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MEAN CURVATURE FLOW ON THE THREE-SPHERE

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Dedicated to my parents

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ABSTRACT

In this thesis we study eternal solutions to the Allen-Cahn equation in the 3-sphere, in view of the connection between the gradient flow of the associated energy functional, and the mean curvature flow. We construct eternal integral Brakke flows that connect Clifford tori to equatorial spheres and study a family of such flows, in particular their symmetry properties. Our approach is based on the realization of Brakke's motion by mean curvature as a singular limit of Allen-Cahn gradient flows, as studied by Ilmanen [40] and Tonegawa [70], and it uses the classification of ancient gradient flows in spheres, by K. Choi and C. Mantoulidis [18], as well as the rigidity of stationary solutions with low Morse index proved by F. Hiesmayr [33]. Many of the results in this thesis are from joint work with Pedro Gaspar [13].

CHAPTER 1

INTRODUCTION

We study low energy solutions of the *parabolic Allen-Cahn equation*

$$\epsilon\,\partial_t u = \epsilon \Delta_g u - \frac{1}{\epsilon} W'(u). \tag{PAC}$$

in the 3-sphere in connection with the *mean curvature flow* (MCF). Here W is a nonnegative double-well potential with two wells at ± 1, such as $W(u) = (1-u^2)^2/4$.

The Allen-Cahn equation models phase transition and separation phenomena [8]. The motion of the diffuse transition region – where u remains bounded away from minima of W – was studied by several authors, who established the convergence of this interface, as $\epsilon \downarrow 0$, to a hypersurface which evolves by the mean curvature flow, in different formulations. We mention here [6, 14, 15, 23, 20, 62, 67] and the references therein. The convergence to measure-theoretic solutions to the MCF (in the sense of Brakke [4]) was studied by Ilmanen [40] and Soner [67], and more recently by Tonegawa [70], Mizuno-Tonegawa [53], Pisante-Punzo [59] and Sato [63].

In this thesis, we study such limit flows in the 3-dimensional round sphere. We show that there are (weak) solutions to the MCF connecting Clifford tori to equatorial spheres constructed as the singular limit of solutions to (PAC) in S^3, as $\epsilon \downarrow 0$, and study a family of such limit flows. Some of the motivations for our study are the connections of the Allen-Cahn equation with minimal hypersurfaces from a variational and a dynamical perspective, and recent developments on the existence and classification of mean curvature flow, as we briefly discuss next.

For stationary solutions of (PAC), the convergence to minimal hypersurfaces, which are stationary solutions to the MCF, was studied by Modica and Mortola [55], Kohn-Sternberg [45], Hutchinson-Tonegawa [39], Tonegawa-Wickramasekera [72], among many others. These results have motivated recent results in the study of minimal hypersurfaces arising as limit interfaces from a variational perspective.

In [29], Guaraco constructed minimal hypersurfaces in closed manifolds using classical variational techniques and the convergence and regularity results of [72, 79] as an alternative to the Almgren-Pitts min-max approach [60]. Very recently, Chodosh and Mantoulidis [17] obtained curvature estimates for level sets of stable stationary solutions of (PAC) and obtained strong convergence results of stationary solutions of (PAC) in 3-dimensional manifolds to minimal surfaces.

The results of Chodosh and Mantoulidis settle a strong version of the *Multiplicity One Conjecture* proposed by F. Codá Marques and A. Neves [50] regarding the multiplicity of min-max minimal hypersurfaces, for generic Riemannian metrics. By applying their results to the stationary solutions of (PAC) constructed by Guaraco and the second named author in [27], Chodosh and Mantoulidis proved the *Morse index conjecture* of Marques and Neves [50], in dimension 3. We point out that this conjecture was recently solved, in dimensions $3 \leq n \leq 7$, by Marques and Neves [51], using the solution to the multiplicity one conjecture, by X. Zhou [81].

Concerning the mean curvature flow, the study of existence, regularity, and classification for certain flows in manifolds has been an active topic in geometric analysis. Huisken and Sinestrari [38] showed that closed mean convex ancient flows in the sphere S^n are shrinking spherical caps, provided it satisfies a curvature pinching condition (see [61] and [48], and the references therein, for generalizations to higher codimension and to other space forms). In [7], P. Bryan, M.N. Ivaki, and J. Scheuer classified ancient convex fully nonlinear flows on S^n. We also mention the work of R. Haslhofer and O. Hershkovitz [31] on the singularities of mean convex MCFs in general ambient manifolds.

In [18], K. Choi and C. Mantoulidis proved that ancient smooth MCFs in the round sphere S^n with low area are steady or shrinking spheres. Under more relaxed area bounds, they also proved that such ancient flows in S^3 are steady or shrinking equators or Clifford tori, along its unstable directions.

From a variational and dynamical viewpoint, the equation (PAC) may be seen as the (negative) gradient flow of the *Allen-Cahn energy functional*

$$E_\epsilon(u) = \int_M \left(\frac{\epsilon}{2} |\nabla_g u|^2 + \frac{1}{\epsilon} W(u) \right).$$

In this thesis, we study orbits of this gradient flow in S^3 which connect low energy stationary solutions, and describe the mean curvature flows they originate, as $\epsilon \downarrow 0$. We prove:

Theorem 1. *For sufficiently small $\epsilon > 0$, there are eternal solutions $\{u_\epsilon\}$ of* (PAC) *such that*

$$u_\epsilon^{-\infty} = \lim_{t \to -\infty} u_\epsilon(\cdot, t) \quad \text{and} \quad u_\epsilon^{+\infty} = \lim_{t \to +\infty} u_\epsilon(\cdot, t)$$

are the symmetric critical points of E_ϵ which accumulate on the Clifford torus and on an equatorial sphere, respectively, as $\epsilon \downarrow 0$. Furthermore, the limit $\{\mu_t\}_{t \in \mathbb{R}}$ of the associated measures

$$\mu_{\epsilon,t} := \left(\frac{\epsilon}{2} |\nabla u_\epsilon(\cdot, t)|^2 + \frac{W(u_\epsilon(\cdot, t))}{\epsilon} \right) d\mu_g$$

is a unit-density Brakke flow on S^3 which converges to an equatorial sphere and to the same Clifford torus, as $t \to \pm\infty$, respectively.

Remark. The limit flow $\{\mu_t\}_{t \in \mathbb{R}}$ can not be smooth everywhere, it must cross through some singularities since the topology changes from a torus to a sphere, that is why we introduce the weak mean curvature flow $--$ Brakke flow.

We observe that the flow μ_t given by the Theorem above is smooth for sufficiently large $|t|$, by Brakke's Local Regularity Theorem [41]. In particular, for large negative t, such flow belongs to the family of ancient flows constructed by Choi-Mantoulidis in [18, Theorems 1.6 and 1.7] which converge exponentially quickly to the Clifford torus as $t \to -\infty$. The

L^1-integrability condition follows from Proposition 5.3 (see also Remark 4.13) in [18], as the Jacobi fields of the Clifford torus are generated by one-parameter families of isometries of S^3.

We also describe a 2-parameter family of solutions that satisfy the conclusions of Theorem 1, using the symmetries of the gradient flows of the energy and the corresponding mean curvature flows.

Theorem 2. *Let* $\{\epsilon_j\}$ *be a sequence of positive parameters such that* $\epsilon_j \downarrow 0$. *Passing to a subsequence, there exist* 2*-parameter families* $\{u_j(a)\}_{a \in S^1 \times S^1}$ *of solutions of* (PAC), *parametrized by a* 2*-torus, which satisfy the conclusions of Theorem 1 (for a fixed Clifford torus as their backward limit). These gradient flows, as well as the family of equatorial spheres which are obtained as forward limits, depend continuously and equivariantly with respect to (rotations of)* $a \in S^1 \times S^1$.

1.1 Outline of the proof

Recall that the least area minimal surfaces on S^3 are the totally geodesic equatorial spheres. Furthermore, by the solution of the Willmore conjecture by Marques and Neves [49], the second least area minimal surface in S^3 is the Clifford torus. Stationary solutions $\{u_\epsilon^{-\infty}\}$ which have this minimal surface as their limit interface can be constructed using a minimization and reflection procedure, see [11]. The rigidity of such solutions was studied by Hiesmayr in [33].

One can construct ancient solutions $\{u_\epsilon(\cdot, t)\}$ of the (negative) gradient flow of the energy functional which quickly converge backward in time to a critical point using a contraction mapping argument, see [17]. These solutions describe the unstable manifold of integrable critical points. Then, a topological argument shows that many of these solutions converge, as $t \to +\infty$, to a least energy unstable solutions of the Allen-Cahn equation, which are symmetric critical points that accumulate on equatorial spheres, by [11].

The convergence of $\{u_\epsilon(\cdot, t)\}$ to an integral Brakke flow on S^3 can be derived from [40] and [63, 70]. This flow is *cyclic mod 2*, in the sense of White [77]. We then use the area bounds for the limit interfaces obtained from $u_\epsilon^{\pm\infty}$, and the symmetries of the negative eigenfunctions to the Jacobi operator of the Clifford torus (which are inherited by the corresponding solutions of (PAC)), to describe the forward and backward limit of this flow using the classification of low area stationary varifolds in S^3.

1.2 Organization

In Section 2, we state some results concerning the minimal surface (with an emphasis on the minimal surface in the 3-sphere), Geometric Measure theory, mean curvature flow (and the weak mean curvature flow $--$ Brakke flow), Allen-Cahn equation (and its relation with minimal surface and rigidity of solutions), and the ancient mean curvature flow which will be used in the sequel. In Section 3, we construct low energy eternal solutions of (PAC) as solutions to the negative gradient flow of the associated energy functional. In Section 4 we study the Brakke flow given as the singular limit of such solutions and provide a preliminary description of its backward and forward limits. In Section 5, we conclude the proof of the main theorems using symmetries of the stability operator of the Clifford torus (and the corresponding solution to the Allen-Cahn equation) to study the asymptotic limits of certain gradient flows of the energy and the corresponding Brakke flow.

1.3 Notation

We use the following notation throughout the paper:

ind()	Morse index
nul()	nullity
S^n_r	$n-$dimensional sphere with radius r
T_c	Clifford torus in S^3, (2.2).
$\mathcal{H}^k$	$k-$dimensional Hausdorff measure
σ	the energy constant $\int_{-1}^{1}\sqrt{W(t)/2}dt$.
ϵ_2	the constant $\lambda_2(S^3)^{-1/2}$ (Section 2.4.3).
$W^{1,2}(M)$	Sobolev space of functions $u \in L^2(M)$ with weak gradient $\lvert\nabla u\rvert \in L^2(M)$.
$u_\epsilon(\cdot,t)$, $\mathscr{S}(a)(\cdot,t)$	gradient flow of E_ϵ, solutions of (PAC) (see Sections 3 and 5).
$u_\epsilon^{\pm\infty}$	forward and backward (subsequential) limits of $u_\epsilon(\cdot,t)$.
$\frac{1}{\sigma}V_{\epsilon,t}$	associated varifold of $u_\epsilon(\cdot,t)$ (Section 2.4).
$\frac{1}{\sigma}\mu_{\epsilon,t}$	corresponding Radon measure of $\frac{1}{\sigma}V_{\epsilon,t}$.
$V^{\pm\infty}$	subsequential limit of the associated varifold of $u_\epsilon^{\pm\infty}$.
Σ_t	limit of $\frac{1}{\sigma}\mu_{\epsilon,t}$, with associated varifold $\frac{1}{\sigma}V_t$.
$V_{\pm\infty}$	subsequential limit of V_t as $t \to \pm\infty$
$\Theta(V,x)$	density of a varifold V at a point x.

CHAPTER 2

PRELIMINARIES

In this chapter, we introduce definitions, notations, results that we will need to use in the next few chapters.

2.1 Area functional and minimal surface

In this section, we review some standard definitions and results on minimal hypersurfaces. We here mention Colding-Minicozzi's book [19] as a good source for readers who want to know details.

The area functional (the volume of codimension one submanifold) is one of the most widely studied functionals in analysis and geometry. A hypersurface $\Sigma^n \subset M^{n+1}$ of a Riemannian manifold (M, g) is said to be *minimal* if it is a critical point for the n-dimensional area functional. More precisely, this means that for every smooth vector field X on M,

$$\delta\Sigma(X) := \frac{d}{dt}\bigg|_{t=0} \operatorname{area}_g(\varphi_t(\Sigma)) = 0,$$

where $\{\varphi_t\}_{t\in\mathbb{R}}$ is a one-parameter family of diffeomorphisms on M generated by X. Moreover, there is a first variation formula that says that

$$\delta\Sigma(X) = -\int_\Sigma \langle X, \vec{H}\rangle,$$

where $\vec{H}$ is the mean curvature vector of Σ. So the mean curvature vector $\vec{H}$ can be interpreted as the (negative) gradient of the area functional. In particular, Σ is a minimal hypersurface for (M, g) if and only if its mean curvature vector $\vec{H}$ vanishes identically.

We say that a minimal hyeprsurface Σ is *stable* if for all vector fields X,

$$\delta^2\Sigma(X,X) := \frac{d^2}{dt^2}\bigg|_{t=0} \mathrm{area}_g(\varphi_t(\Sigma)) \geq 0.$$

If in addition, we assume that Σ is two-sided, then we can pick a choice of the unit normal vector field ν to Σ. Consider normal variations which are generated by (extensions of) vector fields of the form $X = \varphi\nu$ for $\varphi \in C^\infty(\Sigma)$.

The second variation of area is a bilinear form on $C^\infty(\Sigma)$ (see e.g. [66]) given by :

$$\delta^2\Sigma(X,X) = \frac{d^2}{dt^2}\bigg|_{t=0} \mathrm{area}_g(\Sigma_t) = \int_\Sigma \varphi J_\Sigma \varphi d\mu, \tag{2.1}$$

where $J : C^\infty(\Sigma) \to C^\infty(\Sigma)$ is the *Jacobi operator*:

$$J_\Sigma\varphi = -\Delta_\Sigma\varphi - (|A_\Sigma|^2 + Ric_g(\nu,\nu))\varphi.$$

Here Δ_Σ is the Laplace-Beltrami operator, A is the second fundamental form on Σ, and *Ric* is the Ricci curvature tensor on M. The Jacobi operator J_Σ is a self-adjoint second-order linear elliptic operator, with a discrete spectrum that is bounded from below. We can therefore define the nullity of Σ as $\mathrm{nul}(\Sigma) = \dim\ker(J_\Sigma)$, and the *Morse index* of Σ as follows.

Definition. The number of negative eigenvalues of the Jacobi operator J_Σ (counted with multiplicity) is called the Morse index of Σ, denote by $\mathrm{ind}(\Sigma)$.

Hence a stable minimal hypersurface has Morse index 0, while we call a minimal hypersurface *unstable* if it has Morse index of at least 1.

A smooth function ϕ on M which solves $J_\Sigma\phi = 0$ is called a *Jacobi field.* We say Σ is a *nondegenerate* minimal hypersurface if J_Σ has a trivial kernel. For a generic Riemannian metric on M (in the Baire sense), any closed minimal hypersurface in M has this property, as proved by B. White in [75, 78].

Since the eighteenth century, minimal submanifolds have been objects of fundamental interest in geometry and analysis, and their study is linked to fields such as geometric measure theory, partial differential equations, and topology. Many problems related to minimal surface have been studied, for example, Bernstein's problem [3], Chern's conjecture [16], Lawson's conjecture (recently solved by S. Brendle [5]) and so on.

The existence and regularity of minimal hypersurfaces is one of the central problems in Riemannian geometry. In order to generalize G.D. Birkhof's existence result [2], F. Almgren [1] and J. Pitts [60] established the Almgren–Pitts min-max theory, which allows the construction of embedded minimal hypersurfaces through variational methods. This theory was further studied by R. Schoen-L. Simon [64], Gromov [28], Guth [30], among others, and it has recently revived by Marques-Neves [49].

In [80], S.-T Yau conjectured that any $3-$manifold contains infinitely many closed, immersed, minimal surfaces. Following a breakthrough in the Ricci positive case [52], the conjecture has recently been proved by Irie-Marques-Neves [43] in the generic case, and by Antoine Song [68] in full generality.

2.1.1 Minimal surface in S^3

Minimal surfaces in spaces of constant curvature, such as the Euclidean space $\mathbb{R}^3$ or the sphere S^3, have been widely studied, and the case of the sphere S^3 has drawn particular interest. In this section, we study minimal surface in S^3 with low area.

The second variation formula (2.1) above shows that any embedded minimal surface in S^3 cannot be stable (that is, its Morse index is ≥ 1). It is well known that totally geodesic equators in S^3 are the closed minimal surfaces of least area.

Recall that the equatorial spheres in S^3 are given by

$$\{x \in S^3 | \langle x, a \rangle = 0\},$$

where $a \in S^3$. Those equators have area 4π.

Simons [66] characterized the equator as the only minimal surface in S^3 with index one. Furthermore, Urbano [73] proved that if $\Sigma \subset S^3$ is a minimal surface that is not an equator, then its index is at least 5, and the *Clifford torus* is the only closed embedded minimal surface whose index is precisely 5. We recall that this is, up to isometry, the minimal surface

$$T_c = \left\{ (x, y, z, w) \in \mathbb{R}^4 : x^2 + y^2 = z^2 + w^2 = \frac{1}{2} \right\} \subset S^3. \tag{2.2}$$

By the solution of the Willmore conjecture, by Marques-Neves [49], this is the embedded, non-totally geodesic, minimal surface of least area $2\pi^2$ in S^3. Moreover, it is the unique minimally embedded torus in S^3 up to isometries, by the Hsiang–Lawson's conjecture, recently solved by S. Brendle [5].

This result is later generalized to stationary integral varifold in S^3. We mention here the following result [18, Lemma 5.8]:

Corollary 1. *Let T be a 2-dimensional stationary integral varifold in S^3. If Area$(T) \leq 2\pi^2$ and its associated $\mathbb{Z}_2$ chain $[T]$ has $\partial[T] = 0$ (see [77] for the definition of associated $\mathbb{Z}_2$ chain), then T is either a multiplicity one equator or a Clifford torus.*

The proof is based on Marques-Neves's [49] resolution of the Willmore conjecture and standard dimension reduction argument.

Hereafter, we will refer to any such minimal torus as a Clifford torus, while reserving the notation T_c for the specific torus described above.

2.2 Geometric Measure Theory

In this section, we introduce some basic definitions and notations in Geometric Measure Theory from [39, 59]. We here mention Simon's book [65] as a good source for readers who want to learn Geometric Measure Theory.

Roughly speaking, a varifold is a measure-theoretic generalization of a differentiable submanifold of Euclidean space, where differentiability has been replaced by rectifiability while maintaining the measure-theoretic structure usually seen in differentiable geometry. Next, we give the formal definition of varifold and relative concepts.

Let (M, g) be a $n-$dimensional Riemannian manifold, recall that by Nash Embedding Theorem, we can assume that M is isometrically embedded in R^N for some $N \geq n$.

Let $G(N, k)$ denote the Grassmann manifold of unoriented $k-$ dimensional planes in $\mathbb{R}^N$ $(k \leq N)$. Let

$$G_k(M) := \{(x, S) \in M \times G(N, k) : S \subset T_x M\}.$$

We say V is a $k-$dimensional *varifold* (or a $k-$*varifold*) on M if V is a Radon measure on $G_k(M)$. Let $V_k(M)$ denote the set of all $k-$dimensional varifolds on M. Convergence in the varifold sense means convergence in the usual sense of measures. We write:

$$V(\psi) = \int_{G_k(M)} \psi(x, S) dV(x, S),$$

where $\psi \in C_c^0(G_k(M))$.

For $V \in V_k(M)$, we let the *weight measure* $||V||$ be the Radon measure on M defined by

$$||V||(A) = V(\Pi^{-1}(A))$$

for each Borel set $A \subset M$. Here $\Pi : G_k(M) \to M$ is the natural projection map $(x, S) \to x$.

Given a positive Radon measure μ on M, for any $x \in M, \lambda > 0, 1 \leq k \leq n$, we can define the scaled Radon measure $\mu_{x,\lambda}$ on $\mathbb{R}^N$ by

$$\mu_{x,\lambda}(A) = \frac{\mu[M \cap (\lambda A + x)]}{\lambda^k}, A \subseteq \mathbb{R}^N.$$

Let $\mathcal{P}$ denote a $k-$plane in T_xM and $\alpha > 0$. We say that $\mathcal{P} \equiv T_x\mu$ is the $k-$dimensional *approximate tangent plane* of μ at x, if

$$\lim_{\lambda \to 0^+} \mu_{x,\lambda} = \alpha \mathcal{H}^k \lfloor \mathcal{P}.$$

The $k-$dimensional *approximate tangent space* of a set $X \subseteq M$ at $x \in M$ is defined by

$$T_xX := T_x(\mathcal{H}^k \lfloor X),$$

if it exists.

If X is a $k-$rectifiable subset of M, we define the associated varifold $v(X) \in V_k(M)$ by

$$v(X)(E) = \mathcal{H}^k(\{x \in M | (x, T_xX) \in E\}), \tag{2.3}$$

for each Borel set $E \in G_k(M)$.

We say $V \in V_k(M)$ is a $k-$dimensional *rectifiable varifold* if there exist positive real numbers $\{c_i\}_{i=1}^{+\infty}$ and $k-$rectifiable sets $\{X_i\}_{i=1}^{+\infty}$ such that

$$V = \sum_{i=1}^{+\infty} c_i v(X_i).$$

We say that $X \subseteq M$ is locally $k-$rectifiable if in addition X has locally finite $\mathcal{H}^k-$measure. If X is locally $k-$rectifiable and $\mathcal{H}^k-$measurable, then T_xX exists $\mathcal{H}^k \lfloor X-$a.e..

The *density* or *multiplicity function* θ for V is given by

$$\theta(x) = \sum \{c_i | x \in X_i\},$$

and then $||V|| = \theta \mathcal{H}^k \lfloor_X$, where $X = \cup_i X_i$. If $\{c_i\}_{i=1}^{+\infty}$ may be taken to be positive integers, we say V is an $k-$dimensional *integral varifold.*

For $V \in V_k(M)$, Let δV be the *first variation* of V, namely,

$$\delta V(g) = \int Dg(x) \cdot S dV(x, S)$$

for any vector field $g \in C_c^1(M; \mathbb{R}^N)$, and we say V is *stationary* if $\delta V(g) = 0$ for all such g. We also denote the total variation of δV by $||\delta V||$. If $||\delta V||$ is a Radon measure and is absolutely continuous with respect to $||V||$ on M, we define the *generalized mean curvature* $H_V(x)$ of V by

$$\delta V(g) = -\int g \cdot H_V d||V||,$$

where H_V is defined $||V||$ a.e. on M.

Finally we remark that if μ is a measure on M (e.g., $||V||$), then by $\operatorname{supp}\mu$ we always denote the support of μ in M.

In Section 2.4, given $\epsilon > 0$ and a sufficiently regular function u on M, we define the associated varifold as written in the equation (2.6). Note that this definition is different from the definition of induced varifold that appeared in [39], but their limit varifolds only differ by a factor of $\frac{1}{2}$ when the discrepancy measures converge to zero. For more details about the difference between the definitions of associated varifold, see Remark 2.4.1.

2.3 Mean curvature flow and Brakke flow

In this section, we introduce a widely-studied geometric flow $--$ mean curvature flow, and a weak version of it $--$ Brakke flow.

Mean curvature flow (MCF) evolves hypersurfaces in their normal direction with speed equal to the mean curvature at each point. Formally, we say that a family of smoothly embedded hypersurfaces $(\Sigma_t)_{t\in I}$ in (M^{n+1}, g) moves by mean curvature if it satisfies the nonlinear parabolic equation

$$\frac{\partial x}{\partial t} = \vec{H}(x) \tag{MCF}$$

for $x \in \Sigma_t$ and $t \in I$. Here $I \subset \mathbb{R}$ is an interval, $\frac{\partial x}{\partial t}$ is the normal velocity at x, and $\vec{H}(x)$ is the mean curvature vector at $x \in \Sigma_t$. MCF is the steepest descent flow for the area functional (gradient flow with respect to the L^2 norm on the surface), with a lot of interesting applications in many different fields. In particular, minimal hypersurfaces are stationary solutions.

Examples of mean curvature flow include shrinking spheres and shrinking cylinders.

Let S_r^n denote the $n-$dimensional sphere with radius r.

Example. For spheres, let $\Sigma_t = S^n_{r_1(t)}$, where $r_1(t) = \sqrt{r^2 - 2nt}$. Then $\{\Sigma_t\}_{t\in[0,\frac{r^2}{2n})}$ is a family of concentric $n-$spheres in $\mathbb{R}^{n+1}$, starting at the sphere S_r^n, moves by mean curvature, and collapses to a point at time $\frac{r^2}{2n}$.

Example. For cylinders, let $\Sigma_t = S^{n+1-k}_{r_2(t)} \times \mathbb{R}^k$, where $r_2(t) = \sqrt{r^2 - 2(n-k)t}$, for $0 \leq k \leq n$ (this includes the previous example when $k = 0$). Then $\{\Sigma_t\}_{t\in[0,\frac{r^2}{2(n-k)})}$ is a family of spherical cylinders, moves by mean curvature, and collapses to a straight line at time $\frac{r^2}{2(n-k)}$.

Mean curvature flow satisfies a comparison principle (see Ecker's book [22]), any two smooth compact solutions of (MCF) which are initially disjoint remain disjoint. The comparison principle and the behavior of the shrinking spheres combine to show that a smooth compact solution of (MCF) has to become non-smooth or vanish altogether before the spheres shrink to a point.

In general, solutions of (MCF) develop singularities in finite time before they disappear. For example, a singularity is expected to form near the neck for the dumbbell surface, so-called the neckpinch.

In the last decades, several methods have been developed to continue the flow after the onset of singularities:

- Brakke flow [4], using the varifolds of geometric measure theory,
- the level-set flow [24, 15], based on the idea of viscosity solutions,
- mean curvature flow with surgery [37], continuing the flow through singularities by cutting along necks, gluing in caps, and continuing the flow of the pieces.

In this thesis, we will focus on Brakke flow, and its singular perturbation: parabolic Allen-Cahn equation (PAC) (we will explain the relation between Brakke flow and (PAC) later in Section 2.4).

In 1978, Brakke [4] studied mean curvature flow in the framework of singular surfaces, so-called integral varifolds as defined above, defining measure-theoretic solutions of the mean curvature flow in terms of varifolds, called Brakke flows. We mention here the book [71] for a thorough introduction to the Brakke flow.

For a family of smooth hypersurfaces $\{\Sigma_t\}_{t\geq 0}$ in (M, g), which is moving smoothly by mean curvature, we have the following identity for any test function $\phi = \phi(x)$:

$$\frac{d}{dt}\int_{\Sigma_t} \phi d\mathcal{H}^{n-1} = \int_{\Sigma_t} -\phi|\vec{H}|^2 + D\phi \cdot \vec{H} d\mathcal{H}^{n-1}. \tag{2.4}$$

Brakke observed that we can generalize the smooth hypersurfaces Σ_t to anything with the area and tangent planes (and first variation), which are varifolds. Motivated by this, we define $B(\mu,\phi)$ for any Radon measure μ as follows:

$$B(\mu,\phi) \equiv \int -\phi|\vec{H}|^2 + D\phi \cdot T_x\mu \cdot \vec{H} d\mu.$$

Definition. A family $\{\mu_t\}_{t\geq 0}$ of Radon measures on M is called a Brakke motion provided

$$\overline{D_t}\mu_t(\phi) \leq B(\mu_t,\phi) \tag{2.5}$$

for all $\phi \in C_c^2(M,\mathbb{R}^+)$ and all $t \geq 0$. Here $\mu_t(\phi) = \int \phi \, d\mu_t$ and $\overline{D_t}f(t)$ is the upper derivative

$$\limsup_{s\to t} \frac{f(s)-f(t)}{s-t}.$$

On the one hand, Brakke flows display a useful property (see the compactness theorem in Ilmanen [41]) that sequences of Brakke flows with bounded area subconverge to a Brakke flow. On the other hand, Brakke flows lack uniqueness of solutions with a given initial condition: if $t_0 \in I$, and we set $\widetilde{V_t} = V_t$ for $t < t_0$ and $\widetilde{V_t} = 0$ otherwise, then $\{\widetilde{V_t}\}_{t\in I}$ is a Brakke flow.

Brakke proved the existence of non-trivial solutions for his flow and developed a regularity theory. His main regularity theorem says that a weak mean curvature flow is smooth in some neighborhoods of almost every point in space-time. Since then, there have been many further developments. Huisken [35] began the study of the smooth mean curvature flow and he formulated his celebrated monotonicity formula in [36], which was further studied by Ecker [21] and Ilmanen [42]. Later, White showed local regularity in [76], in case the Gaussian density ratios are close to one.

2.4 The Allen-Cahn equation, induced varifolds and convergence

In this section, we introduce the Allen-Cahn setting and its relation with the minimal surface.

Definition. A function $W \in C^\infty(\mathbb{R})$ is a *(symmetric) double-well potential* if:

(1) W is nonnegative and vanishes precisely at ± 1;

(2) W satisfies $W'(0) = 0$, $W''(0) \neq 0$, and $tW'(t) < 0$ for $|t| \in (0,1)$;

(3) $W''(\pm 1) > 0$;

(4) $W(t) = W(-t)$.

The standard example of a double-well potential is the function $W(t) = \frac{1}{4}(1-t^2)^2$. Hereafter, we fix such a potential W.

Definition. Let (M^n, g) be a Riemannian manifold. We define the *Allen-Cahn energy* on Ω by:

$$E_\epsilon(u) := \int_\Omega \left(\frac{\epsilon}{2} |\nabla_g u|^2 + \frac{1}{\epsilon} W(u) \right) d\mu_g, \quad u \in W^{1,2}(M),$$

where $d\mu_g$ is the volume measure with respect to g. Note that this quantity is finite provided $W(u) \in L^1(M)$.

Remark. We implicitly assume, in addition to (1)-(4) above, that W is bounded. This ensures that the energy functional E_ϵ is smooth in $W^{1,2}(M)$ and allows us to use existence results and standard estimates for critical points and gradient flows of E_ϵ. We emphasize that this does not affect the arguments explored in this work (in which M is assumed to be compact), as the objects we consider satisfy a priori bounds $|u| < 1$, so any double-well potential can be modified away from $[-2, 2]$ to meet this requirement.

Energy functionals of the form described above were first considered by Van der Waals [74] in 1893 and then rediscovered by Cahn–Hillard [9] in 1958. The "double well" corresponds to the assumption that the minimum local energy occurs precisely when $u(x) \in \{\pm 1\}$. S. Allen and J. Cahn [8] observed in 1977 that there is a basic link between the location of the interface between the two phases and the mean curvature of the interface. They introduced the parabolic Allen-Cahn equation (PAC), which is the gradient flow of the energy functional E_ϵ, as a model for the evolution of the process of separation of multicomponents systems.

One can check that u is a critical point of E_ϵ on a closed manifold (M^n, g) if and only if u (weakly) solves the *elliptic Allen-Cahn equation*:

$$\epsilon^2 \Delta_g u - W'(u) = 0 \quad \text{on } M. \tag{AC}$$

For the standard double well potential, the Allen-Cahn equation becomes $\epsilon^2 \Delta_g u = u^3 - u$.

We write $\sigma = \int_{-1}^{1} \sqrt{W(t)/2}\,dt$. This is the energy of the *heteroclinic solution* $\mathbb{H}_\epsilon(t)$ of (AC) on $\mathbb{R}$, namely, the unique bounded solution in $\mathbb{R}$ (modulo translation) such that $\mathbb{H}_\epsilon(t) \to \pm 1$ when $t \to \pm\infty$. We refer to [17, Section 1.3] for more on this one-dimensional solution.

Recall that the *Morse index* of a solution u of (AC) (as a critical point of E_ϵ), denoted $\mathrm{ind}_\epsilon(u)$, is the index of the quadratic form given by the second variation of the energy E_ϵ at u, namely

$$d^2 E_\epsilon[u](\phi, \psi) := \int_M \epsilon \langle \nabla\phi, \nabla\psi \rangle + \frac{1}{\epsilon} W''(u) \phi\psi \, d\mu_g,$$

for $\psi, \phi \in C^\infty(M)$. Note that $\mathrm{ind}_\epsilon(u)$ is the number of negative eigenvalues of the linear operator

$$\mathcal{L}_{\epsilon,u}(f) = \Delta f - \frac{W''(u)}{\epsilon^2} f,$$

counted with multiplicity, and the nullity $\mathrm{nul}_\epsilon(u)$ is the dimension of the kernel of $\mathcal{L}_{\epsilon,u}$. In particular, $\mathrm{ind}_\epsilon(u)$ is finite (note we assumed M to be compact). We also recall that u is said to be a *stable* solution if $\mathrm{ind}_\epsilon(u) = 0$.

In order to describe some convergence results for solutions of (PAC) and its elliptic counterpart, we will use some notions and notation from Section 2.2; see also the Notation table above.

The classical variational convergence for solutions of (AC) was studied in the works of Modica and Mortola [54, 55], who proved that the Allen–Cahn energy functional Γ-converges to the *perimeter functional*, a generalization of the $(n-1)$-dimensional volume defined on the space of domains of finite perimeter. In particular, the interfaces of locally minimizing solutions of (AC) (namely the sets where these functions are bounded away from ± 1) converge, as $\epsilon \downarrow 0$, to local minimizers of the area of the perimeter (and are thus regular away from a singular set of dimension $\leq (n-8)$).

A convergence result for families of solutions with uniformly bounded energy and index follows from the combined work of J. Hutchinson, Y. Tonegawa, N, Wickramasekera and M. Guaraco [39, 72, 29], which is based on the deep regularity theory developed by Wickramasekera [79].

Before recalling this convergence result, we note that given $\epsilon > 0$ and a sufficiently regular function u on M (so that almost every level set is a regular hypersurface), we can consider the *associated* $(n-1)$*-varifolds* $V_{\epsilon,u}$ defined by

$$V_{\epsilon,u}(\phi) = \frac{1}{2}\int_{M\cap\{\nabla u\neq 0\}} \phi(x, T_x\{u = u(x)\}) \cdot \left(\frac{\epsilon|\nabla u(x)|^2}{2} + \frac{W(u(x))}{\epsilon}\right) d\mu_g(x) \qquad (2.6)$$

for any continuous function ϕ defined in the Grassmannian manifold $G_{n-1}(M)$, where $V_{\epsilon,u}(\phi)$ denotes the integral of ϕ on $G_{n-1}(M)$ with respect to $V_{\epsilon,u}$. We write $\mu_{\epsilon,u} = \|V_{\epsilon,u}\|$ for the associated Radon measure on M (the *weight measure* of $V_{\epsilon,u}$). In the case where $\{u_j\}$ are solutions to (AC) or (PAC), with $\epsilon = \epsilon_j \downarrow 0$, we will write $V_{\epsilon_j,t} = V_{\epsilon_j,u_j(\cdot,t)}$ and $\mu_{\epsilon_j,t} := \mu_{\epsilon_j,u_j(\cdot,t)}$.

The following $\epsilon-$regularization of area theorem roughly said that if the energy $\{E_\epsilon(u_\epsilon)\}_\epsilon$ is bounded, then the diffuse interfaces $\{u_\epsilon \simeq 0\}$ for solutions of $DE_\epsilon(u_\epsilon) = 0$ converge to a limit interface Σ (critical point of area) in the varifold sense as $\epsilon \downarrow 0$.

Theorem ([39, 72, 29])**.** *Let* (M^n, g) *be a closed Riemannian manifold. Let* $\{u_j\}$ *be a sequence of solutions of* (AC) *with* $\epsilon = \epsilon_j \downarrow 0$*. Suppose that* $\sup_j E_{\epsilon_j}(u_j) < \infty$*. Then we can find a (not relabeled) subsequence of* u_j *such that* V_{ϵ_j} *converge to a stationary* $(n-1)$

varifold V on M such that $\frac{1}{\sigma}V$ is integral. Moreover,

$$\frac{1}{\sigma}\|V\|(M) = \lim_{j\to\infty}\frac{1}{\sigma}\|V_{\epsilon_j}\|(M) = \lim_{j\to\infty}\frac{1}{2\sigma}E_{\epsilon_j}(u_j),$$

and u_j converges uniformly to ± 1 in compact subsets of $M \setminus \operatorname{supp}\|V\|$.

Furthermore, if $n \geq 3$ and if $\sup_j \operatorname{ind}_{\epsilon_j}(u_j) < \infty$, then $\operatorname{supp}\|V\|$ is a smooth, embedded, minimal hypersurface in M away from a closed set of Hausdorff dimension $\leq (n-8)$.

The minimal surface $\operatorname{supp}\|V\|$ is often called a *limit interface* obtained from u_j.

The parabolic counterpart of the measure-theoretic result above was investigated by T. Ilmanen [40], and H.M. Soner [67], among many others. For solutions of the parabolic equation (PAC), the weak limit as $\epsilon \downarrow 0$ is a weak solution of the mean curvature flow – Brakke flow.

We state below the main convergence result we will use in the present thesis, which follows from [40] and the work of Tonegawa [70] (see also [63] and [69]):

Theorem. *Let (M^n, g) be a closed Riemannian manifold. Let $\{u_j\}$ be a sequence of solutions to* (PAC) *on $M \times [t_0, \infty)$ with $\epsilon = \epsilon_j \downarrow 0$. Suppose that there exist constants $c_0, E_0 > 0$ such that*

(a) $\sup_{M\times[t_0,\infty)} |u_j| \leq c_0$, *for all j,*

(b) $E_{\epsilon_j}(u_j(\cdot,t)) \leq E_0$, *for all $t \geq t_0$ and all j, and*

(c) $\int_{M\times(t_0,\infty)} \epsilon_j |\partial_t u_j|^2 \, d\mu_g \leq E_0$, *for all j.*

Write $\mu_{\epsilon_j,t} = \mu_{\epsilon,u_j(\cdot,t)}$, for every $t \geq t_0$ and every j. Then, passing to a subsequence (not relabeled), there are Radon measures $\{\mu_t\}_{t\geq t_0}$ such that

(i) $\mu_{\epsilon_j,t} \to \mu_t$ as Radon measures on M, and

$$\frac{1}{2\sigma}\lim_{j\to\infty} E_{\epsilon_j}(u_j(\cdot,t)) = \frac{1}{\sigma}\lim_{j\to\infty}\|\mu_{\epsilon_j,t}\|(M) = \frac{1}{\sigma}\|\mu_t\|(M),$$

for every $t \in [t_0, \infty)$.

(ii) For a.e. $t > t_0$, μ_t is $(n-1)$-rectifiable, and its density is $N(x)\sigma$, for μ_t-a.e. $x \in M$, where $N(x)$ is a nonnegative integer.

(iii) μ_t satisfies the mean curvature flow in the sense of Brakke (2.5), namely:

$$\overline{D}_t \int_M \phi \, d\mu_t \leq \int_M (-\phi)\|H_t\|^2 + \langle \nabla\phi, H_t \rangle \, d\mu_t,$$

for any C^2 function $\phi \geq 0$. Here H_t is the generalized mean curvature vector of μ_t.

Remark 2.4.1. The normalization chosen in (2.6) differs by a factor of $\frac{1}{2}$ when compared to [40, 67], and it agrees with [59, 63] (observe the different definition of the normalization constant σ). Moreover, even though the associated varifold defined in [70] apparently differs from (2.6), the convergence of the discrepancy measures $\left|\frac{\epsilon|\nabla u|^2}{2} - \frac{W(u)}{\epsilon}\right| d\mu_g$ to zero in the elliptic case [39] and for a.e. time in the parabolic case [40] ensure that the limit varifolds, as $\epsilon \downarrow 0$ agree.

Remark. We mention here the recent results of [56, 57], by H.T. Nguyen and S. Wang, regarding the strong convergence of solutions to (PAC) in the Euclidean space. This parallels the results of [17] for the parabolic setting, under entropy bounds or multiplicity one condition. Even though we do not use these results in the present thesis, we point out that one could employ them to obtain uniform curvature bounds (for sufficiently negative time) for the transition layers of the solutions to (PAC) studied here.

2.4.1 Equilibrium points from minimal surfaces

In this section, we will discuss solutions to (AC) which are constructed from minimal surfaces and rigidity of low index/energy solutions.

Let (M, g) be a $(n+1)-$dimensional compact Riemannian manifold without boundary, Pacard-Ritoré and Caju-Gaspar studied solutions to (AC) whose nodal sets accumulate around a minimal hypersurface $\Sigma \subset M$ which is nondegenerate up to ambient isometries for small $\epsilon > 0$. They proved:

Theorem ([58, 10]). *Let (M, g) be a $(n+1)-$dimensional compact Riemannian manifold without boundary and $\Sigma \subset M$ be a minimal hypersurface which separates M, with $M \setminus \Sigma = M_+ \cup M_-$. Suppose that all Jacobi fields are generated by global isometries of M. Then there exists $\epsilon > 0$ such that for all $\epsilon \in (0, \epsilon_0)$ there is a solution u_ϵ of (AC) such that u_ϵ converges uniformly to 1 (respectively to -1) on compact subsets of M_+ (respectively M_-), and*

$$E_\epsilon(u_\epsilon) \to \frac{1}{2\sigma} Area(\Sigma), \qquad as \;\; \epsilon \to 0.$$

Moreover, the Morse index $\operatorname{ind}_\epsilon(u_\epsilon)$ and the nullity $nul_\epsilon(u_\epsilon)$ of u_ϵ satisfy

$$\operatorname{ind}_\epsilon(u_\epsilon) = \operatorname{ind}(\Sigma) \qquad and \qquad nul_\epsilon(u_\epsilon) = nul(\Sigma).$$

Roughly speaking, the solution u_ϵ they constructed is close to 1 on M_+, and is close to -1 on M_-, while near a neighborhood of Σ, the solution agrees with the composition of the $1-$dimensional solution compose with the distance function.

Recall in Section 2.1.1, we give a classification of minimal surface in S^3 with low area. We know that the equatorial spheres in S^3 are the closed minimal surfaces of least area, they have area 4π and index 1. Furthermore, among all minimal surfaces in S^3 which are not equators, the Clifford torus has the least area $2\pi^2$ with index 5.

We now describe the counterparts of some of these results in the context of the Allen-Cahn equation. First, we recall that $u = \pm 1$ are the unique global minimizers for E_ϵ. Low energy solutions often display variational characterizations and inherit many geometrical properties from the domain. The symmetry properties of *least energy unstable* critical points of E_ϵ –

also referred as *ground state solutions* – in a sphere were studied in [11]. Such solutions are radially symmetric with respect to some point, and they vanish precisely on an equatorial sphere.

Solutions of the Allen-Cahn equation whose energy densities accumulate on T_c can be constructed using gluing techniques (see [10]), or by minimization and reflection. Caju-Gaspar-Guaraco-Matthiesen construct the following example in [11]:

Example. For small $\epsilon > 0$, there exists $u_\epsilon^{-\infty} : S^3 \to \mathbb{R}$ such that $DE_\epsilon(u_\epsilon^{-\infty}) = 0$, the index of the operator $D^2E_\epsilon(u_\epsilon)$ is 5,

$$\{u_\epsilon^{-\infty} = 0\} = T_c, \qquad \text{and} \qquad E_\epsilon(u_\epsilon^{-\infty}) \to 2\sigma \cdot 2\pi^2.$$

The function $u_\epsilon^{-\infty}$ has the following properties:

- It minimizes E_ϵ on $\{x \in S^3 | x_1^2 + x_2^2 \geq \frac{1}{2}\}$ (one region bounded by T_c) among functions that vanish on T_c.
- It is invariant by (x_1, x_2) and (x_3, x_4) rotations.

Remark. We call this function $u_\epsilon^{-\infty}$ the Allen-Cahn approximation of the Clifford torus T_c.

Concretely, for each $\epsilon \in (0, \epsilon_2)$, where

$$\epsilon_2 = 1/\lambda_2(S^3)^{1/2}, \tag{2.7}$$

there is a unique positive function that minimizes E_ϵ on one of the two isometric domains in S^3 bounded by T_c among functions that vanish on the boundary. Using the reflection that maps one of the domains onto the other, one extends this minimizer to a solution of (AC) in S^3 which vanishes precisely on T_c. We refer to [11] or [33] for the detailed construction. Note that the uniqueness of these minimizers implies that this solution inherits the symmetries of the Clifford torus.

2.4.2 Morse index of the limit interface

Lower semicontinuity of the Morse index along the singular limit $\epsilon \downarrow 0$ of a sequence of solutions to the equation (AC) is proven by Le [46, 47]) (assuming the single-multiplicity property of the limiting energy), Hiesmayr [32] (for two-sided surfaces) and Gaspar [26] without assuming two-sidedness. We provide Gaspar's Theorem below:

Theorem ([26])**.** *Let M be a closed Riemannian manifold of dimension $n \geq 3$, and $\{u_{\epsilon_k}\}$ a sequence of solutions to (AC) with $\epsilon = \epsilon_k \downarrow 0$. Assume that there are positive constants c_0 and E_0, and a nonnegative integer p such that*

$$\limsup_k \sup_M |u_{\epsilon_k}| \leq c_0, \quad \limsup_k E_{\epsilon_k}(u_{\epsilon_k}) \leq E_0 \quad \textit{and} \quad \limsup_k m(u_{\epsilon_k}) \leq p.$$

Then the regular part of the limit interface has Morse index at most p.

Upper semicontinuity of the index does not hold in general (see Example 5.2 in [17]). Chodosh-Mantoulidis [17] established the following important upper semicontinuity result in the multiplicity one case:

Theorem 3 ([17])**.** *Suppose that a smooth embedded minimal hypersurface $\Sigma^n \subset (M^{n+1}, g)$ is the multiplicity one limit as $\epsilon \to 0$ of a sequence of solutions u to the Allen-Cahn equation. Then for $\epsilon > 0$ sufficiently small,*

$$\mathit{nul}(\Sigma) + \operatorname{ind}(\Sigma) \quad \geq \quad \mathit{nul}(u) + \operatorname{ind}(u).$$

For the nullity of minimal surface in S^3, by [34], the Clifford torus has nullity 4, and the equatorial sphere has nullity 3.

2.4.3 Rigidity of solutions

As noted in [10], the index estimates from [18] and [26] imply that the solutions produced by either of these methods also have Morse index 5, for sufficiently small $\epsilon > 0$. Very recently, F. Hiesmayr [33] characterized solutions whose nodal sets are equators or Clifford torus as the unique nonradial solutions of Morse index ≤ 5 in S^3 with bounded energy, namely:

Theorem 4 ([33]). *Given any $C > 1$, there exists $\epsilon_3(C) \in (0, \epsilon_2)$ with the following property. For any $\epsilon \in (0, \epsilon_3)$, any solution of* (AC) *with Morse index ≤ 5 and energy $\leq C$ is a ground state solution or a symmetric solution with nodal set on some Clifford torus.*

The proof of this rigidity theorem is based on a Frankel-type result on the nodal sets under mild topological hypothesis. We state the theorem below:

Theorem ([33]). *Let (M, g) be a closed manifold of dimension $n + 1 \geq 2$ with* Ric > 0. *let $u^1_\epsilon, u^2_\epsilon \neq \pm 1$ be two solutions of (AC) on M. If*

- *either $\{u^1_\epsilon = 0\}$ is separating and $\{u^2_\epsilon = 0\}$ is connected,*
- *or $\{u^1_\epsilon = 0\}$, $\{u^2_\epsilon = 0\}$ are both connected,*

then $\{u^1_\epsilon = 0\} \cap \{u^2_\epsilon = 0\} \neq \phi$.

Hiesmayr formulated another Frankel-type theorem concerning the solutions themselves that we will use later:

Proposition 1 ([33]). *Let (M, g) be a closed manifold of dimension $n+1 \geq 2$ with* Ric > 0. *Let $\epsilon > 0$ and $u^1_\epsilon \neq u^2_\epsilon$ be two solutions of* (AC) *on M. If $u^1_\epsilon \leq u^2_\epsilon$, then one of the two is constant.*

In the same paper, Hiesmayr proved the uniqueness of solutions with nodal set restriction and energy bound, up to rotation and change of sign, namely:

Theorem 5 ([33]). *Let $p, g \in \mathbb{Z}_{>0}$ and $n = p + q \geq 2$. There exist $\epsilon_0 > 0, \delta > 1$ so that for all $0 < \epsilon < \epsilon_0$ there is a unique solution u to (AC) on S^{n+1} with*

$$\{u = 0\} \subset (T_{p,q})_\delta \qquad \textit{and} \qquad (1 - \delta)\mathcal{H}^n(T_{p,q}) \leq E_\epsilon(u) \leq (1 + \delta)\mathcal{H}^n(T_{p,q}),$$

up to rotation and change of sign. Moreover, u is $SO(p) \times SO(q)-$symmetric, up to conjugation.

Here $T_{p,q}$ is the Clifford-type minimal hypersurfaces:

$$T_{p,q} = S^{p}_{\sqrt{p/n}} \times S^{q}_{\sqrt{q/n}} \subset S^{n+1},$$

and $(T_{p,q})_\delta \subset S^{n+1}$ is the tubular neighborhood of $T_{p,q}$ with size $\delta > 0$.

In the more symmetric case where $p = q$ and $n = 2p$, we have the following corollary:

Corollary 2 ([33])**.** *Let $p \in \mathbb{Z}_{>0}$ and $n = 2p$. Let $\epsilon_j \to 0$ and u_j be solutions to (AC) with $\epsilon = \epsilon_j$ so that $V_{\epsilon_j,u_j} \to v(T_{p,p})$ ($v(\cdot)$ is defined in equation (2.3)). For large j, the nodal set of u_j is a rigid copy of $T_{p,p}$, and u_j is a symmetric solution around it.*

2.5 Ancient mean curvature flow

In this section, we list the classification results for ancient mean curvature flows by Choi-Mantoulidis.

In [18], K. Choi and C. Mantoulidis classified low area ancient solutions in S^n in arbitrary codimension:

Theorem ([18])**.** *There exists a $\delta = \delta(n) > 0$ such that if $(\Sigma_t)_{t\leq 0}$ is an ancient mean curvature flow of closed $m-$dimensional surfaces embedded in a round sphere S^n, with*

$$\lim_{t\to-\infty Area(\Sigma_t)} < (1+\delta)Area(S^m),$$

then $(\Sigma_t)_{t\leq 0}$ is a steady or shrinking equatorial S^m one along one of $(n-m)$ orthogonal directions.

This theorem implies a complete classification of ancient embedded curve shortening flows in S^2 with bounded length:

Corollary ([18])**.** *Let* $(\Gamma_t)_{t\leq 0}$ *be an ancient curve shortening flow of embedded curves inside a round* $2-$*sphere with*

$$\lim_{t\to-\infty} Length(\Sigma_t) < +\infty.$$

Then $(\Sigma_t)_{t\leq 0}$ *is a steady or a shrinking equator along circles of latitude.*

They also classified smooth ancient MCFs in S^3 with area below $2\pi^2$ plus a small $\delta > 0$, using the fact that the Clifford torus in S^3 (with area $2\pi^2$) is the second smallest area among smooth minimal surfaces, following the equatorial sphere (with area 4π), as we mentioned in Section 2.1.1. They showed:

Corollary ([18])**.** *Let* $(\Sigma_t)_{t\leq 0}$ *be an ancient mean curvature flow of closed surfaces in a round* S^3*, with*

$$\lim_{t\to-\infty} Area(\Sigma_t) < 2\pi^2 + \delta.$$

If $\delta > 0$ *is sufficiently small, then either:*

- $\lim_{t\to-\infty} Area(\Sigma_t) = 4\pi$*, and* $(\Sigma_t)_{t\leq 0}$ *is a steady or shrinking equator along spheres of latitude; or,*
- $\lim_{t\to-\infty} = 2\pi^2$*, and* $(\Sigma_t)_{t\leq 0}$ *is a steady or shrinking Clifford torus along one of its* 5 *linearly unstable directions.*

In the same article, Choi and Mantoulidis constructed ancient solutions to certain quasilinear gradient flows that converge backward in time — that is, as $t \downarrow -\infty$ — to a critical point u for the associated energy with finite Morse index p. They proved:

Theorem. *[18, Theorem 3.3] Let* 0 *be a critical point of* $\mathcal{A}$ *with Morse index* $I \in \mathbb{N}$*. There exists an* $I-$*parameter family of ancient solutions to the "gradient flow" of* $\mathcal{A}$*, which are uniquely determined by their trace at* $t = 0$ *and which converge to* 0 *exponentially as* $t \to$

$-\infty$. *The space of their traces at* $t = 0$ *is tangent to the* $I-$*dimensional space of negative eigenfunctions.*

Here $\mathcal{A}$ is a functional of the form

$$\mathcal{A}(f) := \int_{\Sigma} A(x, f(x), \nabla_g f(x)) d\mu_g(x),$$

which satisfies some additional conditions (for details, see [18, section 1.2]).

These solutions locally describe the unstable manifold of this critical point, hence they are parametrized by a p-dimensional disk, and they are the unique solutions that converge backward to u that satisfy an integrability condition related to the *Łojasiewicz-Simon inequality*, as proved in [18, Theorem 4.1 and Proposition 4.12]. In particular, this assumption holds true for analytic functionals, and also whenever the energy functional is Morse-Bott at the corresponding energy level near the critical point u, see [25].

Here a Morse–Bott function is a smooth function on a manifold whose critical set is a submanifold and whose Hessian is non-degenerate in the normal direction. An example for the Morse-Bott function is $f : \mathbb{R}^3 \to \mathbb{R}, f(x, y, z) = x^2 - y^2$, the critical set is $\{z = 0\}$, one can easily check that the Hessian is non-degenerate in the normal direction. We point out that the fact that the Allen-Cahn energy functional E_ϵ is Morse-Bott on the first two energy levels (the sets of solutions to (AC) whose nodal sets are equatorial spheres and Clifford tori) was proved in [11, 32].

We refer to Sections 3 and 4 in [18] for the complete statements, and to Sections 3 and 5 for the main consequences of the (negative) gradient flow of the Allen-Cahn energy.

CHAPTER 3

GRADIENT FLOWS OF THE ENERGY FUNCTIONAL

3.1 Introduction

As noted above in Section 2.4.1, the Clifford torus $T_c \subset S^3$ can be obtained as the limit of a sequence of solutions to the Allen-Cahn equation (AC), extracted from a family $\{u_\epsilon^{-\infty}\}_{\epsilon\in(0,\epsilon_2)}$ of solutions whose nodal sets are precisely T_c (which are called the Allen-Cahn approximation of T_c), and such that $E_\epsilon(u_\epsilon^{-\infty}) \to 2\sigma \cdot \mathrm{Area}(T_c)$ as $\epsilon \downarrow 0$.

Even though starting from most points in a neighborhood of $u_\epsilon^{-\infty}$ on the unstable manifold of $u_\epsilon^{-\infty}$, the negative gradient flow of the energy functional E_ϵ will converge to the constant solution ±1. We want to show that we can connect these solutions $u_\epsilon^{-\infty}$ to a *nonconstant* critical point of E_ϵ with smaller energy using the Allen-Cahn flow. The key ingredient is the following result:

Proposition 2. *Let (M^n, g) be a compact Riemannian manifold with $\mathrm{Ric}_g > 0$, and let $u_\epsilon^{-\infty}$ be a nonconstant solution of the Allen-Cahn equation on M with Morse index $\mathrm{ind}_\epsilon(u_\epsilon^{-\infty}) \geq 2$. There exist (infinitely many) eternal solutions $u : M\times\mathbb{R} \to \mathbb{R}$ of the parabolic equation* (PAC) *on (M, g) such that $E_\epsilon(u_\epsilon(\cdot, t))$ is strictly decreasing,*

$$\|u_\epsilon(\ \cdot\ , t) - u_\epsilon^{-\infty}\|_{W^{1,2}(M)} \to 0, \quad \text{as} \quad t \to -\infty,$$

and, for any sequence $t_k \uparrow +\infty$, the functions $u_\epsilon(\cdot, t_k)$ do not converge to the constant critical points ±1 of E_ϵ.

The proof is based on the fact that the first eigenfunction has a sign, the maximum principle for parabolic equations, the Frankel-type property (Proposition 1) proved by Hiesmayr, and a topological argument.

Remark. We require the Ricci curvature on the manifold (M, g) to be positive since this is a necessary assumption to use the Frankel-type property.

3.2 Ancient Allen-Cahn flow

Now we present the Choi-Mantoulidis argument for the ancient gradient flow in the Allen-Cahn setting. Roughly speaking, their result showed the existence of ancient gradient flows and the uniqueness under integrability conditions. Then by using the theory of parabolic PDEs, we can show the existence of a family of gradient flows of the Allen-Cahn energy E_ϵ parametrized by $B_\eta(0) \subset \mathbb{R}^I$, converges to $u_\epsilon^{-\infty}$ backward in time, where I is the Morse index of $u_\epsilon^{-\infty}$. The curve corresponding to the gradient flow described above is tangent at $u_\epsilon^{-\infty}$ to eigenfunctions with corresponding coefficients in $\mathbb{R}^I$.

Observe that a time-dependent function u is a (weak) solution to (PAC) if, and only if, is a (weak) solution to the negative L^2-gradient flow of $-\frac{1}{\epsilon}E_\epsilon$, that is

$$\partial_t u = -\frac{1}{\epsilon}\nabla E_\epsilon(u). \tag{3.1}$$

A possible parametrization of solutions of such gradient flows near critical points of the associated energy functional is developed in [18, Theorem 3.3]. In order to describe their results, we will introduce some notation.

Denote by $\mathcal{L}_\epsilon$ the linearization of $-\frac{1}{\epsilon}\nabla E_\epsilon$:

$$\mathcal{L}_\epsilon(f) = -\tfrac{1}{\epsilon}\tfrac{d}{ds}\big|_{s=0}\nabla E_\epsilon(sf).$$

This is related to the linearized Allen-Cahn operator at $u_\epsilon^{-\infty}$ by:

$$\mathcal{L}_\epsilon(f) = \Delta f - \frac{W''(u_\epsilon^{-\infty})}{\epsilon^2}f.$$

Let $\lambda_1^\epsilon < \lambda_2^\epsilon \leq \ldots \leq \lambda_I^\epsilon$ be the negative eigenvalues of $\mathcal{L}_\epsilon$, where I is the Morse index of $u_\epsilon^{-\infty}$, and let $\varphi_1, \ldots, \varphi_I$ be corresponding L^2-orthonormal eigenfunctions with $\varphi_1 > 0$.

In [18], K. Choi and C. Mantoulidis proved that there exists $\eta = \eta(\epsilon, M) > 0$ and a $C^{2,\alpha}$-continuous family $\{\mathscr{S}(a) : M \times (-\infty, 0] \to \mathbb{R}\}$, for $a \in B_\eta(0) \subset \mathbb{R}^I$, of ancient solution to the parabolic equation (3.1) with controlled exponential decay (as $t \downarrow -\infty$). The solution $\mathscr{S}(a)$ is the unique $C^{1,\theta}$ solution of (3.1) with finite $L^1(M \times (-\infty, 0])$ norm, modulo translation in time, that converges to $u_\epsilon^{-\infty}$ backward in time, in the $C^{2,\theta}$ norm, and has $u_\epsilon^{-\infty} + \sum_{j=1}^{I} a_j \varphi_j$ as its projection in the space generated by $\{\varphi_j\}_{j=1}^{I}$ at time $t = 0$. Moreover, it satisfies

$$\left\| \mathscr{S}(a)(\cdot, 0) - \left(u_\epsilon^{-\infty} + \sum_{j=1}^{I} a_j \varphi_j \right) \right\|_{C^{2,\alpha}(M)} \leq C|a|^2, \tag{3.2}$$

for some $C > 0$ (depending on ϵ) – see Theorems 3.3 and 4.1 in [18] for a more precise and complete statement. By picking a sufficiently small $\eta > 0$, we may assume that $\sup_M |\mathscr{S}(a)(\cdot, 0)| < 1$ for all $a \in B_\eta(0)$, and that

$$\mathscr{S}(r, 0 \ldots, 0)(\cdot, 0) > u_\epsilon^{-\infty} > \mathscr{S}(-r, 0, \ldots, 0)(\cdot, 0) \text{ for all } r \in (0, \eta).$$

This is possible because the first eigenfunction φ_1 is positive and by

$$\left\| \frac{\mathscr{S}(a)(\cdot, 0) - u_\epsilon^{-\infty}}{r} - (\pm\varphi_1) \right\|_{C^{2,\alpha}(M)} \leq Cr$$

for $a = (\pm r, 0, \ldots, 0)$, which follows from the estimate (3.2). We are now in a position to prove Proposition 2.

3.3 Proof of Proposition 2

Proof. Since the constant functions ±1 are isolated global minimizers of E_ϵ, there exist disjoint neighborhoods $B_\pm$ of ± 1 in $W^{1,2}(M)$ and $d_\epsilon > 0$ such that ± 1 are the only solutions of (AC) in $B_\pm$, and

$$E_\epsilon(u) < E_\epsilon(\pm 1) + d_\epsilon = d_\epsilon \quad \text{if, and only if,} \quad u \in B_\pm.$$

Let $\mathcal{S} = \{u \in C^2(M) \mid |u| \leq 1\}$. By Lemma 2.3 in [27] (and the continuous dependence of initial data, see e.g. Cazenave-Haraux [12]), there is a continuous map

$$\Phi : \mathcal{S} \times [0, \infty) \to W^{1,2}(M)$$

such that $\Phi(u, \cdot) : M \times [0, \infty) \to \mathbb{R}$ is a solution of (PAC) defined for all $t \geq 0$ with $\Phi(u, 0) = u$, and such that $\Phi(u, t) \in \mathcal{S}$, for all such t. Since $E_\epsilon(\Phi(u, t))$ is decreasing with respect to t, for any $u \in \mathcal{S}$ and for any $T > 0$,

$$\text{if} \quad \Phi(u, T) \in B_\pm, \quad \text{then} \quad \Phi(u, t) \in B_\pm, \text{ for all } t \geq T.$$

We claim that the sets

$$U_\pm = \{u \in \mathcal{S} \mid \|\Phi(t, u) - (\pm 1)\|_{W^{1,2}(M)} \to 0, \text{ as } t \to +\infty\}$$

are open. In fact, if $u \in U_\pm$, then there exists $T > 0$ such that $\Phi(u, t) \in B_\pm$ for all $t \geq T$. By the continuity of $\Phi(\cdot, T)$, there exists $\delta > 0$ such that $\Phi(w, T) \in B_\pm$ for all $w \in B_\delta(u) \cap \mathcal{S}$. This implies $\Phi(w, t) \in B_\pm$ for all $t \geq T$, so $\Phi(w, t)$ must converge to ±1 as $t \to +\infty$.

With the notation introduced above, let $r \in (0, \eta)$ and consider $K = \{\mathscr{S}(a)(\cdot, 0)\}_{a \in \partial B_r(0)}$. Write $w^\pm = \mathscr{S}(\pm r, 0 \ldots, 0)(\cdot, 0)$. By our choice of η, we have $w^+ > u_\epsilon^{-\infty} > w^-$. As noted in [27, Lemma 2.3], there exist sequences $(t_k^{\pm})$ such that $t_k^{\pm} \to \pm\infty$, and solutions $u^{\pm}$ of the Allen-Cahn equation such that $\Phi(w^{\pm}, t_k^{\pm}) \to u^{\pm}$. By the maximum principle for parabolic equations, these solutions satisfy $u^+ \geq u_\epsilon^{-\infty} \geq u^-$, as well as $E_\epsilon(w^{\pm}) < E_\epsilon(u_\epsilon^{-\infty})$. Since $u_\epsilon^{-\infty}$ is nonconstant, by Proposition 1, we see that $u^{\pm} \equiv \pm 1$. Since ± 1 are nondegenerate solutions, we get $\Phi(w^{\pm}, t) \to \pm 1$ as $t \to \pm\infty$ (see Remark 4.13 in [18]) and $w^{\pm} \in U_{\pm}$. Therefore, the sets $U_+ \cap K$ and $U_- \cap K$ are nonempty.

By connectedness, it follows that along any path in K that connects $w^{\pm}$, there exist $w = \mathscr{S}(a)(\cdot, 0) \in K$ also in this path, for some $a \in \partial B_r(0)$, such that $w \notin U_{\pm}$. This means that $\Phi(w, t)$ does not converge to ± 1, as $t \to +\infty$. The desired solution of the parabolic equation (PAC) is then given by $u_\epsilon(\cdot, t) = \mathscr{S}(a)(\cdot, t)$, for $t \leq 0$, and $u_\epsilon(\cdot, t) = \Phi(w, t)$, for $t \geq 0$. Since $u_\epsilon^{-\infty}$ has Morse index ≥ 2, the set K is a continuous injective image of a sphere of dimension $(\mathrm{ind}_\epsilon(u_\epsilon^{-\infty}) - 1) \geq 1$, hence there are infinitely many such paths, and this concludes the proof. □

Remark. For any point $(a_2, a_3, \cdots, a_I) \in B_\eta(0) \subset \mathbb{R}^{I-1}$, we consider the map f which maps a point x to the forward limit of the solution $\mathscr{S}(x, a_2, a_3, \cdots, a_I)$.

By the same argument as above (using the fact that $\phi_1 > 0$ and the maximum principle), we know that $\mathscr{S}(x_1, a_2, a_3, \cdots, a_I) < \mathscr{S}(x_2, a_2, a_3, \cdots, a_I)$ when $x_1 < x_2$. Therefore if $f(x_1), f(x_2)$ are both nonconstant solutions, we conclude $f(x_1) = f(x_2)$ by Proposition 1. Hence the image of f contains at most 3 solutions, ± 1, and a nonconstant solution u.

If $f(x_1) = f(x_2)$ for $x_1 < x_2$. Then for any $x_3 \in (x_1, x_2)$, we know

$$\mathscr{S}(x_1, a_2, a_3, \cdots, a_I) < \mathscr{S}(x_3, a_2, a_3, \cdots, a_I) < \mathscr{S}(x_2, a_2, a_3, \cdots, a_I),$$

which implies $f(x_1) \leq f(x_3) \leq f(x_2)$, and therefore $f(x_3) = f(x_1)$.

Using the fact that the set $U_{\pm}$ are open, we know that for function f that we defined above, the preimages of ± 1 are open intervals, and the preimage of u is a closed interval.

In the next section, we will focus on the case where $u_\epsilon^{-\infty}$ is the solution of (AC) in S^3 which vanishes precisely at the Clifford torus T_c, has this minimal surface as its limit interface. We hope to prove that $u_\epsilon(\cdot, t)$ converges to a solution of the elliptic Allen-Cahn equation which vanishes precisely on an equatorial sphere. We will need to work around the fact that any (subsequential) limit of $u_\epsilon(\cdot, t)$ as $t \to +\infty$ may have a larger Morse index, so the regularity results from [72, 29] does not readily apply. Without imposing any further conditions on the gradient flow of the functional, this phenomenon may happen even in the finite-dimensional setting, as we illustrate with an example below.

3.4 Index change related to the gradient flow

Let F be a C^1 function on a manifold M. If $f : \mathbb{R} \to M$ is a complete solution to the negative gradient flow of F on M which joins a critical point x to another critical point y, we know that $F(x) > F(y)$, but we may not have a relation between the Morse indexes of x and y. In fact, even in finite dimensions, there are examples in which the Morse index increases along the flow, that is, such that $\text{index}(x) < \text{index}(y)$.

Example. Let $M = S^1 \times S^2 \subset \mathbb{R}^5$ equipped with coordinates (x, y, z, w, u) where

$$x^2 + y^2 = z^2 + w^2 + u^2 = 1.$$

Let $F(x, y, z, w, u) = y(u + 2)$, then F has 4 critical points on M:

$$\begin{aligned}
&(0, -1, 0, 0, 1), \text{ with Morse index } 0,\\
&(0, 1, 0, 0, -1), \text{ with Morse index } 1,\\
&(0, -1, 0, 0, -1), \text{ with Morse index } 2,\\
&(0, 1, 0, 0, 1), \text{ with Morse index } 3.
\end{aligned}$$

Then $f(t) = (\cos(-t), \sin(-t), 0, 0, -1), t \in [-\frac{\pi}{2}, \frac{\pi}{2}]$ is a gradient flow of F from the index 1 critical point $(0, 1, 0, 0, -1)$ to the index 2 critical point $(0, -1, 0, 0, -1)$.

Remark. The index change for mean curvature flow is still unknown, i.e. we don't know whether or not there exists a mean curvature flow from a low index minimal hypersurface to a high index minimal hypersurface.

CHAPTER 4

LIMIT FLOWS AND INTERFACES

4.1 Introduction

The main goal for this section is to show that we take the limit of the solution u_ϵ given by Proposition 2 as $\epsilon \to 0$ to get a Brakke flow of integral varifolds. The forward limit and the backward limit of the limit Brakke flow are a Clifford torus and an equatorial sphere, respectively.

Recall that if u_ϵ denotes a solution to (PAC) on $S^3 \times I$, $I \subset \mathbb{R}$, we can define associated 2-varifolds $V_{\epsilon,t} = V_{\epsilon,u_\epsilon(\cdot,t)}$ given by (2.6) (with $u = u_\epsilon(\cdot,t)$), and Radon measures $\mu_{\epsilon,t} = \mu_{\epsilon,u_\epsilon(\cdot,t)}$ on S^3 given by the weight measure $\mu_{\epsilon,t} = \|V_{\epsilon,t}\|$ of $V_{\epsilon,t}$.

Throughout this section, for $\epsilon \in (0, \epsilon_2)$, we consider any solution $u_\epsilon \colon S^3 \times \mathbb{R} \to \mathbb{R}$ which satisfies the conclusions of Proposition 2, where $u_\epsilon^{-\infty}$ is a solution of (AC) whose nodal set is the Clifford torus, $\{u_\epsilon^{-\infty} = 0\} \cap S^3 = T_c$ (recall that this solution is unique, modulo sign).

In Section 4.2, we show that the forward limit u_ϵ is a solution with an equator as its nodal set, based on energy and index estimate and Hiesmayr's rigidity results. In Section 4.3, we study the limit of the gradient flow u_ϵ as $e \downarrow 0$, and point out it is a Brakke flow of integral varifold, by using Ilmanen and Tonegawa's convergence result. In Section 4.4, we analyze the multiplicity of the limit varifold. In Section 4.5, we prove that the limit Brakke flow converges to a multiplicity one equatorial sphere and a multiplicity one Clifford torus as $t \to \pm\infty$. In Section 4.6, we describe the energy limit of u_ϵ.

4.2 Asymptotic convergence of the gradient flow

In what follows, we denote by $u_\epsilon^{+\infty}$ an arbitrary subsequential limit of $u_\epsilon(\cdot, t_k)$, for a sequence $t_k \to +\infty$. Note that this can be extracted from any such sequence, as a consequence of the uniform energy bounds along the flow and the compactness properties of the Sobolev space $W^{1,2}$ [27, Lemma 2.3]. Our main goal in this subsection is to prove that, for sufficiently small $\epsilon > 0$, this limit is a ground state and it is unique, using information about its limit interface.

We introduce the ground states in the sense defined in [11]. Ground state solutions correspond to unstable solutions of the least energy. In convex domains of $\mathbb{R}^n$ and manifolds with Ric ≥ 0, ground states are always of mountain-pass type and have Morse index 1. For the Allen-Cahn equation (AC) on S^n, the ground state is unique up to rotations and corresponds to the equator as a minimal hypersurface.

The main ingredient is the following rigidity result.

Lemma 1. *There exists $\epsilon_4 \in (0, \epsilon_2)$ with the following property. For any $\epsilon \in (0, \epsilon_4)$, the solution $u_\epsilon^{-\infty}$ has Morse index 5, and the only nonconstant solutions of* (AC) *with energy $< E_\epsilon(u_\epsilon^{-\infty})$ are ground states.*

Proof. The existence of ϵ_4 satisfying the first property follows from the index bounds of [26, 32] and [17] under the multiplicity one hypothesis (see Section 4.3 in [10]), and the fact that the Clifford torus has Morse index 5.

Suppose that there does not exist such $\epsilon_4 > 0$. Then there exist a sequence $\epsilon_j > 0$ such that $\epsilon_j \downarrow 0$, and a sequence u_j of nonconstant solutions to (AC) with $\epsilon = \epsilon_j$ which are not ground states and have energy $< E_{\epsilon_j}(u_{\epsilon_j}^{-\infty})$. Since u_j is unstable, if a_j is the energy of a ground state, then $E_{\epsilon_j}(u_j) > a_j$.

Since these solutions have uniformly bounded energy, by passing to a subsequence, we may assume that the varifolds $\frac{1}{\sigma}V_{\epsilon_j,u_j}$ converge to a stationary integral varifold $\frac{1}{\sigma}V$ in S^3 with area in $[4\pi, 2\pi^2]$. We claim that $\frac{1}{\sigma}V$ has density $\mathcal{H}^2$-a.e. equal to 1 on its support. In fact, $\Theta(\frac{1}{\sigma}V, x) \in \mathbb{Z}_+$ for $\mathcal{H}^2$-almost every such x. From $\frac{1}{\sigma}\|V\|(S^3) \leq 2\pi^2$ and the density estimate in [49, Lemma A.2], we see that the density of $\frac{1}{\sigma}V$ is everywhere strictly less than 2, proving the claim.

By the remarks about energy loss in [39], we see that $\frac{1}{\sigma}V$ is the boundary of a region.

More precisely, the varifold $\frac{1}{\sigma}V$ agrees with the multiplicity one varifold induced by the reduced boundary of $\{u = 1\}$, where u is the function of bounded variation on S^3 given by the a.e. limit of u_j. Consequently, the boundary of the $\mathbb{Z}_2$ chain associated to $\frac{1}{\sigma}V$ (in the sense of White [77]) vanishes. By Corollary 1, it follows that $\frac{1}{\sigma}V$ is either a multiplicity one equatorial sphere or a Clifford torus.

If the limit interface $\operatorname{supp}||V||$ is an equator, then by Theorem 3, we know that for sufficiently large j,

$$\operatorname{ind}(u_j) \leq \operatorname{nul}(u_j) + \operatorname{ind}(u_j) \leq \operatorname{nul}(\operatorname{supp}||V||) + \operatorname{ind}(\operatorname{supp}||V||) = 3 + 1 = 4.$$

If the limit interface $\operatorname{supp}||V||$ is a Clifford torus, then by Corollary 2, the nodal set of u_j is a Clifford torus, hence it has Morse index 5.

Combining these index bounds, we know that u_j has Morse index at most 5 for sufficiently large j. Since u_j has energy $< E_{\epsilon_j}(u_{\epsilon_j}^{-\infty})$, which is bounded, therefore by Theorem 4, we know that u_j is either a ground state solution or a symmetric solution with nodal set on some Clifford torus.

Since u_j is not a ground state solution, by Theorem 5, we know that u_j equal to $u_{\epsilon_j}^{-\infty}$ up to isometry. This contradicts the energy bounds $E_{\epsilon_j}(u_j) < E_{\epsilon_j}(u_{\epsilon_j}^{-\infty})$. □

We apply this result to the subsequential limit $u_\epsilon^{+\infty}$, assuming $\epsilon \in (0, \epsilon_4)$. Since the energy of $E_\epsilon(u_\epsilon(\cdot, t))$ is strictly decreasing, we have

$$\sup_\epsilon E_\epsilon(u_\epsilon^{+\infty}) < \sup_\epsilon E_\epsilon(u_\epsilon^{-\infty}) < \infty,$$

so $u_\epsilon^{+\infty}$ is a ground state solution, by Lemma 1. In particular, if $V_\epsilon^{+\infty}$ is the varifold associated to $u_\epsilon^{+\infty}$, then $V_\epsilon^{+\infty} \to V^{+\infty}$ subsequentially, in the varifold sense, where $\frac{1}{\sigma}V^{+\infty}$ is a multiplicity one equatorial sphere.

We are ready to prove the full convergence of $u_\epsilon(\cdot, t)$ to $u_\epsilon^{+\infty}$ using the previous Lemma and the Łojasiewicz-Simon inequality. This is stated, more precisely in the following proposition, which summarizes the results obtained in this subsection.

Proposition 3. *Let $\epsilon_4 > 0$ be given by Lemma 1, and consider the critical points $u_\epsilon^{-\infty}$ of E_ϵ described above. Given $\epsilon \in (0, \epsilon_4)$, suppose that $u_\epsilon : S^3 \times \mathbb{R} \to \mathbb{R}$ is a sequence of eternal solutions to* (PAC) *on S^3 such that*

- *(i)* *$E_\epsilon(u_\epsilon(\cdot, t))$ is strictly decreasing;*
- *(ii)* *$\|u_\epsilon(\ \cdot\ , t) - u_\epsilon^{-\infty}\|_{W^{1,2}(M)} \to 0$ as $t \to -\infty$;*
- *(iii)* *for any sequence $t_k \to +\infty$, the functions $u_\epsilon(\cdot, t_k)$ do not converge to the constant critical points ± 1 of E_ϵ.*

Then, there exists a (nonconstant) least energy unstable solution $u_\epsilon^{+\infty}$ of the Allen-Cahn equation (AC) *such that*

$$\|u_\epsilon(\ \cdot\ , t) - u_\epsilon^{\pm\infty}\|_{W^{1,2}(S^3)} \to 0, \quad \text{as} \quad t \to \pm\infty.$$

In particular, any limit interface obtained from the limits $u_\epsilon^{+\infty}$ is a multiplicity one equatorial sphere, and it holds $E_\epsilon(u_\epsilon^{+\infty}) \to (2\sigma) \cdot 4\pi$.

Proof. It remains to prove the full convergence of $u_\epsilon(\cdot, t)$, as $t \to +\infty$. By [11], least energy unstable critical points of the energy functional E_ϵ are unique up to ambient isometries. In particular, E_ϵ is a *Morse-Bott* functional at this critical level. Thus, the convergence of $u_\epsilon(\cdot, t)$ to $u_\epsilon^{+\infty}$ in $W^{1,2}$ is a consequence of the Łojasiewicz-Simon gradient inequality for such functionals, see e.g. [25]. □

4.3 The limit flow

Now we analyze the limit of the gradient flow given by Proposition 3 as $\epsilon \downarrow 0$. For simplicity, we will omit the index j in sequences $\epsilon_j \downarrow 0$ and in the corresponding objects (the energy functional, functions, and varifolds). First, we aim to prove that the gradient flow satisfies the necessary conditions to take the limit as $\epsilon \downarrow 0$ and obtain a codimension one Brakke flow in the sphere S^3.

We showed that $E_\epsilon(u_\epsilon^{-\infty}) \to 2\sigma(2\pi^2)$, and $E_\epsilon(u_\epsilon^{+\infty}) \to 2\sigma(4\pi)$. Thus given a small $\delta > 0$, for sufficiently small $\epsilon > 0$ (depending on δ), we have

$$2\sigma(2\pi^2) - \delta \leq E_\epsilon(u_\epsilon^{-\infty}) \leq 2\sigma(2\pi^2) + \delta \quad \text{and} \quad E_\epsilon(u_\epsilon^{+\infty}) \leq 2\sigma(4\pi) + \delta.$$

Recall that the energy $E_\epsilon(u_\epsilon(\cdot, t))$ is a continuous strictly decreasing function of t. By picking a sufficiently small $\delta > 0$ and by noting that this solution joins $u_\epsilon^{-\infty}$ to $u_\epsilon^{+\infty}$, we see that there exists $t(\epsilon) \in \mathbb{R}$ such that $E_\epsilon\left(u_\epsilon(\cdot, t(\epsilon)) \right) = 2\sigma(5\pi)$ (as $4\pi < 5\pi < 2\pi^2$). By translating the gradient flow $u_\epsilon(\cdot, t)$ to $u_\epsilon(\ \cdot\ , t + t(\epsilon))$, we can assume that $E_\epsilon\left(u_\epsilon(\cdot, 0) \right) = 2\sigma(5\pi)$ for all small ϵ.

Remark. In this section, we will abuse notation and also denote by u_{ϵ_j} (or simply u_ϵ) the time-translated solution. When considering a *family* of solutions to (PAC) – in particular the family $\mathscr{S}$ constructed in [18] and described in Section 3 – it will be useful to keep track of this translation to contrast it with the initial condition $u_\epsilon(\cdot, 0)$.

Now we check the eternal solutions to (PAC) given by Proposition 3 satisfies the assumption in Tonegawa [70] in order to get the integrality of the limit varifold.

Lemma 2. *There exists $\epsilon_5 \in (0, \epsilon_4)$ with the following property. Let $u_\epsilon \colon M \times \mathbb{R} \to \mathbb{R}$ be a solution of* (PAC) *that satisfies the hypotheses of Proposition 3, for $\epsilon \in (0, \epsilon_5)$. Then $u_\epsilon \in C^3(S^3 \times \mathbb{R})$ and $\sup_{S^3 \times \mathbb{R}} |u_\epsilon| < 1$. Moreover,*

1. *For any $t \in \mathbb{R}$, it holds*

$$E_\epsilon\left(u_\epsilon(\ \cdot\ , t) \right) \leq 2\sigma(2\pi^2) + 1$$

2. It holds

$$\int_{S^3\times\mathbb{R}} \epsilon|\partial_t u_\epsilon|^2 \leq 2\sigma(2\pi^2)+1.$$

Proof. Once again, we will omit the index j. Since $u_\epsilon^{-\infty}$ and $u_\epsilon^{+\infty}$ are solutions of (AC), we have that $|u_\epsilon^{-\infty}| \leq 1$ and $|u_\epsilon^{+\infty}| \leq 1$. Since $\{u_\epsilon\}$ solves the parabolic Allen-Cahn equation (PAC), by a maximum principle argument (i.e. Lemma 2.3 (2) in Gaspar-Guaraco [27]), we know that $|u_\epsilon| < 1$.

As noted above, for sufficiently small $\epsilon > 0$ (depending only on $E_\epsilon(u_\epsilon^{+\infty})$), the energies of $u_\epsilon(\cdot,t)$ are bounded above by $2\sigma(2\pi^2)+1$, which we will denote by E_0. By the monotonicity of the energy, this proves (1). Furthermore, $E_\epsilon(u_\epsilon)$ is differentiable with respect to t and

$$\begin{aligned}\frac{d}{dt}E_\epsilon(u_\epsilon) &= \int_{S^3} \epsilon\left\langle \nabla u_\epsilon, \nabla(\partial_t u_\epsilon)\right\rangle + \frac{1}{\epsilon}W'(u_\epsilon)\partial_t u_\epsilon \\ &= \int_{S^3} -\epsilon(\partial_t u_\epsilon)(\Delta u_\epsilon - \frac{1}{\epsilon^2}W'(u_\epsilon)) \\ &= -\int_{S^3} \epsilon|\partial_t u_\epsilon|^2.\end{aligned}$$

Thus, for all $T_1 < T_2$,

$$\begin{aligned}\int_{S^3\times(T_1,T_2)} \epsilon|\partial_t u_\epsilon|^2 &= -\int_{T_1}^{T_2} \frac{d}{dt}E_\epsilon(u_\epsilon)dt = E_\epsilon(u_\epsilon(\cdot,T_1)) - E_\epsilon(u_\epsilon(\cdot,T_2)) \\ &\leq E_\epsilon(u_\epsilon^{-\infty}) < E_0.\end{aligned}$$

This proves (2). □

By the convergence result for solutions of (PAC) of Ilmanen [40] and Tonegawa [70] (see also Sato [63]), after passing to a subsequence (not relabeled) with $\epsilon \downarrow 0$, the varifolds $V_{\epsilon,t}$ associated to $u_\epsilon(\cdot,t)$ converge, for every $t \in \mathbb{R}$, to a 2-varifold V_t, and the underlying Radon measures $\frac{1}{\sigma}\mu_{\epsilon,t} = \frac{1}{\sigma}\|V_{\epsilon,t}\|$ converge to a Radon measure

$$\Sigma_t := \frac{1}{\sigma}\mu_t = \frac{1}{\sigma}\|V_t\| \tag{4.1}$$

which satisfies the mean curvature flow equation in the sense of Brakke. Moreover,

$$\frac{1}{2\sigma}E_\epsilon\left(\,u_\epsilon(\,\cdot\,,t)\,\right) \to \|\Sigma_t\|(S^3), \quad \text{as} \quad \epsilon \downarrow 0,$$

and, for almost every $t \in \mathbb{R}$, the varifold $\frac{1}{\sigma}V_t$ is an integral varifold.

More precisely, we apply the convergence result to u_ϵ on $S^3 \times [-m, +\infty)$ for each $m \in \mathbb{N}$ to obtain the (subsequential) convergence of $\frac{1}{\sigma}V_{\epsilon,t}$ for all $t \geq -m$. By picking a diagonal subsequence, we get the convergence described above.

Since $\frac{1}{2\sigma}E_\epsilon\left(\,u_\epsilon(\,\cdot\,,0)\,\right) = 5\pi$ for all small ϵ, we see that

$$\|\Sigma_{-t}\|(S^3) = \lim_{\epsilon\downarrow 0}\frac{1}{2\sigma}E_\epsilon(u_\epsilon(\cdot,-t)) \geq \lim_{\epsilon\downarrow 0}\frac{1}{2\sigma}E_\epsilon(u_\epsilon(\cdot,0)) = 5\pi$$

and, similarly, $\|\Sigma_t\|(S^3) \leq 5\pi$, for every $t \geq 0$. Note also that $\|\Sigma_t\|(S^3) \leq 2\pi^2$ for all $t \in \mathbb{R}$. In fact, since $\frac{1}{2\sigma}E_\epsilon(u_\epsilon^{-\infty}) \to 2\pi^2$ as $\epsilon \downarrow 0$, for each $\delta' > 0$, there exists a $\epsilon' > 0$ small, such that

$$\frac{1}{2\sigma}E_\epsilon(u_\epsilon^{-\infty}) < 2\pi^2 + \delta', \quad \text{for all } \epsilon \in (0, \epsilon').$$

By noting that $E_\epsilon\left(\,u_\epsilon(\,\cdot\,,t)\,\right) \leq E_\epsilon(u_\epsilon^{-\infty})$ for all t, we obtain $\|\Sigma_t\|(S^3) \leq 2\pi^2 + \delta'$, for every t. Since δ' is arbitrary, this implies that $\|\Sigma_t\|(S^3) \leq 2\pi^2$.

By abuse of notation, we will identify Σ_t with its support, which is, for almost every $t \in \mathbb{R}$, a 2-dimensional rectifiable set. We want to show that Σ_t (and the associated varifolds $\frac{1}{\sigma}V_t$) converge to a multiplicity one Clifford torus, as $t \to -\infty$ along subsequences, and to a multiplicity one equatorial sphere, as $t \to \infty$, also along subsequences. Note that in general, we don't know whether or not Σ_t converges to the initial torus $T_c = \{u_\epsilon^{-\infty} = 0\} \cap S^3$ as

$t \to -\infty$, and Σ_t converges to $\frac{1}{\sigma}V^{+\infty}$ (the ϵ-limit of $V_{\epsilon, u_\epsilon^{+\infty}}$) as $t \to +\infty$ (the limits in t and ϵ do not commute). This question will be addressed in Section 5 using symmetries of S^3.

4.4 Parity of the multiplicity

We will need further information about the parity of the multiplicity of the limit varifold V_t, as described in the following lemma. It intuitively says that the interfaces fold an even number of times near a point in supp $\|V_t\|$ if, and only if, $u_\epsilon(\cdot, t)$ converges to the same value on the two sides of this surface. This was proved by Hutchinson-Tonegawa [39] in the elliptic case; see also Takasao-Tonegawa [69] in the parabolic case, for an equation with a transport term in Euclidean domains or in a torus.

Lemma 3. *For almost every $t \in \mathbb{R}$, the density of the varifold Σ_t satisfies*

$$\Theta(\Sigma_t, x) = \begin{cases} \textit{odd} & \mathcal{H}^2\text{-}a.e.\ x \in M_t, \\ \textit{even} & \mathcal{H}^2\text{-}a.e.\ x \in \operatorname{supp}\|\Sigma_t\| \setminus M_t, \end{cases}$$

where M_t is the reduced boundary of $\{u_0(\cdot, t) = 1\}$, and $u_0(\cdot, t) = u_0^t$ is the bounded variation function given by the weak-$*$ limit of $u_\epsilon(\cdot, t)$, as functions of bounded variation.

We give here a brief explanation of the proof of the lemma above.

Fix a point $(y, s) \in S^3 \times (0, +\infty)$, the Huisken's monotonicity kernel is:

$$\rho = \rho_{y,s}(x, t) \equiv \frac{1}{(4\pi(s-t))^{(n-1)/2}} e^{-|x-y|^2/4(s-t)}, \qquad 0 \le t < s, x \in S^3.$$

Recall the Clearing-out Lemma of Ilmanen [40] (see also Pisante-Punzo [59, Lemma 4.1]):

Lemma 4 ([40])**.** *(1) There is $\eta > 0$ depending on W such that $\int \rho_{y,s} < \eta$ implies $(y, s) \notin \overline{\cup_{t' \ge 0} \operatorname{supp} \Sigma_{t'} \times \{t'\}}$.*

(2) If $(y,s) \notin \overline{\cup_{t' \geq 0} \operatorname{supp} \Sigma_{t'} \times \{t'\}}$*, then there is a neighborhood* U *of* (y,s) *in* $S^3 \times [0,+\infty)$ *such that* $u_{\epsilon_i} \to u$ *uniformly on* U *to either* $+1$ *or* -1.

This result roughly says that as $\epsilon \downarrow 0$, u_ϵ converges locally uniformly to either 1 or -1 as $\epsilon \downarrow 0$ at any point in $\{|u_0^t| > \alpha\}$, where $0 < \alpha < 1$ is a constant.

In addition, the proof of the integrality [70] of the varifold $\frac{1}{\sigma}V_t$ is based in an a.e.-graphical decomposition of the transition layers of u_ϵ, similarly to the elliptic counterpart in [39] (Hutchinson-Tonegawa proved the parity of the density of the limit interface, which suggests that folding of the interface as $\epsilon \downarrow 0$ occurs locally as an integer multiple of $1-$D traveling wave solutions). Hence, the argument about the oddness and evenness of the density in the proof of Theorem 1 in [39] can be carried out in the parabolic setting. We refer to [69] for detailed proof of the parity of the density.

4.5 Limits of the Brakke flow

Finally, we can describe the limits of the Brakke flow Σ_t in S^3.

Theorem 6. *As* $t \to -\infty$*, the varifold* $\frac{1}{\sigma}V_t$ *converges to some multiplicity one minimal torus in* S^3*, and its support converges graphically to this torus. As* $t \to +\infty$*, the varifold* $\frac{1}{\sigma}V_t$ *subconverges to a multiplicity one equatorial sphere.*

Proof. Consider any sequence $\theta_i \uparrow \infty$, and the sequence of translated Brakke flows $\{\Sigma_t^{(i)} := \Sigma_{t+\theta_i}\}_{t \geq 0}$. By Brakke's compactness theorem and the uniform boundedness of areas, $(\Sigma_t^{(i)})_{t \geq 0}$ converges subsequentially to an integral Brakke flow with constant area. Therefore, this Brakke flow is supported on a stationary integral varifold $\frac{1}{\sigma}V_{+\infty}$. Similarly, Σ_t subconverges, as $t \to -\infty$, to a stationary integral varifold $\frac{1}{\sigma}V_{-\infty}$.

By Lemma 3, the associated $\mathbb{Z}_2$ chain of $\frac{1}{\sigma}V_t$ is M_0^t is the reduced boundary of $\{u_0^t = 1\}$, thus it has vanishing boundary. By White [77] Theorem 4.2, we know that the associated $\mathbb{Z}_2$ chain of any subsequential limit varifold $\frac{1}{\sigma}V_{+\infty}$ or $\frac{1}{\sigma}V_{-\infty}$ has zero boundary.

We have shown the area estimate $\|\Sigma_t\|(S^3) \leq 2\pi^2$. By [18, Lemma 5.8], it follows that any such limit $\frac{1}{\sigma}V_{-\infty}$ or $\frac{1}{\sigma}V_{+\infty}$ is either a multiplicity one equatorial sphere or a multiplicity one Clifford torus. On the other hand, we have the inequality $\|\Sigma_{-t}\|(M) \geq 5\pi \geq \|\Sigma_t\|(M)$ for every $t \geq 0$. Thus $\frac{1}{\sigma}\|V_{-\infty}\|(S^3) \geq 5\pi \geq \frac{1}{\sigma}\|V_{+\infty}\|(S^3)$, and we conclude that any subsequential limit $\frac{1}{\sigma}V_{-\infty}$ is a Clifford torus, and any subsequential limit $\frac{1}{\sigma}V_{+\infty}$ is a multiplicity one equatorial sphere. The convergence as $t \to -\infty$ and the graphical convergence of Σ_t to the minimal torus, follows from [18], by means of Brakke's local regularity for the mean curvature flow [4] (see also [44]), as the characterization of the backward limit allows us to obtain the smoothness of Σ_t for sufficiently negative time. □

4.6 Energy limit

As a final remark, we note that the energy $u_{\epsilon_j}(\cdot, t)$ has a limit as $(\epsilon_j, t) \to (0, -\infty)$ (where $\epsilon_j \downarrow 0$ is a subsequence for which the associated varifolds converge):

Lemma 5. *Given $\delta > 0$, there exist a positive integer $J_0 = J_0(\delta)$ and $T_0 = T_0(\delta) > 0$ (independent of ϵ) such that*

$$0 < E_{\epsilon_j}(u_{\epsilon_j}^{-\infty}) - E_{\epsilon_j}(u_{\epsilon_j}(\,\cdot\,, t)) < \delta, \qquad \textit{for all } j \geq J_0 \textit{ and all } t < -T_0.$$

Consequently,

$$\lim_{(j,t)\to(+\infty,-\infty)} E_{\epsilon_j}(u_{\epsilon_j}(\cdot, t)) = 2\sigma \cdot 2\pi^2.$$

Proof. If not, since $\{u_{\epsilon_j}(\cdot, t)\}$ are nonconstant gradient flows for $-E_{\epsilon_j}$, there exists a $\delta > 0$, a subsequence $\{\epsilon_i = \epsilon_{j_i}\}$ of $\{\epsilon_j\}$ and sequence $t_i \to -\infty$ such that $E_{\epsilon_i}(u_{\epsilon_i}^{-\infty}) - E_{\epsilon_i}(u_{\epsilon_i}(\cdot, t_i)) \geq \delta$ for all i. We claim that this leads to a contradiction, for sufficiently large i.

By construction, $E_\epsilon(u_\epsilon^{-\infty}) \to 2\sigma(2\pi^2)$, so there exists $\epsilon' > 0$ such that for any $\epsilon < \epsilon'$, it holds $E_\epsilon(u_\epsilon^{-\infty}) < 2\sigma(2\pi^2 + \frac{\delta}{6\sigma})$. On the other hand, Σ_t (and the associated varifolds $\frac{1}{\sigma}V_t$) converge to a Clifford torus, thus there exists $t' \ll 0$ such that $\|\Sigma_{t'}\|(S^3) > 2\pi^2 - \frac{\delta}{6\sigma}$. For this fixed t', using $\frac{1}{2\sigma}E_{\epsilon_i}(u_{\epsilon_i}(\cdot, t')) \to \|\Sigma_{t'}\|(S^3)$ as $i \to +\infty$, we get some positive integer I such that

$$\left|\frac{1}{2\sigma}E_{\epsilon_i}(u_{\epsilon_i}(\cdot, t')) - \|\Sigma_{t'}\|(S^3)\right| < \frac{\delta}{6\sigma}$$

for any $i \geq I$. Hence $E_\epsilon(u_\epsilon(\cdot, t')) > 2\sigma(\|\Sigma_{t'}\|(S^3) - \frac{\delta}{6\sigma})$.

Since $\epsilon_i \downarrow 0$ and $t_i \to -\infty$, we may assume I is such that $\epsilon_i < \epsilon'$ and $t_i < t'$ whenever $i \geq I$. Then (using again that the energy decreases along the flow),

$$\begin{aligned} E_{\epsilon_i}(u_{\epsilon_i}^{-\infty}) - \delta \geq E_{\epsilon_i}(u_{\epsilon_i}(\cdot, t_i)) > E_{\epsilon_i}(u_{\epsilon_i}(\cdot, t')) > 2\sigma\left(\|\Sigma_{t'}\|(S^3) - \frac{\delta}{6\sigma}\right) \\ > 2\sigma\left(2\pi^2 - \frac{\delta}{3\sigma}\right) = 2\sigma\left(2\pi^2 + \frac{\delta}{6\sigma}\right) - \delta > E_{\epsilon_i}(u_{\epsilon_i}^{-\infty}) - \delta, \end{aligned}$$

so we get a contradiction. □

CHAPTER 5

SYMMETRIES AND PROOF OF MAIN RESULTS

5.1 Introduction

We use the notation introduced in Section 3 to describe the ancient solutions of the gradient flow of $-E_\epsilon$ given by the results of Choi-Mantoulidis [18], for the critical point $u_\epsilon^{-\infty}$ which has the Clifford torus T_c as its nodal set.

In this section, we study the backward limit and the forward limit of the limit Brakke flow, in order to prove the main result. The main difficulty as we mentioned before is the fact that the limit in t and in ϵ do not commute. So even though for each ϵ, we have the gradient flow of $-E_\epsilon$ connecting $u_\epsilon^{-\infty}$ to a ground state solution with nodal set S_ϵ, and S_ϵ are the same equator for all small ϵ, the limit Brakke flow may converge to a different equator and a different Clifford torus as $t \to \pm\infty$, respectively.

The technique we use to solve this obstacle is a symmetry argument. More precisely, by Choi-Mantoulidis rigidity of ancient gradient flows, we observe that if $a \in B_\eta(0)$ is fixed by an isometry, then so are $S_\epsilon(a)(\cdot, t)$, the limit Brakke flow V_t, and the limits $V_{\pm\infty}$.

Then we show that there are only finitely many possibilities for the backward and forward limit $V_{\pm\infty}$, and use a continuity argument to show that the backward limit needs to be the same Clifford torus T_c, and give a clear description of the forward limit.

Recall that, for $\epsilon \in (0, \epsilon_4)$ (as given by Lemma 1), we denote by $\{\varphi_i\}_{i=1}^5$ an L^2-orthonormal basis for the eigenspaces of the linearized Allen-Cahn operator at $u_\epsilon^{-\infty}$ corresponding to negative eigenvalues, where we assume $\varphi_1 > 0$. We will denote by $\mathcal{V}_1$ the first eigenspace, which is spanned by φ_1, and by $\mathcal{V}$ the eigenspace spanned by $\{\varphi_i\}_{i=2,3,4,5}$.

We also recall the solution map $\mathscr{S}_\epsilon = \mathscr{S} \colon B_\eta(0) \subset \mathbb{R}^5 \to C^{2,\alpha}(S^3 \times \mathbb{R})$, defined for some $\eta = \eta_\epsilon > 0$ (these solutions are also defined for all $t > 0$ by the long-time existence result described in the proof of Proposition 2). We will use the action of the isometry group of S^3 by pre-composition to study the limit in time of some of these solutions. As a consequence, we will establish the existence of a two-parameter family of solutions to this parabolic equation joining $u_\epsilon^{-\infty}$ to ground states, as well as Brakke flows joining the Clifford torus T_c to equatorial spheres. This will conclude the proof of the main results.

5.2 Isometries and invariance

We introduce some isometries of S^3 that leave $u_\epsilon^{-\infty}$ invariant. First, for $\theta \in \mathbb{R}$, let $\rho^\theta, \tau^\theta \colon S^3 \to S^3$ denote the rotations

$$\rho^\theta(x) = (x_1 \cos\theta - x_2 \sin\theta, x_1 \sin\theta + x_2 \cos\theta, x_3, x_4)$$
$$\tau^\theta(x) = (x_1, x_2, x_3 \cos\theta - x_4 \sin\theta, x_3 \sin\theta + x_4 \cos\theta)$$

We also regard ρ^θ and τ^θ naturally as isometries of $S^3 \times \mathbb{R}$, acting on the first factor, and as isometries of $B_\eta(0) \subset \mathbb{R}^5$ acting on the last 4 coordinates, using the same notation for simplicity, as in $R(a_1, \ldots, a_5) = (a_1, R(a_2, a_3, a_4, a_5))$.

Observe that $u_\epsilon^{-\infty}$ is invariant by both ρ^θ and τ^θ. Thus, these rotations leave φ_1 invariant and act linearly and isometrically on the left on the eigenspace $\mathcal{V}$ by pre-composition, that is

$$\varphi \mapsto \varphi \circ \rho^{-\theta} \quad \text{and} \quad \varphi \mapsto \varphi \circ \tau^{-\theta}.$$

We can describe this action as rotations in $\mathcal{V}$. In fact, this describes a representation of $\mathrm{SO}(2) \times \mathrm{SO}(2)$ on $\mathcal{V}$ taking values in $\mathrm{SO}(\mathcal{V}) \simeq \mathrm{SO}(4)$. We will need the following observation:

Lemma 6. *Let $\epsilon \in (0, \epsilon_4)$, where ϵ_4 is given by Lemma 1. The kernel of the representation described above is trivial. That is, if $(\theta, \zeta) \in \mathbb{R}^2$ is such that $\varphi \in \mathcal{V} \mapsto \varphi \circ (\rho^{-\theta} \circ \tau^{-\zeta})$ is the identity map, then $\frac{\theta}{2\pi}, \frac{\zeta}{2\pi} \in \mathbb{Z}$.*

We postpone the proof of the lemma above for later. As a consequence, we can choose the eigenfunctions φ_i in a way that $\{\rho^\theta\}$ acts on $\{\varphi_2, \varphi_3\}$ by rotation and fixes $\{\varphi_4, \varphi_5\}$ pointwise, while $\{\tau^\theta\}$ acts on the latter by rotations and fixes φ_2, φ_3. Concretely, we may assume

$$\begin{cases} \varphi_2 \circ \rho^{-\theta} &= \quad (\cos\theta)\, \varphi_2 + (\sin\theta)\, \varphi_3 \\ \varphi_3 \circ \rho^{-\theta} &= (-\sin\theta)\, \varphi_2 + (\cos\theta)\, \varphi_3 \\ \varphi_i \circ \rho^{-\theta} &= \varphi_i, \qquad i = 4, 5, \end{cases},$$
$$\begin{cases} \varphi_4 \circ \tau^{-\theta} &= \quad (\cos\theta)\, \varphi_4 + (\sin\theta)\, \varphi_5 \\ \varphi_5 \circ \tau^{-\theta} &= (-\sin\theta)\, \varphi_4 + (\cos\theta)\, \varphi_5 \\ \varphi_i \circ \tau^{-\theta} &= \varphi_i, \qquad i = 2, 3. \end{cases} \tag{5.1}$$

This follows from the lemma above, as the injective image of this representation in SO(4) is a closed 2-torus subgroup. Then, it suffices to note that any maximal torus in SO(4) is conjugated to the standard torus SO(2) × SO(2) ⊂ SO(4). Up to changing the basis of $\mathcal{V}$, we obtain (5.1). We emphasize that the minus sign in the angles is chosen so that we have an action on the left, similarly to the actions on S^3, $S^3 \times \mathbb{R}$, and $B_\eta(0)$.

The uniqueness of solutions of (PAC) that converge back to $u_\epsilon^{-\infty}$ can be used to establish the following equivariance property:

Lemma 7. *The solution map $\mathscr{S}$ satisfies*

$$\mathscr{S}(a)(\rho^{-\theta}(x), t) = \mathscr{S}(\rho^\theta(a))(x, t), \quad \textit{for any} \quad (x, t) \in S^3 \times \mathbb{R},$$

for every $\theta \in \mathbb{R}$, and similarly for τ^θ.

Proof. Since $u_\epsilon^{-\infty}$ is invariant by $\rho^{-\theta}$, the solution $\mathscr{S}(a)(\rho^{-\theta}(x), t)$ still converges to $u_\epsilon^{-\infty}$ as $t \to -\infty$. By the uniqueness of such solutions, it suffices to check that the L^2-projections of $\mathscr{S}(a)(\rho^{-\theta}(x), 0) - u_\epsilon^{-\infty}(x)$ and $\mathscr{S}(\rho^\theta(a))(x, 0) - u_\epsilon^{-\infty}(x)$ onto $\mathcal{V}_1 \oplus \mathcal{V}$ coincide. By the invariance of $u_\epsilon^{-\infty}$ and φ_1, a direct computation using (5.1) shows that

$$\begin{aligned}
&\int_{S^3} \Big(\mathscr{S}(a)(\rho^{-\theta}(x),0) - u_\epsilon^{-\infty}(x)\Big)\, \varphi_i(x)\, d\mu_g(x) \\
&\qquad = \int_{S^3} \big(\mathscr{S}(a)(x,0) - u_\epsilon^{-\infty}(x)\big)\, \varphi_i(x)\, d\mu_g(x) = a_i \\
&\qquad = \int_{S^3} \Big(\mathscr{S}(\rho^{\theta}(a))(x,0) - u_\epsilon^{-\infty}(x)\Big)\, \varphi_i(x)\, d\mu_g(x)
\end{aligned}$$

for $i = 1, 4, 5$, while

$$\begin{aligned}
&\int_{S^3} \Big(\mathscr{S}(a)(\rho^{-\theta}(x),0) - u_\epsilon^{-\infty}(x)\Big)\, \varphi_2(x)\, d\mu_g(x) \\
&\quad = (\cos\theta) \int_{S^3} \big(\mathscr{S}(a)(x,0) - u_\epsilon^{-\infty}(x)\big)\, \varphi_2(x)\, d\mu_g(x) \\
&\qquad - (\sin\theta) \int_{S^3} \big(\mathscr{S}(a)(x,0) - u_\epsilon^{-\infty}(x)\big)\, \varphi_3(x)\, d\mu_g(x) \\
&\quad = (\cos\theta) a_2 - (\sin\theta) a_3 = \int_{S^3} \Big(\mathscr{S}(\rho^{\theta}(a))(x,0) - u_\epsilon^{-\infty}(x)\Big)\, \varphi_2(x)\, d\mu_g(x)
\end{aligned}$$

and similarly for the projection of $\Big(\mathscr{S}(a)(\rho^{-\theta}(x),0) - u_\epsilon^{-\infty}(x)\Big)$ onto φ_3. The proof for τ^θ is similar. □

We also consider the isometry $s(x) = (x_3, x_4, x_1, x_2)$ acting on S^3. From the construction of $u_\epsilon^{-\infty}$ (see [11] or [33]), this isometry satisfies $u_\epsilon^{-\infty} \circ s = -u_\epsilon^{-\infty}$, hence it preserves the linearized Allen-Cahn operator at $u_\epsilon^{-\infty}$. It follows that the linear operator $\varphi \mapsto -\varphi \circ s$ defined on $\mathcal{V}_1 \oplus \mathcal{V}$ maps φ_1 to $(-\varphi_1)$ and, possibly after changing the basis of $\mathcal{V}$ by rotations, it maps $\varphi_2, \varphi_3, \varphi_4, \varphi_5$ into $(-\varphi_4)$, $(-\varphi_5)$, $(-\varphi_2)$ and $(-\varphi_3)$, respectively. Arguing as in the proof of Lemma 7, we see that

$$-\mathscr{S}(a)(s(x),t) = \mathscr{S}(-s(a))(x,t), \tag{5.2}$$

where write $s(a)$ to mean $(a_1, s(a_2, a_3, a_4, a_5)) = (a_1, a_4, a_5, a_2, a_3)$.

Remark. The orbits of points $b \in B_\eta(0)$ by the action by ρ^θ and τ^θ are the tori

$$\{a \in B_\eta(0) \mid a_1 = b_1, a_2^2 + a_3^2 = b_2^2 + b_3^2, a_4^2 + a_5^2 = b_4^2 + b_5^2\},$$

which degenerate to circles (or points) when $(b_2, b_3) = 0$ or $(b_4, b_5) = 0$. Note that if the corresponding solution $\mathscr{S}(a)$ converges to a constant c as $t \to +\infty$, for some point a in this orbit, then the solutions corresponding to any point in the same orbit converge to c, as these two solutions to (PAC) differ by pre-composition with an isometry of the sphere.

For each small $\epsilon > 0$, we fix $r = r(\epsilon) \in (0, \eta)$ depending continuously on ϵ. We will be particularly interested in the orbit

$$\mathcal{O}_\epsilon = r\,\mathcal{O}, \quad \text{where } \mathcal{O} = \left\{p \in S^4 \mid p_1 = 0, p_2^2 + p_3^2 = \tfrac{1}{2} = p_4^2 + p_5^2\right\} \subset \mathbb{R}^5, \tag{5.3}$$

which is invariant by the symmetry s. Note that this orbit may possibly change with $\epsilon > 0$, but only by a dilation. Geometrically, each direction in $\mathcal{O}$ corresponds to some deformation of the Clifford torus that decreases its area (this normalization will be useful later when we study the forward limit of $\mathscr{S}(a)$ and the corresponding Brakke flow).

Finally, we consider reflections $r_v \colon S^3 \to S^3$ given by $r_v(x) = x - 2\langle x, v\rangle v$, for unit vectors $v \in \mathbb{R}^4$, and the corresponding action on (the four last coordinates of) $B_\eta(0)$. If $v_1 = 0 = v_2$ (respectively if $v_3 = 0 = v_4$) then $u_\epsilon^{-\infty} \circ r_v = u_\epsilon^{-\infty}$ and $\varphi_1 \circ r_v = \varphi_1$. Moreover, $\varphi \mapsto \varphi \circ r_v$ is a linear involution on $\mathcal{V}$ and it commutes with τ^θ (respectively, with ρ^θ), so it defines an operator on $\operatorname{span}\{\varphi_2, \varphi_3\}$ (respectively, on $\operatorname{span}\{\varphi_4, \varphi_5\}$). Using basic properties of linear involutions, one deduces:

Lemma 8. *Suppose $v \in S^3$ and $v_3 = 0 = v_4$ (respectively, $v_1 = 0 = v_2$). Denote by $E_v^\pm$ the (± 1)-eigenspaces of r_v in the space E spanned by φ_2, φ_3 (respectively, φ_4, φ_5). Then* *(i)* $\dim E_v^+ = 1 = \dim E_v^-$.

(ii) For any $\varphi \in E$*, there is such a unit vector* v *satisfying* $\varphi \circ r_v = \varphi$.

(iii) $E^{\pm}_{\rho^{\pi/2}(v)} = E_v^{\mp}$ *(respectively,* $E^{\pm}_{\tau^{\pi/2}(v)} = E_v^{\mp}$*).*

Proof. We assume $v_3 = 0 = v_4$, the other case being analogous. Denote by E the space spanned by φ_2 and φ_3, the fixed space of τ^θ in $\mathcal{V}$. The only possible eigenvalues of $\varphi \mapsto \varphi \circ r_v$ are ± 1. We cannot have $E_v^- = \{0\}$ for all such v, otherwise any $\varphi \in E$ would satisfy $\varphi \circ r_v = \varphi$. This leads to a contradiction, as we can find such $u, v \in S^3$ so that $r_u \circ r_v = \rho^{-\pi/2}$, implying $\varphi \circ \rho^{-\pi/2} = \varphi$, which is impossible for nonzero $\varphi \in E$.

This proves that, for some $v \in S^3$ with $v_3 = 0 = v_4$, the eigenspace E_v^- is nontrivial. This conclusion then holds for every such v, since $r_{\rho^\theta(v)} \circ \rho^\theta = \rho^\theta \circ r_v$. Similarly, we see that E_v^+ is nontrivial, so these eigenspaces must be one-dimensional. Finally, one readily concludes (ii) and (iii) from the relation $r_{\rho^\theta(v)} \circ \rho^\theta = \rho^\theta \circ r_v$ and the fact that E_v^+ and E_v^- are orthogonal. □

In order to study the backward and forward limits of the Brakke flow constructed from sequences of solutions to (PAC), it will be useful to describe $E_v^{\pm}$ more explicitly. For each $\epsilon \in (0, \epsilon_4)$, after possibly rotating φ_2 and φ_3, we can assume that φ_2 spans the $+1$ eigenspace of r_{e_2}, that is $E_{e_2}^+$. Then by Lemma 8, the reflection r_{e_1} reverses the sign of φ_2, r_{e_2} reverses the sign of φ_3, and φ_3 is preserved by r_{e_1}. The eigenfunctions φ_2 and φ_3 we get satisfy the following equations

$$\begin{aligned}
\varphi_2(x_1, x_2, x_3, x_4) &= \varphi_2(x_1, -x_2, x_3, x_4),\\
\varphi_2(x_1, x_2, x_3, x_4) &= -\varphi_2(-x_1, x_2, x_3, x_4),\\
\varphi_3(x_1, x_2, x_3, x_4) &= -\varphi_3(x_1, -x_2, x_3, x_4),\\
\varphi_3(x_1, x_2, x_3, x_4) &= \varphi_3(-x_1, x_2, x_3, x_4).
\end{aligned}$$

Moreover, we can assume that $\varphi_4 = \varphi_2 \circ s$ and $\varphi_5 = \varphi_3 \circ s$ to get a similar property with respect to reflections on x_3 and x_4. This choice is compatible with the action by rotations ρ^θ and τ^θ on the space generated by these eigenfunctions so that (5.1) remains valid. Furthermore, using this choice, we see that $\mathscr{S}$ is equivariant with respect to reflections that preserve T_c, namely

$$\mathscr{S}(a)(r_v(x), t) = \mathscr{S}(r_v(a))(x, t) \tag{5.4}$$

for any unit vector v with $v_1 = 0 = v_2$ or $v_3 = 0 = v_4$. In fact, the computation is straightforward for $v = e_i$, $i = 1, 2, 3, 4$, and the general case follows from

$$\begin{aligned}\mathscr{S}(a)(r_{\rho^\theta(e_1)}(x), t) &= \mathscr{S}(a)(\rho^\theta \circ r_{e_1} \circ \rho^{-\theta}(x), t)\\ &= \mathscr{S}(\rho^{-\theta} \circ r_{e_1} \circ \rho^\theta(a))(x, t)\\ &= \mathscr{S}(r_{\rho^\theta(e_1)}(a))(x, t),\end{aligned}$$

and similarly for $\tau^\theta(e_3)$.

Proof of Lemma 6. Suppose that $\theta, \zeta \in \mathbb{R}$ satisfy the hypothesis in the statement of the lemma. By Propositions 2 and 3, there exists a point $a = (a_1, a_2, a_3, a_4, a_5) \in \partial B_r(0)$ such that $\mathscr{S}(a)(\cdot, t)$ converges in $W^{1,2}$ to a ground state solution w of (AC), as $t \to +\infty$. Let $y \in S^3$ be such that $S^3 \cap \{w = 0\} = y^\perp = \{x \in S^3 \mid \langle x, y\rangle = 0\}$. We have either $y_1^2 + y_2^2 \neq 0$ or $y_3^2 + y_4^2 \neq 0$. Suppose the former holds; the other case can be argued similarly.

If we write $\varphi = a_1\varphi_1 + \ldots + a_5\varphi_5$, then $\varphi \circ (\rho^{-\theta} \circ \tau^{-\zeta}) = \varphi$, by assumption. By the uniqueness of solutions to (PAC) that converge backward to $u_\epsilon^{-\infty}$, computing projections as in the proof of Lemma 7 (without using the formulas in (5.1)), we see that $\mathscr{S}(a)$ and hence w are invariant by $(\rho^{-\theta} \circ \tau^{-\zeta})$. In particular, the nodal set of w is preserved by this composition of rotations, so $(\rho^{-\theta} \circ \tau^{-\zeta})(y) = \pm y$. Since $y_1^2 + y_2^2 \neq 0$ and $\tau^{-\zeta}$ does not change (y_1, y_2), this implies $\frac{\theta}{2\pi} \in \mathbb{Z}$.

Now consider the solution $\mathscr{S}(-s(a)) = -\mathscr{S}(a) \circ s$, which converges, as $t \to +\infty$, to a ground state solution with nodal set $s(y^\perp) = z^\perp$, where $z = s(y) \in S^3$ is such that $z_3^2 + z_4^2 \neq 0$. This solution is also invariant by $(\rho^{-\theta} \circ \tau^{-\zeta})$, by assumption, allowing us to conclude $\frac{\zeta}{2\pi} \in \mathbb{Z}$. $\square$

We conclude this subsection by noting that the equivariance of the solution map $\mathscr{S}$ implies the equivariance of the varifolds $V_{\epsilon,\mathscr{S}(a)(\cdot,t)}$ with respect to a. This follows from the fact that, for any isometry P of S^3 and any C^1 function u defined on S^3, the pushforward $P_\#$ by the isometry P satisfies

$$P_\# V_{\epsilon,u} = V_{\epsilon,u\circ P^{-1}}. \tag{5.5}$$

This relation follows from a change of variables; we refer to [33] for detailed proof. Noting also that $V_{\epsilon,u} = V_{\epsilon,-u}$, we obtain

$$\begin{aligned} V_{\epsilon,\mathscr{S}(\rho^\theta(a))(\cdot,t)} &= (\rho^\theta)_\# V_{\epsilon,\mathscr{S}(a)(\cdot,t)}, & V_{\epsilon,\mathscr{S}(-s(a))(\cdot,t)} &= s_\# V_{\epsilon,\mathscr{S}(a)(\cdot,t)}, \\ V_{\epsilon,\mathscr{S}(\tau^\theta(a))(\cdot,t)} &= (\tau^\theta)_\# V_{\epsilon,\mathscr{S}(a)(\cdot,t)}, & V_{\epsilon,\mathscr{S}(r_v(a))(\cdot,t)} &= (r_v)_\# V_{\epsilon,\mathscr{S}(a)(\cdot,t)}, \end{aligned}$$

for every $\theta \in \mathbb{R}$, and for every $v \in S^3$ such that $v_1 = 0 = v_2$ or $v_3 = 0 = v_4$. This also implies similar relations for the varifolds associated to any translated solution $\mathscr{S}(a)(x, t + t_0)$, for $t_0 \in \mathbb{R}$, and shows the invariance of the solutions and their limit as $t \to +\infty$, as well as the limits of these varifolds as $\epsilon \downarrow 0$, whenever a is fixed by the respective isometry.

5.3 Backward limit

As mentioned in Section 4, the arguments described so far only guarantee that the limit Brakke flow extracted from a sequence of translated solutions of (PAC) converges back to *some* Clifford torus in S^3 (provided these solutions join $u_\epsilon^{-\infty}$ to some nonconstant critical point of E_ϵ). Our main goal in this subsection is to use the symmetries r_v and s and a continuity argument to show that any Brakke flow constructed from $\mathscr{S}(a)$ has T_c as its backward limit whenever a lies in the orbit $\mathcal{O}_\epsilon$ described in (5.3). This will conclude the proof of Theorem 1.

Let $v = \frac{e_1 - e_2}{\sqrt{2}} \in S^3$. Since $v = \rho^{-\frac{\pi}{4}}(e_1)$, we have $r_v = \rho^{-\frac{\pi}{4}} \circ r_{e_1} \circ \rho^{\frac{\pi}{4}}$. Hence we know that r_v is the reflection $(x_1, x_2, x_3, x_4) \mapsto (x_2, x_1, x_3, x_4)$. Let also $w = s(v) = \tau^{-\frac{\pi}{4}}(e_3)$, and note that $r_w(x_1, x_2, x_3, x_4) = (x_1, x_2, x_4, x_3)$.

For $b = r(\epsilon) \cdot (0, \frac{1}{2}, \frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}) \in \mathcal{O}_\epsilon$, we have $r_v(b) = b$, $r_w(b) = b$, and $s(b) = -b$. Hence the solution $\mathscr{S}(b)$ is invariant by the reflections r_v and r_w, and it changes its sign under the isometry s. Consequently, the limit Brakke flow and its backward limit are also invariant under the reflections r_v, r_w and the isometry s.

Lemma 9. *There are precisely three Clifford tori in S^3 that are invariant under the reflections $r_{\frac{e_1 - e_2}{\sqrt{2}}}$ and $r_{\frac{e_3 - e_4}{\sqrt{2}}}$, and the isometry s. These are the tori $T_c = \{x \in S^3 \mid x_1^2 + x_2^2 = x_3^2 + x_4^2\}$, $T_+ = \{x \in S^3 \mid x_1 x_2 = x_3 x_4\}$, and $T_- = \{x \in S^3 \mid x_1 x_2 = -x_3 x_4\}$.*

We only sketch the proof here. Disregarding orientations, any Clifford torus is uniquely determined by the choice of a 2-plane P in $\mathbb{R}^4$, so that it is explicitly given by

$$\{x \in S^3 \mid \|P(x)\|^2 = \|P^\perp(x)\|^2\},$$

where we use the same notation P for the orthogonal projection $\mathbb{R}^4 \to P$. One checks that if this is preserved by a reflection r_y, for some unit vector y, then $y \in P \cup P^\perp$.

This means that any torus preserved by the reflections r_v and r_w described above are such that $v, w \in P \cup P^\perp$. Unless $v, w \in P$ (in which case P is spanned by $\{v, w\}$), the plane P is determined by the choice of a unit vector orthogonal to v and w. By writing their explicit equations and checking which of these tori are invariant by the isometry s, one concludes the proof.

Proposition 4. *Let $a \in \mathcal{O}_\epsilon$, as defined in* (5.3)*, and let u_ϵ by the solution of* (PAC) *given by the time translation of $\mathscr{S}(a)$ such that $u_\epsilon(\cdot, 0)$ has energy 5π. Then any (subsequential)* *limit Brakke flow* Σ_t *obtained from* u_ϵ *as* $\epsilon \downarrow 0$ *converges to* T_c *as* $t \to -\infty$.

Proof. We will prove the result for $q = r(\epsilon) \cdot (0, \frac{1}{2}, \frac{1}{2}, -\frac{1}{2}, -\frac{1}{2})$. Since any $a \in \mathcal{O}_\epsilon$ is obtained from b by rotations that preserve T_c, the claimed result will then follow from Lemma 7.

For a 2-varifold (or a surface) V in S^3, we will write $V(f)$ to mean the integral of a continuous function f on the Grassmannian $G_2(S^3)$ with respect to V (or the multiplicity 1 varifold induced by such surface). Since T_c and $T_\pm$ are distinct tori, we can pick a smooth real-valued function f defined on $G_2(S^3)$ such that

$$-1 = T_+(f) = T_-(f) < 0 < T_c(f) = 1.$$

By Theorem 6 and the previous Lemma, we know that Σ_t converges, as $t \to -\infty$, to either T_c, T_- or T_+. Suppose, by contradiction, that it converges to T_+. Pick a sequence $t_j \to -\infty$ such that $|\Sigma_{t_j}(f) - (-1)| < 1/6$. We can find a sequence $\{\epsilon_j\}$ of positive numbers such that $\epsilon_j \downarrow 0$,

- the varifold distance between $\frac{1}{\sigma} V_{\epsilon_j, t_j}$ and $\frac{1}{\sigma} V_{t_j}$ is bounded above by $1/j$ (where Σ_{t_j} is the weight measure of $\frac{1}{\sigma} V_{t_j}$), and
- $\left| \frac{1}{\sigma} V_{\epsilon_j, t_j}(f) - (-1) \right| < 1/3$.

Since the varifold $V_\epsilon^{-\infty}$ associated to $u_\epsilon^{-\infty}$ converges to $\sigma \cdot T_c$ as $\epsilon \downarrow 0$, we can also assume

$$\left| \frac{1}{\sigma} V_{\epsilon_j}^{-\infty}(f) - 1 \right| < \frac{1}{3}.$$

Note that $V_{\epsilon_j, t}$ varies continuously in the varifold topology for $t \in [-\infty, 0]$, where we write $V_{\epsilon, -\infty} = V_\epsilon^{-\infty}$. Thus, $t \in [-\infty, 0] \mapsto \frac{1}{\sigma} V_{\epsilon_j, t}(f)$ is a continuous function such that

$$\frac{1}{\sigma} V_{\epsilon_j, t_j}(f) < -\frac{2}{3} < \frac{2}{3} < \frac{1}{\sigma} V_{\epsilon_j}^{-\infty}(f),$$

so there exists $s_j \in (-\infty, t_j) \subset (-\infty, 0)$ such that $\frac{1}{\sigma} V_{\epsilon_j, s_j}(f) = 0$.

Claim. After possibly passing to a (non relabeled) subsequence, $\{\frac{1}{\sigma}V_{\epsilon_j,t+s_j}\}_t$ converges, as $j\to+\infty$, to a Brakke flow which is integral for a.e. t, and which has constant area $2\pi^2$, for every $t\leq 0$. Consequently, this Brakke flow is supported on some Clifford torus T', and $V_{\epsilon_j,s_j}\to T'$ as varifolds.

To prove this claim, let $\tilde{u}_j(x,t)=u_{\epsilon_j}(x,t+s_j)$. Then $\tilde{u}_j$ are solutions to (PAC) such that $\tilde{u}_j(\cdot,t)\to u_{\epsilon_j}^{-\infty}$ as $t\to-\infty$, and that have the same limit as $u_{\epsilon_j}(\cdot,t)$ when $t\to+\infty$. One checks that the sequence $\{\tilde{u}_j\}$ satisfies the hypotheses of Lemma 2, and

$$E_{\epsilon_j}(\tilde{u}_j(\cdot,t))\leq E_{\epsilon_j}(u_{\epsilon_j}^{-\infty}),\quad \text{for all}\quad t\in\mathbb{R}.$$

Hence, after possibly passing to a subsequence, $\{\frac{1}{\sigma}V_{\epsilon_j,\tilde{u}_j(\cdot,t)}=\frac{1}{\sigma}V_{\epsilon_j,t+s_j}\}_{t\in\mathbb{R}}$ converges, as $j\to+\infty$, to a 2-dimensional Brakke flow $\{\Gamma_t\}_t$ which is integral for almost every $t\in\mathbb{R}$, and such that

$$\|\Gamma_t\|(S^3)=\lim_{j\to+\infty}\frac{1}{2\sigma}E_{\epsilon_j}(\tilde{u}_j(\cdot,t))\leq\lim_{j\to+\infty}\frac{1}{2\sigma}E_{\epsilon_j}(u_{\epsilon_j}^{-\infty})=2\pi^2$$

for all $t\in\mathbb{R}$. Furthermore,

$$\begin{aligned}2\pi^2=\|T_+\|(S^3)&=\lim_{j\to+\infty}\frac{1}{2\sigma}E_{\epsilon_j}(u_{\epsilon_j}(\cdot,t_j))\leq\lim_{j\to+\infty}\frac{1}{2\sigma}E_{\epsilon_j}(u_{\epsilon_j}(\cdot,s_j))\\&\leq\lim_{j\to+\infty}\frac{1}{2\sigma}E_{\epsilon_j}(\tilde{u}_j(\cdot,t))=\lim_{j\to+\infty}\frac{1}{\sigma}\|V_{\epsilon_j,\tilde{u}_j(\cdot,t)}\|(S^3)=\|\Gamma_t\|(S^3)\end{aligned}$$

for all $t\leq 0$. This implies $\|\Gamma_t\|(S^3)=2\pi^2$ for every $t\leq 0$. We conclude that $\{\Gamma_t\}_{t\in\mathbb{R}}$ must be supported on a Clifford torus T'. Since Γ_t is integral for a.e. t, this proves the claim.

We can now obtain a contradiction by recalling that the choice of a implies that $\mathscr{S}(a)$ is invariant by the reflections with respect to $v=\frac{e_1-e_2}{\sqrt{2}}$ and $w=\frac{e_3-e_4}{\sqrt{2}}$, as well as the isometry s. Consequently, the translated solutions u_{ϵ_j} and $\tilde{u}_j$, the limit Brakke flows Σ_t and Γ_t, and the torus T' given by the Claim above are invariant by the same isometries. But this contradicts

$$T'(f) = \lim_{j \to +\infty} V_{\epsilon_j, s_j}(f) = 0 \notin \{T_c(f), T_{\pm}(f)\},$$

where we used $\frac{1}{\sigma} V_{\epsilon_j, s_j} \to \Gamma_0 = T'$. Similarly, Σ_t does not converge to T_- as $t \to -\infty$. $\square$

This Proposition establishes the existence of solutions of (PAC), for small $\epsilon > 0$, such that any limit Brakke flow converges back in time to the (same) Clifford torus T_c, thus proving Theorem 1.

5.4 Forward limit

We are in a position to study the forward limit of solutions $\mathscr{S}(a)$, for a in the orbit $\mathcal{O}_\epsilon$ described in (5.3), and of their limit Brakke flow. The key observation is that the invariance by the isometry s, which holds for some points in $\mathcal{O}_\epsilon$, allows one to obtain information about the nodal set of the limit critical point. By arguing as in the previous subsection, we can then describe the convergence and the forward limit of the Brakke flows constructed as limits of these solutions.

For the next result, we recall the small positive $\epsilon_4 > 0$ given by Lemma 1, which ensures the convergence of certain solutions of PAC to ground states.

Proposition 5. *Let $\epsilon \in (0, \epsilon_4)$. For each $p \in \mathcal{O}$, there is $y_p \in S^3$ (possibly depending on ϵ) such that*

(i) Let u_p be the unique ground state solution of (AC) *such that $\{u_p = 0\}$ is the equatorial sphere $S^3 \cap y_p^{\perp}$, and u_p is positive in $\{x \in S^3 \mid \langle x, y_p \rangle > 0\}$. Then*

$$\lim_{t \to +\infty} \|\mathscr{S}(r(\epsilon)p)(\cdot, t) - u_p\|_{W^{1,2}(S^3)} = 0.$$

(ii) The map $p \mapsto y_p$ is injective, continuous, and odd.

Moreover, we can find a sequence $\epsilon_j \downarrow 0$ such that, for every $p \in \mathcal{O}$, the vector y_p is independent of j and, if Σ_t is the Brakke flow in S^3 obtained from (a time-translation of) $\mathscr{S}(r(\epsilon)p)$, as described in Section 4, then Σ_t converges to $y_p^\perp \cap S^3$, as $t \to +\infty$, in the varifold sense.

First, we note that solutions to (PAC) that correspond to initial conditions in $\mathcal{O}_\epsilon$ always join $u_\epsilon^{-\infty}$ to ground states.

Lemma 10. *Let $\epsilon \in (0, \epsilon_4)$, as given by Lemma 1. For any $a \in \mathcal{O}_\epsilon$, the solutions $\mathscr{S}(a)(\cdot, t)$ do not subsequentially converge (in the $W^{1,2}$ or Hölder norm) to the constants ± 1 as $t \to +\infty$. Consequently, $\mathscr{S}(a)(\cdot, t)$ converges, in $W^{1,2}$ norm, to a ground state solution.*

Proof. Suppose some $\mathscr{S}(a)(\cdot, t)$ converges to the constant stationary solution $c \in \{-1, +1\}$ along a subsequence $t_j \to +\infty$. Then $\mathscr{S}(a)(s(x), t) = -\mathscr{S}(-s(a))(x, t)$ is a solution of (PAC) that converges subsequentially to c as $t = t_j \to +\infty$. Consequently, $\mathscr{S}(-s(a))$ also solves (PAC) and converges to $-c$ as $t_j \to +\infty$. On the other hand, $-s(a) \in \mathcal{O}_\epsilon$, so we obtain a contradiction, as $\mathscr{S}(a)$ and $\mathscr{S}(-s(a))$ differ by an isometry by Lemma 7 (see also the Remark above about the orbits of ρ^θ and τ^θ). The last conclusion follows from Proposition 3. $\square$

Proof of Proposition 5. Let $\epsilon \in (0, \epsilon_4)$. For every $p \in \mathcal{O}$, we let $u_{p,\epsilon}$ be the limit of $\mathscr{S}(r(\epsilon)p)$, which is a ground state solution, by Lemma 10. This means we can find $y_{p,\epsilon} \in S^3$ such that $\{u_{p,\epsilon} > 0\} = \{x \in S^3 \mid \langle x, y_{p,\epsilon} \rangle > 0\}$, and $u_{p,\epsilon}$ vanishes precisely at $y_{p,\epsilon}^\perp \cap S^3$. This proves (i). By Lemma 7, we have

$$u_{(\rho^{\theta_1} \circ \tau^{\theta_2})(p),\epsilon} = u_{p,\epsilon} \circ \rho^{-\theta_1} \circ \tau^{-\theta_2},$$

for every $\theta_1, \theta_2 \in \mathbb{R}$. This proves the continuity of $p \mapsto y_{p,\epsilon}, u_{p,\epsilon}$.

Again, we pick $q = (0, \frac{1}{2}, \frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}) \in \mathcal{O}$, so that $s(q) = -q$. We also consider the eigenfunction $\varphi = \frac{r(\epsilon)}{2}(\varphi_2 + \varphi_3 - \varphi_4 - \varphi_5)$. As noted in the previous subsection, $v = \frac{e_1 - e_2}{\sqrt{2}}$

and $w = s(v) = \frac{e_3 - e_4}{\sqrt{2}}$ are such that $\varphi \circ r_v = \varphi = \varphi \circ r_w$. The equivariance of the solution map implies $u_{q,\epsilon} \circ s = -u_{q,\epsilon}$ and thus $y_{q,\epsilon}^{\perp}$ is invariant by the isometry s. This shows that $s(y_{q,\epsilon}) = \pm y_{q,\epsilon}$ and $y_1^2 + y_2^2 = y_3^2 + y_4^2 \neq 0$, where $y_{q,\epsilon} = (y_1, y_2, y_3, y_4)$. We also have $u_{q,\epsilon} \circ r_v = u_{q,\epsilon} = u_{q,\epsilon} \circ r_w$, and ${y_{q,\epsilon}}^{\perp}$ is invariant by both r_v and r_w.

We can now prove the injectivity and oddness. Observe that if $\theta_1, \theta_2 \in \mathbb{R}$, and if the rotation $R = (\rho^{-\theta_1} \circ \tau^{-\theta_2})$ preserves $y_{q,\epsilon}^{\perp}$, then $R(y_{q,\epsilon}) = \pm y_{q,\epsilon}$. Since (y_1, y_2) and (y_3, y_4) are nonzero, this can only happen if R is either the identity or the antipodal map in S^3. Now any $p \in \mathcal{O}$ distinct from q is of the form $p = (\rho^{\theta_1} \circ \tau^{\theta_2})(q)$, for some θ_1, θ_2 not both in $2\pi\mathbb{Z}$, which implies, by Lemma 7, that $u_{p,\epsilon}$ is either $-u_{q,\epsilon}$, in the case $p = -q$ (when θ_1 and θ_2 are odd multiples of π), or a critical point of E_ϵ having a different equator as its nodal set. This shows that this limit is injective and odd with respect to p, and concludes the proof of (ii).

To prove the last statement, note that there are precisely two equators that are invariant by r_v and s, namely

$$\{x \in S^3 \mid x_1 + x_2 + x_3 + x_4 = 0\} \quad \text{and} \quad \{x \in S^3 \mid x_1 + x_2 - x_3 - x_4 = 0\}.$$

Hence, for every $\epsilon \in (0, \epsilon_4)$, we have $y_{q,\epsilon} \in \left\{ \pm \left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2} \right), \pm \left(\frac{1}{2}, \frac{1}{2}, -\frac{1}{2}, -\frac{1}{2} \right) \right\}$. This means we can find a sequence $\epsilon_j \downarrow 0$ such that $y_{q,\epsilon_j} = y$ is constant, and $\mathscr{S}(r(\epsilon_j)q)$ converge to ground states u_{q,ϵ_j} having the same nodal set and fixed signs on the hemispheres bounded by this equatorial sphere. By passing to a further subsequence, we may assume that the appropriate time-translation of $\mathscr{S}(r(\epsilon_j)q)$ gives rise to a Brakke flow Σ_t which satisfies the conditions described in Section 4. We can now use the continuity argument employed to characterize the limit of Σ_t as $t \to -\infty$ to conclude that $\Sigma_t \to S^3 \cap y^{\perp}$ as $t \to +\infty$, in the varifold sense. The conclusion can be extended to every direction in $\mathcal{O}$ using the equivariance of the solution map and of the induced varifolds. $\square$

Since the sequence $\epsilon_j \downarrow 0$ in Proposition 5 can be extracted from any sequence of $\epsilon \downarrow 0$, this finishes the proof of Theorem 2.

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