * 1 ← (28) 2 3 4 4 1 1 * 1

(51, 14)

(1) ....4 5 5 5 3 6 6 5 3 ← (4) ...4 5 5 5 3 6 6 5 3
(9) 3 5 6 .....4 5 3 3 3 ← (10) 5 6 .....3 5 7 3 3
(11) ....3 3 6 6 5 3 ← (12) 3 6 .....3 5 7 3 3
(14) ....3 5 7 3 3 ← (19) 8 1 1 * 2 4 3 3 3
(15) 4 ....4 5 3 3 3 ← (16) 6 ......4 5 3 3 3
(17) ....4 5 3 3 3 ← (18) 4 ......4 5 3 3 3
(21) 5 1 2 3 4 4 1 1 * 1 ← (22) 6 2 3 4 4 1 1 * 1

(51, 15)

(3) 1 * 2 4 7 3 3 6 6 5 3 ← (5) * 2 4 7 3 3 6 6 5 3
(5) 1.........4 5 3 3 3
(9) ....3 3 6 6 5 3 ← (10) 3 5 6 .....4 5 3 3 3
(12) ....3 5 7 3 3 ← (18) 4 5 3 3 3
(15) ....4 5 3 3 3 ← (16) 4 ....4 5 3 3 3
(19) 1 * * 2 4 3 3 3 ← (21) * * 2 4 3 3 3
(21) 3 4 4 1 1 * * 1 ← (22) 5 1 2 3 4 4 1 1 * 1

(51, 16)

(2) 1 1 * 2 4 7 3 3 6 6 5 3 ← (4) 1 * 2 4 7 3 3 6 6 5 3
(3) 6 ....3 3 6 6 5 3
(5) 10 ....4 5 3 3 3 ← (6) 1.........4 5 3 3 3
(9) 6 ....4 5 3 3 3 ← (16) 4 5 3 3 3
(18) 1 1 * * 2 4 3 3 3 ← (20) 1 * * 2 4 3 3 3
(19) 2 3 4 4 1 1 * * 1 ← (22) 3 4 4 1 1 * * 1

(51, 17)

(1) 5 6 ....3 5 7 3 3
(3) 3 6 ....3 5 7 3 3 ← (4) 6 ....3 3 6 6 5 3
(5) 7 13 1 * * * 1 ← (6) 10 ....4 5 3 3 3
(7) 6 ....4 5 3 3 3 ← (13) 13 1 * * * 1
(9) 4 ....4 5 3 3 3 ← (10) 6 ....4 5 3 3 3
(18) 1 2 3 4 4 1 1 * * 1 ← (20) 2 3 4 4 1 1 * * 1

(51, 18)

(1) 3 5 6 ....4 5 3 3 3 ← (2) 5 6 ....3 5 7 3 3
(3) ....3 3 6 6 5 3 ← (4) 3 6 ....3 5 7 3 3
(5) 5 8 1 1 * * 2 4 3 3 3 ← (6) 7 13 1 * * * 1
(6) ....3 5 7 3 3 ← (11) 8 1 1 * * 2 4 3 3 3
(7) 4 ....4 5 3 3 3 ← (8) 6 ....4 5 3 3 3
(9) ....4 5 3 3 3 ← (10) 4 ....4 5 3 3 3
(13) 5 1 2 3 4 4 1 1 * * 1 ← (14) 6 2 3 4 4 1 1 * * 1

(51, 19)

(1) ....3 3 6 6 5 3 ← (2) 3 5 6 ....4 5 3 3 3
(4) ....3 5 7 3 3 ← (10) 4 5 3 3 3
(5) 4 ....4 5 3 3 3 ← (6) 5 8 1 1 * * 2 4 3 3 3
(7) ....4 5 3 3 3 ← (8) 4 ....4 5 3 3 3
(11) 1 * * * 2 4 3 3 3 ← (13) * * * 2 4 3 3 3
(13) 3 4 4 1 1 * * * 1 ← (14) 5 1 2 3 4 4 1 1 * * 1

(51, 20)

(1) 6 ....4 5 3 3 3 ← (8) 4 5 3 3 3
(5) ....4 5 3 3 3 ← (6) 4 ....4 5 3 3 3
(10) 1 1 * * * 2 4 3 3 3 ← (12) 1 * * * 2 4 3 3 3
(11) 2 3 4 4 1 1 * * * 1 ← (14) 3 4 4 1 1 * * * 1

(51, 21)

(1) 4 ....4 5 3 3 3 ← (2) 6 ....4 5 3 3 3
(10) 1 2 3 4 4 1 1 * * * 1 ← (12) 2 3 4 4 1 1 * * * 1

(51, 22)

(1) ....4 5 3 3 3 ← (2) 4 ....4 5 3 3 3
(5) 5 1 2 3 4 4 1 1 * * * 1 ← (6) 6 2 3 4 4 1 1 * * * 1

(51, 23)

(3) 1 * * * * 2 4 3 3 3 ← (5) * * * * 2 4 3 3 3
(5) 3 4 4 1 1 * * * * 1 ← (6) 5 1 2 3 4 4 1 1 * * * 1

(52, 2)

(51) 1 ← (53)

(52, 3)

(21) 30 1 ← (22) 31
(37) 14 1 ← (38) 15
(45) 6 1 ← (46) 7
(49) 2 1 ← (50) 3
(50) 1 1 ← (52) 1

(52, 4)

(7) 15 15 15
(11) 11 15 15 ← (39) 7 7
(15) 23 7 7 ← (19) 27 7
(19) 27 3 3 ← (21) 29 3
(21) 29 1 1 ← (22) 30 1
(31) 7 7 7 ← (35) 11 7
(35) 11 3 3 ← (37) 13 3
(37) 13 1 1 ← (38) 14 1
(43) 3 3 3 ← (45) 5 3
(45) 5 1 1 ← (46) 6 1
(49) 1 1 1 ← (50) 2 1

(52, 5)

(2) 9 11 15 15
(4) 7 11 15 15
(7) 14 13 11 7 ← (8) 15 15 15
(11) 10 13 11 7 ← (12) 11 15 15
(12) 19 7 7 7 ← (16) 23 7 7
(18) 25 3 3 3 ← (20) 27 3 3
(19) 14 5 7 7 ← (32) 7 7 7
(21) 28 1 1 1 ← (22) 29 1 1
(30) 3 5 7 7 ← (34) 5 7 7
(34) 9 3 3 3 ← (36) 11 3 3
(37) 12 1 1 1 ← (38) 13 1 1
(43) 1 2 3 3 ← (45) 2 3 3
(45) 4 1 1 1 ← (46) 5 1 1

(52, 6)

(1) 7 13 13 11 7
(3) 9 11 15 7 7
(5) 7 11 15 7 7
(7) 9 15 7 7 7 ← (9) 13 13 11 7
(7) 13 13 5 7 7 ← (14) 20 5 7 7
(11) 9 11 7 7 7 ← (12) 10 13 11 7
(13) 7 13 5 7 7 ← (21) 13 5 7 7
(15) 14 4 5 7 7 ← (30) 4 5 7 7
(19) 11 3 5 7 7 ← (20) 14 5 7 7
(21) 24 4 1 1 1 ← (22) 28 1 1 1
(29) 3 6 6 5 3 ← (35) 5 7 3 3
(31) 3 5 7 3 3 ← (33) 6 6 5 3
(35) 3 6 2 3 3 ← (36) 8 3 3 3
(37) 8 4 1 1 1 ← (38) 12 1 1 1
(41) 4 4 1 1 1 ← (44) 1 2 3 3
(43) * 1 ← (46) 4 1 1 1

(52, 7)

(1) 6 9 15 7 7 7 ← (2) 7 13 13 11 7
(2) 5 9 15 7 7 7
(3) 8 9 11 7 7 7 ← (4) 9 11 15 7 7
(5) 6 9 11 7 7 7 ← (6) 7 11 15 7 7
(6) 11 5 9 7 7 7 ← (13) 18 3 5 7 7
(7) 12 11 3 5 7 7 ← (8) 13 13 5 7 7
(8) 7 14 4 5 7 7 ← (29) 2 3 5 7 7
(12) 5 5 9 7 7 7 ← (20) 11 3 5 7 7
(14) 3 5 9 7 7 7 ← (18) 5 9 7 7 7
(15) 13 2 3 5 7 7 ← (16) 14 4 5 7 7
(21) 22 * 1 ← (22) 24 4 1 1 1
(26) 3 3 6 6 5 3 ← (32) 3 5 7 3 3
(29) 2 3 5 7 3 3 ← (30) 3 6 6 5 3
(35) ..4 3 3 3 ← (36) 3 6 2 3 3
(37) 6 * 1 ← (38) 8 4 1 1 1
(41) 2 * 1 ← (42) 4 4 1 1 1
(42) 1 * 1 ← (44) * 1

(52, 8)

(1) 4 6 9 11 7 7 7 ← (2) 6 9 15 7 7 7
(4) 8 5 5 9 7 7 7
(5) 9 3 5 9 7 7 7 ← (11) 12 12 9 3 3 3
(6) 8 3 5 9 7 7 7 ← (23) 12 4 5 3 3 3
(7) 7 8 12 9 3 3 3 ← (8) 12 11 3 5 7 7
(11) 11 12 4 5 3 3 3 ← (13) 17 3 6 6 5 3
(13) 3 5 9 3 5 7 7 ← (19) 5 7 3 5 7 7
(15) 3 5 7 3 5 7 7 ← (17) 5 9 3 5 7 7
(15) 7 12 4 5 3 3 3 ← (16) 13 2 3 5 7 7
(19) 5 5 3 6 6 5 3 ← (21) 9 3 6 6 5 3
(21) 21 1 * 1 ← (22) 22 * 1
(23) 6 2 3 5 7 3 3 ← (30) 2 3 5 7 3 3
(35) 1 1 2 4 3 3 3 ← (36) ..4 3 3 3
(37) 5 1 * 1 ← (38) 6 * 1
(41) 1 1 * 1 ← (42) 2 * 1

(52, 9)

(1) 3 5 10 11 3 5 7 7
(5) 4 5 3 5 9 7 7 7 ← (21) 6 3 3 6 6 5 3
(6) 9 3 5 7 3 5 7 7 ← (16) 3 5 7 3 5 7 7
(10) 9 5 5 3 6 6 5 3 ← (12) 11 12 4 5 3 3 3
(15) 4 7 3 3 6 6 5 3 ← (20) 5 5 3 6 6 5 3
(15) 5 6 3 3 6 6 5 3 ← (16) 7 12 4 5 3 3 3
(17) 3 6 3 3 6 6 5 3 ← (18) 6 5 2 3 5 7 7
(21) 5 6 2 4 5 3 3 3 ← (27) 6 2 4 5 3 3 3
(21) 20 1 1 * 1 ← (22) 21 1 * 1
(23) 3 6 2 4 5 3 3 3 ← (24) 6 2 3 5 7 3 3
(29) 12 1 1 * 1 ← (30) 13 1 * 1
(35) 1 2 3 4 4 1 1 1 ← (37) 2 3 4 4 1 1 1
(37) 4 1 1 * 1 ← (38) 5 1 * 1

(52, 10)

(1) 5 9 3 5 7 3 5 7 7
(4) 2 3 5 3 5 9 7 7 7 ← (19) 5 6 2 3 5 7 3 3
(4) 4 5 3 5 9 3 5 7 7
(7) 5 5 6 5 2 3 5 7 7 ← (13) 5 6 5 2 3 5 7 7
(14) 2 3 5 5 3 6 6 5 3 ← (16) 4 7 3 3 6 6 5 3
(15) 3 5 6 2 3 5 7 3 3 ← (16) 5 6 3 3 6 6 5 3
(17) 2 4 3 3 3 6 6 5 3 ← (18) 3 6 3 3 6 6 5 3
(20) ...3 3 6 6 5 3 ← (25) 6 ..4 5 3 3 3
(21) 3 6 ..4 5 3 3 3 ← (22) 5 6 2 4 5 3 3 3
(21) 16 4 1 1 * 1 ← (22) 20 1 1 * 1
(23) ....3 5 7 3 3 ← (24) 3 6 2 4 5 3 3 3
(27) 3 6 2 3 4 4 1 1 1 ← (28) 8 1 1 2 4 3 3 3
(29) 8 4 1 1 * 1 ← (30) 12 1 1 * 1
(33) 4 4 1 1 * 1 ← (36) 1 2 3 4 4 1 1 1
(35) * * 1 ← (38) 4 1 1 * 1

(52, 11)

(2) 5 5 5 6 5 2 3 5 7 7
(5) 4 3 5 6 5 2 3 5 7 7 ← (8) 5 5 6 5 2 3 5 7 7
(11) ..4 7 3 3 6 6 5 3 ← (14) 2 4 7 3 3 6 6 5 3
(15) ..4 3 3 3 6 6 5 3 ← (16) 3 5 6 2 3 5 7 3 3
(18) ....3 3 6 6 5 3 ← (24) 3 5 7 3 3
(21) .....3 5 7 3 3 ← (22) 3 6 ..4 5 3 3 3
(21) 14 * * 1 ← (22) 16 4 1 1 * 1
(27) 2 * 2 4 3 3 3 ← (28) 3 6 2 3 4 4 1 1 1
(29) 6 * * 1 ← (30) 8 4 1 1 * 1
(33) 2 * * 1 ← (34) 4 4 1 1 * 1
(34) 1 * * 1 ← (36) * * 1

(52, 12)

(3) 2 4 3 5 6 5 2 3 5 7 7 ← (6) 4 3 5 6 5 2 3 5 7 7
(9) 6 ..4 3 3 3 6 6 5 3 ← (13) 1 2 4 7 3 3 6 6 5 3
(11) 1 1 2 4 7 3 3 6 6 5 3 ← (12) ..4 7 3 3 6 6 5 3
(15) 6 .....3 5 7 3 3 ← (22) 3 5 7 3 3
(21) 13 1 * * 1 ← (22) 14 * * 1
(27) 1 1 * 2 4 3 3 3 ← (28) 2 * 2 4 3 3 3
(29) 5 1 * * 1 ← (30) 6 * * 1
(33) 1 1 * * 1 ← (34) 2 * * 1

(52, 13)

(1) ..4 3 5 6 5 2 3 5 7 7 ← (4) 2 4 3 5 6 5 2 3 5 7 7
(7) 4 1 1 2 4 7 3 3 6 6 5 3 ← (12) 1 1 2 4 7 3 3 6 6 5 3
(9) 3 6 ....3 3 6 6 5 3 ← (10) 6 ..4 3 3 3 6 6 5 3
(13) 5 6 .....4 5 3 3 3 ← (19) 6 .....4 5 3 3 3
(15) 3 6 .....4 5 3 3 3 ← (16) 6 .....3 5 7 3 3
(21) 12 1 1 * * 1 ← (22) 13 1 * * 1
(27) 1 2 3 4 4 1 1 * 1 ← (29) 2 3 4 4 1 1 * 1
(29) 4 1 1 * * 1 ← (30) 5 1 * * 1

(52, 14)

(2) 4 5 5 5 3 6 6 5 3
(5) 24 4 1 1 * * 1 ← (8) 4 1 1 2 4 7 3 3 6 6 5 3
(9) .....4 3 3 3 6 6 5 3 ← (10) 3 6 ....3 3 6 6 5 3
(12) ....3 3 6 6 5 3 ← (17) 6 ......4 5 3 3 3
(13) 3 6 ......4 5 3 3 3 ← (14) 5 6 .....4 5 3 3 3
(15) ....3 5 7 3 3 ← (16) 3 6 .....4 5 3 3 3
(19) 3 6 2 3 4 4 1 1 * 1 ← (20) 8 1 1 * 2 4 3 3 3
(21) 8 4 1 1 * * 1 ← (22) 12 1 1 * * 1
(25) 4 4 1 1 * * 1 ← (28) 1 2 3 4 4 1 1 * 1
(27) * * * 1 ← (30) 4 1 1 * * 1

(52, 15)

(3) 2 * 2 4 7 3 3 6 6 5 3 ← (6) * 2 4 7 3 3 6 6 5 3
(5) 22 * * * 1 ← (6) 24 4 1 1 * * 1
(10) ....3 3 6 6 5 3 ← (16) 3 5 7 3 3
(13) ....3 5 7 3 3 ← (14) 3 6 ......4 5 3 3 3
(19) 2 * * 2 4 3 3 3 ← (20) 3 6 2 3 4 4 1 1 * 1
(21) 6 * * * 1 ← (22) 8 4 1 1 * * 1
(25) 2 * * * 1 ← (26) 4 4 1 1 * * 1
(26) 1 * * * 1 ← (28) * * * 1

(52, 16)

(1) 6 ......4 3 3 3 6 6 5 3 ← (5) 1 * 2 4 7 3 3 6 6 5 3
(3) 1 1 * 2 4 7 3 3 6 6 5 3 ← (4) 2 * 2 4 7 3 3 6 6 5 3
(5) 21 1 * * * 1 ← (6) 22 * * * 1
(7) 6 ....3 5 7 3 3 ← (14) 3 5 7 3 3
(19) 1 1 * * 2 4 3 3 3 ← (20) 2 * * 2 4 3 3 3
(21) 5 1 * * * 1 ← (22) 6 * * * 1
(25) 1 1 * * * 1 ← (26) 2 * * * 1

(52, 17)

(1) 3 6 ....3 3 6 6 5 3 ← (2) 6 ......4 3 3 3 6 6 5 3
(5) 5 6 ....4 5 3 3 3 ← (11) 6 ....4 5 3 3 3
(5) 20 1 1 * * * 1 ← (6) 21 1 * * * 1
(7) 3 6 ....4 5 3 3 3 ← (8) 6 ....3 5 7 3 3
(13) 12 1 1 * * * 1 ← (14) 13 1 * * * 1
(19) 1 2 3 4 4 1 1 * * 1 ← (21) 2 3 4 4 1 1 * * 1
(21) 4 1 1 * * * 1 ← (22) 5 1 * * * 1

(52, 18)

(1) ....4 3 3 3 6 6 5 3 ← (2) 3 6 ....3 3 6 6 5 3
(4) ....3 3 6 6 5 3 ← (9) 6 ....4 5 3 3 3
(5) 3 6 ....4 5 3 3 3 ← (6) 5 6 ....4 5 3 3 3
(5) 16 4 1 1 * * * 1 ← (6) 20 1 1 * * * 1
(7) ....3 5 7 3 3 ← (8) 3 6 ....4 5 3 3 3
(11) 3 6 2 3 4 4 1 1 * * 1 ← (12) 8 1 1 * * 2 4 3 3 3
(13) 8 4 1 1 * * * 1 ← (14) 12 1 1 * * * 1
(17) 4 4 1 1 * * * 1 ← (20) 1 2 3 4 4 1 1 * * 1
(19) * * * * 1 ← (22) 4 1 1 * * * 1

(52, 19)

(2) ....3 3 6 6 5 3 ← (8) 3 5 7 3 3
(5) ....3 5 7 3 3 ← (6) 3 6 ....4 5 3 3 3
(5) 14 * * * * 1 ← (6) 16 4 1 1 * * * 1
(11) 2 * * * 2 4 3 3 3 ← (12) 3 6 2 3 4 4 1 1 * * 1
(13) 6 * * * * 1 ← (14) 8 4 1 1 * * * 1
(17) 2 * * * * 1 ← (18) 4 4 1 1 * * * 1
(18) 1 * * * * 1 ← (20) * * * * 1

(52, 20)

(5) 13 1 * * * * 1 ← (6) 14 * * * * 1
(6) 4 5 3 3 3
(11) 1 1 * * * 2 4 3 3 3 ← (12) 2 * * * 2 4 3 3 3
(13) 5 1 * * * * 1 ← (14) 6 * * * * 1
(17) 1 1 * * * * 1 ← (18) 2 * * * * 1

(52, 21)

(3) 8 1 1 * * * 2 4 3 3 3
(5) 12 1 1 * * * * 1 ← (6) 13 1 * * * * 1
(11) 1 2 3 4 4 1 1 * * * 1 ← (13) 2 3 4 4 1 1 * * * 1
(13) 4 1 1 * * * * 1 ← (14) 5 1 * * * * 1

(52, 22)

(2) 4 5 3 3 3
(3) 3 6 2 3 4 4 1 1 * * * 1 ← (4) 8 1 1 * * * 2 4 3 3 3
(5) 8 4 1 1 * * * * 1 ← (6) 12 1 1 * * * * 1
(9) 4 4 1 1 * * * * 1 ← (12) 1 2 3 4 4 1 1 * * * 1
(11) * * * * * 1 ← (14) 4 1 1 * * * * 1

(53, 3)

(15) 23 15 ← (23) 31
(23) 15 15 ← (39) 15
(47) 3 3 ← (51) 3
(51) 1 1 ← (53) 1

(53, 4)

(14) 21 11 7 ← (22) 29 3
(15) 22 13 3 ← (16) 23 15
(19) 26 5 3 ← (20) 27 7
(22) 13 11 7 ← (38) 13 3
(23) 14 13 3 ← (24) 15 15
(35) 10 5 3 ← (36) 11 7
(39) 6 5 3 ← (40) 7 7
(44) 3 3 3 ← (48) 3 3
(50) 1 1 1 ← (52) 1 1

(53, 5)

(3) 9 11 15 15
(5) 7 11 15 15
(8) 14 13 11 7 ← (21) 27 3 3
(13) 11 15 7 7 ← (37) 11 3 3
(13) 19 7 7 7 ← (17) 23 7 7
(15) 17 7 7 7 ← (33) 7 7 7
(15) 21 11 3 3 ← (16) 22 13 3
(19) 25 3 3 3 ← (20) 26 5 3
(23) 13 11 3 3 ← (24) 14 13 3
(31) 3 5 7 7 ← (35) 5 7 7
(35) 9 3 3 3 ← (36) 10 5 3
(39) 6 2 3 3 ← (46) 2 3 3

(53, 6)

(1) 4 7 11 15 15
(5) 10 17 7 7 7 ← (20) 25 3 3 3
(8) 9 15 7 7 7 ← (36) 9 3 3 3
(9) 11 14 5 7 7 ← (10) 13 13 11 7
(12) 9 11 7 7 7 ← (22) 13 5 7 7
(13) 7 14 5 7 7 ← (14) 11 15 7 7
(14) 7 13 5 7 7 ← (16) 17 7 7 7
(15) 20 9 3 3 3 ← (16) 21 11 3 3
(22) 9 3 5 7 7 ← (32) 3 5 7 7
(23) 12 9 3 3 3 ← (24) 13 11 3 3
(33) 4 7 3 3 3 ← (34) 6 6 5 3
(35) 4 5 3 3 3 ← (36) 5 7 3 3
(38) 2 4 3 3 3 ← (45) 1 2 3 3
(39) 5 1 2 3 3 ← (40) 6 2 3 3

(53, 7)

(1) 5 7 11 15 7 7
(3) 5 9 15 7 7 7
(4) 8 9 11 7 7 7 ← (16) 20 9 3 3 3
(5) 9 7 13 5 7 7 ← (6) 10 17 7 7 7
(6) 6 9 11 7 7 7 ← (19) 5 9 7 7 7
(7) 11 5 9 7 7 7 ← (9) 13 13 5 7 7
(9) 7 14 4 5 7 7 ← (10) 11 14 5 7 7
(13) 5 5 9 7 7 7 ← (14) 7 14 5 7 7
(15) 3 5 9 7 7 7 ← (17) 14 4 5 7 7
(15) 8 12 9 3 3 3 ← (24) 12 9 3 3 3
(27) 3 3 6 6 5 3 ← (33) 3 5 7 3 3
(33) 2 4 5 3 3 3 ← (34) 4 7 3 3 3
(37) 1 2 4 3 3 3 ← (43) 4 4 1 1 1
(39) 3 4 4 1 1 1 ← (40) 5 1 2 3 3
(43) 1 * 1 ← (45) * 1

(53, 8)

(2) 4 6 9 11 7 7 7 ← (14) 17 3 6 6 5 3
(5) 5 10 11 3 5 7 7 ← (16) 3 5 9 7 7 7
(5) 8 5 5 9 7 7 7 ← (6) 9 7 13 5 7 7
(6) 9 3 5 9 7 7 7 ← (8) 11 5 9 7 7 7
(7) 8 3 5 9 7 7 7 ← (10) 7 14 4 5 7 7
(8) 7 8 12 9 3 3 3 ← (22) 9 3 6 6 5 3
(11) 10 9 3 6 6 5 3 ← (12) 12 12 9 3 3 3
(14) 3 5 9 3 5 7 7 ← (18) 5 9 3 5 7 7
(15) 6 9 3 6 6 5 3 ← (16) 8 12 9 3 3 3
(23) 10 2 4 5 3 3 3 ← (24) 12 4 5 3 3 3
(36) 1 1 2 4 3 3 3 ← (40) 3 4 4 1 1 1
(42) 1 1 * 1 ← (44) 1 * 1

(53, 9)

(1) 4 8 5 5 9 7 7 7
(2) 3 5 10 11 3 5 7 7
(5) 4 6 3 5 9 7 7 7 ← (6) 5 10 11 3 5 7 7
(6) 4 5 3 5 9 7 7 7 ← (8) 8 3 5 9 7 7 7
(6) 8 3 5 9 3 5 7 7 ← (19) 6 5 2 3 5 7 7
(7) 9 3 5 7 3 5 7 7 ← (17) 3 5 7 3 5 7 7
(11) 9 5 5 3 6 6 5 3 ← (12) 10 9 3 6 6 5 3
(15) 3 6 5 2 3 5 7 7 ← (16) 6 9 3 6 6 5 3
(21) 3 6 2 3 5 7 3 3 ← (22) 6 3 3 6 6 5 3
(23) 7 13 1 * 1 ← (24) 10 2 4 5 3 3 3
(27) 4 ..4 5 3 3 3 ← (28) 6 2 4 5 3 3 3
(31) 6 2 3 4 4 1 1 1 ← (38) 2 3 4 4 1 1 1

(53, 10)

(2) 5 9 3 5 7 3 5 7 7 ← (8) 9 3 5 7 3 5 7 7
(5) 2 3 5 3 5 9 7 7 7 ← (6) 4 6 3 5 9 7 7 7
(5) 4 5 3 5 9 3 5 7 7 ← (17) 4 7 3 3 6 6 5 3
(11) 4 5 5 5 3 6 6 5 3 ← (14) 5 6 5 2 3 5 7 7
(15) 2 3 5 5 3 6 6 5 3 ← (16) 3 6 5 2 3 5 7 7
(18) 2 4 3 3 3 6 6 5 3
(19) 3 5 6 2 4 5 3 3 3 ← (20) 5 6 2 3 5 7 3 3
(21) ...3 3 6 6 5 3 ← (22) 3 6 2 3 5 7 3 3
(23) 5 8 1 1 2 4 3 3 3 ← (24) 7 13 1 * 1
(25) 4 ...4 5 3 3 3 ← (26) 6 ..4 5 3 3 3
(27) ....4 5 3 3 3 ← (28) 4 ..4 5 3 3 3
(30) * 2 4 3 3 3 ← (37) 1 2 3 4 4 1 1 1
(31) 5 1 2 3 4 4 1 1 1 ← (32) 6 2 3 4 4 1 1 1

(53, 11)

(3) 5 5 5 6 5 2 3 5 7 7 ← (9) 5 5 6 5 2 3 5 7 7
(9) 2 4 5 5 5 3 6 6 5 3 ← (12) 4 5 5 5 3 6 6 5 3
(16) ..4 3 3 3 6 6 5 3
(19) ....3 3 6 6 5 3 ← (20) 3 5 6 2 4 5 3 3 3
(23) 4 ....4 5 3 3 3 ← (24) 5 8 1 1 2 4 3 3 3
(25) .....4 5 3 3 3 ← (26) 4 ...4 5 3 3 3
(29) 1 * 2 4 3 3 3 ← (35) 4 4 1 1 * 1
(31) 3 4 4 1 1 * 1 ← (32) 5 1 2 3 4 4 1 1 1
(35) 1 * * 1 ← (37) * * 1

(53, 12)

(1) 4 3 5 5 6 5 2 3 5 7 7 ← (4) 5 5 5 6 5 2 3 5 7 7
(7) ..4 5 5 5 3 6 6 5 3 ← (10) 2 4 5 5 5 3 6 6 5 3
(13) 6 ....3 3 6 6 5 3
(23) ......4 5 3 3 3 ← (24) 4 ....4 5 3 3 3
(28) 1 1 * 2 4 3 3 3 ← (32) 3 4 4 1 1 * 1
(34) 1 1 * * 1 ← (36) 1 * * 1

(53, 13)

(2) ..4 3 5 6 5 2 3 5 7 7
(5) ...4 5 5 5 3 6 6 5 3 ← (8) ..4 5 5 5 3 6 6 5 3
(11) 5 6 .....3 5 7 3 3
(13) 3 6 .....3 5 7 3 3 ← (14) 6 ....3 3 6 6 5 3
(19) 4 ......4 5 3 3 3 ← (20) 6 .....4 5 3 3 3
(23) 6 2 3 4 4 1 1 * 1 ← (30) 2 3 4 4 1 1 * 1

(53, 14)

(3) ....4 5 5 5 3 6 6 5 3 ← (6) ...4 5 5 5 3 6 6 5 3
(10) 4 3 3 3 6 6 5 3
(11) 3 5 6 .....4 5 3 3 3 ← (12) 5 6 .....3 5 7 3 3
(13) ....3 3 6 6 5 3 ← (14) 3 6 .....3 5 7 3 3
(17) 4 ....4 5 3 3 3 ← (18) 6 ......4 5 3 3 3
(19) ....4 5 3 3 3 ← (20) 4 ......4 5 3 3 3
(22) * * 2 4 3 3 3 ← (29) 1 2 3 4 4 1 1 * 1
(23) 5 1 2 3 4 4 1 1 * 1 ← (24) 6 2 3 4 4 1 1 * 1

(53, 15)

(1) .....4 5 5 5 3 6 6 5 3 ← (4) 4 5 5 5 3 6 6 5 3
(7) 1.........4 5 3 3 3
(11) ....3 3 6 6 5 3 ← (12) 3 5 6 .....4 5 3 3 3
(17) ....4 5 3 3 3 ← (18) 4 ....4 5 3 3 3
(21) 1 * * 2 4 3 3 3 ← (27) 4 4 1 1 * * 1
(23) 3 4 4 1 1 * * 1 ← (24) 5 1 2 3 4 4 1 1 * 1
(27) 1 * * * 1 ← (29) * * * 1

(53, 16)

(4) 1 1 * 2 4 7 3 3 6 6 5 3
(5) 6 ....3 3 6 6 5 3
(7) 10 ....4 5 3 3 3 ← (8) 1.........4 5 3 3 3
(20) 1 1 * * 2 4 3 3 3 ← (24) 3 4 4 1 1 * * 1
(26) 1 1 * * * 1 ← (28) 1 * * * 1

(53, 17)

(3) 5 6 ....3 5 7 3 3
(5) 3 6 ....3 5 7 3 3 ← (6) 6 ....3 3 6 6 5 3
(7) 7 13 1 * * * 1 ← (8) 10 ....4 5 3 3 3
(11) 4 ....4 5 3 3 3 ← (12) 6 ....4 5 3 3 3
(15) 6 2 3 4 4 1 1 * * 1 ← (22) 2 3 4 4 1 1 * * 1

(53, 18)

(2) 4 3 3 3 6 6 5 3
(3) 3 5 6 ....4 5 3 3 3 ← (4) 5 6 ....3 5 7 3 3
(5) ....3 3 6 6 5 3 ← (6) 3 6 ....3 5 7 3 3
(7) 5 8 1 1 * * 2 4 3 3 3 ← (8) 7 13 1 * * * 1
(9) 4 ....4 5 3 3 3 ← (10) 6 ....4 5 3 3 3
(11) ....4 5 3 3 3 ← (12) 4 ....4 5 3 3 3
(14) * * * 2 4 3 3 3 ← (21) 1 2 3 4 4 1 1 * * 1
(15) 5 1 2 3 4 4 1 1 * * 1 ← (16) 6 2 3 4 4 1 1 * * 1

(53, 19)

(3) ....3 3 6 6 5 3 ← (4) 3 5 6 ....4 5 3 3 3
(6) 3 5 7 3 3
(7) 4 ....4 5 3 3 3 ← (8) 5 8 1 1 * * 2 4 3 3 3
(9) ....4 5 3 3 3 ← (10) 4 ....4 5 3 3 3
(13) 1 * * * 2 4 3 3 3 ← (19) 4 4 1 1 * * * 1
(15) 3 4 4 1 1 * * * 1 ← (16) 5 1 2 3 4 4 1 1 * * 1
(19) 1 * * * * 1 ← (21) * * * * 1

(53, 20)

(3) 6 ....4 5 3 3 3
(7) ....4 5 3 3 3 ← (8) 4 ....4 5 3 3 3
(12) 1 1 * * * 2 4 3 3 3 ← (16) 3 4 4 1 1 * * * 1
(18) 1 1 * * * * 1 ← (20) 1 * * * * 1

(53, 21)

(1) 6 ....4 5 3 3 3
(3) 4 ....4 5 3 3 3 ← (4) 6 ....4 5 3 3 3
(7) 6 2 3 4 4 1 1 * * * 1 ← (14) 2 3 4 4 1 1 * * * 1

(54, 2)

(47) 7 ← (55)

(54, 3)

(23) 30 1 ← (24) 31
(39) 14 1 ← (40) 15
(46) 5 3 ← (54) 1
(47) 6 1 ← (48) 7
(51) 2 1 ← (52) 3

(54, 4)

(9) 15 15 15 ← (21) 27 7
(13) 11 15 15 ← (37) 11 7
(15) 21 11 7 ← (17) 23 15
(23) 13 11 7 ← (25) 15 15
(23) 29 1 1 ← (24) 30 1
(39) 13 1 1 ← (40) 14 1
(40) 6 5 3 ← (53) 1 1
(45) 3 3 3 ← (49) 3 3
(47) 5 1 1 ← (48) 6 1
(51) 1 1 1 ← (52) 2 1

(54, 5)

(4) 9 11 15 15
(6) 7 11 15 15 ← (36) 5 7 7
(9) 14 13 11 7 ← (10) 15 15 15
(13) 10 13 11 7 ← (14) 11 15 15
(14) 19 7 7 7 ← (16) 21 11 7
(15) 20 5 7 7 ← (18) 23 7 7
(21) 14 5 7 7 ← (24) 13 11 7
(23) 28 1 1 1 ← (24) 29 1 1
(31) 4 5 7 7 ← (34) 7 7 7
(37) 8 3 3 3 ← (52) 1 1 1
(39) 12 1 1 1 ← (40) 13 1 1
(47) 4 1 1 1 ← (48) 5 1 1

(54, 6)

(1) 5 7 11 15 15
(2) 4 7 11 15 15
(3) 7 13 13 11 7
(5) 9 11 15 7 7 ← (10) 14 13 11 7
(7) 7 11 15 7 7 ← (11) 13 13 11 7
(9) 9 15 7 7 7 ← (23) 13 5 7 7
(13) 9 11 7 7 7 ← (14) 10 13 11 7
(14) 18 3 5 7 7 ← (16) 20 5 7 7
(15) 7 13 5 7 7 ← (17) 17 7 7 7
(21) 11 3 5 7 7 ← (22) 14 5 7 7
(23) 9 3 5 7 7 ← (33) 3 5 7 7
(23) 24 4 1 1 1 ← (24) 28 1 1 1
(30) 2 3 5 7 7 ← (32) 4 5 7 7
(31) 3 6 6 5 3 ← (35) 6 6 5 3
(36) 4 5 3 3 3 ← (48) 4 1 1 1
(37) 3 6 2 3 3 ← (38) 8 3 3 3
(39) 2 4 3 3 3 ← (41) 6 2 3 3
(39) 8 4 1 1 1 ← (40) 12 1 1 1

(54, 7)

(2) 5 7 11 15 7 7
(3) 6 9 15 7 7 7 ← (4) 7 13 13 11 7
(4) 5 9 15 7 7 7 ← (20) 5 9 7 7 7
(5) 8 9 11 7 7 7 ← (6) 9 11 15 7 7
(7) 6 9 11 7 7 7 ← (8) 7 11 15 7 7
(9) 12 11 3 5 7 7 ← (10) 13 13 5 7 7
(14) 5 5 9 7 7 7 ← (16) 7 13 5 7 7
(17) 13 2 3 5 7 7 ← (18) 14 4 5 7 7
(20) 5 7 3 5 7 7 ← (24) 9 3 5 7 7
(23) 22 * 1 ← (24) 24 4 1 1 1
(28) 3 3 6 6 5 3 ← (34) 3 5 7 3 3
(31) 2 3 5 7 3 3 ← (32) 3 6 6 5 3
(34) 2 4 5 3 3 3 ← (46) * 1
(37) ..4 3 3 3 ← (38) 3 6 2 3 3
(38) 1 2 4 3 3 3 ← (40) 2 4 3 3 3
(39) 6 * 1 ← (40) 8 4 1 1 1
(43) 2 * 1 ← (44) 4 4 1 1 1

(54, 8)

(1) 3 5 9 15 7 7 7
(3) 4 6 9 11 7 7 7 ← (4) 6 9 15 7 7 7
(6) 8 5 5 9 7 7 7 ← (19) 5 9 3 5 7 7
(7) 9 3 5 9 7 7 7 ← (9) 11 5 9 7 7 7
(9) 7 8 12 9 3 3 3 ← (10) 12 11 3 5 7 7
(13) 11 12 4 5 3 3 3
(15) 3 5 9 3 5 7 7 ← (17) 8 12 9 3 3 3
(17) 7 12 4 5 3 3 3 ← (18) 13 2 3 5 7 7
(21) 5 5 3 6 6 5 3 ← (25) 12 4 5 3 3 3
(23) 21 1 * 1 ← (24) 22 * 1
(25) 6 2 3 5 7 3 3 ← (32) 2 3 5 7 3 3
(31) 13 1 * 1 ← (45) 1 * 1
(37) 1 1 2 4 3 3 3 ← (38) ..4 3 3 3
(39) 5 1 * 1 ← (40) 6 * 1
(43) 1 1 * 1 ← (44) 2 * 1

(54, 9)

(2) 4 8 5 5 9 7 7 7 ← (16) 3 5 9 3 5 7 7
(3) 3 5 10 11 3 5 7 7 ← (8) 9 3 5 9 7 7 7
(7) 4 5 3 5 9 7 7 7 ← (9) 8 3 5 9 7 7 7
(7) 8 3 5 9 3 5 7 7 ← (10) 7 8 12 9 3 3 3
(12) 9 5 5 3 6 6 5 3
(17) 5 6 3 3 6 6 5 3 ← (18) 7 12 4 5 3 3 3
(19) 3 6 3 3 6 6 5 3 ← (20) 6 5 2 3 5 7 7
(23) 5 6 2 4 5 3 3 3 ← (29) 6 2 4 5 3 3 3
(23) 20 1 1 * 1 ← (24) 21 1 * 1
(25) 3 6 2 4 5 3 3 3 ← (26) 6 2 3 5 7 3 3
(29) 8 1 1 2 4 3 3 3 ← (44) 1 1 * 1
(31) 12 1 1 * 1 ← (32) 13 1 * 1
(39) 4 1 1 * 1 ← (40) 5 1 * 1

(54, 10)

(1) 2 3 5 10 11 3 5 7 7
(3) 5 9 3 5 7 3 5 7 7 ← (9) 9 3 5 7 3 5 7 7
(6) 2 3 5 3 5 9 7 7 7 ← (8) 4 5 3 5 9 7 7 7
(6) 4 5 3 5 9 3 5 7 7 ← (8) 8 3 5 9 3 5 7 7
(15) 2 4 7 3 3 6 6 5 3 ← (18) 4 7 3 3 6 6 5 3
(16) 2 3 5 5 3 6 6 5 3
(17) 3 5 6 2 3 5 7 3 3 ← (18) 5 6 3 3 6 6 5 3
(19) 2 4 3 3 3 6 6 5 3 ← (20) 3 6 3 3 6 6 5 3
(22) ...3 3 6 6 5 3 ← (27) 6 ..4 5 3 3 3
(23) 3 6 ..4 5 3 3 3 ← (24) 5 6 2 4 5 3 3 3
(23) 16 4 1 1 * 1 ← (24) 20 1 1 * 1
(25) ....3 5 7 3 3 ← (26) 3 6 2 4 5 3 3 3
(28) ....4 5 3 3 3 ← (40) 4 1 1 * 1
(29) 3 6 2 3 4 4 1 1 1 ← (30) 8 1 1 2 4 3 3 3
(31) * 2 4 3 3 3 ← (33) 6 2 3 4 4 1 1 1
(31) 8 4 1 1 * 1 ← (32) 12 1 1 * 1

(54, 11)

(1) 18 2 4 3 3 3 6 6 5 3
(7) 4 3 5 6 5 2 3 5 7 7 ← (10) 5 5 6 5 2 3 5 7 7
(13) ..4 7 3 3 6 6 5 3
(14) 1 2 4 7 3 3 6 6 5 3 ← (16) 2 4 7 3 3 6 6 5 3
(17) ..4 3 3 3 6 6 5 3 ← (18) 3 5 6 2 3 5 7 3 3
(20) ....3 3 6 6 5 3 ← (26) 3 5 7 3 3
(23) .....3 5 7 3 3 ← (24) 3 6 ..4 5 3 3 3
(23) 14 * * 1 ← (24) 16 4 1 1 * 1
(26) .....4 5 3 3 3 ← (38) * * 1
(29) 2 * 2 4 3 3 3 ← (30) 3 6 2 3 4 4 1 1 1
(30) 1 * 2 4 3 3 3 ← (32) * 2 4 3 3 3
(31) 6 * * 1 ← (32) 8 4 1 1 * 1
(35) 2 * * 1 ← (36) 4 4 1 1 * 1

(54, 12)

(1) 16 ..4 3 3 3 6 6 5 3 ← (2) 18 2 4 3 3 3 6 6 5 3
(2) 4 3 5 5 6 5 2 3 5 7 7
(5) 2 4 3 5 6 5 2 3 5 7 7 ← (8) 4 3 5 6 5 2 3 5 7 7
(11) 6 ..4 3 3 3 6 6 5 3
(13) 1 1 2 4 7 3 3 6 6 5 3 ← (14) ..4 7 3 3 6 6 5 3
(17) 6 .....3 5 7 3 3 ← (24) 3 5 7 3 3
(23) 13 1 * * 1 ← (24) 14 * * 1
(24) ......4 5 3 3 3 ← (37) 1 * * 1
(29) 1 1 * 2 4 3 3 3 ← (30) 2 * 2 4 3 3 3
(31) 5 1 * * 1 ← (32) 6 * * 1
(35) 1 1 * * 1 ← (36) 2 * * 1

(54, 13)

(1) 13 6 ....3 3 6 6 5 3 ← (2) 16 ..4 3 3 3 6 6 5 3
(3) ..4 3 5 6 5 2 3 5 7 7 ← (6) 2 4 3 5 6 5 2 3 5 7 7
(9) 4 1 1 2 4 7 3 3 6 6 5 3
(11) 3 6 ....3 3 6 6 5 3 ← (12) 6 ..4 3 3 3 6 6 5 3
(15) 5 6 .....4 5 3 3 3 ← (21) 6 .....4 5 3 3 3
(17) 3 6 .....4 5 3 3 3 ← (18) 6 .....3 5 7 3 3
(21) 8 1 1 * 2 4 3 3 3 ← (36) 1 1 * * 1
(23) 12 1 1 * * 1 ← (24) 13 1 * * 1
(31) 4 1 1 * * 1 ← (32) 5 1 * * 1

(54, 14)

(1) ...4 3 5 6 5 2 3 5 7 7 ← (4) ..4 3 5 6 5 2 3 5 7 7
(1) 11 5 6 .....3 5 7 3 3 ← (2) 13 6 ....3 3 6 6 5 3
(7) * 2 4 7 3 3 6 6 5 3 ← (10) 4 1 1 2 4 7 3 3 6 6 5 3
(7) 24 4 1 1 * * 1
(11) .....4 3 3 3 6 6 5 3 ← (12) 3 6 ....3 3 6 6 5 3
(14) ....3 3 6 6 5 3 ← (19) 6 ......4 5 3 3 3
(15) 3 6 ......4 5 3 3 3 ← (16) 5 6 .....4 5 3 3 3
(17) ....3 5 7 3 3 ← (18) 3 6 .....4 5 3 3 3
(20) ....4 5 3 3 3 ← (32) 4 1 1 * * 1
(21) 3 6 2 3 4 4 1 1 * 1 ← (22) 8 1 1 * 2 4 3 3 3
(23) * * 2 4 3 3 3 ← (25) 6 2 3 4 4 1 1 * 1
(23) 8 4 1 1 * * 1 ← (24) 12 1 1 * * 1

(54, 15)

(1) 10 .....4 3 3 3 6 6 5 3 ← (2) 11 5 6 .....3 5 7 3 3
(2) 4 5 5 5 3 6 6 5 3
(5) 2 * 2 4 7 3 3 6 6 5 3
(6) 1 * 2 4 7 3 3 6 6 5 3 ← (8) * 2 4 7 3 3 6 6 5 3
(7) 22 * * * 1 ← (8) 24 4 1 1 * * 1
(12) ....3 3 6 6 5 3 ← (18) 3 5 7 3 3
(15) ....3 5 7 3 3 ← (16) 3 6 ......4 5 3 3 3
(18) ....4 5 3 3 3 ← (30) * * * 1
(21) 2 * * 2 4 3 3 3 ← (22) 3 6 2 3 4 4 1 1 * 1
(22) 1 * * 2 4 3 3 3 ← (24) * * 2 4 3 3 3
(23) 6 * * * 1 ← (24) 8 4 1 1 * * 1
(27) 2 * * * 1 ← (28) 4 4 1 1 * * 1

(54, 16)

(1) 7 1.........4 5 3 3 3 ← (2) 10 .....4 3 3 3 6 6 5 3
(3) 6 ......4 3 3 3 6 6 5 3
(5) 1 1 * 2 4 7 3 3 6 6 5 3 ← (6) 2 * 2 4 7 3 3 6 6 5 3
(7) 21 1 * * * 1 ← (8) 22 * * * 1
(9) 6 ....3 5 7 3 3 ← (16) 3 5 7 3 3
(15) 13 1 * * * 1 ← (29) 1 * * * 1
(21) 1 1 * * 2 4 3 3 3 ← (22) 2 * * 2 4 3 3 3
(23) 5 1 * * * 1 ← (24) 6 * * * 1
(27) 1 1 * * * 1 ← (28) 2 * * * 1

(54, 17)

(1) 4 1 1 * 2 4 7 3 3 6 6 5 3
(1) 5 6 ....3 3 6 6 5 3 ← (2) 7 1.........4 5 3 3 3
(3) 3 6 ....3 3 6 6 5 3 ← (4) 6 ......4 3 3 3 6 6 5 3
(7) 5 6 ....4 5 3 3 3 ← (13) 6 ....4 5 3 3 3
(7) 20 1 1 * * * 1 ← (8) 21 1 * * * 1
(9) 3 6 ....4 5 3 3 3 ← (10) 6 ....3 5 7 3 3
(13) 8 1 1 * * 2 4 3 3 3 ← (28) 1 1 * * * 1
(15) 12 1 1 * * * 1 ← (16) 13 1 * * * 1
(23) 4 1 1 * * * 1 ← (24) 5 1 * * * 1

(54, 18)

(1) 3 5 6 ....3 5 7 3 3 ← (2) 5 6 ....3 3 6 6 5 3
(3) ....4 3 3 3 6 6 5 3 ← (4) 3 6 ....3 3 6 6 5 3
(6) ....3 3 6 6 5 3 ← (11) 6 ....4 5 3 3 3
(7) 3 6 ....4 5 3 3 3 ← (8) 5 6 ....4 5 3 3 3
(7) 16 4 1 1 * * * 1 ← (8) 20 1 1 * * * 1
(9) ....3 5 7 3 3 ← (10) 3 6 ....4 5 3 3 3
(12) ....4 5 3 3 3 ← (24) 4 1 1 * * * 1
(13) 3 6 2 3 4 4 1 1 * * 1 ← (14) 8 1 1 * * 2 4 3 3 3
(15) * * * 2 4 3 3 3 ← (17) 6 2 3 4 4 1 1 * * 1
(15) 8 4 1 1 * * * 1 ← (16) 12 1 1 * * * 1

(54, 19)

(1) ....4 3 3 3 6 6 5 3 ← (2) 3 5 6 ....3 5 7 3 3
(4) ....3 3 6 6 5 3 ← (10) 3 5 7 3 3
(7) ....3 5 7 3 3 ← (8) 3 6 ....4 5 3 3 3
(7) 14 * * * * 1 ← (8) 16 4 1 1 * * * 1
(10) ....4 5 3 3 3 ← (22) * * * * 1
(13) 2 * * * 2 4 3 3 3 ← (14) 3 6 2 3 4 4 1 1 * * 1
(14) 1 * * * 2 4 3 3 3 ← (16) * * * 2 4 3 3 3
(15) 6 * * * * 1 ← (16) 8 4 1 1 * * * 1
(19) 2 * * * * 1 ← (20) 4 4 1 1 * * * 1

(54, 20)

(1) 6 ....3 5 7 3 3 ← (8) 3 5 7 3 3
(7) 13 1 * * * * 1 ← (8) 14 * * * * 1
(8) ....4 5 3 3 3 ← (21) 1 * * * * 1
(13) 1 1 * * * 2 4 3 3 3 ← (14) 2 * * * 2 4 3 3 3
(15) 5 1 * * * * 1 ← (16) 6 * * * * 1
(19) 1 1 * * * * 1 ← (20) 2 * * * * 1

(55, 3)

(23) 29 3 ← (25) 31
(39) 13 3 ← (41) 15
(41) 7 7 ← (53) 3
(47) 5 3 ← (49) 7

(55, 4)

(17) 22 13 3 ← (18) 23 15
(21) 26 5 3 ← (22) 27 7
(22) 27 3 3 ← (24) 29 3
(25) 14 13 3 ← (26) 15 15
(37) 10 5 3 ← (38) 11 7
(38) 11 3 3 ← (40) 13 3
(41) 6 5 3 ← (42) 7 7
(46) 3 3 3 ← (48) 5 3
(47) 2 3 3 ← (50) 3 3

(55, 5)

(5) 9 11 15 15 ← (19) 23 7 7
(7) 7 11 15 15 ← (35) 7 7 7
(15) 11 15 7 7 ← (25) 13 11 7
(15) 19 7 7 7 ← (17) 21 11 7
(17) 21 11 3 3 ← (18) 22 13 3
(21) 25 3 3 3 ← (22) 26 5 3
(25) 13 11 3 3 ← (26) 14 13 3
(37) 5 7 3 3 ← (42) 6 5 3
(37) 9 3 3 3 ← (38) 10 5 3
(46) 1 2 3 3 ← (48) 2 3 3

(55, 6)

(2) 5 7 11 15 15
(3) 4 7 11 15 15 ← (34) 3 5 7 7
(7) 10 17 7 7 7 ← (16) 19 7 7 7
(10) 9 15 7 7 7 ← (18) 17 7 7 7
(11) 11 14 5 7 7 ← (12) 13 13 11 7
(14) 9 11 7 7 7 ← (23) 14 5 7 7
(15) 7 14 5 7 7 ← (16) 11 15 7 7
(15) 18 3 5 7 7 ← (17) 20 5 7 7
(17) 20 9 3 3 3 ← (18) 21 11 3 3
(22) 11 3 5 7 7 ← (24) 13 5 7 7
(25) 12 9 3 3 3 ← (26) 13 11 3 3
(31) 2 3 5 7 7 ← (33) 4 5 7 7
(35) 4 7 3 3 3 ← (36) 6 6 5 3
(37) 4 5 3 3 3 ← (38) 5 7 3 3
(41) 5 1 2 3 3 ← (42) 6 2 3 3

(55, 7)

(1) 2 4 7 11 15 15
(3) 5 7 11 15 7 7 ← (5) 7 13 13 11 7
(5) 5 9 15 7 7 7 ← (19) 14 4 5 7 7
(6) 8 9 11 7 7 7 ← (11) 13 13 5 7 7
(7) 9 7 13 5 7 7 ← (8) 10 17 7 7 7
(8) 6 9 11 7 7 7 ← (17) 7 13 5 7 7
(11) 7 14 4 5 7 7 ← (12) 11 14 5 7 7
(13) 12 12 9 3 3 3 ← (16) 18 3 5 7 7
(15) 5 5 9 7 7 7 ← (16) 7 14 5 7 7
(15) 17 3 6 6 5 3 ← (18) 20 9 3 3 3
(17) 3 5 9 7 7 7 ← (33) 3 6 6 5 3
(21) 5 7 3 5 7 7 ← (25) 9 3 5 7 7
(23) 9 3 6 6 5 3 ← (26) 12 9 3 3 3
(29) 3 3 6 6 5 3 ← (32) 2 3 5 7 7
(35) 2 4 5 3 3 3 ← (36) 4 7 3 3 3
(39) 1 2 4 3 3 3 ← (41) 2 4 3 3 3
(41) 3 4 4 1 1 1 ← (42) 5 1 2 3 3

(55, 8)

(2) 3 5 9 15 7 7 7 ← (16) 5 5 9 7 7 7
(4) 4 6 9 11 7 7 7 ← (10) 11 5 9 7 7 7
(7) 5 10 11 3 5 7 7 ← (12) 7 14 4 5 7 7
(7) 8 5 5 9 7 7 7 ← (8) 9 7 13 5 7 7
(13) 10 9 3 6 6 5 3 ← (14) 12 12 9 3 3 3
(14) 11 12 4 5 3 3 3 ← (16) 17 3 6 6 5 3
(17) 6 9 3 6 6 5 3 ← (18) 8 12 9 3 3 3
(18) 3 5 7 3 5 7 7 ← (20) 5 9 3 5 7 7
(22) 5 5 3 6 6 5 3 ← (24) 9 3 6 6 5 3
(23) 6 3 3 6 6 5 3 ← (30) 3 3 6 6 5 3
(25) 10 2 4 5 3 3 3 ← (26) 12 4 5 3 3 3
(38) 1 1 2 4 3 3 3 ← (40) 1 2 4 3 3 3
(39) 2 3 4 4 1 1 1 ← (42) 3 4 4 1 1 1

(55, 9)

(1) 13 11 12 4 5 3 3 3 ← (8) 8 5 5 9 7 7 7
(3) 4 8 5 5 9 7 7 7 ← (17) 3 5 9 3 5 7 7
(4) 3 5 10 11 3 5 7 7 ← (10) 8 3 5 9 7 7 7
(7) 4 6 3 5 9 7 7 7 ← (8) 5 10 11 3 5 7 7
(13) 9 5 5 3 6 6 5 3 ← (14) 10 9 3 6 6 5 3
(15) 5 6 5 2 3 5 7 7 ← (21) 6 5 2 3 5 7 7
(17) 3 6 5 2 3 5 7 7 ← (18) 6 9 3 6 6 5 3
(21) 5 6 2 3 5 7 3 3 ← (27) 6 2 3 5 7 3 3
(23) 3 6 2 3 5 7 3 3 ← (24) 6 3 3 6 6 5 3
(25) 7 13 1 * 1 ← (26) 10 2 4 5 3 3 3
(29) 4 ..4 5 3 3 3 ← (30) 6 2 4 5 3 3 3
(38) 1 2 3 4 4 1 1 1 ← (40) 2 3 4 4 1 1 1

(55, 10)

(1) 12 9 5 5 3 6 6 5 3 ← (2) 13 11 12 4 5 3 3 3
(2) 2 3 5 10 11 3 5 7 7 ← (9) 4 5 3 5 9 7 7 7
(4) 5 9 3 5 7 3 5 7 7
(7) 2 3 5 3 5 9 7 7 7 ← (8) 4 6 3 5 9 7 7 7
(7) 4 5 3 5 9 3 5 7 7 ← (9) 8 3 5 9 3 5 7 7
(13) 4 5 5 5 3 6 6 5 3 ← (16) 5 6 5 2 3 5 7 7
(17) 2 3 5 5 3 6 6 5 3 ← (18) 3 6 5 2 3 5 7 7
(20) 2 4 3 3 3 6 6 5 3 ← (25) 5 6 2 4 5 3 3 3
(21) 3 5 6 2 4 5 3 3 3 ← (22) 5 6 2 3 5 7 3 3
(23) ...3 3 6 6 5 3 ← (24) 3 6 2 3 5 7 3 3
(25) 5 8 1 1 2 4 3 3 3 ← (26) 7 13 1 * 1
(27) 4 ...4 5 3 3 3 ← (28) 6 ..4 5 3 3 3
(29) ....4 5 3 3 3 ← (30) 4 ..4 5 3 3 3
(33) 5 1 2 3 4 4 1 1 1 ← (34) 6 2 3 4 4 1 1 1

(55, 11)

(1) 1 2 3 5 10 11 3 5 7 7 ← (2) 12 9 5 5 3 6 6 5 3
(1) 16 2 3 5 5 3 6 6 5 3
(5) 5 5 5 6 5 2 3 5 7 7
(11) 2 4 5 5 5 3 6 6 5 3 ← (14) 4 5 5 5 3 6 6 5 3
(15) 1 2 4 7 3 3 6 6 5 3 ← (17) 2 4 7 3 3 6 6 5 3
(18) ..4 3 3 3 6 6 5 3 ← (24) ...3 3 6 6 5 3
(21) ....3 3 6 6 5 3 ← (22) 3 5 6 2 4 5 3 3 3
(25) 4 ....4 5 3 3 3 ← (26) 5 8 1 1 2 4 3 3 3
(27) .....4 5 3 3 3 ← (28) 4 ...4 5 3 3 3
(31) 1 * 2 4 3 3 3 ← (33) * 2 4 3 3 3
(33) 3 4 4 1 1 * 1 ← (34) 5 1 2 3 4 4 1 1 1

(55, 12)

(1) 2 4 2 3 5 3 5 9 7 7 7
(1) 13 ..4 7 3 3 6 6 5 3 ← (2) 16 2 3 5 5 3 6 6 5 3
(3) 4 3 5 5 6 5 2 3 5 7 7 ← (6) 5 5 5 6 5 2 3 5 7 7
(9) ..4 5 5 5 3 6 6 5 3 ← (12) 2 4 5 5 5 3 6 6 5 3
(14) 1 1 2 4 7 3 3 6 6 5 3 ← (16) 1 2 4 7 3 3 6 6 5 3
(15) 6 ....3 3 6 6 5 3 ← (22) 3 3 6 6 5 3
(25) ......4 5 3 3 3 ← (26) 4 ....4 5 3 3 3
(30) 1 1 * 2 4 3 3 3 ← (32) 1 * 2 4 3 3 3
(31) 2 3 4 4 1 1 * 1 ← (34) 3 4 4 1 1 * 1

(55, 13)

(1) 2 4 3 5 5 6 5 2 3 5 7 7 ← (4) 4 3 5 5 6 5 2 3 5 7 7
(1) 11 6 ..4 3 3 3 6 6 5 3 ← (2) 13 ..4 7 3 3 6 6 5 3
(7) ...4 5 5 5 3 6 6 5 3 ← (10) ..4 5 5 5 3 6 6 5 3
(13) 5 6 .....3 5 7 3 3 ← (19) 6 .....3 5 7 3 3
(15) 3 6 .....3 5 7 3 3 ← (16) 6 ....3 3 6 6 5 3
(21) 4 ......4 5 3 3 3 ← (22) 6 .....4 5 3 3 3
(30) 1 2 3 4 4 1 1 * 1 ← (32) 2 3 4 4 1 1 * 1

(55, 14)

(1) 9 4 1 1 2 4 7 3 3 6 6 5 3 ← (2) 11 6 ..4 3 3 3 6 6 5 3
(2) ...4 3 5 6 5 2 3 5 7 7
(5) ....4 5 5 5 3 6 6 5 3 ← (8) ...4 5 5 5 3 6 6 5 3
(12) .....4 3 3 3 6 6 5 3 ← (17) 5 6 .....4 5 3 3 3
(13) 3 5 6 .....4 5 3 3 3 ← (14) 5 6 .....3 5 7 3 3
(15) ....3 3 6 6 5 3 ← (16) 3 6 .....3 5 7 3 3
(19) 4 ....4 5 3 3 3 ← (20) 6 ......4 5 3 3 3
(21) ....4 5 3 3 3 ← (22) 4 ......4 5 3 3 3
(25) 5 1 2 3 4 4 1 1 * 1 ← (26) 6 2 3 4 4 1 1 * 1

(55, 15)

(1) 8 ..1 ..4 7 3 3 6 6 5 3 ← (2) 9 4 1 1 2 4 7 3 3 6 6 5 3
(3) .....4 5 5 5 3 6 6 5 3 ← (6) 4 5 5 5 3 6 6 5 3
(7) 1 * 2 4 7 3 3 6 6 5 3 ← (9) * 2 4 7 3 3 6 6 5 3
(9) 1.........4 5 3 3 3 ← (16) 3 3 6 6 5 3
(13) ....3 3 6 6 5 3 ← (14) 3 5 6 .....4 5 3 3 3
(19) ....4 5 3 3 3 ← (20) 4 ....4 5 3 3 3
(23) 1 * * 2 4 3 3 3 ← (25) * * 2 4 3 3 3
(25) 3 4 4 1 1 * * 1 ← (26) 5 1 2 3 4 4 1 1 * 1

(55, 16)

(1) ......4 5 5 5 3 6 6 5 3 ← (4) 4 5 5 5 3 6 6 5 3
(1) 5 2 * 2 4 7 3 3 6 6 5 3 ← (2) 8 ..1 ..4 7 3 3 6 6 5 3
(6) 1 1 * 2 4 7 3 3 6 6 5 3 ← (8) 1 * 2 4 7 3 3 6 6 5 3
(7) 6 ....3 3 6 6 5 3 ← (14) 3 3 6 6 5 3
(9) 10 ....4 5 3 3 3 ← (10) 1.........4 5 3 3 3
(22) 1 1 * * 2 4 3 3 3 ← (24) 1 * * 2 4 3 3 3
(23) 2 3 4 4 1 1 * * 1 ← (26) 3 4 4 1 1 * * 1

(55, 17)

(1) 3 6 ......4 3 3 3 6 6 5 3 ← (2) 5 2 * 2 4 7 3 3 6 6 5 3
(2) 4 1 1 * 2 4 7 3 3 6 6 5 3
(5) 5 6 ....3 5 7 3 3 ← (11) 6 ....3 5 7 3 3
(7) 3 6 ....3 5 7 3 3 ← (8) 6 ....3 3 6 6 5 3
(9) 7 13 1 * * * 1 ← (10) 10 ....4 5 3 3 3
(13) 4 ....4 5 3 3 3 ← (14) 6 ....4 5 3 3 3
(22) 1 2 3 4 4 1 1 * * 1 ← (24) 2 3 4 4 1 1 * * 1

(55, 18)

(1) ..1 2 * 2 4 7 3 3 6 6 5 3 ← (2) 3 6 ......4 3 3 3 6 6 5 3
(4) ....4 3 3 3 6 6 5 3 ← (9) 5 6 ....4 5 3 3 3
(5) 3 5 6 ....4 5 3 3 3 ← (6) 5 6 ....3 5 7 3 3
(7) ....3 3 6 6 5 3 ← (8) 3 6 ....3 5 7 3 3
(9) 5 8 1 1 * * 2 4 3 3 3 ← (10) 7 13 1 * * * 1
(11) 4 ....4 5 3 3 3 ← (12) 6 ....4 5 3 3 3
(13) ....4 5 3 3 3 ← (14) 4 ....4 5 3 3 3
(17) 5 1 2 3 4 4 1 1 * * 1 ← (18) 6 2 3 4 4 1 1 * * 1

(55, 19)

(2) ....4 3 3 3 6 6 5 3 ← (8) 3 3 6 6 5 3
(5) ....3 3 6 6 5 3 ← (6) 3 5 6 ....4 5 3 3 3
(9) 4 ....4 5 3 3 3 ← (10) 5 8 1 1 * * 2 4 3 3 3
(11) ....4 5 3 3 3 ← (12) 4 ....4 5 3 3 3
(15) 1 * * * 2 4 3 3 3 ← (17) * * * 2 4 3 3 3
(17) 3 4 4 1 1 * * * 1 ← (18) 5 1 2 3 4 4 1 1 * * 1

(56, 2)

(55) 1 ← (57)

(56, 3)

(25) 30 1 ← (26) 31
(41) 14 1 ← (42) 15
(49) 6 1 ← (50) 7
(53) 2 1 ← (54) 3
(54) 1 1 ← (56) 1

(56, 4)

(11) 15 15 15 ← (19) 23 15
(15) 11 15 15 ← (27) 15 15
(23) 27 3 3 ← (25) 29 3
(25) 29 1 1 ← (26) 30 1
(37) 5 7 7 ← (51) 3 3
(39) 11 3 3 ← (41) 13 3
(41) 13 1 1 ← (42) 14 1
(47) 3 3 3 ← (49) 5 3
(49) 5 1 1 ← (50) 6 1
(53) 1 1 1 ← (54) 2 1

(56, 5)

(6) 9 11 15 15 ← (18) 21 11 7
(8) 7 11 15 15 ← (26) 13 11 7
(11) 14 13 11 7 ← (12) 15 15 15
(15) 10 13 11 7 ← (16) 11 15 15
(22) 25 3 3 3 ← (24) 27 3 3
(25) 28 1 1 1 ← (26) 29 1 1
(38) 9 3 3 3 ← (40) 11 3 3
(39) 8 3 3 3 ← (48) 3 3 3
(41) 12 1 1 1 ← (42) 13 1 1
(47) 1 2 3 3 ← (49) 2 3 3
(49) 4 1 1 1 ← (50) 5 1 1

(56, 6)

(3) 5 7 11 15 15 ← (17) 19 7 7 7
(4) 4 7 11 15 15 ← (19) 17 7 7 7
(7) 9 11 15 7 7 ← (12) 14 13 11 7
(9) 7 11 15 7 7 ← (17) 11 15 7 7
(11) 9 15 7 7 7 ← (13) 13 13 11 7
(15) 9 11 7 7 7 ← (16) 10 13 11 7
(21) 5 9 7 7 7 ← (25) 13 5 7 7
(23) 11 3 5 7 7 ← (24) 14 5 7 7
(25) 24 4 1 1 1 ← (26) 28 1 1 1
(35) 3 5 7 3 3 ← (37) 6 6 5 3
(38) 4 5 3 3 3 ← (43) 6 2 3 3
(39) 3 6 2 3 3 ← (40) 8 3 3 3
(41) 8 4 1 1 1 ← (42) 12 1 1 1
(45) 4 4 1 1 1 ← (48) 1 2 3 3
(47) * 1 ← (50) 4 1 1 1

(56, 7)

(2) 2 4 7 11 15 15 ← (16) 9 11 7 7 7
(4) 5 7 11 15 7 7 ← (9) 10 17 7 7 7
(5) 6 9 15 7 7 7 ← (6) 7 13 13 11 7
(6) 5 9 15 7 7 7 ← (12) 9 15 7 7 7
(7) 8 9 11 7 7 7 ← (8) 9 11 15 7 7
(9) 6 9 11 7 7 7 ← (10) 7 11 15 7 7
(11) 12 11 3 5 7 7 ← (12) 13 13 5 7 7
(18) 3 5 9 7 7 7 ← (24) 11 3 5 7 7
(19) 13 2 3 5 7 7 ← (20) 14 4 5 7 7
(22) 5 7 3 5 7 7 ← (26) 9 3 5 7 7
(25) 22 * 1 ← (26) 24 4 1 1 1
(33) 2 3 5 7 3 3 ← (34) 3 6 6 5 3
(36) 2 4 5 3 3 3 ← (42) 2 4 3 3 3
(39) ..4 3 3 3 ← (40) 3 6 2 3 3
(41) 6 * 1 ← (42) 8 4 1 1 1
(45) 2 * 1 ← (46) 4 4 1 1 1
(46) 1 * 1 ← (48) * 1

(56, 8)

(1) 1 2 4 7 11 15 15 ← (8) 8 9 11 7 7 7
(3) 3 5 9 15 7 7 7 ← (10) 6 9 11 7 7 7
(5) 4 6 9 11 7 7 7 ← (6) 6 9 15 7 7 7
(9) 9 3 5 9 7 7 7 ← (19) 8 12 9 3 3 3
(11) 7 8 12 9 3 3 3 ← (12) 12 11 3 5 7 7
(15) 11 12 4 5 3 3 3 ← (17) 17 3 6 6 5 3
(19) 3 5 7 3 5 7 7 ← (21) 5 9 3 5 7 7
(19) 7 12 4 5 3 3 3 ← (20) 13 2 3 5 7 7
(23) 5 5 3 6 6 5 3 ← (25) 9 3 6 6 5 3
(25) 21 1 * 1 ← (26) 22 * 1
(33) 13 1 * 1 ← (41) 1 2 4 3 3 3
(39) 1 1 2 4 3 3 3 ← (40) ..4 3 3 3
(41) 5 1 * 1 ← (42) 6 * 1
(45) 1 1 * 1 ← (46) 2 * 1

(56, 9)

(4) 4 8 5 5 9 7 7 7
(5) 3 5 10 11 3 5 7 7 ← (9) 5 10 11 3 5 7 7
(10) 9 3 5 7 3 5 7 7
(14) 9 5 5 3 6 6 5 3 ← (16) 11 12 4 5 3 3 3
(19) 4 7 3 3 6 6 5 3 ← (24) 5 5 3 6 6 5 3
(19) 5 6 3 3 6 6 5 3 ← (20) 7 12 4 5 3 3 3
(21) 3 6 3 3 6 6 5 3 ← (22) 6 5 2 3 5 7 7
(25) 20 1 1 * 1 ← (26) 21 1 * 1
(27) 3 6 2 4 5 3 3 3 ← (28) 6 2 3 5 7 3 3
(31) 8 1 1 2 4 3 3 3 ← (40) 1 1 2 4 3 3 3
(33) 12 1 1 * 1 ← (34) 13 1 * 1
(39) 1 2 3 4 4 1 1 1 ← (41) 2 3 4 4 1 1 1
(41) 4 1 1 * 1 ← (42) 5 1 * 1

(56, 10)

(3) 2 3 5 10 11 3 5 7 7 ← (6) 3 5 10 11 3 5 7 7
(5) 5 9 3 5 7 3 5 7 7
(8) 2 3 5 3 5 9 7 7 7
(8) 4 5 3 5 9 3 5 7 7
(11) 5 5 6 5 2 3 5 7 7 ← (17) 5 6 5 2 3 5 7 7
(18) 2 3 5 5 3 6 6 5 3 ← (20) 4 7 3 3 6 6 5 3
(19) 3 5 6 2 3 5 7 3 3 ← (20) 5 6 3 3 6 6 5 3
(21) 2 4 3 3 3 6 6 5 3 ← (22) 3 6 3 3 6 6 5 3
(25) 3 6 ..4 5 3 3 3 ← (26) 5 6 2 4 5 3 3 3
(25) 16 4 1 1 * 1 ← (26) 20 1 1 * 1
(27) ....3 5 7 3 3 ← (28) 3 6 2 4 5 3 3 3
(30) ....4 5 3 3 3 ← (35) 6 2 3 4 4 1 1 1
(31) 3 6 2 3 4 4 1 1 1 ← (32) 8 1 1 2 4 3 3 3
(33) 8 4 1 1 * 1 ← (34) 12 1 1 * 1
(37) 4 4 1 1 * 1 ← (40) 1 2 3 4 4 1 1 1
(39) * * 1 ← (42) 4 1 1 * 1

(56, 11)

(2) 1 2 3 5 10 11 3 5 7 7 ← (4) 2 3 5 10 11 3 5 7 7
(3) 18 2 4 3 3 3 6 6 5 3
(9) 4 3 5 6 5 2 3 5 7 7 ← (12) 5 5 6 5 2 3 5 7 7
(15) ..4 7 3 3 6 6 5 3 ← (18) 2 4 7 3 3 6 6 5 3
(19) ..4 3 3 3 6 6 5 3 ← (20) 3 5 6 2 3 5 7 3 3
(25) .....3 5 7 3 3 ← (26) 3 6 ..4 5 3 3 3
(25) 14 * * 1 ← (26) 16 4 1 1 * 1
(28) .....4 5 3 3 3 ← (34) * 2 4 3 3 3
(31) 2 * 2 4 3 3 3 ← (32) 3 6 2 3 4 4 1 1 1
(33) 6 * * 1 ← (34) 8 4 1 1 * 1
(37) 2 * * 1 ← (38) 4 4 1 1 * 1
(38) 1 * * 1 ← (40) * * 1

(56, 12)

(1) 5 5 5 5 6 5 2 3 5 7 7
(2) 2 4 2 3 5 3 5 9 7 7 7
(3) 16 ..4 3 3 3 6 6 5 3 ← (4) 18 2 4 3 3 3 6 6 5 3
(7) 2 4 3 5 6 5 2 3 5 7 7 ← (10) 4 3 5 6 5 2 3 5 7 7
(13) 6 ..4 3 3 3 6 6 5 3 ← (17) 1 2 4 7 3 3 6 6 5 3
(15) 1 1 2 4 7 3 3 6 6 5 3 ← (16) ..4 7 3 3 6 6 5 3
(25) 13 1 * * 1 ← (26) 14 * * 1
(26) ......4 5 3 3 3 ← (33) 1 * 2 4 3 3 3
(31) 1 1 * 2 4 3 3 3 ← (32) 2 * 2 4 3 3 3
(33) 5 1 * * 1 ← (34) 6 * * 1
(37) 1 1 * * 1 ← (38) 2 * * 1

(56, 13)

(1) 1 2 4 2 3 5 3 5 9 7 7 7
(2) 2 4 3 5 5 6 5 2 3 5 7 7
(3) 13 6 ....3 3 6 6 5 3 ← (4) 16 ..4 3 3 3 6 6 5 3
(5) ..4 3 5 6 5 2 3 5 7 7 ← (8) 2 4 3 5 6 5 2 3 5 7 7
(11) 4 1 1 2 4 7 3 3 6 6 5 3 ← (16) 1 1 2 4 7 3 3 6 6 5 3
(13) 3 6 ....3 3 6 6 5 3 ← (14) 6 ..4 3 3 3 6 6 5 3
(19) 3 6 .....4 5 3 3 3 ← (20) 6 .....3 5 7 3 3
(23) 8 1 1 * 2 4 3 3 3 ← (32) 1 1 * 2 4 3 3 3
(25) 12 1 1 * * 1 ← (26) 13 1 * * 1
(31) 1 2 3 4 4 1 1 * 1 ← (33) 2 3 4 4 1 1 * 1
(33) 4 1 1 * * 1 ← (34) 5 1 * * 1

(56, 14)

(3) ...4 3 5 6 5 2 3 5 7 7 ← (6) ..4 3 5 6 5 2 3 5 7 7
(3) 11 5 6 .....3 5 7 3 3 ← (4) 13 6 ....3 3 6 6 5 3
(9) 24 4 1 1 * * 1 ← (12) 4 1 1 2 4 7 3 3 6 6 5 3
(13) .....4 3 3 3 6 6 5 3 ← (14) 3 6 ....3 3 6 6 5 3
(17) 3 6 ......4 5 3 3 3 ← (18) 5 6 .....4 5 3 3 3
(19) ....3 5 7 3 3 ← (20) 3 6 .....4 5 3 3 3
(22) ....4 5 3 3 3 ← (27) 6 2 3 4 4 1 1 * 1
(23) 3 6 2 3 4 4 1 1 * 1 ← (24) 8 1 1 * 2 4 3 3 3
(25) 8 4 1 1 * * 1 ← (26) 12 1 1 * * 1
(29) 4 4 1 1 * * 1 ← (32) 1 2 3 4 4 1 1 * 1
(31) * * * 1 ← (34) 4 1 1 * * 1

(56, 15)

(1) ....4 3 5 6 5 2 3 5 7 7 ← (4) ...4 3 5 6 5 2 3 5 7 7
(3) 10 .....4 3 3 3 6 6 5 3 ← (4) 11 5 6 .....3 5 7 3 3
(7) 2 * 2 4 7 3 3 6 6 5 3 ← (10) * 2 4 7 3 3 6 6 5 3
(9) 22 * * * 1 ← (10) 24 4 1 1 * * 1
(17) ....3 5 7 3 3 ← (18) 3 6 ......4 5 3 3 3
(20) ....4 5 3 3 3 ← (26) * * 2 4 3 3 3
(23) 2 * * 2 4 3 3 3 ← (24) 3 6 2 3 4 4 1 1 * 1
(25) 6 * * * 1 ← (26) 8 4 1 1 * * 1
(29) 2 * * * 1 ← (30) 4 4 1 1 * * 1
(30) 1 * * * 1 ← (32) * * * 1

(56, 16)

(2) 4 5 5 5 3 6 6 5 3
(3) 7 1.........4 5 3 3 3 ← (4) 10 .....4 3 3 3 6 6 5 3
(5) 6 ......4 3 3 3 6 6 5 3 ← (9) 1 * 2 4 7 3 3 6 6 5 3
(7) 1 1 * 2 4 7 3 3 6 6 5 3 ← (8) 2 * 2 4 7 3 3 6 6 5 3
(9) 21 1 * * * 1 ← (10) 22 * * * 1
(17) 13 1 * * * 1 ← (25) 1 * * 2 4 3 3 3
(23) 1 1 * * 2 4 3 3 3 ← (24) 2 * * 2 4 3 3 3
(25) 5 1 * * * 1 ← (26) 6 * * * 1
(29) 1 1 * * * 1 ← (30) 2 * * * 1

(56, 17)

(3) 4 1 1 * 2 4 7 3 3 6 6 5 3 ← (8) 1 1 * 2 4 7 3 3 6 6 5 3
(3) 5 6 ....3 3 6 6 5 3 ← (4) 7 1.........4 5 3 3 3
(5) 3 6 ....3 3 6 6 5 3 ← (6) 6 ......4 3 3 3 6 6 5 3
(9) 20 1 1 * * * 1 ← (10) 21 1 * * * 1
(11) 3 6 ....4 5 3 3 3 ← (12) 6 ....3 5 7 3 3
(15) 8 1 1 * * 2 4 3 3 3 ← (24) 1 1 * * 2 4 3 3 3
(17) 12 1 1 * * * 1 ← (18) 13 1 * * * 1
(23) 1 2 3 4 4 1 1 * * 1 ← (25) 2 3 4 4 1 1 * * 1
(25) 4 1 1 * * * 1 ← (26) 5 1 * * * 1

(56, 18)

(1) * * 2 4 7 3 3 6 6 5 3
(2) ..1 2 * 2 4 7 3 3 6 6 5 3 ← (4) 4 1 1 * 2 4 7 3 3 6 6 5 3
(3) 3 5 6 ....3 5 7 3 3 ← (4) 5 6 ....3 3 6 6 5 3
(5) ....4 3 3 3 6 6 5 3 ← (6) 3 6 ....3 3 6 6 5 3
(9) 3 6 ....4 5 3 3 3 ← (10) 5 6 ....4 5 3 3 3
(9) 16 4 1 1 * * * 1 ← (10) 20 1 1 * * * 1
(11) ....3 5 7 3 3 ← (12) 3 6 ....4 5 3 3 3
(14) ....4 5 3 3 3 ← (19) 6 2 3 4 4 1 1 * * 1
(15) 3 6 2 3 4 4 1 1 * * 1 ← (16) 8 1 1 * * 2 4 3 3 3
(17) 8 4 1 1 * * * 1 ← (18) 12 1 1 * * * 1
(21) 4 4 1 1 * * * 1 ← (24) 1 2 3 4 4 1 1 * * 1
(23) * * * * 1 ← (26) 4 1 1 * * * 1

(57, 3)

(23) 27 7 ← (27) 31
(39) 11 7 ← (43) 15
(43) 7 7 ← (51) 7
(55) 1 1 ← (57) 1

(57, 4)

(19) 22 13 3 ← (20) 23 15
(20) 23 7 7 ← (26) 29 3
(23) 26 5 3 ← (24) 27 7
(27) 14 13 3 ← (28) 15 15
(36) 7 7 7 ← (42) 13 3
(38) 5 7 7 ← (50) 5 3
(39) 10 5 3 ← (40) 11 7
(43) 6 5 3 ← (44) 7 7
(54) 1 1 1 ← (56) 1 1

(57, 5)

(7) 9 11 15 15 ← (13) 15 15 15
(9) 7 11 15 15 ← (17) 11 15 15
(18) 20 5 7 7 ← (25) 27 3 3
(19) 21 11 3 3 ← (20) 22 13 3
(23) 25 3 3 3 ← (24) 26 5 3
(27) 13 11 3 3 ← (28) 14 13 3
(34) 4 5 7 7 ← (41) 11 3 3
(35) 3 5 7 7 ← (49) 3 3 3
(39) 5 7 3 3 ← (44) 6 5 3
(39) 9 3 3 3 ← (40) 10 5 3

(57, 6)

(4) 5 7 11 15 15 ← (8) 9 11 15 15
(5) 4 7 11 15 15 ← (10) 7 11 15 15
(13) 11 14 5 7 7 ← (14) 13 13 11 7
(17) 7 14 5 7 7 ← (18) 11 15 7 7
(17) 18 3 5 7 7 ← (24) 25 3 3 3
(18) 7 13 5 7 7 ← (20) 17 7 7 7
(19) 20 9 3 3 3 ← (20) 21 11 3 3
(22) 5 9 7 7 7 ← (26) 13 5 7 7
(27) 12 9 3 3 3 ← (28) 13 11 3 3
(33) 2 3 5 7 7 ← (40) 9 3 3 3
(36) 3 5 7 3 3 ← (41) 8 3 3 3
(37) 4 7 3 3 3 ← (38) 6 6 5 3
(39) 4 5 3 3 3 ← (40) 5 7 3 3
(43) 5 1 2 3 3 ← (44) 6 2 3 3

(57, 7)

(3) 2 4 7 11 15 15 ← (6) 4 7 11 15 15
(5) 5 7 11 15 7 7 ← (9) 9 11 15 7 7
(7) 5 9 15 7 7 7 ← (13) 9 15 7 7 7
(9) 9 7 13 5 7 7 ← (10) 10 17 7 7 7
(11) 11 5 9 7 7 7 ← (13) 13 13 5 7 7
(13) 7 14 4 5 7 7 ← (14) 11 14 5 7 7
(15) 12 12 9 3 3 3 ← (20) 20 9 3 3 3
(17) 5 5 9 7 7 7 ← (18) 7 14 5 7 7
(19) 3 5 9 7 7 7 ← (21) 14 4 5 7 7
(23) 5 7 3 5 7 7 ← (27) 9 3 5 7 7
(27) 12 4 5 3 3 3
(31) 3 3 6 6 5 3 ← (35) 3 6 6 5 3
(34) 2 3 5 7 3 3 ← (40) 4 5 3 3 3
(37) 2 4 5 3 3 3 ← (38) 4 7 3 3 3
(43) 3 4 4 1 1 1 ← (44) 5 1 2 3 3
(47) 1 * 1 ← (49) * 1

(57, 8)

(2) 1 2 4 7 11 15 15 ← (4) 2 4 7 11 15 15
(4) 3 5 9 15 7 7 7 ← (8) 5 9 15 7 7 7
(6) 4 6 9 11 7 7 7
(9) 8 5 5 9 7 7 7 ← (10) 9 7 13 5 7 7
(10) 9 3 5 9 7 7 7 ← (12) 11 5 9 7 7 7
(11) 8 3 5 9 7 7 7 ← (14) 7 14 4 5 7 7
(12) 7 8 12 9 3 3 3 ← (18) 17 3 6 6 5 3
(15) 10 9 3 6 6 5 3 ← (16) 12 12 9 3 3 3
(18) 3 5 9 3 5 7 7
(19) 6 9 3 6 6 5 3 ← (20) 8 12 9 3 3 3
(20) 3 5 7 3 5 7 7 ← (24) 5 7 3 5 7 7
(25) 6 3 3 6 6 5 3 ← (32) 3 3 6 6 5 3
(27) 10 2 4 5 3 3 3 ← (28) 12 4 5 3 3 3
(31) 6 2 4 5 3 3 3 ← (38) 2 4 5 3 3 3
(46) 1 1 * 1 ← (48) 1 * 1

(57, 9)

(3) 13 11 12 4 5 3 3 3
(5) 4 8 5 5 9 7 7 7 ← (10) 8 5 5 9 7 7 7
(9) 4 6 3 5 9 7 7 7 ← (10) 5 10 11 3 5 7 7
(10) 4 5 3 5 9 7 7 7 ← (12) 8 3 5 9 7 7 7
(10) 8 3 5 9 3 5 7 7 ← (17) 11 12 4 5 3 3 3
(11) 9 3 5 7 3 5 7 7
(15) 9 5 5 3 6 6 5 3 ← (16) 10 9 3 6 6 5 3
(19) 3 6 5 2 3 5 7 7 ← (20) 6 9 3 6 6 5 3
(23) 5 6 2 3 5 7 3 3 ← (29) 6 2 3 5 7 3 3
(25) 3 6 2 3 5 7 3 3 ← (26) 6 3 3 6 6 5 3
(27) 7 13 1 * 1 ← (28) 10 2 4 5 3 3 3
(29) 6 ..4 5 3 3 3 ← (35) 13 1 * 1
(31) 4 ..4 5 3 3 3 ← (32) 6 2 4 5 3 3 3

(57, 10)

(1) 4 4 8 5 5 9 7 7 7
(3) 12 9 5 5 3 6 6 5 3 ← (4) 13 11 12 4 5 3 3 3
(6) 5 9 3 5 7 3 5 7 7 ← (12) 9 3 5 7 3 5 7 7
(9) 2 3 5 3 5 9 7 7 7 ← (10) 4 6 3 5 9 7 7 7
(9) 4 5 3 5 9 3 5 7 7 ← (16) 9 5 5 3 6 6 5 3
(15) 4 5 5 5 3 6 6 5 3 ← (18) 5 6 5 2 3 5 7 7
(19) 2 3 5 5 3 6 6 5 3 ← (20) 3 6 5 2 3 5 7 7
(22) 2 4 3 3 3 6 6 5 3 ← (27) 5 6 2 4 5 3 3 3
(23) 3 5 6 2 4 5 3 3 3 ← (24) 5 6 2 3 5 7 3 3
(25) ...3 3 6 6 5 3 ← (26) 3 6 2 3 5 7 3 3
(27) 5 8 1 1 2 4 3 3 3 ← (28) 7 13 1 * 1
(28) ....3 5 7 3 3 ← (33) 8 1 1 2 4 3 3 3
(29) 4 ...4 5 3 3 3 ← (30) 6 ..4 5 3 3 3
(31) ....4 5 3 3 3 ← (32) 4 ..4 5 3 3 3
(35) 5 1 2 3 4 4 1 1 1 ← (36) 6 2 3 4 4 1 1 1

(57, 11)

(1) 5 5 9 3 5 7 3 5 7 7
(1) 8 4 5 3 5 9 3 5 7 7
(3) 1 2 3 5 10 11 3 5 7 7 ← (4) 12 9 5 5 3 6 6 5 3
(3) 16 2 3 5 5 3 6 6 5 3
(7) 5 5 5 6 5 2 3 5 7 7 ← (13) 5 5 6 5 2 3 5 7 7
(13) 2 4 5 5 5 3 6 6 5 3 ← (16) 4 5 5 5 3 6 6 5 3
(20) ..4 3 3 3 6 6 5 3 ← (26) ...3 3 6 6 5 3
(23) ....3 3 6 6 5 3 ← (24) 3 5 6 2 4 5 3 3 3
(26) .....3 5 7 3 3 ← (32) 4 5 3 3 3
(27) 4 ....4 5 3 3 3 ← (28) 5 8 1 1 2 4 3 3 3
(29) .....4 5 3 3 3 ← (30) 4 ...4 5 3 3 3
(35) 3 4 4 1 1 * 1 ← (36) 5 1 2 3 4 4 1 1 1
(39) 1 * * 1 ← (41) * * 1

(57, 12)

(1) 4 4 2 3 5 3 5 9 7 7 7 ← (2) 8 4 5 3 5 9 3 5 7 7
(2) 5 5 5 5 6 5 2 3 5 7 7
(3) 2 4 2 3 5 3 5 9 7 7 7 ← (5) 18 2 4 3 3 3 6 6 5 3
(3) 13 ..4 7 3 3 6 6 5 3 ← (4) 16 2 3 5 5 3 6 6 5 3
(5) 4 3 5 5 6 5 2 3 5 7 7 ← (8) 5 5 5 6 5 2 3 5 7 7
(11) ..4 5 5 5 3 6 6 5 3 ← (14) 2 4 5 5 5 3 6 6 5 3
(17) 6 ....3 3 6 6 5 3 ← (24) 3 3 6 6 5 3
(23) 6 .....4 5 3 3 3 ← (30) 4 5 3 3 3
(27) ......4 5 3 3 3 ← (28) 4 ....4 5 3 3 3
(38) 1 1 * * 1 ← (40) 1 * * 1

(57, 13)

(1) ..4 2 3 5 3 5 9 7 7 7 ← (2) 4 4 2 3 5 3 5 9 7 7 7
(2) 1 2 4 2 3 5 3 5 9 7 7 7 ← (4) 2 4 2 3 5 3 5 9 7 7 7
(3) 2 4 3 5 5 6 5 2 3 5 7 7 ← (6) 4 3 5 5 6 5 2 3 5 7 7
(3) 11 6 ..4 3 3 3 6 6 5 3 ← (4) 13 ..4 7 3 3 6 6 5 3
(9) ...4 5 5 5 3 6 6 5 3 ← (12) ..4 5 5 5 3 6 6 5 3
(15) 5 6 .....3 5 7 3 3 ← (21) 6 .....3 5 7 3 3
(17) 3 6 .....3 5 7 3 3 ← (18) 6 ....3 3 6 6 5 3
(21) 6 ......4 5 3 3 3 ← (28) 4 5 3 3 3
(23) 4 ......4 5 3 3 3 ← (24) 6 .....4 5 3 3 3

(57, 14)

(1) 1 1 2 4 2 3 5 3 5 9 7 7 7 ← (2) ..4 2 3 5 3 5 9 7 7 7
(1) ..4 3 5 5 6 5 2 3 5 7 7 ← (4) 2 4 3 5 5 6 5 2 3 5 7 7
(3) 9 4 1 1 2 4 7 3 3 6 6 5 3 ← (4) 11 6 ..4 3 3 3 6 6 5 3
(7) ....4 5 5 5 3 6 6 5 3 ← (10) ...4 5 5 5 3 6 6 5 3
(14) .....4 3 3 3 6 6 5 3 ← (19) 5 6 .....4 5 3 3 3
(15) 3 5 6 .....4 5 3 3 3 ← (16) 5 6 .....3 5 7 3 3
(17) ....3 3 6 6 5 3 ← (18) 3 6 .....3 5 7 3 3
(20) ....3 5 7 3 3 ← (25) 8 1 1 * 2 4 3 3 3
(21) 4 ....4 5 3 3 3 ← (22) 6 ......4 5 3 3 3
(23) ....4 5 3 3 3 ← (24) 4 ......4 5 3 3 3
(27) 5 1 2 3 4 4 1 1 * 1 ← (28) 6 2 3 4 4 1 1 * 1

(57, 15)

(2) 4 3 5 6 5 2 3 5 7 7
(3) 8 ..1 ..4 7 3 3 6 6 5 3 ← (4) 9 4 1 1 2 4 7 3 3 6 6 5 3
(5) .....4 5 5 5 3 6 6 5 3 ← (8) 4 5 5 5 3 6 6 5 3
(11) 1.........4 5 3 3 3 ← (18) 3 3 6 6 5 3
(15) ....3 3 6 6 5 3 ← (16) 3 5 6 .....4 5 3 3 3
(18) ....3 5 7 3 3 ← (24) 4 5 3 3 3
(21) ....4 5 3 3 3 ← (22) 4 ....4 5 3 3 3
(27) 3 4 4 1 1 * * 1 ← (28) 5 1 2 3 4 4 1 1 * 1
(31) 1 * * * 1 ← (33) * * * 1

(57, 16)

(3) ......4 5 5 5 3 6 6 5 3 ← (6) 4 5 5 5 3 6 6 5 3
(3) 5 2 * 2 4 7 3 3 6 6 5 3 ← (4) 8 ..1 ..4 7 3 3 6 6 5 3
(9) 6 ....3 3 6 6 5 3 ← (16) 3 3 6 6 5 3
(11) 10 ....4 5 3 3 3 ← (12) 1.........4 5 3 3 3
(15) 6 ....4 5 3 3 3 ← (22) 4 5 3 3 3
(30) 1 1 * * * 1 ← (32) 1 * * * 1

(57, 17)

(1) ....4 5 5 5 3 6 6 5 3 ← (4) 4 5 5 5 3 6 6 5 3
(3) 3 6 ......4 3 3 3 6 6 5 3 ← (4) 5 2 * 2 4 7 3 3 6 6 5 3
(7) 5 6 ....3 5 7 3 3 ← (13) 6 ....3 5 7 3 3
(9) 3 6 ....3 5 7 3 3 ← (10) 6 ....3 3 6 6 5 3
(11) 7 13 1 * * * 1 ← (12) 10 ....4 5 3 3 3
(13) 6 ....4 5 3 3 3 ← (19) 13 1 * * * 1
(15) 4 ....4 5 3 3 3 ← (16) 6 ....4 5 3 3 3

(58, 2)

(55) 3 ← (59)

(58, 3)

(27) 30 1 ← (28) 31
(43) 14 1 ← (44) 15
(51) 6 1 ← (52) 7
(52) 3 3 ← (58) 1
(55) 2 1 ← (56) 3

(58, 4)

(19) 21 11 7 ← (21) 23 15
(21) 23 7 7 ← (25) 27 7
(27) 13 11 7 ← (29) 15 15
(27) 29 1 1 ← (28) 30 1
(37) 7 7 7 ← (41) 11 7
(39) 5 7 7 ← (45) 7 7
(43) 13 1 1 ← (44) 14 1
(50) 2 3 3 ← (57) 1 1
(51) 5 1 1 ← (52) 6 1
(55) 1 1 1 ← (56) 2 1

(58, 5)

(13) 14 13 11 7 ← (14) 15 15 15
(17) 10 13 11 7 ← (18) 11 15 15
(18) 19 7 7 7 ← (20) 21 11 7
(19) 20 5 7 7 ← (22) 23 7 7
(25) 14 5 7 7 ← (28) 13 11 7
(27) 28 1 1 1 ← (28) 29 1 1
(35) 4 5 7 7 ← (38) 7 7 7
(36) 3 5 7 7 ← (40) 5 7 7
(43) 12 1 1 1 ← (44) 13 1 1
(49) 1 2 3 3 ← (56) 1 1 1
(51) 4 1 1 1 ← (52) 5 1 1

(58, 6)

(5) 5 7 11 15 15 ← (9) 9 11 15 15
(7) 7 13 13 11 7 ← (14) 14 13 11 7
(11) 7 11 15 7 7 ← (19) 11 15 7 7
(17) 9 11 7 7 7 ← (18) 10 13 11 7
(18) 18 3 5 7 7 ← (20) 20 5 7 7
(19) 7 13 5 7 7 ← (21) 17 7 7 7
(23) 5 9 7 7 7 ← (27) 13 5 7 7
(25) 11 3 5 7 7 ← (26) 14 5 7 7
(27) 24 4 1 1 1 ← (28) 28 1 1 1
(28) 12 9 3 3 3
(34) 2 3 5 7 7 ← (36) 4 5 7 7
(37) 3 5 7 3 3 ← (41) 5 7 3 3
(41) 3 6 2 3 3 ← (42) 8 3 3 3
(43) 2 4 3 3 3 ← (45) 6 2 3 3
(43) 8 4 1 1 1 ← (44) 12 1 1 1
(47) 4 4 1 1 1 ← (52) 4 1 1 1

(58, 7)

(6) 5 7 11 15 7 7 ← (11) 10 17 7 7 7
(7) 6 9 15 7 7 7 ← (8) 7 13 13 11 7
(9) 8 9 11 7 7 7 ← (10) 9 11 15 7 7
(11) 6 9 11 7 7 7 ← (12) 7 11 15 7 7
(13) 12 11 3 5 7 7 ← (14) 13 13 5 7 7
(18) 5 5 9 7 7 7 ← (20) 7 13 5 7 7
(20) 3 5 9 7 7 7 ← (24) 5 9 7 7 7
(21) 13 2 3 5 7 7 ← (22) 14 4 5 7 7
(22) 5 9 3 5 7 7 ← (28) 9 3 5 7 7
(26) 9 3 6 6 5 3
(27) 22 * 1 ← (28) 24 4 1 1 1
(35) 2 3 5 7 3 3 ← (36) 3 6 6 5 3
(41) ..4 3 3 3 ← (42) 3 6 2 3 3
(42) 1 2 4 3 3 3 ← (44) 2 4 3 3 3
(43) 6 * 1 ← (44) 8 4 1 1 1
(44) 3 4 4 1 1 1 ← (50) * 1
(47) 2 * 1 ← (48) 4 4 1 1 1

(58, 8)

(1) 27 12 4 5 3 3 3
(3) 1 2 4 7 11 15 15 ← (5) 2 4 7 11 15 15
(5) 3 5 9 15 7 7 7 ← (9) 5 9 15 7 7 7
(7) 4 6 9 11 7 7 7 ← (8) 6 9 15 7 7 7
(11) 9 3 5 9 7 7 7 ← (13) 11 5 9 7 7 7
(13) 7 8 12 9 3 3 3 ← (14) 12 11 3 5 7 7
(19) 3 5 9 3 5 7 7 ← (21) 8 12 9 3 3 3
(21) 3 5 7 3 5 7 7 ← (25) 5 7 3 5 7 7
(21) 7 12 4 5 3 3 3 ← (22) 13 2 3 5 7 7
(23) 6 5 2 3 5 7 7 ← (29) 12 4 5 3 3 3
(25) 5 5 3 6 6 5 3
(27) 21 1 * 1 ← (28) 22 * 1
(41) 1 1 2 4 3 3 3 ← (42) ..4 3 3 3
(42) 2 3 4 4 1 1 1 ← (49) 1 * 1
(43) 5 1 * 1 ← (44) 6 * 1
(47) 1 1 * 1 ← (48) 2 * 1

(58, 9)

(1) 6 4 6 9 11 7 7 7 ← (4) 1 2 4 7 11 15 15
(1) 18 3 5 9 3 5 7 7 ← (2) 27 12 4 5 3 3 3
(6) 4 8 5 5 9 7 7 7 ← (12) 9 3 5 9 7 7 7
(7) 3 5 10 11 3 5 7 7 ← (11) 5 10 11 3 5 7 7
(11) 4 5 3 5 9 7 7 7 ← (13) 8 3 5 9 7 7 7
(11) 8 3 5 9 3 5 7 7 ← (14) 7 8 12 9 3 3 3
(21) 4 7 3 3 6 6 5 3 ← (27) 6 3 3 6 6 5 3
(21) 5 6 3 3 6 6 5 3 ← (22) 7 12 4 5 3 3 3
(23) 3 6 3 3 6 6 5 3 ← (24) 6 5 2 3 5 7 7
(27) 20 1 1 * 1 ← (28) 21 1 * 1
(29) 3 6 2 4 5 3 3 3 ← (30) 6 2 3 5 7 3 3
(35) 12 1 1 * 1 ← (36) 13 1 * 1
(41) 1 2 3 4 4 1 1 1 ← (48) 1 1 * 1
(43) 4 1 1 * 1 ← (44) 5 1 * 1

(58, 10)

(1) 4 2 4 6 9 11 7 7 7 ← (2) 6 4 6 9 11 7 7 7
(1) 11 9 3 5 7 3 5 7 7 ← (2) 18 3 5 9 3 5 7 7
(2) 4 4 8 5 5 9 7 7 7 ← (8) 3 5 10 11 3 5 7 7
(5) 2 3 5 10 11 3 5 7 7
(7) 5 9 3 5 7 3 5 7 7 ← (13) 9 3 5 7 3 5 7 7
(10) 2 3 5 3 5 9 7 7 7 ← (12) 4 5 3 5 9 7 7 7
(10) 4 5 3 5 9 3 5 7 7 ← (12) 8 3 5 9 3 5 7 7
(19) 2 4 7 3 3 6 6 5 3 ← (22) 4 7 3 3 6 6 5 3
(20) 2 3 5 5 3 6 6 5 3 ← (25) 5 6 2 3 5 7 3 3
(21) 3 5 6 2 3 5 7 3 3 ← (22) 5 6 3 3 6 6 5 3
(23) 2 4 3 3 3 6 6 5 3 ← (24) 3 6 3 3 6 6 5 3
(27) 3 6 ..4 5 3 3 3 ← (28) 5 6 2 4 5 3 3 3
(27) 16 4 1 1 * 1 ← (28) 20 1 1 * 1
(29) ....3 5 7 3 3 ← (30) 3 6 2 4 5 3 3 3
(33) 3 6 2 3 4 4 1 1 1 ← (34) 8 1 1 2 4 3 3 3
(35) * 2 4 3 3 3 ← (37) 6 2 3 4 4 1 1 1
(35) 8 4 1 1 * 1 ← (36) 12 1 1 * 1
(39) 4 4 1 1 * 1 ← (44) 4 1 1 * 1

(58, 11)

(1) ...4 6 9 11 7 7 7 ← (2) 4 2 4 6 9 11 7 7 7
(2) 5 5 9 3 5 7 3 5 7 7 ← (8) 5 9 3 5 7 3 5 7 7
(4) 1 2 3 5 10 11 3 5 7 7
(11) 4 3 5 6 5 2 3 5 7 7 ← (14) 5 5 6 5 2 3 5 7 7
(17) ..4 7 3 3 6 6 5 3 ← (24) 2 4 3 3 3 6 6 5 3
(18) 1 2 4 7 3 3 6 6 5 3 ← (20) 2 4 7 3 3 6 6 5 3
(21) ..4 3 3 3 6 6 5 3 ← (22) 3 5 6 2 3 5 7 3 3
(27) .....3 5 7 3 3 ← (28) 3 6 ..4 5 3 3 3
(27) 14 * * 1 ← (28) 16 4 1 1 * 1
(33) 2 * 2 4 3 3 3 ← (34) 3 6 2 3 4 4 1 1 1
(34) 1 * 2 4 3 3 3 ← (36) * 2 4 3 3 3
(35) 6 * * 1 ← (36) 8 4 1 1 * 1
(36) 3 4 4 1 1 * 1 ← (42) * * 1
(39) 2 * * 1 ← (40) 4 4 1 1 * 1

(58, 12)

(1) 4 5 4 5 3 5 9 3 5 7 7
(3) 5 5 5 5 6 5 2 3 5 7 7 ← (9) 5 5 5 6 5 2 3 5 7 7
(5) 16 ..4 3 3 3 6 6 5 3 ← (6) 18 2 4 3 3 3 6 6 5 3
(9) 2 4 3 5 6 5 2 3 5 7 7 ← (12) 4 3 5 6 5 2 3 5 7 7
(15) 6 ..4 3 3 3 6 6 5 3 ← (22) ..4 3 3 3 6 6 5 3
(17) 1 1 2 4 7 3 3 6 6 5 3 ← (18) ..4 7 3 3 6 6 5 3
(27) 13 1 * * 1 ← (28) 14 * * 1
(33) 1 1 * 2 4 3 3 3 ← (34) 2 * 2 4 3 3 3
(34) 2 3 4 4 1 1 * 1 ← (41) 1 * * 1
(35) 5 1 * * 1 ← (36) 6 * * 1
(39) 1 1 * * 1 ← (40) 2 * * 1

(58, 13)

(1) 4 3 5 5 5 6 5 2 3 5 7 7 ← (4) 5 5 5 5 6 5 2 3 5 7 7
(3) 1 2 4 2 3 5 3 5 9 7 7 7 ← (5) 2 4 2 3 5 3 5 9 7 7 7
(5) 13 6 ....3 3 6 6 5 3 ← (6) 16 ..4 3 3 3 6 6 5 3
(7) ..4 3 5 6 5 2 3 5 7 7 ← (10) 2 4 3 5 6 5 2 3 5 7 7
(13) 4 1 1 2 4 7 3 3 6 6 5 3 ← (19) 6 ....3 3 6 6 5 3
(15) 3 6 ....3 3 6 6 5 3 ← (16) 6 ..4 3 3 3 6 6 5 3
(21) 3 6 .....4 5 3 3 3 ← (22) 6 .....3 5 7 3 3
(27) 12 1 1 * * 1 ← (28) 13 1 * * 1
(33) 1 2 3 4 4 1 1 * 1 ← (40) 1 1 * * 1
(35) 4 1 1 * * 1 ← (36) 5 1 * * 1

(58, 14)

(2) 1 1 2 4 2 3 5 3 5 9 7 7 7 ← (4) 1 2 4 2 3 5 3 5 9 7 7 7
(2) ..4 3 5 5 6 5 2 3 5 7 7
(5) ...4 3 5 6 5 2 3 5 7 7 ← (8) ..4 3 5 6 5 2 3 5 7 7
(5) 11 5 6 .....3 5 7 3 3 ← (6) 13 6 ....3 3 6 6 5 3
(11) * 2 4 7 3 3 6 6 5 3 ← (14) 4 1 1 2 4 7 3 3 6 6 5 3
(11) 24 4 1 1 * * 1 ← (17) 5 6 .....3 5 7 3 3
(15) .....4 3 3 3 6 6 5 3 ← (16) 3 6 ....3 3 6 6 5 3
(19) 3 6 ......4 5 3 3 3 ← (20) 5 6 .....4 5 3 3 3
(21) ....3 5 7 3 3 ← (22) 3 6 .....4 5 3 3 3
(25) 3 6 2 3 4 4 1 1 * 1 ← (26) 8 1 1 * 2 4 3 3 3
(27) * * 2 4 3 3 3 ← (29) 6 2 3 4 4 1 1 * 1
(27) 8 4 1 1 * * 1 ← (28) 12 1 1 * * 1
(31) 4 4 1 1 * * 1 ← (36) 4 1 1 * * 1

(58, 15)

(3) ....4 3 5 6 5 2 3 5 7 7 ← (6) ...4 3 5 6 5 2 3 5 7 7
(5) 10 .....4 3 3 3 6 6 5 3 ← (6) 11 5 6 .....3 5 7 3 3
(9) 2 * 2 4 7 3 3 6 6 5 3 ← (16) 4 3 3 3 6 6 5 3
(10) 1 * 2 4 7 3 3 6 6 5 3 ← (12) * 2 4 7 3 3 6 6 5 3
(11) 22 * * * 1 ← (12) 24 4 1 1 * * 1
(19) ....3 5 7 3 3 ← (20) 3 6 ......4 5 3 3 3
(25) 2 * * 2 4 3 3 3 ← (26) 3 6 2 3 4 4 1 1 * 1
(26) 1 * * 2 4 3 3 3 ← (28) * * 2 4 3 3 3
(27) 6 * * * 1 ← (28) 8 4 1 1 * * 1
(28) 3 4 4 1 1 * * 1 ← (34) * * * 1
(31) 2 * * * 1 ← (32) 4 4 1 1 * * 1

(58, 16)

(1) .....4 3 5 6 5 2 3 5 7 7 ← (4) 4 3 5 6 5 2 3 5 7 7
(5) 7 1.........4 5 3 3 3 ← (6) 10 .....4 3 3 3 6 6 5 3
(7) 6 ......4 3 3 3 6 6 5 3 ← (13) 1.........4 5 3 3 3
(9) 1 1 * 2 4 7 3 3 6 6 5 3 ← (10) 2 * 2 4 7 3 3 6 6 5 3
(11) 21 1 * * * 1 ← (12) 22 * * * 1
(25) 1 1 * * 2 4 3 3 3 ← (26) 2 * * 2 4 3 3 3
(26) 2 3 4 4 1 1 * * 1 ← (33) 1 * * * 1
(27) 5 1 * * * 1 ← (28) 6 * * * 1
(31) 1 1 * * * 1 ← (32) 2 * * * 1

(59, 3)

(27) 29 3 ← (29) 31
(43) 13 3 ← (45) 15
(51) 5 3 ← (53) 7
(53) 3 3 ← (57) 3

(59, 4)

(21) 22 13 3 ← (22) 23 15
(25) 26 5 3 ← (26) 27 7
(26) 27 3 3 ← (28) 29 3
(29) 14 13 3 ← (30) 15 15
(41) 10 5 3 ← (42) 11 7
(42) 11 3 3 ← (44) 13 3
(45) 6 5 3 ← (46) 7 7
(50) 3 3 3 ← (52) 5 3
(51) 2 3 3 ← (54) 3 3

(59, 5)

(11) 7 11 15 15 ← (19) 11 15 15
(15) 13 13 11 7 ← (29) 13 11 7
(19) 19 7 7 7 ← (21) 21 11 7
(21) 21 11 3 3 ← (22) 22 13 3
(25) 25 3 3 3 ← (26) 26 5 3
(29) 13 11 3 3 ← (30) 14 13 3
(37) 3 5 7 7 ← (41) 5 7 7
(39) 6 6 5 3 ← (46) 6 5 3
(41) 9 3 3 3 ← (42) 10 5 3
(50) 1 2 3 3 ← (52) 2 3 3

(59, 6)

(6) 5 7 11 15 15 ← (10) 9 11 15 15
(7) 4 7 11 15 15 ← (12) 7 11 15 15
(14) 9 15 7 7 7 ← (27) 14 5 7 7
(15) 11 14 5 7 7 ← (16) 13 13 11 7
(18) 9 11 7 7 7 ← (22) 17 7 7 7
(19) 7 14 5 7 7 ← (20) 11 15 7 7
(19) 18 3 5 7 7 ← (21) 20 5 7 7
(21) 20 9 3 3 3 ← (22) 21 11 3 3
(26) 11 3 5 7 7 ← (28) 13 5 7 7
(29) 12 9 3 3 3 ← (30) 13 11 3 3
(35) 2 3 5 7 7 ← (37) 4 5 7 7
(38) 3 5 7 3 3 ← (43) 8 3 3 3
(39) 4 7 3 3 3 ← (40) 6 6 5 3
(41) 4 5 3 3 3 ← (42) 5 7 3 3
(45) 5 1 2 3 3 ← (46) 6 2 3 3

(59, 7)

(1) 28 12 9 3 3 3 ← (8) 4 7 11 15 15
(7) 5 7 11 15 7 7 ← (9) 7 13 13 11 7
(10) 8 9 11 7 7 7 ← (23) 14 4 5 7 7
(11) 9 7 13 5 7 7 ← (12) 10 17 7 7 7
(12) 6 9 11 7 7 7 ← (21) 7 13 5 7 7
(15) 7 14 4 5 7 7 ← (16) 11 14 5 7 7
(17) 12 12 9 3 3 3 ← (20) 18 3 5 7 7
(19) 5 5 9 7 7 7 ← (20) 7 14 5 7 7
(19) 17 3 6 6 5 3 ← (22) 20 9 3 3 3
(21) 3 5 9 7 7 7 ← (25) 5 9 7 7 7
(23) 5 9 3 5 7 7 ← (29) 9 3 5 7 7
(27) 9 3 6 6 5 3 ← (30) 12 9 3 3 3
(33) 3 3 6 6 5 3 ← (36) 2 3 5 7 7
(36) 2 3 5 7 3 3 ← (42) 4 5 3 3 3
(39) 2 4 5 3 3 3 ← (40) 4 7 3 3 3
(43) 1 2 4 3 3 3 ← (45) 2 4 3 3 3
(45) 3 4 4 1 1 1 ← (46) 5 1 2 3 3

(59, 8)

(1) 26 9 3 6 6 5 3 ← (2) 28 12 9 3 3 3
(6) 3 5 9 15 7 7 7 ← (10) 5 9 15 7 7 7
(8) 4 6 9 11 7 7 7 ← (16) 7 14 4 5 7 7
(11) 8 5 5 9 7 7 7 ← (12) 9 7 13 5 7 7
(17) 10 9 3 6 6 5 3 ← (18) 12 12 9 3 3 3
(18) 11 12 4 5 3 3 3 ← (20) 17 3 6 6 5 3
(20) 3 5 9 3 5 7 7 ← (26) 5 7 3 5 7 7
(21) 6 9 3 6 6 5 3 ← (22) 8 12 9 3 3 3
(22) 3 5 7 3 5 7 7 ← (24) 5 9 3 5 7 7
(26) 5 5 3 6 6 5 3 ← (28) 9 3 6 6 5 3
(29) 10 2 4 5 3 3 3 ← (30) 12 4 5 3 3 3
(33) 6 2 4 5 3 3 3 ← (40) 2 4 5 3 3 3
(42) 1 1 2 4 3 3 3 ← (44) 1 2 4 3 3 3
(43) 2 3 4 4 1 1 1 ← (46) 3 4 4 1 1 1

(59, 9)

(1) 25 5 5 3 6 6 5 3 ← (2) 26 9 3 6 6 5 3
(5) 13 11 12 4 5 3 3 3 ← (14) 8 3 5 9 7 7 7
(7) 4 8 5 5 9 7 7 7 ← (12) 8 5 5 9 7 7 7
(11) 4 6 3 5 9 7 7 7 ← (12) 5 10 11 3 5 7 7
(17) 9 5 5 3 6 6 5 3 ← (18) 10 9 3 6 6 5 3
(19) 5 6 5 2 3 5 7 7 ← (25) 6 5 2 3 5 7 7
(21) 3 6 5 2 3 5 7 7 ← (22) 6 9 3 6 6 5 3
(27) 3 6 2 3 5 7 3 3 ← (28) 6 3 3 6 6 5 3
(29) 7 13 1 * 1 ← (30) 10 2 4 5 3 3 3
(31) 6 ..4 5 3 3 3 ← (37) 13 1 * 1
(33) 4 ..4 5 3 3 3 ← (34) 6 2 4 5 3 3 3
(42) 1 2 3 4 4 1 1 1 ← (44) 2 3 4 4 1 1 1

(59, 10)

(2) 11 9 3 5 7 3 5 7 7 ← (9) 3 5 10 11 3 5 7 7
(3) 4 4 8 5 5 9 7 7 7 ← (8) 4 8 5 5 9 7 7 7
(5) 12 9 5 5 3 6 6 5 3 ← (6) 13 11 12 4 5 3 3 3
(6) 2 3 5 10 11 3 5 7 7
(11) 2 3 5 3 5 9 7 7 7 ← (12) 4 6 3 5 9 7 7 7
(11) 4 5 3 5 9 3 5 7 7 ← (13) 8 3 5 9 3 5 7 7
(17) 4 5 5 5 3 6 6 5 3 ← (20) 5 6 5 2 3 5 7 7
(21) 2 3 5 5 3 6 6 5 3 ← (22) 3 6 5 2 3 5 7 7
(25) 3 5 6 2 4 5 3 3 3 ← (26) 5 6 2 3 5 7 3 3
(27) ...3 3 6 6 5 3 ← (28) 3 6 2 3 5 7 3 3
(29) 5 8 1 1 2 4 3 3 3 ← (30) 7 13 1 * 1
(30) ....3 5 7 3 3 ← (35) 8 1 1 2 4 3 3 3
(31) 4 ...4 5 3 3 3 ← (32) 6 ..4 5 3 3 3
(33) ....4 5 3 3 3 ← (34) 4 ..4 5 3 3 3
(37) 5 1 2 3 4 4 1 1 1 ← (38) 6 2 3 4 4 1 1 1

(59, 11)

(2) ...4 6 9 11 7 7 7 ← (4) 4 4 8 5 5 9 7 7 7
(3) 5 5 9 3 5 7 3 5 7 7 ← (9) 5 9 3 5 7 3 5 7 7
(3) 8 4 5 3 5 9 3 5 7 7
(5) 1 2 3 5 10 11 3 5 7 7 ← (6) 12 9 5 5 3 6 6 5 3
(5) 16 2 3 5 5 3 6 6 5 3 ← (12) 4 5 3 5 9 3 5 7 7
(15) 2 4 5 5 5 3 6 6 5 3 ← (18) 4 5 5 5 3 6 6 5 3
(19) 1 2 4 7 3 3 6 6 5 3 ← (21) 2 4 7 3 3 6 6 5 3
(25) ....3 3 6 6 5 3 ← (26) 3 5 6 2 4 5 3 3 3
(28) .....3 5 7 3 3 ← (34) 4 5 3 3 3
(29) 4 ....4 5 3 3 3 ← (30) 5 8 1 1 2 4 3 3 3
(31) .....4 5 3 3 3 ← (32) 4 ...4 5 3 3 3
(35) 1 * 2 4 3 3 3 ← (37) * 2 4 3 3 3
(37) 3 4 4 1 1 * 1 ← (38) 5 1 2 3 4 4 1 1 1

(59, 12)

(2) 4 5 4 5 3 5 9 3 5 7 7
(3) 4 4 2 3 5 3 5 9 7 7 7 ← (4) 8 4 5 3 5 9 3 5 7 7
(5) 13 ..4 7 3 3 6 6 5 3 ← (6) 16 2 3 5 5 3 6 6 5 3
(7) 4 3 5 5 6 5 2 3 5 7 7 ← (10) 5 5 5 6 5 2 3 5 7 7
(13) ..4 5 5 5 3 6 6 5 3 ← (16) 2 4 5 5 5 3 6 6 5 3
(18) 1 1 2 4 7 3 3 6 6 5 3 ← (20) 1 2 4 7 3 3 6 6 5 3
(25) 6 .....4 5 3 3 3 ← (32) 4 5 3 3 3
(29) ......4 5 3 3 3 ← (30) 4 ....4 5 3 3 3
(34) 1 1 * 2 4 3 3 3 ← (36) 1 * 2 4 3 3 3
(35) 2 3 4 4 1 1 * 1 ← (38) 3 4 4 1 1 * 1

(59, 13)

(2) 4 3 5 5 5 6 5 2 3 5 7 7
(3) ..4 2 3 5 3 5 9 7 7 7 ← (4) 4 4 2 3 5 3 5 9 7 7 7
(5) 2 4 3 5 5 6 5 2 3 5 7 7 ← (8) 4 3 5 5 6 5 2 3 5 7 7
(5) 11 6 ..4 3 3 3 6 6 5 3 ← (6) 13 ..4 7 3 3 6 6 5 3
(11) ...4 5 5 5 3 6 6 5 3 ← (14) ..4 5 5 5 3 6 6 5 3
(19) 3 6 .....3 5 7 3 3 ← (20) 6 ....3 3 6 6 5 3
(23) 6 ......4 5 3 3 3 ← (30) 4 5 3 3 3
(25) 4 ......4 5 3 3 3 ← (26) 6 .....4 5 3 3 3
(34) 1 2 3 4 4 1 1 * 1 ← (36) 2 3 4 4 1 1 * 1

(59, 14)

(3) 1 1 2 4 2 3 5 3 5 9 7 7 7 ← (4) ..4 2 3 5 3 5 9 7 7 7
(3) ..4 3 5 5 6 5 2 3 5 7 7 ← (6) 2 4 3 5 5 6 5 2 3 5 7 7
(5) 9 4 1 1 2 4 7 3 3 6 6 5 3 ← (6) 11 6 ..4 3 3 3 6 6 5 3
(9) ....4 5 5 5 3 6 6 5 3 ← (12) ...4 5 5 5 3 6 6 5 3
(17) 3 5 6 .....4 5 3 3 3 ← (18) 5 6 .....3 5 7 3 3
(19) ....3 3 6 6 5 3 ← (20) 3 6 .....3 5 7 3 3
(22) ....3 5 7 3 3 ← (27) 8 1 1 * 2 4 3 3 3
(23) 4 ....4 5 3 3 3 ← (24) 6 ......4 5 3 3 3
(25) ....4 5 3 3 3 ← (26) 4 ......4 5 3 3 3
(29) 5 1 2 3 4 4 1 1 * 1 ← (30) 6 2 3 4 4 1 1 * 1

(59, 15)

(1) ...4 3 5 5 6 5 2 3 5 7 7 ← (4) ..4 3 5 5 6 5 2 3 5 7 7
(5) 8 ..1 ..4 7 3 3 6 6 5 3 ← (6) 9 4 1 1 2 4 7 3 3 6 6 5 3
(7) .....4 5 5 5 3 6 6 5 3 ← (10) 4 5 5 5 3 6 6 5 3
(11) 1 * 2 4 7 3 3 6 6 5 3 ← (13) * 2 4 7 3 3 6 6 5 3
(17) ....3 3 6 6 5 3 ← (18) 3 5 6 .....4 5 3 3 3
(20) ....3 5 7 3 3 ← (26) 4 5 3 3 3
(23) ....4 5 3 3 3 ← (24) 4 ....4 5 3 3 3
(27) 1 * * 2 4 3 3 3 ← (29) * * 2 4 3 3 3
(29) 3 4 4 1 1 * * 1 ← (30) 5 1 2 3 4 4 1 1 * 1

(60, 2)

(59) 1 ← (61)

(60, 3)

(29) 30 1 ← (30) 31
(45) 14 1 ← (46) 15
(53) 6 1 ← (54) 7
(57) 2 1 ← (58) 3
(58) 1 1 ← (60) 1

(60, 4)

(15) 15 15 15 ← (23) 23 15
(23) 23 7 7 ← (27) 27 7
(27) 27 3 3 ← (29) 29 3
(29) 29 1 1 ← (30) 30 1
(39) 7 7 7 ← (43) 11 7
(43) 11 3 3 ← (45) 13 3
(45) 13 1 1 ← (46) 14 1
(51) 3 3 3 ← (53) 5 3
(53) 5 1 1 ← (54) 6 1
(57) 1 1 1 ← (58) 2 1

(60, 5)

(15) 14 13 11 7 ← (16) 15 15 15
(19) 10 13 11 7 ← (20) 11 15 15
(20) 19 7 7 7 ← (24) 23 7 7
(26) 25 3 3 3 ← (28) 27 3 3
(29) 28 1 1 1 ← (30) 29 1 1
(38) 3 5 7 7 ← (42) 5 7 7
(42) 9 3 3 3 ← (44) 11 3 3
(45) 12 1 1 1 ← (46) 13 1 1
(51) 1 2 3 3 ← (53) 2 3 3
(53) 4 1 1 1 ← (54) 5 1 1

(60, 6)

(7) 5 7 11 15 15 ← (11) 9 11 15 15
(11) 9 11 15 7 7 ← (23) 17 7 7 7
(13) 7 11 15 7 7 ← (21) 11 15 7 7
(15) 9 15 7 7 7 ← (17) 13 13 11 7
(15) 13 13 5 7 7 ← (22) 20 5 7 7
(19) 9 11 7 7 7 ← (20) 10 13 11 7
(27) 11 3 5 7 7 ← (28) 14 5 7 7
(29) 24 4 1 1 1 ← (30) 28 1 1 1
(37) 3 6 6 5 3 ← (43) 5 7 3 3
(39) 3 5 7 3 3 ← (41) 6 6 5 3
(43) 3 6 2 3 3 ← (44) 8 3 3 3
(45) 8 4 1 1 1 ← (46) 12 1 1 1
(49) 4 4 1 1 1 ← (52) 1 2 3 3
(51) * 1 ← (54) 4 1 1 1

(60, 7)

(6) 2 4 7 11 15 15
(8) 5 7 11 15 7 7 ← (16) 9 15 7 7 7
(9) 6 9 15 7 7 7 ← (10) 7 13 13 11 7
(11) 8 9 11 7 7 7 ← (12) 9 11 15 7 7
(13) 6 9 11 7 7 7 ← (14) 7 11 15 7 7
(14) 11 5 9 7 7 7 ← (21) 18 3 5 7 7
(15) 12 11 3 5 7 7 ← (16) 13 13 5 7 7
(20) 5 5 9 7 7 7
(22) 3 5 9 7 7 7 ← (26) 5 9 7 7 7
(23) 13 2 3 5 7 7 ← (24) 14 4 5 7 7
(29) 22 * 1 ← (30) 24 4 1 1 1
(34) 3 3 6 6 5 3 ← (40) 3 5 7 3 3
(37) 2 3 5 7 3 3 ← (38) 3 6 6 5 3
(43) ..4 3 3 3 ← (44) 3 6 2 3 3
(45) 6 * 1 ← (46) 8 4 1 1 1
(49) 2 * 1 ← (50) 4 4 1 1 1
(50) 1 * 1 ← (52) * 1

(60, 8)

(3) 27 12 4 5 3 3 3
(5) 1 2 4 7 11 15 15
(7) 3 5 9 15 7 7 7 ← (11) 5 9 15 7 7 7
(9) 4 6 9 11 7 7 7 ← (10) 6 9 15 7 7 7
(13) 9 3 5 9 7 7 7 ← (19) 12 12 9 3 3 3
(15) 7 8 12 9 3 3 3 ← (16) 12 11 3 5 7 7
(19) 11 12 4 5 3 3 3 ← (21) 17 3 6 6 5 3
(21) 3 5 9 3 5 7 7 ← (27) 5 7 3 5 7 7
(23) 3 5 7 3 5 7 7 ← (25) 5 9 3 5 7 7
(23) 7 12 4 5 3 3 3 ← (24) 13 2 3 5 7 7
(27) 5 5 3 6 6 5 3 ← (29) 9 3 6 6 5 3
(29) 21 1 * 1 ← (30) 22 * 1
(31) 6 2 3 5 7 3 3 ← (38) 2 3 5 7 3 3
(43) 1 1 2 4 3 3 3 ← (44) ..4 3 3 3
(45) 5 1 * 1 ← (46) 6 * 1
(49) 1 1 * 1 ← (50) 2 * 1

(60, 9)

(2) 25 5 5 3 6 6 5 3
(3) 6 4 6 9 11 7 7 7 ← (16) 7 8 12 9 3 3 3
(3) 18 3 5 9 3 5 7 7 ← (4) 27 12 4 5 3 3 3
(13) 4 5 3 5 9 7 7 7
(14) 9 3 5 7 3 5 7 7 ← (24) 3 5 7 3 5 7 7
(18) 9 5 5 3 6 6 5 3 ← (20) 11 12 4 5 3 3 3
(23) 4 7 3 3 6 6 5 3 ← (28) 5 5 3 6 6 5 3
(23) 5 6 3 3 6 6 5 3 ← (24) 7 12 4 5 3 3 3
(25) 3 6 3 3 6 6 5 3 ← (26) 6 5 2 3 5 7 7
(29) 5 6 2 4 5 3 3 3 ← (35) 6 2 4 5 3 3 3
(29) 20 1 1 * 1 ← (30) 21 1 * 1
(31) 3 6 2 4 5 3 3 3 ← (32) 6 2 3 5 7 3 3
(37) 12 1 1 * 1 ← (38) 13 1 * 1
(43) 1 2 3 4 4 1 1 1 ← (45) 2 3 4 4 1 1 1
(45) 4 1 1 * 1 ← (46) 5 1 * 1

(60, 10)

(3) 4 2 4 6 9 11 7 7 7 ← (4) 6 4 6 9 11 7 7 7
(3) 11 9 3 5 7 3 5 7 7 ← (4) 18 3 5 9 3 5 7 7
(7) 2 3 5 10 11 3 5 7 7 ← (10) 3 5 10 11 3 5 7 7
(12) 2 3 5 3 5 9 7 7 7
(15) 5 5 6 5 2 3 5 7 7 ← (21) 5 6 5 2 3 5 7 7
(22) 2 3 5 5 3 6 6 5 3 ← (24) 4 7 3 3 6 6 5 3
(23) 3 5 6 2 3 5 7 3 3 ← (24) 5 6 3 3 6 6 5 3
(25) 2 4 3 3 3 6 6 5 3 ← (26) 3 6 3 3 6 6 5 3
(28) ...3 3 6 6 5 3 ← (33) 6 ..4 5 3 3 3
(29) 3 6 ..4 5 3 3 3 ← (30) 5 6 2 4 5 3 3 3
(29) 16 4 1 1 * 1 ← (30) 20 1 1 * 1
(31) ....3 5 7 3 3 ← (32) 3 6 2 4 5 3 3 3
(35) 3 6 2 3 4 4 1 1 1 ← (36) 8 1 1 2 4 3 3 3
(37) 8 4 1 1 * 1 ← (38) 12 1 1 * 1
(41) 4 4 1 1 * 1 ← (44) 1 2 3 4 4 1 1 1
(43) * * 1 ← (46) 4 1 1 * 1

(60, 11)

(1) 6 2 3 5 10 11 3 5 7 7
(3) ...4 6 9 11 7 7 7 ← (4) 4 2 4 6 9 11 7 7 7
(4) 5 5 9 3 5 7 3 5 7 7
(6) 1 2 3 5 10 11 3 5 7 7 ← (8) 2 3 5 10 11 3 5 7 7
(7) 18 2 4 3 3 3 6 6 5 3
(13) 4 3 5 6 5 2 3 5 7 7 ← (16) 5 5 6 5 2 3 5 7 7
(19) ..4 7 3 3 6 6 5 3 ← (22) 2 4 7 3 3 6 6 5 3
(23) ..4 3 3 3 6 6 5 3 ← (24) 3 5 6 2 3 5 7 3 3
(26) ....3 3 6 6 5 3 ← (32) 3 5 7 3 3
(29) .....3 5 7 3 3 ← (30) 3 6 ..4 5 3 3 3
(29) 14 * * 1 ← (30) 16 4 1 1 * 1
(35) 2 * 2 4 3 3 3 ← (36) 3 6 2 3 4 4 1 1 1
(37) 6 * * 1 ← (38) 8 4 1 1 * 1
(41) 2 * * 1 ← (42) 4 4 1 1 * 1
(42) 1 * * 1 ← (44) * * 1

(60, 12)

(1) 4 7 4 5 3 5 9 3 5 7 7 ← (2) 6 2 3 5 10 11 3 5 7 7
(3) 4 5 4 5 3 5 9 3 5 7 7 ← (5) 8 4 5 3 5 9 3 5 7 7
(5) 5 5 5 5 6 5 2 3 5 7 7
(6) 2 4 2 3 5 3 5 9 7 7 7
(7) 16 ..4 3 3 3 6 6 5 3 ← (8) 18 2 4 3 3 3 6 6 5 3
(11) 2 4 3 5 6 5 2 3 5 7 7 ← (14) 4 3 5 6 5 2 3 5 7 7
(17) 6 ..4 3 3 3 6 6 5 3 ← (21) 1 2 4 7 3 3 6 6 5 3
(19) 1 1 2 4 7 3 3 6 6 5 3 ← (20) ..4 7 3 3 6 6 5 3
(23) 6 .....3 5 7 3 3 ← (30) 3 5 7 3 3
(29) 13 1 * * 1 ← (30) 14 * * 1
(35) 1 1 * 2 4 3 3 3 ← (36) 2 * 2 4 3 3 3
(37) 5 1 * * 1 ← (38) 6 * * 1
(41) 1 1 * * 1 ← (42) 2 * * 1

(60, 13)

(1) 2 4 5 4 5 3 5 9 3 5 7 7 ← (2) 4 7 4 5 3 5 9 3 5 7 7
(3) 4 3 5 5 5 6 5 2 3 5 7 7 ← (6) 5 5 5 5 6 5 2 3 5 7 7
(5) 1 2 4 2 3 5 3 5 9 7 7 7
(7) 13 6 ....3 3 6 6 5 3 ← (8) 16 ..4 3 3 3 6 6 5 3
(9) ..4 3 5 6 5 2 3 5 7 7 ← (12) 2 4 3 5 6 5 2 3 5 7 7
(15) 4 1 1 2 4 7 3 3 6 6 5 3 ← (20) 1 1 2 4 7 3 3 6 6 5 3
(17) 3 6 ....3 3 6 6 5 3 ← (18) 6 ..4 3 3 3 6 6 5 3
(21) 5 6 .....4 5 3 3 3 ← (27) 6 .....4 5 3 3 3
(23) 3 6 .....4 5 3 3 3 ← (24) 6 .....3 5 7 3 3
(29) 12 1 1 * * 1 ← (30) 13 1 * * 1
(35) 1 2 3 4 4 1 1 * 1 ← (37) 2 3 4 4 1 1 * 1
(37) 4 1 1 * * 1 ← (38) 5 1 * * 1

(60, 14)

(1) 2 4 3 5 5 5 6 5 2 3 5 7 7 ← (4) 4 3 5 5 5 6 5 2 3 5 7 7
(4) 1 1 2 4 2 3 5 3 5 9 7 7 7
(7) ...4 3 5 6 5 2 3 5 7 7 ← (10) ..4 3 5 6 5 2 3 5 7 7
(7) 11 5 6 .....3 5 7 3 3 ← (8) 13 6 ....3 3 6 6 5 3
(13) 24 4 1 1 * * 1 ← (16) 4 1 1 2 4 7 3 3 6 6 5 3
(17) .....4 3 3 3 6 6 5 3 ← (18) 3 6 ....3 3 6 6 5 3
(20) ....3 3 6 6 5 3 ← (25) 6 ......4 5 3 3 3
(21) 3 6 ......4 5 3 3 3 ← (22) 5 6 .....4 5 3 3 3
(23) ....3 5 7 3 3 ← (24) 3 6 .....4 5 3 3 3
(27) 3 6 2 3 4 4 1 1 * 1 ← (28) 8 1 1 * 2 4 3 3 3
(29) 8 4 1 1 * * 1 ← (30) 12 1 1 * * 1
(33) 4 4 1 1 * * 1 ← (36) 1 2 3 4 4 1 1 * 1
(35) * * * 1 ← (38) 4 1 1 * * 1

(61, 3)

(31) 15 15 ← (47) 15
(47) 7 7 ← (55) 7
(55) 3 3 ← (59) 3
(59) 1 1 ← (61) 1

(61, 4)

(22) 21 11 7
(23) 22 13 3 ← (24) 23 15
(27) 26 5 3 ← (28) 27 7
(30) 13 11 7 ← (46) 13 3
(31) 14 13 3 ← (32) 15 15
(40) 7 7 7 ← (54) 5 3
(43) 10 5 3 ← (44) 11 7
(47) 6 5 3 ← (48) 7 7
(52) 3 3 3 ← (56) 3 3
(58) 1 1 1 ← (60) 1 1

(61, 5)

(13) 7 11 15 15 ← (21) 11 15 15
(16) 14 13 11 7 ← (45) 11 3 3
(21) 19 7 7 7 ← (25) 23 7 7
(23) 21 11 3 3 ← (24) 22 13 3
(27) 25 3 3 3 ← (28) 26 5 3
(29) 13 5 7 7
(31) 13 11 3 3 ← (32) 14 13 3
(38) 4 5 7 7 ← (48) 6 5 3
(39) 3 5 7 7 ← (43) 5 7 7
(43) 9 3 3 3 ← (44) 10 5 3
(47) 6 2 3 3 ← (54) 2 3 3

(61, 6)

(8) 5 7 11 15 15
(9) 4 7 11 15 15
(13) 10 17 7 7 7 ← (44) 9 3 3 3
(17) 11 14 5 7 7 ← (18) 13 13 11 7
(20) 9 11 7 7 7
(21) 7 14 5 7 7 ← (22) 11 15 7 7
(22) 7 13 5 7 7 ← (24) 17 7 7 7
(23) 20 9 3 3 3 ← (24) 21 11 3 3
(28) 11 3 5 7 7
(30) 9 3 5 7 7 ← (40) 3 5 7 7
(31) 12 9 3 3 3 ← (32) 13 11 3 3
(37) 2 3 5 7 7 ← (45) 8 3 3 3
(41) 4 7 3 3 3 ← (42) 6 6 5 3
(43) 4 5 3 3 3 ← (44) 5 7 3 3
(46) 2 4 3 3 3 ← (53) 1 2 3 3
(47) 5 1 2 3 3 ← (48) 6 2 3 3

(61, 7)

(3) 28 12 9 3 3 3
(7) 2 4 7 11 15 15 ← (10) 4 7 11 15 15
(9) 5 7 11 15 7 7 ← (17) 9 15 7 7 7
(12) 8 9 11 7 7 7 ← (32) 12 9 3 3 3
(13) 9 7 13 5 7 7 ← (14) 10 17 7 7 7
(14) 6 9 11 7 7 7
(15) 11 5 9 7 7 7 ← (17) 13 13 5 7 7
(17) 7 14 4 5 7 7 ← (18) 11 14 5 7 7
(21) 5 5 9 7 7 7 ← (22) 7 14 5 7 7
(23) 3 5 9 7 7 7 ← (25) 14 4 5 7 7
(23) 8 12 9 3 3 3
(31) 12 4 5 3 3 3 ← (44) 4 5 3 3 3
(35) 3 3 6 6 5 3 ← (41) 3 5 7 3 3
(41) 2 4 5 3 3 3 ← (42) 4 7 3 3 3
(45) 1 2 4 3 3 3 ← (51) 4 4 1 1 1
(47) 3 4 4 1 1 1 ← (48) 5 1 2 3 3
(51) 1 * 1 ← (53) * 1

(61, 8)

(1) 6 2 4 7 11 15 15
(3) 26 9 3 6 6 5 3 ← (4) 28 12 9 3 3 3
(6) 1 2 4 7 11 15 15 ← (8) 2 4 7 11 15 15
(8) 3 5 9 15 7 7 7
(10) 4 6 9 11 7 7 7 ← (30) 9 3 6 6 5 3
(13) 5 10 11 3 5 7 7 ← (24) 3 5 9 7 7 7
(13) 8 5 5 9 7 7 7 ← (14) 9 7 13 5 7 7
(14) 9 3 5 9 7 7 7 ← (16) 11 5 9 7 7 7
(15) 8 3 5 9 7 7 7 ← (18) 7 14 4 5 7 7
(19) 10 9 3 6 6 5 3 ← (20) 12 12 9 3 3 3
(22) 3 5 9 3 5 7 7 ← (26) 5 9 3 5 7 7
(23) 6 9 3 6 6 5 3 ← (24) 8 12 9 3 3 3
(29) 6 3 3 6 6 5 3 ← (42) 2 4 5 3 3 3
(31) 10 2 4 5 3 3 3 ← (32) 12 4 5 3 3 3
(44) 1 1 2 4 3 3 3 ← (48) 3 4 4 1 1 1
(50) 1 1 * 1 ← (52) 1 * 1

(61, 9)

(1) 5 1 2 4 7 11 15 15 ← (2) 6 2 4 7 11 15 15
(3) 25 5 5 3 6 6 5 3 ← (4) 26 9 3 6 6 5 3
(7) 13 11 12 4 5 3 3 3 ← (27) 6 5 2 3 5 7 7
(9) 4 8 5 5 9 7 7 7
(13) 4 6 3 5 9 7 7 7 ← (14) 5 10 11 3 5 7 7
(14) 4 5 3 5 9 7 7 7 ← (16) 8 3 5 9 7 7 7
(14) 8 3 5 9 3 5 7 7
(15) 9 3 5 7 3 5 7 7 ← (25) 3 5 7 3 5 7 7
(19) 9 5 5 3 6 6 5 3 ← (20) 10 9 3 6 6 5 3
(23) 3 6 5 2 3 5 7 7 ← (24) 6 9 3 6 6 5 3
(27) 5 6 2 3 5 7 3 3 ← (39) 13 1 * 1
(29) 3 6 2 3 5 7 3 3 ← (30) 6 3 3 6 6 5 3
(31) 7 13 1 * 1 ← (32) 10 2 4 5 3 3 3
(35) 4 ..4 5 3 3 3 ← (36) 6 2 4 5 3 3 3
(39) 6 2 3 4 4 1 1 1 ← (46) 2 3 4 4 1 1 1

(61, 10)

(1) 3 6 4 6 9 11 7 7 7 ← (2) 5 1 2 4 7 11 15 15
(1) 13 4 5 3 5 9 7 7 7
(4) 11 9 3 5 7 3 5 7 7
(5) 4 4 8 5 5 9 7 7 7 ← (25) 4 7 3 3 6 6 5 3
(7) 12 9 5 5 3 6 6 5 3 ← (8) 13 11 12 4 5 3 3 3
(10) 5 9 3 5 7 3 5 7 7 ← (16) 9 3 5 7 3 5 7 7
(13) 2 3 5 3 5 9 7 7 7 ← (14) 4 6 3 5 9 7 7 7
(13) 4 5 3 5 9 3 5 7 7
(19) 4 5 5 5 3 6 6 5 3 ← (22) 5 6 5 2 3 5 7 7
(23) 2 3 5 5 3 6 6 5 3 ← (24) 3 6 5 2 3 5 7 7
(26) 2 4 3 3 3 6 6 5 3 ← (37) 8 1 1 2 4 3 3 3
(27) 3 5 6 2 4 5 3 3 3 ← (28) 5 6 2 3 5 7 3 3
(29) ...3 3 6 6 5 3 ← (30) 3 6 2 3 5 7 3 3
(31) 5 8 1 1 2 4 3 3 3 ← (32) 7 13 1 * 1
(33) 4 ...4 5 3 3 3 ← (34) 6 ..4 5 3 3 3
(35) ....4 5 3 3 3 ← (36) 4 ..4 5 3 3 3
(38) * 2 4 3 3 3 ← (45) 1 2 3 4 4 1 1 1
(39) 5 1 2 3 4 4 1 1 1 ← (40) 6 2 3 4 4 1 1 1

(61, 11)

(1) 1..3 5 3 5 9 7 7 7 ← (2) 13 4 5 3 5 9 7 7 7
(4) ...4 6 9 11 7 7 7 ← (9) 2 3 5 10 11 3 5 7 7
(5) 5 5 9 3 5 7 3 5 7 7
(7) 1 2 3 5 10 11 3 5 7 7 ← (8) 12 9 5 5 3 6 6 5 3
(7) 16 2 3 5 5 3 6 6 5 3
(11) 5 5 5 6 5 2 3 5 7 7 ← (17) 5 5 6 5 2 3 5 7 7
(17) 2 4 5 5 5 3 6 6 5 3 ← (20) 4 5 5 5 3 6 6 5 3
(24) ..4 3 3 3 6 6 5 3 ← (36) 4 5 3 3 3
(27) ....3 3 6 6 5 3 ← (28) 3 5 6 2 4 5 3 3 3
(31) 4 ....4 5 3 3 3 ← (32) 5 8 1 1 2 4 3 3 3
(33) .....4 5 3 3 3 ← (34) 4 ...4 5 3 3 3
(37) 1 * 2 4 3 3 3 ← (43) 4 4 1 1 * 1
(39) 3 4 4 1 1 * 1 ← (40) 5 1 2 3 4 4 1 1 1
(43) 1 * * 1 ← (45) * * 1

(61, 12)

(1) ..4 5 9 3 5 9 7 7 7 ← (8) 1 2 3 5 10 11 3 5 7 7
(1) 8 4 2 3 5 3 5 9 7 7 7 ← (2) 1..3 5 3 5 9 7 7 7
(4) 4 5 4 5 3 5 9 3 5 7 7
(5) 4 4 2 3 5 3 5 9 7 7 7 ← (6) 8 4 5 3 5 9 3 5 7 7
(7) 2 4 2 3 5 3 5 9 7 7 7 ← (9) 18 2 4 3 3 3 6 6 5 3
(7) 13 ..4 7 3 3 6 6 5 3 ← (8) 16 2 3 5 5 3 6 6 5 3
(9) 4 3 5 5 6 5 2 3 5 7 7 ← (12) 5 5 5 6 5 2 3 5 7 7
(15) ..4 5 5 5 3 6 6 5 3 ← (18) 2 4 5 5 5 3 6 6 5 3
(21) 6 ....3 3 6 6 5 3 ← (34) 4 5 3 3 3
(31) ......4 5 3 3 3 ← (32) 4 ....4 5 3 3 3
(36) 1 1 * 2 4 3 3 3 ← (40) 3 4 4 1 1 * 1
(42) 1 1 * * 1 ← (44) 1 * * 1

(61, 13)

(1) 5 5 5 5 5 6 5 2 3 5 7 7
(1) 6 2 4 2 3 5 3 5 9 7 7 7 ← (2) 8 4 2 3 5 3 5 9 7 7 7
(2) 2 4 5 4 5 3 5 9 3 5 7 7
(5) ..4 2 3 5 3 5 9 7 7 7 ← (6) 4 4 2 3 5 3 5 9 7 7 7
(6) 1 2 4 2 3 5 3 5 9 7 7 7 ← (8) 2 4 2 3 5 3 5 9 7 7 7
(7) 2 4 3 5 5 6 5 2 3 5 7 7 ← (10) 4 3 5 5 6 5 2 3 5 7 7
(7) 11 6 ..4 3 3 3 6 6 5 3 ← (8) 13 ..4 7 3 3 6 6 5 3
(13) ...4 5 5 5 3 6 6 5 3 ← (16) ..4 5 5 5 3 6 6 5 3
(19) 5 6 .....3 5 7 3 3 ← (32) 4 5 3 3 3
(21) 3 6 .....3 5 7 3 3 ← (22) 6 ....3 3 6 6 5 3
(27) 4 ......4 5 3 3 3 ← (28) 6 .....4 5 3 3 3
(31) 6 2 3 4 4 1 1 * 1 ← (38) 2 3 4 4 1 1 * 1

(62, 2)

(31) 31

(62, 3)

(30) 29 3
(31) 30 1 ← (32) 31
(47) 14 1 ← (48) 15
(55) 6 1 ← (56) 7
(59) 2 1 ← (60) 3

(62, 4)

(17) 15 15 15 ← (45) 11 7
(23) 21 11 7 ← (25) 23 15
(29) 27 3 3
(31) 13 11 7 ← (33) 15 15
(31) 29 1 1 ← (32) 30 1
(41) 7 7 7 ← (49) 7 7
(47) 13 1 1 ← (48) 14 1
(53) 3 3 3 ← (57) 3 3
(55) 5 1 1 ← (56) 6 1
(59) 1 1 1 ← (60) 2 1

(62, 5)

(1) 22 21 11 7
(12) 9 11 15 15
(14) 7 11 15 15
(17) 14 13 11 7 ← (18) 15 15 15
(21) 10 13 11 7 ← (22) 11 15 15
(22) 19 7 7 7 ← (24) 21 11 7
(23) 20 5 7 7 ← (26) 23 7 7
(28) 25 3 3 3
(29) 14 5 7 7 ← (32) 13 11 7
(30) 13 5 7 7 ← (44) 5 7 7
(31) 28 1 1 1 ← (32) 29 1 1
(39) 4 5 7 7 ← (42) 7 7 7
(47) 12 1 1 1 ← (48) 13 1 1
(55) 4 1 1 1 ← (56) 5 1 1

(62, 6)

(9) 5 7 11 15 15
(11) 7 13 13 11 7 ← (23) 11 15 7 7
(13) 9 11 15 7 7 ← (18) 14 13 11 7
(15) 7 11 15 7 7 ← (19) 13 13 11 7
(21) 9 11 7 7 7 ← (22) 10 13 11 7
(22) 18 3 5 7 7 ← (24) 20 5 7 7
(23) 7 13 5 7 7 ← (25) 17 7 7 7
(24) 20 9 3 3 3
(27) 5 9 7 7 7
(29) 11 3 5 7 7 ← (30) 14 5 7 7
(31) 9 3 5 7 7 ← (41) 3 5 7 7
(31) 24 4 1 1 1 ← (32) 28 1 1 1
(38) 2 3 5 7 7 ← (40) 4 5 7 7
(39) 3 6 6 5 3 ← (43) 6 6 5 3
(45) 3 6 2 3 3 ← (46) 8 3 3 3
(47) 2 4 3 3 3 ← (49) 6 2 3 3
(47) 8 4 1 1 1 ← (48) 12 1 1 1

(62, 7)

(1) 20 9 11 7 7 7
(1) 28 11 3 5 7 7 ← (22) 9 11 7 7 7
(10) 5 7 11 15 7 7 ← (18) 9 15 7 7 7
(11) 6 9 15 7 7 7 ← (12) 7 13 13 11 7
(12) 5 9 15 7 7 7
(13) 8 9 11 7 7 7 ← (14) 9 11 15 7 7
(15) 6 9 11 7 7 7 ← (16) 7 11 15 7 7
(17) 12 11 3 5 7 7 ← (18) 13 13 5 7 7
(22) 5 5 9 7 7 7 ← (24) 7 13 5 7 7
(22) 17 3 6 6 5 3
(25) 13 2 3 5 7 7 ← (26) 14 4 5 7 7
(28) 5 7 3 5 7 7 ← (32) 9 3 5 7 7
(31) 22 * 1 ← (32) 24 4 1 1 1
(36) 3 3 6 6 5 3 ← (42) 3 5 7 3 3
(39) 2 3 5 7 3 3 ← (40) 3 6 6 5 3
(45) ..4 3 3 3 ← (46) 3 6 2 3 3
(46) 1 2 4 3 3 3 ← (48) 2 4 3 3 3
(47) 6 * 1 ← (48) 8 4 1 1 1
(51) 2 * 1 ← (52) 4 4 1 1 1

(62, 8)

(1) 14 6 9 11 7 7 7 ← (2) 20 9 11 7 7 7
(1) 23 8 12 9 3 3 3 ← (2) 28 11 3 5 7 7
(5) 27 12 4 5 3 3 3
(7) 1 2 4 7 11 15 15 ← (9) 2 4 7 11 15 15
(9) 3 5 9 15 7 7 7
(11) 4 6 9 11 7 7 7 ← (12) 6 9 15 7 7 7
(14) 8 5 5 9 7 7 7
(15) 9 3 5 9 7 7 7 ← (17) 11 5 9 7 7 7
(17) 7 8 12 9 3 3 3 ← (18) 12 11 3 5 7 7
(21) 11 12 4 5 3 3 3
(23) 3 5 9 3 5 7 7 ← (25) 8 12 9 3 3 3
(25) 7 12 4 5 3 3 3 ← (26) 13 2 3 5 7 7
(29) 5 5 3 6 6 5 3 ← (33) 12 4 5 3 3 3
(31) 21 1 * 1 ← (32) 22 * 1
(33) 6 2 3 5 7 3 3 ← (40) 2 3 5 7 3 3
(45) 1 1 2 4 3 3 3 ← (46) ..4 3 3 3
(47) 5 1 * 1 ← (48) 6 * 1
(51) 1 1 * 1 ← (52) 2 * 1

(62, 9)

(1) 2 4 3 4 7 11 15 15 ← (2) 23 8 12 9 3 3 3
(4) 25 5 5 3 6 6 5 3
(5) 6 4 6 9 11 7 7 7 ← (8) 1 2 4 7 11 15 15
(5) 18 3 5 9 3 5 7 7 ← (6) 27 12 4 5 3 3 3
(10) 4 8 5 5 9 7 7 7
(11) 3 5 10 11 3 5 7 7 ← (16) 9 3 5 9 7 7 7
(15) 4 5 3 5 9 7 7 7 ← (17) 8 3 5 9 7 7 7
(15) 8 3 5 9 3 5 7 7 ← (18) 7 8 12 9 3 3 3
(20) 9 5 5 3 6 6 5 3
(25) 5 6 3 3 6 6 5 3 ← (26) 7 12 4 5 3 3 3
(27) 3 6 3 3 6 6 5 3 ← (28) 6 5 2 3 5 7 7
(31) 5 6 2 4 5 3 3 3 ← (37) 6 2 4 5 3 3 3
(31) 20 1 1 * 1 ← (32) 21 1 * 1
(33) 3 6 2 4 5 3 3 3 ← (34) 6 2 3 5 7 3 3
(39) 12 1 1 * 1 ← (40) 13 1 * 1
(47) 4 1 1 * 1 ← (48) 5 1 * 1

(62, 10)

(1) 14 8 3 5 9 3 5 7 7
(2) 3 6 4 6 9 11 7 7 7
(5) 4 2 4 6 9 11 7 7 7 ← (6) 6 4 6 9 11 7 7 7
(5) 11 9 3 5 7 3 5 7 7 ← (6) 18 3 5 9 3 5 7 7
(6) 4 4 8 5 5 9 7 7 7
(11) 5 9 3 5 7 3 5 7 7 ← (17) 9 3 5 7 3 5 7 7
(14) 2 3 5 3 5 9 7 7 7 ← (16) 4 5 3 5 9 7 7 7
(14) 4 5 3 5 9 3 5 7 7 ← (16) 8 3 5 9 3 5 7 7
(23) 2 4 7 3 3 6 6 5 3 ← (26) 4 7 3 3 6 6 5 3
(24) 2 3 5 5 3 6 6 5 3
(25) 3 5 6 2 3 5 7 3 3 ← (26) 5 6 3 3 6 6 5 3
(27) 2 4 3 3 3 6 6 5 3 ← (28) 3 6 3 3 6 6 5 3
(30) ...3 3 6 6 5 3 ← (35) 6 ..4 5 3 3 3
(31) 3 6 ..4 5 3 3 3 ← (32) 5 6 2 4 5 3 3 3
(31) 16 4 1 1 * 1 ← (32) 20 1 1 * 1
(33) ....3 5 7 3 3 ← (34) 3 6 2 4 5 3 3 3
(37) 3 6 2 3 4 4 1 1 1 ← (38) 8 1 1 2 4 3 3 3
(39) * 2 4 3 3 3 ← (41) 6 2 3 4 4 1 1 1
(39) 8 4 1 1 * 1 ← (40) 12 1 1 * 1

(62, 11)

(1) 13 4 5 3 5 9 3 5 7 7 ← (2) 14 8 3 5 9 3 5 7 7
(3) 6 2 3 5 10 11 3 5 7 7
(5) ...4 6 9 11 7 7 7 ← (6) 4 2 4 6 9 11 7 7 7
(6) 5 5 9 3 5 7 3 5 7 7 ← (12) 5 9 3 5 7 3 5 7 7
(15) 4 3 5 6 5 2 3 5 7 7 ← (18) 5 5 6 5 2 3 5 7 7
(21) ..4 7 3 3 6 6 5 3
(22) 1 2 4 7 3 3 6 6 5 3 ← (24) 2 4 7 3 3 6 6 5 3
(25) ..4 3 3 3 6 6 5 3 ← (26) 3 5 6 2 3 5 7 3 3
(28) ....3 3 6 6 5 3 ← (34) 3 5 7 3 3
(31) .....3 5 7 3 3 ← (32) 3 6 ..4 5 3 3 3
(31) 14 * * 1 ← (32) 16 4 1 1 * 1
(37) 2 * 2 4 3 3 3 ← (38) 3 6 2 3 4 4 1 1 1
(38) 1 * 2 4 3 3 3 ← (40) * 2 4 3 3 3
(39) 6 * * 1 ← (40) 8 4 1 1 * 1
(43) 2 * * 1 ← (44) 4 4 1 1 * 1

(62, 12)

(1) 5 5 5 9 3 5 7 3 5 7 7
(1) 8 5 4 5 3 5 9 3 5 7 7 ← (2) 13 4 5 3 5 9 3 5 7 7
(2) ..4 5 9 3 5 9 7 7 7
(3) 4 7 4 5 3 5 9 3 5 7 7 ← (4) 6 2 3 5 10 11 3 5 7 7
(5) 4 5 4 5 3 5 9 3 5 7 7 ← (9) 16 2 3 5 5 3 6 6 5 3
(7) 5 5 5 5 6 5 2 3 5 7 7 ← (13) 5 5 5 6 5 2 3 5 7 7
(9) 16 ..4 3 3 3 6 6 5 3 ← (10) 18 2 4 3 3 3 6 6 5 3
(13) 2 4 3 5 6 5 2 3 5 7 7 ← (16) 4 3 5 6 5 2 3 5 7 7
(19) 6 ..4 3 3 3 6 6 5 3
(21) 1 1 2 4 7 3 3 6 6 5 3 ← (22) ..4 7 3 3 6 6 5 3
(25) 6 .....3 5 7 3 3 ← (32) 3 5 7 3 3
(31) 13 1 * * 1 ← (32) 14 * * 1
(37) 1 1 * 2 4 3 3 3 ← (38) 2 * 2 4 3 3 3
(39) 5 1 * * 1 ← (40) 6 * * 1
(43) 1 1 * * 1 ← (44) 2 * * 1

(63, 1)

(63)

(63, 2)

(62) 1

(63, 3)

(1) 31 31
(29) 27 7
(31) 29 3 ← (33) 31
(47) 13 3 ← (49) 15
(55) 5 3 ← (57) 7
(61) 1 1

(63, 4)

(1) 30 29 3 ← (2) 31 31
(25) 22 13 3 ← (26) 23 15
(29) 26 5 3 ← (30) 27 7
(30) 27 3 3 ← (32) 29 3
(33) 14 13 3 ← (34) 15 15
(45) 10 5 3 ← (46) 11 7
(46) 11 3 3 ← (48) 13 3
(49) 6 5 3 ← (50) 7 7
(54) 3 3 3 ← (56) 5 3
(55) 2 3 3 ← (58) 3 3
(60) 1 1 1

(63, 5)

(1) 29 27 3 3 ← (2) 30 29 3
(2) 22 21 11 7
(13) 9 11 15 15 ← (33) 13 11 7
(15) 7 11 15 15 ← (23) 11 15 15
(23) 19 7 7 7 ← (25) 21 11 7
(25) 21 11 3 3 ← (26) 22 13 3
(29) 25 3 3 3 ← (30) 26 5 3
(31) 13 5 7 7 ← (45) 5 7 7
(33) 13 11 3 3 ← (34) 14 13 3
(45) 5 7 3 3 ← (50) 6 5 3
(45) 9 3 3 3 ← (46) 10 5 3
(54) 1 2 3 3 ← (56) 2 3 3
(56) 4 1 1 1

(63, 6)

(1) 12 9 11 15 15 ← (16) 7 11 15 15
(1) 14 7 11 15 15
(1) 28 25 3 3 3 ← (2) 29 27 3 3
(10) 5 7 11 15 15 ← (26) 17 7 7 7
(11) 4 7 11 15 15
(15) 10 17 7 7 7 ← (24) 19 7 7 7
(19) 11 14 5 7 7 ← (20) 13 13 11 7
(23) 7 14 5 7 7 ← (24) 11 15 7 7
(23) 18 3 5 7 7 ← (25) 20 5 7 7
(25) 20 9 3 3 3 ← (26) 21 11 3 3
(28) 5 9 7 7 7 ← (42) 3 5 7 7
(30) 11 3 5 7 7 ← (32) 13 5 7 7
(33) 12 9 3 3 3 ← (34) 13 11 3 3
(39) 2 3 5 7 7 ← (41) 4 5 7 7
(43) 4 7 3 3 3 ← (44) 6 6 5 3
(45) 4 5 3 3 3 ← (46) 5 7 3 3
(49) 5 1 2 3 3 ← (50) 6 2 3 3
(54) * 1

(63, 7)

(1) 9 5 7 11 15 15
(1) 24 20 9 3 3 3 ← (2) 28 25 3 3 3
(5) 28 12 9 3 3 3
(11) 5 7 11 15 7 7 ← (13) 7 13 13 11 7
(13) 5 9 15 7 7 7
(14) 8 9 11 7 7 7 ← (19) 13 13 5 7 7
(15) 9 7 13 5 7 7 ← (16) 10 17 7 7 7
(16) 6 9 11 7 7 7
(19) 7 14 4 5 7 7 ← (20) 11 14 5 7 7
(21) 12 12 9 3 3 3 ← (24) 18 3 5 7 7
(23) 5 5 9 7 7 7 ← (24) 7 14 5 7 7
(23) 17 3 6 6 5 3 ← (26) 20 9 3 3 3
(25) 3 5 9 7 7 7
(27) 5 9 3 5 7 7 ← (41) 3 6 6 5 3
(29) 5 7 3 5 7 7 ← (33) 9 3 5 7 7
(31) 9 3 6 6 5 3 ← (34) 12 9 3 3 3
(37) 3 3 6 6 5 3 ← (40) 2 3 5 7 7
(43) 2 4 5 3 3 3 ← (44) 4 7 3 3 3
(47) 1 2 4 3 3 3 ← (49) 2 4 3 3 3
(49) 3 4 4 1 1 1 ← (50) 5 1 2 3 3
(53) 1 * 1

(63, 8)

(1) 22 17 3 6 6 5 3 ← (2) 24 20 9 3 3 3
(2) 14 6 9 11 7 7 7
(3) 6 2 4 7 11 15 15
(5) 26 9 3 6 6 5 3 ← (6) 28 12 9 3 3 3
(10) 3 5 9 15 7 7 7
(12) 4 6 9 11 7 7 7 ← (18) 11 5 9 7 7 7
(15) 5 10 11 3 5 7 7 ← (20) 7 14 4 5 7 7
(15) 8 5 5 9 7 7 7 ← (16) 9 7 13 5 7 7
(21) 10 9 3 6 6 5 3 ← (22) 12 12 9 3 3 3
(22) 11 12 4 5 3 3 3 ← (24) 17 3 6 6 5 3
(24) 3 5 9 3 5 7 7
(25) 6 9 3 6 6 5 3 ← (26) 8 12 9 3 3 3
(26) 3 5 7 3 5 7 7 ← (28) 5 9 3 5 7 7
(30) 5 5 3 6 6 5 3 ← (32) 9 3 6 6 5 3
(31) 6 3 3 6 6 5 3 ← (38) 3 3 6 6 5 3
(33) 10 2 4 5 3 3 3 ← (34) 12 4 5 3 3 3
(46) 1 1 2 4 3 3 3 ← (48) 1 2 4 3 3 3
(47) 2 3 4 4 1 1 1 ← (50) 3 4 4 1 1 1
(52) 1 1 * 1

(63, 9)

(1) 9 3 5 9 15 7 7 7
(1) 14 8 5 5 9 7 7 7
(1) 21 11 12 4 5 3 3 3 ← (2) 22 17 3 6 6 5 3
(2) 2 4 3 4 7 11 15 15
(3) 5 1 2 4 7 11 15 15 ← (4) 6 2 4 7 11 15 15
(5) 25 5 5 3 6 6 5 3 ← (6) 26 9 3 6 6 5 3
(9) 13 11 12 4 5 3 3 3 ← (16) 8 5 5 9 7 7 7
(11) 4 8 5 5 9 7 7 7
(12) 3 5 10 11 3 5 7 7 ← (18) 8 3 5 9 7 7 7
(15) 4 6 3 5 9 7 7 7 ← (16) 5 10 11 3 5 7 7
(21) 9 5 5 3 6 6 5 3 ← (22) 10 9 3 6 6 5 3
(23) 5 6 5 2 3 5 7 7 ← (29) 6 5 2 3 5 7 7
(25) 3 6 5 2 3 5 7 7 ← (26) 6 9 3 6 6 5 3
(29) 5 6 2 3 5 7 3 3 ← (35) 6 2 3 5 7 3 3
(31) 3 6 2 3 5 7 3 3 ← (32) 6 3 3 6 6 5 3
(33) 7 13 1 * 1 ← (34) 10 2 4 5 3 3 3
(37) 4 ..4 5 3 3 3 ← (38) 6 2 4 5 3 3 3
(46) 1 2 3 4 4 1 1 1 ← (48) 2 3 4 4 1 1 1
(48) 4 1 1 * 1

(63, 10)

(1) 20 9 5 5 3 6 6 5 3 ← (2) 21 11 12 4 5 3 3 3
(3) 3 6 4 6 9 11 7 7 7 ← (4) 5 1 2 4 7 11 15 15
(3) 13 4 5 3 5 9 7 7 7
(6) 11 9 3 5 7 3 5 7 7
(7) 4 4 8 5 5 9 7 7 7 ← (12) 4 8 5 5 9 7 7 7
(9) 12 9 5 5 3 6 6 5 3 ← (10) 13 11 12 4 5 3 3 3
(10) 2 3 5 10 11 3 5 7 7 ← (17) 4 5 3 5 9 7 7 7
(15) 2 3 5 3 5 9 7 7 7 ← (16) 4 6 3 5 9 7 7 7
(15) 4 5 3 5 9 3 5 7 7 ← (17) 8 3 5 9 3 5 7 7
(21) 4 5 5 5 3 6 6 5 3 ← (24) 5 6 5 2 3 5 7 7
(25) 2 3 5 5 3 6 6 5 3 ← (26) 3 6 5 2 3 5 7 7
(28) 2 4 3 3 3 6 6 5 3 ← (33) 5 6 2 4 5 3 3 3
(29) 3 5 6 2 4 5 3 3 3 ← (30) 5 6 2 3 5 7 3 3
(31) ...3 3 6 6 5 3 ← (32) 3 6 2 3 5 7 3 3
(33) 5 8 1 1 2 4 3 3 3 ← (34) 7 13 1 * 1
(35) 4 ...4 5 3 3 3 ← (36) 6 ..4 5 3 3 3
(37) ....4 5 3 3 3 ← (38) 4 ..4 5 3 3 3
(41) 5 1 2 3 4 4 1 1 1 ← (42) 6 2 3 4 4 1 1 1
(46) * * 1

(63, 11)

(1) 2 3 6 4 6 9 11 7 7 7
(1) 6 4 4 8 5 5 9 7 7 7
(1) 24 2 3 5 5 3 6 6 5 3
(3) 1..3 5 3 5 9 7 7 7 ← (4) 13 4 5 3 5 9 7 7 7
(6) ...4 6 9 11 7 7 7 ← (8) 4 4 8 5 5 9 7 7 7
(7) 5 5 9 3 5 7 3 5 7 7 ← (13) 5 9 3 5 7 3 5 7 7
(7) 8 4 5 3 5 9 3 5 7 7 ← (16) 2 3 5 3 5 9 7 7 7
(9) 1 2 3 5 10 11 3 5 7 7 ← (10) 12 9 5 5 3 6 6 5 3
(19) 2 4 5 5 5 3 6 6 5 3 ← (22) 4 5 5 5 3 6 6 5 3
(23) 1 2 4 7 3 3 6 6 5 3 ← (25) 2 4 7 3 3 6 6 5 3
(26) ..4 3 3 3 6 6 5 3 ← (32) ...3 3 6 6 5 3
(29) ....3 3 6 6 5 3 ← (30) 3 5 6 2 4 5 3 3 3
(33) 4 ....4 5 3 3 3 ← (34) 5 8 1 1 2 4 3 3 3
(35) .....4 5 3 3 3 ← (36) 4 ...4 5 3 3 3
(39) 1 * 2 4 3 3 3 ← (41) * 2 4 3 3 3
(41) 3 4 4 1 1 * 1 ← (42) 5 1 2 3 4 4 1 1 1
(45) 1 * * 1

(64, 2)

(1) 63
(61) 3
(63) 1 ← (65)

(64, 3)

(1) 62 1 ← (2) 63
(33) 30 1 ← (34) 31
(49) 14 1 ← (50) 15
(57) 6 1 ← (58) 7
(61) 2 1 ← (62) 3
(62) 1 1 ← (64) 1

(64, 4)

(1) 29 27 7
(1) 61 1 1 ← (2) 62 1
(19) 15 15 15 ← (35) 15 15
(27) 23 7 7
(31) 27 3 3 ← (33) 29 3
(33) 29 1 1 ← (34) 30 1
(43) 7 7 7 ← (51) 7 7
(47) 11 3 3 ← (49) 13 3
(49) 13 1 1 ← (50) 14 1
(55) 3 3 3 ← (57) 5 3
(57) 5 1 1 ← (58) 6 1
(61) 1 1 1 ← (62) 2 1

(64, 5)

(1) 60 1 1 1 ← (2) 61 1 1
(3) 22 21 11 7
(14) 9 11 15 15 ← (34) 13 11 7
(19) 14 13 11 7 ← (20) 15 15 15
(23) 10 13 11 7 ← (24) 11 15 15
(30) 25 3 3 3 ← (32) 27 3 3
(31) 14 5 7 7 ← (46) 5 7 7
(33) 28 1 1 1 ← (34) 29 1 1
(46) 9 3 3 3 ← (48) 11 3 3
(47) 8 3 3 3 ← (56) 3 3 3
(49) 12 1 1 1 ← (50) 13 1 1
(55) 1 2 3 3 ← (57) 2 3 3
(57) 4 1 1 1 ← (58) 5 1 1

(64, 6)

(1) 56 4 1 1 1 ← (2) 60 1 1 1
(2) 12 9 11 15 15 ← (4) 22 21 11 7
(2) 14 7 11 15 15
(11) 5 7 11 15 15 ← (27) 17 7 7 7
(12) 4 7 11 15 15
(15) 9 11 15 7 7 ← (20) 14 13 11 7
(17) 7 11 15 7 7
(19) 9 15 7 7 7 ← (21) 13 13 11 7
(23) 9 11 7 7 7 ← (24) 10 13 11 7
(25) 7 13 5 7 7
(27) 14 4 5 7 7 ← (43) 3 5 7 7
(29) 5 9 7 7 7 ← (33) 13 5 7 7
(31) 11 3 5 7 7 ← (32) 14 5 7 7
(33) 24 4 1 1 1 ← (34) 28 1 1 1
(43) 3 5 7 3 3 ← (45) 6 6 5 3
(46) 4 5 3 3 3 ← (51) 6 2 3 3
(47) 3 6 2 3 3 ← (48) 8 3 3 3
(49) 8 4 1 1 1 ← (50) 12 1 1 1
(53) 4 4 1 1 1 ← (56) 1 2 3 3
(55) * 1 ← (58) 4 1 1 1

(64, 7)

(1) 11 4 7 11 15 15
(1) 54 * 1 ← (2) 56 4 1 1 1
(2) 9 5 7 11 15 15
(3) 20 9 11 7 7 7
(3) 28 11 3 5 7 7
(10) 2 4 7 11 15 15
(12) 5 7 11 15 7 7 ← (17) 10 17 7 7 7
(13) 6 9 15 7 7 7 ← (14) 7 13 13 11 7
(14) 5 9 15 7 7 7 ← (20) 9 15 7 7 7
(15) 8 9 11 7 7 7 ← (16) 9 11 15 7 7
(17) 6 9 11 7 7 7 ← (18) 7 11 15 7 7
(19) 12 11 3 5 7 7 ← (20) 13 13 5 7 7
(24) 5 5 9 7 7 7
(26) 3 5 9 7 7 7 ← (32) 11 3 5 7 7
(27) 13 2 3 5 7 7 ← (28) 14 4 5 7 7
(30) 5 7 3 5 7 7 ← (34) 9 3 5 7 7
(33) 22 * 1 ← (34) 24 4 1 1 1
(41) 2 3 5 7 3 3 ← (42) 3 6 6 5 3
(44) 2 4 5 3 3 3 ← (50) 2 4 3 3 3
(47) ..4 3 3 3 ← (48) 3 6 2 3 3
(49) 6 * 1 ← (50) 8 4 1 1 1
(53) 2 * 1 ← (54) 4 4 1 1 1
(54) 1 * 1 ← (56) * 1

(64, 8)

(1) 13 5 9 15 7 7 7
(1) 25 3 5 9 7 7 7
(1) 53 1 * 1 ← (2) 54 * 1
(3) 14 6 9 11 7 7 7 ← (4) 20 9 11 7 7 7
(3) 23 8 12 9 3 3 3 ← (4) 28 11 3 5 7 7
(7) 27 12 4 5 3 3 3
(9) 1 2 4 7 11 15 15 ← (16) 8 9 11 7 7 7
(11) 3 5 9 15 7 7 7 ← (18) 6 9 11 7 7 7
(13) 4 6 9 11 7 7 7 ← (14) 6 9 15 7 7 7
(17) 9 3 5 9 7 7 7
(19) 7 8 12 9 3 3 3 ← (20) 12 11 3 5 7 7
(23) 11 12 4 5 3 3 3 ← (25) 17 3 6 6 5 3
(25) 3 5 9 3 5 7 7 ← (35) 12 4 5 3 3 3
(27) 3 5 7 3 5 7 7 ← (29) 5 9 3 5 7 7
(27) 7 12 4 5 3 3 3 ← (28) 13 2 3 5 7 7
(31) 5 5 3 6 6 5 3 ← (33) 9 3 6 6 5 3
(33) 21 1 * 1 ← (34) 22 * 1
(41) 13 1 * 1 ← (49) 1 2 4 3 3 3
(47) 1 1 2 4 3 3 3 ← (48) ..4 3 3 3
(49) 5 1 * 1 ← (50) 6 * 1
(53) 1 1 * 1 ← (54) 2 * 1

(64, 9)

(1) 3 6 2 4 7 11 15 15 ← (2) 25 3 5 9 7 7 7
(1) 52 1 1 * 1 ← (2) 53 1 * 1
(2) 9 3 5 9 15 7 7 7
(2) 14 8 5 5 9 7 7 7
(3) 2 4 3 4 7 11 15 15 ← (4) 23 8 12 9 3 3 3
(6) 25 5 5 3 6 6 5 3
(7) 6 4 6 9 11 7 7 7 ← (14) 4 6 9 11 7 7 7
(7) 18 3 5 9 3 5 7 7 ← (8) 27 12 4 5 3 3 3
(13) 3 5 10 11 3 5 7 7 ← (17) 5 10 11 3 5 7 7
(18) 9 3 5 7 3 5 7 7
(22) 9 5 5 3 6 6 5 3 ← (24) 11 12 4 5 3 3 3
(27) 4 7 3 3 6 6 5 3 ← (32) 5 5 3 6 6 5 3
(27) 5 6 3 3 6 6 5 3 ← (28) 7 12 4 5 3 3 3
(29) 3 6 3 3 6 6 5 3 ← (30) 6 5 2 3 5 7 7
(33) 20 1 1 * 1 ← (34) 21 1 * 1
(35) 3 6 2 4 5 3 3 3 ← (36) 6 2 3 5 7 3 3
(39) 8 1 1 2 4 3 3 3 ← (48) 1 1 2 4 3 3 3
(41) 12 1 1 * 1 ← (42) 13 1 * 1
(47) 1 2 3 4 4 1 1 1 ← (49) 2 3 4 4 1 1 1
(49) 4 1 1 * 1 ← (50) 5 1 * 1

(64, 10)

(1) ..4 3 4 7 11 15 15 ← (2) 3 6 2 4 7 11 15 15
(1) 11 4 8 5 5 9 7 7 7
(1) 48 4 1 1 * 1 ← (2) 52 1 1 * 1
(2) 20 9 5 5 3 6 6 5 3
(3) 14 8 3 5 9 3 5 7 7
(4) 3 6 4 6 9 11 7 7 7 ← (11) 13 11 12 4 5 3 3 3
(7) 4 2 4 6 9 11 7 7 7 ← (8) 6 4 6 9 11 7 7 7
(7) 11 9 3 5 7 3 5 7 7 ← (8) 18 3 5 9 3 5 7 7
(11) 2 3 5 10 11 3 5 7 7 ← (14) 3 5 10 11 3 5 7 7
(16) 4 5 3 5 9 3 5 7 7
(19) 5 5 6 5 2 3 5 7 7 ← (25) 5 6 5 2 3 5 7 7
(26) 2 3 5 5 3 6 6 5 3 ← (28) 4 7 3 3 6 6 5 3
(27) 3 5 6 2 3 5 7 3 3 ← (28) 5 6 3 3 6 6 5 3
(29) 2 4 3 3 3 6 6 5 3 ← (30) 3 6 3 3 6 6 5 3
(33) 3 6 ..4 5 3 3 3 ← (34) 5 6 2 4 5 3 3 3
(33) 16 4 1 1 * 1 ← (34) 20 1 1 * 1
(35) ....3 5 7 3 3 ← (36) 3 6 2 4 5 3 3 3
(38) ....4 5 3 3 3 ← (43) 6 2 3 4 4 1 1 1
(39) 3 6 2 3 4 4 1 1 1 ← (40) 8 1 1 2 4 3 3 3
(41) 8 4 1 1 * 1 ← (42) 12 1 1 * 1
(45) 4 4 1 1 * 1 ← (48) 1 2 3 4 4 1 1 1
(47) * * 1 ← (50) 4 1 1 * 1

(65, 3)

(1) 61 3
(3) 31 31
(27) 23 15
(31) 27 7 ← (35) 31
(47) 11 7 ← (51) 15
(59) 3 3
(63) 1 1 ← (65) 1

(65, 4)

(2) 29 27 7
(3) 30 29 3 ← (4) 31 31
(26) 21 11 7
(27) 22 13 3 ← (28) 23 15
(28) 23 7 7 ← (34) 29 3
(31) 26 5 3 ← (32) 27 7
(35) 14 13 3 ← (36) 15 15
(44) 7 7 7 ← (50) 13 3
(47) 10 5 3 ← (48) 11 7
(51) 6 5 3 ← (52) 7 7
(62) 1 1 1 ← (64) 1 1

(65, 5)

(1) 27 23 7 7
(3) 29 27 3 3 ← (4) 30 29 3
(15) 9 11 15 15 ← (21) 15 15 15
(17) 7 11 15 15
(25) 11 15 7 7
(25) 19 7 7 7
(26) 20 5 7 7 ← (33) 27 3 3
(27) 21 11 3 3 ← (28) 22 13 3
(31) 25 3 3 3 ← (32) 26 5 3
(35) 13 11 3 3 ← (36) 14 13 3
(42) 4 5 7 7 ← (49) 11 3 3
(47) 5 7 3 3 ← (52) 6 5 3
(47) 9 3 3 3 ← (48) 10 5 3

(65, 6)

(1) 10 9 15 15 15
(3) 12 9 11 15 15 ← (5) 22 21 11 7
(3) 14 7 11 15 15
(3) 28 25 3 3 3 ← (4) 29 27 3 3
(12) 5 7 11 15 15 ← (16) 9 11 15 15
(13) 4 7 11 15 15 ← (18) 7 11 15 15
(21) 11 14 5 7 7 ← (22) 13 13 11 7
(24) 9 11 7 7 7
(25) 7 14 5 7 7 ← (26) 11 15 7 7
(25) 18 3 5 7 7 ← (32) 25 3 3 3
(26) 7 13 5 7 7 ← (28) 17 7 7 7
(27) 20 9 3 3 3 ← (28) 21 11 3 3
(30) 5 9 7 7 7 ← (34) 13 5 7 7
(35) 12 9 3 3 3 ← (36) 13 11 3 3
(41) 2 3 5 7 7 ← (48) 9 3 3 3
(44) 3 5 7 3 3 ← (49) 8 3 3 3
(45) 4 7 3 3 3 ← (46) 6 6 5 3
(47) 4 5 3 3 3 ← (48) 5 7 3 3
(51) 5 1 2 3 3 ← (52) 6 2 3 3

(65, 7)

(1) 25 7 13 5 7 7 ← (4) 12 9 11 15 15
(2) 11 4 7 11 15 15 ← (4) 14 7 11 15 15
(3) 9 5 7 11 15 15
(3) 24 20 9 3 3 3 ← (4) 28 25 3 3 3
(7) 28 12 9 3 3 3
(11) 2 4 7 11 15 15 ← (14) 4 7 11 15 15
(13) 5 7 11 15 7 7 ← (17) 9 11 15 7 7
(15) 5 9 15 7 7 7 ← (21) 9 15 7 7 7
(17) 9 7 13 5 7 7 ← (18) 10 17 7 7 7
(19) 11 5 9 7 7 7 ← (21) 13 13 5 7 7
(21) 7 14 4 5 7 7 ← (22) 11 14 5 7 7
(23) 12 12 9 3 3 3 ← (28) 20 9 3 3 3
(25) 5 5 9 7 7 7 ← (26) 7 14 5 7 7
(27) 3 5 9 7 7 7 ← (29) 14 4 5 7 7
(27) 8 12 9 3 3 3 ← (36) 12 9 3 3 3
(31) 5 7 3 5 7 7 ← (35) 9 3 5 7 7
(39) 3 3 6 6 5 3 ← (43) 3 6 6 5 3
(42) 2 3 5 7 3 3 ← (48) 4 5 3 3 3
(45) 2 4 5 3 3 3 ← (46) 4 7 3 3 3
(51) 3 4 4 1 1 1 ← (52) 5 1 2 3 3
(55) 1 * 1 ← (57) * 1

(65, 8)

(1) 10 2 4 7 11 15 15
(1) 24 5 5 9 7 7 7 ← (2) 25 7 13 5 7 7
(2) 13 5 9 15 7 7 7
(3) 22 17 3 6 6 5 3 ← (4) 24 20 9 3 3 3
(4) 14 6 9 11 7 7 7
(5) 6 2 4 7 11 15 15
(7) 26 9 3 6 6 5 3 ← (8) 28 12 9 3 3 3
(10) 1 2 4 7 11 15 15 ← (12) 2 4 7 11 15 15
(12) 3 5 9 15 7 7 7 ← (16) 5 9 15 7 7 7
(17) 8 5 5 9 7 7 7 ← (18) 9 7 13 5 7 7
(18) 9 3 5 9 7 7 7 ← (20) 11 5 9 7 7 7
(19) 8 3 5 9 7 7 7 ← (22) 7 14 4 5 7 7
(20) 7 8 12 9 3 3 3 ← (26) 17 3 6 6 5 3
(23) 10 9 3 6 6 5 3 ← (24) 12 12 9 3 3 3
(26) 3 5 9 3 5 7 7 ← (34) 9 3 6 6 5 3
(27) 6 9 3 6 6 5 3 ← (28) 8 12 9 3 3 3
(28) 3 5 7 3 5 7 7 ← (32) 5 7 3 5 7 7
(33) 6 3 3 6 6 5 3 ← (40) 3 3 6 6 5 3
(35) 10 2 4 5 3 3 3 ← (36) 12 4 5 3 3 3
(39) 6 2 4 5 3 3 3 ← (46) 2 4 5 3 3 3
(54) 1 1 * 1 ← (56) 1 * 1

(65, 9)

(1) 5 10 8 9 11 7 7 7 ← (2) 10 2 4 7 11 15 15
(1) 10 20 5 7 3 5 7 7
(1) 17 9 3 5 9 7 7 7 ← (2) 24 5 5 9 7 7 7
(3) 9 3 5 9 15 7 7 7
(3) 14 8 5 5 9 7 7 7
(3) 21 11 12 4 5 3 3 3 ← (4) 22 17 3 6 6 5 3
(4) 2 4 3 4 7 11 15 15 ← (9) 27 12 4 5 3 3 3
(5) 5 1 2 4 7 11 15 15 ← (6) 6 2 4 7 11 15 15
(7) 25 5 5 3 6 6 5 3 ← (8) 26 9 3 6 6 5 3
(13) 4 8 5 5 9 7 7 7 ← (18) 8 5 5 9 7 7 7
(17) 4 6 3 5 9 7 7 7 ← (18) 5 10 11 3 5 7 7
(18) 4 5 3 5 9 7 7 7 ← (20) 8 3 5 9 7 7 7
(18) 8 3 5 9 3 5 7 7 ← (25) 11 12 4 5 3 3 3
(19) 9 3 5 7 3 5 7 7 ← (33) 5 5 3 6 6 5 3
(23) 9 5 5 3 6 6 5 3 ← (24) 10 9 3 6 6 5 3
(27) 3 6 5 2 3 5 7 7 ← (28) 6 9 3 6 6 5 3
(31) 5 6 2 3 5 7 3 3 ← (37) 6 2 3 5 7 3 3
(33) 3 6 2 3 5 7 3 3 ← (34) 6 3 3 6 6 5 3
(35) 7 13 1 * 1 ← (36) 10 2 4 5 3 3 3
(37) 6 ..4 5 3 3 3 ← (43) 13 1 * 1
(39) 4 ..4 5 3 3 3 ← (40) 6 2 4 5 3 3 3

(66, 2)

(3) 63
(59) 7
(63) 3 ← (67)

(66, 3)

(2) 61 3
(3) 62 1 ← (4) 63
(35) 30 1 ← (36) 31
(51) 14 1 ← (52) 15
(58) 5 3
(59) 6 1 ← (60) 7
(60) 3 3 ← (66) 1
(63) 2 1 ← (64) 3

(66, 4)

(1) 27 23 15
(1) 59 3 3
(3) 29 27 7 ← (5) 31 31
(3) 61 1 1 ← (4) 62 1
(25) 11 15 15
(27) 21 11 7 ← (29) 23 15
(29) 23 7 7 ← (33) 27 7
(35) 13 11 7 ← (37) 15 15
(35) 29 1 1 ← (36) 30 1
(45) 7 7 7 ← (49) 11 7
(47) 5 7 7 ← (53) 7 7
(51) 13 1 1 ← (52) 14 1
(57) 3 3 3
(58) 2 3 3 ← (65) 1 1
(59) 5 1 1 ← (60) 6 1
(63) 1 1 1 ← (64) 2 1

(66, 5)

(1) 26 21 11 7 ← (2) 27 23 15
(2) 27 23 7 7 ← (4) 29 27 7
(3) 60 1 1 1 ← (4) 61 1 1
(21) 14 13 11 7 ← (22) 15 15 15
(25) 10 13 11 7 ← (26) 11 15 15
(26) 19 7 7 7 ← (28) 21 11 7
(27) 20 5 7 7 ← (30) 23 7 7
(33) 14 5 7 7 ← (36) 13 11 7
(35) 28 1 1 1 ← (36) 29 1 1
(43) 4 5 7 7 ← (46) 7 7 7
(44) 3 5 7 7 ← (48) 5 7 7
(51) 12 1 1 1 ← (52) 13 1 1
(57) 1 2 3 3 ← (64) 1 1 1
(59) 4 1 1 1 ← (60) 5 1 1

(66, 6)

(1) 25 19 7 7 7 ← (2) 26 21 11 7
(2) 10 9 15 15 15
(3) 56 4 1 1 1 ← (4) 60 1 1 1
(13) 5 7 11 15 15 ← (17) 9 11 15 15
(15) 7 13 13 11 7 ← (22) 14 13 11 7
(19) 7 11 15 7 7
(25) 9 11 7 7 7 ← (26) 10 13 11 7
(26) 18 3 5 7 7 ← (28) 20 5 7 7
(27) 7 13 5 7 7 ← (29) 17 7 7 7
(31) 5 9 7 7 7 ← (35) 13 5 7 7
(33) 11 3 5 7 7 ← (34) 14 5 7 7
(35) 24 4 1 1 1 ← (36) 28 1 1 1
(42) 2 3 5 7 7 ← (44) 4 5 7 7
(45) 3 5 7 3 3 ← (49) 5 7 3 3
(49) 3 6 2 3 3 ← (50) 8 3 3 3
(51) 2 4 3 3 3 ← (53) 6 2 3 3
(51) 8 4 1 1 1 ← (52) 12 1 1 1
(55) 4 4 1 1 1 ← (60) 4 1 1 1

(66, 7)

(1) 6 11 7 11 15 15
(3) 11 4 7 11 15 15 ← (5) 14 7 11 15 15
(3) 54 * 1 ← (4) 56 4 1 1 1
(4) 9 5 7 11 15 15
(5) 20 9 11 7 7 7
(5) 28 11 3 5 7 7
(14) 5 7 11 15 7 7 ← (19) 10 17 7 7 7
(15) 6 9 15 7 7 7 ← (16) 7 13 13 11 7
(17) 8 9 11 7 7 7 ← (18) 9 11 15 7 7
(19) 6 9 11 7 7 7 ← (20) 7 11 15 7 7
(21) 12 11 3 5 7 7 ← (22) 13 13 5 7 7
(26) 5 5 9 7 7 7 ← (28) 7 13 5 7 7
(28) 3 5 9 7 7 7 ← (32) 5 9 7 7 7
(29) 13 2 3 5 7 7 ← (30) 14 4 5 7 7
(30) 5 9 3 5 7 7 ← (36) 9 3 5 7 7
(35) 22 * 1 ← (36) 24 4 1 1 1
(43) 2 3 5 7 3 3 ← (44) 3 6 6 5 3
(49) ..4 3 3 3 ← (50) 3 6 2 3 3
(50) 1 2 4 3 3 3 ← (52) 2 4 3 3 3
(51) 6 * 1 ← (52) 8 4 1 1 1
(52) 3 4 4 1 1 1 ← (58) * 1
(55) 2 * 1 ← (56) 4 4 1 1 1

(66, 8)

(1) 5 7 5 7 11 15 15
(1) 12 8 9 15 7 7 7 ← (4) 11 4 7 11 15 15
(3) 13 5 9 15 7 7 7
(3) 25 3 5 9 7 7 7 ← (18) 8 9 11 7 7 7
(3) 53 1 * 1 ← (4) 54 * 1
(5) 14 6 9 11 7 7 7 ← (6) 20 9 11 7 7 7
(5) 23 8 12 9 3 3 3 ← (6) 28 11 3 5 7 7
(11) 1 2 4 7 11 15 15 ← (13) 2 4 7 11 15 15
(13) 3 5 9 15 7 7 7 ← (17) 5 9 15 7 7 7
(15) 4 6 9 11 7 7 7 ← (16) 6 9 15 7 7 7
(19) 9 3 5 9 7 7 7 ← (21) 11 5 9 7 7 7
(21) 7 8 12 9 3 3 3 ← (22) 12 11 3 5 7 7
(27) 3 5 9 3 5 7 7 ← (29) 8 12 9 3 3 3
(29) 3 5 7 3 5 7 7 ← (33) 5 7 3 5 7 7
(29) 7 12 4 5 3 3 3 ← (30) 13 2 3 5 7 7
(31) 6 5 2 3 5 7 7 ← (37) 12 4 5 3 3 3
(35) 21 1 * 1 ← (36) 22 * 1
(49) 1 1 2 4 3 3 3 ← (50) ..4 3 3 3
(50) 2 3 4 4 1 1 1 ← (57) 1 * 1
(51) 5 1 * 1 ← (52) 6 * 1
(55) 1 1 * 1 ← (56) 2 * 1

(67, 3)

(1) 59 7
(3) 61 3 ← (5) 63
(35) 29 3 ← (37) 31
(51) 13 3 ← (53) 15
(59) 5 3 ← (61) 7
(61) 3 3 ← (65) 3

(67, 4)

(1) 58 5 3 ← (2) 59 7
(2) 59 3 3 ← (4) 61 3
(5) 30 29 3 ← (6) 31 31
(29) 22 13 3 ← (30) 23 15
(33) 26 5 3 ← (34) 27 7
(34) 27 3 3 ← (36) 29 3
(37) 14 13 3 ← (38) 15 15
(49) 10 5 3 ← (50) 11 7
(50) 11 3 3 ← (52) 13 3
(53) 6 5 3 ← (54) 7 7
(58) 3 3 3 ← (60) 5 3
(59) 2 3 3 ← (62) 3 3

(67, 5)

(1) 57 3 3 3 ← (2) 58 5 3
(3) 27 23 7 7 ← (5) 29 27 7
(5) 29 27 3 3 ← (6) 30 29 3
(6) 22 21 11 7
(19) 7 11 15 15
(23) 13 13 11 7
(27) 11 15 7 7 ← (37) 13 11 7
(27) 19 7 7 7 ← (29) 21 11 7
(29) 21 11 3 3 ← (30) 22 13 3
(33) 25 3 3 3 ← (34) 26 5 3
(37) 13 11 3 3 ← (38) 14 13 3
(45) 3 5 7 7 ← (49) 5 7 7
(47) 6 6 5 3 ← (54) 6 5 3
(49) 9 3 3 3 ← (50) 10 5 3
(58) 1 2 3 3 ← (60) 2 3 3

(67, 6)

(2) 25 19 7 7 7 ← (4) 27 23 7 7
(3) 10 9 15 15 15
(5) 12 9 11 15 15
(5) 28 25 3 3 3 ← (6) 29 27 3 3
(14) 5 7 11 15 15 ← (18) 9 11 15 15
(15) 4 7 11 15 15 ← (20) 7 11 15 15
(22) 9 15 7 7 7
(23) 11 14 5 7 7 ← (24) 13 13 11 7
(26) 9 11 7 7 7 ← (30) 17 7 7 7
(27) 7 14 5 7 7 ← (28) 11 15 7 7
(27) 18 3 5 7 7 ← (29) 20 5 7 7
(29) 20 9 3 3 3 ← (30) 21 11 3 3
(34) 11 3 5 7 7 ← (36) 13 5 7 7
(37) 12 9 3 3 3 ← (38) 13 11 3 3
(43) 2 3 5 7 7 ← (45) 4 5 7 7
(46) 3 5 7 3 3 ← (51) 8 3 3 3
(47) 4 7 3 3 3 ← (48) 6 6 5 3
(49) 4 5 3 3 3 ← (50) 5 7 3 3
(53) 5 1 2 3 3 ← (54) 6 2 3 3

(67, 7)

(1) 19 7 11 15 7 7
(2) 6 11 7 11 15 15
(3) 25 7 13 5 7 7
(5) 9 5 7 11 15 15 ← (19) 9 11 15 7 7
(5) 24 20 9 3 3 3 ← (6) 28 25 3 3 3
(9) 28 12 9 3 3 3 ← (16) 4 7 11 15 15
(15) 5 7 11 15 7 7 ← (17) 7 13 13 11 7
(19) 9 7 13 5 7 7 ← (20) 10 17 7 7 7
(20) 6 9 11 7 7 7
(23) 7 14 4 5 7 7 ← (24) 11 14 5 7 7
(25) 12 12 9 3 3 3 ← (28) 18 3 5 7 7
(27) 5 5 9 7 7 7 ← (28) 7 14 5 7 7
(27) 17 3 6 6 5 3 ← (30) 20 9 3 3 3
(29) 3 5 9 7 7 7 ← (33) 5 9 7 7 7
(31) 5 9 3 5 7 7 ← (37) 9 3 5 7 7
(35) 9 3 6 6 5 3 ← (38) 12 9 3 3 3
(41) 3 3 6 6 5 3 ← (44) 2 3 5 7 7
(44) 2 3 5 7 3 3 ← (50) 4 5 3 3 3
(47) 2 4 5 3 3 3 ← (48) 4 7 3 3 3
(51) 1 2 4 3 3 3 ← (53) 2 4 3 3 3
(53) 3 4 4 1 1 1 ← (54) 5 1 2 3 3

(68, 2)

(67) 1 ← (69)

(68, 3)

(5) 62 1 ← (6) 63
(37) 30 1 ← (38) 31
(53) 14 1 ← (54) 15
(61) 6 1 ← (62) 7
(65) 2 1 ← (66) 3
(66) 1 1 ← (68) 1

(68, 4)

(3) 27 23 15
(3) 59 3 3 ← (5) 61 3
(5) 61 1 1 ← (6) 62 1
(23) 15 15 15 ← (39) 15 15
(27) 11 15 15
(31) 23 7 7 ← (35) 27 7
(35) 27 3 3 ← (37) 29 3
(37) 29 1 1 ← (38) 30 1
(47) 7 7 7 ← (51) 11 7
(51) 11 3 3 ← (53) 13 3
(53) 13 1 1 ← (54) 14 1
(59) 3 3 3 ← (61) 5 3
(61) 5 1 1 ← (62) 6 1
(65) 1 1 1 ← (66) 2 1

(68, 5)

(2) 57 3 3 3 ← (4) 59 3 3
(3) 26 21 11 7 ← (4) 27 23 15
(5) 60 1 1 1 ← (6) 61 1 1
(7) 22 21 11 7
(23) 14 13 11 7 ← (24) 15 15 15
(27) 10 13 11 7 ← (28) 11 15 15
(28) 19 7 7 7 ← (32) 23 7 7
(34) 25 3 3 3 ← (36) 27 3 3
(35) 14 5 7 7 ← (48) 7 7 7
(37) 28 1 1 1 ← (38) 29 1 1
(46) 3 5 7 7 ← (50) 5 7 7
(50) 9 3 3 3 ← (52) 11 3 3
(53) 12 1 1 1 ← (54) 13 1 1
(59) 1 2 3 3 ← (61) 2 3 3
(61) 4 1 1 1 ← (62) 5 1 1

(68, 6)

(1) 19 7 11 15 15
(1) 23 13 13 11 7
(3) 25 19 7 7 7 ← (4) 26 21 11 7
(4) 10 9 15 15 15
(5) 56 4 1 1 1 ← (6) 60 1 1 1
(6) 12 9 11 15 15 ← (8) 22 21 11 7
(6) 14 7 11 15 15
(15) 5 7 11 15 15 ← (19) 9 11 15 15
(21) 7 11 15 7 7
(23) 9 15 7 7 7 ← (25) 13 13 11 7
(23) 13 13 5 7 7 ← (30) 20 5 7 7
(27) 9 11 7 7 7 ← (28) 10 13 11 7
(29) 7 13 5 7 7 ← (37) 13 5 7 7
(31) 14 4 5 7 7 ← (46) 4 5 7 7
(35) 11 3 5 7 7 ← (36) 14 5 7 7
(37) 24 4 1 1 1 ← (38) 28 1 1 1
(45) 3 6 6 5 3 ← (51) 5 7 3 3
(47) 3 5 7 3 3 ← (49) 6 6 5 3
(51) 3 6 2 3 3 ← (52) 8 3 3 3
(53) 8 4 1 1 1 ← (54) 12 1 1 1
(57) 4 4 1 1 1 ← (60) 1 2 3 3
(59) * 1 ← (62) 4 1 1 1

(69, 3)

(3) 59 7
(7) 31 31
(31) 23 15 ← (39) 31
(55) 7 7
(63) 3 3 ← (67) 3
(67) 1 1 ← (69) 1

(69, 4)

(3) 58 5 3 ← (4) 59 7
(6) 29 27 7
(7) 30 29 3 ← (8) 31 31
(30) 21 11 7 ← (38) 29 3
(31) 22 13 3 ← (32) 23 15
(35) 26 5 3 ← (36) 27 7
(38) 13 11 7
(39) 14 13 3 ← (40) 15 15
(51) 10 5 3 ← (52) 11 7
(55) 6 5 3 ← (56) 7 7
(60) 3 3 3 ← (64) 3 3
(66) 1 1 1 ← (68) 1 1

(69, 5)

(1) 27 11 15 15
(3) 57 3 3 3 ← (4) 58 5 3
(5) 27 23 7 7
(7) 29 27 3 3 ← (8) 30 29 3
(21) 7 11 15 15
(24) 14 13 11 7 ← (37) 27 3 3
(29) 11 15 7 7
(29) 19 7 7 7 ← (33) 23 7 7
(31) 17 7 7 7 ← (49) 7 7 7
(31) 21 11 3 3 ← (32) 22 13 3
(35) 25 3 3 3 ← (36) 26 5 3
(39) 13 11 3 3 ← (40) 14 13 3
(47) 3 5 7 7 ← (51) 5 7 7
(51) 9 3 3 3 ← (52) 10 5 3
(55) 6 2 3 3 ← (62) 2 3 3

(70, 2)

(7) 63
(55) 15
(63) 7 ← (71)

(70, 3)

(6) 61 3
(7) 62 1 ← (8) 63
(39) 30 1 ← (40) 31
(54) 13 3
(55) 14 1 ← (56) 15
(62) 5 3 ← (70) 1
(63) 6 1 ← (64) 7
(67) 2 1 ← (68) 3

(70, 4)

(1) 55 7 7
(5) 27 23 15
(5) 59 3 3
(7) 29 27 7 ← (9) 31 31
(7) 61 1 1 ← (8) 62 1
(25) 15 15 15 ← (37) 27 7
(29) 11 15 15
(31) 21 11 7 ← (33) 23 15
(39) 13 11 7 ← (41) 15 15
(39) 29 1 1 ← (40) 30 1
(53) 11 3 3
(55) 13 1 1 ← (56) 14 1
(56) 6 5 3 ← (69) 1 1
(61) 3 3 3 ← (65) 3 3
(63) 5 1 1 ← (64) 6 1
(67) 1 1 1 ← (68) 2 1