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IMPROVING COMMUNITY HEALTH SERVICES IN REMOTE SETTINGS: PAPERS ON OPERATIONS MANAGEMENT APPROACHES TO GLOBAL HEALTH

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Abstract

We present Operations Management work on overcoming healthcare delivery challenges in remote settings in two chapters. We begin with work on inventory management of a last-mile supply chain. This work is motivated by the need for a reliable and cost-effective delivery system to resupply new mosquito-repellent technologies to combat malaria in the Lake Tanganyika region of The Democratic Republic of the Congo. The primary delivery methods used around Lake Tanganyika can become inoperable during periods of floods, machinery breakdowns, and armed conflict. To ensure continuous coverage, our model allows for emergency delivery methods, with high variable costs, to be used during outages. In normal operating times, we use a large-capacity fixed-cost delivery method. We model the changes in the operating environment between normal and outage states using a Markov chain with disruption-specific parameters. We use an infinite-horizon discounted objective, which consists of ordering costs and a linear holding cost. We show that a novel hybrid policy is optimal and we provide the means to calculate the policy parameters. Additionally, for managerial considerations, we propose two heuristic policies and show their relative effectiveness through a series of numerical studies.

In the second chapter, we present a constrained optimization model to aid in selecting maternal and newborn community health interventions in previously underserved rural and remote populations through Community Health Worker (CHW) programs. The first application of the model is in Dhusamareeb, Somalia, and the surrounding area within the state of Galmudug. After establishing the resource requirements (financial, time, commodity, and policy) for 25 maternal and neonatal interventions with local stakeholders in Somalia, we used the Lives Saved Tool (LiST) to calculate the number of projected lives saved as a function of services delivered. We then used optimization techniques to sort through the feasible combinations of interventions satisfying the resource constraints to determine the package of care that leads to the most projected lives saved in Galmudug. With a cadre of 1,450 Female Health Workers and a budget of $435,000 for health commodities, we calculate that the optimized set of interventions for Galmudug could avert 15 percent of the 4,132 projected maternal and neonatal deaths in 2023. Sensitivity analysis shows how the optimal combination of interventions, and the number of lives saved, change as the constraints and parameters used in the model change. We focus on Somalia for the first application of this model; however, the combination of LiST's database with a simple framework for sourcing data on local health system resources, allows the application to other country health programs. The model provides practitioners with a new tool and accompanying approach to evaluate possible packages of community health interventions with competing resource requirements.

Chapter 1

Last-mile delivery of malaria prevention products in the DRC: An inventory management model under supply chain disruptions

1.1 Introduction

In 2019 there were 215 million cases of malaria across Africa (WHO 2020). According to the World Health Organization, the Democratic Republic of the Congo (DRC) accounted for more than 28 million of these cases. For reference, Venezuela's 467 thousand malaria cases in 2019 are listed in the same report as the most of any country in the Western Hemisphere. The malaria health crisis may be most pronounced in the Lake Tanganyika region of the Eastern Congo, where malaria leads a list of several diseases, including diarrhea and pneumonia, which contribute to a national under-five mortality rate of 8.5% (World Bank 2020c). While decades of global health work focused on childhood illnesses has meaningfully improved the conditions for much of the world, the Congolese villages on the shores of Lake Tanganyika have remained largely untouched by these efforts.

The aforementioned neglect of the Lake Tanganyika region (specifically, the southern South Kivu and Tanganyika Provinces) is the result of multiple factors. These include limited and lacking infrastructure vulnerable to flooding from extreme seasonal weather patterns, armed conflict between militia groups and government forces, and a large marginalized population of both refugees fleeing other countries and internally displaced persons (IDPs) from other parts of the DRC. Notably, The United Nations High Commission for Refugees estimates that as of February 2022, around 4 million of the 5 million internally displaced persons in the DRC were located in the South Kivu Province (UNHCR 2022). These sources of difficulty for both providing and accessing health services are supported by the findings of a 2021 healthcare worker survey conducted in the region, citing "violence, mobility restrictions, and resource availability" as the primary disruptions (Altare et al. 2021). Together and separately, these factors have formed a logistically complex operational setting, and have consequently led the region to become a morass of the global aid space, where health and economic support is occasionally deployed but often lacks staying power.

Around the villages of Lake Tanganyika, and in much of the developing world, aid organizations have favored bed nets as the primary tool for malaria prevention. Bed nets are often effective in reducing malaria rates when used properly, particularly in urban and peri-urban areas, yet less so in refugee and IDP settings. Notably, they are an inexpensive option when compared to other mosquito control strategies. Much of the low cost attributed to bed nets results from the infrequent need to resupply the product, which is recommended to occur every three years (WHO 2013). The low resupply rates are especially beneficial in regions with limited infrastructure, where it is difficult to maintain a reliable supply chain.

Although bed nets have many favorable attributes and have reduced malaria rates in varied settings, in others, the willingness and ability to use the product has been tepid.

Furthermore, in water-based communities with surplus bed nets and consistently high rates of malaria, there is concern that the chemically treated nets are being used as fishing nets instead of bed coverings. This practice, termed “bed net fishing,” threatens the fragile aquatic ecosystems which are being dredged by these tight weave indiscriminate nets and can lead to juvenile fish die-offs from the eluting of water-soluble chemicals. The practice also represents a potential source of harm to human health in those consuming fish with concentrated levels of pyrethroids, the class of chemicals used in insecticide-treated mosquito nets (Larsen et al. 2021). The deleterious effects of bed net fishing are detailed in a New York Times article on communities in Zambia that participate in this practice and states that “the unsparing mesh, with holes smaller than mosquitoes, traps much more life than traditional fishing nets do. Scientists say that could imperil already stressed fish populations, a critical food source for millions of the world’s poorest people” (Gettleman 2015). In several papers related to the topic, the geographic reach of bed net fishing practices is shown to extend throughout much of Sub-Saharan Africa and across the global malaria belt (Short et al. 2018; Trisos et al. 2019; Larsen et al. 2018; Bush et al. 2017; Minakawa et al. 2008; Lover et al. 2011). For communities around Lake Tanganyika, which depend on the lake for income and sustenance, the consequences of bed net fishing can be devastating, and when combined with the persistently high rates of malaria in the region necessitate a reassessment of malaria control techniques and practices.

With an auxiliary aim of understanding bed net utilization, and the primary goal of measuring the health, security, and economic conditions of the communities around Lake Tanganyika, our partner organization, the Lake Tanganyika Floating Health Clinic (LTFHC), a global health nonprofit focused exclusively on the region, conducted a series of household level surveys. The information was collected over a period of three months in 2016 and included 800 households across 41 villages. A map of these locations is shown in the right panel of Figure 1.1, and an overview of the survey results is provided in Appendix A.9.

Regarding malaria, the surveys showed that in the three months prior, households had an average of 2.07 malaria cases in adults and 1.67 cases in children under the age of 5.

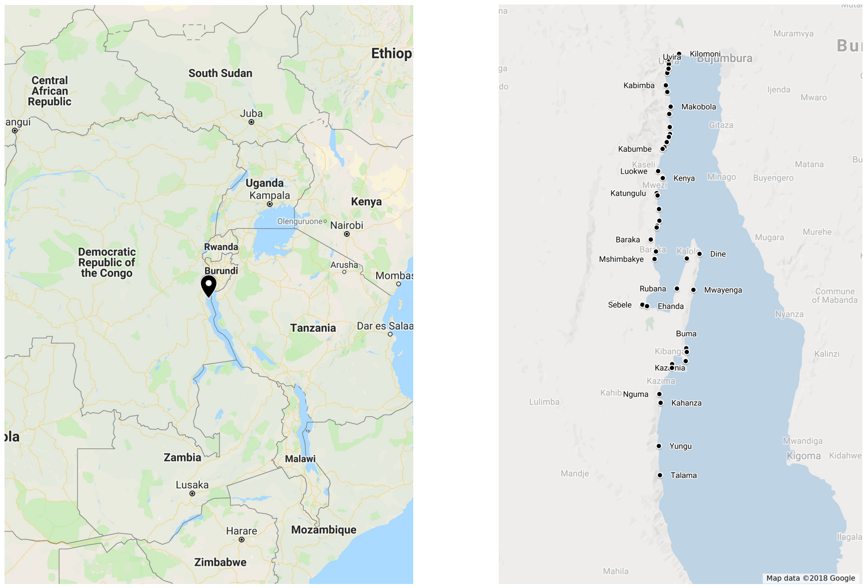

Figure 1.1: Maps of Lake Tanganyika's location along the borders of the DRC and Tanzania (left panel) and the 41 villages included in the survey (right panel).

A possible theory for these high rates of malaria is a lack of access to bed nets within these communities; however, this was shown to not be the case as the surveys reported 3.27 bed nets per household, with 92.2% of households having at least one bed net. Furthermore, it was apparent that only a portion of these bed nets were being used in the intended manner, as 65.7% of children under 15 and 68.0% of adults were reported to have slept under a mosquito net the night before. A 2017 study in the Amhara region of Ethiopia found that 91.9% of the net-owning interviewed households "utilized insecticide-treated bed net on the night preceding the interview" (Alemu et al. 2018). This utilization rate offers a point of comparison; although, it may be on the high end, as utilization rates vary across studies in different regions and different years. There are several anecdotal reasons why the Lake Tanganyika region reports low utilization rates, and these are consistent with findings from the literature on other high-endemic areas (Pulford et al. 2011). Heat is often cited as a cause for abandonment, as the nets can block airflow and lead to discomfort in hot climates. Additionally, given the small dimensions of many of the houses measured in the survey, and the large number of people residing in each dwelling, with an average of 7.97 people per household, permanently affixing bed nets within houses is sometimes eschewed in favor of conserving limited space. Instead of using the nets as intended, 22.6% of households in the survey acknowledged using bed nets for fishing. The prevalence of bed net fishing was corroborated by a component of the survey, which focused on fishing practices and found that of the 2,094 fishing groups observed, 30.4% were using bed nets as their primary method of take. Both the high incidence of malaria and low rates of proper utilization confirm that a bed net-focused strategy alone is unlikely to meaningfully decrease the rates of malaria in communities around Lake Tanganyika.

The difficulties of controlling malaria in these villages are emblematic of wider challenges within the global malaria prevention effort. In particular, the low price per bed net has limited the consideration of new products with higher relative costs. These cost comparisons between bed nets and other vector controls often fail to include product preferences among users, particularly those who have demonstrated an unwillingness to use bed nets, but who would consider other products if presented with the option. Furthermore, mosquito bites often occur in spaces and times where bed nets do not offer protection. This includes both indoors, when people do not, or are unable to, use bed nets (e.g. when awake and active), as well as outside of the house, a primary transmission space (Killeen 2014; Russell et al. 2013; Sherrard-Smith et al. 2019). There has been little attention paid to the benefits of intervention paradigms that offer continual and passive mosquito protection both indoors and outdoors throughout the day and night, instead of only around sleeping spaces. With these factors in mind, it seems insufficient to measure costs on a per-treatment basis instead of attempting to measure the cost per disease incidence prevented. This is a more challenging metric to evaluate, but we feel it holds the key to unlocking better setting-specific strategies for malaria prevention which may justify the use of a wider range of vector controls.

In their 2020 Malaria Eradication Report, the World Health Organization (WHO) acknowledged the diminishing marginal gains from sending additional bed nets to certain regions which have already benefited from their introduction, but have reached a point of limited take-up. The report predicts that "using the current tools, we will still have 11 million cases of malaria in Africa in 2050" (WHO 2020). Additionally, it states that "although existing tools have achieved remarkable impact, the world currently lacks the transformative tools needed to achieve malaria elimination in the highest-burden areas." The WHO echoed a similar need to pivot towards new malaria control products in one of the resolutions of their 2021 *Recommitting to Accelerate Progress Towards Malaria Elimination*, which commits "to extend investment in the development of new tools and support for implementation research and innovation to enable the efficient delivery and equitable access with a view to maximize impact and cost-effectiveness" (WHO 2021).

In response to similar concerns which prompted the WHO reports, there has been a surge in new mosquito control technologies for use in low-resource settings. The resulting products include several lightweight volatile pyrethroid products, which provide a constant spatial barrier to keep mosquitoes out of dwellings and create extended perimeters outside of houses leading to significantly diminished biting incidences (Achee et al. 2012; Syafruddin et al. 2020; see also Cucchiaro et al. 2020). If such products are shown to be effective in high-endemic areas, they could become a vital component of a comprehensive malaria elimination strategy.

As mentioned earlier, in villages like those around Lake Tanganyika, space within each house is limited. Whereas bed nets are affixed to ceilings over sleeping areas, and require designated space, these products are mostly unobtrusive to the limited space within each house. We consider the use of two products nearing market introduction, both similarly lightweight and roughly the size of a sheet of paper, which need to be resupplied more often than bed nets due to their limited product life of around one month once in use. They each have a longer, but still limited, shelf life while in storage. Although promising in its potential to increase the number of houses with some form of mosquito protection, the distribution of these products would require a reworking of the current aid channels deployed in logistically challenging settings like those present around Lake Tanganyika.

Invariably, the introduction of products with shorter active lives and therefore needing recurrent replenishment will increase the long-run cost of the last-mile portion of a supply chain. However, in a sequence of research projects, we aim to show that a properly managed system can minimize this cost, and allow such products to become competitive on a reduction of disease prevalence per dollar spent basis with more durable mosquito vector controls such as bed nets. This paper is the first step in this effort and demonstrates that there is not heretofore an adequate model for this type of setting in the inventory management literature, before proposing such a model and proving the optimal policies that minimize the logistical costs of the last-mile portion of the supply chain. In order to reduce costs, we will manage inventory at the village level by installing small sheds at the health facilities in each village. These point-of-service storage locations supplement the warehouses located in the nearby larger towns and cities. They are intended to reduce the frequency with which deliveries need to be made and provide local safety stocks available for use during road closures when delivery options are more expensive. In ongoing work that serves as a follow-up to this paper, we focus on implementing a pilot program for this delivery system. The pilot program is designed using the inventory management policies presented in this paper to lower the last-mile delivery costs (Ata et al. 2023). In that work, not only are we able to track the costs associated with providing new products to these households, but using our baseline data and subsequent surveys conducted after the product introduction, we will be able to measure the resulting changes in malaria rates as well as transmission patterns within the region.

This paper formulates the aforementioned last-mile delivery problem as an inventory management model with multiple state-dependent delivery methods, fixed demand, fixed storage capacity, and no backlogging. The state-dependent delivery methods are a distinguishing feature of the model. We consider a setting with two classes of delivery modes. The first employs a large capacity fixed cost method, meant to represent a truck, which requires unobstructed roadways to operate. We assume a fixed cost for this method because the truck needs to drive the same distance and make the same number of deliveries regardless of the size of the restocking amount. The product itself is light enough that different load sizes have negligible impact on the time required for each stop. The second class of delivery methods is any one of a series of smaller capacity delivery modes with per-unit costs. These include motorbikes, pedestrians, and canoes, all of which can operate at different times when primary roadways are unusable. The suitability of these secondary delivery methods in hard-to-reach areas is supported by other papers, in particular Mehta et al. (2021), which highlights the benefits of using and maintaining motorcycles to extend coverage for healthcare provisions in Zambia. The multiple delivery method formulation is necessary for settings like those around Lake Tanganyika, where disruptions such as floods, machinery breakdowns, and armed conflict occur frequently and can persist for indeterminately long periods of time. Fixed demand is a common feature of the distribution of goods in the aid space, where users often sign up for or are assigned a good or service instead of choosing to purchase the product. We relax the fixed demand assumption in Appendix A.5 to provide results for other settings with general demand. We use a fixed storage capacity to represent the minimum of the limited storage space at health facility locations and the shelf life of the product. Finally, we prohibit backlogging to maintain continuous coverage, as a lapse in product availability will leave households unprotected, thereby susceptible to illness, and will hamper our ability to properly measure the products' efficacy.

Although there has been extensive work on the point-of-service inventory aspect of last-mile delivery, to our knowledge, an optimal policy had previously not been provided for models with increasing expected fixed ordering costs in future periods (Sethi and Cheng 1997). As our setting fits this cost structure, in this paper we provide such a policy, prove it is optimal, and supply the means to calculate the policy parameters. We also argue that the aspects of this model are present in other humanitarian aid settings and the results thus have wider applications beyond this specific setting. Using an infinite-horizon discounted dynamic programming model, we show that a hybrid $(s^k_A, S^k_A, S^k_{B_e})$ policy, which is a combination of (s, S) and order-up-to policies, is optimal for our model. The multiple (s, S) and order-up-to policies, which comprise this hybrid policy are indexed by k, to indicate the availability of a periodic structure that allows for incorporating parametric variation due to seasonal weather changes into the model. We also conduct sensitivity analysis of the optimal policy and its performance over a range of problem parameters. Furthermore, we compare the optimal policy to several simpler heuristics: an (s, S)-type policy and an EOQ-like policy, both of which can also be used in practice. We find that in the Lake Tanganyika setting the EOQ-like heuristic performs as well as the optimal policy. This observation holds more broadly for different settings where both the EOQ-like and (s, S)-type heuristics perform close to optimal. However, we also demonstrate other realistic settings where the hybrid policy significantly outperforms the heuristics.

Our paper is structured as follows: Section 1.2 reviews the existing literature; Section 1.3 introduces our model; Section 1.4 conducts a numerical analysis of the policy recommendations; Section 1.5 derives heuristic policies and compares them with the optimal policy; and Section 1.6 concludes the paper. The appendix includes relevant definitions of K-convexity for our problem setup, the finite-horizon version of our problem and its analysis, the exercise of passing to the limit in order to connect the finite horizon case to our infinite horizon results, the proofs of the paper's key results, the extension of the model results to a general demand setting, the derivation of our heuristic comparisons, examples of settings where the optimal policy outperforms the heuristic policies and an overview of the survey results.

1.2 Literature Review

Our work builds on and contributes to both the inventory management and humanitarian logistics literatures. In Section 1.2.1, we review inventory models under supply-side uncertainty, as they provide the foundation for our work. In Section 1.2.2, we provide a survey of papers dealing with humanitarian operations, especially those focused on inventory management in low-resource settings. As our model is influenced by the unique conditions of operating in the healthcare space, we emphasize humanitarian papers with a similar focus.

1.2.1 Inventory Management Under Supply Side Uncertainty

The inventory management literature has a long history, see Zipkin (2000) and Porteus (2002) for an overview circa 2000. Our model belongs to the literature on inventory management under supply-side uncertainty. For a comprehensive overview of Operations Research literature on supply chain disruptions, which includes a section on inventory models, see Snyder et al. (2016). Models which incorporate supply-side uncertainty can be further divided into several classes. Three of the primary classes are random yield, where only a portion of each order is filled, random lead times, where the length of time between when an order is placed and when it arrives is uncertain, and random availability, where a supplier alternates between periods of being able to and not being able to fulfill orders. Our model belongs to the random availability class, which traces its general structure back to Parlar and Berkin (1991). This was one of the first papers to model an inventory management problem as having ON and OFF states, then referred to as "wet" and "dry" periods, which determine, at a given time, whether or not the supplier is available to fulfill an order. Parlar and Berkin make the case for such a model having practical applications and use renewal theory to solve an average cost formulation under the EOQ framework. Parts of their model, namely an error in properly defining a cycle, were later corrected by Berk and Arreola-Risa (1994). This modeling convention was expanded upon to allow for a wide range of assumptions, with the EOQ framework remaining a popular policy structure (Weiss and Rosenthal 1992; Qi et al. 2009; Sargut and Qi 2012; Snyder 2014).

Most papers on inventory models with supply-side disruptions do not attempt to prove the optimality of a particular ordering policy; instead, they select any one of the classes of established policies, find the best parameters for that policy class, and study its performance. We begin by reviewing papers that evaluate (s, S) policies used in supply-side disruption models. Moinzadeh and Aggarwal (1997) assumes an (s,S) inventory policy for a system with exponential time between OFF periods and constant restoration time between ON periods. Arreola-Risa and DeCroix (1998) uses an (s, S) policy and include partial backordering during a supplier outage, where a fraction of unmet demand is lost and a fraction is backordered. Poormoaied and Demirci (2021) allow for emergency orders to be placed at the start of a disruption period as a way to model hoarding behavior, which is an option for some firms with advance knowledge of a likely disruption, and use an (s_1, S_1, s_2, S_2) policy to study the benefits of this practice. Similarly, Bakal et al. (2017) measures the value of allowing emergency orders to be placed immediately preceding a supply outage. Ahiska et al. (2013) optimizes an (s_r, S_r, s_u, S_u) policy under different cost and disruption parameters for each supplier in a system that includes both a reliable and unreliable supplier.

We now review papers in the supply-side disruption literature which employ a (Q, r) or similar fixed order quantity type inventory policy. Parlar and Perry (1995) study a periodic review inventory model that includes an identification cost to view the system state. Gupta (1996) evaluates a continuous review model with Poisson demand, exponentially distributed lengths of ON and OFF periods, and no backordering. Parlar (1997) uses a k-stage Erlang distribution to model the ON periods and a general distribution to model the OFF periods. Using an (s, Q) control policy, Mohebbi (2004) provides the long-run average cost expression for a model with lost sales, random demand, and varying lead times. Azimi et al. (2012) evaluates the performance of the (Q, r) inventory policy with customers separated into different classes by their likelihood of placing backorders as opposed to leaving and resulting in lost sales, should the inventory be depleted during an outage period. Islam et al. (2020) structures a single product model with random capacity for both the supplier and retailer, and separate ON and OFF periods for each. Mohebbi and Hao (2006), in addition to modeling supply outages, includes lead-time variability as a form of supply uncertainty. Using random lead times for this purpose hearkens back to earlier work by Song and Zipkin (1996) which models replenishment lead times as a Markov process in place of ON and OFF periods. Song and Zipkin proves that when a fixed order cost is used, a state-dependent (s, S) policy is optimal. Parlar et al. (1995) similarly proves the optimality of an (s, S)-type policy in the finite horizon setting for an extension of the ON and OFF supply disruption model with periodic review and random demand.

Although the convention of assuming an established policy and evaluating its performance was still carried forward in several works, Song and Zipkin (1996) and Parlar et al. (1995) started a stream of literature, to which ours belongs, which attempts to prove the optimality of an (s, S) policy under different modeling conditions. Özekici and Parlar (1999) proves the optimality of an (s, S) policy in an infinite-horizon formulation with state-dependent supplier availability and random demand. Although it does not model supply disruptions, Sethi et al. (2003) proves the optimality of an (s, S)-type policy for a periodic review inventory model with the availability of both a fast, but more expensive, and a slow, but cheaper, delivery mode.

A key feature of our model is the use of multiple delivery methods, which is analogous to sourcing from multiple suppliers. This convention is used by Parlar and Perry (1996) and Gürler and Parlar (1997) both of which allow for multiple suppliers with ON and OFF periods. A version of the model more closely resembling our own is found in Tomlin (2006), which includes two suppliers, one susceptible to outages and the other always available but more expensive. This differs from the earlier supply-side disruption models, where multiple suppliers all have the same associated costs. Tomlin does not assume a fixed ordering cost and incorporates both mitigation and contingency tactics to account for the risks associated with the unreliable supplier. Tomlin's work is related to the literature on manufacturing flexibility, as it attempts to classify sourcing strategies under different parameter regimes. Tomlin and Snyder (2007) add to the earlier model by incorporating disruption threat levels, which help determine the level of safety stock to hold in preparation for possible supply disruptions. Saghafian and Van Oyen (2012) further investigate the value of obtaining risk signals for a primary supplier prone to disruptions, while also evaluating the benefits of a secondary backup supplier.

Another key reference for us is Sethi and Cheng (1997), the results of which, along with those from other variations on the model, can also be found in Beyer et al. (2010). The model presented by Sethi and Chang is similar to ours in that their inventory system is embedded in an environment that evolves according to a Markov chain. Sethi and Cheng prove the optimality of a state-dependent (s, S) policy. However, the results of Sethi and Cheng do not apply to our setting, because a key assumption of theirs is violated in our setting, see condition (4.1) of Sethi and Cheng (1997), which states, "...the fixed cost of ordering in a given period with demand state i should be no less than the expected fixed cost of ordering in the next period." As mentioned in the introduction of our paper, this assumption is restrictive in our setting and others where emergency deliveries need to be made to ensure continuous coverage. Therefore, we prove our results from first principles.

1.2.2 Humanitarian supply chains and logistics

The motivating setting for our model, in a logistically challenging region of the DRC, is central to its design. As such, we highlight several papers which focus on the unique challenges of operating in the global health and aid spaces. Surveys of the work in Operations Research, as it applies to logistics in low- and middle-income countries, cover a range of subjects. Overstreet et al. (2011), Kunz and Reiner (2012), Leiras et al. (2014), and Ozdamar and Ertem (2015) all provide surveys of research on humanitarian logistics. Disaster and relief operations surveys are covered by Altay and Green (2006), Natarajarathinam et al. (2009), Galindo and Batta (2013), and Anaya-Arenas et al. (2014). There has been substantial cross-disciplinary work on vaccine supply chains around the world, especially in Sub-Saharan Africa. Summaries of vaccine supply chain research can be found in Duijzer et al. (2018) and De Boeck (2020).

A topic often overlooked in earlier Operations Research work, several papers emphasize the need for the rigorous treatment of inventory management and supply chain design specific to humanitarian operations (e.g. Van Wassenhove (2006), Whybark (2007), Kovacs and Spens (2009), Zaffram et al. (2013), Privett and Gonsalvez (2014) and Yadav (2015)). Kovacs and Tatham (2009), in their call to action on humanitarian supply chain management, nicely summarize the two-fold objective within this space of "achieving cost efficiency while maintaining the flexibility to respond to an unforeseen disruption in a timely manner."

Stockouts of essential medicines and vaccines are often identified as significant roadblocks to improving health outcomes in low-resource settings. Gallien et al. (2017) looks at how the uncertain nature of grant funding leads to drug stockouts in countries across Africa. Daff et al. (2014) focuses on the causes of stockouts of contraceptives in Senegal. Hwang et al. (2019) conducts a survey of health facilities in South Africa to measure the rates of stockouts of antiretroviral and tuberculosis medicines. A key component of our model is the continuous coverage requirement. In the interest of avoiding stockouts, Harries et al. (2007) incorporates this same requirement into a model of antiretroviral drug distribution in Malawi. Leung et al. (2016) uses data from Zambia to run simulations measuring the impact of different inventory policies on medication stock-outs. Leung et al. points to "demand seasonality and facility access interruptions" as important considerations for any supply chain design in countries susceptible to these issues.

Several papers consider the impacts of seasonal differences on humanitarian operations. This work is especially important in countries, like the DRC, with pronounced rainy seasons. De Boeck et al. (2021) combine several disciplines by performing flood hazard simulation to determine prepositioned safety stock levels in order to continue providing vaccines during Madagascar's rainy season. Sodhi and Tang (2014) propose methods for supply chain adaptability to flooding in Southeast Asian countries. Lodree and Taskin (2009) provide a model for inventory planning in anticipation of the US hurricane season. An extensive overview of issues related to inventory management with seasonality considerations is provided in the USAID project report *Malaria Seasonality and Calculating Resupply* Watson et al. (2014).

As is particularly relevant to our work, there are several papers focused on supply chains for malaria medications and prevention products. Barrington et al. (2010) addresses the issue of unknown inventory stock levels of antimalarial drugs at local health facilities by exploring the use of SMS updates. Similarly, Githinji et al. (2013) focuses on strategies of preempting malaria medication stock-outs in Kenya through the use of phone messaging services. A survey conducted in Kangwana et al. (2009) finds that facilities that provide therapeutic malaria drugs suffered from significant stock-outs, which greatly impacted their ability to provide care. Parvin et al. (2018) develops a multi-tiered model for malaria treatment distribution in Malawi, which incorporates demand uncertainty and allows for transshipments between clinics.

Other papers in the space apply inventory models directly to humanitarian settings.

Deo et al. (2022) focuses on supply uncertainty in HIV treatment clinics. Gallien et al. (2021) presents a simulation model comparing different inventory policies for medications in Zambia. Natarajan and Swaminathan (2014) and Natarajan and Swaminathan (2017) study inventory management with funding constraints, which are commonly encountered in humanitarian work. Perry and Stadje (2001) consider a Markovian system with perishable items, where the occurrence of a random event can lead to spoilage of the entire inventory. Rajgopal et al. (2011) uses simulation to evaluate the performance of different buffer stock policies for various arrival rates and vaccine vial sizes. Bam et al. (2017) conducts inventory safety stock analysis through simulation involving tuberculosis drugs. Deo and Sohini (2015) perform simulations to study the geographic distribution of HIV diagnostic tools in resource-limited locations in Mozambique. Building on this earlier work in Mozambique, Jónasson et al. (2017) measures the extent to which optimal reassigning of clinics, where samples are collected, to labs, where results are processed, can improve HIV diagnosis delays.

There are several additional reports worth highlighting here, as they help in understanding the current practices of organizations and governments managing health supply chains. Jahre et al. (2012) conducts supply chain-specific interviews of health facilities in Uganda to better understand the processes and failures of the drug supply chain. Balcik et al. (2010) and Oloruntoba and Gray (2006) both outline practices used in commercial supply chains and connect these to possible solutions for the challenges of humanitarian relief chains. Matowe et al. (2008) outlines an initiative to improve skills in pharmaceutical supply management in East Africa. As an aside, safety stock determination shares much in common with disaster response and humanitarian relief applications, namely coordination, prepositioning of aid products, and resource-based prioritization policies (e.g. Duran et al. (2011), Mills et al. (2013), Ergun et al. (2014), and Chen et al. (2018)). Lastly, a growing number of papers use the traditional operations research models of vehicle routing and network planning in humanitarian relief applications, see for example Campbell et al. (2008), McCoy and Lee

(2014), Eftekhar and Van Wassenhove (2016), Mills et al. (2018), and Chen et al. (2021).

1.3 Model

We advance a discrete time inventory model to study the last-mile delivery problem described in the Introduction. The sequence of events in each period are as follows: the system manager views the system state at the start of a period. The available delivery modes depend on the state of the environment. She decides how many units to order, using the most cost-efficient delivery method available to her. If an order is placed, it is received instantaneously and the inventory level is updated to reflect the addition of this new order. Since demand corresponds to the number of households served, which is determined *a priori*, we treat demand as fixed and normalize it to one. One unit is removed from the current inventory to meet demand. She incurs the costs for that period's actions. A state transition occurs and the period ends.

Our model's system state has two variables: x_k and e_k for $k \geq 1$. We let x_k denote the inventory at the beginning of period k, for $k \geq 1$, before an order decision is made. The inventory level evolves according to the system dynamics equation:

$$x_{k+1} = x_k + u_k - 1 \text{ for } k \geq 1,$$

where u_k is the order amount in period k. We let e_k denote the environment state at the beginning of period k. The environment state e_k can take any value in the set $\{A, B_1, ..., B_E\}$. State A represents the undisrupted state, during which the large-capacity fixed-cost delivery option is available and preferred. State B_e (for $e = 1, ..., E$) represents a disrupted state, during which, for various reasons presented in the introduction, the large capacity delivery option is unavailable.

The environment state e_k evolves according to an ergodic (irreducible and aperiodic) Markov chain. As mentioned in the introduction, weather events in the DRC follow seasonal patterns, and the likelihood of ending up in a disrupted state from flooding is much higher during the rainy season than during the dry season. This characteristic is shared by many tropical countries and thus incorporating this feature into the model is of wider use beyond this setting. In the Lake Tanganyika region, the rainy season occurs from October through April. We model seasonality by allowing for a periodic structure in the underlying Markov chain and specify the transition probabilities for the environment to depend on time, i.e. P_{ij}^k, where the superscript k denotes the time dependence. More formally, $P_{ij}^k = \mathbb{P}(e_{k+1} = j | e_k = i)$. To be specific, letting π denote the period length, we assume

$$P_{ij}^k = P_{ij}^{k+l\pi} \quad \text{for } k = 1, ..., \pi \text{ and } l = 1, 2, ...$$

For our application, it is natural to set $\pi = 12$ months.

We include a storage constraint, N, meant to represent the minimum of the storage capacity and shelf life of the product in storage. We stipulate that the inventory in the system cannot exceed this level. Thus, we have that $x_k + u_k \le N$ and $e_k \in \{A, B_1, ..., B_E\}$.

From the storage limit, we impose the following constraint on the order quantity for period k:

$$u_k \in U(x_k) = [0, N - x_k] \quad \text{for } k \ \ge 1.$$

For simplicity, we allow fractional orders. However, because the demand is integer-valued, the orders under the optimal policy are also integer-valued. This outcome can be observed in the numerical examples in Section 1.4.

The cost of ordering is a function of the order quantity and the environment state. In state A, order costs are incurred according to the following cost function:

$$C(u, A) = \begin{cases} K, & \text{if } u > 0, \\ 0, & \text{if } u = 0. \end{cases}$$

This cost function reflects the availability of our larger capacity fixed cost option, i.e. delivery by truck. Trucks are assigned from storage depots to a set of individual health facilities.

We assume each truck's capacity exceeds the combined storage constraints of its assigned delivery locations. As such, the distance traveled by a truck, and thus the resulting fuel costs, are the same regardless of the order size. Since the product is lightweight, variable loading and unloading costs are negligible. Under these conditions, which hold for our motivating application, the fixed cost assumption is appropriate. In contrast, when the system is in state $B_e \in \{B_1, ..., B_E\}$, costs are incurred using the linear cost function:

$$C(u, B_e) = c_e \cdot u \text{ for } u \geq 0.$$

Because the large capacity delivery option is unavailable in states B_e, we resort to other options available in those disrupted states, whether it be pedestrians, motorcycles, or canoes, all of which have limited capacity. The delivery method used in an environment state, and thus its per unit cost (c_e), is determined by the method appropriate for the type of disruption that state represents. For example, during flooding, a canoe delivery will likely be appropriate; whereas, we prefer motorbikes during machinery breakdowns blocking roadways. The costs of these delivery methods differ, but all scale roughly linearly with the order size. As such, we assume the ordering cost $C(u, B_e)$.

At the end of each period, after demand has been met but prior to the environment state transition, the system manager incurs an inventory holding cost rate of $h > 0$ for each unit in stock. To satisfy our condition of continuous coverage, we require that all demand is met and not backlogged. However, for analytical convenience, our model initially allows backlogging, where the backlog cost rate is $p > 0$ per period per unit. Ultimately, we will set p large enough so that the inventory never falls below zero under the optimal policy. Thus, the system manager avoids backlogging while meeting all demand; see Corollary 1 below.

Given the inventory level x at the beginning of a period and order quantity u within that period, we combine the holding and backlogging costs into the function $H(x + u)$, where

$$H(y) = h(y - 1)^{+} + p(y - 1)^{-}, \quad y \in (-\infty, N],$$

and where $x^+ = \max\{x, 0\}$ and $x^- = \max\{-x, 0\}$.

We focus on the infinite-horizon discounted cost criterion and as a means to characterize the optimal policy analytically, we next consider the Bellman equation. To be specific, we let $J_k(x, e)$ denote the value function starting in state (x, e) and period $k \geq 1$. We note for $k = 1, ..., \pi$ that

$$J_k(x, e) = J_{k+\pi l}(x, e) \quad \text{for all } x, e \text{ and } l \geq 1.$$

Therefore, it suffices to restrict our attention to $J_1, ..., J_\pi$ to solve the Bellman equation. In particular, letting α denote the discount rate we write the Bellman equation for $x \in (-\infty, N]$ and $k = 1, ..., \pi$ as follows:

$$\begin{aligned} J_k(x, A) = \min_{u \in [0, N-x]} \Big\{ & C(u, A) + H(x+u) + \alpha \sum_{i=1}^{E} P^k_{AB_i} J_{k+1}(x+u-1, B_i) \\ & + \alpha P^k_{AA} J_{k+1}(x+u-1, A) \Big\}, \end{aligned} \tag{1.1}$$

$$\begin{aligned} J_\pi(x, A) = \min_{u \in [0, N-x]} \Big\{ & C(u, A) + H(x+u) + \alpha \sum_{i=1}^{E} P^\pi_{AB_i} J_1(x+u-1, B_i) \\ & + \alpha P^\pi_{AA} J_1(x+u-1, A) \Big\}, \end{aligned} \tag{1.2}$$

$$\begin{aligned} J_k(x, B_e) = \min_{u \in [0, N-x]} \Big\{ & C(u, B_e) + H(x+u) + \alpha P^k_{B_eA} J_{k+1}(x+u-1, A) \\ & + \alpha \sum_{i=1}^{E} P^k_{B_eB_i} J_{k+1}(x+u-1, B_i) \Big\}, \end{aligned} \tag{1.3}$$

$$\begin{aligned} J_\pi(x, B_e) = \min_{u \in [0, N-x]} \Big\{ & C(u, B_e) + H(x+u) + \alpha P^\pi_{B_eA} J_1(x+u-1, A) \\ & + \alpha \sum_{i=1}^{E} P^\pi_{B_eB_i} J_1(x+u-1, B_i) \Big\}, \end{aligned} \tag{1.4}$$

where $e = 1, ..., E$. Note that we use $J_{\pi+1} = J_1$ to write Equations (1.2) and (1.4).

In order to simplify the analysis, we introduce the variable $y = x + u$, which denotes the inventory level after the order is received but before demand is met. We also define the auxiliary functions $G_k(\cdot, A)$ and $G_k(\cdot, B_e)$ for $k = 1, ..., \pi$ and $e = 1, ..., E$ as follows:

$$G_k(y, A) = H(y) + \alpha \sum_{i=1}^{E} P^k_{AB_i} J_{k+1}(y-1, B_i) + \alpha P^k_{AA} J_{k+1}(y-1, A), \tag{1.5}$$

$$G_\pi(y, A) = H(y) + \alpha \sum_{i=1}^{E} P^\pi_{AB_i} J_1(y-1, B_i) + \alpha P^\pi_{AA} J_1(y-1, A), \tag{1.6}$$

$$G_k(y, B_e) = c_e y + H(y) + \alpha P^k_{B_e A} J_{k+1}(y-1, A) + \alpha \sum_{i=1}^{E} P^k_{B_e B_i} J_{k+1}(y-1, B_i), \tag{1.7}$$

$$G_\pi(y, B_e) = c_e y + H(y) + \alpha P^\pi_{B_e A} J_1(y-1, A) + \alpha \sum_{i=1}^{E} P^\pi_{B_e B_i} J_1(y-1, B_i). \tag{1.8}$$

Proposition 1 expresses the Bellman equation in an equivalent but more convenient form using the auxiliary functions $G_k(\cdot, A)$ and $G_k(\cdot, B_e)$. Its proof is provided in Appendix A.4.

Proposition 1. *For $x \in (-\infty, N]$ and $k = 1, ..., \pi$, we write the Bellman Equation (1.1) equivalently as follows:*

$$J_k(x, A) = \min\{G_k(x, A), \min_{x<y\leq N}[K + G_k(y, A)]\},$$

$$J_k(x, B_e) = \min_{x\leq y\leq N}\{G_k(y, B_e)\} - c_e x.$$

Next, we define the hybrid policy, which we will later show is optimal for our infinite-horizon discounted formulation. Additionally, in Appendix A.2 we show that a finite-horizon version of this policy is optimal for the finite-horizon discounted case.

Definition 1. *The hybrid policy.* Given parameters $(s^k_A, S^k_A, S^k_{B_1}, ..., S^k_{B_E})$, for $k = 1, ..., \pi$ we call a policy $\{u_{k+l\pi}(x, e) : l = 0, 1, ...\}$ a hybrid $(s^k_A, S^k_A, S^k_{B_e})$ policy if the following hold:

$$u_{k+l\pi}(x, A) = \begin{cases} S^k_A - x, & \text{if } x < s^k_A, \\ 0, & \text{otherwise,} \end{cases} \qquad u_{k+l\pi}(x, B_e) = \begin{cases} S^k_{B_e} - x, & \text{if } x < S^k_{B_e}, \\ 0, & \text{otherwise,} \end{cases}$$

where $u_{k+l\pi}(x, e)$ denotes the order quantity in period $k + l\pi$ as a function of the inventory level x and the environment state e.

The hybrid policy combines two policies that are commonly encountered in the supply chain management literature: i) the (s, S) policy and ii) the order-up-to policy. Namely, when in environment state A, it uses an (s, S) policy. And, when in any of the environment states B_e for $e = 1, ..., E$, it uses an order-up-to policy with an order-up-to point.

The following theorem is the main result for the infinite-horizon discounted case with backlogging and relies on Definition 1. The proof of Theorem 1 can be found in Appendix A.4.

Theorem 1. *We have a solution for J (and the corresponding G) of the Bellman equation such that $J_k(\cdot, A), J_k(\cdot, B_e), G_k(\cdot, A),$ and $G_k(\cdot, B_e)$ for $k = 1, ..., \pi$ are continuous and K-convex with $\lim_{y\to-\infty} J_k(y, A) < \infty$ and*

$$\lim_{y\to-\infty} G_k(y, A) = \lim_{y\to-\infty} G_k(y, B_e) = \lim_{y\to-\infty} J_k(y, B_e) = \infty \quad \text{for } e = 1, ..., E.$$

Moreover, the hybrid policy with parameters $(s_A^k, S_A^k, S_{B_e}^k)$ for $k = 1, ..., \pi$ is optimal where

$$S_e^k = \inf \arg\min_{x \le N} G_k(x, e) \quad \text{for } e = A, B_1, ..., B_E$$

and s_A^k is the smallest value $z < S_A^k$ such that $G_k(z, A) = K + G_k(S_A^k, A)$.

Next, we argue that $S_{B_e}^k \ge 1$ and $s_A^k \ge 1$. To this end, we first note that because demand is integer-valued (in particular, it is one in all periods), if we assume the initial inventory amount is an integer, then only the integer inventory state values are relevant. Thus, without loss of generality, we assume that s_A^k, S_A^k and $S_{B_e}^k$ are integer-valued. Assuming p is large, i.e. $p > \max\{K, c_{\max}\}$ with $c_{\max} = \max\{c_1, ..., c_E\}$, we prove that $S_{B_e}^k \ge 1$ and $s_A^k \ge 1$. Then, the following results (also see Lemmas 4 and 5 in Appendix A.4) establish that demand is never backlogged.

Proposition 2. *When $p > \max\{K, c_{\max}\}$, we have that $S_{B_e}^k \ge 1$ and $s_A^k \ge 1$.*

The following Corollary is immediate from Proposition 2 and from the optimality of the hybrid $(s_A^k, S_A^k, S_{B_e}^k)$ policy.

Corollary 1. *Under the optimal policy, when* $p > \max\{K, c_{\max}\}$*, the inventory level* $x_k \geq 0$ *for* $k \geq 1$ *almost surely.*

Thus, it follows that the hybrid policy derived above is optimal for the setting that does not allow backlogging. Proofs of Proposition 2 and Corollary 1 are provided in Appendix A.4.

1.4 Numerical Study

In this section, we provide the numerical values of the hybrid policy parameters for the Lake Tanganyika setting. For the purpose of comparison and to derive generalizable managerial insights, we begin with only two states: the normal operating state A and the disrupted state B. This constitutes our base case example. After we provide the solution for our base case, we then perform a sensitivity analysis to study the optimal policies under different disruption probabilities. Next, we propose two additional settings to consider for our model. These settings correspond to the operating conditions of two extremes: those of the developed world and those of a severely unstable environment. We define these settings using fixed disruption probability parameters, solve for their optimal inventory policies, and subsequently conduct a sensitivity analysis by varying their cost parameters. To conclude the section, we provide numerical results for the Lake Tanganyika setting with multiple disruption states.

We begin by establishing values for the parameters which describe the Lake Tanganyika supply chain. A summary of these parameters can be found below in Table 1.1. Our surveys indicate that there are $47,736$ households within the 41 villages that LTFHC serves. We normalize this demand to one. Our product has a shelf life of two years. Under a first-in first-out inventory policy, we set a limit on storage capacity of $N = 24$.

Table 1.1: Lake Tanganyika parameter regime

Parameter	Notation	Value
Storage Capacity	N	24
State A Transition Probability	P_A	1/4
State B Transition Probability	P_B	1/2
Large Capacity Fixed Ordering Cost	K	$\$1,000$
Limited Capacity Linear Ordering Cost	c	\$950
Holding Cost	h	\$154
Discount Rate	α	0.95

As outlined previously, there are three major classes of disruptions that we anticipate encountering: weather-related road closures, large machinery breakdowns leading to road closures, and armed conflict which prohibits large vehicle traffic. We begin by simplifying these events into one state (B) and will expand these into multiple states for later numerical analysis. Each of these scenarios can result in various ranges of potential outage times. We estimate that a normal, undisrupted operating period lasts 4 months on average so that $P_A = 1/4$. Additionally, we set a general outage period expectation of 2 months, so that $P_B = 1/2$. The two-month outage expectation of road networks in the region may seem long; however, when considering the annual devastation caused by the seven-month-long rainy season on the already weak infrastructure, it may in fact be optimistic. Recently, the 2020 flooding in Uvira, the staging city for these operations, caused damage to more than 15,000 homes. The UN Refugee Agency reported that "these floods have further affected a weak road network in and around South Kivu with many bridges being destroyed or damaged" (UNHCR 2020). Rebuilding bridges amid the humanitarian crisis of widespread homelessness caused by flooding is likely to take several months to complete. Likewise, the prolonged unrest in the region can also have a lingering effect on safe operating conditions. In June 2021, The

Washington Post reported on the government declaring a "state of siege" in eastern Congo due to increased violence surrounding protests which started at the end of March (Nagel and Fin 2021). With such uncertainty in the region, it is not only essential that we include the disruption probabilities in our model but that we also consider various scenarios beyond our base case by performing sensitivity analysis on the disruption parameters.

Having established the disruption and demand parameters for our base case, we next set the cost parameters. The weight, dimensions, and costs we use for these calculations are for a product that is not yet on the market, but which the LTFHC has worked with the manufacturer to use in a field trial. In order to cover the delivery routes of the system, we require 4 truck days' worth of time. This requirement is independent of the size of orders and only reflects the distance of routes needed to reach the villages included in the project. At \$250 per day to rent a truck in the main cities of Uvira and Baraka, we set $K = \$1,000$. For deliveries in the disrupted state, we elect to use motorbikes, which each have a capacity of 50kg. We would need to hire 19 motorbike days, at a cost of \$50 per day, to meet the single period demand of 927kg worth of product. Thus, we set $c = \$950$. For the holding cost, we charge a rate of 5% annualized on the cost to purchase the product each period. At a cost of $\$37,065$ for a month's worth of mosquito control vectors, the holding cost is $h = \$154$. We note that for this setting the transportation costs and product costs occur in two different markets. The purchase of the products takes place in the United States and contributes to the majority of the cost in the system, whereas transportation and workers are hired in the Congolese economy. We point this out not as a concern, but rather as a note for other settings where the relative transportation cost and holding cost, which depends on the product price, could be quite different. Per Corollary 1, we set $p > \max\{K, c_{\max}\} + hN$, as such, we use $p = 30,000$. We also set the discount factor as $\alpha = 0.95$.

Given the disruption and cost parameters in Table 1.1, we find that the optimal ordering policy for the Lake Tanganyika setting is, $s_A = 1$ $S_A = 4,$ and $S_B = 1$. In words, under the optimal policy, when inventory drops to zero an order should be placed. If the system is in state A at this time, the order should include four periods' worth of demand ($S_A = 4$). If instead the system is in state B when inventory reaches zero, the order is placed using the secondary delivery option, and should only satisfy one period worth of demand ($S_B = 1$).

As we discussed earlier, the disruption parameters P_A and P_B are influenced by several event classes, all with varying lengths. Because of this uncertainty, it is important to understand the sensitivity of the optimal policy to changes in P_A and P_B. Keeping the cost parameters of the Lake Tanganyika setting fixed, we vary both P_A and P_B from 0.01 to 0.99 in twelve increments. The results shown in Figure 1.2 indicate that the ordering policy of $s_A = 1$, $S_A = 4$, and $S_B = 1$ is optimal for a wide range of transition probabilities. This finding allows us to confidently employ the optimal policy in the Lake Tanganyika setting, even with uncertainty about the true values of P_A and P_B. We also note that S_B, the reorder point in state B, is one for all disruption parameter combinations considered using the Lake Tanganyika cost parameters. We return to this observation later, as it helps motivate our first heuristic, which presupposes one as the order-up-to amount in state B.

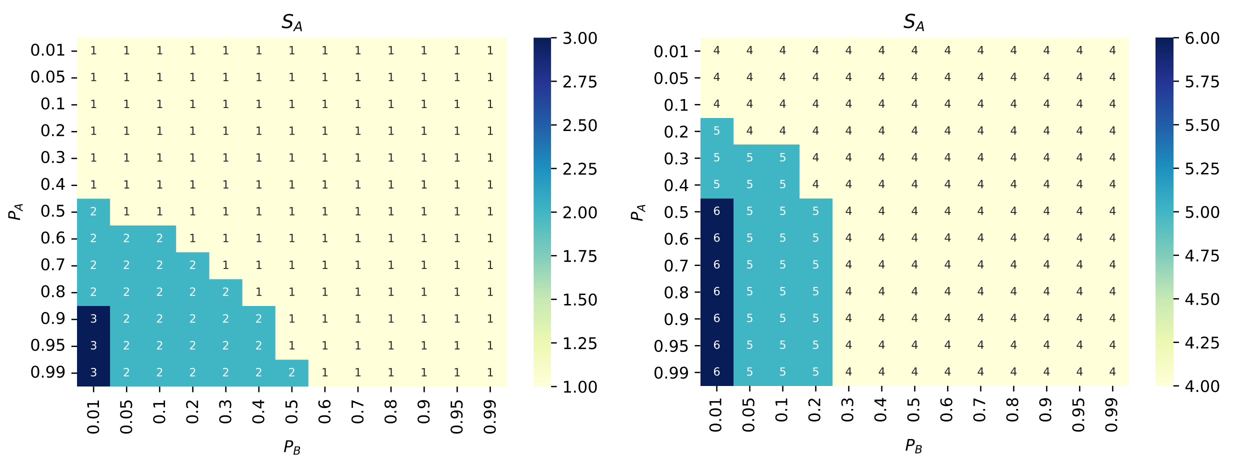

Figure 1.2: Reorder point (left panel) and order-up-to level (right panel) for the Lake Tanganyika cost parameters with varying transition probabilities.

Having shown the results for our base case under a wide range of disruption parameters, we consider the optimal policies which result from the parameters of two additional settings. We refer to the first as the heavily disrupted setting, where outages last for several months and periods of normal operations are infrequent and short-lived. In doing so, we are attempting to replicate conditions akin to an active conflict, where brief periods of ceasefire may allow for humanitarian aid to enter areas otherwise inaccessible. We set $P_A = 11/12$ and $P_B = 1/12$. The second, referred to as the low disruption setting, is meant to represent the other extreme, where disruptions occur infrequently and are quickly resolved. We set $P_A = 1/12$ and $P_B = 11/12$. This is a closer approximation to the operating conditions of the developed world. For orienting purposes, we place the transition probabilities of these settings in relation to one another on the graph shown in Figure 1.3.

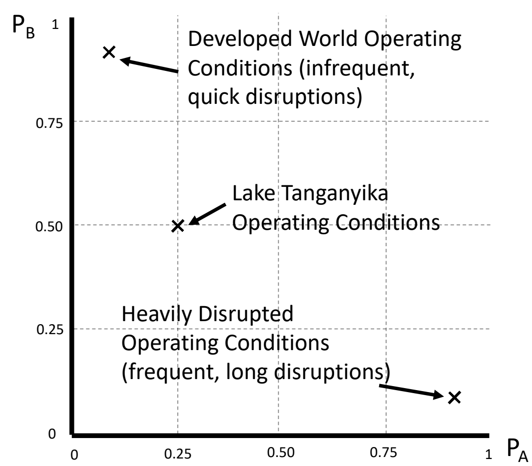

Figure 1.3: Three disruption probability settings, where increases in P_A correspond to a higher frequency of disruptions and increases in P_B correspond to shorter disruption lengths.

We test the sensitivity of the results of the two additional settings by varying both the holding cost and the state B ordering cost. Whereas in the Lake Tanganyika setting the cost parameters were known with certainty and the transition probabilities could reasonably take a wide range of values, in these constructed settings, we consider the converse, where transition probabilities are fixed and the cost parameters are permitted to vary. We fix the following parameters: $K = 1,000$, $\alpha = 0.95$, and $N = 24$. The results for these two settings are shown below in Figures 1.4 and 1.5.

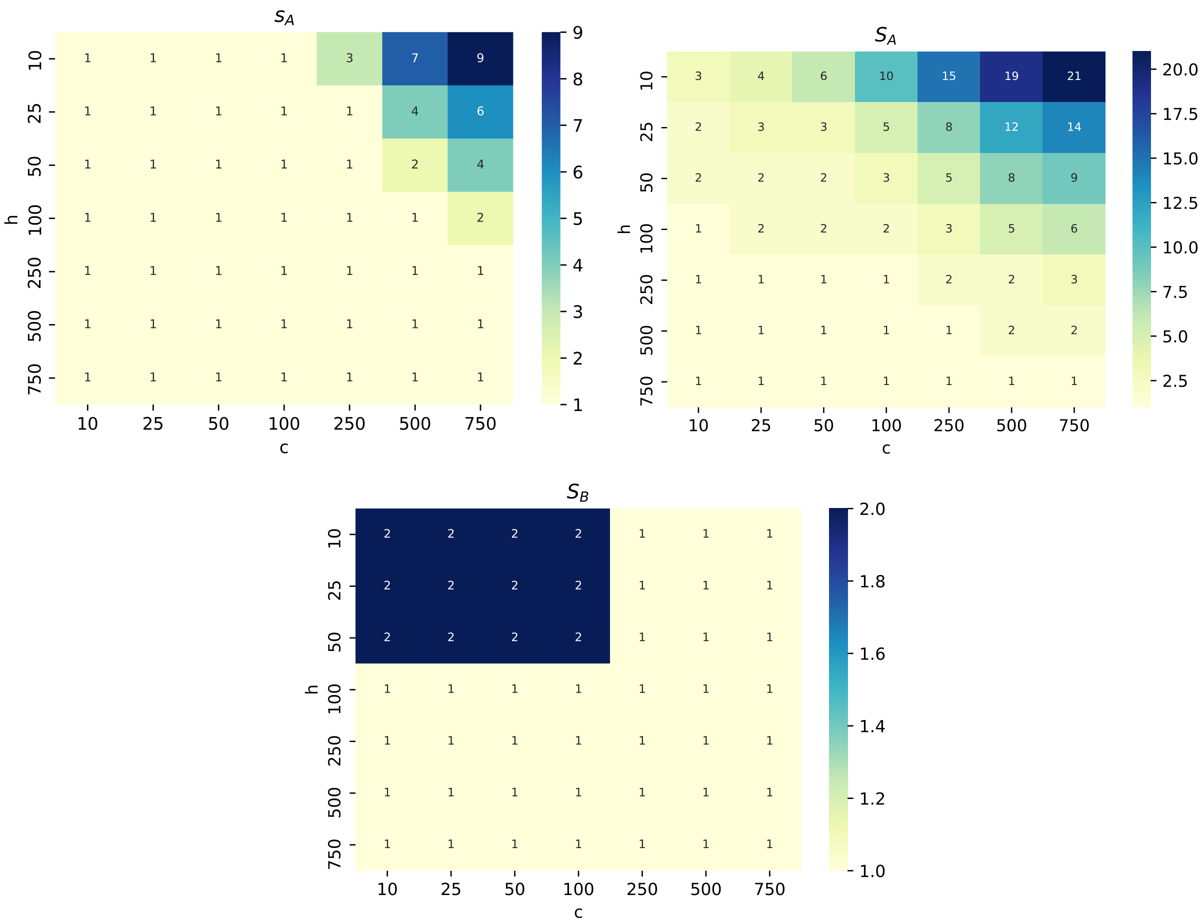

Figure 1.4: Reorder points and order-up-to levels for the heavily disrupted setting (P_A = 11/12 and P_B = 1/12) with varying holding and state B ordering costs.

In the heavily disrupted setting, s_A and S_A increase as h decreases and c increases. We also note that $s_A = 1$ for all but 8 of the 49 instances. Although $S_B = 1$ for all combinations of transition probabilities we test in the Lake Tanganyika setting, it is equal to one for close to 3/4 of the instances in the heavily disrupted system, and equal to two for the remaining instances. The cluster of instances where S_B is greater than one occurs when both the holding costs and state B ordering costs are low.

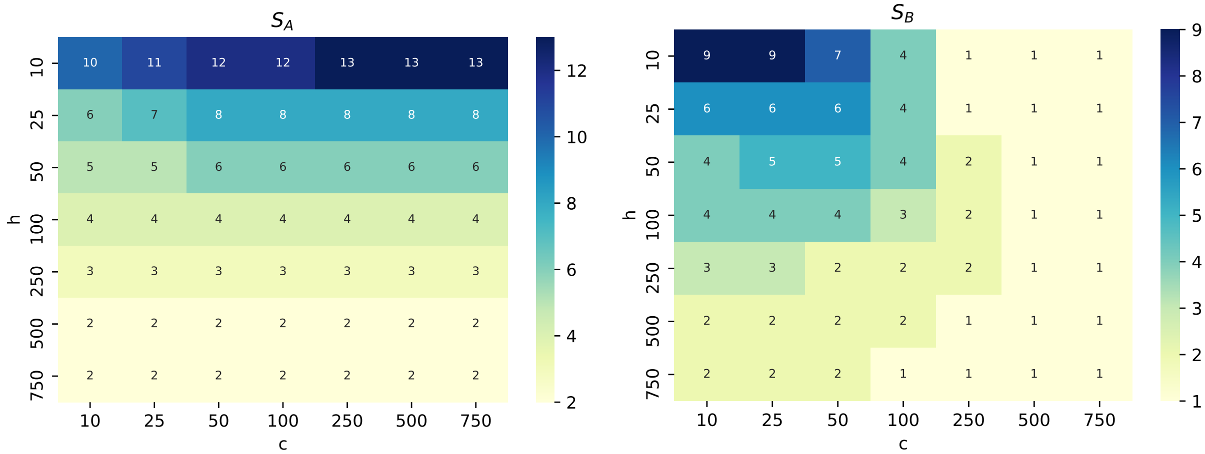

Figure 1.5: Order-up-to levels for the low disruption setting ($P_A = 1/12$ and $P_B = 11/12$) with varying holding and state B ordering costs.

For the low disruption setting, we first note that the reorder point s_A chart is not shown because all the instances we include are equal to one. We see that S_A increases as h decreases and c increases. We also observe that S_B increases as both h and c decrease. We note that in the low disruption setting, S_B accounts for a higher proportion of values greater than one and a much wider range of values than in the other settings we consider.

Overall, we identify the following trends based on the sensitivity analysis performed in the three settings: first, higher values of P_A result in larger values of s_A and S_A and smaller values of S_B. Second, lower values of P_B result in larger values of s_A and S_A and smaller values of S_B. Taken together, these two observations mean that as the system spends more time in state B and less time in state A, the optimal policy instructs the manager to both order more and maintain a larger safety stock when in the fixed cost environment. Third, as one would expect, lower values of h result in larger values of s_A, S_A and S_B. This simply means that when the cost to hold inventory decreases, the optimal policy directs the manager to hold more inventory. Fourth and finally, higher values of c result in larger values of s_A

and S_A and smaller values of S_B. Simply put, the fixed cost order option available in state A is more attractive when the state B linear ordering cost is higher.

With these insights for the two-state system, we now provide a solution for a three-state system using the same parameters from Table 1.1. We maintain the undisrupted state A, and segment state B into two types of disrupted states. State B_1 represents flood conditions, where the road is impassable. Whereas, state B_2 represents active conflict conditions, where the road is available but unsafe for large truck shipments. Each of these disruptions has an expected duration as well as a per-unit order cost. The flood state B_1 occurs more frequently and is shorter lived but is more expensive because deliveries need to be made by boats and pedestrians. The conflict state B_2 is rare but can last for long periods of time. Since the road is still passable by motorbikes during times of conflict, the per unit delivery cost c_2 is less than c_1. We set $c_1 = \$1,500$ and $c_2 = \$950$. The matrix representation of the Markov chain is as follows:

$$\mathbf{P_{ij}} = \begin{array}{c} \begin{matrix} A & B_1 & B_2 \end{matrix} \\ \begin{bmatrix} 2/3 & 1/4 & 1/12 \\ 3/8 & 1/2 & 1/8 \\ 3/5 & 3/10 & 1/10 \end{bmatrix} \end{array} \begin{matrix} A \\ B_1 \\ B_2. \end{matrix}$$

Under this set of parameters the optimal policy is $s_A = 1$, $S_A = 4$, $S_{B_1} = 1$, and $S_{B_2} = 1$. This result is consistent with the earlier results from the two-state system.

1.5 Two-State Heuristics

Although the hybrid policy is optimal, calculating its parameters, namely (s_A, S_A, S_{B_e}), is nontrivial for most workers directly managing supply chains, especially in low-resource environments. As such, we compare the performance of the hybrid policy to two simple heuristics. In Section 1.5.1 we derive a two-parameter heuristic policy for the two-state version of the model. In Section 1.5.2 we do the same for an EOQ-like heuristic. We compare the performance of both heuristic policies to that of the optimal policy in Section 1.5.3.

1.5.1 A two-parameter (*s*,*S*) policy

Having studied the optimal policy in different parametric regimes, we turn our attention to heuristics which are comparatively simple to compute and thus easy to implement by practitioners. Recall that the optimal order-up-to level for the environment state B is one for the majority of cases considered in our numerical study. Motivated by this observation, we set $S_B = 1$ for the heuristics we derive in this and the next section. This allows us to focus our attention on setting the policy parameters for the environment state A.

Specifically, we seek to find the reorder point and the order-up-to level in the environment state A, which we denote by s and S, respectively, for notational brevity. In order to derive effective policy parameters (s, S), we proceed as follows: given (s, S), we compute the corresponding long-run average cost using the theory of regenerative stochastic processes. We then minimize this expression by conducting a grid search over the integer values of s between 0 and $N-1$, and the integer values of S for $s < S \leq N$. The resulting optimal values of s and S are the heuristic policy parameters.

To begin our derivation, let (s, S) be given. We define the system state $(x_t, e_t) = (S, A)$ as the regeneration point, which denotes the beginning of each cycle. A cycle ends when the environment state is A and the inventory level is at or below the reorder point s. At the end of a cycle, an order is placed to bring the inventory level up to S, which incurs the cost K, and the next cycle begins. The inventory profile, and consequently the holding cost, is identical during the first $S - s$ time units in each cycle regardless of how the cycle evolves thereafter. This profile is shown in the shaded region of Figure 1.6.

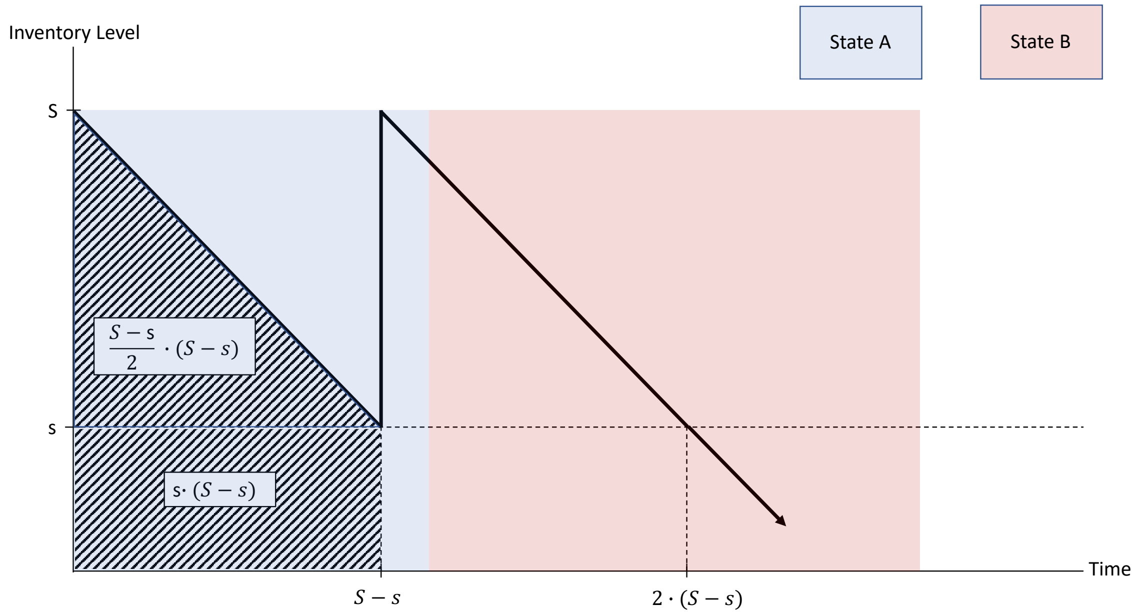

Figure 1.6: Inventory profiles for the first $S-s$ time units of a cycle.

The shaded region of Figure 1.6 has an area of $(S^2-s^2)/2$. Therefore, the combination of the inventory holding cost and the ordering cost incurred during the first $S-s$ time units of the cycle is $h(S^2-s^2)/2+K$. Next, we consider the possible outcomes of a cycle after time $S-s$.

If the environment state is A in period $S-s$, i.e. $e_{S-s}=A$, then the cycle ends. This scenario, where the regeneration point occurs at time $S-s$, is described above. If instead $e_{S-s}=B$, we have two additional scenarios to consider. These scenarios are displayed in Figure 1.7. We use τ_B to denote the additional number of periods for which the environment state remains B beyond time $S-s$, given that $e_{S-s}=B$. Formally, τ_B is a geometric random variable with $\mathbb{E}[\tau_B]=1/P_B$. It can be expressed as follows:

$$\tau_B=\inf\{k\geq 1: e_{S-s+k}=A\}.$$

At time $S-s$, if the system is in state B, there are s periods worth of inventory remaining. Thus, the relative length of τ_B to s defines the two scenarios of interest. The left panel of

Figure 1.7 shows the scenario where $\tau_B \leq s$, whereas the right panel of Figure 1.7 shows the other scenario where $\tau_B > s$.

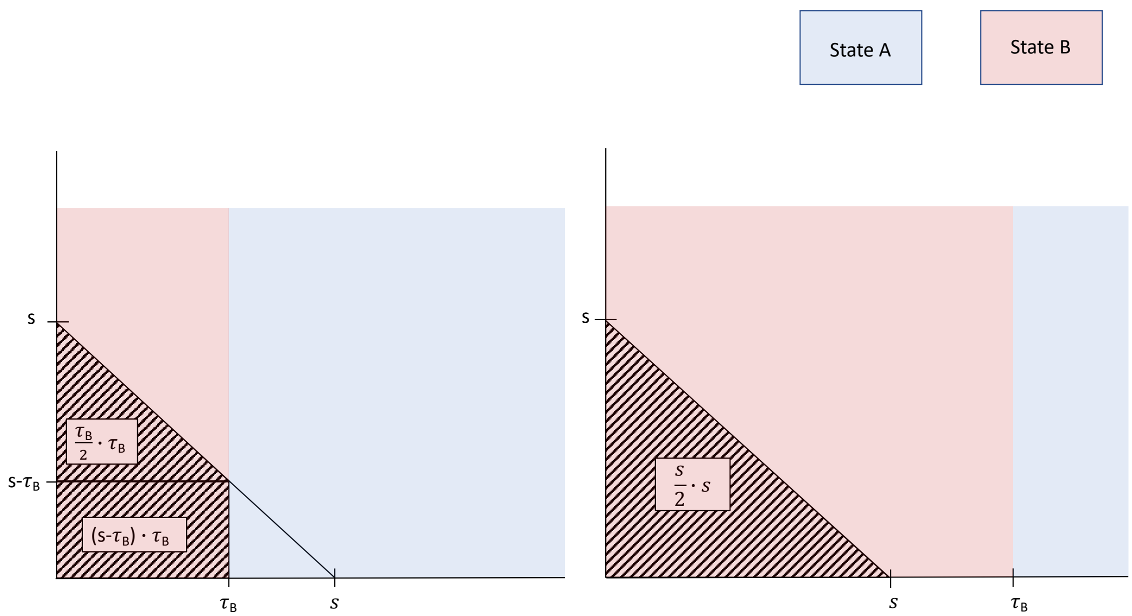

Figure 1.7: Inventory profile starting at time $S - s$ for $\tau_B \leq s$ (left panel) and $\tau_B > s$ (right panel).

To facilitate the analysis to follow we let $P_{AA}(t) = P(e_t = A|e_0 = A)$, which can be written explicitly as follows, see page 5 of Norris (1995):

$$P_{AA}(t) = \frac{P_B}{P_A + P_B} + \frac{P_A}{P_A + P_B}[1 - (P_A + P_B)]^t, \ t \geq 1. \tag{1.9}$$

In what follows, we compute the remaining cycle costs and the length for each of the two scenarios shown in Figure 1.7. We then combine them using the explicit formula in Equation (1.9) in order to derive the expected cycle cost and the expected cycle length. The ratio of the two yields the average cost of the policy with parameters (s, S).

The following proposition characterizes the expected cycle length and the expected cycle cost. Its proof is provided in Appendix A.6.

Proposition 3. *We have*

$\mathbb{E}[\mathit{Cycle\ length}] = S - s + \big(1 - P_{AA}(S-s)\big)/P_B$ *and that*

$$\mathbb{E}[\mathit{Cycle\ cost}] = h\frac{S^2 - s^2}{2} + K + [1 - P_{AA}(S-s)]\frac{\big((2c-h)P_B + 2h\big)(1-P_B)^s + 2h\big((s+\frac{1}{2})P_B - 1\big)}{2P_B^2},$$

where $P_{AA}(t)$ is given by Equation (1.9).

Corollary 2. *We have that*

$$C(s,S) = \frac{\mathbb{E}[\mathit{Cycle\ cost}]}{\mathbb{E}[\mathit{Cycle\ length}]}$$

In order to further simplify the expressions in Proposition 3, we assume that $[1-(P_A + P_B)]^2 << 1$, which is satisfied by the parameters of the Lake Tanganyika example. This allows us to approximate $P_{AA}(t)$ as follows:

$$P_{AA}(t) \approx \frac{P_B}{P_A + P_B} \quad \text{for } t \geq 1. \tag{1.10}$$

Using this approximation, Corollary 3 provides the simplified version of the average cost expression, $C(s,S)$.

Corollary 3. *Under the approximation in Equation (1.10), we have that*

$$\begin{aligned}\tilde{C}(s,S) = & \frac{(P_A+P_B)P_B}{(S-s)P_B^2 + P_A(S-s)P_B + P_A}K + \frac{P_A(1-P_B)^s}{(S-s)P_B^2 + P_A(S-s)P_B + P_A}c \\ & + \frac{-P_A(P_B-2)(1-P_B)^s + (S^2-s^2)P_B^3 + P_A(S^2-s^2)P_B^2 + P_A(2s+1)P_B - 2P_A}{2P_B\big((S-s)P_B^2 + P_A(S-s)P_B + P_A\big)}h\end{aligned}$$

Performing a grid search on $C(s,S)$ and $\tilde{C}(s,S)$ in Corollaries 2 and 3 respectively, allows us to arrive at the long-run average cost and optimal parameters (s,S) for the complete and simplified versions of the two-parameter heuristic.

1.5.2 An EOQ-like Policy

Similar to our observation from the Numerical Study Section, which allowed us to set $S_B = 1$ for the heuristic in Section 1.5.1, we note that the majority of cases from the three settings we study have an optimal reorder point of one, i.e. $s_A = 1$. See Figures 1.2 and 1.4 as well as the justification for not including s_A in Figure 1.5. Motivated by this observation, in addition to setting $S_B = 1$, we further simplify the policy by setting $s_A = 1$. This leaves only one parameter to be optimized, the order-up-to level S_A. The simplification to one parameter represents a significant reduction in complexity. Our approach ultimately yields an EOQ-like heuristic. In order to emphasize this connection, we denote the policy parameter by Q, corresponding to the order quantity.

In order to characterize the average cost of a policy denoted by $C(Q)$, we once again proceed with regenerative analysis. The system state $(x_t, e_t) = (0, A)$ is the regeneration point, marking the beginning of a cycle. At the beginning of each cycle, an order for Q units is placed. A cycle ends when the system state reaches $(0, A)$. We consider two scenarios, which will allow us to compute the cycle length and the cycle cost. The left panel of Figure 1.8 shows the first scenario in which at time $t = Q$ the inventory is depleted and the environment state is A, i.e. $(x_Q, e_Q) = (0, A)$. Thus, the cycle ends at time Q. The right panel of Figure 1.8 shows the second scenario in which at time $t = Q$ the inventory level is depleted but the environment state is B, i.e. $(x_Q, e_Q) = (0, B)$. In this scenario, because we set $S_B = 1$, the system manager orders, and then uses a single unit each period until the environment state transitions to state A, at which point the cycle ends.

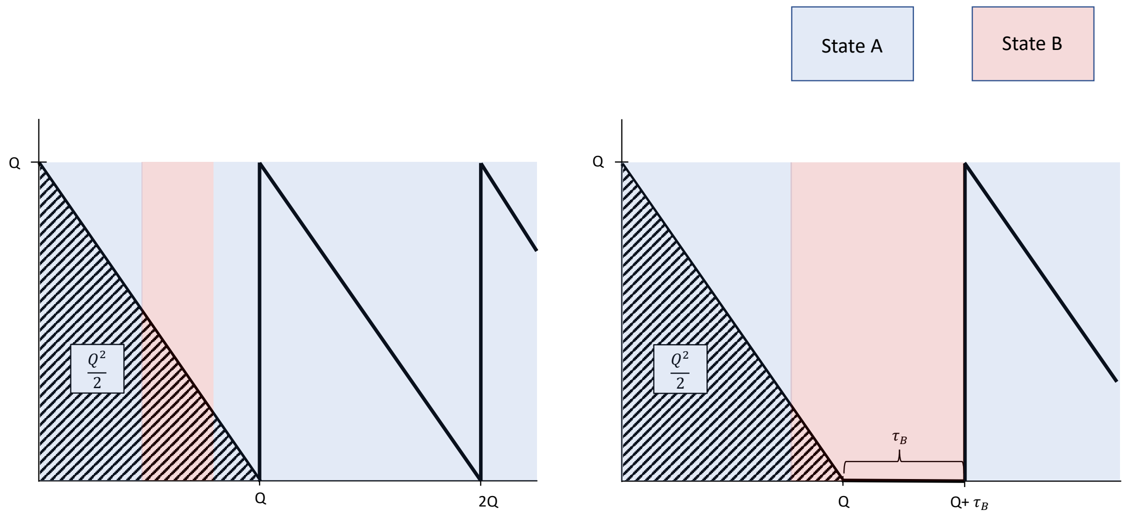

Figure 1.8: Inventory profile of EOQ-like policy when inventory is depleted in state A (left panel) and when inventory is depleted in state B (right panel).

We use τ_B to denote the additional number of periods, after time Q, during which the environment state remains B. Once again, τ_B is a geometric random variable with $\mathbb{E}[\tau_B] = 1/P_B$. The following proposition characterizes the expected cycle length and expected cycle cost. The proof of Proposition 4 is provided in Appendix A.7.

Proposition 4. *For order quantity Q, the expected cycle length and the expected cycle cost are:*

$$\mathbb{E}[\textit{Cycle length}] = Q + (1 - P_{AA}(Q))\frac{1}{P_B},$$
$$\mathbb{E}[\textit{Cycle cost}] = K + \frac{hQ^2}{2} + (1 - P_{AA}(Q))\frac{c}{P_B}.$$

The average cost, denoted by $C(Q)$, is then characterized by Corollary 4 as the ratio of the expected cycle cost to the expected cycle length. Its proof is immediate upon substituting Equation (1.9) into the expressions given in Proposition 4.

Corollary 4. *We have that*

$$C(Q) = \frac{(P_A + P_B)P_B}{QP_BP_A + QP_B^2 - P_A(1 - P_A - P_B)^Q + P_A}K - \frac{P_A\big((1 - P_A - P_B)^Q - 1\big)}{QP_BP_A + QP_B^2 - P_A(1 - P_A - P_B)^Q + P_A}c$$
$$+ \frac{Q^2(P_A + P_B)P_B}{2QP_BP_A + 2QP_B^2 - 2P_A(1 - P_A - P_B)^Q + 2P_A}h$$

Using the earlier approximation from Equation (1.10) that $P_{AA}(Q) \approx P_B/(P_A + P_B)$, we further simplify the average cost equation, $C(Q)$. We denote the average cost approximation as $\tilde{C}(Q)$. Corollary 5 defines $\tilde{C}(Q)$ and establishes its convexity. The proof of Corollary 5 is provided in Appendix A.7.

Corollary 5. *We have that*

$$\tilde{C}(Q) = \frac{K + \frac{hQ^2}{2} + \frac{P_A c}{(P_A+P_B)P_B}}{Q + \frac{P_A}{(P_A+P_B)P_B}}.$$

Moreover, $\tilde{C}(Q)$ is a convex function.

In order to explicitly define the optimal order amount for Corollary 5, we use a further approximation by allowing $Q \geq 0$ to be a continuous variable. As such, the first-order condition, $\tilde{C}'(Q) = 0$, gives the following optimal order quantity in Equation (1.11). The derivation of this result is provided in Appendix A.7.

$$Q^* = \sqrt{\frac{2}{h}\Big(K + \frac{P_A c}{(P_A + P_B)P_B}\Big) + \Big(\frac{P_A}{(P_A + P_B)P_B}\Big)^2} - \frac{P_A}{(P_A + P_B)P_B} \tag{1.11}$$

In practice, we propose using $\lceil Q^* \rceil$ as the order quantity.

1.5.3 Comparison of Numerical Results

In this section, we compare the performance of the two heuristics to the optimal policy. For the set of parameters under the Lake Tanganyika setting, we find that both heuristics along with the optimal hybrid policy all prescribe the same inventory management approach. The policy parameters for the (s, S)-type heuristic are $s = 1$ and $S = 4$, and the parameter for the EOQ-like heuristics is $Q = 4$. In Section 1.4, we presented optimal parameters for the hybrid policy as $(s_A, S_A, S_B) = (1, 4, 1)$. Since the (s, S)-type policy sets $S_B = 1$ and the EOQ-like policy assumes $s_A = 1$ and $S_B = 1$, the infinite horizon discounted costs of all three policies are the same. These results show that the EOQ policy will perform just as well as the optimal hybrid policy in our base case setting.

We similarly evaluate the relative performance of the heuristic policies for the heavily disrupted and low disruption settings using the same cost parameters as our base case setting. The parameters for these two settings are shown in Table 1.2.

Table 1.2: Heavy disruption and low disruption parameters

Parameter	Heavy Disruption Value	Low Disruption Value
N	24	24
P_A	11/12	1/12
P_B	1/12	11/12
K	$\$1,000$	$\$1,000$
c	\$950	\$950
h	\$154	\$154
α	0.95	0.95

In the low disruption setting, as was the case in the Lake Tanganyika setting, all three approaches yield the same inventory management policy. The parameters for the low disruption setting are $s = 1$ and $S = 4$ for the (s, S)-type policy and $Q = 4$ for the EOQ-like policy. Similarly, the optimal hybrid policy parameters for the low disruption setting are $(s_A, S_A, S_B) = (1, 4, 1)$.

In the heavily disrupted setting, we once again find that the hybrid and (s, S)-type policies are the same; however, they both differ from the EOQ-like policy. The parameters for the heavily disrupted setting are $s = 1$ and $S = 5$ for the (s, S)-type policy and $Q = 5$ for the EOQ-like policy. The optimal hybrid policy parameters for the heavily disrupted setting are $(s_A, S_A, S_B) = (1, 5, 1)$. The difference between these policies is the reorder point, which is one for both the (s,S)-like and hybrid policies, and one, as it is fixed in our simplifying assumption, for the EOQ-type policy. The differences in the discounted costs resulting from this deviation of the EOQ-like policy from the other two policies are shown in Figure 1.9. As it is difficult to see these differences, given the scale of the discounted costs, we show the percentage improvement of the other two policies over the EOQ-like policy in Figure 1.10. The EOQ-like policy performs only slightly worse if the starting inventory is low and the environment state is A. Otherwise, its performance is near optimal.

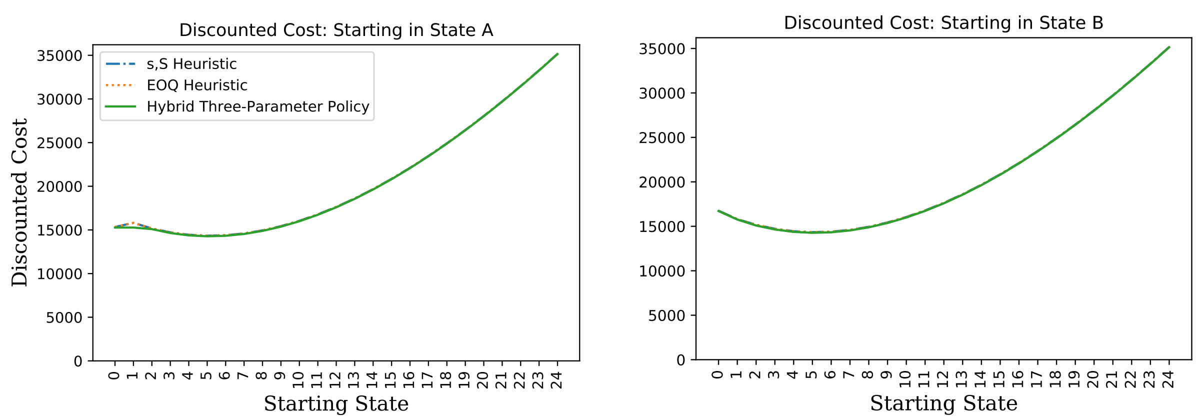

Figure 1.9: Infinite-horizon discounted costs for three policies in the heavily disrupted setting.

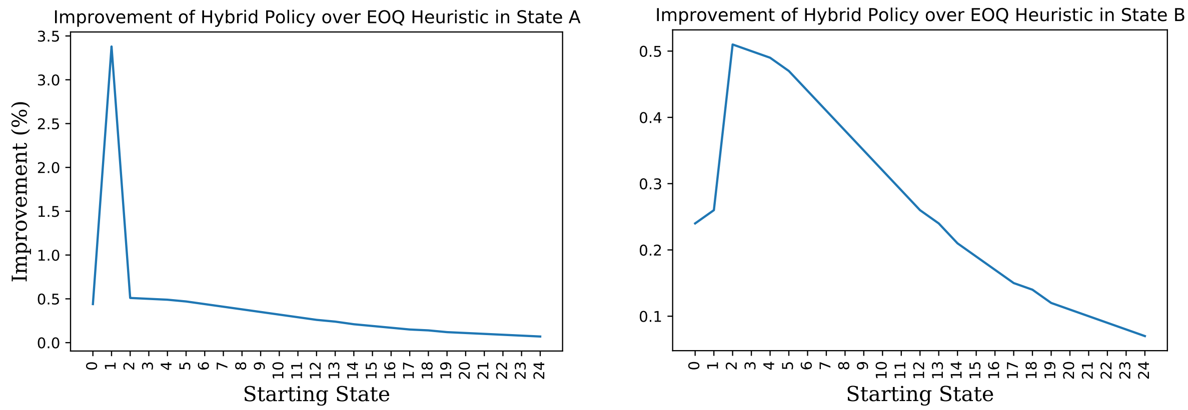

Figure 1.10: Improvement of the discounted costs for the (s, S)-type and hybrid policies over the EOQ-like policy in the heavily disrupted setting.

In the three settings we outline in the numerical results section, we see that there is no cost improvement from calculating the parameters for the hybrid policy instead of calculating those for the heuristic policies in two of the settings. Additionally, using the hybrid policy provides only a small improvement over the EOQ heuristic in the third setting. These findings present a strong case for using either the (s, S)-type policy or the EOQ-like policy in our setting. However, in other varied settings, the ranges of possible cost and disruption parameters contain instances where the cost improvements of the optimal hybrid policy over the two heuristic policies are significant enough to justify its use. We provide two such cases in Appendix A.8.

1.6 Concluding Remarks

We approached the problem of last-mile delivery in the Lake Tanganyika region of the Democratic Republic of the Congo with the goal of designing a supply chain, which would allow for the uninterrupted delivery of preventative medical products in a cost-effective manner. The need for such a project arose from the conjecture that a properly managed supply chain with disruption considerations to reflect the realities of operating in a fragile environment, would allow for setting specific cost comparisons, as they relate to reductions in malaria cases, between bed nets and other spatial vector controls.

Using our model, we proved the optimality of a state-dependent hybrid $(s_A^k, S_A^k, S_{B_e}^k)$ inventory policy. In studying the numerical results and subsequent sensitivity analysis, we provided insights into how changes in the setting parameters influence the optimal policy parameters. We presented the optimal inventory policy for a set of parameters describing the Lake Tanganyika setting, along with those for two other setting types. In addition, we derived two heuristic policies for managing the inventory system and compared their performance to that of the optimal policy. We found that in our setting, and several others, the heuristics perform close to, if not just as well as, the hybrid policy. As a follow-up to this paper, we are working with the LTFHC on a pilot program in the Lake Tanganyika region to evaluate the improvements in health outcomes from the use of a last-mile supply chain for the delivery of spatial mosquito control vectors. The program uses our model to minimize costs in its implementation.

Chapter 2

A Constrained Optimization Approach to Refining a Maternal and Neonatal Community Health Program in Somalia Using the Lives Saved Tool

2.1 Introduction

Community Health Worker (CHW) programs are a widely-used approach to deliver health interventions and counseling to populations often without regular access to conventional health services, including remote, isolated, and mobile populations. While the terminology used and training received varies across cadres and countries, CHWs typically receive training to diagnose and treat or refer common illnesses that is tailored to various levels of literacy and numeracy. An expansive evidence base has demonstrated the effectiveness of CHW cadres to improve care-seeking, health literacy, and health behaviors in numerous contexts, leading to an increasing reliance on them to perform an expanding list of tasks (Perry et al.

2014).

Recognition of CHW programs' potential has also led to many countries establishing ambitious community health programs that are intended to provide a wide variety of services across many areas of health. In contexts where health systems are stressed by low resources, conflict, or state fragility, these ambitious goals can be difficult to meet. Observational research on CHW programs regularly cites the difficulty of recruiting and retaining qualified CHWs as a primary reason why community health programs fail to meet their objectives (Tshering et al. 2018). Although there are benefits to increasing the number of CHWs, an emphasis solely on recruitment and retention represents an overly narrow view of the challenges facing CHW programs. In addition to the critical task of increasing the human and material resources available for CHW programs, improvements could be achieved by re-assessing the process by which interventions are selected for inclusion in community health packages. In particular, in emergency contexts where both human and material resources are likely to be stretched, this approach could help provide CHWs with objectives that are achievable given the resources available. With this goal in mind, we developed a context-adaptable constrained optimization model to select the set of community-based maternal and newborn health (CBMNH) interventions that seeks to maximize the number of lives saved, by weighing the resource requirements of each intervention against the resources available in that context.

Although the constrained optimization model is intended to be flexibly applied in different contexts, it was first constructed to apply to the Female Health Worker (FHW) program in Somalia. As the World Bank continues to partner with the Somalia Ministry of Health to invest in FHWs, national-level stakeholders believe there is value in examining how to best deliver services to remote communities that had not previously been reached by the program and for whom facility access is especially limited. The International Rescue Committee (IRC) partnered with stakeholders including the Somali Ministry of Health, state government officials, professional associations, and partner implementing agencies to define a package of CBMNH interventions for the FHW program in Galmudug, Somalia. This consultation allowed us to source firsthand information on the practical constraints which often hamper the delivery of CBMNH services—particularly staff time, training capacity, commodity availability, and the policy environment. By incorporating the stakeholder input throughout the modeling process, we aimed to both improve the accuracy of the model and develop trust in the resulting recommendations.

The optimization model, using the Somalia health parameters and resource constraints, yields a set of eight suggested interventions to increase the number of maternal and neonatal lives saved (Table 2.1). The package of care is projected to avert 612 maternal and neonatal deaths, which would account for 16 percent of the projected maternal and neonatal deaths in Galmudug, Somalia, in 2023.

Table 2.1: Optimal package of interventions for the Somalia setting

1.	Kangaroo mother care
2.	Clean cord care
3.	Thermal regulation
4.	Prevention of malaria in pregnancy
5.	Breastfeeding promotion
6.	Basic sanitation
7.	Hand washing with soap
8.	Iron Supplementation

Of the 25 interventions identified as viable to offer at the community level for MNH care, only nine are under consideration within the Somalia program (see Table 2.4 of Section 2.2.3). The reasons for excluding the majority of interventions range from logistical to cultural and were determined during stakeholder consultations in Mogadishu, Somalia. Policy feasibility is thus a major determinant of the optimal package of services in Somalia.

Given the planned expansion of the FHW program and the uncertainty surrounding future budget support, we conducted sensitivity analysis and found that increasing the number of FHWs and expanding the list of interventions permitted at the community level are the most effective ways to increase the number of lives saved within the Somalia setting. Given the questionable quality of many secondary data sources that would feed into optimization models, we feel that sensitivity analysis is an important benefit of this approach, and allows users to gain additional insights into programs in fragile or conflict-affected contexts.

The rest of the paper is structured as follows: Section 2.2 presents an overview of the Lives Saved Tool and describes the general model before applying it to the Somalia context; Section 2.3 Provides the solution to the model along with sensitivity analysis; Section 2.4 discusses the results along with the strengths and limitations of this modeling approach; and Section 2.5 concludes the paper with an eye towards future applications.

2.2 Methods

2.2.1 The Lives Saved Tool

One of the great challenges with creating healthcare optimization models in developing countries is the lack of reliable, context-specific data. We address this issue by incorporating data from the Lives Saved Tool (LiST), a combined effort by the Johns Hopkins University Bloomberg School of Public Health and the Bill & Melinda Gates Foundation. This tool is widely used by global health organizations and is regularly revised to reflect changes to country demographic profiles and updates on the efficacy of interventions (Walker et al. 2013). LiST provides both equations to predict the impact of increases in healthcare coverage of various interventions, and a set of demographic and health data to populate those equations in different settings. We specify the parameters from LiST that we use in our model in Table 2.2. Reliance on LiST sets a clear objective of maximizing lives saved in favor of other health metrics such as DALYs and QALYs averted. We find this to be an appropriate objective for our work; however, we do acknowledge an ongoing debate about the strengths and shortcomings of this target, see Spigelman et al. (2018).

Table 2.2: Parameters from LiST used in the model

1. The current level of coverage for interventions
2. The annual number of deaths in four different population groups
3. The percentage of deaths attributed to each cause of death within each population group
4. Efficacy of each intervention for each cause of death
5. Affected fraction of each intervention for each cause of death

LiST provides an estimate for the current level of coverage for each intervention, displayed as a percentage of the country's population. By subtracting the current level of coverage from the maximum percentage of the population that could reasonably be expected to receive that intervention, referred to as full coverage and often listed as 90%, we arrive at the percentage of the population available to receive that intervention. This difference between full coverage and current coverage provides a cap on the available increase in coverage. In doing so, it ensures that the number of possible lives saved for each intervention matches the number of deaths, for which the intervention addresses the underlying cause, within the untreated portion of the population.

LiST separates outcomes among four population groups: Maternal, Neonatal (0-1 month), Stillbirth, and Children (1 to 59 months). It uses a mortality rate for each population group, along with the projected size of each group, to estimate the number of deaths within each group. These deaths within each group are further categorized by cause of death, for which LiST provides country-specific percentages based on demographic and public health surveys. This data is typically available at the national and sub-national unit, but values for smaller geographic designations are often spatially interpolated based on covariates of different causes of mortality.

LiST identifies nine maternal causes of death and eight neonatal causes of death. These are shown for the Somalia setting in Tables 2.5 and 2.6. Each possible intervention then addresses at least one, and in many cases several, of these causes of death. Although this model is intended for use in the design and evaluation of maternal and neonatal programs, the availability of country-specific mortality rates for children and stillbirths allows for a straightforward extension of this model to health programs focusing on these additional populations.

To calculate the number of lives saved attributed to each intervention, LiST provides affected fraction and efficacy values for each intervention and cause-of-mortality pairing. The affected fraction of an intervention is defined in LiST as the proportion of cause-specific deaths which are susceptible to treatment with a given intervention (Walker et al. 2013). Likewise, the efficacy of an intervention is the proportion of cause-specific deaths which can be averted with an intervention. In other words, the affected fraction measures what proportion of a cause of death an intervention targets, and efficacy measures how well a treatment works at reducing the relevant fraction of that cause of death. Using these treatment parameters in conjunction with the demographic parameters listed in Table 2.2, we provide a country-adaptable lives-saved function in Section 2.2.2.

2.2.2 The General Model

We now present the model in its general form. We begin by describing each component of the lives-saved calculation. Next, we write out the mathematical formulation. We then explain each of the constraints included in the model.

Our model has two sets of decision variables, the first denotes the incremental coverage level for different interventions, while the second indicates whether or not each intervention is selected for implementation. To be specific, decision variable x_j represents the incremental coverage of intervention j for $j = 1, ..., J$ where J is the number of interventions. Decision variable y_j is binary and set to one if intervention j is selected (i.e. identified by the solved model as being included in an optimal package of services) and zero otherwise. We let $x = (x_j)$ and $y = (y_j)$ denote the vectors of decision variables for notational convenience.

Using these decision variables, we can calculate the number of projected lives saved using the parameters shown in Table 2.3. To do so for an individual intervention and a specific cause of death, we multiply the percentage change in coverage by the affected fraction, efficacy, and annual number of deaths from that cause. By expanding this calculation across all selected interventions, causes of death, and the maternal and neonatal population groups, we replicate the lives-saved function from LiST, which yields the objection function in Equation (2.1) of the model.

Table 2.3: Parameters included in the model on a country-level basis

c_j	Current level of coverage for intervention j
D_N	Annual number of neonatal deaths
D_M	Annual number of maternal deaths
r_i	Percentage of deaths attributed to cause of death i
f_{ij}	Affected fraction for cause of death i and intervention j
e_{ij}	Efficacy for cause of death i and intervention j
u_j	Level of coverage which equates to full coverage for intervention j
d	Available increase in coverage determined by the number of workers available
α_j	Annual number of commodities required to reach full coverage of intervention j
β_j	Annual commodities available for intervention j

The resulting mixed-integer program for maternal and newborn healthcare optimization is given below.

$$\max_{x,y} \; D_N \sum_{i=1}^{8} r_i \Big[1 - \prod_{j=1}^{J} \Big(1 - f_{ij} \frac{e_{ij} x_j}{1 - e_{ij} c_j}\Big)\Big] + D_M \sum_{i=9}^{17} r_i \Big[1 - \prod_{j=1}^{J} \Big(1 - f_{ij} \frac{e_{ij} x_j}{1 - e_{ij} c_j}\Big)\Big] \quad (2.1)$$

subject to

$$Ax \le a \quad (2.2)$$

$$By \le b \quad (2.3)$$

$$x_j \le \min(u_j - c_j,\ d,\ \beta_j/\alpha_j),\ \forall\ j \quad (2.4)$$

$$x_j \le y_j, \quad \forall\ j \quad (2.5)$$

$$y_j \in \{0,1\}, \quad \forall\ j \quad (2.6)$$

$$x_j \ge 0, \quad \forall\ j \quad (2.7)$$

Matrices A and B in Equations (2.2) and (2.3) are both 3xJ dimensional consumption matrices; their columns correspond to different interventions. Likewise, a and b are 3-dimensional capacity vectors—these capture the relationships between the factors necessary to implement CBMNH programs, and the total resource budget for each of those factors. Let A_i and B_i denote the i^{th} row of matrices A and B, respectively for $i = 1, 2, 3$. And, let a_i and b_i denote the i^{th} component of each vector.

- Delivery Time: The constraint $A_1 x \leq a_1$ ensures the total time required for CHWs to deliver the level of coverage for the selected interventions does not exceed the available time to do so. The resource budget for delivery time, a_1, is a key factor to understand in overall program planning. It can be shifted up or downwards to reflect factors like travel time, the target number of households each CHW serves, and the number of other health modules they are responsible to deliver.

- Supervision Capacity: $A_2 x \leq a_2$ ensures the supervisors' time needed to properly oversee the interventions requiring in-service training does not exceed the available time to train CHWs. Particularly where CHWs may be asked to deliver new and technically complex interventions, the availability of supervisors to provide supportive supervision and ensure quality is often a limiting factor.

- Scaling Coverage Cost: $A_3 x \leq a_3$ ensures the total cost to increase intervention coverage, based on the price of commodities necessary for different interventions, is less than or equal to the budget available to scale up interventions.

- Curriculum Time: The constraint $B_1 y \leq b_1$ ensures that the curriculum time required to train interventions does not exceed the curriculum time available to train maternal and neonatal care. Particularly when new cadres of CHWs are hired, the feasible time for in-service training can limit the roll-out of new components of a service package.

- Intervention Start Up Cost: $B_2 y \leq b_2$ ensures that the total cost to establish new interventions does not exceed the budget available to establish new interventions.
- Permission to Offer at the Community Level: $B_3 y \leq b_3$ ensures that an intervention can only be included in the recommended package if it is permitted for delivery at the community level by the country's government. Some interventions have been shown to be feasible for community-level delivery, but concerns about CHW capacity or control of commodities mean that policies may not allow community delivery in every country.

Equation (2.4) ensures that the change in coverage for intervention j does not exceed the lesser of: one, full coverage (u_j) minus the current coverage of the intervention (c_j); two, the maximum available coverage (d) determined by the size of the CHW cadre; and three, the increase in coverage for intervention j which is achievable based on the number of commodities available. The maximum available coverage (d) in Equation (2.4) is calculated by multiplying the number of CHWs by the average number of people they each serve, then dividing this number by the total population. This provides the percentage of the population which can be covered by the given number of CHW in a cadre. We calculate the increase in coverage which is achievable from the number of commodities available (β_j/α_j) by dividing the number of commodities available (β_j) in a given country by the number of commodities required for full coverage (α_j) of that country's population. Equation (2.5) states that the coverage level of intervention j (x_j) can only be increased if the intervention is selected (i.e. if $y_j = 1$). Equations (2.6) and (2.7) ensure y_j is binary and x_j is nonnegative, respectively, for $j = 1, \ldots, J$.

To properly measure the resources used for certain constraints, we need to translate the level of coverage for each intervention into the equivalent number of treatments needed to achieve that increase in coverage. As the individuals identified as requiring treatment will not be the same for all interventions (for instance, iron fortification is administered to pregnant women but not newborns, infants, or children), we match each intervention with a target population. A target population is a subset of the country's population defined by a common feature, e.g. pregnant women. The sizes of relevant target populations are provided by LiST for each country. Examples of these target populations can be found in Table 2.7 of Section 2.2.3. The number of treatments needed to achieve full coverage for an intervention is calculated by multiplying the target population by the percent of the target population requiring that intervention. Multiplying the number of treatments required for full coverage of intervention j by the change in coverage (x_j) provides the number of treatments required.

To underscore the need for optimization processes in solving this problem, if all 25 interventions are considered only using only the y decision variables of including or not including each intervention in the package, there are 2^{25} (or 33,554,432) possible combinations to evaluate. Having established the components of the general model, we can now apply the analysis to specific settings, beginning with the CBMNH program in Somalia.

2.2.3 Model Applied to Somalia

In Somalia, the Female Health Worker (FHW) cadre provides healthcare to rural populations with minimal access to health facilities. They have been deployed in select sub-national areas to deliver MNH services at the community level, with broader ambitions to scale the plan nationally over the coming years. In 2023, the government estimates having 1,450 FHWs throughout the country, largely around more urban areas, with plans to expand to 3,000 FHWs by 2025.

After consultation with MNH professionals, we have selected 25 interventions to include in the model. These interventions were included because of their demonstrated impact on maternal and neonatal mortality and evidence of the feasibility of their delivery at the community level. However, within Somalia's health program, several of these interventions are deemed infeasible to provide at the community level–this is captured in the constraint for B_3. The interventions included in the model, their current levels of coverage in Somalia according to LiST, and their statuses as either feasible or infeasible for delivery in Somalia are shown in Table 2.4.

Table 2.4: MNH interventions with current coverage levels and status of Somalia feasibility

Intervention	Current Coverage	Feasible at the Community Level
Tetanus toxoid vaccination	67%	no
Multiple micronutrient supplementation in pregnancy	0%	no
Iron supplementation in pregnancy	0%	yes
Calcium supplementation	0%	yes
Balanced energy supplementation	0%	no
KMC - Kangaroo mother care	5%	yes
Breastfeeding promotion	9%	yes
Hand washing with soap	10%	yes
Prevention of malaria in pregnancy	2%	yes
Basic sanitation	38%	yes
Clean birth environment	7%	no
Clean cord care	8%	yes
Immediate drying and additional stimulation	8%	no
Neonatal resuscitation	4%	no
Uterotonics for postpartum hemorrhage	7%	no
Syphilis detection and treatment	4%	no
Thermal regulations	9%	yes
Manual removal of placenta	3%	no
Point-of-use filtered water	4%	no
Malaria case management	1%	no
Injectable antibiotics for neonatal sepsis	9%	no
Safe abortion services	3%	no
Oral antibiotics for neonatal sepsis	0%	no
Post-abortion case management	0%	no
Antibiotics for treatment of dysentery	0%	no

We use the 2023 projected number of births in Somalia according to LiST $(716, 518)$ as a baseline for calculating the number of Neonatal and Maternal Deaths (D_N and D_M). We also use the same neonatal mortality rate of 3.68% as provided by LiST, but a different maternal mortality rate of 0.692% as supplied by the Ministry of Health, to reflect their estimates. Together, these numbers provide a projection of 26,413 neonatal deaths and 4,958 maternal deaths in 2023 across the entire country.

Tables 2.5 and 2.6 list the various causes of death, along with their specific contributions to predicted neonatal and maternal mortality in Somalia. The proportions of neonatal deaths attributed to each cause are from 2017 and sourced from the WHO estimates between 2000-2017 (WHO 2017). The proportions of maternal deaths attributed to each cause are from 2013 (Say et al. 2014).

Table 2.5: Somalia neonatal mortality by cause

Cause of Death	Percent of Neonatal Deaths by Cause	2023 Projected Deaths by Cause
Diarrhea	0.94%	248
Sepsis	9.88%	2,609
Pneumonia	9.58%	2,530
Asphyxia	37.05%	9,785
Prematurity	22.62%	5,974
Tetanus	6.41%	1,693
Congenital anomalies	6.61%	1,745
Other	6.92%	1,827

Table 2.6: Somalia maternal mortality by cause

Cause of Death	Percent of Maternal Deaths by Cause	2023 Projected Deaths by Cause
Antepartum hemorrhage	8.95%	443
Intrapartum hemorrhage	0.88%	43
Postpartum hemorrhage	15.71%	778
Hypertensive disorders	16.94%	840
Sepsis	10.54%	522
Abortion	10.97%	543
Embolism	1.89%	93
Other direct causes	8.46%	419
Indirect causes	25.66%	1,272

The final sets of parameters needed for the objective function of the model are the Affected Fraction (f_{ij}) and Efficacy (e_{ij}) values. With 17 causes of death and 25 interventions, we list these values in two 17×25 matrices. We use the values from the LiST database for these parameters, except for a select number, which we adjust to reflect the estimates provided by the public health experts who participated in consultations in Somalia. The matrices for Affected Fraction and Efficacy values are provided in Appendix B.1. Most interventions only affect one cause of death. A smaller number of interventions affect two to four causes of death. As a result, 85% of entries in the matrices are zero.

The number of FHWs available for CBMNH programming impacts the capacity for delivering services through the feasible increase in coverage (d). Since Somalia employs 1,450 FHWs, who are each responsible for roughly 200 households with an average of six people per household, the current level of employed FHWs can serve 1.74 million of the 17.6 million people in Somalia or 9.88% of the population. Thus, the model does not allow total coverage for each intervention to exceed current coverage plus d = 9.88%. The maximum total coverage levels for each intervention are listed in Table B.5 in Appendix B.1. There are plans within Somalia to grow the program by increasing the number of FHWs employed. In order to understand how this change will alter the optimal package of interventions and the number of lives saved, we perform sensitivity analysis in Section 2.3.2.

To calculate the number of treatments needed, as discussed in Section 2.2.2, there are six population groups that are designated for treatment. These target populations are supplied in LiST and shown in Table 2.7. In Table B.6 of Appendix B.1, we pair each intervention with its target population and provide estimates for the percentage of the target population meeting the criteria for each intervention.

Table 2.7: Target populations in Somalia

Target Population	Size
Pregnant Women	973,781
Live Births	716,518
Number of Households	2,935,990
Pregnancies Carried to Term	843,109
Children 0-59 months	3,036,352
Abortions	130,672

The values for the resource constraints of the three-dimensional vectors a and b are shown in Table 2.8. The calculations for these values are included in Appendix B.2. For the Somalia case, we have decided to remove the constraint regarding the cost to establish new interventions, and instead focus on the cost to scale interventions. Similarly, we would ordinarily provide the annual number of commodities attainable and the commodities required for full coverage for each intervention in Somalia, as shown in Equation (2.4). However, we do not have exact numbers for these values in time to include them in program planning. Instead, we elect to use a binary constraint for whether or not the commodities for each intervention were deemed available in stakeholder consultations. If these values do become available, the constraint can easily be adjusted to reflect this update. For now, the only intervention with a commodity that is unable to be provided in Somalia is calcium supplementation. Calcium supplementation is thus removed from consideration via this constraint.

Table 2.8: Constraints on Female Health Workers (FHWs)

Constraint	Budget
(a_1) Total time available per month for FHWs to perform MNH care	22,602 hours
(a_2) Total supervisor time per month available to oversee FHW in-service training	3,927 hours
(a_3) Total budget available for scaling interventions	\$ 435,000
(b_1) Total time available in FHW M&N curriculum	20 hours
(b_2) Total budget available for establishing new interventions	–
Allowable increase of intervention based on number of FHWs	9.88%

2.3 Results

2.3.1 Model Results

This section provides the model recommendations for the Somalia setting and explains their implications. We examine the resources used to achieve these levels of coverage, paying special attention to the commodities budget available for scaling the interventions, which serves as the binding constraint under the Somalia model parameters. We then relax the restrictions on permissible interventions and provide the optimal solution to the Somalia setting under these relaxations.

The model is a mixed-integer nonlinear program. We express the model in AMPL and use the KNitro solver to arrive at an optimal solution. We provide additional details about the performance of this optimization in Appendix B.4. As we hope there will be interest in adapting and scaling this modeling exercise to other settings, we plan to make this model available for use and welcome inquiries about the results of the model with other sets of parameters. The model results are displayed in Table 2.9.

Table 2.9: The model's selection of interventions for the Somalia program with 1,450 FHWs

Intervention	y_j	Coverage Change (x_j)	Coverage Target	Current Coverage
Tetanus toxoid vaccination	0		67%	67%
MM supplementation in pregnancy	0		0%	0%
Iron supplementation in pregnancy	1	4.15%	4.15%	0%
Calcium supplementation	0		0%	0%
Balanced energy supplementation	0		0%	0%
KMC - Kangaroo mother care	1	9.88%	14.88%	5%
Breastfeeding promotion	1	9.88%	18.88%	9%
Hand washing with soap	1	9.88%	19.88%	10%
Prevention of malaria in pregnancy	1	9.88%	11.88%	2%
Basic sanitation	1	9.88%	47.88%	38%
Clean birth environment	0		7%	7%
Clean cord care	1	9.88%	17.88%	8%
Immediate drying and stimulation	0		8%	8%
Neonatal resuscitation	0		4%	4%
Uterotonics for postpartum hemorrhage	0		7%	7%
Syphilis detection and treatment	0		4%	4%
Thermal regulations	1	9.88%	18.88%	9%
Manual removal of placenta	0		3%	3%
Point-of-use filtered water	0		4%	4%
Malaria case management	0		1%	1%
Injectable antibiotics for neonatal sepsis	0		9%	9%
Safe abortion services	0		3%	3%
Oral antibiotics for neonatal sepsis	0		0%	0%
Post-abortion case management	0		0%	0%
Antibiotics for treatment of dysentery	0		0%	0%

The optimal solution predicts 612 lives saved, with 604 neonatal and 8 maternal deaths averted. This result accounts for 16% of the predicted maternal and neonatal deaths in the Galmudug state in 2023. As indicated in Table 2.4 of Section 2.2.3, only nine interventions are feasible for FHWs to provide in Somalia based on policies currently in place. Eight of those feasible nine interventions are included in the optimal set of interventions. Iron supplementation is limited to only a 4.15% increase because of its relatively high cost per treatment. All other permissible interventions, except for calcium supplementation, are increased by 9.88%, which is the maximum allowable increase given the number of FHWs (d). The single binding constraint under these conditions is the commodity cost to scale up interventions. The amounts of each constrained resource used in the model solution are listed in Table 2.10.

Table 2.10: Constraint budget usage for the Somalia program

Constraint	Budget	Usage
Total time available per month for care	22,602 hours	13,588 hours
Total supervisor time per month for training	3,927 hours	713 hours
Total budget available for scaling interventions	$ 435,000	$ 435,000
Total time available in curriculum	1,200 mins	800 mins

The results in Table 2.9 are heavily influenced by national-level policies. To inform advocacy efforts or future projections, the restrictions within the model can be lifted to understand the impact on the number of lives saved of a currently excluded intervention, or package of interventions, should they become permissible by the MOH. As an example of this type of analysis, we lift the restrictions on most of the interventions through constraint B_3. In this expanded analysis we do not include the two interventions related to abortion services, as the procedure is illegal in Somalia, nor do we include manual removal of the placenta provided by FHWs, as the Somali Ministry of Health wants to encourage all births to take place at a health facility. The results from relaxing these restrictions, while keeping all other constraints as before, are shown in Table 2.11.

Table 2.11: The model's selection of interventions with relaxed restrictions for Somalia

Intervention	y_j	Coverage Change (x_j)	Coverage Target	Current Coverage
Tetanus toxoid vaccination	1	9.88%	76.88%	67%
MM supplementation in pregnancy	1	9.88%	9.88%	0%
Iron supplementation in pregnancy	0		0%	0%
Calcium supplementation	0		0%	0%
Balanced energy supplementation	1	9.88%	9.88%	0%
KMC - Kangaroo mother care	1	9.88%	14.88%	5%
Breastfeeding promotion	0		9%	9%
Hand washing with soap	0		10%	10%
Prevention of malaria in pregnancy	1	9.80%	11.80%	2%
Basic sanitation	1	9.88%	47.88%	38%
Clean birth environment	1	9.88%	16.88%	7%
Clean cord care	1	9.88%	17.88%	8%
Immediate drying and stimulation	1	9.88%	17.88%	8%
Neonatal resuscitation	0		4%	4%
Uterotonics for postpartum hemorrhage	0		7%	7%
Syphilis detection and treatment	0		4%	4%
Thermal regulations	1	9.88%	18.88%	9%
Manual removal of placenta	0		3%	3%
Point-of-use filtered water	0		4%	4%
Malaria case management	0		1%	1%
Injectable antibiotics for neonatal sepsis	1	9.88%	18.88%	9%
Safe abortion services	0		3%	3%
Oral antibiotics for neonatal sepsis	1	9.88%	9.88%	0%
Post-abortion case management	0		0%	0%
Antibiotics for treatment of dysentery	0		0%	0%

The objective function of this solution has a value of 1,990 projected lives saved or 48% of maternal and neonatal deaths in Galmudug state in 2023. This is roughly three times as many lives saved as the 612 in Table 2.9. The exchange of interventions between the two versions of the model is shown in Table 2.12. This expanded version maintains the budget to scale interventions as a binding constraint, while also including curriculum time available as an additional binding constraint. The resources used in this solution are shown in Table 2.13. The estimated increase in lives saved can provide valuable insights regarding how to prioritize future investments and policy efforts.

Table 2.12: Changes to the optimal package of interventions for the Somalia program

Removed	Added
Breastfeeding promotion	Tetanus toxoid vaccination
Hand washing with soap	Multiple micronutrient supplementation in pregnancy
Basic sanitation	Balanced energy supplementation
	Clean birth environment
	Immediate drying and stimulation
	Injectable antibiotics for neonatal sepsis
	Oral antibiotics for neonatal sepsis

Table 2.13: Constraint budget usage for expanded Somalia program

Constraint	Budget	Usage
Total time available per month for care	22,602 hours	16,828 hours
Total supervisor time per month for training	3,927 hours	1,184 hours
Total budget available for scaling interventions	$ 435,000	$ 435,000
Total time available in curriculum	1,200 mins	1,170 mins

Endless adaptations can be made to the model to reflect various situations. Another possibility in Appendix B.3 includes the three additional interventions shown in Table 2.11 but removes those that require injections. We consider this to be a more realistic policy, as injections can be beyond the skill set or comfort level of CHWs, and the proper storage of injections is often impractical in remote settings. This version of the model results in 1,335 lives saved, which still represents a more than doubling in the number of lives saved over the original results under the limited number of permitted interventions.

2.3.2 Sensitivity Analysis

After establishing the optimal package of interventions for the Somalia setting, given current resource constraints, we now show how this package and the resulting number of lives saved change as key parameters change. Sensitivity analysis is one of the benefits of the constrained optimization exercise, as it allows for an improved understanding of how responsive the results are to the selection of certain values in the model. Additionally, it provides the ability to project the expected impact of changes in the operating environment—for instance, shocks to the amount of CHW time available for CBMNH work, or cuts to the commodities budget if donor financing shifts. In this section, we focus specifically on the number of FHWs and the budget available, as they are the parameters deemed most likely to fluctuate in the near future in Somalia. We perform additional sensitivity analysis by varying other parameters in Appendix B.6.

In the following tables, we consider three different quantities of FHWs. We include the current level of FHWs (1,450) and increase this number to 3,000 and then 8,000, to match the projected growth targets for the FHW cadre in 2025 and 2030 respectively.

We alter the budget for commodities in two different ways. In Table 2.14, we provide the budget on a per-FHW basis. This convention assumes that the amount of money available for the program increases as the number of FHWs increases, and is consistent with how the size of community health budgets is often discussed. However, in practice, it is rarely the case that the budget and workforce grow in lockstep. As such, in Table 2.15 we treat the budget as independent of the number of FHWs and explore how four different budgets impact the number of lives saved. For the calculations in Tables 2.14 and 2.15, we use the original model assumptions used to produce the results shown in Table 2.9 and perform the optimization with the budget and FHW values shown.

Table 2.14: Projected lives saved when scaling budgets with number of health workers

	1,450 FHWs	3,000 FHWs	8,000 FHWs
$15/FHW	637	1,300	3,300
$25/FHW	651	1,326	3,361
$40/FHW	672	1,367	3,452

Table 2.15: Projected lives saved for fixed budgets

	1,450 FHWs	3,000 FHWs	8,000 FHWs
$250,000	637	1,217	2,928
$435,000	651	1,291	3,006
$750,000	676	1,315	3,137
$1,00,000	696	1,334	3,242

In Table 2.14, we see that the number of FHWs greatly impacts the number of lives saved. Interestingly, increasing the commodities budget only has a small effect on the number of lives saved. This result underlines the significance of the parameter d, which grows linearly with the number of FHWs. This same effect is repeated in the results shown in Table 2.15. Together, these analyses suggest that with the Somalia parameters used, and the current restrictions on the available interventions, growing the number of FHWs through recruitment and retention should be prioritized over increasing the commodities budget.

2.4 Discussion

The model, informed by the accompanying parameter gathering process, estimates that with 1,450 Female Health Workers (FHWs) the set of interventions selected can avert 16% (or 612) of maternal and neonatal deaths in the Galmudug state of Somalia in one year. The optimal set of interventions, given the resource limitations outlined in Section 2.2.3, includes eight of the nine interventions permitted to offer at the community level within Galmudug state.

Despite the existence of generalized recommendations for interventions to offer in community health programs, the implementation of these recommendations in individual settings requires careful consideration, particularly of the available resources for such programs. We believe there exists a need for a structured approach for adapting these broad policy objectives to country and sub-national levels. Our model, and the process undertaken to inform the model inputs, provides one example of such an approach. In particular, it provides a framework into which local expertise and perspectives can be readily incorporated. Additionally, if the program results are tracked after implementation, the model can serve as a reference to help understand where prior assumptions vary from the on-the-ground realities. In this way, it can help diagnose program issues, particularly instances where outcomes may differ from expectations.

The model presented in this paper is adaptable for several purposes. It can be used in the process of starting a new community MNH program within a country, as demonstrated in Galmudug, Somalia. If such a program already exists in the intended setting, it can serve as a means to assess the current services offered and identify potential changes. Furthermore, it can function as a support tool to advocate for interventions not currently offered, specifically if the interventions are projected to lead to additional lives saved and the available resources are better used for those purposes.

Using a similar approach, we can test shocks to the program, such as budget changes or the redirecting of resources (see Section 2.3.2). In doing so, we can provide options for interventions to include in normal operating times so that during such disruptions, the program is still able to function at some capacity and is not crippled by events such as a conflict or epidemic. This ability to understand how the program performs in multiple scenarios helps elucidate how the prioritized package of interventions would change with variations in the model parameters and emphasizes the strength of this modeling exercise, especially for settings with operational challenges and uncertainty.

Although we highlight the potential benefits we foresee in using the proposed model, it is important to acknowledge its limitations. As with any model, the quality of the results depends on the accuracy of the data used. Specific to our program, LiST uses national-level demographic surveys from conflict-affected countries, of which Somalia is one, where we know these methodologies rarely, if ever, capture the realities. For Somalia in particular, at the time of the exercise, LiST was using national-level estimates, and IRC had no subnational mortality or demographic information to use for Galmudug specifically. The accuracy of our model results thus depends on how closely the country-wide mortality and disease incidence rates approximate those of Galmudug.

With our reliance on LiST, our decision variables were selected because LiST identifies them as incrementally contributing to the lives saved calculation. Notably, this criterion leaves out the routine service delivery systems through which these interventions are delivered. The model is thus focused on specific interventions and not on the program infrastructures in which they exist. For instance, many of these interventions can only be delivered within a system that provides routine antenatal care (ANC) and prenatal care (PNC). Those ANC visits themselves are not qualified in LiST as a lifesaving intervention and therefore are not specifically identified by the model. Yet, it will be imperative that any policymaker using this model to design a package acknowledges routine service delivery platforms (like ANC and PNC visits) that offer care-seeking counseling, behavioral counseling, health literacy and promotion messages –in addition to the interventions selected by the model. Plainly, the model does not consider all aspects of a program and is only looking at the incremental resources required by the interventions which are captured as decision variables.

The model is currently designed with the objective of optimizing lives saved, policymakers who wish to optimize some other health outcome or metric could still use this model, but would need to specify a new objective function to capture this focus. This would also require an additional database that contains values measuring how increases in coverage translate into gains in the objective function. Additionally, we list as one of the strengths of this modeling process the ability to track the program prescribed by the model and gain insights from the deviation of projections by the model from the operating realities. Although the reexamination of the model is a useful exercise, it does require resources and data tracking which presents its own set of difficulties, and which may only be available in limited settings. There is a legitimate question about whether follow-up evaluation is the best use of funds for organizations using the model, and these additional benefits may not be realized in all settings.

Lastly, any type of modeling is limited and does not capture all the considerations in the setting where it will be applied. Adjustments will need to be made to translate this model to the community level. Furthermore, skepticism often abounds when attempting to apply mathematical modeling to established practices and systems. It is thus vital to connect users with the process from the start, not least because local stakeholders will be a critical source of information on resource constraints which are rarely captured elsewhere. Correctly framing the process and results as not telling those involved what to do, but instead providing guidance to aid in their decision-making is essential.

2.5 Conclusion

In this paper, we provide a model for strategically selecting community health interventions in resource-constrained environments and use the Somalia FHW program as the model's first application. We rely on the Lives Saved Tool as a data source to calculate our objective of maximizing the number of maternal and neonatal lives saved and consulted with Somali health experts to understand the resources necessary to deliver those interventions in this context. We use the model to identify the optimal combination of interventions under a set of constraints selected and adapted to the Somalia setting by this paper's authors, with significant input from practitioners within the Somali health system.

The package of CBMNH services which could avert the greatest number of lives saved in Somalia, given current constraints, includes eight interventions. The commodities budget available to scale up interventions is the binding constraint within the model. With a significant number of interventions excluded from the original analysis due to country-specific policy restrictions, we solve a version of the model with an expanded list of interventions to show the increase in the number of lives saved if additional interventions are considered. We also identify which interventions would be excluded from the package of care to incorporate these new interventions. We finally show how the number of lives saved in Somalia is projected to change if the program budget and the number of FHWs change. This analysis suggests that increasing the number of FHWs is one of the best uses of resources for increasing the number of lives saved.

Relying on the work of our in-country partners, the program prescribed by our model is expected to launch in 2023. We plan to perform the same modeling exercise in other country contexts, using the same general model from Section 2.2.2 as our starting point. We believe that this type of analysis can play a valuable role in program design and reevaluation of community health programs.

2.6 Acknowledgements

The participants in the Constrained Optimization Workshop in Mogadishu, Somalia, held October 18-20, 2022 are listed in Table 2.16. This work would not have been possible without their expertise and input into the process.

Table 2.16: Constrained Optimization Workshop attendee list

Participant	Position at the Time of Workshop
Dr. Ubah Farah	Director of Family Health Department
Abdisalaam abdullahi mohamud	Director Community Health
Dr. Abdulkadir Wehliye	Director of medical services
Yusuf Omar	Head of Supply chain
Nur Ali Mohamud	PCIU Senior Programme Coordinator, World Bank
Abdihakim Mohamed Diriye	Director of Policy and Planning
Dr. Yahye Shoole	Director
Dr Ibrahim Guled	Secretary General
Abukar Mohamud	IRC Deputy Director of Programs
Abdijamal Mire	IRC Dhusamreed Field Coordinator
Muna Jama	IRC Research Project Coordinator
Elaine Scudder	IRC Maternal & Newborn Health Advisor
Caitlin Tulloch	IRC Directory of Best Use of Resources

Appendix A

Appendices of Chapter 1

A.1 Auxiliary K-Convexity Results

In this Appendix, we provide the K-convexity results used to prove Theorems 1 and 2 in Appendix A.2. The contents of the Appendix include Definition 2 and Lemmas 1-3.

Definition 2. A function $f:(-\infty,L]\to\mathbb{R}$ is K-convex if $K\geq 0$ and for each $x,y\in(-\infty,L]$ with $x\leq y$, $0\leq\theta\leq 1$ and $\bar{\theta}:=1-\theta$, it satisfies

$$f(\theta x+\bar{\theta}y)\leq \theta f(x)+\bar{\theta}[K+f(y)]. \tag{A.1}$$

Lemmas 1 and 3 will help facilitate the proofs of our main results. In particular, part c) of Lemma 1 provides the basis for the optimality of an (s,S) policy.

Lemma 1. *Given $L<\infty$, we have the following:*

a. A real-valued convex function f on $(-\infty,L]$ is K-convex for all $K\geq 0$.

b. If f_i is K_i-convex with $K_i\geq 0$, $i=1,2$, then $\alpha f_1+\beta f_2$ is $(\alpha K_1+\beta K_2)$-convex for $\alpha\geq 0$, $\beta\geq 0$.

c. If f is a continuous K-convex function on $(-\infty,L]$ and $f(y)\to\infty$ as $y\to-\infty$, then there exists $s,S\in(-\infty,L]$ with $s<S$ such that the following hold:

i. $f(S) \leq f(y) \quad \forall y \in (-\infty, L]$;

ii. $f(S) + K = f(s) < f(y) \quad \forall y < s$;

iii. $f(y)$ *is decreasing on* $(-\infty, s)$;

iv. $f(y) \leq f(z) + K \quad \forall y, z$ *with* $s \leq y \leq z$.

Proof. Parts a) and b) are immediate consequences of the definition of K-convexity. So, we focus on proving part c). There exists a minimizer of f, because it is continuous and $f(y) \to \infty$ as $y \to -\infty$. Let S be such a point. Also, let s be the smallest value z such that $z \leq S$ and $f(S) + K = f(z)$. For all $y < s$, we have the following from the definition of K-convexity (by applying it at y, s, and S such that $\theta y + \bar{\theta} S = s$):

$$f(s) \leq \theta f(y) + \bar{\theta}[K + f(S)] = \theta f(y) + \bar{\theta} f(s),$$

where the equality follows because $f(s) = K + f(S)$. From this, we conclude that $f(y) \geq f(s)$ for $y < s$. Moreover, it follows from the definition of s that $f(y) \neq f(S) + K = f(s)$ for $y < s$. Combining these, we conclude $f(y) > f(s)$ for $y < s$, which completes the proof of part c-ii).

In order to prove part c-iii), consider $y_1 < y_2 < s$. We wish to prove $f(y_1) \geq f(y_2)$. We have the following from the definition of K-convexity (by applying it at y_1, y_2 and S such that $y_2 = \theta y_1 + \bar{\theta} S)$:

$$f(y_2) \leq \theta f(y_1) + \bar{\theta}[K + f(S)] = \theta f(y_1) + \bar{\theta} f(s).$$

Because $f(y_1) > f(s)$ by part c-ii), we deduce from the preceding inequality that $f(y_2) < f(y_1)$ as desired.

In order to prove part c-iv), we note that it holds for $y = z$ as well as $y = S$ or $y = s$. There remain two cases to consider:

Case 1: $S < y$,

Case 2: $s < y < S$.

In case 1, we have that $z > y > S$. It follows from the definition of K-convexity (by applying it at S, y, and z and θ such that $\theta S + \bar{\theta} z = y$) that

$$f(y) \leq \theta f(S) + \bar{\theta}(f(z) + K) \leq \theta f(y) + \bar{\theta}(f(z) + K,$$

where the second inequality follows because $f(y) \geq f(S)$, then we conclude $f(y) \leq f(z) + K$ as desired in this case.

In case 2, we have that $s < y < z$. It follows from the definition of K-convexity (applying it at s, y, z such that $y = \theta s + \bar{\theta} z$) that

$$\begin{aligned} f(y) &\leq \theta f(s) + \bar{\theta}(f(z) + K) \\ &= \theta(f(S) + K) + \bar{\theta}(f(z) + K) \\ &\leq \theta(f(z) + K) + \bar{\theta}(f(z) + K) \\ &= f(z) + K, \end{aligned}$$

where the first equality follows because $f(s) = f(S) + K$, and the second inequality follows because $f(S) \leq f(z)$. ∎

We next consider a function $f : (-\infty, L] \to \mathbb{R}$ that is continuous and K-convex with $\lim_{y \to -\infty} f(y) = \infty$. We let

$$S \in \arg\min_{y \in (-\infty, L]} f(y), \tag{A.2}$$

and $s \in (-\infty, L]$ be the smallest value such that $f(s) = f(S) + K$. Lemma 1 ensures the existence of such an $s < S$. Note from part c-iv) of Lemma 1 that for a, z such that $s \leq a \leq z$,

$$f(a) \leq f(z) + K. \tag{A.3}$$

Given $a \geq s$, we define the piece-wise function g using the K-convex function f as follows:

$$g(x) = \begin{cases} f(a), & x \leq a, \\ f(x), & x \in [a, L]. \end{cases} \tag{A.4}$$

Lemma 3 establishes the K-convexity of the function g. Its proof involves studying several different cases. Next, we introduce auxiliary functions h_1 and h_2, and prove the auxiliary Lemma 2, which facilitates the proof of Lemma 3, see case 3b of the proof of Lemma 3. To this end, fix x, y such that $x < a < y$, and define functions h_1 and h_2 as follows:

$$h_1(z) = \frac{y-z}{y-x} f(a) + \frac{z-x}{y-x}[K + f(y)], \quad \text{for } z \in [x, y],$$
$$h_2(z) = \begin{cases} f(a), & \text{if } z \leq a, \\ \frac{y-z}{y-a} f(a) + \frac{z-a}{y-a}[K + f(y)], & \text{if } z \in [a, y]. \end{cases}$$

Functions h_1 and h_2 are depicted in Figure A.1 for the case of $x < a < z < y$, which corresponds to case 3b of Lemma 3.

Lemma 2. *Given $x < a < y$, we have that $h_1(z) \geq h_2(z)$ for $z \in [x, y]$.*

Proof. First, consider the case $z \leq a$, and note in this case that

$$h_1(z) = \frac{y-z}{y-x} f(a) + \frac{z-x}{y-x}[K + f(y)], \tag{A.5}$$
$$h_2(z) = f(a). \tag{A.6}$$

Moreover, because $s \leq a < y$ and f is K-convex, we conclude from part c-iv) of Lemma 1 that $f(a) \leq f(y) + K$. Combining this with Equations (A.5) and (A.6), we conclude that $h_1(z) \geq h_2(z)$. Note, in particular, that

$$h_1(a) \geq h_2(a) = f(a). \tag{A.7}$$

When $z > a$, we have by linearity of h_1 that

$$h_1(z) = \frac{y-z}{y-a} h_1(a) + \frac{z-a}{y-a}[K + f(y)]. \tag{A.8}$$

Also, by definition of h_2, we have that

$$h_2(z) = \frac{y-z}{y-a} f(a) + \frac{z-a}{y-a}[K + f(y)]. \tag{A.9}$$

Then we conclude from Equations (A.7)-(A.9) that $h_2(z) \geq h_1(z)$ in this case too. ■

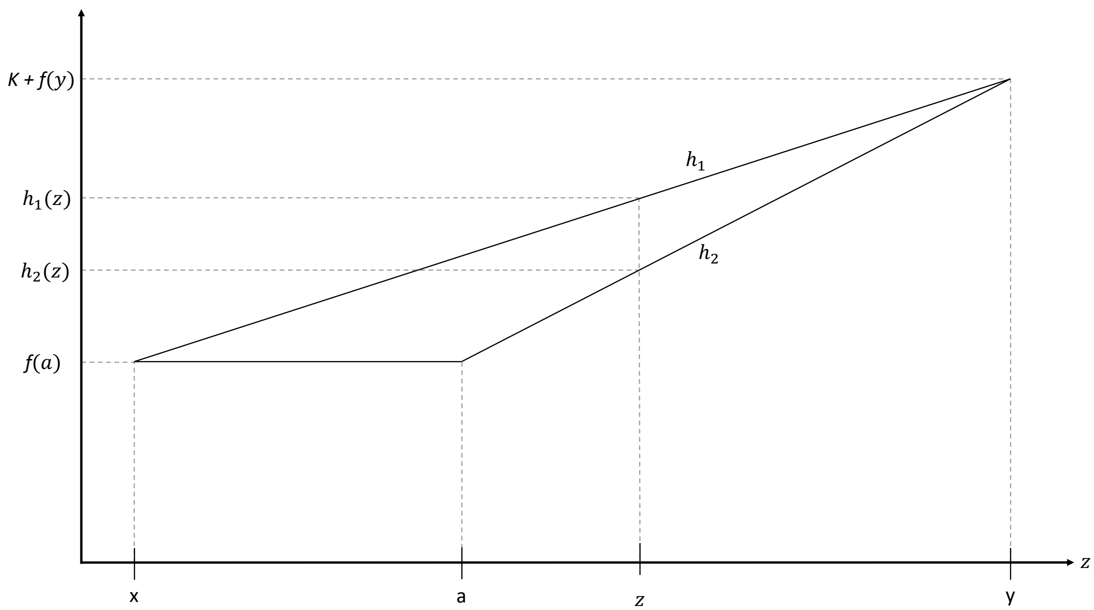

Figure A.1: Illustrative h_1 and h_2 functions for the case of $x < a < z < y$.

Lemma 3. *We have that g is a continuous, K-convex function on $(-\infty, L]$.*

Proof. For $x \leq y \leq L$, $\theta \in [0,1]$ and $\bar{\theta} = 1 - \theta$, we wish to verify

$$g(\theta x + \bar{\theta} y) \leq \theta g(x) + \bar{\theta}[K + g(y)]. \tag{A.10}$$

There are three cases to consider:

Case 1: $x \leq y \leq a$. When $x \leq y \leq a$, $g(x) = g(y) = f(a)$ and the result follows.

Case 2: $a \le x \le y$. When $a \le x \le y$, Equation (A.10) reduces to Equation (A.1); and the result follows from K-convexity of f.

Case 3: $x < a < y$. We have two subcases to consider:

Case 3a: $x < \theta x + \bar{\theta} y < a$

Case 3b: $x < a < \theta x + \bar{\theta} y$

Let's first consider case 3a. In this case, Equation (A.10) reduces to $f(a) \le \theta f(a) + \bar{\theta}[K + f(y)]$. Equivalently, we wish to prove $f(a) \le K + f(y)$, which holds by Equation (A.3).

Now, we consider case 3b. Using Lemma 2, we verify Equation (A.10) for case 3b) as follows:

$$
\begin{aligned}
g(\theta x + \bar{\theta} y) &= f(\theta x + \bar{\theta} y) \\
&\le \frac{\theta(y-x)}{y-a} f(a) + \left[1 - \frac{\theta(y-x)}{y-a}\right][f(y) + K] \\
&= h_2(\theta x + \bar{\theta} y) \\
&\le h_1(\theta x + \bar{\theta} y) \\
&= \theta f(a) + (1-\theta)[f(y) + K] \\
&= \theta g(x) + (1-\theta)[g(y) + K],
\end{aligned}
$$

where the first inequality follows from K-convexity of f applied at the points a, $\theta x + \bar{\theta} y$, and y. The second inequality follows from Lemma 2. The other steps follow from the definitions of h_1, h_2, and g, completing the proof. ■

A.2 A Finite Horizon Formulation

In this Appendix, we consider a finite-horizon version of the model introduced in Section 1.3. Indeed, the only difference between the two formulations is that here the planning horizon is $\{1, ..., \tau\}$, in contrast to the infinite horizon formulation of Section 1.3. Therefore, we omit some of the explanation assuming that the reader has first read through Section 1.3.

Letting $V_k(x,e)$ denote the expected discounted cost over periods $k, k+1, ..., \tau$ under the optimal policy starting in state (x,e), we write down the Bellman equation in the finite horizon case as follows: for $x \in (-\infty, N]$ and $k = 1, ..., \tau$

$$V_k(x,A) = \min_{u \in [0,N-x]} \{C(u,A) + H(x+u) + \alpha \sum_{i=1}^{E} P^k_{AB_i} V_{k+1}(x+u-1, B_i) + \alpha P^k_{AA} V_{k+1}(x+u-1, A)\}, \tag{A.11}$$

$$V_k(x,B_e) = \min_{u \in [0,N-x]} \{C(u,B_e) + H(x+u) + \alpha P^k_{B_eA} V_{k+1}(x+u-1, A) + \alpha \sum_{i=1}^{E} P^k_{B_eB_i} V_{k+1}(x+u-1, B_i)\}, \tag{A.12}$$

subject to the terminal condition that $V_{\tau+1}(x,e) = 0$ for $x \in (-\infty, N]$ and $e \in \{A, B_1, ..., B_E\}$.

As done in section 1.3, we introduce the variable $y = x + u$, which denotes the inventory level after the order is received but before demand is met. Then we define the auxiliary functions $W_k(\cdot, A)$ and $W_k(\cdot, B_e)$ for $k = 1, ..., \tau$ as follows:

$$W_k(y,A) = H(y) + \alpha \sum_{i=1}^{E} P^k_{AB_i} V_{k+1}(y-1, B_i) + \alpha P^k_{AA} V_{k+1}(y-1, A),$$

$$W_k(y,B_e) = c_e y + H(y) + \alpha P^k_{B_eA} V_{k+1}(y-1, A) + \alpha \sum_{i=1}^{E} P^k_{B_eB_i} V_{k+1}(y-1, B_i).$$

The next proposition expresses the value functions in terms of the auxiliary functions $W_k(y,A)$ and $W_k(y,B_e)$.

Proposition 5. *For $k = 1, ..., \tau$ and $x \in (-\infty, N]$, we have that*

$$V_k(x,A) = \min \left\{W_k(x,A), \min_{x<y\le N} [K + W_k(y,A)]\right\},$$

$$V_k(x,B_e) = \min_{x\le y\le N} \{W_k(y,B_e)\} - c_e x.$$

Proof. We write from Equation (A.11) that

$$\begin{aligned} V_k(x,A) &= \min_{u \in [0,N-x]} \left\{C(u,A) + H(x+u) + \alpha \mathbb{E}_A[V_{k+1}(x+u-1, e_{k+1})]\right\} \\ &= \min_{u \in [0,N-x]} \left\{C(u,A) + W_k(x+u,A)\right\} \end{aligned}$$

$$= \min_{y \in [x,N]} \left\{ C(u,A) + W_k(y,A) \right\}$$

$$= \min \left\{ W_k(x,A), \min_{x<y\le N} \{K + W_k(y,A)\} \right\}.$$

Similarly, we write from Equation (A.12) that

$$V_k(x,B_e) = \min_{u \in [0,N-x]} \left\{ c_e u + H(x+u) + \alpha \mathbb{E}_{B_e}[V_{k+1}(x+u-1, e_{k+1})] \right\}$$

$$= \min_{x \le y \le N} \left\{ c_e y - c_e x + H(y) + \alpha \mathbb{E}_{B_e}[V_{k+1}(y-1, e_{k+1})] \right\}$$

$$= \min_{x \le y \le N} \left\{ W_k(y,B_e) \right\} - c_e x.$$

■

The next definition introduces the finite horizon hybrid policy.

Definition 3. *A finite-horizon hybrid policy.* Given parameters $(s_A^k, S_A^k, S_{B_1}^k, ..., S_{B_E}^k)$ for $k = 1, ..., \tau$, we call a policy $\{u_k(x,e) : k = 1, ..., \tau\}$ a finite-horizon hybrid policy if the following holds for $k = 1, ..., \tau$:

$$u_k(x,A) = \begin{cases} S_A^k - x, & \text{if } x < s_A^k, \\ 0, & \text{otherwise,} \end{cases} \qquad u_k(x,B_e) = \begin{cases} S_{B_e}^k - x, & \text{if } x < S_{B_e}^k, \\ 0, & \text{otherwise,} \end{cases}$$

where $u_k(x,e)$ denotes the order quantity in period k as a function of the inventory level x and the environment state e.

The next result establishes the optimality of the finite-horizon hybrid policy.

Theorem 2. *For $k = 1, ..., \tau$, we have a solution for the Bellman equation such that $V_k(\cdot, A)$, $V_k(\cdot, B_e)$, $W_k(\cdot, A)$ and $W_k(\cdot, B_e)$ are continuous and K-convex with $\lim_{y\to-\infty} V_k(y,A) < \infty$ and*

$$\lim_{y\to-\infty} W_k(y,A) = \lim_{y\to-\infty} W_k(y,B_e) = \lim_{y\to-\infty} V_k(y,B_e) = \infty \text{ for } e = 1, ..., E.$$

Moreover, a finite-horizon hybrid policy with parameters $(s_A^k, S_A^k, S_{B_e}^k)$ for $k = 1, ..., \tau$ is opti-

mal, where

$$S_e^k \in \arg\min_{x \le N} W_k(x,e) \textit{ for } e \in \{A, B_1, ..., B_E\}$$

and s_A^k *is the smallest value* $z \le N$ *such that* $W_k(z,A) = K + W_k(S_A^k, A)$.

Proof. We proceed by induction. For the induction basis, we consider $i = \tau$. For the induction step, we assume the result holds for $i = k+1, ..., \tau$ $(i \ge 1)$ and show that it also holds for $i = k$.

In order to establish the induction basis, we begin by noting that $W_\tau(y,A) = H(y) = p(y-1)^- + h(y-1)^+$. Thus, $W_\tau(\cdot, A)$ is continuous and convex. Hence, it is K-convex. We also have that $\lim_{y \to -\infty} W_\tau(y,A) = \infty$. Similarly, we note $W_\tau(y,B_e) = c_e y + H(y)$, which is continuous and convex. Hence, by Lemma 1a) it is K-convex. Moreover, because $p > c_e$, we have that $\lim_{y \to -\infty} W_\tau(y,B_e) = \infty$.

Next, we consider $V_\tau(\cdot, A)$. Recall that

$$V_\tau(x,A) = \min\{W_\tau(x,A), \min_{x < y \le N}[K + W_\tau(x,A)]\}.$$

By part c) of Lemma 1, we set $S_A^\tau \in \arg\min_{y \le N} W_\tau(y,A)$ and we set s_A^τ as the smallest value of y for which $W_\tau(y,A) = K + W_\tau(S_A^\tau, A)$. Then, also by part c) of Lemma 1, we conclude that

$$V_\tau(x,A) = \begin{cases} K + W_\tau(S_A^\tau, A), & x < s_A^\tau, \\ W_\tau(x,A), & x \ge s_A^\tau, \end{cases} \tag{A.13}$$

which also proves that the order quantity $u_\tau(x,A)$ is optimal, where

$$u_\tau(x,A) = \begin{cases} S_A^\tau - x, & \text{if } x < s_A^\tau \\ 0, & \text{otherwise.} \end{cases}$$

Because $W_\tau(s_A^\tau, A) = K + W_\tau(S_A^\tau, A)$ by construction, we conclude that

$$V_\tau(x,A) = \begin{cases} W_\tau(s_A^\tau, A), & x < s_A^\tau, \\ W_\tau(x,A), & x \geq s_A^\tau, \end{cases} \tag{A.14}$$

from which the continuity of $V_\tau(\cdot, A)$ is immediate. Moreover, because $W_\tau(\cdot, A)$ is K-convex, we apply Lemma 3 with $a = s_A^\tau$ to deduce $V_\tau(\cdot, A)$ is K-convex. By Equation (A.14), $V_\tau(\cdot, A)$ is constant for $x \leq s_A^\tau$. Thus, $\lim_{y\to-\infty} V_\tau(y,A) < \infty$.

Now, we turn to $V_\tau(\cdot, B_e)$. Recall that

$$V_\tau(x, B_e) = \min_{x\leq y\leq N} W_\tau(y, B_e) - c_e x. \tag{A.15}$$

Let's first consider the first term on the right-hand side of (A.15). To this end, we define

$$\tilde{V}_\tau(x, B_e) = \min_{x\leq y\leq N} W_\tau(y, B_e), \quad x \leq N. \tag{A.16}$$

Because $W_\tau(\cdot, B_e)$ is continuous and K-convex on $(-\infty, N]$ with $\lim\limits_{y\to-\infty} W_\tau(y, B_e) = \infty$, by Lemma 1, we set

$$S_{B_e}^\tau \in \operatorname*{arg\,min}_{x\leq y\leq N} W_\tau(y, B_e).$$

Then it follows that

$$\tilde{V}_\tau(x, B_e) = \begin{cases} W_\tau(S_{B_e}^\tau, B_e), & x < S_{B_e}^\tau, \\ W_\tau(x, B_e), & x \geq S_{B_e}^\tau, \end{cases} \tag{A.17}$$

which proves that the order quantity $u_\tau(x, B_e)$ is optimal, where

$$u_\tau(x, B_e) = \begin{cases} S_{B_e}^\tau - x, & \text{if } x < S_{B_e}^k, \\ 0, & \text{otherwise.} \end{cases}$$

Equation (A.17) also proves that $\tilde{V}_\tau$ is continuous. Because $W_\tau(\cdot, B_e)$ is continuous and K-convex, and $S_{B_e}^\tau$ is its minimizer, we conclude from Lemma 3 and Equation (A.17) that $\tilde{V}_\tau$ is

K-convex. To be specific, we apply Lemma 3, with $a = S^{\tau}_{B_e}$. Clearly, $\tilde{V}_{\tau}(x, B_e)$ is constant for $x \le S^{\tau}_{B_e}$. Thus, $\lim_{y\to-\infty} \tilde{V}_{\tau}(x, B_e) < \infty$. Combining these observations with Equations (A.15)-(A.16), we conclude that $V_{\tau}(\cdot, B_e)$ is continuous, K-convex with $\lim_{y\to-\infty} V_{\tau}(y, B_e) = \infty$.

Next, we turn to the induction step and assume for $i = k+1, ..., \tau$ that $V_i(\cdot, A)$, $V_i(\cdot, B_e)$, $W_i(\cdot, A)$ and $W_i(\cdot, B_e)$ are continuous and K-convex with $\lim_{y\to-\infty} V_i(y, A) < \infty$ and

$$\lim_{y\to-\infty} W_i(y, A) = \lim_{y\to-\infty} W_i(y, B_e) = \lim_{y\to-\infty} V_i(y, B_e) = \infty \text{ for } B_e = B_1, ..., B_E. \tag{A.18}$$

We also assume that a finite-horizon hybrid policy with parameters $(s^i_A, S^i_A, S^i_{B_e})$ for $i = k+1, ..., \tau$ is optimal for the problem over the period $i = k+1, ..., \tau$.

Now, we want to show that $V_k(\cdot, A)$, $V_k(\cdot, B_e)$, $W_k(\cdot, A)$ and $W_k(\cdot, B_e)$ are continuous and K-convex with $\lim_{y\to-\infty} V_k(y, A) < \infty$, and that

$$\lim_{y\to-\infty} W_k(y, A) = \lim_{y\to-\infty} W_k(y, B_e) = \lim_{y\to-\infty} V_k(y, B_e) = \infty.$$

In accordance with Definition 3 we also want to prove that there exist parameters $(s^k_A, S^k_A, S^k_{B_e})$ such that the order quantity $u_k(x, e)$ is optimal, where

$$u_k(x, A) = \begin{cases} S^k_A - x, & \text{if } x < s^k_A, \\ 0, & \text{otherwise,} \end{cases} \qquad u_k(x, B_e) = \begin{cases} S^k_{B_e} - x, & \text{if } x < S^k_{B_e}, \\ 0, & \text{otherwise.} \end{cases}$$

In order to prove this, we first consider $W_k(y, A)$ and note that

$$W_k(y, A) = H(y) + \alpha \sum_{i=1}^{E} P^k_{AB_i} V_{k+1}(y-1, B_i) + \alpha P^k_{AA} V_{k+1}(y-1, A), \quad y \le N. \tag{A.19}$$

Because $V_{k+1}(\cdot, A)$ and $V_{k+1}(\cdot, B_e)$ are K-convex by the induction hypothesis and H is convex, part b) of Lemma 1 yields that $W_k(\cdot, A)$ is αK-convex, and therefore also K-convex. Moreover, we conclude from Equations (A.18)-(A.19) that $\lim_{y\to-\infty} W_k(y, A) = \infty$. Next, we note that

$$W_k(y, B_e) = c_e y + H(y) + \alpha P^k_{B_e A} V_{k+1}(y-1, A) + \alpha \sum_{i=1}^{E} P^k_{B_e B_i} V_{k+1}(y-1, B_i), \quad y \le N. \tag{A.20}$$

As above, because $V_{k+1}(\cdot, A)$ and $V_{k+1}(\cdot, B_e)$ are K-convex by the induction hypothesis and the convexity of $cy + H(y)$, part b) of Lemma 1 implies that $W_k(y, B_e)$ is αK-convex. Hence, it is also K-convex. Moreover, because $p > c_e$, we conclude from Equations (A.18) and (A.20) that $\lim_{y \to -\infty} W_k(y, B_e) = \infty$. Next, we recall that

$$V_k(x, B_e) = \min_{x \le y \le N} W_k(y, B_e) - c_e x, \quad x \le N. \tag{A.21}$$

As done above, let's first consider the first term on the right-hand side of (A.21). To this end, let's define

$$\tilde{V}_k(x, B_e) = \min_{x \le y \le N} W_k(y, B_e) \quad \text{for } x \le N. \tag{A.22}$$

Because $W_k(\cdot, B_e)$ is continuous and K-convex on $(\infty, N]$ with $\lim_{y \to -\infty} W_k(y, B_e) = \infty$, by virtue of Lemma 1, we set

$$S_{B_e}^k \in \arg\min_{x \le y \le N} W_k(y, B_e).$$

Then it follows that

$$\tilde{V}_k(x, B_e) = \begin{cases} W_k(S_{B_e}^k, B_e), & x < S_{B_e}^k, \\ W_k(x, B_e), & x \ge S_{B_e}^k, \end{cases} \tag{A.23}$$

which proves that the order quantity $u_k(x, B_e)$ is optimal, where

$$u_k(x, B_e) = \begin{cases} S_{B_e}^k - x, & x < S_{B_e}^k, \\ 0, & \text{otherwise.} \end{cases}$$

Equation (A.23) also proves that $\tilde{V}_k$ is continuous. Because $W_k(\cdot, B_e)$ is continuous and K-convex, and $S_{B_e}^k$ is its minimizer, we conclude from Lemma 3 that $\tilde{V}_k$ is K-convex. Clearly, $\tilde{V}_k(x, B_e)$ is constant for $x \le S_{B_e}^k$. Thus, $\lim_{y \to -\infty} \tilde{V}_k(x, B_e) < \infty$. Combining these observations with Equations (A.21)-(A.22), we conclude that $V_k(\cdot, B_e)$ is continuous and K-convex with $\lim_{y \to -\infty} V_k(y, B_e) = \infty$.

Lastly, we consider $V_k(y, A)$, and recall that

$$V_k(x, A) = \min\{W_k(x, A), \min_{x<y\le N}[K + W_k(x, A)]\}.$$

By part c) of Lemma 1, we set $S_A^k \in \arg\min_{y\le N} W_k(y, A)$ and we set s_A^k as the smallest value of y for which $W_k(y, A) = K + W_k(S_A^k, A)$. Then, also by part c) of Lemma 1, we conclude that

$$V_k(x, A) = \begin{cases} K + W_k(S_A^k, A), & x < s_A^k, \\ W_k(x, A), & x \ge s_A^k, \end{cases} \tag{A.24}$$

which also proves that the order quantity $u_k(x, A)$ is optimal, where

$$u_k(x, A) = \begin{cases} S_A^k - x, & \text{if } x < s_A^k, \\ 0, & \text{otherwise.} \end{cases}$$

Because $W_k(s_A^k, A) = K + W_k(S_A^k, A)$ by construction, we conclude that

$$V_k(x, A) \begin{cases} W_k(s_k, A), & x < s_A^k, \\ W_k(x, A), & x \ge s_A^k. \end{cases}$$

The continuity of $V_k(\cdot, A)$ follows from this. Moreover, because $W_k(\cdot, A)$ is continuous and K-convex, we conclude from Lemma 3 that $V_k(\cdot, A)$ is K-convex. Clearly, $V_k(\cdot, A)$ is constant for $x \le s_A^k$. Thus, $\lim_{y\to-\infty} V_k(y, A) < \infty$. ■

A.3 Passing to the limit as $\tau \to \infty$

In this Appendix, we study the finite-horizon discounted formulation and its value function as the planning horizon gets longer, i.e. as $\tau \to \infty$. Going forward, we will attach a superscript τ to various quantities of interest in order to emphasize their dependence on the length τ of the planning horizon for the finite-horizon discounted formulation. The results established here help us prove our main result, Theorem 1.

In order to facilitate the analysis to follow, we let

$$U(x) = (N - x)c_{\max} + K + \frac{c_{\max} + K}{1 - \alpha} + \frac{hN}{1 - \alpha}, \quad x \le N.$$

Note that this is an upper bound on the infinite-horizon discounted cost of the following feasible policy: in all periods, order up to N. Recall that $c_{\max} = \max\{c_1, ..., c_E\}$.

It is more or less obvious that $V_k^\tau(x, e)$ is increasing in the length τ of the planning horizon and that $0 \le V_k^\tau(x, e) \le U(x)$ for $x \in (-\infty, N]$, $e = A, B_1, ..., B_E$, $k = 1, ..., \tau$ and $\tau \ge 1$. Thus, one can argue that $V_k^\tau(x, e)$ converges to a limit as $\tau \to \infty$, and that the limit for $k > j$ is the same as the limit for j whenever $k = j + l\pi$ for $l \ge 1$ because of the periodic structure. Therefore, it suffices to study this limit for $k = 1, ..., \pi$. We denote this limit by the vector-valued function $\bar{V}_k(x, e)$:

$$\bar{V}_k(x, e) = \left(\bar{V}_1(x, e), ..., \bar{V}_\pi(x, e)\right) \text{ for } x \in (-\infty, N],\ e = A, B_1, ..., B_E.$$

Then we define the auxiliary function $\bar{W}_k(x, e) = \left(\bar{W}_1(x, e), ..., \bar{W}_\pi(x, e)\right)$ as follows: For $k = 1, ..., \pi$

$$\bar{W}_k(x, A) = H(x) + \alpha \sum_{i=1}^{E} P_{AB_i}^k \bar{V}_{k+1}(x - 1, B_i) + \alpha P_{AA}^k \bar{V}_{k+1}(x - 1, A), \tag{A.25}$$

$$\bar{W}_k(x, B_e) = c_e x + H(x) + \alpha P_{B_e A}^k \bar{V}_{k+1}(x - 1, A) + \alpha \sum_{i=1}^{E} P_{B_e B_i}^k \bar{V}_{k+1}(x - 1, B_i). \tag{A.26}$$

The following result establishes that the convergence is uniform on compact sets.

Proposition 6. *We have for* $e = A, B_1, ..., B_E$, $k = 1, ..., \pi$, *and* $l \ge 1$ *that*

$$V_{k+l\pi}^\tau(\cdot, e) \to \bar{V}_k(\cdot, e) \quad \textit{uniformly on compact sets as } \tau \to \infty,$$

$$W_{k+l\pi}^\tau(\cdot, e) \to \bar{W}_k(\cdot, e) \quad \textit{uniformly on compact sets as } \tau \to \infty.$$

Proof. The uniform convergence of $V_{k+l\pi}^\tau(\cdot, e)$ follows from Dini's Theorem (see for example page 242 of Billingsley (1999)) and that $V_{k+l\pi}^\tau(x, e)$ increases to its limit, $\bar{V}_k(x, e)$, as $\tau \to \infty$.

The uniform convergence of $W^{\tau}_{k+l\pi}(\cdot,e)$ follows from its definition that

$$W^{\tau}_{k+l\pi}(x,A) = H(x) + \alpha \sum_{i=1}^{E} P^{k}_{AB_i} V^{\tau}_{k+1+l\pi}(x-1,B_i) + \alpha P^{k}_{AA} V^{\tau}_{k+1+l\pi}(x-1,A),$$
$$W^{\tau}_{k+l\pi}(x,B_e) = H(x) + \alpha P^{k}_{B_eA} V^{\tau}_{k+1+l\pi}(x-1,A) + \alpha \sum_{i=1}^{E} P^{k}_{B_eB_i} V^{\tau}_{k+1+l\pi}(x-1,B_i),$$

the uniform convergence of $V^{\tau}_{k+l\pi}(\cdot,e)$ and the definition of $\bar{W}_k(\cdot,e)$ see Equations (A.25) and (A.26). ∎

Proposition 7. *For $k = 1,...,\pi$, functions $\bar{V}_k$ and $\bar{W}_k$ are continuous, K-convex with* $\lim_{y\to-\infty} \bar{V}_k(y,A) < \infty$ *and* $\lim_{y\to-\infty} \bar{V}_k(y,B_e) = \lim_{y\to-\infty} \bar{W}_k(y,A) = \lim_{y\to-\infty} \bar{W}_k(y,B_e) = \infty$.

Proof. Consider $\bar{V}_k(\cdot,e)$ on $[-M,N]$ where $M > 0$ is an arbitrary number. Because $V^{\tau}_{k+l\pi}(\cdot,e)$ is continuous and by Proposition 6.

$$V^{\tau}_{k+l\pi}(\cdot,e) \to \bar{V}_k(\cdot,e) \text{ u.o.c. as } \tau \to \infty,$$

we conclude that $\bar{V}_k(\cdot,e)$ is continuous on $[-M,N]$. Because $M > 0$ was arbitrary, we conclude that $\bar{V}_k(x,e)$ is continuous on $(-\infty,N]$ for $e = A,B_1,...,B_E$. The continuity of $\bar{W}_k(\cdot,e)$ for $e = A,B_1,...,B_E$ follows similarly.

To prove $\bar{V}_k(\cdot,e)$ is K-convex, we need to show for every $x,y \in (-\infty,N]$ with $x \le y$, $0 \le \theta \le 1$ and $\bar{\theta} = 1-\theta$, the following holds:

$$\bar{V}_k(\theta x + \bar{\theta} y, e) \le \theta \bar{V}_k(x,e) + \bar{\theta}[K + \bar{V}_k(y,e)]. \tag{A.27}$$

Recall from Theorem 2 that $V^{\tau}_k(\cdot,e)$ is K-convex. Thus, we have that

$$V^{\tau}_k(\theta x + \bar{\theta} y, e) \le \theta V^{\tau}_k(x,e) + \bar{\theta}[K + V^{\tau}_k(y,e)] \tag{A.28}$$

Because $V^{\tau}_k(\cdot,e) \to \bar{V}_k(\cdot,e)$ as $\tau \to \infty$, by passing to the limit as $\tau \to \infty$ on both sides of Equation (A.28), we arrive at Equation (A.27). Note that K-convexity of $\bar{W}_k$ follows similarly.

Moreover, note that because $V^{\tau}_k(y,B_e)$ increases to its limit as $\tau \to \infty$ monotonically, we have that

$$\bar{V}_k(y, B_e) \ge V_1^\tau(y, B_e).$$

Then, passing to the limit as $y \to -\infty$ on both sides and using that $\lim_{y\to-\infty} V_k^\tau(y, B_e) = \infty$ by Theorem 2 we conclude that $\lim_{y\to-\infty} \bar{V}_k(y, B_e) = \infty$. The proofs of $\lim_{y\to-\infty} \bar{W}_k(y, A) = \lim_{y\to-\infty} \bar{W}_k(y, B_e) = \infty$ follow similarly.

To complete the proof, we next consider $\limsup_{y\to-\infty} \bar{V}_k(y, A)$. Let $y < 0$ and following the initial state (y, A) in the finite-horizon problem, we consider the policy that orders up to zero initially and follows the optimal policy thereafter. Comparing this policy with the optimal policy starting at (y, A), we conclude

$$V_1^\tau(y, A) \le K + \alpha \sum_{i=1}^{E} P_{AB_i}^k V_2^\tau(0, B_i) + \alpha P_{AA}^k V_2^\tau(0, A) \tag{A.29}$$

By virtue of passing to the limit on both sides of Equation (A.29) gives the following:

$$\bar{V}_k(y, A) \le K + \alpha \sum_{i=1}^{E} P_{AB_i}^k \bar{V}_{k+1}(0, B_i) + \alpha P_{AA}^k \bar{V}_{k+1}(0, A), \;\; y < 0.$$

Now, letting $y \to -\infty$ gives

$$\limsup_{y\to-\infty} \bar{V}_k(y, A) \le K + \alpha \sum_{i=1}^{E} P_{AB_i}^k \bar{V}_{k+1}(0, B_i) + \alpha P_{AA}^k \bar{V}_{k+1}(0, A) < \infty.$$

Note from part c-iii) of Lemma 1 and the K-convexity of $\bar{V}_k(\cdot, A)$ that $\bar{V}_k(y, A)$ is monotone decreasing for y sufficiently small. Thus, $\lim_{y\to-\infty} \bar{V}_k(y, A)$ exists and we conclude that

$$\lim_{y\to-\infty} \bar{V}_k(y, A) = \limsup_{y\to-\infty} \bar{V}_k(y, A) < \infty.$$

■

Proposition 8. *Functions $\bar{V}_k$ and $\bar{W}_k$ satisfy the following:*

$$\bar{V}_k(x, A) = \min\left\{\bar{W}_k(x, A), \min_{x<y\le N}[K + \bar{W}_k(x, A)]\right\}, \tag{A.30}$$

$$\bar{V}_k(x, B_e) = \min_{x\le y\le N} \left\{\bar{W}_k(y, B_e)\right\} - c_e x. \tag{A.31}$$

Proof. Fix $x < N$ and note for $k = 1, ..., \pi$, $e = A, B_1, ..., B_E$ that

$$V_k^\tau(y,e) \to \bar{V}_k(y,e) \text{ uniformly on } y \in [x,N] \text{ as } \tau \to \infty, \tag{A.32}$$

$$W_k^\tau(y,e) \to \bar{W}_k(y,e) \text{ uniformly on } y \in [x,N] \text{ as } \tau \to \infty. \tag{A.33}$$

Also, recall from the finite horizon analysis that

$$V_k^\tau(x,A) = \min\left\{W_k^\tau(x,A), \min_{x\le y\le N}[K + W_k^\tau(y,A)]\right\}, \tag{A.34}$$

$$V_k^\tau(x,B_e) = \min_{x\le y\le N}\left\{W_k^\tau(y,B_e)\right\} - c_e x. \tag{A.35}$$

Now, we note that because the convergence in Equations (A.32) and (A.33) is uniform over compact sets, we have

$$\min_{x\le y\le N}\{K + W_k^\tau(y,A)\} \to \min_{x\le y\le N}\{K + \bar{W}_k(y,A)\}, \tag{A.36}$$

$$\min_{x\le y\le N}\left\{W_k^\tau(y,B_e)\right\} \to \min_{x\le y\le N}\{\bar{W}_k(y,B_e)\}. \tag{A.37}$$

Then passing to the limit on both sides of Equations (A.34) and (A.35) and using Equations (A.36) and (A.37) gives the result. ■

A.4 Proofs of Results in Section 1.3

In this Appendix, we prove Proposition 1, Theorem 1, and Lemmas 4 and 5.

Proof of Proposition 1. We write from Equations (1.1)-(1.2) that: For $k = 1, ..., \pi - 1$

$$\begin{aligned}
J_k(x,A) &= \min_{u\in[0,N-x]}\Big\{C(u,A) + H(x+u) + \alpha\sum_{i=1}^{E} P_{AB_i}^k J_{k+1}(x+u-1,B_i) \\
&\quad + \alpha P_{AA}^k J_{k+1}(x+u-1,A)\Big\} \\
&= \min_{u\in[0,N-x]}\Big\{C(u,A) + G_k(x+u,A)\Big\} \\
&= \min\{G_k(x,A), \min_{x<y\le N}\{K + G_k(y,A)\}\}.
\end{aligned}$$

In addition, we have for $k = \pi$ that

$$\begin{aligned}
J_\pi(x, A) &= \min_{u \in [0, N-x]} \left\{ C(u, A) + H(x+u) + \alpha \sum_{i=1}^{E} P^\pi_{AB_i} J_1(x+u-1, B_i) + \alpha P^\pi_{AA} J_1(x+u-1, A) \right\} \\
&= \min_{u \in [0, N-x]} \left\{ C(u, A) + G_\pi(x+u, A) \right\} \\
&= \min\{ G_\pi(x, A), \min_{x<y\le N} \{ K + G_\pi(y, A) \} \}.
\end{aligned}$$

Similarly, we write from Equations (1.3)-(1.4) for $k = 1, ..., \pi - 1$ that

$$\begin{aligned}
J_k(x, B_e) &= \min_{u \in [0, N-x]} \{ c_e u + H(x+u) + \alpha P^k_{B_e A} J_{k+1}(x+u-1, A) \\
&\quad + \alpha \sum_{i=1}^{E} P^k_{B_e B_i} J_{k+1}(x+u-1, B_i) \} \\
&= \min_{x \le y \le N} \{ c_e y - c_e x + H(y) + \alpha P^k_{B_e A} J_{k+1}(x+u-1, A) + \alpha \sum_{i=1}^{E} P^k_{B_e B_i} J_{k+1}(x+u-1, B_i) \} \\
&= \min_{x \le y \le N} \{ G_k(y, B_e) \} - c_e x.
\end{aligned}$$

and for $k = \pi$ that

$$\begin{aligned}
J_\pi(x, B_e) &= \min_{u \in [0, N-x]} \{ c_e u + H(x+u) + \alpha P^\pi_{B_e A} J_1(x+u-1, A) + \alpha \sum_{i=1}^{E} P^\pi_{B_e B_i} J_1(x+u-1, B_i) \} \\
&= \min_{x \le y \le N} \{ c_e y - c_e x + H(y) + \alpha P^\pi_{B_e A} J_1(x+u-1, A) + \alpha \sum_{i=1}^{E} P^\pi_{B_e B_i} J_1(x+u-1, B_i) \} \\
&= \min_{x \le y \le N} \{ G_\pi(y, B_e) \} - c_e x.
\end{aligned}$$

■

Proof of Theorem 1. Recall the finite-horizon discounted formulation that is introduced and solved in Appendix A.2. Appendix A.3 studies its value function as the number of periods tends to infinity. Proposition 6 establishes that the limit functions $\bar{V}_k$ and $\bar{W}_k$ exist and are well-defined. Proposition 7 shows that these functions are continuous and K-convex with $\lim_{y \to -\infty} \bar{V}_k(y, A) < \infty$ and

$$\lim_{y \to -\infty} \bar{V}_k(y, B_e) = \lim_{y \to -\infty} \bar{W}_k(y, A) = \lim_{y \to -\infty} \bar{W}_k(y, B_e) = \infty.$$

Moreover, Propositions 1 and 8 show that functions $\bar{V}_k$ and $\bar{W}_k$ solve the Bellman Equations (1.1)-(1.4) for the infinite horizon discounted formulation setting. Therefore, $(J_k, G_k) = (\bar{V}_k, \bar{W}_k)$ for $k = 1, ..., \pi$ solve the Bellman equation and have the desired properties.

To complete the proof, we let $S_A^k = \inf \arg\min_{y \le N} G_k(y, A)$ and set s_A^k as the smallest value of y for which $G_k(y, A) = K + G_k(s_A^k, A)$ for $k = 1, 2, ..., \pi$. Then by part c) of Lemma 1 (see Appendix A.1), we conclude that

$$J_k(x, A) = \begin{cases} K + G_k(S_A^k, A), & x < s_A^k, \\ G_k(x, A), & x \ge s_A^k, \end{cases} \tag{A.38}$$

which proves the optimality of $u_k(x, A)$, where

$$u_k(x, A) = \begin{cases} S_A^k - x, & \text{if } x < s_A^k, \\ 0, & \text{otherwise.} \end{cases}$$

Equation (A.38) also proves that $J_k(x, A)$ is constant for $x < s_A^k$. Thus, we conclude that $\lim_{x \to -\infty} J_k(x, A) < \infty$. Similarly, letting $S_{B_e}^k = \inf \arg\min_{x \le N} G_k(x, B_e)$, it follows that

$$J_k(x, B_e) = \begin{cases} G_k(S_{B_e}^k, B_e), & x < S_{B_e}^k, \\ G_k(x, B_e), & x \ge S_{B_e}^k, \end{cases}$$

which proves the optimality of $u_k(x, B_e)$, where

$$u_k(x, B_e) = \begin{cases} S_{B_e}^k - x, & \text{if } x < S_{B_e}^k, \\ 0, & \text{otherwise.} \end{cases}$$

Therefore, the hybrid policy with parameters $(s_A^k, S_A^k, S_{B_1}^k, ..., S_{B_E}^k)$ for $k = 1, ..., \pi$ is optimal.

■

In the remainder of this section, we prove Lemmas 4 and 5. In turn, these two lemmas prove Proposition 2. Recall that Proposition 2 assumes the initial inventory amount is integer-valued. Then because demand is integer-valued (i.e. it is one in all periods), only the integer inventory state values are relevant. Thus, without loss of generality we assume that s_A^k, S_A^k, and $S_{B_e}^k$ for all $e \in \{1, ..., E\}$ are integer valued as done in Proposition 2.

Lemma 4. *When $p > c_{\max}$, where $c_{\max} := \max\{c_1, ..., c_E\}$, we have that $S_{B_e}^k \geq 1$ for all $e \in \{1, ..., E\}$ and $k \in \{1, 2, ..., \pi\}$.*

Proof. We proceed by contradiction. Suppose not, i.e. there exists some state B_ι and $\kappa \in \{1, ..., \pi\}$ for which $S_{B_\iota}^\kappa \leq 0$, and define the stopping time

$$T_1 = \inf\{t = \kappa + l\pi \geq 1;\ x_t = 0,\ e_t = B_\iota\}.$$

Using this stopping time, we construct an alternative policy described next. We denote the inventory level process under the alternative policy $\tilde{x}_t$.

The alternative policy is as follows: at period T_1, order one unit and incur the ordering cost c_ι. This means that $x_{T_1+1} = \tilde{x}_{T_1+1} - 1$. Thereafter, the alternative policy follows the optimal policy. Let

$$T_2 = \inf\{t > T_1 : \text{an order is placed under the optimal policy}\}.$$

At time T_2 the processes couple. In order to compare the costs of the two policies, we let $\tilde{J}(x, e)$ denote the cost of the alternative policy. We let $\hat{T} = T_2 - T_1$ and note that $\hat{T} \geq 1$. Then we write

$$\begin{aligned} J_k(0, B_\iota) - \tilde{J}_k(0, B_\iota) &\geq \mathbb{E}\Big[\sum_{t=1}^{\hat{T}} \alpha^t p - c_\iota\Big] \\ &\geq p - c_\iota \\ &> 0, \end{aligned}$$

where the expectation is taken over $\hat{T}$. The first inequality holds because i) during $[T_1, T_2]$ the system manager incurs an additional backlog penalty of p per period under the original policy, corresponding to the first term inside the expectation; ii) she incurs an additional ordering cost of c_ι at time T_1 under the alternative policy; and iii) the order cost she incurs at time T_2 under the optimal policy is greater than (if $e_{T_2} \neq A$) or equal to (if $e_{T_2} = A$) that order cost under the alternative policy. We ignore that order cost to arrive at the first inequality. The second inequality follows by ignoring the penalty costs except for the one at time T_1. The last inequality follows because $p > c_\iota$ and it contradicts the optimality of the original policy. ∎

Next, we prove that $s_A^k \geq 1$. This along with $S_{B_e}^k \geq 1$ for all e will ensure $x_t \geq 0$ almost surely under the optimal policy, see Proposition 2 and Corollary 1.

Lemma 5. *When $p > K$, we have that $s_A^k \geq 1$ for all $k \in \{1, ..., \pi\}$.*

Proof. We proceed by contradiction. Suppose not, i.e. there exists $\kappa \in \{1, ..., \pi\}$ with $s_A^k \leq 0$. Define the stopping time:

$$T_1 = \inf\{t = \kappa + l\pi \geq 0 : \ x_t = 0, \ e_t = A\}.$$

Using this stopping time, we construct an alternative policy described next. We denote the inventory level process under the alternative policy $\tilde{x}_t$.

The alternative policy is as follows: at period T_1, order one unit and incur the fixed ordering cost K. This means that $x_{T_1+1} = \tilde{x}_{T_1+1} - 1$. Thereafter, the alternative policy follows the optimal policy. Let

$$T_2 = \inf\{t > T_1 : \text{an order is placed under the optimal policy}\}.$$

At time T_2 the processes couple. In order to compare the costs of the two policies, we let $\tilde{J}(x, e)$ denote the cost of the alternative policy. We let $\hat{T} = T_2 - T_1$ and note that $\hat{T} \geq 1$. Then we write

$$J_k(0, A) - \tilde{J}_k(0, A) \geq \mathbb{E}\Big[\sum_{t=1}^{\hat{T}} \alpha^t p - K\Big]$$

$$\geq p - K$$

$$> 0,$$

where the expectation is taken over $\hat{T}$. The first inequality follows because i) during $[T_1, T_2]$ the system manager incurs an additional backlog penalty of p per period under the optimal policy, corresponding to the first term inside the expectation; ii) she incurs an additional order cost K at time T_1 under the alternative policy; and iii) the order cost she incurs at time T_2 under the original policy is greater than (if $e_{T_2} \neq A$) or equal to (if $e_{T_2} = A$) that under the alternative policy. We ignore that order cost which gives the first inequality. The second inequality follows because it ignores all penalty costs except the one in the first period. The last inequality follows because $p > K$ by assumption, which contradicts the optimality of the original policy. ∎

A.5 General Demand

In this Appendix, we relax the assumption of fixed demand and show that the hybrid policy remains optimal when demand is modeled as an integer-valued random variable with finite support. We follow similar steps to Appendices A.2-A.4 to show the optimal policy for the infinite horizon discounted cost model.

We let D_k denote the demand in period k $(k \geq 1)$ and assume that $\{D_k : k \geq 1\}$ is an i.i.d. sequence of (integer-valued) random variables. To be specific, we let $D = \{d_1, d_2, ..., d_M\}$, where $d_M > ... > d_2 > d_1 = 1$, denote the support of the demand distribution. We also let

$$q_l = \mathbb{P}[D_l = d_l] > 0, \;\; l = 1, ..., M.$$

The system inventory then evolves according to the equation:

$$x_{k+1} = x_k + u_k - D_k, \;\; k \geq 1. \tag{A.39}$$

The holding and backlogging costs are captured in the function:

$$\begin{aligned} H(y) &= \mathbb{E}[h(y - D_1)^+ + p(y - D_1)^-] \\ &= \sum_{i=1}^{m} q_i [h(y - d_i)^+ + p(y - d_i)^-]. \end{aligned} \tag{A.40}$$

As done in the fixed demand case, we let $J_k(x, e)$ denote the value function starting in state (x, e) and period $k \geq 1$. We note for $k = 1, ..., \pi$ that

$$J_k(x, e) = J_{k+\pi l}(x, e) \text{ for all } x, e \text{ and } l \geq 1. \tag{A.41}$$

We restrict our attention to $J_1, ..., J_\pi$, and solve the Bellman equation for $x \in (-\infty, N]$, discount rate α and $k = 1, 2, ..., \pi$.

$$\begin{aligned} J_k(x, A) = \min_{u \in [0, N-x]} &\Big\{ C(u, A) + H(x+u) + \alpha \sum_{i=1}^{E} P^k_{AB_i} \sum_{l=1}^{m} q_l J_{k+1}(x + u - d_l, B_i) \\ &+ \alpha P^k_{AA} \sum_{l=1}^{m} q_l J_{k+1}(x + u - d_l, A) \Big\}, \end{aligned} \tag{A.42}$$

$$\begin{aligned} J_\pi(x, A) = \min_{u \in [0, N-x]} &\Big\{ C(u, A) + H(x+u) + \alpha \sum_{i=1}^{E} P^\pi_{AB_i} \sum_{l=1}^{m} q_l J_1(x + u - d_l, B_i) \\ &+ \alpha P^\pi_{AA} \sum_{l=1}^{m} q_l J_1(x + u - d_l, A) \Big\}, \end{aligned} \tag{A.43}$$

$$\begin{aligned} J_k(x, B_e) = \min_{u \in [0, N-x]} &\Big\{ C(u, B_e) + H(x+u) + \alpha P^k_{B_e A} \sum_{l=1}^{m} q_l J_{k+1}(x + u - d_l, A) \\ &+ \alpha \sum_{i=l}^{E} P^k_{B_e B_i} \sum_{l=1}^{m} q_l J_{k+1}(x + u - d_l, B_i) \Big\}, \end{aligned} \tag{A.44}$$

$$\begin{aligned} J_\pi(x, B_e) = \min_{u \in [0, N-x]} &\Big\{ C(u, B_e) + H(x+u) + \alpha P^\pi_{B_e A} \sum_{l=1}^{m} q_l J_1(x + u - d_l, A) \\ &+ \alpha \sum_{i=l}^{E} P^\pi_{B_e B_i} \sum_{l=1}^{m} q_l J_1(x + u - d_l, B_i) \Big\}, \end{aligned} \tag{A.45}$$

where $e = 1, ..., E$. Note that we use $J_{\pi+1} = J_1$ to write Equations (A.43) and (A.45).

Using the variable $y = x + u$, the auxiliary function G for $k = 1, 2, ..., \pi$ and $e = 1, ..., E$ are as follows:

$$G_k(y,A) = H(y) + \alpha \sum_{i=1}^{E} P^k_{AB_i} \sum_{l=1}^{m} q_l J_{k+1}(y-d_l,B_i) + \alpha P^k_{AA} \sum_{l=1}^{m} q_l J_{k+1}(y-d_l,A), \tag{A.46}$$

$$G_\pi(y,A) = H(y) + \alpha \sum_{i=1}^{E} P^\pi_{AB_i} \sum_{l=1}^{m} q_l J_1(y-d_l,B_i) + \alpha P^\pi_{AA} \sum_{l=1}^{m} q_l J_1(y-d_l,A), \tag{A.47}$$

$$G_k(y,B_e) = c_e y + H(y) + \alpha P^k_{B_eA} \sum_{l=1}^{m} q_l J_{k+1}(y-d_l,A) + \alpha \sum_{i=1}^{E} P^k_{B_eB_i} \sum_{l=1}^{m} q_l J_{k+1}(y-d_l,B_i), \tag{A.48}$$

$$G_\pi(y,B_e) = c_e y + H(y) + \alpha P^\pi_{B_eA} \sum_{l=1}^{m} q_l J_1(y-d_l,A) + \alpha \sum_{i=1}^{E} P^\pi_{B_eB_i} \sum_{l=1}^{m} q_l J_1(y-d_l,B_i). \tag{A.49}$$

As done above, we first study an auxiliary finite-horizon formulation in the next subsection (Section A.5.1). Then letting the horizon length tend to infinity in Section A.5.2 we establish the optimality of the hybrid policy in Section A.5.3.

A.5.1 An auxiliary finite-horizon formulation with general demand

This section considers a finite-horizon version of the problem with general demand. Letting $V_k(x,e)$ denote the expected discounted cost over periods $k, k+1, ..., \tau$ under the optimal policy starting in state (x,e). The finite-horizon Bellman equation for the general demand case becomes: For $x \in (-\infty, N]$ and $k = 1, ..., \tau$

$$\begin{aligned} V_k(x,A) = \min_{u\in[0,N-x]} \Big\{ & C(u,A) + H(x+u) + \alpha \sum_{i=1}^{E} P^k_{AB_i} \sum_{l=1}^{M} q_l V_{k+1}(x+u-d_l,B_i) \\ & + \alpha P^k_{AA} \sum_{l=1}^{M} q_l V_{k+1}(x+u-d_l,A) \Big\}, \end{aligned} \tag{A.50}$$

$$\begin{aligned} V_k(x,B_e) = \min_{u\in[0,N-x]} \Big\{ & C(u,B_e) + H(x+u) + \alpha P^k_{B_eA} \sum_{l=1}^{M} q_l V_{k+1}(x+u-d_l,A) \\ & + \alpha \sum_{i=1}^{E} P^k_{B_eB_i} \sum_{l=1}^{M} q_l V_{k+1}(x+u-d_l,B_i) \Big\}, \end{aligned} \tag{A.51}$$

subject to the terminal condition that $V_{\tau+1}(x,e) = 0$ for $x \in (-\infty, N]$ and $e \in \{A, B_1, ..., B_E\}$.

We define the auxiliary functions $W_k(\cdot,A)$ and $W_k(\cdot,B_e)$ for $k = 1, ..., \tau$ as follows:

$$W_k(y,A) = H(y) + \alpha \sum_{i=1}^{E} P^k_{AB_i} \sum_{l=1}^{m} q_l V_{k+1}(y-d_l,B_i) + \alpha P^k_{AA} \sum_{l=1}^{m} q_l V_{k+1}(y-d_l,A),$$

$$W_k(y,B_e) = c_e y + H(y) + \alpha P^k_{B_eA}\sum_{l=1}^{m} q_l V_{k+1}(y-d_l,A) + \alpha\sum_{i=1}^{E} P^k_{B_eB_i}\sum_{l=1}^{m} q_l V_{k+1}(y-d_l,B_i).$$

We proceed with proving a modified version of Theorem 1 for the general demand version of the model. To do so, we follow similar steps as those followed in Appendices A.2-A.4.

Proposition 9. *For $k = 1,...,\tau$ and $x \in (-\infty, N]$, we have that*

$$V_k(x,A) = \min\{W_k(x,A), \min_{x<y\le N}[K + W_k(y,A)]\}$$

$$V_k(y,B_e) = \min_{x\le y\le N}\{W_k(y,B_e)\} - c_e x.$$

Proof. We write from Equation (A.50) that

$$\begin{aligned}
V_k(x,A) &= \min_{u\in[0,N-x]}\Big\{C(u,A) + H(x+u) + \alpha\sum_{l=1}^{m} q_l \mathbb{E}_A[V_{k+1}(x+u-d_l,e_{k+1})]\Big\} \\
&= \min_{u\in[0,N-x]}\Big\{C(u,A) + W_k(x+u,A)\Big\} \\
&= \min_{y\in[x,N]}\Big\{C(u,A) + W_k(y,A)\Big\} \\
&= \min\Big\{W_k(x,A), \min_{x<y\le N}\{K + W_k(y,A)\}\Big\}.
\end{aligned}$$

Similarly, we write from Equation (A.51) that

$$\begin{aligned}
V_k(x,B_e) &= \min_{u\in[0,N-x]}\Big\{c_e u + H(x+u) + \alpha\sum_{l=1}^{m} q_l \mathbb{E}_{B_e}[V_{k+1}(x+u-d_l,e_{k+1})]\Big\} \\
&= \min_{x\le y\le N}\Big\{c_e y - c_e x + H(y) + \alpha\sum_{l=1}^{m} q_l \mathbb{E}_{B_e}[V_{k+1}(y-d_l,e_{k+1})]\Big\} \\
&= \min_{x\le y\le N}\Big\{W_k(y,B_e)\Big\} - c_e x.
\end{aligned}$$

■

The next definition introduces the finite horizon hybrid policy.

Definition 4. *A finite-horizon hybrid policy.* Given parameters $(s^k_A, S^k_A, S^k_{B_1}, ..., S^k_{B_E})$ for $k = 1,...,\tau$, we call a policy $\{u_k(x,e) : k = 1,...,\tau\}$ a finite-horizon hybrid policy if the following holds for $k = 1, ..., \tau$:

$$u_k(x, A) = \begin{cases} S_A^k - x, & \text{if } x < s_A^k, \\ 0, & \text{otherwise}, \end{cases} \qquad u_k(x, B_e) = \begin{cases} S_{B_e}^k - x, & \text{if } x < S_{B_e}^k, \\ 0, & \text{otherwise}, \end{cases}$$

where $u_k(x, e)$ denotes the order quantity in period k as a function of the inventory level x and the environment state e.

The next result establishes the optimality of the finite-horizon hybrid policy.

Theorem 3. *For $k = 1, ..., \tau$, we have a solution for the Bellman equation such that $V_k(\cdot, A)$, $V_k(\cdot, B_e)$, $W_k(\cdot, A)$ and $W_k(\cdot, B_e)$ are continuous and K-convex with $\lim_{y \to -\infty} V_k(y, A) < \infty$ and*

$$\lim_{y \to -\infty} W_k(y, A) = \lim_{y \to -\infty} W_k(y, B_e) = \lim_{y \to -\infty} V_k(y, B_e) = \infty, \quad \textit{for } e = 1, ..., E$$

Moreover, a finite-horizon hybrid policy with parameters $(s_A^k, S_A^k, S_{B_e}^k)$ for $k = 1, ..., \tau$ is optimal, where

$$S_e^k \in \arg\min_{x \le N} W_k(x, e), \quad \textit{for } e \in \{A, B_1, ..., B_E\}$$

and s_A^k is the smallest number $z \le S_A^k$ such that $W_k(z, A) = K + W_k(S_A^k, A)$.

Proof. We proceed by induction. For the induction basis, we consider $i = \tau$. For the induction step, we assume the result holds for $i = k + 1, ..., \tau$ $(i \ge 1)$ and show that it also holds for $i = k$.

In order to establish the induction basis, we begin by noting that

$$W_\tau(y, A) = H(y) = \sum_{l=1}^{m} q_l[p(y - d_l)^- + h(y - d_l)^+].$$

Thus, $W_\tau(\cdot, A)$ is continuous and convex. Hence, it is K-convex. We also have that $\lim_{y \to -\infty} W_\tau(y, A) = \infty$. Similarly, we note that $W_\tau(y, B_e) = c_e y + H(y)$, which is continuous and convex. Hence, by Lemma 1a) it is K-convex. Moreover, because $p > c_e$, we have that $\lim_{y \to -\infty} W_\tau(y, B_e) = \infty$.

Next, we consider $V_\tau(\cdot, A)$. Recall that

$$V_\tau(x, A) = \min\{W_\tau(x, A), \min_{x<y\leq N}[K + W_\tau(x, A)]\}.$$

By part c) of Lemma 1, we set $S_A^\tau \in \arg\min_{y\leq N} W_\tau(y, A)$ and we set s_A^τ as the smallest value of y for which $W_\tau(y, A) = K + W_\tau(S_A^\tau, A)$. Then, also by part c) of Lemma 1, we conclude that

$$V_\tau(x, A) = \begin{cases} K + W_\tau(S_A^\tau, A), & x < s_A^\tau, \\ W_\tau(x, A), & x \geq s_A^\tau, \end{cases} \tag{A.52}$$

which also proves that the order quantity $u_\tau(x, A)$ is optimal, where

$$u_\tau(x, A) = \begin{cases} S_A^\tau - x, & \text{if } x < s_A^\tau \\ 0, & \text{otherwise.} \end{cases}$$

Because $W_\tau(s_A^\tau, A) = K + W_\tau(S_A^\tau, A)$ by construction, we conclude that

$$J_\tau(x, A) = \begin{cases} W_\tau(s_\tau, A), & x < s_A^\tau, \\ W_\tau(x, A), & x \geq s_A^\tau, \end{cases} \tag{A.53}$$

from which the continuity of $V_\tau(\cdot, A)$ is immediate. Moreover, because $W_\tau(\cdot, A)$ is K-convex, we apply Lemma 3 with $a = s_A^\tau$ to deduce $V_\tau(\cdot, A)$ is K-convex. By Equation (A.53), $V_\tau(\cdot, A)$ is constant for $x \leq s_A^\tau$. Thus, $\lim_{y\to-\infty} V_\tau(y, A) < \infty$.

Now, we turn to $J_\tau(\cdot, B_e)$. Recall that

$$V_\tau(x, B_e) = \min_{x\leq y\leq N} W_\tau(y, B_e) - c_e x. \tag{A.54}$$

Let's first consider the first term on the right-hand side of (A.54). To this end, we define

$$\tilde{V}_\tau(x) = \min_{x\leq y\leq N} W_\tau(y, B_e), \quad x \leq N. \tag{A.55}$$

Because $W_\tau(\cdot, B_e)$ is continuous and K-convex on $(-\infty, N]$ with $\lim_{y\to-\infty} W_\tau(y, B_e) = \infty$, by

Lemma 1, we set

$$S_{B_e}^{\tau} \in \arg\min_{x \le y \le N} W_{\tau}(y, B_e).$$

Then it follows that

$$\tilde{V}_{\tau}(x) = \begin{cases} W_{\tau}(S_{B_e}^{\tau}, B_e), & x < S_{B_e}^{\tau}, \\ W_{\tau}(x, B_e), & x \ge S_{B_e}^{\tau}, \end{cases} \tag{A.56}$$

which proves that the order quantity $u_{\tau}(x, B_e)$ is optimal, where

$$u_{\tau}(x, B_e) = \begin{cases} S_{B_e}^{\tau} - x, & \text{if } x < S_{B_e}^{k}, \\ 0, & \text{otherwise.} \end{cases}$$

Equation (A.56) also proves that $\tilde{V}_{\tau}$ is continuous. Because $W_{\tau}(\cdot, B_e)$ is continuous and K-convex, and $S_{B_e}^{\tau}$ is its minimizer, we conclude from Lemma 3 and Equation (A.56) that $\tilde{V}_{\tau}$ is K-convex. To be specific, we apply Lemma 3, with $a = S_{B_e}^{\tau}$ Clearly, $\tilde{V}_{\tau}(x)$ is constant for $x \le S_{B_e}^{\tau}$. Thus, $\lim_{y \to -\infty} \tilde{V}_{\tau}(x) < \infty$. Combining these observations with Equations (A.54)-(A.55), we conclude that $V_{\tau}(\cdot, B_e)$ is continuous, K-convex with $\lim_{y \to -\infty} V_{\tau}(y, B_e) = \infty$.

Next, we turn to the induction step and assume for $i = k+1, ..., \tau$ that $V_i(\cdot, A)$, $V_i(\cdot, B_e)$, $W_i(\cdot, A)$ and $W_i(\cdot, B_e)$ are continuous and K-convex with $\lim_{y \to \infty} V_i(y, A) < \infty$ and

$$\lim_{y \to -\infty} W_i(y, A) = \lim_{y \to -\infty} W_i(y, B_e) = \lim_{y \to -\infty} V_i(y, B_e) = \infty, \quad \text{for } B_e = B_1, ..., B_E. \tag{A.57}$$

We also assume that a finite-horizon hybrid policy with parameters $(s_A^i, S_A^i, S_{B_e}^i)$ for $i = k+1, ..., \tau$ is optimal for the problem over the period $i = k+1, ..., \tau$.

Now, we want to show that $V_k(\cdot, A)$, $V_k(\cdot, B_e)$, $W_k(\cdot, A)$ and $W_k(\cdot, B_e)$ are continuous and K-convex with $\lim_{y \to -\infty} V_k(y, A) < \infty$, and that

$$\lim_{y \to -\infty} W_k(y, A) = \lim_{y \to -\infty} W_k(y, B_e) = \lim_{y \to -\infty} V_k(y, B_e) = \infty.$$

In accordance with Definition 4 we also want to prove that there exist parameters $(s_A^k, S_A^k, S_{B_e}^k)$

for $e \in \{1, ..., E\}$ such that the order quantity $u_k(x, e)$ is optimal, where

$$u_k(x, A) = \begin{cases} S_A^k - x, & \text{if } x < s_A^k, \\ 0, & \text{otherwise,} \end{cases} \qquad u_k(x, B_e) = \begin{cases} S_{B_e}^k - x, & \text{if } x < S_{B_e}^k, \\ 0, & \text{otherwise.} \end{cases}$$

In order to prove this, we first consider $W_k(y, A)$ and note that

$$W_k(y, A) = H(y) + \alpha \sum_{i=1}^{E} P_{AB_i}^k \sum_{l=1}^{M} q_l V_{k+1}(y - d_l, B_i) + \alpha P_{AA}^k \sum_{l=1}^{M} q_l V_{k+1}(y - d_l, A), \quad y \leq N. \tag{A.58}$$

Because $V_{k+1}(\cdot, A)$ and $V_{k+1}(\cdot, B_e)$ are K-convex by the induction hypothesis and H is convex, part b) of Lemma 1 yields that $W_k(\cdot, A)$ is αK-convex, and therefore also K-convex. Moreover, we conclude from Equations (A.57)-(A.58) that $\lim_{y \to -\infty} W_k(y, A) = \infty$. Next, we note that

$$W_k(y, B_e) = c_e y + H(y) + \alpha P_{B_e A}^k \sum_{l=1}^{M} q_l V_{k+1}(y - d_l, A) + \alpha \sum_{i=1}^{E} P_{B_e B_i}^k \sum_{l=1}^{M} q_l V_{k+1}(y - d_l, B_i), \quad y \leq N. \tag{A.59}$$

As above, because $V_{k+1}(\cdot, A)$ and $V_{k+1}(\cdot, B_e)$ are K-convex by the induction hypothesis and the convexity of $c_e y + H(y)$, part b) of Lemma 1 implies that $W_k(y, B_e)$ is αK-convex. Hence, it is also K-convex. Moreover, because $p > c_e$, we conclude from Equations (A.57) and (A.59) that

$$\lim_{y \to -\infty} W_k(y, B_e) = \infty.$$

Next, we recall that

$$V_k(x, B_e) = \min_{x \leq y \leq N} W_k(y, B_e) - c_e x, \quad x \leq N. \tag{A.60}$$

As done above, let's first consider the first term on the right-hand side of (A.60). To this end, we define

$$\tilde{V}_k(x, B_e) = \min_{x \leq y \leq N} W_k(y, B_e) \quad \text{for } x \leq N. \tag{A.61}$$

Because $W_k(\cdot, B_e)$ is continuous and K-convex on $(\infty, N]$ with $\lim_{y\to-\infty} W_k(y, B_e) = \infty$, by virtue of Lemma 1, we set

$$S^k_{B_e} \in \arg\min_{x\le y\le N} W_k(y, B_e).$$

Then it follows that

$$\tilde{V}_k(x, B_e) = \begin{cases} W_k(S^k_{B_e}, B_e), & x < S^k_{B_e}, \\ W_k(x, B_e), & x \ge S^k_{B_e}, \end{cases} \tag{A.62}$$

which proves that the order quantity $u_k(x, B_e)$ is optimal, where

$$u_k(x, B_e) = \begin{cases} S^k_{B_e} - x, & x < S^k_{B_e}, \\ 0, & \text{otherwise.} \end{cases}$$

Equation (A.62) also proves that $\tilde{V}_k$ is continuous. Because $W_k(\cdot, B_e)$ is continuous and K-convex, and $S^k_{B_e}$ is its minimizer, we conclude from Lemma 3 that $\tilde{V}_k$ is K-convex. Clearly, $\tilde{V}_k(x, B_e)$ is constant for $x \le S^k_{B_e}$. Thus, $\lim_{y\to-\infty} \tilde{V}_k(x, B_e) < \infty$. Combining these observations with Equations (A.60)-(A.61), we conclude that $V_k(\cdot, B_e)$ is continuous and K-convex with $\lim_{y\to-\infty} V_k(y, B_e) = \infty$.

Lastly, we consider $V_k(y, A)$, and recall that

$$V_k(x, A) = \min\{W_k(x, A), \min_{x<y\le N}[K + W_k(x, A)]\}.$$

By part c) of Lemma 1, we set $S^k_A \in \arg\min_{y\le N} W_k(y, A)$ and we set s^k_A as the smallest value of y for which $W_k(y, A) = K + W_k(S^k_A, A)$. Then, also by part c) of Lemma 1, we conclude that

$$V_k(x, A) = \begin{cases} K + W_k(S^k_A, A), & x < s^k_A, \\ W_k(x, A), & x \ge s^k_A, \end{cases} \tag{A.63}$$

which also proves that the order quantity $u_k(x, A)$ is optimal, where

$$u_k(x, A) = \begin{cases} S_A^k - x, & \text{if } x < s_A^k, \\ 0, & \text{otherwise.} \end{cases}$$

Because $W_k(s_A^k, A) = K + W_k(S_A^k, A)$ by construction, we conclude that

$$V_k(x, A) \begin{cases} W_k(s_k, A), & x < s_A^k, \\ W_k(x, A), & x \geq s_A^k. \end{cases}$$

The continuity of $V_k(\cdot, A)$ follows from this. Moreover, because $W_k(\cdot, A)$ is continuous and K-convex, we conclude from Lemma 3 that $V_k(\cdot, A)$ is K-convex. Clearly, $V_k(\cdot, A)$ is constant for $x \leq s_A^k$. Thus, $\lim_{y \to -\infty} V_k(y, A) < \infty$. ∎

Next, we consider letting $\tau \to \infty$ in the following section.

A.5.2 Passing to the limit as $\tau \to \infty$ in the general demand case

In this section, we turn our attention to passing to the limit as $\tau \to \infty$. In order to facilitate the analysis to follow, we let

$$U(x) = (N - x)c_{\max} + K + \frac{c_{\max} + K}{1 - \alpha} + \frac{hN}{1 - \alpha}, \quad x \leq N.$$

Note that this is an upper bound on the infinite-horizon discounted cost of the following feasible policy: in all periods, order up to N. Recall that $c_{\max} = \max\{c_1, ..., c_E\}$.

It is more or less obvious that $V_k^\tau(x, e)$ is increasing in the length τ of the planning horizon and that $0 \leq V_k^\tau(x, e) \leq U(x)$ for $x \in (-\infty, N]$, $e = A, B_1, ..., B_E$, $k = 1, ..., \tau$ and $\tau \geq 1$. Thus, one can argue that $V_k^\tau(x, e)$ converges to a limit as $\tau \to \infty$, and that the limit for $k > j$ is the same as the limit for j whenever $k = j + l\pi$ for $l \geq 1$ because of the periodic structure. Therefore, it suffices to study this limit for $k = 1, ..., \pi$. We denote the vector-valued function

$\bar{V}_k(x,e)$:

$$\bar{V}_k(x,e) = \big(\bar{V}_1(x,e), ..., \bar{V}_\pi(x,e)\big) \text{ for } x \in (-\infty, N] \text{ and } e = A, B_1, ..., B_E.$$

Then we define auxiliary function $\bar{W}_k(x,e) = \big(\bar{W}_1(x,e), ..., \bar{W}_\pi(x,e)\big)$ as follows:

$$\bar{W}_k(x,A) = H(x) + \alpha \sum_{i=1}^{E} P_{AB_i}^k \sum_{l=1}^{M} q_l \bar{V}_{k+1}(x-d_l, B_i) + \alpha P_{AA}^k \sum_{l=1}^{M} q_l \bar{V}_{k+1}(y-d_l, A), \tag{A.64}$$

$$\bar{W}_k(x,B_e) = c_e x + H(x) + \alpha P_{B_eA}^k \sum_{l=1}^{M} q_l \bar{V}_{k+1}(x-d_l, A) + \alpha \sum_{i=1}^{E} P_{B_eB_i}^k \sum_{l=1}^{M} q_l \bar{V}_{k+1}(x-d_l, B_i). \tag{A.65}$$

The following result establishes that the convergence is uniform on compact sets.

Proposition 10. *We have for $e = A, B_1, ..., B_E$, $k = 1, ..., \pi$, and $l \geq 1$ that*

$$V_{k+l\pi}^\tau(\cdot,e) \to \bar{V}_k(\cdot,e) \quad \textit{uniformly on compact sets as } \tau \to \infty,$$

$$W_{k+l\pi}^\tau(\cdot,e) \to \bar{W}_k(\cdot,e) \quad \textit{uniformly on compact setsas } \tau \to \infty.$$

Proof. The uniform convergence of $V_{k+l\pi}^\tau(\cdot,e)$ follows from Dini's Theorem (see for example page 242 of Billingsley (1999)) and that $V_{k+l\pi}^\tau(x,e)$ increases to its limit, $\bar{V}_k(x,e)$, as $\tau \to \infty$. The uniform convergence of $W_{k+l\pi}^\tau(\cdot,e)$ follows from its definition that

$$W_{k+l\pi}^\tau(x,A) = H(x) + \alpha \sum_{i=1}^{E} P_{AB_i}^k \sum_{l=1}^{M} q_l V_{k+1+l\pi}^\tau(y-d_l, B_i) + \alpha P_{AA}^k \sum_{l=1}^{M} q_l V_{k+1+l\pi}^\tau(x-d_l, A),$$

$$W_{k+l\pi}^\tau(x,B_e) = H(x) + \alpha P_{B_eA}^k \sum_{l=1}^{M} q_l V_{k+1+l\pi}^\tau(x-d_l, A) + \alpha \sum_{i=1}^{E} P_{B_eB_i}^k \sum_{l=1}^{M} q_l V_{k+1+l\pi}^\tau(x-d_l, B_i),$$

the uniform convergence of $V_{k+l\pi}^\tau(\cdot,e)$ and the definition of $\bar{W}(\cdot,e)$ see Equations (A.64) and (A.65). ■

Using the convergence of the relevant functions to the vector of limiting functions from Proposition 10, the following proposition establishes the K-convexity of the functions comprising the vectors $\bar{V}_k(x,e)$ and $\bar{W}_k(x,e)$, as well as their limiting values as the inventory tends to $-\infty$.

Proposition 11. *For $k = 1, ..., \pi$, functions $\bar{V}_k$ and $\bar{W}_k$ are continuous, K-convex with* $\lim_{y \to -\infty} \bar{V}_k(y, A) < \infty$ *and* $\lim_{y\to-\infty} \bar{V}_k(y, B_e) = \lim_{y\to-\infty} \bar{W}_k(y, A) = \lim_{y\to-\infty} \bar{W}_k(y, B_e) = \infty$.

Proof. Consider $\bar{V}_k(\cdot, e)$ on $[-M, N]$ where $M > 0$ is an arbitrary number. Because $V^{\tau}_{k+l\pi}(\cdot, e)$ is continuous and by Proposition 10. $V^{\tau}_{k+l\pi}(\cdot, e) \to \bar{V}_k(\cdot, e)$ u.o.c. as $\tau \to \infty$, we conclude that $\bar{V}(\cdot, e)$ is continuous on $[-M, N]$. Because $M > 0$ was arbitrary, we conclude that $\bar{V}_k(x, e)$ is continuous on $(-\infty, N]$ for $e = A, B_1, ..., B_E$. The continuity of $\bar{W}_k(\cdot, e)$ for $e = A, B_1, ..., B_E$ follows similarly.

To prove $\bar{V}_k(\cdot, e)$ is K-convex, we need to show for every $x, y \in (-\infty, N]$ with $x \leq y$, $0 \leq \theta \leq 1$ and $\bar{\theta} = 1 - \theta$, the following holds:

$$\bar{V}_k(\theta x + \bar{\theta} y, e) \leq \theta \bar{V}_k(x, e) + \bar{\theta}[K + \bar{V}_k(y, e)]. \tag{A.66}$$

Recall from Theorem 3 that $V^{\tau}_k(\cdot, e)$ is K-convex. Thus, we have that

$$V^{\tau}_k(\theta x + \bar{\theta} y, e) \leq \theta V^{\tau}_k(x, e) + \bar{\theta}[K + V^{\tau}_k(y, e)] \tag{A.67}$$

Because $V^{\tau}_k(\cdot, e) \to \bar{V}_k(\cdot, e)$ as $\tau \to \infty$, by passing to the limit as $\tau \to \infty$ on both sides of Equation (A.67), we arrive at Equation (A.66). Note that K-convexity of $\bar{W}_k$ follows similarly.

Moreover, note that because $V^{\tau}_k(y, B_e)$ increases to its limit as $\tau \to \infty$ monotonically, we have that

$$\bar{V}_k(y, B_e) \geq V^{\tau}_1(y, B_e).$$

Then, passing to the limit as $y \to -\infty$ on both sides and using that $\lim_{y \to -\infty} V^{\tau}_k(y, B_e) = \infty$ by Theorem 3 we conclude that $\lim_{y\to-\infty} \bar{V}_k(y, B_e) = \infty$. The proofs of $\lim_{y \to -\infty} \bar{W}_k(y, A) = \lim_{y \to -\infty} \bar{W}_k(y, B_e) = \infty$ follow similarly.

To complete the proof, we next consider $\limsup_{y \to -\infty} \bar{V}_k(y, A)$. Let $y < 0$ and following the initial state (y, A) in the finite-horizon problem, we consider the policy that orders up to zero initially and follows the optimal policy thereafter. Comparing this policy with the optimal policy starting at (y, A), we conclude

$$V_1^\tau(y,A) \leq K + \alpha \sum_{i=1}^{E} \left[P_{AB_i}^k V_2^\tau(0,B_i) \right] + \alpha P_{AA}^k V_2^\tau(0,A) \tag{A.68}$$

By virtue of passing to the limit on both sides of Equation (A.68) gives the following:

$$\bar{V}_k(y,A) \leq K + \alpha \sum_{i=1}^{E} \left[P_{AB_i}^k \bar{V}_{k+1}(0,B_i) \right] + \alpha P_{AA}^k \bar{V}_{k+1}(0,A), \; y < 0.$$

Now, letting $y \to -\infty$ gives

$$\limsup_{y \to -\infty} \bar{V}_k(y,A) \leq K + \alpha \sum_{i=1}^{E} \left[P_{AB_i}^k \bar{V}_{k+1}(0,B_i) \right] + \alpha P_{AA}^k \bar{V}_{k+1}(0,A) < \infty.$$

Note from part c-iii) of Lemma 1 and the K-convexity of $\bar{V}_k(\cdot, A)$ that $\bar{V}_k(y, A)$ is monotone decreasing for y sufficiently small. Thus, $\lim_{y \to -\infty} \bar{V}_k(y, A)$ exists and we conclude that

$$\lim_{y \to -\infty} \bar{V}_k(y,A) = \limsup_{y \to -\infty} \bar{V}_k(y,A) < \infty.$$

∎

We use Propositions 9 and 10 to show that the value functions $\bar{V}_k(x, e)$ for $k = 1, ..., \pi$ can be written in terms of the auxiliary functions $\bar{W}_k(x, e)$.

Proposition 12. *Functions $\bar{V}_k$ and $\bar{W}_k$ satisfy the following:*

$$\bar{V}_k(x,A) = \min\{\bar{W}_k(x,A), \min_{x<y\leq N}[K + \bar{W}_k(x,A)]\}, \tag{A.69}$$

$$\bar{V}_k(x,B_e) = \min_{x\leq y\leq N}\{\bar{W}_k(y,B_e)\} - c_e x \tag{A.70}$$

Proof. Fix $x < N$ and note for $k = 1, 2, ..., \pi$, $e = A, B_1, ..., B_E$ that

$$V_k^\tau(y,e) \to \bar{V}_k(y,e) \text{ uniformly on } y \in [x,N] \text{ as } \tau \to \infty, \tag{A.71}$$

$$W_k^\tau(y,e) \to \bar{W}_k(y,e) \text{ uniformly on } y \in [x,N] \text{ as } \tau \to \infty. \tag{A.72}$$

Also, recall from the finite horizon analysis that

$$V_k^\tau(x,A) = \min\left\{W_k^\tau(x,A), \min_{x\le y\le N}[K + W_k^\tau(y,A)]\right\}, \tag{A.73}$$

$$V_k^\tau(x,B_e) = \min_{x\le y\le N}\left\{W_k^\tau(y,B_e)\right\} - c_e x. \tag{A.74}$$

Now, we note that because the convergence in Equations (A.71) and (A.72) is uniform over compact sets, we have

$$\min_{x\le y\le N}\{K + W_k^\tau(y,A)\} \to \min_{x\le y\le N}\{K + \bar{W}_k(y,A)\}, \tag{A.75}$$

$$\min_{x\le y\le N}\left\{W_k^\tau(y,B_e)\right\} \to \min_{x\le y\le N}\bar{W}_k(y,B_e). \tag{A.76}$$

Then passing to the limit on both sides of Equations (A.73) and (A.74) and using Equations (A.75) and (A.76) gives the result. ■

Using the limiting functions $\bar{V}_k$ and $\bar{W}_k$ $(k = 1,...,\pi)$, the next section establishes the optimality of the hybrid policy.

A.5.3 Proof of the optimality of the hybrid policy under general demand

The following theorem is the main result for the case of general demand distribution.

Theorem 4. *We have a solution for the Bellman equation such that $V_k(\cdot,A)$, $V_k(\cdot,B_e)$, $W_k(\cdot,A)$ and $W_k(\cdot,B_e)$ for $k = 1,...,\pi$ are continuous and K-convex with $\lim_{y\to-\infty}V_k(y,A) < \infty$ and*

$$\lim_{y\to-\infty}W_k(y,A) = \lim_{y\to-\infty}W_k(y,B_e) = \lim_{y\to-\infty}V_k(y,B_e) = \infty, \quad \textit{for } e = 1,...,E.$$

Moreover, a hybrid policy with parameters $(s_A^k, S_A^k, S_{B_e}^k)$ is optimal, where

$$S_e^k \in \arg\min_{x\le N} W_k(x,e), \quad \textit{for } e \in \{A, B_1,...,B_E\}$$

and s_A^k is the smallest number $z \le S_A^k$ such that $W_k(z,A) = K + W_k(S_A,A)$.

Proof. From the finite-horizon discounted formulation, Proposition 10 establishes that the limit functions $\bar{V}_k$ and $\bar{W}_k$ exist and are well defined. Proposition 11 shows that these function are continuous and K-convex with $\lim_{y \to -\infty} \bar{V}_k(y, A) < \infty$ and

$$\lim_{y \to -\infty} \bar{V}_k(y, B_e) = \lim_{y \to -\infty} \bar{W}_k(y, A) = \lim_{y \to -\infty} \bar{W}_k(y, B_e) = \infty.$$

Moreover, Propositions 9 and 12 show that functions $\bar{V}_k$ and $\bar{W}_k$ solve the Bellman Equations (A.50) and (A.51) for the infinite horizon discounted formulation setting, $(J_k, G_k) = (\bar{V}_k, \bar{W}_k)$ for $k = 1, ..., \pi$. Therefore, J_k and G_k solve the Bellman equation and have the desired properties.

To complete the proof, we let $S_A^k = \inf \arg\min_{y \leq N} G_k(y, A)$ and set s_A as the smallest value of y for which $G_k(y, A) = K + G_k(s_A^k, A)$. Then by part c) of Lemma 1 (see Appendix A.1), we conclude that

$$J_k(x, A) = \begin{cases} K + G_k(S_A, A), & \text{if } x < s_A^k, \\ G_k(x, A), & \text{if } x \geq s_A^k, \end{cases} \tag{A.77}$$

which proves the optimality of $u_k(x, A)$ for $k = 1, ..., \pi$, where

$$u_k(x, A) = \begin{cases} S_A^k - x, & \text{if } x < s_A^k, \\ 0, & \text{otherwise.} \end{cases}$$

Equation (A.77) also proves that $J_k(x, A)$ is constant for $x < s_A^k$. Thus, we conclude that $\lim_{x \to -\infty} J(x, A) < \infty$. Similarly, letting $S_{B_e}^k = \inf \arg\min_{x \leq N} G_k(x, B_e)$, it follows that

$$J_k(x, B_e) = \begin{cases} G_k(S_{B_e}, B_e), & \text{if } x < S_{B_e}^k, \\ G_k(x, B_e), & \text{if } x \geq S_{B_e}^k, \end{cases}$$

which proves the optimality of $u_k(x, B_e)$, where

$$u_k(x, B_e) = \begin{cases} S^k_{B_e} - x, & \text{if } x < S^k_{B_e}, \\ 0, & \text{otherwise.} \end{cases}$$

Therefore, the hybrid policy with parameters $(s^k_A, S^k_A, S^k_{B_1}, ..., S^k_{B_E})$ for $k = 1, ..., \pi$ is optimal.

■

To ensure no backlogging, we prove an analog of Proposition 2 next. Recall that $q_M = \mathbb{P}(D_1 = d_M)$ and $c_{\max} = \max\{c_1, ..., c_E\}$.

Proposition 13. *When* $p > \frac{1}{q_M}\left[\frac{h}{1-\alpha} + \max\{K, c_{\max}\}\right]$, *we have that* $s^k_A \geq d_M$ *and* $S^k_{B_e} \geq d_M$ *for* $k = 1, ..., \pi$ *and* $e = 1, ..., E$.

Lemmas 6 and 7 comprise the proofs necessary for Proposition 13.

Lemma 6. *When* $p > \frac{1}{q_M}(c_{\max} + \frac{h}{1-\alpha})$, *we have that* $S^k_{B_e} \geq d_M$ *for* $k = 1, ..., \pi$ *and* $e = 1, ..., E$.

Proof. Arguing by contradiction, suppose there exists $\kappa \in \{1, ..., \pi\}$ and an environment state B_ι such that $S^\kappa_{B_\iota} \leq d_M - 1$. Then define the stopping time T_1 as follows:

$$T_1 = \inf\{t = \kappa + l\pi \geq 0 : \; x_t = d_M - 1, \; e_t = B_\iota\}.$$

Because $d_1 = 1$, one can show that $\mathbb{P}(T_1 < \infty) = 1$. Next, we consider an alternative policy that performs strictly better than the hybrid policy, which is optimal. Under the optimal policy proposed, we don't order at period $t = T_1$ and with probability $\mathbb{P}(D = d_M)$, $x_{T+1} = (d_M - 1) - d_M = -1$. Consider an alternative policy under which we order 1 unit at time T_1, but before time T_1 and after time T_1, follow the optimal hybrid policy. We denote the resulting inventory process under this alternative process by $\tilde{x}_t$. We have that $x_{T+1} = \tilde{x}_{T+1} - 1$. We also define the optimal stopping time $T_2 > T_1$ as follows:

$$T_2 = \inf\{t > T_1 : \text{an order is placed under the hybrid policy}\}.$$

At time T_2, the processes $\tilde{x}_t$ and x_t couple, but until then, (i.e. $t \in \{T_1 + 1, ..., T_2 - 1\}$) $\tilde{x}_t = x_t + 1$. In order to compare the costs of the two policies, we let $\tilde{J}(x, e)$ denote the expected discounted cost under the alternative policy starting in state (x, e). We then have that

$$\tilde{J}(d_M - 1, B_\iota) - J(d_M - 1, B_\iota) \leq \alpha[c_{\max} + \frac{h}{1-\alpha} - q_M p]$$
$$< 0.$$

consider the first inequality. We note that this is a conservative bound. We over-count costs associated with the alternative policy and under-count costs associated with the optimal policy. There are four costs to consider when comparing the two policies. The cost $c_{\max}$ accounts for the most expensive possible ordering cost for the one unit of inventory ordered in the alternative policy. The holding cost for the excess unit in the alternative policy is upper bounded by the cost to hold it in perpetuity. The penalty cost for backlogging is weighted by the probability of having negative inventory under the optimal policy at time $T_1 + 1$, although the penalty cost can persist beyond this period, so we only account for the penalty costs accrued in period $T_1 + 1$. Lastly, the order cost the system manager incurs at time T_2 under the original (i.e. hybrid) policy is greater than (if $e_{T_2} \neq A$) or equal to (if $e_{T_2} = A$) that under the alternative policy; we ignore that cost in writing the first inequality. These jointly yield the first inequality. The second inequality follows because $p > \frac{1}{q_M}(c_{\max} + \frac{h}{1-\alpha})$, implying $\tilde{J}(d_M - 1, B_\iota) < J(d_M - 1, B_\iota)$ which contradicts the optimality of the hybrid policy. ■

Lemma 7. *When $p > \frac{1}{q_M}(c_{\max} + \frac{h}{1-\alpha})$, we have that $s_A^k \geq d_M$ for $k = 1, ..., \pi$.*

Proof. Arguing by contradiction, suppose there exists $\kappa \in \{1, ..., \pi\}$ such that $s_A^\kappa \leq d_M - 1$. Then we define the stopping time T_1 as follows:

$$T_1 = \inf\{t = \kappa + l\pi \geq 0 : \ x_t = d_M - 1, \ e_t = A\}.$$

Because $d_1 = 1$, one can show that $\mathbb{P}(T_1 < \infty) = 1$. Next, we construct an alternative policy that performs strictly better. We denote the resulting inventory process under this alternative process by $\tilde{x}_t$. The alternative policy follows the hybrid policy until period T_1, but orders one unit in period T_1 and incurs a fixed cost K. This means that $x_{T+1} = \tilde{x}_{T+1} - 1$ and with probability $\mathbb{P}(D = d_M)$, $x_{T+1} = (d_M - 1) - d_M = -1$ and $\tilde{x}_{T+1} = 0$. Thereafter, the alternative policy follows the optimal policy. We also define the stopping time $T_2 > T_1$ as follows:

$$T_2 = \inf\{t > T_1 : \text{an order is placed under the optimal policy}\}.$$

At time T_2 the processes x_t and $\tilde{x}_t$ couple. In order to compare the costs of the two policies, we let $\tilde{J}(x, e)$ denote the cost of the alternative policy. We then have that

$$\begin{aligned} \tilde{J}(d_M - 1, A) - J(d_M - 1, A) &\leq \alpha\Big[K + \frac{h}{1-\alpha} - pq_M\Big] \\ &< 0 \end{aligned}$$

For the first inequality, consider x_t and $\tilde{x}_t$ for $t = T_1, T_1 + 1, ..., T_2$ and note that $\tilde{x}_t = x_t + 1$ for $t = T_1 + 1, ..., T_2 - 1$ and $\tilde{x}_t = x_t + 1$ otherwise. The first inequality follows because i) at $t = T+1$, the alternative policy places an order and incurs the ordering cost K, corresponding to the first term in the parentheses on the right-hand side; ii) it also carries an extra unit of inventory during periods $T_1+1, ..., T_2-1$. The incremental holding cost over the hybrid policy is bounded by $h/(1-\alpha)$; iii) with probability q_M we have $D_{T_1} = d_M$ and $x_{T+1} = -1$ whereas $\tilde{x}_{T+1} = 0$. The last term on the right-hand side (pq_M) is a lower bound on the difference in the backlog penalty, which is higher for the hybrid policy; and iv) the order cost the system manager incurs at time T_2 under the original (i.e. the hybrid) policy is greater than (if $e_{T_2} \neq A$) or equal to (if $e_{T_2} = A$) that under the alternative policy. We ignore this in writing the first inequality. The second inequality follows because $p > \frac{1}{q_M}(c_{\max} + \frac{h}{1-\alpha})$, implying $\tilde{J}(d_M - 1, A) < J(d_M - 1, A)$, which contradicts the optimality of the hybrid policy. ■

A.6 An Analysis of the two-parameter (s,S) Heuristic Policy

In this Appendix, we prove Proposition 3. In doing so, we also present and prove Lemmas 8-10.

Proof of Proposition 3. First, we note that we have already established that the minimum cycle length is $S - s$. The remaining length of a cycle depends on Equation (1.9) and τ_B, which we previously defined as a geometric random variable with success parameter P_B, and for which $\mathbb{E}[\tau_B] = 1/P_B$. We can thus write the expected cycle length as we do in Equation (A.78).

$$\mathbb{E}[\text{Cycle length}] = S - s + [1 - P_{AA}(S - s)]\frac{1}{P_B} \tag{A.78}$$

We now focus on deriving the expected cycle cost. That is, we wish to show the following:

$$\begin{aligned}\mathbb{E}[\text{Cycle cost}] =& h\frac{S^2 - s^2}{2} + K \\ &+ [1 - P_{AA}(S - s)]\frac{\big((2c - h)P_B + 2h\big)(1 - P_B)^s + 2h\big((s + \frac{1}{2})P_B - 1\big)}{2P_B^2}.\end{aligned}$$

In order to do so, we first note that all cycles will include the cost $h(S^2 - s^2)/2 + K$, as is shown in Figure 1.6. We now turn our attention to the possible scenarios displayed in Figure 8. We study each case separately to calculate the remaining costs during a cycle, starting with $1 \le \tau_B \le s$. Since inventory is not depleted in this scenario, there is no order cost in state B. The holding cost in this scenario is given by the shaded area of the left panel of Figure 1.7, and is written out as follows:

$$\text{Holding cost} = \frac{s + s - \tau_B}{2}\tau_B h = \Big(s\tau_B - \frac{\tau_B^2}{2}\Big)h, \quad 1 \le \tau_B \le s. \tag{A.79}$$

The order and holding costs for the case where $s < \tau_B$, shown on the right panel of Figure 1.7, are written out as follows:

$$\text{Holding cost} = \frac{s^2}{2}h, \qquad s < \tau_B, \tag{A.80}$$

$$\text{Ordering cost} = (\tau_B - s)c, \qquad s < \tau_B. \tag{A.81}$$

Combing Equations (A.79)-(A.81) with the common cycle cost of $h\frac{S^2-s^2}{2} + K$, we write the expected cycle cost as,

$$\begin{aligned}\mathbb{E}[\text{Cycle cost}] =& h\frac{S^2 - s^2}{2} + K + [1 - P_{AA}(S-s)]\Big[\sum_{k=1}^{s}\big(sk - \frac{k^2}{2}\big)h\mathbb{P}(\tau_B = k) \\ &+ \sum_{k=s+1}^{\infty}\frac{s^2}{2}h\mathbb{P}(\tau_B = k) + \sum_{k=s+1}^{\infty}(k-s)c\mathbb{P}(\tau_B = k)\Big].\end{aligned} \tag{A.82}$$

We make the following rearrangements to the terms in Equation (A.82) and provide a series of expressions we address in Lemmas 8-10:

$$\begin{aligned}&\sum_{k=1}^{s}\big(sk - \frac{k^2}{2}\big)h\mathbb{P}(\tau_B = k) + \sum_{k=s+1}^{\infty}\frac{s^2}{2}h\mathbb{P}(\tau_B = k) + \sum_{k=s+1}^{\infty}(k-s)c\mathbb{P}(\tau_B = k) \\ &= \big(\frac{hs^2}{2} - sc\big)\mathbb{P}(\tau_B \geq s+1) + sh\sum_{k=1}^{s}k\mathbb{P}(\tau_B = k) + c\sum_{k=s+1}^{\infty}k\mathbb{P}(\tau_B = k) - \frac{h}{2}\sum_{k=1}^{s}k^2\mathbb{P}(\tau_B = k) \\ &= \big(\frac{hs^2}{2} - sc\big)\mathbb{P}(\tau_B \geq s+1) + (sh - c)\sum_{k=1}^{s}k\mathbb{P}(\tau_B = k) + c\sum_{k=1}^{\infty}k\mathbb{P}(\tau_B = k) - \frac{h}{2}\sum_{k=1}^{s}k^2\mathbb{P}(\tau_B = k).\end{aligned} \tag{A.83}$$

The series $\sum_{k=1}^{\infty} k\mathbb{P}(\tau_B = k)$ is simply the expectation of the geometric variable τ_B, and can thus be written as $\sum_{k=1}^{\infty} k\mathbb{P}(\tau_B = k) = \mathbb{E}[\tau_B] = 1/P_B$. Incorporating this simplification provides the expected cycle cost in Equation (A.84).

$$\begin{aligned}\mathbb{E}[\text{Cycle Cost}] =& K + h\frac{S^2 - s^2}{2} + [1 - P_{AA}(S-s)]\Big[\frac{c}{P_B} + \big(\frac{hs^2}{2} - sc\big)\mathbb{P}(\tau_B \geq s+1) \\ &+ (sh - c)\sum_{k=1}^{s}k\mathbb{P}(\tau_B = k) - \frac{h}{2}\sum_{k=1}^{s}k^2\mathbb{P}(\tau_B = k)\Big]\end{aligned} \tag{A.84}$$

Using the marginal probability formula $\mathbb{P}(\tau_B = k) = (1 - P_B)^{k-1}P_B$ for $k \geq 1$, Lemmas 9 and 10 derive formulas for the partial expectations appearing on the right-hand side of Equation (A.84). Then, substituting the results of Lemmas 8-10 into Equation (A.84) and rearranging terms completes the proof. ■

Since τ_B is geometrically distributed with parameter P_B, in Lemmas 9 and 10 we use the following expression:

$$\mathbb{P}(\tau_B = k) = (1 - P_B)^{k-1} P_B, \quad k \geq 1.$$

Next, we present some auxiliary lemmas to further simplify the right-hand side of Equation (A.84).

Lemma 8. *We have that* $\mathbb{P}(\tau_B \geq s + 1) = (1 - P_B)^s$.

Proof. By substituting $\mathbb{P}(\tau_B = k) = (1 - P_B)^{k-1} P_B$, we have the following:

$$\begin{aligned}\mathbb{P}(\tau_B \geq s+1) &= \sum_{k=s+1}^{\infty} (1 - P_B)^{k-1} P_B = P_B (1 - P_B)^s \sum_{j=0}^{\infty} (1 - P_B)^j \\ &= P_B (1 - P_B)^s \frac{1}{1 - (1 - P_B)} = (1 - P_B)^s.\end{aligned}$$

■

Lemma 9. *We have that*

$$\sum_{k=1}^{s} k \mathbb{P}(\tau_B = k) = \frac{1 - (1 + sP_B)(1 - P_B)^s}{P_B}$$

Proof. First, note that

$$\sum_{k=1}^{s} k \mathbb{P}(\tau_B = k) = \sum_{k=1}^{s} k (1 - P_B)^{k-1} P_B = P_B \sum_{k=1}^{s} k (1 - P_B)^{k-1}.$$

To facilitate the analysis, we let $f(P_B) = \sum_{k=1}^{s} k(1 - P_B)^{k-1}$ so that

$$\sum_{k=1}^{s} k \mathbb{P}(\tau_B = k) = P_B f(P_B). \tag{A.85}$$

We now characterize $F(P_B)$, by defining

$$\begin{aligned}F(P_B) &= \int f(P_B) dP_B \\ &= \sum_{k=1}^{s} \int k (1 - P_B)^{k-1} dP_B\end{aligned}$$

$$= -\sum_{k=1}^{s}(1-P_B)^k + C$$
$$= \frac{(1-P_B)^{s+1}}{P_B} - \frac{1-P_B}{P_B} + C$$

$$F(P_B) = \Big[1 - \frac{1}{P_B} + \frac{(1-P_B)^{s+1}}{P_B} + C\Big]. \tag{A.86}$$

By differentiating both sides of Equation (A.86) we have that,

$$f(P_B) = F'(P_B) = \frac{1}{P_B^2} + \Big[\frac{-(s+1)(1-P_B)^s P_B - (1-P_B)^{s+1}}{P_B^2}\Big]$$
$$= \frac{1}{P_B^2} + \Big[\frac{-(1-P_B)^s P_B - (1-P_B)^s P_B s - (1-P_B)(1-P_B)^s}{P_B^2}\Big]$$
$$= \frac{1}{P_B^2} - \Big[(1-P_B)^s \frac{sP_B+1}{P_B^2}\Big].$$

Therefore,

$$f(P_B) = \frac{1-(1+sP_B)(1-P_B)^s}{P_B^2}. \tag{A.87}$$

Substituting Equation (A.87) into Equation (A.85) gives the result. ■

Lemma 10. *We have that*

$$\sum_{k=1}^{s} k^2 \mathbb{P}[\tau_B = k] = \frac{2}{P_B^2}[1-(1-P_B)^s] - \frac{1}{P_B}[1-(1-P_B)^s(1-2s)] - s^2(1-P_B)^s.$$

Proof. First, note that

$$\sum_{k=1}^{s} k^2 \mathbb{P}(\tau_B = k) = P_B \sum_{k=1}^{s} k^2 (1-P_B)^{k-1}.$$

We let $g(P_B) = \sum_{k=1}^{s} k^2(1-P_B)^{k-1}$, and write

$$\sum_{k=1}^{s} k^2 \mathbb{P}(\tau_B = k) = P_B g(P_B). \tag{A.88}$$

We also define $G(P_B) = \int g(P_B) dP_B$ and characterize $G(P_B)$ as follows:

$$G(P_B) = \int \sum_{k=1}^{s} k^2 (1-P_B)^{k-1} dP_B$$
$$= \sum_{k=1}^{s} \int k^2 (1-P_B)^{k-1} dP_B$$

$$= -\sum_{k=1}^{s} k(1-P_B)^k + C$$
$$= -(1-P_B)\sum_{k=1}^{s} k(1-P_B)^{k-1} + C$$
$$= -(1-P_B)f(P_B) + C.$$

Substituting Equation (A.87) from Lemma 9 into this gives

$$G(P_B) = \frac{-(1-P_B) + (1+sP_B)(1-P_B)^{s+1}}{P_B^2} + C$$
$$= \frac{1}{P_B} - \frac{1}{P_B^2} + \frac{(1+sP_B)(1-P_B)^{s+1}}{P_B^2} + C.$$

Differentiating both sides gives the following:

$$g(P_B) = G'(P_B) = \frac{2}{P_B^3} - \frac{1}{P_B^2} + \Big[\frac{(1+sP_B)(1-P_B)^{s+1}}{P_B^2}\Big]'$$
$$= \frac{2}{P_B^3} - \frac{1}{P_B^2}$$
$$+ \frac{\big[s(1-P_B)^{s+1} - (s+1)(1-P_B)^s(1+sP_B)\big]P_B^2 - 2P_B(1+sP_B)(1-P_B)^{s+1}}{P_B^4}$$
$$= \frac{2}{P_B^3} - \frac{1}{P_B^2} + \frac{(1-P_B)^s}{P_B^3}(P_B - 2 - s^2P_B^2 - 2sP_B).$$

Substituting this into Equation (A.87) and rearranging the terms gives the result. ■

A.7 An Analysis of the EOQ-like Heuristic Policy

In this Appendix, we provide the proofs of Proposition 4 and Corollary 5 as well as the derivation of Equation (1.11).

Proof of Proposition 4. Combining the two cases in Figure 1.8 with τ_B being a geometric random variable with the expected value $1/P_B$, yields the expected cycle length formula:

$$\mathbb{E}\big[\text{Cycle length}\big] = Q + (1 - P_{AA}(Q))\frac{1}{P_B}$$

In Figure 1.8, it is clear that all cycles include the combined ordering and holding costs,

$K + hQ^2/2$. Moreover, with probability $(1 - P_{AA}(Q))$ the environment state is B when the inventory is depleted. In this instance, the expected order cost for the remaining cycle time is c/P_B. Thus, the expected cycle cost is written as follows:

$$\mathbb{E}\big[\text{cycle cost}\big] = K + \frac{hQ^2}{2} + (1 - P_{AA}(Q))\frac{c}{P_B}.$$

∎

Proof of Corollary 5 and Derivation of Q^*. By substituting the approximation $P_{AA}(Q) \approx P_B/(P_A + P_B)$ into the average cost expression given in Corollary 4 yields $\tilde{C}(Q)$, which contains the following components:

$$\mathbb{E}\big[\text{cycle cost}\big] = K + \frac{hQ^2}{2} + \frac{P_A c}{(P_A + P_B)P_B},$$

$$\mathbb{E}\big[\text{cycle length}\big] = Q + \frac{P_A}{(P_A + P_B)P_B}.$$

Now, consider a change of variable using $x = Q + P_A/(P_A + P_B)P_B$

$$\mathbb{E}\big[\text{cycle cost}\big] = K + \frac{h}{2}\Big(x - \frac{P_A}{(P_A + P_B)P_B}\Big)^2 + \frac{P_A c}{(P_A + P_B)P_B}$$

$$\mathbb{E}\big[\text{cycle length}\big] = x$$

Letting

$$\tilde{C}(x) = \frac{K + \frac{h}{2}\Big(x - \frac{P_A}{(P_A+P_B)P_B}\Big)^2 + \frac{P_A c}{(P_A+P_B)P_B}}{x},$$

we perform the following optimization,

$$\max \ \tilde{C}(x) \ \text{ subject to } x \geq \frac{P_A}{(P_A + P_B)P_B}.$$

In doing so, we arrive at the following expression:

$$\tilde{C}(x) = \Big(K + \frac{P_A c}{(P_A + P_B)P_B}\Big)\frac{1}{x} + \frac{h}{2}\frac{\Big(x^2 - \frac{2P_A}{(P_A+P_B)P_B}x + \frac{P_A^2}{(P_A+P_B)^2P_B^2}\Big)}{x},$$

$$\tilde{C}(x) = \Big(K + \frac{P_A c}{(P_A + P_B)P_B} + \frac{hP_A^2}{2(P_A + P_B)^2 P_B^2}\Big)\frac{1}{x} + \frac{h}{2}x - \frac{hP_A}{(P_A + P_B)P_B}.$$

The first order conditions give

$$\tilde{C}'(x) = \frac{h}{2} - \Big(K + \frac{P_A c}{(P_A + P_B)P_B} + \frac{hP_A^2}{2(P_A + P_B)^2 P_B^2}\Big)\frac{1}{x^2}. \tag{A.89}$$

Taking the second derivative of $\tilde{C}$ gives

$$\tilde{C}''(Q) = \frac{h\Big(\frac{P_A}{(P_A+P_B)P_B}\Big)^2 + 2\Big[K + \frac{cP_A}{(P_A+P_B)P_B}\Big]}{\Big(Q + \frac{P_A}{P_B(P_A+P_B)}\Big)^3} > 0.$$

Thus, $\tilde{C}$ is convex. Therefore, setting $\tilde{C}'(Q) = 0$ in Equation (A.89) and selecting its positive root yields Q^* given in Equation (1.11).

For

$$x^2 = \frac{2}{h}\Big(K + \frac{P_A c}{(P_A + P_B)P_B} + \frac{hP_A^2}{2(P_A + P_B)^2 P_B^2}\Big),$$

we have that

$$Q^* = \sqrt{\frac{2}{h}\Big(K + \frac{P_A c}{(P_A + P_B)P_B}\Big) + \Big(\frac{P_A}{(P_A + P_B)P_B}\Big)^2} - \frac{P_A}{(P_A + P_B)P_B}$$

■

A.8 Examples of Cost Differences Between the Heuristics and Hybrid Policies

In this Appendix, we present two examples where the hybrid policy significantly outperforms the two heuristics. We provide the parameters for the examples used in Table A.1 and display the relative performance of the three policies in Figures A.2-A.3.

We first note that both of our examples use already discussed disruption parameters. Namely, Example 1 uses the same disruption parameters as the Lake Tanganyika setting, and Example 2 uses the same disruption parameters as the Heavily Disrupted setting. We also use the same fixed order cost of $\$1,000$, which we have used in all of the numerical examples included in the paper. In Example 1, both the state B ordering cost and holding cost are small relative to K. In Example 2, the state B ordering cost is only half that of the fixed order cost and the holding cost remains small relative to K. In various examples we tested throughout the course of our research, we find that lower holding cost parameters lead to a significant performance deterioration of the heuristic policies relative to the hybrid policy.

Table A.1: Parameter regimes for Appendix A.7 examples

Parameter	Example 1	Example 2
N	24	24
P_A	1/4	11/12
P_B	1/2	1/12
K	$\$1,000$	$\$1,000$
c	\$25	\$500
h	\$10	\$25
α	0.95	0.95

In Example 1, we find that the policy parameters for the (s,S)-type heuristic are $s = 1$ and $S = 14$, and the parameter for the EOQ-like heuristics is $Q = 14$. We further find that the parameters for the hybrid policy are $(s_A, S_A, S_B) = (1, 9, 6)$. The differences in the discounted costs between the hybrid policy and the two heuristics are shown in Figure A.2. We elucidate these differences by displaying the percentage improvement of the hybrid policy over the two policies in Figure A.3.

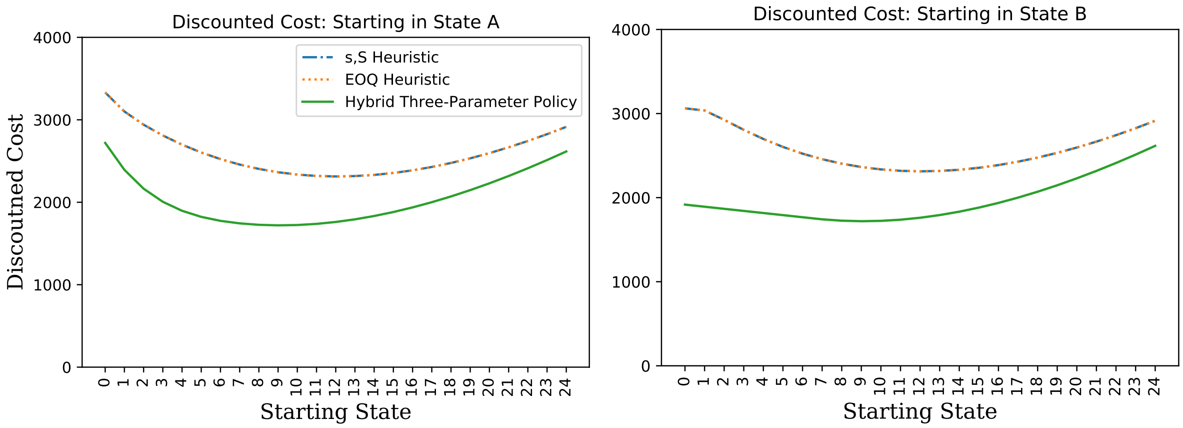

Figure A.2: Infinite-horizon discounted costs for the hybrid and two heuristic policies in Example 1.

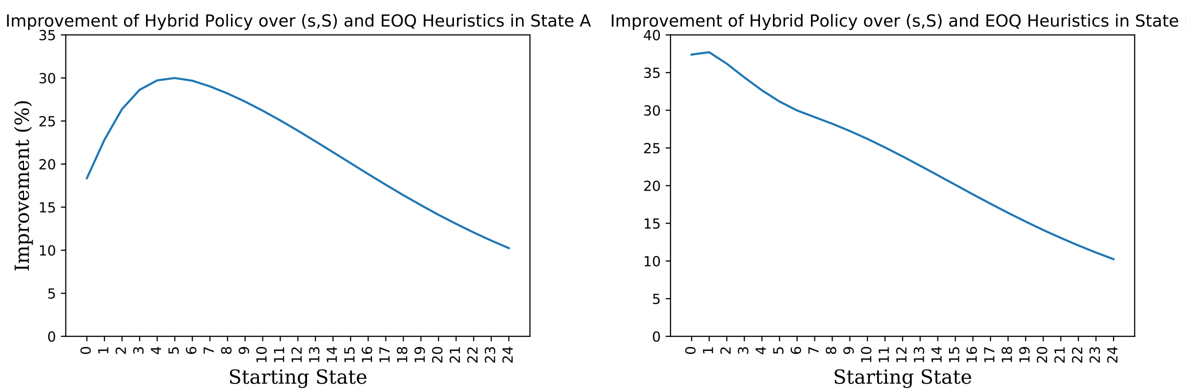

Figure A.3: Improvement of the discounted costs for the hybrid policy over the (s,S)-type and EOQ-like policies in Example 1.

In Example 2, we find that the policy parameters for the (s,S)-type heuristic are $s = 4$ and $S = 14$, and the parameter for the EOQ-like heuristics is $Q = 14$. We further find that the parameters for the hybrid policy are $(s_A, S_A, S_B) = (3, 12, 1)$. The differences in the discounted costs between all three policies are shown in Figure A.4. Once again, we elucidate these differences by displaying the percentage improvement of the hybrid policy over the other two policies in Figure A.5.

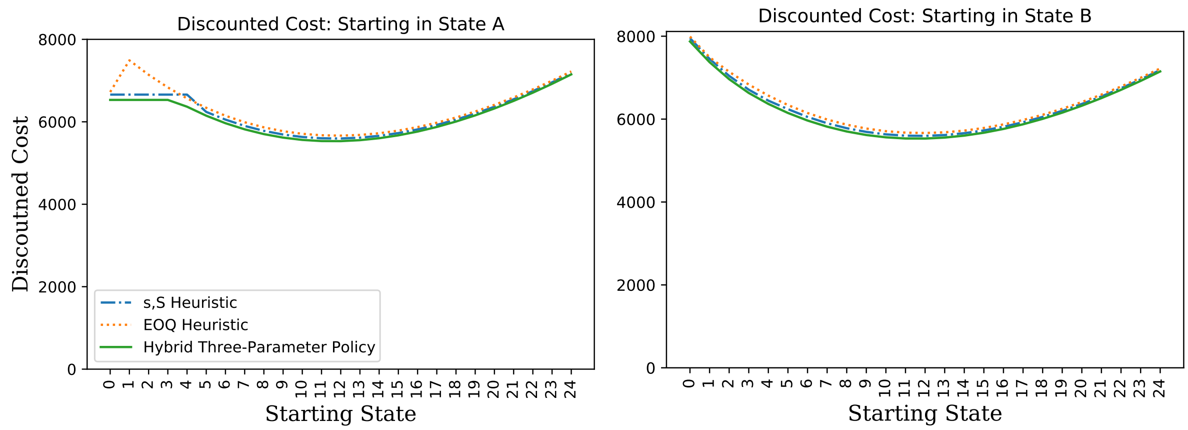

Figure A.4: Infinite-horizon discounted costs for the hybrid and the two heuristic policies in Example 2.

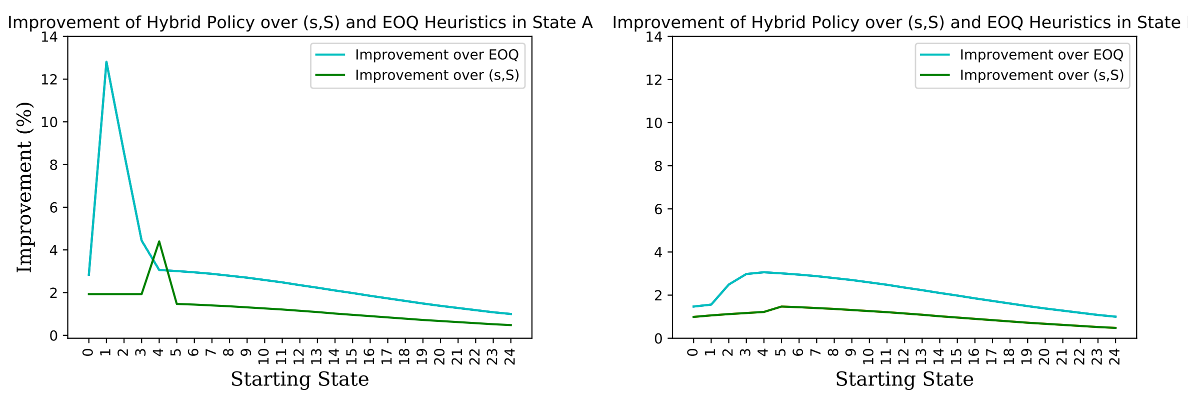

Figure A.5: Improvement of the discounted costs for the hybrid policy over the (s, S)-type and EOQ-like policies in Example 2.

A.9 Summary of Survey Conducted in Target Villages

In this Appendix, we present summary information from the 2016 household-level survey conducted by the Lake Tanganyika Floating Health Clinic. In Table A.2 we provide household composition information which also includes average malaria case counts. In Table A.3 we provide further household information, in this case as the percentage of households satisfying certain criteria related to malaria, bed net delivery frequency, diarrhea, and water access. In Table A.4 we provide self-reported bed net counts per household. In Table A.5 we provide information on bed nets and fishing nets related to their usage, for those observed both inside and outside houses. And, in Table A.6 we provide summary information for observed fishing groups, notably the primary tools used.

Table A.2: Household descriptors

Category	Average	Std. Deviation
Head of Household Age	43.87	14.01
Number of Children Under Age 15	4.73	2.56
Number of Children Under Age 5	1.89	1.37
Number of Adults	3.36	1.77
Number of Rooms in House	3.42	1.45
Adult Malaria Cases in Past 3 Months	2.07	1.83
Under 5 Malaria Cases in Past 3 Months	1.67	1.59

Table A.3: Aggregate household percentages

Category	N	Percentage
Adult Malaria Cases Diagnosed at Health Center	705	48.3%
Under 5 Malaria Cases Diagnosed at Health Center	651	56.2%
Households where Head of Household Sleeps under a Bed Net	750	71.3%
Last Received a Free Bed Net within a Month	705	35.6%
Last Received a Free Bed Net within 6 Months	705	47.5%
Last Received a Free Bed Net within the Last Year	705	9.65%
Last Received a Free Bed Net Over a Year Ago	705	7.23%
Households with Under 5 Diarrhea in Past 2 Weeks	701	48.4%
Households with Access to Toilet In-House or Nearby	277	69.3%
Households with Running Water	427	6.1%
Households which Treat Drinking Water	691	4.5%

Table A.4: Net reported during household visits

Type of Net	Total Count	Household Average	Std. Deviation
Households Included	811		
Mosquito Nets Owned	2649	3.27	2.17
Mosquito Nets Received for Free	2502	3.09	2.08

Table A.5: Nets observed during household visits

Type of Net	Total Count	Household Average	Std. Deviation
Households Included	569		
Mosquito Nets Inside	1223	2.15	1.58
Mosquito Nets Outside Used for Fishing	560	0.99	1.26
Mosquito Nets Outside Not Used for Fishing	418	0.74	1.07
Mosquito Nets Outside with Uncertain Usage	1848	3.25	2.51
Other Nets Outside Used for Fishing	245	0.43	0.81

Table A.6: Fishing groups observed

Category	Count	Percentage
Number of Fishing Groups	2094	
Total People	13481	
Average Group Size	6.44	
Men Comprising Groups	8052	59.7%
Women Comprising Groups	1648	12.2%
Children Comprising Groups	3781	28.0%
Groups Using Bed Nets as Primary Tool	635	30.4%
Groups Using Other Nets as Primary Tool	984	47.1%
Groups Using Other Methods as Primary Tool	471	22.5%

Appendix B

Appendices of Chapter 2

B.1 Somalia Model Parameters

In this Appendix, we list the model parameters used in the Somalia setting. We begin with matrices of Neonatal Affected Fractions and Efficacy values for each intervention and cause of death pair in Tables B.1 and B.2. We then provide matrices for Maternal Affected Fractions and Efficacy values in Tables B.3 and B.4. The values in these matrices match those provided by LiST, except for the italicized values, which were adjusted to reflect the input from Somali health professionals.

Table B.1: Neonatal affected fraction

Intervention	Diarrhea	Sepsis	Pneumonia	Asphyxia	Prematurity	Tetanus	Congenital Anomalies	Other
Tetanus toxoid vaccination	0	0	0	0	0	1	0	0
Multiple micronutrient supplementation	0	1	1	1	1	0	0	0
Iron supplementation in pregnancy	0	0	0	0	0	0	*0.200*	0
Calcium supplementation	0	0	0	0	1	0	0	0
Balanced energy supplementation	0	1	1	1	1	0	0	0
KMC - Kangaroo mother care	0	0	0	0	*0.580*	0	0	0
Breastfeeding promotion	1	1	1	0	0	0	0	0
Hand washing with soap	1	0	0	0	0	0	0	0
Prevention of malaria in pregnancy	0	1	1	1	1	0	0	0
Basic sanitation	1	0	0	0	0	0	0	0
Clean birth environment	0	1	0	0	0	1	0	0
Clean cord care	0	1	0	0	0	1	0	0
Immediate drying and additional stimulation	0	0	0	1	1	0	0	0
Neonatal resuscitation	0	0	0	1	1	0	0	0
Uterotonics for postpartum hemorrhage	0	0	0	0	0	0	0	0
Syphilis detection and treatment	0	1	1	1	1	0	0	0
Thermal regulations	0	1	0	0	1	0	0	0
Manual removal of placenta	0	0	0	0	0	0	0	0
Point-of-use filtered water	1	0	0	0	0	0	0	0
Malaria case management	0	0	0	0	0	0	0	0
Injectable antibiotics for neonatal sepsis	0	1	1	0	0	0	0	0
Safe abortion services	0	0	0	0	0	0	0	0
Oral antibiotics for neonatal sepsis	0	1	1	0	0	0	0	0
Post-abortion case management	0	0	0	0	0	0	0	0
Antibiotics for treatment of dysentery	0.120	0	0	0	0	0	0	0

Table B.2: Neonatal effectiveness

Intervention	Diarrhea	Sepsis	Pneumonia	Asphyxia	Prematurity	Tetanus	Congenital Anomalies	Other
Tetanus toxoid vaccination	0	0	0	0	0	0.940	0	0
Multiple micronutrient supplementation	0	0.029	0.029	0.029	0.050	0	0	0
Iron supplementation in pregnancy	0	0	0	0	0	0	0.460	0
Calcium supplementation	0	0	0	0	0.120	0	0	0
Balanced energy supplementation	0	0.048	0.048	0.048	0.037	0	0	0
KMC - Kangaroo mother care	0	0	0	0	0.510	0	0	0
Breastfeeding promotion	0.025	0.022	0.022	0	0	0	0	0
Hand washing with soap	0.070	0	0	0	0	0	0	0
Prevention of malaria in pregnancy	0	0.031	0.031	0.031	0.074	0	0	0
Basic sanitation	0.160	0	0	0	0	0	0	0
Clean birth environment	0	0.270	0	0	0	0.380	0	0
Clean cord care	0	0.400	0	0	0	0.400	0	0
Immediate drying and additional stimulation	0	0	0	0.100	0.100	0	0	0
Neonatal resuscitation	0	0	0	0.300	0.100	0	0	0
Uterotonics for postpartum hemorrhage	0	0	0	0	0	0	0	0
Syphilis detection and treatment	0	0.011	0.010	0.010	0.023	0	0	0
Thermal regulations	0	0.100	0	0	0.200	0	0	0
Manual removal of placenta	0	0	0	0	0	0	0	0
Point-of-use filtered water	0.400	0	0	0	0	0	0	0
Malaria case management	0	0	0	0	0	0	0	0
Injectable antibiotics for neonatal sepsis	0	0.650	0.750	0	0	0	0	0
Safe abortion services	0	0	0	0	0	0	0	0
Oral antibiotics for neonatal sepsis	0	0.299	0.452	0	0	0	0	0
Post-abortion case management	0	0	0	0	0	0	0	0
Antibiotics for treatment of dysentery	0.820	0	0	0	0	0	0	0

Table B.3: Maternal affected fraction

Intervention	Antepartum Hemorrhage	Intrapartum Hemorrhage	Postpartum Hemorrhage	Hypertensive Disorders	Sepsis	Abortion	Embolism	Other Direct Causes	Indirect Causes
Tetanus toxoid vaccination	0	0	0	0	0	0	0	0	0.005
Multiple micronutrient supplementation	0	0	0	0	0	0	0	0	0
Iron supplementation in pregnancy	0	0	0	0	0	0	0	0	0
Calcium supplementation	0	0	0	1	0	0	0	0	0
Balanced energy supplementation	0	0	0	0	0	0	0	0	0
KMC - Kangaroo mother care	0	0	0	0	0	0	0	0	0
Breastfeeding promotion	0	0	0	0	0	0	0	0	0
Hand washing with soap	0	0	0	0	0	0	0	0	0
Prevention of malaria in pregnancy	0	0	0	0	0	0	0	0	0.045
Basic sanitation	0	0	0	0	0	0	0	0	0
Clean birth environment	0	0	0	0	1	0	0	0	0
Clean cord care	0	0	0	0	0	0	0	0	0
Immediate drying and additional stimulation	0	0	0	0	0	0	0	0	0
Neonatal resuscitation	0	0	0	0	0	0	0	0	0
Uterotonics for postpartum hemorrhage	0	0	1	0	0	0	0	0	0
Syphilis detection and treatment	0	0	0	0	0	0	0	0	0
Thermal regulations	0	0	0	0	0	0	0	0	0
Manual removal of placenta	0	0	1	0	0	0	0	0	0
Point-of-use filtered water	0	0	0	0	0	0	0	0	0
Malaria case management	0	0	0	0	0	0	0	0	0.045
Injectable antibiotics for neonatal sepsis	0	0	0	0	0	0	0	0	0
Safe abortion services	0	0	0	0	0	0.816	0	0	0
Oral antibiotics for neonatal sepsis	0	0	0	0	0	0	0	0	0
Post-abortion case management	0	0	0	0	0	0.816	0	0	0
Antibiotics for treatment of dysentery	0	0	0	0	0	0	0	0	0

Table B.4: Maternal effectiveness

Intervention	Antepartum Hemorrhage	Intrapartum Hemorrhage	Postpartum Hemorrhage	Hypertensive Disorders	Sepsis	Abortion	Embolism	Other Direct Causes	Indirect Causes
Tetanus toxoid vaccination	0	0	0	0	0	0	0	0	0.980
Multiple micronutrient supplementation	0	0	0	0	0	0	0	0	0
Iron supplementation in pregnancy	0	0	0	0	0	0	0	0	0
Calcium supplementation	0	0	0	0.200	0	0	0	0	0
Balanced energy supplementation	0	0	0	0	0	0	0	0	0
KMC - Kangaroo mother care	0	0	0	0	0	0	0	0	0
Breastfeeding promotion	0	0	0	0	0	0	0	0	0
Hand washing with soap	0	0	0	0	0	0	0	0	0
Prevention of malaria in pregnancy	0	0	0	0	0	0	0	0	0.725
Basic sanitation	0	0	0	0	0	0	0	0	0
Clean birth environment	0	0	0	0	0.600	0	0	0	0
Clean cord care	0	0	0	0	0	0	0	0	0
Immediate drying and additional stimulation	0	0	0	0	0	0	0	0	0
Neonatal resuscitation	0	0	0	0	0	0	0	0	0
Uterotonics for postpartum hemorrhage	0	0	0.775	0	0	0	0	0	0
Syphilis detection and treatment	0	0	0	0	0	0	0	0	0
Thermal regulations	0	0	0	0	0	0	0	0	0
Manual removal of placenta	0	0	0.300	0	0	0	0	0	0
Point-of-use filtered water	0	0	0	0	0	0	0	0	0
Malaria case management	0	0	0	0	0	0	0	0	0.800
Injectable antibiotics for neonatal sepsis	0	0	0	0	0	0	0	0	0
Safe abortion services	0	0	0	0	0	0.950	0	0	0
Oral antibiotics for neonatal sepsis	0	0	0	0	0	0	0	0	0
Post-abortion case management	0	0	0	0	0	0.800	0	0	0
Antibiotics for treatment of dysentery	0	0	0	0	0	0	0	0	0

In Table B.5, we display the maximum achievable coverage for each intervention with a fixed number of 1,450 FHWs. The calculation for these values is described in Section 2.2.2.

Table B.5: Maternal and neonatal interventions

Intervention	Maximum allowable level of coverage (1,450 FHWs)
Tetanus toxoid vaccination	76.88%
Multiple micronutrient supplementation	9.88%
Iron supplementation in pregnancy	9.88%
Calcium supplementation	9.88%
Balanced energy supplementation	9.88%
KMC - Kangaroo mother care	15.23%
Breastfeeding promotion	18.69%
Hand washing with soap	19.71%
Prevention of malaria in pregnancy	12.36%
Basic sanitation	48.21%
Clean birth environment	16.55%
Clean cord care	18.15%
Immediate drying and additional stimulation	17.61%
Neonatal resuscitation	13.61%
Uterotonics for postpartum hemorrhage	17.24%
Syphilis detection and treatment	13.65%
Thermal regulations	19.06%
Manual removal of placenta	12.38%
Point-of-use filtered water	14.37%
Malaria case management	11.16%
Injectable antibiotics for neonatal sepsis	19.24%
Safe abortion services	13.18%
Oral antibiotics for neonatal sepsis	9.88%
Post abortion case management	9.88%
Antibiotics for treatment of dysentery	9.88%

As outlined in Section 2.2.1, we need to provide estimates for the number of treatments required to reach full coverage. To do so, in Table B.6, for each intervention, we provide the title of the target population, the size of the target population, the percentage of the target population requiring treatment, and the resulting number of treatments needed to reach full coverage. As before, the values in Table B.6 are provided by LiST, with the exception of the italicized values.

Table B.6: Calculations for the estimated number of treatments

Intervention	Target Population	Size	Percentage Requiring Treatment	Number of Treatments Needed
Tetanus toxoid vaccination	Pregnant Women	973,781	100%	973,781
Multiple micronutrient supplementation	Pregnant Women	973,781	100%	973,781
Iron supplementation in pregnancy	Pregnant Women	973,781	100%	973,781
Calcium supplementation	Pregnant Women	973,781	100%	973,781
Balanced energy supplementation	Pregnant Women	973,781	7%	65,243
KMC - Kangaroo mother care	Live Births	716,518	12%	85,982
Breastfeeding promotion	Live Births	716,518	100%	716,518
Hand washing with soap	Number of Households	3,320,000	100%	3,320,000
Prevention of malaria in pregnancy	Pregnant Women	973,781	100%	973,781
Basic sanitation	Number of Households	3,320,000	100%	3,320,000
Clean birth environment	Pregnancies Carried to Term	843,109	100%	843,109
Clean cord care	Live Births	716,518	100%	716,518
Immediate drying and additional stimulation	Live Births	716,518	100%	716,518
Neonatal resuscitation	Live Births	716,518	7%	50,156
Uterotonics for postpartum hemorrhage	Pregnancies Carried to Term	843,109	1%	10,117
Syphilis detection and treatment	Pregnant Women	973,781	100%	973,781
Thermal regulations	Live Births	716,518	100%	716,518
Manual removal of placenta	Pregnancies Carried to Term	843,109	2%	16,862
Point-of-use filtered water	Number of Households	3,320,000	100%	3,320,000
Malaria case management	Pregnant Women	973,781	20%	194,756
Injectable antibiotics for neonatal sepsis	Live Births	716,518	8%	54,455
Safe abortion services	Abortions	130,672	100%	130,672
Oral antibiotics for neonatal sepsis	Live Births	716,518	8%	54,455
Post-abortion case management	Abortions	130,672	24%	31,623
Antibiotics for treatment of dysentery	Children 0-59 Months	101,211	10%	10,121

B.2 Somalia Model Constraints

In this Appendix, we provide detailed calculations for the constraints used in the Somalia setting. We begin with the time required by FHWs to deliver interventions. In order to calculate the amount of time each intervention requires, we multiply the number of treatments needed to reach full coverage for that intervention by the frequency of providing that intervention and the amount of time each treatment takes to provide. We then divide this number by twelve to produce the time required to reach full coverage of an intervention in one month. The Somalia setting values for this calculation are provided in Table B.7.

The first constraint in our model is the total time available to offer maternal and neonatal services per month. As discussed earlier, the over-tasking of FHWs often makes available time one of the limiting factors to providing adequate care. There is a certain degree of flexibility in how the FHWs divide their time between households. It is reasonable to expect that FHWs will prioritize households with neonates and new mothers, while still attending to their required number of households per month. It would thus seem improper to place a strict time constraint on MNH care. However, if you were to show a list of interventions you expect an FHW to provide as part of her service package, she would likely be able to say right away whether the number of interventions is reasonable, too many, or too few. We attempt to capture this, to a degree, in the time constraint. Unlike funding budgets, the interpretation of the time constraint should be less rigid. This uncertainty reaffirms the need to have buy-in from all levels of program stakeholders. The community providers and FHWs will have a better feel for what is reasonable for their setting and thus need to be part of an iterative modeling process, whereby feedback informs changes to the implementation of the model's insights if not the model overall.

The right-hand side of the constraint (or time budget) is calculated by multiplying together the number of FHWs (1,450), by the time allotted for MNH care which is given by the time FHWs spend on tasks per month (6 days x 4 hours x 4.33 weeks) minus the travel time per month (60 mins per day) and time spent on non-MNH tasks. We allow for the time spent on non-MNH tasks to be represented as the percentage of time spend on non-MNH responsibilities (70%).

$$1450\left[(6 \times 240 \times 4.33) - \left(6 \times 60 \times 4.33 + 0.7\left(6 \times 240 \times 4.33\right)\right)\right] = 1{,}356{,}156 \text{ minutes/month}$$

1,356,156 minutes per month is equivalent to 22,602 hours per month.

Table B.7: Time requirements for providing interventions

Intervention	Number of Treatments Needed	Annual Frequency	Time to Provide Intervention (mins)	Monthly Full Coverage Requirement (mins)
Tetanus toxoid vaccination	973,781	2	5	811,484
Multiple micronutrient supplementation	973,781	1	10	811,484
Iron supplementation in pregnancy	973,781	1	8	649,187
Calcium supplementation	973,781	1	8	649,187
Balanced energy supplementation	65,243	1	8	43,496
KMC - Kangaroo mother care	85,982	1	40	286,607
Breastfeeding promotion	716,518	1	30	1,791,295
Hand washing with soap	3,320,000	1	5	1,383,333
Prevention of malaria in pregnancy	973,781	1	5	405,742
Basic sanitation	3,320,000	1	5	1,383,333
Clean birth environment	843,109	1	60	4,215,545
Clean cord care	716,518	1	15	895,648
Immediate drying and additional stimulation	716,518	1	10	597,098
Neonatal resuscitation	50,156	1	20	83,594
Uterotonics for postpartum hemorrhage	10,117	1	90	75,880
Syphilis detection and treatment	973,781	1	15	1,217,226
Thermal regulations	716,518	1	30	1,791,295
Manual removal of placenta	16,862	1	100	140,518
Point-of-use filtered water	3,320,000	1	10	2,766,667
Malaria case management	194,756	1	15	243,445
Injectable antibiotics for neonatal sepsis	54,455	1	40	181,518
Safe abortion services	130,672	1	30	326,680
Oral antibiotics for neonatal sepsis	54,455	1	5	22,690
Post-abortion case management	31,623	1	90	237,170
Antibiotics for treatment of dysentery	10,121	1	2	1,687

We next calculate the supervision time available to oversee FHW in-service training. In order to calculate the amount of supervision time each intervention requires we multiply the number of observations required by the time required per observation, and the number of FHWs (1,450). These calculations are shown in Table B.8.

The right-hand side of the constraint (or supervision time budget) is calculated by multiplying the number of supervisors by the minutes per year each supervisor has to observe or support FHWs. We assume that each supervisor has 4 hours a day for supervision tasks, works 5 days a week, and spends 25% of her time traveling and 50% of the remaining time on MNH supervision. In Somalia, there is a supervisor for every 12 CHWs. The calculation of the supervision time budget is shown in Equation (B.1).

Table B.8: Supervision time required to certify intervention proficiency

Intervention	Number of Observations Required (per year)	Time per Observation	Supervision Time Required for Full Coverage
Tetanus toxoid vaccination	12	5	87,000
Multiple micronutrient supplementation	24	10	348,000
Iron supplementation in pregnancy	24	8	278,400
Calcium supplementation	24	8	278,400
Balanced energy supplementation	24	8	278,400
KMC - Kangaroo mother care	24	50	1,740,000
Breastfeeding promotion	12	40	696,000
Hand washing with soap	12	5	87,000
Prevention of malaria in pregnancy	12	5	87,000
Basic sanitation	12	5	87,000
Clean birth environment	12	60	1,044,000
Clean cord care	24	30	1,044,000
Immediate drying and additional stimulation	12	30	522,000
Neonatal resuscitation	12	60	1,044,000
Uterotonics for postpartum hemorrhage	24	90	3,132,000
Syphilis detection and treatment	24	20	696,000
Thermal regulations	24	50	1,740,000
Manual removal of placenta	24	100	3,480,000
Point-of-use filtered water	12	15	261,000
Malaria case management	48	15	1,044,000
Injectable antibiotics for neonatal sepsis	24	40	1,392,000
Safe abortion services	24	30	1,044,000
Oral antibiotics for neonatal sepsis	24	5	174,000
Post-abortion case management	48	30	2,088,000
Antibiotics for treatment of dysentery	24	2	69,600

$$\frac{1450}{12} \times 0.5(1-0.25)(5 \times 240 \times 52) = 2{,}827{,}500 \text{ minutes per year} \tag{B.1}$$

2,827,500 minutes per year is equivalent to 3,927 hours per month.

We now calculate the costs for increasing the coverage for intervention. In order to calculate the cost to offer each intervention at full coverage, we multiply the number of treatments needed for full coverage by the cost of one unit of that treatment's commodity and the number of commodities needed per treatment. These calculations for the Somalia setting are shown in Table B.9. The annual budget to scale interventions ($435,000), which serves as the right-hand side of the constraint, is provided by the Somali Ministry of Health.

Table B.9: Costs to increase coverage of interventions

Intervention	Treatments	Cost per Commodity	Commodities per Treatment	Total Cost
Tetanus toxoid vaccination	973,781	$0.22	2	$428,464
Multiple micronutrient supplementation	973,781	$0.50	1	$486,891
Iron supplementation in pregnancy	973,781	$0.04	150	$5,842,686
Calcium supplementation	973,781	$11.00	1	$10,711,591
Balanced energy supplementation	65,243	$5.00	1	$326,217
KMC - Kangaroo mother care	85,982	$0	1	$0
Breastfeeding promotion	716,518	$0	1	$0
Hand washing with soap	3,320,000	$0	1	$0
Prevention of malaria in pregnancy	973,781	$2.00	1	$1,947,562
Basic sanitation	3,320,000	$0	1	$0
Clean birth environment	843,109	$1.37	1	$1,155,059
Clean cord care	716,518	$0	1	$0
Immediate drying and additional stimulation	716,518	$0	1	$0
Neonatal resuscitation	50,156	$0.38	1	$19,059
Uterotonics for postpartum hemorrhage	10,117	$0.23	1	$2,327
Syphilis detection and treatment	973,781	$4.39	1	$4,274,899
Thermal regulations	716,518	$0	1	$0
Manual removal of placenta	16,862	$3.99	1	$67,280
Point-of-use filtered water	3,320,000	$10.00	1	$33,200,000
Malaria case management	194,756	$2.55	1	$496,628
Injectable antibiotics for neonatal sepsis	54,455	$1.24	1	$67,525
Safe abortion services	130,672	$1.26	1	$164,647
Oral antibiotics for neonatal sepsis	54,455	$0.12	1	$6,535
Post-abortion case management	31,623	$15.60	1	$493,313
Antibiotics for treatment of dysentery	10,121	$0.12	1	$1,215

Now that we have covered the three constraints that comprise Matrix A and Vector a of Equation (2.2), we move to Equation (2.3) and provide the values for the Somalia setting in Matrix B and Vector b. The first constraint in Matrix B requires the curriculum time required for each intervention. We list these values in Table B.10. As in the previous constraint, the amount of curriculum time available to teach MNH care (1,200 minutes) is provided by the Ministry of Health.

Table B.10: Curriculum time required to teach maternal and neonatal interventions

Intervention	Curriculum Time (minutes)
Tetanus toxoid vaccination	60
Multiple micronutrient supplementation	60
Iron supplementation in pregnancy	60
Calcium supplementation	60
Balanced energy supplementation	60
KMC - Kangaroo mother care	120
Breastfeeding promotion	180
Hand washing with soap	120
Prevention of malaria in pregnancy	60
Basic sanitation	60
Clean birth environment	180
Clean cord care	60
Immediate drying and additional stimulation	240
Neonatal resuscitation	960
Uterotonics for postpartum hemorrhage	180
Syphilis detection and treatment	120
Thermal regulations	120
Manual removal of placenta	240
Point-of-use filtered water	10
Malaria case management	45
Injectable antibiotics for neonatal sepsis	120
Safe abortion services	240
Oral antibiotics for neonatal sepsis	60
Post-abortion case management	240
Antibiotics for treatment of dysentery	60

The application of the model to Somalia does not include considerations for a budget to launch interventions and is thus not included in this Appendix. Additionally, the government permission status for each intervention is included in Table 2.4 of Section 2.2.3.

B.3 Expanded Version of the Model without Injections

In this Appendix, we provide the results from the expanded model discussed in Section 2.3.1. In this version of the model, we perform the same analysis we did in Table 2.11 but with interventions involving injections removed. As such, tetanus toxoid vaccination, syphilis detection and treatment, manual removal of placenta, injectable antibiotics for neonatal sepsis, safe abortion services, and post-abortion case management are not eligible for inclusion. The results are shown in Table B.11. The objective function has a value of 1,335 lives saved.

Table B.11: Optimal package of interventions with relaxed restrictions excluding injections

Intervention	y_j	Coverage Change (x_j)	Coverage Target	Current Coverage
Tetanus toxoid vaccination	0		67%	67%
MM supplementation in pregnancy	1	9.88%	9.88%	0%
Iron supplementation in pregnancy	0		0%	0%
Calcium supplementation	0		0%	0%
Balanced energy supplementation	1	9.88%	9.88%	0%
KMC - Kangaroo mother care	1	9.88%	14.88%	5%
Breastfeeding promotion	0		9%	9%
Hand washing with soap	0		10%	10%
Prevention of malaria in pregnancy	1	9.88%	11.88%	2%
Basic sanitation	0		38%	38%
Clean birth environment	1	9.88%	16.88%	7%
Clean cord care	1	9.88%	17.88%	8%
Immediate drying and stimulation	1	9.88%	17.88%	8%
Neonatal resuscitation	0		4%	4%
Uterotonics for postpartum hemorrhage	1	9.88%	16.88%	7%
Syphilis detection and treatment	0		4%	4%
Thermal regulations	1	9.88%	18.88%	9%
Manual removal of placenta	0		3%	3%
Point-of-use filtered water	1	0.13%	4.13%	4%
Malaria case management	0		1%	1%
Injectable antibiotics for neonatal sepsis	0		9%	9%
Safe abortion services	0		3%	3%
Oral antibiotics for neonatal sepsis	1	9.88%	9.88%	0%
Post-abortion case management	0		0%	0%
Antibiotics for treatment of dysentery	0		0%	0%

B.4 Optimization Methods

In this Appendix, we discuss the methods used to solve the math program. The difficulty with this model rests in the structure of the program, which, as noted earlier, is a mixed integer non-linear formulation. With such a formulation, a solution is not guaranteed to be optimal. To address this issue, we rely on the techniques used by Qin et al. (2022). In their paper, they require a fast method for solving an integer program for automatic guided vehicles transporting shelves within a warehouse. To do so, they show that their problem is separable when a Lagrangian relaxation is introduced. As their program and ours share a similar structure, we are able to follow the same steps to create two subproblems (one for each set of decision variables x and y) and solve them in sequential order.

For convenience, we define the *mixed-integer program for maternal and newborn health-care optimization* comprised of Equations (2.1)-(2.7) as (P), and $V(P)$ as the optimal objective value of the problem (P). We introduce $\lambda := \{\lambda_j \geq 0, j \in J\}$ as the Lagrangian multipliers associated with constraint (2.5). This allows us to write the following partially relaxed problem:

$$\hat{P}(\Lambda): \max_{x,y} D_n \sum_{i=1}^{8} r_i \Big[1 - \prod_{j=1}^{J} \Big(1 - f_{ij} \frac{e_{ij}x_j}{1 - e_{ij}c_j}\Big)\Big] + D_M \sum_{i=9}^{17} r_i \Big[1 - \prod_{j=1}^{J} (1 - f_{ij} \frac{e_{ij}x_j}{1 - e_{ij}c_j})\Big]$$
$$- \sum_{j=1}^{J} \lambda_j \big(x_j - y_j\big)$$

subject to

$$Ax \leq a \tag{B.2}$$

$$By \leq b \tag{B.3}$$

$$x_j \leq \min(u_j - c_j,\ d,\ \beta_j/\alpha_j),\ \forall j \tag{B.4}$$

$$y_j \in \{0, 1\},\ \forall\ j \tag{B.5}$$

$$x_j \geq 0,\ \forall\ j \tag{B.6}$$

Under duality, for any Λ we have that $V(\hat{P}(\Lambda))$ is an upper bound on the original problem $V(P)$, meaning $V(\hat{P}(\Lambda)) \geq V(P)$. The optimal dual variable Λ^*, is given by $\Lambda^* = \arg\min_\Lambda V(\hat{P}(\Lambda))$.

For any Λ, x_j and y_j become totally separable within the problem $(\hat{P}(\Lambda))$. Because of this separability, any solution to the two subproblems $(P^1(\Lambda))$ and $(P^2(\Lambda))$, is also a solution to $(\hat{P}(\Lambda))$. However, this solution may violate constraint Equation (2.5) of problem (P), and thus be infeasible for the original problem of interest.

$$P^1(\Lambda): \max_x \; D_n \sum_{i=1}^{8} r_i\Big[1 - \prod_{j=1}^{J}\big(1 - f_{ij}\frac{e_{ij}x_j}{1 - e_{ij}c_j}\big)\Big] + D_M \sum_{i=9}^{17} r_i\Big[1 - \prod_{j=1}^{J}(1 - f_{ij}\frac{e_{ij}x_j}{1 - e_{ij}c_j})\Big] - \sum_{j=1}^{J}\lambda_j x_j$$

subject to

$$Ax \le a \tag{B.7}$$

$$x_j \le \min(u_j - c_j,\; d,\; \beta_j/\alpha_j),\;\; \forall j \tag{B.8}$$

$$x_j \ge 0,\;\; \forall\; j \tag{B.9}$$

$$P^2(\Lambda): \max_y \; \sum_{j=1}^{J}\lambda_j y_j$$

subject to

$$By \le b \tag{B.10}$$

$$y_j \in \{0,1\},\;\; \forall\; j \tag{B.11}$$

To arrive at a feasible solution to (P), we approach this problem by solving for problem (P^2) and using the solution for y^* to solve (P^1), which is a modified version of $(P^1(\Lambda))$, which is written as follows:

$$P^1 \; \max_x \; D_n \sum_{i=1}^{8} r_i\Big[1 - \prod_{j=1}^{J}\big(1 - f_{ij}\frac{e_{ij}x_j}{1 - e_{ij}c_j}\big)\Big] + D_M \sum_{i=9}^{17} r_i\Big[1 - \prod_{j=1}^{J}(1 - f_{ij}\frac{e_{ij}x_j}{1 - e_{ij}c_j})\Big]$$

subject to

$$Ax \le a \tag{B.12}$$

$$x_j \le \min(u_j - c_j,\; d,\; \beta_j/\alpha_j),\;\; \forall j \tag{B.13}$$

$$x_j \geq 0, \ \forall \ j \tag{B.14}$$

$$x_j \leq y_j^* \ \forall j \tag{B.15}$$

B.5 Ethiopia Version of the Model

In this Appendix, we provide an additional point of comparison to better understand the improvements from performing optimization on community-health program design. Even with the extensions of the model seen in Tables 2.11 and B.11, we are still comparing projections from models that are optimized, with no quantified understanding of how much is gained by applying this optimization approach to the decision-making process. To better interpret the results in this context, we compare the program recommended by the model to the current structure of the CHW program in Ethiopia. Ethiopia is used as a point of comparison both for its geographical proximity and its regard as a success story for community-health programs in developing countries. As of 2015, there are an estimated 128,000 CHWs operating in Ethiopia (Perry et al. 2014). From 2004, when the CHW program was first introduced in Ethiopia, to 2020 the infant mortality rate fell from 73 per 1000 live births to 35 (World Bank 2020a). During the same time in Somalia, the infant mortality rate only fell from 104 to 73, a 52.05% decrease in Ethiopia with only a 29.81% decrease in Somalia (World Bank 2020b).

Because of the success in Ethiopia, a reasonable policy when establishing a community-health program in Somalia would be to copy the interventions used in Ethiopia. We artificially replicate this approach by taking the list of interventions and their corresponding levels of coverage in Ethiopia, according to the LiST database, and applying them to the Somalia setting. In this way, we offer the same treatments in Somalia as are offered in Ethiopia. Since the CHW program in Ethiopia is much larger than the nascent program in Somalia, we cap the levels of coverage for the interventions with the FHW capacity (d). We then calculate the number of lives saved and the demand for the resources available. The coverage levels for this application are shown in Table B.12.

Table B.12: Ethiopia CHW intervention coverage levels applied to Somalia FHW program

Intervention	Coverage Change (x_j)	Coverage Target	Current Coverage
Tetanus toxoid vaccination	9.88%	76.88%	67%
MM supplementation in pregnancy		0%	0%
Iron supplementation in pregnancy	9.88%	9.88%	0%
Calcium supplementation		0%	0%
Balanced energy supplementation		0%	0%
KMC - Kangaroo mother care		5%	5%
Breastfeeding promotion	9.88%	18.88%	9%
Hand washing with soap		10%	10%
Prevention of malaria in pregnancy		2%	2%
Basic sanitation		38%	38%
Clean birth environment	9.88%	16.88%	7%
Clean cord care	9.88%	17.88%	8%
Immediate drying and stimulation	9.88%	17.88%	8%
Neonatal resuscitation	9.88%	13.88%	4%
Uterotonics for postpartum hemorrhage		7%	7%
Syphilis detection and treatment	9.88%	13.88%	4%
Thermal regulations	9.88%	17.88%	9%
Manual removal of placenta		3%	3%
Point-of-use filtered water		4%	4%
Malaria case management	9.88%	10.88%	1%
Injectable antibiotics for neonatal sepsis	9.88%	18.88%	9%
Safe abortion services		3%	3%
Oral antibiotics for neonatal sepsis		0%	0%
Post-abortion case management		0%	0%
Antibiotics for treatment of dysentery	9.88%	9.88%	0%

The package of interventions shown in Table B.12 is projected to lead to 1,835 lives saved. This is 155 fewer lives saved than the optimized version in Table 2.11. Yet, it is achieved through the use of resources that far outstrip those available in Somalia at this time. In Table B.13 we list the amount of each constraint used to achieve the levels of coverage offered in Table B.12.

Table B.13: Constraints on FHWs

Constraint	Budget	Usage
Total time available per month for care	22,602 hours	18,119 hours
Total supervisor time per month for training	3,927 hours	1,497 hours
Total budget available for scaling interventions	$ 435,000	$ 1,218,010
Total time available in curriculum	1,200 mins	2,225 mins

It is clear that by simply applying the interventions used in Ethiopia to Somalia, the projection of 1,835 lives saved is unrealistic, as the resources required in terms of money and curriculum time are greater than those available under our optimization projection. Furthermore, if we do relax the constraints to reflect the capacity implied by the Ethiopia level of coverage (i.e. increase the budget to $1,218,010 and the curriculum time to 2,225 minutes), the optimized model achieves 2,417 projected lives saved.

B.6 Expanded Sensitivity Analysis

In this Appendix, we add to the sensitivity analysis performed in Section 2.3.2 by adjusting the efficacy of select interventions. One limitation of our model is the lack of data on the relative efficacy of interventions in the community setting relative to the health facility setting. The data provided by LiST is mostly informed by studies of efficacy in health facility-type settings. Many of these interventions can reasonably be expected to perform as well when delivered in the field; however, there are several for which the procedures involved in implementation would need to be altered in order to be provided in the community setting. As a means to better understand how differences between efficacy in the field compared to efficacy at the health facility could alter the projected number of lives saved and the composition of the optimal solution of the model, we conduct sensitivity analysis on the six interventions we feel might differ in their efficacy between the two settings. These interventions are kangaroo mother care, clean cord care, immediate drying and additional stimulation, neonatal resuscitation, uterotonics for postpartum hemorrhage, and thermal regulation.

In Table B.14, we list the interventions identified for this analysis and their efficacy values for the health facility setting. We then indicate whether or not an intervention is included in the optimal package of interventions shown in Table 2.11. Using the same constraints and the expanded list of permissible interventions, we re-run the model for each intervention with the efficacy values for that intervention adjusted down by half and then three-quarters, holding the efficacy values of the other interventions in the model unchanged. We then indicate if the intervention is still included in the optimal set of interventions, and how many lives saved the model predicts will result from this change.

Table B.14: Intervention efficacy sensitivity analysis

Intervention	Current efficacy	Included with Current Efficacy	0.5×Efficacy Included	Lives Saved	0.25×Efficacy Included	Lives Saved
Kangaroo mother care	0.510	Yes	Yes	1903	No	1878
Clean cord care	0.400*	Yes	Yes	1918	Yes	1883
Immediate drying and additional stimulation	0.100*	Yes	Yes	1914	Yes	1902
Neonatal resuscitation	0.300*	No	No	1990	No	1990
Uterotonics for postpartum hemorrhage	0.775	No	No	1990	No	1990
Thermal regulation	0.200*	Yes	Yes	1922	No	1914

We place an asterisk next to the efficacy values of the interventions that impact multiple causes of death. For these interventions, we list the larger of the two efficacy values (all of these interventions affect at most two causes of death). When we perform the analysis of decreasing the efficacy values of one intervention at a time, we adjust all the efficacy values for that intervention by the same weight. For example, thermal regulation has an efficacy value of 0.1 on the neonatal sepsis cause of death and an efficacy value of 0.2 on the neonatal prematurity cause of death. When we run the model with 0.5 times the efficacy of thermal regulation we adjust both thermal regulation's efficacy value on neonatal sepsis to 0.05 and its efficacy value on neonatal prematurity to 0.1 at the same time.

Of the six interventions identified, four are included in the optimal set of interventions from Table 2.11. We do not perform sensitivity analysis on the two interventions not included (neonatal resuscitation and uterotonics for postpartum hemorrhage), as they will still not be included in the set of optimal interventions once their efficacy values are reduced. With a baseline of 1,990 lives saved, we see that all four of the interventions included in the original optimal set are still included if their efficacy values are reduced by 50%. Once the efficacy of these interventions is reduced by 75%, we see that kangaroo mother care and thermal regulation are removed from the set of optimal interventions, whereas clean cord care and immediate drying and additional stimulation remain in the optimal set. This analysis should prove useful to practitioners who express uncertainty about the relative efficacy of these interventions when performed in the field as opposed to in a health facility setting. Furthermore, this analysis can be extended to other interventions with finer gradations of efficacy adjustments.

Bibliography

Achee NL, Bangs MJ, Farlow R, Killeen GF, Lindsay S, Logan JG, Moore SJ, Rowland M, Sweeney K, Torr SJ, Zwiebel LJ, Grieco JP (2012) Spatial repellents: from discovery and development to evidence-based validation. *Malaria Journal*. 11(164).

Ahiska SS, Appaji SR, King RE, Warsing DP (2013) A Markov decision process-based policy characterization approach for a stochastic inventory control problem with unreliable sourcing. *International Journal of Production Economics*. 144:485–496.

Alemu MB, Asnake MA, Lemma MY, Melak MF, Yenit MK (2018) Utilization of insecticide treated bed net and associated factors among households of Kola Diba town, North Gondar, Amhara region, Ethiopia. *BMC Research Notes*. 11:575.

Altare C, Castelgrande V, Tosha M, Malembaka EB, Spiegel P (2021) From insecurity to health service delivery: Pathways and system response strategies in the Eastern Democratic Republic of the Congo. *Global Health: Science and Practice*. 9(4):915–927.

Altay N, Green WG III (2006) OR/MS research in disaster operations management. *European Journal of Operations Research*. 175:475–493.

Anaya-Arenas AM, Renaud J, Ruiz A (2014) Relief distribution networks: a systematic review. *Annals of Operations Research*. 223:53–79.

Arreola-Risa A, Decroix G (1998) Inventory management under random supply disruptions and partial backorders. *Naval Research Logistics*. 45:687–703.

Ata B, Lobo NF, Montgomery RJ, Lehman A (2023) A last mile supply chain pilot program for mosquito spatial repellents in the DRC. Working paper, University of Chicago, Chicago.

Azimi P, Ghanbari MR, Mohammadi H (2012) Simulation modeling for analysis of a (Q, r) inventory system under supply disruption and customer differentiation with partial backordering. *Modelling and Simulation in Engineering*.

Bakal IS, Bayindir ZP, Emer DE (2017) Value of disruption information in an EOQ environment. *European Journal of Operational Research*. 263:46–460.

Balcik B, Beamon BM, Krejci CC, Muramatsu KM, Ramirez M (2010) Coordination in humanitarian relief chains: Practices, challenges and opportunities. *International Journal of Production Economics*. 126:22-34.

Bam L, McLaren ZM, Coetzee E, von Leipzig KH (2017) Reducing stock-outs of essential tuberculosis medicines: A system dynamics modelling approach to supply chain management. *Health Policy and Planning.* 32:1127–1134.

Barrington J, Wereko-Brbby O, Ward P, Mwafongo W, Kungulwe S (2010) SMS for life: A pilot project to improve anti-malarial drug supply management in rural Tanzania using standard technology. *Malaria Journal.* 9:298.

Berk E, Arreola-Risa A (1994) Note on "Future Supply Uncertainty in EOQ Models." *Naval Research Logistics.* 41:129–132.

Beyer D, Chen F, Sethi SP, Taksar M (2010) *Markovian Demand Inventory Models.* (Springer, New York, NY).

Billingsley P (1999) *Convergence of Probability Measures*, 2nd ed. (John Wiley & Sons).

Bush ER, Short RE, Milner-Gulland EJ, Lennox K, Samoilys M, Hill N (2017) Mosquito net use in an artisanal east african fishery. *Conservation Letters.* 10(4):451–459.

Campbell AM, Vandenbussche D, Hermann W (2008) Routing for relief efforts. *Transportation Science.* 42(2):127–145.

Chen J, Liang L, Yao D (2018) Pre-positioning of relief inventories: a multi-product newsvendor approach. *International Journal of Production Research.* 56(18):6294–6313.

Chen L, Kim S, Lee HL (2021) Vehicle mainteance contracting in developing economiers: The role of social enterprise. *Manufacturing & Service Operations Management.* 23(6):1651–1668.

Cucchiaro G, Van Leeuwen J, Goodridge Y (2020) Case report: The role of spatial repellant devices to prevent malaria in low-income countries. *Am. J. Trop. Med. Hyg.* 102(5):1033–1036.

Daff BM, Seck C, Belkhayat H, Sutton P (2014) Informed push distribution of contraceptives in Senegal reduces stockouts and improves quality of family planning. *Global Health: Science and Practice.* 2(2):245–252.

Darmstadt, G.L., Bhutta, Z.A., Cousens, S., Adam, T., Walker, N., de Bernis, L. (2005). Evidence-based, cost-effective interventions: how many newborn babies can we save? *Lancet. 365*, 977–988.

Darmstadt, G.L., Haws, R.A., Thomas, A.L., Bhutta, Z.A., Darmstadt, G.L. (2007). Impact of packaged interventions on neonatal health: a review of the evidence. *Health Policy and Planning, 22*, 93–215.

De Boeck K, Decouttere C, Vandaele N (2020) Vaccine distribution chains in low- and middle-income countries: A literature review. *Omega.* 97.

De Boeck K, Decouttere C, Jónasson JO, Vandaele N (2021) Vaccine supply chains in resource-limited settings: Mitigating the impact of rainy season disruptions. *European Journal of Operational Research.* Forthcoming.

Doe S, Sohoni M (2015) Optimal decentralization of early infant diagnosis of HIV in resource-limited settings. *Manufacturing & Service Operations Management.* 17(2):191–207.

Deo S, Mehta S, Corbett CJ (2022) Optimal scale-up of HIV treatment programs in resource-limited settings under supply uncertainty. *Production and Operations Management.* 31(3):883–905.

Duijzer LE, van Jaarsveld W, Dekker R (2018) Literature review: The vaccine supply chain. *European Journal of Operational Research.* 268:174–192.

Duran S, Gutierrez MA, Keskinocak P (2011) Pre-Positioning of Emergency Items for CARE International. *Interfaces.* 41(3):223–237.

Eftekhar M, Van Wassenhove LN (2016) Fleet management policies for humanitarian organizations: Beyond the utilization-residual value trade-off. *Journal of Operations Management.* 44:1–12.

Ergun Ö, Gui L, Heier Stamm JL, Keskinocak P, Swann J (2014) Improving humanitarian operations through technology-enabled collaboration. *Production and Operations Management.* 23(6):1002–1014.

Galindo G, Batta R (2013) Review of recent developments in OR/MS research in disaster operations management. *European Journal of Operational Research.* 230:201–211.

Gallien J, Rashkova I, Atun R, Yadav P (2017) National drug stockout risks and the global fund disbursement process for procurement. *Production and Operations Management.* 26(6):997–1014.

Gallien J, Leung NZ, Yadav P (2021) Inventory policies for pharmaceutical distribution in Zambia: Improving availability and access equity. *Production and Operations Management.* 30(12):4501–4521.

Gettleman J (2015) Meant to Keep Malaria Out, Mosquito Nets Are Used to Haul Fish In. *The New York Times* (January 24), https://www.nytimes.com/2015/01/25/world/africa/mosquito-nets-for-malaria-spawn-new-epidemic-overfishing.html.

Githinji S, Kigen S, Memusi D, Nyandigisi A, Mbithi AM, Wamari A, Muturi AN, Jagoe G, Barrington J, Snow RW, Zurovac D (2013) Reducing stock-outs of life saving malaria commodities using mobile phone text-messaging: SMS for life study in Kenya. *PLoS ONE.* 8(1).

Gupta D (1996) The (Q,r) inventory system with an unreliable supplier. *Operations Research.* 34(2):59–76.

Gürler Ü, Parlar M (1997) An inventory problem with two randomly available suppliers. *Operations Research.* 45(6):904–918.

Harries AD, Schouten EJ, Makombe SD, Libamba E, Neuville HN, Some E, Kadewere G, Lungu D (2007) Ensuring uninterrupted supplies of antiretroviral drugs in resource-poor settings: An example from Malawi. *Bulletin of the World Health Organization.* 85(2):152–155.

Hwang B, Shroufi A, Gils T, Steele SJ, Grimsrud A, Boulle A, Yawa A, Stevenson S, Jankelowitz L, Versteeg-Mojanaga M, Govender E, Stephens J, Hill J, Duncan K, van Cutsem G (2019) Stock-outs of antiretroviral and tuberculosis medicines in South Africa: A national cross-sectional survey. *PLuS One*. 14(3).

Islam MT, Azeem A, Jabir M, Paul A, Paul SK (2020) An inventory model for a three-stage supply chain with random capacities considering disruptions and supplier reliability. *Annals of Operations Research*.

Jahre M, Dumoulin L, Greenhalgh LB, Hudspeth C, Limlim P, Spindler A (2012) Improving health in developing countries: Reducing complexity of drug supply chains. *Journal of Humanitarian Logistics and Supply Chain Management*. 2(1):54–84.

Jónasson JO, Deo S, Gallien J (2017) Improving HIV early infant diagnosis supply chains in Sub-Saharan Africa: Models and application to Mozambique. *Operations Research*. 65(6):1479–1493.

Kangwana BB, Njogu J, Wasunna B, Kedenge SV, Memusi DN, Goodman CA, Zurovac D, Snow RW (2009) Short report: Malaria drug shortages in Kenya: A major failure to provide access to effective treatment. *American Journal of Tropical Medicine & Hygiene*. 80(5):737–738.

Karsu O, Morton A (2021) Trading off health and financial protection benefits with multiobjective optimization. *Health Economics, 30*. 55–69.

Killeen GF (2014) Characterizing, controlling and eliminating residual malaria transmission. *Malaria Journal*. 13(330).

Kovács G, Spens KM (2009) Humanitarian logistics and supply chain management: the start of a new journal. *Journal of Humanitarian Logistics and Supply Chain Management*. 1(1):5–14.

Kovács G, Tatham P (2009) Responding to disruptions in the supply network-From dormant to action. *Journal of Business Logistics*. 30(2):215–228.

Kunz N, Reiner G (2012) A meta-analysis of humanitarian logistics research. *Journal of Humanitarian Logistics and Supply Chain Management*. 2(2):116–147.

Larsen DA, Welsh R, Mulenga A, Reid R (2018) Wisespread mosquito net fishing in the Barotse floodplain: Evidence from qualitative interviews. *PLOS ONE*. 13(5).

Larsen DA, Makaure J, Ryan SJ, Stewart D, Traub A, Welsh R, Love DH, Bisesi JH (2021) Implications of insecticide-treated mosquito net fishing in lower income countries. *Environmental Health Perspectives*. 129(1).

Leiras A, de Brito I, Peres EQ, Bertazzo TR, Tsugunobu H, Yoshizaki Y (2014) Literature review of humanitarian logistics research: trends and challenges. *Journal of Humanitarian Logistics and Supply Chain Management*. 4(1):95–130.

Leung NZ, Chen A, Yadav P, Gallien J (2016) The impact of inventory management on stock-outs of essential drugs in Sub-Saharan Africa: Secondary analysis of a field experiment in Zambia. *PLoS ONE*. 11(5).

Lodree EJ, Taskin S (2009) Supply chain planning for hurricane response with wind speed information updates: Operations research approaches for disaster recovery planning. *Computers & operations research.* 36(1):2–15.

Lover AA, Sutton BA, Asy AJ (2011) An exploratory study of treated-bed nets in Timor-Leste: Patterns of intended and alternative usage. *Malaria Journal.* 10(199).

Marseille E, Larson B, Kazi DS, Kahn JG, Rosen S (2015) Thresholds for the cost–effectiveness of interventions: alternative approaches. *Bulletin of the World Health Organization, 93.* 118–124.

Matowe L, Waako P, Adome RO, Kibwage I, Minzi O, Bienvenu E (2008) A strategy to improve skills in pharmaceutical supply management in East Africa: The regional technical resource collaboration for pharmaceutical management. *Human Resources for Health.* 6(30).

McCoy JH, Lee HL (2014) Using fairness models to improve equity in health delivery fleet management. *Production and Operations Management.* 23(6):965–977.

McLachlin R, Larson PD, Khan S (2009) Not-for-profit supply chains in interrupted environments: The case of a faith-based humanitarian relief organisation. *Management Research News.* 32(11):1050–1064.

Mehta KM, Rerolle F, Rammohan SV, Albohm DC, Muwowo G, Moseson H, Sept L, Lee HL, Bendavid E (2016) Systematic motorcycle management and health care delivery: A field trial. *AJPH Practice.* 106(1):87–94.

Mills AF, Argon NT, Ziya S (2013) Resource-based patient prioritization in mass-casualty incidents. *Manufacturing & Service Operations Management.* 15(3):361–377.

Mills AF, Argon NT, Ziya S (2018) Dynamic distribution of patients to medical facilities in the aftermath of a disaster. *Operations Research.* 66(3):716–732.

Minakawa N, Dida GO, Sonye GO (2008) Unforeseen misuses of bed nets in fishing villages along Lake Victoria. *Malaria Journal.* 7(165).

Mohebbi E (2004) A replenishment model for the supply-uncertainty problem. *International Journal of Production Economics.* 87:25–37.

Mohebbi E, Hao D (2006) When supplier's availability affects the replenishment lead time– An extension of the supply interruption problem. *European Journal of Operations Research.* 175:992–1008.

Moinzadeh K, Aggarwal P (1997) Analysis of a production/inventory system subject to random disruptions. *Management Science.* 43(11):1577–1588.

Nagel RU, Fin K (2021) 5 things to know about the instability in Eastern Congo. *The Washington Post.* (June 11), https://www.washingtonpost.com/politics/2021/06/11/5-things-know-about-instability-eastern-congo/

Nandi, A., Colson, A.R., Verma, A., Megiddo, I., Ashok, A., & Laxminarayan, R. (2016). Health and economic benefits of scaling up a home-based neonatal care package in rural India: a modelling analysis. *Health Policy and Planning. 31* 634–644.

Natarajan KV, Swaminathan JM (2014) Inventory management in humanitarian operations: Impact of amount, schedule, and uncertainty in funding. *Manufacturing & Service Operations Management.* 16(4):595–603.

Natarajan KV, Swaminathan JM (2017) Multi-treatement inventory allocation in humanitarian health settings under funding constraints. *Production and Operations Management.* 26(6):1015–1034.

Natarajarathinam M, Capar I, Narayanan A (2009) Managing supply chains in times of crisis: a review of literature and insights. *International Journal of Physical Distribution & Logistics Management.* 39(7):535–573.

Norris JR (1997) *Markov Chains* (Cambridge University Press).

Ochalek J, Revill P, Manthalu G, McGuire F, Nkhoma D, Rollinger A, Sculpher M, Claxton K (2018) Supporting the development of a health benefits package in Malawi. *BMJ Global Health, 3.*

Ochalek J, Lomas J, Claxton K (2019) Estimating health opportunity costs in low-income and middle-income countries: a novel approach and evidence from cross-country data. *BMJ Global Health. 3.*

Oloruntoba R, Gray R (2006) Humanitarian aid: An agile supply chain? *Supply Chain Management: An International Journal.* 11(2):115–120.

Overstreet RE, Hall D, Hanna JB, Rainer RK (2011) Research in humanitarian logistics. *Journal of Humanitarian Logistics and Supply Chain Management.* 1(2):114–131.

Özdamar L, Ertem MA (2015) Models, solutions and enabling technologies in humanitarian logistics. *European Journal of Operational Research.* 244:55–65.

Parlar M (1997) Continuous-review inventory problem with random supply interruptions. *European Journal of Operations Research.* 99(2):366–385.

Parlar M, Berkin D (1991) Future supply uncertainty in EOQ models. *Naval Research Logistics.* 38(1):107–121.

Parlar M, Perry D (1995) Analysis of a (Q,r,T) inventory policy with deterministic and random yields when future supply is uncertain. *European Journal of Operations Research.* 84(2):431–443.

Parlar M, Perry D (1996) Inventory models of future supply uncertainty with single and multiple suppliers. *Naval Research Logistics.* 43(2):191–210.

Parlar M, Wang Y, Gerchak Y (1995) A periodic review inventory model with Markovian supply availability. *International Journal of Production Economics.* 42:131–136.

Perry D, Stadje W (2001) Disasters in a Markovian inventory system for perishable items. *Advances in Applied Probability.* 33:61–75.

Perry HB, Zulliger R, Rogers MM (2014) Community Health Workers in Low-, Middle-, and High-Income Countries: An Overview of Their History, Recent Evolution, and Current Effectiveness. *Annu. Rev. Public Health. 35*, 399–421.

Poormoaied S, Demirci EZ (2021) Analysis of an inventory system with emergency ordering option at the time of supply disruption. *OR Spectrum.* 43:1007–1045.

Porteus EL (2002) *Foundations of Stochastic Inventory Theory* (Stanford Business Books, Stanford, CA)

Privett N, Gonsalvez D (2014) The top ten global health supply chain issues: Perspectives from the field. *Operations Research for Health Care.* 3(4):226–230.

Pulford J, Hetzel MW, Bryant M, Siba PM, Mueller I (2011) Reported reasons for not using a mosquito net when one is available: A review of the published literature. *Malaria Journal.* 10.

Qi L, Shen ZM, Snyder LV (2009) A continuous-review inventory model with disruptions at both supplier and retailer. *Production and Operations Management.* 18(5):516–532.

Qin H, Xiao J, Ge D, Xin L, Gao J, He S, Hu H, Carlsson JG (2022) JD.com: Operations Research Algorithms Drive Intelligent Warehouse Robots to Work. *Informs Journal on Applied Analytics. 52*(1), 42–55.

Rajgopal J, Connor DL, Assi T, Norman BA, Chen S, Bailey RR, Long AR, Wateska AR, Bacon KM, Brown ST, Burke DS, Lee BY (2011) The optimal number of routine vaccines to order at health clinics in low or middle income countries. *Vaccine.* 29:5512–5518.

Russell TL, Beebe NW, Cooper RD, Lobo NF, Burkot TR (2013) Successful malaria elimination strategies require interventions that target changing vector behaviours. *Malaria Journal.* 12(56).

Saghafian S, Van Oyen MP (2012) The value of flexible backup suppliers and disruption risk information: Newsvendor analysis with recourse. *IIE Transactions.* 44:834–867.

Sargut FZ, Qi L (2012) Analysis of a two-party supply chain with random disruptions. *Operations Research Letters.* 40:114–122.

Say L, Chou D, Gemmill A, Tuncalp O, Moller A, Daniels J, Gulmezoglu AM, Temmerman M, Alkema L (2014) Global causes of maternal death: a WHO systematic analysis. *Lancet Global Health. 2*, 323–333.

Sethi SP, Cheng F (1997) Optimality of (s,S) policies in inventory models with markovian demand. *Operations Research.* 45(6):931–939.

Sethi SP, Yan H, Zhang H (2003) Inventory models with fixed costs, forecast updates, and two delivery modes. *Operations Research.* 51(2):321–328.

Sherrard-Smith E, Skarp JE, Beale AD, Fornadel C, Norris LC, Moore SJ, Mihreteab S, Charlwood JD, Bhatt S, Winskill P, Griffin JT, Churcher TS (2019) Mosquito feeding behavior and how it influences residual malaria transmission across Africa. *PNAS*. 116(30):15086–15095.

Short R, Gurung R, Rowcliffe M, Hill N, Milner-Gulland EJ (2018) The use of mosquito nets in fisheries: A global perspective. *PLoS ONE*. 13(1).

Snyder LV (2014) A tight approximation for an EOQ model with supply disruptions. *International Journal of Production Economics*. 155:91–108.

Snyder LV, Atan Z, Peng P, Rong Y, Schmitt AJ, Sinsoysal B (2016) OR/MS models for supply chain disruptions: A review. *IIE Transactions*. 48(2):89–109.

Sodhi MS, Tang CS (2014) Buttressing supply chains against floods in Asia for humanitarian relief and economic recovery. *Production and Operations Management*. 23(6):938–950.

Song J, Zipkin P (1996) Inventory control with information about supply conditions. *Management Science*. 42(10):1409–1419.

Spiegelman D, Khudyakov P, Wang M, Vanderweele TJ (2018) Evaluating Public Health Interventions: 7. Let the Subject Matter Choose the Effect Measure: Ratio, Difference, or Something Else Entirely. *American Journal of Public Health Policy. 108*(1), 73–76.

Stinnett AA, Paltiel AD (1996) Mathematical programming for the efficient allocation of health care resources. *Journal of Health Economics. 15*, 641–653.

Stuart RM, Fraser-Hurt N, Shubber Z, Vu L, Cheik N, Kerr CC, Wilson DP (2023) How to do (or not to do)... health resource allocations using constrained mathematical optimization. *Health Policy and Planning. 38*(1), 122–128.

Syafruddin D, Asih PBS, Rozi IE, Permana, DH, Hidayati APN, Syahrani L, Zubaidah S, Sidik D, Bangs MJ, Bogh C, Liu F, Eugenio EC, Hendrickson J, Burton T, Baird JK, Collins F, Grieco JP, Lobo NF, Achee NL (2020) Efficacy of spatial repellent for control of malaria in Indonesia: A cluster-randomized controlled trial. *Am. J. Trop. Med. Hyg.* 103(1):344–358.

Tomlin B (2006) On the value mitigation and contingency strategies for managing supply chain disruption risks. *Management Science*. 52(5):639–657.

Tomlin BT, Snyder LV (2007) On the value of a threat advisory system for managing supply chain disruptions. Working paper, Tuck School of Business, Hanover.

Trisos CH, Alexander SM, Gephart JA, Gurung R, McIntyre PB, Short R (2019) Mosquito net fishing exemplifies conflict among Sustainable Development Goals. *Nature Sustainability*. 2:5–7.

Tshering D, Tejativaddhana P, Briggs D, Wangmo N (2018) Factors Affecting Motivation and Retention of Village Health Workers and Recommended Strategies: A Systematic Review from 11 Developing Countries. *Asia Pacific Journal of Health Management. 13*(2).

UNHCR (2020) Massive floods in DRC's South Kivu impact 80,000 people, kill dozens. (April 21), https://www.unhcr.org/en-us/news/briefing/2020/4/5e9ea96f4/massive-floods-drcs-south-kivu-impact-80000-people-kill-dozens.html

UNHCR (2022) Democratic Republic of the Congo-Monthly statistics of refugees and asylum seeker. (March 11), https://data2.unhcr.org/en/documents/details/91301

van Baal P, Morton A, Severens JL (2018) Health care input constraints and cost effectiveness analysis decision rules. *Social Science & Medicine. 200*, 59–64.

Van Wassenhove LN (2006) Humanitarian aid logistics: Supply chain management in high gear. *Journal of the Operational Research Society.* 57:475–489.

Walker N, Tam Y, Friberg IK (2013) Overview of the Lives Saved Tool (LiST). *BMC Public Health. 13*, S1.

Watson N, Bausell L, Ingles A, Printz N (2014) Malaria seasonality and calculating resupply: Applications of the look-ahead seasonality indices in Zambia, Burkina Faso, and Zimbabwe. Report, USAID, Arlington, Va.

Weiss HJ, Rosenthal EC (1996) Optimal ordering policies when anticipating a disruption in supply or demand. *European Journal of Operations Research.* 59(3):370–382.

World Health Organization (2013) Guidelines for laboratory and field-testing of long-lasting insecticidal nets. Report, World Health Organization & WHO Pesticide Evaluation Scheme, Geneva.

World Health Organization (2020) World malaria report 2020: 20 years of global progress and challenges. Report, Geneva.

World Health Organization (2021) Recommitting to accelerate progress towards malaria elimination. *The Seventy-fourth World Health Assembly* (World Health Organization, Geneva).

Whybark DC (2007) Issues in managing disaster relief inventories. *International Journal of Production Economics.* 108:228–235.

The World Bank (2020a) *Mortality rate, infant (per 1,000 live births)- Ethiopia.* https://data.worldbank.org/indicator/SP.DYN.IMRT.IN?end=2020&locations=ET&start=1966&view=chart

The World Bank (2020b) *Mortality rate, infant (per 1,000 live births)- Somalia.* https://data.worldbank.org/indicator/SP.DYN.IMRT.IN?end=2020&locations=SO&start=1966&view=chart

The World Bank (2020c) *Mortality rate, under-5.* Accessed March 30, 2022, https://data.worldbank.org/indicator/SH.DYN.MORT.

Yadav P (2015) Health product supply chains in developing countries: Diagnosis of the root causes of underperformance and an agenda for reform. *Health Systems & Reform.* 1(2):142–154.

Zaffran M, Vandelaer J, Kristensen D, Melgaard B, Yadav P, Antwi-Agyei KO, Lasher H (2013) The imperative for stronger vaccine supply and logistics systems. *Vaccine.* 31S:B73–B80.

Zipkin PH (2000) *Foundations of Inventory Management* (McGraw-Hill).