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HYPERKÄHLER KUMMER RIGIDITY AND THE VIETA INVOLUTIONS ON TROPICAL MARKOV CUBICS

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To my Parents;

and to my Grandparents, resting in peace

1. The Easiest Dynamics are Linear Ones.
2. Tropical Markov Cubics dream Hyperbolic Origami.

—a poetic summary of the thesis

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ABSTRACT

In this paper, we study holomorphic and algebraic dynamics on complex manifolds. It consists of two parts: one that discusses Kummer rigidity on hyperkähler manifolds, and another that discusses tropicalized actions of algebraic dynamics.

In the part on Kummer rigidity, based on the fundamental structures that any hyperkähler manifold carries, we generalize some classical works done for complex surfaces. In particular, motivated by the study of Green currents on K3 surfaces, we have establised 'Green-like currents' extracted from a singular metric under the assumption that the Green measure equals to the volume. By doing so, we have shown that the only such projective manifolds and dynamics should be constructed from linear actions on (complex) tori.

In the part on tropical dynamics, we discuss the family of Markov cubics, which is a family of degree 3 affine surfaces that lives in the character variety of spheres with four punctures. By defining this family over a non-archimedean field, we define tropicalizations of a natural family of algebraic involutions: the Vieta involutions. These involutions, after tropicalization, exhibit a structure that resembles the hyperbolic plane with three independent reflections. By that we split the system in two parts: one that corresponds to hyperbolic reflections, and another that mimics the Euclidean algorithm on pairs of integers. We also include some introduction to the machinary used to perform the analysis.

Part I

Hyperkähler Kummer Rigidity

CHAPTER 1

INTRODUCTION

The first part of this work is devoted to the analytic study of holomorphic automorphisms on hyperkähler manifolds, with certain assumptions on the Green currents.

Some initial use of currents in the study of complex dynamical systems can be seen in Bedford et al. [1993] for $\mathbb{C}^2$ and Cantat [2001] for K3 surfaces, and the name *Green currents* are used since then. The idea has been further expanded by Dinh and Sibony [2010]De Thélin and Dinh [2012] and others to deal with holomorphic automorphisms on complex manifolds of dimensions greater than one.

Green currents are differential geometric encodings of stable and unstable distributions. Their power can be understood through various methods to study them, such as pluripotentials, cohomologies, and so forth.

One interesting result in this vein is the Kummer rigidity on projective or K3 surfaces Cantat and Dupont [2020]Filip and Tosatti [2021]. There, one studies holomorphic automorphisms $f\colon X \to X$ where X is a projective or K3 surface, and f has positive topological entropy. If one wedges Green currents and has the measure of maximal entropy in the volume class, then f must be Kummer: that is, X must be a Kummer K3 surface and f is induced from a linear map on the torus used to build X.

An analogous result can be established for hyperkähler manifolds, a higher-dimensional analogue of K3 surfaces. In particular, we prove the following

Theorem 1.1. *Let X be a projective hyperkähler manifold. Let $f\colon X \to X$ be a holomorphic automorphism that has positive topological entropy. Suppose the volume form is an f-invariant measure of maximal entropy. Then the underlying hyperkähler manifold X is a normalization of a torus quotient, and f is induced from a hyperbolic affine-linear transformation on that torus quotient.*

That is, if $\dim_{\mathbb{C}} X = 2n$*, there exists a complex torus* $\mathbb{T} = \mathbb{C}^{2n}/\Lambda$ *and a finite group of toral isomorphisms* Γ *in which* X *normalizes* $\mathbb{T}/\Gamma$*, and* f *is induced from a hyperbolic affine-linear transformation* $A\colon \mathbb{T}/\Gamma \to \mathbb{T}/\Gamma$*.*

The proof is essentially analogous to the case of surfaces Cantat and Dupont [2020]Filip and Tosatti [2021], but some modifications are required to generalize things for higher dimensions.

Outline of the Part In chapter 2, we introduce the notion of hyperkähler manifolds and discuss their known dynamical properties. Additionally, analogous to the K3 surface case [Filip and Tosatti, 2021, §2.1.7], we have a singular Ricci-flat metric ω_0 on X. In chapter 3, assuming that the volume measure is the measure of maximal entropy, we establish that the singular metric ω_0 behaves well with the dynamics. In chapter 4, we show that the singular metric defines stable and unstable distributions, thus giving a holomorphic foliation of stable and unstable manifolds. Some flatness follows from this. In chapter 5, we use the observations made so far and prove Theorem 1.1, the claimed Kummer rigidity.

CHAPTER 2

HYPERKÄHLER MANIFOLDS AND THEIR DYNAMICAL STRUCTURES

In this chapter, we introduce the notion of hyperkähler manifolds, which are one of the higher-dimensional analogue of K3 surfaces. Hyperkähler manifolds are rich in structures, and in this chapter we present various notions and notations, along with a list of basic facts needed to study them.

2.1 Hyperkähler Structures

Definition 2.1 (Hyperkähler Manifolds)**.** Suppose (X, ω) is a simply connected compact Kähler manifold and let Ω be a nondegenerate holomorphic $(2,0)$-form on X that generates the Hodge group $H^{2,0}(X, \mathbb{C})$. Then we say a triple (X, ω, Ω) a *hyperkähler manifold*. We call the form Ω a *holomorphic symplectic form* on X.

Another name for this manifold in literatures is an *irreducible holomorphic symplectic manifold* [Gross et al., 2003, Def 21.1]. That ω is hyperkähler may be understood that the tensor Ω is flat with respect to ω. For $n = 1$, this X is nothing but a K3 surface.

This Ω generates the $(2,0)$-Hodge group: $H^{2,0}(X, \mathbb{C}) = \mathbb{C}.\Omega$ [Gross et al., 2003, Proposition 23.3]. Moreover, we declare the *volume form* $\mathrm{vol} = (\Omega \wedge \overline{\Omega})^n$ associated to the holomorphic symplectic form; we normalize Ω so that $\mathrm{vol}(X) = 1$. This volume form is same as that of the Riemannian geometry on X. That is, $\omega^{2n} = c \cdot \mathrm{vol}$ for some constant $c > 0$.

A hyperkähler manifold has the natural quadratic form on H^2 that generalizes the intersection form of K3 surfaces. This is called the Beauville–Bogomolov–Fujiki quadratic form q [Gross et al., 2003, Definition 22.8] and is defined as follows. Suppose $\alpha \in H^2(X, \mathbb{C}) = H^{2,0}(X) \oplus H^{1,1}(X) \oplus H^{0,2}(X)$ is decomposed as $\alpha = c_1\Omega + \beta + c_2\overline{\Omega}$, where $c_1, c_2 \in \mathbb{C}$ and

$\beta \in H^{1,1}(X)$. Then, the number $q(\alpha)$ is defined as

$$q(\alpha) := c_1 c_2 + \frac{n}{2} \int_X \beta^2 (\Omega \wedge \overline{\Omega})^{n-1}. \tag{2.1.1}$$

We will abuse the notation and denote $q(\alpha, \alpha')$ for the symmetric bilinear form from the quadratic form q. The form q, if restricted to $H^{1,1}(X, \mathbb{R})$, has signature $(1, h^{1,1} - 1)$ by [Gross et al., 2003, Corollary 23.11]. One may view this as the Hodge Index Theorem for hyperkähler manifolds.

The Beauville–Bogomolov–Fujiki quadratic form q, together with the Beauville–Fujiki relation [Gross et al., 2003, Proposition 23.14], gives the following formula. Given a Ricci-flat Kähler metric ω' on X, we have

$$(\omega')^{2n} = q(\omega')^n \binom{2n}{n} \cdot (\Omega \wedge \overline{\Omega})^n, \tag{2.1.2}$$

as differential forms.

2.2 Invariance

Given a hyperkähler manifold (X, ω, Ω), let $f \colon X \to X$ be a holomorphic automorphism that has positive topological entropy $h_{\text{top}}(f) > 0$. Various structures of X are preserved under f.

Obviously, as a holomorphic map, f preserves the complex structure I. Therefore we preserve the holomorphic sheaves Ω_X^p, Hodge groups $H^{p,q}(X)$, etc. by f. We also note that f preserves the Kähler cone.

Although the hyperkähler metric ω is seldom invariant under f, but the holomorphic symplectic form Ω is almost invariant, in the following sense.

Lemma 2.2. *There exists a constant k_f, with modulus 1, such that $f^*\Omega = k_f \Omega$.*

Proof. By Dolbault isomorphism, we can identify the group $H^{2,0}(X) = \mathbb{C}.\Omega$ with $H^0(X, \Omega^2_X) = \mathbb{C}.\Omega$. This tells that, any holomorphic section of the vector bundle Ω^2_X on X is proportional to $\Omega\colon X \to \Omega^2_X$.

Now f induces another holomorphic section of Ω^2_X, by $f^*\Omega\colon X \xrightarrow{f} X \xrightarrow{\Omega} \Omega^2_X$. This has to be proportional to $\Omega\colon X \to \Omega^2_X$, thus we have $f^*\Omega = k_f\Omega$ for some $k_f \in \mathbb{C}$. Because f preserves the volume $(\Omega \wedge \overline{\Omega})^n$, $|k_f| = 1$ follows. $\square$

Now recall that Beauville–Bogomolov–Fujiki form is defined only with the form Ω. Because of that, it is natural to guess the

Lemma 2.3. *The Beauville–Bogomolov–Fujiki form q is preserved under f. That is, $q(f^*\alpha) = q(\alpha)$ for any closed 2-form α.*

Proof. First we note that $f^*(\Omega\overline{\Omega}) = |k_f|^2\Omega\overline{\Omega} = \Omega\overline{\Omega}$. Thus f^* acts on $H^{4n}(X) = \mathbb{C}.(\Omega\overline{\Omega})^n$ trivially, which gives the identity $\int_X f^*\gamma = \int_X \gamma$ for any $4n$-form γ.

Write $\alpha = c_1\Omega + \beta + c_2\overline{\Omega}$, where $c_1, c_2 \in \mathbb{C}$ and $\beta \in H^{1,1}(X)$. Then $f^*\alpha = c_1(f^*\Omega) + (f^*\beta) + c_2(f^*\overline{\Omega}) = c_1k_f\Omega + (f^*\beta) + c_2\overline{k}_f\overline{\Omega}$. By (2.1.1),

$$
\begin{aligned}
q(f^*\alpha) &= (c_1k_f)(c_2\overline{k}_f) + \frac{n}{2}\int_X (f^*\beta)^2(\Omega\overline{\Omega})^{n-1} \\
&= c_1c_2|k_f|^2 + \frac{n}{2}\int_X (f^*\beta)^2(f^*(\Omega\overline{\Omega}))^{n-1} \\
&= c_1c_2 + \frac{n}{2}\int_X f^*[\beta^2(\Omega\overline{\Omega})^{n-1}] \\
&= c_1c_2 + \frac{n}{2}\int_X \beta^2(\Omega\overline{\Omega})^{n-1} = q(\alpha),
\end{aligned}
$$

the equation demanded. $\square$

2.3 The Eigenclasses

Lemma 2.3 readily implies that f^* is an *isometry* of the hyperbolic space $\mathbb{H}_X$, the connected component of the hyperboloid $\{x \in H^{1,1}(X, \mathbb{R}) \mid q(x) = 1\}$ that contains a Kähler class. We equip the metric q to make $\mathbb{H}_X$ a Riemannian manifold.

The isometry f^* is loxodromic in the sense of [Cantat, 2014b, §2.3.2], if $h_{\text{top}}(f) > 0$. This is a consequence of the following estimate of the first dynamical degree, by Oguiso.

Theorem ([Oguiso, 2009, Thm. 1.1])**.** *Let X be a hyperkähler manifold of dimension $2n$ and f be a holomorphic automorphism of X. If $d_i(f)$ is the i-th dynamical degree of f, i.e., the spectral radius of f^* on $H^{i,i}(X, \mathbb{R})$, then we have $d_{2n-i}(f) = d_i(f) = d_1(f)^i$ for $0 \le i \le n$. Moreover, the topological entropy $h_{\text{top}}(f)$ is $n \log d_1(f)$.*

We note that $h_{\text{top}}(f) = n \log d_1(f)$ is iterating the Gromov–Yomdin theorem Yomdin [1987]Gromov [2003] for topological entropies of holomorphic automorphisms on compact Kähler manifolds.

Thus if $h_{\text{top}}(f) > 0$, then $\log d_1(f) > 0$ as well. Define the numbers

$$h := \log d_1(f), \quad \lambda := d_1(f). \tag{2.3.1}$$

(A caveat is that the entropy is $h_{\text{top}}(f) = nh$, not h as the notation may suggest.) Thus we have an exponential estimate $\|(f^*)^N\| = O(\lambda^N)$, so f^* must be loxodromic.

Because f^* is loxodromic, we can find eigenclasses $[\eta_+], [\eta_-] \in H^{1,1}(X, \mathbb{R})$ with real single eigenvalues λ, λ^{-1} respectively (cf. [Oguiso, 2007, Theorem 1]). They are isotropic vectors, and has the following further properties.

Proposition 2.4. *The eigenclasses $[\eta_+], [\eta_-] \in H^{1,1}(X, \mathbb{R})$ satisfy the followings.*

(a) Isotropy: $q([\eta_+]) = q([\eta_-]) = 0$.

(b) Nilpontent: $[\eta_+]^{n+1} = [\eta_-]^{n+1} = 0$.

(c) Big and Nef: $[\eta_+] + [\eta_-]$ is a big and nef class.

(d) Spectral convergence: for any Kähler class $\alpha \in H^{1,1}(X, \mathbb{R})$ we have, as $n \to \infty$,

$$\lambda^{-n}(f^*)^n \alpha \to \frac{q(\alpha, [\eta_-])}{q([\eta_+], [\eta_-])}[\eta_+],$$
$$\lambda^{-n}(f^*)^{-n} \alpha \to \frac{q(\alpha, [\eta_+])}{q([\eta_+], [\eta_-])}[\eta_-].$$

Proof. The isotropy property is obvious: $q([\eta_\pm]) = q(f^*[\eta_\pm]) = q(\lambda^{\pm 1}[\eta_\pm]) = \lambda^{\pm 2} q([\eta_\pm])$ and $\lambda > 1$ gives the result.

The nilpotence follows from [Gross et al., 2003, Proposition 24.1][Verbitsky, 1995, Theorem 15.1]: for any $\alpha \in H^2(X, \mathbb{C})$, $\alpha^{n+1} = 0$ iff $q(\alpha) = 0$.

Classes $[\eta_\pm]$ are nef in the following reasons. Because $\mathbb{H}_X$ contains a Kähler class $[\omega]$, we see that $[\eta_\pm]$ are found as limits of classes $\lambda^{-n}(f^*)^{\pm n}[\omega]$, $n \to +\infty$.

We claim that the nef class $[\eta_+] + [\eta_-]$ is big. Note the Beauville–Fujiki relation $q(\alpha)^n = \binom{2n}{n}^{-1} \int_X \alpha^{2n}$ [Gross et al., 2003, Proposition 23.14]. If $[\eta_+] + [\eta_-]$ is not big, $\int_X (\eta_+ + \eta_-)^{2n} = 0$, then we have $q([\eta_+] + [\eta_-]) = 0$. But then $q([\eta_+], [\eta_-]) = 0$ follows, so $q(c_1[\eta_+] + c_2[\eta_-]) = 0$ for all $c_1, c_2 \in \mathbb{R}$. This contradicts with that the cone $\{q = 0\}$ cannot contain a linear space of dimension > 1.

The spectral convergence is in general the case whenever $q(\alpha) > 0$: see [Cantat, 2014b, §2.3.2]. □

We notice that $[\eta_+]$ and $[\eta_-]$ can be rescaled arbitrary, as long as they satisfy the properties listed in Proposition 2.4. For sake of pinning down a class that $[\eta_\pm]$ represents, we impose a normalization

$$\int_X (\eta_+ + \eta_-)^{2n} = 1. \tag{2.3.2}$$

By this, we have $q([\eta_+] + [\eta_-]) = 2q([\eta_+], [\eta_-]) = \binom{2n}{n}^{-1/n}$.

2.4 Null Locus and Metric Approximations

Even though the class $[\eta_+] + [\eta_-]$ is big and nef, the sum is very unlikely to be a Kähler class. Because of that, we introduce the obstructions studied in Collins and Tosatti [2015] for a big and nef class to be Kähler.

For sake of introducing an obstruction, one introduces the *null locus* $E \subset X$ of the class $[\eta_+] + [\eta_-]$, which is the union of all subvarieties $V \subset X$ such that $\int_V (\eta_+ + \eta_-)^{\dim V} = 0$ Collins and Tosatti [2015].

Proposition 2.5. *The null locus is f-invariant.*

Proof. We check that it satisfies $f(E) \subset E$. Recall that E is defined as the union of subvarieties V in which $\int_V (\eta_+ + \eta_-)^{\dim V} = 0$. Because $[\eta_+]$ and $[\eta_-]$ are nef classes, thanks to [Demailly and Paun, 2004, Theorem 4.5] and approximation of $[\eta_\pm]$ by Kähler classes, we have $\int_V (\eta_+)^a (\eta_-)^b \geq 0$ whenever $a + b = \dim V$. Hence if $\int_V (\eta_+ + \eta_-)^{\dim V} = 0$, we have $\int_V (\eta_+)^a (\eta_-)^b = 0$. Because $f^*[\eta_\pm] = \lambda^{\pm 1}[\eta_\pm]$, this implies $\int_{f(V)} (\eta_+)^a (\eta_-)^b = 0$ as well, now integrating on $f(V)$. Collecting them we have $\int_{f(V)} (\eta_+ + \eta_-)^{\dim V} = 0$, verifying $f(V) \subset E$. $\square$

Although the following is an immediate appliation of [Collins and Tosatti, 2015, Theorem 1.6], we state this as a lemma, to introduce some notations for later use.

Lemma 2.6. *There exists a smooth, incomplete Ricci-flat Kähler metric ω_0 on $X \setminus E$, and a sequence (ω_k) of complete, smooth Ricci-flat* hyperkähler *metrics on X that converges to ω_0 in the following senses: (i) $[\omega_k] \to [\eta_+] + [\eta_-]$ in $H^{1,1}(X, \mathbb{R})$, and (ii) $\omega_k \to \omega_0$ in $C^\infty_{\text{loc}}(X \setminus E)$.*

Moreover, ω_k's may be set to have the unit volume. That is, $\omega_k^{2n} = \mathrm{vol}$.

Proof. [Collins and Tosatti, 2015, Theorem 1.6] tells the existence of a smooth, incomplete Ricci-flat Kähler metric ω_0 on $X \setminus E$ in which, for any sequence $[\alpha_k] \to [\eta_+] + [\eta_-]$ in $H^{1,1}(X)$, and the Ricci-flat metric $\omega_k \in [\alpha_k]$, ω_k's converge to ω_0 in $C^\infty_{\text{loc}}(X \setminus E)$ topology.

For hyperkähler X, each Kähler class $[\alpha_k]$ contains a unique hyperkähler metric in it [Gross et al., 2003, Theorem 23.5]. Such a metric is necessarily Ricci-flat [Gross et al., 2003, Proposition 4.5]. By the uniqueness of Ricci-flat metric in a Kähler class, ω_k must be hyperkähler as well.

Because $[\alpha_k]$'s are converging to $[\eta_+] + [\eta_-]$, a big and nef class, the volume $[\alpha_k]^{2n}$ by ω_k, also converges to a positive number. Thus one can normalize ω_k's so that they have unit volume, i.e., $[\alpha_k]^{2n} = 1$ for all k. Consequently, as each ω_k is Ricci-flat, we have $\omega_k^{2n} = (\Omega \wedge \overline{\Omega})^{2n} = \mathrm{vol}$, due to (2.1.2). (Note that this normalization is compatible with (2.3.2).) $\square$

2.5 Ergodicity and Lyapunov Exponents

As a consequence of bigness (Proposition 2.4) of $[\eta_+] + [\eta_-]$, we have the following

Proposition 2.7. *(a) There exists closed positive $(1,1)$-currents S^+ and S^- with Hölder continuous potentials, that are in classes $[\eta_+]$ and $[\eta_-]$ respectively, and satisfy $f^*S^+ = \lambda S^+$ and $f^*S^- = \lambda^{-1}S^-$.*

(b) The wedge $(S^+)^n \wedge (S^-)^n$ in the sense of Bedford–Taylor Bedford and Taylor [1982] is the unique measure μ of maximal entropy, which is mixing.

The currents $S^{\pm}$ mentioned above are called *Green currents* of order 1, according to [Dinh and Sibony, 2010, §4.2]. We will call S^+ an *unstable Green $(1,1)$-current* and S^- a *stable Green $(1,1)$-current.*

Proof. Note that $\mathbb{R}.[\eta_+] \subset H^{1,1}(X, \mathbb{R})$ is a strictly dominant space for $f^* \colon H^{1,1}(X, \mathbb{R}) \circlearrowleft$, in the sense of [Dinh and Sibony, 2010, Theorem 4.2.1]. Then we have a closed $(1,1)$-current S^+ in the class $[\eta_+]$ that has Hölder continuous potentials and satisfies $f^*S^+ = \lambda S^+$. (Recall that $\lambda = d_1(f)$.) We may set this S^+ to be positive, because $[\eta_+]$ itself is a nef class.

Likewise, by considering f^{-1}, we have a closed positive $(1,1)$-current S^- in the class $[\eta_-]$ with the required properties. This shows (a).

Because potentials of $S^\pm$ are locally bounded, the Bedford–Taylor theory applies and defines $(S^\pm)^n$ as a closed positive (n,n)-currents. Denote $T^+ = (S^+)^n$ and $T^- = (S^-)^n$. Note that each current is in class $[\eta_+]^n$ and $[\eta_-]^n$, respectively.

Let $V_+ = \mathbb{R}.[\eta_+]^n \subset H^{n,n}(X,\mathbb{R})$. Then f^* acts as multiplying λ^n on V_+, and we have $\lambda^n = d_n(f) > \lambda^{n-1} = d_{n-1}(f)$. Thus the hypotheses of [Dinh and Sibony, 2010, Theorem 4.3.1] are met, and T^+ is the unique closed positive (n,n)-current in the class $[\eta_+]^n$. We argue likewise with $V_- = \mathbb{R}.[\eta_-]^n \subset H^{n,n}(X,\mathbb{R})$ and $(f^{-1})^*$, to have that T^- is the unique closed positive (n,n)-current in the class $[\eta_-]^n$.

It turns out that T^+ and T^- are Green currents of f and f^{-1} respectively, of order n. We claim that $T^+ \wedge T^-$ is a positive nonzero measure. To have so, note first that $T^+ \wedge T^-$ is cohomologous to $\eta_+^n \wedge \eta_-^n$ (as currents), where $\eta_\pm$ are any smooth representative of the class $[\eta_\pm]$. Then we have

$$\begin{aligned}\int_X T^+ \wedge T^- &= \int_X \eta_+^n \wedge \eta_-^n \\ &= \binom{2n}{n}^{-1} \int_X (\eta_+ + \eta_-)^n \\ &= q([\eta_+] + [\eta_-]) = 1,\end{aligned}$$

by the normalization (2.3.2). As $T^+ \wedge T^-$ is already a positive $(2n,2n)$-current, this suffices to show that $T^+ \wedge T^-$ is a positive nonzero measure. By [Dinh and Sibony, 2010, Proposition 4.4.1], this tells that the eigenvalues $\lambda^{\pm n}$ of f^* on $H^{n,n}(X,\mathbb{C})$ have multiplicities 1.

Now we apply [De Thélin and Dinh, 2012, Theorem 1.2]. We remark that the proof of the theorem only requires f to be 'simple on $\bigoplus_{p=0}^{2n} H^{p,p}(X,\mathbb{C})$,' i.e., admits a unique, multiplicity 1 eigenvalue of modulus $d_n(f)$, at the subring $\bigoplus_{p=0}^{2n} H^{p,p}(X,\mathbb{C})$ of the cohomology ring. Because f^* on other $H^{p,p}$ groups have spectral radius strictly less than $d_n(f)$, we see that the multiplicity of $d_n(f)$ on the subring is precisely that on the group $H^{n,n}(X, \mathbb{C})$. This was observed to be multiplicity 1 above.

Thus the theorem applies, and the Green measure $\mu = T^+ \wedge T^-$ is the unique invariant measure of maximal entropy. By [Dinh and Sibony, 2010, Theorem 4.4.2], we furthermore know that μ is mixing. □

We also claim that, if the Green measure μ is in the volume class, i.e., absolutely continuous with repsect to the volume measure $\mathrm{vol} = \omega^{2n}$, then the Lyapunov exponents are very simple.

Lemma 2.8. *If the Green measure $\mu = (S^+)^n \wedge (S^-)^n$ is in the volume class, the Lyapunov exponents are $\pm h/2$ with multiplicity $2n$ each.*

Proof. Let $\chi_1 \geq \cdots \geq \chi_{2n}$ be the Lyapunov exponents of μ for the cocyle Df on the complexified tangent bundle $T_{\mathbb{C}}X$, listed with multiplicities (cf. [Filip, 2019a, Theorem 2.2.6][Ruelle, 1979, Theorem 1.6]), which are μ-a.e. constant due to ergodicity. As f is invertible, we have a symmetry $\chi_i + \chi_{2n+1-i} = 0$; in particular, the first n exponents are nonnegative and the last n exponents are nonpositive.

Because the measure μ is in the volume class, by Ledrappier–Young formula [Ledrappier and Young, 1985b, Corollary G], the entropy $h_\mu(f)$ equals

$$h_\mu(f) = nh = \chi_1 + \cdots + \chi_n. \tag{2.5.1}$$

Here, $h_\mu(f) = nh$ follows from $h_\mu(f) = h_{\mathrm{top}}(f) = nh$.

Thanks to Oguiso [Oguiso, 2009, Theorem 1.1], we have an increasing-decreasing relation

$$1 = d_0(f) < d_1(f) < \cdots < d_{n-1}(f) < d_n(f) > d_{n+1}(f) > \cdots > d_{2n}(f) = 1$$

of dynamical degrees. The bounds of Lyapunov exponents by Thélin [de Thélin, 2008, Corollaire 3] then yields,

$$\chi_1 \geq \cdots \geq \chi_n \geq \frac{1}{2} \log \frac{d_n(f)}{d_{n-1}(f)} = \frac{h}{2}(> 0). \tag{2.5.2}$$

Combining (2.5.1) and (2.5.2), we have $\chi_1 = \cdots = \chi_n = h/2$. By the symmetry $\chi_i + \chi_{2n+1-i} = 0$, we in addition have $\chi_{n+1} = \cdots = \chi_{2n} = -h/2$. $\square$

2.6 Examples

We finish this chapter by mentioning the examples that we can keep in mind while following the arguments. These examples are brought from [Lo Bianco, 2017, §3.3-3.4], and we seek for whether each example

- is actually a (hyperkähler) Kummer example, and
- has the volume as a measure of maximal entropy.

Throughout this section, following [Lo Bianco, 2017, §3], we denote T as a 2-dimensional complex torus, $f_T \colon T \to T$ a hyperbolic automorphism on it; by hyperbolic we mean by $h := h_{\text{top}}(f_T) > 0$ (cf. [Lo Bianco, 2017, Corollary 1.23]). Let $f_T^{\times n} = (f_T, \cdots, f_T) \colon T^n \to T^n$ be the product of n copies of f_T. From [Lo Bianco, 2017, Lemma 3.1], it is known that $f_T^{\times n}$ has unstable and stable foliations $\mathcal{F}^+$ and $\mathcal{F}^-$, obtained by making the n-product of those for (T, f_T). The topological entropy of $(T^n, f_T^{\times n})$ is nh; cf. [Katok and Hasselblatt, 1995, Proposition 3.1.7(4)].

Moreover, combining [Lo Bianco, 2017, Proposition 1.12] and a theorem of Gromov–Yomdin [Lo Bianco, 2017, Theorem 1.10], a Kummer example (X, f) with associated toral automorphism $(\mathbb{T}, A)$ has the same topological entropies $h_{\text{top}}(X, f) = h_{\text{top}}(\mathbb{T}, A)$.

2.6.1 Generalized Kummer Variety

Denote $K_n(T)$ for the $2n$-dimensional generalized Kummer variety, the notation following [Gross et al., 2003, §21.2].

Following [Lo Bianco, 2017, §3.4], f_T induces an automorphism $K_n(f_T)\colon K_n(T) \to K_n(T)$. By [Lo Bianco, 2017, Lemma 3.12], the pair $(X, f) = (K_n(T), K_n(f_T))$ is a Kummer example, by the following data:

- $(Y, \widetilde{f}) = (T^n/\mathfrak{S}_{n+1}, f_T^{\times n}/\mathfrak{S}_{n+1})$,
- $(\mathbb{T}, A) = (T^n, f_T^{\times n})$, and
- the quotient map $q\colon \mathbb{T} \to Y$, which is birationally equivalent to a generically finite meromorphic map $\pi\colon \mathbb{T} \dashrightarrow X$. (That birational equivalence $\phi\colon X \to Y$ may be defined on whole X.)

Here, $\mathfrak{S}_{n+1}$ is the symmetry group of $(n+1)$ letters, acting on T^n by restricting the natural action $\mathfrak{S}_{n+1} \curvearrowright T^{n+1}$ on

$$T^n = \{(t_0, t_1, \cdots, t_n) \in T^{n+1} \mid t_0 + t_1 + \cdots + t_n = 0\}.$$

Therefore (X, f) is a Kummer example, with its associated toral automorphism $(T^n, f_T^{\times n})$.

By loc.cit., induced from foliations $\mathcal{F}^+$ and $\mathcal{F}^-$ on T^n, we have foliations $\mathcal{F}_X^+$ and $\mathcal{F}_X^-$ on X, called unstable and stable foliations respectively. (Here $\mathcal{F}_X^\pm$ may have singular loci.) By the action of $f_T^{\times n}$, each vector tangent to the foliations $\mathcal{F}^+$ and $\mathcal{F}^-$ on T^n are dilated by $e^{h/2}$ and $e^{-h/2}$ respectively. The same rates apply for $\mathcal{F}_X^+$ and $\mathcal{F}_X^-$ on $X \setminus \mathrm{Sing}(\mathcal{F}_X^\pm)$.

This gives that the Lyapunov exponents of (X, f) under the volume is $\pm h/2$, with multiplicity n each. The Ledrappier–Young formula [Ledrappier and Young, 1985b, Corollary G] then yields $h_{\mathrm{vol}}(X, f) = nh$. Now since (X, f) is a Kummer example, its topological entropy is also nh, as that of $(T^n, f_T^{\times n})$ is nh. Hence the volume measure is a measure of maximal entropy.

2.6.2 Hilbert Scheme of a Kummer Surface

Denote $K_1(T)$ for the Kummer surface of the 2-dimensional complex torus T. Then one has the Hilbert scheme $X := K_1(T)^{[n]}$ of n points and induced map $f := K_1(f_T)^{[n]} \colon X \to X$, from $f_T \colon T \to T$ (cf. [Lo Bianco, 2017, Proposition 3.10]).

The hyperkähler manifold $X = K_1(T)^{[n]}$ is obtained by normalizing T^n/Γ, where Γ is generated by

- an involution $\theta \colon T^n \to T^n$, $\theta(t_1, t_2, \cdots, t_n) = (-t_1, t_2, \cdots, t_n)$, and
- the symmetry group $\mathfrak{S}_n \curvearrowright T^n$ on coordinates.

(The group Γ, generated by involutions, forms the Weyl group of the Lie algebra B_n.) The map $(f_T, \cdots, f_T)$ commutes with Γ, thus induces a map $\widetilde{f} \colon T^n/\Gamma \to T^n/\Gamma$. The map $f = K_1(f_T)^{[n]}$ then satisfies, with the normalization map $\phi \colon X \to T^n/\Gamma$, $\widetilde{f} \circ \phi = \phi \circ f$. Thus (X, f) is a Kummer example, with associated toral automorphism $(T^n, f_T^{\times n})$.

The foliations $\mathcal{F}^{\pm}$ on T^n are Γ-invariant, hence carrying this to the regular locus of T^n/Γ and inducing (singular) foliations on X, we obtain unstable and stable foliations $\mathcal{F}_X^+$ and $\mathcal{F}_X^-$. Arguing as in §2.6.1, we see that for (X, f), the volume measure is a measure of maximal entropy.

Remark. So far we have seen that both of the examples required some extra structures (stable/unstable holomorphic foliations) to make sure that the volume is an invariant measure of maximal entropy. A partial converse of this will be shown in this paper (Proposition 4.2): if the volume is an invariant measure of maximal entropy, the unstable and stable holomorphic foliations are defined, and they dilate by some factors $e^{h/2}$ and $e^{-h/2}$. (This converse is partial because the singularities have codimension ≥ 1.) We note that this converse does not require projectivity assumption.

For a general hyperkähler Kummer example (X, f), we have a restriction on the eigenvalues of its associated toral automorphism $(\mathbb{T}, A)$. That is, the matrix part of A has n eigenvalues of modulus $e^{h/2}$ and n eigenvalues of modulus $e^{-h/2}$, counted with multiplicities. The proof of this fact is based on comparing the dynamical degrees of (X, f) found by two distinct means, [Lo Bianco, 2017, Proposition 1.12] and [Oguiso, 2009, Theorem 1.1].

CHAPTER 3

ANALYSIS ON APPROXIMATE METRICS

Starting from this chapter, we now aim to prove Theorem 1.1. In the sequels we let X to be a projective hyperkähler manifold of dimension $\dim_{\mathbb{C}} X = 2n$, $f\colon X \to X$ a holomorphic automorphism with positive topological entropy $h_{\text{top}}(f) = nh > 0$, whose measure of maximal entropy $\mu = (S^+)^n \wedge (S^-)^n$ is same as the (normalized) volume measure vol $= \omega^{2n}$.

We initiate the study looking at metrics ω_k's defined in Lemma 2.6 above. Ultimately we show that, for the limit metric ω_0 and each $x \in X \setminus E$, we have n-dimensional complex subspaces $E^{\pm}_{N,x} \subset T^{1,0}_x X$ (where $T^{1,0}X$ is the holomorphic tangent space) so that, applying $(f^N)^*$ to $\omega_{0,x}|E^{\pm}_{N,x}$, we rescale it by $\lambda = e^{Nh}$ (Corollary 3.7).

3.1 Local Setups

As claimed in Lemma 2.6, we pick up hyperkähler metrics ω_k that converges to ω_0 in $C^\infty_{\text{loc}}(X \setminus E)$ topology.

Fix $N \in \mathbb{Z}_{>0}$. Along the dynamics, we declare the quantitites $\left(\sigma_i^{(k)}(x, Nh)\right)_{i=1}^{2n}$, to be called *log-singular values of* $(f^N)^*\omega_k$ *relative to* ω_k, as follows. At a point $x \in X$, one can write

$$\omega_k = \frac{\sqrt{-1}}{2} \sum_{i=1}^{2n} dz_i \wedge d\overline{z}_i, \tag{3.1.1}$$

with an appropriate holomorphic coordinate $(z_1, \cdots, z_{2n})$ of X at x. Moreover, for

$$h_{ij}^{(k)} = (f^N)^*\omega_k\left(\frac{\partial}{\partial z_i}, \frac{\partial}{\partial z_j}\right), \tag{3.1.2}$$

we have a self-adjoint matrix $\left(h_{ij}^{(k)}\right)$, thus one can adjust the holomorphic basis vectors $\frac{\partial}{\partial z_1}, \cdots, \frac{\partial}{\partial z_{2n}}$ unitarily, to keep the form (3.1.1) of ω_k, yet make the matrix $\left(h_{ij}^{(k)}\right)$ diagonal.

This enables us to write $(f^N)^*\omega_k$ at $x \in X$ as

$$(f^N)^*\omega_k = \frac{\sqrt{-1}}{2} \sum_{i=1}^{2n} \exp\left(\sigma_i^{(k)}(x, Nh)\right) \, dz_i \wedge d\overline{z}_i. \tag{3.1.3}$$

We moreover choose the base index i so that the numbers $\sigma_1^{(k)}, \cdots, \sigma_{2n}^{(k)}$ are in the decreasing order, i.e.,

$$\sigma_1^{(k)}(x, Nh) \geq \sigma_2^{(k)}(x, Nh) \geq \cdots \geq \sigma_{2n}^{(k)}(x, Nh). \tag{3.1.4}$$

These quantities exhibit the following symmetry. The assumption that ω_k is hyperkähler, is crucial here.

Lemma 3.1. *Log-singular values $\sigma_i^{(k)}$'s exhibit symmetry at 0. That is, for all x and $i = 1, 2, \cdots, n$,*

$$\sigma_i^{(k)}(x, Nh) + \sigma_{2n+1-i}^{(k)}(x, Nh) = 0.$$

Proof. Replacing f to f^N if necessary, we may assume that $N = 1$. Also, for notational simplicity, denote $\omega := \omega_k$.

As ω is a hyperkähler metric, for each point $x \in X$ one has a holomorphic coordinate $(z_1, \cdots, z_{2n})$ that enables representations (valid only at x)

$$\omega = \sum_{i=1}^{2n} dz_i \wedge d\overline{z}_i,$$
$$\Omega = \sum_{\mu=1}^{n} dz_\mu \wedge dz_{n+\mu}.$$

Denote $(w_1, \cdots, w_{2n})$ for another holomorphic coordinate near $f(x)$ with analogous expressions of ω and Ω, at $f(x)$. By this coordinate, describe the map $D_x f \colon T_x X \to T_{f(x)} X$

as a matrix $A = \big(a_{ij}\big)$, where a_{ij}'s are determined by the relation

$$D_x f\left(\frac{\partial}{\partial z_i}\right) = \sum_{j=1}^{2n} a_{ij}\frac{\partial}{\partial w_j}.$$

Now Lemma 2.2, $f^*\Omega = k_f\Omega$, implies

$$\sum_{k,\ell=1}^{2n} a_{ik}\Omega_{k\ell}a_{j\ell} = k_f\Omega_{ij}$$

(where $\Omega_{ij} = \Omega\left(\frac{\partial}{\partial z_i}, \frac{\partial}{\partial z_j}\right)$; in other words, $(\Omega_{ij}) = \begin{bmatrix} 0 & I_n \\ -I_n & 0 \end{bmatrix}$), which entails $A\Omega A^\top = k_f\Omega$. This gives $k_f^{-1/2}A^\top \in \mathrm{Sp}(2n,\mathbb{C})$, which implies $AA^\dagger \in \mathrm{Sp}(2n,\mathbb{C})$.

We then describe how $f^*\omega$ is represented at x. Since

$$\begin{aligned} f^*\omega\left(\frac{\partial}{\partial z_i}, \frac{\partial}{\partial z_j}\right) &= \sum_{k,\ell=1}^{2n} \omega\left(a_{ik}\frac{\partial}{\partial w_k}, a_{j\ell}\frac{\partial}{\partial w_\ell}\right) \\ &= \sum_{k,\ell=1} a_{ik}\overline{a_{j\ell}}\omega\left(\frac{\partial}{\partial w_k}, \frac{\partial}{\partial w_\ell}\right) \\ &= \sum_{k=1}^{2n} a_{ik}\overline{a_{jk}} = (AA^\dagger)_{ij}, \end{aligned}$$

we have $f^*\omega = \frac{\sqrt{-1}}{2}\sum_{i,j=1}^{2n}(AA^\dagger)_{ij}\, dz_i \wedge d\bar{z}_j$ at x. Thus $\sigma_i^{(k)}$ consists of log of eigenvalues of $AA^\dagger$. An exercise, say [de Gosson, 2011, Problem 22], yields that a self-adjoint positive-definite symplectic matrix like $AA^\dagger$ has eigenvalues that are (multiplicatively) symmetric at 1. If this fact is translated to the list $\sigma_1^{(k)}, \cdots, \sigma_{2n}^{(k)}$, we get the desired symmetry. $\square$

3.2 Local computations

Based on the setups established in the previous section 3.1, we now compute the forms ω_k^{2n} and $\omega_k^{2n-1} \wedge (f^N)^*\omega_k$, at x, and compare them.

$$\omega_k^{2n} = \left(\frac{\sqrt{-1}}{2}\right)^{2n} (2n)!(dz_1 \wedge d\overline{z}_1 \wedge \cdots \wedge dz_{2n} \wedge d\overline{z}_{2n}), \tag{3.2.1}$$

$$\begin{aligned}\omega_k^{2n-1} \wedge (f^N)^*\omega_k &= \left(\frac{\sqrt{-1}}{2}\right)^{2n} \left(\sum_{i=1}^{2n} dz_i \wedge d\overline{z}_i\right)^{2n-1} \left(\sum_{j=1}^{2n} e^{\sigma_j^{(k)}(x,Nh)} dz_j \wedge d\overline{z}_j\right) \\ &= \left(\frac{\sqrt{-1}}{2}\right)^{2n} \left[\sum_{j=1}^{2n} e^{\sigma_j^{(k)}(x,Nh)}\right] (2n-1)!(dz_1 \wedge d\overline{z}_1 \wedge \cdots \wedge dz_{2n} \wedge d\overline{z}_{2n}).\end{aligned} \tag{3.2.2}$$

Now applying (3.2.1),

$$= \left[\sum_{j=1}^{2n} e^{\sigma_j^{(k)}(x,Nh)}\right] \frac{(2n-1)!}{(2n)!} \omega_k^{2n}, \tag{3.2.3}$$

and applying Lemma 3.1, we can 'fold' the sum of exponentials as

$$= \left[\sum_{j=1}^{n} 2\cosh(\sigma_j^{(k)}(x, Nh))\right] \frac{\omega_k^{2n}}{2n},$$

and obtain the following

Proposition 3.2. *As differential forms,*

$$\omega_k^{2n-1} \wedge (f^N)^*\omega_k = \left[\frac{1}{n} \sum_{j=1}^{n} \cosh\left(\sigma_j^{(k)}(x, Nh)\right)\right] \omega_k^{2n}.$$

3.3 Cohomological Analysis

The form $\omega_k^{2n-1} \wedge (f^N)^*\omega_k$ can also be understood cohomologically, but with some approximations. Think of the integral of the form, represented as a cup product of cohomology classes:

$$\int_X \omega_k^{2n-1} \wedge (f^N)^*\omega_k = [\omega_k]^{2n-1}.(f^N)^*[\omega_k].$$

Denote $[\omega_0] := [\eta_+] + [\eta_-]$. Then, by Lemma 2.6, we have that $[\omega_k] \to [\omega_0]$ as $k \to \infty$. Thus it leads us to consider the product $[\omega_0]^{2n-1}.(f^N)^*[\omega_0]$, which is evaluated as in the following

Proposition 3.3. *We have the following equation in cohomology:*

$$[\omega_0]^{2n-1}.(f^N)^*[\omega_0] = \cosh(Nh) \cdot [\omega_0]^{2n}.$$

Proof. The proof is a manual computation with $[\omega_0] = [\eta_+] + [\eta_-]$ and Corollary 2.4.

$$\begin{aligned}
[\omega_0]^{2n} &= ([\eta_+] + [\eta_-])^{2n} = \sum_{k=0}^{2n} \binom{2n}{k} [\eta_+]^k [\eta_-]^{2n-k} \\
&= \binom{2n}{n} [\eta_+]^n [\eta_-]^n. && (3.3.1) \\
[\omega_0]^{2n-1}.(f^N)^*[\omega_0] &= ([\eta_+] + [\eta_-])^{2n-1}.(e^{Nh}[\eta_+] + e^{-Nh}[\eta_-]) \\
&= \left[\sum_{k=0}^{2n-1} \binom{2n}{k} [\eta_+]^k [\eta_-]^{2n-1-k} \right] .(e^{Nh}[\eta_+] + e^{-Nh}[\eta_-]) \\
&= \left[\binom{2n-1}{n-1} [\eta_+]^{n-1} [\eta_-]^n + \binom{2n-1}{n} [\eta_+]^n [\eta_-]^{n-1} \right] \\
&\qquad .(e^{Nh}[\eta_+] + e^{-Nh}[\eta_-]) \\
&= \binom{2n-1}{n-1} e^{Nh} [\eta_+]^n [\eta_-]^n + \binom{2n-1}{n} e^{-Nh} [\eta_+]^n [\eta_-]^n, && (3.3.2)
\end{aligned}$$

and because $\binom{2n-1}{n-1} = \binom{2n-1}{n} = \frac{1}{2}\binom{2n}{n}$, we have

$$= \frac{1}{2}\binom{2n}{n}(e^{Nh} + e^{-Nh})[\eta_+]^n[\eta_-]^n. \tag{3.3.3}$$

(3.3.1) then applies to give

$$= \frac{1}{2}(e^{Nh} + e^{-Nh})[\omega_0]^{2n} = \cosh(Nh)[\omega_0]^{2n}. \tag{3.3.4}$$

$\square$

We continue our discussion, now for $[\omega_k]$'s. Because $[\omega_k]$'s converge to $[\omega_0]$ in cohomology, we see that $[\omega_k]^{2n}$ and $[\omega_k]^{2n-1}.(f^N)^*[\omega_k]$ respectively converge to $[\omega_0]^{2n}$ and $[\omega_0]^{2n-1}.(f^N)^*[\omega_0]$. That is, we have the following limit:

$$\begin{aligned}\lim_{k\to\infty}[\omega_k]^{2n-1}.(f^N)^*[\omega_k] - \cosh(Nh)[\omega_k]^{2n} &= [\omega_0]^{2n-1}.(f^N)^*[\omega_0] - \cosh(Nh)[\omega_0]^{2n} \\ &= 0,\end{aligned}$$

where the last zero is the result of Proposition 3.3. If this is rewritten in the integral form, and combined with the local computation in Proposition 3.2, we have

$$\int_X \left[\frac{1}{n}\sum_{j=1}^{n}\cosh\left(\sigma_j^{(k)}(x, Nh)\right)\right]\omega_k^{2n} - \cosh(Nh)\int_X \omega_k^{2n} \xrightarrow{k\to\infty} 0. \tag{3.3.5}$$

By Lemma 2.6, as we have $\omega_k^{2n} = \text{vol}$, (3.3.5) is equivalently written as

$$\int_X \left[\frac{1}{n}\sum_{j=1}^{n}\cosh\left(\sigma_j^{(k)}(x, Nh)\right)\right] d\text{vol}(x) \xrightarrow{k\to\infty} \cosh(Nh). \tag{3.3.6}$$

3.4 Relation with Lyapunov Exponents

Although log-singular values $\sigma_i^{(k)}(x, Nh)$ come from analytic interests on forms $(f^N)^*\omega_k$, these quantities are closely related with Lyapunov exponents (cf. Lemma 2.8). In particular, we have the following

Proposition 3.4. *The log-singular values $\sigma_i^{(k)}(x, Nh)$'s in* (3.1.3) *satisfy*

$$\frac{1}{N}\sum_{j=1}^{n}\int_X \sigma_j^{(k)}(x, Nh)\,\mathrm{dvol}(x) \geq 2\sum_{i=1}^{n}\chi_i = nh. \tag{3.4.1}$$

Consequently, we have

$$\int_X \left[\frac{1}{n}\sum_{j=1}^{n}\sigma_j^{(k)}(x, Nh)\right]\,\mathrm{dvol}(x) \geq Nh. \tag{3.4.2}$$

We start by establish the following computational lemma to clear the potential confusion caused by the definition of log-singular values.

Lemma 3.5. *Endow the holomorphic tangent bundle $T^{1,0}X$ with the metric ω_k. Then*

$$\log\left\|(D_x f^N)^{\wedge n}\right\|_{op} = \frac{1}{2}\left(\sigma_1^{(k)}(x, Nh) + \cdots + \sigma_n^{(k)}(x, Nh)\right),$$

where the wedge is taken over $\mathbb{C}$.

Proof. Fix a local coordinate $(z_1, \ldots, z_{2n})$ as in (3.1.1) and (3.1.3). Denote

$$\partial_i := \frac{\partial}{\partial z_i}, \quad \overline{\partial}_i := \frac{\partial}{\partial \overline{z}_i},$$

so that ∂_i's form a $\mathbb{C}$-basis of $T_x^{1,0}X$. Let $\|\cdot\|_{k,j}$ be the norm notation for the metric $(f^j)^*\omega_k$.

Then one evaluates the operator norm as

$$\|(D_x f^N)^{\wedge n}\|_{op} = \sup_{\substack{v\in\bigwedge^n T_x^{1,0}X \\ v\neq 0}} \frac{\|(D_x f^N)^{\wedge n}(v)\|_k}{\|v\|_k} = \sup_{\substack{v\in\bigwedge^n T_x^{1,0}X \\ v\neq 0}} \frac{\|v\|_{k,N}}{\|v\|_{k,0}}. \tag{3.4.3}$$

By (3.1.3), we have

$$\|\partial_i\|_{k,N} = e^{\sigma_i^{(k)}(x,Nh)/2}, \tag{3.4.4}$$

and by (3.1.1), we have $\|\partial_i\|_{k,0} = 1$. Note also that the vectors ∂_i's are orthogonal at x, with respect to the metrics ω_k and $(f^N)^*\omega_k$. Because $\sigma_i^{(k)}$ is decreasing in i, we see that $v = \partial_1 \wedge \cdots \wedge \partial_n$ maximizes the fraction in (3.4.3). Hence we evaluate

$$\begin{aligned} \|(D_x f^N)^{\wedge n}\|_{op} &= \frac{\|\partial_1 \wedge \cdots \wedge \partial_n\|_{k,N}}{\|\partial_1 \wedge \cdots \wedge \partial_n\|_{k,0}} \\ &= \frac{\exp\left(\frac{1}{2}\sum_{i=1}^n \sigma_i^{(k)}(x,Nh)\right)}{1}, \end{aligned}$$

so taking the logarithm we have our claim. □

Now we prove Proposition 3.4.

Proof. The quantity $\log\left\|(D_x f^N)^{\wedge n}\right\|_{op}$ gains the interest because of its relation with Lyapunov exponents [Ruelle, 1979, §2.1]:

$$\lim_{N\to\infty} \frac{1}{N} \log \|(D_x f^N)^{\wedge 2n}\|_{op} = \chi_1 + \cdots + \chi_n = \frac{nh}{2}, \tag{3.4.5}$$

for μ-a.e. x. Now setting

$$I_N := \int_X \log \|(D_x f^N)^{\wedge 2n}\|_{op} \, d\mathrm{vol}(x), \tag{3.4.6}$$

via Proposition 3.5, we have

$$I_N = \frac{1}{2}\int_X (\sigma_1^{(k)}(x, Nh) + \cdots + \sigma_n^{(k)}(x, Nh))\, d\mathrm{vol}(x). \tag{3.4.7}$$

The intergrals $\frac{1}{N}I_N$'s are having nonnegative integrands. Fatou's Lemma $\int \liminf f_n \leq \liminf \int f_n$ thus applies, which induces the following from (3.4.5):

$$\begin{aligned} \lim_{N\to\infty} \frac{1}{N} I_N &\geq \int_X \lim_{N\to\infty} \frac{1}{N} \log \|(D_x f^N)^{\wedge 2n}\|_{op}\, d\mathrm{vol}(x) \\ &= \frac{nh}{2}. \end{aligned} \tag{3.4.8}$$

Now the inequality $\|(D_x f^{N+M})^{\wedge 2n}\|_{op} \leq \|(D_x f^N)^{\wedge 2n}\|_{op} \cdot \|(D_{f^N(x)} f^M)^{\wedge 2n}\|_{op}$ induces subadditivity $I_{N+M} \leq I_N + I_M$; by Fekete's Lemma Fekete [1923],

$$\inf_{N\geq 1} \frac{1}{N} I_N = \lim_{N\to\infty} \frac{1}{N} I_N, \tag{3.4.9}$$

thus for any N, we have

$$\frac{1}{N} I_N \geq \lim_{N\to\infty} \frac{1}{N} I_N \geq \frac{nh}{2}. \tag{3.4.10}$$

This finishes the proof, thanks to (3.4.7). □

3.5 Jensen's Inequality and Log-singular Values at the Limit Metric

Now we demonstrate how to combine (3.3.6) and Proposition 3.4. The trick is to use Jensen's inequality, combined with the (strong) convexity of the cosh function.

The upshot of this combination is a result on the log-singular values of the 'limit metrics' $(f^N)^*\omega_0$, relative to ω_0 (Corollary 3.7). By this, we get a simple local representations of these metrics.

Let B be a probability space, whose underlying space is $(X \setminus E) \times \{1, \cdots, n\}$, and whose probability measure is $\mathrm{vol} \times (\frac{1}{n}\#)$, where $\#$ is the counting measure. For $(x, j) \in B$, define a random variable $\Sigma^{(k)}$ as $\Sigma^{(k)}(x, j) = \sigma_j^{(k)}(x, Nh)$.

Then (3.3.6) can be rewritten as

$$\mathbb{E}[\cosh(\Sigma^{(k)})] - \cosh(Nh) \xrightarrow{k\to\infty} 0, \tag{3.5.1}$$

and (3.4.2) following Proposition 3.4 can be rewritten as

$$\mathbb{E}[\Sigma^{(k)}] \geq Nh. \tag{3.5.2}$$

To motivate what follows, we note that Jensen's inequality applied to the convex function cosh gives: $\mathbb{E}[\cosh(\Sigma^{(k)})] \geq \cosh(\mathbb{E}[\Sigma^{(k)}]) \geq \cosh(Nh)$. Then (3.5.1) implies that the inequality asymptotically collapses as $k \to \infty$. This observation is the root of the following

Proposition 3.6. *As $k \to \infty$, the variance of $\Sigma^{(k)}$ is converging to 0, and the expected value of $\Sigma^{(k)}$ is converging to Nh. That is,*

$$\int_X \frac{1}{n} \sum_{j=1}^{n} \left(\sigma_j^{(k)}(x, Nh) - Nh\right)^2 \, d\mathrm{vol}(x) \xrightarrow{k\to\infty} 0.$$

Proof. We start with an elementary inequality, which holds for any $x, a \in \mathbb{R}$:

$$\cosh(x) \geq \cosh(a) + \sinh(a) \cdot (x - a) + \frac{1}{2}(x - a)^2. \tag{3.5.3}$$

Apply $x = \Sigma^{(k)}$ and $a = Nh$ into (3.5.3). Taking the average, we then get

$$\mathbb{E}[\cosh(\Sigma^{(k)})] \geq \cosh(Nh) + \sinh(Nh) \cdot \underbrace{\left(\mathbb{E}[\Sigma^{(k)}] - Nh\right)}_{\geq 0 \text{ by } (3.5.2)} + \frac{1}{2}\mathbb{E}[(\Sigma^{(k)} - Nh)^2]$$

$$\geq \cosh(Nh) + \frac{1}{2}\mathbb{E}[(\Sigma^{(k)} - Nh)^2].$$

This implies

$$0 \leq \mathbb{E}[(\Sigma^{(k)} - Nh)^2] \leq 2 \cdot \Big(\mathbb{E}[\cosh(\Sigma^{(k)})] - \cosh(Nh)\Big) \xrightarrow{k\to\infty} 0,$$

where we have used the limit fact (3.5.1). This implies $\mathbb{E}[(\Sigma^{(k)} - Nh)^2] \to 0$. Our proposition restates this limit fact. $\square$

Passing to a subsequence of (ω_k) if necessary, we further have that $\sigma_j^{(k)}(x, Nh) \to Nh$, for vol-a.e. x.

The implication of Proposition 3.6 to the metric ω_0, the Kähler metric on $X\setminus E$ introduced in Lemma 2.6, is the following

Corollary 3.7. *For each $x \in X \setminus E$, the log-singular values of $(f^N)^*\omega_0$ relative to ω_0 are Nh and $-Nh$, counted n times respectively.*

That is, for each $x \in X\setminus E$, one can find a holomorphic coordinate $(z_1, \cdots, z_{2n})$ in which the following expressions hold in the tangent space at x.

$$\omega_0 = \frac{\sqrt{-1}}{2}\sum_{i=1}^{2n} dz_i \wedge d\overline{z}_i, \tag{3.5.4}$$

$$(f^N)^*\omega_0 = \frac{\sqrt{-1}}{2}\sum_{\mu=1}^{n} e^{Nh}\, dz_\mu \wedge d\overline{z}_\mu + e^{-Nh}\, dz_{n+\mu} \wedge d\overline{z}_{n+\mu}. \tag{3.5.5}$$

Proof. First, fix $x \in X \setminus E$ in which $\sigma_j^{(k)}(x, Nh) \to Nh$ as $k \to \infty$.

As $(\omega_k) \to \omega_0$ in $C^\infty_{\mathrm{loc}}(X \setminus E)$ topology, focusing on the compact set $\{x, f^N(x)\}$, we see that the matrices $\left(h_{ij}^{(k)}\right)$ (3.1.1) at x converge to the analogous matrix $\left(h_{ij}\right)$ for ω_0, at x. As eigenvalues behave continuously with perturbing matrix [Kato, 1995, Theorem II.5.1], this implies that the numbers $\exp\left(\sigma_j^{(k)}(x, Nh)\right)$'s approximate eigenvalues of $\left(h_{ij}\right)$ as well.

Thanks to our assumption on x, this implies that $\left(h_{ij}\right)$ has eigenvalues e^{Nh} and e^{-Nh}, counted n times for each.

Now to claim this for *all* $x \in X \setminus E$, we note that ω_0 is smooth; thus the matrices $\left(h_{ij}\right)$ also vary smoothly with respect to x. As eigenvalues thus behaves continuously, we still have the constant eigenvalues for all $x \in X \setminus E$. $\square$

CHAPTER 4

STABLE AND UNSTABLE MANIFOLDS

The local expressions of $(f^N)^*\omega_0$ and ω_0 (Corollary 3.7) vividly shows that f^N is expanding and contracting along certain directions in a uniform rate. Technically, these directions are dependent on the time N, but one can show that these directions are actually time independent (Lemma 4.1). By this, we establish the uniform hyperbolicity (Proposition 4.2).

This is perhaps one of the rarest moment where one can describe Oseledets splitting (cf. [Filip, 2019a, Theorem 2.2.6][Ruelle, 1979, Theorem 1.6]) without limits and thus can verify that it is smooth. But we have a better trait: the distributions define *holomorphic* foliations (Proposition 4.3). One can establish this using upper and lower estimates on the growth rate of f^N along stable or unstable distributions, which is typically more than what we know even with uniformly hyperbolic settings.

4.1 Stable and Unstable Distributions

Corollary 3.7 tells that, for every point $x \in X$ and $N \in \mathbb{Z}$, there exists n-subspaces $E^+_{N,x}, E^-_{N,x} \subset T^{1,0}_x X$ such that every $v \in E^\pm_{N,x}$ has $((f^N)^*\omega_0)_x(v, v) = e^{\pm Nh}\omega_{0,x}(v, v)$.

We first show that these subspaces $E^\pm_{N,x}$ are not dependent on N. This is essentially due to that log-singular values of $(f^N)^*\omega_0$ uniformly cumulates by $\pm h$ as we proceed N; to have the 'optimal cumulation,' we find that the directions that expands e^{Nh} in $(f^N)^*\omega_0$ should be also expanding $e^{(N+1)h}$ in $(f^{N+1})^*\omega_0$, and simliarly for contracting directions.

Lemma 4.1. *Denote $\|\cdot\|_j$ for the norm associated to the metric $(f^j)^*\omega_0$. For any $x \in X \setminus E$ and $N \in \mathbb{Z}_{>0}$, define the following subsets E^{+N}_x, E^{-N}_x and subspaces F^{+N}_x, F^{-N}_x in the* *holomorphic tangent space* $T_x^{1,0}X$*:*

$$E_x^{\pm N} = \{v \in T_x^{1,0}X : \|v\|_N = e^{\pm Nh/2}\|v\|_0\}, \tag{4.1.1}$$

$$F_x^{\pm N} = \{v \in T_x^{1,0}X : (f^N)^*\omega_0(v,w) = e^{\pm Nh}\omega_0(v,w)\ \forall w \in T_xX\}. \tag{4.1.2}$$

Then we have $E_x^{+N} = F_x^{+N} = E_x^{+1} = F_x^{+1}$ *and* $E_x^{-N} = F_x^{-N} = E_x^{-1} = F_x^{-1}$ *for all* $N > 0$ *and* $x \in X \setminus E$*. Furthermore, the distributions* $E^{\pm 1}$ *defined in these fashion are* f*-invariant, i.e.,* $D_xf(E_x^{\pm 1}) = E_{f(x)}^{\pm 1}$*, and have complex dimension* n*.*

Proof. Fix $N > 0$. Fix a holomorphic coordinate $(z_1, \cdots, z_{2n})$ at x that appears in the conclusion of the Corollary 3.7. The corresponding holomorphic vectors will have shorthand notations

$$\frac{\partial}{\partial z_i} =: \partial_i.$$

Denote G_x^{+N} for the (complex) span of the vectors $\partial_1, \ldots, \partial_n$, and G_x^{-N} for the span of the vectors $\partial_{n+1}, \ldots, \partial_{2n}$.

We first claim that $E_x^{+N} = F_x^{+N} = G_x^{+N}$ and $E_x^{-N} = F_x^{-N} = G_x^{-N}$. We show $F_x^{+N} \subset E_x^{+N} \subset G_x^{+N} \subset F_x^{+N}$ to establish the former; the latter can be dealt similarly.

- Let $v \in F_x^{+N}$. In the identity $(f^N)^*\omega_0(v,w) = e^{Nh}\omega_0(v,w)$, we plug in $w = v$. By that we obtain $\|v\|_N^2 = e^{Nh}\|v\|_0^2$, so $v \in E_x^{+N}$.

- Let $v \in E_x^{+N}$. Decompose $v = v^+ + v^-$ where $v^\pm \in G_x^{\pm N}$. Since both ω_0 and $(f^N)^*\omega_0$ view that $G_x^{\pm N}$ are orthogonal, we have

  $$\|v\|_N^2 = \|v^+\|_N^2 + \|v^-\|_N^2.$$

  By (3.5.5), we can directly evaluate $\|v^\pm\|_N^2$ relative to $\|v^\pm\|_0^2$ and obtain

  $$= e^{Nh}\|v^+\|_0^2 + e^{-Nh}\|v^-\|_0^2$$

$$= e^{Nh}\|v\|_0^2 + (e^{-Nh} - e^{Nh})\|v^-\|_0^2. \tag{4.1.3}$$

Thus if we have $\|v\|_N = e^{Nh/2}\|v\|_0$, we necessarily have $\|v^-\|_0 = 0$. Hence $v = v^+ \in G_x^{+N}$.

- Let $v \in G_x^{+N}$. Then for any $w^+ \in G_x^{+N}$, we have

$$(f^N)^*\omega_0(v, w^+) = e^{Nh}\omega_0(v, w^+)$$

(by (3.5.5)). On the other hand, for any $w^- \in G_x^{-N}$, as this is orthogonal to G_x^{+N}, we have

$$(f^N)^*\omega_0(v, w^-) = 0 = e^{Nh}\omega_0(v, w^-).$$

These implies $v \in F_x^{+N}$.

We note that, as $G_x^{\pm N}$ have dimensions n, so are $E_x^{\pm N}$. Furthermore, by the equation (4.1.3), we have an inequality

$$\|v\|_N \leq e^{Nh/2}\|v\|_0, \tag{4.1.4}$$

with equality holding if and only if $v \in E_x^{+N}$.

It remains to show that $E_x^{+N} = E_x^{+1}$, for $N > 1$. Let $v \in E_x^{+N}$. Then,

$$\begin{aligned} \frac{Nh}{2} = \log\frac{\|v\|_N}{\|v\|_0} &= \log\frac{\|v\|_N}{\|v\|_1} + \log\frac{\|v\|_1}{\|v\|_0} \\ &= \log\frac{\|f_*v\|_{N-1}}{\|f_*v\|_0} + \log\frac{\|v\|_1}{\|v\|_0} \\ &\leq \frac{(N-1)h}{2} + \frac{h}{2} = \frac{Nh}{2}. \end{aligned} \tag{4.1.5}$$

In (4.1.5), we have used the inequality (4.1.4). Comparing two sides, we find that we have the equality for (4.1.5). But then the equality condition of (4.1.4) tells that (a) $v \in E_x^{+1}$ and (b) $f_*v = D_xf(v) \in E_{f(x)}^{+(N-1)}$. Therefore (a′) $E_x^{+N} \subset E_x^{+1}$ and (b′) $D_xf(E_x^{+N}) \subset E_{f(x)}^{+(N-1)}$

hold, which turn out to be equalities thanks to dimension comparisons. This establishes both $E_x^{+N} = E_x^{+1}$ for all $N > 1$ and the f-invariance.

That $E_x^{-N} = E_x^{-1}$ and their f-invariance are shown in a similar way, but we need to change (4.1.4) to $\|v\|_0 \leq e^{Nh/2}\|v\|_N$, with equality iff $v \in E_x^{-N}$. A further change is required for (4.1.5), this time focusing on $Nh/2 = \log(\|v\|_0/\|v\|_N)$ whenever $v \in E_x^{-N}$. $\square$

Denote E^+, E^- for distriutions E^{+1}, E^{-1} in the above Lemma, respectively. They then serve as unstable and stable distributions, respectively.

Proposition 4.2. *We have the following operator norms, with respect to ω_0:*

$$\|Df^N|E^{\pm}\|_{op} = e^{\pm Nh/2}, \quad \text{and} \quad \|Df^{-N}|E^{\pm}\|_{op} = e^{\mp Nh/2},$$

applied for any integer $N \in \mathbb{Z}$. *In particular, f is uniformly hyperbolic on $X \setminus E$, and E^+, E^- respectively denotes unstable and stable distributions on $X \setminus E$.*

Proof. We have shown that $\|Df^N|E^{\pm}\|_{op} = e^{\pm Nh/2}$ in Lemma 4.1, for $N > 0$. It remains to show the same identity for $N < 0$. As usual, for $j \in \mathbb{Z}$, we denote $\|\cdot\|_j$ by the norm associated to the metric $(f^j)^*\omega_0$.

Fix $N > 0$. To estimate $\|Df^{-N}|E^{\pm}\|_{op}$, we pick up $v \in E_x^{\pm}$ and estimate $\|v\|_{-N}/\|v\|_0$. Here, by definition of $(f^{-N})^*\omega_0$, we have

$$\frac{\|v\|_{-N}}{\|v\|_0} = \frac{\|f_*^{-N}v\|_0}{\|f_*^{-N}v\|_N} = \frac{1}{e^{\pm Nh/2}} = e^{\mp Nh/2},$$

since $f_*^{-N}v \in E^{\pm}_{f^{-N}(x)}$ as well. This shows the claim. $\square$

The expression (4.1.2) shows that, $E^{\pm}$ are characterized by C^∞ conditions (essentially because ω_0 is smooth). Therefore they are C^∞ distributions. One can appropriately multiply matrices for $f^*\omega_0$ and ω_0^{-1} to find explicit descriptions for C^∞ vector field generators of them.

4.2 Holomorphicity of the Stable Foliation

So far we have studied about the stable and unstable distributions, and concluded that they are defined in a C^∞ manner. What can be claimed furthermore is that, stable and unstable foliations are actually holomorphic.

As shown in [Ghys, 1995, Lemma 2.1], each leaf of the foliations $W^\pm$ generated by $E^\pm$ (respectively), are holomorphic manifolds. In particular, unstable vector fields are holomorphic along the unstable manifolds, and it thus remains to show its holomorphicity along the transverse direction. (Similar claim may be made for stable vector fields.)

The trick is to use the *Poincaré map*, as described in [Mañé, 1987, §III.3]. For a generic Poincaré map ϕ between local holomorphic unstable manifolds, we consider the commutator $[D\phi, I]$:

$$[D\phi, I]_x = D_x\phi \circ I_x - I_{\phi(x)} \circ D_x\phi.$$

If $[D\phi, I] = 0$ can be shown, then it shows that Poincaré maps are holomorphic, showing the desired claim.

This goal setup is encoded in the following

Proposition 4.3. *Let U, U' be unstable local manifolds, not intersecting with one another, and close enough to induce a Poincaré map $\phi\colon U \to U'$. Then $[D\phi, I]_x = 0$, for all $x \in U$.*

4.2.1 On a Foliated Chart

Fix a foliated chart $(V, \mathbf{x})$, for both $E^\pm$. This is possible because $E^\pm$ are both generated by some C^∞ vector fields. Now suppose x and $\phi(x)$ are in V. Then a neighborhood of $x \in U$ and $\phi(x) \in U'$ are laid along the coordinate directions for E^-, and hence ϕ near x is described as a coordinate shift. This concludes, $D\phi$ is represented as an identity matrix on $(V, \mathbf{x})$.

The complex structure I is understood as a family of linear maps on real tangent spaces

$T_{\mathbb{R},y}X \to T_{\mathbb{R},y}X$. So if we fix coordinates, say the foliated coordinates on V, we have a matrix representation $I_y \in \mathrm{GL}(4n, \mathbb{R})$ for each $y \in V$. Collecting these remarks, we conclude as follows.

Lemma 4.4. *Let U, U' be unstable local manifolds as in Proposition 4.3, and suppose $x \in U$ is such that $x, \phi(x) \in V$. For the matrix representation $\{I_y\}_{y\in V}$ of the complex structure I on V, we then have*

$$[D\phi, I]_x = I_x - I_{\phi(x)}. \tag{4.2.1}$$

Furthermore, shrinking V if necessary, we assume that the family $\{I_y\}_{y\in V}$ satisfies the Lipschitz condition, i.e.,

$$\|I_p - I_q\| \leq C \cdot \mathrm{dist}(p, q). \tag{4.2.2}$$

4.2.2 From Ergodicity

Apparently, $(V, \mathbf{x})$ is set on an arbitrary place of X, thus it is hard to expect $x, \phi(x) \in V$ in most cases. We have shown that, in Proposition 2.7, the volume measure (which is the Green measure) is ergodic with respect to $f\colon X \to X$. We then claim the following.

Lemma 4.5. *Let U, U' be unstable local manifolds as in Proposition 4.3. Then for* vol*-a.e. x, there are infinitely many N such that $f^N(x), f^N\phi(x) \in V$.*

Proof. Let $V' \Subset V$ be a nonempty precompact open subset. Let $\epsilon = \inf_{y\in V'} \mathrm{dist}(y, \partial V)$, a positive number.

By the uniform contraction of E^- along f, we have

$$\mathrm{dist}_{E^-}(f^N(x), f^N\phi(x)) \leq C(U, U', V) \cdot e^{-Nh/2}\mathrm{dist}_{E^-}(x, \phi(x)), \tag{4.2.3}$$

where dist_{E^-} is the distance measured along the stable leaves, and $C(U, U', V) > 0$ is a constant depending on U, U' and V. As $\mathrm{dist}(p, q) \leq \mathrm{dist}_{E^-}(p, q)$, this implies that we have

$\mathrm{dist}(f^N(x), f^N\phi(x)) < \epsilon$ for sufficiently large $N \geq N_0$. In particular, if $f^N(x) \in V'$, $N \geq N_0$, then $f^N(x), f^N\phi(x) \in V$.

Now because V' is a nonempty open set, it has a nonzero volume. Thanks to Birkhoff ergodicity, we thus conclude that there are infinitely many N such that $f^N(x) \in V'$, for vol-a.e. x. The lemma then follows. $\square$

4.2.3 Future Estimates

To show Proposition 4.3, we use the trick of 'sending to the future,' as commonly seen in [Ghys, 1995, Theorem 2.2] and [Mañé, 1987, Theorem III.3.1]. The trick starts from the following split:

$$[D\phi, I]_x = D_{f^N\phi(x)} f^{-N} \circ [D(f^N\phi f^{-N}), I]_{f^N(x)} \circ D_x f^N. \tag{4.2.4}$$

We then estimate each factor: $(Df^{-N}|Tf^N(U')) = Df^{-N}|E^+$, $(Df^N|TU) = Df^N|E^+$, and $[D(f^N\phi f^{-N}), I]$. (Below, $C_1, C_2 > 0$ are constants that only depend on the Poincaré map ϕ.)

- For $Df^{-N}|E^+$, we recall that Df^{-1} is (under ω_0) uniformly contracting E^+ with the rate $e^{-h/2}$. Applying such, we get

$$\|Df^{-N}|E^+\| \leq C_1 \cdot e^{-Nh/2}. \tag{4.2.5}$$

- For $Df^N|E^+$, we recall that Df is (under ω_0) uniformly expanding E^+ with the rate $e^{h/2}$. Applying such, we get

$$\|Df^N|E^+\| \leq C_2 \cdot e^{Nh/2}. \tag{4.2.6}$$

- Finally, for $[D(f^N\phi f^{-N}), I]$, we pick N such that $f^N(x), f^N\phi(x) \in V$, by Lemma 4.5.

(Note that this may be done only for vol-a.e. x.)

Note that $f^N \phi f^{-N}$ is a Poincaré map $f^N(U) \to f^N(U')$. Thus Lemma 4.4 applies to give,

$$\begin{aligned}\left\| [D(f^N \phi f^{-N}), I]_{f^N(x)} \right\| &= \left\| I_{f^N(x)} - I_{f^N \phi(x)} \right\| \\ &\leq C \cdot \mathrm{dist}(f^N(x), f^N \phi(x)).\end{aligned}$$

Via (4.2.3), we further estimate,

$$\begin{aligned} &\leq C \cdot \mathrm{dist}_{E^-}(f^N(x), f^N \phi(x)) \\ &\leq C \cdot e^{-Nh/2} \mathrm{dist}_{E^-}(x, \phi(x)). \end{aligned} \tag{4.2.7}$$

Combining all three estimates (4.2.5), (4.2.6), and (4.2.7), we obtain, in (4.2.4),

$$\|[D\phi, I]_x\| \leq C_1 C_2 C \cdot e^{-Nh/2} \mathrm{dist}_{E^-}(x, \phi(x)),$$

whenever N satisfies $f^N(x), f^N \phi(x) \in V$. For vol-a.e. x, there are infinitely many such N's; sending $N \to \infty$, we have $[D\phi, I]_x = 0$, vol-a.e. x. Appealing to the continuity of $x \mapsto [D\phi, I]_x$, we prove the Proposition 4.3.

4.3 Flatness

Thanks to the holomorphicity of stable and unstable foliations, we have the following flatness result. This serves as a key ingredient to infer that the initial manifold X is induced from a torus.

We note here that the proof below can be applied to shorten some known arguments for K3 surfaces, e.g., [Filip and Tosatti, 2021, Proposition 3.2.1].

Proposition 4.6. *The metric ω_0 on $X \setminus E$ is flat.*

Proof. Because E^+, and by a likewise proof with time sent backwards, E^-, are all holomorphic, a standard differential geometry trick builds a holomorphic coordinate $(w_1, \cdots, w_{2n})$ such that

- $E^+ = \bigcap_{i=1}^n \ker(dw_i)$, and
- $E^- = \bigcap_{j=n+1}^{2n} \ker(dw_j)$.

Moreover, it is easy to check that E^- and E^+ are orthogonal under ω_0, as follows. For $v \in E^-$ and $w \in E^+$, we get

$$e^{-h}\omega_0(v, w) = f^*\omega_0(v, w) = -f^*\omega_0(w, v) = -e^h\omega_0(w, v) = e^h\omega_0(v, w),$$

and thus $\omega_0(v, w) = 0$. Therefore, one can re-write ω_0 as

$$\omega_0 = \frac{\sqrt{-1}}{2}\left[\sum_{i,j=1}^{n} a_{i\bar{j}}\, dw_i \wedge d\overline{w}_j + \sum_{k,\ell=n+1}^{2n} b_{k\bar{\ell}}\, dw_k \wedge d\overline{w}_\ell\right],$$

with some positive-definite matrix-valued functions $(a_{i\bar{j}})$ and $(b_{k\bar{\ell}})$.

That $d\omega_0 = 0$ then implies, $w_{n+1}, \overline{w}_{n+1}, \cdots, w_{2n}, \overline{w}_{2n}$-derivatives of $a_{i\bar{j}}$ shall vanish and $w_1, \overline{w}_1, \cdots, w_n, \overline{w}_n$-derivatives of $b_{k\bar{\ell}}$ shall vanish. Consequently, ω_0 is split completely into:

$$\omega_0 = \frac{\sqrt{-1}}{2}\omega_0^-(w_1, \cdots, w_n) + \frac{\sqrt{-1}}{2}\omega_0^+(w_{n+1}, \cdots, w_{2n}).$$

In short, the metric ω_0 decomposes as $\omega_0 = \omega_0^- \times \omega_0^+$ (locally). Now consider the Levi-Civita connection ∇ of the metric ω_0. This connection satisfies the followings:

1. $\nabla\Omega = 0$. As ω_k's satisfy this, and $\nabla\Omega = 0$ is expressed with Christoffel symbols of the metric, that $\omega_k \to \omega_0$ in C^∞_{loc} certifies this for ω_0 as well.

2. $\nabla E^+ \subset E^+$ and $\nabla E^- \subset E^-$. This follows from the local product structure of $\omega_0 = \omega_0^- \times \omega_0^+$, where each $\omega_0^\pm$ is supported on $E^\pm$, respectively.

3. For vector fields Z^+ on E^+ and Z^- on E^-, the following holds:

$$\nabla_{Z^-} Z^+ = p^+([Z^-, Z^+]),$$
$$\nabla_{Z^+} Z^- = p^-([Z^+, Z^-]),$$

   where $p^\pm$ denotes the parallel projections $E^- \oplus E^+ \to E^\pm$. This follows from the torsion-free property $[Z^-, Z^+] = \nabla_{Z^-} Z^+ - \nabla_{Z^+} Z^-$, as well as $\nabla E^\pm \subset E^\pm$ verified above.

According to [Benoist et al., 1992, Lemme 3.4.4], connections satisfying all three above are unique. Now by the proof of [Benoist et al., 1992, Lemme 2.2.3(b)], we get that ∇ is a flat connection, i.e., ω_0 itself is flat on $X \setminus E$. (The cited theorems are all local, thus the non-compact nature of $X \setminus E$ does not matter here.) $\square$

CHAPTER 5

THE KUMMER RIGIDITY

In this chapter, we prove Theorem 1.1, collecting the ingredients that we have collected so far. At Proposition 4.6, we have shown that for a singular metric ω_0 on X, it is flat on its non-singular locus. Thus if we can contract the singular locus and view X as a flat manifold with some singularities, perhaps this will let us to say that X is a torus, or at least something close to that.

It is the contraction construction that have required projectivity assumption. If we have X projective, then there is an algebro-geometric construction which admits us to contract the null locus E. Then the contracted X gives rise to a variety Y which is flat outside its singularity, which must be a torus quotient (cf. [Greb et al., 2016, Corollary 1.16][Claudon et al., 2020, Theorem D]). By that we show that X is desingularization of Y and $f\colon X \to X$ is induced from a linear map.

5.1 Normal Varieties which are Torus Quotients

The following is a consequence of [Greb et al., 2016, Corollary 1.16].

Theorem 5.1. *Suppose Y is a normal complex projective variety that has klt singularities. If $T|Y_{\mathrm{reg}}$ is a flat (in the analytic sense), then Y is a quotient of a complex torus by a finite group acting freely in codimension one.*

Proof. Apply [Greb et al., 2016, Corollary 1.16] to the klt pair $(Y, \varnothing)$. As Y is projective, there exists an abelian variety $\mathbb{T}$ and a finite Galois morphism $\pi\colon \mathbb{T} \to Y$ that is étale in codimension 1. That is, there exsits a finite group $\Gamma \subset \mathrm{Aut}(\mathbb{T})$ such that π is the quotient morphism $\mathbb{T} \to \mathbb{T}/\Gamma = Y$ (cf. [Greb et al., 2016, Definition 3.6]). $\square$

In the followings, we introduce the normal space Y to plug in the above Theorem 5.1. Morally, it is constructed by contracting $E \subset X$ by a contraction $\phi\colon X \to Y$, and the construction of this requires X to be projective (which is also the only place that we use projectivity).

Remark. A recent result [Claudon et al., 2020, Theorem D] gives rise to a generalization of [Greb et al., 2016, Corollary 1.16], applicable for compact Kähler normal complex spaces with klt singluarities (i.e., drops the projectivity assumption). Thus Theorem 5.1 may be extended to the case of a non-projective Y. Nonetheless, we still need X to be projective for constructing Y, in this paper.

5.2 Construction of the Contraction

This section is aimed to prove the following

Proposition 5.2. *There exists a contraction $\phi\colon X \to Y$ in which Y is a normal projective variety, and its regular locus Y_{reg} is the image of $X \setminus E$. Moreover, Y has canonical singularities, and has a Kähler current $\phi_*\omega_0$ on Y_{reg} that is a flat metric on Y_{reg}.*

The proof of this fact extensively uses the fact that X is projective. As a preparation, we first show that the eigenclasses $[\eta_+]$ and $[\eta_-]$ are in fact $(1,1)$-classes of nef Cartier $\mathbb{R}$-divisors. Appealing to the projectivity of X, fix an ample class $[A]$. Then by Proposition 2.4(d), as $n \to \infty$,

$$\frac{\lambda^{-n}}{2q([A],[\eta_-])}(f^n)^*[A] \to [\eta_+]; \quad \frac{\lambda^{-n}}{2q([A],[\eta_+])}(f^{-n})^*[A] \to [\eta_-].$$

Therefore, $\alpha = [\eta_+] + [\eta_-]$ is also (the class of) a Cartier $\mathbb{R}$-divisor, which is big and nef. Note that α is not ample; otherwise, then its null locus $E = \varnothing$, and by Proposition 4.6, we have X a compact flat manifold. The only such manifolds are tori Bieberbach [1911]Bieberbach [1912], which contradicts to that X is simply connected.

The first step of proving Proposition 5.2 is to construct the contraction ϕ. This is essentially done by [Boucksom et al., 2014, Theorem A], but generalized to $\mathbb{R}$-Cartier classes. We present it in the following

Lemma 5.3. *Let α be the $(1,1)$-class of a big and nef Cartier $\mathbb{R}$-divisor, and E be its null locus Collins and Tosatti [2015]. Then one can construct a contraction $\phi\colon X \to Y$ in which $X \setminus E$ is the maximal Zariski open subset in which ϕ maps it isomorphically onto its image. (That is, $E = \operatorname{Exc}(\phi)$.)*

Proof. Denote $\operatorname{Amp}(X)$ and $\operatorname{Big}(X)$ for the cone of $(1,1)$-classes of ample and big Cartier $\mathbb{R}$-divisors, respectively. By Kawamata's Rational Polyhedral Theorem [Kawamata, 1988, Theorem 5.7], the face F of the cone $\operatorname{Big}(X) \cap \overline{\operatorname{Amp}}(X)$ in which α lies on, is represented by a rational linear equation. Consequently, one can write $\alpha = \sum_{\text{finite}} a_i c_1(L_i)$ where each $c_i > 0$ and each L_i is a big and nef line bundle in which $c_1(L_i) \in F$.

Because each L_i is big and nef, by basepoint-free theorems [Birkar et al., 2010, Theorem 3.9.1][Kawamata, 1988, Theorem 1.3], it is semiample. Because all L_i's lie on the same face of the big and nef cone $\operatorname{Big}(X) \cap \overline{\operatorname{Amp}}(X)$, the images of the morphism

$$\Phi_{mL_i}\colon X \to \mathbb{P}H^0(X, mL_i)$$

are isomorphic to each other, when $m \gg 0$ (cf. [Kawamata et al., 1987, Definition 3-2-3]).

For each L_i, denote its augmented base loci as E_i, denote the image of Φ_{mL_i} as Y_i, and the restriction $\phi_i := \Phi_{mL_i}|Y_i$.

We claim that $E_i = E_j$. Fix an isomorphism $\psi\colon Y_i \to Y_j$ in which $\psi \circ \phi_i = \phi_j$. By [Boucksom et al., 2014, Theorem A], the complement $X \setminus E_i$ of the locus is characterized as the maximal Zariski open subset in which Φ_{mL_i} isomorphically sends the subset into its image. Composing ψ to Φ_{mL_i}, we thus see that $X \setminus E_i$ is sent isomorphically into its image via Φ_{mL_j}. Consequently, $X \setminus E_i \subset X \setminus E_j$. Arguing symmetrically, we have the claim.

Fix a bundle L_i and denote $\phi := \phi_i$. An upshot of the above paragraph is, ϕ is a contraction in which $X \setminus E_i$ is the maximal Zariski open subset in which ϕ maps it isomorphically onto its image.

We claim that $E = E_i$. Let $L = \sum L_i$. Denote E' for the augmented base locus of L. Then we have (i) $E' = E_i$, by the same token of showing $E_i = E_j$, and (ii) $E' \subset E \subset E_i$, by the followings. (By [Collins and Tosatti, 2015, Corollary 1.2], it suffices to compare the null loci.)

($E \subset E_i$) Fix any subvariety $V \subset X$ in which $\int_V \alpha^{\dim V} = 0$. By $\alpha = \sum a_i c_1(L_i)$ and as the multinomial theorem gives nonnegative terms, we obtain $\int_V c_1(L_i)^{\dim V} = 0$. Thus $V \subset E_i$, and $E \subset E_i$ follows.

($E' \subset E$) For any subvariety $V \subset X$ in which $\int_V \left(\sum c_1(L_i)\right)^{\dim V} = 0$, expand it with the multinomial theorem with nonnegative terms, to have $\int_V \prod c_1(L_i)^{e_i} = 0$, whenever $\sum e_i = \dim V$. As $\alpha = \sum a_i c_1(L_i)$, again by multinomial theorem, we have $\int_V \alpha^{\dim V} = 0$. This shows $E' \subset E$.

Combining (i) and (ii) we have $E = E_i$, as required. □

Define the *exceptional set* $\operatorname{Exc}(\phi)$ as the minimal Zariski closed subset $E' \subset X$ in which $X \setminus E'$ is mapped isomorphically onto its image by ϕ. By what is stated in Lemma 5.3, $E = \operatorname{Exc}(\phi)$ is thus just the definition. This set is, by inverse function theorem, same as the set of $x \in X$ in which $D_x\phi$ is invertible.

Proof of Proposition 5.2. Construct the contraction map $\phi\colon X \to Y$ by Lemma 5.3, for $\alpha = [\eta_+] + [\eta_-]$. Then $Y_{\text{reg}} = \phi(X \setminus \operatorname{Exc}(\phi)) = \phi(X \setminus E)$ follows.

Recall that there is a flat metric ω_0 on $X \setminus E$ (Proposition 5.1). Pushforwarding this to Y_{reg}, we have a Kähler current $\phi_*\omega_0$ which is also a flat metric, on Y_{reg}.

To see why Y has canonical singularities, we use the remark in [Wierzba, 2003, Remark 1]. As ϕ is a hyperkähler resolution, ϕ is crepant, i.e., $\phi^* K_Y = K_X$. □

5.3 Proof of the Kummer Rigidity

What was claimed about Y in Proposition 5.2 additionally yields that $T|Y_{\text{reg}}$ is flat, hence the hypotheses of the Theorem 5.1 above are met. Indeed, the metric $\phi_*\omega_0|Y_{\text{reg}}$ produces a flat connection on $T|Y_{\text{reg}}$.

Therefore Y is a torus quotient; that is, there exists a complex torus $\mathbb{T} = \mathbb{C}^{2n}/\Lambda$ and a finite group of toral isomorphisms Γ in which $Y = \mathbb{T}/\Gamma$.

To show that f is induced from a hyperbolic linear transform, recall that ϕ isomorphically sends $X \setminus E$ to Y_{reg}. Conjugating $f|_{X\setminus E}$ via ϕ, we then have a map $\widetilde{f}\colon Y_{\text{reg}} \to Y$. This $\widetilde{f}$ lifts to a rational map $\mathbb{T} \dashrightarrow \mathbb{T}$, defined in codimension 1. The only such map is affine-linear [Lo Bianco, 2017, Lemma 1.25], and this descends down to a morphism $\widetilde{f}\colon Y \to Y$. This verifies the desired classification of f, and finishes the proof of Theorem 1.1.

Part II

Vieta Involutions on Tropical Markov Cubics

CHAPTER 6

INTRODUCTION

The second part of this thesis will be devoted to a study of algebraic dynamics on *Markov cubic surfaces*. The focus will be on the family of affine surfaces

$$S_{ABCD}\colon X_1^2 + X_2^2 + X_3^2 + X_1X_2X_3 = AX_1 + BX_2 + CX_3 + D,$$

where A, B, C and D are fixed parameters. In particular, the case where these surfaces are defined over a non-archimedean field K will be of interest, so that our parameters A, B, C and D are in K.

One context in which these (algebraic) surfaces arise naturally is in the character varieties of 1-punctured torus or 4-punctured sphere, which are (real) surfaces whose fundamental group is a free group of rank two or three. The geometric background of this is discussed in Goldman Goldman [2003], Cantat & Loray Cantat and Loray [2009] (see also [Cantat, 2009, §2]), and Rebelo & Roeder [Rebelo and Roeder, 2021, §2.3]. The works listed above also discuss the complex dynamical aspect of algebraic automorphisms on Markov cubics.

Markov surfaces S_{ABCD}, viewed as character varieties, have a natural $\mathrm{PGL}_2(\mathbb{Z})$-action. Moreover, for the 'congruence subgroup of level 2' $\Gamma = \ker(\mathrm{PGL}_2(\mathbb{Z}) \twoheadrightarrow \mathrm{PGL}_2(\mathbb{Z}/2\mathbb{Z})) \leq \mathrm{PGL}_2(\mathbb{Z})$, one can easily describe the algebraic action by Γ concretely. If one looks closely at the formula of the surface S_{ABCD} in question, one can see that they are written as quadratic equations for each variable X_1, X_2, and X_3. Thus, it is natural to think of involutions that interchange the roots of each quadratic equations (after fixing the other variables). These generators can be called Vieta involutions, as they come from Vieta relations. The group they generate can be called the *Vieta group*. This group is, as abstract groups, isomorphic to the 3-fold free product $(\mathbb{Z}/2\mathbb{Z})^{*3}$ of the group of order 2, by a theorem of Èl′-Huti Èl′-Huti [1974].

The results of Spalding & Veselov Spalding and Veselov [2020] and Filip Filip [2019b] put the Vieta group action in a tropical perspective. Specifically, they compare the Vieta group action on the Cayley cubic S_{0004} to the toral linear actions of Γ (the congruence subgroup mentioned above). It is interesting to note that the matrices seen in the toral linear action [Filip, 2019b, Proposition 6.1.2] are the same matrices that $\Gamma \leq \mathrm{PGL}_2(\mathbb{Z})$ acts on the character variety [Cantat, 2009, §2.4]. Additionally, another work by Spalding & Veselov Spalding and Veselov [2017] describes the Lyapunov spectrum of the Vieta group action on the Cayley cubic S_{0004}, relating the infinite walk in Γ to a corresponding continued fraction.

In this part, we will extend the tropical perspective and analyze the actions of Vieta involutions on a surface S_{ABCD} defined over a non-archimedean field. If at least one of the parameters has a negative valuation, then this action will be found to have a dense open invariant set that conjugates to the hyperbolic reflection group on the hyperbolic plane $\mathbb{H}^2$. Otherwise, after some radial projections, we still have a trace of hyperbolic reflections but on the boundary $\partial\mathbb{H}^2$ of the hyperbolic plane. The goal of this part is to establish the results, with further information that helps us to understand how the conjugacy above works on the tropicalization of the surface S_{ABCD}.

Outline of the Part This part consists of the following chapters. Chapter 7 introduces the origin of the Markov cubics S_{ABCD} in the study of character varieties and introduces the Vieta involutions of it. Chapter 8 lists some preliminary knowledge on the tropicalization. Chapter 9 discusses the 'skeleton' of the surface S_{ABCD}, which is a subset of the tropicalization that is canonically determined by the variety. Chapter 10 then starts a detailed description of the Vieta involution on a single skeleton. Chapter 11 discusses the two modes of Vieta involutions on skeleta, which are determined by the valuations of the parameters. This is a chapter that establishes the main comparison results. Chapter 12 discusses what happens to the complement of the open set that corresponds to the hyperbolic plane; this turns out to be a countable union of rays in the skeleton.

CHAPTER 7

MARKOV CUBIC SURFACE

In this chapter, we introduce the Markov cubic surface

$$S_{ABCD}\colon X_1^2 + X_2^2 + X_3^2 + X_1X_2X_3 = AX_1 + BX_2 + CX_3 + D,$$

in the context of character varieties of the 4-punctured 2-sphere. We then study the polynomial automorphisms on this variety, which is virtually generated by Vieta involutions.

7.1 Character Variety of the 4-punctured 2-sphere

We follow Cantat [2009] for this and the next section. Let $\mathbb{S}_4^2$ be the four punctured sphere, whose fundamental group is a free group of rank 3,

$$\pi = \pi_1(\mathbb{S}_4^2) = \langle \alpha, \beta, \gamma, \delta \mid \alpha\beta\gamma\delta = 1 \rangle.$$

Here, the homotopy classes $\alpha, \dots, \delta$ correspond to loops around the puncture. Let $\mathrm{Rep}(\mathbb{S}_4^2)$ be the set of representations of π into $\mathrm{SL}_2(\mathbb{C})$, which can be identified with Cartesian product of three $\mathrm{SL}_2(\mathbb{C})$'s, thus an algebraic variety.

Associated to each $\rho \in \mathrm{Rep}(\mathbb{S}_4^2)$, we consider the following traces:

$$\begin{array}{lllllll} a = \mathrm{tr}(\rho(\alpha)) & ; & b = \mathrm{tr}(\rho(\beta)) & ; & c = \mathrm{tr}(\rho(\gamma)) & ; & d = \mathrm{tr}(\rho(\delta)) \\ x = \mathrm{tr}(\rho(\alpha\beta)) & ; & y = \mathrm{tr}(\rho(\beta\gamma)) & ; & z = \mathrm{tr}(\rho(\gamma\alpha)). & & \end{array}$$

These traces define a polynomial map $\chi\colon \mathrm{Rep}(\mathbb{S}_4^2) \to \mathbb{C}^7$ by $\chi(\rho) = (a, b, c, d, x, y, z)$. By postcomposing a $\mathrm{SL}_2(\mathbb{C})$-conjugation $i_g(x) = gxg^{-1}\colon \mathrm{SL}_2(\mathbb{C}) \circlearrowleft$, we find χ is conjugate invariant, i.e., $\chi(i_g \circ \rho) = \chi(\rho)$.

It turns out that the algebra of polynomial functions on $\mathrm{Rep}(\mathbb{S}^2_4)$ invariant under $\mathrm{SL}_2(\mathbb{C})$-conjugations is generated by components of χ, i.e., by variables a, b, c, d, x, y, z. These variables are not algebraically independent but rather satisfies a quartic relation

$$x^2 + y^2 + z^2 + xyz = Ax + By + Cz + D,$$

where

$$A = ab + cd, \quad B = ad + bc, \quad C = ac + bd,$$
$$D = 4 - a^2 - b^2 - c^2 - d^2 - abcd.$$

We thus observe that S_{ABCD} is the equation for the algebraic quotient $\mathrm{Rep}(\mathbb{S}^2_4)/\!\!/\mathrm{SL}_2(\mathbb{C})$, denoted $\chi(\mathbb{S}^2_4)$, of $\mathrm{Rep}(\mathbb{S}^2_4)$ modulo the $\mathrm{SL}_2(\mathbb{C})$-conjugacy action. This quotient is called the *character variety* of $\mathbb{S}^2_4$.

7.2 Mapping Class Group Actions

Next, we consider the (extended) mapping class group $\mathsf{Mod}^{\pm}(\mathbb{S}^2_4)$ of $\mathbb{S}^2_4$. That is, we think the group $\mathsf{Homeo}^{\pm}(\mathbb{S}^2_4)$ of homeomorphisms[i] on the sphere $\mathbb{S}^2$ that preserves 4 puncture points (but may permute the points within), and consider $\mathsf{Mod}^{\pm}(\mathbb{S}^2_4) = \pi_0(\mathsf{Homeo}^{\pm}(\mathbb{S}^2_4))$ the group of isotopy classes of such homeomorphisms. Because $\mathbb{S}^2_4$ is a $K(\pi, 1)$-space, one has a natural homomorphism $\mathsf{Mod}^{\pm}(\mathbb{S}^2_4) \to \mathsf{Out}(\pi)$ which is an isomorphism, by Dehn–Nielson–Baer theorem [Farb and Margalit, 2012, §8.1].

This extended mapping class group $\mathsf{Mod}^{\pm}(\mathbb{S}^2_4)$ acts on the character variety $\chi(\mathbb{S}^2_4)$. Algebraically speaking, this is induced from precomposing group automorphisms $\mathsf{Aut}(\pi)$, i.e., $g.\rho = \rho \circ g^{-1}$. However, as

$$\rho(\eta x \eta^{-1}) = (i_{\rho(\eta)} \circ \rho)(x),$$

we factor out inner automorphisms and hence obtain the (left) action of the outer automor-

i. possibly orientation-reserving

phism group $\mathsf{Out}(\pi) = \mathsf{Aut}(\pi)/\mathsf{Inn}(\pi)$ on $\chi(\mathbb{S}^2_4)$.

There is no reason for $\mathsf{Out}(\pi)$ to preserve the fibers of the projection $(a, b, c, d, x, y, z) \mapsto (a, b, c, d)$, i.e., the individual surface S_{ABCD}. However, one can find a subgroup Γ of $\mathsf{Out}(\pi)$ whose action preserves each fiber, i.e., each S_{ABCD}.

To describe the copy, we first find a copy of $\mathrm{PGL}_2(\mathbb{Z})$ within $\mathsf{Out}(\pi)$. Recall the antipodal map $\sigma\colon \mathbb{T} = \mathbb{R}^2/\mathbb{Z}^2 \circlearrowright$, $\sigma(x, y) = (-x, -y)$ and its quotient $\mathbb{T}/\sigma = \mathbb{S}^2(2, 2, 2, 2)$. Removing the ramified points we obtain the surface $\mathbb{S}^2_4$. The natural $\mathrm{GL}_2(\mathbb{Z})$-action on $\mathbb{T}$ then induces the natural $\mathrm{PGL}_2(\mathbb{Z})$-action on $\mathbb{T}/\sigma$ which preserves the set of ramified points. Hence each element of $\mathrm{PGL}_2(\mathbb{Z})$ defines a mapping class in $\mathbb{S}^2_4$, in fact injectively.

Let Γ be the kernel of the projection $\mathrm{PGL}_2(\mathbb{Z}) \twoheadrightarrow \mathrm{PGL}_2(\mathbb{Z}/2\mathbb{Z})$. On the torus, this Γ fixes each fixed point of σ. Consequently, Γ preserves the fibers, i.e., individual surfaces S_{ABCD} through the action $\mathsf{Out}(\pi) \curvearrowright \chi(\mathbb{S}^2_4)$.

7.3 Vieta Involutions

Although the description of $\Gamma \curvearrowright S_{ABCD}$ was rather topological, one can describe this action algebraically by the following Vieta involutions.

If one observes the formula of S_{ABCD}, one observes that it is a monoic quadratic equation of each X_i's. For instance, in X_1, we rewrite

$$X_1^2 + (X_2X_3 - A)X_1 + (X_2^2 + X_3^2 - BX_2 - CX_3 - D) = 0$$

to describe the surface S_{ABCD}. If we consider flipping the quadratic roots $X_1 = \dots$ of this polynoimal equation, we obtain an involution

$$\begin{aligned} s_1(X_1, X_2, X_3) &= (-X_1 - X_2X_3 + A, X_2, X_3) \\ &= \left(\frac{X_2^2 + X_3^2 - BX_2 - CX_3 - D}{X_1}, X_2, X_3 \right). \end{aligned}$$

Likewise, we can write involutions corresponding to variables X_2 and X_3 as follows:

$$\begin{aligned}
s_2(X_1, X_2, X_3) &= (X_1, -X_2 - X_1X_3 + B, X_3) \\
&= \left(X_1, \frac{X_1^2 + X_3^2 - AX_1 - CX_3 - D}{X_2}, X_3\right), \\
s_3(X_1, X_2, X_3) &= (X_1, X_2, -X_3 - X_1X_2 + C) \\
&= \left(X_1, X_2, \frac{X_1^2 + X_2^2 - AX_1 - BX_2 - D}{X_3}\right).
\end{aligned}$$

These algebraic automorphisms on S_{ABCD} will be called the *Vieta involutions*.

By [Cantat and Loray, 2009, §2], one can explicitly find elements of Γ that corresponds to each Vieta involution. So $s_1, s_2, s_3 \in \Gamma$ are written as

$$s_1 = \begin{bmatrix} -1 & 2 \\ 0 & 1 \end{bmatrix}, \quad s_2 = \begin{bmatrix} 1 & 0 \\ 2 & -1 \end{bmatrix}, \quad s_3 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.$$

Furthermore, the group Γ is generated by s_1, s_2, s_3 and forms a finite-index ≤ 24 subgroup of algebraic automorphisms $\mathsf{Aut}(S_{ABCD})$ [Cantat and Loray, 2009, Theorem 3.1]. Furthermore, as an abstract group, Γ is isomorphic to the free product $(\mathbb{Z}/2\mathbb{Z})^{*3}$ (loc. cit.).

Hence studying the *Vieta group* $\Gamma = \langle s_1, s_2, s_3 \rangle$ action on S_{ABCD} virtually gives information (i.e., gives a finite-indexed information) on all algebraic actions of S_{ABCD}. This describes our primary object of study.

CHAPTER 8

TROPICALIZATION

This chapter is to provide a minimal information required to understand the tropicalization trick. Because of that nature, the introduction here is by no means capturing all the essential theory; one should consult to standard texts like Maclagan and Sturmfels [2015] or Brugallé et al. [2015] for further information.

The tropicalization convention that we will use here is the 'minimum' convention, following Maclagan and Sturmfels [2015]. This means we are using tropical semiring operations

$$x \oplus y = \min(x, y),$$
$$x \otimes y = x + y$$

in the discussions below.

8.1 The Idea of Tropicalizations

One way to describe the idea of tropicalization is to find points of a complex (algebraic) variety in an approximate way. To elaborate, suppose $V = V(F) \subset \mathbb{A}^d$ is an affine hypersurface over $\mathbb{C}$, given by an polynomial equation $F(X_1, \ldots, X_d) = 0$. Write $F(X) = \sum_\alpha c_\alpha X^\alpha$ to get an access to the coefficients of F.

Instead of finding $\mathbb{C}$-points of V precisely, we rather model the asymptotics of a $\mathbb{C}$-point $(X_1, \ldots, X_d) \in V(\mathbb{C})$ as $X_i = t^{x_i}$, where $0 < t < 1$. Likewise, we model the coefficients as $c_\alpha = t^{v_\alpha}$. (Morally, the exponents x_i's and v_α's are complex numbers. But as we are only interested in the asymptotics, only real parts of them are gaining attention; so we abuse the notations to denote x_i, v_α for the real parts of x_i, v_α, etc.) We then seek for a *necessary* condition for $F(X) = 0$ to hold under this model.

By $X = t^{x_i}$ and $c_\alpha = t^{v_\alpha}$, we have $F(X) = \sum_\alpha t^{v_\alpha + \alpha \cdot x}$, where $\alpha \cdot x = \alpha_1 x_1 + \cdots + \alpha_d x_d$. As $0 < t < 1$, $|F(X)|$ is $O(t^\eta)$, where $\eta = \min_\alpha(v_\alpha + \alpha \cdot x)$. We see if this upper bound also works as a lower bound.

Suppose α_0 is one of the index α such that $\eta = v_\alpha + \alpha \cdot x$. If α_0 is the only index enjoying the property, then by triangle inequality we have

$$\begin{aligned} |F(X)| &\geq t^\eta - \sum_{\alpha \neq \alpha_0} t^{v_\alpha + \alpha \cdot x} \\ &\geq t^\eta - (N-1) \cdot t^\mu, \end{aligned}$$

where $\mu = \min_{\alpha \neq \alpha_0}(v_\alpha + \alpha \cdot x)$ and N is the number of nonzero terms that F has. Since $\eta < \mu$, for $t \ll 1$ sufficiently small we have $t^\eta - (N-1)t^\mu > 0$, concluding that $F(X) \neq 0$ for t small enough.

By this we observe that, under our asymptotic setups, $F(X) = 0$ implies that the minimum $\min_\alpha(v_\alpha + \alpha \cdot x)$ is attained by two or more indices α. Define

$$V(\mathrm{trop}(F)) = \{x \in \mathbb{R}^d : \min_\alpha(v_\alpha + \alpha \cdot x) \text{ is attained by two or more indices } \alpha\}.$$

Then what we have observed suggests the following. For any $\mathbb{C}$-point $(X_1, \ldots, X_d) \in V(\mathbb{C})$ set $(x_1, \ldots, x_d) = (\log_t |x_1|, \ldots, \log_t |x_d|)$. Then we have $(x_1, \ldots, x_d)$ in a small neighborhood of $V(\mathrm{trop}(F))$, which will shirnk to $V(\mathrm{trop}(F))$ as $t \to 0+$. Given this necessary condition, we see that $V(\mathrm{trop}(F))$ is an encoding of the scales of $\mathbb{C}$-points, possibly with some extranous points.

What is stated below is to rigorously establish the sketch above. With the language to be introduced below, what we have done is to find the set of valuations of $K = \mathbb{C}((t^*))$-points in a hypersurface $F = 0$ defined over K. It turns out that, by Kapranov's Theorem 8.12, the closure of such valuations coincides with $V(\mathrm{trop}(F))$ defined above.

8.2 Non-Archimedean Field

Definition 8.1 (Non-archimedean field)**.** By a *valued field*, we mean a pair (K, val) of a field K and a function $\mathrm{val}\colon K \to \mathbb{R} \cup \{+\infty\}$, called a *valuation*, that satisfies the following axioms.

(i) For all $x \in K$, $\mathrm{val}(x) = +\infty$ if and only if $x = 0$.

(ii) For all $x, y \in K$, $\mathrm{val}(xy) = \mathrm{val}(x) + \mathrm{val}(y)$.

(iii) For all $x, y \in K$, $\mathrm{val}(x + y) \geq \min(\mathrm{val}(x), \mathrm{val}(y))$.

(iv) If $\mathrm{val}(x) \neq \mathrm{val}(y)$, then $\mathrm{val}(x + y) = \min(\mathrm{val}(x), \mathrm{val}(y))$.[i]

We say the valued field (K, val) is *nontrivially valued* or *non-archimedean* if there exists $t \in K$ such that $\mathrm{val}(t) \neq 0, +\infty$.

We conventionally use just K when its valuation val is clear. Now given a valued field (K, val) and a real $q > 1$, we define the *absolute value* by val as $|x| = q^{-\mathrm{val}(x)}$ (note the sign in the exponent). This absolute value satisfies $|x| = 0$ iff $x = 0$, $|xy| = |x| \cdot |y|$, but also $|x + y| \leq \max(|x|, |y|)$. The latter inequality is called the *ultrametric inequality* and distinguishes K with other fields with absolute values, e.g., $\mathbb{R}$ or $\mathbb{C}$.

For a non-archimedean field K, that $|\pm 1| = 1$ is immediate from multiplicativity. But by the ultrametric inequality, for any positive integer $n \in \mathbb{Z}_{>0}$ we have

$$|n| = \underbrace{|1 + 1 + \cdots + 1|}_{n \text{ 1's}} \leq \max(|1|, \ldots, |1|) = 1.$$

Therefore the field K satisfies $|n| \leq 1$ for all integer n. This gives us an exact opposite to the archimedean properties on $\mathbb{R}$ or $\mathbb{C}$, justifying the name 'non-archimedean.'

i. It is known that this axiom follows from other three axioms (see, e.g., [Perez-Garcia and Schikhof, 2010, p. 3]). However, we mention this as an axiom, as it will be frequently used in computations.

Example 8.2 (Trivial valuation)**.** Let K be *any* field, perhaps with positive characteristic. The valuation

$$\mathrm{val}_0(x) = \begin{cases} 0 & (x \neq 0), \\ +\infty & (x = 0) \end{cases}$$

satisfies all the axioms in Definition 8.1 and hence makes (K, val) a valued field.

This means that the usual archimedean fields like $\mathbb{R}$ can be viewed as a valued field. So to avoid potential confusions, we restrict non-archimedean fields (K, val) to have a valuation val which is not the trivial one val_0. This setup turns out to be more useful in the study of tropicalizations.

There are two examples of non-archimedean fields that we will keep in mind throughout.

Example 8.3 (p-adic fields)**.** Consider the absolute valuation val_p on $\mathbb{Q}$ given as follows.

(i) For zero, set $\mathrm{val}_p(0) := +\infty$.

(ii) For a positive integer $n \in \mathbb{Z}_{>0}$, $v = \mathrm{val}_p(n) \in \mathbb{Z}_{\geq 0}$ is a nonnegative integer such that p^v divides n but p^{v+1} does not.

(iii) For any rational number $a/b \in \mathbb{Q}$ in a reduced form, we define $\mathrm{val}_p(a/b) = \mathrm{val}_p(|a|) - \mathrm{val}_p(|b|)$.

This well-defines a function $\mathrm{val}_p \colon \mathbb{Q} \to \mathbb{Z} \cup \{+\infty\}$, called the *p-adic valuation* of $\mathbb{Q}$. The induced norm $|x|_p = p^{-\mathrm{val}_p(x)}$ is called the *p-adic norm* on $\mathbb{Q}$.

By the p-adic valuation, we have $(\mathbb{Q}, \mathrm{val}_p)$ already a non-archimedean field, which is not metric complete with respect to the p-adic norm. But interestingly, if we make a metric completion on $\mathbb{Q}$ then we have the *field of p-adic numbers* $\mathbb{Q}_p$ which still gives a non-archimedean field.

One can further extend this p-adic valuation to the algebraic closure $\overline{\mathbb{Q}}_p$ of $\mathbb{Q}_p$. The closure is, however, not complete with respect to the p-adic norm, which makes us to consider the further metric completion $\widehat{\overline{\mathbb{Q}}}_p =: \mathbb{C}_p$. This field $\mathbb{C}_p$ is still algebraically closed, giving our first example of a non-archimedean field $(\mathbb{C}_p, \mathrm{val}_p)$ which is algebraically closed and metric complete. See [Koblitz, 1984, §III.3-4] for further details.

Example 8.4 (Formal/Convergent Puiseux series)**.** Recall the field $\mathbb{C}((t))$ of formal Laurent series over $\mathbb{C}$. For an element $f(t) = \sum_{j \geq j_0} c_j t^j \in \mathbb{C}((t))$, we define its valuation $\mathrm{val}(f(t))$ as

$$\mathrm{val}(f(t)) = \inf\{j \geq j_0 : c_j \neq 0\},$$

i.e., the least exponent with nonzero coefficients. Here, we conventionally think $\inf \varnothing = +\infty$ to have $\mathrm{val}(0) = +\infty$. Then $(\mathbb{C}((t)), \mathrm{val})$ is a non-archimedean field.

Although the non-archimedean field $\mathbb{C}((t))$ is metric complete, it is not algebraically closed; even the simplest equation $x^n - t = 0$ does not have a solution in $\mathbb{C}((t))$. But if we adjoin the solutions and build a field $K = \bigcup_{n \geq 1} \mathbb{C}((t^{1/n}))$, then we have an algebraically closed field [Serre, 1979, §IV.2, Proposition 8][Maclagan and Sturmfels, 2015, Theorem 2.1.5].

This field K is called the *field of (formal) Puiseux series* in t, denoted $\mathbb{C}((t^*))$ (following Ruiz [1993]). Elements of $\mathbb{C}((t^*))$ are formal series of the form $g(t) = \sum_{m \geq m_0} c_m t^{m/n}$. The valuation $\mathrm{val}(g(t))$ is defined to be

$$\mathrm{val}(g(t)) = \inf\{m/n : c_m \neq 0\},$$

which gives a non-archimedean field $(\mathbb{C}((t^*)), \mathrm{val})$. Unfortunately, the field $\mathbb{C}((t^*))$ is not metric complete.

The field of Puiseux series has an interesting subfield, the field of convergent Puiseux series. Say a Puiseux series $f(t) = \sum_{m \geq m_0} c_m t^{m/n} \in \mathbb{C}((t^*))$ is *convergent* if the ordinary power series $\sum_{m \geq 0} c_m z^m$ has a positive radius of convergence (equivalently, $\limsup\limits_{m \to \infty} |c_m|^{1/m} < \infty$). Denote the field of convergent Puiseux series as $\mathbb{C}(\{t^*\}) \subset \mathbb{C}((t^*))$. Then this field is non-archimedean and algebraically closed [Ruiz, 1993, §III, Proposition 4.4]. This field can be also viewed as the algebraic closure of the field of germs of meromorphic functions at $t = 0$, giving a bridge to complex analysis.

8.3 Value Group

Definition 8.5 (Value group)**.** The additive subgroup $\Gamma_{\text{val}} = \text{val}(K^\times) \subset \mathbb{R}$ is called the *value group* of (K, val).

Notice that $\Gamma_{\text{val}} = \{0\}$ if and only if v is a trivial valuation on K.

Remark. A more abstract account to valued field requires the valuations to be a group homomorphism $\text{val}\colon (K^\times, \cdot) \to (\Gamma, +)$, where $(\Gamma, +)$ is an ordered abelian group. Nonetheless, as long as Γ has cardinality at most the continuum $2^{\aleph_0}$, one can find an order-preserving group monomorphism $(\Gamma, +) \hookrightarrow (\mathbb{R}, +)$ [Rudin, 1962, Theorem 8.1.2]. So practically, setting $\Gamma \subset \mathbb{R}$ is sufficient for our purpose.

Proposition 8.6. *Suppose a valued field (K, val) is algebraically closed. Then the value group Γ_{val} is divisible. That is, for every $g \in \Gamma_{\text{val}}$ and $n \in \mathbb{Z}_{>0}$ we have $h \in \Gamma_{\text{val}}$ such that $n.h = g$.*

Proof. Let $g = \text{val}(x)$ for some $x \in K^\times$. For a solution $y \in K$ of $y^n = x$, we clearly have $n.\text{val}(y) = \text{val}(x) = g$. Set $h = \text{val}(y)$. $\square$

In particular, Γ_{val} must have a copy of $\mathbb{Q}$ if $\Gamma_{\text{val}} \neq \{0\}$. This derives the following

Corollary 8.7. *An algebraicaly closed non-archimedean field (K, val) has a dense value group $\Gamma_{\text{val}} \subset \mathbb{R}$.*

8.4 Tropicalization of a Hypersurface

Let (K, val) be an algebraically closed non-archimedean field. Suppose $F(X_1, \ldots, X_d) \in K[X_1^{\pm 1}, \ldots, X_d^{\pm 1}]$ is a Laurent polynomial. Write, in multiindex notations, $F(X_1, \ldots, X_d) =$

$\sum_\alpha c_\alpha X^\alpha$, where $\alpha = (\alpha_1, \dots, \alpha_d) \in \mathbb{Z}^d$, $X^\alpha = X_1^{\alpha_1} \cdots X_d^{\alpha_d}$, and $c_\alpha = 0$ for all but finitely many $\alpha \in \mathbb{Z}^d$'s.

Consider the hypersurface $V \subset \mathbb{G}_m^d$ in the algebraic torus $\mathbb{G}_m^d$ defined by the equation $F(X_1, \dots, X_d) = 0$. We assume that V is not void, i.e., there is a K-point $(Z_1, \dots, Z_d) \in V(K)$ such that $Z_i \neq 0$ for all $i = 1, \dots, d$. (Equivalently, $F(X)$ is not a monomial.) Given a hypersurface, or more generally a variety, in the algebraic torus defined by a Laurent polynomial, there are some ways to define the tropicalization of V.

8.4.1 Tropicalization by Values

Definition 8.8 (Value Tropicalization)**.** Let $V \subset \mathbb{G}_m^d$ be a variety in the algebraic torus. Consider the subset

$$\mathrm{val}(V(K)) = \{(\mathrm{val}(Z_1), \dots, \mathrm{val}(Z_d)) \in \mathbb{R}^d : (Z_1, \dots, Z_d) \in V(K) \cap (K^\times)^d\}$$

of $\mathbb{R}^d$. We will call its (Euclidean) closure $\overline{\mathrm{val}(V(K))} =: \mathbf{Trop}(V)$ the *value tropicalization* of V.

That is, $\mathbf{Trop}(V)$ collects valuation vectors $\in \Gamma_{\mathrm{val}}^d \subset \mathbb{R}^d$ for K-points of V, together with limit points of these vectors. Taking closure would be a natural decision to make, because $\Gamma_{\mathrm{val}}^d \subset \mathbb{R}^d$ is dense.

Remark. The term 'value tropicalization' is not standard and meant to be a temporal term to introduce distinct ways to tropicalize a variety.

8.4.2 Tropicalization by that of Polynomials

Definition 8.9 (Tropicalization of a Laurent Polynomial)**.** Suppose $F(X) = \sum_\alpha c_\alpha X^d \in K[X_1^{\pm 1}, \dots, X_d^{\pm 1}]$ is a nonzero Laurent polynomial. Define its *tropicalization* as a function

$\mathrm{trop}(F)\colon \mathbb{R}^d \to \mathbb{R}$,

$$\begin{aligned}\mathrm{trop}(F)(x_1, \dots, x_d) &= \min_{\alpha} \left(\mathrm{val}(c_\alpha) + \alpha \cdot (x_1, \dots, x_d)\right) \\ &= \min_{\alpha} \left(\mathrm{val}(c_\alpha) + \sum_{i=1}^{d} \alpha_i x_i\right).\end{aligned}$$

Note that $c_\alpha = 0$ for all but finitely many α, so $\mathrm{val}(c_\alpha) = +\infty$ for all but finitely many α. Consequently, $\mathrm{trop}(F)$ find the minimum over a finite set.

The expression $\mathrm{val}(c_\alpha) + \alpha \cdot x$ is called the *tropical monomial* of $c_\alpha X^\alpha$. As tropical monomials are linear functions on x, a tropical polynomial is a piecewise linear concave function which is differentiable except possibly on a $(d-1)$-dimensional (simplicial) complex in $\mathbb{R}^d$. This exceptional set also defines a tropicalization of the variety $V(F)\colon F(X) = 0$.

Definition 8.10 (Tropical Variety)**.** Given a tropical polynomial $\mathrm{trop}(F)(x)$, define its *tropical variety* as

$$V(\mathrm{trop}(F)) = \{x \in \mathbb{R}^d : \mathrm{trop}(F)(x) \text{ is not differentiable at } x\}.$$

Although the definition is based on non-differentiability, the actual computation of the tropical variety $V(\mathrm{trop}(F))$ is done by asking whether two or more tropical monomial attains the minimum defining $\mathrm{trop}(F)$.

Proposition 8.11. *For $x \in \mathbb{R}^d$, we have $x \in V(\mathrm{trop}(F))$ if and only if there are distinct $\alpha \neq \beta$ such that* $\mathrm{val}(c_\alpha) + \alpha \cdot x = \mathrm{val}(c_\beta) + \beta \cdot x = \mathrm{trop}(F)(x)$.

Proof. Suppose there are distinct $\alpha \neq \beta$ such that two tropical monomials $\mathrm{val}(c_\alpha) + \alpha \cdot x$ and $\mathrm{val}(c_\beta) + \beta \cdot x$ attains $\mathrm{trop}(F)(x)$, but $\mathrm{trop}(F)$ is differentiable at x. Because $\mathrm{trop}(F)$ is a concave function differentiable at x, it has the unique supporting hyperplane at x. But as $\mathrm{val}(c_\alpha) + \alpha \cdot (-)$ and $\mathrm{val}(c_\beta) + \beta \cdot (-)$ are two distinct supporting hyperplanes at x, it contradicts.

Suppose there is a unique α such that $\mathrm{trop}(F)(x) = \mathrm{val}(c_\alpha) + \alpha \cdot x$. Then $\mathrm{trop}(F)$ is locally same as the tropical monomial $\mathrm{val}(c_\alpha) + \alpha \cdot (-)$, thus differentiable at x. $\square$

A consequence of Proposition 8.11 is that $V(\mathrm{trop}(F)) \subset \mathbb{R}^d$ is a closed subset.

8.4.3 Kapranov's Theorem

By far we have discussed two tropicalizations of a hypersurface $V = V(F)$ in an algebraic torus:

- by value tropicalizations, $\mathbf{Trop}(V(F))$, or
- by tropical varieties, $V(\mathrm{trop}(F))$.

Kapranov's Theorem Einsiedler et al. [2006][Maclagan and Sturmfels, 2015, Theorem 3.1.3] states that two tropicalizations coincide, for algebraically closed base fields.

Theorem 8.12 (Kapranov's Theorem)**.** *Let* (K, val) *be an algebraically closed non-archimedean field. For a nonzero Laurent polynomial* $F(X)$ *which is not a monomial, we have*

$$\mathbf{Trop}(V(F)) = V(\mathrm{trop}(F)).$$

Although we have only discussed about hypersurfaces up to this point, one can generalize Kapranov's theorem for higher codimensions, producing a theorem known as the *fundamental theorem of tropical algebraic geometry* [Maclagan and Sturmfels, 2015, Theorem 3.2.3].

Theorem 8.13 (Fundamental Theorem of Tropical Algebraic Geometry)**.** *Let* (K, val) *be an algebraically closed non-archimedean field. Let* I *be an ideal in* $K[X_1^{\pm 1}, \ldots, X_d^{\pm 1}]$*, and let* $X = V(I)$ *be its variety in the algebraic torus* $\mathbb{G}_m^d$*. Then the value tropicalization* $\mathbf{Trop}(X)$ *equals to the tropical variety* $\bigcap_{F \in I} V(\mathrm{trop}(F))$*. That is,*

$$\mathbf{Trop}(V(I)) = \bigcap_{F \in I} V(\mathrm{trop}(F)).$$

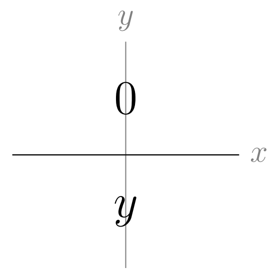

Figure 8.1: Describing $\min(y, 0)$ on xy-plane

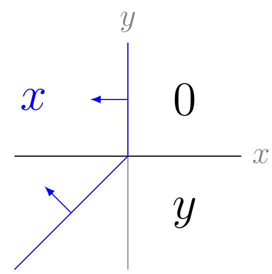

Figure 8.2: Describing $\min(y, 0, x)$ on xy-plane

If K is algebraically closed and X is a variety in the algebraic torus, we simply say the *tropicalization* of X as the set $\mathbf{Trop}(X) = \bigcap_{F \in I(X)} V(\mathrm{trop}(F))$.

8.5 Examples

8.5.1 Planar Curves

Example 8.14 (Line)**.** Perhaps the easiest example is the line $F(X, Y) = X + Y - 1$. Because the tropicalization of F is

$$\mathrm{trop}(F)(x, y) = \min(x, y, 0),$$

one divides the plane by regions in which $\mathrm{trop}(F)(x, y)$ is attained by x, y, and 0 respectively. This is easily done by first sketching $\min(y, 0)$ (Figure 8.1) and extending the picture to $\min(y, 0, x)$ (Figure 8.2). (Here, the order of monoimals does not matter and is chosen randomly for this demo.)

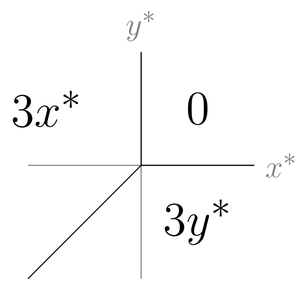

Figure 8.3: Describing $\min(3x^*, 3y^*, 0)$ on x^*y^*-plane

Example 8.15 (Elliptic Curves)**.** Consider the following *Hessian model* of elliptic curves Hesse [1844]Smart [2001]:

$$E_D\colon X^3 + Y^3 + Z^3 = D \cdot XYZ.$$

The equation is written as a projective variety, so we first boil this down to an affine curve:

$$E_D\colon x^3 + y^3 + 1 = Dxy.$$

Tropicalizing $F(x, y) = x^3 + y^3 + 1 - Dxy$, we have

$$\operatorname{trop}(F)(x^*, y^*) = \min(3x^*, 3y^*, 0, \operatorname{val}(D) + x^* + y^*).$$

Here, we use $x^* = \operatorname{val}(x)$ and $y^* = \operatorname{val}(y)$ to avoid abuse of notations.

To study the tropical variety $V(\operatorname{trop}(F)) = \mathbf{Trop}(E_D)$, we first sketch the regions indicating which monomial dominated $\min(3x^*, 3y^*, 0)$ (Figure 8.3) and then adjoining $\operatorname{val}(D) + x^* + y^*$ into it (Figure 8.4). It turns out that the triangle in the middle appears if $\operatorname{val}(D) < 0$ and disappears if $\operatorname{val}(D) \geq 0$. It is known that the picture with the middle triangle reflects the fact that E_D compactifies to give a genus 1 curve.

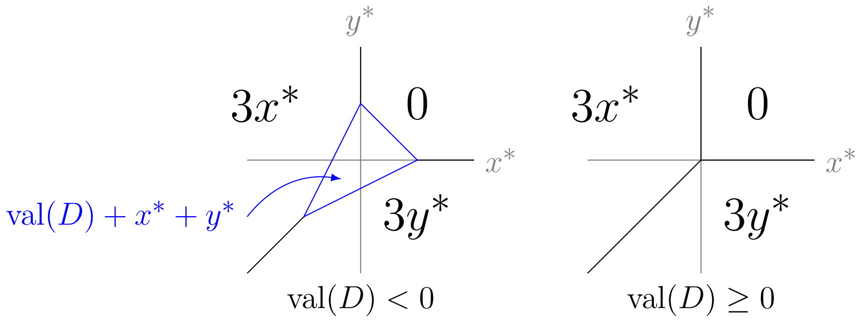

Figure 8.4: Describing $\min(3x^*, 3y^*, 0, \mathrm{val}(D) + x^* + y^*)$ on x^*y^*-plane

8.5.2 Markov Cubic

As we are interested in Markov cubic surfaces, defined over a non-archimedean field,

$$S_{ABCD}\colon X_1^2 + X_2^2 + X_3^2 + X_1X_2X_3 = AX_1 + BX_2 + CX_3 + D,$$

we are to find non-differentiable points of the tropicalization

$$f(x) = \min \left\{ \begin{array}{l} 2x_1, 2x_2, 2x_3, x_1 + x_2 + x_3, \\ a + x_1, b + x_2, c + x_3, d \end{array} \right\},$$

where a, b, c, d are $\mathrm{val}(A), \mathrm{val}(B), \mathrm{val}(C), \mathrm{val}(D)$, respectively. It turns out that sketching the tropical variety $V(f) = \mathbf{Trop}(S_{ABCD})$ is in general a hard task to do in hand, so we rather introduce some computer sketches of this tropicalization. See Figures 8.5 and 8.6.

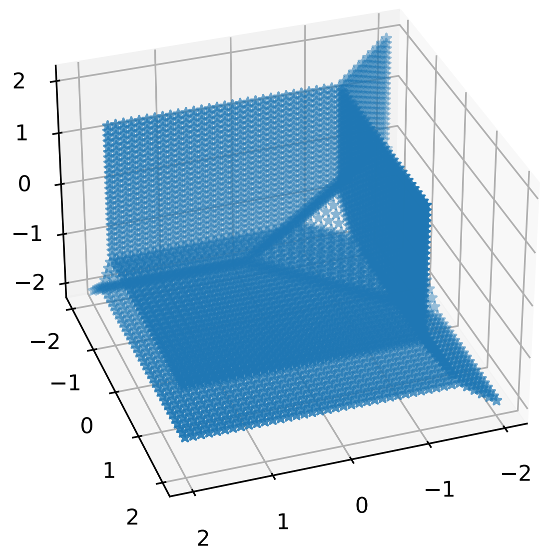

Figure 8.5: $\mathbf{Trop}(S_{ABCD})$ with $(a, b, c, d) = (\infty, \infty, -1.5, -2)$.

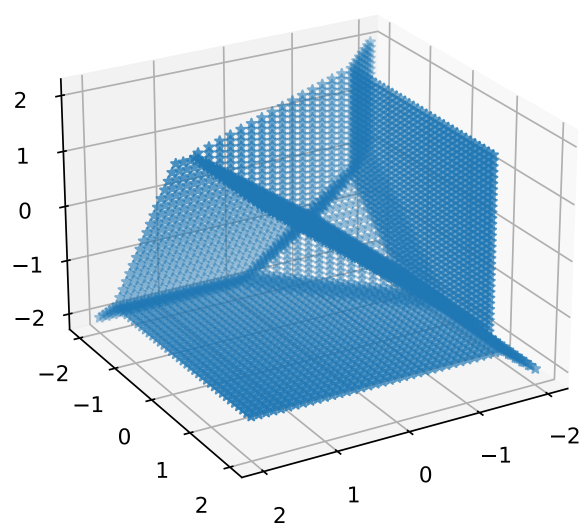

Figure 8.6: $\mathbf{Trop}(S_{ABCD})$ with $(a, b, c, d) = (-1.3, \infty, -1.5, -2.3)$.

CHAPTER 9

THE SKELETON OF MARKOV CUBICS

Although one can compute and sketch the tropicalization of the Markov cubic S_{ABCD} on $\mathbb{R}^3$, not all points of the tropicalization is involved in the tropicalized Vieta dynamics. (Recall, from Chapter 7, the most interesting dynamics on S_{ABCD} is the one generated by Vieta involutions.)

The goal of this chapter is to introduce the *skeleton* subset $Sk(a, b, c, d)$ of the tropicalized surface $\mathbf{Trop}(S_{ABCD})$ and study their invariance properties with respect to the dynamics. Furthermore, we introduce a foliation of $\mathbb{R}^3$ by skeleta, which happens to give a family of dynamical systems that S_{ABCD} carry.

Throughout this chapter and after, we assume that S_{ABCD} is defined over a fixed non-archimedean field K, so that our parameters A, B, C and D are now in K.

9.1 Tropical Vieta Involutions and the Skeleton

Recall the Vieta involutions

$$\begin{aligned}
s_1(X_1, X_2, X_3) &= (-X_1 - X_2X_3 + A, X_2, X_3) \\
&= \left(\frac{X_2^2 + X_3^2 - BX_2 - CX_3 - D}{X_1}, X_2, X_3\right), \\
s_2(X_1, X_2, X_3) &= (X_1, -X_2 - X_1X_3 + B, X_3) \\
&= \left(X_1, \frac{X_1^2 + X_3^2 - AX_1 - CX_3 - D}{X_2}, X_3\right), \\
s_3(X_1, X_2, X_3) &= (X_1, X_2, -X_3 - X_1X_2 + C) \\
&= \left(X_1, X_2, \frac{X_1^2 + X_2^2 - AX_1 - BX_2 - D}{X_3}\right).
\end{aligned}$$

While tropicalizing these involutions, we use the multiplicative expression of each and obtain

$$\mathrm{trop}(s_1)(x_1, x_2, x_3) = (\min(2x_2, 2x_3, b + x_2, c + x_3, d) - x_1, x_2, x_3), \tag{9.1.1}$$

$$\mathrm{trop}(s_2)(x_1, x_2, x_3) = (x_1, \min(2x_1, 2x_3, a + x_1, c + x_3, d) - x_2, x_3), \tag{9.1.2}$$

$$\mathrm{trop}(s_3)(x_1, x_2, x_3) = (x_1, x_2, \min(2x_1, 2x_2, a + x_1, b + x_2, d) - x_3). \tag{9.1.3}$$

These maps are in general not well-defined on all of $\mathbf{Trop}(S_{ABCD})$. Nonetheless, there is a subset of the tropicalization $\mathbf{Trop}(S_{ABCD})$ which is invariant under the tropicalized involutions. This subset is analogous to the Kontsevich–Soibelman skeleton [Kontsevich and Soibelman, 2006, §6.6] of a smooth proper algebraic variety over a non-archimedean field, hence we call it the *skeleton* subset.

Denote the defining equation of S_{ABCD} as $F(X_1, X_2, X_3) = 0$ and let f be the tropicalization of F. That is,

$$f(x_1, x_2, x_3) = \min\left(2x_1, 2x_2, 2x_3, x_1 + x_2 + x_3, a + x_1, b + x_2, c + x_3, d\right). \tag{9.1.4}$$

Definition 9.1. Consider the subset of $\mathbf{Trop}(S_{ABCD})$ defined as

$$\{(x_1, x_2, x_3) \in \mathbf{Trop}(S_{ABCD}) : x_1 + x_2 + x_3 = f(x_1, x_2, x_3)\}, \tag{9.1.5}$$

where f is the tropical polynomial (9.1.4). Call this subset the *skeleton* of S_{ABCD}, and denote it $Sk(a, b, c, d)$, where $a = \mathrm{val}(A)$, $b = \mathrm{val}(B)$, $c = \mathrm{val}(C)$, and $d = \mathrm{val}(D)$.

As a subset of $\mathbb{R}^3$, the skeleton subset is only dependent on the valuations a, b, c, d of parameters A, B, C, D, so the notation $Sk(a, b, c, d)$. Technically, these numbers a, b, c, d are in $\Gamma_{\mathrm{val}} \cup \{\infty\}$, but because the set is defined purely in the language of the semiring $(\mathbb{R} \cup \{\infty\}, \min, +)$, we abuse the notation and sometimes let a, b, c, d to be real numbers or ∞. Note that, however, Γ_{val}-valued points are dense in $Sk(a, b, c, d)$ for some limited cases, e.g., when $a, b, c, d \in \Gamma_{\text{val}} \cup \{\infty\}$.

One can describe $Sk(a,b,c,d)$ without referencing the tropicalization $\mathbf{Trop}(S_{ABCD})$. Consider a variant of f that drops $x_1 + x_2 + x_3$ among the minimands:

$$\min(2x_1, 2x_2, 2x_3, a + x_1, b + x_2, c + x_3, d).$$

If this minimum equals to $x_1 + x_2 + x_3$, then (a) we automatically have $x_1 + x_2 + x_3 = f(x_1, x_2, x_3)$, and (b) the minimum for f is attained by at least two tropical monomials. Therefore we have $x \in Sk(a,b,c,d)$. But instead of formulating it by a comparison of two functions, it is more convenient to just take the difference

$$f_0(x_1, x_2, x_3) = \min(2x_1, 2x_2, 2x_3, a + x_1, b + x_2, c + x_3, d) - x_1 - x_2 - x_3, \tag{9.1.6}$$

and view $Sk(a,b,c,d)$ as the level set $\{f_0 = 0\}$.

Remark. It is not a surprise that the other level sets $\{f_0 = w\}$ appear as a skeleton, especially when w is in the value group Γ_{val}. To see so, suppose $w \in \Gamma_{\text{val}}$ and let $t^w \in K^\times$ has the value $\text{val}(t^w) = w$. Consider the surface $X_w = S_{t^w A, t^w B, t^w C, t^{2w} D}$ and the polynomial map

$$\begin{aligned} \Phi \colon X_w &\to \mathbb{A}^3, \\ (X_1, X_2, X_3) &\mapsto (t^{-w} X_1, t^{-w} X_2, t^{-w} X_3). \end{aligned}$$

Then the image $\Phi(X_w)$ is an affine surface given by the equation

$$\Phi(X_w) \colon X_1^2 + X_2^2 + X_3^2 + t^w X_1 X_2 X_3 = A X_1 + B X_2 + C X_3 + D.$$

If one thinks the 'skeleton' of $\Phi(X_w)$, one can describe the set by the equation

$$\begin{aligned} w + x_1 + x_2 + x_3 &= \min(2x_1, 2x_2, 2x_3, a + x_1, b + x_2, c + x_3, d) \\ &= f_0(x_1, x_2, x_3) + (x_1 + x_2 + x_3), \end{aligned}$$

i.e., the level set $\{f_0 = w\}$. Since $X_w = S_{t^w A, t^w B, t^w C, t^{2w} D}$ itself has the skeleton $Sk(a + w, b + w, c + w, d + 2w)$, we get a formula relating two:

$$\{f_0(x_1, x_2, x_3) = w\} = Sk(a + w, b + w, c + w, d + 2w) - (w, w, w)\,, \tag{9.2.12}$$

where the right hand side is the Minkowski sum $A - p = \{a - p : a \in A\}$.

We will later show, in Corollary 9.4, that (9.2.12) holds even for $w \in \mathbb{R}$, and the tropicalized involutions on $\{f_0 = w\}$ and $Sk(a+w, b+w, c+w, d+2w)$ conjugates via the map $\phi = \mathrm{trop}(\Phi)\colon x \mapsto x - (w, w, w)$.

We then aim to show that $Sk(a, b, c, d)$ is invariant under Vieta involutions (see Proposition 9.3 below). But to have this in elementary means, we need to observe some more facts about the skeleton.

9.2 Inequality Description of the Skeleton

Recall that $Sk(a, b, c, d) = \{f_0 = 0\}$, with f_0 in (9.1.6). One way to characterize the function $w = f_0(x_1, x_2, x_3)$ is to let f_0 to be the supremum of w's satisfying the inequalities

$$w + x_1 + x_2 + x_3 \le 2x_1, \tag{9.2.1}$$

$$w + x_1 + x_2 + x_3 \le 2x_2, \tag{9.2.2}$$

$$w + x_1 + x_2 + x_3 \le 2x_3, \tag{9.2.3}$$

$$w + x_1 + x_2 + x_3 \le a + x_1, \tag{9.2.4}$$

$$w + x_1 + x_2 + x_3 \leq b + x_2, \tag{9.2.5}$$

$$w + x_1 + x_2 + x_3 \leq c + x_3, \tag{9.2.6}$$

$$w + x_1 + x_2 + x_3 \leq d. \tag{9.2.7}$$

Equivalently, to have $w = f_0(x_1, x_2, x_3)$, we demand inequalities (9.2.1)–(9.2.7) to hold, and at least one of them holding as an equality.

One immediate corollary to these inequalities is this.

Corollary 9.2. *If $f_0(x_1, x_2, x_3) = w$, then we have $w + x_1 \leq -|x_2 - x_3|$, and similarly for $w + x_2$ and $w + x_3$. In particular, whenever $w \geq 0$, any point in the level set $\{f_0 = w\}$ has nonpositive coordinates.*

Proof. From inequalities (9.2.2) and (9.2.3), we obtain

$$w + x_1 \leq x_2 - x_3,$$
$$w + x_1 \leq x_3 - x_2.$$

It remains to use $\min(x_2 - x_3, x_3 - x_2) = -|x_2 - x_3|$ to conclude $w + x_1 \leq -|x_2 - x_3|$. Same arguments apply for other coordinates. Furthermore, if $w \geq 0$, then $x_1 \leq -|x_2 - x_3| - w \leq -w \leq 0$, etc., so the second claim follows. □

Furthermore, we can make use of the inequality description to prove a

Proposition 9.3. *The level set $\{f_0 = w\}$ is invariant under tropicalized involutions* $\mathrm{trop}(s_i)$*'s,* (9.1.1)–(9.1.3)*, viewed as an involution $\mathbb{R}^3 \to \mathbb{R}^3$.*

Proof. We focus on $\mathrm{trop}(s_1)$. The other two involutions are dealt similarly.

Denote $f_1 = f_1(x_2, x_3) = \min(2x_2, 2x_3, b + x_2, c + x_3, d)$, so that $\mathrm{trop}(s_1)(x_1, x_2, x_3) = (f_1 - x_1, x_2, x_3)$. With this f_1, we see that the conjunction (i.e., 'and') of inequalities (9.2.2), (9.2.3), (9.2.5), (9.2.6) and (9.2.7) reads as

$$w + x_1 + x_2 + x_3 \le f_1(x_2, x_3). \tag{9.2.8}$$

We recall the other two inequalities for reference:

$$w + x_1 + x_2 + x_3 \le 2x_1, \tag{9.2.1}$$

$$w + x_1 + x_2 + x_3 \le a + x_1. \tag{9.2.4}$$

Then the level set $\{f_0 = w\}$ is characterized by inequalities (9.2.1), (9.2.4), and (9.2.8) but at least one of them holding as an equality.

Let $x_1' = f_1 - x_1$. Then it turns out that

$$(9.2.1) \Longleftrightarrow w + x_1' + x_2 + x_3 \le f_1(x_2, x_3), \tag{9.2.9}$$

$$(9.2.4) \Longleftrightarrow w + x_1' + x_2 + x_3 \le a + x_1', \tag{9.2.10}$$

$$(9.2.8) \Longleftrightarrow w + x_1' + x_2 + x_3 \le 2x_1', \tag{9.2.11}$$

and the equivalences of corresponding equalities as well. This proves that $(x_1', x_2, x_3) = \mathrm{trop}(s_1)(x_1, x_2, x_3) \in \{f_0 = w\}$ whenever $f_0(x_1, x_2, x_3) = w$. $\square$

An immediate corollary is that $Sk(a, b, c, d) = \{f_0 = 0\}$ is invariant under the Vieta involutions, letting us to focus on the dynamics on the skeleton.

Remark. The inequality trick applies for other subsets of $\mathbf{Trop}(S_{ABCD})$ to show that it is *not* invariant under the involutions. For instance, if we think the subset $\{2x_2 = f(x_1, x_2, x_3)\} \subset \mathbf{Trop}(S_{ABCD})$ (cf. (9.1.5) for f), then the analogous step showing the equivalence (9.2.9) gets stuck, causing $\mathrm{trop}(s_1)$ to be not well-defined on there.

Since we now know that the level sets $\{f_0 = w\}$ and skeleta $Sk(a+w, b+w, c+w, d+2w)$

are invariant under the Vieta involutions, we can state the following

Corollary 9.4. *For $w \in \mathbb{R}$, let $\phi^w \colon \mathbb{R}^3 \to \mathbb{R}^3$ be $\phi^w(x_1, x_2, x_3) = (x_1 + w, x_2 + w, x_3 + w)$. Then we have*

$$\begin{aligned} \{f_0(x_1, x_2, x_3) = w\} &= Sk(a + w, b + w, c + w, d + 2w) - (w, w, w) \\ &= \phi^{-w}(Sk(a + w, b + w, c + w, d + 2w)). \end{aligned} \tag{9.2.12}$$

Furthermore, let $\mathrm{trop}(s_i')$ be the tropicalized Vieta involution on $Sk = Sk(a + w, b + w, c + w, d + 2w)$. Then we have the following commutative diagram, for $i = 1, 2, 3$.

$$\begin{tikzcd} \{f_0 = w\} \arrow[r, "\mathrm{trop}(s_i)"] \arrow[d, "\phi^w"] & \{f_0 = w\} \arrow[d, "\phi^w"] \\ Sk \arrow[r, "\mathrm{trop}(s_i')"] & Sk \end{tikzcd}$$

Proof. If we add $2w$ on both sides of the inequalities (9.2.1)–(9.2.7), we have

$$(x_1 + w) + (x_2 + w) + (x_3 + w) \le 2(x_1 + w),$$

$$(x_1 + w) + (x_2 + w) + (x_3 + w) \le 2(x_2 + w),$$

$$(x_1 + w) + (x_2 + w) + (x_3 + w) \le 2(x_3 + w),$$

$$(x_1 + w) + (x_2 + w) + (x_3 + w) \le (a + w) + (x_1 + w),$$

$$(x_1 + w) + (x_2 + w) + (x_3 + w) \le (b + w) + (x_2 + w),$$

$$(x_1 + w) + (x_2 + w) + (x_3 + w) \le (c + w) + (x_3 + w),$$

$$(x_1 + w) + (x_2 + w) + (x_3 + w) \le d + 2w,$$

with at least one of them holding as an equality. Equivalently, $(x_1 + w, x_2 + w, x_3 + w) = \phi^w(x_1, x_2, x_3) \in Sk(a + w, b + w, c + w, d + 2w)$. This proves (9.2.12).

For the conjugation part, we simply compare $\mathrm{trop}(s_i') \circ \phi^w$ and $\phi^w \circ \mathrm{trop}(s_i)$. Say for

$i = 1$:

$$\begin{aligned}
&\operatorname{trop}(s_1') \circ \phi^w(x_1, x_2, x_3) = \operatorname{trop}(s_1')(x_1 + w, x_2 + w, x_3 + w) \\
&= (\min \begin{pmatrix} 2(x_2 + w), 2(x_3 + w), \\ (b + w) + (x_2 + w), (c + w) + (x_3 + w), \\ d + 2w \end{pmatrix} - (x_1 + w), \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad x_2 + w, x_3 + w) \\
&= (\min(2x_2, 2x_3, b + x_2, c + x_3, d) - x_1 + w, x_2 + w, x_3 + w) \\
&= \phi^w \circ \operatorname{trop}(s_1)(x_1, x_2, x_3).
\end{aligned}$$

Similar algebra works for $i = 2, 3$. □

9.3 Foliating the space by Level sets of Partial Tropical Polynomials

One interesting aspect of the level sets $\{f_0 = w\}$ is that it 'foliates' $\mathbb{R}^3$ in the following sense.

Proposition 9.5. *Let* $\Pi = \{(x_1, x_2, x_3) \in \mathbb{R}^3 : x_1 + x_2 + x_3 = 0\}$ *be a plane in* $\mathbb{R}^3$*, and* $v \colon \mathbb{R}^3 \to \Pi$ *be the orthogonal projection onto that plane, i.e.,*

$$v = (v_1, v_2, v_3) = \left(\frac{2x_1 - x_2 - x_3}{3}, \frac{-x_1 + 2x_2 - x_3}{3}, \frac{-x_1 - x_2 + 2x_3}{3} \right).$$

Then the product map $f_0 \times v \colon \mathbb{R}^3 \to \mathbb{R} \times \Pi$ *is a piecewise-linear homeomorphism.*

Proof. Let $\alpha = \frac{1}{3}(x_1 + x_2 + x_3)$. Then the map $\alpha \times v \colon \mathbb{R}^3 \to \mathbb{R} \times \Pi$ is a linear isomorphism, with the inverse $x = (x_1, x_2, x_3) = (\alpha + v_1, \alpha + v_2, \alpha + v_3)$.

We describe the inverse $(f_0 \times v)^{-1}$ by describing $(\alpha \times v) \circ (f_0 \times v)^{-1}$ instead. That is, we suggest how to recover α from $w \in \mathbb{R}$ and $v \in \Pi$ given. Plug in $(x_1, x_2, x_3) =$

$(\alpha + v_1, \alpha + v_2, \alpha + v_3)$ into inequalities (9.2.1)–(9.2.7) and get

$$\begin{aligned} w + 3\alpha &\le 2\alpha + 2v_1, & w + 3\alpha &\le a + \alpha + v_1, \\ w + 3\alpha &\le 2\alpha + 2v_2, & w + 3\alpha &\le b + \alpha + v_2, \\ w + 3\alpha &\le 2\alpha + 2v_3, & w + 3\alpha &\le c + \alpha + v_3, \\ & & w + 3\alpha &\le d, \end{aligned}$$

with at least one of them holding as an equality. Solving these inequalities in α, we obtain

$$\begin{aligned} \alpha &\le 2v_1 - w, & \alpha &\le \frac{1}{2}(a + v_1 - w), \\ \alpha &\le 2v_2 - w, & \alpha &\le \frac{1}{2}(b + v_2 - w), \\ \alpha &\le 2v_3 - w, & \alpha &\le \frac{1}{2}(c + v_3 - w), \\ & & \alpha &\le \frac{1}{3}(d - w), \end{aligned}$$

with at least one of them holding as an equality. This is equivalent to

$$\alpha = \min \left(\begin{array}{l} 2v_1 - w, 2v_2 - w, 2v_3 - w, \\ \frac{1}{2}(a + v_1 - w), \frac{1}{2}(b + v_2 - w), \frac{1}{2}(c + v_3 - w), \\ \frac{1}{3}(d - w) \end{array} \right), \tag{9.3.1}$$

which solves the inverse map required. $\square$

Corollary 9.6. *The orthogonal projection map $v\colon \mathbb{R}^3 \to \Pi$ restricted to a skeleton,*

$$v|Sk(a, b, c, d) \to \Pi,$$

is a piecewise-linear homeomorphism.

9.4 Transversality of Fixed Sets

By the description of tropicalized Vieta involutions (9.1.1)–(9.1.3), as a map $\mathbb{R}^3 \to \mathbb{R}^3$, the fixed sets of $\mathrm{trop}(s_1), \mathrm{trop}(s_2), \mathrm{trop}(s_3)$ appear as

$$\mathsf{Fix}(\mathrm{trop}(s_1))\colon\, 2x_1 = \min(2x_2, 2x_3, b + x_2, c + x_3, d),$$
$$\mathsf{Fix}(\mathrm{trop}(s_2))\colon\, 2x_2 = \min(2x_1, 2x_3, a + x_1, c + x_3, d),$$
$$\mathsf{Fix}(\mathrm{trop}(s_3))\colon\, 2x_3 = \min(2x_1, 2x_2, a + x_1, b + x_2, d).$$

Hence these fixed sets appear as graphs of some piecewise-linear functions $\mathbb{R}^2 \to \mathbb{R}$, thus piecewise-linear homeomorphic to $\mathbb{R}^2$.

It turns out that by the foliation $f_0 \times v\colon \mathbb{R}^3 \to \mathbb{R} \times \Pi$ described in Proposition 9.5 above, the fixed sets are 'transverse' to each level set $\{f_0 = w\}$ in the following sense.

Proposition 9.7. *There is a coordinate $(w; v_1, u_1)$ of $\mathbb{R} \times \Pi$ such that the image of the fixed set $\mathsf{Fix}(\mathrm{trop}(s_1)) \subset \mathbb{R}^3$ under the map $f_0 \times v\colon \mathbb{R}^3 \to \mathbb{R} \times \Pi$ is represented as a graph $v_1 = G(w, u_1)$ of a piecewise-linear continuous function.*

Proof. Specifically, we put $u_1 = v_2 - v_3 = x_2 - x_3$ in our claim. We recall $v_1 = \frac{1}{3}(2x_1 - x_2 - x_3)$ from Proposition 9.5 above. Then $(w; v_1, u_1)$ forms a coordinate of $\mathbb{R} \times \Pi$. Because w and u_1 are invariant under $\mathrm{trop}(s_1)$, by intermediate value theorem, there must be a unique $v_1 = G(w, u_1)$ in which $(w; v_1, u_1)$ represents a point in $\mathsf{Fix}(\mathrm{trop}(s_1))$. So we focus on showing that G is piecewise-linear continuous.

Recall the function $f_1(x_2, x_3) = \min(2x_2, 2x_3, b + x_2, c + x_3, d)$. Then one can describe $f_0(x_1, x_2, x_3) = \min(2x_1, a + x_1, f_1(x_2, x_3)) - (x_1 + x_2 + x_3)$. If $(x_1, x_2, x_3) \in \mathsf{Fix}(\mathrm{trop}(s_1))$, then $f_1(x_2, x_3) = 2x_1$, so we have

$$\begin{aligned} f_0(x_1, x_2, x_3) &= \min(2x_1, a + x_1, 2x_1) - (x_1 + x_2 + x_3) \\ &= \min(x_1, a) - (x_2 + x_3). \end{aligned}$$

For a point $(x_1, x_2, x_3) \in \mathsf{Fix}(\mathrm{trop}(s_1))$, suppose $(w; v_1, u_1) = (w; G(w, u_1), u_1)$ represent the point. Because $f_0(x_1, x_2, x_3) = \min(x_1, a) - (x_2 + x_3) = w$, we have $w + x_2 + x_3 = \min(x_1, a)$. So we have

$$\begin{aligned} x_1 &= \frac{1}{2} f_1(x_2, x_3) \\ &= \min(x_2, x_3, \frac{b + x_2}{2}, \frac{c + x_3}{2}, \frac{d}{2}), && (9.4.1) \\ w + x_2 + x_3 &= \min(x_1, a). && (9.4.2) \end{aligned}$$

Plug in (9.4.1) into (9.4.2). Then we have

$$w + x_2 + x_3 = \min(x_2, x_3, \frac{b + x_2}{2}, \frac{c + x_3}{2}, \frac{d}{2}, a).$$

Set $\tilde{v}_1 = x_2 + x_3$. With $u_1 = x_2 - x_3$, we simplify

$$2\tilde{v}_1 + 2w = \min(\tilde{v}_1 + u_1, \tilde{v}_1 - u_1, b + \frac{1}{2}(\tilde{v}_1 + u_1), c + \frac{1}{2}(\tilde{v} - u_1), d, 2a).$$

This equality can be expanded as a list of inequalities, with at least one of them holding as an equality. Solving the inequalities in $\tilde{v}_1$, one then obtains

$$\tilde{v}_1 = \min \begin{pmatrix} u_1 - 2w, -u_1 - 2w, \\ \frac{1}{3}(2b - 4w + u_1), \frac{1}{3}(2c - 4w - u_1), \\ \frac{1}{2}d - w, a - w \end{pmatrix}.$$

This represents $\tilde{v}_1$ in w and u_1. From this, we represent $x_2 = \frac{1}{2}(\tilde{v}_1 + u_1)$ and $x_3 = \frac{1}{2}(\tilde{v}_1 - u_1)$ in w and u_1. By (9.4.1), we represent x_1 in w and u_1, therefore representing $v_1 = \frac{1}{3}(2x_1 - x_2 - x_3)$ in w and u_1. All these representations are piecewise-linear and continuous, hence the function $G(w, u_1)$ in question is piecewise-linear and continuous. $\square$

In particular, if we intersect a fixed set $\mathsf{Fix}(\mathrm{trop}(s_i))$ with a level set $\{f_0 = w\}$ we get a piecewise-linear image of $\mathbb{R}$. Later, we will verify that on a level set $\{f_0 = w\}$, the tropicalized Vieta involutions are proper and each is topologically conjugate to a line reflection on a plane. By (9.2.12), we turn this claim to an analogous claim on a skeleton. This motivates us to take a closer look at the actions on each skeleton.

CHAPTER 10

VIETA INVOLUTIONS ON A SKELETON

In this chapter, we consider the group $\Gamma = \langle s_1, s_2, s_3 \rangle$ generated by Vieta involutions, and analyze its action

$$\Gamma \curvearrowright Sk(a, b, c, d)$$

on a skeleton. A central tool for this analysis is to view Vieta involutions on $Sk(a, b, c, d)$ as line reflections, and see how these reflections act on $Sk(a, b, c, d)$. Proposition 10.7 is the summary of all these analysis, which will be a starting point of all subsequent discussions.

10.1 Cells

Recall that the equation $f_0 = 0$, i.e.,

$$\min(2x_1, 2x_2, 2x_3, a + x_1, b + x_2, c + x_3, d) = x_1 + x_2 + x_3$$

characterizes the skeleton $Sk(a, b, c, d)$. Hence it is natural to decompose the skeleton into subsets, depending on which tropical monomial among $2x_1$, $2x_2$, $2x_3$, $a + x_1$, $b + x_2$, $c + x_3$, or d does the tropical monomial $x_1 + x_2 + x_3$ equals to. Formally, we define such subsets as follows.

Definition 10.1 (Cells)**.** Let $m(X) \in \{X_1^2, X_2^2, X_3^2, AX_1, BX_2, CX_3, D\}$ be a monomial. Define the *cell* $\mathcal{C}(m(X))$ as a subset of the skeleton $Sk = Sk(a, b, c, d)$ defined as

$$\mathcal{C}(m(X)) = \{x \in Sk : \mathrm{trop}(X_1X_2X_3)(x) = \mathrm{trop}(m(X))(x)\}.$$

It is clear that

$$Sk(a, b, c, d) = \bigcup\{\mathcal{C}(m(X)) : m(X) = X_1^2, X_2^2, X_3^2, AX_1, BX_2, CX_3, D\}.$$

By the light of (9.2.12), we also define cells $\mathcal{C}(m(X))$ on each level set $\{f_0 = w\}$. One can explicitly describe these cells as

$$\mathcal{C}(m(X)) = \{x \in \{f_0 = w\} : w + \mathrm{trop}(X_1X_2X_3)(x) = \mathrm{trop}(m(X))(x).\}$$

Some more immediate facts follow.

Lemma 10.2. *For cells in a skeleton $Sk(a,b,c,d)$, we have the followings.*

(a) Each cell $\mathcal{C}(m(X))$ sits in a plane in $\mathbb{R}^3$ and forms a closed convex subset.

(b) The (planar) interior $\mathcal{C}(m(X))^\circ$ is precisely the set of points $x \in \mathcal{C}(m(X))$ that does not belong to any other cells.

(c) If $x = (x_1, x_2, x_3) \in Sk(a,b,c,d)$ has $x_1 + x_2 + x_3 < \min(2a, 2b, 2c, d)$, then $x \in \mathcal{C}(X_i^2)$ for some $i = 1, 2, 3$.

(d) In particular, the interiors of quadratic cells, $\mathcal{C}(X_i^2)^\circ$ with $i = 1, 2, 3$, are nonempty.

Proof. (a) Say, for instance, if $m(X) = X_1^2$, then we have $(x_1, x_2, x_3) \in \mathcal{C}(X_1^2)$ if and only if (i) $x_1 + x_2 + x_3 = 2x_1$ and (ii) $2x_1 - (x_1 + x_2 + x_3) = f_0(x_1, x_2, x_3)$. It follows that $\mathcal{C}(X_1^2)$ lies on the plane (i). Additionally, one can rewrite (ii) as

$$2x_1 \leq \min(2x_2, 2x_3, a + x_1, b + x_2, c + x_3, d). \tag{10.1.1}$$

This inequality can be viewed as an inequality of the form $0 \leq H$ for some concave function H on $\mathbb{R}^3$. As a superlevel set of a concave function, (10.1.1) defines a convex set. The cell $\mathcal{C}(X_1^2)$ is the intersection of a plane (i) with a closed convex set $0 \leq H$, thus closed convex.

(b) Observe that the concave function H above is the minimum of various affine-linear functions, none of them being a constant. Hence the interior of the set $\{0 \leq H\}$ is $\{0 < H\}$. Therefore the strict inequality for (10.1.1) characterizes the (planar) interior of $\mathcal{C}(X_1^2)$. But then, by (i), we have $x_1 + x_2 + x_3 < \min(2x_2, 2x_3, a + x_1, b + x_2, c + x_3, d)$. So any interior point $x \in \mathcal{C}(X_1^2)$ cannot belong to any other cells. All the above steps were equivalent formulations, so the converse is also the case.

Similar argument applies for other monomials to show (a) and (b).

(c) We show the contrapositive. That is, assume that $x \in \mathcal{C}(AX_1) \cup \mathcal{C}(BX_2) \cup \mathcal{C}(CX_3) \cup \mathcal{C}(D)$ and show that $x_1 + x_2 + x_3 \geq \min(2a, 2b, 2c, d)$.

If $x \in \mathcal{C}(AX_1)$, we have $x_1 + x_2 + x_3 = a + x_1 \leq 2x_1$, so $x_2 + x_3 = a$ and $x_1 \geq a$. Thus $x_1 + x_2 + x_3 \geq a + a = 2a \geq \min(2a, 2b, 2c, d)$ follows. If $x \in \mathcal{C}(BX_2)$ or $x \in \mathcal{C}(CX_3)$, we argue analogously. If $x \in \mathcal{C}(D)$, we have $x_1 + x_2 + x_3 = d \geq \min(2a, 2b, 2c, d)$.

(d) Let $M = -1 + \min(0, 2a, 2b, 2c, d) < 0$. Then $(-2M, -M, -M)$, $(-M, -2M, -M)$, and $(-M, -M, -2M)$ are points that respectively lie on $\mathcal{C}(X_1^2)^\circ$, $\mathcal{C}(X_2^2)^\circ$, and $\mathcal{C}(X_3^2)^\circ$. $\square$

There are two (mutually exclusive) cases for the skeleton $Sk(a, b, c, d)$.

Holomorphic Parameters It consists of quadratic cells, i.e., $Sk(a, b, c, d) = \mathcal{C}(X_1^2) \cup \mathcal{C}(X_2^2) \cup \mathcal{C}(X_3^2)$.

Meromorphic Parameters At least one of the non-quadratic cells, $\mathcal{C}(AX_1)$, $\mathcal{C}(BX_2)$, $\mathcal{C}(CX_3)$, or $\mathcal{C}(D)$, has nonempty interior.

The dynamics on each case are quite different. However, there is a simple numerical criterion distinguishing the two: whether $\min(a, b, c, d) \geq 0$ or not.

Remark. Taking this criterion as granted, we coin the terms 'holomorphic/meromorphic parameter' in reference to the convergent Puiseux field $\mathbb{C}(\{t^*\})$ (see Example 8.4 for notations). If $\min(a, b, c, d) \geq 0$, then all parameters A, B, C, D are holomorphic (in some $t^{1/n}$). Otherwise, at least one of them is meromorphic (in some $t^{1/n}$).

Proposition 10.3. *Let $a, b, c, d \in \mathbb{R}$ be parameters. The followings are equivalent.*

(a) We have $Sk(a, b, c, d) = \mathcal{C}(X_1^2) \cup \mathcal{C}(X_2^2) \cup \mathcal{C}(X_3^2)$.

(b) The origin $(0,0,0)$ *is in* $Sk(a,b,c,d)$.

(c) We have $\min(a,b,c,d) \geq 0$.

Proof. We show this by showing (a)⇒(b), (b)⇔(c), and (c)⇒(a).

((a)⇒(b)) We know that, if $x \in Sk(a,b,c,d)$, we have the equality for (9.2.1), (9.2.2), and (9.2.3) (with $w=0$) if and only if $x \in \mathcal{C}(X_1^2)$, $x \in \mathcal{C}(X_2^2)$, and $x \in \mathcal{C}(X_3^2)$, respectively. Thus, (a) is equivalent to that $Sk(a,b,c,d)$ equals to the boundary of the region

$$S = \{x_1+x_2+x_3 \leq 2x_1\} \cap \{x_1+x_2+x_3 \leq 2x_2\} \cap \{x_1+x_2+x_3 \leq 2x_3\}. \qquad (10.1.2)$$

So if (a) holds, that $(0,0,0) \in \partial S$ gives (b).

((b)⇔(c)) Suppose $(0,0,0) \in Sk(a,b,c,d)$. This means $f_0(0,0,0) = 0$ (cf. (9.1.6)). Because $f_0(0,0,0) = \min(0,a,b,c,d)$, this is zero iff $\min(a,b,c,d) \geq 0$.

((c)⇒(a)) In general, we find the set $Sk(a,b,c,d)$ by intersecting S (10.1.2) with more regions I_j's defined by inequalities

$$I_1\colon x_1+x_2+x_3 \leq a+x_1,$$
$$I_2\colon x_1+x_2+x_3 \leq b+x_2,$$
$$I_3\colon x_1+x_2+x_3 \leq c+x_3,$$
$$I_4\colon x_1+x_2+x_3 \leq d.$$

Call a,b,c,d the parameters of I_1, I_2, I_3, I_4 respectively. We claim, S is a subset of the region I_j if and only if the parameter of I_j is nonnegative.

Note that all inequalities I_j can be written in the form $\epsilon_1 x_1 + \epsilon_2 x_2 + \epsilon_3 x_3 \leq k_j$, where $\epsilon_i = 0, 1$ and k_j is the parameter of I_j. (For instance, I_1 iff $0 \cdot x_1 + 1 \cdot x_2 + 1 \cdot x_3 \leq a$.)

If the parameter k_j of I_j is ≥ 0, then as any element of S has nonpositive coordinates (cf. Corollary 9.2), we have $\epsilon_1 x_1 + \epsilon_2 x_2 + \epsilon_3 x_3 \leq 0 \leq k_j$. So $S \subset I_j$.

If the parameter k_j of I_j is < 0, then $(0,0,0) \in S$ does not satisfy the inequality I_j, as

$\epsilon_1 \cdot 0 + \epsilon_2 \cdot 0 + \epsilon_3 \cdot 0 = 0 > k_j$. Hence $S \not\subset I_j$.

Hence if $\min(a,b,c,d) \geq 0$, then $S \subset I_j$ for all $j = 1,2,3,4$. Thus $Sk(a,b,c,d)$ is the boundary of $S \cap \bigcap_{j=1}^{4} I_j = S$, giving (a). $\square$

Corollary 10.4. *The followings are equivalent.*

(a) We have $w \geq -\min(a,b,c,\frac{1}{2}d)$.

(b) The level set $\{f_0 = w\}$ *is the union of cells* $\mathcal{C}(X_1^2) \cup \mathcal{C}(X_2^2) \cup \mathcal{C}(X_3^2)$.

(c) The level set $\{f_0 = w\}$ *is a translate of* $Sk(\infty,\infty,\infty,\infty)$.

Proof. ((a)$\Leftrightarrow$(b)) Note that $\min(a,b,c,d) \geq 0$ iff $\min(a,b,c,\frac{1}{2}d) \geq 0$. The skeleton $Sk(a+w, b+w, c+w, d+2w)$ is the union of cells $\mathcal{C}(X_i^2)$'s, $i = 1,2,3$, iff $\min(a+w, b+w, c+w, \frac{1}{2}(d+2w)) = w + \min(a,b,c,\frac{1}{2}d) \geq 0$. This shows the equivalence.

((b)$\Leftrightarrow$(c)) As observed in the proof of Proposition 10.3, we have (b) if and only if the corresponding skeleton is the boundary of the region S in (10.1.2). This is equivalent to that we can 'standardize' $\{f_0 = w\} + (w,w,w) = Sk(a+w, b+w, c+w, d+2w) = \partial S = Sk(\infty,\infty,\infty,\infty)$. $\square$

10.2 Reflections

Given the language of cells, we now can claim the followings. Suppose x is a point in the skeleton $x \in Sk(a,b,c,d) = \{f_0 = 0\}$.

- We have
  - equality for (9.2.1) iff $x \in \mathcal{C}(X_1^2)$,
  - equality for (9.2.4) iff $x \in \mathcal{C}(AX_1)$,
  - equality for (9.2.8) iff $x \in \mathcal{C}(X_2^2) \cup \mathcal{C}(X_3^2) \cup \mathcal{C}(BX_2) \cup \mathcal{C}(CX_3) \cup \mathcal{C}(D)$.

- The equivalence (9.2.10) tells, for $x' = \mathrm{trop}(s_1)(x)$, $x \in \mathcal{C}(AX_1)$ iff $x' \in \mathcal{C}(AX_1)$.
- The equivalences (9.2.9) and (9.2.11) tell, for $x' = \mathrm{trop}(s_1)(x)$, $x \in \mathcal{C}(X_1^2)$ iff $x' \in \mathcal{C}(X_2^2) \cup \mathcal{C}(X_3^2) \cup \mathcal{C}(BX_2) \cup \mathcal{C}(CX_3) \cup \mathcal{C}(D)$, and vice versa.

We summarize the above observation below.

Proposition 10.5. *The tropicalized Vieta involution* $\mathrm{trop}(s_1)$ *leaves* $\mathcal{C}(AX_1)$ *invariant, and sends* $\mathcal{C}(X_1^2)$ *onto the union* $\mathcal{C}(X_2^2) \cup \mathcal{C}(X_3^2) \cup \mathcal{C}(BX_2) \cup \mathcal{C}(CX_3) \cup \mathcal{C}(D)$.

Next, we show that $\mathrm{trop}(s_i)$'s are plane reflections on each level set $\{f_0 = w\} \cong \Pi$. To have so, we first need a

Lemma 10.6. *The tropicalized involutions* (9.1.1)–(9.1.3) *are proper on each level set* $\{f_0 = w\}$. *Hence they extend to an involution on the 1-point compactification* $\{f_0 = w\} \cup \{\infty\} \cong \mathbb{S}^2$, *and each involution topologically conjugates to the reflection by a great circle on the sphere.*

Proof. By Corollary 9.4, it suffices to show this on a skeleton $Sk(a,b,c,d)$, with $a,b,c,d \in \mathbb{R}$ any. Fix the parameters a,b,c,d and let $Sk = Sk(a,b,c,d)$.

One way to show properness of $\mathrm{trop}(s_i)$ is to think a sequence $(x^n)_{n=1}^\infty$ in Sk that goes to infinity (i.e., exits any compact subset of Sk eventually), and claim that $(\mathrm{trop}(s_i)(x^n))_{n=1}^\infty$ also goes to infinity. Let $i = 1$, for simplicity.

Write $x^n = (x_1^n, x_2^n, x_3^n)$. Denote the norm $\|x^n\| = |x_1^n| + |x_2^n| + |x_3^n| = -(x_1^n + x_2^n + x_3^n)$ (cf. Corollary 9.2) on Sk.

By Lemma 10.2(c), for $n \gg 0$ we have $x^n \in \mathcal{C}(X_j^2)$ for some j. If $j = 1$, then as $x_2^n + x_3^n = x_1^n \leq \min(x_2^n, x_3^n)$,

$$\begin{aligned}
\|\mathrm{trop}(s_1)(x^n)\| &= -\min(2x_2^n, 2x_3^n) + x_1^n - x_2^n - x_3^n \\
&= -2\min(x_2^n, x_3^n) \\
&\geq -2x_1^n = -(x_1^n + x_2^n + x_3^n) = \|x^n\|.
\end{aligned}$$

If $j = 2$, then as $x_1^n + x_3^n = x_2^n \le \min(x_1^n, x_3^n)$,

$$\begin{aligned}\|\mathrm{trop}(s_1)(x^n)\| &= -2x_2^n + x_1^n - x_2^n - x_3^n \\ &= -2x_2^n - 2x_3^n \ge -x_1^n - x_2^n - 2x_3^n \\ &\ge -x_1^n - x_2^n - x_3^n = \|x^n\|.\end{aligned}$$

Similar argument gives $\|\mathrm{trop}(s_1)(x^n)\| \ge \|x^n\|$ for $j = 3$, with the roles of indices 2 and 3 interchanged. In all cases, we see that $\|\mathrm{trop}(s_1)(x^n)\|$ is no less than $\|x^n\|$ and thus goes to infinity if $(x^n)_{n=1}^{\infty}$ does.

Now we have $\mathrm{trop}(s_i)$'s extended to a sphere homeomorphism $Sk \cup \{\infty\} \to Sk \cup \{\infty\}$. By Kerékjártó's theorem Kerékjártó [1941]Kolev [2006], $\{1, \mathrm{trop}(s_i)\}$ form a compact subgroup of sphere homeomorphisms, so it conjugates to a closed subgroup of the orthogonal group O(3). Now there are three kinds of order 2 subgroups in O(3): one generated by the antipodal map, one generated by rotating π along an axis, and one generated by the reflection on a great circle. We can classify it according to the number of fixed points of the map. For $\mathrm{trop}(s_i)$, say $i = 1$, points $(-t, -t, 0)$ with $t \gg 0$ are fixed by $\mathrm{trop}(s_1)$ and hence we have infinitely many fixed points. Therefore $\mathrm{trop}(s_i)$ topologically conjugates with the reflection by a great circle, which fixes infinity as well. □

Proposition 10.7. *Denote A_1, A_2, A_3 for A, B, C respectively.*

Each tropicalized Vieta involution $\mathrm{trop}(s_i)$ *on $\{f_0 = w\}$ is topologically conjugate to a line reflection. There exists a topological (open) half-space $D_i \subset \{f_0 = w\}$ such that*

(i) $\mathrm{trop}(s_i)$ *sends D_i onto $\{f_0 = w\} \setminus \overline{D}_i$,*

(ii) the boundary ∂D_i is same as the fixed set of $\mathrm{trop}(s_i)$*, and*

(iii) each D_i can be set as a subset of $\mathcal{C}(X_i^2) \cup \mathcal{C}(A_i X_i)$.

Consequently, the domains D_1, D_2, D_3 are mutually disjoint. The closures $\overline{D}_i$'s union to give all of $\{f_0 = w\}$ if and only if $w \geq -\min(a, b, c, \frac{1}{2}d)$.

Proof. On $\{f_0 = w\}$, each tropicalized involution $\operatorname{trop}(s_i)$ is topologically conjugate to a line reflection on a plane. Thus the fixed set of $\operatorname{trop}(s_i) \cong \mathbb{R}$ splits $\{f_0 = w\} \cong \Pi$ into two pieces, D_i and $\{f_0 = w\} \setminus \overline{D}_i$ and interchanges them. This shows (i) and (ii).

For (iii), note that $\operatorname{trop}(s_i)$ maps $\mathcal{C}(X_i^2)$ onto a union of cells and leaves $\mathcal{C}(A_iX_i)$ invariant (Proposition 10.5). Hence $\mathcal{C}(X_i^2) \cup \mathcal{C}(A_iX_i)$ and its image under $\operatorname{trop}(s_i)$ covers all of $\{f_0 = w\}$. Thus one can choose D_i to be an open subset of $\mathcal{C}(X_i^2) \cup \mathcal{C}(A_iX_i)$.

By (iii), we can set D_i's so that

$$D_1 \subset \mathcal{C}(X_1^2) \cup \mathcal{C}(AX_1),$$

$$D_2 \subset \mathcal{C}(X_2^2) \cup \mathcal{C}(BX_2),$$

$$D_3 \subset \mathcal{C}(X_3^2) \cup \mathcal{C}(CX_3).$$

As the cells only intersect at their boundaries, the open subsets D_i's must be mutually disjoint.

Furthermore, we have

$$\bigcup_{i=1}^{3} \mathcal{C}(X_i^2) \subset \bigcup_{i=1}^{3} \overline{D}_i.$$

Thus if $\{f_0 = w\}$ equals to the union $\bigcup_{i=1}^3 \mathcal{C}(X_i^2)$, or equivalently $w \geq -\min(a, b, c, \frac{1}{2}d)$, then we have $\{f_0 = w\} = \bigcup_{i=1}^3 \overline{D}_i$. Otherwise, we have two cases.

1. The cell $\mathcal{C}(D)$ has nonempty interior.

2. One of the cells $\mathcal{C}(A_iX_i)$ has nonempty interior.

For the first case, the interior $\mathcal{C}(D)^\circ$ does not intersect any of $\overline{D}_i$'s, so $\{f_0 = w\} \neq \bigcup_{i=1}^3 \overline{D}_i$. For the second case, suppose $\mathcal{C}(AX_1)$ has nonempty interior. Then as $\operatorname{trop}(s_1)$ leaves this cell invariant, $\mathcal{C}(AX_1)$ is not a subset of $\overline{D}_1$, i.e., $\mathcal{C}(AX_1) \setminus \overline{D}_1$ is nonempty. This verifies $\{f_0 = w\} \neq \bigcup_{i=1}^{3} \overline{D}_i$ for this case. $\square$

CHAPTER 11

TWO MODES OF VIETA INVOLUTIONS ON SKELETA

By a theorem of Èl′-Huti Èl′-Huti [1974] (see also [Cantat and Loray, 2009, Theorem 3.1]), the group $\Gamma = \langle s_1, s_2, s_3 \rangle$ generated by Vieta involutions (acting on S_{ABCD}, defined over $\mathbb{C}$) has a presentation

$$\Gamma = \langle s_1, s_2, s_3 \mid s_1^2 = s_2^2 = s_3^2 = 1 \rangle.$$

The ping-pong lemma for this group Γ is read as follows. (cf. [de la Harpe, 2000, §II.B Lemma 24])

Theorem (Ping-pong Lemma)**.** *Suppose Γ acts on a set X, and there exist disjoint nonempty subsets $X_1, X_2, X_3 \subset X$ such that $s_j.X_i \subset X_j$ whenever $i \neq j$. Then Γ acts faithfully on X.*

Although this lemma is usually used to determine whether a group is a free product of its subgroups, we may use this in a 'reverse' fashion and use it to compare two Γ-spaces (topological spaces admitting continuous Γ-actions). The trick is to compare two Γ-spaces with 'ping-pong structures' by comparing the complements $X \setminus \bigcup_{i=1}^{3} X_i$. By that we can establish the followings.

- For *meromorphic parameters* ($\min(a, b, c, d) < 0$), there exists a Γ-invariant open subset $U \subset Sk(a, b, c, d)$ such that $U \cong \mathbb{H}^2$ as Γ-spaces (Corollary 11.19).
- For *holomorphic parameters* ($\min(a, b, c, d) \geq 0$), there exists a Γ-equivariant surjection $Sk(a, b, c, d) \setminus \{(0, 0, 0)\} \to \partial\mathbb{H}^2$ (Theorem 11.21).

11.1 Ping-pong Theory

Let

$$\Sigma = \langle \sigma_1, \sigma_2, \sigma_3 \mid \sigma_1^2 = \sigma_2^2 = \sigma_3^2 = 1 \rangle$$

be a group modeling the free product $(\mathbb{Z}/2\mathbb{Z})^{*3}$. If one considers the group $\Gamma = \langle s_1, s_2, s_3 \rangle$ of Vieta involutions on complex Markov surface $S_{ABCD}(\mathbb{C})$, then groups Γ and Σ are isomorphic, by Èl′-Huti [1974][Cantat and Loray, 2009, Theorem 3.1].

We claim that the same model group works for algebraically closed non-archimedean fields (K, val), by looking at tropicalizations. To establish this, we recall the *Ping-Pong Lemma*, a key topological tool for this study.

Theorem 11.1 (Ping-pong Lemma)**.** *Suppose Σ acts on a set X, and there exist disjoint nonempty subsets $X_1, X_2, X_3 \subset X$ such that $\sigma_j.X_i \subset X_j$ whenever $i \neq j$. Then Σ acts faithfully on X.*

Remark. Although the statement is almost the same as the conventional Ping-Pong Lemma (e.g., [de la Harpe, 2000, §II.B Lemma 24]), the conventional lemma requires additional criterion that one of the groups involved in the free product has order ≥ 3. This is because of the counterexample $\langle \sigma_1, \sigma_2 \mid \sigma_1^2 = \sigma_2^2 = 1 \rangle \curvearrowright \{1, 2\}$, where both σ_1, σ_2 acts by flipping 1 and 2. Setting $X_1 = \{1\}$ and $X_2 = \{2\}$ gives $\sigma_1.X_2 = X_1$ and $\sigma_2.X_1 = X_2$, but clearly the group does not act faithfully.

Such an issue is no longer the case when there are more than two free factors, like our Σ. The proof below thus demonstrates that the conventional ping-pong lemma still holds for Σ.

Proof. Let $w \in \Sigma$ be any nontrivial reduced word, so that $w = \sigma_{i(n)}\sigma_{i(n-1)} \cdots \sigma_{i(1)}$, where each $i(j) \in \{1, 2, 3\}$ and $i(j+1) \neq i(j)$ for $j = 1, \ldots, n-1$. If $n > 1$ and $i(n) = i(1)$ then conjugate w by $\sigma_{i(1)}$ to trim $\sigma_{i(n)}$ and $\sigma_{i(1)}$ out from the word. By this we may assume $n = 1$ or $\sigma_{i(n)} \neq \sigma_{i(1)}$.

If $n = 1$ then we may assume $w = \sigma_1$ (i.e., $i(1) = 1$). If w acts on any point $x \in X_2$ then we have $w.x \in X_1$, which is clearly $\neq x$. Thus w acts nontrivially.

Suppose $n > 1$. Reindexing the indices if necessary, we may assume that $i(n) = 2$ and $i(1) = 1$. Pick any point $x \in X_3$. Then one can inductively show that $\sigma_{i(j)}\sigma_{i(j-1)} \cdots \sigma_{i(1)}.x \in$

$X_{i(j)}$. In particular,

$$w.x = \sigma_{i(n)}\sigma_{i(n-1)}\cdots\sigma_{i(1)}.x \in X_{i(n)} = X_2,$$

so $w.x \neq x$ and thus w acts nontrivially. $\square$

Corollary 11.2. *Suppose X is a topological space and $\Sigma \curvearrowright X$ acts continuously. Suppose further that σ_i maps X_i onto $X \setminus \overline{X}_i$. Then for a reduced word $\sigma_{i(n)}\cdots\sigma_{i(1)} \in \Sigma$, we have*

$$\sigma_{i(n)}\cdots\sigma_{i(1)}.(X \setminus X_{i(1)}) \subset \overline{X}_{i(n)}, \tag{11.1.1}$$

$$\sigma_{i(n)}\cdots\sigma_{i(1)}.(X \setminus \overline{X}_{i(1)}) \subset X_{i(n)}. \tag{11.1.2}$$

Proof. This follows from $\sigma_i.(X \setminus \overline{X}_i) = X_i$, $\sigma_i.(X \setminus X_i) = \overline{X}_i$, $\sigma_j.X_i \subset X_j$, and $\sigma_j.\overline{X}_i \subset \overline{X}_j$, together with some induction. $\square$

Corollary 11.3. *Suppose $\Sigma \curvearrowright Sk(a,b,c,d)$ by tropicalized actions, i.e., σ_i acts as* $\mathrm{trop}(s_i)$, *$i = 1,2,3$. Then this action is faithful.*

Proof. By Proposition 10.7, we have disjoint nonempty subsets $D_1, D_2, D_3 \subset Sk(a,b,c,d)$ such that $\sigma_j.D_i \subset \sigma_j.(Sk \setminus \overline{D}_j) = D_j$ whenever $i \neq j$. The ping-pong lemma applies. $\square$

Therefore the group $\Gamma = \langle s_1, s_2, s_3 \rangle$ generated by Vieta involutions is isomorphic to $\Sigma = (\mathbb{Z}/2\mathbb{Z})^{*3}$, when it acts on a Markov cubic defined over an algebraically closed non-archimedean field.

Now we illuminate the ping-pong lemma on other perspective, especially as a way to compare two Σ-spaces (topological space that admits a continuous Σ-action). Unfortunately, we do not have a uniform viewpoint for this and we ended up presenting two approaches on comparing Σ-spaces.

11.1.1 The Ping-pong Structure

Definition 11.4 (Ping-pong structure)**.** Let X be a Σ-space. Suppose we have disjoint open subsets $X_1, X_2, X_3 \subset X$ such that

(i) $\sigma_j.X_i \subset X_j$ whenever $i \neq j$,

(ii) σ_i fixes the boundary ∂X_i, pointwise, and

(iii) σ_i sends X_i onto $X \setminus \overline{X}_i$.

Then we say the triple (X_1, X_2, X_3) the *ping-pong structure* on X.

In general, the boundaries ∂X_i may have points with complicated stabilizers. Comparing them is essential when we compare two Σ-spaces with ping-pong structures.

Definition 11.5 (Fat, Thin)**.** Let (X_1, X_2, X_3) be a ping-pong structure on $\Sigma \curvearrowright X$. Define $L_i = \partial X_i \setminus \bigcup_{j \neq i} \overline{X}_j$ and $X_0 = (X \setminus \bigcup_{i=1}^3 \overline{X}_i) \cup \bigcup_{i=1}^3 L_i$. We say the ping-pong structure is *fat* if X_0 has a nonempty interior. Otherwise, we say the structure is *thin*.

For fat ping-pong structures, we say X_0 is the *ping-pong table* and sets L_i's are *ping-pong nets*. We may denote a fat ping-pong structure with its table, as $(X_1, X_2, X_3; X_0)$.

Note that for fat ping-pong structures, we can recover the nets L_i by $L_i = X_0 \cap \overline{X}_i$. So it is not necessary to indicate the nets in the description of a fat ping-pong space.

Fat ping-pong structures are much easier to study, because only looking at the tables we are done with all the major comparisons. We will also study thin ping-pong structures, but only for some limited cases.

Figure 11.1 sketches a fat ping-pong structure. Figure 11.2 sketches a thin ping-pong structure on a circle.

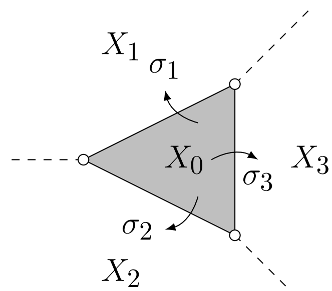

Figure 11.1: A sketch of a fat ping-pong structure.

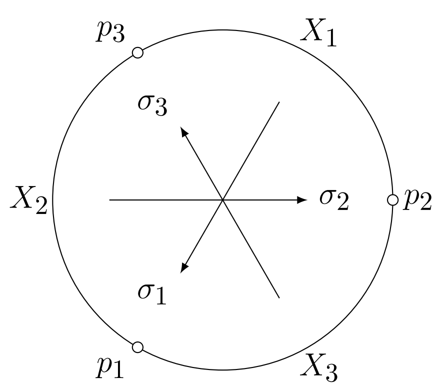

Figure 11.2: A sketch of a thin ping-pong structure.

11.1.2 Comparing Fat Ping-pong Structures

Theorem 11.6 (Comparing Fat Ping-pong Structures)**.** *Let X, Y be Σ-spaces with fat ping-pong strutures $(X_1, X_2, X_3; X_0)$ and $(Y_1, Y_2, Y_3; Y_0)$ respectively. Any continuous map $\psi\colon X_0 \to Y_0$ that respects the ping-pong nets, i.e., that $\psi(X_0 \cap \overline{X}_i) \subset Y_0 \cap \overline{Y}_i$, extends Σ-equivariantly to a map $\Psi\colon \Sigma.X_0 \to \Sigma.Y_0$.*

Lemma 11.7. *Suppose X is a Σ-space with a fat ping-pong structure $(X_1, X_2, X_3; X_0)$. Any point of X_0 has the stabilizer group which is either trivial or one of $\langle \sigma_i \rangle$. The subset $X_0 \cap \partial X_i$ is precisely the set of points $\in X_0$ that have the stabilizer $\langle \sigma_i \rangle$.*

Proof. Suppose $x \in X_0 \setminus \bigcup_{i=1}^3 \partial X_i$. Then for any nontrivial $g = \sigma_{i(n)} \cdots \sigma_{i(1)} \in \Sigma$, we have $x \in X \setminus \overline{X}_{i(1)}$, thus $g.x \in X_{i(n)}$ by (11.1.2). Hence $g.x \notin X_0$, so g cannot stabilize x.

Suppose $x \in X_0 \cap \partial X_i$. We may assume that $i = 1$. Suppose $g = \sigma_{i(n)} \cdots \sigma_{i(1)} \in \Sigma$ is nontrivial and $i(1) \neq 1$. Then we have $x \in X \setminus \overline{X}_{i(1)}$, thus $g.x \in X_{i(n)} \subset X \setminus X_0$ by (11.1.2). So g cannot stabilize x unless $i(1) = 1$. But even if $i(1) = 1$, unless $g = \sigma_{i(1)}$, $g\sigma_{i(1)}^{-1}.x \notin X_0$ by the same reason above. Thus the stabilizer of x is $\langle \sigma_1 \rangle$. □

Lemma 11.8. *Suppose X is a Σ-space with a fat ping-pong structure $(X_1, X_2, X_3; X_0)$. The set $X_0 \cup \sigma_i.X_0$ is a neighborhood of the ping-pong net $L_i = X_0 \cap \overline{X}_i$.*

Proof. Let $i = 1$, without loss of generality. We know that $X \setminus \bigcup_{i=1}^3 \overline{X}_i$ is the interior of X_0, and we denote that set X_0°. The set $G = X_1 \cup L_1 \cup X_0^\circ$ then equals to $X \setminus (\overline{X}_2 \cup \overline{X}_3)$ and is thus an open neighborhood of L_1.

Next, the set $F = X_1 \setminus \sigma_1.X_0^\circ$ is a closed subset of X_1 which is a subset of $\sigma_1.(\overline{X}_2 \cup \overline{X}_3)$. Since $\sigma_1.L_1 = L_1$ is disjoint with $\sigma_1.(\overline{X}_2 \cup \overline{X}_3)$, the set $G \setminus F$ is an open neighborhood of L_1. We evaluate

$$
\begin{aligned}
G \setminus F &= (X_1 \cup L_1 \cup X_0^\circ) \setminus (X_1 \setminus \sigma_1.X_0^\circ) \\
&= L_1 \cup X_0^\circ \cup \sigma_1.X_0^\circ,
\end{aligned}
$$

and we see that this is a subset of $X_0 \cup \sigma_1.X_0$. The claim is shown. $\square$

Lemma 11.9. *Suppose X is a Σ-space with a fat ping-pong structure $(X_1, X_2, X_3; X_0)$. The action map $A\colon \Sigma \times X_0 \to \Sigma.X_0$. is a quotient map.*

Proof. Suppose $U \subset \Sigma.X_0$ is a subset whose preimage $A^{-1}(U) \subset \Sigma \times X_0$ is open. We aim to show that $U \subset \Sigma.X_0$ is open.

Note that $X_0 \setminus \bigcup_{i=1}^{3} \partial X_i$ is precisely the interior X_0° of X_0. By Lemma 11.7, if we restrict A to $\Sigma \times X_0^\circ$, then the action map is a homeomorphism onto the image $\Sigma.X_0^\circ$. Thus if $U \subset \Sigma.X_0^\circ$, $A^{-1}(U)$ is open iff U is open.

Suppose U intersects $\Sigma.(X_0 \setminus X_0^\circ)$ at $g.x_0$. Switching to $g^{-1}.U$ if necessary, we may assume $g = 1$. From $X_0 \setminus X_0^\circ = \bigcup_{i=1}^{3} \partial X_i$, we may assume $x_0 \in \partial X_1$. We aim to show that U is a neighborhood of x_0 (in $\Sigma.X_0$).

Because $A^{-1}(x_0) = \{(1, x_0), (\sigma_1, x_0)\}$ by Lemma 11.7, we see that the open set $A^{-1}(U)$ is a neighborhood of this preimage. Thus there is a neighborhood U' of x_0 in X_0 such that $\{1, \sigma_1\} \times U' \subset A^{-1}(U)$. As A is surjective, $U' \cup \sigma_1.U' \subset U$ follows. Hence it suffices to see if $U' \cup \sigma_1.U'$ is a neighborhood of x_0 in $\Sigma.X_0$ (to prove that U itself is open).

Suppose otherwise. Recall that $X_0 \cup \sigma_1.X_0$ is a neighborhood of x_0 in X and in $\Sigma.X_0$ (Lemma 11.8). For every neighborhood $V \subset X_0 \cup \sigma_1.X_0$ of x_0 in $\Sigma.X_0$, $\sigma_1.V$ is also a neighborhood of x_0. Thus $V \cap \sigma_1.V$ is a neighborhood of x_0 as well. Because x_0 is not on the interior of $U' \cup \sigma_1.U'$ in $\Sigma.X_0$ we have an element $y_V \in V \cap \sigma_1.V$ such that $y_V \notin U' \cup \sigma_1.U'$. Here, we may set $y_V \in X_0$, by replacing $y_V \leftarrow \sigma_1.y_V$ if necessary. But as U' is a neighborhood of x_0 in X_0, $y_V \in U'$ if V gets small enough. This contradicts. $\square$

We now prove Theorem 11.6.

Proof. Denote the action maps $A\colon \Sigma \times X_0 \to \Sigma.X_0$ and $A'\colon \Sigma \times Y_0 \to \Sigma.Y_0$, which are quotient maps. We aim to build a map $\Psi\colon \Sigma.X_0 \to \Sigma.Y_0$ which fits into the following commutative diagram:

$$\begin{tikzcd}
\Sigma \times X_0 \arrow[r, "A"] \arrow[d, "\mathrm{Id}\times\psi"'] & \Sigma.X_0 \arrow[d, dashed, "\exists!\Psi"] \\
\Sigma \times Y_0 \arrow[r, "A'"] & \Sigma.Y_0
\end{tikzcd}$$

Because A is a quotient map, if we know

$$A(g, x_0) = A(g', x_1) \quad \text{implies} \quad A'(g, \psi(x_0)) = A'(g', \psi(x_1)),$$

then the map Ψ is constructed by the universal property. Suppose $A(g, x_0) = A(g', x_1)$, i.e., $g.x_0 = g'.x_1$. By Lemma 11.7, we have $x_0 = x_1$ and $(g')^{-1}g$ stabilizes x_0. Because ψ respects 'ping-pong nets,' $(g')^{-1}g$ also fixes $\psi(x_0)$. Therefore $A'(g, \psi(x_0)) = A'(g', \psi(x_1))$. $\square$

11.1.3 Comparing Ping-pong Structures on Circles

Suppose $X = S^1$ is a circle, which is a Σ-space that has a ping-pong structure (X_1, X_2, X_3). For sake of our potential applications, we restrict the ping-pong structure to satisfy the followings.

Definition 11.10. Call the ping-pong structure (X_1, X_2, X_3) *adapted to the circle* $X = S^1$ if we have the followings.

(i) Each open set X_i is an interval.

(ii) The complement $X \setminus \bigcup_{i=1}^{3} X_i$ has three points.

(iii) The boundary ∂X_i is precisely the fixed locus of σ_i.

(iv) Each σ_i sends X_i onto $X \setminus \overline{X}_i$.

In this case, the open sets X_i's and intersections $\{p_1\} = \overline{X}_2 \cap \overline{X}_3$, $\{p_2\} = \overline{X}_1 \cap \overline{X}_3$, and $\{p_3\} = \overline{X}_1 \cap \overline{X}_3$ exhaust X, so the structure must be thin. Nonetheless, we will call each p_i a *ping-pong net*. See Figure 11.2 to see how intervals X_i's and nets p_i's are typically configured.

Note that the indexing convention of p_i's implies $\partial X_1 = \{p_2, p_3\}$, $\partial X_2 = \{p_1, p_3\}$, and $\partial X_3 = \{p_1, p_2\}$. By this convention, one can describe the stabilizer of p_i as the subgroup generated by $\{\sigma_1, \sigma_2, \sigma_3\} \setminus \{\sigma_i\}$. Furthermore, ping-pong nets p_i are not in the orbit of another.

Lemma 11.11. *If $w.p_i = p_j$, then $i = j$ and w is in the stabilizer of p_i.*

Proof. Write w in the reduced word, $w = \sigma_{i(n)} \cdots \sigma_{i(1)}$. If $i(1) \neq i$, then as $\sigma_{i(1)}$ stabilizes p_i, we reduce the letter and get a shorter w. Continue on, until we reach to $w = 1$ (thus the conclusion) or we have $w \neq 1$ with the rightmost letter $\sigma_{i(1)} = \sigma_i$.

Because $p_i \in X \setminus \overline{X}_i$, we have $\sigma_i.p_i \in X_i$. Together with $\sigma_j.X_i \subset X_j$ if $i \neq j$, we have $w.p_i \in X_{i(n)}$. But as $p_j \notin X_{i(n)}$, we have a contradiction. □

To compare ping-pong structures (X_1, X_2, X_3), (Y_1, Y_2, Y_3) adapted to circles X, Y, we first correspond ping-pong nets $(p_1, p_2, p_3) \mapsto (q_1, q_2, q_3)$ and extend it to the whole space $X \to Y$, Σ-equivariantly. To make this sketch work, we first need to understand the orbit $\Sigma.\{p_1, p_2, p_3\}$ of ping-pong nets.

Denote $|\cdot|$ for the word length of the group Σ with respect to the generators $\sigma_1, \sigma_2, \sigma_3$. Denote $\Sigma_{\leq n}$ for the set of words $w \in \Sigma$ with word length $|w| \leq n$. We say w' is a *left (right) subword* of w if we have $w = w'w''$ ($w = w''w'$) for some reduced word w'' with length $|w| - |w'|$.

We fix an orientation of X and for $a, b \in X$, denote $[a, b]$ ((a, b)) for the closed (open) interval obtained by starting from a, go along the orientation, and end at b. In particular, we have $[a, b] \cup [b, a] = X$ unless $a = b$. For a finite subset $S = \{s_1, \ldots, s_n\} \subset X$, we say s_i is *adjacent* to s_j if one of the open intervals (s_i, s_j) or (s_j, s_i) does not contain any other point in S. Because each σ_i acts as reflections, it preserves intervals (flipping the endpoints) and adjacency (on $\sigma_i.S$). As σ_i filps endpoints, we introduce a notation $-[a,b]=[b,a]$ for flipping an interval.

Consider the partial orbit $P_n = \Sigma_{\leq n}.\{p_1,p_2,p_3\}$ of ping-pong nets. This is a finite set of points in X that "partitions" X by closed intervals. We will call these intervals *partition intervals by* P_n.

Proposition 11.12. *Let* $n \in \mathbb{Z}_{>0}$.

(a) There are $3 \cdot 2^n$ *points in* P_n.

(b) Let $w'.p_j, w''.p_k \in P_n$ *be adjacent points, where rightmost letters of* w', w'' *are* σ_j, σ_k *respectively. Then either* w' *is a left subword of* w'' *or vice versa. If* w' *is a subword of* w''*, then* $(w')^{-1}w''$ *stabilizes* p_j.

(c) Any partition interval by P_n *has an endpoint in* $P_n \setminus P_{n-1}$ *and another in* P_{n-1}.

That is, the partitions by P_n strictly refines that of P_{n-1}, for $n=1,2,3,\ldots$, and each partition interval by P_{n-1} contains a unique point in $P_n \setminus P_{n-1}$.

Proof. First, note that partition intervals of $P_0 = \{p_1,p_2,p_3\}$ are precisely $\overline{X}_1, \overline{X}_2, \overline{X}_3$ and each interval has endpoints in P_0.

We induct on n. For $n=1$, we observe that $\sigma_i.p_i \in X_i$ for $i=1,2,3$. In particular, each $\sigma_i.p_i$ cuts the interval $\overline{X}_i$ into two pieces. Hence P_1 adds three more points from P_0, any adjacent points in P_1 must have a form p_j and $\sigma_k.p_k$ with $k \neq j$, and any partition interval has endpoints from $P_1 \setminus P_0$ and P_0.

Suppose we have the claims (a)–(c) for all $n \leq m$. For any $w.p_i \in P_m \setminus P_{m-1}$, write $w = \sigma_{i(m)} \cdots \sigma_{i(1)}$ with $i(1)=i$. Then we have two choices of σ_j with $j \neq i(m)$. We claim that $\sigma_j w.p_i \notin P_m$.

If otherwise, we have $\sigma_j w.p_i = w'.p_k$ for some $w' \in \Sigma_{\leq m}$. By Lemma 11.11, we must have $k=i$ and $w^{-1}\sigma_j w'$ is in the stabilizer of p_i. But if the leftmost letter of $w^{-1}\sigma_j w'$ is still $\sigma_{i(1)} = \sigma_i$, then the word cannot stabilize p_i; so $\sigma_j w'$ must cancel all letters of w^{-1}. Since $j \neq i(m)$, σ_j must cancel the leftmost letter of w'. But then $|\sigma_j w'| < m$ and we have $w.p_i = \sigma_j w'.p_i \in P_{m-1}$, which contradicts.

Hence we obtain two new points for each element of $P_m \backslash P_{m-1}$. As $|P_m \backslash P_{m-1}| = 3{\cdot}2^{m-1}$, we obtain $3 \cdot 2^m$ new points in P_{m+1}, proving (a).

Now we prove (b). Let $w'.p_j, w''.p_k \in P_{m+1}$ be adjacent, and rightmost letters of w', w'' are σ_j, σ_k respectively. If any of w' or w'' is trivial, then we are done. Suppose $w', w'' \neq 1$ and the leftmost letter of w' and w'' are distinct, say σ_1 and σ_2 respectively. Then we have $w'.p_j \in X_1$ and $w''.p_k \in X_2$. Because p_3 or p_1 is a point in P_{m+1} that can lie between these points, $w'.p_j$ and $w''.p_k$ are not adjacent. Hence w', w'' share the leftmost letters, say σ_1. Then $\sigma_1^{-1} w'.p_j, \sigma_1^{-1} w''.p_k \in P_m$ are adjacent, so we invoke the induction hypothesis to prove (b).

We prove (c). Suppose $[w'.p_j, w''.p_k]$ is a partition interval by P_{m+1}. We may set the rightmost letter of w', w'' are σ_j, σ_k respectively. By (b), we may assume that w' is a left subword of w'', and $(w')^{-1}w''$ stabilizes p_j. Hence $w'.p_j = w'(w')^{-1}w''.p_j = w''.p_j$ follows.

Note that $w'.p_j$ and $w''.p_k$ are adjacent in the subset $w''.P_{m+1-|w''|}$. Thus p_j, p_k are adjacent in $P_{m+1-|w''|}$. But then $m + 1 - |w''|$ must be 0; otherwise we can pick $\sigma_i.p_i$ between p_j, p_k (where $i \neq j, k$). Thus $|w''| = m + 1$ and $w''.p_k \in P_{m+1} \setminus P_m$. Since $w' \neq w''$ (compare the rightmost letters), $|w'| < |w''| = m + 1$ follows and $w'.p_j \in P_m$. This proves (c). $\square$

What was argued in the proof of (c) gives rise to a

Corollary 11.13. *If $\pm[w'.p_j, w''.p_k]$ is a partition interval by P_n with $w''.p_k \in P_n \setminus P_{n-1}$, then for the 3rd index $i \neq j, k$, we have $\pm[w'.p_j, w''.p_k] = w''.\overline{X}_i$.*

Furthermore, the interval $\pm[w'.p_j, w''.p_k]$ has the unique intersection with $P_{n+1} \setminus P_n$ at $w''\sigma_i.p_i$.

Proof. For the latter claim, we note that $\sigma_i.p_i \in X_i$, so on the open interval $\pm(w'.p_j, w''.p_k) =$

$w''.X_i$ we have $w''\sigma_i.p_i$ in that interval. So each partition interval by P_n has at least one element of $P_{n+1} \setminus P_n$. As there are $3 \cdot 2^n$ parititon intervals by P_n and $3 \cdot 2^n$ elements in $P_{n+1} \setminus P_n$, uniqueness follows from counting. $\square$

Recall that a metric d_X on $X = S^1$ is *intrinsic* if $d_X(x, x')$ equals to the infimum of lengths of all paths from x to x'. Then it is natural to define the *total length* L_X of X, which is the intrinsic length of the circle X. In that case, one can measure the length $\mathsf{length}(I)$ of an interval $I = [a, b]$ by either $d_X(a, b)$ or $L_X - d_X(a, b)$, depending on the orientation.

In particular, the lengths of partition intervals of P_n's sum to L_X. Hence we have

$$\delta_X(n) := \inf\{\mathsf{length}(I) : I \text{ is a partition interval by } P_n\} \leq \frac{L_X}{3 \cdot 2^n}. \tag{11.1.3}$$

However, there might be some weird case where the maximal length of partition intervals by P_n is kept long. To avoid so, we impose a condition on the ping-pong dynamics of $\Sigma \curvearrowright X$.

Definition 11.14 (Dense Net Orbits)**.** Suppose X is a circle which has an adapted ping-pong structure (X_1, X_2, X_3). By X *has dense net orbits*, we mean that $\Sigma.\{p_1, p_2, p_3\} \subset X$ is dense.

This criterion suffices to control the maximal length of partition intervals by P_n.

Proposition 11.15. *If X is a circle with an adapted ping-pong structure that has dense orbits, and $P_n = \Sigma_{\leq n}.\{p_1, p_2, p_3\}$ as usual, then the quantity*

$$\Delta_X(n) := \sup\{\mathsf{length}(I) : I \textit{ is a partition interval by } P_n\} \tag{11.1.4}$$

has the limit 0 as $n \to \infty$.

Because partition intervals by P_n refines as we increase n, we see that $\Delta_X(n)$ is a decreasing sequence in n. Hence we have $\lim_{n\to\infty} \Delta_X(n) = \inf_{n\geq 0} \Delta_X(n)$.

Proof. We show the contrapositive. That is, suppose we have $c > 0$ such that $\Delta_X(n) \geq c$ for all $n \geq 0$; we show that there is an open subset $I \subset X$ that does not intersect any P_n.

For each n, pick a partition interval I_n by P_n that has length $\geq c$. Let x_n be the midpoint of I_n, with respect to a fixed intrinsic metric d_X on X.

Because X is compact, we have a limit point $x \in X$ of (x_n)'s and a subsequence (x_{n_k}) converging to x. Suppose we pick a subsequence so that $d_X(x_{n_k}, x) < \frac{1}{4}c$ for all k. Then the distance from x to the boundary of I_{n_k} estimates $\mathsf{dist}(x, \partial I_{n_k}) > \frac{1}{4}c$. Thus there is an interval I centered at x, radius $\frac{1}{4}c$ such that I is in the interior of all I_{n_k}.

Since the interior of I_{n_k} does not intersect P_{n_k}, so it will not intersect P_n too whenever $n \leq n_k$. Since I is a subset of interiors of I_{n_k}'s, I does not intersect any P_n whenever $n \leq n_k$ for some k, i.e., any $n \in \mathbb{Z}_{\geq 0}$. This shows the claim. □

We now state the comparison theorem.

Theorem 11.16 (Ping-pong Comparison on Circles)**.** *Suppose X, Y are circles that have adapted ping-pong structures that has dense net orbits. Denote p_1, p_2, p_3 and q_1, q_2, q_3 for ping-pong nets of X and Y, respectively.*

Then there is a (pointed) homeomorphism $f\colon (X, p_1, p_2, p_3) \to (Y, q_1, q_2, q_3)$ which is Σ-equivariant.

Proof. We first define a map $f\colon \Sigma.\{p_1, p_2, p_3\} \to \Sigma.\{q_1, q_2, q_3\}$ by $f(w.p_i) = w.q_i$. This map is well-defined because the stabilizer of p_i and $f(p_i) = q_i$ coincide.

The map f preserves the cyclic order, in the following sense. Recall $P_n = \Sigma_{\leq n}.\{p_1, p_2, p_3\}$. If $w'.p_j$ and $w''.p_k$ are adjacent points in P_n, then $w'.q_j$ and $w''.q_k$ are also adjacent in $Q_n = \Sigma_{\leq n}.\{q_1, q_2, q_3\}$.

We prove this claim, inductively on n. For $n = 0$, this is obvious. Assume the claim for $n \geq 0$. If $w'.p_j$ and $w''.p_i$ are adjacent points in P_{n+1}, with $w''.p_i \in P_{n+1} \setminus P_n$, then there is a word w in the stabilizer of p_j such that (a) $w'.p_j$ and $w'w.p_k$ (where $k \neq i, j$) are adjacent in P_n, and (b) $w''.p_i = w'w\sigma_i.p_i$ (cf. Corollary 11.13). By induction hypothesis and (a),

$w'.q_j$ and $w'w.q_k$ are adjacent in Q_n. By Corollary 11.13, $w'w\sigma_i.q_i = w''.q_i$ is adjacent to $w'.q_j$ in Q_{n+1}. This clears the induction step.

Next, we show that f is uniformly continuous (fix intrinsic metrics d_X and d_Y for X, Y respectively). Recall that $P_n = \Sigma_{\leq n}.\{p_1, p_2, p_3\}$, and let $\delta_X(n)$ and $\Delta_Y(n)$ be as in (11.1.3) and (11.1.4), respectively.

Fix $n > 0$. We show that, whenever $x, x' \in \Sigma.\{p_1, p_2, p_3\}$,

$$d_X(x, x') < \delta_X(n) \quad \text{implies} \quad d_Y(f(x), f(x')) \leq 2\Delta_Y(n). \tag{11.1.5}$$

If $d_X(x, x') < \delta_X(n)$, we have adjacent points $x_0, x_1, x_2 \in P_n$ (in that cyclic order) such that the parititon intervals $[x_0, x_1] \cup [x_1, x_2]$ contains both x, x'. Same property holds for the intervals $[f(x_0), f(x_1)] \cup [f(x_1), f(x_2)]$ and points $f(x), f(x')$. Because $\pm[f(x), f(x')]$ is a subinterval of $[f(x_0), f(x_2)]$, we estimate

$$\begin{aligned} d_Y(f(x), f(x')) &\leq d_Y(f(x_0), f(x_2)) \\ &= d_Y(f(x_0), f(x_1)) + d_Y(f(x_1), f(x_2)) \leq 2\Delta_Y(n), \end{aligned}$$

by (11.1.4). Now for any $\epsilon > 0$, there is n in which $\Delta_Y(n) < \frac{1}{2}\epsilon$, by Proposition 11.15. Then we choose $\delta = \delta_X(n)$ to have

$$d_X(x, x') < \delta \quad \text{implies} \quad d_Y(f(x), f(x')) < \epsilon,$$

fulfilling the uniform continuity of our interest.

Thanks to uniform continuity, we can extend $f\colon \overline{\Sigma.\{p_1, p_2, p_3\}} = X \to \overline{\Sigma.\{q_1, q_2, q_3\}} = Y$. For any $x \in X$ and $g \in \Sigma$, fix a sequence (x_n) in $\Sigma.\{p_1, p_2, p_3\}$ that converges to x. Then $f(g.x) := \lim_{n\to\infty} f(g.x_n) = \lim_{n\to\infty} g.f(x_n) = g.f(x)$ verifies that f is Σ-equivariant. Because we can construct f^{-1} by the same way, we see that f is a homeomorphism. $\square$

11.2 Meromorphic Parameters

We say a skeleton $Sk(a, b, c, d)$ has *meromorphic parameters* if $\min(a, b, c, d) < 0$. If we think the original Markov cubic S_{ABCD} has parameters from the field of convergent Puiseux series $\mathbb{C}(\{t^*\})$ (see Example 8.4 for notations), then negatively valued elements precisely correspond to meromorphic functions in some $t^{1/n}$. That is why we coin the term.

The ping-pong structure depicted in Proposition 10.7 can be compared with the (∞, ∞, ∞)-triangle reflection action on the hyperbolic plane. To elaborate, let $\mathbb{H}^2 = \{z \in \mathbb{C} : \Im z > 0\}$ be the upper half plane and define the maps $r_1, r_2, r_3 \colon \mathbb{H}^2 \to \mathbb{H}^2$ as

$$r_1(z) = 2 - \bar{z}, \quad r_2(z) = \frac{\bar{z}}{2\bar{z} - 1}, \quad r_3(z) = -\bar{z}, \tag{11.2.1}$$

where $\bar{z}$ is the complex conjugate. Each defines an involution which is an isometry in hyperbolic metrics, i.e., a hyperbolic reflection. The reflections r_1, r_2, r_3 generate a group isomorphic to $\Sigma = (\mathbb{Z}/2\mathbb{Z})^{*3}$, thanks to the ping-pong structure they exhibit.

Proposition 11.17 (Hyperbolic ping-pong structure). *Let $\Sigma \curvearrowright \mathbb{H}^2$ by $\sigma_i.z = r_i(z)$. This action admits a ping-pong structure with*

$$H_1 = \{z \in \mathbb{H}^2 : \Re z > 1\},$$
$$H_2 = \{z \in \mathbb{H}^2 : |2z - 1| < 1\},$$
$$H_3 = \{z \in \mathbb{H}^2 : \Re z < 0\}.$$

The ping-pong table H_0 is then topologically a closed disk minus three points on the boundary. (See Figure 11.3.)

Proof. Computations. ∎

To compare this with meromorphic parameter cases, we establish a

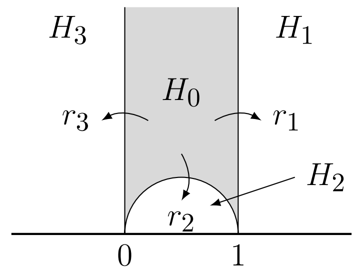

Figure 11.3: The ping-pong structure described in Proposition 11.17.

Lemma 11.18 (Topological description of the Ping-pong table)**.** *Let $Sk = Sk(a, b, c, d)$ be a skeleton defined with parameters a, b, c, d with $\min(a, b, c, d) < 0$. Let D_1, D_2, D_3 be the domains defined in Proposition 10.7. Then the domains define a fat ping-pong structure $(D_1, D_2, D_3; D_0)$ for the action $\Gamma \curvearrowright Sk$. Furthermore, D_0 is topologically a closed disk minus three points on the boundary.*

Proof. Conclusions of Proposition 10.7 show that the definition of ping-pong structure is met. To show the topological description of D_0, note that the interior of D_0 is simply connected, as its complement plus infinity is connected (cf. [Ahlfors, 1978, §4.4.2 Def. 1, §6.1.1 Theorem 1]). Furthermore, $\overline{D}_0$ is a topological disk.

So we ask how many points are in $\overline{D}_0 \setminus D_0$. Such point corresponds to intersections $\mathcal{C}(X_i^2) \cap \mathcal{C}(X_j^2)$, $i \neq j$, which is found as a ray in Sk. These rays start from the boundary of D_0 but does not intersect D_0 (or, 'got removed' by the definition of D_0). Hence $\overline{D}_0 \setminus D_0$ has 3 points, all found at the boundary. Thus the topological description. □

Corollary 11.19 (One Model for the Meromorphic Parameters)**.** *In the skeleton $Sk(a, b, c, d)$ with $\min(a, b, c, d) < 0$, the Γ-orbit $U = \Gamma.D_0$ of the ping-pong table D_0 is open. Furthermore, by the group isomorphism $f \colon \Gamma \to \langle r_1, r_2, r_3 \rangle$, $s_i \mapsto r_i$, we have a f-equivariant homeomorphism $U \to \mathbb{H}^2$.*

Proof. Because the ping-pong tables, D_0 and H_0, are homeomorphic, by Theorem 11.6, we can extend this homeomorphism to $U \cong \mathbb{H}^2$, Σ-equivariantly. (Recall that $\Gamma \cong \Sigma$ by

Corollary 11.3, and so is $\langle r_1, r_2, r_3 \rangle \cong \Sigma$.) This is an open subset of the skeleton $Sk(a, b, c, d)$ because $D' = D_0 \cup \bigcup_{i=1}^{3} \mathrm{trop}(s_i).D_0^\circ$ is open and $U = \Sigma.D'$. $\square$

The machinery developed above is by no means optimal, and we expect a room to improve descriptions for models of Vieta actions. Some expected behaviors are stated as follows.

Question 11.20.

1. Does the Σ-space isomorphism $U \cong \mathbb{H}^2$ extend to a semiconjugacy $Sk(a, b, c, d) \to \mathbb{H}^2 \cup \partial\mathbb{H}^2$?
2. Do we have a Σ-space isomorphism between level sets $\{f_0 = w\}$, as long as $w < -\min(a, b, c, \frac{1}{2}d)$?

The current machinery, especially Lemma 11.9, does not extend well in a way that will readily resolve the above questions. It seems like we need a way to Σ-equivariantly compare our skeleta with the space obtained by first adding rational boundary points $\mathbb{QP}^1 \subset \partial\mathbb{H}^2$ to $\mathbb{H}^2$, then blowing up these points to rays (cf. Theorem 12.4 below).

11.3 Holomorphic Parameters

We say a skeleton $Sk(a, b, c, d)$ has *holomorphic parameters* if $\min(a, b, c, d) \geq 0$. The reason for the terminology is same as for the meromophic parameter cases: in the field of convergent Puiseux series, elements of nonnegatively values precisely correspond to holomorphic functions in some $t^{1/n}$.

By Corollary 10.4, we know that $Sk(a, b, c, d)$ is isomorphic to $Sk_\infty = Sk(\infty, \infty, \infty, \infty)$ as Γ-spaces. Furthermore, the skeleton Sk_∞ has an Γ-invariant point $\mathbf{0} = (0, 0, 0)$. Because of that, we focus on $Sk_\infty \setminus \{\mathbf{0}\}$ instead.

Note that $Sk_\infty \setminus \{\mathbf{0}\}$ has the $\mathbb{R}_{>0}$-action induced from the scalar multiplication on $\mathbb{R}^3$. Because of that, one can think the quotient $(Sk_\infty \setminus \{\mathbf{0}\})/\mathbb{R}_{>0} \cong S^1$. One can check that the tropicalized action of $\Gamma = \langle s_1, s_2, s_3 \rangle$ on Sk_∞ commutes with the $\mathbb{R}_{>0}$-action. Thus one induces $\Gamma \curvearrowright (Sk_\infty \setminus \{\mathbf{0}\})/\mathbb{R}_{>0}$. It is then natural to ask a model of this action.

Theorem 11.21 (Holomorphic Parameters)**.** *If* $\min(a, b, c, d) \geq 0$*, then* $Sk(a, b, c, d)$ *is isomorphic to* $Sk_\infty = Sk(\infty, \infty, \infty, \infty)$ *as* Γ*-spaces.*

Let $S = (Sk_\infty \backslash \{\mathbf{0}\})/\mathbb{R}_{>0}$ *be the spherical projection of the punctured skeleton* $Sk_\infty \backslash \{\mathbf{0}\}$*. The induced action* $\Gamma \curvearrowright S$ *is isomorphic, as* Γ*-spaces, to the hyperbolic reflection action* $\Gamma \curvearrowright \partial\mathbb{H}^2$ *on the boundary* $\partial\mathbb{H}^2$ *of the hyperbolic plane.*

We will ultimately use Theorem 11.16 to establish this comparison. However, to see how $\Gamma \curvearrowright S$ admits a ping-pong structure adatped to the circle S, we need some detailed understanding of the action $\Gamma \curvearrowright Sk_\infty$.

11.3.1 Cell Structures

Recall that the orthogonal projection $v = (v_1, v_2, v_3)\colon Sk_\infty \to \Pi$ is a piecewise-linear homeomorphism (Corollary 9.6), and the skeleton Sk_∞ is the union of quadratic cells $\mathcal{C}(X_i^2)$.

It is then reasonable to ask what are the images $v(\mathcal{C}(X_i^2))$ in Π. They have to be closed convex unbounded subsets, as shown in Lemma 10.2. Not only that, the images span a proper cone[i] in Π.

To describe these cones, rather than using coordinates v_1, v_2, v_3, it is more convenient to use the coordinates

$$u_1 = x_1 - x_3 = v_1 - v_3,$$

$$u_2 = x_2 - x_3 = v_2 - v_3.$$

From (u_1, u_2), we recover the point $(v_1, v_2, v_3) \in \Pi$ by $(v_1, v_2, v_3) = (\frac{2}{3}u_1 - \frac{1}{3}u_2, -\frac{1}{3}u_1 + \frac{2}{3}u_2, -\frac{1}{3}(u_1 + u_2))$.

Proposition 11.22. *In* (u_1, u_2)*-coordinates of* Π*, the image of cells* $v(\mathcal{C}(X_i^2))$ *are described*

i. By a *proper cone* we mean a closed convex cone that is full-dimensional and is salient, i.e., does not contain nonzero antipodal vectors.

as

$$v(\mathcal{C}(X_1^2)) = \{(u_1, u_2) : u_1 \leq u_2,\ u_1 \leq 0\}, \tag{11.3.1}$$

$$v(\mathcal{C}(X_2^2)) = \{(u_1, u_2) : u_2 \leq u_1,\ u_2 \leq 0\}, \tag{11.3.2}$$

$$v(\mathcal{C}(X_3^2)) = \{(u_1, u_2) : u_1, u_2 \geq 0\}. \tag{11.3.3}$$

Proof. Recall the quantity $\alpha = \frac{1}{3}(x_1+x_2+x_3)$ and the formula $x_i = \alpha+v_i$, $i = 1, 2, 3$. Recall that, on Sk_∞, $x \in \mathcal{C}(X_i^2)$ if and only if inequalities (9.2.1)–(9.2.3) hold and $x_1+x_2+x_3 = 2x_i$.

So if $x \in \mathcal{C}(X_1^2)$ say, then $3\alpha = 2x_1 = 2\alpha + 2v_1$ gives $\alpha = 2v_1$. The other two inequalities translate to $3\alpha \leq 2\alpha + 2v_2$ and $3\alpha \leq 2\alpha + 2v_3$. By $\alpha = 2v_1$, they simplify to $v_1 \leq v_2$ and $v_1 \leq v_3$ respectively. Equivalently,

$$\begin{aligned} v_1 \leq v_2 \text{ and } v_1 \leq v_3 &\Leftrightarrow v_1 - v_3 \leq v_2 - v_3 \text{ and } v_1 - v_3 \leq 0 \\ &\Leftrightarrow u_1 \leq u_2 \text{ and } u_1 \leq 0. \end{aligned}$$

The other two cells are dealt similarly. $\square$

Figure 11.4 sketches the regions that (11.3.1)–(11.3.3) describes.

11.3.2 Vieta Involutions

Now we evaluate Vieta involutions on each cell. On Sk_∞ one can simplify tropicalized Vieta involutions $\mathrm{trop}(s_i)$'s as

$$\mathrm{trop}(s_1)(x_1, x_2, x_3) = (\min(2x_2, 2x_3) - x_1, x_2, x_3),$$

$$\mathrm{trop}(s_2)(x_1, x_2, x_3) = (x_1, \min(2x_1, 2x_3) - x_2, x_3),$$

$$\mathrm{trop}(s_3)(x_1, x_2, x_3) = (x_1, x_2, \min(2x_1, 2x_2) - x_3).$$

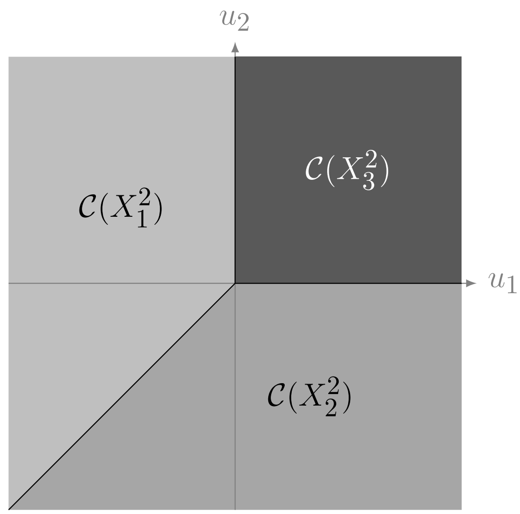

Figure 11.4: Images of $\mathcal{C}(X_i^2) \subset Sk_\infty$ on Π.

The minima above can be resolved if one thinks each $\mathrm{trop}(s_i)$ on a cell $\mathcal{C}(X_j^2)$, $i \neq j$. For instance, $\mathrm{trop}(s_1)$ on $\mathcal{C}(X_2^2)$ appears as

$$\mathrm{trop}(s_1)(x_1, x_2, x_3) = (2x_2 - x_1, x_2, x_3),$$

while on $\mathcal{C}(X_3^2)$, we have

$$\mathrm{trop}(s_1)(x_1, x_2, x_3) = (2x_3 - x_1, x_2, x_3).$$

If we conjugate this map by $v\colon Sk_\infty \to \Pi$, we have

$$v \circ \mathrm{trop}(s_1) \circ v^{-1} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} -1 & 2 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \quad \text{on } \mathcal{C}(X_2^2), \tag{11.3.4}$$

and

$$v \circ \mathrm{trop}(s_1) \circ v^{-1} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \quad \text{on } \mathcal{C}(X_3^2). \tag{11.3.5}$$

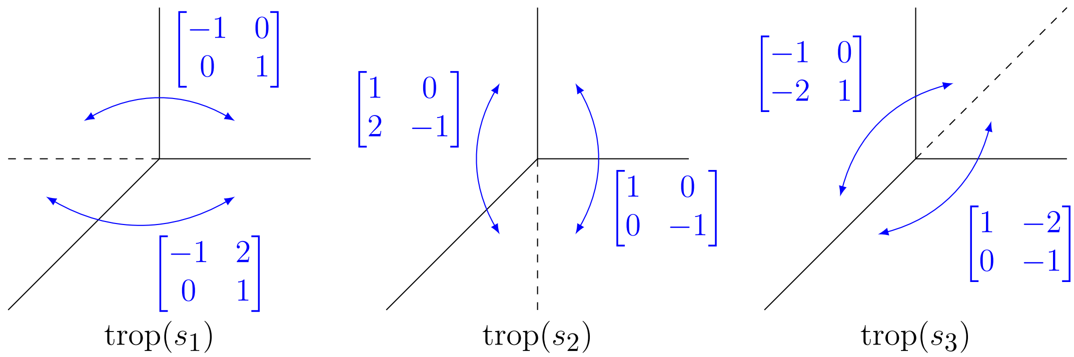

Figure 11.5: Descriptions of trop(s_i) on $\Pi = v(Sk_\infty)$.

Similar computations can be repeated to produce

$$v \circ \mathrm{trop}(s_2) \circ v^{-1} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \quad \text{on } \mathcal{C}(X_1^2), \tag{11.3.6}$$

$$v \circ \mathrm{trop}(s_2) \circ v^{-1} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \quad \text{on } \mathcal{C}(X_3^2), \tag{11.3.7}$$

$$v \circ \mathrm{trop}(s_3) \circ v^{-1} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ -2 & 1 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \quad \text{on } \mathcal{C}(X_1^2), \tag{11.3.8}$$

$$v \circ \mathrm{trop}(s_3) \circ v^{-1} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} 1 & -2 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \quad \text{on } \mathcal{C}(X_2^2). \tag{11.3.9}$$

Note that each 2×2 matrix A involved in the descriptions has trace 0 and determinant -1, so by Cayley–Hamilton theorem we have $A^2 = I$, i.e., the matrices have order 2.

Figure 11.5 describes how the above transformations act on Π. Each trop(s_i) splits the plane Π into two regions, with different matrices used on each region.

11.3.3 Comparison with Hyperbolic Reflections

We first demonstrate that the Vieta action $\langle s_1, s_2, s_3 \rangle \curvearrowright S$ and the induced hyperbolic action $\langle r_1, r_2, r_3 \rangle \curvearrowright \partial\mathbb{H}^2$ both have ping-pong structures adapted to the circles.

Example 11.23 (Vieta side)**.** For the Vieta action $\langle s_1, s_2, s_3\rangle \curvearrowright S = (Sk_\infty \setminus \{\mathbf{0}\})/\mathbb{R}_{>0}$, each cell $\mathcal{C}(X_i^2)$ projects to a closed interval in S. Taking their interiors and calling them S_i's, we have (S_1, S_2, S_3) a ping-pong structure adapted to the circle S.

Example 11.24 (Hyperbolic side)**.** For the hyperbolc reflection $\langle r_1, r_2, r_3\rangle \curvearrowright \partial\mathbb{H}^2$, let B_i be the set of points $x \in \partial\mathbb{H}^2$ that admits a sequence $(x_n)_{n=1}^\infty$ in H_i (see Proposition 11.17) that converges to x. Each B_i then defines a closed interval in $\partial\mathbb{H}^2$, so their interiors $(B_1^\circ, B_2^\circ, B_3^\circ)$ form a ping-pong structure adapted to the circle $\partial\mathbb{H}^2$.

So to prove Theorem 11.21, it suffices to show that they have dense net orbits, thanks to Theorem 11.16.

Example 11.25 (Vieta side)**.** Consider the set of 'rational rays' $S_\mathbb{Q} = (\mathbb{Q}^2 \setminus \{\mathbf{0}\})/\mathbb{Q}_{>0} \subset S$. Fixed points of $\mathrm{trop}(s_i)$ are in the set $\{[(-1,-1)], [(1,0)], [(0,1)]\} \subset S_\mathbb{Q}$. Their length 1 orbits are listed $[(1,1)], [(-1,0)]$, and $[(0,-1)]$.

Now let $[(p,q)] \in S_\mathbb{Q}$ be any point with $p, q \in \mathbb{Z}$, both nonzero and coprime. We claim that we can reduce its height, $\max(|p|,|q|)$, by some appropriate applications of $\mathrm{trop}(s_i)$'s.

If $|p| > |q|$, apply $\mathrm{trop}(s_1)$, $\mathrm{trop}(s_3)\circ\mathrm{trop}(s_1)$, or $\mathrm{trop}(s_2)$ to ensure that $p, q > 0$ (and not changing the height). Apply $\mathrm{trop}(s_2) \circ \mathrm{trop}(s_3)$. By this, if $2q \le p$, then we get $[(p-2q, q)]$ which, by $0 \le p-2q < p$, strictly decreases the height. If $2q > p$, then we get $[(p-2q, 2p-3q)]$ which, by $-q < p-2q < 0$ and $-q < 2p-3q < q$, strictly decreases the height.

If $|p| < |q|$, apply $\mathrm{trop}(s_1)$, $\mathrm{trop}(s_3)\circ\mathrm{trop}(s_2)$, or $\mathrm{trop}(s_2)$ to ensure that $p, q > 0$ (and not changing the height). Apply $\mathrm{trop}(s_1) \circ \mathrm{trop}(s_3)$. By this, if $q \ge 2p$, then we get $[(p, q-2p)]$ which, by $0 \le q-2p < q$, strictly decreases the height. if $q < 2p$, then we get $[(2q-3p, q-2p)]$ which, by $-p < q-2p < 0$ and $-p < 2q-3p < p$, strictly decreases the height.

If $|p| = |q|$, then applying $\mathrm{trop}(s_1)$ or $\mathrm{trop}(s_2)$, we reduce to $[(1,1)]$ or $[(-1,-1)]$. So one can run the induction on heights to demonstrate that the ping-pong nets have orbits containing all of $S_\mathbb{Q}$.

Example 11.26 (Hyperbolic side)**.** We claim that the $\langle r_1, r_2, r_3 \rangle$-orbit of $\{0, 1, \infty\}$ contains all rational boundary points $\mathbb{QP}^1 \subset \partial\mathbb{H}^2 = \mathbb{RP}^1$. Given a rational $p/q \in \mathbb{QP}^1$, with p, q coprime, define its *height* as $\max(|p|, |q|)$. As $r_3(1) = -1$, we have all rational numbers of height ≤ 1 in the $\langle r_1, r_2, r_3 \rangle$-orbit.

For general rational numbers, consider

$$r_3 r_1(z) = z - 2, \quad r_2 r_3(z) = \frac{z}{2z + 1}.$$

Let p/q be any rational with height > 1; we may assume $q > 0$. Applying $r_3 r_1$ or $r_1 r_3$ sufficiently many times, we may also assume that $-1 < p/q < 1$ (so the height equals q). If $p < 0$, apply $r_2 r_3$ to have $p/(2p + q)$; since $-q < 2p + q < q$, it has height $\leq \max(|p|, |q| - 1) < q$. If $p > 0$, apply $r_3 r_2$ to have $p/(q - 2p)$; since $-q < q - 2p < q$, it has height $\leq \max(|p|, |q| - 1) < q$. Either way we strictly reduce the height. By induction on height, we have the claim.

By above checks, Theorem 11.21 follows from the comparison theorem 11.16 applied for systems $\langle s_1, s_2, s_3 \rangle \curvearrowright S = (Sk_\infty \setminus \{\mathbf{0}\})/\mathbb{R}_{>0}$ and $\langle r_1, r_2, r_3 \rangle \curvearrowright \partial\mathbb{H}^2$.

Remark. We know that, from our discussion on ping-pong structures on circles (§11.1.3), the following 'greedy algorithm' works: given $x \in \Sigma.\{p_1, p_2, p_3\}$, apply σ_i if $x \in X_i$. There is no reason to not use this algorithm to show that rational points ($S_\mathbb{Q}$ or $\mathbb{QP}^1$) reduce to ping-pong nets. It turns out that the Euclidean algorithm for greatest common divisor works to study the behavior of the reduction, cf. Proposition 12.9.

In the next chapter, we will discuss some theories suitable to see the connection of the Euclidean algorithm to the skeletal Vieta action $\langle s_1, s_2, s_3 \rangle \curvearrowright Sk_\infty$. A similar connection can be made for rational boundary points $\mathbb{QP}^1 \subset \partial\mathbb{H}^2$ as well, but we will not discuss the details here.

CHAPTER 12

THE EXCEPTION SET

In the statement of Corollary 11.19, we have corresponded an open subset $U = \Gamma.D_0$ of a meromorphic skeleton $Sk(a,b,c,d)$ and the upper half plane $\mathbb{H}^2$, but did not discussed about the nature of U inside $Sk(a,b,c,d)$—say, how large is it in the skeleton. In this chapter we discuss the *exception set* $E = Sk(a,b,c,d) \setminus U$ and claim that E is a countable union of half-rays (affine-linear copy of $\mathbb{R}_{\geq 0} = [0,\infty)$) in $Sk(a,b,c,d)$.

In a special case—called *punctured torus parameters*—when $a = b = c = \infty$ but $d < 0$, then we can explicitly describe this exception set as a superlevel set of an upper-semicontinuous function, which generalizes $\gcd(p,q)$ over the integers. In that case, we can explicitly describe the rays involved in exception sets.

12.1 The GCD Function

One way to analytically define the greatest common divisor (GCD) of integers is to introduce a function on $\mathbb{R}$ that is designed to return "$\gcd(1,x)$." This function will be called *Thomae's function*, following Beanland et al. [2009].

Definition 12.1 (Thomae's Function)**.** Let $f\colon \mathbb{R} \to [0,1]$ be a function defined as follows. If $x = p/q$ is a rational number with $p,q \in \mathbb{Z}$ coprime, then we define $f(x) = 1/|q|$. If x is irrational, we define $f(x) = 0$.

This function f is called the *Thomae's function.* We will denote this function as $\gcd(1,x)$.

We think that 0 is coprime with 1, but not with other nonnegative intergers. Thus we declare $\gcd(1,0) = 1$. We also have the following classical fact.

Theorem 12.2. *Thomae's function is continuous at irrational numbers, and discontinuous at rational numbers. More precisely, we have* $\lim_{x\to c} \gcd(1,x) = 0$ *for all real c.*

Given Thomae's function, one can define $\gcd(a, b)$ as

$$\gcd(a,b) = \begin{cases} |a| \cdot \gcd(1, b/a) & (a \neq 0), \\ |b| & (a = 0). \end{cases} \tag{12.1.1}$$

This is compatible with the usual GCD of integers. To see why, we suppose $a, b \in \mathbb{Z}$ and both are nonzero. If $d > 0$ is the GCD of a and b, then we can write $a = \pm pd$ and $b = \pm qd$ for $p, q \in \mathbb{Z}_{>0}$ coprime. Thus $|a| \cdot \gcd(1, b/a) = pd \cdot \gcd(1, q/p) = pd(1/p) = d$ follows.

Not only $\gcd(a, b)$ above generalizes GCD of integers, but for a, b reals, it measures how two numbers are rationally linearly dependent. We list some interesting properties in this vein, without proofs.

Proposition 12.3. *(a) We have* $\gcd(cx, cy) = |c| \cdot \gcd(x, y)$ *for all* $c \in \mathbb{R}$.

(b) Suppose a, b are $\mathbb{Q}$-linearly dependent with $b \neq 0$. If $a/b = p/q$ with $p, q \in \mathbb{Z}$ coprime, we have

$$\gcd(a,b) = \left|\frac{a}{p}\right| = \left|\frac{b}{q}\right|.$$

(c) If a, b are $\mathbb{Q}$-linearly independent, then $\gcd(a, b) = 0$.

(d) Whenever $A = \begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix}$ *is in* $\mathrm{GL}_2(\mathbb{Z})$, *we have* $\gcd(\alpha x + \beta y, \gamma x + \delta y) = \gcd(x, y)$.

From Theorem 12.2, one can show that $\lim_{(x,y)\to(a,b)} \gcd(x, y) = 0$ for all $(a, b) \in \mathbb{R}^2$. As $\gcd(a, b) \geq 0$ by definition, this limit indicates that gcd is an upper-semicontinuous function. In particular, the superlevel sets $\{(a, b) \in \mathbb{R}^2 : \gcd(a, b) \geq \eta\}$ are closed. In fact, if $\eta > 0$, we can explicitly describe this set as a family of rays $\bigcup_{(p,q)=1} \mathbb{R}_{\geq 1} \cdot \eta \cdot (p, q)$, where (p, q) runs through all pairs of integers that are coprime.

Remark. The concept and notation of the gcd function is not original; it has previously been encountered in a graphing software Software, where the function is denoted $\gcd(x, y)$.

12.2 The Exception Set

We state a theorem to be established in this chapter.

Theorem 12.4. *Suppose* $\min(a,b,c,d) < 0$ *and let* $Sk = Sk(a,b,c,d)$*. Then for the* $\Gamma = \langle s_1, s_2, s_3 \rangle$*-orbit* $U = \Gamma.D_0$ *in Corollary 11.19, its complement* $Sk \backslash U$ *is in a countable union of rays (affine-linear image of* $\mathbb{R}_{\geq 0}$*).*

Observe that U contains the interior of the union of non-quadratic cells $\mathcal{C}(AX_1) \cup \mathcal{C}(BX_2) \cup \mathcal{C}(CX_3) \cup \mathcal{C}(D)$. This is because the 1st-step orbit $\{1, s_1, s_2, s_3\}.D_0$ covers all of the non-quadratic cells, modulo 3 points on the boundary which also lie on quadratic cells. Denote S° for the interior of a subset $S \subset Sk$, and define

$$QC = \left(\bigcup_{i=1}^{3} \mathcal{C}(X_i^2) \right)^{\circ} \cup \bigcup_{i \neq j} \left(\mathcal{C}(X_i^2) \cap \mathcal{C}(X_j^2) \right), \tag{12.2.1}$$

the interior of quadratic cells plus three points. (Notation QC stands for 'Quadratic Cells.')

Thus the only interesting part (in the study of $Sk \setminus U$) comes from the set QC above. There, we have the following dichotomy:

(i) If $x \in QC$ has $g \in \Gamma$ such that $g.x \notin QC$, then $x \in U$.

(ii) If $x \in QC$ is a point whose orbit $\Gamma.x$ is a subset of QC, then $x \in Sk \setminus U$.

For the latter, this comes from that D_0 and the interior of QC are disjoint.

To see how these cases distinguish, we intend to choose an "optimal path" from $x \in \bigcup_{i=1}^{3} \mathcal{C}(X_i^2)$ to U. The choice is aimed to systematically reduce the ℓ^1-norm of x, $|x_1| + |x_2| + |x_3|$. If no further reduction of ℓ^1-norm is possible, then we have a possibility to be in the exception set.

The 'optimal path' can be described as follows. Given $x \in \mathcal{C}(X_i^2)$, we apply $\operatorname{trop}(s_i)$ to it. Then either $\operatorname{trop}(s_i)(x)$ is in a non-quadratic cell (in which we have $x \in U$), or goes to another quadratic cell. For the latter, we keep apply $\operatorname{trop}(s_j)$ which follows the index of the current quadratic cell, and hope that we eventually fall into U or got 'trapped' in one of the boundary rays $\mathcal{C}(X_i^2) \cap \mathcal{C}(X_{i+1}^2)$.

Remark. This introduction of QC is required because D_0 may intersect the quadratic cells. The algorithm sketched above works for all $x \in \bigcup_{i=1}^3 \mathcal{C}(X_i^2)$, but if a point reaches to $D_0 \cap \bigcup_{i=1}^3 \mathcal{C}(X_i^2)$, it becomes ambiguous whether we should stop or not by just looking its position in $\bigcup_{i=1}^3 \mathcal{C}(X_i^2)$. If we restrict our attention in QC, we do not face such an issue since already $D_0 \cap QC = \varnothing$.

To implement the strategy, we first need to how one can describe $\operatorname{trop}(s_i)$ on $\mathcal{C}(X_i^2)$ in a way independent to the index i.

12.3 On Quadratic Cells

Recall the discussions in §11.3, where we have defined coordinates

$$u_1 = x_1 - x_3,$$
$$u_2 = x_2 - x_3,$$

and described the image of cells $v(\mathcal{C}(X_i^2))$'s in (u_1, u_2)-coordiantes of the plane $\Pi = \{x_1 + x_2 + x_3 = 0\}$ in (11.3.1)–(11.3.3). Even for meromorphic parameters, the only change is that we append some inequalities that represents (9.2.4)–(9.2.7). Hence in general, we have

$$v(\mathcal{C}(X_1^2)) \subset \{(u_1, u_2) \in \Pi : u_1 \leq u_2,\ u_1 \leq 0\},$$
$$v(\mathcal{C}(X_2^2)) \subset \{(u_1, u_2) \in \Pi : u_2 \leq u_1,\ u_2 \leq 0\},$$
$$v(\mathcal{C}(X_3^2)) \subset \{(u_1, u_2) \in \Pi : u_1, u_2 \geq 0\}.$$

Furthermore, the descriptions of tropicalized Vieta involutions $\mathrm{trop}(s_i)$ in (11.3.4)–(11.3.9) are also valid, as long as we start from $\bigcup_{i=1}^{3} \mathcal{C}(X_i^2)$. A key success here is that $\mathrm{trop}(s_i)$ is represented as a linear map.

12.3.1 Special Coordinates and Representations of Vieta Involutions

We further develop this idea and introduce coordinates $u^{(i)} = (u_1^{(i)}, u_2^{(i)})$ on Π as

$$u_1^{(1)} = x_2 - x_1, \qquad u_1^{(2)} = x_3 - x_2, \qquad u_1^{(3)} = x_1 - x_3, \tag{12.3.1}$$

$$u_2^{(1)} = x_3 - x_1, \qquad u_2^{(2)} = x_1 - x_2, \qquad u_2^{(3)} = x_2 - x_3, \tag{12.3.2}$$

(note that $(u_1^{(3)}, u_2^{(3)})$ is our old (u_1, u_2) here) and depict the cells in the 1st quadrant:

$$\begin{aligned}
u^{(1)}(\mathcal{C}(X_1^2)) &\subset \{(u_1^{(1)}, u_2^{(1)}) : u_1^{(1)}, u_2^{(1)} \geq 0\}, \\
u^{(2)}(\mathcal{C}(X_2^2)) &\subset \{(u_1^{(2)}, u_2^{(2)}) : u_1^{(2)}, u_2^{(2)} \geq 0\}, \\
u^{(3)}(\mathcal{C}(X_3^2)) &\subset \{(u_1^{(3)}, u_2^{(3)}) : u_1^{(3)}, u_2^{(3)} \geq 0\}.
\end{aligned}$$

Just as how we treated the projection $v\colon Sk \to \Pi$, we treat $u^{(i)}$ as a linear function $Sk \to \mathbb{R}^2$, but $u_1^{(i)}, u_2^{(i)}$ as coordinates of $\mathbb{R}^2$.

These coordinates benefit in describing some tropicalzed Vieta involutions by a 'uniform expression,' as sketched in the following

Proposition 12.5. *Let $i = 1, 2, 3$. On $u^{(i)}(\mathcal{C}(X_i^2) \cap s_i^{-1}.\mathcal{C}(X_{i\pm1}^2))$, we describe the transformations $u^{(i\pm1)} \circ \mathrm{trop}(s_i) \circ (u^{(i)})^{-1}$ as follows.*

(i) On $u^{(i)}(\mathcal{C}(X_i^2) \cap s_i.\mathcal{C}(X_{i-1}^2))$, we have $u_1^{(i)} \geq u_2^{(i)}$, and we find the image of $\mathrm{trop}(s_i)$ in $u^{(i-1)}(\mathcal{C}(X_{i-1}^2))$ by the formula

$$u_1^{(i-1)} = u_2^{(i)},$$

$$u_2^{(i-1)} = u_1^{(i)} - u_2^{(i)}.$$

(ii) *On* $u^{(i)}(\mathcal{C}(X_i^2) \cap s_i.\mathcal{C}(X_{i+1}^2))$, *we have* $u_1^{(i)} \le u_2^{(i)}$, *and we find the image of* $\mathrm{trop}(s_i)$ *in* $u^{(i+1)}(\mathcal{C}(X_{i+1}^2))$ *by the formula*

$$u_1^{(i+1)} = -u_1^{(i)} + u_2^{(i)},$$
$$u_2^{(i+1)} = u_1^{(i)}.$$

Here, we understand indices vary mod 3, so that $\mathcal{C}(X_0^2) = \mathcal{C}(X_3^2)$ *or* $u_2^{(4)} = u_2^{(1)}$, *etc.*

Proof. Computations, just as in derivations of (11.3.4)–(11.3.9). □

Observe that the formulae of $\mathrm{trop}(s_i)$ can be described with a uniform expression. Let $Q_1 = \{(u_1, u_2) \in \mathbb{R}^2 : u_1, u_2 \ge 0\}$ be the first quadrant and define $f\colon Q_1 \to Q_1$ by

$$f(u_1, u_2) = \begin{cases} (u_2, u_1 - u_2) & (u_1 \ge u_2), \\ (-u_1 + u_2, u_1) & (u_1 < u_2) \end{cases} \tag{12.3.3}$$

(note that f is discontinuous at the line $u_1 = u_2$). This f works as the 'uniform expression' mentioned.

12.3.2 Synchronous Index Sequence

We now establish how f can be used to represent the tropical dynamics on quadratic cells. Recall the subset $QC \subset \bigcup_{i=1}^{3} \mathcal{C}(X_i^2)$ in (12.2.1).

Definition 12.6 (Synchronous Index Sequence)**.** For $x \in QC$, define the *synchronous index sequence of* x, $i(1), i(2), \ldots$ as follows.

(i) Let $i(1)$ be an index i in which $x \in \mathcal{C}(X_i^2)$.

(ii) For $j > 1$, if $s_{i(j-1)} \cdots s_{i(1)}.x \in QC$, then let $i(j)$ be an index i in which $s_{i(j-1)} \cdots s_{i(1)}.x \in \mathcal{C}(X_i^2)$.

If otherwise, i.e., $s_{i(j-1)} \cdots s_{i(1)}.x \notin QC$, terminate the sequence.

Note that both steps (i) and (ii) may have a multiple choice for $i(j)$'s. Choose any of the possibility in that case.

We say a synchronous index sequence is *unique* if we do not face a multiple choice in its construction; *ambiguous* if otherwise. If a synchronous index sequence terminates where $i(n)$ is the last defined term, we say it *terminates at* $i(n)$.

Corollary 12.7. *Suppose $x \in QC$. Let $i(1), i(2), \ldots$ be its synchronous index sequence. As long as $i(n+1)$ is defined, we have a formula*

$$f^n(u^{(i(1))}(x)) = u^{(i(n+1))}(s_{i(n)} \cdots s_{i(1)}.x), \quad n \geq 1.$$

If a synchronous index sequence of x terminates at $i(n)$, then by $g = s_{i(n)} \cdots s_{i(1)}$ we have $g.x \notin QC$. Thus we have $x \in U$.

Remark. One can also define synchronous index sequence for points in $x \in \bigcup_{i=1}^{3} \mathcal{C}(X_i^2)$, by continuing the sequence as long as $s_{i(j-1)} \cdots s_{i(1)}.x \in \mathcal{C}(X_{i(j)}^2)$. One drawback is that there might be some points $x \in U$ that does not have terminating synchronous index sequence: if it falls into the intersection $D_0 \cap \mathcal{C}(X_i^2)$, then we will continue i (unqiely!) forever. However, we still have the conclusion of Corollary 12.7 true in this generalization.

If $x \in U$, then the synchronous index sequence of x is unique, as shown in the following

Lemma 12.8. *(a) The map* $\mathrm{trop}(s_j)$ *on $\mathcal{C}(X_i^2)$ maps into $\mathcal{C}(X_j^2)$, if $j \neq i$.*

(b) Any point on intersections $\mathcal{C}(X_i^2) \cap \mathcal{C}(X_{i+1}^2)$ has its Γ-orbit inside QC.

(c) If $x \in U \cap QC$, then its synchronous index sequence is unique.

Proof. For (a), this follows from Proposition 10.5.

For (b), note that any point in $Sk \setminus QC$ has the stabilizer either trivial or order 2. For any a point $x \in \mathcal{C}(X_i^2) \cap \mathcal{C}(X_{i+1}^2)$ and an element $g \in \Gamma$, $g.x$ is the opposite: its stabilizer is infinite and even isomorphic to the group $(\mathbb{Z}/2\mathbb{Z})^{*2}$. Thus $g.x \notin Sk \setminus QC$.

For (c), we show the contrapositive. That is, suppose we have $x' = s_{i(j-1)} \cdots s_{i(1)}.x \in \mathcal{C}(X_i^2) \cap \mathcal{C}(X_{i+1}^2)$ for some j. Then by (b), we have $x' \notin U$, so $x \notin U$. $\square$

We now study the iterates of f, to use Corollary 12.7 better.

Lemma 12.9. *On the first quadrant Q_1, define the norm $\|(u_1, u_2)\| = |u_1| + |u_2|$. Then with function $f\colon Q_1 \to Q_1$ in* (12.3.3)*, for each nonzero $(u_1, u_2) \in Q_1$, we have*

$$\lim_{n\to\infty} \|f^n(u_1, u_2)\| = \gcd(u_1, u_2),$$

where gcd *is the function defined on* (12.1.1)*. If the limit is nonzero, then f^n eventually oscillates between $(0, \gcd(u_1, u_2))$ and $(\gcd(u_1, u_2), 0)$.*

Proof. Note first that f decreases the norm in every iterates: indeed, we have $\|f(u_1, u_2)\| = \max(u_1, u_2) \leq \|(u_1, u_2)\|$. Also, f selectively operates the matrices

$$\begin{bmatrix} 0 & 1 \\ 1 & -1 \end{bmatrix}, \quad \begin{bmatrix} -1 & 1 \\ 1 & 0 \end{bmatrix} \in \mathrm{GL}_2(\mathbb{Z})$$

on its input, whose selection is based on a linear inequality. So we have that f is homogeneous, i.e., $f(tu_1, tu_2) = t \cdot f(u_1, u_2)$ for $t \geq 0$.

Suppose u_1, u_2 are rationally linearly dependent, so that for $\gamma = \gcd(u_1, u_2) > 0$, we have that $p = u_1/\gamma$ and $q = u_2/\gamma$ are coprime integers. Then we have

$$f^n(u_1, u_2) = \gamma \cdot f^n(p, q).$$

Iterating f over positive integers (p,q) means we are running the Euclidean algorithm for GCD of the pair (p,q). So for p,q coprime we must reach to $(1,0)$ or $(0,1)$. In fact we oscillate between these, as $f(1,0)=(0,1)$ and $f(0,1)=(1,0)$. The same phenomenon is observed for $f^n(u_1,u_2)$, except that it reaches to $(\gamma,0)$ or $(0,\gamma)$ this time.

Suppose u_1,u_2 are rationally linearly independent, so that the slope $m=u_2/u_1\in\mathbb{R}$ is an irrational number. Let $C=u_1+u_2$ be the norm of $(u_1,u_2)\in Q_1$. Then we can re-coordinate $(u_1,u_2)\in Q_1$ to $[m,C]$, where we use square brackets $[,]$ to indicate that we are using a new coordinate. Then we describe f as

$$f[m,C]=\begin{cases}\left[\frac{1}{m}-1,\frac{C}{1+m}\right] & (m\le 1),\\ \left[\frac{1}{m-1},\frac{mC}{1+m}\right] & (m>1).\end{cases}$$

This lets us to think a transformation and a cocycle,

$$T(m)=\begin{cases}\frac{1}{m}-1 & (m\le 1),\\ \frac{1}{m-1} & (m>1),\end{cases}\qquad A(m)=\begin{cases}1/(1+m) & (m\le 1),\\ m/(1+m) & (m>1),\end{cases}$$

so that $f^n[m,C]=[T^n(m),A_T^n(m)C]$, where $A_T^n(m)=A(T^{n-1}(m))\cdots A(T(m))A(m)$.

Let $m=[a_0;a_1,a_2,\ldots]$ be the continued fraction expansion of m. Then one can show that

$$T^{a_0+\cdots+a_i-1}(m)=[1;a_{i+1},a_{i+2},\ldots]\text{ or }[0;1,a_{i+1},a_{i+2},\ldots],$$

depending on the parity of the sum $\sum_{j=0}^{i}a_j$. These numbers fall into the interval $[\frac{1}{2},2]$. This helps, because we have $A(m)\le 2/3$ for $m\in[\frac{1}{2},2]$, and $A(m)\le 1$ for all $m\in[0,\infty]$. Therefore we have $A_T^n(m)$ decreasing in n and

$$A_T^{a_0+\cdots+a_i}(m)\le\left(\frac{2}{3}\right)^i.$$

Since m is irrational, we see that $A_T^n(m)$ can be arbitrarily small. Hence it follows that

$$\|f^n(u_1, u_2)\| = \|(u_1, u_2)\| \cdot A_T^n\left(\frac{u_2}{u_1}\right) \to 0 = \gcd(u_1, u_2),$$

when u_2/u_1 is irrational. This shows the claimed. $\square$

12.3.3 Exception Set as Countably many Rays

We now can prove Theorem 12.4 from what we have developed above.

Proof of Theorem 12.4. As remarked after Theorem 12.4, if $x \in Sk \backslash U$ then we have $x \in QC$. If x has an ambiguous synchronous index sequence, then there is $g \in \Gamma$ and j such that $g.x \in \mathcal{C}(X_j^2) \cap \mathcal{C}(X_{j+1}^2)$, so x lies on a (countable collection of) ray in that case.

Suppose x has the unique synchronous index sequence $i(1), i(2), \ldots$. If the sequence terminates then $x \in U$, contradiction. So the sequence does not terminate. Let $w_n = s_{i(n)} \cdots s_{i(1)}$. Denote $\|x\|_{Sk} = |x_1| + |x_2| + |x_3|$ for the ℓ^1-norm on the skeleton. Whenever $x \in \mathcal{C}(X_i^2)$, we have $\|u^{(i)}(x)\| = \frac{1}{2}\|x\|_{Sk}$, as may be verified as follows (demo with $i = 1$):

$$\begin{aligned}
\|(u_1^{(1)}, u_2^{(1)})\| &= u_1^{(1)} + u_2^{(1)} \\
&= (x_2 - x_1) + (x_3 - x_1) = (x_1 + x_2 + x_3) - 3x_1 \\
&= (x_1 + x_2 + x_3) - \frac{3}{2}(x_1 + x_2 + x_3) \\
&= -\frac{1}{2}(x_1 + x_2 + x_3) = \frac{1}{2}\|x\|_{Sk}.
\end{aligned}$$

Hence we have $\|w_n.x\|_{Sk} = 2\|f^n(u^{(i(1))}(x))\|$.

We claim that there is $\epsilon > 0$ such that for any $x \in Sk$, that $\|x\|_{Sk} < 2\epsilon$ implies $x \in U$. As we assume $\min(a, b, c, d) < 0$, by Proposition 10.3, $Sk \subset \mathbb{R}^3$ is a closed subset that does not contain the origin. Therefore we can set $\epsilon > 0$ small enough to make the assumption $\|x\|_{Sk} < 2\epsilon$ to fail for every $x \in Sk$, proving the implication vacuously.

By this setup of ϵ, we have $\gcd(u^{(i(1))}(x)) \geq \epsilon$, since otherwise $\|w_n.x\|_{Sk} < 2\epsilon$ for n large enough, which implies $w_n.x \in U$ and $x \in U$, contradiction. But then $u^{(i(1))}(x)$ lies on a union of rays in $u^{(i(1))}(\mathcal{C}(X^2_{i(1)}))$, as that is how the superlevel set $\gcd(x,y) \geq \epsilon$ on $\mathbb{R}^2$ is sketched. Hence $Sk \setminus U$ lies on a countable union of rays. $\square$

Remark. Of course, the family of rays found here is more than what we need for $Sk \setminus U$. However, one can invoke our understanding of tropical dynamics on $Sk(\infty,\infty,\infty,\infty)$ to study the dynamics of these rays, and figure out which part of the ray is in U. This will verify that $Sk \setminus U$ is a family of rays (rather than a subset of it).

12.3.4 An Effective GCD Bound

We note that the bound $\gcd(u^{(i)}(x)) \geq \epsilon$ is by no means meant to be effective. There, we introduce an effective bound for a subset of $Sk \setminus U$, as follows. Recall, from Lemma 10.2(c), if $\|x\|_{Sk} = |x_1| + |x_2| + |x_3| > -\min(2a, 2b, 2c, d)$ then $x \in QC \subset \bigcup_{i=1}^3 \mathcal{C}(X_i^2)$.

Corollary 12.10. *Suppose* $x \in \bigcup_{i=1}^3 \mathcal{C}(X_i^2)$, *and let*

$$R = \{(x_1, x_2, x_3) \in Sk : |x_1| + |x_2| + |x_3| \leq -\min(2a, 2b, 2c, d)\}. \tag{12.3.4}$$

If

$$\gcd(x_1 - x_3, x_2 - x_3) < -\min(a, b, c, \frac{1}{2}d), \tag{12.3.5}$$

there is $g \in \Gamma$ *such that* $g.x \in R$. *Otherwise, there is* $g \in \Gamma$ *and* $i = 1, 2, 3$ *such that* $g.x \in \mathcal{C}(X_i^2) \cap \mathcal{C}(X_{i+1}^2)$.

Proof. Note that, by $\mathrm{GL}_2(\mathbb{Z})$-invariance, we have

$$\begin{aligned}\gcd(x_2 - x_1, x_3 - x_1) &= \gcd(x_1 - x_2, x_3 - x_2)\\ &= \gcd(x_1 - x_3, x_2 - x_3).\end{aligned}$$

Hence by coordinates (12.3.1)–(12.3.2), whenever $(u_1, u_2) = u^{(i)}(x)$ we have

$$\gcd(u^{(i)}(x)) = \gcd(u_1, u_2) = \gcd(x_1 - x_3, x_2 - x_3).$$

Suppose $x \in \mathcal{C}(X_1^2)$. Let $i(1) = 1, i(2), \ldots$ be a synchronous index sequence of x. Let $g_n = s_{i(n)} \cdots s_{i(1)}$, $\gamma = \gcd(u^{(1)}(x))$, and $T_R = -\min(a, b, c, \frac{1}{2}d)$.

If $\gamma < T_R$ (i.e., (12.3.5)) then by Proposition 12.9, we have $\|f^n(u^{(1)}(x))\| \leq T_R$ for $n \gg 1$. So the norm $\|g_n.x\|_{Sk}$ cannot be $\geq 2T_R$ for all $n > 0$, so $\|g_m.x\|_{Sk} \leq -\min(2a, 2b, 2c, d)$ for some m.

If $\gamma \geq T_R$, then as $\gamma > 0$, $f^n(u^{(1)}(x))$ eventually becomes $(0, \gamma)$ or $(\gamma, 0)$. This concludes that $g_n.x$ is a point on $\mathcal{C}(X_i^2) \cap \mathcal{C}(X_{i+1}^2)$, so the claim follows. □

It would be a good drill to actually describe the complement of (12.3.5) on Sk. Tracing the inverses $(u^{(i)})^{-1} \colon u^{(i)}(\mathcal{C}(X_i^2)) \subset Q_1 \to \mathcal{C}(X_i^2)$, we have that

$$(x_1, x_2, x_3) \in \bigcup_{\substack{p,q \geq 0 \\ \gcd(p,q)=1}} \mathbb{R}_{\geq 1} \cdot (-T_R) \cdot \{(p+q, q, p), (p, p+q, q), (q, p, p+q)\} \tag{12.3.6}$$

is the description of the complement of (12.3.5) in Sk. Here, $T_R = -\min(a, b, c, \frac{1}{2}d)$.

If the region $R \subset U$ (modulo 3 points), then (12.3.6) will be a precise description of $Sk \setminus U$. This is because we establish a further dichotomy (modulo Γ-orbit of 3 points) if $R \subset U$, thanks to Corollary 12.10: either the Γ-orbit of x intersects R, or intersects one of the rays $\mathcal{C}(X_i^2) \cap \mathcal{C}(X_{i+1}^2)$.

12.4 Punctured Torus Parameters

Call the parameters $a = b = c = \infty$ and $d < 0$ a punctured torus parameter, following Rebelo and Roeder [2021]. However, the condition $a = b = c = \infty$ is more than enough and it suffices to have $a, b, c \geq \frac{1}{2}d$ and $d < 0$ for the discussions below.

This is a special case when (12.3.6) is precisely the exception set $Sk \setminus U$. This is because R in (12.3.4), the cell $\mathcal{C}(D)$, and the ping-pong table D_0 are all the same (modulo vertices). Furthermore, we can describe U precisely as the open set $\gcd(x_1 - x_3, x_2 - x_3) < -\frac{1}{2}d$, and the exception set $Sk \setminus U$ precisely as a union of rays (12.3.6) with $T_R = -\frac{1}{2}d$.

It is also interesting to sketch the orbit of D_0 directly. Figure 12.1 sketches images $u^{(i)}(s_i.D) \subset u^{(i)}(\mathcal{C}(X_i^2))$. We denote j for s_j on figures 12.1 to 12.3, for a better readibility.

Now applying s_1 to $s_j.D_0 \subset \mathcal{C}(X_j^2)$, $j = 2, 3$, we append more triangles $s_1 s_2.D_0$ and $s_1 s_3.D_0$ in $\mathcal{C}(X_1^2)$, in addition to $s_1.D_0$. This is sketched in Figure 12.2, along with other index combinations. Figure 12.3 is sketched by adding more triangles by further applying the involutions.

The figures hint us that the rational rays should appear as the limit of these triangles. For instance, the triangles $(s_1 s_2)^n.D_0$ in $\mathcal{C}(X_1^2)$ reaches to the line $u_1^{(1)} = 0$ which contains the intersection $\mathcal{C}(X_1^2) \cap \mathcal{C}(X_2^2)$. Another way to approach to the intersection is $(s_2 s_1)^n.D_0$, which approaches from the other side of the ray. This observation motivates us a further discussion on the behavior of 'infinite-length trajectories' of D_0, which will be saved for further researches.

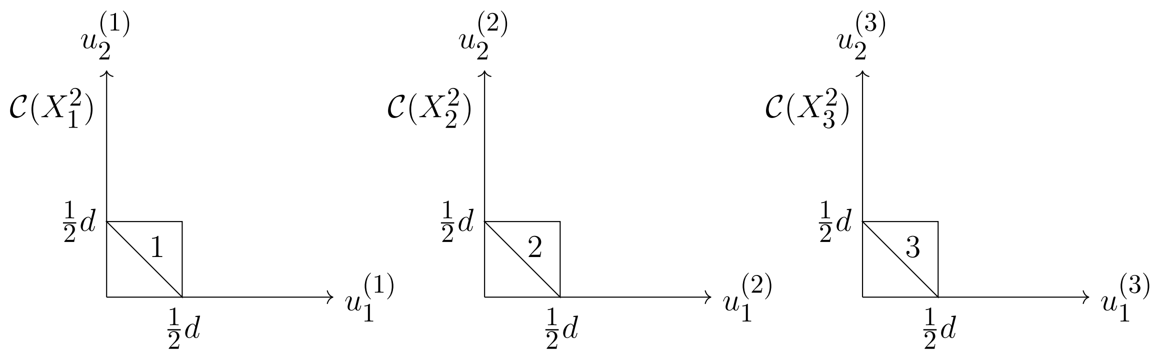

Figure 12.1: Length 1 Images.

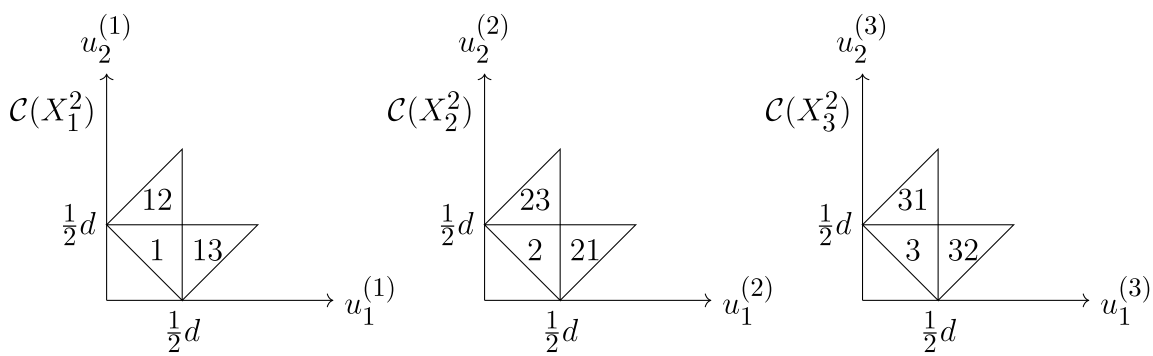

Figure 12.2: Length 2 Images.

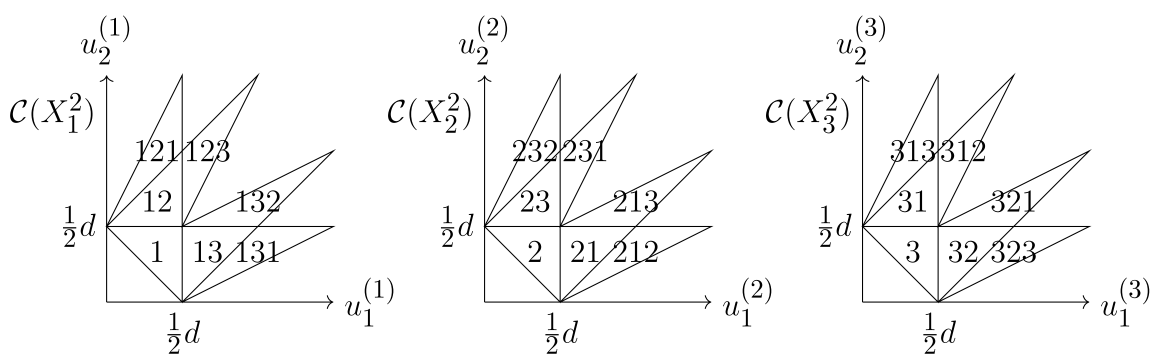

Figure 12.3: Length 3 Images.

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