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QUANTITATIVE HOMOGENIZATION FOR HAMILTON–JACOBI EQUATIONS

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ABSTRACT

Hamilton–Jacobi equations form a broad class of first-order partial differential equations. Some such equations model physical phenomena, such as the evolution of a system of particles according to classical mechanics, or the combustion of a flammable gas. On the other hand, any Hamilton–Jacobi equation can be viewed as the evolution of the value function of a two-player differential game. When obtaining sharp quantitative estimates, the viewpoint of differential games (or, in the convex setting, optimal control) is central to our approach.

This thesis is primarily concerned with the *homogenization*, or large-scale behavior, of such equations when the underlying environment exhibits small-scale structure, which we model by either periodicity or randomness.

In the periodic setting, we investigate the homogenization rate, which we prove depends on the convexity (or lack thereof) of the Hamiltonian. In the random setting, we focus on the G equation, a convex but noncoercive equation which models combustion. In general, noncoercive equations (and even coercive nonconvex equations in a stationary ergodic enviroment) may not homogenize. However, the G equation is coercive in expectation, which we show is sufficent to analyze the large-scale structure of solutions.

In the case of the G equation, our quantitative approach allows us to prove new qualitative results, such as the continuous dependence of the effective Hamiltonian on the law of the environment, and stochastic homogenization when the environment is compressible.

CHAPTER 1

INTRODUCTION

Let $d \geq 2$. Given a function $H \colon \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}$ which is called the Hamiltonian, and some Lipschitz initial data $u_0 \colon \mathbb{R}^d \to \mathbb{R}$, we are interested in the large-scale behavior of the initial-value problem

$$\begin{cases} D_t u(t,x) + H(x, D_x u(t,x)) = 0 & \text{in } \mathbb{R}_{>0} \times \mathbb{R}^d \\ u(0,x) = u_0(x) & \text{in } \mathbb{R}^d. \end{cases} \tag{1.1}$$

We typically interpret (1.1) in the sense of viscosity solutions, since $u(t,\cdot)$ may not remain everywhere differentiable, even for smooth initial data.

There are several interesting special cases.

Example 1 (Classical mechanics). Let $H(x,p) := V(x) + \frac{1}{2}|p|^2$, where $V \colon \mathbb{R}^d \to \mathbb{R}$. The characteristics of (1.1) form the trajectories of a unit mass in d-dimensional space, evolving according to classical mechanics, where V denotes the potential energy.

Example 2 (Front propagation). Let $H(x,p) := a(x)|p|$, where $a \colon \mathbb{R}^d \to (0,\infty)$. The sublevel sets of $u(t,\cdot)$ evolve as *front propagation*, where the speed of the front in the normal direction at position $x \in \mathbb{R}^d$ is given by $a(x)$.

Example 3 (The G equation). Generalizing Example 2, let $H(x,p) := b(x)\cdot p + a(x)|p|$, where $b \colon \mathbb{R}^d \to \mathbb{R}^d$. The sublevel sets of $u(t,\cdot)$ evolve as front propagation, while simultaneously being advected with velocity b. The analysis is significantly more complicated when the "wind speed" $|b(x)|$ is greater than the expansion speed $a(x)$ of the front; in this case the equation is noncoercive. In the random setting, this case is the focus of Chapter 4 and Chapter 5.

To study the large-scale behavior of the equation, we typically replace the Hamiltonian in (1.1) with $H\left(\frac{\cdot}{\varepsilon}, \cdot\right)$, where $\varepsilon > 0$ is small. With this convenient rescaling, we can hope that the corresponding solutions u^ε converge, as $\varepsilon \to 0^+$, to the solution $\bar{u}$ of some *effective* equation

$$
\begin{cases}
D_t\bar{u}(t,x) + \overline{H}(D_x\bar{u}(t,x)) = 0 & \text{in } \mathbb{R}_{>0} \times \mathbb{R}^d \\
\bar{u}(0,x) = u_0(x) & \text{in } \mathbb{R}^d.
\end{cases}
\tag{1.2}
$$

corresponding to some effective Hamiltonian $\overline{H}\colon \mathbb{R}^d \to \mathbb{R}$, which should be determined by the microscopic Hamiltonian H.

If we indeed have the convergence $u^\varepsilon \to \bar{u}$, then we say that the problem (1.1) *homogenizes* to (1.2). As we will see, the effective problem is not a simple average of the microscopic problem, and in fact some properties of the microscopic problem (for instance, isotropy of $H(x,p)$ in p for each x) do not persist in the effective problem.

Our primary interest in this thesis is to establish quantitative rates at which u^ε converges to $\bar{u}$. Each chapter is self-contained, and involves different assumptions on H. In Chapter 2, we prove fast homogenization the case where $H(x,p)$ is periodic in x and convex and coercive in p. In Chapter 3, we drop the convexity assumption and construct examples that homogenize slowly. In Chapters 4 and 5, we consider the G equation, where $H(x,p) = b(x) \cdot p + |p|$ for some random $b\colon \mathbb{R}^d \to \mathbb{R}^d$, which is convex in p but not coercive.

CHAPTER 2

FAST PERIODIC HOMOGENIZATION FOR CONVEX HAMILTONIANS

2.1 Introduction

Let the Hamiltonian $H\colon \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}$ be continuous, $\mathbb{Z}^d$-periodic in the first variable, x, and coercive in the second variable, p. We assume that the coercivity is uniform in x; that is,

$$\liminf_{|p|\to\infty} \inf_{x\in\mathbb{R}^d} H(x,p) = +\infty.$$

Let $u_0\colon \mathbb{R}^d \to \mathbb{R}$ be continuous. Our goal is to study, as $\varepsilon \to 0^+$, the behavior of the unique viscosity solution $u^\varepsilon\colon \mathbb{R}_{\geq 0} \times \mathbb{R}^d \to \mathbb{R}$ to the initial-value problem

$$\begin{cases} D_t u^\varepsilon(t,x) + H(\frac{x}{\varepsilon}, D_x u^\varepsilon(t,x)) = 0 & \text{in } \mathbb{R}_{>0} \times \mathbb{R}^d \\ u^\varepsilon(0,x) = u_0(x) & \text{in } \mathbb{R}^d. \end{cases} \tag{2.1}$$

Lions–Papanicolaou–Varadhan [33] proved that $u^\varepsilon \to \bar{u}$ locally uniformly as $\varepsilon \to 0^+$, where $\bar{u}\colon \mathbb{R}_{\geq 0} \times \mathbb{R}^d \to \mathbb{R}$ is the solution to the effective problem

$$\begin{cases} D_t \bar{u}(t,x) + \overline{H}(D_x \bar{u}(t,x)) = 0 & \text{in } \mathbb{R}_{>0} \times \mathbb{R}^d \\ \bar{u}(0,x) = u_0(x) & \text{in } \mathbb{R}^d. \end{cases} \tag{2.2}$$

Here, $\overline{H}\colon \mathbb{R}^d \to \mathbb{R}$ is called the effective Hamiltonian; we define $\overline{H}(p)$ as the unique constant such that the cell problem

$$H(x, p + D_x v_p) = \overline{H}(p) \tag{2.3}$$

has some $\mathbb{Z}^d$-periodic continuous viscosity solution $v_p\colon \mathbb{R}^d \to \mathbb{R}$, called a corrector.

Our main result is the following rate of convergence, under additional assumptions on H and u_0.

Theorem 1. *If H is convex in p and u_0 is Lipschitz, then there is a constant $C(H, \mathrm{Lip}(u_0)) > 0$ such that, for all $t > 0$ and $x \in \mathbb{R}^d$,*

$$|u^\varepsilon(t,x) - \bar{u}(t,x)| \leq C\varepsilon \log(C + t\varepsilon^{-1}).$$

Additionally, in the case of dimension $d = 2$, we provide a new proof of a result of Mitake–Tran–Yu.

Theorem 2 (Mitake–Tran–Yu [30]). *If $d = 2$, H is convex in p, and u_0 is Lipschitz, then there is a constant $C = C(H, \mathrm{Lip}(u_0)) > 0$ such that, for all $t > 0$ and $x \in \mathbb{R}^d$,*

$$|u^\varepsilon(t,x) - \bar{u}(t,x)| \leq C\varepsilon.$$

The proofs exploit the control formulation of the initial value problem (2.1), which reduces homogenization to a question about convergence of a subadditive function. In both the $d = 2$ and $d \geq 3$ case, results of Alexander [1] [2] apply to quantify the convergence.

Two months after we posted this article, Hung Tran and Yifeng Yu pointed out that, by replacing Step 1 in our proof of Lemma 7 with Lemma 2 of Burago [6], one obtains the optimal $O(\varepsilon)$ rate in all dimensions. In fact, this key lemma is exactly the Hobby–Rice theorem [24], proved in 1965.

2.2 Prior work

After Lions–Papanicolaou–Varadhan proved qualitative homogenization, there have been two main quantitative results. Under the assumptions that u_0 is Lipschitz and H is locally Lipschitz, Capuzzo-Dolcetta–Ishii [11] proved a rate of $O(\varepsilon^{1/3})$, using the perturbed test function method with approximate correctors. Under the additional assumption that H is convex in p, Mitake–Tran–Yu [30] proved a rate of $O(\varepsilon)$ in dimension $d = 2$ and a rate of $O(\varepsilon^{1/2})$ in dimensions $d \geq 3$ using weak KAM theory.

From the definition (2.3) of $\overline{H}$, we can heuristically hope for the expansion

$$u^{\varepsilon}(t,x) \approx \bar{u}(t,x) + \varepsilon v_{D_x \bar{u}(t,x)}(\varepsilon^{-1}x),$$

which suggests a rate of $O(\varepsilon)$. However, the correctors are not unique, u is not C^1 but only Lipschitz, and a continuous selection $p \mapsto v_p$ of correctors (let alone a Lipschitz selection) does not exist in general (see section 5 of [30] for an example). The assumptions on the initial data and the Hamiltonian help by giving additional structure to the problem, in the form of the control formulation.

2.3 Subadditive convergence

We begin by presenting a result of Alexander. In this section, we let $\Omega \subseteq \mathbb{R}^N$ denote an open convex cone. First, we make a few definitions.

Definition 4. A function $f \colon \Omega \cap \mathbb{Z}^N \to \mathbb{R}_{\geq 0}$ has *approximate geodesics* if there is a constant $K > 0$ such that, for every $x \in \Omega \cap \mathbb{Z}^N$, there are $x_0, x_1, \ldots, x_n \in \Omega \cap \mathbb{Z}^N$ with $x_0 = 0$, $x_n = x$, $x_{i+1} - x_i \in \Omega$, $|x_{i+1} - x_i| \leq K$, and

$$|f(x_k - x_i) - f(x_k - x_j) - f(x_j - x_i)| \leq K$$

for all $i \leq j \leq k$.

Definition 5. A function $f \colon \Omega \cap \mathbb{Z}^N \to \mathbb{R}_{\geq 0}$ is subadditive if $f(x+y) \leq f(x) + f(y)$ for all $x, y \in \Omega \cap \mathbb{Z}^N$.

Definition 6. A function $f\colon \Omega \cap \mathbb{Z}^N \to \mathbb{R}_{\geq 0}$ has linear growth if there is a constant $K \geq 1$ such that $K^{-1}|x| - K \leq f(x) \leq K|x| + K$ for all $x \in \Omega \cap \mathbb{Z}^N$.

Theorem 3 (Alexander [1])**.** *If $f\colon \Omega \cap \mathbb{Z}^N \to \mathbb{R}_{\geq 0}$ is subadditive, has linear growth, and has approximate geodesics, then there is a constant $C > 0$ such that, for all $x \in \Omega \cap \mathbb{Z}^N$,*

$$|f(x) - \lim_{n\to\infty} n^{-1} f(nx)| \leq C \log(C + |x|).$$

Proof. Without loss of generality, we assume $K \geq 2$. Define $\overline{f}\colon \Omega \cap \mathbb{Q}^N \to \mathbb{R}$ by

$$\overline{f}(x) := \lim_{n\to\infty} n^{-1} f([nx]),$$

where $[\cdot]$ denotes coordinate-wise rounding to integers. Then $\overline{f}$ is also subadditive with linear growth. From the scaling, it is immediate that $t\overline{f}(x) = \overline{f}(tx)$ for all $t \geq 0$. From subadditivity of f, we see that $\overline{f} \leq f$.

For the rest of the argument, we let $C > 1 > c > 0$ be constants which depend only on K and N and may differ from line to line.

For each $x \in \Omega \cap \mathbb{Q}^N$, define $\overline{f}_x$ to be a supporting affine functional of $\overline{f}$ at x, chosen consistently so that $\overline{f}_{tx} = \overline{f}_x$ for all $t > 0$. We think of $\overline{f}_x(v)$ as the amount of progress that a step v makes in the direction x. Given $x \in \Omega \cap \mathbb{Q}^N$, define the set of "good" increments

$$Q_x = \left\{ v \in \Omega \cap \mathbb{Z}^N \mid f(v) - 5K^2 \leq \overline{f}_x(v) \leq \overline{f}(x) \right\}.$$

We think of $f(v) - \overline{f}_x(v)$ as the amount of inefficiency in the increment v on a path toward x, so a good increment is one which has inefficiency at most $5K^2$. The second part of the inequality means that good increments don't "overshoot" in the direction of x, which implies (from linear growth) that good increments have length at most $C|x|$.

Step 1. We show that if $x \in \Omega \cap \mathbb{Q}^N$ with $|x| \geq C$, then there is $\alpha \in [c, 1]$ such that αx

lies in the convex hull of Q_x. Let $n \in \mathbb{N}$ be large enough so that

$$|n^{-1}f(nx) - \overline{f}(x)| \leq 1.$$

Let $x_0, x_1, \ldots, x_m$ be an approximate geodesic for nx. We iteratively define a subsequence $y_k = x_{j_k}$ by letting $j_0 = 0$ and, as long as $j_k < m$, we define $j_{k+1} \in [j_k + 1, \ldots, m]$ to be maximal such that $y_{k+1} - y_k \in Q_x$. By linear growth of f and $\overline{f}$, we have

$$\overline{f}_x(x_{j_k+1} - x_{j_k}) - 5K^2 \leq K^2 + K - 5K^2 \leq f(x_{j_k+1} - x_{j_k}) \leq K^2 + K \leq K^{-1}C - K \leq \overline{f}(x),$$

as long as C was chosen large enough, so $j_k + 1$ is admissible and therefore the subsequence exists, and we let $p \in \mathbb{N}$ be the index where $j_p = m$. If k is such that $j_{k+1} < m$ and $\overline{f}_x(x_{j_{k+1}+1} - x_{j_k}) > \overline{f}(x)$, then the fact that $|x| \geq C$, linear growth, and the approximate geodesic property yields

$$f(y_{k+1} - y_k) \geq (K^{-1}|x| - K) - (K^2 + 2K).$$

Choosing C large enough and summing over k (using the approximate geodesic property again) shows that there are $O(n)$ many such k.

On the other hand, let ℓ be the number of k such that $j_{k+1} < m$ and

$$f(x_{j_{k+1}+1} - x_{j_k}) - 5K^2 > \overline{f}_x(x_{j_{k+1}+1} - x_{j_k}).$$

For such k, we have

$$\overline{f}_x(y_{k+1} - y_k) \leq f(y_{k+1} - y_k) - 5K^2 + 2(K^2 + K) \leq f(y_{k+1} - y_k) - K^2.$$

Linearity of $\overline{f}_x$ and the approximate geodesic property shows that

$$\overline{f}_x(nx) = n\overline{f}(x) \leq f(nx) + pK - \ell K^2,$$

so the choice of n implies that $\ell K^2 - pK \leq n$ and therefore $\ell \leq \frac{1}{4}n + \frac{1}{2}p$. All together, we have shown that $p \leq Cn$. We conclude this step by noting that

$$x = \frac{1}{n}\sum_{k=1}^{p}(y_k - y_{k-1}),$$

and $n \leq p \leq Cn$ (the first part of the inequality follows from applying $\overline{f}_x$ to both sides of the equation).

Step 2. We show that if $x \in \Omega \cap \mathbb{Q}^N$, $|x| \geq C$, $t \geq 1$, and $tx \in \mathbb{Z}^N$, then there is a $z \in \Omega \cap \mathbb{Z}^N$ with $|z| \leq C|x|$ and

$$f(tx) - \overline{f}(tx) \leq f(z) - \overline{f}(z) + Ct.$$

Using the previous step, write $tx = z + \sum_{k=1}^{m} v_k$, where $|z| \leq C|x|$, $\overline{f}(z) \leq \overline{f}_x(z) + C$, $v_k \in Q_x$, and $m \leq Ct$. Indeed, for some $\alpha \in [c, 1]$ we first write

$$\alpha x = \sum_{i=1}^{N+1} p_i v_i,$$

where $v_i \in Q_x$ and $p_i \geq 0$, $\sum_i p_i = 1$. Note that the sum only requires $N+1$ terms by Caratheodory's theorem on convex hulls, since we are working in $\mathbb{R}^N$. To decompose tx, we write

$$tx = \sum_{i=1}^{N+1}(t\alpha^{-1}p_i - \lfloor t\alpha^{-1}p_i\rfloor)v_i + \sum_{i=1}^{N+1}\lfloor t\alpha^{-1}p_i\rfloor v_i =: z + (tx - z),$$

so z satisfies the required properties. By subadditivity of f and linearity of $\overline{f}_x$,

$$\begin{aligned} f(tx) &\le f(z) + \sum_{k=1}^m f(v_k) \\ &\le f(z) + \sum_{k=1}^m \left(\overline{f}_x(v_k) + 5K^2\right) \\ &= f(z) + \overline{f}_x(tx - z) + 5CK^2 t \\ &\le f(z) + \overline{f}_x(tx - z) + Ct. \end{aligned}$$

Finally, we write $\overline{f}(tx) = \overline{f}_x(z) + \overline{f}_x(tx - z)$ and subtract from both sides of the inequality above to get

$$f(tx) - \overline{f}(tx) \le f(z) - \overline{f}(z) + Ct,$$

where we used the fact that $\overline{f}(z) \le \overline{f_x}(z) + C$.

Step 3. For some large $M > 1$, the previous step yields

$$\sup_{|x| \le M^{k+1}C} f(x) - \overline{f}(x) \le \sup_{|x| \le M^k C} f(x) - \overline{f}(x) + CM.$$

We conclude by induction on k. □

2.4 Homogenization via the metric problem

Let $C > 1 > c > 0$ denote constants which depend on H and $\mathrm{Lip}(u_0)$ and may differ from line to line. If $a \in \mathbb{R}$, then replacing H by $H - a$ replaces solutions u^ε by $u^\varepsilon + ta$, so we lose no generality in assuming that $H(x, 0) \le -1$ for all $x \in \mathbb{R}^d$. It is well-known (see, e.g. Theorem 1.34 from [35]) that the solutions u^ε are Lipschitz, with bound $\mathrm{Lip}(u^\varepsilon) \le C$ independent of ε. In particular, only the values of $H(x, p)$ for $|p| \le C$ are needed to solve the initial-value problem (2.1). Therefore, we lose no generality in assuming that $H(x, p) = |p|^2$

for $|p| \geq C$. We write $L(x, v)$ to denote the Lagrangian

$$L(x, v) := \sup_{p \in \mathbb{R}^d} p \cdot v - H(x, p),$$

which we use to define the metric

$$m(t, x, y) := \inf_{\gamma \in \Gamma(t,x,y)} \int_0^t L(\gamma(s), \gamma'(s)) \, ds, \tag{2.4}$$

where $\Gamma(t, x, y)$ is the set of paths $\gamma \in W^{1,1}([0, t]; \mathbb{R}^d)$ with $\gamma(0) = x$ and $\gamma(t) = y$. We also define the homogeneous metric

$$\overline{m}(t, x, y) := \lim_{n \to \infty} n^{-1} m(nt, nx, ny). \tag{2.5}$$

Given a path $\gamma \in \Gamma(t, x, y)$, we refer to $\int_0^t L(\gamma(s), \gamma'(s)) \, ds$ as the cost of γ. Noting that the assumption on H implies that $L(x, v) = |v|^2$ for $|v| \geq C$, it is a standard fact that a minimizer $\gamma \in \Gamma(t, x, y)$ exists for the infimum in equation (2.4) which satisfies

$$\mathrm{Lip}(\gamma) \leq C + Ct^{-1}|x - y|. \tag{2.6}$$

The optimal control formulation of (2.1) is

$$u^\varepsilon(t, y) = \inf_{|x-y| \leq Ct} u_0(x) + \varepsilon m(\varepsilon^{-1}t, \varepsilon^{-1}x, \varepsilon^{-1}y). \tag{2.7}$$

Define the cone $\Omega := \{(t, x) \in \mathbb{R}_{\geq 0} \times \mathbb{R}^d \mid |x| \leq Ct\}$. For any $(t, y - x) \in \Omega$, we have

$$|m(t, x, y) - m(\lceil t \rceil, [x], [y])| \leq C,$$

where $[\cdot]$ denotes coordinate-wise rounding to integers in a way that stays inside Ω. By $\mathbb{Z}^d$-periodicity,

$$m(\lceil t\rceil, [x], [y]) = m(\lceil t\rceil, 0, [y] - [x]).$$

The Lipschitz estimate (2.6) for minimizers shows that $f(t,x) := m(t,0,x)$ has approximate geodesics. Indeed, we find an approximate geodesic by chopping up a minimizing path, and the Lipschitz estimate (2.6) shows that the pieces lie in Ω. Since $L(x,v) \geq 1$ for all $x, v \in \mathbb{R}^d$, it is clear that f has linear growth (when restricted to Ω) and is subadditive and nonnegative. We finish by applying Alexander's theorem, which yields

$$|\varepsilon m(\varepsilon^{-1}t, \varepsilon^{-1}x, \varepsilon^{-1}y) - \overline{m}(t,x,y) \leq C\varepsilon \log(C + \varepsilon^{-1}t + \varepsilon^{-1}|x-y|),$$

for all (t,x,y) with $(t, y-x) \in \Omega$. The main result follows.

2.5 The case $d = 2$

In this section, we assume $d = 2$. Rather than working with approximate geodesics as before, it will be more convenient to work directly with the minimizers for the metric m. We follow the same method as Alexander [2], who proved an analogous result in the context of Bernoulli percolation. We first show that m is approximately superadditive.

Lemma 7. *If $(t,x) \in \Omega \cap \mathbb{Z}^{d+1}$, then $2m(t,0,x) \leq m(2t,2x) + C$.*

Proof. Let $\gamma \in \Gamma(2t, 0, 2x)$ be a minimizing path.

Step 1. We show that we can form a path from 0 to x as the concatenation of at most 4 non-overlapping segments of γ. Let $\gamma^1, \gamma^2 \colon [0,t] \to \mathbb{R}^d$ be the first and second halves of γ respectively, given by

$$\gamma^1(s) := \gamma(s)$$

and

$$\gamma^2(s) := \gamma(t+s) - \gamma(t)$$

respectively. Then $\gamma^1(t) + \gamma^2(t) = 2x$, so $\gamma^1(t) = x - y$ and $\gamma^2(t) = x + y$ for some $y \in \mathbb{R}^d$. By a linear transformation of $\mathbb{R} \times \mathbb{R}^2$, we lose no generality in assuming that $x = 0$ and $y = (A, 0)$ for some $A > 0$. Consider the paths

$$\eta^1 \colon s \mapsto (s, \gamma^1(s) + (A, 0))$$

and

$$\eta^2 \colon s \mapsto (s, \gamma^2(s)),$$

so $\eta^1(0) = (0, A, 0)$, $\eta^1(t) = (t, 0, 0)$, $\eta^2(0) = (0, 0, 0)$, and $\eta^2(t) = (t, A, 0)$.

For $k \in \{1, 2\}$ and $c \in [0, t]$, we define the *cyclic shift*

$$\eta^{k,c}(s) := \begin{cases} \eta^k(c+s) - \eta^k(c) & \text{if } c + s \leq t \\ \eta^k(s - (t-c)) + \eta^k(t) - \eta^k(c) & \text{otherwise.} \end{cases}$$

We claim that some cyclic shifts of η^1 and η^2 intersect. Indeed, we can cyclically shift either path so that it is contained in the half-space $H^{\pm} := \{x \in \mathbb{R}^3 \mid \pm x \cdot (0, 0, 1) \geq 0\}$. The claim then follows from continuity, starting with η^1 in H^- and η^2 in H^+, and cyclically shifting them into H^+ and H^- respectively, as we will now explain in detail.

Indeed, suppose that the cyclic shifts η^{1,c_1} and η^{2,c_2} do not intersect for any $c_1, c_2 \in [0, t]$. Then form the map $\varphi^{c_1,c_2} \colon [0, t] \to S^1$, where S^1 is the unit circle (identified in $\mathbb{C} = \mathbb{R}^2$ for concreteness) by

$$\varphi^{c_1,c_2}(s) := P\left(\frac{\eta^{1,c_1}(s) - \eta^{2,c_2}(s)}{|\eta^{1,c_1}(s) - \eta^{2,c_2}(s)|}\right),$$

where $P(x, y, z) := (y, z)$ denotes projection onto the last two coordinates. Since the denominator is always nonzero, shifting c_1 and c_2 continuously produces a homotopy. As a homotopy invariant, the winding number of φ^{c_1,c_2} is constant with respect to c_1, c_2. Choosing

$$c_1 := \arg\max_c \eta^1(c) \cdot (0,0,1) \quad \text{and} \quad c_2 := \arg\min_c \eta^2(c) \cdot (0,0,1)$$

ensures $\eta^{1,c_1}(s) \in H^-$ and $\eta^{2,c_2}(s) \in H^+$ for all s. Since $(\eta^{1,c_1}(s) - \eta^{2,c_2}(s)) \cdot (0,0,1) \leq 0$, the map φ^{c_1,c_2} is homotopic to $s \mapsto e^{-i\pi s/t}$, which has winding number $-1/2$. On the other hand, choosing

$$c_1 := \arg\min_c \eta^1(c) \cdot (0,0,1) \quad \text{and} \quad c_2 := \arg\max_c \eta^2(c) \cdot (0,0,1)$$

makes φ^{c_1,c_2} homotopic to $s \mapsto e^{i\pi s/t}$, which has winding number $1/2$, a contradiction.

Finally, we form a new path following (a cyclic shift of) η^2 from $(0,0,0)$ to the point of intersection, and following η^1 the rest of the way to $(t,0,0)$.

To summarize, we found a path from 0 to x which is composed of a segment of a cyclic shift of γ^1 and a segment of a cyclic shift of γ^2, so the segments don't overlap. Since we took cyclic shifts, this equates to at most 4 segments from γ.

Step 2. Use Step 1 to find an approximate geodesic with subsequence

$$0 = (t_0, x_0), (t_1, x_1), \ldots, (t_9, x_9) = (2t, 2x)$$

for m along γ, such that there are indices $i_1, \ldots, i_4$ with $\sum_{k=1}^4 (t_{i_k} - t_{i_k - 1}, x_{i_k} - x_{i_k - 1}) = (t, x)$. Rearranging the indices, we find a path $\widetilde{\gamma} \in \Gamma(2t, 0, 2x)$ with $\widetilde{\gamma}(t) = x$ and cost at most C more than the cost of γ. The conclusion follows. $\square$

The previous lemma and subadditivity show that

$$m(2t, 0, 2x) \leq 2m(t, 0, x) \leq m(2t, 0, 2x) + C$$

for all $(t, x) \in \Omega \cap \mathbb{Z}^{d+1}$. Then $m(t, 0, x) - C \leq 2^{-k}(m(2^k t, 0, 2^k x) - C)$ for all $k \in \mathbb{N}$ by induction, so letting $k \to \infty$ shows

$$|m(t, x, y) - \overline{m}(t, x, y)| \leq C$$

for all $(t, x) \in \Omega \cap \mathbb{Z}^{d+1}$, so the same holds for all $(t, x) \in \Omega$ since m is Lipschitz. The result in dimension $d = 2$ follows.

CHAPTER 3

SLOW PERIODIC HOMOGENIZATION FOR NONCONVEX HAMILTONIANS

3.1 Introduction

Since Lions–Papanicolaou–Varadhan [33] proved periodic homogenization for coercive Hamilton–Jacobi equations, quantifying the rate of convergence has been a well-known open problem in both periodic and random settings. In a periodic environment, without additional structural assumptions on the Hamiltonian, the best known result so far is the $O(\varepsilon^{1/3})$ rate, proven by Capuzzo-Dolcetta–Ishii [11], which was also the first quantitive bound. On the other hand, when the Hamiltonian is convex in the momentum variable, the optimal rate of $O(\varepsilon)$ can be deduced from the optimal control formulation and an argument of Burago [6], who proved a corresponding rate for homogenization of $\mathbb{Z}^d$-periodic metrics on $\mathbb{R}^d$. It is therefore natural to ask whether, in the absence of convexity, the $O(\varepsilon)$ rate still holds. Indeed, Ziliotto's [37] example of stochastic non-homogenization (see also Feldman–Souganidis [21]) suggests that saddle points of the Hamiltonian may play a key role in slowing down periodic homogenization.

We answer this question in the negative by constructing examples which homogenize at a rate of $\Theta(\varepsilon^{1/2})$. In dimensions $d \geq 3$, the example can even be constructed so that the effective Hamiltonian is convex.

Suppose that the Hamiltonian $H \colon \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}$ is locally Lipschitz, $\mathbb{Z}^d$-periodic in the first variable, $x \in \mathbb{R}^d$, and uniformly coercive in the second variable, $p \in \mathbb{R}^d$; that is,

$$\liminf_{|p|\to\infty} \inf_{x\in\mathbb{R}^d} H(x,p) = +\infty.$$

The microscopic problem at scale $\varepsilon > 0$ is the initial-value problem

$$\begin{cases} D_t u^\varepsilon(t,x) + H(\varepsilon^{-1}x, D_x u^\varepsilon(t,x)) = 0 & \text{for } t > 0 \text{ and } x \in \mathbb{R}^d, \\ u^\varepsilon(0,x) = u_0(x) & \text{for } x \in \mathbb{R}^d, \end{cases} \tag{3.1}$$

where the initial data u_0 is Lipschitz.

Lions–Papanicolaou–Varadhan [33] proved that there is an effective Hamiltonian $\overline{H} \colon \mathbb{R}^d \to \mathbb{R}$, uniquely determined by H, such that $u^\varepsilon \to \bar{u}$ uniformly on compact sets, where $\bar{u}$ solves the effective problem

$$\begin{cases} D_t \bar{u}(t,x) + \overline{H}(D_x \bar{u}(t,x)) = 0 & \text{for } t > 0 \text{ and } x \in \mathbb{R}^d, \\ \bar{u}(0,x) = u_0(x) & \text{for } x \in \mathbb{R}^d. \end{cases} \tag{3.2}$$

Under these general assumptions, the only known quantitative upper bound on the rate of homogenization is due to Capuzzo-Dolcetta–Ishii [11], who proved that $\|u^\varepsilon - \bar{u}\|_{L^\infty([0,T]\times\mathbb{R}^d)} = O(\varepsilon^{1/3})$ for $T > 0$.

Theorem 4. *There exists a locally Lipschitz Hamiltonian $H \colon \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}$, $\mathbb{Z}^d$-periodic in the first variable and uniformly coercive in the second variable, along with initial data $u_0 \in C^{0,1}(\mathbb{R}^d)$, such that, for sufficiently small ε,*

$$c\varepsilon^{1/2} \le u^\varepsilon(1,0) \le C\varepsilon^{1/2}$$

where u^ε is the solution to the microscopic problem (3.1)*. If $d \ge 3$, then, furthermore, H can be chosen so that $\overline{H}$ is convex.*

3.2 Examples of slow homogenization

To construct the examples, we first recall some facts from the theory of differential games. For a more thorough treatment, see Isaacs [25] and Evans–Souganidis [18].

Let $A, B \subseteq \mathbb{R}^d$ be compact sets. We consider a differential game between two players named **I** and **II**. The game has a score, which **I** tries to minimize and **II** tries to maximize.

Definition 8. A control for **I** (resp. **II**) is a measurable function $a\colon \mathbb{R}_{\geq 0} \to A$ (resp. $b\colon \mathbb{R}_{\geq 0} \to B$). We write $\mathcal{C}_A, \mathcal{C}_B$ to denote the set of controls for **I** and **II** respectively.

Definition 9. A strategy for **I** is a function $\alpha\colon \mathcal{C}_B \to \mathcal{C}_A$ with the nonanticipative property: if $t > 0$ and $b_1, b_2 \in \mathcal{C}_B$ with $b_1(s) = b_2(s)$ for almost all $s \in [0,t]$, then $\alpha(b_1)(s) = \alpha(b_2)(s)$ for almost all $s \in [0,t]$ also. We write $\mathcal{S}_A$ to denote the set of strategies for **I**, and define the set of strategies $\mathcal{S}_B$ for **II** correspondingly.

A differential game is specified by the sets A, B, some Lipschitz initial data $u_0 \in C^{0,1}(\mathbb{R}^d)$, a running cost $R \in L^\infty(\mathbb{R}^d \times A \times B)$, and a transition function $f \in L^\infty(\mathbb{R}^d \times A \times B; \mathbb{R}^d)$ which is Carathéodory, i.e. $f(x,a,b)$ is continuous in a, b for fixed x, and measurable in x for fixed a, b. The game is based on the evolution of the state, $\sigma\colon \mathbb{R}_{\geq 0} \to \mathbb{R}^d$. Given a strategy α, a control b, and a starting state x, the state evolves to satisfy the ordinary differential equation

$$\begin{cases} \dot\sigma(t) = f(\sigma(t), \alpha(b)(t), b(t)) & \text{for } t > 0, \\ \sigma(0) = x & \text{at } t = 0. \end{cases} \tag{3.3}$$

Sometimes, we will write $\sigma(t) = \sigma(t, \alpha, b)$ to emphasize the dependence on α and b.

Given $t \geq 0$ and $x \in \mathbb{R}^d$, the upper value of the game starting at x after time t is defined by

$$u^+(t,x) := \inf_{\alpha \in \mathcal{S}_A} \sup_{b \in \mathcal{C}_B} \int_0^t R(\sigma(s), \alpha(b)(s), b(s))\, ds + u_0(\sigma(t)). \tag{3.4}$$

The upper value of the game (see Evans–Souganidis [[18], Theorem 4.1] and Lions [28]) is the viscosity solution of the initial-value problem

$$\begin{cases} D_t u^+(t,x) + H^+(x, D_x u^+(t,x)) = 0 & \text{for } t > 0 \text{ and } x \in \mathbb{R}^d, \\ u^+(0,x) = u_0(x) & \text{for } x \in \mathbb{R}^d, \end{cases} \tag{3.5}$$

where the upper Hamiltonian $H\colon \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}$ is given by

$$H^+(x,p) := -\min_{a \in A} \max_{b \in B} R(x,a,b) + p \cdot f(x,a,b). \tag{3.6}$$

It is worth noting that, by interchanging the order of the players, we can similarly define the lower value of the game by

$$u^-(t,x) := \sup_{\beta \in \mathcal{S}_B} \inf_{a \in \mathcal{C}_A} \int_0^t R(\sigma(s), a(s), \beta(a)(s))\, ds + u_0(\sigma(t)), \tag{3.7}$$

which solves a similar initial-value problem corresponding to the lower Hamiltonian

$$H^-(x,p) := -\max_{b \in B} \min_{a \in A} R(x,a,b) + p \cdot f(x,a,b). \tag{3.8}$$

In all of our examples, the Isaacs condition $H^+ = H^-$ will be satisfied and therefore the upper and lower values of the game coincide. For brevity, we put $H := H^- = H^+$.

3.2.1 An example in two dimensions

Let $\varphi\colon \mathbb{R}/\mathbb{Z} \to [0,1]$ be smooth such that $\varphi(0) = 1$, and $\varphi(x) = 0$ if $|x| \geq \frac{1}{100}$. We take $d = 2$ and $A = \overline{B_1(0)}$, the closed unit ball centered at the origin, and $B = [0,1] \times \{0\}$. Define the running cost by

$$R(x,a,b) := 100 \left(1 - \varphi(x_2) - \varphi\left(x_2 + \frac{1}{2}\right)\right) + 100|a|^2, \tag{3.9}$$

the transition function by

$$f(x,a,b) := 2a + b\left(\varphi(x_2) - \varphi\left(x_2 + \frac{1}{2}\right)\right), \tag{3.10}$$

and the initial data by

$$u_0(x) := \min\{|x_1|, 1\}.$$

Although it's unnecessary for the proof, we note that the Isaacs condition

$$H(x,p) = -\min_{a\in A}\max_{b\in B} R(x,a,b) + p\cdot f(x,a,b) = -\max_{b\in B}\min_{a\in A} R(x,a,b) + p\cdot f(x,a,b)$$

is satisfied, so the upper and lower values of this game coincide.

Intuitively, the microscopic environment consists of horizontal "highways" at every height in $\frac{\varepsilon}{2}\mathbb{Z}$. Outside of the highways, the running cost is punishingly large, so **I** is forced to spend most of the time inside the highways. Outside the highways, **II**'s control has no affect on the state. Inside highways at height in $\varepsilon\mathbb{Z}$, **II** has the option to push the state in the $+e_1$ direction, and inside highways at height in $\varepsilon\left(\mathbb{Z} + \frac{1}{2}\right)$, **II** has the option to push the state in the $-e_1$ direction. **II**'s control has no effect on the running cost, and **I** is heavily penalized for pushing the state in any direction. If **I** wants to stay close to the origin (where the terminal cost is lowest), then one strategy is to enter a highway and wait until **II** pushes the state a distance of $\varepsilon^{1/2}$ from the origin. Then, **I** can switch to a highway that leads back to the origin, and repeat. By the same reasoning, **II** can force **I** to switch highways at least $\varepsilon^{-1/2}$ many times, or else **I** risks paying a terminal cost of at least $\varepsilon^{1/2}$. Each highway switch adds running cost proportional to ε to the total, so the error terms balance.

Now, we prove the $d = 2$ case of Theorem 4.

Proof. We use the differential game characterization (3.4) of u^ε. For the upper bound, we construct a strategy $\alpha\colon \mathcal{C}_B \to \mathcal{C}_A$ for **I** piecewise as follows.

1. At the beginning of this step, suppose that the strategy has already been constructed up to time $t_i \geq 0$ and $\sigma(t_i, \alpha, b) \in [-\frac{\varepsilon}{4}, \frac{\varepsilon}{4}] \times \{0\}$. Given $b \in \mathcal{C}_B$, let

$$t_{i+1} := \min\{t > 0 \mid u_0(\sigma(t, \widehat{\alpha}_{t_i}, b)) \geq \varepsilon^{1/2}\},$$

   where

$$\widehat{\alpha}_s(t) := \begin{cases} \alpha(t) & \text{for } t < s, \\ 0 & \text{for } t \geq s. \end{cases}$$

   From the structure of f and u_0, we deduce that $\sigma(t_{i+1}, \widehat{\alpha}_{t_i}, b) = (\varepsilon^{1/2}, 0)$. Set $\alpha(b)(t) := \widehat{\alpha}_{t_i}(b)(t)$ for $t < t_{i+1}$.

2. Write $t_{i+2} := t_{i+1} + \frac{\varepsilon}{4}$ and set $\alpha(b)(t) := (0, 1)$ for $t_{i+1} \leq t < t_{i+2}$.

3. We deduce that $\sigma(t_{i+2}, \alpha, b) \in \left[\varepsilon^{1/2} - \frac{\varepsilon}{4}, \varepsilon^{1/2} + \frac{\varepsilon}{4}\right] \times \left\{\frac{\varepsilon}{2}\right\}$. Now, set

$$t_{i+3} := \min\{t > 0 \mid u_0(\sigma(t, \widehat{\alpha}_{t_{i+2}}, b)) \leq 0\}$$

   and set $\alpha(b)(t) := \widehat{\alpha}_{t_{i+2}}(b)(t)$ for $t_{i+2} \leq t < t_{i+3}$. As in the first step, we deduce that $\sigma(t_{i+3}, \alpha, b) = \left(0, \frac{\varepsilon}{2}\right)$.

4. Write $t_{i+4} := t_{i+3} + \frac{\varepsilon}{4}$ and set $\alpha(b)(t) := (0, -1)$ for $t_{i+3} \leq t < t_{i+4}$.

5. We deduce that $\sigma(t_{i+4}, \alpha, b) \in \left[-\frac{\varepsilon}{4}, \frac{\varepsilon}{4}\right] \times \{0\}$. Now, go back to step 1 and repeat, but starting at time t_{i+4} instead of time t_i.

The construction maintains the invariant that $\sigma(t, \alpha, b) \in \left[0, \frac{\varepsilon}{4}\right] \times \left[-\frac{\varepsilon}{4}, \varepsilon^{1/2} + \frac{\varepsilon}{4}\right]$, so we ensure that the terminal cost is at most $u_0(\sigma(1)) \leq \varepsilon^{1/2} + \frac{\varepsilon}{4} \leq 2\varepsilon^{1/2}$.

It remains to show that the running cost given by α is at most $C\varepsilon^{1/2}$. For any $i = 0, 1, 2, \ldots$, we claim

$$\int_{t_i}^{t_{i+1}} R(\varepsilon^{-1}\sigma(t), \alpha(b)(t), b(t))\, \mathrm{d}t \leq 50\varepsilon.$$

Indeed, an interval created in step 2 or step 4 above satisfies this bound, as $t_{i+1} - t_i = \frac{\varepsilon}{4}$ and $R \leq 200$ everywhere. On the other hand, intervals created in step 1 or step 3 above have running cost 0, since $\sigma(t) \in \mathbb{R} \times \frac{\varepsilon}{2}\mathbb{Z}$ for all t in the interval.

We have shown that each step adds at most 50ε to the running cost. On the other hand, every interval created by step 1 or step 3 runs for time at least $\varepsilon^{1/2}$, so there can be at most $\varepsilon^{-1/2}$ such intervals in $[0,1]$. We conclude that the total running cost is at most $50\varepsilon^{1/2}$, so $C = 50$ satisfies the claim.

Next, we turn to the lower bound $u^\varepsilon(1,0) \geq c\varepsilon^{1/2}$. Given a strategy $\alpha \colon \mathcal{C}_B \to \mathcal{C}_A$, we construct the following control for **II**.

$$b(t) := \begin{cases} (1,0) & \text{if } u_0(\sigma(t,\alpha,b)) \leq 2\varepsilon^{1/2} \text{ or } \sigma(t,\alpha,b) \cdot f(\varepsilon^{-1}\sigma(t,\alpha,b),0,b) > 0, \\ 0 & \text{otherwise.} \end{cases}$$

We claim that this control yields a value of at least $c\varepsilon^{1/2}$. Indeed, consider the set of times

$$E := \{t \in [0,1] \mid R(\varepsilon^{-1}\sigma(t,\alpha,b),0,0) \geq 1\}.$$

First, note that

$$\int_0^1 R(\varepsilon^{-1}\sigma(t,\alpha,b),\alpha(b)(t),b(t))\,\mathrm{d}t \geq |E|, \tag{3.11}$$

where $|E|$ denotes the Lebesgue measure. On the other hand, let $U_+ := \{t \in [0,1] \setminus E \mid \varphi(\varepsilon^{-1}\sigma(t,\alpha,b)_2) > 0\}$ and $U_- := [0,1] \setminus (E \cup U_+)$. In the set of times U_+ (resp. U_-), **II** can control the state to push in the $+e_1$ (resp. $-e_1$) direction with magnitude at least 0.99.

There are two cases.

1. Suppose that there exist $0 \leq t_0 \leq t_1 \leq 1$ such that $|[t_0,t_1] \cap U_-| \geq 6\varepsilon^{1/2}$ and

$|[t_0,t_1]\cap U_+|=0$. Then either

$$\int_{t_0}^{t_1} R(\sigma(t,\alpha,b),\alpha(b)(t),b(t))\ \mathrm{d}t \geq \varepsilon^{1/2},$$

in which case we conclude, or we have $u_0(\sigma(t_1,\alpha,b)) \geq 3\varepsilon^{1/2}$, because **I** has a total effect of less than $\frac{\varepsilon^{1/2}}{100}$ on the state over the interval $[t_0,t_1]$. If t is such that $u_0(\sigma(t,\alpha,b)) \geq 2\varepsilon^{1/2}$, then by definition of b we have $\sigma(t,\alpha,b)\cdot f(\varepsilon^{-1}\sigma(t,\alpha,b),0,b) \geq 0$, so the control of **II** never pushes the state toward smaller values of u_0. So, either $u(\sigma(1,\alpha,b)) \geq 2\varepsilon^{1/2}$ or **II** spends

$$\int_{t_l}^{1} R(\sigma(t,\alpha,b),\alpha(b)(t),b(t))\ \mathrm{d}t \geq \varepsilon^{1/2},$$

and in either case we conclude. We note that if $|[t_0,t_1]\cap U_+| \geq 6\varepsilon^{1/2}$ and $|[t_0,t_1]\cap U_-| = 0$, then we conclude by the same argument.

2. Otherwise, we may assume by (3.11) that $|E| \leq \varepsilon^{1/2}$, so $|U_1 \cup U_2| \geq 1-\varepsilon^{1/2}$. Take $t_- \in U_-$ and $t_+ \in U_+$. Since $|\sigma(t_-,\alpha,b)_2 - \sigma(t_+,\alpha,b)_2| \geq \varepsilon\left(\frac{1}{2}-\frac{1}{50}\right)$ (using the fact that φ is supported in $\left(-\frac{1}{100},\frac{1}{100}\right)$), we conclude that $|[t_-,t_+]\cap E| \geq \frac{\varepsilon}{5}$ and therefore

$$\int_{t_-}^{t_+} R(\sigma(t,\alpha,b),\alpha(b)(t),b(t))\ \mathrm{d}t \geq \frac{\varepsilon}{5}.$$

On the other hand, since the hypotheses of the previous case do not apply, we have $[t_0,t_1]\cap U_- \neq \emptyset$ and $[t_0,t_1]\cap U_+ \neq \emptyset$ whenever $t_1 - t_0 \geq 6\varepsilon^{1/2}$. So, there are at least $\frac{\varepsilon^{-1/2}}{6}-1$ many such disjoint intervals in $[0,1]$, and we conclude that

$$|E| \geq \frac{\varepsilon}{5}\cdot\left(\frac{\varepsilon^{-1/2}}{6}-1\right) \geq \frac{\varepsilon^{1/2}}{35}$$

as long as $\varepsilon \le \frac{1}{42^2}$, and therefore

$$\int_0^1 R(\sigma(t,\alpha,b),\alpha(b)(t),b(t))\,\mathrm{d}t \ge \frac{\varepsilon^{1/2}}{35}.$$

In any of the cases, $c = \frac{1}{35}$ satisfies the claim. $\square$

3.2.2 An example in three and higher dimensions

Next, we show that if $d \ge 3$, we can construct an example where the effective Hamiltonian $\overline{H}$ is convex. Without loss of generality, let $d = 3$. Inspired by the example of Hedlund [23], let

$$\mathcal{L} := \bigcup_{i=1}^{3} \ell_i + \mathbb{Z}^3,$$

where $\ell_1 := \mathbb{R} \times \{0\} \times \{0\}$, $\ell_2 := \{0\} \times \mathbb{R} \times \{\frac{1}{4}\}$, and $\ell_3 := \{\frac{1}{4}\} \times \{\frac{1}{4}\} \times \mathbb{R}$, and for $i \in \{1,2,3\}$ let $\varphi_i \colon \mathbb{R}^3/\mathbb{Z}^3 \to [0,1]$ be smooth such that $\varphi_i(x) = 1$ for $x \in \ell_i$ and $\varphi_i(x) = 0$ if $\operatorname{dist}(x,\ell_i) \ge \frac{1}{100}$. Write $\varphi := \varphi_1 + \varphi_2 + \varphi_3$ and $\widetilde{\varphi} := (\varphi_1,\varphi_2,\varphi_3)$.

Let $A = \overline{B_2(0)}$ and $B = [0,1]^3$. Define the running cost by

$$R(x,a,b) := 100\left(1 - \varphi(x) - \varphi\left(x + \frac{1}{2}\right)\right) + 100|a|, \tag{3.12}$$

and the transition function by

$$f(x,a,b) := 2\left(1 + 99\left(\varphi(x) + \varphi\left(x + \frac{1}{2}\right)\right)\right)a + b \odot \left(\widetilde{\varphi}(x) - \widetilde{\varphi}\left(x + \frac{1}{2}\right)\right), \tag{3.13}$$

where we write $\frac{1}{2}$ to denote the vector $(\frac{1}{2},\frac{1}{2},\frac{1}{2})$ and $\odot$ to denote the pointwise product, i.e. $x \odot y = (x_1y_1, x_2y_2, x_3y_3)$.

As before, the Isaacs condition is satisfied and therefore the upper and lower values of the game coincide. This environment is very similar to the previous example, except that now the highways go in every coordinate direction $\pm e_i$. We take advantage of the fact that this is possible in three dimensions while ensuring that each highway is far away from any other highway. The running cost is nearly identical to the previous example, except that the penalty for **I** is $100|a|$ instead of $100|a|^2$. The transition function is slightly different; **II**'s controls work similarly, but **I** is now able to move faster inside the highways and slower outside.

Now, we prove the $d \geq 3$ case of Theorem 4.

Proof. Using the same initial data $u_0(x) := \min\{|x_1|, 1\}$, the argument for the $\varepsilon^{1/2}$ rate is identical to the previous example, so we omit it. To show that $\overline{H}$ is convex, we find, for each $p \in \mathbb{R}^d$ and $\lambda > 0$, bounds for the long-time corrector v^p, defined as the solution to the initial-value problem

$$\begin{cases} D_t v^p(t,x) + H(x, D_x v^p(t,x)) = 0 & \text{for } t > 0 \text{ and } x \in \mathbb{R}^2, \\ v^p(0,x) = p \cdot x & \text{for } x \in \mathbb{R}^2. \end{cases} \tag{3.14}$$

Then, using the fact that $\overline{H}(p) = -\lim_{t\to\infty} t^{-1} v^p(t,0)$, we obtain a formula for $\overline{H}$.

First, for $\gamma \in \mathbb{R}$ we claim that

$$\overline{H}(\gamma e_i) = h(\gamma) := \max\{0, 400|\gamma| - 200\}.$$

Indeed, we immediately have the lower bound

$$v^{\gamma e_i}(t,x) \geq \min_{a \in A} t(100|a| + 200\gamma a \cdot e_i) = th(\gamma),$$

which follows from considering the constant 0 control for **II** and ignoring the space-dependent part of the running cost.

On the other hand, to obtain the upper bound, **I** can use the following strategy: immediately move the state (in constant time) into $\ell_i + \mathbb{Z}^3$, if $\gamma < 0$, or into $\ell_i + \frac{1}{2} + \mathbb{Z}^3$ otherwise. For the rest of time, use the constant strategy $\alpha = -(\operatorname{sgn}\gamma)\min(2, |\gamma|)\, e_i$. Computing the result of the game with this strategy shows that

$$v^{\gamma e_i}(t, x) \leq C + \min_{a \in A} t(100|a| + 200\gamma a \cdot e_i) = C + th(\gamma).$$

We have shown that $\overline{H}(\gamma e_i) = h(\gamma)$, and $h(\gamma)$ is convex. To conclude, we will show that, for each $p \in \mathbb{R}^d$,

$$\overline{H}(p) = \max_{i=1}^{3} \overline{H}(p_i e_i).$$

The inequality $\overline{H}(p) \geq \overline{H}(p_i e_i)$ follows immediately from using the strategy outlined above for $i \in \{1, 2, 3\}$.

For the other inequality, let $\alpha\colon \mathcal{C}_B \to \mathcal{C}_A$ be a strategy for the initial data $p \cdot x$ and starting state 0. Suppose for contradiction that for all $t > 0$ large, we have

$$\sup_{b \in \mathcal{C}_B} \int_0^t R(\sigma(s, \alpha, b), \alpha(b)(s), b(s))\, \mathrm{d}s + p \cdot \sigma(t, \alpha, b) \leq -t \max_{i=1}^{3} h(p_i) - ct,$$

for some small $c > 0$. We claim that this cannot hold even for the constant control $b = 0$. First, we note that the terminal cost can be interpreted as a kind of running cost, in the sense that

$$p \cdot \sigma(t, \alpha, b) = \int_0^t p \cdot f(\sigma(s, \alpha, b), \alpha(b)(s), b(s))\, \mathrm{d}s,$$

since the starting state is $\sigma(0) = 0$. Let $E := \{t > 0 \mid R(\sigma(t, \alpha, 0), 0, 0) \geq 99\}$. Then if

$t \in E$,

$$\begin{aligned} J(s) &:= R(\sigma(s,\alpha,b),\alpha(b)(s),b(s)) + p \cdot f(\sigma(s,\alpha,b),\alpha(b)(s),b(s)) \\ &\geq 99 + \min_{a \in A} [100|a| + 4a \cdot p] \\ &\geq 99 - \frac{1}{50} h(|p|) \\ &\geq 99 - \frac{\sqrt{3}}{50} \max_{i=1}^{3} h(p_i) - 4(\sqrt{3}-1) \\ &\geq 95 - \frac{\sqrt{3}}{50} \max_{i=1}^{3} h(p_i), \end{aligned}$$

where we use the fact that **I** can only move at much slower the speed away from the highways.

On the other hand, we say that an interval $[t_0, t_1]$ stays close to a highway if there is a line ℓ in $\mathcal{L}$ such that, for every $s \in [t_0, t_1]$, the line ℓ is the closest line in $\mathcal{L}$ to $\sigma(s, \alpha, b)$. In any such interval $[t_0, t_1]$, we have

$$\int_{t_0}^{t_1} J(s) \,\mathrm{d}s \geq -|p| - (t_1 - t_0) \max_{i=1}^{3} h(p_i),$$

where the first term accounts for the (constant-sized) movement in the direction orthogonal to ℓ, and the second term accounts for the movement in the direction parallel to ℓ.

Write $U := [0,t] \setminus E$. Given $t_0, t_1 \in U$, we write $t_0 \sim t_1$ iff $[t_0, t_1]$ stays close to a highway. Let $I_1, I_2, \ldots, I_n$ denote the equivalence classes of $U/\sim$, ordered by the usual order on $\mathbb{R}$. We claim that $n \leq 40t$. Indeed, $[\max I_i, \min I_{i+1}] \subseteq E$, and $\min I_{i+1} - \max I_i \geq \frac{1}{40}$, since the distance $\frac{1}{100}$ neighborhoods of highways are at least distance $\frac{1}{5}$ apart, and the speed limit in E is at most 8.

Putting everything together, we write

$$\begin{aligned}
&\int_0^t R(\sigma(s,\alpha,b),\alpha(b)(s),b(s)) + p\cdot f(\sigma(s,\alpha,b),\alpha(b)(s),b(s))\ \mathrm{d}s\\
&\qquad = \int_0^{\min I_1} J(s)\ \mathrm{d}s + \sum_{j=1}^{n-1}\left(\int_{\min I_j}^{\max I_j} J(s)\ \mathrm{d}s + \int_{\max I_j}^{\min I_{j+1}} J(s)\ \mathrm{d}s\right)\\
&\qquad\quad + \int_{\min I_n}^{\max I_n} J(s)\ \mathrm{d}s + \int_{\max I_n}^{t} J(s)\ \mathrm{d}s\\
&\qquad \geq -\frac{\sqrt{3}\min I_1}{50}\max_{i=1}^{3} h(p_i) + 95(\min I_1)\\
&\quad + \sum_{j=1}^{n-1} -|p| - |I_j|\max_{i=1}^{3} h(p_i) - \frac{\sqrt{3}(\min I_{j+1} - \max I_j)}{50}\max_{i=1}^{3} h(p_i) + 95(\min I_{j+1} - \max I_j)\\
&\qquad\quad - |p| - |I_n|\max_{i=1}^{3} h(p_i) - \frac{\sqrt{3}(t - \max I_n)}{50}\max_{i=1}^{3} h(p_i)\\
&\qquad \geq -t\max_{i=1}^{3} h(p_i),
\end{aligned}$$

where the sum telescopes and we use the fact that

$$2|p| \leq \frac{1}{40}\left(95 + \left(1 - \frac{\sqrt{3}}{50}\right)\max_{i=1}^{3} h(p_i)\right).$$

$\square$

CHAPTER 4

STOCHASTIC HOMOGENIZATION FOR THE G EQUATION

4.1 Introduction

We consider the behavior, as $\varepsilon \to 0^+$, of the family $\{u_\varepsilon\}_{\varepsilon>0}$ of solutions to the G equation,

$$\begin{cases} D_t u^\varepsilon(t,x) - |D_x u^\varepsilon(t,x)| + V(\varepsilon^{-1}x) \cdot D_x u^\varepsilon(t,x) = 0 & \text{in } \mathbb{R}_{>0} \times \mathbb{R}^d \\ u^\varepsilon(0,x) = u_0(x) & \text{in } \mathbb{R}^d, \end{cases} \tag{4.1}$$

where $d \geq 2$, $V \in C^{1,1}(\mathbb{R}^d; \mathbb{R}^d)$ is a divergence-free vector field and $u_0 \colon \mathbb{R}^d \to \mathbb{R}^d$ is Lipschitz. The level sets of u^ε model a flame front which expands at unit speed in the normal direction while being advected by V, which models the wind velocity. When compared with homogenization of other Hamilton–Jacobi equations, the main difficulty with the G equation is that, since we do not assume that $\|V\|_{L^\infty} < 1$, the equation may not be coercive. On the other hand, if $\mathbb{E}[V] = 0$, then the equation is still "coercive on average", so we can recover some large-scale controllability.

Cardaliaguet–Souganidis [13] proved, under the assumption that the environment V is stationary ergodic, that the equation homogenizes; i.e. we have the locally uniform convergence of solutions $u^\varepsilon \to \bar{u}$ as $\varepsilon \to 0$ almost surely, where $\bar{u}$ is the solution to the effective equation

$$\begin{cases} D_t \bar{u}(t,x) = \overline{H}(D_x \bar{u}(t,x)) & \text{in } \mathbb{R}_{>0} \times \mathbb{R}^d \\ \bar{u}(0,x) = u_0(x) & \text{in } \mathbb{R}^d, \end{cases} \tag{4.2}$$

and $\overline{H} \colon \mathbb{R}^d \to \mathbb{R}$ is a degree 1 positively homogeneous coercive function, called the effective Hamiltonian.

Under the purely qualitative ergodicity assumption, there is no hope of proving a rate at which u^ε converges to $\bar{u}$. In this paper, we make the stronger assumption that V has unit range of dependence, which is a continuous analogue of i.i.d. Using this more quantitative assumption, we classify regions as "good" if the controllability estimate from Cardaliaguet–Souganidis holds locally, in a quantitative sense. Then, we use percolation estimates to construct paths which stay inside the good regions. Our main result is the following rate of homogenization.

Theorem 5. *Let $\mathbb{P}$ be a probability measure on $C^{1,1}(\mathbb{R}^d;\mathbb{R}^d)$ which has unit range of dependence and is $\mathbb{R}^d$-translation invariant. Assume that $\operatorname{div} V = 0$ and $\|V\|_{C^{1,1}} \le L$ almost surely for some $L > 0$. Then there are constants $C(d,L) > 1 > c(d,L) > 0$ and a random variable T_0, with $\mathbb{E}[\exp(c\log^{3/2} T_0)] \le C$, such that*

$$|u^\varepsilon(t,x) - \bar{u}(t,x)| \le C\|u_0\|_{C^{0,1}}(t\varepsilon)^{1/2}\log^3(\varepsilon^{-1}t)$$

for all $t \ge \varepsilon T_0$ and $|x| \le t$.

Note that, in particular, the bound on T_0 implies $\mathbb{E}[T_0^n] \le C^{2^n}$ for all $n \in \mathbb{N}$. In the interest of completeness, we record the almost-sure version of the rate of homogenization, which follows immediately from Theorem 5 and the Borel–Cantelli lemma.

Corollary 10. *Under the same assumptions as Theorem 5, there is a constant $C(d,L) > 0$ such that, for all $T > 0$,*

$$\limsup_{\varepsilon\to 0^+} \sup_{(t,x)\in[0,T]\times B_T} \frac{|u^\varepsilon(t,x) - \bar{u}(t,x)|}{(T\varepsilon)^{1/2}\log^3(\varepsilon^{-1}T)} \le C\|u_0\|_{C^{0,1}}$$

almost surely.

As an application of the quantitative rate, we prove that the effective Hamiltonian depends continuously on the law of the environment. We first recall the definition of the Lévy–Prokhorov metric, which quantifies weak convergence of probability measures.

Definition 11. The Lévy–Prokhorov metric $\pi\colon \mathcal{P}(\Omega)\times\mathcal{P}(\Omega)\to\mathbb{R}_{\geq 0}$ on probability measures μ,ν is given by

$$\pi(\mu,\nu) := \inf\{\varepsilon>0 \mid \mu(A)\leq \nu(A+B_\varepsilon)+\varepsilon \text{ and } \nu(A)\leq \mu(A+B_\varepsilon)+\varepsilon \text{ for all Borel } A\subseteq\Omega\},$$

where B_ε denotes the ball in Ω of radius ε centered at 0.

Theorem 6. *Let $\mathbb{P},\mathbb{P}^*$ be probability measures on $C^{1,1}(\mathbb{R}^d;\mathbb{R}^d)$ which are $\mathbb{R}^d$-translation invariant, have unit range of dependence, and satisfy* $\operatorname{div} V=0$ *and* $\|V\|_{C^{1,1}}\leq L$*, $\mathbb{P}$-almost surely and $\mathbb{P}^*$-almost surely. Let $\overline{H}^*$ and $\overline{H}$ be the effective Hamiltonians for $\mathbb{P}^*$ and $\mathbb{P}$ respectively. Assume that $\pi(\mathbb{P},\mathbb{P}^*)\leq\varepsilon$. Then*

$$|\overline{H}(p)-\overline{H}^*(p)|\leq |p|\varepsilon^{1/3}\log^3\varepsilon^{-1}.$$

In particular, the effective Hamiltonian $\overline{H}$ is a weakly-star continuous function of the law $\mathbb{P}$ of the environment, where the topology on $\overline{H}$ is given by uniform convergence on compact sets.

4.1.1 Prior work

Since Lions–Papanicolau–Varadhan [33] proved qualitative homogenization of coercive Hamilton–Jacobi equations in a periodic environment, there has been a rich body of work (see Tran [35]) studying homogenization of Hamilton–Jacobi equations in both periodic and stochastic environments.

In a periodic environment, when the Hamiltonian is coercive, Capuzzo-Dolcetta–Ishii [11] gave the first proof of a quantitative rate $O(\varepsilon^{1/3})$ of homogenization. Although the G equation may not be coercive when $\|V\|_{L^\infty}\geq 1$, it has a particularly simple optimal control formulation, making it an ideal first candidate to study homogenization of noncoercive equations. Cardaliaguet–Nolen–Souganidis [12] proved homogenization, along with a quantitative rate $O(\varepsilon^{1/3})$, for the G equation in a periodic environment.

In a stochastic environment, the situation is more complicated. An example of Ziliotto [37] shows that there exist coercive Hamiltonians and stationary ergodic environments in which homogenization does not hold. Feldman–Souganidis [21] generalized this example by showing that for any Hamiltonian with a strict saddle point, there exists a stationary ergodic environment in which homogenization does not hold. On the other hand, when the Hamiltonian is coercive and convex, Souganidis [34] proved that homogenization holds in a stationary ergodic environment. Later, under the stronger assumption that the environment has finite range of dependence, Armstrong–Cardaliaguet–Souganidis [3] proved a quantitative rate of homogenization when the Hamiltonian is coercive and level-set convex.

In the case of the G equation, Nolen–Novikov [32] used a geometric argument in dimension $d = 2$ to prove qualitative homogenization when the environment is stationary ergodic and satisfies an additional integrability condition. Later, Cardaliaguet–Souganidis [13] improved this result to qualitative homogenization in a stationary ergodic environment for all dimensions $d \geq 2$, without assuming the integrability condition. In the case where the vector field $V = V(t, x)$ depends on time as well as space, Burago–Ivanov–Novikov [7] proved homogenization using a completely different, mostly deterministic argument.

In this paper, we combine ideas from [3] and [13] with percolation theory techniques to find a quantitative rate of homogenization for the G equation, under the stronger finite range of dependence assumption.

4.1.2 Assumptions

We now explicitly specify the assumptions in Theorems 5 and 6. Let $\mathbb{P}$ be a probability measure over $\Omega := C^{1,1}(\mathbb{R}^d; \mathbb{R}^d)$ with $d \geq 2$. We write $V\colon \Omega \to \Omega$ to denote the identity random variable, so V has distribution $\mathbb{P}$. We assume that, $\mathbb{P}$-almost surely, V is divergence-free and $\|V\|_{C^{1,1}} \leq L$, where $L > 0$ is a deterministic constant. Given $A \subseteq \mathbb{R}^d$, we write

$\mathcal{G}(A)$ to denote the σ-algebra generated by V restricted to A. That is, $\mathcal{G}(A)$ is the smallest σ-algebra for which the random variables $V(x)$ are measurable for every $x \in A$. We assume that $\mathbb{P}$ has unit range of dependence, which means that if $A, B \subseteq \mathbb{R}^d$ with $\operatorname{dist}(A, B) > 1$, then the σ-algebras $\mathcal{G}(A)$ and $\mathcal{G}(B)$ are independent. We also assume that $\mathbb{P}$ is $\mathbb{R}^d$-translation invariant, which means that $V(\cdot + x)$ has the same distribution as V for any $x \in \mathbb{R}^d$. Note that this assumption is not much different than $\mathbb{Z}^d$-translation invariance for our purposes; we can simply add a random vector in $[0,1]^d$ to go from $\mathbb{Z}^d$-invariance to full $\mathbb{R}^d$-invariance, which will not have an effect on our results. Finally, we assume that $\mathbb{E}[V] = 0$ (this is the same as $\mathbb{E}[V(0)] = 0$ by $\mathbb{R}^d$-translation invariance).

4.1.3 Structure of the paper

In Section 4.2, we collect some estimates from percolation theory. The bulk of the paper is Section 4.3, in which we prove homogenization of the metric problem, which we later apply via the optimal control formulation. Section 4.3 has three main parts. First, we prove a controllability estimate for the metric problem, using results of Cardaliaguet–Souganidis [13] and percolation estimates from Section 4.2. With the controllability estimate in hand, we split the difference between the microscopic and macroscopic metric problems into two pieces; the random fluctuations and the nonrandom scaling bias. We handle the first using a martingale argument originally due to Kesten [26], and the second with an argument adapted from Alexander [1]. Finally, we apply our estimates for the metric problem in Section 4.4 to deduce our main results.

4.1.4 Notation

Throughout the paper, $C > 1 > c > 0$ will denote constants which may depend on the dimension d and the bound L on $\|V\|_{C^{1,1}}$, but may vary from line to line. We write $B_r := \{x \in \mathbb{R}^d \mid |x| \leq r\}$ for the Euclidean ball of radius r centered at the origin. We write $\operatorname{cl}(\cdot)$

(resp. $\operatorname{int}(\cdot)$) to denote the closure (resp. interior) of a subset of some topological space. The Euclidean distance between two subsets $E, F \subseteq \mathbb{R}^d$ is given by $\operatorname{dist}(E, F) := \inf\{|x - y| \mid (x, y) \in U \times V\}$. The Hausdorff distance between two subsets $E, F \subseteq \mathbb{R}^d$ is given by

$$\operatorname{dist}_H(E, F) := \inf\{\varepsilon > 0 \mid V \subseteq U + B_\varepsilon \text{ and } U \subseteq V + B_\varepsilon\}.$$

For a random variable X, we use the subscript X_ω to denote the value of X at $\omega \in \Omega$.

4.1.5 Acknowledgements

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4.2 Percolation estimates

In this section, we prove a few well-known estimates from supercritical percolation theory, but with a finite range of dependence assumption. This context is similar to the usual one, where the environment is i.i.d., except that the underlying probability space lacks a product structure.

Let $d \geq 2$ and let $G\colon \mathbb{Z}^d \to \{0, 1\}$ be a random function on $\mathbb{Z}^d$ which has finite range $C_{\text{dep}} > 0$ of dependence, which means that the σ-algebras induced by the values of G on sets that are Euclidean distance at least C_{dep} apart are independent. We assume that G is $\mathbb{Z}^d$-translation invariant, i.e., $G(\cdot + v)$ has the same distribution as $G(\cdot)$ for all $v \in \mathbb{Z}^d$. The function G models site percolation, where a site x is open if $G(x) = 1$ and closed otherwise.

We put edges on $\mathbb{Z}^d$ between nearest neighbors in the ℓ^∞ metric. We write $\operatorname{dist}(\cdot, \cdot)$ to indicate the graph distance, and refer to maximal connected components on which G is constant as *clusters*. Clusters composed of open (resp. closed) sites are called open (resp.

closed) clusters. Let $p := \mathbb{P}[G(0) = 1]$ be the probability that a site is open (which is the same for every site, by $\mathbb{Z}^d$-translation invariance). We write $\mathcal{Q}_R(x) \subseteq \mathbb{Z}^d$ to denote the axis-aligned cube of side length $2R$ centered at x.

Lemma 4.2.1. *Let $S \subseteq \mathbb{Z}^d$ be a finite set. Define the closed sites connected to S by*

$$\mathrm{closed}(S) := \left\{x \in \mathbb{Z}^d \mid \textit{there is a path of closed sites from } x \textit{ to a site in } S\right\}.$$

For any $\varepsilon > 0$ there is $p_0(d, C_{dep}, \varepsilon) < 1$ and $C(d, C_{dep}, \varepsilon) > 0$ such that if $p > p_0$ then

$$\mathbb{P}\left[|\,\mathrm{closed}(S)| > \varepsilon|S| + \delta\right] \leq C \exp(-C^{-1}\delta)$$

for every $\delta \geq 0$.

Note in particular that $x \in \mathrm{closed}(S)$ implies that x is a closed site.

Proof. Let $T \subseteq \mathbb{Z}^d$ be a finite set of n vertices. If $T \subseteq \mathrm{closed}(S)$, then every site in T is closed and every cluster in T contains a point in S. For fixed n, the number of sets T which satisfy the latter condition is at most $(3^d + 1)^{|S|+2n}$ (we can encode a spanning tree of T with an alphabet of $3^d + 1$ letters). For a fixed set T, we see that

$$\mathbb{P}[\text{every site in } T \text{ is closed}] \leq (1-p)^{\left\lfloor n/(2C_{\mathrm{dep}})^d \right\rfloor},$$

since we can choose at least $\left\lfloor n/(2C_{\mathrm{dep}})^d \right\rfloor$ sites in T which are far enough to be independent. From the union bound, we have

$$\mathbb{P}[\text{there is a set } T \subseteq \mathrm{closed}(S) \text{ with } n \text{ vertices}] \leq (3^d+1)^{|S|+2n}(1-p)^{\left\lfloor n/(2C_{\mathrm{dep}})^d \right\rfloor}.$$

Now let $n := \lceil \varepsilon |S| + \delta \rceil$ and choose $(1 - p_0)$ small enough so that

$$(1-p_0)^{\varepsilon/(2C_{\text{dep}})^d} (3^d+1)^{1+2\varepsilon} < 1$$

and

$$(1-p_0)^{1/(2C_{\text{dep}})^d} (3^d+1)^{2} < 1$$

and the claim follows. □

The next lemma has nothing to do with the percolation environment; it is simply a property of the graph structure of $\mathbb{Z}^d$. It follows from a topological property of $\mathbb{R}^d$ known as unicoherence (see Kuratowski [27] or Dugundji [17]). In order to state the lemma, we need to define the boundary of a subset of $E \subseteq \mathbb{Z}^d$. Because $\mathbb{Z}^d$ is discrete, there are two choices for our definition.

Definition 4.2.2. The inner (resp. outer) boundary of E, denoted $\partial^- E$ (resp. $\partial^+ E$), is the set

$$\partial^- E := \{x \in E \mid \operatorname{dist}(x, \mathbb{Z}^d \setminus E) = 1\} \qquad (\text{resp. } \partial^+ E := \{x \in \mathbb{Z}^d \setminus E \mid \operatorname{dist}(x, E) = 1\}).$$

Lemma 4.2.3. *Let $\mathcal{Q}_R$ be any cube of side length $2R$ and let $\mathfrak{C} \subseteq \mathcal{Q}_R$ be a connected set. Let $\mathfrak{D} \subseteq \mathcal{Q}_R \setminus \mathfrak{C}$ be a connected component of $\mathcal{Q}_R \setminus \mathfrak{C}$. Then the inner (resp. outer) boundary of $\mathfrak{D}$ is connected.*

Proof. This is part (i) of Lemma 2.1 from Deuschel–Pisztora [16]. The proof is a standard application of Urysohn's lemma. □

The next lemma shows that, with high probability, there is a large open cluster which is near every site.

Lemma 4.2.4. *Let $n, R > 0$ and consider $\mathcal{Q}_R$, a cube of side length $2R$. Let E_n be the event that there exists an open cluster $\mathfrak{C} \subseteq \mathcal{Q}_{R+n}$ such that every connected component of $\mathcal{Q}_{R+n} \setminus \mathfrak{C}$ which intersects $\mathcal{Q}_R$ is of size at most n. Then there are constants $p_0(d, C_{dep}) < 1$ and $C = C(d, C_{dep}) > 0$ such that if $p > p_0$ then*

$$\mathbb{P}[E_n] \geq 1 - CR^d \exp(-C^{-1} n^{(d-1)/d}).$$

Proof. Fix $n, R > 0$ as in the statement. Work in the event that every closed cluster in $\mathcal{Q}_{R+n}$ has size less than $C^{-1} n^{(d-1)/d}$, where $C = C(d)$ comes from the isoperimetric constant (chosen later in the proof). By Lemma 4.2.1 (applied to each site in $\mathcal{Q}_{R+n}$ individually, with (say) $\varepsilon = 1$), this event has probability at least $1 - CR^d \exp(-C^{-1} n^{(d-1)/d})$.

Let $\mathfrak{C}$ be the largest open cluster (breaking ties arbitrarily) in $\mathcal{Q}_{R+n}$. As long as $C \geq 1$, it follows from the isoperimetric inequality that there is an open path between opposite faces of $\mathcal{Q}_{R+n}$, so $|\mathfrak{C}| \geq 2(R+n) + 1 > n$.

Let $\mathfrak{D}$ be any connected component of $\mathcal{Q}_{R+n} \setminus \mathfrak{C}$ which intersects $\mathcal{Q}_R$. The inner boundary of $\mathfrak{D}$ is composed of two kinds of sites: (i) those bordering $\mathfrak{C}$ and (ii) those in the inner boundary of $\mathcal{Q}_{R+n}$. The sites of type (i) are all closed (by definition of $\mathfrak{C}$). We claim that there are no sites of type (ii). Indeed, if there was a site of type (ii) then we could follow the inner boundary of $\mathfrak{D}$ (it is connected by Lemma 4.2.3) from the inner boundary of $\mathcal{Q}_{R+n}$ to a site in $\mathcal{Q}_R$, which would yield a path of length more than $n \geq n^{(d-1)/d}$ of type (i) (and hence closed) sites, contradicting our assumption.

Since the inner boundary of $\mathfrak{D}$ is connected and composed entirely of closed sites, it has size less than $C^{-1} n^{(d-1)/d}$. The isoperimetric inequality then shows that either $\mathfrak{D}$ or $\mathcal{Q}_{R+n} \setminus \mathfrak{D}$ has size at most n. Since $\mathfrak{C} \subseteq \mathcal{Q}_{R+n} \setminus \mathfrak{D}$ and $|\mathfrak{C}| > n$, it follows that $|\mathfrak{D}| \leq n$ as desired. $\square$

4.3 The metric problem

We now shift our focus to the metric problem associated with the G equation, which comes from the optimal control formulation. As we will see, homogenization of solutions to the G equation is implied by convergence of the associated metric to its large-scale limit.

Given $t > 0$ and a measurable function $\alpha\colon [0,t] \to B_1$, define the *controlled path* $X_x^\alpha\colon [0,t] \to \mathbb{R}^d$ to be the solution to the initial-value problem

$$\begin{cases} \dot{X}_x^\alpha = \alpha + V(X_x^\alpha) \\ X_x^\alpha(0) = x. \end{cases} \tag{4.3}$$

As in Barles [5], for each $x \in \mathbb{R}^d$, define the *reachable set at time t* by

$$\mathcal{R}_t(x) := \{y \in \mathbb{R}^d \mid \exists\, \alpha\colon [0,t] \to B_1 \text{ such that } X_x^\alpha(t) = y\}. \tag{4.4}$$

Note that this definition still makes sense for $t < 0$, if we interpret $[0,t]$ as $[t,0]$ and treat the initial-value problem as a terminal-value problem. Equivalently, if $t < 0$ then we define $\mathcal{R}_t(x)$ as the reachable set at time $|t|$ for the negated vector field $-V$. For convenience, we also define the sets

$$\mathcal{R}_t^-(x) := \bigcup_{0 \le s \le t} \mathcal{R}_s(x)$$

for $t \geq 0$ and

$$\mathcal{R}_t^+(x) := \bigcup_{t \le s \le 0} \mathcal{R}_s(x)$$

for $t \leq 0$. Define the *first passage time*

$$\theta(x,y) := \inf\{t \geq 0 \mid y \in \mathcal{R}_t(x)\}. \tag{4.5}$$

Finally, if $E \subseteq \mathbb{R}^d$ is a set, we define

$$\mathcal{R}_t(E) = \bigcup_{e \in E} \mathcal{R}_t(e),$$

and we do the same for $\mathcal{R}_t^-$ and $\mathcal{R}_t^+$.

4.3.1 Controllability

First, we import a few results from Cardaliaguet–Souganidis [13], which hold in the more general ergodic setting.

Lemma 4.3.1 (Cardaliaguet–Souganidis [13], Lemma 4.2)**.** *There is a deterministic constant $\beta = \beta(d) > 0$ such that $\mathcal{R}_t(x) \geq \beta |t|^d$ (and hence the same holds for $\mathcal{R}_t^-(x)$ and $\mathcal{R}_t^+(x)$).*

Theorem 4.3.1 (Cardaliaguet–Souganidis [13], Theorem 4.1)**.** *For every $\varepsilon > 0$, there is a random variable $0 < T(\varepsilon) < \infty$ such that*

$$\theta(x, y) \leq T(\varepsilon) + \varepsilon|x| + (1 + \varepsilon)|x - y| \qquad \text{for all } x, y \in \mathbb{R}^d$$

holds almost surely.

For our quantitative purposes, we would like to have a version of Theorem 4.3.1 where the distribution of the error term $T(\varepsilon)$ has exponential tail bounds. Our strategy is to partition space into cubes where a controllability estimate holds nearby with high probability. Then, using the percolation estimates, we can construct paths which mostly stay in these cubes.

Lemma 4.3.2. *For each $0 < p < 1$ there is $C = C(d, p, L) \in \mathbb{N}$ such that*

$$\mathbb{P}\left[\sup_{x,y \in B_{\sqrt{d}}} \theta(x, y) \leq C\right] \geq p.$$

Proof. By Theorem 4.3.1, the statement of the lemma holds when C is also allowed to depend on the distribution of V, since $T(\varepsilon)$ is almost surely finite. We claim that we can remove this dependence. Indeed, if not, then there would be some sequence $\{\mathbb{P}_n\}_{n\in\mathbb{N}}$ of probability measures which satisfy the same assumptions as $\mathbb{P}$, as well as

$$\mathbb{P}_n\left[E(n)\right] < p,$$

where $E(C)$ is the event that $\theta(x,y) \le C$ for all $x, y \in \overline{B_{\sqrt{d}}}$. Write $\widetilde{\theta}(x,y)$ to denote the first-passage time where the control α is constrained to lie in $B_{1/2}$ instead of B_1 as in the definition (4.4) of the reachable set. Similarly, define $\widetilde{E}(C)$ to be the event that $\widetilde{\theta}(x,y) \le C$ for all $x, y \in \overline{B_{\sqrt{d}}}$, noting that

$$\mathrm{cl}(\widetilde{E}(C)) \subseteq \mathrm{int}(E(C)),$$

where $\mathrm{cl}(\cdot)$ and $\mathrm{int}(\cdot)$ denote the closure and interior respectively, taken in the space $C^1(\mathbb{R}^d;\mathbb{R}^d)$. Taking a subsequence (not relabelled for brevity), we find a probability measure $\mathbb{P}_\infty$, which satisfies all the same assumptions as $\mathbb{P}$, such that $\mathbb{P}_n|_{B_R} \overset{*}{\rightharpoonup} \mathbb{P}_\infty|_{B_R}$ in the space of probability measures on $C^1(B_R;\mathbb{R}^d)$ for every $R > 0$. Then, for any $C > 0$,

$$1 > p \ge \limsup_{n\to\infty} \mathbb{P}_n[E(C)] \ge \mathbb{P}_\infty[\widetilde{E}(C)],$$

which violates Theorem 4.3.1 (noting that Theorem 4.3.1 holds just as well for $\widetilde{\theta}$ as for θ by a change of variables). $\square$

Lemma 4.3.3. *For each $0 < p < 1$, there exists a constant $C = C(d,p,L) > 0$ such that the* *function* $G\colon \mathbb{Z}^d \to \{0,1\}$*, defined by*

$$G(v) = \begin{cases} 1 & \theta(x,y) \le C \text{ for all } x, y \in B_{\sqrt{d}}(v) \\ 0 & \text{otherwise,} \end{cases}$$

is Z^d*-translation invariant with finite range of dependence* $C_{dep} = C_{dep}(d,p,L) > 0$ *and* $\mathbb{P}[G(0) = 1] \ge p$*.*

In other words, G is an environment in which all of our percolation estimates apply.

Proof. By Lemma 4.3.2, there is $C > 0$ such that $\mathbb{P}[G(0) = 1] \ge p$. The $\mathbb{Z}^d$-translation invariance of G follows from that of $\mathbb{P}$. Finite range of dependence follows from the fact that the value of $G(v)$ depends only on controlled paths starting in $B_{\sqrt{d}}(v)$ which run for time at most C. Since $\|V\|_{L^\infty} \le L$, these paths stay inside of $\overline{B_{\sqrt{d}+(L+1)C}(v)}$, so $C_{\mathrm{dep}} := 2(\sqrt{d} + (L+1)C) + 1$ satisfies the claim. □

Definition 4.3.4. To translate between $\mathbb{Z}^d$ and $\mathbb{R}^d$, for each set $E \subseteq \mathbb{Z}^d$ we introduce the "solidification"

$$\mathrm{solid}(E) := E + \left[-\frac{1}{2}, \frac{1}{2}\right]^d.$$

We are now in a position to prove our improved controllability estimate. We refer to clusters as before, using the same notion of adjacency. We write $W_R := \mathrm{solid}(\mathcal{Q}_R)$

Theorem 4.3.2. *There is a constant* $C = C(d,L) > 0$ *such that, for each* $R \ge 1$*, the "extra waiting time"*

$$\mathcal{E}(R) := \sup_{x,y \in W_R} \theta(x,y) - (C + C|x-y|)$$

satisfies

$$\mathbb{P}[\mathcal{E}(R) > n] \le CR^d \exp(-C^{-1}n).$$

Proof. We partition $\mathbb{R}^d$ into cubes of side length 1, centered at points in $\mathbb{Z}^d$. Given

Lemma 4.3.3, there is a constant $C = C(d,p,L) > 0$ such that $\mathbb{P}[G(v) = 1] \geq p$ for some $p > p_0(d, C_{\text{dep}}, \varepsilon)$, where $\varepsilon = 1$. For $v \in \mathbb{Z}^d$, we say that the site v is open if $G(v) = 1$ and closed otherwise. We say that a point $x \in \mathbb{R}^d$ lies near an open site if there is some open $v \in \mathbb{Z}^d$ such that $x \in \operatorname{solid}(\{v\})$. Let $S = \lceil R \rceil$ and $x, y \in W_S$. For convenience, we will prove that $\mathbb{P}[\mathcal{E}(R) > Cn] \leq CR^d \exp(-C^{-1}n)$; this easily implies the original claim by changing C to C^2.

Step 1. We show that we can assume without loss of generality that x and y lie in the same open cluster, by which we mean that there is an open cluster $\mathfrak{C}$ such that $x, y \in \operatorname{solid}(\mathfrak{C})$. Indeed, suppose they lie in different open clusters. Then by Lemma 4.2.4, we have an open cluster $\mathfrak{C} \subseteq \mathcal{Q}_{S+n^d}$, which depends on the environment, such that

$$\mathbb{P}\Big[\text{every connected } \mathfrak{D} \subseteq \mathcal{Q}_{S+n^d} \setminus \mathfrak{C} \text{ which intersects } \mathcal{Q}_S \text{ satisfies } |\mathfrak{D}| \leq n^d\Big]$$
$$\geq 1 - CS^d \exp(-C^{-1} n^{d-1}).$$

Working in this event, let $\mathfrak{D}$ be the connected component of $\mathcal{Q}_{S+n^d} \setminus \mathfrak{C}$ whose solidification $\operatorname{solid}(\mathfrak{D})$ contains x. Then Lemma 4.3.1, along with the fact that $\mathcal{R}_t^-(x)$ is connected, shows that $\mathcal{R}_t^-(x) \setminus \operatorname{solid}(\mathfrak{D}) \neq \emptyset$ if $\beta|t|^d \geq n^d$ and $t > 0$. Since the controlled paths are continuous, there is some $t \leq n\beta^{-1/d} + 1$ such that there is some $z \in \mathcal{R}_t^-(x) \cap \operatorname{solid}(\mathfrak{C})$. Since the loss in probability and travel time can be controlled by enlarging C, we may as well assume that x was z to begin with. Repeating the same argument for y (except running time backwards, so $t < 0$ and we use $\mathcal{R}_t^+(y)$ instead) shows that we may assume that x and y lie in the same open cluster, $\mathfrak{C}$.

Step 2. To show that $\theta(x,y) - C|x-y| \leq Cn$, we will build a "skeleton" of points $x = x_0, x_1, \ldots, x_k = y$ which all lie near open sites and satisfy $|x_{i+1} - x_i| \leq \sqrt{d}$ for each $0 \leq i < k$. Then, by connecting the points with paths given by Lemma 4.3.3, we can build a controlled path of length at most Ck which follows the skeleton. It remains to show that

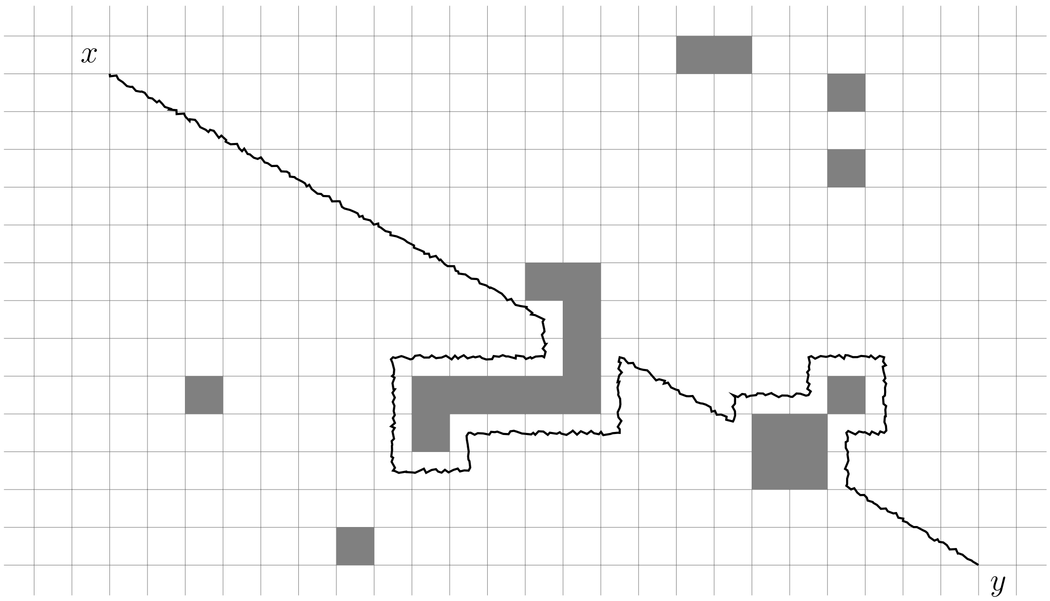

Figure 4.1: An example of our controlled path from x to y; cubes corresponding to closed sites are shaded

we can build such a skeleton with $k \leq C(|x-y|+n)$. Let

$$A := \{v \in \mathbb{Z}^d \mid \text{solid}(\{v\}) \cap \overline{xy} \neq \emptyset\}$$

be the set of centers of cubes which intersect the line segment connecting x and y. Note that $|A| \leq 2^d(1+|x-y|)$. By Lemma 4.2.1, we can choose $\varepsilon = 1$ to work in the event that $\text{closed}(A) \leq |A| + n$.

Our strategy is to go from x to y in a straight line, taking necessary detours around closed clusters. We use Lemma 4.2.1 to bound the total length of our detour.

We can build the skeleton iteratively. Start with $x_0 = x$, and assume we have built the skeleton up to x_i, for $i \geq 0$. We maintain the invariant that, at the end of each step, x_i lies on the line segment $\overline{xy}$, and it is closer to y than any other x_j which lies on $\overline{xy}$ with $0 \leq j < i$. There are three cases.

1. If $|y - x_i| \leq \sqrt{d}$, define $x_{i+1} := y$ and finish.

2. If $z := x_i + \sqrt{d}\frac{y-x}{|y-x|}$ lies near an open site, define $x_{i+1} := z$ and continue to the next step.

3. Otherwise, let $\widetilde{x}$ be the point on $\overline{x_i z}$ which lies near an open site and is closest to z. Then $\widetilde{x}$ also lies near a closed site, which is part of some connected component $\mathfrak{F}$ of $\mathcal{Q}_S \setminus \mathfrak{C}$. By Lemma 4.2.3, the outer boundary of $\mathfrak{F}$ is connected. Since $\widetilde{x} \in \overline{xy}$, we see that $\partial^- \mathfrak{F} \subseteq \text{closed}(A)$. Besides, since every vertex in $\mathbb{Z}^d$ has degree $3^d - 1$, we have $|\partial^+ \mathfrak{F}| \leq 3^d |\partial^- \mathfrak{F}|$. So, let $p_1, \ldots, p_\ell \in \mathbb{Z}^d$ be a path along the outer boundary of $\mathfrak{F}$, where $\text{solid}(\{p_1\}) \ni \widetilde{x}$ and $p_\ell \in A$ is a point on the outer boundary of $\mathfrak{F}$ which maximizes $p_\ell \cdot (y - x)$. Finally, we extend our path by setting $x_{i+1} := \widetilde{x}$ and $x_{i+j+1} := p_j$ for each $1 \leq j \leq \ell$, and set $x_{i+\ell+2}$ to be a point on the line segment $\overline{xy}$ which lies in $\text{solid}(\{p_\ell\})$.

It remains to analyze the length of this skeleton by looking at each of the three cases. The first case happens at most once, so it can be ignored. The second case reduces the distance $|x_i - y|$ by $\sqrt{d}$ and the third case does not increase the distance $|x_i - y|$, so there can be at most $\frac{|x-y|}{\sqrt{d}}$ points in the skeleton coming from the second case. The third case adds $\ell + 2$ points, where $\ell \leq |\partial^+ \mathfrak{F}|$. Since we finish an instance of the third case at a point as close to y as possible on the segment $\overline{xy}$, we never witness the same cluster $\mathfrak{F}$ twice in different instances of the third case. Therefore the third case adds at most

$$C|\,\text{closed}(A)| \leq C(|A| + n) \leq C(|x - y| + n)$$

points to our skeleton. □

4.3.2 *Random fluctuations in first passage time*

Next, we consider how much $\theta(0, y)$ deviates from its expectation. Our proof will follow roughly the same path as the proof of Proposition 4.1 from Armstrong–Cardaliaguet–

Souganidis [3], with some modifications which are made possible by the controllability estimate.

To get started, we introduce a "guaranteed" version of first passage time. For any $\rho > 0$, define the ρ-guaranteed reachable set recursively by

$$\mathcal{R}_t^\rho(x) := \begin{cases} \mathcal{R}_t^-(x) & \text{if } t < \rho \\ \mathcal{R}_\rho^-(\mathcal{R}_{t-\rho}^\rho(x)) \cup (\mathcal{R}_{t-\rho}^\rho(x) + \overline{B_1}) & \text{otherwise.} \end{cases}$$

The ρ-guaranteed reachable set is similar to the reachable set, except that we enforce expansion at a rate of at least $1/\rho$ in a certain discrete sense. We similarly define the ρ-guaranteed first passage time

$$\theta^\rho(x,y) = \min\{t \geq 0 \mid y \in \mathcal{R}_t^\rho(x)\}.$$

Note that the ρ-guaranteed first passage time coincides with the usual first passage time if we have sufficient control on the extra waiting time $\mathcal{E}$ (from Theorem 4.3.2) in a suitable domain.

Fix some $y \in \mathbb{R}^d$ and define the random variable $\{Z_t^\rho\}_{t\geq 0}$ by

$$Z_t^\rho := \mathbb{E}\left[\theta^\rho(0,y) \mid \mathcal{F}_t\right],$$

where $\mathcal{F}_t$ is the σ-algebra generated by the environment $V(x)$ restricted to the ρ-guaranteed reachable set $\mathcal{R}_t^\rho(0)$. In other words, $\mathcal{F}_t$ is the smallest σ-algebra so that the functions $V(x)\mathbb{1}_{x\in\mathcal{R}_t^\rho(0)}$ are $\mathcal{F}_t$-measurable for every $x \in \mathbb{R}^d$. Since $\mathcal{R}_t^\rho(0)$ are increasing sets, $\{\mathcal{F}_t\}_{t\geq 0}$ is a filtration, so $\{Z_t^\rho\}_{t\geq 0}$ is a martingale.

We first show that Z_t^ρ depends mostly on the shape of $\mathcal{R}_t^\rho(0)$, without regard for the values of V inside $\mathcal{R}_t^\rho(0)$. In order to condition on the approximate shape of the reachable set, for any $E \subseteq \mathbb{R}^d$ we introduce the discretization

$$\mathrm{disc}(E) := \{z \in d^{-1/2}\mathbb{Z}^d \mid B(z,1) \cap E \neq \emptyset\}.$$

Lemma 4.3.5. *For any $t \geq 0$, we have*

$$\left|\max(Z_t^\rho, t) - f(t, \mathrm{disc}(\mathcal{R}_t^\rho(0)))\right| \leq 3\rho,$$

where we define $f(t,S)$, for any $t \geq 0$ and any finite set $S \subseteq d^{-1/2}\mathbb{Z}^d$, by

$$f(t,S) := t + \mathbb{E}\left[\theta^\rho(S,y)\right].$$

Proof. Fix some $t \geq 0$. Using the speed limit $L+1$ for controlled paths, we see that

$$\mathcal{R}_t^\rho(0) \subseteq B(0, 1 + \lceil L + 1 + \rho^{-1}\rceil t)$$

almost surely. Define the set of possible discretized reachable sets at time t by

$$C_t := \left\{S \subseteq d^{-1/2}\mathbb{Z}^d \cap B\left(0, \lceil L + 1 + \rho^{-1}\rceil t\right)\right\}. \tag{4.6}$$

For any $S \in C_t$, the event that $\mathrm{disc}(\mathcal{R}_t^\rho(0)) = S$ is $\mathcal{F}_t$-measurable, so we have

$$\begin{aligned}
&\left|\max\left(Z_t^\rho, t\right) - f\left(t, \mathrm{disc}(\mathcal{R}_t^\rho(0))\right)\right| \\
&\qquad = \sum_{S \in C_t} \max\left(\left|\mathbb{E}\left[\left(\theta^\rho(0,y) - f(t,S)\right)\mathbb{1}_{\mathrm{disc}(\mathcal{R}_t^\rho(0))=S} \mid \mathcal{F}_t\right]\right|, |t - f(t,S)|\right).
\end{aligned}$$

Fix some $S \in C_t$. If $B(y,2) \cap S \neq \emptyset$, then $y \in \mathcal{R}_{2\rho}^\rho(S)$, so $\theta^\rho(0,y) \leq t + 3\rho$ and $t \leq f(t,S) \leq t + 3\rho$ and the conclusion holds.

Otherwise, define the set $E := S + B(0,2)$. Note that $\mathcal{R}_t^\rho(0) \subseteq E$ and $\mathrm{dist}(\mathcal{R}_t^\rho(0), \partial E) \geq$

1. Using the definition of the ρ-guaranteed reachable set,

$$\theta^\rho(0,\partial E)+\theta^\rho(\partial E,y)-\rho\le\theta^\rho(0,y)\le\theta^\rho(0,\partial E)+\theta^\rho(\partial E,y). \tag{4.7}$$

Using the definitions of S and E,

$$t\le\theta^\rho(0,\partial E)\le t+3\rho.$$

On the other hand, the term $\theta^\rho(\partial E,y)$ is $\mathcal{G}(\mathbb{R}^d\setminus E)$-measurable. Taking the conditional expectation of (4.7), we have

$$t-\rho+\mathbb{E}[\theta^\rho(\partial E,y)]\le Z_t^\rho\le t+3\rho+\mathbb{E}[\theta(\partial E,y)].$$

To finish, we use the definition of the ρ-guaranteed reachable set to find that

$$0\le\theta^\rho(S,y)-\theta^\rho(\partial E,y)\le 2\rho.$$

Combining the previous two displays yields the conclusion of the lemma. □

Next, we show that our approximation for Z_t^ρ, given by $f(t,\operatorname{disc}(\mathcal{R}_t^\rho(0)))$, has bounded increments.

Lemma 4.3.6. *Let $t,s\ge 0$. Then*

$$\left|f(t,\operatorname{disc}(\mathcal{R}_t^\rho(0)))-f(s,\operatorname{disc}(\mathcal{R}_s^\rho(0)))\right|\le 2\rho+|t-s|(L\rho+\rho+2).$$

Proof. Without loss of generality, assume $s<t$. Let C_s and C_t be as defined in (4.6). We

need to prove that

$$\sum_{A_s \in C_s} \sum_{A_t \in C_t} |f(t, A_t) - f(s, A_s)| \, \mathbb{1}_{\mathrm{disc}(\mathcal{R}_s^\rho(0)) = A_s} \mathbb{1}_{\mathrm{disc}(\mathcal{R}_t^\rho(0)) = A_t} \leq 2\rho + |t-s|(L\rho + \rho + 2).$$

So, fix any $A_s \in C_s$ and $A_t \in C_t$ such that

$$\mathbb{P}\left[\mathrm{disc}(\mathcal{R}_s^\rho(0)) = A_s \text{ and } \mathrm{disc}(\mathcal{R}_t^\rho(0)) = A_t\right] > 0.$$

The speed limit for controlled paths shows that $A_s \subseteq A_t$ and

$$\mathrm{dist}_H(A_s, A_t) \leq 2 + |t-s| \lceil L + 1 + \rho^{-1} \rceil,$$

where dist_H denotes the Hausdorff distance. Using the definition of the ρ-guaranteed reachable set, this yields

$$\theta^\rho(A_t, y) \leq \theta^\rho(A_s, y) \leq \theta^\rho(A_t, y) + \rho \left(2 + |t-s| \lceil L + 1 + \rho^{-1} \rceil\right).$$

The conclusion of the lemma follows from the definition of f. □

Now we put these lemmas together and apply Azuma's inequality to $\{Z_t^\rho\}_{t \geq 0}$, choosing ρ carefully to balance competing error terms.

Proposition 4.3.7. *There is a constant $C = C(d, L) > 0$ such that, if $y_1, y_2 \in \mathbb{R}^d$ and*

$$\lambda \geq C + C|y_1 - y_2|^{1/2} \log^2 |y_1 - y_2|,$$

then

$$\mathbb{P}[|\theta(y_1, y_2) - \mathbb{E}[\theta(y_1, y_2)]| > \lambda] \leq C \exp\left(\frac{-C^{-1}\lambda^{1/2}}{|y_1 - y_2|^{1/4}}\right).$$

Proof. By $\mathbb{R}^d$-translation invariance, we assume without loss of generality that $y_1 = 0$; let

$y := y_2$.

Note that $\theta(0,y) = \theta^\rho(0,y)$ as long as $\theta(u,v) \le \rho$ if $|u-v| \le 1$, for all $|u|,|v| \le \lceil (L+1+\rho^{-1}) \rceil \lceil \rho^{-1}|y| \rceil$ (a ball of this radius contains the ρ-guaranteed reachable set at time $\lceil \rho^{-1}|y|\rceil$). By Lemma 4.3.5 and Lemma 4.3.6, the martingale $\{Z_t^\rho\}_{t\ge 0}$ has bounded increments of

$$|Z_t - Z_s| \le (\rho L + \rho + 2)|t-s| + 8\rho.$$

Also, $Z_t^\rho = \theta(0,y)$ for all $t \ge \rho|y|$. Apply the union bound with Theorem 4.3.2 and Azuma's inequality to the sequence $\{Z_n^\rho\}_{0\le n\le \lceil \rho^{-1}|y|\rceil}$ to see that

$$\begin{aligned}\mathbb{P}\left[|\theta(0,y) - \mathbb{E}[\theta^\rho(0,y)]| > \lambda\right] &\le \mathbb{P}[\theta(0,y) \ne \theta^\rho(0,y)] + \mathbb{P}\left[|\theta(0,y) - \mathbb{E}[\theta^\rho(0,y)]| > \lambda\right] \\ &\le C\Big[(L+1+\rho^{-1})\rho^{-1}|y|\Big]^d \exp(-C^{-1}\rho) \\ &\quad + 2\exp\left(\frac{-C^{-1}\lambda^2}{[(\rho L + \rho + 2) + 8\rho]^2 \left\lceil \rho^{-1}|y|\right\rceil}\right),\end{aligned}$$

as long as $\rho \ge 2C$. Choosing $\rho = \lambda^{1/2}|y|^{-1/4}$ yields the bound

$$\mathbb{P}[|\theta(0,y) - \mathbb{E}[\theta^\rho(0,y)]| > \lambda] \le C\exp\left(\frac{-C^{-1}\lambda^{1/2}}{|y|^{1/4}}\right), \tag{4.8}$$

as long as $\lambda \ge C|y|^{1/2}\log^2|y|$ (we absorb all the polynomials into the exponential by changing the constant appropriately). It remains to replace $\mathbb{E}[\theta^\rho(0,y)]$ in (4.8) with $\mathbb{E}[\theta(0,y)]$. Indeed, we can bound

$$\begin{aligned}\mathbb{E}[\theta(0,y)] &= \mathbb{E}\left[\theta^\rho(0,y) \mid E_\rho\right] \cdot \mathbb{P}[E_\rho] + \mathbb{E}\left[\theta(0,y) \mid E_\rho^c\right] \cdot (1 - \mathbb{P}[E_\rho]) \\ &= \mathbb{E}\left[\theta^\rho(0,y)\right] + O\Big(\rho|y|(1-\mathbb{P}[E_\rho]) + (\rho|y|)^{d+1}\exp(-C^{-1}\rho)\Big),\end{aligned} \tag{4.9}$$

where E_ρ denotes the event that $\theta(0,y) = \theta^\rho(0,y)$ and the last part of the last line comes from the controllability bound in Theorem 4.3.2. Since $\rho > C\log|y|$, we can ensure that the error term in (4.9) is at most $\frac{1}{2}\lambda$ and can therefore be absorbed into the constant. $\square$

4.3.3 Nonrandom scaling bias

Given the controllability estimate in Theorem 4.3.2, we can apply Fekete's lemma [19] to extract a limit, for every $v \in \mathbb{R}^d$, of

$$\overline{\theta}(v) := \lim_{\alpha\to\infty} \frac{\mathbb{E}[\theta(0, \alpha v)]}{\alpha}.$$

We are now interested in bounding the nonrandom scaling bias, given by

$$\mathbb{E}[\theta(0, v)] - \overline{\theta}(v),$$

for $v \in \mathbb{R}^d$. The controllability estimate implies that $\mathbb{E}[\theta(0, y)] - \mathbb{E}[\theta(0, z)] \leq C$ if $|y - z| \leq 1$, so we lose nothing by assuming that $v \in \mathbb{Z}^d$.

First, we find some initial sublinear bound for the nonrandom scaling bias. Then, we follow an argument of Alexander [1] to improve the bound so it matches the estimate (up to a log-factor) for the random fluctuations from Proposition 4.3.7. We need the initial bound to estimate the region where we apply the controllability bound in the later argument.

Since $\theta(x, y)$ is subadditive, we know that $\mathbb{E}[\theta(0, v)] \geq \mathbb{E}[\overline{\theta}(v)]$. We are looking for a bound of the form

$$\mathbb{E}[\theta(0, v)] \leq \mathbb{E}[\overline{\theta}(v)] + \mathrm{error}(|v|).$$

Our strategy will be to take a controlled path from 0 to v, chop it into pieces, and rearrange the pieces into a new path which stays close to the line connecting 0 to v. If the random fluctuations are small, the rearranged path will not take much more time than the original path. In this way, we show that

$$\mathbb{E}[\theta(0, nv)] + \mathbb{E}[\theta(0, mv)] \approx \mathbb{E}[\theta(0, (n + m)v)]$$

for $n, m \in \mathbb{N}$, which will give us the desired bound.

First, we need a lemma about the rearrangement which is purely combinatorial; we include a proof based on a nearly identical result from Matoušek [29], proven originally by Grinberg [22].

We note that the arguments of this section can be simplified considerably via a curve-cutting lemma used by Tran–Yu [36] to achieve the optimal rate of convergence for periodic homogenization of convex Hamilton–Jacobi equations. The simplified version is contained in a future work [14]. Notably, this simplification also improves the bound in Theorem 5 by a factor of $\log(\varepsilon^{-1}t)$.

Lemma 4.3.8. *Let $v_1, \ldots v_n \in B_1$ be vectors lying in the unit ball $B_1 \subseteq \mathbb{R}^d$ with $\sum_{i=1}^n v_i = nx$. Then there is a permutation $\sigma\colon \{1, \ldots, n\} \to \{1, \ldots, n\}$ such that $\left|\sum_{i=1}^k v_{\sigma(i)} - kx\right| \leq 2d$ for every $1 \leq k \leq n$.*

Proof. We say that a set $\{v_1, \ldots, v_n\}$ of vectors is *good* if there are coefficients $\alpha_1, \ldots, \alpha_n$ such that $\alpha_i \in [0, 1]$, with

$$\sum_{i=1}^n \alpha_i v_i = (n-d)x$$

and

$$\sum_{i=1}^n \alpha_i = n - d.$$

Note that our original set of vectors is good, and that if a set of vectors is good then

$$\left|\sum_{i=1}^n v_i - nx\right| \leq 2d.$$

To prove the lemma, it is enough to show that if $n > d$ then there is i such that $\{v_1, \ldots, v_n\} \setminus \{v_i\}$ is also good (then we build σ by putting $\sigma(n) = i$ and proceeding recursively). Consider the following system of equations in n unknowns $x_1, \dots, x_n$:

$$\sum_{i=1}^{n} x_i v_i = (n-d-1)x, \qquad \sum_{i=1}^{n} x_i = n-d-1.$$

Note that $x_i = \alpha_i(n-d-1)/(n-d)$ shows that there is a solution with $x_i \in [0,1]$ for all i. Since there are $d+1$ equations, we can modify $x_1, \dots, x_n$ so that all but $n-d-1$ of them are either 0 or 1. If none of the x_i were 0, then $n-d-1$ of them would 1, so we would have $\sum_{i=1}^{n} x_i > n-d-1$, a contradiction. To conclude, we take i such that $x_i = 0$ and now the remaining x_i witness the fact that $\{v_1, \dots, v_n\} \setminus \{v_i\}$ is good. $\square$

Next, we apply Lemma 4.3.8 together with the fluctuation bound in Proposition 4.3.7 to show that $\mathbb{E}[\theta(0, \cdot)]$ is approximately additive.

Lemma 4.3.9. *There is a constant $C = C(d, L) > 0$ such that for any $n, m \in \mathbb{N}$ and $v \in \mathbb{R}^d$ with $|v| \geq 2$, we have*

$$\mathbb{E}[\theta(0,(n+m)v)] \leq \mathbb{E}[\theta(0,nv)] + \mathbb{E}[\theta(0,mv)] \leq \mathbb{E}[\theta(0,(n+m)v)] + C|v|^{2/3}\log^2|v|.$$

Proof. Let E denote the event that

1. $|\theta(x,y) - \mathbb{E}[\theta(x,y)]| \leq C|v|^{1/3}\log^2|v|$ for all $x, y \in B(0, C(n+m)v)$ with $|x-y| \leq C|v|^{2/3}$, and

2. $|\theta(0,(n+m)v) - \mathbb{E}[\theta(0,(n+m)v)]| \leq C|v|^{1/2}\log^2|v|$.

If we choose C large enough, Proposition 4.3.7 implies that E has probability at least $\frac{1}{2}$. In particular, E is nonempty, so work in some fixed environment V in E. Let $\gamma\colon [0, \theta(0,(n+m)v)] \to \mathbb{R}^d$ be a controlled path from 0 to $(n+m)v$. By chopping γ into pieces of length at most $|v|^{2/3}$, we find a sequence of points $0 = x_0, x_1, \dots, x_k = (n+m)v$ with $k \leq C|v|^{1/3}$

which lie in $\mathbb{Z}^d$ with

$$\sum_{i=1}^{k} \theta(x_{i-1}, x_i) \le \theta(0, (n+m)v) + Ck.$$

Using the definition of E, we conclude that

$$\sum_{i=1}^{k} \mathbb{E}[\theta(x_{i-1}, x_i)] \le \mathbb{E}[\theta(0, (n+m)v)] + C|v|^{2/3} \log^2 |v|.$$

Using $\mathbb{Z}^d$-translation invariance and rearranging using Lemma 4.3.8, we assume that $|x_i - \frac{i}{k}(n+m)v| \le C|v|^{2/3}$. So, letting $p := \lfloor nk/(n+m) \rfloor$, we have

$$\begin{aligned} \mathbb{E}[\theta(0, nv)] + \mathbb{E}[\theta(0, mv)] &\le \sum_{i=1}^{p} \theta(x_{i-1}, x_i) + C|v|^{2/3} + \sum_{i=p+1}^{k} \theta(x_{i-1}, x_i) \\ &\le \mathbb{E}[\theta(0, (n+m)v)] + C|v|^{2/3} \log^2 |v|, \end{aligned}$$

where the first inequality comes from considering the path that goes from 0 to x_p, takes a detour to nv, then continues from x_p to x_k. □

We finish by using Lemma 4.3.9 and subadditivity to bound the nonrandom scaling bias.

Proposition 4.3.10. *There is a constant $C = C(d, L) > 0$ such that $|\mathbb{E}[\theta(0, v)] - \overline{\theta}(v)| \le C|v|^{2/3} \log^2 |v|$ for all $|v| \ge 2$.*

Proof. Using Lemma 4.3.9, we note that $\mathbb{E}[\theta(0, v)] + C|v|^{2/3} \log^2 |v|$ is superadditive in v. Since $C|v|^{2/3} \log^2 |v|$ is strictly sublinear in $|v|$, we conclude that

$$\mathbb{E}[\theta(0, v)] + C|v|^{2/3} \log^2 |v| \ge \lim_{n \to \infty} n^{-1} \left(\mathbb{E}[\theta(0, nv)] + C|nv|^{2/3} \log^2 |nv| \right) = \overline{\theta}(v),$$

and the conclusion follows. □

Now that we have some initial sublinear bound on the scaling bias, we will follow Alexander's [1] argument to show that $\mathbb{E}[\theta(0, v)]$ satisfies the convex hull approximation property (defined below) with exponent $\frac{1}{2}$. The argument is a bit different in our setting because (i) the controllability estimate only holds with high probability (not almost surely) and (ii) we do not have access to the van den Berg–Kesten inequality, so we replace it with the finite range of dependence assumption together with the controllability estimate. The speed limit $L+1$ on controlled paths provides the necessary locality to apply the finite range of depencence assumption.

Let $f\colon \mathbb{Z}^d \to \mathbb{R}$ be a nonnegative subadditive function with sublinear growth; that is, there is some constant $r>0$ such that $f(x) \le r|x|$ (in practice we will set $f(x) := \mathbb{E}[\theta(0,x)]$). Define $\overline{f}\colon \mathbb{R}^d \to \mathbb{R}$ by

$$\overline{f}(x) := \lim_{n\to\infty} n^{-1} f([nx]),$$

where $[\cdot]$ denotes coordinate-wise rounding to integers. Then $\overline{f}$ is positively homogeneous and convex, so we define $\overline{f}_x$ to be a supporting affine functional to $\overline{f}$ at x, chosen consistently so that $\overline{f}_x = \overline{f}_{\alpha x}$ for all $\alpha > 0$. We can think of $\overline{f}_x(v)$ as representing the progress that an increment v makes in the direction of x. Since $\overline{f}$ is positively homogenous, we have $\overline{f}_x(0) = 0$ and $\overline{f}_x(x) = f(x)$.

In the following, $\varphi\colon (1,\infty) \to \mathbb{R}$ is a positive nondecreasing function (in practice we will set $\varphi(x) := \log^3 x$).

Definition 4.3.11. We say that f satisfies the general approximation property with exponent $\nu \ge 0$ and correction factor φ if there are constants $C, M > 0$ such that if $|x| \ge M$ then

$$\overline{f}(x) \le f(x) \le \overline{f}(x) + C|x|^\nu \varphi(|x|).$$

Next, we define the set of "good" increments toward x by

$$G_x(\nu,\varphi,C,K) := \left\{ v \in \mathbb{Q}^d \mid |v| \le K|x|,\ \overline{f}_x(v) \le \overline{f}_x(x),\ f(v) \le \overline{f}_x(v) + C|x|^\nu \varphi(|x|) \right\}, \tag{4.10}$$

where $C, K \geq 0$. The conditions in (4.10) say that good increments are not too much larger than $|x|$, do not overshoot in the direction of x, and are not too wasteful along the way (we can think of $f(v) - \overline{f}_x(v)$ as the inefficiency in a step v towards x). If we could write every x as the sum of a bounded number of good increments, this would imply the general approximation property. Unfortunately, this is not so easy to accomplish. Instead, we approximate nx by $O(n)$-many good increments for some large n; this is the content of the next definitions. In the proofs, we will need to apply the controllability estimate along a path from 0 to nx, so we need some upper bound on n. The sublinear bound in Proposition 4.3.10 allows us to bound n by some power of $|x|$, so we can apply controllability.

Definition 4.3.12. A sequence $v_0, v_1, \ldots, v_m$ is a $G_x(\nu, \varphi, C, K)$-skeleton if $v_i - v_{i-1} \in G_x(\nu, \varphi, C, K)$ for all $1 \leq i \leq m$.

Definition 4.3.13. A function f satisfies the skeleton approximation property with exponent $\nu \geq 0$ and correction factor φ if there are constants $M, C, K > 0$ and $a > 1$ such that, for every $x \in \mathbb{Q}^d$ with $|x| \geq M$, there exists $n \geq 1$ and a $G_x(\nu, \varphi, C, K)$-skeleton $0 = v_0, v_1, \ldots, v_m = nx$ of length $m \leq an$.

As we will see, the skeleton approximation property is more convenient to verify. The following theorem of Alexander [1] provides a link between the skeleton and general approximation properties.

Theorem 4.3.3 (Alexander [1])**.** *Suppose that $f\colon \mathbb{Z}^d \to \mathbb{R}$ is a nonnegative subadditive function and there is a constant $r \geq 1$ such that $f(x) \leq r|x|$ for all $x \in \mathbb{Z}^d$. If f satisfies the skeleton approximation property with exponent $\nu > 0$ and correction factor φ, then f satisfies the general approximation property with the same exponent and correction factor.*

Proof. Step 1. We show that if $x \in \mathbb{Q}^d$ with $|x| \geq C$, then there is $\alpha \in [c, 1]$ such that αx lies in the convex hull of G_x. Let $n \in \mathbb{N}$ be large enough so that

$$|n^{-1} f(nx) - \overline{f}(x)| \leq 1.$$

Let $x_0, x_1, \ldots, x_m$ be an G_x-skeleton for nx. Then

$$x = \frac{1}{n}\sum_{k=1}^{m}(x_k - x_{k-1}),$$

and $n \le m \le Cn$, where the first part of the inequality follows from applying $\overline{f}_x$ to both sides of the equation.

Step 2. We show that if $x \in \mathbb{Q}^d$, $|x| \ge K$, $t \ge 1$, and $tx \in \mathbb{Z}^d$, then there is a $z \in \mathbb{Z}^d$ with $f(tx) - \overline{f}(tx) \le f(z) - \overline{f}(z) + tC|x|^\nu \varphi(|x|)$. Using the previous step, write $tx = z + \sum_{k=1}^m v_k$, where $|z| \le C|x|$, $\overline{f}(z) \le \overline{f}_x(z) + C$, $v_k \in G_x$, and $m \le Ct$. Indeed, for some $\alpha \in [c, 1]$ we first write

$$\alpha x = \sum_{i=1}^{d+1} p_i v_i,$$

where $v_i \in G_x$ and $p_i \ge 0$, $\sum_i p_i = 1$. Note that the sum only requires $d+1$ terms by Caratheodory's theorem on convex hulls, since we are working in $\mathbb{R}^d$. To decompose tx, we write

$$tx = \sum_{i=1}^{d+1}(t\alpha^{-1}p_i - \lfloor t\alpha^{-1}p_i \rfloor)v_i + \sum_{i=1}^{d+1}\lfloor t\alpha^{-1}p_i \rfloor v_i =: z + (tx - z),$$

so z satisfies the required properties. By subadditivity of f and linearity of $\overline{f}_x$,

$$\begin{aligned} f(tx) &\le f(z) + \sum_{k=1}^{m} f(v_k) \\ &\le f(z) + \sum_{k=1}^{m} \overline{f}_x(v_k) + C|x|^\nu \varphi(|x|) \\ &= f(z) + \overline{f}_x(tx - z) + tC|x|^\nu \varphi(|x|). \end{aligned}$$

Finally, we write $\overline{f}(tx) = \overline{f}_x(z) + \overline{f}_x(tx - z)$ and subtract from both sides of the inequality above to get

$$f(tx) - \overline{f}(tx) \le f(z) - \overline{f}(z) + Ct,$$

where we used the fact that $\overline{f}(z) \le \overline{f_x}(z) + C$.

Step 3. For some large $M > 1$ (and possibly enlarged K), the previous step yields

$$\begin{aligned}\sup_{|x| \le M^{k+1}K} f(x) - \overline{f}(x) &\le \sup_{|x| \le M^k K} f(x) - \overline{f}(x) + CM|x|^\nu \varphi(|x|) \\ &\le \sup_{|x| \le M^k K} f(x) - \overline{f}(x) + C(M^\nu - 1)|x|^\nu \varphi(|x|),\end{aligned}$$

where we made C larger in the second line to account for the change in constant. By induction on k, we conclude that

$$f(x) - \overline{f}(x) \le C|x|^\nu \varphi(|x|).$$

□

It remains to verify that $f(x) := \mathbb{E}[\theta(0, x)]$ satisfies the skeleton approximation property.

Proposition 4.3.14. *The function $f(x) := \mathbb{E}[\theta(0, x)]$ satisfies the skeleton approximation property with exponent $\nu = \frac{1}{2}$ and correction factor $\varphi(x) = \log^3 |x|$.*

Proof. Given a skeleton $\{v_i\}_{0 \le i \le m}$, we define its error to be

$$\operatorname{err}\left(\{v_i\}_{0 \le i \le m}\right) := \sum_{i=0}^{m-1} \max\left(0, \mathbb{E}[\theta(v_i, v_{i+1})] - \theta(v_i, v_{i+1})\right),$$

measuring how much faster a controlled path can traverse the skeleton than expected.

Given $\eta \in \mathbb{N}$, we say that a skeleton $\{v_i\}_{0 \le i \le m}$ is η-reasonable if, for every $s > 0$ and $1 \le i \le j \le m$, if $s \le |v_i - v_{i-1}|, |v_j - v_{j-1}| \le 2s$ and there are at least $\eta - 1$ other indices $i < k < j$ such that if $s \le |v_k - v_{k-1}| \le 2s$, then

$$(L + 1)(\mathbb{E}|v_i - v_{i-1}| + \mathbb{E}|v_j - v_{j-1}|) + 1 \le |v_i - v_j|.$$

This definition means that legs of a reasonable skeleton contribute terms to the error which are independent random variables, as long as they are of the same scale and far enough apart.

Step 1. We show that, with probability which tends to 1 as $|x| \to \infty$, there is $\eta \in \mathbb{N}$ such that every η-reasonable G_x-skeleton has small error. Let $\{v_i\}_{0 \le i \le m}$ be an η-reasonable G_x-skeleton. Then partition the indices $1, 2, \ldots, m$ into $O(\log |x|)$ buckets numbered starting at 0, where the ith bucket is given by

$$B_i = \{j \mid 2^i \le |v_j - v_{j-1}| < 2^{i+1}\}.$$

Within a single bucket, indices which are at least η apart (in sorted order) contribute independent terms to the error, so splitting the bucket into η sub-buckets containing independent terms, exponentiating the random fluctuation bound from Proposition 4.3.7, and using Markov's inequality yields

$$\begin{aligned}\mathbb{P}&\left[\sum_{j \in B_i} \max\big(0, \mathbb{E}[\theta(v_j, v_{j+1})] - \theta(v_j, v_{j+1})\big) > \zeta m \eta |x|^{1/2} \log^2 |x|\right] \\ &\qquad \le \eta \Big(C \exp(-C^{-1} \zeta \log^2 |x|)\Big)^m\end{aligned}$$

for some constant $C > 0$ and any $\zeta \ge 1$. Summing over the buckets, we get

$$\begin{aligned}\mathbb{P}&\left[\sum_{j=1}^{m} \max\big(0, \mathbb{E}[\theta(v_j, v_{j+1})] - \theta(v_j, v_{j+1})\big) > \zeta m \eta |x|^{1/2} \log^3 |x|\right] \\ &\qquad \le \eta \log |x| \Big(C \exp(-C^{-1} \zeta \log^2 |x|)\Big)^m \\ &\qquad \le C \exp(-\zeta C^{-1} m \log^2 |x|), \end{aligned} \tag{4.11}$$

where the last inequality holds as long as $|x|$ is sufficiently large.

Estimate (4.11) holds for a particular skeleton $\{v_i\}_{0 \le i \le m}$. Since there are at most $C|x|^{dm}$

many G_x-skeletons of length m, we can set ζ large enough to sum over all G_x skeletons and get

$$\mathbb{P}\left[\text{every } \eta\text{-reasonable } G_x\text{-skeleton of length } m \text{ has error at most } Cm|x|^{1/2}\log^3|x|\right]$$
$$\geq 1 - C\exp(-C^{-1}\log^2|x|).$$

Here we make a note to choose M large enough so that $C\exp(-C^{-1}\log^2 M) < \frac{1}{3}$.

Step 2. Choose $n \in \mathbb{N}$ such that $\frac{1}{n}f(nx) - \overline{f}(x) < 1$. In view of Proposition 4.3.10, we can ensure that $n \leq C|x|^3$. Note that there always exists a G_x-skeleton from 0 to nx, since sufficiently short increments are in G_x. The challenge is to find a G_x-skeleton without too many vertices.

We show that, with probability which tends to 1 as $|x| \to \infty$, there exists a reasonable G_x-skeleton of length $m \leq n$ from 0 to nx. Proposition 4.3.7 applied to $\theta(0, nx)$ shows that

$$\mathbb{P}\left[\theta(0, nx) - \mathbb{E}[\theta(0, nx)] > n\right] < \frac{1}{3},$$

as long as M (and therefore $|x|$) is large enough. Similarly, we can choose M large enough to ensure that

$$\mathbb{P}\left[\mathcal{E}\left(2(L+1)\mathbb{E}[\theta(0, nx)]\right) > |x|^{1/4}\right] < \frac{1}{3}.$$

By the union bound, there is some $\omega \in \Omega$ such that

$$\theta_\omega(0, nx) \leq \mathbb{E}[\theta(0, nx)] + n$$

and

$$\mathcal{E}_\omega(2(L+1)\mathbb{E}[\theta(0, nx)]) \leq |x|^{1/4} \tag{4.12}$$

and so that for every $m \geq 1$, every G_x-skeleton of length m has error no more than

$Cm|x|^{1/2}\log^3|x|$.

We now build a G_x-skeleton greedily from $\gamma\colon [0,\theta_\omega(0,nx)] \to \mathbb{R}^d$, the shortest controlled path from 0 to nx with respect to this particular ω. Let $v_0 := 0$ and $t_0 := 0$. Given v_i, t_i for $i \geq 0$, define

$$t_{i+1} := \min\left(\theta_\omega(0,nx), \sup\{t \geq t_i \mid (\gamma(t) - v_i + B_1) \cap G_x \neq \emptyset\}\right).$$

Then, define v_{i+1} to be any point in $(\gamma(t_{i+1}) + \overline{B_1}) \cap (G_x + v_i)$. Continue until $v_{i+1} = nx$, and set $m := i+1$ in this case.

We claim that the skeleton $v_0, \ldots, v_m$ is η-reasonable for some constant $\eta = \eta(d,L) > 0$. Indeed, suppose not. Then there would be some $s > 0$ and indices $i_0 < i_1 < \cdots < i_\eta$ such that $s \leq |v_{i_{j+1}} - v_{i_j}| \leq 2s$ for all $0 \leq j < \eta$ and $|v_{i_\eta} - v_{i_0}| \leq Cs + 1$, where $C(d,L) > 0$ is a constant; for example we could take $C = 4(L+1)\sup_{|v|=1}\mathbb{E}[\theta(0,v)]$, which depends only on d and L by Theorem 4.3.2. Since we chose the skeleton greedily, we can be sure that

$$s \geq \frac{1}{2}|v_{i+1} - v_i| \geq C^{-1}|x|^{1/2}, \tag{4.13}$$

where we may have to enlarge the constant $C > 0$. Define $\widetilde{\gamma}$ to be the path which follows γ from 0 to $\gamma(t_{i_0})$, then follows a shortest controlled path from $\gamma(t_{i_0})$ to $\gamma(t_{i_\eta})$, then follows γ the rest of the way to nx. The controllability bound (4.12) implies that

$$\text{length}(\widetilde{\gamma}) \leq \text{length}(\gamma) - \eta(L+1)^{-1}s + Cs + 1 + |x|^{1/4}.$$

From (4.13) we see that $\text{length}(\widetilde{\gamma})$ is strictly smaller than $\text{length}(\gamma)$ if $\eta \geq 4C(L+1)$ (here we make a note to choose $C, M \geq 1$). This contradicts the fact that γ is a shortest path, so we conclude that the skeleton is η-reasonable.

Now we use the error bound for reasonable skeletons to see that

$$\operatorname{err}\left(\{v_i\}_{1 \leq i \leq m}\right) \leq Cm|x|^{1/2} \log^3 |x|,$$

and so using the fact that γ is a shortest path we see that

$$\begin{aligned}\sum_{i=0}^{m-1} \mathbb{E}[\theta_\omega(v_i, v_{i+1})] &\leq \theta_\omega(0, nx) + m(4 + |x|^{1/4}) + Cm|x|^{1/2} \log^3 |x| \\ &\leq \mathbb{E}[\theta_\omega(0, nx)] + n + m(4 + |x|^{1/4}) + Cm|x|^{1/2} \log^3 |x|. \end{aligned} \tag{4.14}$$

On the other hand, we can split the indices $0, 1, 2, \ldots, m-1$ into two groups: define

$$L := \{0 \leq i < m \mid |v_{i+1} - v_i| + \sqrt{d} \geq K|x| \text{ or } \overline{f}_x(v_{i+1} - v_i) + r\sqrt{d} \geq \overline{f}(x)\},$$

and let $S := \{0, \ldots, m-1\} \setminus L$. We think of S as the indices of short increments and L as the indices of long increments. Note that every index $i \in S$ of a short increment satisfies

$$\mathbb{E}[\theta_\omega(v_i, v_{i+1})] - \overline{f}_x(v_{i+1} - v_i) \geq C|x|^{1/2} \log^3 |x| - O(1).$$

So summing over $i \in S$ and choosing $C > 0$ large enough relative to the bound (4.14) shows that we can ensure $|S| \leq \frac{m}{4}$. On the other hand, the linear growth of $\overline{f}$ shows that there are at most $C|x|^{-1}\theta_\omega(0, nx) \leq Cn$ long increments, so $m = O(n)$ as desired. □

4.3.4 A shape theorem

To conclude the section, we put together our bounds on random and nonrandom error to get a quantitative shape theorem for the metric problem. Define the large-scale reachable set at time t by

$$\mathcal{S}_t := \{x \in \mathbb{R}^d \mid \overline{\theta}(x) \leq t\}.$$

Theorem 4.3.4. *Let $V\colon \mathbb{R}^d \to \mathbb{R}^d$ be a random divergence-free vector field with unit range of dependence and $\|V\|_{C^{1,1}} \le L$ almost surely. Then there are constants $C(d,L) > 1 > c(d,L) > 0$ such that, for all $t \ge 0$,*

$$\mathbb{P}\left[\operatorname{dist}_H(\mathcal{R}_t(0), \mathcal{S}_t) > Ct^{1/2}\log^3 t + \lambda\right] \le C\exp\left(\frac{-C^{-1}\lambda^{1/2}}{t^{1/4}}\right),$$

where dist_H *denotes the Hausdorff distance. Furthermore, there is a random variable T_0, with*

$$\mathbb{E}[\exp(c\log^{3/2} T_0)] < \infty,$$

such that

$$\sup_{(t,x)\in[0,T]\times B_T} \frac{\operatorname{dist}_H(\mathcal{R}_t(x), x+\mathcal{S}_t)}{T^{1/2}\log^3 T} \le C$$

for all $T \ge T_0$.

Proof. For the first claim, let $t \ge 0$. Apply Theorem 4.3.2 to $B_{(L+1)t+1}$ and Proposition 4.3.7 to every $x \in \mathbb{Z}^d \cap \overline{B_{(L+1)t}}$ and use the union bound to see that as long as $\lambda \ge Ct^{1/2}\log^2 t$ we have

$$\mathbb{P}\left[\forall x \in \overline{B_{(L+1)t}} :\ |\theta(0,x) - \mathbb{E}[\theta(0,x)]| > \lambda\right] \le C\exp\left(\frac{-C^{-1}\lambda^{1/2}}{t^{1/4}}\right), \tag{4.15}$$

where we absorbed polynomials into the exponential by enlarging the constant C. Note also that Theorem 4.3.2 implies that if $0 \le r \le s$, then

$$\mathbb{P}\left[\mathcal{R}_r(0) \not\subseteq \mathcal{R}_s(0) + B_\lambda\right] \le C\exp\left(\frac{-C^{-1}\lambda^{1/2}}{t^{1/4}}\right). \tag{4.16}$$

This bounds the random error. On the other hand, Proposition 4.3.14 and Theorem 4.3.3

combine to show that

$$0 \leq \mathbb{E}[\theta(0,x)] - \overline{\theta}(x) \leq C|x|^{1/2} \log^3 |x|, \tag{4.17}$$

which bounds the nonrandom error. The estimates (4.15) and (4.17) say that, with high probability, the first passage time $\theta(0,x)$ from 0 to any point x is close to the large-scale average $\overline{\theta}(x)$. Furthermore, the estimate (4.16) says that once a controlled path reaches x, the reachable set stays close to x for all later times (the controllability estimate guarantees the existance of controlled paths in the form of short loops). Unwrapping the definition of Hausdorff distance yields the first claim.

For the second claim, apply the first claim to every $(t,x) \in (\mathbb{Z} \cap [0,T]) \times (\mathbb{Z}^d \cap B_T)$ and the union bound to conclude that

$$\mathbb{P}\left[\sup_{\substack{t \in \mathbb{Z}\cap[0,T] \\ x \in \mathbb{Z}^d \cap B_T}} \operatorname{dist}_H(\mathcal{R}_t(x), x + \mathcal{S}_t) > CT^{1/2}\log^3 T + \lambda\right] \leq CT^{d+1} \exp\left(\frac{-C^{-1}\lambda^{1/2}}{T^{1/4}}\right).$$

Next, apply the controllability estimate in B_T to see that the same holds for all $(t,x) \in [0,T] \times B_T$, by enlarging the constant C. Plugging in $\lambda = CT^{1/2}\log^3 T$ shows that

$$\mathbb{P}\left[\sup_{(t,x)\in[0,T]\times B_T} \frac{\operatorname{dist}_H(\mathcal{R}_t(x), x+\mathcal{S}_t)}{T^{1/2}\log^3 T} > C\right] \leq C\exp(-C^{-1}\log^{3/2} T),$$

and the conclusion follows. $\square$

4.4 Proofs of the main results

4.4.1 Homogenization of solutions

Now that we have a quantitative shape theorem for the metric problem, we use the control formulation to extract a rate of convergence for solutions of the G equation to the large-scale limit. Let $u_0 \in C^{0,1}(\mathbb{R}^d)$ and let $\bar{u} \colon \mathbb{R}_t \times \mathbb{R}^d_x \to \mathbb{R}$ be a solution to the effective problem (4.2), where the effective Hamiltonian is given by

$$\overline{H}(p) := \sup_{v \in \mathbb{R}^d} p \cdot \frac{v}{\overline{\theta}(v)}. \tag{4.18}$$

Note that $\overline{\theta}$ is positively homogeneous, so the supremum can be restricted to the unit sphere and is in fact a maximum. We have the representation formula

$$\bar{u}(t,x) = \sup_{x+\mathcal{S}_t} u_0. \tag{4.19}$$

On the other hand, let u^ε be a solution to the G equation (4.1). We have the representation formula

$$u^\varepsilon(t,x) = \sup_{\varepsilon \mathcal{R}_{\varepsilon^{-1}t}(\varepsilon^{-1}x)} u_0. \tag{4.20}$$

See Barles [5] for proofs of these representation formulas.

Proof of Theorem 5. Using the representation formulas (4.19) and (4.20), we see that for every $0 \le t \le T$ and $x \in B_T$ we have

$$|u^\varepsilon(t,x) - \bar{u}(t,x)| = \left| \sup_{\varepsilon \mathcal{R}_{\varepsilon^{-1}t}(\varepsilon^{-1}x)} u_0 - \sup_{x+\mathcal{S}_t} u_0 \right| \le \|u_0\|_{C^{0,1}} \operatorname{dist}_H(\varepsilon \mathcal{R}_{\varepsilon^{-1}t}(\varepsilon^{-1}x), x + \mathcal{S}_t).$$

Rescaling by ε^{-1} and applying Theorem 4.3.4 (using the fact that $\varepsilon^{-1}\mathcal{S}_t = \mathcal{S}_{\varepsilon^{-1}t}$) yields

$$\sup_{(t,x)\in[0,T]\times B_T} \operatorname{dist}_H(\varepsilon\mathcal{R}_{\varepsilon^{-1}t}(\varepsilon^{-1}x), x+\mathcal{S}_t) \leq C(T\varepsilon)^{1/2}\log^3(\varepsilon^{-1}T)$$

for all $T \geq \varepsilon T_0$. Combining these gives

$$|u^\varepsilon(t,x) - \bar{u}(t,x)| \leq C\|u_0\|_{C^{0,1}}(t\varepsilon)^{1/2}\log^3(\varepsilon^{-1}t)$$

for all $t \geq \varepsilon T_0$ as desired. □

4.4.2 Continuity of the effective Hamiltonian

Now we apply Theorem 4.3.4 to show that the effective Hamiltonian $\overline{H}$ depends continuously on the law of the environment.

Proof of Theorem 6. Throughout this proof we will use the superscript $*$ to denote the corresponding object (Hamiltonian, first passage time, reachable set, etc.) for the environment $\mathbb{P}^*$. Since the effective Hamiltonians are positively homogeneous, it suffices to show uniform convergence on B_1. From the formula (4.18) for the effective Hamiltonian, we deduce that

$$\overline{H}(p) = \sup_{v\in\mathcal{S}_1} v\cdot p$$

and

$$\overline{H}^*(p) = \sup_{v\in\mathcal{S}_1^*} v\cdot p.$$

So, it would suffice to show that $\operatorname{dist}_H(\mathcal{S}_1^*, \mathcal{S}_1) \leq C\varepsilon^{1/3}\log^3\varepsilon^{-1}$. From Theorem 4.3.4, we have

$$\mathbb{E}[\operatorname{dist}_H(t^{-1}\mathcal{R}_t(0), \mathcal{S}_1)] \leq Ct^{-1/2}\log^3 t \tag{4.21}$$

and

$$\mathbb{E}^*[\mathrm{dist}_H(t^{-1}\mathcal{R}_t^*(0), \mathcal{S}_1^*)] \leq Ct^{-1/2}\log^3 t \tag{4.22}$$

for all $t \geq 0$.

On the other hand, if $\|V - V^*\|_{C^{1,1}} \leq \varepsilon$ for two vector fields V, V^*, then by using the same controls for each environment and Gronwall's inequality we see that the corresponding reachable sets satisfy

$$\mathrm{dist}_H(\mathcal{R}_t(0), \mathcal{R}_t^*(0)) \leq \varepsilon t \exp(\varepsilon t).$$

If $\pi(\mathbb{P}, \mathbb{P}^*) < \varepsilon$, then by applying Markov's inequality to (4.21) and (4.22), we conclude that such V, V^* exist with

$$\mathrm{dist}_H(t^{-1}\mathcal{R}_t(0), \mathcal{S}_1), \mathrm{dist}_H(t^{-1}\mathcal{R}_t^*(0), \mathcal{S}_1^*) \leq Ct^{-1/2}\log^3 t.$$

By the triangle inequality, we conclude that

$$\mathrm{dist}(\mathcal{S}_1, \mathcal{S}_1^*) \leq \varepsilon t \exp(\varepsilon t) + Ct^{-1/2}\log^3 t.$$

We choose $t = \varepsilon^{-2/3}$ to balance the error terms and conclude. □

CHAPTER 5

THE RANDOM G EQUATION WITH NONZERO DIVERGENCE

5.1 Introduction

We consider the behavior, as $\varepsilon \to 0^+$, of the family $\{u_\varepsilon\}_{\varepsilon>0}$ of solutions to the G equation,

$$\begin{cases} D_t u^\varepsilon(t,x) - |D_x u^\varepsilon(t,x)| + V(\varepsilon^{-1}x) \cdot D_x u^\varepsilon(t,x) = 0 & \text{in } \mathbb{R}_{>0} \times \mathbb{R}^d \\ u^\varepsilon(0,x) = u_0(x) & \text{in } \mathbb{R}^d, \end{cases} \tag{5.1}$$

where $d \geq 2$, $V \colon \mathbb{R}^d \to \mathbb{R}^d$ is a random vector field and the initial data $u_0 \colon \mathbb{R}^d \to \mathbb{R}^d$ is Lipschitz. The level sets of u^ε model a flame front which expands at unit speed in the normal direction while being advected by V, which models the wind velocity. When compared with homogenization of other Hamilton–Jacobi equations, the main difficulty with the G equation is that, since we do not assume that $\|V\|_{L^\infty} < 1$, the equation may not be coercive. On the other hand, if $\mathbb{E}[V] = 0$, then the equation is still "coercive on average", so we can hope to recover some large-scale controllability.

When $\operatorname{div} V = 0$, the wind cannot form "traps" where the flame can be contained, and so a controllability bound holds [13]. The main novelty of this paper is a more quantitative controllability bound, which allows for the possibility that $\operatorname{div} V$ is nonzero but small, and rules out the existence of such traps.

Cardaliaguet–Souganidis [13] proved, under the assumption that the environment $V \in C^{1,1}(\mathbb{R}^d; \mathbb{R}^d)$ is stationary ergodic and divergence-free, that the equation homogenizes; i.e. we have the locally uniform convergence of solutions $u^\varepsilon \to \bar{u}$ as $\varepsilon \to 0$ almost surely, where

$\bar{u}$ is the solution to the effective equation

$$\begin{cases} D_t\bar{u}(t,x) = \overline{H}(D_x\bar{u}(t,x)) & \text{in } \mathbb{R}_{>0} \times \mathbb{R}^d \\ \bar{u}(0,x) = u_0(x) & \text{in } \mathbb{R}^d, \end{cases} \tag{5.2}$$

and $\overline{H} \colon \mathbb{R}^d \to \mathbb{R}$, called the effective Hamiltonian, is positively homogeneous of degree one and coercive.

Our main result is the following.

Theorem 5.1.1. *Let $V \colon \mathbb{R}^d \to \mathbb{R}^d$ be a random Lipschitz vector field which has unit range of dependence and is $\mathbb{Z}^d$-translation invariant. Then there is a function $\overline{H} \colon \mathbb{R}^d \to \mathbb{R}$, which is positively homogeneous of degree one and coercive, and a constant $C = C(d) > 0$ such that, if*

$$|\operatorname{div} V| \leq C^{-1}(\|V\|_{C^{0,1}} + 1)^{-C}$$

almost surely, then there is a random variable T_0, with

$$\mathbb{E}[\exp(C^{-1}(\|V\|_{C^{0,1}} + 1)^{-C} \log^{3/2} T_0)] \leq C,$$

such that

$$|u^\varepsilon(t,x) - \bar{u}(t,x)| \leq C(\|V\|_{C^{0,1}} + 1)^C (T\varepsilon)^{1/2} \log^2(\varepsilon^{-1}T) \tag{5.3}$$

for all $T \geq \varepsilon T_0$ and $t, |x| \leq T$, where u^ε is the solution to the G equation 5.1 and $\bar{u}$ is the solution of the effective equation 5.2.

5.1.1 How quantitative is Theorem 5.1.1?

There are two main quantitative features of Theorem 5.1.1: the bound on $|u^\varepsilon - \bar{u}|$, and the random variable T_0, which represents how long we must wait before the bound takes effect. As for the former, the exponent $\frac{1}{2}$ of $(t\varepsilon)$ matches with the best known bound for convergence of the limiting shape in first-passage percolation [4]. Indeed, first-passage percolation is an easier problem, since controllability is free and the Hamiltonian is i.i.d., so we cannot hope for a better bound without improving the result for first-passage percolation as well.

As for the bound on T_0, we note that the distribution of T_0 has subpolynomial tails and therefore all moments of T_0 are finite. However, our only bound on the typical value of T_0 is

$$\mathbb{E}[T_0] \leq \exp\left(C(\|V\|_{C^{0,1}} + 1)^C\right).$$

We note that while $\|V\|_{C^{0,1}}$ appears to be a random variable, the finite range of dependence assumption implies that it is constant almost surely. The exponential dependence on $\|V\|_{C^{0,1}}$ is an artifact of the fact that the exponent $\frac{1}{2}$, discussed above, is the tightest possible with our current argument. Indeed, by the same proof it would follow that, if we replace the exponent $\frac{1}{2}$ in 5.3 with an exponent of $\frac{1}{2} - \delta$, the corresponding T_0 would instead depend polynomially on $\|V\|_{C^{0,1}}$, with the bound

$$\mathbb{E}[T_0] \leq C(\|V\|_{C^{0,1}} + 1)^{C/\delta}.$$

5.1.2 Prior work

There is a rich body of literature studying homogenization of the G equation and enhancement of the front speed (see [13, 8, 12, 31, 9] for example), so we limit our focus to work most closely related to the current situation. Inspired by Cardaliaguet–Souganidis [13], the author [15] showed that, if $V \in C^{1,1}(\mathbb{R}^d; \mathbb{R}^d)$ has unit range of dependence and is divergence-free, then there is a constant $C = C(d, \|V\|_{C^{1,1}}) > 0$ and a random variable T_0 with subpolynomial tail bound $\mathbb{E}[\exp(C^{-1} \log^{3/2} T_0)] \leq C$, such that

$$|u^\varepsilon(t,x) - \bar{u}(t,x)| \leq C\|u_0\|_{C^{0,1}}(t\varepsilon)^{1/2} \log^3(\varepsilon^{-1}t) \tag{5.4}$$

for all $t \geq \varepsilon T_0$ and $|x| \leq t$.

Since the bound (5.4) is bootstrapped from local controllability estimates by Cardaliaguet–Souganidis [13], the dependence of the constant C on $\|V\|_{C^{1,1}}$ was unspecified. Besides, the work of Cardaliaguet–Souganidis [13] used the divergence-free condition in a critical way, which was necessary under their weaker assumption of stationary ergodicity.

On the other hand, when the environment is periodic instead of random, Cardaliaguet–Nolen–Souganidis [12] proved quantitative homogenization of the G equation without the divergence-free condition. Indeed, they made only the weaker assumption that $|\operatorname{div} V| \leq \varepsilon$ for some $\varepsilon = \varepsilon(d) > 0$, which is related to the constant in the isoperimetric inequality for periodic sets.

We also note that Feldman [20] extended work of Burago–Ivanov–Novikov [8] to prove quantitative estimates on the waiting time in an environment which satisfies a mixing condition in both space and time variables, under the assumption $\operatorname{div} V = 0$.

In this paper, we extend the author's work [15] to the case where $\operatorname{div} V$ may be nonzero but small and V is only Lipschitz. Along the way, we quantify the dependence of the constant in (5.4) on the Lipschitz norm of V. The proof adapts an argument of Burago–Ivanov–Novikov [6], as well as a new argument to show that, even in the presence of nonzero divergence, the reachable set continues to grow quickly.

5.1.3 Definitions, assumptions, and conventions

We use $C > 0$ to denote a (large) constant which may vary from line to line, but (unless otherwise specified) depends only on the dimension, d. For the sake of brevity, we write $\operatorname{Lip}(V)$ to denote the maximum of 1 and the smallest Lipschitz constant for V.

For convenience, we will assume that $V \in C^{1,1}(\mathbb{R}^d; \mathbb{R}^d)$ qualitatively; since every bound we prove depends only on the Lipschitz norm of V, this condition can be dropped by approximating V by its mollification. We also assume that there there is zero average drift, i.e. $\mathbb{E}[V] = 0$.

If $E \subseteq \mathbb{R}^d$, we write $\mathcal{G}(E)$ to denote the σ-algebra generated by V restricted to E. That is, $\mathcal{G}(E)$ is the smallest σ-algebra such that the random variables $V(x)$ are $\mathcal{G}(E)$-measurable for each $x \in E$. We assume that V has unit range of dependence, which means that if $A, B \subseteq \mathbb{R}^d$ are sets with $\operatorname{dist}(A, B) \geq 1$, then $\mathcal{G}(A)$ and $\mathcal{G}(B)$ are independent.

Given $t > 0$ and a measurable function $\alpha\colon [0,t] \to B_1$, define the *controlled path* $X_x^\alpha\colon [0,t] \to \mathbb{R}^d$ to be the solution to the initial-value problem

$$\begin{cases} \dot{X}_x^\alpha = \alpha + V(X_x^\alpha) \\ X_x^\alpha(0) = x. \end{cases} \tag{5.5}$$

For each $x \in \mathbb{R}^d$, define the *reachable set at time t* by

$$\mathcal{R}_t(x) := \{y \in \mathbb{R}^d \mid \exists\, \alpha\colon [0,t] \to B_1 \text{ such that } X_x^\alpha(t) = y\}. \tag{5.6}$$

Note that this definition still makes sense for $t < 0$, if we interpret $[0,t]$ as $[t,0]$. For convenience, we also define the sets

$$\mathcal{R}_t^-(x) := \bigcup_{0 \leq s \leq t} \mathcal{R}_s(x)$$

for $t \geq 0$ and

$$\mathcal{R}_t^+(x) := \bigcup_{t \leq s \leq 0} \mathcal{R}_s(x)$$

for $t \leq 0$. Define the *first passage time*

$$\theta(x,y) := \inf\{t \mid y \in \mathcal{R}_t(x)\}.$$

Finally, if $E \subseteq \mathbb{R}^d$ is a set, we define

$$\mathcal{R}_t(E) = \bigcup_{e \in E} \mathcal{R}_t(e),$$

and we do the same for $\mathcal{R}_t^-$ and $\mathcal{R}_t^+$.

5.1.4 Acknowledgment

I would like to thank my advisor, Charles Smart, for suggesting the problem and for many helpful conversations.

5.2 Local waiting time estimates

In this section, we adapt the proof of Burago–Ivanov–Novikov [6] to estimate the waiting time for the metric problem associated to the G equation.

5.2.1 The incompressible case

First, we prove that, with high probability, sufficiently large $(d-1)$-dimensional cubes have very little flux.

Let $E(R_1, R_0, \varepsilon)$ be the event that every axis-aligned $(d-1)$-dimensional cube B, of radius between R_0 and R_1, which intersects Q_{R_1} satisfies

$$\left| \int_B V(x) \cdot \nu(x) \, \mathrm{d}x \right| \leq \varepsilon |B|, \tag{5.7}$$

where $\nu \colon B \to \mathbb{R}^d$ denotes a unit normal to B.

Lemma 5.2.1. *The event $E(R_1, R_0, \varepsilon)$ has probability at least*

$$\mathbb{P}[E(R_1, R_0, \varepsilon)] \geq 1 - C \left(\frac{R_1 \operatorname{Lip}(V)}{\varepsilon} \right)^d \left(\frac{R_1 (1 + \|V\|_{L^\infty})}{\varepsilon} \right) \exp\left(\frac{-\varepsilon^2 R_0^{d-1}}{C \|V\|_{L^\infty}} \right).$$

Proof. Step 1. Let B be an axis-aligned $(d-1)$-dimensional cube of radius $r \geq R_0$. Partition B into at least $C^{-1}r^{d-1}$ equally sized $(d-1)$-dimensional cubes, called $B_1, \ldots, B_n$, of radius between 1 and 2. For each i, the random variable $\int_{B_i} V(x) \cdot \nu(x) \, \mathrm{d}x$ has expectation zero and absolute value at most $C\|V\|_{L^\infty}$. The random variables for non-neighboring cubes are independent, so we can group the sum

$$\int_B V(x) \cdot \nu(x) \, \mathrm{d}x = \sum_{i=1}^n \int_{B_i} V(x) \cdot \nu(x) \, \mathrm{d}x$$

into 2^{d-1} separate sums, each of which contains mutually independent random variable summands, which correspond to non-neighboring cubes. By Azuma's inequality, we conclude that (5.7) holds with probability at least

$$1 - \exp\left(\frac{-\varepsilon^2 R_0^{d-1}}{C\|V\|_{L^\infty}}\right).$$

Step 2. Use the union bound to apply Step 1 to every cube B which has a vertex in $\left(\frac{\varepsilon}{C \operatorname{Lip}(V)}\mathbb{Z}\right)^d \cap Q_{R_1+1}$ and radius in $\left(\frac{\varepsilon}{C(1+\|V\|_{L^\infty})}\mathbb{Z}\right) \cap [R_0 - 1, R_1 + 1]$, and conclude using the Lipschitz bound on V, translating and rescaling any cube B so that it has a vertex in this set. $\square$

Next, we show that a subset of ∂Q_R which has small boundary must also have small flux.

Lemma 5.2.2. *In the event $E(R_1, R_0, \varepsilon)$, if $D \subseteq \partial Q_R$ has a $(d-2)$-rectifiable boundary for some $0 \leq R \leq R_1$, then*

$$\left| \int_D V(x) \cdot \nu(x) \, \mathrm{d}x \right| \leq C\|V\|_{L^\infty} R_0 |\partial D| + \varepsilon |\partial Q_R|,$$

where $\nu \colon \partial Q_R \to \mathbb{R}^d$ denotes the outward unit normal to ∂Q_R.

Proof. This is exactly Lemma 3.3 from Burago–Ivanov–Novikov [6]; for completeness, we include the proof here.

Partition ∂Q_R into at least $C^{-1}\left(\frac{R}{R_0}\right)^{d-1}$ equally sized $(d-1)$-dimensional cubes, called $B_1, \ldots, B_n$, of radius between R_0 and $2R_0$. For each i, define $P_i := |\partial D \cap B_i|$ and $S_i := \min\{|B_i \cap D|, |B_i \setminus D|\}$. The isoperimetric inequality says that

$$S_i \leq CP_i^{(d-1)/(d-2)}$$

and the fact that $S_i \subseteq B_i$ implies that

$$S_i \leq CR_0^{d-1}.$$

Interpolating between these bounds, we conclude that

$$S_i \leq CP_iR_0.$$

On the other hand, in the event $E(R_1, R_0, \varepsilon)$ we have

$$\left| \int_{B_i \cap D} V(x) \cdot \nu(x) \, \mathrm{d}x + \int_{B_i \setminus D} V(x) \cdot \nu(x) \, \mathrm{d}x \right| \leq \varepsilon |B_i|.$$

Since one of the sets $B_i \cap D$ or $B_i \setminus D$ has measure S_i, we conclude that one of the integrals above has absolute value at most $\|V\|_{L^\infty} S_i$, so they both have absolute value at most $\varepsilon|B_i| + \|V\|_{L^\infty} S_i$. We conclude that

$$\left| \int_{B_i \cap D} V(x) \cdot \nu(x) \, \mathrm{d}x \right| \leq \varepsilon |B_i| + C\|V\|_{L^\infty} P_i R_0.$$

We conclude by summing over i. □

The next lemma shows a weak form of controllability. It can be found in Cardaliaguet–Souganidis [13] and we include it here as well for completeness.

Lemma 5.2.3. *The reachable set $\mathcal{R}_1^-(x_0)$ contains the cone*

$$\left\{x_0 + tv \mid v \in B_{1/2}(V(x_0)),\, 0 \le t \le (2\operatorname{Lip}(V)(1 + \|V\|_{L^\infty}))^{-1}\right\}.$$

Furthermore, suppose that $x_0 \notin \left(\mathcal{R}_{T+1}^-(x_0)\right)^{\mathrm{o}}$. Then $\mathcal{R}_T^-(x)$ is disjoint from the cone

$$\left\{x_0 + tv \mid v \in B_{1/2}(V(x_0)),\, -(2\operatorname{Lip}(V)(1 + \|V\|_{L^\infty}))^{-1} \le t < 0\right\}.$$

Proof. Let $v \in B_{1/2}(V(x_0))$ and $t \in (0, (2\operatorname{Lip}(V)(1 + \|V\|_{L^\infty}))^{-1})$. For any $t' \in (0,t)$, we have

$$|V(x_0) - V(x_0 + t'v)| \le t'|v| \operatorname{Lip}(V) \le \frac{1}{2}.$$

Therefore, $v \in B_1(V(x_0 + t'v))$ for all $t \in (0,t)$, so $x_0 + tv \in \mathcal{R}_t^-(x_0) \subseteq \mathcal{R}_1^-(x_0)$, which was the first claim. The contrapositive of the second claim follows by the same argument. □

Finally, we show that, on most of the boundary of the reachable set, the vector field V points toward the interior of the reachable set.

Lemma 5.2.4. *Let $R > 0$ and $T_0 \ge 0$. Then for every $\varepsilon > 0$, there is some $T_0 < T \le T_0 + CR^d/\varepsilon$ such that*

$$\left|\left\{x \in (\partial\mathcal{R}_T^-(0)) \cap Q_R \mid V(x)\cdot\nu(x) \ge -\frac{1}{2}\right\}\right| \le \varepsilon,$$

where $|\cdot|$ above denotes the Hausdorff $(d-1)$-measure.

Proof. We use the fact that $t \mapsto |\mathcal{R}_t^-(0) \cap Q_R|$ is Lipschitz with derivative

$$\partial_t|\mathcal{R}_t^-(0)| = \int_{(\partial\mathcal{R}_t^-(0))\cap Q_R} (1 + V(x)\cdot\nu(x))_+ \,\mathrm{d}x$$

almost everywhere, where ν denotes the outward unit normal to $\mathcal{R}_t^-(0)$. Also, for any $t \ge 0$

we have $|\mathcal{R}_t^-(0) \cap Q_R| \le |Q_R| \le (2R)^d$. The claim follows with $C = 2^{d+1}$ by the mean value theorem. □

All the ingredients for the proof of our local waiting time estimate are now in place.

Theorem 5.2.1. *Suppose that* $\operatorname{div} V = 0$ *almost surely. Let* $W := \inf\{t > 0 \mid \mathcal{R}_t^-(0) \supseteq B_{1/2}\}$. *Then for any* $\lambda \ge 1$,

$$\mathbb{P}[W \ge \lambda] \le C \exp\left(-C^{-1}\lambda^{(d-1)/d}\operatorname{Lip}(V)^{3-3d}(1 + \|V\|_{L^\infty})^{3-5d}\right).$$

Proof. Assume $d \ge 3$ (if $d = 2$, just add another dimension in which everything is constant). We follow the proof of Burago–Ivanov–Novikov [6], keeping track of an extra error term to get a quantitative estimate.

Let $T > 0$ and $R > 0$. The boundary $\partial(\mathcal{R}_T^-(0) \cap Q_R)$ has two main parts: we define

$$S_R := (\partial\mathcal{R}_T^-(0)) \cap Q_R$$

and

$$D_R := \mathcal{R}_T^-(0) \cap (\partial Q_R).$$

Further, we let

$$L_R := (\partial\mathcal{R}_T^-(0)) \cap (\partial Q_R).$$

Generically, S_R and D_R are $(d-1)$-dimensional and L_R is $(d-2)$-dimensional. Cannarsa–Frankowska [10] proved that the boundary of the reachable set $\mathcal{R}_T^-(0)$ is $C^{1,1}$ everywhere except at the origin, so, as in Burago–Ivanov–Novikov [6] (see the remark after Lemma 2.4), it can be equipped with a continuous unit normal and the divergence theorem holds.

If $x \in S_R$, let $\nu(x)$ denote the outward unit normal to $\mathcal{R}_T^-(0)$ and if $x \in \partial Q_R$, let $\nu(x)$ denote the outward unit normal to Q_R. We also define the subset P_R of S_R to be the part

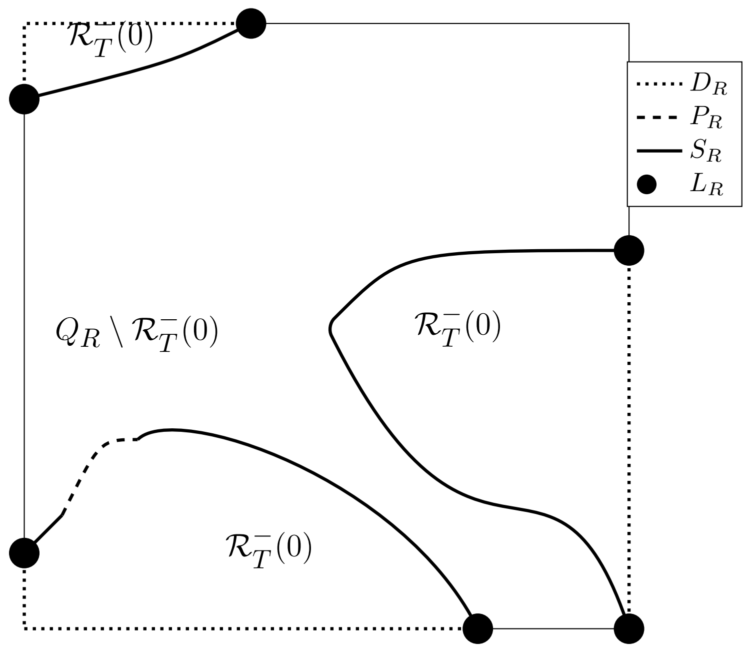

Figure 5.1: Parts of the reachable set $\mathcal{R}_T^-(0)$, the cube Q_R, and boundaries S_R, D_R, L_R, and P_R.

of the boundary of the reachable set which is growing at a speed of more than $\frac{1}{2}$:

$$P_R := \left\{ x \in S_R \mid V(x) \cdot \nu(x) \geq -\frac{1}{2} \right\}.$$

Integrate $\operatorname{div} V = 0$ in $\mathcal{R}_T^-(0) \cap Q_R$ to find

$$|S_R \setminus P_R| \leq 2 \int_{P_R \cup D_R} V(x) \cdot \nu(x) \, \mathrm{d}x \leq 2\|V\|_{L^\infty}(|P_R| + |D_R|), \tag{5.8}$$

where $|\cdot|$ denotes the Hausdorff measure of appropriate dimension (here it's $d-1$).

On the other hand, the co-area inequality yields

$$|S_R| \geq \int_0^R |L_r| \, \mathrm{d}r. \tag{5.9}$$

We also apply the isoperimetric inequality in ∂Q_R:

$$\min(|D_R|, |\partial Q_R \setminus D_R|) \leq C|L_R|^{\frac{d-1}{d-2}}, \tag{5.10}$$

and so the divergence theorem applied to Q_R yields

$$\left| \int_{D_R} V(x) \cdot \nu(x) \, \mathrm{d}x \right| \leq C|L_R|^{\frac{d-1}{d-2}}. \tag{5.11}$$

Combining (5.8), (5.9), and 5.11 yields

$$|L_R| \geq C^{-1} \left(\int_0^R |L_r| \, \mathrm{d}r \; - \; (1 + 2\|V\|_{L^\infty})|P_R| \right)^{\frac{d-2}{d-1}}. \tag{5.12}$$

Everything so far has only used incompressibility and boundedness of V in L^∞, and applies for all $T, R > 0$. From now on, we start selecting parameters to show that $B_{1/2} \subseteq \mathcal{R}_T^-(0)$. For the rest of the proof, we assume for contradiction that $B_{\frac{1}{2}} \not\subseteq \mathcal{R}_{T+1}^-(0)$.

First, choose $\varepsilon > 0$ such that $\int_0^1 |L_r| \; dr \geq \varepsilon$. By Lemma 5.2.3 and the isoperimetric inequality, we can choose

$$\varepsilon := C_0^{-1}(\mathrm{Lip}(V)(1 + \|V\|_{L^\infty}))^{1-d}, \tag{5.13}$$

where $C_0 = C_0(d) > 2^{d-1}$ will be chosen by the end of the proof.

Next, choose

$$R_1 := \left(\frac{\lambda \varepsilon}{1 + \|V\|_{L^\infty}} \right)^{1/d}$$

and

$$R_0 := \frac{R_1}{C_1(1 + \|V\|_{L^\infty})^2},$$

where $C_1 = C_1(d) > 0$ will also be chosen by the end of the proof. By Lemma 5.2.1, the event $E(R_1, R_0, \varepsilon)$ has probability at least

$$\mathbb{P}[E(R_1, R_0, \varepsilon)] \geq 1 - CR_0^{d+1}\|V\|_{C^{0,1}}^{C} \exp\left(-C^{-1}\lambda^{(d-1)/d}\varepsilon^{2+(d-1)/d}(1 + \|V\|_{L^\infty})^{-2d}\right)$$
$$\geq 1 - C\exp\left(-C^{-1}\lambda^{(d-1)/d}\mathrm{Lip}(V)^{3-3d}(1 + \|V\|_{L^\infty})^{3-5d}\right).$$

Work in the event $E(R_1, R_0, \varepsilon)$. By Lemma 5.2.4, we can choose $1 \leq T \leq CR_1^d(1 + \|V\|_{L^\infty})\varepsilon^{-1} \leq C\lambda$ such that $(1+2\|V\|_{L^\infty})|P_R| \leq \frac{\varepsilon}{2}$. Since we assume that $B_{1/2} \notin \mathcal{R}^-_{T+1}(0)$, our choice of T does not affect ε. Plug our choice of ε and T into (5.12) to see that

$$\frac{d}{dR}\int_0^R |L_r| \,\mathrm{d}r \geq C^{-1}\left(\int_0^R |L_r| \,\mathrm{d}r\right)^{\frac{d-2}{d-1}}$$

for all $1 \leq R \leq R_1$ and

$$\int_0^1 |L_r| \,\mathrm{d}r \geq \frac{\varepsilon}{2},$$

which implies

$$\int_0^R |L_r| \,\mathrm{d}r \geq C^{-1}(R-1)^{d-1} \tag{5.14}$$

for all $1 \leq R \leq R_1$. We make a note that the constant C in 5.14 does not depend on C_0 or C_1.

Combining (5.14) with (5.8) and (5.9), we conclude that

$$C^{-1}(R-1)^{d-1} \leq (1 + 2\|V\|_{L^\infty})|P_R| + \int_{D_R} V(x) \cdot \nu(x) \,\mathrm{d}x. \tag{5.15}$$

Apply Lemma 5.2.2 to D_R and combine with (5.15) to obtain

$$C^{-1}(R-1)^{d-1} \leq (1 + 2\|V\|_{L^\infty})|P_R| + C(1 + \|V\|_{L^\infty})R_0|L_R| + 2d\varepsilon R^{d-1}. \tag{5.16}$$

As long as $\varepsilon \leq \frac{1}{2^{d+2}dC}$ and $|P_R| \leq \frac{1}{4C(1+\|V\|_{L^\infty})}$, which we ensure by choosing $C_0 > 0$

sufficiently large in (5.13), then for $R \geq 2$ we have

$$R^{d-1} \leq CR_0(1 + \|V\|_{L^\infty})|L_R|. \tag{5.17}$$

We integrate and apply (5.9) to conclude that

$$|S_R| \geq \frac{R^d - 1}{CR_0(1 + \|V\|_{L^\infty})} \tag{5.18}$$

for every $2 \leq R \leq R_1$.

At $R = R_1$, this yields

$$|S_{R_1}| \geq C_1 C^{-1}(1 + \|V\|_{L^\infty}){R_1}^{d-1}.$$

To conclude, we choose C_1 large enough so that $|S_{R_1}| \geq 2(1 + \|V\|_{L^\infty})(1 + d2^d R_1^{d-1})$, which contradicts (5.8), as $|P_{R_1}| \leq 1$ and $D_{R_1} \subseteq \partial Q_{R_1}$ and hence $|D_{R_1}| \leq |\partial Q_{R_1}| = d2^d R_1^{d-1}$. $\square$

5.2.2 The compressible case

Next, we adapt the proof in the incompressible case to allow $|\operatorname{div} V|$ to be nonzero but small.

Proposition 5.2.5. *Let $W := \inf\{t > 0 \mid \mathcal{R}_t^-(0) \supseteq B_{1/2}\}$. For each $p > 0$, there is $\varepsilon(\mathrm{Lip}(V), \|V\|_{L^\infty}, p, d) > 0$ such that, if $|\operatorname{div} V| \leq \varepsilon$ almost surely, then*

$$\mathbb{P}\left[W \geq C\mathrm{Lip}(V)^{3d}(1 + \|V\|_{L^\infty})^{5d+4}\left(\log p^{-1}\right)^{d/(d-1)}\right] \leq p.$$

Furthermore, we can choose

$$\varepsilon \geq C^{-1}\mathrm{Lip}(V)^{-3}(1 + \|V\|_{L^\infty})^{-6}\left(\log p^{-1}\right)^{1/(d-1)}.$$

Proof. The proof is nearly identical to the proof of Theorem 5.2.1. The only difference is the the addition of $\varepsilon C R^d$ error terms in (5.8)

$$|S_R \setminus P_R| \leq 2 \int_{P_R \cup D_R} V(x) \cdot \nu(x)\, \mathrm{d}x + \varepsilon C R^d \leq 2\|V\|_{L^\infty}(|P_R| + |D_R|) + \varepsilon C R^d, \quad (5.19)$$

and (5.11)

$$\left| \int_{D_R} V(x) \cdot \nu(x)\, \mathrm{d}x \right| \leq C|L_R|^{\frac{d-1}{d-2}} + \varepsilon C R^d. \quad (5.20)$$

As long as $\varepsilon \leq C^{-1} R_1^{-1}$, where R_1 is defined in the proof of Theorem 5.2.1, the extra error term is at most $C^{-1} R^{d-1}$, and therefore does not affect any of the calculations. □

We conclude the section by showing that, even without assuming incompressibility, the reachable set at time t grows proportionally to t^d. This lemma plays a key role in ensuring that homogenization occurs in the sense of uniform convergence, by showing that no traps can arise where the reachable set stays bounded for a long time.

Proposition 5.2.6. *There is some $\varepsilon = \varepsilon(d) > 0$ such that, if $|\operatorname{div} V| \leq \varepsilon$ almost surely, then*

$$|\mathcal{R}_t^-(x_0)| \geq \frac{t^d}{2d}$$

for every $x_0 \in \mathbb{R}^d$ and $t \geq 0$ almost surely.

Proof. Assume for simplicity that $x_0 = 0$ and let $K := t(1 + \|V\|_{L^\infty})$. Then $\mathcal{R}_t^-(0) \subseteq Q_K$.

Since $|\mathcal{R}_t^-(0)| \geq \frac{1}{2}|B_t|$ for sufficiently small $t \geq 0$, it suffices to show that

$$\partial_t |\mathcal{R}_t^-(0)| = \int_{\partial \mathcal{R}_t^-(0)} (1 + V(x) \cdot \nu(x))_+ \, \mathrm{d}x \geq \frac{1}{2} |\partial \mathcal{R}_t^-(0)|,$$

where ν denotes the outward unit normal to $\mathcal{R}_t^-(0)$.

By the divergence theorem,

$$\int_{\partial \mathcal{R}_t^-(0)} (1 + V(x) \cdot \nu(x))\, \mathrm{d}x = |\partial \mathcal{R}_t^-(0)| + \int_{\mathcal{R}_t^-(0)} \operatorname{div} V(x)\, \mathrm{d}x.$$

To get rid of small parts of the boundary of the reachable set, we define its discretized version by

$$E := \bigcup \left\{ x + \overline{Q_{1/2}} \; : \; x \in \mathbb{Z}^d \text{ and } |\mathcal{R}_t^-(0) \cap (x + Q_{1/2})| \geq \frac{1}{2} \right\}.$$

We will estimate the divergence term by integrating over E instead and using the unit range of dependence. First, we bound the symmetric difference by

$$\big|(E \setminus \mathcal{R}_t^-(0)) \cup (\mathcal{R}_t^-(0) \setminus E)\big| \leq C|\partial \mathcal{R}_t^-(0)|,$$

by the isoperimetric inequality applied in each integer-centered unit cube. The bound on $\operatorname{div} V$ then implies

$$\left| \int_E \operatorname{div} V(x) \, \mathrm{d}x - \int_{\mathcal{R}_t^-(0)} \operatorname{div} V(x) \, \mathrm{d}x \right| \leq \varepsilon C |\partial \mathcal{R}_t^-(0)|. \tag{5.21}$$

On the other hand, the isoperimetric inequality also yields

$$|\partial E| \leq C|\partial \mathcal{R}_t^-(0)|. \tag{5.22}$$

We claim that $\int_E \operatorname{div} V(x) \, \mathrm{d}x \leq \varepsilon C|\partial E|$ almost surely. Indeed, if this is true, then since there are only countably many possible values for E, the inequality holds for all possible E almost surely. We finish by combining the claim with (5.21) and (5.22) to conclude that

$$\int_E \operatorname{div} V(x) \, \mathrm{d}x \leq \varepsilon C|\partial \mathcal{R}_t^-(0),$$

so choosing $\varepsilon := \frac{1}{2}C^{-1}$ allows us to conclude.

It remains to prove the claim. Let

$$D := \{x \in E \mid \operatorname{dist}(x, \partial E) \leq 1\}.$$

Then $|D| \leq C|\partial E|$ (as before, we abuse notation by using $|\cdot|$ to denote the d-dimensional measure on the left and $(d-1)$-dimensional measure on the right-hand side), since every integer-centered unit cube in E which intersects ∂E is adjacent to an integer-centered unit cube in E which has at least one of its faces contained in ∂E.

By the divergence theorem,

$$\int_E \operatorname{div} V(x)\, \mathrm{d}x = \int_{\partial E} V(x) \cdot \nu(x)\, \mathrm{d}x,$$

where ν denotes the outward unit normal to E. The integral on the right-hand side depends only on V restricted to ∂E, and is therefore independent from the random variable

$$\int_{E \setminus D} \operatorname{div} V(x)\, \mathrm{d}x.$$

However, we have

$$\int_E \operatorname{div} V(x)\, \mathrm{d}x = \int_D \operatorname{div} V(x)\, \mathrm{d}x + \int_{E \setminus D} \operatorname{div} V(x)\, \mathrm{d}x \leq \varepsilon C |\partial E| + \int_{E \setminus D} \operatorname{div} V(x)\, \mathrm{d}x.$$

Taking the conditional expectation with respect to $\mathcal{G}(E \setminus D)$ and using independence yields the claim. $\square$

5.3 Global waiting time estimates

Next, we improve our local waiting time bounds to global bounds, by showing that the region where the local waiting time is small contains a supercritical percolation cluster. The argument is identical to that in [15], so we only sketch the proofs.

Lemma 5.3.1. *There is a constant $C = C(d) > 0$ such that for each $0 < p < 1$, if*

$$|\operatorname{div} V| \leq C^{-1}(\|V\|_{C^{0,1}} + 1)^{-C}\left(\log{(1-p)^{-1}}\right)^{-1/(d-1)}$$

almost surely, then the function $G\colon \mathbb{Z}^d \to \{0,1\}$, defined by

$$G(v) = \begin{cases} 1 & \theta(x,y) \leq C(\|V\|_{C^{0,1}} + 1)^C \log{(1-p)^{-1}} \text{ for all } x, y \in B_{\sqrt{d}}(v) \\ 0 & \text{otherwise,} \end{cases}$$

is Z^d-translation invariant with finite range of dependence

$$C_{dep} \leq C(\|V\|_{C^{0,1}} + 1)^C \log{(1-p)^{-1}}$$

and $\mathbb{P}[G(0) = 1] \geq p$.

Proof. By Proposition 5.2.5, applied to V and $-V$ separately, there is $C > 0$ such that $\mathbb{P}[G(0) = 1] \geq p$. The $\mathbb{Z}^d$-translation invariance of G follows from that of $\mathbb{P}$. Finite range of dependence follows from the fact that the value of $G(v)$ depends only on controlled paths starting in $B_{\sqrt{d}}(v)$ which run for time at most $C\|V\|_{C^{0,1}}^C \log{(1-p)^{-1}}$. The bound on C_{dep} follows, noting that the top speed of a path is $1 + \|V\|_{L^\infty}$. □

Definition 5.3.2. To translate between $\mathbb{Z}^d$ and $\mathbb{R}^d$, for each set $E \subseteq \mathbb{Z}^d$ we introduce the "solidification"

$$\sigma(E) := E + \left[-\frac{1}{2}, \frac{1}{2}\right]^d.$$

Theorem 5.3.1. *There is a constant $C = C(d) > 0$ such that, if*

$$|\operatorname{div} V| \leq C^{-1}(\|V\|_{C^{0,1}} + 1)^{-C},$$

then for each $R \geq 1$, the "extra waiting time"

$$\mathcal{E}(R) := \sup_{x,y \in Q_R} \frac{\theta(x,y) - C(1+|x-y|)}{C(\|V\|_{C^{0,1}}+1)^C}$$

satisfies

$$\mathbb{P}[\mathcal{E}(R) > n] \leq CR^d \exp(-C^{-1}(\|V\|_{C^{0,1}}+1)^{-C} n).$$

Proof. We partition $\mathbb{R}^d$ into cubes of side length 1, centered at points in $\mathbb{Z}^d$. Fix $p := 1 - \exp(-CC_{\text{dep}}^d)$, where C_{dep} is given by Lemma 5.3.1. By Lemma 5.3.1, if

$$|\operatorname{div} V| \leq C^{-1}(\|V\|_{C^{0,1}}+1)^{-C},$$

then $\mathbb{P}[G(v) = 1] \geq p$. For $v \in \mathbb{Z}^d$, we say that the site v is open if $G(v) = 1$ and closed otherwise. We say that a point $x \in \mathbb{R}^d$ lies near an open site if there is some $v \in \mathbb{Z}^d$ such that $x \in \sigma(\{v\})$. Let $S = \lceil R \rceil$ and $x, y \in Q_S$. For convenience, we will prove that $\mathbb{P}[\mathcal{E}(R) > Cn] \leq CR^d \exp(-C^{-1}(\|V\|_{C^{0,1}}+1)^{-C} n)$; this easily implies the original claim by changing C to C^2.

Step 1. We claim that, without loss of generality, we may assume that x and y lie near sites in the same open cluster. Indeed, if not, then (say) x lies in a connected component, $\mathfrak{D}$, of $\mathcal{Q}_{S+\delta} \setminus \mathfrak{C}$, where $\delta > 0$ is chosen appropriately and $\mathcal{C}$ is the largest open cluster contained in $\mathfrak{C}_{S+\delta}$. With high probability, $\mathfrak{D} \cap \partial \mathcal{Q}_{S+\delta} = \emptyset$, so Proposition 5.2.6 implies that $\mathcal{R}_t^-(x) \cap \sigma(\mathfrak{D}) \neq \emptyset$ for some time $t > 0$ which is not too large, and thus we replace x by any member of this intersection. Repeating for y if necessary, the claim follows.

Step 2. To show that

$$\theta(x,y) - C|x-y| \leq C(\|V\|_{C^{0,1}}+1)^C n,$$

we will build a "skeleton" of points $x = x_0, x_1, \ldots, x_k = y$ which all lie near open sites and

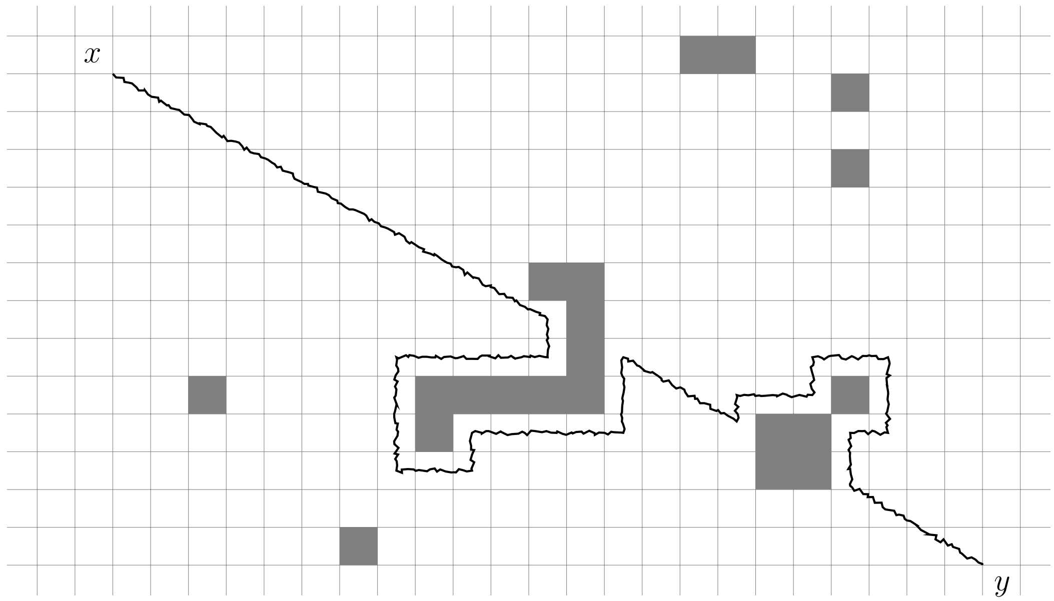

Figure 5.2: An example of our controlled path from x to y; cubes corresponding to closed sites are shaded

satisfy $|x_{i+1} - x_i| \leq \sqrt{d}$ for each $0 \leq i < k$. Then, by connecting the points with paths given by Lemma 5.3.1, we can build a controlled path of length at most $C(\|V\|_{C^{0,1}} + 1)^C k$ which follows the skeleton.

Our strategy is to go from x to y in a straight line, taking necessary detours around closed clusters, as shown in Figure 5.2. We omit the remaining details, as they are identical to those in the proof of Theorem 3.2 in [15].

□

5.4 Random fluctuations in first-passage time

Next, we consider how much $\theta(0, y)$ deviates from its expectation. Our proof will follow roughly the same path as the proof of Proposition 4.1 from Armstrong–Cardaliaguet–Souganidis [3], with some modifications which are made possible by the controllability estimate. As in the previous section, the proofs are nearly identical to those in [15], with a bit of care taken to keep track of the dependence of the constants on V, so we omit them.

To get started, we introduce a "guaranteed" version of first passage time. For any $\rho > 0$, define the ρ-guaranteed reachable set recursively by

$$\mathcal{R}_t^\rho(x) := \begin{cases} \mathcal{R}_t^-(x) & \text{if } t < \rho \\ \mathcal{R}_\rho^-(\mathcal{R}_{t-\rho}^\rho(x)) \cup (\mathcal{R}_{t-\rho}^\rho(x) + \overline{B_1}) & \text{otherwise.} \end{cases}$$

The ρ-guaranteed reachable set is similar to the reachable set, except that we enforce expansion at a rate of at least $1/\rho$ in a certain discrete sense. We similarly define the ρ-guaranteed first passage time

$$\theta^\rho(x,y) = \min\{t \geq 0 \mid y \in \mathcal{R}_t^\rho(x)\}.$$

Note that the ρ-guaranteed first passage time coincides with the usual first passage time if we have sufficient control on the extra waiting time $\mathcal{E}$ (from Theorem 5.3.1) in a suitable domain.

Fix some $y \in \mathbb{R}^d$ and define the random variable $\{Z_t^\rho\}_{t\geq 0}$ by

$$Z_t^\rho := \mathbb{E}\left[\theta^\rho(0,y) \mid \mathcal{F}_t\right],$$

where $\mathcal{F}_t$ is the σ-algebra generated by the environment $V(x)$ restricted to the ρ-guaranteed reachable set $\mathcal{R}_t^\rho(0)$. In other words, $\mathcal{F}_t$ is the smallest σ-algebra so that the functions $V(x)\mathbb{1}_{x\in\mathcal{R}_t^\rho(0)}$ are $\mathcal{F}_t$-measurable for every $x \in \mathbb{R}^d$. Since $\mathcal{R}_t^\rho(0)$ are increasing sets, $\{\mathcal{F}_t\}_{t\geq 0}$ is a filtration, so $\{Z_t^\rho\}_{t\geq 0}$ is a martingale.

We first show that Z_t^ρ depends mostly on the shape of $\mathcal{R}_t^\rho(0)$, without regard for the values of V inside $\mathcal{R}_t^\rho(0)$. In order to condition on the approximate shape of the reachable set, for any $E \subseteq \mathbb{R}^d$ we introduce the discretization

$$\operatorname{disc}(E) := \{z \in d^{-1/2}\mathbb{Z}^d \mid B(z,1) \cap E \neq \emptyset\}.$$

Lemma 5.4.1. *For any $t \geq 0$, we have*

$$\left|\max(Z_t^\rho, t) - f(t, \operatorname{disc}(\mathcal{R}_t^\rho(0)))\right| \leq 3\rho,$$

where we define $f(t, S)$, for any $t \geq 0$ and any finite set $S \subseteq d^{-1/2}\mathbb{Z}^d$, by

$$f(t, S) := t + \mathbb{E}\left[\theta^\rho(S, y)\right].$$

Next, we claim that our approximation for Z_t^ρ, given by $f(t, \operatorname{disc}(\mathcal{R}_t^\rho(0)))$, has bounded increments.

Lemma 5.4.2. *Let $t, s \geq 0$. Then*

$$\left|f(t, \operatorname{disc}(\mathcal{R}_t^\rho(0))) - f(s, \operatorname{disc}(\mathcal{R}_s^\rho(0)))\right| \leq 2\rho + |t - s|(\|V\|_{L^\infty}\rho + \rho + 2).$$

Together, the previous lemmas show that the martingale $\{Z_t^\rho\}_{t \geq 0}$ has bounded increments. Applying Azuma's inequality to $\{Z_t^\rho\}_{t \geq 0}$, choosing ρ carefully to balance competing error terms, we deduce a tail bound on the distribution of

$$Z_t^\infty = \mathbb{E}[\theta(0, y) \mid \mathcal{F}_t].$$

Proposition 5.4.3. *There is a constant $C = C(d) > 0$ such that, if $y_1, y_2 \in \mathbb{R}^d$,*

$$\lambda \geq C(\|V\|_{C^{0,1}} + 1)^C |y_1 - y_2|^{1/2} \log^2 |y_1 - y_2|,$$

and

$$|\operatorname{div} V| \leq C^{-1}(\|V\|_{C^{0,1}} + 1)^{-C},$$

then

$$\mathbb{P}[|\theta(y_1, y_2) - \mathbb{E}[\theta(y_1, y_2)]| > \lambda] \le C \exp\left(\frac{-C^{-1}(\|V\|_{C^{0,1}} + 1)^{-C} \lambda^{1/2}}{|y_1 - y_2|^{1/4}} \right).$$

5.5 Nonrandom scaling bias

In this section, we use the bounds on random fluctuations of θ to bound the difference between $\mathbb{E}[\theta(0, y)]$ and $\lim_{\varepsilon \to 0^+} \varepsilon \mathbb{E}[\theta(0, \varepsilon^{-1} y)]$, which we refer to as the *nonrandom scaling bias.* We follow a similar argument as in Alexander [2], who proved an analogous result for Bernoulli percolation in two dimensions. Happily, the argument goes through in any dimension with the help of the Hobby–Rice theorem [24], a version of which we quote below. We include their proof, because it is short and beautiful.

Theorem 5.5.1 (Hobby–Rice)**.** *Let* $\gamma\colon [0, 1] \to \mathbb{R}^d$ *be continuous. Then there is a partition*

$$0 = t_0 < t_1 < \cdots < t_{d+1} = 1,$$

along with signs

$$\delta_1, \ldots, \delta_{d+1} \in \{-1, +1\},$$

such that

$$\sum_{k=1}^{d+1} \delta_k(\gamma(t_k) - \gamma(t_{k-1})) = 0.$$

Proof. We parameterize signed partitions by points on the d-sphere as follows. Given a point $x \in S^d$, we define associated signs by $\delta_k^x = \operatorname{sgn}(x_k)$ and define $\{t_k\}_k$ to be the unique partition of $[0, 1]$ such that $t_k^x - t_{k-1}^x = x_k^2$. As such, we define the map $f\colon S^d \to \mathbb{R}^d$ by

$$f(x) := \sum_{k=1}^{d+1} \delta_k^x(\gamma(t_k^x) - \gamma(t_{k-1}^x)).$$

By the Borsuk–Ulam theorem, there is some $x \in S^d$ such that $f(x) = f(-x)$. However, f is odd, so $f(x) = 0$, which proves the claim. $\square$

We now bound the nonrandom scaling bias. Given a function $f\colon \mathbb{R}^d \to \mathbb{R}$, we define the large-scale limit $\overline{f}\colon \mathbb{R}^d \to \mathbb{R}$ by

$$\overline{f}(x) := \lim_{\varepsilon \to 0^+} \varepsilon f\left(\varepsilon^{-1} x\right).$$

Proposition 5.5.1. *Assume that the law of V is $\mathbb{Z}^d$-translation invariant and that*

$$|\operatorname{div} V| \le C^{-1}(\|V\|_{C^{0,1}} + 1)^{-C}.$$

Let $f(x) := \mathbb{E}[\theta(0,x)]$. Then

$$|f(x) - \overline{f}(x)| \le C(\|V\|_{C^{0,1}} + 1)^C |x|^{1/2} \log^2 |x|$$

for all $|x| \ge 1$.

Proof. First, note that translation invariance and the controllability bound in Theorem 5.3.1 implies that f is subadditive up to a constant, that is,

$$f(x+y) \le f(x) + f(y) + C(\|V\|_{C^{0,1}} + 1)^C$$

for all $x, y \in \mathbb{R}^d$, so it follows immediately that

$$f(y) \ge \overline{f}(y) + C(\|V\|_{C^{0,1}} + 1)^C.$$

Our goal is to show that f is superadditive up to some small error, after which we apply an argument similar to that in Fekete's lemma to bound the difference between f and its large-scale limit.

Fix any $y \in \mathbb{R}^d$. By Proposition 5.4.3, Theorem 5.3.1, and the union bound, the event that

$$|\theta(v,w) - \mathbb{E}[\theta(v,w)]| \leq C(\|V\|_{C^{0,1}} + 1)^C |y|^{1/2} \log^2 |y| \tag{5.23}$$

for all

$$|v|, |w| \leq C(\|V\|_{C^{0,1}} + 1)^C |y|$$

has positive probability. By translation invariance, this implies that

$$|\theta(w,x) - \theta(y,z)| \leq C(\|V\|_{C^{0,1}} + 1)^C |y|^{1/2} \log^2 |y| \tag{5.24}$$

whenever $|(x-w)-(z-y)| \leq C$ and

$$|w|, |x|, |y|, |z| \leq C(\|V\|_{C^{0,1}} + 1)^C |y|.$$

In an instance of this event, let $\gamma\colon [0, \theta(0,y)] \to \mathbb{R}^d$ be a controlled path from 0 to y. Applying Theorem 5.5.1 to γ, we conclude that there are points

$$0 \leq s_1 < t_1 \leq s_2 < t_2 \leq \cdots \leq s_\ell < t_\ell \leq 1,$$

where $\ell \leq \frac{d+1}{2}$, such that

$$\sum_{k=1}^{\ell} \gamma(t_k) - \gamma(s_k) = \frac{1}{2} y.$$

Applying (5.23) and (5.24), we conclude that

$$
\begin{aligned}
2f\left(\frac{1}{2}y\right) &\leq C(\|V\|_{C^{0,1}}+1)^C|y|^{1/2}\log^2|y| + \theta\left(0,\frac{1}{2}y\right) + \theta\left(\frac{1}{2}y,y\right) \\
&\leq C(\|V\|_{C^{0,1}}+1)^C|y|^{1/2}\log^2|y| + \left(\sum_{k=1}^{\ell}\theta(\gamma(s_k),\gamma(t_k))\right) \\
&\qquad + \left(\theta(0,\gamma(s_1)) + \theta(\gamma(t_1),y) + \sum_{k=2}^{\ell}\theta(\gamma(t_{k-1}),\gamma(s_k))\right) \\
&\leq C(\|V\|_{C^{0,1}}+1)^C|y|^{1/2}\log^2|y| + \theta(0,y) \\
&\leq C(\|V\|_{C^{0,1}}+1)^C|y|^{1/2}\log^2|y| + f(y).
\end{aligned}
$$

It follows by induction that

$$2^n f(y) \leq f\left(2^n y\right) + \sum_{k=0}^{n-1} 2^{n-1-k} C(\|V\|_{C^{0,1}}+1)^C|2^k y|^{1/2}\log^2|2^k y|.$$

Dividing by 2^n on both sides and taking the limit as $n \to \infty$ yields

$$f(y) \leq \overline{f}(y) + C(\|V\|_{C^{0,1}}+1)^C|y|^{1/2}\log^2|y|.$$

$\square$

5.6 Homogenization

In this section, we prove our main homogenization results for the shape of the reachable set and for solutions of the G equation, using our bounds on convergence of first-passage time to the large-scale average.

5.6.1 The reachable set

We combine the random fluctuation bound and nonrandom bias bound to deduce a rate of convergence of the rescaled reachable sets.

Proposition 5.6.1. *Assume that the law of V is $\mathbb{Z}^d$-translation invariant and that*

$$|\operatorname{div} V| \leq C^{-1}(\|V\|_{C^{0,1}} + 1)^{-C}.$$

Then there is a closed set $\mathcal{S} \subseteq \mathbb{R}^d$ such that, for all $t \geq 0$,

$$\begin{aligned}\mathbb{P}\left[\operatorname{dist}_H(\mathcal{R}_t(0), t\mathcal{S}) > C(\|V\|_{C^{0,1}} + 1)^C t^{1/2} \log^2 t + \lambda\right] \\ \leq C \exp\left(\frac{-C^{-1}(\|V\|_{C^{0,1}} + 1)^{-C} \lambda^{1/2}}{t^{1/4}}\right),\end{aligned}$$

where dist_H denotes the Hausdorff distance. Furthermore, there is a random variable T_0, with

$$\mathbb{E}[\exp(C^{-1}(\|V\|_{C^{0,1}} + 1)^C \log^{3/2} T_0)] < \infty,$$

such that

$$\sup_{(t,x)\in[0,T]\times B_T} \frac{\operatorname{dist}_H(\mathcal{R}_t(x), x + t\mathcal{S})}{T^{1/2}\log^2 T} \leq C(\|V\|_{C^{0,1}} + 1)^C$$

for all $T \geq T_0$.

Proof. For the first claim, let $t \geq 0$. Apply Theorem 5.3.1 to $B_{(1+\|V\|_{L^\infty})t+1}$ and Proposition 5.4.3 to every $x \in \mathbb{Z}^d \cap \overline{B_{(1+\|V\|_{L^\infty})t}}$ and use the union bound to see that as long as $\lambda \geq C(\|V\|_{C^{0,1}} + 1)^C t^{1/2} \log^2 t$ we have

$$\mathbb{P}\left[\forall x \in \overline{B_{(1+\|V\|_{L^\infty})t}} :\ |\theta(0,x) - \mathbb{E}[\theta(0,x)]| > \lambda\right] \leq C \exp\left(\frac{-C^{-1}(\|V\|_{C^{0,1}} + 1)^{-C} \lambda^{1/2}}{t^{1/4}}\right), \tag{5.25}$$

where we absorbed polynomials into the exponential by enlarging the constant C. Note also that Theorem 5.3.1 implies that if $0 \leq r \leq s$, then

$$\mathbb{P}\left[\mathcal{R}_r(0) \nsubseteq \mathcal{R}_s(0) + B_\lambda\right] \leq C \exp\left(\frac{-C^{-1}(\|V\|_{C^{0,1}}+1)^{-C}\lambda^{1/2}}{t^{1/4}}\right). \tag{5.26}$$

This bounds the random error. On the other hand, Proposition 5.5.1 shows that

$$0 \leq \mathbb{E}[\theta(0,x)] - \overline{\theta}(x) \leq C(\|V\|_{C^{0,1}}+1)^C |x|^{1/2} \log^2 |x|, \tag{5.27}$$

where

$$\overline{\theta}(x) := \lim_{\varepsilon \to 0^+} \varepsilon \mathbb{E}[\theta(0, \varepsilon^{-1} x)].$$

We define $\mathcal{S} := \{x \in \mathbb{R}^d \mid \overline{\theta}(x) \leq 1\}$. The estimates (5.25) and (5.27) combine to say that, with high probability, the first passage time $\theta(0,x)$ from 0 to any point x is close to the large-scale average $\overline{\theta}(x)$. Furthermore, the estimate (5.26) says that once a controlled path reaches x, the reachable set stays close to x for all later times (the controllability estimate guarantees the existence of controlled paths in the form of short loops). Unwrapping the definition of Hausdorff distance, along with the fact that $\overline{\theta}$ is positively homogeneous of degree one, i.e. $\theta(tx) = t\theta(x)$ for $t \geq 0$, yields the first claim.

For the second claim, apply the first claim to every $(t,x) \in (\mathbb{Z} \cap [0,T]) \times (\mathbb{Z}^d \cap B_T)$ and the union bound to conclude that

$$\begin{aligned}\mathbb{P}\left[\sup_{t \in \mathbb{Z} \cap [0,T]} \sup_{x \in \mathbb{Z}^d \cap B_T} \operatorname{dist}_H(\mathcal{R}_t(x), x + t\mathcal{S}) > C(\|V\|_{C^{0,1}}+1)^C T^{1/2} \log^2 T + \lambda\right] \\ \leq CT^{d+1} \exp\left(\frac{-C^{-1}(\|V\|_{C^{0,1}}+1)^{-C}\lambda^{1/2}}{T^{1/4}}\right).\end{aligned}$$

Next, apply the controllability estimate in B_T to see that the same holds for all $(t,x) \in$

$[0,T] \times B_T$, by enlarging the constant C. Plugging in $\lambda = CT^{1/2}\log^2 T$ shows that

$$\mathbb{P}\left[\sup_{(t,x)\in[0,T]\times B_T} \frac{\operatorname{dist}_H(\mathcal{R}_t(x), x+t\mathcal{S})}{T^{1/2}\log^2 T} > C(\|V\|_{C^{0,1}}+1)^C\right] \\ \leq C\exp(-C^{-1}(\|V\|_{C^{0,1}}+1)^{-C}\log^{3/2} T),$$

and the conclusion follows. □

5.6.2 Solutions of the G equation

We now turn to the proof of Theorem 5.1.1.

Proof. Let u^ε be a solution to the G equation (5.1) with initial data u_0, and let $\bar{u}$ be the solution the the effective equation (5.2) with the same initial data. The effective Hamiltonian is given by

$$H(p) := \sup_{v\in\mathcal{S}} p\cdot v \tag{5.28}$$

The optimal control formulations are

$$u^\varepsilon(t,x) = \sup_{\varepsilon\mathcal{R}_{\varepsilon^{-1}t}(\varepsilon^{-1}x)} u_0 \tag{5.29}$$

and

$$u^\varepsilon(t,x) = \sup_{x+t\mathcal{S}} u_0 \tag{5.30}$$

respectively.

Using the representation formulas (5.30) and (5.29), we see that for every $0 \leq t \leq T$ and $x \in B_T$ we have

$$|u^\varepsilon(t,x) - \bar{u}(t,x)| = \left|\sup_{\varepsilon\mathcal{R}_{\varepsilon^{-1}t}(\varepsilon^{-1}x)} u_0 - \sup_{x+t\mathcal{S}} u_0\right| \leq \operatorname{Lip}(u_0)\operatorname{dist}_H(\varepsilon\mathcal{R}_{\varepsilon^{-1}t}(\varepsilon^{-1}x), x+t\mathcal{S}).$$

Rescaling by ε^{-1} and applying Proposition 5.6.1 yields

$$\sup_{(t,x)\in[0,T]\times B_T} \operatorname{dist}_H(\varepsilon\mathcal{R}_{\varepsilon^{-1}t}(\varepsilon^{-1}x), x + t\mathcal{S}) \leq C(\|V\|_{C^{0,1}} + 1)^C (T\varepsilon)^{1/2} \log^2(\varepsilon^{-1}T)$$

for all $T \geq \varepsilon T_0$, and the result follows. $\square$

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