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CONNECTIONS BETWEEN EULER CHARACTERISTIC INVARIANTS, H-VECTOR PROBLEMS, AND ALGEBRO-GEOMETRIC PROPERTIES OF (POLY)MATROIDS

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To my parents

“Know thyself...”

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ABSTRACT

We explore combinatorial questions using tools from algebraic geometry/topology (or the converse). The first direction we start with involves combinatorial constructions approximating and characterizing properties of varieties. More specifically, we started out with (approximate) relations in the Grothendieck ring of varieties. This involved an arithmetic statistics-type result showing that the Fano variety of k-planes contained in a given variety are determined mostly by symmetric products of points in the initial variety. This involves using a motivic limit/approximate relation of Galkin–Shinder in the Grothendieck ring of varieties. Moving to *exact* relations, we showed that the original relation of Galkin–Shinder can be used to characterize cubic hypersurfaces using a projective geometry construction (and intersections of two quartic or quadric hypesurfaces after weakening assumptions). Exact relations in this ring also gave a transition to combinatorial invariants.

Expressions in the Grothendieck ring of varieties for configuration spaces of points led to our transition to combinatorial problems. More specifically, certain generalizations of chromatic polynomials are uniquely defined (up to normalization) by Cooper-de Silva-Sazdanovic. We showed that these can be expressed using h-vectors of certain simplicial complexes (under certain conditions). Note that **any** h-vector appears in such a construction. In addition, we show that there are no nontrivial bounds on ranks of proper flats that cover the underlying set of a matroid satisfying the matroidal analogue of the Cayley-Bacharach property. This gives a negative answer to a question in recent work of Levinson and Ullery. Finally, we also explore connections to combinatorial invariance and Chow rings of matroids.

CHAPTER 1

INTRODUCTION

In this thesis, we explore connections between the algebraic geometry/topology of varieties and combinatorics. We mainly work with (universal) Euler characteristic-like invariants and combinatorial analogues of questions about independence conditions of points on spaces of hypersurfaces. Using a relation in the Grothendieck ring of varieties $K_0(\mathrm{Var}_k)$, we found combinatorial methods of characterizing (invariants of) varieties and topological invariants of configuration spaces uniquely defined up to normalization by a deletion-contraction relation. This includes characterizations of varieties or asymptotic properties of linear subspaces contained in them. The specialization to configuration spaces has led to close connections to combinatorial questions about h-vectors of simplicial complexes such as unimodality questions.

We also consider combinatorial analogues of geometric problems in place of invariants. The specific question we studied is the Cayley-Bacharach property on points failing to impose independent conditions on hypersurfaces. Recently, Levinson and Ullery showed that points failing to impose independent conditions often lie on a union of low-dimensional linear subspaces and ask whether a matroidal analogue holds. We show that this is not the case for covers by proper flats. There is a strange relationship with combinatorial invariance even when the property has a simple description. However, there are more direct connections with the underlying algebraic structures (Chow rings) or the independence conditions for paving matroids and supersolvable hyperplane arrangements. In many cases, the geometric questions here are known to be determined by the failure of a Lefschetz property. Relating back to $K_0(\mathrm{Var}_k)$ suggests connections with mathematical physics.

In summary, we show that Euler characteristic invariants are a natural framework for bridging questions in algebraic geometry/topology with those in combinatorics. Using a relatively abstract invariant (the Grothendieck ring of varieties), we find projective geometry characterizations of varieties and their invariants which are more elementary than what one might expect from existing results in the literature on these objects. Specializing to configuration spaces (which also appear in other context such as discriminant complements or spaces of rational maps), we find a relation which gives a geometric framework (involving Chow rings of polymatroids or characteristic polynomials of linear subspace arrangements) to study general questions on h-vectors (e.g. unimodality or alternating sums), which are currently poorly understood outside of specific contexts (e.g. from matroids). Finally, Cayley-Bacharach problems adapt algebro-geometric questions which have not previously been studied in the context of matroids and have interesting interactions with existing combinatorial structures. In particular, it would be very interesting if there were a combinatorial analogue of the the failure of points to impose independent conditions on hypersurfaces (termed unexpected hypersurfaces) being characterized by the failure of a Lefschetz property. Keeping in mind that many interesting related objects are based on the geometry of fans (e.g. toric varieties, Chow rings of matroids) are parametrized by fans, we can show that certain families of Poincare duality algebras can be transformed in a natural way to obtain algebras satisfying Lefschetz properties using resolutions of singularities of toric varieties.

1.0.1 Projective geometry characterizations of varieties and invariants

Using a relation in the *Grothendieck ring of varieties* [16], we provide characterizations of varieties and their invariants via concrete combinatorial methods. The main tool we use is the $Y - F(Y)$ relation $[Y^{[2]}] = [\mathbb{P}^m][Y] + \mathbb{L}^2[F(Y)]$ of Galkin–Shinder (Theorem 5.1 of [16]) which relates the class of a cubic hypersurface $Y \subset \mathbb{P}^n$ of dimension m to its Fano variety of lines $F(Y)$ contained in it. The bijection involved is sketched in Figure 1.1 on p. 2 of [46]. It is motivated by rationality problems involving cubic hypersurfaces. We obtain an approximate version of the $Y - F(Y)$ relation for Fano varieties of k-planes.

Theorem 1.0.1. *(Theorem 1.7 of [50]) Let $Y_{d,n,m}$ be a "typical" sequence of varieties of dimension m and degree d in $\mathbb{P}^n$. Treating the initial parameters as functions of the codimension $r := n - m$, a weighted probability of an $(n-m)$-plane being contained in $Y_{d,n,m}$ is determined by symmetric products of points on the original variety, Grassmannians, polynomials in the class of a line, and incidence correspondences of points and linear subspaces of complementary dimension for large r. Analogous statements hold for cut and paste-compatible invariants.*

Conversely, one can also ask how uniquely these cut and paste relations determine varieties. We address a question of Farb (Question 0.1 on p. 2 of [46]) on the case of the $Y - F(Y)$ relation.

Theorem 1.0.2. *(Theorem 0.2 and Corollary 0.5 of [46]) Under certain numerical/genericity conditions, a variety satisfying the $Y - F(Y)$ relation must be a cubic hypersurface. Weakening these conditions expands the cases to complete intersections of two quadric hypersurfaces or two quartic hypersurfaces. On the other hand,* any *generic hypersurface generic among those of its degree satisfying the relation must be a cubic hypersurface.*

1.0.2 Combinatorics and configuration space invariants

Specializing to Euler characteristics of configuration spaces, we obtain connections to many interesting combinatorial problems involving h-vectors of simplicial complexes. Starting with our observation that Euler characteristics of ordered configuration spaces (studied by Eastwood–Huggett (Theorem 2 on p. 155 of [6])) yield chromatic polynomials of graphs (1.1 on p. 2 of [51]), we study a generalization of Cooper–de Silva–Sazdanovic [13] to simplicial complexes via **simplicial chromatic polynomials** $\chi_c(S)(t)$ (Definition 2.1 on p. 725 and

p. 738 of [13]). They are uniquely defined up to normalization by a deletion-contraction type relation (Corollary 6.1 and Proposition 6.4 on p. 738 of [13]). For many families of simplicial complexes, we express the simplicial chromatic polynomial in terms of the h-vector of an auxiliary simplicial complex.

Theorem 1.0.3. *(Theorem 1.5 and Proposition 2.8 of [51]) If the minimal nonfaces of S do not intersect very much, there is an auxiliary simplicial complex $T(S)$ such that $\frac{\chi_c(S)(t)}{t^{d-m}(t-1)^{n-d}} = t^m h_{T(S)}(t^{-1})$ where $m = \dim T(S)$. A similar formula holds if the minimal nonfaces of S have pairwise nonempty intersections. Note that* any *simplicial complex can be set to be the auxiliary simplicial complex $T(S)$ for a suitable simplicial complex S.*

Applications to combinatorial problems

This result connects the simplicial chromatic polynomial to various combinatorial questions. For example, we can show that there are families of graphs such that the number of edge colorings where designated subgraphs are *not* monochromatic are determined by lattice point counts on (dilations of) polytopes (Theorem 3.5 on p. 9 and Theorem 4.9 on p. 15 – 16 of [45]) or Hodge numbers of a toric variety. Note that the simplicial chromatic polynomial is as the characteristic polynomial of a linear subspace arrangement. Combining Theorem 1.0.3 with recent results of Pagaria–Pezzoli (Theorem 6.4 on p. 34 of [44]) and Crowley–Huh–Larson–Simpson–Wang [20] on Chow ring structures on polymatroids, we have that h-vectors of simplicial complexes in Theorem 1.0.3 are determined by a Chow ring structure. This does *not* imply that the characteristic polynomials of linear subspace arrangements have log concave coefficients (Remark 6.6 on p. 35 of [44]). However, natural modifications yield unimodal h_i by a result of Brown–Cameron (Theorem 2.4 on p. 1140 of [13]).

Corollary 1.0.4. *Let S be a simplicial complex on $[n]$ with minimal nonfaces of the form $\{i, j, n\}$ for some $1 \le i < j \le n-1$. Adding minimal nonfaces and increasing n to N, there is a simplicial complex S' such that the polynomial $\frac{\left(\frac{1+2u}{u}\right)^{d'-1}}{\left(\frac{1+u}{u}\right)^{d'}} h_{T(S')}\left(\frac{u}{1+u}\right)$ is unimodal, where*

$d' = \dim S'$.

Question 1.0.5. *1. Are there transformations of simplicial complexes such that their simplicial chromatic polynomials must always have a log concave or unimodal characteristic polynomial? What does this mean for Chow ring structure on the associated polymatroids?*

2. Can we use geometric properties of characteristic polynomials of linear subspace arrangements to find simplicial complexes with log concave or unimodal h-vectors?

Finally, we explored a relation to colorings of *directed* graphs and generating functions of Hodge–Deligne polynomials of configuration spaces and GL_n-character varieties in terms of finitely presented groups (Theorem 1.1 on p. 2 – 3 and Theorem 1.3 on p. 3 – 4, Theorem 1.3 on p. 3 – 4, Part 2 of Proposition 1.5 on p. 4 – 5 of [47]) via work of Crew–Spirkl [19]. It would be interesting if the discussion above carries over to analogues of directed graph colorings.

1.0.3 Independence conditions of points on hypersurfaces

For combinatorial analogues of geometric problems, we considered the Cayley–Bacharach property ($CB(v)$ on p. 1 of [17]) in recent work of Levinson and Ullery, who show that $CB(v)$ points often lie on a union of low-dimensional linear subspaces when the number of points is linearly bounded by the degree (Theorem 1.3 on p. 2 and Conjecture 1.2 on p. 2 of [17]). Since their proof could often be reduced to matroid-theoretic arguments, they define a matroid-theoretic analogue of $CB(v)$ and ask whether a matroidal analogue of their result holds (Question 7.6 on p. 14 of [17]). We give a negative answer to this question.

Theorem 1.0.6. *(Theorem 1.6 of [48], Theorem 1.8 of [49]) The matroidal analogue of Conjecture 1.2 on p. 2 of [17] for covers by proper flats does* not *hold. In addition, there is no nontrivial upper bound on ranks of flats covering the ground set of a matroid M satisfying*

$MCB(a)$.

There are interesting independence/dependence relations between $MCB(a)$ and geometric properties of matroids. If the associated polytopes are nestohedra, the $MCB(a)$ property does not seem to affect the combinatorial equivalence class of the matroidal polytope.

Theorem 1.0.7. *(Theorem 2.4 of [48]) The $MCB(a)$ property for nestohedra depends on the maximal elements of the building set used to construct it. While this often has an interpretation as the number of connected components of an auxiliary matroid, the $MCB(a)$ property is not necessarily a combinatorial invariant of the associated polytopes.*

However, paving matroids (conjecturally generic among matroids of rank r), have a relation between $MCB(a)$ and the Hilbert series of the Chow ring and supersolvable line arrangements give matroids (apart from the representable examples) where $MCB(a)$ is closely related to whether points impose independent conditions on degree d hypersurfaces (Proposition 3.14 of [48]) via work of Hanumanthu–Harbourne (Theorem on p. 3 of [14]).

Theorem 1.0.8. *(Theorem 3.2 of [48], Corollary 3.3 of [48], Proposition 3.7 of [48])*

1. *If the maximal hyperplanes covering a paving matroid are large and of similar size, $MCB(a)$ must hold for a in an interval depending on the initial parameters. The minimal degree a where $MCB(a)$ holds decreases with the as the dimension of the quotients by the annihilators of each x_H for the largest hyperplanes H in $A^*(M)$ of M increase.*

2. *The degrees a where a supersolvable line arrangement satisfies $MCB(a)$ are determined*

by degrees of the modular points and minimal degree a decreases as the number of possible degrees of unexpected curves increases.

In general, the $MCB(a)$ property for supersolvable hyperplane arrangements of rank d is determined by lower degree MCB conditions and ground set covers of specified subarrangements. The ones satisfying $MCB(d)$ can attain all possible characteristic polynomials (Propostiion 3.14 on p. 13 of [48]). Connecting the cases in Part 2 of Theorem 5.2.7, we can ask the following:

Question 1.0.9. *Does the relation to independence conditions imposed by actual points extend to this higher dimensional setting (called "unexpected hypersurfaces" in work of Cook–Harbourne–Migliore–Nagel [7] and Harbourne–Migliore–Nagel–Teitler [15])?*

Question 1.0.10. *In general, unexpected hypersurfaces are related to the failure of a weak Lefschetz property (p. 309 of [15]). Is there a matroidal version (e.g. from hyperplane arrangements)?*

Some steps towards understanding the matroidal version or other combinatorial questions would be to look at the relation tot he geometry of fans. This can include how far an algebra is from satisfying a Lefschetz property. More specifically, studying combinatorial properties of (labeled) fans and their associated toric varieties allows us to use work of Ayzenberg–Masuda [10] to find transformations (adding variables and modifying existing linear relations using the new variable) changing any Poincaré duality algebra generated by degree 1 elements into ones studying the (strong) Lefschetz property.

Proposition 1.0.11. *Given any Poincaré duality algebra generated by degree 1 elements, we can use the following pair of steps repeatedly to obtain one which satisfies the (strong) Lefschetz property:*

- *Add an extra variable c_{m+1} to the existing m variables*

- *Add an extra linear term* $\lambda_{m+1,j}c_{m+1}$ *to the* j^{th} *using an appropriate scalar* $\lambda_{m+1,j}$

It would be interesting if we could relate the (strong) Lefschetz property itself with something which can be used to explicitly measure how singular a toric variety is using properties of the fan defining it (e.g. the multiplicity of a fan used in the proof of resolutions of singularities via fan subdivisions). Also, another possible option for possible applications to combinatorial/geometric properties is to look at the relation between Lefschetz properties and the Chow ring of the associated fan using work of Feichtner-Yuzvinsky [13].

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Part I

Combinatorial approximations and characterizations of varieties via Euler characteristic invariants

CHAPTER 2

MOTIVIC LIMITS FOR FANO VARIETIES OF K-PLANES

Abstract

We study the probability that an $(n-m)$-dimensional linear subspace in $\mathbb{P}^n$ or a collection of points spanning such a linear subspace is contained in an m-dimensional variety $Y \subset \mathbb{P}^n$. This involves a strategy used by Galkin–Shinder to connect properties of a cubic hypersurface to its Fano variety of lines via cut and paste relations in the Grothendieck ring of varieties. Generalizing this idea to varieties of higher codimension and degree, we can measure growth rates of weighted probabilities of k-planes contained in a sequence of varieties with varying initial parameters over a finite field. In the course of doing this, we move an identity motivated by rationality problems involving cubic hypersurfaces to a motivic statistics setting associated with cohomological stability.

2.1 Introduction

Given a variety $Y \subset \mathbb{P}^n$ of dimension m and degree d, the *Fano variety of k-planes* is the subscheme $F_k(Y) \subset \mathbb{G}(k,n)$ parametrizing the set of k-planes contained in Y. This can be taken to be the Hilbert scheme structure ([2], Proposition 6.6 on p. 203 of [12]) or the reduced structure on it (p. 12 of [16]). Since we end up working in the Grothendieck ring of varieties $K_0(\mathrm{Var}_k)$ and $[X] = [X_{\mathrm{red}}]$ in $K_0(\mathrm{Var}_k)$, the nonreduced structure does not play a role our setting and it does not matter which structure we take. In addition, we will take the term "variety" to mean an irreducible scheme of finite type. We would like to study the relationships between the following questions:

Question 2.1.1.

1. *How do properties of $F_k(Y)$ such as arithmetic/geometric invariants vary with initial conditions on Y (e.g. degree, dimension, codimension)?*

2. *Given that the Fano variety of k-planes has a simple definition as a subvariety of $\mathbb{G}(k,n)$, is there a concrete method (e.g. using a projective geometry construction) other than giving explicit defining equations which give an approach to the first question?*

   For example, how can we relate the Fano variety of k-planes with symmetric products of Y corresponding to unordered k-tuples of points in Y?

3. *Over $\mathbb{F}_q$, how "likely" is a k-plane to be contained in Y? How does this probability change as we vary q?*

The main idea in our approach to Question 2.1.1 (Theorem 2.1.7, Corollary 2.1.10) is to combine two different perspectives on cut–and–paste relations between algebraic varieties. By cut–and–paste relations, we mean the *Grothendieck ring of varieties* $K_0(\mathrm{Var}_K)$ over a field K. This is the ring generated by isomorphism classes of algebraic varieties over K quotiented out by relations $[X] = [Z] + [X \setminus Z]$ for closed subvarieties $Z \subset X$ and by $[X \times Y] = [X][Y]$.

Our starting point is Galkin–Shinder's $Y - F(Y)$ relation, which connects the geometry of a cubic hypersurface with the space of lines on it.

Theorem 2.1.2 ($\mathbf{Y - F(Y)}$ relation). *(Galkin–Shinder, Theorem 5.1 on p. 16 of [16]) Let $Y \subset \mathbb{P}^{m+1}$ be a smooth cubic hypersurface of dimension m over an algebraically closed field K of characteristic 0 and $F(Y) \subset \mathbb{G}(1, m+1)$ be its (reduced) Fano scheme of lines (p. 12 of [16]). Then in $K_0(\mathrm{Var}_k)$:*

$$[Y^{[2]}] = [\mathbb{P}^m][Y] + \mathbb{L}^2[F(Y)] \tag{2.1.1}$$

where $Y^{[2]}$ denotes the Hilbert scheme of two points.

The relation (2.1.1) is obtained using a map sending a pair of points (p, q) on Y (considered as an element of $Y^{[2]}$) to the pair $(r, \overline{pq}) \in Y \times \mathbb{G}(1, m+1)$, where r is the residual point of intersection of $\overline{pq}$ with Y (see Figure 2.1.1). This construction gives a correspondence between points of $Y^{[2]}$ spanning a line *not* contained in Y and $(r, \ell) \in \mathbb{G}(1, m+1)$ such that $r \in \ell$ and $\ell \not\subset Y$. Collecting the non-generic terms coming from lines $\ell \subset Y$ from $Y^{[2]}$ and the incidence correspondence $W = \{(r, \ell) \in Y \times \mathbb{G}(1, m+1) : r \in \ell\}$ yields the $Y - F(Y)$ relation.

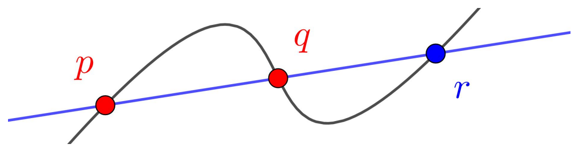

Figure 2.1.1: Sketch of the proof of the $Y - F(Y)$ relation.

As an averaged statement over the space $\mathbb{G}(1, m+1)$ of lines in $\mathbb{P}^{m+1}$, the $Y - F(Y)$ relation in Theorem 2.1.2 can be rewritten as

$$\frac{[Y^{[2]}]}{[\mathbb{G}(1, m+1)]} = \frac{[\mathbb{P}^m][Y]}{[\mathbb{G}(1, m+1)]} + \frac{([(\mathbb{P}^1)^{(2)}] - [\mathbb{P}^1])[F(Y)]}{[\mathbb{G}(1, m+1)]} \tag{2.1.2}$$

in $\widehat{\mathcal{K}}$, where $\widehat{\mathcal{K}}$ is a modification of the usual completion with respect to the dimension filtration (Section 2.2.1).

The "denominator" $\mathbb{G}(1, m+1)$ is the total space of lines in $\mathbb{P}^{m+1}$. Using the argument above, the term of 2.1.2 on the right involving $F(Y)$ gives a weighted probability that a line fails to satisfy the correspondence indicated in Figure 2.1.1 and the comments above it.

Since the structure of $K_0(\mathrm{Var}_k)$ is compatible with a wide range of invariants including Poincaré polynomials and point counts over $\mathbb{F}_q$, the $Y - F(Y)$ relation has many interesting consequences. For example, substituting the Poincaré polynomials of Y and the second symmetric product $Y^{(2)}$ into the relation

$$[Y^{(2)}] = (1 + \mathbb{L}^m)[Y] + \mathbb{L}^2[F(Y)], \tag{2.1.3}$$

which is equivalent to 2.1.1 for $m = 2$, yields a proof that there are 27 lines on a smooth cubic surface when $\operatorname{char} k \neq 2$. If $\operatorname{char} k = 2$, this relation holds modulo universal homeomorphisms since the diagonal morphism $Y \hookrightarrow Y^{(2)}$ is a universal injection (Example 1.1.12 on p. 372 – 374 of [4], p. 114 of [4]). Alternatively, we can localize at radically surjective morphisms as in Section 2.1 of [10].

The approach that we take to studying Fano varieties of k-planes of a sequence of varieties is to generalize the $Y - F(Y)$ relation 2.1.1. In Proposition 2.2.1, we can obtain an analogous relation in $K_0(\mathrm{Var}_k)$ for Fano varieties of $(n-m)$-planes contained in a m-dimensional variety $Y \subset \mathbb{P}^n$ of degree d. In other words, this is a generalization from lines to k-planes of complementary dimension. Examples of varieties $Y \subset \mathbb{P}^n$ of dimension m containing $(n-m)$-planes are complete intersections of general hypersurfaces of degree $(d_1, \ldots, d_{n-m})$ such that $m \gg \binom{d_i+n-m}{n-m}$ for each $1 \leq i \leq n-m$ (Theorem 2.4 on p. 4 of [10]).

The idea is to match up $(k+1)$-tuples of points of Y lying on a fixed generic $(n-m)$-plane $\Lambda \in \mathbb{G}(n-m, n)$ with the remaining $d - k - 1$ points of $Y \cap \Lambda$ paired with the same $(n-m)$-plane Λ. Figure 2.1.2 illustrates this correspondence for a 2-plane intersecting a variety in $\mathbb{P}^n$ which has codimension 2 and degree 6. Incidence correspondences and maps involved in this construction are given in more detail in the proof of Proposition 2.2.1.

Roughly speaking, the $(n-m)$-dimensional linear subspaces contained in a variety parametrize elements of $\mathbb{G}(n-m, n)$ which do *not* give a correspondence between complementary points of intersection. In order to address the complexity of terms involved as the initial parameters are increased, we consider *sequences* of varieties and find an average result. This moves a cut and paste relation coming from a rationality problem into a setting mostly associated with homological stability. We can also give a natural connection between invariants of the variety Y and the space of $(n-m)$-planes contained in it.

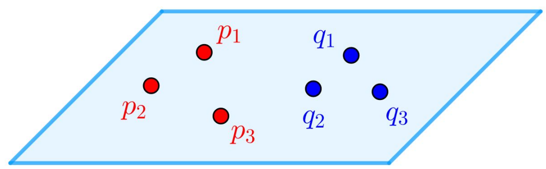

Figure 2.1.2: Suppose that $n-m=2$, $d=6$, and $k=2$. The points drawn are the intersection of Y with a generic $(n-m)$-plane. On the left, the $(k+1)$-tuples of points are drawn in red. The remaining $(d-k-1)$ points of intersection are drawn in blue.

As in the original $Y-F(Y)$ relation (Theorem 2.1.2), the terms of the higher–dimensional generalization (Proposition 2.2.1) involving elements of $F_{n-m}(Y) \subset \mathbb{G}(n-m,n)$ parametrizing $(n-m)$-planes contained in Y are contained in the non-generic loci of incidence correspondences (Section 2.2.1). However, there are new non-generic terms in this higher dimensional analogue coming from linear dependence between points which did not occur in the $Y-F(Y)$ relation since there are at most two distinct points at once in that case. As a result, the complexity of the terms in $K_0(\mathrm{Var}_K)$ that are used in this extended $Y-F(Y)$ relation increase quickly with the starting parameters to the point of making a simple closed form relation as in the $Y-F(Y)$ relation seems impossible. Our main goal is to extract a meaningful generalization of the $Y-F(Y)$ relation.

In order to do this, we consider "average" classes in $K_0(\mathrm{Var}_K)$ over varieties of varying initial parameters (codimension, dimension, degree) and work in a modification $\widehat{\mathcal{K}}$ of the usual completion of $K_0(\mathrm{Var}_k)[\mathbb{L}^{-1}]$ with respect to the dimension filtration (Section 2.2.1).

More specifically, we show that "dividing" by $[\mathbb{G}(n-m,n)]$ gives a sequence of terms in this filtration where the contribution of terms encoding linear dependence approaches 0. This boils down to dimension computations of the non-generic loci. These terms can be eliminated in the limit since the codimensions in the total spaces increase quickly as the sizes of the inital parameters increase. Before stating the limits which we obtain in this completion, here is some notation.

Definition 2.1.3. The ***approximate weighted average*** *of linearly independent u-tuples of points on $Y = Y_{d,n,m}$* is

$$A_{n,m,u} := \frac{[Y^{(u)}][\mathbb{G}(n-m-u, n-u)]}{[\mathbb{G}(n-m,n)]} \text{ in } \widehat{\mathcal{K}} \text{ (Section 2.2.1)},$$

where $Y^{(u)}$ is the u^{th} symmetric product of Y. Note that d, n, m, u are all functions of a single variable r and the limit in the completion is taken as $r \to \infty$ The term $[\mathbb{G}(n-m-u, n-u)]$ parametrizes the set of $(n-m)$-planes that pass through a particular u-tuple of linearly independent points of Y. If we replace $[Y^{(u)}]$ with the subset $U \subset Y^{(u)}$ consisting of linearly independent u-tuples of points, the term $[U][\mathbb{G}(n-m-u, n-u)]$ is the class of

$$\{((p_1, \ldots, p_u), \Lambda) \in Y^{(u)} \times \mathbb{G}(n-m-u, n-u) : p_i \in \Lambda \text{ for each } i$$
$$\text{and the } p_i \text{ are linearly independent}\}.$$

As stated above, the limits will involve several variables that are all functions of a single variable. The geometric meaning of variables d, n, m, u with $k+1$ substituted for u is given in Table 2.1.1.

Definition 2.1.4. Given a convergent sequence of elements $G_{d,m,n,k}$ of $\widehat{\mathcal{K}}$ with $d = d(r), m = m(r), n = r + m(r)$, and $k = k(r)$ approaching infinity as $r \to \infty$, we will use the following notation for the limit.

$$\widetilde{\lim_{d,m,n,k\to\infty}} G_{d,m,n,k} := \lim_{r\to\infty} G_{d(r),m(r),n(r),k(r)}$$

The limits will be taken over sequences of varieties of the following form.

Definition 2.1.5. A **typical sequence** of smooth, closed nondegenerate varieties $Y_{d,m,n} \subset \mathbb{P}^n$ of degree d and dimension n is one where the variables in Table 2.1.1 satisfy the following conditions:

1. For every r, $Y_{d(r),m(r),n(r)}$ is contained in a generic hypersurface.
2. There is some $a > 1$ such that $ar \leq \dim Y_{d(r),m(r),n(r)}$ for all r.
3. u-linearly generic for $u \leq d-k-2$ (Proposition 2.2.22, Proposition 2.2.25, Definition 2.2.24). This is a genericity condition on hypersurfaces defining certain non-generic linear subspaces of a given dimension in $\mathbb{P}^n$.

Now that the notation is fixed, the limiting extensions of the $Y - F(Y)$ relation in $\widehat{\mathcal{K}}$ can be studied. The cases we will consider are split into the size of the degree relative to the number of sampled points (dots of a single color in Figure 2.1.2) and the codimension. In both cases, the $(n-m)$-planes lying inside the given sequence of varieties make up most of the discrepancy between weighted averages of linearly independent $(k+1)$-tuples and $(d-k-1)$-tuples under suitable conditions (Definition 2.1.3).

We first consider the case where the degree d of the m-dimensional variety $Y \subset \mathbb{P}^n$ is not very large with respect to the codimension. In this case, it turns out that $d \leq 2(n-m)+1$ for such varieties. Possible varieties with such a degree have been classified by Ionescu [22]. The example below (Example 2.1.6) discusses properties of Fano varieties of k-planes of such varieties which can also have arbitrarily large dimension and degree in more detail.

Example 2.1.6. (Low degree examples for Part 1 of Theorem 2.1.7)
Examples occuring in the low degree case include scrolls and (hyper)quadric fibrations. Further comments on existence and explicit constructions are given in Example 2.3.2 at the end of Section 2.3.1.

Although the discussion in Example 2.1.6 shows that Fano varieties of k-planes of many low-degree varieties can be understood using direct computations, we will study them using an averaged $Y - F(Y)$ relation as a way to look at the "generic" higher degree case covering *all* other varieties. As in the case of the averaged version 2.1.2 of the original $Y - F(Y)$ relation 2.1.2, the average is a weighted average of k-planes taking tuples of points contained in these planes into account. For this higher degree case, we need some additional notation. Given a variety $Y \subset \mathbb{P}^n$ of dimension m and degree d, let

$$J = \{((p_1, \ldots, p_{d-k-1}), \Lambda) \in Y^{(d-k-1)} \times \mathbb{G}(n-m, n) :$$
$$|\Lambda \cap Y| = d \text{ or } \Lambda \subset Y, \ p_i \text{ distinct}, \ \dim \overline{p_1, \ldots, p_{d-k-1}} = n - m\},$$

and let $D \subset (\mathbb{P}^{n-m})^{(d-k-1)}$ be the set of $(d-k-1)$-tuples of points spanning a linear subspace of dimension $\leq n - m - 1$. The set J parametrizes the set of $(d-k-1)$-tuples of points in Y which span an $(n-m)$-plane. Note that $[D]$ is a polynomial in $\mathbb{L}$ (Proposition 2.2.27). The expressions in the result below give an approximate relation in the high degree case using these objects.

Theorem 2.1.7 (**Averaged $\mathbf{Y - F(Y)}$ relations).**

In the expressions below, we consider a sequence of elements of $\mathcal{K}$ which depend on the variables d, m, n, k. Each of these variables are functions of a single variable r satisfying certain properties and can be written $d = d(r), m = m(r), n = n(r), k = k(r)$. The limits in $\widehat{\mathcal{K}}$ are taken with respect to r as $r \to \infty$ (Definition 2.1.4). Precise statements on relative dimensions (Definition 2.2.10) involved in the limit are listed on p. 39 for the low degree *case and on p. 37 – 38 for the high degree case.*

The limits below are are of sequences indexed by variables which are functions of r that approach infinity as $r \to \infty$. In other words, they can be taken to be limits as $r \to \infty$.

1. ***(Low degree case)*** *: Let $Y_{d,n,m}$ be a typical sequence of varieties (Definition 2.1.5) of dimension m and degree d such that $\operatorname{codim}_{\mathbb{P}^n} Y_{d,n,m} > 2 \dim Y_{d,n,m} + \Theta(r)$, $\deg Y_{d,n,m} - (k+1) + \Theta(\sqrt{r}) \le \operatorname{codim}_{\mathbb{P}^n} Y_{d,n,m} - 1$, and $\dim Y_{d,n,m} > 2 \deg Y_{d,n,m}$, where $\Theta(f(r))$ denotes being bounded below and above by a constant multiple of $f(r)$ as $r \to \infty$ as usual. If the point sample size $k+1$ is small with $k \le b \deg Y_{d(r),n(r),m(r)}$ for some $b < 1$, then the limit of this sequence (Definition 2.1.4) is*

$$\widetilde{\lim_{d,m,n,k \to \infty}} \left(\frac{2[F_{n-m}(Y_{d,n,m})]([(\mathbb{P}^{n-m})^{(k+1)}] - [(\mathbb{P}^{n-m})^{(d-k-1)}])}{[\mathbb{G}(n-m,n)]} \right. \\ \left. - A_{n,m,k+1} + A_{n,m,d-k-1} \right) = 0$$

   in $\widehat{\mathcal{K}}$. The left hand side gives a sequence of elements in $\mathcal{K}$ which approach 0 in the completion $\widehat{\mathcal{K}}$. The precise relative dimensions of terms involved are listed in Section 2.3.1.

2. ***(High degree case)*** *: Let $Y_{d,n,m}$ be a sequence of typical varieties (Definition 2.1.5) of degree $\deg Y_{d,n,m} - (k+1) - 1 > n$ and small point sample size $k+1$ with $k \le br$ for some $b < 1$. Suppose that each $Y_{d,n,m}$ is contained in a complete intersection of s hypersurfaces such that $(n-m)k + k - 1 \ll \sum_{i=1}^{s} \binom{d_i+n-m}{n-m}$. Then the limit of this*

sequence (Definition 2.1.4) is

$$\widetilde{\lim_{d,m,n,k\to\infty}} \frac{2[F_{n-m}(Y_{d,n,m})]([(\mathbb{P}^{n-m})^{(k+1)}] - [\mathrm{UConf}_{d-k-1}\,\mathbb{P}^{n-m}])}{[\mathbb{G}(n-m,n)]} - A_{n,m,k+1}$$
$$+\frac{[J]}{[\mathbb{G}(n-m,n)]} - \frac{2[F_{n-m}(Y_{d,n,m})][D]}{[\mathbb{G}(n-m,n)]} = 0 \text{ in } \widehat{\mathcal{K}} \text{ if } k-2 \ll n-m,$$

where $\mathrm{UConf}_e\, X \subset X^{(e)}$ *denotes unordered e-tuples of distinct points of* X*. Note that* $[D]$ *and* $\mathrm{UConf}_{d-k-1}\,\mathbb{P}^{n-m}$ *are polynomials in* $\mathbb{L}$ *(Proposition 2.2.30, Lemma 2.2.31).*

As noted in part 1, this sequence of elements in $\mathcal{K}$ *which approach* 0 *in the completion* $\widehat{\mathcal{K}}$*. The relative dimensions for this case are listed at the end of Section 2.2.3 after Example 2.2.32.*

Remark 2.1.8.

1. One consequence is that properties of J compatible with the usual completion of $K_0(\mathrm{Var}_K)$ (e.g. point counts) can be expressed in terms of polynomials in $\mathbb{L}$ and $F_{n-m}(Y)$.

2. In the low degree case, we add the lower bound of $2\dim Y$ to avoid cases where Y is forced to be a complete intersection if Hartshorne's conjecture (p. 1017 of [20]) holds. This is to ensure that the varieties considered in Part 1 actually have the "low degree property" defining that case. Note that this conjecture cannot be strengthened to force a complete intersection outside the range of its original statement (p. 1022 of [20]). There is also a specific bound in Corollary 3 on p. 588 of [3] towards this conjecture and a proof of the conjecture when $n \gg d$ in [13].

3. There are large parentheses around the first term in the low degree case since it may actually end up vanishing in the completion with the relative dimension (Definition

2.2.10) approaching $-\infty$. It is stated here since this analogue of the averaged $Y - F(Y)$ is the template for the proof and interpretation of the high degree case. The first term in Part 1 of Theorem 2.1.7 contains information on $(k+1)$-tuples or $(d-k-1)$-tuples contained in an $(n-m)$-plane contained in $Y_{d,n,m}$. On the other hand, the second and third terms parametrize the set of linearly independent $(k+1)$-tuples and $(d-k-1)$-tuples paired with an $(n-m)$-plane containing them. The higher degree case also compares maximally linearly independent $(k+1)$-tuples and $(d-k-1)$-tuples of points on Y lying on an $(n-m)$-plane.

When $\deg Y$ is much larger than $\operatorname{codim}_{\mathbb{P}^n} Y$, note that generic $(d-k-1)$-tuples do *not* lie on an $(n-m)$-plane. In this case, general complete intersections of hypersurfaces of degree $(d_1, \ldots, d_{n-m})$ such that $m \gg \binom{d_i+n-m}{n-m}$ end up being compatible with restrictions on the variables d, m, n, k involved in generalizations of the $Y - F(Y)$ relation (see Example 2.1.9 for more details). Substituting these values into the relative dimensions listed in Section 2.3.1, we can see that this limit in Part 1 of Theorem 2.1.7 can be obtained without assuming $m \gg n - m$ when we take $k = \lfloor \frac{m}{2} \rfloor$.

The proof of the decomposition in the limit in Part 2 of Theorem 2.1.7 is similar to that of Part 1 of Theorem 2.1.7. Note that sufficiently generic complete intersections provide many examples of this higher degree case of Theorem 2.1.7.

Example 2.1.9. (High degree examples for Part 2 of Theorem 2.1.7: Complete intersections of generic hypersurfaces of large degree)
In this case, we an use complete intersections of hypersurfaces which are generic among those of their given degrees. Numerical conditions on possible degrees and their relation to the other variables are explained in more detail in Example 2.3.8 at Section 2.3.2.

Applying the point counting motivic measure and a modified Lang–Weil bound to the limit in Part 2 of Theorem 2.1.7, Corollary 2.1.10 gives an upper bound point counts under certain divisibility conditions. While it is not compatible with the dimension filtration (e.g. disjoint union of a curve with a finite collection of points), it is still compatible with the completion (Definition 2.3.3). This interprets Part 2 of Theorem 2.1.7 as an answer to Question 2.1.1 on point counts over $\mathbb{F}_q$ as q increases. For example, the probability point count involving tuples of points can be expressed as coefficients of exponential generating functions in terms of point counts of Y.

More specifically, we find a point counting counterpart (Corollary 2.1.10) to Part 2 of Theorem 2.1.7 which uses a similar argument. Each term of the latter result comes from a $(k+1)$-tuple or $(d-k-1)$-tuple of points on Y paired with an $(n-m)$-plane containing them. For the first and third terms in the result over $\widehat{\mathcal{K}}$, this $(n-m)$-plane is assumed to be contained in Y. This is a higher degree analogue of the argument used in Theorem 2.1.2 for the $Y - F(Y)$ relation. The main difference is that $\#J(\mathbb{F}_q)$ can be expressed in terms of the $\mathrm{Gal}(\overline{\mathbb{F}_q}/\mathbb{F}_q)$-action on $(d-k-1)$-tuples of points on Y.

Corollary 2.1.10 (Averaged Fano $(\mathbf{n}-\mathbf{m})$-plane point count).

Suppose that $d-k-1 \geq n-m$ and m, n, d, k satisfy the conditions in Part 2 of Theorem 2.1.7 and $Y \subset \mathbb{P}^n$ is smooth over $\mathbb{F}_q$. Note that all the variables are functions of r and that $n(r) = m(r) + r$. As in Theorem 2.1.7, the limits are taken with respect to a single variable r of a sequence of m-dimensional varieties $Y_{d,m,n} \subset \mathbb{P}^n$ of degree d. Each of the variables is a function of r. Given $\Lambda \in \mathbb{G}(n-m, n)$, let

$$T_\Lambda = \{\, N : N \text{ is the number of } \mathrm{Gal}(\overline{\mathbb{F}_q}/\mathbb{F}_q)\text{-orbits of } (d-k-1)\text{-tuples } (p_1, \ldots, p_{d-k-1}) \in Y^{(d-k-1)} \text{ in } Y \cap \Lambda \text{ for some } \Lambda \}.$$

Fix a prime power q. Let $e = e(r)$ be a positive integer and a function of r such that

$e(r) \to \infty$ as $r \to \infty$ and $e > \binom{d}{d-k-1}$ for all r. There is a range of values for $\mathbb{F}_{q^e}$-point counts depending on divisibility properties of $\mathbb{F}_q$-irreducible components of slices by linear subspaces of complementary dimension.

Given a variety X, write $\#_{q,e}X := \#X(\mathbb{F}_{q^e})$. Note that we will consider point counts over varying fields $\mathbb{F}_{q^e}$ with $e \to \infty$ as $r \to \infty$. This means that

$$\lim_{r\to\infty} \frac{\#_{q,e}F_{n-m}(Y_{d,n,m})(\#_{q,e}\,\mathrm{UConf}_{d-k-1}(\mathbb{P}^{n-m}) - \#_{q,e}D - u\#_{q,e}(\mathbb{P}^{n-m})^{(k+1)} + \#_{q,e}C)}{\#_q\mathbb{G}(n-m,n)} + (1-u)\alpha + \gamma = 0.$$

in the limit for $r = n-m$ and $d = d(r), n = n(r), m = m(r).k = k(r)$, where

- *$0 \leq \alpha \leq \binom{d}{d-k-1}$ with $\alpha = 0$ if $N \nmid e$ for each $N \in T_\Lambda$ from $\Lambda \in \mathbb{G}(n-m,n)$ such that $|Y \cap \Lambda| = d$ and $\alpha = \binom{d}{k+1}$ if e is divisible by $\binom{d}{d-k-1}!$ (Proposition 2.3.5)*
- *$u = 1 - \beta + \beta f$, where $\beta = \Theta(q^{e((k+1)(n-m)+\dim F_{n-m}(Y)-m(n-m+1))})$ and f is a rational function in q^e (mostly) determined by $\frac{[\widetilde{R}]}{[\widetilde{A}]}$, which is a rational function in $\mathbb{L}$ of degree $(k+1)(n-m) + \dim F_{n-m}(Y) - m(n-m+1)$*
- *$\gamma = \Theta(q^{e(km-(n-m-k+1)-m(n-m+1))})$*

are functions that vary with the initial parameters, which depend on $n-m$. Note that the classes $[C]$ and $[D]$ of linearly dependent tuples are polynomials in $\mathbb{L}$ (Proposition 2.2.27).

As a byproduct of the decompositions above, we find a relation between "how likely" it is for a k-plane to be contained in a variety with the initial parameters such as degree and dimension. This combines some rather different perspectives on applications of the Grothendieck ring of varieties. For example, Galkin–Shinder's work [16] is motivated by rationality problems involving cubic hypersurfaces while motivic statistics results ([35], [10])

Parameters used in averaged $\mathbf{Y} - \mathbf{F}(\mathbf{Y})$ relations					
Variable	r	m	d	n	$k+1$
Definition	$\operatorname{codim}_{\mathbb{P}^n} Y$	$\dim Y$	$\deg Y$	dimension of projective space in which Y is embedded	number of points on Y on the $(n-m)$-plane for extension of $Y - F(Y)$ construction

Table 2.1.1: Parameters used in Part 1 and Part 2 of Theorem 2.1.7.

tend to be associated with problems related to cohomological stabilization or point counting.

The terms of the direct generalization of the $Y - F(Y)$ relation in $K_0(\mathrm{Var}_k)$ (Proposition 2.2.1 in Section 2.2.1) can be split into generic configurations and non-generic configurations which can be arbitrarily complicated as we increase the parameters involved. For this reason, the limits in Part 1 of Theorem 2.1.7, Part 2 of Theorem 2.1.7, and Corollary 2.1.10 are obtained via upper bounds on the dimensions of the non-generic loci since the completion $\widehat{\mathcal{K}}$ is defined with respect to a dimension filtration (Section 2.2.2). In the dimension counts of Section 2.2.3, the key idea is to bound the dimensions of the non-generic loci in the extended $Y - F(Y)$ relation Proposition 2.2.1. These computations are split into the low and high degree cases in Section 2.2.3 and Section 2.2.3 respectively. Finally, these dimensions are used to prove Theorem 2.1.7 by showing that the relative dimensions (Definition 2.2.10) approach 0 in Section 2.3. These limiting classes in $\widehat{\mathcal{K}}$ are used to obtain approximate point counts over in $\mathbb{F}_q$ in Corollary 2.1.10.

Acknowledgements

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2.2 Dimension computations in $K_0(\mathrm{Var}_k)$ for an extended $Y - F(Y)$ relation

In this section, we compute dimensions of varieties in the extended $Y - F(Y)$ relation (Theorem 2.1.2) for a smooth nondegenerate irreducible, closed, subvariety $Y \subset \mathbb{P}^n$ of dimension m and degree d that is defined over an algebraically closed field of characteristic 0. The initial terms to be used in the generalized $Y - F(Y)$ relation are given on p. 11 – 12 of Section 2.2.1. As in the statement of Theorem 2.1.7, the cases considered are split into those of low degree (Section 2.2.3) and high degree (Section 2.2.3) with respect to the codimension. For the low degree terms, the terms are defined in Proposition 2.2.12 on p. 18 with dimension counts listed on p. 20. The terms used in the high degree case are defined on p. 34 – 35.

2.2.1 The extended $Y - F(Y)$-relation

Before computing dimensions of the generic and degenerate loci, we first explain components/definitions in an extended $Y - F(Y)$-relation (Proposition 2.2.1). The idea is to match up linearly independent $(k+1)$-tuples on the intersection of an $(n-m)$-plane with the residual (linearly independent) $(d-k-1)$-tuples after removing non-generic loci. These specific constructions assume that $d - k - 1 \leq n - m - 1$. The analogous terms for the case $d - k - 1 > n - m - 1$ are listed in Section 2.2.3.

For each of the total spaces of incidence correspondences (V and W defined below) and non-generic loci inside them, there is a diagram giving the intersection of an $(n-m)$-plane with Y with q_i belonging to a $(d-k-1)$-tuple and p_j belonging to the residual $(k+1)$-tuple in the case $d = 6$ and $k = 2$. In the figures below, the points parametrized by the sets defined are filled in. On the other hand, complementary points of intersection of Y with the $(n-m)$-plane are hollow/unfilled (Figure 2.2.1, Figure 2.2.2, Figure 2.2.4). The case involving *both* a $(d-k-1)$-tuple and its complementary $(k+1)$-tuple (Figure 2.2.3) does not have any hollow/unfilled holes. Finally, the set involving $(n-m)$-planes Λ such that $\Lambda \subset Y$ is indicated by having the plane shaded in a new color (Figure 2.2.4).

We now define the total space W of incidence correspondences of $(d-k-1)$-tuples in Y paired with an $(n-m)$-plane and stratify the "non-generic" loci inside W coming for linear dependence of points or containment of an $(n-m)$-plane in Y.

- $W := \{((q_1, \ldots, q_{d-k-1}), \Lambda) \in Y^{(d-k-1)} \times \mathbb{G}(n-m, n) : q_i \in \Lambda,$ distinct, and $|Y \cap \Lambda| = d$ or $\Lambda \subset Y\}$ (Figure 2.2.1)

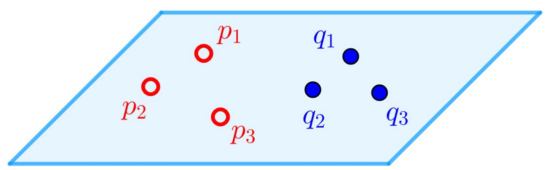

Figure 2.2.1: An example configuration. Note that W only considers the triple q_1, q_2, q_3 and the $(n-m)$-plane containing them.

- $\widetilde{B} := \widetilde{B}_1 \sqcup \widetilde{B}_2$, where

  $\widetilde{B}_1 := \{((q_1, \ldots, q_{d-k-1}), \Lambda) \in W : q_1, \ldots, q_{d-k-1}$ linearly dependent$\}$ (Figure 2.2.2)

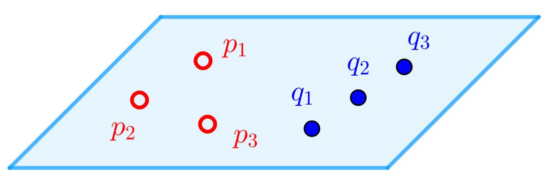

Figure 2.2.2: This is an example of $\widetilde{B}_1$ where q_1, q_2, q_3 are linearly dependent. Note that $\widetilde{B}_1$ only imposes conditions on the q_i and not the p_j.

$$\widetilde{B}_2 := \{((q_1, \ldots, q_{d-k-1}), \Lambda) \in W : q_1, \ldots, q_{d-k-1} \text{ linearly independent but}$$
$$(Y \cap \Lambda) \setminus \{q_1, \ldots, q_{d-k-1}\} \textit{ not } \text{a linearly independent } (k+1)\text{-tuple}, \Lambda \not\subset Y\}$$

(Figure 2.2.3)

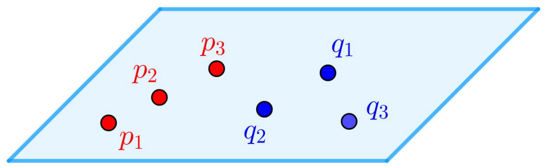

Figure 2.2.3: For $\widetilde{B}_2$, we consider *both* the $(d-k-1)$-tuple $q_1, \ldots, q_{d-k-1}$ and the residual points of intersection. While the q_i are linearly independent, the remaining points of $Y \cap \Lambda$ are not.

- $\widetilde{A} := \{((q_1, \ldots, q_{d-k-1}), \Lambda) \in W : q_1, \ldots, q_{d-k-1}$ linearly independent, $\Lambda \subset Y\}$ (Figure 2.2.4)

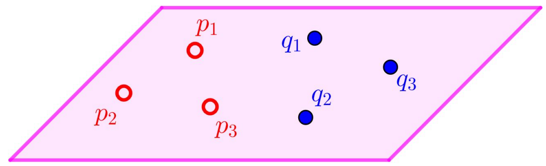

Figure 2.2.4: In $\widetilde{A}$, the only linear independence/dependence condition is on the q_i (which we assume to be linearly independent). This is a similar condition to the one defining $\widetilde{B}_1$. Unlike all the terms defined earlier, we assume that the $(n-m)$-plane is contained in Y. This is indicated by the change in the color of the $(n-m)$-plane.

Similar incidence correspondences for $(k+1)$-tuples of points in Y are defined in the same way except that we switch $k+1$ and $d-k-1$ (i.e. switch the q_i with the p_j).

- $V := \{((p_1, \ldots, p_{k+1}), \Lambda) \in Y^{(k+1)} \times \mathbb{G}(n-m, n) : p_i \in \Lambda,\ p_i$ distinct, and either $|Y \cap \Lambda| = d$ or $\Lambda \subset Y\}$

  This is an analogue of W.

- $\widetilde{T} := \widetilde{T}_1 \sqcup \widetilde{T}_2$, where

$$\widetilde{T}_1 := \{((p_1, \ldots, p_{k+1}), \Lambda) \in V : p_1, \ldots, p_{k+1} \text{ linearly dependent}\}$$

  and

$$\widetilde{T}_2 := \{((p_1, \ldots, p_{k+1}), \Lambda) \in V : p_1, \ldots, p_{k+1} \text{ linearly independent but } (Y \cap \Lambda) \setminus \{p_1, \ldots, p_{k+1}\} \textit{ not} \text{ a linearly independent } (d-k-1)\text{-tuple}, \Lambda \not\subset Y\}.$$

  This is an analogue of $\widetilde{B} = \widetilde{B}_1 \sqcup \widetilde{B}_2$.

- $\widetilde{R} := \{((p_1, \ldots, p_{k+1}), \Lambda) \in V : p_1, \ldots, p_{k+1}$ linearly independent, $\Lambda \subset Y\}$

  This is an analogue of $\widetilde{A}$.

- Variable size restrictions:

  - $d \geq k+3$
  - $d-k-1 \leq n-m-1$
  - (only in Section 2.2.3) $k+1 \leq n-m-1$
  - $n-m \leq m-1$
  - $d \geq (n-m)+2$

  This last condition on the bottom right is used to ensure that Y is nondegenerate variety that is not a rational normal scroll or Veronese surface (Proposition 0 and Theorem 1 on p. 3 of [10]). The remaining conditions come from the incidence correspondences involved in the proof of Proposition 2.2.1.

Under the variable restrictions listed above, these incidence correspondences can be used to obtain a higher–dimensional version of the $Y-F(Y)$ relation. Note that the same reasoning implies a higher degree analogue using the analogous objects from Section 2.2.3. In both cases, the idea is to match up maximally linearly independent points on each side. The following proposition relates the various strata of V and W.

Proposition 2.2.1. *(Extended $Y-F(Y)$-relation)*
Suppose that $\overline{k}=k$ and $\operatorname{char} k = 0$. Then,

$$[W]-[\widetilde{B}]-[\widetilde{A}] = [V]-[\widetilde{R}]-[\widetilde{T}] \textit{ in } K_0(\mathrm{Var}_k).$$

Proof. As with the original $Y-F(Y)$ relation (Theorem 5.1 on p. 16 of [16]), we show that the residual intersection map induces an equality in $K_0(\mathrm{Var}_k)$ of the non-degenerate loci which are considering. Suppose that $Y \subset \mathbb{P}^n$ is a variety of dimension m of degree $d \leq n-m$ over a field K such that $\overline{k}=k$ and $\operatorname{char} k = 0$. By the definitions on p. $10-11$, the term $[W]-[\widetilde{B}]-[\widetilde{A}]$ gives the class of $((p_1,\ldots,p_{d-k-1}),\Lambda) \in W \subset Y^{(d-k-1)} \times \mathbb{G}(n-m,n)$ such that $p_1,\ldots,p_{d-k-1}$ are linearly independent, $\Lambda \not\subset Y$, and $(Y\cap\Lambda)\setminus\{p_1,\ldots,p_{d-k-1}\}$ form a linearly independent $(k+1)$-tuple of points. Let $J \subset W$ be the open subvariety of $((p_1, \ldots, p_{d-k-1}), \Lambda)$ satisfying these conditions. Similarly, let $K \subset V$ be the subset of pairs $((p_1, \ldots, p_{k+1}), \Lambda) \subset V \subset Y^{(k+1)} \times \mathbb{G}(n-m, n)$ such that $p_1, \ldots, p_{k+1}$ are linearly indendent, $\Lambda \not\subset Y$, and $(Y \cap \Lambda) \setminus \{p_1, \ldots, p_{k+1}\}$ form a linearly independent $(d-k-1)$-tuple of points. In the notation of the definitions listed on p. 11 – 12, $[K] = [V] - [\widetilde{R}] - [\widetilde{T}]$.

Below, we show that $[J] = [K]$. While the the residual intersection map of the type given in the proof of the $Y - F(Y)$ relation (Theorem 5.1 on p. 16 of [16], Example 1.1.12 on p. 372 of [4]) gives a bijection between points of J and K, it isn't completely obvious why the residual intersection map should give a well-defined morphism/isomorphism. However, we can use projections from an incidence correspondence to show that there are indeed morphisms which induce a bijection of k-rational points between J and K. The following result implies that this is enough to show equality in $K_0(\mathrm{Var}_k)$:

Proposition 2.2.2. *(Proposition 1.4.11 on p. 65 of [4])*
Let K be a of characteristic 0 and let $S = \operatorname{Spec} K$. Let $\overline{K}$ be an algebraically closed extension of k. Let $f : Y \longrightarrow X$ be a morphism of k-varieties such that the induced map $f(\overline{K}) : Y(\overline{K}) \longrightarrow X(\overline{K})$ is bijective. Then f is a piecewise isomorphism.

Let

$$S = \{((p_1, \ldots, p_{d-k-1}), (q_1, \ldots, q_{k+1}), \Lambda) : ((p_1, \ldots, p_{d-k-1}), \Lambda) \in J$$
$$, ((q_1, \ldots, q_{k+1}), \Lambda) \in K, p_i \neq q_j \text{ for all } i, j\}$$

Note that this is a subset of $Y^{(d-k-1)} \times Y^{(k+1)} \times \mathbb{G}(n-m, n)$. Consider the projections

$$\begin{array}{ccc} & S & \\ \swarrow^{\varphi} & & \searrow^{\psi} \\ J \subset W & & K \subset V \end{array}$$

given by

$$\varphi : ((p_1, \ldots, p_{d-k-1}), (q_1, \ldots, q_{k+1}), \Lambda) \mapsto ((p_1, \ldots, p_{d-k-1}), \Lambda)$$

and

$$\psi : ((p_1, \ldots, p_{d-k-1}), (q_1, \ldots, q_{k+1}), \Lambda) \mapsto ((q_1, \ldots, q_{k+1}), \Lambda).$$

Since $\overline{k} = k$, each of the morphisms φ and ψ induce bijections of k-rational points. Since $\operatorname{char} k = 0$, Proposition 2.2.2 implies that $[S] = [J]$ and $[S] = [K]$. Thus, we have that $[J] = [K]$ and

$$[W] - [\widetilde{B}] - [\widetilde{A}] = [V] - [\widetilde{R}] - [\widetilde{T}] \text{ in } K_0(\mathrm{Var}_k)$$

as desired. □

Remark 2.2.3.

1. The methods used in the proof of Proposition 2.2.1 indicate why we made the variable restrictions on p. 12. More specificially, they ensure the existence of linearly independent k-tuples of points spanning a linear subspace of dimension $< n-m$ intersecting a variety $Y \subset \mathbb{P}^n$ of degree d and dimension m.

2. Applying the proof of Proposition 2.2.1 to analogues of S with linear dependence among the p_i or q_j implies that the subsets $\widetilde{B}_2$ and $\widetilde{T}_2$ used in the definition of the extended $Y - F(Y)$ relation are constructible since the image of a constructible set under a morphism is constructible. The other sets involved are locally closed via intersections of suitable subsets.

Apart from working with incidence correspondences matching complementary points of intersection with an $(n-m)$-plane and varying the number of points under consideration, the main difference from the analysis for the original $Y - F(Y)$ relation is that we remove terms involving $(n-m)$-dimensional planes that are tangent to Y. We can work out the modified relation in the case of a cubic hypersurface in more detail. Essentially, the idea is to subtract the terms involving tangencies from the initial pairs of points and incidence correspondences.

Example 2.2.4 (Cubic hypersurfaces).

Let $Y \subset \mathbb{P}^{r+1}$ be a smooth cubic hypersurface of dimension r. In the notation of the extended $Y - F(Y)$ relation, we are setting in $k = 1$, $d = 3$, $m = r$, and $n = r + 1$. Substituting these values into the terms of Proposition 2.2.1 defined on p. 10 – 12, we have that

$$W = \{(p, \ell) \in Y \times \mathbb{G}(1, r+1) : p \in \ell, \text{ and either } |Y \cap \ell| = 3 \text{ or } \ell \subset Y\}$$

and

$$V = \{((p_1, p_2), \ell) \in Y^{(2)} \times \mathbb{G}(1, r+1) : p_1 \neq p_2, p_i \in \ell, \text{ and either } |Y \cap \ell| = 3 \text{ or } \ell \subset Y\}.$$

This amounts to removing elements such that ℓ is tangent to Y and $\ell \not\subset Y$.

Next, we can show that $\widetilde{B} = \emptyset$. Note that $\widetilde{B}_1 = \emptyset$ since a single point of $\mathbb{P}^{r+1}$ cannot be linearly dependent. We also have that $\widetilde{B}_2 = \emptyset$ since the lines involved must be those which are tangent to Y and *not* contained in Y (which we omitted from W). Similarly, we have that $\widetilde{T}_2 = \emptyset$ since the only instance where the third point of intersection of Y with a line spanned by distinct points of Y is *not* a "linearly independent point" is when one exists. In other words, they must span a line tangent to Y. However, we already omitted such lines in the definition of V.

Thus, we have that

$$W \setminus \widetilde{A} = \{(p, \ell) \in Y \times \mathbb{G}(1, r+1) : p \in \ell, |Y \cap \ell| = 3\}$$

and

$$(V \setminus \widetilde{T}) \setminus \widetilde{R} = \{((p_1, p_2), \ell) \in Y^{(2)} \times \mathbb{G}(1, r+1) : p_1 \neq p_2, |Y \cap \overline{p_1, p_2}| = 3\}.$$

Since two distinct points determine a unique line, the second subset can be interpreted as a subset of $Y^{(2)}$. This reduces to the situation of the original $Y - F(Y)$ relation (Theorem 5.1 on p. 16 of [16])

2.2.2 A modified completion of $K_0(\mathrm{Var}_k)$

Motivic limits of terms in the extended $Y - F(Y)$ relation can be defined in a modified completion $\widehat{\mathcal{K}}$ of $K_0(\mathrm{Var}_k)$. While the "averaging" expressions involving this relation can be defined using a localization by $[\mathbb{G}(n-m,n)]$, this is not necessary and we only need a small modification of the usual completion $\widehat{\mathcal{M}_k}$ to do this. In addition, limits can be defined in a natural way, as is explained in more detail below.

Recall that $\widehat{\mathcal{M}_k} := \varprojlim \mathcal{M}_k / F^r \mathcal{M}_k$ is the completion of $\mathcal{M}_k := K_0(\mathrm{Var}_k)[\mathbb{L}^{-1}]$ with respect to the dimension filtration

$$\cdots \subset F^m \mathcal{M}_k \subset F^{m-1} \mathcal{M}_k \subset \cdots,$$

where $F^r \mathcal{M}_k$ is the subgroup of $\mathcal{M}_k$ spanned by classes of the form $\frac{[V]}{\mathbb{L}^i}$ with $\dim V - i \leq -r$ (p. 8 of [32], p. 111 – 112 of [4]).

Definition 2.2.5. Let $\mathcal{K}$ be the extension of scalars of $K_0(\mathrm{Var}_k)$ to $\mathbb{Q}$ made up of $\mathbb{Q}$-linear combinations of classes of varieties over k modulo the same additive relations $[X] = [Y] + [X \setminus Y]$ for closed subvarieties $Y \subset X$ and multiplicative relations $[X] \cdot [Y] = [X \times Y]$. We will write $\mathcal{K}_{\mathbb{C}}$ for the extension of scalars to $\mathbb{C}$.

Definition 2.2.6. Let $\widehat{\mathcal{K}}$ be the completion of $\mathcal{K}[\mathbb{L}^{-1}]$ with respect to the dimension filtration $\cdots \subset F^r \mathcal{K}[\mathbb{L}^{-1}] \subset F^{r-1} \mathcal{K}[\mathbb{L}^{-1}] \subset \cdots$, where $F^r \mathcal{K}[\mathbb{L}^{-1}]$ is the (additive) subgroup of $\mathcal{K}[\mathbb{L}^{-1}]$ spanned by elements of the form $c\frac{[V]}{\mathbb{L}^i}$ with $c \in \mathbb{Q}$ and $\dim V - i \leq -r$. Define the completion $\widehat{\mathcal{K}_{\mathbb{C}}}$ of $\mathcal{K}_{\mathbb{C}}[\mathbb{L}^{-1}]$ similarly with the same kind of filtration taking $c \in \mathbb{C}$ instead.

Using properties of formal power series, we can show that inverting polynomials in $\mathbb{L}$ such as $[\mathbb{G}(n-m,n)]$ is well-defined in the completion $\widehat{\mathcal{K}}$ with respect to the same dimension filtration as the one used for $\widehat{\mathcal{M}_k}$. We may also consider elements in $\widehat{\mathcal{K}_{\mathbb{C}}}$ while attempting to find an explicit expressions for the inverses used.

Proposition 2.2.7. *Given a nonzero polynomial $P \in \mathbb{Q}[T]$, the element $P(\mathbb{L})$ is invertible in $\widehat{\mathcal{K}}$.*

Proof. Since $\mathbb{L}$ is invertible in $\mathcal{K}$, we can assume without loss of generality that $P(0) \neq 0$. Let $d = \deg P$ and write

$$P(T) = a_d T^d + a_{d-1} T^{d-1} + \ldots + a_1 T + a_0.$$

If we "divide" by $\mathbb{L}^d$ (i.e. multiply by $\mathbb{L}^{-d}$), the resulting expression is a polynomial in $\mathbb{L}^{-1}$ with a nonzero constant term since

$$\begin{aligned} \mathbb{L}^{-d} P(\mathbb{L}) &= \mathbb{L}^{-d}(a_d \mathbb{L}^d + a_{d-1} \mathbb{L}^{d-1} + \ldots + a_1 \mathbb{L} + a_0) \\ &= a_d + a_{d-1} \mathbb{L}^{-1} + \ldots + a_1 \mathbb{L}^{-(d-1)} + a_0 \mathbb{L}^{-d}. \end{aligned}$$

Recall that a formal power series with coefficients in a field is invertible as a power series if and only if its constant term is nonzero. Since a polynomial is a (finite) power series and the leading coefficient $a_d \neq 0$, the polynomial $Q(U) = a_d + a_{d-1} U + \ldots + a_1 U^{d-1} + a_0 U^d$ has a power series inverse of the form

$$R(U) = c_0 + c_1 U + c_2 U^2 + \ldots$$

with $c_i \in \mathbb{Q}$.

We claim that $R(\mathbb{L}^{-1})$ gives an expression that is well-defined in $\widehat{\mathcal{K}}$. Since the filtration construction and inverse limit used to define $\widehat{\mathcal{K}}$ is essentially the same as the one used to define $\widehat{\mathcal{M}_k}$, an infinite sum converges in $\widehat{\mathcal{K}}$ if and only if the dimensions of the terms approaches $-\infty$ for the same reasoning as sums in $\mathbb{Q}_p$ (Exercise 2.5 on p. 9 of [32]). Since $\dim c_m \mathbb{L}^{-m} = -m$ for each m, this clearly holds for $R(\mathbb{L}^{-1})$ and this infinite sum is well-defined in $\widehat{\mathcal{K}}$. Thus, the term $\mathbb{L}^{-d}P(\mathbb{L})$ has an inverse in $\widehat{\mathcal{K}}$. Since $\mathbb{L}$ is taken to be invertible, this implies that $P(\mathbb{L})$ itself is invertible in $\widehat{\mathcal{K}}$. $\square$

Remark 2.2.8.

1. For our purposes, it suffices to consider $P \in \mathbb{Z}[T]$ since the denominators in the formal expression for $\frac{[F_k(Y)]}{[\mathbb{G}(k,n)]}$ are polynomials in $\mathbb{L}$ with integer coefficients.

2. While the proof of Proposition 2.2.7 shows that an inverse of $P(\mathbb{L})$ exists in $\widehat{\mathcal{K}}$, it does not say something explicit about what the inverse should look like. In order to obtain some kind of (formal) decomposition, we will work with coefficients in $\mathbb{C}$ using $\widehat{\mathcal{K}_{\mathbb{C}}}$.

   As in the proof of Proposition 2.2.7, we will work with polynomials in $\mathbb{L}^{-1}$. Let $Q(U) = a_0 + a_1 U + \ldots + a_{m-1}U^{m-1} + a_m U^m$. Without loss of generality, we can assume that $a_m = 1$. Since we are working over $\mathbb{C}$, this (formally) means that

$$\begin{aligned} \frac{1}{Q(\mathbb{L})} &= \frac{1}{(\mathbb{L} - a_1)\cdots(\mathbb{L} - a_m)} \\ &= \prod_{r=1}^{m} \frac{1}{\mathbb{L} - a_r} \end{aligned}$$

   for some $a_i \in \mathbb{C}$.

   For each factor with $a_r \neq 0$, note that

$$\begin{aligned} \frac{1}{\mathbb{L} - c} &= \frac{1}{\mathbb{L}} \cdot \frac{1}{1 - c\mathbb{L}^{-1}} \\ &= \frac{1}{\mathbb{L}} \cdot \sum_{i=0}^{\infty} c^i \mathbb{L}^{-i} \end{aligned}$$

by the same reasoning as Exercise 2.7 on p. 9 of [32]. Substituting this back into our expression for $\frac{1}{Q(\mathbb{L})}$ gives a product of infinite sums with $\mathbb{L}^{-b}$ for some b.

Corollary 2.2.9. *The formal expression for $\frac{[F_k(Y)]}{[\mathbb{G}(k,n)]}$ is well-defined in $\widehat{\mathcal{K}}$.*

Proof. This follows from applying Proposition 2.2.7 to the denominators in the formal expression for $\frac{[F_k(Y)]}{[\mathbb{G}(k,n)]}$, which are polynomials in $\mathbb{L}$. Since $[\mathbb{P}^k] = 1 + \mathbb{L} + \ldots + \mathbb{L}^k$, the fact that symmetric products of sums in $K_0(\mathrm{Var}_k)$ can be expressed as products of symmetric products indexed by partitions (Remark 4.2 on p. 617 of [17], p. 6 of [16]) implies that $[(\mathbb{P}^k)^{(k+1)}]$ is a polynomial in $\mathbb{L}$. Alternatively, we can use the motivic zeta function $Z_{\mathbb{P}^k}(t) = \frac{1}{(1-t)(1-\mathbb{L}t)\cdots(1-\mathbb{L}^k t)}$ for $\mathbb{P}^k$ (p. 375 of [4]) as a generating function for the symmetric product.

As for $\mathbb{G}(k,n)$, we use the fact that

$$[\mathbb{G}(k,n)] = [G(k+1,n+1)] = \prod_{j=1}^{k+1} \frac{\mathbb{L}^{n-k+j} - 1}{\mathbb{L}^j - 1},$$

which follows from a row reduction/Schubert cell argument (Example 2.4.5 on p. 72 – 73 of [4]). $\square$

The grading of a term in $\widehat{\mathcal{K}}$ in the dimension filtration will be called the relative dimension.

Definition 2.2.10. The *relative dimension* of a term $\frac{[P]}{F(\mathbb{L})}$ in $\mathcal{K}[\mathbb{L}^{-1}]$ is $\dim P - \deg F$. This can be extended uniquely to the relative dimension of a term in $\widehat{\mathcal{K}}$ (Remark 2.2.11).

Remark 2.2.11. Given a fixed r, It is clear how to define the dimension for an element of $\mathcal{K}[\mathbb{L}^{-1}]/F^r\mathcal{K}[\mathbb{L}^{-1}]$. Writing each element of $\widehat{\mathcal{K}}$ as a compatible system of elements of $\mathcal{K}[\mathbb{L}^{-1}]/F^r\mathcal{K}[\mathbb{L}^{-1}]$ for varying r, the dimension in $\widehat{\mathcal{K}}$ is defined as the maximum among the dimensions in $\mathcal{K}[\mathbb{L}^{-1}]/F^r\mathcal{K}[\mathbb{L}^{-1}]$ of each nonzero component. Each component is nonzero and of the same dimension if there is *some* nonzero component with positive dimension since the $F^r\mathcal{K}[\mathbb{L}^{-1}]$ keep track of elements with negative dimensions. If all the nonzero components have negative dimensions, all of them actually have the "maximal" one $-a$ since any remaining nonzero components only differ by elements of $F^r\mathcal{K}[\mathbb{L}^{-1}]$ for some $r \geq a+1$. Finally, we take $\dim 0 = 0$ since $\dim c = 0$ for a nonzero constant c.

2.2.3 Dimension computations

Now that we have shown that the extended $Y - F(Y)$ relation (Proposition 2.2.1) holds and defined where motivic limits are taken, we will start to compute the dimensions of terms involved. The computations are split into two subsections according to whether the degree is small or large relative to the codimension. We will first consider the case where $d-k-1 \leq n-m-1$ in Section 2.2.3. In Section 2.2.3, similar ideas will be used to obtain dimension counts when $d-k-1 > n-m-1$.

Low degree nondegenerate varieties ($d-k-1 \leq n-m-1$)

There are decompositions of $\widetilde{B}$ and $\widetilde{T}$ that induce a simplification of the identity $[W]-[\widetilde{B}]-[\widetilde{A}] = [V]-[\widetilde{R}]-[\widetilde{T}]$.

Proposition 2.2.12.

1. *The identities*

$$[W]-[\widetilde{B}_1] = ([Y^{(d-k-1)}]-[N])[G(n-m+1-(d-k-1), n+1-(d-k-1))]-[\widetilde{A}]-[P]$$

and

$$[V]-[\widetilde{T}] = ([Y^{(k+1)}]-[M])[G(n-m+1-(k+1), n+1-(k+1))]-[\widetilde{R}]-[Q]$$

hold in $K_0(\mathrm{Var}_k)$*, where*

- $N \subset Y^{(d-k-1)}$ *is the set of* $(d-k-1)$*-tuples that are linearly* dependent
- $M \subset Y^{(k+1)}$ *is the set of* $(k+1)$*-tuples that are linearly* dependent
- $P \subset Y^{(d-k-1)} \times \mathbb{G}(n-m,n)$ *is the set of* $((p_1,\ldots,p_{d-k-1}),\Lambda)$ *such that* $p_i \in \Lambda$ *for each* i*,* Λ *is* not *transversal to* Y*,* $\Lambda \not\subset Y$*, and* $p_1,\ldots,p_{d-k-1}$ *are linearly independent*
- $Q \subset Y^{(k+1)} \times \mathbb{G}(n-m,n)$ *is the set of* $((p_1,\ldots,p_{k+1}),\Lambda)$ *such that* $p_i \in \Lambda$ *for each* i*,* Λ *is* not *transversal to* Y*,* $\Lambda \not\subset Y$*, and* $p_1,\ldots,p_{k+1}$ *are linearly independent.*

2. *Part 1 implies that*

$$\begin{aligned}
&([Y^{(d-k-1)}]-[N])[G(n-m+1-(d-k-1),n+1-(d-k-1))]\\
&\qquad -[P]-[\widetilde{B}_2]-2[F_{n-m}(Y)]([(\mathbb{P}^{n-m})^{(d-k-1)}]-[D])\\
&= ([Y^{(k+1)}]-[M])[G(n-m+1-(k+1),n+1-(k+1))] \qquad (2.2.1)\\
&\qquad -[Q]-[\widetilde{T}_2]-2[F_{n-m}(Y)]([(\mathbb{P}^{n-m})^{(k+1)}]-[C]),
\end{aligned}$$

(2.2.2)

where $D \subset (\mathbb{P}^{n-m})^{(d-k-1)}$ *and* $C \subset (\mathbb{P}^{n-m})^{(k+1)}$ *are linearly dependent* $(d-k-1)$*-tuples and* $(k+1)$*-tuples of points in* $\mathbb{P}^{n-m}$ *respectively.*

Proof. 1. The assertions that

$$[W]-[\widetilde{B}_1]+[P] = ([Y^{(d-k-1)}]-[N])[\mathbb{G}(n-m-(d-k-1),n+1-(d-k-1))]-[\widetilde{A}]$$

and

$$[V]-[\widetilde{T}_1]+[Q] = ([Y^{(k+1)}]-[M])[\mathbb{G}(n-m-(k+1),n+1-(k+1))]-[\widetilde{R}]$$

in $K_0(\mathrm{Var}_k)$ do not immediately follow from the fibers "looking the same". For example, there is no obvious isomorphism from $C \setminus \{p\}$ to $C \setminus \{q\}$ for an arbitrary choice of $p, q \in C$ if C is a curve of genus $g \geq 2$ since $\mathrm{Aut}\, C$ is finite. Also, the map $\mathbb{G}_m \longrightarrow \mathbb{G}_m$ sending $z \mapsto z^d$ seems to indicate that $K_0(\mathrm{Var}_k)$ behaves poorly with respect to covers. However, the map $V \setminus \widetilde{T} \sqcup Q \longrightarrow U$ sending $((p_1, \ldots, p_{k+1}), \Lambda) \mapsto (p_1, \ldots, p_{k+1})$ is a piecewise trivial fibration with fiber $\mathbb{G}(n-m-(k+1), n-(k+1))$ (Proposition 2.3.4 on p. 70 of [4]).

Alternatively, we can build a bijection of rational points. Let $U \subset Y^{(k+1)}$ be the subset consisting of linearly independent $(k+1)$-tuples of points. Consider the map

$$\varphi : U \times \mathbb{G}(n-m-k-1, n-k-1) \longrightarrow (V \setminus \widetilde{T}_1) \sqcup Q$$

sending $((p_1, \ldots, p_{k+1}), \Gamma) \mapsto ((p_1, \ldots, p_{k+1} l), \langle \overline{p_1, \ldots, p_{k+1}}, \Gamma \rangle)$, where Γ is taken to parametrize $(n-m-k)$-dimensional linear subspaces of the orthogonal complement of $\overline{p_1, \ldots, p_{k+1}}$ in $\mathbb{A}^{n+1}$. Note that two elements of $U \times \mathbb{G}(n-m-k-1, n-k-1)$ mapping to the same element need to start with the same element of U. The second coordinate is the same if and only if the Γ-coordinates parametrize the same $(n-m-k)$-dimensional linear subspaces of $\mathbb{A}^{n+1}$. Then, the morphism φ induces an injection on k-rational points. The morphism φ also induces a surjection on k-rational points since $\Gamma \cong \overline{p_1, \ldots, p_{k+1}} \oplus (\Gamma / \overline{p_1, \ldots, p_{k+1}})$ for any affine linear subspace $\Gamma \supset \overline{p_1, \ldots, p_{k+1}}$.

Proposition 2.2.2 then implies that

$$[V] - [\widetilde{T}_1] + [Q] + [\widetilde{R}] = ([Y^{(k+1)}] - [M])[G(n-m+1-(k+1), n+1-(k+1))].$$

The same reasoning implies that

$$[W]-[\widetilde{B}_1]+[P]+[\widetilde{A}] = ([Y^{(d-k-1)}]-[N])[G(n-m+1-(d-k-1), n+1-(d-k-1))].$$

2. This follows from rewriting the extended $Y - F(Y)$ relation

$$[W] - [\widetilde{B}] - [\widetilde{A}] = [V] - [\widetilde{R}] - [\widetilde{T}]$$

as

$$([W]-[\widetilde{B}_1]+[P]+[\widetilde{A}])-[\widetilde{A}]-[P]-[\widetilde{B}_2]-[\widetilde{A}] = ([V]-[\widetilde{T}_1]+[Q]+[\widetilde{R}])-[\widetilde{R}]-[Q]-[\widetilde{T}_2]-[\widetilde{R}]$$

and making substition from Part 1.

$\square$

If we work in $\widehat{\mathcal{K}}$ instead, the relation 2.2.1 in Part 2 of Proposition 2.2.12 can be converted to

$$\begin{aligned}
&\frac{2[F_{n-m}(Y)]([(\mathbb{P}^{n-m})^{(k+1)}] - [(\mathbb{P}^{n-m})^{(d-k-1)}])}{[\mathbb{G}(n-m,n)]} \\
&= \frac{([Y^{(k+1)}] - [M])[G(n-m+1-(k+1), n+1-(k+1))]}{[\mathbb{G}(n-m,n)]} \\
&- \frac{([Y^{(d-k-1)}] - [N])[G(n-m+1-(d-k-1), n+1-(d-k-1))]}{[\mathbb{G}(n-m,n)]} \\
&+ \frac{[P]-[Q]}{[\mathbb{G}(n-m,n)]} + \frac{[\widetilde{B}_2]-[\widetilde{T}_2]}{[\mathbb{G}(n-m,n)]} \\
&+ \frac{2[F_{n-m}(Y)]([C]-[D])}{[\mathbb{G}(n-m,n)]}.
\end{aligned}$$

We will now compute the relative dimensions (i.e. dimensions in $\widehat{\mathcal{K}}$) of the generic and degenerate terms in this identity. The objective of the remainder of this section is to prove the upper bounds for the dimensions of the degeneracy loci listed below. Note that the relative dimensions are listed at the end of this section on p. 22 – 24.

- $\dim N \leq m(d-k-2)-1$ (Proposition 2.2.14)
- $\dim M \leq mk-1$ (Proposition 2.2.14)
- $\dim P \leq m(n-m+1)-m-1+(d-k-1)$ (Proposition 2.2.15)
- $\dim Q \leq m(n-m+1)-m-1+(k+1)$ (Proposition 2.2.15)
- $\dim \widetilde{B}_2 \leq -2(n-m-(k-2)-1)$ as a *relative* dimension in $\widehat{\mathcal{K}}$ (Proposition 2.2.25)
- $\dim \widetilde{T}_2 \leq -2(n-m-(d-k-4)-1)$ as a *relative* dimension in $\widehat{\mathcal{K}}$ (Proposition 2.2.26)
- $\dim C = (n-m)k+k-1$ (Lemma 2.2.13)
- $\dim D = (n-m)(d-k-2)+(d-k-2)-1$ (Lemma 2.2.13)

We first compute $\dim C$ and $\dim D$. The following lemma implies that $\dim D = (n-m)(d-k-2)+(d-k-2)-1$ and $\dim C = (n-m)k+k-1$.

Lemma 2.2.13. *Let $R_k \subset (\mathbb{P}^n)^{(k+1)}$ of $(k+1)$-tuples which form the columns of a $(n+1)\times(k+1)$ matrix of rank $\leq k$. The dimension of R_k is $nk+k-1$.*

Proof. By an incidence correspondence argument, the dimension of the variety $M \subset \mathbb{P}^{(n+1)(k+1)-1}$ of $(n+1)\times(k+1)$ matrices of rank k up to scalars is $(nk+n+k)-(n-k+1) = nk+2k-1$ (Proposition 12.2 on p. 151 of [18]). The quotients by $\mathbb{C}^\times$ and S_{k+1} indicated in the diagram below imply that $\dim R_k = nk+k-1$.

$$\begin{tikzcd}
(\mathbb{A}^{n+1})^{k+1} \arrow[r, "g"] \arrow[d, "f"'] & (\mathbb{P}^n)^{k+1} \arrow[d, "\pi"] \\
M \subset \mathbb{P}^{(n+1)(k+1)-1} & (\mathbb{P}^n)^{(k+1)}
\end{tikzcd}$$

□

For our purposes, it suffices to give relatively coarse upper bounds on the dimensions of the degeneracy loci M and N.

Proposition 2.2.14. $\dim M \leq mk-1$ *and* $\dim N \leq m(d-k-2)-1$.

Proof. A generic $(k-1)$-plane or $(d-k-3)$-plane spanned by distinct k-tuples or $(d-k-2)$-tuples does *not* intersect an additional point of Y. This follows from our assumptions that $k+1 \leq n-m-1$ and $d-k-1 \leq n-m-1$ by an argument using the Uniform Position Theorem (p. 370 – 371 of [12]). □

The dimension bounds for P and Q follow from the definition of a tangent linear subspace.

Proposition 2.2.15.

$$\dim P \leq m(n-m+1)-m-1+(d-k-1)$$

with "relative dimension" (Definition 2.2.10) $-m-1+(d-k-1)$

and

$$\dim Q \leq \dim Q \leq m(n-m+1)-m-1+(k+1)$$

with relative dimension $-m-1+(k+1)$.

Proof. Without loss of generality, we look at the case of P since the dimension bound for Q has the same proof. Note that P consists of elements of the form $((p_1,\ldots,p_{d-k-1}),\Lambda)$ with p_i linearly independent, $p \in \Lambda$, $p \in Y$, and Λ *not* transversal to Y. We can partition the possible $(n-m)$-planes in question into the dimension of the intersection $Y \cap \Lambda$. Let $L_u \subset \mathbb{G}(n-m,n)$ be the space of such $(n-m)$-planes.

The space L_u is a subset of the space of $(n-m)$-planes $\Lambda \subset \mathbb{P}^n$ such that $\dim(T_qY \cap T_q\Lambda) \geq u$ for some $q \in Y \cap \Lambda$. In some sense, this measures how far the intersection q

is from being transverse. To find the dimension of the latter space, consider the incidence correspondence $J_u = \{(q, \Lambda) \in Y \times \mathbb{G}(n-m, n) : q \in \Lambda, \dim(T_q Y \cap T_q \Lambda) \geq u\}$ and the projection $\alpha : J_u \longrightarrow \mathbb{G}(n-m, n)$ sending $(q, \Lambda) \mapsto \Lambda$. The definition of L_u implies that $\dim \alpha^{-1}(\Lambda) = u$ for $\Lambda \in L_u$ and $\dim L_u = u + \dim \alpha^{-1}(L_u)$. Using these definitions, our earlier observation can be rewritten as the statement that $\dim L_u \leq \dim \alpha(J_u)$. Note that $J_u = \bigcup_{v \geq u} \alpha^{-1}(L_v)$.

To find $\dim J_u$, consider the projection $\beta_u : J_u \longrightarrow Y$ sending $(q, \Lambda) \mapsto q$. Then, we have that $\dim \beta_u^{-1}(q)$ is equal to the dimension of the space of $(n-m)$-planes containing q whose intersection with the tangent plane to q has dimension $\geq u$. To find the dimension of this fiber, we look at a map/projection which parametrizes these $(n-m)$-planes in terms of possible u-planes contained in the intersection $Y \cap \Lambda$ (i.e. elements of $\mathbb{G}(u, m)$ which give u-dimensional linear subspaces Γ of $T_q M \cong \mathbb{P}^m$). For a particular choice of Γ, the possible $(n-m)$-planes in $\mathbb{P}^n$ containing them is parametrized by elements of $\mathbb{G}(n-m-u, n-u)$. Since $\dim \mathbb{G}(u, m) = (u+1)(m-u)$ and $\dim \mathbb{G}(n-m-u, n-u) = m(n-m-u+1)$, we have that $\dim \beta^{-1}(q) = (u+1)(m-u) + m(n-m-u+1)$.

Putting these together, we have that $\dim J_u \leq \dim Y + (u+1)(m-u) + m(n-m-u+1) = m + (u+1)(m-u) + m(n-m-u+1)$. This means that $\dim \alpha(J) = \dim J - u = m + (u+1)(m-u) + m(n-m-u+1) - u$. Recall that $\alpha(J)$ is the space of $(n-m)$-planes in $\mathbb{P}^n$ whose intersection with Y has dimension $\geq u$.

We return to the original incidence correspondence P. Consider the projection $\gamma : P \longrightarrow \mathbb{G}(n-m, n)$ sending $((p_1, \ldots, p_{d-k-1}), \Lambda) \mapsto \Lambda$. The image $\gamma(P)$ can be partitioned into elements of the form $J = J_u$ for some $1 \leq u \leq \min(m-1, n-m-1)$. Note that we will actually take the upper bound is equal to $m-1$ under the assumptions of Theorem

2.1.7. We would like to study $\dim \gamma^{-1}(J_u)$ and see how this varies as we increase u since the J_u partition $\alpha(P)$ and their preimages under γ cover P. After going from u to $u+1$, we find that the dimension of the base *decreases* by $2m-u$. In other words, we have that $J(u)-J(u+1)=2u+3$. For the preimages, we find that they *increase* by $d-k-1$ since the space of possible p_i increases by 1 from u to $u+1$ for each $1 \leq i \leq d-k-1$. The net change in dimension is then $\dim \gamma^{-1}(\alpha(J_u) - \dim \gamma^{-1}(\alpha(J_{u+1})) = 2m-u-(d-k-1)$. If $d-k-1$ is smaller than $2m$, this means that $\dim J_u$ is a decreasing function in u and the value at $u=1$ gives the upper bound $\dim P \leq m(n-m+1)-m-1+(d-k-1)$. Note that the former condition is satisfied under the conditions of Part 1 of Theorem 2.1.7. Replacing $d-k-1$ with $k+1$, the same reasoning implies that $\dim Q \leq m(n-m+1)-m-1+(k+1)$. Note that the sample size here is very small as given in Part 1 of Theorem 2.1.7.

$\square$

The remaining degeneracy loci whose dimensions we need to compute are $\widetilde{B}_2 \subset W$ and $\widetilde{T}_2 \subset V$. The same method will be used to study each space.

We will first study the behavior of $\dim \widetilde{B}_2$. Recall that

$$\begin{aligned}\widetilde{B}_2 = \{((p_1,\dots,p_{d-k-1}),\Lambda) \in W : p_1,\dots,p_{d-k-1} \text{ linearly independent but}\\ (Y \cap \Lambda) \setminus \{p_1,\dots,p_{d-k-1}\} \textit{ not} \text{ a linearly independent } (k+1)\text{-tuple}, \Lambda \not\subset Y\}.\end{aligned}$$

Before making dimension computations, here is a small observation which shows that the result we will use applies to *any* $(k+1)$-tuple spanning a linear subspace of dimension $\leq k-1$. Afterwards, we state the definition of a term used in the result.

Lemma 2.2.16. *Any μ-plane $\Gamma \subset \mathbb{P}^n$ that intersects a variety $X \subset \mathbb{P}^m$ at a finite number of points contains a $(\mu-1)$-plane disjoint from X.*

Proof. To obtain such an $(\mu - 1)$-plane, we can intersect Γ with a hyperplane which does *not* contain any points of $\Gamma \cap X$. For example, we can use the hyperplane $x_i = c$ for some c which is not the i^{th} coordinate of any points of $\Gamma \cap X$. $\square$

Definition 2.2.17. (Ran, p. 716 of [35])

1. Given a subvariety $X \subset \mathbb{P}^m$ and a linear λ-plane Λ disjoint from X, denote by $X_k^\Lambda \subset \mathbb{P}^{m-\lambda-1}$ the locus of fibers of length k or more of the projection $\pi_\Lambda : X \longrightarrow \mathbb{P}^{m-\lambda-1}$. Thus, X_k^Λ is the locus of $(\lambda + 1)$-planes containing Λ which meet X in a scheme of length $\geq k$.

2. The analogous projection and locus of fibers for *generic* Λ is denoted $X_k^\lambda \subset \mathbb{P}^{m-\lambda-1}$. In practice, we state that some property holds for X_k^λ when it holds for X_k^Λ given a generic choice of $\Lambda \in \mathbb{G}(\lambda, m)$. The meaning of "generic" is further explained below in Remark 2.2.18.

Remark 2.2.18.

By "generic" choice of $\Lambda \in \mathbb{G}(\lambda, m)$, we mean the complement of a nowhere dense analytic subset (p. 699 of [35]). For example, let $H \subset \mathbb{G}(n-m, n)$ be the set of $n-m$-planes which intersect the m-dimensional variety $Y \subset \mathbb{P}^n$ at $d := \deg X$ distinct points. This is also an open subset in the Zariski open topology (Corollaire 2.3 on p. 259 of arXiv link and p. 318 in [17]). Since the intersection of a dense subset with an open subset is dense in the open subset, the intersection of H with the generic locus in Definition 2.2.17 is dense in H.

By Lemma 2.2.16, the locus of *all* $(\lambda+1)$-planes which meet X in a scheme of length $\geq k$ is a union of subsets of the form X_k^Λ for some λ-plane Λ disjoint from X. Here is the main result which we use to prove our claim.

Theorem 2.2.19. *(Ran, Theorem 5.1 on p. 716 of [35])*
Let $X \subset \mathbb{P}^m$ be an irreducible closed subvariety of codimension $c > \lambda \geq 0$. Then X_k^λ is smooth of codimension $k(c-\lambda-1)$ in $\mathbb{P}^{m-\lambda-1}$, in a neighborhood of any point image of a fiber of length exactly k that is disjoint from the singular locus of X and has embedding dimension 2 or less.

Remark 2.2.20. 1. Subsets that have codimension strictly larger than the dimension of the ambient space are taken to be empty (p. 699 of [35]).

2. In the definition of V and W from the extended $Y-F(Y)$ relation (Proposition 2.2.1), we assumed that $|Y \cap \Lambda| = d$ for $(n-m)$-planes . Since the curvilinear subscheme of a Hilbert scheme of r points on a smooth projective variety is formed by the closure unordered tuples of r distinct points, we will study curvilinear schemes in our setting. Any λ-plane ($\lambda \leq n-m-1$) contained in these $(n-m)$ planes satisfies the embedding dimension condition of Theorem 2.2.19 since curvilinear schemes have local embedding dimension ≤ 1 (p. 703 of [35]).

3. While n is not explicitly defined in the statement of Corollary 5.6 on p. 717 of [35] (or anywhere in Section 5 of [35]), it is indicated that Theorem 2.2.19 is a partial extension of Theorem 4.1 on p. 713 of [35], which makes use of this notation.

Given $\lambda \leq k-2$, we can use this to compute the dimension of the space of non-tangent $(\lambda+1)$-planes intersecting Y at a linearly dependent $(k+1)$-tuple of points. Similarly, the same method can be used for $\lambda \leq d-k-2$ and $(d-k-1)$-tuples of points.

Proposition 2.2.21. *Suppose that $Y \subset \mathbb{P}^n$ is a smooth closed irreducible variety of dimension m and degree d.*

1. *Given $\lambda \leq k-2$, the space of $(\lambda+1)$-planes in $\mathbb{P}^n$ intersecting Y at $\geq k+1$ points and contain* some *generic λ-plane (in the sense of Theorem 2.2.19) which are* not *tangent to Y or contained in Y has dimension* $\dim \mathbb{G}(\lambda, n) + \dim Y^{\lambda}_{k+1} - (\lambda+1)$.

2. *Given $\lambda \leq d-k-2$, the space of $(\lambda+1)$-planes in $\mathbb{P}^n$ intersecting Y at $\geq d-k-1$ points and contain* some *generic λ-plane which are* not *tangent to Y or contained in Y has dimension* $\dim \mathbb{G}(\lambda, n) + \dim Y^{\lambda}_{d-k-1} - (\lambda+1)$.

Proof. 1. Let

$$\begin{aligned}\mathcal{A} = \{(\Lambda, \Gamma) \in \mathbb{G}(\lambda, n) \times \mathbb{G}(\lambda+1, n) : \Lambda \subset \Gamma, \Lambda \text{ generic and not tangent to } Y, \\ |Y \cap \Lambda| \geq k+1, \Lambda \not\subset Y\}.\end{aligned}$$

Consider the projections

$$\begin{tikzcd} & \mathcal{A} \arrow[dl, "\varphi"'] \arrow[dr, "\psi"] & \\ \mathbb{G}(\lambda, n) & & \mathbb{G}(\lambda+1, n). \end{tikzcd}$$

In this diagram, the space of $(\lambda+1)$-planes intersecting Y at $\geq k+1$ and contain *some* generic λ-plane which are *not* tangent to Y or contained in Y is given by $\psi(\mathcal{A})$. Thus, it suffices to compute $\dim \psi(\mathcal{A})$.

Let $U \subset \mathbb{G}(\lambda, n)$ be the space of generic λ-planes not tangent to Y. Then, the definition of $\mathcal{A}$ implies that $U = \varphi(\mathcal{A})$. For each $\Lambda \in U$, we have that $\dim \varphi^{-1}(\Lambda) = \dim Y_{k+1}^{\lambda}$. On the other hand, we have that $\psi^{-1}(\Gamma) \subset \mathbb{G}(\lambda, \lambda+1) \cong \mathbb{P}^{\lambda+1}$ is a nonempty open subset for each $\Gamma \in \psi(\mathcal{A})$. Since $\mathbb{P}^{\lambda+1}$ is irreducible, this implies that $\psi^{-1}(\Gamma)$ forms a dense open subset. Thus, we have that $\dim \psi^{-1}(\Gamma) = \lambda + 1$ for each $\Gamma \in \psi(\mathcal{A})$. Putting these together, Corollary 11.13 on p. 139 of [18] implies that

$$\begin{aligned}
\dim \psi(\mathcal{A}) &= \dim \mathcal{A} - (\lambda+1) \\
&= \dim U + \dim Y_{k+1}^{\lambda} - (\lambda+1) \\
&= \dim \mathbb{G}(\lambda, n) + \dim Y_{k+1}^{\lambda} - (\lambda+1).
\end{aligned}$$

2. This uses the same steps as part 1 except that k is replaced with $d-k-2$.

□

Under appropriate conditions, we can omit $(n-m)$-planes that do *not* contain any generic $(n-m-1)$-planes in $\mathbb{P}^n$.

Proposition 2.2.22. *Let $Z \subset \mathbb{G}(n-m, n)$ be the complement of the locus of generic $(n-m-1)$-planes (Theorem 2.2.19) and $\mathbb{G}(n-m-1, n) \hookrightarrow \mathbb{P}^N$ be the Plücker embedding. If Z is contained in some hypersurface generic in its degree, then the set of $(n-m)$-planes in $\mathbb{P}^n$ whose $(n-m-1)$-subplanes are all contained in Z form a finite subset which is empty if the degree is ≥ 2. Note that* some *condition is necessary in order to have such a codimension.*

Proof. Each polynomial F in the ideal defining $Z \subset \mathbb{G}(n-m-1, n)$ can be considered as a polynomial in affine (Plücker) coordinates $(a_{i,j})_{\substack{1\leq i\leq n-m\\ 1\leq j\leq m+1}}$ with F modified depending on the specific chart $\begin{pmatrix} I_{n-m} & A \end{pmatrix}$ by precomposing with right multiplication by some element of GL_{n+1}. Recall that the standard affine chart of $G(r,n)$ corresponds to r-dimensional linear subspaces which do *not* intersect a specific $(n-r)$-plane nontrivially and transition maps are given by GL_r-actions. In the statement above, the $(n-m)$-planes $\Gamma \in \mathbb{G}(n-m,n)$ are exactly those such that $\Gamma \cap H \in Z$ for *all* hyperplanes $H \subset \mathbb{P}^n$. We can relate this back to the usual affine chart on $\mathbb{G}(n-m,n)$.

On a standard chart for $\mathbb{G}(n-m,n) = G(n-m+1,n+1)$, we can represent $\Gamma \in \mathbb{G}(n-m,n)$ as a matrix of the form $\begin{pmatrix} I_{n-m+1} & B \end{pmatrix}$, where $B = (b_{r,s})$ is an $(n-m+1)\times m$ matrix corresponding to an element of $\mathbb{A}^{m(n-m+1)}$. Recall that we wanted to have $\Gamma \cap H \in Z$ for each hyperplane $H \subset \mathbb{P}^n$. Note that *rows* of $\begin{pmatrix} I_{n-m+1} & B \end{pmatrix}$ represent elements of $\mathbb{P}^n$. Let $\beta_i = (b_{i,1},\ldots,b_{i,m})$ be the i^{th} row of B. Given $\Gamma \in \mathbb{G}(n-m,n)$ and its chart representation $\begin{pmatrix} I_{n-m+1} & B \end{pmatrix}$, the $(n-m-1)$-dimensional subspaces of Γ correspond to elements of the form

$$\begin{pmatrix} 1 & 0 & \cdots & 0 & \alpha_1 \\ 0 & 1 & \cdots & 0 & \alpha_2 \\ \vdots & \vdots & \cdots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & \alpha_{n-m} \end{pmatrix} \cdot g \cdot \begin{pmatrix} 1 & 0 & \cdots & 0 & 0 & \text{---} & \beta_1 & \text{---} \\ \vdots & \vdots & \cdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & 1 & \text{---} & \beta_{n-m+1} & \text{---} \end{pmatrix}$$

for some $g \in GL_{n-m+1}$. The rows of the product of the first matrix with g give bases of $(n-m-1)$-dimensional subspaces of Γ with respect to the basis given by the rows of the last matrix. The first matrix can be rewritten as the $(n-m)\times(n-m+1)$ matrix $\begin{pmatrix} I_{n-m} & \alpha \end{pmatrix}$ for some $\alpha \in \mathbb{A}^{n-m}$ and the second one is the $(n-m+1)\times n$ matrix $\begin{pmatrix} I_{n-m+1} & B \end{pmatrix}$ representing Γ with $\beta_i = (b_{i,1}, \ldots, b_{i,m})$ the i^{th} row of B. The term $\begin{pmatrix} I_{n-m} & \alpha \end{pmatrix}$ comes from considering representations of $(n-m-1)$-dimensional subspaces of $\mathbb{P}^{n-m}$ and g is gives a change of basis/change of coordinates which moves between charts in the affine covers of $\mathbb{G}(n-m,n)$ and $\mathbb{G}(n-m-1,n)$ which we are using here. We will first consider the case $g = I_{n-m+1}$ and reduce the general case to this afterwards.

In these coordinates, the $(n-m-1)$-dimensional subspaces of the $(n-m)$-dimensional linear subspace Γ represented by $\begin{pmatrix} I_{n-m+1} & B \end{pmatrix}$ satisfy $(F = 0)$ (under the appropriate chart/multiplication by an element of GL_{n-m}) if and only if $F = 0$ on the $(n-m) \times (m+1)$ submatrix

$$\begin{pmatrix} 1 & 0 & \cdots & 0 & \alpha_1 \\ 0 & 1 & \cdots & 0 & \alpha_2 \\ \vdots & \vdots & \cdots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & \alpha_{n-m} \end{pmatrix} \begin{pmatrix} 0 & — & \beta_1 & — \\ \vdots & \vdots & \vdots & \vdots \\ 1 & — & \beta_{n-m+1} & — \end{pmatrix}$$
$$= \begin{pmatrix} \alpha_1 & — & \beta_1 + \alpha_1 \beta_{n-m+1} & — \\ \vdots & \vdots & \vdots & \vdots \\ \alpha_{n-m} & — & \beta_{n-m} + \alpha_{n-m}\beta_{n-m+1} & — \end{pmatrix}$$
$$= \begin{pmatrix} \alpha_1 & — & \beta_1 + \alpha_1 \beta_{n-m+1} & — \\ \vdots & \vdots & \vdots & \vdots \\ \alpha_{n-m-1} & — & \beta_{n-m-1} + \alpha_{n-m-1}\beta_{n-m+1} & — \\ 0 & — & 0 & — \end{pmatrix}$$
$$+ \begin{pmatrix} 0 & — & 0 & — \\ \vdots & \vdots & \vdots & \vdots \\ \alpha_{n-m} & — & \beta_{n-m} + \alpha_{n-m}\beta_{n-m+1} & — \end{pmatrix}$$

for *all* choices of $\alpha_1, \ldots, \alpha_{n-m}$.

The same reasoning can be applied to other charts (i.e. other choices of $g \in GL_{n-m+1}$) by replacing

$$\begin{pmatrix} 0 & — & \gamma_1 & — \\ \vdots & \vdots & \vdots & \vdots \\ 1 & — & \gamma_{n-m+1} & — \end{pmatrix}$$

with

$$\begin{pmatrix} — & \gamma_1 & — \\ \vdots & \vdots & \vdots \\ — & \gamma_{n-m+1} & — \end{pmatrix} = g \cdot \begin{pmatrix} 0 & — & \beta_1 & — \\ \vdots & \vdots & \vdots & \vdots \\ 1 & — & \beta_{n-m+1} & — \end{pmatrix}$$

for appropriate $\gamma_i \in \mathbb{A}^{m+1}$.

Before computing the dimension of solutions to explicit polynomial equations, we will consider heuristics from expected dimensions of Fano varieties of k-planes contained in general hypersurfaces. For example, suppose that $n - m = 2$. Fixing γ_1 and γ_2, the solutions γ_3 to $F(\gamma_1 + \alpha_1\gamma_3, \gamma_2 + \alpha_2\gamma_3) = 0$ for all α_1, α_2 correspond to planes in $\mathbb{A}^m$ contained in the intersection of a hypersurface in $\mathbb{A}^{2m}$ with the complete intersection of hypersurfaces of the form $\widetilde{H}_j = x_1 x_{m+j} - x_j x_{m+1}$ for $2 \leq j \leq m+3$. If this is a complete intersection with $(F = 0)$, then the fact that planes in $\mathbb{A}^{2m+2}$ correspond to lines in $\mathbb{P}^{2m+1}$ implies that the expected dimension of lines (p. 4 of [10]) contained in this complete intersection is $2(2m+1) - 3(m-1) - c = m - c + 5$, where c is the degree of F as a polynomial in $2m+2$ variables. However, the condition that the lines are of the type $(x, 0) + (0, x) \cdot \beta$ gives a codimension m condition and we would generically expect the set to be empty for sufficiently large c.

Starting with a fixed γ_1, γ_2 as above, we can work out the (usually) codimension m condition on γ_3 more explicitly. Again, we would like to find $x \in \mathbb{A}^m$ such that $F(x, \beta x) = 0$ for all β. This boils down to coefficients in using terms involving β being set equal to 0. Generically, this reduces the dimension by e, where e is the degree of F with respect to the final m coefficients. In general, the equations involved can be analyzed using the Taylor expansion of F at a particular point. Given $\gamma_i = (\gamma_{i,1}, \ldots, \gamma_{i,m}) \in \mathbb{A}^m$, we study solutions to

$$F((\gamma_1, \ldots, \gamma_r) + (\alpha_1 \gamma_{r+1}, \ldots, \alpha_r \gamma_{r+1})) = F(\gamma_1, \ldots, \gamma_r) + \sum_{i,j} \frac{\partial F}{\partial x_{ij}}(\gamma)(\alpha_i \gamma_{r+1,j})$$

$$+\frac{1}{2!} \sum_{i,j,k,l} \frac{\partial^2 F}{\partial x_{ij} \partial x_{kl}}(\gamma)(\alpha_i \gamma_{r+1,j})(\alpha_k \gamma_{r+1,l})$$

$$+\frac{1}{3!} \sum_{i,j,k,l,p,q} \frac{\partial^3 F}{\partial x_{ij} \partial x_{kl} \partial x_{pq}}(\gamma)(\alpha_i \gamma_{r+1,j})(\alpha_k \gamma_{r+1,l})(\alpha_p \gamma_{r+1,q}) + \ldots = 0$$

which hold for all $\alpha_1, \ldots, \alpha_r$. If $\alpha_1 \neq 0$, we can assume without loss of generality that $\alpha_1 = 1$. Note that there will be a total of $\deg F$ sums.

Interpreting F as a polynomial in the α_i with coefficients which are polynomials in the $\gamma_{i,j}$, we need all the coefficients in $\gamma_{r,s}$ to be equal to 0. Each term is a sum of the form

$$\sum_{\substack{1 \leq i_a \leq r \\ 1 \leq j_b \leq m+1}} \frac{\partial^u F}{\partial x_{i_1 j_1} \cdots \partial x_{i_u j_u}}(\gamma)(\alpha_{i_1} \gamma_{r+1,j_1}) \cdots (\alpha_{i_u} \gamma_{r+1,j_u}).$$

This gives the degree u terms as a polynomial in the α_i. Now consider the degree 1 term. The coefficient of α_i being 0 requires $m+1$ polynomials to vanish. Repeating this for each i already gives a total of $r(m+1)$ conditions. Since all the other coefficients are also equal to 0, the set of solutions is empty for a generic choice of F.

$\square$

Remark 2.2.23. Given a particular $(n-m)$-plane $\Gamma \in \mathbb{G}(n-m, n)$, the space of $(n-m-1)$-planes contained in Γ forms an $(n-m)$-dimensional linear subspace of $\mathbb{G}(n-m-1)$ (Theorem 3.16 on p. 110 and proof of Theorem 3.20 (ii) on p. 114 of [21]). Thus, the space of $(n-m)$-planes in $\mathbb{P}^n$ whose $(n-m-1)$-dimensional linear subspaces are contained in Z is contained in the space of maximal $(n-m)$-planes in $\mathbb{G}(n-m-1, n)$ which are contained in the hypersurface $Z \subset \mathbb{G}(n-m-1, n)$. With this interpretation, there are a couple more options for genericity conditions which imply Z is empty.

1. Let M be the subvariety of $F_{n-m}(\mathbb{G}(n-m-1, n)) \subset \mathbb{G}(n-m, N)$ consisting of $(n-m)$-planes in $\mathbb{G}(n-m-1, n)$ maximal with respect to inclusion. If M is a general GL_{N+1}-translate of M and Z is contained in some hypersurface of degree e in $\mathbb{P}^N$ of sufficiently large degree, then the subvariety of $\mathbb{G}(n-m, n)$ consisting of $(n-m)$-planes Γ such that $\Lambda \in Z$ for all $(n-m-1)$-planes $\Lambda \subset \Gamma$ is empty if e is sufficiently large compared to $n-m$ by Kleiman's transversality theorem (Theorem on p. 290 of [24]) while taking $\mathbb{G}(n-m, N) = G(n-m+1, N+1)$ to be a homogeneous space with a transitive GL_{N+1}-action.

2. We can follow the usual proof of the generic dimension estimates of Fano varieties of k-planes to show that the subvariety of $\mathbb{G}(n-m, n)$ consisting of $(n-m)$-planes Γ such that $\Lambda \in Z$ for all $(n-m-1)$-planes $\Lambda \subset \Gamma$ is empty if $n \gg 0$ and Z is contained in *some* generic hypersurface $A \subset \mathbb{P}^N$ *not* containing $\mathbb{G}(n-m-1, n)$.

   Let $\mathbb{G}(n-m-1, n) \hookrightarrow \mathbb{P}^N$ be the Plücker embedding and $\mathbb{P}^M$ with $M = \binom{N+e}{e} - 1$ be the space of degree e hypersurfaces in $\mathbb{P}^N$. Let

   $$\Phi = \{(\Gamma, A) \in \mathbb{G}(n-m, N) \times \mathbb{P}^M : \Gamma \subset A \cap \mathbb{G}(n-m-1, n), \mathbb{G}(n-m-1, n) \not\subset A\}.$$

Consider the projections

$$\begin{tikzcd}
& \Phi \arrow[dl, "\varphi"'] \arrow[dr, "\psi"] & \\
\mathbb{G}(n-m, N) & & \mathbb{P}^M.
\end{tikzcd}$$

Then, we have that $\psi^{-1}(A) \cong F_{n-m}(A \cap \mathbb{G}(n-m-1, n))$ and

$$\varphi^{-1}(\Gamma) = \{A \in \mathbb{P}^M : \Gamma \subset A \cap \mathbb{G}(n-m-1, n), \mathbb{G}(n-m-1, n) \not\subset A\}$$

for each $\Gamma \in \varphi(\Phi)$. Note that $\varphi(\Phi)$ consists of $(n-m)$-planes in $\mathbb{P}^N$ which are contained in $A \cap \mathbb{G}(n-m-1, n)$ for some degree e hypersurface $A \subset \mathbb{P}^N$.

Fix $\Gamma \in \varphi(\Phi)$. The $(n-m)$-plane Γ in $\mathbb{P}^N$ is contained in $A \cap \mathbb{G}(n-m-1, n)$ for *some* degree e hypersurface $A \subset \mathbb{P}^N$ *not* containing $\mathbb{G}(n-m-1, n)$ if and only if there is some $f \in H^0(\mathbb{G}(n-m-1, n), \mathcal{O}_{\mathbb{G}(n-m-1,n)}(e))$ such that $f|_\Gamma = 0$. In other words, f is in the kernel of the restriction map $\rho : H^0(\mathbb{G}(n-m-1, n), \mathcal{O}_{\mathbb{G}(n-m-1,n)}(e)) \longrightarrow H^0(\Gamma, \mathcal{O}_\Gamma(e))$. Note that this map is surjective since we assumed that Γ is contained in $\mathbb{G}(n-m-1, n)$. Since $\Gamma \cong \mathbb{P}^{n-m}$, we have that $\dim H^0(\Gamma, \mathcal{O}_\Gamma(e)) = \binom{n-m+e}{e}$. On the other hand, we have that

$$\dim H^0(\mathbb{G}(n-m-1, n), \mathcal{O}_{\mathbb{G}(n-m-1,n)}(e)) = \prod_{j=n-m+1}^{n+1} \frac{\binom{e+j-1}{e}}{\binom{e+j-(n-m)-1}{e}}$$

since the determinant of the dual of the tautological bundle is the pullback of $\mathcal{O}_{\mathbb{G}(n-m-1,n)}(1)$ by the Plücker embedding (Proposition 5.2 on p. 388 of [8]). This means that

$$\dim \varphi^{-1}(\Gamma) = \prod_{j=n-m+1}^{n+1} \frac{\binom{e+j-1}{e}}{\binom{e+j-(n-m)-1}{e}} - \binom{n-m+e}{e}$$

for each $\Gamma \in \varphi(\Phi) \subset \mathbb{G}(n-m, N)$.

Given suitable parameters, we have that $\dim \Phi < M$ and a generic element of $\mathbb{P}^M$ is *not* in the image of $\psi : \Phi \longrightarrow \mathbb{P}^M$.

The assumptions of Proposition 2.2.22 will be denoted using the following term.

Definition 2.2.24. A variety $Y \subset \mathbb{P}^n$ is *k-linearly generic* if the locus of non-generic k-planes in $\mathbb{P}^n$ in the sense of Proposition 2.2.22 is contained in a hypersurface generic in its degree.

Using Proposition 2.2.22 and Proposition 2.2.21, we can be bound the relative dimension (Definition 2.2.10) of $\widetilde{B_2}$.

Proposition 2.2.25. *If $Y \subset \mathbb{P}^n$ is u-linearly generic for $u \leq k-2$, the relative dimension (Definition 2.2.10) of* $\dim \widetilde{B}_2 \leq -2(n-m-(k-2)-1)$ *if* $k-2 \leq n-k+2$.

Proof. Since we assume that $Y \subset \mathbb{P}^n$ is u-linearly generic for $u \leq k-2$, we can assume that the $(\lambda+1)$-planes in question always contain some generic λ-plane. Let

$$S = \{(\Lambda, \Gamma) \in \mathbb{G}(\lambda+1, n) \times \mathbb{G}(n-m, n) : \Lambda \subset \Gamma, \Lambda \in Q, |\Gamma \cap Y| = d\}.$$

Consider the projections

$$\begin{tikzcd} & S \arrow[dl, "\varphi"'] \arrow[dr, "\psi"] & \\ \mathbb{G}(\lambda+1, n) & & \mathbb{G}(n-m, n). \end{tikzcd}$$

In this diagram, the space of $(n-m)$-planes containing some element of Q that is *not* tangent to Y is given by $\psi(S)$ and $Q = \varphi(S)$ (writing Q to mean $\psi(Q)$ from Proposition 2.2.21). Thus, it suffices to give an upper bound for $\dim \psi(S)$. Working over each irreducible component of S, Theorem 11.12 and Corollary 11.13 on p. 138 – 139 of [18] imply that $\dim S \geq \dim \psi(S) + M$ if $\dim \psi^{-1}(\Gamma) \geq M$ for each $\Gamma \in \psi(S)$. Rearranging this inequality gives the upper bound $\dim \psi(S) \leq \dim S - M$.

If we fix $\Lambda \in Q = \varphi(S)$, we have that $\varphi^{-1}(\Lambda) \subset \mathbb{G}(n-m-\lambda-2, n-\lambda-2)$ is a nonempty open subset for each $\Lambda \in \varphi(S) = Q$. Since $\mathbb{G}(n-m-\lambda-2, n-\lambda-2)$ is irreducible, this is a dense open subset and $\dim \varphi^{-1}(\Lambda) = \dim \mathbb{G}(n-m-\lambda-2, n-\lambda-2)$ for each $\Lambda \in \varphi(S)$. Although the fibers can be more complicated for ψ, we can still find a (relatively) uniform method of bounding the dimension.

Given a fixed $(n-m)$-plane $\Gamma \in \psi(S)$, we have that $\psi^{-1}(\Gamma)$ consists of $(\lambda+1)$-planes Λ such that $\Lambda \subset \Gamma$ and $|\Lambda \cap Y| \geq k+1$. In other words, we are looking for $(\lambda+1)$-planes contained in $\Lambda \cong \mathbb{P}^{n-m}$ that intersect Y in $\geq k+1$ points. Note that $Y \cap \Lambda \subset Y \cap \Gamma$. Since we take these k-planes to be contained in Y and $Y \cap \Lambda \subset Y \cap \Gamma$, we only have finitely many choices for their points of intersection with Y. In particular, we can express $\psi^{-1}(\Gamma)$ as the union of elements containing each $(k+1)$-tuple in $Y \cap \Gamma$. Given an unordered $(k+1)$-tuple of points in $Y \cap \Gamma$, let T_p be the elements of $\psi^{-1}(\Gamma)$ containing p. This implies that

$$\psi^{-1}(\Gamma) = \bigcup_p T_p \Rightarrow \dim \psi^{-1}(\Gamma) = \max_p \dim T_p,$$

where p varies over $(k+1)$-tuples of points in $Y \cap \Gamma$ which span a linear subspace of dimension $\mu \leq \lambda + 1$. These $(k+1)$-tuples can be further partitioned into locally closed subspaces corresponding to $(k+1)$-tuples spanning a linear subspace of a given dimension. Since there is a finite number of possible dimensions, it suffices to look at individual $(k+1)$-tuples and take the maximum dimension.

Given a fixed $(k+1)$-tuple in $Y \cap \Gamma$ spanning a linear subspace of dimension $\mu \leq \lambda + 1$, the space of $(\lambda+1)$-planes in $\Gamma \cong \mathbb{P}^{n-m}$ which contain these points is isomorphic to $\mathbb{G}(\lambda + 1 - \mu - 1, n - m - \mu - 1) = \mathbb{G}(\lambda - \mu, n - m - \mu - 1)$. We actually have a lower bound for the space of such $(\lambda+1)$-planes since

$$\begin{aligned}\dim \mathbb{G}(\lambda - \mu, n - m - \mu - 1) &= (n - m - \lambda - 1)(\lambda - \mu + 1) \\ &\geq n - m - \lambda - 1.\end{aligned}$$

Since this lower bound does not depend on μ, it applies to *any* $(k+1)$-tuple of points p. Thus, we have that $\dim \psi^{-1}(\Gamma) \geq n - m - \lambda - 1$ for each $\Gamma \in \psi(S)$ and we can set $M = n - m - \lambda - 1$ above. By Proposition 2.2.21 and Theorem 2.2.19, this implies that

$$\begin{aligned}\dim \psi(S) &\leq \dim S - M \\ &= \dim S - (n - m - \lambda - 1) \\ &= \dim Q + \dim \mathbb{G}(n - m - \lambda - 2, n - \lambda - 2) \\ &- (n - m - \lambda - 1) \\ &= \dim \mathbb{G}(\lambda, n) + \dim Y_{k+1}^{\lambda} - (\lambda + 1) + \dim \mathbb{G}(n - m - \lambda - 2, n - \lambda - 2) \\ &- (n - m - \lambda - 1) \\ &= (\lambda + 1)(n - \lambda) + (n - \lambda - 1) - (k + 1)((n - m) - \lambda - 1) - (\lambda + 1) \\ &+ m(n - m - \lambda - 1) - (n - m - \lambda - 1) \\ &= (\lambda + 1)(n - \lambda) - (k + 1)((n - m) - \lambda - 1) + m(n - m - \lambda - 1) + (m - \lambda - 1).\end{aligned}$$

Thus, the space of $(n-m)$-planes containing a $(\lambda+1)$-plane intersecting Y at $\geq k+1$

points which contains *some* generic λ-plane has dimension at most

$$(\lambda+1)(n-\lambda)-(k+1)((n-m)-\lambda-1)+m(n-m-\lambda-1)+(m-\lambda-1).$$

This implies the same bound for those which intersect Y at exactly $k+1$ points.

Let

$$D=(\lambda+1)(n-\lambda)-(k+1)((n-m)-\lambda-1)+m(n-m-\lambda-1)+(m-\lambda-1).$$

The relative dimension of these $(n-m)$-planes in $\widehat{\mathcal{K}}$ is

$$\begin{aligned} D-m(n-m+1) &= (\lambda+1)(n-\lambda)-(k+1)((n-m)-\lambda-1) \\ &\quad + m(n-m-\lambda-1)+(m-\lambda-1)-m(n-m+1) \\ &= (\lambda+1)(n-m-\lambda)-(k+1)(n-m-\lambda)+(k+1)-(\lambda+1) \\ &= -(k-\lambda)(n-m-\lambda)+(k-\lambda) \\ &= -(k-\lambda)(n-m-\lambda-1) \\ &\leq -2(n-m-(k-2)-1) \end{aligned}$$

since $\lambda \leq k-2$ and $k+1 \leq n-m-1$.

$\square$

The same reasoning with $d-k-2$ replacing k implies the following bound for upper bound for the dimension of $\widetilde{T}_2 \subset V$.

Proposition 2.2.26. *If $Y \subset \mathbb{P}^n$ is u-linearly generic for $u \leq d-k-2$, the relative dimension $\dim \widetilde{T}_2 \leq -2(n-m-(d-k-4)-1)$.*

Here is a summary of dimensions of the degeneracy loci:

- $\dim N \leq m(d-k-2)-1$ (Proposition 2.2.14)

- $\dim M \leq mk-1$ (Proposition 2.2.14)

- $\dim P \leq m(n-m+1)-m-1+(d-k-1)$ (Proposition 2.2.15)

- $\dim Q \leq m(n-m+1)-m-1+(k+1)$ (Proposition 2.2.15)

- $\dim \widetilde{B}_2 \leq -2(n-m-(k-2)-1)$ as a *relative* dimension in $\widehat{\mathcal{K}}$ (Proposition 2.2.25)

- $\dim \widetilde{T}_2 \leq -2(n-m-(d-k-4)-1)$ as a *relative* dimension in $\widehat{\mathcal{K}}$ (Proposition 2.2.26)

- $\dim C = (n-m)k+k-1$ (Lemma 2.2.13)

- $\dim D = (n-m)(d-k-2)+(d-k-2)-1$ (Lemma 2.2.13)

The remaining terms to analyze are $C \subset (\mathbb{P}^{n-m})^{(k+1)}$ and $D \subset (\mathbb{P}^{n-m})^{(d-k-1)}$ of linearly dependent $(k+1)$-tuples and $(d-k-1)$-tuples of $\mathbb{P}^{n-m}$. By Lemma 2.2.13, we have that $\dim C = (n-m)k+k-1$ and $\dim D = (n-m)(d-k-2)+(d-k-2)-1$.

Proposition 2.2.27. *In $K_0(\mathrm{Var}_k)$, the classes $[C]$ and $[D]$ are polynomials in $\mathbb{L}$.*

Proof. We will show this by finding a recursive formula. Given $u \leq r$, let $I_{u,n,r} \subset (\mathbb{P}^n)^{(r)}$ be the locally closed subset of r-tuples of points of $\mathbb{P}^n$ which form the columns of an $(n+1)\times r$ matrix of rank u. We claim that $[I_{u,n,r}] = [\mathbb{G}(u-1,n)][I_{u,u-1,r}]$ in $K_0(\mathrm{Var}_k)$. The idea is to fix the linear subspace spanned by the columns of the matrix and consider coordinates of the columns with respect to a fixed basis for this linear subspace. We can either use a piecewise trivial fibration from a morphism sending the r-tuples of points to their span or form a morphism inducing a bijection of rational points. For each $\Lambda \in \mathbb{G}(u-1,n)$, let A_Λ be a $(n+1)\times u$ matrix whose columns form a basis of Λ. Consider the morphism $\pi : \mathbb{G}(u-1,n) \times I_{u,u-1,r} \longrightarrow I_{u,n,r}$ defined by $(\Lambda, B) \mapsto A_\Lambda \cdot B$, where B is taken under quotients by permutations of columns and division of the columns by nonzero scalars.

Since the columns of A_Λ are linearly independent and B has u linearly independent columns, the span of $A_\Lambda \cdot B$ is Λ. Since two identical matrices have the same span and the columns of A_Λ are linearly independent, the map π is injective on k-rational points. The surjectivity of π comes from setting Λ to be the span of an element of $C \in I_{u,n,r}$ and B to be the matrix whose columns (up to quotienting) are the coordinates of the columns of C with respect to the columns of A_Λ. Thus, π induces a bijection on k-rational points and Proposition 2.2.2 implies that $[I_{u,n,r}] = [\mathbb{G}(u-1,n)][I_{u,u-1,r}]$ in $K_0(\mathrm{Var}_k)$.

By definition, we have that

$$[I_{u,u-1,r}] = [(\mathbb{P}^{u-1})^{(r)}] - \sum_{v=1}^{u-1} [I_{v,u-1,r}]. \tag{2.2.3}$$

For each $1 \leq v \leq u-1$, the same reasoning as above implies that $[I_{v,u-1,r}] = [\mathbb{G}(v-1,u-1)][I_{v,v-1,r}]$ and

$$[I_{v,v-1,r}] = [(\mathbb{P}^{v-1})^{(r)}] - \sum_{w=1}^{u-1} [I_{w,v-1,r}].$$

In each step of this recursion, the indices a, b in $I_{a,b,r}$ are strictly smaller than those in the previous step. So, this process must stop after a finite number of steps. Since $[I_{1,b,r}] = [\mathbb{P}^b]$ and $[I_{2,b,r}] = [\mathbb{G}(1,b)][I_{2,1,r}] = [\mathbb{G}(1,b)]([(\mathbb{P}^1)^{(r)}] - [\mathbb{P}^1])$, the reduction $[I_{u,n,r}] = [\mathbb{G}(u-1,n)][I_{u,u-1,r}]$ followed by induction on u in $I_{u,u-1,r}$ via the recursion 2.2.12 implies that $[I_{u,n,r}]$ is a polynomial in $\mathbb{L}$ for each $u \leq r \leq n+1$.

Since $[C] = [(\mathbb{P}^{n-m})^{(k+1)}] - [I_{k+1,n-m,r}]$ and $[D] = [(\mathbb{P}^{n-m})^{(d-k-1)}] - [I_{d-k-1,n-m,r}]$, they must also be polynomials in $\mathbb{L}$. $\square$

Remark 2.2.28. 1. The degrees of polynomials in $\mathbb{L}$ giving the classes of C and D in $K_0(\mathrm{Var}_k)$ are given by $\dim C = (n-m)k + k - 1$ and $\dim D = (n-m)(d-k-2) + (d-k-2) - 1$.

2. The coefficients of $\mathbb{L}^k$ can be expressed in terms of multinomial coefficients and sizes of partitions corresponding to certain Young tableaux (Example 2.4.5 on p. 72 – 73). These come from the classes of symmetric products $(\mathbb{P}^a)^{(b)}$ and Grassmannians $\mathbb{G}(c,d)$ respectively.

Next, we use the computations earlier in this section to find dimensions of terms of degeneracy loci in the expression

$$\frac{2[F_{n-m}(Y)]([(\mathbb{P}^{n-m})^{(k+1)}] - [(\mathbb{P}^{n-m})^{(d-k-1)}])}{[\mathbb{G}(n-m,n)]} \tag{2.2.4}$$

$$= \underbrace{\frac{([Y^{(k+1)}] - [M])[G(n-m+1-(k+1), n+1-(k+1))]}{[\mathbb{G}(n-m,n)]}}_{\text{Term 1}} \tag{2.2.5}$$

$$- \underbrace{\frac{([Y^{(d-k-1)}] - [N])[G(n-m+1-(d-k-1), n+1-(d-k-1))]}{[\mathbb{G}(n-m,n)]}}_{\text{Term 2}} \tag{2.2.6}$$

$$+ \underbrace{\frac{[P] - [Q]}{[\mathbb{G}(n-m,n)]}}_{\text{Term 3}} + \underbrace{\frac{[\widetilde{B}_2] - [\widetilde{T}_2]}{[\mathbb{G}(n-m,n)]}}_{\text{Term 4}} \tag{2.2.7}$$

$$+ \underbrace{\frac{2[F_{n-m}(Y)]([C] - [D])}{[\mathbb{G}(n-m,n)]}}_{\text{Term 5}}. \tag{2.2.8}$$

Here, we will take "degeneracy loci" to be non-generic subsets of incidence correspondences involved in the simplified higher dimensional $Y - F(Y)$ relation. Let $\alpha = \dim \mathbb{G}(n-$

$m, n) = m(n-m+1)$. Substituting in the (upper bounds of) dimensions of the degeneracy loci above yields the following dimensions in $\widehat{\mathcal{K}}$:

- Terms 1 and 2 from 2.2.5 and 2.2.7:
  - Main terms $[Y^{(k+1)}][G(n-m+1-(k+1), n+1-(k+1))]$ and $[Y^{(d-k-1)}][G(n-m+1-(d-k-1), n+1-(d-k-1))]$:

$$\begin{aligned}
\dim Y^{(k+1)} + \dim G(n-m+1-(k+1), n+1-(k+1)) - \alpha \\
= m(k+1) + m(n-m-k) - m(n-m+1) \\
= m(k+1+n-m-k-n+m-1) \\
= 0
\end{aligned}$$

$$\begin{aligned}
\dim Y^{(d-k-1)} + \dim G(n-m+1-(d-k-1), n+1-(d-k-1)) - \alpha \\
= m(d-k-1) \\
+m(n-m+1-(d-k-1)) \\
-m(n-m+1) \\
= m(n-m+1) - m(n-m+1) \\
= 0
\end{aligned}$$

  - Degenerate terms $[M][G(n-m+1-(k+1), n+1-(k+1))]$ and

$$[N][G(n-m+1-(d-k-1), n+1-(d-k-1))]:$$

$$
\begin{aligned}
\dim M + \dim G(n-m+1-(k+1), n+1-(k+1)) - \alpha \\
\leq mk - 1 + m(n-m-k) - m(n-m+1) \\
= m(k+n-m-k-n+m-1) - 1 \\
= m(-1) - 1 \\
= -m-1
\end{aligned}
$$

$$
\begin{aligned}
\dim N + \dim G(n-m+1-(d-k-1), n+1-(d-k-1)) - \alpha \\
\leq m(d-k-2) - 1 \\
+m(n-m+1-(d-k-1)) \\
-m(n-m+1) \\
= m((d-k-2) + (n-m+1) - (d-k-1) \\
-(n-m+1)) - 1 \\
= -m-1
\end{aligned}
$$

- Term 3 (tangent planes) from 2.2.7

$$
\begin{aligned}
\dim P - \alpha \leq S = m(n-m+1) - m - 1 + (d-k-1) - m(n-m+1) \\
= -m-1+(d-k-1)
\end{aligned}
$$

since $n-m > 2m$ under the conditions of Theorem 2.1.7.

The same reasoning with Proposition 2.2.15 implies that

$$
\dim Q - \alpha \leq -m-1+(k+1).
$$

- Term 4 (degenerate incidence correspondences) from 2.2.7:

  By Proposition 2.2.25 and Proposition 2.2.26, we have that

$$\dim \widetilde{B}_2 - \alpha \leq -2(n - m - (k - 2) - 1)$$

  and

$$\dim \widetilde{T}_2 - \alpha \leq -2(n - m - (d - k - 4) - 1).$$

- Term 5 (degeneracies involving $F_{n-m}(Y)$) from 2.2.8: If Y is contained in a smooth hypersurface $X \subset \mathbb{P}^n$ of degree r, Theorem 4.3 on p. 266 of [25] implies that

$$\begin{aligned}
\dim F_{n-m}(Y) + \dim C - \alpha &\leq \dim F(Y) + \dim C - \alpha \\
&= 2n - 3 - r + (n - m)k + k - 1 - m(n - m + 1) \\
&= (n - m)(k - m + 1) + n + k - r - 4 \\
&= -(n - m)(m - k - 1) + n + k - r - 4 \\
&\leq -(n - m)(m - k - 1) + n + k
\end{aligned}$$

and

$$
\begin{aligned}
\dim F_{n-m}(Y) + \dim D - \alpha \leq \dim F(Y) + \dim C - \alpha \\
&= 2n - 3 - r + (n-m)(d-k-2) \\
&+ (d-k-2) - 1 - m(n-m+1) \\
&= (n-m)((d-k-2) - m + 1) + n \\
&+ (d-k-2) - r - 4 \\
&= -(n-m)(m - (d-k-2) - 1) + n \\
&+ (d-k-2) - r - 4 \\
&\leq -(n-m)(m - (d-k-2) - 1) + n + (d-k-2).
\end{aligned}
$$

- Variable size restrictions:
  - $d \geq k+3$
  - $k+1 \leq n-m-1$
  - $d \geq (n-m)+2$
  - $d-k-1 \leq n-m-1$
  - $n-m \leq m-1$

Higher degree varieties ($d-k-1 > n-m-1$)

Most of the ideas in Section 2.2.3 carry over for the dimension estimates in the case where $d-k-1 > n-m-1$. The key difference is that the extended $Y - F(Y)$ relation involves different sets since a generic $(d-k-1)$-tuple lying on an $(n-m)$-plane is not linearly independent, but spans a linear subspace of dimension $n-m$ in $\mathbb{P}^n$. Let

$$
J = \{((p_1, \ldots, p_{d-k-1}), \Lambda) \in V : \; p_i \text{ distinct}, \; \dim \overline{p_1, \ldots, p_{d-k-1}} = n-m\}
$$

and $\widetilde{J} \subset J$ be the subset where $\Lambda \not\subset Y$. Note that $J = V \setminus \widetilde{T}_1$. Finally, let $\widetilde{T}_{11} \subset \widetilde{T}_1$ be the subset with $\Lambda \not\subset Y$ and $T_{12} \subset \widetilde{T}_1$ be the subset with $\Lambda \subset Y$. In the notation below, we have that $[T_{12}] = [F_{n-m}(Y)][D]$ in $K_0(\mathrm{Var}_k)$.

The setup in Section 2.2.3 (p. 17 – 18) implies that

$$\begin{aligned}([Y^{(k+1)}] - [M])[G(n-m+1-(k+1), n+1-(k+1))] - [Q] - [\widetilde{T}_2] \\ -2[F_{n-m}(Y)]([(\mathbb{P}^{n-m})^{(k+1)}] - [C]) \\ = [V] - [\widetilde{T}] - [\widetilde{R}] \\ = ([V] - [\widetilde{T}_1]) - [\widetilde{T}_2] - [\widetilde{R}] \\ = ([V] - [\widetilde{T}_1] + [Q] + [\widetilde{R}]) - [\widetilde{R}] - [Q] - [\widetilde{T}_2] - [\widetilde{R}]\end{aligned}$$

Taking this into account and using the proof of Proposition 2.2.1 for the variables listed below gives the following expression in $\widehat{\mathcal{K}}$:

$$\frac{2[F_{n-m}(Y)]([(\mathbb{P}^{n-m})^{(k+1)}] - [\mathrm{UConf}_{d-k-1}\,\mathbb{P}^{n-m}])}{[\mathbb{G}(n-m,n)]} \tag{2.2.9}$$

$$= \underbrace{\frac{([Y^{(k+1)}] - [M])[G(n-m+1-(k+1), n+1-(k+1))]}{[\mathbb{G}(n-m,n)]}}_{\text{Term 1}} \tag{2.2.10}$$

$$- \underbrace{\frac{[J]}{[\mathbb{G}(n-m,n)]}}_{\text{Term 2}} - \underbrace{\frac{[Q]}{[\mathbb{G}(n-m,n)]}}_{\text{Term 3}} + \underbrace{\frac{[\widetilde{B}_2] - [\widetilde{T}_2]}{[\mathbb{G}(n-m,n)]}}_{\text{Term 4}} + \underbrace{\frac{2[F_{n-m}(Y)]([C] - [D])}{[\mathbb{G}(n-m,n)]}}_{\text{Term 5}} \tag{2.2.11}$$

where

$$J = \{((p_1, \ldots, p_{d-k-1}), \Lambda) \in V : \; p_i \text{ distinct}, \; \dim \overline{p_1, \ldots, p_{d-k-1}} = n-m\}.$$

Note that $J = W \setminus \widetilde{B}_1$ using the definition of $\widetilde{B}_1$ below. In this higher degree setting, we take

- $D \subset (\mathbb{P}^{n-m})^{(d-k-1)}$ is the set of $(d-k-1)$-tuples of distinct points spanning a linear subspace of dimension $\leq n-m-1$. This can also be embedded inside $\mathrm{UConf}_{d-k-1}\,\mathbb{P}^{n-m}$, where $\mathrm{UConf}_r\,X \subset X^{(r)}$ denotes unordered r-tuples of distinct points on X.

- $C \subset (\mathbb{P}^{n-m})^{(k+1)}$ is the set of linearly dependent $(k+1)$-tuples of points in $\mathbb{P}^{n-m}$

- $\widetilde{B} = \widetilde{B}_1 \sqcup \widetilde{B}_2$, where

$$\widetilde{B}_1 = \{((p_1, \ldots, p_{d-k-1}), \Lambda) \in W : \ p_i \text{ distinct}, , \dim \overline{p_1, \ldots, p_{d-k-1}} \leq n-m-1\}$$

and

$$\begin{aligned}\widetilde{B}_2 = \{((p_1, \ldots, p_{d-k-1}), \Lambda) \in W : \ p_i \text{ distinct}, \ \dim \overline{p_1, \ldots, p_{d-k-1}} = n-m \text{ but} \\ (Y \cap \Lambda) \setminus \{p_1, \ldots, p_{d-k-1}\} \textit{ not} \text{ a linearly independent } (k+1)\text{-tuple}, \Lambda \not\subset Y\}\end{aligned}$$

- $\widetilde{T} = \widetilde{T}_1 \sqcup \widetilde{T}_2$, where

$$\widetilde{T}_1 = \{((p_1, \ldots, p_{k+1}), \Lambda) \in V : p_1, \ldots, p_{k+1} \text{ linearly dependent}\}$$

and

$$\begin{aligned}\widetilde{T}_2 = \{((p_1, \ldots, p_{k+1}), \Lambda) \in V : p_1, \ldots, p_{k+1} \text{ linearly independent but} \\ (Y \cap \Lambda) \setminus \{p_1, \ldots, p_{k+1}\} \textit{ not} \text{ a } (d-k-1)\text{-tuple spanning} \\ \text{an } (n-m)\text{-dimensional linear subspace }, \Lambda \not\subset Y\}\end{aligned}$$

As in Section 2.2.3, our goal of this section is to compute the dimensions listed below. The relative dimensions in $\widehat{\mathcal{K}}$ are listed on p. 28 – 29.

- $\dim M \leq mk - 1$ (Proposition 2.2.14)
- $\dim Q \leq m + (m - (n-m))(n-m)$ (Proposition 2.2.15)
- $\dim \widetilde{B}_2 \leq -2(n - m - (k-2) - 1)$ as a *relative* dimension in $\widehat{\mathcal{K}}$ (Proposition 2.2.25)
- $\dim \widetilde{T}_2 = \emptyset$ (Proposition 2.2.29)
- $\dim C = (n-m)k + k - 1$ (Lemma 2.2.13)
- $\dim D = (n-m-1)(d-k) - (d-k-1)$ This follows from the proof of Lemma 2.2.13 in Section 2.2.3.

It suffices to show that there are m, n, d, k satisfying these inequalities along with the following variable restrictions:

- $d \geq k+3$
- $k + 1 \leq n - m - 1$
- $n - m \leq m - 1$ (implies that $d - k - 1 > n - m - 1$)
- $d \geq (n-m) + 2$

Since the variable restrictions are compatible with the setting of Proposition 2.2.25, we only need to compute a bound for the relative dimension of $\widetilde{T}_2$. This follows from repeating the same steps with a change in parameters.

Proposition 2.2.29. *If $Y \subset \mathbb{P}^n$ is u-linearly generic for $u \leq d - k - 1$ (Definition 2.2.24), the first part of Proposition 2.2.21 implies that $\widetilde{T}_2 = \emptyset$.*

Proof. In Theorem 2.2.19, we will take $\lambda \leq n - m - 2$. Since we assumed that $d - k - 1 > n$ in Part 2 of Theorem 2.1.7, the locus in question is empty since the total space is $\mathbb{P}^{n-\lambda-1}$ and the codimension is $(d-k-1)((n-m)-\lambda-1)$. The convention in Remark 2.2.20 implies that $\widetilde{T}_2 = \emptyset$. $\square$

Combining this with Proposition 2.2.22 and Proposition 2.2.25, we obtain the following dimensions for the degeneracy loci:

- $\dim M \leq mk - 1$ (Proposition 2.2.14)
- $\dim Q \leq m + (m - (n - m))(n - m)$ (Proposition 2.2.15)
- $\dim \widetilde{B}_2 \leq 2(n - m - (k - 2) - 1)$ as a *relative* dimension in $\widehat{\mathcal{K}}$ (Proposition 2.2.25)
- $\widetilde{T}_2 = \emptyset$ (Proposition 2.2.29)
- $\dim C = (n - m)k + k - 1$ (Lemma 2.2.13)
- $\dim D = (n - m - 1)(d - k) - (d - k - 1)$ This follows from the proof of Lemma 2.2.13 in Section 2.2.3.

Before computing the relative dimensions, we write give a higher degree counterpart to Proposition 2.2.27 for $(d - k - 1)$-tuples.

Proposition 2.2.30. *In $K_0(\mathrm{Var}_k)$, the classes $[C]$ and $[D]$ are polynomials in $\mathbb{L}$.*

Proof. Since C is defined in the same way as the low degree case, it remains to consider D, which considers $(d-k-1)$-tuples which aren't necessarily linearly independent. This means that we need to add the condition that the points of $\mathbb{P}^{n-m}$ corresponding to columns of the matrices considered are distinct. However, the underlying recursion argument is identical to that used in Proposition 2.2.27.

Given $u \leq r$, let $K_{u,n,r} \subset (\mathbb{P}^n)^{(r)}$ be the locally closed subset of r-tuples of *distinct* points of $\mathbb{P}^n$ which form the columns of an $(n + 1) \times r$ matrix of rank u. The reasoning in the proof of Proposition 2.2.27 $[K_{u,n,r}] = [\mathbb{G}(u - 1, n)][K_{u,u-1,r}]$ in $K_0(\mathrm{Var}_k)$. We fix the linear subspace spanned by the columns of the matrix and consider coordinates of the columns with respect to a fixed basis of this linear subspace.

As in Proposition 2.2.27, the definition of $K_{u,n,r}$ implies that

$$[K_{u,u-1,r}] = [\mathrm{UConf}_{u-1}\,\mathbb{P}^{n-m}] - \sum_{v=1}^{u-1}[I_{v,u-1,r}]. \tag{2.2.12}$$

where $\mathrm{UConf}_r\,X \subset X^{(r)}$ denotes unordered r-tuples of distinct points on X.

For each $1 \leq v \leq u-1$, the same reasoning as above implies that $[K_{v,u-1,r}] = [\mathbb{G}(v-1, u-1)][K_{v,v-1,r}]$ and

$$[K_{v,v-1,r}] = [\mathrm{UConf}_r\,\mathbb{P}^{v-1}] - \sum_{w=1}^{u-1}[K_{w,v-1,r}].$$

In each step of this recursion, the indices a, b in $K_{a,b,r}$ are strictly smaller than those in the previous step. So, this process must stop after a finite number of steps. Since $[K_{1,b,r}] = 0$ as we're considering distinct points of $\mathbb{P}^{n-m}$ and $[K_{2,b,r}] = [\mathbb{G}(1,b)][K_{2,1,r}] = [\mathbb{G}(1,b)][\mathrm{UConf}_r\,\mathbb{P}^1]$, the reduction $[K_{u,n,r}] = [\mathbb{G}(u-1,n)][K_{u,u-1,r}]$ followed by induction on u in $K_{u,u-1,r}$ via the recursion 2.2.12 implies that $[K_{u,n,r}]$ is a polynomial in $\mathbb{L}$ for each $u \leq r \leq n+1$ if the unordered configuration spaces $\mathrm{UConf}_r\,\mathbb{P}^{n-m}$ are polynomials in $\mathbb{L}$.

We can show this using the standard decomposition of projective space into affine spaces. A bijection of rational points implies that

$$[\mathrm{UConf}_r\,X] = \left[\bigsqcup_{i+j=r}(\mathrm{UConf}_i\,A \times \mathrm{UConf}_j\,B)\right] = \sum_{i+j=r}[\mathrm{UConf}_i\,A][\mathrm{UConf}_j\,B]$$

if $X = A \sqcup B$ with A and B locally closed in X. This reduces the question to showing that $\mathrm{UConf}_r\,\mathbb{L}^k$ is a polynomial in $\mathbb{L}$, which follows from Lemma 2.2.31.

Since

$$[C] = [(\mathbb{P}^{n-m})^{(k+1)}] - [K_{k+1,n-m,r}]$$

and

$$[D] = [\operatorname{UConf}_{d-k-1} \mathbb{P}^{n-m}] - [K_{d-k-1,n-m,r}],$$

they must also be polynomials in $\mathbb{L}$. □

Here is the proof of the lemma used in the proof of Proposition 2.2.30. The main idea is to split to squared and squarefree parts.

Lemma 2.2.31. *Let K be a field of characteristic 0 and X be an affine variety over K. There is a recursive formula for the class of $\operatorname{UConf}^n X$ in $K_0(\mathrm{Var}_K)$:*

$$[\operatorname{UConf}^n X] = [\operatorname{Sym}^n X] - \sum_{k \geq 1} [\operatorname{UConf}^{n-2k} X][\operatorname{Sym}^k X]$$

Note that we use the convention $[\operatorname{UConf}_0 X] = 1$.

Proof. This follows the strategy outlined in the proof of Theorem 1.2 on p. 4 of [14] (proof on p. 7 – 9). The main difference is that $\operatorname{Sym}^n X \not\cong X^n$ for arbitrary varieties X if we don't assume $X = \mathbb{A}^n$ or $X = \mathbb{P}^n$. Our assumption that X is affine is used to show that its image under the diagonal map is closed (i.e. X is separated) and that the topology on X^n/S_n is the quotient topology.

Given an element of X^n, let r be the number of distinct elements (written $x_1, \ldots, x_r$) and m_i be the multiplicity of x_i (i.e. the number of times x_i appears). Let Q_k be the subset of X^n such that $\sum_{i=1}^r \left\lfloor \frac{m_i}{2} \right\rfloor \geq k$. This is an analogue of polynomials of degree n such that the squarefree part has degree $\leq n - 2k$ (preimage of $m = 1$ and $n = 2$ case of $R_{n,k}^{d,m}$ in p. 7 of [14]). Note that this is preserved under the action of S_n on X^n which permutes the coordinates.

We claim that $Q_k \subset X^n$ is closed. Continuing to put $m = 1$ and $n = 2$ in the proof in [14], let $\mathcal{S}$ be the set of injections $\sigma : \{1, 2\} \times \{1, \dots, k\} \hookrightarrow \{1, \dots, n\}$ such that $\sigma(1, a) < \sigma(1, b)$ if $a < b$ and $\sigma(1, j) < \sigma(2, j)$ for each $1 \leq j \leq k$. The first coordinate corresponds to the "copy" of the squared polynomial h in $f = gh^2$ with g squarefree for a particular polynomial f. The condition that $\sigma(1, a) < \sigma(1, b)$ means that we only count which k-tuples of slots occupied by the roots rather than the particular order that the roots are placed. Similarly, the relative ordering of roots in the first and second copy of h is fixed by the condition $\sigma(1, j) < \sigma(2, j)$.

Consider the sets $L_\sigma := \{x_{\sigma(1,b)} = x_{\sigma(2,b)} \forall 1 \leq b \leq k\} \subset X^n$. This matches up k of the roots in the two copies of h in some particular collection of slots corresponding to the embedding σ. For a fixed value of b, the points of X^n satisfying the condition are isomorphic to a product of X^{n-2} with the diagonal $\Delta_X \subset X^2$. Note that $\Delta_X \subset X^2$ is closed since X is affine (and therefore separated). This means that L_σ is an intersection of closed subsets of X^n, which is closed. The connection to the claim above is that $Q_k = \bigcup_{\sigma \in \mathcal{S}} L_\sigma$. This is because we need to match up at least k pairs to obtain an element of Q_k and the condition defining L_σ implies that $\sum_{i=1}^{r} \left\lfloor \frac{m_i}{2} \right\rfloor \geq k$ for any element of $\bigcup_{\sigma \in \mathcal{S}} L_\sigma \subset X^n$. Since Q_k is a finite union of closed sets, it is closed. Finally, this implies that Q_k/S_n is closed in $X^n/S_n = \operatorname{Sym}^n X$ since the topology on quotients of affine varieties by finite groups is the quotient topology (e.g. see Proposition 1.1 in [30]).

Let $R_k = Q_k/S_n \subset X^n/S_n$. We claim that the map

$$\varphi : \operatorname{UConf}^{n-2k} X \times \operatorname{Sym}^k X \subset \operatorname{Sym}^{n-2k} X \times \operatorname{Sym}^k X \longrightarrow R_k \setminus R_{k+1} \subset \operatorname{Sym}^n X$$

induced by the map

$$X^{n-2k} \times X^k \longrightarrow X^n$$

sending $((x_1,\dots,x_{n-2k}),(y_1,\dots,y_k)) \mapsto ((x_1,\dots,x_{n-2k}),(y_1,\dots,y_k),(y_1,\dots,y_k))$ gives a bijection of rational points in $\overline{k}$. Note that φ is *not* an isomorphism even in the case $X=\mathbb{A}^1$ as mentioned on p. 8 of [14]. It is clear that φ is surjective over k. The map φ is also injective since the (unordered) repeated k-tuple and additional $n-2k$ points (of multiplicity 1) added uniquely determine an element of $R_k \setminus R_{k+1}$. Thus, φ gives a bijection of $\overline{k}$-rational points. By Proposition 1.4.11 on p. 65 of [4], this implies that φ is a piecewise isomorphism and $[\mathrm{UConf}^{n-2k} X][\mathrm{Sym}^k X] = [R_k]-[R_{k+1}]$. Since $[\mathrm{UConf}^n X]=[R_0]-[R_1]$ and $R_0=\mathrm{Sym}^n X$, we can add all the terms to obtain the statement in the proposition. $\square$

Example 2.2.32.

$$\begin{aligned}
[\mathrm{UConf}^1 X] &= [X] \\
[\mathrm{UConf}^2 X] &= [\mathrm{Sym}^2 X]-[X] \\
[\mathrm{UConf}^3 X] &= [\mathrm{Sym}^3 X]-[\mathrm{Conf}^2 X]-[X] \\
&= [\mathrm{Sym}^3 X]-[X]^2+[X]-[X] \\
&= [\mathrm{Sym}^3 X]-[X]^2
\end{aligned}$$

All the dimensions here are equal to or bounded above by the dimensions of the analogous degeneracy loci from Section 2.2.3. Taking $\alpha=\dim\mathbb{G}(n-m,n)$, substituting in these bounds gives the following relative dimensions in $\widehat{\mathcal{K}}$ of terms in the beginning of Section 2.2.3:

- Term 1 and 2 from lines 2.2.10 and 2.2.11:
  - Main terms $[Y^{(k+1)}][G(n-m+1-(k+1),n+1-(k+1))]$ and $[J]$:

$$\begin{aligned}
\dim Y^{(k+1)} + \dim G(n-m+1-(k+1),n+1-(k+1)) - \alpha \\
= m(k+1)+m(n-m-k)-m(n-m+1) \\
= 0
\end{aligned}$$

For the second term, we have that

$$\dim J - \alpha = \alpha - \alpha = 0$$

since the projection map $J \longrightarrow \mathbb{G}(n-m, n)$ sending

$$((p_1, \ldots, p_{d-k-1}), \Lambda) \mapsto \Lambda$$

is surjective and has finite fibers.

– Degenerate terms $[M][G(n-m+1-(k+1), n+1-(k+1))]$ and $[K]$:

$$\begin{aligned} \dim M + \dim G(n-m+1-(k+1), n+1-(k+1)) - \alpha \\ \leq mk - 1 + m(n-m-k) - m(n-m+1) \\ = -m-1 \end{aligned}$$

- Term 3 (tangent planes) from line 2.2.11

$$\begin{aligned} \dim Q - \alpha \leq S &= m + (m-(n-m))(n-m) - m(n-m+1) \\ &= m + (m-(n-m))(n-m) - m(n-m) - m \\ &= -(n-m)^2 \end{aligned}$$

- Term 4 (degenerate incidence correspondences) from line 2.2.11:

  By Proposition 2.2.25, Proposition 2.2.29, and Proposition 2.2.22, we have that

$$\dim \widetilde{B}_2 - \alpha \leq -2(n-m-(k-2)-1)$$

and

$$\dim \widetilde{T}_2 - \alpha = -m(n-m+1).$$

- Term 5 (degeneracies involving $F_{n-m}(Y)$) from line 2.2.11: Suppose that Y is contained in a general complete intersection X of hypersurfaces of degrees $d_1, \ldots, d_s$ in $\mathbb{P}^n$, covered by lines, and that a finite number of lines pass through a general point of Y. Then, Theorem 2.3 on p. 4 of [10] implies that

$$\begin{aligned} \dim F_{n-m}(Y) + \dim C - \alpha &\leq m(n-m+1) - \sum_{i=1}^{s} \binom{d_i + n - m}{n-m} \\ &\quad + (n-m)k + k - 1 - m(n-m+1) \\ &= -\sum_{i=1}^{s} \binom{d_i + n - m}{n-m} + (n-m)k + k - 1 \end{aligned}$$

and

$$\begin{aligned} \dim F_{n-m}(Y) + \dim D - \alpha \leq\ & m(n-m+1) - \sum_{i=1}^{s} \binom{d_i + n - m}{n-m} \\ & + (n-m-1)(d-k) - (d-k-1) \\ & - m(n-m+1) \\ & = -\sum_{i=1}^{s} \binom{d_i + n - m}{n-m} + (n-m-1)(d-k) - (d-k-1) \end{aligned}$$

Note that $s \leq n - m - 1$. We can use bounds on binomial coefficients to study the sizes of the dimensions above.For example, suppose that $d_i \gg n - m$ for each i and use the inquality $\binom{mn}{n} \geq \frac{m^{(n-1)+1}}{(m-1)^{(m-1)(n-1)}} n^{-\frac{1}{2}} \geq m^n n^{-\frac{1}{2}}$ for each $m \geq 2$ and $n \geq 1$ (Problem 10819 on p. 652 of [26], p. 2 of [34]) to obtain bounds on suitable variables. Let $r = n - m$. If $d_i = ar$ for each i, it suffices to have $k < s \cdot a^r r^{-\frac{3}{2}}$ since $s \cdot a^r r^{-\frac{1}{2}} > r \cdot k \iff k < s \cdot a^r r^{-\frac{3}{2}}$. This means that the degeneracy term involving C in 2.2.11 approaches 0 in $\widehat{\mathcal{K}}$ as $(n-m) \to \infty$. The relative dimension of the term involving D in 2.2.11 may still be large unless $n-m=1$, in which case we can substitute in $(ar)^s$ for d and obtain suitable bounds.

- Variable size restrictions:

  - $d \geq k+3$
  - $k+1 \leq n-m-1$
  - $d \geq (n-m)+2$
  - $d-k-1 > n-m-1$
  - $n-m \leq m-1$

2.3 Limits in $\widehat{\mathcal{K}}$

The purpose of this section is to combine the dimension computations from Section 2.2.3 and Section 2.2.3 (the relative dimension versions at the end of these sections (Definition 2.2.10)) to obtain the limits in Part 1 and Part 2 of Theorem 2.1.7 in $\widehat{\mathcal{K}}$.

2.3.1 Low degree nondegenerate varieties ($d-k-1 \leq n-m-1$)

We first apply dimension counts to limits in the low degree setting $d-k-1 \leq n-m-1$. Afterwards, we substitute the dimensions into the extended $Y - F(Y)$ relation to obtain a limit in $\widehat{\mathcal{K}}$ (Part 1 of Theorem 2.1.7). Recall that we have the following relative dimensions (i.e. dimensions in $\widehat{\mathcal{K}}$) and restrictions on variables involved:

- Terms 1 and 2 from lines 2.2.5 and 2.2.7:

  - Main term: 0
  - Degenerate terms:

    * Using M: $\leq -m-1$
    * Using N: $\leq -m-1$

- Term 3 from line 2.2.7:

- $\dim P \le -m - 1 + (d-k-1)$ (Proposition 2.2.15)
- $\dim Q \le -m - 1 + (k+1)$ (Proposition 2.2.15)

- Term 4 (degenerate incidence correspondences) from line 2.2.7:
  - $\dim \widetilde{B}_2 \le -2(n-m-(k-2)-1)$
  - $\dim \widetilde{T}_2 \le -2(n-m-(d-k-4)-1)$

- Term 5 (degeneracies involving $F_{n-m}(Y)$) from line 2.2.8:

  If Y is contained in some general hypersurface of degree e, then

$$\begin{aligned}\dim F_{n-m}(Y) + \dim C &\le -(n-m)(m-k-1) + n + k \\ &< -(n-m)(m-k-1) + 2m + k\end{aligned}$$

  and

$$\begin{aligned}\dim F_{n-m}(Y) + \dim D &\le -(n-m)(m-(d-k-2)-1) + n + (d-k-2) \\ &< -(n-m)(m-(d-k-2)-1) + 2m + (d-k-2)\end{aligned}$$

  since we assumed that $n-m \le m-1$. Note that our variable restrictions imply that $m > k+1$ and $m > d-k-2$.

- Variable size restrictions:
  - $d \ge k+3$
  - $k+1 \le n-m-1$
  - $d \ge (n-m)+2$
  - $d-k-1 \le n-m-1$
  - $n-m \le m-1$

In order for the dimensions of the degenerate loci in Terms 1, 2, and 4 (from lines 2.2.5, 2.2.7) to approach $-\infty$, it suffices to have $m \to \infty$ and $n-m \to \infty$ as $k \to \infty$ "reasonably quickly". Note that the dimension of Term 3 approaches $-\infty$ since $n-m$ is much larger than $d-k-1$, $k+1$, or $m-1$ under the assumptions of Part 1 of Theorem 2.1.7. For Term 5, it suffices to take $m-k \to \infty$ and $m-(d-k-2) \to \infty$ as $k \to \infty$ if we assume that $(n-m)-k \to \infty$. Note that these are consistent with our variable restrictions since substituting in the fourth restriction to the second and third ones imply that $d-k-1 \leq m-2$ and $k+1 \leq m-2$. Putting the ranges above together gives the limit from Part 1 of Theorem 2.1.7 in $\widehat{\mathcal{K}}$.

Remark 2.3.1. In the example values, we chose $d = (n-m) + \lfloor \sqrt{k} \rfloor$ to ensure that Y is a nondegenerate variety since $d \geq 2 + (n-m)$ if Y is nondegenerate and not a rational normal scroll or Veronese surface. Note that the Veronese surfaces do not affect what happens in the limit. The main purpose is to find parameters which may apply to a more varied collection of varieties. It is clear that the sample values given satisfy the variable restrictions above.

We end with further details on Example 2.1.6 from the introduction.

Example 2.3.2. (Low degree examples for Part 1 of Theorem 2.1.7 from Example 2.1.6: Linear subspaces contained in scrolls and (hyper)quadric fibrations)
There is a classification of smooth m-dimensional varieties $Y \subset \mathbb{P}^n$ of degree $d \leq 2(n-m)+1$ not contained in a hyperplane (i.e. nondegenerate). We will only consider varieties where the dimension m can be arbitrarily large and contain $(n-m)$-planes (Theorem I on p. 339 of [23]). Two of the three families of such varieties (excluding quadric hypersurfaces and $\mathbb{P}^n$) which can take an arbitrarily large dimension with $d \leq 2(n-m)+1$ are scrolls over curves or surfaces and (hyper)quadric fibrations. When the base of these scrolls and (hyper)quadric fibrations is $\mathbb{P}^1$, we can make some concrete observations on the Fano varieties of k-planes on these varieties.

In the case of scrolls over $\mathbb{P}^1$, we have a complete description (Proposition 2.2 on p. 4066 of [27]). The k-planes contained in such a scroll are either contained in a $(k-1)$-plane inside the fiber of the defining projection map or the span of lines involved in the construction of the scroll as the span of a collection of rational normal curves. Theorem 1.5 on p. 511 of [29] gives a fiberwise embedding of any (hyper)quadric fibration $X \longrightarrow \mathbb{P}^1$ (paired with a very ample line bundle $\mathcal{L}$) over $\mathbb{P}^1$ into a projective bundle $\mathbb{P}(\mathcal{E})$ over $\mathbb{P}^1$ which connects these (hyper)quadric fibrations to scrolls over $\mathbb{P}^1$.

Recall that a scroll $\mathbb{P}(\mathcal{E})$ over $\mathbb{P}^1$ can also be defined as the image of a vector bundle over $\mathbb{P}^1$ with a particular embedding $\mathcal{O}_{\mathbb{P}(\mathcal{E})}(1)$ into projective space (p. 5 of [10]). We also have that the restriction of the tautological line bundle on $\mathbb{P}(\mathcal{E})$ to X is equal to $\mathcal{L}$ (equation 1.0.3 on p. 509 of [29]). These two embeddings can be combined to study linear subspaces of X via their images in the scroll over $\mathbb{P}^1$. This can likely be translated into a concrete problem since scrolls over $\mathbb{P}^1$ can also be defined as the vanishing locus of 2×2 minors of a certain matrix (Exercise 9.11 on p. 106 of [18]).

2.3.2 Higher degree varieties ($d - k - 1 > n - m - 1$)

The same reasoning as Section 2.3.1 can be used to obtain the limit from Part 2 of Theorem 2.1.7 in $\widehat{\mathcal{K}}$. Also, the limit in Part 2 of Theorem 2.1.7 has a particularly simple expression when we apply the point counting motivic measure and assume some divisibility conditions.

Definition 2.3.3. (Definition 4.3.4 on p. 112 of [4])
A motivic measure μ with values in a ring A is *separated* if there is a morphism of rings $\overline{\mu} : \overline{\mathcal{M}_k} \longrightarrow A$ such that $\mu(X) = \overline{\mu}([X])$ for each k-variety X. This is equivalent to a motivic measure $\mu : K_0(\mathrm{Var}_k) \longrightarrow A$ satisfying the following conditions:

- $\mu(\mathbb{L}) \in A^\times$
- $\widetilde{\mu}(F^\infty \mathcal{M}_k) = 0$, where $\widetilde{\mu} : \mathcal{M}_k \longrightarrow A$ is the unique ring homomorphism such that

$$\widetilde{\mu}([X]\mathbb{L}^{-i}) = \mu(X)\mu(\mathbb{L})^{-i}.$$

Proposition 2.3.4. *The extended point counting motivic measure* $\mu : K_0(\mathrm{Var}_k) \longrightarrow \mathbb{Q}$ *sending* $[X] \mapsto \#X(\mathbb{F}_q)$ *is separated.*

Now that we have some idea of how motivic measures of $K_0(\mathrm{Var}_K)$ interact with the completion $\widehat{\mathcal{K}}$, we will give a proof of Corollary 2.1.10.

Proof. (Proof of Corollary 2.1.10)
We will write $\#$ in place of the notation $\#_{q,e}$ from Corollary 2.1.10 for the $\mathbb{F}_{q^e}$-point count. Recall from Section 2.2.3 that

$$J = \{((p_1, \ldots, p_{d-k-1}), \Lambda) \in W : p_i \text{ distinct, } \dim \overline{p_1, \ldots, p_{d-k-1}} = n - m\}.$$

First consider the subset $\widetilde{J} \subset J$ coming from points of $\mathbb{G}(n-m,n) \setminus B$, where $B = \{\Lambda \in \mathbb{G}(n-m,n) : \Lambda \text{ tangent to } Y\}$. By Proposition 2.2.15, we have that

$$\dim B \leq m + (m - (n-m))(n-m).$$

This means that $\frac{\#B(\mathbb{F}_{q^e})}{\#\mathbb{G}(n-m,n)(\mathbb{F}_{q^e})} = O(q^{-e((n-m)^2})$. Our assumption that $e > \binom{d}{d-k-1}$ implies that $\binom{d}{d-k-1}\frac{\#B(\mathbb{F}_{q^e})}{\#\mathbb{G}(n-m,n)(\mathbb{F}_{q^e})} \to 0$ in the limit (which takes $(n-m) \to \infty$). This also takes care of elements of $J \setminus \widetilde{J}$ where $\Lambda \not\subset Y$. Before moving to elements of $J \setminus \widetilde{J}$, we will continue to obtain point counts for elements of J with $\Lambda \not\subset Y$.

The preimage of each point of $\mathbb{G}(n-m,n) \setminus B$ is a collection of d points over $\overline{\mathbb{F}_q}$. What we would like to find are $(d-k-1)$-tuples of points on Y lying on a given $(n-m)$-plane that are invariant under the action of $\mathrm{Gal}(\overline{\mathbb{F}_q}/\mathbb{F}_q)$. Our assumption on $\mathbb{F}_q$-irreducible components of $(d-k-1)$-tuples lying on $Y \cap \Lambda$ implies that there are no $\mathbb{F}_{q^e}$-points coming from J by the following modification of the Lang–Weil bound applied to $\mathbb{F}_q$-irreducible components of

$Y \cap \Lambda$ for each $\Lambda \in \mathbb{G}(n-m, n) \setminus B$.

Proposition 2.3.5. *(Modified Lang–Weil bound, Proposition 3.1 on p. 6 of [31]) Suppose that $K = \mathbb{F}_q$ is a finite field and let $X \subset \mathbb{P}_k^n$ be an irreducible closed subvariety of degree d and dimension r. Denote by $\Gamma = \{W_1, \ldots, W_m\}$ the set of irreducible components of $X_{\overline{k}} = X \times_k \overline{k}$.*

There are positive constants c_X and c'_X such that for every $e \geq 1$, we have

$$\begin{cases} |\#X(\mathbb{F}_{q^e}) - mq^{er}| \leq \frac{(d-m)(d-2m)}{m} q^{e\left(r-\frac{1}{2}\right)} + c_X q^{e(r-1)} & \text{if } m|e \text{ and} \\ \#X(\mathbb{F}_{q^e}) \leq c'_X q^{e(r-1)} & \text{if } m \nmid e. \end{cases}$$

Furthermore, if X is smooth over $\mathbb{F}_q$, then we may take $c'_X = 0$ and c_X to only depend on $n, d,$ and r (but not on X or on k).

If $N|e$ in Corollary 2.1.10, then we use the first part of Proposition 2.3.5. Note that the degree of a finite set as a variety is its cardinality. Alternatively, a simpler method for finding $\mathbb{F}_q$-points of $(d-k-1)$-tuples of Y lying on $Y \cap \Lambda$ for some fixed $\Lambda \in \mathbb{G}(n-m, n) \setminus B$ is to count collections of $\mathrm{Gal}(\overline{\mathbb{F}_q}/\mathbb{F}_q)$-orbits of points on d $Y \cap \Lambda$ which have cardinality adding to $d-k-1$.

In the 0-dimensional case, the number of $\mathbb{F}_q$-irreducible components (denoted m in Proposition 2.3.5) is the number of $\mathbb{F}_q$-points. This applies to our setting since $\pi^{-1}(\Lambda)$ is finite for all $\Lambda \in \mathbb{G}(n-m, n) \setminus B$ even over the algebraic closure. Since $1 \leq \#\pi^{-1}(\Lambda)(\mathbb{F}_q) \leq \binom{d}{d-k-1}$ for each $\Lambda \in \mathbb{G}(n-m, n) \setminus B$, we have that $m|e$ for any Λ if e is divisible by $\binom{d}{d-k-1}!$. After base changing $\pi^{-1}(\Lambda)_{\mathbb{F}_q}$ to $\mathbb{F}_{q^e}$, we end up with the same number of points as in the algebraic closure $\overline{\mathbb{F}_q}$ (see proof of Proposition 3.1 on p. 6 of [31]). Putting together the "irreducible components" (which are really just $\mathbb{F}_q$-points), we find that the number of $\mathbb{F}_{q^e}$-points in $\pi^{-1}(\Lambda)$ is $\binom{d}{d-k-1}$ for each $\Lambda \in \mathbb{G}(n-m, n) \setminus B$. In general, we add the number of geometric points in each $\mathrm{Gal}(\overline{\mathbb{F}_q}/\mathbb{F}_q)$-orbit whose size divides e to get the total number of

$\mathbb{F}_{q^e}$-points in $\pi^{-1}(\Lambda)$.

We finally consider point counts of elements of J such that $\Lambda \subset Y$. These come from $\Lambda \in F_{n-m}(Y)$ and the p_i represent $(d-k-1)$-tuples on an $(n-m)$-plane $\Lambda \in F_{n-m}(Y)$ which span the entire plane Λ. To do this, we compare point counts of linearly independent $(k+1)$-tuples and $(d-k-1)$-tuples lying on an $(n-m)$-plane to find an approximation.

Since $\widetilde{T}_1 = \emptyset$ by Proposition 2.2.29, the $(d-k-1)$-tuples in question must span an $(n-m)$-plane. We will also omit $(n-m)$-planes which contain linearly dependent $(k+1)$-tuples since terms associated to them approach 0 in the completion. Note that every element of $\widetilde{A}$ (linearly independent $(k+1)$-tuples in Y contained in an element of $F_{n-m}(Y)$) comes a $(d-k-1)$-tuple in Y spanning an $(n-m)$-plane of $\widetilde{J} \setminus J$. For each element of $\widetilde{R}$, we have $\binom{d-k-1}{k+1}$ possible choices of $(k+1)$-tuples. However, we also need to take into account redundancies by determining the space of $(d-k-1)$-tuples in Y spanning an $(n-m)$-plane in $F_{n-m}(Y)$ which contain a given linearly independent $(k+1)$-tuple in Y paired with an $(n-m)$-plane in $F_{n-m}(Y)$ containing it. This is the space of $(d-2k-2)$-tuples in $\mathbb{P}^{n-m} \setminus \{k \text{ points}\}$ spanning an $(n-m)$-plane.

Given the point count for $\widetilde{A}$, this gives an approximate count

$$\#\widetilde{A}(\mathbb{F}_{q^e})$$

$$\approx \frac{\binom{d-k-1}{k+1}\#\widetilde{R}(\mathbb{F}_{q^e})}{\#\{((p_1, \ldots, p_{d-2k-2}), \Lambda) : \text{ distinct } p_i \in \mathbb{P}^{n-m}, \dim \overline{p_1, \ldots, p_{d-2k-2}} = n-m\}(\mathbb{F}_{q^e})},$$

where the denominator parametrizes $(d-k-1)$-tuples containing a fixed $(k+1)$-tuple. This denominator can be approximated by $\mathrm{UConf}_{d-2k-2}(\mathbb{P}^{n-m} \setminus \{k+1 \text{ points}\}$.

From the point of view of $\#\widetilde{R}(\mathbb{F}_{q^e})$, this means that

$$\#\widetilde{R}(\mathbb{F}_{q^e})$$

$$\approx \frac{\#\{((p_1,\ldots,p_{d-2k-2}),\Lambda):\ p_i \text{ distinct in } \mathbb{P}^{n-m},\ \dim\overline{p_1,\ldots,p_{d-2k-2}} = n-m\}(\mathbb{F}_{q^e})}{\binom{d-k-1}{k+1}}$$

$$\cdot\#\widetilde{A}(\mathbb{F}_{q^e}).$$

An upper bound would be given by $\#\operatorname{UConf}_{d-2k-2}(\mathbb{P}^{n-m}\setminus\{k+1 \text{ points}\})(\mathbb{F}_{q^e})$. We compute this recursively in Lemma 2.2.31. For a more precise estimate, one could attempt to follow the steps of Proposition 2.2.30 to compute the class of distinct points of $\mathbb{P}^{n-m}\setminus\{k+1 \text{ points}\}$ spanning an $(n-m)$-plane.

We can consider the ratio between point counts of $V\setminus\widetilde{\widetilde{R}}$ and $W\setminus\widetilde{\widetilde{A}}$, where $\widetilde{\widetilde{R}}\subset V$ is the subset of $(d-k-1)$-tuples of distinct points lying in an $(n-m)$-plane contained in Y paired with this plane and $\widetilde{\widetilde{A}}\subset W$ denotes $(k+1)$-tuples of distinct points lying in an $(n-m)$-plane contained in Y. Applying the proof of Proposition 2.2.1, we obtain a correspondence between elements of $W\setminus\widetilde{\widetilde{A}}$ and those of $V\setminus\widetilde{\widetilde{R}}$.

The two ratios of point counts can be used to compare the point count of V with that of W. Writing $\#X := X(\mathbb{F}_{q^e})$, the dimension counts in the proof of Theorem 2.1.7 imply that

$$
\begin{aligned}
\frac{\#V}{\#W} &= \frac{\#V - \#\widetilde{\widetilde{R}}}{\#W} + \frac{\#\widetilde{\widetilde{R}}}{\#W} \\
&= \frac{\#V - \#\widetilde{\widetilde{R}}}{\#W - \#\widetilde{\widetilde{A}}} \cdot \frac{\#W - \widetilde{\widetilde{A}}}{\#W} + \frac{\widetilde{\widetilde{R}}}{\#\widetilde{\widetilde{A}}} \cdot \frac{\#\widetilde{\widetilde{A}}}{\#W} \\
&= 1 \cdot \left(1 - \frac{\#\widetilde{\widetilde{A}}}{\#W}\right) + \frac{\widetilde{\widetilde{R}}}{\#\widetilde{\widetilde{A}}} \cdot \frac{\#\widetilde{\widetilde{A}}}{\#W} \\
&= \Big(1 - \Theta(q^{e((k+1)(n-m)+\dim F_{n-m}(Y)-m(n-m+1))})\Big) \\
&+ \Theta(q^{e(n-m)(d-2k-2)}) \cdot \Theta(q^{e((k+1)(n-m)+\dim F_{n-m}(Y)-m(n-m+1))}) \\
&= 1 + \Theta(q^{e(n-m)(d-2k-2)})(\Theta(q^{e((k+1)(n-m)+\dim F_{n-m}(Y)-m(n-m+1))}) - 1) \\
&= \Theta(q^{e((n-m)(d-k-1)+\dim F_{n-m}(Y)-m(n-m+1))}).
\end{aligned}
$$

Note that $\#\widetilde{\widetilde{A}}$ and $\#\widetilde{\widetilde{R}}$ are polynomials in q which can be written in terms of multinomial coefficients (see Remark 2.2.28, proofs of Proposition 2.2.27 and Proposition 2.2.30). Recall that $d - k - 1 \gg n$ in Part 2 of Theorem 2.1.7. This means that $\frac{\#V}{\#W}$ is very large. Since the limit is taken as $n - m \to \infty$ and all the other variables are taken to be functions of the codimension $n - m$, the term is dominated by large powers of q. Let $u = \frac{\#V}{\#W}$. Substituting $\#V = u\#W$ into in $[W] - [\widetilde{B}] - [\widetilde{A}] = [V] - [\widetilde{R}] - [\widetilde{T}]$ in Proposition 2.2.1, the point counts are

$$
\begin{aligned}
\#W - \#\widetilde{A} - \#\widetilde{B} &= \#V - \#\widetilde{R} - \#T \\
\#W - \#\widetilde{A} - \#\widetilde{B} &= u\#W - \#\widetilde{R} - \#T \\
-\#\widetilde{R} - \#\widetilde{T} &= (1-u)\#W - \#\widetilde{A} - \#\widetilde{B}.
\end{aligned}
$$

At this point, we can either group the terms W, $\widetilde{A}$, and $\widetilde{B}_1$ together to use the point count $J \setminus \widetilde{J}$ above or estimate point counts of individual terms on the right hand side. The first method gives the following decomposition.

$$
\begin{aligned}
-\#\widetilde{R} - \#\widetilde{T} &= (1-u)\#W - \#\widetilde{A} - \#\widetilde{B} \\
&= (1-u)(\#W - \#\widetilde{A} - \#\widetilde{B}_1) \\
&\quad -u\#\widetilde{A} - u\#\widetilde{B}_1 - \#\widetilde{B}_2 \\
\Rightarrow -\frac{\#\widetilde{R}}{\#\mathbb{G}(n-m)} - \frac{\#\widetilde{T}_1}{\#\mathbb{G}(n-m,n)} - \frac{\#\widetilde{T}_2}{\#\mathbb{G}(n-m,n)} & \\
= \frac{(1-u)(\#W - \#\widetilde{A} - \#\widetilde{B}_1)}{\#\mathbb{G}(n-m,n)} - \frac{u\#\widetilde{A}}{\#\mathbb{G}(n-m,n)} - \frac{u\#\widetilde{B}_1}{\#\mathbb{G}(n-m,n)} - \frac{\#\widetilde{B}_2}{\#\mathbb{G}(n-m,n)} &
\end{aligned}
$$

The terms $\#\widetilde{R}$ and $\widetilde{A}$ can be expressed in terms of $\#F_{n-m}(Y)$ and polynomials in q since $[\widetilde{R}]$ and $[\widetilde{A}]$ are $[F_{n-m}(Y)]$ multiplied by a product in $\mathbb{L}$. Also, note that the limiting estimate $\frac{\#W - \#\widetilde{A} - \#\widetilde{B}_1}{\#\mathbb{G}(n-m,n)}$ is given by that of $\frac{\#J \setminus \widetilde{J}}{\#\mathbb{G}(n-m,n)}$ above. Since $\frac{\#\widetilde{T}_2}{\#\mathbb{G}(n-m,n)}$ and $\frac{\#\widetilde{B}_2}{\#\mathbb{G}(n-m,n)}$ vanish in the limit as $(n-m) \to \infty$, it remains to find estimates for $\#\widetilde{T}_1$ and $\#\widetilde{B}_1$ (which parametrize incidence correspondences of linearly dependent $(k+1)$-tuples and $(d-k-1)$-tuples with linear span of dimension $\leq n-m-1$ respectively). Note that terms which involve $(n-m)$-planes $\Lambda \subset Y$ can be absorbed into $\#\widetilde{R}$ and $\#\widetilde{A}$ to get $\#\widetilde{\widetilde{R}}$ and $\#\widetilde{\widetilde{A}}$. Then, the proof of Proposition 2.2.29 implies that there are no terms of $\widetilde{B}_1$ with $\Lambda \not\subset Y$ and $\#\widetilde{T}_1 = \Theta(q^{km-(n-m-k+1)})$ by Proposition 2.2.21 with $\lambda = k-2$.

Substituting in the dimension estimates along with the subset D of $(d-k-1)$-tuples in $\mathbb{P}^{n-m}$ spanning a linear subspace of dimension $\leq n-m-1$ and the subset C of linearly independent $(k+1)$-tuples (whose classes in $K_0(\mathrm{Var}_K)$ a polynomial in $\mathbb{L}$ by Proposition 2.2.27 and Proposition 2.2.30), we find that

$$-\frac{\#\widetilde{R}}{\#\mathbb{G}(n-m)} - \frac{\#\widetilde{T}_1}{\#\mathbb{G}(n-m,n)} - \frac{\#\widetilde{T}_2}{\#\mathbb{G}(n-m,n)} = \frac{(1-u)(\#W - \#\widetilde{A} - \#\widetilde{B}_1)}{\#\mathbb{G}(n-m,n)}$$
$$-\frac{u\#\widetilde{A}}{\#\mathbb{G}(n-m,n)} - \frac{u\#\widetilde{B}_1}{\#\mathbb{G}(n-m,n)} - \frac{\#\widetilde{B}_2}{\#\mathbb{G}(n-m,n)}$$

and

$$-\frac{\#_{q,e}F_{n-m}(Y)(\#_{q,e}\,\mathrm{UConf}_{d-k-1}(\mathbb{P}^{n-m}) - \#_{q,e}D)}{\#_{q,e}\mathbb{G}(n-m,n)} - \Theta(q^{km-(n-m-k+1)-m(n-m+1)})$$
$$-\frac{\#_{q,e}\widetilde{T}_2}{\#_{q,e}\mathbb{G}(n-m,n)}$$
$$= (1-u)\alpha - \frac{u\#_{q,e}F_{n-m}(Y)((\#_{q,e}\mathbb{P}^{n-m})^{(k+1)} - \#_{q,e}C)}{\#_{q,e}\mathbb{G}(n-m,n)}$$
$$-\frac{\#_{q,e}\widetilde{B}_2}{\#_{q,e}\mathbb{G}(n-m,n)},$$

where $0 \leq \alpha \leq \binom{d}{d-k-1}$ with $\alpha = 0$ if $N \nmid e$ for each $N \in T_\Lambda$ from $\Lambda \in \mathbb{G}(n-m,n)$ such that $|Y \cap \Lambda| = d$ (see Corollary 2.1.10) and $\alpha = \binom{d}{k+1}$ if e is divisible by $\binom{d}{d-k-1}!$ (Proposition 2.3.5). Note that $u = 1 - \beta + \beta f$, where $\beta = \Theta(q^{e((k+1)(n-m)+\dim F_{n-m}(Y) - m(n-m+1))})$ and f is a rational function in q^e determined by $\frac{[\widetilde{\widetilde{R}}]}{[\widetilde{\widetilde{A}}]}$, which is a rational function in $\mathbb{L}$ of degree $(k+1)(n-m) + \dim F_{n-m}(Y) - m(n-m+1)$. Given a fixed q, this implies that

$$\lim_{n-m\to\infty} \frac{\#_{q,e}F_{n-m}(Y)(\#_{q,e}\,\mathrm{UConf}_{d-k-1}(\mathbb{P}^{n-m}) - \#_{q,e}D - u\#_{q,e}(\mathbb{P}^{n-m})^{(k+1)} + \#_{q,e}C)}{\#_{q,e}\mathbb{G}(n-m,n)}$$
$$+(1-u)\alpha + \gamma = 0$$

in the limit for some $\gamma = \Theta(q^{e(km-(n-m-k+1)-m(n-m+1))})$ that varies with the initial parameters, which are all functions of $n-m$.

$\square$

Remark 2.3.6.

1. This reasoning with divisibility conditions does *not* imply that $\#Y^{(k+1)}(\mathbb{F}_{q^e}) = 0$ since the divisibility conditions in Corollary 2.1.10 from Proposition 2.3.5 are used on $\mathbb{F}_q$-irreducible components of $(d-k-1)$-tuples, *not* $(k+1)$-tuples.

2. The divisibility condition in Corollary 2.1.10 from Proposition 2.3.5 is least restrictive when e is prime. Equality holds exactly when the size of each $\mathbb{F}_q$-irreducible component is 1 for each $\Lambda \in \mathbb{G}(n-m,n)$ such that $|Y \cap \Lambda| = d$. For example, this would occur if the map $J \longrightarrow \mathbb{G}(n-m,n)$ is a piecewise trivial fibration above its image instead of just being a covering map. Note that this requires checking images over non-closed points of $\mathbb{G}(n-m,n)$.

3. In Theorem 2.1.7, the connection between variations of $\mathbb{F}_q$-point counts with the covering map $J \longrightarrow \mathbb{G}(n-m,n)$ is that the monodromy action induced could be involved in studying how the point count above is distributed among conjugacy classes of the action of Frobenius on general $(n-m)$-plane sections of Y (see [12]).

4. A natural question to ask is what the distribution of point counts of J behave if we impose additional geometric restrictions on the type of variety Y while varying the codimension, dimension and degree.

5. Part 2 of Theorem 2.1.7 can be applied to any separated motivic measure in place of finite field point counts. For example, the following result implies that we can use the étale representation for the Euler characteristic (interpreted as graded respresentations):

   Proposition 2.3.7. *(Corollary 4.3.9 on p. 113 of [4])*

The étale motivic measure $\chi_{ét} : K_0(\mathrm{Var}_k)[\mathbb{L}^{-1}] \longrightarrow K_0(\mathrm{Rep}_{G_k} \mathbb{Q}_l)$ *given by*

$$\chi_{ét}(X) = \sum_{n \geq 0} (-1)^n [H^n_{ét}(X \otimes_k k^s, \mathbb{Q}_l)]$$

for separated k-varieties X (p. 95 of [4]) is a separated motivic measure.

Here is an example of varieties where the point count above and Part 2 of Theorem 2.1.7 applies.

Example 2.3.8. (High degree examples for Part 2 of Theorem 2.1.7 from Example 2.1.9: Complete intersections of generic hypersurfaces of large degree)
We can analyze the relative sizes of the variables to show that there are many terms where the relative dimensions of the main term involving the Fano $(n-m)$-planes do *not* vanish in the completion unlike the linear dependence non-generic terms. Suppose that $Y \subset \mathbb{P}^n$ is general a complete intersection. Since $\dim Y = m$, this means that Y is a complete intersection of $n-m$ hypersurfaces. By Theorem 2.4 on p. 4 of [10], we have that

$$\dim F_{n-m}(Y) = m(n-m+1) - \sum_{i=1}^{n-m} \binom{d_i + n - m}{n-m}.$$

The relative dimension (Definition 2.2.10) of

$$\frac{[F_{n-m}(Y)]([(\mathbb{P}^{n-m})^{(k+1)}] - [(\mathbb{P}^{n-m})^{(d-k-1)}])}{[\mathbb{G}(n-m, n)]}$$

is then

$$(d-k-1)(n-m) - \sum_{i=1}^{n-m} \binom{d_i + n - m}{n-m}.$$

If the degrees d_i are sufficiently large, then the first term involving Fano $(n-m)$-planes Part 2 of Theorem 2.1.7 (a multiple of the term above) does *not* vanish in the limit. Taking the point sample size $k \ll n-m$ means that complete intersections of generic hypersurfaces of large degree (relative to $n-m$) which are u-linearly generic for $u \leq d-k-2$ give examples where Part 2 of Theorem 2.1.7 applies.

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CHAPTER 3

CHARACTERIZING CUBIC HYPERSURFACES VIA PROJECTIVE GEOMETRY

We use the cut and paste relation $[Y^{[2]}] = [Y][\mathbb{P}^m] + \mathbb{L}^2[F(Y)]$ in $K_0(\mathrm{Var}_k)$ of Galkin–Shinder for cubic hypersurfaces arising from projective geometry to characterize cubic hypersurfaces of sufficiently high dimension under certain numerical or genericity conditions. Removing the conditions involving the middle Betti number from the numerical conditions used extends the possible Y to cubic hypersurfaces, complete intersections of two quadric hypersurfaces, or complete intersections of two quartic hypersurfaces. The same method also gives a family of other cut and paste relations that can only possibly be satisfied by cubic hypersurfaces.

3.1 Introduction

The main objective of this note is to provide a characterization of cubic hypersurfaces using projective geometry under certain numerical/genericity conditions. This characterization is based on satisfying a relation of Galkin–Shinder (Theorem 5.1 on p. 16 of [16]) in the Grothendieck ring of varieties $K_0(\mathrm{Var}_k)$ called the $Y - F(Y)$ relation. Given a reduced cubic hypersurface $Y \subset \mathbb{P}^{m+1}$ and its Fano variety of lines $F(Y) \subset \mathbb{G}(1, m+1)$, it states that

$$[Y^{[2]}] = [Y][\mathbb{P}^m] + \mathbb{L}^2[F(Y)] \text{ (equivalently } [Y^{(2)}] = (1 + \mathbb{L}^m)[Y] + \mathbb{L}^2[F(Y)] \text{)}$$

in $K_0(\mathrm{Var}_k)$, where $Y^{[2]}$ is the Hilbert scheme of two points on Y and $Y^{(2)}$ is the second symmetric product. The general idea is to pair distinct points of Y parametrizing a line with the third pont of intersection and the line defined by the initial pair of points. Here, the terms involving $F(Y)$ come from removing instances where the lines involved are contained in Y. We assume that k is an algebraically closed field of characteristic 0. As it turns out, the method we use is not unique to this relation and we indicate other relations in $K_0(\mathrm{Var}_k)$

which can fill this role. Some examples in lower dimensions are given in the final section.

We give some context connecting this characterization to the overall structure of $K_0(\mathrm{Var}_k)$. Most work on the general structure of $K_0(\mathrm{Var}_k)$ or specific subrings generated by certain types of varieties focus on the space of possible relations (e.g. whether $\mathbb{L}$ is a zerodivisor). Instead, we focus on the converse question of what varieties satisfy a given relation. While this can be approached using existing work on varieties known to satisfy Larsen–Lunts' cut and paste property (e.g. varieties containing finitely many rational curves in Theorem 6.3.7 on p. 142 of [4] although known to be false in general) or the graded ring associated to $K_0(\mathrm{Var}_k)$ [26] via birational equivalence classes, there does not seem to be much known outside of these settings. We start to explore this in the case of Galkin–Shinder's $Y - F(Y)$ relation (Theorem 5.1 on p. 16 of [16]).

Question 3.1.1. *(Farb)*
Given a closed subvariety $Y \subset \mathbb{P}^n$ of dimension $m \geq 1$, let $F(Y) \subset \mathbb{G}(1,n)$ be the Fano variety of lines parametrizing lines in $\mathbb{P}^n$ contained in Y. If Y satisfies the $Y - F(Y)$ relation, is it a cubic hypersurface?

In Section 3.2.1, we address this question given some numerical/concrete topological conditions on the variety (Theorem 3.1.2) which can be relaxed if we assume Hartshorne's conjecture or the Debarre–de Jong conjecture (Remark 3.1.3).

Theorem 3.1.2. *Let k be a field of characteristic 0 such that $\overline{k} = k$. Let $Y \subset \mathbb{P}^n$ be a nondegenerate, irreducible, smooth projective variety over k of dimension m and degree d satisfying one of the following conditions:*

a. $d \leq \frac{n}{4}$, or

b. Y can be defined by $\leq \frac{n}{2}$ equations of degree $\leq \frac{n}{2}$.

Suppose that Y is not *a 1-dimensional family of quadrics or a 2-dimensional family of (projective) $(m-2)$-planes (each isomorphic to $\mathbb{P}^{m-2}$). If $m \geq 7$, then the first condition implies the second.*

1. *Y satisfies the $Y - F(Y)$ relation $[Y^{[2]}] = [Y][\mathbb{P}^m] + \mathbb{L}^2[F(Y)]$*

2. *Y is a cubic hypersurface, the intersection of two quadric hypersurfaces, or the intersection of two quartic hypersurfaces. The middle Betti number b_m gives additional constraints:*

    - *If b_m is exponential in m, Y is either a cubic hypersurface or the intersection of two quartics.*
    - *If $m + 5 \leq b_m < 2 \cdot 3^m - 5$, then Y is a cubic hypersurface.*

    If the middle Betti number b_m is exponential in m, it must either be a cubic hypersurface or the intersection of two quartic hypersurfaces.

Note that $F(Y)$ is connected if Y is a cubic hypersurface of dimension ≥ 3 (p. 12 of [16]).

Remark 3.1.3. Here are some comments on the assumptions of Theorem 3.1.2.

1. If Hartshorne's conjecture (part 1 of Remark 3.2.7) holds in codimension 2, then the conditions on Y can be weakened to $2d - 4 \leq n$ and $n \geq 7$. Note that the uniruledness property already implies $d \leq n$ if Y is a hypersurface and $d_1 + d_2 \leq n$ if Y has codimension 2 and d_1, d_2 are the degrees of the hypersurfaces whose intersection is equal to Y. Another possible replacement of the conditions is to take Y contained in *some* hypersurface of degree d such that $2d - 4 \leq n$.

These numerical conditions can be removed if we assume that the analogue of Debarre–de Jong conjecture (Conjecture 1.2 on p. 1 of [11]) for complete intersections holds. This would imply that

$$\dim F(Y) = 2n - d - (n - m) - 2 = n - d + m - 2,$$

where $d = \sum_{i=1}^{n-m} d_i$ is the sum of the degrees of the hypersurfaces. As with the original Debarre–de Jong conjecture, it is well-known to hold in the generic case (e.g. Proposition 2.1 on p. 2 of arXiv version of [7]). When $n - m = 2$, this implies that $\dim F(Y) = m+2-d+m-2 = 2m-d$, which is equal to $2m-4$ only if $d = d_1+d_2 = 4$. The proof of Theorem 3.1.2 eliminates this case.

2. Moving to very generic properties, there are many examples of varieties that are not necessarily complete intersections which cannot satisfy the $Y-F(Y)$ relation (Example 3.2.20).

3. If Y is a variety with a connected Fano variety of lines $F(Y)$ satisfying the $Y - F(Y)$ relation (e.g. the case of cubic hypersurfaces), it is *not* a 1-dimensional family of quadrics or a 2-dimensional family of (projective) $(m - 2)$-planes since such varieties satisfying the $Y - F(Y)$ relation cannot be connected. This is shown in Proposition 3.2.8.

Remark 3.1.4. The uniqueness of cubic hypersurfaces as varieties satisfying the $Y - F(Y)$ relation also applies in the localization rather than $K_0(\mathrm{Var}_k)$. This is because the same logic applies whenever the desired relation among varieties in $K_0(\mathrm{Var}_k)$ forces $\dim F(Y) = 2m-4$ (Theorem 2 on p. 207 – 208 of [35]).

The proof of this result uniquely characterizes cubic hypersurface among generic hypersurfaces of a given degree.

Corollary 3.1.5. *(Corollary 3.2.3) Suppose that $Y \subset \mathbb{P}^n$ is a hypersurface generic among those of its degree. If Y satisfies the $Y - F(Y)$ relation, then Y is a cubic hypersurface.*

Genericity also plays a role in a different approach involving uniruledness properties in Section 3.2.2 (Proposition 3.2.17, Corollary 3.2.19). Here are some examples of results on varieties satisfying the $Y - F(Y)$ relation:

Corollary 3.1.6. *(Corollary 3.2.16)*
Assume that $\overline{k} = k$ and $\operatorname{char} k = 0$ as above and suppose that $Y \subset \mathbb{P}^n$ is a d-dimensional variety satisfying the $Y - F(Y)$ relation. Then, the variety Y is not *contained in a general hypersurface of degree $r > 2n - 3$.*

Corollary 3.1.7. *(Corollary 3.2.19)*
Assume that $\overline{k} = k$ and $\operatorname{char} k = 0$ as usual. If $Y \subset \mathbb{P}^n$ is a variety of dimension ≥ 2 which is the complete intersection of $m \geq 2$ hypersurfaces W_i which are generic among those of their degrees r_i for each i, then it does not *satisfy the $Y - F(Y)$ relation.*

Along the way, we obtain restrictions on varieties which satsify relations that share some properties with the $Y - F(Y)$ relation (Proposition 3.2.22, Corollary 3.2.23) and answer a question of Cadorel–Campana–Rousseau [6] related to a connection between uniruledness and symmetric products of varieties (Example 3.2.21). The steps used give an alternative method of restricting varieties satsifying the $Y - F(Y)$ relation (Proposition 3.2.17, Corollary3.2.19). We also note that the $Y - F(Y)$ relation does *not* generate all polynomial relations between terms involved in the $Y - F(Y)$ relation (Remark 3.2.10).

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3.2 Varieties satisfying the $Y - F(Y)$ relation

In this section, we give some numerical and genericity conditions under which the $Y - F(Y)$ relation uniquely characterizes cubic hypersurfaces.

3.2.1 Using numerical conditions

Our first main result (Theorem 3.1.2) is a characterization of when varieties satisfy the $Y - F(Y)$ relation under certain numerical conditions. Assuming that the degree is sufficiently small compared to the dimension of the projective space it is embedded in, we can show that only cubic hypersurfaces satsify the $Y - F(Y)$ relation if Y is *not* a 1-dimensional family of quadrics or a 2-dimensional family of $(m-2)$-planes ($m = \dim Y$). It turns out that the only possible smooth projective varieties $Y \subset \mathbb{P}^n$ which can satisfy the $Y - F(Y)$ relation are those which have codimension ≤ 2. Before giving the full proof of Theorem 3.1.2, we give an outline of the arguments used.

Recall that the $Y - F(Y)$ relation is given by $[Y^{[2]}] = [Y][\mathbb{P}^m] + \mathbb{L}^2[F(Y)]$ in $K_0(\mathrm{Var}_k)$. This can be rewritten as $[Y^{[2]}] - [Y][\mathbb{P}^m] = \mathbb{L}^2[F(Y)]$ (Theorem 5.1 on p. 16 of [16]). Substituting in Poincaré polynomials to this rearranged relation, each side has terms of degree ≥ 4 since each side is a multiple of $p_{\mathbb{A}^1}(t)^2 = t^4$. In particular, this means that the coefficients of t^2 and t is 0 on each side and $b_1 = 0$. The first nonzero term is from t^{4m-4} since the terms of degree $4m, 4m-1, 4m-2$, and $4m-3$ are all equal to 0. Dividing by t^4, we find that $\deg p_{F(Y)}(t) = 4m-8$ and $\dim F(Y) = 2m-4$. Both of these (and most of the computation in general) are shown in Lemma 3.2.2. Work of Rogora [35] then implies that Y has codimension ≤ 2 if n is sufficiently large. The theorem then follows from applying results (e.g. work on Hartshorne's conjecture) which imply that Y is a complete intersection under the given conditions.

The first observation we make before proving Theorem 3.1.2 is that taking $m \geq 1$ implies that $F(Y) \neq \emptyset$.

Lemma 3.2.1. *Suppose that $Y \subset \mathbb{P}^n$ is a connected m-dimensional variety satisfying the $Y - F(Y)$ relation. If $m \geq 1$, the Fano variety of lines $F(Y) \neq \emptyset$.*

Proof. Suppose that $F(Y) = \emptyset$. In order for Y to satisfy the $Y - F(Y)$ relation, we would need to have $[Y^{(2)}] = (1+\mathbb{L}^d)[Y]$, which means that $p_{Y^{(2)}}(t) = \frac{1}{2}p_Y(t)^2 + \frac{1}{2}p_Y(t^2) = (1+t^{2d})p_Y(t)$.

Writing $p_Y(t) = t^{2m} + b_{2m-1}t^{2m-1} + \ldots + b_1 t + b_0$, this would imply that $b_1 = b_2 = \cdots = b_{2d-1} = 0$. We can prove this by induction. The coefficient of t^2 in $\frac{1}{2}p_Y(t)^2 + \frac{1}{2}p_Y(t^2)$ is $\frac{1}{2}((b_1^2 + 2b_0 b_2) + b_1)$ and its coefficient in $(1+t^{2m})p_Y(t)$ is b_2. Since $b_0 = 1$, this means that $b_1^2 + 2b_2 + b_1 = 2b_2 \Rightarrow b_1^2 + b_1 = 0$. Since $b_i \geq 0$ for each i, this implies that $b_1 = 0$. Suppose that $b_1 = b_2 = \cdots = b_{k-1} = 0$ for some $k \leq 2m-1$. Consider the coefficient of t^{2k} on each side. This implies that $\frac{1}{2}(b_k^2 + 2b_0 b_{2k} + 2b_1 b_{2k-1} + \ldots + 2b_{k-1}b_{k+1} + b_k) = b_{2k}$. Since $b_0 = 1$ and $b_1 = b_2 = \cdots = b_{k-1} = 0$, this means that $b_k^2 + 2b_{2k} + b_k = 2b_{2k} \Rightarrow b_k^2 + b_k = 0$. Since $b_k \geq 0$, this implies that $b_k = 0$. Thus, we have that $b_1 = b_2 = \cdots = b_{2m-1} = 0$ and $p_Y(t) = t^{2m}$.

Substituting this back into $p_{Y^{(2)}}(t) = \frac{1}{2}p_Y(t)^2 + \frac{1}{2}p_Y(t^2) = (1+t^{2m})p_Y(t)$ gives $t^{4m} +$

$t^{4m} = 2(1 + t^{2m}) \cdot t^{2m}$, which is clearly false. $\square$

Afterwards, the initial reduction is in determining $\dim F(Y)$. This involves looking at degrees of terms in the Poincaré polynomials.

Lemma 3.2.2. *If a variety $Y \subset \mathbb{P}^n$ in Theorem 3.1.2 satisfies the $Y - F(Y)$ relation, then* $\dim F(Y) = 2m - 4$.

Proof. Before substituting in Poincaré polynomials, we will rewrite $[Y^{[2]}] = [Y][\mathbb{P}^m] + \mathbb{L}^2[F(Y)]$ as $[Y^{[2]}] - [Y][\mathbb{P}^m] = \mathbb{L}^2[F(Y)]$. Since $Y^{[2]}$ is the blowup of $Y^{(2)}$ along the diagonal $\Delta \cong Y$, we have that $[Y^{[2]}] = [Y^{(2)}] + ([\mathbb{P}^{m-1}] - 1)[Y]$. Since $Y^{(2)}(t) = \frac{1}{2}p_Y(t)^2 + \frac{1}{2}p_Y(t^2)$ (p. 188 of [9] with $x = y = t$), the fact that $p_{\mathbb{P}^{m-1}}(t) = 1 + t^2 + \ldots + t^{2m-2}$ and $p_{\mathbb{A}^1}(t) = t^2$ implies that

$$\begin{aligned} p_{Y^{[2]}}(t) - p_Y(t)p_{\mathbb{P}^m}(t) &= p_Y^{(2)}(t) + (t^2 + \ldots + t^{2m-2})p_Y(t) - (1 + t^2 + \ldots + t^{2m})p_Y(t) \\ &= \frac{1}{2}p_Y(t)^2 + \frac{1}{2}p_Y(t^2) - (1 + t^{2m})p_Y(t). \end{aligned}$$

If the $Y - F(Y)$ relation holds, then

$$\frac{1}{2}p_Y(t)^2 + \frac{1}{2}p_Y(t^2) - (1 + t^{2m})p_Y(t) = t^4 p_{F(Y)}(t).$$

Since each side is a multiple of t^4, this means that the t^2 term on each side is equal to 0.

Note that $b_0 = b_{2m} = 1$ since Y is connected. On the left hand side, this means that

$$\begin{aligned}
\frac{1}{2}(b_0 b_2 + b_1^2 + b_2 b_0) + \frac{1}{2}b_1 - b_2 &= \frac{1}{2}(2b_0 b_2 + b_1^2) + \frac{1}{2}b_1 - b_2 \\
&= \frac{1}{2}(2b_2 + b_1^2) + \frac{1}{2}b_1 - b_2 \\
&= b_2 + \frac{b_1^2}{2} + \frac{1}{2}b_1 - b_2 \\
&= \frac{b_1^2}{2} + \frac{1}{2}b_1 \\
&= 0 \\
\Rightarrow b_1 &= 0.
\end{aligned}$$

Poincaré duality implies that $b_{2m-1} = 0$.

Also, we have that the coefficient of t^4 is nonzero since $F(Y) \neq \emptyset$ by Lemma 3.2.1. Then, the coefficient of t^4 (given by $b_0(F(Y))$) is equal to

$$\begin{aligned}
\frac{1}{2}(2b_0 b_4 + 2b_1 b_3 + b_2^2) + \frac{1}{2}b_2 - b_4 &= b_0 b_4 + b_1 b_3 + \frac{b_2^2}{2} + \frac{b_2}{2} - b_4 \\
&= b_4 + 0 + \frac{b_2^2}{2} + \frac{b_2}{2} - b_4 \\
&= \frac{b_2^2}{2} + \frac{b_2}{2} \\
&\neq 0 \\
\Rightarrow b_2 &\neq 0.
\end{aligned}$$

We use this to look at the coefficients of $t^{4m}, t^{4m-1}, t^{4m-2}, t^{4m-3}$, and t^{4m-4}. Since $\frac{1}{2}p_Y(t^2)$ can only contribute to even degree terms, we first look at the odd degree terms. The coefficient of t^{4m-1} on the left hand side is $\frac{1}{2}(2b_{2m-1}b_{2m}) - b_{2m-1} = b_{2m-1} - b_{2m-1} = 0$ since $4m-1 > 2m$ when $m \geq 4$ and Y is connected. This means that the coefficient of t^{4m-1} is 0. Similarly, the coefficient of t^{4m-3} is equal to $\frac{1}{2}(2b_{2m-3}b_{2m} + 2b_{2m-2}b_{2m-1}) - b_{2m-3} =$

$b_{2m-3} - b_{2m-3} = 0$ since $b_{2m-1} = b_1 = 0$.

Next, we find the coefficients of t^{4m}, t^{4m-2}, and t^{4m-4}. The coefficient of t^{4m} is $\frac{1}{2}b_{2m}^2 + \frac{1}{2}b_{2m} - b_{2m} = \frac{1}{2} + \frac{1}{2} - 1 = 0$. The coefficient of t^{4m-2} is

$$\begin{aligned}
\frac{1}{2}(2b_{2m-2}b_{2m} + b_{2m-1}^2) + \frac{1}{2}b_{2m-1} - b_{2m-2} &= b_{2m-2}b_{2m} + \frac{1}{2}b_{2m-1}^2 + \frac{1}{2}b_{2m-1} - b_{2m-2} \\
&= b_{2m-2} + 0 + 0 - b_{2m-2} \\
&= 0
\end{aligned}$$

since $b_{2m} = 1$ and $b_{2m-1} = b_1 = 0$.

However, the coefficient of t^{4m-4} is nonzero since it is equal to

$$\begin{aligned}
\frac{1}{2}(2b_{2m-4}b_{2m} + 2b_{2m-3}b_{2m-1} + b_{2m-2}^2) + \frac{1}{2}b_{2m-2} - b_{2m-4} &= \frac{1}{2}(2b_{2m-4} + b_{2m-2}^2) \\
&\quad + \frac{1}{2}b_{2m-2} - b_{2m-4} \\
&= b_{2m-4} + \frac{1}{2}b_{2m-2}^2 \\
&\quad + \frac{1}{2}b_{2m-2} - b_{2m-4} \\
&= \frac{1}{2}b_{2m-2}^2 + \frac{1}{2}b_{2m-2} \\
&= \frac{1}{2}b_2^2 + \frac{1}{2}b_2 \\
&\neq 0.
\end{aligned}$$

This implies that $\deg t^4 p_{F(Y)}(t) = 4m-4$ and $\deg p_{F(Y)} = 4m-8 \Rightarrow \dim F(Y) = 2m-4$ since $F(Y)$ is projective (and therefore compact). $\square$

The proof of Lemma 3.2.2 and Remark 3.1.3 imply the following:

Corollary 3.2.3. *Suppose that $Y \subset \mathbb{P}^n$ is a hypersurface generic among those of its degree.*

If Y satisfies the $Y - F(Y)$ relation, then Y is a cubic hypersurface.

Combining the work above with a result of Rogora [35], we now obtain a restriction on the codimension of Y in $\mathbb{P}^n$ which gives Theorem 3.1.2 after combining this with results on Hartshorne's conjecture.

Proof. (Proof of Theorem 3.1.2)
Since $\dim F(Y) = 2m - 4$ by Lemma 3.2.2, the initial conditions of Theorem 3.1.2 and the following result of Rogora [35] imply that the codimension of Y in $\mathbb{P}^n$ is ≤ 2.

Theorem 3.2.4. *(Rogora, Theorem 2 on p. 207 – 208 of [35], Theorem 2' on p. 209 of [35])*
Let $k \geq 4$ and $X \subset \mathbb{P}^n$ be a k-dimensional irreducible subvariety of a projective space of dimension n. Let $\Sigma \subset \mathbb{G}(1, n)$ be a component of maximal dimension of the variety of lines containe din X. If $\dim \Sigma = 2k - 4$, one of the following holds:

1. *X is a 1-dimensional infinite family of quadrics*
2. *X is a 2-dimensional infinite family of projective $(k-2)$-planes*
3. *X is a linear section of $\mathbb{G}(1, 4)$*
4. *$\dim X \geq n - 2$*

We split the remaining cases into hypersurfaces and codimension 2 varieties.

Suppose that Y is a hypersurface. Since $Y^{[2]}$ and Y are both smooth and projective, Larsen–Lunts' stable birational equivalence result implies that Y must be uniruled (Theorem 6.1.5 on p. 134 of [4], Corollary 2.6.3 on p. 476 of [4] with $Z = \mathbb{P}^r$). This implies that $\deg Y \leq n$ if Y is a hypersurface. In this case, a result of Beheshti–Riedl [11] implies that

$\dim F(Y)$ is equal to the "expected dimension" $2(m+1)-d-3 = 2m-d-1$ if $n \geq 2\deg Y - 4$ (Theorem 1.3 of [11]):

Theorem 3.2.5. *(Beheshti–Riedl, Theorem 1.3 on p. 2 of [11])*
Let $X \subset \mathbb{P}^n$ be a smooth hypersurface. Then $F_k(X)$ will be irreducible of the expected dimension provided that

$$n \geq 2\binom{d+k-1}{k} + k.$$

In the special case $k = 1$, we can improve the bound, proving that $F_1(X)$ is of the expected dimension $2n-d-3$ if $n \geq 2d-4$ and irreducible if $n \geq 2d-1$ and $n \geq 4$.

Since $n \geq 2$ and the "expected dimension" is equal to $2m-4$ if and only if $d = 3$, Y must be a cubic hypersurface.

It remains to consider the codimension 2 case. If Y is a complete intersection, the uniruledness restriction implies that $d_1 + d_2 \leq n$ (Section 4.4 on p. 99 of [11]). For example, $Y \subset \mathbb{P}^n$ is a complete intersection if it is defined scheme-theoretically by $\leq \frac{n}{2}$ equations (p. 588 of [3], [14]). Suppose that these equations have degree $\leq \frac{n}{2}$ as in condition b in Theorem 3.1.2. Let Z_i be the hypersurface of degree d_i. Since $Y \subset \mathbb{P}^n$ has codimension 2, we have that $n = m+2$. Since $d_i \leq \frac{n}{2}$, Theorem 3.2.5 applies and $\dim F(Z_i) = 2(m+2)-d_i-3 = 2m-d_i+1$ (i.e. the expected dimension). If $d_i \geq 6$ for *either* of $i = 1, 2$, then $\dim F(Y_i) \leq 2m-5$ and $\dim F(Y) \leq \dim F(Y_i) \leq 2m-5$. This would make it impossible for Y to satisfy the $Y - F(Y)$ relation. Thus, a codimension 2 complete intersection of hypersurfaces of degree (d_1, d_2) can only satisfy the $Y - F(Y)$ relation only if $d_1, d_2 \leq 5$.

We can further narrow down the remaining degree cases using a result of Canning (Theorem 1.3 on p. 2128 of [7]), which implies that the Debarre–de Jong conjecture holds for Fano complete intersections (d_1, d_2) with $d_1 + d_2 \leq 7$. In other words, $F(Y)$ has the expected dimension $2n - d_1 - d_2 - 4$ if $d_1 + d_2 \leq 7$. In order to have $\dim F(Y) = 2m - 4$, we need $d_1 + d_2 = 4$. Since we assumed that Y is nondegenerate (i.e. not contained in a hyperplane), we have that $d_i \geq 2$ for $i = 1, 2$ and the only instance where $d_1 + d_2 = 4$ is $d_1 = d_2 = 2$. It remains to study the cases where $d_1 + d_2 \geq 8$ and $d_1, d_2 \leq 5$. These are the pairs $(d_1, d_2) = (3, 5), (4, 4), (4, 5)$.

Let Z_i be a hypersurface of degree d_i for $i = 1, 2$. Since $n \geq m \geq 7 > 2(5) - 4 = 6$, Theorem 3.2.5 implies that Fano varieties of lines of hypersurfaces of degree 5 in $\mathbb{P}^n$ have the expected dimension $2m - 4$. Suppose that $d_2 = 5$. If $F(Z_1) \cap F(Z_2)$ is a nontrivial intersection, $Y = Z_1 \cap Z_2$ does *not* satisfy the $Y - F(Y)$ relation. We can show that this is indeed the case when $d_1 = 3, 4, 5$. If the intersection $F(Z_1) \cap F(Z_2)$ is trivial, then we either have $F(Z_1) \subset F(Z_2)$ or $F(Z_2) \subset F(Z_1)$. A dimension count implies that the only possibility is $F(Z_2) \subset F(Z_1)$. However, the fact that Z_1 and Z_2 are covered by lines implies that $Z_2 \subset Z_1$, which is impossible if Z_1 intersects Z_2 transversely. Thus, the only possible degrees (d_1, d_2) of codimension 2 varieties satisfying the $Y - F(Y)$ relation are $(2, 2)$ and $(4, 4)$.

When $\deg Y \ll n$, existing results on Hartshorne's conjecture actually force Y to be a complete intersection. If the codimension of Y in $\mathbb{P}^n$ is 2, it suffices to take $\deg Y \leq \frac{n}{4}$ in the codimension 2 case by the following result of Bertram–Ein–Lazarsfeld [3]. The reasoning above on the codimension 2 case can then be repeated.

Theorem 3.2.6. *(Bertram–Ein–Lazarsfeld, Corollary 3 on p. 588 of [3]) Assume that $X \subset \mathbb{P}^r$ is a smooth variety of degree d, dimension n, and codimension e. If*

$$d \leq \frac{r}{2e} \left[= \frac{n}{2e} + \frac{1}{2} \right],$$

then X is a complete intersection.

Finally, we consider additional restrictions among cubic hypersurfaces and codimension 2 complete intersections of degree $(2,2)$ or $(4,4)$ coming from the middle Betti number. Since b_m is either equal to $m+1$ or $m+4$ if Y is a complete intersection of two quadrics (p. 20 of [34]), we can exclude this case if we assume that b_m is exponential in m. We can also remove complete intersections of two quartics $(4,4)$ if we assume that $b_m < 2 \cdot 3^m$. Let $H(a_1, \ldots, a_r)$ be the two-variable generating function for Hodge numbers of complete intersections of hypersurfaces of degree $(a_1, \ldots, a_r)$ in $\mathbb{P}^n$ (Théorème 2.3 on p. 52 and Corollaire 2.4 on p. 53 of [18], p. 19 of [34]). Note that $H(a,a) = 2H(a) + (1+y)(1+z)H(a)$ by the reasoning on p. 20 of [34] and the middle primitive Betti number of a smooth hypersurface of degree a and dimension $u > 0$ is $\frac{(-1)^u}{a}(a - 1 + (1-a)^{u+2})$ (Corollary 1.8 on p. 14 of [22]). Thus, we have that $b_m(Y) > 2 \cdot 3^m - 5$ if Y is the complete intersection $(4,4)$ of two quadrics.

$\square$

Remark 3.2.7.

1. Hartshorne's original conjecture states that a smooth variety $Y \subset \mathbb{P}^n$ of dimension m is a complete intersection if $(n-m) < \frac{1}{3}n$ (p. 1017 of [20]). An overview of work related to this conjecture is given in the introduction to [13]. If the original conjecture holds, then a codimension 2 variety Y is a complete intersection if $n \geq 7$.

2. A possible method of approaching the codimension 2 case of Theorem 3.1.2 using more explicit examples without using results on Hartshorne's conjecture is related to work of Lanteri–Palleschi [27] (Proposition 2.1 (j) on p. 863 of [27], Theorem 2.1 on p. 153 – 154 of [5]) according to the result of Rogora [35] with the codimension restriction on varieties $Y \subset \mathbb{P}^n$ of dimension m such that $\dim F(Y) = 2m - 4$ (Remark 3 on p. 208 of [35]).

If we restricted to varieties with a *connected* Fano variety of lines, we can exclude 1-dimensional families of quadrics and 2-dimesnional families of (projective) $(m-2)$-planes from those varieties characterized by the $Y - F(Y)$ relation.

Proposition 3.2.8. *There are* 1*-dimensional families of quadrics or* 2*-dimensional families of (projective)* $(m-2)$*-planes with connected Fano varieties of lines which do* not *satisfy the* $Y - F(Y)$ *relation.*

Proof. Suppose that $F(Y)$ is connected. If Y is taken to be a 1-dimensional family of quadrics or 2-dimensional family of (projective) $(m-2)$-planes, the $Y - F(Y)$ relation might not necessarily be satisfied. While checking this, we will use the version of the $Y - F(Y)$ relation which states that $[Y^{(2)}] = (1 + \mathbb{L}^m)[Y] + \mathbb{L}^2[F(Y)]$. In the constructions below, we will use the fact that there linear subspaces contained in the the Grassmannian (with respect to the Plücker embedding) which can be constructed out of pencils of linear subspaces of $\mathbb{P}^n$ containing a fixed linear subspace A and contained in a fixed linear subspace A and containing a fixed linear subspace B of $\mathbb{P}^n$ (Theorem 3.16 on p. 110 and Theorem 3.22 on p. 115 of [21]). Given a Grassmannian $\mathbb{G}(r, n)$, these can be used to construct linear subspaces of dimension $\leq \max(n - r, r + 1)$.

1. Suppose that $2m \leq n$ and consider the trivial 1-dimensional family of quadrics $\mathbb{P}^1 \times Q$, where $Q \subset \mathbb{P}^m$ is a quadric hypersurface. Note that we take family to mean a (projective) line in the $\left(\binom{m+2}{2} - 1\right)$-dimensional projective space giving the Hilbert scheme of quadrics of dimension $m - 1$. By Example 2.4.7 on p. 73 – 74 of [4], we have that $[Q] = [\mathbb{P}^{m-1}]$ if m is even and $[Q] = [\mathbb{P}^{m-1}] + \mathbb{L}^{\frac{m-1}{2}}$ if m is odd.

   If m is even, this implies that $[Y] = (\mathbb{L} + 1)(\mathbb{L}^{m-1} + \ldots + \mathbb{L} + 1) = \mathbb{L}^m + 2\mathbb{L}^{m-1} +$

$\ldots + 2\mathbb{L} + 1$ and

$$(1+\mathbb{L}^m)[Y] = 1 + 2\mathbb{L} + \ldots + 2\mathbb{L}^{2m-1} + 2\mathbb{L}^{2m}.$$

Since $([X_1]+\ldots+[X_\ell])^{(n)} = \sum_{n_1+\ldots+n_\ell=n}\prod_{i=1}^{\ell}[X_i]^{(n_i)}$ (Remark 4.2 on p. 5 of [17]), we have that

$$\begin{aligned}[Y^{(2)}] &= 1 + 3\mathbb{L}^2 + \ldots + 3\mathbb{L}^{2m-2} + \mathbb{L}^{2m} + 3\mathbb{L} + 3\mathbb{L}^2 + \ldots + 3\mathbb{L}^{2m-2} \\ &\quad + \mathbb{L}^{2m} + 4\mathbb{L}^3 + \ldots + 4\mathbb{L}^m + 2\mathbb{L}^{m+1} + \ldots + 2\mathbb{L}^{2m-1}\end{aligned}$$

Since the quadratic term of $(1+\mathbb{L}^m)[Y]$ is $2\mathbb{L}^2$ and the quadratic term of $[Y^{(2)}]$ is $5\mathbb{L}^2$, the quadratic term of $[Y^{(2)}] - (1+\mathbb{L}^m)[Y]$ is $3\mathbb{L}^2$. Since the $Y - F(Y)$ relation states that $[Y^{(2)}] = (1+\mathbb{L}^m)[Y] + \mathbb{L}^2[F(Y)] \Longrightarrow \mathbb{L}^2[F(Y)] = [Y^{(2)}] - (1+\mathbb{L}^m)[Y]$, we can apply the Poincaré polynomial motivic measure to see that this contradicts our assumption that $F(Y)$ is connected since the constant term is 3 instead of 1.

Suppose that m is odd. Then, we have that $[Q] = [\mathbb{P}^{m-1}] + \mathbb{L}^{\frac{m-1}{2}}$. This implies that $[Y] = 1 + 2\mathbb{L} + 2\mathbb{L}^2 + \ldots + 2\mathbb{L}^{\frac{m-3}{2}} + 3\mathbb{L}^{\frac{m-1}{2}} + 3\mathbb{L}^{\frac{m+1}{2}} + 2\mathbb{L}^{\frac{m+3}{2}} + \ldots + 2\mathbb{L}^{m-1} + \mathbb{L}^m$ and

$$(1+\mathbb{L}^m)[Y] = 1 + 2\mathbb{L} + \ldots + 2\mathbb{L}^{\frac{m-3}{2}} + 3\mathbb{L}^{\frac{m-1}{2}} + 3\mathbb{L}^{\frac{m+1}{2}} + 2\mathbb{L}^{\frac{m+3}{2}} + \ldots + 2\mathbb{L}^{m-1} + \mathbb{L}^m.$$

Using the same method as above, we have that

$$\begin{aligned}[Y^{(2)}] &= 1 + 3\mathbb{L}^2 + \ldots + 3\mathbb{L}^{m-3} + 4\mathbb{L}^{m-1} + 4\mathbb{L}^{m+1} + 3\mathbb{L}^{m+3} + \ldots + 3\mathbb{L}^{m-1} + \mathbb{L}^{2m} \\ &\quad + 1 + 2\mathbb{L} + 2\mathbb{L}^2 + \ldots + 2\mathbb{L}^{\frac{m-3}{2}} + 3\mathbb{L}^{\frac{m-3}{2}} + 3\mathbb{L}^{\frac{m-1}{2}} + 2\mathbb{L}^{\frac{m+3}{2}} + \ldots + 2\mathbb{L}^{m-1} \\ &\quad + \mathbb{L}^m + \ldots + 2\mathbb{L}^{2m-1}.\end{aligned}$$

As with the previous case, we find that the quadratic term of $(1 + \mathbb{L}^m)[Y]$ is $2\mathbb{L}^2$ and the quadratic term of $[Y^{(2)}]$ is $5\mathbb{L}^2$. If $\mathbb{P}^1 \times Q$ satisfies the $Y - F(Y)$ relation, we would find that the constant term of the Poincaré polynomial of $F(Y)$ is equal to 3. This contradicts our assumption that $F(Y)$ is connected.

2. As mentioned above, we consider $(m-2)$-planes that we consider are $(m-2)$-planes Γ such that $A \subset \Gamma \subset B$ for some fixed (projective) $(m-4)$-plane A and $(m-1)$-plane B such that $A \subset B$. The family considered here is the union of such $(m-2)$-planes. Using an inclusion-exclusion argument, we see that the class of this union of $(m-2)$-planes in $K_0(\mathrm{Var}_k)$ is a polynomial in $\mathbb{L}$. Thus, it suffices to study point counts over $\mathbb{F}_q$. Since this is determined by quotients $\Gamma/A \subset B/A$ and A, we have that $\#Y(\mathbb{F}_q) = \#A(\mathbb{F}_q) \cdot \#(B/A)(\mathbb{F}_q)$. This means that $\#Y(\mathbb{F}_q) = (1 + q + \ldots + q^{m-4})(1 + q + q^2) = (1 + 2q + 3q^2 + \ldots)$ and the coefficient of q^2 is equal to 3. On the other hand, the fact that $([X_1] + \ldots + [X_\ell])^{(n)} = \sum_{n_1+\ldots+n_\ell=n} \prod_{i=1}^{\ell} [X_i]^{(n_i)}$ (Remark 4.2 on p. 5 of [17]) implies that the $\#Y^{(2)}(\mathbb{F}_q) = 1 + 2q + 6q^2 + \ldots$. Matching coefficients, this implies that it is impossible for $F(Y)$ to be connected.

In general, it is difficult to check whether a particular relation in $K_0(\mathrm{Var}_k)$ is satisfied since the projections from the incidence correspondences used to define the families of varieties used here (p. 208 – 209 of [35]) are not necessarily Zariski locally trivial fibrations or piecewise trivial fibrations.

□

The methods above have similar impliciationsfor other relations in $K_0(\mathrm{Var}_k)$.

Corollary 3.2.9. *Consider a polynomial relation in $K_0(\mathrm{Var}_k)$ with integer coefficients involving symmetric products of an m-dimensional smooth projective variety $Y \subset \mathbb{P}^n$, its second symmetric product $Y^{(2)}$, the Fano variety of lines $F(Y) \subset \mathbb{G}(1,n)$ and $\mathbb{A}^1$. If it can only be* *satisfied by Y such that $\dim F(Y) = 2m-4$, the only varieties defined by $\leq \frac{n}{2}$ equations or of degree $\leq \frac{n}{4}$ that can satisfy the relation are cubic hypersurfaces.*

Remark 3.2.10. While the $Y - F(Y)$ is close to characterizing cubic hypersurfaces, the $Y - F(Y)$ relation does *not* necessarily generate possible polynomial relations between a cubic hypersurface $Y \subset \mathbb{P}^{m+1}$ and its Fano variety of lines $F(Y) \subset \mathbb{G}(1, m+1)$. This can be checked explicitly for the case $m = 2$ or $m = 3$ when Y is taken to have an ordinary double point (Example 5.8 on p. 20 of [16]).

3.2.2 Approach via uniruledness property and an application to a stable rationality question

Our initial approach to Question 3.1.1 involved using uniruledness properties of varieties which can satisfy the given relation. The most recent/general restriction using this approach (Proposition 3.2.17, Corollary 3.2.19) is given below. These methods can be generalized to analyze other relations in $K_0(\mathrm{Var}_k)$ with some properties similar to the $Y - F(Y)$ relation. More specifically, we will apply the logic of the proof of Proposition 3.2.11 to related stable birationality problem (Example 3.2.21) and obtain geometric restrictions to varieties satisfying relations sharing certain properties with the $Y - F(Y)$ relation (Example 3.2.22).

Using a result on unirationality of symmetric products (Proposition 3.2.11), we will first show that complete intersections of ≥ 2 general hypersurfaces (in their given degrees) of dimension ≥ 2 do *not* satisfy Galkin–Shinder's $Y - F(Y)$ relation (Corollary 3.2.19). Afterwards, we will use the proof of Proposition 3.2.11 to rule out the $Y - F(Y)$ for the intersection of an integral variety with a *very general* hypersurface (Example 3.2.20). Before doing this, we will state the birational geometry results used.

Proposition 3.2.11. *Suppose that $\overline{k} = k$ and $\operatorname{char} k = 0$. If Y is a smooth projective variety of dimension $d \geq 1$ such that $[Y^{(2)}] \equiv [Y] \pmod{\mathbb{L}}$, the Kodaira dimension of Y is $\kappa(Y) = -\infty$. If $d \geq 2$, the variety Y must be uniruled.*

Remark 3.2.12. In general, it is known that uniruled varieties have Kodaira dimension $-\infty$ (Corollary 1.11 on p. 189 of [24]). However, the converse is only known in dimension ≤ 3 (Conjecture 1.12 on p. 189 of [24]).

Proof. The blowup at the diagonal $\mathrm{Bl}_{\Delta_Y}(Y \times Y) \longrightarrow Y \times Y$ induces the blowup $Y^{[2]} \longrightarrow Y^{(2)}$, which is also a blowup at the diagonal. Since Y is smooth and projective, the variety $Y^{[2]}$ is also smooth and projective (Example 7.3.1 on p. 169 of [15], Theorem 3.1 on p. 5 of [29], Theorem 1.4 on p. 10 of [24]).

Since $Y^{[2]}$ and Y are smooth projective varieties, the isomorphism $K_0(\mathrm{Var}_k)/(\mathbb{L}) \longrightarrow \mathbb{Z}[SB]$ from Larsen–Lunts' motivic measure (Theorem 2.3 on p. 87 of [28], Theorem 6.1.5 on p. 134 of [4]) implies that $Y^{[2]}$ and Y are stably birational to each other. Since we assumed that $d \geq 1$, we have that $\dim Y < \dim Y^{[2]} = 2 \dim Y$. Since Y and $Y^{[2]}$ are *not* birational, the following results imply that Z is uniruled:

Lemma 3.2.13. *(Lemma 33.25.10 of [36])*
Let k be a field. Let X be a variety over k which has a k-rational point x such that X is smooth at x. Then X is geometrically integral over k.

Corollary 3.2.14. *(Corollary 2.6.4 on p. 477 of [4])*
Let k be a field of characteristic 0, and X and Y be stably birational integral k-varieties such

that X is not *uniruled and* $\dim Y \leq \dim X$*. Then X and Y are birational.*

Thus, the variety $Y^{[2]}$ is uniruled and its Kodaira dimension is $\kappa(Y^{[2]}) = -\infty$. Since $Y^{[2]}$ is a desingularization of $Y^{(2)}$, we have that

$$-\infty = \kappa(Y^{[2]}) = \kappa(Y^{(2)}) = 2\kappa(Y) \text{ (Theorem 1 on p. 1369 of [1]).}$$

This implies that $\kappa(Y) = -\infty$.

Since any variety dominated by a uniruled variety is uniruled, the variety $Y^{(2)}$ is also uniruled. In fact, the following result implies that Y itself is uniruled if $d \geq 2$:

Corollary 3.2.15. *(Cadorel–Campana–Rousseau, Corollary 4.2 on p. 9 – 10 of [6]) Suppose that X is compact and Kähler.*

1. *X is rationally connected if and only if $X^{(m)}$ is for some $m \geq 1$.*

2. *X is uniruled if and only if $X^{(m)}$ for some $m \geq 1$, unless X is a curve of genus $g > 0$ and $m > g$. In that case, $X^{(m)}$ is uniruled and X is* not *uniruled.*

□

Before applying the methods used here to study varieties satsifying the $Y - F(Y)$ relation, we obtian some initial restrictions on degrees of generic hypersurfaces containing them. Recall from Lemma 3.2.1 that $F(Y) \neq \emptyset$ if $Y \subset \mathbb{P}^n$ satsifies the $Y - F(Y)$ relation. Applying Theorem 4.3 on p. 266 of [24] gives a bound on (general) hypersurfaces containing a variety satisfying the $Y - F(Y)$ relation.

Corollary 3.2.16. *Suppose that $Y \subset \mathbb{P}^n$ is a d-dimensional variety satisfying the $Y - F(Y)$ relation. Then, the variety Y is* not *contained in a general hypersurface of degree $r > 2n-3$.*

Proof. As mentioned above, this is an application of Lemma 3.2.1 and the fact that the Fano variety of lines for a general hypersurface of degree $r > 2n-3$ is empty (Theorem 4.3 on p. 266 of [24]).

□

Under these restrictions, we can obtain restrictions on varieties $Y \subset \mathbb{P}^n$ satisfying the $Y-F(Y)$ relation which are intersections of generic hypersurfaces (among given degrees). This is done using uniruledness properties and repeated applications of the projective dimension theorem (Theorem 7.2 on p. 48 of [19]) and the dimension of $F(Y)$ for a generic hypersurface of dimension r (Theorem 4.3 on p. 266 of [24]):

Proposition 3.2.17. *Suppose that $Y \subset \mathbb{P}^n$ is a m-dimensional variety satisfying the $Y - F(Y)$ relation.*

1. *If Y is the intersection of u* general *hypersurfaces W_i of degree $r_i \leq 2n-3$ for each i, we have that*

$$(u+1)(n-(R+2)) + (R+1) \leq 2m-4,$$

where $R = \max(r_1, \ldots, r_u)$.

2. *If $\dim Y \geq 2$ and Y is the complete intersection of $m \geq 2$* general *hypersurfaces W_i of degree $r_i \leq 2n-3$ for each i, then it does* not *satisfy the $Y - F(Y)$ relation.*

Remark 3.2.18.

1. We inserted the condition on the r_i to keep $F(Y)$ from being empty (Corollary 3.2.16).

2. Hartshorne's conjecture 3.2.7 implies that Part 2 can be rewritten as generic condition on varieties of a given degree when the codimension is small (proof of Lemma 3.2.2).

3. The dimension bound in Part 1 uses Lemma 3.2.2.

Proof. Both parts use the same initial setup. By definition, the Fano scheme $F(Y)$ is the intersection of $F(W)$ for hypersurfaces W containing Y (p. 196 of [12]). Writing $Y = W_1 \cap \cdots W_u$,this means that $F(Y) = F(W_1) \cap \cdots \cap F(W_u)$. Then, induction on u and repeatedly using the projective dimension theorem (Theorem 7.2 on p. 48 of [19]) implies that

$$\dim F(Y) \geq \dim F(W_1) + \ldots + \dim F(W_u) - (u-1)(n-1)$$

since $F(W_i) \subset \mathbb{G}(1,n) \cong \mathbb{P}^{n-1}$.

We first prove the dimension inequality in Part 1.

1. Let $R = \max(r_1, \ldots, r_n)$. Since the hypersurfaces $W_i \subset \mathbb{P}^n$ are taken to be general among those of degree r_i in $\mathbb{P}^n$ and $\dim F(W_i) = 2n - 3 - r_i$ for such degree r_i hypersurfaces (Theorem 4.3 on p. 266 of [23]), this means that

$$
\begin{aligned}
\dim F(Y) &\geq \dim F(W_1) + \ldots + \dim F(W_u) - (u-1)(n-1) \\
&= (2n - 3 - r_1) + \ldots + (2n - 3 - r_u) - (u-1)(n-1) \\
&= un - 2u + n - 1 - (r_1 + \ldots + r_u) \\
&\geq un - 2u + n - 1 - Ru \\
&= un - (R+2)u + n - 1 \\
&= u(n - (R+2)) + (n - (R+2)) + (R+1) \\
&= (u+1)(n - (R+2)) + (R+1).
\end{aligned}
$$

Since $\dim F(Y) = 2m - 4$ by Lemma 3.2.2, this implies that

$$(u+1)(n - (R+2)) + (R+1) \leq 2m - 4.$$

Next, we prove Part 2 by combining the methods above with a uniruledness property.

2. Since the complete intersection $Y = W_1 \cap \cdots \cap W_u$ is uniruled by Proposition 3.2.11, we have that $r_1 + \ldots + r_u \leq n$ (Section 4.4 on p. 99 of [11]).

Recall from the proof of Part 1 that $\dim F(Y) \geq \dim F(W_1) + \ldots + \dim F(W_u)$.

Since the hypersurfaces $W_i \subset \mathbb{P}^n$ are taken to be general among those of degree r_i in $\mathbb{P}^n$, a standard result on dimensions of Fano varieties of lines on generic hypersurfaces of a given degree (Theorem 4.3 on p. 266 of [24]) implies that

$$
\begin{aligned}
\dim F(Y) &\geq \dim F(W_1) + \ldots + \dim F(W_u) - (u-1)(n-1) \\
&= (2n-3-r_1) + \ldots + (2n-3-r_u) - (u-1)(n-1) \\
&= 2un - 3u - (r_1 + \ldots + r_u) - (un - u - n + 1) \\
&\geq 2un - 3u - n - (un - u - n + 1) \\
&= u(n-2) - 1 \\
&\geq 2(n-2) - 1 \\
&= 2n - 5
\end{aligned}
$$

However, this is impossible since $\dim F(Y) = 2m - 4$ by Lemma 3.2.2.

□

Using Proposition 3.2.11, we can consider the case where Y is a complete intersection of general hypersurfaces.

Corollary 3.2.19. *If $Y \subset \mathbb{P}^n$ is a variety of dimension ≥ 2 which is the complete intersection of $m \geq 2$* general *hypersurfaces W_i of degree r_i for each i, then it does* not *satisfy the $Y - F(Y)$ relation.*

Proof. By Corollary 3.2.16, a variety $Y \subset \mathbb{P}^n$ satisfying the $Y - F(Y)$ is *not* contained in a general hypersurface of degree $r > 2n-3$. Combining this with Part 2 of Proposition 3.2.17, we cover general hypersurfaces of arbitrary degrees. □

If we look at *very general* hypersurfaces, we can use the logic of the uniruled restriction to rule out the $Y - F(Y)$ relation in possibly non-complete intersections.

Example 3.2.20. (**The $Y - F(Y)$ relation and possibly non-complete intersections**)

Recall that we used a result comparing stable birational and birational equivalence classes of varieties (Corollary 3.2.14) to show that smooth projective varieties of dimension ≥ 2 which satisfy the $Y - F(Y)$ must be uniruled. After substituting in the Poincaré polynomials, we use this restriction to show that a complete intersection of general hypersurfaces (of their given degrees) does *not* satisfy the $Y - F(Y)$ relation. However, the varieties that fail to satisfy the $Y - F(Y)$ relation are not necessarily complete intersections. By Proposition 1 on p. 1 of [23], the intersection of an integral k-variety $X \subset \mathbb{P}^N$ and a very general hypersurface $H \subset \mathbb{P}^N$ is *not* uniruled. Thus, such an intersection $X \cap H$ of dimension ≥ 2 does *not* satisfy the $Y - F(Y)$ relation.

The comparison between stable birational and birational equivalence in Corollary 3.2.14 can also be used to study stable birationality questions.

Example 3.2.21. (**Symmetric products and stable birationality**)
In Question 1 on p. 10 of [6], it is asked whether $X^{(m)}$ being unirational (resp. rational, stably rational) for some $m \geq 2$ implies that X is unirational (resp. rational, stably rational). Applying Corollary 2.6.4 on p. 477 of [4] to a smooth projective resolution of $X^{(m)}$, we can show that this is false for stable birationality if X has Kodaira dimension $\kappa(X) \geq 0$.

A more "straightforward" application of the argument of Proposition 3.2.11 from the previous section is to restrict varieties which satisfy other relations which share certain properties with the $Y - F(Y)$ relation.

Suppose a smooth projective variety Y of dimension ≥ 1 satisfies $[Y^{(a)}] \equiv [Y^{(b)}] \pmod{\mathbb{L}}$ for some $a > b$ (e.g. $a = 2$ and $b = 1$ in the $Y - F(Y)$ relation). If $\dim Y \geq 2$, the same we can use Corollary 4.2 on p. 9 – 10 of [6] again to find that such a Y must be uniruled. Here is a generalization of uniruledness arguments restricting varieties satisfying the $Y - F(Y)$ relation taking these observations into account:

Proposition 3.2.22. *(Restatement of Kodaira dimension/uniruled restriction)*
If Y is a smooth projective variety of dimension $d \geq 1$ such that $[Y^{(a)}] \equiv [Y^{(b)}] \pmod{\mathbb{L}}$ for some $a > b \geq 1$, the Kodaira dimension of Y is $\kappa(Y) = -\infty$. If $d \geq 2$, the variety Y must be uniruled.

Again, the restriction on the Kodaira dimension immediately implies the following:

Corollary 3.2.23. *The following hold:*

1. *If $Y \subset \mathbb{P}^{d+1}$ is a d-dimensional hypersurface of degree r such that $[Y^{(a)}] \equiv [Y^{(b)}] \pmod{\mathbb{L}}$ in $K_0(\mathrm{Var}_k)$ for some $a > b \geq 1$, then $r \leq d+1$.*

2. *If Y is a smooth projective surface such that $[Y^{(a)}] \equiv [Y^{(b)}] \pmod{\mathbb{L}}$ in $K_0(\mathrm{Var}_k)$, then Y must be a ruled surface (over a curve of genus ≥ 1) or a rational surface.*

Splitting varieties by whether they are uniruled or not (or by Kodaira dimension being nonnegative or not) can also give a general perspective on restricting varieties which satisfy given relations in $K_0(\mathrm{Var}_k)$:

Remark 3.2.24. (Uniruled varieties and problematic birational automorphisms)
Results of Kuber (Theorem 4.1 and Theorem 5.1 on p. 482 – 485 of [26]) imply that the

structure of (the graded ring associated to) $K_0(\mathrm{Var}_k)$ essentially depends on birational equivalence classes of varieties when we consider classes in $K_0(\mathrm{Var}_k)$ where Larsen–Lunts' cut and paste conjecture holds. By results of Liu and Sebag, this includes varieties containing only finitely many rational curves (Theorem 6.3.7 on p. 142 of [4]) and algebraic surfaces whose 1-dimensional components are rational curves (Corollary 6.3.8 on p. 144 of [4]). In some sense, this is the "opposite" of the uniruled varieties mentioned above since we were looking at when a pair of stably birational varieties are not actually birational to each other in Proposition 3.2.22.

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Part II

h-vector problems and combinatorial analogues of hypersurface independence conditions

CHAPTER 4

GENERALIZATIONS OF CHROMATIC POLYNOMIALS AND H-VECTORS OF SIMPLICIAL COMPLEXES

4.1 Simplicial chromatic polynomials as Hilbert series of Stanley–Reisner rings

We find families of simplicial complexes where the simplicial chromatic polynomials defined by Cooper–de Silva–Sazdanovic [13] are Hilbert series of Stanley–Reisner rings of auxiliary simplicial complexes. As a result, such generalized chromatic polynomials are determined by h-vectors of auxiliary simplicial complexes. In addition to generalizing related results on graphs and matroids, the simplicial complexes used allow us to consider problems that are not necessarily analogues of those considered for graphs. Some examples include supports of cyclotomic polynomials, log concavity properties, and symmetry relations between a polynomial and its reciprocal polynomial.

If the h-vectors involed have sufficiently large entries, the Hilbert series are Hilbert polynomials of some k-algebra. As a consequence of connections between h-vectors and simplicial chromatic polynomials, we also find simplicial complexes whose h-vectors are determined by addition-contraction relations of simplicial complexes analogous to deletion-contraction relations of graphs. The constructions used involve generalizations of relations Euler characteristics of configuration spaces and chromatic polynomials of graphs.

4.1.1 Introduction

The main objective of this paper is to show that Euler characteristics of certain generalized configuration spaces are Hilbert series of Stanley–Reisner rings of associated simplicial complexes. The latter interpretation implies that these Euler characteristics are determined by h-vectors of some auxiliary simplicial complexes. In the course of doing this, we find that these Euler characteristics specialize to invariants satisfying the following properties:

- Polynomials with constant terms determining whether a cyclotomic polynomial of degree $n = p_1 \cdots p_d$ (p_i distinct primes) has a nonzero term of degree j for $0 \le j \le \varphi(n)$ (Corollary 4.1.27)

- Log concavity properties of simplicial chromatic polynomials and their translates (Corollary 4.1.32, Example 4.1.35)

- Symmetric relations between these polynomials and their reciprocal polynomials (Example 4.1.37)

Each problem listed above is associated with an appropriate choice of simplicial complexes.

Let X be a manifold and $\mathrm{Conf}^n X = \{(x_1, \ldots, x_n) \in X^n : x_i \neq x_j \text{ for all } i \neq j\}$. The configuration spaces we study generalize a relation between Euler characteristics of compactications of $\mathrm{Conf}^n X$ and chromatic polynomials of graphs. More specifically, our starting point is a combinatorial interpretation of the (compactly supported) Euler characteristic of the ordered configuration space $\mathrm{Conf}^n X$ and proper colorings of the complete graph with n vertices K_n.

$$\chi_c(\mathrm{Conf}^n X) = \chi_c(X)(\chi_c(X) - 1) \cdots (\chi_c(X) - (n-1)). \tag{4.1.1}$$

Comparing this expression to the chromatic polynomial

$$p_{K_n}(\lambda) = \lambda(\lambda - 1) \cdots (\lambda - (n-1)) \tag{4.1.2}$$

of the complete graph K_n for λ available colors, we find that $\chi_c(\operatorname{Conf}^n X) = p_{K_n}(\chi_c(X))$.

We can consider how to generalize the relation $\chi_c(\operatorname{Conf}^n X) = p_{K_n}(\chi_c(X))$ to chromatic polynomials of arbitrary graphs G. Since adding edges to a graph G introduces additional restrictions to proper colorings of G, it is natural to expect that an associated configuration space would be a partial compactification of $\operatorname{Conf}^n X$. In this context, we would allow the same points of X to occupy slots corresponding to non-adjacent vertices. Indeed, Eastwood–Huggett (Theorem 2 on p. 155 of [6]) found such a generalization for a modified configuration space parametrizing such configurations.

Given a graph G with vertices $V = \{v_1, \ldots, v_n\}$, they consider Euler characteristics of the configuration space

$$M_G = M^n \setminus \bigcup_{e \in E} \Delta_e,$$

for $M = \mathbb{CP}^{\lambda-1}$, where

$$\Delta_e = \{(x_1, \ldots, x_n) \in M^n : x_i = x_j\}$$

for each edge $e = v_i v_j$ of G. The authors then show that

$$p_G(\lambda) = \chi_c((\mathbb{CP}^{\lambda-1})_G) \text{ (Theorem 2 on p. 1155 of [6])}, \tag{4.1.3}$$

which generalizes the correspondence between 4.1.1 and 4.1.2.

Recent work of Cooper–de Silva–Sazdanovic [13] further generalize the construction of Eastwood–Huggett [6] with simplicial complexes replacing graphs. They define the simplicial chromatic polynomial (Definition 6.1 on p. 738 of [13], Definition 4.2.2), which is the compactly supported Euler characteristic of a certain configuration space 4.2.2 which is a higher dimensional version of M_G for simplicial complexes.

Definition 4.1.1. (Definition 2.1 on p. 725 and p. 738 of [13])
Let S be a simplicial complex whose 0-skeleton is given by the vertex set $V = V(S) = \{v_1, \ldots, v_n\}$. Let M be a topological space. For each simplex $\sigma = [v_{i_1} \cdots v_{i_k}]$, define the diagonal corresponding to σ to be

$$D_\sigma = \{(x_1, \ldots, x_n) \in M^n : x_{i_1} = \cdots = x_{i_k}\}.$$

We define the **simplicial configuration space** as

$$M_S = M^n \setminus \bigcup_{\sigma \in \Delta^V \setminus S} D_\sigma \tag{4.1.4}$$

where Δ^V is the simplicial complex containing all subsets of the vertices v_i (analogous to a simplex generated by independent vectors corresponding to the v_i) and $\Delta^V \setminus S$ denotes tuples of vertices in V which do *not* occur as simplices in S.

The simplicial chromatic polynomial $\chi_c(S)(t)$ associated to a simplicial complex S is the compactly supported Euler characteristic of M_S with $M = \mathbb{CP}^{t-1}$ (Definition 4.2.2). This polynomial is characterized (up to normalization) by an analogue of the deletion-contraction property for chromatic polynomials (Proposition 4.1.11, Corollary 6.1 and Proposition 6.4 on p. 738 of [13]). We would like to explore the combinatorial side of the simplicial chromatic polynomial.

In Section 7 on p. 740 – 742 of [13], the authors show that the simplicial chromatic polynomial differs from a number of known polynomial invariants of graphs. They state that it is not a specialization of known polynomial invariants of graphs or simplicial complexes. However, we will show that $\chi_c(S)$ can often be built out of invariants of an auxiliary simplicial complex $T(S)$. The main tool we use to do this is the Stanley–Reisner ring (Definition 4.1.14).

Definition 4.1.2. (Stanley–Reisner ring, p. 53 – 54 of [27])
Let k be a field and S be a simplicial complex with vertex set $V = \{1, \ldots, n\}$. We will call the subsets of V belonging to S **faces** and those that do *not* belong to S **nonfaces**.

For a subset A of the vertex set V, write $x_A = \prod_{i \in A} x_i \in k[x_1, \ldots, x_n]$. Let $I(A)$ denote the ideal in $k[x_1, \ldots, x_n]$ generated by all monomials x_σ such that $\sigma \notin S$ (i.e. $\sigma \in \Delta^V \setminus S$ in the notation above). Note that $I(S)$ is generated by the *minimal* nonfaces of S since σ being a nonface and $\alpha \supset \sigma$ implies that α is also a nonface of S. The quotient ring $k[S] := k[V]/I(S)$ is called the **Stanley–Reisner ring (or face ring)** of S.

The Stanley–Reisner ring $k[S]$ is the natural setting for many combinatorial problems (e.g. Stanley's proof of the upper bound theorem for simplicial spheres in Section 5.4 on p. 237 – 240 of [10]). An overview of algebraic properties of this ring and their applications is given in a survey of Franscisco–Mermin–Schweig [15]. Some examples are also given on p. 7 – 8 of [11].

Our main result constructs families of simplicial complexes S such that $\chi_c(S)$ can be expressed as a specialization of the Hilbert series of the Stanley–Reisner ring associated an auxiliary simplicial complex $T(S)$ (Theorem 4.1.5).

The families of simplicial complexes that will consider satisfy certain intersection properties. We list the properties below and give examples of simplicial complexes satisfying them.

Definition 4.1.3. Let S be a simplicial complex with minimal nonfaces $\sigma_1, \ldots, \sigma_r$. A simplicial complex S satisfies **property I** if there is a collection of finite sets α_i such that $|\alpha_i| = |\sigma_i| - 1$ for each $1 \le i \le r$ and $\alpha_I \cap \alpha_p = \sigma_I \cap \sigma_p = \emptyset$ if $\sigma_I \cap \sigma_p = \emptyset$ and $|\alpha_I \cap \alpha_p| = |\sigma_I \cap \sigma_p| - 1$ if $|I| \ge 2$ and $\sigma_I \cap \sigma_p \neq \emptyset$ for each subset $I \subset [r]$ and $p \notin I$.

Example 4.1.4. Here are two examples where property I is satisfied.

- The minimal nonfaces of S are disjoint from each other by a simplicial complex S.
- There is a point $a \in V$ such that $\sigma_i \cap \sigma_j = \{a\}$ for each i, j.

The general idea is that the minimal nonfaces either intersect at a small number of points or each σ_i contains many points which are not contained in any σ_j for $j \neq i$. Some specific simplicial complexes satisfying property I are given in Example 4.1.19.

Theorem 4.1.5. *Let S be a simplicial complex and $V = V(S) = \{1, \ldots, n\}$ be the vertex set of S. Given $K \subset [n]$, let $x^K = \prod_{i \in K} x_i$. Let $I(S) = \langle x^{\sigma_1}, \ldots, x^{\sigma_r} \rangle$ with the σ_i equal to the minimal nonfaces of S. Given a collection of subsets c_u with $u \in K$ for some finite set K, let $c_K = \bigcup_{u \in K} c_u$.*

Suppose that S satisfies property I. Then, the simplicial chromatic polynomial $\chi_c(S, t)$ of S is a normalization (in the sense of Proposition 4.1.16) of the Hilbert–Poincaré series of an auxiliary simplicial complex $T = T(S)$ which is determined by its h. Let $h_{T(S)}$ be the generating function of the h-vector of $T(S)$. This is the polynomial where the coefficient of t^i is h_i. Tracing through the definitions, we have that $\chi_c(S)(t) - t^n = t^n(h_{T(S)}(t^{-1}) - 1) \Longrightarrow \chi_c(S)(t) = t^n h_{T(S)}(t^{-1})$.

Remark 4.1.6. Proposition 4.1.16 gives an alternate set of conditions on the minimal nonfaces involving the number of connected components of a graph determined by the minimal nonfaces. For example, these conditions are satisfied by simplicial complexes where any pair of minimal nonfaces has a nonempty intersection.

In this setting of Theorem 4.1.5, the simplicial chromatic polynomial is entirely determined by the h-vector of S (Corollary 4.1.21). We also note that "normalized" version (without taking the reciprocal polynomial) of the simplicial chromatic polynomial is a specialization of Hilbert series of the canonical module of the Stanley–Reisner ring.

Under suitable assumptions, there is a natural relation between simplicial chromatic polynomials and Hilbert *polynomials* of other rings (Corollary 4.2.7) when the h_i are sufficiently large.

Definition 4.1.7. Given a k-element subset $I = \{\sigma_1, \ldots, \sigma_k\} \subset \Delta^V \setminus S$, let G_I be the graph whose vertices are the σ_i with two vertices corresponding to σ_i, σ_j being connected by an edge if and only if $\sigma_i \cap \sigma_j \neq \emptyset$. Let $c(I)$ be the number of connected components of G_I.

Corollary 4.1.8. *Let S be a simplicial complex such that $c(I) = a$ for all subsets $I \subset [r]$ as in Proposition 4.1.16. If $h_{a+r} \geq 1$, $h_{a+1}, h_{a+2} \geq 3$, and $h_i \geq 1$ for all $i \geq a$, then*

$$t^{-n} - \chi_c(S, t^{-1}) = t^{-n} P(t^{-1})$$

for the Hilbert polynomial $P = P(x)$ of some k-algebra.

These relations are analogous to some results on chromatic polynomials and Hilbert polynomials in the literature (e.g. Theorem 13 on p. 79 of [16], Proposition 3.3 on p. 9 of [1]) although specializations of our results to these settings seem to yield different simplicial complexes.

Finally, we show that there are some natural connections between simplicial chromatic polynomials and certain simplicial complexes whose structure depends on the coefficients of cyclotomic polynomials in Section 4.1.3. The applications involved include supports of coefficients of cyclotomic polynomials (Corollary 4.1.27), log concavity of polynomials (Corollary

4.1.32, Example 4.1.33), and symmetric relations between the simplicial chromatic polynomial and its reciprocal (Example 4.1.37).

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4.1.2 Simplicial chromatic polynomials and Hilbert series of Stanley–Reisner rings

In this section, we express simplicial chromatic polynomials as Hilbert series (Theorem 4.1.5, Proposition 4.1.16) and Hilbert polynomials (Corollary 4.2.7) of auxiliary simplicial complexes.

Transformation from Hilbert series of (auxiliary) simplicial complexes and h-vectors

Before showing the main reinterpretation of the simplicial chromatic polynomial, we first go over some basic definitions used on both sides of the correspondence.

Definition 4.1.9. (Definition 2.1 on p. 725 and p. 738 of [13])

Let S be a simplicial complex whose 0-skeleton is given by the vertex set $V = V(S) = \{v_1, \ldots, v_n\}$. Let M be a topological space. For each simplex $\sigma = [v_{i_1} \cdots v_{i_k}]$, define the diagonal corresponding to σ to be

$$D_\sigma = \{(x_1, \dots, x_n) \in M^n : x_{i_1} = \dots = x_{i_k}\}.$$

We define the **simplicial configuration space** as

$$M_S = M^n \setminus \bigcup_{\sigma \in \Delta^V \setminus S} D_\sigma \tag{4.1.5}$$

where Δ^V is the simplicial complex containing all subsets of the vertices v_i (analogous to a simplex generated by independent vectors corresponding to the v_i) and $\Delta^V \setminus S$ denotes tuples of vertices in V which do *not* occur as simplices in S.

Definition 4.1.10. (Definition 6.1 on p. 738 of [13])

Let S be a simplicial complex and let M be a manifold. Given S and M, let

$$\chi_c(S, M) := \sum (-1)^k \operatorname{rank} H_c^k(M_S).$$

The **simplicial chromatic polynomial** of a simplicial complex S is the polynomial defined by the assignment $\chi_c(S) : t \mapsto \chi_c(S, \mathbb{CP}^{t-1})$.

Note that the union 4.1.5 is determined by the *minimal* nonfaces $\sigma \in \Delta^V \setminus S$ since $\tau \subset \sigma \Rightarrow D_\tau \supset D_\sigma$. Under an appropriate normalization, the simplicial chromatic polynomial is uniquely defined by the following addition-contraction relation involving the minimal nonfaces:

Proposition 4.1.11. *(Proposition 6.4 on p. 738, Definition 2.2 on p. 727 of [13])*

Let σ be a minimal nonface of a simplicial complex S, and let $S_{/\sigma}$ be the tidied contraction, which removes every element sharing a vertex with σ from S.

The normalization $\chi_c(\Delta^{t-1}, t) = t^n$ and the addition-contraction formula

$$\chi_c(S, t) - \chi_c(S \cup \{\sigma\}, t) + \chi_c(S_{/\sigma}, t) = 0$$

determine a unique polynomial invariant of simplicial complexes.

Given a manifold M, let $[M] = \chi_c(M)$. Using an inclusion-exclusion argument and the fact that $\chi_c(\mathbb{CP}^{t-1}) = t$, we can obtain a more explicit expression for the simplicial chromatic polynomial with $\mathbb{CP}^{t-1}$ substituted in for M below. Note that $[M] = t$ if $M = \mathbb{CP}^{t-1}$.

$$\begin{aligned}
[M_S] &= \left[M^n \setminus \bigcup_{\sigma \in \Delta^V \setminus S} D_\sigma \right] \\
&= [M]^n - \left[\bigcup_{\sigma \in \Delta^V \setminus S} D_\sigma \right] \\
&= [M]^n - \sum_{\sigma_1, \ldots, \sigma_k \in I \subset [\Delta^V \setminus S] : |I| = k} (-1)^k [D_{\sigma_1} \cap \cdots \cap D_{\sigma_k}]
\end{aligned}$$

To simplify this final expression, we need to think about the number of "independent values" in an element of $D_{\sigma_1} \cap \cdots \cap D_{\sigma_k}$. This depends on the number of connected components of a certain graph (Definition 4.2.5).

Definition 4.1.12. Given a k-element subset $I = \{\sigma_1, \ldots, \sigma_k\} \subset \Delta^V \setminus S$, let G_I be the graph whose vertices are the σ_i with two vertices corresponding to σ_i, σ_j being connected by an edge if and only if $\sigma_i \cap \sigma_j \neq \emptyset$. Let $c(I)$ be the number of connected components of G_I.

We can express the class $[D_{\sigma_1} \cap \cdots \cap D_{\sigma_k}]$ in $K_0(\mathrm{Var}_k)$ as a power of $[M]$ whose exponent depends on $c(I)$. More specifically, the class above simplifies to

$$[M_S] = [M]^n - \sum_{\sigma_1,\ldots,\sigma_k \in I \subset [\Delta^V \setminus S] : |I| = k} (-1)^k [M]^{n - |\sigma_1 \cup \cdots \cup \sigma_k| + c(I)} \tag{4.1.6}$$

for each simplicial complex S. Note that we can replace $\sigma \in \Delta^V \setminus S$ with *minimal* nonfaces of S in all the sums above.

Example 4.1.13. (Nonfaces with pairwise nonempty intersections)
If $\sigma_i \cap \sigma_j \neq \emptyset$ for all minimal nonfaces σ_i, σ_j of S, we have that $c(I) = 1$ everywhere and

$$[M_S] = [M]^n - \sum_{\sigma_1,\ldots,\sigma_k \in I \subset [\Delta^V \setminus S] : |I| = k} (-1)^k [M]^{n - |\sigma_1 \cup \cdots \cup \sigma_k| + 1}. \tag{4.1.7}$$

The individual terms of the sum 4.1.7 really only depend on $|\sigma_1 \cup \cdots \cup \sigma_k|$ and not the full information on the nonfaces $\sigma_1, \ldots, \sigma_k$ of S. An example where this occurs is Example 1.14 on p. 7 – 8 of [11]. Note that this example applies since we can reduce to the case where the indices vary over *minimal* nonfaces. It also gives a hint to a connection we will make with Stanley–Reisner rings which applies to arbitrary simplicial complexes.

We would like to relate properties of this polynomial in $[M]$ with certain invariants of an algebraic structure associated to simplicial complexes which are parametrized by tuples of points which do *not* belong to a specified simplicial complex as in the case of the simplicial chromatic polynomials above.

Definition 4.1.14. (p. 201 – 202 of [13])
Let k be a field and S be a simplicial complex. For a subset A of the vertex set $V = \{x_1, \ldots, x_n\}$, write $x_A = \prod_{x_i \in A} x_i \in k[x_1, \ldots, x_n]$. Let $I(A)$ denote the ideal in the polynomial ring $k[x_1, \ldots, x_n]$ generated by all monomials x_σ such that $\sigma \notin S$ (i.e. $\sigma \in \Delta^V \setminus S$ in the notation above).

Note that $I(S)$ is generated by the *minimal* nonfaces of S since σ being a nonface and $\alpha \supset \sigma$ implies that α is also a nonface of S. The quotient ring $k[S] := k[V]/I(S)$ is called the **Stanley–Reisner ring (or face ring)** of S.

Throughout the proofs in this section, we implicitly use the following result to build simplicial complexes out of collections of finite sets.

Theorem 4.1.15. *(Stanley–Reisner correspondence, Definition 2.1, Definition 2.5, and Proposition 2.6 p. 211 – 212 of [15])*
Let $X = \{x_1, \ldots, x_n\}$. *Each* x_i *will be treated as the element* i *of* $\{1, \ldots, n\}$ *when subsets are written below. Given a squarefree monomial ideal* I, *let*

$$\Delta_I = \{m \subset X : m \notin I\}$$

be the simplicial complex consisting of squrefree monomials not *in* I.

Given a simplicial complex Δ, *let*

$$I_\Delta = \langle m \subset X : m \notin \Delta \rangle.$$

If I *is a squarefree monomial ideal, then* $I_{\Delta_I} = I$. *If* Δ *is a simplicial complex, then* $\Delta_{I_\Delta} = \Delta$.

The maps

$$\{\textit{squarefree monomial ideals}\} \longrightarrow \{\textit{simplicial complexes}\}, I \mapsto \Delta_I$$

and

$$\{\textit{simplicial complexes}\} \longrightarrow \{\textit{squarefree monomial ideals}\}, \Delta \mapsto I_\Delta$$

induce a correspondence

$$\{\textit{squarefree monomial ideals}\} \longleftrightarrow \{\textit{simplicial complexes}\}.$$

The connection of this ring with the simplicial chromatic polynomial comes from its Hilbert series. Let $S = k[x_1, \ldots, x_n]$ and let $J = \langle m_1, \ldots, m_r \rangle$ be a monomial ideal of S. The main idea is that the numerator of the Hilbert series $H(S/J, x)$ is the sum of the monomials which are *not* in J (top of p. 7 of [11]). One way to find these is to subtract the union of monomials which are in J from the total set of *all* monomials while counting the former using inclusion-exclusion. Normalizing (by division by $(1-x_1)\cdots(1-x_n)$), this gives the following polynomial in the numerator of the Hilbert series as shown on p. 1295 – 1296 of [8]:

$$H(S/J, x) = H(S, x) - H(J, x) = \sum_{\alpha \in \mathbb{N}^n} x^\alpha - \sum_{\substack{x^\alpha \in \langle m_I \rangle \\ I \subset [r] \\ |I|=1}} x^\alpha + \sum_{\substack{x^\alpha \in \langle m_I \rangle \\ I \subset [r] \\ |I|=2}} x^\alpha - \ldots + (-1)^r \sum_{\substack{x^\alpha \in \langle m_I \rangle \\ I \subset [r] \\ I=[r]}} x^\alpha, \tag{4.1.8}$$

where $[r] = \{1, \ldots, r\}$ and $m_I = \mathrm{lcm}(m_i : i \in I)$ for a subset $I \subset \{1, 2, \ldots, r\}$. In this case, the alternating sum of least common multiples of tuples of generators m_i of M. Taking the lcm ensures that repeating indices aren't counted twice. Applying the normalization $(1-x_1)\cdots(1-x_n)$, 4.1.8 can be rewritten as

$$H(S/J, x) = \frac{1 - \sum_{\substack{I \subset [r] \\ |I|=1}} m_I + \sum_{\substack{I \subset [r] \\ |I|=2}} m_I - \ldots + (-1)^r \sum_{\substack{I \subset [r] \\ I=[r]}} m_I}{(1-x_1)\cdots(1-x_n)}, \tag{4.1.9}$$

where $[r] = \{1, \ldots, r\}$ and $m_I = \mathrm{lcm}(m_i : i \in I)$ for a subset $I \subset \{1, 2, \ldots, r\}$ as above. Let $K(S/M, x)$ be the numerator of 4.1.9. Since $m_I = \mathrm{lcm}(m_i : i \in I)$ and each of the m_i are squarefree, we have that $\operatorname{supp} m_I = \bigcup_{i \in I} \operatorname{supp} m_i$.

Putting everything together, we start finding the main relations between simplicial chromatic polynomials and Stanley–Reisner rings. We can start with the case where $c(I) = 1$ for all subsets $I \subset \Delta^V \setminus S$.

Proposition 4.1.16. *Given a manifold M, let $[M] = \chi_c(M)$. Let S be a simplicial complex with vertex set $V = \{1, \ldots, n\}$. If $c(I) = a$ for all $I \subset [n]$ (Definition 4.2.5), we have that $[M_S] - [M]^n = [M]^{n+a}(K(S/J, [M]^{-1}) - 1)$, where $K(S/J, x)$ is the numerator of 4.1.9 and $J = I(S)$ is the Stanley–Reisner ideal of S. Substituting in $M = \mathbb{CP}^{t-1}$, this implies that $\chi_c(S)(t) - t^n = t^{n+a}(K(S/J, t^{-1} - 1))$ with $J = I(S)$.*

Remark 4.1.17. In Example 4.1.13, we have $c(I) = 1$ for all I.

Proof. Since we take $x = (x_1, \ldots, x_n)$ in $K(S/J, x)$, the expression on the right hand side means substituting $x_i = [M]^{-1}$ for each $i \in [n]$. This is because the number of intersections taken in the M_S setting and the number of simplices used in the inclusion-exclusion argument match up. Note that taking the lcm amounts to taking a union with the degree of each monomial in $H(S/M, x)$ giving the size of the set $|\sigma_1 \cup \cdots \cup \sigma_k|$. $\square$

The following result implies that the "interesting" part of the simplicial chromatic polynomial (i.e. the term obtained from subtracting the leading term) only depends on the h-vector of the simplicial complex:

Proposition 4.1.18. *(Corollary 1.15 on p. 8 of [11])*
Let f_i be the number of i-faces in the simplicial complex S. Then,

$$H(S/I(S); t, \ldots, t) = \frac{1}{(1-t)^n} \sum_{n=0}^{d} f_{i-1} t^i (1-t)^{n-i} = \frac{h_0 + h_1 t + h_2 t^2 + \ldots + h_d t^d}{(1-t)^d},$$

where $d = \dim S + 1$.

There are analogues of the results above which we can give for general simplicial complexes. For a general simplicial complex, terms of the sum describing the class $[M_S]$ in $K_0(\mathrm{Var}_k)$ corresponding to a subset $I = \{\sigma_1, \ldots, \sigma_k\} \subset \Delta^V \setminus S$ depends on more than $|\sigma_1 \cup \cdots \cup \sigma_k|$.

We can find a counterpart of Proposition 4.1.16 for simplicial complexes satisfying a (mild) assumption on finite set covers. This enables us to express the nontrivial part of $[M_S]$ in terms of the Hilbert–Poincaré series of the Stanley–Reisner ring of an auxiliary simplicial complex depending on S (Theorem 4.1.5).

Proof. (of Theorem 4.1.5)

Compare the terms of the expansion

$$[M_S] = [M]^n - \sum_{\sigma_1, \ldots, \sigma_k \in I \subset [\Delta^V \setminus S] : |I| = k} (-1)^k [M]^{n - |\sigma_1 \cup \cdots \cup \sigma_k| + c(I)}$$

with those of

$$H(S/J, x) = \frac{1 - \sum_{\substack{I \subset [r] \\ |I|=1}} m_I + \sum_{\substack{I \subset [r] \\ |I|=2}} m_I - \ldots + (-1)^r \sum_{\substack{I \subset [r] \\ I=[r]}} m_I}{(1 - x_1) \cdots (1 - x_n)},$$

where $[r] = \{1, \ldots, r\}$ and $m_I = \mathrm{lcm}(m_i : i \in I)$ for a subset $I \subset \{1, 2, \ldots, r\}$. Let $d = \dim S + 1$. Recall that

$$H(S/I(S); t, \ldots, t) = \frac{1}{(1-t)^n} \sum_{n=0}^{d} f_{i-1} t^i (1-t)^{n-i} = \frac{h_0 + h_1 t + h_2 t^2 + \ldots + h_d t^d}{(1-t)^d} \quad (4.1.10)$$

by Proposition 4.1.18.

In the sum 4.1.10 (which results from the specialization $x_1 = x_2 = \cdots = x_n = t$), terms of degree u correspond to $\sigma_1, \ldots, \sigma_k \in \Delta^V \setminus S$ such that $|\sigma_1 \cup \cdots \cup \sigma_k| = u$ since $\mathrm{lcm}(x^{\sigma_1}, \ldots, x^{\sigma_k}) = x^{\sigma_1 \cup \cdots \cup \sigma_k}$ (which is equal to $t^{|\sigma_1 \cup \cdots \cup \sigma_k|}$ after substituting $x_1 = x_2 = \cdots = x_n = t$). The class $[M_S]$ is more closely related to the expression we obtain by replacing t with t^{-1} in $H(S/I(S); t, \ldots, t)$ in a way analogous to Proposition 4.1.16.

The idea is to find subsets $\alpha_1, \ldots, \alpha_r \subset [n]$ such that $|\sigma_{i_1} \cup \cdots \cup \sigma_{i_k}| - c(I) = |\alpha_{i_1} \cup \cdots \cup \alpha_{i_k}|$. Setting $\sigma_I = \sigma_{i_1} \cup \cdots \cup \sigma_{i_k}$ and $\alpha_I = \alpha_{i_1} \cup \cdots \cup \alpha_{i_k}$ for $I = \{i_1, \ldots, i_k\}$, the claim can be rewritten as $|\sigma_I| - c(I) = |\alpha_I|$. As a polynomial, this would let us use the sum from Proposition 4.1.18 if the $\alpha_1, \ldots, \alpha_r$ give the Stanley–Reisner ideal of some simplicial complex. By the Stanley–Reisner correspondence (Theorem 4.1.15), such a simplicial complex does exist. The auxiliary simplicial complex $T(S)$ in the statement of Theorem 4.1.5 would be the simplicial complex whose minimal nonfaces correspond to the α_i via the Stanley–Reisner correspondence. Thus, the problem is reduced to showing that the assumptions in the statement allow the existence of such α_i.

We use induction on $|I|$ to show that the sets α_i in the definition of property I (Definition 4.2.3) satisfy the relation $|\sigma_I| - c(I) = |\alpha_I|$. Since $c(I) = 1$ if $|I| = 1$, the case where $|I| = 1$ is really the statement that $|\alpha_i| = |\sigma_i| - 1$. Suppose that $|\alpha_I| = |\sigma_I| - c(I)$. Given $p \notin I$, let $J = I \cup \{p\}$. We would like to show that $|\alpha_{J_p}| = |\sigma_{J_p}| - c(J_p)$. Since σ_p adds a new component if and only if $\sigma_p \cap \sigma_i = \emptyset$ for all $i \in I$, we have that

$$c(J_p) = \begin{cases} c(I) & \text{if } \sigma_i \cap \sigma_p \neq \emptyset \text{ for some } i \in I \\ c(I) + 1 & \text{if } \sigma_i \cap \sigma_p = \emptyset \text{ for all } i \in I \end{cases}$$

Using the notation $\sigma_I = \bigcup_{i \in I} \sigma_i$, this simplifies to

$$c(J_p) = \begin{cases} c(I) & \text{if } \sigma_I \cap \sigma_p \neq \emptyset \\ c(I) + 1 & \text{if } \sigma_I \cap \sigma_p = \emptyset. \end{cases}$$

If $\sigma_I \cap \sigma_p = \emptyset$, then $|\sigma_{J_p}| = |\sigma_I| + |\sigma_p|$ and

$$\begin{aligned} |\sigma_{J_p}| - c(J_p) &= |\sigma_I| + |\sigma_p| - c(I) - 1 \\ &= (|\sigma_I| - c(I)) + (|\sigma_p| - 1) \\ &= |\alpha_I| + |\alpha_p| \\ &= |\alpha_{J_p}|, \end{aligned}$$

where the last line follows from the assumption that $\alpha_I \cap \alpha_p = \emptyset$ if we have $I \subset [r]$ and $p \notin I$ such that $\sigma_I \cap \sigma_p = \emptyset$ and we are considering the case where $\sigma_I \cap \sigma_p = \emptyset$ (which means that $\alpha_I \cap \alpha_p = \emptyset$).

Now consider the case where $\sigma_I \cap \sigma_p \neq \emptyset$. In this case, we have that $c(J_p) = c(I)$. Note that $|\sigma_{J_p}| = |\sigma_I| + |\sigma_p| - |\sigma_I \cap \sigma_p|$. Similarly, we have that

$$\begin{aligned} |\alpha_{J_p}| &= |\alpha_I| + |\alpha_p| - |\alpha_I \cap \alpha_p| \\ &= |\sigma_I| - c(I) + |\sigma_p| - 1 - |\alpha_I \cap \alpha_p| \\ &= |\sigma_I| - c(I) + |\sigma_p| - 1 - |\sigma_I \cap \sigma_p| + 1 \\ &= |\sigma_I| + |\sigma_p| - |\sigma_I \cap \sigma_p| - c(I) \\ &= |\sigma_{J_p}| - c(I), \end{aligned}$$

where we used the assumption that $|\alpha_I \cap \alpha_p| = |\sigma_I \cap \sigma_p| - 1$ if $\sigma_I \cap \sigma_p \neq \emptyset$ in the third line. Thus, the desired conclusion follows from induction on the size of $I \subset [r]$.

$\square$

Example 4.1.19. We give some examples where the assumptions of Theorem 4.1.5 hold. Note that it suffices to start with an initial collection of subsets $\sigma_i \subset [n]$ (taken to be generators of a squarefree monomial ideal) since they can always be taken to be the minimal nonfaces of some simplicial complex by the Stanley–Reisner correspondence (p. 212 of [15]).

1. Suppose that $r = 2$ (i.e. that there are two minimal nonfaces) and $\sigma_1 \cap \sigma_2 = \emptyset$. For example, this applies to the simplicial complex Δ corresponding to the square graph $abcd$ for the vertex set a, b, c, d, which has $I(\Delta) = \langle ac, bd \rangle$ (Example 1.15 on p. 2 of [17]). Then, we can simply take α_i to be any subset of σ_i with one element removed. Since $\alpha_1 \subset \sigma_1$ and $\alpha_2 \subset \sigma_2$, we have that $\alpha_1 \cap \alpha_2 = \emptyset$. If we take $I = \{1, 2\}$, then $|\sigma_I \cap \sigma_i| = |\sigma_i|$ and $|\alpha_I \cap \alpha_i| = |\alpha_i| = |\sigma_i| - 1$ and the assumptions of Theorem 4.1.5 are satisfied. These arguments apply in general when $\sigma_i \cap \sigma_j$ for $i \neq j$.

2. Let $r = 3$ and take subsets $\sigma_1, \sigma_2, \sigma_3 \subset [n]$ of size ≥ 2 such that $\sigma_i \cap \sigma_j = \sigma_1 \cap \sigma_2 \cap \sigma_3 = \{a\}$ for some point a. Let $\alpha_i = \sigma_i \setminus \{a\}$. Since the $|I| = 3$ case is trivial, it suffices to consider the cases $|I| = 1$ and $|I| = 2$. The $|I| = 1$ case has to do with pairwise intersections. This works since $|\alpha_i \cap \alpha_j| = 0 = |\sigma_i \cap \sigma_j| - 1$ for $i \neq j$. As for the $|I| = 2$ case, the nontrivial instance is from noting that $|\alpha_1 \cap \alpha_2 \cap \alpha_3| = 0 = |\sigma_1 \cap \sigma_2 \cap \sigma_3| - 1 = 1$. This satisfies the assumptions of Theorem 4.1.5. Adding a subset σ_4 such that $\sigma_4 \cap \sigma_i = \emptyset$ for $i = 1, 2, 3$ still gives a collection of subsets satisfying the assumptions of Theorem 4.1.5.

3. Another example with $r = 3$ uses $\sigma_1, \sigma_2, \sigma_3 \subset [n]$ such that $|\sigma_1 \cap \sigma_2| = |\sigma_1 \cap \sigma_3| = |\sigma_2 \cap \sigma_3| = 1$ and $\sigma_1 \cap \sigma_2 \cap \sigma_3 = \emptyset$. Let $a = \sigma_1 \cap \sigma_2$, $b = \sigma_1 \cap \sigma_3$, and $c = \sigma_2 \cap \sigma_3$. Then, take $\alpha_1 = \sigma_1 \setminus \{a\}$, $\alpha_2 = \sigma_2 \setminus \{c\}$, and $\alpha_3 = \sigma_3 \setminus \{b\}$. As mentioned in the previous

example, the conditions with $|I| = 3$ are trivial. It remains to consider the cases with $|I| = 1$ and $|I| = 2$. We are done with the $|I| = 1$ case since the α_i are pairwise disjoint. The nontrivial part of the $|I| = 2$ case comes from noting that $\sigma_1 \cap \sigma_2 \cap \sigma_3 = \emptyset$ and $\alpha_1 \cap \alpha_2 \cap \alpha_3 = \emptyset$.

Remark 4.1.20.

1. If we don't invert the variables in $\chi_c(S, t)$ and S is a Cohen–Macaulay simplicial complex, the expression obtained for the simplicial chromatic polynomial $\chi_c(S, t)$ is also closely related to a specialization of the Hilbert–Poincaré series of the *canonical module of $k[S]$ (Exercise 5.6.6 on p. 246 of [10]):

$$H_{\omega_{k[S]}}(t) = \sum_{F \in S} \dim_k \widetilde{H}_{\dim \operatorname{lk} F}(\operatorname{lk} F; k) \prod_{X_i \in F} \frac{t_i}{1 - t_i} = (-1)^d H_{k[S]}(t_1^{-1}, \ldots, t_n^{-1})$$

   The intermediate expression $\operatorname{lk} F = \{G : F \cup G \in S, F \cap G = \emptyset\}$ (Definition 5.3.4 on p. 232 of [10]) is also involved in determining when a simplicial complex is Cohen–Macaulay (Theorem 12.27 on p. 211 of [13]). There are even simpler relations with $H_{k[S]}(t_1, \ldots, t_n) = (-1)^d H_{k[S]}(t_1^{-1}, \ldots, t_n^{-1})$ if the simplicial complexes involved are Euler complexes (Exercise 5.6.7 on p. 246 of [10]) and $t^a H_{k[S]}(t_1, \ldots, t_n) = (-1)^d H_{k[S]}(t_1^{-1}, \ldots, t_n^{-1})$ if they are Gorenstein complexes (Exercise 5.6.9 on p. 246 of [10]).

2. The Hilbert series of auxiliary simplicial complexes determined by the simplicial chromatic polynomials in the setting of Proposition 4.1.16 and Theorem 4.1.5 can be used to generate Hilbert series of other rings using operations such as addition, partial sums, and multiplication related to joins of simplicial complexes as indicated in Corollary 6.6 on p. 738 of [13] (Proposition 2.4 on p. 131 of [2]).

We can combine Proposition 4.1.18 with the main results (Proposition 4.1.16 or Theorem 4.1.5) and the defining property of the simplicial chromatic polynomial (Proposition 4.1.11) to make an observation on simplicial chromatic polynomials of simplicial complexes S in the setting of the results above.

Corollary 4.1.21. *Suppose that S is a simplicial complex satisfying the assumptions of Proposition 4.1.16 or Theorem 4.1.5.*

1. *The simplicial chromatic polynomial $\chi_c(S, t)$ is entirely determined by the h-vector of S.*

2. *The h-vector of the associated simplicial complex (S in Proposition 4.1.16 and $T = T(S)$ in Theorem 4.1.5) is entirely determined by the defining addition-contraction relation and normalization for simplicial chromatic polynomials from Proposition 6.4 on p. 738 of [13].*

Simplicial chromatic polynomials as Hilbert polynomials of other rings

We can also connect the original simplicial chromatic polynomials $\chi_c(S, t)$ associated to a simplicial complex S to Hilbert series of rings without making any modifications on the latter ring. More specifically, we will work with conditions under which a polynomial is a Hilbert polynomial (i.e. "approximately a Hilbert series" – see p. 131 and Theorem 2.2 on p. 129 of [2]) which are listed below.

Proposition 4.1.22. *(Corollary 3.10 on p. 138 of [2])*
Let $P(x) = \sum_{i=0}^{d} a_i x^i$. If $a_0, \ldots, a_d \in \mathbb{N}$ and $a_1, a_2 \geq 3$, then $P(x)$ is a Hilbert polynomial associated to some k-algebra.

Combining Proposition 4.1.22 with the inclusion-exclusion arguments used earlier, we find some conditions under which $[M_S] - [M]$ is a Hilbert polynomial with $[M]$ substituted in for the variable (Corollary 4.2.7).

Proof. (of Corollary 4.2.7)

The expansion

$$[M_S] = [M]^n - \sum_{\substack{\sigma_1,\dots,\sigma_k \in I \subset [\Delta^V \setminus S] \\ |I|=k}} (-1)^k [M]^{n-|\sigma_1 \cup \cdots \cup \sigma_k|+c(I)}$$

implies that

$$[M]^n - [M_S] = [M]^n \sum_{\substack{\sigma_1,\dots,\sigma_k \in I \subset [\Delta^V \setminus S] \\ |I|=k}} (-1)^k [M]^{-|\sigma_1 \cup \cdots \cup \sigma_k|+c(I)}.$$

Replacing t with t^{-1}, we find that

$$t^{-n} - \chi_c(S, t^{-1}) = t^{-n} \sum_{\substack{\sigma_1,\dots,\sigma_k \in I \subset [\Delta^V \setminus S] \\ |I|=k}} (-1)^k t^{-(|\sigma_1 \cup \cdots \cup \sigma_k|-c(I))}.$$

The coefficient of t^{-r} on the right hand side is

$$\sum_{\substack{\sigma_1,\dots,\sigma_k \in I \subset [\Delta^V \setminus S] \\ |I|=k \\ |\sigma_1 \cup \cdots \cup \sigma_k|-c(I)=r}} (-1)^k.$$

Since we assumed that $c(I) = a$ for all I, the coefficient of t^{-r} is

$$\sum_{\substack{\sigma_1,\dots,\sigma_k \in I \subset [\Delta^V \setminus S] \\ |I|=k \\ |\sigma_1 \cup \cdots \cup \sigma_k|=a+r}} (-1)^k. \tag{4.1.11}$$

Recall that

$$h_s = \sum_{\substack{\sigma_1,\ldots,\sigma_k \in I \subset [\Delta^V \setminus S] \\ |I|=k \\ |\sigma_1 \cup \cdots \cup \sigma_k| = s}} (-1)^k \tag{4.1.12}$$

by a correspondence between the numerators of 4.1.9 and 4.1.10.

Comparing 4.1.12 to the coefficient of t^{-r} given by 4.1.11, we have that the coefficient of t^{-r} is equal to h_{a+r}. Combining the assumptions that $h_{a+r} \geq 1$, $h_{a+1}, h_{a+2} \geq 3$, and $h_i \geq 0$ for all $i \geq a$ with Proposition 4.1.22, we find that $t^{-n} - \chi_c(S, t^{-1}) = t^{-n}P(t^{-1})$ for some Hilbert polynomial $P = P(x)$.

□

4.1.3 Applications

In this section, we outline applications of the main results to properties of simplicial chromatic polynomials in various contexts. This includes properties that generalize those of chromatic polynomials or matroids. The fact that we work with simplicial complexes are not necessarily independence complexes of graphs allows us to obtain symmetric properties in other contexts. Here are the main types of applications considered:

- The support of cyclotomic polynomials (Corollary 4.1.27)
- Simplicial complexes with log concave simplicial chromatic polynomials (Corollary 4.1.32, Example 4.1.35)
- Symmetric relations among coefficients of simplicial chromatic polynomials induced by polytopes (Example 4.1.37)

4.1.4 Connection to cyclotomic polynomials

We consider simplicial complexes whose topology encodes information on coefficients of cyclotomic polynomials.

Definition 4.1.23. (p. 114 of [12])
Let K_p denote the 0-dimensional abstract simplicial complex consisting of p vertices. Given d distinct primes $p_1, \dots, p_d$, we set

$$K_{p_1,\dots,p_d} = K_{p_1} * \cdots * K_{p_d},$$

where $*$ denotes join. Setting $n = p_1 \cdots p_d$, we note that each facet of $K_{p_1,\dots,p_d}$ corresponds to a residue $\pmod{n}$ by the Chinese Remainder Theorem.

Given a subset $A \subset \{0, 1, \dots, \varphi(n)\}$, let K_A be the subcomplex of $K_{p_1,\dots,p_d}$ containing the entire $(d-2)$-simplex whose facets correspond to elements of

$$A \cup \{\varphi(n) + 1, \varphi(n) + 2, \dots, n\}.$$

Here is the result of Musiker–Reiner [12] connecting this simplicial complex to the coefficients of simplicial chromatic polynomials.

Theorem 4.1.24. *(Musiker–Reiner, Theorem 1.1 on p. 114 of [12])*
For a squarefree postiive integer $n = p_1 \cdots p_d$, with cyclotomic polynomial

$$\Phi_n(x) = \sum_{j=0}^{\varphi(n)} c_j x^j,$$

one has

$$\widetilde{H}_i(K_{\{j\}};\mathbb{Z}) = \begin{cases} \mathbb{Z}/c_j\mathbb{Z} & \text{if } i = d-2 \\ \mathbb{Z} & \text{if both } i = d-1 \text{ and } c_j = 0 \\ 0 & \text{otherwise.} \end{cases}$$

for each $0 \leq j \leq \varphi(n)$.

While we only account for non-torsion components in the Euler characteristic, the Euler characteristic associated with this simplicial complex still records information on the *parity* of nonzero coefficients.

Before outlining the connection of Theorem 4.1.5 with cyclotomic polynomials via Theorem 4.1.24, we define some notation.

Definition 4.1.25. Given a simplicial complex S, let $h_S(t) = h_0 + h_1 t^2 + \ldots + h_d t^d$, where $h = (h_0, \ldots, h_d)$ is the h-vector associated to S.

There are simplicial complexes satisfying the intersection conditions in the main result.

Example 4.1.26. There are simplicial complexes $K_{p_1,\ldots,p_d}$ such that $K_{\{j\}}$ satisfies the conditions of Theorem 4.1.5. This means that their simplicial complexes are determined by h-vectors of auxiliary simplicial complexes. For example, consider the subcomplex $K_{\{1\}} \subset K_{2,3}$. Writing $K_{2,3} = K_2 * K_3$, let $K_2 = \{a, b\}$ and $K_3 = \{c, d, e\}$. The minimal nonfaces of $K_{\{1\}}$ are $\beta_1 = \{a, b\}, \beta_2 = \{c, d\}, \beta_3 = \{c, e\}, \beta_4 = \{d, e\}$, and $\beta_5 = \{a, e\}$. The nonfaces $\beta_1, \beta_2, \beta_3, \beta_4$ correspond to pairs of points from the same vertex set K_i and the nonface β_5 is the facet removed in the construction of $K_{\{1\}}$. Setting $\sigma_i = \beta_i$ for each $1 \leq i \leq 5$ in Theorem 4.1.5, we can set $\alpha_1 = \{b\}, \alpha_2 = \{c\}, \alpha_3 = \{d\}, \alpha_4 = \{e\}$, and $\alpha_5 = \{a\}$.

The simplicial complexes $K_{\{j\}}$ are also connected to simplicial chromatic polynomials whose constant terms detect whether a cyclotomic polynomial contains a term of a given degree.

Corollary 4.1.27. *Given a cyclotomic polynomial* $\Phi_n(x) = \sum_{j=0}^{\varphi(n)} c_j x^j$*, there are simplicial complexes* S_j *such that the constant term of* $\chi_c(S_j)$ *is equal to* $1 + (-1)^d$ *or* $(-1)^d$ *depending on whether* $c_j = 0$ *or* $c_j \neq 0$ *respectively. This means that* $\chi_c(S_j)$ *determines whether* $c_j = 0$ *or not.*

Proof. Recall that our Theorem 4.1.5 states that

$$\chi_c(S)(t) - t^n = t^n(h_{T(S)}(t^{-1}) - 1) \Longrightarrow \chi_c(S)(t) = t^n h_{T(S)}(t^{-1})$$

for simplicial complexes S satisfying appropriate intersection properties (Definition 4.2.3).

If we add an extraneous element (vertex) not in the ground set of $K_{\{j\}}$ to every minimal nonface of $K_{\{j\}}$, then we obtain the minimal nonfaces of a simplicial complex S such that the auxiliary simplicial complex in Theorem 4.1.5 is $T(S) = K_{\{j\}}$. In other words, we consider the simplicial complex whose minimal nonfaces are $\sigma_i = \alpha_i \cup \{q\}$ for some minimal nonface α_i of $K_{\{j\}}$ and extraneous vertex q *not* in the ground set/vertex set of $K_{\{j\}}$. Then, $\sigma_I \cap \sigma_p \neq \emptyset$ for any choice of I and p in Definition 4.2.3 and removing the extra vertex q would decrease the size of the intersection $\sigma_I \cap \sigma_p$ by 1 since it is contained in every minimal nonface of the simplicial complex S. The following result implies that the constant term of $\chi_c(S)$ is equal to $1 + (-1)^d$ or $(-1)^d$ depending on whether $c_j = 0$ or $c_j \neq 0$ respectively.

Proposition 4.1.28. *(Corollary 5.1.9 on p. 213 of [10])*
Given a simplicial complex S*, we have that* $h_d = (-1)^{d-1}(\chi(S) - 1)$*.*

□

Remark 4.1.29. The construction used in the proof of Corollary 4.1.27 can be repeated to construct σ_i satisfying the intersection property in Definition 4.2.3 given any collection of sets α_i.

Log concavity of simplicial chromatic polynomial coefficients

The example below also gives a family of simplicial complexes whose simplicial chromatic polynomials are log concave. We will specialize to the case where the simplicial complex considered is the independence complex of a matroid and consider log concave sequences of integers, which we both define below.

Definition 4.1.30. (Part 1 on p. 51 of [9])
Given a matroid M with underlying set $E = \{1, \ldots, n\}$, the **independence complex** of M is the simplicial complex with independent subsets of E as faces of the simplicial complex.

Definition 4.1.31. (p. 49 of [9])
A sequence of integers $e_0, e_1, \ldots, e_n$ is **log concave** if $e_{i-1}e_{i+1} \leq e_i^2$ for all $0 < i < n$.

Corollary 4.1.32.

1. *Suppose that S is any simplicial complex S satisfying the conditions of Theorem 4.1.5 (i.e. satisfies property I in Definition 4.2.3). If $T(S)$ is the independence complex of a representable matroid, then the coefficients (a normalization of) the simplicial chromatic polynomial $\chi_c(S)$ are log concave.*

2. *If S satisfies the conditions of Proposition 4.1.16 and is the independence complex of a representable matroid, then the coefficients (a normalization of) sufficiently high degree terms of the simplicial chromatic polynomial $\chi_c(S)$ are log concave. Writing b_j for the coefficient degree j term of $\chi_c(S)$, we have that $b_{j-1}b_{j+1} \leq b_j^2$ for sufficiently large j.*

Proof.

1. If S satisfies property I (Definition 4.2.3), then $\chi_c(S)(t) = t^n h_{T(S)}(t^{-1})$ by Theorem 4.1.5. Let b_i be the coefficient of t^i in $\chi_c(S)(t)$ and h_j be the degree j term of the

polynomial $h_{T(S)}(u)$. Since $\chi_c(S)(t) = t^n h_{T(S)}(t^{-1})$, we have that $a_i = h_{n-i}$. The log concavity of h-vectors associated to the independence complex of representable matroids (Theorem 3 on p. 52 of [9]) implies that $h_{j-1}h_{j+1} \leq h_j^2$ for $0 < j < n$. Substituting in the identity $b_i = h_{n-i}$, this inequality can be rewritten as $b_{n-j+1}b_{n-j-1} = b_{n-j-1}b_{n-j+1} \leq b_{n-j}^2$ for $0 < j < n$. This is the statement of log concavity for the coefficients a_i of the degree i term of $\chi_c(S)(t)$.

2. Suppose that S satisfies the conditions of Proposition 4.1.16. In other words, this means that there is some $a \geq 1$ such that $c(I) = a$ for all subsets $I \subset [n]$ (Definition 4.2.5). Then, we have that $\chi_c(S)(t) - t^n = t^{n+a}h_S(t^{-1})$ for some $a \geq 1$. Let $\widetilde{h}_j$ be the degree j term of $h_S(v)$. The identity $\chi_c(S)(t) - t^n = t^{n+a}h_S(t^{-1})$ implies that $\widetilde{h}_i = b_{n+a-i}$. Using the $\widetilde{h}_i$ in place of the h_i, the log concavity result on independence complexes implies that $\widetilde{h}_{j-1}\widetilde{h}_{j+1} \leq \widetilde{h}_j^2$ for $0 < j < n-1$. Since $\widetilde{h}_i = b_{n+a-i}$, this implies that $b_{n+a-j-1}b_{n+a-j+1} \leq b_{n+a-j}^2$. This implies log concavity for terms of degree $\geq a$.

□

Example 4.1.33. Consider the independence complex of the uniform matroid U_n^r with $n \geq 6$ and $r > \frac{n}{2} + 1$. This is a representable matroid with the elements of the underlying set corresponding to a "generic" n-tuple of vectors spanning a linear subspace of dimension r. We can set $U_n^r = T(S)$ for a suitable S (i.e. one satisfying property I from Definition 4.2.3 and Theorem 4.1.5) using the same construction as the one used in the proof of Corollary 4.1.27 (Remark 4.1.29). More specifically, we can take S to have underlying set $\{1, \ldots, n, n+1\}$ and the minimal nonfaces of S equal to $I \cup \{n+1\}$ for subsets $I \subset [n]$ such that $|I| = r+1$. The minimal nonfaces of U_n^r are obtained by removing $n+1$ from each of the minimal nonfaces of S.

In general, it appears that the log concavity of f-vectors is better understood than that of h-vectors. The relation between f-vectors and h-vectors naturally takes the form of a translation when we consider the reciprocal polynomials of their generating functions.

Proposition 4.1.34. *(Lemma 5.1.8 on p. 213 of [10], p. 321 of [15])*
The f-vector and h-vector of a $(d-1)$-dimensional simplicial complex S are related by

$$\sum_i h_i t^i = \sum_{i=0}^{d} f_{i-1} t^i (1-t)^{d-i} \tag{4.1.13}$$

In particular, the h-vector has length at most d, and for $j = 0, \ldots, d$,

$$h_j = \sum_{i=0}^{j} (-1)^{j-i} \binom{d-i}{j-i} f_{i-1}$$

and

$$f_{j-1} = \sum_{i=0}^{j} \binom{d-i}{j-i} h_i.$$

Alternatively, we can write

$$\sum_{i=0}^{d} f_{i-1}(t-1)^{d-i} = \sum_{i=0}^{d} h_i t^{d-i}. \tag{4.1.14}$$

Note that 4.1.14 can be obtained from 4.1.13 by substituting in t^{-1} for t and multiplying by t^d afterwards (i.e. computing the reciprocal polynomials).

The second version of the relation between h-vectors and f-vectors can be used to build further examples of simplicial chromatic polynomials satisfying log concavity properties.

Example 4.1.35. (Log concavity of translated simplicial chromatic polynomials via f-vectors)
Using the strategy in the proof of Corollary 4.1.27, we can show that *any* simplicial complex is the auxiliary simplicial complex $T(S)$ of some simplicial complex S from (Theorem 4.1.5,

Definition4.2.3). The fact the simplicial chromatic polynomials considered in Theorem 4.1.5 or Proposition 4.1.16 are essentially *reciprocal* polynomials (multiplied by a power of t) of the generating function for h-vectors of the auxiliary simplicial complex $T(S)$ means that we can simply replace t by $t-1$ to get the generating function for an f-vector. This would enable us to input log concavity results on f-vectors of simplicial complexes to this translation of the simplicial chromatic polynomial. For example, the f-vector of the independence complex of a representable matroid is known to be log concave [10]

Symmetric relations among simplicial chromatic coefficients

If the auxiliary simplicial complex $T(S)$ is isomorphic to the simplicial complex given by the vertices of proper faces of a simplicial polytope, then the following relations imply that the simplicial chromatic polynomial $\chi_c(S)$ is equal to its reciprocal polynomial.

Theorem 4.1.36. *(Dehn–Sommerville, Theorem 5.2.16 on p. 229 of [10])*
Let $(h_0, \ldots, h_m)$ be the h-vector of a simplicial polytope. Then $h_i = h_{m-i}$ for $0 \leq i \leq d$.

Here is an example where the chromatic symmetric polynmial is equal to its reciprocal polynomial.

Example 4.1.37. (Symmetries in simplicial chromatic polynomial coefficients)

1. Let A be the simplicial complex corresponding to the boundary of the octahedron. Let $\{a, b, c, d, e, f\}$ be the vertex set with e and f labeling the "antipodal" vertices on the top and bottom and a, b, c, d labeling the vertices of the square "base". The minimal nonfaces come from the pair of "antipodal" vertices $\{e, f\}$ and the pair of diagonals $\{a, c\}$ and $\{b, d\}$ from the square in the middle. Since these are all disjoint from each other, adding 3 new extraneous vertices g, h, i to each of these minimal nonfaces gives

the set of minimal nonfaces of a simplicial complex S such that $A = T(S)$ in Theorem 4.1.5. Applying Theorem 4.1.5, the coefficient of t^i in $\chi_c(S)$ is the coefficient h_{5-i} of t^{5-i} in $h_{T(S}(t)$. Then, the Dehn–Sommerville relations $h_i = h_{m-i}$ (Theorem 5.2.16 on p. 229 of [10]) with $m = 2$ (i.e. $h_0 = h_2$) implies that imply that t^5 and t^3 have the same coefficient in $\chi_c(S, t)$.

2. Let S be a simplicial complex with vertex set $V = \{1, \ldots, n\}$ satisfying property I (Definition 4.2.3, Theorem 4.1.5). Suppose that the auxiliary simplicial complex $T(S)$ (Definition 4.2.3, Theorem 4.1.5) is isomorphic to the boundary complex of a simplicial polytope of dimension m. Let b_i be the coefficient of t^i in the simplicial chromatic polynomial $\chi_c(S, t)$. By Theorem 4.1.5, we have that $\chi_c(S)(t) = t^n h_{T(S)}(t^{-1})$. Then, we have that $b_i = h_{n-i}$. Then, the Dehn–Sommerville relations $h_i = h_{m-i}$ imply that $b_{n-i} = b_{n-m+i} \implies b_i = b_{m-i}$. If $n = m$, this implies that $\chi_c(S)(t)$ is a reciprocal polynomial.

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4.2 Anti-Ramsey theory problems, lattice point counts on polytopes, and Hodge structures on the cohomology of toric varieties

We find families of graphs G and subgraphs H of G such that the number of edge colorings of G avoiding a monochromatic coloring of H is determined by lattice point counts or a Hodge structure on the cohomology of a certain toric variety. In general, this gives a class of "anti-Ramsey theory problems" with a geometric structure. For example, we find one for Ramsey numbers of classes of such graphs. The key observation is that our previous result expressing simplicial chromatic polynomials in terms of h-vectors of auxiliary simplicial complexes [19] can be reinterpreted as one on edge colorings of graphs avoiding monochromatic colorings of specified forbidden subgraphs. Specializing to simplicial complexes arising from triangulations of polytopes (e.g. unimodular triangulations), we obtain families of graphs and forbidden subgraphs where edge colorings avoiding monochromatic colorings of the forbidden subgraphs depend on lattice point counts or Hodge structures on the cohomology of toric varieties.

4.2.1 Introduction

Roughly speaking, Ramsey theory studies structures which are forced to contain a "regular" substructure when they are large enough. For example, Ramsey numbers (and graph-theoretic instances in general) give a threshold for this size in order for an edge coloring of a complete graph of some size to contain a monochromatic subgraph (e.g. a clique). There has been extensive work on related to these kinds of questions and an overview is given in a survey of Conlon–Fox–Sudakov [12].

The specific types of problems which we will focus on have to do with *avoiding* monochromatic structures in edge colorings. We will term these “anti-Ramsey problems”. An overview of such “forbidden graph” problems is given by Bollobás in [8]. Some examples of results in this direction include those of Alon–Balogh–Keevash–Sudakov [2] on avoiding monochromatic cliques and Yuster [29] on avoiding monochromatic triangles. Fujita–Liu–Magnant [16] and Kano–Li [21] give surveys involving results on edge colorings avoiding monochromatic colorings of specified structures. A common property of results of this type (and extremal graph theory problems in general) is that they are often asymptotic or involve some kind of quantitative bound. In this work, we give families of graphs where this problem can be studied from a *structural* perspective coming from topology. This enables us to obtain specializations where the colorings are parametrized by lattice points or generating functions from Hodge structures on the cohomology of toric varieties.

Our starting point is a reinterpretation of the simplicial chromatic polynomial (Definition 4.2.2, Proposition 4.2.8) as a count of edge colorings of graphs avoiding monochromatic colorings of subgraphs corresponding to the minimal nonfaces of the given simplicial complex. Note that the simplicial chromatic polynomial of *any* simplicial complex has such an interpretation. The topological perspective comes from previous results which express the simplicial chromatic polynomials of simplicial complexes whose minimal nonfaces satisfy appropriate intersection properties in terms of h-vectors of auxiliary simplicial complexes (Theorem 4.2.6, Corollary 4.2.7). This gives a family of graphs G and subgraphs $\{H_i\}$ where edge colorings avoiding monochromatic colorings of H_i are parametrized by h-vectors of simplicial complexes. A consequence is an interpretation of Ramsey numbers of classes of graphs in terms of the topology of certain configuration spaces (Corollary 4.2.11) Specializing to instances where the auxiliary simplicial complexes arise from unimodular triangulations of polytopes, we find cases where such colorings are parametrized by lattice point counts of (dilations of) polytopes (Theorem 4.2.16). Finally, a specialization to compressed polytopes

(Theorem 4.2.25) gives cases where they are parametrized by (truncated) generating functions of Hodge structures on the cohomology of toric varieties.

The expression of simplicial chromatic polynomials in terms of h-vectors of auxiliary simplicial complexes (Theorem 4.2.6, Corollary 4.2.7) and their reinterpretation in terms of edge colorings of graphs (Proposition 4.2.8) is given in Section 4.2.2. Afterwards, we specialize to instances where the auxiliary simplicial complexes arise from polytopes with unimodular triangulations to obtain find subfamilies of graphs with colorings considered coming from lattice point counts of polytopes (Theorem 4.2.16) in Section 4.2.3. Finally, we consider edge colorings avoiding monochromatic colorings of specified subgraphs parametrized by truncated generating functions depending on Hodge structures on the cohomology of certain toric hypersurfaces (Theorem 4.2.25) in Section 4.2.4.

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4.2.2 Connections between simplicial chromatic polynomials and anti-Ramsey-type problems

Simplicial chromatic polynomials were originally introduced by Cooper–de Silva–Sazdanovic [13] as a "categorification" of the observation that Euler characteristics of ordered configuration spaces of points can often be parametrized using chromatic polynomials. When the minimal nonfaces of the simplicial complex satsify appropriate intersection properties, we studied this polynomial from a combinatorial point of view [19] and expressed simplicial chromatic polynomials in terms of Hilbert series of Stanley–Reisner rings [19]. The notation we will use writes S for the simplicial complex and V for the vertex set of the simplicial complex. Before discussing new material, we will first review the old result and the definitions used there.

Here is the definition of a simplicial chromatic polynomial (from [19] in reference to [13]):

Definition 4.2.1. (Definition 2.1 on p. 725 and p. 738 of [13])
Let S be a simplicial complex whose 0-skeleton is given by the vertex set $V = V(S) = \{v_1, \ldots, v_n\}$. Let M be a topological space. For each simplex $\sigma = [v_{i_1} \cdots v_{i_k}]$, define the diagonal corresponding to σ to be

$$D_\sigma = \{(x_1, \ldots, x_n) \in M^n : x_{i_1} = \cdots = x_{i_k}\}.$$

We define the **simplicial configuration space** as

$$M_S = M^n \setminus \bigcup_{\sigma \in \Delta^V \setminus S} D_\sigma \tag{4.2.1}$$

where Δ^V is the simplicial complex containing all subsets of the vertices v_i (analogous to a simplex generated by independent vectors corresponding to the v_i) and $\Delta^V \setminus S$ denotes tuples of vertices in V which do *not* occur as simplices in S.

Definition 4.2.2. (Definition 6.1 on p. 738 of [13])
Let S be a simplicial complex and let M be a manifold. Given S and M, let

$$\chi_c(S, M) := \sum (-1)^k \operatorname{rank} H_c^k(M_S).$$

The **simplicial chromatic polynomial** of a simplicial complex S is the polynomial defined by the assignment $\chi_c(S) : t \mapsto \chi_c(S, \mathbb{CP}^{t-1})$.

For simplicial complexes S whose minimal nonfaces satisfy appropriate intersection properties, we can show that their simplicial chromatic polynomials are determined by h-vectors of auxiliary simplicial complexes $T(S)$.

Definition 4.2.3. Let S be a simplicial complex with minimal nonfaces $\sigma_1, \ldots, \sigma_r$. A simplicial complex S satisfies **property I** if there is a collection of finite sets α_i such that $|\alpha_i| = |\sigma_i| - 1$ for each $1 \leq i \leq r$ and $\alpha_I \cap \alpha_p = \sigma_I \cap \sigma_p = \emptyset$ if $\sigma_I \cap \sigma_p = \emptyset$ and $|\alpha_I \cap \alpha_p| = |\sigma_I \cap \sigma_p| - 1$ if $|I| \geq 2$ and $\sigma_I \cap \sigma_p \neq \emptyset$ for each subset $I \subset [r]$ and $p \notin I$.

Remark 4.2.4. Note that *any* simplicial complex can be set equal to $T(S)$ for some simplicial complex S satisfying property I. For example, add a new extraneous vertex to every minimal nonface of $T(S)$.

Definition 4.2.5. Given a k-element subset $I = \{\sigma_1, \ldots, \sigma_k\} \subset \Delta^V \setminus S$, let G_I be the graph whose vertices are the σ_i with two vertices corresponding to σ_i, σ_j being connected by an edge if and only if $\sigma_i \cap \sigma_j \neq \emptyset$. Let $c(I)$ be the number of connected components of G_I.

Theorem 4.2.6. *(Theorem 1.5 on p. 4 and Proposition 2.8 on p. 9 of [19])*

1. *Let r be the number of minimal nonfaces of S. If $c(I) = 1$ for all $I \subset [r]$, then*

$$\begin{aligned}
\chi_c(S)(t) - t^n &= t^{n+1}((1 - t^{-1})^{n-d} h_S(t^{-1}) - 1) \\
&= t^{d+1}((t-1)^{n-d} h_S(t^{-1}) - t^{n-d}) \\
\implies \frac{\chi_c(S)(t) - t^n + t^{n+1}}{t(t-1)^{n-d}} &= t^d h_S(t^{-1}).
\end{aligned}$$

2. *Suppose that S satisfies property I. Then, there is some auxiliary simplicial complex $T(S)$ such that*

$$
\begin{aligned}
\chi_c(S)(t) &= t^n(1-t^{-1})^{n-d}h_{T(S)}(t^{-1}) \\
&= t^d(t-1)^{n-d}h_{T(S)}(t^{-1}) \\
\Longrightarrow \frac{\chi_c(S)(t)}{t^{d-m}(t-1)^{n-d}} &= t^m h_{T(S)}(t^{-1})
\end{aligned}
$$

formally, where $m = \dim T(S)$*.*

There are also other intersection patterns of the minimal nonfaces of S where the simplicial chromatic polynomial is determined by the h-vector of an auxiliary simplicial complex $T(S)$.

Corollary 4.2.7. *(Corollary 1.8 on p. 5 of [19])*
Let S *be a simplicial complex and* $\sigma_1, \ldots, \sigma_r$ *be the minimal nonfaces of* S*. Suppose that* $c(I) = a$ *for all subsets* $I \subset [r]$ *with* $|I| \geq 2$*. If* $h_{a+r} \geq 1$*,* $h_{a+1}, h_{a+2} \geq 3$*, and* $h_i \geq 1$ *for all* $i \geq a$*, then*

$$t^{-n} - \chi_c(S)(t^{-1}) + (t^{n+1} - t^{n+a}) \sum_{\sigma_i} t^{-|\sigma_i|} = t^{-n}P(t^{-1})$$

for the Hilbert polynomial $P = P(x)$ *of some* k*-algebra.*

If $a = 1$*, this specializes to*

$$t^{-n} - \chi_c(S)(t^{-1}) = t^{-n}P(t^{-1}).$$

As noted in Proposition 2.2 on p. 726 of [13], the simplicial chromatic polynomial specializes to the "usual" chromatic polynomial of a graph when the simplicial complex S is the independence complex $I(G)$ of some graph G. However, we can consider the terms involved from a different perspective if S is taken to parametrize "monotone" properties of graphs (i.e. P such that P true for $G \implies P$ true for H for any subgraph H of G). Other such examples are on p. 99 – 100 of [20].

Proposition 4.2.8. *Let $[n] = \{1, \ldots, n\}$ and G be a graph whose vertices are labeled by $[n]$. Consider a simplicial complex $S(G)$ whose vertices are labeled by the edges of G. Then, $\chi_c(S)(t)$ gives the number of edge colorings of G using $\leq t$ colors that avoid monochromatic colorings of collections of edges parametrized by the minimal nonfaces of $S(G)$. In some sense, the minimal nonfaces of $S(G)$ correspond to "minimal forbidden subgraphs".*

Note that any *simplicial complex can be written as $S(G)$ for some graph G. Also, any property P of a graph preserved by its subgraphs can be parametrized by a simplicial complex (which we denote by $S(P)$).*

Proof. The first part follows from the definition of $\chi_c(S)$ since $\sigma \subset \tau \implies D_\sigma \supset D_\tau$, which means that it suffices to consider minimal nonfaces. The omitted subsets D_σ parametrize colorings of the edges of G where the edges corresponding to elements of σ all have the same color. The second statement follows from since we can take the smallest "forbidden" graphs to be minimal nonfaces. $\square$

Observation 4.2.9. Proposition 4.2.8 indicates that the simplicial chromatic polynomial $\chi_c(S)(t)$ counts edge colorings which are the "complement" of what is studied in graph Ramsey theory. In general, combining Proposition 4.2.8 with Theorem 4.2.6 and Corollary 4.2.7 gives a family of extremal graph theory problems (i.e. edge colorings avoiding monochromatic "forbidden subgraphs") which are determined by h-vectors of simplicial complexes (equivalently by f-vectors of simplicial complexes). This gives a topological point of view on extremal graph problems, which have results that are mainly stated in terms of inequalities or focus on specific types of subgraphs where we want to avoid monochromatic colorings. Some examples include Bollobás' overview in [8], results of Alon–Balogh–Keevash–Sudakov

[2] on edge colorings avoiding monochromatic cliques, results on Yuster [29] on edges avoiding monochromatic triangles, and surveys of Kano–Li [21] and Fujita–Liu–Magnant [16] involving monochromatic structures in edge colorings. We also provide some methods of studying monochromatic structures in edge colorings of graphs which are *not* complete graphs, which do not appear to be studied as frequently in Ramsey theory-related literature.

For example, the interpretation of $\chi_c(S)(t)$ given in Proposition 4.2.8 can be connected to both the "usual" Ramsey numbers $r(H)$ for a graph with vertices labeled by $[n] = \{1, \ldots, n\}$ (p. 49 of [12]) and Ramsey numbers of *classes of graphs*, which are analogues of Ramsey numbers restricted to some collection of graphs. The following definition is an extension of that used by Belmonte–Heggernes–van't Hof–Rafiey–Saei in [6].

Definition 4.2.10. (Belmonte–Heggernes–van't Hof–Rafiey–Saei in [6])
Fix a positive integer t. Given a graph class $\mathcal{G}$ (i.e. some finite collection of graphs), the Ramsey number $R_{\mathcal{G}}(i, j)$ is the smallest number such that if the edges of a graph in $\mathcal{G}$ is colored with 2 different colors (say red and blue), then it contains a monochromatic blue clique on i vertices or a red clique on j vertices.

Corollary 4.2.11.

Setting $S = S(G)$ from Proposition 4.2.8, we find that the simplicial chromatic polynomial has a natural relationship with Ramsey numbers and its generalizations studied in the literature.

1. *$G = K_n$ case*

   (a) *The simplicial chromatic polynomial $\chi_c(S(K_n))(t)$ gives the number of edge colorings of G using $\le t$ colors which do* not *have any monochromatic cliques of size i. Substituting $t = 2$ into the simplicial chromatic polynomial $\chi_c(S(K_n))(t)$,* *we find that the Ramsey number $R(i,i)$ is given by the minimal n such that $\chi_c(S(K_n))(2) = 0$.*

(b) *More generally, consider a graph H on the vertex set $[n]$ in the setting of part 2. The Ramsey number $r(H)$ is the smallest number n such that any coloring of the edges K_n using 2 colors contains a monochromatic coloring of the edges of H. Given a graph G and a subgraph H of G, let $S_H(G)$ be the simplicial complex with vertices given by the edges of G and minimal nonfaces given by edges coming from copies of H in G. Then, we have that $r(H)$ is the smallest number n such that $\chi_c(S_H(K_n))(2) = 0$.*

2. *Other graphs G*

   (a) *For each graph G in a graph class $\mathcal{G}$, let $S(G)$ be the simplicial complex described in Proposition 4.2.8 with the minimal nonfaces given by edges of cliques on i vertices contained in G. Substituting $t = 2$ into the polynomials $\chi_c(S(G))(t)$ for $G \in \mathcal{G}$, we have that $R_{\mathcal{G}}(i,i)$ is the smallest number N such that $\chi_c(S(G))(2) = 0$ for all $G \in \mathcal{G}$.*

   (b) *In the setting of Proposition 4.2.8, take $S = S(G)$ with the forbidden graphs given by cycles of length ℓ. Let n be the number of vertices in a graph G and ℓ be a positive integer such that $4 \leq \ell \leq \frac{n}{8}$. If the minimum degree of among the vertices the graph is $\geq \frac{3n}{4}$, then $\chi_c(S)(2) = 0$.*

   (c) *As in Part 2, consider the simplicial complex $S = S(G)$ from Proposition 4.2.8. Let N be the number of vertices of G and M be the number of edges of G. Suppose that $M \geq N$ and fix a positive integer t such that $M \geq tN$.*

   - *If the minimal nonfaces of $S(G)$ are given by paths of length $\geq \left\lceil \frac{2M}{tN} \right\rceil$, then $\chi_c(S(G))(t) = 0$.*
   - *If the minimal nonfaces of $S(G)$ are given by cycles of length $\geq \left\lceil \frac{2M}{t(N-1)} \right\rceil$,*

then $\chi_c(S(G))(t) = 0$.

Proof. 1. Parts a and b are applications of Proposition 4.2.8 with the minimal nonfaces taken to be the edges corresponding to i-cycles and copies of H contained in G respectively.

2. (a) This is an application of Definition 4.2.10 to Proposition 4.2.8

   (b) This is an application of Theorem 2.2.8 on p. 13 and Theorem 2.2.9 on p. 14 of [16] to Proposition 4.2.8.

   (c) This is an application of Theorem 40 on p. 249 of [21] to Proposition 4.2.8.

□

4.2.3 Connection with lattice point counts of polytopes

In this section, we expand the connection between Ramsey-type problems and topological/geometric properties summarized in Observation 4.2.9 in Section 4.2.2. Recall that this came from an interpretation (Proposition 4.2.8) of the simplicial chromatic polynomial $\chi_c(S)(t)$ in terms of edge colorings avoiding monochromatic colorings of certain "forbidden subgraphs" (e.g. cycles, cliques, or paths of a certain size) and its expression in terms of h-vectors of auxiliary simplicial complexes $T(S)$ when the minimal nonfaces of S satisfy certain intersection properties (Theorem 4.2.6 and Corollary 4.2.7). Note that *any* simplicial complex can be set to be $T(S)$ for *some* simplicial complex S satisfying property I.

Background on lattice point counts and heuristics

Before we start stating specific expressions/identities, we go over definitions of the objects used.

Definition 4.2.12. (p. 275 of Bruns–Herzog) Let $P \subset \mathbb{R}^N$ be a convex bounded polytope of dimension r. The **Ehrhart function** is defined as

$$E(P,m) = \left| \left\{ z \in \mathbb{Z}^n : \frac{z}{m} \in P \right\} \right| = |\{z \in \mathbb{Z}^n : z \in mP\}|$$

for $m \in \mathbb{N}$ and $m > 0$. Note that $E(P,0) = 1$.

Its generating function is the **Ehrhart series**

$$E_P(t) := \sum_{m \in \mathbb{N}} E(P,m)t^m.$$

The expression of the Ehrhart series as a Hilbert series comes from a more general framework using Hilbert functions to describe generating functions associated to combinatorial objects arising as solutions to homogeneous linear Diophantine equations in n variables (p. 274 – 276 of [10]). By Lemma 4.1.4 on p. 149 of [10] we can use to write

$$E_P(t) = \frac{h_0^* + h_1^* t + \ldots + h_r^* t^r}{(1-t)^{r+1}} \text{ (p. 3 of [25]).}$$

This can either done by using the fact that $E_P(t)$ is the Hilbert series of a k-algebra of dimension $r+1$ (p. 276 of [10]) or a direct computation using the fact that $E(P,m)$ takes integer values for every $m \in \mathbb{Z}$ (Lemma 4.1.4 on p. 149 of [10], p. 3 of [25]). The vector $h^* = (h_0^*, \ldots, h_r^*)$ is called the **h^*-vector** of the polytope P. Let $h_P^*(t) = h_0^* + h_1^* t + \ldots + h_r^* t^r = (1-t)^{r+1} E_P(t)$.

We can use this expression of $E_P(t)$ as a Hilbert series to study formal analogues of the simplicial chromatic polynomial. The families of simplicial complexes that we have considered make use of the Hilbert series of the Stanley–Reisner ring with t^{-1} substituted in place of t. In the case of Ehrhart series, there is a natural symmetry between such polynomials since $(t-1)^{r+1}E_P(t^{-1}) = t^{r+1}h_P(t^{-1})$. Making the corresponding substitutions, Part 1 of Theorem 4.2.6 implies that

$$(t-1)^{n+1}E_P(t^{-1}) \text{ ``}=\text{'' } \chi_c(S)(t) - t^n + t^{n+1} \tag{4.2.2}$$

formally, where $r = \dim P$ (used in place of $d = \dim S$) and $n = |V|$ (size of the vertex set). Similarly, applying Part 2 of Theorem 4.2.6 implies that

$$(t-1)^{n+1}E_P(t^{-1}) \text{ ``}=\text{'' } \frac{\chi_c(S)(t)}{t^{d-r}}, \tag{4.2.3}$$

where $d = \dim S$, $r = \dim P$ and $n = |V|$ as above. In this case, we use r in place of $m = \dim T(S)$. Note that $\widetilde{E}_P(t) = -E_P(t^{-1})$ in the expressions above, where $\widetilde{E}_P(t) = \sum_{m \geq 1} E(P, -m)t^m$. By a reciprocity result of Ehrhart ((0.3) on p. 166 of [18], Theorem 6.3.11 on p. 276 of [10]), we have that $(-1)^r E(P, -m) = \#(m(P - \partial P) \cap \mathbb{Z}^N)$ for every integer $m > 0$. This implies that the simplicial chromatic polynomial is formally a normalization of the generating function for lattice points of integer factor dilations of the polytope P with its boundary removed.

Edge colorings avoiding forbidden subgraphs and lattice point counts

In this subsection, we consider some instances where the h-vectors of the simplicial complexes such as those considered in Part 1 of Theorem 4.2.6 are actually *equal* to the h-vectors associated to convex polytopes so that this gives an equality. As a consequence, we find that lattice point counts of certain polytopes are determined by the number of ways to color the vertex set of some simplicial complex (within $\leq t$ colors for some t when considered as a polynomial in t) so that no two vertices lying in the same minimal nonface have the same color (Corollary 4.2.16). The latter follows from the definition of the simplicial chromatic polynomial.

We will now cover some known cases where the h-vector associated to a polytope is equal to that of an actual simplicial complexes.

We first recall a result giving an equality between δ-vectors of polytopes and h-vectors of approrpriate simplicial complexes. More specifically, there is also a result which give an equality between h-vectors of polytopes and h-vectors of certain simplicial complexes if there is a unimodular triangulation.

Definition 4.2.13. (p. 694 of [9])
A lattice polytope P satisfies the **integer decomposition property (IDP)** if

$$\text{span}_{\mathbb{Z}_{\geq 0}}\{(1, P) \cap \mathbb{Z}^{n+1}\} = \text{cone}(P) \cap \mathbb{Z}^{n+1}.$$

Theorem 4.2.14. *(Bruns and Römer, Theorem 1 on p. 67 of Theorem 4 on p. 698 of [9]) If P is Gorenstein and IDP, then h_P^* is the h^*-vector of an IDP reflexive polytope. Further, if P admits a regular unimodular triangulation, then there exists a simplicial polytope Q such that h_P^* is the h-vector of Q, and hence h_P^* is unimodal as a consequence of the g-theorem.*

Remark 4.2.15. Although it is possible for cP to admit a unimodular triangulation while $(c+1)P$ does not, every sufficiently large dilation of an integral polytope admits a unimodular triangulation by a recent result of Liu (Theorem 1.2 on p. 2 of [24]). This builds on an older result of Kempf–Knudsen–Mumford–Saint-Donat in [23]. Also, note that the IDP property is preserved under dilation by definition. Results on related combinatorial invariants are in work of Cox–Haase–Hibi–Higashitani [14].

Putting this together with the previous results on simplicial polytopes, these imply the following:

Theorem 4.2.16.

1. *Suppose that P is a Gorenstein integer polytope admitting a regular unimodular triangulation. Then, the boundary complex of P is abstractly isomorphic to a simplicial complex T such that*

$$\frac{\chi_c(S)(t)}{t^{e-u-1}(t-1)^{n-e+r+1}} = (-1)^{r+1} E_P^+(t)$$

for any S is a simplicial complex satisfying property I (Theorem 4.2.6) such that $T = T(S)$, where $e = \dim S$, $r = \dim P$, $u = r - u + 1$, and

$$E^+(P, m) = \left| \left\{ z \in \mathbb{Z}^n : \frac{z}{m} \in P \setminus \partial P \right\} \right| = |\{z \in \mathbb{Z}^n : z \in m(P \setminus \partial P)\}|$$

by p. 275 of [10]. .

The definition of $\chi_c(S)$ then implies that the number of colorings of the vertices of S using at most t colors such that no two vertices of S lie in the same minimal non-face of S divided by $t^{e-m-1}(t-1)^{n-e+m+1}$ is equal to the generating series of lattice point counts on integer dilations of $P \setminus \partial P$. Also, the coefficients of these lattice point counts/colorings are determined by the (primitive) cohomology of hypersurfaces on algebraic tori.

2. *The class of polytopes in Part 1 induces a class of graphs G and specified subgraphs $\{H_i\}$ where the edge colorings of G using $\leq t$ colors where the H_i are* not *monochromatic are parametrized by $t^a(t-1)^b$ multiplied by the Ehrhart function of a polytope. This implies that the colorings in question are essentially parametrized by lattice point counts of integer dilations of some polytope.*

Proof. 1. By Theorem 4.2.6, the initial conditions imply that $E_P(t) = \frac{h_T(t)}{(1-t)^{u+1}}$. Since *any* simplicial complex is equal to $T(S)$ for some simplicial complex satisfying property I (e.g. by adding a single particular new vertex to each minimal nonface), we can set $T = T(S)$ for some simplicail complex S satisfying property I from Theorem 4.2.6. Let $e = \dim S$, n be the number of vertices of S, and $m = \dim T(S)$. Part 2 of Theorem 4.2.6 implies that

$$\begin{aligned}
\frac{\chi_c(S)(t)}{t^e(t-1)^{n-e}} &= h_{T(S)}(t^{-1}) = (1-t^{-1})^{r+1}E_P(t^{-1}) \\
&= t^{-r-1}(t-1)^{r+1}E_P(t^{-1}) \\
\implies \frac{\chi_c(S)(t)}{t^{e-r-1}(t-1)^{n-e+r+1}} &= E_P(t^{-1}) \\
&= (-1)^{r+1}E_P^+(t),
\end{aligned}$$

where $E^+(P,m) = \left|\left\{z \in \mathbb{Z}^n : \frac{z}{m} \in P \setminus \partial P\right\}\right| = |\{z \in \mathbb{Z}^n : z \in m(P \setminus \partial P)\}|$ (p. 275 of [10]).

The last equality follows from the Ehrhart reciprocity relation

$$E_P(t^{-1}) = (-1)^{r+1}E_P^+(t),$$

which is known to be a generating function for lattice points on positive integer dilations of $P \setminus \partial P$ ((0.3) on p. 166 of [18], Theorem 6.3.11 on p. 276 of [10]). This also implies the connection between coloring and lattice point counts in the last part of the statement. Finally, the statement connecting coefficients to the (primitive) cohomology of hypersurfaces on algebraic tori follows from replacing t by t^{-1} and applying Corollary 4.2.24.

2. This is a combination of Part 1 and Proposition 4.2.8. Note that *any* simplicial complex an be written as $S(G)$ for some graph G (Remark 4.2.4) and a collection of forbidden subgraphs corresponding to the minimal nonfaces of S where monochromatic edge colorings are not allowed.

□

4.2.4 A Hodge structure on the cohomology of toric varieties

Using a similar analysis to the one in Section 4.2.3, we give classes of graphs whose Ramsey numbers are determined by the mixed Hodge structure on certain toric varieties.

Background on δ-invariants

The expressions above expressing the simplicial chromatic polynomial $\chi_c(S)$ "formally" in terms of $E_P(t^{-1})$ are very close to a natural existing invariant of convex polytopes (the δ-vector, p. 166 of [18]). Given a convex polytope $P \subset \mathbb{R}^N$ of dimension N, its δ-vectors can be expressed in terms of Hodge–Deligne numbers form the primitive part of the middle cohomology of hypersurfaces in algebraic tori $Z(f) = (f = 0) \subset (\mathbb{C}^*)^N$ for f such that $N(f) = P$ (i.e. with Newton polytope P). In some sense, this implies that a formalization of the simplicial chromatic polynomial is determined by a portion of the "primitive" Hodge–Deligne polynomial of a hypersurface in an algebraic torus. Note that *any* simplicial complex can be set as $T(S)$ in Part 2 of Theorem 4.2.6 for *some* simplicial complex S. In addition, the h-vectors associated to convex polytopes are associated to h-vectors of actual simplicial complexes under suitable conditions. This turns the formalization into an equality which is realized by an actual simplicial chromatic polynomial.

Definition 4.2.17. (p. 166 of [18])

Let $P \subset \mathbb{R}^N$ be an integral convex polytope (i.e. with vertices given by integer coordinates),

$r = \dim P$, and ∂P be the boundary of P. We define the sequence of integers $\delta_0, \delta_1, \delta_2, \ldots$ by the formula

$$(1-t)^{r+1}\left(1 - \sum_{m=1}^{\infty} E(P,m)t^m\right) = (1-t)^{r+1}(1 - E_P(t)) = \sum_{i=0}^{\infty} \delta_i t^i. \tag{4.2.4}$$

By a fundamental result on generating functions (equivalence between i and iii in Corollary 4.3.1 on p. 543 of [28]), we have that $\delta_i = 0$ for every $i > r$.

When $P \subset \mathbb{R}^N$ is an integral convex polytope of dimension r, we say that the sequence $\delta(P) = (\delta_0, \delta_1, \ldots, \delta_r)$ is the **δ-vector** of P. In particular, we have that $\delta_0 = 1$ and $\delta_1 = \#(P \cap \mathbb{Z}^N) - (r+1)$.

There is a subclass of polytopes P whose δ-vectors are equal to h-vectors of the simplicial complex corresponding to some triangulation of the boundary ∂P.

Definition 4.2.18. (p. 168 – 169 of [18])

1. A polytope P of dimension r is of **standard type** if $P \subset \mathbb{R}^r$ and the origin of $\mathbb{R}^r$ is contained in the interior $P \setminus \partial P$ of P. For each integer $r > 1$, let $\mathcal{C}_0(r)$ be the set of integral convex polytopes in $\mathbb{R}^r$ of standard type.

2. Given a polytope P of standard type, its **polar set (or dual polytope)** is defined as

$$P^* := \{(\alpha_1, \ldots, \alpha_r) \in \mathbb{R}^r : \alpha_1\beta_1 + \ldots + \alpha_r\beta_r \leq 1 \text{ for all } (\beta_1, \ldots, \beta_r) \in P\}.$$

   For each $r > 1$, let $\mathcal{C}^*(r)$ be the set of $P \in \mathcal{C}_0(r)$ such that P^* is an integral convex polytope.

Note that $P^* \subset \mathbb{R}^r$ is also a convex polytope of standard type and $(P^*)^* = P$. Moreover, if P is rational, then P^* is rational.

3. A triangulation S of the boundary ∂P of $P \in \mathcal{C}^*(r)$ with vertex set $V = \partial P \cap \mathbb{Z}^r$ is called **compressed**.

Proposition 4.2.19. *(Stanley, Betke–McMullen, Proposition 2.2 on p. 171 of [18]) Suppose that S is a triangulation of the boundary ∂P of $P \in \mathcal{C}^*(r)$ with vertex set $V = \partial P \cap \mathbb{Z}^r$. Let $h(S) = (h_0, \ldots, h_r)$ be the h-vector of S and $\delta(P)$ be the δ-vector of P. Then, $\delta(P) \geq h(S)$ (i.e. $\delta_i \geq h_i$ for each $1 \leq i \leq r$). Moreover, $h(S) = \delta(P)$ if and only if S is compressed.*

We can combine this with Part 2 of Theorem 4.2.6 to state the following:

Corollary 4.2.20. *Suppose that S is a simplicial complex satsifying property I such that $T(S)$ (Theorem 4.2.6) is a compressed triangulation of some polytope $P \in \mathcal{C}^*(m+1)$. Let $d = \dim S$, $n = |V|$ (the size of the vertex set), and $m = \dim T(S)$. Then, we have that*

$$\frac{\chi_c(S)(t)}{t^{d-m-2}(t-1)^{n-d}} = (t^{m+2} - 1)(1 + \widetilde{E}_P(t)),$$

where

$$\widetilde{E}_P(t) = \sum_{m \geq 1} E(P, -m)t^m.$$

Note that $E(P, -m) = (-1)^{m+2}\#(m(P \setminus \partial P) \cap \mathbb{Z}^{m+2})$ and any *simplicial complex can be set equal to $T(S)$ for some simplicial complex S satsifying property I.*

Proof. This is a combination of Part 2 of Theorem 4.2.6, the definition of the δ-vector form 4.2.4, and Proposition 4.2.19. By Theorem 4.2.6, we have that

$$\frac{\chi_c(S)(t)}{t^d(t-1)^{n-d}} = h_{T(S)}(t^{-1}). \tag{4.2.5}$$

Since we assumed that $T(S)$ is a *compressed* triangulation, we have that $h_i(T(S)) = \delta_i(P)$ for each $1 \leq i \leq m$. By the definition of the δ-vector in 4.2.4, we have that

$$(1-t)^{m+2}\left(1-\sum_{a=1}^{\infty} E(P,a)t^a\right) = \sum_{u=0}^{\infty} \delta_u t^u = \sum_{u=0}^{\infty} h_u t^u = h_{T(S)}(t)$$

since $\dim P = \dim T(S) + 1 = m+1$. Note that the "infinite" sum actually terminates since terms of degree $> m$ are equal to 0. We can rewrite this as

$$(1-t)^{m+2}(1-E_P(t)) = h_{T(S)}(t). \tag{4.2.6}$$

Substituting in t^{-1} in place of t, we can combine 4.2.5 with 4.2.6 to find that

$$\begin{aligned}
\frac{\chi_c(S)(t)}{t^d(t-1)^{n-d}} &= (1-t^{-1})^{m+2}(1-E_P(t^{-1})) \\
&= (1-t^{-1})^{m+2}(1+\widetilde{E}_P(t)) \\
&= t^{-m-2}(t^{m+2}-1)(1+\widetilde{E}_P(t)) \\
\implies \frac{\chi_c(S)(t)}{t^{d-m-2}(t-1)^{n-d}} &= (t^{m+1}-1)(1+\widetilde{E}_P(t)),
\end{aligned}$$

where

$$\widetilde{E}_P(t) = \sum_{m\geq 1} E(P,-m)t^m.$$

This follows from the identity $\widetilde{E}_P(t) = -E_p(t^{-1})$ (p. 3 of [25]).

As mentioned previously, a reciprocity result of Ehrhart ((0.3) on p. 166 of [18]) implies that $\widetilde{E}_p(t)$ is a generating function for integer dilations of $P \setminus \partial P$.

□

Hodge structures on toric varieties vs. edge colorings avoiding monochromatic forbidden subgraphs

More specifically, we can analyze the connection between δ-vectors of polytopes and simplicial chromatic polynomials in a more geometric method. More specifically, we make use of the fact that the δ-vectors of a polytope P are determined by Hodge–Deligne components of the primitive part of the middle cohomology of a hypersurface (depending on P) in a certain algebraic torus by work of Batyrev (Section 3 of [5]). Let $P \subset \mathbb{R}^N$ be a polytope of dimension r and $T = (\mathbb{C}^*)^N$.

Definition 4.2.21. (p. 357 – 358 of [5])

1. Given a Laurent polynomial $f \in \mathbb{C}[x_1^{\pm}, \ldots, x_N^{\pm}]$, let $Z_f \subset (\mathbb{C}^*)^N$ be the affine hypersurface determined by f and $P(f)$ be the Newton polytope of f.

2. Given a polytope P, let $L(P) = \{f \in \mathbb{C}[x_1^{\pm}, \ldots, x_N^{\pm}] : P = N(f)\}$.

3. Let P be a polytope and $f = \sum c_m x^m$ be a Laurent polynomial such that $N(f) = P$ (i.e. $f \in L(P)$) Given a face $P' \subset P$, define

$$f^{P'}(x) = \sum_{m' \in P'} c_{m'} x^{m'}.$$

4. Given a Laurent polynomial $g = g(x) \in \mathbb{C}[x_1^{\pm}, \ldots, x_N^{\pm}]$, let g_i for $1 \le i \le N$ be the logarithmic derivatives
$$g_i = x_i \frac{\partial}{\partial x_i} g(x)$$
of g.

5. A Laurent polynomial $f \in L(P)$ and the corresponding affine hypersurface $Z_f \subset T$ are called P**-regular** if $P(f) = P$, and for every ℓ-dimensional edge $P' \subset P$ ($\ell > 0$),

the polynomial equaitons

$$f^{P'}(x) = f_1^{P'}(x) = \cdots = f_n^{P'}(x) = 0$$

have no common solutions in T.

In order to study the relationship between the mixed Hodge structure of Z_f and the Laurent polynomial f, Batyrev defines the following "primitive" cohomology group.

Definition 4.2.22. (Definition 3.13 on p. 361 of [5])
The **primitive part of the cohomology group** $H^{N-1}(Z_f)$ (denoted $PH^{N-1}(Z_f)$) is the cokernel of the injective mapping $H^{N-1}(T) \hookrightarrow H^{N-1}(Z_f)$.

Theorem 4.2.23. *(Batyrev,p. 359, Remark 2.13 on p. 357, Corollary 3.12, Corollary 3.14, and Remark 3.15 on p. 361 of [5])*
Given a smooth affine algebraic variety V and $0 \leq k \leq \dim V$, let

$$H^k(V) = \mathcal{F}^0 H^k(V) \supset \mathcal{F}^1 H^k(V) \supset \cdots \supset \mathcal{F}^{k+1} H^k(V)$$

be the Hodge filtration. We will use $h^{p,q}(H^k(V))$ to denote the Hodge–Deligne numbers (which also involves the weight filtration – see p. 359 of [5]). This will be done similarly (modulo quotients) for the primitive cohomology (as defined in Definition 4.2.22).

Let $Z_f \subset T$ be a P-regular affine hypersurface (Part 5 of Definition 4.2.21).

1. *The dimensions of the quotients of consecutive terms of the Hodge filtration are given*

by

$$\dim \mathcal{F}^i H^{N-1}(Z_f)/\mathcal{F}^{i+1}H^{N-1}(Z_f) = \sum_{q\geq 0} h^{i,q}(H^{N-1}(Z_f))$$
$$= \begin{cases} \delta_{N-i}(P) & \text{if } i < N-1 \\ \delta_1(P) + N & \text{if } i = N-1. \end{cases}$$

2. If $i \leq N-1$, we have that

$$\dim \mathcal{F}^i PH^{N-1}(Z_f)/\mathcal{F}^{i+1}PH^{N-1}(Z_f) = \sum_{q\geq 0} h^{i,q}(PH_c^{N-1}(Z_f)) = \delta_{N-i}(P).$$

Note that the generating function of the δ_i give the Hilbert-Poincaré series of coordinate ring the toric variety associated to P (Definition 2.4 on p. 355 of [5]) by a regular sequence of linear terms in the coordinate ring.

Using Theorem 4.2.23, we can see that the heuristics from 4.2.2 and 4.2.3 can be combined with 4.2.4 to find "formal" expressions for the simplicial chromatic polynomial involving Hodge–Deligne polynomials from the cohomology of hypersurfaces in algebraic tori corresponding to Part 1 and Part 2 of Theorem 4.2.6 given by

$$\begin{aligned} \sum_{i=0}^{r} \delta_i t^i \text{ ``=''} \ (1-t)^{r+1}\left(1 - \frac{\chi_c(S)(t^{-1}) - t^{-n} + t^{-n-1}}{(t^{-1}-1)^{n+1}}\right) \\ = (1-t)^{r+1}\left(1 - \frac{t^{n+1}\chi_c(t^{-1}) - t + 1}{(t-1)^{n+1}}\right) \end{aligned} \tag{4.2.7}$$

from 4.2.2 and 4.2.4 and

$$\sum_{i=0}^{r} \delta_i t^i \text{ ``=''} \ (1-t)^{r+1}\left(1-\frac{t^{d-r}\chi_c(S)(t^{-1})}{(t^{-1}-1)^{n+1}}\right) = (1-t)^{r+1}\left(1-\frac{t^{n+d-r+1}\chi_c(S)(t^{-1})}{(t-1)^{n+1}}\right) \tag{4.2.8}$$

from 4.2.3 and 4.2.4, where $r = \dim P$ and $n = \dim S$. Note that we "truncated" the initial sums above since $\delta_i = 0$ for $i > r$.

Heuristically, the expressions 4.2.7 and 4.2.8 indicate that the simplicial chromatic polynomials in Theorem 4.2.6 are formally determined by mixed Hodge structures of hypersurfaces on algebraic tori (via a truncated Hodge–Deligne polynomial) after we replace the h-vectors of S or $T(S)$ by those of a convex polytope. In the setting of Corollary 4.2.20, the simplicial chromatic polynomials themselves are determined by the mixed Hodge structure of these hypersurfaces on algebraic tori. We can state this more explicitly.

Corollary 4.2.24. *Suppose that S is a simplicial complex satsifying property I such that $T(S)$ (Theorem 4.2.6) is a compressed triangulation of some polytope P. Let $d = \dim S$, $n = |V|$ (the size of the vertex set), and $m = \dim T(S)$.*

Let $f \in \mathbb{C}[x_1^{\pm}, \ldots, x_m^{\pm}]$ a Laurent polynomial which is P-regular (Part 5 of Definition 4.2.21) and $Z_f \subset (\mathbb{C}^*)^m$ be the affine hypersurface determined by f. For each $1 \le i \le m$, the coefficient of t^i in $\frac{t^n \chi_c(S)(t^{-1})}{(t-1)^{n-d}}$ is

$$\dim \mathcal{F}^{m+1-i} PH^m(Z_f)/\mathcal{F}^{m+2-i} PH^m(Z_f) = \sum_{q \ge 0} h^{m+1-i,q}(PH^m(Z_f))$$

Note that this is equal to

$$\dim \mathcal{F}^{m+1-i} H^m(Z_f)/\mathcal{F}^{m+2-i} H^m(Z_f) = \sum_{q \ge 0} h^{m+1-i,q}(H^m(Z_f))$$

if $i < m$.

Proof. We start by substituting in t^{-1} in place of t in 4.2.5 from the proof of Corollary 4.2.20. This gives

$$h_{T(S)}(t) = \frac{\chi_c(S)(t^{-1})}{t^{-d}(t^{-1}-1)^{n-d}} = \frac{t^d\chi_c(S)(t^{-1})}{(t^{-1}-1)^{n-d}} = \frac{t^n\chi_c(S)(t^{-1})}{(t-1)^{n-d}}. \tag{4.2.9}$$

Under the assumptions on the simplicial complex, we have that $h_{T(S)}(t) = \sum_{u=0}^{\infty} \delta_u t^u$ ($\delta_u = 0$ for $u > r$). For each $1 \le i \le m$, Theorem 4.2.23 implies that the coefficient of t^i is

$$\dim \mathcal{F}^{m+1-i} PH^m(Z_f)/\mathcal{F}^{m+2-i} PH^m(Z_f) = \sum_{q \ge 0} h^{m+1-i,q}(PH^m(Z_f))$$

since $\dim P = m+1$.

The second expression comes from Part 2 of Theorem 4.2.23. □

We can summarize the discussion above as follows:

Theorem 4.2.25.

1. *Let $P \subset \mathbb{P}^N$ be a convex polytope of dimension r and $Z_f \subset T$ be a P-regular affine hypersurface. If we replace the h-vector of $T(S)$ in Theorem 4.2.6 by that of P, the expression*

$$(1-t)^{r+1}\left(1 - \frac{t^{n+d-r+1}\chi_c(S)(t^{-1})}{(t-1)^{n+1}}\right)$$

is replaced by a truncated generating function for the Hodge numbers of $H^{N-1}(Z_f)$. By Theorem 4.2.14, this gives an actual equality when $T(S)$ is the unimodular triangulation of some Gorenstein IDP polytope with a unimodular triangulation. Also, note that $\chi_c(S)$ paramtrizes edge colorings avoiding monochromatic colorings of the minimal *nonfaces of S by Proposition 4.2.8.*

The coefficients of the reciprocal polynomial of the resulting function are given by the dimensions of quotients of consecutive terms of the Hodge filtration on $H^{N-1}(Z_f)$.

2. *Suppose that S is a simplicial complex satsifying property I such that $T(S)$ (Theorem 4.2.6) is a compressed triangulation of some polytope P. Let $d = \dim S$, $n = |V|$ (the size of the vertex set), and $m = \dim T(S)$.*

   Let $f \in \mathbb{C}[x_1^{\pm}, \ldots, x_m^{\pm}]$ a Laurent polynomial which is P-regular (Part 5 of Definition 4.2.21) and $Z_f \subset (\mathbb{C}^*)^m$ be the affine hypersurface determined by f. For each $1 \leq i \leq m$, the coefficient of t^i in

$$\frac{t^n \chi_c(S)(t^{-1})}{(t-1)^{n-d}}$$

   is

$$\dim \mathcal{F}^{m+1-i} PH^m(Z_f)/\mathcal{F}^{m+2-i} PH^m(Z_f) = \sum_{q \geq 0} h^{m+1-i,q}(PH^m(Z_f))$$

   Note that this is equal to

$$\dim \mathcal{F}^{m+1-i} H^m(Z_f)/\mathcal{F}^{m+2-i} H^m(Z_f) = \sum_{q \geq 0} h^{m+1-i,q}(H^m(Z_f))$$

   if $i < m$.

3. *As a consequence of the addition-deletion relation satisfied by simplicial chromatic polynomials, a similar one holds for a normalization of the Hodge–Deligne polynomials parametrizing mixed Hodge structures of the affine hypersurfaces on algebraic tori from Part 2.*

4. *As a consequence of Part 1, the edge colorings of a graph G arising from the boundary*

complex of a simplicial complex which do not *give monochromatic colorings of specified subgraphs H_i determine a truncated generating function for the Hodge numbers of $H^{N-1}(Z_f)$ (a toric hypersurface).*

Remark 4.2.26. Since we are making use of the δ-invariant, there are additional symmetry properties in the case where an polytope and its dual/polar polytope are both integral (Theorem 35.8 on p. 105 of [19]).

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CHAPTER 5

COMBINATORICS AND GEOMETRY OF A MATROIDAL ANALOGUE OF INDEPENDENCE CONDITIONS ON HYPERSURFACES

5.1 Matroids satisfying the matroidal Cayley–Bacharach property and ranks of covering flats

Let M be a matroid satisfying a matroidal analogue of the Cayley–Bacharach condition. Given a number $k \geq 2$, we show that there is no nontrivial bound on ranks of a k-tuple of flats covering the underlying set of M. This addresses a question of Levinson–Ullery motivated by earlier results which show that bounding the number of points satisfying the Cayley–Bacharach condition forces them to lie on low-dimensional linear subspaces. We also explore the general question what matroids satisfy the matroidal Cayley–Bacharach condition of a given degree and its relation to the geometry of generalized permutohedra and graphic matroids.

5.1.1 Introduction

A finite subset $\Gamma \subset \mathbb{P}^n$ satisfies the Cayley–Bacharach condition of degree r if a homogeneous polynomial of degree r vanishing on all but one point of Γ vanishes on all of Γ. In recent work, Levinson–Ullery [17] show that a finite subset $\Gamma \subset \mathbb{P}^n$ satisfying the Cayley–Bacharach condition of degree r is covered by low-dimensional linear subspaces if $|\Gamma|$ is not very large compared to r (Theorem 1.3 on p. 2 of [17]). The result is motivated by constructions relating to degrees of irrationality of smooth complete intersections.

More specifically, the varieties X considered as motivating examples are those with generically finite dominant rational maps $X \dashrightarrow \mathbb{P}^n$ connected to (a generalization of) the Cayley–Bacharach property and special configurations of points. It is known that the generic fiber of the rational map $X \dashrightarrow \mathbb{P}^n$ satisfies the Cayley–Bacharach property with respect to the linear system $|K_X|$ (replacing homogeneous degree r polynomials by sections of K_X). If K_X is sufficiently positive, then the fibers also lie in special positions. For example, a result of Bastianelli–Cortini–De Poi (Theorem 1.1 on p. 2 of [17]) states that a finite subset $\Gamma \subset \mathbb{P}^n$ satisfying the degree r Cayley–Bacharach property of degree r such that $|\Gamma| \leq 2r + 1$ lies on a line. The results of Levinson–Ullery (Theorem 1.3 on p. 2 of [17]) are analogues which show that Γ still lies on a *union* of low-dimensional linear subspaces when we impose a weaker linear upper bound in r on the size of Γ. They are part of a more general conjectured statement which is listed below along with the result.

Conjecture 5.1.1. *(Levinson–Ullery, Conjecture 1.2 on p. 2 of [17])*
Let $\Gamma \subset \mathbb{P}^n$ be a finite set of points satisfying $CB(v)$. If $|\Gamma| \leq (d+1)v + 1$, then Γ can be covered by a union of positive–dimensional linear subspaces $P_1 \cup \cdots \cup P_k$ such that $\sum_{i=1}^k \dim P_i = d$.

Theorem 5.1.2. *(Levinson–Ullery, Theorem 1.3 on p. 2 of [17])*
Conjecture 5.1.1 holds in the following cases:

1. *For all $v \leq 2$ and all d. Moreover, we can take $k = 1$.*

2. *For all v and for $d \leq 3$. Moreover, we may take $k \leq 2$.*

3. *For $d = 4$ and $v = 3$. Moreover, we may take $k \leq 2$.*

Since many of the arguments used in the result of Levinson–Ullery (Theorem 1.3 on p. 2 of [17]) are combinatorial in a way that can sometimes be rewritten in terms of matroid theory (p. 14 of [17]), the authors define a matroid-theoretic analogue of the Cayley–Bacharach property.

Definition 5.1.3. (p. 14 of [17])

A matroid M with underlying set E satisfies the **matroidal Cayley–Bacharach property of degree** a **(denoted** $MCB(a)$**)** if, whenever a union of a flats contains all but one point of E, the union contains the last point. In other words:

$$\bigcup_{i=1}^{a} F_i \supset E \setminus p \Longrightarrow \bigcup_{i=1}^{a} F_i = E$$

for any $p \in E$ and any flats $F_1, \ldots, F_a$ of M. We will work with matroids M such that all falts of rank 1 have size 1 since flats of rank 1 correspond to a single point in $\mathbb{P}^n$.

Note that the finite sets satisfying the Cayley–Bacharach property are represented by the underlying (finite) set of the matroid and the flats are analogous to linear subspaces.

Using these flats of matroids, the authors ask whether an analogue of their main result holds for the matroidal Cayley–Bacharach property. We will discuss this question and a variant.

Question 5.1.4.

1. *(Levinson–Ullery, Question 7.6 on p. 14 of [17]) Does the statement of Conjecture 5.1.1 with $CB(v)$ replaced by $MCB(v)$ and dimensions of linear subspaces replaced by ranks of flats hold? Here is a more explicit statement:*

   Let M be a matroid with underlying set E such that all flats of rank 1 have size 1. Suppose that M satisfies $MCB(a)$ and $|E| \leq (d+1)a+1$. Let $d_i = r_i - 1$ if $r_i \geq 2$ and $d_i = 1$ if $r_i = 1$. Is it possible to cover M by a union of (possibly improper) flats $\bigcup_i F_i$ of ranks r_i respectively such that $\sum_i d_i \leq d$?

2. *We can consider a variant of the question in Part 1 since the original source refers to covering matroids by a union of flats of "specified dimensions". Let M be a matroid of rank r with underlying set E of size n. Fix a positive integer $N = N(M)$. Suppose that M satisfies $MCB(a)$. Must M (meaning the underlying set E) be covered by a union of $\leq N$ proper flats where at least one of the flats has rank $\leq r-2$?*

Remark 5.1.5.

1. In Part 1, we replace "dimensions" d_i in the original statement of Question 7.6 on p. 14 of [17] with $r_i - 1$ if $r_i \geq 2$, where r_i is the rank of a flat F_i. If $r_i = 1$, we will take $d_i = 1$. This is because the analogous geometric condition considers dimensions of spans of points in projective space and "dimension" does not seems to be a standard term for flats of a matroid unless we are discussing representable matroids. In the latter setting, the rank is equal to the dimension of the linear subspace spanned by the vectors corresponding to the points of the flat. Also, we consider both an interpretation of the problem using ranks of individual flats (for "flats of specified

ranks" for Theorem 5.1.17) and a direct analogue of Conjecture 5.1.1 (Theorem 5.1.15).

2. The bounds on sizes of finite set Γ (modeled by E above) satisfying the (geometric) Cayely–Bacharach property in Theorem 1.1, Conjecture 1.2, and Theorem 1.3 on p. 2 of [17] are on the size of the finite set (analogous to $n = |E|$) relative to the degree (given by a above). The dimension of a plane configuration is the sum of the dimension of the linear subspaces used to cover the finite set Γ on p. 2 of [17].

3. For each of the questions above, we find a counterexample using a matroid satisfying $MCB(a)$ where the flats involved in the definition of $MCB(a)$ must be hyperplanes (i.e. maximal proper flats).

In Section 5.1.2, we find some examples of nontrivial flats satisfying the matroidal Cayley–Bacharach condition whose nontrivial covers by flats only use hyperplanes (i.e. maximal proper flats) (Example 5.1.16) when $N \leq a$. Since this would mean taking $b_i = r - 1$ for all i in Question 5.1.4, this means that there is no nontrivial bound on the ranks of flats covering the ground set of a matroid satsifying the degree a matroidal Cayley–Bacharach condition (Theorem 5.1.17).

Theorem 5.1.6. *(Theorem 5.1.17)*

Take an even number $B \geq 2m + 2$ with $B | n$ and $\frac{n}{B} < B$. Fix $k, a \leq \frac{n}{B} + \frac{B}{m} - 3$.

1. *Let M be a matroid of rank $m + 1$ satisfying $MCB(a)$ with underlying set E of size n. Suppose that $F_1, \ldots, F_k$ is a k-tuple of flats covering E with each proper flat of size at most B. There is no covering by $\leq k$ flats where at least one of the flats has rank $\leq r - 2$. In other words, it is possible for all the flats F_i to be hyperplanes. Question*

5.1.4 contains an explanation of why this indicates that there is no "nontrivial" bound.

2. *In fact, the matroid from the proof of Part 1 gives a negative answer to Question 5.1.4 using the case $a \leq \frac{n}{B} + \frac{B}{m} - 3$. More specifically, there is no upper bound on the ranks of a collection of $\leq a$ flats which cover the underlying set E of a matroid of rank r satisfying $MCB(a)$.*

Remark 5.1.7. The counterexamples we study have some recursive properties regarding the matroidal Cayley–Bacharach property and *some* upper bound is required in order for $MCB(a)$ to be satisfied (Proposition 5.1.18).

The example above also implies a direct translation of Conjecture 1.2 on p. 2 of [17] does *not* hold.

Theorem 5.1.8. *(Theorem 5.1.15)*
There is a matroid M satisfying $MCB(a)$ with ground set E ($n := |E|$) such that there is some d such that $n \leq (d+1)a + 1$ but E cannot *be covered by a union of flats of total rank d. In other words, we have that $\bigcup_{i=1}^k F_i = E \Longrightarrow \sum_{i=1}^k \operatorname{rank} F_i \geq d+1$.*

Afterwards, we study general properties of matroids satisfying the matroidal Cayley–Bacharach condition in Section 5.1.3. The main tool used here is the matroid polytope determined by the basis elements. This gives a characterization of "generic" matroids that have appropriate connectivity properties (Theorem 5.1.27).

Theorem 5.1.9. *(Theorem 5.1.27)*
Suppose that $M = \sum_{I \subset [n]} y_I \Delta_I$ for some $y_I \geq 0$.

Then, the existence of a matroid N satsifying the following conditions can be checked using a set-theoretic condition involving $(n-1)$-element subsets of $[n]$ or the sets I:

- *N has an underlying set of the same size $n = |E|$ such that the flats inducing facets of P_N satisfy the conditions $MCB(a)$*
- *P_N and the matroid polytope of P_M are nondegenerate deformations of each other (i.e. those not passing through vertices)*

Under additional assumptions on the collection of subsets $I \subset [n]$ such that $y_I > 0$ and connectedness-related properties of I, we can show that checking whether the matroid M itself satsifies the matroidal Cayley–Bacharach property is equivalent to checking whether the set-theoretic analogue holds for the subsets I considered (Part 2 of Theorem 5.1.25). Note that the terms below are defined in Section 5.1.3 (Definition 5.1.24, Definition 5.1.22, Definition 5.1.29).

Theorem 5.1.10. *(Theorem 5.1.25)*

Suppose that M is a connected matroid satisfying the following conditions:

- *$M[F, G]$ is connected for all flats F, G such that $F \subset G$ or every flat A of M is both connected and coconnected. For example, consider the graphic matroid $M(K_n)$ of spanning trees in the complete graph K_n (Remark 5.4 on p. 459 of [12]).*
- *$P_M = \sum_{I \subset [n]} y_I \Delta_I$ for some $y_I \ge 0$ such that $y_{[n]} > 0$. As mentioned in Observation 5.1.21, the condition here is really one on the ranks of the flats since $z_I = \sum_{J \subset I} y_J$ and $y_I = \sum_{J \subset I} (-1)^{|I|-|J|} z_J$, where $z_I = r - \text{rank}(\text{Span}\, I)$ with $r = \text{rank}\, M$ and $\text{Span}\, I$ being the smallest flat containing the elements of I (Proposition 2.2 and Proposition 2.3 on p. 843 of [3]).*

Let B be the collection of subsets $I \subset [n]$ such that $y_I > 0$. Then, the following statements hold:

1. *The matroid M satisfies the matroidal Cayley–Bacharach property $MCB(a)$ if and only if the set-theoretic analogue of $MCB(a)$ is satisfied by the elements of building closure $\widehat{B}$ of B. By the "set-theoretic analogue", we mean the matroidal Cayley–Bacharach condition holds with the flats replaced by elements of the building closure (Definition 5.1.29).*

2. *If B is a building set, then the matroid M satisfies the matroidal Cayley–Bacharach property $MCB(a)$ if and only if the set-theoretic analogue of $MCB(a)$ (Definition 5.1.29) is satisfied by the subsets $I \subset [n]$ such that $y_I > 0$.*

We end with some constructions which use (directed) graphs to (recursively) determine what the sets involved would look like (Proposition 5.1.30, Proposition 5.1.31, Proposition 5.1.33).

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5.1.2 Ranks of flats covering a matroid satisfying $MCB(a)$

Given a fixed positive integer a, we show that there are no nontrivial bounds on the dimensions of a proper flats covering a matroid satisfying $MCB(a)$. Let $n = |E|$ and $r = \operatorname{rank} M$. The "worst" possible situation is when the collection of flats considered must be hyperplanes, which are the flats of rank $r-1$. These occur when we consider paving matroids with appropriate initial parameters. Recall that a paving matroid is one where any set of size $\leq r-1$

is both independent and closed. In other words, a dependent set (equivalently a circuit since considering lower bound) must have size $\geq r$.

Since *any* subset of size $\leq r-1$ is also a flat, we want to eliminate these from consideration since having F_i equal to such a flat would automatically imply that M does *not* satisfy $MCB(a)$ since we can use repeated copies of the same flat. Given an upper bound B on the size of the hyperplanes, we can find a condition which implies that flats F_i such that $\bigcup_{i=1}^{a} F_i \supset E \setminus p$ for some p must have $|F_i| \geq r$. Note that this is really a condition on size of finite sets and doesn't have anything to do with the matroid structure.

Lemma 5.1.11. *Let $F_1, \ldots, F_a \subset E$ be a collection of subsets of E with $|F_i| \leq B$ for each $1 \leq i \leq a$.*

If $n - 1 - B(a-1) \geq r$, then

$$\left| \bigcup_{i=1}^{a} F_i \right| \geq n - 1 \Longrightarrow |F_i| \geq r \text{ for each } 1 \leq i \leq a.$$

If the F_i are proper flats of a paving matroid M with underlying set E and rank r, this implies that the F_i considered must be hyperplanes M (i.e. flats of rank $r-1$).

Proof. Suppose that $|F_1| \leq r - 1$. Then, we have that

$$\begin{aligned} n - 1 &\leq \left| \bigcup_{i=1}^{a} F_i \right| \\ &\leq |F_1| + \left| \bigcup_{i=2}^{a} F_i \right| \\ &\leq r - 1 + (a-1)B, \end{aligned}$$

which contradicts the assumption that $n - 1 - B(a-1) \geq r$.

$\square$

Before we study the covering question, we give an example of a matroid satisfying the degree a matroidal Cayley–Bacharach condition $MCB(a)$ which doesn't have a nontrivial bound on ranks of flats covering the underlying finite set E. The construction we will use involves paving matroids, which are defined below.

Definition 5.1.12. (p. 24 of [18])
A matroid M is **paving** if it has no circuits of size $\leq \operatorname{rank} M$. In particular, flats of rank $\leq r-2$ are always independent sets, where $r = \operatorname{rank} M$.

Definition 5.1.13. (p. 71 of [18])
Let k and m be integers with $k > 1$ and $m > 0$. Suppose that $\mathcal{T}$ is a collection $\{T_1, \ldots, T_k\}$ of subsets of a set E such that each member of $\mathcal{T}$ has $\geq m$ elements, and each m-element subset of E is contained in a unique member of $\mathcal{T}$. Such a set is called an **m-partition of E.**

Proposition 5.1.14. *(Proposition 2.1.24 on p. 71 of [18])*
If $\mathcal{T}$ is an m-partition $\{T_1, \ldots, T_k\}$ of a set E, then $\mathcal{T}$ is the set of hyperplanes of a paving matroid of rank $m+1$ on E. Moreover, for $r \geq 2$, the set of hyperplanes of every $\operatorname{rank} r$ *paving matroid on E is an $(r-1)$-partition of E.*

We can show that the first statement of Definition 5.1.4 does *not* hold (especially if we want a small number of linear subspaces) (Theorem 5.1.17). Note that *some* bound on the number of linear subspaces involved is needed since we can always end up with some collection of lines or planes if we use a sufficient number of flats in the cover. The example which we used (paving matroids with appropriate parameters) can be used to show that the first part of the question (i.e. the direct matroid-theoretic translation of Conjecture 5.1.1 in Part

2 of Question 5.1.4) also does *not* hold.

Theorem 5.1.15. *There is a matroid M satisfying $MCB(a)$ with ground set E ($n := |E|$) such that there is some d such that $n \leq (d+1)a+1$ but E* cannot *be covered by a union of flats of total rank d. In other words, we have that $\bigcup_{i=1}^{k} F_i = E \Longrightarrow \sum_{i=1}^{k} \operatorname{rank} F_i \geq d+1$.*

Proof. We will take $r_i \operatorname{rank} F_i \geq 2$ for each flat F_i. In this context, Part 1 of Question 5.1.4 can be rephrased as whether we can keep $\sum_{i=1}^{a} d_i = \sum_{i=1}^{a} r_i - a \leq d$. To construct a counterexample, it suffices to produce an M satisfying $MCB(a)$ such that $n \leq (d+1)a+1$ and *any* cover $\bigcup_{i=1}^{a} F_i = E$ has $\sum_{i=1}^{a} r_i > d+a$. Note that such an example suffices when we take some of the F_i to have rank 1 (i.e. that $r_i = 1$ for some i) since the required lower bound for $\sum_{i=1}^{a} r_i$ only get smaller.

We will construct a paving matroid M satisfying $MCB(a)$ with $n \leq (d+1)a+1$ such that any cover $\bigcup_{i=1}^{a} F_i = E$ has $\sum_{i=1}^{a} r_i > d+a$. Let $m+1$ be the rank of the paving matroid. By Proposition 2.1.24 on p. 71 of [18], the hyperplanes are given by elements of m-partitions of the ground set $E = [n]$. Since any set of size $\leq m-1$ is closed, any flats F_i involved in the $MCB(a)$ definition must be hyperplanes. In this setting, the condition $\sum_{i=1}^{a} r_i > d+a$ can be rewritten as $am > d+a \iff d < (m-1)a$. The condition $n \leq (d+1)a+1$ is equivalent to having $(d+1)a \geq n-1 \iff \frac{n-1}{a} \leq d+1$. Then, having both $n \leq (d+1)a+1$ and $\sum_{i=1}^{a} r_i > d+a$ simultaneously is equivalent to having $\frac{n-1}{a} < d < (m-1)a$. The existence of a d such that this is true is equivalent to having $\frac{n-1}{a} < (m-1)a \iff \frac{n-1}{a^2} < m-1$.

The arguments above are on the possible initial parameters. We still need to show that there is a paving matroid of rank $m+1$ with ground set $E = [n]$ satisfying $MCB(a)$ such that $\frac{n-1}{a^2} < m-1$. The idea is to consider a paving matroid where there is a collection of "big" hyperplanes (evenly) partitioning the ground set E and the remaining elements of the m-

partition being subsets of size m (i.e. subsets of size m with elements from at least 2 distinct blocks/big hyperplanes). It suffices to take the big hyperplanes to have size $\frac{n}{a}$ partitioning $E = [n]$ into a parts and $\frac{n}{a} \gg m$ since losing a single block means replacement by $\geq \frac{n}{am}$ small hyperplanes to fill in the resulting gap. This implies that $MCB(a)$ is satisfied since using any smaller number of big hyperplanes of size $\frac{n}{a}$. The condition $\frac{n}{a} \gg m$ is satisfied when $m = \frac{n}{a^{\frac{3}{2}}}$ when a is a cube and $a^{\frac{3}{2}}$ divides n. It suffices to consider n such that n is divisible by a^2 and a is a square. For sufficiently large n, m, a it is clear that we can both have $\frac{n}{a} \gg m$ and $\frac{n-1}{a^2} < m - 1$ if $m = \frac{n}{a^{\frac{3}{2}}}$.

$\square$

For Part 2 of Question 5.1.4, we would like to find a bound on possible hyperplanes involved in the cover defining the matroidal Cayley–Bacharach property $MCB(a)$ of degree a. Note that Theorem 5.1.15 implies that there is a negative answer when $N(M) = a$ (the degree used in the matroidal Cayley–Bacharach proeprty).

Example 5.1.16. Take an even number $B \geq 4$ with $B|n$ and $\frac{n}{B} < B$. We construct a rank 2 paving matroid satisfying $MCB(a)$ for $a \leq \min(\frac{n}{B} + \frac{B}{2} - 3, \frac{n-3}{B} + 1)$. By Lemma 5.1.11, the second term in the pair on the right hand side reduces the flats under consideration to hyperplanes. Note that paving matroids satisfying $MCB(a)$ must have the flats involved in covers by $\leq a$ distinct flats equal to hyperplanes if the flats here are proper. This uses the following characterization of paving matroids by possible subsets of the underlying set giving rise to hyperplanes.

Let $n = |E|$. We can set $m = 1$ above and make $T_1, \ldots, T_a$ equal to a subcollection of distinct subsets of $\{1, \ldots, n\}$ from the following families:

- $\frac{n}{B}$ subsets of size $|B|$ partitioning $\{1, \ldots, n\}$ into $\frac{n}{B}$ parts

- $\binom{B}{2}B^2$ subsets of size 2 consisting of pairs of points from distinct blocks of size B

This collection of subsets yields hyperplanes of a paving matroid M satisfying $MCB(a)$. If some subcollection of these subsets is missing an element of E, it is missing ≥ 2 elements. We first find collections of a subsets we can use so that $a \leq \frac{n}{B} + \frac{B}{2} - 3$. Since $|B| > 2$, the number of subsets used is minimized when we maximize the number of subsets of size B and minimize the number of size 2 used. Also, we use at most $\frac{n}{B} - 1$ of the subsets of size B and there are at least B elements of E left to fill using the collection of pairs. This would mean using $\frac{n}{B} - 1$ subsets of size B and at most $\frac{B}{2} - 2$ pairs. However, this would leave us with at least 4 missing elements. Using fewer subsets of size B and more pairs would mean that we would use too many (i.e. more than a) subsets. Thus, the matroid M is a rank 3 paving matroid satisfying $MCB(a)$ for $a \leq \frac{n}{B} + \frac{B}{2} - 3$. Examples where this procedure goes through is $n = 20$, $B = 5$, and $a = 4, 5$.

A generalization of Example 5.1.16 can be used to give a negative answer to Part 1 of Question 5.1.4 for a fixed length a. In the context of the comments below Question 5.1.4, the flats F_i have rank $\leq r - 1$.

Theorem 5.1.17. *Take an even number $B \geq 2m + 2$ with $B|n$ and $\frac{n}{B} < B$. Fix $k, a \leq \frac{n}{B} + \frac{B}{m} - 3$.*

1. *Let M be a matroid of rank $m + 1$ satisfying $MCB(a)$ with underlying set E of size n. Suppose that $F_1, \ldots, F_k$ is a k-tuple of flats covering E with each proper flat of size at most B. There is no nontrivial upper bound on the ranks of proper flats F_i which applies to all such matroids M satisfying $MCB(a)$. In other words, it is possible for all the flats F_i to be hyperplanes. Question 5.1.4 contains an explanation of why this indicates that there is no "nontrivial" bound.*

2. *In fact, the matroid from the proof of Part 1 gives a negative answer to Question 5.1.4 using the case $N \leq \frac{n}{B} + \frac{B}{m} - 3$. More specifically, there is no upper bound on the ranks of a collection of $\leq a$ flats which cover the underlying set E of a matroid of rank r satisfying $MCB(a)$.*

Proof. 1. Let $E = \{1, \ldots, n\}$ be the underlying set of the matroid. The statements above follow from adapting the argument used in Example 5.1.16 to subsets of size m and paving matroids of rank $m+1$ in place of subsets of size 2 and paving matroids of rank 3. Fix $B \geq 2m+2$ with $\frac{n}{B} < B$ If $k, a \leq \frac{n-m-1}{B} + 1$, then Lemma 5.1.11 shows that the flats under consideration must be hyperplanes and we are done. Suppose that this is *not* the case. Consider the paving matroid of rank $m+1$ with the following subsets of size $\geq m$ as hyperplanes:

- $\frac{n}{B}$ subsets of size $|B|$ partitioning $\{1, \ldots, n\}$ into parts
- $A_{m, \frac{n}{B}}$ subsets of size m, where $A_{m,u}$ denotes the number of ordered partitions of m into B distinct parts with at least 2 nonempty parts. This corresponds to m-tuples with points with elements coming from at least 2 different blocks of size B from the first bullet.

We claim that this collection of subsets yield the hyperplanes of a matroid M satisfying $MCB(a)$. In other words, we would like to show that a subscollection of a subsets *not* covering E is missing ≥ 2 elements. Note that we assumed that $a \leq \frac{n}{B} + \frac{B}{m} - 3$. Since $|B| > m$, the number of subsets is minimized when we maximize the number of subsets of size B and minimize the number of size m subsets. For a non-covering collections of subsets, we use at most $\frac{n}{B} - 1$ of the subsets of size B and there are at least B elements of E left to fill using the m-tuples of points. This would mean using $\frac{n}{B} - 1$ elements of size B and at most $\frac{B}{m} - 2$ pairs. However, this would leave us with at least $2m$ missing elements. Using a smaller number of subsets of size B and more m-elements would increase the number of subsets used by at least $\frac{B}{m}$. Thus, the matroid M we obtain is a rank $m+1$ paving matroid satisfying $MCB(a)$ for $a \leq \frac{n}{B} + \frac{B}{m} - 3$. Note that the same arguments that we have just used imply that at least $\frac{n}{B} - 1$ of the blocks of size B must be used.

2. The same reasoning as Part 1 applies since having $\leq a$ covering E would require all of them to be hyperplanes. This is because any flat of rank $\leq m-2$ would have size $\leq m-2$. The argument in Part 1 implies that we need at least $\frac{n}{B} - 1$ hyperplanes of size B in order to cover E with $\leq a$ flats. Since there are only $\leq \frac{B}{m} - 2$ available flats to use for the cover, any remaining flats (even when we use hyperplanes) do not have enough elements of E to cover the remaining elements of E not covered by the earlier $\frac{n}{B}$ hyperplanes of size B.

□

We can make some statements on which paving matroids yield hyperplanes compatible with the matroidal Cayley–Bacharach condition $MCB(a)$. They show that the restriction of the matroidal Cayley–Bacharach property to paving matroids has a recursive property and that some upper bound on a is necessary in order for $MCB(a)$ to hold for a paving matroid of a given rank.

Proposition 5.1.18.

1. *Let M be a paving matroid of rank $m+1$ with underlying set $E = \{1, \ldots, n\}$. Suppose that $|F| \leq B$ for all flats F of M. Fix $a \geq 3$. Suppose that M satisfies $MCB(a)$. Fix a hyperplane A of M. Let R be the paving matroid on $E \setminus A$ of rank $m+1$ with hyperplanes given by $H \cap (E \setminus A)$ for hyperplanes H of M containing $\geq m$ elements of $E \setminus A$. Then, R satisfies $MCB(a-1)$.*

2. *Fix an integer $m \geq 3$. If a is sufficiently large, there is a paving matroid M of rank $m+1$ with underlying set $E = \{1, \ldots, n\}$ that does* not *satisfy $MCB(a)$.*

Proof. 1. Suppose that R does *not* satisfy $MCB(a-1)$. Then, there are flats F_i of R such that $\bigcup_{i=1}^{a-1} F_i = (E \setminus A) \setminus p$ for some $p \in E \setminus A$. By definition, there are hyperplanes H_i of M such that $|H_i \cap (E \setminus A)| \geq m$ and $F_i = H_i \cap (E \setminus A)$. Consider the union of flats of M given by $A \cup \left(\bigcup_{i=1}^{a-1} H_i\right)$. Since $F_i = H_i \cap (E \setminus A)$, the only "new" elements added to A come from those of F_i. This means that $A \cup \left(\bigcup_{i=1}^{a-1} H_i\right) = E \setminus p$ and M does *not* satisfy $MCB(a)$.

2. Let A be a subset of E of size $\geq m$. The hyperplanes of a paving matroid M of rank $m+1$ with ground set E with A as a hyperplane split into the following categories (first, second, third cateogries):

   - **Type 1**: The hyperplane A itself
   - **Type 2**: Hyperplanes containing $\geq m$ elements of $E \setminus A$
   - **Type 3**: Hyperplanes containing v elements of A and w elements of $E \setminus A$, where $1 \leq v, w \leq m-1$

   The last category gives the rest of the hyperplanes since each m-tuple of points of E is contained in a *unique* hyperplane. This means that we avoid repeating m-tuples coming from the first and second category. The conditions listed in the last category are given by this reasoning.

   In the last category, hyperplanes with $m-1$ elements of $E \setminus A$ give rise to a partition of the subset of A not used by hyperplanes in the second category. This is because m-tuples cannot be repeated among different hyperplanes of M. If a hyperplane contains $m-2$ elements of $E \setminus A$, it contains ≥ 2 elements of A which do *not* appear among the elements of the partition given by the hyperplanes of M with $m-1$ elements of $E \setminus A$. In general, hyperplanes of M with $m-P$ elements of $E \setminus A$ have $\geq P$

elements of A which are have not appeared in hyperplanes using more elements of $E \setminus A$.

Consider a paving matroid M of rank $m+1$ with ground set E containing A as a hyperplane satisfying the following conditions:

- **Condition 1**: The Type 2 hyperplanes do *not* contain any elements of A. In other words, suppose that hyperplanes of the second type form an m-partition of $E \setminus A$.
- **Condition 2**: There is a collection of Type 3 hyperplanes (e.g. those with $m-1$ elements of $E \setminus A$) such that the union of the elements of elements of $E \setminus A$ from each hyperplane A has size $|E \setminus A| - 1$.

Given a paving matroid M satisfying both Condition 1 and Condition 2, let $\mathcal{C}$ be a collection of hyperplanes satisfying the properties listed in Condition 2. Taking the union of the hyperplanes in $\mathcal{C}$ with A, we obtain a union of hyperplanes of size $|E| - 1$. This implies that M does *not* satisfy $MCB(a)$ for $a \geq |\mathcal{C}| + 1$ since adding more hyperplanes either keeps the size of the union equal to $|E| - 1$ or makes it equal to E. The size is equal to $|E| - 1$ if we either keep repeating hyperplanes which were already used or only add new hyperplanes which do *not* contain the element left out by the union of the hyperplanes in $\mathcal{C}$ and A. Thus, it suffices to show that there is a paving matroid satisfying both Condition 1 and Condition 2.

As stated in the definition of Condition 1, we start by forming an m-partition of $E \setminus A$. Focusing on Type 3 hyperplanes with $m-1$ elements of $E \setminus A$, we find that we need to use *all* $(m-1)$-element subsets of $E \setminus A$ in order to account for m-tuples in E with $m-1$ elements of $E \setminus A$ since the hyperplanes in the second category (i.e. those with $\geq m$ elements of $E \setminus A$) do not contribute any m-tuples containing elements of

A. For the Type 3 hyperplanes, we take the hyperplanes to be the m-tuples which are *not* covered by the m-tuples contained in a hyperplane of Type 1 (i.e. the hyperplane A) or one of Type 2. The resulting paving matroid satisfies Condition 2 since we can choose the collection in the definition of Condition 2 to be the m-tuples contained in a fixed $(|E \setminus A| - 1)$-element subset of $E \setminus A$.

$\square$

5.1.3 Matroids satisfying $MCB(a)$

This section studies families of matroids satisfying $MCB(a)$ including "generic" cases and those arising from graphs.

Matroid polytopes and $MCB(a)$

In this section, we outline results on "generic" (connected) matroids satisfying the matroidal Cayley–Bacharach property $MCB(r)$ (of degree r) which can be determined set-theoretically (Theorem 5.1.25 and Theorem 5.1.21). We can translate this into properties of ranks of flats that cover the underlying set of such matroids (Corollary 5.1.28). Finally, we give some more concrete information on the structure of the sets involved (Proposition 5.1.30 and Proposition 5.1.31).

We will study matroids satisfying these properties via polytopes built out of them which are uniquely defined by the starting matroids.

Proposition 5.1.19. *(Feichtner–Sturmfels, Proposition 2.3 on p. 441 of [12])*
The matroid polytope P_M associated to a matroid M with an underlying set E of size n

(written as $[n] := \{1, \ldots, n\}$*) is*

$$P_M = \left\{ x \in \Delta : \sum_{i \in F} x_i \leq \operatorname{rank} F \textit{ for all flats } F \subset [n] \right\},$$

where $\Delta = n\Delta_E$*.*

Alternatively, this is the convex hull of vectors $e_B := \sum_{i \in B} e_i$ for bases B of the matroid M. Note that P_M is uniquely determined by M (Theorem 4.1 on p. 311 of [7]) and that this property has even been used to define a matroid in Definition 2.1 on p. 440 of [12]. Each of these can be taken to be a *signed* Minkowski sum of simplices.

Proposition 5.1.20. *(Ardila–Benedetti–Doker and Postnikov, Proposition 2.3 on p. 843 of [3] and Proposition 6.3 and Remark 6.4 on p. 17 – 18 of [21]) Any generalized permutohedron (e.g. matroid polytopes) has a decomposition as* ***signed Minkowski sums of simplices*** *with*

$$P_n(\{z_I\}) = \sum_{I \subset [n]} y_I \Delta_I,$$

where $y_I = \sum_{J \subset I} (-1)^{|I| - |J|} z_J$ *for each* $I \subset [n]$ *and* $z_I = \sum_{I \subset J} y_J$*.*

In addition, any such Minkowski sum gives a generalized permutohedron (Proposition 2.2.3 on p. 14 of [5]). The latter condition is equivalent to having the z_I *satisfy submodular inequalities equivalent to the definition of some rank function on a matroid (Theorem 2.21 on p. 13 of [5]).*

Observation 5.1.21. *For an open/generic/top-dimensional subset of the deformation cone parametrizing generalized permutohedra (i.e. deformations of the usual permutohedron), one we can take* $y_I \geq 0$ *for each* I *(Remark 6.4 on p. 1043 of [21]). The fact that* $y_I =$

$\sum_{J \subset I} (-1)^{|I|-|J|} z_J$ *for each* $I \subset [n]$ *implies that the condition* $y_I \geq 0$ *for each* $I \subset [n]$ *is really an inequality on the ranks* $r - z_I$ *of the flats.*

In the setting of Observation 5.1.21, we can make some characterizations of matroids satisfying the matroidal Cayley-Bacharach property. When the $y_I \geq 0$ for all $I \subset [n]$. the facets have natural connections with nested sets and buildings.

Definition 5.1.22. (Building set and closure, Lemma 3.9 and Lemma 3.10 on p. 450 of [12], Definition 7.1 on p. 1044 of [21])

1. A collection B of nonempty subsets of $[n] = \{1, \ldots, n\}$ is a **building set** on $[n]$ if it satisfies the following conditions:

    - If $I, J \in B$ and $I \cap J \neq \emptyset$, then $I \cup J \in B$.
    - B contains all singletons $\{i\}$ for $i \in [n]$.

2. Given a collection $\mathcal{F}$ of subsets of $[n]$, let $\widehat{\mathcal{F}}$ be the unique minimal collection containing $\mathcal{F}$ of subsets such that $\widehat{\mathcal{F}}$ is a building set on $[n]$. The collection $\widehat{\mathcal{F}}$ is called the **building closure**. Note that this exists for *any* family of subsets $\mathcal{F}$ of $[n]$.

These properties are connected to an alternate description of the facets of the matroid polytope P_M when the generic property described in Observation 5.1.21 holds (i.e. when $y_I \geq 0$ for all $i \subset [n]$).

Proposition 5.1.23. *(Proposition 3.12 and Corollary 3.13 on p. 151 of [12])*
Given a Minkowski sum of (scaled) simplices $\sum_{I \subset [n]} y_I \Delta_I$ *for some* $y_I \geq 0$*, let* B *be the collection of subsets* $I \subset [n]$ *such that* $y_I > 0$*. The polytope* $\sum_{I \subset [n]} y_I \Delta_I$ *consists of vectors* $(x_1, \ldots, x_n) \in \mathbb{R}^n_{\geq 0}$ *such that* $x_1 + \ldots + x_n = |B|$ *and*

$$\sum_{i \in G} x_i \geq |\{I \in B : I \subset G\}|$$

for all subsets $G \subset [n]$*.*

It suffices to take subsets G *in the building closure* $\widehat{B}$ *of* B*. If* $[n] \in B$*, the condition that the linear form* $\sum_{i \in G} x_i$ *is minimized on a facet is* equivalent *to* G *being in the building closure* $\widehat{B}$*.*

Under an additional connectivity assumption, this is entirely determined by set-theoretic considerations corresponding to the ranks of the flats of the given matroid.

Definition 5.1.24. (p. 457 of [12], p. 183 of [10])
Given flats F, G of M with $F \subset G$, the subsets

$$M[F, G] := \{b \cap (G \setminus F) : b \in M, |b \cap F| = \operatorname{rank} F, |b \cap G| = \operatorname{rank} G\}$$

of the underlying set define a matroid with ground set $G \setminus F$.

We now state a structural result connecting the matroid polytope with the matroidal Cayley–Bacharach condition $MCB(a)$.

Theorem 5.1.25. *Suppose that* M *is a connected matroid satisfying the following conditions:*

- $M[F, G]$ *is connected for all flats* F, G *such that* $F \subset G$ *or every flat* A *of* M *is both connected and coconnected. For example, consider the graphic matroid* $M(K_n)$ *of spanning trees in the complete graph* K_n *(Remark 5.4 on p. 459 of [12]).*

- $P_M = \sum_{I \subset [n]} y_I \Delta_I$ *for some* $y_I \geq 0$ *such that* $y_{[n]} > 0$*. As mentioned in Observation 5.1.21, the condition here is really one on the ranks of the flats since* $z_I = \sum_{J \subset I} y_J$ *and* $y_I = \sum_{J \subset I} (-1)^{|I| - |J|} z_J$*, where* $z_I = r - \operatorname{rank}(\operatorname{Span} I)$ *with* $r = \operatorname{rank} M$ *and* Span I *being the smallest flat containing the elements of* I *(Proposition 2.2 and Proposition 2.3 on p. 843 of [3]).*

Let B be the collection of subsets $I \subset [n]$ such that $y_I > 0$. Then, the following statements hold:

1. *The matroid M satisfies the matroidal Cayley–Bacharach property $MCB(a)$ if and only if the set-theoretic analogue of $MCB(a)$ is satisfied by the elements of building closure $\widehat{B}$ of B. By the "set-theoretic analogue", we mean the matroidal Cayley–Bacharach condition holds with the flats replaced by elements of the building closure (Definition 5.1.29).*

2. *If B is a building set, then the matroid M satisfies the matroidal Cayley–Bacharach property $MCB(a)$ if and only if the set-theoretic analogue of $MCB(a)$ (Definition 5.1.29) is satisfied by the subsets $I \subset [n]$ such that $y_I > 0$.*

Remark 5.1.26. If we remove the initial (co)connectivity assumption, counterparts of Part 1 and Part 2 hold with the matroidal Cayley–Bacharach property replaced by its restriction to flats which define facets of the matroid polytope P_M ("flacets" in Proposition 2.6 on p. 443 of [12]).

Proof. Since $\widehat{B} = B$ if B is a building set, Part 2 follows from Part 1. Thus, it suffices to prove Part 1. If $M[F, G]$ is connected for all flats $F \subset G$, the hyperplanes giving the boundary of the half-spaces $\sum_{i \in F} x_i \le \operatorname{rank} F$ for each flat F each yield facets of the matroid polytope P_M. This is because there is an an equivalence between complexes whose vertices correspond to *all* connected flats and those that yield facets of the matroid polytope respectively (Theorem 5.3 on p. 459 of [12])). Alternatively, one can assume that each flat is both connected and co-connected (Proposition 2.4 on p. 184 of [10]).

The observations made above imply that each of the flats of M define a facet of its matroid polytope P_M. By Proposition 5.1.23, the decomposition into a Minkowski sum $\sum_{I \subset [n]} y_I \Delta_I$ with $y_I \ge 0$ and $y_{[n]} > 0$ implies that elements of the building closure $\widehat{B}$ correspond to facets of P_M. Tracing through the correspondences, we find that that the flats of M are given by subsets of $[n]$ in the building closure $\widehat{B}$. Comparing the corresponding normal vectors implies that the matroidal Cayley–Bacharach condition $MCB(a)$ is equivalent to its set-theoretic counterpart applied to the elements of the building closure (Definition 5.1.29).

□

Even without the connectedness assumption of Theorem 5.1.25, we can still characterize "generic" polytopes coming from those satisfying $MCB(r)$ up to a deformations of the the matroid polytopes involved. By "deformation", we mean parallel translations of facets passing through the vertices (p. 1041 of [21]). An example is shown in Figure 2 on p. 1979 of [3].

Theorem 5.1.27. *Suppose that* $M = \sum_{I \subset [n]} y_I \Delta_I$ *for some* $y_I \geq 0$.

Then, the existence of a matroid N satsifying the following conditions can be checked using a set-theoretic condition involving $(n-1)$-element subsets of $[n]$ or the sets I:

- *N has an underlying set of the same size $n = |E|$ such that the flats inducing facets of P_N satisfy the conditions $MCB(a)$*
- *P_N and the matroid polytope of P_M are nondegenerate deformations of each other (i.e. those not passing through vertices)*

Proof. By Proposition 2.6 on p. 1980 of [3], it suffices to check when the matroid polytopes have the same normal fan. In the comparisons of normal cones, note that two collections of vectors define the same cone if and only if they can be transformed to each other using weighted permutation matrices. Since M is a Minkowski sum of the simplices Δ_I, its normal fan is the common refinement of those of the simplices Δ_I. Since the normal fans consist of cones generated by the (outer) normal vectors of the facets, it suffices to find the facets of a matroid polytope P_N. On the other hand, the facets of P_N consist of those coming from flats of $[n]$ and those from $(n-1)$-element subsets of the ground set N (Proposition 2.3 on p. 441 of [12] and p. 930 of [8]). The $(n-1)$-element subsets don't affect the $MCB(a)$ condition and the only ones inducing a nontrivial condition involving the flats is the restriction of the $MCB(a)$ condition to the flats which induce facets of the matroid polytope.

□

Finally, we discuss the implications of Theorem 5.1.25 and Theorem 5.1.27 for a question of Levinson–Ullery (Question 7.6 on p. 14 of [17]) for possible ranks of flats satisfying the $MCB(r)$ property.

Corollary 5.1.28. *Under the conditions of Part 2 of Theorem 5.1.25, the possible sizes of $|I|$ from subsets $I \subset [n]$ with $y_I > 0$ in Theorem 5.1.25 and Theorem 5.1.27 determine the possible ranks of flats covering the underlying set of a matroid M in Theorem 5.1.25 and the matroid N we "deform" into in Theorem 5.1.27. This essentially addresses Question 7.6 on p. 14 of [17] for the generic (connected) matroids discussed in these results.*

We end with some comments on the sets involved.

Definition 5.1.29. Let E be a finite set of size n. A collection of subsets of E satisfies the **set-theoretic matroidal Cayley–Bacharach property** $sMCB(r)$ if $\bigcup_{i=1}^r F_i \supset E \setminus p \Longrightarrow \bigcup_{i=1}^r F_i = E$ for the given collection of proper subsets $F_1, \ldots, F_r$ of E and $p \in E$.

Note that the condition does *not* impose a restriction on r-tuples of subsets $F_1, \ldots, F_r$ such that $|\bigcup_{i=1}^r F_i| \leq n-2$ since it is not possible for these to contain $E \setminus p$ for any $p \in E$. We can understand possible underlying sets of subsets of E satsifying $sMCB(r)$ recursively where the condition is nontrivial. This depends on a counting argument.

Proposition 5.1.30. *The subsets of E satsifying $sMCB(r)$ can be determined recursively using minimal covers of subsets of E.*

Proof. The subsets $F_1, \ldots, F_r$ satisfying the $sMCB(r)$ property depends on the following parts:

- A collection of "ambient sets" $A \subset E$ of size $\leq n-2$ or n (which will eventually be taken to be the union of $F_1, \ldots, F_r$)
- Subsets $F_1, \ldots, F_r \subset A$ such that $\bigcup_{i=1}^{r} F_i = A$. This really depends on the number of subsets F_i used in a *minimal* cover of A (say $m \leq r$) since the remaining $r-m$ subsets can be any subsets of A and still give a cover of A. By "minimal", we mean that removing any of the F_i will give a collection of subsets of A whose union is no longer equal to A. Thus, it suffices to consider the *minimal* covers of A by $\leq r$ subsets.

  This can be constructed recursively. Let $T_{a,b}$ be the number of minimal covers of a set of size a by a collection of b subsets (will take $b \leq r$ in this case). We can split this into cases depending on the number of elements *not* covered by a collection of $b-1$ subsets. For particular number of missing elements r, we set the union of the $b-1$ subsets equal to a particular subset of A with $|A|-r$ elements. There are $\binom{|A|}{|A|-r}$ choices for such a subset. Fixing a subset $U \subset A$ with $|A|-r$ elements, we have that F_r can be any subset of A containing the remaining r elements (giving 2^{n-r} choices) and the number of choices of (unordered) collections of (nonempty) subsets $F_1, \ldots, F_{r-1}$ of A whose union is equal to U is $T_{|A|-r,b-1}$. This gives us the recursive relation

  $$T_{a,b} = \sum_{r=1}^{a-1} \binom{a}{a-r} 2^{n-r} T_{a-r,b-1}.$$

  If we know $T_{u,b-1}$ for all u, then we can compute $T_{a,b}$. In other words, we can treat this as induction on the second index b (eventually setting $b = r$). As base cases, we can use $T_{a,1} = 1$ (single subset equal to A). If $b = 2$, this means choosing a (nonempty) subset of A and having the second set contain its complement. For each m, there are $\binom{a}{m}$ choices of m-element subsets of A and 2^{a-m} choices for subsets of A containing the complement of the first subset in A. This means that $T_{a,2} = \sum_{m=1}^{a} \binom{a}{m} 2^{a-m}$.

It may also be possible to relate this to *disjoint* covers by some collection of elements. After a disjoint cover, we can add whatever elements of A we want to each of the subsets $F_1, \ldots, F_r$ involved (possibly adding nothing to one or more subsets). After choosing a disjoint cover, this is a matter of choosing any r (possibly empty) subsets of A (which gives $(2^a)^r = 2^{ar}$ choices). By Proposition 2.6 on p. 1032 – 1033 of [21], the disjoint covers of A by r elements correspond to the $(a - r)$-dimensional faces of the permutohedron $P_a(x_1, \ldots, x_a)$ (for some choice of fixed $x_1 > \cdots > x_a$) formed by the convex hull of the points formed by permuting the coordinates of the point $(x_1, \ldots, x_a)$.

□

Proposition 5.1.31.

Suppose that M is a matroid such that any r-tuple of flats $F_1, \ldots, F_r$ satsifies the following property: For any p, there is an x_p such that $p \notin F_i \Longrightarrow x_p \notin F_i$.

Then, the matroid M satsifies the matroid Cayley–Bacharach property $MCB(r)$. Also, the flats F_i must come from path covers of some directed graph with the paths being maximal among those sharing the same starting point. The lengths of maximal paths bound the ranks of the flats involved. The structure of the graph also gives an upper bound on the number of points involved.

Proof. A special case where $sMCB(r)$ is satisfied is the case where p *not* being contained in a subset $F_i \subset E$ among $F_1, \ldots, F_r$ means that there is some $x_p \in E$ such that $x_p \notin F_i$. This is equivalent to the statement that $x_p \in F_i \Longrightarrow p \in F_i$. Note that the choice of x_p might not necessarily be unique. Then, we can build a directed graph with an edge $i \longrightarrow j$ if and only if we can set $i = x_j$. Since the $sMCB(r)$ condition is *not* affected by situations where $|\bigcup_{i=1}^r F_i| \leq n-2$, we will restrict ourselves to the situation where $\bigcup_{i=1}^r F_i = E$. This means that the graphs under consideration are those that involve *all* the elements of $\{1, \ldots, n\}$.

Each vertex corresponds ot an element of E. Note that the directed graphs which arise aren't completely arbirtrary. Split the graph into connected components of the underlying undirected graph. Fix a maximal directed path going in one direction. Then, any remaining vertices (which correspond to elements of E) must come from paths that *enter* the maximal directed path at a vertex whcih is *not* an endpoint since joining the new paths at such points would contradict the maximality assumption. Given a particular possible connected graph, the subsets F_i of E must come from paths which keep going until we encounter a loop. In other words, we are looking for paths which are maximal among those with the same starting point. This means that $sMCB(r)$ is equivalent to determining possible covers of directed graphs by such paths. As a consequence of this construction, we find that a particular graph gives upper bounds for the ranks of a collection of r flats which cover E. $\square$

Special case of graphs

We consider the case where the sets in question are disjoint and M is a graphic matroid. Since the flats of a direct sum of matroids $M_1 \oplus \cdots \oplus M_r$ with disjoint underlying sets $E_1, \ldots, E_r$ are of the form $F_1 \cup \cdots \cup F_r$ for flats F_i of M_i (p. 125 of [18]), we can think about this as the case where $M_1 = \cdots = M_r = M$ for some matroid M. Note that the underlying set of the matroid $M_1 \cup \cdots \cup M_r$ is $E_1 \cup \cdots \cup E_r$, where E_i is the underlying set of M_i. For $M^{\oplus r}$, this means taking the copies of the underlying set E of M to be disjoint from each other. When the direct sum is a graphic matroid, the flats and closures have a simple interpretation.

A result of Lovász–Recski [10] indicates when a repeated direct sum is a graphic matroid.

Theorem 5.1.32. *(Theorem 2 on p. 332 of [10])*
*Given a matroid M with underlying set S, we call it a k-**circuit** if $|S| = kr(S) + 1$ and $|T| = kr(T)$ for all $T \subset S$. A repeated matroid direct sum $M^{\oplus k}$ is a graphic matroid if and only if any two k-circuits of (S, M) are disjoint.*

We can interpret a union of r flats of a matroid as a single face F of $M^{\oplus r}$. Let E^i be the copy of E in the i^{th} copy M_i of M. Let A be a subset of $E^1 \cup \cdots \cup E^r$ such that removing the labels i gives the full subset E. This corresponds to some disjoint union of r sets $A_1, \ldots, A_r \subset E$ whose union is equal to E. Then, a variant of $MCB(r)$ can be phrased as the statement that $F \supset A \setminus \{p\} \Longrightarrow F \supset A$. Since flats are the sets preserved under the closure operation, the first statement implies that $F \supset \overline{E \setminus \{p\}}$. If $\overline{A \setminus \{p\}} = A$, then we have the desired conclusion. Under the conditions of Theorem 5.1.32, the resulting direct sum matroid is a graph. Then, the condition that $\overline{A \setminus \{p\}} = A$ is equivalent to the statement that for any $p \in A$, the endpoints of p are connected by a path in $A \setminus \{p\}$. In other words, the subgraph of $M^{\oplus r}$ induced by A is 2-connected.

The reason why we stated that the above is a "variant" is that the "distribution" of E over the different flats $F_1, \ldots, F_r$ in the $MCB(r)$ condition can vary. This means that we need to have the condition satisfied for *all* possible A satisfying the given condition. Also, note that the $MCB(r)$ condition is satisfied if the defining statement holds for all *minimal* flats $F_1, \ldots, F_r$. Then, flats that are minimal under inclusion among those which can be used to give a union of r flats covering $E \setminus \{p\}$ for $p \in E$. Putting everything together, the observations above can be summarized as follows:

Proposition 5.1.33. *Let M be a graphic matroid with underlying set E such that any two r-circuits of M are disjoint and r-tuples of flats which are minimal among those covering single point complements $E \setminus \{p\}$ are disjoint. Then, the degree r matroidal Cayley–Bacharach condition is equivalent to the statement that the subgraph of $M^{\oplus r}$ induced by any subset $A \subset E^1 \cup \cdots \cup E^r$ giving a partition of E as the disjoint union of r subsets yields a 2-connected subgraph of $M^{\oplus r}$.*

Remark 5.1.34.

1. Since the objects used to define a matroid are often analogous to those used to define topological spaces, we can study what statements can be extended to higher dimensional objects. If we continue the assumption that the minimal r-covers by flats are disjoint, the exact argument above applies. If we remove this disjointness condition and only consider the matroid M itself instead of disjoint sums, we need to consider unions of r flats, which aren't necessarily flats. This complicates the argument above.

2. If the definition of a matroid also defines a topological space (e.g. the case of uniform matroids $U_{n,n}$), we have that $\bigcup_{i=1}^r F_i$ is a flat if $F_1, \ldots, F_r$ are flats. This means that $\bigcup_{i=1}^r F_i \supset E \setminus \{p\} \Longrightarrow \overline{\bigcup_{i=1}^r F_i} \supset \overline{E \setminus \{p\}}$. Since we're working with the uniform matroid $U_{n,n}$, we have that $\overline{\bigcup_{i=1}^r F_i} = \bigcup_{i=1}^r F_i$ and $\bigcup_{i=1}^r F_i \supset \overline{E \setminus \{p\}}$. Then, it suffices to show that $\overline{E \setminus \{p\}} = E$. This is *not* the case if and only if $\overline{E \setminus \{p\}} = E \setminus \{p\}$. In particular, this means that $\text{rank}(E) = \text{rank}(E \setminus \{p\}) + 1$ and $E \setminus \{p\}$ is a hyperplane of M. If $A \subset E \setminus \{p\}$, then it cannot be a basis element of M since it must be of maximal

rank (i.e. has rank $\text{rank}(E)$). This means that any basis element must be of the form $R \cup \{p\}$ for some $R \subset E \setminus \{p\}$. Then, we have that $\text{rank}(R \cup \{p\}) \leq \text{rank}(E \setminus \{p\}) + 1$.

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5.2 Matroidal Cayley–Bacharach and independence/dependence of geometric properties of matroids

We consider the relationship between a matroidal analogue of the degree a Cayley-Bacharach property (finite sets of points failing to impose independent conditions on degree a hypersurfaces) and geometric properties of matroids. If the matroid polytopes in question are nestohedra, we show that the minimal degree matroidal Cayley-Bacharach property denoted $MCB(a)$ is determined by the structure of the building sets used to construct them. This analysis also applies for other degrees a. Also, it does not seem to affect the combinatorial equivalence class of the matroid polytope.

However, there are close connections to minimal nontrivial degrees a and the geometry of the matroids in question for paving matroids (which are conjecturally generic among matroids of a given rank) and matroids constructed out of supersolvable hyperplane arrangements. The case of paving matroids is still related to with properties of building sets since it is closely connected to (Hilbert series of) Chow rings of matroids, which are combinatorial models of the cohomology of wonderful compactifications. Finally, our analysis of supersolvable line and hyperplane arrangements give a family of matroids which are natrually related to independence conditions imposed by points one plane curves or can be analyzed recursively.

5.2.1 Introduction

Motivated by recent rationality-related results, Levinson and Ullery [17] recently defined the degree r Cayley–Bacharach property $CB(r)$ of finite sets $\Gamma \subset \mathbb{P}^n$ to mean ones that fail to impose independent conditions on the space of degree r homogeneous polynomials. For a family of cases, they show that such a set Γ lies on a union of low-dimensional linear subspaces)Theorem 1.3 on p. 2 of [17]. In Question 7.6 on p. 14 of [17], they asked whether a matroidal analogue of their result holds. In [19], we show that this does not hold (Theorem 1.6 and Theorem 1.8 on p. 4 of [19]) and explore combinatorial criteria for $MCB(a)$ to hold.

We consider different directions where the matroidal Cayley-Bacharach condition $MCB(a)$ is independent of or dependent on the geometry of the matroids involved or the objects they are constructed from. For minimal a, we analyze how $MCB(a)$ relates to properties of building sets used to construct the nestohedron (Theorem 5.2.4). However, this result also shows that there the $MCB(a)$ property does not measure a form of combinatorial equivalence for matroid polytopes which form nestohedra. We can consider what happens in a (conjecturally) generic setting among matroids of a given rank using paving matroids. In this setting, we study cases where the minimal degree a where $MCB(a)$ is satisfied nontrivial is small (Theorem 5.2.7) and show that lowering such a correlates to larger degree terms in the Hilbert series of the Chow ring of the matroid (Corollary 5.2.8), which is a combinatorial model for the cohomology of wonderful compactifications.

Finally, we use supersolvable arrangements of linear subspaces to find compare minimal degrees a where $MCB(a)$ can be satisfied with degrees D where a collection of points fail to impose independent conditions on plane curves of degree D (Proposition 5.2.12) and a recursive argument for the $MCB(a)$ property on supersolvable hyperplane arrangements (Proposition 5.2.19). Note that the case of line arrangements gives a family of matroids other than the representable case which naturally parametrizes questions about independence of conditions imposed by points on hypersurfaces. This result also shows that $MCB(a)$ properties of supersolvable hyperplane arrangements can be analyzed recursively and that the flats satisfy special clustering properties.

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5.2.2 Independence from geometry and nestohedra

Based on the matroidal Cayley-Bacharach property $MCB(a)$ of degree a, we define $MCB(a)$ for a building set B (Definition 7.1 on p. 1044 of [21]).The motivation/connection to the "usual" $MCB(a)$ property (Question 7.6 on p. 14 of [17]) comes from the fact that the half-space description of matroid polytopes is determined by flats of the matroid (Proposition 2.3 on p. 441 of [12]).

Definition 5.2.1. (Levinson–Ullery, p. 14 of [17], p. 2 of [19])
A matroid M with underlying set E satisfies the **matroidal Cayley-Bacharach property of degree** a if $\bigcup_{i=1}^{a} F_i \supset E \setminus p \Longrightarrow \bigcup_{i=1}^{a} F_i = E$ for any $p \in E$ and flats $F_1, \ldots, F_a$ of M.

Definition 5.2.2. Let $[n] = \{1 \ldots, n\}$ A building set B (Definition 7.1 on p. 1044 of [21]) **satisfies MCB(a)** if $\bigcup_{i=1}^{a} I_i \supset [n] \setminus \{k\} \Longrightarrow \bigcup_{i=1}^{a} I_i = [n]$ for all $k \in [n]$.

In the case of nestohedra constructed out of connected building sets containing the ground set $[n]$, this is identical to the original matroidal Cayley–Bacharch property since the facets correspond to maximal elements of $B \setminus [n]$ (Proposition 3.12, Corollary 3.13, and Theorem 3.14 on p. 451 – 452 of [12]).

We can use this to show that the $MCB(a)$ is "independent" of combinatorial equivalence properties of matroids whose polytopes are nestohedra.

Definition 5.2.3. (p. 450 of [12] Definition 7.1 on p. 1044 of [21], Proposition 7.5 on p. 1046 of [21])

1. Given a family $\mathcal{F}$ of subsets of $[n]$, we associated the following Minkowski sum of simplices

$$\Delta_{\mathcal{F}} = \sum_{F \in \mathcal{F}} \Delta_F.$$

2. A collection B of nonempty subsets in S is a **building set on** S if it satisfies the following conditions:

   - If $I, J \in B$ and $I \cap J \neq \emptyset$, then $I \cup J \in B$.
   - B contains all singletons $\{i\}$ for $i \in S$.

3. A **nestohedron** is a polytope from Part 1 where $\mathcal{F}$ is a building set.

Theorem 5.2.4.

1. *The minimal degree a such that a nestohedron $P = \sum_{I \in B} \Delta_I$ constructed from a connected building set B on $[n] = \{1, \ldots, n\}$ can satisfy $MCB(a)$ nontrivially is given by $n - \dim P$. This is satisfied if and only if each maximal element $I \in B_{\max} \subset B \setminus [n] \subset 2^{[n]}$ has ≥ 2 subsets which are maximal among those contained in I. Finally, it is the only degree where this is possible.*

2. *In Part 1, the degree a is given by $n-c$, where c is the number of connected components of the nestohedron built out of $B \setminus [n]$. If this latter polytope is a matroid polytope P_M for some matroid M, it is also the equal to $n - c(M)$, where $c(M)$ is the number of connected components of M.*

3. *For nestonedra, the $MCB(a)$ property is not a combinatorial invariant. In other words, there are matroids which yield combinatorially equivalent matroid polytopes where one satisfies $MCB(a)$ for some a and the other does not.*

Remark 5.2.5. By Lemma 3.10 on p. 450 of [12], *any* collection of subsets of $[n]$ has a unique minimal extension which is a building set called the **building set closure**. The statements above and arguments used below can be repeated with the building sets replaced by building set closures.

Proof. 1. There is a correspondence between nested sets $N \subset B \subset 2^{[n]}$ and faces of a generalized permutohedron (Proposition 7.5 on p. 1046 of [21], proof of Proposition 7.9 on p. 1048 of [21], Proposition 3.12, Corollary 3.13, and Theorem 3.14 on p. 451 – 452 of [12]). In this correspondence, the facets are parametrized by elements $I \in N \subset 2^{[n]}$ for nested sets N. This is the comes from the same reasoning which shows that flats give the half-space description of a matroid polytope P_M. Next, we use the fact that maximal nested sets correspond to B-forests (Proposition 7.8 on p. 1048 of [21]). Note that any subcollection of a nested set $N \subset 2^{[n]}$ containing the elements of $B_{\max}$ (i.e. maximal elements) is still a nested set and that maximal subcollections is still a nested set. Also, any nested set is contained in a unique maximal building set.

The number of minimal nested subsets (i.e. $T_{\leq i}$ for nodes i) show that the unions of smaller nested sets are missing ≥ 1 subset for each case if and only if there are ≥ 2 "almost maximal" subsets. Finally, we use the fact that $\dim P = n - |B_{\max}|$. The degree a is the only one allowed since allowing larger degrees would include cases where two "almost maximal" building sets are used in place of a single maximal building set (a degree $a + 1$ case).

2. This is an application of Proposition 2.4 on p. 442 and Remark 3.11 on p. 450 of [12].

3. Using Part 1, we see that $MCB(a)$ depends on the number of maximal elements in the building set. Note that $MCB(a)$ cannot be satisfied when the facets do not come from maximal elements of the building sets since they come from those of the a dilation of standard simplex on $\mathbb{R}^n$. However, *any* nestohedron is combinatorially equivalent one from a connected building set even if the facets do not necessarily come from maximal elements of the building sets (Corollary 5 on p. 189 of [22], p. 122 of [11]). Then, the situation in Part 1 applies. For degrees a less than the number of building sets maximal among those excluding $[m]$ (the ground set of the new connected building set),

the $MCB(a)$ property is trivially satisfied. However, this is not true for the original one even when $a = 1$. Both cases come from combinatorially equivalent polytopes.

$\square$

5.2.3 Geometry determined by $MCB(a)$

Paving matroids

We will focus on the case of paving matroids (which are conjectured to make up almost all matroids while known logarithmically [20]) and their geometric structure. More specifically, we will explore connections to the Chow rings of these matroids. Note that these rings are still connected to properties of building sets since they are a combinatorial model for the cohomology of wonderful compactifications, which are built out of building sets.

In the case of a paving matroid of rank $m+1$ with ground set $E = [n] = \{1, \ldots, n\}$, the hyperplanes are given by m-partitions of $[n]$ (Proposition 2.1.24 on p. 71 of [18]). These are collections of subsets of E such that any m-element subset of E is contained in a unique member of this collection. If we take $F_1, \ldots, F_a$ in the definition of $MCB(a)$ to be *any* collection of flats (possibly with repeats), we need to consider *minimal* numbers of flats which cover all but possibly one element of E.

This means that we look at the unions of the smallest number of hyperplanes covering E. Depending on how small the degree a is and how much the sizes of the hyperplanes varies, it may be easier or more difficult to build a paving matroid of rank $m+1$ satisfying $MCB(a)$. For example, setting a large lower bound for hyperplanes which do *not* have size m implies that a lower degree a can be satisfied since a larger gap is left behind by removing a large hyperplane. Note that *any* collection of covering sets to be used in the definition of $MCB(a)$ has an associated paving matroid M of rank $m+1$ with these "covering sets"

F_i as a subcollection of the hyperplanes of M (Proposition 5.2.7). Also, this corresponds to lengths of chains/number of terms used and the sizes of the coefficients in the Hilbert function of the Chow ring of the matroid (Corollary 2 and comments on p. 525 – 526 of [13]) (Corollary 5.2.8). Generalizations to arbitrary matroids satisfying $MCB(a)$ are outlined in Remark 5.2.9.

Proposition 5.2.6. *If a paving matroid M with ground set $E = [n] = \{1, \ldots n\}$ satisfies $MCB(b)$ for some b, then it must satisfy $MCB(a)$ with a equal to the smallest number of hyperplanes that can cover E.*

Proof. Since we are allowed to repeat flats, any matroid failing to satisfy $MCB(a)$ will *not* satisfy $MCB(b)$ for any $b \geq a$. The minimal possible degree where this occurs is the smallest number of hyperplanes that can cover E. $\square$

Theorem 5.2.7.

1. *Let M be a paving matroid of rank $m+1$ such that the largest k hyperplanes $H_1, \ldots, H_k$ of M form a cover of the ground set $E = [n] = \{1, \ldots, n\}$. Consider a family of such paving matroids.*

   Suppose that there is a constant $C \in \mathbb{Z}_{>0}$ such that $\frac{\max |H_i|}{\min |H_i|} < C$. If $\frac{n}{Ck^2(m-1)} >> k$ (i.e. $\frac{n}{Ck^3(m-1)} \to \infty$ as $m \to \infty$ treating the variables as functions of m), then M satisfies $MCB(a)$ for $a \leq k - 1 + \frac{n}{2Ck^2(m-1)}$. In general, this is true whenever $\min |H_i| >> k(m-1)$.

2. *Consider paving matroids M such that the largest k hyperplanes $H_1, \ldots, H_k$ form a cover of the ground set $[n]$ as in Part 1. If $a < 1 + (k-1)\frac{\min |H_i|}{k(m-1)}$, then M satisfies $MCB(a)$.*

Proof. 1. The assumption that $\frac{\max|H_i|}{\min|H_i|} < C$ implies that the sizes of the H_i don't vary much. We would like to use the bound $\frac{n}{Ck^2(m-1)} >> k$ to show that the only flats to consider for the $MCB(a)$ condition are the maximal hyperplanes H_i covering the ground set $[n]$.

Note that $\max|H_i| \geq \frac{n}{k}$ since $H_1, \ldots, H_k$ cover $[n]$,. Since $\frac{\max|H_i|}{\min|H_i|} < C$, we have that $\min|H_i| > \frac{\max|H_i|}{C} \geq \frac{n}{Ck}$. We can use this to study covers of $[n]$ by hyperplanes of the paving matroid. Given that any hyperplane is contained in a maximal hyperplane, the size of the next largest hyperplane after the first k is $\leq k(m-1)$. This means that removing a single H_i and attempting to cover $\geq |H_i| - 1$ elements with smaller hyperplanes would require $\geq \frac{|H_i|}{k(m-1)}$ new hyperplanes. Since $\frac{\min|H_i|}{k(m-1)} \geq \frac{n}{Ck}$, the number of additional hyperplanes required is $\geq \frac{n}{Ck^2(m-1)}$. Removing more of the H_i would give an even greater increase in the number of hyperplanes used. This means that the only cover of $\geq n-1$ elements of $[n]$ by $a \leq k-1+\frac{n}{2Ck^2(m-1)} < k-1+\frac{n}{Ck^2(m-1)}$, hyperplanes is the cover of $[n]$ by $H_1, \ldots, H_k$. Thus, $MCB(a)$ must be satisfied by M for $a \leq k-1+\frac{n}{2Ck^2(m-1)}$.

2. We can use similar reasoning as in Part 1. In general, removing ℓ of the H_i and replacing them with hyperplanes *not* belonging to the $H_1, \ldots, H_k$ uses up $\geq k - \ell + \ell\frac{\min|H_i|}{k(m-1)}$ hyperplanes since the remaining hyperplanes have size $\leq k(m-1)$. Note that this lower bound increases as ℓ increases since $\min|H_i| >> k(m-1)$. Setting $\ell = 1$ gives the lower bounds and this is the reflect

$\square$

For the paving matroids considered in Proposition 5.2.7, the Chow ring has a precise relation to the minimal degree a such that the paving matroids satisfy $MCB(a)$. Note that the Chow ring of a matroid is equal to the Chow ring of an actual toric variety constructed out of a fan (p. 5 of [5]).

Corollary 5.2.8. *Consider paving matroids on E of rank $m+1$ where the maximal hyperplanes are much larger than $\frac{n}{m}$ and don't vary much (in the sense described in Part 1 of Theorem 5.2.7). For example, this includes paving matroids where the hyperplanes are given by very large blocks partitioning E and the rest of the hyperplanes given by sets of size m. Then, the minimal degree a where $MCB(a)$ is satisfied and the upper bound in Part 1 of Thoerem 5.2.7 decreases with the as the dimension of the quotients by the annihilators of each x_{H_i} in the Chow ring $A^*(M)$ of M increase.*

Proof. This is an application of the formula

$$H(D(\mathcal{L}, t) = 1 + \sum_r \left(\prod_{i=1}^{k(r)} \frac{t(1 - t^{r_i - r_{i-1} - 1})}{1 - t} \right) f_{\mathcal{L}}(r)$$

for the Hilbert series of the Chow ring of the matroid on p. 526 of [13] after Corollary 2 on p. 525 of [13], where $\mathcal{L}$ denotes the lattice of flats and $D(\mathcal{L})$ denotes the Chow ring construction, $r = (0 = r_0 < r_1 < \cdots < r_k \leq \operatorname{rank} \mathcal{L})$ gives rank sequences of flags of flats, and $k = k(r)$ is the length of the rank sequence. We can remove the $f_{\mathcal{L}}(r)$ term involving the number of flags with a given rank sequence r if we index over flags instead of rank sequences r.

If we index the formula by flags of flats instead of indices themselves, we can see that the degree k term of the Hilbert series corresponds to the number of flags with the given ranks. Note that the only flags affected by the $MCB(a)$ are those where which end with a hyperplane or the ground set itself (i.e. $r_k = r$ or $r_k = r - 1$, where $r = m + 1$). This increases with the sizes of the maximal hyperplane since this increases the number of smaller rank objects. In other words, the length ℓ ending with $[n]$ correspond to those of length ℓ ending with some flat of smaller rank. While the degrees of the variables considered stay the same, the change comes from the number of possible variables to consider (which correspond to possible flags of flats using the given ranks). We fix the degree and look for flats with given differences in ranks. There are more flats of rank $\leq m-1$ to substitute in. Note that the behavior entirely depends on those of the hyperplanes since *any* subset of E of size $\leq m-1$ has rank equal to its size. This implies that the number of chains of hyperplane or a hyperplane and the ground set $[n]$ entirely depends on the size of the given hyperplane. Since the $x_{F_1}^{\alpha_1} \cdots x_{F_\ell}^{\alpha_\ell}$ from flags of flats $F_1 < \cdots < F_\ell$ and α_i such that $1 \leq \alpha_i \leq \operatorname{rank} F_{i+1} - \operatorname{rank} F_i$ and $\sum \alpha_i = k$ form a basis for $A^k(M)$ as a vector space (p. 526 of [13], Corollary 3.3.3 on p. 18 of [5]), the degree is given by $(m+1) - 1 - \operatorname{rank} F_1$ if F_ℓ is a hyperplane of M.

Given a hyperplane H_i, the condition that x_{H_i} is *not* an annihilator is equivalent to stating that the flats corresponding to the variables in the monomials involved are either strictly contained in H_i or strictly contain H_i by the definition of the Chow ring of a matroid. This restricts the flags under consideration to a collection of flats contained in H_i, one ending at H_i, or one ending with H_i and the ground set $[n]$. The analysis above then implies the conclusion after applying the arguments above.

□

Remark 5.2.9.

1. When the minimal degree a for $MCB(a)$ decrases, the increase in coefficient size can be interpreted in terms of "local" complexity of the Chow ring of toric varieties built out of the Bergman fan of the matroids (p. 5 of [5], Proposition 7.13 and Definition 7.14 on p. 431 – 432 of [1]).

2. The general relation between having a low degree a for the minimal degree such that the matroidal Cayley–Bacharach property $MCB(a)$ is satisfied and sizes of coefficients of the Hilbert series of the Chow ring also seems to apply to the case of arbitrary

matroids when a is small. The main difference appears to be in characterizing which collections of subsets can actually appear as hyperplanes of some matroid. There has been previous work studying such questions about possible subsets (e.g. [23]. [8]).

Supersolvable arrangements

Given an arrangement $\mathcal{A} = \{H_1, \ldots, H_n\}$ of hyperplanes in k^d for some field k, the associated matroid $M_{\mathcal{A}}$ has flats built out of intersections of the hyperplanes involved. In particular, the rank function is defined as $r(B) = n - \dim \bigcap_{i \in B} H_i$ and the flats are given by maximal sets of indices corresponding to intersections of hyperplanes equal to a particular linear subspace (formed by intersections of hyperplanes). We study a such that these matroids $M_{\mathcal{A}}$ satisfy $MCB(a)$ when $n >> a^3$ and $\mathcal{A}$ is a line arrangement. This is in addition to a general description of $MCB(a)$ for $M_{\mathcal{A}}$ for arbitrary hyperplane arrangements $\mathcal{A}$ (Proposition 5.2.10). In addition, we show that the "nontrivial" supersolvable line arrangements give a family of line arrangements where the number of possible degrees of unexpected curves decreases as the minimal a such that $MCB(a)$ is satisfied increases (Proposition 5.2.12). Finally, we end with some comments to topological properties of the arrangements in Remark 5.2.13.

While the degree a matroidal Cayley-Bacharach property $MCB(a)$ is defined as $\bigcup_{i=1}^a F_i \supset E \setminus p \Longrightarrow \bigcup_{i=1}^a F_i = E$ for any $p \in E = [n] = \{1, \ldots, n\}$, this can be rephrased in a simple way for hyperplane arrangements.

Proposition 5.2.10.

1. *Suppose that $a^2 \ll \frac{n}{a}$. Then, the matroid $M_{\mathcal{L}}$ associated to an arrangement $\mathcal{L}$ of n lines with a points of degree close to $\frac{n}{a}$ satisfies $MCB(a)$. Note that a collection of such "high degree points" is necessary in order for $MCB(a)$ to be satisfied nontrivially. In general, $MCB(a)$ is satisfied when a is very small compared to the number of lines*

or maximum multiplicity.

2. *In general, a matroid $M_{\mathcal{A}}$ built out of a hyperplane arrangement $\mathcal{A} = \{H_1, \ldots, H_n\}$ satisfies $MCB(a)$ if and only if a is less than or equal to the minimal number of intersections of elements of $\mathcal{A}$ (i.e. elements of the intersection lattice or intersection points in the case of a line arrangement) such that the indices cover $\mathcal{A} \setminus H_i$ for some $i \in [n] = \{1, \ldots, n\}$. By intersections of elements, we mean linear subspaces of the form $\bigcap_{i \in B} H_i$ for some $B \subset [n]$ with such subspaces written using the largest possible such subset B with respect to inclusion.*

Proof. 1. In the generic case, we can start with a suitable collection of lines parametrized by subsets F_i of the index set $[n]$ (each giving rise to distinct intersection points). Since any two lines only intersect at one point, the remaining points of intersection (not coming from the F_i) have multiplicity $\leq a$. Suppose that $a^2 << \frac{n}{a}$ and that the F_i are not far from being evenly distributed in size (at least much larger than a^2 treating the variables as functions of a). Then, $MCB(a)$ must be satisfied since using a points of intersection not all coming from the F_i will not be able to be used to cover the ground set $[n] = \{1, \ldots, n\}$ indexing the hyperplanes of the arrangement $\mathcal{A}$. In general, the fact that $t_2 + t_3 \geq k + t_5 + 2t_6 + 3t_7 + \ldots$ with t_i equal to the number of intersection points of multiplicity/degree i (Hirzebruch [16]) implies that there are many more low degree points than high degree ones, which implies that $MCB(a)$ must be satisfied if a is small in general. A similar argument can be repeated if we consider the case of hyperplane arrangements and sets parametrizing intersctions of hyperplanes (and linear subspaces in general).

2. We need a such that a collection of a intersections of hyperplanes in $\mathcal{A}$ either use up all the hyperplanes or miss ≥ 2 of them. In the case of line arrangements, this means that taking $\leq a$ intersection points of lines either uses up all the lines or we are missing

≥ 2 of the lines.

□

The case of line arrangements yields further connections to degees of unexpected curves arising from supersolvable line arrangements. This makes use of the following thereom of Hanumanthu–Harbourne [14] on supersolvable line arrangements wth a given number of modular points and their connections to degrees of unexpected curves.

Theorem 5.2.11. *(Hanumanthu–Harbourne, p. 3 of [14])*

Let $\mathcal{L}$ be a line arrangement (over any field) with a modular point (i.e. an intersection point connected to all other intersection points by a line in $\mathcal{L}$).

1. *If $\mathcal{L}$ is* not *homogeneous, then either $\mathcal{L}$ is a near pencil or it has two modular points. If it has two modular points, then $\mathcal{L}$ consists of $a \geq 2$ lines through one modular point and $b > a$ lines throug the other one. This means that there are $a + b - 1$ lines in $\mathcal{L}$ and $(a - 1)(b - 1)$ intersection points of multiplicity 2.*

2. *If $\mathcal{L}$ has a modular point of multiplicity 2, then $\mathcal{L}$ is trivial.*

3. *If $\mathcal{L}$ is complex and homogeneous (i.e. each intersection point has the same multiplicity/degree) with the maximum multiplicity > 2, there are ≤ 4 modular points. If there are 3 or 4 modular points, we have the following possiblities:*

    - *If there are 4 modular points, then there are 6 lines in $\mathcal{L}$, the common multiplicity is $m = 3$, $t_2 = 3$, $t_3 = 4$, and $t_k = 0$ otherwise. Up to a change of coordinates, $\mathcal{L}$ consists of the lines $x = 0, y = 0, z = 0, x - y = 0, x - z = 0$, and $y - z = 0$. The intersection pattern is like that of an equilateral triangle and its angle bisectors.*

- *If there are* 3 *modular points, then the common multiplicity is* $m > 3$ *and up to change of coordinates,* $\mathcal{L}$ *consists of the lines defined by the linear factors of* $xyz(x^{m-2} - y^{m-2})(x^{m-2} - z^{m-2})(y^{m-2} - z^{m-2})$*. This means that there are* $3(m-1)$ *lines,* $t_2 = 3(m-2)$*,* $t_3 = (m-2)^2$*,* $t_m = 3$*, and* $t_k = 0$ *otherwise.*

While we will focus on the final case since it has the most interesting structure, we will also consider the non-homogeneous case.

Proposition 5.2.12.

1. *If* $\mathcal{L}$ *is a non-homogeneous supersolvable line arrangement and satisfies* $MCB(a)$ *for some* a*, then the corresponding matroid satisfies* $MCB(a)$ *if and only if* $a \leq \frac{A+B-1}{2}$*, where* A *and* B *are the degrees of the modular points.*

2. *Given a homogeneous supersolvable line arrangement* $\mathcal{L}$ *with* 3 *modular points, the minimal degree* a *such that the matroid corresponding to* $\mathcal{L}'$ *satisfies* $MCB(a)$ *nontrivially decreases as the number of posisble degrees of unexpected curves increases.*

Proof. 1. The theorem above implies that we either have a near-pencil or two modular points. In the first case, $MCB(a)$ cannot be satisfied for any a since $MCB(1)$ is *not* satisfied. This is because the failure of $MCB(a)$ implies the failure of $MCB(b)$ for any $b > a$. As for the case of two modular points, let A and B be the degrees of the modular points. Then, the conclusion follows from labeling the individual lines of the arrangement by pairs of the form (i, j) with $1 \leq i \leq A$ and $1 \leq j \leq B$. We find the minimal number of pairs such that the coordinates i and j use up all the elements of $[A + B] = \{1, \ldots, A + B - 1\}$.

2. In the final case, note that the counts of the t_i in the case of 3 modular points comes from the fact that intersection points of lines of $\mathcal{L}$ which involve 2 factors *not* involving xyz actually intersect at 3 such factors. Checking for possible a where $MCB(a)$ can

be satisfied by $\mathcal{L}'$ with the linear factors x, y, z of xyz removed corresponds to the possible degrees of unexpected curves throguh points corresponding to the duals of lines of $\mathcal{L}$. More precisely, $\mathcal{L}'$ satisfies $MCB(a)$ for $a \leq \frac{m}{3}$ and D is a possible degree of an unexpected curve if and only if $m \leq D \leq n - m - 1$ (Theorem 3.8 on p. 173, p. 180 – 181 of [9]). This gives a negative correlation between $MCB(a)$ degrees a for $\mathcal{L}'$ and the number of possible degrees of unexpected curves arising from $\mathcal{L}$.

$\square$

Remark 5.2.13.

1. Recall that a central arrangement of linear subspaces is one where all the intersection of all of the linear subspaces is nonempty. In the case of line arrangements, the number of indices covered by a collection of intersection points can be expressed by the number of regions the corresponding central subarrangement splits the plane into. This can be expressed as the a specilization (substituting $t = -1$ into the variable) of the characteristic polynomial (Theorem 4.1 on p. 7 of [2]) of the matroid corresponding to the central subarrangement. Using an inclusion-exclusion argument, the number of elements covered by a collection of intersection points can be bounded above by the sum of specializations of characteristic polynomials of matroids associated to central line arrangements.

2. The arguments of Part 1 also apply in the case of hyperplane arrangements.

3. Using the lattice of flats while representing each flat by a single point and connecting two points by a line if one flat is contained in the other, the $MCB(a)$ condition can be phrased in a graph-theoretic manner. It means that a collection of points connected to all but possibly one point $i \in [n]$ is connected to every point of $[n]$.

We continue to analyze supersolvable arrangements, but move from lines to the more general setting of hyperplanes. As in [6], most of the arrangements considered will be assumed to be *central*. These hyperplane arrangements give a clear connection between the lattice of flats of the associated matroid (i.e. lattice formed by intersections of hyperplanes) and the connected components/regions bounded by the collection of hyperplanes (called chambers).

Definition 5.2.14. (Definition 4.1 and Definition 4.2 on p. 273 – 274 of [6])

1. Writing d for the rank, a **supersolvable geometric lattice** is defined as one having a maximal chain of form $\widehat{0} = V_0 \prec V_1 \prec \cdots \prec V_{d-1} \prec V_d = \widehat{1}$, where $\widehat{0}$ and $\widehat{1}$ are minimal and maximal elements of the lattice (p. 273 of [6]) and $x \prec y$ means that $x < y$ and $x < z \leq y \Longrightarrow z = y$. In our case, we take the elements of the lattice to be intersections of the hyperplanes of the arrangement and the ordering is given by reverse inclusion.

2. A central arrangement $\mathcal{A}$ is **supersolvable** if its lattice $L(\mathcal{A})$ of intersections is a supersolvable lattice.

3. For $1 \leq i \leq d$, let e_i be the number of atoms of $L = L(\mathcal{A})$ that lie below V_i, but not V_{i-1}. We have $e_1 = 1$ and $\sum_{i=1}^d e_i$ is the number of atoms in $L(\mathcal{A})$. Also, the characteristic polynomial of L is $\chi(L,t) = \prod_{i=1}^d (t - e_i)$.

One of the three initial examples considered in [6] is the graph hyperplane. We consider the computations in more detail below.

Example 5.2.15. (Matroids of graph hyperplane arrangements and $MCB(a)$)
Given a graph G with vertex set $[n] = \{1, \ldots, n\}$, consider the hyperplane arrangement $\mathcal{A}_G$ formed by hyperplanes of the form $x_i = x_j$ for each $(i,j) \in E(G)$ (i.e. pairs forming an edge of G). Intersections of hyperplanes that are considered are of the form $x_{i_1} = \cdots = x_{i_k}$

for some set $\{i_1, \ldots, i_k\}$. Since flats consist of maximal collections of hyperplanes from the arrangement considered ($\mathcal{A}_G$ in this case) giving rise to a specific linear subspace, the flats of the matroid $M_{\mathcal{A}_G}$ associated to $\mathcal{A}_G$ has ground set given by the elements of $E(G)$ (edges of G) and the flats are $E(G|_{V_i})$, where $V_i \subset [n]$ and $G|_{V_i}$ is the restriction of G to the vertex subset V_i.

In this particular setting, checking whether $MCB(a)$ can be satisfied doesn't seem to depend on the degree a.

Proposition 5.2.16. *The matroid $M_{\mathcal{A}_G}$ of a hyperplane arrangement $\mathcal{A}_G$ in $\mathbb{R}^n$ associated to a graph G with vertex set $[n]$ satisfies $MCB(a)$ for some a if and only if every edge is bounded by vertices of degree ≥ 2.*

Proof. To see this, we look at what happens when we omit a specific edge from the union of edges coming from some collection of flats (which can be taken to be a). Note that the flats come from edges inside the restriction of the graph G to some subset of the vertex set $[n] = \{1, \ldots, n\}$. We can split into cases according to the degrees of the vertices bounding the missing edge e.

1. **Case 1:** There is an edge e where each bounding vertex has degree 1.

   In this case, it doesn't seem like $MCB(a)$ is satisfied for *any* a. This is because the flats $F_i = E(G|_{V_i})$ can be taken to come from any collection of vertex sets V_i with union equal to $A \setminus \partial e$, where $A \subset [n]$ is the set of vertices of degree ≥ 1 and ∂e denotes the pair of vertices bounding the missing edge e. This would contain all the edges except e. Note that this case would be omitted if the graph G is assumed to be connected.

2. **Case 2:** There is an edge e where one bounding vertex has degree 1 and the other has degree ≥ 2.

The matroid $M_{\mathcal{A}_G}$ still does *not* satisfy $MCB(a)$ in this case. This is because the vertex subsets $V_i \subset [n]$ can be taken to have union equal to $A \setminus p$, where A is defined in the same way as in Case 1 and p is the vertex in ∂e of degree 1. In that case, the restriction to the given set of vertices is still missing the edge e but contains all others.

3. **Case 3:** Each edge e is bounded by vertices of degree ≥ 2.

   In this case, the matroids $M_{\mathcal{A}_G}$ *do* satisfy $MCB(a)$ regardless of the choice of a. By including the edges connected to each of the two vertices in $\partial e = \{p, q\}$, any edges induced by restriction to a subset of the vertices $[n]$ including edges other than e containing p or q in the boundary must include p and q as well. Thus, a collection of edges coming from restrictions of vertex sets misisng at most one edge of G contains all of the edges of G.

$\square$

Remark 5.2.17.

1. A graphic arrangement is supersolvable if and only if the graph in question is *chordal* (i.e. for any cycle with ≥ 4 vertices, there is an edge of G connecting two vertices which are not adjacent in the cycle – see Remark 2.5 on p. 9 of [4]).

2. Some other examples to consider are polytopal arrangements from hyperplanes built out of facets of polytopes and Coxeter arrangements from finite subsets of $GL_d(\mathbb{R})$ (orthogonal reflections through hyperplanes) (p. 268 – 269 of [6]).

In general, the computation above and the definition of $MCB(a)$ for matroids $M_{\mathcal{A}}$ associated to hyperplane arrangements $\mathcal{A}$ seems to indicate some kind of forced connectivity since a "missing hyperplane" must intersect collections of intersections of other hyperplanes in some way. One way to do this would be to impose a dependency on the hyperplane intersections depending on the indices considered. However, we still need to check whether such a condition is necessary.

For supersolvable arrangements, checking $MCB(a)$ can "generically" be reduced to a question on a smaller hyperplane arrangement.

Theorem 5.2.18. *(Björner–Edelman–Ziegler, Theorem 4.3 on p. 274 of [6])*
Every arrangement $\mathcal{A}$ of rank ≤ 2 *is supersolvable. An arrangement $\mathcal{A}$ of rank $d \geq 3$ is supersolvable if and only if $\mathcal{A} = \mathcal{A}_0 \sqcup \mathcal{A}_1$, where $\mathcal{A}_0$ is a supersolvable arrangement of rank $d-1$ and, for any $H', H'' \in \mathcal{A}_1$ with $H' \neq H''$, there is an $H \in \mathcal{A}_0$ such that $H' \cap H'' \subset H$.*

Using this result, we can make the following observations.

Proposition 5.2.19. *Let $\mathcal{A}$ be a central supersolvable hyperplane arrangement.*

1. *Writing $\mathcal{A} = \mathcal{A}_0 \sqcup \mathcal{A}_1$ as in Theorem 5.2.18, let $d = \operatorname{rank} \mathcal{A}$ and $\mathcal{B}_0 = \{V_{d-1} \vee (H' \wedge H'') : H', H'' \in \mathcal{A}_1\}$ be the hyperplanes in $\mathcal{A}_0$ containing the pairwise intersections of elements of $\mathcal{A}_0$. Given a collection of hyperplane intersections/flats P, write $P = P_0 \sqcup P_1$ with P_0 only from hyperplanes in $\mathcal{A}_0$ and P_1 involving hyperplanes from $\mathcal{A}_1$ in each intersection.*

   In this setting, $M_{\mathcal{A}}$ satisfies $MCB(a)$ if and only if the following conditions hold each collection $P = P_0 \sqcup P_1$ ($k := |P_1|$) using up $\geq |\mathcal{A}| - 1$ hyperplanes and $1 \leq k \leq a$:

   - *Let BP_0 be the counterpart of $\mathcal{B}_0$ for P built out pairwise intersections of elements of P_1. M_{P_1} satisfies $MCB(k)$ and the $\leq a-k$ hyperplanes in BP_0 use up all of the hyperplanes in $\mathcal{A}_0$.*
   - *M_{BP_0} satisfies $MCB(a-k)$ and P_1 uses up all the hyperplanes in $\mathcal{A}_1$.*

   In particular, it suffices to have $M_{\mathcal{A}_1}$ satisfy $MCB(k)$ and $M_{\mathcal{A}_0 \setminus \mathcal{B}_0}$ satisfy $MCB(k)$ and $M_{\mathcal{A}_0 \setminus \mathcal{B}_0}$ for each $0 \leq k \leq a$.

2. *The central supersolvable hyperplane arrangements such that $MCB(d)$ is satisfied for the minimal nontrivial degree a can take* any *possible characteristic polynomial or rank generating function. This means that any central supersolvable hyperplane arrangement of rank d has the same characteristic polynomial as one satisfying $MCB(d)$.*

3. *Let $u = \operatorname{rank} \mathcal{A}$ and Ω_u be the intersection of all the hyperplanes in $\mathcal{A}$, and Ω_{u-1} be the intersection of the hyperplanes in $\mathcal{A}_0$. For any $R \in \mathcal{A}_1$, the intersection $R \cap \Omega_{u-1} = \Omega_u$. In particular, this implies that any pair of flats of $M_{\mathcal{A}}$ where one of them is the ground set $\mathcal{A}_0$ of $M_{\mathcal{A}_0}$ and the other contains an element of $\mathcal{A}_1$ covers the entire ground set of $M_{\mathcal{A}}$.*

Proof. 1. Consider a collection of intersections of a hyperplanes of $\mathcal{A}$ which is "missing" at most hyperplane. Let P be a colection of such hyperplane intersections with P_0 only involving hyperplanes in $\mathcal{A}_0$ and P_1 involving hyperplanes in $\mathcal{A}_1$ (and possibly hyperplanes in $\mathcal{A}_0$). We can partition the cases involved into ones where $|P_1| = k$ as k varies over $0 \leq k \leq a$. This potential missing hyperplane is either in $\mathcal{A}_1$ or $\mathcal{A}_0$. If we start indexing the hyperplane intersections by ones that involve elements of $\mathcal{A}_1$, the intersections involved induce a collection of intersections of elements of $\mathcal{A}_1$. These intersections must also include the (unique) hyperplanes in $\mathcal{A}_0$ contain pairwise intersections of hyperplanes in P_1. Omitting these from the elements of $\mathcal{A}_1$, the remaining $a - k$ hyperplane intersections (from P_0) form BP_0. If the potential missing element is in $\mathcal{A}_1$, the elements of BP_0 use up all the elements of $\mathcal{A}_0$. In order for $MCB(a)$ to be satisfied, the missing element in $\mathcal{A}_1$ should actually be covered by P_0. This is the statement that M_{P_1} satisfies $MCB(k)$. If the potential missing element is in $\mathcal{A}_0$, we have that P_1 uses up all the hyperplanes in $\mathcal{A}_1$. This means that the elements of BP_0 satisfy $MCB(a-k)$ as we already have a cover.

2. Note that $e_d = |\mathcal{A}_1|$ (p. 275 of [6]). If $\mathcal{A}_1$ is a pencil of hyperplanes containing a single $(d-2)$ linear subspace of some fixed $H \in \mathcal{A}_0$ which do *not* contain the line V_{d-1}, then

intersecting any two of the hyperplanes in $\mathcal{A}_1$ means intersecting *all* of the hyperplanes in $\mathcal{A}_1$. This means that any collection of intersections of hyperplanes in $\mathcal{A}$ which where at most 1 hyperplane is not involved actually involves *all* of the hyperplanes in $\mathcal{A}$ and the $MCB(a)$ property is satisfied for any a such that this question is nontrivial. This can be done at each step of the construction of a supersolvable hyperplane arrangement of rank ≥ 3. For the base case of a rank 2 supersolvable hyperplane arrangement, there are no restrictions on the "base" characteristic polynomial since *any* central hyperplane arrangement of rank ≤ 2 is supersolvable by Theorem 5.2.18. The conclusion follows from noting that $\chi(L,t) = \prod_{i=1}^{d}(t - e_i)$ (Part 3 of Definition 5.2.14).

3. In general, V_{d-1} can be taken to be a line contained in the common intersection Ω_{d-1} of the hyperplanes in $\mathcal{A}_0$. Given a central hyperplane arrangement of rank u, let Ω_u be the intersection of all the hyperplanes in the arrangement. The new hyperplanes $A_i \in \mathcal{A}_1$ are those do *not* contain V_{d-1}. Choosing an initial such hyperplane A_1 to put in $\mathcal{A}_1$, we actually have that $\Omega_u = \Omega_{u-1} \cap A_1$. Since $A_1 \not\supset V_{d-1}$, we have that $A_1 \not\supset \Omega_{u-1}$ and $\dim A_1 \cap \Omega_{u-1} = d - u + 1 - 1 = d - u$. Since $A_1 \cap \Omega_{u-1}$ contains the intersection of *all* the hyperplanes in the arrangement although it is of the same dimension (due to the rank), we have that $\Omega_u = \Omega_{u-1} \cap A_1$. The remaining choices involve which $(d-2)$-planes to use for the intersections of pairs of elements of $\mathcal{A}_1$ and what hyperplanes to place in them. Since the $(d-2)$-planes must contain Ω_u (which is Ω_3 in this case), the $(d-2)$-planes depend on a choice of $d-2-(d-u) = u-2$-planes (which are lines in this case).

$\square$

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