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FIRST-ORDER THEORY OF FUNCTION FIELDS OF P-ADIC CURVES

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To Aaron Silberstein, my constant source of inspiration in both life and mathematics.

Table of Contents

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Abstract

Let K be a function field over a local field k. We show that the valuation rings of K/k are definable. Furthermore, if $\operatorname{char} k = 0$ and $\operatorname{td}(K/k) = 1$, then any function field L over a local field λ elementarily equivalent to K must be isomorphic to K (as function fields).

Model Theory provides a way to understand the essence of mathematical objects, by abstracting away from their specific properties and studying their underlying structure.

Macintyre, Angus

Model Theory = Universal Algebra + Logic
Model Theory = Algebraic Geometry - Field

Keisler, H. Jerome & Chang, Chen Chung

1 Introduction

Logic is frequently viewed as a secluded isle, its findings rarely deemed relevant to the bustling conversations on the main mathematical continent. An exception to this is Model Theory. Introduced by Tarski in the 1950s, Model Theory has since demonstrated its formidable power as a tool to study other classical branches of mathematics, in particular Number Theory and Algebraic Geometry. For example, as of this writing, the only known proof of the celebrated Mordell-Lang Conjecture in Algebraic Geometry is model-theoretic.

The reason for this is that many of the algebraic, geometric and arithmetical properties of fields are determined by their first-order theory. One naturally wonders, to which extent does the first-order theory $\mathrm{Th}(K)$ of a field K determines its algebraic structure? Two fields K and L are said to be **elementarily equivalent**, denoted $K \equiv L$, if they have the same first-order theory. A fundamental question of the Model Theory of fields is: *For which class of fields does elementary equivalence implies isomorphism?*

This thesis shows that elementary equivalence implies isomorphism for the class of function fields K/k of p-adic curves. Actually it proves something slightly stronger. Theorem 6.1.1 states that if K/k is a function field of some p-adic curve, then any p-adic function field L/λ (not necessarily of transcendence degree 1) that is elementary equivalent to K/k, must be isomorphic to K/k as function fields. This is the consequence of our main theorem on the definability of valuation rings of function fields of curves over local fields (cf. Theorem 7.3.1).

In this Introduction, we give a brief outline of this thesis and the proof of the main result.

1.1 Organization of This Text

This work lies at the intersection of Model Theory, Number Theory and Algebraic Geometry. To accommodate readers of various backgrounds, as well as to motivate the use of certain concepts and ideas, Chapter 2-5 will be dedicated to introducing the various tools from these three areas. Note that though some degree of motivation is given for most concepts, only those whose precise definitions are important to the proof of the main theorem will be explicitly defined. The content of these preliminary chapters are as follow.

- Chapter 2 introduces the basic concepts of Model Theory and explains why we care about the *first*-order theory of fields.
- Chapter 3 motivates and states the Elementary Equivalence vs. Isomorphism

problem. It also includes a brief history of work on this problem.

- Chapter 4 reveals an interface between Model-Theory and Number Theory, namely via norms of cyclic algebras. In particular, Theorem 4.3.2 of Merkurjev-Susslin could be employ to give a first-order characterization of vanishing of certain cocycles.

- The star of Chapter 5 is Kato's Local Global Principle (Theorem 5.4.2), which relate the global behavior of cocycles to its local behavior. To capture "local" behavior, we review Krull's generalization of the concept of discrete rank 1 valuations, as well as the definition of Riemann-Zariski space $\mathcal{V}(K)$ of valuations of a field K. The topology of $\mathcal{V}(K)$ will be important in the proof of Proposition 7.3.1, the most technical proof in this text.

The proof of the main theorem is covered in the last two chapters. (See the next section for an outline of the proof).

1.2 Outline of the Proof

Let K/k be the function field of some smooth projective curve X over a local field k. In Chapter 6, we show that to the main theorem will follow from the definability of the valuation rings of K/k (i.e. valuation rings of K containing k). Thus the rest of the thesis (Chapter 7) is devoted to proving that the valuation rings of K/k are definable. The main ideas are as follows.

Suppose without loss of generality that K contains a primitive l-th root of unity. Let $H^i(K) := H^i(K;\mu_l)$ denote the Galois cohomology groups of K with coefficients in μ_l. We utilize the following insights of Pop in [8].

1. There is a first-order characterization of the vanishing of certain types of cocyles in $H^i(K)$.

2. Kato's Local-Global Principle in turn allows us to relate the vanishing of such cocycles to vanishing of its restriction to certain Henselizations of K.

The rough outline of our proof of definability of valuation rings of K/k is as follows.

1. To each "nice" triple $\mathbf{a} \in (K^*)^3$, we assign a cocyle $(\!|\mathbf{a}|\!)$ and a set $\text{supp}_X((\!|\mathbf{a}|\!))$ of valuations of K/k. The ring $\mathfrak{O}_X(\mathbf{a})$ of functions regular on $\text{supp}_X((\!|\mathbf{a}|\!))$ can also be defined in an obvious way.

2. It turns out that every valuations w_x of K/k could be realized as the common support of some nice tuples $\mathbf{a}$ and $\mathbf{b}$. For such choice of $\mathbf{a}$ and $\mathbf{b}$, the valuation ring O_{w_x} is definable in terms of the rings of regular functions $\mathfrak{O}_X(\mathbf{a})$ and $\mathfrak{O}_X(\mathbf{b})$. Thus definability of O_{w_x} will follow from definability of the latter two.

3. The definability of $\mathfrak{O}_X(\mathbf{a})$ can in turn be reduced to the definability of certain set $\Gamma(\mathbf{a})$ consisting of $\gamma \in K^*$ whose valuations on $\text{supp}_X((\!|\mathbf{a}|\!))$ satisfy certain constraint.

4. The most technical part of the text is the proof of definability of $\Gamma(\mathbf{a})$ (cf. Proposition 7.3.1). The rough idea is as follows. Fix a nice triple $\mathbf{a}$. To each γ one can assign three families $\mathcal{K}_{\gamma,i}, i = 0, 1, 2$ of extensions of K. These definitions of $\mathcal{K}_{\gamma,i}$ and $\Gamma(\mathbf{a})$ are chosen such that $\Gamma(\mathbf{a})$ is exactly the set of γ such that $(\!|\mathbf{a_i}|\!)$ is nontrivial over one of the family $\mathcal{K}_{\gamma,i}$. The latter is definable by Theorem 4.3.2 of Merkurjev-Susslin.

5. The key tools employed in the proof of the above characterization of $\Gamma(\mathbf{a})$ are Kato's Local Global Principle (Theorem 5.4.2) and the topology of the Riemann-Zariski's space of K.

2

Preliminaries on Model Theory

2.1 Language and Structures

Definition 2.1.1. *A* ***language*** $\mathcal{L} = ((f)_{f\in\mathcal{F}}; (R)_{R\in\mathcal{R}}; c_{c\in\mathcal{C}})$ *consists of three types of data:*

- *a set $\mathcal{F}$ of (n-nary, for any $n \in \mathbb{Z}_{\geq 0}$) function symbols;*
- *a set $\mathfrak{R}$ of (n-nary, for any $n \in \mathbb{Z}_{\geq 0}$) relation symbols;*
- *a set $\mathcal{C}$ of constant symbols.*

We will be working with the language $\mathcal{L}_{\text{Rings}} = (+, \cdot; 0, 1)$ of Rings which consists of two binary function symbols + and $\cdot$, together with two constant symbols 0 and 1.

Definition 2.1.2 ($\mathcal{L}$-Formulas and Sentences).

- *Informally, a* ***first-order $\mathcal{L}$-formula*** *is a finite sequence formed by the symbols of $\mathcal{L}$, together with variables as well as logical quantifiers $(\forall, \exists)$ and logical connectives $(\vee, \wedge, \neg)$.*

- *A variable not preceded by a quantifier is called a* ***free*** *variable. Those that are preceded by quantifiers are called* ***bounded*** *variables.*

- *A* ***first-order $\mathcal{L}$ sentence*** *is a* ***first-order $\mathcal{L}$-formula*** *in which all variables are bounded.*

Convention 2.1.1.

1. *From now on, unless otherwise specified, all formulas are assumed to be first-order.*

2. *When we want to emphasize that $x_1, \dots, x_n$ are (all of) the free variables in a formula ϕ, we will write $\phi(x_1, \dots, x_n)$.*

3. *In this thesis, formulas are defined using the following syntax.*

$$\phi(x_1, \dots, x_n) \longleftrightarrow \text{[Definition]}.$$

4. *For ease of reading, we will often use descriptive English name, instead of Greek letters, for formulas.*

Example 1. The following are $\mathcal{L}_{\text{Rings}}$ formulas. Only the last one is a sentence.

1. $\text{AreInverses}(x, y) \longleftrightarrow x \cdot y = 1;$

2. $\text{isInvertible}(x) \longleftrightarrow \exists y, x \cdot y = 1;$

3. $\text{isZeroOrInvertible}(x) \longleftrightarrow (x = 0) \vee (\exists y, x \cdot y = 1);$

4. EveryNonZeroElementIsInvertible $\longleftrightarrow \forall x, (x = 0) \vee (\exists y, x \cdot y = 1)$.

Definition 2.1.3. *An $\mathcal{L}$-**structure** $\mathcal{A}$ is a pair consisting of a set A (called its **universe**) and a function (called an **interpretation**) mapping*

- *each n-nary function symbol f of $\mathcal{L}$ to a function $f^{\mathcal{A}} : A^n \to A$;*
- *each n-nary relation symbol R of $\mathcal{L}$ to an n-nary relation $R^{\mathcal{A}}$ on A (i.e. a subset of A^n);*
- *each constant symbols c of $\mathcal{L}$ to an element $c^{\mathcal{A}}$ of A.*

We often denote such a structure by $\mathcal{A} = (A; (f^{\mathcal{A}})_{f\in\mathcal{F}}; (R^{\mathcal{A}})_{R\in\mathfrak{R}}; (c^{\mathcal{A}})_{c\in\mathcal{C}})$.

Example 2.

Every ring R is equipped with a canonical $\mathcal{L}_{\text{Rings}}$-structure, in which the function symbols $+, \cdot$ are interpreted as the additive and multiplicative operations on R, while the constant symbols 0 and 1 are interpreted as the additive and multiplicative identities of R, respectively.

On the other hand, not all $\mathcal{L}_{\text{Rings}}$-structures are rings. Recall that an $\mathcal{L}_{\text{Rings}}$ structure $\mathcal{A} = (A; +^{\mathcal{A}}, \cdot^{\mathcal{A}}; 0^{\mathcal{A}}, 1^{\mathcal{A}})$ is simply a set A together with a choice of binary operations $+^{\mathcal{A}}$ and $\cdot^{\mathcal{A}}$ and constants $0^{\mathcal{A}}$ and $1^{\mathcal{A}}$ of A. To guarantee that the interpreted operations endows A with a ring structure, we need to further require that $\mathcal{A}$ "satisfies" certain sentences. The definition of "satisfaction" will be given in the next section.

Convention 2.1.2. *From now on, when talking about a ring A as $\mathcal{L}_{Rings}$-structure, we will always assume the canonical interpretation. Thus for convenience, we will use the same notation A to denote both the underlying structure and the universe. Similarly, we will use the same symbols $(+; \cdot; 1; 0)$ to denote their interpretations in A.*

2.2 Definable Sets

Definition 2.2.1 ($\mathcal{L}$-Definability without Parameters)**.** *Let $\mathcal{A}$ be an $\mathcal{L}$ structure with universe A and let $\phi(\mathbf{t}) \coloneqq \phi(t_1, \dots, t_n)$ be an $\mathcal{L}$-formula. Given a tuple $\mathbf{a} \coloneqq (a_1, \dots, a_n) \in A^n$, we can substitute all the symbols in ϕ by their interpretations under $\mathcal{A}$ and all the variables t_i's by the corresponding a_i's to get a meaningful statement about A.*

1. *If this resulting statement is true for A, we say that $\mathcal{A}$ **satisfies** $\phi(\mathbf{a})$, and denote this by $\mathcal{A} \vDash \phi(\mathbf{a})$. Such a tuple $\mathbf{a}$ is called a **realization** of ϕ in $\mathbb{A}$.*

2. *The set of all realizations of ϕ in $\mathcal{A}$ is denoted by $\phi(\mathcal{A})$.*

3. *A subset $S \subset A^n$ is said to be **$\mathcal{L}$-definable (in $\mathcal{A}$) without parameter** if $S = \phi(\mathcal{A})$ for some $\mathcal{L}$- formula ϕ. We call ϕ a formula **defining** or **cutting out** S in $\mathcal{A}$.*

Example 1. Let $\mathcal{L} = \mathcal{L}_{\text{Rings}}$.

1. The positive real numbers are definable in $\mathbb{R}$ by the formula

$$\text{Pos}(t) \longleftrightarrow \exists y\Big(\big(y \neq 0\big) \wedge \big(t = y^2\big)\Big).$$

2. Similarly, by Lagrange's Four Square Theorem, $\mathbb{N}$ is definable inside $\mathbb{Z}$ by the formula

$$\text{Natural}(t) \longleftrightarrow \exists x_1, \dots, x_4, \Big[\big(x_1 \neq 0\big) \wedge \big(x_1^2 + x_2^2 + x_3^2 + x_4^2 = t\big)\Big].$$

3. Less obvious is the fact that for $p \neq 2$, $\mathbb{Z}_p$ can be defined inside $\mathbb{Q}_p$ by the formula

$$\text{Integer}(t) \longleftrightarrow \exists y,\, y^2 = pt^2 + 1.$$

More generally, by [5], the valuation ring of any henselian (cf. Definition 5.3.2) valued field k is definable in k.

4. A beautiful result by Julia Robinson shows that $\mathbb{Z}$ is definable inside $\mathbb{Q}$.

Definition 2.2.2 ($\mathcal{L}$-Definability with Parameters)**.** *Let $\mathcal{A}$ be an $\mathcal{L}$ structure with universe A. A subset $S \subset A^n$ is said to be **$\mathcal{L}$-definable (in $\mathcal{A}$) with parameters** if there exists some $\mathcal{L}$-formula $\Phi(\mathbf{x};\mathbf{t})$ and some tuple $\mathbf{c} \in A^m$ such that $S = \Phi(\mathbf{c};\mathcal{A}) := \{\mathbf{a} \in A^n \mid \mathcal{A} \vDash \Phi(\mathbf{c};\mathbf{a})\}$. In other words, S is definable without parameters over the language $\mathcal{L}(\mathbf{c}) := \mathcal{L} \cup \{\mathfrak{c}_1, \ldots, \mathfrak{c}_m\}$ obtained by adjoining to $\mathcal{L}$ a constant symbol $\mathfrak{c}_i$ for each c_i. (Note that $\mathcal{A}$ could naturally be viewed as an $\mathcal{L}(\mathbf{c})$-structure with ${\mathfrak{c}_i}^{\mathcal{A}} = c_i$.)*

*To stay within the language $\mathcal{L}$, we will often refer to $\Phi(c_1, \ldots, c_m; \mathbf{t})$ as an **$\mathcal{L}$-formula with parameters** (instead of "an $\mathcal{L}(\mathbf{c})$ formula").*

Example 2. Let $\mathcal{L} = \mathcal{L}_{\text{Rings}}$. Let k be a field, viewed as an $\mathcal{L}$-structure. Then for every polynomial $f(x) = \sum_{i=0}^{d} a_i x^i \in k[x]$, its zero sets in k is definable by the following formula with parameters a_i.

$$V_f(a_0, \ldots, a_d; t) \longleftrightarrow \sum_{i=0}^{d} a_i t^i = 0.$$

Similarly, we can define the set of $b \in k$ that can be represented by f over k using the following formula with parameters a_i.

$$\Phi(\mathbf{a}; b) \longleftrightarrow \exists t, \sum_{i=0}^{d} a_i t^i = b.$$

Definition 2.2.3 (Definable Sets and Functions)**.** *Let $\mathcal{A}$ be an $\mathcal{L}$ structure with universe A.*

1. *An **$\mathcal{L}$-definable** set of $\mathcal{A}$ is a subset $S \subset A^n$ for some n such that is $\mathcal{L}$-definable in $\mathcal{A}$ either with or without parameters.*

2. *A function $F : A^m \to A^n$ is said to be **definable** if its graph is definable (as a subset of A^{m+n}). Clearly if F is definable then so is the image $\operatorname{im} F$ and the fiber $F^{-1}(\mathbf{b})$ for any $\mathbf{b} \in A^n$.*

Observation 2.2.1. *Let $\mathcal{L} = \mathcal{L}_{Rings}$. Let k be a field, viewed as an $\mathcal{L}$-structure. Then any polynomial $F(x_1, \dots, x_n) \in k[x_1, \dots, x_n]$ could be viewed as a definable function (with parameters).*

Example 3. Quadratic forms over k and "norm forms" of certain cyclic extensions or cyclic algebras over k (cf. Section 4.3) are definable as functions (with parameters). An insight by Pop [8] is we can utilize Milnor Conjecture and Merkurjev-Susslin's theorem translate statements about such forms into cohomological terms. In particular, we get a first order characterization of vanishing of certain cocycles (cf. Theorem 4.3.1 and Theorem 4.3.2). This is one of the key tools in proving our main theorem.

2.3 Theories and Models

Definition 2.3.1. *A set $\mathcal{T}$ of first-order $\mathcal{L}$-sentences is called a first-order $\mathcal{L}$-**theory**. The sentences in $\mathcal{T}$ are also often referred to as the **axioms** of $\mathcal{T}$. We say that an $\mathcal{L}$-structure $\mathcal{A}$ is a **model** of $\mathcal{T}$ and write $\mathcal{A} \vDash \mathcal{T}$ if $\mathcal{A} \vDash \phi$ for all formulas ϕ of $\mathcal{T}$.*

Example 1. $\mathcal{T}_{\textbf{Rings}}$, $\mathcal{T}_{\textbf{Commutative Rings}}$ and $\mathcal{T}_{\textbf{Fields}}$ The theory $\mathcal{T}_{\text{Rings}}$ is defined as the set consisting of the following $\mathcal{L}$-sentences.

1. $\forall x, y,\ x + y = y + x$

2. $\forall x, x + 0 = x$

3. $\forall x, \exists y, x + y = 0$

4. $\forall x, x \cdot 1 = 1 \cdot x = x$

5. $\forall x, y, z\, (x + y) \cdot z = x \cdot z + y \cdot z$

6. $\forall x, y, z\, (x \cdot y) \cdot z = x \cdot (y \cdot z)$

7. $1 \neq 0$.

Example 2. Theory of Fields By definition, the theory $\mathcal{T}_{\text{Commutative Rings}}$ is obtained from $\mathcal{T}_{\text{Rings}}$ by adding a single axiom $\forall x, y, \,(x \cdot y) = (y \cdot x)$. Similarly the theory $\mathcal{T}_{\text{Fields}}$ is obtained from $\mathcal{T}_{\text{Commutative Rings}}$ by adding the axiom $\forall x, (x = 0) \vee (\exists y,\ x \cdot y = 1)$.

Example 3. Recall that an $\mathcal{L}$-ring structure $\mathcal{A} = (A; +^{\mathcal{A}}, \cdot^{\mathcal{A}}; 0^{\mathcal{A}}, 1^{\mathcal{A}})$ is a set A together with a choice of binary operations $+^{\mathcal{A}}$ and $\cdot^{\mathcal{A}}$ and constants $0^{\mathcal{A}}$ and $1^{\mathcal{A}}$ of A.

Observe that $\mathcal{A} \vDash \mathcal{T}_{\text{Rings}}$ iff the operations $+^{\mathcal{A}}, \cdot^{\mathcal{A}}$ gives A a ring structure with additive constant $0^{\mathcal{A}}$ and multiplicative constant $1^{\mathcal{A}}$. In other words, true to its name, all the models of $\mathcal{T}_{\text{Rings}}$ are rings. Similarly $\mathcal{A} \vDash \mathcal{T}_{\text{Fields}}$ iff $\mathcal{A}$ is a field under the interpreted operations.

Definition 2.3.2. *The $\mathcal{L}$-theory of an $\mathcal{L}$-structure $\mathcal{A}$, denoted* $\mathrm{Th}_{\mathcal{L}}(\mathcal{A})$*, is defined as the set of all $\mathcal{L}$ sentences that holds for $\mathcal{A}$. We say that two $\mathcal{L}$-structures $\mathcal{A}$ and $\mathcal{B}$ are* ***elementarily equivalent*** *(over $\mathcal{L}$) if* $\mathrm{Th}_{\mathcal{L}}(\mathcal{A}) = \mathrm{Th}_{\mathcal{L}}(\mathcal{B})$ *and write $\mathcal{A} \equiv_{\mathcal{L}} \mathcal{B}$. This defines an equivalence relation on the class of $\mathcal{L}$-structures.*

Example 4. Any two algebraically closed field of the same characteristic are elementarily equivalent over the language $\mathcal{L}_{\text{Rings}}$. In other words, if we let ACF_p denote the set of first order $\mathcal{L}_{\text{Rings}}$ sentences that hold for *all* algebraically closed field of characteristic p, then $\mathrm{ACF}_p = \mathrm{Th}(k)$ for any *particular* algebraically closed field k of characteristic p. This could be used to give a model theoretic proof of Hilbert Nullstellensatz.

There is yet another equivalence relations on $\mathcal{L}$-structures called **isomorphism**. In the case of rings viewed as $\mathcal{L}_{\text{Rings}}$ structures, this reduces to the usual notion of ring isomorphism. That is, two rings R and S are isomorphic as $\mathcal{L}_{\text{Rings}}$ structures iff they are isomorphic as rings. Observe that clearly if two rings R and S are isomorphic, then they are also elementary equivalence. (Actually, for structures over *any* language $\mathcal{L}$, isomorphism always implies elementary equivalence.) A natural question is, is the converse true? We will discuss this question specifically in the context of fields, viewed as $\mathcal{L}_{\text{Rings}}$ structures, in the next chapter.

2.4 Why "First" Order

Roughly speaking, the main distinction between "first" and higher order logics is that in the former, one can quantify *only* over elements, not subsets. Put differently, in an explicit structure $\mathcal{A}$ with universe A, all variables of a first-order formula could be interpreted only as elements of A; whereas higher-order logic allows for variables that can be interpreted as subsets of A. This restriction on first-order logic protects it from Russell-paradox type of pathologies. As a consequence, for first order logic, truth is equivalent to provability. More precisely, given an appropriate definition of "proofs", we have the following.

Fact 2.4.1 (Godel Completeness Theorem). *Fix a language $\mathcal{L}$. Let $\mathcal{T}$ be a first order $\mathcal{L}$-theory. Then for any a first order $\mathcal{L}$-formula Φ, Φ holds in all models of $\mathcal{T}$ iff Φ can be "proven" from the axioms in $\mathcal{T}$.*

In fact, by Lindstrom, first-order logic is the "best possible one", in the sense that it is the "strongest" logic in which both Completeness and (downward) Löwenheim–Skolem (see Section 3.1) hold.

The following striking example demonstrate the power of Completeness.

Fact 2.4.2 (Principle of Characteristic Transfer.)**.** *Let Φ be a first order $\mathcal{L}_{Ring}$-sentence. Then $\mathbb{C} \vDash \Phi$ iff for all but finitely many prime p, Φ holds for all algebraically closed fields of characteristic p, .*

Idea of Proof. Let ACF be the set of sentences that hold for all algebraically fields. For each prime p, let char_p denote the sentence $\underbrace{1 + \cdots + 1}_{p} = 0$. Then observe that for every prime p, ACF_p is generated by the "logical consequences" of $\mathrm{ACF} \cup \{\mathrm{char}_p\}$, whereas ACF_0 is generated by the "logical consequences" of $\mathrm{ACF} \cup \left\{ \neg\, \mathrm{char}_p \,\middle|\, p \text{ prime} \right\}$.

Now Example 4, Φ holds for $\mathbb{C}$ iff it holds for all algebraically closed field of characteristic 0. By Completeness, there must exists a proof of Φ from the axioms of ACF_0. Since such a proof must be "finite", it could only use finitely many of the axioms $\neg\, \mathrm{char}_p$. In other words, the set $\mathcal{P} := \{p \mid \neg\, \mathrm{char}_p \text{ is "used" in the proof of } \Phi\}$ is finite. Thus the same proof must apply to any algebraically closed field of characteristic $p \notin \mathcal{P}$. Again by Completeness, Φ must hold for all such fields. □

Remark 2.4.1. *The above principle could be used to give an extremely short model-theoretic proof of the Ax-Grothendieck Theorem in Algebraic Geometry.*

3

Model Theory of Fields and The EEIP Problem

Convention 3.0.1. *For the rest of this dissertation, we will only be talking about the language $\mathcal{L}_{Rings}$. Thus we will drop the prefix $\mathcal{L}$ when referring to formulas, structures and definable sets etc.*

3.1 The EEIP Problem for Fields

Example 4 and Fact 2.4.2 demonstrate the power of model theory as a tools for studying fields. Naturally one wonders: to which extent does first order theory of a field K determines its algebraic properties? For example, does $\mathrm{Th}(K)$ determine the isomorphism class of K? That is, is it true that if two fields K and L are elementarily equivalent, then they must also be isomorphic?

The astute readers might have already notice that Example 4 completely annihilate this naive conjecture. Worse still, since $\overline{Q} \equiv \mathbb{C}$, not only does elementary equivalence not imply isomorphism as fields, but it does not even imply isomorphisms as *sets*! More generally, it turns out, by a theorem of Lowenheim and Skolem, if a theory $\mathcal{T}$ has one infinite model, then it must have infinite models of every cardinality.

Thus maybe the question we should ask instead is, for which class of fields does elementary equivalence implies isomorphism?

Definition 3.1.1 (EEIP). *Let $\mathcal{K}$ and $\mathcal{F}$ be two classes of fields. We say that the pair $(\mathcal{K}, \mathcal{F})$ is* ***EEIP*** *if for every $K \in \mathcal{K}$ and $F \in \mathcal{F}$, we have $K \equiv F$ iff $K \cong F$.*

When $\mathcal{F} = \mathcal{K}$, we will simply say $\mathcal{K}$ instead of $(\mathcal{K}, \mathcal{K})$.

Example 1. The pair $(\{\text{Finite Fields}\}, \{\text{Fields}\})$ is EEIP since the isomorphism class of every finite field is uniquely determined by its cardinality. More explicitly, if K is a finite field with q elements, then we can take Θ_K be to be the following sentence.

$$\exists x_1, \ldots, x_q, \quad \Big(\bigwedge_{i \neq j} (x_i \neq x_j) \Big) \wedge \Big(\forall t, \bigvee_i (t = x_i) \Big).$$

Of particular interest is when the classes $\mathcal{K}$ and $\mathcal{F}$ are subclasses of the class of finitely generated field extensions K/k. This is because every such field can be realized as the function field of some variety over k and thus allow for investigation using geometrical methods. In particular, the so-called **Pop Conjecture** claims that

that the class of *finitely generated fields* (i.e. fields finitely generated over their prime subfield) is EEIP. This was recently proven in by Pop and Dittman, assuming strong desingularization in the case of char 2.

Convention 3.1.1.

- *For the rest of this dissertation, a* ***function field*** K/k *will mean a finitely generated with extension* K/k *with* k relatively algebraically closed *in* K. *We call* k ***the constant field*** *of* K/k.

- *When* $\mathrm{td}(K/k) = 1$, *there exists a unique projective regular (hence smooth if* k *is pefect) curve* X *over* k *whose function field is* K. *We will refer to* X *as the (unique)* ***smooth model*** *of* K/k. *This smooth model can be constructed using the Riemann-Zariski space (cf. Definition 5.2.1) of* K/k *which as a set consists of equivalence classes of valuations (cf. Definition 5.1.1) of* K/k.

- *Two function fields* K/k *and* L/λ *are said to be* ***isomorphic (as function fields)****, denoted* $K/k \cong L/\lambda$, *iff there exists an isomorphism* $K \cong L$ *which restricts to an isomorphism* $k \cong \lambda$ *on the constant fields.*

- *When* $\mathcal{K}$ *is a class consisting only of function fields, the isomorphism in the definitions of EEIP and Strong EEIP are required to be* function field *isomorphisms.*

3.2 EEIP Problem for Function Fields: History and Ideas

1. In 1979, Rumely [10] proved that the class of global fields is EEIP, using class field theory.

2. The first class of *function fields* known to be EEIP is that of curves of over an algebraically closed field, of genus not equal to 1 (Duret 1986) [2]. Duret later managed to extend the result to curves of genus 1 with complex multiplication.

The case of curves of genus 1 continued to be explored by Pierce and Vidaux but only over languages extending $\mathcal{L}_{\text{Rings}}$.

3. The so-called **Pop Conjecture** was first posed by Pop in 2002 [8]. In this paper, Pop employs *Milnor Conjecture* to prove definability of *algebraic dependence.* This allows him to prove that EEIP holds for the class of finitely generated function fields of general type, as well as the class of function fields over a given algebraically closed field k.

   See the next chapter for a brief explanation of the role of Milnor conjecture in the proof.

4. Employing the *Pfister Foms* and cohomological techniques introduced by Pop, Poonen [6] proved in 2007 that for finitely generated fields, *characteristic* and *algebraic independence* are definable. Later that year Pop and Poonen [7] extent the result to function fields over large fields. Additionally, they also prove that the *constant fields* of such fields are definable.

5. In 2008, Scanlon attempted a proof of the Pop Conjecture [11]. Though ultimately found incorrect, the proof introduces the idea of using definability of valuation rings to prove bi-interpretability with the natural numbers, which for finitely generated fields would imply EEIP.

6. In 2017, using higher *Local Global Principle*, Pop managed to prove the Pop Conjecture for fields of Kronecker dimension at most 3. [9]

7. The full Pop Conjecture was finally proven by Pop and Dittman, assuming strong desingularization in char 2. [1]

4

First-Order Characterization of Vanishing of 3-cocycles

4.1 Overview

A significant insight of Pop in the 2002 paper is that, for a field F of characteristic not equal to 2, Milnor conjecture could be use to give first order characterization of vanishing of cocycles of $H^m(F, \mu_2)$. The main ideas are as follows.

1. The Kummer isomorphism $F^*/(F^*)^2 \cong H^1(F, \mu_2)$ associates to each element $a \in F^*$ a one-cocyle $(\!|a|\!) \in H^1(F, \mu_2)$. This in turn allows one to associate to

each tuple $\mathbf{a} = (a_0, \ldots, a_m) \in (F^*)^{m+1}$ an m-cocyle $(\!|\mathbf{a}|\!) := (\!|a_0|\!) \cup \cdots \cup (\!|a_m|\!) \in H^{m+1}(F, \mu_2)$.

2. On the other hand, to each tuple $\mathbf{a}$, one can also associate a quadratic form $\langle\!\langle \mathbf{a} \rangle\!\rangle$ (called a **Pfister Form**), which as a function is definable with parameters $\mathbf{a}$.

3. By Milnor Conjecture, the m-cocyle $(\!|\mathbf{a}|\!)$ is trivial iff the quadratic form $\langle\!\langle \mathbf{a} \rangle\!\rangle$ represents 0 nontrivially over F. This gives a first-order characterization of the vanishing of $(\!|\mathbf{a}|\!)$.

In this chapter, we extend the same idea to include fields of characteristic 2. This can be done as follows. Let l be a any prime and let F be a field containing a primitive l-th root of unity.

1. Again, the Kummer isomorphism $F^*/(F^*)^l \cong H^1(F, \mu_l)$ associates to each element $a \in F^*$ a one-cocyle $(\!|\mathbf{a}|\!) \in H^1(F, \mu_l)$. (Note that this is the base case of the Bloch-Kato conjecture, which generalizes the Milnor Conjecture). This in turn allows one to associate to each tuple $\mathbf{a} = (a_0, \ldots, a_m) \in (F^*)^{m+1}$ an m-cocyle $(\!|\mathbf{a}|\!) = (\!|a_0|\!) \cup \cdots \cup (\!|a_m|\!) \in H^{m+1}(F, \mu_l)$.

2. In the case when $m = 1$ or 2, we can associate to $a_1, \ldots, a_m$ a "norm form" $N(a_1, \ldots, a_m)$ which is definable (as a function) with parameters $\mathbf{a}$.

   In particular, for $m = 2$, one can associate to (a_1, a_2) an *l-cyclic algebra* A_{a_1,a_2} define $N(a_1, a_2)$ in terms of its "reduced norm". When $l = 2$, A_{a_1,a_2} is simply the generalized quaternion algebra $(a_1, a_2)_F$ and whose norm is given by the Pfister form $\langle\!\langle a_1, a_2 \rangle\!\rangle$, as above.

3. For such m, by Merkurjev-Susslin, $(\!|\mathbf{a}|\!)$ is trivial $\iff$ a_0 can be represented by $N(\mathbf{a})$ over F. This gives a first-order characterization of vanishing of 2 and 3 cocyles with coefficients in μ_l.

Below is the organization for this Chapter.

- The first section explicitly defines the cocycle $(\!|\mathbf{a}|\!)$ associates to a tuple $\mathbf{a} \in (F^*)^m$.

- The next/last section defines the norm form $N(\mathbf{a})$ for $\mathbf{a} \in (F^*)^m$, where $m = 2$ or $m = 3$.

4.2 The Cocyle associated to a tuple of elements of F

For the rest of this chapter, fix a prime l. Let F be a field containing a primitive l-th root of unity ζ. Then the absolute Galois group $G_F \coloneqq \mathrm{Gal}(F_s/F)$ acts trivially on the group of roots unity μ_l, so the map $\zeta \mapsto 1$ gives a G_F-module isomorphism $\mu_l \cong \mathbb{Z}/l\mathbb{Z}$. In particular, for every $i \in \mathbb{Z}_{\geq 0}$, we have $H^{i+1}(F, \mu_l^{\otimes i}) = H^{i+1}(F, \mathbb{Z}/l\mathbb{Z}) =: H^{i+1}(F)$.

The **Kummer Exact Sequence** $0 \to \mu_l \to F_s^* \xrightarrow{x \mapsto x^l} F_s^* \to 0$. induces the following Long Exact Sequence in cohomology

$$
\begin{tikzcd}
\cdots \arrow[r] & H^0(F, F_s^*) \arrow[r] \arrow[d, equal] & H^0(F, F_s^*) \arrow[r] \arrow[d, equal] & H^1(F, \mu_l) \arrow[r] \arrow[d, equal] & H^1(F, F_s^*) \arrow[r] \arrow[d, equal] & \cdots \\
\cdots \arrow[r] & F^* \arrow[r, "x \mapsto x^l"] & F^* \arrow[r] & H^1(F, \mu_l) \arrow[r] & 0 \arrow[r] & \cdots
\end{tikzcd}
$$

As $H^1(F, F_s^*) = 0$ by Hilbert 90, the connecting homomorphism induces the following **Kummer Isomorphism** which associates to each $a \in F^*$ a one-cocyle $(\!|a|\!)$ in $H^1(F, \mu_l)$.

$$
\begin{aligned}
(F^*)/(F^*)^l &\to H^1(F, \mu_l) \\
a &\mapsto (\!|a|\!) : \left(\sigma \mapsto \frac{\sigma(\sqrt[l]{a})}{\sqrt[l]{a}} \right).
\end{aligned}
$$

Using cup product, one can extend the above map to associate to every n-tuple $\mathbf{a}$

some n-cocyle $(\mathbf{a})$, as follows.

$$\left(\frac{F^*}{(F^*)^l}\right)^n \to H^n(F, \mu_l)$$
$$\mathbf{a} = (a_0, \ldots, a_{n-1}) \mapsto (\!|\mathbf{a}|\!) := (\!|a_0|\!) \cup (\!|a_1|\!) \cup \cdots \cup (\!|a_{n-1}|\!).$$

For later use in Chapter 7, we introduce the following notations and observations.

Notation 4.2.1. *We will denote the image of the above map by $(\!|\dot{F}^n|\!)$. Given $\alpha \in (\!|\dot{F}^n|\!)$ and $\mathbf{a} \in (F^*)^n$, we say $\mathbf{a}$ **represents** α if $\alpha = (\mathbf{a})$ in $H^n(F)$.*

Observation 4.2.1. *Obviously, if $\mathbf{a}$ represents α and so does any tuple $\mathbf{c}$ such that $c_i \in a_i(F^*)^l$, for all i.*

Fact 4.2.1. *The cocyles in $(\!|\dot{F}^n|\!)$ generate the whole of $H^n(F)$.*

4.3 First-Order Characterization of Vanishing of Cocycles Using Norm

4.3.1 Main Idea

When $n = 1$ or 2, for every $(\!|\mathbf{a}|\!) \in H^n(F)$, one can associate certain "norm form" $N(\mathbf{a})$, which is definable (as a function) with parameters $\mathbf{a}$. This is because $H^1(F)$ classifies cyclic extensions of F of degree dividing l while $H^2(F, \mu_l)$ classifies central simple F-algebras of exponent dividing l, and such extensions and algebras come equipped with canonical norm maps. It turns out that for all cocyles $(\!|\mathbf{a}|\!) \in H^n(F)$ and all $b \in F^*$, $(\!|\mathbf{a}|\!) \cup (\!|b|\!)$ is trivial in $H^{n+1}(F)$ iff $N(\mathbf{a})$ represents b over F. This gives us a first-order way to express the vanishing of $(\!|\mathbf{a}|\!) \cup (\!|b|\!)$

4.3.2 The Norm Group Associated to a one-cocyle $(\!|a|\!)$

$$\begin{tikzcd} F^l - \{0\} \arrow[d, "\sim"] \arrow[dr, bend left, "N_a"] & \\ F(\sqrt[l]{a})^* \arrow[r, "N_{F(\sqrt[l]{a})/F}"] \arrow[d, "\rho_1"] & F^* \arrow[d, equal] \\ GL_F(F(\sqrt[l]{a})) \arrow[r, "\det"] & F^*. \end{tikzcd}$$

To each cocyle of the form $(\!|a|\!) \in H^1(F)$, one can associate the cyclic extension $F(\sqrt[l]{a})/F$ (note that this extension is Galois since we assumed that F contains a primitive l-th root of unity). The Kummer Isomorphism $F^*/(F^*)^l \cong H^1(F)$ ensures that this map is well-defined.

Recall the following basic facts about $F(\sqrt[l]{a})$.

1. $F(\sqrt[l]{a})$ is an l-dim vector space with basis $\{1, \sqrt[l]{a}, \dots, \sqrt[l]{a}^{l-1}\}$. In other words, we have a bijection $F^l \to F(\sqrt[l]{a})$ which is defined on the standard basis by $\mathbf{e}_i := (0, \dots, 1, \dots, 0) \mapsto \sqrt[l]{a}^i$.

2. We have a homomorphism $\rho_1 : F(\sqrt[l]{a})^* \to GL_F(F(\sqrt[l]{a})$ sending every $b \in F(\sqrt[l]{a})^*$ to the "multiplication by b" map. The pullback of the determinant map on $GL_F(F(\sqrt[l]{a})$ under ρ_1 gives a norm map $N_{F(\sqrt[l]{a})/F} : F(\sqrt[l]{a})^* \to F^*$.

Now define $N_a : (F)^l - \{\underline{0}\} \to F^*$ to be the pullback of $N_{F(\sqrt[l]{a})/F} : F(\sqrt[l]{a})^* \to F^*$ under the identification $F^l - \{\underline{0}\} \to F(\sqrt[l]{a})^*$. Explicitly, we have

$$N_a(c_0, \dots, c_{l-1}) := N_{F(\sqrt[l]{a}/F)}\Big(c_0\sqrt[l]{a} + \dots + c_{l-1}(\sqrt[l]{a})^{l-1}\Big).$$

4.3.3 The Norm Group Associated to a two-cocyle $(\!|a_1, a_2|\!)$

The following part of the Long Exact Sequence associated to the above Kummer Sequence implies that $H^2(F) \cong {}_l\operatorname{Br}(F)$.

$$\begin{tikzcd}
H^1(F, F_s^*) \arrow[r] \arrow[d, equal] & H^2(F, \mu_l) \arrow[r] \arrow[d, "\cong"] & H^2(F, F_s^*) \arrow[r] \arrow[d, "\cong"] & H^2(F, F_s^*) \arrow[d, "\cong"] \\
0 \arrow[r] & H^2(F, \mu_l) \arrow[r] & \mathrm{Br}(F) \arrow[r, "A \mapsto A^{\otimes l}"] & \mathrm{Br}(F)
\end{tikzcd}$$

In particular, to each two-cocyle of the form $(\!|a_1, a_2|\!) := (\!|a_1|\!) \cup (\!|a_2|\!) \in H^2(F)$, one can associate a central simple F-algebra $A_{a_1,a_2} \in {}_l\mathrm{Br}(F)$.

Recall the following basic facts about A_{a_1,a_2}.

1. A_{a_1,a_2} is an F-vector space of dim l^2, with basis $\{\alpha_1^i \alpha_2^j \mid 0 \le i, j \le l-1\}$, where $\alpha_i := \sqrt[l]{a_i}$.

2. A_{a_1,a_2} "splits" over $F(\sqrt[l]{a_1})$, that is $A_{a_1,a_2} \otimes_F F(\sqrt[l]{a_1}) \cong M_l(F(\sqrt[l]{a_1}))$. The pullback of the determinant map on $M_l(F(\sqrt[l]{a_1}))$ to $A_{a_1,a_2} \hookrightarrow A_{a_1,a_2} \otimes_F F(\sqrt[l]{a_1})$ defines a *reduced norm* map $\mathrm{Nrd} : A_{a_1,a_2} \to F$.

Now, define $N_{a_1,a_2} : F^{l^2} \to F$ to be the pullback of the reduced norm map under the bijection $F^{l^2} \to A_{a_1,a_2}$ corresponding to the above choice of basis.

$$\begin{tikzcd}
F^{l^2} \arrow[d, "\sim"] \arrow[dr, bend left, "N_{a_1,a_2}"] & \\
A_{a_1,a_2} \arrow[r, "\mathrm{Nrd}"] \arrow[d, hook] & F \arrow[d, hook] \\
A_{a_1,a_2} \otimes F(\sqrt[l]{a}) \arrow[r] \arrow[d, "\sim"] & F(\sqrt[l]{a}) \arrow[d, equal] \\
M_l(F(\sqrt[l]{a})) \arrow[r, "\det"] & F(\sqrt[l]{a})
\end{tikzcd}$$

4.3.4 Vanishing of two-cocyles and three-cocyles in terms of Norms

The following well-known result from the theory of Brauer Groups gives us a first-order way to express the vanishing of a two cocyle $(\!|a, b|\!) \in H^2(F)$.

Theorem 4.3.1. *For all $a, b \in F^*$, we have $(\!|a|\!) \cup (\!|b|\!) = 0 \in H^2(F) \iff A_{a,b} = 0 \in {}_l\operatorname{Br}(F) \iff N_a$ represents b.*

For a first-order characterization of the vanishing of three-cocyles, we use the following celebrated Theorem 12.1 in [4] by Merkurjev and Susslin.

Theorem 4.3.2 (Merkurjev-Susslin)**.** *For all $a_1, a_2, b \in F^*$, we have $(\!|a_1, a_2|\!) \cup (\!|b|\!) = 0$ in $H^3(F)$ iff N_{a_1,a_2} represents b.*

5

Valuations and Local Global Principle

5.1 Valuations as Order of Vanishing

Definition 5.1.1. *A **valued field** (K, v) is a field K endowed with a **valuation** v, i.e. a surjective map $v : K \to (\Gamma, +, \leq) \cup \{\infty\}$ from K to some* ordered abelian group $(\Gamma, +, \leq)$ *satisfying*

- $v(0) = \infty$;
- *v restricts to a group homomorphism $K^* \to \Gamma$;*
- *(non-archimedean Property) For all $a, b \in K$, $v(a + b) \geq \min(v(a), v(b))$.*

*We call Γ the **value group** of v.*

1. *Let $O_v := \{f \in K \mid v(f) \geq 0_\Gamma\}$. The non-archimedean property ensures that O_v is a local subring of K with maximal ideal $\mathfrak{m}_v := \{f \in K \mid v(f) > 0\}$. The quotient $Kv := O_v/\mathfrak{m}_v$ is called the **residue field** of v.*

2. *The ring O_v is an example of a **valuation ring** of K. The latter is defined as a subring $R \subset K$ such that for all $f \in K$, either $f \in R$ or $f^{-1} \in R$. It turns out that every valuation ring of K is of the form O_v for some valuation v. In particular, this induces a bijection*

$$\left\{\text{valuation rings of } K\right\} \longleftrightarrow \left\{\text{(equivalence classes) of valuations on } K\right\},$$

   *where two valuations v and w on K are considered **equivalent** iff $O_v = O_w$.*

   *We also call O_v the **valuation ring of** v.*

3. *If v is trivial on some subfield k of K (equivalently, if $O_v \supset k$), we call v a **valuation of** K/k and O_v a **valuation ring of** K/k.*

We are particularly interested in (normalized) discrete rank 1 valuations, define as follows.

Definition 5.1.2. *A **discrete rank** 1 **valuation** v on a field K is a valuation with value group $(\mathbb{Z}, \leq)$ with the ordering inherited from $\mathbb{R}$. We denote the set of (equivalence classes of) discrete rank 1 valuations of K (resp. of K/k) by $\mathbb{P}(K)$ (resp. $\mathbb{P}(K/k)$).*

*The valuation ring O_v of a discrete rank 1 valuation v is a principal ideal domain all of whose nonzero ideals are powers of $\mathfrak{m}_v$. An element $\pi_v \in O_v$ generates $\mathfrak{m}_v$ iff $v(\pi_v) = 1$. Any such π_v is called a **uniformizer** of v.*

Example 1. $\mathfrak{p}$-adic Valuations

1. Let K be the field of fraction of a domain O with unique factorization into prime ideals (i.e. a **Dedekind domain**). Then for every nonzero prime ideal $\mathfrak{p} \triangleleft O$, we can define a discrete rank 1 valuation $v_{\mathfrak{p}}$, called the **$\mathfrak{p}$-adic valuation** of K, as follows. First, we define $v_{\mathfrak{p}}$ on $0 \neq r \in O$ by letting $v_{\mathfrak{p}}(r)$ be the exponent of $\mathfrak{p}$ in the unique factorization of the ideal rO into prime ideals. Then we extend $v_{\mathfrak{p}}$ to K^* by defining $v_{\mathfrak{p}}(f) = v_{\mathfrak{p}}(r) - v_{\mathfrak{p}}(s)$ for any $r, s \in O$ such that $f = r/s$. It is easy to check that $v_{\mathfrak{p}}$ is well-defined and that its valuation ring the localization $O_{\mathfrak{p}}$ of O at $\mathfrak{p}$. Note that when $\mathfrak{p}$ is a maximal ideal, the residue field $Kv_{\mathfrak{p}} = O_{\mathfrak{p}}/\mathfrak{p}O_{\mathfrak{p}} = (O/\mathfrak{p})_{(\overline{0})}$ is isomorphic $O/\mathfrak{p}$.

   Below we look at some particular examples.

2. Let $K = \mathbb{Q}, O = \mathbb{Z}$ and $\mathfrak{p} = p\mathbb{Z}$ for some prime p. Then the valuation $v_p := v_{\mathfrak{p}}$ is called the **p-adic valuation** on $\mathbb{Q}$. It has valuation ring $(\mathbb{Z}_{\mathfrak{p}}, p\mathbb{Z}_{\mathfrak{p}})$, uniformizer p and residue field $\mathbb{Z}/p\mathbb{Z} \cong \mathbb{F}_p$.

3. Let $K = k(C)$ and $O = k[C]$ be the function field and the coordinate ring of some smooth affine curve C. Let $P \in C$ and let $\mathfrak{p}_P = \mathbb{I}(P)$ be the ideal of all functions $f \in k[C]$ vanishing at P. Then for every $f \in K^*$, $w_P(f) := v_{\mathfrak{p}}(f)$ is simply the "order of vanishing of f at P" and the valuation ring $(O_{w_P}, \mathfrak{m}_{w_P})$ is simply the ring $(\mathfrak{O}_P, \mathfrak{m}_P)$ of functions regular at P.

   We can think of $\mathfrak{O}_P$ as the "ring of functions that can be evaluated at P" and the residue map $\mathfrak{O}_P \to \mathfrak{O}_P/\mathfrak{m}_P =: \kappa(P) = Kv_P$ as the "evaluation at P" map. To see the intuition behind this, consider the simple case of $K = k(t) = k(\mathbb{A}^1_k)$ of field of rational functions on the affine line. Then for any $a \in \mathbb{A}^1_k$, since $\mathfrak{p}_a = (t-a)$ is a maximal ideal of $k[C] = k[t]$, by the last comment in part 1, we have $K_{v_a} = k[t]/(t-a) \cong k$. Under this identification, the residue map $\mathfrak{O}_a = k[t]_{(t-a)} \to K_{v_a} = k[t]/(t-a) \cong k$ is given by $f \mapsto f \pmod{(t-a)} \mapsto f(a)$.

4. The above example of course generalizes to function fields $K = k(C)$ of smooth *projective* curves. Thus to each point $P \in C$, one can associate a valuation w_P (capturing "order of vanishing at P") whose valuation ring is the ring $(\mathfrak{O}_P, \mathfrak{m}_P)$ of functions regular at P. It turns out that every valuation ring of K/k arise this way. In particular, we have the following bijections.

$$\begin{array}{ccccc} \Big\{\text{closed points of } C\Big\} & \xrightarrow{\sim} & \Big\{\text{Valuation Rings of } K/k\Big\} & \xrightarrow{\sim} & \mathbb{P}(K/k) \\ P & \longmapsto & (\mathfrak{O}_P, \mathfrak{m}_P) & \longmapsto & w_P \end{array}$$

   In particular, we could recover the unique smooth model C from $\mathbb{P}(K/k)$ (cf. Definition 5.2.1 below).

5. More generally, let $K = k(X)$ is the function field of a smooth (or simply *normal*) projective variety X over k. Then to every "codim 1 point" $x \in X$ (i.e. a codim 1 irreducible subvariety of X, aka. *prime divisor* of X), we can associate a discrete normalized rank one valuation w_x of K/k in a similar way.

5.2 The Riemann-Zariski Space of K

As mentioned in part 4 of Example 1 above, when K/k is a function field of transcendence degree 1, we can construct a projective smooth model of K/k from the valuations of K/k. More generally, for any field K, the valuations on K naturally form a topological space called the Riemann-Zariski Space of K, defined as follows.

Definition 5.2.1 (The Riemann-Zariski Space of K). *The Riemann-Zariski Space of K is the topological space space whose underlying set and topology are defined as follows.*

- *Underlying set:* $\mathcal{V}(K) :=$ *the set of (equivalence classes) of valuations (not necessarily discrete rank* 1*) on* K.

- *Topology: the coarsest topology in which sets of the forms* $\{[w] \in \mathcal{V}(K) \mid w(a) \geq 0\}$, $a \in K$ *are both closed and open.*

The following fact about the Riemann-Zariski Space of K will be used in the proof of Proposition 7.3.1, which is key to the proof of definability of valuation rings.

Fact 5.2.1. 1. *The Riemann-Zariski Space of* K *is compact Hausdorff.*

2. *For all* $a \in K$, *the sets* $\{[w] \in \mathcal{V}(K) \mid w(a) > 0\}$ *and* $\{[w] \in \mathcal{V}(K) \mid w(a) = 0\}$ *are also both open and closed.*

5.3 Completion and Henselization

To each valued field (K, v), one can associate a valued field $(\widehat{K_v}, \widehat{v})$ called the **completion of** K **at** v where $K \hookrightarrow K_v$. The following examples demonstrate how the completion $\widehat{K_v}$ captures local behavior of elements of K at v.

Example 1.

1. Again, let K be the function field of the Riemann Sphere $\mathbb{C}_\infty$, viewed as a curve over $\mathbb{C}$. Let $P \in \mathbb{C}_\infty$ and let $\mathfrak{O}_P, \mathfrak{m}_P$ and w_P be as in Part 4 of Example 1. Let z_P be a local coordinate at P. Then z_P is a uniformizer of v_P and the completion $\widehat{K}_{w_P}$ is simply the field $K((z_P))$ of Laurent series in z_P. In particular, the inclusion $K \hookrightarrow \widehat{K}_{w_P}$ assigns to every function $f \in K$ its Laurent Expansion at P.

2. Let $K = \mathbb{Q}$ and let $v = v_p$ be a p-adic valuation, as defined in Part 2 of Example 1. Then the completion $(\widehat{K_v}, \widehat{v})$ is the field $\mathbb{Q}_p$ of p-adic numbers.

Since our dissertation is about function fields over local fields, let us quickly recall the definition of the latter.

Definition 5.3.1 (Local Fields)**.** *1. A* ***global field*** *is either a number field or the function field of some curve over some finite field.*

2. A ***local field*** *(or to some authors, a* ***nonArchimedean local field****, is the completion of a global field at some nontrivial valuation.*

Fact 5.3.1. *1. The local fields of positive characteristic are exactly the field of Laurent Series $\mathbb{F}_q((t))$ over some finite field $\mathbb{F}_q$.*

2. The local fields k of characteristic 0 are exactly the finite extensions of $\mathbb{Q}_p$ as p ranges over rational primes. By Krasner's Lemma, which itself is a consequence of ***Hensel's Lemma****, every such k could be generated over $\mathbb{Q}_p$ by some element whose minimal polynomials has* rational *coefficients. That is, there exists an irreducible polynomial $f(x) \in \mathbb{Q}[x]$ such that $k \cong \mathbb{Q}_p[x]/(f(x))$.*

In general, when v is a discrete rank 1 , the completion $(\widehat{K_v}, \widehat{v})$ always satisfies the **Hensel's Lemma**, i.e. it is **henselian**.

Definition 5.3.2. *A valued field (K, v) is called* ***henselian*** *if the following holds. For every polynomial $f \in O_v[x]$ and $a \in O_v$ such that $f(a) \equiv 0$ in $\kappa(v)$ but $f'(a) \not\equiv 0$ in $\kappa(v)$, there exists $\alpha \in O_v$ such that $f(\alpha) = 0$ and $\alpha \equiv a$ in $\kappa(v)$. In other words, every simple root of $\overline{f} \in \kappa(v)[x]$ lifts to a root of f in O_v.*

For a general valuation v on K, however, the completion of (K, v) might no longer be Henselian. In such cases, we work instead with the **henselization** $(K_v, \mathfrak{v})$ of (K, v), which, roughly speaking, is the "smallest" valued field extending K that is henselian.

5.4 From Local to Global

Sometimes, local information at certain valuations of K can be patch together to give global information about K. One way to do so is via the so-called Local Global Principles. Informally, a Local Global Principle (LGP) for K is any theorem of the form

$$\text{Property } P \text{ holds for } K \iff P \text{ holds for } K_v \text{ (or } \widehat{K_v}\text{) for all } v \text{ in some } V \subset \mathcal{V}(K).$$

5.4.1 Historical Motivation

The first theorem to be officially called an LGP is the Hasse-Minkowski's Principle for ternary quadratic forms over $\mathbb{Q}$. It states that such a quadratic form represents 0 over $\mathbb{Q}$ iff it represents 0 over $\mathbb{R}$ and over $\mathbb{Q}_p$ for every prime p. Hasse later generalizes this principle to certain norm forms in the following theorem.

Theorem 5.4.1 (Hasse Norm Theorem)**.** *Let F be an algebraic number field. Let $a \in F$ and $L = K(\sqrt[l]{a})$. Let $N(a)$ denote the norm form induced by $N_{L/F}$, as defined in Section 4.3.2. Then for every $b \in F$, b is represented by N_a over F iff it is represented by $N(a)$ over all completions of F.*

Since by Theorem 4.3.1 , the cyclic algebra $A_{a,b}$ splits over F iff N_a represents b over F, an immediate consequence of the above is that a cyclic algebra splits over F iff it splits over all completions of F.

Working together with Noether and Brauer, Hasse finally managed to generalize this result to *any* central simple algebra over number field, not necessarily cyclic ones. (This is in turn used to show that all central division algebras over number fields are cyclic.) More precisely, the Hasse-Brauer-Noether LGP states that for a number field

F, the following sequence is exact.

$$0 \to \mathrm{Br}(F) \to \bigoplus_{v \in \mathbb{P}(F)} \mathrm{Br}(\widehat{F_v}) \xrightarrow{\sum \mathrm{Inv}_v} \mathbb{Q}/\mathbb{Z} \to 0.$$

Remark 5.4.1. *Using **Hasse Invariants** Inv_v in the above map, Hasse managed to define a **Hasse Symbol** which generalizes **Artin Symbol**, **Hilbert Symbol** and **Legendre Symbol**. Using the Hasse symbols, Hasse was able to generalize **Artin Reciprocity** (which itself is a generalization of **Hilbert Reciprocity** and **Quadratic Reciprocity**). Thus in a sense, Quadratic Reciprocity itself could be considered a LGP - the original LGP for Class Field Theory.*

5.4.2 Cohomological LGP

For the rest of this chapter, we use the same notation as in Chapter 4.

Kato's insight is to view the Hasse-Brauer-Noether LGP as a "cohomological LGP" (our terminology).

Definition 5.4.1 (Cohomological LGP)**.** *Let F be a field. For each $w \in \mathbb{P}(F)$, let F_w denote the completion of F at w if F is a number field and the Henselization of F at w otherwise. Recall that for each such w, the quotient map from the absolute Galois group of F_w to that of F induces **restriction maps** $\mathrm{res}_w : H^n(F) \to H^n(F_w)$ in every degree n.*

*We say that a field F **admits a Cohomological LGP in degree** d (with respect to the fixed prime l) if the following two conditions hold.*

1. *For each cocycle $\alpha \in H^d(F)$, $\mathrm{res}_w(\alpha) = 0 \in H^d(F_w)$ for all but finitely many $w \in \mathbb{P}(F)$.*

2. *The **Global Restriction Map** (in degree d) $\mathrm{res}_d : H^d(F) \xrightarrow{\oplus \mathrm{res}_w} \oplus_{w \in \mathbb{P}(F)} H^d(F_w)$*

is injective. Explicitly, this means that for all $\alpha \in H^d(F)$, we have

$$\alpha = 0 \in H^d(F) \iff \mathrm{res}_w(\alpha) = 0 \in H^d(F_w),\ \forall w \in \mathbb{P}(F).$$

Example 1. Cohomological Version of Hasse-Brauer-Noether LGP Let F be an algebraic number field. The injection $\mathrm{Br}(F) \to \bigoplus_{v \in \mathbb{P}(F)} \mathrm{Br}(\widehat{F_v})$ in the Hasse-Brauer-Noether's SES restricts to an injection on the l-torsion subgroups ${}_l\mathrm{Br}(F) \hookrightarrow \bigoplus_{v \in \mathbb{P}(F)} {}_l\mathrm{Br}(\widehat{F_v})$. By Chapter 4, this is equivalent to saying that the global restriction map $H^2(F) \hookrightarrow \bigoplus_{v \in \mathbb{P}(F)} H^2(F_v)$ is injective. In particular, F admits a cohomological LGP in degree 2.

Observing that every number field has Kronecker dimension 1, Kato suspects that for fields of Kronecker dimension d admits a LGP in degree $d+1$. From this starting point, he managed to give a sufficient condition for existence of certain LGP in higher degrees. This is expressed in terms of the boundary homomorphisms and the Arithmetical Bloch-Ogus Complex, explained in the next section.

5.4.3 Kato's LGP

Let $(\mathcal{X}, \mathfrak{O}_{\mathcal{X}})$ be an excellent normal integral scheme of dimension d with function field F. Under some hypotheses, Kato showed in [3] that one has the following **Arithmetical Bloch-Ogus Complex**

$$C_n^0(\mathcal{X}) : H^{d+1}(F) \xrightarrow{\partial_1} \bigoplus_{x \in \mathcal{X}^1} H^d(\kappa(x)) \xrightarrow{\partial_2} \cdots \bigoplus_{x \in \mathcal{X}^{d-1}} H^2(\kappa(x)) \xrightarrow{\partial_d} \bigoplus_{x \in \mathcal{X}^d} H^1(\kappa(x))$$

where here $\mathcal{X}^i$ denote the set of points $x \in \mathcal{X}$ of codimension i.

The map ∂_1 is defined as follows. As $(\mathcal{X}, \mathfrak{O}_{\mathcal{X}})$ is *normal*, for $x \in \mathcal{X}^1$, the local ring $O_x := \mathfrak{O}_{\mathcal{X},x}$ is a valuation ring with canonical discrete rank 1 valuation w_x.

Observe that the corresponding residue field Kw_x is exactly the residue field $\kappa(x) := O_{\mathcal{X},x}/\mathfrak{m}_x$ of the structure sheaf $\mathfrak{O}_{\mathcal{X}}$ at x. Under some hypotheses, Kato showed in [3] that for each such codim 1 point x and for every $i \geq 0$, there exists a **boundary homomorphism** $\partial_x : H^{i+1}(F) \to H^i(\kappa(x))$ with the following properties.

Fact 5.4.1. *1. ∂_x factors through the residue map as*

$$\partial_x : H^{i+1}(F) \xrightarrow{\mathrm{res}_w} H^{i+1}(F_w) \xrightarrow{\widehat{\partial_x}} H^i(\kappa(x)).$$

Additionally, when l is prime to the characteristic of $\kappa(x)$, the map $H^{i+1}(F_w) \xrightarrow{\widehat{\partial_x}} H^i(\kappa(x))$ is known to be an isomorphism.

2. If $a_1, \cdots, a_i \in F^*$ are w_x units then for all $a_0 \in F^*$, we have $\partial_x((\!(a_0, \ldots, a_n)\!)) = w_x(a_0)(\!(\overline{a_1}, \ldots, \overline{a_n})\!) \in H^{i+1}(\kappa(x))$.

The map ∂_1 is defined as the direct sum of ∂_x as x ranges over $\mathcal{X}_1$. By the first property above, we see that the injectivity of ∂_1 would imply injectivity of $H^{d+1}(F) \xrightarrow{\oplus \mathrm{res}_x} \bigoplus_{x \in \mathcal{X}^1} H^{d+1}(F_x)$. In particular, if ∂_1 is injective, then F admits a LGP in degree $d+1$. In Proposition 5.2 in [3], Kato proved that this is the case if $\mathcal{X}$ be a regular proper flat scheme of dim 2 over the valuation ring of some discretely valued field. This gives the following LGP in degree 3.

Theorem 5.4.2 (Kato's LGP in degree 3)**.** *Let (k, v) be a complete discretely valued field with finite residue field. Let $\mathcal{X}$ be a regular proper flat scheme over the valuation ring O_v such that $\mathcal{X} \otimes_{O_v} k$ is smooth over k. If $\dim \mathcal{X} = 2$ then the function field of $\mathcal{X}$ admits an LGP in degree 3.*

6

EEIP for Function Fields of P-adic Curves

6.1 Introduction and Overview

Definition 6.1.1. *We say that K/k is a* ***local function field****, if k is a local field, relatively algebraically closed in K and K is finitely generated over k. We call k the* ***constant subfield*** *of K.*

By Fact 5.3.1, k must be one of two "types": either $k = \mathbb{F}_q((t))$ for a unique finite field $\mathbb{F}_q$ or k is a finite extension of $\mathbb{Q}_p$ for some unique prime p. In the latter case, we also call K/k a p-adic function field.

Please note that some authors used the term **local function field** to refer instead to local fields of positive characteristic.

The goal of this thesis is to prove the following theorem.

Theorem 6.1.1 (EEIP for Function Fields of p-adic Curves)**.**

$$\textit{The pair } \Big(\{p\text{-adic function fields of } \mathrm{td}\ 1\}, \{p\text{-adic function fields}\}\Big) \textit{ is EEIP.}$$

Explicitly, let K/k and L/l be p-adic function fields, elementarily equivalent over the language of Rings. If $\mathrm{td}(L/k) = 1$ *then K/k must be isomorphic to L/l as function fields.*

In this chapter, we explain how this theorem will follow from the definability of the valuation rings of K/k. The main work lies in the next chapter, where we prove that the valuation rings of K/k are parametrizable by a formula $\mathrm{Val}_K(\mathbf{a};t)$ dependent on K.

6.2 Preliminaries: Definability of the Constant Field and Its "Type"

It turns out that for local function fields, the constant subfield and transcendence degree are uniformly definable. In other words, we have the following.

Proposition 6.2.1. *There exist formulas* $\mathrm{Const}(t)$ *and* Curve *such that for all local function fields K/k, we have* $k = \mathrm{Const}(K)$ *and* $K \vDash \mathrm{Curve} \iff \mathrm{td}(K/k) = 1$.

Proof. Since k is a 2-cohomologically well-behaved large field (in the sense defined in [7]), this follows from Theorem 1.4. and Proposition 5.5. of [7]. □

Since (k, v) is henselian, its valuation ring O_v is definable inside k (and hence inside K) by [5]. As a consequence, we have the following.

Corollary 6.2.1 (Definability of "Types" of Local Fields). *For every prime p and prime power q, there exist formulas $BaseQ_p$ and $BaseF_q$ satisfying the following. For every local field (k, v), we have*

- $k \models BaseQ_p$ *iff k is a finite extension of $\mathbb{Q}_p$;*
- $k \models BaseF_q$ *iff k is the Laurent Series field $\mathbb{F}_q((t))$.*

Proof. First, observe that as valuation rings are uniformly definable in local fields, so are their maximal ideals and the cardinality and characteristic of the residue field. This is because if a valuation ring O_v of a field k is definable, then we can characterize its maximal ideals $\mathfrak{m}_v$ as well as the characteristic and cardinality of the residue field kv in the following way. (For examples of explicit formulas, see the proof of Proposition 6.3.1.)

- $\mathfrak{m}_v$ is the set of all $t \in O$ such that either $t = 0$ or $t^{-1} \notin O$;
- $\operatorname{char} kv = p$ iff $(1 + 1 + \cdots + 1) \in \mathfrak{m}_v$;
- $\#|kv| = q$ iff there exist $f_1, \ldots, f_q$ satisfying
  1. $(f_i - f_j) \notin \mathfrak{m}_v$ for all $i \neq j$.
  2. For all $g \in O$, there exists i such that $(g - f_i) \in \mathfrak{m}_v$.

The formulas BaseQ_p and BaseF_q can now be defined using the following characterizations.

- A local field (k, v) is a finite extension of $\mathbb{Q}_p$ iff $\operatorname{char} kv = p$ but $\operatorname{char} k \neq p$.
- A local field (k, v) is the Laurent Series field $\mathbb{F}_q((t))$ for some prime power $q = p^d$ iff $\operatorname{char} kv = \operatorname{char} k = p$ and $\#|kv| = q$.

$\square$

Corollary 6.2.2 (Strong EEIP for Local Fields)**.** *Let* (k,v) *be a local field. Then there exists a first order formula* $Type_k$ *such that for all local fields* $(\lambda, \mathfrak{v})$ *we have* $\lambda \models Type_k$ *iff* $\lambda \cong k$*.*

Proof. First, consider the case when k is a finite extension of $\mathbb{Q}_p$. By Fact 5.3.1, we can write $k = \mathbb{Q}_p(\alpha)$ for some α whose minimal polynomial f over $\mathbb{Q}_p$ has rational coefficients. □

6.3 Definability of Valuation Rings implies EEIP

Definition 6.3.1. *Let* K/k *be a local function field of transcendence degree* 1*.*

1. *Let* $\mathrm{Val}(\mathbf{z};t) \coloneqq \mathrm{Val}(z_1,\ldots,z_m;t)$ *be a first-order formula.*

   We say that $\mathrm{Val}(\mathbf{a};t)$ ***contains all the valuation (rings)*** *of* K/k *if for every valuation ring* O *of* K/k*, there exists* $(a_1,\ldots a_m) \in K^m$*, such that* $\mathrm{Val}(\mathbf{a};K) = O$*.*

   We say that $\mathrm{Val}(\mathbf{z};t)$ ***parametrizes all the valuation rings*** *of* K/k *if it contains all the valuation ring of* K/k*, and for every tuple* $\mathbf{a} \in K^m$*, the set of realizations* $\mathrm{Val}(\mathbf{a};K)$ *is a valuation ring of* K/k*. If such a formula exists, we say that the valuation rings of* K/k *are* ***parametrizable****. We also call* $\mathrm{Val}(\mathbf{z};t)$ *a* ***parametrization*** *of the valuation rings of* K/k*.*

2. *We say that the collection of formula* $\{\deg_N(t) \mid N \in \mathbb{N}\}$ ***defines degrees in*** K/k *if for all* $N \in \mathbb{N}$ *and* $t \in K$*, we have* $K \models \deg_N(t) \iff [K : k(t)] = N$*.*

It turns out that if the valuation rings of K/k (i.e. valuation rings of K containing k) are parametrizable, then there is a collection of formula defining degrees in K/k.

Proposition 6.3.1. *Given any formulas* $\mathrm{Val}(\mathbf{z};t) \coloneqq \mathrm{Val}(z_1,\ldots,z_m;t)$ *and any* $N \in \mathbb{N}$*, there exist formulas* $\boldsymbol{Ring}(t;u)$ *and a collection of formulas* $\{\deg_N(t) \mid N \in \mathbb{N}\}$ *such* *that for every p-adic function field K/k, if* $\mathrm{Val}(\mathbf{z};t)$ *parametrizes all the valuation rings of K/k, then we have the following.*

1. *For all $t \in K$, $\boldsymbol{Ring}(t;K)$ is the relative algebraic closure of $k[t]$ in K.*

2. *$\{\deg_N(t) \mid N \in \mathbb{N}\}$ defines degrees in K/k.*

Proof. 1. Recall that for every $t \in K$, the relative algebraic closure $\widetilde{k[t]}$ of $k[t]$ in K is the intersection of all valuation rings of K/k containing t. In other words, $\widetilde{k[t]} = \mathbf{Ring}(t;K)$ where

$$\mathbf{Ring}(t;u) \longleftrightarrow \forall \mathbf{a}, \Big[\mathrm{Val}(\mathbf{a};t) \implies \mathrm{Val}(\mathbf{a};u) \Big].$$

2. For readability, we will build $\deg_N(t)$ from a several simpler formulas, defined as follows.

   1. **Formula for Maximal Ideals**

   $$\mathcal{M}(\mathbf{a};t) \longleftrightarrow \Bigg[(t \neq 0) \implies \Big(\mathrm{Val}(\mathbf{a};t) \wedge \neg\, \mathrm{Val}(\mathbf{a};t^{-1}) \Big) \Bigg].$$

   Observe that for all $\mathbf{a} \in K^m$, $\mathcal{M}(\mathbf{a};K)$ is the maximal ideal of the valuation ring $\mathrm{Val}(\mathbf{a};K)$.

   2. **Formula for Uniformizing Parameters**

   $$\begin{aligned}\pi(\mathbf{a};t) \longleftrightarrow \ \forall f, \Bigg[&\Big(\mathrm{Val}(\mathbf{a};f) \implies \mathcal{M}(\mathbf{a};ft) \Big) \\ &\wedge \Big(\mathcal{M}(\mathbf{a};f) \implies \exists g, \big[\mathrm{Val}(\mathbf{a};g) \wedge (f = gt) \big] \Big) \Bigg].\end{aligned}$$

   Observe that for all $\mathbf{a} \in K^m$ and all $t \in K$, $K \vDash \pi(\mathbf{a};t)$ iff t is a uniformization parameter of $\mathrm{Val}(\mathbf{a};K)$.

3. **Formula for Branch Points**

$$\mathrm{BranchPoint}(f,c) \longleftrightarrow \exists \mathbf{a}, \Big(\mathcal{M}(\mathbf{a}; f-c) \wedge \neg\pi(\mathbf{a}; f-c)\Big).$$

   Observe that for all $f \in K$ and $c \in k$, $K \vDash \mathrm{BranchPoint}(f,c)$ iff c is a "branch point" of f, i.e. there exists some closed point x in a smooth model X of K/k such that the order of vanishing of $f-c$ at x is greater than 1.

4. **Formula for Dimension of the Residue Fields** For each $d \in \mathbb{N}$, define

$$\begin{aligned}\dim_d(\mathbf{a}) \longleftrightarrow\ & \exists t_1,\dots,t_d, \\ & \left[\forall c_1,\dots,c_d, \left(\bigwedge_i \mathrm{Const}(c_i) \wedge \mathcal{M}\Big(\mathbf{a}; \sum_i c_i t_i\Big)\right) \implies \bigwedge_i \mathcal{M}(\mathbf{a}; c_i)\right] \\ & \wedge \left[\forall f, \mathfrak{O}(\mathbf{a}; f) \implies \left(\exists c_1,\dots,c_d, \bigwedge_i \mathrm{Const}(c_i) \wedge \mathcal{M}\Big(\mathbf{a}; \sum_i (f - c_i t_i)\Big)\right)\right].\end{aligned}$$

   Observe that for every $\mathbf{a} \in K^m$ and every $d \in \mathbb{N}$, we have $K \vDash \dim_d(\mathbf{a}) \iff$ the residue field of $\mathrm{Val}(\mathbf{a}; K)$ has dimension d over k.

Finally, we are now ready to define $\deg_N(t)$. Let $\mathcal{P}(N) := \{(d_1,\dots,d_N) \in \mathbb{Z}_{\geq 0} \mid \sum_i d_i = N\}$ be the set of partitions of N. Let $\deg_N(t)$ be the formula

$$\forall c, \left(\neg\,\mathrm{BranchPoint}(f,c) \implies \left[\bigvee_{\mathbf{d} \in \mathcal{P}(N)} \Big(\exists (\mathbf{a})_1, \dots (\mathbf{a})_N, \bigwedge_i \dim_{d_i}(\mathbf{a})\Big)\right]\right).$$

□

Now we show how our main theorem would follow from definability of valuation rings of K/k.

Proposition 6.3.2. *Let K/k be a local function field of transcendence degree* 1. *Sup-*

pose the valuation rings of K/k are parametrizable by a formula $\mathrm{Val}_K(\mathbf{z}, t)$ *such that for all field L and all tuples* $\mathbf{b} \in L^m$, $\mathrm{Val}_K(\mathbf{b}, L)$ *is a valuation ring of L. Let* $\{\deg_N(x) \mid N \in \mathbb{N}\}$ *be defined as in Proposition 6.3.1. Then the following must hold.*

1. *There exists a sentence ψ_K such that for all local function fields L/λ, if $L \models \psi_K$ then we have*

   - $\mathrm{td}(L/\lambda) = 1$*;*

   - *For every valuation ring O of L/λ, there exist $\mathbf{b} \in L^m$ such that $O =$* $\mathrm{Val}_K(\mathbf{b}, L)$*. In other words, if $L \models \psi_K$ then the formula* $\mathrm{Val}_K(\mathbf{z}, t)$ *also parametrizes the valuation rings of L/λ. Consequently, by Proposition* **??***,* $\{\deg_N(x) \mid N \in \mathbb{N}\}$ *also defines degrees in L/λ.*

2. *Let L/λ be a p-adic function field elementarily equivalent to K. Then L/λ is isomorphic to K/k as function fields.*

Proof. 1. Define ψ_K by $\psi_K \longleftrightarrow \mathrm{Curve} \wedge \left[\forall c, \left(\mathrm{Const}(c) \iff \forall \mathbf{a}, \mathrm{Val}(\mathbf{a}; c) \right) \right]$.

   Let L/λ be a local function field such that $L \models \psi_K$. Then by $\lambda = \mathrm{Const}(L)$ and L is the function field of a smooth projective curve Y over λ. By definition of ψ_K, we also have $\lambda = \bigcap_{\mathbf{b} \in L^m} \mathrm{Val}_K(\mathbf{b}; L)$. Suppose by contradiction that there exists some valuation ring O of L/λ, such that $O \neq \mathrm{Val}(\mathbf{b}; L)$ for all $\mathbf{b} \in L^m$. Note that O is the local ring O_y at some point $y \in Y$. By Riemann-Roch, there exists a function $g \in L$ with a single pole at y. Since g has a pole, it cannot be a constant. On the other hand, since it is regular at all points other than y, it lies in $\bigcap_{\mathbf{b} \in L^m} \mathrm{Val}_K(\mathbf{b}; L)$ (contradiction).

2. Observe that Corollary 6.2.2, we must have $k \cong \lambda$.

   Now, let d the degree of any irreducible planar affine curve with function field K/k. Let S be the set of irreducible polynomials $F(X, Y) \in k[X, Y]$ of $\deg F =$

$\deg_X F = \deg_Y F = d$ such that K is the function field of the affine curve cut out by F (i.e. $K = \mathrm{Frac}\big(k[X,Y]/(F)\big)$. Note that $S \neq \varnothing$, by our choice of d. For each $F \in S$, let A_F denote the **ring of definition of** F, i.e. the ring generated over $\mathbb{Q}$ by the coefficients of F.

Let $m := \min\left\{ \mathrm{td}\left(\mathrm{Frac}(A_F)/\mathbb{Q}\right) \middle| F \in S \right\}$ (note: we allow $m = 0$). Let F be an element of S corresponding to m. Then there exists algebraically independent elements $a_1, \ldots, a_m \in A_F$ such that A_F is algebraic over the polynomial ring $\mathbb{Q}[a_1, \ldots, a_m] = \mathbb{Q}[\mathbf{a}]$. Thus we can write

$$A_F = \mathbb{Q}[a_1, \ldots, a_n; \alpha_1, \ldots, \alpha_m] = \frac{\mathbb{Q}[\mathbf{a}; \mathbf{t}]}{\big(g_1(\mathbf{a}; \mathbf{t}), \ldots, g_s(\mathbf{a}; \mathbf{t})\big)}$$

for some $\alpha_i \in A_F$ and some (finitely many) polynomials $g_i \in \mathbb{Q}(\mathbf{z}; \mathbf{t})$.

Now, observe that there must exists some polynomial $\mathcal{F} \in \mathbb{Q}[\mathbf{x}, \mathbf{t}; X, Y]$ such that $\mathcal{F}(\mathbf{a}, \underline{\alpha}; X, Y) = F(X, Y)$ and some pair $\xi_1, \xi_2 \in K$ satisfying $F(\xi_1, \xi_2) = 0$. Thus the tuple $(\mathbf{a}; \underline{\alpha}; \xi_1, \xi_2,)$ satisfies the formula $\Phi(\mathbf{z}, \mathbf{t}; x, y)$ defined as the conjunction of the following formulas.

(a) $\mathbf{z}$ and $\mathbf{t}$ lie in the constant field.

(b) $g_1(\mathbf{z}; \mathbf{t}) = \cdots = g_s(\mathbf{z}; \mathbf{t}) = 0$;

(c) $\mathcal{F}(\mathbf{z}; \mathbf{t}; X, Y) \in k[X, Y]$ is irreducible (as a polynomial in X and Y);

(d) $\mathcal{F}(\mathbf{z}, \mathbf{t}; x, y) = 0$;

(e) $[K : k(x)] = [K : k(y)] = d$.

In other words, $K \vDash \exists \mathbf{z}, \mathbf{t}; x, y, \Phi(\mathbf{z}, \mathbf{t}; x, y)$. Since $L \equiv K$, Φ must have some realizations $(\mathbf{b}, \underline{\beta}; \eta_1, \eta_2)$ in L. In other words, there exist $\mathbf{b}, \underline{\beta}; \eta_1, \eta_2 \in L$ such that the following hold.

(a) $b_1, \ldots, b_m; \beta_1, \ldots, \beta_n \in \lambda$.

(b) $g_1(\mathbf{b};\underline{\beta}) = \cdots = g_s(\mathbf{b};\underline{\beta}) = 0$;

(c) $\mathcal{F}(\mathbf{b},\underline{\beta};X,Y) \in \lambda[X,Y]$ is irreducible (as a polynomial in X and Y);

(d) $\mathcal{F}(\mathbf{b},\underline{\beta};\eta_1,\eta_2) = 0$;

(e) $[L : \lambda(\eta_1)] = [L : \lambda(\eta_2)] = d$.

In particular, we have a surjection $\mathbb{Q}[\mathbf{a},\underline{\alpha}] = \dfrac{\mathbb{Q}[\mathbf{a};\mathbf{t}]}{\big(g_1(\mathbf{a};\mathbf{t}),\ldots,g_s(\mathbf{a};\mathbf{t})\big)} \twoheadrightarrow \mathbb{Q}[\mathbf{b},\underline{\beta}]$. We claim that this is an isomorphism. It suffices to show that $\mathbf{b}$ are algebraically independent over $\mathbb{Q}$. Indeed, suppose there exists some nonzero polynomial $h \in \mathbb{Q}[\mathbf{z}]$ such that $h(\mathbf{b}) = 0$. Then $L \vDash \exists \mathbf{z},\mathbf{t};x,y,\ \Phi(\mathbf{z},\mathbf{t};x,y) \wedge h(\mathbf{z})$. Thus so must K. Let $(\mathbf{c},\underline{\gamma},\epsilon_1,\epsilon_2)$ be a realization of this formula in K. Then the polynomial $H(X,Y) \coloneqq \mathcal{F}(\mathbf{c},\underline{\gamma};X,Y)$ is a member of S whose ring of definition A_H satisfies $\operatorname{td}\big(\operatorname{Frac}(A_H)\big) < m$, contradicting the minimality of m.

Since the valuation rings of $k \cong \lambda$ are definable, we can arrange for the isomorphism $\mathbb{Q}(\mathbf{a},\underline{\alpha}) \to \mathbb{Q}(\mathbf{b},\underline{\beta})$ to be an "isometry". In particular, it extends to an isomorphism $k \cong \lambda$. Thus we have an embedding $K = k(\xi_1,\xi_2) \cong \lambda(\eta_1,\eta_2) \subset L$.

$$\begin{tikzcd}
K \cong k(\xi_1,\xi_2) \arrow[r, "\sim"] & k \otimes \dfrac{\mathbb{Q}(\mathbf{a},\underline{\alpha})[X,Y]}{\big(\mathcal{F}(\mathbf{a},\underline{\alpha};X,Y)\big)} \arrow[r, "\sim"] & \lambda \otimes \dfrac{\mathbb{Q}(\mathbf{b},\underline{\beta})[X,Y]}{\big(\mathcal{F}(\mathbf{b},\underline{\beta};X,Y)\big)} \arrow[r, "\sim"] & \lambda(\eta_1,\eta_2) \subset L \\
k \arrow[u, hook] \arrow[r, "\sim"] & k \otimes \mathbb{Q}(\mathbf{a},\underline{\alpha}) \arrow[u, hook] \arrow[r, "\sim"] & \mathbb{Q}(\mathbf{b},\underline{\beta}) \otimes \lambda \arrow[u, hook] \arrow[r, "\sim"] & \lambda \arrow[u, hook]
\end{tikzcd}$$

On the other hand, by definition of η_i's and F, we have $d = [K : k(\xi_1)] =$

$$[k(\xi_1,\xi_2):k(\xi_1)] = [\lambda(\eta_1,\eta_2):\lambda(\eta_1)] \le [L:\lambda(\eta_1)] = d.$$

$$\begin{tikzcd}
K \cong k(\xi_1,\xi_2) \arrow[r, "\sim"] & \lambda(\eta_1,\eta_2) \arrow[r, hook] & L \\
k(\xi_1) \arrow[u, "\deg d"] \arrow[r, "\sim"] & \lambda(\eta_1) \arrow[u] \arrow[ur, "\deg d"'] &
\end{tikzcd}$$

Thus equality must hold throughout. In particular, we must have $\lambda(\eta_1,\eta_2) = L$.

□

7

Definability of Valuation Rings

7.1 Set-Up

For the rest of this chapter, let (k, v) be a local field with valuation ring $(O_v, \mathfrak{m}_v)$ and residue field kv. Let $K = k(X)$ be the function field of a smooth curve X over K.

Let $l = 2$ if $\operatorname{char} k \neq 2$ and $l = 3$ otherwise.

In this chapter we want to show that the valuation rings if K/k are parametrizable. Suppose without loss of generality that K contains a primitive $2l$-th root of unity. Our main tool is Kato's Local Global Principle. To apply it, we fix a regular projective

O_v-model $\mathcal{X}$ of X. In particular, $X = \mathcal{X}_k := \mathcal{X} \times_{O_v} k$ is the generic fiber of $\mathcal{X}$. As mentioned in part 5 of Example 1, every codim 1 point $x \in \mathcal{X}^1$ induces a discrete rank 1 valuation w_x on K. Thus, letting K_x denote the corresponding Henselization, by Kato's LGP (Theorem 5.4.2), we have the following.

A cocyle $\alpha \in H^3(K)$ vanishes iff its restriction $\mathrm{res}_w(\alpha) = 0 \in H^3(K_x)$ for every $x \in \mathcal{X}^1$.

Fact 7.1.1. *Let $x \in \mathcal{X}^1$. Then there are two possibilities for x.*

1. *If x is the the generic points of some special fiber $\mathcal{X}_s$ of $\mathcal{X}$, then $w_x|_k = v$ and $\kappa(x)$ is a global function field over the finite field kv.*
2. *If x is a closed point of the generic fiber X of $\mathcal{X}$, then $w_x|_k$ is trivial and and $\kappa(x)/k$ is finite. In particular, $\kappa(x)$ is a local field.*

In particular, let X_0 denote the set of closed points of the generic fiber X. Then the valuation rings of K/k are exactly the local rings $\mathfrak{O}_x$ for $x \in X_0$.

7.2 Overview

Let K/k be a local function field of transcendence degree 1 .

Our main goal in this chapter is to show that the valuation rings of K/k are parametrizable. Since k is definable in K by Proposition 6.2.1, the proposition below implies that it suffices to find a formula *containing* all the valuation rings of K/k.

Proposition 7.2.1. *Given any formulas $\mathcal{O}(\mathbf{z};t) := \mathcal{O}(z_1,\dots,z_m;t)$ and* $\mathrm{Const}(t)$*, there exists a formula* $\mathrm{Val}(\mathbf{z};t)$ *such that for every local function field K/k, if $\mathcal{O}(\mathbf{z};t)$ contains all the valuation rings of K/k and* $\mathrm{Const}(t)$ *defines k in K, then* $\mathrm{Val}(\mathbf{z};t)$ *parametrizes all the valuation rings of K/k.*

Proof. For readability, we will built the formula $\mathrm{Val}(\mathbf{z};t)$ from a series of simpler formulas, defined below.

$$\begin{aligned}
\mathrm{isRing}(\mathbf{a}) \longleftrightarrow\ & \mathcal{O}(\mathbf{a};1) \wedge \mathcal{O}(\mathbf{a};0) \\
& \wedge \left[\forall s,t, \left(\left[\mathcal{O}(\mathbf{a};t) \wedge \mathcal{O}(\mathbf{a};s) \right] \implies \left[\mathcal{O}(\mathbf{a};s-t) \wedge \mathcal{O}(\mathbf{a};st) \right] \right) \right]; \\
\mathrm{isValRing}(\mathbf{a}) \longleftrightarrow\ & \mathrm{isRing}(\mathbf{a}) \wedge \left[\forall t, \left(t \neq 0 \implies \left[\mathcal{O}(\mathbf{a};t) \vee \mathcal{O}(\mathbf{a};t^{-1}) \right] \right) \right]; \\
\mathrm{isTrivialOnConst}(\mathbf{a}) \longleftrightarrow\ & \mathrm{isValRing}(\mathbf{a}) \wedge \left[\forall c,t, \left(\mathrm{Const}(c) \implies \mathcal{O}(\mathbf{a};c) \right) \right].
\end{aligned}$$

Finally, we define $\mathrm{Val}(\mathbf{a},\mathbf{b};t)$ as

$$\mathrm{Val}(\mathbf{a},\mathbf{b};t) \longleftrightarrow \left[\mathrm{isTrivialOnConst}(\mathbf{a},\mathbf{b}) \implies \mathcal{O}(\mathbf{a},\mathbf{b};t) \right].$$

□

Our plan for constructing a formula containing all valuations of K/k is as follows.

1. For each three cocyle $\alpha \in (\!|\dot{K}^3|\!)$ (cf. Notations 4.2.1), we define the notion of the **support of** α **on** X, denoted $\mathrm{supp}_X(\alpha)$.

2. It turns out that for all α, the ring $\mathfrak{O}_X(\alpha) := \mathfrak{O}_X(\mathrm{supp}_X(\alpha))$ of regular functions on $\mathrm{supp}_X(\alpha)$ is definable by a formula with parameters $\mathbf{a}$, where $\mathbf{a}$ is any "nice" representative of α. (cf. Lemma 7.3.2)

3. Moreover, for every $x \in X_0$, if $\{x\} = \mathrm{supp}_X(\alpha) \cap \mathrm{supp}(\beta)$ for some $\alpha, \beta \in (\!|\dot{K}^3|\!)$, then the local ring $\mathfrak{O}_x$ is definable in terms of $\mathfrak{O}_X(\alpha)$ and $\mathfrak{O}_X(\beta)$. In particular, we will define a formula $\mathcal{O}(z_1, \ldots, z_6; t)$ such that if $\mathbf{a}$ and $\mathbf{b}$ are any "nice" representatives of α and β, respectively, then $\mathcal{O}(\mathbf{a},\mathbf{b};K) = O_x$.

4. Finally, we show that indeed every $x \in X_0$ could be realized as the common support of some α and β in $(\!|\dot{K}^3|\!)$. Thus the formula $\mathcal{O}(\mathbf{z};t)$ contains all valuation rings of K/k.

7.3 Definability of Rings of Regular Functions on the Supports of 3-cocyles

7.3.1 The Support of a cocyle on X

Definition 7.3.1. *For each cocyle $\alpha \in H^3(K)$, we define the* ***support of*** *α* ***on*** *X to be the set* $\mathrm{supp}_X(\alpha)$ *of $x \in X_0$ at which (α) is nontrivial; i.e.* $\mathrm{res}_x(\alpha) \neq 0 \in H^3(K_x) := H^3(K_x, \mu_l)$.

Lemma 7.3.1. *Let $\alpha \in (\!|\dot{K}^3|\!)$. Let $\mathbf{a} \in (K^*)^3$ be any representative of α. Then for all $x \in \mathrm{supp}(\alpha)$, there exists i such that $l \nmid w_x(a_i)$. In particular,* $\mathrm{supp}_X(\alpha)$ *is finite.*

Proof. Suppose by contradiction that $l \mid w_x(a_i)$ for all a_i. Then we can write $a_i = b_i^l u_i$ for some $b_i \in K_x$ and $u_i \in O_x^*$. By Observation 4.2.1 that $\mathrm{res}_x(\mathbf{a}) = \mathrm{res}_x(\mathbf{u})$. On the other hand, by Fact 5.4.2, $\partial_x(\mathbf{u}) = w_x(u_0)(\overline{u_1} \cup \overline{u_2}) = 0$. The same fact now implies that $\mathrm{res}_x(\mathbf{u})$, and hence $\mathrm{res}_x(\mathbf{a})$ is trivial (contradiction).

Thus $l \nmid w_x(a_i)$ for some i. In particular, for such i, $w_x(a_i) \neq 0$ so x must be a pole or a zero of a_i on X. As $a_i \neq 0$, there are only finitely many possibilities for x. □

Lemma 7.3.2. *Let $\alpha \in (\!|\dot{K}^3|\!)$. Then α can be represented by a* ***nice*** *tuple, i.e. a tuple $\mathbf{a} \in (K^*)^3$ such that if for all $x \in \mathrm{supp}_X((\!|\mathbf{a}|\!))$, we have $0 \leq \min_i\{w_x(a_i)\}$ is not divisible by l. We also such a tuple a* ***nice*** *representative of α.*

Proof. First we claim that there is representative $\mathbf{b}$ of α such that for all $x \in \mathrm{supp}_X(\mathbf{b})$, $w_x(b_i) \geq 0$. Indeed, Let $\mathbf{c}$ be a representative of α. By Lemma 7.3.1 above, $\mathrm{supp}(\alpha)$

is finite. Thus Weak Approximation implies that for every $P \in \mathrm{supp}(\alpha)$, there exists $f_P \in K$ such that $w_P(f_P) > 0$ and $w_x(f_P) = 0$ for all $P \neq x \in \mathrm{supp}_X(\alpha)$.

Now for each $i = 0, 1, 2$, let $\mathrm{Poles}(c_i) := \{P \in \mathrm{supp}_X(\alpha) \mid w_P(c_i) < 0\}$. Since for each i, this is a finite set, we can define $b_i := \prod_{P \in \mathrm{Poles}(c_i)} f_P^{l|w_P(c_i)|} c_i$. By Observation 4.2.1, $(\mathbf{b})$ is also a representative of α (with no "poles" on $\mathrm{supp}_X(\alpha)$, as desired).

Finally, to construct a nice representative $\mathbf{a}$ of α, we first define a function $\mathbf{e} = (e_0, e_1, e_2) : \mathrm{supp}(\alpha) \to (l\mathbb{Z})^3$ as follows. For every $x \in \mathrm{supp}_X(\alpha)$, let $m_x := \min\{i \mid l \nmid w_x(b_i)\}$. Observe that m_x is well-defined by Lemma 7.3.1 above. Define $e_j(x) = 0$ for $j \neq m_x$ and $e_{m_x}(x) = \left(\sum_{j \neq m_x} w_x(b_j) \right)$.

Now, for $i = 0, 1, 2$, let $a_i = \prod_{x \in \mathrm{supp}_X(\alpha)} f_P^{le_i(x)} b_i$.

□

7.3.2 *Definability of the Ring of Regular Functions on* $\mathrm{supp}_X(\alpha)$

Let $\mathbf{a} \in (K^*)^3$ be a nice tuple representing $(\!|\mathbf{a}|\!) =: \alpha$. For each $x \in \mathrm{supp}_X(\alpha)$, define

$$\Gamma_x := \left\{ \gamma \in K \mid w_x(\gamma^l) > \min\left(w_x(a_0), w_x(a_1), w_x(a_2) \right) \right\}.$$

Observe that, the local ring $\mathfrak{O}_x$ at x can be recovered from Γ_x by $\mathfrak{O}_x = \{f \in K \mid f\Gamma_x \subseteq \Gamma_x\}$.

Let $\Gamma(\mathbf{a}) := \bigcup_{x \in \mathrm{supp}_X(\alpha)} \Gamma_x$ and $O(\mathbf{a}) = \{\gamma \in K \mid \gamma\Gamma(\mathbf{a}) \subseteq \Gamma(\mathbf{a})\}$. The following lemma shows that $O(\mathbf{a})$ is exactly the ring of regular functions $\mathfrak{O}_X(\alpha) := \bigcap_{x \in \mathrm{supp}_X(\alpha)} \mathfrak{O}_x$ on $\mathrm{supp}_X(\alpha)$.

Lemma 7.3.3.

$$O(\mathbf{a}) = \bigcap_{x \in \mathrm{supp}_X(\alpha)} \mathfrak{O}_x$$

Proof. Clearly, $\bigcap_{x \in \mathrm{supp}_X(\alpha)} \mathfrak{O}_x \subseteq O(\mathbf{a})$. Now let $f \in K$ such that $f \notin \mathfrak{O}_P$ for some

$P \in \mathrm{supp}_X(\alpha)$. We show that $f \notin O(\mathbf{a})$.

Indeed, let m be the minimal valuation of an element of Γ_P. Since by Lemma 7.3.1, $\mathrm{supp}_X(\alpha)$ is finite, Weak Approximation implies that the existence of $\gamma \in K$ such that $w_P(\gamma) = m$ and $lw_x(f\gamma) \leq \min_i(w_x(a_i))$ for all $P \neq x \in \mathrm{supp}_X(\alpha)$. Then $\gamma \in \Gamma_x \subset \Gamma(\mathbf{a})$, but $f\gamma \notin \bigcup_{x \in \mathrm{supp}_X(\alpha)} \Gamma_x = \Gamma(\mathbf{a})$.

$\square$

Our goal is to prove definability of $O(\mathbf{a})$. It is clear that this would follow from the definability of $\Gamma(\mathbf{a})$, which we plan to prove as follows. To each $\gamma \in K$ we will associate three families $\mathcal{K}_{\gamma,a_i}, i = 0,1,2$ of field extensions of K. Then Proposition 6.1.1 below shows that $\Gamma(\mathbf{a})$ could be characterize as the set of $\gamma \in K$ for which $(\!|\mathbf{a}|\!)$ is nontrivial on *all* members of one of the family $\mathcal{K}_{\gamma,i}$.

To prepare for Proposition 6.1.1, we need the following notations and lemmas. For each fixed $a, f \in K^*$, let $K_{f,a} := K\left(\sqrt[l]{1 - \frac{f^l}{a}}\right)$.

Lemma 7.3.4. *Fix $a, f \in K^*$. Let $\mathfrak{w}$ be a discrete rank 1 valuation on $L := K_{f,a}$ and let w be its restriction to K.*

1. *If a is not an l-th power in the Henselization $L_{\mathfrak{w}}$ of L at $\mathfrak{w}$, then $w(\frac{f^l}{a}) \geq 0$, i.e. $lw(f) \geq w(a)$.*

2. *If $lw(f) > w(a)$ then $K_w = L_{\mathfrak{w}} \supset K_{f,a}$.*

Proof. Let $\epsilon := \sqrt[l]{1 - \frac{f^l}{a}} \in L$ so that we have $\epsilon^l = 1 - \dfrac{f^l}{a}$.

1. First, we claim that ϵ is an $\mathfrak{w}$-unit. To see this, observe that the group $U^1_{\mathfrak{w}}$ of principal $\mathfrak{w}$ units is contained in $L^l_{\mathfrak{w}}$. Thus if $\mathfrak{w}(\epsilon) > 0$ then $\frac{f^l}{a} \in U^1_{\mathfrak{w}} \subset L^l_{\mathfrak{w}}$ so $a \in L^l_{\mathfrak{w}}$ (contradiction). On the other hand, if if $\mathfrak{w}(\epsilon) < 0$ then $1-\epsilon^{-l} \in\in U^1_{\mathfrak{w}} \subset L^l_{\mathfrak{w}}$. Since $(-1) \in K^l \subseteq L^l_{\mathfrak{w}}$, we have $a = (-1)f^l\epsilon^{-l}(1-\epsilon^{-l}) \in L^l_{\mathfrak{w}}$ (contradiction).

   Now, if $\mathfrak{w}(f^l/a) < 0$ then $\mathfrak{w}(\epsilon) = \min(\mathfrak{w}(1), \mathfrak{w}(f^l/a)) < 0$, contradicting the above claim. Thus $\mathfrak{w}(f^l/a)$, and hence $w(f^l/a)$, must be positive.

2. Indeed, for such a w, $1 - \frac{f^l}{a}$ is a principal w-unit so w splits completely in L.

$\square$

Now for each γ and a in K, we associate a family of extensions $\mathcal{K}_{\gamma,a} := \{K_{\gamma c,a}K_{\gamma c^{-1},a} \mid c \in k^*\}$. Given any cocyle $\alpha \in H^3(K)$, we say that α is **nontrivial over the family** $\mathcal{K}_{\gamma,a}$ if it is nontrivial over every member of the family.

Proposition 7.3.1. *Let* $\mathbf{a} \in (K^*)^3$ *be a nice tuple. Then we have the following characterization of* $\Gamma(\mathbf{a})$*.*

$$\Gamma(\mathbf{a}) = \{\gamma \in K \mid \exists i, (\!|\mathbf{a}|\!) \textit{ is nontrivial over } \mathcal{K}_{\gamma,a_i}\}.$$

Proof. Let $\gamma \in \Gamma(\mathbf{a})$. In other words, suppose there exists x in the support of $(\!|\mathbf{a}|\!)$ such that

$$w_x(\gamma^l) > \min\Big(w_x(a_0), w_x(a_1), w_x(a_2)\Big) =: w_x(a_0) \geq 0, \text{ say.}$$

Then for all $c \in k^*$, we have $w_x(\gamma^l c) = w_x(\gamma^l c^{-1}) = w_x(\gamma^l) > w_x(a_0)$. By Lemma 7.3.4, we must have K_x must contains $K_{\gamma c,a}$ and $K_{\gamma c^{-1},a}$ and hence their compositum. Since $x \in \mathrm{supp}_X((\!|\mathbf{a}|\!))$, we must have $(\!|\mathbf{a}|\!)$ is nontrivial over $K_{\gamma c,a}K_{\gamma c^{-1},a}$.

Now conversely, suppose $(\!|\mathbf{a}|\!)$ is nontrivial over $\mathcal{K}_{\gamma,a_i}$ for say $a_i = a_0 =: a$. Let c be a nonzero power of a uniformizer of k. For each $N \in \mathbb{N}$, as $(\!|\mathbf{a}|\!)$ is nontrivial over $L_N := K_{\gamma c^N,a}K_{\gamma c^{-N},a}$, by Kato's LGP (Theorem 5.4.2), there must exist some nontrivial discrete valuation $\mathfrak{w}_N$ of L_N such that $(\!|\mathbf{a}|\!)$ is nontrivial over the Henselization of L_N at $\mathfrak{w}_N$. Then in particular, by Lemma 7.3.1, $a = a_0$ cannot be an l-th power in this Henselization. Consequently, Lemma 7.3.4 implies that the restriction w_N of $\mathfrak{w}_N$ to K must satisfy the following.

- $w_N(\gamma^l) + lNw_N(c) \geq w_N(a)$;

- $w_N(\gamma^l) - lNw_N(c) \geq |w_N(a)$.

By replacing c with its inverse, we can assume that $w_N(c) \geq 0$ for infinitely many $N \in \mathbb{N}$. This means that $\mathcal{V}_N \neq \varnothing$ for infinitely many N, where $\mathcal{V}_N$ denote the set of valuations $w \in \mathbb{P}(K)$ satisfying the following three conditions.

1. α is nontrivial over K_w.
2. $w(c) \geq 0$;
3. $w(\gamma^l) - lNw(c) \geq w_N(a)$.

Observe that $\mathcal{V}_N$ form a descending nested sequence of closed sets in the Riemann-Zariski space of K (cf. Definition 5.2.1). Since the latter is compact Hausdorff (cf. Fact 5.2.1) and since $\mathcal{V}_N$ are not all empty, the intersection $\bigcap_{\mathcal{V}_N \neq \varnothing} \mathcal{V}_N$ must contain some $w_x \in \mathbb{P}(K)$.

For such an x, we have

1. $(\!|\mathbf{a}|\!)$ is nontrivial over K_x.
2. $w_x(c) \geq 0$;
3. For every $m \in \mathbb{N}$, there exists $N > m$ such that $w_x(\gamma^l) - lNw_x(c) \geq \left| w_x(a) \right|$.

The last two conditions are possible only if $w_x(c) = 0$. Since c is not a unit of k, this implies that $w_x|_k$ is trivial, i.e. $x \in X_0$ and hence $x \in \mathrm{supp}_X(\alpha)$ (cf. Fact 7.1.1). On the other hand, we have $w_x(\gamma^l) = w_x(\gamma^l) - lNw_x(c) \geq w_x(a) \geq \min_i \left(w_x(a_i) \right)$ As $\mathbf{a}$ is nice, i.e. $l \nmid \min_i \left(w_x(a_i) \right)$, the inequality must be strict. In particular, $\gamma \in \Gamma(\mathbf{a})$, as desired. $\square$

Corollary 7.3.1. *There exists a first order formula $\mathbb{O}(\mathbf{z}; t) := \mathbb{O}_K(\mathbf{z}; t)$, dependent on K, such that for any nice tuples $\mathbf{a}$, the set of realizations $\mathbb{O}(\mathbf{a}; K)$ is exactly the ring $O(\mathbf{a}) = \mathfrak{O}_X((\!|\mathbf{a}|\!))$ of functions regular on* $\mathrm{supp}_X((\!|\mathbf{a}|\!))$.

Proof. By Section 4.3.4, there exists a formula $\Phi(\mathbf{a},\gamma,c)$ such that for all $(\mathbf{a},\gamma,c)\in K^5$ we have $K\vDash\Phi(\mathbf{a},\gamma,c)$ iff $(\!|\mathbf{a}|\!)$ is nontrivial over $K_{a_0,\gamma,c}$.

Define the formula

$$G(\mathbf{a};\gamma)\longleftrightarrow\ \forall c,\big(\big[(\mathrm{Const}(c)\wedge(c\neq 0)\big]\implies\big[\Phi(\mathbf{a},\gamma,c)\wedge\Phi(\mathbf{a},\gamma,c^{-1})\big]\big).$$

Then for every nice tuple $\mathbf{a}$, the set of realizations $G(\mathbf{a};K)$ is simply $\Gamma(\mathbf{a})$.

Finally, let $\mathbb{O}(\mathbf{a};t)$ be the formula $\forall\gamma,[G(\mathbf{a};\gamma)\implies G(\mathbf{a};t\gamma)]$. □

7.3.3 Definability of valuation rings of K/k

In this section, we prove the following theorem.

Theorem 7.3.1. *Let K/k be a local function field of transcendence degree* 1*. Then there exists a formula $\mathrm{Val}_K(\mathbf{z};t):=\mathrm{Val}_K(z_1,\cdots,z_6;t)$ parametrizing all the valuation rings of K/k. Furthermore, for every field L and every tuple $\mathbf{b}\in L^6$, the set of realizations $\mathrm{Val}_K(\mathbf{b};L)$ is a valuation ring of L.*

By Proposition 7.2.1, it suffices for us to construct a formula $\mathcal{O}(\mathbf{z};t)$ containing all the valuation rings of K. To motivate the definition of $\mathcal{O}(\mathbf{z};t)$, we need the following proposition.

Proposition 7.3.2. *Let $\alpha,\beta\in(\!|\dot{K}^3|\!)$ and $x\in X$ be such that $\{x\}=\mathrm{supp}_X(\alpha)\cap\mathrm{supp}_X(\beta)$. Then $\mathfrak{O}_x^*$ could be characterized in terms of $R:=\mathfrak{O}_X(\alpha)$ and $S:=\mathfrak{O}_X(\beta)$ as follows*

$$\mathfrak{O}_x^*=\left\{f\in K\mid\exists r\in R,r_0\in R^*,s\in S,s_0\in S^*\ \textit{such that}\ f=\frac{r_0}{r}=\frac{s}{s_0}\right\}.$$

Proof. Let $f\in\mathfrak{O}_x^*$. Since $\mathfrak{O}_x$ is the localization of R at $\mathfrak{m}_x\cap R$, we can write $f=r_0/r$ for $r_0,r\in R,r\notin\mathfrak{m}_x$. Thus $r\in\mathfrak{O}_x^*$ so as $f\in\mathfrak{O}_x^*$, r_0 must also lie in $\mathfrak{O}_x^*$.

Similarly, as $\mathfrak{O}_x$ is the localization of S at $\mathfrak{m}_x \cap S$, we can write $f = s/s_0$ for $s, s_0 \in R, s_0 \notin \mathfrak{m}_x$. Thus $s_0 \in \mathfrak{O}_x^*$.

Conversely, suppose $f = r_0/r = s/s_0$ for some $r_0 \in R^*$, $r \in R$, $s \in S$ and $s_0 \in S^*$. Then Since $R^*, S^* \subseteq \mathfrak{O}_x^*$, we must have $w_x(r_0) = w_x(s_0) = 0$. On the other hand, since $R, S \subseteq \mathfrak{O}_x$, we also have $w_x(r) \geq 0$ and $w_x(s) \geq 0$. Thus $0 \geq w_x(r_0/r) = w_x(f) = w_x(s/s_0) \geq 0$ so equality must hold throughout. In particular, $w_x(f) = 0$, as desired.

□

This motivates the definition of the following formulas.

$$\mathcal{O}^*(\mathbf{a}, \mathbf{b}; t) \longleftrightarrow \exists r_0, s_0, s, \Bigg(\Big[t = \frac{r_0}{r} = \frac{s}{s_0} \Big] \wedge \\ \Big[\mathbb{O}(\mathbf{a}; r) \wedge \mathbb{O}(\mathbf{a}; r_0) \wedge \mathbb{O}(\mathbf{a}; r_0^{-1}) \Big] \wedge \Big[\mathbb{O}(\mathbf{b}; s) \wedge \mathbb{O}(\mathbf{b}; s_0) \wedge \mathbb{O}(\mathbf{b}; s_0^{-1}) \Big] \Bigg).$$

$$\mathcal{O}(\mathbf{a}, \mathbf{b}; t) \longleftrightarrow \Big[\mathcal{O}^*(\mathbf{a}, \mathbf{b}; t) \vee \mathcal{O}^*(\mathbf{a}, \mathbf{b}; t+1) \Big].$$

Corollary 7.3.2. *Let $\alpha, \beta \in (\!|\dot{K}^3|\!)$ and $x \in X$ be such that $\{x\} = \mathrm{supp}_X(\alpha) \cap \mathrm{supp}_X(\beta)$. Let $\mathbf{a}$ and $\mathbf{b}$ be nice representatives of α and β, respectively. Then $\mathcal{O}^*(\mathbf{a}, \mathbf{b}; K) = O_x^*$ and $\mathcal{O}(\mathbf{a}, \mathbf{b}; K) = O_x$.*

Proof. The first claim follows immediately from Proposition 7.3.2. The second claim comes from observing that $O_x = O_x^* \cup \mathfrak{m}_x \subseteq O_x^* \cup (O_x^* - 1) \subseteq O_x$. Thus equality must hold throughout. In particular, $O_x = O_x^* \cup (O_x^* - 1)$. □

Proposition 7.3.3. *The formula $\mathcal{O}(\mathbf{a}, \mathbf{b}; t)$ defined above contains all valuation rings of K/k.*

Proof. Let x be a closed point of X. By Lemma 7.3.2 and Corollary 7.3.2,it suffices for us to show that there exists $\mathbf{a}$ and $\mathbf{b} \in (K^*)^3$ such that $\mathrm{supp}_X((\!|\mathbf{a}|\!)) \cap \mathrm{supp}_X((\!|\mathbf{b}|\!)) = \{x\}$.

As $0 \neq {}_l H^2(\kappa(x))$, Fact 4.2.1 implies that there must exist some nonzero $\overline{a_1}, \overline{a_2} \in \kappa(x)$. Let a_1 and a_2 denote any of their respective lifts to O_x. Observe that for both i, as $\overline{a_i} \neq 0$, we must have $w_x(a_i) = 0$.

By Riemann-Roch, there exists $f \in K$ such that f has a unique pole at x of order not divisible by l. Let $a_0 = 1/f$. By Lemma 7.3.1, we have $\mathrm{res}_x(\mathbf{a}) \neq 0$ over K_x, i.e. $x \in \mathrm{supp}_{\mathcal{X}}(\mathbf{a})$.

Finally, let $b_1 = a_1$ and $b_2 = a_2$ and $b_0 = \frac{1}{f+1}$. By Lemma 7.3.1 (change), for any $P \in X_0$, we have $P \in \mathrm{supp}_X(⦇\mathbf{a}⦈) \cap \mathrm{supp}_X(⦇\mathbf{b}⦈)$ iff $l \nmid w_P(a_1)$ and $l \nmid w_P(b_1)$ iff $P = x$, as claimed.

$\square$

Theorem 7.3.1 now follows immediately from the above Proposition and Proposition 7.2.1. Together with Proposition 6.3.2, this in turn implies the Main Theorem 6.1.1.

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