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AVERAGE AREA RATIO AND MINIMAL SURFACE ENTROPY OF HYPERBOLIC MANIFOLDS

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Dedicated to my cat Ron in celebration of his first birthday

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ABSTRACT

This thesis studies the average area ratio and minimal surface entropy of hyperbolic manifolds.

On closed hyperbolic manifolds of dimension $n \geq 3$, we review the definition of the average area ratio of a metric h whose scalar curvature is bounded below by $-n(n-1)$ in comparison to the hyperbolic metric h_0. We prove that it reaches its local minimum value of one at h_0, which solves a localized version of Gromov's conjecture.

Furthermore, in the case of odd n, assuming h is a metric with sectional curvature no greater than -1, we introduce the concept of minimal surface entropy of h, which quantifies the number of surface subgroups. It achieves its minimum value if and only if the metric is hyperbolic.

Additionally, we explore the relationship between the average area ratio and the normalized total scalar curvature for hyperbolic n-manifolds. We also discuss its connection to the minimal surface entropy when n is odd.

CHAPTER 1

INTRODUCTION

One of the fundamental problems in geometry is to construct minimal surfaces in Riemannian manifolds. Fortunately, we have seen many significant developments in recent years. The Almgren-Pitts min-max theory [2], [52], established in the 1980s, aims to find a smooth closed embedded minimal hypersurface in every closed manifold M^{n+1} of dimension $3 \leq n+1 \leq 6$ (Schoen-Simon [56] improved this to $3 \leq n+1 \leq 7$). This theory has proven to be a powerful tool in the past few years. Marques-Neves [47] and Song [63] solved Yau's conjecture [70] regarding the existence of infinitely many closed embedded minimal hypersurfaces in a closed manifold M^{n+1} of dimension $3 \leq n+1 \leq 7$. Li [40] investigated the higher dimensional case for generic metrics, where the minimal hypersurfaces exhibit optimal regularity. Furthermore, using the Weyl law for the volume spectrum (Liokumovich-Marques-Neves [41]), Irie-Marques-Neves [29], and Marques-Neves-Song [49] discussed the density and equidistribution properties of these hypersurfaces, respectively, when M^{n+1} admits a generic metric. On the other hand, Zhou [71] proved the Multiplicity One Conjecture raised by Marques-Neves [46]. Combined with [48], it implies the existence of minimal hypersurfaces with any Morse index for generic metrics.

Meanwhile, the ambient metric affects the minimal surfaces in various ways, including their existence, distribution, and asymptotic behavior. In particular, our motivation for exploring manifolds with strictly negative curvature stems from the fact that it enables the study of minimal surfaces from both a variational and dynamical standpoint.

On a closed negatively curved manifold M, a helpful dynamical fact is that the unit tangent bundle admits a one-dimensional foliation whose leaves are orbits of the geodesic flow. Let h and h_0 be a pair of Riemannian metrics on M. The comparisons of the geometric objects associated with h and h_0 have been studied extensively. For instance, the geodesic stretch $I_\mu(h/h_0)$ measures the stretching of the metric h relative to the reference metric h_0 and the measure μ. Comparisons related to $I_\mu(h/h_0)$ were discussed by Knieper [36]. Another example is the volume entropy, which can be expressed using the following form as shown by Manning [44] and Margulis [45].

$$E_{vol}(h) = \lim_{L\to\infty} \frac{\ln \#\{\text{length}_h(\gamma) \leq L : \gamma \text{ is a closed geodesic in } (M,h)\}}{L}.$$

Besson-Courtois-Gallot [7] showed that the *locally symmetric metric*, which refers to metrics on closed hyperbolic manifolds, complex hyperbolic manifolds, quaternionic hyperbolic manifolds, and the Cayley plane, achieves the minimum value among all metrics on M with the same volume. It is natural to explore the extent to which one-dimensional objects (geodesics) can be expanded and applied to two-dimensional scenarios (minimal surfaces).

1.1 Definitions

1.1.1 Average Area Ratio

Gromov discussed an attractive two-dimensional analogue in his work [19]. A two-dimensional foliation of Gr_2M is formed by a family of stable minimal surfaces of M. In particular, let h_0 denote the hyperbolic metric. In this case, there exists a canonical foliation of Gr_2M whose leaves are totally geodesic planes. Gromov introduced the average area ratio $\text{Area}_F(h/h_0)$, which, similar in essence to the geodesic stretch, measures the "stretching" of the area of leaves on Gr_2M of the metric h relative to the hyperbolic metric h_0 under the map F.

Let (M, h_0) be a closed hyperbolic manifold of dimension $n \geq 3$, and let (N, g) be a closed Riemannian manifold of the same dimension. Suppose that $F : (N, g) \to (M, h_0)$

is a smooth map with degree $d > 0$. The average area ratio of F is as follows.

$$\mathrm{Area}_F(g/h_0) = \int_{(y,P)\in Gr_2(M)} \sum_{x\in F^{-1}(y)} \lim_{\delta\to 0} \frac{\mathrm{area}_g((dF_x)^{-1}(D_\delta))}{\delta}\, d\mu_{h_0},$$

where D_δ is a subset of the totally geodesic disc of $\mathbb{H}^n$ which is tangential to P at x, D_δ has an area equal to δ, and μ_{h_0} stands for the unit volume measure on $Gr_2(M)$ with respect to the metric induced by h_0. We refer to Chapter 2 for a more detailed definition.

When $n = 3$ and the scalar curvature of N satisfies $R_g \geq -6$, Gromov proved the following inequality using stability [19].

$$\mathrm{Area}_F(g/h_0) \geq \frac{d}{3}.$$

The proof indicates that if equality holds, then (N, g) should be hyperbolic. However, it means the inequality is not sharp, and therefore, Gromov conjectured that the lower bound could be replaced by d. Moreover, the higher dimensional case might share the same property.

Conjecture 1.1.1 (Gromov, [19])**.** Let (M, h_0) be a closed hyperbolic manifold of dimension $n \geq 3$, and let N be a closed n-manifold with Riemannian metric g, and its scalar curvature satisfies $R_g \geq -n(n-1)$. $F : N \to M$ is a smooth map with degree $d > 0$. We have

$$\mathrm{Area}_F(g/h_0) \geq d.$$

The equality holds if and only if F is a local isometry.

Most recently, Lowe-Neves [43] verified the special case where $n = 3$ and F is a local diffeomorphism.

1.1.2 Minimal Surface Entropy

Calegari-Marques-Neves [11] introduced the concept of the minimal surface entropy, denoted as $E(h)$, which is based on the construction of surface subgroups by Kahn-Markovic [33]. This concept is further supported by the equidistribution property of $\mathrm{PSL}(2,\mathbb{R})$-action on the space of minimal laminations, which was initially proposed by Ratner [53] and Shah [59], and recently formalized by Labourie [38]. The value of $E(h)$ serves as a measurement for the number of essential minimal surfaces of M with respect to h, thus shifting attention from a one-dimensional entity (volume entropy) to an object of dimension two.

Let $\mathbb{H}^n$ denote the hyperbolic n-space, where $n \geq 3$. In the Poincaré ball model, the asymptotic boundary $\partial_\infty \mathbb{H}^n$ can be considered as the $(n-1)$-unit sphere S^{n-1}_∞. Let $M = \mathbb{H}^n/\pi_1(M)$ be a closed orientable n-manifold ($n \geq 3$) that admits a hyperbolic metric h_0, A closed surface immersed in M with genus at least 2 is said to be *essential* if the immersion is π_1-injective, and the image of its fundamental group in $\pi_1(M)$ is called a *surface subgroup*. Let $S(M,g)$ denote the set of surface subgroups of genus at most g up to conjugacy, and let the subset $S(M,g,\epsilon) \subset S(M,g)$ consist of the conjugacy classes whose limit sets are $(1+\epsilon)$-quasicircles (i.e., images of round circles under $(1+\epsilon)$-quasiconformal maps on S^{n-1}_∞). Moreover, $S_\epsilon(M) = \bigcup_{g\geq 2} S(M,g,\epsilon)$. Suppose h is an arbitrary Riemannian metric on M. For any $\Pi \in S(M,g)$, we set

$$\mathrm{area}_h(\Pi) = \inf\{\mathrm{area}_h(\Sigma) : \Sigma \in \Pi\}.$$

Then the *minimal surface entropy with respect to h* is defined as follows.

$$E(h) = \lim_{\epsilon\to 0} \liminf_{L\to\infty} \frac{\ln \#\{\mathrm{area}_h(\Pi) \leq 4\pi(L-1) : \Pi \in S_\epsilon(M)\}}{L \ln L}. \tag{1.1.1}$$

According to Calegari-Marques-Neves [11], when $n = 3$, and among metrics with sectional curvature less than or equal to -1, $E(h)$ attains its minimum value at the hyperbolic metric h_0, and $E(h_0) = 2$. On the other hand, Lowe-Neves [43] showed that when $n = 3$, $E(h)$ is maximized at h_0 among all metrics with scalar curvature greater than or equal to -6.

1.2 Results and Ideas

In this section, we state the main results and provide a brief introduction to the ideas. The detailed proofs are presented in subsequent sections. Additionally, the organization of the thesis is outlined in this section.

1.2.1 Average Area Ratio

In this thesis, we investigate Gromov's conjecture from two perspectives in Theorem 1.2.1 and Theorem 1.2.3. First, if $n \geq 3$ and $F : N \to M$ is a diffeomorphism, it simplifies to the following scenario.

Theorem 1.2.1 (Jiang, [31]). *Let (M, h_0) be a closed hyperbolic manifold of dimension $n \geq 3$. There exists a small neighborhood $\mathcal{U}$ of h_0 in the metric space of M, such that for any Riemannian metric $h \in \mathcal{U}$ on M with $R_h \geq -n(n-1)$, we have*

$$Area_{Id}(h/h_0) \geq 1.$$

The equality holds if and only if $h = h_0$.

Such a neighborhood is "cylindrical", that is, if $h \in \mathcal{U}$, then any metric h' in the conformal class of h with $R_{h'} \geq -n(n-1)$ also belongs to $\mathcal{U}$. This theorem provides a solution to a local version of Gromov's conjecture concerning diffeomorphisms.

We now brief present the idea of the proof, which can be divided into two parts. Suppose h is an arbitrary metric conformal to a metric $\bar{h}$ with constant scalar curvature.

In that case, the assumption $R_h \geq -n(n-1)$ provides estimates of the conformal factor, and thus the area of any surface attains the minimum at $\bar{h}$ in the conformal class.

The second part considers the metrics set on M with constant scalar curvature $-n(n-1)$. We define a functional $\mathcal{A}$ from the space of Riemannian metrics on M to $\mathbb{R}$ as follows.

$$\mathcal{A}(h) = \int_{(x,P)\in Gr_2M} R_h(x)\, |\Lambda^2 \mathrm{Id}|_h^{-1}(x,P)\, d\mu_{h_0}.$$

The theorem is equivalent to show that $\mathcal{A}$ attains a local maximum at h_0. It is not hard to check that h_0 is a critical point of $\mathcal{A}$. Next, in order to evaluate $\mathcal{A}''(h_0)$, we compare it with the second variation of the *normalized total scalar curvature* (or *normalized Einstein-Hilbert functional*)

$$\mathcal{E}(h) = (\mathrm{vol}_h(M))^{\frac{2}{n}-1} \int_M R_h\, dV_h,$$

and apply the estimates in [7].

Notice that unlike $\mathcal{E}$, the functional $\mathcal{A}$ is not invariant under diffeomorphisms; therefore, we need to use the decomposition of space of symmetric tensors [6] and discuss the signs of $\mathcal{A}''(h_0)$ on the conformal deformation, diffeomorphism part, and the transverse-traceless part, respectively.

Besides, let (N, g_0) be another closed hyperbolic manifold of the same dimension n, the theorem leads to the following corollary.

Corollary 1.2.2. *There exists a small neighborhood $\mathcal{U}$ of g_0 in the metric space of N, such that for any metric $g \in \mathcal{U}$ with $R_g \geq -n(n-1)$, and for any local diffeomorphism $F : (N, g) \to (M, h_0)$ with degree $d > 0$, we have*

$$Area_F(g/h_0) \geq d.$$

The equality holds if and only if F is a local isometry between g and h_0, i.e., $g = F^*(h_0)$.

Proof. Since F is a local diffeomorphism,

$$\mathrm{Area}_F(g/h_0) = d\,\mathrm{Area}_{\mathrm{Id}}(g/F^*(h_0)).$$

$F^*(h_0)$ is a hyperbolic metric on N, so it's isometric to g_0 due to Mostow rigidity. Thus we obtain from the previous theorem the following inequality.

$$\mathrm{Area}_F(g/h_0) = d\,\mathrm{Area}_{\mathrm{Id}}(g/g_0) \geq d.$$

□

On the other hand, when the dimension $n = 3$, we establish a more general result applicable to any smooth maps near the diffeomorphisms between homeomorphic 3-manifolds, regardless of whether it is a diffeomorphism or not.

Theorem 1.2.3 (Jiang, [31]). *Let (M, h_0) and (N, g_0) be closed hyperbolic 3-manifolds. If $\pi_1(M) \cong \pi_1(N)$, then there exists a small neighborhood $\mathcal{U}$ of g_0 in the C^2-topology, such that for any metric $g \in \mathcal{U}$ with $R_g \geq -6$, and for any smooth map $F : (N, g) \to (M, h_0)$ with positive degree, we have*

$$deg\,F = 1 \quad and \quad Area_F(g/h_0) \geq 1.$$

In this case, the inverse of a closed surface $S \subset M$ via F is not necessarily homotopic to S. So we need to compare the areas of surfaces with respect to the induced metric of g in different homotopic classes, this is hard in general. Fortunately, if there is a minimal surface with suitable curvature conditions, we can find a global area-minimizing surface. To see this, we prove a key lemma adapting Uhlenbeck's method in [66], and when combined with Lemma 5.1.2 in Section 5.1.1, it generalizes to all dimensions $n \geq 3$.

Lemma 1.2.4 (Uniqueness of minimal surface). *Let (M, h) be a closed n-manifold with strictly negative sectional curvature, and let Σ be a minimal surface in (M, h) whose fundamental group injectively includes in $\pi_1(M)$, the norm squared of the second fundamental form of Σ with respect to h, denoted by $|A|^2$, and the sectional curvature K of (M, h) satisfy that*

$$|A|^2_{L^\infty(\Sigma)} < -2 \sup K.$$

Let $\tilde{M}$ be the cover of M with $\pi_1(\tilde{M}) \cong \pi_1(\Sigma)$. Then Σ is the unique closed minimal surface in $\tilde{M}$ of any type.

The rest of the proof follows from the comparison of $\text{area}_h(\pi_1(\Sigma))$ and $\text{area}_{h_0}(\pi_1(\Sigma))$ in [43].

In a different situation, suppose that $\pi_1(N)$ is isomorphic to an index d subgroup $G < \pi_1(M)$. Let $\tilde{M}$ denote the covering space of M with $\pi_1(\tilde{M}) = G$ and let $p : \tilde{M} \to M$ denote the covering map, the hyperbolic metric on $\tilde{M}$ is still represented by h_0. We have the following corollary.

Corollary 1.2.5. *There exists a small neighborhood $\mathcal{U}$ of g_0 in the C^2-topology, such that for any metric $g \in \mathcal{U}$ with $R_g \geq -6$, and for any smooth map $F : (N, g) \to (M, h_0)$ satisfying $F_*\pi_1(N) < p_*\pi_1(\tilde{M})$ with positive degree, we have*

$$deg\, F = d \quad and \quad Area_F(g/h_0) \geq d.$$

Proof. Since $F_*\pi_1(N) < p_*\pi_1(\tilde{M})$, F can be lifted to a smooth map $\tilde{F} : N \to \tilde{M}$, so that $p \circ \tilde{F} = F$. Applying Theorem 1.2.3 to $\tilde{F}$, we have

$$\deg \tilde{F} = 1 \quad \text{and} \quad \text{Area}_{\tilde{F}}(g/h_0) \geq 1.$$

Thus,

$$\deg F = d \quad \text{and} \quad \mathrm{Area}_F(g/h_0) = d\,\mathrm{Area}_{\tilde{F}}(g/h_0) \geq d.$$

$\square$

1.2.2 Minimal Surface Entropy

When the dimension of M is an odd number, Hamenstädt [22] verified the existence of surface subgroups and constructed an essential surface Σ_ϵ which is sufficiently well-distributed and $(1+\epsilon)$-quasigeodesic (in other words, the geodesics on the surface with respect to intrinsic distance are $(1+\epsilon, \epsilon)$-quasigeodesics in M). Based on this result, we expand the definition of minimal surface entropy to encompass a broader range of scenarios.

Theorem 1.2.6 (Jiang, [30])**.** *Let (M, h_0) be a closed hyperbolic manifold whose dimension $n \geq 3$ is odd, and let h be another metric on M with sectional curvature less than or equal to -1, then*

$$E(h) \geq E(h_0) = 2.$$

The equality holds if and only if h is isometric to h_0.

The strategy of the proof is summarized as follows. The lower and upper bounds of the cardinality of $S(M, g, \epsilon)$ ($\epsilon > 0$) follow easily from Hamenstädt [22] and Kahn-Markovic [33], respectively. The challenge of showing $E(h_0) = 2$ is the following. The argument by Schoen-Yau [57] and Sacks-Uhlenbeck [55] says for any surface subgroup $\Pi_i \in S(M, g_i, \frac{1}{i})$, there exists an immersed minimal surface S_i in M with finitely many branch points, such that it minimizes the area in the corresponding homotopy class up to conjugacy. The appearance of branch points is the primary distinction from the case of dimension three. The strategy to rule out branch points is to consider a sequence of area-minimizing surfaces mod 2 in $\mathbb{H}^n$, denoted by D_i, such that $\partial_\infty D_i = \partial_\infty \tilde{S}_i$, where

$\tilde{S}_i$ is the lift of S_i to $\mathbb{H}^n$. D_i is free of branch points by Almgren [3], and some arguments in geometric measure theory indicate that D_i converges smoothly to a totally geodesic disc in $\mathbb{H}^n$. Thus $|A|^2_{L^\infty(D_i)} \to 0$, and Lemma 1.2.4 (a universal cover version) implies that D_i is identical to $\tilde{S}_i$. As a result, $|A|^2_{L^\infty(S_i)} \to 0$ (when $n = 3$, this was prove by Seppi [58]), which leads to $E(h_0) = 2$.

For metric h with $K_h \leq -1$, now we briefly describe the main obstacle to proving the rigidity. Due to Ratner [53] or Shah [59], the closure of the projection of a totally geodesic disc of $\mathbb{H}^n$ to M is either the whole manifold M or a finite union of closed totally geodesic proper submanifolds of M. When $n > 3$, it is impossible to find a dense orbit using the "dense or isolated" argument in [11]. To deal with this issue, we can use the observation by Mozes-Shah [50] or Lee-Oh [39], which says that any infinite sequence of properly immersed totally geodesic submanifolds either becomes dense in M, or it has a subsequence contained in a higher dimensional proper totally geodesic submanifold, and our result follows by induction.

Furthermore, using the equidistribution property formalized by Labourie [38], we extend Theorem 1.2 of [43] to encompass odd dimensions greater than or equal to 3. This theorem establishes a link between minimal surface entropy and the average area ratio.

Theorem 1.2.7 (Jiang, [31]). *Let (M, h_0) be a closed hyperbolic manifold of an odd dimension $n \geq 3$. For any Riemannian metric h on M,*

$$Area_{Id}(h/h_0)E(h) \geq E(h_0) = 2,$$

the equality holds if and only if $h = ch_0$ for some constant $c > 0$.

1.3 Organization

The thesis is organized as follows:

First, in Chapter 2, we introduce the notations and definitions that are used throughout the thesis.

In Chapter 3, we establish the equidistribution property for closed hyperbolic manifolds with odd dimensions. We also derive the formula for the average area ratio, which will be used in Section 4.2 and Chapters 5 and 6.

Chapter 4 discusses the average area ratio from a dimensional perspective. Specifically, in Section 4.1, we restrict the maps to diffeomorphisms and consider all dimensions greater than or equal to three. We provide the proof for Theorem 1.2.1 in this context. Then, in Section 4.2, we analyze a broader scenario of the maps but only focus on three-dimensional manifolds. We prove Theorem 1.2.3 in this specific case. In Section 4.3, we expand the definition of average area ratio (where $k = 2$) to average k-volume ratio, where $2 \leq k \leq n-1$. We also explore the negatively curved Einstein manifolds to which we can extend the result.

Chapter 5 contains a discussion of the minimal surface entropy for different types of manifolds. In Section 5.1, we focus on the case where the dimension of a closed hyperbolic manifold is an odd number. We compute the minimal surface entropy of the hyperbolic metric and provide the proof for Theorem 1.2.6. Additionally, we explore certain findings in other locally symmetric manifolds in Section 5.2 and cusped hyperbolic three-manifolds in Section 5.3.

Finally, Chapter 6 examines the relationship between average area ratio and minimal surface entropy. We present the proof of Theorem 1.2.7.

CHAPTER 2

PRELIMINARIES

This chapter contains the notations and definitions that are utilized in the subsequent chapters.

2.1 Gromov's Average Area Ratio

In this section, we provide a more detailed explanation of the definition of the average area ratio.

Let (M, h_0) be a closed hyperbolic manifold of dimension $n \geq 3$, and let (N, g) be another closed Riemannian manifold of the same dimension. Suppose that $F : (N, g) \to (M, h_0)$ is a smooth map. For any (x, L) in the Grassmannian bundle $Gr_2(N)$, $|\Lambda^2 F(x, L)|_g$ denotes the Jacobian of dF_x at plane L. Take an arbitrary regular value $y \in M$ of F, for $(y, P) \in Gr_2(M)$, we let

$$|\Lambda^2 F|_g^{-1}(y, P) = \sum_{x \in F^{-1}(y)} \frac{1}{|\Lambda^2 F(x, (dF_x)^{-1}(P)|_g}. \tag{2.1.1}$$

This definition ([43]) is equivalent to Gromov's definition ([19]):

$$|\Lambda^2 F|_g^{-1}(y, P) = \sum_{x \in F^{-1}(y)} \lim_{\delta \to 0} \frac{\mathrm{area}_g((dF_x)^{-1}(D_\delta))}{\delta},$$

where D_δ is a subset of the totally geodesic disc $D \subset \mathbb{H}^n$ which is tangential to P at x, and D_δ has area equal to δ.

The *average area ratio of F* is defined in [19] by

$$\mathrm{Area}_F(g/h_0) = \int_{(y,P) \in Gr_2(M)} |\Lambda^2 F|_g^{-1}(y, P)\, d\mu_{h_0}, \tag{2.1.2}$$

where μ_{h_0} stands for the unit volume measure on $Gr_2(M)$ with respect to the metric induced by h_0.

2.2 Normalized Total Scalar Curvature

In this section, we introduce the normalized total scalar curvature and calculate its second variation. The second variation is an important tool in proving Theorem 1.2.1. We refer to Section 4.1 for more details.

Let (M, h_0) be a closed hyperbolic manifold of dimension $n \geq 3$, and let $\mathcal{M}$ be the space of Riemannian metrics on M. The *total scalar curvature* (or *Einstein-Hilbert functional*) $\tilde{\mathcal{E}} : \mathcal{M} \to \mathbb{R}$ is

$$\tilde{\mathcal{E}}(h) = \int_M R_h \, dV_h.$$

It is a *Riemannian functional* in the sense that it's invariant under diffeomorphisms, but it's not scale-invariant. To resolve this issue, we consider

$$\mathcal{E}(h) = (\mathrm{vol}_h(M))^{\frac{2}{n}-1} \int_M R_h \, dV_h.$$

This is called the *normalized total scalar curvature* (or *normalized Einstein-Hilbert functional*) of M. Under conformal deformations, the first variation of $\mathcal{E}$ is

$$\mathcal{E}'(h) \cdot l = \frac{n-2}{2n} \mathrm{vol}_h(M)^{\frac{2}{n}-1} \int_M \langle R_h - \fint_M R_h \, dV_h, \, l \rangle_h \, dV_h.$$

It equals zero provided that (M, h) has constant scalar curvature. Assuming R_h is constant, we can simplify the full variation to

$$\mathcal{E}'(h) \cdot l = \mathrm{vol}_h(M)^{\frac{2}{n}-1} \int_M \langle \frac{1}{n} R_h - Ric_h, l \rangle_h \, dV_h.$$

Thus, a metric h is critical if and only if (M, h) is Einstein. In particular, the hyperbolic metric h_0 is a critical point for $\mathcal{E}$. Furthermore, since $\mathcal{E}$ is scale-invariant, from now on we may assume that for $h_t = h_0 + tl$,

$$\int_M \frac{d}{dt}|_{t=0}(\sqrt{\det_{h_0}(h_t)})\, dV_{h_0} = \frac{1}{2}\int_M \mathrm{tr}_{h_0} l\, dV_{h_0} = 0.$$

Taking this into account, we obtain the second variation of $\mathcal{E}$ at h_0 as follows.

$$\begin{aligned}\mathcal{E}''(h_0)(l,l) =& \mathrm{vol}_{h_0}(M)^{\frac{2}{n}-1}\int_M \frac{d^2}{dt^2}|_{t=0}R_{h_t} - 2(n-1)\frac{d^2}{dt^2}|_{t=0}(\sqrt{\det_{h_0}(h_t)}) \\ &+ 2\frac{d}{dt}|_{t=0}R_{h_t}\,\frac{d}{dt}|_{t=0}(\sqrt{\det_{h_0}(h_t)})\, dV_{h_0},\end{aligned} \tag{2.2.1}$$

Substituting the formulas

$$\frac{d^2}{dt^2}\sqrt{\det_{h_0}(h_t)} = \big(\frac{1}{4}(\mathrm{tr}_{h_t} l)^2 - \frac{1}{2}\mathrm{tr}_{h_t}(l^2)\big)\sqrt{\det_{h_0}(h_t)},$$

$$\frac{d}{dt}R_{h_t} = -\Delta_{h_t}(\mathrm{tr}_{h_t} l) + \delta^2_{h_t} l - \langle Ric_{h_t}, l\rangle_{h_t},$$

$$\frac{d}{dt}Ric_{h_t} = -\frac{1}{2}\Delta_{h_t} l + \frac{1}{2}Ric_{h_t}(l) + \frac{1}{2}l(Ric_{h_t}) - Rm_{h_t} * l - \delta^*_{h_t}(\delta_{h_t} l) - \frac{1}{2}\nabla^2_{h_t}(\mathrm{tr}_{h_t} l),$$

where $(Rm_{h_t} * l)_{ij} = R_{ikjm}l^{km}$, we obtain that

$$\begin{aligned}\mathcal{E}''(h_0)(l,l) =& \mathrm{vol}_{h_0}(M)^{\frac{2}{n}-1}\int_M \langle \frac{1}{2}\Delta_{h_0} l + Rm_{h_0} * l + \delta^*_{h_0}(\delta_{h_0} l) \\ &+ \frac{n-1}{2}(\mathrm{tr}_{h_0} l)h_0 - \frac{1}{2}(\Delta_{h_0}(\mathrm{tr}_{h_0} l))h_0 + (\delta^2_{h_0} l)h_0\,, l\rangle_{h_0}\, dV_{h_0}.\end{aligned}$$

According to Ebin's Slice Theorem (see [15]), for any $h \in \mathcal{M}$ lying in a small neighborhood of h_0, there exist $\phi \in \mathrm{Diff}(M)$, $f \in C^\infty(M)$, and a *transverse-traceless* tensor l_{TT}, i.e., $\delta_{h_0}(l_{TT}) = 0$ and $\mathrm{tr}_{h_0}(l_{TT}) = 0$, such that $\phi^* h = h_0 + fh_0 + l_{TT}$. Then we can simplify the second variation.

$$\mathcal{E}''(h_0)(fh_0, fh_0) = -\frac{(n-1)(n-2)}{2}\mathrm{vol}_{h_0}(M)^{\frac{2}{n}-1}\int_M \langle \Delta_{h_0} f - nf,\, f\rangle_{h_0}\, dV_{h_0}. \quad (2.2.2)$$

And we also get

$$\mathcal{E}''(h_0)(l_{TT}, l_{TT}) = \mathrm{vol}_{h_0}(M)^{\frac{2}{n}-1}\int_M \langle \frac{1}{2}\Delta_{h_0} l_{TT} + Rm_{h_0} * l_{TT}, l_{TT}\rangle_{h_0}\, dV_{h_0}. \quad (2.2.3)$$

Thus,

$$\begin{aligned}\mathcal{E}(h) &= \mathcal{E}(\phi^* h) = \mathcal{E}(h_0 + fh_0 + l_{TT}) \\ &= \mathcal{E}(h_0) + \mathcal{E}''(h_0)(fh_0, fh_0) + \mathcal{E}''(h_0)(l_{TT}, l_{TT}) + \text{higher order variations.}\end{aligned}$$

Taking the estimates of (2.2.2) and (2.2.3) into consideration, we can use this expansion to discuss the local behavior of $\mathcal{E}$. In particular, as discussed in [7], $\mathcal{E}$ reaches a local maximum at h_0, and there exists $C > 0$, so that for any metric h in a small neighborhood of h_0 in $\mathcal{M}$,

$$\mathcal{E}(h_0) - \mathcal{E}(h) \geq C\, d(h, h_0)^2, \quad \text{where } d(h, h_0) = \inf_{\phi \in \mathrm{Diff}(M)} |\phi^* h - h_0|_{H^1(M, h_0)}.$$

The inequality is sharp unless h and h_0 are isometric via some $\phi \in \mathrm{Diff}(M)$.

2.3 Equidistribution

Suppose that the hyperbolic manifold M has an odd dimension $n \geq 3$. According to the construction by Hamenstädt [22], for any small number $\epsilon > 0$, there is an essential surface Σ_ϵ in M which is sufficiently well-distributed and $(1+\epsilon)$-*quasigeodesic* (i.e. the geodesics on the surface with respect to intrinsic distance are $(1+\epsilon, \epsilon)$-quasigeodesics in M). Additionally, as discussed later in Section 5.1.2, Σ_ϵ determines an $(1+O(\epsilon))$-quasiconformal map on S^{n-1}_∞, and thus associated with an element in $S_\epsilon(M)$.

Furthermore, let $G(M,g,\epsilon)$ denote the subset of $S(M,g,\epsilon)$ consisting of homotopy classes of finite covers of Σ_ϵ that have genus at most g. It has cardinality comparable to g^{2g}. Moreover, let S_i denote the minimal representative of an element in $G(M,g_i,\frac{1}{i})$, then it is homotopic to an $(1+\frac{1}{i})$-quasigeodesic surface Σ_i. From Lemma 4.3 of [11], for any continuous function f on M, the unit Radon measure induced by integration over S_i satisfies

$$\lim_{i\to\infty} \frac{1}{\mathrm{area}_{h_0}(S_i)} \int_{S_i} f \, dA_{h_0} = \nu(f),$$

where the limiting measure ν is positive on any non-empty open set of M.

Notice that the measure ν is not necessarily identified with the unit Radon measure on M induced by integration over M, the latter measure is denoted by μ. However, in order to prove Theorem 1.2.3 and Theorem 1.2.7, we need to find a sequence of minimal surfaces whose Radon measures defined above converge to μ. To solve this problem, we introduce Labourie's construction [38] in Chapter 3. And we stress that both methods of Hamenstädt and Labourie require that M has an odd dimension.

CHAPTER 3

EQUIDISTRIBUTION PROPERTY

In this chapter, we extend the notations of laminations and associated properties for hyperbolic 3-manifolds (see Labourie [38] and Lowe-Neves [43]) to the higher, odd-dimensional case. The purpose of this section is to deduce the average area ratio formula in Lemma 3.3.1, as an important tool in the proofs of Theorem 1.2.3 and Theorem 1.2.7.

3.1 Laminations and Laminar Measures

Let $M = \mathbb{H}^n/\pi_1(M)$ be a closed hyperbolic manifold of dimension $n \geq 3$. And let $\mathcal{F}(\mathbb{H}^n, \epsilon)$ be the space of conformal minimal immersions $\Phi : \mathbb{H}^2 \to \mathbb{H}^n$, such that $\Phi(\partial_\infty \mathbb{H}^2)$ is an $(1+\epsilon)$-quasicircle. As discussed in Section 5.1.1, when ϵ is sufficiently small, $\Phi(\mathbb{H}^2)$ is a stable embedded disc in $\mathbb{H}^n$. The space $\mathcal{F}(\mathbb{H}^n, \epsilon)$ equips with the topology of uniform convergence on compact sets, and we take

$$\mathcal{F}(M, \epsilon) := \mathcal{F}(\mathbb{H}^n, \epsilon)/\pi_1(M)$$

with the quotient topology. The space $\mathcal{F}(M, \epsilon)$ together with the action of $\mathrm{PSL}(2, \mathbb{R})$ by pre-composition

$$\mathcal{R}_\gamma : \mathcal{F}(M, \epsilon) \to \mathcal{F}(M, \epsilon), \quad \mathcal{R}_\gamma(\phi) = \phi \circ \gamma^{-1}, \quad \forall \gamma \in \mathrm{PSL}(2, \mathbb{R}) \tag{3.1.1}$$

is called the *conformal minimal lamination of* M. A *laminar measure* on $\mathcal{F}(M, \epsilon)$ stands for a probability measure which is invariant under the $\mathrm{PSL}(2, \mathbb{R})$-action defined as above. The space $\mathcal{F}(M, \epsilon)$ is sequentially compact, but the space of laminar measures is not necessarily weakly compact. In light of that, we consider a continuous map from $\mathcal{F}(M, \epsilon)$ to the frame bundle $F(M)$ of M, the latter space is compact, so the space of probability measures on $F(M)$ is compact in weak-$*$ topology.

Firstly we define a map from $\mathcal{F}(M,\epsilon)$ to the 2-vector bundle $F_2(M)$ on M consisting of $(x,v_1,v_2) \in M \times S_xM \times S_xM$, where S_xM denotes the unit sphere in the tangent space to M at x. Let $\{e_1,e_2\}$ be an orthonormal basis of $\mathbb{H}^2$, and for any $\phi \in \mathcal{F}(M,\epsilon)$, let $\phi^*(h_0) = C_\phi^2 h_{\mathbb{H}^2}$, where $C_\phi^2 > 0$ denotes the conformal factor between the hyperbolic metric on $\mathbb{H}^2$ and the pull-back metric of h_0 by ϕ. Let $e_1(\phi) = \dfrac{d\phi(e_1)}{C_\phi}$ and $e_2(\phi) = \dfrac{d\phi(e_2)}{C_\phi}$. We define the following continuous map.

$$\Omega : \mathcal{F}(M,\epsilon) \to F_2(M), \quad \Omega(\phi) = (\phi(i), e_1(\phi), e_2(\phi)).$$

Furthermore, it induces a map from $\mathcal{F}(M,\epsilon)$ to the frame bundle $F(M)$ by parallel transport:

$$\overline{\Omega} : \mathcal{F}(M,\epsilon) \to F(M), \quad \overline{\Omega}(\phi) = (\phi(i), e_1(\phi), e_2(\phi), \cdots, e_n(\phi)).$$

And define the projection

$$P : F(M) \to F_2(M), \quad P(x, e_1, e_2, \cdots, e_n) = (x, e_1, e_2).$$

We consider the subspace $\mathcal{F}(\mathbb{H}^n, 0) \subset \mathcal{F}(\mathbb{H}^n, \epsilon)$, it contains isometric immersions $\phi_0 : \mathbb{H}^2 \to \mathbb{H}^n$ whose images are totally geodesic discs in $\mathbb{H}^n$. Conversely, each totally geodesic disc is uniquely determined by $\phi(i)$, and tangent vectors $e_1(\phi)$, $e_2(\phi)$. Let $\Omega_0 : \mathcal{F}(M,0) \to F_2(M)$ be the restriction of Ω to $\mathcal{F}(M,0)$, it's therefore a bijection. Using (3.1.1), We can define the $\mathrm{PSL}(2,\mathbb{R})$-action on $F(M)$ as follows.

$$R_\gamma : F(M) \to F(M), \quad R_\gamma(x) = \overline{\Omega} \circ \mathcal{R}_\gamma \circ \Omega_0^{-1} \circ P(x), \quad \forall \gamma \in \mathrm{PSL}(2,\mathbb{R}).$$

This definition coincides with the homogeneous action of $\mathrm{PSL}(2,\mathbb{R})$ on $F(M)$. Following the discussion of Lemma 3.2 of [43], we conclude the following result.

Proposition 3.1.1. *Given any sequence of laminar measures μ_i on $\mathcal{F}(M,\frac{1}{i})$, the sequence of induced measures $\overline{\Omega}_*\mu_i$ on $F(M)$ converges weakly to a probability measure ν, then ν is invariant under the homogeneous action of $PSL(2,\mathbb{R})$.*

Let $G < \mathrm{PSL}(2,\mathbb{R})$ be a Fuchsian subgroup, then $\mathbb{H}^2/G$ is a closed hyperbolic surface with genus ≥ 2 whose fundamental domain is represented by U. And let $\phi \in \mathcal{F}(M,\epsilon)$ equivariant with respect to a representation Π of G in $\pi_1(M) < SO(n,1)$. The image of $\phi(\mathbb{H}^2)$ in M is a closed minimal surface in M whose fundamental group is Π. We define a laminar measure associated with ϕ as follows.

$$\delta_\phi(f) = \frac{1}{vol(U)} \int_U f(\phi \circ \gamma) d\nu_0(\gamma), \quad \forall f \in C^0(\mathcal{F}(M,\epsilon)), \tag{3.1.2}$$

where ν_0 denotes the bi-invariant measure on $\mathrm{PSL}(2,\mathbb{R})$.

3.2 Equidistribution

In this section, we assume the dimension of M is odd. Adapting the methods of Proposition 6.1 of [43] and Theorem 5.7 in [38], we prove the following result.

Proposition 3.2.1. *For any $i \in \mathbb{N}$, there is a lamination ϕ_i in $\mathcal{F}(M,\frac{1}{i})$ equivariant with respect to a representation of a Fuchsian group $G_i < PSL(2,\mathbb{R})$ in $\pi_1(M)$, such that $\overline{\Omega}_*\delta_{\phi_i}$ converges to the Lebesgue measure μ_{Leb} on $F(M)$ as $i \to \infty$.*

Sketch of the proof. Let $\tilde{\mathcal{T}}$ be the space of tripods $\tilde{X} = (x_1, x_2, x_3)$, where $x_1, x_2, x_3 \in \partial_\infty \mathbb{H}^n$. Each element $\tilde{X}$ determines an ideal triangle $\Delta(\tilde{X})$ in $\mathbb{H}^n$. Let $b(\tilde{X})$ be the barycenter of $\Delta(\tilde{X})$. Denote by $(e_1(\tilde{X}), e_2(\tilde{X}))$ the orthonormal basis of $\Delta(\tilde{X})$, and denote by $(e_3(\tilde{X}, \cdots, e_n(\tilde{X}))$ the orthonormal basis of the normal bundle of $\Delta(\tilde{X})$ in

$\mathbb{H}^n$. Thus, each tripod $\tilde{X}$ determines a point $(b(\tilde{X}), e_1(\tilde{X}), \cdots, e_n(\tilde{X}))$ in $F(\mathbb{H}^n)$, which represents the frame bundle of $\mathbb{H}^n$.

Consider the closed manifold $M = \mathbb{H}^n/\pi_1(M)$. Let X be the corresponding point of $\tilde{X}$ in $\mathcal{T} := \tilde{\mathcal{T}}/\pi_1(M)$, and let $F(X)$ be the point in the frame bundle $F(M)$ corresponding to $(b(\tilde{X}), e_1(\tilde{X}), \cdots, e_n(\tilde{X}))$ in $F(\mathbb{H}^n)$.

A quintuple (X, Y, l_1, l_2, l_3) is called a *triconnected pair of tripods* (Definition 10.1.1 of [32]) if $X, Y \in \mathcal{T}$ and l_1, l_2, l_3 are three distinct homotopy classes of paths connecting X to Y. The space of triconnected pair of tripods is denoted by $\mathcal{TT}$. Let π^1 and π^2 be the forgetting maps from $\mathcal{TT}$ to $\mathcal{T}$

$$\pi^1 : (X, Y, l_1, l_2, l_3) \mapsto X, \quad \pi^2 : (X, Y, l_1, l_2, l_3) \mapsto Y.$$

Moreover, let $\overline{\pi^1}$ and $\overline{\pi^2}$ be the corresponding maps from $\mathcal{TT}$ to $F(M)$

$$\overline{\pi^1} : (X, Y, l_1, l_2, l_3) \mapsto F(X), \quad \overline{\pi^2} : (X, Y, l_1, l_2, l_3) \mapsto F(Y).$$

In addition, there is a weighted measure $\mu_{\epsilon,R}$ on $\mathcal{TT}$ as defined in Definition 11.2.3 of [32]. If (X, Y, l_1, l_2, l_3) is in the support of $\mu_{\epsilon,R}$, then the ideal triangles determined by X and Y can be glued to an (ϵ, R)-*almost closing pair of pants* (see Definition 9.1.1 of [32]). Moreover, it follows from the mixing property that for fixed ϵ, as $R \to \infty$, $\overline{\pi^1}_* \mu_{\epsilon,R}$ and $\overline{\pi^2}_* \mu_{\epsilon,R}$ both converge to the Lebesgue measure μ_{Leb} on $F(M)$.

Arguing like Theorem 5.7 of [38], we can choose a sequence $R_j \to \infty$ as $j \to \infty$, and a sequence of measures $\mu_{\frac{1}{j},R_j}$. Then we approximate each $\mu_{\frac{1}{j},R_j}$ by another weighted measure ν_j supported in finitely many pleated pair of pants $P_j^1, \cdots, P_j^{N_j}$, which can be glued together to get essential surfaces $\Sigma_j^1, \cdots, \Sigma_j^{M_j}$ in M. When j is sufficiently large and for each $1 \leq k \leq M_j$, Σ_j^k is $(1 + \frac{1}{j})$-quasigeodesic, and the projection from Σ_j^k to the unique minimal surface S_j^k homotopic to Σ_j^k is $(1 + \frac{1}{j})$-bi-Lipschitz and it has distance uniformly bounded by $O(\frac{1}{j})$. For this reason, we can further approximate ν_j by a weighted measure supported in $S_j^1 \cup \cdots \cup S_j^{M_j}$. S_j^k is obtained by a lamination $\phi_j^k \in \mathcal{F}(M, \frac{1}{j})$, in fact, it's the image of $\phi_j^k(\mathbb{H}^2)$ in M, and thus associated with the laminar measure $\delta_{\phi_j^k}$, we have the following lemma.

Lemma 3.2.2. *For any $j \in \mathbb{N}$, there exist a finite sequence of laminations $\phi_j^1, \cdots, \phi_j^{M_j}$ in $\mathcal{F}(M, \frac{1}{j})$, and $\theta_j^1, \cdots, \theta_j^{M_j} \in (0,1)$ with $\theta_j^1 + \cdots + \theta_j^{M_j} = 1$, such that each ϕ_j^k is equivariant with respect to a representation of a Fuchsian group in $\pi_1(M)$, and the laminar measure*

$$\mu_j = \sum_{k=1}^{M_j} \theta_j^k \delta_{\phi_j^k}$$

satisfies that $\overline{\Omega}_* \mu_j$ converges to the Lebesgue measure μ_{Leb} on $F(M)$ as $j \to \infty$.

Next, for $2 \leq l \leq n-1$, we define

$$\mathcal{P}_l := \{F(P) \subset F(M), \text{ where } P \text{ is a } l\text{-dimensional closed totally geodesic submanifold of } M\}.$$

Then $\mathcal{P} : \bigcup_{l=2}^{n-1} \mathcal{P}_l$ contains at most countably many candidates. Therefore, we can find a decreasing sequence of tubular neighborhoods $\{B_k\} \subset F(M)$, so that for any $k \in \mathbb{N}$, B_k covers $\mathcal{P}$ and it satisfies $\mu_{Leb}(B^k) < 2^{-2k-1}$ and $\mu_{Leb}(\partial B_k) = 0$. In consequence of previous lemma, after passing to a subsequence, we have $\overline{\Omega}_* \mu_j(B_k) < 2^{-2k}$. Additionally, as argued in Lemma 6.2 of [43], we can find a subsequence $\{j_i\}$, and $\phi_i \in \{\phi_{j_i}^1, \cdots, \phi_{j_i}^{M_{j_i}}\}$, such that $\overline{\Omega}_*(\delta_{\phi_i})(B_k) < 2^{-k}$.

As a result of Proposition 3.1.1, as $i \to \infty$, $\overline{\Omega}_* \delta_{\phi_i}$ converges weakly to a probability measure ν on $F(M)$. ν is invariant under the homogeneous action of $\mathrm{PSL}(2, \mathbb{R})$, and it satisfies that

$$\nu(B_k) < 2^{-k}. \tag{3.2.1}$$

To finish the proof, we need the following lemma.

Lemma 3.2.3. $\nu = \mu_{Leb}$.

Proof. According to the ergodic decomposition theorem ([26]), ν can be expressed by a linear combination of the ergodic measures for $\mathrm{PSL}(2,\mathbb{R})$-action on $F(M)$. Moreover, Ratner's measure classification theorem (see [53] or [59]) says that any ergodic $\mathrm{PSL}(2,\mathbb{R})$-invariant measure on $F(M)$ is either an invariant probability measure supported on a finite union of $\{P_k\} \subset \mathcal{P}$, or it is identical to μ_{Leb}. Thus, we can write ν as

$$\nu = a_1\mu_{Leb} + a_2\mu_{\mathcal{P}_2} + \cdots + a_{n-1}\mu_{\mathcal{P}_{n-1}},$$

where $a_1 + a_2 + \cdots + a_{n-1} = 1$ and $\mu_{\mathcal{P}_l}$ represents an ergodic measure supported on $\mathcal{P}_l$, $2 \le l \le n-1$. By (3.2.1), for all $k \in \mathbb{N}$,

$$a_2 + \cdots + a_{n-1} = a_2\mu_{\mathcal{P}_2}(B_k) + \cdots + a_{n-1}\mu_{\mathcal{P}_{n-1}}(B_k) \le \nu(B_k) < 2^{-k}.$$

So

$$a_1 = 1 - a_2 - \cdots - a_{n-1} > 1 - 2^{-k}, \quad \forall k \in \mathbb{N}.$$

We must have $a_1 = 1$, and therefore $\nu = \mu_{Leb}$. □

Proposition 3.2.1 follows immediately from the lemma. □

3.3 Average Area Ratio Formula

Lemma 3.3.1 (average area ratio formula)**.** *Let (N, g) be a closed Riemannian manifold that also has odd dimension n, and let F be a smooth map that takes (N, g) to (M, h_0). For $i \in \mathbb{N}$, we pick a lamination $\phi_i \in \mathcal{F}(M, \frac{1}{i})$ equivariant with respect to a representation of $G_i < PSL(2,\mathbb{R})$ in $\pi_1(M)$, and it satisfies Proposition 3.2.1. Let S_i be the image of*

$\phi_i(\mathbb{H}^2)$ *in* M. *Then we have*

$$Area_F(g/h_0) = \lim_{i\to\infty} \frac{area_g(F^{-1}(S_i))}{4\pi(g_i-1)}.$$

Proof. Recall that $|\Lambda^2 F|_g^{-1}$ is a function defined almost everywhere on $Gr_2(M)$. Since $|\Lambda^2 F|_g$ can be regarded as a smooth function on $F(M)$ by

$$|\Lambda^2 F|_g : F(M) \to \mathbb{R}, \quad (x, e_1, e_2 \cdots, e_n) \mapsto |\Lambda^2 F|_g(x, \mathrm{span}(e_1, e_2)),$$

based on the definition (2.1.1), $|\Lambda^2 F|_g^{-1}$ is also seen as a function defined almost everywhere on $F(M)$. Thus, Proposition 3.2.1 implies that

$$\mathrm{Area}_F(g/h_0) = \mu_{Leb}(|\Lambda^2 F|_g^{-1}) = \lim_{i\to\infty} \overline{\Omega}_* \delta_{\phi_i}(|\Lambda^2 F|_g^{-1}).$$

In light of the definition of laminar measure δ_{ϕ_i} in (3.1.2), we have

$$\overline{\Omega}_* \delta_{\phi_i}(|\Lambda^2 F|_g^{-1}) = \frac{1}{vol(U_i)} \int_{U_i} |\Lambda^2 F|_g^{-1} \circ \overline{\Omega}(\phi_i \circ \gamma) d\nu_0(\gamma),$$

where U_i is the fundamental domain of $\mathrm{PSL}(2,\mathbb{R})/G_i$. Set $x = \gamma(i)$. Since the hyperbolic surface $\mathbb{H}^2/G_i$ has area equal to $4\pi(g_i - 1)$, where $g_i \geq 2$ denotes the genus. The above expression also can be written as

$$\begin{aligned}
\overline{\Omega}_* \delta_{\phi_i}(|\Lambda^2 F|_g^{-1}) &= \frac{1}{4\pi(g_i-1)} \int_{\mathbb{H}^2/G_i} |\Lambda^2 F|_g^{-1}(\phi_i(x), (d\phi_i)_x T_x \mathbb{H}^2) dA_{h_{\mathbb{H}^2/G_i}}(x) \\
&= \frac{1}{4\pi(g_i-1)} \int_{S_i} \frac{|\Lambda^2 F|_g^{-1}(y, T_y S_i)}{C_i^2(\phi_i^{-1}(y))} dA_{h_0}(y),
\end{aligned}$$

where $C_i^2 > 0$ denotes the conformal factor between the hyperbolic metric on $\mathbb{H}^2/G_i$ and the pull-back metric of h_0 by ϕ_i, namely $\phi_i^*(h_0) = C_i^2 h_{\mathbb{H}^2/G_i}$. Since the Gaussian curvature on S_i has the form $-1 - \frac{1}{2}|A|^2(x)$, we have

$$1 \leq \frac{1}{C_i^2} \leq 1 + \frac{1}{2}|A|^2_{L^\infty(S_i)}.$$

On the other hand, the co-area formula yields that

$$\int_{S_i} |\Lambda^2 F|_g^{-1}(y, T_y S_i) dA_{h_0}(y) = \text{area}_g(F^{-1}(S_i)).$$

Combining these formulas, we have

$$\frac{\text{area}_g(F^{-1}(S_i))}{4\pi(g_i - 1)} \leq \overline{\Omega}_* \delta_{\phi_i}(|\Lambda^2 F|_g^{-1}) \leq (1 + \frac{1}{2}|A|^2_{L^\infty(S_i)}) \frac{\text{area}_g(F^{-1}(S_i))}{4\pi(g_i - 1)}.$$

Since $|A|^2_{L^\infty(S_i)} \to 0$ as $i \to \infty$ (see Section 5.1.1), the lemma follows immediately from the squeeze theorem. $\square$

CHAPTER 4

AVERAGE AREA RATIO

In this chapter, we assess the average area ratio of closed hyperbolic manifolds. First, we begin by considering the general case, including manifolds with any dimension greater than or equal to three.

4.1 General Dimensions

Throughout this section, we can choose the dimension of M to be any integer $n \geq 3$. The proof of Theorem 1.2.1 separates into two parts. If h is an arbitrary metric conformal to a metric $\bar{h}$ with constant scalar curvature, we compare their average area ratios in Theorem 4.1.1. And if h is a metric with constant scalar curvature different from h_0, we make use of the evaluations of normalized Einstein-Hilbert functional in [7].

4.1.1 Conformal Deformations

Firstly, given a metric h on M, we look at the conformal class of h. Since M admits a hyperbolic metric, every conformal class $[h]$ must be *scalar negative*, i.e., it has a metric with negative scalar curvature. And according to the Yamabe problem, after rescaling, there exists a unique metric $\bar{h} \in [h]$ with constant scalar curvature $c < 0$. In Theorem 1.2.1, we assume that $c \geq -n(n-1)$. Set $\bar{h}' = \frac{c}{-n(n-1)}\bar{h}$, so $R_{\bar{h}'} \equiv -n(n-1)$, and for any surface or 2-dimensional subset S in M, we have

$$\text{area}_{\bar{h}'}(S) = \frac{c}{-n(n-1)}\text{area}_{\bar{h}}(S) \leq \text{area}_{\bar{h}}(S).$$

Therefore, to prove the theorem, we may assume that $R_{\bar{h}} \equiv -n(n-1)$.

Theorem 4.1.1. *Suppose the scalar curvature $R_h \geq -n(n-1)$, then we have*

$$Area_{Id}(h/h_0) \geq Area_{Id}(\bar{h}/h_0). \tag{4.1.1}$$

Furthermore, for any surface subgroup $\Pi \in S_\epsilon(M)$,

$$area_h(\Pi) \geq area_{\bar{h}}(\Pi), \tag{4.1.2}$$

and as an immediate result,

$$E(h) \leq E(\bar{h}). \tag{4.1.3}$$

Each of the above equalities holds if and only if $h = \bar{h}$.

Proof. Set $h = e^{2\phi}\bar{h}$. The conformal factor ϕ satisfies that

$$\begin{aligned} e^{2\phi}R_h &= R_{\bar{h}} - 2(n-1)\Delta_{\bar{h}}\phi - (n-2)(n-1)|d\phi|^2_{\bar{h}} \\ &= -n(n-1) - 2(n-1)\Delta_{\bar{h}}\phi - (n-2)(n-1)|d\phi|^2_{\bar{h}}. \end{aligned}$$

Let $\phi_{min} := \min\limits_{x \in M} \phi(x)$, we obtain $\Delta_{\bar{h}}\phi_{min} \geq 0$, and it yields that

$$-n(n-1)e^{2\phi_{min}} \leq e^{2\phi_{min}}R_h \leq -n(n-1). \tag{4.1.4}$$

Thus, for any subset D_δ of a totally geodesic disc in $\mathbb{H}^n$ with hyperbolic area equal to δ, we have

$$e^{2\phi_{min}} \geq 1 \quad \Longrightarrow \quad \text{area}_h(D_\delta) = \int_{D_\delta} e^{2\phi}\, dA_{\bar{h}} \geq \text{area}_{\bar{h}}(D_\delta).$$

The inequality (4.1.1) follows from the definition of the average area ratio (2.1.2).

In addition, it also shows that for any surface $S \in M$,

$$\text{area}_h(S) = \int_S e^{2\phi}\, dA_{\bar{h}} \geq \text{area}_{\bar{h}}(S).$$

Thus we conclude (4.1.2), and the comparison of entropy (4.1.3) is a direct corollary of the definition (1.1.1).

Moreover, if any equality in the theorem holds, the inequality 4.1.4 implies that $R_h \equiv -n(n-1)$, due to the uniqueness of the solution to Yamabe problem, h must be identical to $\bar{h}$. $\square$

4.1.2 Definition of Functional $\mathcal{A}$

Let $\mathcal{M}$ be the space of all Riemannian metrics on M, and let $\mathcal{M}_R$ be the subset consisting of metrics with constant scalar curvature $-n(n-1)$. From the previous section, it remains to consider metrics in $\mathcal{M}_R$.

Define a functional $\mathcal{A}$ from the space of Riemannian metrics on M to $\mathbb{R}$ as follows.

$$\begin{aligned}
\mathcal{A}(h) &= \int_{(x,P)\in Gr_2 M} R_h(x)\, |\Lambda^2 \mathrm{Id}|_h^{-1}(x,P)\, d\mu_{h_0} \\
&= \lim_{\delta\to 0} \int_{(x,P)\in Gr_2 M} R_h(x)\, \frac{\text{area}_h(D_\delta(P))}{\delta}\, d\mu_{h_0} \\
&= \lim_{\delta\to 0} \int_{(x,P)\in Gr_2 M} R_h(x)\, \fint_{D_\delta(P)} \sqrt{\det_{h_0}(h|_{D_\delta(P)})}\, dA_{h_0}\, d\mu_{h_0},
\end{aligned}$$

where $D_\delta(P)$ is a subset of the totally geodesic disc in $\mathbb{H}^n$ that is tangential to P at x, and $D_\delta(P)$ has hyperbolic area equal to δ, μ_{h_0} is the unit volume measure on $Gr_2 M$

with respect to the metric induced by h_0. Notice that the metric $h \in \mathcal{M}_R$ satisfies

$$\begin{aligned}\mathcal{A}(h) &= -n(n-1) \int_{(x,P)\in Gr_2M} |\Lambda^2\mathrm{Id}|_h^{-1}(x,P)\, d\mu_{h_0} \\ &= \mathcal{A}(h_0) \int_{(x,P)\in Gr_2M} |\Lambda^2\mathrm{Id}|_h^{-1}(x,P)\, d\mu_{h_0}.\end{aligned}$$

Therefore, from the definition of the average area ratio (2.1.2), to deduce that if h is in a small neighborhood of h_0 in $\mathcal{M}_R$, then

$$\mathrm{Area}_{\mathrm{Id}}(h/h_0) = \int_{(x,P)\in Gr_2M} |\Lambda^2\mathrm{Id}|_h^{-1}(x,P)\, d\mu_{h_0} \geq 1,$$

we only need to show that $\mathcal{A}$ attains a local maximum at h_0.

4.1.3 Proof of Theorem 1.2.1

To see this, we'll discuss the first and second variations of $\mathcal{A}$ in detail. Before start, we notice that the functional $\mathcal{A}$ is scale-invariant, so it suffices to assume $\mathrm{vol}_{h_0}(M) = 1$, and the symmetric tensor $l = h - h_0$ satisfies $\int_M \mathrm{tr}_{h_0} l\, dV_{h_0} = 0$. Let $h_t = h_0 + tl$, we rewrite $\mathcal{A}(h_t)$ as

$$\mathcal{A}(h_t) = \int_{x\in M} R_{h_t}(x)\, a_{h_t}(x)\, dV_{h_0},$$

where

$$a_{h_t}(x) = \lim_{\delta\to 0} \fint_{P\in Gr_2M_x} \fint_{D_\delta(P)} \sqrt{\det_{h_0}(h_t|_{D_\delta(P)})}\, dV_{h_0} d\nu_{h_0}.$$

Then we have

$$\frac{d}{dt}\mathcal{A}(h_t) = \int_M \frac{d}{dt}R_{h_t}\, a_{h_t} + R_{h_t}\frac{d}{dt}a_{h_t}\, dV_{h_0}. \tag{4.1.5}$$

In addition,

$$\frac{d}{dt}R_{h_t} = -\Delta_{h_t}(\mathrm{tr}_{h_t} l) + \delta_{h_t}^2 l - \langle Ric_{h_t}, l\rangle_{h_t}, \tag{4.1.6}$$

where Δ_{h_t} is the rough Laplacian with negative eigenvalues, and $\delta^2_{h_t}$ is the double divergence operator. In particular, when $t = 0$, applying the Stokes' theorem, we have

$$\int_M \frac{d}{dt}|_{t=0}\, R_{h_t}\, a_{h_0}\, dV_{h_0} = \int_M -\langle Ric_{h_0}, l\rangle_{h_0}\, dV_{h_0} = (n-1)\int_M \mathrm{tr}_{h_0} l\, dV_{h_0}.$$

On the other hand, since

$$\frac{d}{dt}\sqrt{\det_{h_0}(h_t|_{D_\delta(P)})} = \frac{1}{2}\mathrm{tr}_{h_t} l|_{D_\delta(P)}\sqrt{\det_{h_0}(h_t|_{D_\delta(P)})},$$

let $\lambda_1, \cdots, \lambda_n$ be the eigenvalues of the matrix l at point $x \in M$ with respect to h_0, we obtain by computation that

$$\begin{aligned} \frac{d}{dt}|_{t=0}\, a_{h_t} &= \lim_{\delta\to 0}\frac{1}{2}\fint_{P\in Gr_2 M_x}\fint_{D_\delta(P)} \mathrm{tr}_{h_0} l|_{D_\delta(P)}\, dA_{h_0} d\nu_{h_0} \\ &= \frac{1}{2}\frac{\sum_{i<j}\lambda_i + \lambda_j}{\binom{n}{2}} = \frac{1}{2}\frac{(n-1)\sum_{i=1}^n \lambda_i}{\binom{n}{2}} \\ &= \frac{1}{n}\mathrm{tr}_{h_0} l. \end{aligned}$$

It follows that for any symmetric 2-tensor l,

$$\mathcal{A}'(h_0)\cdot l = (n-1)\int_M \mathrm{tr}_{h_0} l\, dV_{h_0} - n(n-1)\frac{1}{n}\int_M \mathrm{tr}_{h_0} l\, dV_{h_0} = 0,$$

thus h_0 is a critical point of $\mathcal{A}$.

Now we proceed to compute the second variation at $t = 0$. Note that $\mathcal{A}$ is an analogue of the normalized total scalar curvature $\mathcal{E}$, it's easier to compare their second variations using the computation in Section 2.2. Based on (4.1.5) and (2.2.1),

$$\begin{aligned}
&\mathcal{A}''(h_0)(l,l) - \mathcal{E}''(h_0)(l,l) \qquad (4.1.7)\\
=&\int_M -n(n-1)\frac{d^2}{dt^2}|_{t=0}\, a_{h_t} + 2(n-1)\frac{d^2}{dt^2}|_{t=0}(\sqrt{\det_{h_0}(h_t)})\, dV_{h_0}\\
&+\int_M 2\frac{d}{dt}|_{t=0} R_{h_t}(\frac{d}{dt}|_{t=0}(a_{h_t} - \sqrt{\det_{h_0}(h_t)})\, dV_{h_0}\\
=&\int_M -n(n-1)\frac{d^2}{dt^2}|_{t=0}\, a_{h_t} + 2(n-1)\big(\frac{1}{4}(\mathrm{tr}_{h_0} l)^2 - \frac{1}{2}\mathrm{tr}_{h_0}(l^2)\big)\, dV_{h_0}\\
&+2\int_M \big(-\Delta_{h_0}(\mathrm{tr}_{h_0} l) + \delta^2_{h_0} l + (n-1)\mathrm{tr}_{h_0} l\big)(\frac{1}{n} - \frac{1}{2})\mathrm{tr}_{h_0} l\, dV_{h_0}.
\end{aligned}$$

Next, using

$$\frac{d^2}{dt^2}\sqrt{\det_{h_0}(h_t|_{D_\delta(P)})} = \big(\frac{1}{4}(\mathrm{tr}_{h_t} l|_{D_\delta(P)})^2 - \frac{1}{2}\mathrm{tr}_{h_t}(l|^2_{D_\delta(P)})\big)\sqrt{\det_{h_0}(h_t|_{D_\delta(P)})},$$

we estimate the first term on the right-hand side of (4.1.7).

$$\begin{aligned}
&-n(n-1)\int_M \frac{d^2}{dt^2}|_{t=0}\, a_{h_t}\, dV_{h_0} \qquad (4.1.8)\\
=&-n(n-1)\lim_{\delta\to 0}\int_{x\in M} ⨍_{P\in Gr_2 M_x} ⨍_{D_\delta(P)} \frac{1}{4}(\mathrm{tr}_{h_0} l|_{D_\delta(P)})^2\\
&-\frac{1}{2}\mathrm{tr}_{h_0}(l|^2_{D_\delta(P)})\, dA_{h_0} d\nu_{h_0} dV_{h_0}\\
=&-n(n-1)\int_M \frac{\sum_{i<j}\frac{1}{4}(\lambda_i+\lambda_j)^2 - \frac{1}{2}(\lambda_i^2+\lambda_j^2)}{\binom{n}{2}}\, dV_{h_0}\\
=&-n(n-1)\int_M \frac{\frac{1}{4}(\sum_{i=1}^n \lambda_i)^2 - \frac{n}{4}\sum_{i=1}^n \lambda_i^2}{\binom{n}{2}}\, dV_{h_0}\\
=&\int_M -\frac{1}{2}(\mathrm{tr}_{h_0} l)^2 + \frac{n}{2}\mathrm{tr}_{h_0}(l^2)\, dV_{h_0}.
\end{aligned}$$

Combining (4.1.7) and (4.1.8), we have

$$\begin{aligned}\mathcal{A}''(h_0)(l,l) =&\mathcal{E}''(h_0)(l,l) - \int_M \frac{n-2}{2}\mathrm{tr}_{h_0}(l^2)\, dV_{h_0} \\ &- \int_M \frac{(n-2)^2}{2n}(\mathrm{tr}_{h_0} l)^2 + \frac{n-2}{n}|\nabla_{h_0}(\mathrm{tr}_{h_0} l)|^2 + \frac{n-2}{n}\delta^2_{h_0} l\, \mathrm{tr}_{h_0} l\, dV_{h_0}.\end{aligned} \tag{4.1.9}$$

To simplify this quadratic form, we decompose l into three parts. Applying the decomposition of space of symmetric tensors for a compact Einstein manifold other than the standard sphere (Theorem 4.60 of [6]), we have

$$T_{h_0}\mathcal{M} = C^\infty(M)\cdot h_0 \oplus T_{h_0}(\mathrm{Diff}(M)(h_0)) \oplus TT_{h_0},$$

where $C^\infty(M)\cdot h_0$ represents the conformal deformations of h_0, $\mathrm{Diff}(M)(h_0)$ is the action of the diffeomorphism group on h_0, and $TT_{h_0} = \ker \mathrm{tr}_{h_0} \cap \ker \delta_{h_0}$ stands for the set of transverse-traceless tensors. Let $h \in \mathcal{M}_R$, and let $l = h - h_0$, it decomposes into $l = fh_0 + l_D + l_{TT}$, where $fh_0 \in C^\infty(M)\cdot h_0$, $l_D \in T_{h_0}(\mathrm{Diff}(M)(h_0))$, and $l_{TT} \in TT_{h_0}$. The second variation of $\mathcal{A}$ has the form

$$\begin{aligned}&\mathcal{A}''(h_0)(l,l)\\ =&\mathcal{A}''(h_0)(l_D + l_{TT}, l_D + l_{TT}) + \mathcal{A}''(h_0)(fh_0, l_D) + \mathcal{A}''(h_0)(l_D, fh_0)\\ &+ \mathcal{A}''(h_0)(fh_0, l_{TT}) + \mathcal{A}''(h_0)(l_{TT}, fh_0) + \mathcal{A}''(h_0)(fh_0, fh_0)\\ \leq&\mathcal{A}''(h_0)(l_D + l_{TT}, l_D + l_{TT}) + C_0|fh_0|_{H^1(M,h_0)}|l|_{H^1(M,h_0)}.\end{aligned}$$

By Theorem 4.1.1, to check that $\mathcal{A}$ reaches a local maximum at h_0 on $\mathcal{M}_R$, it remains to analyze the sign of $\mathcal{A}''(h_0)(l_D + l_{TT}, l_D + l_{TT})$ and estimate $|fh_0|_{H^1(M,h_0)}$. To see these, we prove the following lemmas.

Lemma 4.1.2. *There exists a constant $C > 0$, such that in the decomposition of l,*

$l_D \in T_{h_0}(\mathit{Diff}(M)(h_0))$ *and* $l_{TT} \in TT_{h_0}$ *satisfy*

$$\mathcal{A}''(h_0)((l_D + l_{TT}, l_D + l_{TT}) \leq -C(|l_{TT}|^2_{H^1(M,h_0)} + |X|^2_{L^2(M,h_0)}),$$

where $\mathcal{L}_X h_0 = l_D$.

Lemma 4.1.3. *There exists a constant* $c > 0$*, such that the following statement is true. For any* $\epsilon > 0$*, we can find a* C^2*-neighborhood* $\mathcal{U}_{\epsilon,R}$ *of* h_0 *on* $\mathcal{M}_R$*, so that for any* $h \in \mathcal{U}_{\epsilon,R}$*,* $fh_0 \in C^\infty(M) \cdot h_0$ *in the decomposition of* $l = h - h_0$ *satisfies*

$$|fh_0|_{H^1(M,h_0)} \leq c\,\epsilon|l|_{H^1(M,h_0)}.$$

Proof of Lemma 4.1.2.

$$\begin{aligned}
&\mathcal{A}''(h_0)((l_D + l_{TT}, l_D + l_{TT}) \\
=&\mathcal{A}''(h_0)(l_{TT}, l_{TT}) + \mathcal{A}''(h_0)((l_D, l_D) + \mathcal{A}''(h_0)(l_D, l_{TT}) + \mathcal{A}''(h_0)(l_{TT}, l_D).
\end{aligned} \tag{4.1.10}$$

To find an upper bound of the first term, we use the estimate in Lemma 2.9 of [7]. There exists a constant $C_1 > 0$, such that

$$\mathcal{E}''(h_0)(l_{TT}, l_{TT}) = \int_M \langle \frac{1}{2}\Delta_{h_0} l_{TT} + Rm_{h_0} * l_{TT}, l_{TT}\rangle_{h_0}\, dV_{h_0} \leq -C_1|l_{TT}|^2_{H^1(M,h_0)}.$$

And thus, substituting the above inequality and $\mathrm{tr}_{h_0} l_{TT} = \delta_{h_0} l_{TT} = 0$ into (4.1.9), we obtain

$$\begin{aligned}
\mathcal{A}''(h_0)(l_{TT}, l_{TT}) &= \mathcal{E}''(h_0)(l_{TT}, l_{TT}) - \frac{n-2}{2}\int_M \mathrm{tr}_{h_0}(l_{TT}^2)\, dV_{h_0} \\
&\leq -C_1|l_{TT}|^2_{H^1(M,h_0)}.
\end{aligned} \tag{4.1.11}$$

To deal with the remaining terms of (4.1.10), we apply the diffeomorphism invariance property of $\mathcal{E}$, which says

$$\mathcal{E}''(h_0)(l_D, \cdot) = \mathcal{E}''(h_0)(\cdot, l_D) = 0. \tag{4.1.12}$$

Moreover, for any $\phi \in \mathrm{Diff}(M)$,

$$R_{\phi^* h_0} \equiv R_{h_0} \equiv -n(n-1) \quad \Longrightarrow \quad R'_{h_0} \cdot l_D = 0. \tag{4.1.13}$$

Therefore, the second variations comparison (4.1.7), in company with (4.1.8), says that

$$\mathcal{A}''(h_0)((l_D, l_D) = \frac{n-2}{2} \int_M (\mathrm{tr}_{h_0} l_D)^2 - \mathrm{tr}_{h_0}(l_D^2)\, dV_{h_0}. \tag{4.1.14}$$

Since $l_D \in T_{h_0}(\mathrm{Diff}(M)(h_0))$, it can be expressed by the Lie derivative of the metric h_0 in the direction X. And using Helmholtz-Hodge decomposition, X decomposes further into $X = \nabla_{h_0} r + Y$, where r is a scalar function, and Y is a vector field with $\mathrm{div}_{h_0} Y = 0$. By computation,

$$\mathrm{tr}_{h_0} l_D = 2\Delta_{h_0} r + 2\mathrm{div}_{h_0} Y = 2\Delta_{h_0} r. \tag{4.1.15}$$

In addition,

$$\begin{aligned} |l_D|^2_{L^2(M,h_0)} =& |2\nabla^2_{h_0} r + \sum_{i,j} (\nabla_i Y_j + \nabla_j Y_i)|^2_{L^2(M,h_0)} \\ =& 4|\nabla^2_{h_0} r|^2_{L^2(M,h_0)} + |\sum_{i,j} (\nabla_i Y_j + \nabla_j Y_i)|^2_{L^2(M,h_0)} \\ &+ 4\langle \nabla^2_{h_0} r, \sum_{i,j} (\nabla_i Y_j + \nabla_j Y_i)\rangle_{L^2(M,h_0)}. \end{aligned} \tag{4.1.16}$$

where the second term is in the form

$$
\begin{aligned}
&|\sum_{i,j}(\nabla_i Y_j+\nabla_j Y_i)|^2_{L^2(M,h_0)} \qquad (4.1.17)\\
=&\int_M 4\sum_i(\nabla_i Y_i)^2+\sum_{i\neq j}(\nabla_i Y_j)^2+(\nabla_j Y_i)^2+2\nabla_i Y_j\nabla_j Y_i\, dV_{h_0}\\
=&\int_M 4(\mathrm{div}_{h_0}Y)^2+\sum_{i\neq j}-2\nabla_i Y_i\nabla_j Y_j+(\nabla_i Y_j)^2+(\nabla_j Y_i)^2\\
&-2\nabla_i Y_i\nabla_j Y_j+2\nabla_i Y_j\nabla_j Y_i\, dV_{h_0}\\
=&\int_M \sum_{i\neq j} Y_j\nabla_j\nabla_i Y_i+Y_i\nabla_i\nabla_j Y_j+(\nabla_i Y_j)^2+(\nabla_j Y_i)^2\\
&+Y_i\nabla_i\nabla_j Y_j-Y_i\nabla_j\nabla_i Y_j+Y_j\nabla_j\nabla_i Y_i-Y_j\nabla_i\nabla_j Y_i\, dV_{h_0}\\
=&\int_M \sum_{i\neq j} Y_j(\nabla_i\nabla_j Y_i-R^i_{ijj}Y_j)+Y_i(\nabla_j\nabla_i Y_j-R^j_{jii}Y_i)+(\nabla_i Y_j)^2+(\nabla_j Y_i)^2\\
&+Y_i\nabla_i\nabla_j Y_j-Y_i(\nabla_i\nabla_j Y_j+R^j_{jii}Y_i)+Y_j\nabla_j\nabla_i Y_i-Y_j(\nabla_j\nabla_i Y_i+R^i_{ijj}Y_j)\, dV_{h_0}\\
=&\int_M \sum_{i\neq j} Y_i^2+Y_j^2-2\nabla_j Y_i\nabla_i Y_j+(\nabla_i Y_j)^2+(\nabla_j Y_i)^2+Y_i^2+Y_j^2\, dV_{h_0}\\
=&4(n-1)|Y|^2_{L^2(M,h_0)}+|\sum_{i,j}(\nabla_i Y_j-\nabla_j Y_i)|^2_{L^2(M,h_0)},
\end{aligned}
$$

and the last term of (4.1.16) vanishes, since

$$\begin{aligned}
&4\langle\nabla^2_{h_0}r,\sum_{i,j}(\nabla_iY_j+\nabla_jY_i)\rangle_{L^2(M,h_0)} \qquad (4.1.18)\\
=&4\int_M 2\sum_i \nabla_i\nabla_i r\nabla_iY_i+\sum_{i\neq j}\nabla_i\nabla_j r(\nabla_iY_j+\nabla_jY_i)\,dV_{h_0}\\
=&4\int_M 2\sum_i \nabla_i\nabla_i r\nabla_iY_i-\sum_{i\neq j}(\nabla_i r\nabla_j\nabla_iY_j+\nabla_j r\nabla_i\nabla_jY_i)\,dV_{h_0}\\
=&4\int_M 2\sum_i \nabla_i\nabla_i r\nabla_iY_i-\sum_{i\neq j}\big(\nabla_i r(\nabla_i\nabla_jY_j+R^j_{jii}Y_i)+\nabla_j r(\nabla_j\nabla_iY_i+R^i_{ijj}Y_j)\big)\,dV_{h_0}\\
=&4\int_M 2\Delta_{h_0}r\,\mathrm{div}_{h_0}Y-\sum_{i\neq j}(\nabla_i\nabla_i r\nabla_jY_j+\nabla_j\nabla_j r\nabla_iY_i)\\
&+\sum_{i\neq j}(\nabla_i\nabla_i r\nabla_jY_j+\nabla_j\nabla_j r\nabla_iY_i)+\sum_{i\neq j}\nabla_i rY_i+\nabla_j rY_j\,dV_{h_0}\\
=&8(n-1)\langle\nabla_{h_0}r,Y\rangle_{L^2(M,h_0)}=0.
\end{aligned}$$

Substituting (4.1.15)-(4.1.18) into (4.1.14), then applying Bochner's formula, we obtain

$$\begin{aligned}
&\mathcal{A}''(h_0)(l_D,l_D) \qquad (4.1.19)\\
=&\frac{n-2}{2}\Big(\int_M 4(\Delta_{h_0}r)^2-4|\nabla^2_{h_0}r|^2\,dV_{h_0}\\
&-4(n-1)|Y|^2_{L^2(M,h_0)}-|\sum_{i,j}(\nabla_iY_j-\nabla_jY_i)|^2_{L^2(M,h_0)}\Big)\\
=&\frac{n-2}{2}\Big(\int_M 4Ric(\nabla_{h_0}r,\nabla_{h_0}r)\,dV_{h_0}-4(n-1)|Y|^2_{L^2(M,h_0)}\\
&-|\sum_{i,j}(\nabla_iY_j-\nabla_jY_i)|^2_{L^2(M,h_0)}\Big)\\
\leq&-2(n-1)(n-2)|X|^2_{L^2(M,h_0)}.
\end{aligned}$$

Furthermore, after eliminating some terms using (4.1.12), (4.1.13) and $tr_{h_0}l_{TT}=0$, we get

$$\mathcal{A}''(h_0)(l_D, l_{TT}) = -\frac{n-2}{2}\int_M \langle l_D, l_{TT}\rangle_{h_0}\, dV_{h_0} = 0, \tag{4.1.20}$$

it vanishes because of the L^2-orthogonality between $T_{h_0}(\mathrm{Diff}(M)(h_0))$ and TT_{h_0}. Similarly, the traceless-transverse property of l_{TT} simplifies (4.1.9 to

$$\mathcal{A}''(h_0)(l_{TT}, l_D) = -\frac{n-2}{2}\int_M \langle l_{TT}, l_D\rangle_{h_0}\, dV_{h_0} = 0. \tag{4.1.21}$$

Substituting (4.1.11), (4.1.19), (4.1.20) and (4.1.21) into (4.1.10), we complete the proof.

□

Proof of Lemma 4.1.3. Since $h \in \mathcal{M}_R$, we have

$$0 = R_h - R_{h_0} = \int_0^1 R'_{h_0+tl} \cdot l\, dt,$$

it leads to

$$\langle R'_{h_0} \cdot l, \mathrm{tr}_{h_0}(fh_0)\rangle_{L^2(M,h_0)} = -\int_0^1 \langle (R'_{h_0+tl} - R'_{h_0}) \cdot l, \mathrm{tr}_{h_0}(fh_0)\rangle_{L^2(M,h_0)}\, dt. \tag{4.1.22}$$

On the left-hand side, we have $R'_{h_0} \cdot l = R'_{h_0} \cdot (fh_0)$ as analyzed earlier, and it follows from (4.1.6) that

$$\begin{aligned}
&\langle R'_{h_0} \cdot l, \mathrm{tr}_{h_0}(fh_0)\rangle_{h_0} \qquad (4.1.23)\\
=&\int_M \langle -\Delta_{h_0}(\mathrm{tr}_{h_0}(fh_0)) + \delta^2_{h_0}(fh_0) + (n-1)\mathrm{tr}_{h_0}(fh_0), \mathrm{tr}_{h_0}(fh_0)\rangle_{h_0}\, dV_{h_0}\\
=&\int_M n(n-1)|\nabla_{h_0} f|^2 + n^2(n-1)|f|^2\, dV_{h_0}\\
\geq&(n-1)|fh_0|^2_{H^1(M,h_0)}.
\end{aligned}$$

On the other hand, the right-hand side of (4.1.22) can be estimated using the continuity of $h \to \mathrm{tr}_h l$, $h \to \langle \cdot, \cdot \rangle_{L^2(M,h)}$, $h \to \nabla_h l$, and $h \to \delta_h l$. For any $\epsilon > 0$, after shrinking the neighborhood of h_0 in $\mathcal{M}_R$, we have

$$-\int_0^1 \langle (R'_{h_0+tl} - R'_{h_0}) \cdot l, \mathrm{tr}_{h_0}(fh_0) \rangle_{L^2(M,h_0)}\, dt \le \epsilon |h - h_0|_{C^2} |l|_{H^1(M,h_0)} |fh_0|_{H^1(M,h_0)}. \tag{4.1.24}$$

Therefore, taken (4.1.23) and (4.1.24) into account, (4.1.22) leads to the result.

□

To end this section, we discuss the equality condition of Theorem 1.2.1. There exists $t \in (0,1)$, such that

$$\mathcal{A}(h) = \mathcal{A}(h_0) + \frac{\mathcal{A}(h_0)''(l,l)}{2} + \frac{\mathcal{A}'''(h_0+tl)(l,l,l)}{6}.$$

Following the same procedure to compute $\mathcal{A}'''(h_0)$ and discuss the continuity near h_0, we can see that $\mathcal{A}'''(h_0+tl)(l,l,l) = O(|l|^3_{H^1(M,h_0)})$. Thus, from the above lemmas, there exists $C' > 0$,

$$\mathcal{A}(h) \le \mathcal{A}(h_0) - C'(|fh_0 + l_{TT}|^2_{H^1(M,h_0)} + |X|^2_{L^2(M,h_0)}) + O(|h-h_0|^3_{H^1(M,h_0)}).$$

From the computation of (4.1.19) and its derivative, we can see that as $h \to h_0$, $|X|_{L^2(M,h_0)}$ and $|l_D|_{H^1(M,h_0)}$ decay at the same rate, so the norm $|fh_0 + l_{TT}|^2_{H^1(M,h_0)} + |X|^2_{L^2(M,h_0)} \sim C''|h-h_0|^2_{H^1(M,h_0)}$. Consequently, the following expansion holds for metric h in a small neighborhood of h_0 with $R_h \ge -n(n-1)$.

$$\mathrm{Area}_{\mathrm{Id}}(h/h_0) \ge 1 + C'''(|fh_0 + l_{TT}|^2_{H^1(M,h_0)} + |X|^2_{L^2(M,h_0)}) + O(|h-h_0|^3_{H^1(M,h_0)}).$$

where $C''' > 0$. Note that $\mathrm{Area}_{\mathrm{Id}}(h/h_0) = 1$ requires that $fh_0 = l_{TT} = X = 0$, and thus

$h = h_0$.

4.2 Dimension Three

In this section, we discuss the proof of Theorem 1.2.3. First of all, the fact that $\deg F = 1$ follows from Corollary 0.3 of [67] and the geometrization theorem for 3-manifolds. Next, it's easy to see, the induced map $F_* : \pi_1(N) \to \pi_1(M)$ is surjective, since otherwise, it factors through a d-fold covering space of M with $d > 1$, and thus $\deg F \geq d > 1$, violating the degree one observation. In addition, $\pi_1(M)$ is a Hopfian group (for example, see 15.13 of [27]), so the surjectivity of F_* can be upgraded to be an isomorphism, which makes F a homotopy equivalence between N and M due to Whitehead's theorem. Furthermore, the Mostow rigidity theorem indicates that F is homotopic to an isometry. For this reason, we can simplify the conditions of Theorem 1.2.3 as follows.

4.2.1 A Simpler Version of Theorem 1.2.3

Theorem 1.4'. *Let (M, h_0) be a closed hyperbolic 3-manifold. There exists a small neighborhood $\mathcal{U}$ of h_0 in the C^2-topology, such that for all Riemannian metric $h \in \mathcal{U}$ with $R_h \geq -6$, and for any smooth map $F : (M, h) \to (M, h_0)$ with positive degree, it has $\deg F = 1$ and it is homotopic to the identity, we have*

$$Area_F(h/h_0) \geq 1.$$

Moreover, the equality holds if and only if F is an isometry between h and h_0.

Let S_i be the minimal surface in M with respect to h_0 defined in Lemma 3.3.1. The inverse $F^{-1}(S_i)$ is also a closed surface in M, but note that $F^{-1}(S_i)$ is not necessarily homotopic to S_i. In fact, we can only find the following relation of their genus. The

Gromov norms of S_i and $F^{-1}(S_i)$ satisfies that

$$|\deg F|||S_i|| \leq ||F^{-1}(S_i)||.$$

Here $\deg F = 1$. And for any closed surface S with genus $g(S)$, $||S|| = \frac{4\pi(g(S)-1)}{v_2}$, where v_2 is a fixed number representing the supreme area of geodesic 2-simplices in $\mathbb{H}^2$. As an immediate result, we have $g(F^{-1}(S_i)) \geq g(S_i)$.

To compare the areas of surfaces with respect to the induced metric of h in different homotopic classes, we hope to find a global area-minimizing surface. In general, the existence and the topology of such a surface are complicated. But if there is a minimal surface with suitable curvature conditions, then adapting Uhlenbeck's method in [66], we can check the uniqueness of a closed minimal surface of any type, which is the key point of the proof.

4.2.2 Proof of Theorem 1.4'

For each $i \in \mathbb{N}$, let $\tilde{M}_i$ be the covering space of M such that $\pi_1(\tilde{M}_i) \cong \pi_1(S_i)$. Let $\tilde{F}_i$ be the corresponding lift of F that maps $\tilde{F}_i^{-1}(\tilde{M}_i) \simeq \tilde{M}_i$ to $\tilde{M}_i$. The lift of S_i in $\tilde{M}_i$ still has fundamental group $\pi_1(S_i)$, so we denote it by S_i as well. By assumption, F is homotopic to identity, thus there is a continuous map $H : M \times [0,1] \to M$ with $H(x,0) = x$ and $H(x,1) = F(x)$ for any $x \in M$. Since M is compact, the length of the path of H between x and $F(x)$ is uniformly bounded by a constant $C > 0$. Now let $\tilde{H}_i$ be the lift of H that connects $\tilde{F}_i$ to the identity map on $\tilde{M}_i$. For all $y \in \tilde{M}_i$, the length of the path between y and $\tilde{F}_i(y)$ is therefore uniformly bounded by the same constant C. So $\tilde{F}_i$ is proper, meaning $\tilde{F}_i^{-1}(S_i)$ is a closed set, and therefore it is a k-fold cover of $F^{-1}(S_i)$ for some finite number k. If $k > 1$, the image of $\tilde{F}_i^{-1}(S_i)$ under $\tilde{F}_i$ is either a closed surface with Euler characteristic equal to $k\chi(S_i)$, and therefore having genus $kg(S_i) - k + 1 > g(S_i)$, or it is a union of at least two surfaces with genus $\geq g(S_i)$. However, both cases are impossible because the image cannot be identified with $S_i \subset \tilde{M}_i$. We must have $k = 1$. Consequently, the covering map from $\tilde{F}_i^{-1}(S_i)$ to $F^{-1}(S_i)$ is one-to-one.

On the other hand, the classical result [57] verifies the existence of area-minimizing surface $\Sigma_i \subset (M, h)$ in the homotopy class of S_i. And based on Theorem 4.3 of [42], there exist a C^2-neighborhood $\mathcal{U}_0$ of h_0 and $N_0 \in \mathbb{N}$, so that when $h \in \mathcal{U}_0$ and $i \geq N_0$, Σ_i is the unique minimal surface in (M, h) homotopic to S_i. Furthermore, let D_i (Ω_i) be the lifts of S_i (Σ_i, respectively) in B^3. These discs D_i and Ω_i are asymptotic and at a uniformly bounded Hausdorff distance to each other, as $h \to h_0$, Ω_i converges uniformly on compact sets to D_i in $C^{2,\alpha}$. Therefore, replacing $\mathcal{U}_0$ by a smaller subset or replacing N_0 by a larger integer if necessary, we can assume that if $h \in \mathcal{U}_0$ and $i \geq N_0$, then there exists a smooth map f_i on D_i with $|f_i|_{C^{2,\alpha}} < 1$, such that Ω_i can be represented by a graph of f_i over D_i. More precisely, Let n_i be the unit normal vector field of D_i, then we have the following diffeomorphism.

$$F_i : D_i \to \Omega_i, \quad F_i(x) = \cosh(f_i(x))x + \sinh(f_i(x))n_i(x).$$

Notice that the minimal disc Ω_i has mean curvature equal to zero with respect to h, so the mean curvature $H_{h_0}(\Omega_i)$ with respect to h_0 has a uniform bound determined by the perturbation of h and ∇h. Since h is C^2-close to h_0, we have

$$|H_{h_0}(\Omega_i)|_{C^{0,\alpha}} = O(|h - h_0|_{C^2}), \quad \forall i \geq N_0. \tag{4.2.1}$$

According to the Schauder estimates for elliptic PDE, there exists a constant $c_0 > 0$, such that for any $i \geq N_0$,

$$|f_i|_{C^{2,\alpha}} \leq c_0\big(|f_i|_{L^\infty} + |H_{h_0}(\Omega_i)|_{C^{0,\alpha}}\big). \tag{4.2.2}$$

Besides, for $i \geq N_0$, suppose the principle curvature of D_i with respect to h_0 satisfies that $\sup\limits_{i\geq N_0} |\lambda(D_i)|_{L^\infty} < 1$. Uhlenbeck ([66]) shows that $\mathbb{H}^3$ is foliated by a sequence of equidistant discs relative to D_i. We denote by D_i^r the disc with a fix distance r to D_i, it has mean curvature

$$H_{h_0}(D_i^r) = \frac{2(1-\lambda(D_i)^2)\tanh r}{1-\lambda(D_i)^2\tanh^2 r}.$$

Let R_i^+ and R_i^- be the supremum and infimum of r such that Ω_i meets D_i^r, and the intersections points are x_i^+, x_i^-, respectively. Since $R_i^+, R_i^- \to 0$ as $h \to h_0$, we may assume

$$\min\{\frac{d}{dr}(\tanh r)|_{r=R_i^+}, \frac{d}{dr}(\tanh r)|_{r=R_i^-}\} = \min\{\frac{1}{\cosh^2 R_i^+}, \frac{1}{\cosh^2 R_i^-}\} \geq \frac{1}{2},$$

then we have

$$\begin{aligned} |H_{h_0}(\Omega_i)|_{L^\infty} &\geq \max\{|H_{h_0}(D_i^{R_i^+}(x_i^+)|, |H_{h_0}(D_i^{R_i^-}(x_i^-)|\} \\ &\geq 2(1-|\lambda(D_i)|_{L^\infty}^2)\max\{|\tanh R_i^+|, |\tanh R_i^-|\} \\ &\geq (1-|\lambda(D_i)|_{L^\infty}^2)\max\{|R_i^+|, |R_i^-|\}. \end{aligned}$$

Since Ω_i, described by the graph f_i, is bounded between $D_i^{R_i^+}$ and $D_i^{R_i^-}$, the above result indicates the existence of a uniform constant $c_1 > 0$, such that

$$|f_i|_{L^\infty} \leq c_1|H_{h_0}(\Omega_i)|_{L^\infty}, \quad \forall i \geq N_0. \tag{4.2.3}$$

Combining (4.2.1)-(4.2.3), we obtain

$$|f_i|_{C^{2,\alpha}} = O(|h - h_0|_{C^2}), \quad \forall i \geq N_0.$$

And therefore, the principal curvatures of Σ_i with respect to h_0 and h satisfy that

$$\begin{aligned} &|\lambda_{h_0}(\Sigma_i)|^2_{L^\infty} = O(|h - h_0|^2_{C^2}), \quad \forall i \geq N_0, \\ \Longrightarrow &|\lambda_h(\Sigma_i)|^2_{L^\infty} = |\lambda_{h_0}(\Sigma_i)|^2_{L^\infty} + O(|h - h_0|^2_{C^2}) = O(|h - h_0|^2_{C^2}), \quad \forall i \geq N_0. \end{aligned}$$

Clearly, the sectional curvature of (M, h) has the property

$$|K_h|_{L^\infty} = -1 + O(|h - h_0|_{C^2}), \quad \forall i \geq N_0.$$

Thus, we can find $\mathcal{U} \subset \mathcal{U}_0$ and $N \geq N_0$, such that if $h \in \mathcal{U}$ and $i \geq N$, then the principal curvatures of Σ_i with respect to h and the sectional curvature of (M, h) satisfy that

$$|\lambda_h(\Sigma_i)|^2_{L^\infty} < -\sup K_h.$$

In the lemma below, we apply Uhlenbeck's method [66], as well as the comparison result associated with Riccati equations, to prove that Σ_i is the unique closed minimal surface in $(\tilde{M}_i, h)$, thus minimizing the area among all closed surfaces.

Lemma 4.2.1. *Let Σ be a minimal surface in (M, h) whose fundamental group injectively includes in $\pi_1(M)$, the principal curvatures of Σ with respect to h, denoted by $\pm\lambda$, and the sectional curvature K of (M, h) satisfy that*

$$|\lambda(\Sigma)|^2_{L^\infty} < -\sup K. \tag{4.2.4}$$

Let $\tilde{M}$ be the cover of M with $\pi_1(\tilde{M}) \cong \pi_1(\Sigma)$. Then Σ is the unique closed minimal *surface in $\tilde{M}$.*

Proof. Denote the supreme of the sectional curvature on (M, h) by $-k^2$, where $k > 0$. Let $f(x)$ be the distance function from a fixed point in $\tilde{M} \setminus \Sigma$ to $x \in \Sigma$. By (4.2.4), for any $X \in T_x\Sigma$,

$$D^2 f(X, X) = \operatorname{Hess} f(X, X) - A(X, X)(f) \geq \frac{k}{\tanh(kf(x))} - k > 0. \tag{4.2.5}$$

It turns out that f is a convex function, thus, there's only one critical point that attains the minimum. As a result, $\exp|_{N\Sigma}$ maps injectively from the normal bundle $N\Sigma$ to $\tilde{M}$.

Furthermore, we show that $\exp|_{N\Sigma}$ is a diffeomorphism, and thus $\tilde{M}$ is foliated by a family of surfaces $\{\Sigma_r\}_{r\in\mathbb{R}}$, where Σ_r is the surface at the fixed distance r to Σ. To see this, we introduce some notations beforehand. For $x \in \Sigma$, choose an oriented, orthonormal basis $\{e_1, e_2\}$ for $T_x\Sigma$, and a unit vector e_3 for $N_x\Sigma$. Then we obtain an orthonormal frame by applying parallel transport along $\exp|_{N\Sigma}$. Since Σ is a minimal surface, the principal curvatures satisfy that $\lambda_1 = -\lambda_2 := \lambda$, we assume $\lambda \geq 0$ in the following computation. Let $V_i(r) = v^i(r)e_i(r)$ be the Jacobi field along $\exp(re_3)$, where $i = 1, 2$, it satisfies that $v^i(0) = 1$, $(v^i)'(0) = \lambda_i = \pm\lambda$.

On the other hand, let $\overline{\Sigma}$ be a minimal surface in $\tilde{M}$ with respect to an ambient metric $\overline{h}$ of constant sectional curvature $-k^2$, and its principal curvature satisfies that $\overline{\lambda} = \lambda$. We do not require the existence of $\overline{\Sigma}$, it's only used for comparison in the computation. Similar to the notations defined above, let $\overline{e}_1, \cdots, \overline{e}_3$ be the corresponding frame on $\tilde{M}$ with respect to $\overline{h}$, and let $\overline{V}_i(r) = \overline{v}^i(r)\overline{e}_i(r)$ be the Jacobi field along $\exp(r\overline{e}_3)$ which shares the same initial data with $V_i(r)$. Since $\lambda < k$, we have

$$\overline{v}^i(r) = \cosh kr \pm \frac{\lambda \sinh kr}{k} > 0, \quad i = 1, 2.$$

From

$$(v^i)'' = -R(e_3, e_i, e_i, e_3)v^i \geq k^2\,\overline{v}^i = (\overline{v}^i)''$$

and the initial data, the graph of v^i lies above that of $\overline{v}^i$, thus above the horizontal axis. The non-vanishing Jacobi fields ensure that the induced metric on Σ_r in $(\tilde{M}, h)$ is nonsingular for all $r \in \mathbb{R}$. In addition, we've seen that $\exp|_{N\Sigma} : N\Sigma \to \tilde{M}$ is injective, and therefore also bijective, so it is a diffeomorphism and $\tilde{M}$ admits a foliation structure.

Next, let $\lambda_i(r)\,(i = 1, 2)$ be the principal curvatures on Σ_r, and denote by $\overline{\lambda}_i(r)\,(i = 1, 2)$ the principal curvatures of the r-equidistant surface to $\overline{\Sigma}$ with respect to $\overline{h}$. Notice that each $\lambda_i(r)$ satisfies the Riccati equation

$$\lambda_i'(r) = \lambda_i^2(r) + R(e_3, e_i, e_i, e_3)(r).$$

Then it follows from the comparison theorem associated with Riccati equations (for instance, see Theorem 3.1 of [68]) that

$$\begin{aligned}
\lambda_1(r) &\geq \overline{\lambda}_1(r) = k\,\frac{k\tanh(kr) + \lambda}{k + \lambda\tanh(kr)} > 0,\\
\lambda_2(r) &\geq \overline{\lambda}_2(r) = k\,\frac{k\tanh(kr) - \lambda}{k - \lambda\tanh(kr)}.
\end{aligned}$$

It follows from $\lambda^2 \leq k^2$ that

$$\lambda_1(r) + \lambda_2(r) \geq 2k\,\frac{(k^2 - \lambda^2)\tanh(kr)}{k^2 - \lambda^2\tanh^2(kr)} > 0.$$

Therefore, for any $r \in \mathbb{R}$, Σ_r is strictly mean convex with respect to the metric induced by h.

Finally, we prove the uniqueness. Assume that Σ' is another closed minimal surface in $(\tilde{M}, h)$, and let R_+ and R_- be the supremum and infimum of r such that Σ' intersects Σ_r, respectively, then R_+ and R_- are both finite. However, due to the maximum principle, Σ' cannot be tangential to any strictly mean convex slice Σ_r with $r \neq 0$. Therefore, we must have $\Sigma' \subset \Sigma_0 = \Sigma$. $\square$

Now we finish the proof of Theorem 1.2.3. From the previous lemma, when $i \in \mathbb{N}$ is sufficiently large, Σ_i is the area-minimizer among all closed surfaces in $\tilde{M}_i$ with respect to the induced metric of h, it yields that

$$\text{area}_h(F^{-1}(S_i)) = \text{area}_h(\tilde{F}_i^{-1}(S_i)) \geq \text{area}_h(\Sigma_i).$$

Combining it with the area comparison in Theorem 5.1 of [43], we have

$$\text{Area}_F(h/h_0) = \lim_{i\to\infty} \frac{\text{area}_h(F^{-1}(S_i))}{4\pi(g_i - 1)} \geq \lim_{i\to\infty} \frac{\text{area}_h(\Sigma_i)}{4\pi(g_i - 1)} \geq 1.$$

Moreover, when the equality holds, it follows from the equality of Theorem 5.1 of [43] that $h = F^*(h_0)$, F is an isometry between h and h_0.

4.3 Generalization of Average K-volume Ratio on Einstein Manifolds

In this section, we explore other Einstein manifolds and attempt to extend Theorem 1.2.1 not only to a wider variety of manifolds but also to include the average k-volume ratio where $2 \leq k \leq n-1$.

4.3.1 Einstein Manifolds of Negative Curvature

Let (M, h_0) be a closed Einstein manifold of dimension $n \geq 3$ satisfying

$$Ric_{h_0} = -\lambda h_0,$$

where the $\lambda > 0$ is a constant. The average area ratio of the map $\mathrm{Id} : (M, h) \to (M, h_0)$ is defined as follows.

$$\mathrm{Vol}^k_{\mathrm{Id}}(h/h_0) = \int_{(x,P)\in Gr_k(M)} \lim_{\delta\to 0} \frac{\mathrm{vol}_h((D_\delta(P))}{\delta}\, d\mu_{h_0}, \tag{4.3.1}$$

where $D_\delta(P)$ is a subset of the totally geodesic k-dimensional subspace in the universal cover of (M, h_0), and it is tangential to P at x, and $D_\delta(P)$ has volume equal to δ with respect to h_0. Additionally, μ_{h_0} is the unit volume measure on $Gr_k M$ with respect to the metric induced by h_0.

Likewise, we aim to find a neighborhood $\mathcal{U}$ of h_0 in the metric space of M. In this neighborhood, for any Riemannian metric $h \in \mathcal{U}$ defined on M where $R_h \geq -n\lambda$, we expect the inequality

$$\mathrm{Vol}^k_{\mathrm{Id}}(h/h_0) \geq 1 \tag{4.3.2}$$

to hold. The proof for the conformal deformations follows directly from Section 4.1.1. In this section, it suffices to consider the metrics with constant scalar curvature equal to $-n\lambda$. To accomplish this, we define the functional $\mathcal{A}^k$ as follows.

$$\begin{aligned}\mathcal{A}^k(h) =& (-1)^{1-\frac{k}{2}} \lim_{\delta\to 0} \int_{(x,P)\in Gr_k M} R_h(x)^{\frac{k}{2}} \frac{\mathrm{vol}_h(D_\delta(P))}{\delta}\, d\mu_{h_0} \\ =& (-1)^{1-\frac{k}{2}} \lim_{\delta\to 0} \int_{(x,P)\in Gr_k M} R_h(x)^{\frac{k}{2}} \fint_{D_\delta(P)} \sqrt{\det_{h_0}(h|_{D_\delta(P)})}\, dV_{h_0}\, d\mu_{h_0}.\end{aligned}$$

In particular, the metric h with constant scalar curvature equal to $-n\lambda$ satisfies

$$\begin{aligned}\mathcal{A}^k(h) &= -(n\lambda)^{\frac{k}{2}} \lim_{\delta\to 0} \int_{(x,P)\in Gr_k M} \frac{\mathrm{vol}_h(D_\delta(P))}{\delta}\, d\mu_{h_0} \\ &= \mathcal{A}^k(h_0) \lim_{\delta\to 0} \int_{(x,P)\in Gr_k M} \frac{\mathrm{vol}_h(D_\delta(P))}{\delta}\, d\mu_{h_0}.\end{aligned}$$

Therefore, to deduce (4.3.2), we only need to show that $\mathcal{A}^k$ attains a local maximum at

h_0.

Assume $\mathrm{vol}_{h_0}(M) = 1$, and the symmetric tensor $l = h - h_0$ satisfies $\int_M \mathrm{tr}_{h_0} l \, dV_{h_0} = 0$. Let $h_t = h_0 + tl$, we rewrite $\mathcal{A}^k(h_t)$ as

$$\mathcal{A}^k(h_t) = (-1)^{1-\frac{k}{2}} \int_{x\in M} R_{h_t}(x)^{\frac{k}{2}} \, a^k_{h_t}(x) \, dV_{h_0},$$

where

$$a^k_{h_t}(x) = \lim_{\delta\to 0} \fint_{P\in Gr_k M_x} \fint_{D_\delta(P)} \sqrt{\det_{h_0}(h_t|_{D_\delta(P)})} \, dV_{h_0} d\nu_{h_0}.$$

Then we have

$$\frac{d}{dt}\mathcal{A}^k(h_t) = (-1)^{1-\frac{k}{2}} \int_M \frac{k}{2}(\frac{d}{dt}R_{h_t}) \, R_{h_t}^{\frac{k}{2}-1} a^k_{h_t} + R_{h_t}^{\frac{k}{2}} \frac{d}{dt} a^k_{h_t} \, dV_{h_0}. \tag{4.3.3}$$

By (4.1.6), when $t = 0$, applying the Stokes' theorem, we rewrite the first term as follows.

$$\begin{aligned}\int_M \frac{k}{2}(\frac{d}{dt}|_{t=0} \, R_{h_t}) \, R_{h_0}^{\frac{k}{2}-1} a^k_{h_0} \, dV_{h_0} &= \frac{k}{2}(-n\lambda)^{\frac{k}{2}-1} \int_M -\langle Ric_{h_0}, l\rangle_{h_0} \, dV_{h_0} \\ &= \frac{k}{2}(-n\lambda)^{\frac{k}{2}-1} \lambda \int_M \mathrm{tr}_{h_0} l \, dV_{h_0}.\end{aligned}$$

Furthermore, let $\lambda_1, \cdots, \lambda_n$ be the eigenvalues of the matrix l at point $x \in M$ with respect to h_0, we obtain by computation that

$$\begin{aligned}\frac{d}{dt}|_{t=0} \, a^k_{h_t} &= \lim_{\delta\to 0} \frac{1}{2} \fint_{P\in Gr_2 M_x} \fint_{D_\delta(P)} \mathrm{tr}_{h_0} l|_{D_\delta(P)} \, dV_{h_0} d\nu_{h_0} \\ &= \frac{1}{2} \frac{\sum_{i_1<\cdots<i_k}(\lambda_{i_1} + \cdots + \lambda_{i_k})}{\binom{n}{k}} = \frac{1}{2} \frac{\binom{n-1}{k-1}\sum_{i=1}^n \lambda_i}{\binom{n}{k}} \\ &= \frac{k}{2n}\mathrm{tr}_{h_0} l.\end{aligned}$$

It follows that for any symmetric 2-tensor l,

$$(\mathcal{A}^k)'(h_0)\cdot l = (-1)^{1-\frac{k}{2}}(-n\lambda)^{\frac{k}{2}-1}\big(\frac{k}{2}\lambda\int_M \mathrm{tr}_{h_0} l\, dV_{h_0} - n\lambda\frac{k}{2n}\int_M \mathrm{tr}_{h_0} l\, dV_{h_0}\big) = 0,$$

thus h_0 is a critical point of $\mathcal{A}^k$.

Then, to estimate the second variation, we specifically focus on the case where the 2-tensor l_{TT} is traceless-transverse. As discussed previously, we compare the second variation of $\mathcal{A}^k$ with that of the normalized total scalar curvature. To simplify the calculation, we write the normalized total scalar curvature as the following form.

$$\mathcal{E}(h) = (-1)^{1-\frac{k}{2}}\,\mathrm{vol}_h(M)^{\frac{k}{n}-1}\int_M R_h^{\frac{k}{2}}\, dV_h.$$

By computation, we have

$$\begin{aligned}&(\mathcal{A}^k)''(h_0)(l_{TT}, l_{TT}) - \mathcal{E}''(h_0)(l_{TT}, l_{TT})\\ =&(-1)^{1-\frac{k}{2}}(-n\lambda)^{\frac{k}{2}}\int_M \frac{d^2}{dt^2}|_{t=0}\, a^k_{h_t} - \frac{k}{n}\frac{d^2}{dt^2}|_{t=0}\sqrt{\det(h_t)}\, dV_{h_0}.\end{aligned}$$

For the first term on the right-hand side,

$$
\begin{aligned}
&\int_M \frac{d^2}{dt^2}|_{t=0}\, a^k_{h_t}\, dV_{h_0} \\
&= \lim_{\delta\to 0} \int_{x\in M} \fint_{P\in Gr_k M_x} \fint_{D_\delta(P)} \frac{1}{4}(\mathrm{tr}_{h_0} l|_{D_\delta(P)})^2 - \frac{1}{2}\mathrm{tr}_{h_0}(l|^2_{D_\delta(P)})\, dV_{h_0} d\nu_{h_0} dV_{h_0} \\
&= \int_M \frac{\sum_{i_1<\cdots<i_k} \frac{1}{4}(\lambda_{i_1}+\cdots+\lambda_{i_k})^2 - \frac{1}{2}(\lambda_{i_1}^2+\cdots+\lambda_{i_k}^2)}{\binom{n}{k}}\, dV_{h_0} \\
&= \int_M \frac{\frac{\binom{n-2}{k-2}}{4}(\sum_{i=1}^n \lambda_i)^2 - \frac{\binom{n-1}{k-1}+\binom{n-2}{k-2}}{4}\sum_{i=1}^n \lambda_i^2}{\binom{n}{k}}\, dV_{h_0} \\
&= \int_M \frac{k(k-1)}{4n(n-1)}(\mathrm{tr}_{h_0} l_{TT})^2 - \frac{k(n+k-2)}{4n(n-1)}\mathrm{tr}_{h_0}(l_{TT}^2)\, dV_{h_0} \\
&= -\int_M \frac{k(n+k-2)}{4n(n-1)}\mathrm{tr}_{h_0}(l_{TT}^2)\, dV_{h_0}.
\end{aligned}
$$

Combining the two inequalities above, we obtain

$$
(\mathcal{A}^k)''(h_0)(l_{TT}, l_{TT}) = \mathcal{E}''(h_0)(l_{TT}, l_{TT}) - (n\lambda)^{\frac{k}{2}} \frac{k(n-k)}{4n(n-1)} \int_M \mathrm{tr}_{h_0}(l_{TT}^2)\, dV_{h_0}. \quad (4.3.4)
$$

Notice that when $k = 2$ and $\lambda = n-1$, the formula coincides with (4.1.11).

If the closed Einstein manifold M satisfies $\mathcal{E}''(h_0)(l_{TT}, l_{TT}) \le 0$ for any traceless-transverse tensor l_{TT}, then it is said to be *stable with respect to the normalized total scalar curvature*. In this case, based on the aforementioned inequality, $(\mathcal{A}^k)''(h_0)(l_{TT}, l_{TT})$ is negative for non-zero l_{TT}.

Concerning the second variation of $\mathcal{A}^k$ with respect to the other components of the symmetric 2-tensor l (specifically, $fh_0 \in C^\infty(M)\cdot h_0$ and $l_D \in T_{h_0}(\mathrm{Diff}(M)(h_0))$), we expect it to be constrained by a small value, which, in turn, ensures that $\mathcal{A}^k$ attains a local maximum.

However, (4.3.4) only provides a negative upper bound of $(\mathcal{A}^k)''(h_0)(l_{TT}, l_{TT})$ by the L^2-norm of l_{TT}. In order to replicate the proof of Lemma 4.1.3, we need a negative upper bound in terms of the H^1-norm of l_{TT}. Fortunately, if the sectional curvature of M is negative everywhere, then according to the proof of Lemma 2.15 in [7], $\mathcal{E}''(h_0)(l_{TT}, l_{TT})$ is bounded from above by $-C|l_{TT}|^2_{H^1(M,h_0)}$. As a result, the theorem remains valid. In particular, this condition is satisfied by locally symmetric spaces of rank one, and we provide a detailed discussion of this case in the following section.

4.3.2 Locally Symmetric Manifolds of Negative Curvature

In the section, we prove that the result is valid for closed locally symmetric manifolds of negative curvature, whose universal covers are symmetric spaces of rank one. These manifolds include hyperbolic manifolds, complex hyperbolic manifolds, quaternionic hyperbolic manifolds, and the Cayley plane. For the convenience of the readers, we present the theorem in the following restated form.

Theorem 4.3.1. *Let (M, h_0) be a closed locally symmetric space of rank one. There exists a small neighborhood $\mathcal{U}$ of h_0 in the metric space of M, such that for any Riemannian metric $h \in \mathcal{U}$ on M with $R_h \geq R_{h_0}$ and for any $2 \leq k \leq n-1$, the average k-volume ratio defined in (4.3.1) satisfies the inequality*

$$Vol^k_{Id}(h/h_0) \geq 1.$$

The equality holds if and only if $h = h_0$.

Let's consider the complex hyperbolic case as an example. Suppose we have a closed manifold (M, h_0) with a complex hyperbolic structure, where the dimension is denoted by $n = 2m$ and $m \geq 2$. Denote by J the complex structure. In $\mathbb{H}^m_{\mathbb{C}}$, for each $v \in T^1\mathbb{H}^m_{\mathbb{C}}$, the plane spanned by v and Jv has constant curvature -4, and the set of all planes with constant curvature -1 containing v has dimension equal to $2m-2 = n-2$. Thus, in this case, the Einstein constant $\lambda = n+2$, and $R_{h_0} = -n(n+2)$.

Notice that the result of [7] is also applicable to the other locally symmetric spaces of rank one. Consequently, there exists a constant $C > 0$, such that

$$\mathcal{E}''(h_0)(l_{TT}, l_{TT}) \leq -C|l_{TT}|^2_{H^1(M,h_0)}.$$

Thus, according to (4.3.4), and using $\mathrm{tr}_{h_0} l_{TT} = \delta_{h_0} l_{TT} = 0$, we obtain

$$\begin{aligned}(\mathcal{A}^k)''(h_0)(l_{TT}, l_{TT}) &= \mathcal{E}''(h_0)(l_{TT}, l_{TT}) - (n(n+2))^{\frac{k}{2}} \frac{k(n-k)}{4n(n-1)} \int_M \mathrm{tr}_{h_0}(l_{TT}^2)\, dV_{h_0} \\ &< -C|l_{TT}|^2_{H^1(M,h_0)}.\end{aligned}$$

Finally, we can continue the discussion by considering the other components of the symmetric 2-tensor l, namely $fh_0 \in C^\infty(M) \cdot h_0$ and $l_D \in T_{h_0}(\mathrm{Diff}(M)(h_0))$, using the same argument as before. Consequently, the result can be derived.

CHAPTER 5

MINIMAL SURFACE ENTROPY

In this chapter, we shift our focus to the second concept, namely the minimal surface entropy. Our first objective is to expand upon the findings in the three-dimensional study in [11] and apply them to all odd dimensions equal to or greater than three.

5.1 Odd-dimensional Hyperbolic Manifolds

Suppose that (M, h_0) is a closed hyperbolic manifold with an odd dimension $n \geq 3$. The goal of this section is to prove Theorem 1.2.6. First, we present an outline of the proof.

Suppose that $\Pi \in S_\epsilon(M)$, and let S be the essential minimal surface of M in the homotopy class Π so that $\text{area}(S) = \text{area}(\Pi)$. In Section 5.1.1, we show that when ϵ is sufficiently small, S is free of branch points. Furthermore, the second fundamental form satisfies the following condition:

$$|A|^2_{L^\infty(S)} = o_\epsilon(1). \tag{5.1.1}$$

Next, in Section 5.1.2, we calculate the minimal surface entropy $E(h_0)$. Finally, the inequality and rigidity are proven in a subsequent section, Section 5.1.3.

5.1.1 Estimates of Second Fundamental Form

In this Section, we prove the equation (5.1.1). More specifically, Let $M = \mathbb{H}^n/\Gamma$ be a closed hyperbolic manifold with dimension $n \geq 4$. Since n is odd, the smallest dimension is 5. From the argument by Schoen-Yau [57] and Sacks-Uhlenbeck [55], for any surface subgroup $\Pi < \Gamma$, there exists an immersed minimal surface S in M with finitely many branch points, such that it minimizes the area in the corresponding homotopy class up to conjugacy. The appearance of branch points is the primary distinction from the case of dimension three. So the key point of (5.1.1) is to rule out branch points.

Trap of Convex Hull

First of all, we need to argue that the convex hull of Π lies in a bounded region. In [58], Seppi proved that for any minimal disc $D \subset \mathbb{H}^3$ asymptotic to a $(1+\epsilon)$-quasicircle γ, every point x in D lies on a geodesic segment α that is orthogonal at the endpoints to planes P_+ and P_-, such that the convex hull of γ is bounded between P_+ and P_-. Additionally, the length of α goes to zero uniformly in x as ϵ approaches zero. In the following lemma, we construct the bound of the convex hull of Π while the ambient manifold has dimension at least 4.

Lemma 5.1.1. *Given $\Pi \in S_\epsilon(M)$ and let γ be the limit set of Π. Then the convex hull of γ is contained in a tube, which converges in Hausdorff distance as ϵ goes to zero, the limit is either empty, or it is contained in a totally geodesic disc.*

Proof. Since γ is an $(1+\epsilon)$-quasicircle, we can find $\epsilon' = o_\epsilon(1)$ and a round circle $c \subset S^{n-1}_\infty$, such that the ϵ'-neighborhood of c in S^{n-1}_∞, denoted by N, contains γ. Any round circle in ∂N bounds a unique totally geodesic disc in $\mathbb{H}^n$, and thus there is a tube $T \subset \mathbb{H}^n$ homeomorphic to $D^2 \times S^{n-3}$ asymptotic to ∂N. Notice that as ϵ goes to zero, the limit of T is either empty, or it is contained in a totally geodesic disc. In addition, T is convex, so the convex hull of γ is bounded in T. $\square$

Uniqueness of Minimal Surfaces

Lemma 5.1.2. *Let D be a minimal surface in $\mathbb{H}^n$ asymptotic to $\gamma \subset S^{n-1}_\infty$, where γ is the limit set of a surface subgroup of M. If the norm squared of the second fundamental* *form of D satisfies that $|A|^2_{L^\infty(D)} < 2$, then*

(a) D is an embedded disc,

(b) If D is the lift of a closed surface in M, then $\gamma = \partial_\infty D$ is a quasicircle,

(c) Assume that γ is a quasicircle, then D is the unique minimal surface with $\partial_\infty D = \gamma$ of all types.

The proof is similar to that of Uhlenbeck's argument [66], but we need to be careful about the noncompactness of D, so it's worth providing the proof here.

Proof. First, (a) can be shown by finding a horosphere S that touches D at a single point, and suppose $\partial_\infty S = y$, namely S is centered at y. Notice that $\mathbb{H}^n$ is foliated by horospheres centered at y, and since the principal curvatures of D restricted to any normal direction are less than that of the horospheres, the distance from a fixed point in $\mathbb{H}^n \setminus D$ to the points on D is a convex function, so there's only one critical point which attains the minimum. This is impossible when the intersection of D with an extrinsic ball is an annulus or it has self-intersections, so it forces D to be an embedded disc.

Furthermore, we can also prove that $\exp|_{ND}$ is a diffeomorphism from ND to $\mathbb{H}^n$, and therefore $\mathbb{H}^n \setminus D$ is foliated by a family of hypersurfaces $\{H^r\}_{r>0}$, where H^r is the hypersurface at the fixed distance r to D. To see this, we need some notations beforehand. For $x \in D$, choose an oriented, orthonormal basis $\{e_1, e_2\}$ for T_xD, and an orthonormal basis $\{e_3, \cdots, e_n\}$ for N_xD. Then we obtain an orthonormal frame by applying the parallel transport along $\exp|_{ND}$. Since D is a minimal surface, for any $3 \le j \le n$, let $\lambda_{ijk} := \langle A(e_j, e_k), e_j\rangle$, then we have $\lambda_{j11} = -\lambda_{j22} := \lambda_j$. We assume $\lambda_j \ge 0$ in the following computation. Moreover, for any $3 \le j \le n$ and $i \ne j$, let $V_{ji}(r) = v^i_j(r)e_i(r)$ be the Jacobi field along $\exp(re_j)$. When $1 \le i \le 2$, it satisfies that $v^i_j(0) = 1$, $(v^i_j)'(0) = \lambda_{jii}(x) = \pm\lambda_j(x)$. And when $3 \le i \le n$, $i \ne j$, we take $v^i_j(0) = 0$,

$(v_j^i)'(0) = 1$. Then

$$V_{ji}(r) = \begin{cases} (\cosh r \pm \lambda_j \sinh r) e_i(r) & \text{if } 1 \leq i \leq 2 \\ \sinh r e_i(r) & \text{if } 3 \leq i \leq n \text{ and } i \neq j \end{cases}$$

So the induced metric on H^r is

$$g_j(x,r) = \begin{bmatrix} (\cosh r + \lambda_j(x)\sinh r)^2 & & & & \\ & (\cosh r - \lambda_j(x)\sinh r)^2 & & & \\ & & \sinh^2 r & & \\ & & & \ddots & \\ & & & & \sinh^2 r \end{bmatrix}$$

It is nonsingular for any $r > 0$ since $\lambda_j(x) < 1$. In addition, we've seen that $\exp|_{ND} : ND \to \mathbb{H}^n$ is a bijection because of the convexity of the distance function, thus it is a diffeomorphism and $\mathbb{H}^n$ admits a foliation structure.

Additionally, assume that D is the lift of a closed surface $S \subset M$. Since the induced metric on S is conformally equivalent to a hyperbolic metric g_{hyp}, the induced metric on $D \subset \mathbb{H}^n$ is also conformally equivalent to a hyperbolic metric, we still denote it by g_{hyp}. Because $\lambda_j < 1$, by [66], the induced metric on H^r is quasi-isometrically equivalent to the metric

$$\begin{bmatrix} \cosh^2 r \, g_{hyp} & \\ & \sinh^2 r \, I_{(n-3)\times(n-3)} \end{bmatrix}$$

The quasi-isometry $F : \mathbb{H}^n \to \mathbb{H}^n$ extends to a quasiconformal map $f : S_\infty^{n-1} \to S_\infty^{n-1}$. It means that γ is the image of a round circle mapped by f, so equivalently, it is a quasicircle. This proves (b).

Next, to prove (c), we denote by $\lambda_{ji}(x,r)$ the principal curvature of H^r. Then

$$\begin{aligned}
\lambda_{j1}(x,r) &= \frac{\tanh r + \lambda_j(x)}{1+\lambda_j(x)\tanh r} > 0,\\
\lambda_{j2}(x,r) &= \frac{\tanh r - \lambda_j(x)}{1-\lambda_j(x)\tanh r},\\
\lambda_{ji}(x,r) &= \frac{1}{\tanh r} > 0, \qquad \forall\, 3 \le i \le n \text{ and } i \neq j.
\end{aligned}$$

$\lambda_{j2}(x,r)$ is the only one that is possibly non-positive. The assumption $\lambda_j(x) < 1$ yields that

$$\begin{aligned}
\lambda_{j1}(x,r) + \lambda_{j2}(x,r) &= \frac{2(1-\lambda_j^2(x))\tanh r}{1-\lambda_j^2(x)\tanh^2 r} > 0,\\
\lambda_{j2}(x,r) + \lambda_{ji}(x,r) &= \frac{\tanh^2 r - 2\lambda_j(x)\tanh r + 1}{(1-\lambda_j(x)\tanh r)\tanh r} > \frac{1-\lambda_j(x)\tanh r}{\tanh r} > 0,
\end{aligned}$$

where $3 \le i \le n$ and $i \neq j$. Therefore, for any $r > 0$, H^r is strictly two-convex. Furthermore, when $r > \tanh^{-1} \frac{|A|^2_{L^\infty(D)}}{2}$, we have $r > \tanh^{-1} |\lambda_j|_{L^\infty(D)}$ for all $3 \le j \le n$, then all the principal curvatures of H^r are positive, thus H^r is strictly convex and it bounds inside the convex hull of γ.

Now let D' be any other minimal surface with $\partial_\infty D' = \gamma$, and let $R > 0$ be the supremum of r such that D' intersects H^r. From [4], D' lies in the convex hull of γ, then R cannot exceed the finite number $\tanh^{-1} \frac{|A|^2_{L^\infty(D)}}{2}$. If the supremum is attained on D', then D' is tangent to the two-convex hypersurface H^R, which contradicts the maximum principle.

Otherwise, if the supremum R is not attained. We can take a sequence $x_i' \in D' \cap H^{r(x_i')}$, such that $r(x_i') \to R$ as $i \to \infty$. For each x_i', there exists an isometry T_i of $\mathbb{H}^n$ sending x_i' to the origin. In the meantime, since γ is a quasicircle, there is a quasiconformal map ϕ_i on S^{n-1}_∞ that maps a round circle to $T_i(\gamma)$. And since the convex hull of $T_i(\gamma)$ contains the origin, there exist three distinct points x, y, z on the round circle, so that $\phi_i(x), \phi_i(y), \phi_i(z)$ are at a uniformly positive distance from one another, it then follows from the compactness theorem of quasiconformal maps that after passing to a subsequence, ϕ_i converges uniformly to a quasiconformal map, and $T_i(\gamma)$ converges to a quasicircle γ_∞ in Hausdorff topology (see, for instance, page 25 in [35]). Moreover, after passing to a subsequence, $T_i(D')$ converges as varifolds to D'_∞ with $\partial_\infty D'_\infty = \gamma_\infty$. We can further assume that $T_i(D)$ converges to D_∞ with $\partial_\infty D_\infty = \gamma_\infty$ (since the points x_i, which are the normal projections of x'_i to D, are mapped into a compact region by T_i, we can take further isometries sending $T_i(x_i)$ to the origin and repeat the same procedure). Standard compactness theorem implies that $T_i(D)$ converges graphically, thus the limit D_∞ is still an embedded minimal surface with $|A|^2_\infty < 2$. Denote by H^r_∞ the hypersurface at the fixed distance r to D_∞, then $\{H^r_\infty\}_{r>0}$ foliates $\mathbb{H}^n \setminus D_\infty$. Moreover, D'_∞ is tangent to the two-convex hypersurface H^R_∞, but this violates the maximum principle.

□

Absence of Branch Points

From now on, we consider a sequence $\Pi_i \in S(M, g_i, \frac{1}{i})$, and let S_i be a minimal surface in the homotopy class Π_i such that $\text{area}(S_i) = \text{area}(\Pi_i)$. We denote by γ_i the limit set of Π_i.

Let $D_i \subset \mathbb{H}^n$ be the area-minimizing surface mod 2 with $\partial_\infty D_i = \gamma_i$. Then due to Theorem 3 of [3], D_i is free of branch points. Moreover, after applying an isometry in $Isom(\mathbb{H}^n)$ and passing to a subsequence, γ_i converges to a round circle γ in Hausdorff topology. From Corollary 2.5 in [5], for any $R > 0$, there exists a constant C_R depending only on n and R, such that

$$\text{area}(D_i \cap B_R(0)) \leq C_R \quad \forall i \geq 1. \tag{5.1.2}$$

It follows that D_i converges as varifolds to V, which is also area-minimizing mod 2 (see 34.5 and 42.7 in [61]). By Lemma 5.1.1, V is either empty or contained in the totally geodesic disc D with $\partial_\infty D = \gamma$.

Furthermore, Alexander duality indicates that there exists a $(n-2)$-dimensional submanifold $K \subset \mathbb{H}^n$, such that the boundary ∂K lies in the complement of a tubular neighborhood of γ in S^{n-1}_∞, and it is linked with every γ_i for large enough i. So every D_i intersects K. Additionally, according to Lemma 5.1.1, there exists a radius $R_0 > 0$, such that

$$C(\gamma_i) \cap K \subset B_{R_0}(0), \quad \forall i >> 1, \tag{5.1.3}$$

where $C(\gamma_i)$ represents the convex hull of γ_i. For any $R > R_0$ and any point $x_i \in D_i \cap B_{R_0}(0)$, we have $B_{R-R_0}(x_i) \subset B_R(0)$. Then (5.1.3), together with the monotonicity formula in [5], produces a uniform constant $c_{R-R_0} > 0$, such that

$$\text{area}(D_i \cap B_R(0)) \geq c_{R-R_0}. \tag{5.1.4}$$

It implies that V is non-empty. Thus by constancy theorem (41.1 in [61]), V is a positive multiple of D. And since V is area-minimizing mod 2, the multiplicity has to be one.

Moreover, Allard regularity theorem (see [1], or Theorem 1.1 in [69] for an easy version) indicates that the convergence is smooth on compact sets, and we obtain that $|A|^2_{L^\infty_{loc}(D_i)} \to 0$. Finally, from Lemma 5.1.2, whenever i is sufficiently large, the lift of S_i must coincide with D_i, so we finish the proof of (5.1.1).

5.1.2 Entropy of Hyperbolic Metrics

Let $s(M,g)$ denote the cardinality of $S(M,g)$, and $s(M,g,\epsilon)$ denote the cardinality of the subset $S(M,g,\epsilon) \subset S(M,g)$. We prove the following inequality in this section.

$$(c_1 g)^{2g} \le s(M,g,\epsilon) \le s(M,g) \le (c_2 g)^{2g}. \tag{5.1.5}$$

Upper Bound of (5.1.5)

When $n = 3$, Kahn-Markovic [33] found an upper bound of $s(M,g)$. Their method also applies to the case of $n > 3$. We only need the following fact.

Let M be a closed hyperbolic manifold of dimension $n \ge 3$, and let S_g denote a closed surface of genus g. For any π_1-injective immersion $f : S_g \to M$, there exist a hyperbolic structure on S_g and a homotopy of f that is pleated with respect to this structure, we still denote it by f (see 8.10 in [64] for Thurston's original proof for $n = 3$, and Lemma 3.6 in [10] for the generalization of all dimensions). Furthermore, $s(M,g)$ can be estimated by counting the number of homotopy classes of the pleated immersions. As shown in [33], there exists a constant c_2 depending only on the injectivity radius of M, so that

$$s(M,g) \le (c_2 g)^{2g}.$$

Lower Bound of (5.1.5)

Suppose that M has an odd dimension $n \ge 3$. According to [22], for any small number $\epsilon' > 0$, there is an essential surface in M which is sufficiently well-distributed and $(1+\epsilon')$-quasigeodesic, namely, the geodesics on the surface with respect to intrinsic distance are $(1+\epsilon', \epsilon')$-quasigeodesics in M. This determines a quasi-isometry that embeds $\mathbb{H}^2$ into $\mathbb{H}^n$, whose boundary extends to a quasisymmetry $f_1 : S^1_\infty \to S^{n-1}_\infty$. Pick two discs

$D_1, D_2 \subset S_\infty^{n-1}$ with $\partial D_1 = \partial D_2 = S_\infty^1$ and $D_1 \cup D_2 = S_\infty^2$. By [65], f_1 extends to quasiconformal maps from D_i into S_∞^{n-1}, $i = 1, 2$, so there exists a quasiconformal extension $f_2 : S_\infty^2 \to S_\infty^{n-1}$. Repeating this process, we can find a quasiconformal extension $f_{n-1} : S_\infty^{n-1} \to S_\infty^{n-1}$. Moreover, it has dilatation $1 + \epsilon$, where ϵ depends on n and ϵ', and $\epsilon \to 0$ as $\epsilon' \to 0$. For this reason, we denote this essential surface by Σ_ϵ. So for any $\epsilon > 0$, we can choose a sufficiently small ϵ' to build an essential surface $\Sigma_\epsilon \subset M$ associated with an element in $S_\epsilon(M)$. Let $G(M, g, \epsilon)$ denote the subset of $S(M, g, \epsilon)$ consisting of homotopy classes of finite covers of Σ_ϵ that have genus at most g. Counting the commensurability classes in $G(M, g, \epsilon)$ and using Müller-Puchta's formula (see [33]), we obtain the following lower bound when g is large.

$$s(M, g, \epsilon) \geq \#G(M, g, \epsilon) \geq (c_1 g)^{2g},$$

where c_1 is a constant that depends only on M and ϵ.

Moreover, let S_i denote the minimal representative in the homotopy class $\Pi_i \in G(M, g_i, \frac{1}{i})$, then it is homotopic to a $(1 + \frac{1}{i})$-quasigeodesic surface Σ_i. Assume that μ_i, ν_i represent the Radon measures induced by integration over S_i and Σ_i, respectively. From Lemma 4.3 of [11], we have

$$\lim_{i \to \infty} \mu_i = \lim_{i \to \infty} \nu_i =: \nu, \tag{5.1.6}$$

Furthermore, following the argument on page 16 in [11], we conclude that the measure ν is positive on any non-empty open set of M. The proof makes use of the property that Σ_i is nearly equidistributed in M (see Section 7 of [22]), and the estimates hold for all odd ambient dimensions. This measure ν plays an important role in the proof of the rigidity in Section 5.1.3.

Computation of Entropy

We prove $E(h_0) = 2$ first. Given $\eta > 0$, for all sufficiently small ϵ, and sufficiently large L which only depend on η, we conclude from (5.1.1) that for $\Pi \in S_\epsilon(M)$, if it has area$(\Pi) \leq 4\pi(L-1)$, then $\Pi \in S(M, \lfloor (1+\eta)L \rfloor, \epsilon)$. On the other hand, $\Pi \in S(M, \lfloor (1-\eta)L \rfloor, \epsilon)$ implies that area$(\Pi) \leq 4\pi(L-1)$. Then consequently,

$$\begin{aligned} 2(1-\eta) &\leq \liminf_{L\to\infty} \frac{\ln s(M, \lfloor (1-\eta)L \rfloor, \epsilon)}{L \ln L} \\ &\leq \liminf_{L\to\infty} \frac{\ln \#\{\mathrm{area}_h(\Pi) \leq 4\pi(L-1) : \Pi \in S_\epsilon(M)\}}{L \ln L} \\ &\leq \liminf_{L\to\infty} \frac{\ln s(M, \lfloor (1+\eta)L \rfloor, \epsilon)}{L \ln L} \leq 2(1+\eta). \end{aligned}$$

It follows directly that $E(h_0) = 2$.

5.1.3 Inequality and Rigidity

The key fact to show the rigidity is the following, which is an extension of Theorem 5.1 in [11] to higher dimensional ambient manifolds.

Theorem 5.1.3. *Assume M is defined as above. Let S_i be the essential surface immersed in M that minimizes the area of a surface subgroup $\Pi_i < \Gamma$ in $G(M, g_i, \frac{1}{i})$ with respect to the hyperbolic metric h_0. And let Σ_i be the essential surface (possibly with branch points) homotopic to S_i that minimizes area with respect to the metric h. Then*

$$\limsup_{i\to\infty} \frac{area_h(\Sigma_i)}{area(S_i)} \leq 1. \tag{5.1.7}$$

If the equality holds, then h is hyperbolic and isometric to h_0.

Proof of Theorem 5.1.3

Let's prove the inequality first. Σ_i may have isolated branch points when $n \geq 4$, we denote by P_i the locus of branch points, and by d_{i_j} the order of each branch point $p_{i_j} \in P_i$. Then a generalized Gauss-Bonnet theorem in [18] states that

$$\text{area}_h(\Sigma_i) = 4\pi(g_i - 1) + \int_{\Sigma_i \backslash P_i} (K_{12} + 1) dA_h - \frac{1}{2} \int_{\Sigma_i \backslash P_i} |A|^2 dA_h - 2\pi \sum_j d_{i_j}. \quad (5.1.8)$$

On the other hand, we have shown in Section 5.1.1 that S_i has no branch points and satisfies that $|A|^2_{L^\infty(S_i)} \to 0$ as $i \to \infty$, so for sufficiently large i, $\text{area}(S_i) \simeq 4\pi(g_i - 1)$, and thus inequality (5.1.7) follows.

Now suppose the equality (5.1.7) holds, it yields that

$$\lim_{i \to \infty} \frac{1}{\text{area}_h(\Sigma_i)} \int_{\Sigma_i} \big(-(K_{12} + 1) + \frac{1}{2}|A|^2 + 2\pi \sum_j d_{i_j} \delta_{p_{i_j}} \big) dA_h = 0,$$

where $\delta_{p_{i_j}}(x) = \delta(x - p_{i_j})$ and $\delta(x)$ is the Dirac delta function.

Let $\mathscr{C}$ be the set of all round circles in S^{n-1}_∞, and define

$$\begin{aligned} \mathscr{L} = &\{\gamma \in \mathscr{C} : \exists \phi_i \in F_i(\epsilon_i, R_i), \epsilon_i \to 0, R_i \to \infty, \text{ such that} \\ &\text{after passing to subsequence, } \Lambda(\phi_i \Pi_i \phi_i^{-1}) \text{ converges to } \gamma\}, \end{aligned}$$

in which

$$F_i(\epsilon, R) = \{\phi \in \Gamma : \int_{\phi(\tilde{\Sigma}_i) \cap B_R(0)} \big(-(K_{12}+1)+\frac{1}{2}|A|^2+2\pi \sum_j d_{i_j} \delta_{q_{i_j}}\big) dA_h \leq \epsilon\}, \quad (5.1.9)$$

where q_{i_j} is a branch point with order d_{i_j}, the locus of branch points in $\tilde{\Sigma}_i$ is denoted by Q_i, and δ represents the Dirac delta function. It's not hard to see that $\mathscr{L}$ is closed and Γ-invariant. Due to Lemma 5.2 in [60], almost every element in $\mathscr{C}$ has a dense Γ-orbit.

And for the elements in the subset $\mathscr{L} \subset \mathscr{C}$, we prove the following lemma.

Lemma 5.1.4. *There is a round circle $\gamma \in \mathscr{L}$, such that $\Gamma\gamma$ is dense in $\mathscr{C}$. Additionally, it deduces a stronger result that $\mathscr{L} = \mathscr{C}$. Therefore, by [60], almost every round circle in $\mathscr{L}$ has a dense Γ-orbit.*

Proof. Theorem 6.1 of [11] proved the following fact using the measure ν defined in (5.1.6).

$$\text{For any } n\text{-dimensional compact subset } K \text{ of } \mathbb{H}^n, \text{ there exists } \gamma \in \mathscr{L}, \text{ such that the unique totally geodesic disc } D(\gamma) \text{ in } \mathbb{H}^n \text{ bounded by } \gamma \text{ intersects } K. \quad (\star)$$

Now suppose by contradiction that $\mathscr{L}$ has no element with a dense Γ-orbit in $\mathscr{C}$. According to Shah's result [59], for each $\gamma \in \mathscr{C}$, the projection of the Γ-orbit of $D(\gamma)$ is either dense in M, or it is dense in a finite union of closed totally geodesic submanifolds in M of codimensions $1 \leq k \leq n-2$. So for $\gamma \in \mathscr{L}$, such submanifolds are proper. If the number of elements $\gamma \in \mathscr{L}$ so that the corresponding $D(\gamma)$ intersects Δ were finite, where Δ denotes the fundamental domain of M, then the union of $D(\gamma)$ for all $\gamma \in \mathscr{L}$ would meet Δ in a finite subset. Therefore, there should have been a compact subset $K \subset \Delta$ never intersecting any $D(\gamma)$, but this case can be excluded by $(\star)$.

It turns out that Δ meets infinitely many $D(\gamma)$ for $\gamma \in \mathscr{L}$. By assumption, none of such elements γ have dense Γ-orbits in $\mathscr{C}$, and thus the closures of the projections of these $D(\gamma)$'s in M are infinitely many proper totally geodesic submanifolds, the set of such manifolds is denoted by $\mathscr{P}$. For any $1 \leq k \leq n-2$, let the subset $\mathscr{P}_k \subset \mathscr{P}$ consist of all totally geodesic submanifolds of codimensions k, and let $\mathscr{L}_k$ represent the collection of all $\gamma \in \mathscr{L}$ whose corresponding projection of $D(\gamma)$ in M is dense in at least one of the submanifolds in $\mathscr{P}_k$.

Lemma 5.1.5. *All the elements in $\mathscr{P}$ are contained in a finite union of proper submanifolds of M.*

Let's consider an example first.

Example 5.1.6. When $n = 5$, since $\mathscr{P} = \mathscr{P}_1 \cup \mathscr{P}_2 \cup \mathscr{P}_3$ has infinitely many candidates, we have the following cases.

(1) If $\mathscr{P}_1$ were infinite, then these totally geodesic submanifolds of codimension 1 would be obviously maximal. By Corollary 1.5 of [50] (or Theorem 1.7(1) of [39]), any infinite sequence of maximal properly immersed totally geodesic submanifolds becomes dense in M. Thus, we could pick an infinite sequence $\{\gamma_i\}$ in $\mathscr{L}_1$, and then the limit of $\overline{\Gamma\gamma_i}$ should have been dense in $\mathscr{C}$. Since $\mathscr{L}$ is closed and Γ-invariant, we conclude that $\mathscr{L} = \mathscr{C}$, violating our assumption. It yields that $\mathscr{P}_1$ must be finite.

(2) Suppose further that $\mathscr{P}_2$ is infinite. If $\mathscr{P}_2$ contained infinitely many maximal submanifolds of M, then the argument in (1) could apply. So we only need to consider the situation where all but finitely many elements in $\mathscr{P}_2$ are non-maximal, denoted by $P_1, P_2, \cdots$. By Corollary 1.5 of [50] (or Theorem 14.1(3) of [39]), since the limit of P_i doesn't become dense in M, any infinite subsequence of $\{P_i\}$ must have a further infinite subsequence $\{P_{j,i}\}$ contained in a proper totally geodesic submanifold $\overline{P}_j \subset M$ of higher dimension, so $\overline{P}_j$ must have codimension 1. All elements of $\{P_{j,i}\}$ are maximal submanifolds of $\overline{P}_j$, thus the limit of the sequence is dense in $\overline{P}_j$. Accordingly, there is a sequence $\{\gamma_{j,i}\}$ in $\mathscr{L}_2$, such that the closure of $\lim\limits_{i\to\infty} \overline{\Gamma\gamma_{j,i}}$ contains all circles in $\mathscr{C}$ that lie in $\partial_\infty\widetilde{\overline{P}_j} \approx S^3$, where $\widetilde{\overline{P}_j}$ is a lift of $\overline{P}_j$ in $\mathbb{H}^5$. It's worth noting that almost every element among these circles has a dense orbit in $\partial_\infty\widetilde{\overline{P}_j}$ (Lemma 5.2, [60]). Then because of the closedness and Γ-invariance of $\mathscr{L}$, there exists $\overline{\gamma}_j \in \mathscr{L}$, so that the projection of $D(\overline{\gamma}_j)$ in M is dense in $\overline{P}_j$. In other words, we have $\overline{\gamma}_j \in \mathscr{L}_1$ and $\overline{P}_j \in \mathscr{P}_1$.

Notice that $\mathscr{P}_1$ is a finite set, we can only extract finitely many subsequences $\{P_{1,i}\}, \cdots, \{P_{l,i}\} \subset \{P_i\}$ to build $\overline{P}_1, \cdots, \overline{P}_l \in \mathscr{P}_1$, the number of the remaining elements in $\mathscr{P}_2$ is finite. We see that $\mathscr{P}_1 \cup \mathscr{P}_2$ is contained in a finite union of proper submanifolds.

(3) If $\mathscr{P}_1, \mathscr{P}_2$ are both finite, then $\mathscr{P}_3$ must be infinite. It suffices to assume that all but finitely many candidates in $\mathscr{P}_3$ are non-maximal, and they are denoted by $\{P_i\}$. According to [50] or [39], any infinite subsequence has a further subsequence which is contained in a totally geodesic submanifold of codimension 1 or 2.

Pick a subsequence $\{P_{1,i}\}$ of $\{P_i\}$, so that the elements are contained in a submanifold $\overline{P}_1$ of the maximal codimension $1 \leq m_1 \leq 2$. Then as $i \to \infty$, $P_{1,i}$ becomes dense in $\overline{P}_1$, because otherwise, the argument in [50] or [39] yields a further infinite subsequence lying in a proper submanifold of $\overline{P}_1$, but it violates the assumption that $\overline{P}_1$ attains the maximal codimension. Similarly, the density ensures that $\overline{P}_1 \in \mathscr{P}_{m_1} \subset \mathscr{P}_1 \cup \mathscr{P}_2$.

Furthermore, if the number of candidates in $\{P_i\}$ intersecting $(M \setminus \overline{P}_1)$ is finite, we deduce that $\mathscr{P}$ can be represented by a finite union of submanifolds. Otherwise, we continue to extract an infinite subsequence $\{P_{2,i}\}$ of $\{P_i\}$ meeting $M \setminus \overline{P}_1$, which is contained in a proper submanifold $\overline{P}_2 \subset M$ of the maximal codimension $1 \leq m_2 \leq m_1 \leq 2$. Similarly, the maximality makes the limit of $P_{2,i}$ dense in $\overline{P}_2$, and therefore $\overline{P}_2 \in \mathscr{P}_{m_2} \subset \mathscr{P}_1 \cup \mathscr{P}_2$.

Finally, since $\mathscr{P}_1 \cup \mathscr{P}_2$ is finite, we can only find finitely many closures $\overline{P}_1, \overline{P}_2, \cdots, \overline{P}_l$, and the number of elements in $\{P_i\}$ intersecting $M \setminus (\bigcup_{j=1}^{l} \overline{P}_j)$ is finite. Thus, $\mathscr{P}$ is contained in a finite union of proper submanifolds.

The same method applies to any ambient dimensions, so similarly, we prove by induction that for each $1 \leq k \leq n-2$, $\bigcup_{j \leq k} \mathscr{P}_j$ is contained in a finite union of proper submanifolds of M, this implies Lemma 5.1.5.

Now we complete the proof of Lemma 5.1.4. By Lemma 5.1.5, all elements in $\mathscr{P}$ lie in a finite union of proper submanifolds of M. So there must be a non-empty compact set K in $\Delta \subset \mathbb{H}^n$ away from the fundamental domain of this union, it means that K is disjoint from all $D(\gamma)$ with $\gamma \in \mathscr{L}$, but this contradicts $(\star)$. Therefore, $\mathscr{L}$ contains an element with a dense Γ-orbit, and $\mathscr{L} = \mathscr{C}$ follows from the closedness and Γ-invariance of $\mathscr{L}$.

$\square$

Fix a round circle $\gamma \in \mathscr{L}$ that has a dense Γ-orbit, γ can be represented by $\lim_{i\to\infty} \Lambda(\phi_i \Pi_i \phi_i^{-1})$, where $\phi_i \in F_i(\epsilon_i, R_i)$, as $i \to \infty$, we have $\epsilon_i \to 0$ and $R_i \to \infty$. Let D_i, Ω_i be the lifts of S_i, Σ_i to B^n preserved by $\phi_i \Pi_i \phi_i^{-1}$. We have proved in Section 5.1.1 that after passing to a subsequence, D_i converges to the totally geodesic disc $D = D(\gamma)$. Moreover, it follows from (5.1.9) that

$$\lim_{i\to\infty} \int_{\Omega_i \cap B_{R_i}(0)} \big(-(K_{12}+1) + \frac{1}{2}|A|^2 + 2\pi \sum_j d_{i_j} \delta_{q_{i_j}} \big) dA_h = 0.$$

Since $K_{12} \leq -1$, we obtain that

$$\lim_{i\to\infty} \int_{\Omega_i \cap B_{R_i}(0)} \big(-(K_{12}+1) + \frac{1}{2}|A|^2 \big) dA_h = 0, \tag{5.1.10}$$

and

$$\lim_{i\to\infty} \int_{\Omega_i \cap B_{R_i}(0)} \sum_j d_{i_j} \delta_{q_{i_j}} dA_h = 0.$$

Recall that Q_i represents the set of branch points in Ω_i, the latter equation implies that

$$\#\{Q_i \cap B_{R_i}(0)\} \to 0 \quad \text{as } i \to \infty. \tag{5.1.11}$$

So for large enough i, Ω_i has no branch points inside $B_{R_i}(0)$.

Lemma 5.1.7. There exists a connected component $\Omega_i^0 \subset \Omega_i$, so that Ω_i^0 is a disc and it converges smoothly to a totally geodesic hyperbolic disc Ω with $\partial_\infty \Omega = \gamma$.

Proof. We can explore the convex hulls in the same way as in Section 3 of [11], then Proposition 2.5.4 in [8] and the Morse lemma give rise to a uniform constant $R_0 > 0$, so that

$$d_H(C_h(\Lambda(\phi_i \Pi_i \phi_i^{-1}), C_{h_0}(\Lambda(\phi_i \Pi_i \phi_i^{-1})) \leq R_0,$$

where C_h and C_{h_0} represent the convex hull with respect to metrics h and h_0, respectively. Moreover, [11] also proves that

$$\Omega_i \subset C_h(\Lambda(\phi_i \Pi_i \phi_i^{-1}). \tag{5.1.12}$$

Let H_i^r be the hypersurface in $\mathbb{H}^n$ with the fixed distance r to D_i. By the proof of Lemma 5.1.2, when $r > \tanh^{-1} \frac{|A|^2_{L^\infty(D_i)}}{2}$, H_i^r is strictly convex and it bounds inside the convex hull of $\Lambda(\phi_i \Pi_i \phi_i^{-1})$, so

$$d_H(C_{h_0}(\Lambda(\phi_i \Pi_i \phi_i^{-1}), D_i) \leq \tanh^{-1} \frac{|A|^2_{L^\infty(D_i)}}{2}.$$

Combining these estimates, we conclude that the Hausdorff distance between D_i and Ω_i is uniformly bounded. So there exists $R > 0$, for $i >> 1$ and generic $r \geq R$, Ω_i intersects $B_r(0)$ by a union of circles. Then we can slightly perturb R_i so that $\Omega_i \cap B_{R_i}(0)$ is a union of circles.

Let Ω_i^0 be a component of $\Omega_i \cap B_{R_i}(0)$ that intersects $B_R(0)$, by (5.1.11), for sufficiently large i, Ω_i^0 is free of branch points, so it is embedded in B^n. We claim that it is a disc. Otherwise, if Ω_i^0 were an annulus, then we could find a larger ball $B_{R_i'}(0)$ with some $R_i' > R_i$ whose boundary met tangentially with Ω_i^0 at some point. However, the convexity of $\partial B_{R_i'}(0)$ and the minimality of Ω_i^0 contradict the maximum principle. Therefore, Ω_i^0 is a disc provided that i is large enough. Furthermore, The small total curvature estimates based on (5.1.10) imply that

$$\limsup_{i\to\infty}\{|K_{12}(x)+1|+\frac{1}{2}|A(x)|^2 : x\in\Omega_i^0\}=0. \tag{5.1.13}$$

From the standard compactness theorem for minimal surfaces with a uniform bound on the second fundamental form, after passing to a subsequence, Ω_i^0 converges smoothly to a minimal disc Ω in (B^n,h). Moreover, Ω is totally geodesic and it has sectional curvature equal to -1.

It remains to show that $\partial_\infty\Omega=\gamma$. Take a sequence $x_i\in\Omega_i^0$ that converges to $x\in\Omega$, and take $y_i\in\Lambda(\phi_i\Pi_i\phi_i^{-1})$. Let α_i be the geodesic arc in (B^n,h) connecting x_i to y_i, and let β_i be the geodesic arc in Ω_i connecting x_i to y_i. Due to (5.1.12) and Proposition 2.5.4 in [8], we can find a uniform number $r>0$, such that β_i is contained in the r-neighborhood of α_i. Additionally, since Ω is totally geodesic, both α_i and β_i converge to the same geodesic arc in Ω that connects x to some $y\in\partial_\infty\Omega$. Then y_i converges to y on S^{n-1}_∞. As a consequence, $\partial_\infty\Omega\subset\gamma$, since $\partial_\infty\Omega$ is a circle, it coincides with γ.

□

As defined in Section 5 of [11], $T^1_D(M)$ and $T^1_\Omega(M)$ denote the projections of the circle bundles of D and Ω to the unit tangent bundles of M with respect to h_0 and h, respectively. Since $T^1_D(M)$ is dense in the unit tangent bundle $T^1M(h_0)$, then via the homeomorphism from $T^1M(h_0)$ to $T^1M(h)$ that maps geodesics to geodesics [20], we obtain that $T^1_\Omega(M)$ is dense in $T^1M(h)$. Thus for any $(x,v)\in T^1M(h)$, there is a sequence $\{\psi_i(\Omega)\}_i$, $\psi_i\in\Gamma$, converging to a totally geodesic hyperbolic disc $\Omega_{(x,v)}$ in (B^n,h), whose projection in M contains a geodesic passing through x with direction v. According to the ergodicity of the 2-frame flows on the negatively curved manifolds of arbitrary odd dimensions (Section 4 in [9]), the set of totally geodesic hyperbolic discs is dense in $Gr_2(M)$.

Therefore, if the equality (5.1.7) holds, then (M, h) is hyperbolic, and it is isometric to (M, h_0) due to the Mostow rigidity theorem.

Proof of Rigidity

First of all, if the metric h has sectional curvature less than or equal to -1, then $\Pi \in S(M, \lfloor L \rfloor, \epsilon)$ implies that $\text{area}_h(\Pi) \leq 4\pi(L-1)$ because of the generalized Gauss equation (5.1.8), thus, $E(h) \geq 2 = E(h_0)$.

Next, suppose $E(h) = 2$. Assume that there exists $\eta > 0$, such that for all $L > 0$ and all increasing sequence $\{k_i\} \subset \mathbb{N}$, the condition $\Pi \in G(M, \lfloor (1+\eta)L \rfloor, \frac{1}{k_i})$ must produce that $\text{area}_h(\Pi) \leq 4\pi(L-1)$. As a result,

$$E(h) \geq \liminf_{L\to\infty} \frac{\ln \# G(M, \lfloor (1+\eta)L \rfloor, \frac{1}{k_i})}{L \ln L} \geq 2(1+\eta),$$

which violates the assumption. Therefore, there exists an increasing sequence $\{k_i\} \subset \mathbb{N}$, a sequence of integers $\{g_i\}$ and $\Pi_i \in G(M, g_i, \frac{1}{k_i})$, so that

$$\text{area}_h(\Pi_i) > 4\pi((1 - \frac{1}{i})g_i - 1).$$

Let Σ_i and S_i be the minimal surfaces that minimize the area in the homotopy class Π_i with respect to metrics h and h_0, respectively. Then from the inequality above and Theorem 5.1.3,

$$1 \geq \limsup_{i\to\infty} \frac{\text{area}_h(\Sigma_i)}{\text{area}(S_i)} \geq \liminf_{i\to\infty} \frac{\text{area}_h(\Sigma_i)}{\text{area}(S_i)} \geq \liminf_{i\to\infty} \frac{4\pi((1-\frac{1}{i})g_i - 1)}{4\pi(g_i - 1)} = 1.$$

The equality holds if and only if h is hyperbolic, and thus it is isometric to h_0.

5.2 Locally Symmetric Manifolds of Negative Curvature

In this section, we extend Theorem 1.2.6 to locally symmetric spaces of rank one apart from real hyperbolic manifolds.

5.2.1 Definition of Entropy

Let M be a closed $2n$ $(4n, 16)$-dimensional complex hyperbolic manifold (quaternionic hyperbolic manifold, Cayley plane, respectively), where $n \geq 2$. Then its sectional curvature is between -4 and -1. In the Siegel domain model of $\mathbb{H}^n_{\mathbb{C}}$ $(\mathbb{H}^n_{\mathbb{H}}, \mathbb{H}^2_{Ca})$, its boundary is identified with the one point compactification of the Heisenberg group. A totally geodesic disc in $\mathbb{H}^n_{\mathbb{C}}$ $(\mathbb{H}^n_{\mathbb{H}}, \mathbb{H}^2_{Ca})$ with constant sectional curvature -1 is called *totally real* and is isometric to $\mathbb{H}^2_{\mathbb{R}}$, whose boundary is a *real circle*.

A K-quasiconformal map on $\partial\mathbb{H}^n_{\mathbb{C}} = S^{2n-1}_\infty$ $(\partial\mathbb{H}^n_{\mathbb{H}} = S^{4n-1}_\infty, \partial\mathbb{H}^2_{Ca} = S^{15}_\infty)$ is defined as in Section 1.1.1 with respect to Carnot-Carathéodory metric (see [37]). In particular, for quaternionic hyperbolic spaces and the Cayley plane, [51] points out that any quasiconformal maps are actually conformal. Then the quasi-real circles are real circles, and each of them determines a unique totally geodesic totally real disc. Therefore, if the manifold M admits quaternionic hyperbolic or Cayley metric, then $S(M, g) = S(M, g, 0)$, and let $S(M) = \bigcup_{g \geq 2} S(M, g)$. For any metric h on M, the corresponding minimal surface entropy is redefined as

$$E(h) = \liminf_{L \to \infty} \frac{\ln \#\{\text{area}_h(\Pi) \leq 4\pi(L-1) : \Pi \in S(M)\}}{L \ln L}.$$

Besides, if M admits a complex hyperbolic metric, we still adopt the definition in (1.1.1). Then the main theorem related to the locally symmetric spaces is stated as follows.

Theorem 5.2.1. *Let (M, h_0) be a closed locally symmetric space of rank one. And let h be another metric on M. If the sectional curvature of h is pointwise less than or equal* *to that of the locally symmetric metric, then*

$$E(h) \geq E(h_0) = 2,$$

If the equality holds, then h is isometric to h_0.

5.2.2 Entropy of Locally Symmetric Metric

Hamenstädt [22] proved the existence of the surface subgroup of cocompact lattice in any simple rank one Lie group of noncompact type distinct from $SO(2m, 1)$. From the perspective of geometry, let M be any closed locally symmetric space except an even-dimensional real hyperbolic manifold, then for sufficiently small ϵ, there exists an essential surface $\Sigma_\epsilon \subset M$, which is $(1 + o_\epsilon(1))$-quasigeodesic. As argued in Section 5.1.2, it is associated with a surface subgroup in $S(M, g, \epsilon)$ for the complex hyperbolic case, or $S(M, g)$ for the quaternionic hyperbolic and Cayley case. Moreover, $s(M, g, \epsilon)$ (or $s(M, g)$) is also bounded below by $(c_1 g)^{2g}$. On the other hand, since the power of the upper bound of $s(M, g)$ in Section 5.1.2 only depends on the topology of closed surfaces, the upper bound $(c_2 g)^{2g}$ also holds after modifying the coefficient c_2.

Firstly, if (M, h_0) is quaternionic hyperbolic or Cayley hyperbolic, it's not hard to show $E(h_0) = 2$ based on the above estimates.

If (M, h_0) is complex hyperbolic, however, it requires more discussion on the second fundamental form as in Section 5.1.1. Lemma 5.1.1 is still true because of the following fact. Let γ be the image of a real circle by an $(1 + \epsilon)$-quasiconformal map on S^{2n-1}_∞. Then there exist $\epsilon' = o_\epsilon(1)$ and a real circle $c \subset S^{2n-1}_\infty$, such that the ϵ'-neighborhood of c in S^{2n-1}_∞, denoted by N, contains γ. Any real circle in ∂N bounds a totally geodesic totally real disc in $\mathbb{H}^n_{\mathbb{C}}$, then there is a hypersurface $T \subset \mathbb{H}^n_{\mathbb{C}}$ homeomorphic to $D^2 \times S^{2n-3}$ asymptotic to ∂N, the diameter of T converges to zero on compact sets as

ϵ goes to zero. Moreover, two of the principal curvatures of T are zero, the others are at least $\dfrac{1}{\tanh 2r(x)} > 0$, where $r(x)$ is the Euclidean radius of T centered at x, therefore the convex hull of γ lies in T. Regarding the proof of Lemma 5.1.2, since all principal curvatures of the equidistant hypersurfaces satisfy the Riccati equations, the comparison theorem associated with Riccati equations (see [17]) ensures the two-convexity of each hypersurface. Moreover, the compactness theorem of the quasiconformal maps on the Heisenberg group can be found in [37]. So Lemma 5.1.2 extends easily to the complex hyperbolic case. Likewise, since the manifold has pinched sectional curvature between -4 and -1, there are analogues of the inequalities (5.1.2) and (5.1.4) (see [5]). Let $S \subset M$ be a surface that minimizes the area in its homotopy class contained in $S_\epsilon(M)$. Then the convergence of area-minimizing surfaces mod 2 and the uniqueness indicate the absence of branch points on S, as well as the property that $|A|^2_{L^\infty(S)} \to 0$ as $\epsilon \to 0$. Following the computation in Section 5.1.2, we deduce that $E(h_0) = 2$ for the complex hyperbolic case.

5.2.3 Proof of Rigidity

Let h be a metric on M with sectional curvature less than or equal to that of the symmetric metric, then it follows from the generalized Gauss equation that $E(h) \geq 2$.

To deduce the rigidity of Theorem 5.2.1, the key idea is to apply the hyperbolic rank rigidity theorem established by Hamenstädt in [21]. Suppose that N is a closed manifold whose sectional curvature is less than or equal to -1. The *hyperbolic rank at* $v \in T^1N$ is the dimension of the space generated by all parallel transports J along geodesic $\gamma_v = \exp(tv)$ such that $J(t) \perp \gamma_v'(t)$ and $J(t), \gamma_v'(t)$ span a plane of sectional curvature -1. The *hyperbolic rank of* N is the minimum of hyperbolic rank at all $v \in T^1N$. Hamenstädt's theorem states that if such manifold N has hyperbolic rank at least 1, then it must be locally symmetric, namely a compact quotient of $\mathbb{H}^n_{\mathbb{R}}, \mathbb{H}^n_{\mathbb{C}}, \mathbb{H}^n_{\mathbb{H}}$

or $\mathbb{H}^2_{Ca}$.

Therefore, It suffices to check that the hyperbolic rank of M is positive. Since the technic in [11] proving ($\star$), Shah's density lemma in [60], as well as the equidistribution theorem by Mozes and Shah [50] all apply to locally symmetric spaces, an analogue of Lemma 5.1.4 can be deduced in the same way. Furthermore, repeating the same proof of Lemma 5.1.7, we obtain a totally geodesic disc Ω in (B^m, h) of section curvature -1, whose limit set has a dense Γ-orbit on the set of all real circles in S^{m-1}_∞, where m is the dimension of M. This result in tandem with Gromov's geodesic rigidity theorem [20] implies that every geodesic along $v \in T^1M$ is contained in a closed totally geodesic submanifold of dimension $2 \leq k \leq n$ and with sectional curvature -1. Therefore (M, h) has a positive hyperbolic rank, and finally the rigidity result of Theorem 5.2.1 follows from Hamenstädt's hyperbolic rank rigidity theorem mentioned above.

5.3 Hyperbolic Three-manifolds of Finite Volume

In this section, we compute the minimal surface entropy corresponding to the non-compact hyperbolic 3-manifolds of finite volume that have a finite number of cusps. Suppose (M, h_0) is a hyperbolic 3-manifold with k cusps, where $k \geq 1$, then M can be realized by the interior of a compact hyperbolic manifold whose boundary consists of k flat tori, we also denote the compact manifold with boundary by M. As before, a closed surface immersed in M is essential if the immersion is π_1-injective. Additionally, a noncompact surface (or a compact surface with boundary) is said to be *essential* if the immersion is π_1-injective and π_1-injective relative to the boundary. By Lemma 2.1 in [24], any π_1 injective noncomapct surface with genus at least 2 is also essential. When $S \subset M$ is an essential surface, the image of $\pi_1(S)$ in $\pi_1(M)$ is called a *surface subgroup.* Let $S_j(M, g)$ denote the set of surface subgroups up to conjugacy so that the corresponding surfaces have genus at most g and at most j simple closed curves on the boundary counting multiplicity, and let $S_j(M,g,\epsilon) \subset S_j(M,g)$ consist of the conjugacy classes whose limit sets are $(1+\epsilon)$-quasicircles. Moreover,

$$S_j(M,\epsilon) = \underset{g\geq 2}{\cup} S_j(M,g,\epsilon).$$

Given an arbitrary Riemannian metric h on M, we define the minimal surface entropy of h on M.

$$E_j(h) = \lim_{\epsilon\to 0} \liminf_{L\to\infty} \frac{\ln \#\{\mathrm{area}_h(\Pi) \leq 4\pi(L-1) : \Pi \in S_j(M,\epsilon)\}}{L\ln L}. \tag{5.3.1}$$

The statement of the theorem is the following.

Theorem 5.3.1. *Let M be a hyperbolic 3-manifolds with k cusps, then for any $j \in \mathbb{N}_{\geq 0}$, we have*

$$E_j(h_0) = 2.$$

Remark 5.3.2. To evaluate the entropy of M with respect to other metrics, we need further evidence concerning the existence of surfaces with the least area. However, this remains an open problem.

Nevertheless, when the metric is asymptotically cusped, we can ensure the existence of such surfaces. Currently, in collaboration with Franco Vargas Pallete, we are working on the minimal surface entropy in this scenario, and comparing it to that of the hyperbolic metric.

5.3.1 Existence of Minimal Surfaces

We've seen that for closed hyperbolic manifolds, the works of Schoen-Yau [57] and Sacks-Uhlenbeck [55] indicate that every surface subgroup produces a least-area surface in the homotopy class. However, the argument fails to hold for some noncompact ambient 3-manifolds (see Example 6.1 in [25]). In this section, we only list the existence results for hyperbolic 3-manifolds with finitely many cusps. First, it was stated by Hass-Rubinstein-Wang [24] and Ruberman [54] that in a cusped hyperbolic 3-manifold, any noncompact essential surface with genus at least 2 can be homotoped to a least-area surface. Then in [13] and [14], Collin-Hauswirth-Mazet-Rosenberg proved the existence of closed essential minimal surfaces embedded in such manifolds. Later on, Huang-Wang addressed the question for immersed essential surfaces in [28], they showed that any immersed essential surface in a cusped hyperbolic 3-manifold with genus at least 2 can be homotoped into an area-minimizer. Therefore, the minimal surface entropy (5.3.1) of cusped hyperbolic 3-manifolds (M, h_0) can be approximated by counting the least-area surfaces up to homotopy, and we'll estimate the upper bound and lower bound of $\#S_j(M, g, \epsilon)$ associated with h_0 and prove the theorem.

5.3.2 Upper Bound

Counting Closed Minimal Surfaces

For any closed surface $\bar{S}$ with genus g and π_1-injective immersion $f : \bar{S} \to M$ that determines a surface subgroup in $S_0(M, g, \epsilon)$, there exist a hyperbolic structure on $\bar{S}$ and a homotopy of f that is pleated with respect to this structure, we still denote it by f (see 8.10 in [64] or Lemma 3.6 in [10]). Additionally, let $S = f(\bar{S})$. When ϵ is small enough, S has no accidental parabolics, hence the systole length of S, denoted by $sl(S)$, is simply twice the injectivity radius $I(S)$ of S. Let s be the systole of M, since f does not increase the length of closed geodesics, we have

$$2I(S) = sl(S) \geq sl(M) = s > 0.$$

Then we say that S is $\frac{s}{2}$-thick, thus from Lemma 2.1 of [33], there exists $k = k(s) > 0$ and a triangulation τ on S, such that

(1) each edge of τ is a geodesic arc of length at most $\frac{s}{8}$,

(2) τ has at most kg vertices and edges,

(3) the degree of each vertex is at most k.

The set of all triangulations on S with genus g satisfying (2) and (3) is denoted by $\mathcal{T}(k, g)$. As shown in [33], there exists a constant c depending only on k, so that

$$\#\mathcal{T}(k, g) \leq (cg)^{2g}. \tag{5.3.2}$$

Furthermore, We claim that $\#S_0(M, g, \epsilon)$ can be estimated by counting the number of homotopy classes of the pleated immersions. Let f_1 and f_2 be two pleated maps of genus g surfaces S_1 and S_2, respectively. Suppose that the triangulations $\tau(S_1)$ and $\tau(S_2)$ are equivalent, i.e., there is a homeomorphism $h : S_1 \to S_2$, such that $h(\tau(S_1)) = \tau(S_2)$. We say that f_1 and f_2 are *homotopic* if f_1 is homotopic to $f_2 \circ h$ in M. Moreover, since M has finite volume, it is covered by finitely many balls of radius $\frac{s}{16}$, say $B_1, \cdots, B_m$. We assume that for any vertex $v \subset \tau(S_1)$, the points $f_1(v)$ and $f_2(h(v))$ of M are contained in the same ball B_i. Then the distance between $f_1(v)$ and $f_2(h(v))$ is at most $\frac{s}{8}$. Given another vertex $v' \in \tau(S_1)$. Let e_1, e_2 be the edges connecting $f_1(v)$ and $f_1(v')$, $f_2(h(v))$ and $f_2(h(v'))$, respectively, the lengths are at most $\frac{s}{8}$. And let s_v, $s_{v'}$ be the segments connecting $f_1(v)$ and $f_2(h(v))$, $f_1(v')$ and $f_2(h(v'))$, respectively, the lengths are at most $\frac{s}{8}$. So we get a closed curve

$$\gamma := e_1 \cup s_v \cup e_2 \cup s_{v'} \quad \text{with} \quad \text{length}(\gamma) \leq \frac{s}{2} < sl(M).$$

We notice that γ cannot shrink homotopically to a closed geodesic γ', since otherwise, it gives rise to a smaller systole length of M. As a result, γ must bound a disc, in other words, any two segments of M with the same endpoints with length less than $\frac{sl(M)}{2}$ must be homotopic. And therefore, repeating this argument for any pair of vertices of $\tau(S_1)$, We conclude that f_1 and f_2 are homotopic.

Let $\tilde{S}_0(M,g)$ be the subset of $S_0(M,g)$ that includes surfaces of fixed genus g. For any $\tau \in \mathcal{T}(k,g)$ satisfying (1)-(3), any vertex $v_1 \in \tau$ is mapped to a ball B_i with m possibilities. For $v_2 \neq v_1$ that bounds an edge e with v_1. By (1), the length of e is at most $\frac{s}{8}$. And since the balls covering M have radius $\frac{s}{16}$, there is a finite number $M > 0$, such that v_2 can be mapped to at most N options of the balls. Therefore, it follows from (2) that

$$\#\tilde{S}_0(M,g) \leq mN^{kg-1}\#\mathcal{T}(k,g). \tag{5.3.3}$$

Finally, combining (5.3.2) and (5.3.3), we can find $c_2 > 0$, such that

$$\#S_0(M,g) \leq \sum_{i=2}^{g} \#\tilde{S}_0(M,i) \leq (c_2 g)^{2g}. \tag{5.3.4}$$

Counting noncompact minimal surfaces

Next, we estimate $\#S_j(M,g,\epsilon)$ for $j \geq 1$. For sufficiently small ϵ, let S be a minimal surface corresponding to an element in $S_j(M,g,\epsilon) \setminus S_0(M,g,\epsilon)$, where $g \geq 2$ and $j \geq 1$, then S can be seen as a compact surface with j boundary curves counting multiplicity, each of which is a simple closed curve C in one of the tori T_i, the homotopy class of C can be identified with a slope in $\mathbb{Q} \cup \{\infty\}$. Due to [24], for each g, there is a finite number $N(g) > 0$, such that for each boundary torus, the number of slopes that can be realized by boundary curves of immersed essential surfaces in M with genus at most g is bounded from above by $N(g)$. Moreover, $N(g)$ grows at most quadratically in g. Cutting off each boundary curve of S and filling it with a disc, we obtain a closed surface in M associated with an element in $S_0(M,g,\epsilon)$ up to homotopy. As a result, $\#S_j(M,g,\epsilon)$ is bounded by a constant depending on k, j and ϵ. More precisely,

$$\#S_j(M, g, \epsilon) \leq \sum_{i=0}^{j} (kN(g))^i \#S_0(M, g, \epsilon) \leq (c_2' g)^{2g+2j+2},$$

where $c_2' > 0$ depends only on M and ϵ, and the last inequality follows from (5.3.4).

5.3.3 Lower Bound

Recently, Kahn-Wright [34] proved that when $M = \mathbb{H}^3/\Gamma$ is noncompact with finite volume, for any sufficiently small $\epsilon > 0$, there exists $(1+\epsilon)$-quasi-Fuchsian surface subgroups of Γ. And the construction gives rise to an essential surface Σ_ϵ, which is also sufficiently well-distributed and $(1+\epsilon)$-quasigeodesic. So as defined in the compact case, we let $G(M, g, \epsilon)$ be the subset of $S_0(M, g, \epsilon)$ consisting of homotopy classes of finite covers of Σ_ϵ with genus at most g, then we also have

$$\#S_j(M, g, \epsilon) \geq \#S_0(M, g, \epsilon) \geq \#G(M, g, \epsilon) \geq (c_1 g)^{2g}, \quad \forall j \in \mathbb{N}$$

where $c_1 > 0$ depends only on M and ϵ.

5.3.4 Proof of Theorem

If a surface subgroup $\Pi < \Gamma$ has $i \leq j$ cusps, then from [18],

$$\text{area}(\Pi) = 4\pi(g-1) + 2\pi i - \frac{1}{2}\int |A|^2 dA.$$

Since it is proved in Section 5.1.1 that $|A|^2 = o_\epsilon(1)$, and $i \leq j$ is uniformly bounded, then for any $\eta > 0$, the following conclusions still hold for sufficiently large L and sufficiently small ϵ. For $\Pi \in S_j(M, \epsilon)$, if it satisfies $\text{area}(\Pi) \leq 4\pi(L-1)$, then we have $\Pi \in S_j(M, \lfloor(1+\eta)L\rfloor, \epsilon)$. On the other hand, if $\Pi \in S_j(M, \lfloor(1-\eta)L\rfloor, \epsilon)$, then

$\text{area}(\Pi) \leq 4\pi(\lfloor(1-\eta)L\rfloor - 1) + 2\pi j \leq 4\pi(L-1)$. Therefore, using the previous estimates, we have

$$
\begin{aligned}
2(1-\eta) &\leq \liminf_{L\to\infty} \frac{\ln s(M, \lfloor(1-\eta)L\rfloor, \epsilon)}{L \ln L} \\
&\leq \liminf_{L\to\infty} \frac{\ln \#\{\text{area}_h(\Pi) \leq 4\pi(L-1) : \Pi \in S_\epsilon(M)\}}{L \ln L} \\
&\leq \liminf_{L\to\infty} \frac{\ln s(M, \lfloor(1+\eta)L\rfloor, \epsilon)}{L \ln L} \leq \liminf_{L\to\infty} \frac{2\lfloor(1+\eta)L\rfloor + 2j + 2}{L} = 2(1+\eta),
\end{aligned}
$$

the last inequality uses the fact that j is a fixed integer. It follows that $E_j(h_0) = 2$.

CHAPTER 6

RELATIONSHIP BETWEEN AVERAGE AREA RATIO AND MINIMAL SURFACE ENTROPY

In this last chapter, we examine the relationship between average area ratio and minimal surface entropy and prove Theorem 1.2.7. Furthermore, we note that all the proofs below also work for the other closed locally symmetric spaces.

6.1 Proof of Inequality

The proof follows directly from [43], but for readers' convenience, it is stated as follows.

We let

$$\alpha := \lim_{i\to\infty} \frac{\mathrm{area}_h(\Pi_i)}{4\pi(g_i - 1)}.$$

For any $\delta > 0$, we can take i sufficiently large, such that

$$\lim_{i\to\infty} \frac{\mathrm{area}_h(\Pi_i)}{4\pi(g_i - 1)} < \alpha + \delta.$$

Let Π_i^k be the k-cover of Π_i. Since Π_i has genus $g_i \geq 2$,

$$4\pi(g_i^k - 1) \geq k\, 4\pi(g_i - 1),$$

then the least area surface in the homotopy class of Π_i^k with respect to h satisfies that

$$\lim_{i\to\infty} \frac{\mathrm{area}_h(\Pi_i^k)}{4\pi(g_i^k - 1)} \leq \lim_{i\to\infty} \frac{k\,\mathrm{area}_h(\Pi_i)}{k\,4\pi(g_i - 1)} = \lim_{i\to\infty} \frac{\mathrm{area}_h(\Pi_i)}{4\pi(g_i - 1)} < \alpha + \delta. \tag{6.1.1}$$

According to Müller-Puchta's formula (see [33]), there exists a constant c_i that depends only on M and i, such that the following is true when g_i is large.

$$s(M, g_i^k, \frac{1}{i}) \geq (c_i g_i^k)^{2g_i^k}. \tag{6.1.2}$$

Define L_i^k in the following way

$$4\pi(L_i^k - 1) = (\alpha + \delta)4\pi(g_i^k - 1) \Longrightarrow \lim_{k\to\infty} \frac{g_i^k}{L_i^k} = \frac{1}{\alpha+\delta}. \tag{6.1.3}$$

Combining (6.1.2) with (6.1.1), we have that

$$\#\{\mathrm{area}_h(\Pi) \leq 4\pi(L_i^k - 1) : \Pi \in S_{\frac{1}{i}}(M)\} \geq (c_i g_i^k)^{2g_i^k},$$

Therefore, by (6.1.3),

$$\begin{aligned}
E(h) &= \lim_{i\to\infty} \liminf_{k\to\infty} \frac{\ln \#\{\mathrm{area}_h(\Pi) \leq 4\pi(L_i^k - 1) : \Pi \in S_{\frac{1}{i}}(M)\}}{L_i^k \ln L_i^k} \\
&\geq \lim_{i\to\infty} \liminf_{k\to\infty} \frac{(c_i g_i^k)^{2g_i^k}}{L_i^k \ln L_i^k} \\
&= \frac{2}{\alpha+\delta}.
\end{aligned}$$

Since δ is an arbitrarily small positive number, we conclude that

$$\mathrm{Area}_{\mathrm{Id}}(h/h_0)E(h) = \lim_{i\to\infty} \frac{\mathrm{area}_h(S_i)}{4\pi(g_i-1)} E(h) \geq \lim_{i\to\infty} \frac{\mathrm{area}_h(\Pi_i)}{4\pi(g_i-1)} E(h) \geq 2. \tag{6.1.4}$$

6.2 Proof of Rigidity

If $\mathrm{Area}_{\mathrm{Id}}(h/h_0)E(h) = 2$, then (6.1.4) yields that

$$\lim_{i\to\infty} \frac{\mathrm{area}_h(\Pi_i)}{\mathrm{area}_h(S_i)} = 1. \tag{6.2.1}$$

To make use of this equality, we run the mean curvature flow in (B^n, h) with initial condition D_i, which is the lift of S_i in $\mathbb{H}^n$, then we estimate the decay rate of the area. First of all, we need to review and establish some tools for complete, noncompact surfaces moving by mean curvature. The classical short-time existence theorem for compact manifolds moving by mean curvature is well-known [23]. However, the general theory for complete, noncompact manifolds has not been established in the literature. There are only several essential contributions in some special cases: Ecker-Huisken [16] proved the codimension one case in which only a local Lipschitz condition on the initial hypersurface was required. For higher codimensions, Chau-Chen-He [12] discussed the case of nonparametric mean curvature flow for flat metrics. The result related to our case is listed as follows.

Lemma 6.2.1. *There exist $T > 0$ and $C > 0$ depending only on M, so that for sufficiently large $i \in \mathbb{N}$, we can find a solution $D_i(t)$ to the mean curvature flow in (M, h) with initial condition $D_i(0) = D_i$, where $0 \leq t \leq T$. Additionally, the mean curvature of $D_i(t)$ and its derivative are both bounded uniformly by C.*

Proof. Notice that after passing to a subsequence, D_i converges smoothly on compact sets to a disc D, and each of them is a cover of a compact surface in M. Take $x \in D$, the standard theory indicates that there is a number $T_0 > 0$, such that for any $k \in \mathbb{N}$, we can find a solution $D_i^k(t)$ to the mean curvature flow with initial condition $\overline{B(x,k)} \cap D_i$, where $0 \leq t \leq T_0$. Since T_0 depends only on the second fundamental form of D, in particular, it's independent of i and k.

Next, in order to apply the Arzela-Ascoli theorem and estimate the mean curvature of $D_i(t)$ and its derivative for any small time t, we need the following preparation. Claim that for any $\delta > 0$, and any spacetime $X_i^k = (x_i^k, t)$ of $D_i^k(t)$, there exists an open neighborhood U_i^k of X_i^k, so that the *Guassian density ratio*

$$\Theta(D_i^k(t), X_i^k, r) := \int_{y\in D_i^k(t-r^2)} \frac{1}{4\pi r^2} \exp\big(-\frac{|y-x_i^k|^2}{4r^2}\big)\, d\mathcal{H}^2(y)$$

satisfies that

$$\Theta(D_i^k(t), X_i^k, r) \leq 1+\delta, \quad \forall\, 0<r<d(X_i^k, U_i^k). \tag{6.2.2}$$

If this wasn't true for some integers i and k, then we could pick a sequence $\lambda_j \to \infty$ as $j\to\infty$, and

$$\Theta(\mathcal{D}_{\lambda_j}(D_i^k(t) - X_i^k), 0, \lambda_j r) > 1+\delta,$$

where $\mathcal{D}_\lambda$ denotes the parabolic dilation $\mathcal{D}_\lambda(y,t) = (\lambda y, \lambda^2 t)$. Since the second fundamental form satisfies $|A|^2_{L^\infty(\mathcal{D}_{\lambda_j}(D_i^k(t))-X_i^k)} \to 0$ as $j\to\infty$, $\mathcal{D}_{\lambda_j}(D_i^k(t) - X_i^k)$ converges smoothly to a disc $\bar{D}_i^k$ whose second fundamental form vanishes. However, the inequality above implies that

$$\lim_{j\to\infty} \Theta(\bar{D}_i^k, 0, \lambda_j r) > 1,$$

which contradicts the topology of $\bar{D}_i^k$.

We've seen that (6.2.2) holds, so due to the local regularity theorem in [69], there is a uniform constant C_0 that is independent of i and k, so that at any spacetime $X_i^k = (x_i^k, t)$,

$$|A|^2(X_i^k) d(X_i^k, U_i^k) \leq C_0. \tag{6.2.3}$$

Therefore, Arzela-Ascoli theorem (see page 1494 of [69]) implies the short-time existence of the mean curvature flow with noncompact initial condition $D_i(0) = D_i$ on time interval $[0, T_0]$. Moreover, the interior estimate (6.2.3) validates the condition of the maximum principle ([16], Theorem 4.3). Arguing like Theorem 4.4 of [16], we can find $T>0$ and $C>0$ that make the lemma hold.

$\square$

Next, following the method of Lemma 6.5 in [43], we prove a similar result for the case of the higher codimensions.

Lemma 6.2.2.

$$\lim_{i\to\infty} \frac{1}{area_h(D_i)} \int_{D_i} |H_h|^2 dA_h = 0,$$

where H_h denotes the mean curvature of each disc D_i in (M,h).

Proof. Suppose by contradiction that there exists $\epsilon > 0$, such that after passing to a subsequence, and for $i \in \mathbb{N}$ large enough,

$$\frac{1}{\text{area}_h(D_i)} \int_{D_i} |H_h|^2 dA_h > 2\epsilon. \tag{6.2.4}$$

Under the mean curvature flow, the mean curvature satisfies the following evolution equation on the time interval $t \in [0,T]$ (see [62]).

$$\begin{aligned}\nabla_{\frac{d}{dt}}|H_h(t)|^2 =&\Delta|H_h(t)|^2 - 2|\nabla H_h(t)|^2 + 4\langle A_h^{jk}(t), H_h(t)\rangle_{h_t}\langle (A_h)_{jk}(t), H_h(t)\rangle_{h_t} \\ &+ 2(R_h)_{jklm}(t)H_h^j(t)(F_i)_p^k(t)H_h^l(t)F_i^{mp}(t),\end{aligned}$$

where $H_h(t)$, $A_h(t)$ and $F_i(t)$ represent the mean curvature, second fundamental form, and the immersion $F_i(t) : D_i(t) \to M$, respectively.

Using the result of Lemma 6.2.1, we can pick a uniform constant $C_1 > 0$, such that for all sufficiently large $i \in \mathbb{N}$, and for any $t \in [0,T]$,

$$\frac{d}{dt}\int_{D_i(t)} |H_h(t)|^2 dA_h \geq -C_1\text{area}_h(D_i(t)) \geq -C_1\text{area}_h(D_i),$$

the latter inequality follows from the fact

$$\frac{d}{dt}\text{area}_h(D_i(t)) = \frac{1}{2}\int_{D_i(t)} tr\langle \frac{d}{dt}h_{jk}, h^{jk}\rangle\, dA_h = -\int_{D_i(t)} |H_h(t)|^2 dA_h \leq 0.$$

We can choose $T_1 < \min\{\frac{\epsilon}{C_1}, T\}$, by assumption (6.2.4), for any $i \in \mathbb{N}$ and $t \in [0, T_1]$,

$$\int_{D_i(t)} |H_h(t)|^2 dA_h \geq \epsilon \, \mathrm{area}_h(D_i) \geq \epsilon \, \mathrm{area}_h(D_i(t)).$$

Then we obtain

$$\frac{d}{dt}\mathrm{area}_h(D_i(t)) = -\int_{D_i(t)} |H_h(t)|^2 dA_h \leq -\epsilon \, \mathrm{area}_h(D_i(t)).$$

Thus, for any sufficiently large $i \in \mathbb{N}$,

$$\frac{\mathrm{area}_h(\Pi_i)}{\mathrm{area}_h(D_i)} \leq \frac{\mathrm{area}_h(D_i(t))}{\mathrm{area}_h(D_i)} \leq \frac{e^{-\epsilon T_1}\mathrm{area}_h(D_i)}{\mathrm{area}_h(D_i)} = e^{-\epsilon T_1} < 1,$$

which violates (6.2.1). $\square$

Furthermore, arguing like Lemma 5.1.4 of Section 5.1.3, we deduce the following result from Lemma 6.2.2. For any round circle $c \subset \partial_\infty \mathbb{H}^n$, it has a dense $\pi_1(M)$-orbit in $\partial_\infty \mathbb{H}^n$. In addition, c can be represented by $\lim_{i\to\infty} \Lambda(\phi_i \Pi_i \phi_i^{-1})$, where $\phi_i \in \pi_1(M)$, and $\Lambda(\phi_i \Pi_i \phi_i^{-1})$ represents the limit set of $\phi_i \Pi_i \phi_i^{-1}$. Redefine D_i by the lifts of S_i to $\mathbb{H}^n$ preserved by $\phi_i \Pi_i \phi_i^{-1}$. It has the property that

$$\lim_{i\to\infty} \int_{D_i \cap B_{R_i}(0)} |H_h|^2 dA_h = 0, \quad R_i \to \infty.$$

Note that after passing to a subsequence, D_i converges to the totally geodesic disc $D(c) \subset \mathbb{H}^n$ that is asymptotic to c. Therefore, the mean curvature H_h vanishes on $D(c)$, namely, $D(c)$ is a minimal disc of B^n with respect to the metric h. And since c is chosen arbitrarily, every totally geodesic disc of $\mathbb{H}^n$ must be minimal for h.

We apply the result below for surfaces in 3-manifolds, the proof can be found in [43].

Lemma 6.2.3. *Every totally geodesic disc in $\mathbb{H}^3$ is minimal with respect to another* *metric h if and only if for any geodesic $\gamma \subset \mathbb{H}^3$, the following function is a constant*

$$t \mapsto |h|_{h_0}^{-\frac{1}{2}} h\left(\gamma'(t), \gamma'(t)\right).$$

Because of the ergodicity of the geodesic flow in (M, h_0), we can choose a geodesic γ of M whose orbit is dense in the unit tangent bundle. Let $\tilde{\gamma}$ be the lift of γ to $\mathbb{H}^n$. $\tilde{\gamma}$ must be contained in a hyperbolic 3-ball $B \approx \mathbb{H}^3$. Applying the previous lemma to the geodesic $\tilde{\gamma}$ and ambient manifold B, we conclude that

$$t \mapsto |h|_B|_{h_0}^{-\frac{1}{2}} h|_B\left(\tilde{\gamma}'(t), \tilde{\gamma}'(t)\right)$$

is constant. So the projection γ in M also satisfies that

$$t \mapsto |h|_{h_0}^{-\frac{1}{2}} h\left(\gamma'(t), \gamma'(t)\right)$$

is constant. Thus due to the density, there is a constant $c > 0$, such that for any vector field X of the unit tangent bundle of M,

$$|h|_{h_0}^{-\frac{1}{2}} h(X, X) = c\, h_0(X, X) \quad \Longrightarrow \quad |h|_{h_0}^{-\frac{1}{2}} h = c\, h_0.$$

As a result, h coincides with a multiple of h_0.

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