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LOCAL AND COVARIANT FLOW RELATIONS FOR OPE COEFFICIENTS IN LORENTZIAN SPACETIMES

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To my parents, John and Kristal.

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ABSTRACT

The singular properties of quantum fields have posed an intransigent obstruction to formulating a mathematically well-defined theory of interacting quantum fields at nonzero coupling in $(3 + 1)$-spacetime dimensions. To date, a systematic renormalization of the unavoidable "ultraviolet" divergences produced by pointwise products of quantum fields has only been achieved order-by-order in perturbation theory. In the coincidence limit, the behavior of products of quantum fields is characterized by the coefficients of their operator product expansion (OPE). For Euclidean quantum field theories, Holland and Hollands have shown operator product expansion (OPE) coefficients satisfy "flow equations": For interaction parameter λ, the partial derivative of any OPE coefficient with respect to λ is given by an integral over Euclidean space of a sum of products of other OPE coefficients. These Euclidean flow equations were proven to hold order-by-order in perturbation theory, but they are well defined non-perturbatively and, thus, provide a possible route towards giving a non-perturbative construction of the interacting field theory. In this thesis, we generalize these results for flat Euclidean space to curved Lorentzian spacetimes in the context of the solvable "toy model" of massive Klein-Gordon scalar field theory, with m^2 viewed as the "self-interaction parameter". Even in Minkowski spacetime, a serious difficulty arises from the fact that all integrals must be taken over a compact spacetime region to ensure convergence but any integration cutoff necessarily breaks Lorentz covariance. We show how covariant flow relations can be obtained by adding compensating "counterterms" in a manner similar to that of the Epstein-Glaser renormalization scheme. We also show how to eliminate dependence on the "infrared-cutoff scale" L, thereby yielding flow relations compatible with almost homogeneous scaling of the fields. In curved spacetime, the spacetime integration will cause the OPE coefficients to depend non-locally on the spacetime metric, in violation of the requirement that quantum fields should depend locally and covariantly on the metric. We show how this potentially serious difficulty can be overcome by replacing the metric with a suitable local polynomial approximation about the OPE expansion point. We thereby obtain local and covariant flow relations for the OPE coefficients of Klein-Gordon theory in curved Lorentzian spacetimes. As a byproduct of our analysis, we prove the field redefinition freedom in the Wick fields (i.e. monomials of the scalar field and its covariant derivatives) can be characterized by the freedom to add a smooth, covariant, and symmetric function $F_n(x_1, \ldots, x_n; z)$ to the identity OPE coefficients, $C^I_{\phi\cdots\phi}(x_1, \ldots, x_n; z)$, for the elementary n-point products. We thereby obtain an explicit construction of any renormalization prescription for the nonlinear Wick fields in terms of the OPE coefficients $C^I_{\phi\cdots\phi}$. The ambiguities inherent in our procedure for modifying the flow relations are shown to be in precise correspondence with the field redefinition freedom of the Klein-Gordon OPE coefficients. In an appendix, we develop an algorithm for constructing local and covariant flow relations beyond our "toy model" based on the associativity properties of OPE coefficients. We illustrate our method by applying it to the flow relations of $\lambda\phi^4$-theory.

CHAPTER 1

INTRODUCTION TO THE OPE AND FLOW RELATIONS FOR OPE COEFFICIENTS

The primary objectives of this chapter are to introduce the operator product expansion (OPE), provide some informal motivation for the existence (and functional form) of the flow relations for the OPE coefficients, and to briefly summarize some of the rigorous results for the flow relations that have already been established in Euclidean quantum field theories. We begin with a discussion of the distributional properties of quantum fields and the inherent difficulties these properties present for defining nonlinear field observables and nonlinear dynamics (i.e. "interactions"). We then define the OPE and describe how the OPE coefficients could be used to non-perturbatively analyze (and remove) the short-distance "ultraviolet" divergences that arise in integrals involving the (unrenormalized) expectation values of time-ordered products. Generalizing an approach sketched by Wilson in [1] (see Remark 3 below), we obtain the basic structure of the flow relations for the OPE coefficients. We conclude this chapter by comparing our informally-obtained relations to those that have been rigorously-derived in Euclidean quantum field theories. A reader who is well-acquainted with the distributional nature of quantum fields, the non-integrable divergences that arise in (position-space) perturbation theory, and the operator product expansion may wish to skip ahead to the discussion of the Euclidean flow relations found at the end of this chapter.

$*\,*\,*$ Singular quantum fields $*\,*\,*$

Suppressing any spinorial/tensorial indices, denote an arbitrary local quantum field by the symbol Φ and a quantum state by $\langle\,\cdot\,\rangle_\Psi$. Disregarding the practical limitations of experimental measurements, one might anticipate that an expectation value of Φ at spacetime event x,

$$\langle\Phi(x)\rangle_\Psi\,, \tag{1.1}$$

would correspond to an elementary physical observable[1], provided it is real-valued[2]. For a renormalizable quantum field theory, there exists a wide class of quantum states and a large (countably-infinite) set of quantum fields such that the quantity denoted by (1.1) is mathematically well-defined as a C^∞-smooth function of x.

However, if the spacetime is modeled as a continuum[3], then there is a sense in which quantum fields are inherently more "singular" (in their spacetime dependence) than their classical counterparts. The first indication of this fact is that the "fluctuations" of a quantum field at any event x are always divergent:

$$\mathrm{Var}[\Phi(x)]_\Psi \equiv \langle \Phi(x)\Phi(x)\rangle_\Psi - \langle \Phi(x)\rangle_\Psi \langle \Phi(x)\rangle_\Psi \sim \infty, \tag{1.2}$$

even when the expected value (1.1) is finite[4]. The fact that all quantum states exhibit infinite field fluctuations implies[5] that there cannot exist renormalizable "eigenstates" corresponding to Φ evaluated at event x. According to the standard postulates of quantum mechanics, immediately subsequent to the measurement of a physical observable the quantum state describing a physical system "collapses" to the eigenspace associated with the measured observable. Since no normalizable state belongs to the eigenspace corresponding to "Φ evaluated at event x", a measurement of the quantum field at a sharply-defined

1. Of course, it is not possible to take repeated measurements at a single event in spacetime. However, measurements could be taken at distinct events each with the same local spacetime structure.

2. i.e., assuming Φ is defined as a Hermitian "quadratic form".

3. If spacetime is modeled as a discrete lattice, divergent fluctuations are avoided at the expense of explicitly breaking (continuous) spacetime isometries.

4. As articulated in the influential paper [2], the divergent fluctuations of non-interacting quantum fields can be inferred from the canonical equal-time commutation relations.

5. Here we use Dirac bra-ket notation for the quantum field and states: Suppose there exists a self-adjoint operator $\widehat{\Phi}(x)$ with dense invariant domain on a Hilbert space. Further suppose there exist normalized eigenstates, $|\Phi_{\text{e.v.}}(x)\rangle$, corresponding to this operator: i.e., $\langle \Phi_{\text{e.v.}}(x)\,|\,\Phi_{\text{e.v.}}(x)\rangle = 1$ and $\widehat{\Phi}(x)\,|\Phi_{\text{e.v.}}(x)\rangle = \Phi_{\text{e.v.}}(x)\,|\Phi_{\text{e.v.}}(x)\rangle$. From these assumptions and (1.2), we immediately obtain a contradiction:

$$\infty > \Phi_{\text{e.v.}}(x)\Phi_{\text{e.v.}}(x) = \Phi_{\text{e.v.}}(x)\Phi_{\text{e.v.}}(x)\,\langle \Phi_{\text{e.v.}}(x)\,|\,\Phi_{\text{e.v.}}(x)\rangle = \langle \Phi_{\text{e.v.}}(x)|\widehat{\Phi}(x)\widehat{\Phi}(x)|\Phi_{\text{e.v.}}(x)\rangle \sim \infty \ \lightning \tag{1.3}$$

spacetime event is theoretically incompatible with the standard postulates of quantum mechanics. This is a close analogue of the well-known fact that eigenstates of the position operator for a non-relativistic particle are non-normalizable[6] and, thus, are not physically realizable. Nevertheless, in close analogy to the non-relativistic case[7], one can construct a physically-sensible quantum observable by "smearing" the field over a finite (but possibly "very small") spacetime region[8]. The smeared field observable is often written informally as

$$\Phi(f) = \int d^D x\, f(x)\Phi(x), \tag{1.4}$$

with f denoting a compactly-supported "bump function"[9]. In contrast to the fluctuations of the pointwise field $\Phi(x)$, the fluctuations of the smeared field observable are finite-valued in any physically-reasonable state. If f is real-valued and normalized such that $\int d^D x f(x) = 1$, then the smeared field (1.4) may be interpreted as a weighted average of the quantum field over the spacetime region in which f is nonzero[10].

Although smeared fields $\Phi(f)$ are satisfactory quantum observables, the singular nature of quantum fields does create difficulties in defining nonlinear observables and nonlinear dynamics that are not present for quantum systems with countable degrees of freedom. Because the quantity (1.1) is finite-valued in any physically-reasonable state, the divergent fluctuations in (1.2) arise from the expectation value of the pointwise product $\Phi(x)\Phi(x)$.

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6. More precisely, the inner product between position eigenstates, $\langle \vec{x} | \vec{x}' \rangle = \delta(\vec{x} - \vec{x}')$, diverges when $\vec{x} = \vec{x}'$.

7. The analogy between (1.4) and the non-relativistic case is that, for bump function ψ on $\mathbb{R}^3$, $|\psi\rangle \equiv \int d^3x\, \psi(\vec{x})\, |\vec{x}\rangle$ is a normalizable state even though the individual position eigenstates $|\vec{x}\rangle$ are not.

8. The microlocal spectrum condition [15, see eqs. 22-23 for definition] implies it is possible to define a quantum field at a sharp "position in space" by smearing only in a timelike direction. For the very special case of the linear field observable for a non-interacting theory, it is also possible to define the field at a sharp "moment in time" by smearing in (all) spacelike directions. Of course, the latter is required to define equal-time commutation relations.

9. In cases where the quantum field is tensor-valued, $\Phi = \Phi^{abcd\cdots}$, it is smeared with a test function element of its dual space, $f_{abcd\cdots}$, so that the smeared field (1.4) is always a scalar quantity.

10. From a practical standpoint, the weighted average of the field is a decidedly more realistic observable than $\Phi(x)$ even in classical field theory, since the actual resolving abilities of any physical detector will be limited and non-uniform in spacetime.

This is a special case of the general “n-point functions”:

$$\langle \Phi_1(x_1)\Phi_2(x_2)\cdots\Phi_n(x_n)\rangle_\Psi\,, \tag{1.5}$$

which characterize the correlations between n quantum fields $\Phi_1, \ldots, \Phi_n$ evaluated at events $x_1, \ldots, x_n$, respectively. The discussion of the previous paragraph implies that, in general, the quantities (1.5) are only mathematically and physically meaningful when the quantum fields have been smeared with bump functions $f_1(x_1)$, …, $f_n(x_n)$. Viewed as maps from bump functions to numbers (i.e., as “functionals”), the expectation values (1.5) are linear (as suggested by the informal integral notation (1.4) for the smeared fields) and suitably continuous[11] and, thus, are mathematically well-defined as distributions. The fact that quantum fields are represented by distributions, which are inherently linear objects in general, is the ultimate origin of all “short-distance”/“ultraviolet” divergences that arise when expressions involving quantum fields are naively manipulated as if they were ordinary functions.

It is worth emphasizing that the n-point distributions (1.5) and their “time-ordered” counterparts contain, in principle, all the types of predictions that are relevant for modern particle physics experiments, including the probability amplitudes (i.e. “S-matrix elements”) that are used to compute decay rates and scattering cross sections. Moreover, in spite of the challenges posed by the singularities of (1.5) at coinciding spacetime events, the strong correlations implied by the n-point distributions between fields at small “spacelike” separations are intimately related to some of the most interesting and phenomologically-important predictions of quantum field theory in curved spacetimes [4, see Section 2.2] : In particular, the black-body radiation predicted in the Hawking effect ultimately originates from quantum correlations between “closely-separated” fields located on opposite sides of a black hole event horizon. The spatial correlations in the n-point distributions also provide a natural explanation for the temperature fluctuations observed in the power spectrum of the

11. For a precise definition of “continuity” of a distribution, see [3, Definition 2.1.1].

cosmic microwave background radiation and imply a mechanism for “structure-formation” in our universe: When amplified by the exponential expansion of an “inflationary” epoch, quantum correlations between fields separated by very short-distances in the early universe will seed the kind of density perturbations that are required for the formation of large-scale structures like galaxies and galaxy clusters in our present universe.

$* * *$ Nonlinear dynamics $* * *$

In (3+1)-spacetime dimensions, the singular nature of quantum fields has thus far obstructed serious attempts to formulate a mathematically well-defined theory of interacting quantum fields, i.e., a theory where the elementary quantum fields satisfy nonlinear equations of motion. The kinds of interacting quantum field theories that constitute the Standard Model of particle physics are Yang-Mills gauge theories, and proving the existence of a non-trivial Yang-Mills theory (with a “mass gap”) in 4-dimensions remains a famous unsolved problem with a sizable bounty [6]. Consequently, in spite of their physical importance, it is not known how to satisfactorily *define*—much less construct—the n-point distributions, eq. (1.5), in $(3 + 1)$-spacetime dimensions except for quantum field theories whose elementary fields satisfy linear equations of motion, i.e., for “non-interacting” quantum fields. To the extent that deviations from linear dynamics are “small”, the effects of interactions have been quantified (with incredible success) using perturbation theory.

In the standard perturbative approach, it is the “time-ordered” version[12] of the n-point distributions denoted by,

$$\langle T\{\Phi_1(x_1)\Phi_2(x_2)\cdots\Phi_n(x_n)\}\rangle_\Psi \,, \tag{1.6}$$

that are typically analyzed rather than the ordinary n-point distributions (1.5). In perturbation theory, the expectation values for these time-ordered products of the interacting theory

12. See Section 4.3 for the definition of “time-ordering”.

are approximated by a formal power series in the theory's "interaction parameter(s)". For concreteness and simplicity, we will discuss a scalar field theory with a quartic potential and interaction parameter λ,

$$\mathcal{L} = -\frac{1}{2}g^{ab}\nabla_a\phi\nabla_b\phi - \frac{1}{2}(m^2+\xi R)\phi^2 - \frac{1}{4!}\lambda\phi^4. \tag{1.7}$$

Classically, this Lagrangian corresponds to the nonlinear equation of motion:

$$\left(-g^{ab}\nabla_a\nabla_b + m^2 + \xi R(x)\right)\phi(x) + \frac{1}{3!}\lambda\phi^3(x) = 0, \tag{1.8}$$

with g_{ab} denoting the spacetime metric, m the mass, ξ a curvature coupling parameter, and R the Ricci scalar. For this model, the n-point time-ordered products involving the elementary field observable ϕ, e.g., are supposed to be approximated by a power series of the form:

$$\langle T\{\phi(x_1)\cdots\phi(x_n)\}\rangle_\Psi = \langle T\{\phi_0(x_1)\cdots\phi_0(x_n)\}\rangle_{\Psi_0} + \sum_{k=1}^{N}\lambda^k D_{\Psi_0,k}(x_1,\ldots,x_n), \tag{1.9}$$

where ϕ_0 denotes a non-interacting field satisfying the linear Klein-Gordon equation (i.e. eq. (1.8) with $\lambda = 0$), Ψ_0 denotes a quantum state of the non-interacting theory, and $D_{\Psi_0,k}$ denote λ-independent distributions that depend on both the quantum state Ψ_0 and the perturbative order k. Even for the simplest of quantum states in flat spacetime, it was highly nontrivial to develop a renormalization program that allowed the distributions, like $D_{\Psi_0,k}$, that appear in perturbation theory to be constructed and given a physical interpretation. Typically, calculations of physical observables in quantum field theory are still carried out to only relatively low (single-digit) orders in perturbation theory. Nevertheless, many of these low-order calculations have resulted in (by far) the most precise agreements between theoretical prediction and experimental measurement that have yet been achieved in empirical science.

At an early stage in the development of renormalized perturbation theory, it was expected/hoped that a power series like (1.9) might converge to a well-defined distribution as $N \to \infty$ for at least some nonzero range of values for the interaction parameter λ. In which case, perturbation theory would, in principle, provide a means to *define* (rather than just approximate) interacting quantum field theories. However, by now, it is generally believed that this kind of infinite series would not converge even when spacetime is flat and the fields are in their vacuum state, $\langle \cdot \rangle_{\text{vac}}$[13]. Thus, it is anticipated that the n-point distributions of an interacting theory (and their time-ordered counterparts) cannot—even in principle—be obtained by summing all ($N \to \infty$) orders in perturbation theory, and it is expected that perturbation theory only approximately describes the behavior of interacting quantum fields in the limit the strength of the interaction tends toward zero.

Remark 1. In one dimension, the Lagrangian (1.7) reduces to that of an anharmonic oscillator. The quartic anharmonic oscillator's ground state correlation functions are known to be non-analytic at $\lambda = 0$, and their standard perturbative series are divergent. Of course, this does not present any obstacle whatsoever to the formulation of the quantum theory of the quartic anharmonic oscillator: In contrast to the $(3+1)$-spacetime field theory case, the correlation functions of the anharmonic oscillator are non-singular and there is no inherent obstruction to defining nonlinear operators, like the theory's Hamiltonian, non-perturbatively.

For the purpose of motivating our application of the operator product expansion and the general form of the flow relations for the OPE coefficients, it is useful to examine the complications that arise when attempting to naively "turn on" the interaction parameter λ for the scalar field theory with quartic potential (1.7) in the very special case that the quantum fields are in the flat spacetime vacuum state. Naive manipulation of the formal

13. A simplistic, but suggestive, argument was given by Dyson in the context of QED [5]. For a scalar field with self-interaction potential $\lambda\phi^4/4!$, his reasoning implies the vacuum expectation values should not be analytic at $\lambda = 0$, since there does not exist a ground state for $\lambda < 0$, which corresponds to a potential that is unbounded from below.

functional integral expressions for the vacuum expectation values would suggest[14],

$$\frac{\partial}{\partial\lambda}\left\langle T\left\{\Phi_1(x_1)\cdots\Phi_n(x_n)\right\}\right\rangle_{\mathrm{vac}} \text{``="} -\frac{i}{4!}\int d^Dy\left\langle T\left\{\phi^4(y)\Phi_1(x_1)\cdots\Phi_n(x_n)\right\}\right\rangle_{\mathrm{vac}}, \quad (1.10)$$

where the integral is taken over all of D-dimensional Minkowski spacetime. The quotations around the equality sign are intended to indicate this expression cannot be taken literally since the integral on the right-hand side does not actually converge. Nevertheless, the heuristic suggested by the informal relation (1.10) is that changes to the interaction parameter's magnitude are induced by "inserting" the interaction operator ($\phi^4/4!$ in this case) and integrating over spacetime. This simple idea is complicated by several inconvenient facts:

Firstly, on account of the distributional nature of quantum fields described previously, it must first be specified what is meant by expressions involving nonlinear fields like ϕ^4, since nonlinear fields cannot be defined by taking pointwise products of the linear field. In particular, we note $\phi^4(y) \neq \phi(y)\phi(y)\phi(y)\phi(y)$. Defining nonlinear fields is a non-trivial task already in the non-interacting limit $\lambda = 0$: Indeed, Chapter 3 of this thesis is dedicated to a review of the renormalized Wick monomials and the characterization of their inherent renormalization ambiguities in generic (globally-hyperbolic) spacetimes. To make sense of (1.10) as a system of non-perturbative differential equations in the interaction parameter λ, nonlinear fields would need to be defined and their renormalization ambiguities understood for nonzero values of λ.

Secondly, as discussed in Section 4.3 of this thesis, time-ordering defines (1.6) as distributions only when all spacetime events are not coinciding, $x_i \neq x_j$ for all $i, j = 1, \ldots, n$. When any two spacetime events are coincident, the expectation value of time-ordered products (1.6) generically possess non-integrable divergences. Hence, the integrand on the right-hand side of (1.10) will generally fail to be *locally* integrable[15] if event y coincides with any one

14. For example, see the derivation given in Section 5.1, starting around eq. (5.6).

15. This is a (non-perturbative) position-space analogue of the "ultraviolet divergences" that appear in

of $x_1, \ldots, x_n$. Provided these isolated divergences are of finite "severity"[16] at any fixed λ-value, it should always be possible to renormalize the integrand of (1.10) to render it locally integrable. However, the severity of divergences in the integrand of (1.10) are related to the dimension of the quantum fields, which are known to depend on the value of the interaction parameter(s). Thus, the renormalization scheme would generally also depend on λ in a potentially-complicated way. Moreover, this renormalization procedure would generally not be unique and, thus, would not unambiguously determine the right-hand side of (1.10). Since (1.10) form an infinite set of coupled equations—each one requiring a λ-dependent renormalization—it is not at all obvious that the ambiguities in these equations could be fixed by only a finite number of physical measurements as is the case in the standard perturbative treatment of $\lambda\phi^4$-theory.

Thirdly, the integral is unbounded and generally does not converge as $y \to \infty$ even for the massive theory, $m^2 > 0$.

Remark 2. The first two complications described in the preceding paragraphs simplify considerably (but remain nontrivial) in perturbation theory. Formal differentiation of the power series (1.9) and the formula (1.10) suggest the k-th order distribution $D_{\text{vac},k}$ will, e.g., involve a term of the form:

$$\frac{i^k}{k!(4!)^k} \int d^4y_1 \cdots d^4y_k \left\langle T\left\{\phi_0^4(y_1)\cdots\phi_0^4(y_k)\phi_0(x_1)\cdots\phi_0(x_n)\right\}\right\rangle_{\text{vac}}, \tag{1.11}$$

in (3+1)-spacetime dimensions. Here $\langle\,\cdot\,\rangle_{\text{vac}}$ denotes the vacuum state of the non-interacting (Klein-Gordon) field. There is a unique prescription[17] for defining "Wick powers" like ϕ_0^4 such that their vacuum expectation value vanishes: i.e., $\langle\phi_0^4\rangle_{\text{vac}} = 0$. If the Wick powers

momentum-space perturbation theory.

16. This can be made precise using the concept of a "scaling degree"; see the discussion preceding eq. (4.5) and Footnote 1 of Section 4.1.

17. This prescription corresponds to replacing $H(x_1, x_2)$ with $\langle\phi(x_1)\phi(x_2)\rangle_{\text{vac}}$ in formula (3.28).

are defined in this canonical way, then the only ambiguities present in (1.11) arise from the renormalization of the time-ordered products for the *non*-interacting Klein-Gordon theory[18]. If the time-ordered products are locally and covariantly defined and satisfy[19] the properties-postulated in [7], then it follows the only ambiguities that arise in the renormalization of (1.11) are inherited from the ambiguities in the time-ordered products involving just ϕ_0^4 :

$$T\left\{\phi_0^4(y_1)\cdots\phi_0^4(y_k)\right\}. \tag{1.12}$$

Once the renormalization prescriptions for (1.12) with $k < p$ have been fixed, the prescription for defining (1.12) with $k = p$ is uniquely determined up to a "contact-term" of the form,

$$\left[c_0 m^4 I + c_1 \eta^{\mu\nu}(\partial_\mu\phi\partial_\nu\phi)_0(y_1) + c_2 m^2\phi_0^2(y_1) + c_3\phi_0^4(y_1)\right]\delta(y_1,\ldots,y_k), \tag{1.13}$$

where c_0, c_1, c_2, c_3 are arbitrary, dimensionless, real-valued, spacetime-independent numbers. Aside from the term proportional to the identity element I, every term appearing in (1.13) is of the same form as a term that appears in the Lagrangian (1.7). It can be shown that making different choices for c_0, c_1, c_2, c_3 is (perturbatively) equivalent to shifting the physical parameters appearing in the original Lagrangian (1.7) by certain λ-dependent functions,

$$\mathcal{L} \to \widetilde{\mathcal{L}} = f_0(\lambda)m^4 + (1 + f_1(\lambda))\,\eta^{\mu\nu}\partial_\mu\phi\partial_\nu\phi + m^2\,(1 + f_2(\lambda))\,\phi^2 + f_3(\lambda)\phi^4, \tag{1.14}$$

prior to differentiating (1.9) and (1.10) with respect to λ. Note functions f_1, f_2, f_3, f_4 vanish at $\lambda = 0$.

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18. See Section 4.3 for further discussion.

19. In fact, the smoothness and analyticity axioms of [7] are not satisfied by the prescription where $\langle\phi_0^4\rangle_{\text{vac}} = 0$, since the vacuum 2-point function is not smooth in m^2 at $m^2 = 0$; see also the discussion surrounding eq. (5.2). To simplify the discussion here, this subtlety will be ignored within the context of Remark 2.

$\ast\ast\ast$ Operator product expansion and informal flow relations for OPE coefficients $\ast\ast\ast$

In spite of the incredible successes of renormalized perturbation theory, there are important quantum field phenomena that cannot be analyzed perturbatively: e.g., it is expected that the observed hadronic "confinement" of quarks and gluons manifests non-perturbatively in the theory of quantum chromodynamics. As discussed previously, the renormalization required to define nonlinear fields and time-ordered products poses a serious technical and conceptual obstacle to defining interacting quantum field theories non-perturbatively. Because the need for renormalization ultimately arises from the singular behavior of products of quantum fields as two or more of their spacetime events approach coincidence, the operator product expansion is an indispensable tool for the systematic analysis of these inherent difficulties: A quantum field theory is said to possess an operator product expansion if, in any physically-acceptable state Ψ, the expectation value of any product of local quantum field observables can be approximated near event z as

$$\left\langle \Phi_{A_1}(x_1)\cdots\Phi_{A_n}(x_n)\right\rangle_\Psi \sim \sum_B C^B_{A_1\cdots A_n}(x_1,\dots,x_n;z)\,\langle\Phi_B(z)\rangle_\Psi\,. \tag{1.15}$$

Here $A_1\dots,A_n,B$ label the renormalized field observables of the theory (see e.g. (3.4)), and the sum over B extends over all observables. The coefficients $C^B_{A_1\cdots A_n}(x_1,\dots,x_n;z)$ of this expansion are ordinary c-valued distributions that are independent of the state Ψ (within the class of allowed states). The "$\sim$" in eq. (1.15) denotes that this relation holds asymptotically in the coincidence limit $x_1,\dots,x_n\to z$; a precise statement of this asymptotic relationship will be given in formula (4.2)) below. The OPE was initially postulated by Wilson in [1] and it is expected to exist for any renormalizable local quantum field theory under very general assumptions [8–14].

Since the 1-point functions $\langle\Phi_B(z)\rangle_\Psi$ are smooth, the OPE implies the "singular behavior" of the $(n>2)$-point distributions (1.5) as $x_1,\dots,x_n\to z$ is entirely contained in the distributional coefficients $C^{B}_{A_1 \cdots A_n}(x_1, \ldots, x_n; z)$. Although the B-sum contains infinitely-many terms, there are typically only finitely-many singular OPE coefficients for any fixed set of $A_1, \ldots, A_n$, with the coefficients becoming less singular (or approaching zero faster) as the dimension of the field Φ_B increases.

As discussed in Section 4.3, the OPE exists also for the expectation values of (un-renormalized) time-ordered products, eq. (1.6). Assuming the OPE exists when the interaction parameters are nonzero, it naturally suggests a relatively simple algorithm for removing the non-integrable divergences appearing in informal expressions involving time-ordered products like eq. (1.10), thereby bypassing the second major issue discussed under eq. (1.10). In particular, for any Φ_A and spacetime dimension D, the OPE of $\left\langle T\{\phi^4(y)\Phi_A(x)\}\right\rangle_\Psi$ is expected to contain only finitely-many terms that are non-integrable at $y = x$. Supposing the OPE coefficients satisfy the scaling degree axiom[20] of [15], see eq. (4.5), these non-integrable terms are,

$$\begin{aligned} &\left\langle T\{\phi^4(y)\Phi_A(x)\}\right\rangle_\Psi \\ &\sim \sum_{[C] \leq [A]+[\phi^4]-D} C^{C}_{T\{\phi^4 A\}}(y, x; x) \left\langle \Phi_C(x) \right\rangle_\Psi + \text{locally-integrable terms}, \end{aligned} \tag{1.16}$$

where $[A]$ denotes the dimension of the field Φ_A as defined in [15, Eq. 10]; see also Footnote 25. Here we have elected to expand about $z = x$. Suppose the non-integrable divergences that occur in $\left\langle T\left\{\phi^4(y)\Phi_{A_1}(x_1)\cdots\Phi_{A_n}(x_n)\right\}\right\rangle_\Psi$ at $y = x_i$ for each $i \in \{1, \ldots, n\}$ are of the same form as (1.16), i.e., suppose that as $y \to x_i$ with all other x-spacetime events held

20. This property has been proven to hold order-by-order in perturbation theory [10].

fixed[21]:

$$\left\langle T\left\{\phi^4(y)\Phi_{A_1}(x_1)\cdots\Phi_{A_n}(x_n)\right\}\right\rangle_\Psi$$
$$\sim \sum_{[C]\leq[A_i]+[\phi^4]-D} C^C_{T\{\phi^4 A_i\}}(y,x_i;x_i)\left\langle T\left\{\Phi_{A_1}(x_1)\cdots\Phi_C(x_i)\cdots\Phi_{A_n}(x_n)\right\}\right\rangle_\Psi + \tag{1.17}$$
$$+ \text{ terms locally integrable at } y = x_i.$$

This implies that, unlike the informal expression (1.10), the following expression should be free of any (local) non-integrable divergences,

$$\frac{\partial}{\partial\lambda}\left\langle T\left\{\Phi_{A_1}(x_1)\cdots\Phi_{A_n}(x_n)\right\}\right\rangle_\Psi \text{ “=”}$$
$$-\frac{i}{4!}\int d^Dy\sqrt{-g(y)}\left[\left\langle T\left\{\phi^4(y)\Phi_{A_1}(x_1)\cdots\Phi_{A_n}(x_n)\right\}\right\rangle_\Psi + \right. \tag{1.18}$$
$$\left. -\sum_{i=1}^n\sum_{[C]\leq[A_i]+[\phi^4]-D} C^C_{T\{\phi^4 A_i\}}(y,x_i;x_i)\left\langle T\left\{\Phi_{A_1}(x_1)\cdots\Phi_C(x_i)\cdots\Phi_{A_n}(x_n)\right\}\right\rangle_\Psi\right],$$

and, thus, it effectively bypasses the second major difficulty discussed directly after eq. (1.10). We emphasize that the summation over each $[C]$ is finite and the dimensions of the fields may depend on λ. Here we have generalized (1.10) from the vacuum state $\langle\,\cdot\,\rangle_{\text{vac}}$ in flat Minkowski spacetime to any state $\langle\,\cdot\,\rangle_\Psi$ in curved spacetime satisfying the OPE relations (1.15).

By introducing an infrared cutoff, the integral in (1.18) could now be made to converge. In which case, provided our assumptions held for $\lambda > 0$, the right-hand side of (1.18) would be mathematically well-defined non-perturbatively. However, in order for formula (1.18) to be useful for obtaining the expectation values $\left\langle T\left\{\Phi_{A_1}(x_1)\cdots\Phi_{A_n}(x_n)\right\}\right\rangle_\Psi$ of the interacting theory, one would first need to know the explicit form of the OPE coefficients,

21. This assumption is a stronger version of the “associativity” properties that are known to hold (perturbatively) for the OPE coefficients [10]; see also the two paragraphs preceding Theorem 3 in Section 4.1.

$C^{C}_{T\{\phi^4 A_i\}}(y, x_i; x_i)$, appearing in formula (1.18) for $\lambda > 0$. Unfortunately, as with the n-point distributions themselves, it is only known how to calculate the OPE coefficients to finite order in perturbation theory.

Although formula (1.18) cannot be solved for the time-ordered expectation values without knowing the OPE coefficients, we can use formula (1.18) to obtain relations for the OPE coefficients themselves: By applying the operator product expansion (1.15) to both sides of (1.18) and equating the coefficients of $\langle \Phi_B(z)\rangle_\Psi$ for each B, we obtain[22]

$$\begin{aligned}
&\frac{\partial}{\partial\lambda} C^{B}_{T\{A_1\cdots A_n\}}(x_1,\ldots,x_n;z) \text{ ``}\sim\text{''} \\
&-\frac{1}{4!}\int d^D y\sqrt{g(y)}\,\Big[C^{B}_{T\{A_1\cdots A_n\}}(x_1,\ldots,x_n;z)+ \\
&-\sum_{i=1}^{n}\sum_{[C]\leq[A_i]+[\phi^4]-D} C^{C}_{T\{\phi^4 A_i\}}(y,x_i;x_i)\,C^{B}_{T\{A_1\cdots C\cdots A_n\}}(x_1,\ldots,x_n;z)\Big]\,,
\end{aligned} \tag{1.19}$$

as all events, $x_1, \ldots, x_n$, approach an event z. Here $g \equiv \det g_{\mu\nu}$ denotes the determinant of the metric tensor. These relations are entirely "self-contained" in the sense that they involve only the OPE coefficients and, in particular, do not contain any state-dependent expectation values. If a suitable integration cutoff could be introduced, then these formulas should be entirely mathematically well-defined. The existence and uniqueness of solutions to an infinite system of differential equations like (1.19) is not presently known. However, since (1.19) are first-order in λ, it is conceivable that solutions are uniquely determined by specifying (all) the OPE coefficients $C^{B}_{T\{A_1\cdots A_n\}}$ at a single value of λ. For $\lambda = 0$, the theory is non-interacting and the OPE coefficients can be directly constructed: see Chapter 4 for explicit formulas. Therefore, by taking the known values for the OPE coefficients at $\lambda = 0$ as the "initial conditions" for the differential equations (1.19), one might then attempt to compute the non-perturbative OPE coefficients for the interacting theory, $\lambda > 0$, by solving

22. The formula (1.19) is obtained by assuming there exists a state Ψ and a renormalization prescription for the quantum fields such that $\partial_\lambda \langle \Phi_B(z)\rangle_\Psi = 0$ for all B.

the equations.

Presuming they exist, the solutions to (1.19) for the time-ordered OPE coefficients $C^{B}_{T\{A_1\cdots A_n\}}(x_1,\ldots,x_n;z)$ should then be useful for determining the expectation values for the time-ordered products $\left\langle T\left\{\Phi_{A_1}(x_1)\cdots\Phi_{A_n}(x_n)\right\}\right\rangle_\Psi$ at $\lambda>0$, recalling the heuristic formula (1.18). However, it is far from obvious how to actually construct time-ordered expectation values from knowledge of just the OPE coefficients (and the time-ordered expectation values at $\lambda=0$), since physical quantum states must satisfy additional nontrivial requirements like "positivity" conditions, $\left\langle\Phi^*_A(f)\Phi_A(f)\right\rangle_\Psi\geq 0$ for all Φ_A (and all test functions f), and they must satisfy the OPE relations, formula (1.15), for all composite fields $A_1,\ldots,A_n$ and at all spacetime events $z\in M$. The highly-nontrivial task of constructing quantum states from the OPE coefficients will not be considered further in this thesis.

Remark 3. The preceding informal derivation of the formula (1.10) for $\partial_\lambda\left\langle T\left\{\Phi_{A_1}(x_1)\cdots\Phi_{A_n}(x_n)\right\}\right\rangle_\Psi$ essentially generalizes an approach sketched by Wilson in the first OPE paper [1, Section V. Mass Terms: Generalities]

> A general perturbation formula can be set up to describe [relevant, marginal, and irrelevant] interactions. To avoid innumerable complications of perturbation theory to all orders one writes only a first-order formula giving the change in any local (Heisenberg) field $O_n(x)$ when any coupling constant is changed. That is, if $\{\lambda_i\}$ are the set of coupling constants associated with the interactions $\mathcal{L}_i$, one obtains a formula for $\partial O_n(x)/\partial\lambda_i$. The usual (unrenormalized) formula is
>
> $$\frac{\partial O_n(x)}{\partial\lambda_i}=i\int_y[O_n(x),\mathcal{L}_i(y)]_{\mathrm{ret}},$$
>
> where $[\quad]_{\mathrm{ret}}$ means the retarded commutator[23] $(y_0<x_0)$. This formula has to be corrected both for nonadiabatic effects (when physical particle masses vary with

23. For $x\neq y$, the retarded commutator can be expressed in terms of the time-ordered product: $[O_n(x),\mathcal{L}_i(y)]_{\mathrm{ret}}=T\{\mathcal{L}_i(y)O_n(x)\}-\mathcal{L}_i(y)O_n(x)$.

> λ_i) and for ultraviolet singularities at $x = y$. The nonadiabatic efects are easily accounted for and will not be considered here. The ultraviolet singularities can be analyzed using the operator-product expansion for the commutator $[O_n(x), \mathcal{L}_i(y)]$ and the singular terms can then be removed by subtraction.

In the very next line, Wilson anticipates the passage from formulas like (1.10) to (perturbative) flow relations for the OPE coefficients:

> One can then show[a] that operator product expansions continue to hold in the presence of the perturbation and obtain formulas for derivatives of expansion functions such as $\partial C_n(z)/\partial\lambda_i$. These formulas will not be quoted here.
>
> *a*. This paragraph summarizes a very complex analysis.

The footnote denoted by "*a*" in the preceding quotation is Footnote 27 in Wilson's paper. To my knowledge, explicit formulas like (1.19) for the flow relations did not appear in any of Wilson's publicly-accessible work.

In the preceding discussion, we described how the OPE coefficients may be used to extract (and remove) the non-integrable divergences of the time-ordered n-point distributions, which was the second issue described below eq. (1.10). Since the OPE coefficients contain more general information about the distributional properties of the quantum fields, they should also be relevant to the first issue described below eq. (1.10) : viz., defining nonlinear quantum fields at nonzero coupling. In standard approaches to defining and constructing quantum fields in Euclidean space and Minkowski spacetime, the existence of a unique vacuum state plays an essential role. However, in the formulation of quantum field theory in a curved Lorentzian spacetime, Hollands and Wald have previously argued [15, 16] that OPEs must play a role of similar importance. In a general, curved Lorentzian spacetime there is no notion of Poincare invariance and no preferred vacuum state, so properties of the quantum field normally formulated in terms of vacuum expectation values in Minkowski spacetime must now be formulated in terms of OPE coefficients. Hollands and Wald have argued that the key relations satisfied by the quantum field observables can be expressed via the OPE, so that, in essence, a quantum field theory in curved spacetime may be viewed as being specified by providing all of its OPE coefficients $C^{B}_{A_1\cdots A_n}$. Thus, it is of considerable interest to determine the OPE coefficients of an interacting quantum field theory. It would be especially of interest to determine the OPE coefficients of an interacting theory by methods that do not rely on perturbation theory, since this would have the potential for providing a non-perturbative definition of the interacting theory.

The third issue identified below eq. (1.10) regarding the "infrared" divergence of the unbounded spacetime integral is, of course, also present in the naive flow relation (1.19) for the OPE coefficients. As we will describe, the need to integrate the OPE coefficients over a compact spacetime region introduces major complications in Lorentzian spacetimes. Since the resolution of these difficulties is the primary objective of this thesis, discussion of the integration cutoff and its associated issues is deferred until the overview chapter, Chapter 2.

The informal manipulations that led to the flow relations (1.19) for the OPE coefficients were based on a crude heuristic and are entirely non-rigorous. Nevertheless, mathematically well-defined flow equations for the OPE coefficients with essentially the same structure as (1.19) have been rigorously derived for a variety of interacting quantum field theories on manifolds with Euclidean-signature metrics. We turn next to a summary of these Euclidean results.

$* * *$ Mathematically well-defined flow equations for Euclidean QFTs $* * *$

For the case of a Euclidean quantum field theory with power-counting renormalizable self interactions, Hollands has argued [17] that the OPE coefficients must satisfy a "flow" relation under changes of the coupling parameters. Such flow equations have been proven to hold order-by-order in perturbation theory for several interacting models, including $\lambda\phi^4$-theory [18, 19], Yang-Mills gauge theories [20], and CFTs with strictly marginal interactions [17]. In particular, Holland and Hollands have proven [17, Theorem 1] that, by making use of the freedom to redefine the quantum field observables, the OPE coefficients of $\lambda\phi^4$-theory in $D = 4$ dimensional (flat) Euclidean space satisfy the following[24] flow equations to any (finite) perturbative order in λ,

$$\begin{aligned}\frac{\partial}{\partial\lambda}C^{B}_{A_1\cdots A_n}(x_1,\ldots,x_n;z) = &-\frac{1}{4!}\int_{|y-z|\leq L} d^4y\left[C^{B}_{\phi^4 A_1\cdots A_n}(y,x_1,\ldots,x_n;z)+\right.\\ &-\sum_{i=1}^{n}\sum_{[C]\leq[A_i]} C^{C}_{\phi^4 A_i}(y,x_i;x_i)C^{B}_{A_1\cdots\widehat{A_i}C\cdots A_n}(x_1,\ldots,x_n;z)+\\ &\left.-\sum_{[C]<[B]} C^{C}_{A_1\cdots A_n}(x_1,\ldots,x_n;z)C^{B}_{\phi^4 C}(y,z;z)\right].\end{aligned} \tag{1.20}$$

Here λ is the renormalized coupling parameter; L is a positive constant with units of length; $\widehat{A_i}C$ indicates the replacement of the label A_i with the label C; and $[A]$ denotes the dimension of the renormalized field Φ_A as defined in [15, Eq. 10][25]. For the spatial integral over y, it is understood that the integration is initially done over the region bounded by $\epsilon \leq |y-x_i|$ and $\epsilon \leq |y-z| \leq L$, the subtractions appearing in the integrand are performed, and the limit as $\epsilon \to 0^+$ is then taken. Holland and Hollands have shown that all ultraviolet divergences that may arise in individual terms as $\epsilon \to 0^+$ precisely cancel between terms[26], so the $\epsilon \to 0^+$ limit is well-defined without any additional regulators or renormalization.

To compare the flow equations (1.20) for Euclidean $\lambda\phi^4$-theory to the informal Lorentzian relations (1.19) obtained above, we first note that $g(y) = 1$ in flat spacetime. In $D = 4$ spacetime dimensions and to any finite order in perturbation theory, we also note that $[\phi^4] = 4$ so the sum over $[C] \leq [A_i]+[\phi^4]-D$ in the Lorentzian relations (1.19) reduces to $[C] \leq [A_i]$. Therefore, apart from the integration cutoff (and the "$\sim$" symbol), the first two lines of the

24. In [17–19], Holland and Hollands set the expansion point $z = x_n$. We prefer to define the coefficients more symmetrically in $x_1,\ldots,x_n$ by using an independent expansion point z.

25. To any finite perturbative order, the dimension defined in [15, Eq. 10] coincides with the standard "engineering dimension" given in our "Notation and conventions" at the end of Chapter 2.

26. For comparison to the terminology that will be used in Chapter 2, note the cancellation of non-integrable divergences at $y = x_i$ (for $i = 1,\ldots,n$) is equivalent to the statement that the integrand of (1.20) is uniquely "extendable" as a distribution to the "partial diagonals" involving y and any single x_i-point.

Holland and Hollands flow equations (1.20) for $\lambda\phi^4$-theory are (perturbatively) equivalent to the "Wick rotated"[27] flat spacetime limit of the informal Lorentzian flow relations (1.19). Terms in the third line of the Euclidean flow equations (1.20) are related to the specific renormalization scheme[28] used in [18, 19] and contain no non-integrable divergences for $|y - z| \leq L$.

Although the flow equations (1.20) were rigorously derived in a perturbative setting, these equations make sense mathematically for any value of λ under very general model-independent assumptions—specifically, if the OPE coefficients satisfy the "associativity" and "scaling degree" axioms postulated in [15]. Thus, it seems reasonable to assume that eq. (1.20) would hold for the OPE coefficients of the non-perturbative theory. That is, if it were possible to integrate eq. (1.20) from $\lambda = 0$ (where the field is free and the OPE coefficients may be computed directly) up to some nonzero λ, we would obtain a non-perturbative construction of the interacting OPE coefficients. As mentioned before, it is not known if there exist solutions to an infinite system of ordinary differential equations like (1.20). Nevertheless, flow relations like eq. (1.20) have the potential to provide a new approach to the formulation of interacting quantum field theory, and may be of considerable "practical" use as well.

Remark 4. For some (or all) values of the coupling parameter, there may exist nontrivial solutions to the flow relations that do not satisfy the OPE relation (1.15) for some (or all) physically-reasonable states. For example, the explicit formulas for the OPE coefficients of massive non-interacting Klein-Gordon fields can be shown to satisfy flow relations, eq. (2.1), with respect the mass squared parameter, m^2, for both positive and negative values of m^2. However, the OPE (1.15) cannot hold for the Klein-Gordon vacuum state when $m^2 < 0$ for the simple reason that no vacuum exists when $m^2 < 0$. For pure $\lambda\phi^4$-theory[29], it has

27. Under a Wick rotation, the volume element picks up a factor of $-i$. Note the Euclidean OPE coefficients are symmetric so "time ordering" does not affect them.

28. For massive fields satisfying the BPHZ renormalization conditions, the terms in the third line ensure the integral converges in the limit the cutoff is removed, $L \to \infty$.

29. Here "pure" means there are no self-couplings besides the quartic term and no couplings between ϕ

been proven under certain assumptions [21] that the non-perturbative lattice-regularized Schwinger functions are "Gaussian"[30] in the continuum limit for all nonzero values of the renormalized interaction parameter λ. This result is sometimes referred to as the quantum "triviality" of $\lambda\phi^4$-theory[31]. Hence, for any nonzero λ, solutions to the flow relations (1.20) clearly would not correspond to the OPE coefficients of the renormalized $\lambda\phi^4$-vacuum that is obtained by this lattice-based construction: cf. eq. (1.20) with the form of the flow relations, eq. (2.1), for the non-interacting OPE coefficients. Regardless of the non-perturbative status of $\lambda\phi^4$-theory, we emphasize that, as mentioned previously, flow equations have also been obtained for "asymptotically free" Yang-Mills gauge theories that are expected [6] to have non-Gaussian vacua at finite nonzero coupling.

and other quantum fields.

30. i.e., the Schwinger n-point functions, $\langle\phi(x_1)\cdots\phi(x_n)\rangle$, vanish for n odd and for n even are given in terms of the 2-point function, $\langle\phi(x_i)\phi(x_j)\rangle$, by the usual Wick combinatorial formulas, eq. (5.5), for a non-interacting theory. Gaussian states are also known as "quasifree states" with vanishing 1-point function.

31. It is not entirely obvious [22, see Section 8: Is Destructive Field Theory Possible?] whether lattice-based results should be considered decisive evidence for determining existence of nontrivially-interacting quantum field theories in the continuum.

CHAPTER 2

OVERVIEW OF RESULTS AND ORGANIZATION OF THESIS

The OPE flow relations (1.20) and their generalization to other interacting theories apply for the case of flat Euclidean space. Recently, Fröb [23] has generalized these relations to quantum fields on curved Riemannian spaces, without, however, imposing the condition that the OPE coefficients be locally and covariantly defined. Since the physical world is Lorentzian, it would be of interest to generalize the flow relations to Lorentzian spacetimes. Furthermore, the requirement that the OPE coefficients be locally and covariantly defined in curved spacetime is the natural generalization of the requirement of Poincaré invariance in Minkowski spacetime [15] and it thereby provides an important requirement on the flow relations. Thus, it is of interest to determine if the flow relations can be formulated for Lorentzian spacetimes in a local and covariant manner.

There are two major obstacles to generalizing flow relations such as eq. (1.20) to the Lorentzian case: (i) In the Euclidean case, the infrared cutoff, L, appearing in the flow relations (1.20) is fully compatible with rotational invariance, and the resulting flow relations are automatically Euclidean invariant. However, in Minkowski spacetime, no bounded region of spacetime can be invariant under Lorentz boosts. Thus, in Minkowski spacetime, either the corresponding integral must be taken over an unbounded region—resulting in serious problems with convergence of the integral in Minkowski spacetime as well as with the definition of the OPE coefficients throughout the region in the generalization to curved spacetime—or the corresponding integral will not be Lorentz invariant, leading to flow relations that are not Poincaré invariant. (ii) There is a fundamental difficulty with obtaining local and covariant results by performing an integral over a spacetime region. If the curved spacetime flow relations take a form similar to eq. (1.20) where the integral is performed over some neighborhood U_z of $z \in M$, this integral would depend on the spacetime metric in all of U_z, not just in an arbitrarily small neighborhood of z. Thus, for a flow relation of the form of eq. (1.20) with an integral performed over a finite spacetime region U_z, the flow of OPE coefficients will necessarily depend non-locally on the metric.

The purpose of this thesis is to show how the above difficulties can be overcome, thereby showing that local and covariant OPE flow relations can be defined in curved Lorentzian spacetimes. We will also show how to modify the flow relations so as to eliminate any dependence on the infrared cutoff scale L. We will restrict consideration in this thesis to the "toy model" of massive, non-minimally-coupled Klein-Gordon theory, with m^2 and the curvature coupling parameter, ξ, viewed as interaction parameters. Of course, this model is a free field for all values of the parameters. Nevertheless, we may treat m^2 and ξ as coupling constants in an interaction Lagrangian, in parallel with the treatment of λ in eq. (1.20). The resulting flow relations have a form that is very similar in its essential features to that of a nonlinearly interacting theory, so this toy model provides a good testing ground for confronting the issues needed to generalize the flow relations to curved Lorentzian spacetimes. For this toy model, in Euclidean space of any dimension $D \geq 2$, the direct analog of eq. (1.20) above is the following flow relation in m^2 for the coefficients[1] $C^I_{\phi\cdots\phi}(x_1, \ldots, x_n; z)$ appearing in the OPE of the n-point product of linear field observables, $\langle \phi(x_1) \cdots \phi(x_n) \rangle_\Psi$:

$$\frac{\partial}{\partial m^2} C^I_{\phi\cdots\phi}(x_1, \ldots, x_n; z) = -\frac{1}{2} \int_{|y-z|^2 \leq L^2} d^D y \, C^I_{\phi^2\phi\cdots\phi}(y, x_1, \ldots, x_n; z) \tag{2.1}$$

Note that in this case the y-integral yields a well-defined distribution in $(x_1, \ldots, x_n)$ with no need for an ultraviolet cutoff ϵ. Our goal is to obtain an analogous flow relation in the Lorentzian case.

The first issue we must address is the "type" of products of fields that must be considered in order for the OPE coefficients to satisfy flow relations. In the Euclidean case, there is a unique notion of the n-point (correlation = Green's = Schwinger) distributions and their

1. As we shall see in Section 4.2, all other OPE coefficients are determined by $C^I_{\phi\cdots\phi}(x_1, \ldots, x_n; z)$, so it suffices to consider only the flow relations for these coefficients.

corresponding OPE coefficients. However, in the Lorentzian case, one can consider Wightman products, time-ordered products, retarded products, etc. Any of these products could be put on the left side of eq. (1.15) and used to define OPE coefficients. The resulting OPE coefficients will possess distinct singular behavior (i.e., "wavefront sets"), and it is not obvious, a priori[2], which—if any—of these Lorentzian objects are viable candidates for satisfying flow relations. Our analysis of this issue in Chapter 5 reveals that the Green's function properties of the n-point distributions play an essential role in the derivation of flow relations. Consequently, as we discuss in Chapter 6, the usual Wightman n-point OPE coefficients as written in eq. (1.15) are not suitable candidates for satisfying flow relations in the Lorentzian case. On the other hand, time-ordered products do possess the requisite Green's function properties for flow relations[3]. The Lorentzian flow relations we shall obtain will thus apply to the OPE coefficients arising from the asymptotic expansion of the time ordered products $\langle T\{\Phi_{A_1}(x_1)\cdots\Phi_{A_n}(x_n)\}\rangle_\Psi$ rather than the Wightman products $\langle \Phi_{A_1}(x_1)\cdots\Phi_{A_n}(x_n)\rangle_\Psi$.

However, working with time-ordered products has the potential to lead to significant additional complications, since time-ordered products possess substantial additional renormalization ambiguities beyond those associated with the definition of Wick powers and their corresponding Wightman functions. Time-ordered products of n field observables are well defined by naive time ordering only when no two points in the n-point distribution coincide, i.e., away from all "diagonals." We denote this well defined, "unextended" time-ordered product by $T_0\{\Phi_{A_1}(x_1)\cdots\Phi_{A_n}(x_n)\}$. Any procedure for extending T_0 to any of the diagonals (i.e. renormalization) is generally non-unique and, therefore, must unavoidably introduce new ambiguities proportional to δ-distributions (i.e. "contact terms"). This will result in corresponding ambiguities on the diagonals of the OPE coefficients defined using time-ordered products. Thus, if we formulate the flow relations in terms of these OPE coefficients, it might

2. the heuristic analysis of the previous chapter notwithstanding

3. Retarded and advanced products also satisfy the Green's function properties.

appear that we will have to deal with substantial additional renormalization ambiguities on the diagonals.

Fortunately, however, we find that this is not the case. In the OPE, eq. (1.15), we may keep all of the x_i distinct, so that the unextended time ordered products and corresponding OPE coefficients are well defined. However, flow relations such as eq. (1.20) involve an integration over a variable y, so we cannot avoid the coincidence of y with the various x_i. Thus, it might appear that the flow relations require us to evaluate the OPE coefficients at points where they are not defined. However, the integrand of the OPE flow relations contains a very special combination of OPE coefficients that has sufficiently mild divergences (i.e., "low scaling degrees") on the "partial diagonals" involving only y and one other spacetime point[4]. Consequently, the integrand *can* be uniquely extended to these—and typically only these—partial diagonals, and the flow relations are well defined for the unextended time-ordered products T_0. Thus, no new renormalization ambiguities arise beyond those occurring for the Wick monomials in the flow relations of the OPE coefficients of unextended time ordered products.

We now explain how the two major obstacles described above to obtaining Lorentzian flow relations are overcome. The first obstacle originates from the fact that no bounded neighborhood of z in Minkowski spacetime can be invariant under Lorentz boosts. To ensure that the integrals appearing in the flow relations are well defined and convergent, we introduce into the integrand a smooth function[5] $\chi(y-z;L)$ such that $\chi = 1$ in a coordinate ball of radius L and $\chi = 0$ outside a coordinate ball of radius $2L$. The presence of χ ensures that the integral extends over only a compact spacetime region, but it also necessarily breaks the Lorentz covariance of the flow relations. Nevertheless, we prove in Chapter 6 that Lorentz

4. see also Footnote 26 of Chapter 1

5. It is preferable to work with a smooth function χ rather than a step function as in (1.20) and (2.1) since in the Lorentzian case the singular behavior of a step function will overlap the singular behavior of the OPE coefficients in the integrand.

covariance can be restored in Minkowski spacetime—to any desired "scaling degree"—by subtracting off finitely many terms in the flow relations with a compensating failure of Lorentz invariance. For the OPE coefficients $C^I_{T_0\{\phi\cdots\phi\}}$, this results in a Minkowski spacetime flow relation of the form,

$$\frac{\partial}{\partial m^2} C^I_{T_0\{\phi\cdots\phi\}}(x_1,\ldots,x_n;z) \sim -\frac{i}{2}\int d^D y\, \chi(y-z;L)\, C^I_{T_0\{\phi^2\phi\cdots\phi\}}(y,x_1,\ldots,x_n;z) + \\ -\sum_C a_C[\chi] C^C_{T_0\{\phi\cdots\phi\}}(x_1,\ldots,x_n;z), \tag{2.2}$$

where a_C are spacetime constant tensors which depend on χ. As described in Appendix C, the existence of such a_C is guaranteed by the same kind of cohomological argument [24] that ensures the Lorentz-covariance of the Epstein-Glaser renormalization scheme. In Appendix C, we also obtain a recursive construction[6] of the coefficients $a_C[\chi]$ required for the Lorentz-covariant flow relations (2.2) in Minkowski spacetime, in parallel with the analysis given in [25, 26] of the covariance-restoring Epstein-Glaser counterterms.

The flow relations (2.2) are Lorentz covariant. However, they contain an infrared cutoff scale L and the presence of L in this formula will spoil the required almost homogeneous scaling of $C^I_{T_0\{\phi\cdots\phi\}}(x_1,\ldots,x_n;z)$ under the scalings $g_{ab} \to \lambda^{-2} g_{ab}$, $m^2 \to \lambda^2 m^2$ of the metric and the mass. This issue also arises for the Euclidean flow relation eq. (2.1). Thus, we must further modify these flow relations so as to eliminate its L dependence up to any desired scaling degree. This can be accomplished in the following manner. As shown in Section 5.2, the partial derivative with respect to L of the right side of the Euclidean flow relation eq. (2.1) is of the form,

$$\frac{\partial}{\partial L}\,[\text{rhs of (2.1)}] \sim \sum_C \beta_C(L) C^C_{\phi\cdots\phi}(x_1,\ldots,x_n;z), \tag{2.3}$$

6. The inductive formula for a_C is given in eq. (C.44) with $\boldsymbol{B}^{\kappa\rho}$ given by eq. (6.30).

where $\beta_C = \beta_{\gamma_1 \cdots \gamma_k}$ denote tensors that are computed from the OPE coefficients and depend on the infrared length scale L. If the divergences in $\beta_C(L)$ were integrable in a neighborhood containing $L = 0$, then the problematic L-dependence of the Euclidean flow relation (2.1) could be removed by simply subtracting the definite integral,

$$\sum_C C^C_{\phi\cdots\phi}(x_1, \ldots, x_n; z) \int_0^L dL' \beta_C(L'), \tag{2.4}$$

from the right-hand side of (2.1). However, the divergences in $\beta_C(L)$ are not, in general, integrable. Nevertheless, we show that, for any finite field dimension $[C]$, all divergences in $\beta_C(L)$ as $L \to 0^+$ can be expressed as a finite linear combination of terms proportional to $L^{-\Delta} \log^N L$ for positive integers Δ, N. Such non-integrable terms are in the kernel of differential operators of the form $(1 + \Delta^{-1} L\partial_L)^{N+1}$, and these differential operators simply act like the identity operator on any L-independent terms. Making use of these facts, we construct a linear differential operator $\mathfrak{L}[L]$ which, when applied to the right-hand side of (2.1), effectively removes the L-dependent terms which lead to non-integrabilities in $\beta_C(L)$, while perfectly preserving all of its L-independent behavior. Once the operator $\mathfrak{L}[L]$ has been applied to the right-hand side of (2.1), any remaining L-dependence is guaranteed to be integrable and, thus, can be eliminated via simple subtraction of a definite integral as described above. In the Euclidean case, this yields the following L-independent flow relations for OPE coefficients defined by Hadamard normal ordering:

$$\begin{aligned}\frac{\partial}{\partial m^2} C^I_{\phi\cdots\phi}(x_1, \ldots, x_n; z) \sim &-\frac{1}{2}\mathfrak{L}[L] \int d^D y \, \chi(y, z; L) \, C^I_{\phi^2\phi\cdots\phi}(y, x_1, \ldots, x_n; z) + \\ &- \sum_C b_C(L) C^C_{\phi\cdots\phi}(x_1, \ldots, x_n; z),\end{aligned} \tag{2.5}$$

with $\mathfrak{L}[L]$ given by eq. (5.33) and the explicit dependence of b_C on the OPE coefficients given in formula (5.48) of Theorem 6. (For comparison with the Euclidean flow relations (1.20) and (2.1), one should take χ to be a step function cutoff, $\chi(y, z) = \theta(L^{-2}|y - z|^2)$.)

In the Minkowski case, the flow relations for the case where the Wick powers are defined by Hadamard normal ordering[7] become (see Theorem 7)

$$\frac{\partial}{\partial m^2} C^I_{T_0\{\phi\cdots\phi\}}(x_1,\ldots,x_n;z) \sim -\frac{i}{2}\int d^D y\, \mathfrak{L}[L]\chi(y,z;L) C^I_{T_0\{\phi^2\phi\cdots\phi\}}(y,x_1,\ldots,x_n;z) + \\ -\sum_C c_C(L) C^C_{T_0\{\phi\cdots\phi\}}(x_1,\ldots,x_n;z), \tag{2.6}$$

where c_C is given by formula (6.29). The ambiguities in the choice of c_C correspond to the inherent renormalization ambiguities in the OPE coefficients of Hadamard normal-ordered Wick monomials.

The second major obstacle to obtaining Lorentzian flow relations arises in curved spacetimes as a result of the nonlocal dependence on the metric caused by integrating over a region of finite size. We overcome this obstacle by replacing the true spacetime metric, $g_{\mu\nu}$, with its Taylor polynomial, $g^{(N)}_{\mu\nu}$, in Riemannian normal coordinates about z, carried to sufficiently high order, N, to achieve equivalence in the flow relations up to the desired scaling degree. This replacement is made prior to evaluating the spacetime integral, so the resulting flow relations will be suitably "local" in the sense that they depend only on finitely-many derivatives of the metric evaluated at the event z. However, we still need to introduce a cutoff function, χ, with an associated length scale L and, thus, these local flow relations will fail to be covariant on account of the presence of χ and fail to scale almost homogeneously due to the presence of L. Nevertheless, we can again introduce compensating local counterterms to render the flow relation covariant and we can construct an operator $\mathfrak{L}$ to eliminate the dependence on L to any desired asymptotic scaling degree. In any Riemannian normal

7. A similar formula holds for the case of a general definition of Wick powers, with the only difference being the presence of additional terms containing factors of the smooth functions F_k that parameterize the field-redefinition freedom of Wick fields.

coordinate system with origin at z, the resulting flow relations take the form,

$$\begin{aligned}
&\frac{\partial}{\partial m^2} C^I_{T_0\{\phi\cdots\phi\}}(x_1,\ldots,x_n;\vec{0}) \sim \\
&-\frac{i}{2}\int_{\mathbb{R}^D} d^D y \sqrt{-g^{(N)}(y)}\,\mathfrak{L}[L]\chi(y,\vec{0};L) C^I_{T_0\{\phi^2\phi\cdots\phi\}}(y,x_1,\ldots,x_n;\vec{0}) + \\
&-\sum_C c_C C^C_{T_0\{\phi\cdots\phi\}}(x_1,\ldots,x_n;\vec{0}),
\end{aligned} \tag{2.7}$$

where the OPE coefficients on all lines and the counterterm coefficients c_C are functionals of the polynomial metric $g^{(N)}_{\mu\nu}$. All dependence of c_C on the polynomial metric at event z can be expressed entirely in terms of totally-symmetric covariant derivatives of the Riemann curvature tensor. The explicit form of c_C is given in terms of the OPE coefficients in formula (7.37). Overall, the key new aspects of the curved spacetime flow relations (2.7) are the replacement of the metric by a polynomial approximation and the presence of additional counterterms involving the curvature.

Finally, we note that our derivations of the flow relations for flat Euclidean space given in Chapter 5, the flow relations for Minkowski spacetime given in Chapter 6, and the flow relations for general curved Lorentzian spacetimes given in Chapter 7 were based upon formulas for OPE coefficients that we obtained explicitly in Chapter 4. However, for nonlinear models, such explicit non-perturbative formulas for the OPE coefficients are not available. However, in Appendix E, we show that for the integrals which appear in the flow relations, one can derive covariance-restoring counterterms using only the associativity property of OPE coefficients, without explicit knowledge of the coefficients. When specialized to Klein-Gordon theory, this general algorithm reproduces the results we derived in Chapters 6-7. When applied to $\lambda\phi^4$-theory in a curved Lorentzian spacetime (M, g_{ab}), the algorithm developed in Appendix E yields

$$
\begin{aligned}
&\frac{\partial}{\partial\lambda} C^{B}_{T_0\{A_1\cdots A_n\}}(x_1,\ldots,x_n;\vec{0}) \sim \\
&-\frac{1}{4!}\int d^4y\sqrt{-g^{(N)}(y)}\,\chi(y,\vec{0};L)\left[C^{B}_{T_0\{\phi^4 A_1\cdots A_n\}}(y,x_1,\ldots,x_n;\vec{0}) + \right.\\
&-\sum_{i=1}^{n}\sum_{[C]\le[A_i]}\left[C^{C}_{T_0\{\phi^4 A_i\}}(y,x_i;x_i) - \sum_{[D]} c^{C}_{D} C^{D}_{T_0\{A_i\}}(x_i;\vec{0})\right] C^{B}_{T_0\{A_1\cdots\widehat{A_i}C\cdots A_n\}}(x_1,\ldots,x_n;\vec{0}) + \\
&\left. -\Big[\sum_{[C]<[B]} C^{B}_{T_0\{\phi^4 C\}}(y,\vec{0};\vec{0}) - \sum_{[C]\ge[B]} c^{B}_{C}\Big]\, C^{C}_{T_0\{A_1\cdots A_n\}}(x_1,\ldots,x_n;\vec{0})\right], \qquad (2.8)
\end{aligned}
$$

where the $[D]$-sum in the third line and the $[C] \geq [B]$ sum in the final line are carried out to sufficiently-large but finite field dimensions[8]. The form of the counterterm coefficients c^{B}_{C} is given in Appendix E for flat Minkowski spacetime. It would be natural to associate the inherent local and covariant ambiguities in c^{B}_{C} with the field-redefinition freedom of $\lambda\phi^4$-theory, but we have not investigated this issue[9].

∗ ∗ ∗ Organization of this document ∗ ∗ ∗

The structure of this thesis is as follows. In Chapter 3, we review the theory of a free Klein-Gordon field on a curved Lorentzian spacetime. The ambiguities in the definition of arbitrary Wick monomials $\Phi_A \equiv \nabla_{\alpha_1}\phi\cdots\nabla_{\alpha_n}\phi$ (where α_i denote spacetime multi-indices) is fully analyzed. The precise form of the "mixing matrix" $\mathcal{Z}^{B}_{A}$ describing allowed field redefinitions is given in Theorem 1, and it is shown in Proposition 1 that the field redefinition freedom is fully characterized by a sequence of smooth, real-valued functions $F_n(x_1,\ldots,x_n;z)$ that are symmetric in $(x_1,\ldots,x_n)$.

8. The coefficient $C^{D}_{T_0\{A_i\}} = C^{D}_{A_i}$ involving a single field factor is given by the geometric factors that appear in an ordinary Taylor expansion (see eq. (E.25)).

9. This analysis would require an understanding of what field-redefinition freedom is allowed for the non-perturbative interacting theory. Note also that we have not attempted to eliminate the L-dependence of the flow relations (2.8). The techniques described in Section 5.2 can be used to eliminate the L-dependence of (2.8) to any finite order in perturbation theory, but it is not obvious how to remove the L-dependence non-perturbatively.

In Chapter 4, we show that the Klein-Gordon field admits an OPE of the form eq. (1.15) for Hadamard states Ψ. In Theorem 2, we obtain an explicit formula for the OPE coefficients for the case where Wick monomials are defined by Hadamard normal ordering. For a general prescription for Wick monomials, we show that the OPE coefficients $C^{B}_{A_1\cdots A_n}$ for products of general Wick monomials are completely determined by the OPE coefficients $C^{I}_{\phi\cdots\phi}$ of the identity operator, I, for the n-point products of the linear field observable, $\phi(x_1)\cdots\phi(x_n)$. Furthermore, $C^{I}_{\phi\cdots\phi}(x_1,\ldots,x_n;z)$ is uniquely determined by the coefficients $C^{I}_{\phi\cdots\phi}$ with smaller n up to the addition of the function $F_n(x_1,\ldots,x_n;z)$ appearing in Proposition 1. The existence and properties of the OPE for a general definition of Wick monomials is summarized in Theorem 4. An inductive construction of the Wick monomials in terms of $C^{I}_{\phi\cdots\phi}$ is given in Proposition 5. As discussed in Section 4.3, all these statements carry over to the OPE for unextended time-ordered products, since the formulas for their OPE coefficients may be obtained in a simple and direct manner from the formulas for $C^{B}_{A_1\cdots A_n}$.

In Chapter 5, we derive the flow relations for the OPE coefficients of the Euclidean version of the Klein-Gordon field. The modification of the flow relations needed to remove the L-dependence is given in Section 5.2.

In Chapter 6, we analyze the flow relations for the OPE coefficients of the Klein-Gordon field in Minkowski spacetime. The counterterms in the flow relations needed to restore Lorentz covariance are obtained, with the technical details given in Appendix C.

The generalization to curved spacetimes is given in Chapter 7. To any specified scaling degree, we replace the spacetime metric by a Taylor approximation in a Riemannian normal coordinate system defined relative to the expansion point z. We then show that suitable counterterms can be introduced to yield local and covariant flow relations that are independent of L.

Finally, although our analysis in this thesis is restricted to the toy model of the free Klein-Gordon field, we show in Appendix E that our construction of the covariance-restoring counterterms requires only the associativity property of the OPE coefficients and thus should be applicable to nonlinearly interacting theories. The algorithm for constructing counterterms given in Appendix E reproduces the results we derived in Chapters 6-7 when applied to Klein-Gordon theory. When applied to $\lambda\phi^4$-theory in Lorentzian spacetime (M, g_{ab}), we obtain the local and covariant Lorentzian analogue (2.8) of the Holland and Hollands Euclidean flow relations (1.20).

Notation and conventions: We use letters from the beginning of the Latin alphabet to denote abstract indices and our spacetime geometry conventions coincide with those of [27]. Tensors are often abbreviated with multi-indices chosen from the beginning of the Greek alphabet $(\alpha, \beta, \gamma, \dots)$—e.g., we denote a tensor $T^{a_1 \cdots a_n}{}_{b_1 \cdots b_m}$ of type (n, m) simply as $T^{\alpha}{}_{\beta}$. In combinatorial formulas involving abstract multi-indices, we use the obvious analogues of the standard multi-index conventions: e.g., for $T^{\alpha} \equiv T^{a_1 \cdots a_n}$, we have $|\alpha| \equiv n$ and $\alpha! \equiv |\alpha|!$. When coordinate components of a tensor are needed, we denote ordinary spacetime indices with letters from the middle of the Greek alphabet $(\mu, \nu, \kappa, \rho, \dots)$ but continue to denote multi-indices with $(\alpha, \beta, \gamma, \dots)$. Throughout, $\mathbb{N}$ denotes the natural numbers (positive integers, excluding 0) and $\mathbb{N}_0 \equiv \{0\} \cup \mathbb{N}$. We use "smooth" to mean infinitely differentiable, i.e. C^{∞}, and the "Taylor coefficients of f evaluated at z" will refer to the set, $\nabla_{\alpha_1} \cdots \nabla_{\alpha_n} f(x_1, \dots, x_n)|_{x_1, \dots x_n = z}$, of covariant derivatives of a multivariate smooth function f evaluated at z *without* the numerical factor $1/(\alpha_1! \cdots \alpha_n!)$. The set of smooth functions of compact support is denoted by C_0^{∞} and the dual space of distributions is denoted by $\mathcal{D}' : C_0^{\infty} \to \mathbb{R}$.

Some notation in the thesis may not always be redefined with each use. For the convenience of the reader, we include here a list of frequently-employed non-standard symbols and their definitions or, in cases where the definition is too lengthy, we reference the equation where the symbol is defined.

field notation

Φ_A	the differentiated scalar field monomial, $\nabla_{\alpha_1}\phi\nabla_{\alpha_2}\phi\cdots\nabla_{\alpha_p}\phi$
Φ^H_A	monomial, $(\nabla_{\alpha_1}\phi\cdots\nabla_{\alpha_p}\phi)_H$, defined via "Hadamard normal ordering", see eq. (3.28)
$\mathcal{Z}_A^B$	field redefinition "mixing matrix" defined in eq. (3.38)
$[A]_\phi$	the number of ϕ-factors appearing in Φ_A (i.e., p, in this case)
$[A]_\nabla$	the number of covariant derivatives acting on ϕ in Φ_A (i.e., $\sum_{i=1}^p \lvert\alpha_i\rvert$, in this case)
$[A]$	"engineering dimension" of Φ_A given by (rational) number $(D/2-1)\times[A]_\phi+[A]_\nabla$
$C^B_{A_1\cdots A_n}$	OPE coefficients defined in relation (1.15)
$(C_H)^B_{A_1\cdots A_n}$	OPE coefficients defined in relation (4.1) for Hadamard normal-ordered fields
$C^B_{T_0\{A_1\cdots A_n\}}$	OPE coefficients of unextended time-ordered products defined in eq. (4.53)
$(C_H)^B_{T_0\{A_1\cdots A_n\}}$	Hadamard normal-ordered version of $C^B_{T_0\{A_1\cdots A_n\}}$

differential operators, parametrices and Greens functions

K	Klein-Gordon operator, $K\equiv -g^{ab}\nabla_a\nabla_b+m^2+\xi R$
H	Hadamard parametrix defined in eq. (3.26)
H_F	Feynman parametrix, $H_F\equiv H-i\Delta^{\text{adv}}$, see also Footnote 10 in Section 4.3
Δ	causal propagator, $\Delta\equiv\Delta^{\text{adv}}-\Delta^{\text{ret}}$
$\Delta^{\text{adv}},\Delta^{\text{ret}}$	advanced and retarded, resp., Greens function of K
$\mathfrak{L}$	operator defined in eq. (5.33) in terms of infrared length scale L and $\partial/\partial L$

geometric notation

D	the spacetime dimension, i.e., #(spatial dimensions) + #(temporal dimensions)
$d\mu_g(x)$	covariant volume element, $d^D x\sqrt{-g(x)}$, on spacetime (M, g_{ab})
$\int_{x_1,x_2,\ldots,x_n}$	abbreviation for $\int_{\times^n M} d\mu_g(x_1)d\mu_g(x_2)\cdots d\mu_g(x_n)$
$S^\beta(x,z)$	bi-tensor defined with respect to the geodesic distance function in eq. (3.58)
Z^*M	zero section of the cotangent bundle T^*M
$V_x^\pm$	future/past lightcone of the cotangent space T_x^*M
$\dot{V}_x^\pm$	boundary of future/past lightcone of cotangent space T_x^*M
$(x_1,k_1)\sim(x_2,k_2)$	equivalence relation defined below eq. (3.13) for $(x_1,k_1),(x_2,k_2)\in T^*M$

asymptotic equivalence relations

$\sim_{\mathcal{T},\delta}$	asymptotic equivalence to scaling degree δ for merger tree $\mathcal{T}$, defined in the paragraph surrounding eq. (4.2)
$\sim_\delta$	shorthand for "$\sim_{\mathcal{T},\delta}$" when $\mathcal{T}$ is the trivial merger tree, i.e., all spacetime points merge at the same rate to z
$\approx$	asymptotic equivalence for all δ and $\mathcal{T}$, defined in the paragraph surrounding eq. (4.2)

CHAPTER 3

KLEIN-GORDON THEORY AND LOCAL WICK FIELDS

The theory of a Klein-Gordon scalar field on a D-dimensional spacetime (M, g_{ab}) with mass m and curvature coupling ξ is given by the action,

$$S_{\mathrm{KG}} \equiv -\frac{1}{2}\int_M d^D x \sqrt{-g(x)} \left[g^{ab}(x)\nabla_a\phi(x)\nabla_b\phi(x) + \left(m^2 + \xi R(x)\right)\phi^2(x)\right]. \tag{3.1}$$

The equation of motion arising from this action is

$$K\phi = 0, \tag{3.2}$$

where the Klein-Gordon operator K is given by

$$K \equiv -g^{ab}\nabla_a\nabla_b + m^2 + \xi R. \tag{3.3}$$

To guarantee well-defined dynamics and to avoid causal pathologies, we will restrict consideration throughout to globally-hyperbolic spacetimes, (M, g_{ab}). Any globally-hyperbolic spacetime admits unique advanced, Δ^{adv}, and retarded, Δ^{ret}, Green's distributions of the Klein-Gordon operator, K [28].

In this chapter, we consider the quantum field theory of the Klein-Gordon field. Our main concern is the ambiguities in the definition of arbitrary Wick monomials, i.e., quantum field observables of the form

$$\Phi_A \equiv \nabla_{\alpha_1}\phi \cdots \nabla_{\alpha_p}\phi. \tag{3.4}$$

Here α_i denotes an abstract multi-index, i.e., $\alpha_i = a_{i,1}\dots a_{i,|\alpha_i|}$ where each $a_{i,j}$ is a spacetime index. Thus, Φ_A corresponds to a tensor constructed from p-factors of ϕ, with $|\alpha_i|$-number of derivatives on the i-th factor. The ambiguities in Φ_A will give rise to corresponding ambiguities in the n-point distributions,

$$\left\langle \Phi_{A_1}(f_1) \cdots \Phi_{A_n}(f_n) \right\rangle_{\Psi}, \tag{3.5}$$

as well as the n-point distributions for the un-extended time-ordered products. This will, in turn, give rise to corresponding ambiguities in the OPE coefficients. The main result of this chapter will be to obtain a simple characterization of the ambiguities in the definition of Wick monomials which will be extremely useful for characterizing the corresponding ambiguities in the OPE coefficients derived in the next section.

In Section 3.1, we review the construction of the abstract algebra[1] containing Wick polynomials and the requirements ("axioms") imposed on the Wick monomials. The known uniqueness theorem for Wick monomials implied by these axioms (see Theorem 1) is then reformulated in Section 3.2 in terms of a choice of smooth functions F_n (see Proposition 1).

3.1 Wick algebra and state space: axioms and existence of Wick polynomials

In this section, we review the definition of the algebra of observables $\mathcal{W}(M, g_{ab})$ for the Klein-Gordon field and the axioms that determine the Wick monomials—up to the uniqueness discussed in the following section. Our discussion closely follows [7] which built on the earlier work of [31–34].

The construction of $\mathcal{W}(M, g_{ab})$ begins with the standard CCR (canonical commutation relation) algebra $\mathcal{A}(M, g_{ab})$ generated by observables that are linear in ϕ. To define $\mathcal{A}$, we start with the free $*$-algebra $\mathcal{A}_0$ generated by the identity I and the fundamental (smeared) field $\phi(f)$ with $f \in C_0^\infty(M)$. We then factor $\mathcal{A}_0$ by all of the relations we wish to impose.

1. The algebraic approach to quantum field theory was initiated in [29]. A comprehensive review may be found in [30, Chapter III].

To do so, we let $\mathcal{I} \subset \mathcal{A}_0$ be the two-sided ideal consisting of all elements in $\mathcal{A}_0$ that contain at least one factor that can be put into any of the following forms:

i) $\phi(c_1 f_1 + c_2 f_2) - c_1\phi(f_1) - c_2\phi(f_2)$, with $c_1, c_2 \in \mathbb{C}$

ii) $\phi(f)^* - \phi(\overline{f})$

iii) $\phi(Kf)$, with the Klein-Gordon operator K given by eq. (3.3).

iv) $\phi(f_1)\phi(f_2) - \phi(f_2)\phi(f_1) - i\Delta(f_1, f_2)I$, where $\Delta[M, g_{ab}]$ denotes the advanced minus retarded Green's distribution for $K[g_{ab}, m^2, \xi]$ on M

The algebra $\mathcal{A}$ is then defined to be the free algebra factored by this ideal,

$$\mathcal{A}(M, g_{ab}) \equiv \mathcal{A}_0/\mathcal{I}(M, g_{ab}). \tag{3.6}$$

Thus, the CCR algebra effectively incorporates (i) the distributional nature of quantum fields, (ii) the Hermiticity of real-valued fields, (iii) the Klein-Gordon field equation, and (iv) the canonical commutation relations. It contains all elements that are finite linear combinations of products of the (smeared) fundamental field. Quantum states of the CCR algebra are then just linear maps $\langle \cdot \rangle_\Psi : \mathcal{A}(M, g_{ab}) \to \mathbb{C}$ which are normalized, $\langle I \rangle_\Psi = 1$, and positive, $\langle A^* A \rangle_\Psi \geq 0$ for all $A \in \mathcal{A}$.

The first step towards enlarging $\mathcal{A}(M, g_{ab})$ to the full algebra of observables $\mathcal{W}(M, g_{ab})$ is to define the normal-ordered product relative to a state $\langle \cdot \rangle_\Psi$ by the formula

$$:\phi(f_1)\cdots\phi(f_n):_\Psi \equiv \sum_P (-1)^{|P|} \prod_{(i,j)\in P} \langle \phi(f_i)\phi(f_j) \rangle_\Psi \prod_{k\in\{1,\ldots,n\}\setminus P} \phi(f_k), \tag{3.7}$$

where the P are sets containing disjoint, ordered pairs taken from $\{1, \ldots, n\}$ such that $i < j$, and $|P|$ denotes the number of pairs in P. Note that normal-ordered elements (3.7) of $\mathcal{A}$ are symmetric under interchange of test functions, i.e., $:\phi(f_1)\cdots\phi(f_n):_\Psi =:$

$\phi(f_{\pi(1)})\cdots\phi(f_{\pi(n)}) :_\Psi$ for any permutation π. Products of normal-ordered elements also satisfy the following important identity ("Wick's theorem"),

$$\begin{aligned} &: \phi(f_1)\cdots\phi(f_n) :_\Psi : \phi(f_{n+1})\cdots\phi(f_{n+m}) :_\Psi \\ &= \sum_{p\leq \min(n,m)} \prod_{(i,j)\in P_p} \big\langle \phi(f_i)\phi(f_j)\big\rangle_\Psi : \prod_{k\in\{1,\ldots,n\}\setminus P_p} \phi(f_k) :_\Psi, \end{aligned} \tag{3.8}$$

where P_p denote a set containing p disjoint, ordered pairs (i,j) such that $i \in \{1,2,\ldots,n\}$ and $j \in \{n+1, n+2, \ldots, n+m\}$. Noting that $: \phi(f) :_\Psi = \phi(f)$, it follows from this identity that normal-ordered elements, in fact, comprise a basis of the CCR algebra in the sense that any element of $\mathcal{A}(M, g_{ab})$ can be expressed via (3.8) as a linear combination of terms of the form (3.7) (see (B.18) for an explicit formula).

It is useful to view $: \phi(f_1)\cdots\phi(f_n) :_\Psi$ as mapping $t_n = f_{(1}\otimes f_2 \otimes \cdots \otimes f_{n)}$ into $\mathcal{A}(M, g_{ab})$. We write

$$\begin{aligned} W_n(t_n) &= : \phi(f_1)\cdots\phi(f_n) :_\Psi \\ &= \int_{\times^n M} d\mu_g(x_1)\cdots d\mu_g(x_n) : \phi(x_1)\cdots\phi(x_n) :_\Psi t_n(x_1,\ldots,x_n), \end{aligned} \tag{3.9}$$

where $d\mu_g(x) \equiv d^D x\sqrt{-g(x)}$. Similarly, denote by $u_m \equiv f_{(n+1} \otimes f_{n+2} \otimes \cdots \otimes f_{n+m)}$ another symmetrized tensor product of smooth test functions. In this notation, we may write eq. (3.8) as

$$W_n(t_n)W_m(u_m) = \sum_{k\leq \min(n,m)} W_{n+m-2k}(t_n \otimes_k u_m), \tag{3.10}$$

where we define, for $n, m \geq k$,

$$(t_n \otimes_k u_m)(x_1, \ldots x_{n+m-2k}) \equiv \tag{3.11}$$
$$\frac{n!m!}{k!((n-k)!)^2((m-k)!)^2} \int_{y_1 \cdots y_{2k}} \Big[\langle \phi(y_1)\phi(y_2) \rangle_\Psi \cdots \langle \phi(y_{2k-1})\phi(y_{2k}) \rangle_\Psi \times$$
$$\times \sum_{\pi \in \Pi_k} \Big[t_n(y_1, y_3, \ldots, y_{2k-1}, x_{\pi(1)}, x_{\pi(2)}, \ldots, x_{\pi(n-k)}) \times$$
$$\times\, u_m(y_2, y_4, \ldots, y_{2k}, x_{\pi(n-k+1)}, \ldots, x_{\pi(n+m-2k)}) \Big] \Big],$$

where we abbreviate,

$$\int_{y_1 \cdots y_{2k}} \equiv \int_{\times^{2k} M} d\mu_g(y_1) \cdots d\mu_g(y_{2k}), \tag{3.12}$$

and where Π_k denotes any permutation of $\{1, \ldots, n+m-2k\}$ such that $\pi(1) < \pi(2) < \cdots < \pi(n-k)$ and $\pi(n-k+1) < \pi(n-k+2) < \cdots < \pi(n+m-2k)$. Note (3.11) is symmetric in $(x_1, x_2, \ldots, x_{n+m-2k})$.

We now require Ψ to be a Hadamard state, i.e a state whose two-point distribution $\Psi_2(f_1, f_2) \equiv \langle \phi(f_1)\phi(f_2) \rangle_\Psi$ has a wavefront set of the form:

$$\mathrm{WF}[\Psi_2] = \Big\{ (x_1, k_1; x_2, k_2) \in \times^2 (T^*M \backslash Z^*M) | (x_1, k_1) \sim (x_2, -k_2), k_1 \in \dot{V}^+_{x_1} \Big\}. \tag{3.13}$$

Here Z^*M denotes the zero section of the cotangent bundle T^*M and $\dot{V}^\pm_x$ denotes the boundary of the future/past lightcone of x. The relation $(x_1, k_1) \sim (x_2, k_2)$ is satisfied iff x_1 and x_2 can be joined by a null-geodesic with respect to which the covectors k_1 and k_2 are cotangent and coparallel. In any convex normal neighborhood, the two-point distribution of a Hadamard state takes the form[2]:

$$\Psi_2(x_1,x_2) = \frac{U(x_1,x_2)}{\left[\sigma(x_1,x_2) + 2i0^+(T(x_1)-T(x_2)) + (0^+)^2\right]^{D/2-1}} + \\ + V(x_1,x_2)\log\left[\ell^{-2}\sigma(x_1,x_2) + 2i0^+(T(x_1)-T(x_2)) + (0^+)^2\right] + W_\Psi(x_1,x_2), \tag{3.14}$$

where T is any local time function; σ is the (signed) squared geodesic distance[3] between points x_1 and x_2; ℓ is an arbitrary length scale; and U, V and W_Ψ are smooth symmetric functions. If D is odd, then $V = 0$. Moreover, U and V are independent of the Hadamard state Ψ and are locally and covariantly determined by the Hadamard recursion relations[4]. It is known that there exist Hadamard states on $\mathcal{A}(M, g_{ab})$ for any globally-hyperbolic spacetime (M, g_{ab}).

Thus far, we have merely rewritten the product rules of $\mathcal{A}(M, g_{ab})$ in terms of normal-ordered products. The enlargement of the algebra $\mathcal{A}(M, g_{ab})$ to the desired algebra $\mathcal{W}(M, g_{ab})$ is accomplished by recognizing that for Hadamard states, eq. (3.11) makes sense not merely when t_n and u_m are products of test functions but also when they are distributions of the following type: Denote by $\mathcal{V}_n(M, g_{ab})$ the set of all elements of the (product) cotangent bundle $\times^n T^*M$ that are entirely contained within either the future or past lightcones,

$$\mathcal{V}_n(M, g_{ab}) \equiv \left\{(x_1, k_1; x_2, k_2; \ldots; x_n k_n) \in \times^n T^*M \,|\, (k_i \in V^+_{x_i}, \forall i \in n) \text{ or } (k_i \in V^-_{x_i}, \forall i \in n)\right\} \tag{3.15}$$

Let $\mathcal{E}'(\times^n M, g_{ab})$ denote the space of compactly-supported symmetric distributions $\mathcal{D}'_0(\times^n M)$

2. For states of the CCR algebra $\mathcal{A}$, the equivalence of the microlocal spectral version (3.13) of the Hadamard condition and the position-space version (3.14) was established by Radzikowski in [35, Theorem 5.1].

3. i.e., σ is equal to twice the "Synge bi-scalar/world function".

4. More precisely, all of the derivatives of U and V at coincidence $x_1 = x_2$ are uniquely as well as locally and covariantly determined by the fact that $K\Psi_2 =$ smooth, with the Klein-Gordon operator K, see eq. (3.3), acting on either spacetime variable.

whose wavefront sets do not intersect $\mathcal{V}_n(M, g_{ab})$,

$$\mathcal{E}'(\times^n M, g_{ab}) \equiv \left\{ t \in \mathcal{D}'_0(\times^n M) \,|\, n \in \mathbb{N} \text{ and } \mathrm{WF}(t) \cap \mathcal{V}_n(g_{ab}, M) = \emptyset \right\}. \tag{3.16}$$

Then formula (3.11) is well defined whenever t_n and u_m are distributions in $\mathcal{E}'$ [7, Theorem 2.1]. This means that we can extend the algebra $\mathcal{A}(M, g_{ab})$ to an algebra $\mathcal{W}(M, g_{ab})$ generated by quantities of the form $W(t_n)$ for all $t_n \in \mathcal{E}'$, with product rule given by eq. (3.10). An example of such a distribution in $\mathcal{E}'$ is $t_n = f(x_1)\delta(x_1, \dots, x_n)$. By eq. (3.9), $W(t_n)$ corresponds to $: \phi^n :_\Psi (f)$. Thus, $\mathcal{W}(M, g_{ab})$ includes elements corresponding to the normal-ordered powers of the field. More generally, it includes all normal-ordered monomials, $: \nabla_{\alpha_1}\phi \cdots \nabla_{\alpha_n}\phi :_\Psi (f^{\alpha_1 \cdots \alpha_n})$, where the α_i denote multi-spacetime-indices and $f^{\alpha_1 \cdots \alpha_n}$ denotes a test tensor field. For notational convenience, we will typically suppress the multi-indices of $f^{\alpha_1 \cdots \alpha_n}$ and write $: \nabla_{\alpha_1}\phi \cdots \nabla_{\alpha_n}\phi :_\Psi (f)$, with it always being understood that f is a tensor field dual to the tensor $: \nabla_{\alpha_1}\phi \cdots \nabla_{\alpha_n}\phi :_\Psi$. Note that all Hadamard states on $\mathcal{A}(M, g_{ab})$ can be naturally extended to states on $\mathcal{W}(M, g_{ab})$. Furthermore, it can be shown that the only continuous states on $\mathcal{W}(M, g_{ab})$ are Hadamard states [36].

The above construction of $\mathcal{W}(M, g_{ab})$ made use of a choice of Hadamard state Ψ. However, it is not difficult to show that, as an abstract algebra, $\mathcal{W}(M, g_{ab})$ does not depend on the choice of Ψ [7, see Lemma 2.1]. Nevertheless, normal-ordered quantities such as $: \phi^n :_\Psi$ do depend on the choice of Ψ for any $n > 1$, i.e., $: \phi^n :_{\Psi'} \neq : \phi^n :_\Psi$ if $\Psi' \neq \Psi$. Which quantity should represent the true field observable ϕ^n and other Wick monomials? In fact, when $n > 1$, $: \phi^n :_\Psi$ for any choice of Hadamard state Ψ is not a suitable candidate to represent ϕ^n since it does not satisfy the requirement of being locally and covariantly defined. Following [7, 37], we determine the Wick monomials by imposing the requirements ("axioms") on their definition. Existence of a definition of Wick monomials satisfying these axioms can then be proven. We will consider the allowed freedom (i.e., non-uniqueness) in the definition of the Wick monomials in the next section.

The following are our axioms[5] for Wick monomials:

W1 Local and covariant The Wick monomials are required to be "local and covariant" in the following sense. Let (M, g_{ab}) and (M', g'_{ab}) denote two globally-hyperbolic spacetimes. Suppose $\psi : M \to M'$ is an isometric embedding (i.e., $g_{ab} = \psi^* g'_{ab}$, where ψ^* denotes the pullback by ψ) that also is causality-preserving: i.e., $\psi(x_1), \psi(x_2) \in M'$ is connected by a causal curve only if $x_1, x_2 \in M$ is connected by a causal curve. Then, as shown in [7, Lemma 3.1], there is a canonical injective unital $*$-homomorphism $\alpha_\psi : \mathcal{W}(M, g_{ab}) \to \mathcal{W}(M', g'_{ab})$. We demand that the definition of any Wick monomial $\Phi_A(f) = (\nabla_{\alpha_1}\phi \cdots \nabla_{\alpha_n}\phi)(f)$ be such that, under this homomorphism, we have $\alpha_\psi [\Phi_A(f)] \mapsto \Phi_A(\psi_* f)$, where f is a test tensor field on M dual to Φ_A and $\psi_* f$ is the push-forward of f via ψ.

W2 Smoothness and joint smoothness For any Wick monomial Φ_A and for any Hadamard state $\langle \cdot \rangle_\Psi$, we require that $\mathrm{WF}[\langle \Phi_A \rangle_\Psi] = \emptyset$, i.e., that $\langle \Phi_A(x) \rangle_\Psi$ is smooth. Furthermore, we require that this quantity be jointly smooth in x, the spacetime metric, and the parameters m^2 and ξ. To define this notion, we must first allow m^2 and ξ to have spacetime dependence. We then consider one parameter variations $g_{ab}(s_1)$, $m^2(s_2)$, and $\xi(s_3)$ in a compact spacetime region $\mathcal{R}$, such that $(M, g_{ab}(s_1))$ is globally hyperbolic for all s_1. As shown in [7, Lemma 4.1], we may naturally identify the algebra $\mathcal{W}$ associated with $(g_{ab}(s_1), m^2(s_2), \xi(s_3))$ with the algebra associated with $(g_{ab}(0), m^2(0), \xi(0))$ by identifying these algebras on a Cauchy surface lying outside the future of $\mathcal{R}$. Consequently, we may identify a Hadamard state $\langle \cdot \rangle_\Psi$ on the algebra for $(g_{ab}(0), m^2(0), \xi(0))$ with a Hadamard state on the algebra associated with $(g_{ab}(s_1), m^2(s_2), \xi(s_3))$. For any Hadamard state $\langle \cdot \rangle_\Psi$, for any Wick monomial Φ, and for any family $(g_{ab}(s_1), m^2(s_2), \xi(s_3))$ as above, we require that $\left\langle \Phi_A[g_{ab}(s_1), m^2(s_2), \xi(s_3)](x) \right\rangle_\Psi$ be jointly smooth in (x, s_1, s_2, s_3).

5. These axioms differ from the ones originally given in [7] in that the Leibniz rule W4 and the conservation of stress-energy W8 have been added as in [37]. In addition, the analytic dependence condition of [7,37] has been replaced by the joint smoothness condition of [38,39].

W3 Commutator The commutator of any Wick monomial $\Phi_A = \nabla_{\alpha_1}\phi \cdots \nabla_{\alpha_n}\phi$ with the fundamental field ϕ is given by,

$$[(\nabla_{\alpha_1}\phi \cdots \nabla_{\alpha_n}\phi)(f_1), \phi(f_2)] \\ = i\sum_{i=1}^{n}(\nabla_{\alpha_1}\phi \cdots \widehat{\nabla_{\alpha_i}\phi} \cdots \nabla_{\alpha_n}\phi)(f_1)\Delta\left((-1)^{|\alpha_i|}\nabla_{\alpha_i^T} f_1, f_2\right), \tag{3.17}$$

where $\Delta = \Delta^{\mathrm{adv}} - \Delta^{\mathrm{ret}}$ is the advanced minus retarded Green's function, $\widehat{\nabla_{\alpha_i}\phi}$ denotes the omission of the $\nabla_{\alpha_i}\phi$ factor and for the multi-index $\alpha \equiv a_1 a_2 \cdots a_{|\alpha|}$, we use the notation $\alpha^T \equiv a_{|\alpha|} a_{|\alpha|-1} \cdots a_1$.

W4 Leibniz rule Any Wick monomial $\Phi_A = \nabla_{\alpha_1}\phi \cdots \nabla_{\alpha_n}\phi$ must satisfy the Leibniz rule in the sense that

$$(\nabla_{\alpha_1}\phi \cdots \nabla_{\alpha_n}\phi)(-\nabla_a f) = ((\nabla_a \nabla_{\alpha_1})\phi \cdots \nabla_{\alpha_n}\phi)(f) + \cdots + (\nabla_{\alpha_1}\phi \cdots (\nabla_a \nabla_{\alpha_n})\phi)(f). \tag{3.18}$$

Here, the left-hand side of this equation is the distributional derivative of Φ_A, whereas the right-hand side is what one would obtain by applying the Leibniz rule to the classical expression $\Phi_A = \nabla_{\alpha_1}\phi \cdots \nabla_{\alpha_n}\phi$.

W5 Hermiticity All Wick monomials are required to be Hermitian in the sense that,

$$(\nabla_{\alpha_1}\phi \cdots \nabla_{\alpha_n}\phi)(f)^* = (\nabla_{\alpha_1}\phi \cdots \nabla_{\alpha_n}\phi)(\overline{f}). \tag{3.19}$$

W6 Symmetry Any Wick monomial is required to be symmetric under interchange of the fields—i.e.,

$$(\nabla_{\alpha_{\pi(1)}}\phi \cdots \nabla_{\alpha_{\pi(n)}}\phi)(f) = (\nabla_{\alpha_1}\phi \cdots \nabla_{\alpha_n}\phi)(f), \tag{3.20}$$

for all permutations π of $\{1, \ldots, n\}$.

W7 Scaling For $\lambda > 0$, let $\sigma_\lambda : \mathcal{W}(M, \lambda^{-2} g_{ab}, \lambda^2 m^2, \xi) \to \mathcal{W}(M, g_{ab}, m^2, \xi)$ be the canonical $*$-isomorphism defined in [7, Lemma 4.2]. The "scaling dimension" d_A of any local, covariant field Φ_A is defined to be the smallest real number δ such that

$$\lim_{\lambda \to 0^+} \lambda^{(D-\delta)} \sigma_\lambda \left[\Phi_A[\lambda^{-2} g_{ab}, \lambda^2 m^2, \xi] \right] (f) = 0, \tag{3.21}$$

for all (g_{ab}, m^2, ξ). The factor of λ^D accounts for the fact that the volume element scales as $d\mu_{\lambda^{-2} g} = \lambda^D d\mu_g$. We require the Wick monomial Φ_A to have scaling dimension,

$$\begin{aligned} d_A = \frac{(D-2)}{2} \times \#(\text{factors of } \phi) + \#(\text{derivatives}) + 2 \times \#(\text{factors of } m^2) + \\ + 2 \times \#(\text{factors of curvature}) + \#(\text{"up" indices}) - \#(\text{"down" indices}). \end{aligned} \tag{3.22}$$

For example, $(\nabla_a \phi \nabla_b \nabla_c \phi)$ has two factors of ϕ, three derivatives, and three "down" indices, and thus has scaling dimension $D - 2$. As another example, $g_{ab} R^{cd} \nabla_d R \phi$ has scaling dimension $D/2 + 3$, because it has one factor of ϕ, one derivative, two factors of curvature (each "R" counting as a curvature factor), two "up" indices, and three "down" indices. Whereas any "R" denoting a scalar or tensor constructed from the Riemann curvature tensor (and its covariant derivatives) counts as a "curvature factor", note the spacetime metric does *not* count as a "curvature factor" for the purposes of formula (3.22). We further require that Φ_A scale homogeneously up to logarithms: i.e., there must exist finite N such that,

$$\frac{\partial^N}{\partial (\log \lambda)^N} \left[\lambda^{(D-d_A)} \sigma_\lambda \left[\Phi_A[\lambda^{-2} g_{ab}, \lambda^2 m^2, \xi \right] (f) \right] = 0. \tag{3.23}$$

W8 Conservation of stress-energy The stress-energy tensor, $T_{ab}(f) \in \mathcal{W}(M, g_{ab})$, is given by

$$T_{ab} = (1-2\xi)(\nabla_a\phi\nabla_b\phi) + \left(2\xi - \frac{1}{2}\right) g_{ab}(\nabla^c\phi\nabla_c\phi) + \\ +2\xi g_{ab}(\phi\nabla^c\nabla_c\phi) - 2\xi(\phi\nabla_a\nabla_b\phi) + \left(\xi G_{ab} - \frac{1}{2}m^2 g_{ab}\right)\phi^2, \tag{3.24}$$

where $G_{ab} \equiv R_{ab} - \frac{1}{2}g_{ab}R$ is the Einstein tensor. We require that T_{ab} is divergence free,

$$0 = T_{ab}(-\nabla^a f) = -(\nabla_b\phi K\phi)(f), \tag{3.25}$$

where $K = K[g_{ab}, m^2, \xi]$ is the Klein-Gordon operator, eq. (3.3), and the second equality in (3.25) follows straightforwardly from differentiating (3.24) and using the Leibniz and symmetry axioms.

Remark 5. Note that even in flat spacetime where $G_{ab} = 0$, the stress-energy tensor (3.24) has nontrivial dependence on the curvature coupling ξ. However, the conservation constraint (3.25) is independent of ξ in any region with vanishing Ricci scalar curvature, since $K[g_{ab}, m^2, \xi = 0] = K[g_{ab}, m^2, \xi]$ at any spacetime point x where $R(x) = 0$.

If we wished to define Wick monomials by normal ordering with respect to a Hadamard state, we would have to choose a Hadamard state $\Psi(M, g_{ab})$ for each globally hyperbolic spacetime (M, g_{ab}). However, as we have already mentioned above, it can be shown [7, see Section 3] that no choice of $\Psi(M, g_{ab})$ can give rise to a prescription for Wick monomials that satisfies the local and covariant condition, W1. Nevertheless, a construction of Wick monomials satisfying all of our requirements W1-W8 can be given by normal ordering with respect to a locally and covariantly constructed Hadamard parametrix, $H(x_1, x_2)$, rather than a Hadamard state. We define $H(x_1, x_2)$ in a sufficiently small neighborhood of the diagonal $x_1 = x_2$ by,

$$H(x_1, x_2) = \frac{U(x_1, x_2)}{\left[\sigma(x_1, x_2) + 2i0^+(T(x_1) - T(x_2)) + (0^+)^2\right]^{D/2-1}} + \\ + V(x_1, x_2) \log\left[\ell^{-2}\sigma(x_1, x_2) + 2i0^+(T(x_1) - T(x_2)) + (0^+)^2\right], \tag{3.26}$$

where the quantities appearing in this equation are defined as in eq. (3.14). Thus, $H(x_1, x_2)$ differs from the two-point function of any Hadamard state, Ψ, by a state-dependent, smooth, symmetric function $W_\Psi(x_1, x_2)$. We refer to $H(x_1, x_2)$ as a "parametrix" because, although it does *not* satisfy the Klein-Gordon equation in either variable, its failure to satisfy the Klein-Gordon equation is smooth. We define the normal-ordered product of field operators with respect to H by,

$$:\phi(x_1)\cdots\phi(x_n):_H \equiv \sum_P (-1)^{|P|} \prod_{(i,j)\in P} H(x_i, x_j) \prod_{k\in\{1,\ldots,n\}\setminus P} \phi(x_k), \tag{3.27}$$

i.e., by the same formula as in eq. (3.7) but with the two-point function, $\langle\phi(x_i)\phi(x_j)\rangle_\Psi$, of a state, Ψ, replaced by the Hadamard parametrix $H(x_i, x_j)$. Note that the Hadamard normal-ordered elements satisfy Wick's theorem (3.10) with, again, $\langle\phi(x_i)\phi(x_j)\rangle_\Psi$ replaced by $H(x_i, x_j)$ in eq. (3.11). Using H, we define the Wick monomial corresponding to $\nabla_{\alpha_1}\phi\cdots\nabla_{\alpha_n}\phi$ by,

$$\begin{aligned} \Phi_A^H(f) &\equiv (\nabla_{\alpha_1}\phi\cdots\nabla_{\alpha_n}\phi)_H(f) \\ &\equiv \int_{y,x_1,x_2,\ldots,x_n} :\phi(x_1)\cdots\phi(x_n):_H \, t_{n+1}[f](y, x_1, \ldots, x_n), \end{aligned} \tag{3.28}$$

with $t_{n+1}[f]$ given by,

$$t_{n+1}[f](y, x_1, \ldots, x_n) = f^{\alpha_1\cdots\alpha_n}(y)(-1)^{[A]_\nabla} \nabla^{(x_1)}_{\alpha_1^T} \cdots \nabla^{(x_n)}_{\alpha_n^T} \delta(y, x_1, \ldots, x_n), \tag{3.29}$$

where we define $[A]_\nabla \equiv \sum_{i=1}^n |\alpha_i|$ and the abbreviation $\int_{y,x_1,\ldots,x_n}$ is defined as in (3.12) and our "Notation and conventions" in Chapter 2. In contrast to normal ordering defined with respect to a Hadamard state, the prescription (3.28) for Φ_H given by normal ordering with respect to the locally and covariantly constructed Hadamard parametrix eq. (3.26) satisfies requirement W1. It also satisfies [37] requirements W2-W7 for Wick monomials[6].

However, the failure of H to be an exact solution of the Klein-Gordon wave equation implies this prescription generally does not satisfy requirement W8,

$$(\nabla_b \phi K \phi)_H(f) = \int d\mu_g(y) f(y) \nabla_b^{(x_1)} K_{x_2} H(x_1, x_2)|_{x_1,x_2=y} \neq 0. \tag{3.30}$$

Odd dimensions are an exception: For D odd, formula (3.26) contains only half-integer powers of $\sigma(x_1, x_2)$, so it follows that for $U(x_1, x_2)$ smooth, $H(x_1, x_2)$ is a parametrix of the Klein-Gordon equation only if,

$$K_{x_2} H(x_1, x_2)|_{x_1,x_2=y} = 0. \tag{3.31}$$

Furthermore, it can be shown [40, Lemma 2.1] that,

$$\nabla_b^{(x_1)} K_{x_2} H(x_1, x_2)|_{x_1,x_2=y} = \frac{D}{2(D+2)} \nabla_b^{(y)} \left[K_{x_2} H(x_1, x_2)\right]_{x_1,x_2=y}, \tag{3.32}$$

so (3.31) implies the left-hand side of (3.30) does, in fact, vanish and, thus, W8 is satisfied in all odd dimensions.

In even dimensions, however, $K_{x_2} H(x_1, x_2)|_{x_1,x_2=y}$ yields a curvature scalar which is non-vanishing in general spacetimes and, thus, normal-ordering with respect to the parametrix

6. The proof in [37] used an analytic dependence assumption in place of the joint smoothness condition of [38] that we have used here in our formulation of W2. In order to prove that W2 holds for the Hadamard normal ordered prescription, we would need to show that the Hadamard normal ordered n-point functions, $\langle : \phi(x_1) \cdots \phi(x_n) :_H \rangle_\Psi$, are jointly smooth in the required sense. We do not anticipate any difficulties in proving this but, as far as we are aware, a proof has not been given in the literature.

(3.26) fails to produce Wick fields satisfying the conservation axiom W8. Nevertheless, we prove in Appendix A that for $D > 2$, there exists a smooth symmetric function $Q(x_1, x_2)$ which is locally and covariantly defined for $x_1 = x_2$ such that

$$\nabla_b^{(x_1)} K_{x_2} H(x_1, x_2)|_{x_1,x_2=y} = -\nabla_b^{(x_1)} K_{x_2} Q(x_1, x_2)|_{x_1,x_2=y} \tag{3.33}$$

Furthermore, Q is smooth in (m^2, ξ) and scales as,

$$Q[\lambda^{-2} g_{ab}, \lambda^2 m^2, \xi] = \lambda^{(D-2)} Q[g_{ab}, m^2, \xi], \tag{3.34}$$

in a sufficiently small neighborhood of $x_1, x_2 = y$. Therefore, normal-ordering instead with respect to the new Hadamard parametrix,

$$H' \equiv H + Q, \tag{3.35}$$

will give a construction of Wick fields satisfying the axioms W1-W8.

It will be understood below that, unless otherwise stated, we are always normal-ordering with respect to a Hadamard parametrix H which is smooth in (m^2, ξ), satisfies

$$\nabla_b^{(x_1)} K_{x_2} H(x_1, x_2)|_{x_1,x_2=y} = 0, \tag{3.36}$$

and scales homogeneously up to logarithms,

$$\lambda^{-(D-2)} H[\lambda^{-2} g_{ab}, \lambda^2 m^2, \xi] = H[g_{ab}, m^2, \xi] + V[g_{ab}, m^2, \xi] \log \lambda^2. \tag{3.37}$$

(Recall $V = 0$ for D odd, so H scales exactly homogeneously in odd spacetime dimensions.) Thus, for any $D > 2$, Hadamard normal ordering yields a prescription for defining Wick monomials that satisfies W1-W8. For $D = 2$, no such Q exists, and condition W8 cannot be satisfied by any prescription that satisfies W1-W7 [37, see Subsection 3.2]. However, Hadamard normal ordering satisfies W1-W7.

We turn our attention now to the characterization of the non-uniqueness of prescriptions satisfying W1-W8 (or W1-W7 for $D = 2$).

3.2 Uniqueness of Wick monomials

In the previous section, we imposed conditions W1-W8 on the definition of Wick monomials and gave a prescription based on "Hadamard normal-ordering" which satisfies these requirements (or requirements W1-W7 for $D = 2$). This prescription is not unique. In this section, we will show that the difference between any two prescriptions Φ_A and $\widetilde{\Phi}_A$ for Wick monomials satisfying W1-W8 (or W1-W7 for $D = 2$) are described by a "mixing matrix" $\mathcal{Z}$ such that

$$\widetilde{\Phi}_A(x) = \sum_B \mathcal{Z}_A^B(x)\Phi_B(x). \tag{3.38}$$

Theorem 1 below explicitly gives the general form of $\mathcal{Z}$ which, thereby, characterizes the freedom to modify any prescription, such as the Hadamard prescription of the previous section.

It will be convenient to use the following notation for $\mathcal{Z}_A^B$. An arbitrary Wick monomial is of the form $\Phi_A = \nabla_{\alpha_1}\phi \cdots \nabla_{\alpha_p}\phi$ and thus is characterized by the multi-indices $\alpha_1, \ldots, \alpha_p$. For $\widetilde{\Phi}_A = \widetilde{\nabla_{\alpha_1}\phi \cdots \nabla_{\alpha_p}\phi}$ and $\Phi_B = \nabla_{\beta_1}\phi \cdots \nabla_{\beta_q}\phi$, we represent $\mathcal{Z}_A^B$ as

$$\mathcal{Z}_A^B = \mathcal{Z}_{\alpha_1 \cdots \alpha_p}^{\beta_1 \cdots \beta_q}. \tag{3.39}$$

Each multi-index, α, is itself a product of spacetime indices, $\alpha = a_1 \cdots a_{|\alpha|}$, so we may, in turn, write $\mathcal{Z}$ as a spacetime tensor field

$$\mathcal{Z}_{\alpha_1 \cdots \alpha_p}^{\beta_1 \cdots \beta_q} = \mathcal{Z}_{\{a_{1,1} \cdots a_{1,|\alpha_1|}\} \cdots \{a_{n,1} \cdots a_{n,|\alpha_p|}\}}^{\{b_{1,1} \cdots b_{1,|\beta_1|}\} \cdots \{b_{k,1} \cdots b_{k,|\beta_q|}\}} \tag{3.40}$$

In this notation, we enclose the spacetime indices corresponding to any given multi-index with a curly bracket. If any multi-index is "empty"—i.e., if any factor of ϕ in the corresponding Wick monomial has no derivatives acting on it, then we insert a "$\{0\}$" as a place-holder. If q is zero, it is understood $\Phi_B = I$ and we simply write "I" in the superscripts of (3.39) and (3.40) as in examples (3.41)-(3.43) below. Similarly, when $p = 0$, it is understood $\Phi_A = I$ and we write "I" in the subscripts of (3.39) and (3.40).

As an example to illustrate this notation, it will follow from the theorem below that the difference between any two prescriptions for Wick monomials that are quadratic in ϕ will be given by a multiple of the identity element, I. In our notation, this would be expressed as

$$(\widetilde{\nabla_{\alpha_1}\phi\nabla_{\alpha_2}\phi})(x) = \sum_{\beta_1,\beta_2} \mathcal{Z}_{\alpha_1\alpha_2}^{\beta_1\beta_2}(x)(\nabla_{\beta_1}\phi\nabla_{\beta_2}\phi)(x) + \mathcal{Z}_{\alpha_1\alpha_2}^{I}(x)I, \tag{3.41}$$

where $\mathcal{Z}_{\alpha_1\alpha_2}^{\beta_1\beta_2} = \delta_{(\alpha_1}^{\beta_1}\delta_{\alpha_2)}^{\beta_2}$ and δ_α^β is the Kronecker delta for the multi-indices defined by $\delta_\alpha^\beta = 1$ if the multi-indices α and β coincide and zero otherwise. As particular examples of (3.41), we have

$$(\widetilde{\nabla_a\phi\nabla_b\nabla_c\phi})(x) = (\nabla_a\phi\nabla_b\nabla_c\phi)(x) + \mathcal{Z}_{\{a\}\{bc\}}^{I}(x)I, \tag{3.42}$$

whereas

$$(\widetilde{\phi\nabla_a\nabla_b\phi})(x) = (\phi\nabla_a\nabla_b\phi)(x) + \mathcal{Z}_{\{0\}\{ab\}}^{I}(x)I. \tag{3.43}$$

With this notation established, we may state our main result in the following theorem. Let $[A]_\phi = p$ and $[B]_\phi = q$ denote the number of factors of ϕ in $\widetilde{\Phi}_A = \widetilde{\nabla_{\alpha_1}\phi\cdots\nabla_{\alpha_p}\phi}$ and $\Phi_B = \nabla_{\beta_1}\phi\cdots\nabla_{\beta_q}\phi$, respectively.

Theorem 1. *The Wick mixing matrix $\mathcal{Z}_A^B$ defined in* (3.38) *is nonzero only when $[B]_\phi \leq [A]_\phi$, i.e. $q \leq p$, and is given in terms of $\mathcal{Z}_A^I$ by,*

$$\mathcal{Z}_{\alpha_1\cdots\alpha_p}^{\beta_1\cdots\beta_q} = \binom{p}{q}\delta_{(\alpha_1}^{\beta_1}\cdots\delta_{\alpha_q}^{\beta_q}\mathcal{Z}_{\alpha_{q+1}\cdots\alpha_p)}^{I}, \tag{3.44}$$

where $\binom{p}{q}$ *denotes the binomial coefficient. Furthermore, we have* $\mathcal{Z}_I^I = 1$ *and* $\mathcal{Z}_{\alpha_1}^I = 0$. *For* $p \geq 2$, *each* $\mathcal{Z}_{\alpha_1 \cdots \alpha_p}^I$ *is a real-valued, smooth tensor field of type* $(0, \sum_{i=1}^p |\alpha_i|)$ *that is symmetric under permutation* π *of multi-indices,*

$$\mathcal{Z}_{\alpha_1 \cdots \alpha_p}^I = \mathcal{Z}_{\alpha_{\pi(1)} \cdots \alpha_{\pi(p)}}^I, \tag{3.45}$$

and is of the form,

$$\mathcal{Z}_A^I = \mathcal{Z}_A^I[g_{ab}, R_{abcd}, \ldots, \nabla_{(e_1} \cdots \nabla_{e_n)} R_{abcd}(x), m^2, \xi]. \tag{3.46}$$

where the right side is a jointly smooth function of its arguments with polynomial dependence on m^2, R_{abcd}, *and finitely many (totally-symmetric) covariant derivatives of* R_{abcd}. *The* $\mathcal{Z}_A^I$ *scale as,*

$$\mathcal{Z}_A^I[\lambda^{-2} g_{ab}, \lambda^2 m^2, \xi] = \lambda^{d_A} \mathcal{Z}_A^I[g_{ab}, m^2, \xi], \tag{3.47}$$

recalling the definition (3.22) *of the scaling dimension* d_A. *Furthermore, the tensor fields* $\mathcal{Z}_A^I$ *satisfy the Leibniz condition,*

$$\nabla_b^{(x)} \mathcal{Z}_{\alpha_1 \cdots \alpha_p}^I(x) = \mathcal{Z}_{\{b\alpha_1\}\alpha_2 \cdots \alpha_p}^I(x) + \mathcal{Z}_{a_1\{b\alpha_2\} \cdots \alpha_p}^I(x) + \cdots + \mathcal{Z}_{\alpha_1 \cdots \{b\alpha_p\}}^I(x), \tag{3.48}$$

where $\{b\alpha\} \equiv b a_1 a_2 \cdots a_{|\alpha|}$ *for* $\alpha \equiv a_1 \cdots a_{|\alpha|}$. *In addition, on account of W8, for* $D > 2$, *the tensor fields* $\mathcal{Z}_{\{b\}\{ac\}}^I$ *and* $\mathcal{Z}_{\{b\}\{0\}}^I$ *must satisfy,*

$$g^{ac} \mathcal{Z}_{\{b\}\{ac\}}^I = (m^2 + \xi R) \mathcal{Z}_{\{b\}\{0\}}^I. \tag{3.49}$$

Conversely, if $\{\Phi_B(x) | B = \beta_1 \cdots \beta_q\}_{q \in \mathbb{N}_0}$ *are any Wick monomials satisfying W1-W8 (or W1-W7 for* $D = 2$*) and* $\mathcal{Z}_A^B$ *satisfy all of the above conditions of this theorem, then the new prescription* $\{\widetilde{\Phi}_A | A = \alpha_1 \cdots \alpha_p\}_{p \in \mathbb{N}_0}$ *defined by eq.* (3.38) *will also satisfy W1-W8 (or*

W1-W7 for $D = 2$). Consequently, the inverse mixing matrix $(\mathcal{Z}^{-1})_A^B$ satisfies the same properties as $\mathcal{Z}_A^B$.

Sketch of Proof. The proof follows [7, Proof of Theorem 5.1], with the main difference being that they did not consider Wick powers involving derivatives and did not impose requirement W4. The key first step is to note that if, inductively, the prescription for Wick monomials involving q-factors of ϕ has been fixed for all $q < p$, then the prescription for any Wick monomial with p-factors of ϕ is uniquely determined by the commutator condition W3 up to the addition of a multiple of the identity I. In our notation, this c-number multiple is denoted by $\mathcal{Z}^I_{\alpha_1 \cdots \alpha_p}$. In particular, eq. (3.41) holds for $p = 2$. We then can prove eq. (3.44) for general p by induction. By condition W6, $\mathcal{Z}^I_{\alpha_1 \cdots \alpha_p}$ must be totally symmetric in its multi-indices. By condition W1, $\mathcal{Z}^I_{\alpha_1 \cdots \alpha_p}$ must be local and covariant, and thus must be constructed from the metric and the Riemann tensor and its derivatives as well as from m^2 and ξ. By the arguments of [38, 39] the joint smoothness requirement, W2, and the scaling requirement, W7, imply polynomial dependence[7] on m^2, R_{abcd}, and finitely many derivatives of R_{abcd}. The remaining properties of $\mathcal{Z}^I_{\alpha_1 \cdots \alpha_p}$ follow directly from the axioms. The verification of the converse is straightforward. □

Remark 6. The fact that $\mathcal{Z}^I_{\alpha_1 \cdots \alpha_p}$ has polynomial dependence on m^2, R_{abcd}, and finitely many of its derivatives and must have the scaling behavior stated in the theorem puts significant constraints on $\mathcal{Z}^I_{\alpha_1 \cdots \alpha_p}$. In particular, (3.47) can hold non-trivially only if $p(D-2)/2$ is even. Hence, $\mathcal{Z}^I_{\alpha_1 \cdots \alpha_p} = 0$ when p is odd and $D \neq 2 + 4k$ for integer k. Furthermore, if D is odd, then we also have $\mathcal{Z}^I_{\alpha_1 \cdots \alpha_p} = 0$ whenever $p = 2 + 4k$.

Remark 7. For the purpose of proving Theorem 3 in Section 4.2, it is useful to note the Wick

7. The corresponding result was obtained in [7, Theorem 5.1] by imposing an additional analytic variation requirement, which we do not impose here.

mixing matrices $\mathcal{Z}_A^B$ satisfy the following recursion relation, for any $r \leq q$,

$$\mathcal{Z}_{\alpha_1 \cdots \alpha_p}^{\beta_1 \cdots \beta_q} = \binom{p}{r} \binom{q}{r}^{-1} \delta_{(\alpha_1}^{\beta_1} \cdots \delta_{\alpha_r}^{\beta_r} \mathcal{Z}_{\alpha_{(r+1)} \cdots \alpha_p)}^{\beta_{(r+1)} \cdots \beta_q}. \tag{3.50}$$

This identity is immediately established by plugging the expression (3.44) for $\mathcal{Z}_A^B$ into both sides of (3.50) and noting,

$$\binom{p}{r} \binom{q}{r}^{-1} \binom{p-r}{q-r} = \binom{p}{q}. \tag{3.51}$$

We now prove the following result that will enable us to characterize in a simple and direct manner the freedom in the prescription for defining Wick monomials specified by Theorem 1. This new characterization will be very useful for characterizing the freedom of the OPE coefficients for products of Wick monomials.

Proposition 1. *For each $n \geq 2$, there exists a smooth, real-valued function $F_n(x_1, \ldots, x_n; z)$ on some neighborhood of $\times^{n+1} M$ containing $(z, \ldots, z)$ such that F_n is symmetric in $(x_1, \ldots, x_n)$ and such that the coefficients $\mathcal{Z}_{\alpha_1 \cdots \alpha_n}^I$ of eq. (3.44) are given by,*

$$\mathcal{Z}_{\alpha_1 \cdots \alpha_n}^I(z) = \nabla_{\alpha_1}^{(x_1)} \cdots \nabla_{\alpha_n}^{(x_n)} F_n(x_1, \ldots, x_n; z)|_{x_1, \ldots, x_n = z}. \tag{3.52}$$

Furthermore, F_n satisfy,

$$\left[\nabla_{\alpha_1}^{(x_1)} \cdots \nabla_{\alpha_n}^{(x_n)} \nabla_\beta^{(z)} F_n(x_1, \ldots, x_n; z)\right]_{x_1, \ldots, x_n = z} = 0. \tag{3.53}$$

Sketch of proof. Let x be in a normal neighborhood of $z \in M$ and let $\sigma(x, z)$ denote the (signed) squared geodesic distance between z and x. Let

$$\sigma_a(x, z) \equiv \frac{1}{2} \nabla_a^{(z)} \sigma(x, z). \tag{3.54}$$

Note that in flat spacetime, in global inertial coordinates, we have

$$\sigma^{\mu}(x,z) = -(x^{\mu} - z^{\mu}). \tag{3.55}$$

Let $f : M \to \mathbb{R}$ be smooth at z. Then the covariant Taylor expansion of f at z is given by [41, see "Addendum to chapter 4: derivation of covariant Taylor expansions"]

$$f(x) \sim \sum_{k} \frac{(-1)^k}{k!} \nabla_{a_1} \cdots \nabla_{a_k} f(x)\big|_{x=z} \sigma^{a_1}(x,z) \cdots \sigma^{a_k}(x,z), \tag{3.56}$$

where the meaning of this equation is that if the sum on the right side is taken from $k = 0$ to $k = N$, then its difference with the right side in any coordinates vanishes to order $(x - z)^N$. Note that $\sigma^{a_1} \cdots \sigma^{a_k} = \sigma^{(a_1} \cdots \sigma^{a_k)}$, so only the totally-symmetric part of f's covariant derivatives contribute non-trivially to (3.56). We may write this equation more compactly as,

$$f(x) \sim \sum_{\beta} \nabla_{\beta} f(x)\big|_{x=z} S^{\beta}(x,z) \tag{3.57}$$

where the sum ranges over all multi-indices β and we have written,

$$S^{\{b_1 \cdots b_{|\beta|}\}}(x;z) \equiv \frac{(-1)^{|\beta|}}{|\beta|!} \sigma^{b_1}(x,z) \cdots \sigma^{b_{|\beta|}}(x,z). \tag{3.58}$$

Note that in flat spacetime in global inertial coordinates, we have

$$S^{\{\mu_1 \cdots \mu_k\}}(x;z) = \frac{1}{k!}(x^{\mu_1} - z^{\mu_1}) \cdots (x^{\mu_n} - z^{\mu_k}) \tag{3.59}$$

Applying the operator $\nabla^{(x)}_{\alpha}$ to both sides of (3.56) and evaluating at $x = z$ should yield the trivial identity, $\nabla^{(z)}_{\alpha} f(z) = \nabla^{(z)}_{\alpha} f(z)$. This will be the case, in general, if and only if,

$$\sum_{|\beta| \leq |\alpha|} \nabla^{(x)}_{\alpha} S^{\beta}(x,z)|_{x=z} \nabla^{(z)}_{\beta} = \nabla^{(z)}_{\alpha}, \tag{3.60}$$

when applied to any smooth scalar field[8]. It follows that if the multivariable series,

$$\sum_{\beta_1\cdots\beta_n} \mathcal{Z}^I_{\beta_1\cdots\beta_n}(z) S^{\beta_1}(x_1;z)\cdots S^{\beta_n}(x_n;z), \tag{3.62}$$

were to converge to a smooth function of $(x_1,\ldots,x_n;z)$, then this function would satisfy eq. (3.52) by construction. However, there is no reason why the series (3.62) need converge. Nevertheless, it is always possible to modify the series (3.62) away from $x_1,\ldots,x_n = z$ so as to render it convergent and C^∞, while preserving the desired identity (3.52). To see this, fix z, choose a tetrad at z, and let $U_z \subset M$ be a convex normal neighborhood of z. In Riemannian normal coordinates x^μ centered at z and based on this tetrad, the (non-convergent) series (3.62) takes the form,

$$\sum_{\beta_1\cdots\beta_n} \mathcal{Z}^I_{\beta_1\cdots\beta_n}(\vec{0})\, x_1^{\beta_1}\cdots x_n^{\beta_n}, \tag{3.63}$$

with $x^\beta \equiv x^{\mu_1}\cdots x^{\mu_{|\beta|}}$. By Borel's Lemma [3, see Corollary 1.3.4], every power series is the Taylor series of some smooth function, so we may always construct $F_n \in C^\infty(\times^n\mathbb{R}^D)$ such that,

$$\begin{aligned}\partial^{(x_1)}_{a_{1,1}}\cdots\partial^{(x_1)}_{a_{1,k_1}}\cdots\partial^{(x_n)}_{a_{n,1}}\cdots\partial^{(x_n)}_{a_{n,k_n}} F_n(x_1,\ldots,x_n)|_{x_1,\ldots,x_n=\vec{0}}&\\ = \mathcal{Z}^I_{\{(a_{1,1}\cdots a_{1,k_1})\}\cdots\{(a_{n,1}\cdots a_{n,k_n})\}}(\vec{0}),&\end{aligned} \tag{3.64}$$

where we note the equality of mixed partials and the index symmetry of the terms which

8. Of course, for any finite $|\alpha|$, this identity could (with substantial computational labor) alternatively be directly derived from the values of the differentiated geodesic distance function $\sigma(x,z)$ at coincidence $x=z$. In global inertial coordinates in flat spacetime, the identity (3.60) holds if and only if,

$$\nabla^{(y)}_\alpha S^\beta(x;z)|_{x=z} = \delta^\beta_\alpha, \tag{3.61}$$

since covariant derivatives commute in this case. Note the identity (3.61) can be directly verified using formula (3.59) for S^β in flat spacetime. However, in curved spacetime, the left-hand side of (3.60) receives non-trivial contributions which depend on the curvature tensor from $|\beta| < |\alpha|$.

contribute non-trivially to (3.63). Without loss of generality, we may assume that the support of F_n is contained in $\times^n U_z$ since, if necessary, we may multiply it by smooth function $\chi(x_1,\ldots,x_n;\vec{0})$ which is equal to unity in a neighborhood of the origin and has support in $\times^n U_z$. However, in any RNC system, the ordinary partial derivatives of a scalar field evaluated at the origin coincide with the totally-symmetrized covariant derivatives of the scalar field evaluated at the origin[9]. It follows then from the identity (3.60) that, in fact,

$$\nabla^{(x_1)}_{\alpha_1}\cdots\nabla^{(x_n)}_{\alpha_n}F_n(x_1,\ldots,x_n)|_{x_1,\ldots,x_n=\vec{0}} = \mathcal{Z}^I_{\alpha_1\cdots\alpha_n}(\vec{0}). \tag{3.66}$$

Thus, we have obtained a function F_n satisfying (3.52) in a neighborhood of one fixed event z. However, by choosing a smooth set of tetrad vector fields and using them to define RNC systems at each event, F_n satisfying (3.52) can be defined as a smooth function of z for any event $z \in M$, noting that $\mathcal{Z}^I_{\alpha_1\cdots\alpha_n}(z)$ is smooth in z by Theorem 1.

Although this construction of F_n will depend on z (and the arbitrarily-chosen tetrad vector fields) away from total coincidence, the "germ" of F_n at $x_1,\ldots,x_n = z$ is independent of z in the sense of (3.53). To prove eq. (3.53) we use the fact that

$$\begin{aligned}\nabla^{(z)}_b&\left[\nabla^{(x_1)}_{\alpha_1}\cdots\nabla^{(x_n)}_{\alpha_n}F_n(x_1,\ldots,x_n;z)|_{x_1,\ldots,x_m=z}\right]\\ &=\left[\left((\nabla^{(x_1)}_{\{b\alpha_1\}}\cdots\nabla^{(x_n)}_{\alpha_n})+\cdots+(\nabla^{(x_1)}_{\alpha_1}\cdots\nabla^{(x_n)}_{\{b\alpha_n\}})\right)F_n(x_1,\ldots,x_n;z)\right]_{x_1,\ldots,x_n=z}+\\ &+\left[\nabla^{(x_1)}_{\alpha_1}\cdots\nabla^{(x_n)}_{\alpha_n}\nabla^{(z)}_b F_n(x_1,\ldots,x_n;z)\right]_{x_1,\ldots,x_n=z},\end{aligned} \tag{3.67}$$

which follows from the ordinary Leibniz rule and the commutativity of derivatives with respect to different variables. The Leibniz condition, eq. (3.48), on $\mathcal{Z}^I_{\alpha_1\cdots\alpha_n}$ then implies

9. In any RNC system, it can be deduced from the geodesic equation for geodesics passing through the origin that

$$\partial_{(\sigma_1}\cdots\partial_{\sigma_n}\Gamma^{\kappa}_{\mu\nu)}(x)|_{x=\vec{0}} = 0, \tag{3.65}$$

with $\Gamma^{\kappa}_{\mu\nu}$ denoting the Christoffel symbols. For scalar fields evaluated at the origin, the equivalence between partial derivatives and totally-symmetrized covariant derivatives can then be inductively established for all n using (3.65).

that the first line of eq. (3.67) is equal to the second line, so the final line must vanish identically. This establishes the result (3.53) for $\beta = \{b\}$. The general case, $|\beta| > 1$ follows via induction. $\square$

Remark 8. By Remark 6 below Theorem 1, F_n and all its derivatives on the total diagonal are greatly constrained by the Wick axioms and will vanish identically unless $n(D-2)/2$ is even. In particular, $\nabla^{(x_1)}_{\alpha_1} \cdots \nabla^{(x_n)}_{\alpha_n} F_n(x_1, \ldots, x_n; z)|_{x_1,\ldots,x_n=z}$ vanish when n is odd and $D \neq 2+4k$ for integer k, and when D is odd and $n = 2+4k$.

Remark 9. Only the germ of $F_n(x_1, \ldots, x_n; z)$ on the total diagonal is relevant to (3.52) and (3.53). Hence, if F_n and F'_n have the same germ on the total diagonal, they are equivalent as far as Proposition 1 is concerned. Note that F_n is not locally and covariantly defined away from the total diagonal on account of the coordinate system and cutoff function used in its construction. However, F_n and its derivatives on the total diagonal are local and covariant.

Remark 10. The property (3.53) implies the germ of $F_n(x_1, \ldots, x_n; z)$ on the total diagonal is independent of its right-most point, z. By the previous remark, $F_n(x_1, \ldots, x_n; z)$ is, therefore, equivalent to, e.g., $F_n(x_1, \ldots, x_n; x_1)$ or $F_n(x_1, \ldots, x_n; x_n)$. Therefore, it is possible to write F_n as functions of only n-spacetime points rather than $(n+1)$-points. However, in anticipation of the role they will play in the Wick OPE coefficients of Section 4.2, it is more convenient to write F_n symmetrically with respect to $x_1, \ldots, x_n$ as we have done here by using the auxiliary point, z.

Remark 11. A notable consequence of Proposition 1 is that all prescriptions for constructing the quadratic Wick fields may be obtained by normal-ordering with respect to some Hadamard parametrix. Suppose H is any Hadamard parametrix such that the prescription for Wick monomials satisfies axioms W1-W8 (or W1-W7 for $D = 2$). Then by the above proposition, any other prescription will satisfy

$$(\widetilde{\nabla_{\alpha_1}\phi\nabla_{\alpha_2}\phi})(z) = (\nabla_{\alpha_1}\phi\nabla_{\alpha_2}\phi)_H(z) + \nabla^{(x_1)}_{\alpha_1}\nabla^{(x_2)}_{\alpha_2} F_2(x_1, x_2; z)|_{x_1,x_2=z} I. \tag{3.68}$$

This prescription for general quadratic Wick monomials can be reproduced by Hadamard normal ordering with respect to the new Hadamard parametrix

$$\widetilde{H}(x_1, x_2) = H(x_1, x_2) + \frac{1}{2}F_2(x_1, x_2; x_1) + \frac{1}{2}F_2(x_1, x_2; x_2) \tag{3.69}$$

This result is special to the quadratic fields. Prescriptions for the higher-order Wick monomials are generally *not* equivalent to Hadamard normal ordering.

Thus, we have shown that the ambiguities between any two definitions of the Wick monomials is completely characterized by a sequence of functions $\{F_n(x_1, \ldots, x_n; z)\}_{n\geq 2}$. As described in the previous section, normal ordering (see eq. (3.28)) with respect to a Hadamard parametrix satisfying (3.36) provides an explicit construction of the Wick monomials compatible with axioms W1-W8 (or W1-W7 in $D = 2$). Our results, therefore, imply any Hadamard normal-ordered monomial Φ_A^H may be expressed as

$$\Phi_A^H = \sum_B \mathcal{Z}_A^B \Phi_B = \sum_{q=0}^{p} \binom{p}{q} (\nabla_{(\alpha_1}\phi \cdots \nabla_{\alpha_q}\phi)[\nabla_{\alpha_{q+1}} \cdots \nabla_{\alpha_p)} F_{p-q}], \tag{3.70}$$

where Φ_B corresponds to a Wick monomial defined via *any* renormalization prescription satisfying the axioms, and we have introduced the shorthand

$$[\nabla_{\alpha_1} \cdots \nabla_{\alpha_n} F_n]_z \equiv \nabla_{\alpha_1}^{(x_1)} \cdots \nabla_{\alpha_n}^{(x_n)} F_n(x_1, \ldots, x_n; z)|_{x_1,\ldots,x_n=z}. \tag{3.71}$$

The right-most equality in (3.70) follows directly from plugging (3.52) of Proposition 1 into the expression (3.44) for $\mathcal{Z}_A^B$ in Theorem 1. Of course, (3.70) can be inverted to express any monomial Φ_A in a general Wick prescription in terms of Hadamard normal-ordered fields

$$\Phi_A = \sum_B (\mathcal{Z}^{-1})_A^B \Phi_B^H. \tag{3.72}$$

Note that Theorem 1 and Proposition 1 apply also to $(\mathcal{Z}^{-1})_A^B$ and $(\mathcal{Z}^{-1})_A^I$. We can obtain expressions for $(\mathcal{Z}^{-1})_A^I$ in terms of the functions $F_n(x_1, \ldots, x_n; z)$ by using $\sum_C (\mathcal{Z}^{-1})_C^I \mathcal{Z}_A^C = \delta_A^I$ together with the expression for $\mathcal{Z}_A^C$ in terms of F_n implied by eqs. (3.44) and (3.52). For $A \neq I$, this yields

$$(\mathcal{Z}^{-1})^I_{\alpha_1 \cdots \alpha_n} = -\sum_{k=0}^{n-2} \binom{n}{k} (\mathcal{Z}^{-1})^I_{(\alpha_1 \cdots \alpha_k} [\nabla_{\alpha_{k+1}} \cdots \nabla_{\alpha_n)} F_{n-k}], \tag{3.73}$$

where we recall the shorthand (3.71) for the Taylor coefficients of F_n at z. This relation allows one to recursively solve for $(\mathcal{Z}^{-1})^I_{\alpha_1 \cdots \alpha_n}$. For example, we have

$$(\mathcal{Z}^{-1})^I_{\alpha_1 \alpha_2} = -[\nabla_{\alpha_1} \nabla_{\alpha_2} F_2] \tag{3.74}$$

$$(\mathcal{Z}^{-1})^I_{\alpha_1 \alpha_2 \alpha_3 \alpha_4} = -[\nabla_{\alpha_1} \nabla_{\alpha_2} \nabla_{\alpha_3} \nabla_{\alpha_4} F_4] + \binom{4}{2} [\nabla_{(\alpha_1} \nabla_{\alpha_2} F_2][\nabla_{\alpha_3} \nabla_{\alpha_4)} F_2]. \tag{3.75}$$

In this way, (3.72) provides a construction of the Wick monomials in any prescription satisfying the axioms in terms of F_n and the Hadamard normal-ordered monomials defined in (3.28). In the next chapter, we will see that the corresponding ambiguities in the OPE coefficients for products of Wick fields can be expressed in a simple way in terms of F_n.

CHAPTER 4

KLEIN-GORDON OPE COEFFICIENTS

A renormalization prescription for the Wick monomials uniquely determines the Wightman products of Wick fields as well as the unextended time-ordered products. In Section 4.1, we obtain the explicit form of the OPE coefficients of the n-point Wightman distributions involving Wick monomials defined via a local Hadamard normal-ordering procedure (see Theorem 2). In Section 4.2, we then give the general form of the (Wightman) OPE coefficients corresponding to any prescription for the Wick monomials satisfying axioms W1-W8 in terms of the smooth functions F_n appearing in the Wick monomial uniqueness theorem (see Theorem 4). In Section 4.3, we show that the OPE coefficients for unextended time-ordered products are given by the same expressions as for the Wightman products with the substitution $H \to H_F$, where H is a locally constructed Hadamard distribution and H_F is a locally-constructed Feynman distribution (see Proposition 6).

4.1 Local Hadamard normal-ordered OPE coefficients

In this section, we show that products of Wick monomials defined by local Hadamard normal ordering admit an operator product expansion (OPE), i.e., we will show that for any Wick monomials $\Phi^H_{A_1}, \ldots, \Phi^H_{A_n}$ defined via the local Hadamard normal ordering prescription (see eq. (3.28)) and in any Hadamard state Ψ we have,

$$\left\langle \Phi^H_{A_1}(x_1) \cdots \Phi^H_{A_n}(x_n) \right\rangle_\Psi \approx \sum_B (C_H)^B_{A_1 \cdots A_n}(x_1, \ldots, x_n; z) \left\langle \Phi^H_B(z) \right\rangle_\Psi , \tag{4.1}$$

where the B-sum runs over all Wick monomials. In Theorem 2 below, we will also obtain explicit expressions for the local and covariant OPE coefficients, $(C_H)^B_{A_1 \cdots A_n}$. For products involving more than two Hadamard normal-ordered monomials (i.e. $n > 2$), the OPE coefficients of (4.1) are found to satisfy important relations called "associativity" conditions which are especially useful for analyzing the OPE for Wick monomials Φ_A that are *not* defined via Hadamard normal ordering. In the next section, we will then show that a general definition of Wick monomials Φ_A also satisfy an OPE, and we will characterize how the freedom in the choice of the definition of Wick monomials affects its OPE coefficients $C^B_{A_1 \cdots A_n}$.

The asymptotic equivalence relation "$\approx$" used in the definition of the OPE (4.1) was precisely formulated in a local and covariant manner in [15] by introducing a family of asymptotic equivalence relations "$\sim_{\mathcal{T},\delta}$" which are parameterized by a positive real number δ and a "merger tree" $\mathcal{T}$. We introduce here the details relevant for our analysis and refer the reader to [15] for the precise definition of $\sim_{\mathcal{T},\delta}$ and for further discussion. Merger trees classify the different ways in which the limit $x_1, \dots, x_n \to z$ may be taken. For instance, when $n = 3$, one possible merger tree would correspond to taking all three points (x_1, x_2, x_3) together to z at the "same rate", while another possible merger tree would correspond to having two of the points, e.g. x_1 and x_2, approach each other "faster" than all three points (x_1, x_2, x_3) approach z. For a given merger tree, $\mathcal{T}$, the positive number δ in "$\sim_{\mathcal{T},\delta}$" indicates how rapidly the difference between both sides of the equivalence relation goes to zero as the spacetime points approach z at their various rates. Altogether, the equivalence relation "$\approx$" in (4.1) means that, for every $\mathcal{T}$ and $\delta > 0$, there exists a real number Δ such that,

$$\left\langle \Phi^H_{A_1}(x_1) \cdots \Phi^H_{A_n}(x_n) \right\rangle_\Psi \sim_{\mathcal{T},\delta} \sum_{[B] \leq \Delta(\mathcal{T},\delta)} (C_H)^B_{A_1 \cdots A_n}(x_1, \dots, x_n; z) \left\langle \Phi^H_B(z) \right\rangle_\Psi , \tag{4.2}$$

where we define the "engineering dimension" of Φ_B,

$$[\Phi_B] \equiv [B] \equiv \frac{(D-2)}{2} \times [B]_\phi + [B]_\nabla , \tag{4.3}$$

with $[B]_\phi$ and $[B]_\nabla$ denoting, respectively, the number of factors of ϕ in Φ_B and the number of covariant derivatives acting on the ϕ factors in Φ_B (e.g., for $\Phi_B = (\nabla_{\beta_1}\phi \cdots \nabla_{\beta_p}\phi)$, we have $[B]_\phi = p$ and $[B]_\nabla = \sum_{i=1}^p |\beta_i|$).

The rate at which a distribution either diverges or converges to zero in the limit all its spacetime points merge to z at the same rate (i.e. for the trivial merger tree) is known as its "scaling degree at z"[1]. By convention, positive scaling degrees indicate divergence and negative scaling degrees imply convergence: For example, the geometric factors $S^{\beta}(x;z)$ have scaling degree $-|\beta|$ at z, while the Hadamard parametrix H has scaling degree $D-2$. As we will see, the engineering dimension $[A]$ of a Wick field Φ_A is related to the scaling degree of the coefficient C_{AA}^{I} as follows[2]:

$$[A] = \frac{1}{2}\mathrm{sd}_z\left[C_{AA}^{I}(x_1, x_2; z)\right]. \tag{4.4}$$

Moreover, we will find the scaling degree of all Wick OPE coefficients are bounded from above by:

$$\mathrm{sd}_z\left[C_{A_1\cdots A_n}^{B}\right] \leq [A_1] + \cdots + [A_n] - [B]. \tag{4.5}$$

The key result needed to show the existence of an OPE for Hadamard normal-ordered Wick monomials is that, in any Hadamard state Ψ, the distribution,

$$h_{n,\Psi}(x_1, \ldots, x_n) \equiv \langle : \phi(x_1)\cdots\phi(x_n) :_H\rangle_\Psi \tag{4.6}$$

is, in fact, a smooth function[3] of $(x_1, \ldots, x_n)$. It then follows immediately from the definition,

1. The "scaling degree" was introduced by Steinmann in the context of Minkowski spacetime [42, Section 5]. See [32] for further discussion in the context of curved manifolds.

2. If the scaling degree varies for different background geometries, then $[A]$ is equal to the supremum of the right-hand side with respect to (M, g_{ab}). If Φ_A is tensor-valued, then the maximum scaling degree of the tensor components is used.

3. It was proven in [36, Lemma III.1] that (4.6) is smooth if and only if Ψ is Hadamard and the truncated n-point functions of Ψ are smooth. However, Sanders later proved that all Hadamard states have smooth truncated n-point functions [43, Proposition 3.1.14] and, therefore, (4.6) is smooth for *all* Hadamard Ψ and *only* Hadamard Ψ.

eq. (3.28), of the Hadamard normal-ordered Wick power $\phi^n_H(z)$ that we have,

$$\langle \phi^n_H(z) \rangle_\Psi = h_{n,\Psi}(z, \dots, z) \tag{4.7}$$

i.e., the expectation value of the Wick power ϕ^n_H evaluated at z is the total coincidence value at z of the smooth function $h_\Psi(x_1, \dots, x_n)$. More generally, we have,

$$\langle (\nabla_{\alpha_1}\phi \cdots \nabla_{\alpha_n}\phi)_H(z) \rangle_\Psi = \nabla^{(x_1)}_{\alpha_1} \cdots \nabla^{(x_n)}_{\alpha_n} h_{n,\Psi}(x_1, \dots, x_n)\big|_{x_1,\dots,x_n=z} \tag{4.8}$$

The simplest example of an OPE is the one for the two point function $\langle \phi(x_1)\phi(x_2) \rangle_\Psi$. From the definition eq. (3.27) of Hadamard normal ordering, we have for x_1 and x_2 in a common convex normal neighborhood,

$$\phi(x_1)\phi(x_2) =: \phi(x_1)\phi(x_2) :_H + H(x_1, x_2) I \tag{4.9}$$

We now take the expectation value of this equation in an arbitrary Hadamard state Ψ. Since $\langle : \phi(x_1)\phi(x_2) :_H \rangle_\Psi$ is smooth, we may take its covariant Taylor expansion (see eq. (3.57) above) for x_1 and x_2 in a normal neighborhood of some arbitrarily chosen point z, thereby obtaining asymptotic relations[4] that hold in the coincidence limit,

$$\langle : \phi(x_1)\phi(x_2) :_H \rangle_\Psi \sim_\delta \sum_{|\beta_1|+|\beta_2| \leq \delta} S^{\beta_1}(x_1; z) S^{\beta_2}(x_1; z) \nabla^{(x_1)}_{\beta_1} \nabla^{(x_2)}_{\beta_2} h_{2,\Psi}(x_1, x_2)\big|_{x_1=x_2=z}, \tag{4.10}$$

using the fact that

$$S^{\beta_1}(x_1; z) S^{\beta_2}(x_2; z) \sim_\delta 0, \qquad \text{for } |\beta_1| + |\beta_2| > \delta. \tag{4.11}$$

4. For $n = 2$, we omit the $\mathcal{T}$ symbol since there is only one possible merger tree in this case.

Substituting expression (4.10) into eq. (4.9) and using eq. (4.8), we find that for any Hadamard state Ψ, we have,

$$\langle \phi(x_1)\phi(x_2)\rangle_\Psi \sim_\delta \sum_{|\beta_1|+|\beta_2|\leq\delta} S^{\beta_1}(x_1;z)S^{\beta_2}(x_2;z)\left\langle(\nabla_{\beta_1}\phi\nabla_{\beta_2}\phi)_H(z)\right\rangle_\Psi + H(x_1,x_2)\left\langle I\right\rangle_\Psi . \tag{4.12}$$

Noting this holds for all $\delta > 0$, this equation takes the form of an OPE, from which we can read off the OPE coefficients,

$$(C_H)^I_{\phi\phi}(x_1,x_2;z) = H(x_1,x_2) \tag{4.13}$$

$$(C_H)^{(\nabla_{\beta_1}\phi\nabla_{\beta_2}\phi)}_{\phi\phi}(x_1,x_2;z) = S^{(\beta_1}(x_1;z)S^{\beta_2)}(x_2;z), \tag{4.14}$$

where we have symmetrized over β_1 and β_2 on the right side of the last expression since $(\nabla_{\beta_1}\phi\nabla_{\beta_2}\phi)_H$ is symmetric in β_1 and β_2, so only the symmetric part of this OPE coefficient contributes. All other OPE coefficients of the form $C^B_{\phi\phi}$ vanish. Given the scaling degree of S^β and H stated above, we indeed find (as anticipated in formula (4.4)),

$$[\phi] = \frac{1}{2}\mathrm{sd}_z\left[(C_H)^I_{\phi\phi}(x_1,x_2;z)\right] = \frac{1}{2}(D-2), \tag{4.15}$$

and (as anticipated in formula (4.5)) the scaling degree of $(C_H)^B_{\phi\phi}$ at z is found to be bounded from above by $[\phi]+[\phi]-[B]$ with the non-trivial coefficients saturating the bound.

In order to illustrate how more general OPEs are obtained for Hadamard normal-ordered monomials and to understand the patterns that emerge in the structure of the general OPE coefficients, it is instructive to consider another simple example, namely $n = 2$ and $\Phi^H_{A_1}, \Phi^H_{A_2} = \phi^2_H$. Wick's theorem (3.10) implies that for x_1, x_2 in a common convex normal neighborhood, we have

$$\left\langle \phi_H^2(x_1)\phi_H^2(x_2)\right\rangle_\Psi = \langle :\phi(x_1)\phi(x_1)\phi(x_2)\phi(x_2):_H\rangle_\Psi + \\ + 4H(x_1,x_2)\,\langle :\phi(x_1)\phi(x_2):_H\rangle_\Psi + 2H(x_1,x_2)H(x_1,x_2). \tag{4.16}$$

Again, all of the "totally normal-ordered" quantities appearing on the right-hand side are smooth functions. Therefore, we may covariantly Taylor expand these terms about $x_1, x_2 = z$, to obtain

$$\begin{aligned} &\left\langle \phi_H^2(x_1)\phi_H^2(x_2)\right\rangle_\Psi \sim_\delta \\ &\sum_{\beta_1,\beta_2,\beta_3,\beta_4} S^{(\beta_1}(x_1;z)S^{\beta_2}(x_1;z)S^{\beta_3}(x_2;z)S^{\beta_4)}(x_2;z)\left\langle(\nabla_{\beta_1}\phi\nabla_{\beta_2}\phi\nabla_{\beta_3}\phi\nabla_{\beta_4}\phi)_H(z)\right\rangle_\Psi + \\ &+ 4H(x_1,x_2)\sum_{\beta_1,\beta_2} S^{(\beta_1}(x_1;z)S^{\beta_2)}(x_2;z)\left\langle(\nabla_{\beta_1}\phi\nabla_{\beta_2}\phi)_H(z)\right\rangle_\Psi + \\ &+ 2H(x_1,x_2)H(x_1,x_2)\,\langle I\rangle_\Psi\,, \end{aligned} \tag{4.17}$$

where the respective sums run over $\sum_i |\beta_i| \le \delta$. Thus, the nonvanishing OPE coefficients are,

$$\begin{aligned} &(C_H)^B_{\phi^2\phi^2}(x_1,x_2;z) \\ &= \begin{cases} S^{(\beta_1}(x_1;z)S^{\beta_2}(x_1;z)S^{\beta_3}(x_2;z)S^{\beta_4)}(x_2;z) & \Phi^H_B = (\nabla_{\beta_1}\phi\nabla_{\beta_2}\phi\nabla_{\beta_3}\phi\nabla_{\beta_4}\phi)_H \\ 4S^{(\beta_1}(x_1;z)S^{\beta_2)}(x_2;z)H(x_1,x_2) & \Phi^H_B = (\nabla_{\beta_1}\phi\nabla_{\beta_2}\phi)_H \\ 2H(x_1,x_2)H(x_1,x_2) & \Phi^H_B = I \end{cases} \end{aligned} \tag{4.18}$$

Thus, we see that all of the nonvanishing OPE coefficients are given by products of the Hadamard parametrix $H(x_1, x_2)$ and the geometrical factors $S^\beta(x; z)$ defined by eq. (3.58).

The existence of an OPE for an arbitrary product of n Hadamard normal-ordered Wick monomials,

$$\Big\langle (\nabla_{\alpha_{(1,1)}}\phi\cdots\nabla_{\alpha_{(1,k_1)}}\phi)_H(x_1)\cdots(\nabla_{\alpha_{(n,1)}}\phi\cdots\nabla_{\alpha_{(n,k_n)}}\phi)_H(x_n)\Big\rangle_\Psi , \tag{4.19}$$

can be established by paralleling the derivation used in the above examples. As in the definition of the engineering dimension, eq. (4.3), we denote the number of factors of ϕ that appear in a Wick monomial Φ_A by $[\Phi_A]_\phi$. Thus, for the factor $\Phi^H_{A_i} = (\nabla_{\alpha_{(i,1)}}\phi\cdots\nabla_{\alpha_{(i,k_i)}}\phi)_H$ in eq. (4.19), we have $[\Phi^H_{A_i}]_\phi = k_i$. We denote by K the total number of factors of ϕ appearing in the expression (4.19),

$$K = \sum_{i=1}^{n} k_i \tag{4.20}$$

We write the quantity (4.19) in terms of products of H and normal ordered products of ϕ's. We then obtain an OPE for (4.19) by Taylor expanding the normal-ordered products of ϕ's. It follows that the general OPE coefficients are given by products of $H(x_1,x_2)$, $S^\beta(x;z)$ and their derivatives. It also can be seen that the only fields $\Phi^H_B = (\nabla_{\beta_1}\phi\nabla_{\beta_2}\phi\cdots\nabla_{\beta_m}\phi)_H$ for which $(C_H)^B_{A_1\cdots A_n}$ can be non-vanishing are such that $[\Phi^H_B]_\phi = m$ takes the values $m = K, K-2, K-4, \ldots$ and $m \geq 0$.

In order to explain the combinatorics of the formula for the general OPE coefficients in terms of $H(x_1,x_2)$, $S^\beta(x;z)$ and their derivatives, it is useful to introduce a uniform notation for all the multi-indices relevant for $(C_H)^B_{A_1\cdots A_n}$ by pairing each β_j multi-index with a "0" and write $\Phi^H_B = (\nabla_{\beta_{(0,1)}}\phi\nabla_{\beta_{(0,2)}}\phi\cdots\nabla_{\beta_{(0,m)}}\phi)_H$. The multi-indices relevant for $(C_H)^B_{A_1\cdots A_n}$ then comprise the set of ordered pairs,

$$\mathcal{S} \equiv \{(0,1),(0,2),\ldots,(0,m),(1,1),(1,2),\ldots,(1,k_1),\ldots\ldots,(n,1),(n,2)\ldots,(n,k_n)\} \tag{4.21}$$

This set has $(m+K)$-elements, which is an even number whenever $(C_H)^B_{A_1\cdots A_n}$ is nonvanishing. In order to describe the combinations of $S^{\beta_j}(x_u)$ and $H(x_v,x_w)$ and their derivatives that appear in the formula for $(C_H)^B_{A_1 \cdots A_n}$, we follow [19, see Section 4.1] by employing the notion of "perfect matchings"[5] for elements of $\mathcal{S}$. By definition, the set, $\mathcal{M}(\mathcal{S})$, of perfect matchings is the set of all partitions of $\mathcal{S}$ into subsets each of which contains precisely two elements. Each pair of distinct elements of $\mathcal{S}$ is of the form $\{(v, i), (w, j)\}$. It is convenient to require that these pairs be ordered so that $v \leq w$. (When $v = w$, we may require $i < j$, but the matrix elements of the matrix $\mathcal{N}$ defined below will vanish in that case, so the ordering is irrelevant.) Since $\mathcal{S}$ has $(m + K)$-elements it follows that $\mathcal{M}(\mathcal{S})$ has $(m + K - 1)!! \equiv (m + K - 1)(m + K - 3)(m + K - 5) \cdots 1$ elements when $m + K$ is even. Thus, for example, if $\mathcal{S} = \{(0, 1), (1, 1), (1, 2), (2, 1)\}$ corresponding to $n = 2$, $K = 3$, and $m = 1$, then $\mathcal{M}(\mathcal{S})$ consists of the three partitions:

$$\mathcal{M}(\mathcal{S}) = \big\{\{(0,1),(1,1);(1,2),(2,1)\},\{(0,1),(1,2);(1,1),(2,1)\},\{(0,1),(2,1);(1,1),(1,2)\}\big\}, \tag{4.22}$$

which are diagrammed in the following figure.

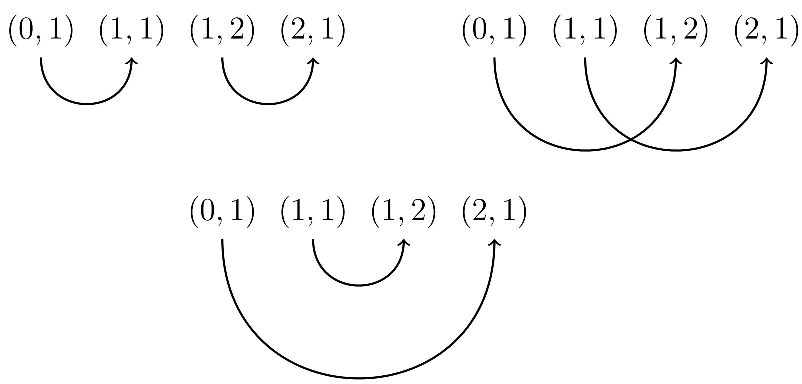

Figure 4.1: Directed graphs representing the three perfect matchings in (4.22). Arrow direction points from a vertex $(v, i) \in \mathcal{S}$ toward a vertex $(w, j) \in \mathcal{S}$ such that $v \leq w$ and $i < j$.

5. The terminology is borrowed from graph theory: The elements of $\mathcal{S}$ can be viewed as labeling the vertices of a graph. (See e.g. Figure 4.1 below). An arrow connecting two vertices of this graph corresponds then to a pairing between two elements of $\mathcal{S}$. A "perfect matching" is achieved when every vertex is connected to exactly one arrow and there are no loops (connecting a vertex to itself): i.e., every element of $\mathcal{S}$ is paired with precisely one other element of $\mathcal{S}$.

It is useful to combine the relevant multi-index derivatives of $S^{\beta_j}(x_u)$ and $H(x_v, x_w)$ into a single $(K+m) \times (K+m)$ matrix $\mathcal{N}$ as follows,

$$\mathcal{N}_{(v,i)(w,j)} \equiv \begin{cases} \nabla^{(x_v)}_{\alpha_{(v,i)}} \nabla^{(x_w)}_{\alpha_{(w,j)}} H(x_v, x_w) & v \neq w; v, w \neq 0 \\ \nabla^{(x_w)}_{\alpha_{(w,j)}} S^{\beta_i}(x_w; z) & v = 0, w \neq 0 \\ 0 & \text{otherwise} \end{cases} \tag{4.23}$$

The *hafnian* of $\mathcal{N}$ is defined by [44]

$$\operatorname{haf} \mathcal{N} \equiv \sum_{M \in \mathcal{M}(\mathcal{S})} \prod_{\{(v,i),(w,j)\} \in M} \mathcal{N}_{(v,i)(w,j)}, \tag{4.24}$$

where the sum is taken over the $(m+K-1)!!$ perfect matchings, M, of the set $\mathcal{S}$, eq. (4.21), and the product is taken over all ordered pairs $\{(v,i),(w,j)\}$ occurring in M. With these definitions, the existence of an OPE for Hadamard normal-ordered products and the general formula for the OPE coefficients may now be stated as the following theorem:

Theorem 2. *For Hadamard normal-ordered fields* $\Phi^H_{A_i}$*, there exists an OPE of the form eq.* (4.1)*, with local and covariant OPE coefficients* $(C_H)^B_{A_1\cdots A_n}(x_1, \ldots, x_n; z)$*. The OPE coefficients* $(C_H)^B_{A_1\cdots A_n}$ *can be nonvanishing only when* $m = [\Phi^H_B]_\phi$ *takes the values* $m = K, K-2, K-4, \ldots$ *and* $m \geq 0$*, where* K *is given by eq.* (4.20)*. Furthermore, the OPE coefficients are explicitly given by*

$$(C_H)^B_{A_1\cdots A_n}(x_1, \ldots, x_n; z) = \frac{1}{m!} \operatorname{haf} \mathcal{N}, \tag{4.25}$$

with $\operatorname{haf} \mathcal{N}$ *given by eq.* (4.24) *and they satisfy the scaling degree properties* (4.4) *and* (4.5)*, saturating the bound whenever* (4.25) *is nonzero.*

A formal proof of the existence of an OPE for scalar field theories with renormalizable interactions on any globally-hyperbolic spacetime was given (to any finite order in perturbation theory) in [10, Theorem 1]. Since at zeroth-order the quantum fields in [10] were defined via Hadamard normal ordering, this result encompasses the case considered here. For the case of flat spacetime, we have $\nabla^{(x)}_\alpha S^\beta(x;z) = \frac{1}{|\beta|!}\partial^{(x)}_\alpha (x-z)^\beta$ and our formula (4.25) for the Hadamard normal-ordered OPE coefficients corresponds[6] to the formula given in [19, Section 4.1] for the vacuum normal-ordered (flat) Euclidean OPE coefficients after replacing H with the Euclidean Green's function G_E (see eq. (5.3) below). The scaling degree properties stated in the theorem follow immediately from eq. (4.25) and the scaling behavior of the Hadamard parametrix and the geometric factors $S^\beta(x;z)$.

Remark 12. In the Euclidean case considered in [19, Section 4.1], $G_E(x_1, x_2)$ is symmetric in (x_1, x_2) so the ordering of (x_1, x_2) is irrelevant. However, in the Lorentzian case, the anti-symmetric part of H is proportional to the causal propagator, $i\Delta \equiv i\Delta^{\text{adv}} - i\Delta^{\text{ret}}$, modulo $C^\infty(M \times M)$, so the ordering of the events matters.

Remark 13. For $B = I$, we have $m = 0$, so we have $(C_H)^I_{A_1 \cdots A_n} = 0$ if K is odd. If K is even, then since $v = 0$ does not arise on the right side of eq. (4.25) when $m = 0$, we may replace $\mathcal{N}_{(v,i)(w,j)}$ by $\nabla^{(x_v)}_{\alpha_{(v,i)}} \nabla^{(x_w)}_{\alpha_{(w,j)}} H(x_v, x_w)$, so $(C_H)^I_{A_1 \cdots A_n}$ is given by

$$(C_H)^I_{A_1 \cdots A_n}(x_1, \ldots, x_n; z) = \sum_{M \in \mathcal{M}(\mathcal{S})} \prod_{\{(v,i),(w,j)\} \in M} \nabla^{(x_v)}_{\alpha_{(v,i)}} \nabla^{(x_w)}_{\alpha_{(w,j)}} H(x_v, x_w), \tag{4.26}$$

i.e., $(C_H)^I_{A_1 \cdots A_n}$ is a sum of products of derivatives of H's. As in the specific examples with $B = I$ given in formulas (4.13) and (4.18) above, it is observed that the right-hand side of the general formula (4.26) does not explicitly depend on the expansion point z. We will sometimes emphasize this independence by omitting z in the notation for the identity OPE coefficients, writing $(C_H)^I_{A_1 \cdots A_n} = (C_H)^I_{A_1 \cdots A_n}(x_1, \ldots, x_n)$.

Remark 14. At the other extreme, when $m = K$, then if any product on the right side of eq. (4.25) contained a factor with both $v \neq 0$ and $w \neq 0$, then it would also have to contain

6. There is a discrepancy of a factor of $1/m!$ between our formula (4.25) and the formula given in [19].

a factor with $v = w = 0$ and thus would vanish. Thus, for $m = K$, the only elements of $\mathcal{M}(\mathcal{S})$ which may contribute nontrivially to (4.25) are those such that $v = 0$ and $w \neq 0$, and the OPE coefficients $(C_H)^B_{A_1 \cdots A_n}$ are given by a sum of terms composed of products of derivatives of S^β's. Explicitly, this formula is,

$$(C_H)^B_{A_1 \cdots A_n}(x_1, \ldots, x_n; z) = \text{sym}_\beta \prod_{i=1}^{n} \prod_{j=1}^{k_i} \nabla^{(x_i)}_{\alpha_{(i,j)}} S^{\beta_{p(i,j)}}(x_i; z), \tag{4.27}$$

where $p(i,j) \equiv j + \sum_{q=1}^{i-1} k_q$ and the symmetrization over the β-multi-indices (already seen in examples (4.14) and (4.18)) is here denoted using "sym_β" as follows,

$$\text{sym}_\beta S^{\beta_1} \cdots S^{\beta_m} \equiv S^{(\beta_1} \cdots S^{\beta_m)} \equiv \frac{1}{m!} \sum_\sigma S^{\beta_{\sigma(1)}} \cdots S^{\beta_{\sigma(m)}}, \tag{4.28}$$

where σ sums over the permutations of $\{1, \ldots, m\}$.

For $0 < m < K$, the OPE coefficient $(C_H)^B_{A_1 \cdots A_n}$ will be a sum of terms involving products of derivatives of both H's and S^β's. In fact, the formula for $(C_H)^B_{A_1 \cdots A_n}$ in this case satisfies very useful recursion relations in terms of a sum of products of OPE coefficients of smaller K. An example of this structure can be seen from eqs. (4.13), (4.14), and (4.18) where, by inspection, we see that

$$(C_H)^{(\nabla_{\beta_1}\phi \nabla_{\beta_2}\phi)}_{\phi^2 \phi^2}(x_1, x_2; z) = 4 (C_H)^{(\nabla_{\beta_1}\phi \nabla_{\beta_2}\phi)}_{\phi\phi}(x_1, x_2; z) (C_H)^I_{\phi\phi}(x_1, x_2). \tag{4.29}$$

To state the general result, let $\mathcal{S}_A$ be the set of the K multi-index labels of the $\Phi^H_{A_1}$, $\ldots$,$\Phi^H_{A_n}$ fields, i.e., $\mathcal{S}_A = \{(1,1), \ldots, (n, k_n)\}$. ($\mathcal{S}_A$ differs from $\mathcal{S}$ by not including the labels $\{(0,1), \ldots, (0,m)\}$ associated with multi-indices of the operator Φ^B_H.) Let p be an integer with $0 < p \leq m$. Partition $\mathcal{S}_A$ into two subsets P_1, P_2, such that P_1 contains p elements and P_2 contains $(K - p)$ elements, i.e., P_1 and P_2 are complements of each other with respect to the set $\mathcal{S}_A$. (There are $\binom{K}{p}$ possible ordered partitions of this sort.) For any such partition, we define,

$$\Phi_{A_i'} \equiv \prod_{(i,j)\in P_1} \nabla_{\alpha_{(i,j)}}\phi \tag{4.30}$$

$$\Phi_{A_i''} \equiv \prod_{(i,j)\in P_2} \nabla_{\alpha_{(i,j)}}\phi \tag{4.31}$$

For any i such that there exists no $(i,j) \in P_1$, then we set $\Phi_{A_i'} = I$ and, similarly, for any i such that there are no $(i,j) \in P_2$, we have $\Phi_{A_i''} = I$. Our result on the Hadamard OPE coefficients $(C_H)^B_{A_1\cdots A_n}$ with $0 < m < K$ is the following:

Proposition 2. *For* $0 < m < K$*, the Hadamard normal-ordered OPE coefficients* (4.25) *of Theorem* 2 satisfy,

$$(C_H)^B_{A_1\cdots A_n}(x_1,\ldots,x_n;z) = \tag{4.32}$$
$$\binom{m}{p}^{-1} \sum_{\{P_1,P_2\}\in\mathcal{P}_p(\mathcal{S}_A)} \left[(C_H)^{(\nabla_{\beta_1}\phi\cdots\nabla_{\beta_p}\phi)}_{A_1'\cdots A_n'}(x_1,\ldots,x_n;z)\,(C_H)^{(\nabla_{\beta_{(p+1)}}\phi\cdots\nabla_{\beta_m}\phi)}_{A_1''\cdots A_n''}(x_1,\ldots,x_n;z) \right]$$

Here p is any integer with $0 < p \leq m$ and the sum is taken over the $\binom{K}{p}$-ordered partitions $\mathcal{P}_p(\mathcal{S}_A)$ into subsets, P_1 and P_2, containing p and $K-p$ elements, respectively. The fields $\Phi^H_{A_i'}$ and $\Phi^H_{A_i''}$ were defined with respect to the partition by eqs. (4.30) and (4.31), respectively.

Proof. From the explicit expression for the Hadamard normal-ordered OPE coefficients (4.25) given in Theorem 2, it can be seen directly that (4.32) is equivalent, for any $0 < p \leq m$ and

$0 < m < K$, to the relation

$$\begin{aligned}
&\sum_{M\in\mathcal{M}(\mathcal{S})}\prod_{\{(v,i),(w,j)\}\in M}\mathcal{N}_{(v,i)(w,j)} \\
&\quad= \sum_{\{P_1;P_2\}\in\mathcal{P}_p(\mathcal{S}_A)}\Bigg\{\left[\sum_{M_1\in\mathcal{M}_1[P_1]}\prod_{\{(v,i),(w,j)\}\in M_1}\mathcal{N}_{(v,i)(w,j)}\right]\times \\
&\qquad\qquad\left[\sum_{M_2\in\mathcal{M}_2[P_2]}\prod_{\{(v,i),(w,j)\}\in M_2}\mathcal{N}_{(v,i)(w,j)}\right]\Bigg\},
\end{aligned} \tag{4.33}$$

with $\mathcal{M}_1 \equiv \mathcal{M}(P_1 \cup \{(0,1),\dots,(0,p)\})$ and $\mathcal{M}_2 \equiv \mathcal{M}(P_2 \cup \{(0,p+1),\dots,(0,m)\})$. To prove this relation, we note that the first line of eq. (4.33) instructs us to take the product of the matrix elements $\mathcal{N}_{(v,i)(w,j)}$ over a perfect matching of $\mathcal{S}$ and then sum over all perfect matchings. By eq. (4.23), in order for any perfect matching to contribute nontrivially, any element of the form $(0,j)$ must be matched with an element of $\mathcal{S}_A$. Fix any integer p with $0 < p \leq m$. For a given perfect matching that contributes nontrivially to eq. (4.33), the elements of $\mathcal{S}_A$ that are paired with $(0,1),\dots,(0,p)$ define a subset, P_1, of $\mathcal{S}_A$ with p elements. Let $P_2 = \mathcal{S}_A \setminus P_1$ so that $\{P_1, P_2\}$ is a partition of $\mathcal{S}_A$ into subsets of p and $K-p$ elements, respectively. When we sum over all perfect matchings, we may first sum over all perfect matchings that respect these partitions. That sum yields the term in large curly braces on the second and third lines of eq. (4.33). It then remains to sum over all partitions $\{P_1, P_2\} \in \mathcal{P}_p(\mathcal{S}_A)$, which yields eq. (4.33). □

Remark 15. An important case is $m = p$ for which relation (4.32) of Proposition 2 reduces to,

$$\begin{aligned}
&(C_H)^B_{A_1\cdots A_n}(x_1,\dots,x_n;z) \\
&= \sum_{\{P_1,P_2\}\in\mathcal{P}_m(\mathcal{S}_A)}(C_H)^B_{A'_1\cdots A'_n}(x_1,\dots,x_n;z)\,(C_H)^I_{A''_1\cdots A''_n}(x_1,\dots,x_n).
\end{aligned} \tag{4.34}$$

This implies that every Hadamard normal-ordered OPE coefficient can be expressed as a sum of products of OPE coefficients with $m = 0$ of the form (4.26) and OPE coefficients with $m = K'$ of the form (4.27). In the second line of (4.34), we note the OPE coefficients with $B = I$ are independent of z; see also Remark 13 above.

While eq. (4.32) was derived here using the particular form (4.25) of the Hadamard normal-ordered coefficients, we will show, in the next section, these identities for the Hadamard normal-ordered OPE coefficients and the field redefinition relations for Wick fields obtained in Section 3.2 can be used to prove relation (4.32) holds also for the OPE coefficients corresponding to completely general constructions of the Wick fields; that is, we will show that (4.32) continues to be a valid formula even when the H-subscripts are removed.

Above, we have given explicit formulas for all of the OPE coefficients occurring for products of Wick monomials of the Klein-Gordon field defined by Hadamard normal ordering. There is an important associativity property satisfied by these OPE coefficients, which will be seen in the next section to hold for general prescriptions for Wick monomials and, indeed, is expected to hold for general interacting theories [10]. As already mentioned at the beginning of this section, for an OPE involving $n > 2$ spacetime points x_i, we have different possible "merger trees," i.e., different possible rates at which the different x_i's may approach z. For example, for an OPE involving three spacetime points (x_1, x_2, x_3), we could let x_1 and x_2, approach each other faster than the remaining point, x_3. In this case, one might expect that the OPE and its coefficients could be alternatively computed by first expanding the expectation value in x_1 and x_2 about an auxiliary point z' and, subsequently, expanding z' and x_3 about z. For this to be self-consistent, the OPE coefficients obtained via this iterated expansion should be asymptotically equivalent (for this merger tree) to the original OPE coefficients. For general merger trees, this implies that OPE coefficients involving $n > 2$ spacetime points must factorize into a sum of products of OPE coefficients involving fewer spacetime points. This property is referred to as "associativity."

The associativity conditions corresponding to the most general possible merger trees may be found in [15, Section 3]. For our purposes, it will be useful to have an explicit formula for the following merger trees: Consider the set of $K = k_1 + k_2 + \cdots + k_n$ spacelike-separated spacetime points,

$$\left\{x_{(1,1)}, \ldots, x_{(1,k_1)}, x_{(2,1)}, \ldots, x_{(2,k_2)}, \ldots, x_{(n,1)}, \ldots, x_{(n,k_n)}\right\}. \tag{4.35}$$

Let $\mathcal{T}$ denote any merger tree where, for all $i \in \{1, \ldots, n\}$, the k_i-spacetime points $x_{(i,1)}, \ldots, x_{(i,k_i)}$ approach each other faster than the remaining points in (4.35). Supposing a Wick field is located at each one of these spacetime points, the associativity condition for this class of merger trees is,

$$\begin{aligned}&(C_H)^B_{A_{(1,1)}\cdots A_{(n,k_n)}}(\vec{x}_1, \ldots, \vec{x}_n; z) \sim_{\mathcal{T},\delta} \\ &\sum_{C_1,\ldots,C_n} (C_H)^{C_1}_{A_{(1,1)}\cdots A_{(1,k_1)}}(\vec{x}_1; z_1) \cdots (C_H)^{C_n}_{A_{(n,1)}\cdots A_{(n,k_n)}}(\vec{x}_n; z_n)(C_H)^B_{C_1\cdots C_n}(z_1, \ldots, z_n; z),\end{aligned} \tag{4.36}$$

where we have introduced the shorthand $\vec{x}_i \equiv x_{(i,1)}, \ldots, x_{(i,k_i)}$. Here the $C_1, \ldots, C_n$-sums are carried out to sufficiently high, but finite, $[C_i]$ for all i. The associativity condition and other properties of the OPE coefficients were established in [10, Section 4]. We state this result in the following theorem:

Theorem 3. *The OPE coefficients* $(C_H)^B_{A_1\cdots A_n}$ *satisfy* (4.36) *and the more general associativity conditions of* [10, 15].

4.2 OPE coefficients for a general definition of Wick monomials

We are now in a position to obtain the expression for the coefficients that arise in the OPE expansion of products of Wick monomials defined using an arbitrary prescription for Wick monomials that satisfies the axioms of Section 3.1. Let Φ^H_A denote the Hadamard normal-ordered prescription for Wick monomials and let Φ_A be an arbitrary prescription. The key equations (3.70) and (3.72) relating Φ^H_A and Φ_A via $\mathcal{Z}$ and its inverse, respectively, were obtained in Section 3.2.

To obtain an OPE for $\langle \Phi_{A_1}(x_1)\cdots\Phi_{A_n}(x_n)\rangle_\Psi$ for our arbitrary prescription for Wick monomials, we now use eq. (3.72) to write

$$\begin{aligned}&\langle \Phi_{A_1}(x_1)\cdots\Phi_{A_n}(x_n)\rangle_\Psi \\ &= \sum_{C_1}\cdots\sum_{C_n}(\mathcal{Z}^{-1})^{C_1}_{A_1}(x_1)\cdots(\mathcal{Z}^{-1})^{C_n}_{A_n}(x_n)\left\langle \Phi^H_{C_1}(x_1)\cdots\Phi^H_{C_n}(x_n)\right\rangle_\Psi .\end{aligned} \tag{4.37}$$

It should be noted that the sums in the second line include only a finite number of terms because $(\mathcal{Z}^{-1})^C_A = 0$ unless $[C]\leq[A]$. Next, we use the OPE, eq. (4.1), for the Hadamard normal-ordered Wick monomials, with OPE coefficients given by eq. (4.25) to obtain

$$\begin{aligned}\langle \Phi_{A_1}(x_1)\cdots\Phi_{A_n}(x_n)\rangle_\Psi \approx &\sum_{C_1}\cdots\sum_{C_n}(\mathcal{Z}^{-1})^{C_1}_{A_1}(x_1)\cdots(\mathcal{Z}^{-1})^{C_n}_{A_n}(x_n)\times \\ &\times\left[\sum_{C_0}(C_H)^{C_0}_{C_1\cdots C_n}(x_1,\dots,x_n;z)\left\langle\Phi^H_{C_0}(z)\right\rangle_\Psi\right]\end{aligned} \tag{4.38}$$

Finally, we use eq. (3.70) to write $\left\langle\Phi^H_B(z)\right\rangle_\Psi$ in terms of one-point Wick monomials in the prescription that we are using,

$$\begin{aligned}\langle \Phi_{A_1}(x_1)\cdots\Phi_{A_n}(x_n)\rangle_\Psi \approx &\sum_{C_1}\cdots\sum_{C_n}(\mathcal{Z}^{-1})^{C_1}_{A_1}(x_1)\cdots(\mathcal{Z}^{-1})^{C_n}_{A_n}(x_n)\times \\ &\times\left[\sum_{C_0}(C_H)^{C_0}_{C_1\cdots C_n}(x_1,\dots,x_n;z)\left[\sum_B \mathcal{Z}^B_{C_0}(z)\langle\Phi_B(z)\rangle_\Psi\right]\right].\end{aligned} \tag{4.39}$$

This provides an OPE expansion for $\langle \Phi_{A_1}(x_1)\cdots\Phi_{A_n}(x_n)\rangle_\Psi$, from which we can read off the OPE coefficients

$$C^B_{A_1\cdots A_n}(x_1,\ldots,x_n;z) \approx \tag{4.40}$$
$$\sum_{C_0} \mathcal{Z}^B_{C_0}(z)\left[\sum_{C_1}\cdots\sum_{C_n}(\mathcal{Z}^{-1})^{C_1}_{A_1}(x_1)\cdots(\mathcal{Z}^{-1})^{C_n}_{A_n}(x_n)(C_H)^{C_0}_{C_1\cdots C_n}(x_1,\ldots,x_n;z)\right].$$

Expressions for the Hadamard normal-ordered coefficients $(C_H)^{C_0}_{C_1\cdots C_n}$ were given in terms of S^β and H by eq. (4.25) of Theorem 2. The mixing matrix $\mathcal{Z}^B_A$ was given in terms of F_n via eq. (3.44) of Theorem 1 and (3.52) of Proposition 1. As described in Section 3.2, $(\mathcal{Z}^{-1})^B_A$ can also be expressed in terms of F_n using eq. (3.44) and eq. (3.73). Thus, as desired, eq. (4.40) yields a formula for the OPE coefficients $C^B_{A_1\cdots A_n}$ in terms of a Hadamard parametrix H, the geometric factors S^β, and the smooth functions F_n (which characterize the difference between Φ_A and Φ^H_A).

Theorem 4. *For any prescription for the Wick monomials* $\{\Phi_A | A \equiv \alpha_1\cdots\alpha_n\}_{n\in\mathbb{N}_0}$ *compatible with axioms W1-W8, there exists an OPE in the sense of* (4.2) *with local and covariant defined OPE coefficients* $C^B_{A_1\cdots A_n}(x_1,\ldots,x_n;z)$ *given by* (4.40). *These OPE coefficients satisfy* (4.36) *(with the H-subscripts removed) as well as the general associativity conditions of* [10, 15]. *The coefficients are also compatible with the scaling degree properties* (4.4) *and* (4.5).

Sketch of proof. See Appendix B. □

Equation (4.40) provides a complete characterization of the OPE coefficients for an arbitrary prescription for Wick monomials and, thus, achieves the primary goal of this section. However, there are important properties of the general Wick coefficients which are not immediately apparent from (4.40) but will be extremely useful for our analysis of the flow relations in future chapters as well as for illuminating the general qualitative structure of the Wick coefficients. In particular, as we will show, the special form of the Wick mixing matrices (3.38) and the factorization properties (4.32) of the Hadamard normal-ordered products together imply knowledge of just the $C^{I}_{\phi\cdots\phi}$-coefficients is sufficient for one to determine all other Wick OPE coefficients. This property of the Wick coefficients will greatly reduce the number of independent flow relations we must consider in future chapters. Moreover, the relative simplicity of the $C^{I}_{\phi\cdots\phi}$-coefficients permits us to obtain an explicit formula for these elementary coefficients in terms of H and F_n, thereby generalizing the Hadamard normal-ordered formula (4.26) to arbitrary prescriptions.

We now outline the steps that allow us to obtain an arbitrary OPE coefficient $C^{B}_{A_1\cdots A_n}$ in terms of S^β and OPE coefficients of the form $C^{I}_{\phi\cdots\phi}$. We will then give an explicit formula (see eq. (4.48)) for $C^{I}_{\phi\cdots\phi}$ in terms of the Hadamard parametrix H and the functions F_n. Finally, we obtain in Proposition 5 an explicit (inductive) construction for the Wick monomials in terms of the OPE coefficients $C^{I}_{\phi\cdots\phi}$.

We first note that eq. (4.40) implies that $C^{B}_{A_1\cdots A_n} = 0$ whenever $m > K$ for $m \equiv [B]_\phi$, since this property holds for $(C_H)^{B}_{A_1\cdots A_n}$ and the mixing matrices $\mathcal{Z}^{B}_{A}$ and $(\mathcal{Z}^{-1})^{B}_{A}$ never increase the number of powers of ϕ appearing in any Wick monomial. For the case $m = K$, the only terms in $\mathcal{Z}^{B}_{A}$ and $(\mathcal{Z}^{-1})^{B}_{A}$ that can contribute nontrivially to eq. (4.40) are δ^{B}_{A}. Thus, for $m = K$ we obtain,

$$\begin{aligned} C^{B}_{A_1\cdots A_n}(x_1,\ldots,x_n;z) &= (C_H)^{B}_{A_1\cdots A_n}(x_1,\ldots,x_n;z) \\ &= \mathrm{sym}_\beta \prod_{i=1}^{n}\prod_{j=1}^{k_i} \nabla^{(x_i)}_{\alpha(i,j)} S^{\beta_{p(i,j)}}(x_i;z), \end{aligned} \tag{4.41}$$

with $p(i,j) \equiv j + \sum_{q=1}^{i-1} k_q$ and sym_β defined as in (4.28). Thus, for $m = K$ the OPE coefficients for a general prescription are the same as for the Hadamard normal ordered prescription, and depend only on the geometrical factors S^β.

Next, we show that the OPE coefficients $C^{B}_{A_1\cdots A_n}$ such that $0 < m < K$ are determined by OPE coefficients with $B = I$ together with OPE coefficients of the form eq. (4.41). More precisely,

$$
\begin{aligned}
&C^{B}_{A_1\cdots A_n}(x_1,\ldots,x_n;z)\\
&=\sum_{\{P_1,P_2\}\in\mathcal{P}_m(\mathcal{S}_A)} C^{B}_{A'_1\cdots A'_n}(x_1,\ldots,x_n;z)C^{I}_{A''_1\cdots A''_n}(x_1,\ldots,x_n),
\end{aligned}
\tag{4.42}
$$

with the notation as in Proposition 2. Since we have $[A'_1]_\phi+\cdots+[A'_n]_\phi=m\equiv[B]_\phi$, the coefficients $C^{B}_{A'_1\cdots A'_n}(x_1,\ldots,x_n;z)$ are of the form (4.41). Thus, (4.42) expresses a general OPE coefficient with $0<m<K$ in terms of OPE coefficients with $B=I$. Formula (4.42) is a special case of the following proposition when $p=m$.

Proposition 3. *For $0<p\le m<K$, the OPE coefficients given by* (4.40) *satisfy the same formula* (4.32) *as derived in Proposition* 2 *for the Hadamard normal-ordered OPE coefficients. i.e., formula* (4.32) *remains a valid formula when the H-subscripts are removed.*

Sketch of proof. See Appendix B. □

The following proposition shows that any Wick OPE coefficient $C^{B}_{A_1\cdots A_n}(x_1,\ldots,x_n;z)$ is ultimately fixed by OPE coefficients of the form $C^{C_i}_{\phi\cdots\phi}(x_1,\ldots,x_{k_i})$ for $[C_i]_\phi\le[A_i]_\phi=k_i$. When combined with the previous proposition, this implies all Wick OPE coefficients may be obtained from a finite number of OPE coefficients of the form $C^{I}_{\phi\cdots\phi}$.

Proposition 4. *The Wick OPE coefficients* (4.40) *satisfy,*

$$
\begin{aligned}
&C^{B}_{A_1\cdots A_n}(x_1,\ldots,x_n;z)\\
&=\lim_{\vec{y}_1\to x_1}\cdots\lim_{\vec{y}_n\to x_n}\nabla^{y_{(1,1)}}_{\alpha_{(1,1)}}\cdots\nabla^{y_{(n,k_n)}}_{\alpha_{(n,k_n)}}\Big[C^{B}_{\phi\cdots\phi}(\vec{y}_1,\ldots,\vec{y}_n;z)+\\
&-\sum_{\substack{[C_1]<[A_1]\\ [C_1]_\phi<[A_1]_\phi}}\cdots\sum_{\substack{[C_n]<[A_n]\\ [C_n]_\phi<[A_n]_\phi}} C^{C_1}_{\phi\cdots\phi}(\vec{y}_1;x_1)\cdots C^{C_n}_{\phi\cdots\phi}(\vec{y}_n;x_n)C^{B}_{C_1\cdots C_n}(x_1,\ldots,x_n;z)\Big],
\end{aligned}
\tag{4.43}
$$

where we define the shorthand $\vec{y}_i\equiv y_{(i,1)},\ldots,y_{(i,k_i)}$ and denote $k_i\equiv[A_i]_\phi$.

Proof. See Appendix B. □

Remark 16. Recall the definition (4.3) of $[C] \equiv (D-2)/2 \times [C]_\phi + [C]_\nabla$. Hence, for a fixed $[A]$, there are only finitely-many $[C]_\phi$ and $[C]_\nabla$ such that $[C] < [A]$ and, thus, the C-sums in (4.43) are all finite sums. Note also $[C] < [A]$ iff

$$[C]_\nabla < \frac{D-2}{2} \times \big([A]_\phi - [C]_\phi\big) + [A]_\nabla, \tag{4.44}$$

where the right-hand side is non-negative for $[C]_\phi < [A]_\phi$ and reduces to $[A]_\nabla$ when $D = 2$.

The preceding proposition enables us to inductively compute all OPE coefficients using only the OPE coefficients $C^B_{\phi\cdots\phi}$ as input. To observe this, first note that, for these elementary OPE coefficients,

$$C^B_{(\nabla_{\alpha_1}\phi)\cdots(\nabla_{\alpha_n}\phi)}(x_1,\ldots,x_n;z) = \nabla^{(x_1)}_{\alpha_1}\cdots\nabla^{(x_n)}_{\alpha_n} C^B_{\phi\cdots\phi}(x_1,\ldots,x_n;z), \tag{4.45}$$

and, thus, knowledge of $C^B_{\phi\cdots\phi}$ implies knowledge of $C^B_{(\nabla_{\alpha_1}\phi)\cdots(\nabla_{\alpha_n}\phi)}$ for all α_i. Hence, by assumption, we begin with knowledge of all OPE coefficients $C^B_{A_1\cdots A_n}$ such that $[A_i]_\phi = 1$ and $[A_i]_\nabla < \infty$ for $i \in \{1,\ldots,n\}$. Noting the bounds on the C_i-sums, we may, therefore, immediately use formula (4.43) to calculate any $C^B_{A_1\cdots A_n}$ such that, for all i, $[A_i]_\phi \leq 2$ and $[A_i]_\nabla < \infty$. Of course, this, in turn, provides enough data to compute any coefficient such that $[A_i]_\phi \leq 3$ and $[A_i]_\nabla < \infty$ and, in this way, we may obtain any OPE coefficient $C^B_{A_1\cdots A_n}$ from formula (4.43) starting from knowledge of just $C^B_{\phi\cdots\phi}$.

Remark 17. For any finite $[A_i]_\phi$ and $[A_i]_\nabla$, we emphasize that the coefficient $C^B_{A_1\cdots A_n}$ can be computed from (4.43) with only a finite number of iterations. In particular, it is *not* required that we compute all $[A_i]_\nabla < \infty$ for a given $[A_i]_\phi$ before incrementing to $[A'_i]_\phi = [A_i]_\phi + 1$. By inequality (4.44), computing $C^B_{A_1\cdots A_n}$ for any $[A_i]_\phi$ and $[A_i]_\nabla$ only requires knowledge of coefficients $C^B_{C_1\cdots C_n}$ such that $[C_i]_\phi < [A_i]_\phi$ and $[C_i]_\nabla < (D-2)/2 \times [A_i]_\phi + [A_i]_\nabla$.

Taken together, the above results allow us to express an arbitrary Wick OPE coefficient $C^{B}_{A_1\cdots A_n}$ in terms[7] of the OPE coefficients $C^{I}_{\phi\cdots\phi}$ and pure geometrical factors S^{β}. Finally, we give an explicit formula for $C^{I}_{\phi\cdots\phi}$. To see how this formula is obtained, consider first the simplest case of $C^{I}_{\phi\phi}$. We have,

$$\begin{aligned} C^{I}_{\phi\phi}(x_1,x_2;z) &\approx \mathcal{Z}^{I}_{I}(C_H)^{I}_{\phi\phi}(x_1,x_2) + \sum_{\gamma_1,\gamma_2} \mathcal{Z}^{I}_{\gamma_1\gamma_2}(z)(C_H)^{(\nabla_{\gamma_1}\phi\nabla_{\gamma_2}\phi)}_{\phi\phi}(x_1,x_2;z) \\ &\approx H(x_1,x_2) + \sum_{\gamma_1,\gamma_2}[\nabla_{\gamma_1}\nabla_{\gamma_2}F_2]_z\, S^{(\gamma_1}(x_1;z)S^{\gamma_2)}(x_2;z) \\ &\approx H(x_1,x_2) + F_2(x_1,x_2;z), \end{aligned} \tag{4.46}$$

where in the last line, we used the fact that the series,

$$\sum_{\gamma_1,\gamma_2}[\nabla_{\gamma_1}\nabla_{\gamma_2}F_2]_z\, S^{(\gamma_1}(x_1;z)S^{\gamma_2)}(x_2;z), \tag{4.47}$$

is simply the covariant Taylor expansion of the smooth function $F_2(x_1,x_2;z)$. Proceeding in a similar manner and recalling formulas (4.40) and (4.41), we obtain the general formula,

$$\begin{aligned} C^{I}_{\phi\cdots\phi}(x_1,\ldots,x_n;z) &\approx F_n(x_1,\ldots,x_n;z) + \\ &+ \sum_{k=1}^{\lfloor n/2\rfloor}\sum_{\pi_k} H(x_{\pi(1)},x_{\pi(2)})\cdots H(x_{\pi(2k-1)},x_{\pi(2k)})F_{(n-2k)}(x_{\pi(2k+1)},\ldots,x_{\pi(n)};z), \end{aligned} \tag{4.48}$$

where the π_k sums over the permutations of $\{1,\ldots,n\}$ such that $\pi(1) < \pi(2), \pi(3) < \pi(4), \cdots, \pi(2k-1) < \pi(2k)$; $\pi(1) < \pi(3) < \cdots < \pi(2k-1)$; and $\pi(2k+1) < \pi(2k+2) < \cdots < \pi(n)$. Formula (4.48) generalizes the normal-ordered formula (4.26) obtained in the previous section to arbitrary prescriptions for the Wick monomials.

7. The only method we have provided for computing coefficients of the form $C^{I}_{A_1\cdots A_n}$ from $C^{I}_{\phi\cdots\phi}$ is via formula (4.43) of Proposition 4. However, coefficients $C^{B}_{A_1\cdots A_n}$ for $B \neq I$ may be computed from $C^{I}_{\phi\cdots\phi}$ either via formula (4.43) or, alternatively, by plugging the values of $C^{I}_{A_1\cdots A_n}$ back into formula (4.32) of Proposition 3.

Formula (4.48) now implies the full renormalization freedom for the Wick fields may be expressed entirely in terms of the identity coefficients $\{C^I_{\phi\cdots\phi}(x_1,\ldots,x_n;z)\}_{n\geq 2}$ and, thus, the set of these coefficients uniquely determines a prescription for the Wick fields $\{\Phi_A | A \equiv \alpha_1\cdots\alpha_n\}_{n\in\mathbb{N}_0}$. To see this, note eq. (4.48) implies $C^I_{\phi\phi}$ is itself a Hadamard parametrix $\widetilde{H} \equiv H + F_2$ (in accordance with Remark 11). If we choose to normal order instead with respect to the parametrix $\widetilde{H}$, i.e. use $\Phi^{\widetilde{H}}_B$ in formula (3.72) rather than Φ^H_B, then the preceding manipulations would again yield formula (4.48) for $C^I_{\phi\cdots\phi}$ but now with all H's replaced by $\widetilde{H} = C^I_{\phi\phi}$. Since this formula depends only on $F_{k\leq n}$ and OPE coefficients of the form $C^I_{\phi\cdots\phi}$, it may be iteratively inverted to express F_n purely in terms of $C^I_{\phi\cdots\phi}(x_1,\ldots,x_k;z)$ for $k\leq n$. The claim is then an immediate consequence of Proposition 1 and the Wick uniqueness theorem (Theorem 1).

Using identities (3.44) and (3.73), our expression for F_n in terms of the OPE coefficients allows us to similarly express $(\mathcal{Z}^{-1})^A_B$ purely in terms of $C^I_{\phi\cdots\phi}(x_1,\ldots,x_k;z)$ for $k\leq n$. A Wick monomial Φ_A in any prescription satisfying axioms W1-W8 can, thus, be expressed via $\Phi_A = \sum_{[B]\leq[A]}(\mathcal{Z}^{-1})^B_A\Phi^{\widetilde{H}}_B$ in terms of just $\{C^I_{\phi\cdots\phi}(x_1,\ldots,x_n;z)\}_{n\leq[A]}$ and products of the linear field observable ϕ, noting that the normal-ordered Wick fields $\Phi^{\widetilde{H}}_B$ are themselves defined in (3.28) with respect to only products of the linear field observable ϕ and the OPE coefficient $\widetilde{H} = C^I_{\phi\phi}$. An explicit inductive formula for Φ_A expressed purely in terms of ϕ, $C^I_{\phi\cdots\phi}$ and the geometric factors S^β is obtained in the following proposition.

Proposition 5. *For the OPE coefficients $C^I_{\phi\cdots\phi}$ given by the formula* (4.48)*, the monomial*

Φ_A *in any prescription satisfying axioms W1-W8 satisfies:*

$$(\nabla_{\alpha_1}\phi \cdots \nabla_{\alpha_n}\phi)(f) =$$

$$\int_{z,x_1,\ldots,x_n} f^{\alpha_1\cdots\alpha_n}(z)\delta(z,x_1,\ldots,x_n)\nabla^{(x_1)}_{\alpha_1}\cdots\nabla^{(x_n)}_{\alpha_n}\Bigg[\phi(x_1)\cdots\phi(x_n) - \tag{4.49}$$

$$\sum_{\substack{m<n \\ [B]<[A]}} \sum_{\pi\in\Pi_m} C^I_{\phi\cdots\phi}(x_{\pi(m+1)},\ldots,x_{\pi(n)};z)\, S^{\beta_1}(x_{\pi(1)};z)\,\cdots S^{\beta_m}(x_{\pi(m)};z)\,(\nabla_{\beta_1}\phi\cdots\nabla_{\beta_m}\phi)(z)\Bigg]$$

where Π_m *denotes the set of all permutations of* $\{1,\ldots,n\}$ *such that* $\pi(1) < \pi(2) < \cdots < \pi(m)$ *and* $\pi(m+1) < \pi(m+2) < \cdots < \pi(n)$*, and the abbreviation* $\int_{z,x_1,\ldots,x_n}$ *is defined as in* (3.12) *and our "Notation and conventions" in Chapter* 2.

Proof. See Appendix B. □

4.3 OPE coefficients of (unextended) time-ordered products

As we shall see in Chapter 6, the flow relations for OPE coefficients that we shall obtain in Lorentzian spacetimes will involve expansions of time-ordered products—rather than ordinary products—of Wick monomials. Away from the diagonals[8], the "unextended time-ordered product" of Wick monomials is defined by

$$T_0\{\Phi_{A_1}(x_1)\cdots\Phi_{A_n}(x_n)\} = \Phi_{A_{P(1)}}(x_{P(1)})\cdots\Phi_{A_{P(n)}}(x_{P(n)}), \tag{4.50}$$

where P is a permutation of $\{1,\ldots,n\}$ such that $x_{P(i)} \notin J^-(x_{P(i+1)})$, where J^- denotes the causal past. In other words, $T_0\{\Phi_{A_1}(x_1)\cdots\Phi_{A_n}(x_n)\}$ re-orders the product $\Phi_{A_1}(x_1)\cdots\Phi_{A_n}(x_n)$ by the "time" at which the Wick monomial is being evaluated. The right side of eq. (4.50) yields a well-defined (algebra-valued) distribution on the product

8. The "diagonals" are the subset of the product manifold, $\{(x_1,\ldots x_n) \in \times^n M |\ x_i = x_j \text{ for any } i,j \in \{1,\ldots,n\}\}$. Thus, "away from the diagonals" means when all points are distinct.

manifold $\times^n M$ minus all of the diagonals.

Renormalization theory is primarily concerned with the "extension of $T_0\{\Phi_{A_1}(x_1)\cdots\Phi_{A_n}(x_n)\}$ to the diagonals": i.e., obtaining (algebra-valued) distributions $T\{\Phi_{A_1}(x_1)\cdots\Phi_{A_n}(x_n)\}$ that are well-defined on all of $\times^n M$, including the diagonals and defined such that,

$$T\{\Phi_{A_1}(x_1)\cdots\Phi_{A_n}(x_n)\} = T_0\{\Phi_{A_1}(x_1)\cdots\Phi_{A_n}(x_n)\}, \tag{4.51}$$

away from all diagonals. In curved spacetime, it has been proven [37,45] that there exist "extensions" of (4.50) that are compatible with a list of axioms that generalize those stated here (W1-W8) for Wick powers[9]. However, generally, there are additional "contact term" ambiguities in these extensions, corresponding to the freedom to add finitely-many "δ-function-type" terms on the diagonals. Although these ambiguities can be fully characterized [37], they greatly complicate the analysis of time-ordered products. For the integral in Lorentzian flow relations such as eq. (2.2) to be well-defined, it is necessary that the unextended time-ordered products be extended to, at least, all partial diagonals involving the integration variable, y. Fortunately, as we shall see in Chapter 6, the unextended time-ordered-products will satisfy flow relations where the extension to the requisite partial diagonals is unambiguous and, thus, independent of contact terms. Therefore, we will only ever need to consider the OPE of unextended time-ordered-products and the field redefinition freedom of its coefficients, and we may thereby bypass all of the usual complications of renormalization theory.

It is clear that the unextended time-ordered products satisfy OPE relations of the form,

$$\left\langle T_0\{\Phi_{A_1}(x_1)\cdots\Phi_{A_n}(x_n)\}\right\rangle_\Psi \approx \sum_B C^B_{T_0\{A_1\cdots A_n\}}(x_1,\ldots,x_n;z)\,\langle\Phi_B(z)\rangle_\Psi\,, \tag{4.52}$$

9. The proof in [37,45] used an analytic dependence assumption in place of W2 (see Footnote 6 in Section 3.1).

where

$$C^B_{T_0\{A_1\cdots A_n\}}(x_1,\ldots,x_n;z) = C^B_{A_{P(1)}\cdots A_{P(n)}}(x_{P(1)},\ldots,x_{P(n)};z), \tag{4.53}$$

with the permutations P as defined in (4.50). It is understood that eq. (4.52) holds only on $\times^n M$ minus all of the diagonals. As described in the following proposition, the explicit form of the time-ordered OPE coefficients (4.53) is readily obtained from our previously-stated expressions for the Wick OPE coefficients in terms of a Hadamard parametrix H, the geometrical quantities S^β, and the smooth functions F_n that characterize the difference between the Hadamard normal-ordering prescription for Wick monomials and the prescription being used.

Proposition 6. For any fixed prescription for the Wick monomials, the time-ordered OPE coefficients (4.53) are simply obtained from the formula for $C^B_{A_1\cdots A_n}(x_1,\ldots,x_n;z)$ by individually time-ordering all Hadamard parametrices: i.e., replacing every occurrence of H with its corresponding Feynman parametrix[10],

$$H_F(x_1,x_x) = H(x_1,x_2) - i\Delta^{\text{adv}}(x_1,x_2), \tag{4.55}$$

with Δ^{adv} denoting the advanced Green's function.

Proof. By applying the definition of time-ordering (4.53) to the formula (4.40) for the general

10. The precise asymptotic behavior of the distribution kernel for H_F is obtained by replacing the "$i\epsilon$-prescription" in the expression (3.26) for H with the usual Feynman prescription: i.e., making the following substitution,

$$[2i0^+\,(T(x_1)-T(x_2)) + (0^+)^2] \to i0^+ \tag{4.54}$$

Wick OPE coefficients, it is straightforwardly shown that we have

$$C^{B}_{T_0\{A_1\cdots A_n\}}(x_1,\ldots,x_n;z) \approx \tag{4.56}$$
$$\sum_{C_0} \mathcal{Z}^{B}_{C_0}(z)\left[\sum_{C_1}\cdots\sum_{C_n}(\mathcal{Z}^{-1})^{C_1}_{A_1}(x_1)\cdots(\mathcal{Z}^{-1})^{C_n}_{A_n}(x_n)(C_H)^{C_0}_{T_0\{C_1\cdots C_n\}}(x_1,\ldots,x_n;z)\right],$$

where we use the notation $(C_H)^{B}_{T_0\{A_1\cdots A_n\}}$ for the OPE coefficients of the unextended time-ordered products (4.53) when the Wick fields are defined via a Hadamard normal-ordering prescription, $\Phi_A = \Phi^H_A$. It then follows from the factorization property (4.34) of the Hadamard normal-ordered OPE coefficients that we have

$$(C_H)^{C_0}_{T_0\{C_1\cdots C_n\}}(x_1,\ldots,x_n;z) \tag{4.57}$$
$$= \sum_{\{P_1,P_2\}\in\mathcal{P}_m(\mathcal{S})}(C_H)^{C_0}_{T_0\{A'_1\cdots A'_n\}}(x_1,\ldots,x_n;z)(C_H)^{I}_{T_0\{A''_1\cdots A''_n\}}(x_1,\ldots,x_n).$$

The first factor in each product is unaffected by time-ordering because they depend only on symmetric combinations of S^β-factors, i.e., we have

$$(C_H)^{C_0}_{T_0\{A'_1\cdots A'_n\}}(x_1,\ldots,x_n;z) = (C_H)^{C_0}_{A'_1\cdots A'_n}(x_1,\ldots,x_n;z), \tag{4.58}$$

where the explicit form of the right-hand side is given by (4.27). Finally, recalling (4.26), we have

$$\begin{aligned}(C_H)^{I}_{T_0\{A''_1\cdots A''_n\}} &= T_0\left\{\sum_{P\in\mathcal{M}(\mathcal{S})}\prod_{\{(v,i),(w,j)\}\in P}\nabla^{(x_v)}_{\alpha_{v,i}}\nabla^{(x_w)}_{\alpha_{w,j}}H(x_v,x_w)\right\}\\ &= \sum_{P\in\mathcal{M}(\mathcal{S})}\prod_{\{(v,i),(w,j)\}\in P}\nabla^{(x_v)}_{\alpha_{v,i}}\nabla^{(x_w)}_{\alpha_{w,j}}T_0\left\{H(x_v,x_w)\right\},\end{aligned} \tag{4.59}$$

where the second line follows from the fact that tensor products of ordinary c-number distributions commute. Altogether, we conclude the time-ordering map acts non-trivially only on the Hadamard parametrices and in the way specified by the proposition. We note, when $x_1 \neq x_2$, the time-ordered Hadamard parametrix $T_0\{H(x_1, x_2)\}$ is equivalent to the Feynman parametrix (4.55). □

Remark 18. Although $T_0\{H(x_1, x_2)\}$ is a priori only defined away from $x_1 = x_2$, its extension to its diagonal $x_1 = x_2$ uniquely yields the Feynman parametrix (4.55), because the scaling degree of $T_0\{H\}$ is $D-2$ which is less than that of the Dirac delta distribution (and all of its distributional derivatives) and, thus, there do not exist any possible "contact terms" with the correct scaling degree.

As examples, from eqs. (4.13) and (4.18), we see that for the Hadamard normal-ordering prescription, we have,

$$(C_H)^{I}_{T_0\{\phi\phi\}}(x_1, x_2; z) = H_F(x_1, x_2) \tag{4.60}$$

$$(C_H)^{I}_{T_0\{\phi^2\phi^2\}}(x_1, x_2; z) = H_F(x_1, x_2)H_F(x_1, x_2). \tag{4.61}$$

The wavefront set calculus implies H^2 is a well-defined distribution on (a convex normal neighborhood of) the product manifold $M \times M$ and, thus, the ordinary OPE coefficient $(C_H)^{I}_{\phi^2\phi^2}$ is similarly well-defined. However, the pointwise product of the Feynman parametrix, H_F^2, is only well-defined as a distribution on the product manifold minus the diagonal so the time-ordered coefficient $(C_H)^{I}_{T_0\{\phi^2\phi^2\}}$ is thus only well-defined as a distribution for $x_1 \neq x_2$.

The advanced Green's function scales almost homogeneously and, thus, H_F defined via (4.55) will scale almost homogeneously if and only if H is compatible with axiom W7. Note (4.55) is symmetric in its spacetime variables and solves the inhomogeneous Klein-Gordon equation with "δ-source" up to a smooth remainder,

$$KH_F(x_1, x_2) = -i\delta(x_1, x_2) \mod C^{\infty}(M \times M) \tag{4.62}$$

Any bi-distribution satisfying (4.62) is referred to as a parametrix of a fundamental solution for the differential operator K. If $H(x_1, x_2)$ is any Hadamard parametrix (of the homogeneous Klein-Gordon equation) in $D > 2$ satisfying the conservation constraint (3.36), then the Feynman parametrix defined via (4.55) will then necessarily satisfy,

$$\left[\nabla_b^{(x_1)} K_{x_2} H_F(x_1, x_2) + i\nabla_b^{(x_1)} \delta(x_1, x_2)\right]_{x_1, x_2 = z} = 0. \tag{4.63}$$

Conversely, for any Feynman parametrix satisfying (4.63), the corresponding Hadamard parametrix, $H = H_F + i\Delta^{\text{adv}}$, will satisfy the conservation constraint (3.36).

CHAPTER 5

FLOW RELATIONS FOR OPE COEFFICIENTS IN FLAT EUCLIDEAN SPACE

In this chapter, we obtain flow equations in m^2 for the Wick OPE coefficients in flat Euclidean space. We initially focus our attention on the flow relation for $C^{I}_{\phi\phi}$, since our analysis of the preceding chapter implies the flow relations for all other Wick OPE coefficients can be readily obtained after the flow relation for $C^{I}_{\phi\phi}$ is known. We begin in Section 5.1 by deriving flow relations for the case where the Euclidean Green's function $G_E(x_1, x_2)$ is used to define a Hadamard normal-ordering prescription, so $(C_G)^{I}_{\phi\phi}(x_1, x_2; z) = G_E(x_1, x_2)$. These flow equations (see eq. (5.9) below) are the direct analogues of the Holland and Hollands flow equations (1.20) for Klein-Gordon theory in the limit as the infrared cutoff L is removed, i.e., $L \to +\infty$. Note that Hadamard normal-ordering with respect to $G_E(x_1, x_2)$ corresponds to ordinary normal ordering with respect to the Euclidean vacuum state. However, $G_E(x_1, x_2; m^2)$ does not have smooth dependence in m^2 at $m^2 = 0$, so it is not acceptable to use it in a Hadamard normal-ordering prescription that is valid in an open interval in m^2 containing $m^2 = 0$. Nevertheless, it can be used outside any open interval in m^2 containing $m^2 = 0$, and it is convenient to begin our consideration of flow relations with it because the flow relation analysis is much simpler when a Green's function (rather than a parametrix) is used in the Hadamard normal-ordering prescription.

We turn then in Section 5.2 to the derivation of Euclidean flow relations for the case of a Euclidean-invariant parametrix that has smooth dependence on m^2 for all m^2, including $m^2 = 0$. To avoid infrared divergences, this requires introducing a cutoff function in the integral over all space appearing in the flow relation. The cutoff function can be chosen to be Euclidean invariant, so it will not spoil the Euclidean invariance of the flow relations. However, it will unavoidably spoil the scaling behavior of the flow relations. Nevertheless, we develop an algorithm for modifying the flow relations which restores proper scaling behavior to any desired scaling degree. We show any ambiguities in our algorithm are in a 1-1 correspondence with the ambiguities of Euclidean OPE coefficients for Hadamard normal-ordered Wick fields (see Theorem 6).

5.1 Vacuum normal ordering without an infrared cutoff ($m^2 > 0$)

The Riemannian version of quantum field theory in curved spacetime has been formulated by [46] in close parallel with the axiomatic formulation for the Lorentzian case given in Section 3.1. An analogue of the "Hadamard normal-ordering" prescription for defining Wick monomials can then be given by choosing a local and covariant Green's parametrix for the (now elliptic) Klein-Gordon operator. OPE coefficients for the Euclidean Wick OPE coefficients can then be obtained in parallel with the Lorentzian case away from the diagonals[1].

In this chapter, we will be concerned only with the case of flat, Euclidean space $(\mathbb{R}^D, \delta_{ab})$. In this case, there is a unique Green's function, $G_E(x_1, x_2; m^2)$, for the operator,

$$K = -\delta^{ab}\partial_a\partial_b + m^2, \tag{5.1}$$

such that G_E vanishes as $|x_1 - x_2| \to \infty$. It would be extremely convenient to use this Green's function in a Hadamard normal-ordering prescription for Wick monomials. Indeed, since this Green's function is the vacuum 2-point function of the Euclidean quantum field theory,

$$\langle\phi(x_1)\phi(x_2)\rangle_{\text{vac}} = G_E(x_1, x_2), \tag{5.2}$$

it follows that Hadamard normal ordering with respect to $G_E(x_1, x_2; m^2)$ corresponds to ordinary normal ordering with respect to the Euclidean vacuum state. However, as previously mentioned, $G_E(x_1, x_2; m^2)$ does not have smooth dependence on m^2 at $m^2 = 0$. Thus, it is

1. Defining products of Euclidean Wick fields on diagonals generally requires renormalization analogous to extending the Lorentzian unextended time-ordered products to their diagonals and, thus, is subject to additional contact-term renormalization ambiguities. See also Footnote 3.

not acceptable to use it in a Hadamard normal-ordering prescription that is valid in an open interval in m^2 containing $m^2 = 0$, since the corresponding Wick monomials defined in this way will not have the required smooth dependence on m^2. In order to obtain an acceptable prescription that includes the case $m^2 = 0$, we therefore must use a Green's function (or parametrix) for K that has smooth dependence on m^2. Nevertheless, there are significant simplifications in the derivation of the flow relations for $G_E(x_1, x_2; m^2)$. Therefore, we will proceed by first obtaining flow relations for normal ordering with respect to $G_E(x_1, x_2; m^2)$ for $m^2 > 0$, and then derive flow relations for normal ordering with respect to a parametrix that is smooth in m^2.

Although we shall not need to make use of its explicit form, we note that for $m^2 > 0$, $G_E(x_1, x_2; m^2)$ is given explicitly by,

$$\begin{aligned} G_E(\Delta x; m^2) &= \int_{\mathbb{R}^D} \frac{d^D p}{(2\pi)^D} \frac{e^{ip\cdot\Delta x}}{p^2 + m^2} \\ &= \frac{m^{(D-2)/2}}{(2\pi)^{D/2} \left(|\Delta x|^2\right)^{(D-2)/4}} K_{(D-2)/2}\left(m\sqrt{|\Delta x|^2}\right), \end{aligned} \tag{5.3}$$

where $\Delta x = x_1 - x_2$ and $K_{(D-2)/2}$ is a modified Bessel function of the second kind [47, see Subsection 10.25 for definition of $K_\nu(z)$ and Subsection 10.30 for its limiting form at the origin]. It should be noted that $G_E(x_1, x_2; m^2)$ is symmetric in x_1 and x_2. The wavefront set of G_E is the same as the wavefront set of a (two-variable) δ-function,

$$\begin{aligned} \mathrm{WF}[G_E] &= \mathrm{WF}[\delta] \\ &\equiv \left\{(x_1, k; x_2, -k) \in \times^2(T^*\mathbb{R}^D \backslash Z^*\mathbb{R}^D) | x_1 = x_2, k \in T^*\mathbb{R}^D \backslash Z^*\mathbb{R}^D\right\}. \end{aligned} \tag{5.4}$$

In particular, G_E is smooth in x_1 and x_2 for $\Delta x \neq 0$. Furthermore, it follows from the form of its wavefront set that, when smeared in one of its variables with any test function

f, $G_E(x_1, f; m^2)$ is smooth[2] in x_1. In other words, as a Schwartz kernel, $G_E(x_1, x_2)$ defines a continuous linear map from $C_0^\infty(\mathbb{R}^D)$ into $C^\infty(\mathbb{R}^D)$. It follows from the explicit formula (5.3) that $G_E(x_1, f; m^2)$ is smooth in m^2 for $m^2 > 0$.

For Wick monomials defined by normal ordering with respect to G_E, the Euclidean OPE coefficients are given, away from their diagonals[3], by formula (4.25) with the replacement $H \to G_E$. In particular, for the OPE coefficient $(C_G)^I_{\phi\cdots\phi}$ with n factors of ϕ, we have $(C_G)^I_{\phi\cdots\phi} = 0$ when n is odd, whereas when n is even, we have,

$$(C_G)^I_{\phi\cdots\phi}(x_1, \ldots, x_n) = \sum_\pi G_E(x_{\pi(1)}, x_{\pi(2)}) \cdots G_E(x_{\pi(n-1)}, x_{\pi(n)}), \tag{5.5}$$

where the π sums over all permutations such that $\pi(1) < \pi(3) < \cdots < \pi(n-1)$ and $\pi(1) < \pi(2), \pi(3) < \pi(4), \ldots, \pi(n-1) < \pi(n)$. As discussed in Section 4.2, the Wick OPE coefficients for any prescription are determined by the values of the $C^I_{\phi\cdots\phi}$-coefficients.

We first motivate the form of the flow equations for $(C_G)^I_{\phi\cdots\phi}$ following Hollands [17]. Since the OPE coefficients $(C_G)^I_{\phi\cdots\phi}$ defined by normal ordering are just n-point "Schwinger functions," they are formally given by the functional integral,

$$(C_G)^I_{\phi\cdots\phi}(x_1, \ldots, x_n) = \int_{\mathcal{S}'(\mathbb{R}^D)} d\mu[\varphi]\, \varphi(x_1) \cdots \varphi(x_n), \tag{5.6}$$

with measure,

$$d\mu[\varphi] \ = \mathcal{D}[\varphi] \frac{1}{Z_0} \exp\left(-S_{\text{KG}}[\delta_{ab}, \mathbb{R}^D]\right). \tag{5.7}$$

2. This is established by a straightforward application of [3, Theorem 8.2.12]. In fact, as explained in the next chapter, this property holds for any translation invariant bi-distribution.

3. Whereas the wavefront set calculus implies the pointwise products of any Lorentzian Hadamard parametrix H appearing in formula (4.25) are guaranteed to be well-defined as distributions, the corresponding pointwise products of G_E are generally ill-defined as distributions on diagonals, since the wavefront set (5.4) of G_E is identical to the wavefront set of the two-variable Dirac delta function. In this respect, G_E is more analogous to a Feynman parametrix H_F rather than to H; see also the discussion following eq. (4.61).

Formal differentiation of eq. (5.6) with respect to m^2 yields,

$$\frac{\partial}{\partial m^2}(C_G)^I_{\phi\cdots\phi}(x_1,\ldots,x_n) = -\frac{1}{2}\int_{\mathbb{R}^D} d^D y \int_{\mathcal{S}'(\mathbb{R}^D)} d\mu\, \varphi^2(y)\varphi(x_1)\cdots\varphi(x_n). \tag{5.8}$$

This suggests that we should have the flow relation

$$\frac{\partial}{\partial m^2}(C_G)^I_{\phi\cdots\phi}(x_1,\ldots,x_n) = -\frac{1}{2}\int_{\mathbb{R}^D} d^D y (C_G)^I_{\phi^2\phi\cdots\phi}(y,x_1,\ldots,x_n). \tag{5.9}$$

That this flow equation, eq. (5.9), does indeed hold will be seen to be a consequence of the following lemma:

Lemma 1. *The Euclidean Green's function G_E satisfies the flow relation*

$$\frac{\partial}{\partial m^2}G_E(x_1,x_2;m^2) = -\int_{\mathbb{R}^D} d^D y\, G_E(y,x_1;m^2)G_E(y,x_2;m^2). \tag{5.10}$$

Proof. We note first that, by a trivial calculation, the commutator of the differential operators $K = -\partial^a\partial_a + m^2$ and ∂_{m^2} is given by,

$$[K, \partial_{m^2}] = -I \tag{5.11}$$

Thus, in particular, we have,

$$\begin{aligned} K_y \frac{\partial}{\partial m^2}G_E(y,x;m^2) &= -G_E(y,x;m^2) + \frac{\partial}{\partial m^2}K_y G_E(y,x;m^2) \\ &= -G_E(y,x;m^2) + \frac{\partial}{\partial m^2}\delta(y,x) \\ &= -G_E(y,x;m^2), \end{aligned} \tag{5.12}$$

where we used the Green's function property, $K_y G_E(y,x;m^2) = \delta(y,x)$, to get the second line and we used the fact that the δ-function has no m^2 dependence to get the last line. As already noted, the wavefront set of $G_E(y,x;m^2)$ (and, hence, of $\partial_{m^2}G_E(y,x;m^2)$) is such that if we smear in x, we obtain a smooth function of y and, on account of the explicit formula (5.3), a smooth function of m^2 for $m^2>0$. Therefore, for any test functions f_1 and f_2, we have

$$\begin{aligned}
\frac{\partial}{\partial m^2}G_E(f_1,f_2;m^2) &= \int_{\mathbb{R}^D} d^D y\, \delta(y,f_1)\frac{\partial}{\partial m^2}G_E(y,f_2;m^2) \\
&= \int_{\mathbb{R}^D} d^D y \left[K_y G_E(y,f_1;m^2)\right]\frac{\partial}{\partial m^2}G_E(y,f_2;m^2) \\
&= \int_{\mathbb{R}^D} d^D y\, G_E(y,f_1;m^2)K_y\frac{\partial}{\partial m^2}G_E(y,f_2;m^2) \\
&= -\int_{\mathbb{R}^D} d^D y\, G_E(y,f_1;m^2)G_E(y,f_2;m^2). \qquad (5.13)
\end{aligned}$$

Here, in going to the third line, we integrated by parts twice, invoking the fall-off behavior[4] of G_E as $y\to\infty$. Equation (5.13) is just the smeared form of eq. (5.10). □

As an immediate consequence of this lemma, we have

Theorem 5. *The flow relation* (5.9) *holds for OPE coefficients* $(C_G)^I_{\phi\cdots\phi}(x_1,\ldots,x_n)$ *corresponding to Euclidean vacuum normal-ordered Wick fields.*

Proof. To obtain the flow equation (5.9), we apply ∂_{m^2} to eq. (5.5), and use eq. (5.10) together with the fact that $(C_G)^I_{\phi^2\phi\cdots\phi}=0$ when n is odd and, when n is even,

$$\begin{aligned}
&(C_G)^I_{\phi^2\phi\cdots\phi}(y,x_1,\ldots,x_n) \\
&= \sum_\pi 2G_E(y,x_{\pi(1)})G_E(y,x_{\pi(2)})G_E(x_{\pi(3)},x_{\pi(4)})\cdots G_E(x_{\pi(n-1)},x_{\pi(n)}), \qquad (5.14)
\end{aligned}$$

where the π-sum runs over the same permutations as in (5.5). Equation (5.9) then follows by inspection. □

4. We have restricted to the case $m^2>0$ here, but it is worth noting that for $m^2=0$, the fall-off of G_E is too slow to justify the integration by parts.

5.2 Hadamard normal ordering with an infrared cutoff

We turn now to the modifications to the Euclidean flow relations that arise when we consider the OPE coefficients corresponding to a Hadamard normal-ordering prescription using a Euclidean invariant Hadamard parametrix, $H_E(x_1, x_2; m^2)$, that varies smoothly with m^2 for all m^2, including $m^2 = 0$. That is, H_E is required to satisfy,

$$(-\partial^2 + m^2) H_E(x_1, x_2; m^2) = \delta(x_1, x_2) + h_E(x_1, x_2; m^2), \tag{5.15}$$

where $h_E(x_1, x_2; m^2)$ is smooth in all of its variables and symmetric in (x_1, x_2). Clearly, the choice of H_E is not unique, but any two such parametrices must differ from each other by addition of a smooth, Euclidean-invariant function, $w(x_1, x_2; m^2)$. If we now try to repeat the calculation of eq. (5.13) to obtain a flow relation for H_E, we will pick up extra terms involving h. In addition, H_E will not, in general, vanish as $|x_1 - x_2| \to \infty$, so we will not be able to carry out the integration by parts of the third line of eq. (5.13) in the preceding section. We can deal with the latter problem in the following manner by introducing a cutoff function $\chi(x_1, x_2)$. We take χ to be Euclidean invariant by choosing it to be of the form,

$$\chi(x_1, x_2; L) = \zeta\left(L^{-2}\sigma(x_1, x_2)\right), \tag{5.16}$$

where $\sigma(x_1, x_2)$ is the squared geodesic distance between x_1 and x_2, L is an arbitrary length scale, and $\zeta(s)$ is a smooth function that is equal to one for $|s| \leq 1$ and vanishes for $|s| \geq 2$. Let z be an arbitrary point in $\mathbb{R}^D$ and let $\mathcal{B}_z$ denote the ball of radius L centered about z. Then for $x_1, x_2 \in \mathcal{B}_z$, we have the identity,

$$\frac{\partial}{\partial m^2} H_E(x_1, x_2; m^2) = \int_{\mathbb{R}^D} d^D y \, \chi(y, z; L)\, \delta(y, x_1) \frac{\partial}{\partial m^2} H_E(y, x_2; m^2). \tag{5.17}$$

Starting with this equation, we can now carry out all the steps of eq. (5.13) including the integration by parts, although we now pick up additional terms where derivatives act on χ. The final result is,

$$
\begin{aligned}
\frac{\partial}{\partial m^2} H_E(x_1, x_2; m^2) = &- \int_{\mathbb{R}^D} d^D y \, \chi(y; z; L) H_E(y, x_1; m^2) H_E(y, x_2; m^2) + \\
&+ \int_{\mathbb{R}^D} d^D y \, \partial_\mu^{(y)} \chi(y; z; L) \left[\partial^\mu_{(y)} H_E(y, x_1; m^2) \frac{\partial}{\partial m^2} H_E(y, x_2; m^2) + \right. \\
&\qquad \left. - H_E(y, x_1; m^2) \partial^\mu_{(y)} \frac{\partial}{\partial m^2} H_E(y, x_2; m^2) \right] + \\
&+ \int_{\mathbb{R}^D} d^D y \, \chi(y, z; L) \left[H_E(y, x_1; m^2) \frac{\partial}{\partial m^2} h_E(y, x_2; m^2) + \right. \\
&\qquad \left. - h_E(y, x_1; m^2) \frac{\partial}{\partial m^2} H_E(y, x_2; m^2) \right] .
\end{aligned}
\tag{5.18}
$$

The first term on the right side corresponds to the final line of eq. (5.13). The second and third lines contain the terms where derivatives from the integration by parts act on χ, and the fourth and fifth lines contain the terms arising from the fact that H_E is a parametrix rather than a Green's function.

Equation (5.18) is unsatisfactory as a flow equation since terms below the first line contain the unknown quantities $\partial_{m^2} H_E$ and $\partial_{m^2} h_E$. Nevertheless, the second through fifth lines must be smooth in $(x_1, x_2; m^2)$ for $x_1, x_2 \in \mathcal{B}_z$. To see this, we note that $H_E(y, x)$ can be singular only when $y = x$. However, $\partial_\mu^{(y)} \chi(y, z)$ is nonvanishing only for $y \notin \mathcal{B}_z$, so the second and third lines are smooth for $x_1, x_2 \in \mathcal{B}_z$. Since h is smooth in all of its variables and χ is of compact support in y, the fourth and fifth lines must be smooth for all x_1, x_2. Recall from its definition (5.15) that H_E is only uniquely determined up to the addition of a smooth function, so we have some freedom to redefine H_E. Thus, a possible way of dealing with the problematic terms in the second through fifth lines of eq. (5.18) would to simply drop these terms from the flow relations, leaving only the first line[5],

$$\frac{\partial}{\partial m^2} H_E(x_1, x_2; m^2) \sim_\delta - \int_{\mathbb{R}^D} d^D y \, \chi(y; z; L) H_E(y, x_1; m^2) H_E(y, x_2; m^2). \tag{5.19}$$

In other words, one might attempt to use the freedom in the choice of H_E to work with flow relation (5.19) rather than (5.18). Indeed, this is a simple analog of the procedure used in [19, Section V] to deal with the infrared difficulties in their Euclidean flow relations for $\lambda\phi^4$-theory. Note that since $(C_H)^I_{\phi\phi} = H_E$ and $(C_H)^I_{\phi^2\phi\phi}(y, x_1, x_2; z) = 2H_E(y, x_1)H_E(y, x_2)$, we see that (5.19) is equivalent to relation (2.1) discussed in Chapter 2 for the special case of $n = 2$, with χ is chosen to be a sharp step function instead of a smooth function.

However, for Klein-Gordon theory, the flow relation (2.1) would give rise to OPE coefficients that are incompatible with the scaling axiom W7. Namely, in order to satisfy this axiom, H_E must have scaling behavior given by eq. (3.37) under the simultaneous rescaling $(\delta_{ab}, m^2) \to (\lambda^{-2}\delta_{ab}, \lambda^2 m^2)$. Here we are working in a fixed global inertial coordinate system defined with respect to metric δ_{ab}. Hence, with the coordinate basis held fixed, rescaling the metric $\delta_{ab} \to \lambda^{-2}\delta_{ab}$ is equivalent to rescaling the metric coordinate components as $\delta_{\mu\nu} \to \lambda^{-2}\delta_{\mu\nu}$ and the volume element as $d^D y \to \lambda^{-D} d^D y$. However, under the rescaling

$$(\delta_{\mu\nu}, d^D y, m^2) \to (\lambda^{-2}\delta_{\mu\nu}, \lambda^{-D} d^D y, \lambda^2 m^2), \tag{5.20}$$

we find the quantity

$$\Omega(x_1, x_2; z; m^2; L) \equiv - \int_{\mathbb{R}^D} d^D y \, \chi(y; z; L) H_E(y, x_1; m^2) H_E(y, x_2; m^2), \tag{5.21}$$

appearing on the right side of (2.1) does not scale almost homogeneously for any fixed power

5. Since only the asymptotic behavior of the parametrix is relevant for the OPE coefficients, we have replaced the equality symbol in (5.18) with the weaker relation "$\sim_\delta$" which implies both sides are asymptotically equivalent to an arbitrary scaling degree δ.

of λ on account of the fact that—due to the presence of the length scale L—χ scales as

$$\chi[\lambda^{-2}\delta_{\mu\nu}](x_1, x_2; L) = \zeta\left((\lambda L)^{-2}\sigma[\delta_{\mu\nu}](x_1, x_2)\right), \tag{5.22}$$

rather than homogeneously. It follows that eq. (2.1) is incompatible with the scaling behavior (3.37) of H_E, and any prescription for defining Wick monomials based on its solutions would fail to satisfy axiom W7.

Although Ω, as defined in (5.21), does not scale almost homogeneously under (5.20), it *does* transform almost homogeneously with an overall factor of λ^{D-4} under the simultaneous rescaling

$$(\delta_{\mu\nu}, d^D y, m^2, L) \to (\lambda^{-2}\delta_{\mu\nu}, \lambda^{-D}d^D y, \lambda^2 m^2, \lambda^{-1}L), \tag{5.23}$$

since

$$\chi[\lambda^{-2}\delta_{\mu\nu}](x_1, x_2; \lambda^{-1}L) = \chi[\delta_{\mu\nu}](x_1, x_2; L). \tag{5.24}$$

It follows that we will obtain a satisfactory flow relation if we can replace the flow relation (5.19), i.e.,

$$\frac{\partial}{\partial m^2}H_E(x_1, x_2; m^2) \sim_\delta \Omega(x_1, x_2; z; m^2; L), \tag{5.25}$$

with the modified flow relation

$$\frac{\partial}{\partial m^2}H_E(x_1, x_2; m^2) \sim_\delta \widetilde{\Omega}_\delta(x_1, x_2; z; m^2; L). \tag{5.26}$$

where $\widetilde{\Omega}_\delta(x_1, x_2; z; m^2; L)$ satisfies the following two properties:

1. $\widetilde{\Omega}_\delta$ is an Euclidean-invariant distribution, symmetric in (x_1, x_2), and depending smoothly on m^2 such that for any $(x_1, x_2) \in \mathcal{B}_z$, the distribution $\widetilde{\Omega}_\delta$ differs from Ω by at most a smooth function in (x_1, x_2) which scales almost homogeneously under (5.23) with an overall factor of $\lambda^{(D-4)}$.

2. To scaling degree δ,

$$\frac{\partial}{\partial L}\widetilde{\Omega}_\delta(x_1, x_2; z; m^2; L) \sim_\delta 0. \tag{5.27}$$

Given the previously-described scaling behavior of Ω, the first property implies $\widetilde{\Omega}_\delta$ is required to scale almost homogeneously under (5.23) with an overall factor of $\lambda^{(D-4)}$. However, since the second property requires $\widetilde{\Omega}_\delta$ to be independent of L up to asymptotic degree δ, it follows immediately that $\widetilde{\Omega}_\delta$ must, in fact, scale almost homogeneously under just (5.20) up to asymptotic degree δ. Note we make no demand that $\widetilde{\Omega}_\delta$ be independent of L at asymptotic orders higher than the chosen δ since this is not relevant to the OPE coefficients. Together, the two properties above therefore formalize the notion that $\widetilde{\Omega}_\delta$ must contain the same L-independent distributional behavior in (x_1, x_2) as Ω and simultaneously scale almost homogeneously with respect to (5.20) up to any arbitrary, but fixed, asymptotic degree. In odd spacetime dimensions, any $\widetilde{\Omega}_\delta$ satisfying the two properties described below eq. (5.26) is necessarily unique up to scaling degree δ. In even spacetime dimensions, any two $\widetilde{\Omega}_\delta$ satisfying the described properties may differ, to asymptotic degree δ, by only a smooth function of the form, $m^{(D-4)}f(m^2\sigma(x_1, x_2))$. Our task is now to find $\widetilde{\Omega}_\delta(x_1, x_2; z; m^2; L)$ satisfying the above two properties.

Since L enters Ω only through the cutoff function χ, it follows that,

$$\frac{\partial}{\partial L}\Omega(x_1, x_2; z; m^2; L) = -\int_{\mathbb{R}^D} d^D y \, \frac{\partial}{\partial L}\chi(y; z; L) H_E(y, x_1; m^2) H_E(y, x_2; m^2). \tag{5.28}$$

From the definition of the cutoff function (5.16), we observe $\partial_L \chi(y; z; L) = 0$ for any $y \in \mathcal{B}_z$. However, since $H_E(y, x)$ is singular only when $y = x$, it follows immediately that (5.28) is, in fact, a smooth function of (x_1, x_2) in the neighborhood $\mathcal{B}_z$ containing z. If the L-dependence of the smooth function of (x_1, x_2) appearing on the right side of eq. (5.28) were integrable in L on the interval $[0, L]$, we could obtain the desired $\widetilde{\Omega}_\delta$ by simply subtracting $\int_0^L$ of the right side of eq. (5.28) from Ω. However, the right side of eq. (5.28) is not integrable in L

on the interval $[0, L]$. Nevertheless, the singular behavior in L of the right side of eq. (5.28) can be characterized as follows.

The quantity Ω scales almost homogeneously with an overall factor of λ^{D-4} under the simultaneous rescaling (5.23). It follows that the quantities

$$\left[\partial_{\gamma_1}^{(x_1)}\partial_{\gamma_2}^{(x_2)}\frac{\partial}{\partial L}\Omega(x_1, x_2; z; L)\right]_{x_1,x_2=z}, \tag{5.29}$$

appearing in the Taylor expansion of $\partial\Omega/\partial L$ scale almost homogeneously with an overall factor of $\lambda^{(D-3+\Gamma)}$, with $\Gamma \equiv |\gamma_1| + |\gamma_2|$, under $(m^2, L) \to (\lambda^2 m^2, \lambda^{-1}L)$ with the metric components and volume element held fixed. The smoothness of Ω in m^2 then implies that any divergent dependence of (5.29) on L (as $L \to 0^+$) is expressible as a finite linear combination of terms of the form,

$$L^{-\Delta}\log^N L, \qquad \text{for integer } \Delta \leq (D-3+\Gamma) \text{ and } N \in \mathbb{N}_0. \tag{5.30}$$

For $\Delta \geq 1$, this gives rise to non-integrable divergences in L in any neighborhood of $L = 0$. Our procedure for eliminating these non-integrable terms is to apply a differential operator $\mathfrak{L}[L]$ to Ω that annihilates these terms but leaves the L-independent parts of Ω untouched. The remaining L dependence can then be eliminated by following the strategy indicated in the previous paragraph.

The desired operator $\mathfrak{L}[L]$ is constructed as follows. Define first the family of differential operators,

$$\mathcal{L}_\Delta^N[L] \equiv \left(1 + \Delta^{-1}L\frac{\partial}{\partial L}\right)^N = \Delta^{-1}L^{-\Delta}\frac{\partial^N}{\partial(\log L)^N}L^\Delta, \qquad \Delta \neq 0, N \in \mathbb{N}. \tag{5.31}$$

These operators are designed so as to act trivially on L-independent terms and annihilate terms of the form (5.30) with the same Δ-value and lower N-values. When acting on a term of the form (5.30) with a different Δ-value, the operator $\mathcal{L}_\Delta^N$ leaves the leading L-behavior unchanged in the sense of eq. (5.42) below. We define also the operators,

$$\mathcal{L}_0^N[L] \equiv \prod_{k=1}^{N} \left(1 - k^{-1} L \log L \frac{\partial}{\partial L}\right) = \prod_{k=1}^{N} \left(1 - k^{-1} \log L \frac{\partial}{\partial (\log L)}\right), \tag{5.32}$$

whose definition is unambiguous because the commutator between every k-factor vanishes. The $\mathcal{L}_0^N$ operator is designed to annihilate terms of the form (5.30) with $\Delta = 0$ and lower N-values. When $\mathcal{L}_0^N$ acts on a term of the form (5.30) with $\Delta \neq 0$, it produces terms of the same form and Δ-value (but generally increases the N-value). We define $\mathfrak{L}[L]$ by[6]

$$\mathfrak{L} \equiv \mathcal{L}_0^N \prod_{\Delta=1}^{D-4+\delta} \mathcal{L}_\Delta^N, \tag{5.33}$$

for any $N > 2$. Note that $[\mathcal{L}_\Delta^N, \mathcal{L}_{\Delta'}^{N'}] = 0$ for any $\Delta, \Delta' \neq 0$, so the order of composition between these operators does not matter. However, $[\mathcal{L}_0^N, \mathcal{L}_{\Delta\neq 0}^{N'}] \neq 0$, so the order of composition for $\mathcal{L}_0^N$ relative to the other operators $\mathcal{L}_{\Delta\neq 0}^N$ *does* matter. Note that (5.33) scales almost homogeneously with an overall factor of λ^0 under $L \to \lambda^{-1} L$ since it is composed of operators (5.31) and (5.32) with this property. By expanding out the product of operators in (5.33), note also that $\mathfrak{L}[L]$ can be rewritten in the general form,

$$\mathfrak{L}[L] = 1 + \sum_{\Delta=1}^{N(D-3+\delta)} \sum_{n=0}^{N} c_{n,\Delta} L^\Delta \log^n L \frac{\partial^\Delta}{\partial L^\Delta}, \tag{5.34}$$

6. The product over $\mathcal{L}_\Delta$-operators with different Δ-values is needed to account for the dependence of the Wick OPE coefficients on the dimensionful parameter m^2. In a theory without dimensionful parameters, we could eliminate the L-dependence of a flow relation by simply using the operator $\mathfrak{L} = \mathcal{L}_\Delta$ with Δ corresponding to the conformal scaling dimension of the flow relation. (Note that Δ would generally depend on the renormalized coupling parameter in an interacting theory.)

where $c_{n,\Delta}$ are L-independent numerical coefficients. Hence,

$$\mathfrak{L}[L]\Omega(x_1,x_2;z;L) - \Omega(x_1,x_2;z;L) = \sum_{\Delta=1}^{N(D-3+\delta)} \sum_{n=0}^{N} c_{n,\Delta} L^{\Delta} \log^n L \frac{\partial^{\Delta}}{\partial L^{\Delta}} \Omega(x_1,x_2;z;L). \tag{5.35}$$

Recalling $\partial_L \Omega$ is smooth in (x_1, x_2, m^2) in a neighborhood containing $z \in \mathbb{R}^D$, it follows the right-hand side of (5.35) is also smooth in (x_1, x_2, m^2) because every term involves at least one L-derivative of Ω. Every term on the right-hand side of (5.35) clearly scales almost homogeneously with an overall factor of $\lambda^{(D-4)}$.

We now define $\widetilde{\Omega}_\delta$ by

$$\widetilde{\Omega}_\delta(x_1,x_2;z;L) \equiv \mathfrak{L}[L]\Omega(x_1,x_2;z;L) - \sum_{|\gamma_1|+|\gamma_2|\le\delta} \frac{1}{\gamma_1!\gamma_2!} b^H_{\gamma_1\gamma_2}(L)(x_1-z)^{\gamma_1}(x_2-z)^{\gamma_2}, \tag{5.36}$$

where

$$b^H_{\gamma_1\gamma_2}(L) \equiv b^0_{\gamma_1\gamma_2} + \int_0^L dL' \left[\partial^{(x_1)}_{\gamma_1}\partial^{(x_2)}_{\gamma_2} \frac{\partial}{\partial L'}\left(\mathfrak{L}[L']\Omega(x_1,x_2;z;L')\right)\right]_{x_1,x_2=z}, \tag{5.37}$$

where $b^0_{\gamma_1\gamma_2}$ corresponds to the inherent ambiguities in our prescription discussed underneath eq. (5.48) below. That $b^H_{\gamma_1\gamma_2}(L)$ is well defined is a consequence of the following proposition:

Proposition 7. *For any $N > 2$, the L-dependent (Euclidean-covariant) tensors,*

$$\left[\partial^{(x_1)}_{\gamma_1}\partial^{(x_2)}_{\gamma_2} \frac{\partial}{\partial L}\left(\mathfrak{L}_\delta[L]\Omega(x_1,x_2;z;L)\right)\right]_{x_1,x_2=z}, \tag{5.38}$$

are integrable in L on a finite interval containing $L = 0$ for any $|\gamma_1| + |\gamma_2| \le \delta$

Proof. It is useful to first commute ∂_L past the operator $\mathfrak{L}_\delta[L]$ in (5.38). To do this, we note,

$$\partial_L \mathcal{L}^N_\Delta = \begin{cases} \Delta^{-N}(\Delta+1)^N \mathcal{L}^N_{(\Delta+1)}\partial_L & \Delta > 0 \\ (-1)^N (N!)^{-1} \log^N(L) \mathcal{L}^N_1 \partial_L & \Delta = 0 \end{cases}, \tag{5.39}$$

and, therefore,

$$\partial_L \mathfrak{L} \propto \log^N(L) \left(\prod_{\Delta=1}^{D-3+\delta} \mathcal{L}^N_\Delta \right) \partial_L . \tag{5.40}$$

Plugging this back into (5.38) and noting the smoothness of $\partial_L \Omega$ in (x_1, x_2), we obtain,

$$\text{formula (5.38)} \propto \log^N(L) \left(\prod_{\Delta=1}^{D-3+\delta} \mathcal{L}^N_\Delta[L] \right) \left[\partial^{(x_1)}_{\gamma_1} \partial^{(x_2)}_{\gamma_2} \frac{\partial}{\partial L} \Omega(x_1, x_2; z; L) \right]_{x_1, x_2 = z} . \tag{5.41}$$

Noting that

$$\mathcal{L}^{N'}_{\Delta'}(L^{-\Delta} \log^N L) = \begin{cases} 0 & \Delta' = \Delta, N' > N \\ \mathcal{O}(L^{-\Delta} \log^N L) & \Delta' \neq \Delta, N' > 0 \end{cases}, \tag{5.42}$$

we see that, for any $N' > N$ and all $\Gamma \leq \delta$, all non-integrable terms of the form (5.30) are annihilated by the string of operators $\mathcal{L}^{N'}_1 \mathcal{L}^{N'}_2 \cdots \mathcal{L}^{N'}_{(D-3+\delta)}$ appearing in (5.29), and that any integrable terms of the form (5.30) remain integrable after application of $\mathcal{L}^{N'}_1 \mathcal{L}^{N'}_2 \cdots \mathcal{L}^{N'}_{(D-3+\delta)}$. Noting that H_E contains at most one power of the logarithm and Ω depends quadratically on H_E, we conclude the right-hand side of (5.41) and, thus, the Taylor coefficients (5.38) must be integrable on an interval containing $L = 0$ for any $N > 2$ as we desired to show. □

Remark 19. By the same reasoning as in (5.28), it follows that $\mathfrak{L}\Omega[\chi] = \Omega[\mathfrak{L}\chi]$. For any cutoff function χ of the form (5.16), it follows from (5.34) that $\mathfrak{L}\chi$ is also a smooth cutoff function. Hence, Proposition 7 implies $\mathfrak{L}$ acts as a map from the set of cutoff functions of the form (5.16) to the set of cutoff functions such that, to scaling degree δ, the asymptotic expansion of $\partial_L(\Omega[\mathfrak{L}\chi])$ diverges, at worst, logarithmically as $L \to 0^+$.

Note the translational symmetry of Ω implies $b^H_{\gamma_1\gamma_2}$ are independent of z and rotational symmetry of Ω implies $b^H_{\gamma_1\gamma_2}$ are composed of products of the Euclidean metric[7] and, thus, vanish unless $|\gamma_1|, |\gamma_2|$ are even. Recalling the definition (5.21) of Ω and the fact that $(C_H)_{\phi\phi}^{(\partial_{\gamma_1}\phi\partial_{\gamma_2}\phi)}(x_1, x_2; z) = (x_1 - z)^{\gamma_1}(x_2 - z)^{\gamma_2}/(\gamma_1!\gamma_2!)$, we note that the $\widetilde{\Omega}_\delta$ defined in (5.36) is identical to the right-hand side of the L-independent flow relation (2.5) claimed in Chapter 2 for the special case that $n = 2$. i.e., b^H_C in formula (2.5) is given explicitly by (5.37) for $[C]_\phi = 2$ and vanishes otherwise.

The required L-independence (5.27) of $\widetilde{\Omega}_\delta$ is verified by differentiating (5.36),

$$\begin{aligned}
&\frac{\partial}{\partial L}\widetilde{\Omega}_\delta(x_1, x_2; z; L) \\
&= \frac{\partial}{\partial L}\left[\mathfrak{L}[L]\Omega(x_1, x_2; z; L)\right] - \sum_{|\gamma_1|+|\gamma_2|\leq\delta} \frac{1}{\gamma_1!\gamma_2!}\frac{\partial}{\partial L} b^H_{\gamma_1\gamma_2}(L)(x_1 - z)^{\gamma_1}(x_2 - z)^{\gamma_2} \\
&\sim_\delta \sum_{|\gamma_1|+|\gamma_2|\leq\delta} \frac{1}{\gamma_1!\gamma_2!}(x_1 - z)^{\gamma_1}(x_2 - z)^{\gamma_2} \times \\
&\qquad\times \left[\left[\partial^{(x_1)}_{\gamma_1}\partial^{(x_2)}_{\gamma_2}\frac{\partial}{\partial L}\left(\mathfrak{L}[L]\Omega(x_1, x_2; z; L)\right)\right]_{x_1, x_2 = z} - \frac{\partial}{\partial L} b^H_{\gamma_1\gamma_2}(L)\right] \\
&\sim_\delta 0,
\end{aligned} \tag{5.43}$$

where in going to the third line we have used the smoothness of $\partial_L(\mathfrak{L}\Omega)$ in (x_1, x_2) to Taylor expand the first term in the second line around $x_1, x_2 = z$. The final line then follows from the definition (5.37) of $b^H_{\gamma_1\gamma_2}$ and the fundamental theorem of calculus. Thus, our construction (5.36) of $\widetilde{\Omega}_\delta$ complies with the required properties.

7. Note that Ω is invariant under the full orthogonal group including improper rotations. Although the Levi-Civita symbols $\epsilon_{\mu_1\cdots\mu_n}$ are invariant under proper rotations, the Euclidean metric is the only tensor invariant under all $R \in O(D)$.

It is worth noting that, using the formulas for the Hadamard-normal ordered coefficients

$$(C_H)^I_{\phi^2\phi\phi}(y,x_1,x_2;z) = 2H_E(y,x_1)H_E(y,x_2) \tag{5.44}$$

$$(C_H)^I_{\phi^2(\partial_{\gamma_1}\phi\cdots\partial_{\gamma_k}\phi)}(y,z;z) = \begin{cases} 2\partial^{(z)}_{\gamma_1}H_E(y,z)\partial^{(z)}_{\gamma_2}H_E(y,z) & k=2 \\ 0 & \text{otherwise} \end{cases} \tag{5.45}$$

$$(C_H)^{(\partial_{\gamma_1}\phi\partial_{\gamma_2}\phi)}_{\phi\phi}(x_1,x_2;z) = \frac{1}{\gamma_1!\gamma_2!}(x_1-z)^{(\gamma_1}(x_2-z)^{\gamma_2)} \tag{5.46}$$

the flow relation (5.26) can be written equivalently as,

$$\begin{aligned}\frac{\partial}{\partial m^2}(C_H)^I_{\phi\phi}(x_1,x_2;z) \sim_\delta & -\frac{1}{2}\int d^Dy\,\mathfrak{L}[L]\chi(y,z;L)(C_H)^I_{\phi^2\phi\phi}(y,x_1,x_2;z)+ \\ & -\sum_{[C]\le\delta+2} b^H_C(L)(C_H)^C_{\phi\phi}(x_1,x_2;z),\end{aligned} \tag{5.47}$$

where, for $L>0$,

$$b^H_C(L) = b^0_C - \frac{1}{2}\int_0^L dL'\int d^Dy\frac{\partial}{\partial L'}\left(\mathfrak{L}[L']\chi(y,z;L')\right)(C_H)^I_{\phi^2C}(y,z;z). \tag{5.48}$$

The only ambiguities in our construction arise from a limited choice for the value of the L-independent b^0_C-tensors when $[C]_\phi = 2$. We have $b^0_C = 0$ unless $[C]_\phi = 2$. The b^0_C are required to depend smoothly on $(\delta_{\mu\nu}, m^2)$ and scale exactly homogeneously under $(\delta_{\mu\nu}, m^2) \to (\lambda^{-2}\delta_{\mu\nu}, \lambda^2 m^2)$ with an overall factor of $\lambda^{(D-4)}$. This implies that b^0_C must vanish identically when D is odd. In even spacetime dimensions, these ambiguities correspond to the freedom to choose the Taylor coefficients of a smooth, Euclidean-invariant function which depends smoothly on $(\delta_{\mu\nu}, m^2)$ and scales exactly homogeneously under $(\delta_{\mu\nu}, m^2) \to (\lambda^{-2}\delta_{\mu\nu}, \lambda^2 m^2)$ with an overall factor of $\lambda^{(D-4)}$. For any fixed cutoff function χ and any choice of Euclidean-invariant Hadamard parametrix H_E which scales almost homogeneously, one can choose $b^0_{\gamma_1\gamma_2}$ such that OPE coefficients obtained via Hadamard normal ordering satisfy (5.47). Conversely, for any fixed χ and admissible choice of $b^0_{\gamma_1\gamma_2}$, one can find an H_E such that the Hadamard-normal-ordered OPE coefficients satisfy (5.47). Hence, the ambiguity in our construction of the L-independent flow relation (5.47) is in a 1-1 correspondence with the inherent freedom to choose a Hadamard parametrix for defining normal-ordered Wick fields compatible with axioms W1-W8.

Remark 20. In flat space and all dimensions $D \geq 2$, we note the conservation axiom W8 places no constraints on the ambiguities in H_E and, thus, does not require any further modifications to the flow relation (5.47). In particular, although H_E is not an exact Greens function of the Euclidean Klein-Gordon operator (5.1), it does automatically satisfy the Euclidean version of the conservation constraint (4.63):

$$\nabla^{(x_1)}_\mu h_E(x_1, x_2)|_{x_1,x_2=z} = \left[\nabla^{(x_1)}_\mu K_{x_2} H_E(x_1, x_2) - \nabla^{(x_1)}_\mu \delta(x_1, x_2)\right]_{x_1,x_2=z} = 0, \quad (5.49)$$

where we recall the smooth function h_E was defined via (5.15). Because H_E is required to be invariant under the inhomogeneous orthogonal group, $\nabla^{(x_1)}_\mu h_E(x_1, x_2)|_{x_1,x_2=z}$ must be invariant under rotations about the point z. However, since there does not exist a rotationally-invariant D-vector, we conclude $\nabla^{(x_1)}_\mu h_E(x_1, x_2)|_{x_1,x_2=z}$ identically vanishes in flat Euclidean space for any dimension, including $D = 2$.

By the same reasoning used in the proof of Theorem 5, the flow relation (5.47) for $(C_H)^I_{\phi\phi}$ straightforwardly implies flow relations for $(C_H)^I_{\phi\cdots\phi}$ as expressed in the following theorem:

Theorem 6. *For any Hadamard parametrix satisfying* (5.47)*, the corresponding Hadamard normal-ordered coefficients* $(C_H)^I_{\phi\cdots\phi}$ *satisfy the flow relation:*

$$\begin{aligned}\frac{\partial}{\partial m^2}(C_H)^I_{\phi\cdots\phi}(x_1, \ldots, x_n; z) \approx &-\frac{1}{2}\int d^D y\, \mathfrak{L}[L]\chi(y, z; L)(C_H)^I_{\phi^2\phi\cdots\phi}(y, x_1, \ldots, x_n; z) + \\ &- \sum_C b^H_C(L)(C_H)^C_{\phi\cdots\phi}(x_1, \ldots, x_n; z),\end{aligned} \quad (5.50)$$

where $b_C^H(L)$ is again given by (5.48) *with the same constraints on b_C^0 as stated below* (5.48)*.*

Finally, the results of Section 4.2 can be used to obtain the flow relations for $C^I_{\phi\cdots\phi}$ for an arbitrary prescription for Wick monomials satisfying W1-W8. We obtain

$$\frac{\partial}{\partial m^2}C^I_{\phi\cdots\phi}(x_1,\ldots,x_n;z) \approx -\frac{1}{2}\int d^Dy\,\mathfrak{L}[L]\chi(y,z;L)\,C^I_{\phi^2\phi\cdots\phi}(y,x_1,\ldots,x_n;z)+ \\ -\sum_C b_C(L)C^C_{\phi\cdots\phi}(x_1,\ldots,x_n;z)+\text{“}F_k\text{-terms”}, \tag{5.51}$$

where

$$b_C(L)\equiv\delta_{n,2}\,b_C^0-\frac{1}{2}\int_0^L dL'\int d^Dy\frac{\partial}{\partial L'}\left(\mathfrak{L}[L']\chi(y,z;L')\right)C^I_{\phi^2C}(y,z;z), \tag{5.52}$$

and “F_k-terms” denotes terms that contain at least one factor of F_k (for $k\le n$). By the discussion in Section 4.2 below eq. (4.48), F_j can, in turn, be written purely in terms of OPE coefficients of the form $C^I_{\phi\cdots\phi}(x_1,\ldots,x_i;z)$ such that $i\le j$. In this way, all terms on the right-hand side of (5.51) are expressible entirely in terms of OPE coefficients and the cutoff function χ, and (5.51) yields the flow relation for the OPE coefficients corresponding to an arbitrary prescription for the Wick fields compatible with the axioms W1-W8. Note, in contrast to $b_C^H(L)$ given in (5.48), here $b_C(L)$ can be nonzero when $[C]_\phi\neq 2$ since, for prescriptions not corresponding to normal ordering, $C^I_{\phi^2C}$ is generally nonzero when $[C]_\phi\neq 2$.

CHAPTER 6

FLOW RELATIONS FOR OPE COEFFICIENTS IN MINKOWSKI SPACETIME

We turn, now, to the derivation of flow relations for OPE coefficients in Minkowski spacetime $(\mathbb{R}^D, \eta_{ab})$. As can be seen from the derivation of the Euclidean flow relations in the preceding chapter, it is essential that the two-point OPE coefficient for which we are obtaining a flow relation be a Green's parametrix for the wave equation. Consequently, we do not believe it is possible to obtain a flow relation for the Lorentzian $C^I_{\phi\phi}$, since it does not have this property. However, as we shall show, a flow relation for $C^I_{T_0\{\phi\phi\}}$ can be obtained, where T_0 denotes the unextended time-ordered-product.

In the Minkowski case, if we choose $C^I_{T_0\{\phi\phi\}}$ to be the exact Feynman propagator for $m^2 > 0$, the spacetime integral that would appear in the flow relation will not converge, so we would need to introduce a cutoff function even in this case. Therefore, in contrast to the Euclidean case, there is no advantage in initially working with the exact Feynman propagator as compared with a Poincaré-invariant Feynman parametrix that is smooth at $m^2 = 0$. As we shall see, a new difficulty arises from a cutoff in the Minkowski case in that there does not exist a nontrivial function of compact support that is Lorentz invariant. Consequently, in the Minkowski case, the introduction of a cutoff spoils the Poincaré invariance of the flow relations. Nevertheless, we shall show that counterterms can be introduced into the flow relations so as to restore Poincaré invariance. The presence of the cutoff function in the flow relations also spoils their scaling behavior. However, this can be fixed using the same procedure as developed for the Euclidean flow relations. Thus, we will, in the end, obtain entirely satisfactory flow relations for the OPE coefficients of unextended time-ordered-products in Minkowski spacetime (see Theorem 7). These flow relations will be unique up to modifications of the counterterms that correspond to the ambiguities in the definitions of the Wick monomials themselves.

The requirement W1 that the Wick monomials be locally and covariantly defined implies that, in Minkowski spacetime, the Wick monomials must be Poincaré covariant [15]. Thus, in a Hadamard normal-ordering prescription, we must use a Poincaré-invariant Hadamard parametrix. Since, in this chapter, we will want to include the case $m^2 = 0$, we will not use the usual choice $\langle\phi(x_1)\phi(x_2)\rangle_{\rm vac}$—which fails to be smooth in m^2 at $m^2 = 0$—but rather will take $H(x_1, x_2; m^2)$ to be given by eq. (3.26), with ℓ fixed (i.e., independent of m^2).

The starting point for our derivation of Euclidean flow relations in the preceding chapter was the preliminary flow-like equation (5.18) for the Euclidean Hadamard parametrix $H_E(x_1, x_2; m^2)$. The key ingredients that went into the derivation of this equation were (i) that H_E is a fundamental solution (5.15) of the Klein-Gordon operator up to smooth remainder and (ii) for any test function f, $H_E(y, f)$ is smooth in y. In Minkowski spacetime, the OPE coefficient $(C_H)^I_{\phi\phi} = H(x_1, x_2)$ will *not* be a Green's parametrix, i.e., it will satisfy $K_{x_1}(C_H)^I_{\phi\phi}(x_1, x_2) = \text{smooth}$ rather than $K_{x_1}(C_H)^I_{\phi\phi}(x_1, x_2) = (\delta(x_1, x_2) + \text{smooth})$. Consequently, the analog of condition (i) will not be satisfied and we cannot expect to obtain flow relations for the ordinary OPE coefficients. However, condition (i) does hold for the Feynman parametrix $H_F(x_1, x_2; m^2)$ given by eq. (4.55). Such a parametrix satisfies,

$$(-\eta^{ab}\partial_a\partial_b + m^2)H_F(x_1, x_2; m^2) = -i\delta(x_1, x_2) + h(x_1, x_2; m^2), \tag{6.1}$$

where h is a smooth function of its arguments. As with the Euclidean parametrix, any two Feynman parametrices H_F and H'_F satisfying (6.1) can differ by a Poincaré invariant smooth function of (x_1, x_2). Since $(C_H)^I_{T_0\{\phi\phi\}} = H_F(x_1, x_2)$, it might be expected that flow relations will hold for the OPE coefficients of time-ordered products[1]. As we shall see below, flow relations do indeed hold for the OPE coefficients of time-ordered products.

Condition (ii) also holds for $H_F(x_1, x_2; m^2)$. Indeed, for any translation invariant dis-

1. Indeed, this also could be anticipated from the fact that a Wick rotation from Euclidean space to Minkowski spacetime will take the Euclidean Green's function G_E to the Feynman propagator G_F.

tribution $\mathcal{D}(x_1, x_2)$ on $\mathbb{R}^D \times \mathbb{R}^D$ and any test function f on $\mathbb{R}^D$, we have that $\mathcal{D}(x_1, f)$ is smooth in x_1. Namely, if we define new variables $X_1 = x_1 + x_2$ and $X_2 = x_1 - x_2$, then, by translation invariance, $\mathcal{D}$ cannot depend on X_1, so the elements of its wavefront set must be of the form $(X_1, 0; X_2, K_2)$ with $K_2 \neq 0$. Therefore, in terms of the original variables (x_1, x_2), the elements of WF$[\mathcal{D}]$ must be of the form $(x_1, k_1; x_2, -k_1)$ with $k_1 \neq 0$. The wavefront set calculus rules then immediately imply that $\mathcal{D}(x_1, f)$ is smooth for any test function f.

Remark 21. Since the unextended time-ordered products are only defined away from all diagonals, applying the Klein-Gordon operator to $(C_H)^I_{T_0\{\phi\phi\}} = T_0\{H(x_1, x_2)\}$ will yield a distribution that is a priori only defined when $x_1 \neq x_2$ and, thus, the OPE coefficient $(C_H)^I_{T_0\{\phi\phi\}}$ is itself not actually a Green's function satisfying (6.1). Nevertheless, as discussed in Remark 18 below Theorem 6, the extension of $T_0\{H(x_1, x_2)\}$ to $x_1 = x_2$ is uniquely given by the Feynman parametrix $H_F = H - i\Delta^{\text{adv}}$. Hence, whenever we need to use the identity (6.1) in what follows below, we may, without introducing any new ambiguities, first extend $(C_H)^I_{T_0\{\phi\phi\}}$ to its diagonal $x_1 = x_2$ and then subsequently apply the Green's function identity (6.1) for the Feynman parametrix. As we will see, this is sufficient to derive all the flow relations for the time-ordered Wick OPE coefficients of the form $C^I_{T_0\{\phi\cdots\phi\}}$. As discussed in Chapter 2 and Section 4.3, unique extensions of the OPE coefficients appearing inside the integral on the right-hand side of the flow relations are only possible, in general, to the "partial diagonals", where the integration variable y coincides with only a single x_i-spacetime variable, so we will continue to write all OPE coefficients appearing in the flow relations with the unextended time-ordering symbol T_0 rather than T, with the understanding that (unique) extensions to the appropriate partial diagonals with y are necessary for evaluating the y-integral. See Remark 23 below Theorem 7 for further discussion regarding the extension of the OPE coefficients appearing in the Minkowski flow relations.

Since conditions (i) and (ii) hold for H_F, we can directly parallel the derivation of the key preliminary Euclidean flow-like equation (6.2) for H_E to obtain a flow-like relation for $H_F(x_1, x_2; m^2)$ by introducing a cutoff function $\chi(y, z; L)$ defined such that $\chi = 1$ for y in some compact neighborhood, $\mathcal{B}_1$, of z and $\chi = 0$ outside of some larger compact neighborhood, $\mathcal{B}_2$, of z. We again denote by L the arbitrary length scale which is required to define a spacetime cutoff. Then, for $x_1, x_2 \in \mathcal{B}_1$, we similarly obtain,

$$
\begin{aligned}
\frac{\partial}{\partial m^2} H_F(x_1, x_2; m^2) = -i \int_{\mathcal{B}_2} d^D y \, \chi(y; z; L) H_F(y, x_1; m^2) H_F(y, x_2; m^2) + \\
+ i \int_{\mathcal{B}_2 \setminus \mathcal{B}_1} d^D y \, \partial_\mu^{(y)} \chi(y; z; L) \left[\partial^\mu_{(y)} H_F(y, x_1; m^2) \frac{\partial}{\partial m^2} H_F(y, x_2; m^2) + \right. \\
\left. - H_F(y, x_1; m^2) \partial^\mu_{(y)} \frac{\partial}{\partial m^2} H_F(y, x_2; m^2) \right] + \\
+ i \int_{\mathcal{B}_2} d^D y \, \chi(y; z; L) \left[H_F(y, x_1; m^2) \frac{\partial}{\partial m^2} h(y, x_2; m^2) + \right. \\
\left. - h(y, x_1; m^2) \frac{\partial}{\partial m^2} H_F(y, x_2; m^2) \right],
\end{aligned}
\tag{6.2}
$$

where h is defined via eq. (6.1). Note that the factor of $\partial_\mu^{(y)} \chi(y; z; L)$ appearing in the second line has support only on $\mathcal{B}_2 \setminus \mathcal{B}_1$ because we require $\chi(y; z; L) = 1$ for $y \in \mathcal{B}_1$. Note also that eq. (6.2) is identical to (5.18) modulo the substitutions $H_E \to i H_F$ and $h_E \to ih$.

As in the Euclidean formula (5.18), the fourth and fifth lines are automatically smooth on account of the smoothness of h and the compact-support of χ. Similarly, in the second and third lines, the differentiated cutoff function $\partial_\mu^{(y)} \chi(y; z; L)$ is only nonzero when $y \in \mathcal{B}_2 \setminus \mathcal{B}_1$ and thus vanishes when $y = x_1, x_2$ if $x_1, x_2 \in \mathcal{B}_1$. However, whereas the Euclidean parametrix $H_E(y, x)$ is singular only when $y = x$, the singular support of the Feynman parametrix $H_F(y, x)$ includes all (y, x) such that y and x can be connected by a null geodesic. Thus, the integrand in the second and third lines of (6.2) will be singular even for $y \in \mathcal{B}_2 \setminus \mathcal{B}_1$ whenever y is lightlike separated from either or both (x_1, x_2). Therefore, it is not at all obvious that the integral will yield a smooth function. However, since the partial m^2-derivative does not alter the wavefront set of H_F, the terms in the second and third lines of

(6.2) will be smooth if and only if the quantity,

$$\Theta[\chi, H_F](x_1, x_2; z; m^2) \equiv \int_{\mathcal{B}_2\backslash\mathcal{B}_1} d^D y\, \partial^{(y)}_\mu \chi(y, z; L) H_F(y, x_1; m^2)\partial^\mu_{(y)} H_F(y, x_2; m^2), \quad (6.3)$$

is smooth. The following proposition establishes smoothness of this quantity:

Proposition 8. *For $x_1, x_2 \in \mathcal{B}_1(z)$, the quantity Θ defined by* (6.3) *is a C^∞ function of (x_1, x_2).*

Proof. A generalized function is smooth if and only if its wavefront set is the empty set. We show the wavefront set of the generalized function (6.3) is contained in the empty set when $x_1, x_2 \in \mathcal{B}_1(z)$ and, thus, $\Theta(x_1, x_2; z)$ must be smooth. Note first the wavefront set of a Feynman parametrix is,

$$\begin{aligned} \mathrm{WF}[H_F] = \mathrm{WF}[\delta] \cup \Big\{ &(x_1, k_1; x_2, k_2) \in \times^2 (T^*\mathbb{R}^D \backslash Z^*\mathbb{R}^D) \mid x_1 \neq x_2, (x_1, k_1) \sim (x_2, -k_2), \\ &k_1 \in \dot{V}^+_{x_1} \text{ if } x_1 \in J^+(x_2), k_1 \in \dot{V}^-_{x_1} \text{ if } x_1 \in J^-(x_2) \Big\}, \end{aligned} \quad (6.4)$$

where we recall the notation: $\dot{V}^\pm_x$ denotes, respectively the boundary of the future/past lightcone at x; $(x, k) \sim (y, p)$ iff points x and y may be joined by a null geodesic γ such that k and p are cotangent and coparallel to γ; and $Z^*\mathbb{R}^D$ denotes the zero section of the cotangent bundle $T^*\mathbb{R}^D$. Recall the wavefront set of the δ-distribution was given in (5.4).

We write $\mathcal{B} \equiv \mathcal{B}_2\backslash\mathcal{B}_1$. Theorem 8.2.14 of [3] immediately implies the wavefront set of the bi-distribution (6.3) is bounded by the union of three sets,

$$\mathrm{WF}[\Theta] \subseteq \left(\mathrm{WF}'[H_F] \circ \mathrm{WF}[H_F]\right) \cup (\mathrm{WF}_{\mathcal{B}}[H_F] \times (\mathcal{B}) \times \{0\}) \cup ((\mathcal{B}) \times \{0\} \times \mathrm{WF}_{\mathcal{B}}[H_F])\,. \quad (6.5)$$

Here the notation is defined as follows: For any $u \in \mathcal{D}'(\mathbb{R}^D \times \mathbb{R}^D)$,

$$\mathrm{WF}'[u] \equiv \{(x,k;y,p)\,|\,(x,k;y,-p) \in \mathrm{WF}[u]\} \tag{6.6}$$

$$\mathrm{WF}_{\mathcal{B}}[u] \equiv \{(x,k)\,|\,(x,k;y,0) \in \mathrm{WF}[u] \text{ for some } y \in \mathcal{B}\}\,. \tag{6.7}$$

For any $u, v \in \mathcal{D}'(\mathbb{R}^D \times \mathbb{R}^D)$, the composition of wavefront sets $\mathrm{WF}'[u]$ and $\mathrm{WF}[v]$ goes as,

$$\mathrm{WF}'[u] \circ \mathrm{WF}[v] \equiv \Big\{(x_1,k_1;x_2,k_2)\,|\,(y,p;x_1,k_1) \in \mathrm{WF}'[u] \text{ and } (y,p;x_2,k_2) \in \mathrm{WF}[v], \\ \text{for some } (y,p) \in (\mathcal{B} \times \mathbb{R}^D \backslash \{0\})\Big\}\,. \tag{6.8}$$

The form of the Feynman wavefront set (6.4) immediately implies that[2],

$$\mathrm{WF}_{\mathcal{B}}[H_F] \subset \emptyset, \tag{6.9}$$

so nontrivial contributions to the right-hand side of (6.5) could only potentially come from the set $\mathrm{WF}'[H_F] \circ \mathrm{WF}[H_F]$. We show now this set is empty. Note, for $y \in \mathcal{B}$, we have $(y,p;x_1,k_1) \in \mathrm{WF}'[H_F]$ and $(y,p;x_2,k_2) \in \mathrm{WF}[H_F]$ only if all three spacetime points (y, x_1, x_2) reside on the same null geodesic. Furthermore, when $x_1, x_2 \in \mathcal{B}_1$, then any $y \in \mathcal{B}$ must be either to the future or to the past of both x_1 and x_2. Consider first the case where y is to the future of x_1: By (6.4), $(y,p;x_1,k_1) \in \mathrm{WF}'[H_F]$ only if $p \in V_y^-$. However, when y is to the future of x_2, then $(y,p;x_2,k_2) \in \mathrm{WF}[H_F]$ only if $p \in V_y^+$. Since $V_y^- \cap V_y^+ \subset \emptyset$, it follows that, when y is to the future of both points, there are no nontrivial elements in (6.8). In the case where y lies instead to the past of both points, one arrives at the same

2. In fact, eq. (6.9) would hold if H_F was replaced with any bi-distribution whose wavefront set contains only covectors such that $k_1 = -k_2$. Hence, by the discussion above, it holds also for all translationally-invariant bi-distributions.

conclusions only with the roles of V_y^+ and V_y^- swapped. Therefore, when $x_1, x_2 \in \mathcal{B}_1$,

$$\mathrm{WF}'[H_F] \circ \mathrm{WF}[H_F] \subset \emptyset, \tag{6.10}$$

and, thus, (6.5) implies,

$$\mathrm{WF}[\Theta] \subseteq \emptyset, \tag{6.11}$$

which is what we sought to show. □

Remark 22. The proof of Proposition 8 would *not* go through if the Feynman parametrix, H_F, was replaced by parametrices for the advanced, G_A, or retarded, G_R, Green's functions. In particular, one finds, $\mathrm{WF}'[G_{A/R}] \circ \mathrm{WF}[G_{A/R}] = \mathrm{WF}[G_{A/R}]$, respectively, so (6.10) would no longer hold. Note also, despite its apparent similarity to Θ, Proposition 8 does *not* apply to the integral on the first line of (6.2) which is *not* a smooth function in (x_1, x_2). In particular, for the result of Proposition 8, it was critical that $y \notin \mathcal{B}_1$; otherwise, it would be possible for y to simultaneously lie to the past of one point and to the future of the other, while being an element of both $(y, p; x_1, k_1) \in \mathrm{WF}'[H_F]$ and $(y, p; x_2, k_2) \in \mathrm{WF}[H_F]$, in which case, $\mathrm{WF}'[H_F] \circ \mathrm{WF}[H_F] = \mathrm{WF}[H_F] \neq \emptyset$ and (6.10) no longer holds.

Since the second through fifth lines of (6.2) are smooth, we may attempt to drop these terms and replace that flow relation with

$$\begin{aligned} \frac{\partial}{\partial m^2} H_F(x_1, x_2; m^2) &= \Omega_M(x_1, x_2; z; m^2; L) \\ &\equiv -i \int_{\mathcal{B}_2} d^D y \, \chi(y; z; L) H_F(y, x_1; m^2) H_F(y, x_2; m^2). \end{aligned} \tag{6.12}$$

As in the Euclidean case, this replacement will lead to difficulties with scaling behavior under $(\eta_{ab}, m^2) \to (\lambda^{-2}\eta_{ab}, \lambda^2 m^2)$. (As previously mentioned, in a fixed global inertial coordinate system, this is equivalent to rescaling $(\eta_{\mu\nu}, d^D y, m^2) \to (\lambda^{-2}\eta_{\mu\nu}, \lambda^{-D} d^D y, \lambda^2 m^2)$.) If this were the only difficulty with (6.12), it could be dealt with in the same manner as in the

Euclidean case. However, a potentially much more serious difficulty arises from the fact that (6.12) fails to be Poincaré-invariant since there do not exist Lorentz-invariant functions of compact support[3],

$$\chi(\Lambda y, \Lambda z) \neq \chi(y, z). \tag{6.13}$$

Hence, for a Lorentzian metric, naively dropping the second through fifth lines of (6.2) would necessarily violate the locality and covariance axiom W1, since this axiom implies Poincaré invariance in the case of flat spacetime.

It follows from the smoothness of the second through fifth lines of (6.2) for all $x_1, x_2 \in \mathcal{B}_1(z)$ that the failure of (6.12) to be Poincaré-invariant on its own must then be given by a smooth function of (x_1, x_2). More precisely, for any $x_1, x_2 \in \mathcal{B}_1(z)$ and any Poincaré transformation P such that $Px_1, Px_2 \in \mathcal{B}_1(Pz)$, the quantity

$$\Omega_M(Px_1, Px_2; Pz) - \Omega_M(x_1, x_2; z) \tag{6.14}$$

is smooth in (x_1, x_2). Therefore, in parallel with our restoration of desired scaling behavior in the Euclidean case, we will restore Poincaré invariance to the flow relation (6.12) if we can replace Ω_M on the right-hand side of that equation with a distribution $\widetilde{\Omega}_{M,\delta}$ which satisfies the following two properties:

1. For $(x_1, x_2) \in \mathcal{B}_1(z)$,

$$\begin{aligned} &\widetilde{\Omega}_{M,\delta}(x_1, x_2; z; m^2; L) \\ &\equiv \Omega_M(x_1, x_2; z; m^2; L) - \sum_{|\gamma_1|+|\gamma_2|\leq\delta} \frac{1}{\gamma_1!\gamma_2!} a_{\gamma_1\gamma_2}(\chi)(x_1 - z)^{\gamma_1}(x_2 - z)^{\gamma_2}, \end{aligned} \tag{6.15}$$

where $a_{\gamma_1\gamma_2} = a_{\gamma_2\gamma_1}$ are constant tensors that scale almost homogeneously under

3. Note the function used in Euclidean space, $\zeta\left(L^{-2}\sigma(y,z)\right)$, is Lorentz invariant but not compactly-supported in Minkowski spacetime, since $\sigma(y, z)$ is zero on the boundary of the entire lightcone of point z.

$(\eta_{ab}, m^2, L) \to (\lambda^{-2}\eta_{ab}, \lambda^2 m^2, \lambda^{-1}L)$ with an overall factor of $\lambda^{(D-4)}$.

2. To asymptotic degree δ, $\widetilde{\Omega}_{M,\delta}$ is asymptotically Poincaré-invariant with respect to (x_1, x_2, z). That is, for any Poincaré transformation P such that $(Px_1, Px_2) \in \mathcal{B}_1(Pz)$,

$$\widetilde{\Omega}_{M,\delta}(Px_1, Px_2; Pz; m^2; L) \sim_\delta \widetilde{\Omega}_{M,\delta}(x_1, x_2; z; m^2; L). \tag{6.16}$$

Note it is not required that $\widetilde{\Omega}_{M,\delta}$ be Poincaré-invariant at asymptotic degrees higher than δ. Any two $\widetilde{\Omega}_{M,\delta}$ satisfying these properties may differ, to scaling degree δ, by at most a quantity of the form $L^{(D-4)} f(m^2\sigma(x_1, x_2), L^{-2}\sigma(x_1, x_2))$, where f is a smooth bi-variate function. Thus, the difference between any two $a_{\gamma_1\gamma_2}$ and $a'_{\gamma_1\gamma_2}$ in (6.15) is necessarily of the form,

$$a_{\gamma_1\gamma_2} - a'_{\gamma_1\gamma_2} = L^{(D-4)}\, \partial^{(x_1)}_{\gamma_1} \partial^{(x_2)}_{\gamma_2} f(m^2\sigma(x_1, x_2), L^{-2}\sigma(x_1, x_2))\Big|_{x_1, x_2 = z}. \tag{6.17}$$

If we can find a distribution $\widetilde{\Omega}_{M,\delta}$ satisfying the above two properties, then the flow relation

$$\frac{\partial}{\partial m^2} H_F(x_1, x_2; m^2) \sim_\delta \widetilde{\Omega}_{M,\delta}(x_1, x_2; z; m^2; L), \tag{6.18}$$

will be Poincaré invariant. This flow relation still fails to scale almost homogeneously with respect to the metric and m^2 due to the dependence of $\widetilde{\Omega}_{M,\delta}$ on L. However, the unwanted L-dependence can then be eliminated by the same procedure as used in the Euclidean case treated in Section 5.2. Thus, we will be able to obtain satisfactory flow relation if we can find a distribution $\widetilde{\Omega}_{M,\delta}$ satisfying the above two properties. We turn now to the construction of the tensors $a_{\gamma_1\gamma_2}$ in the definition (6.16) of $\widetilde{\Omega}_M$ such that $\widetilde{\Omega}_{M,\delta}$ is Poincaré invariant to scaling degree δ in the sense of (6.16).

Although we cannot choose the cutoff function $\chi(y, z)$ to be Lorentz invariant, we can require that it be invariant under a simultaneous translation of (y, z). In particular, we can choose a global inertial coordinate system on Minkowski spacetime and take χ to be given by

$$\chi(y, z; L; t^\mu) = \zeta\left(L^{-2}\left(\eta_{\mu\nu} + 2t_\mu t_\nu\right)(y-z)^\mu (y-z)^\nu\right), \tag{6.19}$$

where t^μ is proportional to the unit time vector field of these coordinates but is required to remain unit normalized with respect to the metric components under the rescaling $\eta_{\mu\nu} \to \lambda^{-2}\eta_{\mu\nu}$, i.e., under this rescaling, it is required that $t_\mu \to \lambda^{-1} t_\mu$. As in the Euclidean case, ζ is a test function and $\zeta(s) = 1$ if $|s| \leq 1$ and $\zeta(z) = 0$ if $|s| > 2$. Note that $\eta_{\mu\nu} + 2t_\mu t_\nu$ is a Riemannian metric with components $\mathrm{diag}(+1, \ldots, +1)$ in the chosen global inertial coordinates, so (6.19) is supported on a D-dimensional coordinate ball of radius $2L$. Equation (6.19) is manifestly translationally invariant under a simultaneous translation of (y, z). It is also invariant under pure spatial rotations $(y, z) \to (Ry, Rz)$ since $(R^{-1}t)_\mu = t_\mu$, but it is not invariant under Lorentz boosts. Note also the cutoff (6.19) is invariant under the rescaling $(\eta_{ab}, L) \to (\lambda^{-2}\eta_{ab}, \lambda^{-1}L)$ with the coordinate basis held fixed.

For any translationally-invariant χ and any Poincaré transformation P composed of an arbitrary Lorentz transformation Λ together with an arbitrary translation, it follows that,

$$\Omega_M(Px_1, Px_2; Pz) = \Omega_M(\Lambda x_1, \Lambda x_2; \Lambda z). \tag{6.20}$$

Plugging this into (6.16) and using the definition (6.15) of $\widetilde{\Omega}_M$, it follows that $\widetilde{\Omega}_M$ will be Poincaré-invariant to the required scaling degree if and only if $a_{\gamma_1\gamma_2}$ can be found such that,

$$\begin{aligned} &\Omega_M(\Lambda x_1, \Lambda x_2; \Lambda z) - \Omega_M(x_1, x_2; z) \\ &\sim_\delta \sum_{|\gamma_1|+|\gamma_2| \leq \delta} \frac{1}{\gamma_1!\gamma_2!}(x_1 - z)^{\gamma_1}(x_2 - z)^{\gamma_2}\left(\Lambda^{\gamma_1'}{}_{\gamma_1}\Lambda^{\gamma_2'}{}_{\gamma_2} - \delta^{\gamma_1'}{}_{\gamma_1}\delta^{\gamma_2'}{}_{\gamma_2}\right) a_{\gamma_1'\gamma_2'}, \end{aligned} \tag{6.21}$$

where $\Lambda^{\alpha'}{}_{\alpha} \equiv \Lambda^{\mu_1'}{}_{\mu_1} \cdots \Lambda^{\mu_{|\alpha|}'}{}_{\mu_{|\alpha|}}$ with the convention $\Lambda^{\alpha'}{}_{\alpha} = 1$ if $|\alpha| = 0$. Since the first line of (6.21) has been shown to be smooth in (x_1, x_2), it is asymptotic to its Taylor series.

Hence, Taylor expanding the first line and equating the coefficients of $(x_1 - z)^{\gamma_1}(x_2 - z)^{\gamma_2}$ appearing on both lines, we see that $a_{\gamma_1\gamma_2}$ must satisfy

$$\left[\partial^{(x_1)}_{\gamma_1}\partial^{(x_2)}_{\gamma_2}\left[\Omega_M(\Lambda x_1, \Lambda x_2; \Lambda z) - \Omega_M(x_1, x_2; z)\right]\right]_{x_1,x_2=z} = \left(\Lambda^{\gamma_1'}{}_{\gamma_1}\Lambda^{\gamma_2'}{}_{\gamma_2} - \delta^{\gamma_1'}{}_{\gamma_1}\delta^{\gamma_2'}{}_{\gamma_2}\right) a_{\gamma_1'\gamma_2'} \tag{6.22}$$

If Ω_M were itself a smooth function of (x_1, x_2), then we could trivially satisfy (6.22) by setting $a_{\gamma_1\gamma_2}$ equal to the Taylor coefficients of $\Omega_M(x_1, x_2; z)$ evaluated at $x_1, x_2 = z$. However, Ω_M is fundamentally distributional, so it is far from obvious that there exist Λ-independent $a_{\gamma_1\gamma_2}$ satisfying (6.22).

In Appendix C we show that (6.22) can always be solved and we obtain explicit solutions. First, we use a cohomological argument to prove existence of solutions $a_{\gamma_1\gamma_2}$ to (6.22). We then obtain the explicit solutions for $a_{\gamma_1\gamma_2}$ in the cases of rank $r = 1, 2$, where $r \equiv |\gamma_1| + |\gamma_2|$. The $r = 1$ solutions are

$$a_{\{\mu\}\{0\}} = a_{\{0\}\{\mu\}} = -i\int d^D y\, \partial^{(y)}_\mu \chi(y, \vec{0})\, H_F(y, \vec{0}) H_F(y, \vec{0}). \tag{6.23}$$

and the $r = 2$ solutions are[4]

$$a_{\{(\mu\}\{\nu)\}} = -i\int d^D y\, \chi(y, \vec{0}) \left[\partial_\mu H_F(y, \vec{0})\partial_\nu H_F(y, \vec{0}) - \frac{1}{D}\eta_{\mu\nu}\partial_\sigma H_F(y, \vec{0})\partial^\sigma H_F(y, \vec{0})\right], \tag{6.24}$$

4. In eqs. (6.24) and (6.25), it is understood that the subtraction inside the integrand must be performed prior to evaluating the integral, since the individual terms in the integrand contain non-integrable divergences at $y = \vec{0}$, i.e., the integrand is well-defined as a distribution in y only when $y \neq \vec{0}$, but its definition can be *uniquely* extended to include the origin.

and

$$a_{\{(\mu\nu)\}\{0\}} = a_{\{0\}\{(\mu\nu)\}} \tag{6.25}$$
$$= -i \int d^D y\, \chi(y,\vec{0}) \left[H_F(y,\vec{0}) \partial_\mu \partial_\nu H_F(y,\vec{0}) - \frac{1}{D} \eta_{\mu\nu} H_F(y,\vec{0}) \partial^2 H_F(y,\vec{0}) \right].$$

Finally, we obtain the recursive solution (C.44) for $a_{\gamma_1\gamma_2}$ for all $r > 2$.

With the above solution for $a_{\gamma_1\gamma_2}$, we obtain $\widetilde{\Omega}_{M,\delta}$ satisfying (6.15) and (6.16). We thereby obtain the Poincaré-invariant flow relation (6.18). However, as in the Euclidean case, the flow relation (6.18) is not compatible with the scaling behavior of the Wick monomials required by the scaling axiom W7. Nevertheless, as in the Euclidean case, we can obtain a flow relation that remains compatible with Poincaré invariance and satisfies the desired scaling behavior by replacing $\widetilde{\Omega}_{M,\delta}$ on the right side of (6.18) with

$$\mathfrak{L}[L]\widetilde{\Omega}_M(x_1, x_2; z; L) - \sum_{|\gamma_1|+|\gamma_2|\leq\delta} \frac{1}{\gamma_1!\gamma_2!} c_{\gamma_1\gamma_2}(L)(x_1 - z)^{(\gamma_1}(x_2 - z)^{\gamma_2)}, \tag{6.26}$$

where $\mathfrak{L}$ was defined by (5.33) and

$$c_{\gamma_1\gamma_2}(L) \equiv \int_0^L dL' \left[\partial^{(x_1)}_{\gamma_1} \partial^{(x_2)}_{\gamma_2} \frac{\partial}{\partial L'} \left(\mathfrak{L}[L']\widetilde{\Omega}_M(x_1, x_2; z; L') \right) \right]. \tag{6.27}$$

The distribution (6.26) is Poincaré-invariant and is asymptotically independent of L up to scaling degree δ. Moreover, the distribution (6.26) differs from Ω_M by a smooth function of (x_1, x_2). Hence, the distribution (6.26) *can* be used in a flow relation for the Feynman parametrix $C^I_{T_0\{\phi\phi\}} = H_F$ which is compatible with all Wick axioms. Recalling the definition (6.15) of $\widetilde{\Omega}_M$ and the explicit formulas (5.44)-(5.46) for the OPE coefficients, the flow relation with (6.26) on the right-hand side can be written in the form:

$$\frac{\partial}{\partial m^2}(C_H)^I_{T_0\{\phi\phi\}}(x_1,x_2;z) \sim_\delta -\frac{i}{2}\int d^D y \mathfrak{L}[L]\chi(y,z;L)(C_H)^I_{T_0\{\phi^2\phi\phi\}}(y,x_1,x_2;z) + \\ -\sum_{[C]\leq\delta+2} c_C (C_H)^C_{T_0\{\phi\phi\}}(x_1,x_2;z), \tag{6.28}$$

where $c_C = 0$ unless $[C]_\phi = 2$, in which case it is given by

$$c_C(L) \equiv \mathfrak{L}[L]a_C(L) + \\ -\int_0^L dL' \frac{\partial}{\partial L'}\left[\frac{i}{2}\int d^D y\, \mathfrak{L}[L']\chi(y,\vec{0};L')\,(C_H)^I_{T_0\{\phi^2 C\}}(y,z;z) + \mathfrak{L}[L']a_C(L')\right], \tag{6.29}$$

for $L > 0$. The tensors a_C are also zero unless $[C]_\phi = 2$, in which case, they are inductively defined via (C.44) in terms of

$$(B^{\kappa\rho})_C \equiv i\int d^D y\, y^{[\kappa}\partial^{\rho]}\chi(y,\vec{0})(C_H)^I_{T_0\{\phi^2 C\}}(y,\vec{0};\vec{0}). \tag{6.30}$$

Note, by writing the y-integral in (6.28), we have implicitly (uniquely) extended the OPE coefficient $(C_H)^I_{T_0\{\phi^2\phi\phi\}}(y,x_1,x_2;z) = 2H_F(y,x_1)H_F(y,x_2)$ to the partial diagonals $y = x_1$ and $y = x_2$ as justified in Remark 21 above.

The inductive solution (C.44) determines a_C up to Lorentz-invariant tensors of the correct rank which scale with an overall factor of $\lambda^{(D-4)}$ under $(\eta_{ab}, m^2, L) \to (\lambda^{-2}\eta_{ab}, \lambda^2 m^2, \lambda^{-2}L)$ and depend smoothly on (η_{ab}, m^2). Although the inherent ambiguities in a_C may depend on L, the $\mathfrak{L}$-operator and L-integral terms in (6.29) ensure that only the L-independent parts of a_C can contribute non-trivially to c_C. Therefore, the only ambiguity in c_C corresponds to the choice of an L-independent tensor in a_C that scales with an overall factor of $\lambda^{(D-4)}$ under $(\eta_{ab}, m^2) \to (\lambda^{-2}\eta_{ab}, \lambda^2 m^2)$. In odd dimensions, there are no tensors that scale in this way and depend smoothly on (η_{ab}, m^2), so a_C is unique. In even dimensions, this ambiguity corresponds to freedom to choose the Taylor coefficients of a Poincaré-invariant smooth function in (x_1, x_2, m^2). We note also, as discussed in Remark 20, that the conservation constraint (3.36) is automatically satisfied in flat spacetime.

The flow relation (6.28) for the (unextended) time-ordered OPE coefficient $(C_H)^I_{T_0\{\phi\phi\}}$ is the Minkowski spacetime analogue of the Euclidean flow relation (5.47) for the ordinary OPE coefficient $(C_H)^I_{\phi\phi}$. In both cases, the inherent ambiguity in the flow relation corresponds to a smooth function that is invariant under the respective isometry group. By Theorem 6, formulas for the (unextended) time-ordered OPE coefficients, $C^B_{T_0\{A_1\cdots A_n\}} \equiv T_0\{C^B_{A_1\cdots A_n}\}$, for any given Wick prescription are obtained from formulas for the corresponding non-time-ordered OPE coefficients, $C^B_{A_1\cdots A_n}$, by simply replacing all occurrences of the Hadamard parametrix H with its corresponding Feynman parametrix $H_F = H(x_1, x_2) - i\Delta^{\text{adv}}(x_1, x_2)$. Hence, from the explicit formulas for the Hadamard normal-ordered OPE coefficients (see (5.5) and (5.14)) and the flow relation (6.28), we immediately obtain the following theorem giving the flow relations for the (unextended) time-ordered OPE coefficients $(C_H)^I_{T_0\{\phi\cdots\phi\}}(x_1, \ldots, x_n; z)$.

Theorem 7. *For any Hadamard parametrix satisfying* (6.28)*, the corresponding OPE coefficients* $(C_H)^I_{T_0\{\phi\cdots\phi\}}$ *satisfy:*

$$\begin{aligned}\frac{\partial}{\partial m^2}(C_H)^I_{T_0\{\phi\cdots\phi\}}(x_1, \ldots, x_n; z) \approx -\frac{i}{2}\int d^D y\, \mathfrak{L}[L]\chi(y, z; L)\, (C_H)^I_{T_0\{\phi^2\phi\cdots\phi\}}(y, x_1, \ldots, x_n; z) \\ - \sum_C c_C(L)(C_H)^C_{T_0\{\phi\cdots\phi\}}(x_1, \ldots, x_n; z),\end{aligned} \tag{6.31}$$

where c_C is given by formula (6.29) *with the same ambiguities arising from a_C.*

Note that the inherent ambiguities in these flow relations are in 1-1 correspondence with the freedom to choose a Hadamard parametrix whose corresponding Hadamard normal-ordered Wick fields are compatible with axioms W1-W8.

Remark 23. As emphasized in Section 4.3, the extension of $T_0\{\Phi_{A_1}(x_1)\cdots\Phi_{A_n}(x_n)\}$ to algebra-valued distributions defined on the diagonals generally introduces additional "contact-term" ambiguities proportional to δ-distributions (and their distributional derivatives). However, the scaling degree of $\delta(x_1, \ldots, x_n)$ is $(n-1) \cdot D$, whereas by Theorem 4 the scaling degree of the coefficients $C^I_{T_0\{\phi\cdots\phi\}}(x_1, \ldots, x_n; z)$ appearing in the flow relation (6.31) is $n \cdot (D-2)/2$. Since $(n-1)D$ is strictly greater than $n(D-2)/2$ for $n \geq 2$ and $D \geq 2$, it follows that there do not exist contact terms with scaling degree less than or equal to the scaling degree of $C^I_{T_0\{\phi\cdots\phi\}}$. By the axioms for time-ordered products in [7, 37], this implies that the extension of the $C^I_{T_0\{\phi\cdots\phi\}}$ coefficients to the diagonals is unique and, therefore, it so happens that we could replace T_0 with T in formula (6.31) without introducing additional contact term ambiguities. Note, however, that this does not occur for the general unextended time-ordered Wick coefficients $C^B_{T_0\{A_1\cdots A_n\}}$ nor, in general, for the coefficients appearing in the flow relations (1.20) of $\lambda\phi^4$-theory.

Relation (6.31) of Theorem 7 applies to the time-ordered OPE coefficients for the Hadamard normal-ordered Wick fields. However, following the steps outlined below Theorem 6 of the preceding chapter, one may straightforwardly obtain flow relations for the time-ordered OPE coefficients corresponding to any prescription for the Wick fields satisfying axioms W1-W8. These relations will similarly take the same general form as the Hadamard normal-ordered relation (6.31) except there will be additional terms containing factors of F_k (with $k \leq n$) as in relation (5.51) above.

Finally, we note that our derivation of the flow relation (6.31) for general n relied heavily on our knowledge of the explicit expressions for the time-ordered OPE coefficients of Hadamard normal-ordered Wick fields, since this knowledge enabled us to obtain (6.31) from the flow relation (6.28) for $n = 2$ via inspection. However, if the OPE coefficients with $n > 2$ had not been related in a simple, known manner to the $n = 2$ OPE coefficients, we would not have been able to construct covariance-restoring terms for the $n > 2$ case using the techniques described in this chapter. In Appendix E, we develop a general method for constructing covariance-restoring counterterms based on the model-independent associativity conditions that *can* be applied to the $n > 2$ case and show that this general algorithm reproduces the results claimed here.

CHAPTER 7

FLOW RELATIONS FOR OPE COEFFICIENTS IN CURVED SPACETIME

In this chapter, we obtain flow relations for the unextended time-ordered Wick OPE coefficients in general globally-hyperbolic Lorentzian spacetimes (M, g_{ab}) in any dimension $D \geq 2$. As in the preceding Minkowski chapter, we focus attention initially to the flow relation for the time-ordered OPE coefficient $(C_H)^I_{T_0\{\phi\phi\}} = H_F(x_1, x_2; m^2; \xi)$, since the flow relations for other time-ordered Wick OPE coefficients may be straightforwardly obtained once the flow relation for $(C_H)^I_{T_0\{\phi\phi\}}$ is known.

In curved spacetime, any Feynman parametrix H_F used for the construction of $(C_H)^I_{T_0\{\phi\phi\}}$ is required to be locally and covariantly defined and have (jointly) smooth dependence on the coupling parameters (m^2, ξ). As already noted in Proposition 6, the relation between H_F and a Hadamard parametrix H (see (3.26)) is given by

$$H_F(x_1, x_x) = H(x_1, x_2) - i\Delta^{\text{adv}}(x_1, x_2), \tag{7.1}$$

with Δ^{adv} denoting the advanced Green's function. Since the forms of H and H_F depend on the squared geodesic distance function $\sigma(x_1, x_2)$, these parametrices are well defined only in convex normal neighborhoods. The Feynman parametrix is a fundamental solution to the Klein-Gordon equation

$$(-g^{ab}\nabla_a\nabla_b + m^2 + \xi R)H_F(x_1, x_2; m^2; \xi) \approx -i\delta(x_1, x_2) + \text{smooth terms}. \tag{7.2}$$

Furthermore, in curved spacetime, the wavefront set of H_F continues to be of the form[1] $(x_1, k; x_2, -k)$ [35]. In particular, when smeared in either of its spacetime variables with a

1. Replacing $\mathbb{R}^D$ with M in the Minkowski formula (6.4) gives the explicit wavefront set of H_F in a curved spacetime.

test function f of sufficiently small compact support, $H_F(y, f)$ is a smooth function in y within a convex normal neighborhood of the support of f.

The above properties of H_F were all that were needed to obtain the initial flow relation (6.12) in Minkowski spacetime, so we can parallel these steps to derive a similar flow relation in any any globally-hyperbolic curved spacetime (M, g_{ab}). To do so, let $U_z \subset M$ be a convex normal neighborhood of the point $z \in M$. It is convenient to work in a Riemannian normal coordinate (RNC) system about z. A RNC system is constructed by introducing an orthonormal basis (i.e., "tetrad") for T_zM,

$$\left\{(e_\mu)^a \in T_zM \,|\, \mu \in \{0, \ldots, D-1\} \text{ and } g_{ab}(e_\mu)^a(e_\nu)^b = \eta_{\mu\nu}\right\}. \tag{7.3}$$

The tetrad allows us to identify T_zM with $\mathbb{R}^D$. We then use the exponential map—which maps $v^a \in T_zM$ into the point in M lying at unit affine parameter along the geodesic determined by (z, v^a)—to provide a diffeomorphism between U_z and a neighborhood $\mathcal{U}_0$ of the origin of $\mathbb{R}^D$. This correspondence provides coordinates x^μ on U_z. We denote by t^μ the RNC components of the timelike vector at z that is proportional to $(e_0)^\mu$ but required to remain unit-normalized with respect to the metric components under the rescaling $g_{\mu\nu} \to \lambda^{-2} g_{\mu\nu}$, i.e., under this rescaling, it is required that $t_\mu \to \lambda^{-1} t_\mu$. Let $\zeta \in C_0^\infty(\mathbb{R})$ again denote a test function that is equal to one for $|s| \leq 1$ and vanishes for $|s| \geq 2$. We then define a cutoff function on U_z by,

$$\chi[g_{\mu\nu}, t_\mu, L](y; \vec{0}) = \zeta\left(L^{-2}\left(g_{\mu\nu}(\vec{0}) + 2t_\mu t_\nu\right) y^\mu y^\nu\right), \tag{7.4}$$

where L is chosen such that the coordinate ball of radius $2L$ lies within U_z. Here y^μ denotes the RNC values of y and we have denoted z by its RNC value $\vec{0}$. Note that the cutoff function (7.4) is invariant under the simultaneous rescaling $(g_{ab}, L) \to (\lambda^{-2} g_{ab}, \lambda^{-1} L)$ with the RNC coordinate basis held fixed.

With these definitions and constructions, we can now straightforwardly generalize the derivation of (6.12) to curved spacetime. We obtain

$$\frac{\partial}{\partial m^2} H_F[g_{\mu\nu}](x_1, x_2) \approx \Omega_C[g_{\mu\nu}, t_\mu, L](x_1, x_2; \vec{0}) + \text{terms smooth in } (x_1, x_2), \tag{7.5}$$

where

$$\begin{aligned} &\Omega_C[g_{\mu\nu}, t_\mu, L](f_1, f_2; \vec{0}) \\ &\equiv -i \int_{\mathbb{R}^D} d^D y \sqrt{-g(y)}\, \chi[g_{\mu\nu}, t_\mu, L](y; \vec{0})\, H_F[g_{\mu\nu}](y, f_1) H_F[g_{\mu\nu}](y, f_2). \end{aligned} \tag{7.6}$$

In curved spacetime, the parameter ξ enters the Klein-Gordon equation in a nontrivial manner and we also seek a flow equation in ξ. Using the fact that the commutator of the differential operator $\partial_\xi \equiv \partial/\partial\xi$ with the Klein-Gordon operator (3.3) is given by

$$[K, \partial_\xi] = -RI. \tag{7.7}$$

we can similarly derive the ξ-flow equation

$$\begin{aligned} &\frac{\partial}{\partial \xi} H_F[g_{\mu\nu}](x_1, x_2; \xi) \\ &\approx -i \int_{\mathbb{R}^D} d^D y \sqrt{-g(y)} \chi(y, z) R(y) H_F[g_{\mu\nu}](y, x_1; \xi) H_F[g_{\mu\nu}](y, x_2; \xi) + \text{smooth}. \end{aligned} \tag{7.8}$$

Note that the integral in the second line vanishes unless the scalar curvature is nonzero. Since the analysis of the flow relations (7.5) and (7.8) are essentially identical, in the following we will focus attention on only the m^2-flow relation (7.5), it being understood that (7.8) can be analyzed in a completely parallel manner, with the minor differences described in Remark 24 below Theorem 8.

If we attempt to drop the smooth terms and use (7.5) as our flow equation we will encounter three major difficulties: (i) Since the quantity Ω_C is defined in (7.6) by an integral over a finite spacetime region, Ω_C depends nonlocally on the metric, which is not compatible with axiom W1. (ii) On account of the presence of the cutoff function χ, Ω_C is not covariantly defined, which also is not compatible with axiom W1. (iii) On account of the cutoff scale L present in χ, the scaling dependence of the OPE coefficients will not be compatible with axiom W7. As we shall now show, these difficulties can be overcome by suitably modifying the flow relation (7.5). Specifically, difficulty (i) can be overcome by replacing (7.6) with a similar expression involving the Taylor coefficients of the metric in an expansion about z rather than the metric itself. Difficulty (ii) then can be overcome by a generalization of the procedure used to restore Lorentz invariance in Minkowski spacetime. Finally, difficulty (iii) can be overcome by the same procedure as used for the Euclidean and Minkowski flow relations. We now discuss, in turn, these difficulties and their resolutions.

(i) Locality. As already indicated above, the key idea needed to convert (7.6) into an expression that depends only on the metric in an arbitrarily small neighborhood of z is to replace the metric by its Taylor approximation about z, carried to sufficiently high order. To scaling degree δ, the RNC components of the metric are asymptotically equivalent to its Taylor polynomial about the origin,

$$\begin{aligned} g_{\mu\nu}(x) \sim_\delta g^{(N)}_{\mu\nu}(x) &\equiv \sum_{k=0}^{N} \frac{1}{k!} x^{\sigma_1} \cdots x^{\sigma_k} \left. \frac{\partial^k g_{\mu\nu}(x)}{\partial x^{\sigma_1} \cdots \partial x^{\sigma_k}} \right|_{x=\vec{0}} \\ &= \eta_{\mu\nu} + \frac{1}{3} R_{\mu\nu\kappa\rho}(\vec{0}) x^\kappa x^\rho - \frac{1}{6} \nabla_\sigma R_{\mu\nu\kappa\rho}(\vec{0}) x^\kappa x^\rho x^\sigma + \cdots , \end{aligned} \tag{7.9}$$

provided that we take $N \geq \delta$. As indicated by the second line of (7.9), the Taylor coefficients are expressible entirely in terms of the Riemann curvature tensor and its totally-symmetric covariant derivatives evaluated at the origin[2]. For sufficiently large x^μ, the Taylor polynomial

2. This follows from a close relative [48, see Lemma 2.1] of the "Thomas replacement theorem" [49].

$g^{(N)}_{\mu\nu}(x)$ need not define a Lorentz metric. However, we can choose L sufficiently small that $|g^{(N)}_{\mu\nu} - \eta_{\mu\nu}| \ll 1$ within a coordinate ball of radius $2L$, so that $g^{(N)}_{\mu\nu}(x)$ is a Lorentz metric wherever χ is nonvanishing.

To proceed, we perform an expansion of $\Omega_C[g_{\mu\nu}, t_\mu, L]$ about $g_{\mu\nu} = \eta_{\mu\nu}$ as a power series in the (symmetrized) covariant derivatives of the Riemann curvature tensor. This curvature expansion as well as the precise bound on the scaling degree of its non-smooth terms is derived in Appendix D. This expansion also will be needed for our construction of covariance-restoring counterterms below. The expansion takes the form[3]

$$\Omega_C[g_{\mu\nu}, t_\mu, L](f_1, f_2; \vec{0}) \sim_\delta \tag{7.10}$$
$$\sum_{k=0}^{\delta+D-4} \sum_{\vec{p}_k} (\Omega_{\vec{p}})^{\{\mu\cdots\sigma_{k-2}\}}[\eta_{\mu\nu}, t_\mu, L](f_1, f_2; \vec{0}) \prod_{j=0}^{k-2} \left[R_{\mu\nu\kappa\rho;(\sigma_1\cdots\sigma_j)}(\vec{0})\right]^{p_j} + \text{smooth terms},$$

where "smooth" refers to the behavior in x_1 and x_2 prior to smearing (cf. formula (7.14) of Proposition 9 below). Here we have defined,

$$(\Omega_{\vec{p}})^{\{\mu\cdots\sigma_{k-2}\}}[\eta_{\mu\nu}, t_\mu, L](f_1, f_2; \vec{0}) \equiv \left. \frac{\partial^P \Omega_C[g^{(k)}_{\mu\nu}, t_\mu, L](f_1, f_2; \vec{0})}{\partial^{p_0} R_{\mu\nu\kappa\rho}(\vec{0}) \cdots \partial^{p_{k-2}} R_{\mu\nu\kappa\rho;(\sigma_1\cdots\sigma_{k-2})}(\vec{0})} \right|_{g^{(k)}_{\mu\nu} = \eta_{\mu\nu}}, \tag{7.11}$$

where $g^{(k)}_{\mu\nu}$ denotes the kth-order polynomial metric (7.9) computed from $g_{\mu\nu}$ and $P \equiv \sum_{j=0}^{k-2} p_j$. In (7.10) the $\vec{p}_k$-sum runs over all non-negative integers $\vec{p}_k \equiv (p_0, \ldots, p_{k-2})$ such that

$$2p_0 + 3p_1 + \cdots + kp_{(k-2)} = k. \tag{7.12}$$

Note (7.11) are tensor-valued distributions defined on a neighborhood of the origin in *flat* Minkowski spacetime, $(\mathcal{N}_0, \eta_{\mu\nu})$. Hence, all of the curvature dependence of the explicit terms

3. To avoid overly cumbersome notation involving multiple subscripts on spacetime indices, we have implicitly re-used some Greek letters in (7.10), but the intended summations should be clear from context.

in the curvature expansion (7.10) for Ω_C comes through a finite product of curvature tensors evaluated at the origin. Note the derivatives in (7.11) with respect to curvature tensors are well-defined because the smeared distribution Ω_C is a smooth function of the metric and the polynomial metric $g^{(k)}_{\mu\nu}$ is a smooth function of finitely-many curvature tensors evaluated at the origin:

$$g^{(k)}_{\mu\nu}(x) = g^{(k)}_{\mu\nu}[x^\sigma, \eta_{\mu\nu}, R_{\mu\nu\kappa\rho}(\vec{0}), \nabla_\sigma R_{\mu\nu\kappa\rho}(\vec{0}), \ldots, \nabla_{(\sigma_1} \cdots \nabla_{\sigma_{k-2})} R_{\mu\nu\kappa\rho}(\vec{0})]. \tag{7.13}$$

The result needed to effectively replace $g_{\mu\nu}$ with $g^{(N)}_{\mu\nu}$ in (7.6) is given in the following proposition:

Proposition 9. *Let H_F be a local and covariant Feynman parametrix which scales almost homogeneously with an overall factor of $\lambda^{(D-2)}$ under $(g_{\mu\nu}, m^2) \to (\lambda^{-2} g_{\mu\nu}, \lambda^2 m^2)$ and which depends smoothly on m^2. Let Ω_C be given by (7.6). Then for all $N \geq \delta + D - 4$, we have*

$$\Omega_C[g_{\mu\nu}, t_\mu, L](x_1, x_2; \vec{0}) \sim_\delta \Omega_C[g^{(N)}_{\mu\nu}, t_\mu, L](x_1, x_2; \vec{0}) + \textit{terms smooth in } (x_1, x_2). \tag{7.14}$$

Proof. The proposition can be deduced from the curvature expansion (7.10) for Ω_C as follows: Note the maximum number of covariant derivatives of $R_{\mu\nu\kappa\rho}$ appearing in the curvature expansion (7.10) is $\delta + D - 6$. Consider first the special case that $g_{\mu\nu} = g^{(P)}_{\mu\nu}$ for arbitrary, but finite, integer P. We want to determine the smallest integer $N < P$ such that the relation (7.14) of the proposition holds. Since both $g_{\mu\nu}$ and $g^{(N)}_{\mu\nu}$ are themselves polynomial metrics of the form (7.9), it follows immediately that their respective polynomial approximations, $g^{(k)}_{\mu\nu}[g_{\mu\nu}]$ and $g^{(k)}_{\mu\nu}[g^{(N)}_{\mu\nu}]$, are identical for any $k \leq N$ and, thus, all the coefficients (7.11) of the curvature expansion computed from their respective polynomial approximations are identical so long as the number of covariant derivatives in $g^{(N)}_{\mu\nu}$ is at least $\delta + D - 6$ (i.e., if $N - 2 \geq \delta + D - 6$). Since their respective curvature expansions (7.10) are thus identical for $N \geq \delta + D - 4$, this then implies the claimed relation (7.14) holds for the special case that $g_{\mu\nu} = g^{(P)}_{\mu\nu}$. To extend the proof of relation (7.14) to arbitrary smooth metrics $g_{\mu\nu}$, we use the fact [45, see proof of Theorem 4.1] that it is always possible to define a 1-parameter family of metrics $h_{\mu\nu}(x; p)$ which depend smoothly on p in a neighborhood of $p = 0$ and such that: i) For any fixed $p \neq 0$, $h_{\mu\nu}(x; p)$ is a polynomial metric of finite order and ii) $h_{\mu\nu}(x; p = 0) = g_{\mu\nu}$. The proposition has already been established for $h_{\mu\nu}(x; p)$ when $p \neq 0$ since these are polynomial metrics, so compatibility with the smoothness axiom W2 then implies the proposition must hold also for $p = 0$. □

Our provisional proposal is to replace (7.5) with

$$\frac{\partial}{\partial m^2} H_F[g_{\mu\nu}](x_1, x_2) \sim_\delta \Omega_C[g^{(N)}_{\mu\nu}, t_\mu, L](x_1, x_2; \vec{0}), \tag{7.15}$$

where $N \geq \delta + D - 4$. The distribution $\Omega_C[g^{(N)}_{\mu\nu}, t_\mu, L]$ appearing on the right-hand side of (7.15) is manifestly local with respect to the original spacetime (M, g_{ab}), since the only dependence of $g^{(N)}_{\mu\nu}$ on the spacetime curvature comes through a finite number of local curvature tensors evaluated at z (see (7.9)). Thus, the flow equation (7.15) is now local in the spacetime metric. However, it fails to be covariant. We turn now to making a further modification of (7.15) to restore covariance.

(ii) Covariance. The distribution $\Omega_C[g^{(N)}_{\mu\nu}, t_\mu, L]$ appearing in (7.15) fails to be covariant because the cutoff function, χ, depends upon a choice of the unit timelike co-vector t_μ at z, which is not determined by the metric. However, any two normalized timelike co-vectors t_μ and t'_μ at z are related via a restricted Lorentz transformation $\Lambda \in \mathcal{L}^\uparrow_+$,

$$t'_\mu = \Lambda^\nu{}_\mu t_\nu = (\Lambda^{-1})_\mu{}^\nu t_\nu \equiv (\Lambda^{-1} t)_\mu. \tag{7.16}$$

Thus, in order to obtain a covariant flow relation, we seek to modify the flow relations by the addition of smooth locally-constructed "counterterms" that compensate for the failure of $\Omega_C[g^{(N)}_{\mu\nu}, t_\mu, L]$ to be invariant under Lorentz transformations of t_μ.

The dependence of $\Omega_C[g^{(N)}_{\mu\nu}, t_\mu, L]$ on Lorentz transformations of t_μ is quantified by the distribution,

$$\begin{aligned} &Q_C[g^{(N)}_{\mu\nu}, t_\mu, L](x_1, x_2; \vec{0}; \Lambda^{-1}) \\ &\equiv \Omega_C[g^{(N)}_{\mu\nu}, (\Lambda^{-1}t)_\mu, L](x_1, x_2; \vec{0}) - \Omega_C[g^{(N)}_{\mu\nu}, t_\mu, L](x_1, x_2; \vec{0}) \\ &= -i \int_y \left[\chi[g^{(N)}_{\mu\nu}, t_\mu, L](\Lambda y; \vec{0}) - \chi[g^{(N)}_{\mu\nu}, t_\mu, L](y; \vec{0})\right] H_F[g^{(N)}_{\mu\nu}](y, x_1) H_F[g^{(N)}_{\mu\nu}](y, x_2), \end{aligned} \tag{7.17}$$

where $\int_y \equiv \int_M d^D y \sqrt{-g^{(N)}(y)}$. By the same arguments as given for quantity Q_M in Minkowski spacetime (see eq. (C.3)), the quantity Q_C has smooth dependence on (x_1, x_2). In Minkowski spacetime, the Taylor coefficients, $\boldsymbol{Q}_M(\Lambda^{-1}) \equiv \partial^{(x_1)}_{\gamma_1} \partial^{(x_2)}_{\gamma_2} Q_M(x_1, x_2; z; \Lambda^{-1})|_{x_1, x_2 = z}$, of $Q_M(x_1, x_2; z; \Lambda^{-1})$ were shown to satisfy (C.9). The existence of the desired counterterms in the flow relations was then established by cohomological arguments. However, in curved spacetime, the Taylor coefficients of Q_C do not satisfy (C.9) for the simple reason that the curved metric $g^{(N)}_{\mu\nu}$ given by (7.9) is not invariant under Lorentz transformations.

Nevertheless, we can use the curvature expansion (7.10) for $Q_C[g^{(N)}_{\mu\nu}, t_\mu, L]$ and consider the behavior under Lorentz transformations of the coefficients $(\Omega_{\vec{p}})^{\{\mu \cdots \sigma_{k-2}\}}[\eta_{\mu\nu}, t_\mu, L]$ appearing in that expansion (see (7.11)). We write

$$(Q_{\vec{p}})^\gamma[\eta_{\mu\nu}, t_\mu, L, \Lambda^{-1}] \equiv (\Omega_{\vec{p}})^\gamma[\eta_{\mu\nu}, (\Lambda^{-1}t)_\mu, L] - (\Omega_{\vec{p}})^\gamma[\eta_{\mu\nu}, t_\mu, L], \tag{7.18}$$

where we use the multi-index notation $\gamma \equiv \{\mu \cdots \sigma_{k-2}\}$. For notational convenience, we will suppress the p-subscripts in the following and write the left side of (7.18) simply as Q^γ.

Since Q^γ is smooth, its asymptotic behavior is determined by its Taylor coefficients,

$$Q^{\gamma}{}_{\gamma_1\gamma_2}(\Lambda^{-1}) \equiv \partial^{(x_1)}_{\gamma_1}\partial^{(x_2)}_{\gamma_2} Q^{\gamma}(x_1, x_2; \vec{0}; \Lambda^{-1})|_{x_1,x_2=\vec{0}} \tag{7.19}$$

The crucial point is that the Taylor coefficients (7.19) depend only on $\eta_{\mu\nu}$, not the spacetime metric $g^{(N)}_{\mu\nu}$—all of the dependence on the spacetime metric in the curvature expansion (7.10) appears in the curvature factors, not in $(\Omega_{\vec{p}})$. Consequently, we obtain,

$$\begin{aligned}
&Q^{\gamma_3}{}_{\gamma_1\gamma_2}(\Lambda_1\Lambda_2) - Q^{\gamma_3}{}_{\gamma_1\gamma_2}(\Lambda_1)\\
&= \left[\partial^{(x_1)}_{\gamma_1}\partial^{(x_2)}_{\gamma_2}\left[\Omega^{\gamma_3}[(\Lambda_1\Lambda_2 t)_\mu](x_1, x_2; \vec{0}) - \Omega^{\gamma_3}[(\Lambda_1 t)_\mu](x_1, x_2; \vec{0})\right]\right]_{x_1,x_2=\vec{0}}\\
&= (\Lambda_1)^{\gamma_3}{}_{\gamma_3'}\left[\partial^{(x_1)}_{\gamma_1}\partial^{(x_2)}_{\gamma_2}\left[\Omega^{\gamma_3'}[(\Lambda_2 t)_\mu](\Lambda_1^{-1}x_1, \Lambda_1^{-1}x_2; \vec{0}) - \Omega^{\gamma_3'}[t_\mu](\Lambda_1^{-1}x_1, \Lambda_1^{-1}x_2; \vec{0})\right]\right]_{x_1,x_2=\vec{0}}\\
&= (\Lambda_1)^{\gamma_3}{}_{\gamma_3'}(\Lambda_1)_{\gamma_1}{}^{\gamma_1'}(\Lambda_1)_{\gamma_2}{}^{\gamma_2'}Q^{\gamma_3'}{}_{\gamma_1'\gamma_2'}(\Lambda_2),
\end{aligned} \tag{7.20}$$

where the second equality follows from the identity:

$$\Omega^{\gamma}[g^{(N)}_{\mu\nu}, (\Lambda^{-1}t)_\mu, L](x_1, x_2; \vec{0}) = (\Lambda^{-1})^{\gamma}{}_{\gamma'}\Omega^{\gamma'}[g^{(N)}_{\mu\nu}, t_\mu, L](\Lambda x_1, \Lambda x_2; \vec{0}), \tag{7.21}$$

where we have used the fact that $H_F[(\Lambda g)^{(N)}_{\mu\nu}](\Lambda y, \Lambda x) = H_F[g^{(N)}_{\mu\nu}](y, x)$, with

$$(\Lambda g)^{(N)}_{\mu\nu}(x) \equiv \Lambda_{\mu_1}{}^{\nu_1}\Lambda_{\mu_2}{}^{\nu_2} g^{(N)}_{\nu_1\nu_2}(\Lambda^{-1}x). \tag{7.22}$$

Equation (7.20) is a close analogue of the equation (C.9). Writing $\boldsymbol{Q}_C(\Lambda) \equiv (Q_{\vec{p}})^{\gamma_3}{}_{\gamma_1\gamma_2}(\Lambda)$, we see that (7.20) corresponds to the cohomological identity

$$\boldsymbol{Q}_C(\Lambda_1) + D(\Lambda_1)\boldsymbol{Q}_C(\Lambda_2) - \boldsymbol{Q}_C(\Lambda_1\Lambda_2) = 0, \tag{7.23}$$

see (C.14). By the same arguments as given in Proposition 10 of Appendix C, it follows that there exist tensors $\boldsymbol{a} \equiv (a_{\vec{p}})^{\gamma_3}{}_{\gamma_1\gamma_2}$ such that

$$\boldsymbol{Q}_C(\Lambda) = (D(\Lambda) - \mathbb{I})\, \boldsymbol{a}. \tag{7.24}$$

We now can restore covariance to the curved spacetime flow equations in close parallel with the procedure we used to restore Lorentz covariance to the Minkowski flow equations. Let $\boldsymbol{a} \equiv (a_{\vec{p}})^{\gamma_3}{}_{\gamma_1\gamma_2}$ denote the solutions to (7.24). Let

$$a_{\gamma_1\gamma_2} \equiv \sum_{k=0}^{\delta+D-4} \sum_{\vec{p}} \sum_{\gamma_3} (R^{\vec{p}})_{\gamma_3}(\vec{0})(a_{\vec{p}})^{\gamma_3}{}_{\gamma_1\gamma_2}, \tag{7.25}$$

with the $\vec{p} = (p_0, p_1, \ldots, p_{k-2})$ sum running over (7.12). Here we have abbreviated the product of curvature tensors appearing in the curvature expansion by writing

$$(R^{\vec{p}})_{\{\mu\nu\kappa\rho;(\sigma_1\cdots\sigma_j)\}}(\vec{0}) \equiv \prod_{j=0}^{k-2} \left[R_{\mu\nu\kappa\rho;(\sigma_1\cdots\sigma_j)}(\vec{0})\right]^{p_j}. \tag{7.26}$$

Now replace $\Omega_C[g^{(N)}_{\mu\nu}, t_\mu, L](x_1, x_2; \vec{0})$ in (7.15) with

$$\begin{aligned} &\widetilde{\Omega}_C[g^{(N)}_{\mu\nu}, t_\mu, L](x_1, x_2; \vec{0}) \\ &\equiv \Omega_C[g^{(N)}_{\mu\nu}, t_\mu, L](x_1, x_2; \vec{0}) - \sum_{|\gamma_1|+|\gamma_2|\leq\delta} \frac{1}{\gamma_1!\gamma_2!} a_{\gamma_1\gamma_2}[g^{(N)}_{\mu\nu}, t_\mu, L] x_1^{(\gamma_1} x_2^{\gamma_2)} \end{aligned} \tag{7.27}$$

Then, to scaling degree δ, $\widetilde{\Omega}_C$ is independent of the choice of unit-normalized timelike t_μ and differs from Ω_C by a smooth function of (x_1, x_2) with the same scaling behavior as Ω_C. Thus, the flow relation

$$\frac{\partial}{\partial m^2} H_F[g_{\mu\nu}](x_1, x_2) \sim_\delta \widetilde{\Omega}_C[g^{(N)}_{\mu\nu}, t_\mu, L](x_1, x_2; \vec{0}), \tag{7.28}$$

is both local and covariant in the metric. However, it does not have the required scaling behavior, so we will make a further modification to this flow relation in the "scaling" paragraph below.

Finally, we note that we can obtain a recursive formula for $\boldsymbol{a}$ by the same procedure as in the Minkowski case discussed in Appendix C. Define

$$\boldsymbol{B}^{\kappa\rho} \equiv (B^{\kappa\rho})^{\gamma_3}{}_{\gamma_1\gamma_2} \equiv -2\Omega^{\gamma_3}{}_{\gamma_1\gamma_2}[\chi = y^{[\kappa}\partial^{\rho]}\chi], \tag{7.29}$$

with $\Omega^{\gamma_3}{}_{\gamma_1\gamma_2}[\chi = y^{[\kappa}\partial^{\rho]}\chi]$ denoting the Taylor coefficients of the smooth function,

$$\Omega^{\gamma_3}[\chi = y^{[\kappa}\partial^{\rho]}\chi](x_1, x_2; \vec{0}). \tag{7.30}$$

(Smoothness of (7.30) in (x_1, x_2) is guaranteed by the fact that $\partial^{\rho}_{(y)}\chi(y, \vec{0})$ vanishes in a neighborhood of $y = \vec{0}$.) Then for any infinitesimal restricted Lorentz transformation Λ_θ, we have

$$\boldsymbol{Q}_C(\Lambda_\theta) = -\theta_{\kappa\rho}\boldsymbol{B}^{\kappa\rho} + \mathcal{O}(\theta^2). \tag{7.31}$$

The analysis of Appendix C then implies that

$$\boldsymbol{a} = \sum_{\substack{j=1 \\ \widetilde{c}_j \neq 0}}^{k} \frac{1}{\widetilde{c}_j}\mathbb{E}_j\left(-\mathbb{L}_{\kappa\rho}\boldsymbol{B}^{\kappa\rho} + 4\sum_{i<j\leq n}\eta_{\mu_i\mu_j}\mathrm{tr}_{ij}\boldsymbol{a}\right), \tag{7.32}$$

with the notation defined in Appendix C, where we have lowered all indices on the tensors so that all tensors in (7.32) are of type $(0, |\gamma_1| + |\gamma_2| + |\gamma_3|)$. As explained in Appendix C, equation (7.32) determines higher rank coefficients $(a_{\vec{p}})^{\gamma_3}{}_{\gamma_1\gamma_2}$ inductively in terms of the equations for the lowest nontrivial ranks with a given symmetry. When $\vec{p} = \vec{0}$, the coefficients $(a_{\vec{p}})^{\gamma_3}{}_{\gamma_1\gamma_2}$ coincide with those appearing in the Minkowski flow relations: i.e., $(a_{\vec{0}})^{\gamma_3}{}_{\gamma_1\gamma_2} = a_{\gamma_1\gamma_2}$, whose rank $r \equiv |\gamma_1| + |\gamma_2| = 0, 1, 2$ cases were stated explicitly in the appendix. When $\vec{p} \neq \vec{0}$, the explicit lower-rank cases can be straightforwardly obtained using the methods of the appendix. For this purpose, it is worth noting the $(a_{\vec{p}})^{\gamma_3}{}_{\gamma_1\gamma_2}$ coefficients have the same symmetries as the Minkowski coefficients $a_{\gamma_1\gamma_2}$ in the lower multi-indices (and their respective spacetime indices $\gamma_1 = \{\mu_1 \cdots \mu_p\}$, $\gamma_2 = \{\nu_1 \cdots \nu_q\}$). However, the symmetries of the upper spacetime indices in $(a_{\vec{p}})^{\gamma_3}{}_{\gamma_1\gamma_2}$ are dictated by the curvature tensors (7.26).

Under the rescaling

$$(\eta_{\mu\nu}, d^D y, m^2, L) \to (\lambda^{-2}\eta_{\mu\nu}, \lambda^{-D} d^D y, \lambda^2 m^2, \lambda^{-1} L), \tag{7.33}$$

the inductive solutions (7.32) for $(a_{\vec{p}_k})^{\gamma_3}{}_{\gamma_1\gamma_2}$ will scale in the manner required for $\widetilde{\Omega}_C$ to have the same scaling behavior as Ω_C.

(iii) Scaling The flow equation (7.28) is local and covariant and scales almost homogeneously with the correct power of λ under $(g_{\mu\nu}, m^2, L) \to (\lambda^{-2} g_{\mu\nu}, \lambda^2 m^2, \lambda^{-1} L)$. However, on account of the nontrivial L dependence, we do not have the required almost homogeneous scaling under $(g_{\mu\nu}, m^2) \to (\lambda^{-2} g_{\mu\nu}, \lambda^2 m^2)$. This is the same difficulty as occurred in the Euclidean and Minkowski cases, and it can be overcome by further modifying the flow relation in the same manner as for those cases. Specifically, we replace the flow relation (7.28) with

$$\frac{\partial}{\partial m^2} H_F[g_{\mu\nu}](x_1, x_2) \sim_\delta \mathfrak{L}[L]\widetilde{\Omega}_C(x_1, x_2; \vec{0}; L) - \sum_{|\gamma_1|+|\gamma_2| \le \delta} \frac{1}{\gamma_1! \gamma_2!} c_{\gamma_1\gamma_2}(L) x_1^{(\gamma_1} x_2^{\gamma_2)}, \tag{7.34}$$

where $\mathfrak{L}$ was defined by (5.33) and where, for $L > 0$,

$$c_{\gamma_1\gamma_2}(L) \equiv \sum_k \sum_{\vec{p}_k} (R^{\vec{p}})_{\gamma_3}(\vec{0}) \Bigg[\mathfrak{L}[L](a_{\vec{p}})^{\gamma_3}{}_{\gamma_1\gamma_2}(L) + \tag{7.35}$$
$$+ \int_0^L dL' \left[\Omega^{\gamma_3}{}_{\gamma_1\gamma_2}\left[\chi = \partial_{L'}\big(\mathfrak{L}[L']\chi\big)\right] - \frac{\partial}{\partial L'}\left(\mathfrak{L}[L'](a_{\vec{p}})^{\gamma_3}{}_{\gamma_1\gamma_2}(L')\right)\right]\Bigg]$$

The flow relation (7.34) is local and covariant and has the proper scaling under $(g_{\mu\nu}, m^2) \to (\lambda^{-2} g_{\mu\nu}, \lambda^2 m^2)$.

Using the definition (7.27) of $\widetilde{\Omega}_C$ and the relations (5.44)-(5.46), we may rewrite (7.34) in terms of the OPE coefficients:

$$\begin{aligned} &\frac{\partial}{\partial m^2}(C_H)^I_{T_0\{\phi\phi\}}(x_1,x_2;\vec{0}) \\ &\sim_\delta -\frac{i}{2}\int \sqrt{-g^{(N)}(y)}\,\mathfrak{L}[L]\chi(y,\vec{0};L)\,(C_H)^I_{T_0\{\phi^2\phi\phi\}}(y,x_1,x_2;\vec{0}) + \\ &\quad - \sum_{[C]\le\delta+2} c_C[g^{(N)}_{\mu\nu},t_\mu,L](C_H)^C_{T_0\{\phi\phi\}}(x_1,x_2;\vec{0}), \end{aligned} \tag{7.36}$$

where $N \geq \delta + D - 4$. Here we have

$$\begin{aligned} c_C \equiv \sum_k \sum_{\vec{p}_k} (R^{\vec{p}})_\gamma(\vec{0}) \Bigg[&\mathfrak{L}[L](a_{\vec{p}})^\gamma{}_C(L) - \int_0^L dL' \frac{\partial}{\partial L'}\Big(\mathfrak{L}[L'](a_{\vec{p}})^\gamma{}_C(L')\Big) + \\ &-\frac{i}{2}\left[\frac{\partial^P}{(\partial^{\vec{p}}R)_\gamma(\vec{0})}\int_0^L dL' \int_y \frac{\partial}{\partial L'}\Big(\mathfrak{L}[L']\chi(y,\vec{0};L')\Big)\, C^I_{T_0\{\phi^2 C\}}(y,\vec{0};\vec{0})\right]_{g^{(k)}_{\mu\nu}=\eta_{\mu\nu}}\Bigg], \end{aligned} \tag{7.37}$$

with the k and $\vec{p}_k$ sums taken as in the curvature expansion (7.10) and we have abbreviated $\int_y \equiv d^D y\sqrt{-g^{(k)}(y)}$ and,

$$(\partial^{\vec{p}}R)_{\{\mu\cdots\sigma_{k-2}\}} \equiv \partial^{p_0}R_{\mu\nu\kappa\rho}\partial^{p_1}R_{\mu\nu\kappa\rho;\sigma}\cdots\partial^{p_{k-2}}R_{\mu\nu\kappa\rho;(\sigma_1\cdots\sigma_{k-2})}. \tag{7.38}$$

It is required that $(a_{\vec{p}})^\gamma{}_C = 0$ unless $[C]_\phi = 2$. For $[C]_\phi = 2$, the tensors $(a_{\vec{p}})^\gamma{}_C$ are given via the inductive formula (7.32) with,

$$(B^{\kappa\rho}_{\vec{p}_k})^\gamma{}_C \equiv i\left[\frac{\partial^P}{(\partial^{\vec{p}}R)_\gamma(\vec{0})}\int d^D y\sqrt{-g^{(k)}(y)}\, y^{[\kappa}\partial^{\rho]}\chi(y,\vec{0};L)\,(C_H)^I_{T_0\{\phi^2 C\}}(y,\vec{0};\vec{0})\right]_{g^{(k)}_{\mu\nu}=\eta_{\mu\nu}} \tag{7.39}$$

Formula (7.32) determines $(a_{\vec{p}})^{\gamma}{}_{C}$ up to Lorentz-invariant tensors that depend smoothly on $(\eta_{\mu\nu}, m^2, \xi)$ and that scale with the same overall factor of λ as Ω^{γ}_{C} under the rescaling (7.33). As in the Minkowski case, the $\mathfrak{L}$-operator and L-integral terms in (7.37) ensure that only the L-independent terms in $(a_{\vec{p}})^{\gamma}{}_{C}$ can contribute to c_C. Therefore, the only ambiguity in c_C corresponds to the choice of an L-independent Lorentz-invariant tensor in $(a_{\vec{p}})^{\gamma}{}_{C}$ that scales with the same overall factor of λ as Ω^{γ}_{C} under $(\eta_{\mu\nu}, m^2) \to (\lambda^{-2}\eta_{\mu\nu}, \lambda^2 m^2)$. In odd dimensions, there are no tensors that scale in this way and depend smoothly on $(\eta_{\mu\nu}, m^2)$ so $(a_{\vec{p}})^{\gamma}{}_{C}$ is unique and, thus, c_C has no ambiguities. In even dimensions, $(a_{\vec{p}})^{\gamma}{}_{C}$ is not unique and this yields the freedom in c_C to choose a local and covariant smooth function in $(x_1, x_2, g^{(N)}_{ab}, m^2, \xi)$. In curved spacetime, compatibility with the Leibniz axiom W4 places additional constraints on the allowed choices of c_C and, in even dimensional curved spacetimes with $D > 2$, there is an additional constraint coming from the conservation axiom W8. These constraints can always be (non-uniquely) satisfied and, for c_C satisfying these conditions, the remaining ambiguities in (7.36) are in 1-1 correspondence with the freedom to choose a Hadamard parametrix whose corresponding Hadamard normal-ordered Wick fields are compatible with axioms W1-W8.

By the same reasoning that led to Theorems 6 and 7, the flow relation eq. (7.36) together with the explicit formulas for the unextended time-ordered OPE coefficients of the Hadamard normal-ordered Wick fields imply flow relations for $(C_H)^{I}_{T_0\{\phi\cdots\phi\}}$, as expressed by the following theorem:

Theorem 8. *For any construction of the Wick monomials by Hadamard normal ordering,* *we have*

$$\begin{aligned}
&\frac{\partial}{\partial m^2}(C_H)^I_{T_0\{\phi\cdots\phi\}}(x_1,\ldots,x_n;\vec{0}) \approx \\
&-\frac{i}{2}\int d^D y\sqrt{-g^{(N)}(y)}\,\mathfrak{L}[L]\chi(y,\vec{0};L)\,(C_H)^I_{T_0\{\phi^2\phi\cdots\phi\}}(y,x_1,\ldots,x_n;\vec{0})+ \\
&-\sum_C c_C[g^{(N)}_{\mu\nu},t_\mu,L](C_H)^C_{T_0\{\phi\cdots\phi\}}(x_1,\ldots,x_n;\vec{0}),
\end{aligned} \tag{7.40}$$

with c_C *defined in* (7.37)*.*

As was the case in the flat spacetime case, the ambiguities in these flow relations are in 1-1 correspondence with the freedom to choose H. The flow relations for general prescriptions for the Wick fields may straightforwardly obtained from (7.40) in the manner discussed below Theorem 6.

Remark 24. The derivation of L-independent local and covariant flow relations with respect to the curvature-coupling parameter ξ proceeds essentially identically as the one presented here for m^2. The ξ flow relations are of the same form as (7.40) with the substitutions $m^2 \to \xi$ and $d^D y \to d^D y R(y)$. Of course, locality requires the Ricci scalar curvature, R, must be computed using the polynomial metric $g^{(N)}_{\mu\nu}$ rather than $g_{\mu\nu}$. Note also ξ is dimensionless and the Ricci scalar curvature scales as $R[\lambda^{-2}g^{(N)}_{\mu\nu}] = \lambda^2 R[g^{(N)}_{\mu\nu}]$ so the ξ flow relations scale with an overall extra power of λ^2 relative to the m^2 flow relations.

APPENDIX A

EXISTENCE OF HADAMARD PARAMETRIX SATISFYING THE CONSERVATION CONSTRAINT

In this appendix, we prove that there exists $Q(x_1, x_2)$ satisfying (3.33) for $D > 2$. Abbreviate $Q_0(y) \equiv Q(y, y)$ and

$$Q_{ab}(y) \equiv \left[\nabla_a^{(x_1)} \nabla_b^{(x_2)} Q(x_1, x_2)\right]_{x_1, x_2 = y}. \tag{A.1}$$

It is straightforward to show that

$$\left[\nabla_b^{(x_1)} K_{x_2} Q(x_1, x_2)\right]_{x_1, x_2 = y} = -\nabla^a Q_{ba} + \frac{1}{2}\nabla_b Q^a{}_a + \frac{1}{2}(m^2 + \xi R)\nabla_b Q_0, \tag{A.2}$$

with $Q^a{}_a \equiv g^{ab} Q_{ab}$. Hence, the conservation condition (3.33) is equivalent to:

$$-\nabla^a Q_{ba} + \frac{1}{2}\nabla_b Q^a{}_a + \frac{1}{2}(m^2 + \xi R)\nabla_b Q_0 = -\frac{D}{2(D+2)}\nabla_b^{(y)} \left[K_{x_2} H(x_1, x_2)\right]_{x_1, x_2 = y}, \tag{A.3}$$

where we have used (3.32). Eq. (A.3) is solved (non-uniquely) for $D > 2$ by setting $Q_0(y) = 0$ and

$$Q_{ab}(y) = -\frac{D}{D^2 - 4} g_{ab} \left[K_{x_2} H(x_1, x_2)\right]_{x_1, x_2 = y}. \tag{A.4}$$

To see that there exists a smooth function $Q(x_1, x_2)$ with these properties, we first note one can always obtain a smooth function $f(x_1, x_2; y)$ with arbitrarily-specified covariant derivatives evaluated at $x_1, x_2 = y$, by the construction described in the proof of Proposition 1. Thus, we may arrange that $f(y, y; y) = 0$ and

$$\nabla_a^{(x_1)} \nabla_b^{(x_2)} f(x_1, x_2; y)|_{x_1, x_2 = y} = -\frac{D}{D^2 - 4} g_{ab} \left[K_{x_2} H(x_1, x_2)\right]_{x_1, x_2 = y}, \tag{A.5}$$

while requiring f and its derivatives at $x_1, x_2 = y$ to depend smoothly on (m^2, ξ) and scale almost homogeneously. Moreover, this construction implies

$$\nabla^{(x_1)}_{\alpha_1}\nabla^{(x_2)}_{\alpha_2}\nabla^{(y)}_{\beta} f(x_1,x_2;y)|_{x_1,x_2=y} = 0, \qquad \text{for all } |\beta| > 0, \tag{A.6}$$

and, thus, the "germ" of f at $x_1, x_2 = y$ is independent of y. Hence, we may construct a y-independent smooth bi-variate Q satisfying (3.33) and (3.34) which depends symmetrically on (x_1, x_2) via:

$$Q(x_1,x_2) = \frac{1}{2} f(x_1,x_2;x_1) + \frac{1}{2} f(x_1,x_2;x_2). \tag{A.7}$$

APPENDIX B

PROOFS FOR SECTION 4.2

We collect here proofs to the theorem and propositions contained in Section 4.2

***Sketch of proof for Theorem* 4**. The manipulations leading to (4.40) establish OPEs are preserved under field redefinitions, so the existence of the OPE for general Wick prescriptions follows from the existence of an OPE for Hadamard normal-ordered Wick fields (see Theorem 2 in Section 4.1). Moreover, the scaling degree of the OPE coefficients are unaffected by the field redefinitions. We now argue the associativity conditions are also preserved under field redefinitions. For notational simplicity, we give the argument for an OPE involving three spacetime points with the merger tree $\mathcal{T}$ corresponding to x_1 and x_2 approaching each other faster than x_3. The argument can then be straightforwardly generalized to n-point OPEs with arbitrary merger trees. From (4.40), we have,

$$\begin{aligned}
&C^B_{A_1A_2A_3}(x_1,x_2,x_3;z) \\
&\approx \sum_{C_0,\ldots,C_3} \mathcal{Z}^B_{C_0}(z)(\mathcal{Z}^{-1})^{C_1}_{A_1}(x_1)(\mathcal{Z}^{-1})^{C_2}_{A_2}(x_2)(\mathcal{Z}^{-1})^{C_3}_{A_3}(x_3)(C_H)^{C_0}_{C_1C_2C_3}(x_1,x_2,x_3;z).
\end{aligned} \tag{B.1}$$

The associativity condition for Hadamard normal-ordered OPE coefficients implies the coefficient in the second line can be expanded as

$$\begin{aligned}
&(C_H)^{C_0}_{C_1C_2C_3}(x_1,x_2,x_3;z) \\
&\sim_{\mathcal{T},\delta} \sum_{D_1}(C_H)^{D_1}_{C_1C_2}(x_1,x_2;z')(C_H)^{C_0}_{D_1C_3}(z',x_3;z) \\
&= \sum_{D_1,D_2}\left[\sum_E \mathcal{Z}^E_{D_2}(z')(\mathcal{Z}^{-1})^{D_1}_E(z')\right](C_H)^{D_2}_{C_1C_2}(x_1,x_2;z')(C_H)^{C_0}_{D_1C_3}(z',x_3;z),
\end{aligned} \tag{B.2}$$

where, in going to the final line, we have used the identity:

$$\sum_E \mathcal{Z}_{D_2}^{E}(z')(\mathcal{Z}^{-1})_E^{D_1}(z') = \delta_{D_2}^{D_1}. \tag{B.3}$$

Plugging (B.3) back into (B.2) and rearranging summations, we find then,

$$\begin{aligned} &C_{A_1A_2A_3}^{B}(x_1,x_2,x_3;z) \\ &\sim_{\mathcal{T},\delta} \sum_E \left[\sum_{D_2,C_1,C_2} \mathcal{Z}_{D_2}^{E}(z')(\mathcal{Z}^{-1})_{A_1}^{C_1}(x_1)(\mathcal{Z}^{-1})_{A_2}^{C_2}(x_2)(C_H)_{C_1C_2}^{D_2}(x_1,x_2;z') \right] \times \\ &\qquad \times \left[\sum_{C_0,D_1,C_3} \mathcal{Z}_{C_0}^{B}(z)(\mathcal{Z}^{-1})_E^{D_1}(z')(\mathcal{Z}^{-1})_{A_3}^{C_3}(x_3)(C_H)_{D_1C_3}^{C_0}(z',x_3;z) \right]. \end{aligned} \tag{B.4}$$

By (4.40), this is equivalent to,

$$C_{A_1A_2A_3}^{B}(x_1,x_2,x_3;z) \sim_{\mathcal{T},\delta} \sum_E C_{A_1A_2}^{E}(x_1,x_2;z')C_{EA_3}^{B}(z',x_3;z). \tag{B.5}$$

All other associativity conditions, including (4.36), for general prescriptions of the Wick powers may similarly be established using the corresponding associativity conditions for Hadamard normal-ordered OPE coefficients and the identity (B.3). □

***Sketch of proof for Proposition* 3.** The proof makes use of the relationship (4.40) between the general Wick OPE coefficients and the Hadamard normal-ordered coefficients, the identity (4.32) for the Hadamard OPE coefficients established in Proposition 2, and the recursion relation (3.50) satisfied by the mixing matrix $\mathcal{Z}_A^B$. By (4.40) and (4.32), we have for any $p \leq m \equiv [B]_\phi$,

$$C^B_{A_1\cdots A_n} = \sum_{C_1,\ldots,C_n} \Bigg[(\mathcal{Z}^{-1})^{C_1}_{A_1} \cdots (\mathcal{Z}^{-1})^{C_n}_{A_n} \times \tag{B.6}$$
$$\times \sum_{\{P_1,P_2\}\in\mathcal{P}_p(\mathcal{S}_C)} \sum_{k\geq m} \underbrace{\binom{k}{p}^{-1} \mathcal{Z}^B_{\gamma_1\cdots\gamma_k} (C_H)^{(\nabla_{\gamma_1}\phi\cdots\nabla_{\gamma_p}\phi)}_{C'_1\cdots C'_n} (C_H)^{(\nabla_{\gamma_{(p+1)}}\phi\cdots\nabla_{\gamma_k}\phi)}_{C''_1\cdots C''_n}}_{\text{(a)}} \Bigg]$$

Now, inserting the recursion relation (3.50) for the mixing matrix,

$$\mathcal{Z}^B_{\gamma_1\cdots\gamma_k} = \mathcal{Z}^{\beta_1\cdots\beta_m}_{\gamma_1\cdots\gamma_k} = \binom{k}{p}\binom{m}{p}^{-1} \delta^{\beta_1}_{(\gamma_1} \cdots \delta^{\beta_p}_{\gamma_p} \mathcal{Z}^{\beta_{(p+1)}\cdots\beta_m}_{\gamma_{(p+1)}\cdots\gamma_k)}, \tag{B.7}$$

into the underbraced factor immediately yields:

$$\text{(a)} = \binom{m}{p}^{-1} (C_H)^{(\nabla_{\beta_1}\phi\cdots\nabla_{\beta_p}\phi)}_{C'_1\cdots C'_n} \mathcal{Z}^{\beta_{(p+1)}\cdots\beta_m}_{\gamma_{(p+1)}\cdots\gamma_k} (C_H)^{(\nabla_{\gamma_{(p+1)}}\phi\cdots\nabla_{\gamma_k}\phi)}_{C''_1\cdots C''_n}. \tag{B.8}$$

Plugging this back into (B.6) gives,

$$C^B_{A_1\cdots A_n} = \binom{m}{p}^{-1} \sum_{C_0} \Bigg[\mathcal{Z}^{\beta_{(p+1)}\cdots\beta_m}_{C_0} \times \tag{B.9}$$
$$\times \sum_{C_1,\ldots,C_n} \underbrace{(\mathcal{Z}^{-1})^{C_1}_{A_1} \cdots (\mathcal{Z}^{-1})^{C_n}_{A_n} \sum_{\{P_1,P_2\}\in\mathcal{P}_p(\mathcal{S}_C)} (C_H)^{(\nabla_{\gamma_1}\phi\cdots\nabla_{\gamma_p}\phi)}_{C'_1\cdots C'_n} (C_H)^{C_0}_{C''_1\cdots C''_n}}_{\text{(b)}} \Bigg].$$

Recalling the definition of $\mathcal{P}_p(\mathcal{S})$ above eq. (4.30), one may use the recursion relation (3.50) for the inverse mixing matrices in a similar manner (as with $\mathcal{Z}^B_{\gamma_1\cdots\gamma_k}$ in the (a)-term above) to rewrite the underbraced term:

$$\text{(b)} = \sum_{\{P_1,P_2\}\in\mathcal{P}_p(\mathcal{S}_A)} \delta^{C'_1}_{A'_1}\cdots\delta^{C'_n}_{A'_1}\,(C_H)^{(\nabla_{\gamma_1}\phi\cdots\nabla_{\gamma_p}\phi)}_{C'_1\cdots C'_n}(\mathcal{Z}^{-1})^{C''_1}_{A''_1}\cdots(\mathcal{Z}^{-1})^{C''_n}_{A''_n}\,(C_H)^{C_0}_{C''_1\cdots C''_n}\,, \tag{B.10}$$

where we note that the sum in (B.10) is now taken over elements of $\mathcal{P}_p(\mathcal{S}_A)$ rather than $\mathcal{P}_p(\mathcal{S}_C)$. Finally, inserting this back into (B.9) yields,

$$C^B_{A_1\cdots A_n} = \binom{m}{p}^{-1} \sum_{\{P_1,P_2\}\in\mathcal{P}_p(\mathcal{S}_A)} \Bigg[(C_H)^{(\nabla_{\gamma_1}\phi\cdots\nabla_{\gamma_p}\phi)}_{A'_1\cdots A'_n} \times \tag{B.11}$$

$$\times \sum_{C''_0,C''_1,\ldots,C''_n} (\mathcal{Z}^{-1})^{C''_1}_{A''_1}\cdots(\mathcal{Z}^{-1})^{C''_n}_{A''_n}\mathcal{Z}^{\beta_{(p+1)}\cdots\beta_m}_{C_0}(C_H)^{C_0}_{C''_1\cdots C''_n}\Bigg],$$

which, by eqs. (4.41) and (4.40), is equivalent to formula (4.32) with the H-subscripts removed. □

***Proof of Proposition* 4**. The proof of (4.43) is based on the associativity conditions (4.36) and the behavior of the Wick OPE coefficients $C^B_{\phi\cdots\phi}(x_1,\ldots,x_n;z)$ on the total diagonal when $[B]_\phi = n$. As established in Theorem 4, the associativity conditions hold for general prescriptions for the Wick powers. In particular, for the class of merger trees $\mathcal{T}$ such that $\vec{y}_i \to x_i$ at a faster rate than $x_i \to z$, we have, cf. formula (4.36),

$$C^B_{\phi\cdots\phi}(\vec{y}_1,\ldots,\vec{y}_n;z) \sim_{\mathcal{T},\delta} \sum_{C_1,\ldots,C_n} C^{C_1}_{\phi\cdots\phi}(\vec{y}_1;x_1)\cdots C^{C_n}_{\phi\cdots\phi}(\vec{y}_n;x_n)C^B_{C_1\cdots C_n}(x_1,\ldots,x_n;z), \tag{B.12}$$

with the summations carried to a sufficiently high, but finite, order. As we shall see, for our purposes, it is sufficient to include only $[C_i] \le [A_i]$ for all i.

We note the OPE coefficients $C^{C_i}_{\phi\cdots\phi}(\vec{y}_i;x_i)$ vanish unless $[C_i]_\phi \le k_i \equiv [A_i]_\phi$ for all i. It is useful to rearrange (B.12), putting all terms such that $[C_i] = k_i$ for all i on one side:

$$\begin{aligned}
&\sum_{[C_1]_\phi = [A_1]_\phi} \cdots \sum_{[C_n] = [A_n]_\phi} C^{C_1}_{\phi\cdots\phi}(\vec{y}_1; x_1) \cdots C^{C_n}_{\phi\cdots\phi}(\vec{y}_n; x_n) C^{B}_{C_1\cdots C_n}(x_1, \ldots, x_n; z) \\
&\sim_{\mathcal{T},\delta} C^{B}_{\phi\cdots\phi}(\vec{y}_1, \ldots, \vec{y}_n; z) + \\
&- \sum_{[C_1]_\phi < [A_1]_\phi} \cdots \sum_{[C_n]_\phi < [A_n]_\phi} C^{C_1}_{\phi\cdots\phi}(\vec{y}_1; x_1) \cdots C^{C_n}_{\phi\cdots\phi}(\vec{y}_n; x_n) C^{B}_{C_1\cdots C_n}(x_1, \ldots, x_n; z).
\end{aligned} \tag{B.13}$$

We now note the limiting behavior of the coefficients:

$$\lim_{\vec{y}_i \to x_i} C^{C_i}_{\phi\cdots\phi}(\vec{y}_i; x) = \begin{cases} 1 & [C_i] = [C_i]_\phi = k_i \\ 0 & [C_i] > k_i \end{cases} \tag{B.14}$$

The second case follows from the fact that $C^{C_i}_{\phi\cdots\phi}(\vec{y}_i; x_i)$ has negative scaling degree when $[C_i] > k_i$ by (4.5). The first case follows from the fact that, when $[C_i]_\phi = k_i$, the $C^{C_i}_{\phi\cdots\phi}(\vec{y}_i; x_i)$ are given by geometric factors (4.41) and these factors satisfy:

$$\lim_{y \to x} S^{\beta}(y; x) = \begin{cases} 1 & |\beta| = 0 \\ 0 & |\beta| > 0 \end{cases}, \tag{B.15}$$

because $\lim_{y\to x} \nabla^{b}_{(y)} \sigma(y; x) = 0$. Evaluating the proposed limit of (B.13), using (B.14), we then find:

$$\begin{aligned}
&C^{B}_{\phi^{k_1}\cdots\phi^{k_n}}(x_1, \ldots, x_n; z) \\
&= \lim_{\vec{y}_1 \to x_1} \cdots \lim_{\vec{y}_n \to x_n} \Big[C^{B}_{\phi\cdots\phi}(\vec{y}_1, \ldots, \vec{y}_n; z) + \\
&- \sum_{\substack{[C_1] < [A_1] \\ [C_1]_\phi < [A_1]_\phi}} \cdots \sum_{\substack{[C_n] < [A_n] \\ [C_n]_\phi < [A_n]_\phi}} C^{C_1}_{\phi\cdots\phi}(\vec{y}_1; x_1) \cdots C^{C_n}_{\phi\cdots\phi}(\vec{y}_n; x_n) C^{B}_{C_1\cdots C_n}(x_1, \ldots, x_n; z) \Big].
\end{aligned} \tag{B.16}$$

This establishes formula (4.43) for the OPE coefficients involving products of Wick powers with no derivatives. To obtain the general case, apply the derivative operator $\nabla^{y_{(1,1)}}_{\alpha_{(1,1)}} \cdots \nabla^{y_{(n,k_n)}}_{\alpha_{(n,k_n)}}$ to both sides of relation (B.13) and take the limits $\vec{y}_i \to x_i$ for all i, using the identity:

$$\lim_{\vec{y}_i \to x_i} \sum_{[C_i]_\phi = [A_i]_\phi} \nabla^{(y_{i,1})}_{\alpha_{(i,1)}} \cdots \nabla^{(y_{i,k_i})}_{\alpha_{(i,k_i)}} C^{C_i}_{\phi\cdots\phi}(\vec{y}_i; x_i) C^{B}_{C_1\cdots C_i \cdots C_n}(x_1, \ldots, x_n; z)$$
$$= C^{B}_{C_1\cdots A_i\cdots C_n}(x_1, \ldots, x_n; z), \tag{B.17}$$

which follows, in turn, from the identity (3.60) for the covariant derivative acting on any scalar field.

Note, in our derivation, no assumption has been made about the rate $x_1, \ldots, x_n$ approach each other in (B.12), so the resulting formula (4.43) is valid under arbitrary merger trees for these points. □

***Proof of Proposition* 5.** Using Wick's theorem (3.8), we find:

$$\phi(x_1)\phi(x_2)\cdots\phi(x_n) \tag{B.18}$$
$$= \sum_{k=0}^{\lfloor n/2 \rfloor} \sum_{\sigma_k} H(x_{\sigma(1)}, x_{\sigma(2)}) \cdots H(x_{\sigma(2k-1)}, x_{\sigma(2k)}) :\phi(x_{\sigma(2k+1)}) \cdots \phi(x_{\sigma(n)}):_H,$$

where σ_k runs over the same permutations as in formula (4.48). Putting all terms on the right-hand side and smearing with the test distribution $t_{n+1} \in \mathcal{E}'(\times^{(n+1)} M, g_{ab})$ defined in eq. (3.29) then yields:

$$0=\int_{z,x_1,\ldots,x_n} f^{\alpha_1\cdots\alpha_n}\delta(z,x_1,\ldots,x_n)\nabla^{(x_1)}_{\alpha_1}\cdots\nabla^{(x_n)}_{\alpha_n}\Bigg[\phi(x_1)\cdots\phi(x_n)+ \tag{B.19}$$

$$-\sum_{k=0}^{\lfloor n/2\rfloor}\sum_{\sigma_k}H(x_{\sigma(1)},x_{\sigma(2)})\cdots H(x_{\sigma(2k-1)},x_{\sigma(2k)})\times$$

$$\times S^{\beta_{2k+1}}(x_{\sigma(2k+1)};z)\cdots S^{\beta_n}(x_{\sigma(n)};z)(\nabla_{\beta_{2k+1}}\phi\cdots\nabla_{\beta_n}\phi)_H(z)\Bigg],$$

with implied summations over β multi-indices. Note only finitely-many terms contribute non-trivially to the sum. In writing (B.19), we have used the definition (3.28) of the Hadamard normal-ordered Wick fields. We may now use formula (3.70) to write $(\nabla_{\beta_1}\phi\cdots\nabla_{\beta_m}\phi)_H$ in terms of $(\nabla_{\beta_1}\phi\cdots\nabla_{\beta_m}\phi)$ and the smooth functions $F_{q\leq m}$. Plugging this into (B.19), one can then use the explicit expression (4.48) for the Wick OPE coefficients $C^I_{\phi\cdots\phi}$ to write (B.19) as:

$$0=\int_{z,x_1,\ldots,x_n} f^{\alpha_1\cdots\alpha_n}\delta(z,x_1,\ldots,x_n)\nabla^{(x_1)}_{\alpha_1}\cdots\nabla^{(x_n)}_{\alpha_n}\Bigg[\phi(x_1)\cdots\phi(x_n)+ \tag{B.20}$$

$$-\sum_{m\leq n}\sum_{\pi\in\Pi_m}C^I_{\phi\cdots\phi}(x_{\pi(m+1)},\ldots,x_{\pi(n)};z)\times$$

$$\times S^{\beta_1}(x_{\pi(1)};z)\cdots S^{\beta_m}(x_{\pi(m)};z)(\nabla_{\beta_1}\phi\cdots\nabla_{\beta_m}\phi)(z)\Bigg],$$

with Π_m and $\int_{z,x_1,\ldots,x_n}$ defined as in (4.49). Note again there are implied finite sums over β multi-indices. Note the $m=n$ term in the sum yields:

$$-\delta(z,x_1,\ldots,x_n)\nabla^{(x_1)}_{\alpha_1}S^{\beta_1}(x_1;z)\cdots\nabla^{(x_n)}_{\alpha_n}S^{\beta_n}(x_n;z)(\nabla_{\beta_1}\phi\cdots\nabla_{\beta_n}\phi)=-(\nabla_{\alpha_1}\phi\cdots\nabla_{\alpha_m}\phi), \tag{B.21}$$

using the identity (3.60). Moving this term to the left-hand side of (B.20), then gives the equation (4.49) we sought to show. □

APPENDIX C

CONSTRUCTION OF $a_{\gamma_1\gamma_2}$ FOR LORENTZ-COVARIANCE-RESTORING TERMS

The goal of this appendix is construct Λ-independent $\boldsymbol{a} \equiv a_{\gamma_1\gamma_2}$ such that (6.22) holds for any choice of cutoff function χ, i.e., to construct $\boldsymbol{a}$ such that $\widetilde{\Omega}_M$ defined via (6.15) is Lorentz-invariant. Our strategy will be to solve (6.22) inductively for infinitesimal Lorentz transformations,

$$\Lambda_\theta = I + \frac{1}{2}\theta_{\kappa\rho} l^{\kappa\rho}, \tag{C.1}$$

which generate the restricted Lorentz group. Here $\theta_{\kappa\rho} = \theta_{[\kappa\rho]}$ parameterize an arbitrary infinitesimal transformation and $l^{\kappa\rho}$ denote the Lorentz generators. Restoring indices, the generators are given explicitly by,

$$(l^{\kappa\rho})^{\mu}{}_{\nu} \equiv 2\eta^{\mu[\kappa}\delta^{\rho]}{}_{\nu}, \tag{C.2}$$

in the vector representation. We define

$$Q_M(x_1, x_2; z; \Lambda^{-1}) \equiv \Omega_M(\Lambda x_1, \Lambda x_2; \Lambda z) - \Omega_M(x_1, x_2; z), \tag{C.3}$$

and we denote the Taylor coefficients which appear on the left-hand side of (6.22) by

$$\boldsymbol{Q}(\Lambda^{-1}) \equiv Q_{\gamma_1\gamma_2}(\Lambda^{-1}) \equiv \partial^{(x_1)}_{\gamma_1}\partial^{(x_2)}_{\gamma_2} Q_M(x_1, x_2; z; \Lambda^{-1})|_{x_1,x_2=z}\,. \tag{C.4}$$

Thus, $\boldsymbol{Q}$ is a spacetime tensor of the same rank $r = |\gamma_1| + |\gamma_1|$ as $\boldsymbol{a}$. Translation invariance implies $\boldsymbol{Q}(\Lambda^{-1})$ is independent of z. With this notation, the set of equations (6.22) that we wish to solve for $\boldsymbol{a}$ can be written as,

$$\boldsymbol{Q}(\Lambda) = (D(\Lambda) - \mathbb{I})\, \boldsymbol{a}, \tag{C.5}$$

where $D(\Lambda)$ denotes the representation of the Lorentz group on tensors of rank r. The kernel of the operator $D(\Lambda) - \mathbb{I}$ is comprised of Lorentz-invariant tensors, so (C.5) determines $\boldsymbol{a}$ up to the addition of a Lorentz-invariant tensor of rank r.

For the purposes of showing existence of a solution, $\boldsymbol{a}$, to (C.5), it is useful to have a manifestly smooth expression for the function $Q_M(x_1, x_2; z; \Lambda^{-1})$ defined in (C.3). From the definition (6.12) of Ω_M, we have,

$$\begin{aligned}\Omega_M(\Lambda x_1, \Lambda x_2; \Lambda z) &= -i \int d^D y\, \chi(y, \Lambda z)\, H_F(y, \Lambda x_1) H_F(y, \Lambda x_2) \\ &= -i \int d^D y'\, \chi(\Lambda y', \Lambda z)\, H_F(\Lambda y', \Lambda x_1)\, H_F(\Lambda y', \Lambda x_2) \\ &= -i \int d^D y'\, \chi(\Lambda y', \Lambda z)\, H_F(y', x_1) H_F(y', x_2).\end{aligned} \tag{C.6}$$

Here, the second equality was obtained by making a change of integration variables $y \to y' = \Lambda^{-1} y$ and the final equality follows from the Lorentz invariance of H_F. Plugging (C.6) into (C.3) yields,

$$Q_M(x_1, x_2; z; \Lambda^{-1}) = -i \int d^D y\, \left[\chi(\Lambda y, \Lambda z) - \chi(y, z)\right] H_F(y, x_1) H_F(y, x_2). \tag{C.7}$$

Since for arbitrary, fixed Λ, we have $\chi(\Lambda y, \Lambda z) - \chi(y, z) = 0$ when y is sufficiently close to z, it follows from Proposition 8 that $Q(x_1, x_2; z; \Lambda^{-1})$ is smooth in (x_1, x_2) when these points are sufficiently close to z. Evaluating the Taylor coefficients of (C.7) yields,

$$Q_{\gamma_1\gamma_2}(\Lambda^{-1}) = (-1)^{(1+|\gamma_1|+|\gamma_2|)} i \int d^D y \left[\chi(\Lambda y, \vec{0}) - \chi(y, \vec{0})\right] \partial^{(y)}_{\gamma_1} H_F(y, \vec{0}) \partial^{(y)}_{\gamma_2} H_F(y, \vec{0}), \tag{C.8}$$

where the translation symmetry has been used to put z at the origin. Note $Q_{\gamma_1\gamma_2}$ are manifestly invariant under interchange of multi-indices $Q_{\gamma_1\gamma_2} = Q_{\gamma_2\gamma_1}$ and symmetric within their respective spacetime indices $Q_{\{\mu_1\cdots\mu_{|\gamma_1|}\}\{\nu_1\cdots\nu_{|\gamma_2|}\}} = Q_{\{(\mu_1\cdots\mu_{|\gamma_1|})\}\{(\nu_1\cdots\nu_{|\gamma_2|})\}}$. The following proposition establishes the existence of $\boldsymbol{a}$ satisfying (C.5) by the same type of cohomology argument as used to prove the existence of counterterms in Epstein-Glaser renormalization [24] :

Proposition 10. *For any translation-invariant cutoff function χ and any restricted Lorentz transformation Λ, the tensors $\boldsymbol{Q}(\Lambda^{-1})$ defined in* (C.8) *are always of the form* (C.5) *for some Λ-independent tensors $\boldsymbol{a}$, which are uniquely determined modulo Lorentz-invariant tensors of rank $r \equiv |\gamma_1| + |\gamma_2|$.*

Proof. Using the explicit formula (C.8), we find:

$$\begin{aligned}
&Q_{\gamma_1\gamma_2}(\Lambda_1\Lambda_2) - Q_{\gamma_1\gamma_2}(\Lambda_1) \\
&= (-1)^{(1+|\gamma_1|+|\gamma_2|)} i \int d^D y \left[\chi(\Lambda_2^{-1}\Lambda_1^{-1} y, \vec{0}) - \chi(\Lambda_1^{-1} y, \vec{0})\right] \partial^{(y)}_{\gamma_1} H_F(y, \vec{0}) \partial^{(y)}_{\gamma_2} H_F(y, \vec{0}) \\
&= (\Lambda_1^{-1})^{\gamma_1'}{}_{\gamma_1} (\Lambda_1^{-1})^{\gamma_2'}{}_{\gamma_2} (-1)^{(1+|\gamma_1|+|\gamma_2|)} i \times \\
&\qquad\qquad \times \int d^D y' \left[\chi(\Lambda_2^{-1} y', \vec{0}) - \chi(y', \vec{0})\right] \partial^{(y)}_{\gamma_1'} H_F(y', \vec{0}) \partial^{(y)}_{\gamma_2'} H_F(y', \vec{0}) \\
&= (\Lambda_1)_{\gamma_1}{}^{\gamma_1'} (\Lambda_1)_{\gamma_2}{}^{\gamma_2'} Q_{\gamma_1'\gamma_2'}(\Lambda_2).
\end{aligned} \tag{C.9}$$

In going to the first equality, we note $(\Lambda_1\Lambda_2)^{-1} = \Lambda_2^{-1}\Lambda_1^{-1}$. The second equality follows from a change of integration variables $y \to y' = \Lambda_1^{-1} y$, noting the parametrix is Lorentz invariant and $\det \Lambda_1 = 1$ so $d^D y' = d^D y$.

Given (C.9), eq. (C.5) can now be established via the following cohomological argument: Denote the restricted Lorentz group $\mathcal{L}_+^\uparrow \equiv SO^+(1, D-1)$ and denote by $C^n(\mathcal{L}_+^\uparrow)$ the set of all tensors $\boldsymbol{T} \equiv T_\alpha(\Lambda_1, \ldots, \Lambda_n)$ which depend continuously on Λ. For each $n \geq 0$, we define the "coboundary operator" $d^n : C^n(\mathcal{L}^\uparrow) \to C^{n+1}(\mathcal{L}_+^\uparrow)$ by[1],

$$
\begin{aligned}
(d^n\boldsymbol{T})(\Lambda_1, \ldots, \Lambda_{n+1}) \equiv (-1)^{(n+1)}\boldsymbol{T}(\Lambda_1, \ldots, \Lambda_n) + D(\Lambda_1)\boldsymbol{T}(\Lambda_2, \ldots, \Lambda_{(n+1)}) + \\
+ \sum_{k=1}^{n}(-1)^k\boldsymbol{T}(\Lambda_1, \ldots, \Lambda_{(k-1)}, \widehat{\Lambda_k}\Lambda_k\Lambda_{(k+1)}, \Lambda_{(k+2)} \ldots, \Lambda_{(n+1)}).
\end{aligned}
\tag{C.10}
$$

For any $\boldsymbol{T} \in C^n(\mathcal{L}_+^\uparrow)$, it follows from the definition (C.10) via a straightforward computation that we have

$$
(d^{n+1} \circ d^n\boldsymbol{T})(\Lambda_1, \ldots, \Lambda_{n+2}) = 0. \tag{C.11}
$$

Hence, for any $\boldsymbol{T}$ such that,

$$
d^n\boldsymbol{T} = 0, \tag{C.12}
$$

it follows immediately from (C.11) that (C.12) is satisfied by,

$$
\boldsymbol{T} = d^{n-1}\boldsymbol{S}, \tag{C.13}
$$

for tensor $\boldsymbol{S} = \boldsymbol{S}(\Lambda_1, \ldots, \Lambda_{n-1})$ with the same rank as $\boldsymbol{T}$. If the only solutions to (C.12) are of the form (C.13), then it is said that the "n-th cohomology group", $H^n(\mathcal{L}_+^\uparrow) \equiv \ker d^n/ \operatorname{im} d^{n-1}$, is empty. It has been proven [50, Subsection 5.C] that the first cohomology group $H^1(\mathcal{L}_+^\uparrow)$ is empty. However, by eq. (C.9), we have

$$
0 = (d^1\boldsymbol{Q})(\Lambda_1, \Lambda_2) = \boldsymbol{Q}(\Lambda_1) + D(\Lambda_1)\boldsymbol{Q}(\Lambda_2) - \boldsymbol{Q}(\Lambda_1\Lambda_2), \tag{C.14}
$$

1. In cohomology theory, C^n are known as the group of "n-cochains". The sequence, $C^0 \xrightarrow{d^0} C^1 \xrightarrow{d^1} C^2 \xrightarrow{d^2} \cdots$, generated by the coboundary operators d^n is called a "cochain complex".

Therefore, the *only* tensors satisfying (C.14) are of the form

$$\boldsymbol{Q}(\Lambda) = (d^0\boldsymbol{a})(\Lambda) = (D(\Lambda) - \mathbb{I})\boldsymbol{a}. \tag{C.15}$$

Thus, for $\boldsymbol{Q}$ given by (C.8), there exists a solution $\boldsymbol{a}$ to (C.5). □

Although Proposition 10 establishes existence of $\boldsymbol{a}$, we wish to obtain an explicit solution for $\boldsymbol{a}$ in order to write the flow relations in an explicit form. In the remainder of this appendix, we derive an explicit solution for $\boldsymbol{a}$ for ranks $r = 0, 1, 2$ and then obtain an inductive solution for $\boldsymbol{a}$ for $r > 2$. Our analysis closely follows the approach taken by [26, Subsection 3.3] in the context of the Epstein-Glaser renormalization scheme, while generalizing to arbitrary spacetime dimension.

For $r = 0$, $D(\Lambda) = 1$ and thus (C.15) implies $Q_{\{0\}\{0\}} = 0$ for any Lorentz-invariant scalar $a_{\{0\}\{0\}}$. For $r = 1$, we have $Q_{\{\mu\}\{0\}} = Q_{\{0\}\{\mu\}}$, so there is only a single independent $\boldsymbol{Q}(\Lambda)$. The dependence of $\boldsymbol{Q}$ on Λ comes entirely through the cutoff function χ. Since we have

$$\chi(\Lambda_\theta^{-1}y, \vec{0}) - \chi(y, \vec{0}) = -\frac{1}{2}\theta_{\kappa\rho}(l^{\kappa\rho})^{\mu}{}_{\nu}y^{\nu}\partial_\mu\chi(y, \vec{0}) + \mathcal{O}(\theta^2) \tag{C.16}$$

it follows from (C.8) that at leading order in θ we have

$$Q_{\{\mu\}\{0\}}(\Lambda_\theta) = -\frac{1}{2}\theta_{\kappa\rho}(B^{\kappa\rho})_{\{\mu\}\{0\}}, \tag{C.17}$$

where

$$(B^{\kappa\rho})_{\{\mu\}\{0\}} \equiv i(l^{\kappa\rho})^{\sigma_1}{}_{\sigma_2}\int d^D y\, y^{\sigma_2}\partial^{(y)}_{\sigma_1}\chi(y, \vec{0})\, \partial^{(y)}_{\mu} H_F(y, \vec{0}) H_F(y, \vec{0}). \tag{C.18}$$

Note that $(B^{\kappa\rho})_{\{\mu\}\{0\}}$ is independent of $\theta_{\kappa\rho}$. On the other hand, for $r = 1$, to leading order in $\theta_{\kappa\rho}$ the right-hand side of (C.5) is simply,

$$(D(\Lambda_\theta) - I)\,\boldsymbol{a} = -\frac{1}{2}\theta_{\kappa\rho} l^{\kappa\rho} \boldsymbol{a}. \tag{C.19}$$

Hence, for $r = 1$, to leading order in $\theta_{\kappa\rho}$ equation (C.5) is equivalent to,

$$\theta_{\kappa\rho} l^{\kappa\rho} \boldsymbol{a} = \theta_{\kappa\rho} \boldsymbol{B}^{\kappa\rho}. \tag{C.20}$$

eq. (C.20) will hold for all infinitesimal $\theta_{\kappa\rho}$ if and only if,

$$l^{\kappa\rho} \boldsymbol{a} = \boldsymbol{B}^{\kappa\rho}, \tag{C.21}$$

for all $\kappa, \rho = 0, 1, \ldots, D-1$. Contracting this equation with $l_{\kappa\rho}$ and using the identity[2]

$$l_{\kappa\rho} l^{\kappa\rho} = -2(D-1)I, \tag{C.22}$$

we obtain the explicit solution

$$\begin{aligned} a_{\{0\}\{\mu\}} = a_{\{\mu\}\{0\}} &= -\frac{1}{2(D-1)} (l_{\kappa\rho} B^{\kappa\rho})_{\{\mu\}\{0\}} \\ &= -i \int d^D y\, \partial_\mu^{(y)} \chi(y, \vec{0})\, H_F(y, \vec{0}) H_F(y, \vec{0}), \end{aligned} \tag{C.23}$$

where we have used (C.2) and (C.18) to obtain the second line.

We proceed now to $r = 2$. There are two independent $\boldsymbol{Q}$ tensors of rank two and they are both symmetric in their spacetime indices: $Q_{\{\mu\}\{\nu\}} = Q_{\{(\mu\}\{\nu)\}}$ and $Q_{\{\mu\nu\}\{0\}} = Q_{\{0\}\{\mu\nu\}} =$

2. The left-hand side of (C.22) is the quadratic Casimir operator of the Lie algebra of the homogeneous Lorentz group.

$Q_{\{0\}\{(\mu\nu)\}}$. For $r = 2$ and infinitesimal $\Lambda = \Lambda_\theta$, one now finds (C.5) takes the form

$$\left(l^{\kappa\rho} \otimes I + I \otimes l^{\kappa\rho}\right) \boldsymbol{a} = \boldsymbol{B}^{\kappa\rho}. \tag{C.24}$$

Here $(B^{\kappa\rho})_{\{\mu\}\{\nu\}}$ and $(B^{\kappa\rho})_{\{\mu\nu\}\{0\}} = (B^{\kappa\rho})_{\{0\}\{\mu\nu\}}$ are defined by a rank 2 generalization of (C.18); the general formula for $\boldsymbol{B}^{\kappa\rho}$ for arbitrary rank is given in equation (C.31) below. Applying the operator $\left(l_{\kappa\rho} \otimes I + I \otimes l_{\kappa\rho}\right)$ to both sides of (C.24) and contracting over the κ, ρ indices yields,

$$-4(D-1)\boldsymbol{a} + 2\left(l_{\kappa\rho} \otimes l^{\kappa\rho}\right) \boldsymbol{a} = \left(l_{\kappa\rho} \otimes I + I \otimes l_{\kappa\rho}\right) \boldsymbol{B}^{\kappa\rho}. \tag{C.25}$$

Using the explicit expression (C.2) for $l^{\kappa\rho}$, it is easily seen that for any rank two tensor $\boldsymbol{T} \equiv T_{\mu\nu}$ we have

$$\left(\left(l_{\kappa\rho} \otimes l^{\kappa\rho}\right) T\right)_{\mu\nu} = 2\left(\mathrm{tr}\left(\boldsymbol{T}\right) \eta_{\mu\nu} - T_{\nu\mu}\right), \tag{C.26}$$

where $\mathrm{tr}(\boldsymbol{T}) \equiv \eta^{\mu\nu} T_{\mu\nu}$. Note that the trace is a Lorentz scalar, so this term is automatically Lorentz invariant. Substituting (C.26) into (C.25) and symmetrizing over (μ, ν), we obtain

$$\begin{aligned} a_{\{(\mu\}\{\nu)\}} &= -\frac{1}{4D}\left((l_{\kappa\rho} \otimes I + I \otimes l_{\kappa\rho}) B^{\kappa\rho}\right)_{\{(\mu_1\}\{\mu_2)\}} \qquad \text{(C.27)} \\ &= -i \int d^D y\, \chi(y, \vec{0}) \left[\partial_\mu H_F(y, \vec{0}) \partial_\nu H_F(y, \vec{0}) - \frac{1}{D} \eta_{\mu\nu} \partial_\sigma H_F(y, \vec{0}) \partial^\sigma H_F(y, \vec{0})\right], \end{aligned}$$

where all derivatives are taken with respect to the spacetime point y. Similarly, we find,

$$\begin{aligned} a_{\{(\mu\nu)\}\{0\}} &= a_{\{0\}\{(\mu\nu)\}} \\ &= -\frac{1}{4D}\left((l_{\kappa\rho} \otimes I + I \otimes l_{\kappa\rho}) B^{\kappa\rho}\right)_{\{0\}\{(\mu_1\mu_2)\}} \qquad \text{(C.28)} \\ &= -i \int d^D y\, \chi(y, \vec{0}) \left[H_F(y, \vec{0}) \partial_\mu \partial_\nu H_F(y, \vec{0}) - \frac{1}{D} \eta_{\mu\nu} H_F(y, \vec{0}) \partial^2 H_F(y, \vec{0})\right]. \end{aligned}$$

Thus, we have explicitly solved for $\boldsymbol{a}$ for all ranks $r \leq 2$.

We turn now to the derivation of an inductive solution to (C.5) for $r > 2$. For infinitesimal $\Lambda = \Lambda_\theta$ and to leading order in θ, eq. (C.5) now yields

$$\mathbb{L}^{\kappa\rho}\boldsymbol{a} = \boldsymbol{B}^{\kappa\rho}, \tag{C.29}$$

where

$$\mathbb{L}^{\kappa\rho} \equiv (l^{\kappa\rho} \otimes I \otimes \cdots \otimes I) + (I \otimes l^{\kappa\rho} \otimes I \otimes \cdots \otimes I) + \cdots + (I \otimes \cdots \otimes I \otimes l^{\kappa\rho}), \tag{C.30}$$

and

$$\begin{aligned}&(B^{\kappa\rho})_{\{\mu_1\cdots\mu_{|\gamma_1|}\}\{\nu_1\cdots\nu_{|\gamma_2|}\}} \equiv \\ &2i(-1)^{(|\gamma_1|+|\gamma_2|)}\int d^D y\, y^{[\kappa}\partial^{\rho]}\chi(y,\vec{0})\partial_{\mu_1}\cdots\partial_{\mu_{|\gamma_1|}}H_F(y,\vec{0})\partial_{\nu_1}\cdots\partial_{\nu_{|\gamma_2|}}H_F(y,\vec{0}),\end{aligned} \tag{C.31}$$

with all derivatives taken with respect to y. As in the $r = 1, 2$ cases, we solve (C.29) by applying the operator $\mathbb{L}_{\kappa\rho}$ to both sides and contracting the κ, ρ-indices. We begin by noting that the operator we obtain on the left-hand side,

$$\mathbb{L}_{\kappa\rho}\mathbb{L}^{\kappa\rho}, \tag{C.32}$$

contains two types of terms: There are r terms of the form,

$$I \otimes \cdots \otimes I \otimes l_{\kappa\rho}l^{\kappa\rho} \otimes I \otimes \cdots \otimes I = -2(D-1)I \otimes \cdots \otimes I, \tag{C.33}$$

where we used (C.22). Similarly, using (C.26), the remaining $r(r-1)$ terms in (C.32) are of the form,

$$I \otimes \cdots I \otimes \underbrace{l_{\kappa\rho}}_{i\text{-th slot}} \otimes I \cdots I \otimes \underbrace{l^{\kappa\rho}}_{j\text{-th slot}} \otimes I \cdots \otimes I = 2(\eta_{\mu_i\mu_j}\mathrm{tr}_{ij} - \mathbb{T}_{ij}), \tag{C.34}$$

where tr_{ij} and $\mathbb{T}_{ij}$ denote, respectively, the trace over the i, j-th spacetime indices and the transposition of the i, j-th indices, i.e., for any tensor $\boldsymbol{T}$ we have

$$(\mathrm{tr}_{ij}T)_{\mu_1\cdots\mu_r} \equiv \eta^{\mu_i\mu_j}T_{\mu_1\cdots\mu_r}, \qquad (\mathbb{T}_{ij}T)_{\mu_1\cdots\mu_r} \equiv T_{\mu_1\cdots\widehat{\mu_i}\mu_j\cdots\widehat{\mu_j}\mu_i\cdots\mu_r}. \tag{C.35}$$

Altogether, therefore, we have

$$\mathbb{L}_{\kappa\rho}\mathbb{L}^{\kappa\rho} = -2r(D-1)\mathbb{I} + 4\sum_{i<j\leq r}(\eta_{\mu_i\mu_j}\mathrm{tr}_{ij} - \mathbb{T}_{ij}), \tag{C.36}$$

where $\mathbb{I} \equiv I^{\otimes r}$. Hence, multiplying both sides of (C.29) by $\mathbb{L}_{\kappa\rho}$ and contracting the κ, ρ-indices yields,

$$2r(D-1)\boldsymbol{a} + 4\sum_{i<j\leq n}\mathbb{T}_{ij}\boldsymbol{a} = -\mathbb{L}_{\kappa\rho}\boldsymbol{B}^{\kappa\rho} + 4\sum_{i<j\leq n}\eta_{\mu_i\mu_j}\mathrm{tr}_{ij}\boldsymbol{a}. \tag{C.37}$$

Now, the trace of (C.29) yields

$$\mathrm{tr}_{ij}\left(\mathbb{L}^{\kappa\rho}_{(r)}\boldsymbol{a}\right) = \mathbb{L}^{\kappa\rho}_{(r-2)}\left(\mathrm{tr}_{ij}\boldsymbol{a}\right) = \mathrm{tr}_{ij}\boldsymbol{B}^{\kappa\rho}, \tag{C.38}$$

where we have inserted a subscript (r) on $\mathbb{L}^{\kappa\rho}_{(r)}$ to indicate the rank of the operator (C.30) being considered. Thus, $\mathrm{tr}_{ij}\boldsymbol{a}$ satisfies an equation of the same form as (C.29) but for the lower rank $r' = r - 2$ and with $\boldsymbol{B}'^{\kappa\rho} = \mathrm{tr}_{ij}\boldsymbol{B}^{\kappa\rho}$. For example, this implies the trace of the

$r = 3$ tensor $a_{\{\mu_1\mu_2\}\{\nu_1\}}$ with respect to its two μ-spacetime indices is given by:

$$\eta^{\mu_1\mu_2} a_{\{\mu_1\mu_2\}\{\nu_1\}} = -\frac{1}{2(D-1)} \eta^{\mu_1\mu_2} (l_{\kappa\rho} B^{\kappa\rho})_{\{\mu_1\mu_2\}\{\nu_1\}}, \tag{C.39}$$

which is obtained by replacing $(B^{\kappa\rho})_{\{0\}\{\nu_1\}}$ with $\eta^{\mu_1\mu_2}(B^{\kappa\rho})_{\{\mu_1\mu_2\}\{\nu_1\}}$ in the $r = 1$ solution (C.23) for $a_{\{0\}\{\nu_1\}}$. Thus, since we are obtaining solutions inductively in r and have already obtained explicit solutions for $r = 1, 2$, we may treat $\mathrm{tr}_{ij}\boldsymbol{a}$ in (C.37) as "known."

Thus, it remains only to extract $\boldsymbol{a}$ from the combination of components of $\boldsymbol{a}$ appearing on the left side of (C.37). To do so, we note that the sum over all transpositions commutes with any permutation. A standard result in the representation theory of finite-dimensional groups implies the set of all elements that commute with the group algebra of the symmetric group S_r is spanned by a complete set of orthogonal (idempotent) elements $\mathbb{E}_i$,

$$\mathbb{E}_i \mathbb{E}_j = \delta_{ij} \mathbb{E}_i \qquad , \qquad \sum_{i=1}^{k} \mathbb{E}_i = \mathbb{I}, \tag{C.40}$$

where k denotes the number of partitions of r. Hence, we may expand the sum over transpositions appearing in (C.37),

$$\sum_{i<j\leq r} \mathbb{T}_{ij} = \sum_{i=1}^{k} c_i \mathbb{E}_i, \tag{C.41}$$

for some real-valued coefficients c_i. Applying the operator $\mathbb{E}_j$ to both sides of (C.37) and using the orthogonality property (C.40), we obtain then,

$$\big(2r(D-1) + 4c_j\big)\, \mathbb{E}_j \boldsymbol{a} = \mathbb{E}_j \left(-\mathbb{L}_{\kappa\rho} \boldsymbol{B}^{\kappa\rho} + 4 \sum_{i<j\leq n} \eta_{\mu_i\mu_j} \mathrm{tr}_{ij} \boldsymbol{a} \right). \tag{C.42}$$

We abbreviate the numerical coefficients,

$$\widetilde{c}_j \equiv 2r(D-1) + 4c_j. \tag{C.43}$$

For any j such that $\widetilde{c}_j = 0$, eq. (C.42) places no constraint on the corresponding $\mathbb{E}_j \boldsymbol{a}$ and, thus, this particular $\mathbb{E}_j \boldsymbol{a}$ must automatically be composed of an Lorentz-invariant combination of the metric and totally-antisymmetric D-dimensional tensor densities (i.e. "Levi-Civita symbols" $\epsilon_{\mu_1 \cdots \mu_n}$). For all j such that $\widetilde{c}_j \neq 0$, we may divide (C.42) through by $\widetilde{c}_j$ and use the completeness relation (C.40) to obtain the inductive solution,

$$\boldsymbol{a} = \sum_{\substack{j=1 \\ \widetilde{c}_j \neq 0}}^{k} \frac{1}{\widetilde{c}_j} \mathbb{E}_j \left(-\mathbb{L}_{\kappa\rho} \boldsymbol{B}^{\kappa\rho} + 4 \sum_{i<j\leq n} \eta_{\mu_i \mu_j} \mathrm{tr}_{ij} \boldsymbol{a} \right), \tag{C.44}$$

modulo arbitrary Lorentz-invariant tensors which may be identified with the value of the sum over the terms which are unconstrained by (C.42):

$$\sum_{\substack{j=1 \\ \widetilde{c}_j = 0}}^{k} \mathbb{E}_j \boldsymbol{a} = \text{Lorentz invariant tensor of rank } r. \tag{C.45}$$

All quantities appearing in our inductive solution (C.44) for $\boldsymbol{a}$ have been explicitly defined here except for the numerical coefficients c_j and the idempotent elements $\mathbb{E}_j$ which may be constructed via standard methods from the representation theory of the symmetric group (see [26, see "Appendix A: Representation of the symmetric groups"] and references therein). Note that the inductive solution, eq. (C.44), with $\boldsymbol{B}^{\kappa\rho}$ defined via

$$\boldsymbol{Q}(\Lambda_\theta) = -\frac{1}{2} \theta_{\kappa\rho} \boldsymbol{B}^{\kappa\rho} + \mathcal{O}(\theta^2), \tag{C.46}$$

holds for *any* tensors $\boldsymbol{Q}(\Lambda)$ satisfying (C.5) not just those defined[3] via (C.4).

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3. In particular, the solution (C.44) for $\boldsymbol{a}$ holds when $\boldsymbol{Q}$ corresponds to the Λ-dependent coefficients of the contact terms,

$$(Q_{A_1 \cdots A_n})^{\alpha_1 \cdots \alpha_n} (\Lambda^{-1}) \partial_{\alpha_1}^{(x_1)} \cdots \partial_{\alpha_n}^{(x_n)} \delta(x_1, \ldots, x_n), \tag{C.47}$$

that quantify the failure of the Epstein-Glaser renormalized (i.e. "extended") time-ordered products, $T\{\Phi_{A_1}(x_1) \cdots \Phi_{A_n}(x_n)\}$, to be Lorentz covariant [26]. Hence, there is a close analogy between the countert-erms required to restore Lorentz covariance in Epstein-Glaser renormalization and our "counterterms" for

Remark 25. In the case where either $|\gamma_1| = 0$ or $|\gamma_2| = 0$, the tensor $a_{\gamma_1\gamma_2}$ is totally symmetric in its spacetime indices and a closed-form solution to the induction equation (C.44) can be obtained (see the solution to the analogous problem in Epstein-Glaser renormalization given in [25, Section 3]).

Remark 26. When $r \leq 2$, the inductive solution (C.44) to eq. (C.5) reproduces the explicit solutions we obtained above. The $r = 0, 1$ cases are trivial. To verify the $r = 2$ case, note the symmetric group S_2 contains two elements: the identity I and the transposition $\mathbb{T}_{12}$. It is easily checked, in this case, that the idempotent decomposition (C.41) is satisfied by,

$$\mathbb{T}_{12} = \mathbb{S} - \mathbb{A}, \tag{C.48}$$

where $\mathbb{S}^2 = \mathbb{S}$ and $\mathbb{A}^2 = \mathbb{A}$ denote, respectively, the projector onto the symmetric part and the anti-symmetric part of any tensor of rank 2. Note these projectors are "orthogonal" in the sense that, for any tensor $\boldsymbol{T}$ of rank 2, $\mathbb{S}(\mathbb{A}\boldsymbol{T}) = 0 = \mathbb{A}(\mathbb{S}\boldsymbol{T})$. Moreover, they are "complete" in the sense that $\mathbb{S} + \mathbb{A} = \mathbb{I}$. Therefore, since they satisfy all the requisite properties, we may identify these projectors with the idempotents (C.40) for $r = 2$. Denoting $\mathbb{E}_1 = \mathbb{S}$ and $\mathbb{E}_2 = \mathbb{A}$, we simply read off the coefficients $c_1 = 1$ and $c_2 = -1$ by comparing (C.48) with (C.41). Hence, the formula (C.43) gives $\widetilde{c}_1 = 4D$ and $\widetilde{c}_2 = 4(D-2)$ in this case. Plugging these into the general formula (C.44) immediately yields: for $D \neq 2$,

$$\boldsymbol{a} = -\frac{1}{4D}\left(\mathbb{S} + \frac{D}{D-2}\mathbb{A}\right)\left(l_{\kappa\rho} \otimes I + I \otimes l_{\kappa\rho}\right)\boldsymbol{B}^{\kappa\rho} + 4\eta_{\mu_1\mu_2}\mathrm{tr}_{12}\boldsymbol{a}, \tag{C.49}$$

which is the most general rank 2 solution to (C.5). For $D = 2$, we have $\widetilde{c}_2 = 0$, so the general formula (C.44) yields (C.49) without the anti-symmetric term: note, in $D = 2$, any anti-symmetric tensor of type $(0, 2)$ is proportional to the Levi-Civita symbol $\epsilon_{\mu_1\mu_2}$ and, thus, is

the flow relations. The primary difference is that our counterterms are not proportional to (differentiated) δ-functions and, in the particular case of the flow relation for $(C_H)^{I}_{T_0\{\phi\phi\}} = H_F$, they are actually smooth functions of the spacetime variables, see eqs. (6.18) and (6.15).

automatically invariant under restricted Lorentz transformations. For our application in any dimension, only the symmetric part of $\boldsymbol{a}$ is of interest. Note also the trace of any rank two tensor is a Lorentz scalar. Hence, (C.49) is consistent with the results given in eqs. (C.27) and (C.28) above, i.e.,

$$\mathbb{S}\boldsymbol{a} = -\frac{1}{4D}\mathbb{S}\left(l_{\kappa\rho} \otimes I + I \otimes l_{\kappa\rho}\right)\boldsymbol{B}^{\kappa\rho} + \text{ Lorentz-invariant tensor.} \tag{C.50}$$

APPENDIX D

CURVATURE EXPANSION OF Ω_C

In this appendix, we derive the curvature expansion, eq. (7.10), for Ω_C. The derivation closely follows the approach of [45, Proof of Theorem 4.1] with modifications to account for the non-local metric dependence of Ω_C and its dependence on t_μ. Let $g_{\mu\nu}$ denote the components of the metric in RNC centered at $z \in M$. Let $S_\lambda : \mathbb{R}^D \to \mathbb{R}^D$ denote the map corresponding to re-scaling the Riemannian normal coordinates $x^\mu \mapsto \lambda x^\mu$. We note S_λ leaves the origin invariant and it is a diffeomorphism for $\lambda \in (0, 1]$. Consider now the smooth 1-parameter family of smooth metrics defined via,

$$h_{\mu\nu}(x;\lambda) \equiv \lambda^{-2}(S^*_\lambda g)_{\mu\nu}(x) = g_{\mu\nu}(\lambda x). \tag{D.1}$$

Note that $h_{\mu\nu}(\lambda)$ smoothly interpolates between the flat spacetime metric, $\eta_{\mu\nu}$, at $\lambda = 0$ and the original curved metric, $g_{\mu\nu}$, at $\lambda = 1$.

For any Feynman parametrix compatible with the joint smoothness axiom W2, the quantity,

$$\Omega_C[h_{\mu\nu}(\lambda), t_\mu, L](f_1, f_2; \vec{0}), \tag{D.2}$$

defined via (7.6) is smooth in λ. Hence, by Taylor's theorem with remainder, for any non-negative integer n, we have

$$\Omega_C[g_{\mu\nu}, t_\mu, L](f_1, f_2; \vec{0}) = \sum_{k=0}^{n} \frac{1}{k!} \left[\frac{d^k}{d\lambda^k} \Omega_C[h_{\mu\nu}(\lambda), t_\mu, L](f_1, f_2; \vec{0}) \right]_{\lambda=0} + R_n(f_1, f_2; \vec{0}), \tag{D.3}$$

where the Taylor remainder is given by

$$R_n(f_1, f_2; \vec{0}) \equiv \frac{1}{n!} \int_0^1 d\lambda (1-\lambda)^n \frac{d^{(n+1)}}{d\lambda^{(n+1)}} \Omega_C[h_{\mu\nu}(\lambda), t_\mu, L](f_1, f_2; \vec{0}). \tag{D.4}$$

We now show that, modulo smooth terms, the remainder (D.4) is of scaling degree $(n - D + 5)$ and, thus, the non-smooth behavior of Ω_C is entirely contained (up to scaling degree δ) in the finite k-sum of (D.3) for $n \geq \delta + D - 4$. We have

$$\big(S_s^* \Omega_C[h_{\mu\nu}(\lambda), t_\mu, L]\big) = \Omega_C[(S_s^* h)_{\mu\nu}(\lambda), (S_s^* t)_\mu, L] = \Omega_C[s^2 h_{\mu\nu}(s\lambda), st_\mu, L]. \tag{D.5}$$

where the first equality follows directly from the definition (7.6) of Ω_C and the second equality follows from the definition (D.1) of $h_{\mu\nu}$,

$$(S_s^* h)_{\mu\nu}(x;\lambda) = \lambda^{-2}(S_s^* \circ S_\lambda^* g)_{\mu\nu}(x) = s^2 (S_{s\lambda}^* g)_{\mu\nu}(x) = s^2 h_{\mu\nu}(x; s\lambda). \tag{D.6}$$

On the other hand, since $\chi(y, \vec{0}; sL) - \chi(y, \vec{0}; L)$ vanishes in a neighborhood of the origin, $y = \vec{0}$, it follows from the same wavefront set arguments used in Proposition 8 that for any $s \in (0, 1]$, we have

$$\Omega_C[h_{\mu\nu}(\lambda), t_\mu, L] = \Omega_C[h_{\mu\nu}(\lambda), t_\mu, sL] + \text{smooth terms}. \tag{D.7}$$

Plugging (D.7) into (D.5) yields,

$$\big(S_s^* \Omega_C[h_{\mu\nu}(\lambda), t_\mu, L]\big) = \Omega_C[s^2 h_{\mu\nu}(s\lambda), st_\mu, sL] + \text{smooth terms}. \tag{D.8}$$

Plugging this back into the remainder (D.4), we find modulo smooth terms,

$$\begin{aligned}(S_s^* R_n)(f_1, f_2; \vec{0}) &= \frac{1}{n!}\int_0^1 d\lambda (1-\lambda)^n \frac{d^{(n+1)}}{d\lambda^{(n+1)}} \Omega_C[s^2 h_{\mu\nu}(s\lambda), st_\mu, sL](f_1, f_2; \vec{0}) \\ &= s^{(n+1)} \frac{1}{n!}\int_0^1 d\lambda (1-\lambda)^n \left[\frac{\partial^{(n+1)}}{\partial q^{(n+1)}} \Omega_C[s^2 h_{\mu\nu}(q), st_\mu, sL](f_1, f_2; \vec{0})\right]_{q=s\lambda}\end{aligned} \tag{D.9}$$

However, from the almost homogeneous scaling behavior of the Feynman parametrix and its smoothness in m^2 together with the invariance of the cutoff function (7.4) under the simultaneous rescaling $(g_{\mu\nu}, t_\mu, L) \to (s^2 g_{\mu\nu}, st_\mu, sL)$, it follows that for any $q \in [0,1]$, we have

$$\Omega_C[s^2 h_{\mu\nu}(q), st_\mu, sL](f_1, f_2; \vec{0}) = \mathcal{O}\left(s^{(-D+4)}\right). \tag{D.10}$$

Consequently, we find modulo smooth terms

$$(S_s^* R_n)(f_1, f_2; \vec{0}) = \mathcal{O}(s^{(n+5-D)}) \tag{D.11}$$

which implies that the scaling degree of any non-smooth contributions to $R_n(f_1, f_2; \vec{0})$ must be at least $n + 5 - D$.

Thus, we have shown that

$$\Omega_C[g_{\mu\nu}, t_\mu, L](f_1, f_2; \vec{0}) \sim_\delta \sum_{k=0}^{\delta-D+4} \frac{1}{k!} \left[\frac{d^k}{d\lambda^k} \Omega_C[\lambda^{-2}(S_\lambda^* g)_{\mu\nu}, t_\mu, L](f_1, f_2; \vec{0}) \right]_{\lambda=0} + \text{smooth.} \tag{D.12}$$

We now rewrite (D.12) in the form of the claimed curvature expansion (7.10) for the special case that the metric has polynomial dependence on the coordinates, $g_{\mu\nu} = g_{\mu\nu}^{(P)}$. Since we have

$$\begin{aligned} &\lambda^{-2}(S_\lambda^* g)_{\mu\nu}^{(P)}(x) \\ &= g_{\mu\nu}^{(P)}[x^\sigma, \eta_{\mu\nu}, \lambda^2 R_{\mu\nu\kappa\rho}(\vec{0}), \lambda^3 \nabla_\sigma R_{\mu\nu\kappa\rho}(\vec{0}), \ldots, \lambda^P \nabla_{(\sigma_1} \cdots \nabla_{\sigma_{(P-2)})} R_{\mu\nu\kappa\rho}(\vec{0})], \end{aligned} \tag{D.13}$$

it follows that

$$\Omega_C[\lambda^{-2}(S^*_\lambda g)_{\mu\nu}, t_\mu, L] = \Omega_C[\eta_{\mu\nu}, \lambda^2 R_{\mu\nu\kappa\rho}(\vec{0}), \ldots, \lambda^P \nabla_{(\sigma_1} \cdots \nabla_{\sigma_{(P-2)})} R_{\mu\nu\kappa\rho}(\vec{0}), t_\mu, L]. \tag{D.14}$$

For any smooth function of the form $f = f(\lambda^2\alpha_0, \lambda^3\alpha_1, \ldots, \lambda^P\alpha_{P-2})$, a straightforward application of the multi-variate chain rule yields,

$$\left.\frac{d^k f}{d\lambda^k}\right|_{\lambda=0} = \sum_{2p_0+3p_1+\cdots+kp_{(k-2)}=k} k!\, \alpha_0^{p_0} \cdots \alpha_{(k-2)}^{p_{(k-2)}} \left.\frac{\partial^{(p_0+\cdots+p_{(k-2)})} f(\alpha_0, \ldots, \alpha_{(P-2)})}{\partial^{p_0}\alpha_0 \cdots \partial^{p_{k-2}}\alpha_{(k-2)}}\right|_{\alpha_0,\ldots,\alpha_{(P-2)}=0} \tag{D.15}$$

Using this formula to evaluate the terms in the k-sum of (D.12) then yields the claimed curvature expansion (7.10). The result can then be extended to general smooth $g_{\mu\nu}$ via compatibility with axiom W2, using the same argument as in the proof of Proposition 9.

APPENDIX E

CONSTRUCTION OF COVARIANCE-RESTORING COUNTERTERMS BASED ON GENERAL ASSOCIATIVITY CONDITIONS

The purpose of this appendix is to develop an algorithm for constructing covariance-restoring counterterms without relying on explicit formulas for the OPE coefficients or any other special model-dependent properties. This algorithm is based on the general associativity properties of OPE coefficients and, thus, should be applicable to flow relations for any renormalizable Lorentzian quantum field theory. At the end of the appendix, we show this algorithm reproduces the counterterms derived in Chapter 6 for the Klein-Gordon OPE coefficients $C^{I}_{T_0\{\phi\cdots\phi\}}$ and we will use the algorithm to generate counterterms for the flow relations of $\lambda\phi^4$-theory. For simplicity, we give a derivation for Lorentz-covariance restoring counterterms in flat spacetime; however, the derivation can be generalized to curved spacetimes using the approach developed in Chapter 7.

Consider a theory arising from a Lagrangian with a self-interaction term $\zeta\Phi_V$, where ζ denotes the coupling parameter. (Note that for power-counting renormalizable theories, the dimension of Φ_V must be less than or equal to the spacetime dimension.) For example, for $\lambda\phi^4$-theory we have $\zeta = \lambda$ and $\Phi_V = \phi^4/4!$. Consider the OPE coefficients arising from products $\Phi_{A_1}(x_1)\ldots\Phi_{A_n}(x_n)$, where the fields Φ_{A_i} are of arbitrary tensorial (or spinorial) type. We assume that the Lorentzian OPE coefficients $C^{B}_{T_0\{A_1,\ldots,A_n\}}$ have been found to satisfy a flow relation of the form

$$\frac{\partial}{\partial\zeta}C^{B}_{T_0\{A_1,\ldots,A_n\}}(x_1,\ldots,x_n;z) \approx -i\int d^D y\,\chi(y,z;L)\,\Omega^{B}_{T_0\{VA_1\cdots A_n\}}(y,x_1,\ldots,x_n;z)+ \\ + \text{covariance-restoring counterterms,} \tag{E.1}$$

where $\chi(y,z;L)$ is a suitable translationally-invariant cutoff function (see (6.19)) and the quantity $\Omega^{B}_{T_0\{VA_1\cdots A_n\}}(y,x_1,\dots,x_n;z)$ is given in terms of OPE coefficients by a formula of the general form

$$\begin{aligned}
&\Omega^{B}_{T_0\{VA_1\cdots A_n\}}(y,x_1,\dots,x_n;z) \\
&= C^{B}_{T_0\{VA_1\cdots A_n\}}(y,x_1,\dots,x_n;z)+ \\
&-\sum_{i=1}^{n}\sum_{[C]\leq[A_i]+[V]-D} C^{C}_{T_0\{VA_i\}}(y,x_i;x_i)C^{B}_{T_0\{A_1\cdots\widehat{A_i}C\cdots A_n\}}(x_1,\dots,x_n;z)+ \\
&-\sum_{[C]<[B]-[V]+D} C^{C}_{T_0\{A_1\cdots A_n\}}(x_1,\dots,x_n;z)C^{B}_{T_0\{VC\}}(y,z;z),
\end{aligned} \tag{E.2}$$

where D denotes the spacetime dimension. For Klein-Gordon theory ($\zeta = m^2$ and $\Phi_V = \phi^2/2$), eq. (E.2) corresponds to the flow relation (2.2) for the Wick OPE coefficient $C^{I}_{T_0\{\phi\cdots\phi\}}$ (where only the second line of eq. (E.2) contributes in this case). For 4-dimensional $\lambda\phi^4$-theory ($\zeta = \lambda$ and $\Phi_V = \phi^4/4!$), eq. (E.2) corresponds to the Wick-rotated integrand of the Euclidean Holland and Hollands flow equation (1.20). For 4-dimensional Yang-Mills gauge theories, eq. (E.2) coincides with the Wick-rotated integrand of the Euclidean flow relations given in [20, Theorem 4]. Thus, eq. (E.2) encompasses all of these cases. Our aim is to explicitly obtain the covariance restoring counterterms in eq. (E.1).

Note that the individual terms in the sum for $\Omega^{B}_{T_0\{VA_1\cdots A_n\}}$ are well-defined as distributions in spacetime variables $y, x_1, \dots, x_n$ only away from all diagonals, i.e., where none of the spacetime events coincide. However, assuming the OPE coefficients satisfy the associativity and scaling axioms postulated in [15], then the scaling degree of $\Omega^{B}_{T_0\{VA_1\cdots A_n\}}$ on any partial diagonal involving y and one other spacetime event x_i is guaranteed to be strictly less than the spacetime dimension D. It follows then that $\Omega^{B}_{T_0\{VA_1\cdots A_n\}}$ can be *uniquely* extended to a distribution on these partial diagonals involving y, so the integral in (E.1) is well defined (even though individual terms in $\Omega^{B}_{T_0\{VA_1\cdots A_n\}}$ generally contain non-integrable divergences at $y = x_i$ for $i = 1, \ldots, n$).

The failure of the integral in (E.1) by itself to be covariant under Lorentz transformation Λ is characterized by the nonvanishing of the quantity

$$-i \int d^D y \, [\chi(\Lambda y, \Lambda z; L) - \chi(y, z; L)] \, \Omega^E_{T_0\{VD_1 \cdots D_n\}}(y, x_1, \ldots, x_n; z) \tag{E.3}$$

Since $\chi(y, z; L) = 1$ in an open neighborhood of z, if the spacetime events x_i are sufficiently near to z, then we have

$$\chi(\Lambda x_i, \Lambda z; L) = \chi(x_i, z; L), \qquad \text{for all } i = 1, \ldots, n. \tag{E.4}$$

It then follows that the integrand in (E.3) vanishes as y approaches the partial diagonals $y = x_i \neq x_j$. Consequently, unlike the integral in (E.1), the expression (E.3) is well defined for each of the individual terms in the sum defining $\Omega^B_{T_0\{VA_1 \cdots A_n\}}$. Note the y-dependence of $\Omega^B_{T_0\{VA_1 \cdots A_n\}}$ is isolated within terms of the form

$$C^E_{T_0\{VD_1 \cdots D_n\}}. \tag{E.5}$$

Specifically, the y-dependence of second line of (E.2) appears in $C^B_{T_0\{VA_1 \cdots A_n\}}$. The y-dependence of the third line of (E.2) appears in $C^C_{T_0\{VA_i\}}$. Finally, the y-dependence of the fourth line of (E.2) appears in $C^B_{T_0\{VC\}}$. It follows that the non-covariance of (E.3) is quantified by integrals of the form:

$$\begin{aligned} &\Upsilon^E_{T_0\{D_1 \cdots D_n\}}(x_1, \ldots, x_n; z; \Lambda) \equiv \\ &- i \int d^D y \, [\chi(\Lambda y, \Lambda z; L) - \chi(y, z; L)] \, C^E_{T_0\{VD_1 \cdots D_n\}}(y, x_1, \ldots, x_n; z). \end{aligned} \tag{E.6}$$

Our task is now to show that the non-covariance of these terms can be compensated by counterterms and thereby to construct the "covariance-restoring counterterms" for the flow relation (E.1).

The integrand of (E.6) is nonvanishing only when y lies outside an open neighborhood of z. The associativity condition (see eq. (4.36)) implies that, for any merger tree $\mathcal{T}$ such that $x_1, \ldots, x_n$ approach an auxiliary point z' faster than z' and y approach z, we have

$$C^{E}_{T_0\{VD_1\cdots D_n\}}(y, x_1, \ldots, x_n; z) \sim_{\mathcal{T},\delta} \sum_{C} C^{C}_{T_0\{D_1\cdots D_n\}}(x_1, \ldots, x_n; z') C^{E}_{T_0\{VC\}}(y, z'; z), \tag{E.7}$$

where both sides are viewed as distributions in $(y, x_1, \ldots, x_n, z')$ but with the left-hand side having trivial dependence on the auxiliary point z'. Plugging (E.7) into (E.6) yields,

$$\begin{aligned} &\Upsilon^{E}_{T_0\{D_1\cdots D_n\}}(x_1, \ldots, x_n; z; \Lambda) \sim_{\mathcal{T}',\delta} \\ &- i \sum_{C} C^{C}_{T_0\{D_1\cdots D_n\}}(x_1, \ldots, x_n; z') \int_y \left[\chi(\Lambda y, \Lambda z) - \chi(y, z)\right] C^{E}_{T_0\{VC\}}(y, z'; z), \end{aligned} \tag{E.8}$$

where $\mathcal{T}'$ denotes any merger tree with $(x_1, \ldots, x_n)$ approaching z' faster than z' approaches z. Here $\int_y \equiv \int d^D y$ and we suppress the L-dependence of χ for notational convenience. Assuming the OPE coefficient C^{E}_{VC} satisfies the general microlocal spectrum condition stated in [15], then all elements $(y, k_1, z', k_2, z, k_3) \in (T^*M)^3$ in the wavefront set of $C^{E}_{T_0\{VC\}}(y, z'; z)$ will be such that $k_1 = -k_2$ and $k_3 = \vec{0}$. It follows then from a straightforward application of [3, Theorem 8.2.12] that the dependence of (E.8) on (z', z) is, in fact, smooth and, thus, we may set $z' = z$:

$$\Upsilon^{E}_{T_0\{D_1\cdots D_n\}}(x_1, \ldots, x_n; z; \Lambda) \approx \sum_{C} Q^{E}_{T_0\{VC\}}(\Lambda^{-1}) C^{C}_{T_0\{D_1\cdots D_n\}}(x_1, \ldots, x_n; z), \tag{E.9}$$

where $Q^{E}_{T_0\{VC\}}(\Lambda^{-1})$ is given by

$$Q^{E}_{T_0\{VC\}}(\Lambda^{-1}) \equiv -i \int d^D y \left[\chi(\Lambda y, \vec{0}; L) - \chi(y, \vec{0}; L)\right] C^{E}_{T_0\{VC\}}(y, \vec{0}; \vec{0}), \tag{E.10}$$

where translation invariance was used to set $z = \vec{0}$. Thus, $Q^E_{T_0\{VC\}}(\Lambda^{-1})$ is independent of spacetime point z. Note that no assumption has been made on how quickly events $x_1, \ldots, x_n$ approach z relative to each other, so (E.9) is valid for all merger trees involving the events $x_1, \ldots, x_n$. Hence, we simply use the notation "$\approx$" that was introduced in the paragraph surrounding eq. (4.2).

We now show that (E.10) satisfies a cohomological identity that enables us to obtain the desired counterterms. Let Λ_1 and Λ_2 be Lorentz transformations. Then we have

$$
\begin{aligned}
& Q^E_{T_0\{VC\}}(\Lambda_1\Lambda_2) - Q^E_{T_0\{VC\}}(\Lambda_1) \\
&= -i \int_y \left[\chi(\Lambda_2^{-1}\Lambda_1^{-1}y, \vec{0}; L) - \chi(\Lambda_1^{-1}y, \vec{0}; L)\right] C^E_{T_0\{VC\}}(y, \vec{0}; \vec{0}) \\
&= -i \int_{y'} \left[\chi(\Lambda_2^{-1}y', \vec{0}; L) - \chi(y', \vec{0}; L)\right] C^E_{T_0\{VC\}}(\Lambda_1 y', \vec{0}; \vec{0}) \\
&= -i \int_{y'} \left[\chi(\Lambda_2^{-1}y', \vec{0}; L) - \chi(y', \vec{0}; L)\right] \sum_{A,B} D^E_A(\Lambda_1) D^B_C(\Lambda_1^{-1}) C^A_{T_0\{VB\}}(y', \vec{0}; \vec{0}) \\
&= \sum_{A,B} D^E_A(\Lambda_1) D^B_C(\Lambda_1^{-1}) Q^A_{T_0\{VB\}}(\Lambda_2),
\end{aligned}
\tag{E.11}
$$

where the second equality follows from a change of integration variables $y \to y' = \Lambda_1^{-1}y$ and third equality follows from the Lorentz covariance of the OPE coefficients (where we recall that Φ_V is a Lorentz scalar). Here we have abbreviated $\int_y \equiv \int d^D y$. Denoting $\boldsymbol{Q} \equiv Q^E_{T_0\{VC\}}$ and suppressing field indices, eq. (E.11) is equivalent to:

$$
0 = (d^1\boldsymbol{Q})(\Lambda_1, \Lambda_2) = \boldsymbol{Q}(\Lambda_1) + D(\Lambda_1)\boldsymbol{Q}(\Lambda_2) - \boldsymbol{Q}(\Lambda_1\Lambda_2),
\tag{E.12}
$$

which is the cohomological identity (C.14). As established in Proposition 10, this identity implies there exists $\boldsymbol{a} \equiv a^B_{T_0\{VC\}}$ such that:

$$
\boldsymbol{Q}(\Lambda) = (d^0\boldsymbol{a})(\Lambda) = (D(\Lambda) - I)\boldsymbol{a}.
\tag{E.13}
$$

For tensor-valued[1] $\boldsymbol{Q}$, the results of Appendix C imply the $\boldsymbol{a}$ can be inductively constructed (modulo Lorentz-invariant tensors) from

$$\boldsymbol{a} = \sum_{\substack{j=1 \\ \widetilde{c}_j \neq 0}}^{k} \frac{1}{\widetilde{c}_j} \mathbb{E}_j \left(-\mathbb{L}_{\kappa\rho} \boldsymbol{B}^{\kappa\rho} + 4 \sum_{i<j\leq n} \eta_{\mu_i \mu_j} \mathrm{tr}_{ij} \boldsymbol{a} \right), \tag{E.14}$$

with

$$\boldsymbol{B}^{\kappa\rho} \equiv (B^{\kappa\rho})^{E}_{T_0\{VC\}} = 2i \int d^D y \, y^{[\kappa} \partial^{\rho]} \chi(y, \vec{0}) \, C^{E}_{T_0\{VC\}}(y, \vec{0}; \vec{0}). \tag{E.15}$$

By reasoning analogous to the arguments of Chapter 6, we obtain counterterms that ensure the Lorentz-covariance of the flow relation (E.1) by making the following substitution in every appearance of $C^{E}_{T_0\{VD_1\cdots D_n\}}$ in $\Omega^{B}_{T_0\{VA_1\cdots A_n\}}$:

$$C^{E}_{T_0\{VD_1\cdots D_n\}} \to C^{E}_{T_0\{VD_1\cdots D_n\}}(y, x_1, \ldots, x_n; z) - \frac{1}{\mathcal{V}} \sum_{C} a^{E}_{T_0\{VC\}} C^{C}_{T_0\{D_1\cdots D_n\}}(x_1, \ldots, x_n; z), \tag{E.16}$$

where we have written

$$\mathcal{V} \equiv \int d^D y \, \chi(y, \vec{0}; \vec{0}). \tag{E.17}$$

It is understood the C-sum in (E.16) is carried to sufficiently-large field dimension $[C]$ to achieve whatever asymptotic precision is desired from the flow relation. The substitution rule (E.16) is the key result of this appendix. We now illustrate it by applying it to the cases of the massive Klein-Gordon field and 4-dimensional $\lambda\phi^4$-theory.

For the case of the flow relations for $C^{I}_{T_0\{\phi\cdots\phi\}}$ obtained in this thesis for the massive Klein-Gordon field, we have $\Phi_V = \phi^2/2$, $\zeta = m^2$, and (E.2) reduces to:

$$\Omega^{I}_{T_0\{(\phi^2/2)\phi\cdots\phi\}}(y, x_1, \ldots, x_n; z) = \frac{1}{2} C^{I}_{T_0\{\phi^2\phi\cdots\phi\}}(y, x_1, \ldots, x_n; z). \tag{E.18}$$

1. A formula analogous to (E.14) can be obtained using the methods of Appendix C when $\boldsymbol{Q}$ is spinor-valued (see also [26, Section 4]).

Our algorithm instructs us to make the substitution (E.16) in $\Omega^I_{T_0\{(\phi^2/2)\phi\cdots\phi\}}$. Plugging the result of this substitution into (E.1) yields the flow relation:

$$\frac{\partial}{\partial m^2} C^I_{T_0\{\phi\cdots\phi\}}(x_1,\ldots,x_n;z) \approx -\frac{i}{2}\int d^D y\,\chi(y,z;L)\,C^I_{T_0\{\phi^2\phi\cdots\phi\}}(y,x_1,\ldots,x_n;z) + \\ -\sum_C a^I_{T_0\{(\phi^2/2)C\}} C^C_{T_0\{A_1\cdots A_n\}}(x_1,\ldots,x_n;z) \tag{E.19}$$

where $a^I_{T_0\{(\phi^2/2)C\}}$ is given recursively by (E.14) with

$$(B^{\kappa\rho})^I_{T_0\{(\phi^2/2)C\}} = i\int d^D y\, y^{[\kappa}\partial^{\rho]}\chi(y,\vec{0})\,C^I_{T_0\{\phi^2 C\}}(y,\vec{0};\vec{0}). \tag{E.20}$$

Comparing (E.19) with (6.31) of Theorem 7 and (E.20) with (6.30), we find that the substitution (E.16) reproduces the covariance-restoring counterterms obtained in Chapter 6 for the flow relations of the Klein-Gordon OPE coefficients $C^I_{T_0\{\phi\cdots\phi\}}$.

For $\lambda\phi^4$-theory, we have $\Phi_V = \phi^4/4!$ and $\zeta = \lambda$. Our algorithm instructs us to make the following substitutions in the formula (E.2) for $\Omega^B_{T_0\{(\phi^4/4!)A_1\cdots A_n\}}$:

$$C^B_{T_0\{\phi^4 A_1\cdots A_n\}} \to C^B_{T_0\{\phi^4 A_1\cdots A_n\}}(y,x_1,\ldots,x_n;z) + \\ -\frac{1}{\mathcal{V}}\sum_C a^B_{T_0\{\phi^4 C\}} C^C_{T_0\{A_1\cdots A_n\}}(x_1,\ldots,x_n;z) \tag{E.21}$$

$$C^C_{T_0\{\phi^4 A_i\}} \to C^C_{T_0\{\phi^4 A_i\}}(y,x_i;z) - \frac{1}{\mathcal{V}}\sum_D a^C_{T_0\{\phi^4 D\}} C^D_{T_0\{A_i\}}(x_i;z) \tag{E.22}$$

$$C^B_{T_0\{\phi^4 C\}} \to C^B_{T_0\{\phi^4 C\}}(y,z;z) - \frac{1}{\mathcal{V}} a^B_{T_0\{\phi^4 C\}}, \tag{E.23}$$

where $a^B_{T_0\{\phi^4 C\}}$ is given inductively by (E.14) in terms of

$$(B^{\kappa\rho})^B_{T_0\{\phi^4 C\}} = 2i\int d^D y\, y^{[\kappa}\partial^{\rho]}\chi(y,\vec{0})\,C^B_{T_0\{\phi^4 C\}}(y,\vec{0};\vec{0}). \tag{E.24}$$

Note that the OPE coefficient $C^B_{T_0\{A\}} = C^B_A$ appearing on the right side of (E.22) is zero unless $[A]_\phi = [B]_\phi \equiv m$ and, in this case, it is given by

$$C_A^B(x;z) = \frac{1}{\beta_1! \cdots \beta_m!} \partial_{\alpha_1}^{(x)} (x-z)^{(\beta_1} \cdots \partial_{\alpha_m}^{(x)} (x-z)^{\beta_m)}, \tag{E.25}$$

where $A = \alpha_1 \cdots \alpha_m$ and $B = \beta_1 \cdots \beta_m$. Making the substitutions (E.21)-(E.23) in $\Omega^B_{T_0\{(\phi^4/4!)A_1 \cdots A_n\}}$ and plugging this back into the flow relation (E.1), we obtain

$$\begin{aligned}
&\frac{\partial}{\partial\lambda} C^B_{T_0\{A_1 \cdots A_n\}}(x_1, \ldots, x_n; z) \approx \\
&- \frac{1}{4!} \int d^4y \, \chi(y,z;L) \Big[C^B_{T_0\{\phi^4 A_1 \cdots A_n\}}(y, x_1, \ldots, x_n; z) + \\
&- \sum_{i=1}^{n} \sum_{[C] \leq [A_i]} \Big[C^C_{T_0\{\phi^4 A_i\}}(y, x_i; x_i) + \\
&\qquad\qquad - \frac{1}{\mathcal{V}} \sum_{[D]} a^C_{T_0\{\phi^4 D\}} C^D_{T_0\{A_i\}}(x_i; z) \Big] \, C^B_{T_0\{A_1 \cdots \widehat{A_i} C \cdots A_n\}}(x_1, \ldots, x_n; z) + \\
&- \Big[\sum_{[C]<[B]} C^B_{T_0\{\phi^4 C\}}(y, z; z) - \frac{1}{\mathcal{V}} \sum_{[C] \geq [B]} a^B_{T_0\{\phi^4 C\}} \Big] \, C^C_{T_0\{A_1 \cdots A_n\}}(x_1, \ldots, x_n; z) \Big].
\end{aligned} \tag{E.26}$$

The generalization of this relation to curved spacetime was already given in eq. (2.8).

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