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KENNETH C. GRIFFIN DEPARTMENT OF ECONOMICS

BY
OĞUZ BAYRAKTAR

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To my family.

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ABSTRACT

This thesis consists of two essays on education market design. Chapter 1, **"Education Market in the Presence of Peer Effects: Theory and Evidence From South Korea"** is a joint work with Sam Il Myoung Hwang. In this paper we evaluate alternative market designs in the presence of peer effect considerations in school choice. Schools can differentiate by peer composition among other characteristics. Different rules public schools are subject to compared to elite or private schools have been heavily debated because of their effects on student distribution. In many countries former group admit students earlier than the latter, which we call Sequential Admissions (SA). Furthermore, the former group can use academic criteria in admissions unlike the latter. We study the effect of these admission aspects on welfare and distribution. Our analysis has important distinguishing features. First, it includes private schools as well as public schools, whereas previous studies on admission rules focused on centralized public school allocation. Second, students can have preferences for peer composition at schools. This is an important factor that can determine sorting behavior which is not carefully studied before. First we theoretically study equilibria of the admission games complicated by the peer effects. Then we estimate a structural model using detailed high school applications data from Seoul to run counterfactual simulations. We show that SA can approximate centralized admission schemes well. This is important since complete centralization is known to increase welfare but hard to implement in many cases. Moreover analysis of SA is informative on the controversial "exploding offers" in labor markets. Regarding admission criteria, we show that use of academic criteria in a subset of schools increases the desirability of these schools. The reason is that high performing students want to coordinate to study together and

academic screening provides this. This suggest that, school choice is also a coordination game, not just an object allocation problem.

Chapter 2, **"The Role of Outside Options under Boston Mechanism"** studies ex ante welfare from centralized public school allocation for students who cannot go to private schools when there are others who have such option. I show that when a private school is preferred to only the last option, students prefer Boston Mechanism (BM) to Deferred Acceptance (DA). Analyzing the case of desirable private schools in a model with three public schools, I demonstrate that students who are marginal in the decision of which school to report as top choice are better off under DA compared to BM; whereas inframarginal students are better off under BM. I show that a distribution of preferences with full support guarantees the existence of students who are better off under BM compared to DA. Assuming uniform distribution of preferences allows one to find the fraction of students who are better off under BM compared to DA. Analysis of the effect of private school entry on the welfare of students who cannot access private schools demonstrates that under mild conditions either there are students who are strictly better off after the entry or welfare of none of the students change.

Chapter 1

EDUCATION MARKET DESIGN IN THE PRESENCE OF PEER EFFECTS: THEORY AND EVIDENCE FROM SOUTH KOREA

Co-authors for This Chapter: Sam Il Myoung Hwang (University of British Columbia)

1.1 Introduction

In school districts across the world, parents and their children are now faced with many options beyond their catchment area public schools. These encompass the following: alternatives like public schools from other neighborhoods, private schools, charter schools, as well as elite public schools that utilize academic screening in admissions. In this context, "tuition" or prices tend to play a limited role in clearing the market for elementary as well as secondary education - unlike in product markets.[1] Thus, admission rules are imperative for education markets to clear. However, there are apparent disparities across school types when it comes to admissions, such as the use of academic screening, the timing of admissions, and tuition.

These disparities in admissions might affect student distribution across schools, and student distribution matters because of potential peer effects. The existing body of economics of education literature has furnished empirical evidence, according to which the attributes of students' peers impact both their academic achievement as well as non-academic outcomes, including smoking behavior (Sacerdote, 2011). Hence, peer effect considerations can also affect students' school choice. This paper assesses the effect of disparities in admissions across school types on student welfare and distribution when students have peer effect considerations. Our paper is the first to study this for any aspect of admissions other than tuition while dealing with the potential endogeneity of school compositions at the same time.

Market design studies have demonstrated the welfare gains arising from centralizing the admission processes in comparison to uncoordinated admissions (Kapor, Karnani, and Neilson, 2022; Abdulkadiroğlu, Agarwal, and Pathak, 2017). Each student submits a rank-ordered list of schools to a centralized clearinghouse in centralized admissions, based on which the allocation is determined. However, very often, centralized admissions are not found to encompass schools from multiple sectors, and in most cases, only include traditional public schools.

Instead, sequentiality in admissions is observed in a number of school districts. In this process, a subset of schools first decides on their admissions, following which the rest of the schools participate in the next stage along with the remaining students. In general, private or elite (exam) public schools are the early-moving ones. Admissions to the initial group are followed by centralized admissions to public schools. To the best of our knowledge, sequentiality of admissions has been

[1] At most times, public schools are impervious to charges; and tuition often does not obliterate the excess demand within private schools. This could be attributed to government regulations out of concern for the affordability of private education. These regulations may be inclusive of tuition caps or vouchers for students with low income. In addition, private schools may have incentives to be over-demanded, which helps them select students based on characteristics apart from income. Student ability is an example to this.

observed in high school matches in countries such as Turkey, Sweden[2], South Korea and Canada[3] where public school admissions follow private school admissions. In a similar manner, admissions to charter schools in New York City (NYC) have been shown to commence after determining public school assignments.[4] Furthermore, in cities like NYC, Boston, and Chicago, public school admissions followed admissions to elite (exam) schools for several years.

In the illustrations concerning the aforementioned countries, sequentiality is accompanied by a commitment structure. Admissions to private schools end prior to the commencement of admissions to public schools in these places. Furthermore, in the first three examples, students with a seat in private schools cannot attend public schools' centralized admission.[5] Such a commitment structure intuitively reduces or eradicates frictions from wait lists since students are prohibited from simultaneously holding more than one offer from various sectors (Andersson, Dur, Ertemel, and Kesten, 2018).[6][7] Other settings also observe such a commitment structure. Among elite colleges and universities in the US, early decisions are getting increasingly popular.[8] At the same time, early applications and exploding offers are also ubiquitous phenomena in certain job markets.[9]

In South Korea, the issue of Sequential Admissions (SA) has remained contentious. Those who oppose such an admission process contend that it tends to enhance the sorting of high-performing students to private schools while adversely impacting public schools' classroom environments.

[2]See Andersson, Dur, Ertemel, and Kesten (2018) for the implementation in Turkey and Sweden.

[3]In Canada, popular private schools tend to have deadlines of admission decisions before public school choice.

[4]In 2022, admissions are announced in March for non-charter public schools, while lotteries for charter schools are held in April.

[5]In Canada, some private schools demand a part of the tuition to finalize the enrollment, which explains why the commitment structure is non-different from other examples.

[6]As a case point Andersson, Dur, Ertemel, and Kesten (2018) posit that Turkey's switch to such a commitment structure took place following a summer with several rounds of admissions that were unable to fill vacant seats across public schools

[7]However, students must decide whether or not they would enroll before learning their match in the market, which renders it a complicated decision (Andersson, Dur, Ertemel, and Kesten, 2018).

[8]Early decision applications imply that students are required to enroll if they are admitted to their desired college. The share of students admitted through an early application or early decision is above 50% for many colleges and universities (Murphy, 2022)

[9]This year, prospective Ph.D. graduates are being hired by many economic consulting firms through two tracks. In one track, students will need to decide before the fly-outs begin in the academic job market. In job markets, exploding offers have similar characteristics. Following an offer, job candidates are asked to take a decision shortly before they come to know the decision of another potential employer. As a case in point, this was seen in a job market for economists with Ph.D. as well as in the Federal judicial law clerk market (Avery, Jolls, Posner, and Roth, 2007).

Due to these discussions, SA had to face a ban in 2019 despite vehement opposition from private schools and students themselves. In this study, SA is compared to Deferred Acceptance (DA) mechanism (Gale and Shapley, 1962), as the latter is a well-known and commonly used school choice mechanism around the world. It is necessary to understand this comparison with respect to welfare and student distribution. Authorities might find it easier to implement SA; which can also transpire on its own, as observed in early college decisions and the exploding offers across job markets. Therefore, comparing it with the benchmark of full centralization has significance.

In many school districts, only some of the schools can use academic screening in admissions. Schools with academic screening tend to be particularly popular among high-performing students, and have intense competition for entrance. The academic screening policy of private and public schools in large cities both in the US and elsewhere globally has been subject to heated debates.[10] In 2022, Latino and Black students have received only five and three percent of the offers, respectively from the eight elite high schools in NYC. In contrast, White and Asian students took the majority of the offers from these schools. Some people are in favor of scrapping exams. According to their contention exams create impediments for Black and Latino students, hence exacerbate the segregation in these schools.[11][12] In Seoul, starting with the 2015 academic year, private schools must allow everyone to enter their admission lotteries. Before that, attendance to their lotteries was subject to academic criteria.[13] Currently, the discussion is about whether or not the elite schools must be less segregated, at the expense of some of the high-performing students within the standardized tests. In this study, the interaction between academic screening and students' preferences for peers is investigated so that the debate can be approached from a different perspective. Our

[10]In large US cities such as Boston, Chicago, and Philadelphia, elite public schools admit students via academic screening. On the contrary, the majority of public schools make use of lottery-based admissions. Also, private schools in US are allowed to select students based on academic screening.

[11]See Bocanegra (2022)

[12]Former mayor of NYC, Bill de Blasio proposed a bill to change the admissions in these elite schools, which face strong opposition from Asian families. The bill never made it to floor vote in the end (Shapiro and Wang, 2019). Bill de Blasio also proposed removing gifted and talented programs for NYC elementary schools (Shapiro, 2021). However, the current mayor Eric Adams reversed this policy, and announced the increase of gifted and talented programs, making it available in every district in NYC (Closson, 2022).

[13]Before 2015, only students in the top 50% with respect to grades within their middle schools were allowed to enter their lottery.

analysis involves comparing the distribution of students under two distinct scenarios: i) when a set of schools does utilize academic screening; and ii) when academic screening is not used by any school.

To carry out a comparison between SA and DA, we first employ a simple model capable of elucidating why the choice between both admission schemes does make a difference. Students have heterogeneous preferences over endogenous student composition in schools under this model. Under both admission rules, we characterize the price range for which there exist an equilibrium with sorting of high-performing students to the private school. We see that the sorting of high-performers to private schools tend to increase with price for both admission rules. According to our results, even in a simple model, the comparison between DA and SA is ambiguous. We show that it is predicated on the private school's tuition. When the tuition is higher (lower), there is more sorting of over-performing students to the private school under SA (DA). Where the tuition levels observed in the real-world stand is an empirical question, which shall be addressed during the empirical part of this paper.

To empirically answer our research questions, we utilize data on high school applications and enrollment (from Seoul) for the 2010-2012 period, which is inclusive of students' rank-ordered lists on public schools. Additionally, the total number of applicants to each private school, as well as the number of students enrolled in private schools are observed from about 70% of middle schools. This dataset is combined with additional data on family income and also questions relating to private school preferences. We employ and estimate a structural model of student preference and sequential decision in the two consecutive markets. Our specification enables students to exercise preferences over student composition in his/her cohort at schools.

Our empirical task is confronted with several challenges. To begin with, Seoul's centralized mechanism is not impervious to strategic decisions, so the truthfulness of submitted preferences cannot be guaranteed. In order to address this challenge, we follow approach of Hwang (2017), which makes a minimal assumption about the ability of students to calculate the probabilities of entry to schools. This strategy yields students' truthful comparisons over many pairs of schools, which we can use in preference parameter estimation.

The second problem pertains to the potential endogeneity of student composition within schools. It is possible for some school-level unobserved variables to exert a heterogeneous effect on the preferences of various kinds of students. Nevertheless, this also impacts schools' student compositions, which then results in omitted variable bias. To resolve the problem, we exploit the 100% increase in the number of private schools during our data period. This alteration, coupled with the aforementioned academic criteria of private schools, helps us develop an instrument for student composition within public schools. The idea behind the instrument is that public schools closer to the these private schools are potentially more affected in terms of incoming student achievement compared to other public schools. Moreover, neighborhood fixed effects in our model eliminate the potential selection problem regarding the instrument. We have found that higher-performing students have stronger preferences for studying with high-performing students. We also demonstrate that heterogeneity regarding preferences over student composition would be over-estimated if endogeneity is not considered.

The identification of preferences for tuition denotes the third challenge. The government sets a price cap for private schools, and most of the schools set their prices close to this cap. Due to the small variation in tuition, usual (Berry, Levinsohn, and Pakes, 1995) (BLP) instruments are inadequate for identifying private school preferences and preferences for tuition separately. To solve this problem, we leverage the presence of affirmative action tracks across private schools and exploit the price difference for affirmative and general track students as an instrument. This gives rise to a concern that students in both tracks could have varying preferences concerning private schools. We deal with this concern by including Regression Discontinuity (RD) moments in structural estimation via indirect inference, as in Larroucau and Rios (2020). To construct these moments, income threshold for affirmative action track eligibility is exploited. This approach ensures that the mean utility difference between the two tracks is caused by price variations as opposed to varying preferences.

The fourth challenge is addressing capacity constraints in demand estimation. Studies using rank-ordered preference lists of students do not depend upon the schools' enrollment shares to identify each school's mean utility. However, this data is unavailable, in many cases. It can be troublesome to use enrollment shares to identify mean utilities within a BLP setting. Such an approach might end up underestimating the mean utilities for at-capacity and over-demanded schools.[14] Individual rank-ordered lists over private schools are not observed, which is why we adopt the BLP approach to estimate these schools' mean utilities. We are not required to rely on enrollment shares as we can observe the number of applicants in all schools. We can back out mean utilities for private schools from application shares as students can apply to only one private school; moreover, we can incorporate the effect of entry probabilities on students' decisions, which is relevant to our setting.

The interdependence between the first step of private school admissions and the second step of public school admissions constitutes the fifth challenge. In the first step, a student needs to consider his expected utility from public school admissions under his optimal play, conditional on participating in the second step. Identifying the optimal strategy entails going through hundreds of thousands of lotteries over public schools for each student, as these lotteries hinge on students' attributes. Recalculating these during the parameter search routine would be computationally challenging, if not infeasible. Thus, we utilize a two-step estimation procedure whereby we initially estimate the preference parameters concerning public schools. Next, we use these estimates in the second phase of BLP estimation that recovers parameters only related to private schools. For this reason, the expected utilities are calculated only once.

The sixth problem involves getting the equilibrium of SA and DA simulated in our counterfactuals. Simulating these mechanisms once would not suffice as students' preferences rely on the composition of peers in schools. Hence, we iterate over the best responses to until the convergence of peer compositions and school lottery cutoffs to arrive at the equilibrium of preference submission game.

According to our results, at the tuition levels observed in the data, the average student welfare is

[14]Similarly, it may result in overestimation of mean utilities for schools with low demand if students rejected from other schools due to capacity constraints enroll in them.

\$27 to \$51 higher under totally centralized DA in comparison to SA, depending on the implementation of SA. In addition, under both regimes, student distributions are very similar. For authorities, SA is potentially easier to implement than total centralization; it can also take place on its own at times. Thus, if complete centralization is costly or impossible to achieve due to legal frictions, the SA's coordination structure may be a worthwhile alternative to consider or keep if it has occurred naturally.[15] Many matching markets around the world tend to be decentralized. This result also suggests that a centralized market could emerge as a viable approximation to decentralized markets that have small frictions. This is significant as it can be easier to simulate a centralized match when investigating the effect of policies. Our counterfactual simulations under different tuition levels for private schools qualitatively align with the predictions of our theoretical analysis. However, the difference between the mechanisms is again quantitatively small.

We simulate the effect of a switch from lottery-based admission to academic screening for a set of schools to comprehend the impact of academic screening. According to our result, there is a significant increase in the shares of high-performers in such schools. The direct impact of the admission criteria is part of this. We also exhibit an indirect impact of altering preferences over schools. This change is caused by students preferences for peers. This indirect effect comprises 38% of the total effect. The results of this study suggest that academically strong students want to coordinate with each other to be in the same school. Permitting academic screening within a subset of schools offers this coordination without any alteration in the schools' intrinsic qualities. This implies that academic screening makes a school popular among high-performers regardless of school identity. Therefore, who gains admission into a particular well-known school does not seem to be a fruitful discussion. According to our research, more importance needs to be paid to understanding whether high-performers should be allowed to coordinate; and, if so, what the strength of that coordination could be.

[15] In our setting admission processes go through in a couple of weeks. If the time between steps is very large, it could lead to other problems, e.g. inefficiencies due to missing information. Moreover, cost of strategic considerations are not addressed by this paper.

1.1.1 Related Literature

Our paper is related to several branches of the literature. First one is the literature applying tools from matching literature to school choice which starts with Abdulkadiroğlu and Sönmez (2003).[16] The papers in this branch of literature take capacity constraints in schools seriously and investigate the trade offs between different admission schemes in terms of efficiency, strategy-proofness and stability.[17] Our paper contributes to this literature as we compare strategy-proof DA and SA which has non-straightforward incentives. There are other recent papers in the literature about parallel or sequential school admission systems. Andersson, Dur, Ertemel, and Kesten (2018) is the first paper examining SA and shows that such mechanism is not straightforward in terms of incentives.[18] Andersson, Dur, Ertemel, and Kesten (2018) finds a way to minimize the wasted seats when different type of schools run their own centralized match sequentially if there is no commitment structure as in SA. Dur and Kesten (2019) shows that sequential admissions systems perform worse than one step centralized admission systems. Ekmekci and Yenmez (2019) show theoretically that schools have incentives to stay outside of the centralized match in a setting without strong commitment structure; and also students are better off under the unified enrollment.[19] An empirical paper closely related to ours is Kapor, Karnani, and Neilson (2022). They show that expansion of the centralized market, i.e. addition of off platform options to the centralized

[16] They compare DA (Gale and Shapley, 1962), Top Trading Cycles (TTC) and Boston Mechanism (BM)

[17] Strategy-proof has been an important concept in the literature. Some papers in this literature advocated use of a strategy-proof mechanism such as DA (Dubins and Freedman, 1981) instead of manipulable BM especially with the motivation of leveling the play field between strategic and sophisticated agents (Abdulkadiroğlu, Pathak, Roth, and Sönmez, 2006; Pathak and Sönmez, 2008, 2013; Ergin and Sönmez, 2006). Later theoretical studies demonstrate that a manipulable mechanism like BM can have favorable efficiency properties (Miralles, 2009; Abdulkadiroğlu, Che, and Yasuda, 2011; Troyan, 2012) compared to DA. This comparison between manipulable and strategy-proof mechanisms has been studied by empirical papers later as well (Agarwal and Somaini, 2018; Calsamiglia, Fu, and Güell, 2020; Kapor, Neilson, and Zimmerman, 2020; Hwang, 2017; He, 2016).

[18] Avery and Levin (2010) is an earlier paper that is also related to sequential admissions. This paper theoretically investigates early action/decision in colleges, which has strategic aspects similar to SA. In Avery and Levin (2010) signaling aspect of early action/decision is emphasized and possible since the early admitting colleges also admit through regular channel. In our setting there is no such aspect since private schools admit once.

[19] Other recent theoretical papers show welfare improvement of unified enrollment compared to parallel admission systems, where in the latter students can get multiple offers from different type of schools simultaneously (Doğan and Yenmez, 2019; Manjunath and Turhan, 2016; Turhan, 2019); and analyze the role of additional admission stages in unified enrollment settings (Haeringer and Iehlé, 2021; Doğan and Yenmez, 2023). See Abdulkadiroglu and Andersson (2022) for a more detailed review of this recent literature.

matching platform leads to improvement in welfare and student outcomes. One of the differences of our setting from the work on sequential or parallel admissions is that, priorities at public schools are determined via lottery numbers instead of being known strict priorities beforehand (e.g., due to exam score). Another related empirical paper is Abdulkadiroğlu, Agarwal, and Pathak (2017) which shows that going from decentralized market in New York public school admissions to fully centralized public school match improves welfare significantly. An important difference from the aforementioned previous studies is that we allow students to have preferences for peers when making school choice. Also, we contribute this literature by comparing a sequential admissions scheme with a completely centralized benchmark.

Our paper is related to literature which examines the effect of school choice on school competition and student sorting behavior. School choice is argued to be increasing the competition among schools (Hoxby, 2000, 2003). However, Rothstein (2006) argues that choice may not be enough to motivate schools to increase their effectiveness if peers in a school are important for parents. Modeling school quality as a function of peer quality leads to sorting and stratification of students according to their income and academic success in equilibrium (Epple and Romano, 1998; Epple, Figlio, and Romano, 2004; Epple, Romano, and Sieg, 2006). Empirical findings indicates that private school voucher programs may be a factor increasing sorting and stratification (Urquiola, 2005; Hsieh and Urquiola, 2006; Altonji, Huang, and Taber, 2015). Allende (2019) empirically shows that students/families prefer to be with peers that have higher socio-economic status while dealing with the endogeneity of peer composition measures. These papers abstract away from the details of admissions by assuming no capacity constraints or in their model housing market clearing implies school market clearing. In contrast, we take capacity constraints seriously and model the admission process explicitly.

There are other studies in the market design literature which includes both capacity constraints and peer dependent preferences over schools. Echenique and Yenmez (2007), Pycia (2012), Dur and Wiseman (2019), Leshno (2022) and Pycia and Yenmez (2023) theoretically study existence of stability in matching markets with externalities such as peer effects.[20] Avery and Pathak (2021) studies the distributional consequences of school choice by allowing for feedback effect from the residential market. Our theoretical model builds upon Calsamiglia, Martínez-Mora, Miralles, et al. (2015) which compares the degree of sorting between public schools under DA versus Boston Mechanism. Our contribution to theoretical literature is to show how private school prices can interact with mechanism choice between SA and DA in determining sorting behavior between private and public schools. Moreover, our empirical analysis verifies the existence of peer preferences in school choice which is the assumption of this theoretical literature. In the empirical literature Abdulkadiroğlu, Pathak, Schellenberg, and Walters (2020) show that after controlling for peer quality, preferences of parents are not related to school effectiveness or match quality. Laverde et al. (2022) and Idoux (2021) investigate how admission rules affect student distribution. These studies are on public schools only, however we look at the effect of admission rules in the context of both public and private schools, hence price also plays a role in our analysis. Also, in these papers endogeneity of peer composition measures were not taken into consideration, whereas we use an instrument to identify students' preferences for peers.

This paper is also related to the empirical literature on school choice that performs structural estimation of preferences.[21] Agarwal and Somaini (2018), Calsamiglia, Fu, and Güell (2020) and Hwang (2017) are the closest papers to the this work; in their and our settings, students may report strategically since the centralized mechanisms in these context are manipulable. We use the same administrative school choice data set as Hwang (2017) and we use the same strategy for identifying truthful school comparisons regarding public schools. Differently from these papers, we also use an instrumental variable to deal with endogeneity of student composition, as in Hastings, Hortaçsu, and Syverson (2017). Our estimation also includes estimation of preferences using aggregate market shares since preference lists is not available for private schools, so this work is also related

[20]Sasaki and Toda (1996), Mumcu and İsmail Sağlam (2010) and Fisher and Hafalir (2016) considers matching with externalities in one to one matching markets. Baccara, İmrohoroğlu, Wilson, and Yariv (2012) quantifies the effect of network externalities in a matching setting

[21] See Agarwal and Somaini (2020) for a survey of the techniques that are used in estimating the preferences of students from rank ordered choice lists.

to Berry, Levinsohn, and Pakes (1995); Neilson (2021); Allende (2019). Our contribution is to modify the BLP framework to take the capacity constraints in school choice into consideration.

Abdulkadiroğlu, Angrist, and Pathak (2014); Cullen, Jacob, and Levitt (2006) investigates the effect of elite schools or popular schools on student achievement. They cannot find the expected effects. In our work, we show that for a school to be popular among high achievers, it is enough to coordinate these students into this school, independent of schools' value added.

Our paper is also related to papers on private autonomous schools of Seoul. Park (2021) compares academic performance of students attending autonomous versus regular schools using a value added model. Shin (2018) studies the existence of spillover effects across grades in high schools exploiting the start of the autonomous school policy.

Section 2 presents our model and theoretical results. Section 3 describes our empirical context, data and important features of Seoul high school market. Section 4 describes our structural empirical model and identification strategy. Section 5 describes our estimation strategy. Section 6 illustrates the estimates of the structural model parameters. Section 7 describes our counterfactual analysis. Section 8 concludes.

1.2 Theoretical Analysis

This section illustrates the existence of equilibria under which high performing students sort into private school under both DA and SA. Also, using additional functional and type distribution assumptions we compare the degree of sorting to private schools under both mechanisms. Our model in this section is simple yet informative about the existence and comparison of sorting equilibria under both mechanisms.

1.2.1 Model Primitives

Primitives of our model builds upon Calsamiglia, Martínez-Mora, Miralles, et al. (2015). There are 3 schools s_1, s_2, s_3; $s_1 \in PR$ and $s_2, s_3 \in PU$ where PR and PU are the sets of private and public schools, respectively; and $S = PR \cup PU$ is the set of schools. There exists continuum of students with a mass of 1. School j has capacity q_j, we assume s_1 and s_2 have $1/3$ measure of capacity; whereas s_3 has $2/3$ measure capacity. Type (ability) of student is denoted by $t \in [\underline{t}, \bar{t}] \equiv T$ and μ is the measure associated with this type space. The cumulative distribution function associated with μ, $F(t) := \mu([\underline{t}, t])$ is continuous and strictly increasing in t. Each school j has a measure μ_j over type distribution which will be determined endogenously in an equilibrium. The quality of student body in school j, ω_j is defined as the expected type in school j, i.e. $\omega_j := \mathbb{E}_{\mu_j}(t)$. s_1 has an exogenous price $p > 0$, whereas s_2 and s_3 are free. Student of type t, gets utility $h(\omega, t)$ from the quality ω of the student body. We assume that $h(\omega, t)$ is strictly increasing in both arguments and $h_{\omega t} > 0$, i.e. school quality and student type are complements in student's preferences. This function might be motivated by social preferences students may have over the student body in the school or their production of human capital in a school with ω quality. Student type t's payoff from attending school j is:

$$v_j(t) = h(\omega_j, t) - 1\{j = 1\}p - 1\{j = 3\}\Delta$$

where $\Delta > 0$. So, a student's payoff from school j depends on his utility from the quality of the student body in school j; the price student pays; and an exogenous disutility student get from being assigned to s_3 for possible reasons such as being far or having undesirable neighborhood characteristics or bad management of the school, which are assumed to be independent from the endogenous student measure μ_{s_3}.

1.2.2 Assignment Rules and Entry Probabilities

Each student is assigned independent fair lottery numbers for each school $l := (l_1, l_2, l_3)$ for $l \in L = [0, 1]^3$ from uniform distribution; l_j indicates the lottery number at school j. Formally, there is a uniform measure Υ on L such that, for any measurable subset $\bar{L} \subseteq L$, a student's probability of getting lottery numbers in $\bar{L}$ is $\Upsilon(\bar{L})$. Conditional distribution of types that draw lottery numbers in

$\bar{L}$ is μ for any $\bar{L} \subseteq L$, i.e. lottery distribution is independent of type distribution. We assume that smaller lottery draw has precedence.

In our model we restrict attention to mechanisms which receive rank ordered lists over three schools from students. Formally a rank ordered list is a permutation $\pi := (\pi_1, \pi_2, \pi_3) \in S^3$ over schools such that $\pi_k \neq \pi_l$ if $k \neq l$ and π_1, π_2, π_3 is the ordering of the schools from top to bottom. The set of all such orderings is Π. Given π, *relative order* of public schools is a permutation $\pi^{pu} \in PU^2$ over s_2 and s_3 such that $\pi(j) < \pi(j')$ if and only if $\pi^{pu}(j) < \pi^{pu}(j')$ for $j, j' \in PU$ and $j \neq j'$.

Next we define Deferred Acceptance and Sequential Admissions mechanisms. In the introduction we have mentioned that Sequential Admissions in our context has a commitment structure. A student cannot attend the centralized match of public schools, if the student holds an assignment of private school. This commitment structure allows us to model SA in a convenient way.

Definition 1.1. Deferred Acceptance (DA):

Step 0: Students submit preferences over schools then each get a lottery number over each school.

Step 1: Each student applies to his first choice. Each school admits applicants tentatively up to its capacity according to lottery order, rejects the rest.

Step k: Students rejected at Step k-1, apply to their kth choice. Schools consider new applicants along with tentatively accepted applicants at k-1; admit according to lottery order up to capacity.

The process converges when the set of students that are rejected has zero measure or all students run out of schools in their list.

Definition 1.2. Sequential Admissions (SA):

Step 0: Students submit preferences over schools then each get a lottery number over each of the schools.

Step 1: Students who ranked the private school as top choice are ordered according to their lottery number at the private school. Private school admits students up to its capacity according to the lottery order and reject others. This is the final assignment for students who are admitted.

Step 2: Students who are not admitted are assigned to public schools through DA. The *relative order* of the public schools in the submitted lists is used for DA.

The process converges when the set of students that are rejected has zero measure or all students run out of schools in their list.

In reality a student who is assigned a private school might drop his private school assignment after being assigned to it and then can join the public school match. The way we model SA does not allow a student to drop his or her private school assignment. In other words the assumption we make when modeling SA is that a student would not apply to a private school that later he would less prefer than attending the centralized allocation of public schools.

Under both mechanisms $m \in \{DA, SA\}$ a pure strategy profile is a function $\sigma : T \to \Pi$ from type space to set of possible rank ordered lists. Defining the equilibrium of this game is challenging. Students' preferences will depend on student quality at schools $\{\omega_j\}_{j=1}^{3}$. This implies that the lists they submit also depends on these qualities. However, these qualities are also determined by the rank ordered lists submitted by students, and the processes of the mechanisms. To define an equilibrium, for a given strategy profile, students' payoffs from all the strategies they can play must be well defined. 1.1 is helpful for showing this is the case.

Lemma 1.1. *(Abdulkadiroglu et al. 2015) Given a strategy profile σ, there exist unique lottery cutoffs $c_j^{DA}(\sigma)$ at each school j such that it indicates the largest lottery number among the lottery numbers of admitted students under DA.*

Existence of unique lottery cutoffs c_j^{SA} for SA immediately follows from Lemma 1.[22] Importantly, the result implies that for a given profile of rank-ordered lists, the largest lottery number that is admitted to each school is the same whatever is the realization of lottery numbers.[23] This will be the key to define payoffs for a given strategy profile.

[22] Given a strategy profile σ, the lottery cutoff $c_1^{SA}(\sigma)$ is equal to the minimum of the following two: (i) 1/3 (capacity of s_1) divided by measure of students who ranked s_1 as top choice; (ii) 1.

[23] Also, this result means that for any profile of rank-ordered lists students may submit, after the lottery numbers are drawn, the assignments are unique.

We define $\sigma^{-1}(\pi)$ as the set of types that submit ranking π under strategy profile σ, i.e., $\sigma^{-1}(\pi) := \{t \in T : \sigma(t) = \pi\}$. Since lottery numbers are drawn independently from types, given cutoffs $c_j^m(\sigma)$ and π, for each student type $t \in \sigma^{-1}(\pi)$ the probability of entry to each school j under each mechanism $m \in \{SA, DA\}$ can be calculated easily. We show the calculation of these probabilities in Appendix A.3.

1.2.3 School Qualities and Equilibria

When the measure of students who submit π is greater than 0, i.e., $\mu(\sigma^{-1}(\pi)) > 0$, by Law of Large Numbers the probabilities of entry is equal to the shares of the types $t \in \sigma^{-1}(\pi)$ admitted to each school. Then school qualities can be calculated using the measure over types that submit π under the strategy profile σ, which we denote as $\mu_{\{\sigma(t)=\pi\}}$; and the entry probability of a student submitting π to school s_j under the strategy profile σ and mechanism m, $\phi_j^m(\pi, \sigma)$. So, the total measure of students admitted to school j in strategy profile σ is equal to $\sum_{\pi \in \Pi} \mu(\sigma^{-1}(\pi))\phi_j^m(\pi, \sigma)$.

Then the quality at each school under strategy profile σ and mechanism m is average of expected types of students submitting different lists weighted by their measures in the school:

$$\omega_j^m(\sigma) = \sum_{\pi \in \Pi} [\underbrace{\mathbb{E}_{\mu_{\{\sigma(t)=\pi\}}}(t)}_{\text{Expected quality of types submitting } \pi} \times \underbrace{\mu(\sigma^{-1}(\pi))\phi_j^m(\pi, \sigma)}_{\text{measure of students submitting } \pi \text{ and admitted to } j}] \times (\sum_{\pi \in \Pi} \mu(\sigma^{-1}(\pi))\phi_j^m(\pi, \sigma))^{-1}$$

In general, it is possible for s_1 or s_2 to remain empty. We assume that quality of empty school is $\mathbb{E}_{\mu}(t)$.

Given the school qualities calculated, the payoff of type t from enrolling to school j under mechanism m and strategy profile σ is

$$v_j^m(t, \sigma) = h(\omega_j^m(\sigma), t) - 1\{j = 1\}p - 1\{j = 3\}\Delta$$

Thus, type t's expected payoff from submitting π when under the strategy profile σ and the mechanism m can be written as:

$$U^m(t,\sigma,\pi) = \sum_{j=1}^{3} v_j^m(t,\sigma)\phi_j^m(\pi,\sigma)$$

Since for a given strategy profile the payoffs are well defined, we can define Nash Equilibrium.

Definition 1.3. Nash Equilibrium:

For each $m \in \{DA, SA\}$, a Nash Equilibrium is a strategy profile σ_m^*, such that

i) School qualities are $\{w_j^m(\sigma_m^*)\}_{j=1}^3$

ii) $U^m(t,\sigma^*,\sigma^*(t))$ are the payoffs of each $t \in T$.

iii) No type deviates: $U^m(t,\sigma^*,\sigma^*(t)) = \max_{\pi\in\Pi} U^m(t,\sigma^*,\pi)\, \forall t \in T$

There can be many different Nash Equilibria of this game. For example, it is possible to have an equilibrium where higher types sort into s_3, the bad public school. To eliminate such equilibria, we restrict attention to strategy profiles in which s_3 is ranked last by all types.

Lemma 1.2 shows that we can eliminate many types of equilibria with this restriction under DA. It illustrates that potential equilibria is a cutoff type equilibrium where types above the cutoff rank s_1 as top choice, or every school should have the same quality.

Lemma 1.2. *Suppose a strategy profile σ_{DA}^* in which s_3 is ranked last by all $t \in T$ is a Nash Equilibrium, then there exist a cutoff $t_1 > \underline{t}$ such that $\sigma_{DA}^*(t) = (s_1, s_2, s_3)$ if $t \geq t_1$, and $\sigma_{DA}^*(t) = (s_2, s_1, s_3)$ if $t < t_1$.*

Lemma 1.2 means that in any equilibrium of DA there can be at most one cutoff type at which behavior changes; and if that cutoff type exists then types above the cutoff type rank private school as top choice whereas types below the cutoff type ranks s_2 as top choice. Moreover, there is no equilibrium where all students first rank s_1. To understand the intuition, first note that under DA when deciding the relative order of the schools students can restrict attention to comparison of payoffs from the two schools. If a type t' finds optimal to rank s_1 as top choice, then quality of s_1 must be higher. Otherwise, no student would find it optimal to rank s_1 as top choice since its price is higher. As quality of ω_1 is higher than ω_2, then all types above t' must strictly prefer to rank s_1 as top choice since they have stronger preferences to be at a higher quality school compared to type t'. Thus, we cannot have a cutoff where types above the cutoff rank s_2 first and types below the cutoff rank s_1 first. This also implies that, we can have at most one cutoff. Also, s_1 cannot be ranked first by all students. In that case qualities at each school will be the same, then everyone would deviate to ranking s_2 first, which is a contradiction.

Similarly, Lemma 1.3 indicates what potential equilibria under SA can be under the restriction of s_3 being last ranked school for everyone. In comparison to DA, we can eliminate less types of equilibria under SA.

Lemma 1.3. *Suppose a strategy profile σ^*_{SA} in which s_3 is ranked last among public schools by all $t \in T$ is a Nash Equilibrium, then $\omega_2(\sigma^*_{SA}) = \omega_3(\sigma^*_{SA})$ and one of the following is true:*

i) there exist a cutoff $t_1 \in (\underline{t}, \bar{t})$ such that $\sigma^*_{SA}(t) = (s_1, s_2, s_3)$ if $t \geq t_1$, and $\sigma^*_{SA}(t) = (s_2, s_3)$ if $t < t_1$

ii) there exist a cutoff $t'_1 \in (\underline{t}, \bar{t})$ such that $\sigma^*_{SA}(t) = (s_2, s_3)$ if $t \geq t'_1$, and $\sigma^*_{SA}(t) = (s_1, s_2, s_3)$ if $t < t'_1$

iii) $\omega_1(\sigma^*_{SA}) = \omega_2(\sigma^*_{SA})$

First, Lemma 1.3 says that quality of s_2 and s_3 must be the same. It also indicates that if an equilibrium is not cutoff type, then it can only have $\omega_1(\sigma^*_{SA}) = \omega_2(\sigma^*_{SA})$. We cannot eliminate many types of equilibria through Lemma 1.3, but it still provides enough restriction to characterize the equilibria we are interested in.

We are mainly interested in equilibria where private school has higher quality than public schools. This is the more interesting case given our empirical context. Lemma 1.4 indicates that any potential equilibrium in which private school has higher quality compared to all public schools can be characterized by a cutoff rule under both mechanisms. There must exist a cutoff type such that, types above the cutoff rank s_1 as top choice and types below the cutoff rank s_2 as top choice.

Lemma 1.4. *Let $m \in \{DA, SA\}$, and suppose σ^*_m is an equilibrium such that s_3 is ranked last by* *all $t \in T$ then following is true:*

$$\omega_1^m(\sigma_m^*) > \max\{\omega_2^m(\sigma_m^*), \omega_3^m(\sigma_m^*)\} \iff \sigma_m^*(t) = \begin{cases} (s_1, s_2, s_3) & \text{if } t \geq t_1 \\ (s_2, s_1, s_3) & \text{if } t < t_1 \end{cases}$$

for some $t_1 \in (\underline{t}, \bar{t})$.

If part of Lemma 1.4 is intuitive, especially for SA. Under the equilibrium described for SA, types lower than the cutoff never apply to s_1, so private school has only the types above the cutoff. Whereas, for public schools it includes types below the cutoff and if private school is oversubscribed it also includes types above the cutoff. For DA, the intuition is less clear. Types below the cutoff first apply to s_2, but if they are rejected then they apply and get into s_1. Therefore all schools have a mix of types below and above the cutoff. However, we can still expect that higher types have more weight in s_1 and lower types have more weight in s_2. Since both students above and below the cutoff go to s_3 only if they are rejected from both schools, this school has quality in between $s_1$and s_2. Therefore, s_1 has higher quality compared to s_2 and s_3 under DA as well. Only if part follows from previous lemmas. Under SA, we have shown that an equilibrium should either be a cutoff type or all schools must have equal quality. It is also expected that the cutoff equilibrium in which low types rank s_1 as top choice cannot yield higher quality for private school.[24] For DA, we have shown that in the equilibrium either there is a cutoff type as described in Lemma 1.4, or every student ranks s_2 as top choice. In the latter case the qualities are equal across all the schools.

In a cutoff equilibrium, only types above the cutoff type produce enough difference between human capitals in s_1 and s_2, which surpasses the price difference. Although, s_1 has higher quality in the cutoff equilibrium described in Lemma 1.4, low types do not rank it as top choice because of the price. Therefore, positive price acts as a coordination mechanism and sustains the sorting equilibrium.

[24] In that case private school's student body will consist of students below the cutoff type and body of s_2 and s_3 will have types above the cutoff, and potentially types below the cutoff if private school is over-demanded.

1.2.4 Existence of Equilibria

In this subsection we are going to provide the conditions under which cutoff equilibrium described in Lemma 1.4 exists for *SA* and *DA*. From now on, by cutoff equilibrium we mean the equilibrium described in Lemma 1.4. Also, we denote the school qualities in an equilibrium σ^* with cutoff $t_1 \in (\underline{t}, \bar{t})$, by $\omega(t_1)$ instead of $\omega(\sigma^*)$; and we denote $U(\sigma^*, ., .)$ by $U(t_1, ., .)$ from now on. We characterize the existence of equilibrium for each possible cutoff under both mechanisms. Moreover, we characterize the price range for which there can exist cutoff equilibrium if certain conditions are satisfied at each price level in this range.

We provide a lemma that is useful to understand the conditions necessary for the existence of cutoff equilibria. Lemma 1.5 shows the comparison of qualities across schools and mechanisms for a given cutoff.

Lemma 1.5. *Given $t_1 \in (\underline{t}, \bar{t})$, for an equilibrium σ^* with cutoff t_1, we have $\omega_1^{SA}(t_1) > \omega_1^{DA}(t_1) > \omega_3^{DA}(t_1) > \omega_2^{DA}(t_1) > \omega_2^{SA}(t_1) = \omega_3^{SA}(t_1)$*

According to Lemma 1.5, in a cutoff equilibrium under DA, s_3 has higher quality compared to s_2. This lemma also shows that when the cutoffs are the same across mechanisms, private school has higher quality under SA; and each public school have higher quality under DA.

To show the existence of equilibrium we first look at the cutoff type. Under DA, student type t_1 satisfies

$$h(\omega_1^{DA}(t_1), t_1) - h(\omega_2^{DA}(t_1), t_1) = p \tag{1.2.1}$$

Let's define the LHS of Equation (1.2.1) as a function of the cutoff t_1: $\Gamma^{DA}(t_1)$. And let $\Gamma^{DA,-1}(p)$ yields the set of cutoffs t_1 such that $\Gamma^{DA}(t_1) = p$, i.e. $\Gamma^{DA,-1} : \mathbb{R} \rightrightarrows (\underline{t}, \bar{t})$ is a correspondence. The maximum value that Γ^{DA} can take will be important to determine the price range for which there can be a cutoff equilibrium.

$$\mathscr{M}^{DA} := \max_{t_1 \in (\underline{t}, \bar{t})} \Gamma^{DA}(t_1)$$

To guarantee the existence of cutoff equilibrium, we need to make sure that for all students s_3 is the least preferred option. Note that we are looking at cutoff equilibria where students alter in their ranking of s_1 and s_2. Therefore it is enough to make sure that no type prefers s_3 as second choice. First, for types below the cutoff we need them to prefer s_1 to s_3. In this group, for the lowest type difference in human capital production is least. So it is enough to convince the lowest type. Therefore for given cutoff t_1 to be an equilibrium cutoff we need $\Delta \geq p - \Delta_{13}(t_1)$ where $\Delta_{13}(t_1) := h(\omega_1^{DA}(t_1), \underline{t}) - h(\omega_3^{DA}(t_1), \underline{t})$. Second, for types above the cutoff we need them to prefer s_2 to s_3. Remember that according to Lemma 1.5, s_3 has higher quality than s_2. In this group the highest type will have greatest difference in terms of human capital production between s_3 and s_2. So it is enough to convince the highest type for that group. Therefore for given cutoff t_1 to be an equilibrium cutoff we need $\Delta \geq \Delta_{32}(t_1)$ where $\Delta_{32}(t_1) := h(\omega_3^{DA}(t_1), \bar{t}) - h(\omega_2^{DA}(t_1), \bar{t})$

The following Proposition 1.1 is on the existence of cutoff equilibrium under DA.

Proposition 1.1. *i) Given $t_1 \in (\underline{t}, \bar{t})$, an equilibrium σ^* with cutoff t_1 exists if and only if $p = \Gamma^{DA}(t_1)$ and $\Delta \geq \max\{p - \Delta_{13}(t_1), \Delta_{32}(t_1)\}$*

ii) $\mathscr{M}^{DA}$ exists, $\mathscr{M}^{DA} > 0$; and $\Gamma^{DA,-1}(p)$ is non-empty if and only if $p \in (0, \mathscr{M}^{DA}]$.

iii) Given a price $p \in (0, \mathscr{M}^{DA}]$, $\Gamma^{DA,-1}(p)$ is the set of cutoff equilibria if and only if $\Delta \geq \max\{\max_{t_1 \in \Gamma^{DA,-1}(p)} \Delta_{32}(t_1), \max_{t_1 \in \Gamma^{DA,-1}(p)} p - \Delta_{13}(t_1)\}$

First item of Proposition 1.1 means that a given candidate cutoff is an equilibrium cutoff if and only if price level leaves the cutoff type indifferent; and Δ is high enough such that for all types s_3 is the least preferred option under the proposed strategy profile. Second item means that, the maximum difference of human capital production of cutoff type between s_1 and s_2 across cutoff types is well-defined and is greater than zero. These are expected since for a given cutoff $t_1 \in (\underline{t}, \bar{t})$ we have already shown that quality is higher in s_1 compared to s_2. So for any cutoff in this range the difference between human capital from two schools is greater than zero for the cutoff type. Existence follows from continuity of human capital function and school qualities; and end points of the type space not being the maximum. Also, it says that, for each price in the range of 0 to $\mathscr{M}^{DA}$, there is a cutoff equilibrium candidate. The last article means that for a given price the candidate set of equilibrium given in the previous article is actually set of cutoff equilibrium when Δ prevents any type deviating from ranking s_3 as last choice, which depends on the equilibrium candidates.

Under SA to show the existence of equilibrium we again look at the cutoff type. In SA, every type when deciding to rank s_1 as top choice, considers the expected utility from Step 2, which includes the possibility of going to s_3, the bad public school. Therefore the calculation includes both the entry probabilities in Step 2 and Δ. Note that in a cutoff equilibrium c_2^{SA} is equal to $\min\{\frac{1}{2}, \frac{1}{3F(t_1)}\}$ and qualities of s_2 and s_3 are the same. Thus, for the cutoff type $t_1 \in (\underline{t}, \bar{t})$ we have:

$$\begin{aligned} &U^{SA}(t_1, t_1, \pi = (s_1, s_2, s_3)) = U^{SA}(t_1, t_1, \pi = (s_2, s_3)) \\ \Longleftrightarrow\ & h(\omega_1^{SA}(t_1), t_1) - h(\omega_2^{SA}(t_1), t_1) + \Delta(1 - \min\{\frac{1}{2}, \frac{1}{3F(t_1)}\}) = p \end{aligned} \tag{1.2.2}$$

Define LHS of the Equation (1.2.2) as a function of t_1: $\Gamma^{SA}(t_1)$. $\Gamma^{SA,-1}(p)$ denotes the set of cutoffs t_1 such that $\Gamma^{SA}(t_1) = p$, i.e. $\Gamma^{SA,-1} : \mathbb{R} \rightrightarrows (\underline{t}, \bar{t})$ is a correspondence. Supremum and maximum of $\Gamma^{SA}(t_1)$ is defined as:

$$\mathscr{S}^{SA} := \sup_{t_1 \in (\underline{t}, \bar{t})} \Gamma^{SA}(t_1)$$

$$\mathscr{M}^{SA} := \max_{t_1 \in (\underline{t}, \bar{t})} \Gamma^{SA}(t_1)$$

The following Proposition 1.2 is on the existence of equilibrium where private school quality is higher than public school qualities under SA.

Proposition 1.2. *i) $\mathscr{S}^{SA}$ exists and $\mathscr{S}^{SA} > \frac{\Delta}{2}$*

ii) Given $t_1 \in (\underline{t}, \bar{t})$, an equilibrium σ^* with cutoff t_1 exists if and only if $p = \Gamma^{SA}(t_1)$

iii) $\Gamma^{SA,-1}(p)$ is non-empty if $p \in (\frac{\Delta}{2}, \mathscr{S}^{SA})$ and empty if $p \leq \frac{\Delta}{2}$ or $p > \mathscr{S}^{SA}$. $\Gamma^{SA,-1}(\mathscr{S}^{SA})$ is non-empty if and only if $\mathscr{M}^{SA}$ exists.

iv) Given $p \in (\frac{\Delta}{2}, \mathscr{S}^{SA}]$, $\Gamma^{-1}(p)$ is the set of cutoff equilibria.

First item tells the existence of the supremum of Γ^{SA}, and that supremum is greater than $\Delta/2$. This is important to derive the price range for which there can be a cutoff equilibrium since the cutoff type must be indifferent between ranking s_1 as top choice versus ranking s_2 as top choice. Second item tells that price must leave the cutoff type indifferent if there exists an equilibrium with that cutoff type. Third item characterizes the price range for which there can exist a cutoff equilibrium. And this price range includes prices from $\Delta/2$ (not included) to $\mathscr{S}^{SA}$. Whether there exist a cutoff equilibrium when the price is $\mathscr{S}^{SA}$ depends on whether the supremum is actually attained within $(\underline{t}, \bar{t})$. The last item gives the set of cutoff equilibria for a given price in the range given by previous item.

1.2.5 Comparing Equilibria of SA and DA

In this subsection we compare the cutoff equilibria of SA and DA in which private school has higher quality. We do this comparison across prices in terms of quality of schools. For this purpose, we need to make some assumptions on functional form and distribution of students. We assume that $h(\omega,t) = \omega^{0.5}t^{0.5}$ and μ is uniform on $[0,1]$. In Figure 1.1 we plot the Γ functions for both mechanisms under those assumptions. Proposition 1.1 says that, a cutoff is an equilibrium for DA when the price is equal to the y-axis value of the blue curve evaluated at the cutoff, and Δ is greater than the maximum of values of blue and orange curves evaluated at the cutoff. Therefore for DA a cutoff equilibrium exists for each price between the highest value of the blue curve and zero when Δ is greater than the y-axis value of the blue curve. When drawing Γ^{SA} we assume that Δ is as large as the tip of the blue curve.[25] Proposition 1.2 says that, an equilibrium for SA exists whenever price is in the y-axis range of Γ^{SA} function, and lower bound of that range is $\Delta/2$.

From Figure 1.1 we see that for a given price the cutoff type is larger for DA. Intuition behind this is as follows. In DA, students compare only s_1 and s_2 when deciding to rank which one as top choice. However, in SA, they compare the payoff from s_1 to payoff from the centralized match of public schools. The latter not only includes payoff from s_2 but also payoff from s_3 as well since there is possibility of going to s_3, the bad school in the second step. This makes ranking s_1 as top choice more attractive for all students. Therefore, the cutoff type is smaller for a given price. Another point worth noticing is that for this specification, there exist multiple equilibria for a given price under DA, but for SA there is unique equilibrium. This is again not surprising. In

[25] It is 0.0704 in this case.

DA, two different cutoffs one closer to lowest type and one closest to highest type can lead to same difference of quality between s_1 and s_2. This is because in the following two cases there is similar mix of types in the two schools. When most students rank s_1 as top choice, most of them will be rejected and get into s_2. When most students rank s_2 as top choice, most of these students will be rejected and enroll to s_1. So in both cases quality of schools will be close to each other. In SA, there is no case of students moving from s_2 to s_1 during the course of the mechanism.

In Figure 1.2, we plot the qualities assuming uniform distribution of types in $[0,1]$. Note that quality of private school increases much faster with cutoff type under SA. The intuition is that, DA allows for students who first apply s_2 to apply s_1 in the next round if they are rejected; whereas SA does not allow this. In the kind of equilibria we look at, the students who first apply s_2 are lower types then students who first apply s_1,and under DA a student rejected from s_2 and next apply to s_1 and get accepted. This leads to decrease in the quality of s_1. However, under SA, types below the cutoff never applies s_1, so as cutoff type increases quality increases in s_1 faster under SA compared to DA.[26] Comparison of qualities for a given cutoff is not enough since equilibria under different mechanisms may not occur at the same cutoff. Therefore we will compare the equilibria under the two mechanisms at a given price.

We compare the qualities for the price range under which both mechanism admits cutoff equilibria in which private school has higher quality. Both graphs in Figure 1.3, shows that for sufficiently high (low) prices $\omega_1^{SA} - \omega_1^{DA} > 0$ ($\omega_1^{SA} - \omega_1^{DA} < 0$). Graph at the top indicates that quality difference between mechanisms for s_2 decrease as price increases. However, the difference increases for s_3 in favor of DA as price increases. The intuition follows from the previous two figures we have analyzed. Remember that for a given price, cutoff type is smaller in SA compared to DA. This leads to higher quality of private school under DA for lower tuition. However, we also show that quality of private school increases faster with cutoff type under SA. And note that from Figure 1.1 it is seen that cutoff type increases with price under SA. Therefore, the second effect

[26]Note that there is always a type below the cutoff that is rejected from s_2. This is because even if the first round applicants of s_2 does not exceed capacity of s_2 that means, first round applicants of s_1 exceed the capacity of s_1 and apply s_2 next and lead to rejections of some of the lower types that first apply s_2.

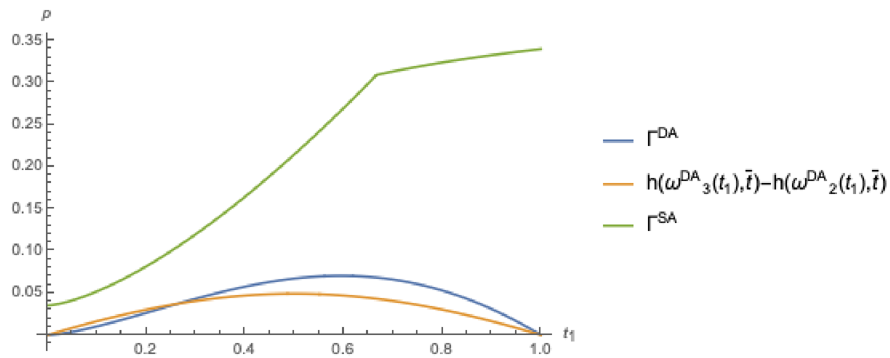
dominates as tuition fee increases for the private school.

It is possible that a schools chooses alternative prices under DA vs. SA. So, does it make sense to compare these mechanisms at the same tuition fee? Admittedly, this model shows what happens at certain price level and certain mechanism. However, here we can interpret the price also as a price cap. We compare these mechanisms at prices in which private school is over-demanded in the equilibrium. So private school can increase the profit by increasing prices up to the price cap, as it will be charging higher price and it still fills its capacity. Moreover, under SA increasing price increases its quality in this price range, and this is similar for most of the price range under DA.[27] Therefore, it is plausible to assume that private school would choose the price equal to price cap that is set by government in this possible range of price caps. This is especially relevant for our empirical context of South Korea that has a price cap for private schools.

In this part, we have shown that the price cap can have important implications. Even without the price cap, often private schools are over-demanded due to reasons like vouchers, or private schools not only caring about profit. Another story why private school might be over-demanded is that, they may consider long term revenue instead of short term. These factors could also allow for combination of price and student selection rules to have an effect on student distribution. However, our model in this section is a simple one with three schools. There are many other factors that can determine students' choices, e.g. distance, neighborhood, other features of the schools. Also, potentially there is heterogeneity of students' in these dimensions or in their preferences regarding these features. These factors motivate us to compare these mechanisms empirically. Also, even in our simple model the comparison between the mechanisms depends on the price level. Another advantage of the empirical analysis would be to make the comparison at the observed tuition fees and quantify the difference at different tuition fees.

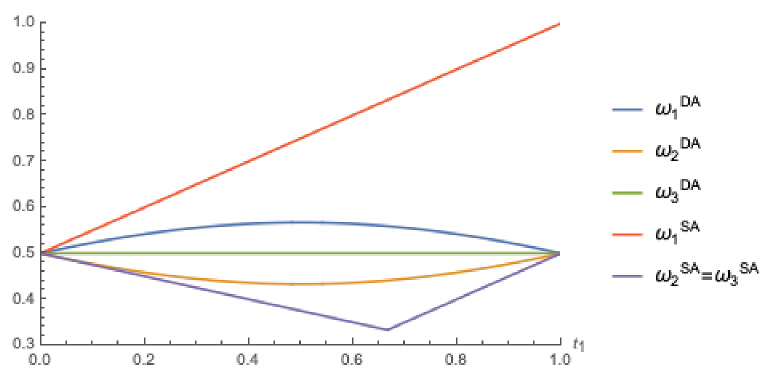
[27] Until the highest portion of the price range

Figure 1.1: Γ^{SA} and Γ^{DA} under $h(\omega,t) = \omega^{0.5}t^{0.5}$ and $\mu \sim U[0,1]$



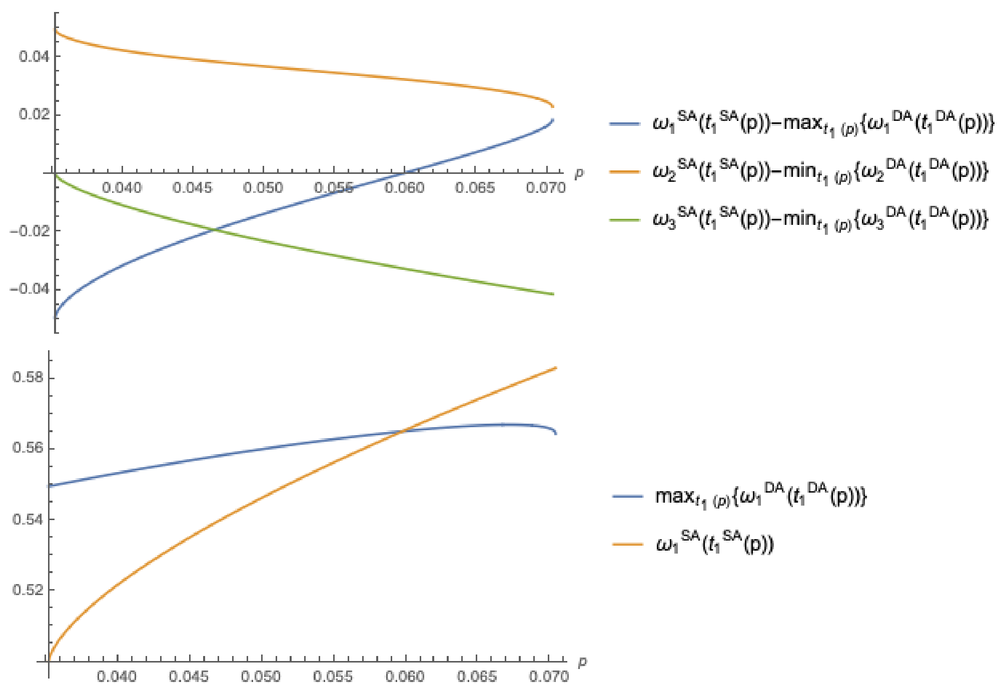
Notes: This figure illustrates the Γ functions defined in the text, for the functional form and type distribution assumptions: $h(\omega,t) = \omega^{0.5}t^{0.5}$ and $\mu \sim U[0,1]$. Also it plots the difference between human capitals produced from s_3 and s_2 by the highest type student under DA. The y-axis corresponds to price, and the x-axis corresponds to the cutoff type.

Figure 1.2: Qualities for given t_1 under $\mu \sim U[0,1]$



Notes: This figure illustrates the school qualities across cutoff types under the assumption of $\mu \sim U[0,1]$. The x-axis corresponds to the cutoff type, and the y-axis corresponds to the school quality.

Figure 1.3: Comparison of SA and DA



Notes: This figure illustrates the quality comparison at schools across levels of prices under the assumptions $h(\omega,t) = \omega^{0.5}t^{0.5}$ and $\mu \sim U[0,1]$. The graph at the top illustrates the difference of school qualities between SA and DA for each school. The graph at the bottom illustrates the absolute quality of private school under both mechanisms. We take the equilibrium which gives the maximum quality for s_1 and minimum quality for public schools as there is multiple equilibria under DA.

1.3 High School Market In Seoul

1.3.1 Institutional Details

High school equalization policy of South Korea, enacted in 1974, aimed to provide equal educational opportunities to all the students by equalizing all the high schools across the nation. To achieve that, the government regulated most schools heavily. The guidelines regarding the operation of private schools were so restrictive that they were operated like public schools after this policy. The schools were not free in choosing their teachers, curriculum, students and even tuition. Since they were not able to charge tuition higher than public schools, they were subsidized by the government to continue their operations (Kim and Lee, 2002). In 2009, government started private autonomous (PA) school program to diversify the education and increase competition among schools (Park, 2021). According to the policy, PA schools would gain more independence regarding their choice of curriculum, academic terms, teachers, student body and tuition, although there were still some restrictions regarding tuition fees and student selection.

At the same time with PA school policy, centralized school choice for high schools began in Seoul. Beginning with students starting high school in 2010, students have been allowed to submit preference lists over schools that participate in the school choice.[28] A centralized mechanism assigns students to schools according to preferences of students, school quotas, a proximity rule and lottery numbers.

Students in Seoul can attend various types of high schools. Main types are science, foreign language, vocational and general schools.[29] Science and foreign language schools admit students through exams. These and vocational schools do not join the centralized match.

Our analysis focuses on general schools market. 224 of the 317 high schools in 2010 are general schools.[30] These include charter schools which are publicly funded, regular schools which can

[28] In South Korea, school year starts in March.

[29] Vocational schools consist of technical, commercial and art schools. Other than these main types of schools there exists one international high school and physical education school in the period we analyze

[30] 3 of the high schools were science, 6 of them were foreign language and remaining 85 were mostly vocational schools. For 2011, 3, 6, 221 and 84 were the numbers of science, foreign language, general, and vocational high schools respectively; and for 2012 these numbers were 3, 6, 224 and 84 respectively.

be public or private, and PA schools. All of these except PA schools join the centralized school choice during our data period of 2010-2012. Until 2019, PA schools did not join the centralized match and admitted their students before the centralized match begins. Sequentiality of admissions was blamed to be the reason of high achieving students' sorting into PA schools; which was also degrading the overall student quality in schools attending the centralized match. As a result, sequentiality is banned in 2019, and PA schools have been included in the centralized match since then.[31] We call non-PA general high schools as NPA schools from now on.

General High Schools

Table 1.1 illustrates the distribution of general school types. Going from 2010 to 2011 the number of PA schools double.[32] Among general public schools, 19 of them are charter schools in 2012. In 2011, 19 of the NPA schools started to offer science programs. These programs have quotas specified for the centralized match and students are admitted to these programs independently from the general part of the school. PA schools are male-only school dominated. In 2011, 19 of them are male-only, 4 are co-ed and 3 of them are female-only.

Within our data period, all NPA schools charge the same tuition. Therefore, even a private school in the centralized match charges the same tuition as the general public schools. This tuition was 1,450,800 KRW in 2010, which is around 1,400 USD.

Designation of Autonomy and Student Admission in Autonomous Schools

A general private school has to apply to SOE for PA school designation. According to SOE officials, sound finance is the most important consideration in the approval process.[33] The tuition of

[31] In 2017, President of South Korea Moon Jae In announced their aim to end sequentiality. In 2018, a group of private schools and parents filed a lawsuit agains ending sequentiality. They have argued that the ending sequentiality violates freedom of schools to differentiate and freedom of students to choose. In 2019, the supreme court upheld the government's policy and starting in 2019 Seoul Office of Education (SEO) included these schools to centralized match.

[32] We exclude two private schools that admit from all of South Korea from our analysis.

[33] In fact, for 2010, out of 67 schools applied only 13 of the schools with sound finance were designated autonomy status and 5 schools were told to work more on improving their finances. In the second year 2011, 13 more schools were designated as autonomous schools. PA schools may voluntarily forgo autonomy status or government can strip autonomy through the approval process.

PA schools cannot be larger than three times the NPA school tuition amount. This corresponds to 4,352,400 KRW.[34] Since PA schools can charge higher tuition, they cannot receive subsidies from the government.

PA schools can choose their own students but not through exams or academic interviews.[35] Within our data period they hold independent lottery procedures to determine the enrollment of the applicants.[36] Students can apply to only one autonomous school and if a student holds a PA school assignment, he/she cannot join the centralized school match. Until 2015, only students who are in the top 50 percent in terms of academic ranking in their middle schools were able to apply these lotteries. After 2015, PA schools had to allow applications from all students.

In addition, PA schools had to reserve 20% of their seats for affirmative action track applicants. Students in this track pay tuition equal to the tuition fee of NPA schools. A student is eligible for the affirmative action track if the income of the students' family is below the 150% of the poverty line. Students who are admitted through general track pay the full tuition amount (sticker price).

Centralized Matching Process

Seoul is divided into 11 school zones. Each zone is divided into several Gus (district); and similarly each Gu is divided into several Dongs (neighborhoods).[37] Zone and Gu of a school play some role in the centralized mechanism as we describe below. Each student fills the application form in Table 1.2 and can leave a line blank if he or she would like to do so. As can be seen from the Table 1.2 students can rank up to 6 schools. Charter schools and science programs have to be ranked at the specified positions.[38] 40% of the seats are reserved for administrative assignment for schools in each zone except Central Zone. In Central Zone schools, no seat is reserved for administrative assignment. (Hwang, 2017) defines Centralized Mechanism of Seoul (CMS), and

[34] which is around 4,300 USD

[35] They are allowed to run lottery or choose students according to the middle school grades.

[36] Before 2019, if a general private school forgoes its autonomy status it had to participate in the centralized match.

[37] There are 25 Gus in total in Seoul and there are 420 Dongs in total. Area of dong ranges between 0.5 square miles and around 2 square miles; and area of a Gu ranges between 5 to 17 square miles.

[38] A student can rank a school more than once: a student can rank a school from his/her choice zone as 3rd or 4th and rank the same school as 5th or 6th choice.

we this definition in Appendix A.2 as well. Note that this mechanism is not strategy-proof. So, students do not necessarily rank-order schools truthfully.

Table 1.1: Type of General High Schools Across Years

Year	2010	2011	2012
Private autonomous	13	26	25
General public	85	88	91
General private	117	104	105
Charter	7	17	19
Science program	0	19	19
Male-only	69	70	91
Female-only	61	61	70
Co-ed	83	84	61

Notes: This table illustrates the type distribution of general high schools in Seoul. Charter schools are included in general public schools. Science programs are part of general public or general private schools.

Table 1.2: Preference List Structure for Centralized Mechanism

Position	Type of School	Name of the School
1st	Charter (optional)	
2nd	Science (optional)	
3rd	Regular in any choice zone	
4th	Regular in any choice zone	
5th	Regular in your choice zone	
6th	Regular in your choice zone	

Notes: This table illustrates the structure of the preference list students submit in Centralized Match of Seoul (CMS).

1.3.2 Data

Our main dataset is from SOE. We observe the preference lists submitted by students who start high school in 2010, 2011 and 2012 if they attended CMS. For these students we also see their final enrollment, gender, religion and percentile of their ranking according to grades within their own middle schools; and their geocodes (with error). A student's percentile of his/her ranking in the middle school indicates the percent of students with lower grades average than the student. Therefore as a student's percentile ranking goes up, he is academically more successful. Number of students that participated the centralized mechanism are over 80,000 for each of these years.[39] In addition, we have a survey data from 2009 that asks students about whether they would like to apply to an PA school for the school year 2010. This dataset can be matched to the dataset of submitted lists by students in 2010.[40] We know the capacity and tuition of each the high schools for 2010 and 2012.[41] We observe school addresses and whether the school is male-only, female-only or co-ed school for all the high schools in Seoul. For PA schools we also observe the number of applicants to these schools since 2010 for general and affirmative action track applications.

We also use Seoul Educational Longitudinal Study (SELS).[42] In this data we observe test scores in Korean, Math, English for each year in school; study and private lesson hours, the middle school of the student, family income and number of people in the household for the surveyed students. This enables us to predict the probabilities of being in affirmative action versus general track for the students in our main dataset, in which we do not observe income and family size.

We also use School Education Condition Analyses: Elementary/Middle/High School (SECA) which is a survey data that asks 2nd year students in middle school for the years 2010 and 2013 their plans after middle school. For both of the years, we can observe income of students' families and number of people in the household, students scores in the tests accompanying the survey, their

[39] 90459, 83499, 81558 are the exact numbers for 2010, 2011 and 2012 respectively

[40] We can match based on the dong student lives, his/her middle school, assigned school and rank of the schools submitted. There are cases where one student in 2009 survey data is matched to more than one person in 2010 data.

[41] We use the tuition of 2012 for the year 2011 as well, since we could not find tuitions of PA schools for the year 2011.

[42] This data starts tracking 3 cohorts (4th, 7th and 10th graders) in 2010 until the end of high school.

answers to questions about school life, their studying and learning habits and other socioeconomic characteristics of the family. For 2010 we can see students' answer to his/her ranking in his middle school, and for 2013 we observe whether they consider applying an autonomous school. This dataset will be useful for showing the effect of tuition on preferences via Regression Discontinuity (RD) approach. Moreover, we will be able to use this information in structural estimation via indirect inference approach, which will help us identifying the preference for tuition.[43]

1.3.3 Relevant Features of Seoul High School Market

Private Schools Are Over-Demanded

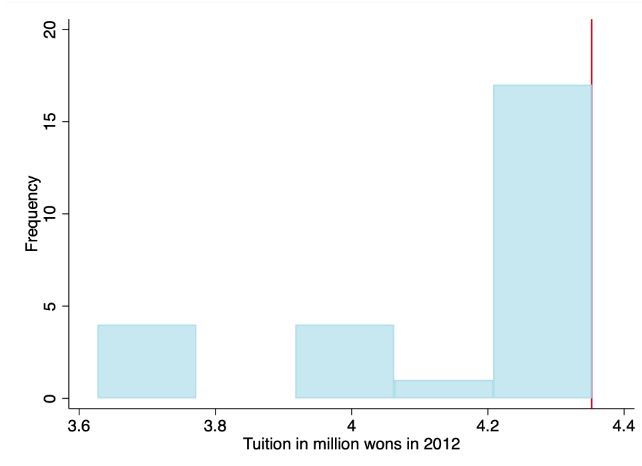
One salient feature of public school markets where students can choose other than neighborhood schools is that some schools are over-demanded. That is, the number of students desiring to go to a school exceeds the school's capacity. This is the main reason behind the existence of school choice algorithms in the world. However, for private schools it is less clear whether they would be over-demanded. One may think that a private school can rise its price up to the point where demand equals capacity which would increase its profit. However, many factors might prevent this reasoning to hold. Price caps, vouchers and subsidies are among the potential reasons. Or private schools may care for incoming student success as well, which may not be perfectly correlated with ability to pay. Tuition caps is the case for PA schools in Seoul. Figure 1.4 illustrates the tuition cap and bunching of schools at the tuition cap; and Figure 1.5 illustrates that PA schools are in general over-demanded for the period we study.

Tuitions of PA Schools Plays A Role

Although the tuition does not clear the market for private schools, this does not mean that tuition has no role in preferences of students. It is natural to expect that higher tuition fees of PA schools

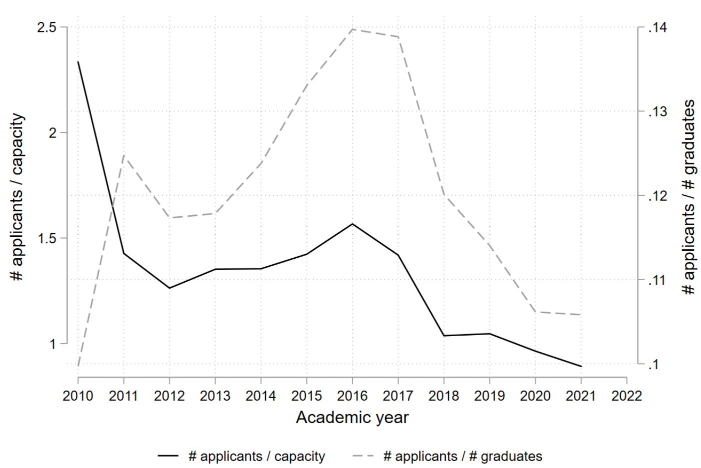
[43] In the 2013 SECA data we do not observe students' answer to their middle school ranking. This ranking is important to restrict the sample to only students in top 50 percent in terms of ranking in their schools. Moreover, ranking information will be important within the structural estimation. Therefore we predict the ranking information for students in the 2013 data. We do this by using a model estimated from 2010 data with the variables in both datasets. The details of construction of RD estimation sample can be found in Appendix A.2.

Figure 1.4: Tuition Cap and Bunching of PA Schools at the Cap



Notes: This graph illustrates the distribution of tuition in 2012 for PA schools in million won (around thousand USD). The red line indicates the tuition cap.

Figure 1.5: Demand for Autonomous Schools



Notes: This graph illustrates the application behavior towards PA schools. The solid line shows the number of total PA applicants divided by total capacity of PA schools. The dashed line illustrates the share of PA school applicants among middle school graduates.

compared NPA schools discourage some students from applying to PA schools. We use a sharp RD design to show evidence of that. Remember that students below 150% of the poverty line pay the tuition amount of 1,450,800 Korean won; whereas students above the income threshold pay the sticker price, which is in general around 3 times of tuition of NPA schools. In our RD design, the running variable is natural logarithm of effective household income[44] and the outcome variable is whether the student plans to apply PA school.[45] Table 1.3 illustrates our RD estimates. According to this, increase in the prices at the cutoff decreases the share of students who wants to apply PA school by around 0.42 and this effect is statistically significant at 5% level. In Section 1.4 we show that application decision is independent of entry chances. Therefore, this is purely the effect of tuition change in the cutoff. Figure 1.6 illustrates the downward jump in the share of students desiring to apply PA schools visually. This evidence strengthens our belief that the observed tuition fees affect the demand.

Middle School Academic Achievement of Students Vary Across General High Schools

All of the middle school graduates can attend CMS. In CMS, students with different middle school performances have same admission chances to any school participating in CMS. However, PA school lotteries were restricted to students in the top 50 percent of their middle schools in terms of grades within our period of analysis. Therefore, the distribution of the previous academic achievements of the enrolled students must be different for the two group of schools. We are interested in an aggregate measure of the academic achievement of each cohort for each high school. The only achievement measure we observe for students is their percentile ranking in middle school. We construct the aggregate measure using these individual measures of the enrolled students for each year. We take the mean of the enrolled students' percentile ranking in their middle schools.

[44] the household income adjusted by household size since poverty line depends on household size

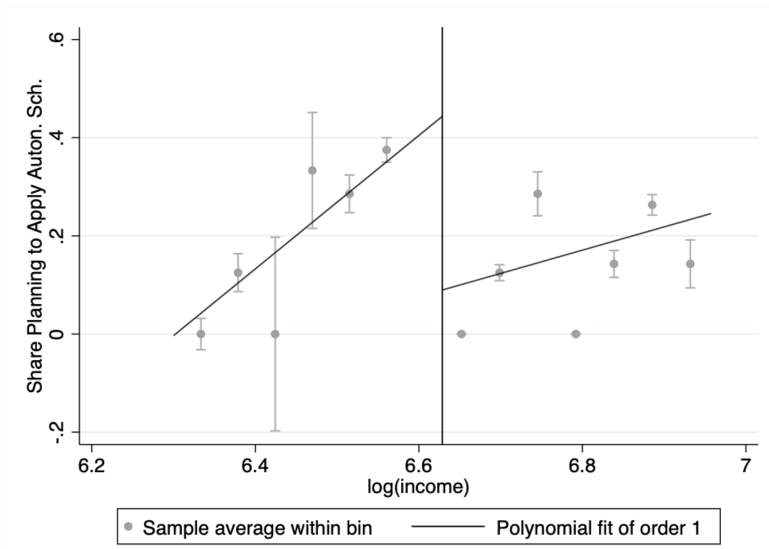
[45] Note that our running variable is constructed from income, therefore due to misreporting some students who are actually to the left of the cutoff might fall to the right of the cutoff and vice versa. To prevent our estimates to be affected from this, we drop some of the observations that are very close to the cutoff from both size. We drop the observations within 30,000KRW (around $30) effective income. This drops only 3 observations. Our estimates without dropping these observations are qualitatively similar and significant at 10% significance level. See Table A.2 for the results without dropping these observations.

Table 1.3: Regression Discontinuity Estimates

Dependent Variable: Share of students wanting to apply autonomous school	
Income>Cutoff Income (conventional)	-0.354 (0.166)
Income>Cutoff Income (bias-corrected)	-0.419 (0.134)
Income>Cutoff Income (robust)	-0.419 (0.203)
Observations Left	86
Observations Right	518
Effective Observations Left	48
Effective Observations Right	83

Notes: This table indicates the effect of the discontinuous change in sticker price of PA schools on the share of students desiring to apply to PA schools at the effective income threshold for affirmative action eligibility. Estimates are calculated using bandwidth calculation proposed by Calonico, Cattaneo, Farrell, and Titiunik (2017). Included covariates are, students' gender, and whether middle school is public/private. Standard errors are in parenthesis. Observations within 0.02 of the cutoff are discarded.

Figure 1.6: Visual Regression Discontinuity Evidence



Notes: This figure illustrates the causal effect of the discontinuous change of sticker price of PA schools on the share of eligible students desiring to apply PA schools at the effective income threshold. We plot this figure as suggested by Calonico, Cattaneo, Farrell, and Titiunik (2017). We use the observations within the optimal bandwidth of the income threshold and fit a polynomial of degree 1 separately for both sides of the cutoff. Observations within 0.02 of the cutoff are discarded.

We are going to refer to this quantity as mean percentile ranking *m.p.r.* for each high school from now on. Since we can observe which students are enrolled to each NPA school, we can construct *m.p.r.* for these schools easily. For PA schools, we infer it from the missing percentile rankings in each middle school. We describe how we infer these mean percentile rankings for PA schools in Appendix A.2.[46]

Summary statistics of *m.p.r.* across NPA schools between 2010 and 2012 are shown in Table 1.4.[47] Mean of the *m.p.r.* for NPA schools increase a little after 2010, which is explained in the next subsection. Table 1.5 illustrates a descriptive regression of *m.p.r.* on characteristics for NPA schools. Compared to co-ed schools female-only and male-only schools have 11.18 and 1.89 higher *m.p.r.* on average, respectively.[48][49]

Table 1.4: Mean Rank Percentile Statistics for NPA Schools

Year	Mean	Std.	Min	Max
2010	51.57	6.29	37.58	66.28
2011	49.38	6.84	32.47	70.48
2012	49.48	6.82	37.54	71.62

Notes: This table illustrates the summary statistics of *m.p.r.* in NPA schools for each year. For each NPA school *m.p.r.* is constructed by taking the average of middle school percentile of ranking of enrolled students.

[46] Remember that we only observe percentile rankings for students who attend CMS. Therefore, most of the students whose percentile rankings are missing and above 50 must have gone to PA schools.

[47] Across the years 2010, 2011 and 2012 we infer average *m.p.r.* in PA schools as 76.45055, 76.546755 and 75.111345 respectively.

[48] This difference between male-only and female-only schools can be explained by distribution of PA schools and their academic criteria for attending their lotteries. Most of the PA schools are male-only. Since PA schools admit higher-achieving students, mean academic achievement of students who would like to attend male-only NPA schools decreases.

[49] Charter school seem to have little higher *m.p.r.* compared to non-charter schools. Schools with science program has 3.18 higher *m.p.r.* on average compared to schools without science program.

Table 1.5: Variation of *m.p.r.* Across Types of NPA Schools

Dependent variable: *m.p.r.* of enrolled students	
Charter School	0.58 (1.69)
School with Science Program	3.18 (0.88)
Male-only School	1.89 (0.65)
Female-only School	11.18 (0.74)
Constant	46.07 (0.44)
R^2	0.5149
Observations	593

Notes: This table describes the relationship between *m.p.r.* of NPA high schools and their types. The table is created by running an OLS regression of *m.p.r.* on indicator variables regarding type of school for the years 2010, 2011 and 2012. Standard errors are in parenthesis and they are calculated by clustering at the school level.

The Effect of Increase in Number of PA Schools on NPA School Market

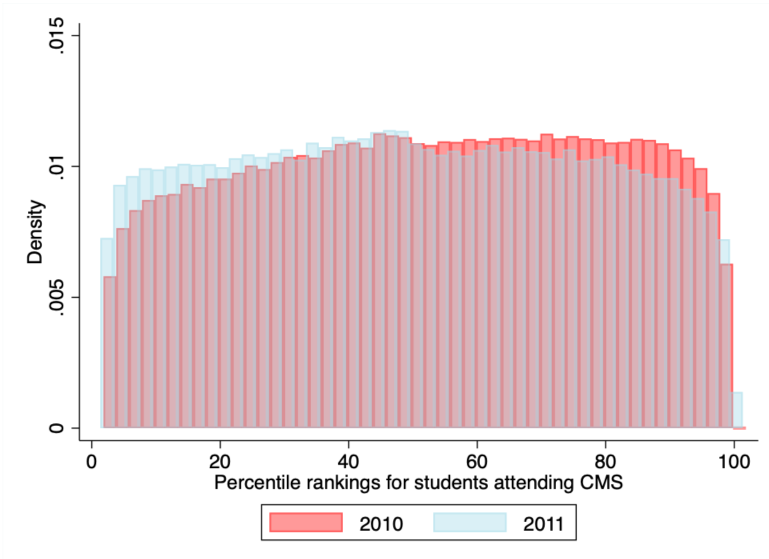
In the second year which our micro data covers (2011), 13 of the NPA schools switch to becoming PA schools and left the CMS. During our sample period only the students who were above the 50th percentile in their middle schools could enter the lottery of these schools. Therefore, these switches create a shock to the entry cohort peer composition at NPA schools. The graph at the top in Figure 1.7 plots the densities of percentile rankings of students who attended the CMS on top of each other for 2010 and 2011. We observe that the share of students who are better ranked in their middle schools has declined in 2011 compared to 2010. However, we do not observe a notable difference between the densities of 2011 and 2012, which is illustrated by the graph at the bottom. This suggest that the change in the distribution of students attending CMS is due to the switch of 13 schools from NPA to PA. Table 1.6 illustrates how much mean of percentile rankings of entering students in NPA schools change across years. In year 2011, mean rank percentiles of entering students change in NPA schools on average by 2.20 compared to 2010. The surge in the number of PA schools in 2011 plays an important role in identification of preferences over peers.

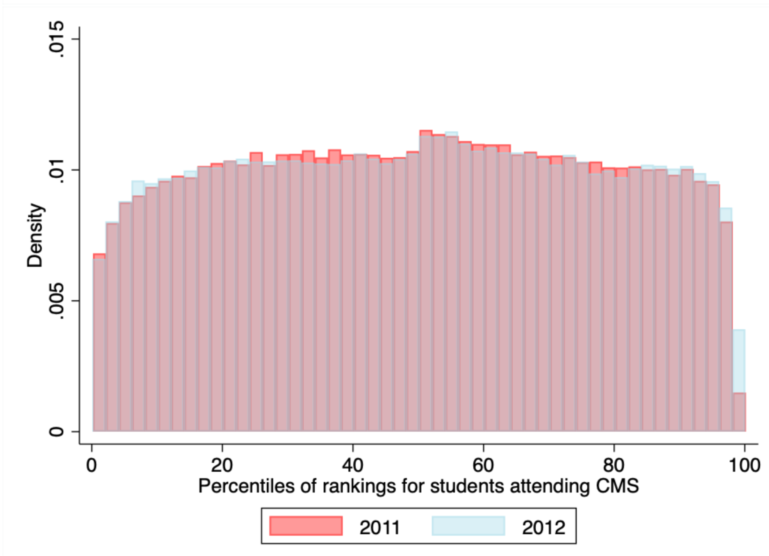
Table 1.6: Decline of Achievement of Students Enrolling in NPA Schools

Dependent variable: $m.p.r.$ of enrolled students	
Year 2011	-2.20 (0.66)
Year 2012	-2.09 (0.66)
Constant	51.57 (0.44)
R^2	0.02
Observations	593

Notes: This table indicates the OLS regression of mean of percentile rankings in their middle schools of enrolled students to high schools on year indicators for 2011 and 2012. Robust standard errors are in parenthesis.

Figure 1.7: Shock to Distribution of Student Achievement





Notes: The graph on the top illustrates the change in the distribution of percentile rankings in middle schools for students who attend the CMS for the years 2010 and 2011. The graph on the bottom illustrates this difference for the years 2011 and 2012. These graphs are created by plotting the densities for the two consecutive years on top of each other.

1.4 Empirical Model

In this section we describe our empirical model, i.e. our specification of student preferences and how we model their decision process in the general high school market. Also, we discuss how we identify the key parameters of our model.

1.4.1 Student Preferences

Let PA_t denotes the set of PA schools, and NPA_t be the set of schools which are NPA in year t. Also let a and g stand for affirmative action and general track, respectively. In this subsection we describe the preferences $v_{i,j,t,k}$ of student i on program $j \in PA_t \cup NPA_t$ such that student i enters high school in year t and he or she is in the track $k \in \{a,g\}$ for PA schools application purposes. Each student has a percentile ranking $p.r_i$ in their middle school which indicates the percent of people who performs worse than student i in his/her middle school. Note that as $p.r.$ increases, student's academic achievement increases. We group student i's $p.r_i$ into deciles dec_i, i.e. $dec_i := \lceil p.r._i/10 \rceil$. We assign students into cells. Cells are determined by interaction of gender, $gen_i \in \{f,m\}$, t and dec_i where f denotes female and m denotes male. Therefore, there are 60 cells in total. Formally, a cell, c is a bijective function defined as follows.

$$c : \{f,m\} \times \{2010, 2011, 2012\} \times \{dec \in \mathbb{Z} : 1 \le dec \le 10\} \to \{x \in \mathbb{Z} : 1 \le x \le 60\}$$

Since for a given student i, arguments of c is determined we denote the cell i is assigned by $c(i)$. Also, let c_χ^{-1} denote the χ of cell c for $\chi \in \{gen, t, dec\}$.

We also aggregate cells into groups $gr(c)$ by student performance in middle school to aggregate the preferences regarding some of the school characteristics. We choose groups such that we still allow preferences to be heterogenous for students with different academic achievement levels, since we are interested in heterogeneity with respect to ability dimension. We use dec_i for grouping the cells in the following way.

Cells which corresponds to students in the first three deciles are in group 3, students in the next four deciles according to $p.r$ are in group 2, and students who are in the last three deciles are in group 1.[50] These groups, differing in terms of students' academic performance in middle school, will allow us to compare students with different academic achievement level in terms of their preferences for peers.

In our specification we allow students' preferences to depend on: distance (in miles) D_{ij} between program j and student i's geocoded address; tuition of the program $p_{j,t,k}$ which also depends on the track student applies; mean of the middle school rank percentiles of students assigned to school j in year t, $m.p.r_{j,t}$ which is our peer composition measure[51]; other observables of program j in year t, $X_{j,t}$, this includes whether school is a charter school, a science program in a school, a co-ed school, a male-only school or a female-only school; the heterogenous effect of neighborhood of the program j , $\gamma_{nb(j),gr(c(i))}$ where $nb(j)$ denotes the neighborhood of program j; cell level intercepts $\gamma_{c(i),0}$; Type 1 Extreme Value with location parameter 0 and scale parameter 1, ε_{ij}; a mean utility term regarding school j which is same for every student entering high school in year t and is in track k for autonomous school admission purposes, $\delta_{j,t,k}$.

Since we can observe individual preferences for the schools in CMS, for these schools we can allow for additional heterogeneity in preferences, $1\{j \in NPA_t\}\lambda_{c(i),j,t}$ which can be interpreted as cell level shocks to match quality between students in cell $c(i)$ and program $j \in NPA_t$.[52] Another detail regarding the preferences is about neighborhood fixed effects. There are 10 neighborhoods in which only PA schools are present during the data period. For these neighborhoods we choose to allow for only homogenous preferences, since it would be hard to identify the heterogeneity without any individual level data provides information about preferences for these neighborhoods. Let $NBC = \{x : \exists (j,t)\, s.t.\, nb(j) = x\, for\, some\, t \in \{2010, 2011, 2012\}\, and\, j \in NPA_t\}$ be the set of neighborhoods of the schools joined the centralized match at least once. Formally we assume the following: $nb(j') \notin NBC \implies \gamma_{nb(j'),gr(c(i))} = \gamma_{nb(j')}\, \forall gr$

[50]Formally we define groups as follows: $gr(c(i)) := \begin{cases} 1 & if\, dec_i \geq 8 \\ 2 & if\, 3 < dec_i < 8 \\ 3 & if\, dec_i \leq 3 \end{cases}$

[51]i.e. $m.p.r_{j,t} = \frac{\sum_i (p.r._i) 1\{i \text{ is enrolled in } j \text{ in year } t\}}{\sum_i 1\{i \text{ is enrolled in } j \text{ in year } t\}}$

[52]Note that $c(i)$ already specifies the year t, but to remind to reader that λ varies by year, we add the t subscript to λ

We specify $v_{i,j,t,k}$ in the following way:

$$v_{ijtk} = \delta_{j,t,k} + \alpha_{c(i)} D_{ij} + (o.r.p_i)\beta_2 p_{j,t,k} + \gamma_{1,gr(c)} m.p.r_{j,t} + \gamma'_{2,gr(c)} X_{j,t} + \gamma_{nb(j),gr(c(i))} 1\{nb(j) \in NBC\} + \gamma_{c(i),0} + 1\{j \in NPA_t\}\lambda_{c(i),j,t,k} + \varepsilon_{ij} \tag{1.4.1}$$

where the mean utility $\delta_{j,t,k}$ depends on $p_{j,t,k}$, whether the school j is autonomous in year t and school, year and track level unobservables $\xi_{j,t,k}$ as in:

$$\delta_{j,t,k} = \beta_0 p_{j,t,k} + \beta_1 1\{\text{j is aut. in } t\} + \gamma_{nb(j)} 1\{nb(j) \notin NBC\} + \xi_{j,t,k} \tag{1.4.2}$$

As will be discussed in Section 1.4.2, students will need to consider their expected utilities from the centralized match when deciding to apply to an autonomous school. Therefore it will be convenient to employ a two step estimation procedure. Therefore, we will be estimating some of the preference parameters from the first step which involves only the schools that attend CMS. Therefore, it will be handy to consider what equation (1.4.1) looks like if we restrict attention only to NPA schools. If we normalize the tuition of non-autonomous schools to 0, then for $j \in NPA_t$ can write the preferences regarding non-autonomous schools only as follows:

$$v_{i,j,t} = \alpha_{c(i)} D_{ij} + \gamma_{1,gr(c)} m.p.r_{j,t} + \gamma'_{2,gr(c)} X_{j,t} + \gamma_{nb(j),gr(c(i))} + \gamma_{c(i),0} + \eta_{c(i),j} + \varepsilon_{ij} \tag{1.4.3}$$

where the match quality of students in cell $c(i)$ and program j in year t is defined as $\eta_{c(i),j} := \xi_{j,t,k} + 1\{j \in NPA_t\}\lambda_{c(i),j,t,k} for k \in \{a,g\}, j \in NPA_t$.

This means that we assume match quality between student and programs in NPA varies at the cell level[53], and through which track student enters autonomous school does not matter for match quality between the student and a program in CMS.

[53] Remember that $c(i)$ also specifies a year t, so variation in cell level includes variation across years

1.4.2 Two Step Decision of Students

1.4.2.1 First Step: Decision to Apply to an Autonomous School

When student i decides to go to a general high school after finishing middle school, he or she faces the following problem. First, student i has to decide whether to apply and which PA school to apply since he or she can apply to only one PA school. Also, the student knows that these schools admit using lotteries. Moreover, we assume that the student knows the capacities of autonomous schools and number of applicants to each school. For student i to apply to a PA school we need:

$$v_{ij} \geq U_c^i \text{ for some } j \in PA \tag{1.4.4}$$

where U_c^i denotes the student's expected payoff from the centralized match under the assumption that he plays optimally in the centralized match given what other students play. This means there must be at least one PA school that gives the student as high utility as his expected payoff from the centralized match. If there is such school, the student should apply the school that gives him the highest expected utility from doing so. Therefore to apply school $j \in PA$ it must be the case that:

$$\min\{\frac{q_j}{\eta_j}, 1\}(v_j^i - U_c^i) \geq \min\{\frac{q_k}{\eta_k}, 1\}(v_k^i - U_c^i) \ \forall k \in PA \tag{1.4.5}$$

where q_k denotes the capacity of school k; and η_k denotes the number of applicants to school j in the track he or she belongs to (affirmative or general). Therefore, the student incorporates his/her entry chances to PA schools when deciding which one to apply since he/she can apply to only one of them.

1.4.2.2 Second Step: Submitting Preference List for the Schools in the Centralized Match

If a student is rejected in the first step or decide not to apply to a PA school, then he or she attends the centralized matching process. Note that the CMS is not strategy-proof. Therefore, students do not necessarily submit their truthful ordering of schools to the mechanism. Therefore students would submit the list that they think that maximizes their expected utilities: i.e. they should choose π^* such that:

$$\sum_{j\in NPA} \hat{\phi}_{ij}(\pi^*)v_{ij} \geq \sum_{j\in NPA} \hat{\phi}_{ij}(\pi)v_{ij} \; \forall \pi \in \Pi$$

where Π is all the possible rank ordered lists student i can submit and $\hat{\phi}_{ij}(\pi')$ denotes the student i's belief of admission to program j when he submits π'.

1.4.3 Identification

1.4.3.1 Identification of Truthful Rank Ordering of Schools From Strategic Reports

Since students do not necessarily submit the truthful rank order of schools to the mechanism, it is challenging to identify the truthful ordinal ranking regarding schools that attend CMS. We follow the approach of Hwang (2017) in identifying the truthful ordinal preferences from the manipulable mechanism CMS. Hwang (2017) identifies (some of) the truthful ordinal preferences regarding schools. This identification strategy relies on a minimal assumption about the sophistication of students in terms of calculating the entry probabilities. Hwang (2017) only assumes that students know that whether an alternative list other than the submitted list increase or decrease their chance to a school. This assumption allows to identify ordinal preferences of students over many pairs of schools.[54] In our estimation, we use these identified truthful ordinal preferences.For the sake of completeness we repeat the identification argument in Appendix A.2.

1.4.3.2 Identification of Preferences Regarding Peer Academic Achievement in High School-s

Our identification strategy for students' preferences for peers relies on an instrumental variable that is defined only for NPA schools and the truthful pairwise comparisons of students over schools joining CMS. To formally show the identification challenge we separate match quality $\eta_{c(i),j}$ into two parts: $\eta_{c(i),j} = \tilde{\xi}_{j,t} + \tilde{\eta}_{c(i),j}$. The first term is program level unobservables, that is same for

[54]In total we identify 34,556,705 pairwise comparisons for 250,819 students.

all students joining the CMS in year t. The second term captures cell level idiosyncratic shocks to preferences.

The challenge in identification of preferences regarding the peers is that the peer quality measure $m.p.r_{j,t}$ is potentially an endogenous variable. That is, we might have $Cov(m.p.r_{j,t}, \tilde{\xi}_{j,t}) \neq 0$. An unobservable feature of school j, which is in $\tilde{\xi}_{j,t}$, might be more appealing to some type of students than others. For example a school with more homework, assignments, harder exams or with better counseling might be more appealing to academically stronger students. This might increase the sorting of such students to those schools which also increases the measure of academic achievement of high school cohort. In this case it would be hard to identify if students like the school because of this unobservable feature or because there are more successful peers in the school. Moreover, we are also interested in heterogeneity in preferences for peers. Therefore, we allow preferences over peer composition to vary by academic strength of students. This implies that, one should also be worried about the correlation between interaction of student group $gr(c(i))$ with $m.p.r_{j,t}$ and heterogeneity of preferences regarding the school unobservables across different $gr(c(i))$, which can be interpreted as match quality between the school and type of the student. This latter type of unobservable heterogeneity of preferences are inside the composite error term, $\eta_{c(i),j}$.

Our IV approach relies on the change of mean achievement levels of students enrolled to NPA schools due to switching of 13 NPA schools to PA school in 2011. We use the differential changes in school peer composition across years due to the events of switching. As shown in Section 1.3, academic achievement of entering cohorts to NPA schools has declined on average. However, it is natural to expect that the switch of schools affected the NPA schools near the switchers of 2011 the most. Note that in 2012 there is no new switcher, and only one of the schools which was designated as autonomous in 2011 has lost its autonomy status. Our instrument is defined for each program joining the CMS for each year.

For program $j \in NPA_t$, the instrument $Z^{peer}_{j,t}$ in year t is the number of PA schools in year t among the switchers of 2011 for which j satisfies all of the following:

i) j is among the closest three NPA schools to the PA school in year t

ii) j is within 2.5 miles distance from the PA school in year t

iii) j have the same school gender with the PA school in year t, i.e. if the switching school is male-only, j has to be male-only, if former is co-ed, then the latter must be co-ed and similarly for female-only schools.

For clarity we define the instrument $Z_{j,t}^{peer}$ formally. Schools switched in 2011 are in the set $SW_{2011} := PA_{2011} \setminus PA_{2010}$ and among these, schools in the set $SW_{2012} := PA_{2012} \setminus PA_{2010}$ remain as PA in 2012. Let $gd(j,t)$ denotes the gender of the school j in year t, where this function takes values from the set $\{fo, mo, co\}$, the elements of which corresponds to female-only, male-only, co-ed schools, respectively.

$$Z_{j,t}^{peer} := \begin{cases} 0 & if\, t = 2010 \\ \sum_{k \in SW_t} (1\{j \in NPA_t : j\, is\, among\, closest\, 3\, NPA\, schools\, to\, k\, at\, year\, t\} & \\ \times 1\{j \in NPA_t : j\, is\, within\, 2.5\, miles\, of\, k\} \times 1\{gd(j,t) = gd(k,t)\}) & if\, t = 2011, 2012 \end{cases}$$

Note that the instrument is zero for all the schools in CMS for the year of 2010. And remember that we are controlling for neighborhood fixed effects. This means that we have two kinds of identifying variation. First one is the one in $m.p.r._{j,t}$ across years for the schools for which the instrument is positive in 2011 or 2012. The second one is variation of $m.p.r_{j,t}$ between the schools in the same neighborhood in year 2011 or 2012 if they had different exposure to switchers.

Exclusion restriction requires that the instrument should affect the students' decision between the schools in CMS only through its effects on entry cohort peer composition measure. In other words, the comparison of two schools in CMS should not depend on whether a school is a close competitor to a switching school for reasons other than the effects of switching on peer composition measure. This would be violated if schools in CMS react to switching of the schools systematically differently, and this reaction was noticed by students. For example schools nearby switching schools may start to change their effort in attracting students or providing education. However, this is unlikely to happen in our setting due to high school equalization policy. All schools in the centralized match are operated like public schools. They cannot change their tuition or curriculum, which limits the set of possible reactions for these schools. Also, they heavily rely on subsidies, and public schools' teachers rotate every couple of years, which can be argued to lead to little motivation to make any change. Therefore, schools participating in CMS seem to have small incentives for reacting to a switch of a school nearby and their reaction capabilities seem limited. Moreover, even if there is considerable change in the effort for providing education, it may take years to be noticed by students/families. And our instrument is using the effect of switchers for at most one year forward.

Exogeneity of our instrument might be harmed if there is selection to the instrument. Remember that, sound finance is the most important criterion for being designated as autonomous school. But sound finance can be affected by the location of the school. For example, rents can depend on location, or schools in richer neighborhoods may attract students of richer families more easily, hence get more donation. Since our instrument depends on location one might be concerned about having instrument positive for schools in more advantageous locations which is potentially not excluded from preferences. We alleviate this concern by controlling for neighborhood fixed effects in our specification. The advantage of our identification strategy compared to a strategy of using fixed effects of schools is discussed in Appendix A.2.

1.4.3.3 Identification of Preferences Regarding Tuition

We cannot identify preferences regarding tuition from the pairwise comparisons obtained from the CMS. This is because, tuition does not vary among the schools joining the CMS. Tuition varies between PA schools, so a natural first candidate for identification is to use BLP (Berry, Levinsohn, and Pakes, 1995) type instruments for the tuition of PA schools. However, it turns out that the variation in the prices of PA schools is very small economically, therefore these type of instruments are not enough to identify preferences for tuition separately from preferences for private autonomy. An alternative identification strategy is to use the price difference for general track versus affirmative track students. However, one has to be careful since these two group of students may differ in their preferences regarding autonomy. We use indirect inference method to add the information from our RD approach to structural estimation, as in Larroucau and Rios (2020). Using these RD moments helps identifying the correct mean utility difference at the income threshold cutoff for the two groups, which makes sure that the mean utility difference is not due to different preferences regarding autonomy for these two groups of students but due to tuition difference they face.

1.5 Estimation of the Model

We use two step procedure to estimate the preferences of students. Such estimation method is convenient given the two step nature of students' decision and necessity to calculate the expected utilities from the CMS. The parameter vector θ we estimate can be grouped into two parts: $\theta = (\theta_1, \theta_2)$. θ_1 includes the preference parameters regarding school and peer characteristics excluding price and autonomy, distance preferences and neighborhood fixed effects (heterogenous across groups) for NPA schools. So, $\theta_1 = (\{\alpha_c\}_{c=1}^{60}, \{\gamma_c\}_{c=1}^{60}, \{\gamma_{1,gr}\}_{gr=1}^{3}, \{\gamma_{2,gr}\}_{gr=1}^{3}, \{\{\gamma_{nb,gr}\}_{gr=1}^{3}\}_{nb \in NBC})$. Second part θ_2 includes preference parameters regarding autonomy, price and 10 neighborhood fixed effects for the neighborhoods that does not belong to a school that participated CMS any time, i.e. $\theta_2 = (\beta_0, \beta_1, \{\gamma_{nb}\}_{nb \notin NBC}, \beta_2)'$.

1.5.1 Estimation From Individual Level Pairwise Rankings

In the first step, we use the pairwise rankings over schools participating in CMS, identified via method of Hwang (2017), to estimate θ_1. We cannot estimate preferences regarding autonomy and price in this step, since all NPA schools charge the same tuition and none of them are autonomous. In this step of estimation we follow the approach of Hastings, Hortaçsu, and Syverson (2017). We can rewrite Equation (1.4.3) as follows:

$$v_{i,j,t} = \alpha_{c(i)} D_{ij} + \zeta_{c(i)j} + \varepsilon_{ij} \tag{1.5.1}$$

$$\zeta_{c(i)j} = \gamma_{c(i),0} + \gamma_{1,gr(c)} m.p.r_{j,t} + \gamma'_{2,g(c)} X_{j,t} + \gamma_{nb(j),gr(c(i))} + \eta_{c(i),j} \tag{1.5.2}$$

We determine one of the programs j' as outside option and set $\zeta_{c(i),j'} = 0$. We estimate Equation (1.5.1) by multinomial logit separately for each of the cells c separately by using pairwise rankings of students in cell c. For example if for student i, pairwise rankings indicates that i prefers A to B and A to C, then we can conclude that student chooses A from the set $\{A, B, C\}$. Then we use estimates of $\zeta_{c(i)j}$, $\hat{\zeta}_{c(i)j}$ combining it with data of covariates $X_{j,t}$, cell level indicators, $m.p.r_{j,t}$ and our instrument $Z^{peer}_{j,t}$ to estimate equation (1.5.2). We do this estimation using Two Stages Least Squares (TSLS). In TSLS, we weight the observations by the inverse of the variance of $\hat{\zeta}_{c(i)j}$, and double cluster standard errors at the program level and cell level.

1.5.2 Estimating the Preferences for Tuition and Autonomy

In the second step of the estimation we treat the parameter estimates θ_1 from the previous step (denote it by θ_1^*) as data and perform a BLP type estimation (Berry, Levinsohn, and Pakes, 1995) using aggregate moments regarding applications to autonomous schools; moments from middle schools regarding the share of students going to autonomous schools and mean of their rank percentiles; tuition data; price instruments; and RD moments simulated using our SECA data. The idea of the estimation for θ_2 is to find θ_2^* which makes the moments simulated using our structural model as close as possible to moments in the data.

1.5.2.1 Empirical Distribution of Students in General High Schools Market

To construct the moments using the structural model and parameter candidates, we need an empirical distribution of students in the combined market of NPA and PA schools to draw students from. We have individual level observations of students, but these data do not include students who went to PA schools. If we only used these data, the distribution of students would be incorrect. We solve this problem by using the auxiliary survey data from 2009, which asks students who will start high school in 2010, whether they plan to apply private autonomous schools. However, this survey data is only available for students attending CMS. We use the fact that the students are admitted to PA schools through random lottery. To construct the empirical distribution we assume that the admitted students are not fundamentally different from the students who applied but could not get in. Further details are in Appendix A.2.

Another issue for the empirical student distribution is that we do not observe family size and and family income in the data from SOE. Hence we do not know whether a student is in the affirmative action track or general track for the purposes of the entry into PA schools. We solve this issue by using the 7th grader cohort in the additional SELS data. As a result we predict the probabilities of being in the affirmative action track for each student in the empirical student distribution. Details are in Appendix A.2.

These probabilities implies weights of each observation in each track of entry to PA schools. Once we know the weight of students in different cells for affirmative action and general tracks, we can estimate $\xi_{j,t,k} \forall j \in NPA_t$ and $\forall k \in \{a,g\}$ using estimates of $\eta_{c(i),j}$, $\eta_{c(i),j}(\theta_1^*)$.[55] We denote these estimates by $\xi_{j,t,k}(\theta_1^*)$.

1.5.2.2 Moments and GMM Estimation

We describe how to construct moments used in BLP estimation, for a parameter candidate $\hat{\theta}_2$. These moments are: middle school share moments $(mssh(\hat{\theta}_2) - mssh)$ which is a vector indicating the difference between model predicted share of students going to PA schools and actual share from the middle schools for which such data is available in 2011 and 2012; $(mspr(\hat{\theta}_2) - mspr)$

[55] Let $w_k(c)$ denote the total weight of all students in cell c for track $k \in \{a,g\}$. We calculate the estimates of $\xi_{j,t,k} \forall j \in NPA_t$, $\xi_{j,t,k}(\theta_1^*)$ as follows:

$$\xi_{j,t,k}(\theta_1^*) = \frac{\sum_c 1\{c_t^{-1} = t \text{ and } c_{dec}^{-1} \leq 5\} w_k(c) \eta_{c,j}(\theta_1^*)}{\sum_c 1\{c_t^{-1} = t \text{ and } c_{dec}^{-1} \leq 5\} w_k(c)}$$

which is a weighted average of estimates $\eta_{c,j}(\theta_1^*)$ where weights are determined by the total weights of students in each cell for each track.

which includes a vector of differences between the predicted mean of the middle school percentile rankings of students who goes to PA schools and actual mean for the same middle schools as above and for the same years; and it also includes the difference between model predicted mean of the middle school percentile rankings of all students going to PA schools and the actual mean for the years 2011 and 2012; $\sum_{j,t,k} Z^{BLP'}_{j,t,k} \xi_{j,t,k}(\hat{\theta}_2)$ where $Z^{BLP}_{j,t,k}$ is constructed using the price instrument and indicators of exogenous variables which are autonomy and 10 neighborhood indicators, and $\xi_{j,t,k}(\hat{\theta}_2)$ is an estimate of $\xi_{j,t,k}$; $(rdest(\hat{\theta}_2) - rdest)$ which is difference between RD estimates generated with the simulated outcomes from the model and reduced form RD estimates from SECA dataset; and moments of application shares for PA schools, i.e. what share of student applied to PA school $j, \forall j \in PA_t$ and what share did not apply them at all. The last moments will be exactly matched to the application shares in the data by Nested Fixed Point (NFP) algorithm to back out the mean utility terms $\delta_{j,t,k}$.

Our goal of constructing the moments is to find θ_2^* that minimizes the GMM criterion function:

$$\theta_2^* = \arg\min_{\hat{\theta}_2} g(\hat{\theta}_2)' W g(\hat{\theta}_2)$$

where $g(\hat{\theta}_2)$ is the stacked moment conditions multiplied by square root of the sample sizes used to construct them and evaluated at $\hat{\theta}_2$[56], W is a positive definite weighting matrix.

Generating Moment Conditions From the Model: As parameters in θ_2 except β_2 enter the utility linearly, we can restrict the parameter search to β_2 since $\theta_2 \setminus \beta_2$ can be estimated by linear GMM estimation. When constructing the middle school moments and regression discontinuity moments, for each student in the empirical distribution and SECA data respectively, we need to calculate whether he/she applies to an autonomous school. Similarly when constructing the application shares we need to calculate the same quantity and which autonomous school the student applies for each drawn student from the empirical distribution. We do this by using estimates of θ_1 from the first step, Equation (1.4.4), Equation (1.4.5), candidate parameter for β_2, candidates

[56]Appendix A.2 illustrates the sample sizes in stacked moments explicitly.

for $\delta_{j,t,k}$ for (j,t) such that $j \in PA_t$ as well as entry chances to PA and NPA schools.[57] This task requires calculation of expected utility from the centralized match, which is a computationally heavy task. Further details on calculating the application behavior can be found in Appendix A.2.

Application Shares and NFP Algorithm: We have six markets since for each year 2010, 2011 and 2012 we also separate the market for affirmative action track and general application track. This is convenient as out of pocket price is different in these two markets; and this way we can use track of student as an instrument for price.[58] To back out mean utility terms $\delta_{j\in PA,t,k}$, we match application shares instead of market shares because application shares would be more informative about preferences in a capacity constrained setting. Not everyone who desires can enter autonomous schools. Moreover, trying to match market shares may not yield unique mean utilities, since any level of mean utility that makes a school over-demanded will match the market share of the school for schools with binding capacity. Further details about matching application shares through NFP Algorithm is given in Appendix A.2.

GMM Moment Conditions: The construction of instrument moments $\sum_{j,t,k} Z_{j,t,k}^{BLP'} \xi_{j,t,k}(\hat{\theta}_2)$; middle school moments $(mssh(\hat{\theta}_2) - mssh)$ and $(mspr(\hat{\theta}_2) - mspr)$; and RD moments $(rdest(\hat{\theta}_2) - rdest)$ is in Appendix A.2.

Weighting Matrix: We run two step GMM estimation to obtain the parameter estimates (Hansen, 1982).[59] We do this by first estimating the model via a positive definite weighting matrix to calculate the optimal weighting matrix. In second iteration we use the optimal weighting matrix to estimate the parameters. Details of constructing weighting matrix are in Appendix A.2.

[57] We also need to draw also ε from T1EV(0,1) since Equation (1.4.4) and Equation (1.4.5) does not yield closed form choice probabilities as in usual BLP setting. We draw these errors once at the beginning of the estimation procedure.

[58] Moreover, price coefficient can remain as linear parameter, this would reduce the computational burden.

[59] We use Nelder-Mead simplex search method to find the optimal parameter that minimizes the GMM objective function. Specifically, we use *fminsearch* function of MATLAB with tolerance levels set to 1e-7.

1.6 Estimation Results

Table 1.7 illustrates the estimates from the first stage. Remember, as *m.p.r.* measure increases for a school, this means students better ranked in their middle schools are enrolled to that high school. As expected distance is estimated to have a negative effect on preferences for schools. The effect is slightly larger for groups which are less successful in middle school, and standard deviations of the estimates are small. The best group of students in terms of middle school performance cares most about the academic success of peers. As peer success measure in a school improves, these group of students get more utility compared to other groups of students. The effect for this group is statistically significant at 5% level and it is equivalent to a decrease in distance by around $1/3$ miles. For the other groups sign is again positive, which means they prefer to be with high performing peers, but these coefficient estimates are not statistically significant at 10% significance level[60]. This finding confirms our assumptions in the theory section, that higher performing students have stronger preferences for academically strong peers. For female-only schools we cannot reject the null hypothesis that female students do not have extra utility from them compared to co-ed schools. Nevertheless, the signs of the coefficients are positive. Male students prefer male-only schools to co-ed schools, and academically better students have stronger preferences for such schools. The group with strongest academics prefer charter schools to other NPA schools, which is equivalent to a decrease in distance around $1/5$ mile. This effect is statistically significant at 10% significance level. In general, all groups have negative preferences for science programs. However, the negative effect is much higher for the least successful two groups.

Table 1.8 illustrates the estimates using Ordinary Least Squares (OLS) instead of TSLS. Most parameter estimates look similar to TSLS estimates. However, the preferences over student composition is overestimated for high-performing students and it is underestimated for other two groups. Therefore the heterogeneity of preferences regarding student composition is over-estimated in OLS.

[60]The effect of m.p.r. is statistically significant at 15% level for the group that have $30 \leq p.r. < 70$

Table 1.9 illustrates the parameter estimates from the second step of the estimation. As expected tuition has negative effect on utility. Increase of tuition by a million KRW is equivalent to around two thirds of a mile increase in distance to school for the students with lowest rank percentile. The effect is a little stronger for students who have higher rank percentiles. As student's rank percentile changes from 0 to 100, his/her disutility from price increases by 0.13. Both of these coefficient estimates are statistically significant at 5% significance level. Autonomy coefficient is estimated as positive and close to the effect of a decrease in distance by one third of a mile. However, it is not significant at conventional significance levels.[61]

1.7 Counterfactual Simulations

In this section we compare alternative ways of allocating school seats to students. In Section 1.2 we have shown that choice between SA and DA might matter in terms of high-performing students' sorting towards PA schools depending on the tuition level. Using simulations with the estimated preferences will allow us to test this theory and also to understand where the observed prices in the data stands.

Moreover, comparing SA and DA is another instance of comparison of mechanisms where students can strategize to get the best outcome for themselves versus mechanisms where truthfully revealing preferences is optimal.[62] In our case, SA is a representative of the former type. Students need to consider their admission chances for the second step while forming their applications for schools admitting in the first step. In contrast, DA with unrestricted list length we consider, is a representative of the latter.[63] Differently from previous work without peer effects, the preferences

[61] Autonomy coefficient has p-value 0.108.

[62] Such comparisons has been a topic of central discussion in theoretical and empirical literature (Abdulkadiroğlu, Che, and Yasuda, 2011; Pathak and Sönmez, 2013; Agarwal and Somaini, 2018).

[63] In our case preferences of students depends on the strategy profile of students. Therefore, we cannot exactly say that DA is strategy-proof in this setting. However, we can argue that we are in a large market given the size of student population; therefore we assume that one student's submitted preferences can not change the distribution of $m.p.r$ across schools. Then, for a given strategy profile, the expected payoff of each student from each school is well defined and is independent of his strategy. For this reason, for a given strategy profile we are under the usual DA case in terms of the optimal decision of a student. Consequently, arguments showing that DA is strategy proof (Dubins and Freedman, 1981) can be used here, as well. This guarantees that the student cannot do better than submitting his/her

Table 1.7: Parameter Estimates From the First Step

	(1) $p.r \geq 70$	(2) $30 \leq p.r. < 70$	(3) $p.r. < 30$
Panel A: Preferences regarding distance, estimated by multinomial logit			
Mean of Dist. coeff. (miles)	-1.51	-1.62	-1.61
Std. of Distance coeff.	0.11	0.09	0.12
Panel B: Preference parameter estimates from TSLS			
Dependent variable: ζ_{cj}			
m.p.r. (standardized)	0.54	0.38	0.21
	(0.27)	(0.26)	(0.41)
Female-only school	0.23	0.30	0.08
	(0.53)	(0.51)	(0.82)
Male-only school	1.22	0.98	0.47
	(0.14)	(0.11)	(0.20)
Charter school	0.33	0.13	0.01
	(0.19)	(0.15)	(0.18)
Science program	-0.31	-1.23	-1.92
	(0.13)	(0.14)	(0.19)
Neighborhood and Cell Level F.E.	Yes	Yes	Yes
First Stage F-stat.	14.93	14.07	11.56
R^2	0.83	0.82	0.74
Observations	2673	3564	2673

Notes: This table illustrates the preference parameter estimates from the first stage of estimation. Distance coefficients are calculated using multinomial logit separately for each of 60 cells, and their means and standard deviations within each cell group are in Panel A. Other parameter estimates from the first step are in Panel B. These are obtained running TSLS separately for each the three cell groups, which is determined by deciles of students ranking within their middle schools. For each column in Panel B, the dependent variable is ζ_{cj} where c belongs to the group specified by percentile rankings specified at the top of the table. ζ_{cj} is also estimated alongside distance parameters for each cell using multinomial logit. We standardize $m.p.r.$ in each school by the mean and standard deviation of $m.p.r.$ of NPA schools across the three years; and we instrument $m.p.r.$ by $Z^{peer}_{j,t}$. For TSLS, each observation is weighted by the inverse of the variance of the estimate of ζ_{cj}. Standard errors are in parenthesis. Double clustering at the cell and school level is used to calculate standard errors. First Stage F-stat is F-stat of excluded instruments in the first stage.

Table 1.8: Parameter Estimates From First Step Using OLS

	(1) $p.r \geq 70$	(2) $30 \leq p.r. < 70$	(3) $p.r. < 30$
Panel A: Preferences regarding distance, estimated by multinomial logit			
Mean of Distance coeff. (miles)	-1.51	-1.62	-1.61
Std. of Distance coeff.	0.11	0.09	0.12
Panel B: Preference parameter estimates using OLS			
Dependent variable: ζ_{cj}			
m.p.r. (standardized)	0.61	0.30	0.00
	(0.06)	(0.05)	(0.05)
Female-only school	0.08	0.46	0.49
	(0.14)	(0.13)	(0.14)
Male-only school	1.19	1.01	0.55
	(0.12)	(0.09)	(0.11)
Charter school	0.29	0.17	0.10
	(0.10)	(0.09)	(0.09)
Science program	-0.33	-1.20	-1.86
	(0.11)	(0.12)	(0.11)
Neighborhood and Cell Level F.E.	Yes	Yes	Yes
R^2	0.83	0.82	0.75
Observations	2673	3564	2673

Notes: This table illustrates the preference parameter estimates from the first stage of estimation using OLS. Distance coefficients are calculated using multinomial logit separately for each of 60 cells, and their means and standard deviation within each cell group are presented in Panel A. Estimates of other preference parameters from the first step are presented in Panel B. These estimates are obtained using OLS separately for each cell of the three cell groups, which is determined by deciles of students ranking within their middle schools. For each column in Panel B, the dependent variable is ζ_{cj} where c belongs to the group specified by percentile rankings specified at the top of the table. Note that ζ_{cj} is also estimated alongside distance parameters for each cell using multinomial logit. We standardize mean rank percentile in each school by the mean and standard deviation of mean rank percentiles of NPA schools across the three years; and we instrument m.p.r. by $Z^{peer}_{j,t}$. In OLS each observation is weighted by the inverse of the variance of $\hat{\zeta}_{cj}$. Standard errors are in parenthesis. Double clustering at the cell and school level is used to calculate standard errors.

Table 1.9: Parameter Estimates From the Second Step

	Parameter	Estimate
Tuition (KRW in millions)	β_0	-1.13 (0.11)
Autonomy	β_1	0.45 (0.28)
Tuition×(p.r./100)	β_2	0.13 (0.05)

Notes: This table illustrates the parameter estimates from the second step of the estimation. In second step we also estimate neighborhood effects for the 10 neighborhoods, the effects of which were not estimated in the first step of estimation. For brevity we do not include estimates of neighborhoods here. Standard errors are in parenthesis.

depend on the given strategy profile in our case.

We consider two variants of SA, which are SA One (SA1) and SA Unrestricted (SAU). The first one is similar to case of Seoul in terms of implementation of the first step. Students can apply only one PA school in the first step. In the second step students are allocated to NPA schools via DA. The second mechanism SAU consists of two consecutive DA mechanisms (both unrestricted length) which is motivated by the case of allocation of exam schools and other public schools in the US. Under both variants the optimal behavior of student is very similar to the optimal behavior we described in the empirical section for the case of Seoul. But it is actually simpler, as now the second step is DA as opposed to manipulable CMS.

Another comparison we would like to make is the case where all schools admit using lotteries versus a only subset of schools is allowed to have a pure score based admission scheme and other schools admit by lottery numbers. This is motivated by the existence of elite high schools in the US and around the world.

Under these mechanisms assignments are deterministic given lottery draws. Therefore we use an ex-ante measure of welfare as in previous literature. Our goal is to find the equilibria of the

truthful preferences.

submitted reports under these mechanisms. Students' preferences depend on peer composition at schools under DA and SA. Moreover, for the case of SA the optimal behavior of a student depends on entry chances to schools. Therefore, it is not enough to simulate these mechanisms only once. To find the equilibria, we iterate over the best responses of students until the peer composition measures and lottery cutoffs converge. In Appendix A.2, we describe how we compute the equilibria of these mechanisms.

1.7.1 Comparison of DA and Variants of SA

First, we compare students' expected utilities, mean middle school achievement of entering cohorts to PA schools and share of students entering PA schools under the prices of 2011 using empirical distribution of students in 2011.[64] We compare DA (with unrestricted length list) to two alternative ways of implementing SA as described in previous subsection. We still impose the 50 percent admission rule; and PA schools still reserve 20% of their capacity for affirmative action. Table 1.10 illustrates welfare comparisons under different mechanisms. The welfare numbers are in million won (around thousand dollars). Comparison of ex-ante welfare of students across different mechanisms shows that ex-ante average welfare is \$27 to \$51 of yearly tuition higher under DA compared to SA.[65] This difference is a little higher for higher performing students; and sign of the difference change for low performing students. However, these differences are not large. Panel A of Table 1.12 illustrates the weighted average of the *m.p.r.* across PA and NPA schools separately across different mechanisms. When taking the average across schools, number of students assigned to schools are used as weights. In terms of *m.p.r.* comparison between the mechanisms we have found that weighted average of *m.p.r.* in PA schools for the general track students is 76.74, 76.88, 76.94 under DA, SA and SAU respectively. And these numbers are 75.66, 75.71 and 75.68 when we also include the students in the affirmative track. For NPA schools, the weighted averages

[64] We increase the capacity of public schools by 2% so that no student remain unassigned during iterations of best responses. Results are not sensitive as trial with 10% increase in capacity constraints are very similar in terms of comparison of the mechanisms.

[65] In terms of miles this is equivalent to 0.02 to 0.04 difference in distance.

of $m.p.r.$ are 50.46, 50.83 and 50.77 respectively.

Table 1.10: Student Welfare in Counterfactuals Using 2011 Tuitions

Panel A: Students among highest 50% in their middle schools with respect to grades

Mechanisms	$p.r. \geq 75$	$50 \leq p.r. < 75$	General track	Affirmative track
DA	0.635	0.366	0.408	1.503
SA1	0.584	0.316	0.348	1.544
SAU	0.512	0.271	0.314	1.226

Panel B: Students among lowest 50% in their middle schools with respect to grades / *Panel C: All students*

Mechanisms	$25 \leq p.r. < 50$	$p.r. < 25$	All students
DA	-0.187	-0.625	0.101
SA1	-0.178	-0.627	0.074
SAU	0.161	-0.609	0.050

Notes: This table illustrates the welfare comparisons (in million won or thousand dollars) across the equilibrium of DA, SA1 and SAU mechanisms. DA corresponds to student proposing Deferred Acceptance Mechanism, SA1 corresponds to Sequential Admissions mechanism where students can apply to only one PA school in the first step. SAU corresponds to the Sequential Admissions mechanism where students apply to PA schools in an unrestricted list Deferred Acceptance fashion in the first step. The table demonstrates the average welfare for different types of students, as well as average across all students. To simulate the equilibria of these mechanisms, we start the iteration of best responses from 3 different initial submissions. Under all 3 starting points iterations converges to very similar equilibria, for each mechanism. In these counterfactuals, we keep the top 50% rule and affirmative action policy.

Second set of counterfactuals compares the mean middle school achievement of entering cohorts to PA schools in DA and variants of SA across different prices.[66] In our theoretical section we had shown that for higher (lower) prices, mean achievement was higher in private school under SA (DA) compared to DA (SA). Figure 1.8 illustrates the change in quality of students across different normalized prices for PA schools. Note that the x-axis is in million won and normalized

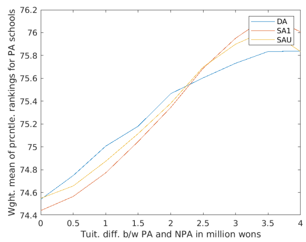
[66] For this comparison across different prices we needed to increase the public school capacity by 9% as when prices get very large number of applicants to public schools exceed the original capacity in public schools.

such that NPA schools charge 0 tuition. The y-axis indicates the student number weighted average of $m.p.r$ of PA schools in graphs (a), (c) and (e); in other graphs only general track students are included from PA schools as these are the students behavior of which the theory predicts. Higher the number in the y-axis, higher is the mean academic achievement of the entering cohort. Graph (b) indicates that counterfactuals agree with theory prediction: as the price increase SA yields higher mean student academic achievement for PA schools compared to DA.

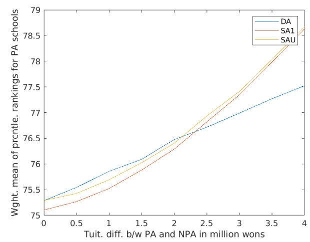
An important point to notice is that the differences in cohort qualities are small in both counterfactuals across mechanisms and prices. We have calculated these counterfactuals by keeping the top 50 percent rule and affirmative action admission policy in autonomous schools. Potentially these rules affect the sorting patterns to a great extent which may not have left much room for difference that may stem from mechanism choice. Therefore we also run these counterfactuals by eliminating the top 50 percentile and affirmative action rule. Panel A of Table 1.12 illustrates the $m.p.r.$ of these counterfactuals. At the observed tuition levels, the weighted $m.p.r.$ of PA schools are 54.22, 55.49, 54.02 under DA, SA1 and SAU, respectively. These numbers are 52.94, 52.98, 53.01 for NPA schools, following the same order of mechanisms. So without top 50 rule, the student achievement distribution is much more similar in PA and NPA schools, although still PA schools have higher $m.p.r.$ Similarly to the previous case, there is not much difference between admission schemes in terms of distribution of students in different sectors. Table 1.11 illustrates the welfare of students in million won, across different admission schemes. Compared to Table 1.10, not surprisingly affirmative action eligible students' welfare went down compared to case with top 50% rule and affirmative action rule. Similarly, there is also a decline in welfare of students in the top 50% in their middle schools, since they have higher probability of enrolling in schools with lower $m.p.r.$ The welfare of students in the bottom 50 percent is higher since high-achieving peers are more equally distributed across types of schools when top 50 percent rule is cancelled. Comparing the welfare across mechanisms, now the difference for the average student went up to around \$260 regarding the comparison between DA and SA, which is also equivalent to traveling 0.18 miles more to school every day, which does not seem large. When we compare to previous

Figure 1.8: Comparison Across Different Prices

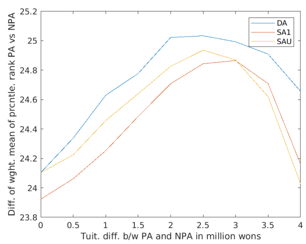
(a) Weighted average of $m.p.r.$ for PA schools



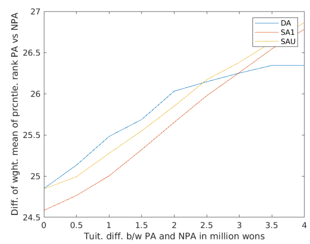
(b) Weighted average of $m.p.r.$ of general track students for PA schools



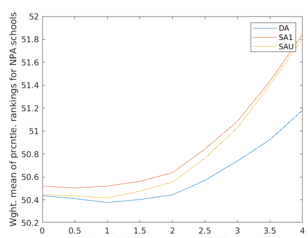
(c) Difference between PA vs. NPA by weighted average of $m.p.r.$ considering all tracks



(d) Difference between PA vs. NPA by weighted average of $m.p.r.$ considering general track for PA



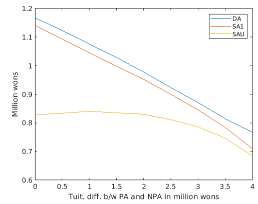
(c) Weighted average of $m.p.r.$ for NPA schools



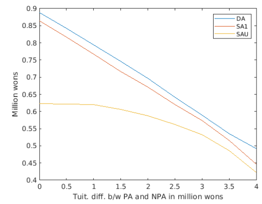
Notes: For each graph in this figure, the x-axis corresponds to the difference of tuition (in million won or thousand dollars) between PA and NPA schools. Across the tuition differences in the x-axis, tuitions of all PA schools are set the same. The y-axis corresponds to the average of the $m.p.r.$ measures of schools weighted by number of students across schools for the considered set of schools across different tuition differences. Each line corresponds to a different mechanism. The equilibria are calculated for 9 different tuitions in this range, with 0.5 million won difference between each. In each simulation top 50 percent rule for admission to lottery of PA school and affirmative action is kept.

Figure 1.9: Welfare Differences Between Admission Rules Across Different PA School Tuition

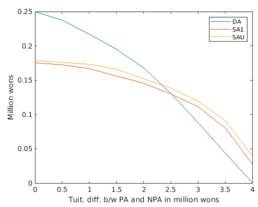
(a) Average welfare of students with $p.r. \geq 75$



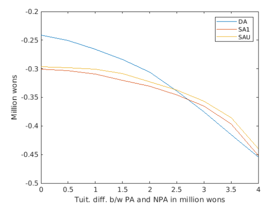
(b) Average welfare of students with $50 \leq p.r. < 75$



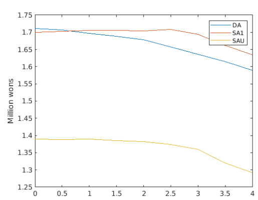
(c) Average welfare of students with $25 \leq p.r. < 50$



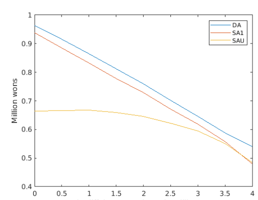
(d) Average welfare of students with $p.r. \leq 25$



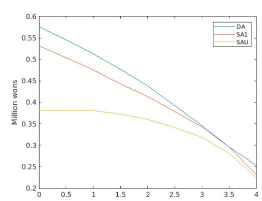
(e) Average welfare of affirmative action eligible students with $p.r. \geq 50$



(f) Average welfare of students not eligible for affirmative action track and with $p.r. \geq 50$



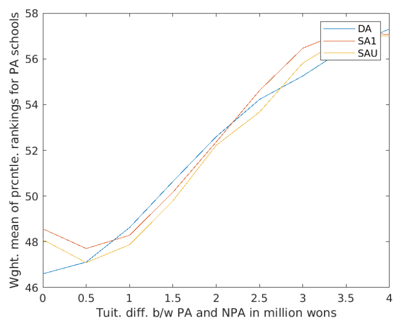
(c) Average welfare of all students



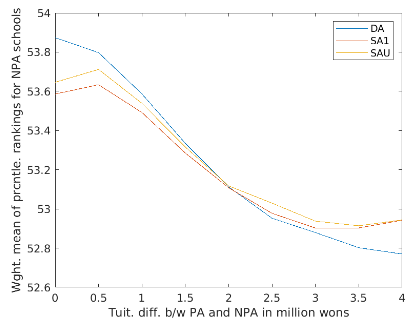
Notes: For each graph in this figure, the x-axis corresponds to the difference of tuition (in million won or thousand dollars) between PA and NPA schools. Tuitions of all PA schools are set the same. The y-axis corresponds to the average of the welfare of students (in million KRW or thousand USD). Each line corresponds to different mechanism. The equilibria are calculated for 9 different tuitions in this range, with 0.5 million KRW difference between each. In each simulation top 50 percent rule for admission to lottery of PA school and affirmative action is kept.

Figure 1.10: Comparison of *m.p.r.* without Top 50% Rule and without Affirmative Action

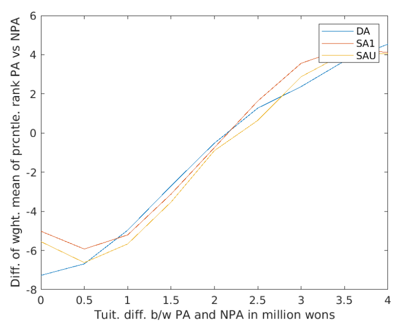
(a) Weighted average of *m.p.r.* for PA schools



(b) Weighted average of *m.p.r.* for NPA schools



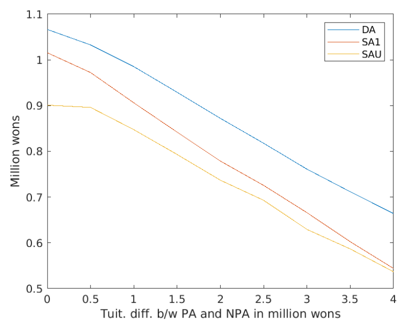
(c) Difference between PA vs. NPA by weighted average of *m.p.r.*



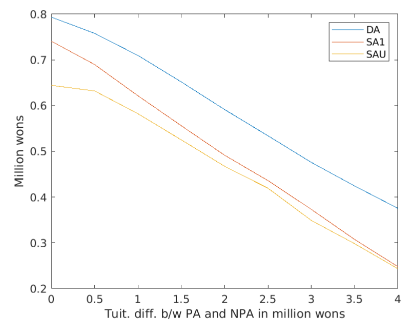
Notes: For each graph in this figure, the x-axis corresponds to the difference of tuition (in million won or thousand dollars) between PA and NPA schools. Across the tuition differences in the x-axis, tuitions of all PA schools are set the same. The y-axis corresponds to the average of the *m.p.r.* measures of schools weighted by number of students across schools for the considered set of schools across different tuition differences. Each line corresponds to different mechanism. The equilibria are calculated for 9 different tuitions in this range, with 0.5 million won difference between each. In all the simulations, top 50 percent rule for admission to lottery of PA school and affirmative action policy are not present.

Figure 1.11: Welfare Across Tuition Levels without Top50 and without Affirmative Action

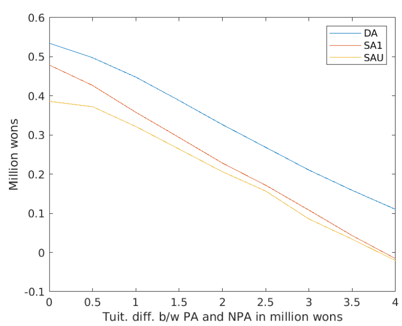
(a) Average welfare of students with $p.r. \geq 75$



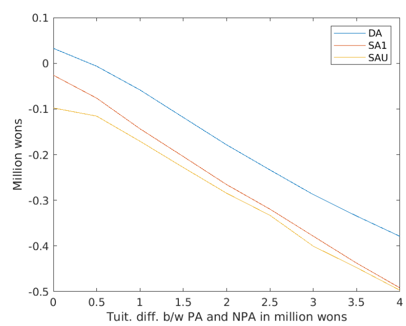
(b) Average welfare of students with $50 \leq p.r. < 75$



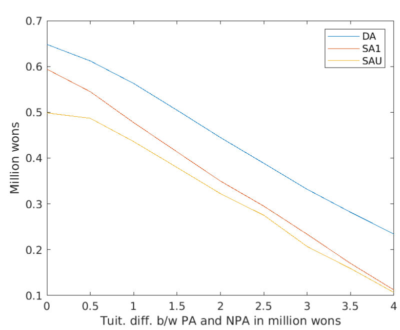
(c) Average welfare of students with $25 \leq p.r. < 50$



(d) Average welfare of students with $p.r. \leq 25$



(e) Average welfare of all students



Notes: For each graph in this figure, the x-axis corresponds to the difference of tuition (in million won or thousand dollars) between PA and NPA schools. Across the tuition differences in the x-axis, tuitions of all PA schools are set the same. The y-axis corresponds to the average of the welfare of students (in million won or thousand dollar). Each line corresponds to different mechanism. The equilibria are calculated for 9 different tuitions in this range, with 0.5 million won difference between each. In all the simulations, top 50 percent rule for admission to lottery of PA school and affirmative action policy are not present.

literatures' comparisons between centralized mechanisms it is smaller than the difference between them.[67] Figure 1.10 illustrates the comparison of weighted *m.p.r.* at PA and NPA schools. The graphs agree with the theory prediction again, but student composition measure in schools closer to each other across different mechanisms compared to case with top 50% rule.

Table 1.11: Welfare Comparison without Top %50 Rule and without Affirmative Action Policy

Panel A: Students among highest 50% in their middle schools with respect to grades

Mechanisms	$p.r. \geq 75$	$50 \leq p.r. < 75$	General track	Affirmative track
DA	0.518	0.244	0.358	0.625
SA1	0.253	-0.034	0.088	0.339
SAU	0.225	-0.044	0.075	0.248

Panel B: Students among lowest 50% in their middle schools with respect to grades / *Panel C: All students*

Mechanisms	$25 \leq p.r. < 50$	$p.r. < 25$	
DA	-0.015	-0.481	0.106
SA1	-0.292	-0.746	-0.165
SAU	-0.300	-0.753	-0.179

Notes: This table illustrates the welfare comparisons (in million won or thousand dollars) across the equilibrium of DA, SA1 and SAU mechanisms. DA corresponds to student proposing Deferred Acceptance Mechanism, SA1 corresponds to Sequential Admissions mechanism where students can apply to only one PA school in the first step. SAU corresponds to the Sequential Admissions mechanism where students apply to PA schools in an unrestricted list Deferred Acceptance fashion in the first step. The table demonstrates the average welfare for different types of students, as well as average across all students. In these counterfactuals, we remove the top 50% rule in admission to PA school lottery and affirmative action policy in PA schools.

[67] When we compare to previous literatures' comparisons between centralized mechanisms it is smaller than the difference between them. Calsamiglia, Fu, and Güell (2020) finds that a switch from Boston Mechanism (which is a centralized mechanism) to DA decreases average welfare by €1,020. Abdulkadiroğlu, Agarwal, and Pathak (2017) finds the average welfare difference between coordinated and uncoordinated assignments as large as 10.6 miles.

1.7.2 Effect of Score Based Admission

Another counterfactual policy is the effect of exam schools. Since we have causally estimated preferences for peers, we can check whether admission through academic performance directly (without any role left for random lottery) is enough to generate strong sorting towards schools admitting students with exams even without any intrinsic preferences for these kinds of schools. We set the tuition of PA schools same as NPA schools in this simulation. We also remove top 50 percent rule and affirmative action rules.

We compare $m.p.r$ of PA schools under score based admission for PA schools and lottery based admission for PA schools. In both schemes the schools are allocated through SAU, where PA schools are allocated in the first step and NPA schools are allocated in the second step. Panel B of Table 1.12 illustrates the results from these counterfactuals. Our simulations show that weighted average of $m.p.r$ for PA schools under the first scheme is 88.98 whereas it is 48.20 under the second scheme. For NPA schools it is 48.43 and 53.66 respectively. Table 1.13 illustrates the utility differences of students. As expected, students in the top 25 percent have much higher expected utility under score based admission since the schools most of them go to has very high share of high-performers. For this group welfare difference is around \$860 which is quite large. Also, as expected all other students lose from the score based admission of the PA schools. Average difference in welfare is only around \$30.

We decompose the total change in weighted average of $m.p.r.$ in PA schools into a direct effect and indirect effect. Direct effect would be the effect that is only due to the change in the admission rules. So we should not allow subsequent changes in the preferences due to new realizations of the student distribution. But there are actually subsequent changes in preferences due to realizations of student distribution. Indirect effect is the extra change in the peer distribution on top of the direct effect. To calculate the direct effect we keep the preferences over schools fixed as in the equilibrium of completely lottery based SAU. Then we simulate SAU in which PA schools admit students using percentile rankings. If there were no peer effects, the score based admission would only yield weighted average of $m.p.r.$ for PA schools as high as 72.53. This means that, the indirect effect accounts for around 38% of the total effect. Therefore, large portion of the total effect of academic screening is actually effect of preferences for peers.

1.8 Conclusion

In this study, we evaluated the discrepancies in admissions to schools belonging to different sectors when students have peer effect considerations in school choice. We have investigated the effects of academic screening, early admission to private/elite schools, as well as differences in tuition. We performed our analysis empirically as well as theoretically, combining several datasets on the Seoul high school market. This includes application shares to private schools and rank-ordered lists of students submitted to public school admissions. The study established a structural model in congruence with the students' decision process. Preference parameters were estimated by combining/adapting multiple techniques from the empirical industrial organization literature, as well as by resolving some complicated computational/identification puzzles. In particular, we identified preferences concerning peer composition by adopting an IV approach, also jointly identified the preferences for tuition and private school via a blend of IV and RD approaches by means of indirect inference. The computational challenge implied by the interdependent nature of decisions in two consecutive markets has been addressed by using a two-step estimation approach. In addition, our paper adapts the BLP approach to a setting where market shares have upper bounds due to capacity constraints. Our estimates allowed us to do counterfactual analysis of different admission regimes. In order to do that, we simulated the iteration of best responses until there was a convergence to an equilibrium of strategies and student compositions across schools.

Around the world, the sequentiality of markets is not confined to school districts. The higher education and labor market contexts has also witnessed such a market structure. In this study, we demonstrated that sequential admissions with a commitment structure yield results similar to complete centralization through Deferred Acceptance (DA) with respect to welfare and student distribution. This finding assumes significance because sequentiality might be an easier to imple-

Table 1.12: Weighted $m.p.r$ at PA and NPA schools

Panel A: Completely Lottery Based Admissions (at the observed 2011 tuition levels)

School Type	PA						NPA		
Mechanism	DA		SA1		SAU		DA	SA1	SAU
Tracks	Gen. only	All trck.	Gen. only	All trck.	Gen. only	All trck.			
With Top 50% rule and affirm. action	76.74	75.66	76.88	75.71	76.94	75.68	50.46	50.83	50.77
Without Top 50% rule or affirm. act.	54.22		55.49		54.02		52.94	52.98	53.01

Panel B: Score Based Admissions vs. lottery based admissions for PA schools: tuitions are set to zero across all PA vs. NPA schools, top 50% and affirmative action rules are removed, NPA schools' admission criteria is lottery

School Type	PA			NPA		
PA Sch. Admiss.	Lottery	Score based		Lottery	Score based	
		Fixed	Not fixed		Fixed	Not fixed
	48.20	73.53	88.98	53.66	50.55	48.43

Notes: Panel A illustrates the weighted $m.p.r.$ of the PA and NPA schools across DA, SA1 and SAU at 2011 tuition fees. DA corresponds to student proposing Deferred Acceptance, SA1 corresponds to Sequential Admissions where students can apply to only one PA school in the first step. SAU corresponds to the Sequential Admissions where students apply to PA schools in an unrestricted list DA fashion in the first step. Panel B illustrates the weighted $m.p.r.$ of the PA and NPA schools across SAU with different admission criteria. "Lottery" means, all schools are using random lotteries; whereas "score based" for PA means that, PA schools only are using $p.r.$ of students. The "Not fixed" column in Panel B corresponds to equilibrium when score based admission is in effect for PA schools. The "Fixed" column corresponds simulating the equilibrium of score based case by fixing the preferences of students over schools as in the equilibrium of lottery case. The difference between the "Fixed" and "Lottery" column gives the direct effect of academic screening; whereas the difference between "Not Fixed" and "Lottery" columns gives the total effect. Equilibrium that corresponds to each mechanism and each admission rule is calculated using iteration of best responses.

Table 1.13: SAU score based for PA schools versus SAU lottery based for PA schools

Panel A: Students among highest 50% in their middle schools with respect to grades			
Mechanisms	$p.r. \geq 75$	$50 \leq p.r. < 75$	
Score based admission for PA	1.517	0.149	
Lottery based admission for PA	0.657	0.414	
Panel B: Students among lowest 50% in their middle schools with respect to grades			*Panel C: All students*
Mechanisms	$25 \leq p.r. < 50$	$p.r. < 25$	
Score based admission for PA	-0.140	-0.553	0.309
Lottery based admission for PA	0.166	-0.285	0.276

Notes: This table illustrates the welfare comparisons (in million won or thousand dollars) across the equilibrium of two SAU mechanisms. SAU corresponds to the Sequential Admissions mechanism where students apply to PA schools in an unrestricted list Deferred Acceptance fashion in the first step. The difference between lottery based SAU and score based SAU is that in the former, students are assigned to PA schools via lottery numbers; whereas for the latter assignments to PA schools is based on *p.r.* Under both rules, admissions to NPA schools are lottery based. The table demonstrates the average welfare for different types of students, as well as average across all students. In these counterfactuals, for the lottery based SAU we remove the top 50% rule in admission to PA school lottery and affirmative action policy in PA schools; similarly for score based SAU we do not impose any of these admission rules.

ment or could also take place naturally in a decentralized setting. According to previous literature, there are several advantages of centralization in school markets along with other matching markets. Our findings indicate that it is possible to attain these benefits without total centralization in case the frictions are small in the matching market, at least in school choice settings. We have demonstrated that a rise in tuition leads to an increase in the high-performing students' sorting to private schools through theoretical analysis. We have also shown how the choice of mechanisms interacts with tuition to determine students' distribution. Moreover, we have verified these predictions of the theory using our counterfactual simulations for varied tuition levels.

Furthermore, we have examined the impact of academic screening. In several countries, unlike the majority of public schools, private/elite schools have the autonomy to implement this policy. According to our analysis, the decision of a school to switch from lottery-based admission to score-based admission significantly increases the share of high-performers in the school, irrespective of the school's intrinsic quality. We have also demonstrated that this cannot be dismissed as just a mechanical impact of admission criteria. A substantial 38% of the effect stems from the altered preferences over these schools. These schools start becoming more popular among all students, particularly for high-achievers because admission criteria mechanically increase the share of high-performers in the schools, which in turn, paves the way for more applications, and as a consequence, a higher cutoff in admissions. Notably, the existing sphere of debate concerning student segregation in media and public centers on what kind of student gains admission into specific schools. According to our assessment, high-performing students are interested in coordinating with other high-performing students to gain admission into the same school. Regardless of the schools' intrinsic qualities, this coordination is provided by academic screening. Therefore, discussion of how much coordination should be permitted between high-performing students would yield more fruitful results.

Chapter 2

THE ROLE OF OUTSIDE OPTIONS UNDER BOSTON MECHANISM

2.1 Introduction

School choice is a major concern for families and students around the world. Many districts around the world, including Amsterdam[1], Barcelona[2], Beijing[3], Boston[4], Chicago, London[5], Minneapolis, New York City[6], New Heaven[7], New Orleans [8], Seattle [9] and Shanghai [10] make use of centralized mechanisms. These mechanisms take the ordered preference lists from families and the priority structures at schools over students as inputs, and produce a matching of students and schools (Abdulkadiroğlu and Sönmez, 2003). Beginning with the work of Abdulkadiroğlu and Sönmez (2003) an important debate around the so called "Boston Mechanism" has started. In 2005, Boston Public Schools (BPS) decided to change its mechanism which is referred to as Boston Mechanis-

[1]Haan, Gautier, Oosterbeek, and van der Klaauw (2016)
[2]Calsamiglia, Fu, and Güell (2020)
[3]He (2016)
[4]Abdulkadiroğlu and Sönmez (2003)
[5]Hind, Pennell, and West (2006)
[6]Abdulkadiroglu, Pathak, and Roth (2009)
[7]Kapor, Neilson, and Zimmerman (2020)
[8]Abdulkadiroglu, Che, Pathak, Roth, and Tercieux (2017)
[9]Pathak and Sönmez (2013)
[10]Chen and Kesten (2013)

m (BM) in the academic literature to Deferred Acceptance (DA) Mechanism (Gale and Shapley, 1962). The former mechanism is manipulable and potentially unstable, whereas the second one is strategy proof and stable. In the process of change, BPS consulted to community and academic experts (Abdulkadiroğlu, Pathak, Roth, and Sönmez, 2005) and decided a change in favor of DA. Despite this change and undesirable aspects of BM mentioned above there are still many districts in U.S. and the world that are using BM. In recent years an important effort in the literature has been devoted to understand BM, answering the question of why it may be appealing both through theoretical and empirical work (Miralles, 2009), (Abdulkadiroğlu, Che, and Yasuda, 2011), (Agarwal and Somaini, 2018), (Calsamiglia, Fu, and Güell, 2020), (Kapor, Neilson, and Zimmerman, 2020).

This project examines BM under the existence of options to some students outside of the centralized public school assignment system i.e., outside options, for which the most important example is private schools. Analyzing BM when there are students who can go to private schools is important because of the strategic and risky nature of BM. In BM, a student decreases the chance of getting into a school as he ranks it lower even conditional on being rejected from schools he listed above it; whereas in a strategy-proof mechanism DA, this is not the case. Reporting preferences truthfully is a dominant strategy. When some students have safe outside options (private schools), this may make them gain an advantage in BM. And it is not clear whether students without these safe outside options would be better off in terms of welfare under BM or DA. This setting is first analyzed by Akbarpour, Kapor, Neilson, van Dijk, and Zimmerman (2022). In this regard my work is closely related to their work. This paper considers two theoretical exercises. First, it compares ex ante payoffs under BM versus DA for students who does not have private school options when there are students who have private school options. It is an important question since this group seems vulnerable to manipulability of BM when some other students have some kind of strategic advantage through safe options. Second, it analyzes how the students who cannot go to private school are affected in terms of ex ante welfare when private school enters the district. This is also an interesting question. First, BM is a manipulable mechanism, so it is interesting to see how the equilibrium play responds to a decrease in competition for schools. Competition will decrease for some schools since some students will lose interest in some public schools after private school entry. Second, entry of private school will provide strategic advantage to some students with B-M through having safe good outside options; so it is ambiguous how students who cannot go to private school will be affected from the entry of private schools.

In my analysis, I assume that all students agree on their ordinal ranking of the schools. This assumption is needed for tractability of the analysis and is critical for getting the results. And if one thinks of the other extreme benchmark which is having extremely heterogenous ordinal preferences for public schools, I am analyzing the interesting benchmark. In the extremely heterogenous preference case, there is no point in comparing any mechanisms; or analyzing the entry of private schools, since everyone will be able to go to his most desirable public school. Also, it is plausible to expect correlation between students' preferences since determinants of those preferences are generally common across families. (Abdulkadiroğlu, Che, and Yasuda, 2011) argues that high correlation of preferences can be seen from BPS data.

First, I establish the result that when private schools is only preferred the least preferred public school in the neighborhood by the students, then all the students who cannot go to private school are weakly better off under BM compared to DA (Proposition 2.1). This result is important since as I will explain when describing the model the least preferred public school of the model can be interpreted as all the public schools in the neighborhood which are not very popular. So, the result says that if private school is not preferable to high stakes public schools students are competing to get in, then BM is weakly better than DA for all students who cannot go to private school. This case may be relevant in some real world settings. For example, there may be very good private schools such that students considering them do not even consider public schools so they do not enter centralized public school allocation mechanism; and rest of the private schools may be worse than the desirable public schools of the neighborhood. In such a case BM would be better for all students who cannot go to private schools compared to DA.

For the rest of the paper I analyze a model with three public schools, where again school three is interpreted as all the unpopular schools. Therefore, I analyze the strategic choice between two popular schools. Such simplification is needed to keep the tractability of the analysis. This is certainly a relevant case as the number of public schools in a neighborhood is not very large and school three is interpreted as all the unpopular schools. There are probably neighborhoods with more than 2 popular schools in reality. Albeit, this model allows one to get a clear intuition about which students are better off under BM compared to DA and that can be useful for other cases, too. When model has three schools comparison of payoffs of BM and DA when private school is preferred to only third school is already done by Proposition 2.1. So, I consider the case in which only most preferred public school is preferred to private school for comparison of payoffs under BM and DA.

Under such a model described in previous paragraph, by Lemma 2.2 and Lemma 2.3, I show that among the students who cannot go to private schools, the ones whose decision between reporting which school as top choice is not a close decision, that is the ones who are inframarginal in terms of deciding which school to report as top choice are better off under BM compared to DA; and the students whose decision is close, that is marginal students, are better off under DA compared to BM. This result is intuitive since risky nature of BM allows students to express their cardinal preferences; whereas for DA cardinal preferences do not matter (Abdulkadiroğlu, Che, and Yasuda, 2011). So one can expect that students with stronger cardinal preferences for one school or the other to be better off under BM compared to DA. Moreover, these lemmas allow me to provide mild conditions on the distributions of preferences that guarantees the existence of students who are strictly better off under BM compared to DA among the students who cannot go to private school. These results do not require explicit assumptions on distributions of preferences. Large classes of preference distributions would satisfy these mild conditions. In Proposition 2.2, I restrict attention to equilibrium in which all popular schools are filled in first round of BM. I can guarantee this through mild assumptions on primitives of the model; and it is an empirically relevant case (Agarwal and Somaini, 2018). Proposition 2.2 establishes that, if there is enough heterogeneity in preferences of students then there exists students who are strictly better off under

BM compared to DA among the students who cannot go to private school. This is expected given the intuition that is established in Lemma 2.2 and Lemma 2.3. Enough heterogeneity ensures that there are students who are not marginal when making the decision between which school to report as top choice; and that is the group of students that will be better off under BM compared to DA as shown in Lemma 2.3. My next result, Theorem 2.1 shows that there are students who are strictly better off under BM compared to DA among students who cannot go to private school, when distribution of preferences has full support. Theorem 2.1 does not restrict attention to a kind of equilibrium unlike Proposition 2.2. It requires stronger heterogeneity than Proposition 2.2, but it is still satisfiable by a large class of distributions of preferences. Again, full support assumption ensures the existence of inframarginal students which is key for having students better off under BM compared to DA.

My next result, Theorem 2.2 yields the fraction of students who are strictly better off under BM compared to DA among the students who cannot go to private school. An assumption on distribution of preferences is required to fully pin down the equilibrium. I assume uniform distribution of preferences to find this fraction. Theorem 2.2 shows that share of students who are strictly better off under BM compared to DA among the students who cannot go to private school is equal to one minus the share of students who can go to private schools in the economy. This means, if less than half of the students can go to private school then under uniform distribution assumption, more than half of the students who cannot go to private schools are better off under BM compared to DA. This is the last result regarding comparison between BM and DA.

For the analysis of the effect of entry of private schools on the centralized allocation mechanism I compare the ex ante welfare of students who cannot go to private school in the equilibria of BM before the entry versus after the entry. First, I analyze the case of entry of very high quality private schools. So, I analyze what happens if the entering private school has quality such that all students who can go to private school leave the centralized mechanism. Since after the entry, number of students applying to the centralized mechanism strictly decreases, one would expect that some of the students who are not able to go to private school must be strictly better off. Surprisingly it is possible to find an example in which all students become weakly worse off and some become strictly worse off. This is given by Example 2.3. However, I show that this is a knife edge case and one can find a student strictly better off after the entry of private school for almost all distributions of preferences (Proposition 2.3).

Next, I analyze the case in which most preferred public school is preferred to the entering private school and private school is preferred to all other other public schools. This case is interesting because entry is expected to affect the centralized mechanism through two channels. First, since private school is preferred to all except the most preferred public school, students who can go to private school lose interest in public schools other than the most preferable one. This decreases the competition for those public schools and can be expected to benefit the students who cannot go to private school. Second, since after the entry students who can go to private school have a safe alternative if they cannot get into most preferred school, they can all apply to the most desired public school (if they were not already doing so). This would weakly increase the competition in the most desirable school which can harm welfare of students who cannot go to private school. First, I silence the second channel. In that case I show that except a knife edge case (Example 2.4) some students who cannot go to private school strictly benefit from entry of private school or none of them is affected (Lemma 2.10). For my last result, I allow the effect coming from the second channel. In this case again I restrict attention to equilibrium in which popular schools are filled in the first round of BM by making mild assumptions on primitives. In that case I show that some of the students who cannot go to private school are strictly better off or none of them is affected if the distribution of preferences have full support and share of students who can go to private schools is not very large (Theorem 2.3). This result has a clear intuition. If students who can go to private school are switching to reporting most desirable school as top choice after the entry, then there will be decrease in competition in schools below the most desirable one. Full support assumption guarantees that there will be students who are applying to second most desirable school before the entry among the students who cannot go to private school.

2.1.1 Related Literature

Abdulkadiroğlu and Sönmez (2003) lays out the model of school choice and analyze BM, Top Trading Cycles (TTC) method and DA mechanisms. Abdulkadiroğlu, Pathak, Roth, and Sönmez (2005) proposes TTC and DA as two alternative (non-manipulable) mechanisms that can be used for school choice instead of BM. The ground on which BM is most criticized is manipulability. Manipulability is not desirable: there may be heterogeneity across parents in terms of ability to manipulate; and this can lead to less desirable outcomes for the group that is less sophisticated in terms of manipulation behavior (Pathak and Sönmez, 2008). Ergin and Sönmez (2006), under strict priority structure show that set of Nash equilibria of BM under complete information is equal to set of stable matchings. Therefore, since DA gives the student optimal stable matching, it dominates BM in terms of welfare. Here, I assume coarse priority structure and in reality there are many students that have same priority in a school. Since their work characterizes all the Nash equilibria of BM and can compare among them, it is also related to second part of this paper. Albeit, my setting is quite different, I compare the equilibria of BM before the entry of private school to equilibria after the entry of private schools.

Abdulkadiroğlu, Che, and Yasuda (2011) is closely related to this paper. They show that when students have common ordinal preferences and priority structure is coarse, all students are weakly better off under BM compared to DA in terms of welfare. Proposition 2.1 of my work is closely related to this result since the assumption of private school being preferred to only last public school eliminates the strategic asymmetry between the students who can go to private schools and who cannot. In fact this result follows from only a slight modification of the proof of the result in Abdulkadiroğlu, Che, and Yasuda (2011). Also, Lemma 2.2 and Lemma 2.3 of current work can be seen as an improvement over their result for the special case of three schools. Their setting does not have any student who can go to private school. In their setting, Lemma 2.2 and Lemma 2.3 of this work implies that when there are more than one type of students, then all students who are not marginal in the decision of reporting which school as top choice (such types must exist when there are multiple types) are strictly better off under BM compared to DA. As stated above closest work to mine is Akbarpour, Kapor, Neilson, van Dijk, and Zimmerman (2022). They show that, students who can go to private schools are better off under BM compared to DA. Also, they show that when all students have same cardinal utilities and only most desired public school is preferred to private school then all students who cannot go to private school are weakly worse off in BM compared to DA. This paper on the contrary, tries to find the conditions that guarantees the existence of students who are better off under BM compared to DA among the ones who cannot go to private school. Proposition 2.1 of this paper, analyze the case in which private school is preferred to only the last public school. Rest of my results on comparison of BM and DA relaxes the same cardinal utilities assumption of the second result of Akbarpour, Kapor, Neilson, van Dijk, and Zimmerman (2022) under a model with three schools. In addition, Lemma 2.2 and Lemma 2.3 of this paper can be seen as an alternative proof for their segregation result for the case with 3 schools. Second part of this paper is also related to theirs since there is a distinction between students who can go to private schools and other students, but this work is the first attempt of comparing BM equilibria with and without private schools as far as I am aware. Miralles (2009) by assuming only one priority class and continuum of types, shows that BM outperforms DA according to several ex ante efficiency criteria. He also shows that BM has nice efficiency properties. Troyan (2012) shows that when weak priorities are introduced BM no longer dominates DA from an interim perspective, but from an ex-ante perspective it still does. Featherstone and Niederle (2016) shows through experiments that agents fail to reach the non-truth-telling equilibria of BM. Pathak and Sönmez (2013) compare the mechanisms in terms of manipulability and document that variants of BM that are ruled out in England were more manipulable compared to their successors.

In recent years, empirical literature worked on answering the question of whether BM or DA is better in terms of welfare by using school choice application data of students. Agarwal and Somaini (2018), using school choice data from Cambridge Public Schools shows that mean utility of students under BM is higher compared to mean utility of students under DA. Calsamiglia, Fu, and Güell (2020) finds similar results by examining school choice data from Barcelona. Kapor, Neilson, and Zimmerman (2020) finds that DA outperforms BM in terms of efficiency when families have mistakes in their beliefs about their admission chances using school choice data from New Heaven and a survey they make to parents about their beliefs.

Section 2 sets up the model and defines DA and BM. Section 3 compares the welfare of students under BM and DA. Section 4 compares the welfare of students in BM equilibria before versus after the entry of private schools. Section 5 concludes. Additional results are in Appendix B.1. Proofs of the results in Section 2.3 and Section 2.4 are in Appendix B.2 and Appendix B.3, respectively.

2.2 Model

Let $S = \{s_1, ..., s_m\}$ denote the finite set of public schools with cardinality m. A school $s_j \in S$ has capacity $0 < q_j < 1$. There is also an unlimited capacity private school s_p. I assume that there is measure 1 of continuum of students. A student is denoted by $i \in \mathscr{I}$. I follow the notation and naming in Akbarpour, Kapor, Neilson, van Dijk, and Zimmerman (2022) for distinguishing between students who can afford private schools and students who cannot afford private schools. Let $\Theta = \{constrained, unconstrained\}$ be the possible type of students. A student i is one of the types *constrained* and *unconstrained*. Students who can afford private schools are *unconstrained* and ones who cannot afford private schools are *constrained*. So private school is s_p is only available to *unconstrained* students. Share of *unconstrained* students is η. I assume $\sum_{j=1}^{m} q_j \geq 1$ since every student has right to get education in public schools. Each student i has VNM utility values over schools, $\mathbf{v^i} = (v_1^i, ..., v_m^i, v_p^i) \in [0,1]^{m+1}$.

I make the simplifying assumption that all students have the same ordinal preferences over schools. This assumption is also made in Abdulkadiroğlu, Che, and Yasuda (2011) and Akbarpour, Kapor, Neilson, van Dijk, and Zimmerman (2022). Although in reality this assumption would not exactly hold, it is a good approximation to reality since there is high correlation between students' ordinal preferences[11]. This assumption is critical for my results, since with enough heterogeneity of preferences there will not be scarcity in any school and in that case DA and BM would yield

[11] Abdulkadiroğlu, Che, and Yasuda (2011) argues that this correlation can be seen from BPS data

the same results; and entry of private schools would not have an effect since everyone would be able to go to their top choice. So under common ordinal preferences assumption I am analyzing the interesting benchmark case. For convenience I assume that $s_1 \succ_i s_2 \succ_i \ldots \succ_i s_m$ for all i.

Assumption 2.1. *(A2.1) $s_1 \succ_i s_2 \ldots \succ_i s_m$ for all i, therefore $\mathbf{v^i} \in \mathscr{V} = \{(v_1,\ldots,v_m) \in [0,1]^{m+1} | v_1 > \ldots > v_p^i > \ldots > v_m\}$*

To have s_m relevant, assume that $\sum_{j=1}^{m-1} q_j < 1$. Density functions of distribution of preferences for *constrained* students and *unconstrained* students are $f(\mathbf{v})$ and $g(\mathbf{v})$ respectively. Note that distribution of preferences does not have to be continuous.

I assume schools have only one priority level, so students have no difference in terms of priorities. This is again not the case in reality for most of the school districts. But this assumption is again a good approximation to the reality since in most of the cases schools have coarse priority structures, there are many students in the same priority categoy (e.g. walk zone).

Assumption 2.2. *(A2.2) Schools have only one priority level, so all students are in the same priority level.*

I would like to interpret this model as an analysis of a part of the whole district: a neighborhood or a ZIP code. All the students and schools in the model belong to the same neighborhood. In that sense neighborhood students having same priorities in the neighborhood schools makes sense. neighborhood students do not consider other schools from other districts in the model. This is again not a very strong assumption since if a student in neighborhood A considers schools in neighborhood B, then schools in B are probably popular schools. Students from A will typically not have chance to get in popular schools of B (regardless of the mechanism) since students of B would also like to get into popular school in their own neighborhood and they would have higher priority for those schools compared to neighborhood A students in reality. Since I am analyzing a neighborhood, cardinality of set of public schools is a small integer.

Let l denote the last public school that is preferred to private school by *unconstrained* students. That is $l = 2$ means *unconstrained* students prefer s_1 and $s_2$to s_p.

A strategy is a mapping $\sigma : \Theta \times \mathscr{V} \to \Delta(\Pi)$, where Π is the set of all rank-order lists of S. I focus on symmetric strategies in which students of the same type follow the same strategy.

An economy with m public schools and unlimited capacity private school is denoted by $E_m = (\{q_j\}_{j=1}^{m}, l, f(v), g(v))$ is a quadruple. An economy with m public schools and no private school is denoted by $P_m = (\{q_j\}_{j=1}^{m}, f(v), g(v))$ is a tuple.

Below I describe how DA and BM works. All students submits rank ordered lists of schools in both mechanisms. Ties are broken in ex ante symmetric way via lottery numbers.

Deferred Acceptance (DA)

Step (1): All students apply to their top choices. Each school looks at the applicant pool, tentatively admits the students according to their lottery numbers starting from the top until either they fill their capacity or they run out of applicants.

Step (k): All rejected students of the Step k-1, apply to their k^{th} preferred choice. Each school considers the applicant pool together with already tentatively admitted students. Each school tentatively admits the students according to their lottery numbers starting from the top until either they fill their capacity or they run out of applicants.

The mechanism terminates when there is no rejected student in a step or all students ran out of schools in their list. All tentative assignments at that step are finalized.

Boston Mechanism (BM)

Step (1): All students apply to their top choices. Each school considers the applicants who have written it as the top choice, each school admits the students according to their lottery numbers starting from the top until either they fill their capacity or they run out of applicants.

Step (k): All students who could not be admitted in previous steps apply to their k^{th} ranked choices. Each school that has remaining capacity at the beginning of k^{th} round considers the applicants who have written it as the kth choice, each school fills its remaining seats with the k^{th} round applicants according to their lottery numbers starting from the top until either they fill their capacity or they run out of applicants.

The mechanism terminates when there is no rejected student at a step or all students ran out of schools in their list.

It is dominant strategy to report true ordinal preferences under DA, whereas under BM one's chance of getting into a school decreases as she ranks it lower even conditional on being rejected from the schools listed above that school. That is, order a school is listed affects the chance of getting into that school even conditional on being rejected from schools listed before that school. So BM is not a strategy proof mechanism. So in the analysis below, I will assume that students play truthfully under DA and they play the symmetric Bayesian Nash Equilibrium under BM.

Another thing to highlight is that, without loss of generality I can interpret s_m as follows. Suppose there are $m-1$ public schools such that all students agree on their ordinal ranking of these schools, and they prefer these schools to all other schools in the neighborhood. These $m-1$ schools can be interpreted as popular schools. For the rest of the public schools WLOG there is enough heterogeneity for preferences over them so they do not get filled in any round of BM or DA. Therefore, a student can get into his most favorite school among these schools wherever he positions such school in DA and BM conditional on being rejected from any of the ones listed above it (all schools above it must be one of the first $m-1$ schools). Such favorite schools below first $m-1$ schools differs among students but since $q_m \geq 1-\Sigma_{j=1}^{m-1} q_j$ in the model, without loss of generality I can collect all such schools into s_m, hence s_m is the most preferable school for each student after the first $m-1$ schools. Without loss of generality we can assume $v_m^i = 0$ for all $i \in \mathscr{I}$. I assume if a *constrained* student cannot be assigned in the centralized mechanism (this can happen only if such student does not submit s_m in his preference list for both mechanisms) reporting, he is either randomly assigned in the post assignment to an unfilled school or he is home schooling. I assume such outcome gives less utility than v_m^i for all $i \in \mathscr{I}$, so it gives negative utility. Therefore all *constrained* students will report s_m at the bottom of their list in BM or DA [12].

Finally, as can be understood from A2.1, I do not consider the case where $l = m$, since this means that private school is not considered by anybody. Such a model does not make sense if

[12]If I simply assume that a student i can go to s_m in the post assignment period if he wishes to, then I can treat s_m as a pure outside option. In that case a student would not necessarily report s_m in his preference list. The results in the paper would again follow in such situation.

we want to analyze a model with private schools since it will not get any student. If we think that such case corresponds to no private school case, then the comparison is already done by Abdulkadiroğlu, Che, and Yasuda (2011) where conclusion is that all students are weakly better off under BM equilibria compared to DA. Also, for the first part of the paper (Section 3) I do not consider the $l = 0$ case since this means any student participating in the centralized assignment procedure is not considering private schools, since all the *unconstrained* students would prefer going to private school to any public school. Conclusion of comparison of BM and DA already follows from Abdulkadiroğlu, Che, and Yasuda (2011) as described above for this case, too.

2.3 Comparison of DA and BM

In this section my goal is to compare the welfare of the students under symmetric equilibria of BM and truthful equilibrium of DA. Theorem 1 of Akbarpour, Kapor, Neilson, van Dijk, and Zimmerman (2022) already shows that *unconstrained* students are weakly better off in any symmetric equilibrium of BM compared to DA [13]. Illustrative example of Akbarpour and van Dijk (2018) shows that one can find an example of capacities, preferences and η such that all *constrained* students are strictly worse off in symmetric equilibria of BM compared to DA. This section provides conditions under which some *constrained* students are strictly better off, and conditions under which all *constrained* students are weakly better off in BM equilibria compared to DA. Also, I provide some necessary conditions for having all constrained students strictly better off under BM equilibria compared to DA.

Following Example 2.1 shows a case in which the students without outside option are always strictly better off under BM compared to DA.

[13] Akbarpour and van Dijk (2018) use the following notion in their theorem: A student ***i always prefers*** an assignment mechanism A to an assignment mechanism B, if he gets weakly higher expected utility under any symmetric equilibrium of the mechanism A than under any symmetric equilibrium of the mechanism B.

Theorem. *Theorem 1 of Akbarpour and van Dijk (2018): A student i always prefers the BM to DA iff he is unconstrained.*

This theorem says that we can find an example of preferences and capacities and η such that for any *constrained* student there is an equilibrium of BM in which a *constrained* student is worse off compared to DA.

Table 2.1: Payoffs of Example 2.1

	1	2	3
s_1	0.81	0.8	0.55
s_2	0.6	0.2	0.4
s_3	0	0	0

Example 2.1. In Table 2.1 preferences of 1 represents the preferences of a type 1 of continuum of students (*constrained*) with mass of 1/3, preferences of 2 represents the preferences of type 2 of continuum of students (*constrained*) with mass of 1/3, preferences of 3 represents the preferences of a type 3 of continuum of students (*unconstrained*) with mass of 1/3. Types are again used to distinguish between students with different valuation vectors. And suppose each school has mass of 1/3 seats. Suppose $v_p^3 = 0.5$. Then equilibria under BM are (unique up to report of 2 for 2nd and 3rd choice) 1 reports s_2 as top choice, s_1 or s_3 as second choice; 2 reports s_1 as top choice and reports $s_2$or s_3 as second choice; 3 reports s_1 as top choice and does not list other public schools[14]. For DA every student reports his preferences truthfully since it is dominant strategy under DA. All payoffs under BM equilibria are higher than payoffs under DA. See Table 2.2 for comparison. So all students are strictly better off under this equilibrium of BM compared to DA.

Table 2.2: Payoff comparison of Example 2.1

	1	2	3
Boston	0.6	0.4	$0.55/2+0.5/2=0.525$
DA	$(0.81)/3+(0.6)/2=0.57$	$(0.8)/3+(0.2)/2=0.37$	$0.55/3+0.5(2/3)=0.517$

Akbarpour and van Dijk (2018) also has an example in which *constrained* students are better off in BM equilibrium compared to DA equilibrium in which they have a common utility vector for *constrained* students assumption and they have $l = 2$ and $m = 3$. So their example is a case

[14]There is no additional payoff changing mixed strategy equilibria since ranking top choice truthfully is dominant strategy for 2nd and 3rd players, note that 2nd player can mix between ranking $s_2$and s_3 as second choice.

of $l = m - 1$ for which I will show more general result in next subsection. My previous example indicates that it is possible to have all *constrained* students better off under BM equilibria compared to DA when there are more than one type of *constrained* students and $l = 1$.

Before going into results, let me remind Theorem 2 of Akbarpour and van Dijk (2018) so that readers can compare the negative results about BM to positive results about BM. Their theorem says that when all *constrained* students have the same valuation vector and $l = 1$ *constrained* students are weakly worse off under all BM equilibria compared to DA.[15]

Note that all *constrained* students having same valuation vector is demanding, it means that all constrained students are same type. Under that assumption cardinal utilities, that is the intensity of preferences does not play a role.

In the next subsection I compare the mechanisms under a special case: $l = m - 1$. Before that following remark gives the entry probabilities of students under DA.

Remark 2.1. Probabilities of students entering to schools in DA is as follows:

For unconstrained students:

$$[q_1, ..., q_l, 0, ..., 0]$$

For constrained students:

$$[q_1, ..., q_l, \frac{q_{l+1}}{1-\eta}, \frac{q_{l+2}}{1-\eta}, ..., 1 - \sum_{i=1}^{l} q_i - \sum_{i=l+1}^{k^*-1} \frac{q_i}{1-\eta}, 0, ..., 0]$$

where k^* is the first j such that

$$q_{k^*} > (1-\eta) - [(1-\eta)(q_1 + ... + q_l) + q_{l+1} + ... + q_{k^*-1}]$$

Remember that in DA everyone reports truthfully, so 1 measure of students wants to enter

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Theorem. *Theorem 2 of Akbarpour and van Dijk (2018): Suppose $l = 1$, and all constrained students have the same valuation vectors. Then, constrained students always prefer the deferred-acceptance mechanism to the Boston Mechanism.*

s_1 with q_1 capacity. So entrance probability for any of the students is q_1. Then conditional on being rejected from s_1 all unassigned students enter apply to s_2, which leads to probability of entry $\frac{q_2}{1-q_1}(1-q_1) = q_2$ if $l > 2$, otherwise it is $\frac{q_2}{(1-\eta)(1-q_1)}(1-q_1) = \frac{q_2}{1-\eta}$ and so on. This is how the probabilities under DA are calculated.

2.3.1 $l = m - 1$

For this case only I assume there is discrete distribution of preferences for convenience. So there are finite payoff types. This result is generalizable to infinite payoff types. In this case *unconstrained* students prefer private school only to the last school s_m. Remember, with my interpretation of the model this means that the popular first $m-1$ schools are preferred to private school and private school is preferred to any other public school in the neighborhood. Notice that s_m is a safe option for *constrained* students. That is, they can get into it for sure as long as they report it in the list conditional on not getting the ones listed above it. This situation removes the strategic asymmetry between the students because in this case *constrained* students have a guaranteed option if they cannot get first $m-1$ public schools just like *unconstrained* students. Guaranteed option for *unconstrained* students is private school s_p. This situation leads to the following proposition.

Proposition 2.1. *Suppose (A2.1)-(A2.2) holds. When $l = m-1$, constrained students are weakly better off compared to DA, in any symmetric equilibrium of the BM.*

The proof is a slight modification of proof of the Theorem 1 of Abdulkadiroğlu, Che, and Yasuda (2011).[16] Here, since *unconstrained* students do not rank s_m, a *constrained* student does not consider exactly mimicking the strategy of the *unconstrained* students, he needs to add s_m to the bottom of the list of *unconstrained* students when mimicking the strategy of *unconstrained* students; whereas in Abdulkadiroğlu, Che, and Yasuda (2011) a student considers mimicking others

[16] Abdulkadiroğlu, Che, and Yasuda (2011) proves it for the case with number of students equal to number of total seats. See unpublished version Abdulkadiroglu, Che, and Yasuda (2009) for the proof of the more general case with number of total seats greater than or equal to number of students. Another slight difference is that these papers have finite number of students, but their proofs would follow exactly in continuum of students case.

strategy in exactly same way since there is no difference between students on the schools that are considered.[17]

This result says that under the assumptions of the model, when the private school is not very desirable in the sense that it is only preferred to schools that students do not need to compete for then all students are weakly better off under any BM equilibria compared to DA. This is not a surprising result since in this case *unconstrained* students do not have any strategic advantage compared to *constrained* students which makes the situation almost same with case of Abdulkadiroğlu, Che, and Yasuda (2011).

Proposition 2.1 guarantees that *constrained* students get payoff under BM equilibria at least as much as their payoff under DA. To see that payoffs for some students can be strictly better for some *constrained* students under BM equilibria compared to DA, let's look at the following Example 2.2.

Table 2.3: Payoffs of Example 2.2

	1	2	3
s_1	0.81	0.8	0.91
s_2	0.6	0.2	0.3
s_3	0	0	0

Example 2.2. As illustrated in Table 2.3 suppose there are 3 types of students with $1/3$ measure each and type 3 students are *unconstrained*. Suppose $v_p^3 = 0.25$. Then under BM unique symmetric equilibrium is: Type 2 and Type 3 students report s_1 as top choice with probability one and Type 1 students report s_2 as top choice with probability one. One can easily check this is the unique symmetric BM since it is strictly dominant strategy to report s_1 as top choice for type 2 and type

[17] If we had assumed s_m is certainly attainable in the post-assignment period, so treat it as pure outside option, then the result would follow without need to distinguish between *constrained* and *unconstrained* students. In that case, we would have $m-1$ public schools considered in the centralized mechanism, so total capacity of schools would be less than total number of students. And this would be the only difference from the Abdulkadiroglu, Che, and Yasuda (2009) as they have total capacity of schools greater than or equal to total number of students.

3 students. Note that type 1 cannot deviate from ranking s_2 as top choice.[18] Also, we do not have another equilibrium.[19]

Then under the unique symmetric BM equilibrium type 1 gets 0.6, type 2 gets 0.4, type 3 gets 0.58. Under DA, type 1 gets: 0.47, type 2 gets 0.33 and type 3 gets 0.49. So everyone is strictly better off under BM equilibrium compared to DA in this example.

Next subsection analyzes the $m=3$ case which has the interpretation that there are two popular public schools in the neighborhood and conditional on not getting the two popular schools each other public school can be easily gotten into by writing it at the bottom of the reported list.

2.3.2 $m=3$

In this subsection I analyze the case with two popular public schools.

Assumption 2.3. *(A2.3) $m=3$*

This assumption is needed to keep the tractability of the analysis. Two popular schools is certainly a relevant case as the number of public schools in a neighborhood is not very large and school three is interpreted as all the unpopular schools. There are probably neighborhoods with more than 2 popular schools in the real world. Albeit, this model allows one to get a clear intuition about which students are better off under BM compared to DA. This intuition can be useful for settings with arbitrary number of schools.

Under $m=3$, case with $l=2$ is already answered in previous subsection. So here I will analyze the case with $l=1$ which is the only remaining case for $m=3$ case.

Since we have 3 schools, without loss of generality we can do a scale normalization for utilities such that $v_1^i = 1$ for all $i \in \mathscr{I}$. So our normalizations imply that $v_3^i = 0$ and $v_1^i = 1$ for all $i \in \mathscr{I}$. Therefore, each student's type can be just characterized by his valuation for s_2 which is a scaler

[18] Note that $0.81\frac{1/3}{2/3} < 0.6$.

[19] Note that $0.81\frac{1/3}{\frac{2}{3}+x\frac{1}{3}} + 0.6\frac{\frac{1}{3}-(1-x)\frac{1}{3}}{\frac{2}{3}+x\frac{1}{3}} \geq 0.6$ since the inequality boils down to $0.27 \geq 0.6$.

v_2^i. Not to carry out extra subscript from now on let v denote the value of a student for s_2, and $v \in (0,1)$.

Assumption 2.4. *(A2.4)* $l = 1$

My first claim shows that in any equilibrium of BM measure of students reporting s_1 as top choice must be larger than capacity of s_1. This result is helpful in calculating the payoffs for students who are reporting s_1 as top choice.

Claim 2.1. Suppose A2.1-A2.4 holds. Let k be the measure of constrained students who report s_1 as top choice in a symmetric equilibrium of BM. Then we must have $k + \eta > q_1$

Logic behind the claim is very simple. If claim does not hold, some constrained student who did not already report s_1 as top choice with probability one (note that there exists such student in that case) can deviate to reporting s_1 as top choice with probability one and get into s_1 for sure. So any equilibrium measure k of constrained students reporting s_1 as top choice must satisfy $k > q_1 - \eta$.

With three school model it is not very meaningful to have an equilibrium in which no one goes to s_3, although s_3 can be seen as an unpopular school. Next assumption rules out equilibria in which no student gets into s_3 and so is necessary to ensure scarcity of popular schools.

Assumption 2.5. *(A2.5)* $q_2 < (1-\eta)(1-q_1)$

First note that this assumption implies $q_2 < 1 - \eta$. This is needed to ensure there is strategic aspect of BM: If measure of *constrained* students were smaller than or equal to capacity of s_2, note that s_2 would never be filled. Then all students would report truthfully their top choice as s_1, since if they cannot get s_1 in the first round they are guaranteed to get s_2 in the subsequent round. So there would not be strategic behavior and in that case DA and BM would yield the same result. I require a stronger condition $q_2 < (1-\eta)(1-q_1)$ because otherwise, there will still be an equilibrium in which all *constrained* students report s_1 as top choice in which case there will be no student entering s_3. To see that such equilibrium exists when $q_2 \geq (1-\eta)(1-q_1)$, note that payoff of *constrained* student reporting s_1 as top choice is:

$$q_1 + (1 - q_1)v$$

since measure of *constrained* students that will be rejected is $(1-\eta)(1-q_1)$ and this is less than or equal to q_2, so it is sure to get into s_2 conditional on being rejected from s_1. Since this payoff is greater than v, such an equilibrium indeed exists and no one gets into s_3. The same thing happens for DA. I rule out this case by A2.5.

Note that A2.5 holds when q_1, q_2 and η are not large. It is intuitive that such situation will lead to s_3 getting no student since *unconstrained* students do not consider schools below s_1, and if q_1 and q_2 are too large, an equilibrium in which all *constrained* students get into s_1 or s_2 can occur.

Under A2.1-A2.5 and knowing Claim 2.1 holds, it is easy to calculate the payoffs from reporting s_1 as top choice and s_2 as top choice in a given equilibrium of BM.

Remark 2.2. Suppose A2.1-A2.5 holds. Note that in this case *unconstrained* students all report s_1 as top choice and do not submit any other school in their list. Take a symmetric equilibrium of BM. *Let k* denote the measure of *constrained* students reporting s_1 as top choice. Then *constrained* students of type v reporting s_1 as top choice get payoff of $\frac{q_1}{k+\eta}$ when measure of *constrained* students reporting s_2 as top choice is more than q_2. And when measure of *constrained* students reporting s_2 as top choice is less than q_2 *constrained* students of type v reporting s_1 as top choice get payoff $\frac{q_1}{k+\eta} + v\frac{q_2-(1-\eta-k)}{k}$. *Constrained* students of type v reporting s_2 as top choice get payoffs of $v\frac{q_2}{1-\eta-k}$ and v when measure of *constrained* students reporting s_2 as top choice is more than or equal to q_2 and less than or equal to q_2 respectively. Also, note that when $k = 1-\eta$, i.e. all *constrained* students report s_1 as top choice, DA and BM yields same payoffs for everyone.

Next claim provides the cutoff below (above) which a *constrained* student reports s_1 (s_2) as top choice for a given BM equilibrium measure k of *constrained* students reporting s_1 as top choice, where measure k is small enough so that there are at least q_2 measure of *constrained* students reporting s_2 as top choice.

Claim 2.2. Suppose A2.1-A2.5 holds. Let k be the measure of *constrained* students who report truthfully the first choice in a symmetric equilibrium of BM. Suppose $1-\eta-k \geq q_2$, then a constrained student with type v reports s_1 as top choice in this equilibrium if $v < c(k)$, reports s_2 as top choice if $v > c(k)$ and is indifferent if $v = c(k)$, where $c(k) := \frac{q_1}{q_2}\frac{1-\eta-k}{k+\eta}$

Claim 2.2 follows from the comparison of expected payoffs for a *constrained* student from reporting s_1 as top choice versus s_2 as top choice for a given k. Note that if first (second) one is strictly larger for a *constrained* student than that student must be reporting s_1 (s_2) as top choice in equilibrium, otherwise he would deviate to increase his payoff. Claim 2.2 is important since it tells how *constrained* students will behave in a given equilibrium with $k \leq 1-\eta-q_2$ and some of the results below analyze particularly equilibria of this kind. First, it is an empirically relevant case since s_2 is a popular school according to the interpretation of the model and such schools are generally filled in the first round (Agarwal and Somaini, 2018). Second, Lemma 2.1 below shows that there is unique equilibrium in this case so it makes analysis much easier. Also, conditions for ruling out equilibria in which there are less than q_2 measure of *constrained* students reporting s_2 as top choice will be satisfied if η and q_2 are not very large. Next claim characterizes when a symmetric equilibrium of BM in which measure of *constrained* students reporting s_2 as top choice is at least q_2 exists. And Claim 2.4 below will provide sufficient conditions to rule out equilibria in which less than q_2 measure of *constrained* students report s_2 as top choice.

Claim 2.3. A symmetric equilibrium of BM in which measure of *constrained* students reporting s_2 as top choice is at least q_2 exists if and only if there are at least q_2 measure of *constrained* students with $v \geq \frac{q_1}{1-q_2}$

Below, Claim 2.4 provides a sufficient condition under which s_2 is guaranteed to be filled in the first round of BM equilibrium. So it rules out the equilibria in which less than q_2 students report s_2 as top choice.

Claim 2.4. Suppose A2.1-A2.5 holds. Let k be the measure of constrained students who report truthfully the first choice in a symmetric equilibrium of BM. Suppose there is at least q_2 measure of constrained students for whom $v > \frac{(1-\eta)q_1}{1-\eta-q_2}$ then measure of constrained students who report s_2

as top choice is greater than or equal to capacity of s_2, i.e. $1-\eta-k \geq q_2$.

The logic behind the proof of this claim is as follows. When the equilibrium measure of students reporting s_2 as top choice is less than q_2, there remains capacity to be filled in second round for s_2, which also means that a list that reports s_2 as top choice will get s_2 for sure. Gain of expected payoff from s_2 from such deviation turns out to be $v\frac{1-\eta-q_2}{k}$; whereas loss will be the expected payoff from s_1: $\frac{q_1}{k+\eta}$. Note that gain decreases faster than the loss. So assuming that gain is larger than loss at the maximum possible k, which is $1-\eta$, is enough to guarantee that deviation is better for all possible $k > 1-\eta-q_2$.

So next assumption guarantees ruling out equilibria in which s_2 is not filled at the end of first round.

Assumption 2.6. *(A2.6) Suppose there is at least q_2 measure of constrained students for whom $v > \frac{(1-\eta)q_1}{1-\eta-q_2}$*

Note that, $\frac{(1-\eta)q_1}{1-\eta-q_2}$ is less than 1 by A2.5, so one can always find a preference distribution that satisfies A2.6. If q_1, q_2 and η is not very large then $\frac{(1-\eta)q_1}{1-\eta-q_2}$ would not be very large, so there would be many preference distributions that would satisfy A2.6. Notice also that $\frac{(1-\eta)q_1}{1-\eta-q_2} > \frac{q_1}{1-q_2}$ since this can be shown to be equivalent to $q_2\eta > 0$. Therefore, A2.6 also guarantees the existence of an equilibrium in which there are at least q_2 *constrained* students reporting s_2 as top choice.

Next result shows that under the assumptions we discussed above, measure of *constrained* students reporting s_1 as top choice is unique across all symmetric equilibria of BM. Also, it provides bounds for that measure.

Lemma 2.1. *Suppose A2.1-A2.6 holds, then symmetric BM equilibrium is unique.*

Let's denote measure of constrained students reporting s_1 as top choice with k. We must have $k \in [\max\{\frac{q_1}{q_1+q_2}-\eta, 0\}, 1-\eta-q_2] \setminus \{\frac{q_1}{q_1+q_2}-\eta\}$.

Moreover for any $x \in [\max\{\frac{q_1}{q_1+q_2}-\eta, 0\}, 1-\eta-q_2] \setminus \{\frac{q_1}{q_1+q_2}-\eta\}$ there exists a distribution of preferences that induces $k = x$.

It is easy to see the logic behind the uniqueness of k which implies uniqueness of equilibrium. From Claim 2.2, as k increases the cutoff *constrained* students below which reports s_1 as top choice decreases and this weakly decreases the number of students reporting s_1 as top choice, but in that case k cannot increase. So there cannot be more than one k.

Note that the upper bound of k in Lemma 2.1 directly follows from Claim 2.4. For the lower bound the logic is as follows. From Claim 2.2, the threshold below which *constrained* students report s_1 as top choice is large when k is low. In fact, when the threshold is above 1, it turns out that k has to be less than measure of *constrained* students. But, when threshold is above 1, all *constrained* students would like to report s_1 as top choice. This is not possible since $k < 1-\eta$. Therefore, threshold must be smaller than 1 in any equilibrium, which is equivalent to having $k > \frac{q_1}{q_1+q_2}$.

Lemma 2.1 also says that any measure $x \in [\max\{\frac{q_1}{q_1+q_2} - \eta, 0\}, 1-\eta-q_2] \setminus \{\frac{q_1}{q_1+q_2} - \eta\}$ of *constrained* students reporting s_1 truthfully across equilibria of BM is possible for some distribution of preferences. This logic behind this result is as follows. Given $x \in [\max\{\frac{q_1}{q_1+q_2} - \eta, 0\}, 1-\eta-q_2] \setminus \{\frac{q_1}{q_1+q_2} - \eta\}$ since threshold $c(.)$ is a strictly decreasing function, $c(.)$ evaluated at $\frac{q_1}{q_1+q_2} - \eta$ is equal to 1 and $c(.)$ evaluated at $1-\eta-q_2$ is greater than 0, then one can always find a distribution that has x measure of students with $v \leq c(x)$.

From Lemma 2.1, one can see that it may be possible to have $k = 0$ in some cases. The following remark provides necessary conditions for having no *constrained* student reporting s_1 as top choice under my assumptions. The first necessary condition $\frac{q_1}{q_1+q_2} < \eta$ directly follows from Lemma 2.1 and second condition follows from Claim 2.2. Right hand side of the second condition is the threshold for playing s_1 as top choice evaluated at $k = 0$.

Remark 2.3. Under our assumptions, if equilibrium measure of *constrained* students reporting s_1 as top choice in BM, $k = 0$ then we must have $\frac{q_1}{q_1+q_2} < \eta$ and there is no *constrained* student with valuation v such that $v < \frac{q_1}{q_2}\frac{1-\eta}{\eta}$

Notice that if $q_1 \geq q_2$ the first condition says that, to have $k = 0$ at least half of the students must be *unconstrained*, i.e. $\eta \geq 0.5$. When q_1 is significantly smaller than q_2, the second condition is hard to be satisfied if η is not considerably large. Note that, even when q_1 is significantly less than q_2 and η is very large it is possible violate the second condition. So, having $k = 0$ requires a restrictive case. This is important since Lemma 2.3 below will tell that when $k = 0$ in equilibria of BM all *constrained* students are strictly worse off in these equilibria compared to DA.

Note that to solve for equilibrium, so also for finding equilibrium k, one needs an explicit distributional assumption. In Subsection 3.2.1, I will adopt uniform distribution to solve for the equilibrium. Before that I will make an analysis without assuming an explicit distribution. I will provide conditions that can be satisfied by a large class of distributions of preferences that will lead to existence of *constrained* students who are strictly better off under BM equilibria compared to DA.

Lemma 2.2 below provides the conditions under which a *constrained* students reporting s_1 as top choice or s_2 as top choice is better off in a symmetric equilibrium of BM compared to DA when the equilibrium measure of constrained students reporting s_1 as top choice is known. Also, it shows that *constrained* students who would gain higher payoff in the symmetric equilibrium of BM compared to DA if they were playing s_1 as top choice, are in fact playing s_1 as top choice in the equilibrium with probability one; and *constrained* students who would gain higher payoff the symmetric equilibrium of BM compared to DA if they were playing s_2 as top choice, are playing s_2 as top choice in the equilibrium with probability one.

Lemma 2.2. *Suppose A2.1-A2.6 holds,*

Take a symmetric equilibrium of BM. Let $k > 0$ be the total measure of constrained students reporting truthfully their first choice.

A constrained student of type v reporting s_1 as top choice is weakly better off in the symmetric BM equilibrium compared to DA if and only if $v \leq \underline{c}(k)$ where $\underline{c}(k) := \frac{(1-\eta-k)(1-\eta)q_1}{(k+\eta)q_2}$ and strictly better off iff the inequality is strict.

A constrained student of type v reporting s_2 as top choice is weakly better off in the symmetric BM equilibrium compared to DA if and only if $v \geq \bar{c}(k)$ where $\bar{c}(k) := \frac{q_1(1-\eta)(1-\eta-k)}{q_2 k}$ and strictly better off iff the inequality is strict.

Moreover, $\underline{c}(k) < c(k) < \bar{c}(k)$.

For given $k > 0$, it is easy to write the payoff in BM equilibrium from reporting s_1 as top choice and reporting s_2 as top choice for a *constrained* student with valuation v. We also know the payoff from DA. So it is easy to get the cutoff for v below (above) which students reporting s_1 (s_2) as top choice are better off in BM compared to DA.

Following Lemma easily follows from the Lemma 2.1 and Lemma 2.2. For a given positive equilibrium measure of *constrained* students reporting s_1 as top choice, it characterizes the condition for a *constrained* student to be better off in the BM equilibrium compared to DA. Also, it says that when there is no constrained student reporting s_1 as top choice in the BM equilibrium, all *constrained* students are worse off in the BM equilibrium compared to DA.

Lemma 2.3. *Suppose A2.1-A2.6 holds,*

Let k be the total measure of constrained *students reporting truthfully their first choice in the symmetric equilibrium of BM.*

If $k = 0$*, then all* constrained *students are strictly worse off under the symmetric equilibrium of BM compared to DA.*

If $k > 0$, constrained *students of type v is weakly better off under the symmetric equilibrium of BM compared to DA if and only if* $v \geq \bar{c}(k)$ *or* $v \leq \underline{c}(k)$ *and strictly better off if and only if one of the inequalities is strictly satisfied.*

Moreover if there is positive measure of constrained *students whose valuation v satisfy* $v \geq \bar{c}(k)$ *or* $v \leq \underline{c}(k)$ *then* $k > 0$*.*

An immediate corollary of Lemma 2.3 is about the existence of *constrained* students who are better off under the symmetric BM equilibrium compared to DA and having all *constrained* students better off under the symmetric BM equilibrium compared to DA for given k.

Remark 2.4. Suppose A2.1-A2.6 holds. Given total measure $k > 0$ of *constrained* students reporting s_1 as top choice in the symmetric equilibrium of BM, there exists constrained students who are weakly (strictly) better off in the symmetric equilibrium of BM compared to DA if and only if there is a *constrained* student type v with $v \geq \bar{c}(k)$ or $v \leq \underline{c}(k)$ ($v > \bar{c}(k)$ or $v < \underline{c}(k)$); and all *constrained* students are weakly (strictly) better off in any symmetric equilibria of BM compared to DA if and only if any *constrained* student type v has $v \geq \bar{c}(k)$ or $v \leq \underline{c}(k)$ ($v > \bar{c}(k)$ or $v < \underline{c}(k)$)

One thing to notice is that for any possible $k > 0$ one can find a distribution of *constrained* student types that induces a *constrained* student being strictly better off under the symmetric BM equilibrium compared to DA. This is because for any $k \in [\max\{\frac{q_1}{q_1+q_2} - \eta, 0\}, 1 - \eta - q_2] \setminus \frac{q_1}{q_1+q_2} - \eta$ we have $\underline{c}(k) > 0$.

Remark 2.5. $0 < \underline{c}(k) < 1$ for any $k \in [\max\{\frac{q_1}{q_1+q_2} - \eta, 0\}, 1 - \eta - q_2] \setminus \frac{q_1}{q_1+q_2} - \eta$

Note that under A2.5, $\bar{c}(k)$ gets below 1 at $k = 1 - \eta - q_2$. But, still for some $k < 1 - \eta - q_2$, $\bar{c}(k)$ is not below 1. So for some distributions of preferences that leads to an equilibrium measure k of *constrained* students reporting s_1 as top choice such that $\bar{c}(k) > 1$, the existence of a *constrained* student better off in BM equilibria compared to DA relies on existence of a *constrained* student type v with $v < \underline{c}(k)$ only for that k.

Remark 2.6. $\bar{c}(k) < 1$ at $k = 1 - \eta - q_2$ if and only if $q_2 < (1-\eta)(1-q_1)$, i.e A2.5 is satisfied.

Another thing to notice is that as k gets larger there are weakly less *constrained* student with $v < \underline{c}(k)$, but there are weakly more *constrained* student with $v > \bar{c}(k)$ at the same time for any distribution of preferences.

Lemma 2.2 and Lemma 2.3 also gives clear intuition about which students are better off under BM compared to DA. For a given equilibrium k, students who are close to the cutoff for reporting s_1 versus s_2 as top choice, $c(k)$ are worse off under BM compared to DA. And students who are enough farther away from the cutoff are better off in BM compared to DA. That it is the ones above $\bar{c}(k)$ or below $\underline{c}(k)$. So that means students who are marginal in the decision of reporting s_1 and s_2 as top choice are worse off and inframarginal students are better off in BM compared to DA. Also, from the proof of Lemma 2.2 one can see that difference between payoffs between BM and DA gets larger for a *constrained* student with value above $\bar{c}(k)$ (below $\underline{c}(k)$) as his value increases (decreases). Note that BM is a mechanism that allows one to express their cardinal values through their preference list since the choices are risky, whereas for DA cardinal values do not matter for given ordinal preferences (Abdulkadiroğlu, Che, and Yasuda, 2011). Abdulkadiroğlu, Che, and Yasuda (2011) shows when only cardinal values matter BM is weakly better for everyone. Lemma 2.2 and 3 shows that existence of *unconstrained* students creates a band around the cutoff such that students with value within that band are better off in DA. But students outside of the band are still better off in BM and those are the students who has stronger cardinal values for a school compared to other schools. Notice also that the wedge between $\bar{c}(k)$ and $\underline{c}(k)$ increases as η increases.

From the results above one can also see Theorem 2 of Akbarpour, Kapor, Neilson, van Dijk, and Zimmerman (2022) for the special case of $m = 3$ and $k \leq 1 - \eta - q_2$. If there were only single type *constrained* student such that $v \geq c(0) = \frac{q_1}{q_2}\frac{1-\eta}{\eta}$ then $k = 0$; if there were single type of *constrained* student with $v < c(0) = \frac{q_1}{q_2}\frac{1-\eta}{\eta}$ then that means equilibrium $0 < k \leq 1 - \eta - q_2$ which means *constrained* student must be mixing, so he must have $v = c(k)$ in the equilibrium. In both cases he is strictly worse off under BM compared to DA. It is also possible to see Theorem 1 of Abdulkadiroğlu, Che, and Yasuda (2011) for the special case of $m = 3$ and existence of at least q_2 measure of students with $v > \frac{q_1}{1-q_2}$. Note that with $\eta = 0$, $\underline{c}(k) = c(k) = \bar{c}(k)$. So any student must be weakly better off in the symmetric BM equilibrium compared to DA. Furthermore, for this special case it also shows that there is a student strictly better off in BM compared to DA when there are more than one type of students; and all students who are not indifferent between reporting s_1 as top choice and s_2 as top choice in the BM equilibrium are better off compared to DA. So, students just at the margin are indifferent between BM and DA, but all students who are not marginal are better off under BM. This is an improvement on the result of Abdulkadiroğlu, Che, and Yasuda (2011) for the special case I am considering. Also, from proof of Lemma 2.2 one can see that even when $\eta = 0$ (so everyone is weakly better off under DA compared to BM), as a student gets farther away from cutoff $c(k)$ in terms of valuation, his utility difference with DA increases. This is another note that can be added on the setting of Abdulkadiroğlu, Che, and Yasuda (2011) for the special case I am considering. Going back to discussion in previous paragraph, this is also why even when *unconstrained* exists there can be *constrained* students who are better off under BM. When $\eta = 0$, utility difference between BM and DA gets higher as value of a student gets farther away from cutoff $c(k)$. So even when $\eta > 0$ students who are enough farther away from the cutoff $c(k)$ are better off in BM compared to DA.

As discussed above for distribution of preferences that leads to BM equilibrium with k such that $\bar{c}(k) > 1$, it is not possible to have any *constrained* student with $v > \bar{c}(k)$. Then, it is not possible to have all *constrained* students better off in the BM equilibrium compared to DA. The reason is in any equilibrium there must be *constrained* student of type v who is reporting s_2 as top choice, and such students can be better off in BM compared to DA if and only if they have $v > \bar{c}(k)$; so k must be such that $\bar{c}(k) < 1$ which means $k > \bar{c}^{-1}(1)$. This implies a restriction on the distribution of preferences. Remark 2.7 below provides necessary conditions for having all *constrained* students better off under symmetric equilibria of BM compared to DA. Note that my first example, Example 2.1 satisfies these necessary conditions.

Remark 2.7. Suppose A2.1-A2.6 holds, and all constrained students are weakly better off in the symmetric equilibrium of BM compared to DA, then there exists more than $\bar{c}^{-1}(1) = \frac{q_1(1-\eta)^2}{q_2+q_1(1-\eta)}$ measure of *constrained* students with valuation $v < \underline{c}(\frac{q_1(1-\eta)^2}{q_2+q_1(1-\eta)}) = \frac{(1-\eta)^2 q_1}{q_1(1-\eta)+\eta q_2}$

Lemma 2.3 says that, if we can find a *constrained* student type v with $v > \bar{c}(k)$ or $v < \underline{c}(k)$ for the equilibrium measure k of constrained students reporting s_1 as top choice, then there is a *constrained* student who is strictly better off in BM equilibrium compared to DA. This implies that a possible sufficient condition to guarantee a *constrained* student being better off in BM compared to DA independently from k is enough heterogeneity across *constrained* students in terms of relative valuations of s_1 and s_2. To get such a sufficient condition, I find the maximum distance across k possible between $\min\{\bar{c}(k), 1\}$ and $\underline{c}(k)$, this distance will be the measure of heterogeneity I need since assuming that there are positive measure of *constrained* students this far from each other guarantees that some of the *constrained* students satisfies one of the inequalities above for all possible k. Proposition 2.2 below formalizes this argument.

Proposition 2.2. *Suppose A2.1-A2.6 holds, additionally suppose that distribution of preferences for* constrained *students is such that* $|v - v'| \geq \frac{\eta(q_1(1-\eta)+q_2)}{q_1(1-\eta)+\eta q_2}$ *for all* v *and* v' *from two differen-t subsets of* $\mathscr{V}$ *each with positive measure, then in the symmetric equilibrium of BM there are* constrained *students who are strictly better off compared to DA.*

Proof shows that, under my assumptions maximum distance between $\min\{1, \bar{c}(k)\}$ and $\underline{c}(k)$ occurs at k that makes $\bar{c}(k) = 1$. This follows from both $\bar{c}(k)$ and $\underline{c}(k)$ being strictly decreasing and $\bar{c}(k) - \underline{c}(k)$ also being strictly decreasing in k. So the maximum distance occurs when $\bar{c}(k)$ just equals 1. Note that, measure of heterogeneity $\frac{\eta(q_1(1-\eta)+q_2)}{q_1(1-\eta)+\eta q_2}$ is decreasing in q_1 and increasing in η and q_2. This condition being independent of k means it is independent of the distribution of preferences (as long as distribution satisfies A2.6) which determines k. So it even guarantees the existence of *constrained* students who are better off under BM compared to DA for the worst case distribution of preferences. Another way to interpret this condition is that this measure of heterogeneity guarantees the existence of inframarginal students which is the group that benefits from BM compared to DA. And it does that for any possible distribution of preferences.

Remember that $\underline{c}(k) > 0$ for any possible k. This helps us to find another sufficient condition for having a *constrained* student better off in BM equilibria compared to DA. If there is a *constrained* student type v with $v < \underline{c}(1-\eta-q_2)$, since $\underline{c}(.)$ is decreasing this type of student will be better off in BM equilibria compared to DA for any given k. Following Lemma formalizes this argument.

Lemma 2.4. *Suppose A2.1-A2.6 holds, then constrained students with type v with $v \leq \frac{(1-\eta)q_1}{(1-q_2)}$ (if exists) are weakly better off in the symmetric equilibrium of BM compared to DA; and they are strictly better off if the inequality is strict.*

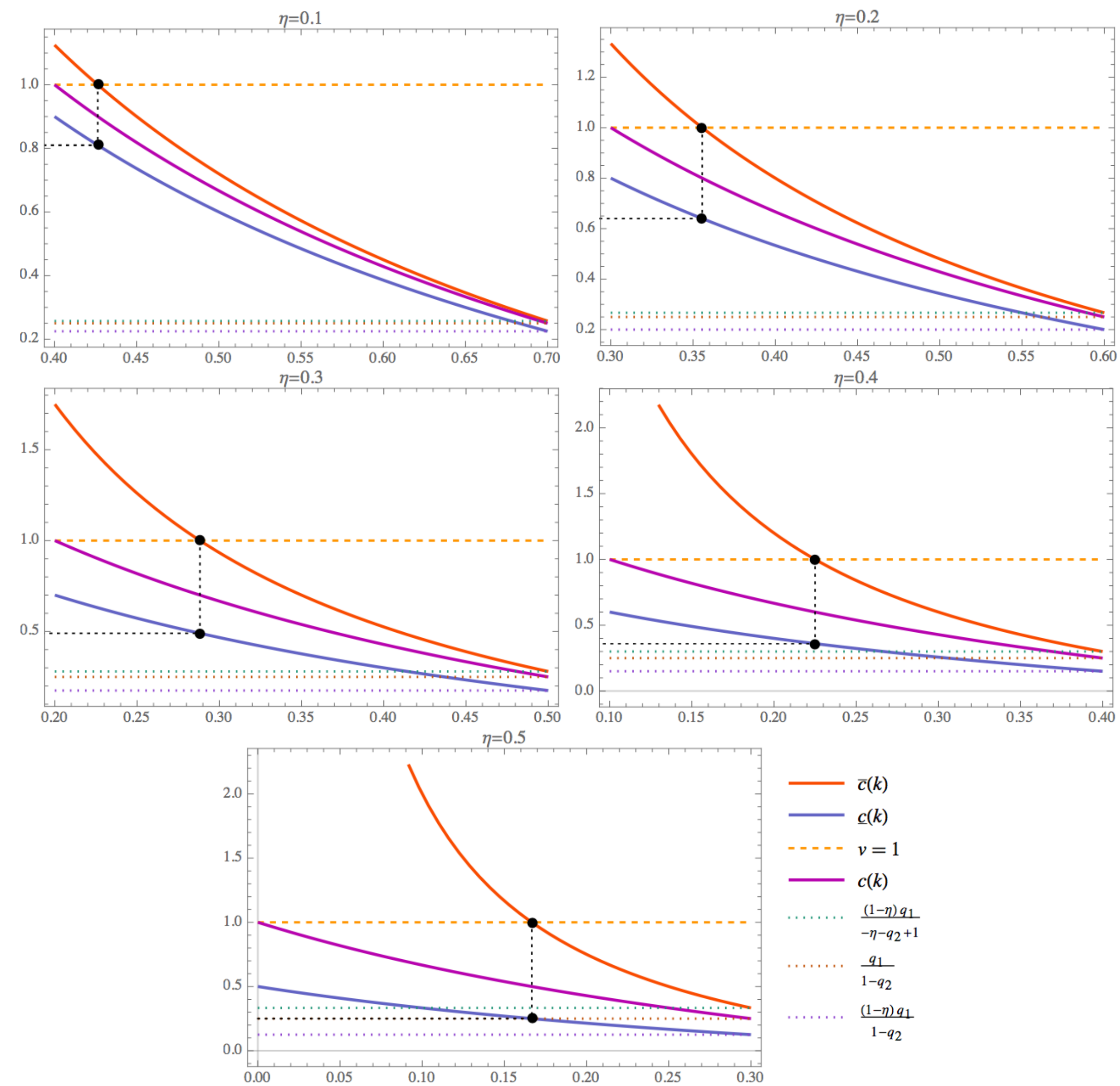
Lemma 2.4 says that if there are *constrained* students who have strong enough preferences for s_1 compared to s_2 then such students are better off under BM compared to DA whatever is the distribution of preferences are as long A2.6 is satisfied and there exists positive measure of *constrained* students with $v < \frac{(1-\eta)q_1}{(1-q_2)}$. Such students are enough away from being marginal student in the decision of which school to report as top choice such that even in the worst case distribution of preferences their valuation is still outside the wedge that is created by existence of *unconstrained* students.

Figure 2.1, Figure 2.2, Figure 2.3 are drawn for given numerical values to parameters q_1, q_2 and η. They show the behavior of key functions of the analysis I have made so far. In the figures

$x-axis$ ranges all possible values of equilibrium measure of *constrained* students reporting s_1 as top choice, k. This range was given by Lemma 2.1. Green dotted lines in the figures indicates the value of $\frac{(1-\eta)q_1}{1-\eta-q_2}$, remember by Claim 2.4 there must be at least q_2 measure of *constrained* students with value above $\frac{(1-\eta)q_1}{1-\eta-q_2}$ to rule out equilibria in which there are less than q_2 measure of *constrained* students reporting s_2 as top choice. Brown dotted line indicates the value of $\frac{q_1}{1-q_2}$, remember by Claim 2.3 there must be at least q_2 measure of *constrained* students with value above $\frac{q_1}{1-q_2}$ to have existence of equilibrium in which there are at least q_2 measure of *constrained* students reporting s_2 as top choice. Orange line shows $\bar{c}(k)$, that is the cutoff *constrained* students with valuation above which are better off under BM compared to DA for given equilibrium measure of *constrained* students reporting s_1 as top choice, k. Blue line shows $\underline{c}(k)$, that is the cutoff *constrained* students with valuation below which are better off under BM compared to DA for given k. Purple solid line shows $c(k)$, that is the cutoff *constrained* students with valuation strictly below (above) which reports s_1 (s_2) as top choice and *constrained* students with valuation equal to $c(k)$ are indifferent for given k. Purple dotted line shows the value of $\frac{(1-\eta)q_1}{(1-q_2)}$ given in Lemma 2.4. Notice that if a *constrained* student has valuation below purple dotted line, he will be better off in the BM equilibrium whatever is the distribution of preferences since he will be below the blue line for all possible k. Remember, as mentioned above distribution of preferences pins down the equilibrium k. So, having a *constrained* student with value $v < \frac{(1-\eta)q_1}{(1-q_2)}$ ensures the existence of *constrained* student better off in BM compared to DA even in the worst case distribution of preferences. The vertical distance between the two black dots is equal to $\frac{\eta(q_1(1-\eta)+q_2)}{q_1(1-\eta)+\eta q_2}$ which was given Proposition 2.2. This vertical distance shows the heterogeneity in preferences sufficient to ensure the existence of *constrained* students who are better off under BM compared to DA for any given distribution of preferences (that satisfy A2.6). So this is the heterogeneity that will be enough even for the worst case distribution of preferences. From the figures the wedge between $\bar{c}(k)$ and $\underline{c}(k)$ that is created by existence of *constrained* students can be seen clearly. And one can see from the figures that people who are marginal in their decision of choosing a school as top choice (ones with value near $c(k)$ for given equilibrium k) are better off in DA compared to BM.

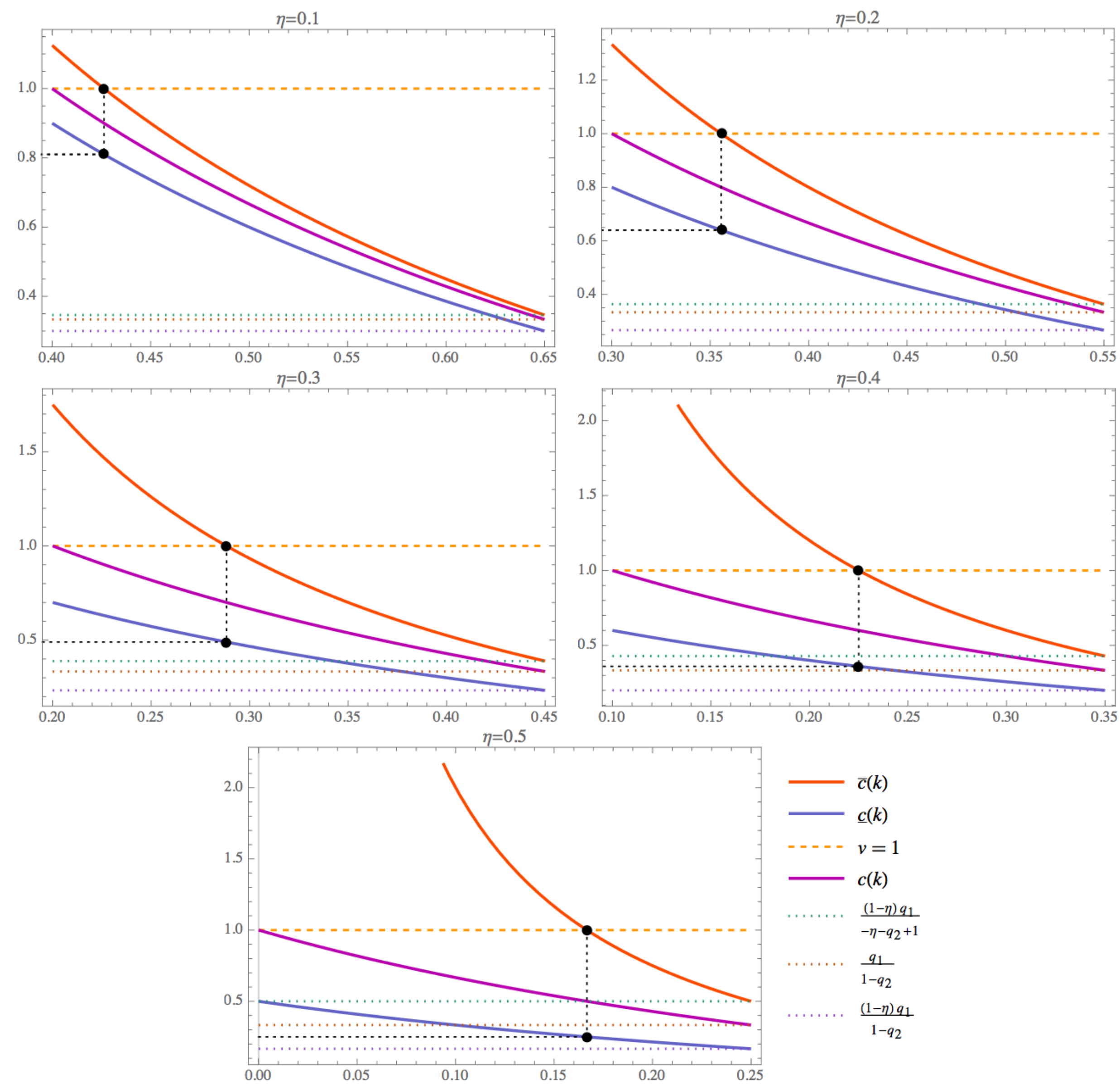
And *constrained* students who are inframarginal (the ones outside the wedge) are better off in BM compared to DA. Another thing to notice is that the wedge between $\bar{c}(k)$ and $\underline{c}(k)$ increases in η. When η is kept constant, as $q_1$and q_2 increases range of possible k decrease, since $k \leq 1-\eta-q_2$; and the dotted lines shift upwards.

Figure 2.1: Case with $q_1 = q_2 = 1/5$ for different values of η



Notes: Range of the x-axis is $[\max\{\frac{q_1}{q_1+q_2}-\eta, 0\}, 1-\eta-q_2] \setminus \{\frac{q_1}{q_1+q_2}-\eta\}$, which are all possible values of k, the equilibrium measure of constrained students reporting s_1 as top choice; the y-axis gives the values of the functions evaluated at each k.

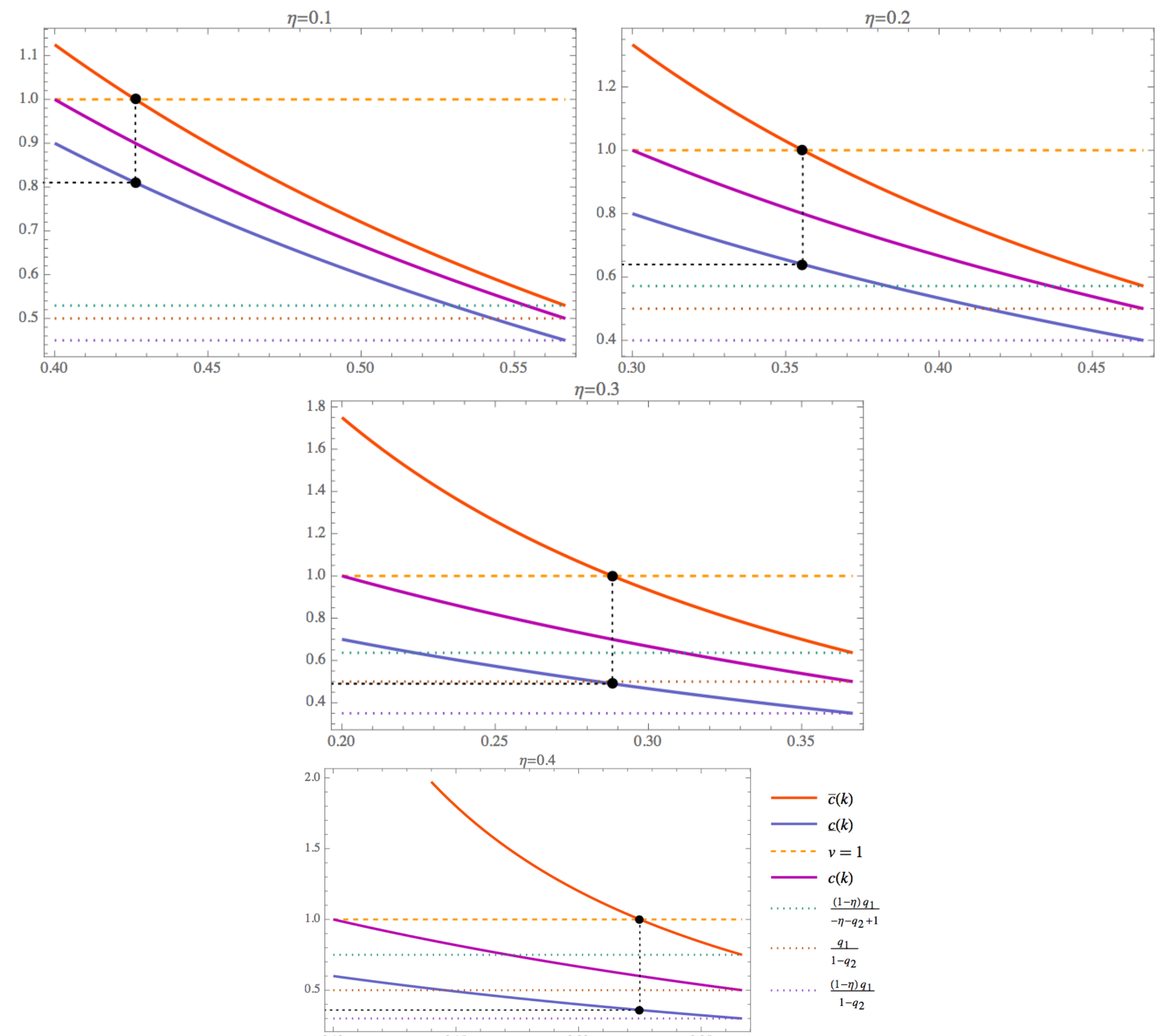
Figure 2.2: Case with $q_1 = q_2 = 1/4$ for different values of η



Notes: Range of the x-axis is $[\max\{\frac{q_1}{q_1+q_2} - \eta, 0\}, 1-\eta-q_2] \setminus \{\frac{q_1}{q_1+q_2} - \eta\}$, which are all possible values of k, the equilibrium measure of constrained students reporting s_1 as top choice; the y-axis gives the values of the functions evaluated at each k.

Going back to Lemma 2.4, it is not surprising that a type that has strong enough preferences for s_1 relative to s_2 is better off under the BM equilibrium compared to DA since in BM equilibrium there are less students reporting $s_1$as top choice compared to DA. Remember the reason of impos-

Figure 2.3: Case with $q_1 = q_2 = 1/3$ for different values of η



Notes: Range of the x-axis is $[\max\{\frac{q_1}{q_1+q_2} - \eta, 0\}, 1 - \eta - q_2] \setminus \{\frac{q_1}{q_1+q_2} - \eta\}$, which are all possible values of k, the equilibrium measure of constrained students reporting s_1 as top choice; the y-axis gives the values of the functions evaluated at each k.

ing condition A2.6 was to rule out the equilibria in which there are less than q_2 students who report s_2 as top choice. Following Lemma shows that even if we do not rule out such equilibria, types of *constrained* students described in Lemma 2.4 would still be weakly better off in any equilibria of BM compared to DA.

Lemma 2.5. *Suppose A2.1-A2.5 holds, then constrained students with type v with* $v < \frac{(1-\eta)q_1}{(1-q_2)}$ *(if exists) are weakly better off in any symmetric equilibrium of BM compared to DA; and they are strictly better off if not everyone is reporting* s_1 *as top choice in which case payoffs are equal for everyone.*

Proof shows that when $k > 1-\eta-q_2$, it turns out that threshold for v below which a *constrained* student reporting s_1 as top choice is better off in BM compared to DA, is increasing in k. So it is guaranteed to be satisfied for all k, if it is satisfied at $k = 1-\eta-q_2$. And at $k = 1-\eta-q_2$ the threshold becomes $\frac{(1-\eta)q_1}{(1-q_2)}$ like in the case of $k \leq 1-\eta-q_2$, although the two thresholds are different functions of k.

This leads to a positive result for BM for distributions of preferences that has rich support without need to restricting attention to equilibrium of the kind $k \leq 1-\eta-q_2$. First, let's show that an equilibrium exists even if the condition in Claim 2.3 is not satisfied.

Claim 2.5. Suppose A2.1-A2.5 holds, then there exist a symmetric equilibrium of BM.

Last lemma above leads to the following result that is favorable for BM. When the distribution of preferences for *constrained* students has full support in $(0,1)$ then there are *constrained* students who strictly prefer BM payoff to their DA payoff.

Theorem 2.1. *Suppose A2.1-A2.5 holds, then there exists positive measure of* constrained *students who are strictly better off under any symmetric equilibrium of BM compared to DA for any distribution of preferences of* constrained *students with full support.*

Note that BM equilibria is no longer guaranteed to be unique since I do not assume A2.6. Nevertheless, there are *constrained* students who are strictly better off in all equilibria of BM compared to DA when the distribution of preferences has rich enough support.

Role of full support assumption is clear. First it eliminates the situation in which everyone reports s_1 as top choice since there are *constrained* students who have strong enough preferences for s_2. Remember in such situation BM and DA would yield same payoffs. Second, since not everyone is reporting s_1 as top choice in the equilibrium of BM, students who have strong enough preferences for s_1 compared to s_2 must be better off in BM compared to DA since everyone reports s_1 as top choice in DA. Another interpretation of full support assumption is that it guarantees the existence of inframarginal *constrained* students outside of the band that is given by Lemma 2.2 and 3. Remember *constrained* students with valuation outside of that band are better off in BM compared to DA.

2.3.2.1 Uniform (Continuous) Distribution of Types

In this subsection I will make assumption of uniform distribution of types to pin down the equilibrium. This will allow me to get the share of *constrained* students who are strictly better off under BM compared to DA. So let $f(.)$ be the density function of uniform distribution on $(0,1)$ and $F(.)$ be its associated c.d.f.

Assumption 2.7. *(A2.7) Distribution of preferences for* constrained *students with p.d.f. $f(.)$, follows $U(0,1)$.*

Under uniform distribution assumption I will show below that we have unique equilibrium whether equilibrium measure of *constrained* students reporting s_2 as top choice is less than q_2 or at least q_2. Moreover, a condition on primitives characterizes the kind of the unique equilibrium that is played.

Lemma 2.6. *Suppose A2.1-A2.5 and A2.7 holds. Then there exists a unique (symmetric) equilibrium of BM. Moreover , in the equilibrium there are at least q_2 measure of* constrained *students reporting s_2 as top choice if and only if $\frac{q_1(1-\eta)}{(1-\eta-q_2)(1-q_2)} \leq 1$.*

Lemma 2.6 says that condition $\frac{q_1(1-\eta)}{(1-\eta-q_2)(1-q_2)} \leq 1$ rules out the equilibria in which there are less than q_2 measure of *constrained* students who are reporting s_2 as top choice and it also guarantees the existence of an equilibrium in which there are at least q_2 measure of *constrained* students reporting s_2 as top choice. Likewise, $\frac{q_1(1-\eta)}{(1-\eta-q_2)(1-q_2)} > 1$ rules out the equilibrium in which there are at least q_2 measure of *constrained* students reporting s_2 as top choice and guarantees the existence of an equilibrium in which there are less than q_2 measure of *constrained* students reporting s_2 as top choice. Since here I assume a distribution for types, this implies a restriction on the primitives through Claim 2.3 and this is how I get the condition $\frac{q_1(1-\eta)}{(1-\eta-q_2)(1-q_2)} \leq 1$. Remember when I did not assume a distribution I had to assume A2.6 to rule out the equilibrium in which $k > 1-\eta-q_2$ and A2.6 was stronger than the condition in Claim 2.3 . Here with uniform distribution assumption, I can pin down the equilibrium measure of k when $k > 1-\eta-q_2$, it turns out the condition that guarantees the existence of equilibrium with $k \leq 1-\eta-q_2$ also rules out equilibrium with $k > 1-\eta-q_2$.

Next Lemma provides the equilibrium measure of *constrained* students reporting s_1 as top choice when $k > 1-\eta-q_2$ and $k \leq 1-\eta-q_2$. Also, it shows which type of *constrained* students reports s_1 or s_2 as top choice and which students are strictly better off under the BM equilibrium compared to DA.

Lemma 2.7. *Suppose A2.1-A2.5 and A2.7 holds. Let $k(q_1,q_2,\eta)$ denote the equilibrium measure of* constrained *students reporting s_1 as top choice.*

If $\frac{q_1(1-\eta)}{(1-\eta-q_2)(1-q_2)} \leq 1$, then

$$k(q_1,q_2,\eta) = \frac{q_1(-1+\eta)+q_2\left(-\eta+\sqrt{\frac{q_1^2(-1+\eta)^2}{q_2^2}+\eta^2+\frac{2q_1(2-3\eta+\eta^2)}{q_2}}\right)}{2q_2};$$

$\frac{k(q_1,q_2,\eta)}{(1-\eta)}$ *is the cutoff that* constrained *students with v below (above) the cutoff reports s_1 (s_2) as top choice; and* constrained *students with valuation v are strictly better off under BM compared to DA if and only if either $v > k(q_1,q_2,\eta)+\eta$ or $v < k(q_1,q_2,\eta)$. Also, $k(q_1,q_2,\eta)+\eta < 1$.*

If $\frac{q_1(1-\eta)}{(1-\eta-q_2)(1-q_2)} > 1$*, then*

$$k(q_1,q_2,\eta) = \frac{q_1(1-\eta)}{1-\eta-q_2} - \eta;$$

$\frac{q_1}{1-\eta-q_2} - \frac{\eta}{1-\eta}$ *is the cutoff that* constrained *students with v below (above) the cutoff reports s_1 (s_2) as top choice;* and constrained *students with valuation v is strictly better off under BM compared to DA if and only if either* $v > \frac{q_1(1-\eta)}{1-\eta-q_2}$ *or* $v < \frac{q_1(1-\eta)}{1-\eta-q_2} - \eta$ *. Also,* $\frac{q_1(1-\eta)}{1-\eta-q_2} < 1$ *and* $\frac{q_1(1-\eta)}{1-\eta-q_2} - \eta > 0$.

Equilibrium measure $k(q_1,q_2,\eta)$ of *constrained* students for the case with $\frac{q_1(1-\eta)}{(1-\eta-q_2)(1-q_2)} \leq 1$ is derived from Claim 2.2 since this is the case in which $k \leq 1-\eta-q_2$; and by the logic that in an equilibrium k there must be k measure of *constrained* students below $c(k)$. Then the cutoffs for playing s_1 versus s_2 and being better off in BM versus being better off in DA are derived from definitions of $c(k)$, $\bar{c}(k)$ and $\underline{c}(k)$ given in Claim 2.2 and Lemma 2.2. For the case with $\frac{q_1(1-\eta)}{(1-\eta-q_2)(1-q_2)} > 1$, $k(q_1,q_2,\eta)$ was already derived in Lemma 2.6 with a similar logic to the case of $\frac{q_1(1-\eta)}{(1-\eta-q_2)(1-q_2)} \leq 1$ and it was found to be unique. Cutoffs are derived from comparing the payoffs of *constrained* students for a given equilibrium k and evaluating these cutoffs at the equilibrium $k(q_1,q_2,\eta)$.

Next, I state the main result of this subsection. It follows from last lemma and having uniform distribution of types.

Theorem 2.2. *Suppose A2.1-A2.5 and A2.7 holds. Then,* $1-\eta$ *fraction of* constrained *students are strictly better off under the BM equilibrium compared to DA; and* η *fraction of* constrained *students are strictly better off under DA compared to the BM equilibrium.*

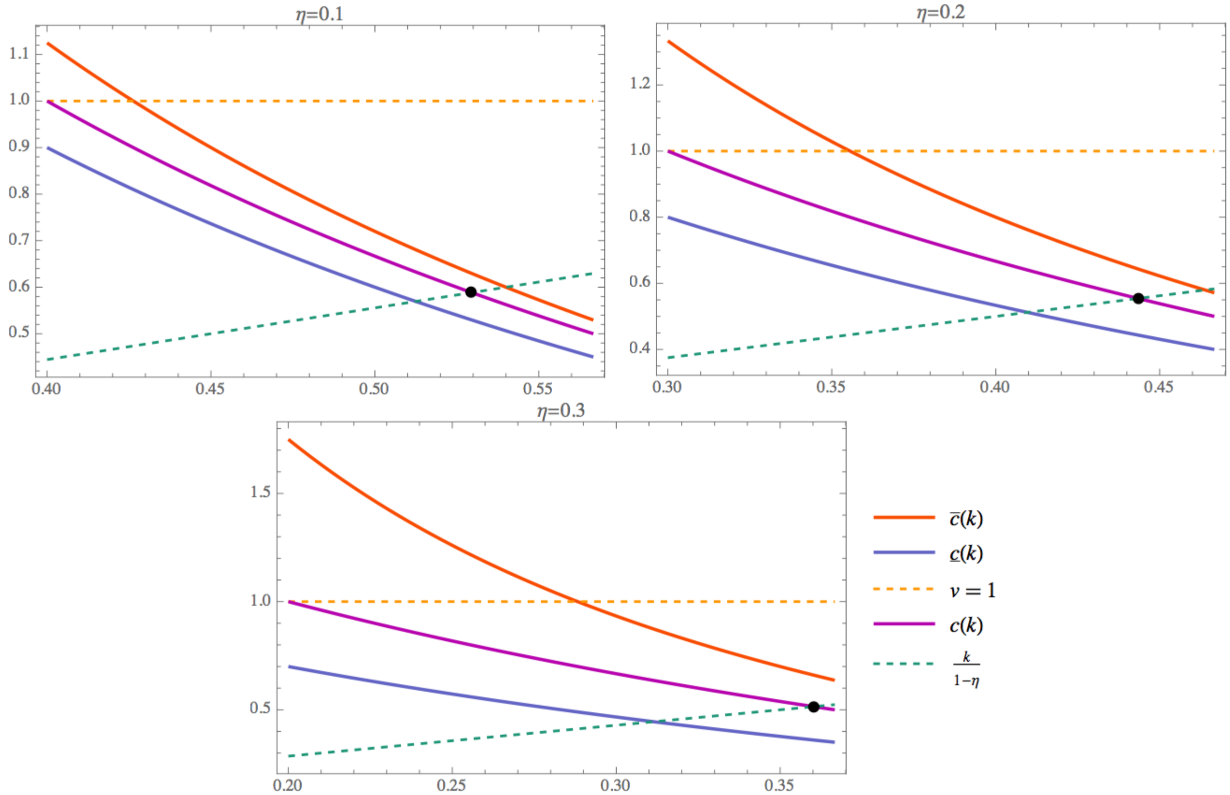
Theorem 2.2 states that when preference distribution for *constrained* students is uniform, fraction of *constrained* students who are strictly better off under the BM equilibrium compared to DA is exactly equal to share of *constrained* students $1-\eta$ in the economy. The result follows from the fact that under uniform distribution the wedge between upper bound and lower bound of the set of values for which students are strictly better off under DA compared to BM is exactly equal to

η, share of *unconstrained* students. This result means that when the distribution of preferences is uniform, if less than half of the students are able to go to private school then more than half of the students who cannot go to private school are better off in the unique BM equilibrium compared to DA.

Figure 2.4 below is drawn for the case of $q_1 = q_2 = 1/3$ for values of $\eta = 0.1, 0.2, 0.3$ and distribution of preferences for *constrained* students is uniform. Note that $\frac{q_1(1-\eta)}{(1-\eta-q_2)(1-q_2)} \leq 1$ for all the cases so equilibrium $k(q_1,q_2,\eta) \leq 1-\eta-q_2$ for all the cases in Figure 2.4. The value of black dot on the the x-axis gives the equilibrium $k(q_1,q_2,\eta)$. Theorem 2.2 tells that vertical distance between orange line and blue line is equal to η at the equilibrium $k(q_1,q_2,\eta)$. Also it shows how the equilibrium is calculated under the uniform distribution. Equilibrium occurs at the intersection of the lines $\frac{k}{1-\eta}$ and $c(k)$. This is because at the equilibrium there must be k measure of *constrained* students below $c(k)$; and measure of *constrained* students below $c(k)$ at a given k is given by $F(c(k))(1-\eta) = c(k)(1-\eta)$.

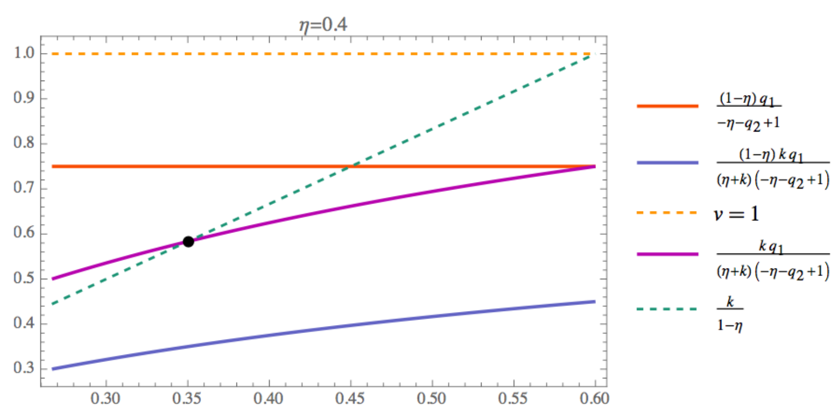
Figure 2.5 indicates the case $q_1 = q_2 = 1/3$ and $\eta = 0.4$. In this case $\frac{q_1(1-\eta)}{(1-\eta-q_2)(1-q_2)} > 1$. So equilibrium $k(q_1,q_2,\eta) > 1-\eta-q_2$. Proof of Lemma 2.7 shows that when $k > 1-\eta-q_2$, a *constrained* student with valuation v is strictly better off under BM compared to DA when $v > \frac{(1-\eta)q_1}{1-\eta-q_2}$ or $v < \frac{(1-\eta)kq_1}{(\eta+k)(1-\eta-q_2)}$. These thresholds as functions of k (first one is constant) are orange and and blue curves in Figure 2.5. Proof of Lemma 2.6 shows that when $k > 1-\eta-q_2$, threshold students with value above (below) which reports s_2 (s_1) as top choice is $\frac{k}{k+\eta}\frac{q_1}{1-\eta-q_2}$. This threshold is the purple curve in Figure 2.5. The value on the the x-axis that the black dot corresponds to is the equilibrium $k(q_1,q_2,\eta)$ again. The calculation of equilibrium is similar to the case with $\frac{q_1(1-\eta)}{(1-\eta-q_2)(1-q_2)} \leq 1$. Again, in this case as told by Theorem 2.2 distance between orange and blue curves is equal to η.

Figure 2.4: Equilibrium under uniform distribution and $q_1 = q_2 = 1/3$ for $\eta = 0.1, 0.2, 0.3$



Notes: The x-axis ranges all possible equilibrium $k \leq 1 - \eta - q_2$; the y-axis shows the values of functions at different all possible $k \leq 1 - \eta - q_2$

Figure 2.5: Equilibrium under uniform distribution and $q_1 = q_2 = 1/3$ for $\eta = 0.4$



Notes: the x-axis ranges all possible equilibrium $k > 1 - \eta - q_2$; the y-axis shows the values of functions at different all possible $k > 1 - \eta - q_2$

2.4 Analysis of Entry of Private Schools: Comparison of Equilibria with and without Private Schools in Boston Mechanism

Many school districts around the world are using BM. If one wants to consider the effect of entry of private schools on the centralized allocation system of public schools, comparing the payoff of students under BM before and after the entry of the private school would be helpful.

Aim of this section is to compare the welfare of *constrained* students under Boston mechanism when unconstrained students have access to outside options and not; in other words comparing payoffs of *constrained* students before and after the entry of private schools. I will provide conditions under which *constrained* students are better off when private school enters the neighborhood. I again analyze the symmetric equilibria of the economies. Also, I continue to have assumptions A2.1-A2.3 as basis assumptions. Therefore, below analysis is for $m = 3$ case. I will state when additional assumptions are used.

First I examine the case when the entering private school is very high quality, so that all *unconstrained* students prefer to leave the centralized mechanism and go to the private school directly, that is $l = 0$.

2.4.1 Entry of High Quality Private School

Assumption 2.8. *(A2.8)* $l = 0$

In this case, I analyze switch from an economy $P_3 = (\{q_j\}_{j=1}^3, f(v), g(v))$ without private schools to economy $E_3 = (\{q_j\}_{j=1}^3, 0, f(v), g(v))$ with private schools under assumptions A2.1-A2.3 and A2.8. The following lemma presents an expected result. When number of students leaving the centralized mechanism is very large enough there will be some *constrained* student strictly better off after private school enters.

Lemma 2.8. *Suppose A2.1-A2.3 and A2.8 holds and $\eta \geq 1 - q_1 - q_2$. Let's take an arbitrary (symmetric) equilibrium σ^* of the economy P_3, and let's take an arbitrary (symmetric) equilibrium $\tilde{\sigma}$ of the economy E_3. Then there is positive measure of* constrained *students who are strictly better off under $\tilde{\sigma}$ compared to σ^*.*

Nevertheless this is not very interesting case since when measure of *unconstrained* students is that large there is no need to strategize for *constrained* students after *unconstrained* students leave the market since if they cannot enter s_1 in first step it is sure to get into s_2 since total measure of *constrained* students is less than or equal to $q_1 + q_2$.

Example 2.3 shows that if number of *unconstrained* students is not large very large, entry of private school, even if it is very good, may not strictly benefit any *constrained* students. In fact, the example shows that all *constrained* students are weakly worse off (some strictly) under some equilibria with private school compared to equilibrium without private school.

Table 2.4: Payoffs of Example 2.3

	Type 1	Type 2	Type 3
s_1	1	1	1
s_2	$\frac{2}{9}$	$10/17$	$\frac{2}{9}$
s_3	0	0	0

Example 2.3. Suppose there are three types of students with payoffs as given in Table 2.4. Type 1 and Type 2 students are *constrained* and Type 3 student is *unconstrained*. There are $13/20$ measure of Type 1 students, $1/4$ measure of Type 2 students and, $1/10$ measure of type 3 students. $q_1 = q_2 = 1/3$ and $q_3 \geq 1/3$. Consider equilibria without private schools. Note that, it is strictly dominant strategy to report s_1 as top choice for type 1 and type 3 students. In that case student type 2 ranking s_2 as top choice is an equilibrium strategy.[20]

[20]Note that $\frac{10}{17} > \frac{1/3}{3/4} + \frac{10}{17}\frac{1/3-1/4}{3/4}$ which boils down to $\frac{10}{17} > \frac{26}{51}$

One can check that this is the unique equilibrium when there is no private school. In this equilibrium type 2 student gets payoff $10/17$ and type 1 student gets $38/81$.[21]

Now suppose private school enters, that is preferred to s_1. So type 3 leaves the centralized mechanism. Again note that it is strictly dominant strategy for type 1 to report s_1 as top choice again. Type 1 reports s_1 as top choice and type 2 reports s_1 as top choice with probability x is an equilibrium for any $x \in [0,1]$. Note that type 2 does not deviate from the equilibrium strategy.[22]

It can be shown that for any equilibrium with $x > 0.76$ when private school exists, type 1 student is worse off compared to the equilibrium in which private school does not exist.[23]

The question that emerges from this example is that whether this is a knife-edge case. It is an unexpected situation to have *constrained* students get worse off if some students leave the market because one can think that competition in the centralized mechanism has declined. Note that there were infinitely many equilibria in the game with private school. And in all the equilibria in which $x < 0.76$, type 1 *constrained* students were strictly better off compared to economy without private school. So indifference and how one breaks the indifference plays an important role in getting this kind of example.

The following result shows that Example 2.3 presents indeed a knife edge case, hence one can actually "expect" to have a *constrained* student benefiting from entry of private school when the private school is very high quality.

Lemma 2.9. *Suppose A2.1-A2.3 and A2.8 holds. For any pair of (symmetric) equilibria σ^* and $\tilde{\sigma}$ such that σ^* is an equilibrium under $P_3 = (\{q_j\}_{j=1}^3, f(v), g(v))$ and $\tilde{\sigma}$ is an equilibrium under $E_3 = (\{q_j\}_{j=1}^3, 0, f(v), g(v))$, there exist positive measure of* constrained *students who are strictly better off in $\tilde{\sigma}$ compared to σ^* if the following condition is satisfied:*

[21] The latter is derived from $\frac{1/3}{3/4} + \frac{1/3-1/4}{3/4}\frac{2}{9}$.

[22] $\frac{1/3}{\frac{13}{20}+\frac{x}{4}} + \frac{10}{17}\frac{1/3-\frac{1-x}{4}}{\frac{13}{20}+\frac{x}{4}} = 10/17$ and note that payoff of type 2 is $10/17$ in all of these equilibria which equals his payoff in the unique equilibrium of the economy without private schools.

[23] To see this first note that payoff of type 1 in these equilibria are $\frac{1/3}{\frac{13}{20}+\frac{x}{4}} + \frac{2}{9}\frac{1/3-\frac{1-x}{4}}{\frac{13}{20}+\frac{x}{4}}$. To find the x for which type 1 is worse off compared to equilibrium in the economy without private schools, it is enough to solve the following inequality $\frac{1/3}{\frac{13}{20}+\frac{x}{4}} + \frac{2}{9}\frac{1/3-\frac{1-x}{4}}{\frac{13}{20}+\frac{x}{4}} < \frac{38}{81}$ which is equivalent to $x > 0.76$.

If there exists positive measure of constrained *students with valuation v such that* $\frac{q_1}{1-q_2} < v \leq \frac{q_1}{1-\eta-q_2}$ *then positive measure of such* constrained *students have valuation* $v \neq \frac{q_1}{1-\eta-q_2}$

Proof of this lemma follows from Claim B.1 and Lemma B.1 in Other Results. Lemma B.1 characterizes the situations when there can exist *constrained* students who are strictly better off after the entry of private school.

Lemma 2.9 shows that not having a *constrained* students strictly better off after the entry of private school is a situation that is hard to occur. Following argument hints to the logic behind this result. Note that $\frac{q_1}{1-q_2}$ is the cutoff that students with valuation strictly above which strictly prefer reporting s_2 as top choice in the case when there are less than q_2 students reporting s_2 as top choice and there is no private school. And $\frac{q_1}{1-\eta-q_2}$ is the cutoff that students with valuation (strictly) below which (strictly) weakly prefers reporting s_1 as top choice when there are less than q_2 *constrained* students reporting s_2 as top choice and there is private school. Therefore, the result says that when there are less than q_2 students reporting s_2 as top choice before and after the entry of the private school, having some of the *constrained* students who switches from reporting s_2 to s_1, strictly prefer switching is enough. Note that it would not be possible to have Example 2.3 in this case since type 2 in the example strictly prefers reporting s_2 as top choice before the entry of private school but indifferent between reporting s_1 as top choice and s_2 as top choice after the entry of private school. This lemma leads to the following proposition immediately.

Proposition 2.3. *Suppose A2.1-A2.3 and A2.8 holds. For any pair of (symmetric) equilibria* σ^* *and* $\tilde{\sigma}$ *such that* σ^* *is an equilibrium under* $P_3 = (\{q_j\}_{j=1}^3, f(v), g(v))$ *and* $\tilde{\sigma}$ *is an equilibrium under* $E_3 = (\{q_j\}_{j=1}^3, 0, f(v), g(v))$*, there exist positive measure of* constrained *students who are strictly better off in* $\tilde{\sigma}$ *compared to* σ^* *for almost all distributions of preferences.*

In the next subsection, I will analyze the case with $l = 1$. Such a case is important since *unconstrained* students do not leave the centralized market after the entry of private school in that case and they switch to reporting s_1 as top choice for sure if they were not already doing so.

2.4.2 Case with $m = 3$ and $l = 1$

In this section I examine the entry of private school when only the most preferred public school is preferred to the private school. Therefore $l = 1$, and I analyze the case of $m = 3$ again. One should compare the (symmetric) equilibria σ^* and $\tilde{\sigma}$ such that σ^* is an equilibrium under $P_3 = (\{q_j\}_{j=1}^3, f(v), g(v))$ and $\tilde{\sigma}$ is an equilibrium under $E_3 = (\{q_j\}_{j=1}^3, 1, f(v), g(v))$ for the analysis.

In this case, one can expect the effect of entry of private school to be realized through two channels. First, since *unconstrained* students would not prefer to go to public schools below s_1, there will be a decrease in competition for s_2, this can be expected to have positive welfare effect at least for some *constrained* students. Second, after the entry of private school, *unconstrained* students will definitely report s_1 as top choice since their second most preferable school which is private school is safe for them. This can increase the competition in s_1 coming from *unconstrained* students side if *unconstrained* students were not already reporting s_1 as top choice with probability one. Thus, it can potentially harm welfare of *constrained* students.

First I will silence the second channel and see if the effect is positive for some *constrained* students as expected. And I will use this result to get the result of the general case. To do so, for given economies P_3 and E_3, among the equilibria of P_3, I pick the ones such that all *unconstrained* students report s_1 as top choice (if such equilibrium exists) to compare with the equilibria from E_3.

2.4.2.1 All unconstrained students report s_1 as top choice with probability one when there is no private school

For this case I can show by an example that it is possible to have *constrained* students weakly worse off and a *constrained* student type strictly worse off in an equilibrium that occurs after entry of private schools compared to an equilibrium that occurs before the entry of private schools. Example 2.4 is similar to Example 2.3 but now it is with the case $l = 1$.

Example 2.4. Let payoff of the three types of students be as in Table 2.5 and all type 3 students are unconstrained and all type 1 and type 2 students are constrained. There is $17/30$ measure

Table 2.5: Payoffs of Example 2.4

	1	2	3
s_1	1	1	1
s_2	$\frac{4}{19}$	$\frac{9}{17}$	$\frac{19}{90}$
s_3	0	0	0

of type 1 students, $1/3$ measure of type 2 students and $1/10$ measure of type 3 students. And $q_3 > q_1 = q_2 = 1/3$. And suppose for type 3 value of private school is $\frac{2}{9}$.

Before entry of private schools, σ^* is the unique equilibrium where in σ^* all type 1 students and type 3 students report s_1 as top choice with probability one and all type 2 students report s_2 as top choice.[24]

After the entry of the private schools, we have an equilibrium $\tilde{\sigma}$ in which all students report s_1 as top choice with probability one.[25] Note that in this equilibrium $\tilde{\sigma}$ type 1 and 3 are worse off compared to σ^* although type 3 students are unconstrained and type 2 are equally well off in σ^* and $\tilde{\sigma}$.

As in the previous case with $l = 0$, the question of whether this is a knife edge case occurs. Lemma 2.10 shows that it is.

Lemma 2.10. *Suppose A2.1-A2.4 hold. Take economies $P_3 = (\{q_j\}_{j=1}^3, f(v), g(v))$ and $E_3 = (\{q_j\}_{j=1}^3, 1, f(v), g(v))$. For any pair of equilibrium σ^* and $\tilde{\sigma}$ such that σ^* is an equilibrium under P_3 in which all* unconstrained *students report s_1 as top choice with probability one and $\tilde{\sigma}$ is an equilibrium under E_3, there are* constrained *students who are strictly better off in $\tilde{\sigma}$ compared to σ^*or all* constrained *students get the same payoff under both equilibria if the following condition is satisfied:*

[24] For type 1 we should check $\frac{1/3}{2/3} \geq \frac{4}{19}$ which boils down to $\frac{1}{2} \geq \frac{4}{19}$. For type 2 we should check $\frac{9}{17} \geq \frac{1/3}{2/3}$ which is true. For type 3 we should check $\frac{1/3}{2/3} \geq \frac{19}{90}$ which is true. It is easy to check there is no other equilibrium (symmetric) before the entry of private schools.

[25] Type 1 needs to satisfy $\frac{1}{3} + \frac{1/3}{9/10}\frac{4}{19} \geq \frac{4}{19}$ which boils down to $0.411306 \geq 0.211$. Type 2 needs to satisfy $\frac{1}{3} + \frac{9}{17}\frac{1/3}{9/10} \geq \frac{9}{17}$ which boils down to $\frac{9}{17} \geq \frac{9}{17}$. Type 3 needs to satisfy $\frac{1}{3} + \frac{2}{3}\frac{2}{9} \geq \frac{19}{90}$ which boils down to $\frac{13}{27} \geq \frac{19}{90}$

If $\tilde{x}_1 > x_1 > 1 - q_2$ and there exists constrained *students with $\frac{q_1}{1-q_2} < v \leq \frac{\tilde{x}_1 - \eta}{\tilde{x}_1} \frac{q_1}{1-\eta-q_2}$ then at least one of such students has $v \neq \frac{\tilde{x}_1 - \eta}{\tilde{x}_1} \frac{q_1}{1-\eta-q_2}$; where $\tilde{x}_1$ and x_1 are the total measure of students reporting s_1 as top choice in $\tilde{\sigma}$ and σ^* respectively.*

The result follows from Claim B.2, Claim B.3 and Lemma B.2 in the Other Results. Lemma B.2 characterizes all the situations in which there exists a *constrained* student who is strictly better off or all *constrained* students get the same payoff after the entry of the private school when all *unconstrained* students report s_1 as top choice before the entry of the private school. Lemma 2.10 says that when all *unconstrained* students report s_1 as top choice before the entry of private schools, it is hard to have all *constrained* students weakly worse off and some strictly worse off after the entry of private school. Condition given in Lemma 2.10 , which is very similar to the key condition in the Lemma 2.9, can be explained as follows. If we are in the case that there are less than q_2 measure of students reporting s_2 as top choice before and after the entry of the private school, then some of the *constrained* students who strictly prefer to report s_2 as top choice in the equilibrium without private school and weakly prefers to report s_1 as top choice after the entry of private schools (if such students exist) must be strictly preferring reporting s_1 as top choice after the entry of the private school. To see this note that $\frac{q_1}{1-q_2}$ is the cutoff that students with value strictly above which strictly prefers reporting s_2 as top choice to reporting s_1 as top choice when there is no private school. And $\frac{\tilde{x}_1 - \eta}{\tilde{x}_1} \frac{q_1}{1-\eta-q_2}$ is the cutoff that students with value (strictly) below which (strictly) prefers reporting s_1 as top choice to reporting s_2 as top choice when there is private school. Note that it would not be possible to have Example 2.4 in this case since type 2 in the example strictly prefers reporting s_2 as top choice before the entry of private school but indifferent between reporting s_1 as top choice and s_2 as top choice after the entry of private school.

Another point that worths mentioning is that Lemma 2.10 means that if we rule out the equilibria in which there are less than q_2 students reporting s_2 as top choice before and after the entry of the private school, then either each *constrained* student get the same payoff before and after the entry of private school or there are *constrained* students who are strictly better off after the entry of private schools. This is important since when I look at the case in which *unconstrained* students can report s_2 as top choice before the entry of private schools, I will restrict attention to equilibria in which there are at least q_2 measure of students reporting s_2 as top choice before and after the entry of private schools.

2.4.2.2 General case for $l = 1$ and $m = 3$

Here, I will consider the more interesting case in which $l = 1$ and in the equilibria I consider without private schools there may be *unconstrained* students reporting s_2 as top choice with positive probability. Note that in the equilibrium after the entry of the private school, all of them report s_1 as top choice with probability one.

First of the two examples, Example 2.5 below shows that it is possible to have all *constrained* students weakly worse off after the entry of private schools and some strictly worse off. Second example, Example 2.6 shows that it is possible to have some *constrained* students strictly better off after the entry of private school.

Table 2.6: Payoffs of Example 2.5

	1	2	3
s_1	1	1	1
s_2	0.4	0.2	0.8
s_3	0	0	0

Example 2.5. Suppose there are $1/4$ measure of type 1 students, $3/8$ measure of type 2 students and $3/8$ measure of type 3 students with payoffs as in Table 2.6 and type 3 students are *unconstrained* and all other students are *constrained.* Also, suppose that $q_1 = q_2 = \frac{1}{4}$ and $q_3 \geq \frac{1}{2}$.

Let's call the equilibrium before the entry of the private school σ^*. In σ^*, all type 1 and type 2 students report s_1 as top choice with probability one and all type 3 students report s_2 as top choice.[26]

[26] To see that this is an equilibrium, see the following. Type 2 needs to satisfy $\frac{1/4}{5/8} \geq 0.2\frac{1/4}{3/8}$ which boils down to $0.4 \geq \frac{4}{30}$. Type 1 needs to satisfy $\frac{1/4}{5/8} \geq 0.4\frac{1/4}{3/8}$ which boils down to $0.4 \geq \frac{4}{15}$. Type 3 needs to satisfy $\frac{1/4}{5/8} \leq 0.8\frac{1/4}{3/8}$ which boils down to $0.4 \leq \frac{8}{15}$. It is easy to see that, this equilibrium is unique.

Let's call the equilibrium after the entry of the private school $\tilde{\sigma}$. In $\tilde{\sigma}$, all type 2 and type 3 students report s_1 as top choice with probability one and all type 1 students report s_2 as top choice.[27]

Note that type 1 gets same payoff 0.4 in both $\tilde{\sigma}$ and σ^*, but type 2 *constrained* student gets $1/3$ in $\tilde{\sigma}$ and gets 0.4 in σ^*. So he is negatively affected from the increase in the competition for s_1 after the entry of private school and decrease in the competition for s_2 was not enough to make any *constrained* student strictly better off.

Now, I present the second example in which there are *constrained* students who are strictly better off after the entry of the private school.

Table 2.7: Payoffs of Example 2.6

	1	2	3
s_1	1	1	1
s_2	0.7	0.2	0.8
s_3	0	0	0

Example 2.6. Let there be $3/10$, $3/5$ and $1/10$ measure of type 1, type 2 and type 3 students respectively, with payoffs as in Table 2.7. Type 1 and type 2 are *constrained* and type 3 are *unconstrained* students. Let $q_1 = q_2 = 1/4$ and $q_3 \geq 1/2$.

Let the equilibrium of BM before (after) the entry of private schools is called σ^* ($\tilde{\sigma}$). One can check that in both cases there is unique equilibrium.

In σ^*, type 2 reports s_1 as top choice and type 1 and 3 report s_2 as top choice, and this is the unique equilibrium.[28] In $\tilde{\sigma}$, type 2 and 3 reports s_1 as top choice and type 1 reports $s_2$as top

[27]To see that this is an equilibrium, see the following. Type 2 needs to satisfy $\frac{1/4}{6/8} \geq 0.2$ which boils down to $\frac{1}{3} \geq \frac{1}{5}$. Type 1 needs to satisfy $\frac{1/4}{6/8} \leq 0.4$ which boils down to $\frac{1}{3} \leq \frac{2}{5}$. Type 3 plays s_1 as top choice, since he prefers private school to s_2.

[28]To see this is the unique equilibrium check the following. Type 1 needs to satisfy $\frac{1/4}{3/5} < \frac{1/4}{4/10}\frac{7}{10}$ which boils down to $\frac{5}{12} < \frac{7}{16}$. Type 3 needs to satisfy $\frac{1/4}{3/5} < \frac{1/4}{4/10}\frac{8}{10}$ which boils down to $\frac{5}{12} < \frac{8}{16}$. Type 2 needs to satisfy $\frac{1/4}{3/5} > \frac{1/4}{4/10}\frac{2}{10}$ which boils down to $\frac{5}{12} > \frac{1}{8}$.

choice.[29]

In $\tilde{\sigma}$ type 2 is worse off compared to σ^* since he gets $5/14$ and $5/12$ in these equilibria respectively. However, type 1 is better off under $\tilde{\sigma}$ compared to σ^*, type 1 *constrained* students get $7/12$ in $\tilde{\sigma}$ and $7/16$ in σ^*.

Note that in the second example differently from first example there was a *constrained* student who reports s_2 as top choice in the equilibrium before the entry of private schools, such student must be better off in the equilibrium after the entry private schools. The reason is that there is a decrease in competition for s_2 when *unconstrained* students change strategy, so reporting s_2 as top choice gives higher payoff to type 1 students compared to their payoff in equilibrium without private schools.

To get conditions on having *constrained* students better off after the entry of private schools I will restrict attention to equilibrium in which there are at least q_2 students reporting s_2 as top choice since it will make my life easier and it is also a relevant case as discussed in Section 3. First I will show the condition for existence of such an equilibrium when there is no private school. It is almost same condition with Claim 2.3 , but since there is no private school requirement is not on *constrained* students per se.

Claim 2.6. An equilibrium of BM without private schools in which measure of students reporting s_2 as top choice is at least q_2 exists if and only if there are at least q_2 measure of students with $v \geq \frac{q_1}{1-q_2}$

Following Claim shows that, A2.1-A2.6 guarantees not only that the equilibrium with private school has at least q_2 students reporting s_2 but also guarantees that the equilibria without private school has at least q_2 students reporting s_2 as top choice. But the key difference is that, A2.6 cannot guarantee that students who are reporting s_2 as top choice are *constrained* students in the equilibrium without private schools. They can be *unconstrained* students.

[29]To see this is an equilibrium check the following. Type 1 needs to satisfy $\frac{1/4}{7/10} < \frac{1/4}{3/10}\frac{7}{10}$ which boils down to $\frac{5}{14} < \frac{7}{12}$. Type 3 plays s_1 as top choice since he prefers private school to s_2. Type 2 needs to satisfy $\frac{1/4}{7/10} > \frac{1/4}{3/10}\frac{2}{10}$ which boils down to $\frac{5}{14} > \frac{1}{6}$.

Claim 2.7. Suppose A2.1-A2.6 holds, then in the economy without private schools, there are at least q_2 students who report s_2 as top choice in all symmetric equilibria of BM.

Note that as in Section 3, A2.6 guarantees existence of equilibrium both in an economy with private school and without private school. Next, I will show that under A2.1-A2.6 equilibrium without private schools is also unique.

Claim 2.8. Suppose A2.1-A2.6 holds, then game without private schools have unique equilibrium. Let k denote the measure of students reporting s_1 as top choice, then $k \in (\frac{q_1}{q_1+q_2}, 1-q_2]$.

Following proposition gives sufficient conditions for existence of *constrained* students who are better off after the entry of private school. The intuition behind the result is as follows. Under assumptions A2.1-A2.6, there are more than q_2 students reporting s_2 as top choice when there is no private school. When private school enter the neighborhood, all *unconstrained* students switch to reporting s_1 as top choice (the ones who were not already doing). This decreases the competition for *constrained* students who were applying to s_2 in the equilibrium without private schools if measure of *constrained* students who are switching from reporting s_1 as top choice to s_2 as top choice are not as much as *unconstrained* students switching to reporting s_1 as top. The conditions given in the proposition below guarantees that there are *constrained* students who are reporting s_2 as top choice when there is no private school. As mentioned above A2.6 does not guarantee that when there is no private school.

Proposition 2.4. *Suppose A2.1-A2.6 holds. Take the equilibrium σ^* from economy without private schools; and take the equilibrium $\tilde{\sigma}$ from economy with private schools. There is positive measure of* constrained *students who are strictly better off in $\tilde{\sigma}$ compared to σ^* or payoffs of each* constrained *student are same across $\tilde{\sigma}$ and σ^* if one of the following conditions is satisfied:*

(1) $\eta < q_2$

(2) there are more than $\max\{\eta - q_2, 0\}$ *measure of* unconstrained *students with valuation vectors $v < \frac{q_1}{1-q_2}$.*

(3) there are constrained *students with valuation $v \geq \frac{\eta}{1-\eta}\frac{q_1}{q_2}$.*

Condition (iii) in Proposition 2.4 immediately implies a result for distributions of preferences with rich support. To have condition (iii) satisfiable one needs to have $\frac{\eta}{1-\eta}\frac{q_1}{q_2} < 1$ which will be my next assumption.

Assumption 2.9. *(A2.9)* $\frac{\eta}{1-\eta}\frac{q_1}{q_2} < 1$

This assumption will satisfied for small η, having it less than 0.5 when $q_1 \leq q_2$ will be sufficient, if q_1 is large compared to q_2, then η needs to be smaller to satisfy the assumption.

Theorem 2.3. *Suppose A2.1-A2.5 and A2.9 holds. Take the equilibrium σ^* from economy without private schools; and take the equilibrium $\tilde{\sigma}$ from economy with private schools. There are* constrained *students who are strictly better off in $\tilde{\sigma}$ compared to σ^* or payoff of each* constrained *students is same across $\tilde{\sigma}$ and σ^* for all distributions of preferences of* constrained *students that have full support and satisfies A2.6.*

The role of full support assumption is intuitive since that means there will be students with strong preferences for s_2 and these *constrained* students will report s_2 as top choice when there is no private school and this is what I wanted to guarantee.

2.5 Conclusion

This paper is interested in two problems. First, it compares the ex ante welfare of students who cannot go to private schools under Boston Mechanism and Deferred Acceptance Mechanism, when some other students are able to go to private schools. I show that when the private school is not very desirable but still desirable to public schools in the neighborhood that are not very popular, then all students who cannot go to private school are weakly better off in BM compared to DA. Also, with a three school model, when only the first public school is preferred to private schools, my results show that students who are marginal on their decision of which school to submit as top choice in BM are worse off under BM compared to DA; and students with stronger cardinal preferences for some schools, that is students who are inframarginal in deciding which school to report as top choice in BM, are better off in BM compared to DA. I show that if the distribution of preferences has full support, then it is guaranteed to have students better off under BM compared to DA among the students who cannot go to private schools. In addition, assuming uniform distribution of preferences allows me to find the fraction of students who are better off under BM compared to DA. My results indicate that fraction of students who are better off under BM compared to DA among students who cannot go to private school is equal to one minus fraction of students who can go to private schools. I believe these results are important they show that existence of students who are better off in BM compared to DA even among the students who cannot afford private schools can be typical. Moreover, it shows which students win and lose from manipulable mechanisms. This can lead to some policy implications on whom to compensate if one or the other mechanism is chosen.

Second task this paper works on is comparing payoffs of students who cannot go to private schools before and after the entry of private schools under BM. I show that, when the entering private school is very high quality since all students who can go to private school leave the market then for almost all the distributions of preferences one can find students who are strictly better off after the entry of private schools among the students who cannot go to private school. When the most desirable public school is preferred to the entering public school, I show that if the distribution of preferences for students who cannot go to private school have full support and number of students who can go to private school is not very large, then either some of those students get strictly higher ex ante welfare or welfare of none of these students changes after the entry of private school. This exercise is also helpful since it allows one to understand the effect of entry of private schools on centralized mechanism. And this effect is not trivial when the centralized mechanism is not strategy proof. Note that this analysis can also be interpreted as an analysis of what can happen if a state decides to start a voucher program.

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APPENDIX A

APPENDIX TO CHAPTER 1

A.1 Additional Tables

Table A.1: Poverty Line in South Korea Across Household Size and Years

Year	1 person	2 people	3 people	4 people	5 people	6 people
2010	504,344	858,747	1,110,919	1,363,091	1,615,263	1,867,435
2011	532,583	906,830	1,173,121	1,439,413	1,705,704	1,971,995
2012	553,354	942,197	1,218,873	1,495,550	1,772,227	2,048,904
2013	572,168	974,231	1,260,315	1,546,399	1.832.482	2,118,566

Notes: This table indicates the poverty line for monthly income (in South Korean Won (KRW) depending on the number of people in the household. 1000 KRW is around $1 during our period of analysis. This table is obtained from Korean Statistical Information Service.

A.2 Appendix For the Empirical Analysis

Description of CMS

Definition. Centralized Mechanism of Seoul (CMS)

Table A.2: Regression Discontinuity Robustness Check

Dependent Variable: Share of students wanting to apply autonomous school	
Income>Cutoff Income (conventional)	-0.255
	(0.134)
Income>Cutoff Income (bias-corrected)	-0.27
	(0.134)
Income>Cutoff Income (robust)	-0.27
	(0.160)
Observations Left	88
Observations Right	519
Effective Observations Left	52
Effective Observations Right	129

Notes: This table indicates the effect of the discontinuous change in sticker price of PA schools on the share of students desiring to apply to PA schools at the effective income threshold for affirmative action eligibility. Estimates are calculated using bandwidth calculation proposed by Calonico, Cattaneo, Farrell, and Titiunik (2017). Included covariates are, students' gender, and whether middle school is public/private. Standard errors are in parenthesis. We do not discard the observations that are very close to cutoff in this RD regression.

Round -1: Each student submits his/preference list

Round 0: Each student draws a tie-breaking lottery number from the uniform distribution on [0,1].

Round 1-1: A student is assigned to his or her first choice schools (i.e. charter schools) if the schools are located in the Gu (district) in which he or she resides. 50% of the capacities can be filled in this round. In this round and the subsequent rounds, break ties with the tie-breaker and assignments are final.

Round 1-2: Assign remaining students to their first choice schools. All remaining capacities may be filled in this round.

Round 2-1: Assign remaining students to their second choice schools (i.e., science magnet) if the schools are located in the same Zone as the one in which students reside. 50% of remaining capacities may be filled in this round.

Round 2-2: Assign remaining students to their second choice schools. All remaining capacities may be filled in this round.

Round 3: Assign remaining students to their third choice schools (i.e., non-charter schools or non-science programs). Up to 20% of their capacities may be filled in this round, except for schools in the Central District, for which 60% may be filled.

Round 4: Assign remaining students to their fourth choice schools if the schools have not filled 20% (60% for schools in the Central District) of their capacities by the end of Round 3.

Round 5: Assign remaining students to their fifth choice schools.

Round 6: Assign remaining students to their sixth choice schools if the schools have not filled their capacities by the end of Round 5.

Administrative Assignment: Assign remaining students based on their ranking, length of commute, and their religion.

Comparison of Our Identification Strategy to Fixed Effects Approach

Another identification strategy for preferences over peers could have been using the variation in $m.p.r_{j,t}$ within schools across time, via using fixed effects of schools without using any instrument. If one decides to continue with fixed effects approach, for identifying the heterogenous preferences regarding peer quality, due to the concern described in the second paragraph of this sub-section, one would need to use school level fixed effects varying at the group level $g(c)$. One could argue that such strategy would be plausible in our case since one can assume that for the schools always remaining in CMS, unobservable characteristics are constant across 3 years, and there are changes in $m.p.r_{j,t}$ measures of these schools across years due to the effect of switches in 2011. However, students may not be able to know the way in which a school is affected in terms of entry cohort academic quality for *all* the schools, or they might not make the exact comparison in terms of the magnitude of the effect for *all* the schools. This is because it is not clear how will drain of high achieving students joining CMS due to switchers would propagate to *all* the schools remaining in CMS. Therefore, such strategy would try to identify preferences for peers from changes in peer composition that student is potentially unaware, which might lead to biased estimates. Our instrument allows us to have a more convincing story of the way students can be aware of the change in peer composition in schools. Students/families living close to a switching school might be aware that academically strong students in their middle school or neighborhood plan applying to a recently switched PA school nearby. This implies that many of the good students will be admitted to the switched school. Thus, they might also anticipate that the proportion of academically strong students who will enroll to close by NPA school is going to decline. Therefore, they might react by choosing a farther school from the closest neighborhood school.

Constructing Empirical Student Distribution for the Market of General High Schools

To construct the moments using the structural model and parameter candidates, we need an empirical distribution of students in the combined market of schools in CMS and PA schools to draw students from. For years 2010, 2011 and 2012 we have individual level observations of students, but these data sets do not include students who went to autonomous schools. If we only used these data sets, the distribution of students would be incorrect. We solve this problem by using the auxiliary survey data from 2009, which asks students who will start high school in 2010, whether they plan to apply PA schools. In the survey data we observe 2335 students who declared that they would like to apply to PA schools. However, this survey data is only available for students who actually attended CMS. Remember the students are admitted to PA schools through random lottery. We assume that students who indicated their desire to apply autonomous schools are the ones who applied and could not enter due to their lottery number. Since who enters PA schools among applicants is determined randomly, we assume that students who applied but could not enroll to PA schools comes from the same distribution with students who enrolled in autonomous schools in terms of observable and unobservable characteristics. Among the students who indicated interest of applying to autonomous school, we randomly duplicate some of them, so that number of these students get close to the number of students enrolled in PA schools in 2010.[1] This survey dataset is also linked to data of students attending CMS. In this survey data set we observe students' middle school, gender, enrolled school, neighborhood he lives and his preference list submitted to actual CMS for 2010. Using these common variables, we match students who had indicated that they would like to apply to an autonomous school to dataset of students who attended CMS in 2010. On average one student (unique non-duplicate student) in the survey data is matched to 4.2 students in student level 2010 CMS data. We weight students in the matched data by the inverse of the number of matches. Then we stack this dataset to the dataset of students who submit reports to CMS in

[1]This number is 4415 to be precise. We duplicated each student showing interest in PA school with probability 416/467.

2010. This way, we get the correct empirical distribution of students in the combined market and we assume that this distribution is same across three years. Note that since only students in the top 50 rank percentile in their middle school is eligible for autonomous schools, we discard the students in the bottom 50 rank percentile for this part of demand estimation.

Determining Probability of Eligibility for Affirmative Action Track

In Section 1.5.2.1 we have constructed the empirical student distribution, but we do not observe income or household size for students in this data. For BLP estimation we need to find the joint distribution of observables in the data and being eligible for affirmative action track. We use SELS data in which we observe family size, monthly family income and some other observables that would allow us to connect this information to empirical student distribution. We use the 7th grader cohort in SELS data. Students who started middle school in 2010 are observed through their middle schools. For each student in this dataset we determine whether he/she is in affirmative action track from family income of student and size of the household. In each middle school we order the students surveyed according to their scores across three years in the tests accompanying the survey; and we calculate the rank percentile of students according to this order. We estimate a probit model where being in the affirmative action track is the outcome variable, and the explanatory variables are rank percentile of student, whether middle school of the student is private, the district (gu) the middle school is in and a constant term. Let student st_in_aff denote whether the student is eligible for affirmative action track. Estimation results are given in Table A.3. Using these coefficient estimates, we can predict the probability of eligibility for affirmative action track for students in the empirical student distribution, since in empirical student distribution we can observe students' own rank percentiles in their middle schools, their middle schools, whether a middle school is private, and the district students live in.

Table A.3: Probit Estimation for Affirmative Action Eligibility

	depvar:st_in_aff
Estimated p.r.	-0.011 (0.001)
Student's middle school is private	-0.200 (0.071)
Constant	-2.593 (0.255)
District (Gu) Indicators	Yes
N	4408
Pseudo R^2	0.1057
Log Likelihood	- 1716.3368

Notes: This table presents the coefficient estimates from probit regression of students being in the affirmative action track on their estimated own rank percentile, indicator for student's middle school being private, a constant term and 24 District (Gu) indicators using 7th grader cohort of 2010 SELS data. Standard errors are in parenthesis. Coefficient estimates for district indicators are available upon request.

Construction of Regression Discontinuity Sample

In 2013 SECA data we observe students' answer to question of whether he/she would like to go to PA school after graduating from middle school. Since we can also observe family income and variables informing us about family size, this sample will be useful for Regression Discontinuity approach. However, eligibility for PA schools depend on students' performance in middle school. Therefore, knowing about students' success in middle school is important. Also, as our structural model parameters change depending on students performance in middle school, knowing students' performance in middle school is also important for Indirect Inference approach. Unfortunately, we do not observe students' evaluation of his success in middle school in 2013 SECA data. To predict this, we use 2010 SECA data. In this dataset we do not observe whether a student plans to go to PA school after middle school (otherwise we could include it in our RD sample). However, we can observe students' evaluation of his/her own performance with respect to other students in the classroom in this dataset. A student is asked a question about his academic standing compared to his/her classmates. The student can choose 9 groups where the best one is 9th group and the worst is 1st group. Moreover, we can observe many other variables which are common between 2010 and 2013 datasets. There are 231 common variables between the two datasets. We can use these variables to predict students' success in middle school in 2013 SECA data once we estimate a model which predicts middle school success using 2010 SECA data. These variables varies from questions related to socioeconomic status of students' family, students' approach/perception to school, studying, friends, teachers, preparation to exams, self confidence, computer usage; their approach to different subjects and study habits; test scores in reading, math and English in the tests accompanying the survey; also we order students according to these grades within each school and generate a rank percentile for them, we also include this variable as a candidate variable for prediction. We create a new variable indicating students' percentile ranking by using the variable indicating students' group in terms of academic success. To construct that variable we assign each student the rank percentile at the midpoint of the group he/she belongs to. So if a student is in 9th group, we assume that the student's rank percentile is the midpoint between 100 and

89; and we try to predict this variable instead of the variable indicating the group. Since number of potential predictor variables is large and sample size is not very large, we use LASSO (Least Absolute Shrinkage and Selection Operator) method in the way proposed by Ahrens, Hansen, and Schaffer (2018), to decide the model which best predicts the students' success in middle school.[2] We run ordinary least squares (OLS) using the selected model by LASSO, then use the coefficient estimates to predict the middle school rank percentile of students in 2013 SECA dataset. Since we are only interested in predicting the middle school rank percentile, not the coefficients we do not present the OLS coefficients here. Measures of model fit R^2 and $Adj-R^2$ are found as 0.6282 and 0.6168 respectively.

After predicting the percentile rankings of students within their middle schools for 2013 SECA data, we discard the students who are not eligible to apply autonomous schools (the students who are in the bottom half of the student ranking in their middle school), the students who plans to go to a school type different from general high schools; and the students for which we do not observe family income. What remains is 607 students from 31 different middle schools in Seoul which are distributed across 22 districts; 20 of these middle schools are public and the rest are private. Note that, poverty line hence the eligibility for affirmative action track depends on household size, not only income. Instead of imposing different thresholds for different family sizes, we construct a measure of effective income for each family size, which is equivalent to income of a one person household in terms of poverty line. Effective income is found by multiplying the income of household of size x, with the ratio of poverty line for household of size 1 and household of size x from the year 2013. Then we turn these effective incomes into 2010 year effective incomes to preserve consistency of income measure with the income measure in SELS data. We do this by multiplying the effective incomes in 2013 dollars with the ratio of poverty lines for one person households from 2010 and 2013. Table A.4 illustrates the descriptive statistics of our estimation

[2]We have 1,975 students in Seoul, but for some observations some variables are missing. We run three LASSOs consecutively: first we run it with all the available common variables, but since some values are missing for some of the variables, the sample size is 1,423 for the first LASSO regression. Then after we discard the variables which are not selected by LASSO, we run the LASSO second time and the sample size increased to 1,609 and one more variable is discarded by LASSO. In the third run, the sample size has increased to 1,610; but the number of variables selected by LASSO did not change.

sample.

Table A.4: Descriptive Statistics of Regression Discontinuity Sample

Variable	Mean	Std. dev.	Min	Max
Would like to apply PA school	0.265	0.442	0	1
Male	0.458	0.499	0	1
Estimated Rank Percentile	65.212	9.441	96.955	50.127
$ln(eff\_income(KRW)/1000)$	7.241	0.696	4.003	11.609
Number affirmative action eligible	Number ineligible	Total students	Threshold for running variable	
88	519	607	6.6287237	

Notes: This table presents the descriptive statistics of the variables in the RD Estimation sample. We use the natural logarithm of 1,000 KRW of effective income as running variable. Effective income is found by multiplying the income of household of size x, with the ratio of poverty line for household of size 1 and household of size x from the year 2013. Then we turn these effective incomes into 2010 year effective incomes to preserve consistency of income measure with the income measure in SELS data. We do this by multiplying the effective incomes in 2013 dollars with the ratio of poverty lines for one person households from 2010 and 2013.

Calculating Mean Percentile Rankings for PA Schools

We observe the middle school rank percentiles only for the students who attend CMS and not for students who enrolled in PA schools. For students attending the CMS, we know their middle school, the number of students graduated with them from their middle school, and their percentile rankings. This means we know what would be the sum of the percentile rankings of *all* the top 50 percent students for each middle school. Since the sum of rank percentiles of students attending CMS is already observable, we can calculate the sum of percentile rankings of students not attending CMS. Using this, it is easy to calculate the mean of percentile rankings of students who do not attend CMS in top 50 percent of each middle school, since we observe the number of students in the graduating cohort of each middle school. These averages are used as moment conditions within the BLP estimation. From this, it is easy to construct the average of the middle school percentile rankings for *all* students who do not attend CMS and are in the top 50 percent within their middle school. We assume that this average is the $m.p.r$ measure for all PA schools.

This approach assumes that for the students in the top 50 percent in their middle school and not attending CMS and students going to PA schools have the same mean of percentile rankings. However, remember that there are other schools that students in top 50 might have gone to, when they do not attend CMS. These were foreign language high schools and science high schools (Seoul had 10 such schools in total). We assume the $m.p.r$ of these schools and PA schools is the same. We can check whether this assumption is sensible. For this we can use the switch of 13 general schools from being NPA to PA schools. Remember there was a change in the percentile ranking distribution of students in CMS and in top 50 percent within their middle schools between 2010 and 2011. We know there is no change in the number of science and foreign language schools between 2010 and 2011. Therefore, this change must be due to students who enrolled to PA schools in 2011 but would not be able to if these 13 schools had not switched. We know the mean percentile ranking of students who are in top 50 percent of their middle school for 2010 and attended CMS. Had there been no change in school composition in 2011, this mean would be same in 2011. We know the actual mean in 2011 and number of students attending CMS. And we know the number of students who enrolled in PA but would not be able to, if the school composition had not changed. Using these information we calculate the mean rank percentile for the latter group of students. We found this number as 75.89 which is very close to our estimates for the years 2010, 2011 and 2012.

Details of Computation for BLP Estimation

Stacked Moment Conditions: We have

$$g(\hat{\theta}_2) := \begin{bmatrix} \sqrt{n_{rd}}(rdest(\hat{\theta}_2) - \hat{rdest}) \\ \sqrt{n_{mid}}(mssh(\hat{\theta}_2) - mssh) \\ \sqrt{n_{mid}}(mspr(\hat{\theta}_2) - mspr) \\ \frac{1}{\sqrt{2\sum_t |PA_t| + |NPA_t|}} \left(\sum_{j,t,k} Z^{BLP'}_{j,t,k} \xi_{j,t,k}(\hat{\theta}_2) \right) \end{bmatrix}$$

where $2\sum_t |PA_t| + |NPA_t|$ is the total number of programs across three years and across tracks, n_{rd} is the sample size of the RD estimation sample, n_{mid} is the total sample size of empirical student distribution in 2011 and 2012 since middle school moments also include average rank percentile of *all* students enrolled in PA schools, and n_{mid} takes weights of observations into consideration.

Further Details on Constructing Moment Conditions From The Empirical Model: This calculation involves entry chances to PA schools because of the terms $\frac{q_j}{\eta_j}$ for $j \in PA_t$, and entry chances to schools in NPA_t. First, we assume that students know these admission chances.[3] Further we assume that students take these admission chances as given, following the literature.[4] Calculation of expected utility from the centralized match under students' best response to the observed equilibrium is a computationally heavy task. The reason is that each student, depending on his neighborhood, gender and religion faces hundreds of thousands unique lotteries over NPA schools in CMS. This is where the two step estimation becomes advantageous. Note that we draw the students (for the case of calculating application shares) and error terms once at the beginning of the estimation procedure. Therefore we calculate the U^i_c for each (drawn) student only once at the

[3]Calsamiglia, Fu, and Güell (2020) uses the same assumption. Agarwal and Somaini (2018) generates rational expectation beliefs by resampling the submitted reports to the mechanism. Such exercise can be implemented in future as a robustness check.

[4]This is assumed by most of the previous literature (Calsamiglia, Fu, and Güell, 2020; Agarwal and Somaini, 2018). The idea behind this assumption is that, since each student is small in a large market it is assumed that a student's decision cannot affect the equilibrium. That is, we are not simulating a game of strategic interactions. Each student takes the observed equilibrium of applications in the data as given and plays his/her best response to the observed application profile in the data.

beginning of the estimation procedure. As we normalized the price of NPA schools to 0, from first step of the estimation we already know $\alpha_{c(i)}D_{ij} + \zeta_{c(i)j}$ and the drawn $\varepsilon_{ij}\,\forall j \in NPA_t$, therefore we can calculate the best lottery for student i in CMS and calculate $i's$ expected payoff from that lottery.

Since Equation (1.4.4) and Equation (1.4.5) do not yield closed form probabilities, the strategy described above leads to discrete outcomes for each student instead of probabilities of application to each PA school. This makes the moments non-smooth functions of the parameter β_2, which increases the computational burden for minimizing the objective function. Therefore, we use a logit smoother which yields application probabilities to each PA school instead of discrete outcomes for each student.

So for each student i given a draw of $\varepsilon_{ij} \in NPA_t \cup PA_t$, the probability application to school $j \in PA_t$ is given by:

$$\mathbb{P}(i\,applies\,to\,j|\varepsilon_i, \theta_1^*, \beta_2, \{\delta_{i,j,t,k}\}_{j\in PA}) = \frac{\exp((\min\{\frac{q_j}{\eta_j},1\}(v_{i,j,t,k} - U_c^i))/\rho)}{1+\sum_{j'\in PA}\exp((\min\{\frac{q_{j'}}{\eta_{j'}},1\}(v_{i,j',t,k} - U_c^i))/\rho)} \quad \text{(A.2.1)}$$

where ρ is the smoothing parameter. See Chapter 5 of Train (2009) for discussion of logit s-moothers. Smaller the ρ is, more close probabilities gets to the discrete decision. We use $\rho = 0.5$ in our main specification, but the results are very similar when we vary ρ between 0.5 and 1.

Application Shares and NFP Algorithm: For each market we draw many pairs of student and vector of ε from the empirical student distribution and T1EV(0,1) independently.[5]When drawing students we use the probabilities for being in the affirmative action track and weights of the observations in the empirical student distribution.

Using equation (A.2.1) we can calculate the shares of applications for each autonomous school in each market given the parameter candidate $\hat{\beta}_2$ and candidate mean utility vector for PA schools

[5]For each market we draw 20,000 students since the application shares of autonomous schools are small

in year t and track k, $\delta_{j\in PA,t,k}$:

$$s_{j,t,k}(\delta_{j\in PA,t,k};\hat{\beta}_2,\theta_1^*)=\frac{\sum_{i=1}^{ns}\mathbb{P}(i\,applies\,to\,j|\varepsilon_i,\theta_1^*,\hat{\beta}_2,\{\delta_{i,j,t,k}\}_{j\in PA})}{ns}\,\forall j,t,k$$

where ns is the number of simulation draws. For given $\hat{\beta}_2$ we find the vector $\delta_{j\in PA,t,k}$which equates the calculated market share to the observed market shares for each PA school for all t,k:

$$s_{.,t,k}(\delta_{j\in PA,t,k};\hat{\beta}_2,\theta_1^*)=S_{.,t,k}\,\forall t,k$$

Solution to this equation can be denoted as $\delta_{j\in PA,t,k}(\hat{\beta}_2)$ for given candidate $\hat{\beta}_2$. Berry, Levinsohn, and Pakes (1995) shows that this equation can be solved using NFP algorithm, which is calculating the series

$$\delta^{h+1}_{j\in PA,t,k}=\delta^{h}_{j\in PA,t,k}+\ln(S_{.,t,k})-\ln(s_{.,t,k}(\delta^{h}_{j\in PA,t,k};\hat{\beta}_2,\theta_1^*))\,\forall t,k$$

until $||\delta^{h+1}_{j\in PA,t,k}-\delta^{h}_{j\in PA,t,k}||$ is smaller than some tolerance level for all t,k. [6] Then we set $\delta_{j\in PA,t,k}(\hat{\beta}_2)=\delta^{H}_{j\in PA,t,k}$ where H is the smallest $h+1$ that satisfies the tolerance level.

Moments of Price Instruments: Through NFP algorithm we have calculated $\delta_{j\in PA,t,k}(\hat{\beta}_2)$ given a parameter candidate $\hat{\beta}_2$. This allows us to estimate the linear parameters $\theta_2\setminus\beta_2$ and $\xi_{j,t,k}(\hat{\beta}_2)$ via linear GMM estimation using equation equation (1.4.2). The estimator for linear parameters is:

$$(\beta_0,\beta_1,\{\gamma_{nb}\}_{nb\notin NBC})'=(\tilde{\mathbb{X}}'\mathbb{Z}_{BLP}W_{BLP}\mathbb{Z}'_{BLP}\tilde{\mathbb{X}})^{-1}\tilde{\mathbb{X}}'\mathbb{Z}_{BLP}W_{BLP}\mathbb{Z}'_{BLP}\delta(\hat{\beta}_2,\theta_1^*)$$

where $\tilde{\mathbb{X}}$ is the tuition, autonomy indicators, and 10 neighborhood indicators stacked for all programs $j\in NPA\cup PA$ and all markets; $\mathbb{Z}_{BLP}$ is $Z^{price}_{j,t,k}$ stacked for all programs $j\in NPA\cup PA$ and markets; W_{BLP} is a positive definite weighting matrix; and $\delta(\hat{\beta}_2,\theta_1^*)$ is constructed by stacking the vector $\delta_{j\in PA,t,k}(\hat{\beta}_2)$ and $\delta_{j\in NPA,t,k}(\theta_1^*)$, where the latter is the vector of mean utilities for NPA

[6] We choose 1e-9 as tolerance level.

schools. These were already available after the first step of the estimation: since all entries of $\tilde{\mathbb{X}}$ are zero for NPA schools, each $\delta_{j,t,k}(\theta_1^*)$ for $j \in NPA_t$ is equal to $\xi_{j,t,k}(\theta_1^*)$.

Then we calculate the vector of all $\xi_{j,t,k}$ for all j,t,k by:

$$\xi(\hat{\beta}_2, \theta_1^*) = \delta(\hat{\beta}_2, \theta_1^*) - \tilde{\mathbb{X}}(\beta_0, \beta_1, \{\gamma_{nb}\}_{nb \notin NBC})'$$

which we can use for constructing $\sum_{j,t,k} Z_{j,t,k}^{price'} \xi_{j,t,k}(\hat{\beta}_2, \theta_1^*)$.

Middle School Moments: For each student i in the empirical student distribution of year $t \in \{2011, 2012\}$ we draw 50 ε_{ij} $\forall j \in PA_t \cup NPA_t$ at the beginning of the algorithm. Using a candidate $\hat{\beta}_2$ and calculated $\delta_{j \in PA,t,k}(\hat{\beta}_2)$ by NFP algorithm, we calculate the probability of enrolling to autonomous schools for each student. We do this for each set of ε draws. To calculate the enrollment probabilities, for each student we first calculate the application probabilities to each PA school as described before, then we multiply these probabilities with the admission probabilities of PA schools in the data. We average these probabilities across set of draws of ε to integrate it out. Also, we integrate out the probability of being an affirmative action eligible student. We then calculate the mean of the middle school rank percentiles of all students enrolling to autonomous school for years 2011 and 2012. We do this by taking taking the average of middle school rank percentiles weighted by probability of enrollment to any PA school and observation weights in the empirical distribution. Similarly, we can calculate the same quantity within each middle school. We calculate it for the middle schools which we can observe the number of students going to PA schools and we can estimate such average from the data. [7] We describe in Appendix A.2 how we estimate mean of the percentile rankings of students going to PA school from the student level data we get from SOE. Taking the difference between the vector constructed from the structural model for given $\hat{\beta}_2$ and vector of the corresponding averages from data, we construct $mspr(\hat{\beta}_2) - mspr$. Similarly, using

[7] Although we can estimate average of middle school rank percentiles for each middle school that exists in the empirical student distribution, we do it for only the schools for which we know the exact number of students going to autonomous school. We do this way since these numbers are necessary to find the covariance between the moment conditions, which will be important in calculation of efficient weighting matrix. These consists of around 70% of middle schools in 2011 and 2012.

the calculated entry probabilities for each student, we calculate the share of students enrolled to any PA school for the middle schools for which we have the corresponding data in years 2011 and 2012. Difference of vector constructed from the structural model and the corresponding shares in the data yields $mssh(\hat{\beta}_2) - mssh$. Note that, to decrease the variance of each moment, we combine middle school moments into several groups.

RD Moments: We use the students in RD estimation sample, in construction of these moments. In this dataset we do not observe students' neighborhood or religion, both of which are needed to simulate the decision of students in the structural model. We observe students' middle schools, gender, estimated rank percentiles in middle school and district of the middle school. We draw 40 unobservables for each of the students: 4 random pairs of neighborhood and religion and 10 ε_i for each pair of neighborhood and religion. We draw the pairs of neighborhood and religion from the empirical student distribution of 2012 using commonly observed variables in RD sample and empirical student distribution.[8] For each set of draw, we calculate the the probability of application to any PA school for all students given a candidate parameter $\hat{\beta}_2$ and calculated $\delta_{j \in PA,t,k}(\hat{\beta}_2)$ by NFP algorithm. When doing so, we use the lottery cutoffs and estimated utilities for NPA schools from the year 2012 since this is the closest year we were able to estimate utilities for NPA schools and lottery cutoffs. But for entry chances to PA schools we use the actual number of applications and capacity from 2013, as we observe those.[9] For each set of draws, using the simulated outcomes of application probabilities and the observables of students in the RD sample, we implement the RD estimation as proposed by Calonico, Cattaneo, Farrell, and Titiunik (2017), and estimate the discontinuity at the income cutoff for being in the affirmative action track.[10] Then we average the

[8]2012 is the closest year for which we can observe empirical student distribution as well as entry chances to NPA schools. When drawing pairs of neighborhood and religion from empirical student distribution, we look at the neighborhood and religion of students: who are within -5 and +5 of the rank percentile of the student in the RD estimation sample; have same middle school, have same gender; living in the same district as student's middle school. In case middle school of student in 2013 data is not in the empirical distribution of 2012, we choose draw a student from the same type of middle school in terms of private/public and keep criteria for other variables same.

[9]One of the PA schools in 2012, becomes NPA school in 2013, therefore we calculate the probability of application to any one of the remaining 24 PA schools.

[10]As in the estimation using the actual outcomes in the SECA dataset we include covariates of gender, indicator of whether middle school is private and the estimated rank percentile of student in his/her middle school.

estimated discontinuities across set of simulation draws and denote it by $rdest(\hat{\beta}_2)$. We construct the moment by taking the difference of this quantity and the RD estimate using the actual outcomes in the RD estimation sample, $r\hat{dest}$.

Weighting Matrix: We run two step GMM estimation to obtain the parameter estimates (Hansen, 1982).[11] We do this by first estimating the model via a positive definite weighting matrix to calculate the optimal weighting matrix. In second iteration we use the optimal weighting matrix to find the optimal parameters. In the first step we minimize the GMM objective function with the following block diagonal matrix:

$$\begin{bmatrix} \mathbb{I} & 0 \\ 0 & (\mathbb{Z}'_{BLP}\mathbb{Z}_{BLP})^{-1} \end{bmatrix}$$

where the first block is the identity matrix, which has number of rows equal to the length of $((rdest(\hat{\beta}_2) - r\hat{dest}), (mssh(\hat{\beta}_2) - mssh)', (mspr(\hat{\beta}_2) - mspr)')'$. Note that in this step to estimate the linear parameters given $\delta(\hat{\beta}_2, \theta_1^*)$, we also use $(\mathbb{Z}'_{BLP}\mathbb{Z}_{BLP})^{-1}$ instead of W_{BLP}, which makes the linear parameter estimation step equivalent to TSLS. Using the consistent estimates of β_2 from this first step, denoted by $\tilde{\beta}_2$, we construct the efficient weighting matrix. Efficient weighting matrix W is block diagonal with three blocks:

$$\begin{bmatrix} W_{RD} & 0 & 0 \\ 0 & W_{mid} & 0 \\ 0 & 0 & W_{BLP} \end{bmatrix}$$

First block corresponds to variance of RD moment; the second one corresponds to variance-covariance matrix of middle school moments; and the third one corresponds to variance covariance of the moments obtained from BLP instruments. For constructing W_{BLP} we first calculate the sample variance covariance of $Z^{BLP'}_{j,t,k}\xi_{j,t,k}(\tilde{\beta}_2)$ where sample observations are indexed by j,t,k; and then invert this variance covariance matrix to obtain W_{BLP}. Middle school moments are con-

[11] We use Nelder-Mead simplex search method to find the optimal parameter that minimizes the GMM objective function. Specifically, we use *fminsearch* function of MATLAB with tolerance levels set to 1e-7.

ditional moments.[12] To obtain variance-covariance matrix of these moments we follow Petrin (2002)[13] who also uses micro data moments within BLP setting. We define random variables that is well defined for each student in the empirical student distribution across years 2011 and 2012. Using variance-covariance matrix of these random variables and Delta Method, we construct the variance-covariance matrix of the conditional moments regarding middle schools and two moments regarding the average rank percentile of all students enrolled to PA in 2011 and 2012.[14] Then we take the inverse of this variance-covariance matrix to obtain W_{mid}. We calculate W_{RD} by bootstrap procedure: For each bootstrap sample, we draw a sample with replacement from the RD estimation sample with sample size equal to RD estimation sample. And for each of these bootstrap samples we draw the unobservables as we do in the estimation procedure, and calculate $rdest(\tilde{\beta}_2) - \hat{rdest}$ for each of these bootstrap samples exactly as in the estimation procedure given the consistent estimates $\tilde{\beta}_2$ and $\delta_{j\in PA,t,k}(\tilde{\beta}_2)$. Next we take the variance of $rdest(\tilde{\beta}_2) - \hat{rdest}$ across bootstrap samples and multiply it with sample size of RD estimation sample and take the inverse of this quantity.

Details for Computing Equilibria in Counterfactuals

Our goal is to find the equilibria of the submitted reports under DA and variants of SA. To do this we iterate over best responses of students. We let everyone submit a preference list at the beginning of iteration. Outcome of the mechanism is deterministic only for given lottery draws which students do not know before applying. So, when best responding to previous strategy profile, students should consider the expected $m.p.r.$ and expected lottery cutoffs implied by the previous strategy profile. To calculate these expected qualities and cutoffs we run the continuum version

[12] Remember they consists of share of students enrolling to PA schools from around 70% percent of middle schools in Seoul for the years 2011 and 2012; the average of rank percentile of students going to PA schools from the same middle schools and same years; and average of rank percentile of students going to autonomous schools in years 2011 and 2012.

[13] For details see the Appendix of working paper version: Petrin (2001)

[14] Additional details are available upon request.

of these mechanisms taking advantage of the large size of the market.[15] In the first iteration we use the initial set of submitted lists to compute the continuum version of these mechanisms. Outcome of the mechanism implies an expected $m.p.r.$ at each school, expected lottery cutoffs. In the next iteration, students' preferences are determined by the expected $m.p.r.$ realized in the previous step.[16] In addition, expected entry probabilities to each school is also determined. Given these, we can calculate each student's best response to the previous strategy profile. We iterate these best responses until the convergence of average of expected $m.p.r.$ measure at schools and convergence of average of cutoffs at schools.

Now we describe how we calculate the best responses of students under variants of SA and DA. Given a strategy profile, expected $m.p.r.$ at each school is determined. Therefore, for any student the preferences over all schools are determined. Under DA, students will just rank all the schools according to their preferences given the previous strategy profile. Under SA, since the second step is unrestricted list DA, students cannot do better than ranking NPA schools truthfully conditional on joining the second step. So, we let that all students rank NPA schools truthfully as their best response to a given strategy profile. Given the previous iteration of strategy profile, lottery cutoffs at each school is also determined. In consequence, the student can calculate his/her expected utility from the DA over NPA schools. In the first step of SA1, as in the empirical model the student chooses to apply a PA school only if there is any PA school with higher utility from the expected utility from the second step. Among such PA schools, the student chooses the school that gives the highest expected utility from doing so, by considering the entry probabilities to PA schools as described in Section 1.4. In the first step of SAU, it is not optimal for student to consider any PA school that has lower utility compared to students' expected utility from the centralized match. The student's optimal best response is to rank order all the other PA schools truthfully according to his/her preferences determined by the given strategy profile. The reason is

[15]In continuum version, each student has measure $1/n$ where n is the total number of students, and each school has capacity measure equal to its capacity divided by n. So using continuum version implies each discrete student is admitted probabilistically to different schools. Previous literature follows a bootstrap procedure of running discrete versions of the mechanisms (Agarwal and Somaini, 2018; Calsamiglia, Fu, and Güell, 2020; Kapor, Neilson, and Zimmerman, 2020) which is much more demanding in terms of computational resources.

[16]For the case of score based admissions in private schools, the score cutoffs are deterministic.

that the first step of SAU is also DA. The intuition is that the expected utility of student from the second step can be considered as student's outside option in the first step.

Identification of Truthful Pairwise Comparisons From Strategic Reports

We suppress the year subscript t below. Given the submitted lists by all the students in the observed equilibrium, each list $\pi \in \Pi$ student i can submit implies chances of admission $\phi_{ij}(\pi)$ to each NPA program $j \in NPA$. Let $\pi^* \in \Pi$ denote the list submitted by student i (suppressing i). For all $\pi \in \Pi \setminus \{\pi^*\}$ we define the following sets.

The set of NPA programs which have higher chance of entry in the lottery implied by π^* compared to lottery implied by π,

$$NPA_{+}^{i,\pi^*,\pi} := \{j \in NPA : \phi_{ij}(\pi^*) > \phi_{ij}(\pi)\}$$

and the set of NPA programs which have higher chance of entry in the lottery implied by π compared to lottery implied by π^*,

$$NPA_{-}^{i,\pi^*,\pi} := \{j \in NPA : \phi_{ij}(\pi) > \phi_{ij}(\pi^*)\}$$

We assume that each student can correctly predict the set of programs $NPA_{+}^{i,\pi^*,\pi}$ and $NPA_{-}^{i,\pi^*,\pi}$ for all $\pi \in \Pi \setminus \{\pi^*\}$. We define the set of lists that are equivalent to π^* in terms of admission chances,

$$L^{\pi^*} = \{\pi \in \Pi : \phi_{ij}(\pi) = \phi_{ij}(\pi^*) \, \forall j \in NPA\}$$

, since we do not compare equivalent lotteries. Let $\hat{\phi}_{ij}(\pi^*), \hat{\phi}_{ij}(\pi) \; \forall \pi \in \Pi \setminus \{\pi^*\}$ denote the student's beliefs on chances of admission, which are not necessarily correct. Formally our assumption is

$$\hat{\phi}_{ij}(\pi) > \hat{\phi}_{ij}(\pi^*) \iff \phi_{ij}(\pi) > \phi_{ij}(\pi^*)$$

Since student i maximizes his/her expected utility given his/her beliefs of admission chances, then π^* submitted by i must satisfy:

$$\sum_{j \in NPA} \hat{\phi}_{ij}(\pi^*) v_{ij} \geq \sum_{j \in NPA} \hat{\phi}_{ij}(\pi) v_{ij} \; \forall \pi \in \Pi \setminus L^{\pi^*}$$

Using our assumption we can write this inequality as:

$$\sum_{j \in NPA^+} [\hat{\phi}_{ij}(\pi^*) - \hat{\phi}_{ij}(\pi)] v_{ij} \geq \sum_{j \in NPA^-} [\hat{\phi}_{ij}(\pi) - \hat{\phi}_{ij}(\pi^*)] v_{ij} \; \forall \pi \in \Pi \setminus L^{\pi^*}$$

Note that

$$\sum_{j \in NPA^+} [\hat{\phi}_{ij}(\pi^*) - \hat{\phi}_{ij}(\pi)] = \sum_{j \in NPA^-} [\hat{\phi}_{ij}(\pi) - \hat{\phi}_{ij}(\pi^*)] > 0 \; \forall \pi \in \Pi \setminus L^{\pi^*}$$

Dividing the penultimate equation by the last equation we obtain:

$$\sum_{j \in NPA^+} \varphi_j^+ v_{ij} \geq \sum_{j \in NPA^-} \varphi_j^- v_{ij} > 0 \; \forall \pi \in \Pi \setminus L^{\pi^*}$$

where $\sum_{j \in NPA^+} \varphi_j^+ = \sum_{j \in NPA^-} \varphi_j^- = 1$. It follows that

$$\max_{j \in NPA^+} v_{ij} \geq \min_{j \in NPA^-} v_{ij}$$

Two programs have different payoffs with almost probability one. There are many pairs of π^* and π such that NPA^+ and NPA^- are singletons. In that case, the last inequality above will yield a pairwise comparison between two programs. Since there are many possible pairs of π^* and comparable π for each student, this strategy identifies many pairwise comparisons for students.

Table A.5: Welfare Without Top 50% and Affirmative Action Rule

Panel A

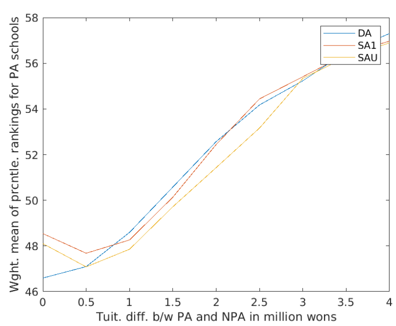
Mechanisms	$o.r.p \geq 75$	$50 \leq o.r.p < 75$	General track	Affirmative track
DA	0.790	0.506	0.632	0.817
SAR1	0.688	0.396	0.527	0.688
SAUR	0.666	0.387	0.519	0.601

Panel B

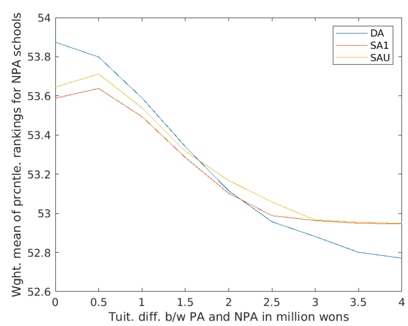
Mechanisms	$25 \leq o.r.p < 50$	$o.r.p < 25$	All students
DA	0.241	-0.258	0.361
SAR1	0.130	-0.357	0.256
SAUR	0.123	-0.366	0.244

Figure A.1: Comparison Across Price Without Top 50% and Affirmative Action Rule

(a) Weighted average of $m.p.r$ for PA schools



(b) Weighted average of $m.p.r$ for NPA schools



(c) Weighted average of $m.p.r$ for NPA schools

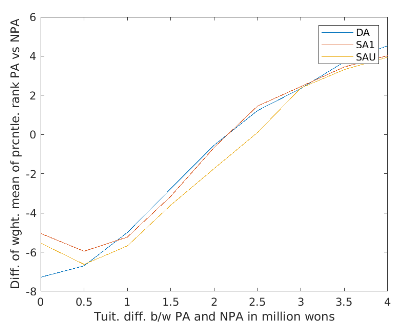
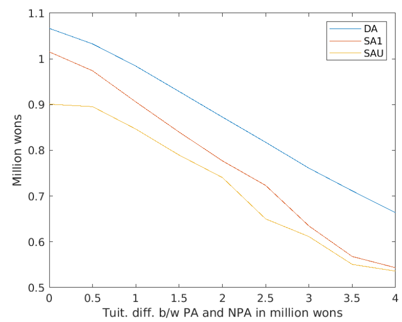
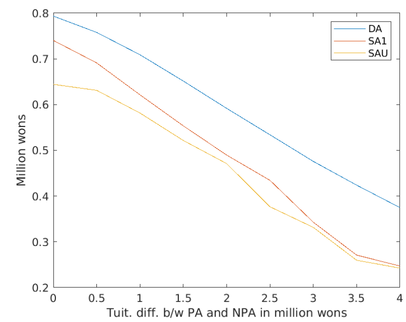


Figure A.2: Welfare Without Top 50% and Affirmative Action Rule

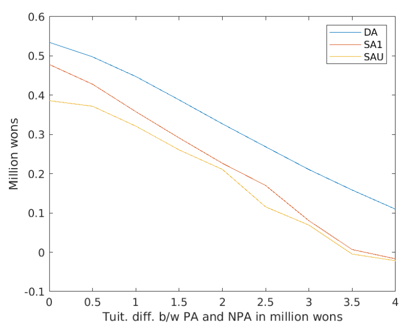
(a) Average welfare of students with $o.p.r \geq 75$



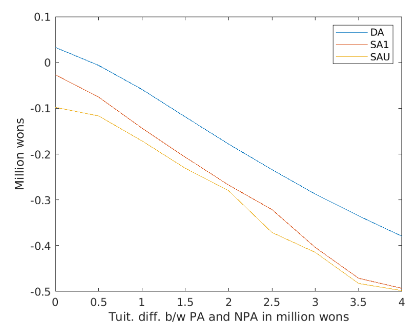
(b) Average welfare of students with $50 \leq o.p.r < 75$



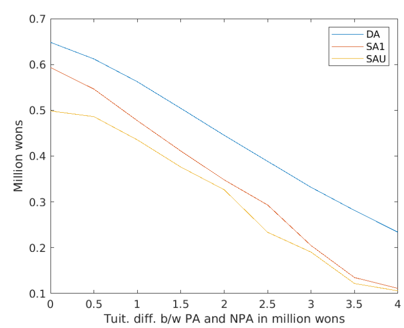
(c) Average welfare of students with $25 \leq o.p.r < 50$



(d) Average welfare of students with $o.p.r \leq 25$



(c) Average welfare of all students



Robustness Checks for Starting Points of Iterations without Top 50 Percent and Without Affirmative Action

A.3 Appendix for Theoretical Analysis

Calculation of Entry Probabilities for Given Strategy Profile and Rank Ordered List

Under DA, a type who submitted π as rank order list i.e. $t \in \sigma^{-1}(\pi)$ has $c^{DA}_{\pi_1}(\sigma)$ probability of entry to his first ranked school π_1; $(1-c^{DA}_{\pi_1}(\sigma))c^{DA}_{\pi_2}(\sigma)$ probability of entry to his second ranked school π_2 and $(1-c^{DA}_{\pi_1}(\sigma))(1-c^{DA}_{\pi_2}(\sigma))c^{DA}_{\pi_3}(\sigma)$ probability of entry to his third ranked school π_3.

Under SA, for a type who submitted π as rank order list i.e. $t \in \sigma^{-1}(\pi)$ entry probabilities to π_1, π_2 and π_3 are as follows:

$$\begin{cases} \pi_1 \quad w.p.\, c^{SA}_{\pi_1}(\sigma) \\ \pi_2 \quad w.p. \begin{cases} (1-c^{SA}_{\pi_1}(\sigma))c^{SA}_{\pi_2}(\sigma) & if\, \pi_2 \neq s_1 \\ 0 & otherwise \end{cases} \\ \pi_3 \quad w.p. \begin{cases} (1-c^{SA}_{\pi_1}(\sigma))(1-c^{SA}_{\pi_2}(\sigma))c^{SA}_{\pi_3}(\sigma)1\{\pi_1 = s_1\} & \\ +(1-c^{SA}_{\pi_1}(\sigma))c^{SA}_{\pi_3}(\sigma)1\{\pi_1 \neq s_1\} & if\, \pi_3 \neq s_1 \\ 0 & otherwise \end{cases} \end{cases}$$

Proof of Lemma 1.2

Case 1: Suppose there are types t, t' with $t > t'$ and $\sigma^*_{DA}(t) = (s_2, s_1, s_3)$ and $\sigma^*_{DA}(t') = (s_1, s_2, s_3)$.

Then type t and t' must satisfy:

$$v^{DA}_2(t, \sigma^*) \geq v^{DA}_1(t, \sigma^*)$$

$$v^{DA}_2(t', \sigma^*) \leq v^{DA}_1(t', \sigma^*)$$

which implies

$$h(\omega_2,t) \geq h(\omega_1,t) - p$$

$$h(\omega_2,t') \leq h(\omega_1,t') - p$$

Since $p > 0$, second inequality implies that $\omega_1 > \omega_2$. Then we must have $h(\omega_1,t'') - h(\omega_2,t'')$ increases in t'' for all $t'' \in T$ since

$$\begin{aligned} \frac{\partial(h(\omega_1,t'') - h(\omega_2,t''))}{\partial t''} &= h_t(\omega_1,t'') - h_t(\omega_2,t'') \\ &> 0 \end{aligned}$$

where the last inequality follows since $h_{\omega t} > 0$. Therefore we must have:

$$h(\omega_1,t) - p > h(\omega_2,t)$$

which is a contradiction.

Case 2: Suppose for all types t, $\sigma^*_{DA}(t) = (s_1,s_2,s_3)$, then $\omega_1 = \omega_2 = \omega_3$. Then for all types we have $h(\omega_2,t) > h(\omega_1,t) - p$. Then each type can deviate to ranking s_2 above s_1.

Proof of Lemma 1.3

Case 1: Suppose there are types t,t',t'' with $t > t' > t''$ and $\sigma^*_{SA}(t) = (s_2,s_3)$, $\sigma^*_{SA}(t') = (s_1,s_2,s_3)$, $\sigma^*_{SA}(t'') = (s_2,s_3)$ and $\omega_1 \neq \omega_2$. Since everyone ranks s_3 as last choice, quality distribution of students in s_2 and s_3 must be the same.

Then types must satisfy:

$$v_1^{SA}(t,\sigma) + (1-c_2)\Delta \leq v_2^{SA}(t,\sigma)$$

$$v_1^{SA}(t',\sigma) + (1-c_2)\Delta \geq v_2^{SA}(t',\sigma)$$

$$v_1^{SA}(t'',\sigma)+(1-c_2)\Delta \leq v_2^{SA}(t'',\sigma)$$

which implies respectively

$$h(\omega_2,t) \geq h(\omega_1,t)-p+(1-c_2)\Delta$$

$$h(\omega_2,t') \leq h(\omega_1,t')-p+(1-c_2)\Delta$$

$$h(\omega_2,t'') \geq h(\omega_1,t'')-p+(1-c_2)\Delta$$

Last two inequalities imply $h(\omega_1,t')-h(\omega_2,t') \geq h(\omega_1,t'')-h(\omega_2,t'')$, then we must have $\omega_1 \geq \omega_2$. Since $\omega_1 \neq \omega_2$, we must have $\omega_1 > \omega_2$. But this also implies that

$$h(\omega_1,t)-h(\omega_2,t)-p+(1-c_2)\Delta > h(\omega_1,t')-h(\omega_2,t')-p+(1-c_2)\Delta \geq 0$$

which is a contradiction.

Case 2: Suppose there are types t,t',t'' with $t > t' > t''$ and $\sigma^*_{SA}(t) = (s_1,s_2,s_3)$, $\sigma^*_{SA}(t') = (s_2,s_3)$, $\sigma^*_{SA}(t'') = (s_1,s_2,s_3)$ and $\omega_1 \neq \omega_2$. Contradiction can be reached similarly here to the case above.

Proof of Lemma lemma 1.4

DA part:

$(\implies)$: By Lemma 1.2 there is another possible equilibrium. For contradiction, suppose all types submit (s_2,s_1,s_3), then qualities become $\omega_1=\omega_2=\omega_3$.

$(\impliedby)$: Students with type $t < t_1$ will submit (s_2,s_1,s_3) whereas students with $t > t_1$ will submit (s_1,s_2,s_3). First we will derive lottery cutoffs c_1 and c_2 which shows the probability of being accepted to s_1 and s_2 conditional on applying to the school during the course of DA. To do that note that $F(t_1)$ is the measure of students who ranked (s_2, s_1, s_3) and $1-F(t_1)$ is the measure of students who ranked (s_1, s_2, s_3). c_1 and c_2 satisfies the following equations:

$$c_1((1-F(t_1)) + (1-c_2)F(t_1)) = \frac{1}{3}$$
$$c_2(F(t_1) + (1-c_1)(1-F(t_1))) = \frac{1}{3}$$

Solving for c_1 and c_2 we get:

$$c_1 = \frac{1}{2-F(t_1)+\sqrt{F(t_1)^2-F(t_1)+1}}$$

$$c_2 = \frac{F(t_1)+1-\sqrt{F(t_1)^2-F(t_1)+1}}{3F(t_1)}$$

From this we can derive the qualities at schools:

$$\omega_1(t_1) = 3[c_1(1-F(t_1))\mathbb{E}(t|t>t_1) + c_1(1-c_2)F(t_1)\mathbb{E}(t|t<t_1)]$$
$$\omega_2(t_1) = 3[c_2F(t_1)\mathbb{E}(t|t<t_1) + c_2(1-c_1)(1-F(t_1))\mathbb{E}(t|t>t_1)]$$
$$\omega_3(t_1) = 3[(1-c_1)(1-c_2)(1-F(t_1))\mathbb{E}(t|t>t_1) + (1-c_2)(1-c_1)F(t_1)\mathbb{E}(t|t<t_1)]$$

By plugging in c_1 and c_2 to qualities, we get:

$$\begin{aligned}
\omega_1(t_1) &= (2-F(t_1)-\sqrt{F(t_1)^2-F(t_1)+1})\mathbb{E}(t|t>t_1) \\
&\quad -(1-F(t_1)-\sqrt{F(t_1)^2-F(t_1)+1})\mathbb{E}(t|t<t_1) \\
\omega_2(t_1) &= (F(t_1)+1-\sqrt{F(t_1)^2-F(t_1)+1})\mathbb{E}(t|t<t_1) \\
&\quad +[-F(t_1)+\sqrt{F(t_1)^2-F(t_1)+1}]\mathbb{E}(t|t>t_1) \\
\omega_3(t_1) &= \mathbb{E}(t|t>t_1) - F(t_1)(\mathbb{E}(t|t>t_1) - \mathbb{E}(t|t<t_1))
\end{aligned}$$

We show that $\omega_1^{DA}(\sigma^*) > \omega_3^{DA}(\sigma^*) > \omega_2^{DA}(\sigma^*)\,\forall t \in (\underline{t},\bar{t})$:

$$
\begin{aligned}
&\omega_1^{DA}(\sigma^*) > \omega_3^{DA}(\sigma^*)\\
\iff &(2-F(t_1)-\sqrt{F(t_1)^2-F(t_1)+1})\mathbb{E}(t|t>t_1)\\
&-(1-F(t_1)-\sqrt{F(t_1)^2-F(t_1)+1})\mathbb{E}(t|t<t_1)\\
&>\mathbb{E}(t|t>t_1)-F(t_1)(\mathbb{E}(t|t>t_1)-\mathbb{E}(t|t<t_1))\\
\iff &(1-\sqrt{F(t_1)^2-F(t_1)+1})\mathbb{E}(t|t>t_1)-(1-\sqrt{F(t_1)^2-F(t_1)+1})\mathbb{E}(t|t<t_1)>0\\
\iff &(1-\sqrt{F(t_1)^2-F(t_1)+1})[\mathbb{E}(t|t>t_1)-\mathbb{E}(t|t<t_1)]>0
\end{aligned}
$$

which is true since $(1-\sqrt{F(t_1)^2-F(t_1)+1})>0$ and $\mathbb{E}(t|t>t_1)-\mathbb{E}(t|t<t_1)>0$

$$
\begin{aligned}
&\omega_3^{DA}(\sigma^*) > \omega_2^{DA}(\sigma^*)\\
\iff &\mathbb{E}(t|t>t_1)-F(t_1)(\mathbb{E}(t|t>t_1)-\mathbb{E}(t|t<t_1))\\
&>(F(t_1)+1-\sqrt{F(t_1)^2-F(t_1)+1})\mathbb{E}(t|t<t_1)\\
&+[-F(t_1)+\sqrt{F(t_1)^2-F(t_1)+1}]\mathbb{E}(t|t>t_1)\\
\iff &(1-\sqrt{F(t_1)^2-F(t_1)+1})\mathbb{E}(t|t>t_1)-(1-\sqrt{F(t_1)^2-F(t_1)+1})\mathbb{E}(t|t<t_1)>0\\
\iff &(1-\sqrt{F(t_1)^2-F(t_1)+1})[\mathbb{E}(t|t>t_1)-\mathbb{E}(t|t<t_1)]>0
\end{aligned}
$$

which is true. Note that quality difference between s_1 and s_3 is same as quality difference between s_2 and s_3.

SA Part:

$(\implies)$: By Lemma 3, if we do not have the asserted equilibrium then there are 2 other possible types of equilibria. Trivially, we cannot have the equilibrium in which $\omega_1 = \omega_2$.

Suppose in equilibrium there is $t_1' \in (\underline{t},\bar{t})$ such that $\sigma_{SA}^*(t) = \begin{cases} (s_2,s_3) & t \geq t_1' \\ (s_1,s_2,s_3) & t < t_1' \end{cases}$

In this case we have:

$$c_1 = \begin{cases} 1 & \text{if } F(t_1') \leq 1/3 \\ \frac{1}{3F(t_1')} & \text{if } F(t_1') > 1/3 \end{cases}$$

$$c_2 = \begin{cases} \frac{1}{3(1-F(t_1'))} & \text{if } F(t_1') \leq 1/3 \\ 1/2 & \text{if } F(t_1') > 1/3 \end{cases}$$

Then qualities become:

$$\omega_1 = \mathbb{E}(t|t < t_1')$$

$$\omega_2 = \begin{cases} \mathbb{E}(t|t > t_1') & \text{if } F(t_1') \leq 1/3 \\ 3[c_2(1-c_1)F(t_1')\mathbb{E}(t|t < t_1') + c_2(1-F(t_1'))\mathbb{E}(t|t > t_1')] & \text{if } F(t_1') > 1/3 \end{cases}$$

For the case $F(t_1') \leq 1/3$, we have $\omega_2 > \omega_1$. For the case with $F(t_1') > 1/3$ we show the following:

$$\begin{aligned} \omega_2 &= 3[c_2(1-c_1)F(t_1')\mathbb{E}(t|t < t_1') + c_2(1-F(t_1'))\mathbb{E}(t|t > t_1')] \\ &= \frac{1}{2}[\mathbb{E}(t|t > t_1') - \mathbb{E}(t|t < t_1')] + \mathbb{E}(t|t > t_1') - \frac{3}{2}F(t_1')[\mathbb{E}(t|t > t_1') - \mathbb{E}(t|t < t_1')] \\ &< \frac{1}{2}[\mathbb{E}(t|t > t_1') - \mathbb{E}(t|t < t_1')] + \mathbb{E}(t|t > t_1') - \frac{3}{2}\frac{1}{3}[\mathbb{E}(t|t > t_1') - \mathbb{E}(t|t < t_1')] \\ &= \mathbb{E}(t|t > t_1') \end{aligned}$$

which means $\omega_2 > \omega_1$.

$(\Longleftarrow)$:

Suppose there is an equilibrium such that type $t > t_1$ submits (s_1, s_2, s_3) and type $t < t_1$ submits (s_2, s_3). In this case we get following equations determining cutoffs:

$$c_1 = \begin{cases} \frac{1}{3(1-F(t_1))} & \text{if } 1-F(t_1) \geq \frac{1}{3} \\ 1 & \text{otherwise} \end{cases}$$

$$c_2 = \begin{cases} \frac{1}{2} & \text{if } 1-F(t_1) \geq \frac{1}{3} \\ \frac{1}{3F(t_1)} & \text{otherwise} \end{cases}$$

Case 1: Suppose $\frac{2}{3} < F(t_1)$. Then qualities will be:

$$\omega_1 = \mathbb{E}(t|t > t_1')$$

$$\omega_2 = \mathbb{E}(t|t < t_1')$$

and obviously $\omega_1 > \omega_2$.

Case 2: Suppose $\frac{2}{3} \geq F(t_1)$. Qualities will be:

$$\begin{aligned}
\omega_1 &= \mathbb{E}(t|t > t_1') \\
\omega_2 &= 3[(1-c_1)\frac{1}{2}(1-F(t_1'))\mathbb{E}(t|t > t_1') + \frac{1}{2}F(t_1')\mathbb{E}(t|t < t_1')] \\
&= \mathbb{E}(t|t > t_1') - \frac{3}{2}F(t_1')[\mathbb{E}(t|t > t_1') - \mathbb{E}(t|t < t_1') \\
&< \mathbb{E}(t|t > t_1')
\end{aligned}$$

Proof of Lemma 1.5

Follows from Lemma 4 and following comparison of $\omega_1^{SA}(t_1)$, $\omega_1^{DA}(t_1)$ and comparison of $\omega_2^{SA}(t_1)$, $\omega_2^{DA}(t_1)$:

$$
\begin{aligned}
\omega_1^{DA}(t_1) &= (2 - F(t_1) - \sqrt{F(t_1)^2 - F(t_1) + 1})\mathbb{E}(t|t > t_1) \\
&\quad - (1 - F(t_1) - \sqrt{F(t_1)^2 - F(t_1) + 1})\mathbb{E}(t|t < t_1) \\
&= \mathbb{E}(t|t > t_1) + (1 - F(t_1) - \sqrt{F(t_1)^2 - F(t_1) + 1})[\mathbb{E}(t|t > t_1) - \mathbb{E}(t|t < t_1)]
\end{aligned}
$$

$$
\omega_1^{SA}(t) = \mathbb{E}(t|t > t_1)
$$

Since $1 - F(t_1) - \sqrt{F(t_1)^2 - F(t_1) + 1} < 0$ and $\mathbb{E}(t|t > t_1) - \mathbb{E}(t|t < t_1) > 0$, we have $\omega_1^{SA}(t_1) > \omega_1^{DA}(t_1)$ for all $t_1 \in (\underline{t}, \bar{t})$

$$
\begin{aligned}
\omega_2^{DA}(t_1) &= (F(t_1) + 1 - \sqrt{F(t_1)^2 - F(t_1) + 1})\mathbb{E}(t|t < t_1) \\
&\quad + [-F(t_1) + \sqrt{F(t_1)^2 - F(t_1) + 1}]\mathbb{E}(t|t > t_1) \\
&= \mathbb{E}(t|t < t_1) + [-F(t_1) + \sqrt{F(t_1)^2 - F(t_1) + 1}][\mathbb{E}(t|t > t_1) - \mathbb{E}(t|t < t_1)]
\end{aligned}
$$

$$
\begin{cases}
\omega_2^{SA}(t_1) = \omega_3^{SA}(t_1) = \mathbb{E}(t|t < t_1) & \text{if } F(t_1) > \frac{2}{3} \\
\omega_2^{SA}(t_1) = \omega_3^{SA}(t_1) = \mathbb{E}(t|t > t_1) - \frac{3}{2}F(t_1)[\mathbb{E}(t|t > t_1) - \mathbb{E}(t|t < t_1)] & \text{otherwise}
\end{cases}
$$

Since $-F(t_1) + \sqrt{F(t_1)^2 - F(t_1) + 1} > 0$ and $\mathbb{E}(t|t > t_1) - \mathbb{E}(t|t < t_1) > 0$ for all $t_1 \in (\underline{t}, \bar{t})$ we have $\omega_2^{DA}(t_1) > \omega_2^{SA}(t_1)$ if $1 > F(t_1) > \frac{2}{3}$

When $F(t_1) \leq \frac{2}{3}$ we have:

$$\begin{aligned}
\omega_2^{DA}(t_1) - \omega_2^{SA}(t_1) &= \mathbb{E}(t|t<t_1) + [-F(t_1) + \sqrt{F(t_1)^2 - F(t_1) + 1}][\mathbb{E}(t|t>t_1) - \mathbb{E}(t|t<t_1)] \\
&\quad - \mathbb{E}(t|t>t_1) + \frac{3}{2}F(t_1)[\mathbb{E}(t|t>t_1) - \mathbb{E}(t|t<t_1)] \\
&= \mathbb{E}(t|t<t_1) - \mathbb{E}(t|t>t_1) \\
&\quad + [\mathbb{E}(t|t>t_1) - \mathbb{E}(t|t<t_1)][\frac{1}{2}F(t_1) + \sqrt{F(t_1)^2 - F(t_1) + 1}] \\
&= -[\mathbb{E}(t|t>t_1) - \mathbb{E}(t|t<t_1)] \\
&\quad + [\mathbb{E}(t|t>t_1) - \mathbb{E}(t|t<t_1)][\frac{1}{2}F(t_1) + \sqrt{F(t_1)^2 - F(t_1) + 1}] \\
&= [\mathbb{E}(t|t>t_1) - \mathbb{E}(t|t<t_1)][-1 + \frac{1}{2}F(t_1) + \sqrt{F(t_1)^2 - F(t_1) + 1}] \\
&> 0
\end{aligned}$$

since $-1 + \frac{1}{2}F(t_1) + \sqrt{F(t_1)^2 - F(t_1) + 1} = 0$ if $t_1 = \underline{t}$ and $-1 + \frac{1}{2}F(t_1) + \sqrt{F(t_1)^2 - F(t_1) + 1}$ is strictly increasing in t_1 and $\mathbb{E}(t|t>t_1) - \mathbb{E}(t|t<t_1) > 0$ for $\underline{t} < t$ and $F(t_1) \leq \frac{2}{3}$

Proof of Proposition 1.1

Remember from proof of Lemma lemma 1.4, the school qualities under DA for a given equilibrium with cutoff t_1 is:

$$\begin{aligned}
\omega_1(t_1) &= (2 - F(t_1) - \sqrt{F(t_1)^2 - F(t_1) + 1})\mathbb{E}(t|t>t_1) \\
&\quad - (1 - F(t_1) - \sqrt{F(t_1)^2 - F(t_1) + 1})\mathbb{E}(t|t<t_1) \\
\omega_2(t_1) &= (F(t_1) + 1 - \sqrt{F(t_1)^2 - F(t_1) + 1})\mathbb{E}(t|t<t_1) \\
&\quad + [-F(t_1) + \sqrt{F(t_1)^2 - F(t_1) + 1}]\mathbb{E}(t|t>t_1) \\
\omega_3(t_1) &= \mathbb{E}(t|t>t_1) - F(t_1)(\mathbb{E}(t|t>t_1) - \mathbb{E}(t|t<t_1))
\end{aligned}$$

Note that when $t_1 \geq \bar{t}$, i.e. everyone submits (s_2, s_1, s_3) we have: $\omega_1 = \omega_2 = \omega_3 = \mathbb{E}_\mu(t)$, and when $t_1 \leq \underline{t}$ i.e. everyone submits (s_1,s_2,s_3) we have: $\omega_1 = \omega_2 = \omega_3 = \mathbb{E}_\mu(t)$. This means $\omega_1, \omega_2, \omega_3$ are continuous at t_1 for all $t_1 \in T$.

i) ($\Longrightarrow$:) Suppose an equilibrium t_1 exists. Then t_1 must be indifferent which means $p = \Gamma^{DA}(t_1)$. Since t_1 is equilibrium, $t \geq t_1$ does not deviate from submitting (s_1,s_2,s_3). This means:

$$h(\omega_1^{DA}(t_1),t) - p \geq h(\omega_2^{DA}(t_1),t) \geq h(\omega_3^{DA}(t_1),t) - \Delta \forall t \geq t_1$$

In particular this implies:

$$h(\omega_2^{DA}(t_1),\bar{t}) \geq h(\omega_3^{DA}(t_1),\bar{t}) - \Delta$$
$$\Longleftrightarrow \Delta \geq h(\omega_3^{DA}(t_1),\bar{t}) - h(\omega_2^{DA}(t_1),\bar{t}) > 0$$

Also, $t < t_1$ does not deviate from submitting (s_2,s_1,s_3). This means:

$$h(\omega_2^{DA}(t_1),t) \geq h(\omega_1^{DA}(t_1),t) - p \geq h(\omega_3^{DA}(t_1),t) - \Delta \forall t < t_1$$

In particular this implies:

$$p \leq h(\omega_1^{DA}(\sigma^*),\underline{t}) - h(\omega_3^{DA}(\sigma^*),\underline{t}) + \Delta$$
$$\Longleftrightarrow \Delta \geq p + h(\omega_3^{DA}(\sigma^*),\underline{t}) - h(\omega_1^{DA}(\sigma^*),\underline{t})$$

Therefore,

$$\Delta \geq \max\{p + h(\omega_3^{DA}(t_1),\underline{t}) - h(\omega_1^{DA}(t_1),\underline{t}), h(\omega_3^{DA}(t_1),\bar{t}) - h(\omega_2^{DA}(t_1),\bar{t})\}$$

($\Longleftarrow$:) Suppose $p = \Gamma^{DA}(t_1)$ and $\Delta \geq \max\{p + h(\omega_3^{DA}(t_1),\underline{t}) - h(\omega_1^{DA}(t_1),\underline{t}), h(\omega_3^{DA}(t_1),\bar{t}) - h(\omega_2^{DA}(t_1),\bar{t})\}$. Then t_1 is a cutoff equilibrium. To see this, note that type t_1 is indifferent between (s_1,s_2,s_3) and (s_2,s_1,s_3) since $p = \Gamma^{DA}(t_1)$. We show that type $t > t_1$ does not deviate from submitting (s_1, s_2, s_3):

$$h(\omega_1^{DA}(t_1), t) - p \geq h(\omega_2^{DA}(t_1), t) \geq h(\omega_3^{DA}(t_1), t) - \Delta \, \forall t \geq t_1$$

The first inequality is already satisfied since $p = \Gamma^{DA}(t_1)$, $\omega_1 > \omega_2$, $t > t_1$ and $h_{\omega t} > 0$. Second one will be satisfied since

$$\Delta \geq h(\omega_3^{DA}(t_1), \bar{t}) - h(\omega_2^{DA}(t_1), \bar{t}) \geq h(\omega_3^{DA}(t_1), t) - h(\omega_2^{DA}(t_1), t) \, \forall t \geq t_1$$

where the first inequality is satisfied by assumption and second one is satisfied since $\omega_3 > \omega_2$ and $\bar{t} \geq t \, \forall t \geq t_1$

Now we show that type $t < t_1$ does not deviate from submitting (s_2, s_1, s_3) :

$$h(\omega_2^{DA}(t_1), t) \geq h(\omega_1^{DA}(t_1), t) - p \geq h(\omega_3^{DA}(t_1), t) - \Delta \, \forall t < t_1$$

The first inequality is already satisfied since $p = \Gamma^{DA}(t_1)$, $\omega_1 > \omega_2$, $t < t_1$ and $h_{\omega t} > 0$. Second one will be satisfied since

$$\Delta \geq p + h(\omega_3^{DA}(t_1), \underline{t}) - h(\omega_1^{DA}(t_1), \underline{t}) \geq p + h(\omega_3^{DA}(t_1), t) - h(\omega_1^{DA}(t_1), t) \, \forall t < t_1$$

where the first inequality is satisfied by assumption and second one is satisfied since $\omega_1 > \omega_3$ and $\underline{t} \leq t \, \forall t \leq t_1$.

ii)

$$\begin{aligned} \mathscr{M}^{DA} &= \max_{t_1 \in (\underline{t}, \bar{t})} \Gamma^{DA}(t_1) \\ &= \max_{t_1 \in T} \Gamma^{DA}(t_1) \end{aligned}$$

where the second equality follows since $\Gamma^{DA}(\bar{t}) = 0$ and $\Gamma^{DA}(\underline{t}) = 0$ since $\omega_1 = \omega_2$ in these cases. And for $t_1 \in (\underline{t}, \bar{t})$, $\omega_1(t_1) > \omega_2(t_1)$ which implies $\Gamma^{DA}(t_1) > 0$. Note that second maximum exists by Weierstrass Theorem, since T is compact set and $\Gamma^{DA}(.)$ is continuous in t_1 for $t_1 \in T$. $\Gamma^{DA}(.)$ is continuous because $h(.,.)$ is continuous in its both arguments and $\omega_1(.)$ and $\omega_2(.)$ are continuous in t_1 at $t_1 \in T$.

($\Longrightarrow$:) Suppose $\Gamma^{DA,-1}(p)$ is non-empty. Then $\exists t_1 \in (\underline{t},\bar{t})$ such that $\Gamma^{DA}(t_1) = p$, i.e.

$$h(\omega_1^{DA}(t_1),t_1) - h(\omega_2^{DA}(t_1),t_1) = p$$

Then $p \leq \mathscr{M}^{DA}$ otherwise we could not find t_1. And note that for $t_1 \in (\underline{t},\bar{t})$ we have $\Gamma^{DA}(t_1) > 0$, therefore $p > 0$, otherwise we could not find t_1.

($\Longleftarrow$:) Suppose $p \in (0, \mathscr{M}^{DA}]$. We need to show that $\exists t_1$ such that $\Gamma(t_1) = p$. Note that $\Gamma(.)$ is continuous in t_1 and it attains $\mathscr{M}^{DA}$. Also we have,

$$\inf_{t_1 \in (\underline{t},\bar{t})} \Gamma^{DA}(t_1) = 0$$

To see this note that $\Gamma^{DA}(t_1) > 0 \, \forall t_1 \in (\underline{t},\bar{t})$ since $\omega_1^{DA}(t_1) > \omega_2^{DA}(t_1)$ for any given t_1. To see that 0 is the greatest lower bound, check the limit as $t_1 \to \underline{t}$:

$$\begin{aligned}
&\lim_{t_1 \to \underline{t}} h(\omega_1^{DA}(t_1),t_1) - h(\omega_2^{DA}(t_1),t_1) \\
=&h(\lim_{t_1 \to \underline{t}} \omega_1^{DA}(t_1), \lim_{t_1 \to \underline{t}} t_1) - h(\lim_{t_1 \to \underline{t}} \omega_2^{DA}(t_1), \lim_{t_1 \to \underline{t}} t_1) \\
=&h(\mathbb{E}_\mu(t),\underline{t}) - h(\mathbb{E}_\mu(t),\underline{t}) \\
=&0
\end{aligned}$$

Therefore, for any given $x \in (0, \mathscr{M}^{DA})$, there exists $t_1 \in (\underline{t},\bar{t})$ such that $\Gamma^{DA}(t_1) = x$. Therefore it also attains p in particular.

iii) Take any price $p \in (0, \mathscr{M}^{DA}]$

($\Longrightarrow$:) Suppose $\Gamma^{DA,-1}(p)$ is the set of cutoff equilibria. We know from (iii) that $\Gamma^{DA,-1}(p) \neq$

$\emptyset$. Take any $t_1 \in \Gamma^{DA,-1}(p)$ we have:

$$h(\omega_1^{DA}(t_1), t_1) - p = h(\omega_2^{DA}(t_1), t_1)$$

We also have that

$$h(\omega_1^{DA}(t_1), t) - p \geq h(\omega_2^{DA}(t_1), t) \geq h(\omega_3^{DA}(t_1), t) - \Delta \, \forall t \geq t_1$$

which implies

$$\Delta \geq h(\omega_3^{DA}(t_1), t) - h(\omega_2^{DA}(t_1), t) \, \forall t \geq t_1$$
$$\implies \Delta \geq h(\omega_3^{DA}(t_1), \bar{t}) - h(\omega_2^{DA}(t_1), \bar{t})$$

Since t_1 is arbitrary this means we have:

$$\Delta \geq h(\omega_3^{DA}(t_1), \bar{t}) - h(\omega_2^{DA}(t_1), \bar{t}) \, \forall t_1 \in \Gamma^{DA,-1}(p)$$
$$\implies \Delta \geq \max_{t_1 \in \Gamma^{DA,-1}(p)} h(\omega_3^{DA}(t_1), \bar{t}) - h(\omega_2^{DA}(t_1), \bar{t})$$

We also have,

$$h(\omega_2^{DA}(t_1), t) \geq h(\omega_1^{DA}(t_1), t) - p \geq h(\omega_3^{DA}(t_1), t) - \Delta \, \forall t < t_1$$

which implies

$$\Delta \geq h(\omega_3^{DA}(t_1), t) - h(\omega_1^{DA}(t_1), t) + p \, \forall t < t_1$$
$$\implies \Delta \geq h(\omega_3^{DA}(t_1), \underline{t}) - h(\omega_1^{DA}(t_1), \underline{t}) + p$$

Since t_1 is arbitrary this means we have:

$$\Delta \geq h(\omega_3^{DA}(t_1),\underline{t}) - h(\omega_1^{DA}(t_1),\underline{t}) + p \,\forall t_1 \in \Gamma^{DA,-1}(p)$$
$$\implies \Delta \geq \max_{t_1 \in \Gamma^{DA,-1}(p)} h(\omega_3^{DA}(t_1),\underline{t}) - h(\omega_1^{DA}(t_1),\underline{t}) + p$$

($\Longleftarrow$) : Suppose $\Delta \geq \max\{\max_{t_1\in\Gamma^{DA,-1}(p)} h(\omega_3^{DA}(t_1),\bar{t}) - h(\omega_2^{DA}(t_1),\bar{t}), \max_{t_1\in\Gamma^{DA,-1}(p)}\{p + h(\omega_3^{DA}(t_1),\underline{t}) - h(\omega_1^{DA}(t_1),\underline{t})\}$. We will show that any $t_1 \in \Gamma^{DA,-1}(p)$ is a cutoff equilibrium, and $t_1' \notin \Gamma^{DA,-1}(p)$ is not cutoff equilibrium. The second one is easy to show since $\Gamma(t_1') \neq p$, hence t_1' cannot be the cutoff type.

Take any $t_1 \in \Gamma^{DA,-1}(p)$, first note that type $t_1$is indifferent between submitting $(s_1, s_2 s_3)$ and (s_2, s_1, s_3) at p and types $t > t_1$ prefers submitting (s_1, s_2, s_3) to (s_2, s_1, s_3)

$$U(t, t_1, \pi = (s_1, s_2, s_3) > U(t, t_1, \pi = (s_2, s_1, s_3) \,\forall t > t_1$$
$$U(t, t_1, \pi = (s_1, s_2, s_3) < U(t, t_1, \pi = (s_2, s_1, s_3) \,\forall t < t_1$$

by definition of $\Gamma^{DA,-1}(p)$.

We also need:

$$h(\omega_2^{DA}(t_1), t) \geq h(\omega_3^{DA}(t_1), t) - \Delta \,\forall t \geq t_1$$
$$\iff \Delta \geq h(\omega_3^{DA}(t_1), t) - h(\omega_2^{DA}(t_1), t) \,\forall t \geq t_1$$

We know that:

$$\Delta \geq \max_{t_1' \in \Gamma^{DA,-1}(p)} h(\omega_3^{DA}(t_1'), \bar{t}) - h(\omega_2^{DA}(t_1'), \bar{t})$$
$$\geq h(\omega_3^{DA}(t_1), \bar{t}) - h(\omega_2^{DA}(t_1), \bar{t})$$
$$\geq h(\omega_3^{DA}(t_1), t) - h(\omega_2^{DA}(t_1), t) \,\forall t \geq t_1$$

We also need:

$$\Delta \geq h(\omega_3^{DA}(t_1), t) - h(\omega_1^{DA}(t_1), t) + p \, \forall t < t_1$$

We know that:

$$\begin{aligned}
\Delta &\geq \max_{t_1' \in \Gamma^{DA,-1}(p)} h(\omega_3^{DA}(t_1'), \underline{t}) - h(\omega_1^{DA}(t_1'), \underline{t}) + p \\
&\geq h(\omega_3^{DA}(t_1), \underline{t}) - h(\omega_1^{DA}(t_1), \underline{t}) + p \\
&\geq h(\omega_3^{DA}(t_1), t) - h(\omega_1^{DA}(t_1), t) + p \, \forall t < t_1
\end{aligned}$$

Proof of Proposition 1.2

From Lemma lemma 1.4, remember that

$$\omega_1(t_1) = \mathbb{E}(t|t > t_1)$$

$$\begin{cases} \omega_2(t_1) = \omega_3(t_1) = \mathbb{E}(t|t < t_1) & \text{if } F(t_1) > \frac{2}{3} \\ \omega_2(t_1) = \omega_3(t_1) = \mathbb{E}(t|t > t_1) - \frac{3}{2}F(t_1)[\mathbb{E}(t|t > t_1) - \mathbb{E}(t|t < t_1)] & \text{otherwise} \end{cases}$$

Also, if $t_1 \geq \bar{t}$, i.e. all students submit (s_2, s_3) we have $\omega_1 = \omega_2 = \omega_3 = \mathbb{E}_\mu(t)$

And if $t_1 \leq \underline{t}$, i.e. all students submit (s_1, s_2, s_3) we have $\omega_1 = \omega_2 = \omega_3 = \mathbb{E}_\mu(t)$

Note that $\omega_2(t_1), \omega_3(t_1)$ are continuous in t_1 for $t_1 \in T$; and $\omega_1(t_1)$ is continuous in t_1 for $t_1 \in [\underline{t}, \bar{t})$

Define $\tilde{\omega}_1^{SA}(t_1) := \begin{cases} \omega_1^{SA}(t_1) & \text{if } t_1 < \bar{t} \\ \bar{t} & \text{if } t_1 = \bar{t} \end{cases}$, and note that $\tilde{\omega}_1^{SA}(t_1)$ is continuous in T

i)

$$
\begin{aligned}
\mathscr{S}^{SA} &= \sup_{t_1 \in (\underline{t},\bar{t})} \{h(\omega_1^{SA}(t_1),t_1) - h(\omega_2^{SA}(t_1),t_1) + \Delta(1-\min\{\frac{1}{2},\frac{1}{3F(t_1)}\})\} \\
&= \sup_{t_1 \in (\underline{t},\bar{t})} \{h(\tilde{\omega}_1^{SA}(t_1),t_1) - h(\omega_2^{SA}(t_1),t_1) + \Delta(1-\min\{\frac{1}{2},\frac{1}{3F(t_1)}\})\} \\
&= \max_{t_1 \in T} \{h(\tilde{\omega}_1^{SA}(t_1),t_1) - h(\omega_2^{SA}(t_1),t_1) + \Delta(1-\min\{\frac{1}{2},\frac{1}{3F(t_1)}\})\}
\end{aligned}
$$

Second equality follows since $\tilde{\omega}_1$ and ω_1 are different only at $\bar{t}$. Third equality follows since we only included the limit points and the functions above are continuous at the limit points. Above maximum exists by Weierstrass Theorem since T is a compact set; $\tilde{\omega}_1^{SA}(t_1), \omega_2^{SA}(t_1)$ are continuous in t_1 for $t_1 \in T$ and $h(.,.)$ is a continuous function in its both arguments.

$\Gamma^{SA}(t_1)$ also attains an infimum:

$$
\begin{aligned}
\mathscr{I}^{SA} &:= \inf_{t_1 \in (\underline{t},\bar{t})} \{h(\omega_1^{SA}(t_1),t_1) - h(\omega_2^{SA}(t_1),t_1) + \Delta(1-\min\{\frac{1}{2},\frac{1}{3F(t_1)}\})\} \\
&= \frac{\Delta}{2}
\end{aligned}
$$

To see this note that $h(\omega_1^{SA}(t_1),t_1) - h(\omega_2^{SA}(t_1),t_1) > 0 \, \forall t_1 \in (\underline{t},\bar{t})$ and this implies $h(\omega_1^{SA}(t_1),t_1) - h(\omega_2^{SA}(t_1),t_1) + \Delta(1-\min\{\frac{1}{2},\frac{1}{3F(t_1)}\}) > \frac{\Delta}{2} \, \forall t_1 \in (\underline{t},\bar{t})$. And this is the greatest lower bound since

$$
\begin{aligned}
&\lim_{t_1 \to \underline{t}} h(\omega_1^{SA}(t_1),t_1) - h(\omega_2^{SA}(t_1),t_1) + \Delta(1-\min\{\frac{1}{2},\frac{1}{3F(t_1)}\}) \\
=&h(\lim_{t_1 \to \underline{t}} \omega_1^{SA}(t_1), \lim_{t_1 \to \underline{t}} t_1) - h(\lim_{t_1 \to \underline{t}} \omega_2^{SA}(t_1), \lim_{t_1 \to \underline{t}} t_1) + \frac{\Delta}{2} \\
=&h(\mathbb{E}_\mu(t),\underline{t}) - h(\mathbb{E}_\mu(t),\underline{t}) + \frac{\Delta}{2} \\
=&\frac{\Delta}{2}
\end{aligned}
$$

Second line follows from continuity of $h(.,.)$ in its both arguments. Third line follows from continuity of $\omega_1^{SA}(t_1)$ and $\omega_2^{SA}(t_1)$ in t_1 at $t_1 = \underline{t}$.

And note that $h(\omega_1^{SA}(\sigma^*),t_1') - h(\omega_2^{SA}(\sigma^*),t_1') + \Delta(1-\min\{\frac{1}{2},\frac{1}{3\Phi(t_1')}\}) > \frac{\Delta}{2} \, \forall t_1' \in (\underline{t},\bar{t})$ implies that

$$\mathscr{S}^{SA} > \frac{\Delta}{2}$$

ii)($\Longrightarrow$) : Take $t_1 \in (\underline{t}, \bar{t})$ such that $\sigma^*(t) = \begin{cases} (s_1, s_2, s_3) & t \geq t_1 \\ (s_2, s_3) & t < t_1 \end{cases}$ is an equilibrium then for t_1 we must have:

$$\Gamma(t_1) = p$$

($\Longleftarrow$) : Suppose $p = \Gamma^{SA}(t_1)$, then at students with type t_1 are indifferent, and

$$\begin{aligned} h(\omega_1^{SA}(t_1), t) - h(\omega_2^{SA}(t_1), t) + \Delta(1 - \min\{\frac{1}{2}, \frac{1}{3F(t)}\}) &= p \text{ for } t = t_1 \\ &> \forall t > t_1 \\ &< \forall t < t_1 \end{aligned}$$

Thus an equilibrium with cutoff t_1 exists.

iii) Since $\Gamma(t_1)$ is continuous in t_1 for $t_1 \in (\underline{t}, \bar{t})$, for all $p \in (\mathscr{I}^{SA}, \mathscr{S}^{SA})$ $\exists t_1 \in (\underline{t}, \bar{t})$ such that $\Gamma^{SA}(t_1) = p$. This means $\Gamma^{SA,-1}(p)$ is non-empty for all $p \in (\frac{\Delta}{2}, \mathscr{S}^{SA})$. If $p > \mathscr{S}^{SA}$ we cannot find t_1 such that $\Gamma^{SA}(t_1) = p$, and similarly for $p \leq \mathscr{I}^{SA} = \frac{\Delta}{2}$.

If $\mathscr{M}^{SA}$ exists then that means $\mathscr{M}^{SA} = \mathscr{S}^{SA}$ which means there exists some $t_1 \in (\underline{t}, \bar{t})$ such that $\Gamma(t_1) = \mathscr{M}^{SA}$ which means $\Gamma^{SA,-1}(p)$ is non-empty for $p = \mathscr{S}^{SA}$.

Suppose $\Gamma^{SA,-1}(p)$ is non-empty for $p = \mathscr{S}^{SA}$. Then take $t_1 \in \Gamma^{SA,-1}(p)$, t_1 satisfies:

$$\Gamma(t_1) = \mathscr{S}^{SA}$$

which means $\mathscr{S}^{SA}$ is attained. This implies

$$\mathscr{M}^{SA} = \mathscr{S}^{SA}$$

hence $\mathscr{M}^{SA}$ exists.

iv) Take $p \in (\frac{\Delta}{2}, \mathscr{S}^{SA}]$. If $p = \mathscr{S}^{SA}$ and $\Gamma^{-1}(p)$ is empty, we will show that there is no cutoff equilibrium. $\Gamma^{-1}(\mathscr{S}^{SA})$ is empty means that there is no $t_1 \in (\underline{t}, \bar{t})$ such that $\Gamma(t_1) = p$ for $p = \mathscr{S}^{SA}$, so there cannot be a cutoff equilibrium.

Suppose $\Gamma^{-1}(p)$ is non-empty for $p = \mathscr{S}^{SA}$. If $t_1 \notin \Gamma^{-1}(p)$ then t_1 cannot be a cutoff when price is p such that $t > t_1$ submits (s_1, s_2, s_3) and $t < t_1$submits (s_2, s_1, s_3) since t_1 is not indifferent when cutoff is t_1.

take $t_1 \in \Gamma^{-1}(p)$ if $\Gamma^{-1}(p)$ is non-empty. If $t_1 \in \Gamma^{-1}(p)$, then

$$
\begin{aligned}
h(\omega_1^{SA}(t_1), t) - h(\omega_2^{SA}(t_1), t) + \Delta(1 - \min\{\frac{1}{2}, \frac{1}{3F(t)}\}) &= p \,\text{for}\, t = t_1 \\
&> \forall t > t_1 \\
&< \forall t < t_1
\end{aligned}
$$

which means t_1 is a cutoff equilibrium.

APPENDIX B

APPENDIX TO CHAPTER 2

B.1 Other Results

Claim B.1. Suppose A2.1-A2.3 and A2.8 holds. For any pair of (symmetric) equilibria σ^* and $\tilde{\sigma}$ such that σ^* is an equilibrium under $P_3 = (\{q_j\}_{j=1}^3, f(v), g(v))$ and $\tilde{\sigma}$ is an equilibrium under $E_3 = (\{q_j\}_{j=1}^3, 0, f(v), g(v))$, if

(i) all constrained student types v with $\pi_1(v) + \pi_2(v) \leq \tilde{\pi}_1(v) + \tilde{\pi}_2(v)$ have $\pi_1(v) + v\pi_2(v) \geq \tilde{\pi}_1(v) + v\tilde{\pi}_2(v)$;

(ii) $\eta < 1 - q_1 - q_2$, $\tilde{x}_1 \geq x_1$ and $\tilde{x}_2 < x_2 \leq q_2$

then $\tilde{\pi}_1(v') + \tilde{\pi}_2(v') < \pi_1(v') + \pi_2(v')$ and $\tilde{\pi}_1(v') > \pi_1(v')$ for all v' in some subset of $\mathscr{V}$ that has positive measure. Moreover such students' valuation v must satisfy $\frac{q_1}{1-q_2} \leq v \leq \frac{q_1}{1-\eta-q_2}$; where $\pi_j(v)$ and $\tilde{\pi}_j(v)$ are probability of type v entering school j in σ^* and $\tilde{\sigma}$ respectively.

Proof. Suppose all types with valuation v that satisfy $\pi_1(v) + \pi_2(v) \leq \tilde{\pi}_1(v) + \tilde{\pi}_2(v)$ have $\pi_1(v) + v\pi_2(v) \geq \tilde{\pi}_1(v) + v\tilde{\pi}_2(v)$, then we must have $\pi_1(v) > \tilde{\pi}_1(v)$, $\tilde{\pi}_2(v) > \pi_2(v)$ for all such types. Then there must be positive measure of *constrained* students that has weakly less probability of entering s_1 or s_2 in $\tilde{\sigma}$ compared to σ^* but has strictly larger entrance probability to s_1 in $\tilde{\sigma}$ compared to σ^*, i.e. $\tilde{\pi}_1(v') + \tilde{\pi}_2(v') < \pi_1(v') + \pi_2(v')$ and $\tilde{\pi}_1(v') > \pi_1(v')$ for all v' in some subset of $\mathscr{V}$ that has positive measure. otherwise s_1 would not be filled in $\tilde{\sigma}$. Note that since $\tilde{x}_1 > x_1$, $\tilde{x}_2 < x_2 \leq q_2$ and

$\eta < 1 - q_1 - q_2$ types for whom $\tilde{\pi}_1(v') > \pi_1(v')$ must be increasing the probability with which they are reporting s_1 as top choice. This means they report s_2 as top choice with positive probability in σ^* and s_1 as top choice with positive probability in $\tilde{\sigma}$. This means their valuation v must satisfy:

$$\frac{q_1}{x_1} + v\frac{q_2 - x_2}{x_1} \leq v$$
$$\iff v \geq \frac{q_1}{1 - q_2}$$

from σ^* and it must satisfy:

$$\frac{q_1}{\tilde{x}_1} + v\frac{q_2 - \tilde{x}_2}{\tilde{x}_1} \leq v$$
$$\iff v \geq \frac{q_1}{1 - \eta - q_2}$$

□

Lemma B.1. *Suppose A2.1-A2.3 and A2.8 holds. For any pair of (symmetric) equilibria σ^* and $\tilde{\sigma}$ such that σ^* is an equilibrium under $P_3 = (\{q_j\}_{j=1}^3, f(v), g(v))$ and $\tilde{\sigma}$ is an equilibrium under $E_3 = (\{q_j\}_{j=1}^3, 0, f(v), g(v))$, there exist a constrained student who is strictly better off in $\tilde{\sigma}$ compared to σ^* if and only if the following condition is satisfied:*

(*) If

(i) all constrained student types v with $\pi_1(v) + \pi_2(v) \leq \tilde{\pi}_1(v) + \tilde{\pi}_2(v)$ have $\pi_1(v) + v\pi_2(v) \geq \tilde{\pi}_1(v) + v\tilde{\pi}_2(v)$;

(ii) $\eta < 1 - q_1 - q_2$, $\tilde{x}_1 \geq x_1$ and $\tilde{x}_2 < x_2 \leq q_2$

then some constrained student type v' with $\tilde{\pi}_1(v') > \pi_1(v')$ have $v' \neq \frac{q_1}{1-\eta-q_2}$ for all $\sigma^*$and $\tilde{\sigma}$ of economies P_3 and E_3 respectively; where $\pi_j(v)$ is the probability of type v entering school j in σ^*; $\tilde{\pi}_j(v)$ is the probability of type v entering school j in $\tilde{\sigma}$; x_j,$\tilde{x}_j$ are the measures of students who report j as first choice in equilibrium σ^* and $\tilde{\sigma}$ respectively.

Proof. ($\Longrightarrow$) : Suppose (*) is not satisfied. That is *(i)* and *(ii)* are satisfied but all constrained student types v' with $\tilde{\pi}_1(v') > \pi_1(v')$ have $v' = \frac{q_1}{1-\eta-q_2}$ for some pairs of σ^* and $\tilde{\sigma}$. Take such pair of

σ^* and $\tilde{\sigma}$, since $\pi_1(v)+v\pi_2(v) \geq \tilde{\pi}_1(v)+v\tilde{\pi}_2(v)$ for all constrained students with $\pi_1(v)+\pi_2(v) \leq \tilde{\pi}_1(v)+\tilde{\pi}_2(v)$, such students are not better off in $\tilde{\sigma}$ compared to σ^*. Now we need to show that constrained student types with v' with $\pi_1(v')+\pi_2(v') > \tilde{\pi}_1(v')+\tilde{\pi}_2(v')$ are not better off, either. Take any such v'. If $\tilde{\pi}_1(v') \leq \pi_1(v')$, since $\tilde{\pi}_2(v')-\pi_2(v') < \pi_1(v')-\tilde{\pi}_1(v')$, $\pi_1(v)+v\pi_2(v) > \tilde{\pi}_1(v)+v\tilde{\pi}_2(v)$, so such constrained students are not better off. If $\tilde{\pi}_1(v') > \pi_1(v')$ since (*) does not hold we have $v' = \frac{q_1}{1-\eta-q_2}$. Note that $\tilde{\pi}_1(v') > \pi_1(v')$ and since $\tilde{x}_1 \geq x_1$ the only way this can happen is v' increasing the probability with which he submits s_1 as top choice. This means he was not submitting it as top choice with probability one in σ^*. Note that constrained student type v' is not indifferent between submitting s_1 as top choice and s_2 as top choice since that would mean for σ^*:

$$\begin{aligned} &\frac{q_1}{x_1} + v'\frac{q_2-x_2}{x_1} = v' \\ \Longrightarrow v' = &\frac{q_1}{x_1+x_2-q_2} = \frac{q_1}{1-q_2} \neq \frac{q_1}{1-\eta-q_2} \end{aligned}$$

which is not possible. Therefore, it must be the case that v' submits s_2 as top choice with probability one in σ^* and gets payoff v'. Whereas for $\tilde{\sigma}$ he is indifferent since:

$$\begin{aligned} &\frac{q_1}{\tilde{x}_1} + v'\frac{q_2-\tilde{x}_2}{\tilde{x}_1} = v \\ \Longleftrightarrow v' = &\frac{q_1}{1-\eta-q_2} \end{aligned}$$

and we know the last line is true. So he gets payoff v' in $\tilde{\sigma}$. So he does not increase his payoff in $\tilde{\sigma}$ compared to σ^*. This finishes the proof of the only if part.

($\Longleftarrow$) :

Suppose that, $\eta \geq 1-q_1-q_2$, then by the Proposition 6 we are done.

Now suppose that $\eta < 1-q_1-q_2$. Note that *unconstrained* students enter to s_1 or s_2 with positive probability in σ^*. Otherwise that means they report s_3 as top choice with probability one. But a student cannot report s_3 as top choice with positive probability p, since any deviation to reporting s_1 or s_2 as top choice with probability p and keeping the strategy followed with $1-p$ probability the same the student will get strictly higher payoff. So in $\tilde{\sigma}$ there must be positive measure of *constrained* students who increases his total probability of getting into s_1 or s_2 in $\tilde{\sigma}$ compared to σ^*, i.e. $\tilde{\pi}_1(v)+\tilde{\pi}_2(v) > \pi_1(v)+\pi_2(v)$ for v in some subset of $\mathscr{V}$ that has positive measure. Otherwise q_1 or q_2 would not be filled in $\tilde{\sigma}$ which is not possible since $1-\eta > q_1+q_2$.

If any of such types has $\pi_1(v)+v\pi_2(v) < \tilde{\pi}_1(v)+v\tilde{\pi}_2(v)$ we are done. So suppose $\pi_1(v)+v\pi_2(v) \geq \tilde{\pi}_1(v)+v\tilde{\pi}_2(v)$ for all such types, then we must have $\pi_1(v) > \tilde{\pi}_1(v)$, $\tilde{\pi}_2(v) > \pi_2(v)$ for all such types. Then there must be positive measure of *constrained* students that has weakly less probability of entering s_1 or s_2 in $\tilde{\sigma}$ compared to σ^* but has strictly larger entrance probability to s_1 in $\tilde{\sigma}$ compared to σ^*, i.e. $\tilde{\pi}_1(v')+\tilde{\pi}_2(v') \leq \pi_1(v')+\pi_2(v')$ and $\tilde{\pi}_1(v') > \pi_1(v')$ for all v' in some subset of $\mathscr{V}$ that has positive measure. otherwise s_1 would not be filled in $\tilde{\sigma}$.

If there is positive measure of types with $\tilde{\pi}_1(v')+\tilde{\pi}_2(v') = \pi_1(v')+\pi_2(v')$ and $\tilde{\pi}_1(v') > \pi_1(v')$ then there are *constrained* students strictly better off under $\tilde{\sigma}$ compared to σ^*. So suppose there are no such types. Hence, $\tilde{\pi}_1(v')+\tilde{\pi}_2(v') < \pi_1(v')+\pi_2(v')$ and $\tilde{\pi}_1(v') > \pi_1(v')$ for all v' in some subset of $\mathscr{V}$ that has positive measure. Take among such *constrained* student ones with v such that $v' \neq \frac{q_1}{1-\eta-q_2}$. Such types must have positive measure by (*). Note that we must have $\pi_2(v) > \tilde{\pi}_2(v)$ for all such types. Take one such type and call it v'. Note also that $1-\eta = \tilde{x}_1+\tilde{x}_2 \leq x_1+x_2 = 1$.

Case 1: $\tilde{x}_1 \geq x_1$ and $\tilde{x}_2 < x_2$

Since $\tilde{\pi}_1(v') > \pi_1(v')$ and $\tilde{x}_1 \geq x_1$, type v' must have increased the probability with which he is submitting s_1 as top choice in $\tilde{\sigma}$ compared to σ^*. So in σ^*, v' was not submitting s_1 as top choice with probability one so either he was indifferent between submitting s_1 and s_2 as top choice or strictly preferring to submit s_2 as top choice.

Subcase 1: $\tilde{x}_2 < x_2 \leq q_2$

By (*) we know that one such v' with $\tilde{\pi}_1(v') > \pi_1(v')$ must have $v' \neq \frac{q_1}{1-\eta-q_2}$. Suppose v' is indifferent in σ^*, then since he is not indifferent in $\tilde{\sigma}$ (because $\frac{q_1}{\tilde{x}_1} + v'\frac{q_2-\tilde{x}_2}{\tilde{x}_1} \neq v'$ by $v' \neq \frac{q_1}{1-\eta-q_2}$)

and increasing the probability of submitting s_1 as top choice in $\tilde{\sigma}$ compared to σ^*, he must be submitting s_1 as top choice with probability one in $\tilde{\sigma}$. Since v' is indifferent in σ^*, he must be getting payoff equal to submitting s_2 as top choice, which is v' since $x_2 \leq q_2$. In $\tilde{\sigma}$, type v' must be getting at least v' since if v' deviates to reporting s_2 as top choice with probability one, he could get v'. Moreover, he must be getting more than v' in $\tilde{\sigma}$ since we know that he is not indifferent between submitting s_1 and s_2 as top choice. So v'' becomes strictly better off in $\tilde{\sigma}$ compared to $\sigma^*$and we are done.

So suppose now that v' strictly prefers to submit s_2 as top choice in σ^*. Hence submits s_2 as top choice with probability one in σ^*. So v' gets v' in σ^*. In $\tilde{\sigma}$, v' does not play s_2 as top choice with probability one and since he is not indifferent, he must be submitting s_1 as top choice with probability one. If he submitted s_2 as top choice with probability one in $\tilde{\sigma}$ he would get v' since $\tilde{x}_2 < q_2$. Thus he must be getting at least v' in $\tilde{\sigma}$ when submitting s_1 as top choice with probability one. If he were getting v' he would be indifferent between submitting s_1 and s_2 as top choice, which cannot happen. So v' must be getting more than v' in $\tilde{\sigma}$ so he is better off compared to σ^* and we are done for this case.

Subcase 2: $\tilde{x}_2 \leq q_2 < x_2$

If v' is strictly preferring to submit s_2 as top choice or indifferent between submitting s_1 and s_2 as top choice, in both cases his payoff in σ^* is $v'\frac{q_2}{x_2} < v'$. In $\tilde{\sigma}$ if v' submits s_2 as top choice with probability one he gets v' since $\tilde{x}_2 \leq q_2$. Since he is submitting s_2 as top choice with probability less than one (may be zero) in $\tilde{\sigma}$,he must be getting at least v'. So v' gets higher payoff in $\tilde{\sigma}$ compared to $\sigma^*$and we are done for this case.

Subcase 3: $q_2 < \tilde{x}_2 < x_2$

In σ^*, since v' is submitting s_2 as top choice with positive probability therefore:

$$v'\frac{q_2}{x_2} \geq \frac{q_1}{x_1}$$
$$\implies v' \geq \frac{x_2}{x_1}\frac{q_1}{q_2}$$

In $\tilde{\sigma}$ since v' increased the probability with which he is submitting s_1 as top choice, this probability must be positive which means v' must satisfy:

$$\frac{q_1}{\tilde{x}_1} \geq v' \frac{q_2}{\tilde{x}_2}$$
$$\implies v' \leq \frac{\tilde{x}_2}{\tilde{x}_1} \frac{q_1}{q_2} < \frac{x_2}{x_1} \frac{q_1}{q_2}$$

So we cannot have this case.

Case 2: $\tilde{x}_1 < x_1$ and $\tilde{x}_2 \geq x_2$

Remember, for all *constrained* students with valuation v such that $\tilde{\pi}_1(v) + \tilde{\pi}_2(v) > \pi_1(v) + \pi_2(v)$, $\pi_1(v) > \tilde{\pi}_1(v)$ and $\tilde{\pi}_2(v) > \pi_2(v)$. Take one such type and call it v. Since $\tilde{\pi}_1(v) < \pi_1(v)$ and $\tilde{x}_1 < x_1$, type v must have decreased the probability with which he is playing s_1 as top choice in $\tilde{\sigma}$ compared to σ^*. So v plays s_1 as top choice with positive probability in σ^* and he plays s_2 as top choice with positive probability in $\tilde{\sigma}$.

Subcase 1: $\tilde{x}_2 \geq x_2 > q_2$.

In this case from σ^*, v must satisfy:

$$\frac{q_1}{x_1} \geq \frac{q_2}{x_2} v$$
$$\implies v \leq \frac{x_2}{x_1} \frac{q_1}{q_2}$$

and from $\tilde{\sigma}$, v must satisfy:

$$v \frac{q_2}{\tilde{x}_2} \geq \frac{q_1}{\tilde{x}_1}$$
$$\implies v \geq \frac{q_1}{q_2} \frac{\tilde{x}_2}{\tilde{x}_1} > \frac{q_1}{q_2} \frac{x_2}{x_1}$$

which is not possible.

Subcase 2: $\tilde{x}_2 > q_2 \geq x_2$

In this case, from σ^* v must satisfy:

$$\begin{aligned} \frac{q_1}{x_1} + \frac{q_2 - x_2}{x_1} v \geq v \\ \implies v \leq \frac{q_1}{1 - q_2} \end{aligned}$$

and from $\tilde{\sigma}$, v must satisfy:

$$\begin{aligned} &\frac{q_2}{\tilde{x}_2} v \geq \frac{q_1}{\tilde{x}_1} \\ \implies &v \geq \frac{q_1}{q_2} \frac{\tilde{x}_2}{\tilde{x}_1} = \frac{q_1}{\tilde{x}_1 \frac{q_2}{\tilde{x}_2}} > \frac{q_1}{\tilde{x}_1} = \frac{q_1}{1 - \eta - \tilde{x}_2} > \frac{q_1}{1 - \eta - q_2} \end{aligned}$$

which is not possible.

Subcase 3: $q_2 \geq \tilde{x}_2 \geq x_2$

In this case, from σ^* v must satisfy:

$$\begin{aligned} \frac{q_1}{x_1} + \frac{q_2 - x_2}{x_1} v \geq v \\ \implies v \leq \frac{q_1}{1 - q_2} \end{aligned}$$

and from $\tilde{\sigma}$ v must satisfy:

$$\begin{aligned} &\frac{q_1}{\tilde{x}_1} + \frac{q_2 - \tilde{x}_2}{\tilde{x}_1} v \leq v \\ \implies &v \geq \frac{q_1}{\tilde{x}_1 + \tilde{x}_2 - q_2} = \frac{q_1}{1 - \eta - q_2} > \frac{q_1}{1 - q_2} \end{aligned}$$

which is not possible.

Case 3: $\tilde{x}_1 < x_1$ and $\tilde{x}_2 < x_2$

Remember for all *constrained* students with type v such that $\tilde{\pi}_1(v)+\tilde{\pi}_2(v) > \pi_1(v)+\pi_2(v)$, $\pi_1(v) > \tilde{\pi}_1(v)$ and $\tilde{\pi}_2(v) > \pi_2(v)$. Take one such type and call it v. Since $\tilde{\pi}_1(v) < \pi_1(v)$ and $\tilde{x}_1 < x_1$, type v must have decreased the probability with which he is reporting s_1 as top choice in $\tilde{\sigma}$ compared to σ^*. So v reports s_1 as top choice with positive probability in σ^* and he reports s_2 as top choice with positive probability in $\tilde{\sigma}$.

Subcase 1: $q_2 < \tilde{x}_2 < x_2$

If type v is indifferent between submitting s_1 and s_2 as top choice than his payoff under σ^* would be $v\frac{q_2}{x_2}$. Since type v is increasing his probability of submitting s_2 as top choice, he must be indifferent between submitting $s_1$and s_2 as top choice or strictly preferring to submit s_2 as top choice in $\tilde{\sigma}$. In both situations payoff of type v is $v\frac{q_2}{\tilde{x}_2}$ which is strictly greater than $v\frac{q_2}{x_2}$, so v increases his payoff strictly in $\tilde{\sigma}$ compared to σ^*.

Suppose type v strictly prefers submitting s_1 as top choice to submitting s_2 as top choice under σ^*. So he submits s_1 as top choice with probability one. Then his payoff is $\frac{q_1}{x_1}$ at σ^*. If he submitted s_1 as top choice with probability one in $\tilde{\sigma}$, then his payoff would be $\frac{q_1}{\tilde{x}_1}$. That means in $\tilde{\sigma}$, v must be getting at least $\frac{q_1}{\tilde{x}_1}$ which is greater than $\frac{q_1}{x_1}$. So v increases his payoff in $\tilde{\sigma}$ compared to σ^*.

Subcase 2: $\tilde{x}_2 \leq q_2 < x_2$

If type v is indifferent between submitting s_1 and s_2 as top choice than his payoff under σ^* would be $v\frac{q_2}{x_2}$. Since type v is increasing his probability of submitting s_2 as top choice, he must be indifferent between submitting $s_1$and s_2 as top choice or strictly preferring to submit s_2 as top choice in $\tilde{\sigma}$. In both situations payoff of type v is v which is strictly greater than $v\frac{q_2}{x_2}$, so v increases his payoff strictly in $\tilde{\sigma}$ compared to σ^*.

Suppose type v strictly prefers submitting s_1 as top choice to submitting s_2 as top choice. So he submits s_1 as top choice with probability one. Then his payoff is $\frac{q_1}{x_1}$ at σ^*. If he submitted s_1 as top choice with probability one in $\tilde{\sigma}$, then his payoff would be $\frac{q_1}{\tilde{x}_1}$. That means in $\tilde{\sigma}$, type v must be getting at least $\frac{q_1}{\tilde{x}_1}$ which is greater than $\frac{q_1}{x_1}$. So type v increases his payoff in $\tilde{\sigma}$ compared to σ^*.

Subcase 3: $\tilde{x}_2 < x_2 \leq q_2$

From σ^*, v must satisfy:

$$\frac{q_1}{x_1} + v\frac{q_2 - x_2}{x_1} \geq v$$
$$\Longrightarrow v \leq \frac{q_1}{1 - q_2}$$

From $\tilde{\sigma}$, v must satisfy:

$$\frac{q_1}{\tilde{x}_1} + v\frac{q_2 - \tilde{x}_2}{\tilde{x}_1} \leq v$$
$$\Longrightarrow v \geq \frac{q_1}{\tilde{x}_1 + \tilde{x}_2 - q_2} = \frac{q_1}{1 - \eta - q_2} > \frac{q_1}{1 - q_2}$$

which is not possible. This completes the proof. □

Claim B.2. Let σ^* be an equilibrium of an economy $P = (\{q_j\}_{j=1}^3, f(v), g(v))$ without private schools in which all *unconstrained* students report s_1 as top choice, and $\tilde{\sigma}$ be an equilibrium of the economy $E = (\{q_j\}_{j=1}^3, 1, f(v), g(v))$. Let x_1 and $\tilde{x}_1$ be the corresponding measure of students who report s_1 as top choice in σ^* and $\tilde{\sigma}$ respectively. If $\tilde{x}_2 < x_2 \leq q_2$, $\tilde{x}_1 > x_1$ and $(\tilde{x}_1 - \eta)\left(1 - \frac{q_1}{\tilde{x}_1}\right) > q_2 - \tilde{x}_2$ then $\exists$ positive measure of *constrained* students with valuation $v \in [\frac{q_1}{1-q_2}, \frac{q_1}{\tilde{x}_1}\frac{\tilde{x}_1-\eta}{1-\eta-q_2}]$

Proof. Since $\tilde{x}_1 > x_1$, there must be positive measure of *constrained* students who increase probability of reporting s_1 as top choice in $\tilde{\sigma}$ compared to σ^*. Such studens must be reporting s_1 as top choice with positive probability in $\tilde{\sigma}$ and s_2 as top choice with positive probability in σ^*. Therefore from σ^* such students' valuation v must satisfy:

$$\frac{q_1}{x_1} + v\frac{q_2 - x_2}{x_1} \leq v$$
$$\Longleftrightarrow v \geq \frac{q_1}{1 - q_2}$$

and from $\tilde{\sigma}$, v must satisfy:

$$\frac{q_1}{\tilde{x}_1} + v\frac{q_2 - \tilde{x}_2}{\tilde{x}_1 - \eta} \geq v$$
$$\iff v \leq \frac{q_1}{\tilde{x}_1}\frac{\tilde{x}_1 - \eta}{1 - \eta - q_2}$$

And finally note that $\frac{q_1}{1-q_2} < \frac{q_1}{\tilde{x}_1}\frac{\tilde{x}_1-\eta}{1-\eta-q_2}$ since $\tilde{x}_1 = 1 - \tilde{x}_2$ and $\tilde{x}_2 < q_2$. □

Claim B.3. Let σ^* be an equilibrium of an economy $P = (\{q_j\}_{j=1}^3, f(v), g(v))$ without private schools in which all *unconstrained* students report s_1 as top choice, and $\tilde{\sigma}$ be an equilibrium of the economy $E = (\{q_j\}_{j=1}^3, 1, f(v), g(v))$. Let x_1 and $\tilde{x}_1$ be the corresponding measure of students who report s_1 as top choice in σ^* and $\tilde{\sigma}$ respectively. If $\tilde{x}_2 < x_2 \leq q_2$, $\tilde{x}_1 > x_1$ and $(\tilde{x}_1 - \eta)\left(1 - \frac{q_1}{\tilde{x}_1}\right) \leq q_2 - \tilde{x}_2$ then $\tilde{x}_1 = 1$ and there are *constrained* students who are strictly better off under $\tilde{\sigma}$ compared to σ^*.

Proof. Since $\tilde{x}_1 > x_1$, there must be positive measure of *constrained* students who increase probability of reporting s_1 as top choice in $\tilde{\sigma}$ compared to σ^*. Such studens must be reporting s_1 as top choice with positive probability in $\tilde{\sigma}$ and s_2 as top choice with positive probability in σ^*. Therefore such student must be indifferent in between reporting s_1 as top choice or s_2 as top choice or strictly prefer reporting s_2 as top choice in σ^*. In both cases such students with valuation v gets payoff v in σ^*.

Since $(\tilde{x}_1 - \eta)\left(1 - \frac{q_1}{\tilde{x}_1}\right) \leq q_2 - \tilde{x}_2$ if a *constrained* student reports s_1 as top choice and gets rejected it is sure for him to get into s_2 in second round. In that case payoff for *constrained* student with valuation v from reporting s_1 as top choice is

$$\frac{q_1}{\tilde{x}_1} + v\left(1 - \frac{q_1}{\tilde{x}_1}\right)$$

which is strictly greater than payoff from reporting s_2 as top choice which is v. And note that this is true for any v. So only equilibrium this can happen is equilibrium in which $\tilde{x}_1 = 1$.

So everyone must play s_1 as top choice in such equilibrium, then this yields payoff $q_1 + v(1 -$

q_1) to any type with valuation vector v and this is greater than v. So *constrained* students who increase the probability of reporting s_1 as top choice in $\tilde{\sigma}$ compared to σ^* are strictly better off under $\tilde{\sigma}$. □

Lemma B.2. *Suppose A2.1-A2.4 holds. Take economies $P_3 = (\{q_j\}_{j=1}^3, f(v), g(v))$ and $E_3 = (\{q_j\}_{j=1}^3, 1, f(v), g(v))$. For any pair of equilibrium σ^* and $\tilde{\sigma}$ such that σ^* is an equilibrium under P_3 in which all* unconstrained *students report s_1 as top choice with probability one and $\tilde{\sigma}$ is an equilibrium under E_3, there exist a* constrained *student who is strictly better off in $\tilde{\sigma}$ compared to σ^*or all* constrained *students get the same payoff under both equilibria if and only if the following condition is satisfied for any such pair of σ^* and $\tilde{\sigma}$:*

(***) If we have

(i)$\tilde{x}_2 < x_2 \leq q_2$, $\tilde{x}_1 > x_1$ and $(\tilde{x}_1 - \eta)(1 - \frac{q_1}{\tilde{x}_1}) > q_2 - \tilde{x}_2$

then one of the following conditions is satisfied

(1) $\exists$ positive measure of constrained *students with valuation $v \in [\frac{q_1}{1-q_2}, \frac{q_1}{\tilde{x}_1}\frac{\tilde{x}_1-\eta}{1-\eta-q_2})$*

(2) there is no constrained *students with valuation $v < \frac{q_1}{1-q_2}$*

(3) positive measure of constrained *students with valuation $v < \frac{q_1}{1-q_2}$ have $\frac{q_1}{x_1} + v\frac{q_2-x_2}{x_1} < \frac{q_1}{\tilde{x}_1} + v\frac{q_2-\tilde{x}_2}{\tilde{x}_1-\eta}$ or all* constrained *students with valuation $v < \frac{q_1}{1-q_2}$ have $\frac{q_1}{x_1} + v\frac{q_2-x_2}{x_1} = \frac{q_1}{\tilde{x}_1} + v\frac{q_2-\tilde{x}_2}{\tilde{x}_1-\eta}$;*

where x_j,$\tilde{x}_j$ are the measures of students who report j as first choice in equilibrium σ^* and $\tilde{\sigma}$ respectively.

Proof. ($\Longrightarrow$): Suppose the condition does not hold. That means: $\tilde{x}_2 < x_2 \leq q_2$, $\tilde{x}_1 > x_1$ and

$(\tilde{x}_1 - \eta)(1 - \frac{q_1}{\tilde{x}_1}) > q_2 - \tilde{x}_2$.

In this case note that a *constrained* student with valuation v strictly prefers reporting s_1 as top choice to s_2 as top choice in σ^* if

$$\frac{q_1}{x_1} + v\frac{q_2 - x_2}{x_1} > v$$
$$\Longleftrightarrow v' < \frac{q_1}{1-q_2}$$

and strictly prefers reporting s_2 as top choice in σ^* if

$$v > \frac{q_1}{1-q_2}$$

and he must be indifferent in σ^* when $v = \frac{q_1}{1-q_2}$

Also, note that if a *constrained* student with valuation v strictly prefers reporting s_1 as top choice to s_2 as top choice in $\tilde{\sigma}$ if

$$\begin{aligned}
&\frac{q_1}{\tilde{x}_1} + v\frac{q_2 - \tilde{x}_2}{\tilde{x}_1 - \eta} > v \\
\iff &v < \frac{q_1}{\tilde{x}_1}\frac{\tilde{x}_1 - \eta}{1 - \eta - q_2}
\end{aligned}$$

and strictly prefers reporting s_2 as top choice in $\tilde{\sigma}$ if

$$v > \frac{q_1}{\tilde{x}_1}\frac{\tilde{x}_1 - \eta}{1 - \eta - q_2}$$

and he must be indifferent in $\tilde{\sigma}$ when $v = \frac{q_1}{\tilde{x}_1}\frac{\tilde{x}_1-\eta}{1-\eta-q_2}$. Also, remember from previous lemma $\frac{q_1}{1-q_2} < \frac{q_1}{\tilde{x}_1}\frac{\tilde{x}_1-\eta}{1-\eta-q_2}$ since $\tilde{x}_1 = 1 - \tilde{x}_2$ and $\tilde{x}_2 < q_2$. And note that in both $\tilde{\sigma}$ and σ^* a *constrained* student reporting s_2 as top choice with positive probability gets v as payoff since $\tilde{x}_2 < x_2 \leq q_2$.

Note that there is no *constrained* student with valuation v such that $v \in [\frac{q_1}{1-q_2}, \frac{q_1}{\tilde{x}_1}\frac{\tilde{x}_1-\eta}{1-\eta-q_2})$.

First, let's check *constrained* students with valuation $v \geq \frac{q_1}{\tilde{x}_1}\frac{\tilde{x}_1-\eta}{1-\eta-q_2}$ if they exist. Such *constrained* students strictly prefer reporting s_2 as top choice to s_1 as top choice in σ^* since $v \geq \frac{q_1}{\tilde{x}_1}\frac{\tilde{x}_1-\eta}{1-\eta-q_2} > \frac{q_1}{1-q_2}$, so they report s_2 as top choice and get v in σ^*. *Constrained* students with valuation $v = \frac{q_1}{\tilde{x}_1}\frac{\tilde{x}_1-\eta}{1-\eta-q_2}$ are indifferent between reporting s_1 and s_2 as top choice in $\tilde{\sigma}$, so they must get payoff equal to reporting s_2 as top choice in $\tilde{\sigma}$ which is v. *Constrained* students with valuation $v > \frac{q_1}{\tilde{x}_1}\frac{\tilde{x}_1-\eta}{1-\eta-q_2}$ strictly prefer reporting s_2 as top choice to reporting s_1 as top choice in $\tilde{\sigma}$, so they report s_2 as top choice and get payoff v in $\tilde{\sigma}$. Therefore, if *constrained* students with valuation $v \geq \frac{q_1}{\tilde{x}_1}\frac{\tilde{x}_1-\eta}{1-\eta-q_2}$ exist, they get the same payoff under both equilibria.

Note that there is positive measure of *constrained* students with valuation v that satisfy $v < \frac{q_1}{1-q_2}$. Note that such *constrained* students strictly prefer reporting s_1 as top choice to s_2 as top choice in both σ^* and $\tilde{\sigma}$. Payoffs of students with valuation v that satisfy $v < \frac{q_1}{1-q_2}$ are $\frac{q_1}{\tilde{x}_1} + v\frac{q_2-\tilde{x}_2}{\tilde{x}_1-\eta}$ and $\frac{q_1}{x_1} + v\frac{q_2-x_2}{x_1}$ in $\tilde{\sigma}$ and σ^* respectively. Then that means there are *constrained* students who are strictly worse off under $\tilde{\sigma}$ compared to σ^* and no *constrained* student is strictly better off under $\tilde{\sigma}$ compared to σ^* since positive measure of *constrained* students with valuation $v < \frac{q_1}{1-q_2}$ have $\frac{q_1}{\tilde{x}_1} + v\frac{q_2-\tilde{x}_2}{\tilde{x}_1-\eta} < \frac{q_1}{x_1} + v\frac{q_2-x_2}{x_1}$; and no *constrained* student with valuation v has $\frac{q_1}{\tilde{x}_1} + v\frac{q_2-\tilde{x}_2}{\tilde{x}_1-\eta} > \frac{q_1}{x_1} + v\frac{q_2-x_2}{x_1}$. This finishes the proof of only if part.

($\Longleftarrow$) :First note that there are 3 cases since $\tilde{x}_1 + \tilde{x}_2 = x_1 + x_2 = 1$: $\tilde{x}_1 < x_1$ and $\tilde{x}_2 > x_2$; $\tilde{x}_1 = x_1$ and $\tilde{x}_2 = x_2$; and finally $\tilde{x}_1 > x_1$ and $\tilde{x}_2 < x_2$.

Case 1: $\tilde{x}_1 < x_1$ and $\tilde{x}_2 > x_2$

In this case, there is positive measure of *constrained* students who decreased the probability reporting s_1 as top choice. Take one such type and call this type v. That means type v must submit s_1 as top choice with positive probability at σ^* and submit s_2 as top choice with positive probability in $\tilde{\sigma}$.

Subcase 1: $\tilde{x}_2 > x_2 > q_2$

Then from σ^* v satisfy:

$$\frac{q_1}{x_1} \geq v\frac{q_2}{x_2}$$
$$\Longrightarrow v \leq \frac{q_1}{x_1}\frac{x_2}{q_2}$$

and from $\tilde{\sigma}$ v satisfy:

$$\frac{q_1}{\tilde{x}_1} \leq v\frac{q_2}{\tilde{x}_2}$$
$$\Longrightarrow v \geq \frac{\tilde{x}_2}{q_2}\frac{q_1}{\tilde{x}_1} > \frac{x_2}{q_2}\frac{q_1}{x_1}$$

which is not possible.

Subcase 2: $\tilde{x}_2 > q_2 \geq x_2$

From σ^* v satisfy:

$$\begin{aligned}
&\frac{q_1}{x_1} + v\frac{q_2 - x_2}{x_1} \geq v \\
\implies &\frac{q_1}{x_1} \geq v\frac{x_1 + x_2 - q_2}{x_1} \\
\implies &v \leq \frac{q_1}{1 - q_2}
\end{aligned}$$

From $\tilde{\sigma}$ v satisfy:

$$\begin{aligned}
&\frac{q_1}{\tilde{x}_1} \leq v\frac{q_2}{\tilde{x}_2} \\
\implies &v \geq \frac{\tilde{x}_2}{q_2}\frac{q_1}{\tilde{x}_1} = \frac{q_1}{\tilde{x}_1\frac{q_2}{\tilde{x}_2}} > \frac{q_1}{\tilde{x}_1} = \frac{q_1}{1-\tilde{x}_2} > \frac{q_1}{1-q_2}
\end{aligned}$$

which is not possible.

Subcase 3: $q_2 \geq \tilde{x}_2 > x_2$

Then from σ^*, v satisfy:

$$\begin{aligned}
&\frac{q_1}{x_1} + v\frac{q_2 - x_2}{x_1} \geq v \\
\implies &v \leq \frac{q_1}{1 - q_2}
\end{aligned}$$

and he gets payoff $\frac{q_1}{x_1} + v\frac{q_2 - x_2}{x_1}$. In $\tilde{\sigma}$ if $(\tilde{x}_1 - \eta)\left(1 - \frac{q_1}{\tilde{x}_1}\right) \leq q_2 - \tilde{x}_2$ then since the following is always true

$$\frac{q_1}{\tilde{x}_1} + v\left(1 - \frac{q_1}{\tilde{x}_1}\right) > v$$

he gets $\frac{q_1}{\tilde{x}_1} + v\left(1 - \frac{q_1}{\tilde{x}_1}\right)$ and reports s_1 as top choice with probability one. Contradiction.

If $(\tilde{x}_1 - \eta)\left(1 - \frac{q_1}{\tilde{x}_1}\right) > q_2 - \tilde{x}_2$ from $\tilde{\sigma}$ then v satisfy:

$$
\begin{aligned}
&\frac{q_1}{\tilde{x}_1} + v\frac{q_2 - \tilde{x}_2}{\tilde{x}_1 - \eta} \leq v \\
\implies &\frac{q_1}{\tilde{x}_1} \leq v\frac{\tilde{x}_1 + \tilde{x}_2 - \eta - q_2}{\tilde{x}_1 - \eta} \\
\implies &v \geq \frac{\tilde{x}_1 - \eta}{\tilde{x}_1}\frac{q_1}{1 - \eta - q_2} = \frac{1 - \tilde{x}_2 - \eta}{1 - \eta - q_2}\frac{q_1}{\tilde{x}_1} \geq \frac{1 - \tilde{x}_2}{1 - q_2}\frac{q_1}{1 - \tilde{x}_2} = \frac{q_1}{1 - q_2}
\end{aligned}
$$

Note that the inequality in the penultimate line is strict unless $\tilde{x}_2 = q_2$, if the inequality is strict then this case is impossible (combining with the inequality we get from σ^*). Then, $\tilde{x}_2 = q_2$ and v is indifferent between ranking s_1 and s_2 as top choice in both σ^* and $\tilde{\sigma}$. Hence, he gets v in both equilibria, and $v = \frac{q_1}{1-q_2}$.

So in the case $x_2 < \tilde{x}_2 = q_2$ all *constrained* students decreasing probability of submitting s_1 as top choice gets the same payoff in σ^* and $\tilde{\sigma}$. Consider a constrained student who increases the probability of submitting s_1 as top choice (if exists). Take any such student and call it type v'. In σ^* since he ranks s_2 as top choice with positive probability, type v' either strictly prefers ranking s_2 as top choice or indifferent between ranking s_1 as top choice or ranking s_2 as top choice. In either case he gets v' in σ^*. In $\tilde{\sigma}$, if he had ranked s_2 as top choice with probability one, he would get payoff v', then equilibrium strategy of v' must give him at least v' in $\tilde{\sigma}$.

Now, check a *constrained* student who does not change the probability of submitting s_1 as top chocie (if exists). Call it type v'. Note that indifference condition for reporting s_1 or s_2 as top choice is

$$
\begin{aligned}
&\frac{q_1}{x_1} + v\frac{q_2 - x_2}{x_1} = v \\
\iff &v = \frac{q_1}{1 - q_2}
\end{aligned}
$$

for σ^* and

$$\frac{q_1}{\tilde{x}_1} = v$$
$$\iff v = \frac{q_1}{1-q_2}$$

for $\tilde{\sigma}$. Therefore, if type v' is indifferent between ranking s_1 as top choice and s_2 as top choice in one of σ^* or $\tilde{\sigma}$, he has $v' = \frac{q_1}{1-q_2}$ and indifferent in both equilibria. Hence, he gets v' in both equilibria and we are done for this case. Suppose he is not indifferent in σ^* or $\tilde{\sigma}$. Since he does not change the probability with which he submits s_1 as top choice between σ^* and $\tilde{\sigma}$, he must strictly prefer submitting s_1 as top choice to submitting s_2 as top choice or vice versa in both equilibria. Let's check the first case. In σ^*, he gets payoff: $\frac{x_1}{q_1} + v'\frac{q_2-x_2}{x_1}$ and in $\tilde{\sigma}$ he gets payoff $\frac{q_1}{\tilde{x}_1}$. We want to show:

$$\frac{q_1}{\tilde{x}_1} \geq \frac{q_1}{x_1} + v'\frac{q_2-x_2}{x_1}$$

which is true iff

$$\frac{q_1(x_1-\tilde{x}_1)}{\tilde{x}_1 x_1} \geq v'\frac{q_2-x_2}{x_1}$$
$$\iff v' \leq \frac{q_1(x_1-\tilde{x}_1)}{\tilde{x}_1(q_2-x_2)} = \frac{q_1(x_1-\tilde{x}_1)}{\tilde{x}_1(\tilde{x}_2-x_2)} = \frac{q_1}{\tilde{x}_1}$$

which is true since from $\tilde{\sigma}$, we have: $\frac{q_1}{\tilde{x}_1} > v'$ which implies the desired condition.

Now, suppose v' strictly prefers ranking s_2 as top choice to ranking s_1 as top choice in both σ^* and $\tilde{\sigma}$. Then, in both equilibria he gets v and we are done for this case, too.

Case 2: $\tilde{x}_1 = x_1$ and $\tilde{x}_2 = x_2$

Note that in this case there is positive measure of *constrained* studenst increasing the probability with which he reports s_1 as top choice in $\tilde{\sigma}$ compared to σ^*, if and only if there is positive measure of of *constrained* students decreasing the probability of reporting s_1 as top choice in $\tilde{\sigma}$ compared to σ^*. Suppose there are such types of constrained students, if not then no *constrained* student changes the probability of reporting s_1 as top choice in $\tilde{\sigma}$ from σ^*. In that case, look at only the *constrained* students who does not change the probability in the analysis below.

Subcase 1: $\tilde{x}_2 = x_2 > q_2$

If there is a type who did not change the probability with which he reports s_1 as top choice his payoff is same across the two equilibria in this case.

Let's look at the type which increases the probability with which he ranks s_1 as top choice in $\tilde{\sigma}$ compared to σ^*. Call this type v. He must be ranking s_2 as top choice in σ^* with positive probability and ranking s_1 as top choice with positive probability in $\tilde{\sigma}$.

So from σ^* v satisfy:

$$v\frac{q_2}{x_2} \geq \frac{q_1}{x_1}$$
$$\implies v \geq \frac{x_2}{x_1}\frac{q_1}{q_2}$$

and from $\tilde{\sigma}$ v satisfy:

$$\frac{q_1}{\tilde{x}_1} \geq v\frac{q_2}{\tilde{x}_2}$$
$$\implies v \leq \frac{\tilde{x}_2}{\tilde{x}_1}\frac{q_1}{q_2} = \frac{x_2}{x_1}\frac{q_1}{q_2}$$

So we must have $v = \frac{x_2}{x_1}\frac{q_1}{q_2}$. Then type v must be indifferent between reporting s_1 as top choice and reporting s_2 as top choice both in σ^* and $\tilde{\sigma}$. So, in σ^* he gets $v\frac{q_2}{x_2}$ which equals $v\frac{q_2}{\tilde{x}_2}$, and the latter is what he gets in $\tilde{\sigma}$.

Now check the type who decreases the probability of reporting s_1 as top choice in $\tilde{\sigma}$ compared to σ^*. Call such student type v. Such student reports s_1 as top choice with positive probability in σ^* and reports s_2 as top choice with positive probability in $\tilde{\sigma}$.

So from σ^* v satisfy:

$$
\begin{aligned}
v\frac{q_2}{x_2} &\leq \frac{q_1}{x_1} \\
\Longrightarrow v &\leq \frac{x_2}{x_1}\frac{q_1}{q_2}
\end{aligned}
$$

and from $\tilde{\sigma}$ v satisfy:

$$
\begin{aligned}
\frac{q_1}{\tilde{x}_1} &\leq v\frac{q_2}{\tilde{x}_2} \\
\Longrightarrow v &\geq \frac{\tilde{x}_2}{\tilde{x}_1}\frac{q_1}{q_2} = \frac{x_2}{x_1}\frac{q_1}{q_2}
\end{aligned}
$$

So we must have $v = \frac{x_2}{x_1}\frac{q_1}{q_2}$. Then type v must be indifferent between reporting s_1 as top choice and reporting s_2 as top choice both in σ^* and $\tilde{\sigma}$. So in σ^* he gets $v\frac{q_2}{x_2}$ which equals $v\frac{q_2}{\tilde{x}_2}$, and the latter is what he gets in $\tilde{\sigma}$.

Subcase 2: $\tilde{x}_2 = x_2 \leq q_2$

If there is a type who does not change the probability of reporting s_1 as top choice in $\tilde{\sigma}$ compared to σ^*, then this type's payoff is v in both σ^* and $\tilde{\sigma}$ if he does not report s_1 as top choice with positive probability. If he reports s_1 with positive probability in these equilibria, then his payoff is higher in $\tilde{\sigma}$ compared to σ^* since $\frac{q_1}{x_1} + v\frac{q_2 - x_2}{x_1} < \frac{q_1}{\tilde{x}_1} + v\frac{q_2 - \tilde{x}_2}{\tilde{x}_1 - \eta}$ for the case $(\tilde{x}_1 - \eta)\left(1 - \frac{q_1}{\tilde{x}_1}\right) > q_2 - \tilde{x}_2$ and $\frac{q_1}{x_1} + v\frac{q_2 - x_2}{x_1} < \frac{q_1}{\tilde{x}_1} + v\left(1 - \frac{q_1}{\tilde{x}_1}\right)$ for the case $(\tilde{x}_1 - \eta)\left(1 - \frac{q_1}{\tilde{x}_1}\right) \leq q_2 - \tilde{x}_2$

Let's look at the type which increases the probability with which he ranks s_1 as top choice in $\tilde{\sigma}$ compared to σ^*. Call this type v. Since this type reports s_2 as top choice with positive probability, either he strictly prefers ranking s_2 as top choice to ranking s_1 as top choice or he is indifferent between the two in σ^*. In either case, he gets payoff v. If he had ranked s_2 as top choice with probability one, he would get v in $\tilde{\sigma}$. So with his equilibrium strategy he must get at least v in $\tilde{\sigma}$.

Let's look at the type which decreases the probability with which he ranks s_1 as top choice in $\tilde{\sigma}$ compared to σ^*. Call this type v'. Type v' submits s_1 as top choice with positive probability in

σ^* and reports s_2 as top choice with positive probability in $\tilde{\sigma}$. For $\tilde{\sigma}$, $(\tilde{x}_1 - \eta)\left(1 - \frac{q_1}{\tilde{x}_1}\right) > q_2 - \tilde{x}_2$, otherwise type v' would not rank s_2 as top choice with positive probability in $\tilde{\sigma}$.

Then from σ^*, v satisfy:

$$\begin{aligned} \frac{q_1}{x_1} + v' \frac{q_2 - x_2}{x_1} &\geq v' \\ \implies v' &\leq \frac{q_1}{1 - q_2} \end{aligned}$$

From $\tilde{\sigma}$, v satisfy:

$$\begin{aligned} v' &\geq \frac{q_1}{\tilde{x}_1} + v' \frac{q_2 - \tilde{x}_2}{\tilde{x}_1 - \eta} \\ \implies v' \left(\frac{\tilde{x}_1 + \tilde{x}_2 - \eta - q_2}{\tilde{x}_1 - \eta} \right) &\geq \frac{q_1}{\tilde{x}_1} \\ \implies v' &\geq \frac{x_1 - \eta}{x_1} \frac{q_1}{1 - \eta - q_2} \geq \frac{x_1}{1 - q_2} \frac{q_1}{x_1} = \frac{q_1}{1 - q_2} \end{aligned}$$

Note that inequality in the penultimate line is strict unless $x_2 = q_2$. So, there is no student increasing or decreasing the probability with which he ranks s_1 as top choice in $\tilde{\sigma}$ compared to σ^* when $x_2 = \tilde{x}_2 < q_2$. So suppose $x_2 = \tilde{x}_2 = q_2$.

Then from σ^*, v satisfy:

$$\begin{aligned} \frac{q_1}{x_1} &\geq v' \\ \implies v' &\leq \frac{q_1}{x_1} \end{aligned}$$

and from $\tilde{\sigma}$, v satisfy:

$$\begin{aligned} \frac{q_1}{\tilde{x}_1} &\leq v' \\ \implies v' &\geq \frac{q_1}{x_1} \end{aligned}$$

Thus, we must have $v' = \frac{q_1}{x_1}$ and v' is indifferent between reporting s_1 as top choice and reporting s_2 as top choice in both σ^* and $\tilde{\sigma}$. So he gets the same payoff in both equilibria.

Case 3: $\tilde{x}_1 > x_1$ and $\tilde{x}_2 < x_2$

Positive measure of *constrained* students must have increased the probability with which they rank s_1 as top choice. Take such a student and call it type v'.

Subcase 1: $x_2 > \tilde{x}_2 > q_2$

In σ^*, type v' ranks s_2 as top choice with positive probability. So either he strictly prefers ranking s_2 as top choice or he is indifferent between ranking s_2 as top choice and s_1 as top choice. In either case, his payoff in σ^* is $v'\frac{q_2}{x_2}$. In $\tilde{\sigma}$, if he ranked s_2 as top choice with probability one, he would get $v'\frac{q_2}{\tilde{x}_2}$, so with his equilibrium strategy he must be getting at least $v'\frac{q_2}{\tilde{x}_2}$ which is greater than what he gets in σ^*. So, this case is done.

Subcase 2: $x_2 > q_2 \geq \tilde{x}_2$

In σ^* type v' ranks s_2 as top choice with positive probability. So either he strictly prefers ranking s_2 as top choice or he is indifferent between ranking s_2 as top choice and s_1 as top choice. In either case, his payoff in σ^* is $v'\frac{q_2}{x_2}$. In $\tilde{\sigma}$, if he ranked s_2 as top choice with probability one, he would get v', so with his equilibrium strategy he must be getting at least v' which is greater than what he gets in σ^*. So, this case is done.

Subcase 3: $q_2 \geq x_2 > \tilde{x}_2$

Suppose (i) is not satisfied: $(\tilde{x}_1 - \eta)\left(1 - \frac{q_1}{\tilde{x}_1}\right) \leq q_2 - \tilde{x}_2$. In this case result follows from the last Claim.

Suppose (i) is satisfied: $(\tilde{x}_1 - \eta)\left(1 - \frac{q_1}{\tilde{x}_1}\right) > q_2 - \tilde{x}_2$. In this case note that a *constrained* student with valuation v strictly prefers reporting s_1 as top choice to s_2 as top choice in σ^* if

$$\frac{q_1}{x_1} + v\frac{q_2 - x_2}{x_1} > v$$
$$\iff v' < \frac{q_1}{1 - q_2}$$

and strictly prefers reporting s_2 as top choice in σ^* if

$$v > \frac{q_1}{1-q_2}$$

and he must be indifferent in σ^* when $v = \frac{q_1}{1-q_2}$

Also, note that if a *constrained* student with valuation v strictly prefers reporting s_1 as top choice to s_2 as top choice in $\tilde{\sigma}$ if

$$\begin{aligned} &\frac{q_1}{\tilde{x}_1} + v\frac{q_2-\tilde{x}_2}{\tilde{x}_1-\eta} > v \\ \iff &v < \frac{q_1}{\tilde{x}_1}\frac{\tilde{x}_1-\eta}{1-\eta-q_2} \end{aligned}$$

and strictly prefers reporting s_2 as top choice in $\tilde{\sigma}$ if

$$v > \frac{q_1}{\tilde{x}_1}\frac{\tilde{x}_1-\eta}{1-\eta-q_2}$$

and he must be indifferent in $\tilde{\sigma}$ when $v = \frac{q_1}{\tilde{x}_1}\frac{\tilde{x}_1-\eta}{1-\eta-q_2}$. Also, remember from previous lemma $\frac{q_1}{1-q_2} < \frac{q_1}{\tilde{x}_1}\frac{\tilde{x}_1-\eta}{1-\eta-q_2}$ since $\tilde{x}_1 = 1-\tilde{x}_2$ and $\tilde{x}_2 < q_2$. And note that in both $\tilde{\sigma}$ and σ^* a *constrained* student reporting s_2 as top choice with positive probability gets v as payoff since $\tilde{x}_2 < x_2 \leq q_2$.

First, let's check *constrained* students with valuation $v \geq \frac{q_1}{\tilde{x}_1}\frac{\tilde{x}_1-\eta}{1-\eta-q_2}$ if they exist. Such *constrained* students strictly prefer reporting s_2 as top choice to s_1 as top choice in σ^* since $v \geq \frac{q_1}{\tilde{x}_1}\frac{\tilde{x}_1-\eta}{1-\eta-q_2} > \frac{q_1}{1-q_2}$, so they report s_2 as top choice and get v in σ^*. *Constrained* students with valuation $v = \frac{q_1}{\tilde{x}_1}\frac{\tilde{x}_1-\eta}{1-\eta-q_2}$ are indifferent between reporting s_1 and s_2 as top choice in $\tilde{\sigma}$, so they must get payoff equal to reporting s_2 as top choice in $\tilde{\sigma}$ which is v. *Constrained* students with valuation $v > \frac{q_1}{\tilde{x}_1}\frac{\tilde{x}_1-\eta}{1-\eta-q_2}$ strictly prefer reporting s_2 as top choice to reporting s_1 as top choice in $\tilde{\sigma}$, so they report s_2 as top choice and get payoff v in $\tilde{\sigma}$. Therefore, if *constrained* students with valuation $v \geq \frac{q_1}{\tilde{x}_1}\frac{\tilde{x}_1-\eta}{1-\eta-q_2}$ exist, they get the same payoff under both equilibria.

Suppose now there exists positive measure of *constrained* students with valuation

$v \in [\frac{q_1}{1-q_2}, \frac{q_1}{\tilde{x}_1}\frac{\tilde{x}_1-\eta}{1-\eta-q_2})$. *Constrained* students with $v = \frac{q_1}{1-q_2}$ are indifferent between reporting s_1 and s_2 as top choice in σ^*, so they must get v in σ^*. *Constrained* students with $\frac{q_1}{1-q_2} < v < \frac{q_1}{\tilde{x}_1}\frac{\tilde{x}_1-\eta}{1-\eta-q_2}$ strictly prefer reporting s_2 as top choice in σ^*, so they must be reporting s_2 as top choice and get payoff v. So all *constrained* students with valuation $v \in [\frac{q_1}{1-q_2}, \frac{q_1}{\tilde{x}_1}\frac{\tilde{x}_1-\eta}{1-\eta-q_2})$ get payoff of v in σ^*. Since such students have $v < \frac{q_1}{\tilde{x}_1}\frac{\tilde{x}_1-\eta}{1-\eta-q_2}$, in $\tilde{\sigma}$ they strictly prefer reporting s_1 as top choice, so they must be getting more than payoff v under $\tilde{\sigma}$ and this case is done.

Suppose there is no *constrained* student with valuation $v \in [\frac{q_1}{1-q_2}, \frac{q_1}{\tilde{x}_1}\frac{\tilde{x}_1-\eta}{1-\eta-q_2})$.

Suppose also that there is no *constrained* student with valuation $v < \frac{q_1}{1-q_2}$. Then that means all *constrained* students get the same payoff and this case is done.

Suppose there are *constrained* students with valuation $v < \frac{q_1}{1-q_2}$. Then all such students must strictly prefer reporting s_1 as top choice to s_2 as top choice in both σ^* and $\tilde{\sigma}$. If positive measure of these students have valuation v' that satisfy

$$\frac{q_1}{x_1} + v'\frac{q_2 - x_2}{x_1} < \frac{q_1}{\tilde{x}_1} + v'\frac{q_2 - \tilde{x}_2}{\tilde{x}_1 - \eta}$$

then this case is done since left hand side and right hand side are payoffs from reporting s_1 as top choice in σ^* and $\tilde{\sigma}$ respectively. Suppose there is no *constrained* student with valuation valuation v that satisfy $v < \frac{q_1}{1-q_2}$ and $\frac{q_1}{x_1} + v\frac{q_2-x_2}{x_1} < \frac{q_1}{\tilde{x}_1} + v\frac{q_2-\tilde{x}_2}{\tilde{x}_1-\eta}$. Then suppose all *constrained* students with valuation v that satisfy $v < \frac{q_1}{1-q_2}$ have

$$\frac{q_1}{x_1} + v\frac{q_2 - x_2}{x_1} = \frac{q_1}{\tilde{x}_1} + v\frac{q_2 - \tilde{x}_2}{\tilde{x}_1 - \eta}$$

Then all *constrained* students with valuation v that satisfy $v < \frac{q_1}{1-q_2}$ get the same payoff under σ^* and $\tilde{\sigma}$, and this case is done. This finishes the proof. $\square$

B.2 Proofs of Section 3

Proof of Proposition 2.1

Proof. (Modified from proof of Theorem 1 Abdulkadiroğlu, Che, and Yasuda (2011)) Take any equilibrium (symmetric) of BM. Let $\sigma_c^*(v)$ and $\sigma_u^*(v)$ be equilibrium strategies under BM of *constrained* and *unconstrained* students respectively. Let $\pi_j^c(v)$ and $\pi_j^u(v)$ be the probabililties of constrained students and unconstrained students with valuation v going to school j respectively when they play the equilibrium strategies. Consider a constrained student with valuation vector v' playing strategy $\sigma_c^*(v)$ with probability $(1-\eta)f(v)$ and playing $\tilde{\sigma}_u(v)$ with probability $\eta g(v)$ where $\tilde{\sigma}_u(v)$ is modification of $\sigma_u^*(v)$ such that when v' is playing strategy $\sigma_u^*(v)$ he modifies it such that he adds s_m to the bottom of his list whereas an unconstrained student does not submit s_m in his preferences since he prefers s_p to it. Then under this strategy probability of going to school $j < m$ for student with v^i is:

$$\sum_v \pi_j^c(v)f(v)(1-\eta) + \pi_j^u(v)g(v)\eta = q_j$$

Note that the equality above must hold in equilibrium: Left hand side is the measure of students who enter school j in equilibrium and right hand side is the measure of seats at school j. Trivially, left hand side cannot be greater than right hand side. Right hand side cannot be greater than left hand side. Suppose it is for some $j < m$. That means no unconstrained student went to s_p in this equilibrium since they could have deviated to writing s_j to the bottom of their list and be assigned to s_j which would make them better compared to s_p. Also we must have that, no one (constrained student) is assigned to $s_m$in this equilibrium, otherwise such student could include j just above $s_m$in his preference list and be assigned to school j. Then since no unconstrained student is assigned to private school, we must have $\sum_{j=1}^{m-1} q_j > 1$ which contradicts one of the assumptions of the model

That means constrained student with v will go to s_m with probability $1-\sum_{j=1}^{m-1} q_j$ under this strategy. So his expected utility under this strategy is:

$$\sum q_j v_j^i + \left(1 - \sum_{j=1}^{m-1} q_j\right) v_m^i$$

which is equal to his utility under DA. Since this is not necessarily an equilibrium strategy under BM, thus his utility under BM equilibrium must be greater than or equal to his utility under DA.

$\square$

Proof of Claim 2.1

Proof. Suppose not, that is $k + \eta \leq q_1$, then we must have $1 - \eta - k \geq q_2$. Otherwise we would have $(1 - \eta) + \eta < q_1 + q_2 < 1$ which is contradiction. So we must have at least q_2 measure of constrained students who report s_2 as top choice. Payoff of such constrained students is

$$v \frac{q_2}{1 - \eta - k}$$

which is less than or equal to v, then such a constrained student can deviate to reporting truthfully and get utility of 1. Hence, we get contradiction. $\square$

Proof of Remark 2.2

Proof. If number of students reporting s_2 as top choice is larger than q_2, then *constrained* students with valuation vector v reporting s_1 as top choice will get payoff of

$$\frac{q_1}{k + \eta}$$

since $k + \eta$ measure of students apply to q_1 measure of seats in the first round so each has $\frac{q_1}{k+\eta}$ probability of entering and they will not have any chance of getting into s_2 (since s_2 is filled in first round). *Constrained* students with valuation v reporting s_2 as top choice gets payoff of

$$v\frac{q_2}{1-\eta-k}$$

since $1-\eta-k$ measure of students apply to q_2 measure of seats in first round and $1-\eta-k>q_2$, each student has $\frac{q_2}{1-\eta-k}$ chance of entering to s_2. And they do not have any chance to get into s_1 since s_1 is filled in first round.

If number of students reporting s_2 as top choice is smaller than q_2 *constrained* students with valuation vector v reporting s_2 as top choice will get payoff of v since they get into s_2 for sure in the first round. And *constrained* students with valuation vector v reporting s_1 as top choice will get payoff of

$$\frac{q_1}{k+\eta}+v\frac{q_2-(1-\eta-k)}{k}$$

First term follows from s_1 being filled in the first round, so any applicant of s_1 in the first round can enter with probability $\frac{q_1}{k+\eta}$. To understand second term note that since s_2 is not filled in the first round, so a student who reported s_1 as top choice must have reported s_2 as second choice. In second round $k(1-\frac{q_1}{k+\eta})$ students apply to remaining $q_2-(1-\eta-k)$ capacity which leads to the (unconditional) probability $\frac{q_2-(1-\eta-k)}{k}$ of entry to s_2. Note that we cannot have $q_2-(1-\eta-k)>k\left(1-\frac{q_1}{k+\eta}\right)$ for any $0\leq k\leq 1-\eta$ since this is equivalent to

$$\frac{k}{k+\eta}q_1>1-\eta-q_2$$

Left hand side is largest when $k=1-\eta$, plugging in $k=1-\eta$, inequality becomes $q_2>(1-\eta)(1-q_1)$. □

Proof of Claim 2.2

Proof. If type v submits the first choice truthfully he gets $\frac{q_1}{k+\eta}$ since s_2 is also filled in the first round.

If type v submits s_2 as top choice truthfully, he gets $v\frac{q_2}{1-\eta-k}$. Then in the equilibrium he submits s_1 as top choice with probability one if

$$\frac{q_1}{k+\eta} > v\frac{q_2}{1-\eta-k}$$
$$\iff v < \frac{q_1}{q_2}\frac{(1-\eta-k)}{k+\eta}$$

Other conditions can be obtained similarly. □

Proof of Claim 2.3

Proof. $(\Longrightarrow)$: Suppose there is less than q_2 measure *constrained* students for whom $v \geq \frac{q_1}{1-q_2}$. Let k denote the measure of *constrained* students who report s_1 as top choice. I want to show that there cannot be an equilibrium with $k \leq 1-\eta-q_2$. For $k = 1-\eta-q_2$ to be equilibrium we need q_2 measure of *constrained* students who has $v \geq c(1-\eta-q_2) = \frac{q_1}{1-q_2}$ which cannot happen. Note that we cannot have an equilibrium with $k < 1-\eta-q_2$, either. To have such an equilibrium, we need to have $1-\eta-k$ measure of *constrained* students for whom $v \geq c(k)$. Note that $1-\eta-k > q_2$ and $c(k) > \frac{q_1}{1-q_2}$ since $k < 1-\eta-q_2$ and $c(.)$ is strictly decreasing. So we need more than q_2 measure of *constrained* students with $v > \frac{q_1}{1-q_2}$ which is not possible.

$(\Longleftarrow)$: If there are q_2 measure of *constrained* students for whom $v \geq \frac{q_1}{1-q_2}$, then there is an equilibrium in which *constrained* students with $v \geq \frac{q_1}{1-q_2}$ report s_2 as top choice and other *constrained* students report s_1 as top choice, so in this equilibrium we have $k = 1-\eta-q_2$.

Suppose there are more than q_2 measure of *constrained* students with $v \geq \frac{q_1}{1-q_2}$. If measure of *constrained* students with $v > \frac{q_1}{1-q_2}$ is equal to q_2 then we have an equilibrium in which *constrained* students with $v > \frac{q_1}{1-q_2}$ report s_2 as top choice and other *constrained* students report s_1 as top choice. If measure of *constrained* students with $v > \frac{q_1}{1-q_2}$ is less than q_2 then we have an equilibrium in which q_2 measure of *constrained* students with $v \geq \frac{q_1}{1-q_2}$ report s_2 as top choice some and other *constrained* students report s_1 as top choice. So suppose measure of *constrained* students with $v > \frac{q_1}{1-q_2}$ is larger than q_2. That means there are less than $1-\eta-q_2$ measure of *constrained* students with $v \leq \frac{q_1}{1-q_2} = c(1-\eta-q_2)$. So we have less than x students for whom $v \leq c(x)$ when $x = 1-\eta-q_2$

Suppose $c(0) < 1$, i.e. $\frac{q_1}{q_2}\frac{1-\eta}{\eta} < 1$, if all *constrained* students have $v \geq \frac{q_1}{q_2}\frac{1-\eta}{\eta}$ then there is an equilibrium in which all *constrained* students report s_2 as top choice. Suppose now some *constrained* students have $v < \frac{q_1}{q_2}\frac{1-\eta}{\eta} = c(0)$. That means we have more than x *constrained* students that has $v < c(x)$ when $x = 0$. Note that $c(.)$ is a decreasing continuous function, so there must be $x > 0$ such that there are $x' \geq x$ students with $v \leq c(x)$ and $x'' \leq x$ students with $v < c(x)$ by the last sentence of the last paragraph. Therefore, we have an equilibrium in which *constrained* students with $v < c(x)$ report s_1 as top choice and $x - x''$ of *constrained* students with $v = c(x)$ report s_1 as top choice.

Suppose that $c(0) > 1$, that means we have more than x *constrained* students that has $v < c(x)$ when $x = 0$ (all *constrained* students to be precise). Then by the same arguments as in previous paragraph, there is an equilibrium. $\square$

Proof of Claim 2.4

Proof. Suppose $1-\eta-k < q_2$. Then some of the constrained students for whom $v > \frac{(1-\eta)q_1}{1-\eta-q_2}$ are reporting s_1 as top choice with positive probability $p > 0$. Their payoff is $\frac{q_1}{k+\eta} + v\frac{q_2-(1-\eta-k)}{k}$ since a list that reports s_1 as top choice must report s_2 as second choice, otherwise such a list will get only expected payoff of $\frac{q_1}{k+\eta}$. We want to show

$$
\begin{aligned}
&\frac{q_1}{k+\eta} + v\frac{q_2-(1-\eta-k)}{k} < v \\
\Longleftrightarrow\ & v > \frac{k}{k+\eta}\frac{q_1}{1-\eta-q_2}
\end{aligned}
$$

which is true by $\frac{(1-\eta)q_1}{1-\eta-q_2} \geq \frac{k}{k+\eta}\frac{q_1}{1-\eta-q_2}$ since right hand side is increasing in k and k can be at most $1-\eta$. So such students would deviate to reporting s_2 as top choice. We get contradiction. Note also that $1-\eta-q_2 > 0$ by A2.5. $\square$

Proof of Lemma 2.1

Proof. Let's take a symmetric equilibrium σ^* of BM, let k denote the total measure of constrained students who report s_1 as top choice in this equilibrium. If there is no other equilibrium, we are done. Suppose there are other equilibria. Take any of them and call it $\tilde{\sigma}$. Suppose for contradiction total measure of constrained students who report s_1 as top choice in $\tilde{\sigma}$ is $k' \neq k$. From Claim 2.2, we know that $\frac{q_1}{q_2}\frac{(1-\eta-k)}{k+\eta}$ is the threshold for v such that, if v is above the threshold for a student, that student submits s_2 as top choice and if v is below the threshold, that student submits s_1 as top choice and the student is indifferent in case of equality in σ^* and similarly for $\tilde{\sigma}$. Let $k' > k$, then $\frac{q_1}{q_2}\frac{(1-\eta-k)}{k+\eta} > \frac{q_1}{q_2}\frac{(1-\eta-k')}{k'+\eta}$, so threshold decreases when k changes to k' and measure of constrained students who are above the threshold weakly increases. That means total measure of constrained students who report s_2 as top choice weakly increases in $\tilde{\sigma}$ compared to σ^*. But this is not possible since $k' > k$. Now, suppose $k' < k$ then $\frac{q_1}{q_2}\frac{(1-\eta-k)}{k+\eta} < \frac{q_1}{q_2}\frac{(1-\eta-k')}{k'+\eta}$, so threshold increases when k changes to k' and measure of constrained students who are above the threshold weakly decreases. That means total measure of constrained students who report s_2 as top choice weakly decreases in $\tilde{\sigma}$ compared to σ^*. But this is not possible since $k' < k$. So σ^* and $\tilde{\sigma}$ have the same $i(k)$. Students with $v \neq c(k)$ must play the same strategy in both equilibria, students with type $v = c(k)$ must play the same strategy in both symmetric equilibria since otherwise k would be different for σ^* and $\tilde{\sigma}$.

By the assumptions and Claim 2.4, total measure of constrained students who report s_2 as top choice must be at least q_2, so k must be less than or equal to $1-\eta-q_2$. By Claim 2.1 we also know $k > q_1 - \eta$.

Also, by Claim 2.2 for given k, a constrained student type v reports s_1 as top choice if we have:

$$v < \frac{q_1}{q_2}\frac{(1-\eta-k)}{k+\eta}$$

Note that if $\frac{q_1}{q_2}\frac{(1-\eta-k)}{k+\eta} \geq 1$ we must have $k = (1-\eta)$ since all constrained students will report s_1 as top choice. This is not possible since

$$\begin{aligned}&\frac{q_1}{q_2}\frac{(1-\eta-k)}{k+\eta}\geq 1\\ \Longrightarrow\, &k\leq\frac{q_1}{q_2+q_1}-\eta\\ \Longrightarrow\, &k<1-\eta\end{aligned}$$

So we must have:

$$\begin{aligned}&\frac{q_1}{q_2}\frac{(1-\eta-k)}{k+\eta}<1\\ \Longleftrightarrow\, &k>\frac{q_1}{q_1+q_2}-\eta\end{aligned}$$

Thus we have $k\in[\max\{\frac{q_1}{q_1+q_2}-\eta,0\},1-\eta-q_2]\setminus\{\frac{q_1}{q_1+q_2}-\eta\}$. Note that this set is always well defined: we cannot have $\frac{q_1}{q_1+q_2}-\eta\geq 1-\eta-q_2$ since this is true iff $q_2+q_1-1\geq 0$ which is not true by assumptions of the model.

Take any $x\in[\max\{\frac{q_1}{q_1+q_2}-\eta,0\},1-\eta-q_2]\setminus\{\frac{q_1}{q_1+q_2}-\eta\}$, then consider the distribution of preferences such that there are x measure of *constrained* students with valuation vectors such that $v\leq c(x)=\frac{q_1}{q_2}\frac{(1-\eta-x)}{x+\eta}$ and $1-\eta-x$ measure of *constrained* students with valuation vectors such that $v\geq c(x)=\frac{q_1}{q_2}\frac{(1-\eta-x)}{x+\eta}$. To see such types can exist note that $i(.)$ is decreasing so it is enough to show $c(\frac{q_1}{q_1+q_2}-\eta)\leq 1$ and $c(1-\eta-q_2)>0$. We have

$$c(\frac{q_1}{q_1+q_2}-\eta)=\frac{q_1}{q_2}\frac{1-\frac{q_1}{q_1+q_2}}{\frac{q_1}{q_1+q_2}}=1$$

and

$$c(1-\eta-q_2)=\frac{q_1}{q_2}\frac{q_2}{1-q_2}=\frac{q_1}{1-q_2}>0$$

$\square$

Proof of Lemma 2.2

Proof. By assumptions A2.1-A2.6 and Claim 2.4 we have $1-\eta-k \geq q_2$. So type v students reporting s_1 as top choice under this equilibrium of BM they get payoff:

$$\frac{q_1}{k+\eta}$$

And under DA they get payoff:

$$q_1 + v\frac{q_2}{(1-\eta)}$$

They are weakly better off in BM compared to DA iff

$$\frac{q_1}{k+\eta} \geq q_1 + v\frac{q_2}{(1-\eta)}$$

(and strictly better off iff inequality is strict) which is equivalent to showing:

$$v \leq \frac{q_1(1-\eta-k)(1-\eta)}{q_2(k+\eta)}$$

which is the condition we gave.

Type v students report s_2 as top choice under this equilibrium of BM they get payoff:

$$v\frac{q_2}{1-\eta-k}$$

and under DA they get payoff:

$$q_1 + v\frac{q_2}{(1-\eta)}$$

They are weakly better off in BM compared to DA iff

$$v\frac{q_2}{1-\eta-k} \geq q_1 + v\frac{q_2}{(1-\eta)}$$

(and strictly better off iff inequality is strict) which is equivalent to showing

$$v \geq \frac{(1-\eta-k)(1-\eta)q_1}{kq_2}$$

which is the condition we provided.

Finally, we have $c(k) = \frac{q_1}{q_2}\frac{1-\eta-k}{k+\eta}$ so $\bar{c}(k) > c(k)$ iff

$$\frac{q_1(1-\eta)(1-\eta-k)}{q_2 k} > \frac{q_1}{q_2}\frac{1-\eta-k}{k+\eta}$$

The last inequality is equivalent to

$$\eta(1-\eta-k) > 0$$

which is true since $1-\eta-k \geq q_2$. And $\underline{c}(k) < c(k)$ iff

$$\frac{(1-\eta-k)(1-\eta)q_1}{(k+\eta)q_2} < \frac{q_1}{q_2}\frac{1-\eta-k}{k+\eta}$$

which is true. □

Proof of Lemma 2.3

Proof. By A2.1-A2.6 and Claim 2.4 we have $1-\eta-k \geq q_2$. Let's prove the first part of the proposition. Suppose $k = 0$, then all constrained students report s_2 as top choice with probability one in this equilibrium. So a constrained student with valuation v gets:

$$v\frac{q_2}{1-\eta}$$

under BM and under DA he gets:

$$q_1 + v\frac{q_2}{(1-\eta)}$$

which is greater than his payoff under BM.

Now, I prove the second part of the Lemma.

($\Longleftarrow$) : Given $k > 0$, if type v has $v \geq \bar{c}(k)$ then by Lemma 2.2 above $v > c(k)$ and so type v plays s_2 as top choice, and so again by Lemma 2.2 and Lemma 2.1 he is weakly better off in the symmetric equilibrium of BM and strictly better off if the inequality is strict. If type v has $v \leq \underline{c}(k)$, then by Lemma 2.2, $v < c(k)$ and so type v plays s_1 as top choice, and so again by Lemma 2.1 and Lemma 2.2 he is weakly better off in the symmetric equilibrium of BM and strictly better off if the inequality is strict.

($\Longrightarrow$) :Given $k > 0$, if type v has $\underline{c}(k) < v < \bar{c}(k)$, then he may be reporting s_1 as top choice or s_2 as top choice, but in any case he is strictly worse off in the symmetric BM equilibria compared to DA by Lemma 2.1 and Lemma 2.2.

Finally, if there are *constrained* students with valuation $v \geq \bar{c}(k)$ then $k > 0$. Suppose $k = 0$ then $\lim_{k\to 0} \bar{c}(k) = \infty$ then all *constrained* students have valuation $v < \bar{c}(k)$. Contradiction. If there are *constrained* students with valuation $v \leq \underline{c}(k)$ then $k > 0$. Suppose $k = 0$, then all *constrained* students' valuations must be above $c(0)$ which is greater than $\underline{c}(0)$, which is a contradiction. □

Proof of Remark 2.5

Proof. $k > \frac{q_1}{q_1+q_2} - \eta$ implies

$$\begin{aligned}
&\frac{1}{k+\eta} < 1 + \frac{q_2}{q_1} \\
\Longrightarrow & -\frac{q_2}{q_1} < \frac{\eta + k - 1}{k+\eta} \\
\Longrightarrow & \frac{1-\eta-k}{k+\eta}\frac{q_1}{q_2} < 1 \\
\Longrightarrow & \frac{(1-\eta-k)(1-\eta)}{k+\eta}\frac{q_1}{q_2} < 1 \\
\Longrightarrow & \underline{c}(k) < 1
\end{aligned}$$

Next, note that $\underline{c}(.)$ is decreasing, so it will be lowest at $k = 1 - \eta - q_2$:

$$\underline{c}(1-\eta-q_2) = \frac{(1-\eta)q_1}{(1-q_2)} > 0$$

$\square$

Proof of Remark 2.6

Proof. $\bar{c}(k) < 1$ at $k = 1 - \eta - q_2$ iff

$$q_1(1-\eta)(1-\eta-k) < q_2 k \text{ at } k = 1-\eta-q_2$$
$$\iff q_2 < (1-\eta)(1-q_1)$$

$\square$

Proof of Remark 2.7

Proof. Let k be the measure of constraiend students reporting s_1 as top choice in the symmetric equilibrium of BM. By Claim 2.4, there are at least q_2 measure of constrained students reporting s_2 as top choice. Let's take any such type and call it type v. Type v must be better off in the symmetric equilibrium of BM compared to DA. By Lemma 2.1 we know that such type v students will be better off in BM equilibrium compared to DA if and only if $v \geq \frac{q_1(1-\eta)(1-\eta-k)}{q_2 k}$. So first we must not have

$$\frac{q_1(1-\eta)(1-\eta-k)}{q_2 k} \geq 1 \text{ for } k = 1-\eta-q_2$$

since $\frac{q_1(1-\eta)(1-\eta-k)}{q_2 k}$ is decreasing. So we must have

$$q_1(1-\eta)(1-\eta-k) < q_2 k \quad \text{for } k = 1-\eta-q_2$$
$$\implies 0 < (1-\eta)(1-q_1) - q_2$$

which is satisfied by A2.5. Also, since we cannot have $v \geq 1$, k must be such that $\frac{q_1(1-\eta)(1-\eta-k)}{q_2 k} < 1$, which implies

$$k > \frac{q_1(1-\eta)^2}{q_2 + q_1(1-\eta)}$$

So there must be $x > \frac{q_1(1-\eta)^2}{q_2+q_1(1-\eta)}$ measure of *constrained* students with $v \leq c(x) < c(\frac{q_1(1-\eta)^2}{q_2+q_1(1-\eta)})$ to have $k > \frac{q_1(1-\eta)^2}{q_2+q_1(1-\eta)}$. But for such students we also need to have $v < \underline{c}(x) < \underline{c}(\frac{q_1(1-\eta)^2}{q_2+q_1(1-\eta)})$ to have such students strictly better off in equilibria of BM compared to DA. Since $\underline{c}(\frac{q_1(1-\eta)^2}{q_2+q_1(1-\eta)}) < c(\frac{q_1(1-\eta)^2}{q_2+q_1(1-\eta)})$ necessary condition we have is: there exists more than $\frac{q_1(1-\eta)^2}{q_2+q_1(1-\eta)}$ measure of *constrained* students with $v < \underline{c}(\frac{q_1(1-\eta)^2}{q_2+q_1(1-\eta)}) = \frac{(1-\eta)^2 q_1}{q_1(1-\eta)+\eta q_2}$ □

Proof of Proposition 2.2

Proof. By Lemma 2.3, to guarantee the existence of a constrained student who are strictly better off in BM compared to DA, a sufficient condition will be: for any $k \in \{x \in \mathbb{R} : \max\{\frac{q_1}{q_1+q_2} - \eta, 0\} < x \leq 1 - \eta - q_2\}$ we need to have positive measure of *constrained* students whose valuation v satisfies $v \leq \underline{c}(k)$ or $v > \bar{c}(k)$. Remember that $k > \frac{q_1}{q_1+q_2} - \eta$ implies $\underline{c}(k) < 1$ by Remark 2.5. And since $\bar{c}(k) > \underline{c}(k)$ for all $0 < k \leq 1 - \eta - q_2$ our sufficient condition is having positive measure of *constrained* students farther away from each other in terms of v: there exists two different subsets of $\mathscr{V}$ each with positive measure such that for all v and v' from first and second set respectively (for discrete distribution this is equivalent to saying there exists constrained student types v and v') we have $|v - v'| \geq \max_{k \in \Gamma}\{\min\{\bar{c}(k), 1\} - \underline{c}(k)\}$ where $\Gamma = \{k \in \mathbb{R} : \max\{\frac{q_1}{q_1+q_2} - \eta, 0\} < k \leq 1 - \eta - q_2\}$.

To find the solution to

$$\max_{k \in \Gamma}\{\min\{\bar{c}(k), 1\} - \underline{c}(k)\}$$

first note that A2.5 implies $\bar{c}(k) < 1$ at $k = 1 - \eta - q_2$. Note that we cannot have k' such that $\bar{c}(k') > 1$ as solution since $\min\{\bar{c}(k'), 1\} = 1 = \min\{\bar{c}(k' + \varepsilon), 1\}$ and $\underline{c}(k' + \varepsilon) < \underline{c}(k')$ for sufficiently small ε. Also, we cannot have k such that $\bar{c}(k) < 1$ solution since $\bar{c}(k) - \underline{c}(k)$ is strictly decreasing in k for $k > 0$ (so objective can be improved by decreasing k in sufficiently small amount):

$$\bar{c}(k) - \underline{c}(k) = \frac{q_1(1-\eta)(1-\eta-k)}{q_2 k} - \frac{(1-\eta-k)(1-\eta)q_1}{(k+\eta)q_2} = \frac{q_1(1-\eta)(1-\eta-k)\eta}{(k+\eta)q_2 k}$$

So $k \in \Gamma$ such that $\bar{c}(k) = 1$ must be the solution in this case. $\bar{c}(k) = 1$ if $\frac{q_1(1-\eta)(1-\eta-k)}{q_2 k} = 1$.

Solution of which is: $\frac{q_1(1-\eta)^2}{q_2+q_1(1-\eta)}$. In this case $\underline{c}(.)$ evaluated at the solution is

$$\frac{(1-\eta)^2 q_1}{q_1(1-\eta)+\eta q_2}$$

and $\min\{\bar{c}(k),1\}=1$ at $k=\frac{q_1(1-\eta)^2}{q_2+q_1(1-\eta)}$. Therefore, solution to the objective function is:

$$\begin{aligned}&1-\frac{(1-\eta)^2 q_1}{q_1(1-\eta)+\eta q_2}\\ =&\frac{\eta(q_1(1-\eta)+q_2)}{q_1(1-\eta)+\eta q_2}\end{aligned}$$

Now, it remains to show k cannot be 0 when we have $|v-v'|\geq\frac{\eta(q_1(1-\eta)+q_2)}{q_1(1-\eta)+\eta q_2}$ for all v and v' from two different subsets of $\mathscr{V}$ each with positive measure.

Note that $1-\underline{c}(0)<\frac{\eta(q_1(1-\eta)+q_2)}{q_1(1-\eta)+\eta q_2}$ which is true since we have $\frac{\eta(q_1(1-\eta)+q_2)}{q_1(1-\eta)+\eta q_2}=1-\underline{c}(\frac{q_1(1-\eta)^2}{q_2+q_1(1-\eta)})$ and $\frac{q_1(1-\eta)^2}{q_2+q_1(1-\eta)}>0$, so there must be positive measure of *constrained* students with valuation below $\underline{c}(0)$, which means there is *constrained* student reporting s_1 as top choice since $\underline{c}(0)<c(0)$. □

Proof of Lemma 2.4

Proof. Let k be the measure of *constrained* students reporting s_1 as top choice. Again, Lemma 2.1 we have $k\in[\max\{\frac{q_1}{q_1+q_2}-\eta,0\},1-\eta-q_2]\setminus\{\frac{q_1}{q_1+q_2}-\eta\}$. By Lemma 2.3 we know that for given $k>0$, type v *constrained* student is weakly better off in this equilibrium of BM compared to DA if $v\leq\underline{c}(k)=\frac{(1-\eta-k)(1-\eta)q_1}{(k+\eta)q_2}$ (strictly better off if inequality is strict). Since $\underline{c}(.)$ is strictly decreasing in its argument, type v will satisfy $v\leq\underline{c}(k)$ for any $k\in(\max\{\frac{q_1}{q_1+q_2}-\eta,0\},1-\eta-q_2]$, if it satisfies $v\leq\underline{c}(1-\eta-q_2)=\frac{(1-\eta)q_1}{(1-q_2)}$. It is easy to see that when there is positive measure of constrained students types v with $v\leq\frac{(1-\eta)q_1}{(1-q_2)}$, we cannot have $k=0$ since $\frac{(1-\eta)q_1}{(1-q_2)}<\underline{c}(0)<c(0)$ implies that there will be positive measure of *constrained* students reporting s_1 as top choice. □

Proof of Lemma 2.5

Proof. For the case when $k \in [\max\{\frac{q_1}{q_1+q_2} - \eta, 0\}, 1-\eta-q_2] \setminus \frac{q_1}{q_1+q_2} - \eta$ the assertion follows from Lemma 2.4.

Suppose $k > 1-\eta-q_2$, in that case note that a *constrained* student of type v reports s_1 as top choice if:

$$\frac{q_1}{k+\eta} + v\frac{q_2-(1-\eta-k)}{k} > v$$
$$\iff v < \frac{k}{k+\eta}\frac{q_1}{1-\eta-q_2}$$

First, note that type v reports s_1 as top choice since:

$$v \leq \frac{(1-\eta)q_1}{(1-q_2)} < \frac{q_1}{1-q_2} < \frac{k}{k+\eta}\frac{q_1}{1-\eta-q_2}$$

for any $1-\eta-q_2 < k < 1-\eta$ where the last inequality above follows since last term is increasing in k and $k > 1-\eta-q_2$.

If $k = 1-\eta$, then symmetric BM equilibrium and DA gives the same payoffs to everyone.

For $k < 1-\eta$, comparing BM and DA payoff, we need to have:

$$\frac{q_1}{k+\eta} + v\frac{q_2-(1-\eta-k)}{k} > q_1 + v\frac{q_2}{1-\eta}$$
$$\iff v < \frac{k}{k+\eta}\frac{q_1(1-\eta)}{(1-\eta-q_2)}$$

And note that this is satisfied since

$$v \leq \frac{(1-\eta)q_1}{(1-q_2)} < \frac{k}{k+\eta}\frac{q_1(1-\eta)}{(1-\eta-q_2)}$$

where the last inequality follows since the last term is increasing in k, and $k > 1-\eta-q_2$ $\square$

Proof of Claim 2.5

Proof. If there are at least q_2 measure of *constrained* students with valuation v that satisfy $v \geq \frac{q_1}{1-q_2}$, then by Claim 2.3 , a symmetric equilibrium exists. If not, then there is not an equilibrium in which at least q_2 measure of *constrained* students report s_2 as top choice. In that case for given equilibrium measure of *constrained* students reporting s_1 as top choice k, cutoff that students with valuation above (below) which reports s_2 (s_1) as top choice becomes:

$$\tilde{c}(k) := \frac{k}{k+\eta}\frac{q_1}{1-\eta-q_2}$$

which can be derived by comparing payoffs $\frac{q_1}{k+\eta} + v\frac{q_2-(1-\eta-k)}{k}$ and v from reporting s_1 as top choice and s_2 as top choice respectively. Note that $\tilde{c}(1-\eta-q_2) = \frac{q_1}{1-q_2}$. Note that there is more than $1-\eta-q_2$ measure of *constrained* students strictly below $\frac{q_1}{1-q_2}$. If for some $x > 1-\eta-q_2$, $k = x$ then there must be $x' \geq x$ measure of *constrained* students that has valuation $v \leq \tilde{c}(x)$ and $x'' \leq x$ measure of *constrained* students with valuation $v < \tilde{c}(x)$. Suppose for contradiction there is no such x. Then it may be the case that for all $x > 1-\eta-q_2$ there are more than x measure of *constrained* students with valuation $v < \tilde{c}(x)$, this is not possible since there can be at most $1-\eta$ *constrained* students strictly below $\tilde{c}(x)$ at $x = 1-\eta$. Or it may be the case that for any $x > 1-\eta-q_2$ there are strictly less than x measure of *constrained* students with valuation $v \leq \tilde{c}(x)$. Note that measure of *constrained* students with $v \leq \tilde{c}(x)$ is weakly decreasing and x is strictly decreasing as x approaches $1-\eta-q_2$. In this case we cannot have strictly more than x measure of *constrained* studens with valuation $v < \tilde{c}(x)$ at $x = 1-\eta-q_2$ which is a contradiction. □

Proof of Theorem 2.1

Proof. Since distribution of preferences for *constrained* students have full support, there is positive measure of *constrained* students with valuation v that satisfy $v < \frac{(1-\eta)q_1}{(1-q_2)}$. By Lemma 2.5, these students are weakly better off in any (symmetric) equilibrium of BM compared to DA. To show that they are strictly better off, I need to show that not everyone reports s_1 as top choice in any equilibrium of BM. To have such equilibrium everyone must have valuation below $\frac{k}{k+\eta}\frac{q_1}{1-\eta-q_2}$ evaluated at $k = 1-\eta$, which is:

$$\frac{q_1(1-\eta)}{1-\eta-q_2}$$

and this is guaranteed to be less than 1 by A2.5. So there is positive measure of *constrained* students with valuation $v > \frac{q_1(1-\eta)}{1-\eta-q_2}$ under full support assumption. So there cannot be an equilibrium in which all *constrained* students report s_1 as top choice. $\square$

Proof of Lemma 2.6

Proof. Suppose $\frac{q_1(1-\eta)}{(1-\eta-q_2)(1-q_2)} \leq 1$. By Claim 2.3, equilibrium in which there are at least q_2 measure of *constrained* sudents reporting s_2 as top choice exists iff there are at least q_2 measure of *constrained* students with $v \geq \frac{q_1}{1-q_2}$. Under uniform distribution assumption, measure of *constrained* students with $v \geq \frac{q_1}{1-q_2}$ is $\left(1-\frac{q_1}{1-q_2}\right)(1-\eta)$. This is greater than or equal to q_2 since

$$\begin{aligned}
&\left(1-\frac{q_1}{1-q_2}\right)(1-\eta) \geq q_2 \\
\iff &\frac{q_1(1-\eta)}{(1-\eta-q_2)(1-q_2)} \leq 1
\end{aligned}$$

Finally remember from Lemma 2.1 that such an equilibrium is unique.

Now, I will show an equilibrium in which less than q_2 measure of *constrained* students report s_2 as top choice exists iff $\frac{q_1(1-\eta)}{(1-\eta-q_2)(1-q_2)} > 1$. Let $k > 1-\eta-q_2$ be equilibrium measure of *constrained* students reporting s_1 as top choice. If a *constrained* student reports s_1 as top choice in this case he must have

$$\frac{q_1}{k+\eta} + v\frac{q_2-(1-\eta-k)}{k} \geq v$$

which is equivalent to

$$v \leq \frac{k}{k+\eta}\frac{q_1}{1-\eta-q_2}$$

Then the share of *constrained* students reporting s_1 as top choice is

$$F(\frac{k}{k+\eta}\frac{q_1}{1-\eta-q_2})$$

which must equal to $\frac{k}{1-\eta}$. Thus solution (note that it is unique since $k > 1-\eta-q_2$) of equality below gives the equilibrium k.

$$\begin{aligned}
\frac{k}{k+\eta}\frac{q_1}{1-\eta-q_2} &= \frac{k}{1-\eta} \\
\Longrightarrow k &= \frac{q_1(1-\eta)}{1-\eta-q_2} - \eta
\end{aligned}$$

Note that $k = \frac{q_1(1-\eta)}{1-\eta-q_2} - \eta < 1-\eta$ by A2.5. So $\frac{q_1(1-\eta)}{1-\eta-q_2} - \eta$ is the equilibrium measure of *constrained* students who reports s_1 as top choice if and only if $\frac{q_1(1-\eta)}{1-\eta-q_2} - \eta > 1-\eta-q_2$ which can shown to be equivalent to

$$\frac{q_1(1-\eta)}{(1-\eta-q_2)(1-q_2)} > 1$$

□

Proof of Lemma 2.7

Proof. By Lemma 2.6, when $\frac{q_1(1-\eta)}{(1-\eta-q_2)(1-q_2)} \leq 1$ there are at least q_2 measure of *constrained* students reporting s_2 as top choice. Note that equilibrium measure $k(q_1,q_2,\eta)$ of *constrained* students reporting s_1 as top choice must satisfy

$$F(c(k(q_1,q_2,\eta))) = \frac{k(q_1,q_2,\eta)}{1-\eta}$$

This means

$$
\begin{aligned}
c(k(q_1,q_2,\eta)) &= \frac{k(q_1,q_2,\eta)}{1-\eta} \\
\iff \frac{q_1}{q_2}\frac{1-\eta-k(q_1,q_2,\eta)}{k(q_1,q_2,\eta)+\eta} &= \frac{k(q_1,q_2,\eta)}{1-\eta}
\end{aligned}
$$

Solving the last equality for $k(q_1,q_2,\eta)$ and taking the larger root (smaller root is less than 0) one can get $k(q_1,q_2,\eta)$ as given in the proposition. Note that $k(q_1,q_2,\eta) \leq 1-\eta-q_2$ in this case. Remember from Claim 2.2 that $c(.)$ evaluated at $k(q_1,q_2,\eta)$ gives the threshold that *constrained* students below which report s_1 as top choice and *constrained* students above which report s_2 as top choice. So threshold is:

$$
\begin{aligned}
c(k(q_1,q_2,\eta)) &= \frac{q_1}{q_2}\frac{1-\eta-k(q_1,q_2,\eta)}{k(q_1,q_2,\eta)+\eta} \\
&= \frac{k(q_1,q_2,\eta)}{1-\eta}
\end{aligned}
$$

where the last line is by definition of the equilibrium.

Remember from Lemma 2.3 that $\bar{c}(.)$ and $\underline{c}(.)$ evaluated at $k(q_1,q_2,\eta)$ gives the bounds such that *constrained* students with valuation v such that $\underline{c}(k(q_1,q_2,\eta)) < v < \bar{c}(k(q_1,q_2,\eta))$ are strictly worse off under BM equilibrium compared to DA and students who has $v < \underline{c}(k(q_1,q_2,\eta))$ or $v > \bar{c}(k(q_1,q_2,\eta))$ are strictly better off under BM equilibrium compared to DA. Let's derive these bounds for the equilibrium $k(q_1,q_2,\eta)$:

$$
\begin{aligned}
\bar{c}(k(q_1,q_2,\eta)) &= \frac{q_1(1-\eta)(1-\eta-k(q_1,q_2,\eta))}{q_2 k(q_1,q_2,\eta)} \\
&= \frac{q_1(1-\eta)(1-\eta-k)}{q_2 k}\frac{k+\eta}{k+\eta} \\
&= c(k)\frac{(1-\eta)(k+\eta)}{k} \\
&= \frac{k}{1-\eta}\frac{(1-\eta)(k+\eta)}{k} \\
&= k(q_1,q_2,\eta)+\eta
\end{aligned}
$$

Note that $\bar{c}(k(q_1,q_2,\eta)) < 1$ since $k(q_1,q_2,\eta) \leq 1-\eta-q_2$ in this case. And

$$\begin{aligned}
\underline{c}(k(q_1,q_2,\eta)) &= \frac{(1-\eta-k(q_1,q_2,\eta))(1-\eta)q_1}{(k(q_1,q_2,\eta)+\eta)q_2} \\
&= \frac{q_1(1-\eta-k)}{(k+\eta)q_2}(1-\eta) \\
&= c(k)(1-\eta) \\
&= \frac{k}{1-\eta}(1-\eta) \\
&= k(q_1,q_2,\eta)
\end{aligned}$$

By Lemma 2.6, when $\frac{q_1(1-\eta)}{(1-\eta-q_2)(1-q_2)} > 1$ there are less than q_2 measure of *constrained* students reporting s_2 as top choice. From proof of Lemma 2.6, $k = \frac{q_1(1-\eta)}{1-\eta-q_2} - \eta$ in this case (it is less than $1-\eta$ by A2.5). Note that for given k, a *constrained* student with valuation v reports s_1 as top choice if

$$\begin{aligned}
&\frac{q_1}{k+\eta} + v\frac{q_2-(1-\eta-k)}{k} > v \\
\iff &v < \frac{k}{k+\eta}\frac{q_1}{1-\eta-q_2}
\end{aligned}$$

evaluating this threshold $\frac{k}{k+\eta}\frac{q_1}{1-\eta-q_2}$ at $k(q_1,q_2,\eta)$:

$$\frac{k(q_1,q_2,\eta)}{k(q_1,q_2,\eta)+\eta}\frac{q_1}{1-\eta-q_2} = \frac{q_1}{1-\eta-q_2} - \frac{\eta}{1-\eta}$$

similarly it can be shown that a *constrained* student with valuation v reports s_2 as top choice if $v > \frac{q_1}{1-\eta-q_2} - \frac{\eta}{1-\eta}$.

A *constrained* student reporting s_1 as top choice in an equilibrium with less than q_2 *constrained* students reporting s_2 as top choice is strictly better off in BM compared to DA iff:

$$\begin{aligned}&\frac{q_1}{k+\eta}+v\frac{q_2-(1-\eta-k)}{k}>q_1+\frac{q_2}{1-\eta}v\\ \iff & v<\frac{k}{k+\eta}\frac{q_1(1-\eta)}{(1-\eta-q_2)}\end{aligned}$$

and note that $\frac{k(q_1,q_2,\eta)}{k(q_1,q_2,\eta)+\eta}\frac{q_1(1-\eta)}{(1-\eta-q_2)} = \frac{q_1(1-\eta)}{1-\eta-q_2} - \eta$. Also note that $\frac{k(q_1,q_2,\eta)}{k(q_1,q_2,\eta)+\eta}\frac{q_1(1-\eta)}{(1-\eta-q_2)} < \frac{k(q_1,q_2,\eta)}{k(q_1,q_2,\eta)+\eta}\frac{q_1}{1-\eta-q_2}$ for all $k>0$. So *constrained* student with $v<\frac{q_1(1-\eta)}{1-\eta-q_2}-\eta$ reports s_1 as top choice. Note that $\frac{q_1(1-\eta)}{1-\eta-q_2}-\eta>0$ since $\frac{k}{k+\eta}\frac{q_1(1-\eta)}{(1-\eta-q_2)}$ is increasing in k, so it is smallest when $k=1-\eta-q_2$ which yields $\frac{q_1(1-\eta)}{1-q_2}>0$

A *constrained* student reporting s_2 as top choice in an equilibrium with less than q_2 *constrained* students reporting s_2 as top choice is strictly better off in BM compared to DA iff:

$$\begin{aligned}&v>q_1+\frac{q_2}{1-\eta}v\\ \iff & v>\frac{q_1(1-\eta)}{(1-\eta-q_2)}\end{aligned}$$

Also note that $\frac{q_1(1-\eta)}{(1-\eta-q_2)} > \frac{k(q_1,q_2,\eta)}{k(q_1,q_2,\eta)+\eta}\frac{q_1}{1-\eta-q_2}$ since $k(q_1,q_2,\eta)<1-\eta$. So *constrained* students with $v>\frac{q_1(1-\eta)}{(1-\eta-q_2)}$ report s_2 as top choice. Note that $\frac{q_1(1-\eta)}{1-\eta-q_2}<1$ by A2.5. This finishes the proof. $\square$

Proof of Theorem 2.2

Proof. Suppose $\frac{q_1(1-\eta)}{(1-\eta-q_2)(1-q_2)}\leq 1$, $\bar{c}(k(q_1,q_2,\eta))-\underline{c}(k(q_1,q_2,\eta))=(k(q_1,q_2,\eta)+\eta)-k(q_1,q_2,\eta)=\eta$. By A2.7, η is the fraction of *constrained* students who does not satisfy $v>\bar{c}(k(q_1,q_2,\eta))$ or $v<\underline{c}(k(q_1,q_2,\eta))$ and $1-\eta$ is the fraction that satisfies one of these inequalities.

Suppose $\frac{q_1(1-\eta)}{(1-\eta-q_2)(1-q_2)}>1$. By previous lemma, *Constrained* students are strictly better off under BM compared to DA iff $v>\frac{q_1(1-\eta)}{(1-\eta-q_2)}$ or $v<\frac{q_1(1-\eta)}{1-\eta-q_2}-\eta$. By A2.7 there are $\frac{q_1(1-\eta)}{(1-\eta-q_2)}-(\frac{q_1(1-\eta)}{1-\eta-q_2}-\eta)=\eta$ fraction of *constrained* students does not satisfy these inequalities. $\square$

B.3 Proofs of Section 4

Proof of Lemma 2.8

Proof. Suppose first that $1-\eta \leq q_1$, then trivially all *constrained* students enter s_1 with probability one in $\tilde{\sigma}$. Also, note that no *constrained* student can enter s_1 with probability one in σ^*. Otherwise, a student who did not report s_1 as top choice with probability one (note that it exists since otherwise it is impossible to get into s_1 with probability one) can deviate to ranking it as top with probability one. Thus, there exist a constrained student better off under $\tilde{\sigma}$ compared to σ^*.

Suppose now that, $1-q_1 > \eta \geq 1-q_1-q_2$. Let $x_j, \tilde{x}_j$ denote the measure of students reporting s_j as top choice in σ^* and $\tilde{\sigma}$ respectively.

In $\tilde{\sigma}$, it must be $\tilde{x}_1 > q_1$ since otherwise a *constrained* student who is not doing already can deviate to reporting s_1 as top choice and get payoff 1. Then $\tilde{x}_2 < q_2$ since otherwise $\tilde{x}_1+\tilde{x}_2 = 1-\eta > q_1+q_2$ which is a contradiction. Note that $\tilde{x}_1 - q_1 \leq q_2 - \tilde{x}_2$ since $\tilde{x}_1+\tilde{x}_2 \leq q_1+q_2$. Thus, all *constrained* students must be reporting s_1 as top choice with probability one, i.e. $\tilde{x}_1 = 1$ and $\tilde{x}_2 = 0$. Otherwise a *constrained* student gets v and deviation would give:

$$\frac{q_1}{\tilde{x}_1} + \left(1 - \frac{q_1}{\tilde{x}_1}\right)v$$

which is greater than v. Thus, in $\tilde{\sigma}$ all *constrained* students get

$$\frac{q_1}{1-\eta} + \left(1 - \frac{q_1}{1-\eta}\right)v$$

Note that this is greater than v.

Case 1: $x_2 \geq q_2$

If there is *constrained* student who increase the probability of reporting s_1 as top choice in $\tilde{\sigma}$ compared to σ^* she must be getting $v\frac{q_2}{x_2}$ in σ^* which is less than v and hence she is strictly better off under $\tilde{\sigma}$.

If there is no *constrained* student who increase the probability of reporting s_1 as top choice in $\tilde{\sigma}$

compared to σ^* then all *constrained* students must be reporting s_1 as top choice in σ^*, that means $x_1 \geq 1-\eta$ and they must be getting $\frac{q_1}{x_1}$ in σ^* which is less than $\frac{q_1}{1-\eta}$ so they are strictly better off under $\tilde{\sigma}$.

Case 2: $x_2 < q_2$

If there is *constrained* student who increase the probability of reporting s_1 as top choice in $\tilde{\sigma}$ compared to σ^* she must be getting v in σ^* which is less than his payoff under $\tilde{\sigma}$ and hence she is strictly better off under $\tilde{\sigma}$.

If there is no *constrained* student who increase the probability of reporting s_1 as top choice in $\tilde{\sigma}$ compared to σ^* then all *constrained* students must be reporting s_1 as top choice in σ^*, that means $x_1 \geq 1-\eta$ and they must be getting

$$\frac{q_1}{x_1} + \frac{q_2 - x_2}{x_1} v$$

It can be shown that

$$\frac{q_1}{1-\eta} + \left(1 - \frac{q_1}{1-\eta}\right) v > \frac{q_1}{x_1} + \frac{q_2 - x_2}{x_1} v$$

If $x_1 = 1-\eta$ then this is true since

$$1 - \frac{q_1}{1-\eta} > \frac{q_2 - \eta}{1-\eta}$$
$$\iff 1 > q_1 + q_2$$

Now, suppose that $x_1 > 1-\eta$. Note that what I want to show is true iff

$$q_1 x_1 - q_1(1-\eta) > v((1-\eta)q_2 - (1-\eta) + q_1 x_1)$$

If $(1-\eta)q_2 - (1-\eta) + q_1 x_1 < 0$ we are done. If not, then last equation will be true iff:

$$v < \frac{q_1 x_1 - q_1(1-\eta)}{(1-\eta)(q_2 - 1) + q_1 x_1}$$

and right hand side is greater than 1 since $q_2 - 1 < -q_1$. So we are done.

$\square$

Proof of Claim 2.6

Proof. Let k denote the measure of students who report s_1 as top choice. Cutoff for reporting s_1 versus s_2 as top choice is

$$\hat{c}(k) := \frac{q_1}{q_2}\frac{1-k}{k}$$

$(\implies)$: Suppose there is less than q_2 measure students for whom $v \geq \frac{q_1}{1-q_2}$. I want to show that there cannot be an equilibrium with $k \leq 1-q_2$. For $k = 1-q_2$ to be equilibrium we need q_2 measure of students who has $v \geq \hat{c}(1-q_2) = \frac{q_1}{1-q_2}$ which cannot happen. Note that there cannot be an equilibrium with $k < 1-q_2$, either. To have such an equilibrium, we need to have $1-k$ measure of students for whom $v \geq \hat{c}(k)$. Note that $1-k > q_2$ and $\hat{c}(k) > \frac{q_1}{1-q_2}$ since $k < 1-q_2$ and $\hat{c}(.)$ is strictly decreasing. So, such equilibrium more than q_2 measure of *constrained* students with $v > \frac{q_1}{1-q_2}$ which is not possible.

$(\impliedby)$: If there are q_2 measure of students for whom $v \geq \frac{q_1}{1-q_2}$, then there is an equilibrium in which students with $v \geq \frac{q_1}{1-q_2}$ report s_2 as top choice and other students report s_1 as top choice, so in this equilibrium we have $k = 1-q_2$.

Suppose there are more than q_2 measure of students with $v \geq \frac{q_1}{1-q_2}$. If measure of students with $v > \frac{q_1}{1-q_2}$ is equal to q_2 then we have an equilibrium in which students with $v > \frac{q_1}{1-q_2}$ report s_2 as top choice and other students report s_1 as top choice. If measure of students with $v > \frac{q_1}{1-q_2}$ is less than q_2 then we have an equilibrium in which q_2 measure of students with $v \geq \frac{q_1}{1-q_2}$ report s_2 as top choice and other students report s_1 as top choice. So suppose measure of students with $v > \frac{q_1}{1-q_2}$ is larger than q_2. That means there are less than $1-q_2$ measure of students with $v \leq \frac{q_1}{1-q_2} = \hat{c}(1-q_2)$. So we have less than x students for whom $v \leq \hat{c}(x)$ when $x = 1-q_2$.

Note that $\lim_{k\to 0}\hat{c}(k) = \infty$, that means we have more than x students that has $v < \hat{c}(x)$ when $x \to 0$ (all students to be precise). Note that $c(.)$ is a decreasing continuous function, so there must be $x > 0$ such that there are $x' \geq x$ students with $v \leq \hat{c}(x)$ and $x'' \leq x$ students with $v < \hat{c}(x)$ by the last sentence of the last paragraph. Therefore, we have an equilibrium in which students with $v < \hat{c}(x)$ report s_1 as top choice and $x - x''$ of students with $v = \hat{c}(x)$ report s_1 as top choice. □

Proof of Claim 2.7

Proof. Let k denote the total measure of students reporting s_1 as top choice. Suppose, there are less than q_2 students who report s_2 as top choice a symmetric equilibrium of BM. Since there are at least q_2 measure of *constrained* students with $v > \frac{(1-\eta)q_1}{1-\eta-q_2}$, that means some of these students report s_1 as top choice in the equilibrium. Then they get:

$$\frac{q_1}{k} + v\frac{q_2 - (1-k)}{k}$$

This is smaller than v since:

$$\begin{aligned} &\frac{q_1}{k} + v\frac{q_2 - (1-k)}{k} < v \\ \iff &\frac{q_1}{k} < v\frac{k - q_2 + (1-k)}{k} \\ \iff &v > \frac{q_1}{1-q_2} \end{aligned}$$

and we know that $\frac{(1-\eta)q_1}{1-\eta-q_2} > \frac{q_1}{1-q_2}$ since this can be shown to be equivalent to $q_2\eta > 0$. □

Proof of Claim 2.8

Proof. Let k denote the measure of students reporting s_1 as top choice in an equilibrium without private schools. Under A2.1-A2.6, by previous claim $1 - k \geq q_2$. In this case the threshold such that students with value strictly below which strictly prefer reporting s_1 as top choice, students with value above which strictly prefer reporting s_2 as top choice and students with value equal to which is indifferent between reporting s_1 and s_2 as top choice is

$$\frac{q_1}{q_2}\frac{1-k}{k}$$

Note that threshold is strictly decreasing in k. Suppose there are two different equilibria σ^* and $\tilde{\sigma}$ and measure of students reporting s_1 as top choice is k and k' in those equilibria respectively. Without loss of generality assume $k > k'$. Since $k > k'$ then $\frac{q_1}{q_2}\frac{1-k}{k} < \frac{q_1}{q_2}\frac{1-k'}{k'}$. So that means there are weakly less students reporting s_1 as top choice in σ^*, but this is not possible since $k > k'$.

Note that $k \leq 1-q_2$ by Claim 2.7. Also, there must be at least q_1 students reporting s_1 as top choice since otherwise a student who is not doing already can deviate to reporting s_1 as top choice and get into s_1 for sure.

Note that if $\frac{q_1}{q_2}\frac{1-k}{k} \geq 1$, then $k = 1$ but this is not possible since $\frac{q_1}{q_2}\frac{1-k}{k}$ equals 0 at $k = 1$. So it must be the case that $\frac{q_1}{q_2}\frac{1-k}{k} < 1 \implies k > \frac{q_1}{q_1+q_2}$. Note that this is possible since $\frac{q_1}{q_1+q_2} < 1-q_2 \iff q_2(q_1+q_2-1) < 0$ which is known to be true. □

Proof of Proposition 2.4

Proof. By A2.1-A2.6 in $\tilde{\sigma}$ and σ^* there are at least q_2 measure of students reporting s_2 as top choice. Let $x_j, \tilde{x}_j$ denote the measure of students reporting s_j as top choice in σ^* and $\tilde{\sigma}$ respectively. Note that we must have $x_1 + x_2 = \tilde{x}_1 + \tilde{x}_2 = 1$.

If σ^* is an equilibrium in which all *unconstrained* students report s_1 as top choice, from Lemma 2.10 , in the case of $x_2 \geq q_2$ and $\tilde{x}_2 \geq q_2$ there are *constrained* students who are strictly better off under $\tilde{\sigma}$ compared to σ^* or all *constrained* students get the same payoff in σ^* and $\tilde{\sigma}$.

So suppose in σ^* at least some *unconstrained* students report s_2 as top choice with positive probability.

Case 1: $\tilde{x}_1 < x_1, \tilde{x}_2 > x_2 \geq q_2$

Since *unconstrained* students who do not report s_1 as top choice with probability one in σ^*, report it as top choice with probability one in $\tilde{\sigma}$, we must have a *constrained* student decrease the probability of reporting s_1 as top choice in $\tilde{\sigma}$ compared to σ^* in this case. Such student must have:

$$\frac{q_1}{x_1} \geq v\frac{q_2}{x_2}$$
$$\implies v \leq \frac{x_2}{x_1}\frac{q_1}{q_2}$$

and

$$\frac{q_1}{\tilde{x}_1} \leq v\frac{q_2}{\tilde{x}_2}$$
$$\implies v \geq \frac{\tilde{x}_2}{\tilde{x}_1}\frac{q_1}{q_2} > \frac{x_2}{x_1}\frac{q_1}{q_2}$$

which is not possible. So this case is not possible.

Case 2: $\tilde{x}_1 = x_1, \tilde{x}_2 = x_2 \geq q_2$

Some *constrained* students must have decreased the probability of reporting s_1 as top choice since there are *unconstrained* students who increased the probability of reporting s_1 as top choice. By the same arguments in the case above, such *constrained* students must have

$$v = \frac{x_2}{x_1}\frac{q_1}{q_2} = \frac{\tilde{x}_2}{\tilde{x}_1}\frac{q_1}{q_2}$$

hence indifferent between reporting s_1 as top choice and s_2 as top choice in both equilibria. So in both of them they get the same payoff: $\frac{q_1}{x_1}$.

If there are *constrained* students who did not change the probability of reporting s_1 as top choice they are either getting $\frac{q_1}{x_1}$ and $\frac{q_1}{\tilde{x}_1}$ in σ^* and $\tilde{\sigma}$ respectively; or $v\frac{q_2}{x_2}$ and $v\frac{q_2}{\tilde{x}_2}$ in in σ^* and $\tilde{\sigma}$ respectively. So their payoffs does not change.

If there are *constrained* students who increased the probability of reporting s_1 as top choice in

$\tilde{\sigma}$ compared to σ^*, then they must have:

$$\frac{q_1}{x_1} \leq v\frac{q_2}{x_2}$$
$$\implies v \geq \frac{x_2}{x_1}\frac{q_1}{q_2}$$

and

$$\frac{q_1}{\tilde{x}_1} \geq v\frac{q_2}{\tilde{x}_2}$$
$$\implies v \leq \frac{\tilde{x}_2}{\tilde{x}_1}\frac{q_1}{q_2} = \frac{x_2}{x_1}\frac{q_1}{q_2}$$

so they must have $v = \frac{x_2}{x_1}\frac{q_1}{q_2} = \frac{\tilde{x}_2}{\tilde{x}_1}\frac{q_1}{q_2}$ and be indifferent between reporting s_1 as top choice and s_2 as top choice in both equilibria. So they should be getting the same payoff $\frac{q_1}{x_1}$ in both equilibria.

Case 3: $\tilde{x}_1 > x_1, x_2 > \tilde{x}_2 \geq q_2$

Note that in this case if there are *constrained* students for whom reporting s_1 as top choice is weakly preferred to reporting s_2 as top choice in σ^*, they are better off in $\tilde{\sigma}$. To see this note that such students get payoff of

$$v\frac{q_2}{x_2}$$

in $\tilde{\sigma}$ and they get payoff of

$$v\frac{q_2}{\tilde{x}_2}$$

since $v\frac{q_2}{x_2} \geq \frac{q_1}{x_1} \implies v \geq \frac{q_1}{q_2}\frac{x_2}{x_1} \implies v \geq \frac{q_1}{q_2}\frac{\tilde{x}_2}{\tilde{x}_1} \implies v\frac{q_2}{\tilde{x}_2} \geq \frac{q_1}{\tilde{x}_1}$.

So guaranteeing existence of *constrained* students with $v\frac{q_2}{x_2} \geq \frac{q_1}{x_1}$ is enough. We know that in σ^* there are at least q_2 measure of students reporting s_2 as top choice. So if $\eta < q_2$, some of these students must be *constrained.* Hence, there are *constrained* students with valuation vector v such that $v\frac{q_2}{x_2} \geq \frac{q_1}{x_1}$.

Suppose $\eta \geq q_2$ and suppose there are more than $\eta - q_2$ measure of *unconstrained* students with $v < \frac{q_1}{1-q_2}$. Remember for given x_1 and x_2 in an equilibrium without private schools a student with valuation v reports s_1 as top choice if $v < \frac{x_2}{x_1}\frac{q_1}{q_2}$. Right hand side decreases in x_1, largest x_1 the equilibrium can have is $1-q_2$. So we have $\frac{q_1}{1-q_2} < \frac{x_2}{x_1}\frac{q_1}{q_2}$. Therefore more than $\eta - q_2$ measure of *unconstrained* students report s_1 as top choice. Thus, there must be positive measure of *constrained* students that reports s_2 as top choice since there are at least q_2 measure of students reporting s_2 as top choice. Hence, there are *constrained* students with valuation vector v such that $v\frac{q_2}{x_2} \geq \frac{q_1}{x_1}$.

Suppose (1) and (2) does not hold. Suppose for contradiction in σ^* there is no *constrained* student with $v\frac{q_2}{x_2} \geq \frac{q_1}{x_1}$. So all students with valuation vector v such that $v\frac{q_2}{x_2} \geq \frac{q_1}{x_1}$ are *unconstrained.* Note that by assumption all *constrained* students have

$$\frac{q_1}{x_1} > v\frac{q_2}{x_2}$$
$$\implies v < \frac{x_2}{x_1}\frac{q_1}{q_2}$$

and note that right hand side is largest when $x_2 = \eta$ since no *constrained* student reports s_2 as top choice. So for all *constrained* students we must have:

$$v < \frac{x_2}{x_1}\frac{q_1}{q_2} \leq \frac{\eta}{1-\eta}\frac{q_1}{q_2}$$

which contradicts (3). This finishes the proof. $\square$