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ABSTRACT

In this thesis we will present several of the author's results on homological stability phenomena for discriminant complements. The contents of this thesis are taken from the author's papers (1) and (2) .

CHAPTER 1

INTRODUCTION

Let X be a smooth projective variety over $\mathbb{C}$. Let $\mathcal{L}$ be an ample line bundle on X. Let $H^0(X, \mathcal{L})$ be the space of global sections of $\mathcal{L}$.

Definition 1. *The discriminant variety*

$$\Sigma(X, \mathcal{L}) := \{f \in H^0(X, \mathcal{L}) | \exists p \in X, f(p) = 0, df(p) = 0\}.$$

We define the discriminant complement $U(X, \mathcal{L}) := H^0(X, \mathcal{L}) \setminus \Sigma(X, \mathcal{L})$. The variety $U(X, \mathcal{L})$ consists of sections of $\mathcal{L}$ with smooth zero locus.

Let us discuss a few special examples of discriminant varieties and discriminant complements.

Example 1. *Let $X = \mathbb{P}^n$, $\mathcal{L} = \mathcal{O}(d)$. Then $U(\mathbb{P}^n, \mathcal{O}(d))$ consists of homogenous polynomials of degree d in $n+1$ variables defining a smooth hypersurface in $\mathbb{P}^n$. The discriminant variety is a hypersurface in $H^0(\mathbb{P}^n, \mathcal{O}(d))$ defined by the classical discriminant polynomial.*

Example 2. *Let X be a smooth algebraic curve. Let $\mathcal{L}$ be an ample line bundle on X. Then $U(X, \mathcal{L})$ consists of sections of $\mathcal{L}$ with reduced zero locus. If $\mathcal{L}$ is very ample and $X \neq \mathbb{P}^1$ then $\Sigma(X, \mathcal{L})$ is a hypersurface in $H^0(X, \mathcal{L})$.*

Remark 1. *There is a $\mathbb{C}^*$ action on $H^0(X, \mathcal{L})$ by scaling, which preserves $\Sigma(X, \mathcal{L})$ and $U(X, \mathcal{L})$. We could therefore study the quotients of $\Sigma(X, \mathcal{L})$ and $U(X, \mathcal{L})$ by this action and indeed many authors deal exclusively with these quotients. However for our purposes studying the quotients will be more or less equivalent to studying the original varieties and we believed that it was a little cleaner to not pass to the quotient.*

In (12) Tommasi proves that $U(\mathbb{P}^n, \mathcal{O}(d))$ satisfies rational homological stability as $d \to \infty$. More precisely she shows that

$$H^k(U(\mathbb{P}^n, \mathcal{O}(d)), \mathbb{Q}) \cong H^k(\mathrm{GL}_{n+1}(\mathbb{C}), \mathbb{Q}) \tag{1.1}$$

for $k < \frac{d}{2}$. This result is interesting both because it establishes homological stability for a natural sequence of spaces but also because it computes the limiting stable cohomology as coming from the underlying (analytic) topology of $\mathbb{P}^n$ and not its structure as a variety (this is made more clear in the proof of the statement).

The results of this thesis are in two parts corresponding to the papers (2) and (1) by the author. In the first part we establish a twisted version of 1.1 . Namely we prove that

$$H^k(X_{d,n}; \mathbb{V}) = \begin{cases} H^k(X_{d,n}; \mathbb{Q}) & \text{if } n \text{ is odd} \\ 0 & \text{if } n \text{ is even} \end{cases} \tag{1.2}$$

Here $\mathbb{V}$ is the local system on $U(\mathbb{P}^n, \mathcal{O}(d))$ whose fibre over a point $f \in U(\mathbb{P}^n, \mathcal{O}(d))$ is $H^*(Z(f); \mathbb{Q})$ where $Z(f)$ is the zero locus of f. The proof combines the techniques of (5) and (12). We first study the space

$$E(\mathbb{P}^n, \mathcal{O}(d)) := \{(f, p) \in U(\mathbb{P}^n, \mathcal{O}(d)) \times \mathbb{P}^n | f(p) = 0\},$$

which can be thought of as a pointed version of $U(\mathbb{P}^n, \mathcal{O}(d))$. We establish homological stability results for $E(\mathbb{P}^n, \mathcal{O}(d))$ and use them to obtain 1.2.

In the second part of this thesis, we focus on the case when X is a curve. Let

$$\mathbb{U}(X, \mathcal{L}) := \{f \in C^\infty(X, \mathcal{L}) | f \text{has regular zeroes of index } 1\}$$

Let $i : U(X, \mathcal{L}) \hookrightarrow \mathbb{U}(\mathcal{L})$ be the inclusion. We prove that

$$i^* : H^k(\mathbb{U}(X, \mathcal{L}), \mathbb{Z}) \to H^k(U(X, \mathcal{L}), \mathbb{Z})$$

is an isomorphism for $2k \leq n - 2g$ (this is Theorem 6).

This can be seen as the cohomology of $U(X, \mathcal{L})$ is determined in a stable range by the C^∞ topology of X and is independent of the algebraic structure of X and $\mathcal{L}$. We also note that our result works integrally and is the first integral result in this area.

CHAPTER 2

STABLE COHOMOLOGY OF THE UNIVERSAL DEGREE d HYPERSURFACE IN $\mathbb{P}^n$

2.1 Introduction

Let $U_{d,n}$ be the *parameter space* of smooth degree d hypersurfaces in $\mathbb{P}^n$. There is a natural inclusion $U_{d,n} \subseteq \mathbb{P}^{\binom{n+d}{d}} = \mathbb{P}(V_{d,n})$, where $V_{d,n}$ is the vector space of homogenous degree d complex polynomials in $n+1$ variables. Let

$$U_{d,n}^* := \{(f,p) \in U_{d,n} \times \mathbb{P}^n | f(p) = 0\}.$$

Let $\phi : U_{d,n}^* \to U_{d,n}$ be defined by $\phi(f,p) = f$. The map $\phi : U_{d,n}^* \to U_{d,n}$ is the *universal family* of smooth degree d hypersurfaces in $\mathbb{P}^n$; it satisfies the following property: given a family $\pi : E \to B$ of smooth degree d hypersurfaces in $\mathbb{P}^n$ there is a unique diagram:

$$\begin{tikzcd} E \arrow[r, "\exists!"] \arrow[d] & U_{d,n}^* \arrow[d] \\ B \arrow[r, "\exists!"] & U_{d,n} \end{tikzcd}$$

In other words, any family of smooth degree d hypersurfaces is pulled back from this one. Our main result is as follows:

Theorem 1. *Let $d, n \geq 1$. Then there is an embedding of graded algebras:*

$$\phi : \mathrm{H}^*(\mathrm{PGL}_{n+1}(\mathbb{C}); \mathbb{Q}) \otimes \mathbb{Q}[x]/(x^n) \hookrightarrow H^*(U_{d,n}^*; \mathbb{Q})$$

where $|x| = 2$. Here $|.|$ denotes the cohomological degree. Let $c_1(E)$ denote the first chern class of the line bundle E.

1. *The element* $\phi(x) = c_1(\mathcal{L})$ *where* $\mathcal{L}$ *is the fiberwise canonical bundle (defined in Section 2).*

2. *Suppose* $d \geq 4n+1$. *Then* ϕ *is surjective in degree less than* $\frac{d-1}{2}$.

Let $X_{d,n} \subseteq V_{d,n}$ be the open subspace of polynomials defining a nonsingular hypersurface. The complement of $X_{d,n}$ in $V_{d,n}$ is known as the *discriminant hypersurface*; it is the zero locus of the classical discriminant polynomial. It is known to be highly singular.

A point of $X_{d,n}$ determines a projective hypersurface up to a scalar. There is a natural action of $\mathbb{C}^*$ on $X_{d,n}$ such that the quotient $X_{d,n}/\mathbb{C}^*$ is $U_{d,n}$. Let

$$X^*_{d,n} := \{(f,p) | f \in X_{d,n}, p \in \mathbb{P}^n, f(p) = 0\}.$$

There is a forgetful map $\pi : X^*_{d,n} \to X_{d,n}$ defined by $\pi(f,p) = f$. The fibres of π are

$$Z(f) := \pi^{-1}(f) = \{p \in \mathbb{P}^n | f(p) = 0\} \subseteq \mathbb{P}^n.$$

It is well known that the map π is a fibre bundle.

$X^*_{d,n}$ also has several interesting quotients. The action of GL_{n+1} on $X_{d,n}$ lifts to one on $X^*_{d,n}$. We obtain $U^*_{d,n} = X^*_{d,n}/\mathbb{C}^*$. The map $\pi : X^*_{d,n} \to X_{d,n}$ is $\mathbb{C}^*$-equivariant and descends to the map $\phi : U^*_{d,n} \to U_{d,n}$.

We define $M_{d,n} := U_{d,n}/PGL_{n+1}(\mathbb{C})$, the *moduli space* of degree d smooth hypersurfaces in $\mathbb{P}^n$. We also define $M^*_{d,n} = X^*_{d,n}/GL_{n+1}(\mathbb{C})$.

We can rewrite our result in terms of $X^*_{d,n}$ and $M^*_{d,n}$ as well. This is important to us as our proof will mostly involve understanding the space $X^*_{d,n}$. The space $M^*_{d,n}$ is important conceptually.

Theorem 2. *Let* $d, n \geq 1$.

1. *There is an embedding of graded algebras:*

$$\psi : (\mathrm{H}^*(\mathrm{GL}_{n+1}(\mathbb{C}); \mathbb{Q}) \otimes \mathbb{Q}[x]/(x^n)) \hookrightarrow \mathrm{H}^*(X^*_{d,n}; \mathbb{Q})$$

where $|x| = 2$.

2. *There is an embedding of graded algebras:*

$$\varphi : \mathbb{Q}[x]/(x^n) \hookrightarrow \mathrm{H}^*(M^*_{d,n}; \mathbb{Q})$$

where $|x| = 2$.

Suppose that $d \geq 4n + 1$. *Then, the maps* ψ *and* φ *are surjective in degree* $\leq \frac{d-1}{2}$.

Theorem 2 is equivalent to Theorem 1 after applying Theorem 2 of (10).

Nature of stable cohomology: Throughout the course of the proof of Theorem 2 we also obtain the following description of the stable cohomology classes of $X^*_{d,n}$- the stable classes are tautological in the following sense: There is a line bundle $\mathcal{L}$ on $M^*_{d,n}$ defined by taking the canonical bundle fibrewise (we rigorously define $\mathcal{L}$ in Section 2). We will show that $c_1(\mathcal{L}), \ldots, c_1(\mathcal{L})^{n-1}$ are nonzero in $H^*(M^*_{d,n}; \mathbb{Q})$ and that stably the entire cohomology ring of $M^*_{d,n}$ is just the algebra generated by $c_1(\mathcal{L})$. By (10),

$$\mathrm{H}^*(X^*_{d,n}; \mathbb{Q}) \cong \mathrm{H}^*(GL_{n+1}(\mathbb{C}); \mathbb{Q}) \otimes \mathrm{H}^*(M^*_{d,n}; \mathbb{Q}).$$

In this way we have some qualitative understanding of the stable cohomology of $X^*_{d,n}$.

Both the statement of Theorem 2 and our proof of it are heavily influenced by (12), in which Tommasi proves analogous theorems for $X_{d,n}$. Our techniques and approach are also similar to that of Das in (5),where he proves

$$\mathrm{H}^*(X^*_{3,3}; \mathbb{Q}) \cong \mathrm{H}^*(GL_3(\mathbb{C}); \mathbb{Q}) \otimes \mathbb{Q}[x]/x^3$$

with $|x| = 2$. We would also like to mention the paper (11) where $\mathrm{H}^*(X_{2,4};\mathbb{Q})$ is computed.

In some sense, this paper shows that in a stable range, something similar to Das's theorem is true for marked hypersurfaces in general.

Some motivation and historical comments At this point we'd like to make some remarks on historical motivations for computing and understanding stable cohomology of moduli spaces of algebraic varieties.

The cohomology of moduli spaces are often interesting because they provide us with invariants for families of varieties. However in many interesting cases the entire cohomology ring of the moduli space may be difficult to understand and compute. An example of such a phenomenon is the moduli space of curves of genus g, $\mathscr{M}_g$. In this setting, $H^*(\mathscr{M}_g;\mathbb{Q})$ is a huge ring which is not fully understood. However, the spaces $\mathscr{M}_g$ are known to satisfy homological stability and the stable cohomology ring can be explicitly described. For a survey, see (4).

Another motivation for computing the stable cohomology of moduli spaces has to do with arithmetic statistics. Let X be an algebraic variety over $\mathbb{Z}$. Often one would like to compute $\#X(\mathbb{F}_p)$ by studying the eigenvalues of $Frob_p$ on $H^*_{et}(X;\mathbb{Q}_l)$. There are often comparison theorems which relate the etale cohomology with the singular cohomology of $X(\mathbb{C})$ and computations of $H^*(X(\mathbb{C});\mathbb{Q})$ can often imply bounds on $\#(X(\mathbb{F}_p))$. For an introduction to this topic, see for instance Section 1 and 2 of (3).

Method of Proof. One could attempt to prove Theorem 2 by applying the Serre spectral sequence to the fibration $\pi : X^*_{d,n} \to X_{d,n}$. To successfully do this however, one would need to understand the groups $\mathrm{H}^p(X_{d,n};\mathrm{H}^q(Z(f);\mathbb{Q}))$. While we do a priori understand what the groups $\mathrm{H}^p(X_{d,n};\mathbb{Q})$ are (This is the main theorem of (12)), this is not sufficient for us to understand what the groups $\mathrm{H}^p(X_{d,n};\mathrm{H}^q(Z(f);\mathbb{Q}))$ are, since $\mathrm{H}^q(Z(f);\mathbb{Q})$ is a *nontrivial* local coefficient system. Instead we use an idea of Das and compute $\mathrm{H}^*(X^p_{d,n};\mathbb{Q})$, where $X^p_{d,n} := \{f \in X_{d,n} | f(p) = 0\}$ to avoid any computations with nontrivial coefficient systems.

After we have proved Theorem 2 we can use it to deduce what these twisted cohomology groups are.

Corollary 1. *Let $d, n > 0$. Suppose $d \geq 4n+1$ and $k < \frac{d-1}{2}$. Then*

$$H^k(X_{d,n}; H^{n-1}(Z(f); \mathbb{Q})) = \begin{cases} H^k(X_{d,n}; \mathbb{Q}) & \text{if } n \text{ is odd} \\ 0 & \text{if } n \text{ is even} \end{cases}$$

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2.2 A lower bound on $\mathrm{H}^k(X^*_{d,n})$

We begin by noting that there is an embedding of algebras $\mathrm{H}^k(GL_{n+1}(\mathbb{C})) \otimes \mathbb{Q}[x]/(x^n) \hookrightarrow \mathrm{H}^k(X^*_d)$ in the stable range. More precisely, we have the following:

Proposition 1. *Let $n \geq 0$, and let $d > n+1$. There is a natural embedding of algebras*

$$i : H^*(GL_{n+1}(\mathbb{C}); \mathbb{Q}) \otimes \mathbb{Q}[x]/(x^n) \hookrightarrow H^*(X^*_{d,n}; \mathbb{Q})$$

where $|x| = 2$.

Proof. We first define the *fiberwise canonical bundle* $\mathcal{L}$ over $M^*_{d,n}$ as follows:

$$\mathcal{L} = \{(f,p,v) | (f,p) \in M^*_d, v \in \wedge^{n-1} T^*_p(Z(f))\}.$$

We can pull back $\mathcal{L}$ to a bundle on $X^*_{d,n}$, which we will also denote by $\mathcal{L}$. By the same argument as in Theorem 1 of (10),

$$\mathrm{H}^*(X^*_{d,n};\mathbb{Q}) \cong \mathrm{H}^*(GL_{n+1}(\mathbb{C});\mathbb{Q}) \otimes \mathrm{H}^*(M^*_{d,n}(\mathbb{C});\mathbb{Q}).$$

Let $f \in X_{d,n}$. Let $i : \mathrm{GL}_{n+1}(\mathbb{C}) \to X_{d,n}$ be the orbit map defined by $i(g) = g \cdot f$. More precisely, Theorem 1 of (10) states that the natural map

$$\pi^* : \mathrm{H}^*(M^*_{d,n};\mathbb{Q}) \to \mathrm{H}^*(X^*_{d,n};\mathbb{Q})$$

makes $\mathrm{H}^*(X^*_{d,n};\mathbb{Q})$ a free $\mathrm{H}^*(M^*_{d,n};\mathbb{Q})-$module with a basis given by some set $\{\alpha_i\}$ such that the pullbacks $\{i^*(\alpha_i)\}$ give a basis of $\mathrm{H}^*(GL_{n+1}(\mathbb{C});\mathbb{Q})$. But since $\mathrm{H}^*(GL_{n+1}(\mathbb{C});\mathbb{Q})$ is a free graded commutative algebra, this forces $H^*(X^*_{d,n};\mathbb{Q})$ to be isomorphic to $\mathrm{H}^*(GL_{n+1}(\mathbb{C});\mathbb{Q}) \otimes \mathrm{H}^*(M^*_{d,n}(\mathbb{C});\mathbb{Q})$ as an algebra.

If we restrict $\mathcal{L}$ to a particular hypersurface Z, the bundle $\mathcal{L}|_Z = \mathcal{O}_Z(d-n-1)$. The chern class of $\mathcal{L}|_Z$ satisfies the following equality:

$$c_1(\mathcal{O}_Z(d-n-1)) = (d-n-1)c_1(\mathcal{O}_Z(1)) = d(d-n-1)\omega_Z,$$

where ω_Z is the Kahler class of the variety Z. This implies that for $d > n+1$, the classes $c_1(\mathcal{L})|Z, \ldots, c_1^{n-1}(\mathcal{L})|Z$ are nonzero since $\omega_Z, \ldots \omega_Z^{n-1}$ are nonzero. Now taking $x = c_1(\mathcal{L})$, this implies that $\mathrm{H}^*(M;\mathbb{Q})$ contains a subalgebra isomorphic to $\mathbb{Q}[x]/x^n$. □

2.3 The space X_d^p and the Vassiliev method

Given a space X, the nth *ordered configuration space* of X denoted $\mathrm{PConf}_n X$ is

$$\mathrm{PConf}_n X := \{(x_1 \ldots, x_n) \in X^n | \forall i \neq j, x_i \neq x_j\}.$$

There is a natural action of the symmetric group on n letters S_n on X by permuting the coordinates. The quotient $\mathrm{PConf}_n X/S_n$ is called the nth *unordered configuration space* and denoted $\mathrm{UConf}_n X$. In order to understand $X_{d,n}$ we will first look at the cohomology of a related space. For a fixed point $p \in \mathbb{P}^n$ we set

$$X_d^p = \{f \in X_{d,n} | f(p) = 0\}.$$

Then

$$X_d^p \subseteq V_d^p = \{f \in V_d | f(p) = 0\}.$$

The space V_d^p is a vector space. The complement of X_d^p in V_d^p will be called $\Sigma_{d,p}$. We will compute its Borel-Moore homology and use Alexander duality to compute $H^*(X_d^p)$.

Let $p \in \mathbb{P}^n$. By definition p is a one-dimensional subspace $p \subseteq \mathbb{C}^{n+1}$. Choose a complementary subspace $W \subseteq \mathbb{C}^{n+1}$ (it is not unique, but we will fix a particular one). We define $G_p := GL(W)$.

Let $x_1, \ldots x_n$ be local coordinates in a neighbourhood U containing p. Pick a local trivialisation s of the line bundle $\mathcal{O}(d)$ in U. There is an induced map

$$f^* : T_0^*(\mathcal{O}(d)_p) \to T_p^*(\mathbb{P}^n).$$

Let us use our local coordinates to identify $T_0^*(\mathcal{O}(d)_p)$ with $\mathbb{C}$ and $T_p^*(\mathbb{P}^n)$ with $\mathbb{C}^n$.

Suppose $f \in X_d^p$. Then the map f^* is nonzero because f has a regular zero locus. Let This defines a map

$$\pi : X_d^p \to T_p^*(\mathbb{P}^n) - \{0\} \cong \mathbb{C}^n - \{0\}$$

defined by $\pi(f) = f^*(1)$.

Proposition 2. *The map $\pi : X_d^p \to \mathbb{C}^n - \{0\}$ is a fibration.*

Proof. The group G_p acts on $\mathbb{P}^n$ fixing p. Therefore it acts on both X_d^p and $\mathbb{C}^n - \{0\}$. The

the map π is equivariant with respect to these actions. The map π therefore is the pullback of a map from $\pi' : X_d^p/G_p$ to $\mathbb{C}^n - \{0\}/G_p$. But $\mathbb{C}^n - \{0\}/G_p$ is a point and since π' is surjective it is a fibration. Since pullbacks of fibrations are fibrations, π is a fibration. $\square$

Let $X_v := \pi^{-1}(v)$ and let

$$V_v := \{f \in V_d | f^*(1) = v\}.$$

Clearly, $X_v \subseteq V_v$. Let $\Sigma_v := V_v - X_v$. We will try to understand the Borel-Moore homology of Σ_v.

To accomplish this the Vassiliev method (13) will be applied. The Vassiliev method to compute Borel-Moore homology involves stratifying a space and using the associated spectral sequence to compute its Borel Moore homology. The space Σ_v will be stratified based on the points at which a section f is singular. The techniques used are very similar to that in (12) which contains many of the technical details.

We denote the k simplex with vertex set $\{a_0, \ldots a_k\}$ by $\Delta_{\{a_0, \ldots a_k\}}$. We denote a k simplex by Δ_k and an open k simplex by Δ_k°

We will now construct a cubical space $\mathcal{C}$ which will be involved in understanding Σ_v. Let $N = \frac{d-1}{2}$. Let I be a subset of $\{1, \ldots, N-1\}$. For $k < N$, let

$$\mathcal{C}_I := \{(f, p) | f \in \Sigma_v, p : I \to \mathbb{P}^n, p(I) \subseteq \text{ Singular zeroes of } f\}.$$

We define $\Sigma_v^{\geq N} = \{f \in \Sigma_v | f$ has at least N singular zeroes$\}$ We define

$$\mathcal{C}_{I \cup \{N\}} := \{(f, p) | f \in \Sigma_v, p : I \to \mathbb{P}^n, p(I) \subseteq \text{ Singular zeroes of } f, f \in \bar{\Sigma}^{\geq N}\}.$$

If $I \subseteq J$ then we have a natural map from $\mathcal{C}_J \to \mathcal{C}_I$ defined by restricting p. This gives $\mathcal{C}$ the structure of a cubical space over the set $\{1, \ldots, N\}$. We can take the geometric realization of

$\mathcal{C}$ denoted by $|\mathcal{C}|$. Then there is a map $\rho : |\mathcal{C}| \to \Sigma_v$, induced by the forgetful maps $\mathcal{C}_I \to \Sigma_v$.

$|\mathcal{C}|$ is topologised in a non-standard way so as to make ρ proper. We topologise it as follows: in (12), a space $|\mathscr{X}|$ is constructed with a map $\rho : |\mathscr{X}| \to \Sigma$. Here, $\Sigma = V_d - X_d$. The topology on $|\mathscr{X}|$ is chosen carefully so as to make ρ proper. The construction of $|\mathscr{X}|$ as a set identical to that of $|\mathcal{C}|$ except we replace Σ_v with Σ. There is a natural inclusion $|\mathcal{C}| \to |\mathscr{X}|$. We give $|\mathcal{C}|$ the subspace topology along this map.

Proposition 3. *The map $\rho : |\mathcal{C}| \to \Sigma_v$ is a proper homotopy equivalence.*

Proof. This proof is nearly identical to that of Lemma 15 in (12). The properness of ρ : $|\mathcal{C}| \to \Sigma_v$ follows from the properness of $\rho : |\mathscr{X}| \to \Sigma$ and the properties of the subspace topology. In our setting having contractible fibres implies that the map ρ is a homotopy equivalence, this follows by combining Theorem 1.1 and Theorem 1.2 of (9). We will now prove that the fibres are contractible. If $f \notin \bar{\Sigma}_{\bar{v}}^{\geq N}$, let $\{p_1, \ldots, p_k\}$ be the singular zeroes of f. In this case the fibre $\rho^{-1}(f)$ is a simplex with vertices given by the images of the points $(f, x_i) \in \mathcal{C}_{\{1\}} \times \Delta_{\{1\}}$. Now suppose $f \in \bar{\Sigma}_{\bar{v}}^{\geq N}$. In this case the fibre $\rho^{-1}(f)$ is a cone over the point $f \in \mathcal{C}_N \times \Delta_{\{N\}}$.

□

Now as in any geometric realization, $|\mathcal{C}|$ is filtered by

$$F_n = \mathrm{im}(\coprod_{|I| \leq n} \mathcal{C}_I \times \Delta_k).$$

The F_n form an increasing filtration of $|\mathcal{C}|$, i.e. $F_1 \subseteq F_2 \ldots F_n \subseteq F_{n+1} \subseteq \ldots$ and $\cup_{n=1}^{\infty} F_n = |\mathcal{C}|$.

Proposition 4. *Let $d, n \geq 1$. Let $N = \frac{d-1}{2}$. For $k < N$, the space $F_k - F_{k-1}$ is a Δ_k°-bundle, over a vector bundle B_k over* $\mathrm{UConf}_k(\mathbb{P}^n - p)$.

Proof. The space $F_k - F_{k-1}$ consists of the interiors of k simplices labeled by $\{f, p_0 \dots p_k\}$. Let

$$B_k = \{(f, \{p_0 \dots p_k\}) \in \Sigma_v \times \mathrm{UConf}_k(\mathbb{P}^n - p) | p_i \text{ are singular zeroes of } f\}.$$

We have a map $\phi : F_k - F_{k-1} \to B_k$, defined by

$$\phi((f, \{p_0 \dots p_k\}), s_0, \dots, s_k) = (f, \{p_0 \dots p_k\}).$$

The map ϕ expresses $F_k - F_{k-1}$ as a fibre bundle over B_k with Δ_k° fibres, i.e we have a diagram as follows:

$$\begin{tikzcd} \Delta_k^\circ \arrow[r] & F_k - F_{k-1} \arrow[d] \\ & B_k \end{tikzcd}$$

We have a map $B_k \to \mathrm{UConf}_k(\mathbb{P}^n - p)$ defined by $\{f, p_0 \dots, p_k\} \mapsto \{p_0, \dots p_k\}$. This is a vector bundle by Lemma 3.2 in (18). $\square$

We have a one-dimensional local coefficient system denoted $\pm\mathbb{Q}$ on $\mathrm{UConf}_k(\mathbb{P}^n - p)$ defined in the following way: Let S_k be the symmetric group on k letters. We have a homomorphism $\pi_1 \mathrm{UConf}_k(\mathbb{P}^n - p) \to S_k$ associated to the covering space $\mathrm{PConf}_k(\mathbb{P}^n - p) \to \mathrm{UConf}_k(\mathbb{P}^n - p)$. Compose this homomorphism with the sign representation $S_k \to \pm 1 = GL_1(\mathbb{Q})$ to obtain our local system.

Proposition 5. *Let $d, n \geq 1$. Let $e_d = \dim_{\mathbb{C}}(V_v)$. For $k < \frac{d-1}{2}$,*

$$\bar{H}_*(F_k - F_{k-1}) \cong H_{*-(k+2e_d-2(n+1)(k+1))}(\mathrm{UConf}_k(\mathbb{P}^n - p), \pm\mathbb{Q}).$$

Proof. By Proposition 4 the space $F_k - F_{k-1}$ is a bundle over $\mathrm{UConf}_k(\mathbb{P}^n - p)$. This fact implies that

$$\bar{H}_*(F_k - F_{k-1}) \cong H_{*-(k+2e_d-2(n+1)(k+1))}(\mathrm{UConf}_k(\mathbb{P}^n - p), \mathbb{Q}(\sigma)).$$

Here $\mathbb{Q}(\sigma)$ is the local sytem obtained by the action of $\pi_1(\mathrm{UConf}_k(\mathbb{P}^n - p))$ on the fibres $\bar{H}_k(\Delta_k^\circ)$ where in this case Δ_k° is the open k simplex corresponding to the fibres of the map $F_k - F_{k-1} \to B_k$. But one observes that the action of $\pi_1(\mathrm{UConf}_k(\mathbb{P}^n - p))$ on this open simplex is by permutation of the vertices which imples that $\mathbb{Q}(\sigma) = \pm\mathbb{Q}$.

□

As with any filtered space, we have a spectral sequence with

$$E_1^{p,q} = \bar{H}_{p+q}(F_p - F_{p-1}; \mathbb{Q})$$

converging to $\bar{H}_*(Y; \mathbb{Q})$. Now for $p < N$ by Proposition 5

$$E_1^{p,q} = \bar{H}_{q-(2e_d-2(n+1)(p+1))}(\mathrm{UConf}_p(\mathbb{P}^n - p); \pm\mathbb{Q}).$$

We would like to claim that $E_1^{N,q}$ doesn't matter in the stable range. To be more precise, we have the following:

Lemma 1. *Let $d, n \geq 1$. Let $N = \frac{d-1}{2}$. Let $k > 2e_d - N$. Then,*

$$\bar{H}_k(|\mathcal{C}| - F_N; \mathbb{Q}) \cong \bar{H}_k(|\mathcal{C}|; \mathbb{Q}).$$

Proof. We first will try to bound the $\bar{H}_*(F_N; \mathbb{Q})$ and then use the long exact sequence of the pair. F_N is the union of locally closed subspaces

$$\phi_k = \{(f, x_1, \ldots, x_k), p | f \in \Sigma^{\geq N}, x_i \text{ are singular zeroes of } f, p \in \Delta_k\}.$$

We have a surjection $\pi : \phi_k \to \mathrm{UConf}_k(\mathbb{P}^n - p)$. This map π is in fact a fibre bundle with fibres $\Delta^k \times \mathbb{C}^{e_d - N(n+1)}$. The space $\mathrm{UConf}_k(\mathbb{P}^n - p)$ is kn dimensional. Therefore

$$\bar{H}_*(\phi_k; \mathbb{Q}) = 0 \text{ if } * > 2(e_d - (n+1)N) + kn < 2e_d - N.$$

This implies that for all k, $\bar{H}_*(\phi_k; \mathbb{Q}) = 0$, if $* > 2e_d - N$. This implies $\bar{H}_*(F_N; \mathbb{Q}) = 0$, if $* > 2e_d - N$. By the long exact sequence in Borel Moore homology associated to the pair $F_N \hookrightarrow Y$, $\bar{H}_k(Y - F_N; \mathbb{Q}) \cong \bar{H}_k(Y; \mathbb{Q})$ for $k > 2e_d - N$. $\square$

2.4 Interlude

In (12), Tommasi proves the following result:

Theorem 3 ((12))**.** *Let $d, n \geq 1$. Let $f \in X_{d,n}$. Let $\psi : GL_{n+1}(\mathbb{C}) \to X_{d,n}$ be the orbit map defined by $\psi(g) = g \cdot f$. Then $\psi^* : H^k(X_{d,n}, \mathbb{Q}) \to H^k(GL_{n+1}(\mathbb{C}), \mathbb{Q})$ is an isomorphism for $k < \frac{d+1}{2}$.*

In this section we shall look at the proof of Theorem 3 in (12) and use it to prove an identity used later on in this paper. One of the ingredients in the proof of Theorem 3 is a Vassiliev spectral sequence. We introduce a new convention, by letting h denote the dimension of H. We also define $Gr(p, n)$ to be the Grassmanian of p-planes in $\mathbb{C}^n$. In what follows we shall need a few basic facts about $H_*(Gr(p, n); \mathbb{Q})$ and Schubert symbols. Let

$$0 = E_0 \subsetneq E_1 \cdots \subsetneq E_{n-1} \subsetneq E_n = \mathbb{C}^n$$

be a complete flag. Given $U \in Gr(p, n)$, we can associate to it a sequence of numbers, $a_i = \dim U \cap E_i$. These a_i satisfy the following conditions:

$$0 \leq a_{i+1} - a_i \leq 1, a_0 = 0 \text{ and } a_n = p.$$

Such sequences are called *Schubert symbols*. Let $\mathbf{a} = (a_0 \dots a_n)$. We call $\mathbf{a}$ a Schubert symbol if $0 \leq a_{i+1} - a_i \leq 1$,$a_0 = 0$ and $a_n = p$. Associated to each Schubert symbol $\mathbf{a}$ we have a subvariety $W_{\mathbf{a}} \subseteq Gr(p, \mathbb{C}^n)$ defined as follows.

$$W_{\mathbf{a}} := \overline{\{U \subseteq \mathbb{C}^n | \dim(U \cap \mathbb{C}^i) = a_i\}}.$$

The main result we will be using is the following.

Theorem 4. *Let* $\mathbf{a}$ *be a Schubert symbol. The classes* $[W_{\mathbf{a}}] \in H_*(Gr(p,n);\mathbb{Q})$ *form a basis.*

For a proof of Theorem 4 see page 1071 of (8).

Proposition 6. *Let* n *be a positive integer. Then*

$$\sum_{k,p} h_k(Gr(p, \mathbb{C}^n); \mathbb{Q}) = 2^n.$$

Proof. By Theorem 4,

$$\begin{aligned}
\sum_{k,p} h_k(Gr(p, \mathbb{C}^n); \mathbb{Q}) &= \sum_{p} \#\{(a_0, \dots a_n) | 0 \leq a_{i+1} - a_i \leq 1, a_0 = 0, a_n = p\} \\
&= \#\{(a_0, \dots a_n) | 0 \leq a_{i+1} - a_i \leq 1, a_0 = 0\} \\
&= \#\{(b_1, \dots b_n) \in \{0, 1\}\}.
\end{aligned}$$

The last equality follows because if we are given a sequence of a_i, we can uniquely obtain a sequence of b_i, by letting $b_i = a_i - a_{i-1}$. $\square$

Our main aim of this section is to prove the following technical result.

Theorem 5. *The Vassiliev spectral sequence in (12) degenerates in the stable range: if* $p < \frac{d+1}{2}$ *and if* $q > 0$*, then* $E_1^{p,q} \cong E^{\infty}_{p,q}$.

Equivalently, for $k < \frac{d+1}{2}$,

$$\sum_p h_{2(p+1)(n+1)-p-k-1}(\mathrm{UConf}_p(\mathbb{P}^n);\mathbb{Q}) = h_k(GL_{n+1};\mathbb{Q}). \tag{2.1}$$

Remark 2. *The statements are equivalent because the group $H^k(GL_{n+1}(\mathbb{C});\mathbb{Q})$ is a subquotient of*

$$\bigoplus H_{2(p+1)(n+1)-p-k-1}(\mathrm{UConf}_p(\mathbb{P}^n);\mathbb{Q})).$$

Proof. We already know that

$$\sum_p h_{2(p+1)(n+1)-p-k-1}(\mathrm{UConf}_p(\mathbb{P}^n);\pm\mathbb{Q}) \geq h_k(GL_{n+1};\mathbb{Q})$$

because the left hand side of (2.1) are the appropriate terms in a spectral sequence converging to the right hand side of (2.1).

It suffices to prove that

$$\begin{aligned}\sum_k \sum_p & h_{2(p+1)(n+1)-p-k-1}(\mathrm{UConf}_p(\mathbb{P}^n);\pm\mathbb{Q}) \\ &= \sum_k h_k(GL_{n+1};\mathbb{Q}) = 2^{n+1}.\end{aligned}$$

Lemma 2 in (13) states that:

$$\begin{aligned}& h_{2(p+1)(n+1)-p-k-1}(\mathrm{UConf}_p(\mathbb{P}^n),\pm\mathbb{Q}) \\ &= h_{2(p+1)(n+1)-p-k-1-p(p-1)}(Gr(p,\mathbb{C}^{n+1});\mathbb{Q}).\end{aligned}$$

Therefore

$$\sum_k \sum_p h_{2(p+1)(n+1)-p-k-1}(\mathrm{UConf}_p(\mathbb{P}^n);\pm\mathbb{Q}) = \sum_k \sum_p h_k(Gr_(p,\mathbb{C}^{n+1});\mathbb{Q}).$$

By Proposition 6 this is equal to 2^{n+1}.

$\square$

2.5 Computation

We would like to know what the groups $\bar{H}_*(\mathrm{UConf}_{k+1}(\mathbb{P}^n - p); \pm\mathbb{Q})$ are. First note that in (13) Vassiliev proves that :

Proposition 7 ((13))**.** *Let* $k, n > 0$. *Then,*

$$H_*(\mathrm{UConf}_k(\mathbb{P}^n); \pm\mathbb{Q}) \cong H_{*-(k)(k-1)}(Gr_k(\mathbb{C}^{n+1}); \mathbb{Q}).$$

Also note that in light of Theorem 4 the homology of Grassmannians is well understood in terms of Schubert cells.

Consider the long exact sequence in Borel Moore homology associated to

$$\mathrm{UConf}_{k+1}(\mathbb{P}^n - p) \subseteq \mathrm{UConf}_{k+1}(\mathbb{P}^n) \hookleftarrow \mathrm{UConf}_k(\mathbb{P}^n - p).$$

The last inclusion is defined by the map $\phi : \mathrm{UConf}_k(\mathbb{P}^n - p) \to \mathrm{UConf}_{k+1}(\mathbb{P}^n)$ where $\phi(\{x_1 \dots x_n\}) = \{x_1 \dots x_n, p\}$.

We consider the long exact sequence in Borel-Moore homology associated to the pair $(\mathrm{UConf}_{k+1}(\mathbb{P}^n), \mathrm{UConf}_{k+1}(\mathbb{P}^n - p))$. Here $\mathrm{UConf}_{k+1}(\mathbb{P}^n - p)$ is an open subset of $\mathrm{UConf}_{k+1}(\mathbb{P}^n)$ with complement homeomorphic to $\mathrm{UConf}_k(\mathbb{P}^n - p)$. A segment of this exact sequence is displayed below:

$$\bar{H}_*(\mathrm{UConf}_k(\mathbb{P}^n - p); \pm\mathbb{Q}) \to \bar{H}_*(\mathrm{UConf}_{k+1}(\mathbb{P}^n); \pm\mathbb{Q}) \to \bar{H}_*(\mathrm{UConf}_{k+1}(\mathbb{P}^n - p); \pm\mathbb{Q}) \quad (2.2)$$

Proposition 8. *Let $k, n > 0$. Then there is a canonical decomposition*

$$\bar{H}_*(\mathrm{UConf}_{k+1}(\mathbb{P}^n); \pm\mathbb{Q}) \cong \bar{H}_*(\mathrm{UConf}_k(\mathbb{P}^n - p); \pm\mathbb{Q}) \oplus \bar{H}_*(\mathrm{UConf}_k(\mathbb{P}^n - p); \pm\mathbb{Q}),$$

due to the fact that (2.2) *splits.*

Proof. Lemma 2 of (13) implies that (2.2) decomposes into split short exact sequences, i.e.

$$\bar{H}_*(\mathrm{UConf}_{k+1}(\mathbb{P}^n); \pm\mathbb{Q}) \cong \bar{H}_*(\mathrm{UConf}_k(\mathbb{P}^n - p); \pm\mathbb{Q}) \oplus \bar{H}_*(\mathrm{UConf}_k(\mathbb{P}^n - p); \pm\mathbb{Q}).$$

□

Remark 3. *In fact the $H_*(\mathrm{UConf}_k(\mathbb{P}^n - p); \pm\mathbb{Q})$ has a basis given by Schubert symbols with $a_1 = 0$.*

Proposition 9. *If the Vassiliev spectral sequence has no nonzero differentials and $k < \frac{d-1}{2}$, then $H^k(X_v) \cong H^k(G_p)$ as vector spaces.*

Proof. Now in our spectral sequence we had

$$E_1^{p,q} = \bar{H}_{q-(2e_d - 2(p+1)(n+1))}(\mathrm{UConf}_{p+1}(\mathbb{P}^n - p); \pm\mathbb{Q}).$$

First collect all terms in the main diagonal, i.e.

$$V := \oplus_{p+q=l} \bar{H}_{q-(2D_n - 2(p+1)(n+1))}(\mathrm{UConf}_{p+1}(\mathbb{P}^n - p); \pm\mathbb{Q})$$

It will suffice to prove that

$$\dim V = \sum_{p \le 2D_n - k} h_{2(p+1)(n+1)-p-k-1}(\mathrm{UConf}_p(\mathbb{P}^n - pt); \pm\mathbb{Q}) = h^k(GL_n; \mathbb{Q}). \tag{2.3}$$

Proposition 5 implies

$$\sum_p h_{2(p+1)(n+1)-p-k-1}(\mathrm{UConf}_p(\mathbb{P}^n); \pm\mathbb{Q}) = h_k(GL_{n+1}; \mathbb{Q}). \tag{2.4}$$

Proposition 7 implies,

$$h_{2(p+1)(n+1)-p-k-1}(\mathrm{UConf}_p(\mathbb{P}^n); \pm\mathbb{Q}) = 0 \text{ if } p > n$$

.

So as long as $n < 2(D_n + n + 1) - k$,

$$\sum_{p \leq 2(D_n+n+1)-k} h_{2(p+1)(n+1)-p-k-1}(\mathrm{UConf}_p(\mathbb{P}^n); \pm\mathbb{Q})$$

$$= \sum_p h_{2(p+1)(n+1)-p-k-1}(\mathrm{UConf}_p(\mathbb{P}^n); \pm\mathbb{Q}).$$

But the condition $n < 2(D_n + n + 1) - k$ is equivalent to $k < 2(D_n + 1) + n$, which is true if $k < N$. We have another equality from Proposition 8,

$$h_k(\mathrm{UConf}_p(\mathbb{P}^n - pt); \pm\mathbb{Q}) + h_k(\mathrm{UConf}_{p-1}(\mathbb{P}^n - pt); \pm\mathbb{Q}) = h_k(\mathrm{UConf}_p(\mathbb{P}^n); \pm\mathbb{Q}).$$

Plugging this into (2.4) we have

$$\begin{aligned} h^k(GL_{n+1}; \mathbb{Q}) &= \sum h_{2(p+1)(n+1)-p-k}(\mathrm{UConf}_p(\mathbb{P}^n); \pm\mathbb{Q}) \\ &= \sum h_{2(p+1)(n+1)-p-k-1}(\mathrm{UConf}_p(\mathbb{P}^n - pt); \pm\mathbb{Q}) \\ &+ h_{2(p+1)(n+1)-p-k-1}(\mathrm{UConf}_{p-1}(\mathbb{P}^n - pt); \pm\mathbb{Q}). \end{aligned}$$

We have the identity

$$h^k(GL_n;\mathbb{Q}) + h^{k-(2n+1)}(GL_n;\mathbb{Q}) = h^k(GL_{n+1};\mathbb{Q}).$$

This implies,

$$\begin{aligned} &h^k(GL_n;\mathbb{Q}) + h^{k-(2n+1)}(GL_n;\mathbb{Q}) \\ &\quad = \sum_p h_{2(p+1)(n+1)-p-k-1}(\mathrm{UConf}_p(\mathbb{P}^n - pt);\pm\mathbb{Q}) \\ &\quad + h_{2(p+1)(n+1)-p-k-1}(\mathrm{UConf}_{p-1}(\mathbb{P}^n - pt);\pm\mathbb{Q}). \end{aligned} \tag{2.5}$$

Now we will try to prove 2.3 by induction on k. For $k = 0$, (2.3) is trivial. By induction

$$h^{k-(2n+1)}(GL_n;\mathbb{Q}) = \sum_p h_{2(p+1)(n+1)-p-k-1}(\mathrm{UConf}_{p-1}(\mathbb{P}^n - pt);\pm\mathbb{Q}).$$

Putting this into 2.5 we obtain

$$\sum_p h_{2(p+1)(n+1)-p-k-1}(\mathrm{UConf}_p(\mathbb{P}^n - pt);\pm\mathbb{Q}) = h^k(GL_n;\mathbb{Q}).$$

□

Now we can look at the Serre Spectral sequence associated to the fibration

$$X_v \hookrightarrow X_p \to \mathbb{C}^n - 0.$$

We observe that if there are no nonzero differentials, then

$$H^*(X_p;\mathbb{Q}) \cong H^*(X_v;\mathbb{Q}) \otimes \mathbb{Q}[e_{2n-1}]/e_{2n-1}^2.$$

This is because the Serre spectral sequence degenerates and since $\mathbb{Q}[e_{2n-1}]/e_{2n-1}^2$ is a free graded commutative algebra the ring structure of the total space is forced to be the tensor product.

Proposition 10. *Let $d > 0$ and $p \in \mathbb{P}^n$. Then,*

$$H^*(X_{d,p};\mathbb{Q}) \cong H^*(G_p;\mathbb{Q}) \otimes A,$$

where A is $H^*(X_d^p/G_p;\mathbb{Q})$.

Proof. This follows immediately from Theorem 2 in (10). □

We will also need the following fact that is a special case of Lemma 2.6 in (5).

Proposition 11. *Let $d > 0$, $k < \frac{d-1}{2}$. Let $U_d^* = X_d^*/\mathbb{C}^*$. Then*

$$H^*(X_d^*;\mathbb{Q}) \cong H^*(U_d^*;\mathbb{Q}) \otimes \mathbb{Q}[e_1]/(e_1^2),$$

where $|e_1| = 1$.

Proposition 11 implies if there are no nonzero differentials in both our Vassiliev spectral sequence and in the Serre spectral sequence associated to the fibration $X_{d,n}^p \to \mathbb{C}^n - 0$ then

$$H^*(U_{d,p};\mathbb{Q}) \cong H^*(G_p;\mathbb{Q}) \otimes \mathbb{Q}[e_{2n-1}]/(e_{2n-1}^2)$$

for $* < \frac{d-1}{2}$. In case there are nonzero differentials in either spectral sequence, then $H^*(U_{d,p};\mathbb{Q}) \cong H^*(G_p;\mathbb{Q})$ for $* < \frac{d-1}{2}$.

2.6 Comparing fibre bundles

In this section we finish the proof of Theorem 2.

Proof of Theorem 2. We compare three related fibre bundles and their associated spectral sequences. This is similar to the Proof of Theorem 1.1 in (5).

Let $PG_p := Stab_{PGL(n+1)}(p)$

$$\begin{tikzcd}
PG_p \arrow[r] \arrow[dr] & U_{d,p} \arrow[r] \arrow[dr] & U_d \arrow[dr] & \\
& PGL_{n+1}(\mathbb{C}) \arrow[r] \arrow[d] & U_d^* \arrow[r] \arrow[d] & U_d \times \mathbb{P}^n \arrow[d] \\
& \mathbb{P}^n \arrow[r, "="] & \mathbb{P}^n \arrow[r, "="] & \mathbb{P}^n
\end{tikzcd} \tag{2.6}$$

We denote the exterior algebra on generators $a_1 \dots a_n$ by $\Lambda\langle a_1 \dots a_n\rangle$. By Proposition 9 and Theorem 1 of (10) there are two possibilities for $H^*(U_{d,p};\mathbb{Q})$: either

$$H^*(U_{d,p};\mathbb{Q}) \cong H^*(PG_p;\mathbb{Q}) \otimes \mathbb{Q}[e_{2n-1}]/(e_{2n-1}^2) \cong \Lambda\langle u_1, u_3, \dots u_{2n-1}, e_{2n-1}\rangle$$

or

$$H^*(U_{d,p};\mathbb{Q}) \cong H^*(PG_p;\mathbb{Q}) = \Lambda\langle u_1, u_3, \dots u_{2n-1}\rangle.$$

Suppose for the sake of contradiction that $H^*(U_{d,p}) = \Lambda\langle u_3, \dots u_{2n-1}\rangle$ for $* < \frac{d-1}{2}$. In this case $H^*(U_{d,p};\mathbb{Q}) \cong H^*(PG_p;\mathbb{Q})$ for $* < \frac{d-1}{2}$. Then since the homology of the base and the fibres are isomorphic, $H^*(U_d^*;\mathbb{Q}) \cong H^*(PGL_{n+1}(\mathbb{C});\mathbb{Q})$ for $* < \frac{d-1}{2}$. However by Proposition 1,

$$H^*(PGL_{n+1}(\mathbb{C});\mathbb{Q}) \otimes \mathbb{Q}[x]/x^n) \subseteq H^*(U_d^*;\mathbb{Q}).$$

But $H^*(PGL_{n+1}(\mathbb{C});\mathbb{Q})$ does not contain a subalgebra isomorphic to

$$H^*(PGL_{n+1}(\mathbb{C});\mathbb{Q}) \otimes \mathbb{Q}[x]/x^n).$$

This is a contradiction. So we must be in the case where

$$H^*(U_{d,p};\mathbb{Q}) \cong H^*(PG_p;\mathbb{Q}) \otimes \mathbb{Q}[e_{2n-1}]/(e_{2n-1}^2).$$

Consider the Serre spectral sequence associated to the fibration $U_d^* \to \mathbb{P}^n$. Its E_2 page has terms

$$E_2^{p,q} = H^p(\mathbb{P}^n, H^q(U_d^p;\mathbb{Q})) \cong H^p(\mathbb{P}^n;\mathbb{Q}) \otimes H^q(U_d^p;\mathbb{Q}).$$

Now

$$H^q(U_d^p;\mathbb{Q}) \cong H^q(PG_p;\mathbb{Q}) \otimes \mathbb{Q}[e_{2n-1}]/(e_{2n-1}^2).$$

Consider the trivial fibre bundle $U_d \times \mathbb{P}^n \to \mathbb{P}^n$. There is a natural inclusion of fibre bundles as shown in (2.6). This induces a map of spectral sequences between the associated Serre spectral sequences.

Note that any class $\alpha \in H^q(U_d^p;\mathbb{Q})$ that lies in the image of $H^q(U_d;\mathbb{Q})$ is mapped to zero under any differential thanks to the fact that all dfferentials are zero in the spectral sequence associated to a trivial fibration. The only possible nonzero differential in the E_2 page of the Serre spectral sequence associated to the fibration $U_d^* \to \mathbb{P}^n$ is $d(e_{2n-1})$.

Suppose for contradiction that $d(e_{2n-1}) = 0$. This implies that

$$H^k(U_d^*;\mathbb{Q}) \cong (H^*(U_{d,p};\mathbb{Q}) \otimes H^*(\mathbb{P}^n;\mathbb{Q}))_k = (H^*(PG_p;\mathbb{Q}) \otimes H^*(\mathbb{P}^n,\mathbb{Q}))_k$$

for $k < \frac{d-1}{2}$.

Let $p(t)$ be the Poincare polynomial of U_d^*. We already know that

$$H^*(U_d^*;\mathbb{Q}) \cong H^*(PGL_{n+1}(\mathbb{C});\mathbb{Q}) \otimes H^*(U_d^*/\mathrm{PGL}_{n+1}(\mathbb{C});\mathbb{Q}).$$

So $(1+t^3)\dots(1+t^{2n+1})|p(t)$. On the other hand, if $de_{2n-1}=0$ then

$$p(t) = (1+t^3)\dots(1+t^{2n-1})(1+t^2+t^4\dots t^{2n}) \mod t^{\frac{d-1}{2}}.$$

If $d \geq 4n+1$, then this implies that $(1+t^{2n+1}) \not| p(t)$. This is a contradiction.

So we must have a differential killing the class in $H^{2n}(\mathbb{P}^n, H^0(U_{d,p}); \mathbb{Q})$. The differential must come from from e_{2n-1}, i.e. $d(e_{2n-1}) = ax^n$ for some $a \in \mathbb{Q}^*$. This (along with multiplicativity of differentials) determines all differentials and implies (1). By Proposition 11 (1) $\implies$ (2). By Theorem 1 of (10)

$$H^*(X^*_{d,n}; \mathbb{Q}) \cong H^*(M^*_{d,n}; \mathbb{Q}) \otimes (H^*(GL_{n+1})(\mathbb{C}); \mathbb{Q}).$$

In light of this (2) $\implies$ (3).

$\square$

Having finished the proof of Theorem 2 we can prove Corollary 1.

Proof of Corollary 1. Consider the fibration:

$$\begin{tikzcd} Z(f) \arrow[r] & X^*_d \arrow[d] \\ & X_d \end{tikzcd}$$

and its associated Serre spectral sequence whose E_2 page is of the form

$$H^p(X_d; H^q(Z(f); \mathbb{Q})) \implies H^*(X^*_d; \mathbb{Q}).$$

By Theorem 3 for $* < \frac{d+1}{2}$

$$H^*(X_d; \mathbb{Q}) \cong H^*(GL_{n+1}(\mathbb{C}); \mathbb{Q}).$$

By Theorem 2, we know that the classes in the E_2 page corresponding to the group

$$H^p(GL_{n+1}(\mathbb{C}); c_1(\mathcal{L})^q)$$

survive till the E^∞ page and in the stable range all other terms are killed by differentials.

Now suppose n is even. Then the only other terms in the spectral sequence are of the form $H^p(X_d; H^{n-1}(Z(f); \mathbb{Q}))$. However it is not possible for any such term to be in the image or in the preimage of a nonzero differential. This is because all other terms survive so any possible nonzero differential must be from $H^{p_1}(X_d; H^{n-1}(Z(f); \mathbb{Q}))$ to $H^{p_2}(X_d; H^{n-1}(Z(f); \mathbb{Q}))$ for some choice of p_1 and p_2. However no differential is of bidegree $(p_2 - p_1, 0)$. This implies that

$$H^p(X_d; H^{n-1}(Z(f); \mathbb{Q})) \cong 0.$$

A similar argument shows that if n is odd, $H^p(X_d; H^{n-1}(Z(f); \mathbb{Q})) \cong H^p(X_d; \mathbb{Q})$. Essentially the only difference between the even case and the odd case is that in the odd case we have a class $c_1(\mathcal{L})^{\frac{n-1}{2}} \in H^{n-1}(Z(f); \mathbb{Q})$. Let $A = \mathbb{Q} - \mathrm{span}(c_1(\mathcal{L})^{\frac{n-1}{2}})$ By Theorem 2, we know that $H^p(X_{d,n}; A)$ survives till the E^∞ page. An argument similar to that in the even case shows that

$$H^p(X_d; H^{n-1}(Z(f); \mathbb{Q})) \cong H^p(X_d; A).$$

$\square$

CHAPTER 3

STABLE COHOMOLOGY FOR DISCRIMINANT COMPLEMENTS FOR AN ALGEBRAIC CURVE

3.1 Introduction

In this paper we are concerned with understanding the difference between spaces of algebraic and C^∞ sections of complex line bundles on a smooth projective algebraic curve X over $\mathbb{C}$. We prove that these spaces have isomorphic cohomology in a range of degrees that grows with the degree of the line bundle.

Let X be a smooth projective algebraic curve over $\mathbb{C}$ of genus g. Let $\mathcal{L}$ be an algebraic line bundle on X of degree n.

Let $C^\infty(X,\mathcal{L})$ be the vector space of *smooth* sections of $\mathcal{L}$. Given $s \in C^\infty(X,\mathcal{L})$, we say $p \in X$ is a *regular zero* of s if $s(p) = 0$ and $s'(p) \neq 0$. If $s \in C^\infty(X,\mathcal{L})$ is an algebraic section, then being a regular zero is equivalent to having index 1. Let

$$\mathbb{U}(\mathcal{L}) = \{s \in C^\infty(X,\mathcal{L}) | \text{all zeroes of } s \text{ are isolated and of index } 1\}.$$

Let

$$U(\mathcal{L}) := \{s \in H^0(X,\mathcal{L}) : \text{All zeroes of } s \text{ are regular}\}.$$

There is a natural inclusion map $i : U(\mathcal{L}) \hookrightarrow \mathbb{U}(\mathcal{L})$. The aim of the present paper is to understand this inclusion at the level of cohomology. Our main theorem is as follows:

Theorem 6. *Let X be a smooth projective complex algebraic curve of genus g. Let $n \geq 0$. Let $\mathcal{L}$ be an algebraic line bundle of degree n on X. Let $i : U(\mathcal{L}) \to \mathbb{U}(\mathcal{L})$ be the inclusion map. Then for all $0 \leq 2k \leq n - 2g$,*

$$i^* : H^k(\mathbb{U}(\mathcal{L});\mathbb{Z}) \to H^k(U(\mathcal{L});\mathbb{Z})$$

is an isomorphism.

We also provide some qualitative understanding of the topology of the space $\mathbb{U}(\mathcal{L})$ and relate it with more classical objects. Let $n = \deg(\mathcal{L})$. Given a space M, define

$$\mathrm{PConf}_n M := \{(x_1, \ldots, x_n) \in M^n | x_i \neq x_j\}.$$

The permutation action of S_n on M^n restricts to an action on $\mathrm{PConf}_n M$. Let

$$\mathrm{UConf}_n M = \mathrm{Pconf}_n M / S_n$$

be the *unordered configuration space* of n points on M.

Define the n stranded *surface braid group* on a surface X as

$$Br_n(X) := \pi_1(\mathrm{UConf}_n(X)).$$

For our purposes we will need to define a group $\tilde{B}r_n(X)$ that we call the extended surface braid group. This will be defined later on in Section 3 as $\pi_1(U_n^{alg})$ where U_n^{alg} is a space defined in Section 3 that is a $\mathbb{C}^*$ bundle over $\mathrm{UConf}_n X$. 1

Let $\pi : \tilde{B}r_n(X) \to Br_n(X)$ be the projection. Now, $\mathrm{UConf}_n(X) \subseteq Sym^n(X)$ has an Abel Jacobi map to $Pic^n(X)$. This induces a map $\alpha : Br_n(X) \to \mathbb{Z}^{2g}$. Let $\mathbb{A} = \alpha \circ \pi$. Let $K_n \subseteq \tilde{B}r_n(X)$ be the kernel of this map.

Theorem 7. *Let $n \geq 1$. Let X be a smooth projective curve. Let $\mathcal{L}$ be a line bundle of degree n on X. The space $\mathbb{U}(\mathcal{L})$ is a $K(\pi, 1)$. Furthermore,*

$$\pi_1(\mathbb{U}(\mathcal{L})) \cong K_n.$$

Motivation In (18) Vakil and Wood consider (among other things) the 'stable class' of the discriminant locus in the Grothendieck group of varieties $K_0(\mathrm{Var})$. Let us recall the definition of the *Grothendieck group of varieties.* Let us fix a base field k. Then we can consider the set

$$\mathrm{Var}_k = \{X : X \text{is a variety over } k\}/isomorphism.$$

We can form a monoid M out of Var_k as follows: let M be generated by elements of Var_k, with the relation, if $Y \subseteq X$, $[X] = [X - Y] + [Y] \in M$. The *Grothendieck group* $K_0(\mathrm{Var}_k)$ is the group completion of M. It has a ring structure coming from the product of varieties. In the literature, the element $\mathbb{A}^1$ (often denoted $\mathbb{L}$) is sometimes inverted. Define $\mathscr{M}_{\mathbb{L}} = K_0(\mathrm{Var}_k)[\frac{1}{\mathbb{L}}]$.

Consider a smooth variety X along with an ample line bundle $\mathcal{L}$ on it.

Theorem 8 (Vakil - Wood (18)). *Let $j \geq 1$. Let $U(\mathcal{L}^{\otimes j})$ be the (open) variety of sections with smooth zero locus. Let ζ_X be the Kapranov motivic zeta function, and let d be the dimension of X. Then,*

$$\lim_{j\to\infty} \frac{[U(\mathcal{L}^j)]}{[H^0(X, \mathcal{L}^j)]} = \frac{1}{\zeta_X(d+1)}.$$

Here the limit is with respect to the dimension filtration in $\mathscr{M}_{\mathbb{L}}$.

While Theorem 8 seems to have nothing to do with the cohomology of the space $U(\mathcal{L}^j)$, there is a specialisation map

$$K_0(\mathrm{Var}_k) \to \{\text{Weighted Euler characteristics}\}.$$

Thus, Theorem 1.3 implies that there is a stabilisation of Euler characteristics and one can hope for a stabilisation in cohomology as well.

In (12), Tommasi proves a cohomological result in the same vein as that of this paper, where she studies discriminant complements on $\mathbb{P}^n$. Her set up is as follows. Let $d, n \geq 1$.

Let $X = \mathbb{P}^n$. Let $\mathcal{L} = \mathcal{O}(d)$. Let

$$U(\mathcal{L}) = \{f \in H^0(X, \mathcal{L}) | f \text{ has only regular zeroes}\}.$$

Then the main theorem of (12) stated in our notation is as follows:

Theorem 9 (Tommasi (12)). *Let $d, n \geq 1$. Let $X = \mathbb{P}^n$, $\mathcal{L} = \mathcal{O}(d)$. Let $0 \leq k \leq \frac{d+1}{2}$. Then*

$$H^k(U(\mathcal{L}); \mathbb{Q}) \cong H^k(GL_{n+1}(\mathbb{C}); \mathbb{Q}).$$

Our motivation for the present paper was to understand if there are stability phenomena for discriminant complements over general varieties and whether cohomology in the stable range is dependent only on the topology of the variety. Theorem 6 shows that at least in the case of an algebraic curve there is some kind of stability phenomenon with cohomology in the stable range being purely topological in nature. We are currently working on extending these results to more general varieties.

Relation to other work: Orsola Tommasi has anounced some results on homological stability for discriminant complements over arbitrary smooth projective varieties. We believe that the results in this paper are substantially different from hers. We focus on relating discriminant complements to spaces of C^∞ sections, which is not the focus of her results.

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3.2 Smooth sections

The space $\mathbb{U}(\mathcal{L})$ is actually easy to understand topologically. There is a fibration $\pi : \mathbb{U}(\mathcal{L}) \to \mathrm{UConf}_n X$ defined by: $\pi(f) = \{a | f(a) = 0\}$. We shall need to understand the fibres $\pi^{-1}(\{a_1, \dots a_n\}) \subseteq \mathbb{U}(\mathcal{L})$, but first we shall introduce some basic objects and prove some more technical lemmas. Let Y, Z be based spaces. Let $C(Y, Z), C_*(Y, Z)$ denote the space of continuous maps from Y to Z and the space of based continuous maps from Y to Z. Let X be a C^∞ manifold. Let $\mathfrak{G} = C^\infty(X, \mathbb{C}^*)$. For $p \in X$, let

$$\mathfrak{G}_p = \{f \in \mathfrak{G} | f(p) = 1\}.$$

Before we begin with stating and proving the propositions in this section, we note that they are mostly applications of the fact that $\mathbb{C}^*$ is a $K(\pi, 1)$ space and is covered by $\mathbb{C}$ a contractible space. The space of continuous based maps into a $K(\pi, 1)$ has been classically studied first by Thom and then by many others. Lemma 3 is a bit more specific to our situation and is not an immediate application of the theory of $K(\pi, 1)$ spaces.

Proposition 12. *with the above notation,*

1. $\mathfrak{G}_p$ *is weak homotopy equivalent to* $H^1(X, \mathbb{Z})$.
2. $\mathfrak{G}$ *is weak homotopy equivalent to* $H^1(X, \mathbb{Z}) \times \mathbb{C}^*$.

Proof. The space $\mathbb{C}^*$ is a $K(\mathbb{Z}, 1)$ and by the Proposition labelled Thom [4] in (16) $C_*(X, \mathbb{C}^*)$ is homotopy equivalent to $H^1(X, \mathbb{Z})$ (i.e. each of its components is contractible and the set of components is in natural bijection with $H^1(X, \mathbb{Z})$). By Theorem 1.5 of (17), $C_*(X, \mathbb{C}^*)$ is weak homotopy equivalent to $C^\infty(X, \mathbb{C}^*)$. This establishes (1). To establish (2) we note that $\mathfrak{G}$ is homeomorphic to $\mathfrak{G}_p \times \mathbb{C}^*$, indeed an explicit homeomorphism is given $(f, \alpha) \in \mathfrak{G}_p \times \mathbb{C}^* \mapsto \alpha f \in \mathfrak{G}$.

□

Proposition 13. *Let* $D = \{z \in \mathbb{C} | |z| \leq 1\}$. *Let*

$$S = \{f \in C^{\infty}(D - \{0\}, \mathbb{C}^*) | f(z) = 1 \text{ for } z \in S^1\}.$$

Then S *is contractible.*

Proof. Let

$$S' = \{f \in C^{\infty}(D - \{0\}, \mathbb{C}) | f(z) = 0 \text{ for } z \in S^1\}.$$

Then S' deformation retracts to the constant function $f(z) = 0$ by a straight line homotopy.

Now note that there is a covering map $\exp : \mathbb{C} \to \mathbb{C}^*$ such that $exp(0) = 1$. We note that if $f \in S$, f is nullhomotopic. Hence by the lifting criterion, there is a unique lift $\tilde{f} : D - \{0\} \to \mathbb{C}$ such that $\tilde{f}(1) = 0$ and $\exp \circ \tilde{f} = f$. We know have a homeomorphism between S and S' $\phi : S \to S'$be defined by $\phi(f) = \tilde{f}$.

This implies that S is contractible. □

Lemma 2. *Let* D *be the closed unit disk in* $\mathbb{C}$. *Let*

$$F = \{f \in C^{\infty}(D - \{0\}, \mathbb{C}^*) | \; f|_{\partial D} = 1\}.$$

Let

$$\tilde{F} = \{f \in C^{\infty}(D - \{0\}, \mathbb{C}^*) | \; f \text{ is nullhomotopic and } f(1) = 1\}.$$

Then $\tilde{F}$ *deformation retracts to the point* f_0, *where* $f_0(x) = 1$ *for all* x. *Furthermore, the deformation retraction preserves the subset* F.

Proof. Given $f \in \tilde{F}$ we can lift it to a unique map $\tilde{f} : D - \{0\} \to \mathbb{C}$, such that $\exp \circ \tilde{f} = f$ and $\tilde{(}f)(1) = 0$. The straight line homotopy in $\mathbb{C}$ defines a homotopy between $\tilde{f}$ and the constant function. This in turn defines a homotopy h between f and f_0. This gives us our deformation retraction. It is easy to see that this preserves F. □

Lemma 3. *Recall that $\mathfrak{G} = C^\infty(X, \mathbb{C}^*)$. Let $n \geq 1$. Let $\{a_1, \ldots, a_n\} \in \mathrm{UConf}_n X$. Then,*

1. *There is a free action of $\mathfrak{G}$ (as a group under multiplication) on $\pi^{-1}(\{a_1, \ldots, a_n\})$.*

2. *The quotient $\pi^{-1}(\{a_1, \ldots, a_n\})/\mathfrak{G}$ is contractible.*

Proof. We define our action as follows: if $s \in \pi^{-1}(\{a_1, \ldots, a_n\})$ and $g \in \mathfrak{G}$, define $g.s(x) = g(x)s(x)$. If $g.s(x) = 0$ then $s(x) = 0$ as $g(x) \neq 0$ for all $x \in X$. Furthermore, if $g.s = s$ then $g(x) = 1$ for $x \in X - \{a_1, \ldots, a_n\}$ and since $X - \{a_1, \ldots a_n\}$ is dense, $g(x) = 1$ for all $x \in X$. This concludes the proof of (1).

Let D_i be small disks surrounding the points a_i. Let $G_i = \{f \in C^\infty(D_i, \mathbb{C}^*) | f|_{\partial D_i} = 1\}$. We can identify each G_i with the set of based maps from S^2 to $\mathbb{C}^*$. Since $\pi_1(S^2) = 0$, any based map $f : S^2 \to \mathbb{C}^*$ lifts to a unique map $\tilde{f} : S^2 \to \mathbb{C}$. Hence G_i is homeomorphic to the space of based maps from S^2 to $\mathbb{C}$ and since $\mathbb{C}$ is contractible, G_i is contractible.

Let $F_i = \{f \in C^\infty(D_i - a_i, \mathbb{C}^*) : f|_{\partial D_i} = 1\}$. By Proposition 13 F_i is contractible. Let

$$\tilde{F}_i = \{f \in C(D_i - a_i, \mathbb{C}^*) | f|_{\partial D_i} \text{ is nullhomotopic}\}.$$

Let

$$\tilde{G}_i = \{f \in C(D_i, \mathbb{C}^*)\}.$$

Since D_i is contractible, the space $\tilde{G}_i \simeq \mathbb{C}^*$ and the quotient $\tilde{F}_i/\tilde{G}_i$ is contractible (this is analogous to the proof of Proposition 13). There is an inclusion map $i : F_i/G_i \hookrightarrow \tilde{F}_i/\tilde{G}_i$ which is a homotopy equivalence as both spaces are contractible. By Lemma 2,there is a map $j : \tilde{F}_i/\tilde{G}_i \to F_i/G_i$ satisfying the following properties.

1. There exists a homotopy $h : \tilde{F}_i \times [0,1] \to \tilde{F}_i$ such that $h(f,1) = f$, $h(f,0) = j(f)$ and $h(f,t)|_{U_i} = f|_{U_i}$.

2. For all $f \in F_i$, $h(f,t) \in F_i$.

Let $A = \pi^{-1}(\{a_1, \ldots a_n\})/\mathfrak{G}$. Fix an $s_0 \in pi^{-1}(\{a_1, \ldots a_n\})$. Let $\phi : A \to \prod_{i=1}^n \tilde{F}_i/\tilde{G}_i$ be defined as follows:

$$\phi(s) = (s/s_0|_{D_1 - a_1}, \ldots s/s_0|_{D_n - a_n}).$$

We claim that ϕ is a homotopy equivalence. To prove this we first define $\psi : \prod_{i=1}^n F_i/G_i \to A$ as follows:.

$$\psi(f_1, \ldots f_n)(x) = \begin{cases} f_i(x)s_0(x) & \text{if } x \in D_i \\ s_0(x) & \text{otherwise.} \end{cases}$$

It is easy to see that $\psi \circ j \circ \phi \simeq \mathrm{Id}$ (the homotopy h mentioned above can be seen to define such a homotopy). Since $\prod_{i=1}^n F_i/G_i$ is contractible this implies (2). $\square$

Remark: The action of $\mathfrak{G}$ on $\pi^{-1}(\{a_1, \ldots, a_n\})$ is in fact *not* transitive for any value of $n \geq 1$. The following example will illustrate this fact. Let D be the closed unit disk in $\mathbb{C}$ which we identify with $\mathbb{R}^2$. Let $f : D \to \mathbb{C}$ be defined as $f(x, y) = (x, y)$. Let $g : g(x, y) = (2x, y)$. Let $E \to \mathbb{P}^1$ be the unique degree 1 line bundle on $\mathbb{P}^1$. Let $\phi : E|_D \to D \times \mathbb{C}$ be a trivialisation. Let $\bar{f}, \bar{g}$ be C^∞ sections on $\mathbb{P}^1$ of E such that for $(x, y) \in D$, $\phi(\bar{f}(x, y)) = ((x, y), f(x, y))$ and $\phi(\bar{g}(x, y)) = ((x, y), g(x, y))$ (it follows from a standard obstruction theoretic argument that there indeed exist such $\bar{f}$ and $\bar{g}$). If there exists $\bar{h} \in C^\infty(X, \mathbb{C}^*)$ such that $\bar{h}\bar{f} = \bar{g}$, then $\bar{h}(0,0) = \lim_{(x,y)\to(0,0)} \frac{g(x,y)}{f(x,y)}$. But this limit does not exist and hence $\bar{f}$ and $\bar{g}$ are not in the same $\mathfrak{G}$ orbit.

Corollary 2. *Let $n \geq 0$. Let X be a smooth projective curve and $\mathcal{L}$ a line bundle on it of degree n. Then $\mathbb{U}(\mathcal{L})$ is a $K(\pi, 1)$.*

Proof. There is a fibration

$$\begin{tikzcd} \pi^{-1}(\mathbb{U}(\mathcal{L})) \arrow[r] & \mathbb{U}(\mathcal{L}) \arrow[d, "\pi"] \\ & \mathrm{UConf}_n X \end{tikzcd}$$

Lemma 3 implies that$\pi^{-1}(\mathbb{U}(\mathcal{L})) \simeq \mathfrak{G}$. The space $\mathfrak{G}$ is a $K(\pi, 1)$ by Proposition 12. Since $\mathbb{U}(\mathcal{L})$ is the total space in a fibration with both base and fibre $K(\pi, 1)$ spaces is itself a $K(\pi, 1)$.

□

3.3 Abel-Jacobi

In this section we will try to understand the space $U(\mathcal{L})$. Our method to understand the topology of $U(\mathcal{L})$ is by making it a subspace of a space U_n^{alg} which we shall construct.

We would like to remind the reader that to give a complex line bundle $\mathcal{L}$, a holomorphic structure h is equivalent to giving a Dolbeault operator $\partial_h : \Gamma(L) \to \Omega^{0,1} \otimes \Gamma(L)$. More details on Dolbeault operators and holomorphic structures may be found in Ch.3 of (14). Let $\mathscr{H}_n$ be the space of holomorphic structures on $\mathcal{L}$. The group $\mathfrak{G}_p$ acts on $\mathscr{H}_n$ with trivial stabilizers. The quotient $\mathscr{H}_n/\mathfrak{G}_p$ is naturally isomorphic to $\mathrm{Pic}_n X$.

Let

$$\mathscr{U}_n = \{(s, h) \in \mathbb{U}(\mathcal{L}) \times \mathscr{H}_n | s \text{ is a algebraic section } of \mathcal{L} \text{ with respect to } h\}.$$

Note that the groups $\mathfrak{G}$ and $\mathfrak{G}_p$ act on this space $\mathscr{U}_n$.

Let $U_n^{alg} := \mathscr{U}_n/\mathfrak{G}_p$. There is a surjective map $\pi : \mathscr{U}_n \to \mathscr{H}_n$ defined by $\pi(s, h) = h$. Since π is equivariant with respect to the action of $\mathfrak{G}_p$, it descends to a surjection

$$\mathbb{A} : U_n^{alg} = \mathscr{U}_n/\mathfrak{G}_p \to \mathscr{H}_n/\mathfrak{G}_p = \mathrm{Pic}_n X.$$

We observe that for $\mathcal{L} \in \mathrm{Pic}_n X$, we have the equality $\mathbb{A}^{-1}(\mathcal{L}) = U(\mathcal{L})$. This map $\mathbb{A}$ can be seen as a section level version of the Abel-Jacobi map. We now wish to understand the topology of U_n^{alg}.

Proposition 14. *Let $n \geq 1$.*

1. *U_n^{alg} is a $K(\pi, 1)$.*

2. *There is a short exact sequence*

$$1 \to \mathbb{Z} \to \pi_1(U_n^{alg}) \to Br_n(X) \to 1.$$

Proof. There is a fibration $\pi : U_n^{alg} :\to \mathrm{UConf}_n X$ defined by

$$\pi(s, h) = \{a \in X | s(a) = 0\}.$$

If $\mathbf{a} = \{a_1 \dots a_n\} \in \mathrm{UConf}_n X$, then $\pi^{-1}(\mathbf{a}) \cong \mathbb{C}^*$, as algebraic sections of a line bundle are uniquely identified with their zeroes up to a scalar. Since $\mathbb{C}^*$ and $\mathrm{UConf}_n X$ are $K(\pi, 1)$ spaces, so is U_n^{alg}. □

3.3.1 An alternative definition of U_n^{alg}

In this subsection we will give an alternative definition of U_n^{alg}. This will not be used in the rest of the paper.

Let X be a smooth projective curve of genus g. Let $n > g$. Let $Sym^n X$ be the nth symmetric power of X. Let $\mathcal{P}$ denote the Poincare line bundle on $X \times \mathrm{Pic}_n X$, this is the unique line bundle on $X \times \mathrm{Pic}_n(X)$ such that $\mathcal{P}|_{X \times \{\mathcal{L}\}} = \mathcal{L}$ and $\mathcal{P}|_{\{p\} \times \mathrm{Pic}_n X}$. Let $\pi : X \times \mathrm{Pic}_n X \to \mathrm{Pic}_n X$ denote the projection. The pushforward $\pi_*(\mathcal{P})$ defines a vector bundle E on $\mathrm{Pic}_n X$, sometimes called the Picard bundle. Let $E_0 \subseteq E$ denote the zero section. We may identify $Sym^n X$ with $E - E_0/\mathbb{C}^*$, i.e. $Sym^n X$ is the projective space bundle associated to the vector bundle E. Let $\rho : E - E_0 \to Sym^n X$ denote the projection map. We then define U_n^{alg} to be $\rho^{-1}(\mathrm{UConf}_n X)$.

Let us emphasize that $E - E_0 \to Sym^n X$ is not a trivial $\mathbb{C}^*$ bundle. Indeed after restricting to a fibre of the projection $Sym^n X \to \mathrm{Pic}_n X$ the bundle ρ restricts to the bundle $\mathbb{C}^{n-g+1} - \{0\} \to \mathbb{P}^{n-g}$ which is classically known to be non trivial. While it is possible that the bundle $U_n^{alg} \to \mathrm{UConf}_n X$ is a trivial $\mathbb{C}^*$ bundle, we are unable to determine whether this is the case.

3.4 Comparing different fibres

To understand $U(\mathcal{L})$ we will analyze the map $\mathbb{A} : U_n^{alg} \to \mathrm{Pic}_n X$. We will prove that the map $\mathbb{A}$ is similar to a homology fibration. More precisely, we have the following.

Theorem 10. *Let $n \geq 0$. Let X be a smooth projective complex algebraic curve of genus g, $\mathcal{L}$ a line bundle of degree n on X. Let $2k \leq n - g$. Let $\mathbb{A}$ be the map defined in Setion 3. Let $W \subseteq Pic_n(X)$ be a small contractible neighbourhood of $\mathcal{L}$ homeomorphic to a ball. Let*

$$i : \mathbb{A}^{-1}(\mathcal{L}) \hookrightarrow \mathbb{A}^{-1}(W)$$

be the inclusion map. Then

$$i^* : H^k(\mathbb{A}^{-1}(W); \mathbb{Z}) \to H^k(\mathbb{A}^{-1}(\mathcal{L}); \mathbb{Z})$$

is an isomorphism.

Before embarking on the proof of Theorem 10 we will need to set up some machinery.

There is a vector bundle $\pi : H^0(X, W) \to W$ defined as follows. Let

$$H^0(X, W) = \{(s, \mathcal{L}) | \mathcal{L} \in W, s \in H^0(X, \mathcal{L})\}.$$

Then $\mathbb{A}^{-1}(W)$ is an open subset of $H^0(X, W)$. The topology of the complement $\Sigma_W = H^0(X, W) - \mathbb{A}^{-1}(W)$ will be important for us to understand. It is immediate that $\Sigma_W =$

$\{(f, \mathcal{L}) | \mathcal{L} \in W, f \in \Sigma(\mathcal{L})\}$, since for any $\mathcal{L} \in W$ $\mathbb{A}^{-1}(\mathcal{L})$ consists of all sections of $\mathcal{L}$ with regular zeroes.

We will create a relative stratification of $\Sigma^{-1}(W)$. This is similar to the stratification in (12). Let

$$\Sigma_{\mathcal{L}}^{\geq k} = \{f \in \Sigma_{\mathcal{L}} | \ |\mathrm{Sing}(f)| \geq k\}.$$

Let $N = \frac{d-g}{2}$. We stratify $\Sigma_{\mathcal{L}}$, the complement of $\mathbb{A}^{-1}(\mathcal{L})$ in $H^0(X, \mathcal{L})$ by

$$\Sigma_{\mathcal{L}}^{\geq k} = \mathbb{P}\{f \in V | f \text{ has atleast } k \text{ distinct singular zeroes}\},$$

for $k \leq N$. So $\Sigma_{\mathcal{L}}^{\geq 1} \supset \Sigma_{\mathcal{L}}^{\geq 2} \supset \ldots$.

Now we construct a cubical space C that will be involved in understanding $\Sigma(\mathcal{L})$. Let $N = \frac{d-1}{2}$. Let I be a subset of $\{1, \ldots, N-1\}$. Let $I = \{i_1, \ldots, i_k\}$ let

$$C_I := \{(f, x_1, \ldots, x_k) | f \in \Sigma(\mathcal{L}), x_j \in \mathrm{UConf}_{i_j}(X) \ x_1 \subseteq x_2, \ldots, \subseteq x_k \subseteq \text{ Singular zeroes of } f\}.$$

We define

$$C_{I \cup \{N\}} := \{(f, x_1, \ldots, x_k) \in C_I | f \in \bar{\Sigma}^{\geq N}\}.$$

If $I \subseteq J$ then we have a natural map from $C_J \to C_I$ defined by restricting p. This gives C the structure of a cubical space over the set $\{1, \ldots, N\}$. We can take the geometric realization of C denoted by $|C|$. Then there is a map $\rho : |C| \to \Sigma(\mathcal{L})$, induced by the forgetful maps $C_I \to \Sigma(\mathcal{L})$.

$|C|$ is topologized in a non-standard way. The topology we give is analogous to the topology on $\mathscr{X}$ in (12). The primary reason we give $|C|$ this topology is to make ρ proper. For $k < N$, there is an inclusion

$$i : \mathrm{UConf}_k(X) \to Gr(h^0(X, \mathcal{L}) - 2k, H^0(X, \mathcal{L})).$$

We define $L_k(\mathcal{L})$ to be the Zariski closure of the image. We will omit the $\mathcal{L}$ in our notation if there is only one line bundle that we are discussing. There is a relation, $<$ on the collection of all L_k, defined by $\lambda_1 < \lambda_2$ if as subspaces of $H^0(X, \mathcal{L})$, $\lambda_2 \subseteq \lambda_1$. Note that this extends the relation $\supset$ on the collection of all $\mathrm{UConf}_k(X)$. Let $I = \{i_1, \ldots, i_k\} \subseteq \{1, \ldots, N-1\}$. Let $\bar{C}_I = \{(f, \lambda_1, \ldots, \lambda_k) | \lambda_j \in L_{i_j}, \lambda_1 < \lambda_2 \cdots < \lambda_k < \mathrm{Sing}(f)\}$. Let $\bar{C}_{I \cup N} = \{f, \lambda_i, \ldots \lambda_k \in C_I | f \in \Sigma^{\geq N}_{\mathcal{L}}\}$. Then $\bar{C}$ forms a cubical space in the same way that C does.

Take the geometric resolution $|\bar{C}|$. Now we will construct a map $|\bar{C}| \to |C|$ that is the identity on $|C| \subseteq |\bar{C}|$. This will exhibit $|C|$ as a quotient of $|\bar{C}|$ and we will give it the quotient topology. Given $\lambda \in L_k$, we can define $\mathrm{supp}(\lambda) \in \mathrm{UConf}_{n(\lambda)}(X)$ by $\mathrm{supp}(\lambda) = \cap_{f \in \lambda} \mathrm{Sing}(f)$.

This defines a map $\mathrm{supp} : |\bar{C}| \to |C|$ given by

$$(f, \lambda_i, s_i) \in \bar{C}_I \times \Delta_I \mapsto (f, \mathrm{supp}(\lambda_i), s'_j) \in C_I \times \Delta_J.$$

Here $J = \{\mathrm{supp}(\lambda_i)\}$ and $s'_j = \sum_{n(\lambda_i)=j} s_i$.

The maps $\bar{C}_I \to \Sigma_{\mathcal{L}}$ are proper and hence so is the induced map $|\bar{C}| \to \Sigma_{\mathcal{L}}$.

Proposition 15. *The map $\rho : |C| \to \Sigma$ is a proper homotopy equivalence.*

Proof. In our setting having proper contractible fibres implies that the map ρ is a proper homotopy equivalence, this follows by combining Theorem 1.1 and Theorem 1.2 of (9). We note that if $f \notin \bar{\Sigma}^{\geq N}$ the fibre $\rho^{-1}(f)$ is the simplex with vertices labelled by the singular points. If $f \in \bar{\Sigma}^{\geq N}$ then the fibre is a cone. We have given $|C|$ the quotient topology.The map ρ is a factor in the composite $|\bar{C}| \to |C| \to \Sigma_{\mathcal{L}}$ which is proper, hence ρ itself is proper. $\square$

Now as in any geometric realization, $|C|$ is filtered by

$$F_n = \mathrm{im}(\coprod_{|I| \leq n} C_I \times \Delta_k).$$

The F_n form an increasing filtration of $|C|$, i.e. $F_1 \subseteq F_2 \ldots F_n \subseteq F_{n+1} \subseteq \ldots$ and $\cup_{n=1}^{\infty} F_n = |C|$.

We define

$$B_n = \{f \in \Sigma_{\mathcal{L}} | \; f \text{ has at least } n \text{ singular zeroes}\}.$$

Proposition 16. *Let $n < N$. Let Δ_n° be the interior of an n simplex. The space $F_n - F_{n-1}$ is a Δ_n°- bundle over the space B_n. This is in turn a vector bundle over* $\mathrm{UConf}_n(X)$.

Proof. The fact that B_n is the total space of a vector bundle over $\mathrm{UConf}_n(X)$ follows from Riemann-Roch, the fibres are all vector subspaces of $H^0(X, \mathcal{L})$ of codimension exactly $2(n+1)$. A point in $F_n - F_{n-1}$ is a pair $((f, x_0, \ldots x_n), s_0, \ldots s_n)$ where the s_i are the simplicial coordinates. We have a map $\pi : F_n - F_{n-1} \to B_n$ defined by

$$(f, (x_1, \ldots, x_n), (s_0, \ldots, s_n)) \mapsto (f, (x_1, \ldots, x_n)).$$

The map π expresses $F_n - F_{n-1}$ as a Δ_n° bundle over B_n. □

Let $e_d = \dim_{\mathbb{C}}(H^0(X, \mathcal{L}))$. We define a local coefficient system on $\mathrm{UConf}_n X$, denoted by $\pm\mathbb{Z}$, in the following way. There is a homomorphism $\pi_1(\mathrm{UConf}_n X) \to S_n$ associated to the covering $\mathrm{PConf}_n X \to \mathrm{UConf}_k X$. We compose this with the sign homomorphism $S_n \to \pm 1 \cong GL_1(\mathbb{Z})$ to obtain our local system on $\mathrm{UConf}_n X$.

Proposition 17. *Let $d \geq 1$. Let $n < N$.*

$$\bar{H}_*(F_{n+1} - F_n; \mathbb{Z}) = \bar{H}_{*-(e_d-(2(n+1)))}(\mathrm{UConf}_{n+1}(X); \pm\mathbb{Z}).$$

Proof: By Proposition 16 the space $F_k - F_{k-1}$ is a bundle over $\mathrm{UConf}_k(X)$. This fact implies that

$$\bar{H}_*(F_k - F_{k-1}) \cong H_{*-(k+2e_d-2(n+1)(k+1))}(\mathrm{UConf}_k(X), \mathbb{Z}(\sigma)).$$

Here $\mathbb{Z}(\sigma)$ is the local sytem obtained by the action of $\pi_1(\mathrm{UConf}_k(X))$ on the fibres $\bar{H}_k(\Delta_k^\circ, \mathbb{Z})$ where in this case Δ_k^{circ} is the open k simplex corresponding to the fibres of the map $F_k - F_{k-1} \to B_k$. But one observes that the action of $\pi_1(\mathrm{UConf}_k(X))$ on this open simplex is by permutation of the vertices which implies that $\mathbb{Z}(\sigma) = \pm\mathbb{Z}$.

As with any filtered space, there is a spectral sequence with $E_1^{p,q} = \bar{H}_{p+q}(F_p - F_{p-1}; \mathbb{Z})$ converging to $\bar{H}_*(|C|; \mathbb{Z})$. Now by Proposition 17 we know what $E_1^{p,q}$ is for $p < N$.

Proposition 18. *$\bar{H}_*(|C| - F_N; \mathbb{Z}) \cong \bar{H}_*(|C|; \mathbb{Z})$ for $* \geq 2e_d - N$.*

Proof. We first will try to bound $\bar{H}_*(F_N; \mathbb{Z})$ and then use the long exact sequence of the pair $(F_N, |C|)$. The space F_N is built out of locally closed subspaces

$$\phi_k = \{(f, x_1, \dots, x_k), p | \; f \in \Sigma^{\geq N}, p \in \Delta^k, x_i \text{ are singular zeroes of } f\}.$$

There exists a surjection $\pi : \phi_k \to \mathrm{UConf}_k X$. The map π is a fibre bundle with fibres $\mathbb{C}^{e_d - 2k} \times \Delta_\circ^k$. The space $\mathrm{UConf}_k X$ has complex dimension k. Therefore $\bar{H}_j(\phi_k; \mathbb{Z}) = 0$ if $j \geq 2e_d - N \geq 2(e_d - 2k) + 3k$. This implies $\bar{H}_j(F_N) = 0$ if $j \geq 2e_d - N$. The long exact sequence of the pair $(F_N, |C|)$ implies that $\bar{H}_*(Y - F_N; \mathbb{Z}) \cong \bar{H}_*(|C|; \mathbb{Z})$ for $* \geq 2e_d - N$. $\square$

Now this simplicial resolution of Σ gives an associated spectral sequence for its Borel Moore homology with

$$E_1^{p,q} = \bar{H}_{p+q}(F_p - F_{p-1}) = H_{p-(e_d-(2)(q+1))}(\mathrm{UConf}_{p+1}(X), \pm\mathbb{Z})$$

, for $p < N$. Also, $E_1^{N,q} = 0$ if $q \geq 2e_d - N$.

We will now construct a cubical space $\mathcal{C}$ which will be involved in understanding $\Sigma(W)$. Our construction of $\mathcal{C}$ will be similar to that of C. Let $N = \frac{d-g}{2}$. Let I be a subset of $\{1, \dots, N-1\}$. Say $I = \{i_1, \dots, i_k\}$ let

$$\mathcal{C}_I := \{(f, x_1, \dots, x_k) | f \in \Sigma(W), x_j \in \mathrm{UConf}_{i_j}(X), x_1 \subseteq \dots x_k \subseteq \text{ Singular zeroes of } f\}.$$

We define

$$\mathcal{C}_{I\cup\{N\}} := \{(f, x_1, \ldots x_k) \in \mathcal{C}_I|, f \in \bar{\Sigma}^{\geq N}(W)\}.$$

If $I \subseteq J$ then we have a natural forgetful map from $\mathcal{C}_J \to \mathcal{C}_I$. This gives $\mathcal{C}.$ the structure of a cubical space over the set $\{1, \ldots, N\}$. We can take the geometric realization of $\mathcal{C}.$ denoted by $|\mathcal{C}|$. Then there is a map $\rho : |\mathcal{C}| \to \Sigma(W)$, induced by the forgetful maps $\mathcal{C}_I \to \Sigma(W)$.

We again topologise $|\mathcal{C}|$ in a nonstandard way, this is entirely analogous to the way we topologise $|C|$, so we will be brief in our description of it. We construct a bigger cubical space $\bar{\mathcal{C}}$ such that for $I = \{i_1, \ldots, i_k\} \subseteq \{1, \ldots, N-1\}$, $\bar{\mathcal{C}}_I = \{(f, x_1 \ldots, x_k)|f \in \Sigma(\mathcal{L}), \mathcal{L} \in W, x_j \in L_{i_j}(\mathcal{L}), x_1 < x_2 \ldots x_k < \mathrm{Sing}(f)\}$. We define $\mathcal{C}_I \cup \{N\}$ analogously. We then form the geometric realisation, $|\bar{\mathcal{C}}|$ and note that there is a surjective map $|\bar{\mathcal{C}}| \to |\mathcal{C}|$ and we give $|\mathcal{C}|$ the quotient topology with respect to this map.

Proposition 19. *$|\mathcal{C}|$ is proper homotopy equivalent to Σ_W.*

Proof. This is analogous to the proof of Proposition 15. □

$\mathscr{Y}$ also has an ascending filtration, $\mathbb{F}_n$ and this filtration gives us a spectral sequence for $\bar{H}_*(\mathscr{Y};\mathbb{Z})$.

Proposition 20. *$\bar{H}_*(|\mathcal{C}| - \mathbb{F}_N;\mathbb{Z}) \cong \bar{H}_*(\mathcal{C};\mathbb{Z})$ for $* \geq 2e_d - N$.*

Proof. This is analogous to the proof of Proposition 18. □

So there is a spectral sequence with

$$\mathbb{E}_1^{p,q} = \bar{H}_{p+q}(\mathcal{C}_{0,\ldots q}) = H_{p-(e_d-(2)(q+1))+2g}(\mathrm{UConf}_{p+1}(X), \pm\mathbb{Z})$$

for $p < N$. Finally we have the main theorem of this section.

Theorem 11. *The map $\mathbb{A}^{-1}(\mathcal{L}) \to \mathbb{A}^{-1}(W)$ induces an isomorphism $H_*(\mathbb{A}^{-1}(\mathcal{L});\mathbb{Z}) \to H_*(\mathbb{A}^{-1}(W);\mathbb{Z})$ for $* < N = \frac{d-g}{2}$*

Proof. This proof involves studying Alexander duality of $A^{-1}(\mathcal{L})$ inside $H^0(X,\mathcal{L})$ and $A^{-1}(W)$ inside $H^0(X,W)$ (the space $H^0(X,W)$ is a topological vector bundle over W and so is at least homeomorphic to an affine space).

Now we use the fact that under Alexander duality, intersection of Borel-Moore cycles turns into pullback in cohomology, namely the map

$$H^*(A^{-1}(W)) \to H^*(A^{-1}(\mathcal{L})),$$

is Alexander dual to the map

$$f : \bar{H}_{*+2g}(\Sigma_W) \to \bar{H}_*(\Sigma_{\mathcal{L}})$$

given by intersecting cycles with $\Sigma_{\mathcal{L}}$, i.e. $f(\sigma) = \sigma \cap \Sigma_{\mathcal{L}}$.

To understand this map in Borel-Moore homology, we turn to our spectral sequences for $\bar{H}_*(\Sigma_{\mathcal{L}};\mathbb{Z})$ and $\bar{H}_*(\Sigma_W;\mathbb{Z})$. Since our stratification of Σ_W is fiberwise we get a map of spectral sequences between the two spectral sequences. We have a map $\mathbb{E}^1_{p,q+2g} \to E^1_{p,q}$. It will suffice to show that this map is an isomorphism for $p < N$. For $p < N$, this map is given by the map

$$\phi : \bar{H}_{p+q+2g}(\mathbb{F}_p - \mathbb{F}_{p-1},\mathbb{Z}) \to \bar{H}_{p+q}(F_p - F_{p-1},\mathbb{Z})$$

induced by intersecting cycles. However, we have a diagram of fiber bundles as follows:

$$\begin{tikzcd}
K \arrow[r] \arrow[dr] & K \times W \arrow[drr] & & \\
& F_p - F_{p-1} \arrow[rr] \arrow[d] & & \mathbb{F}_p - \mathbb{F}_{p-1} \arrow[d] \\
& \mathrm{UConf}_p(X) \arrow[rr, equal] & & \mathrm{UConf}_p X
\end{tikzcd}$$

Using this diagram and the fact that the intersection map $\bar{H}_{*+2g}(K \times W) \to \bar{H}_*(K)$

is an isomorphism (as W is homeomorphic to $\mathbb{C}^g$), ϕ is an isomorphism. This implies the theorem. $\square$

3.5 Homology fibration theorem

Let us recall the usual homology fibration theorem:

Theorem 12 ((15))**.** *Let $f : X \to Y$ be a map. Let $Hf^{-1}(y)$ be the homotopy fibre of f. Suppose $f^{-1}(y) \hookrightarrow f^{-1}(U)$ is a homology equivalence for sufficiently small U open then $f^{-1}(y) \to Hf^{-1}(y)$ is a homology equivalence.*

For this paper we need the following analogous theorem:

Theorem 13. *Let $n \geq 0$. Let $f : X \to Y$ be a map such that for all $y \in Y$ there exists an open neighbourhood U such that the inclusion $j : f^{-1}(y) \hookrightarrow f^{-1}(U)$ induces an isomorphism $j_* : H_k(f^{-1}(y); \mathbb{Z}) \to H_k(f^{-1}(U); \mathbb{Z})$ for all $k \leq n$.*

Then the natural map $i : f^{-1}(y) \to Hf^{-1}(y)$ induces an isomorphism $i_* : H_k(f^{-1}(y); \mathbb{Z}) \to H_k(Hf^{-1}(y); \mathbb{Z})$ for $k \leq n$.

Proof. This follows from the proof of Proposition 5 (which is the same as Theorem 12 of this paper) in (15). $\square$

This implies the following theorem.

Theorem 14. *For $* \leq N$,*

$$H^*(A^{-1}(\mathcal{L}); \mathbb{Z}) \cong H^*(HA^{-1}(\mathcal{L}); \mathbb{Z}) \cong H^*(\pi; \mathbb{Z})$$

where π is the previously described subgroup of the extended surface braid group.

Proof. First we note that $HA^{-1}(\mathcal{L})$ is a $K(\pi, 1)$, where π is our previously described subgroup of the extended surface braid group. The map $\mathbb{A} : U_n^{alg} \to Pic_n X$ has $K(G, 1)$s for both source and target and the induced map at the level of π_1 is given by $\tilde{Br}_nX \to Br_nX \to \mathbb{Z}^{2g}$, Since $H\mathbb{A}^{-1}\mathcal{L}$ is the homotopy fibre it is also a $K(G,1)$ with fundamental group equal to

$$K_n := \ker(\tilde{Br}_nX \to \mathbb{Z}^{2g}).$$

By Theorem 13 and Theorem 11 $H^*(A^{-1}(\mathcal{L});\mathbb{Z}) \cong H^*(H\mathbb{A}^{-1}(\mathcal{L});\mathbb{Z})$ for $* \leq N$. □

3.6 Relating $U(\mathcal{L})$ and $\mathbb{U}(\mathcal{L})$

In this section we will relate the two spaces $U(\mathcal{L})$ and $\mathbb{U}(\mathcal{L})$. We begin first with the following result.

Proposition 21. *Let X be an algebraic curve. Let $\mathcal{L}$ be a line bundle on X of degree n. Let $p \in X$ be a point. Then, $\mathbb{U}(\mathcal{L})/\mathfrak{G}_p \simeq U_n^{alg}$.*

Proof. Let

$$S_{\{x_1,\ldots,x_n\}} = \{f \in \mathbb{U}(\mathcal{L}) :\ x_i \text{ are regular zeroes of } f\}/\mathfrak{G}_p.$$

We have a diagram of fiber bundles as follows.

$$\begin{tikzcd}
\mathbb{C}^* \arrow[r, "f"] \arrow[dr] & S_{\{x_1,\ldots,x_n\}} \arrow[dr] & \\
& U_n^{alg} \arrow[r] \arrow[d] & \mathbb{U}(\mathcal{L})/\mathfrak{G}_p \arrow[d] \\
& \mathrm{UConf}_nX \arrow[r, equal] & \mathrm{UConf}_nX
\end{tikzcd}$$

By considering the long exact sequences of homotopy groups associated to these fiber bundles, it suffices to prove that the map $f : \mathbb{C}^* \to S$ is a homotopy equivalence.

To prove this it suffices to prove that $S/\mathbb{C}^*$ is contractible where $\mathbb{C}^*$ is acting on S by

$z \cdot f(x) = z(f(x))$. But

$$S/\mathbb{C}^* = \{f \in \mathbb{U}(\mathcal{L}) |\ x_i \text{ are regular zeroes of } f\}/\mathfrak{G}.$$

This is contractible by (2) of Proposition 3. $\square$

We will need the following lemma to obtain our results.

Lemma 4. *Let X be an algebraic curve. Let $n \geq 1$. Let $\mathbf{a} = \{a_1, \ldots a_n\} \in \mathrm{UConf}_n X$. Let $\alpha \in \pi_1(\mathrm{UConf}_n X, \mathbf{a})$. Suppose $\mathbb{A}_*(\alpha) \neq 0 \in H_1(X; \mathbb{Z})$. Let P_α denote the point-pushing map associated to α. Let $c_i \in H_1(X - \{a_1 \ldots a_n\}; \mathbb{Z})$ be the puncture classes. Then there exists a class*

$$\gamma \in H_1(X - \{a_1 \ldots a_n\}; \mathbb{Z})$$

such that $\gamma \cap \mathbb{A}_*(\alpha) = 1$ and

$$P_\alpha(\gamma) - \gamma = \sum m_i c_i,$$

where $\sum m_i \neq 0$.

This can be deduced from a computation by Bena Tshishiku. For a refererence see (19).

Theorem 15. *The natural map $\rho : \mathbb{U}(\mathcal{L}) \to Pic_n X$ is nullhomotopic.*

Proof. As $\mathrm{Pic}_n X$ is a $K(\pi, 1)$ it suffices to prove that

$$\rho_* : \pi_1(\mathbb{U}(\mathcal{L})) \to \pi_1(\mathrm{Pic}_n X) \cong H_1(X; \mathbb{Z})$$

is trivial. Let $\pi : \mathbb{U}(\mathcal{L}) \to \mathrm{UConf}_n X$ be defined by $\pi(s) = \{a \in X | s(a) = 0\}$. Let $\mathbb{A} : \mathrm{UConf}_n X \to \mathrm{Pic}_n X$ be the Abel-Jacobi map. Let $\mathbf{a} = \{a_1, \ldots a_n\} \in \mathrm{UConf}_n X$. Let $\alpha \in \pi_1(\mathrm{UConf}_n X, \mathbf{a})$. Suppose $\mathbb{A}^*(\alpha) \neq 0 \in H_1(X; \mathbb{Z})$. It suffices to show that $\alpha \notin \rho_*(\pi_1(\mathbb{U}(\mathcal{L})))$. This is because the map ρ factors through $\mathbb{A}$.

Let $Mod(X - \{a_1, \ldots, a_n\})$ be the mapping class group of the punctured surface $X - \{a_1, \ldots, a_n\}$. Associated to α there exists a point pushing map $P_\alpha \in Mod(X - \{a_1, \ldots a_n\})$. Let $c_i \in H_1(X - \{a_1 \ldots a_n\}; \mathbb{Z})$ be the puncture classes.

Then by Lemma 4there exists a class $[\gamma] \in H_1(X - \{a_1 \ldots a_n\})$ such that

$$P_{\alpha*}([\gamma]) - [\gamma] = \sum m_i c_i,$$

where $m_i \in \mathbb{Z}$ satisfying $\sum m_1 \neq 0$.

Let $f \in \mathbb{U}(\mathcal{L})$ be such that $\pi(f) = \mathbf{a}$. Suppose for the sake of contradiction that $\alpha \in \mathrm{im}(\pi_1(\mathbb{U}(\mathcal{L})), f)$ with $\alpha \neq 0$. Then there exists a loop in $\mathbb{U}(\mathcal{L})$, which we will call F_α such that $\pi(F_\alpha) = \alpha$, i.e. F_α is a lift of α.

Now for $s \in (0, 1)$, let P^s_α be the point-pushing homeomorphism along the path $\alpha|_{[0,s]}$. It is a well-defined element of

$$\pi_0(\mathrm{Homeo}((X, \alpha(0)), (X, \alpha(s)))).$$

Now $P^t_\alpha(f)$ is a lift of α as a path (not a loop) to $\mathbb{U}(\mathcal{L})$. Since the map π is a fibration, any two paths that are lifts of α must have endpoints in the same component of $\pi^{-1}(\bar{a})$. This would imply that f (the endpoint of F_α) and $P_{\alpha*}(f)$ (the endpoint of $P^t_\alpha(f)$) would be in the same path component of $\pi^{-1}(\mathbf{a})$. If that were so, then we would have

$$\int_\gamma \frac{P_\alpha(f)}{|P_\alpha(f)|} - \int_\gamma \frac{f}{|f|} = 0.$$

However we will now show that this is not the case.

We'd like to remind the reader that $\int_{c_i} f/|f| = 1$. This is because the section f has a zero of index 1 at each of the a_is. Then we know that

$$\int_\gamma \frac{P_\alpha(f)}{|P_\alpha(f)|} - \int_\gamma \frac{f}{|f|} = \int_{P_{\alpha*}\gamma} \frac{f}{|f|} - \int_\gamma \frac{f}{|f|}$$

$$= \sum_i m_i \int_{c_i} f/|f| = \sum m_i \neq 0.$$

This proves that any α that lifts is forced to be trivial, which completes the proof. □

Proposition 22. *let $n \geq 1$, $p \in X$. There exists a homotopy equivalence $f : Pic_n X \to B\mathfrak{G}_p$ that makes the following diagram commute upto homotopy:*

$$\begin{tikzcd}
 & \mathbb{U}(\mathcal{L}) \arrow[d, "\pi"] \\
U_n^{alg} \arrow[r, "i"] \arrow[d, "\mathbb{A}"] & \mathbb{U}(\mathcal{L})/\mathfrak{G}_p \arrow[d, "g"] \\
Pic_n X \arrow[r, "f"] & B\mathfrak{G}_p
\end{tikzcd}$$

Here the map g is the classifying map for the fibration $\mathbb{U}(\mathcal{L}) \to \mathbb{U}(\mathcal{L})/\mathfrak{G}_p$.

Proof. The situation is a follows: $\pi : \mathbb{U}(\mathcal{L}) \to \mathbb{U}(\mathcal{L})/\mathfrak{G}_p$ is a principal $\mathfrak{G}_p$ bundle. Since $\mathfrak{G}_p \simeq \mathbb{Z}^2 g$, we have an associated principal $\mathbb{Z}^{2g}$ bundle $E \to \mathbb{U}(\mathcal{L})/\mathfrak{G}_p$, where

$E := \mathbb{U}(\mathcal{L})/(f_1 \sim f_2$ if $\pi(f_1) = \pi(f_2)$ and $f_1, f_2$are in the same path component of $\pi^{-1}(\pi(f_1)))$.

Equivalently, if $(\mathfrak{G}_p)_0$ is the identity component of $\mathfrak{G}_p$, $E = \mathbb{U}(\mathcal{L})/(\mathfrak{G}_p)_0$.

The quotient map $p : \mathbb{U}(\mathcal{L}) \to E$ is naturally a homotopy equivalence as the group $(\mathfrak{G}_p)_0$ is contractible. We then have a diagram as follows:

$$\begin{tikzcd}
\mathbb{U}(\mathcal{L}) \arrow[r, "p"] \arrow[d] & E \arrow[d] \\
\mathbb{U}(\mathcal{L})/\mathfrak{G}_p \arrow[r, "="] & \mathbb{U}(\mathcal{L})/\mathfrak{G}_p
\end{tikzcd}$$

It suffices to prove that the natural map

$$\alpha : \mathbb{U}(\mathcal{L})/\mathfrak{G}_p \to \mathrm{Pic}_n X$$

satisfies the classifying space property for the fibration $E \to \mathbb{U}(\mathcal{L})/\mathfrak{G}_p$. However by Proposition 15 the composite map $E \to Pic_n X$ is nullhomotopic and we can lift it to $\mathrm{Pi}\tilde{\mathrm{c}}_n X$, the universal cover of $\mathrm{Pic}_n X$. So we have a commutative diagram as follows.

$$\begin{tikzcd}
E \arrow[r] \arrow[d] & \mathrm{Pi}\tilde{\mathrm{c}}_n X \arrow[d] \\
\mathbb{U}(\mathcal{L})/\mathfrak{G}_p \arrow[r, "\alpha"] & \mathrm{Pic}_n X
\end{tikzcd}$$

Hence α is a classifying map and we are done. □

Theorem 16. $\mathbb{U}(\mathcal{L})$ *is homotopy equivalent to* $H\mathbb{A}^{-1}(\mathcal{L})$*, the homotopy fibre of* $\mathbb{A}$*.*

Proof. By Propositions 15 and 22 there is a diagram as follows:

$$\begin{tikzcd}
H\mathbb{A}^{-1}(\mathcal{L}) \arrow[d] & \mathbb{U}(\mathcal{L}) \arrow[d] \\
U_n^{alg} \arrow[r, "i"] \arrow[d, "\mathbb{A}"] & \mathbb{U}(\mathcal{L})/\mathfrak{G}_p \arrow[d] \\
\mathrm{Pic}_n X \arrow[r, "f"] & B\mathfrak{G}_p
\end{tikzcd}$$

Since the composite map $H\mathbb{A}^{-1}(\mathcal{L}) \to B\mathfrak{G}_p$ is null homotopic, by the properties of a fibre sequence we have a map $g : H\mathbb{A}^1(\mathcal{L}) \to \mathbb{U}(\mathcal{L})$ that commutes with the maps of the diagram. Since the maps i and f are homotopy equivalences, so is g. □

Now we can finally prove the theorems in the introduction of this paper.

Proof of Theorem 7. By Theorem 16 $\mathbb{U}(\mathcal{L}) \simeq H\mathbb{A}^{-1}(\mathcal{L})$. So it suffices to prove that $H\mathbb{A}^{-1}(\mathcal{L})$ is a $K(\pi, 1)$ for K_n. However this follows from Theorem 14. □

Proof of Theorem 6. By Theorem 13 and Theorem 11 the map $f : U(\mathcal{L}) \to H\mathbb{A}^{-1}(\mathcal{L})$ induces an isomorphism $H^*(U(\mathcal{L})) \cong H^*(H\mathbb{A}^{-1}(\mathcal{L}))$ for $* < n - (2g))$. But by Theorem 16

$H\mathbb{A}^{-1}(\mathcal{L}) \simeq \mathbb{U}(\mathcal{L})$ and it is easy to see that

$$i^* : H^*(\mathbb{U}(\mathcal{L}); \mathbb{Z}) \to H^*(U(\mathcal{L}); \mathbb{Z})$$

is the composition

$$H^*(\mathbb{U}(\mathcal{L}); \mathbb{Z}) \cong H^*(H\mathbb{A}^{-1}(\mathcal{L}); \mathbb{Z}) \to^{f^*} H^*(U(\mathcal{L}); \mathbb{Z}).$$

□

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