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THE GEOMETRY OF THE FRACTIONAL QUANTUM HALL EFFECT: EXPOSING THE GRAVITATIONAL ANOMALY

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ABSTRACT

We study quantum Hall states on curved surfaces with the aim of exposing the gravitational anomaly. We develop two general methods for computing correlation functions of the fractional quantum Hall effect (FQHE) on curved surfaces - the Ward Identity and Field Theory. Using these methods we show that on surfaces with conical singularities, the electronic fluid near the tip of the cone has an intrinsic angular momentum due solely to the gravitational anomaly. This is effect occurs because quantum Hall states behave as conformal primaries near singular points, with a conformal dimension equal to the angular momentum. We argue that the gravitational anomaly and conformal dimension determine the fine structure of the electronic density at the conical point. The singularities emerge as quasi-particles with spin and exchange statistics arising from adiabatically braiding conical singularities. Thus, the gravitational anomaly, which appears as a finite size correction on smooth surfaces, dominates geometric transport on singular surfaces.

CHAPTER 1

INTRODUCTION

1.1 Overview

In the last 30 years, topological states of matter have played an important role in condensed matter physics. What started with the discovery of the role of the topology in the quantum Hall effect (QHE) extended to myriad other discoveries - the spin quantum Hall effect, topological insulators, non-Abelian anyons, and more. The one thing in common between topological states of matter is that they are characterized by observables that are topologically protected - meaning they are insensitive to small perturbations of the Hamiltonian. Understanding topological materials comes down to understanding the topologically invariant observables that characterize them, so-called universal characteristics.

We study topological invariants in the fractional quantum Hall effect (FQHE). There are three such invariants for the FQHE - the Hall conductance σ_H, the Hall viscosity η_H, and the gravitational anomaly c_H. The Hall conductance σ_H and viscosity η_H have been studied extensively, and there are physical arguments that explain their quantization and topological nature. As a result, we know a lot about the Hall conductance and viscosity - the conductance is easily measured in experiments and there are proposals for measuring the Hall viscosity as well. However, little was known about the third universal characteristic of FQH states - the gravitational anomaly c_H - until recently. The reason is that the gravitational anomaly arises as a response to geometric stress in the FQHE, and prior to our work as well as [116, 71], there were no frameworks for calculating correlation functions for the fractional quantum Hall effect on geometric surfaces.

In this paper, we develop tools necessary to compute correlation functions for the FQHE in curved space. We introduce two methods - a Ward identity and a field theory for FQHE states on curved surfaces. These methods provide a formulaic way of computing correlation functions for Laughlin states of the FQHE. Each method prevails for certain types of calculation, so it's useful to know both. After developing the Ward identity and field theory for FQH states on curved surfaces, we apply these methods to study FQHE on singular surfaces.

We find that singular surfaces are the setting where the gravitational anomaly is understood best. It is only in this setting, that certain observables like angular momentum and exchange statistics between cones are determined completely by the gravitational anomaly. On smooth geometric surfaces, the gravitational anomaly always appears as a finite-sized correction. For this reason, geometric response of the FQHE on singular surface provides, to date, the best theoretical and experimental setting for studying the gravitational anomaly.

1.2 Geometry in the FQHE

The reason for studying the geometry of the FQHE is that it provides a probe for measuring the gravitational anomaly. Generally, important universal properties of fractional quantum Hall (FQH) states are evident in the quantization of kinetic coefficients in terms of the filling fraction. The most well-known kinetic coefficient is the Hall conductance [1], a transversal response to the electromagnetic field. Beside this, FQH states possess a richer structure evident through their response to changes in spatial geometry and topology, both captured by the gravitational response.

A kinetic coefficient which reflects a transversal response to the gravitational field is the odd viscosity (also referred as anomalous viscosity, Hall viscosity or Lorentz shear modulus) [2, 3, 4]. This coefficient also exhibits a quantization and reveals universal features of FQH states as much as the Hall conductance. While the Hall conductance is seen in an adiabatic response to homogeneous flux deformation [1], the anomalous viscosity is seen as an adiabatic response to homogeneous metric deformations [2]. However, even more universal features become apparent when one considers the adiabatic response to inhomogeneous deformations of the flux and the metric. This is the reason for studying the geometric response of the

FQHE.

In the first part of this paper, we develop the Ward identity for FQHE states on curved surfaces. This leads us to compute the response of the FQH states to local curvature and show that this response reveals corrections to physical quantities in a flat space that remain hidden otherwise. Our results fall into two categories. First, we compute the particle density through a gradient expansion in local curvature, explain the relation of the leading terms to the gravitational anomaly, and show that they are geometrical in nature. For this reason, we expect these terms to be universal (i.e. insensitive to the details of the underlying electronic interaction as long as the interaction gives rise to the FQH state). We develop a general method to compute these terms. Additionally, we show that the dependence on curvature determines the long wavelength expansion of the static structure factor in a flat background, linking the electromagnetic response to the gravitational anomaly. Furthermore, correlation functions computed on arbitrary surfaces provide information about the properties of FQH states under general covariant and, in particular, conformal transformations.

We consider only Laughlin states for which the filling fraction ν is the inverse of an integer, but comment on how our results could be extended to other FQH states, such as the $\nu = 5/2$ Pfaffian state [5]. We restrict our analysis to FQH states without boundaries. Though our analysis is limited to the Laughlin wave function, we believe that our results capture the geometric properties of FQH states. As such, they may serve as universal bounds for response functions in realistic materials exhibiting the FQH effect.

1.3 The Role of Field Theory

While the Ward identity method is a useful way to compute correlation functions for the FQHE on a curved surface, it is cumbersome and lacks physical interpretation. The Ward identity is a powerful mathematical toolkit. To better understand the origin of anomalies in the FQHE, we need to study it in a more physical framework. For this reason, we develop a field theory for FQH states on curved surfaces. This allows us to to relate the quantities we compute with well-understood physical structures studied extensively in traditional quantum field theory.

In the second part of this paper, we develop a field theory for Laughlin states. This approach naturally captures universal features of the QHE, and emphasizes the geometric aspects of QH-states. We demonstrate how the field theory encompasses recent developments in the field [**?** 85, 30, 50, 71, 61, 72, 47, 86] and obtain some properties of quasi-hole excitations. A preliminary treatment of this approach appears in [50].

The universal properties of the QHE are encoded in the dependence of the ground state wave function on electromagnetic and gravitational backgrounds (see e.g., [**?**]). For that reason, as with the Ward identity, we use the field theoretic approach to study QH states on a Riemann surface and for simplicity focus on genus zero surfaces.

As with the Ward identity, restrict our analysis to the Laughlin states. Our approach is closely connected to the hydrodynamic theory of QH states of Ref [134, 132] and the collective field theory approach of Gervais, Sakita and Jevicki developed in [68, 81, 82] and extended in [40, 39, 31, 44]. The action of the field theory for Laughlin states is written in Sec.(3). The leading part, Eq.(3.10), is equivalent to the classical energy of a 2D neutralized Coulomb plasma when the discreteness of particles is not taken into account. This is used in the familiar plasma analogy of Ref.[93] to deduce the equilibrium density, as well as properties of the quasi-hole state such as charge and statistics. The other terms in the action are more subtle but equally significant, and give rise to important effects including the gravitational anomaly.

1.4 Singular Surfaces

In the last part of the paper, we study the FQHE on singular surfaces. Results from the Ward identity and field theory methods, as well as early developments in the theory of the quantum Hall effect [38, 94, 95, 130, 66], and a recent resurgence [76, 109, 111, 58, 85, 49, 50, 30, 71, 72, 91, 61, 47, 86] point to the role of geometric response as a fundamental probe of quantum liquids with topological characterization, complementary to the more familiar electromagnetic response. Such liquids exhibit non-dissipative transport as a response to variations of the spatial geometry, controlled by quantized transport coefficients. This *geometric transport* is distinct from the transport caused by electromotive forces. It is determined by the geometric transport coefficients which are independent characteristics of the state and can not be read from the electromagnetic response.

Surfaces with a singular geometry, such as isolated conical singularities, or disclination defects, highlight the geometric properties of the state. For this reason, they serve as an ideal setting to probe the geometry of QH states. In the last chapter (prior to the Conclusion), we demonstrate this by examining Laughlin states on a singular surface, where geometric transport is best understood. We compare spatial curvature singularities to magnetic ones (flux tubes) and emphasize the difference. While the QH state imbues both types of singularities with local structure such as charge, spin, and statistics, *only* the curvature singularities reflect the geometric transport.

The gravitational anomaly is central to understanding the geometry of topological states [49, 50, 71, 91, 61, 47, 86, 127]. This effect encodes the geometric characterization of such states, and is often referred to as the *central charge*.

On a smooth surface the gravitational anomaly is a sub-leading effect. For example, the central charge, c_H, appears as a finite size correction to the angular momentum of the electronic fluid. In [86] it is shown that the angular momentum on a small region D of the fluid on a surface with rotational symmetry is

$$\mathrm{L} = -\frac{1}{2\pi}\int_D \left(\mu_H(eB) - \hbar\frac{c_H}{24}R\right) dV, \tag{1.1}$$

The last term in (2.1) represents the gravitational anomaly. We are ultimately interested in features which do not depend on the size of the domain D. For the j-spin Laughlin states (see [49], and (4.16) below for the definition of spin) the transport coefficient μ_H and the 'central charge' c_H were found to be

$$c_H = 1 - \frac{12\mu_H^2}{\nu}, \quad \mu_H = \frac{1}{2}(1 - 2j\nu), \tag{1.2}$$

where ν is the filling fraction.

On a smooth surface, the geometric transport is hard to detect, since c_H typically enters transport formulas as a small correction. We want to identify a setting where the gravitational anomaly c_H is the dominant feature, as opposed to being a finite-size correction overshadowed by a larger electromagnetic contribution. We demonstrate that a surface with conical singularities brings geometric transport to the fore.

We show that a small parcel of electronic liquid centered at a singularity spins around it with an *intensive* angular momentum. It is unchanged when the parcel volume D shrinks as long as it stays larger than the magnetic length. The intensive angular momentum is universal and proportional to c_H, the geometric characteristic of the state (see (4.2,4.3)). Similarly, the parcel maintains the excess or deficit of electric charge and moment of inertia in the limit of vanishing D. That is to say, the moments are localized near the conical point, see (4.10,4.11). Besides the charge, this property leads to the notions of spin and exchange statistics of singularities. We compute both, see (4.6,4.7), and show that they are also determined by the gravitational anomaly.

Our argument stems from the observation that a state near the singularity has conformal symmetry. Specifically, we find that it transforms as a primary field.

Singularities elucidate the uneasy relation of QH-states to conformal field theory. In general, QH-states do not possess conformal symmetry. They feature a scale - the magnetic length. As a result, physical observables do not transform conformally. However, the states appear to be conformal in the vicinity of a singularity. In this paper, we show that the state is primary with a conformal dimension identical to the angular momentum in units $\hbar$. We emphasize that the dimensions have the *opposite* sign to the dimensions of similar primary fields in conformal field theory

Conical singularities are not as exotic as they may seem, and occur naturally in several experimental settings. Disclination defects in a regular lattice can be described by metrics with conical singularities [84], and occur generically in graphene [90, 128]. In a recent photonic experiment, synthetic Landau levels on a cone were designed in an optical resonator [114].

A conical singularity of order $\alpha < 1$ is an isolated point ξ_0 on the surface with a concentration of curvature

$$R(\xi) = R_0 + 4\pi\alpha\delta(\xi - \xi_0), \tag{1.3}$$

where R_0 is the background curvature, a smooth function describing the curvature away from the singularity and the delta-function is defined such that its integral over the volume element gives one. We refer ξ_0 as the conical point.

Examples of genus-zero surfaces with constant curvature and conical singularities include: $R_0 > 0$ - an 'american football' with two antipodal conical singularities [124], $R_0 = 0$ - a polyhedron [125, 121], $R_0 < 0$ - a pseudo-sphere (see e.g. [119] and references therein). For the purpose of this paper it suffices to consider conical singularities which locally are flat surfaces of revolution. If $\alpha > 0$ the singularity is equivalent to an embedded cone with the apex angle $2 \arcsin \gamma$, where $2\pi\gamma = 2\pi(1 - \alpha)$ is called cone angle (see Fig.4.1). If $\alpha < 0$, the singularity can be seen as a branch point of multi-sheeted Riemann surface (though non-convex polyhedra also can contain cone points with degree $\alpha < 0$).

An especially interesting case occurs when γ or $1/\gamma$ is an integer. In this case, it may be possible to represent the surface as an orbifold, a surface quotiented by a discrete group of automorphisms. Then the conical singularities arise as fixed points of the group action

[121].

Most of the formulas below are valid regardless of whether concentrated curvature is positive or negative (given by the sign of α), but strictly do not apply for cusp singularities for which $\alpha = 1$. Braiding of singularities on orbifolds is more involved (see [87, 43] for a similar issue in the context of CFT). We do not address them here.

Conical singularities affect QH states differently than magnetic singularities (flux tubes)

$$eB(\xi) = eB_0 - 2\pi\hbar\, a\delta(\xi - \xi_0), \tag{1.4}$$

To emphasize the difference between geometric and magnetic singularities we consider both simultaneously: a magnetic flux a threaded through the conical singularity α. We take eB_0 to be positive throughout the paper.

Lastly, we comment on the inclusion of spin j. As discussed in [50, 91, 86] Laughlin states are characterized not only by the filling fraction but also by the spin. Spin does not enter electromagnetic transport. Nor does it enter local bulk correlation functions, such as the structure factor. The spin enters the geometric transport as seen in (3.13).

To the best of our knowledge, there is no experimental or numerical evidence that determines the spin in QH materials, nor are there any arguments that $j = 0$, as it silently assumed in earlier papers. For this reason, we keep spin as a parameter. It affects the physics of the QHE. For example, at the filling $\nu = 1/3$, the central charge vanishes at $j = 1, \text{and}\, j = 2$. The central charge equals -2 if $\nu = 1$ and $j = 0$ or 1. If $j = \frac{1}{2\nu}$, the coefficient μ_H vanishes and $c_H = 1$.

1.5 Summary

To conclude, there are two main physical results of this paper. The first is the calculation of the angular momentum near the tip of a conical singularity. We'll show that the angular momentum of a small parcel of fluid rotating about the origin of a conical singularity is proportional to c_H, the topological invariant that encodes the geometric response of the state. The second result is that the braiding statistics of two cones are also proportional to c_H. Thus, we demonstrate for the first time a physical setting where the geometric response coefficient c_H can be measured independently of other universal characteristics of the FQH state.

These results can be summarized in two short formulas. Let α be a variable that parametrizes the cone angle (steepness of the cone). Then the angular momentum has the form

$$L = \hbar c_H f(\alpha). \tag{1.5}$$

where $f(\alpha)$ is a smooth function that only depends on the conical angle and nothing else. This result allows for direct measurement of the geometric coefficient in an experimental setting. For the second result, let α_1 and α_2 be variables that parametrize the deficit angles of two cones. Then the exchange statistics, when one cone is adiabatically transported around the other one in a closed loop is

$$\Phi_{12} = \pi \frac{c_H}{24} f(\alpha_1, \alpha_2). \tag{1.6}$$

where again $f(\alpha_1, \alpha_2)$ is a smooth function that only depends on the conical angles. This result contrasts the more familiar exchange statistics of two quasiholes. The exchange statistics between two quasiholes are proportional to the filling factor ν and are therefore fractional. For cones, however, the statistics are proportional to the geometric characteristic of the state c_H.

In order to arrive at these results, we needed to built frameworks for studying the FQHE on curved surfaces from the ground up. In what follows, we start by developing these frameworks - the Ward identity and field theory - and apply them to compute correlation functions of the FQHE on smooth surfaces first. After that, we apply these frameworks to study the FQHE on singular surfaces and re-write the Ward identity in a form more suitable for singular geometries. We conclude by deriving the main results of the paper - the angular momentum of a parcel of QH fluid rotating around a conical singularity and the exchange statistics between two cones.

CHAPTER 2

WARD IDENTITY FOR THE FQHE

2.1 Main Results and Problem Setup

We consider electrons placed on a closed oriented curved surface, such as a deformed sphere, and assume that the magnetic flux $d\Phi$ through a differential volume element of the surface is uniformly proportional to the volume so that $d\Phi = BdV$ where $B > 0$ is a uniform magnetic field. The total number of flux quanta $N_\phi = V(2\pi l^2)^{-1}$ piercing the surface is an integer equal to the area V of the surface in units of $2\pi l^2$, where $l = \sqrt{\hbar/eB}$ is the magnetic length. In this setting, and in the case of free particles, the lowest Landau level remains degenerate on a curved surface [7], even in the presence of isolated conical singularities, and remains separated from the rest of the spectrum by an energy of the order of the cyclotron energy. The degeneracy of the level is determined by the Riemann-Roch theorem. Assuming that the surface possesses no singularities so that the Euler characteristic χ is an even integer, the degeneracy is $N_1 = N_\phi + \frac{\chi}{2}$ [6, 7]. If the number of particles is chosen exactly equal to N_1, the electronic droplet completely covers the surface and, lacking a boundary, admits no edge states.

This result readily extends to Laughlin states (for a sphere and torus see [8], for a general Riemann surface see [9, 10, 7]): the droplet has no boundary if the number of particles N is equal to

$$N_\nu = \nu N_\phi + \frac{\chi}{2}, \tag{2.1}$$

assuming that N_ν is integer. We consider this case.

We focus on the particle density ρ defined such that $\rho\, dV$ is the number of particles in the volume element dV. A locally coordinate invariant quantity, the density must be expressed locally through the (scalar) curvature R. In this paper we compute the leading terms in the gradient expansion of the density of the ground state

$$\langle\rho\rangle = \rho_0 + \frac{1}{8\pi}R - \frac{b}{8\pi}(-l^2\Delta_g)R, \quad b = \frac{1}{3} + \frac{\nu-1}{4\nu}, \tag{2.2}$$

where $\rho_0 = \nu(2\pi l^2)^{-1}$ and Δ_g is the Laplace-Beltrami operator. We omit higher order terms in l^2. They are computable, but may not have a universal meaning beyond the Laughlin wave function. Higher order terms consist of higher order derivatives of the curvature, as well as higher degrees of curvature, whereas the first three terms remain linear in curvature.

The first two terms are a local version of the global relation (2.1) between the maximum particle number and the number of flux quanta. Eq. (2.1) is obtained by integrating (2.2) over the surface with the help of the Gauss-Bonnet theorem $\int R dV = 4\pi\chi$. Higher order terms do not contribute to this expression.

The second term indicates that particles accumulate in regions of positive curvature and repell from regions of negative curvature. For example, it shows the excess number of particles accumulating at the tip of a cone. If the conical singularity is of the order $\alpha > -1$ such that it metric locally is $|z|^{2\alpha}dzd\bar{z}$, the excess number of particles at the tip is $-\alpha/2$. The term appears in equivalent form in [9, 10].

The last term encodes the gravitational anomaly, which we explain in the body of the paper. A noticeable feature of this term is the shift from $1/3$ in the coefficient b when $\nu \neq 1$. This shift is yet another signature of states at fractional filling. We discuss its implications below.

For the case of integer filling $\nu = 1$, described by free electrons, Eq. (2.2) was obtained in [14, 15, 16]. In equivalent form, it is known in mathematical literature as an asymptotic expansion of the Bergman kernel [17]. The formula (2.2) allows us to write the linear response to curvature in flat space. Defining

$$\eta = (\rho_0 l^2)^{-1} \frac{\delta\rho}{\delta R}\Big|_{R=0}, \tag{2.3}$$

and passing to Fourier modes, Eq. (2.2) implies

$$\eta(q) = \frac{1}{4\nu}(1 - bq^2 + \mathcal{O}(q^4)), \qquad q = kl. \tag{2.4}$$

The momentum dependence of various correlation functions in flat space is closely related to the linear response to curvature . In [12], one of the authors argued that the kinetic coefficient defined by (4.33) enters the hydrodynamics of a FQH incompressible quantum liquid (see also [132]) as the anomalous term in the momentum flux tensor representing kinematic odd-viscosity. The homogeneous part of the odd-viscosity, computed through alternative methods in [2, 3, 11, 4], corresponds to the first term in (2.4). The leading gradient corrections to the odd-viscosity for the integer case $\nu = 1$ was recently computed in [16]. It corresponds to the second term in Eq. (2.4), and as we show below, receives a contribution from the gravitational anomaly.

We will show the following general relation between the static structure factor, $s(k) = \langle \rho_k \rho_{-k} \rangle_c / \rho_0$, and the response to curvature that is a feature of Laughlin states, and is likely valid for more general FQH states as well

$$\frac{q^4}{2}\eta(q) = -\frac{q^2}{2} + \Big(1 + \frac{q^2}{2}\Big)s(q), \quad q = kl. \tag{2.5}$$

Using these relations we obtain

$$s(q) = \frac{1}{2}q^2 + s_2 q^4 + s_3 q^6 + \mathcal{O}(q^8) \tag{2.6}$$

where $s_2 = (\nu^{-1} - 2)/8$ and $s_3 = (3\nu^{-1} - 4)(\nu^{-1} - 3)/96$.

The term of order q^4 in the structure factor goes back to [18]. We find that it is controlled by $\eta(0)$ and $\lim_{q\to 0} s(q)/q^2$. The next correction s_3 was recently obtained in [20] by means of a Mayer expansion. We provide an alternative derivation which emphasizes its connection to the gravitational anomaly. Curiously, s_2 vanishes at $\nu^{-1} = 2$, the bosonic Laughlin state, while s_3 vanishes for the Laughlin state at 1/3 filling. The higher order coefficients are polynomials of increasing degree in ν^{-1}.

We also mention another general relation between the structure factor and the Hall conductance valid for the Laughlin wave function

$$\sigma_{xy}(k) = \frac{e^2}{\hbar}\frac{2\rho_0}{k^2}s(k) \tag{2.7}$$

We clarify it in the body of the paper (see also [132]). This relation links the Hall conductance to the response to curvature through (2.5) [19]. Furthermore, knowledge of s_3 determines the Hall conductance $\sigma_{xy}(k)$ up to order k^4.

These results follow from iteration of a Ward identity obtained for the Laughlin wave function in [22], combined with the gravitational anomaly. An important ingredient of the Ward identity is the two point function of the "Bose" field φ at merged points. The Bose field is defined as a potential of charges created by particles through the Poisson equation

$$-\Delta_g\varphi = 4\pi\nu^{-1}\rho. \tag{2.8}$$

In the paper, we show that in the leading $1/N$ approximation, the Bose field has Gaussian correlations. This means that (i) the connected correlation function of φ at large distances between points is the Green function of the Laplace-Beltrami operator $\Delta_g G(z, z') = -4\pi[\frac{1}{\sqrt{g}}\delta^{(2)}(z - z') - \frac{1}{V}]$, and that (ii) at small distances between points the correlation function is the regularized Green function,

$$\langle\varphi(1)\varphi(2)\rangle_c = \nu^{-1}\begin{cases} G(1,2) & \text{at large separation} \\ G_R(1,2) & \text{at short distances.} \end{cases} \tag{2.9}$$

The regularized Green function is defined as

$$G_R(1,2) = G(1,2) + 2\log d(1,2) \tag{2.10}$$

where $d(1,2)$ is the geodesic distance between the two points.

The apparent metric dependence of the two point correlation function at short distances is referred to as the gravitational anomaly.

2.2 The Laughlin State on a Riemann Surface

It is convenient to work in holomorphic coordinates where the metric is conformal to the Euclidean metric $ds^2 = \sqrt{g}dzd\bar{z}$. In these coordinates, the scalar curvature reads $R = -\Delta_g \log\sqrt{g}$, where the Laplace-Beltrami operator takes the form $\Delta_g = (4/\sqrt{g})\partial\bar{\partial}$. The Kähler potential K, defined through the equation $\partial\bar{\partial}K = \sqrt{g}$, also plays an important role.

A convenient gauge is one in which the antiholomorphic component of the gauge potential for a uniform magnetic field B is given by $\bar{A} = \frac{1}{2}(A_1 + iA_2) = iB\bar{\partial}K/4$, such that $\nabla \times \mathbf{A} = B\sqrt{g}$. The states in the lowest Landau level are defined as those annihilated by the antiholomorphic component of the covariant momentum operator (see e.g., [7])

$$\bar{\Pi} = -i\hbar\bar{\partial} - e\bar{A}. \tag{2.11}$$

The solutions to $\bar{\Pi}\psi_n = 0$ are the single particle eigenstates given by $\psi_n(z) = s_n(z)e^{-K(z,\bar{z})/4l^2}$, where the functions $\{s_n\}$ are called holomorphic sections, defined as solutions to $\bar{\partial}s_n = 0$ such that ψ_n is normalizable (i.e. $\int dV|\psi_n|^2 < \infty$).

The many-body ground state wave function for free fermions is the Slater determinant of the single particle eigenstates, and for the filled lowest Landau level on a curved surface it is just $\Psi_1(z_1, ..., z_N) \propto e^{-\sum_i^N K(z_i,\bar{z}_i)/4l^2}\det[s_n(z_i)]$. In this form it appears in [15], and in equivalent form in [7].

We construct Laughlin states at the filling fraction ν by raising the determinant to the power equal to the inverse fraction.

$$\Psi_\beta(z_1,...,z_N) \propto e^{-\frac{1}{4l^2}\sum_i^N K(z_i,\bar z_i)} \Big(\det[s_n(z_i)]\Big)^\beta, \qquad \beta \equiv \nu^{-1}$$

We denote the inverse filling fraction as β for the majority of what follows, but it is interchangeable with ν^{-1}. This wave function is normalizable only for $N \leq N_\nu$ given by (2.1). We consider states with $N = N_\nu$, the only case in which the wave-function is modular invariant. This indicates that the surface is completely filled with particles and there is no boundary. The area of such a surface $V = 2\pi\beta l^2(N - \chi/2)$ is quantized in units of $2\pi l^2$.

For simplicity, we work in the case of genus zero. However, our formulas are local and therefore apply to more general surfaces. For a comprehensive discussion of the lowest Landau level on a surface of arbitrary genus see [7]. In the case of genus zero, we choose a marked point at infinity where $K \sim (V/\pi)\log|z|^2 + o(1)$ and $\log\sqrt{g} \sim -2\log|z|^2$. In this case the holomorphic sections $s_n(z)$ are polynomials of degree $n = 0, 1, ..., N_\phi$. Therefore, the Vandermonde identity $\det[s_n(z_i)] \propto \prod_{i<j}^N (z_i - z_j)$ yields

$$\Psi_\beta(z_1,...,z_N) = \frac{1}{\sqrt{\mathcal{Z}[g]}} \prod_{i<j}^N (z_i - z_j)^\beta e^{-\frac{1}{4l^2}\sum_i^N K(z_i,\bar z_i)}, \tag{2.12}$$

where $\mathcal{Z}[g]$ is a normalization factor. The asymptotic behavior of the wave-function at a marked point (such as $z \to \infty$) is $|z|^{\beta(N-1)-N_\phi}$, which determines the maximal number of particles in the state (2.1).

As an example, consider the case of a sphere of radius r. Then, $K = 4r^2\log(1 + |z|^2/4r^2)$, $\sqrt{g} = (1 + |z|^2/4r^2)^{-2}$, $R = 2/r^2$ and the orthonormal holomorphic sections are monomials $s_n(z) = [(N/V)C^n_{N-1}]^{-1/2}(z/2r)^n$. Inserting this Kähler potential K into Eq.(2.12) reproduces the well-known wave-function on a sphere in stereographic coordinates

[8]. In the limit that $r \to \infty$, $K = |z|^2$ and the planar wave function is recovered.

With this setup, we wish to evaluate equal time correlation functions in the limit $l \to 0, N_\phi \to \infty$ such that the area $V = 2\pi l^2 N_\phi$ is fixed.

2.3 Generating functional for the FQHE

The normalization factor $\mathcal{Z}[g]$ encodes the geometry of the surface through its dependence on the metric. It can be used to generate response functions to surface deformations. From (2.12) it is defined as

$$\mathcal{Z}[g] = \int \prod_{i<j}^{N} |z_i - z_j|^{2\beta} \prod_{i}^{N} e^{W(z_i,\bar{z}_i)} d^2 z_i, \tag{2.13}$$

where

$$W = -\frac{1}{2l^2} K + \log \sqrt{g}. \tag{2.14}$$

Each variation of $\log \mathcal{Z}$ over $W(z, \bar{z})$ inserts a factor of $\sum_i \delta^{(2)}(z - z_i)$ proportional to the density

$$\rho(z) = \frac{1}{\sqrt{g}} \sum_i \delta^{(2)}(z - z_i) \tag{2.15}$$

into the integral (2.13). Such variations produce connected correlation functions of the density such as

$$\sqrt{g}\langle\rho\rangle = \delta \log \mathcal{Z}/\delta W, \tag{2.16}$$

and $\sqrt{g(z)}\sqrt{g(z')}\langle\rho(z)\rho(z')\rangle_c = \delta^2 \log \mathcal{Z}/\delta W(z)\delta W(z')$. More generally, if $\mathcal{A}(z_1, ..., z_N)$ is a symmetric function of the coordinates, which does not depend on the metric, then

$\delta\langle\mathcal{A}\rangle/\delta W = \sqrt{g}\langle\mathcal{A}\,\rho\rangle_c$. This method for computing correlation functions is detailed in [22].

2.4 Relations between linear responses on the lowest Landau level

Using the explicit dependence of W on $\sqrt{g}$, we observe a general relation for a linear response to area preserving variations of the metric

$$\frac{1}{2}(-l^2\Delta_g)\frac{\delta\langle\mathcal{A}\rangle}{\delta\sqrt{g(\zeta)}} = \Big(1+\frac{1}{2}(-l^2\Delta_g)\Big)\langle\mathcal{A}\rho(\zeta)\rangle_c. \tag{2.17}$$

This relation is valid for any N and any β (including the integer case). It follows from the identity $-2l^2\delta\langle\mathcal{A}\rangle/\delta K = -2l^2\sqrt{g}(\delta W/\delta K)\langle\rho\,\mathcal{A}\rangle_c$, where the Jacobian $-2l^2\delta W/\delta K = 1+\frac{1}{2}(-l^2\Delta_g)$ acts as an operator on $\langle\rho\,\mathcal{A}\rangle_c$. Then the transformation $\sqrt{g}\Delta_g\delta\langle\mathcal{A}\rangle/\delta\sqrt{g} = 4\delta\langle\mathcal{A}\rangle/\delta K$ brings it to the form of (4.3). With the choice of $\mathcal{A} = \sum_i \delta^{(2)}(z-z_i)$ and the functional identity $\delta\langle\rho\rangle/\delta\sqrt{g}\Big|_{R=0} = -\Delta\left[\delta\langle\rho\rangle/\delta R\right|_{R=0}$, we obtain (2.5).

This relation reflects a symmetry between gravity and electromagnetism specific to the lowest Landau level. It can be traced back to properties of zero modes of the operator (2.11).

Similar arguments lead to the relation between the static structure factor and the Hall conductance expressed in (2.7). The generating functional (2.13) can be seen as the normalization factor of the Laughlin wave function in a flat space, but in a weakly inhomogeneous magnetic field. A key assumption is that the form of the wave function is the same as in the case of a uniform magnetic field where $B = -\frac{\hbar}{2e}\Delta W$, as in [32]. With this, the two-point density correlation function, computed as a variation of the density $\langle\rho(z)\rho(z')\rangle_c = \delta\langle\rho(z)\rangle/\delta W(z')$, can also be understood as a variation of the density over magnetic field under condition that the filling fraction is kept fixed. In Fourier modes, this functional identity leads to $\rho_0 s(k) = \frac{\hbar}{2e}k^2(\delta\rho_k/\delta B_k)$. The inhomogeneous version of the Streda formula $e\delta\rho_k/\delta B_k = \sigma_{xy}(k)$, yields the relation (2.7) [21]. Then, computing $\eta(k)$

allows us to extract $s(k)$, and thus $\sigma_{xy}(k)$, from (2.5). Moreover, once we compute $\langle\rho\rangle$, we can recover the generating functional which we present in the end of the paper.

To compute the response to curvature we employ the Ward identity explained in the next section.

2.5 Ward identity

The generating functional $\mathcal{Z}[g]$ is invariant under any transformation of coordinates of the integrand (2.13). In particular, a holomorphic infinitesimal diffeomorphism $z_i \to z_i + \epsilon/(z - z_i)$ where z is a parameter, invokes a change of the integrand (2.13) by the factor $\sum_i \frac{\partial_{z_i} W}{z - z_i} + \sum_{j \neq i} \frac{\beta}{(z-z_i)(z_i-z_j)} + \sum_i \frac{1}{(z-z_i)^2}$. The Ward identity states that the expectation value of this factor vanishes. Expressing the sum as an integral over the density $\sum_i \to \int d^2\xi \sqrt{g(\xi)}\rho(\xi)$, yields the relation connecting one- and two-point functions

$$-2\beta \int \frac{\partial W}{z - \xi} \langle\rho\rangle \sqrt{g} d^2\xi = \langle(\partial\varphi)^2\rangle + (2 - \beta)\langle\partial^2\varphi\rangle, \tag{2.18}$$

where the density is given by (4.16) and the Bose field $\varphi = -\beta \sum_i \log|z - z_i|^2$. Eq. (2.18) was obtained in [22]. Furthermore, it is convenient to define the field

$$\tilde{\varphi}(z) = \varphi + \frac{K}{2l^2} - \frac{\beta}{2}\log\sqrt{g}, \tag{2.19}$$

that vanishes at $z \to \infty$. The anti-holomorphic derivative of Eq.(2.18) eliminates the integral, by virtue of the ∂-bar formula $\bar{\partial}(\frac{1}{z}) = \pi\delta^{(2)}(z)$, to give

$$\langle\rho\rangle\partial\langle\tilde{\varphi}\rangle + \Big(1 - \frac{\beta}{2}\Big)\partial\langle\rho\rangle = \frac{1}{2\pi\beta\sqrt{g}}\bar{\partial}\langle(\partial\tilde{\varphi})^2\rangle_c. \tag{2.20}$$

2.6 Iterating the Ward identity: the leading order

The Ward identity consists of terms of different order in N, and can be solved iteratively order by order. The first term on the l.h.s. is of the order N^2, the other two are of the order N. To leading order we thus have $\langle\tilde{\varphi}\rangle = 0$, which yields $\langle\varphi\rangle = -\frac{K}{2l^2} + \frac{\beta}{2}\log\sqrt{g} + \mathcal{O}(l^2)$. From this, using (2.8) we recover the first two terms in (2.2).

To proceed with the next iteration, we need to know $\langle(\partial\tilde{\varphi})^2\rangle_c$ or rather the short distance behavior of the connected two-point correlation function $\langle\varphi(z)\varphi(z')\rangle_c$.

2.7 The Gravitational Anomaly

We obtain the two-point function by varying the one-point function of φ with respect to W: $\delta\langle\varphi(z)\rangle/\delta W(z') = \sqrt{g(z')}\langle\varphi(z)\rho(z')\rangle_c$. Since we already know the leading order of $\langle\varphi\rangle$, we can obtain the leading order of the two-point function $\langle\varphi(z)\rho(z')\rangle_c = \frac{1}{\sqrt{g}}\delta^{(2)}(z-z') - \frac{1}{V}$, or equivalently $\Delta_g\langle\varphi(z)\varphi(z')\rangle_c = -4\pi\beta[\frac{1}{\sqrt{g}}\delta^{(2)}(z-z') - \frac{1}{V}]$ [27]. With this, we see that the two-point function is the Green function of the Laplace-Beltrami operator as in (2.9).

However, this formula is only valid at distances much larger the magnetic length. At short distances, the two-point correlation function $\langle\varphi(z)\varphi(z')\rangle_c$ is regular. General covariance requires regularization of the two point function to be as in Eq. (2.9). The regularization procedure, although plausible, is not immediately evident. However, it can be proved rigorously. We save further discussion of this subtle point for a subsequent paper.

We are now in a position to compute the missing ingredient of the Ward identity (2.20). Taking derivatives and merging points we obtain the known result

$$\begin{aligned}\langle(\partial\tilde{\varphi}(z))^2\rangle_c &= \beta\lim_{z\to z'}\partial_z\partial_\zeta G_R(z,z') \\ &= \frac{\beta}{6}\left[\partial^2\log\sqrt{g} - \frac{1}{2}\left(\partial\log\sqrt{g}\right)^2\right]. \end{aligned} \tag{2.21}$$

This describes the gravitational anomaly. In a curved space we obtain

$$\frac{1}{\sqrt{g}}\bar{\partial}\langle(\partial\tilde{\varphi}(z))^2\rangle_c = -\frac{\beta}{24}\partial R. \tag{2.22}$$

This is the anomalous part of the Ward identity.

2.8 Iterating the Ward identity: subsequent orders

The anomalous contribution (2.22) allows us to extract b by computing the next order in the Ward identity. Inserting (2.22) into (2.20) we obtain the equation

$$\langle\rho\rangle\partial\langle\tilde{\varphi}\rangle + \left(1-\frac{\beta}{2}\right)\partial\langle\rho\rangle = -\frac{1}{48\pi}\partial R. \tag{2.23}$$

Matching terms of the same order (by replacing the first $\langle\rho\rangle$ in (2.23) with its leading order) reduces the equation to a linear form, which readily integrates to

$$\rho_0\langle\tilde{\varphi}\rangle + \left(1-\frac{\beta}{2}\right)(\langle\rho\rangle-\rho_0) = -\frac{1}{48\pi}R. \tag{2.24}$$

In this equation, all of the terms are proportional to the curvature. The r.h.s. is proportional to the trace anomaly of the free Gaussian field. Matching the coefficients determines the coefficient b in (2.2).

2.9 The Pfaffian state

Our results can be generalized to other holomorphic FQH states. We present heuristic arguments for the Pfaffian state attributed to $\nu = 1/2$ filled spin polarized second Landau level [5]. The holomorphic part of the wave-function for this state is proportional to

$\prod_{i<j}(z_i - z_j)^2 \mathrm{Pf}\left(\frac{1}{z_i - z_j}\right)$, where $\mathrm{Pf}(M_{ij})$ is the Pfaffian of the matrix M. We assume that the geometry is encoded entirely in the exponentiated Kähler potential, as in (2.12). In this state the maximal number of particles is $N = \nu(N_\phi + \mathcal{S}\frac{\chi}{2})$ [3] where $\mathcal{S}$ (equal to 3 in this case) is often referred to as the "shift" for FQH states. This formula fixes the leading terms in the density $\rho = \nu(\frac{1}{2\pi l^2} + \frac{\mathcal{S}}{8\pi}R) + \mathcal{O}(l^2)$, where $\nu = 1/2$. The Ward identity fixes the rest of the expansion.

Without derivation we assume that the only change in the Ward identity is the coefficient reflecting the change of the leading order. Explicitly, we assume that the linearized version of the Ward identity (2.24) reads

$$\rho_0 \langle \tilde{\varphi} \rangle + \left(1 - \frac{\mathcal{S}}{2}\right) (\langle \rho \rangle - \rho_0) = -\frac{1}{48\pi} R, \tag{2.25}$$

where $\tilde{\varphi} = \varphi + \frac{1}{2l^2}K - \frac{\mathcal{S}}{2}\log\sqrt{g}$. This equation gives a relation between the coefficient of the leading and next to the leading gradient expansion of the density, and leads us to conjecture

$$b = \frac{1}{12} + \frac{\nu}{4}\mathcal{S}\,(2 - \mathcal{S}) = -\frac{7}{24}. \tag{2.26}$$

Furthermore, since Eqs.(2.5) and (2.7) are generally valid, we find the gradient expansion of $\eta(k)$, and consequently the static structure factor (and Hall conductance) for the Pfaffian state

$$s(q) = \frac{q^2}{2} + \frac{\mathcal{S} - 2}{8} q^4 + \frac{3(2 - \mathcal{S})^2 - \nu^{-1}}{96} q^6 + \mathcal{O}(q^8). \tag{2.27}$$

This expression also applies to the bosonic Pfaffian state at $\nu = 1$ with $\mathcal{S} = 2$. In this case $b = 1/12$.

The q^4 coefficient was previously argued to be robust within a FQH phase [11]. We find that it follows directly from the generalization of Eq.(2.1) to the Pfaffian state, under minimal assumptions on the form of the wave function. The q^6 coefficient is connected to the conjectured b. It can be tested numerically .

2.10 Generating functional and Polyakov's Liouville action

Once we know the density (2.2), the generating functional can be computed by integrating (2.16) in a similar manner to what has been done in [22]. The result for $\beta = 1$ was presented in the recent paper [15]. We proceed by using the relation following from (4.3)

$$\frac{1}{2}(-l^2\Delta_g)\frac{\delta \log \mathcal{Z}[g]}{\delta\sqrt{g}} = \Big(1 + \frac{1}{2}(-l^2\Delta_g)\Big)\langle\rho\rangle = \rho_0 + \frac{1}{8\pi}R + \frac{1}{8\pi}(-b+\frac{1}{2})(-l^2\Delta_g)R. \tag{2.28}$$

The generating functional for an arbitrary filling fraction, developed as an expansion in $1/N_\phi$, reads

$$\begin{aligned}
\log\frac{\mathcal{Z}[g]}{\mathcal{Z}[g_0]} &= \frac{N_\phi(N_\nu+1)}{2} + N_\phi^2 A^{(2)}[g] + N_\phi A^{(1)}[g] + A^{(0)}[g],\\
A^{(2)} &= -\frac{\pi}{2\beta}\frac{1}{V^2}\int K dV, \quad A^{(1)} = \frac{1}{2V}\int \log\sqrt{g}\, dV,\\
A^{(0)} &= \frac{1}{16\pi}\left(\frac{1}{3}+\frac{\beta-1}{2}\right)\left(\int \log\sqrt{g} R\, dV + 16\pi\right),
\end{aligned}$$

where $\mathcal{Z}[g_0]$ is the generating functional of a FQH state on a sphere.

The functionals $A^{(2)}$ and $A^{(1)}$ are familiar objects in Kähler geometry [25, 15]. Unlike the higher order terms, the first three terms cannot be expressed locally through the scalar curvature R. For this reason, they obey non-trivial co-cycle properties explained in [25, 15]. The variations of the first two functionals over the Kähler potential are the volume form and the curvature.

The functional $A^{(0)}$ is Polyakov's Liouville action representing the logarithm of the partition function of a free Bose field [24]. Recall that Polyakov's action appears as a normalized spectral determinant of the Laplace-Beltrami operator, or as a partition function of the Gaussian Bose field

$$-\frac{1}{2}\log\frac{\det\left(-\Delta_g\right)}{\det\left(-\Delta_{g0}\right)}=\frac{1}{96\pi}\int\log\sqrt{g}R\,dV+\frac{1}{6}. \tag{2.29}$$

It is instructive to to normalize the generating functional to β copies of the generating functional of the integer filling case with the proper adjustment of the magnetic field $\mathcal{Z}^{\mathrm{reg}}[g] = \mathcal{Z}[g,\beta,N_\phi]/\left(\mathcal{Z}[g,1,N_\phi/\beta]\right)^{\beta}$, where we emphasize the dependence on β and N_ϕ [28]. This ratio remains finite in the $N_\phi\to\infty$ limit

$$\mathcal{Z}^{reg}[g]=\mathcal{Z}^{\mathrm{reg}}[g_0]\left[\frac{\det\left(-\Delta_g\right)}{\det\left(-\Delta_{g0}\right)}\right]^{-\frac{1}{2}(\beta-1)} \tag{2.30}$$

The regularized part reflects the gravitational anomaly. The generating functional encodes the gravitational and electromagnetic response of the FQH states. It shows how various correlation functions transform under variations of the geometry such as conformal transformations. In particular, the regularized part of the generating function transforms covariantly.

CHAPTER 3

FIELD THEORY FOR THE FQHE

3.1 Collective Field Theory

We start with some general remarks about the collective field theoretical approach.

To compute the expectation value of an observable $\mathcal{O}(z_1, ..., z_N)$ within the ground state $\Psi(z_1, \ldots, z_N)$, one has to evaluate a multiple integral over the individual particle coordinates

$$\langle \mathcal{O} \rangle = \int \Psi^* \mathcal{O} \Psi \, dV_1 \ldots dV_N \,, \quad dV_i = \sqrt{g(z_i)} d^2 z_i, \tag{3.1}$$

and then proceed with the large N limit. The field theory approach assumes instead that the appropriate variables are collective modes. In the QH systems the ground state at a fixed background gauge potential is a holomorphic function of coordinates. On a Riemann surface this means that the wave function is holomorphic in complex (or isothermal) coordinates where the metric is $ds^2 = \sqrt{g} dz d\bar{z}$. Therefore holomorphic collective modes suffice for a complete field theory of the QHE. On genus-0 surfaces they are power sums

$$a_{-k} = \sum_{i=1}^{N} z_i^k, \quad k \geq 1, \quad D\varphi = \prod_{k>0} da_{-k} d\bar{a}_{-k},$$

The sum is taken in the $N \to \infty$ limit and the measure of integration $D\varphi$ represents a functional integration over the real *collective field* $\varphi(\xi)$, where we denote $\xi = (z, \bar{z})$. For further discussion of the measure, see Sec.(6). The field is defined such that its current, the holomorphic derivative $\partial_z \varphi$, is a generating function of the modes a_{-k}

$$\mathrm{i} \partial_z \varphi \equiv -\mathrm{i} \sum_{k \geq 1} a_{-k} z^{-k-1}. \tag{3.2}$$

In this definition we assume that the field has no zero modes $\int \varphi \, dV = 0$ and is therefore globally defined on the Riemann surface. Expectation values are obtained by a functional integral over the field with the appropriate action

$$\langle \mathcal{O} \rangle = \frac{\int \mathcal{O}[\varphi] e^{-\Gamma[\varphi]} D\varphi}{\int e^{-\Gamma[\varphi]} D\varphi} \tag{3.3}$$

as opposed to the multiple integral in (3.1). The collective field φ defined by its expansion at infinity (3.2) can be extended to the finite part of the plane excluding the positions of particles where the current has poles $\partial\varphi|_{z\to z_i} \sim -1/(z - z_i)$. This field is defined as

$$\varphi(\xi) = 4\pi \sum_i G(\xi, \xi_i), \tag{3.4}$$

where G is the Green function of the Laplace-Beltrami operator Δ with the zero mode removed, and which satisfies

$$-\Delta G(\xi, \xi') = \delta^{(2)}(\xi - \xi') - \frac{1}{V}.$$

By definition, the collective field is a solution of the Poisson equation

$$-\Delta\varphi = 4\pi(\rho - \frac{N}{V}), \tag{3.5}$$

where $\rho(\xi)$ is the particle density.

We now specialize our discussion to the Laughlin state on genus-0 surfaces, but the final results hold for any genus. The Laughlin wave function reads

$$\Psi = \frac{1}{\sqrt{\mathcal{Z}}} \prod_{i<j} (z_i - z_j)^m e^{\frac{1}{2}\sum_i Q(\xi_i)}, \tag{3.6}$$

$$\hbar\Delta Q = -2eB, \tag{3.7}$$

where $m = 1/\nu$ is an integer, ν is the filling fraction, and Q is the 'magnetic' potential of a slow varying magnetic field B. Below we set $e = \hbar = 1$.

The normalization $\mathcal{Z}$, known as the generating functional, was studied in [**?** 50]. The generating functional is independent of the choice of coordinates and depends only on the geometry of the surface through functionals of the metric.

At a given magnetic field the state is normalizable if the maximal number of particles is

$$N = \nu N_\phi + \frac{1}{2}\chi, \tag{3.8}$$

where χ is the Euler characteristic of the surface ($\chi = 2$ for a sphere) and $N_\phi = \frac{1}{2\pi}\int B\, dV$ is the total number of magnetic flux quanta. We assume that the state contains a maximal number of particles so the surface is completely filled and the particle density has no boundary.

Our goal is to represent the probability density $dP = |\Psi|^2 \prod_i dV_i$ as a functional integral over the collective field Eq.(4.17) such that $dP \to e^{-\Gamma[\varphi]} D\varphi$.

3.2 Main Results for Field Theory

Now we can formulate some results for the Laughlin state. We compute the action $\Gamma[\varphi]$ in (3.3) in the leading $1/N$ approximation. The action consists of three parts

$$\Gamma[\varphi] = \Gamma_G[\varphi] + \Gamma_B[\varphi] + \Gamma_L[\varphi] \tag{3.9}$$

which are conveniently written in terms of the field φ and related field $\sigma = \log\sqrt{\rho/(N/V)}$

$$\Gamma_G[\varphi] = \frac{1}{8\pi\nu}\int\Big[(\nabla\varphi)^2 - R\varphi - 4\nu B\varphi\Big]\,dV, \tag{3.10}$$

$$\Gamma_B[\varphi] = \frac{2}{\nu}\left(\nu - \frac{1}{2}\right)\frac{N}{V}\int e^{2\sigma}\sigma\,dV, \tag{3.11}$$

$$\Gamma_L[\varphi] = \frac{1}{24\pi}\int\Big[(\nabla\sigma)^2 + R\sigma\Big]\,dV. \tag{3.12}$$

where R is a scalar curvature of the surface. The actions (3.10-3.12) are derived in sections 4-7. We remind that the field φ is defined such that $\int\varphi\,dV = 0$, so the coupling with the curvature R and magnetic field B in (3.10) occurs only if the curvature and magnetic field are not uniform. If they are uniform, the magnetic field enters only through relation (3.8).

The action is non-linear since σ and φ are connected by the Eq. (3.5). It consists of three distinct terms at different orders in $1/N$, in descending order. This can be seen by noticing that φ defined by (4.17) is of the order N, while σ is of the order 1.

The leading term (3.10) of the action is the Gaussian free field with a *background charge* which describes the coupling to curvature, cf. [64, 35, 136] The background charge is directly related to the shift $\chi/2$ in (3.8). Perturbatively, the action (3.10) is equivalent to the Liouville theory of gravity (see e.g., [60]) in the sense that the background charge increases the central charge of the Gaussian field from 1 to $1 + 3\nu^{-1}$. As a consequence the conformal dimension of the vertex operator $e^{-a\varphi}$ is

$$h_a = \frac{1}{2}a(1 - a\nu). \tag{3.13}$$

The conformal dimension is equal to the spin of the quasi-hole. This result refines the erroneous notion that the spin of a quasi-hole matches its mutual statistics and the charge deficit, both equal the filling fraction ν at $a = 1$.[1]

1. To the best of our knowledge the spin of the quasi-hole was correctly computed in [96], see also [89] and [129].

Formally the action (3.10) is that of a Gaussian free field and possesses conformal invariance. This invariance breaks at the next order of the action (3.11), except in the case of the Bosonic Laughlin state $\nu = 1/2$ at which (3.11) vanishes.

Finally, the Polyakov-Liouville action (3.12) manifests the gravitational anomaly. This part of the action alone is identical to the action of the Liouville theory of gravity if the density $\rho = (N/V)e^{2\sigma}$ is identified as a random metric (from this point of view, the field φ plays the role of a random Kähler potential (cf.[62])). The action does not posses the cosmological term since the number of particles is fixed and $\int e^{2\sigma} dV = V$.

We can check the consistency of the action against some known results.

Minimizing the action we find the first three leading terms of the $1/N$ expansion of the ground state value of the particle density previously obtained in [**?**]. If the magnetic field is uniform it is also a gradient expansion in curvature

$$\langle\rho\rangle = \bar{\rho} + \left[\frac{1}{2\nu}\left(\nu - \frac{1}{2}\right) + \frac{1}{12}\right](l^2\Delta)\frac{R}{8\pi}, \quad \bar{\rho} = \frac{\nu B}{2\pi} + \frac{R}{8\pi}, \tag{3.14}$$

where $l = \sqrt{\hbar/eB}$ is the magnetic length.

The $\bar{\rho}$ term in (3.14) comes from (3.10). Integrating over the density yields the particle number (3.8), where the $R/(8\pi)$ term yields the background charge of $\chi/2$ due to the Gauss-Bonnet theorem $\int R dV = 4\pi\chi$. The order l^2 term in (3.14), which receives contributions from both (3.11) and (3.12) does not contribute to the particle number.

Linearizing the action on a flat space yields the propagator of density modes

$$\Gamma[\varphi] \approx \frac{V}{2N} \sum_k S^{-1}(k)|\rho_k|^2, \tag{3.15}$$

where $S(k)$ is the static structure factor expanded to order k^6, first computed in [83] (see also [?])

$$S^{-1}(k) = \frac{2}{\nu(kl)^2}\left(\nu+(\nu-\frac{1}{2})(kl)^2+\frac{1}{48}(kl)^4\dots\right) \tag{3.16}$$

Other results are described below.

3.3 Boltzmann Weight

The first step in constructing the collective field theory is expressing the wave function (3.6) as a functional of the collective field. The amplitude of (3.6) is interpreted as the Boltzmann weight of the neutralized Coulomb plasma $|\Psi|^2 \sim e^{-E}$, with temperature set to unity. We express the energy in terms of the Green function and the Kähler potential K defined by the conditions $\partial_z\partial_{\bar z}K = (\pi/V)\sqrt{g}$ and $K \sim \log|z|^2 + \mathcal{O}(1/|z|)$ at infinity. Note that for constant B, the potential becomes $Q = -N_\phi K$. The energy reads

$$\begin{aligned} E = &-2\iint \rho(\xi)G(\xi,\xi')B(\xi')dV_\xi dV_{\xi'} - N\int Q\frac{dV}{V} \\ &-\frac{1}{2}NN_\phi\int K\frac{dV}{V} + \frac{2\pi}{\nu}\sum_{i\neq j}G(\xi_i,\xi_j). \end{aligned} \tag{3.17}$$

The last term in (3.17) takes into account the discreteness of particles.

In the continuum limit, we have to replace the sums over particle positions $\sum_{i\neq j} G(\xi_i,\xi_j)$ by integrals over the density taking into account the excluded self-interaction at $i = j$. We must therefore regularize Green function $G(\xi_i,\xi_j)$ at coinciding points. The regularized Green function is defined by subtracting the logarithm of the geodesic distance $|\xi-\xi'|g^{1/4}$ between the points in units of the typical separation between particles, which is of the order of $\rho^{-1/2}$

$$G_R(\xi) = \lim_{\xi\to\xi'}\left(G(\xi,\xi') + \frac{1}{4\pi}\log[|\xi-\xi'|^2\rho\sqrt{g}].\right) \tag{3.18}$$

Thus $\sum_{i\neq j} G(\xi_i,\xi_j)$ must be replaced by

$$\int\left[\int G(\xi,\xi')\rho(\xi')dV_{\xi'} - G_R(\xi)\right]\rho(\xi)dV_\xi.$$

Bringing all pieces together and integrating by parts

$$E = E_0 + \Gamma_G[\varphi] - \frac{1}{2\nu}\int \rho\log\rho\, dV, \tag{3.19}$$

where $\Gamma_G[\varphi]$ is given by (3.10), and

$$E_0 = \frac{N}{\nu V}\int\int \log|\xi-\xi'|^2\left(\bar\rho(\xi') - \frac{1}{2}\frac{N}{V}\right)dV_\xi dV_{\xi'}$$

where $\bar\rho$ is defined in (3.14). This gives the field theoretical representation of the wave function. We comment that the short distance regularization is determined by the density ρ and for that reason depends on the state of the plasma. A similar regularization scheme was employed for a 1D plasma in Ref.[59].

3.4 Entropy

The next step is to pass from integration over coordinates of individual particles to integration over the macroscopic density. This is a standard method in statistical mechanics (used in a setting similar to ours in [59]). The transformation defines the Boltzmann entropy $S_B[\rho] = -\int \rho\log(\rho/\bar\rho)\, dV$

$$\prod_i \sqrt{g(\xi_i)}d^2\xi_i \to e^{S_B} D\rho.$$

Combining the Boltzmann weight and the entropy together we obtain the probability density

$$dP \to e^{-E[\rho]+S_B[\rho]} D\rho.$$

Here, the free energy of local equilibrium is

$$E - S_B = E_0 + \Gamma_G + \Gamma_B.$$

We observe that the Boltzmann entropy and the short distance regularization of the Coulomb energy (3.19) combine to form Γ_B.

3.5 Ghosts

The next step is to determine the measure $D\rho$. Passing from $\rho \to \varphi$ comes at the price of a Jacobian, which is given by the spectral determinant of the Laplace-Beltrami operator

$$D\rho \sim \mathrm{Det}(-\Delta) D\varphi. \tag{3.20}$$

The determinant can be represented by $(1,0)$ Faddeev-Popov ghosts by the following equation $\mathrm{Det}(-\Delta) = \int e^{-\int \bar{\eta}(-\Delta)\eta\, dV} D\eta D\bar{\eta}$, where η are complex fermionic modes.

3.6 Gravitational Anomaly

The last step involves the functional measure in (3.20). The procedure we outline below is commonly used in the theory of quantum gravity. Let us denote by X a field φ or ghosts $\eta, \bar{\eta}$ and consider the deviation δX from a given value of the field, say its mean. We define the norm of the deviation as

$$||\delta X||^2 = \sum_{i=1}^{N} (\delta X(\xi_i))^2 = \int (\delta X)^2 \rho dV \tag{3.21}$$

and assume that the measure is normalized as $\int DX \exp[-||\delta X||^2] = 1$. Such normalization is supported by calculations based on the Ward identity for Laughin states [50]. Thus the measure for both φ and the ghost fields depends in a nontrivial fashion on the density, and thus on φ itself. So although the ghosts appear decoupled from the rest of the action, in fact they are not.

The density ρ appearing in (4.20) can be treated as a conformal factor of the metric and thus removed from the measure by a conformal transformation of coordinates $dV \to \rho^{-1} dV$. It is known, however, that under conformal transformation the measure transforms anomalously as

$$DX \to e^{c_X \, \Gamma_L[\sigma]} DX,$$

where c_X is the central charge of the field X, where $\Gamma_L[\rho]$ is the Polyakov-Liouville action (3.5) [106], see also [55]. This is the *Weyl* or *gravitational anomaly* which appears here in a similar fashion as in the quantum theory of gravity. Applying this to the collective field φ with the central charge $+1$ and ghost with the central charge -2 we obtain the measure

$$e^{-\Gamma_L[\rho]} D\varphi D\eta D\bar{\eta}.$$

After the Polyakov-Liouville action is taken into account the short distance regularization of the field φ and ghosts does not depend on density. Since the ghosts are decoupled their contribution is the spectral determinant of the Laplace operator. Summing up, the probability distribution is

$$dP = \mathcal{Z}^{-1}\ \mathrm{Det}(-\Delta) e^{-E_0-\Gamma[\varphi]} D\varphi. \tag{3.22}$$

The ghosts determinant contributes to the finite size correction to the free energy of the Coulomb plasma [79, 50].

Now we turn to some applications.

3.7 Density and generating functional

We start from computing the generating functional - the normalization factor of the Laughlin wave function or (3.22).

The integral of the lhs of (3.22) is 1. The relevant contribution to the integral of the rhs of (3.22) comes from the Gaussian approximation. It consists of the on-shell action $\Gamma[\varphi_c]$ computed on the "classical" solution φ_c, which minimizes the action. Computing Gaussian fluctuations it is sufficient to take into account only the leading part of the action (3.10)

$$\int e^{-\Gamma[\varphi]} D\varphi = [\ \mathrm{Det}(-\Delta)]^{-\frac{1}{2}} e^{-\Gamma[\varphi_c]}.$$

Thus integrating (3.22) gives

$$\mathcal{Z} = [\ \mathrm{Det}(-\Delta)]^{\frac{1}{2}} e^{-\Gamma_0}, \quad \Gamma_0 = E_0 + \Gamma[\varphi_c]. \tag{3.23}$$

In the three first leading orders in $1/N$ solution of $\delta\Gamma[\varphi]/\delta\varphi = 0$ is the ground state value of the field $\varphi_c = \langle\varphi\rangle$, which, through (3.5) determines the ground state value of the density. Solving in the leading order in $1/N$ we obtain Eq.(3.14).

Inserting (3.14) back into (3.9) we find

$$\Gamma[\varphi_c] = -\frac{2\pi}{\nu} \int\int \bar{\rho}(\xi')G(\xi,\xi')\bar{\rho}(\xi')dV_\xi dV_{\xi'}.$$

The final result for the functional Γ_0 in (3.23) is best expressed in terms of the gauge potential and spin connection. Their complex components are defined by

$$2\mathrm{i}(\partial_{\bar z}A_z - \partial_z A_{\bar z}) = B\sqrt{g}, \quad 2\mathrm{i}(\partial_{\bar z}\omega_z - \partial_z\omega_{\bar z}) = \frac{1}{2}R\sqrt{g}.$$

In the transverse gauge $\partial_{\bar z}A_z = -\partial_z A_{\bar z}$, $\partial_{\bar z}\omega_z = -\partial_z\omega_{\bar z}$ the functional Γ_0 has a compact form

$$\Gamma_0 = -\frac{2}{\pi\nu}\int \left|\left(\nu A_z + \frac{1}{2}\omega_z\right)\right|^2 dzd\bar z.$$

It remains to recall the value of the spectral determinant of the Laplace operator in (3.23). Up to a metric independent terms it is given by the Polyakov formula [104]

$$\log \mathrm{Det}(-\Delta) = -\frac{1}{3\pi}\int |\omega_z|^2 dzd\bar z.$$

As a result (cf.,[50])

$$\log \mathcal{Z} = \int \left[\frac{2}{\pi\nu}\left|\left(\nu A_z + \frac{1}{2}\omega_z\right)\right|^2 - \frac{1}{6\pi}|\omega_z|^2\right] dzd\bar z. \tag{3.24}$$

In the form (3.24) it is valid on a surface with any genus.

The authors of Ref.[86] argued that the elements of the Hessian matrix of the generating functional

$$\sigma_H = \frac{\pi}{2}\frac{\delta^2 \log \mathcal{Z}}{\delta A_z \delta A_{\bar z}}, \; 2\varsigma_H = \frac{\pi}{2}\frac{\delta^2 \log \mathcal{Z}}{\delta \omega_z \delta A_{\bar z}}, \; -\frac{c_H}{12} = \frac{\pi}{2}\frac{\delta^2 \log \mathcal{Z}}{\delta \omega_z \delta \omega_{\bar z}}$$

are universal transport coefficients precisely quantized on QH-plateaus. Here σ_H is the Hall conductance, ς_H determines the current caused by changing of the metric and the third coefficient, c_H, describes forces exerted on the fluid as a result of a changing the metric. We refer to [86] for further details. For Laughlin states these coefficients are encoded in (3.24)

$$\sigma_H = \nu, \quad \varsigma_H = 1/4, \quad c_H = 1 - 3/\nu \tag{3.25}$$

3.8 Quasi-holes - gauge anomaly.

Introduced by Laughlin [93], a quasi-hole state with charge a on a compact surface reads

$$\Psi_a = \frac{e^{\frac{1}{2}\nu a[Q(w) - aK(w)]}}{\sqrt{\mathcal{Z}_a[w, \bar{w}]}} \left[\prod_{i=1}^{N} (z_i - w)^a e^{-\frac{a}{2}K(z_i, \bar{z}_i)} \right] \Psi, \tag{3.26}$$

where w is a holomorphic coordinate of the quasi-hole, Ψ is the ground state (3.6) with N particles subject to the condition (3.8), a is a positive integer less than $m = 1/\nu$, and K is defined above (3.17). The factor of $\exp\left(-\frac{a}{2}K(\xi_i)\right)$ neutralizes the insertion of the quasi-hole. This state covers the entire surface. The exponential factor of $\frac{a\nu}{2}[Q - aK]$ in (3.26) is added for a convenience.

A quasi-hole is represented by the vertex operator $V_a(w, \bar{w}) = e^{-a\varphi(w, \bar{w})}$. In particular the normalization factor $\mathcal{Z}_a$, the generating functional for a quasi-hole state, reads up to constants

$$\mathcal{Z}_a[w, \bar{w}] \sim \Big\langle V_a(w, \bar{w}) \Big\rangle,$$

where the average is taken over the ground state (3.6) without the quasi-hole. As such the quasi-hole may be seen as a source for the action (3.10) $\Gamma \to \Gamma + a\varphi(w)$. However, there is a caveat. The quasi-hole disturbs the electronic density around itself in a vicinity of the size of magnetic length. At the limit of a vanishing magnetic length the density becomes singular. At the same time the derivation of the action was based on the assumption that the density is smooth. Therefore the derivation must be reexamined to take into account the feedback of the singularity.

The leading $1/N$ value of (3.24) is given by the Gaussian part of the action (3.10)

$$\mathcal{Z}_a \approx \exp\left(-a\langle\varphi\rangle + \frac{a^2}{2}\langle\varphi^2\rangle_c\right). \tag{3.27}$$

The mean of the field φ determined by (3.10) is

$$\langle\varphi(\xi)\rangle \approx 4\pi \int G(\xi,\xi')\bar{\rho}(\xi')dV_{\xi'} = \nu Q + \frac{1}{2}\log\sqrt{g(\xi)},$$

the variance is $\langle\varphi^2\rangle_c \equiv \langle\varphi^2\rangle - \langle\varphi\rangle^2 = 4\pi\nu G_R$, where the regularized Green function is given by (3.18). But the G_R depends on the density itself, and in the leading approximation one replaces the density by its mean such that $\langle\varphi^2\rangle_c = \nu\log\left(\langle\rho\rangle\sqrt{g}\right)$. Putting this together we obtain

$$\mathcal{Z}_a \approx \left(\sqrt{\langle\rho\rangle}\right)^{\nu a^2} (\sqrt{g})^{-h_a}, \tag{3.28}$$

where $h_a = \frac{a}{2}(1-\nu a)$ is the conformal dimension as in (3.13).

In the leading approximation the factor $\langle\rho\rangle$ in (4.29) can be treated as a constant. Then (3.22) suggests that h_a is the conformal dimension of the quasi-hole state: the quasi-hole state transforms as a primary field under a holomorphic transformation. Symbolically

$$w \to f(w), \quad V_a \to (f'(w))^{h_a} V_a$$

Because the state is holomorphic (up to the normalization factors in (3.26))the holomorphic dimension h_a is also the spin of the state. Later we show this in a more direct manner.

In the next to the leading approximation we cannot assume the density is (4.29) to be a constant. As with the gravitational anomaly above, the field transforms as $\varphi \to \varphi - a\nu \log\sqrt{\rho}$, which modifies the vertex operator

$$V_a = (\sqrt{\rho})^{\nu a^2} e^{-a\varphi},$$

such that the regularization of the two-point correlation function at coincident points is independ on the state density. Alternatively, we may say that the quasi-hole contributes to the action as a source $\Gamma \to \Gamma + a\varphi - a^2\nu\log\sqrt{\rho}$. Thus the stationary point of the action reads

$$\frac{\delta\Gamma}{\delta\varphi(\xi)} = -a\left(1 + \frac{\nu a}{8\pi\rho}\Delta\right)\delta(w-\xi). \tag{3.29}$$

In the linear approximation we treat ρ in (4.31) as a constant $\approx \nu/(2\pi l^2)$ and use (3.15). As a result we obtain the first two terms of the expansion in $(kl)^2$

$$\rho_k \approx \frac{2\nu a}{(kl)^2}\left(-1 + \frac{a}{4}(kl)^2\right)S(k) \approx -\nu a + \frac{(kl)^2}{2}(a\nu - h_a).$$

Equivalently the first two moments of the density $\delta\rho = \langle\rho\rangle - \frac{N}{V}$ are

$$m_0 = \int \delta\rho\, dV = -\nu a, \tag{3.30}$$

$$m_2 = \frac{1}{2l^2}\int r^2\delta\rho\, dV = -\nu a + \frac{1}{2}a(1-\nu a). \tag{3.31}$$

The first formula describes the fractional charge deficit $-\nu a$. This result goes back to [93]. The second moment is more involved [126, 80, 50]. Curiously, the second moment vanishes at $\nu = \frac{1}{3}$ and $a = 1$.

Having determined the generating functional, we compute the adiabatic phase $\gamma_{\mathcal{C}}$ acquired by the quasi-holes by transporting one around a closed path $\mathcal{C}$.

For simplicity we compute the adiabatic phase when one hole with coordinate w_1 moves around a closed path $\mathcal{C}$ enclosing another quasi-hole with coordinate w_2. The extension of (3.27,4.29) to the case of two quasi-holes is

$$\mathcal{Z}_{a_1 a_2}(w_1, w_2) = \mathcal{Z}_{a_1}(w_1)\mathcal{Z}_{a_2}(w_2)e^{4\pi\nu a_2 a_1 G(w_1,w_2)}, \tag{3.32}$$

where we used $\langle\varphi(w_2)\varphi(w_2)\rangle_c = 4\pi\nu G(w_1, w_2)$ and (3.27).

The adiabatic phase reads

$$\gamma_{\mathcal{C}} = 2\mathrm{i} \int \left[\oint_{\mathcal{C}} \overline{\Psi} \partial_{w_1} \Psi dw_1 \right] dV_1 \ldots dV_N.$$

Since the state is a holomorphic function of position of the quasi-holes, only normalization factor in (3.26) contributes to the phase

$$\gamma_{\mathcal{C}} = -2\pi a_1 \nu\, \Phi_{\mathcal{C}} + \mathrm{i} \oint_{\mathcal{C}} \partial_{w_1} \log \mathcal{Z}_{a_1 a_2} dw_1.$$

The first term is the Aharonov-Bohm phase picked up by a particle with charge $-a_1\nu$ enclosing the magnetic flux $\Phi_{\mathcal{C}} = (N_\Phi + a_1 + a_2)\mathrm{Area}(\mathcal{C})/V$ in units of the flux quantum. The contribution of the second term follows from (4.30)

$$\mathrm{i} \oint_{\mathcal{C}} \partial_{w_1} \log \mathcal{Z}_{a_1 a_2} dw_1 = -h_{a_1} \Omega_{\mathcal{C}} + 2\pi\nu a_1 a_2. \tag{3.33}$$

It contains the solid angle $\Omega_{\mathcal{C}} = \mathrm{i} \oint d \log \sqrt{g} = \frac{1}{2} \int_{\mathcal{C}} R dV$. The coefficient in front of it is the spin of the quasi-hole, equal to the holomorphic dimension (3.13). This formula extends the result of Refs.[96], which was for the adiabatic phase of a single quasi-hole ($a = 1$) on a sphere.

The last term in (3.33) $4\pi i\nu a_2 a_1 \oint dG(w_1, w_2)$, which vanishes if the contour $\mathcal{C}$ does not enclose w_2, is commonly referred to as the mutual statistics of the quasi-holes. When the quasi-holes are identical, it is equal to νa^2, and differs from the spin.

3.9 Effect of spin

Lastly, we comment on the effect of spin of quantum Hall states. The spin, yet another characterization of the QH state was introduced in Ref. [50]. The inclusion of spin comes as a generalization of the lowest Landau level (LLL). We recall that the LLL are defined as zero modes of the anti-holomorphic component of the kinetic momentum operator $\bar{\pi} = -i\hbar\bar{\partial} + \hbar s\bar{\omega} - e\bar{A}$ where $\bar{\omega} = -(i/2)\bar{\partial}\log\sqrt{g}$, where parameter s is the spin. Throughout the paper we set the spin to zero. Inclusion of spin effectively shifts the potential Q in (3.7) by $-s\log\sqrt{g}$, such that the modified Q now satisfies the Poisson equation $\Delta Q = -\frac{2e}{\hbar}B + sR$. As a result, the action acquires an additional term $\frac{s}{4\pi}\int \varphi R dV$, which shifts the background charge in the Gaussian action

$$\Gamma_G[\varphi] = \frac{1}{8\pi\nu}\int \Big[(\nabla\varphi)^2 - (1-2\nu s)R\varphi - 4\nu B\varphi\Big]\, dV.$$

The Boltzmann entropy (3.11) and the Polyakov-Liouville action (3.12) remain the same. Below we list some effects of spin.

Spin does not appear in local properties evaluated at distances where change of curvature is negligible, for example in a flat space. In particular the structure factor $S(k)$ (3.16), the charge of the quasi-hole m_0 (3.30) and its moment m_2 (3.31) are independent of spin.

However geometric characteristics depend on spin. As such, the relation (3.8) between the total number of particles and magnetic flux becomes

$$N = \nu N_\phi + \frac{1}{2}(1 - 2\nu s)\chi. \tag{3.34}$$

The spin modifies the conformal dimension (3.13) defined in (4.29) and appearing in the adiabatic phase (3.33)

$$h_a = \frac{1}{2}a(1 - 2\nu s - a\nu).$$

However, the second moment (3.31) will not acquire any spin dependence, and will maintain its relation to the conformal dimension $m_2 = (1 - s)m_0 + h_a$.

Spin also enters the generating functional (3.24)

$$\log \mathcal{Z} = \int \left[\frac{2}{\pi\nu} \left| \left(\nu A_z + \frac{1}{2}(1 - 2\nu s)\omega_z \right) \right|^2 - \frac{1}{6\pi} |\omega_z|^2 \right] dz d\bar{z}$$

Consequently, the Hall conductance does not depend on spin, but the geometric transport coefficients in (3.25) do

$$\varsigma_H = \frac{1}{4}(1 - 2\nu s), \; c_H = 1 - 3\nu^{-1} (1 - 2\nu s)^2$$

For more details regarding the inclusion of spin into the FQHE on a curved space, see [50].

To conclude, we reformulated the FQHE on a curved surface in terms of a field theory. The field theory consists of the Gaussian action with the background charge and the sub-leading corrections representing the gravitational anomaly. We demonstrated that this theory captures conformal properties of quasi-holes, the adiabatic transport, and clarifies the effect of the gravitational anomaly.

Finally we comment that the action similar to (3.9) has been considered in [62] as an admissible action for a random metric. The actions become analogous upon identifying the fluctuating density as a random metric and the field φ as a fluctuating Kähler potential. We

thank S. Klevtsov for bringing Ref. [62] to our attention.

CHAPTER 4

FQHE ON SINGULAR SURFACES

4.1 Main results for FQHE on Singular Surfaces

a. Conformal dimensions. We show that the states are conformal primary in the vicinity of a singularity, magnetic or geometric. In [49, 91] (see also [89]) it was shown that the magnetic singularity is a conformal primary with the dimension

$$h_a = \frac{1}{2}a(2\mu_H - \nu a). \tag{4.1}$$

In this paper we show that the geometric singularity is also conformal primary, but in this case its dimension is controlled solely the gravitational anomaly

$$\Delta_\alpha = \frac{c_H}{24}(\gamma^{-1} - \gamma), \quad \gamma = 1 - \alpha. \tag{4.2}$$

The formula (4.2) is familiar in the conformal field theory: $-\Delta_\alpha$ (mind the opposite sign!) is the dimension of a vertex operator of a conical point in conformal field theory with the central charge c_H [87, 43]. The same formula enters the finite size correction to the free energy of critical systems on a conical surface [52] and equivalently the formula for the determinant of the Laplace operator (e.g., [88, 37]). These are not coincidences. In the neighborhood of a singularity, QH-states and conformal field theory share the same mathematics, but are by no means identical: the conformal dimension of QH states is opposite to that in conformal field theory with the central charge given by (3.13).

Conformal dimensions are important characteristics which enter physical observables.

b. Dimension, gyration, and spin We show that the dimension determines transport in the neighborhood of the singularity. The electronic fluid gyrates around the conical point with an intensive angular momentum, independent of volume. We will show that the intensive part of the angular momentum is exactly the dimension (4.2)

$$\mathrm{L}_\alpha = \hbar \Delta_\alpha. \tag{4.3}$$

A reason for this is that the angular momentum of the gyrating fluid gives the spin to the singularity - an adiabatic rotation of the state by 2π results in a phase $(2\pi/\hbar)L_{a,\alpha}$. But because the state is holomorphic, its spin is identical to the dimension.

A similar formula holds for the angular momentum of the combined magnetic and geometric singularities

$$\mathrm{L}_{\alpha,a} = \hbar \left(\frac{1}{\gamma} h_a + \Delta_\alpha \right). \tag{4.4}$$

The intensive angular momentum (4.4) is added to (2.1) Hence, if the area D of the fluid parcel is taken to zero, only the angular momentum (4.4) remains.

c. Braiding singularities Just like Laughlin's quasi-holes (which are closely related to flux tubes (1.4)), conical singularities can be braided. The phase acquired by adiabatically exchanging two singularities is called the exchange statistics. Braiding two quasi-holes with charges a_1 and a_2 yields the phase

$$\Phi_{12} = \pi(\nu a_1 a_2). \tag{4.5}$$

This result is known since early days of QHE [36].

Braiding conical singularities is more involved. We argue that braiding phase of two cones of the order α_1 and α_2 are determined exclusively by the central charge

$$\Phi_{12} = -\pi \frac{c_H}{24} \alpha_1 \alpha_2 \left(\frac{1}{\gamma_1} + \frac{1}{\gamma_2} \right) = \pi \left(\alpha_2 \Delta_{\alpha_1} + \alpha_1 \Delta_{\alpha_2} \right) + \pi \frac{c_H}{12} \alpha_1 \alpha_2. \tag{4.6}$$

Here, we assume that the path is sufficiently small, so conical singularities are the only contributions to the solid angle swept out by the path. The first two terms in (4.6) are the phase acquired by a particle with spin Δ_{α_1} (or Δ_{α_2}) going half way around a solid angle $4\pi\alpha_1$(or α_2). The last term

$$\frac{c_H}{12}\alpha_1\alpha_2$$

is the exchange statistics.

On an orbifold, where either γ or $1/\gamma$ is an integer n the phase for identical cones is

$$\Phi_{12} = \pi\frac{c_H}{12}\left(\sqrt{n} - \frac{1}{\sqrt{n}}\right)^2. \tag{4.7}$$

It appears rational, even in the case of the integer QHE.

The formulae (4.2-4.6) are our main results: the braiding statistics of the singularities and the angular momentum of the electronic fluid around a cone are given solely by the gravitational anomaly. Other results such as the transport and the fine structure of the density profile at the singularity are shown below.

d. Moment of inertia The conformal dimension can be also read-off from the fine structure of the density profile in the neighborhood of the singularity. On a singular surface the density changes abruptly on the scale of magnetic length and in the limit of vanishing magnetic length is a singular function. It is properly characterized by the moments

$$m_{2n} = \int (r^2/2l^2)^n(\rho - \rho_\infty)dV. \tag{4.8}$$

Here, $\rho_\infty = \nu(e/h)B$ is the asymptotic value of the density away from the singularity and $l = \sqrt{\hbar/(eB)}$ is the magnetic length. In the integral (4.8) r is the Euclidean distance to the singularity and $dV = 2\pi\gamma r dr$ is the volume element.

The first moment, the 'charge' m_0, follows from the generalized Středa formula – the number of particles in an area dV is saturated by $\bar{\rho} dV$ where

$$\bar{\rho} = \nu(eB/h) + (\mu_H/4\pi)R. \tag{4.9}$$

We will obtain this relation in the next section.

Hence

$$m_0 = \int (\bar{\rho} - \rho_\infty)\, dV = -\nu a + \mu_H \alpha. \tag{4.10}$$

Eq. (4.10) says that if $\mu_H > 0$, the apex accumulates electrons when $\alpha > 0$. It gives an alternative definition of the transport coefficient μ_H.This result for $j = 0$ is well known (see, e.g., [45, 134, 30]) and there is even a recent claim of experimental observation [114]. However, the gravitational anomaly does not enter here. It emerges in the next moment, the *moment of inertia* of the gyrating parcel m_2. We will see that

$$m_2 = (1 - j)m_0 + \gamma^{-1} h_a + \Delta_\alpha, \tag{4.11}$$

where h_a and Δ_α are the dimension (4.1,4.2). We check this formula against the integer QH effect, $\nu = 1$, where all the moments are computed exactly. We do this in the last section of the paper.

The relation between the moment of inertia (4.11) and the angular moment (4.3) is not surprising. In a QH state, positions of particles determine their velocities. Consequently, the density determines the momentum P of the flow. The relation between the momentum and the density has been obtained in [134, 131, 132], in the next section we recall its origin. The relation reads

$$\nabla \times \mathrm{P} = -eB\,(\rho - \bar{\rho}) + \frac{\hbar}{2}(1 - j)\Delta\rho, \tag{4.12}$$

where $(\nabla\times)_i = \epsilon_{ij}\nabla_j$, ∇_j is a covariant derivative, Δ is the Laplace-Beltrami operator, and $\bar{\rho}$ is given by (4.9). In the next section we recall its origin.

With the help of this formula we express the angular momentum in terms of the density. In order to avoid unnecessary complications with the definition of the angular momentum on a curved surface, we assume that close to the singularity the surface locally can be approximated by a flat surface with rotational symmetry. Then the angular momentum about the conical point, expressed in local coordinates ξ, is given by the standard formula $\mathrm{L} = \int (\xi \times \mathrm{P})\, dV$.

Using (4.12) we obtain

$$\mathrm{L} = (eB) \int \frac{r^2}{2} (\rho - \bar{\rho}) dV + \hbar (j-1) \int \rho dV. \tag{4.13}$$

Interpreting this formula we notice that the first term is the diamagnetic effect of fluid gyrating in magnetic field the second term is the paramagnetic contribution.

The formula for the charge of the cone (4.10) is a consequence of (4.12). Away from the singularity the momentum rapidly vanishes. As a result the integral $\int (\nabla \times \mathrm{P})\, dV$ vanishes. Then (4.12) yields (4.10).

The integral (4.13) over the bulk of the surface gives the extensive part (2.1), while the integral over a patch at the singularity is $m_2 - (1-j) m_0$. Then (4.11) yields (4.4). It remains to compute (4.1,4.2).

e. Transport at the singularity. Since the work of Laughlin [92] it was known that an adiabatic change of the magnetic flux $a(t)$ in (1.4) threading through the puncture of a disk causes a radial electric current flowing outward $\mathrm{I} = -\nu e \dot{a}$.

Adiabatically evolving the order of the conical singularity $\alpha(t)$ also induces a current. It follows from (4.10) that the current flowing away from the apex $\mathrm{I} = e\dot{m}_0$ is $\mathrm{I} = e\mu_H \dot{\alpha}$.

More interestingly, both evolving flux and the cone angle accelerate the gyration of the fluid, and produce a torque. The torque is the moment of the force exerted on a fluid parcel

$\mathrm{M} = \int (\mathrm{r} \times \mathrm{F}) dV$. Since $\dot{\mathrm{P}} = \mathrm{F}$, the torque is the rate of change of the angular momentum $\mathrm{M} = \dot{\mathrm{L}}$. From (4.3) it then follows that the torque is proportional to the rate of change of the conformal dimension. We collect the formulae for electric and geometric transport

$$\text{e-transport: current} = -e\nu\dot{\alpha}, \qquad \text{torque} = \hbar\dot{h}_a, \tag{4.14}$$

$$\text{g-transport: current} = e\mu_H\dot{\alpha}, \qquad \text{torque} = \hbar\dot{\Delta}_\alpha. \tag{4.15}$$

These formulas put geometric transport in a nutshell.

In the remaining part of the paper we obtain the dimensions (4.1,4.2) and the statistics (4.6) by employing the conformal Ward identity, a framework developed in [136, 49].

4.2 QH-states on a Riemann surface

Before turning to singular surfaces, we recall some key facts about Laughlin states on a Riemann surface [85, 49].

The most compact form of the state appears in locally chosen complex coordinates $(z, \bar{z})$, where the metric is conformal $ds^2 = e^{\phi}|dz|^2$. In these coordinates the Laplace-Beltrami operator is $\Delta = 4e^{-\phi}\partial_z\partial_{\bar{z}}$ and the volume form is $dV = e^{\phi}d^2z$.

In the conformal metric we can always choose coordinates such that the unnormalized spin-j state reads

$$\Psi = \prod_{1 \le i < k}^{N} (z_i - z_k)^{\beta} \exp \sum_{i=1}^{N} \frac{1}{2} \left[Q(z_i, \bar{z}_i) - j\phi(z_i, \bar{z}_i) \right] \tag{4.16}$$

t where, the integer $\beta = \nu^{-1}$ is the inverse filling fraction and Q is the magnetic potential defined by $-\hbar\Delta Q = 2eB$.

While the wave function (4.16) explicitly depends on the choice of coordinates, the normalization factor

$$\mathcal{Z}[Q,\phi] = \int |\Psi|^2 \prod_i \exp\left[\phi(z_i,\bar{z}_i)\right] d^2 z_i \tag{4.17}$$

does not. It is an invariant functional depending on the geometry of the surface, and in particular on the positions and orders of singularities.

The functional encodes the correlations and the transport properties of the state and for that reason is referred to as a generating functional. For example, a variation of the generating functional over the magnetic potential Q at a fixed conformal factor ϕ is the particle density

$$\rho\, dV = \left(\frac{\delta \log \mathcal{Z}}{\delta Q}\right) d^2 z.$$

In [86], it was shown that that a variation of the generating functional over ϕ at a fixed volume and a fixed gauged potential implies the variational formula for the momentum of the fluid

$$\mathrm{P} = \frac{\hbar}{2}\nabla \times \left(\frac{\delta \log \mathcal{Z}}{\delta \phi}\right). \tag{4.18}$$

Then for surfaces of revolution the quantity

$$\mathrm{L} = -\hbar \int \left(\frac{\delta \log \mathcal{Z}}{\delta \phi}\right) d^2 z. \tag{4.19}$$

is interpreted as angular momentum. In (4.18,4.19) the variation is taken at a constant magnetic field and curvature.

With the help of these formulas we can obtain the relations (4.12,4.13). They follow from the observation that the magnetic potential and the conformal factor appear in (4.16,4.17) on almost equal footing, besides that under a variation over the conformal factor magnetic potential varies as $-\hbar\Delta\delta Q = 2\delta\phi(eB)$. This contributes to the diamagnetic part of the relation (4.12).

4.3 QH-state on a cone

A surface has a conical singularity of order $-\alpha$ ($\alpha < 1$) if in the neighborhood of the conical point z_0 the conformal factor behaves as

$$\phi \sim -\alpha \log |z - z_0|^2. \tag{4.20}$$

Locally a cone is thought as a wedge of a plane with the deficit angle $2\pi\alpha$, whose sides are isometrically glued together (see the Fig. 4.1). Let denote the complex coordinate on the

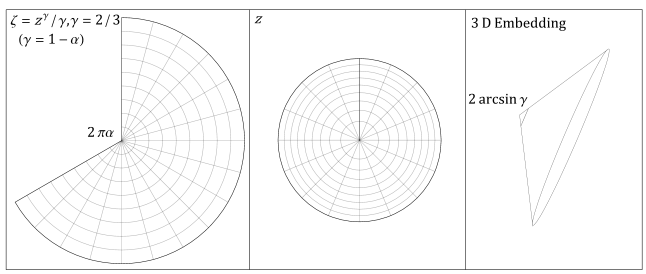

Figure 4.1: Schematic diagram of a cone and its 3D embedding

plane as ξ and the cone angle $2\pi\gamma = 2\pi(1-\alpha)$. The wedge is a domain $0 \leq \arg \xi < 2\pi\gamma$ with the Euclidean metric $ds^2 = |d\xi|^2$. A pullback of a singular conformal map

$$z \to \xi(z) = (z - z_0)^\gamma / \gamma \tag{4.21}$$

maps the wedge to a punctured disk. The map introduces the complex coordinates $(z, \bar{z})$ where the metric is conformal

$$ds^2 = |z - z_0|^{-2\alpha} |dz|^2. \tag{4.22}$$

The quantum mechanics on the cone assumes the 'wedge-periodic' condition. The lowest Landau level on a cone is spanned by the holomorphic polynomials of z (see, (4.48)) in the metric (4.22).

Eq. (4.16) is valid on any genus-zero surface. Specifically, in the neighborhood of the conical singularity the the conformal factor in (4.16) behaves as (4.20) and locally the state reads

$$\Psi_\alpha = \prod_{i<k}(z_i - z_k)^\beta \prod_i |z_0 - z_i|^{j\alpha} e^{-|z_i - z_0|^{2\gamma}/(4l^2\gamma^2)}.$$

Then the generating functional $\mathcal{Z}_\alpha$ is the expectation value of this operator.

A singularity can be interpreted as an insertion of the 'vertex operator' such that the generating functional $\mathcal{Z}_\alpha$ is the expectation value of this operator. We will show that this operator is conformal primary. This means that under a dilatation transformation of the metric close to the singularity, the functional transforms conformally

$$-\delta \log \mathcal{Z}_\alpha = \Delta_\alpha \delta\phi.$$

Here Δ_α is the conformal dimension. Eq. (4.19) identifies the conformal dimension with the angular momentum (4.3). We compute it in the remaining part of the paper.

Calculations are the most convenient in complex notation. We will need the formula for the angular momentum written in complex coordinates. The momentum in complex coordinates reads $P_z dz + P_{\bar z} d\bar z = P_\xi d\xi + P_{\bar\xi} d\bar\xi$ in local coordinates related by the conformal map (4.21). Then with the help of (4.21), the angular momentum density in flat coordinates reads $\mathrm{Im}(\xi P_\xi) = \mathrm{Im}(\xi \frac{dz}{d\xi} P_z) = \gamma^{-1}\mathrm{Im}(z P_z)$. Hence

$$\mathrm{L} = -\gamma^{-1} \int \mathrm{Im}(z P_z)\, dV. \tag{4.23}$$

4.4 Conformal Ward identity

Moments of the density and the angular momentum are computed via the Ward identity. The Ward identity reflects the invariance of the integral (4.17) under the infinitesimal holomorphic change of variables $z_i \to z_i + \epsilon/(z - z_i)$. It claims that the function of coordinates z_i and a complex parameter z

$$\sum_i \frac{\partial_{z_i} Q + (1-j)\partial_{z_i}\phi}{z - z_i} + \frac{\beta}{2}\left(\sum_i \frac{1}{z - z_i}\right)^2 + \sum_i \frac{1 - \frac{\beta}{2}}{(z - z_i)^2} \tag{4.24}$$

vanishes under averaging over the state.

The identity is closely related to the Ward identity of conformal field theory. In order to flesh out this analogy, we introduce the scalar field φ

$$\varphi = -2\beta \sum_i \log|z - z_i| - Q. \tag{4.25}$$

We also need the holomorphic component of the conformal 'stress tensor'

$$T = \frac{\nu}{2}\langle(\partial_z\varphi)^2\rangle - \mu_H\langle\partial_z^2\varphi\rangle. \tag{4.26}$$

Then the Ward identity can be brought to the form connecting the momentum and the conformal 'stress tensor'

$$\frac{1}{\hbar}\int \frac{\mathrm{i}P_{z'} - \frac{\mu_H}{2\pi}\partial_{z'}(eB)}{z - z'} dV_{z'} = T. \tag{4.27}$$

We describe the algebra elsewhere.

4.5 Trace of the conformal stress tensor

The meaning of the Ward identity becomes transparent if we complete the stress tensor by its trace Θ, defined through the conservation law equation

$$\partial_{\bar{z}} T + e^{\phi}\partial_z\Theta = 0. \tag{4.28}$$

Together the components T, $\bar{T}$, Θ form a quadratic differential $T_{ij}dx^i dx^j = T(dz)^2 + \bar{T}(d\bar{z})^2 + 2e^{\phi}\Theta dz d\bar{z}$. Then $\partial_{\bar{z}}$ derivative brings (4.27) to the form

$$P_z = \frac{1}{2\pi \mathrm{i}}\partial_z\left(\mu_H(eB) - 2\hbar\Theta\right). \tag{4.29}$$

The formula (4.29) identifies the trace Θ with the intensive part of the angular momentum

$$\mathrm{L} = -\frac{\mu_H}{2\pi\gamma}\int (eB)dV + \frac{\hbar}{\pi\gamma}\int \Theta\, dV.$$

Gravitational Anomaly On its own, the Ward identity is a relation between one and two-point correlation functions. The two-point function $\langle(\partial_z\varphi)^2\rangle$ in the Ward identity is evaluated coincident points. The connected part of this function $T^A = \frac{\nu}{2}\langle(\partial_z\varphi)^2\rangle_c = \frac{\nu}{2}\left[\langle(\partial_z\varphi)^2\rangle - \langle\partial_z\varphi\rangle^2\right]$ is the entry which converts the identity into a meaningful equation. In [49] it was argued that the connected two-point function is proportional to the Schwarzian of the metric

$$T^A = \frac{1}{12}\mathcal{S}[\phi], \quad \mathcal{S}[\phi] \equiv -\frac{1}{2}(\partial_z\phi)^2 + \partial_z^2\phi. \tag{4.30}$$

Thus $T = T^C + T^A$ consists of the 'classical' part

$$T^C = \frac{\nu}{2} \left(\langle \partial_z \varphi \rangle \right)^2 - \mu_H \partial_z^2 \langle \varphi \rangle \tag{4.31}$$

and the anomalous part (4.30). This explicit representation of T converts the Ward identity to the equation.

This equation consists of terms of a different order in magnetic length and has to be solved iteratively. The leading approximation, where $\langle \rho \rangle \approx \bar{\rho}$ suffices. From (4.25) it follows that $\langle \varphi \rangle \approx (\mu_H/\nu)\phi$. Up to this order the classical part of the stress tensor is $T^C = -(\mu_H^2/\nu) \left[-\frac{1}{2}(\partial_z \phi)^2 + \partial_z^2 \phi \right]$. Together with the anomalous part (4.30) the stress tensor reads

$$T = \frac{c_H}{12} \mathcal{S}[\phi]. \tag{4.32}$$

On smooth surfaces by virtue of (4.28)

$$\Theta = \frac{c_H}{48} R.$$

This is the trace anomaly.

Thus, to leading order the Ward identity is equivalent to the conformal Ward identity. Eq. (4.29) then yields the result obtained in [86] for the momentum on a smooth surface

$$P_z = \frac{1}{2\pi \mathrm{i}} \partial_z \left(\mu_H \left(eB \right) - \hbar \frac{c_H}{24} R \right). \tag{4.33}$$

In its turn it yields the angular momentum given by (2.1). For reference we present the formula for the density on a smooth surface, which follows from (4.33) and (4.12). It was previously obtained in [50]

$$\rho = \bar{\rho} + \frac{1}{4\pi B}\left[(\nu - \frac{1}{2})\Delta B + \left((1-j)\frac{\mu_H}{2} + \frac{c_H}{24}\right)\frac{\hbar}{e}\Delta R\right].$$

4.6 Geometric singularity

On a singular surface the formula (4.33) is not valid, since the curvature is singular. However, the results promptly follow from the dilatation sum rule we now obtain.

We multiply (4.27) by $\frac{zdz}{2\pi\mathrm{i}}$ and integrate it along a boundary of an infinitesemaly small parcel. Using $\frac{1}{2\pi\mathrm{i}}\oint\frac{zdz}{z-z'} = z'$, (4.23) we reduce the Ward identity to the sum rule for the intensive part of the angular momentum

$$\gamma \mathrm{L}_{\alpha,a} = \hbar \oint T(z)\frac{zdz}{2\pi\mathrm{i}}. \tag{4.34}$$

and notice that in the neighborhood of singularity $T(z)$ is a holomorphic function. Thus $\oint T(z)\frac{zdz}{2\pi\mathrm{i}} = \mathrm{res}(zT)$.

We compute the singular part of the stress tensor by evaluating the Schwarz derivative on the singular metric (4.20). Equivalently, we treat a conical singularity as a conformal map (4.21) and compute the Schwarz derivative of the map

$$\mathcal{S}[\phi] \equiv \{\xi,\, z\} = \frac{\xi'''}{\xi'} - \frac{3}{2}\left(\frac{\xi''}{\xi'}\right)^2 = \frac{\alpha(2-\alpha)}{2z^2}.$$

We obtain

$$T = \frac{c_H}{24}\frac{\alpha(2-\alpha)}{z^2}. \tag{4.35}$$

Using (4.34), we arrive at our main result (4.3).

4.7 Magnetic singularity

In this case, the gravitational anomaly does not contribute to the the singularity of the the stress tensor. Rather, the stress tensor receives an additional contribution from the magnetic potential of the flux tube $Q_a = 2a\log|z|$

$$T = -\frac{\nu}{2}(\partial_z Q_a)^2 - \mu_H \partial_z^2 Q_a = \frac{h_a}{z^2}, \tag{4.36}$$

where h_a is the conformal dimension (4.1).

Finally, when the flux tube sits on top of a conical singularity, the stress tensor is the sum of (4.36) and (4.35). Near the singularity $T \sim (\gamma\Delta_\alpha + h_a)/z^2$. This implies the relation (4.4).

4.8 Exchange statistics

Now consider adiabatically exchanging two singularities. The state will acquire a phase. Since the state is a holomorphic function of singularity position, its holonomy is encoded in the normalization factor. The phase is then $\Phi_{12} = \frac{\mathrm{i}}{2}\oint d\log\mathcal{Z}$, where the integral in positions of singularities goes along the adiabatic path. The adiabatic connection $d\log\mathcal{Z}$ treated as a differential of the position, say, the first singularity has a pole when two singularities coincide, so the phase is the residue of the pole $\Phi_{12} = -\pi\,\mathrm{res}[d\log\mathcal{Z}]$.

For conical singularities, the residue arises entirely from the gravitational anomaly. We notice that the calculation of the normalization factor for multiple singularities is closely related to the determinant of the Laplacian $\mathrm{Det}(-\Delta)$ on singular surfaces. This relation was discussed in [50, 91, 86]. A reason for this is that the stress tensor for $\log\mathcal{Z}$ and $\frac{1}{2}c_H\log\mathrm{Det}(-\Delta)$, share the same singularities.

The result is summarized by the formula

$$\log \mathcal{Z}|_{p_1 \to p_2} = \frac{c_H}{12} \alpha_1 \alpha_2 \left(\frac{1}{\gamma_1} + \frac{1}{\gamma_2} \right) \log |p_1 - p_2|. \tag{4.37}$$

where p_1 and p_2 are positions of two merging singularities.

Then the adiabatic connection is

$$d \log \mathcal{Z} = \frac{c_H}{24} \alpha_1 \alpha_2 \left(\frac{1}{\gamma_1} + \frac{1}{\gamma_2} \right) \frac{dp_1 - dp_2}{p_1 - p_2}. \tag{4.38}$$

It prompts the formula (4.6) for the exchange statistics.

We illustrate the calculation of the generating functional on the example of a genus-0 polyhedral surface, such as a cube, tetrahedral, etc., whose vertices are separated by distances well exceeding the magnetic length. In this calculation we focus on the geometric singularities setting the fluxes $a = 0$. The metric describing a polyhedron is piece-wise flat with multiple conical singularities. It is obtained from the Schwarz-Christoffel map $\xi(z)$ unfolding the polyhedron

$$e^{\phi} = |\xi(z)'|^2 = \prod_i |z - p_i|^{-2\alpha_i}. \tag{4.39}$$

The metric describes a flat surface with conical singularities of the order α_i, conditioned by the Gauss-Bonnet theorem $-\sum_i \alpha_i + 2 = 0$, and located at points p_i. The magnetic potential $-\hbar \Delta Q = 2eB$ corresponding to a uniform magnetic field will be $Q = -|\xi(z)|^2/2l^2$.

The Schwarzian of this metric is

$$\mathcal{S}[\phi] = \sum_i \frac{-\frac{1}{2}\alpha_i^2 + \alpha_i}{(z - p_i)^2} + \frac{\gamma_i}{z - p_i}, \quad \gamma_i \equiv -\sum_{j \neq i} \frac{\alpha_i \alpha_j}{p_i - p_j}. \tag{4.40}$$

Using the explicit form of the metric, we find the following asymptotic behavior of the magnetic potential and the metric at singularities

$$\begin{aligned}
\partial_{p_i}\phi|_{z\to p_i} &= -\partial_z\phi + \frac{\gamma_i}{\alpha_i} + ..., \\
\partial_{p_i}\phi|_{z=p_j} &= -\frac{\alpha_i}{p_i - p_j}, \\
\partial_{p_i}Q|_{z\to p_i} &= (-\partial_z + \frac{\gamma_i}{\alpha_i})Q|_{z\to p_i} + ..., \\
\partial_{p_i}Q|_{z=p_j} &= -\frac{\alpha_i}{p_i - p_j}Q|_{z=p_j}
\end{aligned} \tag{4.41}$$

Next, we use the formula for the angular momentum of a that for a single cone (4.3), which we write in the form

$$\int_{D_\epsilon(p_i)} (-Q + (j-1))\,(\rho - \rho_\infty)\,dV = \Delta_{\alpha_i}. \tag{4.42}$$

To make sense of this formula for multiple cones, we take the domain of integration $D_\epsilon(p_i)$ to be a small disk of radius ϵ centered on p_i which is much larger than the magnetic length but much smaller than the Euclidean distance to the closest neighboring cone point. In this case, $\rho_\infty = \frac{\nu}{2\pi l^2}$ is the density far from any cone points.

For multiple cones, there is also another important non-vanishing sum rule that comes from the examining the simple poles of the Ward identity, which come entirely from the Schwarzian of the metric. This sum rule reads

$$\int_{D_\epsilon(p_i)} \partial_z\,(Q + (1-j)\phi)\,(\rho - \rho_\infty)\,dV = -\frac{c_H}{12}\gamma_i. \tag{4.43}$$

The proof for these sum rules comes directly from the Ward identity (4.27) specified for a uniform magnetic field

$$\frac{1}{\hbar}\int \frac{i\langle P_{z'}\rangle}{z - z'}dV_{z'} = \langle T\rangle = \frac{c_H}{12}\mathcal{S}[\phi] + O(l^2). \tag{4.44}$$

Since the RHS is proportional to the Schwarzian, it will have second order pole and simple poles at the cone points. This implies that the integral on the LHS has a Laurent expansion around each cone point. Comparing the residues of the poles, we find (4.42) comes from the second order pole of the Schwarzian, whereas (4.43) follows from the simple pole.

Now we are equipped to derive the variational formula for the generating functional.

Derivatives with respect to the cone points will act only the single-particle factors of the wave function (4.16), and thus lead to

$$\partial_{p_i} \log Z = \int \partial_{p_i} \left(Q + (1-j)\phi\right) \rho) dV. \tag{4.45}$$

In the integrand we write $\rho = (\rho - \rho_\infty) + \rho_\infty$. The contribution of ρ_∞ is of the order l^{-2}. We ignore it and focus on the contribution of $\rho - \rho_\infty$ which has a finite support at the conical singularity points p_i. Therefore, we convert the integral over the entire surface to a sum of integrals over small disks $D_\epsilon(p_j)$ centered on singularities

$$\partial_{p_i} \log Z = \sum_j \int_{D_\epsilon(p_j)} \partial_{p_i} \left(Q + (1-j)\phi\right) (\rho - \rho_\infty)\, dV. \tag{4.46}$$

With this, we utilize the asymptotic expressions for Q and ϕ in (41), combined with the formula (4.42) for the moment of inertia and the sum rule (4.43). We obtain

$$\begin{aligned} \partial_{p_i} \log Z &= \frac{c_H}{12}\gamma_i - \frac{\gamma_i}{\alpha_i}\Delta_{\alpha_i} + \sum_{j\neq i}\left(\frac{\alpha_i}{p_i - p_j}\right)\Delta_{\alpha_j} \\ &= \frac{c_H}{24}\sum_{j\neq i}\frac{\alpha_i\alpha_j}{p_i - p_j}\left[\frac{1}{\gamma_i} + \frac{1}{\gamma_j}\right]. \end{aligned} \tag{4.47}$$

This formula represents the adiabatic connection with respect to adiabatic displacement of conical singularity announced in the main text in Eq (30).

4.9 Integer QH-state on a cone

The formulae for the charge of the singularity (4.10) and the moment of inertia (4.11) are readily checked against the direct calculations for the integer case $\nu = 1$. See [67, 100, 107] for a study of Landau levels on a cone. In the case where a flux $(h/e)a$ threads the cone the Landau level is spanned by one-particle states $k = 0, \dots, N-1$

$$\psi_k = \frac{e^{-|\xi|^2/4l^2}}{l\sqrt{2\pi\gamma\Gamma(\frac{k}{\gamma} + \frac{q}{\gamma} + 1)}} \left(\frac{|\xi|}{\sqrt{2}l}\right)^{\frac{q}{\gamma}} \cdot \left(\frac{\xi}{\sqrt{2}l}\right)^{\frac{k}{\gamma}}, \tag{4.48}$$

where $q = a + \alpha j$, and the density is the sum of densities of each one-particle state

$$\rho = \sum_{k=0}^{N-1} |\psi_k|^2. \tag{4.49}$$

We observe that in the integer case the magnetic singularity and spin come together in a combination $q = a + \alpha j$.

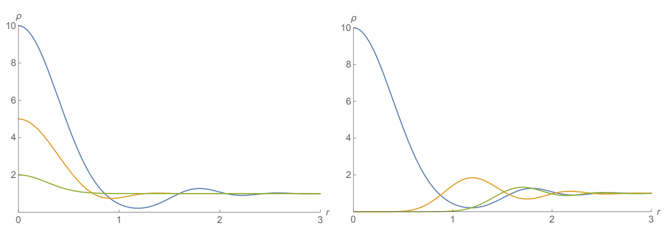

Figure 4.2: Left panel: Density on a cone of angle $\gamma = 1/10, 1/5, 1/2$ (blue, yellow, green) with spin $j = 0$. Right panel: Density on a cone of angle $\gamma = 1/10$ and spin $j = 0, 1/2, 1$ (blue, yellow, green).

At $N \to \infty$ the density is expressed in terms of Mittag-Leffler function

$$\rho = \rho_\infty \gamma^{-1} e^{-x} x^{\frac{q}{\gamma}} E_{\frac{1}{\gamma}, \frac{q}{\gamma}+1}\left(x^{\frac{1}{\gamma}}\right),$$

where we denoted $x = |\xi|^2/2l^2$, and $\rho_\infty = 1/2\pi l^2$. We recall the definition of the Mittag-

Leffler function

$$E_{a,b}(z) = \sum_{k\geq 0} \frac{z^k}{\Gamma(ak+b)}.$$

We find the moments from the arguments used to obtain the conformal Ward identity. Under re-scaling the magnetic length $l^2 \to \lambda^{-1} l^2$ the state (4.48) scales

$$\psi_k \to \lambda^{\frac{1}{2}+\frac{k+q}{2\gamma}} e^{(1-\lambda)|\xi|^2/4l^2} \psi_k.$$

but remains normalized. The normalization condition for the new state yields the identity

$$\int e^{(1-\lambda)\frac{|\xi|^2}{2l^2}} |\psi_k|^2 dV = \lambda^{-1-(q+k)/\gamma}.$$

Then, summing over all modes and taking $N \to \infty$ at $|\lambda| > 1$ we obtain the exact Laplace transform of the density

$$\int e^{(1-\lambda)\frac{|\xi|^2}{2l^2}} (\rho - \rho_\infty) dV = \frac{\lambda^{-1-\frac{q}{\gamma}}}{1-\lambda^{-1/\gamma}} - \frac{\gamma}{\lambda - 1} \tag{4.50}$$

$$= \frac{\lambda^{\frac{\alpha-2}{2\gamma}+\frac{m_0}{\gamma}}}{1-\lambda^{-1/\gamma}} - \frac{\gamma}{\lambda - 1} \tag{4.51}$$

where $m_0 = -a + (1-2j)\alpha/2$ is the charge of the cone. This formula can be seen as a generating function of moments (4.8). Expanding around $\lambda = 1$ yields the charge m_0 (4.10) and the moment of inertia m_2 (4.11).

In the orbifold setting the density is a finite sum where $\gamma(s,x) = \int_0^x t^{s-1} e^{-t} dt$ is the lower incomplete gamma function. At $\gamma = 1/n$ and $q = 0$, the density is Riccati's *generalized hyperbolic function* - the sum over n-roots of unity $\omega_n^k = e^{\mathrm{i}\frac{2k\pi}{n}}$

$$\rho = \rho_\infty e^{-x} \sum_{k=0}^{n-1} \exp\left(\omega_n^k x\right) \tag{4.52}$$

In Fig.4.2, we illustrate how the cone angle and spin affect the density, respectively. In both figures, the density far away from the singularity is normalized to 1 and the distance to the conical point is measured in units of magnetic length. On the left panel we set $j = 0$ and plot ρ for $\gamma = 1/2, 1/5, 1/10$. The values of density near the origin reflect the charge m_0 accumulated at the tip of the cone and feature oscillations on the order of magnetic length away from the apex. We comment that magnetic singularity does not feature oscillations at $\nu = 1$. On the right panel we set $\gamma = 1/10$ and show the effect of spin at $j = 0, 1/2, 2$. Unless spin is zero the charge m_0 is negative. The cone repels electrons. Spin also suppresses oscillations, but for small values of spin oscillations persist. For larger values of spin (not plotted) oscillations are suppressed entirely.

Apart from the flat cone, there exist exact results for the "American football" geometry, Fig. 4.3, a unique surface with positive constant curvature and two conical singularities [124]. The singularities have the same order and antipodal. The football metric reads

$$ds^2 = \frac{(1-\alpha)^2|z|^{-2\alpha}}{\left(1+|z|^{2\gamma}/4r^2\right)^2}|dz|^2, \quad \gamma = 1-\alpha. \tag{4.53}$$

The volume and the curvature of this surface are

Figure 4.3: "American football": a surface of constant positive curvature and two antipodal conical singularities

$V = 4\pi r^2\gamma$ and $R = 2/r^2$, respectively. Identical conical singularities of order α are located at $z = 0$ and $z = \infty$. The number of states is $N = N_\Phi + (1 - 2j) - 2a$, where N_Φ, is the total magnetic flux, a is the flux threaded through the singularities, and

$$\rho_\infty = \frac{e}{h} B + \frac{(1-2j)}{8\pi} \frac{2}{r^2}, \tag{4.54}$$

is the bulk density away from the conical singularities.

In coordinates $\xi = z^\gamma/(2r\gamma)$, the normalized eigenstates are

$$\psi_k = \frac{1}{\sqrt{V\mathcal{N}_k}} \frac{\xi^{\frac{1}{\gamma}(k+q)}}{(1+|\xi|^2)^{\frac{1}{2\gamma}N_\Phi - j}}, \quad q = a + \alpha j, \tag{4.55}$$

where the normalization factor is

$$\mathcal{N}_k = B\left(\frac{1}{\gamma}N_\Phi - 2j + 1 - \frac{1}{\gamma}(k+q), \frac{1}{\gamma}(k+q) + 1\right) \tag{4.56}$$

and $B(x, y) = \Gamma(x)\Gamma(y)/\Gamma(x + y)$ is the beta-function. The formula (4.49) gives the density

$$\begin{aligned}\rho =& \rho_\infty \gamma^{-1} (1+|\xi|^2)^{-2(N_\Phi/\gamma - 2j)} \times \\ & \sum_{k=0}^{N-1} \frac{\Gamma(N_\Phi/\gamma - 2j + 1)|\xi|^{\frac{2}{\gamma}(k+q)}}{\Gamma(N_\Phi/\gamma - 2j + 1 - (k+q)/\gamma)\Gamma((k+q)/\gamma + 1)}.\end{aligned} \tag{4.57}$$

As before, the density simplifies when $q = 0$ and $\gamma^{-1} = n$ is integer. Writing as before the n^{th} root of unity $\omega_n = e^{\mathrm{i}2\pi/n}$, we find

$$\rho = \rho_\infty \left(1+|\xi|^2\right)^{-nN_\Phi} \sum_{k=0}^{n-1} \left(1+\omega_n^k |\xi|^2\right)^{nN_\Phi}. \tag{4.58}$$

The charge moment follows from simple integration

$$m_0 = \int (\langle\rho\rangle - \rho_\infty)\, dV = N - N_\Phi + (1-2j)(1-\alpha) \tag{4.59}$$

$$= -(1-2j)\alpha - 2a. \tag{4.60}$$

This is twice the charge of a single flat cone given by (4.10).

CHAPTER 5

CONCLUSION

In summary, in this paper we did three things. We developed the Ward identity framework for computing correlation functions for the FQHE in curved space, we formulated the theory of the Laughlin QH-states as a field theory of a scalar Bose field, and we showed that the geometric response of the FQHE is understood best in the context of singularities on the surface.

The Ward identity provided a toolkit for computing correlation functions of the FQHE on curved surfaces. This allowed us to compute the particle density and see corrections to it and other observables that had been invisible in flat space calculations. In particular, we saw that the geometric characteristic of the QH state entered the density as a finite-sized correction. We saw that to measure the geometric coefficient c_H, one needs to compute the second moment of the density. While the Ward identity provided a powerful mathematical framework for computing observables of the FQHE on curved surfaces, it lacked physical interpretation. For this reason, we developed a field theoretic approach to studying the geometry of the FQHE.

Once we formulated the field theory, we saw how parts of the field theoretic action affect the structure of observables. In particular, we saw that the geometric coefficient which appears as a finite-size correction to observables of the QH fluid, has two sources. First, there is an anomaly in the measure of the action. The anomaly comes from the divergence of the two point correlation function of the scalar field φ as two field merge. The connected correlation function diverges and needs to be regularized, which is the source of the anomaly. The second contribution to the finite-sized correction is Fadeev-Popov ghosts in the action. These determine the sign of the correction.

After developing sufficient toolkits for computing correlation functions of the FQHE on curved surfaces, we applied this machinery to study the FQHE on singular surfaces. We showed that singular surfaces are the ideal setting for studying the geometric characteristic of QH states c_H. The reason is that, while on smooth surfaces c_H is always a finite-sized correction to other quantities which dominate transport, on singular surfaces c_H dominates transport near the singularities.

We showed this with two main results. First we showed that the angular momentum near a singularity is directly proportional to c_H. This implies that angular momentum near a conical singularity for a QH fluid is an experimentally viable quantity for detecting c_H. Then we showed that adiabatically transporting one singularity around another produced a geometric phase, and the exchange statistics of that phase are proportional to c_H as well. Therefore, the second setting where c_H can be measured directly is in the adiabatic transport of conical singularities.

We hope that this work opens up ways for detecting the topological invariant c_H that characterizes the geometric response of the FQHE in an experimental setting. In addition, we hope the tools developed in this work - the Ward identity and field theory - help others explore the geometric properties of the FQHE and other topological states of matter. Although it is stated in the acknowledgements, I'd like to thank my collaborators Y.H. Chiu, T. Can, and my advisor P. Wiegmann for an exhilarating hunt for the gravitational anomaly.

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[27] The factor of $1/V$ ensures that connected correlation function with density has no zero mode $\int\langle\mathcal{O}\rho(z)\rangle_c dV = 0$

[28] Discussions with S. Klevtsov are acknowledged.

[29] In a general orbifold setting, where $\gamma = m/n$ is rational, and $q \neq 0$, the density is a finite sum $\rho = \rho_\infty e^{-x} \sum_{k=0}^{n-1} e^{x\omega_n^k} \sum_{k'=0}^{m-1} \omega_n^{-k(k'+q)/\gamma} \frac{\gamma\left((k'+q)/\gamma, x\omega_n^k\right)}{\Gamma((k'+q)/\gamma)}$ where $\gamma(s,x) = \int_0^x t^{s-1} e^{-t} dt$ is the lower incomplete gamma function.

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