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MULTIPLICATIVE HITCHIN FIBRATION AND FUNDAMENTAL LEMMA

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To my wife Jenchi

Student: “Why is it called *endoscopy*?”

Ngô: “Because it’s painful?”

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ABSTRACT

In this thesis, we thoroughly study the multiplicative Hitchin fibration, which is the group analogue of the usual Hitchin fibration by replacing the Lie algebra with a reductive monoid. As an application, we use it to prove the standard endoscopic fundamental lemma for adjoint groups. Although similar to its Lie algebra counterpart in many ways, the multiplicative Hitchin fibration has a lot more new features and is much more complicated. There are three main highlights in this thesis: the geometrization of endoscopic transfer, including the construction of endoscopic monoids; a local model of singularity that connects with representations of the dual group; a generalized support theorem which not only is the key to prove fundamental lemma but also reveals some potential new phenomenon that is not explained by endoscopy.

CHAPTER 1

INTRODUCTION

The goal of this paper is to establish a solid foundation for the multiplicative version of Hitchin-type fibrations in the case of the adjoint action of a reductive group. We will call it the multiplicative Hitchin fibration, or mH-fibration for short. As an application, we use it to prove the standard endoscopic fundamental lemma for the adjoint groups. In fact, we are able to provide a direct proof of the endoscopic transfer for the spherical Hecke algebra of an adjoint group. The following result is Theorem 2.6.11.

Theorem 1.0.1. *Let $k = \mathbb{F}_q$ be a finite field with q element and characteristic p, $\mathcal{O} = k[[\pi]]$ the ring of power series of one variable over k and $F = k((\pi))$ its field of fractions. Let $G = G^{\mathrm{ad}}$ be a reductive group scheme of adjoint type over $\mathcal{O}$ whose Coxeter number is less than $p/2$. Let $(\kappa, \vartheta_\kappa)$ be an elliptic endoscopic datum of G over $\mathcal{O}$ and H the corresponding endoscopic group. Then we have equality in orbital integrals*

$$q^{-d(a)/2}\,\mathbf{O}_a^{\kappa}(f^{\lambda}, \mathrm{d}t) = q^{-d_H(a_H)/2}\,\mathbf{SO}_{a_H}\left(\sum_i f_H^{\lambda_{H,i}}, \mathrm{d}t\right),$$

where a and a_H are matching strongly regular semisimple conjugacy classes in $G(F)$ and $H(F)$ respectively, d and d_H are the respective discriminant valuations on G and H, f^{λ} is the IC-function associated with an irreducible $\check{G}$-representation V_λ of highest weight λ, and $\sum_i f_H^{\lambda_{H,i}}$ is the corresponding sum of IC-functions of H obtained by restricting V_λ to $\check{H}$.

Although the proof of Theorem 1.0.1 may be viewed as the multiplicative analogue of the proof in Lie algebra case in [Ngô10], it has a lot of new features absent in the latter case and potentially leads to a much larger field for future exploration. Therefore the current paper serves more as a proof of concept for future development than an alternative tool to solve the same problem.

To this end, in this introduction we will try to give an overview of this paper with an emphasis on the big picture, new features and potential development points. The geometry of mH-fibrations is incredibly rich and beautiful and has profound connection with representation theory, therefore it is important to give enough spotlight to all the key features in this introduction so that the reader can have a grasp of what is to come. However, to keep the introduction concise and clear enough, we will also try to avoid using too many notations when we can due to the technical complexity in this subject. Unfortunately, this means it is best that the reader has at least some vague sense of how Hitchin fibration works in the context of endoscopy. To compensate that, we will include a bullet point list at the beginning of § 1.3 of the things discussed up to that point.

1.1 On Hitchin-type Fibrations

One of the motivations behind introducing Hitchin fibration to arithmetic problems like fundamental lemma is that global objects tend to behave much better than local ones, even if they are less computationally accessible at times. Therefore it would be beneficial to understand the general principles on those Hitchin-type global constructions. We will use mH-fibrations as the primary example for this section.

1.1.1 There has been many attempts to prove the fundamental lemma for Lie algebras in the past focusing on the local picture. Over Archimedean local fields Shelstad was able to prove the general statement directly at group level, see [She79]. Over non-Archimedean fields, however, the problem seems much harder, and most proofs before [Ngô10] were on one specific group at a time. The reader can see the introduction of [Ngô10] for a full historical picture.

One of the most successful general results on the local front is perhaps the conditional proof for unramified conjugacy classes by Goresky, Kottwitz and MacPherson in [GKM04]. Although quite an impressive framework in itself, it also shows the limitation of local geometric method. For one, as Ngô already pointed out in [Ngô10], it depends on a purity conjecture of the cohomology of related affine Springer fibers, which is only partially known due to affine Springer fibers usually being highly singular. Another serious obstacle pointed out by [Ngô10] is that it crucially depends on the conjugacy class being unramified so that there is a large torus acting on the affine Springer fibers.

1.1.2 Roughly speaking, the local geometric method is based on the fact that orbital integrals may be interpreted as certain kind of point-counting on related affine Springer fibers, which in turn reduce to some cohomological statement using a variant of Grothendieck-Lefschetz trace formula. The affine Springer fiber in question, can be roughly understood as certain subset of maps from a formal disc $X_v = \operatorname{Spec} \mathcal{O}$ to the quotient stack $[\mathfrak{g}/G]$, together with some other naturally attached data. To move from the local picture to the global one one only needs to replace the formal disc by a smooth projective curve X. Because $\mathfrak{g}$ is affine and X is projective, one needs to add some auxiliary twist, otherwise the global object we end up getting will be trivial. In Lie algebra case, such twisting is provided by the natural $\mathbb{G}_m$-action on $\mathfrak{g}$ viewed as a vector space.

Without going further into the global geometry, we must first ask what is the analogue for the above setup in the group (i.e., multiplicative) case. Clearly we are now interested in the quotient stack $[G/G]$ (where G acts on itself by adjoint action) in place of $[\mathfrak{g}/G]$, but we also need a natural twisting similar to the $\mathbb{G}_m$-action on $\mathfrak{g}$. However, there does not appear to be any natural symmetry we can use that also commutes with the adjoint action. If $G = \mathrm{GL}_n$, for example, we have the scaling action of $\mathbb{G}_m$ by viewing G as a subspace of Mat_n, and for a group G with a non-trivial central torus Z_0, we might use the action of Z_0, viewed as a product of $\mathbb{G}_m$'s. But first, these actions do not look very natural (one can replace Z_0 by a subtorus for example), and second, if G is semisimple then we seem truly at a loss. There is also a third difficulty that even for $G = \mathrm{GL}_1 = \mathbb{G}_m$, if we twist G by a $\mathbb{G}_m$-torsor, we will just get the same $\mathbb{G}_m$-torsor which will have no global sections unless the torsor is trivial. Comparing to the Lie algebra case when $\mathfrak{g} = \mathbb{A}^1$: by twisting we obtain the line bundle associated with the $\mathbb{G}_{\mathrm{m}}$-torsor which does have a lot of sections provided the degree is large enough.

1.1.3 The answer to our first obstacle above turns out to be quite profound. Since we cannot guarantee that G contains a central torus, we can certainly add one to it by augmenting the group. The question is of course what torus to add. Moreover, the third difficulty above shows we need to add some "boundaries" to G, similar to embedding $\mathbb{G}_{\mathrm{m}}$ into $\mathbb{A}^1$ so that we do get a lot of global sections. We may rephrase this question in the following way: in Lie algebra case, we use a line bundle to twist $\mathfrak{g}$, so we are really just studying rational maps from X to $[\mathfrak{g}/G]$ with a systematic constraint on the poles. Therefore in group case, we are really trying to find a systematic way to control the poles of rational maps from X to $[G/G]$. This turns out to be the thought behind some prototype of mH-fibrations in the field of mathematical physics (see, for example, [HM02]), as well as its first appearance in arithmetic setting in [FN11].

Temporarily going back to the local setting, we can see that the effect of the $\mathbb{G}_{\mathrm{m}}$-twisting in the Lie algebra case is that we are not really considering the characteristic function on $\mathfrak{g}(\mathcal{O})$, but rather those on scaled sets $\pi^{-d}\mathfrak{g}(\mathcal{O}) \subset \mathfrak{g}(F)$ for arbitrary $d \geq 0$. This corresponds to the fact that the spherical Hecke algebra of $\mathfrak{g}$ is generated by translations and scalings of the characteristic function on $\mathfrak{g}(\mathcal{O})$, and they do not look very differently from the latter. In group case, however, things get complicated because the analogue would be the Cartan decomposition and characteristic functions on double cosets $G(\mathcal{O})\pi^{\lambda}G(\mathcal{O})$, and they certainly do not look alike. Therefore, we need to add "cocharacter-valued poles" to match the Cartan double cosets in the local setting.

1.1.4 So far all look just like a purely technical nuisance, at least if one follows the formulation in [FN11]. However, the method in [FN11] does not generalize well, and we can only elaborate later since the correct way to add those poles is far from being just a technical tool. This is where reductive monoids enter the picture.

A primary example of reductive monoid is $n \times n$ matrices Mat_n viewed as a multiplicative monoid. The group GL_n embeds in Mat_n as an open subset and is its group of units. Therefore $\mathrm{Mat}_n - \mathrm{GL}_n$ is the "boundary" we could add to GL_n, and by considering the mapping stack from X to $[\mathrm{Mat}_n / \mathrm{GL}_n \times \mathbb{G}_\mathrm{m}]$, we may form our first mH-fibration using monoids. In general, we would want to consider the maps from X to stack $[\mathfrak{M}/G \times Z_{\mathfrak{M}}]$ where $\mathfrak{M}$ is a *very flat* (see Definition 2.3.10) reductive monoid whose unit group $\mathfrak{M}^\times$ has derived subgroup isomorphic to G^{sc} (the simply-connected cover of the derived group of G), and $Z_{\mathfrak{M}}$ is the center of $\mathfrak{M}^\times$.

Such a monoid $\mathfrak{M}$ comes with an abelianization map (introduced by Vinberg in [Vin95]) $\mathfrak{M} \to \mathfrak{A}_{\mathfrak{M}}$ by taking the invariant quotient by $G^{\mathrm{sc}} \times G^{\mathrm{sc}}$-multiplication on the left and right. Roughly speaking $\mathfrak{M}$ and $\mathfrak{A}_{\mathfrak{M}}$ are in bijective correspondence to each other, with $\mathfrak{A}_{\mathfrak{M}}$ keeping track of what kind of boundary (i.e., cocharacters or Cartan double cosets) we want to add, and $\mathfrak{M}$ being the actual space itself to replace $\mathfrak{g}$. It turns out there is a universal object $\mathrm{Env}(G^{\mathrm{sc}})$ among those $\mathfrak{M}$ for each semisimple type, called the universal semigroup or Vinberg monoid of G^{sc}. For $\mathrm{Env}(G^{\mathrm{sc}})$, the boundary or cocharacters added is the set of all fundamental coweights.

In some early development of mH-fibrations after [FN11], most notably [Bou15,Bou17, Chi19], the monoid is fixed as $\mathrm{Env}(G^{\mathrm{sc}})$, while in [FN11] itself the corresponding monoid is so-called L-monoid where only one fixed (but arbitrarily chosen) cocharacter λ is allowed. Neither turns out to be the completely correct formulation.

1.1.5 Utilizing $\mathrm{Env}(G^{\mathrm{sc}})$ and the associated mH-fibration, J. Chi in [Chi19] was able to prove some very good local results for G including the dimension formula for the relevant multiplicative affine Springer fibers. However, once one attempts to connect mH-fibrations of G to those of H, things quickly fall apart if one sticks with $\mathrm{Env}(G^{\mathrm{sc}})$ and $\mathrm{Env}(H^{\mathrm{sc}})$.

In Lie algebra case, assuming H is a subgroup of G, then we have the natural map $[\mathfrak{h}/H] \to [\mathfrak{g}/G]$, and since both $\mathfrak{h}$ and $\mathfrak{g}$ are reductive Lie algebras, the general results for Hitchin fibrations can be applied to both sides. In general, there is still a natural map from $\mathfrak{c}_H = \mathfrak{h}/\!/H$ to $\mathfrak{c}_G = \mathfrak{g}/\!/G$, and it is all we need for establishing the endoscopic transfer.

In group case, however, there is usually no way to embed $\mathrm{Env}(H^{\mathrm{sc}})$ into $\mathrm{Env}(G^{\mathrm{sc}})$ as a closed submonoid even if H is a subgroup of G, and more generally $\mathfrak{C}_{\mathrm{Env}(H^{\mathrm{sc}})} = \mathrm{Env}(H^{\mathrm{sc}}) /\!/ H$ will not map to $\mathfrak{C}_{\mathrm{Env}(G^{\mathrm{sc}})} = \mathrm{Env}(G^{\mathrm{sc}}) /\!/ G$ either. Using the general theory of reductive monoids, one can still produce some monoid $\mathfrak{M}'_H$ for H such that $\mathfrak{M}'_H /\!/ H$ does map to $\mathfrak{C}_{\mathrm{Env}(G^{\mathrm{sc}})}$, and if H is a subgroup of G, $\mathfrak{M}'_H$ would just be the closure of the group $H' \subset \mathfrak{M}^\times$ where H' has the same semisimple type as H. However, such monoid $\mathfrak{M}'_H$ in general would not be a very flat monoid, so our general theory of mH-fibrations could not be applied to such.

This, of course, is not a coincidence, and (in the author's opinion) is where the story becomes really interesting. The picture is better understood once we move to the dual group side, and the solution to this difficulty will manifest itself once we do so. We also note that if any other Hitchin-type fibration is to be developed, the same analysis on the dual side should also be indispensable. We will discuss the story about dual groups in the next section, and close this one with a few comments on some potential generalizations.

1.1.6 The most obvious generalization is perhaps the twisted conjugation of G on itself, which corresponds to the theory of twisted endoscopy (see [KS99]). In fact, one of the motivations behind the current paper is to prove the twisted-weighted fundamental lemma using mH-fibrations. Significant work has been done by the author at the time of writing and will probably be published in a future paper.

In [SV17], Sakellaridis and Venkatesh established a unified framework for the relative trace formula using spherical varieties. In the group case, G is viewed as a $G\times G$-spherical variety, and $\mathrm{Env}(G^{\mathrm{sc}})$ is a horospherical contraction of G^{sc} as a spherical variety and also the spectrum of the Cox ring of the wonderful compactification of G^{ad} (one may consult [Tim11] for terminologies). Another possible way to generalize the current paper is to consider the Hitchin-type fibration derived from the spectrum of the Cox ring of some other wonderful varieties.

1.2 Dual Groups and Geometric Transfer Map

The fundamental lemma is usually stated as an equality between the evaluation of orbital integrals at one hyper special function. Since the function is already given, all we need is to figure out how to match conjugacy classes. As we already pointed out in the previous section, in proving the fundamental lemma using global method, we are more or less directly proving the transfer for spherical Hecke algebras. In Lie algebra case, it is not very different from using just one function, but in group case one function is not enough, and the endoscopic transfer has to incorporate both the matching of conjugacy classes and the matching of functions. In the group case the latter has close connection to the representations of the dual groups thanks to geometric Satake isomorphism.

1.2.1 Recall that the construction of mH-fibration involves choosing a monoid $\mathfrak{M}$, which in turn corresponds to choosing what kind of Cartan double coset that can appear. To make the discussion more precise, we replace Cartan double cosets with their image Gr_G^{λ} in affine Grassmannian Gr_G, in other words, the affine Schubert cells. In choosing $\mathfrak{M}$, or equivalently the λ's, we are encoding the geometry of Gr_G^{λ}, a local object, into mH-fibration, a global one. It is not entirely accurate: in the end we are in fact encoding the geometry of the closure $\mathsf{Gr}_G^{\leq\lambda}$ of Gr_G^{λ}, i.e., the affine Schubert varieties, into mH-fibrations. The precise statement is Theorem 6.10.2, called the local model of singularities. It is essentially conjectured in [FN11, Conjecture 4.1], and Bouthier in [Bou17] gives the first proof with some additional technical constraint using an *ad hoc* method. Bouthier's result, as it turns out, will be too weak for proving fundamental lemma due to the technical constraint therein being too strict. In this paper we will give a more natural proof with less constraint using deformation theory that partially solves the problem (e.g., when G is adjoint). It seems our proof can be easily further optimized to fully solve the issue but nevertheless we have not fully worked out the details at the time of writing.

Because mH-fibration will naturally encode $\mathsf{Gr}_G^{\le\lambda}$ instead of Gr_G^{λ} (one can still take an open substack so that the fibration really encodes Gr_G^{λ}, but the resulting fibration is not proper, making it harder to study cohomologically), it is also natural to consider the IC-functions f^λ instead of the characteristic functions of the double cosets. To recall, the intersection complex IC^λ on $\mathsf{Gr}_G^{\le\lambda}$ induces a function on its k-points by Grothendieck's function-sheaf dictionary, whose pull back to $G(F)$ is the IC-function f^λ. All the IC-functions form an alternative basis of the spherical Hecke algebra to the characteristic functions. On the other hand, by geometric Satake isomorphism, each IC^λ corresponds to an irreducible representation V_λ of $\check{G}$ of highest weight λ. Since $\check{H}$ is a subgroup of $\check{G}$, the matching between functions in endoscopic transfer is easy to guess: the function f^λ should be matched with the sum of IC-functions $f_H^{\lambda_{H,i}}$ of H, where $\lambda_{H,i}$ are the highest weights (allowing repetitions) of $\check{H}$ obtained from restriction V_λ to $\check{H}$.

1.2.2 This connection with the restriction functor from $\check{G}$ to $\check{H}$ turns out to be the clue to solve the difficulty left near the end of previous section. Something we have avoided talking about to this point is that the abelianization $\mathfrak{A}_{\mathfrak{M}}$ does not just record the λ's, but also record their "multiplicities", in other words, it records a multiset of cocharacters. Note that it is different from the notion of "degree" or equivalently at how many points on the curve does λ appear (e.g., the repeating number d in [FN11]), because $\mathfrak{A}_{\mathfrak{M}}$ is an absolute object and has nothing to do with the curve X. Rather, this "multiplicity" simply corresponds to the multiplicity of an irreducible representation in a given representation. It is very nice because an irreducible $\check{G}$-representation is unlikely to stay irreducible when restricted to $\check{H}$, and the irreducible $\check{H}$-representations in the decomposition will also have multiplicities in general. Therefore we will not lose multiplicities when utilizing monoids.

The upshot is that for any given monoid $\mathfrak{M}$ of G, there will be a canonically associated monoid $\mathfrak{M}_H$ (see Definition 2.5.15) of H that after some natural manipulations will induce the correct transfer map that simultaneously take care of both the matching of conjugacy classes and of functions.

1.2.3 Finding the correct monoid, however, is not the end of the story. In earlier papers such as [Bou17, Chi19], there is another variant of mH-fibrations, called the *restricted* mH-fibrations. The idea is instead of using the full monoid, we simply fix a number of points on the curve X, and at each such point x there is also a fixed cocharacter λ_x attached, avoiding the whole monoid business. In Lie algebra setting, it is akin to instead of using a line bundle $\mathcal{O}(D)$ to twist the Lie algebra $\mathfrak{g}$ (so that poles can appear as any divisor linearly equivalent to D), we fix the divisor D and only allow poles to appear at D up to designated degrees.

This, of course, is not the right way to solve fundamental lemma in Lie algebra case, and neither in the group case. Although it has the clear advantage of seeing how λ and $\lambda_{H,i}$'s are related to each other when both G and H are split, it quickly becomes a mess when they are not. More importantly, the naive transfer maps between restricted mH-fibrations obtained using this formulation does not seem to exhaust the entire endoscopic locus but only part of it. The seemingly correct way, analogous to the Lie algebra case, is then to use the monoid $\mathfrak{M}$, and pick a $Z_{\mathfrak{M}}$-torsor $\mathcal{L}$ of large "degree" (whatever it means), and so on. What is shocking, however, is that even this method is *not* correct in the group case.

The reason is very simple: the monoids $\mathfrak{M}$ and $\mathfrak{M}_H$ have different centers. It can already be seen in groups themselves: H has less roots than G, so naturally its center is larger. At monoid level, because we need to take care of the restriction functor $\mathrm{Res}_{\check{H}}^{\check{G}}$, every $\lambda_{H,i}$ in the decomposition will contribute to the rank of the center $Z_{\mathfrak{M},H}$ of $\mathfrak{M}_H$, making it much, much larger than $Z_{\mathfrak{M}}$. Moreover, there is no map from $Z_{\mathfrak{M},H}$ to $Z_{\mathfrak{M}}$ whatsoever, just like the center of H, which, although embeds into G, does not map to the center of G.

Fortunately, the solution to this problem is also very simple: one simply takes the preimage (which in fact will be well-defined) of $Z_{\mathfrak{M}}$ in $\mathfrak{M}_H$, denoted by $Z_{\mathfrak{M}}^{\kappa}$. Still, $Z_{\mathfrak{M}}^{\kappa}$ is also much larger than $Z_{\mathfrak{M}}$, but we at least will have a map

$$[\mathfrak{C}_{\mathfrak{M},H}/Z_{\mathfrak{M}}^{\kappa}] \to [\mathfrak{C}_{\mathfrak{M}}/Z_{\mathfrak{M}}], \tag{1.2.1}$$

compatible with the map between classifying stacks $\mathbb{B}Z_{\mathfrak{M}}^{\kappa} \to \mathbb{B}Z_{\mathfrak{M}}$. This is analogue to the map $[\mathfrak{c}_H/\mathbb{G}_{\mathrm{m}}] \to [\mathfrak{c}_G/\mathbb{G}_{\mathrm{m}}]$ in Lie algebra case, which further maps down to $\mathbb{B}\mathbb{G}_{\mathrm{m}}$. Choosing a line bundle $\mathcal{O}(D)$ is the same as taking the fiber over $\mathcal{O}(D) \in \mathbb{B}\mathbb{G}_{\mathrm{m}}(X)$, and such choice is possible because on both G-side and H-side we use $\mathbb{G}_{\mathrm{m}}$. In group case, we cannot afford to fix any torsor because $Z_{\mathfrak{M}}^{\kappa}$ and $Z_{\mathfrak{M}}$ are different, and all we can do is to use the whole map (1.2.1). It turns out to be the correct answer. In Lie algebra case, it is analogue to using the entire moduli of effective divisors instead of only using divisors linearly equivalent to D. Of course, one has to throw away some components of small degrees so that we have enough ampleness for geometric arguments, but that is more of a technical issue.

1.2.4 The geometric transfer map in the group case is absolutely beautiful to look at. To illustrate the point, we briefly discuss how it ties in with some features in the local geometry and combine them with representation theory in a neat package.

In Lie algebra case, even though affine Springer fibers are highly singular, they still has a lot of symmetries. Such symmetry can be described using the so-called regular centralizer, first introduced in [DG02] and later extensively studied in [Ngô10]. In particular, there is an open dense subset (the regular locus) on which the symmetry group acts simply transitively. This is one of the key ingredients in Ngô's support theorem, which is used to prove fundamental lemma.

In group case, however, the story is more complicated. In [Chi19], Chi described the irreducible components of multiplicative affine Springer fibers for groups in the case of unramified conjugacy classes. He found that there are in general more than one open orbit of the symmetry group, and the union of those orbits is not dense in the multiplicative affine Springer fiber. This makes basic questions like describing irreducible components much harder, and one of the highlights of [Chi19] is to use global method to prove local results like the dimension formula of multiplicative affine Springer fibers (contrary to the case where the dimension formula of the usual affine Springer fiber is proved purely locally). Because of the existence of the irregular components, it seems very hard to determine the number of them, and similar phenomenon also appears in the study of affine Deligne-Lusztig varieties (see [Chi19, § 3.9] for a more detailed discussion).

This phenomenon may be originally perceived as an issue, but now it appears to be more of a feature rather than a bug. Indeed, fortunately Chi was able to describe the irreducible components for unramified conjugacy classes, and it has close connection with the Mirković-Vilonen (MV) cycles. For an unramified class γ, one can associate a W-orbit of cocharacters ν_γ called the Newton point of γ, and for convenience we regard it as a dominant cocharacter. One necessary condition for the multiplicative affine Springer fiber associated with γ and highest coweight λ to be non-empty is $\nu_\gamma \leq \lambda$, in other words, ν_γ appears as a weight in the highest weight $\check{G}$-representation V_λ. Chi showed that the number of irreducible components modulo the symmetry group is exactly $m_{\lambda\nu_\gamma}$, the weight multiplicity of ν_γ in V_λ. The presence of MV-cycles in Chi's proof suggests that not only do the numbers line up with weight multiplicities, but those components actually *are* the weight vectors. This is indeed the case because it turns out one can describe the Galois action (the same as some twisted Frobenius action since the conjugacy class is unramified) on those components, and it can be identified with twisted Frobenius action on V_λ induced by L-action.

With this input in mind, let us now look at the geometric transfer map. In Lie algebra case, the geometric transfer map is a closed embedding (see [Ngô10, § 6.3]), but in group case, it is only a finite map (and far from being flat). The phenomenon we want to describe can already be observed at the local level so we will substitute X with a formal disc X_v for convenience. Recall that over the formal disc, there is the pole determined by a dominant cocharacter λ, and the unramified conjugacy class supplies us with its Newton point ν_γ. Together they are described as a point a in the multiplicative Hitchin base (mH-base). The geometric transfer map is induced by a canonical monoid (depending on the starting monoid for G) for H, and since the transfer map is finite, there will be finitely many points $a_{H,i}$ in the mH-base for H lying over the given point a (we assume that a does lie in the image of transfer map from H). We also know that V_λ decomposes into a bunch of irreducible $\check{H}$-representations $V'_{\lambda_{H,i}}$. The set of $a_{H,i}$'s that maps to a is in bijection with those $V_{\lambda_{H,i}}$ in which ν_γ appears as a weight. Moreover, the weight multiplicity of ν_γ in V_λ is of course the sum of the multiplicities of ν_γ in each $V'_{\lambda_{H,i}}$. This fact plays very nicely with both the connection with the dual group through geometric Satake isomorphism and our proof of the support theorem for mH-fibrations (see § 9.9), which we will summarize in the next section.

1.3 Support Theorem and Beyond

The proof of Theorem 1.0.1 follows the same general strategy in [Ngô10]. We have already discussed several difficulties and features unique to the group case, and we summarize them below:

- We use reductive monoids as replacement for Lie algebras, and the central group of the monoid as replacement for $\mathbb{G}_m$-action.
- We cannot stick to one "universal" monoid for a group G, instead, we have to develop our theory for all very flat monoids.

- For each chosen monoid $\mathfrak{M}$ for G, and each endoscopic group H, there is a canonical associated monoid $\mathfrak{M}_H$. It is essentially induced by the restriction functor $\mathrm{Res}_{\check{H}}^{\check{G}}$ of representations.

- Unlike in Lie algebra case where a $\mathbb{G}_{\mathrm{m}}$-torsor (or line bundle) is fixed, in group case we cannot fix any torsor of the center $Z_{\mathfrak{M}}$. Instead, we should consider the whole moduli of $Z_{\mathfrak{M}}$-torsors.

- The geometric transfer map has very deep connection with the dual group through geometric Satake isomorphism, which also helps taking care of the matching of (sum of) IC-functions.

- The choice of monoid corresponds to which IC-function we want to consider.

- It also relates to the perceived issue where there are irregular components in the multiplicative affine Springer fibers.

1.3.1 Ngô's support theorem in [Ngô10] has three main ingredients outside of the geometric transfer map already established: a Picard stack acting on the Hitchin total space induced by regular centralizer, a codimension estimate of the so-called δ-strata, and a description of the irreducible components of Hitchin fibers using so-called product formula.

The action of the Picard stack makes the Hitchin fibration into a *weak abelian fibration*, so that the cohomology of Hitchin fibers can be described using the Tate module of the Picard stack and is a free module in some sense. Such freeness result allows us to provide an "upper bound" on the set of supports of the perverse summands appearing in the decomposition theorem of [BBD82]. Such upper bound is essentially a vastly optimized Goresky-MacPherson inequality in this special case (see [Ngô10, § 7.3]).

The codimension estimate of δ-strata is proved using a codimension formula for the root valuation strata studied in [GKM09]. It further improves the upper bound, so that only certain kind of δ-strata can appear as supports, and the description of irreducible components then implies that the only possible support is the unique largest one, hence completely pinning down the supports.

1.3.2 In group case, we also have the three ingredients. First, the regular centralizer can be defined in a similar way, albeit with more technical efforts, so that there is a Picard action. Second, the codimension of δ-strata is done by a parallel argument, in which we need to study the multiplicative analogue of root valuation strata in a way similar to [GKM09].

Finally, we need to describe the irreducible components of mH-fibers using product formula (which does exist for mH-fibrations). As we noted, since locally there is no such description for multiplicative affine Springer fibers (only a conjectural one), it is also impossible to derive the description for mH-fibers using product formula. Fortunately, it is not too bad. Upon analyzing the codimension estimate of δ-strata in more details, we found that in the end we only need to know the irreducible components of multiplicative affine Springer fibers in the unramified case, which we do know about. Combining it with an inductive argument, we are able to prove Theorem 1.0.1.

1.3.3 We would like to point out that the reason why we are unable to prove Theorem 1.0.1 for general groups is only temporary. There are two main issues that we expect to be patchable but have not been completely worked out.

The first reason is already mentioned in the previous section, namely the local model of singularity Theorem 6.10.2 is still not strong enough to cover all the cases we want. However, it is likely more due to expediency than the difficulty of the issue. The argument for Theorem 6.10.2 as written has some obvious optimization to be done and we expect a more elaborate argument will solve the remaining issue. See Remark 6.10.15.

The second reason is that we have not studied transfer factor itself in this paper in any capacity. In [Ngô10] such problem is avoided because it has been already studied in

[Kot99], and can be conveniently skipped over if one uses the Kostant section. In group case, there does not seem to be any corresponding study, and in many cases one can get away with using the section constructed by Steinberg, but if the group contains a semisimple factor of type A_{2n} and is not split, then such section does not exist. In the case of adjoint group and elliptic endoscopy, however, such deficiency will not cause any essential problem seemingly due to coincidence.

Although transfer factor itself is extremely complicated, we do not expect the second issue to be too bad either. Because in the end, we most likely only need to study the transfer factor for the twisted quasi-split form of SL_3 in order to solve our problem, which may be doable by direct computations.

1.3.4 Even though mH-fibration already showed its usefulness in proving fundamental lemma, at least in the adjoint case, we expect it leads to an even bigger field to explore, and it is motivated by the result of our support theorem.

Unlike the Lie algebra case, we are unable to completely determine the set of supports, but only an upper bound, in other words, a set of potential supports. These potential supports have a very interesting property: given a "degree" in appropriate sense (similar to $\deg(D)$ in Lie algebra case) and consider the mH-fibration with given degree, then potential support in it either come from endoscopy (in which case they do appear as supports), or it looks like the embedding of the mH-fibration of a smaller degree, thus forming an inductive structure on the potential supports.

Although it would be nice to determine whether those potential, inductive supports appear as actual support or not, it is more exciting to think about its implication in a bigger picture. We will avoid talking about anything too concrete since the author does not really understand the phenomenon well, but at least we would like to predict that there is a limit version of mH-fibrations. It will be similar to the relationship between Beilinson-Drinfeld affine Grassmannians of various degrees and the Ran Grassmannian over the Ran space. So we may call this conjectural object Ran-mH-fibration. This object should be able to geometrize the entire trace formula, while mH-fibration is the truncated version. We also expect such object will be an object of interest for the so-called Beyond Endoscopy program.

1.4 Structure of This Paper

Here is a summary of the content in each chapter. Many chapters are arranged very much like the way in [Ngô10] not only because the logical structure therein is well-organized, but it is also more convenient to compare with Lie algebra case.

In Chapter 2 we introduce some basic notations and setups in the absolute setting. We will review some standard facts about reductive monoids and its invariant theory under the adjoint action. The highlight of this chapter is the construction of the endoscopic monoid and the transfer map at invariant-theoretic level. We will also give a formal statement of the fundamental lemma at the end.

Chapter 3 studies the multiplicative valuation strata. It is a purely technical tool for this paper and is mostly self-contained and can be read by itself. Chapter 5 is also auxiliary. It collects many global constructions needed for our discussions, including the moduli of boundary divisors and a construction of global affine Schubert schemes using reductive monoids. The latter is somewhat interesting by itself as by using monoids we are able to define affine Schubert schemes without referring to affine Grassmannians.

Chapter 4 and Chapter 6 are more or less parallel to each other, except the latter has a lot more content. Chapter 4 reviews constructions and properties of multiplicative affine Springer fibers for groups. Chapter 6 studies the constructions of mH-fibrations and its basic properties. Most of these two chapters are very similar to what is done in the Lie algebra case, with two most notable new additions: the connections with MV-cycles in the local setting, and the local model of singularities in the global setting. We also further discuss how the transfer map works in those settings.

In Chapter 7 and Chapter 8, we discuss various useful stratifications on the mH-base and study cohomologies based on those stratifications. We will statement the (conjectural) geometric stabilization theorem, of which we will be able to prove a weaker version. There are two particularly new features: one is the notion of inductive strata, which will be important in our support theorem; the other is a new kind of Hecke-type stack, from which we can upgrade the traditional product formula over a point into a family. The latter will be particularly useful when we study the top ordinary cohomologies, as studying them beyond just the rank of stalks will be necessary.

Chapter 9 contains the proof of the support theorem. The proof itself is surprisingly close to the case of Lie algebra with perhaps the only important new ingredient being the local model of singularity proved in Chapter 6. However, due to mH-fibers having more complicated description of irreducible components, the implication of the support theorem differs from the Lie algebra case, and this is where inductive strata and the connection with MV-cycles come into play.

In Chapter 10, we review the point-counting framework done by the last chapter of [Ngô10], and extend it slightly to make it more convenient. After that we will be able to prove the fundamental lemma for adjoint groups.

Finally, in Chapter 11, we will discuss projects that are currently under construction based on this paper, as well as promising future projects.

CHAPTER 2

REDUCTIVE MONOIDS AND INVARIANT THEORY

We start by reviewing some facts about the invariant theories of the adjoint action of a reductive group G on the simply-connected cover G^{sc} of its derived subgroup and on a very flat reductive monoid $\mathfrak{M}$ associated with G^{sc}. The new results in this chapter are mostly contained in § 2.5, where the canonical endoscopic monoid $\mathfrak{M}_H$ associated with any given monoid $\mathfrak{M}$ and endoscopic group H is defined. The endoscopic monoid plays a key role throughout this paper. Proofs are mostly omitted for well-known facts but references will be given when possible.

2.1 Quasi-split Forms

2.1.1 Let k be a finite field with q elements. Let $\bar{k}$ be a fixed algebraic closure of k. Let $\mathbf{G}$ be a split connected reductive group defined over k of rank n_G and semisimple rank r. We assume $p = \mathrm{char}(k)$ is larger than twice the Coxeter number of $\mathbf{G}$. We fix once and for all a split maximal k-torus $\mathbf{T}$ of $\mathbf{G}$ and a Borel subgroup $\mathbf{B}$ containing $\mathbf{T}$. Let $\mathbf{U}$ be the unipotent radical of $\mathbf{B}$. Let $(\mathbb{X}, \Phi, \check{\mathbb{X}}, \check{\Phi})$ be the root datum associated with $(\mathbf{G}, \mathbf{T})$, and $\Delta = \{\alpha_1, \dots, \alpha_r\}$ (resp. Φ_+) the set of simple (resp. positive) roots determined by $\mathbf{B}$, and let $\check{\Delta} = \{\check{\alpha}_1, \dots, \check{\alpha}_r\}$ (resp. $\check{\Phi}_+$) their duals. Let $\mathbb{X}_+$ (resp. $\check{\mathbb{X}}_+$) be the set of dominant characters (resp. cocharacters). Let $\mathbf{W} = W(\mathbf{G}, \mathbf{T})$ be the Weyl group, and let $w_0 \in \mathbf{W}$ be the longest element with respect to $\mathbf{B}$.

2.1.2 Let $\mathbf{G}^{\mathrm{der}}$ be the derived subgroup of $\mathbf{G}$, and let $\mathbf{G}^{\mathrm{sc}}$ (resp. $\mathbf{G}^{\mathrm{ad}}$) be the universal cover (resp. adjoint quotient) of $\mathbf{G}^{\mathrm{der}}$. We also use H^{sc} (resp. H^{ad}) to denote the preimage (resp. image) of any subset $H \subset \mathbf{G}$ in $\mathbf{G}^{\mathrm{sc}}$ (resp. $\mathbf{G}^{\mathrm{ad}}$). Let $\{\varpi_1, \dots, \varpi_r\}$ be the set of fundamental weights of $\mathbf{G}^{\mathrm{sc}}$, and (ρ_i, V_{ϖ_i}) be the corresponding Weyl module with highest weight ϖ_i. Let $\{\check{\varpi}_1, \dots, \check{\varpi}_r\}$ be the set of fundamental coweights. Let ρ (resp. $\check{\rho}$) be the half-sum of all positive roots (resp. coroots).

2.1.3 We fix a k-pinning $\mathbf{spl} = (\mathbf{T}, \mathbf{B}, \mathbf{x}_+ \coloneqq \{\mathbf{U}_\alpha\}_{\alpha\in\Delta})$ of $\mathbf{G}$, where $\mathbf{U}_\alpha \colon \mathbb{G}_\mathrm{a} \to \mathbf{G}$ is a one-parameter unipotent group associated with simple root α. The associated groups $\mathbf{G}^{\mathrm{der}}$, $\mathbf{G}^{\mathrm{sc}}$ and $\mathbf{G}^{\mathrm{ad}}$ all come with pinnings induced by $(\mathbf{T}, \mathbf{B}, \mathbf{x}_+)$. Using this k-pinning, we may identify the group $\mathrm{Out}(\mathbf{G})$ of outer automorphisms of G with a subgroup of $\mathrm{Aut}_k(\mathbf{G})$. This is a discrete group, possibly infinite. Given any k-scheme X, we can consider étale $\mathrm{Out}(\mathbf{G})$-torsors over X.

2.1.4 Given such a $\mathrm{Out}(\mathbf{G})$-torsor ϑ_G, we may obtain a quasi-split twisted form

$$G = \mathbf{G} \times^{\mathrm{Out}(\mathbf{G})} \vartheta_G$$

of $\mathbf{G}$ on X. This is a reductive group scheme over X together with a pinning $\mathrm{spl} = (T, B, x_+)$ relative to X. The torsor ϑ_G also induces a $\mathrm{Out}(\mathbf{G}^{\mathrm{ad}}) = \mathrm{Out}(\mathbf{G}^{\mathrm{sc}})$-torsor, still denoted by ϑ_G. It induces quasi-split forms G^{ad} and G^{sc} over X. The Weyl group $\mathbf{W}$ induces a Weyl group scheme W over X, which is an étale group scheme.

2.1.5 If we fix a geometric point $x \in X$, then any étale $\mathrm{Out}(\mathbf{G})$-torsor can be given by a continuous homomorphism

$$\vartheta_G^\bullet \colon \pi_1(X, x) \longrightarrow \mathrm{Out}(\mathbf{G}).$$

In this way the group G comes with a canonical geometric point x_G over x, and we use (G, x_G) for this pointed twisted form.

2.2 Invariant Theory of the Group

2.2.1 The group $\mathbf{G}$ acts on $\mathbf{G}^{\mathrm{sc}}$ by adjoint action. Let $\chi\colon \mathbf{G}^{\mathrm{sc}} \to \mathbf{G}^{\mathrm{sc}} /\!\!/ \mathrm{Ad}(\mathbf{G})$ be the GIT-quotient map.

Theorem 2.2.2 ([Ste65])**.** *The inclusion $\mathbf{T}^{\mathrm{sc}} \hookrightarrow \mathbf{G}^{\mathrm{sc}}$ induces an isomorphism*

$$\mathbf{C} \coloneqq \mathbf{T}^{\mathrm{sc}} /\!\!/ \mathbf{W} \xrightarrow{\sim} \mathbf{G}^{\mathrm{sc}} /\!\!/ \mathrm{Ad}(\mathbf{G}^{\mathrm{sc}}) = \mathbf{G}^{\mathrm{sc}} /\!\!/ \mathrm{Ad}(\mathbf{G}).$$

In addition, both schemes are isomorphic to affine space $\mathbb{A}^r$ whose coordinates are given by the traces χ_i of the fundamental representations (ρ_i, V_{ϖ_i}) of $\mathbf{G}^{\mathrm{sc}}$.

Definition 2.2.3. A $\mathbf{G}$-orbit $\mathrm{Ad}(\mathbf{G})(\gamma)$ ($\gamma \in \mathbf{G}^{\mathrm{sc}}$) is called *regular* if its stabilizer has minimal dimension among all stabilizes. It is called *semisimple* if it contains an element in $\mathbf{T}^{\mathrm{sc}}$. It is called *regular semisimple* if it is both regular and semisimple. The union of regular (resp. semisimple, resp. regular semisimple) orbits is denoted by $\mathbf{G}^{\mathrm{sc}}_{\mathrm{reg}}$ (resp. $\mathbf{G}^{\mathrm{sc}}_{\mathrm{ss}}$, resp. $\mathbf{G}^{\mathrm{sc}}_{\mathrm{rs}}$).

2.2.4 The regular locus $\mathbf{G}^{\mathrm{sc}}_{\mathrm{reg}}$ is open and the restriction of χ to $\mathbf{G}^{\mathrm{sc}}_{\mathrm{reg}}$ is smooth and surjective. The semisimple locus is dense but not open in $\mathbf{G}^{\mathrm{sc}}$. Their intersection $\mathbf{G}^{\mathrm{sc}}_{\mathrm{rs}}$, however, remains open dense in $\mathbf{G}^{\mathrm{sc}}$, and its image in $\mathbf{C}$ is denoted by $\mathbf{C}^{\mathrm{rs}}$. Consider the *discriminant function* on $\mathbf{T}$:

$$\mathrm{Disc} \coloneqq \prod_{\alpha\in\Phi} (1 - e^{\alpha}).$$

This function is $\mathbf{W}$-invariant, hence descends to a function on $\mathbf{C}$, still denoted by Disc. It defines an effective principal divisor $\mathbf{D}$ on $\mathbf{C}$, called the *discriminant divisor*. In fact, $\mathbf{D}$ is a reduced divisor, so it makes sense to talk about its singular locus $\mathbf{D}^{\mathrm{sing}}$.

Proposition 2.2.5 ([Ste65])**.** *The regular semisimple locus $\mathbf{C}^{\mathrm{rs}}$ is equal to the complement* *of the divisor* $\mathbf{D}$ *and thus is an open subset of* $\mathbf{C}$. *Moreover, it is exactly the locus over which the fiber of* χ *consists of a single* $\mathbf{G}$*-orbit.*

2.2.6 Steinberg quasi-section The quotient map χ admits many sections, just like in the Lie algebra case. The difference is that the section we want to use lacks an explicit formula, unlike for example the Kostant section for Lie algebras. Instead, what Steinberg explicitly constructed is a quasi-section. A morphism f in a category is called a quasi-section to morphism g if $f \circ g$ is an *automorphism* (not necessarily identity).

Definition 2.2.7. Fix our choice of simple roots Δ earlier. A *Coxeter datum* is a pair $(\xi, \dot{S})$, where

(1) $\xi\colon \{1, \ldots, r\} \to \Delta$ is a bijection (i.e. a total ordering on the set of simple roots),

(2) $\dot{S}$ is a set of representatives $\dot{s}_\alpha$ in $\mathbf{N} = \mathrm{N}_{\mathbf{G}}(\mathbf{T})$ of simple reflections determined by Δ.

A *Coxeter element* (after fixing a set of simple reflections) of $\mathbf{W}$ is one that can be written as $w = w_\xi = s_{\xi(1)} \cdots s_{\xi(r)}$ for some ξ. Denote by $\mathrm{Cox}(\mathbf{W}, \Delta)$ the set of all Coxeter elements of $\mathbf{W}$.

Fix a Coxeter datum $(\xi, \dot{S})$. Let $\beta_j = \xi(j)$ be the simple root corresponding to $s_{\xi(j)}$. Recall we have one-parameter groups $\mathbf{U}_\alpha$ in the pinning **spl**, such that $\mathrm{Ad}_t(\mathbf{U}_\alpha(x)) = \mathbf{U}_\alpha(\alpha(t)x)$ for all $t \in \mathbf{T}$ and $x \in \mathbb{G}_\mathrm{a}$. Let

$$\begin{aligned} \epsilon^{(\xi,\dot{S})}\colon \mathbf{C} \cong \mathbb{A}^r &\longrightarrow \mathbf{G}^{\mathrm{sc}} \\ (x_1, \ldots, x_r) &\longmapsto \prod_{j=1}^{r} \mathbf{U}_{\beta_j}(x_j)\dot{s}_{\xi(j)}, \end{aligned}$$

where the product on the right-hand side is considered taken in the specified order. This is the Steinberg quasi-section associated with Coxeter datum $(\xi, \dot{S})$. We also call the image $\mathrm{Im}\,\epsilon^{(\xi,\dot{S})}$ a *Steinberg cross-section*. We summarize the results in the following theorem.

Theorem 2.2.8 ([Ste65]). *For each pair $(\xi, \dot{S})$, the map $\epsilon^{(\xi,\dot{S})}$ is a quasi-section of χ. Moreover, $\operatorname{Im}\epsilon^{(\xi,\dot{S})}$ is contained in the regular locus, and has transversal intersection with each regular orbit.*

Once **spl** is fixed, the construction of Steinberg quasi-section relies on two choices: a total ordering of the simple roots, and representatives of simple reflections. The influence of the choices is summarized below.

Proposition 2.2.9 ([Ste65, Lemma 7.5 and Proposition 7.8]). *For any Coxeter data $(\xi, \dot{S})$ and $(\xi', \dot{S}')$,*

(1) if $\xi = \xi'$, then there exists $t, t' \in \mathbf{T}$ such that $\operatorname{Im}\epsilon^{(\xi,\dot{s}')} = t' \operatorname{Im}\epsilon^{(\xi,\dot{S})} = t \operatorname{Im}\epsilon^{(\xi,\dot{S})} t^{-1}$.

(2) if $\dot{S} = \dot{S}'$, then for any $x, x' \in \mathbb{A}^r$ such that $x_{\xi(j)} = x'_{\xi'(j)}$ for $1 \le j \le r$, $\epsilon^{(\xi,\dot{S})}(x)$ and $\epsilon^{(\xi',\dot{S})}(x')$ are $\mathbf{G}$-conjugate. In fact, such conjugation can be made functorially for any k-algebra R. In other words, the transporter from $\epsilon^{(\xi,\dot{S})}(\mathbf{C})$ to $\epsilon^{(\xi',\dot{S})}(\mathbf{C})$ is a trivial torsor under the centralizer group scheme over $\epsilon^{(\xi,\dot{S})}(\mathbf{C})$.

2.2.10 Regular centralizer The universal centralizer group scheme $\mathbf{I} \to \mathbf{G}^{\mathrm{sc}}$ restricts to a smooth group scheme over the regular locus $\mathbf{I}^{\mathrm{reg}} \to \mathbf{G}^{\mathrm{sc}}_{\mathrm{reg}}$. Since generically the centralizer is a torus, $\mathbf{I}^{\mathrm{reg}} \to \mathbf{G}^{\mathrm{sc}}_{\mathrm{reg}}$ is a commutative group scheme. Therefore one can utilize the descent argument in [Ngô10, Lemme 2.1.1] to obtain the following result.

Proposition 2.2.11. *There is a unique smooth commutative group scheme $\mathbf{J} \to \mathbf{C}$ with a $\mathbf{G}$-equivariant isomorphism*

$$\chi^*\mathbf{J}|_{\mathbf{G}^{\mathrm{sc}}_{\mathrm{reg}}} \xrightarrow{\sim} \mathbf{I}^{\mathrm{reg}},$$

which can be extended to a homomorphism $\chi^*\mathbf{J} \to \mathbf{I}$.

Definition 2.2.12. The group scheme $\mathbf{J} \to \mathbf{C}$ is called the *regular centralizer.*

There is another description of $\mathbf{J}$ using the cameral cover $\pi: \mathbf{T}^{\mathrm{sc}} \to \mathbf{C}$, similar to the Lie algebra case as in [DG02] and [Ngô10]. Consider the trivial family of torus $p_2: \mathbf{T} \times \mathbf{T}^{\mathrm{sc}} \to \mathbf{T}^{\mathrm{sc}}$ (p_2 means the second projection), and we have Weil restriction

$$\Pi_{\mathbf{G}} := \pi_*(\mathbf{T} \times \mathbf{T}^{\mathrm{sc}}),$$

on which $\mathbf{W}$ acts diagonally. Since π is finite flat, $\Pi_{\mathbf{G}}$ is representable. Moreover, since $\mathbf{T} \times \mathbf{T}^{\mathrm{sc}}$ is smooth, so is Π_G. Over $\mathbf{C}^{\mathrm{rs}}$, π is Galois étale with Galois group $\mathbf{W}$, hence $\Pi_{\mathbf{G}}^{\mathrm{rs}}$ is a torus. Since $\mathrm{char}(k)$ does not divide the order of $\mathbf{W}$, we have a smooth group scheme $\mathbf{J}^1$ over $\mathbf{C}$

$$\mathbf{J}^1 := \Pi_{\mathbf{G}}^{\mathbf{W}}.$$

Let $\mathbf{J}^0 \subset \mathbf{J}^1$ be the open subgroup scheme of fiberwise neutral component.

Similar to [Ngô10, Définition 2.4.5], we consider this subfunctor $\mathbf{J}'$ of $\mathbf{J}^1$: for a $\mathbf{C}$-scheme S, $\mathbf{J}'(S)$ consists of points

$$f: S \times_{\mathbf{C}} \mathbf{T}^{\mathrm{sc}} \to \mathbf{T}$$

such that for every geometric point $x \in S \times_{\mathbf{C}} \mathbf{T}^{\mathrm{sc}}$, if $s_\alpha(x) = x$ for a root α, then $\alpha(f(x)) \neq -1$. Notice that on $\mathbf{T}^{\mathrm{sc}}$, the condition $s_\alpha(x) = x$ is the same as $x \in \ker \alpha$.

Lemma 2.2.13. *The subfunctor $\mathbf{J}'$ is representable by an open subgroup scheme of $\mathbf{J}^1$ containing $\mathbf{J}^0$.*

Proof. The proof is entirely parallel to [Ngô10, Lemme 2.4.6]. Indeed, it suffices to prove this claim after finite flat base change to $\mathbf{T}^{\mathrm{sc}}$.

On $\mathbf{T}^{\mathrm{sc}}$, the discriminant divisor is the union (with multiplicities) of subgroups $\ker \alpha \subset \mathbf{T}^{\mathrm{sc}}$ for all roots $\alpha \in \Phi$. By adjunction, we have map $\mathbf{J}^1 \times_{\mathbf{C}} \mathbf{T}^{\mathrm{sc}} \to \mathbf{T} \times \mathbf{T}^{\mathrm{sc}}$, whose restriction to $\ker\alpha$ factors through fixed-point subgroup $\mathbf{T}^{s_\alpha}\times\ker\alpha$. Therefore we have map

$$\mathbf{J}^1\times_{\mathbf{C}}\ker\alpha\xrightarrow{\alpha}\{\pm 1\}.$$

The inverse image of -1 is an open and closed subset of $\mathbf{J}^1\times_{\mathbf{C}}\ker\alpha$, hence a Cartier divisor on $\mathbf{J}^1\times_{\mathbf{C}}\mathbf{T}^{\mathrm{sc}}$. The subfunctor $\mathbf{J}'\times_{\mathbf{C}}\mathbf{T}^{\mathrm{sc}}$ is the complement of these Cartier divisors, hence an open subgroup scheme. In addition, $\mathbf{J}'$ contains $\mathbf{J}^0$. ∎

Proposition 2.2.14. *There exists a canonical open embedding of group schemes $\mathbf{J}\to\mathbf{J}^1$ that identifies $\mathbf{J}$ with subgroup scheme $\mathbf{J}'$.*

Proof. The claim about open embedding is proved in [Chi19, Proposition 2.3.2]. In fact, the argument in *loc. cit.* shows that if $\mathbf{G}=\mathbf{G}^{\mathrm{sc}}$, then $\mathbf{J}\to\mathbf{J}^1$ is an isomorphism. Moreover, since on $\mathbf{T}^{\mathrm{sc}}$, $\ker\alpha$ and $\mathbf{T}^{\mathrm{sc},s_\alpha}$ coincide for any root α, we also see that $\mathbf{J}'=\mathbf{J}^1$.

In general, we have a canonical map $\mathbf{J}^{\mathrm{sc}}\to\mathbf{J}$ and $\mathbf{J}$ is generated by the image of $\mathbf{J}^{\mathrm{sc}}$ and $\mathbf{Z}_{\mathbf{G}}$. On the other hand, the map $\mathbf{T}^{\mathrm{sc}}\to\mathbf{T}$ induces canonical map $\mathbf{J}^{\mathrm{sc},1}\to\mathbf{J}^1$ compatible with the map $\mathbf{J}^{\mathrm{sc}}\to\mathbf{J}$. By definition of $\mathbf{J}'$ we also have a third map $\mathbf{J}^{\mathrm{sc},\prime}\to\mathbf{J}'$. This means that the image of $\mathbf{J}^{\mathrm{sc},1}$ is contained in $\mathbf{J}'$. Since on $\mathbf{Z}_{\mathbf{G}}$ all roots are trivial, we know that the image of $\mathbf{J}$ is contained in $\mathbf{J}'$.

It remains to show that $\mathbf{J}\to\mathbf{J}'$ is an isomorphism. Then we can repeat the "codimension 2" argument of [Chi19, Proposition 2.3.2] (see also [Ngô10, Proposition 2.4.7]) to reduce the problem to the case where $\mathbf{G}=\mathrm{SL}_2$, GL_2, or PGL_2. The direct calculation is omitted. ∎

Remark 2.2.15. The proof we present here is subtly different from [Ngô10, Proposition 2.4.7], where there is a surjection $\pi_0(\mathbf{Z}_{\mathbf{G}})\to\pi_0(\mathbf{J}_{\mathfrak{g}})=\mathbf{J}_{\mathfrak{g}}/\mathbf{J}_{\mathfrak{g}}^0$ (here $\mathbf{J}_{\mathfrak{g}}$ stands for the regular centralizer for the Lie algebra of $\mathbf{G}$). It is not so in group case. For example, suppose $\mathbf{G}$ is simple of type G_2. Then $\mathbf{G}=\mathbf{G}^{\mathrm{ad}}=\mathbf{G}^{\mathrm{sc}}$. Let $x\in\mathbf{T}$ be such that $\mathrm{C}_{\mathbf{G}}(x)_0\cong\mathrm{SL}_3$. Let $u\in U_3:=\mathbf{U}\cap\mathrm{SL}_3$ be a regular unipotent element in SL_3. Assuming $\mathrm{char}(k)$ is large enough, then $\mathrm{C}_{\mathbf{G}}(xu)$ contains $\mathrm{C}_{\mathrm{SL}\,3}(u)=Z_{\mathrm{SL}_3}\times\mathrm{C}_{U_3}(u)$ with finite index. So $C_{\mathbf{G}}(xu)$ is disconnected but $\mathbf{Z}_{\mathbf{G}}$ is trivial.

2.2.16 Let G be a quasi-split form of $\mathbf{G}$ over k-scheme X, induced by an $\mathrm{Out}(\mathbf{G})$-torsor ϑ_G. We can twist almost every construction in the invariant theory of $\mathbf{G}$ by ϑ_G and obtain a twisted form over X. First we still have adjoint action of G on G^{sc}, with invariant quotient $\chi\colon G^{\mathrm{sc}} \to \mathfrak{C} = G^{\mathrm{sc}} /\!\!/ \mathrm{Ad}(G)$. We also have the natural isomorphism

$$T /\!\!/ W \xrightarrow{\sim} G^{\mathrm{sc}} /\!\!/ \mathrm{Ad}(G).$$

The discriminant is invariant under $\mathrm{Out}(\mathbf{G})$, so we have a divisor $\mathfrak{D} \subset \mathfrak{C}$ relative to X. The open loci of regular and regular semisimple orbits are stable under $\mathrm{Out}(\mathbf{G})$, and so we still have the twisted form of regular centralizer $\mathfrak{J} \to \mathfrak{C}$ over X, as well as its Galois construction $\mathfrak{J}^1$.

2.2.17 An important difference here compared to Lie algebra case is that Steinberg quasi-section does not necessarily exist, since it may not be stable under $\mathrm{Out}(\mathbf{G})$. Nevertheless, one may obtain a weaker result as follows. The Steinberg quasi-sections depends on a choice of representatives of simple reflections in $\mathbf{G}^{\mathrm{sc}}$. Using the pinning **spl**, one can make such a choice that is stable under $\mathrm{Out}(\mathbf{G})$. Indeed, the root vectors in $\mathbf{x}_+$ can each be extended to a unique $\mathfrak{sl}_2$-triple, hence an opposite pinning with root vectors denoted by $\mathbf{x}_-$. For each simple root $\alpha_i \in \Delta$ (under $\mathbf{B}$), we let

$$\dot{s}_i = \mathbf{U}_{\alpha_i}(1)\mathbf{U}_{-\alpha_i}(-1)\mathbf{U}_{\alpha_i}(1),$$

where $\mathbf{U}_{\alpha_i}$ (resp. $\mathbf{U}_{-\alpha_i}$) are the one-parameter unipotent subgroups determined by $\mathbf{x}_+$ (resp. $\mathbf{x}_-$).

The Steinberg quasi-section also depends on a choice of ordering on Δ. When $\mathbf{G}^{\mathrm{sc}}$ does not contain any simple factor of type A_{2m}, any two simple roots conjugate under

Out($\mathbf{G}$) are not linked in the Dynkin diagram. Therefore by grouping together Out($\mathbf{G}$) orbit when making the ordering, we can make such Steinberg quasi-section equivariant under Out($\mathbf{G}$). Thus a section to χ always exists as long as $\mathbf{G}^{\mathrm{sc}}$ does not have any simple factor of type A_{2m}.

In fact, we have a slightly stronger result: let $\mathbf{G}'$ be the direct factor of $\mathbf{G}^{\mathrm{sc}}$ consisting of all its simple factors of types A_{2m} (for various m), which is preserved by Out($\mathbf{G}$). Then a Steinberg quasi-section exists for G^{sc} as long as the twist G' of $\mathbf{G}'$ is split.

In the remaining cases (i.e., G' is a non-trivial outer twist), we cannot hope to construct a section whose image lies in the regular locus. Rather, we have a weaker result by Steinberg, which is still very helpful later.

Theorem 2.2.18 ([Ste65, Theorem 9.8]). *Let G be a quasi-split semisimple and simply-connected group over a perfect field K, then $G(K) \to \mathfrak{C}(K)$ is surjective.*

The statement in *loc. cit.* requires K to be perfect, but the proof in fact works for arbitrary field provided that $\mathrm{char}(K)$ is not too small for G (e.g., larger than twice the Coxeter number of G). If K is perfect, then the result refines to that $G^{\mathrm{ss}}(K) \to \mathfrak{C}(K)$ is surjective. We will postpone the details until Theorem 2.4.24 since we need to extend the above result to reductive monoids.

2.3 Review of Very Flat Reductive Monoids

The Lie algebra of $\mathbf{G}$ carries a natural $\mathbb{G}_{\mathrm{m}}$-action from its vector space structure, which is useful in global constructions. The group $\mathbf{G}^{\mathrm{sc}}$, however, has no such symmetry built in. Therefore, we must embed $\mathbf{G}^{\mathrm{sc}}$ into a $\mathbf{G}$-space where a similar "scaling" is possible. The quintessential example of this is the embedding of SL_n into Mat_n. In general, we use the theory of very flat reductive monoids.

2.3.1 An algebraic semigroup over k is just a k-scheme of finite type M together with a multiplication morphism $m\colon M \times M \to M$, such that the usual commutative diagram of associativity holds. If there exists a multiplicative identity $e\colon \operatorname{Spec} k \to M$, then M is an algebraic monoid over k. We will only consider monoids that are affine, integral and normal as schemes. If the subgroup $M^{\times}$ of invertible elements of M is a reductive group, then we call M a *reductive monoid*. In this case, we denote the derived subgroup of $M^{\times}$ by M^{der}, and call it the *derived subgroup* of M.

Example 2.3.2. A (normal) toric variety M is a reductive monoid, whose unit group $M^{\times}$ is the torus acting on it, and whose derived subgroup is the trivial group. The variety Mat_n with matrix multiplication is also a reductive monoid, whose unit group is GL_n and derived subgroup is SL_n.

The category of normal reductive monoids is classified by Renner (see e.g., [Ren05]) with the help of their unit groups. It is well known that over an algebraically closed field K, the category of normal affine toric varieties A of a fixed torus T is equivalent to that of strictly convex and saturated cones $\mathscr{C} \subset \check{\mathbb{X}}(T)$. If $T \subset G$ is a maximal torus in a reductive group over K with Weyl group W, and suppose $\mathscr{C}$ is stable under W-action, then the W-action extends over A. Renner's classification theorem states that reductive monoids M with unit group G is classified by such cones $\mathscr{C}$. More precisely, we have the following result.

Theorem 2.3.3 ([Ren05, Theorems 5.2 and 5.4])**.** *Let G be a reductive group over an algebraically closed field with maximal torus T and Weyl group $W = W(G,T)$.*

(1) Let $T \subset A$ be a normal affine toric variety such that the W-action on T extends over A. Then there exists a normal monoid M with unit group G such that $\overline{T}$ is isomorphic to A.

(2) Let M be any normal reductive monoid with unit group G, then the submonoid $\overline{T}$ is normal.

(3) If M_1 and M_2 are such that we have a commutative diagram of monoids

$$\begin{array}{ccccc} \bar{T}_1 & \longleftarrow & T_1 & \longrightarrow & G_1 \\ \downarrow & & \downarrow & & \downarrow \\ \bar{T}_2 & \longleftarrow & T_2 & \longrightarrow & G_2 \end{array}$$

Then this diagram extends to a unique homomorphism $M_1 \to M_2$. Moreover, if the vertical arrows are isomorphisms, then $M_1 \cong M_2$.

Remark 2.3.4. The result is proved over an algebraically closed field, but it is not hard to see that if G is split over a perfect field k, T is a split maximal torus, and all the maps in the commutative diagram are defined over k, then $M_1 \to M_2$ is also defined over k by looking at $\mathrm{Gal}(\bar{k}/k)$-action. The existence of M given G and A is proved using k-rational representations which are well-defined since G is split. See *loc. cit.* for details.

2.3.5 We first give an explicit description of a very important monoid $\mathbf{M}$ with $\mathbf{M}^{\mathrm{der}} \simeq \mathbf{G}^{\mathrm{sc}}$. Consider the group

$$\mathbf{G}_+ \coloneqq (\mathbf{T}^{\mathrm{sc}} \times \mathbf{G}^{\mathrm{sc}})/\mathbf{Z}^{\mathrm{sc}},$$

where the center $\mathbf{Z}^{\mathrm{sc}}$ of $\mathbf{G}^{\mathrm{sc}}$ acts on $\mathbf{T}^{\mathrm{sc}} \times \mathbf{G}^{\mathrm{sc}}$ anti-diagonally. There is a maximal torus $\mathbf{T}_+ = (\mathbf{T}^{\mathrm{sc}} \times \mathbf{T}^{\mathrm{sc}})/\mathbf{Z}^{\mathrm{sc}}$ in $\mathbf{G}_+$, and we denote $\mathbf{Z}_+ = \mathbf{Z}_{\mathbf{G}_+} \cong \mathbf{T}^{\mathrm{sc}}$. The character lattice of $\mathbf{T}_+$ is identified as

$$\mathbb{X}(\mathbf{T}_+) = \left\{(\lambda, \mu) \in \mathbb{X}(\mathbf{T}^{\mathrm{sc}}) \times \mathbb{X}(\mathbf{T}^{\mathrm{sc}}) \;\middle|\; \lambda - \mu \in \mathbb{X}(\mathbf{T}^{\mathrm{ad}})\right\},$$

and its cocharacter lattice is identified as

$$\check{\mathbb{X}}(\mathbf{T}_+) = \left\{(\check{\lambda}, \check{\mu}) \in \check{\mathbb{X}}(\mathbf{T}^{\mathrm{ad}}) \times \check{\mathbb{X}}(\mathbf{T}^{\mathrm{ad}}) \;\middle|\; \check{\lambda} + \check{\mu} \in \check{\mathbb{X}}(\mathbf{T}^{\mathrm{sc}})\right\}.$$

So we have characters $(\alpha, 0)$ $(\alpha \in \Phi)$ of $\mathbf{T}_+$ which are also one-dimensional representations of $\mathbf{G}_+$. On the other hand, any k-rational representation ρ of highest weight λ of $\mathbf{G}^{\mathrm{sc}}$ extends to $\mathbf{G}_+$ by the formula

$$\rho_+(z, g) = \lambda(z)\rho(g),$$

which is easily seen well-defined. Consider the representation of $\mathbf{G}_+$

$$\rho_\star \coloneqq \bigoplus_{i=1}^{r} (\alpha_i, 0) \oplus \bigoplus_{i=1}^{r} \rho_{i+}.$$

We take the normalization of the closure of $\mathbf{G}_+$ in $\rho_\star$, and denote it by $\mathrm{Env}(\mathbf{G}^{\mathrm{sc}})$. This is the *universal monoid* associated with $\mathbf{G}^{\mathrm{sc}}$, constructed by Vinberg in characteristic 0 and Rittatore in all characteristics.

2.3.6 Given any reductive monoid $\mathbf{M}$ with derived subgroup $\mathbf{G}^{\mathrm{sc}}$, we have a action of $\mathbf{G}$ by multiplication on the left, and another one by inverse-multiplication on the right. These two (left) actions commute, hence combine to a $\mathbf{G} \times \mathbf{G}$-action on $\mathbf{M}$. The GIT quotient

$$\alpha_{\mathbf{M}} \colon \mathbf{M} \longrightarrow \mathbf{A}_{\mathbf{M}} \coloneqq \mathbf{M} /\!\!/ (\mathbf{G}^{\mathrm{sc}} \times \mathbf{G}^{\mathrm{sc}})$$

is called the *abelianization* of $\mathbf{M}$, in the sense that it is the largest quotient monoid of $\mathbf{M}$ that is commutative.

Theorem 2.3.7 ([Vin95, Rit01]). *Let $\mathbf{Z}_{\mathbf{M}}$ be the center of $\mathbf{M}^{\times}$, $\mathbf{Z}_{\mathbf{M},0}$ its neutral component, and $\mathbf{Z}_0 = \mathbf{Z}_{\mathbf{M},0} \cap \mathbf{G}^{\mathrm{sc}}$, then we have that*

(1) $\mathbf{A}_{\mathbf{M}}$ is a normal commutative monoid with unit group $\alpha_{\mathbf{M}}(\mathbf{Z}_{\mathbf{M}}) \simeq \mathbf{Z}_{\mathbf{M},0}/\mathbf{Z}_0$,

(2) $\alpha_{\mathbf{M}}^{-1}(1) = \mathbf{G}$ and $\alpha_{\mathbf{M}}^{-1}(\alpha_{\mathbf{M}}(\mathbf{Z}_{\mathbf{M},0})) = \mathbf{G}_+$,

(3) $\alpha_{\mathbf{M}}(\overline{\mathbf{Z}_{\mathbf{M},0}}) = \mathbf{A}_{\mathbf{M}}$, and $\mathbf{A}_{\mathbf{M}} \simeq \overline{\mathbf{Z}_{\mathbf{M},0}}/\mathbf{Z}_0$, where $\overline{\mathbf{Z}_{\mathbf{M},0}}$ denotes the Zariski closure of

$\mathbf{Z}_{\mathbf{M},0}$ *in* $\mathbf{M}$.

2.3.8 Given a homomorphism of two reductive monoids $\phi\colon \mathbf{M}' \to \mathbf{M}$, by the universal property of GIT quotient it induces a homomorphism $\phi_{\mathbf{A}}\colon \mathbf{A}_{\mathbf{M}'} \to \mathbf{A}_{\mathbf{M}}$ that fits into the commutative square

$$\begin{array}{ccc} \mathbf{M}' & \xrightarrow{\phi} & \mathbf{M} \\ \downarrow{\scriptstyle \alpha_{\mathbf{M}'}} & & \downarrow{\scriptstyle \alpha_{\mathbf{M}}} \\ \mathbf{A}_{\mathbf{M}'} & \xrightarrow{\phi_{\mathbf{A}}} & \mathbf{A}_{\mathbf{M}} \end{array} . \tag{2.3.1}$$

Definition 2.3.9. The homomorphism $\phi\colon \mathbf{M}' \to \mathbf{M}$ is called *excellent* if (2.3.1) is Cartesian.

Definition 2.3.10. A reductive monoid $\mathbf{M}$ is called *very flat* if $\alpha_{\mathbf{M}}\colon \mathbf{M} \to \mathbf{A}_{\mathbf{M}}$ is flat with integral fibers. Let $\mathcal{FM}(\mathbf{G}^{\mathrm{sc}})$ be the category in which:

(1) objects are very flat reductive monoids $\mathbf{M}$ with $\mathbf{M}^{\mathrm{der}} \cong \mathbf{G}^{\mathrm{sc}}$,

(2) morphisms from $\mathbf{M}'$ to $\mathbf{M}$ are excellent homomorphisms $\phi\colon \mathbf{M}' \to \mathbf{M}$.

Let $\mathcal{FM}_0(\mathbf{G}^{\mathrm{sc}})$ be the full subcategory of $\mathcal{FM}(\mathbf{G}^{\mathrm{sc}})$ whose objects are very flat monoids with element 0 (one such that $0x = x0 = 0$ for all $x \in \mathbf{M}$).

Theorem 2.3.11 ([Vin95, Rit01])**.** *If* $\mathbf{M} \in \mathcal{FM}(\mathbf{G}^{\mathrm{sc}})$*, then*

$$\mathrm{Hom}_{\mathcal{FM}(\mathbf{G}^{\mathrm{sc}})}(\mathbf{M}, \mathrm{Env}(\mathbf{G}^{\mathrm{sc}})) \neq \emptyset,$$

and is a singleton if $\mathbf{M} \in \mathcal{FM}_0(\mathbf{G}^{\mathrm{sc}})$*. In other words,* $\mathrm{Env}(\mathbf{G}^{\mathrm{sc}})$ *is a versal (resp. universal) object in* $\mathcal{FM}(\mathbf{G}^{\mathrm{sc}})$ *(resp.* $\mathcal{FM}_0(\mathbf{G}^{\mathrm{sc}})$*).*

Remark 2.3.12*.* Theorem 2.3.11 implies that a homomorphism $\mathbf{M} \to \mathrm{Env}(\mathbf{G}^{\mathrm{sc}})$ will induce a homomorphism $\phi_{\mathbf{Z}_{\mathbf{M}}}\colon \mathbf{Z}_{\mathbf{M}} \to \mathbf{T}^{\mathrm{sc}}$ whose restriction on $\mathbf{Z}^{\mathrm{sc}}$ is the identity. In case that $\mathbf{M} = \mathrm{Env}(\mathbf{G}^{\mathrm{sc}})$, $\phi_{\mathbf{Z}_{\mathbf{M}}} = \mathrm{id}_{\mathbf{T}^{\mathrm{sc}}}$.

2.3.13 The abelianization of the universal monoid $\mathrm{Env}(\mathbf{G}^{\mathrm{sc}})$ is an affine space of dimension r, whose coordinate functions can be exactly given by $e^{(\alpha,0)}$ where $\alpha \in \Delta$ are simple roots. In the language of toric varieties, $\mathbf{A}_{\mathrm{Env}(\mathbf{G}^{\mathrm{sc}})}$ is the $\mathbf{T}^{\mathrm{ad}}$-toric variety associated with cone generated by fundamental coweights $\check{\varpi}_i \in \check{\mathbb{X}}(\mathbf{T}^{\mathrm{ad}})$.

If $\mathbf{A}$ is any affine normal toric variety, and $\phi\colon \mathbf{A} \to \mathbf{A}_{\mathrm{Env}(\mathbf{G}^{\mathrm{sc}})}$ is a homomorphism of algebraic monoids, then the pullback of $\mathrm{Env}(\mathbf{G}^{\mathrm{sc}})$ through ϕ gives an object $\mathbf{M} \in \mathcal{FM}(\mathbf{G}^{\mathrm{sc}})$ with abelianization $\mathbf{A}_{\mathbf{M}} = \mathbf{A}$, and by Theorem 2.3.11, every $\mathbf{M} \in \mathcal{FM}(\mathbf{G}^{\mathrm{sc}})$ arises in this way.

An important class of monoids consists of those corresponding to the map $\mathbb{A}^1 \to \mathbf{A}_{\mathrm{Env}(\mathbf{G}^{\mathrm{sc}})}$ given by a *dominant* cocharacter $\check{\lambda} \in \check{\mathbb{X}}(\mathbf{T}^{\mathrm{ad}})$. We will denote such monoid by $\mathbf{M}(\check{\lambda})$. More generally, we may also consider the monoid formed using a tuple of dominant cocharacters $\underline{\check{\lambda}} = (\check{\lambda}_1, \dots, \check{\lambda}_m)$ (allowing repetitions), or equivalently, a multiplicative map $\mathbb{A}^m \to \mathbf{A}_{\mathrm{Env}(\mathbf{G}^{\mathrm{sc}})}$. We shall denote this monoid by $\mathbf{M}(\underline{\check{\lambda}})$.

More generally, if $\underline{\check{\lambda}}$ is a tuple of dominant cocharacters in $\check{\mathbb{X}}(\mathbf{T})$, it induces a tuple $\underline{\check{\lambda}}_{\mathrm{ad}}$ of dominant cocharacters in $\check{\mathbb{X}}(\mathbf{T}^{\mathrm{ad}})$. The induced map $\mathbb{A}^m \to \mathbf{A}_{\mathrm{Env}(G^{\mathrm{sc}})}$ also produces a monoid, still denoted by $\mathbf{M}(\underline{\check{\lambda}})$.

Example 2.3.14. When $\mathbf{G}^{\mathrm{sc}} = \mathrm{SL}_n$ and $\mathbf{M} = \mathrm{Mat}_n$, the abelianization map is the determinant, and the excellent map $\mathbf{M} \to \mathrm{Env}(\mathrm{SL}_n)$ is the pullback of

$$\mathbf{A}_{\mathbf{M}} \cong \mathbb{A}^1 \longrightarrow \mathbf{A}_{\mathrm{Env}(\mathrm{SL}_n)} \cong \mathbb{A}^{n-1}$$

corresponding to cocharacter $\check{\varpi}_{n-1}$, and the map $\phi_{\mathbf{Z}_{\mathbf{M}}}$ is the cocharacter $\check{\alpha}_1 + 2\check{\alpha}_2 + \cdots + (n-1)\check{\alpha}_{n-1} = n\check{\varpi}_{n-1}$. In other words, $\mathrm{Mat}_n = \mathbf{M}(\check{\varpi}_{n-1})$.

2.3.15 It is sometimes convenient to define a *numerical boundary divisor* $\mathbf{B}_{\mathbf{M}}$ on $\mathbf{C}_{\mathbf{M}}$ as the complement of $\mathbf{C}_{\mathbf{M}}^{\times} \coloneqq \mathbf{A}_{\mathbf{M}}^{\times} \times \mathbf{C}$, with reduced scheme structure. It is a Weil divisor and when $\mathbf{C}_{\mathbf{M}}$ is factorial, it is an effective Cartier divisor. In this paper we will mostly be interested in those $\mathbf{M}$ such that $\mathbf{A_M}$ is isomorphic to an affine space $\mathbb{A}^m$, in which case $\mathbf{B_M}$ is cut out by the product of the m coordinate functions, hence principal. Note that $\mathbf{B_M}$ is in general *not* the pullback of $\mathbf{B}_{\mathrm{Env}(\mathbf{G}^{\mathrm{sc}})}$, but they have the same underlying topological space.

2.3.16 The abelianization map admits a section as follows: for $\mathbf{M} = \mathrm{Env}(\mathbf{G}^{\mathrm{sc}})$, let $\mathbf{T}^{\mathrm{ad}} \to \mathbf{T}_+$ be the map $t \mapsto (t, t^{-1})$. It is well-defined and extends to a map $\delta_\mathbf{M} \colon \mathbf{A_M} \to \mathbf{M}$, which is in fact a section of α_M. For general monoid in $\mathcal{FM}(\mathbf{G}^{\mathrm{sc}})$, the formula is $z \mapsto (z, \phi_{\mathbf{Z_M}}(z)^{-1})$.

2.3.17 Let $\mathbf{T_M}$ be the maximal torus of $\mathbf{M}^\times$ containing $\mathbf{T}^{\mathrm{sc}}$, and $\bar{\mathbf{T}}_\mathbf{M}$ its closure in $\mathbf{M}$. It is a normal affine toric variety under $\mathbf{T_M}$, and when $\mathbf{M} = \mathrm{Env}(\mathbf{G}^{\mathrm{sc}})$, its character cone $\mathscr{C}^*_{\bar{\mathbf{T}}_\mathbf{M}}$ and cocharacter cone $\mathscr{C}_{\bar{\mathbf{T}}_\mathbf{M}}$ have the following description:

$$\begin{aligned}
\mathscr{C}^*_{\bar{\mathbf{T}}_\mathbf{M}} &= \{(\lambda, \mu) \in \mathbb{X}(\mathbf{T}_+) \mid \lambda \geq w(\mu), w \in \mathbf{W}\} \\
&= \mathbb{N}\left(\{(\alpha, 0) \mid \alpha \in \Delta\} \cup \{(\varpi_i, w(\varpi_i)) \mid 1 \leq i \leq r \text{ and } w \in \mathbf{W}\}\right), \\
\mathscr{C}_{\bar{\mathbf{T}}_\mathbf{M}} &= \left\{(\check{\lambda}, \check{\mu}) \in \check{\mathbb{X}}(\mathbf{T}_+) \,\middle|\, \check{\lambda} \in \check{\mathbb{X}}(\mathbf{T}^{\mathrm{ad}})_+ \text{ and } \check{\lambda} \geq -w(\mu), w \in \mathbf{W}\right\}.
\end{aligned}$$

It is known that $\bar{\mathbf{T}}_\mathbf{M}$ is Cohen-Macaulay.

2.4 Invariant Theory of Reductive Monoids

2.4.1 Let $\mathbf{M} \in \mathcal{FM}(\mathbf{G}^{\mathrm{sc}})$ be any monoid. We have the adjoint action of $\mathbf{G}^{\mathrm{sc}}$ on $\mathbf{M}$. It can be viewed as the restriction of the $\mathbf{G}^{\mathrm{sc}} \times \mathbf{G}^{\mathrm{sc}}$-action to the diagonal embedding of $\mathbf{G}^{\mathrm{sc}}$. The center $\mathbf{Z}^{\mathrm{sc}}$ acts trivially, so the action factors through $\mathbf{G}^{\mathrm{ad}}$, hence lifts to a $\mathbf{G}$-action on $\mathbf{M}$. The GIT quotient space $\mathbf{M} /\!\!/ \mathrm{Ad}(\mathbf{G})$ maps to $\mathbf{A_M}$. We also have the cameral cover $\pi_\mathbf{M} \colon \bar{\mathbf{T}}_\mathbf{M} \to \bar{\mathbf{T}}_\mathbf{M} /\!\!/ \mathbf{W}$.

On the other hand, the fundamental representation ρ_i extends to $\mathrm{Env}(\mathbf{G}^{\mathrm{sc}})$ by its definition, still denoted by ρ_{i+}. Therefore, the character function χ_{i+} makes sense for arbitrary $\mathbf{M} \in \mathcal{FM}(\mathbf{G}^{\mathrm{sc}})$ after realizing it as a pullback of $\mathrm{Env}(\mathbf{G}^{\mathrm{sc}})$. Thus, we have a map $\mathbf{M} \to \mathbf{A_M} \times \mathbf{C}$, which factors through the GIT quotient map $\chi_{\mathbf{M}} \colon \mathbf{M} \to \mathbf{C_M} \coloneqq \mathbf{M} /\!\!/ \mathrm{Ad}(\mathbf{G})$.

Theorem 2.4.2. *The maps* $\bar{\mathbf{T}}_{\mathbf{M}} \to \mathbf{M} \to \mathbf{A_M} \times \mathbf{C}$ *induce canonical isomorphisms*

$$\bar{\mathbf{T}}_{\mathbf{M}} /\!\!/ \mathbf{W} \simeq \mathbf{C_M} \simeq \mathbf{A_M} \times \mathbf{C}.$$

In fact, the first isomorphism holds for any *(not necessarily very flat) monoid with* $\mathbf{M}^{\mathrm{der}} \cong \mathbf{G}^{\mathrm{sc}}$*.*

Proof. The first isomorphism is proved in [Ren88], but the reader can also find it in [Ren05] for a more modern reference. We do not know where the second isomorphism was first proved, but it was at least proved in [Bou15]. However, the reader should also see [BC18, Chi19] since [Bou15] contains some error unrelated to the current theorem. ∎

Corollary 2.4.3. *The cameral cover* $\pi_{\mathbf{M}}$ *is a Cohen-Macaulay morphism, i.e., it is flat and has Cohen-Macaulay fibers.*

Proof. Since being a Cohen-Macaulay morphism is stable under base change, it suffices to prove the claim for $\mathbf{M} = \mathrm{Env}(\mathbf{G}^{\mathrm{sc}})$, in which case $\mathbf{C_M}$ is regular. Since $\bar{\mathbf{T}}_{\mathbf{M}}$ is Cohen-Macaulay, and $\pi_{\mathbf{M}}$ is finite, it is a flat morphism. Then it is a general result that flat morphism between locally Noetherian schemes with a Cohen-Macaulay source must be Cohen-Macaulay. See [Sta22, Tag 0C0X]. ∎

Proposition 2.4.4. *Given a Steinberg quasi-section* $\epsilon^{(\xi,\dot{S})}$ *of the group* $\mathbf{G}^{\mathrm{sc}}$*, the map*

$$\begin{aligned} \epsilon_{\mathbf{M}}^{(\xi,\dot{S})} \colon \mathbf{A_M} \times \mathbf{C} &\longrightarrow \mathbf{M} \\ (a, x) &\longmapsto \delta_{\mathbf{M}}(a)\epsilon^{(\xi,\dot{S})}(x) \end{aligned}$$

defines a quasi-section of $\chi_{\mathbf{M}}$ *whose image lies in the regular locus* $\mathbf{M}^{\mathrm{reg}}$*.*

2.4.5 The Weyl group $\mathbf{W}$ acts on $\bar{\mathbf{T}}_{\mathbf{M}}$. Similar to the group case, a $\mathbf{G}$-orbit in $\mathbf{M}$ is called *regular* if the stabilizer achieves minimal dimension, *semisimple* if it contains an element in $\bar{\mathbf{T}}$, and *regular semisimple* if it's both regular and semisimple.

2.4.6 The regular semisimple locus $\mathbf{M}^{\mathrm{rs}}$ is open and smooth, and can be characterized by a discriminant function extending Disc from the group case. Indeed, we only need to define it for $\mathbf{M} = \mathrm{Env}(\mathbf{G}^{\mathrm{sc}})$ and pullback it to general $\mathbf{M}$. For $\mathbf{M} = \mathrm{Env}(\mathbf{G}^{\mathrm{sc}})$, let

$$\mathrm{Disc}_{+} \coloneqq e^{(2\rho,0)} \prod_{\alpha\in\Phi} (1 - e^{(0,\alpha)}).$$

This function extends to $\bar{\mathbf{T}}_{\mathbf{M}}$ and is $\mathbf{W}$-Invariant, hence descends to a function on $\mathbf{C}_{\mathbf{M}}$. It is called the *extended discriminant function*, and defines an effective divisor $\mathbf{D}_{\mathbf{M}} \subset \mathbf{C}_{\mathbf{M}}$. The complement of $\mathbf{D}_{\mathbf{M}}$ is the regular semisimple locus $\mathbf{C}_{\mathbf{M}}^{\mathrm{rs}}$, whose preimage under $\chi_{\mathbf{M}}$ is $\mathbf{M}^{\mathrm{rs}}$.

Lemma 2.4.7. *The divisor $\mathbf{D}_{\mathbf{M}}$ intersects $\mathbf{C}_{\mathbf{M}} - \mathbf{C}_{\mathbf{M}}^{\times}$ properly. In other words, the codimension of $\mathbf{D}_{\mathbf{M}}^{\times} \coloneqq \mathbf{D}_{\mathbf{M}} \cap (\mathbf{C}_{\mathbf{M}} - \mathbf{C}_{\mathbf{M}}^{\times})$ in $\mathbf{C}_{\mathbf{M}}$ is at least 2. In particular, set-theoretically $\mathbf{D}_{\mathbf{M}}$ is the closure of $\mathbf{D}_{\mathbf{M}}^{\times}$.*

Proof. The proof is essentially in [Chi19, Lemma 2.4.2] and we reproduce it here. Without loss of generality, we may assume that $\mathbf{M}$ has 0 (since we can always find a very flat reductive monoid with 0 containing $\mathbf{M}$ by enlarging the cocharacter cone). When $\mathbf{M} = \mathrm{Env}(\mathbf{G}^{\mathrm{sc}})$, consider the idempotent $e_{\emptyset,\Delta}$ (see § 2.4.15). It is known that $e_{\emptyset,\Delta}$ is regular semisimple. Its image in $\mathbf{A}_{\mathbf{M}}$ is 0 which is contained in every irreducible component of $\mathbf{A}_{\mathbf{M}} - \mathbf{A}_{\mathbf{M}}^{\times}$. Using the universal property of $\mathrm{Env}(\mathbf{G}^{\mathrm{sc}})$, we see that for any $\mathbf{M}$ the regular semisimple locus is dense in every irreducible component of $\mathbf{A}_{\mathbf{M}} - \mathbf{A}_{\mathbf{M}}^{\times}$ and we are done. ∎

Lemma 2.4.8. *The divisor $\mathbf{D}_{\mathbf{M}}$ is a reduced divisor.*

Proof. Since $\mathbf{D_M}$ is a principal divisor in a Cohen-Macaulay scheme, it is itself Cohen-Macaulay. By the previous lemma we only need to prove that it is reduced over the invertible locus $\mathbf{D}_\mathbf{M}^\times$. But then we can further reduce the divisor in $\mathbf{T_M}$ cut out by $(1-e^{(0,\alpha)})(1-e^{(0,-\alpha)})$ for a single root α, and it reduces to groups with derived subgroup SL_2. Then we can directly compute. The argument is parallel to for example [Ngô10, Lemme 1.10.1]. We leave the details to the reader. ∎

2.4.9 Contrary to the Lie algebra case, there are in general more than one open orbits in the fibers of $\chi_\mathbf{M}$, in other words, the fibers of $\chi_\mathbf{M}^{\mathrm{reg}} : \mathbf{M}^{\mathrm{reg}} \to \mathbf{C_M}$ are no longer homogeneous $\mathbf{G}$-spaces.

2.4.10 We would like to define the regular centralizer group scheme $\mathbf{J_M} \to \mathbf{C_M}$ similarly to the group case, but the original descent argument needs some adaptation, due to the fact that a fiber of $\chi_\mathbf{M}$ may contain multiple regular orbits.

The key observation used by [Chi19] is that the numerical boundary divisor $\mathbf{B_M}$ and the discriminant divisor $\mathbf{D_M}$ intersect properly. The descent argument works over $\mathbf{M}^\times \cup \mathbf{M}^{\mathrm{rs}}$ which is an open subset whose complement has codimension at least 2. We leave the details to [Chi19].

Proposition 2.4.11. *There is a unique smooth commutative group scheme* $\mathbf{J_M} \to \mathbf{C_M}$ *with a* $\mathbf{G}$*-equivariant isomorphism*

$$\chi_\mathbf{M}^*\mathbf{J_M}|_{\mathbf{M}^{\mathrm{reg}}} \xrightarrow{\sim} \mathbf{I}_\mathbf{M}^{\mathrm{reg}},$$

which can be extended to a homomorphism $\chi_\mathbf{M}^*\mathbf{J_M} \to \mathbf{I_M}$*.*

There is also a description of the regular centralizer using cameral cover $\pi_\mathbf{M} : \bar{\mathbf{T}}_\mathbf{M} \to \mathbf{C_M}$. As in the group case, let

$$\Pi_\mathbf{M} = \pi_{\mathbf{M}*}(\mathbf{T} \times \bar{\mathbf{T}}_\mathbf{M}),$$

and

$$\mathbf{J}^1_{\mathbf{M}} = \Pi^{\mathbf{W}}_{\mathbf{M}}.$$

Similarly, we have subfunctor $\mathbf{J}'_{\mathbf{M}}$: for a $\mathbf{C}_{\mathbf{M}}$-scheme S, $\mathbf{J}'_{\mathbf{M}}(S)$ consists of points

$$f\colon S \times_{\mathbf{C}_{\mathbf{M}}} \bar{\mathbf{T}}_{\mathbf{M}} \to \mathbf{T}$$

such that for every geometric point $x \in S \times_{\mathbf{C}_{\mathbf{M}}} \bar{\mathbf{T}}_{\mathbf{M}}$, if $s_\alpha(x) = x$ for a root α, then $\alpha(f(x)) \neq -1$. With the same proof as in the group case, this is an open subgroup scheme of $\mathbf{J}^1_{\mathbf{M}}$ containing the fiberwise neutral component $\mathbf{J}^0_{\mathbf{M}}$.

Proposition 2.4.12. *There is a canonical open embedding*

$$\mathbf{J}_{\mathbf{M}} \longrightarrow \mathbf{J}^1_{\mathbf{M}}$$

that identifies $\mathbf{J}_{\mathbf{M}}$ *with* $\mathbf{J}'_{\mathbf{M}}$.

Proof. The open embedding claim is proved in [Chi19, Proposition 2.4.7]. We even have $\mathbf{J}_{\mathbf{M}} = \mathbf{J}^1_{\mathbf{M}} = \mathbf{J}'_{\mathbf{M}}$ if $\mathbf{G} = \mathbf{G}^{\mathrm{sc}}$. The point is that the complement of $\mathbf{C}^{\times}_{\mathbf{M}} \cup \mathbf{C}^{\mathrm{rs}}_{\mathbf{M}}$ has codimension 2, thus we only need to prove the claim over this open locus, but then it is a consequence of the group case and regular semisimple case, either of which is easy. ∎

Corollary 2.4.13. *The map* $[\mathbf{M}^{\mathrm{reg}}/\operatorname{Ad}(\mathbf{G})] \to \mathbf{C}_{\mathbf{M}}$ *is a finite union of* $\mathbf{J}$*-gerbes.*

2.4.14 Big-cell locus There is an open subset $\mathbf{M}^{\circ} \subset \mathbf{M}$ containing $\mathbf{M}^{\mathrm{reg}}$ that has significant representation-theoretic meaning. We will call it the *big-cell locus*. We will define it for $\operatorname{Env}(\mathbf{G}^{\mathrm{sc}})$, and for general $\mathbf{M} \in \mathcal{FM}(\mathbf{G}^{\mathrm{sc}})$ one simply takes the preimage under any morphism $\mathbf{M} \to \operatorname{Env}(\mathbf{G}^{\mathrm{sc}})$ in the same category.

For $\mathrm{Env}(\mathbf{G}^{\mathrm{sc}})$, the big-cell locus is the set

$$\mathrm{Env}(\mathbf{G}^{\mathrm{sc}})^{\circ} = \{x \in \mathrm{Env}(\mathbf{G}^{\mathrm{sc}}) \mid \rho_{i+}(x) \neq 0, 1 \leq i \leq r\}. \tag{2.4.1}$$

This is readily seen an open locus and can be shown to contain the regular locus. The central torus $\mathbf{Z}_+$ acts on $\mathrm{Env}(\mathbf{G}^{\mathrm{sc}})^{\circ}$, in fact freely, and the quotient is isomorphic to the wonderful compactification $\overline{\mathbf{G}^{\mathrm{ad}}}$ of $\mathbf{G}^{\mathrm{ad}}$ (c.f., [Bou15, Proposition 2.2]). Therefore, $\mathrm{Env}(\mathbf{G}^{\mathrm{sc}})^{\circ}$ is smooth. Roughly speaking, $\mathrm{Env}(\mathbf{G}^{\mathrm{sc}})^{\circ}$ may be viewed as the affine cone associated with projective variety $\overline{\mathbf{G}^{\mathrm{ad}}}$ "without the vertex", while $\mathrm{Env}(\mathbf{G}^{\mathrm{sc}})$ is the entire affine cone.

2.4.15 There is another description of the big-cell locus using the idempotents. Idempotents are very important in studying reductive monoids because they represent $\mathbf{M}^{\times} \times \mathbf{M}^{\times}$-orbits in $\mathbf{M}$. For $\mathrm{Env}(\mathbf{G}^{\mathrm{sc}})^{\circ}$, idempotents allows a more conceptually pleasing description while (2.4.1), although concise, is not very revealing.

Definition 2.4.16. A pair (I, J) of subsets of Δ is called *essential* if no connected components of $\Delta - J$ in the Dynkin diagram is entirely contained in I

Pairs (I, J) of subsets of Δ can be partially ordered by inclusion condition. There is a finite lattice (in the sense of a partial ordering, not of abelian groups) of idempotents $e_{I,J} \in \mathrm{Env}(\mathbf{G}^{\mathrm{sc}})$ labeled by essential pairs (I, J). The $\mathbf{G}_+ \times \mathbf{G}_+$-orbits in $\mathrm{Env}(\mathbf{G}^{\mathrm{sc}})$ are in bijection with $e_{I,J}$ in an order-preserving way. In other words, $\mathbf{G}_+ e_{I,J} \mathbf{G}_+$ is contained in the closure of $\mathbf{G}_+ e_{I',J'} \mathbf{G}_+$ if and only if $I \subset I'$ and $J \subset J'$. Similarly, the $\mathbf{T}^{\mathrm{ad}}$-orbits in $\mathbf{A}_{\mathrm{Env}(\mathbf{G}^{\mathrm{sc}})}$ is in bijection with idempotents e_I for $I \subset \Delta$, and readers can easily figure out what they are. We have that $\alpha_{\mathrm{Env}(\mathbf{G}^{\mathrm{sc}})}(e_{I,J}) = e_I$. In particular, $e_{\Delta,\Delta} = 1$ and $e_{\varnothing,\varnothing} = 0$. The big-cell locus is the union

$$\mathrm{Env}(\mathbf{G}^{\mathrm{sc}})^{\circ} = \bigcup_{I \subset \Delta} \mathbf{G}_+ e_{I,\Delta} \mathbf{G}_+.$$

Then it is readily seen that $\mathrm{Env}(\mathbf{G}^{\mathrm{sc}})^{\circ}$ is smooth because the restriction of $\alpha_{\mathrm{Env}(\mathbf{G}^{\mathrm{sc}})}$ is flat with smooth fibers (homogeneous spaces of of $\mathbf{G}^{\mathrm{sc}} \times \mathbf{G}^{\mathrm{sc}}$), and the base $\mathbf{A}_{\mathrm{Env}(\mathbf{G}^{\mathrm{sc}})}$ is itself smooth.

2.4.17 More properties of the idempotents $e_{I,J}$ are described by Vinberg in [Vin95] and we recall it here for future reference.

For any subset $I \subset \Delta$, let $\mathbf{P}_I \subset \mathbf{G}^{\mathrm{sc}}$ be the standard parabolic subgroup containing $\mathbf{B}^{\mathrm{sc}}$ corresponding to I, and $\mathbf{P}_I^-$ its opposite. Let $\mathbf{U}_I$ (resp. $\mathbf{U}_I^-$) the unipotent radical of $\mathbf{P}_I$ (resp. $\mathbf{P}_I^-$), and $\mathbf{L}_I = \mathbf{P}_I \cap \mathbf{P}_I^-$ the standard Levi subgroup. Let pr_I (resp. pr_I^-) the projection from $\mathbf{P}_I$ (resp. $\mathbf{P}_I^-$) to $\mathbf{L}_I$. Let $\mathbf{P}_{I+}, \mathbf{U}_{I+}, \mathbf{L}_{I+}$, etc. be the same constructions in $\mathbf{G}_+$.

For each $I, J \subset \Delta$, define I^c to be the set $\Delta - I$, I° to be the *interior* of I, that is, those in I that is not joint with I^c by an edge in the Dynkin diagram, and $\Sigma_{I,J} = (I \cap J^{\circ}) \cup J^c$. Let D_I be the abelian monoid in $\mathbb{X}(\mathbf{T}^{\mathrm{sc}})$ generated by I, and C_J the one generated by such ϖ_j that $\alpha_j \in J$. Using identification $\mathbb{X}(\mathbf{T}_+) \subset \mathbb{X}(\mathbf{Z}_+) \times \mathbb{X}(\mathbf{T}^{\mathrm{sc}}) = \mathbb{X}(\mathbf{T}^{\mathrm{sc}}) \times \mathbb{X}(\mathbf{T}^{\mathrm{sc}})$, we let

$$F_{I,J} = \{(\lambda_1, \lambda_2) \in \mathbb{X}(\mathbf{T}_+) \mid \lambda_1 - \lambda_2 \in D_I, \lambda_2 \in C_J\},$$
$$\mathbf{T}_{I,J} = \{t \in \mathbf{T}_+ \mid \lambda(t) = 1 \text{ for all } \lambda \in F_{I,J}\}.$$

The stabilizer of $e_{I,J}$ in $\mathbf{G}_+ \times \mathbf{G}_+$ for an essential pair (I, J) is the subgroups of $\mathbf{P}_{\Sigma_{I,J}+} \times \mathbf{P}^-_{\Sigma_{I,J}+}$ consisting of elements (g, g^-) such that

$$\mathrm{pr}_{\Sigma_{I,J}+}(g) \equiv \mathrm{pr}^-_{\Sigma_{I,J}+}(g^-) \bmod \mathbf{L}^{\mathrm{der}}_{J^c}\mathbf{T}_{I,J}.$$

The idempotent $e_{I,J}$ itself is characterized by

$$(\alpha_i, 0)(e_{I,J}) = \begin{cases} 1 & \alpha_i \in I \\ 0 & \alpha_i \notin I \end{cases},$$

$$(\varpi_j, \varpi_j)(e_{I,J}) = \begin{cases} 1 & \alpha_j \in J \\ 0 & \alpha_j \notin J \end{cases},$$

$$(\varpi_j, w(\varpi_j))(e_{I,J}) = \begin{cases} 1 & \alpha_j \in J, \varpi_j - w(\varpi_j) \in D_{I \cap J^\circ} \\ 0 & \text{otherwise} \end{cases}.$$

2.4.18 Central equivariance The central diagonalizable group $\mathbf{Z_M}$ acts on $\mathbf{M}$ by translation. This action commutes with the adjoint action of $\mathbf{G}$, hence descends to an action on $\mathbf{C_M}$, making $\chi_{\mathbf{M}}$ a $\mathbf{Z_M}$-equivariant map. The $\mathbf{Z_M}$-action on $\mathbf{C_M}$ has a simple description as follows: on $\mathbf{A_M}$ it is simply the translation by torus $\alpha_{\mathbf{M}}(\mathbf{Z_M}) = \mathbf{Z_M}/\mathbf{Z}^{\mathrm{sc}}$, while on $\mathbf{C}$ it is the translation action given by weights $\varpi_i \circ \phi_{\mathbf{Z_M}}$ if we use the coordinates χ_{i+} on $\mathbf{C}$ (recall that $\phi_{\mathbf{Z_M}}$ is the map $\mathbf{Z_M} \to \mathbf{T}^{\mathrm{sc}}$ induced by a choice of morphism $\mathbf{M} \to \mathrm{Env}(\mathbf{G}^{\mathrm{sc}})$ in $\mathcal{FM}(\mathbf{G}^{\mathrm{sc}})$). This action can be lifted to an action of $\mathbf{Z_M}$ on $\mathbf{J_M}$ compatible with the group scheme structure by looking at the construction of $\mathbf{J_M}$.

On the other hand, any choice of a Steinberg quasi-section is far from being $\mathbf{Z_M}$-equivariant. However, one can rectify this with some technical modification. This modification first appears in [Bou15] and later used by [Chi19]. However, the proof in [Bou15] contains an elementary but serious mistake, so we include a corrected proof here. None of the results in [Chi19] is affected by this error.

Proposition 2.4.19. *For each Coxeter datum $(\xi, \dot{S})$, one can define an action $\tau_{\mathbf{M}}^{(\xi,\dot{S})}$ of $\mathbf{Z_M}$ on $\mathbf{M}$ such that*

$$\epsilon_{\mathbf{M}}^{(\xi,\dot{S})} \circ \tau_{\mathbf{C_M}}(z^c) = \tau_{\mathbf{M}}^{(\xi,\dot{S})}(z) \circ \epsilon_{\mathbf{M}}^{(\xi,\dot{S})},$$

where $\tau_{\mathbf{C_M}}$ is the natural action of $\mathbf{Z_M}$ on $\mathbf{C_M}$, and $c = |\mathbf{Z}^{\mathrm{sc}}|$. Moreover, for a fixed z, $\tau_{\mathbf{M}}^{(\xi,\dot{S})}(z)$ is a composition of translation by z^c and conjugation by an element in $\mathbf{T}^{\mathrm{sc}}$ determined by a homomorphism $\mathbf{Z_M} \to \mathbf{T}^{\mathrm{sc}}$ independent of z.

Proof. We re-label $\{\varpi_i\}$ in such a way that ϖ_j is the weight corresponding to root $\beta_j = \alpha_{\xi(j)}$. To simplify notations, for $z \in \mathbf{Z_M}$, we will denote $\phi_{\mathbf{Z_M}}(z)$ simply by z_T.

Let $(a,x) \in \chi_{\mathbf{M}}(\mathbf{M}^{\times}) \subset \mathbf{C_M}$. Fix a ϖ_i $(1 \le i \le r)$, and a weight vector $0 \neq v \in V_{\varpi_i}[\mu]$, where $\mu \le \varpi_i$ is a weight of ρ_i such that $\mu = \sum_{j=1}^{r} m_j \varpi_j$. We have that (to simplify notations we omit ρ_{i+} in the computations)

$$\begin{aligned}\left[\epsilon^{(\xi,\dot{S})}(x)\right](v) &= \left(\prod_{j=1}^{r} U_{\beta_j}(x_j)\dot{s}_{\xi(j)}\right)(v)\\ &= \sum_{\substack{k_j \ge -m_j\\ 1\le j\le r}} \left(\prod_{l=1}^{r} x_l^{k_l+m_l}\right) v_k,\end{aligned}$$

where $\underline{k} = (k_1,\dots,k_r)$ is a multi-index, $v_{\underline{k}}$ is some vector, independent of any x_j, of weight

$$\mu_{\underline{k}} \coloneqq \mu + 0_{\underline{k}},$$

and

$$0_{\underline{k}} \coloneqq \sum_{d=1}^{r} \sum_{1\le l_1<\cdots<l_d\le r} \left[k_{l_d} \prod_{e=1}^{d-1} \left(-\langle \beta_{l_{e+1}}, \beta_{l_e}^{\vee}\rangle\right) \beta_{l_1}\right].$$

Note that $(k_1,\dots,k_r) \mapsto 0_{\underline{k}}$ is a group isomorphism from $\mathbb{Z}^r$ to root lattice $\mathbb{Z}\Phi$.

Thus we have that

$$\begin{aligned}\left[\epsilon_{\mathbf{M}}^{(\xi,\dot{S})}(a,x)\right](v) &= \delta_{\mathbf{M}}(a)\left[\epsilon^{(\xi,\dot{S})}(x)\right](v)\\ &= (z_a, z_{a,T}^{-1}) \sum_{\substack{k_j \ge -m_j\\ 1\le j\le r}} \left(\prod_{l=1}^{r} x_l^{k_l+m_l}\right) v_{\underline{k}}\\ &= \varpi_i(z_{a,T}) \sum_{\substack{k_j \ge -m_j\\ 1\le j\le r}} \left(\prod_{l=1}^{r} x_l^{k_l+m_l}\right) \mu_k(z_{a,T})^{-1} v_{\underline{k}},\end{aligned}$$

where $z_a \in \mathbf{Z_M}$ is some element that $\alpha_{\mathbf{M}}(z_a) = a$; and that

$$\begin{aligned}\left[\epsilon_{\mathbf{M}}^{(\xi,\dot{S})}\left(\tau_{\mathbf{C_M}}(z)(a,x)\right)\right](v) &= (zz_a, (z_T z_{a,T})^{-1})\epsilon^{(\xi,\dot{S})}(\varpi_j(z_T)x_j)(v) \\ &= \varpi_i(z_T z_{a,T}) \sum_{\substack{k_j \geq -m_j \\ 1 \leq j \leq r}} \left(\prod_{l=1}^{r} (\varpi_l(z_T)x_1)^{k_l+m_l}\right) \mu_k(z_T z_{a,T})^{-1} v_{\underline{k}}, \end{aligned} \tag{2.4.2}$$

for any $z \in \mathbf{Z_M}$.

On the other hand, for $t \in \mathbf{T}^{\mathrm{sc}}$,

$$\begin{aligned}\left[\mathrm{Ad}_t\left(\epsilon_{\mathbf{M}}^{(\xi,\dot{S})}(a,x)\right)\right](v) &= t\left[\epsilon_{\mathbf{M}}^{(\xi,\dot{S})}(a,x)\right]t^{-1}(v) \\ &= \varpi_i(z_{a,T})\mu(t)^{-1} \sum_{\substack{k_j \geq -m_j \\ 1 \leq j \leq r}} \left(\prod_{l=1}^{r} x_l^{k_l+m_l}\right) \mu_k(t z_{a,T}^{-1}) v_{\underline{k}} \\ &= \varpi_i(z_{a,T}) \sum_{\substack{k_j \geq -m_j \\ 1 \leq j \leq r}} \left(\prod_{l=1}^{r} x_l^{k_l+m_l}\right) 0_{\underline{k}}(t)\mu_k(z_{a,T})^{-1} v_{\underline{k}}. \end{aligned} \tag{2.4.3}$$

Consider commutative diagram

$$\begin{array}{ccccc} \mathbf{Z_M} & \xrightarrow{\varpi_\bullet} & \mathbb{G}_{\mathrm{m}}^r & \xleftarrow{0_{(\bullet)}} & \mathbf{T}^{\mathrm{sc}} \\ & & \sim\downarrow & \swarrow_{\beta_\bullet} & \downarrow{\scriptstyle z \mapsto z^c} \\ & & \beta_\bullet(\mathbf{T}^{\mathrm{sc}}) & \longrightarrow & \mathbf{T}^{\mathrm{sc}} \end{array}, \tag{2.4.4}$$

where $\beta_\bullet$ is the map

$$t \mapsto (\beta_1(t), \dots, \beta_r(t)),$$

$\varpi_\bullet$ is the map

$$z \mapsto (\varpi_1(z_T), \dots, \varpi_r(z_T)),$$

and $0_{(\bullet)}$ is the one

$$t \mapsto (0_{(1,0,\dots,0)}(t), \dots, 0_{(0,\dots,1,\dots,0)}(t), \dots, 0_{(0,\dots,0,1)}(t)).$$

Denote by ψ the map $\mathbf{Z_M} \to \mathbf{T}^{\mathrm{sc}}$ from the upper-left to the lower-right in (2.4.4), then we have that $\varpi_\bullet(z^c) = 0_{(\bullet)}(\psi(z))$.

Now For $z \in \mathbf{Z_M}$, define $\tau_{\mathbf{M}}^{(\xi,\dot{S})}(z)$ to be the composition of translation by $(z^c, 1)$ and conjugation by $\psi(z)z_T^{-c}$, in other words, for $(t,g) \in \mathbf{M}^\times$,

$$\tau_{\mathbf{M}}^{(\xi,\dot{S})}(z) \colon (t,g) \mapsto (z^c t, \mathrm{Ad}_{\psi(z)z_T^{-c}}(g)).$$

Then one sees from (2.4.2) and (2.4.3) that

$$\begin{aligned}
&\left[\tau_{\mathbf{M}}^{(\xi,\dot{S})}(z)\left(\epsilon_{\mathbf{M}}^{(\xi,\dot{S})}(a,x)\right)\right](v) \\
&\qquad = (z^c,1)\varpi_i(z_{a,T}) \sum_{\substack{k_j \geq -m_j \\ 1\leq j\leq r}} \left(\prod_{l=1}^r x_l^{k_l+m_l}\right) 0_{\underline{k}}(\psi(z)z_T^{-c})\mu_k(z_{a,T})^{-1} v_{\underline{k}} \\
&\qquad = \varpi_i(z_T^c z_{a,T}) \sum_{\substack{k_j \geq -m_j \\ 1\leq j\leq r}} \left(\prod_{l=1}^r x_l^{k_l+m_l}\right) 0_{\underline{k}}(\psi(z)z_T^{-c})\mu_k(z_{a,T})^{-1} v_{\underline{k}} \\
&\qquad = \varpi_i(z_T^c z_{a,T}) \sum_{\substack{k_j \geq -m_j \\ 1\leq j\leq r}} \left(\prod_{l=1}^r (\varpi_l(z_T)^c x_l)^{k_l+m_l}\right) \mu_k(z_T^c z_{a,T})^{-1} v_{\underline{k}} \\
&\qquad\qquad = \left[\epsilon_{\mathbf{M}}^{(\xi,\dot{S})}\left(\tau_{\mathbf{C_M}}(z^c)(a,x)\right)\right](v).
\end{aligned}$$

Finally, clearly the images of $\tau_{\mathbf{M}}^{(\xi,\dot{S})}(z)\left(\epsilon_{\mathbf{M}}^{(\xi,\dot{S})}(a,x)\right)$ and $\epsilon_{\mathbf{M}}^{(\xi,\dot{S})}\left(\tau_{\mathbf{C_M}}^{(\xi,\dot{S})}(z^c)(a,x)\right)$ under $\alpha_{\mathbf{M}}$ are the same, being $\alpha_{\mathbf{M}}(z^c,1)a$.

Therefore, since v, μ, and ϖ_i are arbitrary, we know that $\tau_{\mathbf{M}}^{(\xi,\dot{S})}(z) \circ \epsilon_{\mathbf{M}}^{(\xi,\dot{S})} = \epsilon_{\mathbf{M}}^{(\xi,\dot{S})} \circ \tau_{\mathbf{C_M}}(z^c)$ when restricted to $\chi_{\mathbf{M}}(\mathbf{M}^\times)$. Since $\chi_{\mathbf{M}}(\mathbf{M}^\times)$ is dense in $\mathbf{C_M}$, we are done. ∎

2.4.20 The GIT quotient map $\chi_{\mathbf{M}}$ induces map

$$\overline{\chi_{\mathbf{M}}}\colon [\mathbf{M}/\mathbf{G}] \longrightarrow \mathbf{C}_{\mathbf{M}},$$

which is $\mathbf{Z}_{\mathbf{M}}$-equivariant. We have the further induced map

$$[\chi_{\mathbf{M}}]\colon [\mathbf{M}/(\mathbf{G}\times\mathbf{Z}_{\mathbf{M}})] \longrightarrow [\mathbf{C}_{\mathbf{M}}/\mathbf{Z}_{\mathbf{M}}].$$

Choose a Coxeter datum $(\xi,\dot{S})$, the quasi-section $\epsilon_{\mathbf{M}}^{(\xi,\dot{S})}$ induces a quasi-section of $\overline{\chi_{\mathbf{M}}}$, but not of $[\chi_{\mathbf{M}}]$ unless $c = |\mathbf{Z}^{\mathrm{sc}}| = 1$.

To fix this, recall we have $\psi\colon \mathbf{Z}_{\mathbf{M}} \to \mathbf{T}^{\mathrm{sc}}$ in the proof of Proposition 2.4.19, also viewed as a morphism into $\mathbf{T}$ by abuse of notations. Let

$$\begin{aligned}\Psi\colon \mathbf{Z}_{\mathbf{M}} &\longrightarrow \mathbf{T}\\ z &\longmapsto \psi(z_T)z_T^{-c}.\end{aligned}$$

We define stack $[\mathbf{M}/(\mathbf{G}\times\mathbf{Z}_{\mathbf{M}})]_{[c]}$ using pullback Cartesian diagram

$$\begin{array}{ccc}[\mathbf{M}/(\mathbf{G}\times\mathbf{Z}_{\mathbf{M}})]_{[c]} & \longrightarrow & [\mathbf{M}/(\mathbf{G}\times\mathbf{Z}_{\mathbf{M}})]\\ \downarrow & & \downarrow\\ \mathbb{B}\mathbf{Z}_{\mathbf{M}} & \xrightarrow{\mu\mapsto\mu^{\otimes c}} & \mathbb{B}\mathbf{Z}_{\mathbf{M}}\end{array},$$

and similarly for $[\mathbf{C}_{\mathbf{M}}/\mathbf{Z}_{\mathbf{M}}]_{[c]}$. By Proposition 2.4.19, we obtain a quasi-section

$$\left[\epsilon_{\mathbf{M}}^{(\xi,\dot{S})}\right]_{[c]}\colon [\mathbf{C}_{\mathbf{M}}/\mathbf{Z}_{\mathbf{M}}]_{[c]} \longrightarrow [\mathbf{M}/(\Psi\times\mathrm{id})(\mathbf{Z}_{\mathbf{M}}))]_{[c]},$$

which, by composition, induces quasi-section

$$\left[\epsilon_{\mathbf{M}}^{(\xi,\dot{S})}\right]_{[c]}\colon [\mathbf{C}_{\mathbf{M}}/\mathbf{Z}_{\mathbf{M}}]_{[c]} \longrightarrow [\mathbf{M}/(\mathbf{G}\times\mathbf{Z}_{\mathbf{M}})]_{[c]}.$$

Finally, note that the images of all these quasi-sections lie inside the regular locus (that is, the images of $\mathbf{M}^{\mathrm{reg}}$ in the respective quotient stacks).

2.4.21 The $\mathrm{Out}(\mathbf{G}^{\mathrm{sc}})$ action on $\mathbf{G}^{\mathrm{sc}}$ induces an action on $\mathrm{Env}(\mathbf{G}^{\mathrm{sc}})$, hence we have a quasi-split universal monoid $\mathrm{Env}(G^{\mathrm{sc}})$ on X. The actions of $\mathrm{Out}(\mathbf{G}^{\mathrm{sc}})$ on maximal toric variety $\bar{\mathbf{T}}_{\mathrm{Env}(\mathbf{G}^{\mathrm{sc}})}$ and on Weyl group $\mathbf{W}$ can be combined into an action of $\mathbf{W}\rtimes\mathrm{Out}(\mathbf{G}^{\mathrm{sc}})$ on $\bar{\mathbf{T}}_{\mathrm{Env}(\mathbf{G}^{\mathrm{sc}})}$ and a compatible action of $\mathrm{Out}(\mathbf{G}^{\mathrm{sc}})$ on $\mathbf{C}_{\mathrm{Env}(\mathbf{G}^{\mathrm{sc}})}$. Let $\mathfrak{T}_{\mathrm{Env}(G^{\mathrm{sc}})}$ and $\mathfrak{C}_{\mathrm{Env}(G^{\mathrm{sc}})}$ be the respective induced twisted forms. After twisting by ϑ_G, we obtain invariant map

$$\chi_{\mathrm{Env}(G^{\mathrm{sc}})}\colon \mathrm{Env}(G^{\mathrm{sc}}) \longrightarrow \mathfrak{C}_{\mathrm{Env}(G^{\mathrm{sc}})},$$

and the quotient space is naturally isomorphic to $\mathfrak{T}_{\mathrm{Env}(G^{\mathrm{sc}})}/\!\!/W$. The regular centralizer $\mathfrak{J}_{\mathrm{Env}(G^{\mathrm{sc}})}$ is also well-defined over X, and so is its Galois description $\mathfrak{J}^1_{\mathrm{Env}(G^{\mathrm{sc}})}$.

The Steinberg quasi-section, however, is not necessarily defined unless G^{sc} either has no simple factor of type A_{2m}, or all its such factors are split. The various choices are carefully made in constructing the quasi-section as in the group case. On the other hand, if such conditions are satisfied and a Steinberg quasi-section $\epsilon^{(\xi,\dot{S})}_{\mathrm{Env}(G^{\mathrm{sc}})}$ is defined, then the equivariant version $[\epsilon^{(\xi,\dot{S})}_{\mathrm{Env}(G^{\mathrm{sc}})}]_{[c]}$ can also be defined since the relevant construction are $\mathrm{Out}(\mathbf{G})$-equivariant.

2.4.22 If $\mathbf{M}\in\mathcal{FM}(\mathbf{G}^{\mathrm{sc}})$ is a very flat monoid such that the $\mathrm{Out}(\mathbf{G})$-action on $\mathbf{G}^{\mathrm{sc}}$ extends over $\mathbf{A}_{\mathbf{M}}$ compatible with map $\mathbf{A}_{\mathbf{M}}\to\mathbf{A}_{\mathrm{Env}(\mathbf{G}^{\mathrm{sc}})}$, then we have twisted forms $\mathfrak{M}$ (resp. $\mathfrak{T}_{\mathfrak{M}}$, resp. $\mathfrak{C}_{\mathfrak{M}}$, etc.) of $\mathbf{M}$ (resp. $\bar{\mathbf{T}}_{\mathbf{M}}$, resp. $\mathbf{C}_{\mathbf{M}}$, etc.) over X. More generally, if $\mathbf{M}$ is such that $\mathbf{A}_{\mathbf{M}}$ is stable under the monodromy determined by $\vartheta^\bullet_G$ (but need not necessarily be stable under $\mathrm{Out}(\mathbf{G})$), then we also have the pointed twisted form $\mathfrak{M}$. We also have non-pointed version by using appropriate étale coverings of X instead. The category of such monoids is denoted by $\mathcal{FM}(G^{\mathrm{sc}})$, and $\mathcal{FM}_0(G^{\mathrm{sc}})$ the full subcategory of monoids with 0.

2.4.23 In case G^{sc} has a non-split factor whose type is a product of types A_{2m} for various m, we want to show the following result extending Steinberg's:

Theorem 2.4.24. *Let* $\mathfrak{M} \in \mathcal{FM}(G^{\mathrm{sc}})$ *for a quasi-split group* G *over* $X = \operatorname{Spec} K$ *of some field* K *whose characteristic is larger than twice the Coxeter number of* G*. Then* $\mathfrak{M}(K) \to \mathfrak{C}_{\mathfrak{M}}(K)$ *is surjective.*

Proof. Since $\mathfrak{M}$ is the fiber product of $\mathfrak{A}_{\mathfrak{M}}$ with $\operatorname{Env}(G^{\mathrm{sc}})$ over $\mathfrak{A}_{\operatorname{Env}(G^{\mathrm{sc}})}$, it suffices to prove the result for $\mathfrak{M} = \operatorname{Env}(G^{\mathrm{sc}})$. The result of Steinberg is the same as setting $\mathfrak{M} = G^{\mathrm{sc}}$. Moreover, it suffices to assume $\mathbf{G}^{\mathrm{sc}}$ consists solely of types A_{2m}. Our proof is a modified version of Steinberg's proof.

Indeed, we start with simple group $\mathbf{G} = \mathrm{SL}_{2m+1}$. We label the simple roots (and fundamental weights, etc.) from one end of Dynkin diagram to the other end, so that the middle two simple roots are α_m and α_{m+1}. Let $\alpha = \alpha_m + \alpha_{m+1}$. One can verify that the set

$$\Delta' = \{\alpha_1, \dots, \alpha_{m-1}, \alpha, \alpha_{m+2}, \dots, \alpha_{2m}\}$$

generates a root subsystem of type A_{2m-1}, and it induces a subgroup $\mathbf{G}' \subset \mathbf{G}$ isomorphic to SL_{2m} which is stable under $\operatorname{Out}(\mathbf{G})$. This also identifies $\operatorname{Out}(\mathbf{G})$ with $\operatorname{Out}(\mathbf{G}')$ using the pinning **spl**. The fundamental representations (ρ_i, V_i) of $\mathbf{G}$ are just i-th exterior products of the standard representation of $\mathbf{G}$, and simple linear algebra shows that the restriction of (ρ_i, V_i) to $\mathbf{G}'$ decomposes into two irreducible representations, one with highest weight ϖ_i, and the other with highest weight which we shall denote by $\varpi_i' \in \mathbb{X}(\mathbf{T})$. The weight ϖ_i' is a weight in V_i, and the difference $\varpi_i - \varpi_i'$ is a linear combination of simple roots of $\mathbf{G}$ with coefficients in $\mathbb{N}$.

Consider map

$$\epsilon'' : \mathbb{A}^{2m-1} \longrightarrow \mathbf{G}' \subset \mathbf{G}$$

$$x = (x_1, \dots, x_{m-1}, x_\alpha, x_{m+2}, \dots, x_{2m}) \\ \longmapsto \mathbf{U}_\alpha(x_\alpha)\dot{s}_\alpha \prod_{i=1}^{m-1} (\mathbf{U}_i(x_i)\dot{s}_i\mathbf{U}_{2m-i+1}(x_{2m-i+1})\dot{s}_{2m-i+1}),$$

where the representatives of reflections $\dot{s}_i$ and $\dot{s}_\alpha = \dot{s}_{m+1}\dot{s}_m\dot{s}_{m+1}$ are so chosen that they are stable under $\mathrm{Out}(\mathbf{G})$. Steinberg shows that it is a closed embedding, and for fixed $1 \le i \le 2m$ and a weight μ in V_i, the trace of $\pi_\mu\rho_i(\epsilon''(x))\pi_\mu$ is zero unless $\mu = \varpi_i$ or $\mu = \varpi_i'$, where $\pi_\mu : V_i \to V_i$ is the projection to the weight space of weight μ, in which cases the weight multiplicities are both 1. Furthermore, letting $x_0 = x_{2m+1} = 1$, we have

$$\mathrm{Tr}(\pi_{\varpi_i}\rho_i(\epsilon''(x))\pi_{\varpi_i}) = \begin{cases} x_i & i \neq m-1, m \\ x_\alpha & \text{otherwise} \end{cases}$$

$$\mathrm{Tr}(\pi_{\varpi_i'}\rho_i(\epsilon''(x))\pi_{\varpi_i'}) = \begin{cases} x_{i-1} & 1 \le i \le m \\ x_{i+1} & m+1 \le i \le 2m \end{cases}$$

The abelianization $\mathbf{A_M}$ is isomorphic to $\mathbb{A}^{2m}$ with coordinates given by simple roots. The subtorus $\mathbf{A}_\mathbf{M}^\times$ is identified with $\mathbf{T}^{\mathrm{ad}}$. The section $\delta_\mathbf{M}$ of abelianization map is induced by the anti-diagonal map

$$\mathbf{T}^{\mathrm{ad}} \longrightarrow \mathbf{T}_+ \\ a \longmapsto (a, a^{-1})$$

Now let $\tilde{\epsilon}'' : \mathbf{A_M} \times \mathbb{A}^{2m-1} \to \mathbf{M}$ be the product $\delta_\mathbf{M}\epsilon''$. The image of $\delta_\mathbf{M}$ lies in $\bar{\mathbf{T}}_+$, thus preserves each weight spaces in V_i, so $\pi_\mu\rho_i(\tilde{\epsilon}''(a,x))\pi_\mu$ is still zero unless $\mu = \varpi_i$ or

ϖ_i', and

$$\operatorname{Tr}(\pi_{\varpi_i}\rho_i(\tilde{\epsilon}''(a,x))\pi_{\varpi_i}) = \begin{cases} x_i & i \neq m-1, m \\ x_\alpha & \text{otherwise} \end{cases}$$

$$\operatorname{Tr}(\pi_{\varpi_i'}\rho_i(\tilde{\epsilon}''(a,x))\pi_{\varpi_i'}) = \begin{cases} x_{i-1}\prod_{j=1}^{2m} a_j^{c_{ij}} & 1 \leq i \leq m \\ x_{i+1}\prod_{j=1}^{2m} a_j^{c_{ij}} & m+1 \leq i \leq 2m \end{cases}$$

where c_{ij} is such that $\varpi_i - \varpi_i' = \sum_j c_{ij}\alpha_j$. Summarizing this part, we have that

$$\chi_i(\tilde{\epsilon}''(a,x)) = \begin{cases} x_i + x_{i-1}\prod_{j=1}^{2m} a_j^{c_{ij}} & 1 \leq i \leq m-1 \\ x_i + x_{i+1}\prod_{j=1}^{2m} a_j^{c_{ij}} & m+2 \leq i \leq 2m \\ x_\alpha + x_{m-1}\prod_{j=1}^{2m} a_j^{c_{ij}} & i = m \\ x_\alpha + x_{m+2}\prod_{j=1}^{2m} a_j^{c_{ij}} & i = m+1 \end{cases}$$

On the other hand, consider another map

$$\epsilon''' : \mathbb{A}^{2m-1} \times \mathbb{G}_{\mathrm{m}} \longrightarrow \mathbf{G}$$
$$(x,t) \longmapsto u_m u_{m+1} \mathbf{U}_\alpha(x_\alpha)\dot{s}_\alpha \check{\alpha}(t) \prod_{i=1}^{m-1} (\mathbf{U}_i(x_i)\dot{s}_i \mathbf{U}_{2m-i+1}(x_{2m-i+1})\dot{s}_{2m-i+1}),$$

where $1 \neq u_m \in \mathbf{U}_m(K)$ and $1 \neq u_{m+1} \in \mathbf{U}_{m+1}(K)$ are two arbitrarily chosen elements. Steinberg shows that this map is also a closed embedding. He also shows that if the commutator $[u_{m+1}, u_m]$ is $\mathbf{U}_\alpha(1)$, $\pi_\mu \rho_i(\epsilon''(x))\pi_\mu$ has trace 0 unless $\mu = \varpi_i$ or $\mu = \varpi_i'$, as well as

$$\mathrm{Tr}(\pi_{\varpi_i}\rho_i(\epsilon'''(x,t))\pi_{\varpi_i}) = \begin{cases} x_i & i \neq m-1, m \\ tx_\alpha & i = m \\ tx_\alpha + t & i = m+1 \end{cases}$$

$$\mathrm{Tr}(\pi_{\varpi_i'}\rho_i(\epsilon'''(x,t))\pi_{\varpi_i'}) = \begin{cases} x_{i-1} & 1 \leq i \leq m \\ x_{i+1} & m+1 \leq i \leq 2m \end{cases}$$

Let $\tilde{\epsilon}''' = \delta_{\mathbf{M}}\epsilon'''$, then we similarly have

$$\mathrm{Tr}(\pi_{\varpi_i}\rho_i(\tilde{\epsilon}''(a,x,t))\pi_{\varpi_i}) = \begin{cases} tx_i & i \neq m-1, m \\ tx_\alpha & i = m \\ tx_\alpha + t & i = m+1 \end{cases}$$

$$\mathrm{Tr}(\pi_{\varpi_i'}\rho_i(\tilde{\epsilon}''(a,x,t))\pi_{\varpi_i'}) = \begin{cases} x_{i-1}\prod_{j=1}^{2m} a_j^{c_{ij}} & 1 \leq i \leq m \\ x_{i+1}\prod_{j=1}^{2m} a_j^{c_{ij}} & m+1 \leq i \leq 2m \end{cases}$$

hence

$$\chi_i(\tilde{\epsilon}'''(a,x,t)) = \begin{cases} tx_i + x_{i-1}\prod_{j=1}^{2m} a_j^{c_{ij}} & 1 \leq i \leq m-1 \\ tx_i + x_{i+1}\prod_{j=1}^{2m} a_j^{c_{ij}} & m+2 \leq i \leq 2m \\ tx_\alpha + x_{m-1}\prod_{j=1}^{2m} a_j^{c_{ij}} & i = m \\ tx_\alpha + t + x_{m+2}\prod_{j=1}^{2m} a_j^{c_{ij}} & i = m+1 \end{cases}$$

Let $\tilde{\epsilon}'$ be the disjoint union of $\tilde{\epsilon}''$ and $\tilde{\epsilon}'''$, then it is not hard to see that $\tilde{\epsilon}'$ is a bijection on points valued in K-fields between $\mathbf{A_M} \times (\mathbb{A}^{2m-1} \cup (\mathbb{A}^{2m-1} \times \mathbb{G}_\mathrm{m}))$ and $\mathbf{C_M}$. Moreover, since $[u_m, u_{m+1}] \in \mathbf{U}_\alpha$, $u_{m+1}u_m\mathbf{U}_\alpha$ is stable under $\mathrm{Out}(\mathbf{G})$. This means that the image of ϵ''' is stable under $\mathrm{Out}(\mathbf{G})$. Thus there is an induced action of $\mathrm{Out}(\mathbf{G})$ on $\mathbf{A_M} \times (\mathbb{A}^{2m-1} \cup (\mathbb{A}^{2m-1} \times \mathbb{G}_\mathrm{m}))$ making $\chi_\mathbf{M} \circ \tilde{\epsilon}'$ an $\mathrm{Out}(\mathbf{G})$-equivariant map. Therefore $\tilde{\epsilon}'$ also induces bijection on K-rational points after any $\mathrm{Out}(\mathbf{G})$-twisting.

If $\mathbf{G}$ is a product of groups SL_{2m+1} for various m (allowing repetitions), then we simply take the direct product of $\tilde{\epsilon}'$ of each simple factor in such way that if $m_1 = m_2$, then $u_{m_1} = u_{m_2}$ and $u_{m_1+1} = u_{m_2+1}$. In this way the image of $\tilde{\epsilon}'$ will be stable under $\mathrm{Out}(\mathbf{G})$. Hence we are done. ∎

2.5 Endoscopic Groups and Endoscopic Monoids

What follows until when we start considering monoids is extracted from [Ngô10, 1.8–1.9] and we omit the proofs. Given the pinning $\mathbf{spl} = (\mathbf{T}, \mathbf{B}, \mathbf{x}_+)$, there is a pinning $(\check{\mathbf{T}}, \check{\mathbf{B}}, \check{\mathbf{x}}_+)$ on the dual group $\check{\mathbf{G}}$. Let $\kappa \in \check{\mathbf{T}}$, and $\check{\mathbf{H}}$ be the *connected* centralizer of κ in $\check{\mathbf{G}}$. Then $\check{\mathbf{H}}$ has a maximal torus $\check{\mathbf{T}}$ and a Borel subgroup $\mathbf{B}_{\check{\mathbf{H}}} = \check{\mathbf{H}} \cap \check{\mathbf{B}}$. Taking the dual root datum of $\check{\mathbf{H}}$ determined by $(\check{\mathbf{T}}, \mathbf{B}_{\check{\mathbf{H}}})$, one obtains a split k-group $\mathbf{H}$. Note that $\mathrm{Out}(\mathbf{G}) = \mathrm{Out}(\check{\mathbf{G}})$ and similarly for $\mathbf{H}$ and $\check{\mathbf{H}}$.

2.5.1 The centralizer $(\check{\mathbf{G}} \rtimes \mathrm{Out}(\mathbf{G}))_\kappa$ of κ in $\check{\mathbf{G}} \rtimes \mathrm{Out}(\mathbf{G})$ fits into a short exact sequence

$$1 \longrightarrow \check{\mathbf{H}} \longrightarrow (\check{\mathbf{G}} \rtimes \mathrm{Out}(\mathbf{G}))_\kappa \longrightarrow \pi_0(\kappa) \longrightarrow 1,$$

where $\pi_0(\kappa)$ is the component group of $(\check{\mathbf{G}} \rtimes \mathrm{Out}(\mathbf{G}))_\kappa$. Because $\mathrm{Out}(\mathbf{G})$ is discrete, the projection $\check{\mathbf{G}} \rtimes \mathrm{Out}(\mathbf{G}) \to \mathrm{Out}(\mathbf{G})$ induces a canonical map

$$\mathbf{o}_\mathbf{G}(\kappa)\colon \pi_0(\kappa) \longrightarrow \mathrm{Out}(\mathbf{G}).$$

The action of $(\check{\mathbf{G}} \rtimes \mathrm{Out}(\mathbf{G}))_\kappa$ on its normal subgroup $\check{\mathbf{H}}$ gives a homomorphism into $\mathrm{Aut}(\check{\mathbf{H}})$, which further maps into $\mathrm{Out}(\check{\mathbf{H}})$. Hence we have another canonical map

$$\mathbf{o}_{\mathbf{H}}(\kappa)\colon \pi_0(\kappa) \longrightarrow \mathrm{Out}(\mathbf{H}).$$

Definition 2.5.2. Let G be a reductive group on X given by a $\mathrm{Out}(\mathbf{G})$-torsor ϑ_G. An *endoscopic datum* $(\kappa, \vartheta_\kappa)$ of G is a pair where $\kappa \in \check{\mathbf{T}}$, and ϑ_κ is a $\pi_0(\kappa)$-torsor such that the $\mathrm{Out}(\mathbf{G})$-torsor induced by it through $\mathbf{o}_{\mathbf{G}}(\kappa)$ is isomorphic to ϑ_G. The *endoscopic group* H associated with $(\kappa, \vartheta_\kappa)$ is the twisted form of $\mathbf{H}$ induced by $\mathrm{Out}(\mathbf{H})$-torsor ϑ_H, itself induced by ϑ_κ through $\mathbf{o}_{\mathbf{H}}(\kappa)$.

2.5.3 There is also a pointed variant using representations of $\pi_1(X, x)$ after fixing a geometric point $x \in X$.

Definition 2.5.4. A *pointed endoscopic datum* of (G, x_G) is a pair $(\kappa, \vartheta_\kappa^\bullet)$, where $\kappa \in \check{\mathbf{T}}$ and $\vartheta_\kappa^\bullet$ is a continuous homomorphism $\pi_1(X, x) \to \pi_0(\kappa)$ lying over $\vartheta_G^\bullet$. The *pointed endoscopic group* (H, x_H) is the pointed twisted form induced by $\vartheta_\kappa^\bullet$ through $\mathbf{o}_{\mathbf{H}}(\kappa)$.

Through $\mathbf{o}_{\mathbf{G}}(\kappa)$ and $\mathbf{o}_{\mathbf{H}}(\kappa)$ respectively, we can define actions of $\mathbf{W} \rtimes \pi_0(\kappa)$ and $\mathbf{W}_{\mathbf{H}} \rtimes \pi_0(\kappa)$ on $\mathbf{T}$. The following lemma is crucial to settling the compatibility questions regarding these two constructions.

Lemma 2.5.5 ([Ngô10, Lemme 1.9.1])**.** *There is a canonical homomorphism*

$$\mathbf{W}_{\mathbf{H}} \rtimes \pi_0(\kappa) \longrightarrow \mathbf{W} \rtimes \pi_0(\kappa)$$

whose restriction to $\mathbf{W}_{\mathbf{H}}$ *is the inclusion* $\mathbf{W}_{\mathbf{H}} \subset \mathbf{W}$*, and induces identity on quotient group* $\pi_0(\kappa)$*. Moreover, such homomorphism is compatible with the actions of the two groups on* $\mathbf{T}$*.*

Remark 2.5.6. Note that on contrary the $\pi_0(\kappa)$-actions on $\mathbf{T}$ induced respectively by $\mathbf{o_G}(\kappa)$ and $\mathbf{o_H}(\kappa)$ are not compatible in general.

2.5.7 Let $\mathbf{M} \in \mathcal{FM}(\mathbf{G}^{\mathrm{sc}})$ be a very flat monoid with maximal toric variety $\bar{\mathbf{T}}_{\mathbf{M}}$. Then the root datum of $\mathbf{H}$ induces a group $\mathbf{H_M}$ with maximal torus $\mathbf{T_M}$ and $\mathbf{H}_{\mathbf{M}}^{\mathrm{sc}} = \mathbf{H}^{\mathrm{sc}}$. According to Theorem 2.3.3 (also see Remark 2.3.4), the diagram

$$\bar{\mathbf{T}}_{\mathbf{M}} \supset \mathbf{T_M} \subset \mathbf{H_M}$$

defines up to isomorphism a unique monoid $\mathbf{M}'_{\mathbf{H}}$ with unit group $\mathbf{H_M}$. However, $\mathbf{M}'_{\mathbf{H}}$ is usually not very flat, and $\mathbf{H}_{\mathbf{M}}^{\mathrm{der}}$ is not necessarily simply-connected. Therefore we need to find a very flat monoid in $\mathcal{FM}(\mathbf{H}^{\mathrm{sc}})$ that remedies this problem. We shall see much later that it has to do with the fact that an irreducible representation of $\check{\mathbf{G}}$ is no longer irreducible when restricted to $\check{\mathbf{H}}$ but decomposes into a direct sum of irreducible ones. The correct monoid for $\mathbf{H}$ is exactly guided by this decomposition.

2.5.8 Suppose $\mathbf{A_M}$ is isomorphic to an affine space whose cone is freely generated by cocharacters $\theta_1, \dots, \theta_m$. In general, we may pick a minimal set of generators (necessarily unique since the cone is strictly convex). We have morphism $\mathbf{A_M} \to \mathbf{A}_{\mathrm{Env}(\mathbf{G}^{\mathrm{sc}})}$ so for simplicity we may also treat θ_i as elements in $\check{\mathbb{X}}(\mathbf{T}^{\mathrm{ad}})_+$. The elements in the cone of $\bar{\mathbf{T}}_{\mathbf{M}}$ are then of the form

$$(c_1\theta_1, \dots, c_m\theta_m, \mu)$$

where $c_i \in \mathbb{N}$, $\mu \leq -w_0(\sum_{i=1}^m c_i\theta_i)$ is a weight in the irreducible representation of $\check{\mathbf{G}}^{\mathrm{sc}}$ with highest weight $-w_0(\sum_{i=1}^m c_i\theta_i) \in \check{\mathbb{X}}(\mathbf{T}^{\mathrm{ad}})$.

Let $\check{\mathbf{H}}'$ be the preimage of $\check{\mathbf{H}}$ in $\check{\mathbf{G}}^{\mathrm{sc}}$. Suppose the highest weight representation $V_{-w_0(\theta_i)}$

of $\check{\mathbf{G}}^{\mathrm{sc}}$ and weight $-w_0(\theta_i)$ decomposes as

$$V_{-w_0(\theta_i)} = \bigoplus_{j=1}^{e_i} V'_{-w_{\mathbf{H},0}(\lambda_{ij})}$$

into irreducible $\check{\mathbf{H}}'$-representations with highest weights $-w_{\mathbf{H},0}(\lambda_{ij}) \in \check{\mathbb{X}}(\mathbf{T}^{\mathrm{ad}})$, where $w_{\mathbf{H},0}$ is the longest element of $\mathbf{W_H}$ corresponding to $\mathbf{B_H}$. Note that $\mathbf{W_H}$ acts on $\check{\mathbb{X}}(\mathbf{T}^{\mathrm{ad}})$ through inclusion $\mathbf{W_H} \subset \mathbf{W}$. Let $\mathbf{M_H} \in \mathcal{FM}_0(\mathbf{H}^{\mathrm{sc}})$ be monoid whose abelianization is the affine space with cone freely generated by λ_{ij}. The cone of the maximal toric variety $\bar{\mathbf{T}}_{\mathbf{M},\mathbf{H}}$ consists of elements of the following form:

$$(a_{ij}\lambda_{ij}, \mu)$$

where $\mu \le_{\mathbf{H}} -w_{\mathbf{H},0}(\sum_{i,j} a_{ij}\lambda_{ij})$ is a weight in the irreducible representation of $\check{\mathbf{H}}^{\mathrm{sc}}$ of highest weight $-w_{\mathbf{H},0}(\sum_{i,j} a_{ij}\lambda_{ij})$.

We now construct a $\mathbf{W_H}$-equivariant homomorphism

$$\tilde{\nu}_{\mathbf{H}} \colon \bar{\mathbf{T}}_{\mathbf{M},\mathbf{H}} \longrightarrow \bar{\mathbf{T}}_{\mathbf{M}},$$

and as a consequence we will obtain by Remark 2.3.4 a homomorphism of monoids $\mathbf{M_H} \to \mathbf{M}'_{\mathbf{H}}$. Indeed, the weight $(a_{ij}\lambda_{ij}, \mu)$ can be uniquely written as

$$\Big(a_{ij}\lambda_{ij}, -w_{\mathbf{H},0}\big(\sum_{i,j} a_{ij}\lambda_{ij}\big) - \sum_{\alpha\in\Delta_{\mathbf{H}}} h_\alpha \alpha\Big).$$

We send this element to

$$\Big(\sum_{j=1}^{e_1} a_{1j}\theta_1, \dots, \sum_{j=1}^{e_m} a_{mj}\theta_m, -w_{\mathbf{H},0}\big(\sum_{i,j} a_{ij}\lambda_{ij}\big) - \sum_{\alpha\in\Delta_{\mathbf{H}}} h_\alpha \alpha\Big).$$

One may check that it is indeed a $\mathbf{W_H}$-equivariant homomorphism. Since for each i the weight spaces in $V'_{-w_{\mathbf{H},0}(\lambda_{ij})}$ $(1 \leq j \leq e_i)$ account for all weight spaces in $V_{-w_0(\theta_i)}$, we see that $\mathbf{T}_{\mathbf{M},\mathbf{H}} \to \mathbf{T}_{\mathbf{M}}$ is surjective with connected fibers.

2.5.9 We have $\mathbf{W}_{\mathbf{H}}$-homomorphisms

$$\mathbf{T}_{\mathbf{M},\mathbf{H}} \longrightarrow \mathbf{T}_{\mathbf{M}} \longrightarrow \mathbf{T}^{\mathrm{ad}} \longrightarrow \mathbf{T}_{\mathbf{H}^{\mathrm{ad}}}.$$

It implies that the diagonalizable group (but not necessarily a torus)

$$\mathbf{Z}_{\mathbf{M}}^{\kappa} := \tilde{\nu}_{\mathbf{H}}^{-1}(\mathbf{Z}_{\mathbf{M}})$$

is contained in the center of $\mathbf{M}_{\mathbf{H}}$. As a result, the map

$$\left[\bar{\mathbf{T}}_{\mathbf{M},\mathbf{H}}/\mathbf{Z}_{\mathbf{M}}^{\kappa}\right] \longrightarrow \left[\bar{\mathbf{T}}_{\mathbf{M}}/\mathbf{Z}_{\mathbf{M}}\right]$$

is generically an isomorphism. In fact, we can improve it to the following statement:

Lemma 2.5.10. *Let $\mathcal{O} = \bar{k}[[\pi]]$ and $F = \bar{k}((\pi))$. Then the map*

$$\bar{\mathbf{T}}_{\mathbf{M},\mathbf{H}}(\mathcal{O}) \cap \mathbf{T}_{\mathbf{M},\mathbf{H}}(F) \longrightarrow \bar{\mathbf{T}}_{\mathbf{M}}(\mathcal{O}) \cap \mathbf{T}_{\mathbf{M}}(F)$$

is surjective. Moreover, suppose $t \in \mathbf{T}_{\mathbf{M},\mathbf{H}}/\mathbf{Z}_{\mathbf{M}}^{\kappa}(F)$ extends to a point $t_{\mathcal{O}} \in \left[\bar{\mathbf{T}}_{\mathbf{M}}/\mathbf{Z}_{\mathbf{M}}\right](\mathcal{O})$, then there exists at least one and finitely many ways to extend t to a point in $\left[\bar{\mathbf{T}}_{\mathbf{M},\mathbf{H}}/\mathbf{Z}_{\mathbf{M}}^{\kappa}\right](\mathcal{O})$ lying over $t_{\mathcal{O}}$.

Proof. The first claim follows from the fact that the cone of $\bar{\mathbf{T}}_{\mathbf{M},\mathbf{H}}$ maps surjectively onto the cone of $\bar{\mathbf{T}}_{\mathbf{M}}$ (because for any fixed $1 \leq i \leq m$, the weight spaces in $V'_{-w_{\mathbf{H},0}(\lambda_{ij})}$ $(1 \leq j \leq e_i)$ account for all weight spaces in $V_{-w_0(\theta_i)}$).

For the second claim, since $\mathcal{O}$ has algebraically closed residue field, $\mathbf{Z}_{\mathbf{M}}$-torsors over $\mathcal{O}$ are trivial, so wer can lift $t_{\mathcal{O}}$ to a point $t'_{\mathcal{O}} \in \bar{\mathbf{T}}_{\mathbf{M}}(\mathcal{O}) \cap \mathbf{T}_{\mathbf{M}}(F)$. Using the first claim just proved, let $\tilde{t}$ be the lift of $t'_{\mathcal{O}}$ in $\bar{\mathbf{T}}_{\mathbf{M},\mathbf{H}}(\mathcal{O}) \cap \mathbf{T}_{\mathbf{M},\mathbf{H}}(F)$, then its image in $\left[\bar{\mathbf{T}}_{\mathbf{M},\mathbf{H}}/\mathbf{Z}^{\kappa}_{\mathbf{M}}\right](\mathcal{O})$ is the desired extension of t.

Since $\mathbf{Z}^{\kappa}_{\mathbf{M}}$-torsors over $\mathcal{O}$ are also trivial, the isomorphism class of any lift of $t_{\mathcal{O}}$ is uniquely determined by the lifting of cocharacter $(c_i\theta_i, \mu)$ to $(a_{ij}\lambda_{ij}, \mu')$. For a fixed t', c_i and μ are fixed, so the set of possible a_{ij} is a finite set, and μ' is uniquely determined by μ. This proves finiteness of possible liftings. ∎

2.5.11 As another comment, let $a_{ij} = \#(\mathbf{W}_{\mathbf{H}} \cdot (-w_{\mathbf{H},0}(\lambda_{ij}))) > 0$ and $c_i = \sum_{j=1}^{e_i} a_{ij}$, then we have

$$0 \leq_{\mathbf{H}} -w_{\mathbf{H},0}\Big(\sum_{ij} a_{ij}\lambda_{ij}\Big) \in \check{\mathbb{X}}(\mathbf{T}^{\mathrm{ad}}),$$

Thus the projection of $\mathbf{Z}^{\kappa}_{\mathbf{M}}$ to each $\mathbb{A}^1$ corresponding to λ_{ij} is dominant. This shows that the subgroup $\mathbf{Z}^{\kappa}_{\mathbf{M}}$ is still quite big compared to $\mathbf{Z}_{\mathbf{M},\mathbf{H}}$.

Lemma 2.5.12. *The map $\tilde{\nu}_{\mathbf{H}}$ induces maps of abelianizations of $\mathbf{M}_{\mathbf{H}}$, $\mathbf{M}'_{\mathbf{H}}$ and $\mathbf{M}$:*

$$\mathbf{A}_{\mathbf{M},\mathbf{H}} \longrightarrow \mathbf{A}'_{\mathbf{M},\mathbf{H}} \longrightarrow \mathbf{A}_{\mathbf{M}}.$$

The preimage of invertible locus $\mathbf{A}^{\times}_{\mathbf{M}}$ in $\mathbf{A}_{\mathbf{M},\mathbf{H}}$ is precisely $\mathbf{A}^{\times}_{\mathbf{M},\mathbf{H}}$.

Proof. The first map is induced by the universal property of taking invariant-theoretic $\mathbf{H}^{\mathrm{sc}} \times \mathbf{H}^{\mathrm{sc}}$-quotient. The second one can be seen as follows: the ring of regular functions $k[\mathbf{A}'_{\mathbf{M},\mathbf{H}}]$ is generated by characters in $k[\bar{\mathbf{T}}_{\mathbf{M}}]$ perpendicular to coroots in $\mathbf{H}$, while $k[\mathbf{A}_{\mathbf{M}}]$ is one generated by characters perpendicular to coroots in $\mathbf{G}$, and $\check{\Phi}_{\mathbf{H}} \subset \check{\Phi}$.

The composition map $\mathbf{A}_{\mathbf{M},\mathbf{H}} \to \mathbf{A}_{\mathbf{M}}$ can be described at cocharacter level by

$$(a_{ij}\lambda_{ij}) \longmapsto \left(\sum_{j=1}^{e_i} a_{ij}\theta_i\right).$$

So it is readily seen that the preimage of $\mathbf{A}^{\times}_{\mathbf{M}}$ is $\mathbf{A}^{\times}_{\mathbf{M},\mathbf{H}}$. ∎

2.5.13 Suppose $\pi_0(\kappa)$ acts on $\mathbf{A_M}$ compatibly with its action on $\mathbf{A}_{\mathrm{Env}(\mathbf{G}^{\mathrm{sc}})}$ induced by $\mathbf{o_G}$. Then by uniqueness of minimal set of generators, it has to permute the basis θ_i. Thus $\mathbf{W} \rtimes \pi_0(\kappa)$ acts on $\bar{\mathbf{T}}_\mathbf{M}$ by combining the actions of $\mathbf{W}$ and $\pi_0(\kappa)$. We also have an action of $\mathbf{W_H} \rtimes \pi_0(\kappa)$ on $\bar{\mathbf{T}}_\mathbf{M}$ induced by the canonical map in Lemma 2.5.5.

Lemma 2.5.14. *There is a canonical action of* $\mathbf{W_H} \rtimes \pi_0(\kappa)$ *on* $\bar{\mathbf{T}}_{\mathbf{M},\mathbf{H}}$ *making* $\tilde{\nu}_\mathbf{H}$ *a* $\mathbf{W_H} \rtimes \pi_0(\kappa)$*-equivariant map.*

Proof. To avoid confusion, we write $\mathbf{W} \rtimes \pi_0(\kappa)$ as $\mathbf{W} \rtimes_\mathbf{G} \pi_0(\kappa)$ and $\mathbf{W_H} \rtimes \pi_0(\kappa)$ as $\mathbf{W_H} \rtimes_\mathbf{H} \pi_0(\kappa)$. It suffices to prove compatibility on subgroup $1 \rtimes_\mathbf{H} \pi_0(\kappa)$. Let $\sigma \in 1 \rtimes_\mathbf{H} \pi_0(\kappa)$ be an element.

The action of σ on $\mathbf{T}$ through $\mathbf{W} \rtimes_\mathbf{G} \pi_0(\kappa)$ is compatible with its natural action on $\mathbf{T}_{\mathbf{H}^{\mathrm{sc}}}$ through $\mathbf{W_H} \rtimes \mathrm{Out}(\mathbf{H})$. In particular, $w_{\mathbf{H},0}$ is fixed by σ. We claim that the action of σ on $\check{\mathbb{X}}(\mathbf{T}^{\mathrm{ad}})$ through $\mathbf{W} \rtimes_\mathbf{G} \pi_0(\kappa)$ preserves subset $\{\lambda_{ij}\}$. Indeed, we already know it maps each weight space in the direct sum of $V_{-w_0(\theta_i)}$ into another weight space, as well as each irreducible $\check{\mathbf{H}}'$-representation (viewed as a $\check{\mathbf{H}}^{\mathrm{sc}}$-representation) therein into another one. We also know that since $1 \rtimes_\mathbf{H} \pi_0(\kappa)$ stabilizes the set of simple coroots in $\mathbf{H}$, it must map a $\check{\mathbf{H}}'$-highest weight to another one. Therefore the set of $\check{\mathbf{H}}'$-highest weights λ_{ij} is preserved by σ hence $1 \rtimes_\mathbf{H} \pi_0(\kappa)$.

This way we obtain an action of $1 \rtimes_\mathbf{H} \pi_0(\kappa)$ on the abelianization $\mathbf{A}_{\mathbf{M},\mathbf{H}}$ that is compatible with its natural action on $\mathbf{A}_{\mathrm{Env}(\mathbf{H}^{\mathrm{sc}})}$. It is also compatible with the induced action of $1 \rtimes_\mathbf{H} \pi_0(\kappa)$ on $\mathbf{A_M}$: indeed, since $\mathbf{W}$ acts trivially on the abelianization, the action of $\mathbf{W_H} \rtimes \pi_0(\kappa)$ factors through the *quotient* group $\pi_0(\kappa)$, but the map $\mathbf{W_H} \rtimes_\mathbf{H} \pi_0(\kappa) \to \mathbf{W} \rtimes_\mathbf{G} \pi_0(\kappa)$ induces identity on the quotient group $\pi_0(\kappa)$. Then the result follows from the definition of map $\tilde{\nu}_\mathbf{H}$. ∎

Combining Lemma 2.5.14 with the action of $\mathrm{Out}(\mathbf{H})$ on $\mathrm{Env}(\mathbf{H}^{\mathrm{sc}})$, we obtains a canonical action of $1 \rtimes_\mathbf{H} \pi_0(\kappa)$ on $\mathbf{M_H}$. If we are given a $\pi_0(\kappa)$-torsor ϑ_κ on X, we have an

$1 \rtimes_{\mathbf{H}} \pi_0(\kappa)$-equivariant map of induced twisted forms

$$\mathfrak{M}_H \longrightarrow \mathrm{Env}(H^{\mathrm{sc}}),$$

where the right-hand side is induced by $\mathbf{o}_{\mathbf{H}}$. However, there is no action of $\mathrm{Out}(\mathbf{H})$ on $\bar{\mathbf{T}}_{\mathbf{M},\mathbf{H}}$ or $\mathbf{M}_{\mathbf{H}}$.

Definition 2.5.15. The monoid $\mathfrak{M}_H$ is called the *endoscopic monoid* associated with monoid $\mathfrak{M}$ and endoscopic group H.

The group $\pi_0(\kappa)$ acts canonically on $\mathbf{Z}_{\mathbf{M}}^{\kappa}$ and $\mathbf{Z}_{\mathbf{M},\mathbf{H}}$ because $\mathbf{W}_{\mathbf{H}}$ and $\mathbf{W}$ always act trivially on central tori, and the map in Lemma 2.5.5 induces identity on quotient group $\pi_0(\kappa)$. The map $\mathbf{Z}_{\mathbf{M}}^{\kappa} \to \mathbf{Z}_{\mathbf{M}}$ is $\pi_0(\kappa)$-equivariant.

2.5.16 We have a commutative diagram of invariant quotients

$$\begin{array}{ccccc} & & \bar{\mathbf{T}}_{\mathbf{M},\mathbf{H}} /\!\!/ \mathbf{W}_{\mathbf{H}} & \xrightarrow{\sim} & \mathbf{M}_{\mathbf{H}} /\!\!/ \mathrm{Ad}(\mathbf{H}) \\ & & \downarrow & & \downarrow \\ \bar{\mathbf{T}}_{\mathbf{M}} /\!\!/ \mathbf{W} & \longleftarrow & \bar{\mathbf{T}}_{\mathbf{M}} / \mathbf{W}_{\mathbf{H}} & \xrightarrow{\sim} & \mathbf{M}'_{\mathbf{H}} /\!\!/ \mathrm{Ad}(\mathbf{H}) \end{array}$$

Thus we have a canonical map

$$\nu_{\mathbf{H}} \colon \mathbf{C}_{\mathbf{M},\mathbf{H}} \coloneqq \mathbf{M}_{\mathbf{H}} /\!\!/ \mathrm{Ad}(\mathbf{H}) \longrightarrow \mathbf{C}_{\mathbf{M}}. \tag{2.5.1}$$

Since the preimage of $\mathbf{A}_{\mathbf{M}}^{\times}$ in $\mathbf{A}_{\mathbf{M},\mathbf{H}}$ is $\mathbf{A}_{\mathbf{M},\mathbf{H}}^{\times}$, we have $\nu_{\mathbf{H}}^{-1}(\mathbf{C}_{\mathbf{M}}^{\times}) = \mathbf{C}_{\mathbf{M},\mathbf{H}}^{\times}$. We also have the big-cell locus $\mathbf{C}_{\mathbf{M}}^{\circ}$, and $\mathbf{C}_{\mathbf{M},\mathbf{H}}^{\mathbf{G}\text{-}\circ}$ its preimage. We necessarily have $\mathbf{C}_{\mathbf{M},\mathbf{H}}^{\mathbf{G}\text{-}\circ} \subset \mathbf{C}_{\mathbf{M},\mathbf{H}}^{\circ}$ from the definition. For convenience, we let $\mathbf{C}'_{\mathbf{M},\mathbf{H}}$ be the quotient $\bar{\mathbf{T}}_{\mathbf{M}} /\!\!/ \mathbf{W}_{\mathbf{H}}$.

Lemma 2.5.17. *The restriction of maps*

$$\left[\mathbf{C}_{\mathbf{M},\mathbf{H}}/\mathbf{Z}_{\mathbf{M}}^{\kappa}\right] \longrightarrow \left[\mathbf{C}'_{\mathbf{M},\mathbf{H}}/\mathbf{Z}_{\mathbf{M}}^{\kappa}\right] \longrightarrow [\mathbf{C}_{\mathbf{M}}/\mathbf{Z}_{\mathbf{M}}].$$

to $\left[\mathbf{C}_{\mathbf{M}}^{\times,\mathrm{rs}}/\mathbf{Z}_{\mathbf{M}}\right]$ *are étale, and the first one is even an isomorphism over this subset. Moreover, let* $\mathcal{O} = \bar{k}[[\pi]]$ *and* $F = \bar{k}((\pi))$*, and suppose* $a_{\mathbf{H}} \in \left[\mathbf{C}_{\mathbf{M},\mathbf{H}}^{\times}/\mathbf{Z}_{\mathbf{M}}^{\kappa}\right](F)$ *extends to a point* $a \in [\mathbf{C}_{\mathbf{M}}/\mathbf{Z}_{\mathbf{M}}](\mathcal{O})$*, then there exists at least one and finitely many ways to extend* $a_{\mathbf{H}}$ *to a point in* $\left[\mathbf{C}_{\mathbf{M},\mathbf{H}}/\mathbf{Z}_{\mathbf{M}}^{\kappa}\right](\mathcal{O})$ *lying over* a*.*

Proof. The first claim is clear from definition; the second is proved using Lemma 2.5.10. Indeed, since $\bar{\mathbf{T}}_{\mathbf{M},\mathbf{H}} \to \mathbf{C}_{\mathbf{M},\mathbf{H}}$ is finite, we may lift $a_{\mathbf{H}}$ to a point $t_{\mathbf{H}} \in \left[\bar{\mathbf{T}}_{\mathbf{M}}/\mathbf{Z}_{\mathbf{M}}\right](F_l)$ for some finite tamely ramified extension F_l of F, and there are only finitely many non-isomorphic ways to do so if we require that F_l is chosen to be as small as possible.

Since $\bar{\mathbf{T}}_{\mathbf{M}} \to \mathbf{C}_{\mathbf{M}}$ is also finite, the image of $t_{\mathbf{H}}$ in $\left[\bar{\mathbf{T}}_{\mathbf{M}}/\mathbf{Z}_{\mathbf{M}}\right]$ extends to a unique $\mathcal{O}_l'$-point t lying over a by valuative criteria for properness, where $\mathcal{O}_l'$ is a finite extension of $\mathcal{O}_l$. By our assumption on $\mathrm{char}(k)$ relative to $\mathbf{G}$, any $\mathbf{Z}_{\mathbf{M}}$-torsor over F_l is trivializable over a tamely ramified extension, so $\mathcal{O}_l'$ can be chosen to be tamely ramified over $\mathcal{O}_l$, hence over $\mathcal{O}$. Thus we may replace $\mathcal{O}_l$ with $\mathcal{O}_l'$ and then $t_{\mathbf{H}}$ extends to a point in $\left[\bar{\mathbf{T}}_{\mathbf{M}}/\mathbf{Z}_{\mathbf{M}}\right](\mathcal{O}_l)$.

Lemma 2.5.10 shows that there is at least one and at most finitely many ways to extend $t_{\mathbf{H}}$ to an $\mathcal{O}_l$-point in $\left[\bar{\mathbf{T}}_{\mathbf{M},\mathbf{H}}/\mathbf{Z}_{\mathbf{M}}^{\kappa}\right]$. The image of such extension in $\left[\mathbf{C}_{\mathbf{M},\mathbf{H}}/\mathbf{Z}_{\mathbf{M}}^{\kappa}\right]$ is an $\mathcal{O}_l$-point over a extending $a_{\mathbf{H}}$. But it is also an F-point, thus it must be an $\mathcal{O}$-point. Moreover, any extension of $a_{\mathbf{H}}$ to an $\mathcal{O}$-point can be obtained in this way: indeed, any such $\mathcal{O}$-point lifts to some $\mathcal{O}_l$-point in $\left[\bar{\mathbf{T}}_{\mathbf{M},\mathbf{H}}/\mathbf{Z}_{\mathbf{M}}^{\kappa}\right]$ (using the fact that $\mathbf{Z}_{\mathbf{M}}^{\kappa}$-torsors over $\mathcal{O}$ are trivial and valuative criteria for finite map of varieties $\bar{\mathbf{T}}_{\mathbf{M},\mathbf{H}} \to \mathbf{C}_{\mathbf{M},\mathbf{H}}$).

Therefore, the set of extensions of $a_{\mathbf{H}}$ to $\mathcal{O}$-points is necessarily finite because each step above yields finitely many possibilities and every possible such extension of $a_{\mathbf{H}}$ can be obtained in this way. ∎

2.5.18 Let $\mathbf{C}_{\mathbf{M},\mathbf{H}}^{\mathbf{G}\text{-rs}}$ be the preimage of $\mathbf{C}_{\mathbf{M}}^{\mathrm{rs}}$ under $\nu_{\mathbf{H}}$. Recall the defining equations of discriminant divisors:

$$\mathrm{Disc}_{+} = e^{(2\rho,0)} \prod_{\alpha\in\Phi} (1 - e^{(0,\alpha)})$$

$$\mathrm{Disc}_{\mathbf{H},+} = e^{(2\rho_{\mathbf{H}},0)} \prod_{\alpha\in\Phi_{\mathbf{H}}} (1 - e^{(0,\alpha)})$$

and note that $\Phi_{\mathbf{H}} \subset \Phi$ and $\Phi_{\mathbf{H},+} \subset \Phi_{+}$. Over the invertible locus, $\mathrm{Disc}_{+}/\mathrm{Disc}_{\mathbf{H},+}$ is a regular function, hence $\nu_{\mathbf{H}}(\mathbf{D}^{\times}_{\mathbf{M},\mathbf{H}}) \subset \mathbf{D}^{\times}_{\mathbf{M}}$. Since $\mathbf{D}^{\times}_{\mathbf{M},\mathbf{H}}$ is dense in $\mathbf{D}_{\mathbf{M},\mathbf{H}}$, we see that $\nu_{\mathbf{H}}(\mathbf{D}_{\mathbf{M},\mathbf{H}}) \subset \mathbf{D}_{\mathbf{M}}$ and as a result

$$\mathbf{C}^{\mathbf{G}\text{-rs}}_{\mathbf{M},\mathbf{H}} \subset \mathbf{C}^{\mathrm{rs}}_{\mathbf{M},\mathbf{H}}.$$

We claim that the formula

$$(r^{\mathbf{G}}_{\mathbf{H}})^2 := e^{(2\rho-2\rho_{\mathbf{H}},0)} \prod_{\alpha\in\Phi-\Phi_{\mathbf{H}}} (1 - e^{(0,\alpha)})$$

defines a $\mathbf{W}_{\mathbf{H}}$-invariant function on $\bar{\mathbf{T}}_{\mathbf{M},\mathbf{H}}$. Indeed, we only need to show for any 1-parameter homomorphism $\mathbb{G}_{\mathrm{m}} \to \mathbf{T}_{\mathbf{M},\mathbf{H}}$ such that it extends to a homomorphism of monoids $\mathbb{A}^1 \to \bar{\mathbf{T}}_{\mathbf{M},\mathbf{H}}$, the rational function $(r^{\mathbf{G}}_{\mathbf{H}})^2$ restricts to a function on $\mathbb{A}^1$, or equivalently, it has non-negative valuation as a function on $\mathbb{G}_{\mathrm{m}}$, which is implied by the fact that $\nu_{\mathbf{H}}(\mathbf{D}_{\mathbf{M},\mathbf{H}}) \subset \mathbf{D}_{\mathbf{M}}$. As a consequence, we have a principal divisor $2\mathbf{R}^{\mathbf{G}}_{\mathbf{H}}$ on $\mathbf{C}_{\mathbf{M},\mathbf{H}}$ such that

$$\nu^{*}_{\mathbf{H}}\mathbf{D}_{\mathbf{M}} = \mathbf{D}_{\mathbf{M},\mathbf{H}} + 2\mathbf{R}^{\mathbf{G}}_{\mathbf{H}}.$$

Moreover, over the invertible locus, $2\mathbf{R}^{\mathbf{G}}_{\mathbf{H}}$, as a divisor, is clearly twice of some reduced divisor D. Since $\mathbf{C}_{\mathbf{M},\mathbf{H}}$ is an affine space, hence in particular integral and factorial, the closure of D in $\mathbf{C}_{\mathbf{M},\mathbf{H}}$ with the reduced subscheme structure is a principal divisor $\overline{D}$, and we have $2\overline{D} = 2\mathbf{R}^{\mathbf{G}}_{\mathbf{H}}$. Therefore we define divisor $\mathbf{R}^{\mathbf{G}}_{\mathbf{H}}$ as $\overline{D}$.

2.5.19 We have regular centralizer $\mathbf{J}_{\mathbf{M},\mathbf{H}}$ of the adjoint $\mathbf{H}$-action on $\mathbf{M}_{\mathbf{H}}$ as well as the group scheme $\mathbf{J}^{1}_{\mathbf{M},\mathbf{H}}$ on $\mathbf{C}_{\mathbf{M},\mathbf{H}}$ using Galois description of regular centralizer. Similar to

Lie algebra case, we have the following results.

Lemma 2.5.20. *There is a canonical commutative diagram over* $\mathbf{C}_{\mathbf{M},\mathbf{H}}$

$$\begin{array}{ccc} \nu_{\mathbf{H}}^{*}\mathbf{J}_{\mathbf{M}} & \longrightarrow & \mathbf{J}_{\mathbf{M},\mathbf{H}} \\ \downarrow & & \downarrow \\ \nu_{\mathbf{H}}^{*}\mathbf{J}_{\mathbf{M}}^{1} & \longrightarrow & \mathbf{J}_{\mathbf{M},\mathbf{H}}^{1} \end{array}$$

such that all arrows are isomorphisms over $\mathbf{C}_{\mathbf{M},\mathbf{H}}^{\mathbf{G}\text{-rs}}$.

Proof. We have canonical homomorphism by adjunction

$$\nu_{\mathbf{H}}^{*}\pi_{\mathbf{M}*}\mathbf{T} \longrightarrow \pi_{\mathbf{M},\mathbf{H}*}\tilde{\nu}_{\mathbf{H}}^{*}\mathbf{T}.$$

Taking invariant under either $\mathbf{W}$ or $\mathbf{W}_{\mathbf{H}}$, we have

$$\nu_{\mathbf{H}}^{*}\mathbf{J}_{\mathbf{M}}^{1} = \nu_{\mathbf{H}}^{*}(\pi_{\mathbf{M}*}\mathbf{T})^{\mathbf{W}} \longrightarrow \nu_{\mathbf{H}}^{*}(\pi_{\mathbf{M}*}\mathbf{T})^{\mathbf{W}_{\mathbf{H}}} \longrightarrow (\pi_{\mathbf{M},\mathbf{H}*}\tilde{\nu}_{\mathbf{H}}^{*}\mathbf{T})^{\mathbf{W}_{\mathbf{H}}} = \mathbf{J}_{\mathbf{M},\mathbf{H}}^{1}.$$

This map is clearly an isomorphism over the $\mathbf{G}$-regular semisimple locus (which is contained in $\mathbf{H}$-regular semisimple locus). Using the identification of $\mathbf{J}_{\mathbf{M}}$ with $\mathbf{J}'_{\mathbf{M}}$, we see that if the condition defining $\mathbf{J}'_{\mathbf{M}}$ is satisfied, then the analogous condition for $\mathbf{J}'_{\mathbf{M},\mathbf{H}}$ is also satisfied since $\Phi_{\mathbf{H}} \subset \Phi$. Therefore the map $\nu_{\mathbf{H}}^{*}\mathbf{J}_{\mathbf{M}} \to \mathbf{J}_{\mathbf{M},\mathbf{H}}^{1}$ factors through $\mathbf{J}_{\mathbf{M},\mathbf{H}}$ and we are done. ∎

2.5.21 To establish the transfer between regular semisimple orbit of G and G-regular semisimple orbit in H, we need to construct a canonical map

$$\nu_{H} : \mathfrak{C}_{\mathfrak{M},H} \longrightarrow \mathfrak{C}_{\mathfrak{M}}.$$

Through $\mathbf{o}_{\mathbf{G}}$, we have twisted form $\mathfrak{T}_{\mathfrak{M}}$ of $\bar{\mathbf{T}}_{\mathbf{M}}$, and the canonical action of $1 \rtimes_{\mathbf{H}} \pi_0(\kappa)$ induces twisted form $\mathfrak{T}_{\mathfrak{M},H}$ of $\bar{\mathbf{T}}_{\mathbf{M},\mathbf{H}}$. However, $\tilde{\nu}_{\mathbf{H}}$ is not $\pi_0(\kappa)$-equivariant since the image of $1 \rtimes_{\mathbf{H}} \pi_0(\kappa)$ is not $1 \rtimes_{\mathbf{G}} \pi_0(\kappa)$, therefore there is no natural map between $\mathfrak{T}_{\mathfrak{M}}$ and $\mathfrak{T}_{\mathfrak{M},H}$. Nevertheless, we still have the canonical map ν_H because the action of $\mathbf{W_H} \rtimes \pi_0(\kappa)$ on $\mathbf{C_{M,H}}$ factors through quotient $\pi_0(\kappa)$ and same is true for $\mathbf{W} \rtimes \pi_0(\kappa)$ acting on $\mathbf{C_M}$.

Similarly, we have relation between twisted discriminant divisors

$$\nu_H^* \mathfrak{D}_{\mathfrak{M}} = \mathfrak{D}_{\mathfrak{M},H} + 2\mathfrak{R}_H^G,$$

where $\mathfrak{R}_H^G$ is a reduced principal divisor on $\mathfrak{C}_{\mathfrak{M},H}$. The natural map between abelianizations also have natural twisted form

$$\underline{\nu}_H : \mathfrak{A}_{\mathfrak{M},H} \longrightarrow \mathfrak{A}_{\mathfrak{M}}.$$

For the same reason, the central diagonalizable group $\mathbf{Z}_{\mathbf{M}}^{\kappa}$ has twisted form $Z_{\mathfrak{M}}^{\kappa}$ and it maps naturally to $\mathfrak{A}_{\mathfrak{M},H}^{\times}$ and surjectively onto $Z_{\mathfrak{M}}$.

2.5.22 Finally, we want to establish a canonical map $\nu_H : \mathfrak{J}_{\mathfrak{M}} \to \mathfrak{J}_{\mathfrak{M},H}$ as follows: regarding ϑ_κ as a finite étale map $\vartheta_\kappa : X_\kappa \to X$, we have a finite flat map

$$\pi_\kappa : X_\kappa \times_X \mathfrak{T}_{\mathfrak{M}} \simeq X_\kappa \times \bar{\mathbf{T}}_{\mathbf{M}} \longrightarrow \mathfrak{C}_{\mathfrak{M}}$$

such that over any geometric point $\bar{v} \in X$ it is generically a $\mathbf{W} \rtimes \pi_0(\kappa)$-torsor. Then we may alternatively describe $\mathfrak{J}_{\mathfrak{M}}^1$ as the fixed-point scheme

$$\mathfrak{J}_{\mathfrak{M}}^1 = \pi_{\kappa *}(X_\kappa \times \bar{\mathbf{T}}_{\mathbf{M}} \times \mathbf{T})^{\mathbf{W} \rtimes \pi_0(\kappa)},$$

and $\mathfrak{J}_{\mathfrak{M}}$ can be identified with the subfunctor whose S-points for a $\mathfrak{C}_{\mathfrak{M}}$-scheme S consists of maps

$$f : S \times_{\mathfrak{C}_{\mathfrak{M}}} (X_\kappa \times \bar{\mathbf{T}}_{\mathbf{M}}) \longrightarrow \mathbf{T}$$

such that for any geometric point $x \in S \times_{\mathfrak{C}_{\mathfrak{M}}} (X_\kappa \times \bar{\mathbf{T}}_{\mathbf{M}})$, if $s_\alpha(x) = x$ for a root α, then $\alpha(f(x)) \neq -1$. We also have

$$\mathfrak{J}^1_{\mathfrak{M},H} = \pi_{\kappa *}(X_\kappa \times \bar{\mathbf{T}}_{\mathbf{M},\mathbf{H}} \times \mathbf{T})^{\mathbf{W_H} \rtimes \pi_0(\kappa)},$$

and a similar description for $\mathfrak{J}_{\mathfrak{M},H}$ using roots $\alpha \in \Phi_{\mathbf{H}}$. Using commutative diagram

$$\begin{array}{ccc} X_\kappa \times \bar{\mathbf{T}}_{\mathbf{M},\mathbf{H}} & \xrightarrow{\mathrm{id} \times \tilde{\nu}_{\mathbf{H}}} & X_\kappa \times \bar{\mathbf{T}}_{\mathbf{M}} \\ \downarrow & & \downarrow \\ \mathfrak{C}_{\mathfrak{M},H} & \xrightarrow{\nu_H} & \mathfrak{C}_{\mathfrak{M}} \end{array}$$

and the same argument in Lemma 2.5.20, we have:

Lemma 2.5.23. *There is a canonical commutative diagram over* $\mathfrak{C}_{\mathfrak{M},H}$

$$\begin{array}{ccc} \nu_H^* \mathfrak{J}_{\mathfrak{M}} & \longrightarrow & \mathfrak{J}_{\mathfrak{M},H} \\ \downarrow & & \downarrow \\ \nu_H^* \mathfrak{J}^1_{\mathfrak{M}} & \longrightarrow & \mathfrak{J}^1_{\mathfrak{M},H} \end{array}$$

such that all arrows are isomorphisms over $\mathfrak{C}^{G\text{-rs}}_{\mathfrak{M},H}$.

2.5.24 Finally, we have induced maps of quotient stacks

$$[\mathfrak{C}_{\mathfrak{M},H}/Z^\kappa_{\mathfrak{M}}] \longrightarrow [\mathfrak{C}'_{\mathfrak{M},H}/Z_{\mathfrak{M}}] \longrightarrow [\mathfrak{C}_{\mathfrak{M}}/Z_{\mathfrak{M}}],$$

where the first map is generically an isomorphism over the invertible locus. The discriminant divisor, the regular centralizer, etc. all descend to this quotient. However, even if one has a Steinberg quasi-section over $\mathfrak{C}_{\mathfrak{M},H}$, it may not descend to $[\mathfrak{C}_{\mathfrak{M},H}/Z^\kappa_{\mathfrak{M}}]$.

2.5.25 Let $\mathbf{S} \subset \mathbf{T}$ be a subtorus, then the centralizer $\mathbf{L}$ of $\mathbf{S}$ in $\mathbf{G}$ is a Levi-type subgroup containing maximal torus $\mathbf{T}$. Using the pinning **spl**, its dual $\check{\mathbf{L}}$ can be identified with a Levi-type subgroup in $\check{\mathbf{G}}$. Choosing a sufficiently general element γ in the center of $\check{L}$, we can realize $\check{\mathbf{L}}$ as the centralizer of γ in $\check{\mathbf{G}}$. Therefore similar to endoscopic group $\mathbf{H}$, we have for any monoid $\mathbf{M} \in \mathcal{FM}(\mathbf{G}^{\mathrm{sc}})$ a canonical associated monoid $\mathbf{M}_{\mathbf{L}} \in \mathcal{FM}(\mathbf{L}^{\mathrm{sc}})$ using the same construction. We also have the non-flat monoid $\mathbf{M}_{\mathbf{L}}'$, and canonical homomorphism of reductive monoids

$$\mathbf{M}_{\mathbf{L}} \longrightarrow \mathbf{M}_{\mathbf{L}}' \longrightarrow \mathbf{M},$$

where the second arrow is induced by inclusion $\mathbf{L} \subset \mathbf{G}$ (which is not necessarily present in endoscopic groups). Let $\mathbf{C}_{\mathbf{M},\mathbf{L}}$ (resp. $\mathbf{C}_{\mathbf{M},\mathbf{L}}'$) be the GIT quotient of $\mathbf{M}_{\mathbf{L}}$ (resp. $\mathbf{M}_{\mathbf{L}}'$) by $\mathbf{L}$, and let $\mathbf{Z}_{\mathbf{M}}^{\mathbf{L}}$ be the preimage of $\mathbf{Z}_{\mathbf{M}}$ in $\mathbf{M}_{\mathbf{L}}$, then we have canonical maps

$$\left[\mathbf{C}_{\mathbf{M},\mathbf{L}}/\mathbf{Z}_{\mathbf{M}}^{\mathbf{L}}\right] \longrightarrow \left[\mathbf{C}_{\mathbf{M},\mathbf{L}}'/\mathbf{Z}_{\mathbf{M}}\right] \longrightarrow [\mathbf{C}_{\mathbf{M}}/\mathbf{Z}_{\mathbf{M}}],$$

where the second map is finite and the first map is generically an isomorphism, and Lemma 2.5.17 holds (after replacing $\mathbf{H}$ by $\mathbf{L}$ and $\mathbf{Z}_{\mathbf{M}}^{\kappa}$ by $\mathbf{Z}_{\mathbf{M}}^{\mathbf{L}}$).

Let $\tilde{\mathbf{L}}$ be the centralizer of $\mathbf{S}$ in $\mathbf{G} \rtimes \mathrm{Out}(\mathbf{G})$, then its neutral component is $\mathbf{L}$, and its quotient group of connected component $\pi_0(\tilde{\mathbf{L}})$ is a subgroup of $\mathrm{Out}(\mathbf{G})$. Similarly we may replace $\mathrm{Out}(\mathbf{G})$ by a subgroup Θ therein, so that $\pi_0(\tilde{\mathbf{L}}) \subset \Theta$. Suppose G is a quasi-split form over X induced by a Θ-torsor $\vartheta\colon X_\vartheta \to X$, and $\mathfrak{M}$ is a very flat monoid in $\mathcal{FM}(G^{\mathrm{sc}})$. The GIT quotient $\mathfrak{M}/\!\!/\mathrm{Ad}(G)$ may be identified with the invariant quotient of $\bar{\mathbf{T}}_{\mathbf{M}} \times_X X_\vartheta$ by $\mathbf{W} \rtimes \Theta$. Suppose that ϑ is induced by a $\pi_0(\tilde{\mathbf{L}})$-torsor, then we have induced maps similar to endoscopic case

$$\left[\mathfrak{C}_{\mathfrak{M},L}/Z_{\mathfrak{M}}^{L}\right] \longrightarrow \left[\mathfrak{C}_{\mathfrak{M},L}'/Z_{\mathfrak{M}}\right] \longrightarrow [\mathfrak{C}_{\mathfrak{M}}/Z_{\mathfrak{M}}].$$

2.6 Fundamental Lemma for Spherical Hecke Algebras

Now we give the statement of fundamental lemmas for spherical Hecke algebras. Most cohomological statements used for describing κ-orbital integrals are extracted from [Ngô10, §§ 1.5–1.7] for reader's convenience and are kept brief. Readers should refer to *loc. cit.* for proofs or references.

2.6.1 Let $F_v = k((\pi_v))$ be the field of Laurent series over k and $\mathcal{O}_v = k[[\pi_v]]$ its ring of integers. Let G be a quasi-split twist of $\mathbf{G}$, and $\gamma \in G(F_v)$ be a semisimple element. Since the derived subgroup of G is not necessarily simply-connected, the centralizer I_γ of γ in G may not be a torus even if γ is regular semisimple (regular means that I_γ achieves minimal dimension). For example, the diagonal matrix with entries 1 and -1 in PGL_2 is one such element. However, the collection of those γ whose centralizer is a torus is still dense in G and those elements are usually referred to as *strongly regular semisimple*. So we will assume γ is strongly regular semisimple.

We still have natural map $T/\!\!/W \to G/\!\!/G$, and the cameral cover $T \to T/\!\!/W$ is a W-torsor over any strongly regular semisimple element. Conversely, any point in T at which the map $T \to T/\!\!/W$ is étale is clearly strongly regular semisimple. Therefore the strongly regular semisimple locus is open in G, and its image in $G/\!\!/G$ is also open. In addition, the map $T/\!\!/W \to G/\!\!/G$ is an isomorphism on strongly regular semisimple locus. We denote the resulting variety by $\mathfrak{C}_G^{\mathrm{srs}}$, and let $a \in \mathfrak{C}_G^{\mathrm{srs}}(F_v)$ be the image of γ.

If $\gamma' \in G(F_v)$ is another F_v-point over a, then there exists some $g \in G(\bar{F}_v)$ such that $\gamma' = g\gamma g^{-1}$. Therefore for any $\sigma \in \Gamma_v = \mathrm{Gal}(\bar{F}_v/F_v)$, we have $g\sigma(g)^{-1} \in I_\gamma(\bar{F}_v)$, hence a cohomology class

$$\mathrm{inv}(\gamma, \gamma') \in \mathrm{H}^1(F_v, I_\gamma),$$

which depends only on the $G(F_v)$-conjugacy class of γ' but not on g. Its image in

$\mathrm{H}^1(F_v, G)$ is trivial, and in this way we have a bijection between the kernel of the map

$$\mathrm{H}^1(F_v, I_\gamma) \longrightarrow \mathrm{H}^1(F_v, G),$$

and the set of F-conjugacy classes inside the $\bar{F}_v$-conjugacy class of γ.

2.6.2 The quasi-split twist G over F_v corresponds to a $\mathrm{Out}(\mathbf{G})$-torsor over F_v, which can be identified with a Galois representation $\vartheta_G^\bullet \colon \Gamma_v \to \mathrm{Out}(\mathbf{G})$ after fixing a separable closure F_v^{s} of F_v. By Tate-Nakayama duality, we have

$$\mathrm{H}^1(F_v, G)^* \simeq \pi_0\big((\mathbf{Z}_{\check{\mathbf{G}}})^{\vartheta_G^\bullet(\Gamma_v)}\big),$$

where the left-hand side is the Pontryagin dual of $\mathrm{H}^1(F_v, G)$.

Choose a geometric point $x_a \in \mathbf{T}(F_v^{\mathrm{s}})$ over a, then it corresponds to a lifting of $\vartheta_G^\bullet$ to a homomorphism

$$\pi_a^\bullet \colon \Gamma_v \longrightarrow \mathbf{W} \rtimes \mathrm{Out}(\mathbf{G}).$$

Although G is not an object in $\mathcal{FM}(G^{\mathrm{sc}})$, the regular centralizer $\mathfrak{J}_a$ can still be defined for strongly regular semisimple elements and its Galois description still hold using the same proof because $T^{\mathrm{srs}} \to \mathfrak{C}_G^{\mathrm{srs}}$ is étale. Thus we have isomorphisms over F_v

$$I_\gamma \simeq \mathfrak{J}_a \simeq \operatorname{Spec} F_v^{\mathrm{s}} \wedge^{\Gamma_v, \pi_a^\bullet} \mathbf{T}.$$

Using Tate-Nakayama duality again, we have

$$\mathrm{H}^1(F_v, \mathfrak{J}_a)^* \simeq \pi_0(\check{\mathbf{T}}^{\pi_a^\bullet(\Gamma_v)}).$$

Note that here the isomorphism depends on the choice of x_a. The inclusion $\iota \colon \check{\mathbf{T}} \to \check{\mathbf{G}}$

is Γ_v-equivariant up to conjugacy in the following sense: for any $t \in \check{\mathbf{T}}$ and any $\sigma \in \Gamma_v$, we always have that $\vartheta^{\bullet}_{G}(\sigma)(\iota(t))$ and $\iota(\pi^{\bullet}_{a}(\sigma)(t))$ are $\check{\mathbf{G}}$-conjugate. Therefore we have an induced map

$$\pi_0\big((\mathbf{Z}_{\check{\mathbf{G}}})^{\vartheta^{\bullet}_{G}(\Gamma_v)}\big) \longrightarrow \pi_0\big(\check{\mathbf{T}}^{\pi^{\bullet}_{a}(\Gamma_v)}\big),$$

whose Pontryagin dual is the map $\mathrm{H}^1(F_v, I_\gamma) \longrightarrow \mathrm{H}^1(F_v, G)$.

2.6.3 We fix a Haar measure $\mathrm{d}g_v$ on $G(F_v)$ such that $G(\mathcal{O}_v)$ has volume 1, as well as a non-zero Haar measure $\mathrm{d}t_v$ on $\mathfrak{J}_a(F_v)$. Using isomorphism $I_\gamma \simeq \mathfrak{J}_a$ for $\gamma \in G(F_v)$ lying over a, we have an induced Haar measure on $I_\gamma(F_v)$.

With the above choices, and any locally constant and compactly supported function f on $G(F_v)$, we may define orbital integral

$$\mathbf{O}_\gamma(f, \mathrm{d}t_v) = \int_{I_\gamma(F_v)\backslash G(F_v)} f(g_v^{-1}\gamma g_v)\frac{\mathrm{d}g_v}{\mathrm{d}t_v}.$$

Definition 2.6.4. Let $\kappa \in \check{\mathbf{T}}^{\pi^{\bullet}_{a}(\Gamma_v)}$. Then we define the *$\kappa$-orbital integral* of a as the sum

$$\mathbf{O}^{\kappa}_{a}(f, \mathrm{d}t_v) = \sum_{\gamma'}\langle \mathrm{inv}(\gamma, \gamma'), \kappa\rangle\, \mathbf{O}_{\gamma}(f, \mathrm{d}t_v),$$

where γ' ranges over all $G(F_v)$-conjugacy classes over a, and γ is a fixed choice. When $\kappa = 1$, we denote $\mathbf{O}^{\kappa}_{a}$ by $\mathbf{SO}_a$, called the *stable orbital integral*.

Note that κ-orbital integral is sensitive to the choice of base point γ if $\kappa \neq 1$, while the stable orbital integral is not. Both also depend on a choice of the geometric point x_a.

2.6.5 Let $(\kappa, \vartheta_\kappa)$ be an endoscopic datum of G and H is the endoscopic group. The canonical homomorphism in Lemma 2.5.5 induces map

$$\vartheta_\kappa \times \mathbf{T}^{\mathrm{srs}} /\!\!/ \mathbf{W_H} \rtimes \pi_0(\kappa) \longrightarrow \vartheta_\kappa \times \mathbf{T}^{\mathrm{srs}} /\!\!/ \mathbf{W} \rtimes \pi_0(\kappa),$$

where the right-hand side is exactly $\mathfrak{C}_G^{\mathrm{srs}}$, and we denote left-hand side by $\mathfrak{C}_H^{G\text{-srs}}$, called the strongly G-regular semisimple locus. It is easy to see that it is an open subset of $\mathfrak{C}_H^{\mathrm{srs}}$. Note that since we only consider the strongly regular semisimple locus, both GIT quotients can be identified with the stack quotient as the actions are free. We say a conjugacy class $a \in \mathfrak{C}_G^{\mathrm{srs}}(F_v)$ and $a_H \in \mathfrak{C}_H^{G\text{-srs}}(F_v)$ match each other if a is the image of a_H. In this case, we have a canonical isomorphism $\mathfrak{J}_a \simeq \mathfrak{J}_{H,a_H}$.

Since κ-orbital integrals depend on the choice of geometric points x_a and x_{a_H}, we shall choose the same point in $\mathbf{T}$ for both. It also depends on the choice of base point γ over a (we do not need to worry about a_H, for which we will only consider stable orbital integrals), which we shall choose as follows in the special case when a Steinberg quasi-section exists for G^{sc}: let G_1 be a fixed z-extension of G, then $G_1 \in \mathcal{FM}(G^{\mathrm{sc}})$. Since a Steinberg quasi-section exists for G^{sc}, it also exists for G_1 and we fix one once and for all if there are multiple. The map $G_1(F_v) \to G(F_v)$ is surjective because G_1 is a z-extension, so $\gamma \in G(F_v)$ is chosen to be the image of any element in $G_1(F_v)$ lying over a that is contained in the image of the Steinberg quasi-section. Clearly, such choice is well-defined and does not depend on the choice of z-extension itself.

2.6.6 Now we make a digression and discuss the IC-functions. For split group $\mathbf{G}$, we may consider its affine Grassmannian defined as $\mathsf{Gr}_{\mathbf{G}} = \mathbb{L}\mathbf{G}/\mathbb{L}^+\mathbf{G}$ (see § 4.1 for details), whose k-points is the quotient set $\mathbf{G}(F_v)/\mathbf{G}(\mathcal{O}_v)$. It is an ind-projective ind-scheme over k. The arc group $\mathbb{L}^+\mathbf{G}$ acts naturally on $\mathsf{Gr}_{\mathbf{G}}$, and its orbits are given by the Cartan decomposition

$$\mathbf{G}(F_v) = \coprod_{[\lambda]} \mathbf{G}(\mathcal{O}_v)\pi_v^{\lambda}\mathbf{G}(\mathcal{O}_v),$$

where $[\lambda]$ ranges over the $\mathbf{W}$-orbits in $\check{\mathbb{X}}(\mathbf{T})$, and we represent each $\mathbf{W}$-orbit $[\lambda]$ by the unique $\mathbf{B}$-dominant coweight within, denoted by λ. The orbits, denoted by $\mathsf{Gr}_{\mathbf{G}}^{\lambda}$ for each λ, form a stratification of $\mathsf{Gr}_{\mathbf{G}}$ such that $\mathsf{Gr}_{\mathbf{G}}^{\mu}$ is contained in the closure of $\mathsf{Gr}_{\mathbf{G}}^{\lambda}$ if and only if $\lambda-\mu$ is an $\mathbb{N}$-combination of simple coroots (denoted by $\mu \leq \lambda$). Let $\mathsf{Gr}_{\mathbf{G}}^{\leq\lambda}$ be such closure.

We may consider the standard intersection complex IC^{λ} with $\overline{\mathbb{Q}}_{\ell}$-coefficients ($\ell$ is coprime to p) on each $\mathsf{Gr}_{\mathbf{G}}^{\leq\lambda}$. By geometric Satake isomorphism, these complexes are exactly the simple objects in the category of $\mathbb{L}^+\mathbf{G}$-equivariant constructible perverse sheaves on $\mathsf{Gr}_{\mathbf{G}}$, and the latter is equivalent to the Tannakian category of finite dimensional representations of $\check{\mathbf{G}}$. Under such equivalence, IC^{λ} corresponds to the irreducible representation of $\check{\mathbf{G}}$ of highest weight λ.

On the other hand, by Grothendieck's sheaf-function dictionary, each IC^{λ} induces a function f^{λ} on the set of k-points of $\mathsf{Gr}_{\mathbf{G}}$, hence also a $\mathbf{G}(\mathcal{O}_v)$-bi-invariant function on $\mathbf{G}(F_v)$, still denoted by f^{λ}. By induction, we can easily see that the collection of f^{λ} form a different basis of the spherical algebra other than the characteristic functions on each $\mathbf{G}(\mathcal{O}_v)\pi_v^{\lambda}\mathbf{G}(\mathcal{O}_v)$. In other words, we have

$$\mathcal{H}_{\mathbf{G},0} = \bigoplus_{\lambda} \overline{\mathbb{Q}}_{\ell} \cdot f^{\lambda},$$

where $\mathcal{H}_{\mathbf{G},0}$ denotes the spherical Hecke algebra of $\mathbf{G}$, i.e., $\mathbf{G}(\mathcal{O}_v)$-bi-invariant locally constant and compactly supported functions on $\mathbf{G}(F_v)$.

Note that each function f^{λ} makes sense as a function on $\mathbf{G}^{\mathrm{ad}}(F_v)$ as well because the natural map $\mathbf{G} \to \mathbf{G}^{\mathrm{ad}}$ maps each double coset $\mathbf{G}(\mathcal{O}_v)\pi_v^{\lambda}\mathbf{G}(\mathcal{O}_v)$ to $\mathbf{G}^{\mathrm{ad}}(\mathcal{O}_v)\pi_v^{\lambda_{\mathrm{ad}}}\mathbf{G}^{\mathrm{ad}}(\mathcal{O}_v)$ (λ_{ad} is the image of λ in $\check{\mathbb{X}}(\mathbf{T}^{\mathrm{ad}})$), and f^{λ} is constant on those double cosets.

2.6.7 When G is a quasi-split twist of $\mathbf{G}$ over $\mathcal{O}_v$, the action of Γ_v on $\check{\mathbb{X}}(T)$ factors through $\mathrm{Gal}(\bar{k}/k)$. Let $\sigma_v \in \mathrm{Gal}(\bar{k}/k)$ be the geometric Frobenius element and also choose a lifting to Γ_v. In this case, IC^λ may not be defined on Gr_G over k, but the direct sum $\bigoplus_{\mu\in\mathrm{Gal}(\bar{k}/k)\cdot\lambda}\mathrm{IC}^\mu$ descends to Gr_G over k, hence it induces a function on $G(F_v)$, which we will denote by

$$\sum_{\mu\in\mathrm{Gal}(\bar{k}/k)\cdot\lambda} f^\mu.$$

These functions generate $\mathcal{H}_{G,0}$ over $\overline{\mathbb{Q}}_\ell$.

Now given a F_v-rational dominant cocharacter λ of G, let V_λ be the corresponding $\check{G}$-representation. It extends to a representation of $\check{G}\rtimes\Gamma_v$, whose restriction to $\check{H}$ decomposes into a bunch of irreducible ones, denoted by $V'_{\lambda_{H,i}}$ ($1 \le i \le e$ for some e). Let $[\lambda_{H,i}]$ be the $\mathbf{W_H}$-orbit of $\lambda_{H,i}$, then the Galois group Γ_v acts on the set of all $[\lambda_{H,i}]$. Therefore the function

$$\sum_{i=1}^{e} f_H^{\lambda_{H,i}}$$

makes sense as an element of $\mathcal{H}_{H,0}$. Note that any function above makes sense as a function on $G^{\mathrm{ad}}(F_v)$ (resp. $H^{\mathrm{ad}}(F_v)$) for the same reason as in the split case.

2.6.8 Now we are ready to state the fundamental lemma.

Conjecture 2.6.9. *We have equality*

$$\mathbf{O}_a^\kappa(f^\lambda, \mathrm{d}t_v) = \Delta(\gamma_H, \gamma)\,\mathbf{SO}_{a_H}\left(\sum_{i=1}^{e} f_H^{\lambda_{H,i}}, \mathrm{d}t_v\right),$$

where $\Delta(\gamma_H,\gamma)$ is a number (transfer factor) depending only on the F_v-conjugacy classes of γ and γ_H.

Remark 2.6.10. (1) Conjecture 2.6.9 is not an actual conjecture by itself since it has been proved by a combination of a series of reduction works most notably by Waldspurger (especially [Wal97, Wal08]) and Kottwitz ([Kot99]), and the proof in Lie algebra case by Ngô ([Ngô10]).

(2) However, it has not been proved directly at group level, so it remains a conjecture for the purpose of this paper.

(3) The transfer factor is defined by Langlands and Shelstad in [LS87] and is extremely complicated, so we do not attempt to give the definition here.

We are able to prove the above conjecture in the following special case, except that we have not checked that the "transfer factor" obtained in this article is the same as the one defined by [LS87].

Theorem 2.6.11. *Suppose $G = G^{\mathrm{ad}}$ and $(\kappa, \vartheta_\kappa)$ is elliptic, then we have*

$$q^{-d(a)/2}\,\mathbf{O}_a^\kappa(f^\lambda, \mathrm{d}t_v) = q^{-d_H(a_H)/2}\,\mathbf{SO}_{a_H}\left(\sum_{i=1}^{e} f_H^{\lambda_{H,i}}, \mathrm{d}t_v\right),$$

where d and d_H are the F_v-valuations of the (non-extended) discriminant functions (see § 2.2) of G and H respectively.

The theorem will be proved in § 10.5 using multiplicative Hitchin fibrations (mH-fibrations), where we will also note what remains to be done in order to extend the result to arbitrary groups.

2.6.12 It will be convenient to make some preliminary reductions so that our statement of fundamental lemma plays better with reductive monoids. First of all, observing the definition of $\mathbf{O}_\gamma$, we see that there is no need to treat the group G doing the action and the *G-space G* as the same object. Indeed, let γ_{ad} be the image of γ in $G^{\mathrm{ad}}(F_v)$ and f_{ad} be a function on $G^{\mathrm{ad}}(F_v)$, then we can define $\mathbf{O}_{\gamma_{\mathrm{ad}},G}$ as follows:

$$\mathbf{O}_{\gamma_{\mathrm{ad}},G}(f_{\mathrm{ad}}, \mathrm{d}t_v) = \int_{I_\gamma(F_v)\backslash G(F_v)} f(g_v^{-1}\gamma_{\mathrm{ad}} g_v)\frac{\mathrm{d}g_v}{\mathrm{d}t_v}.$$

The important thing to note is that g_v still ranges over $G(F_v)$, not $G^{\mathrm{ad}}(F_v)$. Since $\mathsf{Gr}_G \to \mathsf{Gr}_{G^{\mathrm{ad}}}$ induces homeomorphism on each geometric connected component of Gr_G, pull-back of IC-functions on $G^{\mathrm{ad}}(F_v)$ will be a (potentially infinite) sum of IC-functions on Gr_G with disjoint supports, by function-sheaf dictionary (and that G-torsors over $\mathcal{O}_v$ are necessarily trivial since k is finite and G is connected).

Let $F_{\bar{v}} = F_v \hat{\otimes}_k \bar{k}$ and $\mathcal{O}_{\bar{v}}$ its ring of integers. Then G is split over $\mathcal{O}_{\bar{v}}$. We shall see later (c.f. Proposition 4.1.7) that for a given Cartan double coset $G^{\mathrm{ad}}(\mathcal{O}_{\bar{v}})\pi^{\lambda_{\mathrm{ad}}}G^{\mathrm{ad}}(\mathcal{O}_{\bar{v}})$ in $G^{\mathrm{ad}}(F_{\bar{v}})$ that has non-trivial intersection with $\mathrm{Ad}_G(\gamma_{\mathrm{ad}})$, there is a unique λ in the preimage of λ_{ad} such that $G(\mathcal{O}_{\bar{v}})\pi^{\lambda}G(\mathcal{O}_{\bar{v}})$ and $\mathrm{Ad}_G(\gamma)$ has non-trivial intersection. Moreover, if γ_{ad} is the image of γ and λ_{ad} is F_v-rational, then so is λ. This shows that

$$\mathbf{O}_{\gamma_{\mathrm{ad}},G}(f_{\mathrm{ad}}^{\lambda_{\mathrm{ad}}}, \mathrm{d}t_v) = \mathbf{O}_{\gamma}(f^{\lambda}, \mathrm{d}t_v),$$

and similarly for any sum of IC-functions in a Galois orbit.

2.6.13 The modified κ-orbital will be defined as

$$\mathbf{O}_{a,\mathrm{ad}}^{\kappa}(f_{\mathrm{ad}}, \mathrm{d}t_v) = \sum_{\gamma'} \langle \mathrm{inv}(\gamma, \gamma'), \kappa \rangle \, \mathbf{O}_{\gamma_{\mathrm{ad}},G}(f_{\mathrm{ad}}, \mathrm{d}t_v),$$

so we may reduce the fundamental lemma to the following equality:

$$\mathbf{O}_{a,\mathrm{ad}}^{\kappa}(f_{\mathrm{ad}}^{\lambda_{\mathrm{ad}}}, \mathrm{d}t_v) = \Delta(\gamma_H, \gamma) \, \mathbf{SO}_{a_H,\mathrm{ad}}\left(\sum_{i=1}^{e} f_{H^{\mathrm{ad}}}^{\lambda_{H,i,\mathrm{ad}}}, \mathrm{d}t_v\right).$$

If $G_1 \to G^{\mathrm{ad}}$ is any central extension, and if there exists some $\gamma_1 \in G_1(F_v)$ lying over γ_{ad}, then we may replace γ_{ad} by γ_1 and $f_{\mathrm{ad}}^{\lambda_{\mathrm{ad}}}$ by its pullback to $G_1(F_v)$, and obtain a similar equality.

As a special case, if $G_1 = \mathfrak{M}^{\times}$ for some reductive monoid $\mathfrak{M} \in \mathcal{FM}(G^{\mathrm{sc}})$, it makes sense to talk about orbital integrals of invertible F_v-points of $\mathfrak{M}$ whose image in G^{ad} is

γ_{ad}. In fact, we do not have to only consider those elements lying over γ_{ad}, because the definition makes sense for any elements in $\mathfrak{M}^{\times,\mathrm{rs}}(F_v)$.

Finally, to simplify notations, since we only care about the evaluation of orbital integrals at IC-functions, not as distributions on the group, we will freely switch between $\mathbf{O}_\gamma$ and $\mathbf{O}_{\gamma_{\mathrm{ad}},G}$ and so on.

CHAPTER 3

MULTIPLICATIVE VALUATION STRATA

In this section we study the *valuation strata* in the multiplicative setting analogous to those in [GKM09] in the Lie algebra setting. The main result of this section is a codimension formula for the valuation strata, which will later become a key ingredient for the application of support theorem to multiplicative Hitchin fibrations.

3.1 Arc Spaces of Tori and Congruent Subgroups

3.1.1 Let $F = \bar{k}((\pi))$ and $\mathcal{O} = \bar{k}[[\pi]]$ be the rings of Laurent series and power series over $\bar{k}$ respectively. Let val_F be the normalized valuation on F such that $\mathrm{val}_F(\pi) = 1$. Let F_∞ be the maximal tamely ramified extension of F inside a fixed algebraic closure $\bar{F}$. For each $l \geq 1$ not divisible by p, we choose $\pi^{1/l} \in F_\infty$ and a primitive root of unity ζ_l in such a compatible way that $(\pi^{1/(lm)})^m = \pi^{1/l}$ and $\zeta_{lm}^m = \zeta_l$. This way one obtains a unique element $\tau_\infty \in \mathrm{Gal}(F_\infty/F)$ such that $\tau_\infty(\pi^{1/l}) = \zeta_l \pi^{1/l}$. Clearly τ_∞ is a topological generator of $\mathrm{Gal}(F_\infty/F)$. Let $F_l = F[\pi^{1/l}]$ and τ_l be the image of τ_∞ in $\mathrm{Gal}(F_l/F)$.

3.1.2 For $m \in \mathbb{N}$, we let $\mathcal{J}_m = \bar{k}[\pi]/(\pi^{m+1})$ be the ring of m-jets over $\bar{k}$. For a scheme X over F, let $\mathbb{L}X$ be the *loop space* of X. In other words, for any $\bar{k}$-algebra R, $\mathbb{L}X(R) = X(R \otimes_{\bar{k}} F)$. If X is defined over $\mathcal{O}$, let $\mathbb{L}_m^+X$ be the *m-jet space* of X so that $\mathbb{L}_m^+X(R) = X(R \otimes_{\bar{k}} \mathcal{J}_m)$, and $\mathbb{L}^+X := \varprojlim \mathbb{L}_m^+X$ be the *arc space* of X. If X is a $\bar{k}$-scheme, we let $\mathbb{L}X := \mathbb{L}X_F$, $\mathbb{L}_m^+X := \mathbb{L}_m^+X_{\mathcal{O}}$, and $\mathbb{L}^+X := \mathbb{L}^+X_{\mathcal{O}}$.

3.1.3 Let T be a torus over $\bar{k}$ and $\mathfrak{h}$ its Lie algebra. Let T_n be the congruent subgroup

$$T_n := \ker\left(\mathbb{L}^+T \to \mathbb{L}_n^+T\right).$$

Since we are primarily interested in the topological properties, we use T_n for both this group scheme and its $\bar{k}$-points.

We have canonical isomorphisms

$$T(\mathcal{O}) \simeq T(\bar{k}) \times T_0,$$
$$T_n/T_{n+1} \simeq \ker\left[T(\mathcal{J}_{n+1}) \to T(\mathcal{J}_n)\right],$$

hence

$$T_n/T_{n+1} \simeq \left\{\phi \in \mathrm{Hom}_{\bar{k}\text{-Alg}}(\bar{k}[T], \mathcal{J}_{n+1})) \,\middle|\, \phi(\lambda) \equiv 1 \bmod \pi^{n+1}, \forall \lambda \in \mathbb{X}(T)\right\}.$$

Therefore for any $\phi \in T_n/T_{n+1}$ and $\lambda \in \mathbb{X}(T)$, $\phi(\lambda) = 1 + a(\lambda)\pi^{n+2}$, where $a \in \mathrm{Hom}_{\mathbb{Z}}(\mathbb{X}(T), \bar{k}) \simeq \mathfrak{h}$. Thus $T_n/T_{n+1} \simeq \mathfrak{h}$ for all $n \geq 0$, such that for any $\lambda \in \mathbb{X}(T)$, $\lambda(t) - 1 \equiv \mathrm{d}\lambda(\bar{t})\pi^{n+1} \bmod \pi^{n+2}$ for all $t \in T_n$ and its image $\bar{t} \in \mathfrak{h}$ under this isomorphism.

3.1.4 Let $l \in \mathbb{Z}_+$ be coprime to p, and suppose A is a cyclic group of order l acting on $\mathbb{L}^+T$ compatible with filtrations T_n and canonical map $T \to \mathbb{L}^+T$. Note then $T_n/T_{n+1} \simeq \mathfrak{h}$ is a $\bar{k}$-linear representation of A. The following lemma is necessary when we later consider some twisted forms of $T(\mathcal{O})$.

Lemma 3.1.5. *The induced maps on fixed points $T_n^A \to (T_n/T_{n+1})^A$ and $T(\mathcal{O})^A \to T(\bar{k})^A$ are surjective.*

Proof. The claim about $T(\mathcal{O}) \to T(\bar{k})$ is trivial as the projection splits. View T_n/T_{n+1} as $\bar{k}$-vector space $\mathfrak{h}$. Let $\sigma \in A$ be a generator. Since l is invertible in $\bar{k}$, we know the $\bar{k}$-linear map

$$\begin{aligned} T_n/T_{n+1} &\longrightarrow (T_n/T_{n+1})^A \\ x &\longmapsto \mathrm{Nm}_\sigma(x) := x + \sigma(x) + \cdots + \sigma^{l-1}(x) \end{aligned}$$

is surjective. Lift x to T_n, denoted by t, then $\mathrm{Nm}_\sigma(t) \coloneqq t\sigma(t)\cdots\sigma^{l-1}(t)$ is A-invariant and maps to $\mathrm{Nm}_\sigma(x)$. ∎

3.2 Root Valuation Functions and Filtrations

3.2.1 Let $t \in T(F_\infty)$, and $\lambda \in \mathbb{X}(T)$. Define

$$r_t(\lambda) = \mathrm{val}_F(1 - \lambda(t)).$$

Then if $t \in T(\mathcal{O}_\infty)$, $r_t(\lambda+\mu) \geq \min\{r_t(\lambda), r_t(\mu)\}$ and reaches equality if $r_t(\lambda) \neq r_t(\mu)$. Note that r_t can take ∞ as value.

Using the fixed system of uniformizers $\pi^{1/l}$, we can decompose $T(F_\infty)$ as

$$T(F_\infty) \cong T(\mathcal{O}_\infty) \times \check{\mathbb{X}}(T)_{\mathbb{Z}_{(p)}} \hookrightarrow T(\mathcal{O}_\infty) \times \check{\mathbb{X}}(T)_{\mathbb{Q}}.$$

Then one can uniquely write $t \in T(F_\infty)$ as a product $t_0\pi^{\lambda/l}$ for some $t_0 \in T(\mathcal{O}_\infty)$, $\lambda \in \check{\mathbb{X}}(T)$, and positive integer l coprime to p.

3.2.2 Now suppose G is a connected reductive group over $\bar{k}$ and T is a maximal torus of G. Let $t \in T(F_\infty)$, then r_t induces a function on Φ by restriction, still denoted by r_t, called the *root valuation function* induced by t. We can also define the *discriminant valuation* of t by

$$d(t) = \sum_{\alpha\in\Phi} r_t(\alpha) \in \mathbb{Q} \cup \{\infty\},$$

which is finite if and only if t is regular semisimple. Note that $d(t)$ is none other than the F-valuation of the discriminant $\mathrm{Disc}(t)$ introduced in § 2.2.

Write $t = t_0\pi^{\lambda/l}$, then the rational cocharacter $\lambda_{\mathrm{ad}}/l \in \check{\mathbb{X}}(T^{\mathrm{ad}})_{\mathbb{Q}}$ can be recovered from r_t as follows. Observe that $r_t(\alpha) = r_t(-\alpha)$ if and only if $\langle\alpha,\lambda\rangle = 0$, and otherwise

$\{r_t(\alpha), r_t(-\alpha)\} = \{0, s\}$ for some $s < 0$. Define a new function

$$r_t^-(\alpha) = \begin{cases} r_t(\alpha) & r_t(\alpha) < 0, \\ 0 & r_t(\alpha) = r_t(-\alpha), \\ -r_t(-\alpha) & r_t(-\alpha) < 0. \end{cases}$$

Then r_t^- extends to a homomorphism $\mathbb{X}(T)_{\mathrm{ad},\mathbb{Q}} \to \mathbb{Q}$. Since any set of simple roots form a basis of $\mathbb{X}(T)_{\mathrm{ad},\mathbb{Q}}$, this extension to a homomorphism is unique. Thus λ_{ad}/l is recovered from r_t^-. As a result, let $\bar{\lambda}$ be the image of λ in $\check{\mathbb{X}}(G/G^{\mathrm{der}})$, then the pairs $(r_t, \lambda/l)$ and $(r_t, \bar{\lambda}/l)$ carry the same amount of information about t.

For later convenience, we also define a modified form $r_t^\star$ of r_t by

$$r_t^\star(\alpha) = \min\{r_t(\alpha), r_t(-\alpha)\}.$$

Note that one can recover r_t from $r_t^\star$ and r_t^-. In fact, we can define r^- and $r^\star$ for any function r on Φ such that either $0 \le r(\alpha) = r(-\alpha) \le \infty$ or $\{r(\alpha), r(-\alpha)\} = \{0, s\}$ for some $s < 0$.

3.2.3 To study the pair $(r_t, \bar{\lambda}/l)$ incurred by $t \in T(F_\infty)$ in general, we first study those $t \in T(\mathcal{O})$, in which case r_t takes values in $\mathbb{N} \cup \{\infty\}$ and λ-part is trivial. For any root system Φ and any functions $r\colon \Phi \to \mathbb{N} \cup \{\infty\}$, we may define filtration (of subsets) of Φ

$$\Phi_n(r) \coloneqq \{\alpha \in \Phi \mid r(\alpha) \ge n\},$$

where $n \in \mathbb{N} \cup \{\infty\}$, and we will simply use Φ_n if the function r is clear from the context. Clearly this filtration stabilizes after finite steps and

$$\bigcap_{0 \le n < \infty} \Phi_n = \Phi_\infty.$$

To study this filtration for those $r = r_t$, we first record two theorems concerning root subsystems of Φ.

Theorem 3.2.4 (Slodowy's criterion)**.** *A root subsystem Φ' of Φ comes from a Levi-type subgroup if and only if Φ' is $\mathbb{Q}$-closed in Φ, in other words, $\mathbb{Q}\Phi' \cap \Phi = \Phi'$.*

Theorem 3.2.5 (Deriziotis' criterion)**.** *A root subsystem Φ' of Φ comes from a pseudo-Levi-type subgroup (i.e. the connected centralizer of a semisimple element of G) if and only if after conjugation by W, Φ' has a basis which is a* proper *subset of the simple* affine *roots of Φ, in other words, simple roots together with negative of the highest root.*

Lemma 3.2.6. *Let $t \in T_0$, then $\Phi_n(r_t)$ is $\mathbb{Q}$-closed.*

Proof. Clearly Φ_n is $\mathbb{Z}$-closed for each n. Following the proof in [GKM09, 14.1.1], if $\alpha \in \Phi$ is a $\mathbb{Q}$-combination of roots in Φ_n, then $d\alpha \in \mathbb{Z}\Phi_n$ for some d coprime to p (here only the fact that Φ_n is $\mathbb{Z}$-closed is needed). Thus $r_t(d\alpha) \geq n$. But since $t \in T_0$ and d is coprime to p, we must have $r_t(\alpha) \geq n$ as well, hence the lemma. ∎

The following corollary is not used in subsequent parts of this paper, but highlights the subtle difference between the multiplicative and additive cases (cf. [GKM09, 3.4.1]).

Corollary 3.2.7. *There is a Zariski-dense open subset of $T(\bar{k})$ such that the conclusion in Lemma 3.2.6 holds for its preimage in $T(\mathcal{O})$.*

Proof. The collection of possible d (chosen to be as small as possible each time) appearing in the proof of Lemma 3.2.6 is a finite set S. Let $U \subset T(\bar{k})$ be the complement of those $t \in T(\bar{k})$ such that $1 \neq \alpha(t) \in \mu_d$ (μ_d is the set of d-th roots of unity) for some $\alpha \in \Phi$ and some $d \in S$. Then its clear that the same proof goes through for elements in the preimage of U. ∎

Remark 3.2.8. It is not true even in characteristic 0 that Lemma 3.2.6 holds for all $t \in T(\mathcal{O})$. For example, suppose G is of type B_2 or C_2, and let t be an element such that

$\alpha(t) = -1 + \pi$ for both positive short root α, then Φ_1 for r_t is exactly the set of all long roots, which is not $\mathbb{Q}$-closed.

Proposition 3.2.9. *Let $t \in T(\mathcal{O})$. Then $\Phi_1(r_t)$ is a root subsystem of pseudo-Levi type, and Φ_n is $\mathbb{Q}$-closed in Φ_1 for all $n \geq 1$.*

Proof. Write $t = xt_0$, where $x \in T(\bar{k})$ and $t_0 \in T_0$. Then

$$\Phi_1(r_t) = \{\alpha \in \Phi \mid \alpha(x) = 1\},$$

which is the same as the roots of the connected centralizer of x in G, hence the first claim.

For the second claim, note that for $n \geq 1$, we have that

$$\Phi_n(r_t) = \Phi_n(r_{t_0}) \cap \Phi_1(r_t) \subset \Phi_n(r_{t_0}).$$

Since $\Phi_n(r_{t_0})$ is $\mathbb{Q}$-closed in Φ by Lemma 3.2.6, we have that

$$\begin{aligned}
\Phi_n(r_t) &\subset \mathbb{Q}\Phi_n(r_t) \cap \Phi_1(r_t) \\
&\subset \mathbb{Q}\Phi_n(r_{t_0}) \cap \Phi_1(r_t) \\
&= \mathbb{Q}\Phi_n(r_{t_0}) \cap [\Phi \cap \Phi_1(r_t)] \\
&= [\mathbb{Q}\Phi_n(r_{t_0}) \cap \Phi] \cap \Phi_1(r_t) \\
&= \Phi_n(r_{t_0}) \cap \Phi_1(r_t) \\
&= \Phi_n(r_t).
\end{aligned}$$

Therefore every inclusion is an equality, and in particular $\Phi_n(r_t) = \mathbb{Q}\Phi_n(r_t) \cap \Phi_1(r_t)$, as claimed. ∎

3.2.10 Let $M \subset G$ be a connected reductive subgroup of Levi type containing T, and $\Phi_M \subset \Phi$ be its root subsystem. Then Φ_M is $\mathbb{Q}$-closed by Slodowy's criterion.

Lemma 3.2.11. *For any* $0 \le n < \infty$*, we can find* $t_n \in T_n$ *such that*

$$r_{t_n}(\alpha) = \begin{cases} n+1 & \alpha \in \Phi \setminus \Phi_M, \\ \infty & \alpha \in \Phi_M. \end{cases}$$

Proof. Let Z_M be the *connected* center of M. Let $X \in \mathfrak{z}_M \coloneqq \mathrm{Lie}(Z_M) \simeq (Z_M)_n/(Z_M)_{n+1}$ be an element that $\alpha(X) \neq 0$ for all $\alpha \in \Phi \setminus \Phi_M$. This X exists since $\mathfrak{z}_M$ is exactly the intersection of kernels of $\alpha \in \Phi_M$ and no other root vanishes identically on $\mathfrak{z}_M$ (cf. [GKM09, 14.2]). Lift X to $t_n \in (Z_M)_n$, and we are done. ∎

Lemma 3.2.12. *Suppose we have a function* $r : \Phi \to \mathbb{Z}_+ \cup \{\infty\}$ *and the associated filtration* Φ_n*. If* Φ_n *is a* $\mathbb{Q}$*-closed root subsystem in* Φ *for each* $1 \le n \le \infty$*, then we can find* $t \in T_0$ *such that* $r_t = r$*.*

Proof. For each (finite) $i \ge 0$, let M_{i+1} be the Levi type subgroup with root system Φ_{i+i}, and let t_i be as in Lemma 3.2.11 for pair $M_{i+2} \subset M_{i+1}$ and $n = i$. Then

$$r_{t_i}(\alpha) = \begin{cases} \text{something} \ge i+1 & \alpha \in \Phi \setminus \Phi_{i+1}, \\ i+1 & \alpha \in \Phi_{i+1} \setminus \Phi_{i+2}, \\ \infty & \alpha \in \Phi_{i+2}. \end{cases}$$

Thus for a fixed $i \ge 0$, and $\alpha \in \Phi_{i+1} \setminus \Phi_{i+2}$, we have that

$$r_{t_j}(\alpha) = \begin{cases} \text{something} \ge j+1 & j > i, \\ j+1 & j = i, \\ \infty & j < i. \end{cases}$$

Let $t = \prod_{i=0}^{\infty} t_i$, which necessarily converges as $t_i \in T_i$. Since for $\alpha \in \Phi_{i+1} \setminus \Phi_{i+2}$, $r_{t_j}(\alpha)$ reaches unique minimum at $j = i$, we have that $r_t(\alpha) = r_{t_i}(\alpha) = i + 1$, and for $\alpha \in \Phi_\infty$, it's clear that $r_t(\alpha) = \infty$, as desired. ∎

Theorem 3.2.13. *Suppose we have a function $r : \Phi \to \mathbb{N} \cup \{\infty\}$ and the associated filtration Φ_n. Then $r = r_t$ for some $t \in T(\mathcal{O})$ if and only if $\Phi_1 \subset \Phi$ is a root subsystem of pseudo-Levi type and Φ_n is a $\mathbb{Q}$-closed subsystem in Φ_1 for each $1 \le n \le \infty$. Moreover, $t \in T(\mathcal{O})$ is such that $r_t = r$ if and only if we can write $t = x \prod_{n=0}^{\infty} t_n$ where $x \in T(\bar{k})$ and $t_n \in T_n$ are such that Φ_1 is the set of roots taking trivial value on x, and*

$$r_{t_n}(\alpha) = \begin{cases} \textit{something} \ge n+1 & \alpha \in \Phi_1 \setminus \Phi_{n+1}, \\ n+1 & \alpha \in \Phi_{n+1} \setminus \Phi_{n+2}, \\ \infty & \alpha \in \Phi_{n+2}. \end{cases} \tag{3.2.1}$$

Proof. For the first claim, the "only if" part is proved in Proposition 3.2.9. Now for the "if" part. Let $x \in T(\bar{k})$ be such that Φ_1 is the root system of its connected centralizer M, which is a connected reductive group with maximal torus T. Apply Lemma 3.2.12 to M and restricted function $r|_{\Phi_1}$, then one can find $t_0 \in T_0$ such that $r_{t_0}|_{\Phi_1} = r|_{\Phi_1}$. This means that $t = xt_0$ is the desired element.

For the second claim, the only nontrivial part is the "only if" part. The existence of x is also clear, so we may replace t by $x^{-1}t$ and G by the connected centralizer of x, and in turn we may assume t is already in T_0. Let M_n be the Levi-type subgroup of G corresponding to Φ_n, Z_{M_n} its connected centralizer, and $\mathfrak{z}_{M_n} = \operatorname{Lie} Z_{M_n}$. Let X_0 be the image of t in $\mathfrak{h}$, then $\alpha(X_0) = 0$ for all $\alpha \in \Phi_2$, so we can lift it to $t_0 \in Z_{M_2,0}$ (again, the subscript 0 here means the congruent subgroup, not neutral component), then t_0 satisfies (3.2.1). Moreover, let $\Phi'_n = \Phi_n(r_{tt_0^{-1}})$, then we have that $\Phi'_2 = \Phi$, $\Phi'_n \supset \Phi_n$ for all n, and $\Phi'_n \setminus \Phi'_{n+1} \supset \Phi_n \setminus \Phi_{n+1}$ for $n \ge 2$. Doing the same argument for tt_0^{-1} but with every index increased by 1, we have some $t_1 \in T_1$ satisfying (3.2.1) (*a priori* only for Φ'_n, but as $\Phi'_n \setminus \Phi'_{n+1} \supset \Phi_n \setminus \Phi_{n+1}$, same is true for Φ_n). Continuing this process, we see that the infinite product

$$t \prod_{n=0}^{\infty} t_n^{-1}$$

converges to some $\mathcal{O}$-point z in the center of G. Absorb z into t_0 and we are done. ∎

3.2.14 Next we consider $t \in T(F)$. Then $t = t_0 \pi^{\lambda}$ for some $t_0 \in T(\mathcal{O})$ and $\lambda \in \check{\mathbb{X}}(T)$. We form the functions r_t^- and $r_t^{\star}$ as in § 3.2. In this case, let $\Phi_n \coloneqq \Phi_n(r_t^{\star})$ for $0 \le n \le \infty$, then we have filtration

$$\Phi \supset \Phi_0 \supset \cdots \supset \Phi_n \supset \cdots \supset \Phi_\infty \supset \varnothing.$$

Since Φ_0 is the set of roots that vanish on λ, it is in particular $\mathbb{Q}$-closed in Φ, and thus is a root subsystem of Levi type. Then for each $0 \le n \le \infty$, it is clear that

$$\Phi_n = \Phi_0 \cap \Phi_n(r_{t_0}). \tag{3.2.2}$$

So Φ_1 is a root subsystem of pseudo-Levi type in Φ_0, and for each $1 \le n \le \infty$, Φ_n is $\mathbb{Q}$-closed in Φ_1. To summarize it, we have

$$\Phi \overset{\text{Levi}}{\supset} \Phi_0 \overset{\text{pseudo-Levi}}{\supset} \Phi_1 \overset{\text{Levi}}{\supset} \Phi_{1 \le n \le \infty}. \tag{3.2.3}$$

On the other hand, given a function $r \colon \Phi \to \mathbb{Z} \cup \{\infty\}$, a necessary condition for r to be equal to some r_t is that r^- and $r^{\star}$ as in § 3.2 are well-defined. Suppose r is such a function.

Theorem 3.2.15. *There exists $t \in T(F)$ such that $r = r_t$ if and only if r^- extends to a homomorphism $\mathbb{X}(T) \to \mathbb{Z}$, and the filtration Φ_n satisfies* (3.2.3).

Proof. The "only if" part is proved by discussions above and in § 3.2. We now prove the "if" part. Let $\lambda \in \check{\mathbb{X}}(T)$ be one of extensions of r^- (which is not unique unless G is semisimple). By Theorem 3.2.13, we can find $t_0 \in T(\mathcal{O})$ such that (3.2.2) holds. Let $t = \pi^\lambda t_0$ and we are done. ∎

3.2.16 As discussed before, we have associated pair $(\bar{\lambda}_t, r_t)$ for any $t \in T(F)$ where $\bar{\lambda}_t \in \check{\mathbb{X}}(G/G^{\mathrm{der}})$. Let S_G^1 be the set of all pairs $(\bar{\lambda}, r)$ where $\bar{\lambda} \in \check{\mathbb{X}}(G/G^{\mathrm{der}})$ and $r\colon \Phi \to \mathbb{Q} \cup \{\infty\}$. Then we have a partition of $T(F)$ by

$$T(F) = \coprod_{(\bar{\lambda},r)\in S_G^1} T(F)_{(\bar{\lambda},r)},$$

where $T(F)_{(\bar{\lambda},r)}$ is the set of all $t \in T(F)$ such that $(\bar{\lambda}_t, r_t) = (\bar{\lambda}, r)$. Then by Theorem 3.2.15, $T(F)_{(\bar{\lambda},r)}$ is nonempty if and only if the following conditions are satisfied:

(1) r takes values in $\mathbb{Z} \cup \{\infty\}$;

(2) r^- and $r^\star$ are defined;

(3) r^- extends to (necessarily unique) $\lambda_{\mathrm{ad}} \in \check{\mathbb{X}}(T^{\mathrm{ad}})$ such that

$$(\bar{\lambda}, \lambda_{\mathrm{ad}}) \in \check{\mathbb{X}}(G/G^{\mathrm{der}}) \times \check{\mathbb{X}}(T^{\mathrm{ad}})$$

lies in the image of $\check{\mathbb{X}}(T)$;

(4) the filtration $\Phi_n(r^\star)$ satisfies (3.2.3).

For convenience, we will call $\bar{\lambda}$ (resp. λ_{ad}) the central (resp. adjoint) component of λ.

3.2.17 We now study $T(F_\infty)$ in general. Let $t \in T(F_\infty)$, then $t \in T(F_l)$ for some l. We know that the image of τ_∞ in cyclic group $\mathrm{Gal}(F_l/F)$ is a generator. Suppose there is some $w \in W$ with order l. Then we know that r_t takes values in $\mathbb{Z}/l \cup \{\infty\}$, and we are mostly interested in the case when $\tau_\infty w(t) = t$ (note that the Galois action commutes with the Weyl group action). To that end we define

$$T_w(F) \coloneqq \{t \in T(F_l) \mid \tau_\infty w(t) = t\}.$$

Then we immediately have that if $t = t'\pi^{\lambda_t/l}$, then λ_t is fixed by w, and $t' \in T(\mathcal{O}_l)$ is such that

$$\tau_\infty w(t')\zeta_l^{\lambda_t} = t'.$$

Let $T_{F_l,n}$ ($n \in \mathbb{N}$) be the congruent subgroup of $T(\mathcal{O}_l)$ with F replaced by F_l, then we can write canonically $t' = xt_0$ for $x \in T(\bar{k})$ and $t_0 \in T_{F_l,0}$. Since $\zeta_l \in \bar{k}$, we must have

$$\tau_\infty w(t_0) = t_0 \text{ and } (1-w)(x) = \zeta_l^{\lambda_t}. \tag{3.2.4}$$

An immediate observation is that $\bar{\lambda}_t/l \in \check{\mathbb{X}}(G/G^{\mathrm{der}})$.

As in the F-rational case, we can define r_t^- and $r_t^\star$, as well as filtration

$$\Phi_n = \Phi_n(r_t^\star) \coloneqq \left\{\alpha \in \Phi \;\middle|\; r_t^\star(\alpha) \geq \frac{n}{l}\right\}, (0 \leq n \leq \infty).$$

By the same argument with F replaced by F_l, we know that Φ_n satisfies (3.2.2). We also have that r_t^- and $\lambda_{t,\mathrm{ad}}/l$ mutually determines each other. Therefore t induces pair $(\bar{\lambda}_t/l, r_t)$ where the first component is an integral cocharacter of G/G^{der}.

3.2.18 To fully characterize what pairs $(\bar{\lambda}/l, r)$ arises in this way, again we first assume $\lambda_t = 0$, in other words, $t \in T(\mathcal{O}_l)$. In fact, it is better to start with $t \in T_{F_l,0}$, in which case $\Phi = \Phi_0 = \Phi_1$.

Given $t \in T_{F_l,0}$, by Theorem 3.2.13, we may write t as the infinite product of $t_n \in T_{F_l,n}$

such that

$$r_{t_n}(\alpha) = \begin{cases} \text{something} \geq \frac{n+1}{l} & \alpha \in \Phi_1 \setminus \Phi_{n+1}, \\ \frac{n+1}{l} & \alpha \in \Phi_{n+1} \setminus \Phi_{n+2}, \\ \infty & \alpha \in \Phi_{n+2}. \end{cases} \tag{3.2.5}$$

Using the same notations in the proof of Theorem 3.2.13, since $\tau_\infty w(t) = t$, $\tau_\infty w$ preserves each Φ_n, and we see that $X_0 \in \mathfrak{z}_{M_2}^{\tau_\infty w}$ (here the action of τ_∞ on $\mathfrak{h}$, when viewed as quotient T_n/T_{n+1}, is multiplication by ζ_l^{n+1}, and the action of w is the usual one). By Lemma 3.1.5, we can require the lift t_0 to be contained in $Z_{M_2,0}^{\tau_\infty w}$, hence tt_0^{-1} is also fixed by $\tau_\infty w$. Inductively doing so in the construction of each t_n, we can make each t_n fixed by $\tau_\infty w$.

Definition 3.2.19. Suppose $w \in W$ is of order l and acting on a $\bar{k}$-vector space V. For any integer i, define $V(w,i) \subset V$ to be the maximal subspace where w acts as ζ_l^{-i}.

Lemma 3.2.20. *Suppose we have a function $r\colon \Phi \to \mathbb{Z}_+/l \cup \{\infty\}$ and the associated filtration Φ_n. Then we may find $t \in T_{w,0} \coloneqq T_w(\mathcal{O}) \cap T_{F_l,0}$ such that $r_t = r$ if and only if Φ_n is $\mathbb{Q}$-closed in Φ for all $1 \leq n \leq \infty$, and the set*

$$\left\{ X_n \in \mathfrak{h}(w, n+1) \;\middle|\; \begin{array}{l} \alpha(X_n) = 0, \forall \alpha \in \Phi_{n+2}, \\ \alpha(X_n) \neq 0, \forall \alpha \in \Phi_{n+1} \setminus \Phi_{n+2} \end{array} \right\} \tag{3.2.6}$$

is nonempty for all $0 \leq n < \infty$.

Proof. For the "only if" direction, simply choose X_n to be the image of t_n in the discussion above. For the "if" direction, by Lemma 3.1.5, we can lift X_n to $t_n \in T_{w,n} \coloneqq T_w(\mathcal{O}) \cap T_{F_l,n}$, and then $t = \prod_{n=0}^{\infty} t_n$ would be as desired. ∎

Remark 3.2.21. Note that the condition (3.2.6) being nonempty for all n automatically implies that each Φ_n (hence also r) is preserved by w, hence also by $\tau_\infty w$.

Corollary 3.2.22. *Suppose we have a function $r\colon \Phi \to \mathbb{N}/l \cup \{\infty\}$ and the associated filtration Φ_n. Then we may find $t \in T_w(\mathcal{O})$ such that $r_t = r$ if and only if the following conditions are satisfied*

(1) $\Phi_1 \subset \Phi$ is a root subsystem of pseudo-Levi type with corresponding reductive subgroup H, such that the set

$$\left\{x \in T(\bar{k})^w \,\middle|\, H \textit{ is the connected centralizer of } x\right\}$$

is nonempty (in particular Φ_1 is preserved by w);

(2) Φ_n is $\mathbb{Q}$-closed in Φ_1 for all $1 \le n \le \infty$;

(3) the set (3.2.6) *is nonempty for all $0 \le n < \infty$.*

3.2.23 Now we consider general pairs $(\bar{\lambda}/l, r)$, where $\bar{\lambda}/l$ is an (integral) element of $\check{\mathbb{X}}(G/G^{\mathrm{der}})$, and $r\colon \Phi \to \mathbb{Q} \cup \{\infty\}$. In other words, $(\bar{\lambda}/l, r) \in S_G^1$. Similar to F-rational case, we stratify $T_w(F)$ by such pairs, denoted by $T_w(F)_{(\bar{\lambda}/l,r)}$.

From discussions in previous subsections, we see that $T_w(F)_{(\bar{\lambda}/l,r)}$ is nonempty if and only if the following conditions are satisfied:

(1) r takes values in $\mathbb{Z}/l \cup \{\infty\}$;

(2) r^- and $r^\star$ are defined, and r^- extends to an element $\lambda_{\mathrm{ad}}/l \in \check{\mathbb{X}}(T^{\mathrm{ad}})^w/l$, such that $\bar{\lambda}$ and λ_{ad} are the central and adjoint components of some $\lambda \in \check{\mathbb{X}}(T)$ respectively;

(3) let M_λ be the Levi-type subgroup of G determined by λ (or equivalently, Φ_0), then Φ_1 is of pseudo-Levi type in Φ_0 corresponding to subgroup $H < M_\lambda$, such that the set

$$\left\{x \in (1-w)^{-1}(\zeta_l^\lambda) \,\middle|\, H \text{ is the connected centralizer of } x \text{ in } M_\lambda\right\} \tag{3.2.7}$$

is nonempty;

(4) Φ_n is $\mathbb{Q}$-closed in Φ_1 for all $1 \le n \le \infty$, and the set (3.2.6) is nonempty for all $0 \le n < \infty$.

Note that these conditions automatically imply that the filtration Φ_n is preserved by w. Moreover, given $t \in T_w(F)$, we have that $t \in T_w(F)_{(\bar{\lambda}/l, r)}$, if and only if we can write

$$t = \pi^{\lambda/l} x \prod_{n=0}^{\infty} t_n,$$

where x is as in (3.2.7), and $t_n \in T_{F_l,n}$ is fixed by $\tau_\infty w$ such that r_{t_n} satisfies (3.2.5) for all $0 \le n < \infty$.

3.3 Cylinders in Reductive Monoids

3.3.1 We briefly review some facts about the arc spaces of an smooth affine scheme X over $\mathcal{O}$. Since X is $\mathcal{O}$-smooth, every jet scheme of X is smooth over $\bar{k}$, and each consecutive map $\mathbb{L}^+_{n+1} X \to \mathbb{L}^+_n X$ is an affine space bundle of relative dimension $\dim_{\mathcal{O}} X$.

Definition 3.3.2. A subset Z of $\mathbb{L}^+ X$ is called *n-admissible* or an *n-cylinder* if it is the preimage of a constructible subset of $\mathbb{L}^+_n X$ for some $n \ge 0$. A cylinder is called *open* (resp. *closed*, *locally-closed*) if it's the preimage of some open (resp. closed, locally closed) subset of $\mathbb{L}^+_n X$.

Definition 3.3.3. Let $Z \subset \mathbb{L}^+ X$ be an n-cylinder. Then the *codimension* of Z in $\mathbb{L}^+ X$ is defined as the codimension of Z_m in $\mathbb{L}^+_m X$ for any (and every) $m \ge n$.

Definition 3.3.4. Suppose Y is a locally-closed n-cylinder of $\mathbb{L}^+ X$. Then we call Y *non-singular* if for any (and every) $m \ge n$ the reduced subscheme $Y_m \in \mathbb{L}^+_m X$ is non-singular. We call a map $g\colon Y \to Z$ of any locally-closed n-cylinders *smooth* if g_m is smooth for all $m \ge n$.

3.3.5 Recall the universal monoid $\mathfrak{M} = \mathrm{Env}(G)$ for a semisimple and simply-connected group G, and the closure $\mathfrak{T}$ of the extended maximal torus T_+. We know $\bar{k}[\mathfrak{T}]$ is spanned by a saturated, strictly convex cone $\mathscr{C}_{\mathfrak{T}}^* \subset \mathbb{X}(T_+)$, whose dual $\mathscr{C}_{\mathfrak{T}} \subset \check{\mathbb{X}}(T_+)$ determines a stratification of non-degenerate arcs

$$\mathfrak{T}(\mathcal{O}) \cap T_+(F) = \coprod_{\lambda \in \mathscr{C}_{\mathfrak{T}}} \mathbb{L}^+\mathfrak{T}^\lambda(\bar{k}),$$

where $\mathbb{L}^+\mathfrak{T}^\lambda = \pi^\lambda \mathbb{L}^+ T_+$. Note that $\mathbb{L}^+\mathfrak{T}^\lambda$ is isomorphic to $\mathbb{L}^+ T_+$ as an abstract $\bar{k}$-scheme.

For $w \in W$ be of order l and λ fixed by w, we also have the w-twisted form of $\mathbb{L}^+\mathfrak{T}^\lambda$ defined in an obvious way:

$$\mathfrak{T}_w^{\lambda/l} \coloneqq \left(\mathrm{Res}_{\mathcal{O}_l/\mathcal{O}}\, \pi^{\lambda/l} T_{+,\mathcal{O}_l}\right)^{\tau_\infty w},$$

and the $\bar{k}$-points of $\mathbb{L}^+\mathfrak{T}_w^{\lambda/l}$ are simply those $t \in \pi^{\lambda/l} T_+(\mathcal{O}_l)$ that are fixed by $\tau_\infty w$. Since l is coprime to $\mathrm{char}(k)$, the scheme $\mathfrak{T}_w^{\lambda/l}$ is $\mathcal{O}$-smooth, hence notions like cylinders and their codimensions make sense for its arc space.

It is clear that any non-empty stratum $T_{+,w}(F)_{(\bar{\lambda}/l,r)}$ as in § 3.2 is entirely contained in a unique $\mathbb{L}^+\mathfrak{T}_w^{\lambda/l}(\bar{k})$ if λ is contained in the cone $\mathscr{C}_{\mathfrak{T}}$. In such case, we denote the stratum by

$$\mathfrak{T}_w(\mathcal{O})_{(\bar{\lambda}/l,r)} \coloneqq T_{+,w}(F)_{(\bar{\lambda}/l,r)}.$$

3.3.6 Choose a geometric point $x_+ \in T_+(F^s)^{\mathrm{rs}}$ lying over some $a \in \mathfrak{C}_{\mathfrak{M}}(\mathcal{O})$, then it induces a homomorphism

$$\rho_a \colon \Gamma_F = \mathrm{Gal}(F^s/F) \longrightarrow W,$$

the conjugacy class of whose image depends only on a. Since a is tamely ramified (by our assumption on $\mathrm{char}(k)$), one sees that the image W_a is cyclic and generated by the image of τ_∞, denoted by w_a. Define the *ramification index* c of x_+ or a to be

$$c(x_+) = c(a) \coloneqq \dim T_+ - \dim T_+^{w_a}.$$

It is in fact equal to the difference $\mathrm{rk}_{\bar{F}}((G_{x_+})_0) - \mathrm{rk}_F((G_{x_+})_0)$, where rk_F means the F-split rank of F-torus $(G_{x_+})_0$. So $c(\gamma)$ can be defined for arbitrary reductive group G and $\gamma \in G(F)^{\mathrm{rs}}$.

Recall we have extended discriminant function $\mathrm{Disc}_+ \in k[\mathfrak{C}_{\mathfrak{M}}]$. The extended discriminant valuation of a is defined by $d_+(a) \coloneqq \mathrm{val}_F(\mathrm{Disc}_+(a))$. We define the *local δ-invariant* of a to be

$$\delta(a) \coloneqq \frac{d_+(a) - c(a)}{2},$$

which is in fact necessarily in $\mathbb{N}$ (due to the fact that it is the dimension of certain multiplicative affine Springer fiber, see Theorem 4.2.1).

3.3.7 Let us clarify some relations between Levi subgroups and pseudo-Levi subgroups in an arbitrary connected reductive group G containing maximal torus T. Suppose M_1 is a pseudo-Levi subgroup containing T with root subsystem Φ_1, and let Φ' be its $\mathbb{Q}$-closure in Φ, which gives a Levi subgroup M of G containing M_1. Let $Z(M_1)$ be the center of M_1, and $Z(M_1)_0$ the neutral component. Then the centralizer of $Z(M_1)_0$ in G is M. Let $x \in Z(M_1)$ be such that the connected centralizer is exactly M_1 and $t \in Z(M_1)_0$. Then $\alpha(xt) \neq 1$ for all $\alpha \notin \Phi_1$ if t is general enough. This means that the closure of the set

$$\{x \in T \mid (G_x)_0 = M_1\}$$

in T is the union of some connected components of $Z(M_1)$. In particular, it has the same dimension as $Z(M_1)$. We denote the set of such components by $\pi_0^\circ(Z(M_1))$.

Proposition 3.3.8. *Suppose r takes values in $\mathbb{N}$ and $T_+(\mathcal{O})_r \neq \varnothing$. Let M_n be the (pseudo-)Levi subgroup of G_+ determined by $\Phi_n(r)$ and $\mathfrak{z}_n = \mathrm{Lie}(Z(M_n))$. Then we have the following:*

(1) The closure of $T_+(\mathcal{O})_r$ in $\mathbb{L}^+T_+$ is the union of some connected components of the subgroup

$$\{t_+ \in T_+(\mathcal{O}) \mid r_{t_+}(\alpha) \geq r(\alpha)\}. \tag{3.3.1}$$

(2) $T_+(\mathcal{O})_r$ is non-singular and $\pi_0(T_+(\mathcal{O})_r)$ is in bijection with $\pi_0^\circ(Z(M_1))$.

(3) The codimension of $T_+(\mathcal{O})_r$ in $\mathbb{L}^+T_+$ is

$$\sum_{n=0}^{\infty} n \dim_{\bar{k}}(\mathfrak{z}_{n+1}/\mathfrak{z}_n) = \sum_{n=1}^{\infty} \dim_{\bar{k}}(\mathfrak{h}_+/\mathfrak{z}_n).$$

Proof. Let X be the set in (3.3.1), which is easily seen a subgroup hence non-singular. Clearly X is closed in $\mathbb{L}^+T_+$ and contains $T_+(\mathcal{O})_r$ as an open subset. So $T_+(\mathcal{O})_r$ is also non-singular.

Similar to the proof of Theorem 3.2.13, let M_n be the connected reductive subgroups of G_+ given by $\Phi_n(r)$, and $\mathfrak{z}_n \subset \mathfrak{h}_+$ be the Lie algebra of the center of M_n. Let $S_1 = Z(M_1)$ and $S_n \subset Z(M_n)_{n-2} \subset \mathbb{L}^+Z(M_n)_0$ be the lift of $\mathfrak{z}_n$ for $n \geq 2$. Then $S_n = \{1\}$ if $n > r(\alpha)$ for all α. Then its clear by the factorization in Theorem 3.2.13 that the multiplication map

$$\phi\colon \prod_{n=1}^{\infty} S_n \to X \tag{3.3.2}$$

is surjective. Therefore $\pi_0(X)$ is bounded by $\pi_0(Z(M_1))$. We also have a projection $X \to$

$Z(M_1)$, which implies the bijection $\pi_0(X) \cong \pi_0(Z(M_1))$. The bijection $\pi_0(T_+(\mathcal{O})_r) \cong \pi_0^\circ(Z(M_1))$ is then clear as the projection of $\phi^{-1}(T_+(\mathcal{O})_r)$ in S_n ($n \geq 2$) is dense.

To prove the codimension formula, note that $\operatorname{codim}_{\mathbb{L}^+T_+}(T_+(\mathcal{O})_r) = \operatorname{codim}_{\mathbb{L}^+T_+}(X)$. So we only need to show the formula for X, which is easily deduced from (3.3.2). ∎

Corollary 3.3.9. *Fix $w \in W$ with $\operatorname{ord}(w) = l$. Suppose r takes values in $\mathbb{N}/l$ and $T_{+,w}(\mathcal{O})_r \neq \emptyset$. Let M_n be the (pseudo-)Levi subgroup of G_+ determined by $\Phi_n(r)$ and $\mathfrak{z}_n = \operatorname{Lie}(Z(M_n))$. Then we have the following:*

(1) *The closure of $T_{+,w}(\mathcal{O})_r$ in $\mathbb{L}^+T_{+,w}$ is the union of some connected components of the subgroup*

$$\{t_+ \in T_{+,w}(\mathcal{O}) \mid r_{t_+}(\alpha) \geq r(\alpha)\}.$$

(2) *$T_{+,w}(\mathcal{O})_r$ is non-singular and $\pi_0(T_{+,w}(\mathcal{O})_r)$ is in bijection with $\pi_0^\circ(Z(M_1)^w)$, the latter defined as the preimage of $\pi_0^\circ(Z(M_1))$ in $\pi_0(Z(M_1)^w)$.*

(3) *The codimension of $T_{+,w}(\mathcal{O})_r$ in $\mathbb{L}^+T_{+,w}$ is*

$$\sum_{n=0}^{\infty} \dim_{\bar{k}}(\mathfrak{h}_+/\mathfrak{z}_{n+1})(w,n).$$

Proof. The first and second claim is a result of (3.2.4) and Proposition 3.3.8. The last claim is deduced from Corollary 3.2.22 and Proposition 3.3.8, and the fact that the action of w on any $\bar{k}$-vector space is semisimple due to our assumption on $\operatorname{char}(k)$. ∎

Corollary 3.3.10. *Fix $w \in W$ with $\operatorname{ord}(w) = l$. Suppose r takes values in $\mathbb{Z}/l$ and $\mathfrak{T}_w(\mathcal{O})_{(\bar{\lambda}/l,r)} \neq \emptyset$. Let M_n be the (pseudo-)Levi subgroup of G_+ determined by $\Phi_n(r^\star)$ and $\mathfrak{z}_n = \operatorname{Lie}(Z(M_n))$. Then we have the following:*

(1) *The closure of $\mathfrak{T}_w(\mathcal{O})_{(\bar{\lambda}/l,r)}$ in $\mathbb{L}^+\mathfrak{T}_w^{\lambda/l}$ is the union of some connected components*

of

$$\left\{t_+ \in \mathbb{L}^+\mathfrak{T}_w^{\lambda/l}(\bar{k}) \;\middle|\; r_{t_+}(\alpha) \geq r(\alpha)\right\},$$

which is itself a union of torsors over a subgroup scheme of $\mathbb{L}^+ T_{+,w}$.

(2) $\mathfrak{T}_w(\mathcal{O})_{(\bar{\lambda}/l,r)}$ *is non-singular and* $\pi_0(\mathfrak{T}_w(\mathcal{O})_{(\bar{\lambda}/l,r)})$ *is in bijection with* $\pi_0^\circ(Z(M_1)^w)$, *the latter defined as the preimage of* $\pi_0^\circ(Z(M_1))$ *in* $\pi_0(Z(M_1)^w)$.

(3) *The codimension of* $\mathfrak{T}_w(\mathcal{O})_{(\bar{\lambda}/l,r)}$ *in* $\mathbb{L}^+\mathfrak{T}_w^{\lambda/l}$ *is*

$$\sum_{n=0}^{\infty} \dim_{\bar{k}}(\mathfrak{h}_+/\mathfrak{z}_{n+1})(w,n),$$

Proof. This is a straightforward result of the discussion at the end of § 3.2, (3.2.4) and Corollary 3.3.9 (the same proof applies). ∎

3.3.11 Recall we have the set S_G^1 of pairs $(\bar{\lambda}, r)$ where $\bar{\lambda} \in \check{\mathbb{X}}(G/G^{\mathrm{der}})$ and $r\colon \Phi \to \mathbb{Q} \cup \{\infty\}$ for any connected reductive group G. The Weyl group W acts on $W \times S_G^1$ by conjugation on the first factor and the action on $(\bar{\lambda}, r)$ is the most obvious one:

$$w(\bar{\lambda}, r) = (\bar{\lambda}, wr \coloneqq \alpha \mapsto r(w^{-1}\alpha)).$$

Let $S_G = W\backslash(W \times S_G^1)$, whose elements will be denoted by $[w, \bar{\lambda}/l, r]$ (where $l = \mathrm{ord}(w)$, and $\bar{\lambda}/l$ is integral). In case where the group is G_+, we will simply use $S = S_{G_+}$. Let $S_G^{\mathrm{rs}} \subset S_G$ be the subsets where r takes finite values. Clearly, if $t \in T_w(F)_{(\bar{\lambda}/l,r)}$, then $u(t) \in T_{uwu^{-1}}(F)_{(\bar{\lambda}/l,ur)}$ for any $u \in W$. This justifies the following definition.

Definition 3.3.12. For $s \in S$, define $\mathfrak{C}_{\mathfrak{M}}(\mathcal{O})_s$ be the image of $\mathfrak{T}_w(\mathcal{O})_{(\bar{\lambda}/l,r)}$ in $\mathfrak{C}_{\mathfrak{M}}(\mathcal{O})$ for any (and every) representative $(w, \bar{\lambda}/l, r)$ of s.

Lemma 3.3.13. *The set of* $\bar{k}$*-points of* $\mathbb{L}^\flat\mathfrak{C} \coloneqq \mathbb{L}^+\mathfrak{C}_{\mathfrak{M}} - \mathbb{L}^+\mathfrak{B}_{\mathfrak{M}} - \mathbb{L}^+\mathfrak{D}_{\mathfrak{M}}$ *is the* disjoint *union* *of strata $\mathfrak{C}(\mathcal{O})_s$ for all $s \in \mathcal{S}^{\mathrm{rs}}$, where $\mathfrak{B}_{\mathfrak{M}}$ (resp. $\mathfrak{D}_{\mathfrak{M}}$) is the numerical boundary divisor (resp. discriminant divisor) as in §§ 2.3 and 2.4.*

Proof. Straightforward; c.f. [GKM09, 7.3]. ∎

3.3.14 Consider $\mathcal{O}_l$-morphism

$$\begin{aligned}\chi_{\lambda/l}\colon T_{+,\mathcal{O}_l} &\longrightarrow \mathfrak{C}_{\mathfrak{M},\mathcal{O}_l}\\ x &\longmapsto \chi_{\mathfrak{M}}(\pi^{\lambda/l}x),\end{aligned}$$

which induces $\mathcal{O}$-morphism

$$\chi_{(w,\lambda/l)}\colon \left(\pi^{\lambda/l}T_{+,\mathcal{O}_l}\right)^{\tau_\infty w} \longrightarrow \mathfrak{C}_{\mathfrak{M},\mathcal{O}}.$$

This further induces map $\mathbb{L}^+\mathfrak{T}_w^{\lambda/l} \to \mathbb{L}^+\mathfrak{C}_{\mathfrak{M}}$, which is precisely the restriction of $\mathbb{L}^+\chi_{\mathfrak{M}}$ to $\mathbb{L}^+\mathfrak{T}_w^{\lambda/l}$. Fix any $x \in \mathbb{L}^+\mathfrak{T}_w^{\lambda/l}(\bar{k})$, let $a = \chi(x) \in \mathbb{L}^+\mathfrak{C}_{\mathfrak{M}}(\bar{k})$, then $\chi_{\mathfrak{M}}$ induces $\tau_\infty w$-equivariant isomorphism of F_l-tangent spaces

$$(\mathrm{d}\chi_{\mathfrak{M},F_l})_x\colon\ \mathrm{T}_x(\pi^{\lambda/l}T_{+,F_l}) \longrightarrow \mathrm{T}_a\,\mathfrak{C}_{\mathfrak{M},F_l} \simeq \mathfrak{C}_{\mathfrak{M},F_l},$$

which restricts to an injection of $\mathcal{O}_l$-modules

$$(\mathrm{d}\chi_{\mathfrak{M},F_l})_x\colon \mathfrak{h}_{\mathcal{O}_l} \longrightarrow \mathfrak{C}_{\mathfrak{M},\mathcal{O}_l},$$

Taking $\tau_\infty w$-invariants, we have $\mathcal{O}$-linear injection

$$(\mathrm{d}\chi_w)_x\colon Q_x \coloneqq \mathfrak{h}_{\mathcal{O}_l}^{\tau_\infty w} \longrightarrow \mathfrak{C}_{\mathfrak{M},\mathcal{O}}.$$

Choosing an $\mathcal{O}$-basis on both sides, then $(\mathrm{d}\chi_w)_x$ is represented by a matrix in $\mathrm{GL}_{2r}(\mathcal{O})$. Changing uniformizer $\pi^{1/l}$ or the $\mathcal{O}$-bases doesn't change the valuation of the determinant of the matrix, so $\mathrm{val}_F \det(\mathrm{d}\chi_w)_x$ is well-defined for x.

Proposition 3.3.15. *Suppose $x \in \mathfrak{C}_w(\mathcal{O})_{(\bar{\lambda}/l,r)}$ is generically regular semisimple, then*

$$\begin{aligned}\mathrm{val}_F \det(\mathrm{d}\chi_w)_x &= \sum_{i=1}^{r} \langle \alpha_i, \bar{\lambda}/l \rangle + \frac{d_+(a)+c(a)}{2} \\ &= \sum_{i=1}^{r} \langle \alpha_i, \bar{\lambda}/l \rangle + \delta(a) + c(a).\end{aligned}$$

Proof. The proof is completely parallel to the argument in [GKM09], but the technical counterparts will occupy several pages. We extend $(\mathrm{d}\chi_w)_x$ to $\mathcal{O}_l$:

$$\mathrm{id}_{F_l} \otimes (\mathrm{d}\chi_w)_x \colon \mathcal{O}_l \otimes_{\mathcal{O}} Q_x \longrightarrow \mathfrak{C}_{\mathcal{O}_l},$$

which is the restriction of map

$$(\mathrm{d}\chi_{\mathfrak{W},F_l})_x \colon \mathfrak{h}_{\mathcal{O}_l} \longrightarrow \mathfrak{C}_{\mathcal{O}_l}$$

to submodule $\mathcal{O}_l \otimes_{\mathcal{O}} Q_x$. Therefore

$$\mathrm{val}_F \det(\mathrm{d}\chi_w)_x = \mathrm{val}_F \det(\mathrm{d}\chi_{\mathfrak{W},F_l})_x + \frac{1}{l}\dim_{\bar{k}} \frac{\mathfrak{h}_{\mathcal{O}_l}}{\mathcal{O}_l \otimes_{\mathcal{O}} Q_x}.$$

So we reduce to the $w = 1$ case (replacing $\mathcal{O}$ with $\mathcal{O}_l$ and val_F with val_{F_l}) and the claim that

$$\dim_{\bar{k}} \frac{\mathfrak{h}_{\mathcal{O}_l}}{\mathcal{O}_l \otimes_{\mathcal{O}} Q_x} = \frac{lc(a)}{2}. \qquad (3.3.3)$$

The equation (3.3.3) is proved using the same argument in [Bez96]. By definition, $c(a)$ is the dimension of the largest subspace of $\mathfrak{h}$ (over $\bar{k}$) that doesn't contain a trivial representation of w. On the other hand, we have

$$Q_x = \left[\bigoplus_{i=0}^{l-1} \mathfrak{h}(i)\pi^{i/l}\right] \otimes_{\bar{k}} \mathcal{O},$$

where $\mathfrak{h}(i)$ is the eigenspace of w in $\mathfrak{h}$ with eigenvalue ζ_l^{-i}. Therefore we can describe the quotient space as follows:

$$\frac{\mathfrak{h}_{\mathcal{O}_l}}{\mathcal{O}_l \otimes_{\mathcal{O}} Q_x} \simeq \bigoplus_{i=0}^{l-1}\left[\bigoplus_{j=i+1}^{l-1} \mathfrak{h}(j)\right],$$

which implies (3.3.3).

The argument for $w = 1$ (and $l = 1$) case is a slight generalization of the argument in [Ste74, pp. 125–127]. For completeness, we include a detailed argument here. Without loss of generality, we may assume λ_{ad} is *anti-dominant*. Let Δ_λ be the simple roots vanishing on λ, Φ_λ the induced root subsystem, and $W_\lambda \subset W$ the subgroup generated by the reflections corresponding to roots in Δ_λ. It's harmless to assume that Δ_λ contains first s simple roots.

Choose generator basis (in the given order)

$$\left\{e^{(\alpha_1,0)},\dots,e^{(\alpha_n,0)},e^{(\varpi_1,\varpi_1)},\dots,\dots,e^{(\varpi_n,\varpi_n)}\right\}$$

of $\bar{k}[T_+]$, and

$$\left\{e^{(\alpha_1,0)},\dots,e^{(\alpha_n,0)},\chi_{1,+},\dots,\chi_{r,+}\right\}$$

of $\bar{k}[\mathfrak{C}_{\mathfrak{M}}]$, where $\chi_{i,+}(z,t) = \varpi_i(z)\chi_i(t)$. Let $e_i = \mathrm{d}e^{(\alpha_i,0)}$, $f_i = e^{-(\varpi_i,\varpi_i)}\mathrm{d}e^{(\varpi_i,\varpi_i)}$, and $g_i = \mathrm{d}\chi_{i,+}$. We need to compute the valuation of linear map

$$\mathrm{d}(\chi_\lambda)_x \colon \mathcal{O}e_1 \wedge \cdots \wedge e_r \wedge g_1 \wedge \cdots \wedge g_n \longrightarrow \mathcal{O}e_1 \wedge \cdots \wedge e_r \wedge f_1 \wedge \cdots \wedge f_r,$$

identified with $A \in \mathrm{Mat}_1(\mathcal{O}) \simeq \mathcal{O}$. We claim that A is W_λ-skew symmetric, in other words, $w(A) = \det(w)A$ for all $w \in W_\lambda$. Indeed, fix any $w \in W_\lambda$, and suppose that

$$w(\varpi_i) = \sum_{i=1}^{r} n_{ij}\varpi_j,$$

then

$$\begin{aligned}
w(f_i) &= e^{-(\varpi_i, w(\varpi_i))}\mathrm{d}e^{(\varpi_i, w(\varpi_i))} \\
&= e^{(w(\varpi_i)-\varpi_i,0)}e^{-(w(\varpi_i),w(\varpi_i))}\mathrm{d}\left[e^{(w(\varpi_i)-\varpi_i,0)}e^{(w(\varpi_i),w(\varpi_i))}\right] \\
&= \left(\prod_{j=1}^{r} e^{-(n_{ij}\varpi_j, n_{ij}\varpi_j)}\right)\mathrm{d}\left(\prod_{j=1}^{r} e^{(n_{ij}\varpi_j, n_{ij}\varpi_j)}\right) \\
&\qquad + \text{terms involving at least one } e_j \\
&= \sum_{j=1}^{r} n_{ij}f_j + \text{terms involving at least one } e_j.
\end{aligned}$$

Therefore since e_i and g_i are fixed by w,

$$\begin{aligned}
w(Ae_1 \wedge \cdots \wedge e_r \wedge f_1 \wedge \cdots \wedge f_r) &= w(A)e_1 \wedge \cdots \wedge e_r \wedge w(f_1) \wedge \cdots \wedge w(f_r) \\
&= w(A)\det(n_{ij})e_1 \wedge \cdots \wedge e_r \wedge f_1 \wedge \cdots \wedge f_r \\
&= w(A)\det(w)e_1 \wedge \cdots \wedge e_r \wedge f_1 \wedge \cdots \wedge f_r.
\end{aligned}$$

This means that $w(A) = \det(w)^{-1}A = \det(w)A$ as claimed.

Let Ω_i (resp. Ω_ν) be the set of weights in the Weyl module of G of highest weight ϖ_i (resp. any dominant ν), and $m_{i,\mu}$ the multiplicity of $\mu \in \Omega_i$. We can expand g_i into linear combinations of e_j and f_j as follows:

$$\begin{aligned}
g_i &= \pi^{\langle(\varpi_i,\varpi_i),\lambda\rangle}\mathrm{d}e^{(\varpi_i,\varpi_i)} + \sum_{\varpi_i \neq \mu \in \Omega_i} m_{i,\mu}\pi^{\langle(\varpi_i,\mu),\lambda\rangle}\mathrm{d}e^{(\varpi_i,\mu)} \\
&= \pi^{\langle(\varpi_i,\varpi_i),\lambda\rangle}e^{(\varpi_i,\varpi_i)}f_i + \sum_{\varpi_i \neq \mu \in \Omega_i} m_{i,\mu}\pi^{\langle(\varpi_i,\mu),\lambda\rangle}e^{(\varpi_i,\mu)}e^{-(\mu,\mu)}\mathrm{d}e^{(\mu,\mu)}
\end{aligned}$$

$$+ \text{ terms involving at least one } e_j$$

$$= \pi^{\langle(\varpi_i,\varpi_i),\lambda\rangle} e^{(\varpi_i,\varpi_i)} f_i + \sum_{\varpi_i \neq \mu \in \Omega_i} \left[m_{i,\mu} \pi^{\langle(\varpi_i,\mu),\lambda\rangle} e^{(\varpi_i,\mu)} \sum_{j=1}^{r} n_{\mu,j} f_j \right]$$

$$+ \text{ terms involving at least one } e_j.$$

This implies that

$$B := \prod_{i=1}^{r} \pi^{-\langle \alpha_i, \bar{\lambda}\rangle} A = \pi^{\langle(\rho,\rho),\lambda\rangle} e^{(\rho,\rho)} + \sum_{\mu \in \Omega_\rho} C_\mu \pi^{\langle(\rho,\mu),\lambda\rangle} e^{(\rho,\mu)}$$

for some integers C_μ. As B is W_λ-skew symmetric (because A is), we have that

$$\begin{aligned} B &= \sum_{w \in W_\lambda} \det(w) \pi^{\langle(\rho,w(\rho)),\lambda\rangle} e^{(\rho,w(\rho))} \\ &\quad + \sum_{\substack{\mu \in \Omega_\rho \\ \Delta_\lambda\text{-dominant}}} C_\mu \left(\sum_{w \in W_\lambda} \det(w) \pi^{\langle(\rho,w(\mu)),\lambda\rangle} e^{(\rho,w(\mu))} \right) \\ &= \pi^{\langle(\rho,\rho),\lambda\rangle} \sum_{w \in W_\lambda} \det(w) e^{(\rho,w(\rho))} \\ &\quad + \sum_{\substack{\mu \in \Omega_\rho \\ \Delta_\lambda\text{-dominant}}} C_\mu \pi^{\langle(\rho,\mu),\lambda\rangle} \left(\sum_{w \in W_\lambda} \det(w) e^{(\rho,w(\mu))} \right). \end{aligned}$$

Observe that if μ is not *strictly* Δ_λ-dominant, then the summation

$$\sum_{w \in W_\lambda} \det(w) \pi^{\langle(\rho,w(\mu)),\lambda\rangle} e^{(\rho,w(\mu))}$$

equals 0 because its summands cancel pairwise.

We then claim that for strictly Δ_λ-dominant μ, $\langle(\rho,\mu),\lambda\rangle$ reaches minimum if and only if $\mu = \rho$. Indeed, suppose

$$\mu = \rho - \sum_{i=1}^{s} p_i \alpha_i - \sum_{i=s+1}^{r} p_i \alpha_i = \rho + \sum_{i=1}^{r} q_i \varpi_i$$

for some $p_i \in \mathbb{N}$ and $q_i \in \mathbb{Z}$. If $p_i > 0$ for any $i > s$, we are done since λ_{ad} is anti-dominant and $\langle \alpha_i, \lambda_{\mathrm{ad}} \rangle < 0$ if $i > s$. Now suppose $p_i = 0$ for all $i > s$. Since μ is strictly Δ_λ-dominant and $\langle \rho, \check{\alpha}_i \rangle = 1$, we must have $q_i \geq 0$ for $1 \leq i \leq s$. Let $\check{\rho}_\lambda$ be the half-sum of the positive coroots in Φ_λ. Then we have

$$\langle \rho - \mu, \check{\rho}_\lambda \rangle = \sum_{i=1}^{s} p_i \geq 0,$$

but on the other hand

$$\langle \rho - \mu, \check{\rho}_\lambda \rangle = -\sum_{i=1}^{s} \langle \varpi_i, \check{\rho}_\lambda \rangle q_i \leq 0,$$

which means $p_i = q_i = 0$ for all $1 \leq i \leq s$ and $\mu = \rho$. Hence the claim.

For strictly Δ_λ-dominant μ, a well-known fact due to Weyl gives that

$$\sum_{w \in W_\lambda} \det(w) e^{(\rho, w(\mu))} = e^{(\rho,\mu)} \prod_{\alpha \in \Phi_\lambda^+} \left(1 - e^{-(0,\alpha)}\right).$$

Thus, we have that

$$\begin{aligned}
\mathrm{val}_F(B) = \mathrm{val}_F \Bigg[& \pi^{\langle (\rho,\rho),\lambda \rangle} \sum_{w \in W_\lambda} \det(w) e^{(\rho, w(\rho))} \\
& + \sum_{\substack{\mu \in \Omega_\rho \\ \text{strictly } \Delta_\lambda\text{-dominant}}} C_\mu \pi^{\langle (\rho,\mu),\lambda \rangle} \left(\sum_{w \in W_\lambda} \det(w) e^{(\rho, w(\mu))} \right) \Bigg] \\
= {} & \mathrm{val}_F \left(\prod_{\alpha \in \Phi_\lambda^+} \left(1 - e^{-(0,\alpha)}\right) \right) + \mathrm{val}_F \Bigg[\pi^{\langle (\rho,\rho),\lambda \rangle} + \sum_{\substack{\mu \in \Omega_\rho \\ \text{strictly } \Delta_\lambda\text{-dominant}}} \pi^{\langle (\rho,\mu),\lambda \rangle} \Bigg] \\
= {} & \mathrm{val}_F \left(\prod_{\alpha \in \Phi_\lambda^+} \left(1 - e^{-(0,\alpha)}\right) \right) + \langle (\rho,\rho),\lambda \rangle \\
= {} & \frac{d_+(a)}{2}.
\end{aligned}$$

So

$$\mathrm{val}_F(A) = \sum_{i=1}^{r} \langle \alpha_i, \bar{\lambda} \rangle + \frac{d_+(a)}{2},$$

as desired. This concludes the proof. ∎

Corollary 3.3.16. *Let M be an arbitrary very flat reductive monoid associated with G such that its abelianization is an affine space $\mathfrak{A}_M = \mathbb{A}^m$ with coordinates $e^{\alpha'_1}, \ldots, e^{\alpha'_m}$. Let $\mathfrak{T}_M$ be the maximal toric variety in M. Let $x \in \mathfrak{T}_{M,w}(\mathcal{O})_{(\lambda/l,r)}$ be generically regular semisimple whose image is $a \in \mathfrak{C}_M(\mathcal{O})$. Then*

$$\begin{aligned} \mathrm{val}_F \det(\mathrm{d}\chi_{M,w})_x &= b(a) + \frac{d_+(a) + c(a)}{2} \\ &= b(a) + \delta(a) + c(a), \end{aligned} \tag{3.3.4}$$

where $b(a)$ is the boundary valuation $\sum_{i=1}^{m} \langle \alpha'_i, \bar{\lambda}/l \rangle$.

Proof. Note that the notion of $d_+(a)$ makes sense as it is the valuation of the pullback of the discriminant function (from the universal monoid to M), and that the notion of $c(a)$ also makes sense because it only depends on w.

The character lattice of the maximal torus in M is freely spanned by the pullbacks of functions $e^{\alpha'_i}$ from $\mathfrak{A}_M$, and $e^{(\varpi_i, \varpi_i)}$ from the toric variety of the universal monoid. The proof then proceeds essentially the same in the proposition above; notably the computation for B is repeated word-by-word. ∎

3.4 Codimension Formula of Valuation Strata

We can now use (3.3.4) to derive a codimension formula for the valuation strata for arbitrary very flat reductive monoid M of G whose abelianization is an affine space.

Lemma 3.4.1. *Let $f\colon \mathbb{G}_{\mathrm{m},\mathcal{O}}^r \to \mathbb{A}_{\mathcal{O}}^r$ be an $\mathcal{O}$-morphism, viewed as an r-vector of rational functions in r variables and $\mathcal{O}$-coefficients. Let $x \in \mathbb{G}_{\mathrm{m}}^r(\mathcal{O})$ such that $d = \operatorname{val}\det D_x f$ is finite. Then for any $m > d$, we have that*

$$f(x + \pi^m) = f(x) + D_x f(\pi^m),$$

where π^m means π^m times the $\mathcal{O}$-tangent space at x.

Proof. This is just a tiny modification of the claim in [GKM09, Lemma 10.3.1], but since they did not provide a general statement, we include the adaptation here.

We may regard $\mathbb{G}_{\mathrm{m}}^r(\mathcal{O})$ as a subset of $\mathbb{A}^r(\mathcal{O})$ and identify the $\mathcal{O}$-tangent space at x with r-copies of $\mathcal{O}$, denoted by L. Take the Taylor expansion of f at x over $\mathcal{O}$, whose linear terms gives the matrix $A = D_x f$. Since $m > d$, we have $m \geq 1$, so in turn we have for any $h \in \pi^m L$,

$$f(x + h) \equiv f(x) + (D_x f)(h) \bmod \pi^{2m} L.$$

The proof then proceeds as in [GKM09, Lemma 10.3.1]. ∎

Lemma 3.4.2. *The valuation strata $\mathfrak{C}_M(\mathcal{O})_{[w,\bar{\lambda}/l,r]}$ are cylinders.*

Proof. It is clear from the fact that $\mathfrak{T}_w(\mathcal{O})_{(\bar{\lambda}/l,r)}$ are cylinders, Corollary 3.3.16, and Lemma 3.4.1 (applied to $\pi^\lambda T_M(\mathcal{O}) \cong T_M(\mathcal{O}) \to \mathfrak{C}_M(\mathcal{O})$). ∎

Theorem 3.4.3. *The codimension of $\mathfrak{C}_M(\mathcal{O})_{[w,\bar{\lambda}/l,r]}$ in $\mathfrak{C}_M(\mathcal{O})$ is given by*

$$b + \frac{d_+ + c}{2} + e = b + \delta + c + e,$$

where b, d_+, c, δ are as in Corollary 3.3.16 (necessarily constant over $\mathfrak{C}_M(\mathcal{O})_{[w,\bar{\lambda}/l,r]}$), and e is the codimension of $\mathfrak{T}_w(\mathcal{O})_{(\bar{\lambda}/l,r)}$ in $\mathbb{L}^+\mathfrak{T}_w^{\lambda/l}$.

Proof. Let $U = \mathfrak{C}_w(\mathcal{O})_{(\bar{\lambda}/l,r)}$ and $V = \mathfrak{C}_M(\mathcal{O})_{[w,\bar{\lambda}/l,r]}$. Since they are both cylinders, we may pass to some jet space modulo π^N for some big N, after which U becomes smooth and V is constructible, hence the notion of codimension makes sense for V. Since U is smooth (Corollary 3.3.10) and the original $\mathcal{O}$-map from U to V have constant valuation (Corollary 3.3.16), we may find an open dense subset of V modulo π^N over which $U \to V$ is smooth of relative dimension $b+\delta+c$. Then the claim follows from standard dimension counting (c.f. [GKM09, § 5]). ∎

Remark 3.4.4. In [GKM09], they proved much more properties about valuation strata, such as its non-singularity and the smoothness of the map between the strata in $\mathfrak{h}$ and those in $\mathfrak{h} /\!\!/ W$. However, we are temporarily satisfied with just the codimension formula and will leave the rest for a future paper.

CHAPTER 4

MULTIPLICATIVE AFFINE SPRINGER FIBERS

In this section we review the multiplicative affine Springer fibers also known as Kottwitz-Viehmann varieties (KV-varieties). We will use these two names interchangeably. Analogous to the Lie algebra case, the point-count of these schemes over k encode information of orbital integrals of G. The main reference we use for this section is [Chi19]. Other important references include [MV07] and [KV12].

In this chapter, we consider $\mathcal{O}_v = k_v[[\pi]]$, and $F_v = k_v((\pi))$ for a finite extension k_v/k. Let $X_v = \operatorname{Spec}\mathcal{O}_v$ be the corresponding formal disc, and $X_v^\bullet = \operatorname{Spec} F_v$ the punctured disc. Let v be the closed point of X_v, and η_v the generic point. Let $\check{X}_v = \operatorname{Spec}\mathcal{O}_v\hat{\otimes}_k\bar{k}$ then we have isomorphism

$$\check{X}_v = \coprod_{\bar{v}\colon k_v\to\bar{k}} \check{X}_{\bar{v}},$$

where $\bar{v}$ ranges over all k-field embeddings of k_v into $\bar{k}$, and $\check{X}_{\bar{v}} \cong \operatorname{Spec}\bar{k}[[\pi]]$. If we choose a geometric point $\bar{\eta}_{\bar{v}}$ on $\check{X}_{\bar{v}}$ over its generic point $\eta_{\bar{v}}$, then we have short exact sequence

$$1 \longrightarrow I_v \longrightarrow \Gamma_v \longrightarrow \operatorname{Gal}(\bar{k}/\bar{v}(k_v)) \longrightarrow 1,$$

where $\Gamma_v = \pi_1(\eta_v, \bar{\eta}_{\bar{v}})$ is the Galois group of F_v, and $I_v = \pi_1(\check{X}_{\bar{v}}, \bar{\eta}_{\bar{v}})$ is the inertia group. We shall use $\check{\mathcal{O}}_{\bar{v}}$ and $\check{F}_{\bar{v}}$ to denote the ring of functions on $\check{X}_{\bar{v}}$ and $\check{X}_{\bar{v}}^\bullet$, respectively.

We retain notations from Chapter 2 and let G be a quasi-split reductive group over $\mathcal{O}_v$ associated with a $\operatorname{Out}(\mathbf{G})$-torsor ϑ_G on X_v and similarly a pointed version $\vartheta_G^\bullet$ if we fix a geometric point $\bar{v}$ of X_v. Equivalently, if we fix a geometric generic point $\bar{\eta}_{\bar{v}}$, we have a homomorphism $\vartheta_G^\bullet\colon \Gamma_v \to \operatorname{Out}(\mathbf{G})$ which factors through $\operatorname{Gal}(\bar{k}/\bar{v}(k_v))$, because ϑ_G is always trivial over $\check{X}_v$.

4.1 Definition and Generalities

Let $\lambda \in \check{\mathbb{X}}(T)_+$ be a dominant F_v-rational cocharacter. We denote by C_G^λ the double coset $G(\mathcal{O}_v)\pi^\lambda G(\mathcal{O}_v)$ in Cartan decomposition of $G(F_v)$. We also denote by $\mathsf{C}_G^{\leq\lambda}$ the union of all C_G^μ such that $\mu \leq \lambda$ (i.e., $\lambda - \mu \in \mathbb{N}\check{\Delta}$). Let $\gamma \in G(F_v)$ be a regular semisimple element. We are interested in the sets

$$\begin{aligned} \mathsf{M}_G^\lambda(\gamma) &:= \left\{ g \in G(F_v)/G(\mathcal{O}_v) \,\middle|\, \mathrm{Ad}_g^{-1}(\gamma) \in \mathsf{C}_G^\lambda \right\}, \\ \mathsf{M}_G^{\leq\lambda}(\gamma) &:= \left\{ g \in G(F_v)/G(\mathcal{O}_v) \,\middle|\, \mathrm{Ad}_g^{-1}(\gamma) \in \mathsf{C}_G^{\leq\lambda} \right\}, \end{aligned} \tag{4.1.1}$$

on which we will later impose structures of k-varieties.

In order to best establish the connection to reductive monoids, it is best to generalize the definition (4.1.1) as follows. Let $G^{\mathrm{ab}} = G/G^{\mathrm{der}}$, then the homomorphism $T \to G^{\mathrm{ab}} \times T^{\mathrm{ad}}$ is étale. We have the induced monomorphism of lattices $\check{\mathbb{X}}(T) \to \check{\mathbb{X}}(G^{\mathrm{ab}}) \times \check{\mathbb{X}}(T^{\mathrm{ad}})$ with finite index, and the same holds at F_v-rational level. Denote the image of λ under this map by $(\lambda_{\mathrm{ab}}, \lambda_{\mathrm{ad}})$. It is clear that for any $\gamma \in G(F_v)$, the condition $\gamma \in \mathsf{C}_G^\lambda$ is the same as $\gamma_{\mathrm{ad}} \in \mathsf{C}_{G^{\mathrm{ad}}}^{\lambda_{\mathrm{ad}}}$ and $\gamma_{\mathrm{ab}} \in \pi^{\lambda_{\mathrm{ab}}} G^{\mathrm{ab}}(\mathcal{O}_v)$. Therefore if we define, for $\lambda \in \check{\mathbb{X}}(T^{\mathrm{ad}})$ and $\gamma \in G^{\mathrm{ad}}(F_v)$,

$$\begin{aligned} \mathsf{M}_G^\lambda(\gamma) &:= \left\{ g \in G(F_v)/G(\mathcal{O}_v) \,\middle|\, \mathrm{Ad}_g^{-1}(\gamma) \in \mathsf{C}_{G^{\mathrm{ad}}}^\lambda \right\}, \\ \mathsf{M}_G^{\leq\lambda}(\gamma) &:= \left\{ g \in G(F_v)/G(\mathcal{O}_v) \,\middle|\, \mathrm{Ad}_g^{-1}(\gamma) \in \mathsf{C}_{G^{\mathrm{ad}}}^{\leq\lambda} \right\}, \end{aligned} \tag{4.1.2}$$

then the sets in (4.1.1), if non-empty, are respectively isomorphic to the sets in (4.1.2). By replacing X_v with $\check{X}_v$, we have analogously defined sets, which we will call the $\bar{k}$-points of the corresponding sets, and they are exactly the set of $\bar{k}$-points of the corresponding schemes once we define them.

4.1.1 Cartan Decomposition Let $\mathfrak{M} \in \mathcal{FM}(G^{\mathrm{sc}})$. Recall we have abelianization $\mathfrak{A}_{\mathfrak{M}}$ and its subtorus $\mathfrak{A}_{\mathfrak{M}}^\times$. We call an F_v-cocharacter $\lambda \in \check{\mathbb{X}}(\mathfrak{A}_{\mathfrak{M}}^\times)$ *dominant* if it is contained in the cocharacter cone $\mathscr{C}(\mathfrak{A}_{\mathfrak{M}})$ or equivalently, $\pi^\lambda \in \mathfrak{A}_{\mathfrak{M}}(\mathcal{O}_v)$. For each dominant λ, we define a reductive monoid $\mathcal{O}_v$-scheme $\mathfrak{M}^\lambda$ by the pullback diagram

$$\begin{array}{ccc} \mathfrak{M}^\lambda & \longrightarrow & \mathfrak{M} \times \mathfrak{A}_{\mathfrak{M}}^\times \\ \big\downarrow & & \big\downarrow {\scriptstyle (x,z)\mapsto \alpha_{\mathfrak{M}}(x)z} \\ \operatorname{Spec} \mathcal{O}_v & \xrightarrow{\pi^\lambda} & \mathfrak{A}_{\mathfrak{M}} \end{array},$$

and by replacing $\mathfrak{M}$ with the big-cell locus $\mathfrak{M}^\circ$, we obtain an open subscheme $\mathfrak{M}^{\circ\lambda}$ of $\mathfrak{M}^\lambda$.

Remark 4.1.2. Note the difference in notation compared to [Chi19]: if $G = G^{\mathrm{sc}}$ and $\mathfrak{M} = \mathrm{Env}(G)$, then our $\mathfrak{M}^\lambda$ is the same as $\mathrm{Vin}_G^{-w_0(\lambda)}$ (not Vin_G^λ) in [Chi19]. The reason is that in [Chi19], they only consider the universal monoid, and taking the involution by $-w_0$ saves a lot of notations later on. Here more general framework is considered where there is no natural involution by $-w_0$ on $\mathscr{C}(\mathfrak{A}_{\mathfrak{M}})$.

Recall for $\mathfrak{M} \in \mathcal{FM}(G^{\mathrm{sc}})$, we can choose an excellent morphism $\mathfrak{M} \to \mathrm{Env}(G^{\mathrm{sc}})$ so that we have induced map $\check{\mathbb{X}}(\mathfrak{A}_{\mathfrak{M}}^\times) \to \check{\mathbb{X}}(T^{\mathrm{ad}})$. Let λ_{ad} be the image of $\lambda \in \check{\mathbb{X}}(\mathfrak{A}_{\mathfrak{M}}^\times)$ under this map. Using the same argument in [Chi19, Lemma 2.5.1], we can show the following results.

Lemma 4.1.3. *We have a disjoint union of $\mathfrak{M}^\times(\mathcal{O}_v)$-stable subsets*

$$\mathfrak{M}(\mathcal{O}_v) \cap \mathfrak{M}^\times(F_v) = \bigcup_{\lambda \in \mathscr{C}(\mathfrak{A}_{\mathfrak{M}})} \mathfrak{M}^\lambda(\mathcal{O}_v).$$

Moreover, let $g \in \mathfrak{M}^\times(F_v)$, then $g \in \mathfrak{M}^\lambda(\mathcal{O}_v)$ (resp. $\mathfrak{M}^{\circ\lambda}(\mathcal{O}_v)$) if and only if $\alpha_{\mathfrak{M}}(g) \in \pi^\lambda \mathfrak{A}_{\mathfrak{M}}^\times(\mathcal{O}_v)$, and the image of g in $G^{\mathrm{ad}}(F_v)$ belongs to $\mathsf{C}_{G^{\mathrm{ad}}}^{\leq -w_0(\lambda_{\mathrm{ad}})}$ (resp. $\mathsf{C}_{G^{\mathrm{ad}}}^{-w_0(\lambda_{\mathrm{ad}})}$).

4.1.4 Hodge-Newton decomposition and Kottwitz map For $\gamma \in G(F_v)$, one can define a dominant element $\nu_\gamma \in \check{\mathbb{X}}(T)_{\mathbb{Q}}$, called the *Newton point* of γ, that captures the F_v-valuations (called the *slopes*) of eigenvalues of γ in G-representations. See, for example, [KV12, § 2]. In [Kot97], Kottwitz defines (after fixing an algebraic closure $F_v \to \bar{F}_v$, or equivalently, a geometric point $\bar{\eta}_{\bar{v}}$ over η_v) a canonical group homomorphism

$$\kappa_G \colon G(F_v) \longrightarrow \mathbb{X}(Z(\check{G})^{I_v})^{\sigma_v},$$

where I_v is the inertia group of F_v. Here our group is unramified, so it simplifies to a homomorphism

$$\kappa_G \colon G(F_v) \longrightarrow (\check{\mathbb{X}}(T)/\mathbb{Z}\check{\Phi})_{F_v} = \pi_1(G)_{F_v}.$$

This is the *Kottwitz map*. One can also see [KV12, § 3.1] for a description in this simplified situation.

Let p_G be the natural quotient $\check{\mathbb{X}}(T) \to \pi_1(G)$. The following observation is crucial.

Lemma 4.1.5. *Let $\gamma \in G^{\mathrm{ad}}(F_v)$ and $\lambda \in \check{\mathbb{X}}_{F_v}(T^{\mathrm{ad}})$. Suppose $\kappa_G(\gamma) = p_G(\lambda)$. Then there exists an element $\gamma_\lambda \in \mathrm{Env}(G^{\mathrm{sc}})^\times(F_v)$ such that*

(1) the image of γ_λ in $G^{\mathrm{ad}}(F_v)$ is γ

(2) $\alpha_{\mathrm{Env}(G^{\mathrm{sc}})}(\gamma_\lambda) \in \pi^{-w_0(\lambda)}\mathfrak{A}^\times_{\mathrm{Env}(G^{\mathrm{sc}})}(\mathcal{O}_v)$.

Proof. This is basically [Chi19, Lemma 3.1.5], except for the fact that residue field k_v is no longer algebraically closed (or G no longer split). The key in Chi's proof is the surjectivity of map $\mathbf{G}_+(F_v) \to \mathbf{G}^{\mathrm{ad}}(F_v)$. After twisting, it is still true that $G_+(F_v) \to G^{\mathrm{ad}}(F_v)$ is surjective, because the kernel is the unramified torus T^{sc} and the Frobenius acts by permuting a basis of $\mathbb{X}(T^{\mathrm{sc}})$, in other words, T^{sc} is an induced torus, thus $\mathrm{H}^1(F_v, T) = 0$. The remaining part of the proof then proceeds easily. ∎

4.1.6 Non-emptiness The first question is when the sets (4.1.1) and (4.1.2) are non-empty. It is settled in the proposition below.

Proposition 4.1.7 ([Chi19, Proposition 3.1.6]). *Suppose $\gamma \in G(F_v)$ (resp. $G^{\mathrm{ad}}(F_v)$) is regular semisimple and $\lambda \in \check{\mathbb{X}}(T)_{F_v}$ (resp. $\check{\mathbb{X}}(T^{\mathrm{ad}})_{F_v}$), then the followings are equivalent:*

(1) *The set of* $\bar{k}$*-points in* $\mathsf{M}_G^\lambda(\gamma)$ *is non-empty;*

(2) *The set of* $\bar{k}$*-points in* $\mathsf{M}_G^{\leq\lambda}(\gamma)$ *is non-empty;*

(3) $\kappa_G(\gamma) = p_G(\lambda)$*, and* $\nu_\gamma \leq_{\mathbb{Q}} \lambda$*, the latter meaning* $\lambda - \nu_\gamma$ *is a non-negative* $\mathbb{Q}$*-combination of positive roots;*

(4) $\kappa_G(\gamma) = p_G(\lambda)$*, and* $\chi_{\mathrm{Env}(G^{\mathrm{sc}})}(\gamma_\lambda) \in \mathfrak{C}_{\mathrm{Env}(G^{\mathrm{sc}})}(\mathcal{O}_v)$*, where* γ_λ *is as defined in Lemma 4.1.5.*

4.1.8 Local affine Grassmannian The *affine Grassmannian* Gr_G is the functor sending a k-scheme S to the set of pairs (E, ϕ) where E is a G-torsor on $X_{v,S} = X_v \hat{\times} S$ and ϕ is a trivialization of E over $X_{v,S}^\bullet = X_v^\bullet \hat{\times} S$. Here $X_{v,S}$ is the completion of $X_v \times_k S$ at $v \times_k S$, and $X_{v,S}^\bullet$ is the (open) complement of $\{v\} \times_k S$. *A priori*, $X_{v,S}$ is only a formal scheme, not a scheme, but in this simple case it is easy to see it is representable by a scheme. It is well-known that Gr_G is represented by an ind-projective ind-scheme of ind-finite-type over k. If we fix a k-embedding $k_v \to \bar{k}$, we have an analogous definition of affine Grassmannian $\mathsf{Gr}_{G,\bar{v}}$ over $\bar{k}$. We have the isomorphism

$$\mathsf{Gr}_G \times_k \bar{k} \simeq \prod_{\bar{v}\colon k_v \to \bar{k}} \mathsf{Gr}_{G,\bar{v}}.$$

Since G has connected fibers, we may also regard Gr_G as the quotient sheaf $\mathbb{L}G/\mathbb{L}^+G$, where $\mathbb{L}^+G$ is the arc space functor of G sending S to the set $G(X_{v,S})$, and $\mathbb{L}G$ is the loop space functor sending S to $G(X_{v,S}^\bullet)$. See, for example, [Zhu17]. See also Chapter 5, where we will review a global construction of affine Grassmannians such that the fibers are products of the local version here.

Remark 4.1.9. The construction and representability of affine Grassmannian Gr_G requires only G to be a smooth affine group scheme, not necessarily reductive. The arc space and loop space functors both make sense for any k-scheme, and in some sense behave “well enough” for normal varieties.

4.1.10 Algebraic structures We now impose algebraic structures on (4.1.1) promised at the beginning. Following [Chi19], we introduce two approaches to this, one of which relates to reductive monoids.

We regard the Cartan double coset C_G^λ as a sub-presheaf $\mathbb{L}^+G\pi^\lambda\mathbb{L}^+G \subset \mathbb{L}G$. We define a sheaf $\mathcal{M}_{G,v}^\lambda(\gamma)$ over k to be the sheaf associated with presheaf

$$\operatorname{Spec} R \longmapsto \left\{g \in \mathsf{Gr}_G(R) \,\middle|\, g^{-1}\gamma g \in \mathsf{C}_G^\lambda(R)\right\},$$

which has an ind-scheme structure. The algebraic structure on $\mathsf{M}_{G,v}^\lambda(\gamma)$ is then the reduced ind-subscheme structure induced by $\mathcal{M}_{G,v}^\lambda(\gamma)$. Similarly, we have an algebraic structure on $\mathsf{M}_{G,v}^{\le\lambda}(\gamma)$.

Another approach to the algebraic structure on $\mathsf{M}_G^\lambda(\gamma)$ is to use reductive monoids. Let $\mathfrak{M} \in \mathcal{FM}(G^{\mathrm{sc}})$ and $\gamma_{\mathfrak{M}} \in \mathfrak{M}^\times(F_v)^{\mathrm{rs}}$ such that $a = \chi_{\mathfrak{M}}(\gamma_{\mathfrak{M}}) \in \mathfrak{C}_{\mathfrak{M}}(\mathcal{O}_v)$.

Definition 4.1.11. We make multiple closely related definitions:

(1) The *multiplicative affine Springer fiber* $\mathcal{M}_{G,v}(\gamma_{\mathfrak{M}})$ associated with $\gamma_{\mathfrak{M}}$ is the functor that associates to k-scheme S the isomorphism classes of pairs (h, ι) where h is an $X_{v,S}$-point of $[\mathfrak{M}/\operatorname{Ad}(G)]$ over a:

$$\begin{array}{ccc} X_{v,S} & \xrightarrow{h} & [\mathfrak{M}/\operatorname{Ad}(G)] \\ \downarrow & & \downarrow{\scriptstyle \chi_{\mathfrak{M}}} \\ X_v & \xrightarrow{a} & \mathfrak{C}_{\mathfrak{M}} \end{array},$$

and ι is an isomorphism between the restriction of h to $X_{v,S}^\bullet$ and the $X_{v,S}^\bullet$-point of $[\mathfrak{M}/\operatorname{Ad}(G)]$ induced by $\gamma_{\mathfrak{M}}$.

(2) The (open) subfunctors $\mathcal{M}_{G,v}^\circ(\gamma_{\mathfrak{M}})$ (resp. $\mathcal{M}_{G,v}^{\mathrm{reg}}(\gamma_{\mathfrak{M}})$) is defined similarly by replacing $\mathfrak{M}$ with $\mathfrak{M}^\circ$ (resp. $\mathfrak{M}^{\mathrm{reg}}$).

(3) For $a \in \mathfrak{C}_{\mathfrak{M}}(\mathcal{O}_v) \cap \mathfrak{C}_{\mathfrak{M}}^{\times}(F_v)$, we denote by $\mathcal{M}_{G,v}(a)$ (resp. $\mathcal{M}_{G,v}(a)$, etc.) a fixed choice of $\mathcal{M}_{G,v}(\gamma_{\mathfrak{M}})$ (resp. $\mathcal{M}_{G,v}(\gamma_{\mathfrak{M}})$, etc.) where $\chi_{\mathfrak{M}}(\gamma_{\mathfrak{M}}) = a$ (which always exists thanks to Theorem 2.4.24). The dependence on such choice will be emphasized if not clear from context.

(4) When the group G is clear from context, we drop it from subscripts.

Let $\gamma \in G^{\mathrm{ad}}(F_v)^{\mathrm{rs}}$ and $\lambda \in \mathscr{E}(\mathfrak{A}_{\mathfrak{M}})$ a dominant F_v-cocharacter. Recall we have $\lambda_{\mathrm{ad}} \in \check{\mathbb{X}}(T^{\mathrm{ad}})_+$. Suppose $\mathsf{M}_{G,v}^{\lambda^{\mathrm{ad}}}(\gamma)$ is nonempty, then by Proposition 4.1.7, we have the element $\gamma_\lambda \in \mathrm{Env}(G^{\mathrm{sc}})^{\times}(F_v)^{\mathrm{rs}}$ defined by Lemma 4.1.5. We may lift γ_λ to an element in $\mathfrak{M}^{\times}(F_v)$, still denoted by γ_λ. Let $a = \chi_{\mathfrak{M}}(\gamma_\lambda)$, then it is easy to see that

$$\mathsf{M}_{G,v}^{\leq\lambda_T}(\gamma) \cong \mathcal{M}_{G,v}(a)^{\mathrm{red}} \text{ and } \mathsf{M}_{G,v}^{\lambda_T}(\gamma) \cong \mathcal{M}_{G,v}(a)^{\circ,\mathrm{red}}. \tag{4.1.3}$$

Conversely, for any $a \in \mathfrak{C}_{\mathfrak{M}}(\mathcal{O}_v) \cap \mathfrak{C}_{\mathfrak{M}}^{\times}(F_v)^{\mathrm{rs}}$ whose image in $\mathfrak{A}_{\mathrm{Env}(G^{\mathrm{sc}})}$ is contained in $\pi^{-w_0(\lambda_{\mathrm{ad}})}T^{\mathrm{ad}}(\mathcal{O}_v)$. Suppose $\gamma_{\mathfrak{M}}$ maps to a under $\chi_{\mathfrak{M}}$ (such element always exists by Theorem 2.4.24). Suppose $\gamma \in G^{\mathrm{ad}}(F_v)$ is the image of $\gamma_{\mathfrak{M}}$ in $G^{\mathrm{ad}}(F_v)$, then (4.1.3) still holds. Note that a k-point may not exist for such functors.

4.1.12 Like affine Grassmannian, we may define everything by replacing X_v with $\check{X}_{\bar{v}}$, denoted by subscript $\bar{v}$, e.g. $\mathsf{M}_{G,\bar{v}}^{\lambda}(\gamma)$, and so on. If we base change $\mathsf{M}_{G,v}^{\lambda}(\gamma)$ to $\bar{k}$, we obtain isomorphism

$$\mathsf{M}_{G,v}^{\lambda}(\gamma) \times_k \bar{k} \simeq \prod_{\bar{v}\colon k_v \to \bar{k}} \mathsf{M}_{G,\bar{v}}^{\lambda}(\gamma),$$

and similarly for $\mathcal{M}_{G,v}(a)$, and so on.

4.1.13 Let $\mathcal{L}$ be a $Z_{\mathfrak{M}}$-torsor over $\mathcal{O}_v$. Since $\mathcal{O}_v$ has finite residue field and $Z_{\mathfrak{M}}$ is a torus, $\mathcal{L}$ is necessarily a trivial torsor. Therefore $\mathfrak{M}$ is isomorphic to $\mathfrak{M}_{\mathcal{L}}$. By replacing $\mathfrak{M}$ with $\mathfrak{M}_{\mathcal{L}}$, we may also define $\mathcal{M}_{G,v}(a)$ for $a \in \mathfrak{C}_{\mathfrak{M},\mathcal{L}}(\mathcal{O}_v) \cap \mathfrak{C}_{\mathfrak{M},\mathcal{L}}^{\times,\mathrm{rs}}(F_v)$, still depending on a choice of a lift of a to $\mathfrak{M}_{\mathcal{L}}^{\times}(F_v)$. If $\mathcal{L}$ has a c-th root (c being the order of $\mathbf{Z}^{\mathrm{sc}}$) and all simple factors of G of types A_{2m} are split, then a Steinberg quasi-section exists, and we use it to define $\mathcal{M}_{G,v}(a)$.

4.2 Dimension Formula

In the following few sections we consider some geometric properties of multiplicative affine Springer fibers, and thus it's harmless to base change to $\bar{k}$ for our discussion.

Suppose we have a non-empty (as a sheaf) multiplicative affine Springer fiber $\mathcal{M}_{G,\bar{v}}^{\lambda}(\gamma)$, then γ_λ as in Lemma 4.1.5 exists, and let $a = \chi_{\mathrm{Env}(G)}(\gamma_\lambda)$. Recall the invariants $d(\gamma)$ from § 3.2 and $c(a) = c(\gamma)$, $d_+(a)$, and $\delta(a)$ from § 3.3. Sometimes we want to emphasize the dependence on $\bar{v}$ and denote these invariants as $c_{\bar{v}}(a)$, $\delta_{\bar{v}}(a)$, etc.

Theorem 4.2.1 ([Chi19, Theorem 1.2.1]). *The functors $\mathcal{M}_{G,\bar{v}}^{\lambda,\mathrm{red}}(\gamma)$ and $\mathcal{M}_{G,\bar{v}}^{\leq\lambda,\mathrm{red}}(\gamma)$ are represented by an equi-dimensional $\bar{k}$-scheme locally of finite type, with dimension*

$$\begin{aligned}\dim \mathcal{M}_{G,\bar{v}}^{\lambda}(\gamma) = \dim \mathcal{M}_{G,\bar{v}}^{\leq\lambda}(\gamma) &= \langle \rho, \lambda\rangle + \frac{d_{\bar{v}}(\gamma) - c_{\bar{v}}(\gamma)}{2} \\ &= \frac{d_{\bar{v}+}(a) - c_{\bar{v}}(a)}{2} \\ &= \delta_{\bar{v}}(a).\end{aligned}$$

4.3 Symmetry and Irreducible Components

Given $a \in \mathfrak{C}_{\mathfrak{M}}(\mathcal{O}_v) \cap \mathfrak{C}_{\mathfrak{M}}^{\times}(F_v)^{\mathrm{rs}}$, one can pullback the regular centralizer group scheme $\mathfrak{J}_{\mathfrak{M}}$ to a, viewed as an $\mathcal{O}_v$-scheme denoted by $\mathfrak{J}_a$. Define the local Picard functor as the affine Grassmannian

$$\mathcal{P}_v(a) \coloneqq \mathsf{Gr}_{\mathfrak{J}_a}.$$

This group functor naturally acts on $\mathcal{M}_v(a)$ as follows: any S-point of $\mathcal{M}_v(a)$ is a tuple (E, ϕ, ι) where E is a G-bundle on $X_{v,S}$, ϕ is a G-equivariant map $E \to \mathfrak{M}$, and ι is an isomorphism $(E_0, \gamma_{\mathfrak{M}}) \to (E, \phi)$ over the punctured disc $X^{\bullet}_{v,S}$, where E_0 is the trivial G-torsor. On the other hand, a point of $\mathcal{P}_v(a)$ is a $\mathfrak{J}_a$-torsor $E_{\mathfrak{J}}$ on $X_{v,S}$ with a trivialization τ over $X^{\bullet}_{v,S}$. It sends (E, ϕ, ι) to the following tuple (E', ϕ', ι'): the pair (E', ϕ') is defined as

$$\phi' : E' \coloneqq E \times^{\mathfrak{J}_a}_{\phi, \mathfrak{M}} E_{\mathfrak{J}} \longrightarrow \mathfrak{M},$$

where $\mathfrak{J}_a$ acts on the fibers of ϕ through canonical map $\chi^*_{\mathfrak{M}} \mathfrak{J}_{\mathfrak{M}} \to I_{\mathfrak{M}}$. The trivialization τ induces an isomorphism

$$E|_{X^{\bullet}_{v,S}} \longrightarrow E'|_{X^{\bullet}_{v,S}},$$

whose composition with ι is ι'.

4.3.1 The sharp contrast to the Lie algebra case is that, unlike the former situation, the open $\mathcal{P}_a$-orbits in $\mathcal{M}(a)$ is in general not a $\mathcal{P}_a$-torsor, but a union of them. Even worse, the union of all open orbits is not dense in $\mathcal{M}(a)$, meaning there are some irreducible components of $\mathcal{M}(a)$ that are stratified into infinitely many orbits. Fortunately, the free action of a lattice quotient of $\mathcal{P}_a$ is still present, in other words, $\mathcal{M}(a)$ still possesses a very large symmetry group. More or less equivalently, the action of $\mathcal{P}_a$ on the set $\mathrm{Irr}(\mathcal{M}(a))$ of irreducible components is still quite nice, and it actually has strong connection with the representations of complex dual group $\check{G}_{\mathbb{C}}$ as we shall see below.

4.3.2 We may also base change to $\bar{k}$, then we have

$$\mathcal{P}_v(a) \times_k \bar{k} \simeq \prod_{\bar{v} :\, k_v \to \bar{k}} \mathcal{P}_{\bar{v}}(a).$$

The action of $\mathcal{P}_v(a)$ on $\mathcal{M}(a)$ is compatible with each direct factor. So starting from this point we will base change everything to $\bar{k}$ and consider $\mathcal{M}_{\bar{v}}(a)$ for a fixed $\bar{v}$ (equivalently, we replace X_v with $\breve{X}_{\bar{v}}$.

4.3.3 Suppose $\gamma \in G(\breve{F}_{\bar{v}})$ is such that the Newton point $\nu = \nu_\gamma$ is an integral cocharacter in $\breve{\mathbb{X}}(T)$. We may consider more generally $\gamma \in G^{\mathrm{ad}}(\breve{F}_{\bar{v}})$. This happens when γ is unramified, but not exclusively. Let $m_{\lambda\nu}$ be the weight multiplicity of the weight space ν in the irreducible representation of $\check{G}_{\mathbb{C}}$ with highest weight λ. If $\nu \in W\lambda$, then $m_{\lambda\nu} = 1$. When λ is a central cocharacter, in other words, $\langle \alpha, \lambda \rangle = 0$ for all $\alpha \in \Phi$, or equivalently, $\lambda_{\mathrm{ad}} = 0$, and $\mathcal{M}_G^\lambda(\gamma) \neq \emptyset$, then we always have $\nu = \lambda$ and so $m_{\lambda\nu} = 1$.

Using (4.1.3), the condition of λ being central corresponds to that a is contained in $\mathfrak{C}_{\mathfrak{M}}^\times$ (i.e. has no intersection with the numerical boundary divisor). The condition γ being unramified corresponds to $c(a) = 0$, because recall that $c(a)$ is the difference between the absolute rank and $\breve{F}_{\bar{v}}$-split rank of the centralizer of γ_λ (which is a torus).

Definition 4.3.4. With the notations above, we call $a \in \mathfrak{C}_{\mathfrak{M}}(\mathcal{O}_v) \cap \mathfrak{C}_{\mathfrak{M}}^\times(F_v)^{\mathrm{rs}}$ *unramified* if $c_{\bar{v}}(a) = 0$ for one (equivalently, all) $\bar{v}$, and *invertible* if it is contained in $\mathfrak{C}_{\mathfrak{M}}^\times(\mathcal{O}_v)$.

Theorem 4.3.5 ([Chi19, Corollaries 3.5.3 and 3.8.2])**.** *Suppose either $a \in \mathfrak{C}_{\mathfrak{M}}(\breve{\mathcal{O}}_{\bar{v}}) \cap \mathfrak{C}_{\mathfrak{M}}^\times(\breve{F}_{\bar{v}})$ is either unramified or invertible, then the number of irreducible components in $\mathcal{M}_{\bar{v}}(a)$ modulo the action of $\mathcal{P}_{\bar{v}}(a)$ is $m_{\lambda\nu}$. Moreover, when a is invertible, there is a unique open dense $\mathcal{P}_{\bar{v}}(a)$-orbit in $\mathcal{M}_{\bar{v}}(a)$ being the regular locus $\mathcal{M}_{\bar{v}}(a)^{\mathrm{reg}}$.*

Corollary 4.3.6 ([Chi19, corollary 3.8.3])**.** *Suppose a is invertible and $d_+(a) \leq 1$, then $\mathcal{M}_{\bar{v}}(a) = \mathcal{M}_{\bar{v}}(a)^{\mathrm{reg}}$ and is itself a $\mathcal{P}_{\bar{v}}(a)$-torsor.*

The weight multiplicity $m_{\lambda\nu}$ in Theorem 4.3.5 is obtained by connecting $\mathcal{M}_{\bar{v}}(a)$ with MV-cycles, which we will elaborate in § 4.5.

4.4 Néron Models and Connected Components

Following [Ngô10] and [Chi19], we have a more precise description of the $\mathcal{P}_{\bar{v}}(a)$ using Néron models. By definition, the Néron model of $\mathfrak{J}_a$ is a unique (up to a unique isomorphism) smooth group scheme $\mathfrak{J}_a^\flat$ over $\breve{\mathcal{O}}_{\bar{v}}$ together with a homomorphism $\mathfrak{J}_a \to \mathfrak{J}_a^\flat$ that is an isomorphism over the generic point and satisfies the following universal property: for any smooth group scheme J over $\breve{\mathcal{O}}_{\bar{v}}$ with an $\breve{F}_{\bar{v}}$-isomorphism $J_{\breve{F}_{\bar{v}}} \to \mathfrak{J}^\flat_{a,\breve{F}_{\bar{v}}}$, there is a canonical lift to an $\breve{\mathcal{O}}_{\bar{v}}$-homomorphism $J \to \mathfrak{J}_a^\flat$. Néron model necessarily exists for smooth commutative $\breve{\mathcal{O}}_{\bar{v}}$-group schemes, and can be explicitly constructed using cameral covers.

4.4.1 Recall we have the cameral cover $\pi_{\mathfrak{M}} \colon \mathfrak{T}_{\mathfrak{M}} \to \mathfrak{C}_{\mathfrak{M}}$. The group scheme $\mathfrak{J}_{\mathfrak{M}}$ is canonically isomorphic to an open subgroup of the fixed point scheme

$$\mathfrak{J}_{\mathfrak{M}}^1 = (\pi_{\mathfrak{M}*}(T \times \mathfrak{T}_{\mathfrak{M}}))^W .$$

The scheme $\mathfrak{J}_a$ is the pullback of $\mathfrak{J}_{\mathfrak{M}}$ via a. Let local cameral cover $\pi_a \colon \tilde{X}_a \to \breve{X}_{\bar{v}}$ be the pullback of $\pi_{\mathfrak{M}}$ through a. Then by proper base change, one can also define $\mathfrak{J}_a$ and $\mathfrak{J}_a^1$ using the same construction applied to π_a.

The cameral cover $\pi_{\mathfrak{M}}$ is flat with a Cohen-Macaulay source and a regular target. Thus $\tilde{X}_a$ is Cohen-Macaulay as well. Since a is generically regular semisimple, $\tilde{X}_a^\bullet$ is regular being a W-étale cover of $\breve{X}_{\bar{v}}^\bullet$. So $\tilde{X}_a$ is a reduced scheme. Since $\tilde{X}_a$ is one-dimensional, its normalization $\tilde{X}_a^\flat$ is regular. The Néron model can be shown to be the group scheme

$$\mathfrak{J}_a^\flat = \pi_{a*}^\flat (T \times \tilde{X}_a^\flat)^W,$$

where $\pi_a^\flat$ is the natural map $\tilde{X}_a^\flat \to \breve{X}_{\bar{v}}$. The same proof in [Ngô10, 3.8.2] applies to the case here.

Lemma 4.4.2. *We have formula*

$$\delta_{\bar{v}}(a) = \dim \mathcal{P}_{\bar{v}}(a) = \dim_{\bar{k}} (\mathfrak{t}_{\breve{\mathcal{O}}_{\bar{v}}} \otimes_{\breve{\mathcal{O}}_{\bar{v}}} (\tilde{\mathcal{O}}_a^{\flat}/\tilde{\mathcal{O}}_a))^W,$$

where $\tilde{\mathcal{O}}_a$ (resp. $\tilde{\mathcal{O}}_a^{\flat}$) is the ring of functions of $\tilde{X}_a$ (resp. $\tilde{X}_a^{\flat}$).

Proof. This is essentially [Ngô10, Corollaire 3.8.3] and [Chi19, Corollary 3.3.4]. The key is that $\mathfrak{J}_a$ and $\mathfrak{J}_a^{\flat}$ are smooth so we only need to compute the dimension of the tangent space at the identity of $\mathcal{P}_{\bar{v}}$, and then it is clear from the Galois descriptions of $\mathfrak{J}_a$ and $\mathfrak{J}_a^{\flat}$. ∎

4.4.3 The morphism $\mathfrak{J}_a \to \mathfrak{J}_a^{\flat}$ induces morphism of group ind-schemes

$$p_{\bar{v}} \colon \mathcal{P}_{\bar{v}}(a) \longrightarrow \mathcal{P}_{\bar{v}}^{\flat}(a) \coloneqq \mathsf{Gr}_{\mathfrak{J}_a^{\flat}}.$$

With exact same proof, we have:

Lemma 4.4.4 ([Ngô10, Lemme 3.8.1]). *The group $\mathcal{P}_{\bar{v}}^{\flat}(a)$ is homeomorphic to a finitely generated free abelian group (viewed as a discrete $\bar{k}$-scheme). The map $p_{\bar{v}}$ is surjective, and its kernel $\mathcal{R}_{\bar{v}}(a)$ is an affine group scheme of finite type over $\bar{k}$.*

Corollary 4.4.5. *The dimension of $\mathcal{R}_{\bar{v}}(a)$ is exactly the local δ-invariant $\delta_{\bar{v}}(a)$.*

4.4.6 Since $\pi_0(\mathcal{P}_{\bar{v}}(a))^{\mathrm{red}}$ is a finitely generated abelian group and $\mathcal{R}_{\bar{v}}(a)$ is affine of finite type, $\pi_0(\mathcal{P}_{\bar{v}}^{\flat}(a))$ must be homeomorphic to the largest free quotient of $\pi_0(\mathcal{P}_{\bar{v}}(a))$. Call this lattice Λ_a. Since Λ_a is free, we can choose a lifting of it to $\mathcal{P}_{\bar{v}}(a)$ to define an action of Λ_a on $\mathcal{M}_{\bar{v}}(a)$.

Proposition 4.4.7. *The action of Λ_a on $\mathcal{M}_{\bar{v}}^{\mathrm{red}}(a)$ is free, and the quotient $\mathcal{M}_{\bar{v}}^{\mathrm{red}}(a)/\Lambda_a$ is a projective $\bar{k}$-scheme.*

Proof. This is essentially [Chi19, Theorem 3.6.2], in which the freeness is proved for an explicitly constructed lifting $\Lambda_a \to \mathcal{M}_{\bar{v}}(a)$, and the $\mathcal{M}_{\bar{v}}^{\mathrm{red}}(a)/\Lambda_a$ is shown to be a proper algebraic space. So we only need to slightly strengthen these statements.

First, the lifting of Λ_a can be arbitrary, because using the result of the said theorem, the stabilizer of any point in $\mathcal{M}_{\bar{v}}(a)$ is a subgroup of Λ_a of finite type, hence trivial because Λ_a is a lattice.

Moreover, the quotient space $\mathcal{M}_{\bar{v}}^{\mathrm{red}}(a)/\Lambda_a$ is a scheme because one can pick a large enough quasi-compact open subscheme $U \subset \mathcal{M}_{\bar{v}}(a)$ such that $\mathcal{M}_{\bar{v}}^{\mathrm{red}}(a)/\Lambda_a$ is the quotient of a finite étale equivalence relation R satisfying this condition: for any $x \in U$, its equivalence class as a subset of U is contained in an affine open subset. Therefore $\mathcal{M}_{\bar{v}}^{\mathrm{red}}(a)/\Lambda_a = U/R$ is a scheme. ∎

Corollary 4.4.8. *The stabilizers of the $\mathcal{P}_{\bar{v}}^{\mathrm{red}}(a)$-action are affine and contained in $\mathcal{R}_{\bar{v}}(a)$.*

4.4.9 Following [Ngô10, § 3.9], we give a precise description of connected components of $\mathcal{P}_{\bar{v}}(a)$. Recall we have open subscheme $\mathfrak{J}_a^0 \subset \mathfrak{J}_a$ of fiberwise neutral component. The quotient $\mathfrak{J}_a/\mathfrak{J}_a^0$ is supported over the closed point of $\check{X}_{\bar{v}}$, with fiber

$$\pi_0(\mathfrak{J}_a) = \mathfrak{J}_a(\check{\mathcal{O}}_{\bar{v}})/\mathfrak{J}_a^0(\check{\mathcal{O}}_{\bar{v}}).$$

We have exact sequence

$$1 \longrightarrow \pi_0(\mathfrak{J}_a) \longrightarrow \mathfrak{J}_a(\check{F}_{\bar{v}})/\mathfrak{J}_a^0(\check{\mathcal{O}}_{\bar{v}}) \longrightarrow \mathfrak{J}_a(\check{F}_{\bar{v}})/\mathfrak{J}_a(\check{\mathcal{O}}_{\bar{v}}) \longrightarrow 1.$$

Since $\bar{k}$ is algebraically closed, this is the same as exact sequence

$$1 \longrightarrow \pi_0(\mathfrak{J}_a) \longrightarrow \mathcal{P}_{\bar{v}}^0(a)(\bar{k}) \longrightarrow \mathcal{P}_{\bar{v}}(a)(\bar{k}) \longrightarrow 1,$$

where $\mathcal{P}_{\bar{v}}^0(a) = \mathsf{Gr}_{\mathfrak{J}_a^0}$. In other words, the morphism $\mathcal{P}_{\bar{v}}^0(a) \to \mathcal{P}_{\bar{v}}(a)$ is surjective. Thus we have exact sequence

$$\pi_0(\mathfrak{J}_a) \longrightarrow \pi_0(\mathcal{P}^0_{\bar{v}}(a)) \longrightarrow \pi_0(\mathcal{P}_{\bar{v}}(a)) \longrightarrow 1. \tag{4.4.1}$$

For any finitely generated abelian group Λ, we let Λ^* be its $\overline{\mathbb{Q}}_\ell$-Cartier dual

$$\Lambda^* = \operatorname{Spec} \overline{\mathbb{Q}}_\ell[\Lambda],$$

and conversely, for any diagonalizable group A over $\overline{\mathbb{Q}}_\ell$, we let $A^* = \mathbb{X}(A)$ be its character group.

Now we fix a trivialization of ϑ_G over $\check{X}_{\bar{v}}$, which identifies G with $\mathbf{G}$ together with associated pinnings. Over $\check{X}^\bullet_{\bar{v}}$, the cameral cover is a $\mathbf{W}$-étale cover. If we fix a geometric point in $\tilde{X}_a$ over the geometric generic point $\bar{\eta}_{\bar{v}} \in \check{X}_{\bar{v}}$, we have a homomorphism $\pi^\bullet_a \colon I_v \to \mathbf{W}$ (recall that I_v is the inertia group of F_v).

Proposition 4.4.10. *After fixing a geometric point of $\tilde{X}_a$ lying over $\bar{\eta}_{\bar{v}}$ as above, we have canonical isomorphisms of diagonalizable groups*

$$\pi_0(\mathcal{P}^0_{\bar{v}})^* \simeq \check{\mathbf{T}}^{\pi^\bullet_a(I_v)},$$
$$\pi_0(\mathcal{P}_{\bar{v}})^* \simeq \check{\mathbf{T}}(\pi^\bullet_a(I_v)),$$

where $\check{\mathbf{T}}(\pi^\bullet_a(I_v))$ is a subgroup of $\check{\mathbf{T}}^{\pi^\bullet_a(I_v)}$ consisting of elements κ such that $\pi^\bullet_a(I_v)$ is contained in the Weyl group $\mathbf{W_H}$ of the neutral component $\check{\mathbf{H}}$ of the centralizer of κ in $\check{\mathbf{G}}$.

The proof is essentially the same as [Ngô10, Proposition 3.9.2], but since the replacement of Lie algebras by reductive monoids is not completely parallel (e.g. one always considers monoids associated with G^{sc} instead of with G), we sketch a proof here. First we have a lemma concerning $\mathcal{P}^{\flat,0}_{\bar{v}}(a)$ associated with the neutral component of Néron model $\mathfrak{J}^{\flat,0}_a$.

Lemma 4.4.11 ([Ngô10, Lemme 3.9.3]). *The homomorphism* $\mathcal{P}^0_{\bar{v}}(a) \to \mathcal{P}^{\flat,0}_{\bar{v}}(a)$ *induces an isomorphism*

$$\pi_0(\mathcal{P}^0_{\bar{v}}(a)) \longrightarrow \mathfrak{J}_a(\check{F}_{\bar{v}})/\mathfrak{J}^{\flat,0}_a(\check{\mathcal{O}}_{\bar{v}}).$$

Proof. Both $\mathfrak{J}^0_a$ and $\mathfrak{J}^{\flat,0}_a$ have connected fibers, thus following the exact same steps for obtaining (4.4.1), we have exact sequence

$$1 = \pi_0(\mathfrak{J}^{\flat,0}_a) \longrightarrow \pi_0(\mathcal{P}^0_{\bar{v}}(a)) \longrightarrow \pi_0(\mathcal{P}^{\flat,0}_{\bar{v}}(a)) = \mathfrak{J}_a(\check{F}_{\bar{v}})/\mathfrak{J}^{\flat,0}_a(\check{\mathcal{O}}_{\bar{v}}) \longrightarrow 1$$

as desired. ∎

Lemma 4.4.12. *We have isomorphism*

$$\mathfrak{J}_a(\check{F}_{\bar{v}})/\mathfrak{J}^{\flat,0}_a(\check{\mathcal{O}}_{\bar{v}}) \simeq \check{\mathbb{X}}(\mathbf{T})_{\pi^\bullet_a(I_v)}.$$

Proof. This is [Ngô10, Lemme 3.9.4] and is a special case of [Kot85, Lemma 2.2]. It only uses certain properties of the functor $A \mapsto A(\check{F}_{\bar{v}})/A^{\flat,0}(\check{\mathcal{O}}_{\bar{v}})$ on the category of $\check{F}_{\bar{v}}$-tori, as well as the fact that $\mathfrak{J}_a|_{\check{F}_{\bar{v}}}$ is an $\check{F}_{\bar{v}}$-torus. ∎

Proof of Proposition 4.4.10. By duality we have $\check{\mathbb{X}}(\mathbf{T})_{\pi^\bullet_a(I_v)} = \mathbb{X}(\check{\mathbf{T}})_{\pi^\bullet_a(I_v)}$, so by combining the two lemmas above, we obtain

$$\pi_0(\mathcal{P}^0_{\bar{v}}(a)) \simeq \check{\mathbb{X}}(\mathbf{T})_{\pi^\bullet_a(I_v)},$$

and by taking Cartier dual we have

$$\pi_0(\mathcal{P}^0_{\bar{v}}(a))^* \simeq \check{\mathbf{T}}^{\pi^\bullet_a(I_v)}.$$

This is the first claim.

For the second claim, we use z-extensions. We have exact sequence of reductive group schemes

$$1 \longrightarrow G \longrightarrow G_1 \longrightarrow C \longrightarrow 1,$$

where G_1 has connected center. It is obtained from its split counterparts twisted by an $\mathrm{Out}(\mathbf{G})$-torsor. We also have exact sequence of tori

$$1 \longrightarrow T \longrightarrow T_1 \longrightarrow C \longrightarrow 1$$

and its split counterpart. Both G and G_1 act on $\mathfrak{M} \in \mathcal{FM}(G^{\mathrm{sc}})$ and they are compatible. In fact, we have $\mathfrak{C}_{\mathfrak{M}} = \mathfrak{M} /\!\!/ \mathrm{Ad}(G) \simeq \mathfrak{M} /\!\!/ \mathrm{Ad}(G_1)$. Let $\mathfrak{J}_1 \to \mathfrak{C}$ be the regular centralizer scheme associated with G_1-action, then it has connected fibers since G_1 has connected center. We also have exact sequence

$$1 \longrightarrow \mathfrak{J} \longrightarrow \mathfrak{J}_1 \longrightarrow C \longrightarrow 1,$$

and the fiber over a

$$1 \longrightarrow \mathfrak{J}_a \longrightarrow \mathfrak{J}_{1,a} \longrightarrow C_a \longrightarrow 1.$$

Since $\mathfrak{J}_a \to \mathfrak{J}_{1,a}$ is proper, $\mathfrak{J}_a(\breve{\mathcal{O}}_{\bar{v}} = \mathfrak{J}_a(\breve{F}_{\bar{v}}) \cap \mathfrak{J}_{1,a}(\breve{\mathcal{O}}_{\bar{v}}$. This implies that the map

$$\mathfrak{J}_a(\breve{F}_{\bar{v}})/\mathfrak{J}_a(\breve{\mathcal{O}}_{\bar{v}}) \longrightarrow \mathfrak{J}_{1,a}(\breve{F}_{\bar{v}})/\mathfrak{J}_{1,a}(\breve{\mathcal{O}}_{\bar{v}})$$

is injective. Since $C_a(\breve{F}_{\bar{v}})/C_a(\breve{\mathcal{O}}_{\bar{v}})$ is a finitely generated free abelian group, the neutral components in $\mathcal{P}_{\bar{v}}(a)$ and $\mathcal{P}_{\bar{v},1}(a)$ are homeomorphic. Thus $\pi_0(\mathcal{P}_{\bar{v}}(a)) \to \pi_0(\mathcal{P}_{\bar{v},1}(a)$ is injective. This means that $\pi_0(\mathcal{P}_{\bar{v}}(a))$ can be identified with the image of $\pi_0(\mathcal{P}_{\bar{v}}^0(a))$ in $\pi_0(\mathcal{P}_{\bar{v},1}(a))$.

Using the same argument as in the first claim, we have

$$\pi_0(\mathcal{P}^0_{\bar{v}}(a)) \simeq \check{\mathbb{X}}(\mathbf{T})_{\pi^\bullet_a(I_v)},$$
$$\pi_0(\mathcal{P}_{\bar{v},1}(a)) \simeq \check{\mathbb{X}}(\mathbf{T}_1)_{\pi^\bullet_a(I_v)},$$

since $\mathfrak{J}_{1,a}$ has connected fibers. Thus $\pi_0(\mathcal{P}_{\bar{v}}(a))^*$ may be identified with the image of homomorphism

$$\check{\mathbf{T}}_1^{\pi^\bullet_a(I_v)} \longrightarrow \check{\mathbf{T}}^{\pi^\bullet_a(I_v)}.$$

Let $\kappa_1 \in \check{\mathbf{T}}_1^{\pi^\bullet_a(I_v)}$ with image $\kappa \in \check{\mathbf{T}}$. Since $\check{\mathbf{G}}_1$ has a simply-connected derived subgroup, the centralizer $\check{\mathbf{H}}_1$ of κ_1 (a semisimple element) in $\check{\mathbf{G}}_1$ is connected, and its image in $\check{\mathbf{G}}$ is $\check{\mathbf{H}}$. Thus $\pi^\bullet_a(I_v)$ is contained in $\mathbf{W}_\mathbf{H}$. ∎

4.5 Connection with MV-cycles

In this section we study the special case of unramified conjugacy classes. In this case, [Chi19] established a bijection between the irreducible components of $\mathcal{M}_{G,\bar{v}}(a)$ modulo $\mathcal{P}_{\bar{v}}(a)$-action and that of certain Mirković-Vilonen (MV) cycles. Here we slightly expand their result to include Frobenius actions.

4.5.1 First, let us consider the case where $G = \mathbf{G}$ is split, and $\gamma \in \mathbf{T}(\breve{F}_{\bar{v}})^{\mathrm{rs}}$. Let λ be a dominant cocharacter of $\mathbf{T}$, and suppose $\gamma_\lambda \in \mathbf{M}$ as in Lemma 4.1.5 exists for some $\mathbf{M} \in \mathcal{FM}(\mathbf{G}^{\mathrm{sc}})$. Since $\gamma \in \mathbf{T}$, we have $\gamma_\lambda \in \bar{\mathbf{T}}_\mathbf{M}$. Let $a = \chi_\mathfrak{M}(\gamma_\lambda)$ and suppose it lies in $\mathbf{C}_\mathbf{M}(\breve{\mathcal{O}}_{\bar{v}})$. Then the $\breve{F}_{\bar{v}}$-torus $\mathbf{J}_{a,\breve{F}_{\bar{v}}}$ is canonically isomorphic to the maximal torus $\mathbf{T}_{\breve{F}_{\bar{v}}}$ itself. Thus its Néron model is just $\mathbf{T}_{\breve{\mathcal{O}}_{\bar{v}}}$. Recall we have the lattice Λ_a being the largest free quotient of $\pi_0(\mathcal{P}_{\bar{v}}(a))^{\mathrm{red}}$. In this case Λ_a is isomorphic to $\check{\mathbb{X}}(\mathbf{T})$, which, after choosing a uniformizer of $\breve{F}_{\bar{v}}$, can be regarded as a subgroup of $\mathcal{P}_{\bar{v}}(a)$.

4.5.2 Following [Chi19, § 3.5.1], let

$$\mathsf{Y}^{\lambda}_{\mathbf{G},\bar{v}}(\gamma) = \left\{ u \in \mathbb{L}_{\bar{v}}\mathbf{U}/\mathbb{L}^{+}_{\bar{v}}\mathbf{U} \,\middle|\, \mathrm{Ad}_{u}^{-1}(\gamma) \in \mathsf{C}^{\lambda}_{\mathbf{G}} \right\},$$

and $\tilde{\mathsf{Y}}^{\lambda}_{\mathbf{G},\bar{v}}(\gamma)$ be its preimage in $\mathbb{L}_{\bar{v}}\mathbf{U}$, both with reduced ind-scheme structure. Let $S_{\mu,\bar{v}}$ be the semi-infinite orbit

$$S_{\mu,\bar{v}} = \mathbb{L}_{\bar{v}}U\pi^{\mu}\mathbb{L}^{+}_{\bar{v}}G/\mathbb{L}^{+}_{\bar{v}}G \subset \mathsf{Gr}_{G,\bar{v}}$$

for any $\mu \in \check{\mathbb{X}}(T)$. Then we have commutative diagram

$$\begin{array}{ccc}
\tilde{\mathsf{Y}}^{\lambda}_{\mathbf{G},\bar{v}}(\gamma) & \xrightarrow{u \mapsto \mathrm{Ad}_{u}^{-1}(\gamma)} & \mathsf{C}^{\lambda}_{\mathbf{G},\bar{v}} \cap \mathbb{L}_{\bar{v}}\mathbf{U}\gamma \\
{\scriptstyle /\mathbb{L}^{+}_{\bar{v}}\mathbf{U}}\downarrow & & \downarrow{\scriptstyle /\mathbb{L}^{+}_{\bar{v}}\mathbf{U}} \\
\mathsf{Y}^{\lambda}_{\mathbf{G},\bar{v}}(\gamma) & & S_{\nu_{\gamma},\bar{v}} \cap \mathsf{Gr}^{\lambda}_{\mathbf{G},\bar{v}} \\
\downarrow & \searrow & \downarrow \\
\check{\mathbb{X}}(\mathbf{T}) \times \mathsf{Y}^{\lambda}_{\mathbf{G},\bar{v}}(\gamma) & & \left[\mathbb{L}^{+}_{\bar{v}}\mathbf{G}\backslash\mathsf{Gr}^{\lambda}_{\mathbf{G},\bar{v}}\right] \\
{\scriptstyle (\mu,u)\mapsto\pi^{\mu}u}\downarrow & \nearrow{\scriptstyle g\mapsto \mathrm{Ad}_{g}^{-1}(\gamma)} & \\
\mathcal{M}^{\lambda}_{\mathbf{G},\bar{v}}(\gamma) & &
\end{array} \quad . \tag{4.5.1}$$

In this diagram, the arrows marked with "$/\mathbb{L}^{+}_{\bar{v}}U$" are $\mathbb{L}^{+}_{\bar{v}}U$-torsors, and the top horizontal arrow, after taking quotient by a sufficiently small congruent subgroup of $\mathbb{L}^{+}_{\bar{v}}U$, is a "homotopy equivalence" in the following sense:

Definition 4.5.3. Let Y_1, Y_2 be k-schemes of finite type. We say Y_1 and Y_2 are $\mathbb{A}$-*homotopy equivalent* if they can be connected by a chain of diagrams between k-schemes

$$Y \longleftarrow \tilde{Y} \longrightarrow Y',$$

where the arrows are locally trivial fibration of affine spaces. In this case, we denote

$Y_1 \leftrightsquigarrow Y_2$.

We continue describing (4.5.1). The lower-left vertical arrow is a bijection on $\bar{k}$-points and induces a stratification compatible with the ind-scheme topology on $\mathcal{M}^{\lambda}_{\mathbf{G},\bar{v}}(\gamma)$ induced by that on $\mathsf{Gr}_{\mathbf{G},\bar{v}}$. So the closure of $\mathsf{Y}^{\lambda}_{\mathbf{G},\bar{v}}(\gamma)$ in $\mathcal{M}^{\lambda}_{\mathbf{G},\bar{v}}(\gamma)$ is a fundamental domain of the free $\check{\mathbb{X}}(\mathbf{T})$-action, and the irreducible components of $[\mathcal{M}_{\mathbf{G},\bar{v}}(a)/\mathcal{P}_{\bar{v}}(a)]$ are identified with those of $\mathsf{Y}^{\lambda}_{\mathbf{G},\bar{v}}(\gamma)$.

4.5.4 Now we move to general G obtained by outer twist ϑ_G of $\mathbf{G}$. Let $\mathfrak{M} \in \mathcal{FM}(G^{\mathrm{sc}})$ and let $a \in \mathfrak{C}^{\heartsuit}_{\mathfrak{M}}(\mathcal{O}_v)$ be generically regular semisimple and unramified. Base change to $\bar{k}$, then as before we have isomorphism

$$\mathcal{M}_{G,v}(a)_{\bar{k}} \simeq \prod_{\bar{v}\colon k_v \to \bar{k}} \mathcal{M}_{G,\bar{v}}(a),$$

where $\bar{v}$ ranges over k-embeddings of k_v into $\bar{k}$. In particular, we have bijection of geometric irreducible components:

$$\mathrm{Irr}[\mathcal{M}_{G,v}(a)_{\bar{k}}] \simeq \prod_{\bar{v}\colon k_v \to \bar{k}} \mathrm{Irr}[\mathcal{M}_{G,\bar{v}}(a)].$$

The Frobenius $\sigma_k \in \mathrm{Gal}(\bar{k}/k)$ acts on the $\bar{k}$-points on the left-hand side and it induces an σ_k-action on the right-hand side, sending a $\bar{k}$-point in the $\bar{v}$-factor to one in the $\sigma_k(\bar{v})$-factor. For convenience, let $\sigma_v = \sigma_k^{[k_v:k]}$ be the Frobenius of $\bar{k}/k_v$, then σ_v acts on each factor $\mathcal{M}_{G,\bar{v}}(a)$.

If we let $\mathcal{M}'_{G,v}(a)$ be the k_v-functor analogously defined as $\mathcal{M}_{G,v}(a)$ except we treat $\mathcal{O}_v$-points of $[\mathfrak{M}/G]$ with trivialization over F_v as k_v-points of $\mathcal{M}'_{G,v}(a)$ rather than k-points, then $\mathcal{M}_{G,v}(a)$ is just the Weil restriction of $\mathcal{M}'_{G,v}(a)$ from k_v to k, and each $\mathcal{M}_{G,\bar{v}}(a)$ is the base change of $\mathcal{M}'_{G,v}(a)$ via $\bar{v}\colon k_v \to \bar{k}$. Therefore, it is clear that the σ_k-action on $\mathrm{Irr}[\mathcal{M}_{G,v}(a)]$ is completely determined by the σ_v-action on any one of $\mathcal{M}_{G,\bar{v}}(a)$.

Consequentially, without loss of generality we may assume $k_v = k$.

4.5.5 Now we assume $k_v = k$. Recall that the definition of $\mathcal{M}_{G,v}(a)$ depends on a choice of $\gamma_a \in \mathfrak{M}(F_v)$ lying over a, which necessarily exists by Theorem 2.4.24. Since a is unramified, over each $\breve{\mathcal{O}}_{\bar{v}}$ it can be lifted to a point $\gamma_{\bar{v}} \in \mathfrak{T}_{\mathfrak{M}}(\breve{\mathcal{O}}_{\bar{v}})$. Let $h_{\bar{v}} \in G^{\mathrm{sc}}(\breve{F}_{\bar{v}})$ such that $h_{\bar{v}}\gamma_a h_{\bar{v}}^{-1} = \gamma_{\bar{v}}$ (necessarily exists by Steinberg's theorem on torsors over $\breve{F}_{\bar{v}}$). Then

$$\dot{w}_{\gamma_{\bar{v}}} = h_{\bar{v}}\sigma_v(h_{\bar{v}})^{-1} \in \mathrm{N}_G(T)(\breve{F}_{\bar{v}})$$

because a is generically regular semisimple, and we let $w_{\gamma_{\bar{v}}}$ be the image of $\dot{w}_{\gamma_{\bar{v}}}$ in $W(\breve{F}_{\bar{v}}) \cong \mathbf{W}$. The element $w_{\gamma_{\bar{v}}}$ depends only on $\gamma_{\bar{v}}$ but not on $h_{\bar{v}}$. Then $\gamma_{\bar{v}}$ is fixed by $\sigma'_v = w_{\gamma_{\bar{v}}} \rtimes \sigma_v$, and $\mathcal{M}_{G,v}(a)$ may be regarded as a k_v-structure on the $\bar{k}$-scheme $\mathcal{M}^{\leq\lambda}_{G,\bar{v}}(\gamma)$ with Frobenius σ'_v, where γ is the image of $\gamma_{\bar{v}}$ in G^{ad}. For convenience, we let $\dot{\sigma}'_v = \dot{w}_{\gamma_{\bar{v}}} \rtimes \sigma_v$.

4.5.6 Over $\bar{k}$ we fix an isomorphism $G \cong \mathbf{G}$ that identifies their pinnings. Then $\gamma_{\bar{v}} \in \bar{\mathbf{T}}_{\mathbf{M}}$. Let

$$\mathbf{B}' = \sigma'_v(\mathbf{B})$$

be another Borel containing $\mathbf{T}$ and $\mathbf{U}'$ its unipotent radical. Then we may obtain functors $\tilde{\mathsf{Y}}^{\lambda,\prime}_{\mathbf{G},\bar{v}}(\gamma)$, $\mathsf{Y}^{\lambda,\prime}_{\mathbf{G},\bar{v}}(\gamma)$, and so on by replacing $\mathbf{U}$ with $\mathbf{U}'$, and $\mathcal{M}^{\lambda}_{\mathbf{G},\bar{v}}(\gamma)$ is a $\breve{\mathbb{X}}(\mathbf{T})$-tiling of $\mathsf{Y}^{\lambda,\prime}_{\mathbf{G},\bar{v}}(\gamma)$ as well.

Since $\dot{\sigma}'_v$ preserves $\breve{\mathbb{X}}(\mathbf{T})$, it acts on the stack $[\mathcal{M}^{\lambda}_{\mathbf{G},\bar{v}}(\gamma)/\breve{\mathbb{X}}(\mathbf{T})]$, which induces an σ'_v-action on the irreducible components of $[\mathcal{M}^{\lambda}_{\mathbf{G},\bar{v}}(\gamma)/\breve{\mathbb{X}}(\mathbf{T})]$. Similarly, the natural $\dot{\sigma}'_v$-

action on $\mathbb{L}_{\bar{v}}\mathbf{G}$ induces an isomorphism $\mathbb{L}_{\bar{v}}\mathbf{U} \to \mathbb{L}_{\bar{v}}\mathbf{U}'$, which descends to an isomorphism

$$\dot{\sigma}_v : \mathsf{Y}^{\lambda}_{\mathbf{G},\bar{v}}(\gamma)\mathbb{L}^{+}_{\bar{v}}\mathbf{G}/\mathbb{L}^{+}_{\bar{v}}\mathbf{G} \xrightarrow{\sim} \mathsf{Y}^{\lambda,\prime}_{\mathbf{G},\bar{v}}(\gamma)\mathbb{L}^{+}_{\bar{v}}\mathbf{G}/\mathbb{L}^{+}_{\bar{v}}\mathbf{G}.$$

Therefore, σ'_v induces a bijection

$$\sigma'_v : \operatorname{Irr}(\mathsf{Y}^{\lambda}_{\mathbf{G},\bar{v}}(\gamma)) \xrightarrow{\sim} \operatorname{Irr}(\mathsf{Y}^{\lambda,\prime}_{\mathbf{G},\bar{v}}(\gamma)),$$

compatible with the action of σ'_v on the irreducible components of $[\mathcal{M}^{\lambda}_{\mathbf{G},\bar{v}}(\gamma)/\check{\mathbb{X}}(\mathbf{T})]$.

4.5.7 On the other hand, (4.5.1) establishes a bijection

$$\operatorname{Irr}(\mathsf{Y}^{\lambda}_{\mathbf{G},\bar{v}}(\gamma)) \simeq \operatorname{Irr}(S_{\nu_\gamma,\bar{v}} \cap \mathsf{Gr}^{\lambda}_{\mathbf{G},\bar{v}}),$$

hence an isomorphism

$$\overline{\mathbb{Q}}_\ell^{\oplus \operatorname{Irr}(\mathsf{Y}^{\lambda}_{\mathbf{G},\bar{v}}(\gamma))} \simeq \overline{\mathbb{Q}}_\ell^{\operatorname{Irr}(S_{\nu_\gamma,\bar{v}} \cap \mathsf{Gr}^{\lambda}_{\mathbf{G},\bar{v}})}.$$

According the theory of geometric Satake isomorphism (see [MV07] or [Zhu17] for example), up to a Tate twist the right-hand side is canonically isomorphic to the cohomology group $\mathrm{H}^{\bullet}_{c}(S_{\nu_\gamma,\bar{v}}, \mathrm{IC}^{\lambda})$, were IC^{λ} is the intersection complex on $\mathsf{Gr}^{\le\lambda}_{\mathbf{G}}$. We also have a direct sum decomposition

$$\mathrm{RT}_{\mathbf{B}} : \mathrm{H}^{\bullet}_{c}(\mathsf{Gr}_{\mathbf{G},\bar{v}}, \mathrm{IC}^{\lambda}) \xrightarrow{\sim} \bigoplus_{\mu\in\check{\mathbb{X}}} \mathrm{H}^{\bullet}_{c}(S_{\mu,\bar{v}}, \mathrm{IC}^{\lambda}).$$

It is known in [MV07, Zhu17] that $\mathrm{H}^{\bullet}_{c}(S_{\mu,\bar{v}}, \mathrm{IC}^{\lambda})$ is concentrated on degree $\langle 2\rho, \mu\rangle$. Similarly, by replacing $\mathbf{U}$ with $\mathbf{U}'$, we have identification

$$\overline{\mathbb{Q}}_\ell^{\oplus \operatorname{Irr}(\mathsf{Y}^{\lambda,\prime}_{\mathbf{G},\bar{v}}(\gamma))} \simeq \overline{\mathbb{Q}}_\ell^{\operatorname{Irr}(S'_{\nu_\gamma,\bar{v}} \cap \mathsf{Gr}^{\lambda}_{\mathbf{G},\bar{v}})},$$

and another decomposition

$$\mathrm{RT}_{\mathbf{B}'}: \mathrm{H}_c^\bullet(\mathrm{Gr}_{\mathbf{G},\bar{v}}, \mathrm{IC}^\lambda) \xrightarrow{\sim} \bigoplus_{\mu \in \check{\mathbb{X}}} \mathrm{H}_c^\bullet(S'_{\mu,\bar{v}}, \mathrm{IC}^\lambda).$$

Here $\mathrm{H}_c^\bullet(S'_{\mu,\bar{v}}, \mathrm{IC}^\lambda)$ is concentrated on degree $\langle 2\rho', \mu\rangle$, where ρ' is the analogue of ρ by replacing $\mathbf{U}$ with $\mathbf{U}'$. As a result, σ'_v induces isomorphism

$$\sigma'_v: \mathrm{H}_c^\bullet(S_{\nu_\gamma,\bar{v}}, \mathrm{IC}^\lambda) \xrightarrow{\sim} \mathrm{H}_c^\bullet(S'_{\nu_\gamma,\bar{v}}, \mathrm{IC}^\lambda). \tag{4.5.2}$$

4.5.8 The lift $\dot{\sigma}'_v \in \mathbf{G} \rtimes \sigma_v$ of σ'_v naturally acts on $\mathrm{Gr}_{\mathbf{G},\bar{v}}$, and by *Lemme d'homotopie* ([LN08, Lemme 3.2.3]) it is the same action as σ_v because $\mathbf{G}$ is connected. We then have commutative diagram

$$\begin{array}{ccc}
\mathrm{H}_c^\bullet(\mathrm{Gr}_{\mathbf{G},\bar{v}}, \mathrm{IC}^\lambda) & \xrightarrow{\mathrm{pr}_{\nu_\gamma}\circ \mathrm{RT}_{\mathbf{B}}} & \mathrm{H}_c^\bullet(S_{\nu_\gamma,\bar{v}}, \mathrm{IC}^\lambda) \\
{\scriptstyle \dot{\sigma}'_v = \sigma_v}\downarrow & & \downarrow{\scriptstyle \sigma'_v} \\
\mathrm{H}_c^\bullet(\mathrm{Gr}_{\mathbf{G},\bar{v}}, \mathrm{IC}^\lambda) & \xrightarrow{\mathrm{pr}'_{\nu_\gamma}\circ \mathrm{RT}_{\mathbf{B}'}} & \mathrm{H}_c^\bullet(S'_{\nu_\gamma,\bar{v}}, \mathrm{IC}^\lambda)
\end{array}$$

where pr_{ν_γ} is the projection to the direct summand given by $\mathrm{H}_c^\bullet(S_{\nu_\gamma}, \mathrm{IC}^\lambda)$ and similarly for $\mathrm{pr}'_{\nu_\gamma}$. As a result, we have inclusion of σ'_v-modules as direct summand

$$\overline{\mathbb{Q}}_\ell^{\oplus \mathrm{Irr}([\mathcal{M}^{\leq\lambda}_{\mathbf{G},\bar{v}}(\gamma)/\check{\mathbb{X}}(\mathbf{T})])} \subset \mathrm{H}_c^0(\mathrm{Gr}_{\mathbf{G},\bar{v}}, \mathrm{IC}^\lambda).$$

4.5.9 Again by geometric Satake isomorphism, the cohomology group $\mathrm{H}_c^\bullet(\mathrm{Gr}_{\mathbf{G},\bar{v}}, \mathrm{IC}^\lambda)$ is identified with the irreducible $\check{\mathbf{G}}$-representation V_λ with highest weight λ through Tannakian formalism, and the map $\mathrm{RT}_{\mathbf{B}}$ gives the weight decomposition with respect to $\check{\mathbf{T}}$. It can be easily upgraded to a representation of $\check{\mathbf{G}} \rtimes \langle\sigma_v\rangle$.

However, it is important to note that this isomorphism depends on the choice of Borel $\mathbf{B}$ in $\mathbf{G}$. To avoid confusion, we let the group given by Tannakian formalism by $\tilde{\mathbf{G}}$, and $\mathrm{RT}_{\mathbf{B}}$

and $\mathrm{RT}_{\mathbf{B}'}$ gives two different Borels $\tilde{\mathbf{B}}$ and $\tilde{\mathbf{B}}'$ in $\tilde{\mathbf{G}}$ respectively. Then we may find some $\tilde{w} \in \mathrm{N}_{\tilde{\mathbf{G}}}(\check{\mathbf{T}})$ whose image in $\mathbf{W}$ is $w_{\gamma_{\bar{v}}}$, such that $\tilde{w}(\tilde{\mathbf{B}}) = \tilde{\mathbf{B}}'$, and the $\tilde{\mathbf{B}}'$-highest weight in $\mathrm{H}_{\mathrm{c}}^{\bullet}(\mathsf{Gr}_{\mathbf{G},\bar{v}}, \mathrm{IC}^{\lambda})$ is no longer λ but $w_{\gamma_{\bar{v}}}(\lambda)$. Fix an isomorphism $\iota_{\mathbf{B}} \colon \check{\mathbf{G}} \to \tilde{\mathbf{G}}$ mapping $\check{\mathbf{B}}$ to $\tilde{\mathbf{B}}$, and let $\iota_{\mathbf{B}'} = \mathrm{Ad}_{\tilde{w}} \circ \iota_{\mathbf{B}}$, then we have isomorphisms

$$\iota_{\mathbf{B}}^{*} \colon \mathrm{H}_{\mathrm{c}}^{\bullet}(\mathsf{Gr}_{\mathbf{G},\bar{v}}, \mathrm{IC}^{\lambda}) \xrightarrow{\sim} V_{\lambda},$$
$$\iota_{\mathbf{B}'}^{*} \colon \mathrm{H}_{\mathrm{c}}^{\bullet}(\mathsf{Gr}_{\mathbf{G},\bar{v}}, \mathrm{IC}^{\lambda}) \xrightarrow{\sim} V_{\lambda}.$$

4.5.10 The Tannakian formalism ensures an action of $\tilde{w}$ on $\mathrm{H}_{\mathrm{c}}^{\bullet}(\mathsf{Gr}_{\mathbf{G},\bar{v}}, \mathrm{IC}^{\lambda})$, such that under the maps $\mathrm{RT}_{\mathbf{B}}$ and $\mathrm{RT}_{\mathbf{B}'}$ respectively, it identifies subspaces

$$\tilde{w} \colon \mathrm{H}_{\mathrm{c}}^{\bullet}(S_{\mu,\bar{v}}, \mathrm{IC}^{\lambda}) \xrightarrow{\sim} \mathrm{H}_{\mathrm{c}}^{\bullet}(S'_{\tilde{w}(\mu),\bar{v}}, \mathrm{IC}^{\lambda}).$$

Composing with the action of σ_v, we have induced map

$$\tilde{\sigma}'_v \coloneqq \tilde{w} \rtimes \sigma_v \colon \mathrm{H}_{\mathrm{c}}^{\bullet}(S_{\mu,\bar{v}}, \mathrm{IC}^{\lambda}) \xrightarrow{\sim} \mathrm{H}_{\mathrm{c}}^{\bullet}(S'_{\tilde{w} \rtimes \sigma_v(\mu),\bar{v}}, \mathrm{IC}^{\lambda}).$$

This map depends on the choice of representative $\tilde{w}$ in general, which we will fix one. Thus we have induced map

$$\tilde{\sigma}'_v \colon \mathrm{H}_{\mathrm{c}}^{0}(S_{\nu_{\gamma},\bar{v}}, \mathrm{IC}^{\lambda}) \xrightarrow{\sim} \mathrm{H}_{\mathrm{c}}^{0}(S'_{\nu_{\gamma},\bar{v}}, \mathrm{IC}^{\lambda}),$$

which is the same as the action (4.5.2). In other words, we have commutative diagram of maps

$$\begin{array}{ccccc}
\mathrm{H}_{\mathrm{c}}^{\bullet}(S_{\nu_{\gamma},\bar{v}}, \mathrm{IC}^{\lambda}) & \xleftarrow{\mathrm{pr}_{\nu_{\gamma}} \circ \mathrm{RT}_{\mathbf{B}}} & \mathrm{H}_{\mathrm{c}}^{\bullet}(\mathsf{Gr}_{\mathbf{G},\bar{v}}, \mathrm{IC}^{\lambda}) & \xrightarrow{\iota_{\mathbf{B}}^{*}[\nu_{\gamma}]} & V_{\lambda}[\nu_{\gamma}] \\
\downarrow \sigma'_v & & \dot{\sigma}'_v = \sigma_v \downarrow & & \downarrow \tilde{\sigma}'_v \\
\mathrm{H}_{\mathrm{c}}^{\bullet}(S'_{\nu_{\gamma},\bar{v}}, \mathrm{IC}^{\lambda}) & \xleftarrow{\mathrm{pr}'_{\nu_{\gamma}} \circ \mathrm{RT}_{\mathbf{B}'}} & \mathrm{H}_{\mathrm{c}}^{\bullet}(\mathsf{Gr}_{\mathbf{G},\bar{v}}, \mathrm{IC}^{\lambda}) & \xrightarrow{\iota_{\mathbf{B}'}^{*}[\nu_{\gamma}]} & V_{\lambda}[\nu_{\gamma}]
\end{array}$$

where $V_\lambda[\nu_\gamma]$ means the ν_γ-weight subspace. As a result, we have isomorphism of σ'_v-modules

$$\overline{\mathbb{Q}}_\ell^{\oplus \operatorname{Irr}([\mathcal{M}^{\leq\lambda}_{\mathbf{G},\bar{v}}(\gamma)/\check{\mathbb{X}}(\mathbf{T})])} \simeq V_\lambda[\nu_\gamma], \tag{4.5.3}$$

where σ'_v acts on the right-hand side through $\tilde{\sigma}'_v$.

4.5.11 Finally, we consider endoscopic groups. Let $a \in [\mathfrak{C}_{\mathfrak{M}}/Z_{\mathfrak{M}}](\mathcal{O}_v)$ be a point in the image of $[\mathfrak{C}_{\mathfrak{M},H}/Z^\kappa_{\mathfrak{M}}]$ for some endoscopic group H given by endoscopic datum $(\kappa, \vartheta_\kappa)$. Suppose a is unramified and $\gamma_{\bar{v}} \in [\bar{\mathbf{T}}_{\mathbf{M}}/\mathbf{Z}_{\mathbf{M}}](\mathcal{O}_{\bar{v}})$ lies over a. As above let γ be the image of $\gamma_{\bar{v}}$ in $\mathbf{G}^{\mathrm{ad}}$, and let the Newton point of γ be ν_γ. We also assume that a maps to a point in $\pi^{-w_0(\lambda)}\mathbf{T}^{\mathrm{ad}}(\mathcal{O}_{\bar{v}}) \subset \mathbf{A}_{\mathrm{Env}(\mathbf{G}^{\mathrm{sc}})}(\mathcal{O}_{\bar{v}})$.

Let $\check{\mathbf{H}}'$ be the preimage of $\check{\mathbf{H}}$ in $\check{\mathbf{G}}^{\mathrm{sc}}$, and $\lambda_{H,1}, \dots, \lambda_{H,m}$ be the $\check{\mathbf{H}}'$ highest weights (allowing repetitions) such that we have decomposition into irreducible $\check{\mathbf{H}}'$-representations

$$\operatorname{Res}^{\check{\mathbf{G}}}_{\check{\mathbf{H}}'} V_{-w_0(\lambda)} = \bigoplus_{i=1}^m V'_{-w_{\mathbf{H},0}(\lambda_{H,i})}.$$

By the proof of Lemma 2.5.10, there are only finitely many lifts of $\gamma_{\bar{v}}$ to $[\bar{\mathbf{T}}_{\mathbf{M},\mathbf{H}}/\mathbf{Z}^\kappa_{\mathbf{M}}](\mathcal{O}_{\bar{v}})$, one for each $\lambda_{H,i}$ such that $V'_{-w_{\mathbf{H},0}(\lambda_{H,i})}[\nu_\gamma] \neq 0$. Denote those lifts by $\gamma_{H,\bar{v},1}, \dots, \gamma_{H,\bar{v},e}$ where $0 < e \leq m$. The canonical map

$$\mathbf{W}_{\mathbf{H}} \rtimes \pi_0(\kappa) \longrightarrow \mathbf{W} \rtimes \pi_0(\kappa)$$

as in Lemma 2.5.5 identifies (the image of) σ'_v determined by $\gamma_{\bar{v}}$ with those determined by each $\gamma_{H,\bar{v},i}$, and in particular $w_{\gamma_{\bar{v}}} \in \mathbf{W}_{\mathbf{H}}$. Pick a lift $\tilde{w}$ of $w_{\gamma_{\bar{v}}} \in \mathbf{W}_{\mathbf{H}}$ in $\mathrm{N}_{\check{\mathbf{H}}'}(\check{\mathbf{T}}) \subset \mathrm{N}_{\check{\mathbf{G}}^{\mathrm{sc}}}(\check{\mathbf{T}})$, then we have decomposition of σ'_v-modules

$$V_{-w_0(\lambda)}[\nu_\gamma] = \bigoplus_{i=1}^m V'_{-w_{\mathbf{H},0}(\lambda_{H,i})}[\nu_\gamma],$$

where σ_v' acts on the right-hand side through $\tilde{\sigma}_v'$. Therefore using (4.5.3), we have the following result:

Proposition 4.5.12. *With the assumptions above, we have a canonical isomorphism of Frobenius modules*

$$\overline{\mathbb{Q}}_\ell^{\oplus \operatorname{Irr}(\mathcal{M}_{G,\bar{v}}(a))/\mathcal{P}_{\bar{v}}(a)} \simeq \bigoplus_{i=1}^{e} \overline{\mathbb{Q}}_\ell^{\oplus \operatorname{Irr}(\mathcal{M}_{H,\bar{v}}(a_{H,i}))/\mathcal{P}_{H,\bar{v}}(a_{H,i})},$$

where $a_{H,i}$ is the image of $\gamma_{H,\bar{v},i}$.

4.6 Matching Orbits

In this section, we demonstrate how to translate the matching of orbits and functions in the group setting into the language of monoids.

4.6.1 We keep the notations in § 2.6. In particular, our endoscopic datum will be pointed. Now given matching elements $\gamma \in G(F_v)$ and $\gamma_H \in H(F_v)$ and matching functions f^λ and $\sum_i f_H^{\lambda_{H,i}}$, we may assume $\gamma_\lambda \in \operatorname{Env}(G^{\mathrm{sc}})$ as in Lemma 4.1.5 exists (otherwise both sides of fundamental lemma will be zero because the relevant multiplicative affine Springer fibers for both G and H will be empty). Since λ is F_v-rational, we may lift γ_λ to some element int $\mathfrak{M}(\mathcal{O}_v)$ where $\mathfrak{M} \in \mathcal{FM}(G^{\mathrm{sc}})$ is a monoid such that λ is one of the generators of the cone of $\mathfrak{A}_{\mathfrak{M}}$. Let $\mathfrak{M}_H$ be the endoscopic monoid.

4.6.2 Recall we have the fixed geometric point $x_a \in \mathbf{T}$ which, together with the base point of endoscopic datum, gives a geometric point $x_a^\bullet$ lying over γ, and similarly $x_{a_H}^\bullet$ for γ_H. Recall also that we choose $x_{a_H}^\bullet = x_a^\bullet$. For any σ in Galois group $\Gamma_v = \operatorname{Gal}(F_v^{\mathrm{s}}/F_v)$, there exists some element $w_\sigma \in \mathbf{W} \rtimes \pi_0(\kappa)$ such that $\sigma(x_a^\bullet) = w_\sigma(x_a^\bullet)$, and a is the image of a_H means that w_σ is actually contained in $\mathbf{W_H} \rtimes \pi_0(\kappa)$. This means that the point $a_{\mathfrak{M}} = \chi_{\mathfrak{M}}(\gamma_\lambda) \in \mathfrak{C}_{\mathfrak{M}}^{\times,\mathrm{rs}}(F_v)$ must come from an element in $\mathfrak{C}_{\mathfrak{M},H}' = T_{\mathfrak{M}} /\!\!/ W_H(F_v)$ because the monodromy groups are insensitive to central extensions. Since we are considering regular semisimple locus, we may replace GIT quotients by W or W_H with stack quotients, so we have

$$[T^{G\text{-rs}}_{\mathfrak{M},H}/W_H \times Z^{\kappa}_{\mathfrak{M}}] \xrightarrow{\sim} [T^{\mathrm{rs}}_{\mathfrak{M}}/W_H \times Z_{\mathfrak{M}}] \simeq [(T^{\mathrm{rs}}_{\mathfrak{M}} /\!\!/ W_H)/Z_{\mathfrak{M}}]$$

by definition of $Z_{\mathfrak{M}}$. Therefore the image of $a_{\mathfrak{M}}$ on the right-hand side may be identified a unique F_v-point on the left-hand side, denoted by $a_{\mathfrak{M},H}$. The torus T does not embed into H in a canonical way, but Z_G still does. It is straightforward to see that the image of a_H in $T^{\mathrm{ad}} /\!\!/ W_H$ is the same as that of $a_{\mathfrak{M}}$ hence also $a_{\mathfrak{M},H}$. Taking further quotient, then the image of $a_{\mathfrak{M},H}$ in $H^{\mathrm{ad}} /\!\!/ H$ is the same as that of a_H. By the comments at the end of § 2.6, it makes sense to talk about orbital integrals of $a_{\mathfrak{M}}$ and $a_{\mathfrak{M},H}$ respectively.

4.6.3 Finally, each $\lambda_{H,i}$ is not necessarily F_v-rational, but must be $F_{\bar{v}}$-rational, and the extensions of $a_{\mathfrak{M},H}$ to $\mathcal{O}_{\bar{v}}$-points of $[\mathfrak{C}_{\mathfrak{M},H}/Z^{\kappa}_{\mathfrak{M}}]$ are in bijection with $\lambda_{H,i}$ by our construction of $\mathfrak{M}_H$.

CHAPTER 5

GLOBAL CONSTRUCTIONS

In this section, we make several useful global constructions. The most important construction is the moduli of boundary divisors, which is an indispensable tool for describing multiplicative Hitchin fibrations. Technically, the moduli of boundary divisors in this chapter is slightly different from what we will use for future chapters: here we mostly formulate them as a space due to its relative simplicity in language, but in practice we need a modified version where the moduli is a Deligne-Mumford stack instead of a space. Such difference can be seen as early as in § 5.3, where we provide a somewhat novel definition of global affine Schubert schemes.

5.1 Boundary Divisors

Before we can discuss multiplicative Hitchin fibrations, we must introduce moduli spaces of divisors in a certain generalized sense. We mainly follow [BNS16, § 3] for this purpose, but generalize it to non-split tori. Due to the technicality of this section and for convenience of the readers, we would like to make all relevant statements in split setting first and then prove the parallel results in general setting. The proofs are for the most part the same as in [BNS16], with some changes.

Let A be a split torus over k, and $\mathfrak{A}$ is an affine normal toric variety of torus A, viewed also as a commutative monoid with $\mathfrak{A}^\times = A$. Such toric variety is determined by a saturated, strictly convex cone $\mathscr{C} \subset \check{\mathbb{X}}(A)$, and $\lambda \in \mathscr{C}$ if and only if $\lambda\colon \mathbb{G}_\mathrm{m} \to A$ extends to a monoidal homomorphism from $\mathbb{A}^1$ to $\mathfrak{A}$. The ring of functions $k[\mathfrak{A}] \subset k[A]$ is spanned by the function determined by characters in the dual cone

$$\mathscr{C}^* = \{\alpha \in \mathbb{X}(A) \mid \langle \alpha, \lambda \rangle \geq 0, \forall \lambda \in \mathscr{C}\}.$$

5.1.1 Let X be a smooth, geometrically connected, projective curve over k. We have k-mapping stack

$$\mathcal{B}_X^{1+} := \underline{\mathrm{Hom}}_{X/k}(X, [\mathfrak{A}/A]).$$

For a k-scheme S, an S-point of $\mathcal{B}_X^{1+}$ consists of a pair (E, ϕ) where E is an A-torsor over $X \times S$ and ϕ is a section of the induced toric bundle $\mathfrak{A}_{\mathcal{L}}$ over $X \times S$. We are interested in the open substack $\mathcal{B}_X^1 \subset \mathcal{B}_X^{1+}$ where for any geometric point $s \in S$ the image of the generic point of $X \times \{s\}$ under ϕ is contained in $A_E \subset \mathfrak{A}_E$. When the monoid is not clear from the context, we will use $\mathcal{B}_{X,\mathfrak{A}}^1$, etc. to emphasize the monoid. The k-points of $\mathcal{B}_X^1$ can be conveniently described by so-called $\mathscr{C}$-valued divisors.

Definition 5.1.2. Given a smooth curve X over k, a *$\mathscr{C}$-valued divisor* or a *boundary divisor* is a formal sum

$$\lambda_X = \sum_{x \in |X|} \lambda_x x,$$

where x ranges over the closed points of X, and $\lambda_x \in \mathscr{C}$ is non-zero for only finitely many x.

Remark 5.1.3. We use the name boundary divisor because it records the intersection type of the curve with the boundary of the unit group inside the monoid. The dependence of this notion on cone $\mathscr{C}$ won't cause any confusion since the cone is usually clear from the context.

5.1.4 Let $(E, \phi) \in \mathcal{B}_X^1(k)$. At some $x \in |X|$ such that $\phi(x)$ is not contained in A, we can choose a local trivialization of E over the formal disk $\hat{X}_x \cong \operatorname{Spec} \mathcal{O}_x$ (since the residue field k_x is finite). After choosing a local uniformizer π_x, we denote $\mathcal{O}_x = k_x[[\pi_x]]$ and

$F_x = k_x((\pi_x))$. Then $\phi(\hat{X}_x)$ is a point in

$$A(F_x) \cap \mathfrak{A}(\mathcal{O}_x) = \coprod_{\lambda \in \mathscr{C}} \pi_x^{\lambda} A(\mathcal{O}_x).$$

This way we obtain a cocharacter $\lambda_x \in \mathscr{C}$, and it does not depend on the trivialization of E. Thus we have an associated boundary divisor on X. It is not hard to see the following using Beauville-Laszlo's gluing theorem.

Lemma 5.1.5 ([BNS16, Lemma 3.4])**.** *The above construction induces a canonical bijection between $\mathcal{B}_X^1(k)$ and boundary divisors on X.*

Example 5.1.6. When $A = \mathbb{G}_{\mathrm{m}}$ and $\mathfrak{A} = \mathbb{A}^1$, the space $\mathcal{B}_X^1$ is simply the moduli space of all effective divisors on X.

5.1.7 Similar to usual divisors on a curve, we would like to have a notion of *degree* to stratify space $\mathcal{B}_X^1$ into more accessible objects. Given a boundary divisor λ_X, we can define its degree to be the following element in $\mathscr{C}$:

$$\deg(\lambda_X) \coloneqq \sum_{x \in |X|} [k_x : k] \lambda_x.$$

This straightforward definition by itself is too coarse, so we have the following definition in [BNS16].

Definition 5.1.8. A *multiset* in $\mathscr{C}$ is an element $\underline{\mu}$ of the free abelian monoid generated by $\mathscr{C} - \{0\}$

$$\underline{\mu} = \sum_{\lambda \in \mathscr{C}} \underline{\mu}(\lambda) e^{\lambda} \in \bigoplus_{\lambda \in \mathscr{C} - \{0\}} \mathbb{N} e^{\lambda},$$

where $\underline{\mu}(0) = 0$ by convention. The *degree* of multiset $\underline{\mu}$ is defined as

$$\deg(\underline{\mu}) \coloneqq \sum_{\lambda \in \mathscr{C}} \underline{\mu}(\lambda) \lambda \in \mathscr{C}.$$

There is an natural partial order on the set of multisets by refinement: we say $\underline{\mu}$ refines $\underline{\mu}'$ or $\underline{\mu} \vdash \underline{\mu}'$ if the difference $\underline{\mu} - \underline{\mu}'$, viewed as an element of the free abelian group generated by $\mathscr{E} - \{0\}$, can be written as a sum of elements of the form $e^{\lambda_1} + e^{\lambda_2} - e^{\lambda_1+\lambda_2}$. Clearly, refinement does not have an effect on degree.

Definition 5.1.9. A multiset $\underline{\mu}$ is called *primitive* if there is no strict refinement in the sense that $\underline{\mu}' \vdash \underline{\mu}$ and $\underline{\mu}' \neq \underline{\mu}$. Equivalently, $\underline{\mu}$ is primitive if $\underline{\mu}(\lambda) = 0$ unless λ is primitive in $\mathscr{E}$ (i.e., λ is not the sum of two non-zero elements in $\mathscr{E}$).

We can associate to a boundary divisor λ_X a multiset

$$\underline{\lambda}_X = \bigoplus_{\lambda \in \mathscr{E}} \left(\sum_{\lambda_x = \lambda} [k_x : k] \right) e^{\lambda},$$

and one sees that $\deg(\lambda_X) = \deg(\underline{\lambda}_X)$. We have the following results.

Proposition 5.1.10 ([BNS16, Proposition 3.5])**.** *There exists a unique stratification of $\mathcal{B}^1_X$ indexed by multisets:*

$$\mathcal{B}^1_X = \coprod_{\underline{\mu}} \mathcal{B}^1_{\underline{\mu}},$$

such that

(1) $\mathcal{B}^1_{\underline{\mu}}(k)$ consists of those boundary divisors whose associated multiset is $\underline{\mu}$.

(2) $\mathcal{B}^1_{\underline{\mu}}$ is isomorphic to the multiplicity-free locus of the (necessarily finite) direct product

$$X_{\underline{\mu}} := \prod_{\lambda \in \mathscr{E} - \{0\}} X_{\underline{\mu}(\lambda)},$$

where $X_{\underline{\mu}(\lambda)}$ is the $\underline{\mu}(\lambda)$-th symmetric power of X. Here $(D_\lambda)_\lambda \in X_{\underline{\mu}}$ being multiplicity-free means that the total divisor $\sum_\lambda D_\lambda$ is a multiplicity-free divisor (not just D_λ individually).

(3) $\mathcal{B}^1_{\underline{\mu}'}$ *lies in the closure of* $\mathcal{B}^1_{\underline{\mu}}$ *if and only if* $\underline{\mu} \vdash \underline{\mu}'$.

Corollary 5.1.11 ([BNS16, Corollary 3.6]). *We have the following description of connected and irreducible components of* $\mathcal{B}^1_X$*:*

(1) *Each connected component of* $\mathcal{B}^1_X$ *contains a unique closed stratum of the form* $\mathcal{B}^1_{e^\lambda}$ *for some* $\lambda \in \mathcal{C}$*. Consequently, there is a canonical bijection* $\pi_0(\mathcal{B}^1_X) \simeq \mathcal{C}$*, and* $\mathcal{B}^1_{\underline{\mu}}$ *lies in the connected component* $\mathcal{B}^{1,\lambda}_X$ *associated with* λ *if and only if* $\deg(\underline{\mu}) = \lambda$.

(2) *Each irreducible component of* $\mathcal{B}^1_X$ *is the closure of a stratum* $\mathcal{B}^1_{\underline{\mu}}$ *where* $\underline{\mu}$ *is a primitive multiset.*

For a primitive multiset $\underline{\mu}$, let $\lambda_1, \dots, \lambda_m$ be the coweights such that $\underline{\mu}(\lambda_i) \neq 0$, then the isomorphism in Proposition 5.1.10 (2) can be extended to a morphism

$$\prod_{i=1}^{m} X_{\underline{\mu}(\lambda_i)} \longrightarrow \overline{\mathcal{B}^1_{\underline{\mu}}}. \tag{5.1.1}$$

Corollary 5.1.12 ([BNS16, Corollary 3.7]). *The disjoint union of morphisms* (5.1.1) *for which* $\underline{\mu}$ *ranges over all primitive multisets is both a normalization and a resolution of singularity of* $\mathcal{B}^1_X$*. In particular, it is a finite map.*

5.1.13 Given $\mathfrak{A}$ with associated cone $\mathcal{C}$, there are two constructions to associate a "standard" monoid to $\mathfrak{A}$. First, let $\lambda_1, \dots, \lambda_m$ be the primitive elements of $\mathcal{C}$, which each gives a monoidal homomorphism $\mathbb{A}^1 \to \mathfrak{A}$. The multiplication map then defines a homomorphism

$$\tilde{\mathfrak{A}} := \mathbb{A}^m \longrightarrow \mathfrak{A}.$$

On the other hand, let $\beta_1, \dots, \beta_m$ be a set of generators of the dual cone $\mathcal{C}^*$ *as an* $\mathbb{N}$*-monoid*, and let $\underline{\mathcal{C}}$ be the $\mathbb{N}$-dual of the free abelian monoid generated by β_i, then we have injective map of cones $\mathscr{C} \hookrightarrow \underline{\mathscr{C}}$ and a homomorphism of monoids

$$\mathfrak{A} \longrightarrow \underline{\mathfrak{A}} := \operatorname{Spec} k[e^{\beta_1}, \dots, e^{\beta_m}] = \mathbb{A}^m.$$

Note that $\tilde{\mathfrak{A}}$ is canonical since $\mathscr{C}$ is strictly convex, while $\underline{\mathfrak{A}}$ depends on the choice of generators β_i.

Both $\mathcal{B}^1_{X,\tilde{\mathfrak{A}}}$ and $\mathcal{B}^1_{X,\underline{\mathfrak{A}}}$ are clearly smooth. By Corollary 5.1.12, the induced morphism $\mathcal{B}^1_{X,\tilde{\mathfrak{A}}} \to \mathcal{B}^1_{X,\mathfrak{A}}$ is a finite resolution of singularity, and by Proposition 5.1.10, the morphism $\mathcal{B}^1_{X,\mathfrak{A}} \to \mathcal{B}^1_{X,\underline{\mathfrak{A}}}$ is a closed embedding.

5.1.14 Now we deal with the case where the torus is not necessarily split or constant. Let A be a torus defined over X, and $\mathfrak{A}$ an étale-locally trivial A-toric X-scheme with affine and normal fibers. Then the stacks $\mathcal{B}^{1+}_X$ and $\mathcal{B}^1_X$ still make sense, and we are primarily interested in $\mathcal{B}^1_X$.

Let $\check{\mathbb{X}}(A)$ be the *étale sheaf* of cocharacters defined as the internal hom-functor in the category of *abelian* sheaves

$$\check{\mathbb{X}}(A) := \underline{\operatorname{Hom}}_X(\mathbb{G}_{\mathrm{m}}, A).$$

The toric scheme $\mathfrak{A}$ is then defined by a sheaf of saturated strictly convex cones $\mathscr{C} \subset \check{\mathbb{X}}(A)$.

Let $x \in |X|$ be any closed point, the decomposition of $A(F_x) \cap \mathfrak{A}(\mathcal{O}_x)$ into $A(\mathcal{O}_x)$-cosets has become

$$A(F_x) \cap \mathfrak{A}(\mathcal{O}_x) = \coprod_{\lambda \in \mathscr{C}(\mathcal{O}_x)} \pi_x^{\lambda} A(\mathcal{O}_x),$$

where the cone $\mathscr{C}(\mathcal{O}_x)$ sits inside the cocharacter lattice of the maximal $\mathcal{O}_x$-split torus of $A_{\mathcal{O}_x}$, which is the same as the lattice $\check{\mathbb{X}}(A_{k_x})$, because the automorphism group of a split torus is discrete. Therefore, $\mathscr{C}(\mathcal{O}_x)$ can be canonically identified with the fixed point of any Frobenius element $\sigma_x \in \mathrm{Gal}(\bar{k}_x/k)$ in the cone $\mathscr{C}(\mathcal{O}_x \otimes_{k_x} \bar{k}_x)$. For this reason, we denote $\mathscr{C}(\mathcal{O}_x)$ by $\mathscr{C}_x$. Thus we have a generalized definition of boundary divisor as follows:

Definition 5.1.15. Given a smooth curve X over k and toric scheme $\mathfrak{A}$ over X, a *boundary divisor* is a formal sum

$$\lambda_X = \sum_{x \in |X|} \lambda_x x,$$

where $\lambda_x \in \mathscr{C}_x$ is non-zero for only finitely many x.

This way any k-point of $\mathcal{B}^1_X$ naturally induces a boundary divisor, and it induces a bijection between the two sets because the Beauville-Laszlo gluing construction still works (see the proof of [BNS16, Lemma 3.4]).

5.1.16 Note that the sheaf $\check{\mathbb{X}}(A)$ can be represented by a countable union of finite étale covers of X, and the cone $\mathscr{C}$ a subscheme consisting of some connected components therein. There is a canonical component in $\mathscr{C}$ corresponding to the zero cocharacter, still denoted by 0. The relative symmetric power $\mathscr{C}_{m/X} \coloneqq \mathrm{Sym}^m_X \mathscr{C}$ is easily seen smooth by looking étale-locally over X. We also have the addition map by definition:

$$+ \colon \mathscr{C}_{m/X} \longrightarrow \mathscr{C}.$$

Since $\mathscr{C}$ is a disjoint union of smooth projective curves, its scheme of connected components $\pi_0(\mathscr{C})$ is an étale scheme over k locally of finite type. For $\lambda \in \pi_0(\mathscr{C})$ a closed point and $\mathscr{C}^\lambda$ the corresponding component in $\mathscr{C}$, we let $k_\lambda = \mathrm{H}^0(\mathscr{C}^\lambda, \mathcal{O}_{\mathscr{C}})$, then $\lambda = \operatorname{Spec} k_\lambda$.

5.1.17 A multiset $\underline{\mu}$ is defined as an element in the free abelian monoid generated by the closed points in $\pi_0(\mathscr{C}) - \{0\}$. The partial order by refinement is as follows: for

$0 \neq \underline{\mu} = \sum_{\lambda \in \pi_0(\mathscr{E})} \underline{\mu}(\lambda)\lambda$ (again $\underline{\mu}(0) = 0$ by convention), we have

$$+ \colon \prod_{\lambda} \mathscr{E}^{\lambda}_{\underline{\mu}(\lambda)/X} \longrightarrow \mathscr{E},$$

in which the direct product is taken over X, and the image in $\pi_0(\mathscr{E})$ is a finite subset. Any point $\underline{\mu}'$ in such image is called *a degree of* $\underline{\mu}$, regarded itself as a multiset. Note that there may be multiple possible degrees in this sense.

If $\underline{\mu} = \underline{\mu}_1 + \underline{\mu}_2$, and $\underline{\mu}'_2$ is a degree of $\underline{\mu}_2$, then $\underline{\mu}' = \underline{\mu}_1 + \underline{\mu}'_2$ is again a multiset. In this case $\underline{\mu}$ can be seen as a refinement of $\underline{\mu}'$. More generally, a multiset $\underline{\mu}$ is said to be a refinement of $\underline{\mu}'$ or $\underline{\mu} \vdash \underline{\mu}'$ if $\underline{\mu}'$ can be obtained from $\underline{\mu}$ after finite steps of taking degrees of its summands. The notion of degree is less useful compared to the case of constant $\mathfrak{A}$, since they no longer correspond bijectively to connected components of $\mathcal{B}^1_X$.

5.1.18 Given a boundary divisor λ_X, we may define an induced multiset $\underline{\lambda}_X$ as follows: let $x_1, \dots, x_m$ be the mutually distinct points of X such that $\lambda_{x_i} \neq 0$. Since λ_{x_i} can be identified with a point in $\mathscr{E}(k_{x_i})$, it is contained in a unique component $\lambda_i \in \pi_0(\mathscr{E})$. Let $d_i = [k_{x_i} : k]$ be the degree of point x_i over k. Then $\underline{\lambda}_X$ is just the formal sum

$$\underline{\lambda}_X \coloneqq \sum_{i=1}^{m} d_i \lambda_i.$$

5.1.19 Now we are ready to prove various properties of $\mathcal{B}^1_X$ parallel to the split case.

Definition 5.1.20. The toric scheme $\mathfrak{A}$ with torus A is called *of standard type* if it is defined as

$$A = p_*(\mathbb{G}_{\mathrm{m}} \times X') \subset \mathfrak{A} = p_*(\mathbb{A}^1 \times X')$$

where $p \colon X' \to X$ is a finite étale cover.

Lemma 5.1.21. *The functor $\mathcal{B}^1_X$ is representable by a countable disjoint union of projective varieties over k if $\mathfrak{A}$ is of standard type.*

Proof. Any A-torsor E over $X \times S$ for any k-scheme S can be obtained as $p_* E'$ for some $\mathbb{G}_m$-torsor on $X' \times S$ as follows: take étale cover $U \to X \times S$ so that E is trivial on U. The descent datum is given by an isomorphism of A-torsors $\sigma \colon p_1^* E \to p_2^* E$ (p_i is the i-th projection map $U \times_{X \times S} U \to U$) satisfying cocycle condition. This σ may be regarded as a point in $A(U \times_{X \times S} U)$ since A is commutative. Let $U' = U \times_X X'$, then by definition $A(U \times_{X \times S} U) = \mathbb{G}_m(U' \times_{X' \times S} U')$, so that σ lifts to a $\mathbb{G}_m$-cocycle on the cover $U' \to X' \times S$. This means that the morphism of algebraic stacks

$$p_* \colon \mathrm{Bun}_{\mathbb{G}_m/X'} \longrightarrow \mathrm{Bun}_{A/X}$$

is essentially surjective. Again because A is commutative, the automorphism group of E is $A(X \times S) = \mathbb{G}_m(X' \times S)$. Thus p_* is a fully faithful map of the stacks. Therefore p_* is an equivalence of algebraic stacks.

The stack $\mathcal{B}^1_{X,\mathfrak{A}}$ classifies pairs (E, ϕ) where $E \in \mathrm{Bun}_{A/X}$ and ϕ is a section of $E \times^A \mathfrak{A}$. Let $E' \in \mathrm{Bun}_{\mathbb{G}_m/X'}$ be such that $p_* E' = E$, then $\phi \in E \times^A \mathfrak{A}(X) = E' \times^{\mathbb{G}_m} \mathbb{A}^1(X')$. This means that $p_* \colon \mathcal{B}^1_{X',\mathbb{A}^1} \to \mathcal{B}^1_{X,\mathfrak{A}}$ is also an equivalence. It is well-known that $\mathcal{B}^1_{X',\mathbb{A}^1}$ is representable by a countable union of projective schemes (being symmetric powers of X'), hence so is $\mathcal{B}^1_{X,\mathfrak{A}}$. ∎

Lemma 5.1.22. *Suppose we have closed embedding $\mathfrak{A} \subset \mathfrak{A}'$ of toric schemes associated with closed embedding of tori $A \subset A'$ over X, and suppose further that both $\mathcal{B}^1_{X,\mathfrak{A}}$ and $\mathcal{B}^1_{X,\mathfrak{A}'}$ are representable by separated schemes locally of finite type, then the induced map $\mathcal{B}^1_{X,\mathfrak{A}} \to \mathcal{B}^1_{X,\mathfrak{A}'}$ is a closed embedding.*

Proof. We show this by showing that it is a proper monomorphism of functors, which is equivalent to the map being a closed embedding for schemes. For injectivity, given $(E', \phi') \in \mathcal{B}^1_{X,\mathfrak{A}'}(S)$ and two liftings to $\mathcal{B}^1_{X,\mathfrak{A}}(S)$ which may be identified with two A-

torsors $E_1, E_2 \subset E'$ with a rational section $\phi = \phi'$. Since over an open dense subset of $X \times S$, ϕ gives a trivialization of both E_1 and E_2, they must coincide over all $X \times S$ by taking scheme-theoretic closure in E'. This means $(E_1, \phi) = (E_2, \phi) \in \mathcal{B}^1_{X,\mathfrak{A}}(S)$.

For properness, as both spaces are separated schemes and locally of finite type over k, we only need to show the existence part of valuative criterion. Let R be a discrete valuation ring and F its fraction field. Let $(E', \phi') \in \mathcal{B}^1_{X,\mathfrak{A}'}(R)$ with F-lifting $(E_F, \phi_F) \in \mathcal{B}^1_{X,\mathfrak{A}}(F)$. Let $U \subset X_R$ be the open subset such that the image of ϕ' is contained in E', then $X_R - U \cup X_F$ has codimension at least 2. The torsor E' is trivialized by ϕ' over U, hence we may glue E_F with the trivial A-torsor over U to obtain a lift of (E', ϕ') over $U \cup X_F$. Since A and X_R are both normal and $X_R - U \cup X_F$ has codimension at least 2, we may extend it into a pair $(E, \phi) \in \mathcal{B}^1_{X,\mathfrak{A}}(R)$ using Hartogs' theorem. This proves properness and we are done. ∎

Proposition 5.1.23. *The functor $\mathcal{B}^1_X$ is representable by a countable disjoint union of projective schemes over k. Moreover, every $\mathcal{B}^1_{X,\mathfrak{A}}$ can be embedded into $\mathcal{B}^1_{X,\mathfrak{A}_P}$ as a closed subscheme for some $\mathfrak{A}_P$ of standard type.*

Proof. We deduce the general case from the case of standard type. Consider the sheaf of characters $\mathbb{X}(A) = \underline{\mathrm{Hom}}_X(A, \mathbb{G}_m)$ and the dual cone $\mathcal{C}^* \subset \mathbb{X}(A)$, both of which can be represented by a countable disjoint union of étale covers of X. Take a finite set $P \subset \pi_0(\mathcal{C}^*)$, viewed as a subsheaf of $\mathcal{C}^*$, such that P generates $\mathcal{C}^*$ as a sheaf of monoids. Consider the constant sheaf $\mathbb{N}$ (resp. $\mathbb{Z}$) on P, whose pushforward to X may be seen as the sheaf of free abelian monoid (resp. group) generated by P, denoted by $\mathcal{C}_P^*$ (resp. $\mathbb{X}_P$). Then we have surjective sheaf maps $\mathcal{C}_P^* \to \mathcal{C}^*$ and $\mathbb{X}_P \to \mathbb{X}(A)$.

The monoid $\mathfrak{A}$ is given by the relative spectrum

$$\underline{\mathrm{Spec}}_{\mathcal{O}_X}\Big(\bigoplus_{\alpha}(p_{\alpha*}\mathcal{O}_\alpha)^\vee\Big),$$

where α ranges over connected components of $\mathcal{C}^*$, $p_\alpha\colon \alpha \to X$ is the natural map, and superscript $\vee$ is the dual as $\mathcal{O}_X$-modules. The $\mathcal{O}_X$-algebra structure on the direct sum can be easily seen étale-locally. The torus A is similarly defined by replacing $\mathscr{E}^*$ with $\mathbb{X}(A)$. We let $\mathfrak{A}_P$ (resp. A_P) be the monoid defined by the same construction, with $\mathscr{E}^*$ replaced by $\mathscr{E}_P^*$ (resp. $\mathbb{X}_P$), then we have closed embeddings $\mathfrak{A} \to \mathfrak{A}_P$ and $A \to A_P$. We also know that $\mathfrak{A}_P$ is of standard type.

We have maps of stacks

$$[\mathfrak{A}/A] \longrightarrow [\mathfrak{A}_P/A] \longrightarrow [\mathfrak{A}_P/A_P]$$

which induces morphisms of k-mapping stacks

$$\mathcal{B}^{1+}_{X,\mathfrak{A}} \longrightarrow \mathcal{B}^{1+}_{\star} := \underline{\mathrm{Hom}}_{X/k}(X, [\mathfrak{A}_P/A]) \longrightarrow \mathcal{B}^{1+}_{X,\mathfrak{A}_P}.$$

The map $\mathcal{B}^{1+}_{\star} \to \mathrm{Bun}_A$ is representable: indeed, since the fiber over an S-point $E_S \in \mathrm{Bun}_A(S)$ is the hom-functor $\underline{\mathrm{Hom}}_{X\times S/S}(X \times S, E_S \times^A \mathfrak{A}_P)$, which is representable by an S-scheme due to the fact that $E_S \times^A \mathfrak{A}_P$ is a vector bundle (see [Gro63, 7.7.8, 7.7.9]). Since Bun_A is an algebraic stack, this means that $\mathcal{B}^{1+}_{\star}$ is also an algebraic stack.

Now we restrict to $\mathcal{B}^1_{X,\mathfrak{A}} \to \mathcal{B}^1_{\star}$ where $\mathcal{B}^1_{\star} \subset \mathcal{B}^{1+}_{\star}$ is defined similarly by requiring the section ϕ in the pair (E,ϕ) to be generically contained in $E \times^A A_P$. Since A acts on both A and A_P freely, we see that both functors have no non-trivial automorphisms, hence are equivalent to sheaves. In fact, $\mathcal{B}^1_{\star} \to \mathcal{B}^1_{X,\mathfrak{A}_P}$ is just the pullback of homomorphism of Picard stacks $\mathrm{Bun}_A \to \mathrm{Bun}_{A_P}$, which is a morphism representable by schemes because A is a subtorus of A_P. Since $\mathcal{B}^1_{X,\mathfrak{A}_P}$ is a scheme by Lemma 5.1.21, so is $\mathcal{B}^1_{\star}$. In fact, $\mathcal{B}^1_{\star}$ must be a countable disjoint union of quasi-projective varieties.

Since $\mathfrak{A} \subset \mathfrak{A}_P$, $\mathcal{B}^1_{X,\mathfrak{A}}$ is a subsheaf of (not necessarily projective) scheme $\mathcal{B}^1_{\star}$. Let $(E,\phi) \in \mathcal{B}^1_{\star}(\mathcal{B}^1_{\star})$ be the universal pair given by the identity map of $\mathcal{B}^1_{\star}$. Let $(\mathcal{B}^1_{\star})'$ be defined by Cartesian diagram

$$\begin{array}{ccc} (\mathcal{B}^1_\star)' & \xrightarrow{\phi'} & E\times^A \mathfrak{A} \\ \downarrow & & \downarrow \\ \mathcal{B}^1_\star & \xrightarrow{\phi} & E\times^A \mathfrak{A}_P \end{array},$$

then $(\mathcal{B}^1_\star)'$ is a closed subscheme of $\mathcal{B}^1_\star$. The inclusion $\mathcal{B}^1_{X,\mathfrak{A}} \to \mathcal{B}^1_\star$ factors through $(\mathcal{B}^1_\star)'$ and $(\mathcal{B}^1_\star)'$ clearly coincides with $\mathcal{B}^1_{X,\mathfrak{A}}$ as subsheaves. This means that $\mathcal{B}^1_{X,\mathfrak{A}}$ is representable by a countable disjoint union of quasi-projective schemes.

Finally, as $\mathcal{B}^1_{X,\mathfrak{A}}$ is representable by a scheme locally of finite type over k, $\mathcal{B}^1_{X,\mathfrak{A}}$ is a closed subscheme of $\mathcal{B}^1_{X,\mathfrak{A}_P}$ by Lemma 5.1.22 and we are done. ∎

Corollary 5.1.24. *Suppose we have homomorphism $\mathfrak{A}\subset\mathfrak{A}'$ of toric schemes associated with homomorphism of tori $A\subset A'$ over X, such that the induced map of cones $\mathscr{C}_{\mathfrak{A}}\to\mathscr{C}_{\mathfrak{A}'}$ is a closed embedding, then the induced map $\mathcal{B}^1_{X,\mathfrak{A}}\to\mathcal{B}^1_{X,\mathfrak{A}'}$ is a closed embedding.*

Proof. As both spaces are representable by schemes locally of finite type over k, it suffices to show the map is a monomorphism and proper. With exactly same argument as in Lemma 5.1.22, we can show that $\mathcal{B}^1_{X,\mathfrak{A}}\to\mathcal{B}^1_{X,\mathfrak{A}'}$ is proper using valuative criterion. So we only need to show injectivity as functors. We first do some reductions.

The cone $\mathscr{C}_{\mathfrak{A}}$ generates a subsheaf of lattices in $\check{\mathbb{X}}(A)$, giving a subtorus $A_1\subset A$. The cone $\mathscr{C}_{\mathfrak{A}}\subset\check{\mathbb{X}}(A_1)$ corresponds to an A_1-toric scheme $\mathfrak{A}_1$ which also embeds as a closed subscheme in $\mathfrak{A}$. The quotient lattice corresponds to the quotient torus A/A_1, and the image of the cone $\mathscr{C}_{\mathfrak{A}}=\mathscr{C}_{\mathfrak{A}_1}$ is 0. Thus we have a Cartesian diagram

$$\begin{array}{ccc} \mathfrak{A}_1 & \longrightarrow & \mathfrak{A} \\ \downarrow & & \downarrow \\ \{1\} & \longrightarrow & A/A_1 \end{array},$$

which induces a Cartesian diagram of moduli spaces

$$\begin{array}{ccc} \mathcal{B}^1_{X,\mathfrak{A}_1} & \longrightarrow & \mathcal{B}^1_{X,\mathfrak{A}} \\ \downarrow & & \downarrow \\ \{b\} & \longrightarrow & \mathcal{B}^1_{X,A/A_1} \end{array},$$

where b is the k-point of $\mathcal{B}^1_{X,A/A_1}$ corresponding to the trivial A/A_1-bundle equipped with the natural trivialization. But since $\mathcal{B}^1_{X,A/A_1}$ is obviously equivalent to a k-point, the map $\mathcal{B}^1_{X,\mathfrak{A}_1} \to \mathcal{B}^1_{X,\mathfrak{A}}$ is an isomorphism.

This way we reduce to the case where the rank of cone $\mathscr{C}_{\mathfrak{A}}$ (resp. $\mathscr{C}_{\mathfrak{A}'}$) equals lattice $\check{\mathbb{X}}(A)$ (resp. $\check{\mathbb{X}}(A')$). This means that $A \to A'$ has finite kernel Z. Suppose (E_1, ϕ_1) and (E_2, ϕ_2) in $\mathcal{B}^1_{X,\mathfrak{A}}(S)$ are both liftings of $(E', \phi') \in \mathcal{B}^1_{X,\mathfrak{A}'}(S)$, then there is an Z-torsor E_Z such that $E_1 \times^Z E_Z \cong E_2$. Let $U \subset X \times S$ be the open dense subset such that the image of ϕ' is contained in E', then over U the quotient ϕ_1/ϕ_2 gives a trivialization of E_Z over U. Since Z is finite, the scheme-theoretic closure of image of ϕ_1/ϕ_2 in E_Z is an isomorphic copy of $X \times S$. Hence E_Z is trivial and we are done. ∎

Proposition 5.1.25. *There exists a unique stratification of $\mathcal{B}^1_X$ indexed by multisets:*

$$\mathcal{B}^1_X = \coprod_{\underline{\mu}} \mathcal{B}^1_{\underline{\mu}},$$

such that

(1) $\mathcal{B}^1_{\underline{\mu}}(k)$ consists of those boundary divisors whose associated multiset is $\underline{\mu}$.

(2) $\mathcal{B}^1_{\underline{\mu}}$ is isomorphic to the multiplicity-free locus of the (necessarily finite) direct product

$$X_{\underline{\mu}} \coloneqq \prod_{\lambda \in \pi_0(\mathscr{C})} \mathscr{C}^{\lambda}_{\underline{\mu}(\lambda)},$$

where $(D_\lambda)_\lambda \in X_{\underline{\mu}}$ being multiplicity-free means that the total divisor $\sum_\lambda D_\lambda$ is a multiplicity-free divisor on X after pushing forward to X.

(3) $\mathcal{B}^1_{\underline{\mu}'}$ lies in the closure of $\mathcal{B}^1_{\underline{\mu}}$ if and only if $\underline{\mu} \vdash \underline{\mu}'$.

Proof. First, in the case of a simple multiset $\underline{\mu} = \lambda \in \pi_0(\mathscr{E}) - \{0\}$, we show $\mathcal{B}^1_{\underline{\mu}} \simeq \mathscr{E}^\lambda$. In the case $\mathfrak{A} = p_*\mathbb{A}^1$, where $p\colon \mathscr{E}^\lambda \to X$ is the natural map, the result is given by the proof of Lemma 5.1.21. In general case, the inclusion $\mathscr{E}^\lambda \to \mathscr{E}$ gives an embedding of cones corresponding to morphism of toric schemes $p_*\mathbb{A}^1 \to \mathfrak{A}$ (with homomorphism of tori $p_*\mathbb{G}_\mathrm{m} \to A$). Then we apply Corollary 5.1.24 and we are done for the case of a simple multiset. Let

$$\theta_\lambda : \mathscr{E}^\lambda \longrightarrow \mathcal{B}^1_{\underline{\mu}}$$

be the isomorphism.

The monoidal structure on $\mathfrak{A}$ induces a monoidal structure on $\mathcal{B}^1_X$: given $\theta_1 = (E_1, \phi_1)$ and $\theta_2 = (E_2, \phi_2)$ in $\mathcal{B}^1_X$, the product $\theta_1 \otimes \theta_2$ is the pair $(E_1 \times^A E_2, \phi_1 \otimes \phi_2)$, and this monoidal structure is clearly commutative. Given a multiset $\underline{\mu}$, we can define a map

$$\begin{aligned} \iota_{\underline{\mu}} : X_{\underline{\mu}} = \prod_{\lambda\in\pi_0(\mathscr{E})} \mathscr{E}^\lambda_{\underline{\mu}(\lambda)} &\longrightarrow \mathcal{B}^1_X \\ \prod_\lambda (x_1^\lambda, \dots, x^\lambda_{\underline{\mu}(\lambda)}) &\longmapsto \bigotimes_\lambda [\theta_\lambda(x_1^\lambda) \otimes \cdots \otimes \theta_\lambda(x^\lambda_{\underline{\mu}(\lambda)})]. \end{aligned} \tag{5.1.2}$$

Note that the ordering of the product has no effect on the resulting map so it is well-defined. The map (5.1.2) is proper because the source is a projective variety and the target is a separated scheme. Let $\overline{\mathcal{B}^1_{\underline{\mu}}}$ be the image of $\iota_{\underline{\mu}}$, which is a reduced closed subscheme of $\mathcal{B}^1_X$ because the map is proper and the source is reduced. Let $X^\circ_{\underline{\mu}}$ be the multiplicity-free locus designated by part (2), then by checking the boundary divisors at $\bar{k}$-point level we can see that $\iota_{\underline{\mu}}^{-1}(\iota_{\underline{\mu}}(X_{\underline{\mu}})) = X_{\underline{\mu}}$ hence it has open image in $\overline{\mathcal{B}^1_{\underline{\mu}}}$, denoted by $\mathcal{B}^1_{\underline{\mu}}$. In fact, one may check that $\overline{\mathcal{B}^1_{\underline{\mu}}} - \mathcal{B}^1_{\underline{\mu}}$ is just the union of the images of $\iota_{\underline{\mu}'}$ such that $\underline{\mu} \vdash \underline{\mu}'$. This stratification exhausts $\mathcal{B}^1_X$ by checking on $\bar{k}$-points which is straightforward because $\mathcal{B}^1_X(\bar{k})$ is in bijection with the set of $\bar{k}$-valued boundary divisors.

It remains to prove that the restriction of $\iota_{\underline{\mu}}$ to $X^\circ_{\underline{\mu}}$ is an isomorphism. For this purpose we embed $\mathcal{B}^1_X$ into $\mathcal{B}^1_{X,\mathfrak{A}_P}$ for some $\mathfrak{A}_P$ of standard type. In the latter case, the map $X^\circ_{\underline{\mu}} \to \mathcal{B}^1_{\underline{\mu},\mathfrak{A}_P}$ is an isomorphism by direct computation. Therefore we have a section $\mathcal{B}^1_{\underline{\mu}} \to X^\circ_{\underline{\mu}}$ to $\iota_{\underline{\mu}}$ by composing with embedding $\mathcal{B}^1_X \to \mathcal{B}^1_{X,\mathfrak{A}_P}$. Since $X^\circ_{\underline{\mu}}$ is integral and $\iota_{\underline{\mu}}$ is a bijection on $\bar{k}$-points, it has to be an isomorphism. ∎

Corollary 5.1.26. *(1) Each connected component of $\mathcal{B}^1_X$ contains a (not necessarily unique) closed stratum $\mathcal{B}^1_{\underline{\mu}}$ where $\underline{\mu} = \lambda$ is a simple multiset. In particular, there is an equivalence relation $\sim$ on the set of closed points of $\pi_0(\mathscr{E})$ such that $\pi_0(\mathcal{B}^1_X)$ (as a set) is in canonical bijection with $\pi_0(\mathscr{E})/\sim$. Each $\mathcal{B}^1_{\underline{\mu}}$ lies in the connected component corresponding to $[\lambda]$ if and only if there is some $\lambda' \sim \lambda$ such that λ' is a degree of $\underline{\mu}$.*

(2) Irreducible components of $\mathcal{B}^1_X$ are the closures of strata $\mathcal{B}^1_{\underline{\mu}}$ where $\underline{\mu}$ are primitive multisets, i.e. $\underline{\mu}(\lambda) \neq 0$ only if λ cannot be further refined.

Proof. For the first part, the minimal elements of the partial order $\vdash$ are the simple ones, and the closure of $\mathcal{B}^1_{\underline{\mu}}$ contains and only contains those $\mathcal{B}^1_\lambda$ where λ is a degree of $\underline{\mu}$.

For the second part, the maximal elements of the partial order $\vdash$ are the ones that cannot be refined any further, and those multisets are exactly the ones described by the statement. ∎

Corollary 5.1.27. *For each primitive multiset $\underline{\mu}$, the map $\iota_{\underline{\mu}}$ gives a normalization which is also a resolution of singularity of $\overline{\mathcal{B}^1_{\underline{\mu}}}$.*

Proof. The map $\iota_{\underline{\mu}}$ is birational and finite by Proposition 5.1.25, thus must be a normalization map. Since the source $X_{\underline{\mu}}$ is smooth, it is also a resolution of singularity. ∎

5.2 Global Affine Grassmannian

The reference of this section is [Zhu17], especially Lecture III therein, where proofs or reference to them can be found.

5.2.1 Let G be a reductive group over smooth curve X induced by $\mathrm{Out}(\mathbf{G})$-torsor ϑ_G. For any positive integer d, we may define the so-called Beilinson-Drinfeld affine Grassmannian over $X_d = \mathrm{Sym}^d X$ as the follows: any S-point $D \in X_d(S)$ may be interpreted as a finite flat S-family of divisors of degree d in X. The Beilinson-Drinfeld affine Grassmannian $\mathsf{Gr}_{G,d}$ sends S to the groupoid

$$\left\{ (D, E, \phi) \;\middle|\; \begin{array}{c} D \in X_d(S), E \in \mathrm{Bun}_G(S) \\ \phi\colon E|_{X\times S-D} \xrightarrow{\sim} E_0|_{X\times S-D} \text{ is a trivialization} \end{array} \right\}.$$

This is known to be an ind-scheme of ind-finite-type over X_d. Since G is reductive, it is also ind-projective. Similarly, we have the X_d-family of jet groups and arc groups defined as follows: for $D \in X_d(S)$, let I_D be the ideal in $\mathcal{O}_{X\times S}$ defining divisor D, then we have infinitesimal neighborhoods D_n defined by I_D^n as well as the formal completion $\hat{X}_D$ of $X \times S$ at D. Then the jet groups and arc group are defined as

$$\begin{aligned} \mathbb{L}^+_{X_d,n} G(S) &= \{(D, g) \mid D \in X_d(S), g \in G(D_n)\}, \\ \mathbb{L}^+_{X_d} G(S) &= \{(D, g) \mid D \in X_d(S), g \in G(\hat{X}_D)\}. \end{aligned}$$

They are known to be representable by an affine scheme over X_d, and the jet groups are of finite type. In addition, using the fact that G is smooth over X and the infinitesimal lifting criterion, we see that the jet group schemes are smooth over X_d, while the arc group is formally smooth over X_d.

5.2.2 The definition of loop groups is trickier. First, one can show that the formal scheme $\hat{X}_D$ is ind-affine relative to S. Without loss of generality let S be affine. Taking its ring R_D of global functions and let $\hat{X}'_D = \mathrm{Spec}\, R_D$, then there is a canonical map $\hat{X}_D \to \hat{X}'_D$ through which the map $\hat{X}_D \to X \times S$ uniquely factors. Therefore it makes sense to define scheme $\hat{X}_D^\circ = \hat{X}_D' - D$. Then the loop group is defined as

$$\mathbb{L}_{X_d}G(S) = \{(D, g) \mid D \in X_d(S), g \in G(\hat{X}_D^\circ)\}.$$

This functor is represented by an ind-scheme over X_d and we have natural isomorphism of k-spaces

$$\mathsf{Gr}_{G,d} \simeq [\mathbb{L}_{X_d}G/\mathbb{L}^+_{X_d}G].$$

5.2.3 If we have curves X^λ parametrized by λ in a finite set Λ and a tuple of positive integers $\underline{d} = (d_\lambda)_{\lambda\in\Lambda}$, then we can let $X = \coprod_\lambda X^\lambda$ and we will have natural map

$$X^\Lambda_{\underline{d}} := \prod_\lambda X^\lambda_{d_\lambda} \longrightarrow X_{|\underline{d}|},$$

where $|\underline{d}| = \sum_\lambda d_\lambda$. Therefore we also have the affine Grassmannian, arc group, etc. over $X^\Lambda_{\underline{d}}$.

5.2.4 The map $G \to G^{\mathrm{ad}}$ compatible with the fixed pinning spl induces a homomorphism $\check{\mathbb{X}}(T) \to \check{\mathbb{X}}(T^{\mathrm{ad}})$ as well as the dominant cones therein. The dominant cone $\check{\mathbb{X}}(T^{\mathrm{ad}})_+$ is locally freely generated by the fundamental coweights, hence it corresponds to a T^{ad}-toric scheme $\mathfrak{A}^{\mathrm{ad}}$ of standard type over X. Viewing $\check{\mathbb{X}}(T^{\mathrm{ad}})$ as a countable étale cover of X, then there is a canonical union of connected components $\mathscr{E}^{\check{\varpi}} \subset \check{\mathbb{X}}(T^{\mathrm{ad}})_+$ corresponding to the set of all fundamental coweights. By Lemma 5.1.21, the monoid $\mathfrak{A}^{\mathrm{ad}}$ is then the pushforward

$$\mathfrak{A}^{\mathrm{ad}} = p^{\check{\varpi}}_* \mathbb{A}^1,$$

where $p^{\check{\varpi}} : \mathcal{C}^{\check{\varpi}} \to X$ is the natural map. The moduli of $\check{\mathbb{X}}(T^{\mathrm{ad}})_+$-valued boundary divisors $\mathcal{B}^1_{X,\mathfrak{A}^{\mathrm{ad}}}$ can thus be identified with union of symmetric powers

$$\mathcal{B}^1_{X,\mathfrak{A}^{\mathrm{ad}}} = \coprod_{d=0}^{\infty} \mathcal{C}^{\check{\varpi}}_d,$$

on which we have the affine Grassmannian, arc group, etc. (using either G or G^{ad}, or any group for that matter) all well-defined. The dominant cone $\check{\mathbb{X}}(T)_+$ for T is not necessarily strictly convex, but any saturated strictly convex subcone $\mathcal{C} \subset \check{\mathbb{X}}(T)_+$ determines a T-toric scheme $\mathfrak{A}$, and the homomorphism of cones $\mathcal{C} \to \check{\mathbb{X}}(T^{\mathrm{ad}})_+$ induces a proper map of boundary moduli

$$\mathcal{B}^1_{X,\mathfrak{A}} \longrightarrow \mathcal{B}^1_{X,\mathfrak{A}^{\mathrm{ad}}},$$

and thus one may pullback the affine Grassmannian, arc group, etc. from $\mathcal{B}^1_{X,\mathfrak{A}^{\mathrm{ad}}}$. Therefore such notions make sense for $\mathcal{B}^1_{X,\mathfrak{A}}$ even if it is not a union of symmetric powers of curves (or even smooth).

5.3 Global Affine Schubert Scheme

Just like the Beilinson-Drinfeld affine Grassmannian is "affine Grassmannians in a family", we also have the corresponding family of affine Schubert schemes.

Usually, the affine Schubert varieties are defined as certain subschemes of affine Grassmannian. However, things get tricky when multiple points are involved due to the complication caused by the torsions in the fundamental group of G (or equivalently, in $\pi_0(\mathrm{Bun}_G)$). See for example, [Var04, Definition 2.10] when the group is split. The definition given in *loc. cit.* is in fact slightly incorrect (further showing its trickiness).

Here we give a definition of affine Schubert scheme using reductive monoids. This method allows us to directly define affine Schubert schemes without referring to affine

Grassmannian at all, and is very straightforward.

5.3.1 Let $\mathfrak{M} = \mathrm{Env}(G^{\mathrm{sc}})$, and we have the Deligne-Mumford stack of boundary divisors $\mathcal{B}_X$ defined as an open substack

$$\mathcal{B}_X \subset \underline{\mathrm{Hom}}_X(X, [\mathfrak{A}_{\mathfrak{M}}/Z_{\mathfrak{M}}]).$$

This is not a space simply because $Z_{\mathfrak{M}} \to \mathfrak{A}_{\mathfrak{M}}^{\times}$ has a finite kernel, but it won't complicate matters too much.

Given $b = (\mathcal{L}, \theta) \in \mathcal{B}_X(S)$, where $\mathcal{L}$ is a $Z_{\mathfrak{M}}$-torsor over $X \times S$ and θ is a section of $\mathfrak{A}_{\mathfrak{M},\mathcal{L}}$ that is generically contained in $\mathfrak{A}_{\mathfrak{M},\mathcal{L}}^{\times}$ over every geometric point $s \in S$. We have Cartesian diagram

$$\begin{array}{ccc} \mathfrak{M}_b & \longrightarrow & \mathfrak{M}_{\mathcal{L}} \\ \downarrow & & \downarrow \\ X \times S & \xrightarrow{\theta} & \mathfrak{A}_{\mathfrak{M},\mathcal{L}} \end{array}$$

Recall that the numerical boundary divisor $\mathfrak{A}_{\mathfrak{M}} - \mathfrak{A}_{\mathfrak{M}}^{\times}$ is a principal divisor cut out by the product of all simple roots

$$\Pi_{\Delta} \coloneqq \prod_{i=1}^{r} e^{\alpha_i},$$

which is well-defined since Π_{Δ} is fixed by $\mathrm{Out}(\mathbf{G}^{\mathrm{sc}})$. The pullback $\theta^{*}\Pi_{\Delta}$ defines a Cartier divisor $\mathfrak{B}_b$ on X. Let $\hat{X}_{\mathfrak{B}_b}$ be the formal completion of $X \times S$ at $\mathfrak{B}_b$. Let $\mathbb{L}_b^{+}G^{\mathrm{sc}}$ be the arc group defined using divisor $\mathfrak{B}_b$, and similarly $\mathbb{L}_b^{+}\mathfrak{M}_b$ the arc space of $\mathfrak{M}_b$. Then we have arc spaces over $\mathcal{B}_X$:

$$\mathbb{L}_{\mathcal{B}_X}^{+}G^{\mathrm{sc}}(S) \coloneqq \{(b, g) \mid b \in \mathcal{B}_X(S), g \in G^{\mathrm{sc}}(\hat{X}_{\mathfrak{B}_b})\},$$

$$\mathbb{L}_{\mathcal{B}_X}^{+}(\mathfrak{M}/\mathfrak{A}_{\mathfrak{M}})(S) \coloneqq \{(b, x) \mid b \in \mathcal{B}_X(S), x \in \mathfrak{M}_b(\hat{X}_{\mathfrak{B}_b})\}.$$

The arc group $\mathbb{L}_{\mathcal{B}_X}^{+}G^{\mathrm{sc}}$ acts on $\mathbb{L}_{\mathcal{B}_X}^{+}(\mathfrak{M}/\mathfrak{A}_{\mathfrak{M}})$ freely both on the left and on the right by translation. If b is a $\bar{k}$-point, with corresponding boundary divisor $\sum_{\bar{v}\in X(\bar{k})}\lambda_{\bar{v}}\bar{v}$, then the set-theoretic fiber $\mathbb{L}_b^+\mathfrak{M}_b(\bar{k})$ over b is isomorphic to the product

$$\prod_{\bar{v}\in X(\bar{k})}\Bigg[\sum_{\substack{\mu\in\check{\mathbb{X}}(T_{\bar{v}}^{\mathrm{ad}})_+\\ \mu\leq -w_0(\lambda_{\bar{v}})}} G^{\mathrm{sc}}(\breve{\mathcal{O}}_{\bar{v}})\pi_{\bar{v}}^{(\lambda_{\bar{v}},\mu)}G^{\mathrm{sc}}(\breve{\mathcal{O}}_{\bar{v}})\Bigg].$$

The *global affine Schubert scheme* of the *adjoint* group G^{ad} is defined as the fpqc quotient

$$\mathsf{Gr}_{G^{\mathrm{ad}}}^{\leq -w_0(\mathcal{B}_X)} \coloneqq \mathbb{L}_{\mathcal{B}_X}^+(\mathfrak{M}/\mathfrak{A}_{\mathfrak{M}})/\mathbb{L}_{\mathcal{B}_X}^+G^{\mathrm{sc}}.$$

Then it is straightforward to see that its fiber at $b\in\mathcal{B}_X(\bar{k})$ is canonically isomorphic to

$$\prod_{\bar{v}\in X(\bar{k})}\mathsf{Gr}_{G_{\bar{v}}^{\mathrm{ad}}}^{\leq -w_0(\lambda_{\bar{v}})}(\bar{k}).$$

5.3.2 By replacing $\mathfrak{M}$ with the big-cell locus $\mathfrak{M}^\circ$, we obtain a generalized notion of affine Schubert cells. More precisely, we define

$$\mathsf{Gr}_{G^{\mathrm{ad}}}^{-w_0(\mathcal{B}_X)} = \mathbb{L}_{\mathcal{B}_X}^+(\mathfrak{M}^\circ/\mathfrak{A}_{\mathfrak{M}})/\mathbb{L}_{\mathcal{B}_X}^+G^{\mathrm{sc}}$$

whose fiber at $b\in\mathcal{B}_X(\bar{k})$ is canonically isomorphic to product

$$\prod_{\bar{v}\in X(\bar{k})}\mathsf{Gr}_{G_{\bar{v}}^{\mathrm{ad}}}^{-w_0(\lambda_{\bar{v}})}(\bar{k}).$$

5.3.3 We may also construct a canonical map of functors

$$\mathsf{Gr}_{G^{\mathrm{ad}}}^{\leq -w_0(\mathcal{B}_X)}\longrightarrow \mathsf{Gr}_{G^{\mathrm{ad}},\mathcal{B}_X}.$$

First, it is easy to see that the Beilinson-Drinfeld affine Grassmannian of G^{ad} over $\mathcal{B}_X$ defined in the previous section is the same as the functor sending S to the groupoid of tuples (b, E, ϕ) where $b \in \mathcal{B}_X(S)$, E is a G^{ad}-bundle over $X \times S$, and ϕ is a trivialization of E over $X \times S - \mathfrak{B}_b$.

Now let $(b, x) \in \mathbb{L}^+_{\mathcal{B}_X}(\mathfrak{M}/\mathfrak{A}_{\mathfrak{M}})(S)$ is an S-point. Similar to affine Grassmannian case, the formal scheme $\hat{X}_{\mathfrak{B}_b}$ uniquely factors through an S-ind-affine scheme $\hat{X}'_{\mathfrak{B}_b}$. Let punctured disc $\hat{X}^{\circ}_{\mathfrak{B}_b} = \hat{X}'_{\mathfrak{B}_b} - \mathfrak{B}_b$. Over the $\hat{X}^{\circ}_{\mathfrak{B}_b}$, the arc in $\mathfrak{M}_b$ is a loop point of $(G_+)_{\mathcal{L}}$, which induces a loop point in $G^{\mathrm{ad}}_{\mathcal{L}}$. Since $\mathcal{L}$ is a $Z_{\mathfrak{M}}$-torsor, the induced G^{ad}-torsor $G^{\mathrm{ad}}_{\mathcal{L}}$ is canonically trivial. Thus we have a loop point in G^{ad}. Using this loop point, we may glue the trivial G^{ad}-torsor on $X \times S - \mathfrak{B}_b$ with the trivial G^{ad}-torsor on $\hat{X}'_{\mathfrak{B}_b}$ and obtain a G^{ad}-torsor E_{ad} on $X \times S$, together with a tautological trivialization on $X \times S - \mathfrak{B}_b$. This gives a morphism

$$\mathbb{L}^+_{\mathcal{B}_X}(\mathfrak{M}/\mathfrak{A}_{\mathfrak{M}}) \longrightarrow \mathsf{Gr}_{G^{\mathrm{ad}}, \mathcal{B}_X}.$$

This morphism is invariant under $\mathbb{L}^+_{\mathcal{B}_X} G^{\mathrm{sc}}$-action, hence we have a well-defined map

$$\mathsf{Gr}^{\leq -w_0(\mathcal{B}_X)}_{G^{\mathrm{ad}}} \longrightarrow \mathsf{Gr}_{G^{\mathrm{ad}}, \mathcal{B}_X}.$$

The details will play out exactly same as in the situation of the "usual" affine Grassmannian, so we leave it to the reader.

5.3.4 For group G itself, as in the affine Grassmannian case, we may use the map $G \to G^{\mathrm{ad}}$ to produce global affine Schubert schemes "of G", although it has little content since it is just multiple copies of the objects in G^{ad} case. We may also change monoid to an arbitrary $\mathfrak{M} \in \mathcal{FM}(G^{\mathrm{sc}})$, and use the Cartesian diagram

$$\begin{array}{ccc} \mathfrak{M} & \longrightarrow & \mathrm{Env}(G^{\mathrm{sc}}) \\ \downarrow & & \downarrow \\ \mathfrak{A}_{\mathfrak{M}} & \longrightarrow & \mathfrak{A}_{\mathrm{Env}(G^{\mathrm{sc}})} \end{array}$$

to pullback and obtain the global affine Schubert scheme relative to the divisor stack associated with $[\mathfrak{A}_{\mathfrak{M}}/Z_{\mathfrak{M}}]$. If $\mathfrak{A}_{\mathfrak{M}}$ is of standard type, then it is the same as replacing $\mathrm{Env}(G^{\mathrm{sc}})$ by $\mathfrak{M}$ in our previous construction. The detail is left to the reader.

5.3.5 An immediate benefit of our definition is the following

Lemma 5.3.6. *Let $\mathfrak{M} \in \mathcal{FM}(G^{\mathrm{sc}})$. The map*

$$\mathbb{L}^+_{\mathcal{B}_X}(\mathfrak{M}/\mathfrak{A}_{\mathfrak{M}}) \longrightarrow \mathsf{Gr}^{\leq -w_0(\mathcal{B}_X)}_{G^{\mathrm{ad}}}$$

is formally smooth.

Proof. Indeed, because the arc group $\mathbb{L}^+_{\mathcal{B}_X}G^{\mathrm{sc}}$ is formally smooth and its action by right translation on $\mathbb{L}^+_{\mathcal{B}_X}(\mathfrak{M}/\mathfrak{A}_{\mathfrak{M}})$ is free. ∎

Remark 5.3.7. (1) If the reader compares the definition in [Var04] with the one we give here, one can see the reason why the traditional definition gets so complicated: when we define the universal monoid $\mathrm{Env}(G^{\mathrm{sc}})$ (c.f., § 2.3), we are not only using the representations of G^{sc}, but also the abelianization space $\mathfrak{A}_{\mathrm{Env}(G^{\mathrm{sc}})} \cong \mathbb{A}^r$; on the other hand, the traditional definition only utilizes the representation part, thus certain "integral" condition is lost and has to be rebuilt using ad hoc formulations.

(2) The family of affine Schubert variety here is not parametrized by a space rather a Deligne-Mumford stack instead. One may view it as a slight downside, but the inclusion of automorphism groups is precisely why it naturally resolves the complication caused by Z^{sc}-twisting.

5.3.8 Let U be a k-scheme and $U \to \mathcal{B}_X$ be a map, then we have induced family of affine Schubert varieties over U, and we denote it by $\mathsf{Gr}_{G^{\mathrm{ad}}}^{\leq -w_0(\lambda_U)}$. The support of boundary divisors λ_U parametrized by U can be viewed as a purely codimension 1 subscheme of $U \times X$. The union of some connected components of such subscheme can be seen as a U-flat family of boundary subdivisors of λ_U. Call this subdivisor λ'_U, then since the arc scheme construction is local, we may also define a U-family of affine Schubert schemes

$$\mathsf{Gr}_{G^{\mathrm{ad}}}^{\leq -w_0(\lambda'_U)} \longrightarrow U, \tag{5.3.1}$$

whose fiber at $u \in U$ is just the direct factor of $\mathsf{Gr}_{G^{\mathrm{ad}}}^{\leq -w_0(\lambda_{U,u})}$ supported on λ'_U. So we have natural projections fitting into commutative diagram

$$\begin{array}{ccc}
\mathbb{L}^+_{\lambda_U}(\mathfrak{M}/\mathfrak{A}_{\mathfrak{M}}) & \longrightarrow & \mathbb{L}^+_{\lambda'_U}(\mathfrak{M}/\mathfrak{A}_{\mathfrak{M}}) \\
\downarrow & & \downarrow \\
\mathsf{Gr}_{G^{\mathrm{ad}}}^{\leq -w_0(\lambda_U)} & \longrightarrow & \mathsf{Gr}_{G^{\mathrm{ad}}}^{\leq -w_0(\lambda'_U)}
\end{array}$$

where the vertical maps are formally smooth.

Definition 5.3.9. The map (5.3.1) is called the *partial affine Schubert scheme supported on* λ'_U.

5.4 Perverse Sheaves

The important aspect about the global affine Grassmannian is the category of equivariant perverse sheaves on it. In the single-point case, i.e., the “usual” affine Grassmannian case, it is well-known that it is equivalent to the Tannakian category of the dual group over $\overline{\mathbb{Q}}_\ell$ through geometric Satake equivalence. The affine Schubert varieties characterize all the simple objects in the that category corresponding to irreducible representations. The results are similar in the case of Beilinson-Drinfeld affine Grassmannians, and we briefly review it. See [Zhu17] (especially its §§ 4 and 5) for example for details.

5.4.1 First of all, it is more convenient to start with de-symmetrized version of Beilinson-Drinfeld affine Grassmannians. In other words, instead of symmetric power X_d, we consider direct product X^d and Gr_{G,X^d} on it. The factorization property of Beilinson-Drinfeld affine Grassmannian states that over the multiplicity-free locus $(X^d)^\circ$, we have canonical isomorphism

$$\mathrm{Gr}_{G,X^d} \times_{X^d} (X^d)^\circ \simeq \left(\prod_{i=1}^{d} \mathrm{Gr}_G\right) \times_{X^d} (X^d)^\circ.$$

We also have the d-fold convolution affine Grassmannian

$$\mathrm{Gr}_G^{\tilde{\times} d} \coloneqq \mathrm{Gr}_G \tilde{\times} \cdots \tilde{\times} \mathrm{Gr}_G$$

whose S-points consist of tuples

$$(x_1, \dots, x_d; E_1, \dots, E_d, \phi_1, \dots, \phi_d),$$

where $x_i \in X(S)$, $E_i \in \mathrm{Bun}_G(S)$, and

$$\phi_i \colon E_i|_{X\times S - x_i} \xrightarrow{\sim} E_{i-1}|_{X\times S - x_i}$$

is an isomorphism. Here again E_0 denotes the trivial G-torsor. We have the convolution map

$$\begin{aligned} m_d \colon \mathrm{Gr}_G^{\tilde{\times} d} &\longrightarrow \mathrm{Gr}_{G,X^d} \\ (x_i, E_i, \phi_i) &\longmapsto (x_1, \dots, x_d, E_d, \phi_1 \circ \cdots \circ \phi_d), \end{aligned}$$

which is also known to be an isomorphism over $(X^d)^\circ$. We may factor m_d into smaller steps

$$\mathsf{Gr}_G^{\tilde{\times} d} \longrightarrow \mathsf{Gr}_{G,X^2} \tilde{\times} \mathsf{Gr}_G^{\tilde{\times}(d-2)} \longrightarrow \cdots \longrightarrow \mathsf{Gr}_{G,X^{d-1}} \tilde{\times} \mathsf{Gr}_G \longrightarrow \mathsf{Gr}_{G,X^d}.$$

Over any point $(x_i) \in X^d$, each step above is stratified semi-small in the sense of [MV07, p. 14] (to see this, one only need to combine factorization property and the well-known fact that over a single point x, the convolution map $\mathsf{Gr}_{G,x} \tilde{\times} \mathsf{Gr}_{G,x} \to \mathsf{Gr}_{G,x}$ is stratified semi-small). Over the whole base X^d, each step above is in fact small because it is an isomorphism over an open dense subset $(X^d)^\circ$, hence making the inequality in the definition of semi-smallness strict.

5.4.2 Let F_i $(1 \le i \le d)$ be a $\mathbb{L}^+G$-equivariant perverse sheaf on Gr_G. Then there is the notion of twisted external product $F_1 \tilde{\boxtimes} \cdots \tilde{\boxtimes} F_d$ on $\mathsf{Gr}_G^{\tilde{\times} d}$ whose restriction to

$$\mathsf{Gr}_G^{\tilde{\times} d} \times_{X^d} (X^d)^\circ \simeq \left(\prod_{i=1}^{d} \mathsf{Gr}_G \right) \times_{X^d} (X^d)^\circ$$

may be identified with external product $F_1 \boxtimes \cdots \boxtimes F_d$. The classical result is that

$$m_{d!}(F_1 \tilde{\boxtimes} \cdots \tilde{\boxtimes} F_d) = m_{d*}(F_1 \tilde{\boxtimes} \cdots \tilde{\boxtimes} F_d) = j_{!*}(F_1 \boxtimes \cdots \boxtimes F_d|_{(X^d)^\circ})$$

where j is the inclusion $\mathsf{Gr}_G^{\tilde{\times} d} \times_{X^d} (X^d)^\circ \to \mathsf{Gr}_G^{\tilde{\times} d}$. Suppose that $F_1 \tilde{\boxtimes} \cdots \tilde{\boxtimes} F_d$ is fiberwise perverse over X^d, then we know that $j_{!*}(F_1 \boxtimes \cdots \boxtimes F_d|_{(X^d)^\circ})$ is fiberwise perverse over X^d.

5.4.3 The scheme Gr_G is locally trivial fibration over X, and so we also have locally trivial fibration of various affine Schubert varieties which can also be defined using reductive monoids in § 5.3. Suppose F_i is the intersection sheaf corresponding to such a locally trivial fibration of affine Schubert varieties, then $F_1 \tilde{\boxtimes} \cdots \tilde{\boxtimes} F_d$ is fiberwise perverse over

X^d because $\mathsf{Gr}_G^{\tilde{\times} d}$ is locally trivial over X^d. Thus we have that $j_{!*}(F_1 \boxtimes \cdots \boxtimes F_d|_{(X^d)^\circ})$ is fiberwise perverse over X^d. Using the Cartesian diagram

$$\begin{array}{ccc} \mathsf{Gr}_{G,X^d} & \xrightarrow{q_d} & \mathsf{Gr}_{G,d} \\ \downarrow & & \downarrow \\ X^d & \longrightarrow & X_d \end{array}$$

we have that the sheaf

$$q_{d*}j_{!*}(F_1 \boxtimes \cdots \boxtimes F_d|_{(X^d)^\circ})^{\mathfrak{S}_d} = j_{!*}\left[(q_d|_{(X^d)^\circ})_*(F_1 \boxtimes \cdots \boxtimes F_d|_{(X^d)^\circ})\right]^{\mathfrak{S}_d}$$

is fiberwise perverse over X_d, where $\mathfrak{S}_d$ is the symmetric group of d elements, and the inclusion $X_d^\circ \to X_d$ is still denoted by j. Thus we have the following result:

Proposition 5.4.4. *Let $\mathfrak{M} \in \mathcal{FM}(G^{\mathrm{sc}})$. Then the sheaf*

$$\mathrm{IC}_{\mathsf{Gr}_{G^{\mathrm{ad}}}^{\leq -w_0(\mathcal{B}_X)}}$$

is fiberwise $\mathbb{L}^+_{\mathcal{B}_X}G^{\mathrm{ad}}$-equivariant and perverse over $\mathcal{B}_X$.

Proof. If $\mathfrak{A}_{\mathfrak{M}}$ is of standard type (i.e., is isomorphic to a vector bundle over X), the result follows directly from the discussion above and the description of $\mathcal{B}_X$ as an étale gerbe over a disjoint union of direct products of symmetric powers of smooth curves. The fact that $\mathcal{B}_X$ has étale automorphism groups has no impact on perversity.

If $\mathfrak{A}_{\mathfrak{M}}$ is not of standard type, then we still have a finite birational cover of $\mathcal{B}_X$ such that each irreducible component is cover by an étale gerbe over a direct product of symmetric power of smooth curves. Therefore we still have the same result, since for a finite birational map $f\colon \mathscr{X} \to \mathscr{Y}$ of Deligne-Mumford stacks locally of finite type, we have that $f_*\mathrm{IC}_{\mathscr{X}} = \mathrm{IC}_{\mathscr{Y}}$. ∎

CHAPTER 6

MULTIPLICATIVE HITCHIN FIBRATIONS

In this section, we study the global construction known as the multiplicative Hitchin fibrations (mH-fibrations), also known as Hitchin-Frenkel-Ngo fibrations in some literature. There has been some earlier study of mH-fibrations in algebraic setting mostly focusing on split groups in, for example, [FN11], [Bou17], and [Chi19, § 4]. The mH-fibrations in those papers come with several variants and will be unified in a single, much generalized framework in this chapter. In fact, we will see starting from § 6.11 that such generalization is crucial in studying fundamental lemma and likely in geometrizing trace formulae in general.

Another important result of this chapter is a local model of singularity in § 6.10. Such result was predicted in [FN11] and a weaker version was proved in [Bou17] with a rather *ad hoc* argument. We will provide a more conceptual proof using deformation theory with a stronger statement.

The remaining part of this chapter is fashioned in a similar way as in [Ngô10, § 4]. The proofs will mostly be similar with some modifications.

6.1 Constructions

Compared to the well-known Hitchin fibration for the Lie algebras, the multiplicative version comes with more flavors, since in essence it is a two-stage fibration, much like the two-stage invariant map $\mathfrak{M} \to \mathfrak{C}_{\mathfrak{M}} \to \mathfrak{A}_{\mathfrak{M}}$ for reductive monoid $\mathfrak{M}$. Each of these variations has its technical advantages and weaknesses, but they do not differ in a very essential way. Although a "versal" construction clearly exists in concept, it has never been thoroughly written down in literature. Here we give such a construction using moduli of boundary divisors.

6.1.1 The construction of the usual Hitchin fibration starts with (smooth, geometrically connected, projective) curve X with genus g_X and reductive group G over X given by a $\mathrm{Out}(\mathbf{G})$-torsor ϑ_G. The scaling action of $\mathbb{G}_{\mathrm{m}}$ on $\mathfrak{g} = \mathrm{Lie}(G)$ commutes with the adjoint action of G hence induces an action on $\mathfrak{c} = \mathfrak{g} /\!\!/ \mathrm{Ad}(G)$. We then have maps

$$[\mathfrak{g}/\,\mathrm{Ad}(G) \times \mathbb{G}_{\mathrm{m}}] \longrightarrow [\mathfrak{c}/\mathbb{G}_{\mathrm{m}}] \longrightarrow \mathbb{B}\mathbb{G}_{\mathrm{m}}.$$

Let $p\colon X \to \operatorname{Spec} k$, then the image of these maps under endofunctor $p_* p^*$ on the big étale site of $\operatorname{Spec} k$ produces morphisms of k-mapping stacks

$$\underline{\mathrm{Hom}}(X, [\mathfrak{g}/\,\mathrm{Ad}(G) \times \mathbb{G}_{\mathrm{m}}]) \longrightarrow \underline{\mathrm{Hom}}(X, [\mathfrak{c}/\mathbb{G}_{\mathrm{m}}]) \longrightarrow \mathcal{P}\mathrm{ic}\, X. \tag{6.1.1}$$

Fix any k-point in $\mathcal{P}\mathrm{ic}\, X$, i.e. a line bundle $\mathcal{L}$ on X, the fiber of (6.1.1) over $\mathcal{L}$ is then the Hitchin fibration $\mathcal{M}_{\mathcal{L}}^{\mathrm{Hit}} \to \mathcal{A}_{\mathcal{L}}^{\mathrm{Hit}}$.

6.1.2 In Frenkel-Ngô's original paper [FN11], there is a primitive definition of mH-fibration with a hint towards the general one. It is later carried out and studied in various later papers (e.g. [Bou15, Bou17, Chi19]). It uses reductive monoids as an analogue of Lie algebra $\mathfrak{g}$, and the translation action of the central torus of the monoid as the analogue of the $\mathbb{G}_{\mathrm{m}}$-action on $\mathfrak{g}$.

Let $\mathfrak{M}$ be a reductive monoid in $\mathcal{FM}(G^{\mathrm{sc}})$ with abelianization $\mathfrak{A}_{\mathfrak{M}}$. The translation action of central torus $Z_{\mathfrak{M}}$ on $\mathfrak{M}$ induces maps

$$[\mathfrak{M}/G \times Z_{\mathfrak{M}}] \longrightarrow [\mathfrak{C}_{\mathfrak{M}}/Z_{\mathfrak{M}}] \longrightarrow [\mathfrak{A}_{\mathfrak{M}}/Z_{\mathfrak{M}}] \longrightarrow \mathbb{B}Z_{\mathfrak{M}},$$

which further induce maps of mapping stacks over k

$$\begin{array}{ccccccc}
\underline{\mathrm{Hom}}(X,[\mathfrak{M}/G\times Z_{\mathfrak{M}}]) & \longrightarrow & \underline{\mathrm{Hom}}(X,[\mathfrak{C}_{\mathfrak{M}}/Z_{\mathfrak{M}}]) & \longrightarrow & \underline{\mathrm{Hom}}(X,[\mathfrak{A}_{\mathfrak{M}}/Z_{\mathfrak{M}}]) & \longrightarrow & \mathrm{Bun}_{Z_{\mathfrak{M}}} \\
\| & & \| & & \| & & \downarrow \\
\mathcal{M}_X^+ & & \mathcal{A}_X^+ & & \mathcal{B}_X^+ & & \downarrow \\
 & & & & \downarrow & & \downarrow \\
 & & & & \mathcal{B}^{1+}_{X,\mathfrak{A}_{\mathfrak{M}}} & \longrightarrow & \mathrm{Bun}_{\mathfrak{A}^\times_{\mathfrak{M}}}
\end{array} \tag{6.1.2}$$

For all practical purposes, we would only consider the open subscheme $\mathcal{B}_X \subset \mathcal{B}_X^+$ being the preimage of the moduli $\mathcal{B}^1_{X,\mathfrak{A}_{\mathfrak{M}}}$ of boundary divisors. Let h_X be the map $\mathcal{M}_X \to \mathcal{A}_X$. Note that $\mathcal{B}_X \to \mathcal{B}^1_{X,\mathfrak{A}_{\mathfrak{M}}}$ is always a Z^{sc}-gerbe over its image, hence a Deligne-Mumford stack, proper and locally of finite type.

Definition 6.1.3. The map $h_X\colon \mathcal{M}_X \to \mathcal{A}_X$ is called the *(universal) multiplicative Hitchin fibration* associated with monoid $\mathfrak{M}$. We will use mH-fibration for short and omit the monoid if it is clear from the context. The stack $\mathcal{M}_X$ is called the *mH-total stack* and $\mathcal{A}_X$ is called the *mH-base*. The stack $\mathcal{B}_X$ is (still) called the *moduli of boundary divisors*. If S is a k-scheme, then an S-point of $\mathcal{M}_X$ is called an *mHiggs-bundle* on S.

6.1.4 Let $\mathfrak{M}' \to \mathfrak{M}$ be a morphism in $\mathcal{FM}(G^{\mathrm{sc}})$, that is, an excellent map of reductive monoids. Then recall we have the following Cartesian diagram

$$\begin{array}{ccccc}
[\mathfrak{M}'/G\times Z_{\mathfrak{M}'}] & \longrightarrow & [\mathfrak{C}_{\mathfrak{M}'}/Z_{\mathfrak{M}'}] & \longrightarrow & [\mathfrak{A}_{\mathfrak{M}'}/Z_{\mathfrak{M}'}] \\
\downarrow & & \downarrow & & \downarrow \\
[\mathfrak{M}/G\times Z_{\mathfrak{M}}] & \longrightarrow & [\mathfrak{C}_{\mathfrak{M}}/Z_{\mathfrak{M}}] & \longrightarrow & [\mathfrak{A}_{\mathfrak{M}}/Z_{\mathfrak{M}}]
\end{array},$$

which induces Cartesian diagram of mH-fibrations

$$\begin{array}{ccccc}
\mathcal{M}'_X & \longrightarrow & \mathcal{A}'_X & \longrightarrow & \mathcal{B}'_X \\
\downarrow & & \downarrow & & \downarrow \\
\mathcal{M}_X & \longrightarrow & \mathcal{A}_X & \longrightarrow & \mathcal{B}_X
\end{array}. \tag{6.1.3}$$

Therefore, studying mH-fibrations in general can largely be reduced to studying the mH-fibration associated with the universal monoid $\mathrm{Env}(G^{\mathrm{sc}})$ using (6.1.3).

As another generalization, we may replace $Z_{\mathfrak{M}}$ by a smooth group Z of multiplicative type together with a finite unramified homomorphism $Z \to Z_{\mathfrak{M}}$. The resulting mapping stacks are obtained by pulling back the ones above via map $\mathrm{Bun}_Z \to \mathrm{Bun}_{Z_{\mathfrak{M}}}$, and so it largely reduce to study the homomorphism of commutative group stacks $\mathrm{Bun}_Z \to \mathrm{Bun}_{Z_{\mathfrak{M}}}$, which is not so hard.

6.1.5 The map $\mathcal{A}_X \to \mathcal{B}_X$ is easy to describe using the isomorphism $\mathfrak{C}_{\mathfrak{M}} \simeq \mathfrak{A}_{\mathfrak{M}} \times \mathfrak{C}$. Over boundary divisor $\lambda_X \in \mathcal{B}_X(\bar{k})$ induced by $Z_{\mathfrak{M}}$-torsor $\mathcal{L}$, the fiber is the vector space $\mathrm{H}^0(X, \mathfrak{C}_{\mathcal{L}})$, where $\mathfrak{C}_{\mathcal{L}} = \mathfrak{C} \times^{Z_{\mathfrak{M}}} \mathcal{L}$. When $\mathfrak{M} = \mathrm{Env}(G^{\mathrm{sc}})$, $\mathcal{L}$ is a T^{sc}-torsor. Let $\varpi \in \mathbb{X}(T^{\mathrm{sc}})$ be a connected component consisting of fundamental weights, then $\mathcal{L}$ induces a line bundle on ϖ denoted by $\varpi(\mathcal{L})$, such that

$$\mathfrak{C}_{\mathcal{L}} \cong \bigoplus_{\varpi} p_{\varpi *}\varpi(\mathcal{L}),$$

where $p_{\varpi} \colon \varpi \to X$ is the natural map. Let d_{ϖ} be the degree of ϖ over X, thus we see that if $\deg \varpi(\mathcal{L}) > d_{\varpi}(2g_X - 2)$ for all ϖ, then $\mathfrak{C}_{\mathcal{L}}$ has no higher cohomology and

$$\dim_{\bar{k}} \mathrm{H}^0(X, \mathfrak{C}_{\mathcal{L}}) = \sum_{\varpi} \deg \varpi(\mathcal{L}) - r g_X + r.$$

Therefore $\mathcal{A}_X \to \mathcal{B}_X$ is a vector bundle (of varying rank) for all but finitely many connected components of $\mathcal{B}_X$. For general monoid $\mathfrak{M} \in \mathcal{FM}(G^{\mathrm{sc}})$, one may use (6.1.3) to draw the same conclusion, or alternatively use the following fact: the $Z_{\mathfrak{M}}$-torsor $\mathcal{L}$ induces a T^{sc}-torsor using any excellent morphism $\mathfrak{M} \to \mathrm{Env}(G^{\mathrm{sc}})$, thus the line bundles $\varpi(\mathcal{L})$ still make sense and so on.

6.1.6 Let us see the construction more explicitly in some special cases. First, we consider the case where the group is split and the abelianization $\mathfrak{A}_{\mathfrak{M}}$ is of standard type. Suppose $\mathfrak{A}_{\mathfrak{M}}$ is is an affine space with coordinates given by characters $e^{\theta_1}, \dots, e^{\theta_m}$. Let $\mathcal{L}$ be a $Z_{\mathfrak{M}}$-torsor such that $\theta_i(\mathcal{L})$ has degree larger than $2g_X - 2$ for all $1 \le i \le m$. Recall $\mathfrak{M}$ corresponds to a homomorphism $\phi_{\mathfrak{M}} \colon Z_{\mathfrak{M}} \to T$, which induces a T-torsor $\mathcal{L}_T$ from $\mathcal{L}$. Assume that $\varpi_i(\mathcal{L}_T)$ also has degree larger than $2g_X - 2$ for all $1 \le i \le r$. Denote the fiber of (6.1.2) over $\mathcal{L}$ by

$$h_{\mathcal{L}}^{+} \colon \mathcal{M}_{\mathcal{L}}^{+} \longrightarrow \mathcal{A}_{\mathcal{L}}^{+}.$$

Here $\mathcal{A}_{\mathcal{L}}^{+}$ is the vector space

$$\mathcal{B}_{\mathcal{L}}^{+} \oplus C_{\mathcal{L}} \coloneqq \bigoplus_{i=1}^{m} \mathrm{H}^0(X, \theta_i(\mathcal{L})) \oplus \bigoplus_{i=1}^{r} \mathrm{H}^0(X, \varpi_i(\mathcal{L}_T)),$$

where $\mathcal{B}_{\mathcal{L}}^{+}$ is the first m summands. The subspace $\mathcal{B}_{\mathcal{L}}$ is the open locus where all m sections to $\theta_i(\mathcal{L})$ is non-zero. The fibration $h_{\mathcal{L}}$ over $\mathcal{B}_{\mathcal{L}}$ is the closest analogue to the Lie algebra case.

6.1.7 Next, as another special case, let us explain here the connection between our current construction and the one in [FN11]. We consider the case where $G = G^{\mathrm{sc}}$ and is split. Let $\mathcal{M}_d$ to be the classifying stack of tuples (D, E, ϕ) where $D \in X_d$, $E \in \mathrm{Bun}_G$, and $\phi \colon E|_{X-D} \to E|_{X-D}$ is an automorphism of G-torsor over $X - D$. For any point $x \in D$, we have a well-defined relative position $\lambda_x = \mathrm{Inv}_x(\phi) \in \check{\mathbb{X}}(T)_+$ by choosing any trivialization of E over the formal disk $\hat{X}_x$. Since the relative position can be arbitrarily large, $\mathcal{M}_d$ must be of infinite type. To obtain a finite-type object, one has to put a restraint on relative positions. The simplest way is to fix a dominant cocharacter $\lambda \in \check{\mathbb{X}}(T)_+$, and let $\mathcal{M}_{d,\lambda}$ to be the (closed) substack of tuples (D, E, ϕ) such that $\lambda_x \le d_x\lambda$ for all $x \in D$, where d_x is the multiplicity of x in D.

The mH-fibration with total stack $\mathcal{M}_{d,\lambda}$ is then constructed as follows: fix a tuple (D, E, ϕ), and recall that $\mathfrak{C} = G /\!/ \mathrm{Ad}(G)$ is an affine r-space whose coordinates are given by the traces of the r fundamental representations of G. Taking the traces χ_i of ϕ, the restraint on relative positions means that the $\chi_i(\phi)$ will have poles bounded by the divisor $\langle \varpi_i, \lambda \rangle D$. This means that there is a map

$$h_{d,\lambda} \colon \mathcal{M}_{d,\lambda} \longrightarrow \mathcal{A}_{d,\lambda},$$

where $\mathcal{A}_{d,\lambda}$ is the line bundle over X_d whose fiber over D is $\bigoplus_{i=1}^{r} \mathrm{H}^0(X, \mathcal{O}_X(\langle \varpi_i, \lambda \rangle D))$. Let $\alpha_{d,\lambda}$ be the map $\mathcal{M}_{d,\lambda} \to X_d$ and $h_D \colon \mathcal{M}_D \to \mathcal{A}_D$ to be the fiber over $D \in X_d$. After [Chi19], we call h_D the *restricted* mH-fibrations.

6.1.8 More generally (still assuming $G = G^{\mathrm{sc}}$ and split), let $\underline{\lambda} = (\lambda_1, \dots, \lambda_m)$ be a tuple of dominant cocharacters (allowing repetitions). Fix a tuple of positive integers $\underline{d} = (d_1, \dots, d_m)$ and let

$$X_{\underline{d}} \coloneqq X_{d_1} \times \cdots \times X_{d_m}.$$

The sum $d = |\underline{d}| \coloneqq d_1 + \cdots + d_m$ is called the *total degree* of $\underline{d}$. For $\underline{D} \in X_{\underline{d}}$, we define D to be the sum of divisors $|\underline{D}| \coloneqq D_1 + \ldots + D_m$, so that $d = \deg D$. We also define $\check{\mathbb{X}}(T)_+$-valued divisor on X

$$\lambda_D = \underline{\lambda} \cdot \underline{D} \coloneqq \sum_{i=1}^{m} \lambda_i \cdot D_i.$$

Let $\mathcal{M}_{\underline{d}}$ be the classifying stack of tuples $(\underline{D}, E, \phi)$ where $\underline{D} \in X_{\underline{d}}$, $E \in \mathrm{Bun}_G$, and $\phi \colon E|_{X-D} \to E|_{X-D}$ is an automorphism of G-torsor. We also define $\mathcal{M}_{\underline{d},\underline{\lambda}}$ to be the closed substack such that $\mathrm{Inv}(\phi) \leq \lambda_D$. We can also define the mH-fibration by taking the trace of ϕ

$$h_{\underline{d},\underline{\lambda}}\colon \mathcal{M}_{\underline{d},\underline{\lambda}} \longrightarrow \mathcal{A}_{\underline{d},\underline{\lambda}},$$

where $\mathcal{A}_{\underline{d},\underline{\lambda}}$ is the vector bundle over $X_{\underline{d}}$ whose fiber over $\underline{D}$ is $\bigoplus_{i=1}^{r} \mathrm{H}^0(X, \mathcal{O}_X(\langle \varpi_i, \lambda_D\rangle))$. Again, let $\alpha_{\underline{d},\underline{\lambda}}$ be the map $\mathcal{M}_{\underline{d},\underline{\lambda}} \to X_{\underline{d}}$, and the restricted mH-fibration $h_{\underline{D}}$ to be the fiber of $h_{\underline{d},\underline{\lambda}}$ over $\underline{D}$.

6.1.9 Consider $\mathfrak{M} = \mathfrak{M}(\underline{\lambda})$ as in § 2.3.13 where each $\lambda_i \in \check{\mathbb{X}}(T)_+$ for $1 \le i \le m$. Then the maximal torus in $\mathfrak{M}$ is $\mathbb{G}_{\mathrm{m}}^m \times T$ and $Z_{\mathfrak{M}} = \mathbb{G}_{\mathrm{m}}^m$ maps to T via $\underline{\lambda}$. In this case, the maps (6.1.2) can be extended into a commutative diagram

$$\begin{array}{ccccccc} \mathcal{M}_{\underline{d},\underline{\lambda}} & \longrightarrow & \mathcal{A}_{\underline{d},\underline{\lambda}} & \longrightarrow & X_{\underline{d}} & & \\ \downarrow & & \downarrow & & \downarrow{\scriptstyle D_i \mapsto (\mathcal{O}(D_i), \sigma_{D_i})} & & \\ \mathcal{M}_X & \longrightarrow & \mathcal{A}_X & \longrightarrow & \mathcal{B}_X & \longrightarrow & (\mathcal{P}\mathrm{ic}\, X)^m \end{array}, \tag{6.1.4}$$

where σ_{D_i} is the canonical section of $\mathcal{O}(D_i)$. Here the stack $\mathcal{B}_X$ classifies m-tuples pairs $(\mathcal{L}_i, s_i)$ where $\mathcal{L}_i$ is a line bundle and s_i a non-zero section therein, and $X_{\underline{d}}$ embeds as an open and closed subspace of pairs with $\deg(\mathcal{L}_i) = d_i$. The squares in (6.1.4) are easily shown to be Cartesian. If we fix for each i a line bundle $\mathcal{L}_i \in \mathcal{P}\mathrm{ic}(X)$ with degree d_i, they assemble into a $Z_{\mathfrak{M}}$-bundle $\mathcal{L}$ by taking direct product, then we have pullback diagram

$$\begin{array}{ccccc} \mathcal{M}_{\mathcal{L}} & \longrightarrow & \mathcal{A}_{\mathcal{L}} & \longrightarrow & \mathcal{B}_{\mathcal{L}} \\ \downarrow & & \downarrow & & \downarrow \\ \mathcal{M}_{\underline{d},\underline{\lambda}} & \longrightarrow & \mathcal{A}_{\underline{d},\underline{\lambda}} & \longrightarrow & X_{\underline{d}} \end{array},$$

with non-empty fibers being $\mathbb{G}_{\mathrm{m}}^m$-torsors. When $\underline{\lambda} = \lambda$ is a single cocharacter and degree d is fixed, we recover the construction in [FN11]. This is the case where the monoid is a so-called L-monoid.

6.2 Symmetry of mH-fibrations

The $Z_{\mathfrak{M}}$-action on $\mathfrak{C}_{\mathfrak{M}}$ lifts to an action on $\mathfrak{J}_{\mathfrak{M}}$ and is compatible with the group scheme structure, therefore we have group scheme

$$[\mathfrak{J}_{\mathfrak{M}}/Z_{\mathfrak{M}}] \longrightarrow [\mathfrak{C}_{\mathfrak{M}}/Z_{\mathfrak{M}}],$$

which further induces a group scheme

$$\mathfrak{J}_X \to X \times \mathcal{A}_X$$

through pulling back along the evaluation map $X \times \mathcal{A}_X \to [\mathfrak{C}_{\mathfrak{M}}/Z_{\mathfrak{M}}]$. We define relative Picard stack

$$p_X \colon \mathcal{P}_X \coloneqq \mathcal{P}\mathrm{ic}(\mathfrak{J}_X/X \times \mathcal{A}_X/\mathcal{A}_X) \longrightarrow \mathcal{A}_X$$

classifying $\mathfrak{J}_X$-torsors over X relative to $\mathcal{A}_X$. This is a relative algebraic stack over $\mathcal{A}_X$. It has a natural group stack structure because $\mathfrak{J}_X$ is commutative, and naturally acts on $\mathcal{M}_X$ relative to $\mathcal{A}_X$ induced by the canonical homomorphism $\chi_{\mathfrak{M}}^{*}\mathfrak{J} \to I$.

Proposition 6.2.1. *The relative Picard stack $p_X \colon \mathcal{P}_X \to \mathcal{A}_X$ of $\mathfrak{J}_X$-torsors is smooth.*

Proof. This is because the obstruction space of deforming $\mathfrak{J}_X$-torsors is $\mathrm{H}^2(X, \mathrm{Lie}(\mathfrak{J}_X)) = 0$ since X is a curve (see [Ngô10, Proposition 4.3.5] and [Chi19, Proposition 4.2.2]). ∎

We denote the pullback of $\mathfrak{J}_X$ (resp. $\mathcal{P}_X$) to $\mathcal{A}_{\underline{d},\underline{\lambda}}$ by $\mathfrak{J}_{\underline{d},\underline{\lambda}}$ (resp. $\mathcal{P}_{\underline{d},\underline{\lambda}}$) and that to $\mathcal{A}_{\mathcal{L}}$ by $\mathfrak{J}_{\mathcal{L}}$ (resp. $\mathcal{P}_{\mathcal{L}}$). Similarly, for any point $a \in \mathcal{A}_X$, we use $\mathcal{P}_a$ to denote the fiber over a.

6.2.2 Similar to the local situation, we can consider the open subset $\mathcal{M}_X^{\mathrm{reg}}$ (resp. $\mathcal{M}_X^{\circ}$) of $\mathcal{M}_X$ consisting of tuples $(\mathcal{L}, E, \phi)$ such that the image of the Higgs field ϕ is contained in $[\mathfrak{M}_{\mathcal{L}}^{\mathrm{reg}}/\mathrm{Ad}(G)]$ (resp. $[\mathfrak{M}_{\mathcal{L}}^{\circ}/\mathrm{Ad}(G)]$). According to Corollary 2.4.13, we have:

Proposition 6.2.3. *The action of $p_X\colon \mathcal{P}_X \to \mathcal{A}_X$ on $h_X^{\mathrm{reg}}\colon \mathcal{M}_X^{\mathrm{reg}} \to \mathcal{A}_X$ has trivial stabilizers.*

6.3 Cameral Curves

Definition 6.3.1. The *universal cameral curve* $\tilde{\pi}\colon \tilde{X} \to X \times \mathcal{A}_X$ is the pullback of cameral cover

$$[\pi]\colon [\mathfrak{T}_{\mathfrak{M}}/Z_{\mathfrak{M}}] \longrightarrow [\mathfrak{C}_{\mathfrak{M}}/Z_{\mathfrak{M}}]$$

to $X \times \mathcal{A}_X$ via the evaluation map.

Definition 6.3.2. The *universal discriminant divisor* $\mathfrak{D}_X$ is the pullback of $[\mathfrak{D}_{\mathfrak{M}}/Z_{\mathfrak{M}}]$ to $X \times \mathcal{A}_X$ via the evaluation map. If the numerical boundary divisor $\mathfrak{B}_{\mathfrak{M}}$ makes sense for the monoid, we define $\mathcal{B}_X$ also by pullback.

Fix $a \in \mathcal{A}_X(S)$, we have the *cameral curve* $\pi_a\colon \tilde{X}_a \to X \times S$ that is the fiber of $\tilde{\pi}\colon \tilde{X} \to X \times \mathcal{A}_X$ over a. We also have the *discriminant divisor* $\mathfrak{D}_a$ and the *numerical boundary divisor* $\mathfrak{B}_a$ by looking at the fibers of $\mathfrak{D}_X$ and $\mathfrak{B}_X$ over a, respectively. Note that despite the name, $\mathfrak{D}_a$ may not be a (proper) divisor.

Definition 6.3.3. Let *reduced locus* $\mathcal{A}_X^{\heartsuit} \subset \mathcal{A}_X$ be the open locus such that $\mathfrak{D}_a$ is either empty or an effective divisor.

Lemma 6.3.4. *Let $a \in \mathcal{A}_X^{\heartsuit}(\bar{k})$ and let b be its boundary divisor such that at each $\bar{v} \in X(\bar{k})$ it gives a dominant cocharacter $\lambda_{\bar{v}}$. Then*

$$\deg(\mathfrak{D}_a) = \sum_{\bar{v}\in X(\bar{k})} \langle 2\rho, \lambda_{\bar{v}}\rangle.$$

Proof. It suffices to prove for $\mathfrak{M} = \mathrm{Env}(G^{\mathrm{sc}})$. The extended discriminant function is a map $\mathfrak{C}_{\mathfrak{M}} \to \mathbb{A}^1$. It is $Z_{\mathfrak{M}}$-equivariant if we let $Z_{\mathfrak{M}}$ act on $\mathbb{A}^1$ by character 2ρ. Let $\mathcal{L}$ be the

$Z_{\mathfrak{M}}$-torsor under b, then we have induced map

$$\mathfrak{C}_{\mathfrak{M},\mathcal{L}} \longrightarrow \mathcal{O}(\mathfrak{D}_{\mathfrak{M},\mathcal{L}}) := \mathcal{O}\left(\sum_{\bar{v}\in X(\bar{k})} \langle 2\rho, \lambda_{\bar{v}}\rangle\right),$$

whose preimage of the zero section of $\mathcal{O}(\mathfrak{D}_{\mathfrak{M},\mathcal{L}})$ is the discriminant divisor. The point a is a map $\breve{X} \to \mathfrak{C}_{\mathfrak{M},\mathcal{L}}$, and so its composition with the map above is a section of $\mathcal{O}(\mathfrak{D}_{\mathfrak{M},\mathcal{L}})$ whose zero divisor is exactly $\mathfrak{D}_a$. ∎

Lemma 6.3.5. *For $a \in \mathcal{A}_X^{\heartsuit}(\bar{k})$, the curve $\tilde{X}_a$ is reduced.*

Proof. Since cameral cover is a Cohen-Macaulay morphism, $\tilde{X}_a$ is Cohen-Macaulay. Since X_a is generically reduced being a finite flat cover over X, it must be reduced. ∎

6.3.6 Using the Galois description of regular centralizer $\mathfrak{J}_{\mathfrak{M}}$, we can reach a similar description of $\mathfrak{J}_X$ using cameral cover $\tilde{\pi}$. Indeed, for any $a \in \mathcal{A}_X(\bar{k})$, we have monomorphism of sheaf of commutative groups

$$\mathfrak{J}_a \longrightarrow \mathfrak{J}_a^1 = \pi_{a,*}(\tilde{X}_a \times_{\breve{X}} T)^W$$

with a finite cokernel of finite support relative to a. Sometimes it is convenient to base-change so that G (and T, etc.) becomes split. After [Ngô10, § 4.5.2], let $\vartheta\colon X_\vartheta \to \breve{X}$ be a finite Galois étale cover with X_ϑ being connected and over which ϑ_G becomes a trivial $\mathrm{Out}(\mathbf{G})$-torsor. Let Θ_ϑ be the Galois group. For any $a \in \mathcal{A}_X(\bar{k})$ whose image in $\mathrm{Bun}_{Z_{\mathfrak{M}}}$ is $\mathcal{L}$, we have Cartesian diagram

$$\begin{array}{ccc} \tilde{X}_{\vartheta,a} & \longrightarrow & X_\vartheta \times_{\breve{X}} \mathfrak{T}_{\mathfrak{M},\mathcal{L}} \\ \downarrow{\scriptstyle \pi_{\vartheta,a}} & & \downarrow \\ \breve{X} & \xrightarrow{\ a\ } & \mathfrak{C}_{\mathfrak{M},\mathcal{L}} \end{array}.$$

Then we may describe $\mathfrak{J}^1_a$ as

$$\mathfrak{J}^1_a = \pi_{\vartheta,a,*}(\tilde{X}_{\vartheta,a} \times \mathbf{T})^{\mathbf{W} \rtimes \Theta_\vartheta}.$$

Similarly, we have a global Néron model $\mathfrak{J}^\flat_a$, and its Galois description

$$\mathfrak{J}^\flat_a = \pi^\flat_{a,*}(\tilde{X}^\flat_a \times_{\check{X}} T)^W = \pi^\flat_{\vartheta,a,*}(\tilde{X}^\flat_{\vartheta,a} \times \mathbf{T})^{\mathbf{W} \rtimes \Theta_\vartheta}, \tag{6.3.1}$$

where $\tilde{X}^\flat_a$ (resp. $\tilde{X}^\flat_{\vartheta,a}$) is the normalization of $\tilde{X}_a$ (resp. $\tilde{X}_{\vartheta,a}$).

6.3.7 Connected cameral curves Using $\tilde{X}_{\vartheta,a}$ we can prove a connectivity result using the same method as in [Ngô10, § 4.6]. First we record a theorem.

Theorem 6.3.8 ([Deb96, Théorème 1.4])**.** *Let M be an irreducible variety,*

$$m\colon M \to \mathbb{P} = \mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_r}$$

is a morphism from M into a product of several projective spaces. Let $H_i \subset \mathbb{P}^{n_i}$ be a fixed linear subspace. Suppose for any subset $I \subset \{1, \dots, r\}$, we have

$$\dim(p_I(m(M))) > \sum_{i \in I} \operatorname{codim}_{\mathbb{P}^{n_i}}(H_i),$$

where p_I is the natural projection with multi-index I. Suppose in addition there is an open subset $V \subset \mathbb{P}$ containing the product $H = H_1 \times \cdots \times H_r$, and $m^{-1}(V)$ is proper over V. Then $m^{-1}(H)$ is connected.

Definition 6.3.9. For any integer $N \geq 0$, a $Z_{\mathfrak{M}}$-torsor $\mathcal{L}$ is called *very (G, N)-ample* if

$$\deg \varpi(\mathcal{L}) > 2d_\varpi(g_X - 1) + 2 + N$$

for every connected component ϖ of fundamental weights of G^{sc}. If $N = 0$, we simply call $\mathcal{L}$ very G-ample. A point $a \in \mathcal{A}_X(\bar{k})$ or a boundary divisor $b \in \mathcal{B}_X(\bar{k})$ is very (G, N)-ample (resp. very G-ample) if its associated bundle $\mathcal{L}$ is. Let $\mathcal{A}_{\gg} \subset \mathcal{A}_X$ and $\mathcal{B}_{\gg} \subset \mathcal{B}_X$ be the respective loci of very G-ample points, and $\mathcal{A}_{\gg N}$, $\mathcal{B}_{\gg N}$ be the respective very (G, N)-ample loci.

Remark 6.3.10. It is clear that $\mathcal{B}_{\gg}$ is both open and closed in $\mathcal{B}_X$. In some situation, a weaker condition on ampleness can be used, i.e., one that requires $\deg \varpi(\mathcal{L}) > 2d_{\varpi}(g_X - 1)$ instead. We may call those points *G-ample* but since obtaining a sharp numerical bound is not important, we will stick to very G-ampleness to save notations. Finally, note that if $\vartheta \colon X_\vartheta \to X$ is a finite étale cover, then if $\mathcal{L}$ is G-ample or very G-ample, so is $\vartheta^*\mathcal{L}$.

Proposition 6.3.11. *If $a \in \mathcal{A}_X^{\heartsuit}(\bar{k})$ is very G-ample, then both $\tilde{X}_a$ and $\tilde{X}_{\vartheta,a}$ are reduced and connected.*

Proof. Reducedness is already proved in Lemma 6.3.5. For connectedness, it suffices to prove for $\tilde{X}_{\vartheta,a}$ hence we may replace $\check{X}$ with X_ϑ and assume that G is split. Let $b \in \mathcal{B}_X$ and $\mathcal{L} \in \mathrm{Bun}_{Z_{\mathfrak{M}}}$ be the image of a. Consider the pullback diagram

$$\begin{array}{ccc} \mathfrak{T}_b & \longrightarrow & \mathfrak{T}_{\mathfrak{M},\mathcal{L}} \\ \downarrow{\scriptstyle \pi_b} & & \downarrow \\ \mathfrak{C}_b & \longrightarrow & \mathfrak{C}_{\mathfrak{M},\mathcal{L}} \\ \downarrow & & \downarrow \\ \check{X} & \xrightarrow{b} & \mathfrak{A}_{\mathfrak{M},\mathcal{L}} \end{array}$$

such that a is a section $\check{X} \to \mathfrak{C}_b$. We claim that $\mathfrak{T}_b$ is an irreducible variety.

Indeed, because b lies generically inside the open part $\mathfrak{A}_{\mathfrak{M},\mathcal{L}}^{\times}$, there is an open dense subset $U \subset \check{X}$ such that the fibers of $\mathfrak{T}_b \to \check{X}$ are T^{sc}-torsors. This means that $\mathfrak{T}_b \times_{\check{X}} U$ is irreducible, hence so is its closure in $\mathfrak{T}_b$. The complement U' of the closure of $\mathfrak{T}_b \times_{\check{X}} U$ is open, hence its image is open in $\check{X}$ by flatness. We know U' must be empty, since otherwise its image in $\check{X}$ would have non-trivial intersection with U, which is impossible by the definition of U'. Therefore $\mathfrak{T}_b$ is irreducible. Similar to cameral curves, since $a \in \mathcal{A}_X^{\heartsuit}$, $\mathfrak{T}_b$ is Cohen-Macaulay and generically reduced, hence reduced. Thus $\mathfrak{T}_b$ is an irreducible variety.

We can now apply Theorem 6.3.8 where $M = \mathfrak{T}_b$. Since G is split, $\mathfrak{C}_b$ is a direct sum of line bundles $\varpi_i(\mathcal{L})$ for $1 \leq i \leq r$. Compactify $\varpi_i(\mathcal{L})$ into a projective line bundle $\overline{\varpi_i(\mathcal{L})}$, whose total space is a projective surface over $\bar{k}$. Let $Z_i = \overline{\varpi_i(\mathcal{L})} - \varpi_i(\mathcal{L})$ be the infinity divisor. Since by assumption $\deg \varpi_i(\mathcal{L}) > 2g_X$, the line bundle $\mathcal{O}(1)$ on $\overline{\varpi_i(\mathcal{L})}$ is very ample, hence induces a closed embedding

$$\overline{\varpi_i(\mathcal{L})} \longrightarrow \mathbb{P}^{n_i}.$$

Let $V \subset \mathbb{P} = \prod_{i=1}^r \mathbb{P}^{n_i}$ be the open subset

$$\mathbb{P} - \bigcup_{i=1}^r \left(Z_i \times \prod_{j \neq i} \overline{\varpi_j(\mathcal{L})} \right),$$

then $\prod_{i=1}^r \varpi_i(\mathcal{L})$ is a closed subscheme of V. Since $\mathfrak{C}_b \subset \prod_{i=1}^r \varpi_i(\mathcal{L})$ is a closed subscheme, it is a closed subscheme of V. Therefore $m \colon M \to V$ is proper since π_b is finite. The i-th component of a is a section $a_i \in \mathrm{H}^0(X, \varpi_i(\mathcal{L}))$. Since

$$(a_i, 1) \in \mathrm{H}^0(\mathbb{P}^{n_i}, \mathcal{O}(1)) = \mathrm{H}^0(X, \varpi_i(\mathcal{L})) \oplus \mathrm{H}^0(X, \mathcal{O}_X),$$

it determines a hyperplane $H_i \subset \mathbb{P}^{n_i}$ that has no intersection with Z_i. Thus $H = \prod_{i=1}^r H_i$ is contained in V.

It is easy to see that for every $I \subset \{1, \dots, r\}$, $p_I(m(M)) = p_I(\mathfrak{C}_b)$ has dimension $|I|+1$ (being the dimension of the direct sum of line bundles $\varpi_i(\mathcal{L})$ for $i \in I$), thus

$$\dim(p_I(m(M))) > \sum_{i \in I} \operatorname{codim}_{\mathbb{P}^{n_i}}(H_i) = |I|.$$

By Theorem 6.3.8 we are done as $\tilde{X}_a = m^{-1}(H)$. ∎

6.3.12 Smooth cameral curves We consider open subset $\mathcal{A}_X^{\diamondsuit}$ of $\mathcal{A}_X$ consisting of points a such that $a(X)$ intersects with the discriminant divisor transversally. We also consider another open subset $\mathcal{A}_X^{\sharp} \subset \mathcal{A}_X^{\diamondsuit}$ of points with the additional condition that $a(X)$ does not intersect with $\mathfrak{D}_X$ and $\mathfrak{C}_X - \mathfrak{C}_X^{\times}$ simultaneously. Clearly both $\mathcal{A}_X^{\diamondsuit}$ and $\mathcal{A}_X^{\sharp}$ is a subset of $\mathcal{A}_X^{\heartsuit}$.

Proposition 6.3.13. *Suppose $b \in \mathcal{B}_X(\bar{k})$ is very G-ample, then $\mathcal{A}_b^{\sharp}$ is non-empty. As a result, both $\mathcal{A}_b^{\diamondsuit}$ and $\mathcal{A}_b^{\heartsuit}$ are also non-empty.*

Proof. The proof is completely parallel to that of [Ngô10, Proposition 4.7.1]. First we show that the discriminant divisor $\mathfrak{D}_b \subset \mathfrak{C}_b$ is reduced. Indeed, as b lies generically (over X) inside $\mathfrak{A}_{\mathfrak{M},\mathcal{L}}^{\times}$, there is an open dense subset U of X over which $\mathfrak{D}_b$ is reduced. Similar to the proof of reducedness of cameral curves, we see that $\mathfrak{D}_b$ is Cohen-Macaulay and generically reduced, hence reduced.

Next we show that for any $x \in \check{X}(\bar{k})$ with ideal $\mathfrak{m}_x \subset \mathcal{O}_{\check{X}}$, the map

$$\mathrm{H}^0(X_{\bar{k}}, \mathfrak{C}_b) \longrightarrow \mathfrak{C}_b \otimes_{\mathcal{O}_{X_{\bar{k}}}} \mathcal{O}_{X_{\bar{k}}}/\mathfrak{m}_x^2 \tag{6.3.2}$$

is surjective. Indeed, let $\vartheta\colon X_\vartheta \to \check{X}$ be a connected finite Galois cover of $\check{X}$ with Galois group Θ. Then $\vartheta^*\mathfrak{C}_b$ is isomorphic to a direct sum of line bundles

$$\vartheta^*\mathfrak{C}_b = \bigoplus_{i=1}^{r} \varpi_i(\vartheta^*\mathcal{L}).$$

Since $\mathfrak{C}_b$ is a direct summand of $\vartheta_*\vartheta^*\mathfrak{C}_b$ being the Θ-fixed subbundle, it suffices to prove the surjectivity of map

$$\mathrm{H}^0(X_\vartheta, \vartheta^*\mathfrak{C}_b) \longrightarrow \vartheta^*\mathfrak{C}_b \otimes_{\mathcal{O}_{\check{X}}} \mathcal{O}_{\check{X}}/\mathfrak{m}_x^2,$$

and in turn it suffices to prove surjectivity after replacing $\vartheta^*\mathfrak{C}_b$ by each $\varpi_i(\vartheta^*\mathcal{L})$. The numerical assumption on $\varpi(\mathcal{L})$ ensures that $\varpi_i(\vartheta^*\mathcal{L})$ has degree greater than $2g_{X_\vartheta}$, hence the claim is implied by Riemann-Roch theorem.

Now since $\mathfrak{D}_b$ is reduced, it has an open dense smooth locus $\mathfrak{D}_b^\diamond = \mathfrak{D}_b - \mathfrak{D}_b^{\mathrm{sing}}$ such that $\mathfrak{D}_b^{\mathrm{sing}}$ has codimension 2 in $\mathfrak{C}_b$. Let $\mathfrak{D}_b^\sharp = \mathfrak{D}_b^\diamond - (\mathfrak{C}_b - \mathfrak{C}_b^\times)$, then $\mathfrak{D}_b - \mathfrak{D}_b^\sharp$ still has codimension at most 2 in $\mathfrak{C}_b$ since $\mathfrak{D}_b \cap (\mathfrak{C}_b - \mathfrak{C}_b^\times)$ is so. Let $Z_1 \subset \mathfrak{D}_b^\sharp \times \mathcal{A}_b$ consisting of pairs (c, a) such that $a(X)$ passes c, and has intersection multiplicity with $\mathfrak{D}_b$ at least 2 at c. Fix any c, then the subset of $a \in \mathcal{A}_b$ such that $(c, a) \in Z_1$ has codimension at least $2r$ in $\mathcal{A}_b$ by the surjectivity of (6.3.2). Hence

$$\dim Z_1 \le \dim \mathcal{A}_b - 2r + \dim \mathfrak{D}_b = \dim \mathcal{A}_b - r - 1 \le \dim \mathcal{A}_b - 1.$$

Thus the image of Z_1 in $\mathcal{A}_b$ has codimension at least 1. Similarly, consider the subset $Z_2 \subset (\mathfrak{D}_b - \mathfrak{D}_b^\sharp) \times \mathcal{A}_b$ of pairs (c, a) such that $a(X)$ passes c. Then we also have that

$$\dim Z_2 \le \dim \mathcal{A}_b - 1.$$

Therefore $\mathcal{A}_b - \overline{Z_1 \cup Z_2} \subset \mathcal{A}_b^\sharp$ is dense in $\mathcal{A}_b$ as desired. ∎

Proposition 6.3.14. *Let $a \in \mathcal{A}_X^\diamond(\bar{k})$, then $\tilde{X}_a$ is smooth.*

We make some preparations before attempting to prove Proposition 6.3.14. Recall that in the absolute setting, $\mathbf{D_M}$ is defined by the extended discriminant function

$$\mathrm{Disc}_+ = e^{(2\rho,0)} \prod_{\alpha\in\Phi} (1 - e^{(0,\alpha)}).$$

For each positive root α, we define a rational function

$$\mathrm{Disc}_\alpha = (1 - e^{(0,\alpha)})(1 - e^{(0,-\alpha)})$$

on $\bar{\mathbf{T}}_{\mathbf{M}}$, which is a regular function on $\mathbf{T}_{\mathbf{M}}$. Let $\mathbf{D}_\alpha$ be the scheme-theoretic closure of the vanishing locus of Disc_α in $\mathbf{T}_{\mathbf{M}}$.

Lemma 6.3.15. *Let $a \in \mathbf{C}_{\mathbf{M}}$ be a geometric point contained in the smooth locus $\mathbf{D}_{\mathbf{M}}^{\mathrm{sm}}$ of the discriminant divisor. Suppose in addition a is invertible, i.e., contained in $\mathbf{C}_{\mathbf{M}}^{\times}$ and $t \in \mathbf{T}_{\mathbf{M}}$ is a preimage of a. Then there is a unique root $\alpha \in \Phi_+$ such that* $\mathrm{Disc}_\alpha(t) = 0$ *and* $\mathrm{Disc}_\beta(t) \neq 0$ *for $\beta \neq \alpha$.*

Proof. Since a is invertible, the condition $a \in \mathbf{D}_{\mathbf{M}}^{\mathrm{sm}}$ is not affected by changing monoid. Thus it suffices to prove in the case $\mathbf{M} = \mathrm{Env}(\mathbf{G}^{\mathrm{sc}})$.

Let $a = \operatorname{Spec} k(a)$ and $\mathcal{O}(a) = k(a)[[\pi]]$ be the ring of formal series with coefficients in $k(a)$. Since $\mathbf{C}_{\mathbf{M}}$ is smooth, $a \in \mathbf{D}_{\mathbf{M}}^{\mathrm{sm}} \cap \mathbf{C}_{\mathbf{M}}^{\times}$ implies that we can find a map

$$\tilde{a}\colon \operatorname{Spec} \mathcal{O}(a) \longrightarrow \mathbf{C}_{\mathbf{M}}^{\times}$$

whose special point is a, and $\tilde{a}$ intersects with $\mathbf{D}_{\mathbf{M}}$ transversally. This means that the $\mathcal{O}(a)$-valuation of $\mathbf{D}_{\mathbf{M}}$ at $\tilde{a}$ is 1. By dimension formula of multiplicative affine Springer fibers, this forces the ramification index $c(\tilde{a}) = 1$ (observe that the local dimension formula still hold if we replace $\bar{k}$ by any algebraically closed field containing $\bar{k}$). This implies that $\tilde{a}$ lifts to a point $\tilde{t} \in \mathbf{T}_{\mathbf{M}}(k(a)[[\pi^{1/2}]])$ specializing to t.

The $\mathcal{O}(a)$-valuation of $1 - e^{(0,\alpha)}$ at $\tilde{t}$ is contained in $\mathbb{N}/2$ for all $\alpha \in \Phi$. Since the valuation of $1-e^{(0,\alpha)}$ equal to that of $1-e^{(0,-\alpha)}$ (as $\tilde{a}$ is invertible), we have our result. ∎

Lemma 6.3.16. *Suppose $a \in \mathbf{D}_{\mathbf{M}}^{\mathrm{sm}} \cap \mathbf{C}_{\mathbf{M}}^{\times}$ and $t \in \mathbf{T}_{\mathbf{M}}$ a lift of a. Let $\mathbf{W}_{\mathbf{L}_\alpha}$ be the Weyl group of the Levi subgroup $\mathbf{L}_\alpha$ generated by root α, where α is the unique positive root such that* $\mathrm{Disc}_\alpha(t) = 0$*. Then the natural map*

$$\mathbf{C}_\alpha \coloneqq \bar{\mathbf{T}}_{\mathbf{M}} /\!\!/ \mathbf{W}_{\mathbf{L}_\alpha} \longrightarrow \mathbf{C}_{\mathbf{M}}$$

is étale over a.

Proof. It suffices to prove that it is étale at the image of t in $\mathbf{C}_\alpha$. Since the derived subgroup of $\mathbf{G}_+$ is simply-connected, the same is true for $\mathbf{L}_\alpha$, hence $\mathbf{T_M} /\!/ \mathbf{W}_{\mathbf{L}_\alpha}$ is smooth (treating $\mathbf{G}_+$ as a very flat reductive monoid containing $\mathbf{L}_\alpha^{\mathrm{der}}$). Then similar to the differential calculation done at the end of Chapter 3, we see that the map $\mathbf{C}_\alpha \to \mathbf{C_M}$ is smooth at the image of t, because the determinant of the differential is given by a non-zero scaling of the product of Disc_β for $\beta \neq \alpha$. Done. ∎

Corollary 6.3.17. *Suppose $t \in \bar{\mathbf{T}}_\mathbf{M}$ is such that there exists a small enough étale (or formal) neighborhood $t \in U$ with non-empty intersection with at most one $\mathbf{D}_\alpha$, then the map*

$$\mathbf{C}_\alpha \longrightarrow \mathbf{C_M}$$

is étale at the image of t.

Proof. By Lemma 6.3.16, the map is étale on U outside of the intersection of numerical boundary divisor and discriminant divisor, which have codimension at least 2. Since the map given is finite, we are done since the branching locus has to have codimension 1. ∎

Proof of Proposition 6.3.14. The statement is local so we may look at the formal disc $\check{X}_{\bar{v}}$ at a point $\bar{v} \in \check{X}$ and the cameral cover over it. Let $\mathcal{O} = \bar{k}[[\pi]]$ and $F = \bar{k}((\pi))$. Since G is necessarily split over $\check{X}_{\bar{v}}$, we may assume without loss of generality that $G = \mathbf{G} \times \operatorname{Spec} \mathcal{O}$ and $\mathfrak{M} = \mathbf{M} \times \operatorname{Spec} \mathcal{O}$ where $\mathbf{M} = \mathrm{Env}(\mathbf{G}^{\mathrm{sc}})$. Suppose the image of a in the abelianization $\mathbf{A}$ is contained in $\pi^\lambda \mathbf{T}^{\mathrm{ad}}(\mathcal{O})$.

Suppose a intersects with $\mathbf{D_M}$ transversally. Let U be a small enough étale neighborhood of the special point of a. Then we can find a root α determined by Lemma 6.3.15, and by Corollary 6.3.17, the map $\mathbf{C}_\alpha \to \mathbf{C_M}$ is étale over U. Thus we may lift a to a point

$$a' \in \mathbf{C}_\alpha(\mathcal{O}),$$

and we only need to show that the preimage $\tilde{X}_{a'}$ of a' in $\bar{\mathbf{T}}_\mathbf{M}$ is smooth.

Let $\mathcal{O}_2 = \bar{k}[[\pi^{1/2}]]$, $F_2 = \bar{k}((\pi^{1/2}))$, and $\gamma \in \bar{\mathbf{T}}_{\mathbf{M}}(\mathcal{O}_2) \cap \mathbf{T}_{\mathbf{M}}(F_2)$ be a point lifting a'. Let ν_γ be the (dominant) Newton point of γ. Let $\gamma_0 \bar{\mathbf{T}}_{\mathbf{M}}(\bar{k})$ be the special point of γ, $\gamma_1 \in \mathbf{T}_{\mathbf{M}}(F_2)$ be the generic point. After $\mathbf{W}$-conjugation, we may assume $\gamma \in \pi^{(\lambda,\nu_\gamma)}\mathbf{T}_{\mathbf{M}}(\mathcal{O}_2)$ (and α may change to another root). The special point of γ is fixed by the reflection s_α corresponding to α, so $\alpha(\gamma) \in \mathcal{O}_2^\times$ and $\langle \alpha, \nu_\gamma \rangle = 0$. Using the formula

$$\begin{aligned} 1 = d_+(a) &= \langle 2\rho, \lambda - \nu_\gamma \rangle + \sum_{\langle \alpha, \nu_\gamma \rangle = 0} \mathrm{val}_F(1 - \alpha(\gamma)) \\ &= \langle 2\rho, -w_0(\lambda) - \nu_\gamma \rangle + \sum_{\langle \alpha, \nu_\gamma \rangle = 0} \mathrm{val}_F(1 - \alpha(\gamma)), \end{aligned}$$

we see that there are two possibilities:

(1) $\nu_\gamma = -w_0(\lambda)$, and for any β with $\langle \beta, \nu_\gamma \rangle = 0$, $1 - \beta(\gamma)$ has non-zero valuation $1/2$ if and only if $\beta = \alpha$, or

(2) $\nu_\gamma = -w_0(\lambda) - 1/2\check{\alpha}_i$ for some simple coroot $\check{\alpha}_i$, and $\mathrm{val}_F(1 - \beta(\gamma)) = 0$ for any root β with $\langle \beta, \nu_\gamma \rangle = 0$

The second possibility is easy to deal with: consider fundamental weight ϖ_i and let

$$\mu = (-w_0(\varpi_i), -\varpi_i) \in \bar{k}[\bar{\mathbf{T}}_{\mathbf{M}}].$$

Then $\langle \mu, (\lambda, \nu_\gamma) \rangle = 1/2$ and so we have a morphism

$$\mu \colon \bar{\mathbf{T}}_{\mathbf{M}} \longrightarrow \mathbb{A}^1$$

such that the composition with γ factors through a point $\operatorname{Spec} \mathcal{O}_2 \to \mathbb{A}^1$ sending the coordinate of $\mathbb{A}^1$ into $\pi^{1/2}\mathcal{O}_2^\times$. Since we have the factorization

$$\operatorname{Spec} \mathcal{O}_2 \longrightarrow \tilde{X}_{a'} \longrightarrow \mathbb{A}^1,$$

we see that the ring of regular functions of $\tilde{X}_{a'}$ is a subring of $\mathcal{O}_2$ that contains an element in $\pi^{1/2}\mathcal{O}_2^{\times}$, hence must be the whole ring $\mathcal{O}_2$, and thus $\tilde{X}_{a'}$ is smooth.

Now we deal with the first possibility. If $\mu \in \mathbb{X}(\mathbf{T_M}) \cap \bar{k}[\bar{\mathbf{T}}_\mathbf{M}]$ is any character such that $\langle \mu, \check{\alpha} \rangle = 1$, then the map

$$\bar{\mathbf{T}}_\mathbf{M} \longrightarrow \operatorname{Spec} \bar{k}[e^{\mu}, e^{\mu-\alpha}] \cong \mathbb{A}^2$$

is $\mathbf{W}_{\mathbf{L}_\alpha}$ equivariant, hence induces commutative diagram

$$\begin{array}{ccc} \bar{\mathbf{T}}_\mathbf{M} & \longrightarrow & \mathbb{A}^2 \\ \downarrow & & \downarrow \\ \mathbf{C}_\alpha & \longrightarrow & \mathbf{C}'' := \mathbb{A}^2 /\!\!/ \mathbf{W}_{\mathbf{L}_\alpha} \cong \mathbb{A}^2 \end{array}$$

Let a'' be the image of a' in $\mathbf{C}''(\mathcal{O})$, and $\tilde{X}_{a''}$ the corresponding SL_2-cameral cover. Then we have commutative diagram

$$\begin{array}{ccc} \tilde{X}_{a'} & \longrightarrow & \tilde{X}_{a''} \\ \downarrow & & \downarrow \\ a' & \longrightarrow & a'' \end{array} \tag{6.3.3}$$

If $\lambda \neq 0$, since $\check{\alpha}$ is perpendicular to $-w_0(\lambda) \neq 0$ which is dominant, it is not the highest coroot. By looking up the table of root systems, one can always find a fundamental weight ϖ such that $\langle \varpi, \check{\alpha} \rangle = 1$. Then the character $\mu = (-w_0(\varpi), -\varpi + \alpha) \in \bar{k}[\bar{\mathbf{T}}_\mathbf{M}]$ is such that

$$\langle \mu, (\lambda, \nu_\gamma) \rangle = 0 \text{ and } \langle \mu, \check{\alpha} \rangle = 1.$$

By direct computation, we see that $\tilde{X}_{a''}$ in (6.3.3) is isomorphic to $\operatorname{Spec} \mathcal{O}_2$, hence smooth. Then the map of 1-dimensional schemes $\tilde{X}_{a'} \to \tilde{X}_{a''}$ is generically an isomorphism and finite, thus must be an isomorphism. This shows that $\tilde{X}_{a'}$ is smooth. If $\lambda = \nu_\gamma = 0$, then find an arbitrary weight ϖ with $\langle \varpi, \check{\alpha} \rangle = 1$ and the dominant weight ϖ_+ in $\mathbf{W}$-orbit of

ϖ. Let $\mu = (\varpi_+, \varpi)$ and we are done by repeating the calculation above. ∎

Corollary 6.3.18. *Let $X_\vartheta \to \check{X}$ be a connected Galois étale covering making G split. Suppose $a \in \mathbf{A}_X^\diamond$ is very G-ample, then $\tilde{X}_{\vartheta,a}$ is smooth and irreducible.*

Proof. We know $\tilde{X}_a$ is smooth, hence so is $\tilde{X}_{\vartheta,a}$. It is also connected by Proposition 6.3.11, thus irreducible. ∎

Contrary to Lie algebra case (see [Ngô10, Lemme 4.7.3]), the converse to Proposition 6.3.14 is not true in general. For example, suppose $\mathbf{G}$ is of type B_3, and let

$$
\begin{aligned}
(\lambda, \nu_\gamma) &= (2\check{\varpi}_2 + 2\check{\varpi}_3 + \check{\alpha}_3, 2\check{\varpi}_2 + 2\check{\varpi}_3), \\
\mu &= (-w_0(\varpi_1), -\varpi_1 + \alpha_1) = (\varpi_1, -\varpi_1 + \alpha_1), \\
\alpha &= \alpha_1,
\end{aligned}
$$

where the labeling on the Dynkin diagram is the "usual one", i.e., the vertex labeled with 1 is not directly joint with the one labeled with 3, and α_3 is the short simple root. Then $\langle \mu, (\lambda, \nu_\gamma) \rangle = 0$, α is the only positive root perpendicular to ν_γ, and $\langle \mu, \check{\alpha} \rangle = 1$. Thus by carefully choosing an element $t \in \mathbf{T}_+(\mathcal{O}_2)$ with $\mathrm{val}_F(1 - \alpha(t)) = 1/2$, we can find $a \in \mathbf{C}_\mathbf{M}(\mathcal{O})$ such that $\gamma = \pi^{(\lambda,\nu_\gamma)} t$ lies over a, $d_+(a) = 3$, and $\tilde{X}_a$ is smooth because in this case a' as in the Proposition 6.3.14 exists and $\tilde{X}_{a''}$ hence $\tilde{X}_{a'}$ is smooth by direct computation. Since we do not use this in the remaining part of this paper, we leave the verification to the reader (c.f. Chapter 3 on how to find such t). Nevertheless, it is still easy to prove a partial converse as follows.

Lemma 6.3.19. *If $\tilde{X}_a$ is smooth for some $a \in \mathcal{A}_X^\heartsuit(\bar{k})$, then for any points $\bar{v} \in \check{X}$, $a(\bar{v})$ is contained in $\mathbf{D}_\alpha$ for at most one α.*

Proof. If $\tilde{X}_a$ is smooth, then the local monodromy group $\pi_a^\bullet(I_v)$ is cyclic. But if $a(\bar{v})$ is contained in two different $\mathbf{D}_\alpha$, then the local monodromy group would contain two different involutions. A contradiction. ∎

6.4 Global Néron Model and δ-Invariant

Similar to the local case, we have seen in (6.3.1) that we have for any $a \in \mathcal{A}_X^{\heartsuit}(\bar{k})$ a Néron model $\mathfrak{J}_a^{\flat}$ of $\mathfrak{J}_a$ described using normalization of cameral curves. Another way of describing the global Néron model is to use Beauville-Laszlo-type gluing theorem: let $U = \check{X} - \mathfrak{D}_a$, then for any closed point $\bar{v} \in \mathfrak{D}_a$, we have a local Néron model over the formal disc $\check{X}_{\bar{v}}$ of the torus $\mathfrak{J}_a|_{\check{X}_{\bar{v}}^{\bullet}}$ over the punctured disc. Gluing the local Néron models with the torus $\mathfrak{J}_a|_U$, we obtain a group scheme $\mathfrak{J}_a^{\flat}$ which is precisely the global Néron model.

Consider the Picard stack $\mathcal{P}_a^{\flat}$ of $\mathfrak{J}_a^{\flat}$-torsors. We have a natural homomorphism of Picard stacks $\mathcal{P}_a \to \mathcal{P}_a^{\flat}$. We will see that $\mathcal{P}_a^{\flat}$ is an abelian stack in the following sense:

Definition 6.4.1 ([Ngô10, Définition 4.7.6])**.** A $\bar{k}$-abelian stack is the quotient of a $\bar{k}$-abelian variety by the trivial action of a diagonalizable group.

Proposition 6.4.2. *(1) The homomorphism $\mathcal{P}_a(\bar{k}) \to \mathcal{P}_a^{\flat}(\bar{k})$ is essentially surjective.*

(2) The neutral component $(\mathcal{P}_a^{\flat})_0$ of $\mathcal{P}_a^{\flat}$ is an abelian stack.

(3) The kernel $\mathcal{R}_a$ of $\mathcal{P}_a \to \mathcal{P}_a^{\flat}$ is representable by the product of some affine algebraic groups of finite type $\mathcal{R}_{\bar{v}}(a)$ appearing in Lemma 4.4.4 for finitely many $\bar{v} \in \check{X}$.

Proof. For the first claim, consider short exact sequence

$$1 \longrightarrow \mathfrak{J}_a \longrightarrow \mathfrak{J}_a^{\flat} \longrightarrow \mathfrak{J}_a^{\flat}/\mathfrak{J}_a \longrightarrow 1,$$

and the induced cohomological exact sequence

$$\mathrm{H}^0(\check{X}, \mathfrak{J}_a^{\flat}/\mathfrak{J}_a) \longrightarrow \mathrm{H}^1(\check{X}, \mathfrak{J}_a) \longrightarrow \mathrm{H}^1(\check{X}, \mathfrak{J}_a^{\flat}) \longrightarrow \mathrm{H}^1(\check{X}, \mathfrak{J}_a^{\flat}/\mathfrak{J}_a) = 0,$$

where the last term vanishes because the sheaf $\mathfrak{J}_a^{\flat}/\mathfrak{J}_a$ is supported on finite subset $\check{X} - U$. The first claim then follows.

For the second claim, recall the Galois description (6.3.1) of $\mathfrak{J}_a^\flat$, and let $\tilde{\mathcal{P}}_a^\flat$ be the stack of torsors under group $\tilde{\mathfrak{J}}_a^\flat \coloneqq \pi_{\vartheta,a,*}^\flat(\tilde{X}_{\vartheta,a}^\flat \times \mathbf{T})$. We know that the neutral component of $\tilde{\mathcal{P}}_a^\flat$ is an abelian stack, because it is just the product of n-copies (n being the rank of G) of the usual Picard stack $\mathcal{P}\mathrm{ic}(\tilde{X}_{\vartheta,a}^\flat)$ of line bundles on curve $\tilde{X}_{\vartheta,a}^\flat$. The averaging homomorphism

$$\begin{aligned} \tilde{\mathfrak{J}}_a^\flat &\longrightarrow \mathfrak{J}_a^\flat \\ t &\longmapsto \prod_{w \in \mathbf{W} \rtimes \Theta_\vartheta} w(t) \end{aligned}$$

induces homomorphism $\tilde{\mathcal{P}}_a^\flat \to \mathcal{P}_a^\flat$ whose composition with the natural map $\mathcal{P}_a^\flat \to \tilde{\mathcal{P}}_a^\flat$ is an isogeny of $\mathcal{P}_a^\flat$ to itself, as long as $\mathrm{char}(k)$ does not divide the order of $\mathbf{W} \rtimes \Theta_\vartheta$. This proves the second claim.

The third claim follows from a Beauville-Laszlo-type gluing theorem. Indeed, by definition, the kernel $\mathcal{R}_a$ consists of pairs of a $\mathfrak{J}_a$-torsor together with a trivialization of its induced $\mathfrak{J}_a^\flat$-torsor. The local $\mathcal{R}_{\bar{v}}(a)$, on the other hand, consists of pairs of $\mathfrak{J}_a|_{\check{X}_{\bar{v}}}$-torsors together with a trivialization of its induced $\mathfrak{J}_a^\flat|_{\check{X}_{\bar{v}}}$-torsor, which also gives a trivialization of the said $\mathfrak{J}_a|_{\check{X}_{\bar{v}}}$-torsor over punctured disc $\check{X}_{\bar{v}}^\bullet$. Therefore by the formal gluing theorem, the map of $\bar{k}$-functors

$$\prod_{\bar{v} \in \check{X} - U} \mathcal{R}_{\bar{v}}(a) \longrightarrow \mathcal{R}_a \tag{6.4.1}$$

obtained from gluing with the trivial torsor over U is an isomorphism. Hence the claim. ∎

Definition 6.4.3. Given $a \in \mathcal{A}_X^\heartsuit(\bar{k})$, the δ*-invariant* associated with a is defined as

$$\delta_a = \dim \mathcal{R}_a.$$

In view of isomorphism (6.4.1), we have that

$$\delta_a = \sum_{\bar{v}\in\check{X}-U} \delta_{\bar{v}}(a).$$

Corollary 6.4.4. *For $a \in \mathcal{A}_X^{\heartsuit}(\bar{k})$, we have formula*

$$\delta_a = \dim \mathrm{H}^0(\check{X}, \mathfrak{t} \otimes_{\mathcal{O}_{\check{X}}} (\pi^{\flat}_{a,*}\mathcal{O}_{\tilde{X}^{\flat}_a}/\pi_{a,*}\mathcal{O}_{\tilde{X}_a}))^W$$

Proof. This is the result of the formula above and Lemma 4.4.2. ∎

6.4.5 Rigidification of Picard stack Fix a point $\infty \in X(\bar{k})$, and consider open set $\mathcal{A}_X^{\infty} \subset \mathcal{A}_{X,\bar{k}}^{\heartsuit}$ consisting of points a such that $a(\infty)$ is contained in $\mathfrak{C}_{\mathfrak{W}}^{\mathrm{rs}}$. If ∞ is defined over a field extension k'/k, then so is $\mathcal{A}_X^{\infty}$.

We may rigidify $\mathcal{P}_X$ over $\mathcal{A}_X^{\infty}$ as follows: let $\mathcal{P}_X^{\infty}$ be the classifying stack over $\mathcal{A}_X^{\infty}$ of $\mathfrak{J}_a$-torsors together with a trivialization at point ∞. Then by [Ngô10, Proposition 4.5.7], $\mathcal{P}_X^{\infty} \to \mathcal{A}_X^{\infty}$ is representable by a smooth group scheme locally of finite type over $\mathcal{A}_X^{\infty}$, and the forgetful morphism $\mathcal{P}_X^{\infty} \to \mathcal{P}_X$ induces a canonical isomorphism

$$[\mathcal{P}_X^{\infty}/\mathfrak{J}_{X,\infty}] \longrightarrow \mathcal{P}_X^{\heartsuit},$$

where $\mathfrak{J}_{X,\infty}$ is the group scheme over $\mathcal{A}_X^{\infty}$ being the fiber of $\mathfrak{J}_X$ at point ∞.

For $a \in \mathcal{A}_X^{\infty}(\bar{k})$, let $P_{a,0}$ be the neutral component of $\mathcal{P}_a^{\infty}$. Chevalley structure theorem implies that there is a canonical short exact sequence

$$1 \longrightarrow R_a \longrightarrow P_{a,0} \longrightarrow A_a \longrightarrow 1,$$

where R_a is a smooth and connected affine algebraic group over $\bar{k}$, and A_a is an abelian variety. Since R_a is smooth and connected, the map $R_a \to \mathcal{P}_a^{\flat}$ is trivial. So the map $P_{a,0} \to \mathcal{P}_a^{\flat}$ factors through A_a and is surjective. If $\mathcal{P}_a^{\flat}$ has finite automorphism groups (cf. § 6.6), then

$$\dim A_a = \dim \mathcal{P}_a^\flat = \dim \mathcal{P}_a - \delta_a.$$

In this situation, we call δ_a the *dimension of the affine part of* $\mathcal{P}_a$. Note that such notion is only well-defined if the automorphism group is finite, otherwise since the automorphism group here is also affine, it introduces ambiguity in such "dimension of affine part".

6.5 Component Group

In this section, we study the fiberwise component group $\pi_0(\mathcal{P}_X)$ defined as an étale sheaf on $\mathcal{A}_X$.

6.5.1 For a point $a \in \mathcal{A}_X^\infty(\bar{k})$, let $U \subset \check{X}$ be the maximal open subset over which the cameral cover is étale. In particular, $\infty \in U$. Recall we have a pointed version of the group G after we fixing the point ∞, given by a continuous homomorphism

$$\vartheta_G^\bullet \colon \pi_1(X, \infty) \longrightarrow \mathrm{Out}(\mathbf{G}).$$

If we also fix a point $\tilde{\infty} \in \tilde{X}_a(\bar{k})$ lying over ∞, we can lift $\vartheta_G^\bullet$ into a commutative diagram

$$\begin{array}{ccc} \pi_1(U, \infty) & \xrightarrow{\pi_{\tilde{a}}^\bullet} & \mathbf{W} \rtimes \mathrm{Out}(\mathbf{G}) \\ \downarrow & & \downarrow \\ \pi_1(\check{X}, \infty) & \xrightarrow{\vartheta_G^\bullet} & \mathrm{Out}(\mathbf{G}) \end{array}, \tag{6.5.1}$$

in which $\tilde{a} = (a, \tilde{\infty})$. Let $W_{\tilde{a}}$ be the image of $\pi_{\tilde{a}}^\bullet$ in $\mathbf{W} \rtimes \mathrm{Out}(\mathbf{G})$, and $I_{\tilde{a}}$ the image of the kernel of $\pi_1(U, \infty) \to \pi_1(\check{X}, \infty)$ under $\pi_{\tilde{a}}^\bullet$. By commutativity of the diagram we have that $I_{\tilde{a}} \subset \mathbf{W}$.

6.5.2 Recall that we may fix a connected Galois cover $X_\vartheta \to \check{X}$ with Galois group Θ_ϑ over which the $\mathrm{Out}(\mathbf{G})$-torsor ϑ_G becomes trivial. Choosing a point ∞_ϑ lying over ∞, we have the pointed variant $(X_\vartheta, \infty_\vartheta)$, and we may identify Θ_ϑ with the quotient of $\pi_1(\check{X}, \infty)$ by $\pi_1(X_\vartheta, \infty_\vartheta)$. By assumption of X_ϑ, the map $\vartheta_G^\bullet\colon \pi_1(\check{X}, \infty) \to \mathrm{Out}(\mathbf{G})$ factors through Θ_ϑ. The normalization $\tilde{X}^\flat_{\vartheta,a}$ of cameral curve $\tilde{X}_{\vartheta,a}$ maps to $\check{X}$, and it is an étale $\mathbf{W} \rtimes \Theta_\vartheta$-cover over U. By replacing $(\tilde{X}_a, \tilde{\infty})$ with $(\tilde{X}_{\vartheta,a}, \tilde{\infty}_\vartheta)$, we may lift $\vartheta_G^\bullet$ to

$$\pi^\bullet_{\tilde{a}_\vartheta}\colon \pi_1(U, \infty) \longrightarrow \mathbf{W} \rtimes \Theta,$$

where Θ is the image of $\vartheta_G^\bullet$ in $\mathrm{Out}(\mathbf{G})$. Thus we have inclusion

$$W_{\tilde{a}} \subset \mathbf{W} \rtimes \Theta. \tag{6.5.2}$$

Let $\mathfrak{J}_a^0 \subset \mathfrak{J}_a$ be the open subgroup such that over any point $x \in \check{X}$ the fiber of $\mathfrak{J}_a^0$ is the neutral component of the fiber of $\mathfrak{J}_a$. We have an induced homomorphism of Picard stacks $\mathcal{P}'_a \to \mathcal{P}_a$.

Lemma 6.5.3 ([Ngô10, Lemme 4.10.2])**.** *The homomorphism $\mathcal{P}'_a \to \mathcal{P}_a$ is surjective with finite kernel. The same is true for induced homomorphism $\pi_0(\mathcal{P}'_a) \to \pi_0(\mathcal{P}_a)$.*

Proof. Since a is generically (over $\check{X}$) regular semisimple, the sheaf $\pi_0(\mathfrak{J}_a) = \mathfrak{J}_a/\mathfrak{J}_a^0$ has finite support on $\check{X}$. The short exact sequence

$$1 \longrightarrow \mathfrak{J}_a^0 \longrightarrow \mathfrak{J}_a \longrightarrow \pi_0(\mathfrak{J}_a) \longrightarrow 1$$

induces cohomological long exact sequence

$$\mathrm{H}^0(\check{X}, \pi_0(\mathfrak{J}_a)) \longrightarrow \mathrm{H}^1(\check{X}, \mathfrak{J}_a^0) \longrightarrow \mathrm{H}^1(\check{X}, \mathfrak{J}_a) \longrightarrow \mathrm{H}^1(\check{X}, \pi_0(\mathfrak{J}_a)) = 0. \tag{6.5.3}$$

Therefore $\mathcal{P}'_a \to \mathcal{P}_a$ is surjective with kernel being the image of $\mathrm{H}^0(\check{X}, \pi_0(\mathfrak{J}_a))$, which is necessarily finite. The induced map $\pi_0(\mathcal{P}'_a) \to \pi_0(\mathcal{P}_a)$ is then also surjective. ∎

6.5.4 Once we fix a point $\infty \in X(\bar{k})$, there is a nice description of the Cartier dual of the diagonalizable groups $\pi_0(\mathcal{P}'_a)$ and $\pi_0(\mathcal{P}_a)$. Let

$$\pi_0(\mathcal{P}_a)^* = \mathrm{Spec}(\overline{\mathbb{Q}}_\ell[\pi_0(\mathcal{P}_a)]),$$
$$\pi_0(\mathcal{P}'_a)^* = \mathrm{Spec}(\overline{\mathbb{Q}}_\ell[\pi_0(\mathcal{P}'_a)]),$$

then the surjectivity of $\pi_0(\mathcal{P}'_a) \to \pi_0(\mathcal{P}_a)$ induces closed embedding $\pi_0(\mathcal{P}_a)^* \subset \pi_0(\mathcal{P}'_a)^*$.

Proposition 6.5.5. *For any $\tilde{a} = (a, \tilde{\infty})$, we have canonical isomorphisms of diagonalizable groups*

$$\pi_0(\mathcal{P}'_a)^* \simeq \check{\mathbf{T}}^{W_{\tilde{a}}},$$
$$\pi_0(\mathcal{P}_a)^* \simeq \check{\mathbf{T}}(I_{\tilde{a}}, W_{\tilde{a}}),$$

where $\check{\mathbf{T}}(I_{\tilde{a}}, W_{\tilde{a}}) \subset \check{\mathbf{T}}^{W_{\tilde{a}}}$ is the subgroup of elements κ such that $I_{\tilde{a}} \subset \mathbf{W_H}$ where $\mathbf{W_H}$ is the Weyl group of the neutral component $\check{\mathbf{H}}$ of the centralizer of κ in $\check{\mathbf{G}}$.

Proof. First, similar to the local case in Lemma 4.4.12, we have a canonical isomorphism

$$\check{\mathbb{X}}(\mathbf{T})_{W_{\tilde{a}}} \longrightarrow \pi_0(\mathcal{P}'_a),$$

see [Ngô06, Lemme 6.6 and Corollaire 6.7] and [Kot85, Lemma 2.2]. Therefore the first isomorphism follows by taking Cartier dual.

Let $U \subset \check{X}$ be the regular semisimple locus of a. Using (6.5.3), we obtain exact sequence

$$\mathrm{H}^0(\check{X}, \pi_0(\mathfrak{J}_a)) \longrightarrow \pi_0(\mathcal{P}'_a) \longrightarrow \pi_0(\mathcal{P}_a) \longrightarrow 0.$$

We may decompose $\mathrm{H}^0(\check{X}, \pi_0(\mathfrak{J}_a))$ as

$$\mathrm{H}^0(\check{X}, \pi_0(\mathfrak{J}_a)) = \bigoplus_{\bar{v}\in\check{X}-U} \pi_0(\mathfrak{J}_{a,\bar{v}}),$$

where $\mathfrak{J}_{a,\bar{v}}$ is the fiber of $\mathfrak{J}_a$ at closed point $\bar{v}$. Recall we also have local Picard groups $\mathcal{P}_{\bar{v}}(a_{\bar{v}})$ (resp. $\mathcal{P}_{\bar{v}}^0(a_{\bar{v}})$) associated with $\mathfrak{J}_{a_{\bar{v}}}$ (resp. $\mathfrak{J}_{a_{\bar{v}}}^0$) induced by a. Note here $\mathfrak{J}_{a_{\bar{v}}}$ is used to denote the restriction of $\mathfrak{J}_a$ to the formal disc $\check{X}_{\bar{v}}$ (contrary to $\mathfrak{J}_{a,\bar{v}}$ which is the special fiber). We then have exact sequence

$$\pi_0(\mathfrak{J}_{a,\bar{v}}) \longrightarrow \pi_0(\mathcal{P}_{\bar{v}}^0(a_{\bar{v}})) \longrightarrow \pi_0(\mathcal{P}_{\bar{v}}(a_{\bar{v}})) \longrightarrow 0,$$

compatible with the forgetful maps $\mathcal{P}_{\bar{v}}(a_{\bar{v}}) \to \mathcal{P}_a$ and $\mathcal{P}_{\bar{v}}^0(a_{\bar{v}}) \to \mathcal{P}_a'$. Taking Cartier dual, we have commutative diagram

$$\begin{array}{ccccccc}
0 \longrightarrow & \pi_0(\mathcal{P}_a)^* & \longrightarrow & \pi_0(\mathcal{P}_a^0)^* & \longrightarrow & \bigoplus_{\bar{v}\in\check{X}-U} \pi_0(\mathfrak{J}_{a,\bar{v}})^* \\
& \downarrow & & \downarrow & & \| \\
0 \to & \bigoplus_{\bar{v}\in\check{X}-U} \pi_0(\mathcal{P}_{\bar{v}}(a_{\bar{v}}))^* & \to & \bigoplus_{\bar{v}\in\check{X}-U} \pi_0(\mathcal{P}_{\bar{v}}^0(a_{\bar{v}}))^* & \to & \bigoplus_{\bar{v}\in\check{X}-U} \pi_0(\mathfrak{J}_{a,\bar{v}})^*
\end{array}$$

Thus $\pi_0(\mathcal{P}_a)^*$ is such subgroup of $\pi_0(\mathcal{P}_a')^*$ that its local restriction is contained in $\pi_0(\mathcal{P}_{\bar{v}}(a_{\bar{v}}))$, which is described in Proposition 4.4.10. Since the group $I_{\tilde{a}}$ is generated by the inertia groups at each $\bar{v}$, such requirement is the same as that $I_{\tilde{a}}$ is contained in $\mathbf{W_H}$ as desired since each inertia group is so. ∎

Corollary 6.5.6. $\pi_0(\mathcal{P}_a)$ *is finite if and only if* $\check{\mathrm{T}}^{W_{\tilde{a}}}$ *(and equivalently* $\mathrm{T}^{W_{\tilde{a}}}$*) is.*

Definition 6.5.7. The *anisotropic locus* $\mathcal{A}_X^{\natural} \subset \mathcal{A}_X^{\heartsuit}$ is the subset consisting of a such that $\pi_0(\mathcal{P}_a)$ is finite.

It is not immediately obvious that $\mathcal{A}_X^{\natural}$ is an open subset of $\mathcal{A}_X^{\heartsuit}$, or it is non-empty, but we shall see in § 7.2.7 that both are true under mild assumptions.

6.6 Automorphism Group

Let $(\mathcal{L}, E, \phi) \in \mathcal{M}_X^{\heartsuit}(\bar{k})$ with image $a \in \mathcal{A}_X^{\heartsuit}$, $b \in \mathcal{B}_X$, and $\mathcal{L} \in \mathrm{Bun}_{Z_{\mathfrak{M}}}$. Obviously, we have maps of automorphism groups

$$\underline{\mathrm{Aut}}(\mathcal{L}, E, \phi) \longrightarrow \underline{\mathrm{Aut}}(a) \longrightarrow \underline{\mathrm{Aut}}(b) \longrightarrow \underline{\mathrm{Aut}}_{Z_{\mathfrak{M}}}(\mathcal{L}).$$

We already know that $\underline{\mathrm{Aut}}(b)$ is finite and described using the kernel of map $Z_{\mathfrak{M}} \to \mathfrak{A}_{\mathfrak{M}}^{\times}$. So the image of $\underline{\mathrm{Aut}}(\mathcal{L}, E, \phi)$ in $\underline{\mathrm{Aut}}_{Z_{\mathfrak{M}}}(\mathcal{L})$ is finite, and we only need to describe

$$\underline{\mathrm{Aut}}(E, \phi) = \ker\left[\underline{\mathrm{Aut}}(\mathcal{L}, E, \phi) \to \underline{\mathrm{Aut}}_{Z_{\mathfrak{M}}}(\mathcal{L})\right],$$

in other words, when the automorphism on $\mathcal{L}$ is identity.

Recall the universal centralizer $I_{\mathfrak{M}} \to \mathfrak{M}$, whose fiber over $x \in \mathfrak{M}$ is the centralizer $G_x \subset G$ of x in G. Since $I_{\mathfrak{M}} \to \mathfrak{M}$ is $\mathrm{Ad}(G) \times Z_{\mathfrak{M}}$-equivariant, it descends to $[\mathfrak{M}/G \times Z_{\mathfrak{M}}]$. The pair (E, ϕ) is a map $\check{X} \to [\mathfrak{M}_{\mathcal{L}}/G]$, and let $I_{(E,\phi)}$ be the pullback of $I_{\mathfrak{M}}$ along this map. It is easy to see that the sheaf of automorphisms of (E, ϕ) is representable by $I_{(E,\phi)}$. The group $I_{(E,\phi)}$ is not flat in general, but according to [BLR90], there exists a unique group scheme $I_{(E,\phi)}^{\mathrm{sm}}$ smooth over $\check{X}$ such that for any $\check{X}$-scheme S smooth over $\check{X}$, we have

$$\mathrm{Hom}_{\check{X}}(S, I_{(E,\phi)}) = \mathrm{Hom}_{\check{X}}(S, I_{(E,\phi)}^{\mathrm{sm}}).$$

The tautological map $I_{(E,\phi)}^{\mathrm{sm}} \to I_{(E,\phi)}$ is an isomorphism over open subset $U = \check{X} - \mathfrak{D}_a$. Since $\mathfrak{J}_a$ is smooth, the canonical map $\mathfrak{J}_a \to I_{(E,\phi)}$ induces canonical map $\mathfrak{J}_a \to I_{(E,\phi)}^{\mathrm{sm}}$. We also have map

$$I_{(E,\phi)}^{\mathrm{sm}} \longrightarrow \mathfrak{J}_a^{\flat}$$

by universal property of Néron models. Both $\mathfrak{J}_a \to I_{(E,\phi)}^{\mathrm{sm}}$ and $I_{(E,\phi)}^{\mathrm{sm}} \longrightarrow \mathfrak{J}_a^{\flat}$ are isomorphisms over U. At any point $\bar{v} \in \check{X} - U$, we have inclusions

$$\mathfrak{J}_a(\check{\mathcal{O}}_{\bar{v}}) \subset I^{\mathrm{sm}}_{(E,\phi)}(\check{\mathcal{O}}_{\bar{v}}) \subset \mathfrak{J}^{\flat}_a(\check{\mathcal{O}}_{\bar{v}}).$$

Using Beauville-Laszlo-type formal gluing theorem, we see that for any étale cover $S \to \check{X}$, the maps of sets (in the category of $\check{X}$-schemes)

$$\mathfrak{J}_a(S) \longrightarrow I^{\mathrm{sm}}_{(E,\phi)}(S) \longrightarrow \mathfrak{J}^{\flat}_a(S)$$

are both injective. This implies that we have inclusions

$$\mathrm{H}^0(\check{X}, \mathfrak{J}_a) \subset \mathrm{Aut}(E,\phi) = \mathrm{H}^0(\check{X}, I_{(E,\phi)}) = \mathrm{H}^0(\check{X}, I^{\mathrm{sm}}_{(E,\phi)}) \subset \mathrm{H}^0(\check{X}, \mathfrak{J}^{\flat}_a).$$

Using the Galois description of Néron model (6.3.1), we have that $\mathrm{H}^0(\check{X}, \mathfrak{J}^{\flat}_a) \subset \mathbf{T}^{W_{\tilde{a}}}$ after fixing $\tilde{a} = (a, \tilde{\infty})$ over a: indeed, let Θ be the image of $\vartheta^{\bullet}_G$ in $\mathrm{Out}(\mathbf{G})$, then we have $W_{\tilde{a}} \subset \mathbf{W} \rtimes \Theta$ by (6.5.2). Thus we proved the analogue to [Ngô10, Corollaire 4.11.3]:

Proposition 6.6.1. *Fix $\tilde{a} = (a, \tilde{\infty})$ over $a \in \mathcal{A}^{\heartsuit}_X(\bar{k})$, and any $(\mathcal{L}, E, \phi) \in \mathcal{M}^{\heartsuit}_X(\bar{k})$ lying over a, the automorphism group $\mathrm{Aut}(E,\phi)$ (after fixing $\mathcal{L}$) can be canonically identified with a subgroup of $\mathbf{T}^{W_{\tilde{a}}}$.*

If $\mathrm{char}(k)$ does not divide the order of $\mathbf{W} \rtimes \Theta$, then $\mathbf{T}^{W_{\tilde{a}}}$ is unramified if it is finite. Therefore, we have:

Corollary 6.6.2. *Assuming $\mathrm{char}(k)$ does not divide the order of $\mathbf{W} \rtimes \Theta$, then $\mathcal{M}^{\natural}_a$ and $\mathcal{P}^{\natural}_a$ are Deligne-Mumford stacks.*

Proof. Over $a \in \mathcal{A}^{\natural}_X(\bar{k})$, $\check{\mathbf{T}}^{W_{\tilde{a}}}$, hence also $\mathbf{T}^{W_{\tilde{a}}}$, is finite. This implies that $\mathrm{Aut}(E,\phi)$ is finite for $(\mathcal{L}, E, \phi)$ lying over a. Since the image of $\mathrm{Aut}(\mathcal{L}, E, \phi)$ in $\mathrm{Aut}_{Z_{\mathfrak{M}}}(\mathcal{L})$ is finite, the group $\mathrm{Aut}(\mathcal{L}, E, \phi)$ is itself finite and the claim for $\mathcal{M}^{\natural}_a$ follows. The claim for $\mathcal{P}^{\natural}_a$ also follows since the automorphism groups (after fixing $\mathcal{L}$) are just $\mathrm{H}^0(\check{X}, \mathfrak{J}_a)$ which is a subgroup of $\mathbf{T}^{W_{\tilde{a}}}$. ∎

In fact, for $\mathcal{P}_X^{\heartsuit}$ we have a stronger result.

Proposition 6.6.3. *Assuming* $\mathrm{char}(k)$ *does not divide the order of* $\mathbf{W} \rtimes \Theta$*, then for any* $a \in \mathcal{A}_{\gg}^{\infty}(\bar{k})$*, we have*

$$\mathrm{H}^0(\check{X}, \mathfrak{J}_a^1) = \mathbf{T}^{\mathbf{W} \rtimes \Theta},$$
$$\mathrm{H}^0(\check{X}, \mathfrak{J}_a) = \mathbf{Z}_{\mathbf{G}}^{\Theta}.$$

In particular, if Z_G *does not contain a split torus over* $\check{X}$ *then* $\mathcal{P}_a$ *is a Deligne-Mumford stack.*

Proof. The curve $\tilde{X}_{\vartheta,a}$ is connected by Proposition 6.3.11. Then we have $\mathrm{H}^0(\check{X}, \mathfrak{J}_a^1) = \mathbf{T}^{\mathbf{W} \rtimes \Theta}$, and $\mathrm{H}^0(\check{X}, \mathfrak{J}_a)$ is a subgroup therein. If $\mathrm{char}(k)$ does not divide the order of $\mathbf{W} \rtimes \Theta$ and Z_G does not contain a split torus, they are both étale $\bar{k}$-groups, so $\mathcal{P}_a$ in this case is a Deligne-Mumford stack.

The description of $\mathrm{H}^0(\check{X}, \mathfrak{J}_a)$ is proved using the identification $\mathfrak{J}_a = \mathfrak{J}_a'$ in Proposition 2.4.12. Since $a(\check{X})$ intersects with every irreducible component of discriminant divisor $\mathfrak{D}_{\mathfrak{M}}$, the definition of $\mathfrak{J}_a'$ implies that it is the subgroup of $\mathbf{T}^{\mathbf{W} \rtimes \Theta}$ with elements lying in the kernel of every root, which is exactly $\mathbf{Z}_{\mathbf{G}}^{\Theta}$. ∎

6.7 Tate Module

Suppose the center of G does not contain a split torus over $\check{X}$. Then over very G-ample locus $\mathcal{B}_{\gg}$, $\mathcal{P}_{\gg}^{\heartsuit}$ is a Deligne-Mumford stack. In this section, we fix a connected component $\mathcal{U}$ of $\mathcal{A}_{\gg}^{\heartsuit}$.

The Picard stack $\mathcal{P}_{\gg} = \mathcal{P}_X|_{\mathcal{U}}$ is smooth over $\mathcal{U}$, and let $\mathcal{P}_{\gg,0}$ be the open substack of fiberwise neutral component. Let $g\colon \mathcal{P}_{\gg,0} \to \mathcal{U}$ be the natural map. Let d be the relative dimension of g. Consider the sheaf of Tate modules

$$\mathrm{T}_{\overline{\mathbb{Q}}_\ell}(\mathcal{P}_{\gg,0}) = \mathrm{H}^{2d-1}(g_!\overline{\mathbb{Q}}_\ell).$$

Over open subset $\mathcal{U}^\infty = \mathcal{A}_X^\infty \cap \mathcal{U}$, we may rigidify $\mathcal{P}_{\gg,0}$ as the quotient of smooth relative group schemes P_0 by P_{-1}. Thus we have short exact sequence

$$1 \longrightarrow P_{-1} \longrightarrow P_0 \longrightarrow \mathcal{P}_{\gg,0} \longrightarrow 1,$$

and the induced short exact sequence of Tate modules (over $\overline{\mathbb{Q}}_\ell$)

$$1 \longrightarrow \mathrm{T}_{\overline{\mathbb{Q}}_\ell}(P_{-1}) \longrightarrow \mathrm{T}_{\overline{\mathbb{Q}}_\ell}(P_0) \longrightarrow \mathrm{T}_{\overline{\mathbb{Q}}_\ell}(\mathcal{P}_{\gg,0}) \longrightarrow 1.$$

For any $a \in \mathcal{U}^\infty$, the Chevalley exact sequence

$$1 \longrightarrow R_a \longrightarrow \mathcal{P}_a^\infty \longrightarrow A_a \longrightarrow 1$$

also induces an exact sequence of Tate modules

$$1 \longrightarrow \mathrm{T}_{\overline{\mathbb{Q}}_\ell}(R_a) \longrightarrow \mathrm{T}_{\overline{\mathbb{Q}}_\ell}(P_0) \longrightarrow \mathrm{T}_{\overline{\mathbb{Q}}_\ell}(A_a) \longrightarrow 1.$$

Since P_{-1} is affine, the morphism $\mathrm{T}_{\overline{\mathbb{Q}}_\ell}(P_0) \to \mathrm{T}_{\overline{\mathbb{Q}}_\ell}(A_a)$ factors through $\mathrm{T}_{\overline{\mathbb{Q}}_\ell}(\mathcal{P}_{a,0})$ (since $\mathrm{T}_{\overline{\mathbb{Q}}_\ell}(\mathfrak{J}_a^\infty)$ and $\mathrm{T}_{\overline{\mathbb{Q}}_\ell}(A_a)$ have incompatible Frobenius weights over any sufficiently large extension k'/k in $\bar{k}$). Therefore we have an exact sequence

$$1 \longrightarrow \mathrm{T}_{\overline{\mathbb{Q}}_\ell}(R_a)/\mathrm{T}_{\overline{\mathbb{Q}}_\ell}(\mathfrak{J}_a^\infty) \longrightarrow \mathrm{T}_{\overline{\mathbb{Q}}_\ell}(\mathcal{P}_{a,0}) \longrightarrow \mathrm{T}_{\overline{\mathbb{Q}}_\ell}(A_a) \longrightarrow 1. \tag{6.7.1}$$

It does not depend on the choice of rigidification, as if $\mathcal{P}_\gg = [P_0'/P_{-1}']$ is another rigidification, one may form a third rigidification $P_0'' = P_0 \times_{\mathcal{P}_\gg} P_0'$ over the two and identify exact sequences (6.7.1) in all three cases. Since we only consider $\overline{\mathbb{Q}}_\ell$-coefficients, an isogeny of smooth commutative Deligne-Mumford group stacks induces isomorphism of corresponding Tate modules. Thus we have canonical isomorphism

$$\mathrm{T}_{\overline{\mathbb{Q}}_\ell}(R_a)/\,\mathrm{T}_{\overline{\mathbb{Q}}_\ell}(P_{-1}) \simeq \mathrm{T}_{\overline{\mathbb{Q}}_\ell}(\mathcal{R}_a).$$

Thus we may call $\mathrm{T}_{\overline{\mathbb{Q}}_\ell}(\mathcal{R}_a)$ (resp. $\mathrm{T}_{\overline{\mathbb{Q}}_\ell}(A_a)$) the *affine part* (resp. *abelian part*) of $\mathrm{T}_{\overline{\mathbb{Q}}_\ell}(\mathcal{P}_a)$.

Proposition 6.7.1. *Over $\mathcal{U}$, there exists an alternating bilinear form*

$$\psi\colon\ \mathrm{T}_{\overline{\mathbb{Q}}_\ell}(\mathcal{P}_\gg) \times \mathrm{T}_{\overline{\mathbb{Q}}_\ell}(\mathcal{P}_\gg) \longrightarrow \overline{\mathbb{Q}}_\ell(-1)$$

such that at any $a \in \mathcal{U}$, the fiber ψ_a is identically zero on the affine part $\mathrm{T}_{\overline{\mathbb{Q}}_\ell}(\mathcal{R}_a)$, and non-degenerate on the abelian part $\mathrm{T}_{\overline{\mathbb{Q}}_\ell}(A_a)$.

Proof. The proof is identical to [Ngô10, § 4.12]. The two key ingredients are one, the homomorphism in the Galois description of $\mathfrak{J}_X$

$$\mathfrak{J}_X \longrightarrow \pi_{\vartheta,*}(\tilde{X}_\vartheta \times \mathbf{T}),$$

and two, the Weil pairing theory, which is a general theory for any flat and proper family of reduced and connected curves. It is applied to the family of curves $\tilde{X}_\vartheta$, which we have shown to have reduced and connected fibers over $\mathcal{U}$ in Proposition 6.3.11. ∎

6.8 Dimensions

Let $\mathcal{L} \in \mathrm{Bun}_{Z_{\mathfrak{M}}}(\bar{k})$ be a very G-ample $Z_{\mathfrak{M}}$-torsor. As before, let $\vartheta\colon X_\vartheta \to \check{X}$ be a connected finite Galois étale cover of $\check{X}$ with Galois group Θ and making G split. We also assume that $\mathrm{char}(k)$ does not divide the order d_ϑ of Θ. Then $\mathfrak{C}_b$ is a direct summand of $\vartheta^*\mathfrak{C}_b$ being the Θ-invariant subspace. We know $g_{X_\vartheta} - 1 = d_\vartheta(g_X - 1)$. So if $\mathcal{L}$ is very G-ample, then for any fundamental weight ϖ_i we have

$$\deg \varpi_i(\vartheta^*\mathcal{L}) > 2g_{X_\vartheta} - 2.$$

So by Riemann-Roch theorem, we have for any $b \in \mathcal{B}_X(\bar{k})$ lying over $\mathcal{L}$,

$$\dim_{\bar{k}} \mathrm{H}^0(X_\vartheta, \vartheta^*\mathfrak{C}_b) = \sum_{i=1}^{r} \deg \varpi_i(\vartheta^*\mathcal{L}) - r(g_{X_\vartheta} - 1),$$

and the first cohomology term vanishes. This means that $\mathfrak{C}_b$ has no first cohomology term either. Taking Θ-invariant, and note that $\mathfrak{C}_b$ is the Θ-trivial isotypic subbundle in $\vartheta_*\vartheta^*\mathfrak{C}_b$ which as a Θ-vector bundle is r-copies of regular ones (i.e., the one whose fibers are regular Θ-representations). Therefore if the boundary divisor of b can be written as $\sum_{\bar{v}\in X(\bar{k})} \lambda_{\bar{v}} \cdot \bar{v}$, we have

$$\dim \mathcal{A}_b = \sum_{\bar{v}\in X(\bar{k})} \langle \rho, \lambda_{\bar{v}} \rangle - r(g_X - 1). \tag{6.8.1}$$

Note that due to $\mathrm{Out}(\mathbf{G})$-twisting the sum $\sum_{\bar{v}} \lambda_{\bar{v}}$ does not make sense, but its pairing with ρ does since ρ is fixed by $\mathrm{Out}(\mathbf{G})$. For convenience, we denote this pairing by $\langle \rho, \lambda_b \rangle$.

In case $\mathcal{L}$ is not very G-ample, we still have estimate by Riemann-Roch theorem:

$$\dim \mathcal{A}_b \leq \langle \rho, \lambda_b \rangle + r, \tag{6.8.2}$$

where the equality is reached only if $\mathcal{L}$ is trivial or $g_X = 0$, and if neither is true we may improve it to

$$\dim \mathcal{A}_b \leq \langle \rho, \lambda_b \rangle.$$

6.8.1 Suppose $\mathfrak{M} \in \mathcal{FM}_0(G^{\mathrm{sc}})$ such that $\vartheta^*\mathfrak{M} \cong \mathbf{M} \times X_\vartheta$ for some $\mathbf{M} \in \mathcal{FM}_0(\mathbf{G}^{\mathrm{sc}})$. Then at each geometric point $\bar{v} \in X(\bar{k})$ we may identify $\mathfrak{M}$ with $\mathbf{M}$. If $\mathbf{A}_\mathbf{M} \cong \mathbb{A}^m$ is a standard monoid with coordinates e^{θ_i} $(1 \leq i \leq m)$, and suppose at each $\bar{v} \in X(\bar{k})$, we have

$$\lambda_{\bar{v}} = \sum_{i=1}^{m} c_{\bar{v},i}\theta_i,$$

then the local dimension of $\mathcal{B}_X$ at b is simply

$$\deg(b) := \dim_b \mathcal{B}_X = \sum_{\bar{v}\in X(\bar{k})} \sum_{i=1}^{m} c_{\bar{v},i},$$

which is locally constant. Combining with (6.8.1), we have the dimension formula when a is very G-ample:

$$\dim_a \mathcal{A}_X = \deg(b) + \langle \rho, \lambda_b \rangle - r(g_X - 1). \tag{6.8.3}$$

We also have dimension estimate in general regardless of ampleness

$$\dim_a \mathcal{A}_X \leq \deg(b) + \langle \rho, \lambda_b \rangle + r.$$

In the case where $\mathbf{A_M}$ is not an affine space, we can replace moduli $\mathcal{B}_X^1$ of boundary divisors by its normalization given by Proposition 5.1.25, and reduce the case to (6.8.3). Since we do not need this result, we leave it to the reader.

6.8.2 Let $\mathfrak{T}_{\mathfrak{M}}^{\circ} = \mathfrak{T}_{\mathfrak{M}} \cap \mathfrak{M}^{\circ}$ and let $\mathfrak{C}_{\mathfrak{M}}^{\circ}$ be the image of $\mathfrak{T}_{\mathfrak{M}}^{\circ}$. It is an open subset of $\mathfrak{C}_{\mathfrak{M}}$ since the cameral cover is flat, and its complement has codimension at least 2 in $\mathfrak{C}_{\mathfrak{M}}$, because it contains both $\mathfrak{C}_{\mathfrak{M}}^{\mathrm{rs}}$ and $\mathfrak{C}_{\mathfrak{M}}^{\times}$. The torus T^{sc} acts freely on the fibers of $\mathfrak{T}_{\mathfrak{M}}^{\circ} \to \mathfrak{A}_{\mathfrak{M}}$ and has open orbits. This implies that we have a description of relative tangent and cotangent bundles

$$\mathrm{T}_{\mathfrak{T}_{\mathfrak{M}}^{\circ}/\mathfrak{A}_{\mathfrak{M}}} \cong \mathfrak{t}^{\mathrm{sc}} \times_X \mathfrak{T}_{\mathfrak{M}}^{\circ},$$

$$\Omega_{\mathfrak{T}_{\mathfrak{M}}^{\circ}/\mathfrak{A}_{\mathfrak{M}}} \cong (\mathfrak{t}^{\mathrm{sc}})^* \times_X \mathfrak{T}_{\mathfrak{M}}^{\circ}.$$

This description is W-equivariant because T^{sc} is commutative. Since $\mathfrak{C}^{\circ}_{\mathfrak{M}}$ is the W-invariant quotient of $\mathfrak{T}^{\circ}_{\mathfrak{M}}$, same is true for the total space of their tangent bundles, in other words,

$$\mathrm{T}_{\mathfrak{T}^{\circ}_{\mathfrak{M}}/\mathfrak{A}_{\mathfrak{M}}}/\!\!/W \xrightarrow{\sim} \mathrm{T}_{\mathfrak{C}^{\circ}_{\mathfrak{M}}/\mathfrak{A}_{\mathfrak{M}}}.$$

It implies that

$$(\pi_{\mathfrak{M}*}\Omega_{\mathfrak{T}^{\circ}_{\mathfrak{M}}/\mathfrak{A}_{\mathfrak{M}}})^{W} = \Omega_{\mathfrak{C}^{\circ}_{\mathfrak{M}}/\mathfrak{A}_{\mathfrak{M}}}.$$

Since $\Omega_{\mathfrak{C}_{\mathfrak{M}}/\mathfrak{A}_{\mathfrak{M}}} = \mathfrak{C}^{*} \times_X \mathfrak{C}_{\mathfrak{M}}$, we have that

$$\pi_{\mathfrak{M}*}((\mathfrak{t}^{\mathrm{sc}})^{*} \times_X \mathfrak{T}^{\circ}_{\mathfrak{M}})^{W} = \mathfrak{C}^{*} \times_X \mathfrak{C}^{\circ}_{\mathfrak{M}}.$$

The Killing form on $\mathfrak{g}^{\mathrm{sc}}$ identifies $\mathfrak{t}^{\mathrm{sc}}$ with its dual as W-spaces. In addition, $\mathfrak{C}_{\mathfrak{M}} - \mathfrak{C}^{\circ}_{\mathfrak{M}}$ has codimension 2, thus we have over $\mathfrak{C}_{\mathfrak{M}}$

$$\pi_{\mathfrak{M}*}(\mathfrak{t}^{\mathrm{sc}} \times_X \mathfrak{T}_{\mathfrak{M}})^{W} = \mathfrak{C}^{*} \times_X \mathfrak{C}_{\mathfrak{M}}.$$

Let $\mathcal{L} \in \mathrm{Bun}_{Z_{\mathfrak{M}}}$, since W commutes with $Z_{\mathfrak{M}}$, the same argument also applies to $\mathcal{L}$-twisted cameral cover $\pi_{\mathfrak{M},\mathcal{L}} : \mathfrak{T}_{\mathfrak{M},\mathcal{L}} \to \mathfrak{C}_{\mathfrak{M},\mathcal{L}}$. Since $\mathfrak{t} = \mathfrak{z}_G \times_X \mathfrak{t}^{\mathrm{sc}}$, we have

$$\pi_{\mathfrak{M},\mathcal{L},*}(\mathfrak{t} \times_X \mathfrak{T}_{\mathfrak{M},\mathcal{L}})^{W} = \mathfrak{z}_G \times_X \mathfrak{C}^{*}_{\mathcal{L}} \times_X \mathfrak{C}_{\mathfrak{M},\mathcal{L}}.$$

Proposition 6.8.3. *For any $a \in \mathcal{A}_X(\bar{k})$ with associated $Z_{\mathfrak{M}}$-torsor $\mathcal{L}$, we have canonical isomorphism*

$$\mathrm{Lie}(\mathfrak{J}_a) = \mathrm{Lie}(\mathfrak{J}^{1}_a) \simeq \mathfrak{z}_G \times \mathfrak{C}^{*}_{\mathcal{L}}.$$

Proof. The cameral cover is flat and finite. Using Cartesian diagram

$$\begin{array}{ccc} \tilde{X}_a & \longrightarrow & \mathfrak{T}_{\mathfrak{M},\mathcal{L}} \\ {\scriptstyle \pi_a}\downarrow & & \downarrow{\scriptstyle \pi_{\mathfrak{M},\mathcal{L}}} \\ X & \xrightarrow{a} & \mathfrak{C}_{\mathfrak{M},\mathcal{L}} \end{array}$$

and the Galois description of $\mathfrak{J}_a^1$, we have the result by proper base change for coherent sheaves. ∎

Corollary 6.8.4. *For any $a \in \mathcal{A}_X^{\heartsuit}(\bar{k})$, we have*

$$\dim(\mathcal{P}_a) = \sum_{\bar{v} \in X(\bar{k})} \langle \rho, \lambda_{\bar{v}} \rangle + n g_X - n,$$

where n is the rank of G.

Proof. Since

$$\dim(\mathcal{P}_a) = \dim_{\bar{k}}(\mathrm{H}^1(\check{X}, \mathrm{Lie}(\mathfrak{J}_a))) - \dim_{\bar{k}}(\mathrm{H}^0(\check{X}, \mathrm{Lie}(\mathfrak{J}_a))),$$

we have the desired equation using Riemann-Roch theorem. ∎

6.9 Product Formula

Let $a \in \mathcal{A}_X^{\heartsuit}(\bar{k})$ and b (resp. $\mathcal{L}$) its image in $\mathcal{B}_X$ (resp. $\mathrm{Bun}_{Z_{\mathfrak{M}}}$). Let $U_a = \check{X} - \mathfrak{D}_a$. For each $\bar{k}$-point $\bar{v} \in \check{X} - U_a$, we have the local multiplicative affine Springer fiber $\mathcal{M}_{\bar{v}}(a)$, defined by a choice of $\gamma_{\bar{v}} \in \mathfrak{M}_{\mathcal{L}}^{\times}(\check{F}_{\bar{v}})$ lying over a.

Let $\tilde{U}_{\vartheta,a} = \tilde{X}_{\vartheta,a} \times_{\check{X}} U_a$. Since U_a is an affine curve over an algebraically closed field, so is $\tilde{U}_{\vartheta,a}$. Therefore

$$\mathrm{H}^2(U_a, \pi_{\vartheta,a,*}(\tilde{U}_{\vartheta,a} \times \mathbf{T})) = \mathrm{H}^2(\tilde{U}_{\vartheta,a}, \mathbf{T}) = 0.$$

Since the map

$$\begin{aligned}\pi_{\vartheta,a,*}(\tilde{U}_{\vartheta,a}\times\mathbf{T}) &\longrightarrow \mathfrak{J}^1_{U_a}=\pi_{\vartheta,a,*}(\tilde{U}_{\vartheta,a}\times\mathbf{T})^{\mathbf{W}\rtimes\Theta_\vartheta}\\ t &\longmapsto \prod_{w\in\mathbf{W}\rtimes\Theta_\vartheta} w(t)\end{aligned}$$

is surjective, a standard argument using long exact sequence shows that $\mathrm{H}^2(U_a,\mathfrak{J}^1_{U_a})=0$. Since $\mathfrak{J}^1_{U_a}=\mathfrak{J}_{U_a}$, we then have

$$\mathrm{H}^2(U_a,\mathfrak{J}_{U_a})=0.$$

The restriction of $[\mathfrak{M}_{\mathcal{L}}/G]\to\mathfrak{C}_{\mathcal{L}}$ to U_a is a $\mathfrak{J}_{U_a}$-gerbe, and it is trivial by vanishing of cohomology above. Choose a trivialization, that is, a G-torsor E_{U_a} over U_a, and a G-equivariant map $\phi_{U_a}\colon E_{U_a}\to\mathfrak{M}_{\mathcal{L}}$.

Over punctured disk $\breve{X}^{\bullet}_{\bar{v}}$, the preimage of the chosen point $\gamma_{\bar{v}}$ under $\phi_{U_a}|_{\breve{X}^{\bullet}_{\bar{v}}}$ is a $\mathfrak{J}_X|_{\breve{X}^{\bullet}_{\bar{v}}}$-torsor, which is trivial because the residue field of $\breve{F}_{\bar{v}}$ is algebraically closed and $\mathfrak{J}_X|_{\breve{X}^{\bullet}_{\bar{v}}}$ is a torus. This implies that we may choose a trivialization $\epsilon^{\bullet}_{\bar{v}}$ of $\mathfrak{J}_X|_{\breve{X}^{\bullet}_{\bar{v}}}$ so that its neutral point is sent to $\gamma_{\bar{v}}$ under $\phi_{U_a}|_{\breve{X}^{\bullet}_{\bar{v}}}$.

Fix a $\bar{k}$-algebra R and an R-point $g_{\bar{v}}\in\mathcal{M}_{\bar{v}}(a)(R)$. This gives a G-torsor $E_{\bar{v}}$ over $\breve{X}_{\bar{v},R}$, a G-equivariant map $\phi_{\bar{v}}\colon E_{\bar{v}}\to\mathfrak{M}_{\mathcal{L}}$, and a commutative diagram over $\breve{X}^{\bullet}_{\bar{v},R}$

$$\begin{array}{ccc} E_{\bar{v}} & \xrightarrow{\phi_{\bar{v}}} & \mathfrak{M}_{\mathcal{L}} \\ \downarrow{\scriptstyle\sim} & & \| \\ E_0 & \xrightarrow{\gamma_{\bar{v}}} & \mathfrak{M}_{\mathcal{L}} \end{array}.$$

Gluing $E_{\bar{v}}$ and E_{U_a} using $\epsilon^{\bullet}_{\bar{v}}$, we obtain a point $(E,\phi)\in\mathcal{M}_a(R)$. Therefore we have a morphism defined over $\bar{k}$

$$\prod_{\bar{v}\in\breve{X}-U_a}\mathcal{M}_{\bar{v}}(a)\longrightarrow\mathcal{M}_a.$$

Similarly, by gluing with trivial torsor over U_a, we have

$$\prod_{\bar{v}\in\check{X}-U_a} \mathcal{P}_{\bar{v}}(a) \longrightarrow \mathcal{P}_a. \tag{6.9.1}$$

The induced morphism

$$\prod_{\bar{v}\in\check{X}-U_a} \mathcal{M}_{\bar{v}}(a) \times \mathcal{P}_a \longrightarrow \mathcal{M}_a$$

is $\prod_{\bar{v}\in\check{X}-U_a} \mathcal{P}_{\bar{v}}(a)$-invariant (acting diagonally), hence we have a morphism

$$\prod_{\bar{v}\in\check{X}-U_a} \mathcal{M}_{\bar{v}}(a) \times^{\prod_{\bar{v}\in\check{X}-U_a} \mathcal{P}_{\bar{v}}(a)} \mathcal{P}_a \longrightarrow \mathcal{M}_a \tag{6.9.2}$$

and its reduced version

$$\prod_{\bar{v}\in\check{X}-U_a} \mathcal{M}_{\bar{v}}^{\mathrm{red}}(a) \times^{\prod_{\bar{v}\in\check{X}-U_a} \mathcal{P}_{\bar{v}}^{\mathrm{red}}(a)} \mathcal{P}_a \longrightarrow \mathcal{M}_a. \tag{6.9.3}$$

It is a equivalence of groupoids over geometric points, and from its construction, we see that if a Steinberg quasi-section can be defined over all X, then these maps can be defined over k.

We shall see in Proposition 8.1.2 that h_X is proper when restricted to anisotropic locus. Therefore, following the same proof in [Ngô10, Proposition 4.15.1], we have

Proposition 6.9.1 (Product Formula)**.** *For any $a \in \mathcal{A}_X^{\natural}(\bar{k})$, the left-hand side of* (6.9.3) *is a proper Deligne-Mumford stack, and* (6.9.3) *is a universal homeomorphism. If a Steinberg quasi-section is defined over all X, then* (6.9.3) *also induces an equivalence on k'-rational points for any field extension k'/k.*

Corollary 6.9.2. *For any $a \in \mathcal{A}_X^{\natural}(\bar{k})$, $\mathcal{M}_a$ is homeomorphic to a projective $\bar{k}$-scheme, and*

$$\dim \mathcal{M}_a = \dim \mathcal{P}_a.$$

Proof. It follows from the product formula, (6.4.1), Theorem 4.2.1 and Proposition 4.4.7. ∎

Another consequence of (6.9.3) is the following:

Proposition 6.9.3. *For any $a \in \mathcal{A}_X^{\heartsuit}(\bar{k})$, the fiber $\mathcal{M}_a$ is non-empty. In fact, $\mathcal{M}_a^{\mathrm{reg}}$ is non-empty.*

Remark 6.9.4. Proposition 6.9.1 is a stronger version of [Chi19, Theorem 4.2.10], since the existence of global Steinberg quasi-section is no longer required. On the other hand, that a being a $\bar{k}$-point is important in the construction of map (6.9.3), so this method only works over a point a. Later we will construct this map as a family in § 8.5.

6.9.5 There is another version of product formula which is also useful. Following [Ngô06], for any closed point $v \in X$, we may consider the stack $\mathcal{M}_{a,v}$ classifying pairs (E_v, ϕ_v) where E_v is a G-torsor over the formal disk X_v and ϕ_v is a G-equivariant map $E_v \to \mathfrak{M}_{\mathcal{L}}$. In other words, it is a sort of stacky version of multiplicative affine Springer fiber. We also let $\mathcal{P}_{a,v}$ be the classifying stack of $\mathfrak{J}_a$-torsors over X_v. The multiplicative affine Springer fiber $\mathcal{M}_v(a)$ naturally maps to $\mathcal{M}_{a,v}$ by forgetting the trivialization part. On the other hand, $\mathcal{M}_a$ also maps to $\mathcal{M}_{a,v}$ by restricting to X_v. Similarly, we may define $\mathcal{M}_{a,\bar{v}}$ and $\mathcal{P}_{a,\bar{v}}$ for any closed points $\bar{v} \in \check{X}$. Therefore we have a commutative diagram

$$\begin{array}{ccc} \prod_{\bar{v}\in\check{X}-U_a} \mathcal{M}_{\bar{v}}(a) & \longrightarrow & \mathcal{M}_a \\ & \searrow \quad \swarrow & \\ & \prod_{\bar{v}\in\check{X}-U_a} \mathcal{M}_{a,\bar{v}} & \end{array}$$

where the horizontal arrow is defined over $\bar{k}$ while the two diagonal ones are defined over k. So we have induced maps of 2-categorical quotients

$$\begin{array}{ccc} \prod_{\bar{v}\in\check{X}-U_a}[\mathcal{M}_{\bar{v}}(a)/\mathcal{P}_{\bar{v}}(a)] & \longrightarrow & [\mathcal{M}_a/\mathcal{P}_a] \\ & \searrow \quad \swarrow & \\ & \prod_{\bar{v}\in\check{X}-U_a}[\mathcal{M}_{a,\bar{v}}/\mathcal{P}_{a,\bar{v}}] & \end{array}$$

and all three maps are equivalences of groupoids after taking $\bar{k}$-points. The map (6.9.1) is always defined over k, hence so are the diagonal maps above. It implies that the horizontal map is defined over k, and induces an equivalence of k'-points for any k'/k regardless whether (6.9.3) is defined over k or not. We summarize it in the following proposition.

Proposition 6.9.6. *We have a natural map of* 2*-stacks over* k

$$\prod_{v\in X-U_a}[\mathcal{M}_v(a)/\mathcal{P}_v(a)] \longrightarrow [\mathcal{M}_a/\mathcal{P}_a]$$

that induces equivalence on k'*-points for any field extension* k'/k.

6.10 Local Model of Singularities

Unlike in the Lie algebra case, the total space $\mathcal{M}_X$ of mH-fibration is no longer smooth. Instead, it admits a stratification induced by affine Schubert cells, which in turn translates to representations via geometric Satake equivalence. Therefore, the existence of singularities is in fact a feature of mH-fibrations.

The main result of this section was essentially first conjectured in [FN11] (see Conjecture 4.1 of *loc. cit.*), and a weaker version was first proved in [Bou17]. Readers can find a more streamlined proof in [Chi19], due to Zhiwei Yun. However, these previous results would turn out to be too weak for studying endoscopy. The main reason behind is probably due to that the core argument in literature is *ad hoc* in nature, and in particular it does not try to establish a tangent-obstruction theory for deformations of mHiggs pairs, contrary to what is done in the Lie algebra case (see [Ngô10, § 4.14]). In this section, by establishing such tangent-obstruction theory, we are able to prove a much stronger result.

6.10.1 Let $h_X\colon \mathcal{M}_X \to \mathcal{A}_X$ be the universal mH-fibration of a monoid $\mathfrak{M} \in \mathcal{FM}(G^{\mathrm{sc}})$. Recall that in 5.3.8 we defined global affine Schubert schemes on $\mathcal{B}_X$:

$$\mathsf{Gr}^{\leq -w_0(\mathcal{B}_X)} = \mathsf{Gr}_{G^{\mathrm{ad}}}^{\leq -w_0(\mathcal{B}_X)} \longrightarrow \mathcal{B}_X.$$

The $\mathcal{B}_X$-family of arc groups $\mathbb{L}^+_{\mathcal{B}_X}G$ around boundary divisors acts on $\mathsf{Gr}^{\leq -w_0(\mathcal{B}_X)}$ by left multiplication, and locally over $\mathcal{B}_X$ this action factors through some jet group $\mathbb{L}^+_{\mathcal{B}_X,N}G$ for sufficiently large N. So we have evaluation map

$$\mathrm{ev}_N\colon \mathcal{M}_X \longrightarrow \left[\mathbb{L}^+_{\mathcal{B}_X,N}G\backslash\mathsf{Gr}^{\leq -w_0(\mathcal{B}_X)}\right],$$

which factors through

$$\mathrm{ev}\colon \mathcal{M}_X \longrightarrow \left[\mathbb{L}^+_{\mathcal{B}_X}G\backslash\mathsf{Gr}^{\leq -w_0(\mathcal{B}_X)}\right].$$

The following is a big improvement of [Bou17] (see also [Chi19]).

Theorem 6.10.2. *Let $m = (\mathcal{L}, E, \phi) \in \mathcal{M}_X^{\heartsuit}(\bar{k})$ be a point and let $a = h_X(m) \in \mathcal{A}_X(\bar{k})$ be its image. If*

$$\mathrm{H}^1(\breve{X}, (\mathrm{Lie}(I^{\mathrm{sm}}_{(E,\phi)})/\mathfrak{z}_G)^*) = 0,$$

then ev_N (resp. ev) is smooth (resp. formally smooth) at m. In particular, it is true when $\mathcal{L}$ is very (G, δ_a)-ample, The proof consists of several steps and will occupy the rest of this section.

6.10.3 We start with some generalities concerning deformations. Consider variety M over $\bar{k}$ with a smooth group G acting on it. Consider the quotient

$$\chi\colon M \longrightarrow [M/G].$$

We have distinguished triangle of cotangent complexes

$$\chi^* L_{[M/G]} \longrightarrow L_M \longrightarrow L_{M/[M/G]} \xrightarrow{+1}.$$

Since χ is a G-torsor, we have $L_{M/[M/G]} \simeq \mathcal{O}_M \otimes \mathfrak{g}^*[0]$, and L_M is G-equivariant. Descending to $[M/G]$, we get

$$L_{[M/G]} \longrightarrow L_M/G \longrightarrow M \times^G \mathfrak{g}^*[0] \xrightarrow{+1}.$$

Let X be a $\bar{k}$-scheme of finite type and $m \coloneqq (E, \phi)$ be an X-point of $[M/G]$, then $\mathbf{L}m^* L_{[M/G]}$ is isomorphic to the cone of the map of complexes

$$(\mathbf{L}\phi^* L_{E\times^G M} \longrightarrow \mathrm{Ad}(E)^*[0])[-1].$$

Let $\mathcal{T}^\bullet = \underline{\mathbf{R}\mathrm{Hom}}_X(\mathbf{L}m^* L_{[M/G]}, \mathcal{O}_X)$, then it is isomorphic to the cone of map

$$\mathrm{Ad}(E)[0] \longrightarrow \underline{\mathbf{R}\mathrm{Hom}}_X(\mathbf{L}\phi^* L_{E\times^G M}, \mathcal{O}_X). \tag{6.10.1}$$

Since L_M is supported on degrees $(-\infty, 0]$, so is $\mathbf{L}\phi^* L_{E\times^G M}$. Since $\underline{\mathrm{Hom}}_X(-, \mathcal{O}_X)$ is left exact, we see that $\underline{\mathbf{R}\mathrm{Hom}}_X(\mathbf{L}\phi^* L_{E\times^G M}, \mathcal{O}_X)$ is supported on $[0, +\infty)$. Therefore we have that $\mathcal{T}^\bullet$ is isomorphic to the complex

$$\mathrm{Ad}(E)[1] \longrightarrow \underline{\mathbf{R}\mathrm{Hom}}_X(\mathbf{L}\phi^* L_{E\times^G M}, \mathcal{O}_X),$$

where the arrow is just the map (of coherent sheaves) $\mathrm{Ad}(E) \longrightarrow \underline{\mathrm{Hom}}_X(\mathbf{L}\phi^* L_{E\times^G M}, \mathcal{O}_X)$ induced by (6.10.1), which can also be seen as the derivative of the G-action. In particular, $\mathcal{T}^\bullet$ is supported on degrees $[-1, +\infty)$.

Similar to the Lie algebra case, the deformation of m in mapping stack $\underline{\mathrm{Hom}}(X, [M/G])$ is controlled by $\mathcal{T}^\bullet$, such that the obstruction space is the hypercohomology group $\mathrm{H}^1(X, \mathcal{T}^\bullet)$, the tangent space is $\mathrm{H}^0(X, \mathcal{T}^\bullet)$ and the infinitesimal automorphism group is $\mathrm{H}^{-1}(X, \mathcal{T}^\bullet)$.

Now we compute these cohomology groups using Čech cohomology when X is a curve. We shall assume that the generic points of X are sent to the smooth locus of $[M/G]$ under m, in which case each quasi-coherent sheaf $\mathrm{H}^i(\mathcal{T}^\bullet)$ is supported on finitely many points on X if $i \geq 1$.

6.10.4 Let $\mathscr{X} = \{X_i\}_{i\in I}$ be a finite Zariski open affine covering of X. The forgetful functor from the category of sheaves of $\mathcal{O}_X$-modules to presheaves of $\mathcal{O}_X$-modules is left exact, whose right-derived functors $\underline{\mathscr{H}}^q$ are given by ($\mathcal{F}$ is any sheaf of $\mathcal{O}_X$-modules):

$$\underline{\mathscr{H}}^q(\mathcal{F})\colon U \subset X \longmapsto \mathrm{H}^q(U, \mathcal{F}).$$

For any finite subset $\{i_0, i_1, \cdots, i_p\}$ of I, we let

$$X_{i_0,\ldots,i_p} = X_{i_0} \cap \cdots \cap X_{i_p}.$$

Then for each $q \geq 0$ we have Čech double complex $\check{C}^\bullet(\mathscr{X}, \underline{\mathscr{H}}^q(\mathcal{T}^\bullet))$

$$\begin{array}{ccccccccc}
 & & \vdots & & \vdots & & \vdots & & \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
0 & \to & \prod_{ijk} \mathrm{H}^q(X_{ijk}, \mathcal{T}^{-1}) & \to & \prod_{ijk} \mathrm{H}^q(X_{ijk}, \mathcal{T}^{0}) & \to & \prod_{ijk} \mathrm{H}^q(X_{ijk}, \mathcal{T}^{1}) & \to & \cdots \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
0 & \longrightarrow & \prod_{ij} \mathrm{H}^q(X_{ij}, \mathcal{T}^{-1}) & \longrightarrow & \prod_{ij} \mathrm{H}^q(X_{ij}, \mathcal{T}^{0}) & \longrightarrow & \prod_{ij} \mathrm{H}^q(X_{ij}, \mathcal{T}^{1}) & \longrightarrow & \cdots \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
0 & \longrightarrow & \prod_{i} \mathrm{H}^q(X_{i}, \mathcal{T}^{-1}) & \longrightarrow & \prod_{i} \mathrm{H}^q(X_{i}, \mathcal{T}^{0}) & \longrightarrow & \prod_{i} \mathrm{H}^q(X_{i}, \mathcal{T}^{1}) & \longrightarrow & \cdots \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

The general theory of Čech cohomologies [Sta22, Tag 01FP] shows that there is a spectral sequence

$$E_2^{p,q} = \mathrm{H}^p(\mathrm{Tot}(\check{C}^\bullet(\mathscr{X}, \underline{\mathscr{H}}^q(\mathcal{T}^\bullet)))) \Longrightarrow \mathrm{H}^{p+q}(X, \mathcal{T}^\bullet),$$

where Tot means taking the total complex associated with a double complex. Since $X_{i_0,\ldots,x_p}$ is affine for any p and any $i_0, \ldots, i_p$, and $\mathcal{T}^i$ is quasi-coherent for any i, we know that the above double complex vanishes completely for all $q > 0$. Thus the above spectral sequence degenerates at E_2-page, and so we have

$$\mathrm{H}^p(X, \mathcal{T}^\bullet) \simeq \mathrm{H}^p(\mathrm{Tot}(\check{C}^\bullet(\mathscr{X}, \underline{\mathscr{H}}^0(\mathcal{T}^\bullet)))).$$

Now we use the spectral sequence of double complexes to compute

$$\mathrm{H}^p(\mathrm{Tot}(\check{C}^\bullet(\mathscr{X}, \underline{\mathscr{H}}^0(\mathcal{T}^\bullet)))).$$

The 0-th page is

$$\begin{array}{ccccccccc}
 & & \cdots & & \cdots & & \cdots & & \\
0 & \to & \prod_{ijk} \mathrm{H}^0(X_{ijk}, \mathcal{T}^{-1}) & \to & \prod_{ijk} \mathrm{H}^0(X_{ijk}, \mathcal{T}^{0}) & \to & \prod_{ijk} \mathrm{H}^0(X_{ijk}, \mathcal{T}^{1}) & \to & \cdots \\
0 & \longrightarrow & \prod_{ij} \mathrm{H}^0(X_{ij}, \mathcal{T}^{-1}) & \longrightarrow & \prod_{ij} \mathrm{H}^0(X_{ij}, \mathcal{T}^{0}) & \longrightarrow & \prod_{ij} \mathrm{H}^0(X_{ij}, \mathcal{T}^{1}) & \longrightarrow & \cdots \\
0 & \longrightarrow & \prod_{i} \mathrm{H}^0(X_{i}, \mathcal{T}^{-1}) & \longrightarrow & \prod_{i} \mathrm{H}^0(X_{i}, \mathcal{T}^{0}) & \longrightarrow & \prod_{i} \mathrm{H}^0(X_{i}, \mathcal{T}^{1}) & \longrightarrow & \cdots
\end{array}$$

Since all $X_{i_0,\dots,i_p}$ are affine, we may directly compute the 1-st page to be

$$\begin{array}{ccc}
\vdots & \vdots & \vdots \\
\uparrow & \uparrow & \uparrow \\
\prod_{ijk} \mathrm{H}^0(X_{ijk}, \mathrm{H}^{-1}(\mathcal{T}^\bullet)) & \prod_{ijk} \mathrm{H}^0(X_{ijk}, \mathrm{H}^{0}(\mathcal{T}^\bullet)) & \prod_{ijk} \mathrm{H}^0(X_{ijk}, \mathrm{H}^{1}(\mathcal{T}^\bullet)) \\
\uparrow & \uparrow & \uparrow \\
\prod_{ij} \mathrm{H}^0(X_{ij}, \mathrm{H}^{-1}(\mathcal{T}^\bullet)) & \prod_{ij} \mathrm{H}^0(X_{ij}, \mathrm{H}^{0}(\mathcal{T}^\bullet)) & \prod_{ij} \mathrm{H}^0(X_{ij}, \mathrm{H}^{1}(\mathcal{T}^\bullet)) \\
\uparrow & \uparrow & \uparrow \\
\prod_{i} \mathrm{H}^0(X_{i}, \mathrm{H}^{-1}(\mathcal{T}^\bullet)) & \prod_{i} \mathrm{H}^0(X_{i}, \mathrm{H}^{0}(\mathcal{T}^\bullet)) & \prod_{i} \mathrm{H}^0(X_{i}, \mathrm{H}^{1}(\mathcal{T}^\bullet)) \\
\uparrow & \uparrow & \uparrow \\
0 & 0 & 0
\end{array}$$

Observe that the j-th column above is exactly the Čech complex of quasi-coherent sheaf $\mathrm{H}^j(\mathcal{T}^\bullet)$ with respect to covering $\mathscr{X}$, so its cohomologies are just $\mathrm{H}^i(X, \mathrm{H}^j(\mathcal{T}^\bullet))$. Since X is a curve, $\mathrm{H}^i(X, \mathrm{H}^j(\mathcal{T}^\bullet)) = 0$ for all $i \geq 2$, and since for any $j \geq 1$ the sheaf $\mathrm{H}^j(\mathcal{T}^\bullet)$ is supported over finitely many points, we also have $\mathrm{H}^1(\mathrm{H}^j(\mathcal{T}^\bullet)) = 0$ for $j \geq 1$. Thus, the 2-nd page has only two non-zero rows:

$$
\begin{array}{ccccc}
\mathrm{H}^1(X,\mathrm{H}^{-1}(\mathcal{T}^\bullet)) & \mathrm{H}^1(X,\mathrm{H}^0(\mathcal{T}^\bullet)) & 0 & 0 & \cdots \\
\mathrm{H}^0(X,\mathrm{H}^{-1}(\mathcal{T}^\bullet)) & \mathrm{H}^0(X,\mathrm{H}^0(\mathcal{T}^\bullet)) & \mathrm{H}^0(X,\mathrm{H}^1(\mathcal{T}^\bullet)) & \mathrm{H}^0(X,\mathrm{H}^2(\mathcal{T})) & \cdots
\end{array}
$$

and all differentials on all pages hereafter are zero, because the vertical component of any arrow will go upwards at least two rows. Thus the spectral sequence degenerates at the second page, and we have canonical isomorphisms

$$\mathrm{H}^i(X,\mathcal{T}^\bullet) \simeq \mathrm{H}^0(X,\mathrm{H}^i(\mathcal{T}^\bullet)), \quad i = -1 \text{ or } i \geq 2,$$

as well as exact sequences

$$
\begin{gathered}
0 \longrightarrow \mathrm{H}^1(X,\mathrm{H}^{-1}(\mathcal{T}^\bullet)) \longrightarrow \mathrm{H}^0(X,\mathcal{T}^\bullet) \longrightarrow \mathrm{H}^0(X,\mathrm{H}^0(\mathcal{T}^\bullet)) \longrightarrow 0, \\
0 \longrightarrow \mathrm{H}^1(X,\mathrm{H}^0(\mathcal{T}^\bullet)) \longrightarrow \mathrm{H}^1(X,\mathcal{T}^\bullet) \longrightarrow \mathrm{H}^0(X,\mathrm{H}^1(\mathcal{T}^\bullet)) \longrightarrow 0.
\end{gathered}
$$

Note that everything still holds in the relative setting where M is defined over a scheme S and G is smooth over S, and X is an S-curve.

6.10.5 We may replace X with a formal disc X_v at a point $v \in X$. In this case $\mathrm{H}^1(X_v,\mathcal{F}) = 0$ for any quasi-coherent sheaf $\mathcal{F}$. Let $\mathcal{T}_v^\bullet$ be the analogue of $\mathcal{T}^\bullet$ on X_v, then we have for all $i \geq -1$

$$\mathrm{H}^i(X_v,\mathcal{T}_v^\bullet) \simeq \mathrm{H}^0(X_v,\mathrm{H}^i(\mathcal{T}_v^\bullet)).$$

Let $\iota\colon X_v \to X$ be the natural map, then it is flat and the (non-derived) functor ι^* is exact. Thus we have $\mathcal{T}_v^\bullet \simeq \iota^*\mathcal{T}^\bullet$, and the natural map

$$\mathrm{H}^i(\mathcal{T}^\bullet) \otimes_{\mathcal{O}_X} \mathcal{O}_v \longrightarrow \mathrm{H}^i(\mathcal{T}_v^\bullet)$$

is an isomorphism of $\mathcal{O}_v$-modules. In particular, when $i \geq 1$, since $\mathrm{H}^i(\mathcal{T}^\bullet)$ is finitely supported, we have injective map

$$\mathrm{H}^0(X, \mathrm{H}^i(\mathcal{T}^\bullet)) \longrightarrow \prod_v \mathrm{H}^0(X_v, \mathrm{H}^i(\mathcal{T}_v^\bullet)),$$

where v ranges over the points $v \in X$ such that $m(v)$ is singular in $[M/G]$. Note that when $i = 1$, the right-hand side is precisely the obstruction space of deforming the $\prod_v X_v$-arc in $[M/G]$ induced by m, in other words, the local obstruction space.

6.10.6 We now look at the obstruction space $\mathrm{H}^1(X, \mathcal{T}^\bullet)$. We have seen above that the quotient $\mathrm{H}^0(X, \mathrm{H}^1(\mathcal{T}^\bullet))$ is precisely the space of local obstructions. As a result, if we can show that

$$\mathrm{H}^1(X, \mathrm{H}^0(\mathcal{T}^\bullet)) = 0,$$

we will be able to prove that the global obstruction to deforming m is completely determined by its local obstructions. More explicitly, consider two-step complex

$$\mathcal{T}^{\leq 0}\colon\ \mathrm{Ad}(E) \longrightarrow \underline{\mathrm{Hom}}_X(\phi^*\Omega^1_{E\times^G M}, \mathcal{O}_X),$$

then we want to show that

$$\mathrm{H}^1(X, \mathrm{coker}(\mathcal{T}^{\leq 0})) = 0. \tag{6.10.2}$$

This statement is a generalization of the one for additive Hitchin fibrations where $M = \mathfrak{g}_D$. In that case, M is smooth, so there is no local obstruction, and if D is sufficiently ample, the global obstructions also vanish using Serre duality (see [Ngô10, § 4.14]). Our goal then is to find a similar duality statement to prove (6.10.2) in multiplicative case.

6.10.7 For $\mathfrak{M} \in \mathcal{FM}(G^{\mathrm{sc}})$, recall we have the big-cell locus $\mathfrak{M}^\circ \subset \mathfrak{M}$ such that the restriction of abelianization map $\alpha_{\mathfrak{M}}^\circ \colon \mathfrak{M}^\circ \to \mathfrak{A}_{\mathfrak{M}}$ is smooth, and its fibers are homogeneous spaces under $G^{\mathrm{sc}} \times G^{\mathrm{sc}}$. The action of $G^{\mathrm{sc}} \times G^{\mathrm{sc}}$ induces injection map of vector bundles on $\mathfrak{M}^\circ$:

$$\Omega^1_{\mathfrak{M}^\circ/\mathfrak{A}_{\mathfrak{M}}} \longrightarrow (\mathfrak{g}^{\mathrm{sc}})^* \times (\mathfrak{g}^{\mathrm{sc}})^*,$$

and its (surjective) dual map

$$\mathfrak{g}^{\mathrm{sc}} \times \mathfrak{g}^{\mathrm{sc}} \longrightarrow \mathrm{T}_{\mathfrak{M}^\circ/\mathfrak{A}_{\mathfrak{M}}}.$$

Choosing a non-degenerate G^{sc}-invariant symmetric bilinear form q on $\mathfrak{g}^{\mathrm{sc}}$, we may identify $\mathfrak{g}^{\mathrm{sc}}$ with its dual. We fix the *anti-diagonal* form $(q, -q)$ on $\mathfrak{g}^{\mathrm{sc}} \times \mathfrak{g}^{\mathrm{sc}}$ and use this form to identify $\mathfrak{g}^{\mathrm{sc}} \times \mathfrak{g}^{\mathrm{sc}}$ with its dual. Note that under such identification, the diagonal subspace

$$\Delta \colon \mathfrak{g}^{\mathrm{sc}} \longrightarrow \mathfrak{g}^{\mathrm{sc}} \times \mathfrak{g}^{\mathrm{sc}}$$

and the quotient $\mathfrak{g}^{\mathrm{sc}} \times \mathfrak{g}^{\mathrm{sc}}/\Delta(\mathfrak{g}^{\mathrm{sc}})$ are G^{sc}-equivariant duals. In other words, we have a short exact sequence that is G^{sc}-equivariantly self-dual:

$$0 \longrightarrow \mathfrak{g}^{\mathrm{sc}} \xrightarrow{\Delta} \mathfrak{g}^{\mathrm{sc}} \times \mathfrak{g}^{\mathrm{sc}} \longrightarrow \mathfrak{g}^{\mathrm{sc}} \times \mathfrak{g}^{\mathrm{sc}}/\Delta(\mathfrak{g}^{\mathrm{sc}}) \longrightarrow 0.$$

The following result is inspired by [Bri09, Example 2.5]:

Lemma 6.10.8. *With the choice of $(q,-q)$, we have a G^{sc}-equivariantly self-dual short exact sequence*

$$0 \longrightarrow \Omega^1_{\mathfrak{M}^\circ/\mathfrak{A}_{\mathfrak{M}}} \longrightarrow \mathfrak{g}^{\mathrm{sc}} \times \mathfrak{g}^{\mathrm{sc}} \longrightarrow \mathrm{T}_{\mathfrak{M}^\circ/\mathfrak{A}_{\mathfrak{M}}} \longrightarrow 0.$$

Proof. It suffices to consider $\mathfrak{M} = \mathrm{Env}(G^{\mathrm{sc}})$ by universal property and show that the fibers of

$$\ker(\mathfrak{g}^{\mathrm{sc}} \times \mathfrak{g}^{\mathrm{sc}} \longrightarrow \mathrm{T}_{\mathfrak{M}^\circ/\mathfrak{A}_{\mathfrak{M}}})$$

are maximal $(q,-q)$-isotropic subspaces of $\mathfrak{g}^{\mathrm{sc}} \times \mathfrak{g}^{\mathrm{sc}}$. Since $Z_{\mathfrak{M}}$ commutes with $G^{\mathrm{sc}} \times G^{\mathrm{sc}}$-action, it suffices to prove the statement for $g e_{I,\Delta} h$, where $g, h \in G^{\mathrm{sc}}$, and $e_{I,\Delta}$ are the system of idempotents associated with subsets of simple roots $I \subset \Delta$ (c.f. § 2.4.15). Moreover, since $(q,-q)$ is $G^{\mathrm{sc}} \times G^{\mathrm{sc}}$-invariant, it suffices to prove the statement for $e_{I,\Delta}$.

Indeed, Let P_I (resp. P_I^-) be the standard parabolic subgroup of G^{sc} containing B (resp. B^-), U_I (resp. U_I^-) be its unipotent radical, and L_I be the Levi factor containing T^{sc}, then the stabilizer of $e_{I,\Delta}$ in $G^{\mathrm{sc}} \times G^{\mathrm{sc}}$ is the semidirect product $(U_I \times U_I^-) \rtimes \Delta(L_I)$, where $\Delta(L_I)$ is the diagonal embedding of L_I in $G^{\mathrm{sc}} \times G^{\mathrm{sc}}$. One then verify by direct computation that $\mathfrak{u}_I \oplus \mathfrak{u}_I^- \oplus \Delta(\mathfrak{l}_I)$ is $(q,-q)$-isotropic and has half the dimension of $\mathfrak{g}^{\mathrm{sc}} \times \mathfrak{g}^{\mathrm{sc}}$. ∎

6.10.9 The monoid $\mathfrak{M}$ is usually not smooth, but since it is normal and $\mathfrak{M} - \mathfrak{M}^\circ$ has codimension at least 2, we still have identification by Hartogs' theorem (here ι is the inclusion map $\mathfrak{M}^\circ \to \mathfrak{M}$):

$$\iota_*(\mathfrak{g}^{\mathrm{sc}} \times \mathfrak{g}^{\mathrm{sc}}) = \mathfrak{g}^{\mathrm{sc}} \times \mathfrak{g}^{\mathrm{sc}},$$

because locally over the curve X, $\mathfrak{g}^{\mathrm{sc}}\times\mathfrak{g}^{\mathrm{sc}}$ is a trivial vector bundle. By functoriality, we have maps

$$\Omega^1_{\mathfrak{M}/\mathfrak{A}_{\mathfrak{M}}}\longrightarrow \iota_*\Omega^1_{\mathfrak{M}^\circ/\mathfrak{A}_{\mathfrak{M}}}\longrightarrow \mathfrak{g}^{\mathrm{sc}}\times\mathfrak{g}^{\mathrm{sc}}\longrightarrow \mathrm{T}_{\mathfrak{M}/\mathfrak{A}_{\mathfrak{M}}},$$

where we already identified $\mathfrak{g}^{\mathrm{sc}}\times\mathfrak{g}^{\mathrm{sc}}$ with its dual using $(q,-q)$. Since $\mathfrak{M}$ is normal, and $\mathrm{T}_{\mathfrak{M}/\mathfrak{A}_{\mathfrak{M}}}$ is reflexive, it may be identified with $\iota_*\mathrm{T}_{\mathfrak{M}^\circ/\mathfrak{A}_{\mathfrak{M}}}$. Now since $Z_{\mathfrak{M}}$ commutes with $G^{\mathrm{sc}}\times G^{\mathrm{sc}}$, everything descends to quotient

$$[\mathfrak{M}/Z_{\mathfrak{M}}]\longrightarrow[\mathfrak{A}_{\mathfrak{M}}/Z_{\mathfrak{M}}].$$

To simplify notations, we denote $[\mathfrak{M}]=[\mathfrak{M}/Z_{\mathfrak{M}}]$ and $[\mathfrak{A}_{\mathfrak{M}}]=[\mathfrak{A}_{\mathfrak{M}}/Z_{\mathfrak{M}}]$. So we have

$$\Omega^1_{[\mathfrak{M}]/[\mathfrak{A}_{\mathfrak{M}}]}\longrightarrow \iota_*\Omega^1_{[\mathfrak{M}^\circ]/[\mathfrak{A}_{\mathfrak{M}}]}\longrightarrow \mathfrak{g}^{\mathrm{sc}}\times\mathfrak{g}^{\mathrm{sc}}\longrightarrow \mathrm{T}_{[\mathfrak{M}]/[\mathfrak{A}_{\mathfrak{M}}]}.$$

Let $m=(\mathcal{L},E,\phi)\in\mathcal{M}_X(\bar{k})^{\heartsuit}$ be an mHiggs bundle, viewed as an $\breve{X}$-point of the stack $[\mathfrak{M}/G\times Z_{\mathfrak{M}}]$, and let $\lambda\in[\mathfrak{A}_{\mathfrak{M}}](\breve{X})$ be its boundary divisor. Then (6.10.2) translates to the following statement:

$$\mathrm{H}^1\Big(\breve{X},\operatorname{coker}\Big[\mathrm{Ad}(E)\xrightarrow{\mathrm{D}_{\mathrm{Ad}}}(\phi^*\Omega^1_{E\times^G[\mathfrak{M}]/[\mathfrak{A}_{\mathfrak{M}}]})^*\Big]\Big)=0,$$

where superscript $*$ means taking $\mathcal{O}_{\breve{X}}$-dual. Since the image of D_{Ad} stays the same if we replace $\mathrm{Ad}(E)$ by $\mathrm{Ad}(E)^{\mathrm{sc}}$, we are reduced to prove that

$$\mathrm{H}^1\Big(\breve{X},\operatorname{coker}\Big[\mathrm{Ad}(E)^{\mathrm{sc}}\xrightarrow{\mathrm{D}_{\mathrm{Ad}}}(\phi^*\Omega^1_{E\times^G[\mathfrak{M}]/[\mathfrak{A}_{\mathfrak{M}}]})^*\Big]\Big)=0,$$

6.10.10 On the other hand, since ι_* is left-exact, we have exact sequence

$$\iota_*\Omega^1_{\mathfrak{M}^\circ/\mathfrak{A}_{\mathfrak{M}}}\longrightarrow \mathfrak{g}^{\mathrm{sc}}\times\mathfrak{g}^{\mathrm{sc}}\longrightarrow \iota_*\mathrm{T}_{\mathfrak{M}^\circ/\mathfrak{A}_{\mathfrak{M}}}\simeq\mathrm{T}_{\mathfrak{M}/\mathfrak{A}_{\mathfrak{M}}},$$

which induces exact sequence

$$\phi^* \iota_* \Omega^1_{E\times^G[\mathfrak{M}^\circ]/[\mathfrak{A}_\mathfrak{M}]} \longrightarrow \mathrm{Ad}(E)^{\mathrm{sc}} \times \mathrm{Ad}(E)^{\mathrm{sc}} \longrightarrow \phi^* \mathrm{T}_{E\times^G[\mathfrak{M}]/[\mathfrak{A}_\mathfrak{M}]}.$$

Taking $\mathcal{O}_{\check{X}}$-dual and using bilinear form $(q,-q)$, we have exact sequence

$$(\phi^* \mathrm{T}_{E\times^G[\mathfrak{M}]/[\mathfrak{A}_\mathfrak{M}]})^* \longrightarrow \mathrm{Ad}(E)^{\mathrm{sc}} \times \mathrm{Ad}(E)^{\mathrm{sc}} \longrightarrow (\phi^* \iota_* \Omega^1_{E\times^G[\mathfrak{M}^\circ]/[\mathfrak{A}_\mathfrak{M}]})^*$$

Since $\check{X}$ is a curve, both $(\phi^* \mathrm{T}_{E\times^G[\mathfrak{M}]/[\mathfrak{A}_\mathfrak{M}]})^*$ and $(\phi^* \iota_* \Omega^1_{E\times^G[\mathfrak{M}^\circ]/[\mathfrak{A}_\mathfrak{M}]})^*$ are locally free, so the above sequence is also exact on the left. Let $K \subset (\phi^* \iota_* \Omega^1_{E\times^G[\mathfrak{M}^\circ]/[\mathfrak{A}_\mathfrak{M}]})^*$ be the cokernel of the first map, then it is again locally free. This implies that for any $v \in \check{X}$, the fiber map

$$(\phi^* \mathrm{T}_{E\times^G[\mathfrak{M}]/[\mathfrak{A}_\mathfrak{M}]})^*_v \longrightarrow (\mathrm{Ad}(E)^{\mathrm{sc}} \times \mathrm{Ad}(E)^{\mathrm{sc}})_v$$

is injective. As a consequence, since generically over $\check{X}$ the fiber of $(\phi^* \mathrm{T}_{E\times^G[\mathfrak{M}]/[\mathfrak{A}_\mathfrak{M}]})$ is a maximal $(q,-q)$-isotropic subspace of $\mathrm{Ad}(E)^{\mathrm{sc}} \times \mathrm{Ad}(E)^{\mathrm{sc}}$, the same must be true over all $\check{X}$. Thus it shows that the short exact sequence

$$0 \longrightarrow (\phi^* \mathrm{T}_{E\times^G[\mathfrak{M}]/[\mathfrak{A}_\mathfrak{M}]})^* \longrightarrow \mathrm{Ad}(E)^{\mathrm{sc}} \times \mathrm{Ad}(E)^{\mathrm{sc}} \longrightarrow K \longrightarrow 0$$

is self-dual. Moreover, since the adjoint action of G^{sc} is simply the restriction of the $G^{\mathrm{sc}} \times G^{\mathrm{sc}}$-action to the diagonal, we see that its derivative D_{Ad} factors through K. Since we have inclusions of locally free sheaves on $\check{X}$

$$K \subset (\phi^* \iota_* \Omega^1_{E\times^G[\mathfrak{M}^\circ]/[\mathfrak{A}_\mathfrak{M}]})^* \subset (\phi^* \Omega^1_{E\times^G[\mathfrak{M}]/[\mathfrak{A}_\mathfrak{M}]})^*,$$

to prove (6.10.2) it suffices to prove that

$$\mathrm{H}^1\left(\breve{X}, \mathrm{coker}(\mathrm{Ad}(E)^{\mathrm{sc}} \to K)\right) = 0.$$

Now consider the following self-dual diagram of locally free sheaves on $\breve{X}$:

$$\begin{array}{ccccccc}
 & & & \mathrm{Ad}(E)^{\mathrm{sc}} & \xrightarrow{\ \mathrm{D}_{\mathrm{Ad}}\ } (\text{to } K) & & \\
 & & & \downarrow \Delta & & & \\
0 \longrightarrow & (\phi^* \mathrm{T}_{E\times^G[\mathfrak{M}]/[\mathfrak{A}_{\mathfrak{M}}]})^* & \longrightarrow & \mathrm{Ad}(E)^{\mathrm{sc}} \times \mathrm{Ad}(E)^{\mathrm{sc}} & \longrightarrow & K & \longrightarrow 0 \\
 & & \searrow & \downarrow & & & \\
 & & & \mathrm{Ad}(E)^{\mathrm{sc}} \times \mathrm{Ad}(E)^{\mathrm{sc}}/\Delta(\mathrm{Ad}(E)^{\mathrm{sc}}) & & &
\end{array}$$

The $\mathcal{O}_{\breve{X}}$-dual of $\mathrm{coker}(\mathrm{D}_{\mathrm{Ad}})$ is identified with the kernel of the lower-left map. But that kernel is none other than the kernel of D_{Ad}, which is simply $\mathrm{Lie}(I^{\mathrm{sm}}_{(E,\phi)})/\mathfrak{z}_G$ (c.f. § 6.6). So we reduce (6.10.2) to

$$\mathrm{H}^1(\breve{X}, (\mathrm{Lie}(I^{\mathrm{sm}}_{(E,\phi)})/\mathfrak{z}_G)^*) = 0. \tag{6.10.3}$$

The canonical inclusion map $I^{\mathrm{sm}}_{(E,\phi)} \to \mathfrak{J}^{\flat}_a$ into the Néron model induces inclusion

$$\mathrm{Lie}(I^{\mathrm{sm}}_{(E,\phi)})/\mathfrak{z}_G \longrightarrow \mathrm{Lie}(\mathfrak{J}^{\flat}_a)/\mathfrak{z}_G,$$

hence we may also further reduce to

$$\mathrm{H}^1(\breve{X}, (\mathrm{Lie}(\mathfrak{J}^{\flat}_a)/\mathfrak{z}_G)^*) = 0.$$

6.10.11 Let $\mathrm{Ob}(\mathcal{L}, E, \phi)$ be the obstruction space of deforming $(\mathcal{L}, E, \phi)$ relative to boundary divisor λ, and for any $v \in \breve{X}$, let $\mathrm{Ob}_v(\mathcal{L}, E, \phi)$ be the obstruction space of deforming the $\breve{X}_v$-arc induced by $(\mathcal{L}, E, \phi)$. We have the following global-local principle for the obstructions:

Proposition 6.10.12. *If* (6.10.3) *holds, then we have canonical injective map*

$$\mathrm{Ob}(\mathcal{L},E,\phi)\longrightarrow\prod_{v}\mathrm{Ob}_v(\mathcal{L},E,\phi),$$

where v ranges over the points in $\check{X}$ that is sent to the singular locus of $[\mathfrak{M}/G\times Z_{\mathfrak{M}}]$. In particular, it is true when $\mathcal{L}$ is very (G,δ_a)-ample.

Proof. We only need to prove the second statement. By Proposition 6.8.3, we know that

$$\mathrm{Lie}(\mathfrak{J}_a)/\mathfrak{z}_G\simeq\mathfrak{C}_{\mathcal{L}}^{*}.$$

The results then follow from the fact that δ_a is the length of $\mathrm{Lie}(\mathfrak{J}_a^{\flat})/\mathrm{Lie}(\mathfrak{J}_a)$ and the definition of $\mathcal{L}$ being very (G,δ_a)-ample. ∎

6.10.13 Next we turn to tangent spaces. As we have seen above, the tangent space of $\mathcal{M}_X$ at $(\mathcal{L},E,\phi)$ relative to $\mathcal{B}_X$ fits in the short exact sequence

$$0\longrightarrow\mathrm{H}^1(\check{X},\ker(\mathrm{D}_{\mathrm{Ad}}))\longrightarrow\mathrm{T}_{(\mathcal{L},E,\phi)}(\mathcal{M}_X/\mathcal{B}_X)\longrightarrow\mathrm{H}^0(\check{X},\mathrm{coker}(\mathrm{D}_{\mathrm{Ad}}))\longrightarrow 0.$$

Restricting to a formal disc $\check{X}_v\to X$, and choosing a trivialization of $\mathcal{L}$ over $\check{X}_v$, we have commutative diagram

$$\begin{array}{ccccccccc}
0 & \to & \mathrm{H}^1(\check{X},\ker(\mathrm{D}_{\mathrm{Ad}})) & \longrightarrow & \mathrm{T}_{(\mathcal{L},E,\phi)}(\mathcal{M}_X/\mathcal{B}_X) & \longrightarrow & \mathrm{H}^0(\check{X},\mathrm{coker}(\mathrm{D}_{\mathrm{Ad}})) & \longrightarrow & 0\\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & \longrightarrow & \mathrm{T}_{(E_v,\phi_v)}(\mathbb{L}_v^{+}[\mathfrak{M}/G]/\mathbb{L}_v^{+}\mathfrak{A}_{\mathfrak{M}}) & \xrightarrow{\sim} & \mathrm{H}^0(\check{X}_v,\mathrm{coker}(\mathrm{D}_{\mathrm{Ad}})) & \longrightarrow & 0
\end{array}$$

Since $\mathrm{coker}(\mathrm{D}_{\mathrm{Ad}})$ is unaffected by the center of G, we may replace G by $G^{\mathrm{ad}}\times G^{\mathrm{ad}}$, and since $\mathfrak{g}^{\mathrm{ad}}\simeq\mathfrak{g}^{\mathrm{sc}}$, we have natural map

$$\mathrm{coker}(\mathrm{D}_{\mathrm{Ad}})\longrightarrow Q\coloneqq\mathrm{coker}\Big(\mathrm{Ad}(E)^{\mathrm{sc}}\times\mathrm{Ad}(E)^{\mathrm{sc}}\to(\phi^{*}\Omega^1_{E\times^G[\mathfrak{M}]/[\mathfrak{A}_{\mathfrak{M}}]})^{*}\Big).$$

Since the map on the right-hand side above is generically surjective, Q is finitely generated and torsion, and so

$$\mathrm{H}^0(X,Q) \simeq \prod_v \mathrm{H}^0(\breve{X}_v,Q),$$

where v ranges over the points in $\breve{X}$ that are sent to the singular locus of $[\mathfrak{M}/G\times Z_{\mathfrak{M}}]$. Moreover, one easily sees that the torsion-free quotient of the kernel of $\operatorname{coker}(\mathrm{D}_{\mathrm{Ad}}) \to Q$ is $(\mathrm{Lie}(I^{\mathrm{sm}}_{(E,\phi)})/\mathfrak{z}_G)^*$. Thus when $\mathcal{L}$ is very (G,δ_a)-ample, we have surjective map

$$\mathrm{H}^0(\breve{X},\operatorname{coker}(\mathrm{D}_{\mathrm{Ad}})) \longrightarrow \prod_v \mathrm{H}^0(\breve{X}_v,Q),$$

because $\mathrm{H}^1(\breve{X},(\mathrm{Lie}(\mathfrak{J}^\flat_a)/\mathfrak{z}_G)^*) = 0$. It induces surjective map

$$\mathrm{T}_{(\mathcal{L},E,\phi)}(\mathcal{M}_X/\mathcal{B}_X) \longrightarrow \prod_v \mathrm{T}_{(E_v,\phi_v)}(\mathbb{L}^+_v[\mathfrak{M}/G^{\mathrm{sc}}\times G^{\mathrm{sc}}]/\mathbb{L}^+_v\mathfrak{A}_{\mathfrak{M}}). \qquad (6.10.4)$$

6.10.14 Now we are ready to prove Theorem 6.10.2.

Proof. Given a small extension of Artinian $\bar{k}$-algebras $R \to R' = R/I$ with residue fields $\bar{k}$ and an $m_{R'} \in \mathcal{M}_X(R')$ specializing to m, let $\overline{m}_{R'}$ be its image in

$$\left[\mathbb{L}^+_{\mathcal{B}_X}G\backslash \mathsf{Gr}^{\le -w_0(\mathcal{B}_X)}\right].$$

Suppose $\overline{m}_R$ is an R-lifting of $\overline{m}_{R'}$, then the local obstruction of deforming $\overline{m}_{R'}$ vanishes. By Proposition 6.10.12, the global obstruction of deforming $m_{R'}$ also vanishes. Here we use the fact that since G^{sc} is smooth, the map $[\mathfrak{M}/\operatorname{Ad}(G^{\mathrm{sc}})] \to [\mathfrak{M}/G^{\mathrm{sc}}\times G^{\mathrm{sc}}]$ is smooth, hence so is the map between induced arc-stacks. Then, by surjectivity of map (6.10.4), there is a lifting of $m_{R'}$ to R lying over $\overline{m}_R$, thus we have the desired theorem. ∎

Remark 6.10.15. Note that Theorem 6.10.2 can still be vastly improved due to its conclusion being unnecessarily strong. Indeed, since we only care about the model of singularity, if the boundary divisor is supported at point $\bar{v}$ while the discriminant divisor is not, then there is no need to include the local affine Schubert variety at $\bar{v}$ in the target of the map ev since at $\bar{v}$ the mHiggs-field automatically lands in the big cell, which has no singularity. Thus by allowing such $\bar{v}$ to move the cohomological condition in Theorem 6.10.2 can be weakened.

6.11 The Case of Endoscopic Groups

Let $(\kappa, \vartheta_\kappa)$ be an endoscopic datum of G on X, and H is the endoscopic group. By § 2.5, there is a canonical monoid $\mathfrak{M}_H \in \mathcal{FM}(H^{\mathrm{sc}})$ associated with $\mathfrak{M}$ and a canonical map $\nu_H\colon \mathfrak{C}_{\mathfrak{M},H} \to \mathfrak{C}_{\mathfrak{M}}$. Let $Z_{\mathfrak{M},H} \subset \mathfrak{M}_H^\times$ be the center, and we have mH-fibration

$$h_{H,X}\colon \mathcal{M}_{H,X} \longrightarrow \mathcal{A}_{H,X}$$

associated with $\mathfrak{M}_H$. However, there is no direct relation between h_X and $h_{H,X}$, because $Z_{\mathfrak{M},H}$ does not map into $Z_{\mathfrak{M}}$. Instead, we need to replace $Z_{\mathfrak{M},H}$ with the preimage $Z_{\mathfrak{M}}^\kappa$ of $Z_{\mathfrak{M}}$. Let

$$h_{H,X}^\kappa\colon \mathcal{M}_{H,X}^\kappa \longrightarrow \mathcal{A}_{H,X}^\kappa$$

be the pullback of $h_{H,X}$ through $\mathrm{Bun}_{Z_{\mathfrak{M}}^\kappa} \to \mathrm{Bun}_{Z_{\mathfrak{M},H}}$. Let $\mathcal{B}_{H,X}^\kappa$ be the same pullback of $\mathcal{B}_{H,X}$. Every result in this chapter about $h_{H,X}$ applies to $h_{H,X}^\kappa$ due to it being defined via pullback from $\mathrm{Bun}_{Z_{\mathfrak{M},H}}$.

The canonical map $\nu_H\colon \mathfrak{C}_{\mathfrak{M},H} \to \mathfrak{C}_{\mathfrak{M}}$ induces commutative diagram

$$\begin{array}{ccc} [\mathfrak{C}_{\mathfrak{M},H}/Z_{\mathfrak{M}}^\kappa] & \longrightarrow & [\mathfrak{C}_{\mathfrak{M}}/Z_{\mathfrak{M}}] \\ \downarrow & & \downarrow \\ [\mathfrak{A}_{\mathfrak{M},H}/Z_{\mathfrak{M}}^\kappa] & \longrightarrow & [\mathfrak{A}_{\mathfrak{M}}/Z_{\mathfrak{M}}] \end{array}$$

hence further induces diagram

$$\begin{array}{ccc} \nu_{\mathcal{A}}\colon \mathcal{A}^{\kappa}_{H,X} & \longrightarrow & \mathcal{A}_X \\ \downarrow & & \downarrow \\ \nu_{\mathcal{B}}\colon \mathcal{B}^{\kappa}_{H,X} & \longrightarrow & \mathcal{B}_X \end{array}$$

The following lemma can be easily deduced from Zariski's Main Theorem.

Lemma 6.11.1 ([Ngô06, Lemme 7.3])**.** *Let S be a normal integral and separated k-scheme. Let $\nu\colon \tilde{V} \to V$ be a finite morphism of S-schemes and $h\colon S \to V$ a section of $V \to S$. Suppose there is an open dense subset $S' \subset S$ over which h lifts to a section $h'\colon S' \to \tilde{V} \times_S S'$. Then h' extends uniquely to S such that $\nu \circ h' = h$.*

Proposition 6.11.2. *The map*

$$\nu^{\heartsuit}_{\mathcal{A}}\colon \mathcal{A}^{\kappa,G\text{-}\heartsuit}_{H,X} \longrightarrow \mathcal{A}^{\heartsuit}_X$$

is finite.

Proof. The $\pi_0(\kappa)$-torsor $\vartheta_\kappa\colon X_\kappa \to X$ induces commutative diagram

$$\begin{array}{ccccc} \bar{\mathbf{T}}_{\mathbf{M},\mathbf{H}} \times X_\kappa & \longrightarrow & \bar{\mathbf{T}}_{\mathbf{M}} \times X_\kappa & = & \bar{\mathbf{T}}_{\mathbf{M}} \times X_\kappa \\ \downarrow & & \downarrow & & \downarrow \\ \mathfrak{C}_{\mathfrak{M},H} & \longrightarrow & \mathfrak{C}'_{\mathfrak{M},H} & \longrightarrow & \mathfrak{C}_{\mathfrak{M}} \end{array}$$

where the left and middle vertical maps are obtained by taking $\mathbf{W}_{\mathbf{H}} \rtimes \pi_0(\kappa)$-quotient, and the right one is by taking $\mathbf{W} \rtimes \pi_0(\kappa)$-quotient, and the two groups are connected by Lemma 2.5.5.

Let $a \in \mathcal{A}^{\heartsuit}_X(\bar{k})$, and let $U_a \subset \check{X}$ be the open subset whose image under a is contained in $\left[\mathfrak{C}^{\times,\mathrm{rs}}_{\mathfrak{M}}/Z_{\mathfrak{M}}\right]$. We also fix a geometric point $\infty \in U_a(\bar{k})$. Over U_a, the vertical maps in the above commutative diagram are respective torsors of groups $\mathbf{W}_{\mathbf{H}} \rtimes \pi_0(\kappa)$ and

$\mathbf{W} \rtimes \pi_0(\kappa)$. Then the restriction of a to U_a lifts to a section $U_a \to \left[\mathfrak{C}'_{\mathfrak{M},H}/Z_{\mathfrak{M}}\right] \times_X U_a$ if and only if the image of the monodromy

$$\vartheta_a^\bullet \colon \pi_1(U_a, \infty) \longrightarrow \mathbf{W} \rtimes \pi_0(\kappa)$$

is conjugate to subgroup $\mathbf{W_H} \rtimes \pi_0(\kappa)$. Since the number of such subgroups is finite, so is the number of such liftings. Also over U_a, the map $\left[\mathfrak{C}_{\mathfrak{M},H}/Z_{\mathfrak{M}}^\kappa\right] \to \left[\mathfrak{C}'_{\mathfrak{M},H}/Z_{\mathfrak{M}}\right]$ is an isomorphism. At each $\bar{v} \in \check{X} - U_a$, there are only finitely many ways to extend a $\check{X}_{\bar{v}}^\bullet$-point of $\left[\mathfrak{C}_{\mathfrak{M},H}/Z_{\mathfrak{M}}^\kappa\right]$ to a $\check{X}_{\bar{v}}$-point lying over a fixed $\check{X}_{\bar{v}}$-point in $\left[\mathfrak{C}'_{\mathfrak{M},H}/Z_{\mathfrak{M}}\right]$, according to Lemma 2.5.17 (note that over $\check{X}_{\bar{v}}$ the monoid $\mathfrak{M}$ is necessarily split). Thus $\nu_{\mathcal{A}}^\heartsuit$ is quasi-finite.

We then show that $\nu_{\mathcal{A}}^\heartsuit$ is proper using valuative criteria. Let R be a discrete valuation ring and $S = \operatorname{Spec} R$. Let $\eta \in S$ be the generic point. Now let a be an S-point instead of a $\bar{k}$-point of $\mathcal{A}_X^\heartsuit$, and $U_a \subset X \times S$ is defined as above. Since U_a is normal and integral and the map

$$\left[\mathfrak{C}_{\mathfrak{M},H}^{\times,G\text{-rs}}/Z_{\mathfrak{M}}^\kappa\right] \longrightarrow \left[\mathfrak{C}_{\mathfrak{M}}^{\times,\mathrm{rs}}/Z_{\mathfrak{M}}\right]$$

is finite (because $T_{\mathfrak{M},H}/Z_{\mathfrak{M}}^\kappa \to T_{\mathfrak{M}}/Z_{\mathfrak{M}}$ is an isomorphism), a lifting of $a|_{U_a}$ over $U_a \times_S \eta$ can be extended to a lifting over U_a by Lemma 6.11.1. Then since the complement of $U_a \cup (X \times \eta)$ has codimension 2 in normal scheme $X \times S$, any $Z_{\mathfrak{M}}^\kappa$-torsor on $U_a \cup (X \times \eta)$ can be uniquely extended over $X \times S$, and since $\mathfrak{C}_{\mathfrak{M},H}$ is affine over $X \times S$, we can uniquely lift a over $X \times S$. This proves properness. ∎

6.11.3 Suppose $b \in \mathcal{B}_{\gg}(\bar{k})$ induces a boundary divisor written as

$$\lambda_b = \sum_{\bar{v} \in X(\bar{k})} \lambda_{\bar{v}} \cdot \bar{v}.$$

Similarly $b_H \in \mathcal{B}^{\kappa}_{H,\gg}(\bar{k})$ induces very H-ample boundary divisor

$$\lambda_{H,b_H} = \sum_{\bar{v} \in X(\bar{k})} \lambda_{H,\bar{v}} \cdot \bar{v}.$$

Suppose b is the image of b_H. For each $\bar{v} \in X(\bar{k})$, $-w_{H,0}(\lambda_{H,\bar{v}})$ is one of the $\check{H}^{\mathrm{sc}}_{\bar{v}}$-highest weights appearing in the decomposition of irreducible $\check{G}^{\mathrm{sc}}_{\bar{v}}$-representation with highest weight $-w_0(\lambda_{\bar{v}})$ into irreducible $\check{H}^{\mathrm{sc}}_{\bar{v}}$-representations. Let

$$r^G_H(b_H) = \langle \rho, \lambda_b \rangle - \langle \rho_H, \lambda_{H,b_H} \rangle.$$

This way we obtain a locally constant function

$$r^G_H : \mathcal{B}^{\kappa}_{H,X} \longrightarrow \mathbb{N}$$

and we use the same notation for its pullback to $\mathcal{A}^{\kappa}_{H,X}$, $\mathcal{M}^{\kappa}_{H,X}$, etc.

By (6.8.1), we have

$$\dim \mathcal{A}_b - \dim \mathcal{A}^{\kappa}_{H,b_H} = r^G_H(b_H) - (r - r_H)(g_X - 1), \tag{6.11.1}$$

where r_H is the semisimple rank of H. So the image of $\mathcal{A}^{\kappa,G\text{-}\heartsuit}_{H,b_H}$ in $\mathcal{A}^{\heartsuit}_b$ is a closed subscheme of codimension $r^G_H(b_H) - (r - r_H)(g_X - 1)$. In case b_H is not very H-ample (but b is still very G-ample), we still have inequality according to (6.8.2):

$$\dim \mathcal{A}_b - \dim \mathcal{A}^{\kappa}_{H,b_H} \geq r^G_H(b_H) - r(g_X - 1) - r_H. \tag{6.11.2}$$

Replacing endoscopic groups by Levi subgroups, then 6.11.2 implies the following:

Corollary 6.11.4. *Suppose Z_G contains no split torus. The codimension of the complement $\mathcal{A}_b - \mathcal{A}^{\natural}_b$ of the anisotropic locus in $\mathcal{A}_b$ goes to ∞ as $\langle \rho, \lambda_b \rangle \to \infty$.*

6.11.5 Suppose $\mathfrak{M} \in \mathcal{FM}_0(G^{\mathrm{sc}})$ and $\mathfrak{A}_{\mathfrak{M}}$ is of standard type. If $\mathcal{L}_\kappa$ is a $Z_{\mathfrak{M}}^\kappa$-torsor, then it induces a $Z_{\mathfrak{M},H}$-torsor $\mathcal{L}_H$. We say $\mathcal{L}_\kappa$ is very H-ample if $\mathcal{L}_H$ is. It also induces a $Z_{\mathfrak{M}}$-torsor which we denote by $\mathcal{L}$. However, even if $\mathcal{L}$ is very G-ample, it is not immediately clear that $\mathcal{L}_\kappa$ is very H-ample, making a precise dimension formula a little tricky. For the time being we settle at providing a precise formula with some assumptions and an estimate in general.

The assumption is as follows: suppose that the vector bundle $\mathfrak{A}_{\mathfrak{M},H,\mathcal{L}'_\kappa}$ has trivial first cohomology for all $\mathcal{L}'_\kappa$ in the connected component of $\mathrm{Bun}_{Z_{\mathfrak{M}}^\kappa}$ containing $\mathcal{L}_\kappa$, then the map

$$\mathcal{B}_{H,X}^\kappa \longrightarrow \mathrm{Bun}_{Z_{\mathfrak{M}}^\kappa}$$

is surjective over that component with relative dimension $\deg(b)$. As a result, we have

$$\dim_{b_H} \mathcal{B}_{H,X}^\kappa = \deg(b) + (\mathrm{rk}\, Z_{\mathfrak{M}}^\kappa - \mathrm{rk}\, Z_{\mathfrak{M},H})(g_X - 1).$$

Assuming very H-ampleness as well, then $\mathcal{A}_{H,X}^\kappa \to \mathcal{B}_{H,X}^\kappa$ is surjective, and we have by (6.8.1)

$$\dim_{a_H} \mathcal{A}_{H,X}^\kappa = \deg(b) + (\mathrm{rk}\, Z_{\mathfrak{M}}^\kappa - \mathrm{rk}\, Z_{\mathfrak{M},H})(g_X - 1) + \langle \rho_H, \lambda_{H,b_H} \rangle - r_H(g_X - 1),$$

where we recall that $\deg(b) = \dim_b \mathcal{B}_X$ by definition. Since $Z_{\mathfrak{M}}^\kappa$ is none other than the preimage of $Z_{\mathfrak{M}}$, we have

$$\mathrm{rk}(Z_{\mathfrak{M},H}) - \mathrm{rk}(Z_{\mathfrak{M}}^\kappa) = r - r_H.$$

Thus we obtain

$$\dim_{a_H} \mathcal{A}^{\kappa}_{H,X} = \deg(b) + \langle \rho_H, \lambda_{H,b_H} \rangle - r(g_X - 1).$$

If $a \in \mathcal{A}_X(\bar{k})$ is the image of a_H and is very G-ample, then

$$\dim_a \mathcal{A}_X - \dim_{a_H} \mathcal{A}^{\kappa}_{H,X} = r^G_H(a_H). \tag{6.11.3}$$

Similarly, if the cohomological assumptions on $\mathcal{L}_\kappa$ and a_H are not true, we still have estimate

$$\dim_a \mathcal{A}_X - \dim_{a_H} \mathcal{A}^{\kappa}_{H,X} \geq r^G_H(a_H) - r g_X. \tag{6.11.4}$$

6.11.6 Due to the rather unsatisfactory estimate we temporarily settled with, we want to emphasize a special case where we do get a precise formula: when $r = r_H$. In this case, $Z^{\kappa}_{\mathfrak{M}}$ and $Z_{\mathfrak{M},H}$ have the same rank. This means that the map $\mathcal{B}^{\kappa}_{H,X} \to \mathcal{B}_{H,X}$ is open and finite, hence so is the map $\mathcal{B}^{\kappa}_{H,X} \to \mathcal{B}_X$. Moreover, suppose $b_H \in \mathcal{B}^{\kappa}_{H,X}$ is very H-ample, then since we have tangent map

$$\mathfrak{C}_{\mathfrak{M},H,b_H} \simeq \mathrm{T}_{a_H} \mathfrak{C}_{\mathfrak{M},H,b_H} \longrightarrow \mathrm{T}_a \mathfrak{C}_{\mathfrak{M},b} \simeq \mathfrak{C}_{\mathfrak{M},b},$$

which is an injective map between vector bundles of rank $r = r_H$, we automatically have

$$\mathrm{H}^1(\check{X}, \mathfrak{C}_{\mathfrak{M},b}) = 0,$$

so that (6.8.1) still holds for b. Therefore in this case we have the following:

Proposition 6.11.7. *Let $a_H \in \mathcal{A}^{\kappa}_{H,X}$ and $a = \nu_{\mathcal{A}}(a_H)$. If $r = r_H$ and a_H is very H-ample, then* (6.11.3) *holds.*

Remark 6.11.8. We expect the technical difficulty in estimating $\dim_{a_H} \mathcal{A}^{\kappa}_{H,X}$ is not very essential and can be solved with more in-depth utilization of deformation theory. In fact, in view of Proposition 6.8.3 and Theorem 6.10.2, we expect that it will be the same problem mentioned in Remark 6.10.15 and can be solved simultaneously.

6.11.9 Although the mH-fibrations of G and H do not have direct connection, there is a canonical map between their Picard stacks. Let $a_H \in \mathcal{A}^{\kappa,G\text{-}\heartsuit}_{H,X}(\bar{k})$ and its image is $a \in \mathcal{A}^{\heartsuit}_X(\bar{k})$. Recall that the map $\nu_H\colon \mathfrak{C}_{\mathfrak{M},H} \to \mathfrak{C}_{\mathfrak{M}}$ induces a homomorphism $\nu_H^*\mathfrak{J}_{\mathfrak{M}} \to \mathfrak{J}_{\mathfrak{M},H}$, therefore we have a homomorphism of commutative group schemes $\mathfrak{J}_a \to \mathfrak{J}_{H,a_H}$ over $\check{X}$. Since $a \in \mathcal{A}^{\heartsuit}_X$, this is generically an isomorphism hence also injective. Therefore we have a surjective map

$$\mathcal{P}_a \longrightarrow \mathcal{P}_{H,a_H}.$$

Its kernel is affine group $\mathrm{H}^0(\check{X}, \mathfrak{J}_{H,a_H}/\mathfrak{J}_a)$, which we denote by $\mathcal{R}^G_{H,a_H}$. Using Corollary 6.8.4, we have

$$\dim \mathcal{R}^G_{H,a_H} = r^G_H(a_H).$$

In particular, it is independent of a_H. Let $\mathfrak{J}^\flat_{H,a_H}$ be the Néron model of $\mathfrak{J}_{H,a_H}$, then the composition $\mathfrak{J}_a \to \mathfrak{J}^\flat_{H,a_H}$ is generically an isomorphism, so it is also the Néron model of $\mathfrak{J}_a$. Thus we have exact sequence

$$1 \longrightarrow \mathcal{R}^G_{H,a_H} \longrightarrow \mathcal{R}_a \longrightarrow \mathcal{R}_{H,a_H} \longrightarrow 1,$$

where $\mathcal{R}_a = \ker(\mathcal{P}_a \to \mathcal{P}^\flat_a)$ and $\mathcal{R}_{H,a_H} = \ker(\mathcal{P}_{H,a_H} \to \mathcal{P}^\flat_{H,a_H})$, and so

$$\delta_a - \delta_{H,a_H} = r^G_H(a_H), \tag{6.11.5}$$

where $\delta_a = \dim \mathcal{R}_a$ and $\delta_{H,a_H} = \dim \mathcal{R}_{H,a_H}$.

Remark 6.11.10. Comparing (6.11.1) with (6.11.5), it looks like so-called δ-regularity (the inequalities in Proposition 7.4.5, see also § 9.1) is violated. However, it is not a contradiction, but merely shows that the dependence of N on δ in Proposition 7.4.5 cannot be eliminated. On the other hand, according to (6.11.3), there is no such "violation" over the whole $\mathcal{A}_X^{\heartsuit}$ (not fixing b) provided certain cohomological conditions are satisfied (although it is not always so).

6.11.11 Similar to the technical difficulty in estimating the dimension of $\mathcal{A}_{H,X}^{\kappa}$, the local model of singularity Theorem 6.10.2 also has some restriction due to possibly insufficient ampleness. For example, suppose a_H is very (H, δ_{H,a_H})-ample, then Theorem 6.10.2 holds for any mHiggs-field over a_H, due to that

$$\mathrm{H}^1(\check{X}, (\mathrm{Lie}(\mathfrak{J}_{H,a_H}^{\flat})/\mathfrak{z}_H)^*) = 0.$$

However, the (sufficient) condition for a is

$$\mathrm{H}^1(\check{X}, (\mathrm{Lie}(\mathfrak{J}_a^{\flat})/\mathfrak{z}_G)^*) = 0,$$

which may not be true because $\mathfrak{z}_H$ may have larger rank than $\mathfrak{z}_G$. Obviously this difficulty vanishes when $r = r_H$ so that $\mathfrak{z}_H$ and $\mathfrak{z}_G$ are canonically isomorphic (both embeds into $\mathfrak{T}$). Thus we have the following result:

Proposition 6.11.12. *Let $a_H \in \mathcal{A}_{H,X}^{\kappa}$ and $a = \nu_{\mathcal{A}}(a_H)$. If $r = r_H$ and a_H is very (H, δ_{H,a_H})-ample, then Theorem 6.10.2 holds for both a and a_H.*

Remark 6.11.13. We expect this restriction requiring $r = r_H$ is merely technical and can be resolved due to reasons laid out in Remarks 6.10.15 and 6.11.8.

CHAPTER 7

STRATIFICATIONS

In this chapter we study two important stratifications on the Hitchin base $\mathcal{A}_X$ in the same fashion as in [Ngô10, § 5]. One of them is associated with δ-invariants δ_a, which is loosely speaking the dimension of the "affine part" of $\mathcal{P}_a$. The other is associated with $\pi_0(\mathcal{P}_a)$, which is essentially the global counterpart of the group of endoscopic characters κ. The endoscopic monoid continues to play a key role here, and the study of δ-strata will be more refined compared to the Lie algebra counterpart due to the effect of boundary divisors. The final section introduces inductive strata, which will become important in stating the support theorem in § 9.9. The outline of a lot of the proofs will be similar to those in [Ngô10] (if they are counterparts), but there can be some small and occasional technical challenges.

7.1 Simultaneous Normalization of Cameral Curves

7.1.1 The existence of aforementioned stratifications depends on a constructibility result of certain subsets of $\mathcal{A}_X^\heartsuit$, whose proof relies on the theory of simultaneous normalization of a family of curves.

Definition 7.1.2. Let $f\colon Y \to S$ be a flat and proper map with reduced 1-dimensional fibers. A *simultaneous normalization* of $Y \to S$ (or just Y if base S is clear from the context) is a proper birational map $\xi\colon Y^\flat \to Y$ such that

(1) There exists open subset $U \subset Y$ over which ξ is an isomorphism and $f(U) = S$.

(2) The composition $f^\flat = f \circ \xi\colon Y^\flat \to S$ is smooth and proper.

Lemma 7.1.3 ([Ngô10, Proposition 5.1.2])**.** *Let $\xi\colon Y^\flat \to Y$ be a simultaneous normalization of $f\colon Y \to S$, then*

(1) $f_*(\xi_*\mathcal{O}_{Y^\flat}/\mathcal{O}_Y)$ *is a locally free* $\mathcal{O}_S$*-sheaf of finite type.*

(2) There is a locally constant étale sheaf $\pi_0(Y^\flat)$ *on* S *whose fiber at a geometric point* $s \in S$ *is the set of connected components of* $Y_s^\flat$*.*

Consider (pseudo-)functor $\mathcal{A}_X^\flat$ whose S-points are triples $(a, \tilde{X}_a^\flat, \xi)$ where $a \in \mathcal{A}_X^\heartsuit(S)$, $\tilde{X}_a^\flat$ is a smooth and proper S-family of curves together with a map $\tilde{X}_a^\flat \to X \times S$ on which W acts, and $\xi\colon \tilde{X}_a^\flat \to \tilde{X}_a$ is a W-equivariant simultaneous normalization. The forgetful functor $\mathcal{A}_X^\flat \to \mathcal{A}_X^\heartsuit$ induces a bijection on $\bar{k}$-points, because for any $a \in \mathcal{A}_X^\heartsuit(\bar{k})$ the normalization of $\tilde{X}_a$ is unique.

The functor $\mathcal{A}_X^\flat$ has another description. Recall that for $a \in \mathcal{A}_X^\heartsuit(S)$, its image $\mathcal{L} \in \mathrm{Bun}_{Z_\mathfrak{M}}(S)$ corresponds to a $Z_\mathfrak{M}$ torsor $\mathcal{L}$ on $X \times S$, which induces map $\pi_\mathcal{L}\colon \mathfrak{T}_{\mathfrak{M},\mathcal{L}} \to \mathfrak{C}_{\mathfrak{M},\mathcal{L}}$. Let $\mathcal{A}_X^{\flat\prime}$ be the functor of triples $(\mathcal{L}, \tilde{X}_a^\flat, \gamma)$ where $\mathcal{L} \in \mathrm{Bun}_{Z_\mathfrak{M}}(S)$, $\tilde{X}_a^\flat$ is a smooth and proper family of curves over S with a finite flat map $\pi_a^\flat\colon \tilde{X}_a^\flat \to X \times S$ on which W acts, and $\gamma\colon \tilde{X}_a^\flat \to \mathfrak{T}_{\mathfrak{M},\mathcal{L}}$ is a W-equivariant map. In addition, for any geometric point $s \in S$ the map $\pi_{a,s}^\flat$ is generically a W-torsor, and the preimage of $\mathfrak{T}_{\mathcal{L},s}^{\mathrm{rs}}$ is dense in $\tilde{X}_{a,s}^\flat$. Given $(a, \tilde{X}_a^\flat, \xi) \in \mathcal{A}_X^\flat(S)$, one may define γ as the composition of ξ and the embedding $\tilde{X}_a \to \mathfrak{T}_{\mathfrak{M},\mathcal{L}}$. Thus we have a natural map $\mathcal{A}_X^\flat \to \mathcal{A}_X^{\flat\prime}$.

Lemma 7.1.4. *The natural map* $\mathcal{A}_X^\flat \to \mathcal{A}_X^{\flat\prime}$ *is an isomorphism.*

Proof. It suffices to define the inverse map. Let $(\mathcal{L}, \tilde{X}_a^\flat, \gamma) \in \mathcal{A}_X^{\flat\prime}(S)$, then we claim

$$\pi_{a*}^\flat(\mathcal{O}_{\tilde{X}_a^\flat})^W = \mathcal{O}_{X\times S}.$$

Indeed, the left-hand side is a finite $\mathcal{O}_{X\times S}$-algebra containing $\mathcal{O}_{X\times S}$ and over any geometric point $s \in S$ they are generically isomorphic on $X \times \{s\}$. Since X is normal, it must be an isomorphism.

Since γ is W-equivariant, it induces a map $a\colon X\times S \to \mathfrak{C}_{\mathfrak{M},\mathcal{L}}$ by taking W-GIT-quotients. Let $\tilde{X}_a$ be the corresponding cameral curve, then γ factors through $\tilde{X}_a \to \mathfrak{T}_{\mathfrak{M},\mathcal{L}}$ hence induces map

$$\xi\colon \tilde{X}_a^\flat \longrightarrow \tilde{X}_a.$$

Clearly over any geometric point $s \in S$, ξ_s is a normalization map. Thus we obtain a point $(a, \tilde{X}_a^\flat, \xi) \in \mathcal{A}_X^\flat(S)$. One can verify it is the inverse map as desired. ∎

Consider $(a, \tilde{X}_a^\flat, \xi) \in \mathcal{A}_X^\flat(\bar{k})$, its automorphism group is a subgroup of $\Gamma(\check{X}, Z^{\mathrm{sc}})$, so in general it cannot be representable by a scheme. However, the étale group Z^{sc} is the only "obstruction" of $\mathcal{A}_X^\flat$ being a sheaf. Therefore, it is reasonable to expect that it is a scheme relative to $\mathcal{B}_X$. In any case, $\mathcal{B}_X$ is still a Deligne-Mumford stack, even a $\underline{\mathrm{Hom}}_X(X, Z^{\mathrm{sc}})$-gerbe, thus most topological and representability result about schemes and sheaves still apply.

Unfortunately, it is not straightforward to prove the representability result we want, but it is enough for our purpose to prove the following slightly weaker result.

Proposition 7.1.5. *There exists two $\mathcal{B}_X$-schemes $(\mathcal{A}_X^\flat)^{\mathrm{red}} \to \mathcal{B}_X$ and $\mathcal{A}_X^\star \to \mathcal{B}_X$ such that we have fully faithful inclusions*

$$(\mathcal{A}_X^\flat)^{\mathrm{red}} \subset \mathcal{A}_X^\flat \subset \mathcal{A}_X^\star,$$

where the first inclusion induces bijection on $\bar{k}$-points, and the second is topologically a closed embedding.

Proof. We first consider functor $\mathscr{H}$ whose S-points are the isomorphism classes of the following data: a smooth and projective S-curve $\tilde{X}_a^\flat$ together with a map $\tilde{X}_a^\flat \to X \times S$ on which W acts, such that over any geometric point $s \in S$ it is generically a W-torsor.

Suppose for the moment that $\mathscr{H}$ is representable by a quasi-projective k-scheme, then we have a universal family of smooth and proper curves

$$\tilde{X}^{\flat}_{\mathscr{H}} \longrightarrow \mathscr{H}$$

together with a W-equivariant map $\tilde{X}^{\flat}_{\mathscr{H}} \to X \times \mathscr{H}$.

This induces a $\mathfrak{T}_{\mathfrak{M}}$-bundle $\mathfrak{T}_{\mathfrak{M},\mathscr{H}}$ over $\tilde{X}^{\flat}_{\mathscr{H}} \times \mathrm{Bun}_{Z_{\mathfrak{M}}}$, and the fibers of forgetful map $\mathcal{A}^{\flat}_X \to \mathscr{H} \times \mathrm{Bun}_{Z_{\mathfrak{M}}}$ is just the W-fixed point of the sections of $\mathfrak{T}_{\mathfrak{M},\mathscr{H}}$ over the curve $\tilde{X}^{\flat}_{\mathscr{H}}$. Using the representation-theoretic description of $\mathrm{Env}(\mathbf{G}^{\mathrm{sc}})$, we know that we may embed $\mathfrak{T}_{\mathfrak{M}}$ into a vector bundle $\mathfrak{v}$ with W-action as a closed subbundle. Moreover, $\mathfrak{v}$ can be so chosen that it maps to $\mathfrak{A}_{\mathfrak{M}}$ compatible with $\mathfrak{T}_{\mathfrak{M}} \to \mathfrak{A}_{\mathfrak{M}}$ and the W-action. One can pull it back to get a W-vector bundle $\mathfrak{v}_{\mathscr{H}}$.

We define a functor $\mathcal{A}^{\star}_X$ same as $\mathcal{A}^{\flat\prime}_X$ except that γ is a W-equivariant map to $\mathfrak{v}$ instead of $\mathfrak{T}_{\mathfrak{M}}$. By the definition of $\mathfrak{v}$, the map $\mathcal{A}^{\star}_X \to \mathrm{Bun}_{Z_{\mathfrak{M}}}$ factors through $\mathcal{B}_X$, and it is clear that $\mathcal{A}^{\star}_X$ is a sheaf relative to $\mathcal{B}_X$. It is also clear that $\mathcal{A}^{\star}_X$ is the W-fixed points of an open subset of a vector bundle on $\mathscr{H} \times \mathrm{Bun}_{Z_{\mathfrak{M}}}$, hence it is representable by Deligne-Mumford stack which is also a $\mathcal{B}_X$-scheme. Moreover, it contains $\mathcal{A}^{\flat}_X$ as a fully faithful subfunctor. Consider Cartesian diagram

$$\begin{array}{ccc} \tilde{X}' & \longrightarrow & \mathfrak{T}_{\mathscr{H}} \\ \downarrow & & \downarrow \\ \tilde{X}^{\flat}_{\mathcal{A}^{\star}_X} & \xrightarrow{\gamma} & \mathfrak{v}_{\mathscr{H}} \end{array}$$

where $\tilde{X}^{\flat}_{\mathcal{A}^{\star}_X}$ is the universal curve on $\mathcal{A}^{\star}_X$ and the bottom horizontal map is the universal map γ over $\mathcal{A}^{\star}_X$. The vertical maps in the above square are closed embeddings. Since $\tilde{X}_{\mathcal{A}^{\star}_X} \to \mathcal{A}^{\star}_X$ is proper, the image of $\tilde{X}'$ in $\mathcal{A}^{\star}_X$ is closed. By upper-semicontinuity of fiber dimensions for proper maps, the locus in $\mathcal{A}^{\star}_X$ where the fiber in $\tilde{X}'$ has dimension 1 is also a closed subset. Let $(\mathcal{A}^{\flat}_X)^{\mathrm{red}}$ be the reduced substack of this closed subset.

We want to show that $\tilde{X}' \to \tilde{X}^{\flat}_{\mathcal{A}^{\star}_X}$ is an isomorphism over $(\mathcal{A}^{\flat}_X)^{\mathrm{red}}$. We already know that their fibers have the same dimension and are proper and reduced (in fact regular), but $\tilde{X}_a^\flat$ is not necessarily irreducible. Nevertheless, since γ is W-equivariant and since $\pi_a^\flat$ is generically a W-torsor, once a fiber of $\tilde{X}'$ contains one irreducible component of $\tilde{X}_a^\flat$, it has to contain all of them. This means that $(\mathcal{A}_X^\flat)^{\mathrm{red}}$ is a subfunctor of $\mathcal{A}_X^\flat$. It is also clear that $(\mathcal{A}_X^\flat)^{\mathrm{red}}$ has the same $\bar{k}$-points as $\mathcal{A}_X^\flat$. Thus it remains to prove that $\mathscr{H}$ is representable a quasi-projective k-scheme. We leave it to a separate proposition below. ∎

Proposition 7.1.6. *The moduli functor $\mathscr{H}$ as in Proposition 7.1.5 is representable by a quasi-projective k-scheme.*

Proof. As commented in [Ngô10, § 5.2], the proof is similar to the representability of Hurwitz schemes. We include a proof for completeness, as it is not in *loc. cit.*

Let S be a k-scheme and let $f\colon \tilde{X}_a^\flat \to X \times S$ be an S-point of $\mathscr{H}$. For simplicity we denote $\tilde{X}_a^\flat$ by Y and $X \times S$ by X_S. The relative cotangent complex Ω_{Y/X_S} is the coherent sheaf $\Omega_{Y/S}/f^*\Omega_{X_S/S}$ since Y and X_S are smooth curves over S and f is finite flat. It is a coherent sheaf of finite length, whose induced divisor on Y is the ramification divisor D_Y of f. Since D_Y is stable under W, it descends to a divisor B_Y on X_S, finite and flat over S. Therefore we obtain a map

$$\begin{aligned} f_B\colon \mathscr{H} &\longrightarrow H_X \coloneqq \coprod_n \mathrm{Hilb}^n X \\ Y &\longmapsto B_Y. \end{aligned}$$

It then suffices to show that the map f_B is representable and étale, and we shall use the same criteria in [Ful69, Theorem 6.5], which is a slightly modified version of a result due to Grothendieck reported in [Mur95]. Explicitly, we need to show the followings:

(1) $\mathscr{H}$ is an fpqc sheaf over H_X.

(2) $\mathscr{H}$ commutes with filtered inductive limit of rings over H_X.

(3) If A is a complete Noetherian local ring over H_X and $\mathfrak{m} \subset A$ is the maximal ideal, then

$$\mathscr{H}(A) \longrightarrow \varprojlim_i \mathscr{H}(A/\mathfrak{m}^i)$$

is bijective. In other words, formal deformation of f_B over H_X can be uniquely promoted to a local deformation.

(4) If R is an Artinian ring over H_X and $I \subset R$ is a nilpotent ideal, then $\mathscr{H}(R) \to \mathscr{H}(R/I)$ is bijective. In other words, f_B is formally étale.

(5) f_B satisfies the uniqueness part of valuative criteria for all complete discrete valuation rings.

Property (1) is immediate from definition.

Property (2): let A_i be a filtered inductive system of rings over H_X, and A is the limit of A_i. Then $Y_{A_i} \to \operatorname{Spec} A_i$ is finitely presented, quasi-compact and quasi-separated. Moreover, H_X is locally quasi-compact and quasi-separated. By [Gro66, Théorèm 8.8.2], the natural map

$$\varinjlim_i \mathscr{H}(A_i) \longrightarrow \mathscr{H}(A)$$

is bijective.

Properties (3): is due to the fact that formal deformation of projective curves can always be uniquely algebraicized ([Gro61, Théorèm 5.4.1]).

Properties (4) is proved using deformation theory. Let $R' = R/I$ and $f' : Y' \to X_{R'}$. Without loss of generality, we may assume that R is a small extension of R'. Let f_0 be the fiber of f' over the residue field k_R of R'. The obstruction of flatly deforming map

f' (without regarding W-action) lies in the group

$$\mathrm{Ext}^2_{\mathcal{O}_{Y'}}(\Omega_{Y'/X_{R'}}, I)$$

which vanishes by Serre duality. Therefore all flat deformations of f' to R is a torsor under vector space

$$\begin{aligned} \mathrm{T}_I &\coloneqq \mathrm{Ext}^1_{\mathcal{O}_{Y'}}(\Omega_{Y'/X_{R'}}, I) \\ &\cong \mathrm{Ext}^1_{\mathcal{O}_{Y'}}(\Omega_{Y'/X_{R'}}, \mathcal{O}_{Y'}) \otimes I. \end{aligned}$$

The group $W_X = \Gamma(X, W)$ acts canonically on the set of deformations by composition, in other words, $w \in W_X$ sends a deformation $\iota\colon Y' \to Y$ to $\iota \circ w^{-1}$. The tangent space T_I also admits a canonical W_X-action, thus the set of formations induces a class in $\mathrm{H}^1(W_X, \mathrm{T}_I)$. Since $\mathrm{char}(k)$ does not divide the order of $\mathbf{W}$, the last cohomology group is trivial, being a vector space annihilated by the order of W_X. Thus we identify the set of deformations with T_I as W_X-sets. Let $f \in \mathrm{T}_I^{W_X}$ be a W_X-invariant deformation, the obstruction of extending W_X-action to f lies in the group

$$\mathrm{H}^2(W_X, \mathrm{Aut}_{f'}(f)) = \mathrm{H}^2(W_X, \mathrm{Aut}_{f_0}(f_0[\epsilon]) \otimes I),$$

where $f_0[\epsilon]$ is the trivial extension of f_0 to $k_R[\epsilon]/\epsilon^2$. This obstruction group also vanishes since it takes value in a vector space.

These construction can also be done locally over X, hence we see that the set of deforming f' together with W-action is identified with the subspace of $\mathrm{T}_I^{W_X}$ such that over any open subset $U \subset X$, it is also fixed by $\Gamma(U, W)$. Since f' is finite flat, we have by adjunction

$$\mathrm{T}_I \cong \mathrm{Ext}^1_{\mathcal{O}_{X_{R'}}}(\Omega^W_{Y'/X_{R'}}, \mathcal{O}_{Y'}) \otimes I,$$

hence the deformation of Y' with W-action can be identified with vector space

$$\mathrm{Ext}^1_{\mathcal{O}_{X_{R'}}}(\Omega^W_{Y'/X_{R'}}, \mathcal{O}_{X_{R'}}) \otimes I.$$

On the other hand, it is well-known that the deformation of $B_{Y'}$ in H_X by I is identified with

$$\mathrm{Hom}_{\mathcal{O}_{X_{R'}}}(J, \mathcal{O}_{B_{Y'}}) \otimes I \simeq \mathrm{Ext}^1_{\mathcal{O}_{X_{R'}}}(\mathcal{O}_{B_{Y'}}, \mathcal{O}_{B_{Y'}}) \otimes I,$$

where $J \subset \mathcal{O}_{X_{R'}}$ is the ideal of $B_{Y'}$. Therefore it boils down to showing that the derivative

$$\mathrm{D}_{f_0}(f_B)\colon \mathrm{Ext}^1_{\mathcal{O}_{X_{R'}}}(\Omega^W_{Y'/X_{R'}}, \mathcal{O}_{X_{R'}}) \longrightarrow \mathrm{Ext}^1_{\mathcal{O}_{X_{R'}}}(\mathcal{O}_{B_{Y'}}, \mathcal{O}_{B_{Y'}})$$

induced by tensoring the source by $(\Omega^W_{Y'})^\vee$ is an isomorphism, but this is simply the definition of $B_{Y'}$ combined with Serre duality, hence we are done.

Property (5) is straightforward: if A is a complete discrete valuation ring and F its function field and $Y \in \mathscr{H}(A)$, then since Y is smooth over A its ring of functions can be characterized as the integral closure of $\mathcal{O}_{X_A}$ in $\mathcal{O}_{Y_F}$. Therefore Y is completely determined by its fiber over F. This finishes the proof. ∎

7.2 Stratification by Monodromy

7.2.1 Using the moduli stack $\mathcal{A}^\flat_X$, we may now study the stratifications on $\mathcal{A}^\heartsuit_X$. Similar to [Ngô10, § 5], it is more convenient to study an étale cover of $\mathcal{A}_X$ since it will simplify the description of $\pi_0(\mathcal{P}_X)$ and the resulting stratifications. Recall that after fixing $\infty \in X(\bar{k})$, we have the open subset $\mathcal{A}^\infty_X \subset \mathcal{A}_X$ consisting of points a such that the cameral cover $\tilde{X}_a$ is étale over ∞. If $\infty \in X(k)$, then $\mathcal{A}^\infty_X$ has a natural k-structure. Consider the functor $\tilde{\mathcal{A}}_X$ whose S-points are pairs $(a, \tilde{\infty})$ where $a \in \mathcal{A}^\infty_X(S)$ and $\tilde{\infty} \in \tilde{X}_a(S)$ lying over ∞_S. It is clear that $\tilde{\mathcal{A}}_X$ is an étale subset of $\mathcal{A}_X$, and has a k-structure if $\infty \in X(k)$.

We have the natural map $\mathcal{A}_X^\infty \to \mathcal{B}_X$, and the point ∞ defines a locally trivial fiber bundle $\mathfrak{C}^\infty$ over $\mathcal{B}_X$ whose fiber at $(\mathcal{L}, \theta)$ is the fiber of $\mathfrak{C}_\mathcal{L}$ at ∞ (recall that $\mathfrak{C}$ is the GIT quotient $G^{\mathrm{sc}} /\!\!/ \mathrm{Ad}(G)$). In addition, $\theta \in \Gamma(X, \mathfrak{A}_{\mathfrak{M},\mathcal{L}})$ induces a cameral cover $\mathfrak{T}_{(\mathcal{L},\theta)}$ of $\mathfrak{C}_\mathcal{L}$, which further induces cameral cover $\mathfrak{T}_\mathfrak{M}^\infty \to \mathfrak{C}^\infty$. Consider Cartesian diagram

$$\begin{array}{ccc} \tilde{\mathcal{A}}_X & \longrightarrow & \mathfrak{T}_\mathfrak{M}^\infty \\ \downarrow & & \downarrow \\ \mathcal{A}_X^\infty & \longrightarrow & \mathfrak{C}_\mathfrak{M}^\infty \end{array}$$

Lemma 7.2.2. *Over the very G-ample locus $\mathcal{B}_\gg$, $\tilde{\mathcal{A}}_\gg \to \mathcal{B}_\gg$ is smooth and induces bijection on irreducible components.*

Proof. We already know $\mathcal{A}_X$ is a vector bundle when restricted to $\mathcal{B}_\gg$, and $\tilde{\mathcal{A}}_\gg$ is étale over $\mathcal{A}_\gg$, hence the smoothness result. Also by very G-ampleness, the bottom arrow in the above Cartesian diagram is a surjective map of vector bundles. This means that the top horizontal map has connected fibers. Since the abelianization map $\mathfrak{T}_\mathfrak{M} \to \mathfrak{A}_\mathfrak{M}$ has irreducible fibers over the invertible locus, $\mathfrak{T}_\mathfrak{M}^\infty$ is irreducible, and so is the preimage of any irreducible component of $\mathcal{B}_\gg$. ∎

Now we define $\tilde{\mathcal{A}}_\gg^\flat = \mathcal{A}_X^\flat \times_{\mathcal{A}_X} \tilde{\mathcal{A}}_\gg$. The map $\tilde{\mathcal{A}}_\gg^\flat \to \tilde{\mathcal{A}}_\gg$ is a bijection on $\bar{k}$-points. With the help of the reduced substack $(\mathcal{A}_X^\flat)^{\mathrm{red}}$, the image of any connected components of $(\tilde{\mathcal{A}}_\gg^\flat)_{\bar{k}}$ in $(\tilde{\mathcal{A}}_\gg)_{\bar{k}}$ is a constructible subset relative to $\mathcal{B}_{\gg,\bar{k}}$.

Over any geometric connected component of $\mathcal{B}_\gg$, one may decompose this constructible subset into a finite union of locally closed subsets. Let $\tilde{\mathcal{A}}_1$ be one of such locally closed subsets and $\tilde{\mathcal{A}}_1^\flat$ be its preimage in $(\tilde{\mathcal{A}}^\flat)_{\bar{k}}$. By Zariski Main Theorem, there exists an open dense subset of $\tilde{\mathcal{A}}_1$ over which the map $(\tilde{\mathcal{A}}_1^\flat)^{\mathrm{red}} \to \tilde{\mathcal{A}}_1$ is finite radical. Using

Noetherian induction, we may refine those $\mathcal{A}_1$ into a locally closed stratification

$$(\tilde{\mathcal{A}}_{\gg})_{\bar{k}} = \coprod_{\psi\in\Psi} \tilde{\mathcal{A}}_{\psi}, \tag{7.2.1}$$

such that the map $(\tilde{\mathcal{A}}^{\flat}_{\psi})^{\mathrm{red}} \to \tilde{\mathcal{A}}_{\psi}$ is finite radical. We may further refine the stratification so that the closure of any $\tilde{\mathcal{A}}_{\psi}$ is a finite union of strata. Hence we have a partial order on Ψ such that $\psi \leq \psi'$ if $\tilde{\mathcal{A}}_{\psi}$ is contained in the closure of $\tilde{\mathcal{A}}_{\psi'}$. Since $\tilde{\mathcal{A}}_{\gg} \to \mathcal{B}_{\gg}$ is smooth and a bijection of irreducible components, over each geometric irreducible component B of $\mathcal{B}_X$ there is a maximal element $\psi_B \in \Psi$.

7.2.3 Recall G comes from an $\mathrm{Out}(\mathbf{G})$-torsor ϑ_G on X, and when we fix a point ∞, we may lift it to a pointed version $\vartheta^{\bullet}_G$, and $\Theta = \Theta_{\vartheta}$ is the image of $\pi_1(\check{X}, \infty)$ in $\mathrm{Out}(\mathbf{G})$ under $\vartheta^{\bullet}_G$. By assumption, Θ is finite and its order is not divided by $\mathrm{char}(k)$.

Let $\tilde{a} = (a, \tilde{\infty}) \in \tilde{\mathcal{A}}_X(\bar{k})$. Recall the commutative diagram (6.5.1) which we reproduce here:

$$\begin{array}{ccc} \pi_1(U,\infty) & \xrightarrow{\pi^{\bullet}_{\tilde{a}}} & \mathbf{W} \rtimes \mathrm{Out}(\mathbf{G}) \\ \downarrow & & \downarrow \\ \pi_1(\check{X},\infty) & \xrightarrow{\vartheta^{\bullet}_G} & \mathrm{Out}(\mathbf{G}) \end{array} . \tag{7.2.2}$$

In this diagram $U = \check{X} - \mathfrak{D}_a$. Let $W_{\tilde{a}}$ be the image of $\pi^{\bullet}_{\tilde{a}}$ and $I_{\tilde{a}}$ the image of the kernel of $\pi_1(U,\infty) \to \pi_1(\check{X},\infty)$ under $\pi^{\bullet}_{\tilde{a}}$. By construction, $W_{\tilde{a}}$ is contained in $\mathbf{W} \rtimes \Theta$, and $I_{\tilde{a}}$ is a normal subgroup of $W_{\tilde{a}}$ and contained in $W_{\tilde{a}} \cap \mathbf{W}$.

An alternative approach to the above setup is as follows. Let $X_{\vartheta} \to \check{X}$ be a connected finite Galois étale cover with Galois group Θ' together with a point ∞_{ϑ} lying over ∞ such that ϑ_G becomes trivial on X_{ϑ}. Such requirement is the same as saying $\vartheta^{\bullet}_G \colon \pi_1(\check{X},\infty) \to \mathrm{Out}(\mathbf{G})$ factors through Θ', thus we may replace X_{ϑ} by a quotient cover so that $\Theta' = \Theta$. Let

$$\tilde{X}_{\vartheta,a} = \tilde{X}_a \times_{\check{X}} X_{\vartheta},$$

then $\mathbf{W}\rtimes\Theta$ acts on $\tilde{X}_{\vartheta,a}$ as well as its normalization $\tilde{X}^{\flat}_{\vartheta,a}$. Let $C_{\tilde{a}} \subset \tilde{X}^{\flat}_{\vartheta,a}$ be the connected component containing $\tilde{\infty}_{\vartheta} = (\tilde{\infty}, \infty_{\vartheta})$. Let $W_{\tilde{a}}$ be the subgroup of $\mathbf{W} \rtimes \Theta$ mapping $C_{\tilde{a}}$ to itself, and $I_{\tilde{a}} \subset W_{\tilde{a}}$ is the subgroup generated by elements with at least one fixed point in $C_{\tilde{a}}$.

Since U is the maximal subset of $\check{X}$ over which the cover $\tilde{X}_{\vartheta,a} \to \check{X}$ is a $\mathbf{W} \rtimes \Theta$-torsor, $W_{\tilde{a}}$ is exactly the image of $\pi_1(U, \infty)$ in $\mathbf{W} \rtimes \Theta$. For any closed point $\tilde{v} \in \tilde{X}^{\flat}_{\vartheta,a}$ with image $\bar{v} \in \check{X}$, the elements in $I_{\tilde{a}}$ fixing $\tilde{v}$ generates local inertia group of $\tilde{X}_{\vartheta,a,\tilde{v}}$ over $\check{X}_v$. Since $I_{\tilde{a}}$ is by definition generated by these elements, by a version of Riemann Existence Theorem (see e.g., [Gro71, Exposé XIII, Corollaire 2.12]), $I_{\tilde{a}}$ is exactly the kernel of map $\pi_1(U, \infty) \to \pi_1(\check{X}, \infty)$. Finally, since $C_{\tilde{a}}$ maps onto X_{ϑ} and since Θ acts freely on X_{ϑ}, the projection of $I_{\tilde{a}}$ to Θ is trivial. Therefore, $I_{\tilde{a}} = W_{\tilde{a}} \cap \mathbf{W}$. Thus we established the equivalence of two formulations of pairs $(I_{\tilde{a}}, W_{\tilde{a}})$ associated with $\tilde{a}$ (and $\vartheta^{\bullet}_{G}$, which is independent of $\tilde{a}$).

Proposition 7.2.4. *The map $\tilde{a} \mapsto (I_{\tilde{a}}, W_{\tilde{a}})$ is constant on every stratum $\tilde{\mathcal{A}}_{\psi}$ in* (7.2.1)*. As a result, we have a well defined map on the set of strata Ψ*

$$\psi \longmapsto (I_{\psi}, W_{\psi}).$$

Proof. We have that $(\tilde{\mathcal{A}}^{\flat}_{\psi})^{\mathrm{red}} \to \tilde{\mathcal{A}}_{\psi}$ is a finite radical morphism. Over $(\tilde{\mathcal{A}}^{\flat}_{\psi})^{\mathrm{red}}$, the normalizations of individual cameral curves form a smooth family $\tilde{X}^{\flat}_{\psi} \to \tilde{\mathcal{A}}^{\flat}_{\psi}$. In fact, using the cover X_{ϑ}, we also have the smooth proper family

$$\tilde{X}^{\flat}_{\vartheta,\psi} \longrightarrow \tilde{\mathcal{A}}^{\flat}_{\psi}$$

on which $\mathbf{W} \rtimes \Theta$ acts, as well as a section $\tilde{\infty}_{\vartheta}$. Using Lemma 7.1.3, we have a locally constant sheaf $\pi_0(\tilde{X}^{\flat}_{\vartheta,\psi}/\tilde{\mathcal{A}}^{\flat}_{\psi})$ whose fibers are the connected components of the fibers of $\tilde{X}^{\flat}_{\vartheta,\psi}$, and $\mathbf{W} \rtimes \Theta$ acts transitively on the fibers. The existence of section $\tilde{\infty}_{\vartheta}$ means that $\pi_0(\tilde{X}^{\flat}_{\vartheta,\psi}/\tilde{\mathcal{A}}^{\flat}_{\psi})$ is constant. This means that the collection of $C_{\tilde{a}}$ forms a smooth and proper family over $\tilde{\mathcal{A}}_\psi^\flat$ with connected fibers. Since $\mathbf{W} \rtimes \Theta$ is discrete, we see that $W_{\tilde{a}}$ and $I_{\tilde{a}}$ are constant. ∎

Definition 7.2.5. We define a partial order on the set of pairs (I_ψ, W_ψ) where W_ψ is a subgroup of $\mathbf{W} \rtimes \Theta$ and $I_\psi \subset W_\psi$ is a normal subgroup contained in $\mathbf{W}$: $(I_\psi, W_\psi) \leq (I'_\psi, W'_\psi)$ if $W_\psi \subset W'_\psi$ and $I_\psi \subset I'_\psi$.

Lemma 7.2.6. *The map $\psi \mapsto (I_\psi, W_\psi)$ is an increasing map on Ψ.*

Proof. Let $S = \operatorname{Spec} R$ be the spectrum of a complete discrete valuation ring, with generic point $\eta = \operatorname{Spec} K$ and special point $s = \operatorname{Spec} k(s)$, with $k(s)$ being algebraically closed. Let $\tilde{a} \colon S \to \tilde{\mathcal{A}}$ be a morphism sending s to $\tilde{\mathcal{A}}_\psi$ and η to $\tilde{\mathcal{A}}_{\psi'}$. We want to show that $(I_\psi, W_\psi) \leq (I_{\psi'}, W_{\psi'})$.

Since finite étale coverings can only be trivially deformed locally, we have canonical cospecialization maps $\pi_1(U_s, \infty_s) \to \pi_1(U_\eta, \infty_\eta)$ and $\pi_1(\check{X}_s, \infty_s) \to \pi_1(\check{X}_\eta, \infty_\eta)$ compatible with diagram (7.2.2). Then the result follows from the definition. ∎

7.2.7 Using Lemma 7.2.6, if (I_-, W_-) is a pair where W_- is a subgroup of $\mathbf{W} \rtimes \Theta$ and $I_- \subset W_-$ is a normal subgroup also contained in $\mathbf{W}$, then the union of such $\tilde{\mathcal{A}}_\psi$ that $(I_\psi, W_\psi) \leq (I_-, W_-)$ is a closed subset of $\tilde{\mathcal{A}}_\psi$, and the union $\tilde{\mathcal{A}}_{(I_-, W_-)}$ of those strata satisfying $(I_\psi, W_\psi) = (I_-, W_-)$ is a open subset of that closed subset. Thus we have a locally closed stratification

$$(\tilde{\mathcal{A}}_X)_{\bar{k}} = \coprod_{(I_-, W_-)} \tilde{\mathcal{A}}_{(I_-, W_-)}.$$

In particular, the union of those strata such that $\mathbf{T}^{W_\psi}$ is finite is an open subset of $\tilde{\mathcal{A}}_X$. By Proposition 6.5.5, we see that $\mathcal{A}_X^\natural$ is an open subset of $\mathcal{A}_X^\heartsuit$. The stratifications on $\tilde{\mathcal{A}}_X$ naturally induce stratifications on $\tilde{\mathcal{A}}_X^\natural = \mathcal{A}_X^\natural \times_{\mathcal{A}_X} \tilde{\mathcal{A}}_X$

Lemma 7.2.8. *Let $B \subset \mathcal{B}_\gg$ be an irreducible component, and ψ_B be the maximal element* *in Ψ corresponding to B. Then*

$$(I_{\psi_B}, W_{\psi_B}) = (\mathbf{W}, \mathbf{W} \rtimes \Theta).$$

In fact, for any $b \in \mathcal{B}_{\gg}(\bar{k})$, we can find $\tilde{a}$ lying over b such that $(I_{\tilde{a}}, W_{\tilde{a}}) = (\mathbf{W}, \mathbf{W} \rtimes \Theta)$.

Proof. For any $b \in \mathcal{B}_{\gg}(\bar{k})$, we know $\mathcal{A}_b^{\diamondsuit}$ is non-empty by Proposition 6.3.13. Let $\tilde{a} \in \tilde{\mathcal{A}}_{\gg}^{\diamondsuit}(\bar{k})$ be a point lying over b. Then we know $\tilde{X}_{\vartheta,a}$ is smooth and irreducible, therefore $W_{\tilde{a}} = \mathbf{W} \rtimes \Theta$. We also know that $\tilde{X}_{\vartheta,a}$ intersects with $\mathbf{D}_\alpha$ for every positive root α. Therefore $I_{\tilde{a}}$ is a normal subgroup of $\mathbf{W}$ containing every reflection, thus must be $\mathbf{W}$ itself. ∎

7.3 The Sheaf $\pi_0(\mathcal{P}_X)$

We have already seen the descriptions of $\pi_0(\mathcal{P}_{\tilde{a}})$ and $\pi_0(\mathcal{P}'_{\tilde{a}})$ for individual $\tilde{a} \in \tilde{\mathcal{A}}_X^{\heartsuit}(\bar{k})$ in Proposition 6.5.5. Here we want to describe the restriction of sheaves $\pi_0(\mathcal{P}_X)$ and $\pi_0(\mathcal{P}'_X)$ to each stratum $\tilde{\mathcal{A}}_{(I_-,W_-)}$.

Proposition 7.3.1. *There exists canonical surjections of sheaves over $\tilde{\mathcal{A}}_X$*

$$\check{\mathbb{X}}(\mathbf{T}) \longrightarrow \pi_0(\mathcal{P}'_X) \longrightarrow \pi_0(\mathcal{P}_X)$$

such that the fiber at any $\tilde{a} \in \tilde{\mathcal{A}}_X(\bar{k})$ are the surjections given in Proposition 6.5.5.

Proof. The section $\tilde{\infty}$ of the cameral curve over $\tilde{\mathcal{A}}_X$ gives a fixed $\tilde{\mathcal{A}}_X$-family of pinnings of G at ∞. Hence using the Galois description of $\mathfrak{J}_X$, we may identify the fiber of $\mathfrak{J}_X$ over $\{\infty\}\times\tilde{\mathcal{A}}_X$ with $\mathbf{T}\times\tilde{\mathcal{A}}_X$. Such identification uniquely extends to the formal disc $\check{X}_\infty\times\tilde{\mathcal{A}}_X$, because $\check{\mathbb{X}}(\mathbf{T})$ is discrete. For each $\mu \in \check{\mathbb{X}} \simeq \mathrm{Gr}_{\mathbf{T}}^{\mathrm{red}}$, we have an induced local $\mathfrak{J}_{\check{X}_\infty}$-torsor E_μ over $\tilde{\mathcal{A}}_X$. Gluing with the trivial $\mathfrak{J}_X$-torsor over the complement of $\{\infty\} \times \tilde{\mathcal{A}}_X$, we obtain a well-defined map $\check{\mathbb{X}}(\mathbf{T}) \to \pi_0(\mathcal{P}_X)$. The same argument works for $\mathcal{P}'_X$ as well since at

∞, the fibers of $\mathfrak{J}_X$ and $\mathfrak{J}_X^0$ are the same. The surjectivity can be checked at stalk level using Proposition 6.5.5. ∎

For any étale open subset $U \to \tilde{\mathcal{A}}_X$, the stratifications of $\tilde{\mathcal{A}}_X$ induce stratifications of U. In particular we have strata $U_{(I_-,W_-)}$ for pairs (I_-,W_-). For a given pair (I_1,W_1), we say that U is *of type* (I_1,W_1) if U_{I_1,W_1} is the unique non-empty closed stratum in U. Clearly, étale open subsets of this kind form a base of the small étale site of $\tilde{\mathcal{A}}_X$. This base allows us to define étale sheaves Π and Π' as follows: for U_1 of type (I_1,W_1), let

$$\Gamma(U_1,\Pi') = (\check{\mathbf{T}}^{W_1})^* = \check{\mathbb{X}}(\mathbf{T})_{W_1},$$
$$\Gamma(U_1,\Pi) = \check{\mathbf{T}}(I_1,W_1)^*,$$

where $\check{\mathbf{T}}(I_1,W_1)$ is as in Proposition 6.5.5. Suppose U_2 is an étale open subset of U_1 of type (I_2,W_2), since $U_{(I_1,W_1)}$ is the unique non-empty closed stratum in U_1, we see that

$$(I_1,W_1) \leq (I_2,W_2).$$

The following lemma is straightforward.

Lemma 7.3.2 ([Ngô10, Lemme 5.5.3])**.** *If* $(I_1,W_1) \leq (I_2,W_2)$*, then* $\check{\mathbf{T}}(I_2,W_2) \subset \check{\mathbf{T}}(I_1,W_1)$*.*

Thus we have canonical restriction maps by dualization

$$\Gamma(U_1,\Pi') \longrightarrow \Gamma(U_2,\Pi'),$$
$$\Gamma(U_1,\Pi) \longrightarrow \Gamma(U_2,\Pi).$$

Thus the sheaves Π and Π' are defined, and we immediately have the following by Proposition 6.5.5 and Proposition 7.3.1.

Corollary 7.3.3. *We have canonical isomorphisms of sheaves over $\tilde{\mathcal{A}}_X$*

$$\pi_0(\mathcal{P}'_X) \simeq \Pi',$$
$$\pi_0(\mathcal{P}_X) \simeq \Pi.$$

7.4 Stratification by δ-invariant

Recall that for any $a \in \mathcal{A}_X^{\heartsuit}(\bar{k})$ we have a global δ-invariant δ_a by Definition 6.4.3.

Lemma 7.4.1. *The function*

$$\begin{aligned} \tilde{\mathcal{A}}_X(\bar{k}) &\longrightarrow \mathbb{N} \\ a &\longmapsto \delta_a \end{aligned}$$

is constant on every stratum $\tilde{\mathcal{A}}_\psi$ ($\psi \in \Psi$).

Proof. Base change to $S = (\tilde{\mathcal{A}}_\psi^\flat)^{\mathrm{red}}$, then the cameral curve $\pi_\psi \colon \tilde{X}_\psi \to \check{X} \times S$ admits a simultaneous normalization $\xi \colon \tilde{X}_\psi^\flat \to \tilde{X}_\psi$. Let p_S be the projection $\check{X} \times S \to S$. By Lemma 7.1.3, the sheaf $F = \pi_{\psi,*}(\xi_* \mathcal{O}_{\tilde{X}_\psi^\flat} / \mathcal{O}_{\tilde{X}_\psi})$ is a coherent sheaf with W-action such that $p_{S,*}F$ is a locally free $\mathcal{O}_S$-sheaf of finite type. Since $\mathrm{char}(k)$ does not divide the order of $\mathbf{W}$, the same is true for

$$(F \otimes_{\mathcal{O}_{\check{X}\times S}} \mathfrak{t})^W.$$

Therefore the result follows from Corollary 6.4.4. ∎

Recall also that we have a rigidification of $\mathcal{P}_X$ over $\mathcal{A}_X^\infty$ hence also over $\tilde{\mathcal{A}}_X$ in 6.4.5. Combined with the following lemma, we see that δ-invariant is upper semi-continuous.

Lemma 7.4.2 ([Ngô10, Lemme 5.6.3]). *Let $P \to S$ is a smooth commutative group scheme of finite type. The function $s \mapsto \tau_s$ sending a geometric point $s \in S$ to the dimension of the* *abelian part of P_s is lower semi-continuous. Equivalently, the function $s \mapsto \delta_s$ sending s to the dimension of the affine part of P_s is upper semi-continuous.*

Corollary 7.4.3. *For any $\delta \in \mathbb{N}$, let $\tilde{\mathcal{A}}_\delta$ (resp. $\tilde{\mathcal{A}}_{\geq\delta}$, resp. $\tilde{\mathcal{A}}_{\leq\delta}$) be the union of all strata $\tilde{\mathcal{A}}_\psi$ such that $\delta_\psi = \delta$ (resp. $\delta_\psi \geq \delta$, resp. $\delta_\psi \leq \delta$). Then it is a locally closed (resp. closed, resp. open) subset of $\tilde{\mathcal{A}}_X$. In particular, we have a stratification by δ-invariant*

$$\tilde{\mathcal{A}} = \coprod_{\delta\in\mathbb{N}} \tilde{\mathcal{A}}_\delta.$$

The stratification by δ-invariant induces a stratification on $\tilde{\mathcal{A}}_X^\natural$, such that for any $\tilde{a} \in \tilde{\mathcal{A}}_\delta^\natural(\bar{k})$, the dimension of the affine part of $\mathcal{P}_a$ is δ. It is also clear from Corollary 6.4.4 that δ-invariant is independent of the choice of points ∞ and $\tilde{\infty}$, thus the stratification by δ-invariant descends to a stratification on $\mathcal{A}_X^\heartsuit$:

$$\mathcal{A}_X^\heartsuit = \coprod_{\delta\in\mathbb{N}} \mathcal{A}_\delta.$$

7.4.4 The important result about δ-strata is its codimension in $\mathcal{A}_X$. We wish to prove that the codimension of $\mathcal{A}_\delta$ is at least δ for all δ. However, there is currently no clear indication on how to do so. Instead, we use the local-global argument in [Ngô10, § 5.7] to prove a weaker result that is likely sufficient for most practical purposes. The proof in *loc. cit.* uses so-called root valuation strata studied in [GKM09]. Its multiplicative counterpart has been studied in Chapter 3 (albeit less thoroughly).

Proposition 7.4.5. *For any $\delta \in \mathbb{N}$, there exists an integer N depending on G and δ such that if $b \in \mathcal{B}_X(\bar{k})$ is very (G,N)-ample, then we have*

$$\operatorname{codim}_{\mathcal{A}_b} \mathcal{A}_{b,\delta} \geq \delta.$$

In particular, for every irreducible component $A \subset \mathcal{A}_X^\heartsuit$ that is very (G,N)-ample, we have

$\operatorname{codim}_A A_\delta \geq \delta$.

Proof. The $\delta = 0$ case is trivial. Suppose $\delta > 0$ and let $\delta_\bullet$ be a partition of δ by positive integers

$$\delta = \delta_1 + \cdots + \delta_n.$$

Consider subset $Z_{\delta_\bullet} \subset \mathcal{A}_b^\heartsuit \times X^n$ consisting of tuples $(a; x_1, \ldots, x_n)$ such that $a \in \mathcal{A}_b^\heartsuit(\bar{k})$, $x_i \in X(\bar{k})$ such that $\delta_{x_i}(a) = \delta_i$. Refine $Z_{\delta_\bullet}$ into a disjoint union of subsets $Z_{[w_\bullet, \bar{\lambda}_\bullet/l_\bullet, r_\bullet]}$ consisting of points $(a; x_1, \ldots, x_n)$ such that the image of a in $\mathfrak{C}_{\mathfrak{M},b}(\mathcal{O}_{x_i})$ lies in the stratum $\mathfrak{C}_{\mathfrak{M},b}(\mathcal{O}_{x_i})_{[w_i, \bar{\lambda}_i/l_i, r_i]}$. Note that G is split over $\mathcal{O}_{x_i}$ so the valuation strata make sense. Also note that $\bar{\lambda}_i/l_i$ is actually fixed since b is. Suppose this stratum is an N_i-admissible cylinder. Let $N' = N_1 + \cdots + N_n$ and suppose b is very (G, N')-ample. Then the linear map

$$\mathcal{A}_\mathcal{L} \longrightarrow \prod_{i=1}^n \mathfrak{C}_{\mathfrak{M},b}(\mathcal{O}_{x_i}/\pi_{x_i}^{N_i}\mathcal{O}_{x_i})$$

is surjective. Using a modified proof of Theorem 3.4.3 by fixing the boundary divisor, we see that the codimension of $Z_{[w_\bullet, \bar{\lambda}_\bullet/l_\bullet, r_\bullet]}$ in $\mathcal{A}_b \times X^n$ is

$$\sum_{i=1}^n (\delta_i + c_i + e_i).$$

Here there is no b_i-term because the boundary divisor b is fixed. At x_i, if $\bar{\lambda}_i/l_i = 0$, then since $\delta_i \neq 0$, we must have either $c_i > 0$ or $e_i > 0$. If $\bar{\lambda}_i/l_i \neq 0$, then x_i has to be one of the finite points in the support of b, so it cannot move freely in X. Thus the codimension of $Z_{[w_\bullet, \bar{\lambda}_\bullet/l_\bullet, r_\bullet]}$ in $\mathcal{A}_b \times X^n$ is always at least

$$\sum_{i=1}^n (\delta_i + 1) = \delta + n,$$

hence its image in $\mathcal{A}_b$ is at least δ. Let N be the supremum of all N' for various $Z_{[w_\bullet, \bar{\lambda}_\bullet/l_\bullet, r_\bullet]}$ (there are only finitely many for a fixed δ) and we are done. ∎

7.4.6 Let $A \subset \mathcal{A}_X$ be an irreducible component. If A is very G-ample, then its image in $\mathcal{B}_X$ is an irreducible component B of $\mathcal{B}_X$. The normalization of B is isomorphic to certain direct product of symmetric power of curves and there is a open dense subset $B^\circ \subset B$ of "multiplicity-free" locus (c.f., Proposition 5.1.25).

Corollary 7.4.7. *Suppose we have a fixed G, $\mathfrak{M}$ and $\delta \in \mathbb{N}$. Let $N = N(\delta)$ be as in Proposition 7.4.5. Then for every irreducible component $Z \subset \mathcal{A}_{X,\delta}$ that is contained in very (G, N)-ample locus, the equality $\operatorname{codim}_{\mathcal{A}_X} Z = \delta$ is achieved if and only if one of the following conditions is met*

(1) $\delta = 0$,

(2) $\delta > 0$, and there exists a geometric point $a \in Z$ lying over a point $b \in B^\circ$ such that at every $\bar{v} \in X(\bar{k})$, one of the followings must be true:

(a) $\lambda_{\bar{v}} = 0$ and $c_{\bar{v}} + e_{\bar{v}} = 1$, or

(b) $\lambda_{\bar{v}} \neq 0$ and $c_{\bar{v}} = e_{\bar{v}} = 0$.

Proof. Straightforward from the proof of Proposition 7.4.5. ∎

7.4.8 The numerical conditions in Corollary 7.4.7 implies that when $\delta > 0$ is not too large, in view of product formula (6.9.3), only those multiplicative affine Springer fibers of sufficiently simple types can occur at a general point of Z, which we summarize below.

In case there is no boundary divisor at $\bar{v}$, i.e., $\lambda_{\bar{v}} = 0$, we have two possibilities. The first possibility is $c_{\bar{v}} = 1$ and $e_{\bar{v}} = 0$. In this case the ramification happens in an Levi subgroup generated by a single pair of roots and no other root has any contribution to δ, and we must have $d_{\bar{v}+}(a) = 1$ and $\delta_{\bar{v}}(a) = 0$. The second possibility is $c_{\bar{v}} = 0$ and $e_{\bar{v}} = 1$. It implies that a is unramified at $\bar{v}$ and the contribution to δ still only comes from a single pair of roots, so we must have $d_{\bar{v}+}(a) = 2$ and $\delta_{\bar{v}}(a) = 1$. In both cases the local computations can be reduced to the case when $G = \mathrm{SL}_2$, and we shall revisit it later in this paper.

On the other hand, when $\lambda_{\bar{v}} \neq 0$, then a must be unramified at $\bar{v}$, and suppose $\mu_{\bar{v}}$ is the local Newton point of a at $\bar{v}$ and $\mu_{\bar{v}}$ is chosen to be dominant, then since $c_{\bar{v}} = e_{\bar{v}} = 0$, we have

$$\begin{aligned} \delta_{\bar{v}}(a) &= \langle \rho, \lambda_{\bar{v},\mathrm{ad}} - \mu_{\bar{v},\mathrm{ad}} \rangle \\ &= \langle \rho, -w_0(\lambda_{\bar{v},\mathrm{ad}}) - \mu_{\bar{v},\mathrm{ad}} \rangle, \end{aligned} \tag{7.4.1}$$

To summarize, the equality in Proposition 7.4.5 is achieved if and only if the following conditions are met:

(1) A general point $a \in A_\delta$ has multiplicity-free boundary divisor;

(2) If the boundary divisor λ_b and the discriminant divisor $\mathfrak{D}_a$ collide at some point $\bar{v} \in X(\bar{k})$, then a is ν-regular semisimple (Definition 7.4.9) at $\bar{v}$.

For convenience, we make two more definitions.

Definition 7.4.9. We say a is *ν-regular semisimple*, or *regular semisimple relative to its Newton point* at $\bar{v}$ if it is unramified at $\bar{v}$ and (7.4.1) holds.

Definition 7.4.10. An irreducible locally-closed subset in $\mathcal{A}_X^\heartsuit$ is called *δ-critical* if its codimension in $\mathcal{A}_X$ equals its minimal δ-invariant. We denote the union of those δ-strata that are δ-critical by $\mathcal{A}_\delta^=$.

7.5 Inductive Strata

In the Lie algebra case, the stratification associated with δ-invariant is one of the key ingredient in the proof of Ngô's support theorem for Hitchin fibrations. Roughly speaking, those strata $\mathcal{A}_\delta$ with codimension exactly δ are the only strata that "matter". Furthermore, in the Lie algebra case, only the $\delta = 0$ stratum ends up being relevant.

However, it is not so in the multiplicative setting due to the presence of boundary divisors. The counterpart of boundary divisor (or the $Z_{\mathfrak{M}}$-torsor) in Lie algebra case is a very ample line bundle on X. The difference is that such twisting has no impact on δ-invariant for Lie algebras, while the boundary divisor does in multiplicative case.

In the previous section, we have already computed the codimension of $\mathcal{A}_\delta$ using local valuation strata, under the assumption that δ is not too large compared to the ampleness of the boundary divisor. It turns out that certain irreducible components in strata $\mathcal{A}_\delta$ with codimension δ can be identified with mH-fibrations of "smaller degrees". This way we obtains an inductive system of mH-fibrations.

7.5.1 We use split group $\mathbf{G} = \mathbf{G}^{\mathrm{sc}}$ and base change to $\bar{k}$ to illustrate the idea. Let $\lambda \in \check{\mathbb{X}}(\mathbf{T})_+$ be a dominant cocharacter and $d > 0$ an integer. Then we may have a moduli space of boundary divisors $\check{X}_d$, viewed as the system of divisors $\sum_{v \in X(\bar{k})} d_v \lambda \cdot v$ such that $\sum_{\bar{v} \in X(\bar{k})} d_{\bar{v}} = d$. We have finite $\mathfrak{S}_d$-cover $\check{X}^d \to \check{X}_d$, where $\mathfrak{S}_d$ is the symmetric group of d elements. The fiber product $\mathcal{A}_{d,\lambda} \times_{\check{X}_d} \check{X}^d \to \check{X}^d$ is a vector bundle whose fiber at $D = (\bar{v}_1, \ldots, \bar{v}_d) \in \check{X}^d$ is section space

$$\bigoplus_{j=1}^{r} \mathrm{H}^0(\check{X}, \mathcal{O}_{\check{X}}(\langle \varpi_j, \lambda \rangle D)),$$

viewed as a vector space.

For each $1 \le i \le d$, let $\mu_i \in \check{\mathbb{X}}(\mathbf{T})_+$ be another dominant cocharacter smaller than λ, i.e., $\lambda - \mu_i$ is an $\mathbb{N}$-combination of simple coroots. Then for each $1 \le j \le r$ we have natural inclusion of divisors on $\check{X}$:

$$D_{\mu,j} \coloneqq \sum_{i=1}^{d} \langle \varpi_j, \mu_i \rangle \cdot \bar{v}_i \subset \langle \varpi_j, \lambda \rangle D.$$

This induces an inclusion of vector bundles

$$\mathcal{A}' \subset \mathcal{A}_{d,\lambda} \times_{\check{X}_d} \check{X}^d,$$

where $\mathcal{A}' \to \check{X}^d$ is the vector bundle whose fiber over D is

$$\bigoplus_{j=1}^{r} \mathrm{H}^0(\check{X}, \mathcal{O}_{\check{X}}(D_{\mu,j})).$$

Since $\check{X}^d \to \check{X}_d$ is finite, the image of $\mathcal{A}'$ in $\mathcal{A}_{d,\lambda}$ is closed, and this is a δ-critical stratum in $\mathcal{A}_{d,\lambda}$ if $\sum_i \mu_i \cdot \bar{v}_i$ is very G-ample.

7.5.2 Now we formulate for arbitrary quasi-split group G. Let $\mathfrak{M} \in \mathcal{FM}_0(G^{\mathrm{sc}})$ be a monoid with 0 and let B be an irreducible component of the very G-ample locus $\mathcal{B}_{\gg} \subset \mathcal{B}_X$. There is a unique open subset B° being the multiplicity-free locus. See Proposition 5.1.25.

A point $b \in B^\circ(\bar{k})$ may be written as a tuple $(\mathcal{L}, \lambda_b)$ where $\mathcal{L} \in \mathrm{Bun}_{Z_{\mathfrak{M}}}$, and

$$\lambda_b = \sum_{i=1}^{d} \lambda_i \cdot \bar{v}_i$$

for some closed points $\bar{v}_i \in \check{X}(\bar{k})$ and dominant cocharacters $\lambda_i \in \check{\mathbb{X}}(T^{\mathrm{ad}}_{\bar{v}_i})_+$. The points $\bar{v}_i$ are pairwise distinct for any i, and if we identify $\mathfrak{M}$ with its split model $\mathbf{M}$ at $\bar{v}_i$, then λ_i must be one of the minimal generators of the cone of $\mathbf{A}_{\mathbf{M}}$ because b is multiplicity-free. Let $\mu_i \in \check{\mathbb{X}}(T^{\mathrm{ad}}_{\bar{v}_i})_+$ be a dominant cocharacter with $\mu_i \leq \lambda_i$.

Let $\vartheta\colon X_\vartheta \to \check{X}$ be a connected finite étale cover over which G becomes split. Let $\mathcal{L}_\vartheta$ be the pullback of $\mathcal{L}$ to X_ϑ. For each fundamental weight ϖ_j on X_ϑ, the divisor

$$D_j \coloneqq \sum_i \langle \varpi_j, \lambda_i - \mu_i \rangle \cdot \vartheta^* \bar{v}_i$$

defines an inclusion of line bundles

$$\varpi_j(\mathcal{L}_\vartheta)(-D_j) \subset \varpi_j(\mathcal{L}_\vartheta),$$

and the inclusion of direct sums

$$\bigoplus_{j=1}^{r} \varpi_j(\mathcal{L}_\vartheta)(-D_j) \subset \bigoplus_{j=1}^{r} \varpi_j(\mathcal{L}_\vartheta)$$

descends to an inclusion of vector bundles on $\check{X}$

$$\mathfrak{C}''_{\mathcal{L}} \subset \mathfrak{C}_{\mathcal{L}}.$$

Suppose each $\varpi_j(\mathcal{L}_\vartheta)(-D_j)$ is still very ample over X_ϑ, then the section space

$$\mathcal{A}''_b = \mathrm{H}^0(\check{X}, \mathfrak{C}''_{\mathcal{L}})$$

is a linear subspace of $\mathcal{A}_b$ with codimension $\delta = \sum_i \langle \rho, \lambda_i - \mu_i \rangle$, and in which a general point has δ-invariant exactly δ.

Let $b \in U^\circ$ vary while keeping μ_i locally constant, and let $\mathcal{A}'$ be the resulting union. Then it is not hard to see that $\mathcal{A}'$ is a locally-trivial fibration over U° whose geometric fiber $\mathcal{A}'_b$ at b is a union of spaces looking like $\mathcal{A}''_b$ (in fact, it's just the union of $\mathcal{A}''_b$ and all its conjugates under monodromy). By construction it is δ-critical and defined over k since it is stable under monodromy of U°.

Definition 7.5.3. The closure of $\mathcal{A}'$ in $\mathcal{A}_X$ is called an *inductive stratum*. We denote the union of all inductive strata with given δ by $\mathcal{A}^{\equiv}_{\geq\delta}$, and its intersection with $\mathcal{A}_\delta$ by $\mathcal{A}^{\equiv}_{\delta}$.

7.5.4 We want to point out that the key difference between being δ-critical and being inductive, assuming enough G-ampleness. Let a be a general point in an inductive stratum, then at any $\bar{v} \in X(\bar{k})$ such that the boundary divisor is 0, one must have either $d_{\bar{v}+}(a) = 0$, or $d_{\bar{v}+}(a) = 1$ and $\delta_{\bar{v}}(a) = 0$. On the other hand, if a is a general point of a δ-critical stratum, there is a third possibility where $d_{\bar{v}+}(a) = 2$ and $\delta_{\bar{v}}(a) = 1$. Therefore we have strict inclusion

$$\{\text{inductive strata}\} \subsetneq \{\delta\text{-critical strata}\}.$$

Proposition 7.5.5. *For any $\delta \in \mathbb{N}$ and $N = N(\delta)$ as in Proposition 7.4.5, let $A' \subset \mathcal{A}_{\gg N}$ be an irreducible δ-critical stratum. Then we can find an inductive stratum A such that*

(1) $A' \subset A$, and

(2) For a general point $a' \in A'_b$ with multiplicity-free boundary divisor b, we may find and a general point $a \in A_b$, such that if we let

$$\delta'' = \sum_{\bar{v} \notin \mathrm{supp}(b)} \delta_{\bar{v}}(a'),$$

then $\delta_{a'} = \delta_a + \delta''$.

Proof. The proof is similar to that of Proposition 6.3.13. With the assumption on G-ampleness, we may simply deform locally near every point $\bar{v} \in \mathfrak{D}_{a'} - \mathrm{supp}(b)$ so that $\check{X}$ intersects with $\mathfrak{D}_{\mathfrak{M}}$ transversally near those points. ∎

CHAPTER 8

COHOMOLOGIES

In this chapter we study the general properties of cohomologies over the anisotropic locus. The first two sections contain results similar to those in [Ngô10, §§ 6.1–6.3]. The main difference is, however, that the global transfer map will no longer be a closed embedding in general but only a finite map. Such difference is not merely technical but is completely explained by representations of the dual groups.

We will give a statement of geometric stabilization which we will prove a weaker version of due to current technical constraint, but we do not doubt that it will not be hard to remove most of the constraint.

The last part of this chapter studies top ordinary cohomologies, which is much more complicated than the Lie algebra case. To this end, we will introduce a new type of Hecke stack which allows us to upgrade the product formula in § 6.9 into a family.

8.1 Properness over Anisotropic Locus

So far we have studied the properties of individual Hitchin fibers $\mathcal{M}_a$ as well as the Picard stack $\mathcal{P}_X \to \mathcal{A}_X$. In this section we turn to the total space $\mathcal{M}_X$ of mH-fibration. The first result is its finiteness properties.

Proposition 8.1.1. *The stack $\mathcal{M}_X$ is locally of finite type, and $h_X^\natural\colon \mathcal{M}_X^\natural \to \mathcal{A}_X^\natural$ over anisotropic locus is a relative Deligne-Mumford stack of finite type. The same is true for $\mathcal{P}_X$.*

Proof. The natural map $\mathcal{M}_X \to \mathrm{Bun}_G \times \mathrm{Bun}_{Z_{\mathfrak{M}}}$ is of finite type because the fibers are sections of a fixed étale-locally trivial fiber bundle with affine fibers. Since X is projective, such section space must be of finite type. Since $\mathrm{Bun}_G \times \mathrm{Bun}_{Z_{\mathfrak{M}}}$ is locally of finite type, so is $\mathcal{M}_X$.

For the second claim, it suffices to prove for $\mathfrak{M} = \mathrm{Env}(G^{\mathrm{sc}})$. The case when G is split is proved by [Chi19, Proposition 4.3.3]. Although the statement in *loc. cit.* is about mH-fibration over a fixed $Z_{\mathfrak{M}}$-torsor $\mathcal{L}$, the role of $\mathcal{L}$ is inconsequential since our claim is about relative finiteness of map $h_X^{\natural}$.

When G is non-split, let $\vartheta\colon X_\vartheta \to X$ be a connected finite Galois étale cover over which G becomes split, and Θ be the Galois group. Then we have a map by pullback

$$\vartheta^*\colon \mathcal{M}_X \longrightarrow \mathcal{M}_{X_\vartheta},$$

the latter being the total space of mH-fibration associated with split monoid $\vartheta^*\mathfrak{M}$. Since ϑ is étale, $G \to G_\vartheta \coloneqq \vartheta_*\vartheta^*G$ is a closed embedding of reductive group schemes over X, hence G_ϑ/G is affine over X. It is straightforward to see that a G-bundle may be identified with a G_ϑ-bundle E together with a section of associated bundle $E \times^{G_\vartheta} G_\vartheta/G$. Since G_ϑ/G is affine over X and X is projective, the section space is of finite type (being a closed subscheme of the section scheme of a vector bundle), hence the map

$$\mathrm{Bun}_{G/X} \longrightarrow \mathrm{Bun}_{G/X_\vartheta}$$

is of finite type. This implies that $\mathcal{M}_X \to \mathcal{M}_{X_\vartheta}$ is of finite type. The proof for $\mathcal{P}_X$ is the same hence we are done. ∎

Proposition 8.1.2. *The map $h_X^{\natural}\colon \mathcal{M}_X^{\natural} \to \mathcal{A}_X^{\natural}$ is proper.*

Proof. Since the map is a morphism of Deligne-Mumford stacks and is of finite type, we use valuative criteria as in [CL10, §§ 8–9] and [Chi19, Proposition 4.3.6]. Let R be a discrete valuation ring and K its fractional field.

For the existence part of valuative criteria, it is harmless to assume that the residue field k_R of R is algebraically closed, because we are allowed to take finite extensions of R. Let $(E,\phi) \in \mathcal{M}_X^{\heartsuit}(K)$ over $a \in \mathcal{A}_X^{\heartsuit}(R)$. Let $\mathcal{L}$ be the $Z_{\mathfrak{M}}$-torsor corresponding to a. Let

$R(X)$ be the local ring of the generic point of the special fiber of $X \times \operatorname{Spec} R$ and $K(X)$ its fractional field. Similar notation is used for any algebraic extension K'/K and R'/R where R' is the integral closure of R in K'.

Since $a \in \mathcal{A}_X^{\heartsuit}$, the restriction of a to $R'(X)$ has image contained in $\mathfrak{M}_{\mathcal{L}}^{\mathrm{rs}}$, thus $\mathfrak{J}_a$ is a torus when restricted to $R(X)$. Since $[\mathfrak{M}^{\mathrm{rs}}/G] \to \mathfrak{C}_{\mathfrak{M}}$ is a gerbe bounded by $\mathfrak{J}$, any trivialization of this gerbe over $k_R(X)$ (necessarily exists as k_R is algebraically closed) gives a trivial G-torsor E_0 together with a G-equivariant map $\phi_0 \colon E_0 \to \mathfrak{M}_{\mathcal{L}}$ over $k_R(X)$. Since $\mathfrak{M}^{\mathrm{rs}} \to \mathfrak{C}_{\mathfrak{M}}$ is smooth, we can extend (E_0, ϕ_0) to a pair (E_1, ϕ_1) over $R(X)$ by formal lifting property of smoothness, where E_1 is a trivial G-torsor over $R(X)$.

The transporter between ϕ and ϕ_1 over $K(X)$ is a $\mathfrak{J}_a$-torsor, which can be trivialized after passing to a finite extension K'/K. This means that (E, ϕ) and (E_1, ϕ_1) can be glued into a pair (E', ϕ') over an open subset of $X \times \operatorname{Spec} R'$ whose complement has codimension at least 2. Since any G-torsor can be extended over a subset of codimension at least 2, and since $\mathfrak{M}_{\mathcal{L}}$ is affine over $X \times \operatorname{Spec} R'$, the pair (E', ϕ') extends to a point in $\mathcal{M}_X(R')$ lying over a. This proves the existence part of valuative criteria.

Now for the uniqueness part. Suppose $(E, \phi), (E', \phi') \in \mathcal{M}_X^{\natural}(R)$ be such that their restriction to K are isomorphic. Let ι_K be such isomorphism, then using codimension-2 argument again it suffices to extend ι_K to $R'(X)$ for some finite extension R'/R, because $K \cap R' = R$. Hence it is still harmless to assume k_R is algebraically closed. Therefore E and E' are both trivial over $R(X)$.

Moreover, as in the existence part, we may pass to a finite extension K'/K and carefully choose trivializations so that both ϕ and ϕ' map the neutral point of $E_{K'(X)} \cong E'_{K'(X)}$ to some $\gamma \in \mathfrak{M}_{\mathcal{L}}^{\mathrm{rs}}(K'(X))$. This means that $\iota_{K'(X)}$ may be represented by some element in $G_\gamma(K'(X))$. Since (E, ϕ) and (E', ϕ') are contained in the anisotropic locus, the centralizer G_γ is an anisotropic torus over $R'(X)$. Since $R'(X)$ is a discrete valuation ring with fractional field $K'(X)$, we have $G_\gamma(R'(X)) = G_\gamma(K'(X))$ and we are done. ∎

8.2 κ-decomposition and Endoscopic Transfer

Let $\tilde{h}_X^\natural\colon \tilde{\mathcal{M}}_X^\natural \to \tilde{\mathcal{A}}_X^\natural$ and $\tilde{p}_X^\natural\colon \tilde{\mathcal{P}}_X^\natural \to \tilde{\mathcal{A}}_X^\natural$ be the restriction of mH-fibration $h_X\colon \mathcal{M}_X \to \mathcal{A}_X$ and Picard stack $p_X\colon \mathcal{P}_X \to \mathcal{A}_X$ to $\tilde{\mathcal{A}}_X^\natural$ respectively. We know that both $\tilde{h}_X^\natural$ and $\tilde{p}_X^\natural$ are morphisms of Deligne-Mumford stacks with $\tilde{h}_X^\natural$ being proper and $\tilde{p}_X^\natural$ being smooth. Let $\mathcal{Q} := \mathrm{IC}_{\tilde{\mathcal{M}}_X^\natural}$ be the intersection complex on $\tilde{\mathcal{M}}_X^\natural$, then we know that $\tilde{h}_{X,*}^\natural\mathcal{Q}$ is a pure complex, hence non-canonically decomposes into a direct sum of shifted perverse sheaves

$$\tilde{h}_{X,*}^\natural\mathcal{Q} \cong \bigoplus_{n\in\mathbb{Z}} {}^{\mathfrak{p}}\mathrm{H}^n(\tilde{h}_{X,*}^\natural\mathcal{Q})[-n],$$

where ${}^{\mathfrak{p}}\mathrm{H}^n(\tilde{h}_{X,*}^\natural\mathcal{Q})$ is a perverse sheaf of weight n over $\tilde{\mathcal{A}}_X^\natural$.

8.2.1 The action of $\tilde{\mathcal{P}}_X^\natural$ on $\tilde{\mathcal{M}}_X^\natural$ relative to $\tilde{\mathcal{A}}_X^\natural$ induces an action on $\tilde{h}_{X,*}^\natural\mathcal{Q}$. According to *Lemme d'homotopie* [LN08, Lemme 3.2.3], this action factors through $\pi_0(\tilde{\mathcal{P}}_X^\natural)$ (although the statement in *loc. cit.* is about schemes and constant sheaf, its proof applies to Deligne-Mumford stacks and equivariant complexes). Over $\tilde{\mathcal{A}}_X^\natural$, we have by Proposition 7.3.1 a canonical epimorphism

$$\check{\mathbb{X}}(\mathbf{T}) \times \tilde{\mathcal{A}}_X^\natural \longrightarrow \pi_0(\tilde{\mathcal{P}}_X^\natural),$$

so we have an inflated action of $\check{\mathbb{X}}(\mathbf{T})$ on $\tilde{h}_{X,*}^\natural\mathcal{Q}$. For any $\kappa \in \check{\mathbf{T}}$, we define ${}^{\mathfrak{p}}\mathrm{H}^n(\tilde{h}_{X,*}^\natural\mathcal{Q})_\kappa$ to be the κ-isotypic subspace of ${}^{\mathfrak{p}}\mathrm{H}^n(\tilde{h}_{X,*}^\natural\mathcal{Q})$, where κ is regarded as a character $\check{\mathbb{X}}(\mathbf{T}) \to \overline{\mathbb{Q}}_\ell^\times$. Therefore we have a (necessarily finite) decomposition

$${}^{\mathfrak{p}}\mathrm{H}^n(\tilde{h}_{X,*}^\natural\mathcal{Q}) = \bigoplus_{\kappa\in\check{\mathbf{T}}} {}^{\mathfrak{p}}\mathrm{H}^n(\tilde{h}_{X,*}^\natural\mathcal{Q})_\kappa.$$

When $\kappa = 1$, we write ${}^{\mathfrak{p}}\mathrm{H}^n(\tilde{h}_{X,*}^\natural\mathcal{Q})_{\mathrm{st}}$ instead of ${}^{\mathfrak{p}}\mathrm{H}^n(\tilde{h}_{X,*}^\natural\mathcal{Q})_1$.

Recall we have the stratification (7.2.1) over G-very ample locus induced by simultaneous normalization of cameral curves

$$\tilde{\mathcal{A}}_{\gg} = \coprod_{\psi \in \Psi} \tilde{\mathcal{A}}_{\psi}.$$

Let $\tilde{\mathcal{A}}^{\natural}_{\gg} = \tilde{\mathcal{A}}_{\gg} \cap \tilde{\mathcal{A}}^{\natural}_{X}$. By Lemmas 7.2.6 and 7.3.2, the union of all ψ such that $\kappa \in \check{\mathbf{T}}(I_{\psi}, W_{\psi})$ for a fixed κ is a closed subset of $\tilde{\mathcal{A}}^{\natural}_{\gg}$. Call this subset $\tilde{\mathcal{A}}^{\natural}_{\kappa}$.

Proposition 8.2.2. *The support of* ${}^{\mathfrak{p}}\mathrm{H}^{n}(\tilde{h}^{\natural}_{\gg *}\mathcal{Q})_{\kappa}$ *is contained in* $\tilde{\mathcal{A}}^{\natural}_{\kappa}$*.*

Proof. This is a direct consequence of the definition of ${}^{\mathfrak{p}}\mathrm{H}^{n}(\tilde{h}^{\natural}_{\gg *}\mathcal{Q})_{\kappa}$ and Corollary 7.3.3. ∎

8.2.3 Now we turn to endoscopic side. Let $(\kappa, \vartheta^{\bullet}_{\kappa})$ is a pointed endoscopic datum with endoscopic group H over $\check{X}$. Let $\vartheta_{\kappa}\colon X_{\kappa} \to \check{X}$ be the corresponding $\pi_0(\kappa)$-torsor. Recall we have finite map

$$\nu^{\heartsuit}_{\mathcal{A}}\colon \mathcal{A}^{\kappa, G\text{-}\heartsuit}_{H,X} \longrightarrow \mathcal{A}^{\heartsuit}_{X},$$

by Proposition 6.11.2.

The pointed endoscopic datum is given by a continuous homomorphism $\pi_1(\check{X}, \infty) \to \pi_0(\kappa)$. There is a natural point $\infty_{\vartheta_{\kappa}}$ lying over ∞. Given $a \in \mathcal{A}^{\infty}_{X}(\bar{k})$, let

$$\tilde{X}_{\vartheta_{\kappa}, a} = \tilde{X}_{a} \times_{\check{X}} X_{\kappa}.$$

Choosing a point $\tilde{\infty} \in \tilde{X}_a$ is the same as choosing a point $\tilde{\infty}_{\vartheta_{\kappa}} = (\tilde{\infty}, \infty_{\vartheta_{\kappa}})$, and let $(a, \tilde{\infty}_{\vartheta_{\kappa}}) \in \tilde{\mathcal{A}}_{\gg}$. Suppose $a_H \in \mathcal{A}^{\kappa}_{H,X}(\bar{k})$ and $\nu_{\mathcal{A}}(a_H) = a$, then we have a finite map $\tilde{X}_{\vartheta_{\kappa}, a_H} \to \tilde{X}_{\vartheta_{\kappa}, a}$ by construction, and if $a \in \mathcal{A}^{\heartsuit}_{X}$, it birationally identifies $\tilde{X}_{\vartheta_{\kappa}, a_H}$ with the union of some irreducible components of $\tilde{X}_{\vartheta_{\kappa}, a}$. Thus we have a finite map

$$\tilde{\nu}_{\mathcal{A}}\colon \tilde{\mathcal{A}}^{\kappa}_{H,X} \longrightarrow \tilde{\mathcal{A}}_{X},$$

and it is defined over k if $(\vartheta_\kappa, \kappa)$ is.

Remark 8.2.4. Unlike Lie algebra case, $\tilde{\nu}_{\mathcal{A}}$ is in general not a closed embedding. This roughly corresponds to the fact that an irreducible representation of $\check{\mathbf{G}}^{\mathrm{sc}}$ is usually not irreducible when restricted to $\check{\mathbf{H}}^{\mathrm{sc}}$. The number of points in a fiber can, roughly speaking, be determined by looking at how many irreducible $\check{\mathbf{H}}^{\mathrm{sc}}$-subrepresentations has the same $\check{\mathbf{T}}^{\mathrm{sc}}$-weight in the said $\check{\mathbf{G}}^{\mathrm{sc}}$-representation. See the proof of Lemma 2.5.10 for details, which already contains this hint.

Proposition 8.2.5. *Over the G-very ample locus, the subset $\tilde{\mathcal{A}}_\kappa \subset \tilde{\mathcal{A}}_{\gg}$ is the disjoint union of various closed subsets $\tilde{\nu}_{\mathcal{A}}(\tilde{\mathcal{A}}_H^\kappa)$, where H is the endoscopic group corresponding to a continuous homomorphism $\vartheta_\kappa^\bullet \colon \pi_1(\check{X}, \infty) \to \pi_0(\kappa)$.*

Proof. Given a geometric point $\tilde{a} = (a, \tilde{\infty}) \in \tilde{\mathcal{A}}_X(\bar{k})$, recall we have diagram (6.5.1) which we reproduce here:

$$\begin{array}{ccc} \pi_1(U, \infty) & \xrightarrow{\pi_{\tilde{a}}^\bullet} & \mathbf{W} \rtimes \mathrm{Out}(\mathbf{G}) \\ \downarrow & & \downarrow \\ \pi_1(\check{X}, \infty) & \xrightarrow{\vartheta_G^\bullet} & \mathrm{Out}(\mathbf{G}) \end{array}$$

Here $U = \check{X} - \mathfrak{D}_a$. Let $W_{\tilde{a}}$ be the image of $\pi_{\tilde{a}}^\bullet$ in $\mathbf{W} \rtimes \mathrm{Out}(\mathbf{G})$ and $I_{\tilde{a}}$ the image of the kernel of $\pi_1(U, \infty) \to \pi_1(\check{X}, x)$.

If $\tilde{a} \in \tilde{\mathcal{A}}_\kappa$, then $W_{\tilde{a}} \subset (\mathbf{W} \rtimes \mathrm{Out}(\mathbf{G}))_\kappa$ and $I_{\tilde{a}} \subset \mathbf{W_H}$. Here we canonically identify $(\mathbf{W} \rtimes \mathrm{Out}(\mathbf{G}))_\kappa$ with $\mathbf{W_H} \rtimes \pi_0(\kappa)$ by [Ngô06, Lemme 10.1]. Then $\pi_{\tilde{a}}^\bullet$ induces a unique homomorphism

$$\vartheta_\kappa^\bullet \colon \pi_1(\check{X}, \infty) \longrightarrow \pi_0(\kappa),$$

and let H be the corresponding endoscopic group. Let $C_{\tilde{a}}^\kappa \subset \tilde{X}_{\vartheta_\kappa, a}$ be the union of irreducible components in the $\mathbf{W_H} \rtimes \pi_0(\kappa)$-orbit of the unique component containing $\tilde{\infty}_{\vartheta_\kappa}$, then $\mathbf{W_H} \rtimes \pi_0(\kappa)$ acts transitively on fibers of $C_{\tilde{a}}^\kappa \to \check{X}$. This means that $\tilde{a}$ comes from a map $\check{X} \to \left[\mathfrak{C}'_{\mathfrak{M},H}/Z_{\mathfrak{M}}\right]$ (recall that over $\check{X}$, $\mathfrak{C}'_{\mathfrak{M},H} = (\bar{\mathbf{T}}_{\mathbf{M}} \times X_\kappa)/(\mathbf{W_H} \rtimes \pi_0(\kappa))$ by definition).

The map

$$\left[\mathfrak{C}_{\mathfrak{M},H}/Z^{\kappa}_{\mathfrak{M}}\right] \longrightarrow \left[\mathfrak{C}'_{\mathfrak{M},H}/Z_{\mathfrak{M}}\right]$$

is an isomorphism over the intersection of invertible and G-regular semisimple loci. The point $\tilde{a}$, viewed as a map $\check{X} \to [\mathfrak{C}_{\mathfrak{M}}/Z_{\mathfrak{M}}]$ that is generically contained in the invertible and regular semisimple locus, can then be lifted to a rational map from $\check{X}$ to $\left[\mathfrak{C}_{\mathfrak{M},H}/Z^{\kappa}_{\mathfrak{M}}\right]$. By Lemma 2.5.17, it can be extended to a morphism $\check{X} \to \left[\mathfrak{C}_{\mathfrak{M},H}/Z^{\kappa}_{\mathfrak{M}}\right]$. This shows that every $\tilde{a} \in \tilde{\mathcal{A}}_{\kappa}(\bar{k})$ comes from some (not necessarily unique) $\tilde{a}_H \in \tilde{\mathcal{A}}^{\kappa}_{H,X}(\bar{k})$. The argument also shows that $\vartheta^{\bullet}_{\kappa}$ hence H is uniquely determined by $\tilde{a}$.

Conversely, let H be an endoscopic group given by a homomorphism $\vartheta^{\bullet}_{\kappa}$ and $\tilde{a}_H \in \tilde{\mathcal{A}}^{\kappa}_{H}(\bar{k})$. Then the H-cameral cover $\tilde{X}_{a_H} \to \check{X}$ is étale over U. Let $U_H \subset \check{X}$ be the largest subset over which this H-cameral cover is étale, then $U \subset U_H$. So the homomorphism

$$\pi^{\bullet}_{\tilde{a}_H} \colon \pi_1(U_H, \infty) \longrightarrow \mathbf{W_H} \rtimes \mathrm{Out}(\mathbf{H})$$

induces homomorphism

$$\pi^{\kappa,\bullet}_{\tilde{a}_H} \colon \pi_1(U, \infty) \longrightarrow \mathbf{W_H} \rtimes \pi_0(\kappa)$$

lying over $\vartheta^{\bullet}_{\kappa}$. Let $\tilde{a} \in \tilde{\mathcal{A}}_{\gg}$ be the image of $\tilde{a}_H$, then $\pi^{\bullet}_{\tilde{a}}$ is the composition of map $\pi^{\kappa,\bullet}_{\tilde{a}_H}$ with canonical map $\mathbf{W_H} \rtimes \pi_0(\kappa) \to \mathbf{W} \rtimes \mathrm{Out}(\mathbf{G})$. Therefore we have $W_{\tilde{a}} \subset (\mathbf{W} \rtimes \mathrm{Out}(\mathbf{G}))_{\kappa}$ and $I_{\tilde{a}} \subset \mathbf{W_H}$, and so $\tilde{a} \in \tilde{\mathcal{A}}_{\kappa}(\bar{k})$. ∎

8.2.6 Similar to κ-strata, one can describe the compliment $\tilde{\mathcal{A}} - \tilde{\mathcal{A}}^{\natural}$ using proper Levi subgroups of $\mathbf{G}$ containing $\mathbf{T}$. Suppose $\tilde{a} \in (\tilde{\mathcal{A}} - \tilde{\mathcal{A}}^{\natural})(\bar{k})$, then $\mathbf{T}^{W_{\tilde{a}}}$ is not finite, hence contains a subtorus $\mathbf{S}$. Let $\mathbf{L}$ be the centralizer of $\mathbf{S}$ in $\mathbf{G}$, then $W_{\tilde{a}}$ is contained in the centralizer of $\mathbf{S}$ in $\mathbf{W} \rtimes \mathrm{Out}(\mathbf{G})$, and $I_{\tilde{a}}$ is contained in $\mathbf{W_L}$. Let $\tilde{\mathbf{L}}$ be the centralizer of $\mathbf{S}$

in $\mathbf{G} \rtimes \mathrm{Out}(\mathbf{G})$ as in § 2.5.25. Therefore there are induced maps

$$\pi_1(\check{X}, \infty) \longrightarrow W_{\tilde{a}}/I_{\tilde{a}} \longrightarrow \pi_0(\tilde{\mathbf{L}}) \longrightarrow \mathrm{Out}(\mathbf{G}).$$

Let $\mathcal{A}_X^L$ be the mH-base corresponding to $\left[\mathfrak{C}_{\mathfrak{M}}^L/Z_{\mathfrak{M}}^L\right]$, then the above maps imply that $\tilde{a}$ comes from a point in $\tilde{\mathcal{A}}_X^L$, similar to endoscopic case.

Let $a_L \in \mathcal{A}_X^L(\bar{k})$ be a point over $a \in \mathcal{A}_X(\bar{k})$, with respective boundary divisor λ_L and λ. Suppose $N > 0$ is such that $\langle \rho, \lambda \rangle - \langle \rho_L, \lambda_L \rangle > N$ for all proper Levi subgroups L and all possible λ_L over λ. It is clear by representation-theoretic construction of $\mathfrak{M}_L$ that N depends only on the connected component of $\mathcal{B}_X$ containing λ. Combining with (6.11.4) (the same argument works for Levi subgroups), we have the following result:

Proposition 8.2.7. *Suppose the center of G does not contain a split torus. Let $B \subset \mathcal{B}_{\gg}$ be an irreducible component and $A \subset \tilde{\mathcal{A}}_X$ be the preimage of B. Then the codimension of $A - A^\natural$ is at least $N - rg_X$ where N is as in the paragraph above.*

Remark 8.2.8. It is not hard to see that if λ is a non-zero dominant cocharacter in any non-trivial direct factor of G^{ad}, then by replacing λ with its multiples, one can always find a component $A \subset \mathcal{A}_X$ such that the codimension of $A - A^\natural$ is larger than a given number.

8.3 Geometric Stabilization

In [Ngô10, Théorèmes 6.4.1, 6.4.2], Ngô established the geometric stabilization theorem for the usual Hitchin fibrations. It is essentially the geometric side of the stabilization process of the trace formula for anisotropic Lie algebras over function fields, and one can deduce from it the endoscopic fundamental lemma for Lie algebras. Our goal for multiplicative Hitchin fibration is the same, however, it is technically more complicated.

8.3.1 In the Lie algebra case, the total stack of Hitchin fibration is smooth, and we are considering the constant sheaf on that stack. This corresponds to the fact that at local places v, the functions we are considering are the characteristic functions on $\pi^d\mathfrak{g}(\mathcal{O}_v)$ for some integer $d \leq 0$.

In the multiplicative case, however, we are considering the basic functions in the spherical Hecke algebra, which corresponds to the fact that we have a local model of singularity Theorem 6.10.2. Ideally, we would like the theorem to be true over all $\mathcal{M}_{\gg}^{\heartsuit}$, but it seems so far some tighter cohomological constraint must be assumed. We also noted in Remark 6.10.15 that the constraint as written in Theorem 6.10.2 can be foreseeably loosened, but probably not to the extent that the whole $\mathcal{M}_{\gg}^{\heartsuit}$ can be covered.

On the other hand, we do not believe this cohomological constraint is a purely technical nuisance either. In fact, we predict that there is a "limit version" of mH-fibrations, and the cohomological constraint is there because mH-fibrations are "truncations" of the limit. We will be discussing this later in more detail, so we do not expand here.

8.3.2 Another issue is the dimension estimate of the endoscopic strata. We already noted in § 6.11 that such estimation is not very straightforward if the semisimple rank of H is not the same as that of G, due to potential insufficiency in appropriate ampleness. We also noted in Remark 6.11.8 that it seems to be a different aspect of the same problem as the issue with local model of singularity, so at least it does not add to our difficulty. One can also see for example § 9.9.5 that in some situations one can deduce the desired dimension estimate if local model of singularity is established.

8.3.3 Due to the reasons discussed above, in this paper we only attempt to prove a weaker version of the geometric stabilization theorem.

Theorem 8.3.4 (Weak geometric stabilization for adjoint groups)**.** *Suppose $G = G^{\mathrm{ad}}$. Let $\mathcal{U} \subset \tilde{\mathcal{A}}_{\kappa}^{\natural}$ be an open subset over which the local model of singularity Theorem 6.10.2 holds* *for all the endoscopic groups appearing in $\mathcal{U}$. Then there exists an isomorphism between graded shifted perverse sheaves over $\mathcal{U}$*

$$\bigoplus_n {}^{\mathfrak{p}}\mathrm{H}^n(\tilde{h}^{\natural}_{X,*}\mathcal{Q})_\kappa \cong \bigoplus_n \bigoplus_{(\kappa,\vartheta^\bullet_{\kappa,\xi})} \tilde{\nu}^{\natural}_{\xi,*}\, {}^{\mathfrak{p}}\mathrm{H}^n(\tilde{h}^{\kappa,\natural}_{H_\xi,X,*}\mathcal{Q}^{\kappa}_{H_\xi})_{\mathrm{st}},$$

where $(\kappa,\vartheta^\bullet_{\kappa,\xi})$ ranges over all pointed endoscopic data for a fixed κ, H_ξ is the corresponding endoscopic group, $\tilde{\nu}_\xi$ is the endoscopic transfer map $\tilde{\nu}_{\mathcal{A}}$ corresponding to H_ξ, and $\mathcal{Q}^{\kappa}_{H_\xi}$ is the intersection complex of $\tilde{\mathcal{M}}^{\kappa,\natural}_{H_\xi,X}$.

Since $\tilde{\nu}_\xi$ for different ξ have disjoint image, we see that Theorem 8.3.4 is a consequence of the following statement:

Theorem 8.3.5. *With the assumptions in Theorem 8.3.4, suppose $(\kappa,\vartheta^\bullet_{\kappa,\xi})$ is defined over k, then there exists a k-isomorphism of semisimplifications of graded shifted perverse sheaves over $\mathcal{U}$*

$$\Big(\bigoplus_n {}^{\mathfrak{p}}\mathrm{H}^n(\tilde{h}^{\natural}_{X,*}\mathcal{Q})_\kappa\Big)\Big|_{\tilde{\nu}^{\natural}_{\xi}(\tilde{\mathcal{A}}^{\kappa,\natural}_{H_\xi,X})} \cong \tilde{\nu}^{\natural}_{\xi,*} \bigoplus_n {}^{\mathfrak{p}}\mathrm{H}^n(\tilde{h}^{\kappa,\natural}_{H_\xi,X,*}\mathcal{Q}^{\kappa}_{H_\xi})_{\mathrm{st}}.$$

These two results will be proved in Chapter 10 after a complicated back-and-forth between global and local arguments, similar to what is done in [Ngô10, § 8].

Conjecture 8.3.6 (Strong geometric stabilization)**.** *Theorems 8.3.4 and 8.3.5 hold over the entire very G-ample anisotropic locus $\tilde{\mathcal{A}}^{\natural}_{\gg}$ for all G and H.*

8.4 Top Ordinary Cohomology

In this section we study the top ordinary cohomology of $\tilde{h}^{\natural}_{X,*}\mathcal{Q}$. We hope to give a description similar to that in [Ngô10, § 6.5] which can be used in tandem with so-called support theorem.

8.4.1 First we consider a general map $f\colon X \to Y$ of k-varieties (or more generally Deligne-Mumford stacks locally of finite types over k) of relative dimension d. The complex $f_!\overline{\mathbb{Q}}_\ell$ will have cohomological amplitude $[0, 2d]$. The stalk of the sheaf of top cohomology $\mathbf{R}^{2d} f_!\overline{\mathbb{Q}}_\ell$ at geometric point $y \in Y$ is a $\overline{\mathbb{Q}}_\ell$-vector space with a canonical basis in bijection with irreducible components of fiber X_y. If X is smooth, then the constant sheaf $\overline{\mathbb{Q}}_\ell$ is a pure complex, and so is $f_!\overline{\mathbb{Q}}_\ell$ if in addition f is proper.

On the other hand, if f is proper but X is not smooth, then $f_*\overline{\mathbb{Q}}_\ell = f_!\overline{\mathbb{Q}}_\ell$ is not necessarily pure, and one needs to replace $\overline{\mathbb{Q}}_\ell$ with intersection complex IC_X in order to restore purity. However, in general $f_*\mathrm{IC}_X$ may have larger cohomological amplitude than $2d$.

Unlike Lie algebra case, the total space of mH-fibration is not smooth in general; rather we have a local model of singularity established in Theorem 6.10.2. Therefore in order to give a nice description of the top ordinary cohomology, we need to establish cohomological amplitude of $\tilde{h}^{\natural}_{X,*}\mathscr{Q}$.

8.4.2 The question is local in $\tilde{\mathcal{A}}^{\natural}_X$, and by proper base change, we may fix $b \in \mathcal{B}_X(\bar{k})$ and restrict to $\tilde{\mathcal{A}}^{\natural}_b$. Recall by Theorem 6.10.2, we have an open dense subset $\mathcal{U} \subset \tilde{\mathcal{A}}^{\natural}_b$ such that $\mathcal{M} \coloneqq \tilde{h}^{\natural,-1}_X(\mathcal{U})$ admits a local model of singularity given by affine Schubert varieties. Here the boundary divisor b is fixed, so the local model is just a finite direct product

$$\mathsf{Q} \coloneqq \prod_{i=1}^{m} \left[\mathbb{L}^+_{\bar{v}_i,N} G \backslash \mathsf{Gr}^{\leq -w_0(\lambda_i)}_{\bar{v}_i}\right],$$

where $\bar{v}_i \in X(\bar{k})$ are distinct points and N is a sufficiently large integer depending on b. Let $\mathrm{ev}_N\colon \mathcal{M} \to \mathsf{Q}$ be the evaluation map as in Theorem 6.10.2, but to save notations we will simply call it ev instead (and it is not the same as the ev in Theorem 6.10.2). Let e be the relative dimension of ev.

By Theorem 6.10.2 and Proposition 5.4.4, we know that

$$\mathcal{Q}|_{\mathcal{M}} = \mathrm{ev}^* \mathcal{F}[e](e/2)$$

for some equivariant perverse sheaf $\mathcal{F}$ on Q. We also know that IC_{Q} is the unique direct summand of $\mathcal{F}$ supported on the entire Q, and if b is multiplicity-free in the sense of Proposition 5.1.25, then $\mathcal{F}$ is exactly IC_{Q}.

The well-known fact from the theory of geometric Satake isomorphism is that simple equivariant perverse sheaves on Q of weight 0 are none other than the intersection complexes of substacks

$$\mathsf{Q}' := \prod_{i=1}^{m} \left[\mathbb{L}^+_{\overline{v}_i,N} G \backslash \mathsf{Gr}_{\overline{v}_i}^{\leq -w_0(\lambda'_i)} \right]$$

for some $\lambda'_i \leq \lambda_i$. Let $\lambda_b = \sum_{i=1}^m \lambda_i \cdot \overline{v}_i$ and $\lambda'_b = \sum_{i=1}^m \lambda'_i \cdot \overline{v}_i$, by smoothness of ev, we can compute the codimension of $\mathcal{M}' = \mathrm{ev}^{-1}(\mathsf{Q}')$:

$$2\delta' := \mathrm{codim}_{\mathcal{M}}(\mathcal{M}') = \mathrm{codim}_{\mathsf{Q}}(\mathsf{Q}') = \sum_{i=1}^{m} \langle 2\rho, \lambda_i - \lambda'_i \rangle = \deg \langle 2\rho, \lambda_b - \lambda'_b \rangle,$$

and that of $\mathcal{U}' = \tilde{h}^\natural_X(\mathcal{M}')$ by Riemann-Roch theorem,

$$\mathrm{codim}_{\mathcal{U}}(\mathcal{U}')) \leq \sum_{i=1}^{m} \langle \rho, \lambda_i - \lambda'_i \rangle = \frac{1}{2} \mathrm{codim}_{\mathcal{M}}(\mathrm{ev}^{-1}(\mathsf{Q}')) = \delta'.$$

The last inequality is similar to the computations of δ-critical and inductive strata in §§ 7.4 and 7.5, so we do not repeat here. Using dimension formula Theorem 4.2.1 and product formula Proposition 6.9.1, we see that the restriction of $\tilde{h}^\natural_X$ to $\mathcal{M}'$ is of relative dimension at most $d - \delta'$, where $d = \dim \mathcal{M} - \dim \mathcal{U}$.

Since all the Q' for different $\lambda'_b \leq \lambda_b$ induces a stratification of Q by smooth substacks, the construction of intermediate extension functor implies that the support of $\mathrm{H}^i(\mathrm{IC}_{\mathsf{Q}})$

is contained in the union of those Q' with dimension at most $-i$, and the equality is achieved if and only if $-i = \dim \mathsf{Q}$. This implies that $\mathrm{IC}_{\mathcal{M}} = \mathrm{ev}^*\mathrm{IC}_{\mathsf{Q}}[e](e/2)$ has its i-th cohomology supported on those $\mathcal{M}'$ with dimension at most $-i$, and the equality is achieved if and only if $-i = \dim \mathcal{M}$. In other words, if $\mathcal{M}'$ is contained in the support of $\mathrm{H}^i(\mathrm{IC}_{\mathcal{M}})$, then $2\delta' \geq i + \dim \mathcal{M}$, with equality achieved if and only if $\delta' = 0$. This shows that $\tilde{h}^{\natural}_{X,*}(\mathrm{H}^i(\mathrm{IC}_{\mathcal{M}}))$ is supported on cohomological degrees

$$[-\dim \mathcal{M}, -\dim \mathcal{M} + 2d],$$

and with the upper bound achieved if and only if $i = -\dim \mathcal{M}$. A standard argument using spectral sequence then shows that $\tilde{h}^{\natural}_{X,*}(\mathrm{IC}_{\mathcal{M}})$ has cohomological degree bounded above by $-\dim \mathcal{M} + 2d$, and the stalk of its top cohomology at any geometric point $a \in \mathcal{U}$ has a basis in bijection with the irreducible components of $\mathcal{M}_a$.

The same argument can be applied to $\mathrm{IC}_{\mathcal{M}'} = \mathrm{ev}^*\mathrm{IC}_{\mathsf{Q}'}[e](e/2)$, and if $\mathrm{codim}_{\mathcal{U}}(\mathcal{U}')$ is exactly δ', then we may have additional contribution to the top cohomology. Using geometric Satake isomorphism, the appearance of $\mathrm{IC}_{\mathsf{Q}'}$ has a representation-theoretic explanation, i.e., the branching rule in the decomposition of (symmetric) tensor products of irreducible $\check{G}^{\mathrm{sc}}$-representations. So such additional contribution to top cohomology is also relatively easy to compute. Thus far we have shown that

Proposition 8.4.3. *Let $\mathcal{U} \subset \tilde{\mathcal{A}}^{\natural}_X$ be any irreducible open substack over which there is a local model of singularity as in Theorem 6.10.2, and $\mathcal{M} = \tilde{h}_X^{-1}(\mathcal{U})$. Then the complex $\tilde{h}^{\natural}_{X,*}(\mathcal{Q})|_{\mathcal{U}}$ is supported on cohomological degrees*

$$[-\dim \mathcal{M}, -\dim \mathcal{M} + 2d],$$

where $d = \dim \mathcal{M} - \dim \mathcal{U}$. In addition, if $\tilde{a} \in \mathcal{U}$ lies over a multiplicity-free boundary divisor in the sense of Proposition 5.1.25, then $\mathrm{H}^{-\dim \mathcal{M} + 2d}(\tilde{h}^{\natural}_{X,*}\mathcal{Q})_{\tilde{a}}$ has a canonical basis being the irreducible components of $\mathcal{M}_{\tilde{a}}$.

8.4.4 For simplicity, we retain the notations in Proposition 8.4.3. We let $p\colon \mathcal{P} \to \mathcal{U}$ be the pullback of $\mathcal{P}_X \to \mathcal{A}_X$ to $\mathcal{U}$, and let h be the restriction of $\tilde{h}_X$ to $\mathcal{U}$. We still use $\mathcal{Q}$ to denote $\mathrm{IC}_{\mathcal{M}} = \mathrm{IC}_{\tilde{\mathcal{M}}_X^\natural}|_{\mathcal{M}}$ and it should not cause any confusion.

According to *Lemme d'homotopie* [LN08, Lemme 3.2.3], the action of $\mathcal{P}$ on $h_*\mathcal{Q}$ factors through sheaf of finite abelian groups $\pi_0(\mathcal{P})$, and so does the action on $\mathbf{R}^{-\dim\mathcal{M}+2d}h_*\mathcal{Q}$. Let $\tilde{a} \in \mathcal{U}(\bar{k})$ be a point lying over a multiplicity-free boundary divisor $\lambda_b = \sum_{i=1}^m \lambda_i \bar{v}_i$. Being multiplicity-free means that at each point $\bar{v}_i$, λ_i is a cocharacter in the set of minimal generators of the cone of $\mathfrak{A}_{\mathfrak{M}}$ at $\bar{v}_i$. By Proposition 8.4.3, the $\mathcal{P}_{\tilde{a}}$-action on $(\mathbf{R}^{-\dim\mathcal{M}+2d}h_*\mathcal{Q})_{\tilde{a}}$ is just the action induced by the $\pi_0(\mathcal{P}_{\tilde{a}})$-action on the set of irreducible components of $\mathcal{M}_{\tilde{a}}$.

Using product formula Propositions 6.9.1 and 6.9.6, we see that $(\mathbf{R}^{-\dim\mathcal{M}+2d}h_*\mathcal{Q})_{\mathrm{st},\tilde{a}}$ has a canonical basis in bijection with the (necessarily finite) direct product

$$\prod_{\bar{v}\in X(\bar{k})} \mathrm{Irr}[\mathcal{M}_{\bar{v}}(\tilde{a})/\mathcal{P}_{\bar{v}}(\tilde{a})].$$

If a is unramified at point $\bar{v}_i$, then its Newton point ν_i is integral, and we know

$$\#\mathrm{Irr}\Big[\mathcal{M}_{\bar{v}_i}(\tilde{a})/\mathcal{P}_{\bar{v}_i}(\tilde{a})\Big] = m_{-w_0(\lambda_i)\nu_i}$$

according to Theorem 4.3.5. For the same reason, when $\bar{v} \neq \bar{v}_i$ for any i, the regular locus $\mathcal{M}_{\bar{v}}(\tilde{a})^{\mathrm{reg}}$ is dense in $\mathcal{M}_{\bar{v}}(\tilde{a})$ and is a $\mathcal{P}_{\bar{v}}(\tilde{a})$-torsor. So in this case

$$\#\mathrm{Irr}[\mathcal{M}_{\bar{v}}(\tilde{a})/\mathcal{P}_{\bar{v}}(\tilde{a})] = 1.$$

Suppose $\tilde{a}$ is very (G,N)-ample for some $N = N(\delta_{\tilde{a}})$ as in § 7.4. This ensures that the δ-stratum in $\mathcal{U}$ containing $\tilde{a}$ has codimension at least $\delta_{\tilde{a}}$ in $\mathcal{U}$. Furthermore, assume $\tilde{a}$ is contained in a δ-critical stratum and is a general enough point therein. Then we know by Corollary 7.4.7 that $\tilde{a}$ is unramified at each $\bar{v}_i$ and its Newton point ν_i at $\bar{v}_i$ is integral.

Thus we have the following:

Proposition 8.4.5. *Let $\tilde{a} \in \mathcal{U}(\bar{k})$ be such that it is unramified at every $\bar{v}_i$. Then the stalk $(\mathbf{R}^{-\dim \mathcal{M}+2d} h_* \mathcal{Q})_{\mathrm{st},\tilde{a}}$ is a $\overline{\mathbb{Q}}_\ell$-vector space of rank $\sum_i m_{-w_0(\lambda_i)\nu_i}$. In particular, it is true for $\tilde{a}$ that is very $(G, N(\delta_{\tilde{a}}))$-ample and is a general enough point of a δ-critical stratum.*

8.4.6 Suppose we have a κ-stratum $\mathcal{U}_{(\kappa,\vartheta^\bullet_{\kappa,\xi})} \subset \mathcal{U}$ corresponding to endoscopic datum $(\kappa, \vartheta^\bullet_{\kappa,\xi})$ and endoscopic group H. Let $\mathcal{U}_H^\kappa$ be the preimage of $\mathcal{U}_{(\kappa,\vartheta^\bullet_{\kappa,\xi})}$ in $\tilde{\mathcal{A}}_{H,X}^\kappa$. Note that $\mathcal{U}_H^\kappa$ may still have multiple irreducible components of various dimensions even if $\mathcal{U}_{(\kappa,\vartheta^\bullet_{\kappa,\xi})}$ is irreducible.

Suppose that $\mathcal{U}_H^\kappa$ is very H-ample, then $\delta_{\tilde{a}_H} = 0$ for any general $\tilde{a}_H$ in each irreducible component of $\mathcal{U}_H^\kappa$. Let $S_H \subset \mathcal{U}_H^\kappa$ be an irreducible component and let S be its image in $\mathcal{U}_{(\kappa,\vartheta^\bullet_{\kappa,\xi})}$. By upper-semicontinuity, the δ-invariant achieves minimal value over an open dense subset of S, and let δ_S be this value.

If $\mathrm{codim}_{\mathcal{U}}(S) = \delta_S$, then by definition S is δ-critical. In fact, we know that the image of any δ-critical stratum in S_H is also δ-critical in $\mathcal{U}$, because the difference $\delta - \delta_H$ is constant throughout S by (6.11.5). For any $\delta' > 0$, if we further assume that S_H is very $(H, N(\delta'))$-ample (note that $N(\delta')$ here depends on both H and δ'), then by Proposition 7.4.5 applied to S_H, we have that

$$\mathrm{codim}_{\mathcal{U}}(S \cap \mathcal{U}_{\delta_S+\delta'}) \geq \delta_S + \delta'.$$

8.4.7 Suppose now $\tilde{a}_H$ is a general $\bar{k}$-point of a δ-critical stratum $S_{H,\delta'}$ in S_H, and suppose it is very $(H, N(\delta'))$-ample. Let

$$\lambda_{H,b_H} = \sum_{i=1}^{m} \lambda_{H,i} \cdot \bar{v}_i$$

be the boundary divisor of $\tilde{a}_H$. Let $\tilde{a}$ be the image of $\tilde{a}_H$ in S, then it is also a general point in a δ-critical stratum in $\mathcal{U}$. Let

$$\lambda_b = \sum_{i=1}^{m} \lambda_i \cdot \bar{v}_i$$

be the boundary divisor of $\tilde{a}$.

There is no guarantee that $\tilde{a}$ is very $(G, N(\delta_{\tilde{a}}))$-ample even assuming S is δ-critical, so we cannot directly apply Corollary 7.4.7 to $\tilde{a}$. Nevertheless, we may apply it to $\tilde{a}_H$ since we assume $\tilde{a}_H$ is $(H, N(\delta'))$-very ample, and it implies that $\tilde{a}_H$ is unramified at every $\bar{v}_i$. Since the local ramification index $c_{\bar{v}_i}(\tilde{a})$ (resp. $c_{H,\bar{v}_i}(\tilde{a}_H)$) depends only on the generic fiber of the regular centralizer $\mathfrak{J}_{\tilde{a}}$ (resp. $\mathfrak{J}_{H,\tilde{a}_H}$), and we know generically $\mathfrak{J}_{\tilde{a}}$ and $\mathfrak{J}_{H,\tilde{a}_H}$ are canonically isomorphic, we have

$$c_{\bar{v}_i}(\tilde{a}) = c_{H,\bar{v}_i}(\tilde{a}_H) = 0.$$

In other words, $\tilde{a}$ must be unramified at every $\bar{v}_i$. Using the same argument as Proposition 8.4.5, and replace the stable constituent by κ-isotypic constituent, we reach a similar description.

Proposition 8.4.8. *Suppose $\tilde{a}_H$ is a general point of a δ-critical stratum in $\tilde{\mathcal{A}}_{H,X}^{\kappa,\natural}$ and is very $(H, N(\delta_{H,\tilde{a}_H}))$-ample. Let $\tilde{a} \in \mathcal{U}$ be the image of $\tilde{a}_H$. Suppose the local model of singularity as in Theorem 6.10.2 exists in a neighborhood of $\tilde{a}$, and suppose $\tilde{a}$ has multiplicity-free boundary divisor. Then $(\mathbf{R}^{-\dim \mathcal{M}+2d} h_* \mathcal{Q})_{\kappa,\tilde{a}}$ has rank $\sum_i m_{-w_0(\lambda_i)\nu_i}$.*

Since at each $\bar{v}_i$ the Newton point ν_i depends only on the generic point of $\tilde{a}$, and similarly for the Newton point $\nu_{H,i}$ of $\tilde{a}_H$, we have that $\nu_i = \nu_{H,i}$ viewed as rational cocharacters of $T^{\mathrm{ad}}(\breve{F}_{\bar{v}_i})$. If $\tilde{a}$ is multiplicity-free, then so is $\tilde{a}_H$ by construction of $\mathfrak{M}_H$. Using Proposition 8.4.5 and suppose local model of singularity exists near $\tilde{a}_H$ too, then we have

$$\dim_{\overline{\mathbb{Q}}_\ell}(\mathbf{R}^{-\dim_{\tilde{a}_H}\mathcal{M}_H+2d_H}h_{H,*}\mathcal{Q}_H)_{\mathrm{st},\tilde{a}_H}=\sum_{i=1}^{m}m_{-w_{H,0}(\lambda_{H,i})\nu_i},$$

where $\dim_{\tilde{a}_H}\mathcal{M}_H$ is the dimension of the component of $\tilde{\mathcal{M}}_{H,X}^{\kappa}$ containing $\tilde{a}_H$, and d_H is the relative dimension of mH-fibration at $\tilde{a}_H$.

8.4.9 Choose compatible identifications $G\cong\mathbf{G}$ and $H\cong\mathbf{H}$ at $\bar{v}_i$ and let $\check{\mathbf{H}}'\subset\check{\mathbf{G}}^{\mathrm{sc}}$ be the preimage of $\check{\mathbf{H}}$. Recall that $-w_{\mathbf{H},0}(\lambda_{\mathbf{H},i})$ is an $\check{\mathbf{H}}'$-highest weight in the decomposition of the irreducible $\check{\mathbf{G}}^{\mathrm{sc}}$-representation with highest weight $-w_0(\lambda_i)$, and we must have

$$\nu_i\leq_{\mathbf{H}}-w_{\mathbf{H},0}(\lambda_{\mathbf{H},i}).$$

With this restriction in mind, if the set of $\tilde{a}_H$ is non-empty for a fixed $\tilde{a}$, then among those $\tilde{a}_H$ mapping to a fixed $\tilde{a}$, they all restrict to the same map $\tilde{a}_H^{\circ}$ from $\check{X}-\mathrm{supp}(b)$ to $[\mathfrak{C}_{\mathfrak{M},H}/Z_{\mathfrak{M}}^{\kappa}]$. The set of ways to extend $\tilde{a}_H^{\circ}$ over $\bar{v}_i$ is in natural bijection with the set of irreducible $\check{\mathbf{H}}'$-representations contained in the $\check{\mathbf{G}}$-representation of highest weight $-w_0(\lambda_i)$, same as in the proof of Lemma 2.5.17 (see also Proposition 6.11.2). Do the same for each $\bar{v}_i$ one by one, we reach equality

$$\dim_{\overline{\mathbb{Q}}_\ell}(\mathbf{R}^{-\dim\mathcal{M}+2d}h_*\mathcal{Q})_{\kappa,\tilde{a}}=\sum_{\tilde{a}_H\mapsto\tilde{a}}\dim_{\overline{\mathbb{Q}}_\ell}(\mathbf{R}^{-\dim_{\tilde{a}_H}\mathcal{M}_H+2d_H}h_{H,*}\mathcal{Q}_H)_{\mathrm{st},\tilde{a}_H}.$$

If furthermore we assume that a Steinberg quasi-section exists for $\mathfrak{M}$, then using product formulae Propositions 6.9.1 and 6.9.6 and Proposition 4.5.12, and suppose both a and a_H are defined over some finite extension k'/k inside $\bar{k}$, we even have canonical isomorphism of $\mathrm{Gal}(\bar{k}/k')$-modules induced by the restriction functor $\mathrm{Res}_{\check{\mathbf{H}}'}^{\check{\mathbf{G}}}$:

$$(\mathbf{R}^{-\dim\mathcal{M}+2d}h_*\mathcal{Q})_{\kappa,\tilde{a}}\simeq\bigoplus_{\tilde{a}_H\mapsto\tilde{a}}(\mathbf{R}^{-\dim_{\tilde{a}_H}\mathcal{M}_H+2d_H}h_{H,*}\mathcal{Q}_H)_{\mathrm{st},\tilde{a}_H}. \tag{8.4.1}$$

Remark 8.4.10. (1) (8.4.1) can be viewed as a sort of primal form of Theorem 8.3.4;

(2) On G-side, we need Proposition 6.9.1 to hold over k' in order to maintain compatible κ-twisting on both sides of the product formula; on the other hand, the stable constituent does not have κ-twisting, so we may simply use Proposition 6.9.6 on H-side, which always holds over k'.

(3) In fact, there is no need to assume that $\tilde{a}$ has multiplicity-free boundary divisor in (8.4.1) due to the representation-theoretic interpretation of $\mathscr{Q}$ and $\mathscr{Q}_H$ (see the discussion preceding Proposition 8.4.3). However, we do not explicitly state it here because the notations would become too involved and it does not provide anything more interesting.

8.4.11 Up until now in this section we made a lot of assumptions. For reader's convenience we summarize the essential ones below:

(1) For both G and H, we require the existence of respective local model of singularity as in Theorem 6.10.2 (only its conclusion, not the cohomological conditions therein), so that we have a representation-theoretic description of $\mathscr{Q}$ and $\mathscr{Q}_H$ respectively.

(2) For stable constituent of the cohomology, some ampleness condition on the boundary divisor depending only on the group and δ-invariant (the curve X is always fixed).

(3) For κ-constituent, only ampleness condition on H-side is required, not for G, although we still require δ-criticality on the G-side.

(4) For a particularly clean formula for the rank of top cohomology, we want the boundary divisor to be multiplicity-free in the sense of Proposition 5.1.25.

(5) For (8.4.1), no multiplicity-free condition is required.

8.4.12 So far our description of the top cohomology $\mathbf{R}^{-\dim\mathcal{M}+2d}h_*\mathscr{Q}$ is at the stalk level. Due the jump in ranks, there does not appear to be an easy description of the top cohomology as a sheaf even just for the stable constituent. The reader can compare to the Lie algebra case where the stable top cohomology is just the constant sheaf of rank 1 over the Hitchin base (see [Ngô10, Proposition 6.5.1]).

Nevertheless, it is still possible to describe $(\mathbf{R}^{-\dim\mathcal{M}+2d}h_*\mathscr{Q})_{\mathrm{st}}$ over each δ-critical stratum. Over the open dense subset $\mathcal{U}_0 \subset \mathcal{U}$ where $\delta = 0$, since $\mathcal{M}$ is a $\mathcal{P}$-torsor, $(\mathbf{R}^{-\dim\mathcal{M}+2d}h_*\mathscr{Q})_{\mathrm{st}}$ is isomorphic to constant sheaf $\overline{\mathbb{Q}}_\ell$ up to a Tate twist. In fact, we can do much better than this.

Lemma 8.4.13. *With the setup in Proposition 8.4.5, let $\mathcal{U}' \subset \mathcal{U}$ be the open dense locus where either the discriminant divisor and the boundary divisor do not collide or $\delta = 0$. Then $(\mathbf{R}^{-\dim\mathcal{M}+2d}h_*\mathscr{Q})_{\mathrm{st}}$ is isomorphic to the constant sheaf $\overline{\mathbb{Q}}_\ell$ up to Tate twist.*

Proof. By Corollary 4.3.6, we have that for any $\tilde{a} \in \mathcal{U}'$, $\mathcal{M}_{\tilde{a}}^{\mathrm{reg}}$ is dense in $\mathcal{M}_{\tilde{a}}$ and is a $\mathcal{P}_{\tilde{a}}$-torsor. Étale-locally we may trivialize this torsor hence have an open embedding $\mathcal{P} \to \mathcal{M}$. This embedding identifies $\mathbf{R}^{-\dim\mathcal{M}+2d}h_*\mathscr{Q}$ with $\mathbf{R}^{2d}p_!\overline{\mathbb{Q}}_\ell$ compatible with $\pi_0(\mathcal{P})$-actions up to Tate twist. If we only consider the stable constituent, then this description is compatible over the intersections of étale neighborhoods, so we have the desired result. ∎

Note that $\mathcal{U}'$ already contains multiple δ-critical loci: there is $\mathcal{U}_0$, but there can also be other strata with $\delta > 0$, and their associated local δ-invariants is described in § 7.4.8.

8.4.14 Now we move on to a δ-critical stratum outside $\mathcal{U}'$. Suppose now $\mathcal{V} \subset \mathcal{U}$ is an irreducible locally-closed subset where the discriminant divisor and the boundary divisor intersects at exactly one point (set-theoretically), and $\mathcal{V}$ is δ-critical. Let $\tilde{a} \in \mathcal{V}(\bar{k})$ be a general point, and $\bar{v} \in X(\bar{k})$ is the unique point where the discriminant and boundary divisors intersect. When $\tilde{a}$ moves in $\mathcal{V}$, $\bar{v}$ moves in X, hence it may be seen as a family of divisors in $X \times \mathcal{V}$ over $\mathcal{V}$. Then $\tilde{a}$ is unramified at $\bar{v}$ (because $\mathcal{V}$ is δ-critical) with local Newton point ν, and $(\mathbf{R}^{-\dim \mathcal{M}+2d} h_* \mathcal{Q})_{\mathrm{st}}$ has rank given by weight multiplicity

$$m_{\mathcal{V}} \coloneqq m_{\lambda_{\bar{v}} \nu}.$$

Let $\delta_{\mathcal{V}} = \delta_{\tilde{a}}$ be the minimum of δ on $\mathcal{V}$. Since $\mathcal{V}$ is irreducible, we know that the sheaf $(\mathbf{R}^{-\dim \mathcal{M}+2d} h_* \mathcal{Q})_{\mathrm{st}}$ has to have constant rank when restricted to $\mathcal{V}_{\delta_{\mathcal{V}}}$. Let $\mathcal{V}' \subset \mathcal{V}$ be an open subset containing $\mathcal{V}_{\delta_{\mathcal{V}}}$ where $(\mathbf{R}^{-\dim \mathcal{M}+2d} h_* \mathcal{Q})_{\mathrm{st}}$ has constant rank $m_{\mathcal{V}}$. If we choose locally over $\mathcal{V}'$ a pinning of G over the formal disc around $\bar{v}$, this is equivalent to saying that the Newton point at $\bar{v}$ does not change.

We would like to show that $(\mathbf{R}^{-\dim \mathcal{M}+2d} h_* \mathcal{Q})_{\mathrm{st}}|_{\mathcal{V}'}$ is a local system with rank $m_{\mathcal{V}}$, and since one can always extend a local system over any subset of codimension 2, it suffices to only consider the subset of $\mathcal{V}'$ such that we either have $\delta = \delta_{\mathcal{V}}$ or $\delta = \delta_{\mathcal{V}} + 1$. In such case, we may assume that a is ν-regular semisimple at $\bar{v}$, because the collection of points not satisfying this condition has codimension at least 2 (see the proof of Proposition 7.4.5) hence can be safely deleted. In particular, the multiplicative affine Springer fiber at $\bar{v}$ is a locally constant fibration over $\mathcal{V}'$.

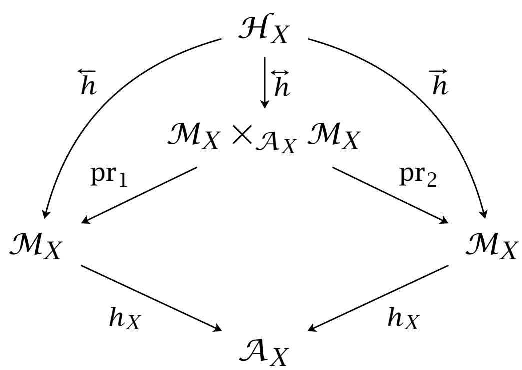
If we look at product formula (6.9.3), it is easy to believe that the jump in number of irreducible components (modulo $\pi_0(\mathcal{P})$-action) is purely a local phenomenon and only comes from the multiplicative affine Springer fiber at $\bar{v}$, and we know the multiplicative affine Springer fiber at $\bar{v}$ is locally constant over $\mathcal{V}'$. The problem is, however, the product formula we have been using so far only works over one point $\tilde{a} \in \mathcal{V}'(\bar{k})$. Therefore we must find another way to extract local geometry from global geometry, and ideally only at $\bar{v}$, not every point in the discriminant divisor. This is achieved using a new kind of Hecke-type stack which we formulate in the next section, and afterwards we shall continue describing top cohomology using those Hecke stacks.

8.5 mH-Hecke Stacks

Given mH-fibration $h_X\colon \mathcal{M}_X \to \mathcal{A}_X$, let $\mathcal{H}_X$ be the stack whose S-points are tuples $(\mathcal{L}, E_1, \phi_1, E_2, \phi_2, \psi)$ where $\mathcal{L} \in \mathrm{Bun}_{Z_{\mathfrak{M}}}(S)$, $(\mathcal{L}, E_i, \phi_i) \in \mathcal{M}_X(S)$ are two points mapping to the same point in $\mathcal{A}_X(S)$ whose boundary divisor is denoted by λ_b, and ψ is an isomorphism

$$\psi\colon (E_1, \phi_1)|_{X\times S-\lambda_b} \xrightarrow{\sim} (E_2, \phi_2)|_{X\times S-\lambda_b}.$$

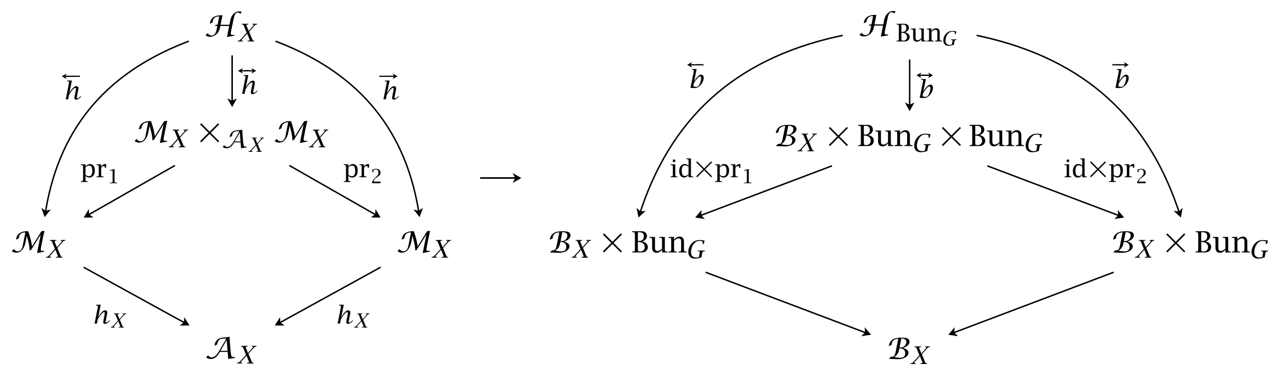
Note that since X is separated, the mHiggs field ϕ_2 (or ϕ_1, but not both) is determined by other data in the tuple, but we still want to keep both ϕ_1 and ϕ_2 to make the definition more symmetric. By its definition $\mathcal{H}_X$ fits into the following diagram where the maps are the obvious ones:



Definition 8.5.1. The stack $\mathcal{H}_X$ is called the *mH-Hecke stack* associated with mH-fibration $h_X\colon \mathcal{M}_X \to \mathcal{A}_X$.

Similarly, let $\mathcal{H}_{\mathrm{Bun}_G}$ be the stack classifying tuples (b, E_1, E_2, ψ), where $b \in \mathcal{B}_X$, and

(E_1, E_2, ψ) is as in $\mathcal{H}_X$, then $\mathcal{H}_X$ fits into the larger diagram



(8.5.1)

It is clear that $\overleftarrow{b}$ (resp. $\overrightarrow{b}$) is a locally trivial fibration of affine Grassmannians of G relative to $\mathcal{B}_X$, and that

$$\mathcal{H}_X \longrightarrow \mathcal{H}_{\mathrm{Bun}_G} \times_{\mathcal{B}_X \times \mathrm{Bun}_G} \mathcal{M}_X$$

induced by $\overleftarrow{h}$ and $\overleftarrow{b}$ (resp. $\overrightarrow{h}$ and $\overrightarrow{b}$) is a closed embedding of $\mathcal{M}_X$-functors. Therefore $\mathcal{H}_X$ is an ind-algebraic stack of ind-finite type that is ind-proper over $\mathcal{M}_X$.

Lemma 8.5.2. *Let $(\mathcal{L}, E, \phi) \in \mathcal{M}_X^{\heartsuit}(\bar{k})$ and let $a \in \mathcal{A}_X^{\heartsuit}$ be its image and λ_b the associated boundary divisor. Then the fiber of $\overleftarrow{h}$ (resp. $\overrightarrow{h}$) is isomorphic to the product of multiplicative affine Springer fibers at the support of λ_b*

$$\mathcal{M}_{\lambda_b}(a) \coloneqq \prod_{\bar{v} \in \lambda_b} \mathcal{M}_{\bar{v}}(a). \tag{8.5.2}$$

Proof. The statements for $\overleftarrow{h}$ and for $\overrightarrow{h}$ are the same so it suffices to prove for $\overrightarrow{h}$. Given $(\mathcal{L}, E, \phi)$, since $\bar{k}$ is algebraically closed, we may choose and fix an isomorphism around the formal disc $\breve{X}_{\lambda_b}$ around the support of λ_b:

$$\tau\colon (\mathcal{L}, E, \phi) \longrightarrow (\mathcal{L}_0, E_0, \gamma_{a,\lambda_b})$$

where $\mathcal{L}_0$ (resp. E_0) is the trivial $Z_{\mathfrak{M}}$-torsor (resp. G-torsor), and $\gamma_{a,\lambda_b} \in \mathfrak{M}(\check{X}_{\lambda_b})$ such that $\chi_{\mathfrak{M}}(\gamma_{a,\lambda_b}) = a(\check{X}_{\lambda_b})$.

For any k-scheme S, we have map

$$\begin{aligned}\overrightarrow{h}^{-1}(\mathcal{L}, E, \phi)(S) &\longrightarrow \mathcal{M}_{\lambda_b}(a)\\(\mathcal{L}, E_1, \phi_1, \psi) &\longmapsto (\mathcal{L}|_{\check{X}_{\lambda_b}}, E_1|_{\hat{X}_{\lambda_b}}, \phi_1|_{\check{X}_{\lambda_b}}, \beta),\end{aligned}$$

where β is the composition of maps

$$\beta\colon (\mathcal{L}, E_1, \phi_1)|_{\check{X}^{\bullet}_{\lambda_b}} \xrightarrow{\psi} (\mathcal{L}, E, \phi)|_{\check{X}^{\bullet}_{\lambda_b}} \xrightarrow{\tau} (\mathcal{L}_0, E_0, \gamma_a)|_{\check{X}^{\bullet}_{\lambda_b}},$$

and $\check{X}^{\bullet}_{\lambda_b}$ is the punctured disc. The map β is clearly injective: if $(\mathcal{L}, E_1, \phi_1, \psi)$ and $(\mathcal{L}, E_1', \phi_1', \psi')$ have isomorphic image under β, they are isomorphic over both $\check{X} \times S - \lambda_b$ and $\check{X}_{\lambda_b}$, together with their gluing data. It implies they are isomorphic tuples in $\overrightarrow{h}^{-1}(\mathcal{L}, E, \phi)(S)$. On the other hand, β is also surjective, because any point in $\mathcal{M}_{\lambda_b}(a)$ can be glued with $(\mathcal{L}, E, \phi)|_{\check{X}\times S-\lambda_b}$ to obtain a point in $\overrightarrow{h}^{-1}(\mathcal{L}, E, \phi)$. This finishes the proof. ∎

8.5.3 There are some useful variants of mH-Hecke stacks. First of all, the mH-base can be replaced by any algebraic $\mathcal{A}_X$-stack $\mathcal{U} \to \mathcal{A}_X$ and $\mathcal{M}_X$ by its pullback to $\mathcal{U}$. Secondly, the boundary divisor λ_b can be replaced by any finite flat family $\lambda'_{\mathcal{U}}$ of Cartier divisors in $X \times \mathcal{U}$ over $\mathcal{U}$, so that the rational map ψ in the definition is now an isomorphism outside $\lambda'_{\mathcal{U}}$. Let $\mathcal{H}'_{\mathcal{U}}$ be the corresponding Hecke stack, together with maps $\overleftarrow{h}'_{\mathcal{U}}, \overrightarrow{h}'_{\mathcal{U}}$, etc.

Definition 8.5.4. Given a tuple $(h_X, \mathcal{U}, \lambda'_{\mathcal{U}})$ as above, we call $\mathcal{H}'_{\mathcal{U}}$ the *generalized mH-Hecke stack* associated with $(h_X, \mathcal{U}, \lambda'_{\mathcal{U}})$.

Remark 8.5.5. Note that in the case of (usual) mH-Hecke stack, the Cartier divisor can be chosen to be the numerical boundary divisor $\mathfrak{B}_{\mathrm{Env}(G^{\mathrm{sc}})}$ (because the actual boundary divisor is not really a Cartier divisor on X).

The representability of $\mathcal{H}'_{\mathcal{U}}$ can be seen using a similar diagram as (8.5.1), with $\mathcal{B}_X$ replaced with appropriate Hilbert scheme of X.

Example 8.5.6. If $U \to \mathcal{B}_X$ is any map of algebraic stacks, we can let $\mathcal{U}$ be the preimage of U. As for the divisor family, suppose that over U the boundary divisor λ_U, viewed as a finite flat U-scheme, has multiple connected components, and we denote one of which by λ'_U, in other words, λ'_U is a finite flat family of boundary subdivisors of λ_U. We call the resulting Hecke stack $\mathcal{H}'_U$ the *partial mH-Hecke stack* associated with (h_X, U, λ'_U).

Example 8.5.7. We can let $\mathcal{U} = \mathcal{A}_X^{\heartsuit}$ and $\lambda'_{\mathcal{U}}$ be the discriminant divisor. In this case we denote the resulting Hecke stack by $\mathcal{H}_{\mathfrak{D}_{\mathfrak{M}}}$, and call it the $\mathfrak{D}$*-Hecke stack*.

If $\lambda''_{\mathcal{U}} \to \lambda'_{\mathcal{U}}$ is a $\mathcal{U}$-morphism of divisor families, then we have natural maps $\mathcal{H}''_{\mathcal{U}} \to \mathcal{H}'_{\mathcal{U}}$. Using the same argument as in Lemma 8.5.2, we have the following result:

Lemma 8.5.8. *Let $(\mathcal{L}, E, \phi) \in \mathcal{M}_X^{\heartsuit}|_{\mathcal{U}}(\bar{k})$ and $a \in \mathcal{U}$ be its image, and λ'_a the associated Cartier divisor induced by $\lambda'_{\mathcal{U}}$. The fiber of $\overleftarrow{h}'_{\mathcal{U}}$ (resp. $\overrightarrow{h}'_{\mathcal{U}}$) over $(\mathcal{L}, E, \phi)$ is isomorphic to*

$$\mathcal{M}_{\lambda'_a} \coloneqq \prod_{\bar{v} \in \lambda'_a} \mathcal{M}_{\bar{v}}(a).$$

Moreover, if $\lambda''_{\mathcal{U}} \to \lambda'_{\mathcal{U}}$ are two divisor families, then this isomorphism is compatible with the natural map $(\overleftarrow{h}''_{\mathcal{U}})^{-1}(\mathcal{L}, E, \phi) \to (\overleftarrow{h}'_{\mathcal{U}})^{-1}(\mathcal{L}, E, \phi)$ and the map induced by (E, ϕ):

$$\mathcal{M}_{\lambda''_a} \longrightarrow \mathcal{M}_{\lambda'_a},$$

and similarly for the $\overrightarrow{h}$ side.

8.5.9 The symmetry of mH-Hecke stacks can be described using regular centralizer just like mH-fibrations. The action of Picard stack $\mathcal{P}_X$ can be pulled back to $\mathcal{M}_X$ to give an action of $\mathcal{M}_X \times_{\mathcal{A}_X} \mathcal{P}_X$ on $\mathcal{M}_X \times_{\mathcal{A}_X} \mathcal{M}_X$, relative to the first projection to $\mathcal{M}_X$. This can also be achieved by pulling back regular centralizer $\mathfrak{J}_X \to X \times \mathcal{A}_X$ to $X \times \mathcal{M}_X$, and form the relative Picard stack over $\mathcal{M}_X$.

Suppose for now $\mathfrak{M} = \mathrm{Env}(G^{\mathrm{sc}})$, then over $\mathcal{B}_X$ there is a finite flat family of Cartier divisors in X, namely the numerical boundary divisor induced by $\mathfrak{B}_{\mathfrak{M}}$. Since the regular centralizer is affine smooth, we may use the same construction as in § 5.2, and define the relative affine Grassmannian

$$\mathcal{P}_{\mathcal{B}_X} \coloneqq \mathsf{Gr}_{\mathfrak{J}_X,\mathcal{B}_X} \longrightarrow \mathcal{A}_X$$

whose fiber at $a \in \mathcal{A}_X$ is exactly the product of local Picard group

$$\mathcal{P}_{\mathcal{B}_X,a} = \prod_{\bar{v}\in\mathfrak{B}_a} \mathcal{P}_{\bar{v}}(a).$$

If $\mathfrak{M} \neq \mathrm{Env}(G^{\mathrm{sc}})$, we may simply pull back the construction for $\mathrm{Env}(G^{\mathrm{sc}})$. The same construction can be done for any monoid $\mathfrak{M}$ such that $\mathfrak{A}_{\mathfrak{M}}$ is of standard type, and it agrees with the pullback of the construction for $\mathrm{Env}(G^{\mathrm{sc}})$ (even though $\mathfrak{B}_{\mathfrak{M}}$ in general is not the pullback of $\mathfrak{B}_{\mathrm{Env}(G^{\mathrm{sc}})}$).

We have the forgetful map $\mathcal{P}_{\mathcal{B}_X} \to \mathcal{P}_X$ by forgetting the trivialization of $\mathfrak{J}_{\mathfrak{M}}$-torsor over $X - \mathfrak{B}_{\mathfrak{M}}$, so $\mathcal{P}_{\mathcal{B}_X}$ naturally acts on $\mathcal{M}_X$. We claim that $\mathcal{P}_{\mathcal{B}_X}$, after pulling back to $\mathcal{M}_X$, acts on $\overleftarrow{h}\colon \mathcal{H}_X \to \mathcal{M}_X$, making $\overleftarrow{h}$ a $\mathcal{P}_{\mathcal{B}_X}$-equivariant map, and similarly for $\overrightarrow{h}$. Indeed, suppose we have tuple $(\mathcal{L}, E_1, \phi_1, E_2, \phi_2, \psi) \in \mathcal{H}_X(S)$ and $(E_{\mathfrak{J}}, \tau) \in \mathcal{P}_{\mathcal{B}_X}(S)$ where $E_{\mathfrak{J}}$ is a $\mathfrak{J}_{\mathfrak{M}}$-torsor over $X \times S$ and τ is a trivialization of $E_{\mathfrak{J}}$ outside $\mathfrak{B}_{\mathfrak{M}}$. The action $E_{\mathfrak{J}}$ on $\mathcal{M}_X$ sends $(\mathcal{L}, E_2, \phi_2)$ to

$$\phi_2'\colon E_2' \coloneqq E_2 \times^{\mathfrak{J}_{\mathfrak{M}}}_{\phi_2,\mathfrak{M}_{\mathcal{L}}} E_{\mathfrak{J}} \longrightarrow \mathfrak{M}_{\mathcal{L}},$$

where the action of $\mathfrak{J}_{\mathfrak{M}}$ on ϕ_2 is induced by the canonical map $\chi_{\mathfrak{M}}^{*}\mathfrak{J}_{\mathfrak{M}} \to I_{\mathfrak{M}}$. The trivialization τ induces isomorphism

$$(E_2, \phi_2)|_{X\times S-\mathfrak{B}_{\mathfrak{M}}} \xrightarrow{\sim} (E_2', \phi_2')|_{X\times S-\mathfrak{B}_{\mathfrak{M}}},$$

whose composition with ψ gives

$$\psi' \colon (E_1, \phi_1)|_{X\times S-\mathfrak{B}_{\mathfrak{M}}} \xrightarrow{\sim} (E_2', \phi_2')|_{X\times S-\mathfrak{B}_{\mathfrak{M}}}.$$

This defines the $\mathcal{P}_{\mathcal{B}_X}$-action and clearly it makes $\overleftarrow{h}$ equivariant. The argument for $\overrightarrow{h}$ is the same.

The story for generalized mH-Hecke stacks $\mathcal{H}'_{\mathcal{U}}$ associated with tuple $(h_X, \mathcal{U}, \lambda'_{\mathcal{U}})$ is also the same, except one replaces $\mathfrak{B}_{\mathfrak{M}}$ by $\lambda'_{\mathcal{U}}$. For future convenience we denote the local Picard group in this case by $\mathcal{P}_{\lambda'_{\mathcal{U}}}$ in place of $\mathcal{P}_{\mathcal{B}_X}$. We leave other details to the reader.

Lemma 8.5.10. *Let $(\mathcal{L}, E, \phi) \in \mathcal{M}_X^{\natural}(\bar{k})$ and $a \in \mathcal{A}_X^{\natural}(\bar{k})$ be its image. The action of $\mathcal{P}_{\mathcal{B}_X}$ on $\mathcal{H}_X$ induces a bijection between sets of irreducible components modulo symmetry on the fibers*

$$\#(\mathrm{Irr}(\mathcal{H}_{(\mathcal{L},E,\phi)})/\mathcal{P}_{\mathcal{B}_X,(\mathcal{L},E,\phi)}(\bar{k})) \xrightarrow{\sim} \#(\mathrm{Irr}(\mathcal{M}_a)/\mathcal{P}_a(\bar{k})).$$

Proof. The map $\mathcal{H}_{(\mathcal{L},E,\phi)} \to \mathcal{M}_a$ factors through the space (8.5.2) in Lemma 8.5.2 using a gluing argument similar to that in product formula: indeed, at $\bar{v} \notin \lambda_b$, (E, ϕ) together with a fixed local trivialization determines a distinguished point in $\mathcal{M}_{\bar{v}}(a)$. It then induces maps

$$\left[\mathcal{H}_{(\mathcal{L},E,\phi)}/\mathcal{P}_{\mathcal{B}_X,(\mathcal{L},E,\phi)}\right] \xrightarrow{\sim} \prod_{\bar{v}\in\lambda_b} [\mathcal{M}_{\bar{v}}(a)/\mathcal{P}_{\bar{v}}(a)] \longrightarrow [\mathcal{M}_a/\mathcal{P}_a],$$

which factors through

$$\prod_{\bar{v}\in\mathfrak{D}_a}[\mathcal{M}_{\bar{v}}(a)/\mathcal{P}_{\bar{v}}(a)].$$

Note that when $\bar{v}\notin\mathfrak{D}_a$, the stack $[\mathcal{M}_{\bar{v}}(a)/\mathcal{P}_{\bar{v}}(a)]$ is just a $\bar{k}$-point. According to Theorem 4.3.5, if $\bar{v}\in X$ does not support the boundary divisor, then $\mathcal{M}_{\bar{v}}^{\mathrm{reg}}(a)$ is dense in $\mathcal{M}_{\bar{v}}(a)$ and is a $\mathcal{P}_{\bar{v}}(a)$-torsor. Combining these facts we obtain the lemma. ∎

8.5.11 Simultaneous product formula We may replace the mH-Hecke stack by $\mathfrak{D}$-Hecke stack $\mathcal{H}_{\mathfrak{D}_{\mathfrak{W}}}$, and obtain a family of maps

$$\begin{array}{ccc}
\mathcal{H}_{\mathfrak{D}}\times^{\mathcal{P}_{\mathfrak{D}}}\mathcal{P}_X^{\heartsuit} & \xrightarrow{\overleftarrow{\Pi}_{\mathfrak{D}}} & \mathcal{M}_X^{\heartsuit}\times_{\mathcal{A}_X^{\heartsuit}}\mathcal{M}_X^{\heartsuit} \\
& \searrow^{\overleftarrow{h}_{\mathfrak{D}}\circ\mathrm{pr}_1} & \downarrow{\scriptstyle \mathrm{pr}_1} \\
& & \mathcal{M}_X^{\heartsuit}
\end{array}$$

where $\overleftarrow{\Pi}_{\mathfrak{D}}$ is the map

$$((\mathcal{L},E_1,\phi_1,E_2,\phi_2,\psi),p\in\mathcal{P}_X^{\heartsuit})\longmapsto(\mathcal{L},E_1,\phi_1,p\cdot(E_2,\phi_2)).$$

Over any $(\mathcal{L},E,\phi)$ whose image is $a\in\mathcal{A}_X^{\heartsuit}(\bar{k})$, the fiber of $\overleftarrow{\Pi}_{\mathfrak{D}}$ is clearly isomorphic to the (non-reduced) product formula (6.9.2), so its reduced version is isomorphic to (6.9.3). If $(\mathcal{L},E,\phi)\in\mathcal{M}_X^{\mathrm{reg}}(\bar{k})$, since $\mathcal{P}_X$ acts on $\mathcal{M}_X^{\mathrm{reg}}$ freely, $\mathcal{M}_X^{\mathrm{reg}}$ is smooth over $\mathcal{A}_X$, then we may locally around a choose a section $\mathcal{A}_X\to\mathcal{M}_X^{\mathrm{reg}}$, and pull back $\overleftarrow{\Pi}_{\mathfrak{D}}$. If a Steinberg quasi-section exists, we may even do this over the entire $\mathcal{A}_X^{\heartsuit}$. This way we obtain a *simultaneous product formula.* There is a symmetric construction for $\overrightarrow{h}_{\mathfrak{D}}$ as well.

Remark 8.5.12. (1) Although fiberwise $\overleftarrow{\Pi}_{\mathfrak{D}}$ is a universal homeomorphism at least over $\mathcal{M}_X^{\natural}$, it is far from being an isomorphism.

(2) When there is no Steinberg quasi-section, the construction of $\overleftarrow{\Pi}_{\mathfrak{D}}$ does not supersede

the construction of (6.9.3), because we need to use the latter to show that $\mathcal{M}_a$ is non-empty first so that $(\mathcal{L}, E, \phi)$ lying over a exists.

(3) Unlike (6.9.3), which may only be defined over $\bar{k}$ (e.g., when there is no Steinberg quasi-sections), $\overleftarrow{\Pi}_{\mathfrak{D}}$ is always defined over k. However, this still does not upgrade (6.9.3) to a k-morphism, unless we already know $\mathcal{M}_a$ contains a k-point.

8.5.13 The construction of map $\overleftarrow{\Pi}_{\mathfrak{D}}$ (and $\overrightarrow{\Pi}_{\mathfrak{D}}$) works for any generalized mH-Hecke stack $\mathcal{H}'_{\mathcal{U}}$, so we have morphism of stacks

$$\begin{array}{ccc} \mathcal{H}'_{\mathcal{U}} \times^{\mathcal{P}_{\lambda'_{\mathcal{U}}}} \mathcal{P}_{\mathcal{U}} & \xrightarrow{\overleftarrow{\Pi}'_{\mathcal{U}}} & \mathcal{M}_{\mathcal{U}} \times_{\mathcal{U}} \mathcal{M}_{\mathcal{U}} \\ & \searrow^{\overleftarrow{h}'_{\mathcal{U}} \circ \mathrm{pr}_1} & \downarrow \mathrm{pr}_1 \\ & & \mathcal{M}_{\mathcal{U}} \end{array}$$

The quotient of $\mathcal{H}'_{\mathcal{U}}$ by $\mathcal{P}_{\lambda'_{\mathcal{U}}}$ is an ind-algebraic stack of ind-finite type over $\mathcal{M}_{\mathcal{U}}$ whose geometric fibers are proper algebraic stacks of finite type. In fact, since the reduced geometric fibers of $\overleftarrow{h}'_{\mathcal{U}}$ are schemes locally of finite type, we can, locally over $\mathcal{M}_{\mathcal{U}}$, find some affine open subset of $\mathcal{H}'_{\mathcal{U}}$ that maps surjectively onto the quotient $\left[(\mathcal{H}'_{\mathcal{U}})^{\mathrm{red}}/\mathcal{P}^{\mathrm{red}}_{\lambda'_{\mathcal{U}}}\right]$. For example, since $\mathcal{H}'_{\mathcal{U}}$ embeds into a locally constant affine Grassmannian over $\mathcal{M}_{\mathcal{U}}$, we can take a sufficiently large truncation in the affine Grassmannian and take the preimage in $\mathcal{H}'_{\mathcal{U}}$. Hence the stack

$$(\mathcal{H}'_{\mathcal{U}})^{\mathrm{red}} \times^{\mathcal{P}^{\mathrm{red}}_{\lambda'_{\mathcal{U}}}} \mathcal{P}_{\mathcal{U}} \tag{8.5.3}$$

is algebraic over $\mathcal{M}_{\mathcal{U}}$, locally of finite type (it is not necessarily of finite type because $\pi_0(\mathcal{P}_{\mathcal{U}})$ may be infinite). When $\mathcal{U} \to \mathcal{A}_X$ has its image contained in $\mathcal{A}_X^{\natural}$, then (8.5.3) is a Deligne-Mumford stack of finite type over $\mathcal{U}$.

8.5.14 We return to the situation at the end of § 8.4 and continue describing the sheaf $(\mathbf{R}^{-\dim \mathcal{M}+2d}h_*\mathcal{Q})_{\mathrm{st}}$. Over $\mathcal{V}'$, choose the divisor family to be $\bar{v}$ (note that $\bar{v}$ can vary in X over $\mathcal{V}'$), and consider the generalized mH-Hecke stack $\mathcal{H}_{\bar{v}}$ associated with $(h_X, \mathcal{V}', \bar{v})$. This is a partial mH-Hecke stack with boundary subdivisor $\lambda_{\bar{v}}$. Recall the Newton point ν at $\bar{v}$ is locally constant on $\mathcal{V}'$, and the local δ-invariant is exactly $\langle \rho, \lambda_{\bar{v}} - \nu \rangle$, which is constant on $\mathcal{V}'$.

Since we want to prove that the stable top cohomology is a local system on $\mathcal{V}'$, we may replace $\mathcal{V}'$ by any strict Henselian neighborhood therein. Since $\mathcal{M}^{\mathrm{reg}}_{\mathcal{V}'}$ is smooth over $\mathcal{V}'$ and $\mathcal{V}'$ is strictly Henselian, we have a section τ of $\mathcal{V}'$ in $\mathcal{M}_{\mathcal{V}'}$. Such section induces a tuple $(\mathcal{L}, E, \phi) \in \mathcal{M}_{\mathcal{V}'}(\mathcal{V}')$. We may trivialize E at $\bar{v}$ over $\mathcal{V}'$, and by smoothness lift it to the formal disc $\check{X}_{\bar{v}}$ around $\bar{v}$. Using the same argument as in Lemma 8.5.2 (and the fact that G-torsors over $\check{X}_{\bar{v}}$ is always trivial), we have that

$$\tau^*\mathcal{H}_{\bar{v}} \cong \mathcal{V}' \times \mathcal{M}_{\bar{v}}(a_0),$$

where a_0 is the unique closed point in $\mathcal{V}'$. Since by assumption $\bar{v}$ is the only point supporting both boundary and discriminant divisors, $\mathcal{M}^{\mathrm{reg}}_{\bar{v}'}(a)$ is dense in $\mathcal{M}_{\bar{v}'}(a)$ for any $a \in \mathcal{V}'$ and $\bar{v}' \neq \bar{v}$. Since the image of ϕ is contained in $\mathfrak{M}^{\mathrm{reg}}_{\mathcal{L}}$ outside $\bar{v}$, the $\mathcal{P}_{\mathcal{V}'}$-equivariant map

$$\tau^*\left(\mathcal{H}_{\bar{v}} \times^{\mathcal{P}_{\bar{v}}} \mathcal{P}_{\mathcal{V}'}\right) \longrightarrow \tau^*\left(\mathcal{M}_{\mathcal{V}'} \times_{\mathcal{V}'} \mathcal{M}_{\mathcal{V}'}\right) \cong \mathcal{M}_{\mathcal{V}'}$$

has fiberwise dense image over $\mathcal{V}'$. This shows that $(\mathbf{R}^{-\dim \mathcal{M}+2d}h_*\mathcal{Q})_{\mathrm{st}}|_{\mathcal{V}'}$ is local system of rank $m_{\lambda_{\bar{v}}\nu}$ over $\mathcal{V}'$. The whole argument clearly generalizes to more points than $\bar{v}$. Thus we have the following result:

Proposition 8.5.15. *Suppose locally-closed substack $\mathcal{V} \subset \tilde{\mathcal{A}}^{\natural}_X$ is such that:*

(1) the local model of singularity as in Theorem 6.10.2 exists,

(2) the boundary divisor is locally constant,

(3) for any $a \in \mathcal{V}$, a is unramified at all points supporting the boundary divisor

(4) if locally over $\mathcal{V}$ we write boundary divisor as $\lambda_b = \sum_{i=1}^{m} \lambda_i \cdot \bar{v}_i$, where $\bar{v}_i$ varies in X, then the Newton point ν_i at $\bar{v}_i$ locally constant over $\mathcal{V}$.

Then $(\mathbf{R}^{-\dim \mathcal{M}+2d} h_* \mathcal{Q})_{\mathrm{st}}|_{\mathcal{V}}$ is a local system. If moreover the boundary divisor stays multiplicity-free in the sense of Proposition 5.1.25, then this local system has rank

$$\prod_{i=1}^{m} m_{\lambda_i \nu_i}.$$

The same argument for Proposition 8.5.15 can be applied to κ-isotypic constituent as well.

Proposition 8.5.16. *For a fixed κ, suppose locally-closed substack $\mathcal{V} \subset \tilde{\mathcal{A}}_{\kappa}^{\natural}$ satisfies all the conditions in Proposition 8.5.15, then $(\mathbf{R}^{-\dim \mathcal{M}+2d} h_* \mathcal{Q})_{\kappa}|_{\mathcal{V}}$ is a local system. If moreover the boundary divisor stays multiplicity-free in the sense of Proposition 5.1.25, then this local system has rank*

$$\prod_{i=1}^{m} m_{\lambda_i \nu_i},$$

where the notations are as in Proposition 8.5.15.

CHAPTER 9

SUPPORT THEOREM

In this section we prove a slightly generalized version of the Support Theorem in [Ngô10, § 7.2]. The method here follows the outline in [Ngô10, § 7.3–7.7], with some modifications.

We then apply our abstract support theorem to mH-fibrations. One key input for this application is the local model of singularity (Theorem 6.10.2), which will provide us with the bound on cohomological amplitude that is only assumed in the abstract support theorem.

9.1 Abelian Fibrations

Let $f\colon M \to S$ be a proper map of varieties over a finite field k. Let $g\colon P \to S$ be a smooth commutative group scheme over S. Suppose P acts on M relative to S and the stabilizers are affine. Let $P^0 \subset P$ be the open group subscheme such that for any geometric point $s \in S$, P^0_s is the neutral component of P_s. We then have the canonical short exact sequence of Chevalley

$$1 \longrightarrow R_s \longrightarrow P^0_s \longrightarrow A_s \longrightarrow 1,$$

where R_s is connected and affine and A_s is an abelian variety. It induces a decomposition of Tate modules

$$0 \longrightarrow \mathrm{T}_{\overline{\mathbb{Q}}_\ell}(R_s) \longrightarrow \mathrm{T}_{\overline{\mathbb{Q}}_\ell}(P^0_s) \longrightarrow \mathrm{T}_{\overline{\mathbb{Q}}_\ell}(A_s) \longrightarrow 0$$

We have an $\mathbb{N}$-valued function $\delta(s) = \dim R_s$ defined for the topological points of S, which is necessarily upper-semicontinuous (see [Ngô10, § 5.6.2]). Suppose δ is constructible, then it induces a locally closed stratification

$$S = \coprod_{\delta \in \mathbb{N}} S^{\delta},$$

so that if $s \in S^{\delta}$, then $\delta(s) = \delta$.

Definition 9.1.1. We call (f, g) a *weak abelian fibration* if the following conditions are satisfied:

(1) f and g have the same relative dimension d.

(2) For any geometric point $s \in S$ and any $m \in M$, it stabilizer in P_s is affine.

(3) The Tate module $\mathrm{T}_{\overline{\mathbb{Q}}_\ell}(P^0)$ is polarizable. In other words, there exists étale locally over S an alternating bilinear form on $\mathrm{T}_{\overline{\mathbb{Q}}_\ell}(P^0)$, such that for any s its restriction to $\mathrm{T}_{\overline{\mathbb{Q}}_\ell}(R_s)$ is zero and it induces a perfect pairing of $\mathrm{T}_{\overline{\mathbb{Q}}_\ell}(A_s)$ with itself.

Definition 9.1.2. We call (f, g) a *δ-regular abelian fibration* if it is a weak abelian fibration, and for any $\delta \in \mathbb{N}$, we have $\mathrm{codim}_S(S^{\delta}) \geq \delta$. (If $S^{\delta} = \emptyset$, then the codimension is ∞ by convention.) Equivalently, we have for any irreducible closed subset $Z \subset S$, $\mathrm{codim}_S(Z) \geq \delta_Z$ where δ_Z is the minimum of δ on Z.

Remark 9.1.3. Note that both Definitions 9.1.1 and 9.1.2 make sense if we replace schemes with Deligne-Mumford stacks.

9.2 Goresky-MacPherson Inequality

Let (f, g) be a weak abelian fibration. Let $\mathcal{F} \in \mathrm{D}^{\mathrm{b}}_{\mathrm{c}}(M, \overline{\mathbb{Q}}_\ell)$ be a self-dual complex hence of pure weight 0, so $f_*\mathcal{F} \in \mathrm{D}^{\mathrm{b}}_{\mathrm{c}}(S, \overline{\mathbb{Q}}_\ell)$ is also of pure weight 0 since f is proper. Thus we have (non-canonical) decomposition of Frobenius modules by [BBD82]:

$$f_*\mathcal{F} \cong \bigoplus_{n \in \mathbb{Z}} {}^{p}\mathrm{H}^{n}(f_*\mathcal{F})[-n]. \tag{9.2.1}$$

Given an irreducible closed subset Z of S, let $\operatorname{occ}(Z) \subset \mathbb{Z}$ be the set of numbers n such that Z appears in the set of supports $\operatorname{supp}{}^{\mathfrak{p}}\mathrm{H}^n(f_*\mathcal{F})$. By Poincaré duality, $\operatorname{occ}(Z)$ is symmetric about 0. Suppose N is the largest number such that $\mathrm{H}^N(f_*\mathcal{F}) \neq 0$. Suppose $Z \neq \emptyset$ and let $0 \leq n \in \operatorname{occ}(Z)$. Then there exists an open subset $U \subset S$ such that $U \cap Z \neq \emptyset$, and a local system L on $U \cap Z$, such that $i_*L[\dim Z]$ (i being the map $U \cap Z \hookrightarrow Z$) is a direct summand of ${}^{\mathfrak{p}}\mathrm{H}^n(f_*\mathcal{F})|_U$, hence also a direct summand of $f_*\mathcal{F}[n]$. Taking the usual cohomology, one has that i_*L is a direct summand of $\mathrm{H}^{n-\dim Z}(f_*\mathcal{F})$. Therefore $n - \dim Z \leq N$. Since $n \geq 0$, we obtain the Goresky-MacPherson inequality:

$$\operatorname{codim}_S(Z) \leq \dim S + N - n \leq \dim S + N.$$

In particular, if $N \leq -\dim M + 2d = -\dim S + d$, then $\operatorname{codim}_S(Z) \leq d$. Suppose further we have that n can be so chosen that $n \geq (d - \delta_Z)$, then we have an improved inequality $\operatorname{codim}_S(Z) \leq \delta_Z$. If equality holds (e.g., it happens when (f,g) is δ-regular), all the inequalities just mentioned are equalities, and in particular $N = -\dim M + 2d$, and the restriction of ${}^{\mathfrak{p}}\mathrm{H}^{n-\dim Z}(f_*\mathcal{F})$ to $U \cap Z$ is a direct summand of the top cohomology $\mathbf{R}^{-\dim M + 2d} f_*\mathcal{F}$.

9.3 Action by Cap Product

Let (f,g) be a weak abelian fibration. Following [LO08a, LO08b], we have the derived category $\mathrm{D}([M/P], \overline{\mathbb{Q}}_\ell)$ of quotient stack $[M/P]$, which we also define to be the P-equivariant derived category $\mathrm{D}_P(M, \overline{\mathbb{Q}}_\ell)$ on M by pulling back through map $q\colon M \to [M/P]$. We also have the full subcategories of various boundedness and constructibility conditions[1]. Note that if a complex is P-equivariant, then it is also P^0 equivariant. For the rest of this section we are going to replace P with P^0 so that P is fiberwise connected.

1. In [LO08a, LO08b], they have two variants of subcategories for each boundedness condition, but the distinction is not very important here.

Following [Ngô10, § 7.4], we define complex in $D_c^b(S, \overline{\mathbb{Q}}_\ell)$

$$\Lambda_P = g_! \overline{\mathbb{Q}}_\ell[2d](d),$$

which is concentrated in non-positive degrees and whose degree -1 cohomology is the sheaf $T_{\overline{\mathbb{Q}}_\ell}(P)$ such that at any geometric point $s \in S$ its stalk is the Tate module $T_{\overline{\mathbb{Q}}_\ell}(P_s)$. We have in fact canonical isomorphisms in $D_c^b(S, \overline{\mathbb{Q}}_\ell)$

$$\Lambda_P \simeq \bigoplus_{i \geq 0} H^{-i}(\Lambda_P)[i] \simeq \bigoplus_{i \geq 0} \wedge^i T_{\overline{\mathbb{Q}}_\ell}(P)[i], \tag{9.3.1}$$

making it a graded algebra.

9.3.1 More generally, suppose $\mathcal{F} \in D_c^b(M, \overline{\mathbb{Q}}_\ell)$ is P-equivariant, in other words, $\mathcal{F} \simeq q^* \overline{\mathcal{F}}$ for some $\overline{\mathcal{F}}$ in $D_c^b([M/P], \overline{\mathbb{Q}}_\ell)$. Then the action morphism

$$a \colon P \times_S M \longrightarrow M$$

is smooth and of relative dimension d. Therefore we have by adjunction a morphism of complexes

$$a_! a^* \mathcal{F}[2d](d) \longrightarrow \mathcal{F}.$$

Further pushing forward using $f_!$, we obtain the morphism

$$(g \times_S f)_! a^* \mathcal{F}[2d](d) \longrightarrow f_! \mathcal{F}.$$

On the other hand, using the Cartesian diagram

$$\begin{array}{ccc} P \times_S M & \xrightarrow{a} & M \\ \downarrow{\scriptstyle p_2} & & \downarrow{\scriptstyle q} \\ M & \xrightarrow{q} & [M/P] \end{array},$$

we see that

$$a^* \mathcal{F} \simeq a^* q^* \overline{\mathcal{F}} \simeq p_2^* q^* \overline{\mathcal{F}} \simeq \overline{\mathbb{Q}}_\ell \boxtimes \mathcal{F}. \tag{9.3.2}$$

So by Künneth formula, we have a morphism of cap products

$$\Lambda_P \otimes f_! \mathcal{F} \longrightarrow f_! \mathcal{F}. \tag{9.3.3}$$

9.3.2 Now let $\mathcal{F} \in \mathrm{D}_c^b(M, \overline{\mathbb{Q}}_\ell)$ be a P-equivariant complex of pure weight 0. Since f is proper, we have $f_! \mathcal{F} \simeq f_* \mathcal{F}$ and its also pure of weight 0. For each $n \in \mathbb{Z}$ we have a canonical isomorphism of Frobenius modules

$$^{\mathfrak{p}}\mathrm{H}^n(f_* \mathcal{F}) \simeq \bigoplus_{\alpha \in \Sigma} K_\alpha^n, \tag{9.3.4}$$

where Σ is the index set of supports of the perverse cohomologies of $f_* \mathcal{F}$. For $\alpha \in \Sigma$, we denote by Z_α corresponding the irreducible closed subset in $S_{\bar{k}}$ and K_α^n the perverse summand supported on Z_α. Since $\mathcal{F}$ is bounded and f is of finite type, Σ is necessarily finite.

By (9.3.1) and (9.3.3), we have a morphism

$$\mathrm{T}_{\overline{\mathbb{Q}}_\ell}(P) \otimes f_! \mathcal{F} \longrightarrow f_! \mathcal{F}[-1].$$

Composing with perverse truncation ${}^{\mathfrak{p}}\tau^{\leq n}(f_!\mathcal{F}) \to f_!\mathcal{F}$, we have the induced map

$$\mathrm{T}_{\overline{\mathbb{Q}}_\ell}(P) \otimes {}^{\mathfrak{p}}\tau^{\leq n}(f_!\mathcal{F}) \longrightarrow f_!\mathcal{F}[-1].$$

Applying functor ${}^{\mathfrak{p}}\mathrm{H}^n$, we have

$${}^{\mathfrak{p}}\mathrm{H}^n\left(\mathrm{T}_{\overline{\mathbb{Q}}_\ell}(P) \otimes {}^{\mathfrak{p}}\tau^{\leq n}(f_!\mathcal{F})\right) \longrightarrow {}^{\mathfrak{p}}\mathrm{H}^{n-1}(f_!\mathcal{F}).$$

Since tensoring with $\mathrm{T}_{\overline{\mathbb{Q}}_\ell}(P)$ is perverse right-exact, we know that

$$\mathrm{T}_{\overline{\mathbb{Q}}_\ell}(P) \otimes {}^{\mathfrak{p}}\tau^{\leq n-1}(f_!\mathcal{F}) \in {}^{\mathfrak{p}}\mathrm{D}_{\mathrm{c}}^{\leq n-1}(S, \overline{\mathbb{Q}}_\ell),$$

hence by tensoring with $\mathrm{T}_{\overline{\mathbb{Q}}_\ell}(P)$ and then taking the n-th perverse cohomology, the exact triangle

$${}^{\mathfrak{p}}\tau^{\leq n-1}(f_!\mathcal{F}) \longrightarrow {}^{\mathfrak{p}}\tau^{\leq n}(f_!\mathcal{F}) \longrightarrow {}^{\mathfrak{p}}\mathrm{H}^n(f_!\mathcal{F})[-n] \xrightarrow{+1}$$

induces an isomorphism

$${}^{\mathfrak{p}}\mathrm{H}^n\left(\mathrm{T}_{\overline{\mathbb{Q}}_\ell}(P) \otimes {}^{\mathfrak{p}}\tau^{\leq n}(f_!\mathcal{F})\right) \xrightarrow{\sim} {}^{\mathfrak{p}}\mathrm{H}^0\left(\mathrm{T}_{\overline{\mathbb{Q}}_\ell}(P) \otimes {}^{\mathfrak{p}}\mathrm{H}^n(f_!\mathcal{F})\right).$$

From this we have a map

$${}^{\mathfrak{p}}\mathrm{H}^0\left(\mathrm{T}_{\overline{\mathbb{Q}}_\ell}(P) \otimes {}^{\mathfrak{p}}\mathrm{H}^n(f_!\mathcal{F})\right) \longrightarrow {}^{\mathfrak{p}}\mathrm{H}^{n-1}(f_!\mathcal{F}).$$

Since $\mathrm{T}_{\overline{\mathbb{Q}}_\ell}(P) \otimes {}^{\mathfrak{p}}\mathrm{H}^n(f_!\mathcal{F}) \in {}^{\mathfrak{p}}\mathrm{D}_{\mathrm{c}}^{\leq 0}(S, \overline{\mathbb{Q}}_\ell)$, it projects to its 0-th perverse cohomology, hence we have a canonical map

$$\mathrm{T}_{\overline{\mathbb{Q}}_\ell}(P) \otimes {}^{\mathfrak{p}}\mathrm{H}^n(f_!\mathcal{F}) \longrightarrow {}^{\mathfrak{p}}\mathrm{H}^{n-1}(f_!\mathcal{F}).$$

The canonical decomposition (9.3.4) gives map

$$\bigoplus_{\alpha\in\Sigma} \mathrm{T}_{\overline{\mathbb{Q}}_\ell}(P) \otimes K_\alpha^n \longrightarrow \bigoplus_{\alpha\in\Sigma} K_\alpha^{n-1},$$

and in particular a canonical map

$$\mathrm{T}_{\overline{\mathbb{Q}}_\ell}(P) \otimes K_\alpha^n \longrightarrow K_\alpha^{n-1} \tag{9.3.5}$$

for each $\alpha \in \Sigma$ and n.

9.4 Statement of Freeness

Following [Ngô10, §§ 7.4.8–7.4.9], for each $\alpha \in \Sigma$, we may find a dense open subset V_α, such that K_α^n can be expressed as $\mathcal{K}_\alpha^n[\dim V_\alpha]$ for some local system $\mathcal{K}_\alpha^n$ of weight n on V_α. One can also so choose V_α that there is a finite radical base change $V'_\alpha \to V_\alpha$ over which the Chevalley exact sequence exists:

$$1 \longrightarrow R_\alpha \longrightarrow P|_{V'_\alpha} \longrightarrow A_\alpha \longrightarrow 1,$$

where R_α is a smooth, fiberwise connected affine group scheme over V'_α, and A_α is an abelian scheme over V'_α. It then induces short exact sequence of sheaves

$$0 \longrightarrow \mathrm{T}_{\overline{\mathbb{Q}}_\ell}(R_\alpha) \longrightarrow \mathrm{T}_{\overline{\mathbb{Q}}_\ell}(P|_{V'_\alpha}) \longrightarrow \mathrm{T}_{\overline{\mathbb{Q}}_\ell}(A_\alpha) \longrightarrow 0,$$

which can be seen as a sequence of sheaves on V_α since $V'_\alpha \to V_\alpha$ is a universal homeomorphism, and $\mathrm{T}_{\overline{\mathbb{Q}}_\ell}(P|_{V'_\alpha})$ is identified with $\mathrm{T}_{\overline{\mathbb{Q}}_\ell}(P|_{V_\alpha})$. One may further shrink V_α so that $\mathrm{T}_{\overline{\mathbb{Q}}_\ell}(R_\alpha)$ is a local system of weight -2 (coming from the multiplicative part) and $\mathrm{T}_{\overline{\mathbb{Q}}_\ell}(A_\alpha)$ is a local system of weight -1. Here the weight means the weight of $\sigma_{\bar{k}/k'}$ for some finite extension k'/k over which Z_α is defined. Finally, one shrinks V_α further so that $V_\alpha \cap Z_{\alpha'} = \varnothing$ unless $Z_\alpha \subset Z_{\alpha'}$.

Now we choose an open subset $U_\alpha \subset S_{\bar{k}}$ such that $i_\alpha \colon V_\alpha \hookrightarrow U_\alpha$ is a closed embedding. Over U_α, (9.3.5) becomes

$$\mathrm{T}_{\overline{\mathbb{Q}}_\ell}(P) \otimes i_{\alpha*}\mathcal{K}_\alpha^n[\dim V_\alpha] \longrightarrow i_{\alpha*}\mathcal{K}_\alpha^{n-1}[\dim V_\alpha].$$

The projection formula gives

$$\mathrm{T}_{\overline{\mathbb{Q}}_\ell}(P) \otimes i_{\alpha*}\mathcal{K}_\alpha^n \simeq i_{\alpha*}\left(i_\alpha^* \mathrm{T}_{\overline{\mathbb{Q}}_\ell}(P) \otimes \mathcal{K}_\alpha^n\right).$$

Since i_α is closed embedding, $i_\alpha^* i_{\alpha*} \simeq \mathrm{id}$, hence we obtain canonical map over V_α

$$\mathrm{T}_{\overline{\mathbb{Q}}_\ell}(P|_{V_\alpha'}) \otimes \mathcal{K}_\alpha^n \longrightarrow \mathcal{K}_\alpha^{n-1}.$$

Because $\mathcal{K}_\alpha^n$ is of weight n and $\mathcal{K}_\alpha^{n-1}$ is of weight $n-1$, the action of $\mathrm{T}_{\overline{\mathbb{Q}}_\ell}(P|_{V_\alpha'})$ factors through $\mathrm{T}_{\overline{\mathbb{Q}}_\ell}(A_\alpha)$ since the affine part has weight -2. Therefore we have a graded module structure on $\mathcal{K}_\alpha = \bigoplus_n \mathcal{K}_\alpha^n[-n]$ over graded algebra Λ_{A_α}

$$\Lambda_{A_\alpha} \otimes \mathcal{K}_\alpha \longrightarrow \mathcal{K}_\alpha.$$

We are going to prove the following result:

Proposition 9.4.1. *Suppose (f,g) is a weak abelian fibration and $\mathcal{F} \in \mathrm{D}_c^b(M, \overline{\mathbb{Q}}_\ell)$ is a self-dual P-equivariant complex. Then for any geometric point $u_\alpha \in V_\alpha$, the stalk $\mathcal{K}_{\alpha,u_\alpha}$ is a free $\Lambda_{A_\alpha,u_\alpha}$-module.*

Proof. It will be proved in §§ 9.5–9.7. ∎

Remark 9.4.2. According to [Ngô10, Lemme 7.4.11], the freeness statement is independent of the geometric point u_α. In other words, if the freeness holds as stated at just one point u_α, then it holds for any point, and in addition one can find some graded local system E (i.e., a direct sum of shifted local systems) on V_α such that $\mathcal{K}_\alpha \cong \Lambda_{A_\alpha} \otimes E$ as graded Λ_{A_α}-modules.

9.5 Freeness over a Point

We first prove a generalized version of [Ngô10, Proposition 7.5.1], which will serve as the base case for the inductive argument later on.

Lemma 9.5.1. *Let M be a projective variety over algebraically closed field $\bar{k}$ with an action of an abelian variety A over $\bar{k}$. Suppose all stabilizers are finite. Then*

$$\bigoplus_n \mathrm{H}_c^n(M, \mathcal{F})[-n]$$

is a free graded Λ_A-module for any A-equivariant complex $\mathcal{F} \in \mathrm{D}_c^b(M, \overline{\mathbb{Q}}_\ell)$.

Proof. Denote by f (resp. $\bar{f}$) the map $M \to \operatorname{Spec} \bar{k}$ (resp. $[M/A] \to \operatorname{Spec} \bar{k}$). Consider quotient map $q\colon M \to [M/A]$. By definition, $\mathcal{F} \simeq q^* \overline{\mathcal{F}}$ for some $\overline{\mathcal{F}} \in \mathrm{D}_c^b([M/A], \overline{\mathbb{Q}}_\ell)$. Here

The map q is smooth and projective, and following [Ngô10, Proposition 7.5.1], we have a non-canonical isomorphism

$$q_* \overline{\mathbb{Q}}_\ell \cong \bigoplus_n \mathrm{H}_c^n(A, \overline{\mathbb{Q}}_\ell)[-n], \tag{9.5.1}$$

where the right-hand side is viewed as constant sheaves on $[M/A]$. Since $[M/A]$ is a Deligne-Mumford stack, the cap product action constructed in 9.3, although not stated explicitly, can also be applied to $S = [M/A]$ with $P = A \times S$. Therefore both sides of (9.5.1) carry actions of Λ_A over $[M/A]$. We then claim that we can choose the isomorphism to be compatible with the respective Λ_A-actions. Indeed, any choice of the isomorphism

(9.5.1) allows us to define a morphism in $\mathrm{D}^{\mathrm{b}}_{\mathrm{c}}([M/A], \overline{\mathbb{Q}}_\ell)$

$$\overline{\mathbb{Q}}_\ell \longrightarrow q_*\overline{\mathbb{Q}}_\ell[2d](d).$$

Applying tensor product, we have morphism

$$\Lambda_A \longrightarrow \Lambda_A \otimes q_*\overline{\mathbb{Q}}_\ell[2d](d).$$

The composition map with cap product $\Lambda_A \longrightarrow q_*\overline{\mathbb{Q}}_\ell[2d](d)$ induces isomorphism on cohomology groups because isomorphism can be checked stalkwise, and fibers of q are just trivial A-torsors (since $\bar{k}$ is algebraically closed). This proves the claim by shifting and twisting back by $[-2d](-d)$.

Now by projection formula,

$$q_*\mathcal{F} \simeq q_*q^*\overline{\mathcal{F}} \simeq q_*\overline{\mathbb{Q}}_\ell \otimes \overline{\mathcal{F}} \cong \left(\bigoplus_n \mathrm{H}^n_{\mathrm{c}}(A, \overline{\mathbb{Q}}_\ell)[-n]\right) \otimes \overline{\mathcal{F}},$$

with the last isomorphism compatible with Λ_A-action. Using projection formula again, we see that

$$f_*\mathcal{F} \cong \left(\bigoplus_n \mathrm{H}^n_{\mathrm{c}}(A, \overline{\mathbb{Q}}_\ell)[-n]\right) \otimes \bar{f}_*\overline{\mathcal{F}}$$

as complexes with Λ_A-action. One can then compute the cohomologies of $f_*\mathcal{F}$ using the total complex of the tensor product, and because of $\mathrm{H}^n_{\mathrm{c}}(A, \overline{\mathbb{Q}}_\ell)[-n]$ is a direct sum of complexes with only one non-trivial term placed at different degrees, we have that

$$\bigoplus_n \mathrm{H}^n_{\mathrm{c}}(M, \mathcal{F})[-n] \cong \Lambda_A[-2d](-d) \otimes \left(\bigoplus_n \mathrm{H}^n_{\mathrm{c}}([M/A], \overline{\mathcal{F}})[-n]\right).$$

This finishes the proof. ∎

Let P be a smooth connected commutative group scheme over a finite field k. Since k is perfect, Chevalley's exact sequence is defined over k, and let A be the abelian quotient of P in that sequence. By [Ngô10, Proposition 7.5.3], there is a *quasi-lifting* homomorphism $a\colon A \to P$ such that the composition with $P \to A$ is the endomorphism of multiplication by some positive integer N on A. This quasi-lifting induces a canonical section of Tate modules

$$N^{-1}\,\mathrm{T}_{\overline{\mathbb{Q}}_\ell}(a)\colon\ \mathrm{T}_{\overline{\mathbb{Q}}_\ell}(A) \longrightarrow \mathrm{T}_{\overline{\mathbb{Q}}_\ell}(P)$$

that is compatible with Galois action.

Corollary 9.5.2. *Let M be a projective variety over algebraically closed field $\bar{k}$ with an action of smooth connected commutative group scheme P over $\bar{k}$. Suppose all stabilizers are affine and P is defined over a finite field, so there is quasi-lifting $a\colon A \to P$. Then*

$$\bigoplus_n \mathrm{H}^n_{\mathrm{c}}(M, \mathcal{F})[-n]$$

is a free graded Λ_A-module for any P-equivariant complex $\mathcal{F} \in \mathrm{D}^{\mathrm{b}}_{\mathrm{c}}(M, \overline{\mathbb{Q}}_\ell)$, where the Λ_A-action is defined using the canonical section $N^{-1}\,\mathrm{T}_{\overline{\mathbb{Q}}_\ell}(a)$.

Proof. Using the quasi-lifting a, we have an action of A on M with finite stabilizers. If $\mathcal{F}$ is P-equivariant, then it is also A-equivariant since $M \to [M/P]$ factors through $[M/A]$. Then the claim follows from Lemma 9.5.1 and scaling by N^{-1}. ∎

9.6 A Degenerate Spectral Sequence

Now we move from one-point base to strict Henselian bases. The setup and argument here are completely parallel to those in [Ngô10, § 7.6], only with constant sheaf $\overline{\mathbb{Q}}_\ell$ replaced by $\mathcal{F}$. Let S be a strict Henselian scheme over $\bar{k}$ with closed $\bar{k}$-point s. Let $\epsilon\colon S \to \operatorname{Spec} \bar{k}$ be the structure morphism. Retain scheme $f\colon M \to S$ with an action of group scheme

$g\colon P \to S$ as before. Then $\epsilon_* \Lambda_P$ is identified with stalk $\Lambda_{P,s}$, and we have by adjunction

$$\epsilon^* \Lambda_{P,s} \longrightarrow \Lambda_P.$$

Let $\mathcal{F}$ be a P-equivariant complex on M. Then the restriction of cap product (9.3.3) gives map

$$\epsilon^* \Lambda_{P,s} \boxtimes f_! \mathcal{F} \longrightarrow f_! \mathcal{F},$$

which defines an action of graded algebra $\Lambda_{P,s}$ on $f_! \mathcal{F}$ (to simplify notation, we now drop ϵ^* and treat $\Lambda_{P,s}$ as a constant sheaf on S). In particular, we have map

$$\mathrm{T}_{\overline{\mathbb{Q}}_\ell}(P_s) \boxtimes f_! \mathcal{F} \longrightarrow f_! \mathcal{F}[-1].$$

Since external tensor product is perverse t-exact, we have induced map

$$\mathrm{T}_{\overline{\mathbb{Q}}_\ell}(P_s) \boxtimes {}^{\mathfrak{p}}\tau^{\leq n}(f_! \mathcal{F}) \longrightarrow {}^{\mathfrak{p}}\tau^{\leq n}(f_! \mathcal{F}[-1]) = {}^{\mathfrak{p}}\tau^{\leq n-1}(f_! \mathcal{F})[-1],$$

hence also a map

$$\mathrm{T}_{\overline{\mathbb{Q}}_\ell}(P_s) \boxtimes {}^{\mathfrak{p}}\mathrm{H}^n(f_! \mathcal{F}) \longrightarrow {}^{\mathfrak{p}}\mathrm{H}^{n-1}(f_! \mathcal{F}).$$

This map is compatible with the map constructed in 9.3 restricted to $\mathrm{T}_{\overline{\mathbb{Q}}_\ell}(P_s)$ as they are both canonical. Therefore we obtain a graded $\Lambda_{P,s}$-module structure on

$${}^{\mathfrak{p}}\mathrm{H}^\bullet(f_! \mathcal{F}) = \bigoplus_n {}^{\mathfrak{p}}\mathrm{H}^n(f_! \mathcal{F})[-n].$$

The canonical decomposition by supports (9.3.4) and properties of external tensor product allows us to express the $\mathrm{T}_{\overline{\mathbb{Q}}_\ell}(P_s)$ action by a matrix indexed by $(\alpha, \alpha') \in \Sigma \times \Sigma$, with entries in

$$\mathrm{T}_{\overline{\mathbb{Q}}_\ell}(P_s)^* \otimes \mathrm{Hom}(K^n_\alpha, K^{n-1}_{\alpha'}),$$

where $\mathrm{T}_{\overline{\mathbb{Q}}_\ell}(P_s)^*$ means the $\overline{\mathbb{Q}}_\ell$-dual vector space. In this case, if $\alpha \neq \alpha'$, then we have $\mathrm{Hom}(K^n_\alpha, K^{n-1}_{\alpha'}) = 0$, so we know the matrix is diagonal.

Again use the fact that tensoring with $\Lambda_{P,s}$ commutes with ${}^{\mathfrak{p}}\tau^{\leq n}$, we have an increasing filtration ${}^{\mathfrak{p}}\tau^{\leq n}(f_!\mathcal{F})$ of $f_!\mathcal{F}$ compatible with $\Lambda_{P,s}$-actions. This gives a spectral sequence, also compatible with $\Lambda_{P,s}$-actions,

$$E_2^{m,n} = \mathrm{H}^m({}^{\mathfrak{p}}\mathrm{H}^n(f_!\mathcal{F})_s) \Rightarrow \mathrm{H}^{m+n}_{\mathrm{c}}(M_s, \mathcal{F}|_{M_s}). \tag{9.6.1}$$

Since $f_!\mathcal{F}$ is bounded, (9.6.1) is convergent. This implies that we have a *decreasing* filtration $F^m\,\mathrm{H}^\bullet_{\mathrm{c}}(M_s, \mathcal{F}|_{M_s})$ of direct sum

$$\mathrm{H}^\bullet_{\mathrm{c}}(M_s, \mathcal{F}|_{M_s}) = \bigoplus_n \mathrm{H}^n_{\mathrm{c}}(M_s, \mathcal{F}|_{M_s})[-n],$$

such that

$$F^m\,\mathrm{H}^\bullet_{\mathrm{c}}(M_s, \mathcal{F}|_{M_s}) / F^{m+1}\,\mathrm{H}^\bullet_{\mathrm{c}}(M_s, \mathcal{F}|_{M_s}) = \bigoplus_n E_\infty^{m,n}[-m-n].$$

The action of $\Lambda_{P,s}$ is compatible with the filtration F^m, hence induces an action on the associated graded $\overline{\mathbb{Q}}_\ell$-vector space

$$\bigoplus_{m,n} E_\infty^{m,n}[-m-n].$$

This action is the same as the $\Lambda_{P,s}$-action induced from the E_2-page, which in turn comes from the action on ${}^{\mathfrak{p}}\mathrm{H}^\bullet(f_!\mathcal{F})$.

Now we assume (f, g) is the strict Henselization at s of a weak abelian fibration and $\mathcal{F}$

is self-dual. Then the non-canonical decomposition (9.2.1) exists on S (as perverse cohomologies commute with étale base change). So the spectral sequence (9.6.1) degenerates at E_2-page and $E_\infty^{m,n} = E_2^{m,n}$.

9.7 Freeness by Induction

We can now finish the proof of Proposition 9.4.1 using the same inductive argument in [Ngô10, § 7.7]. The proof is by induction on dimension of Z_α. Let $\alpha_0 \in \Sigma$ be the unique maximal element so that $Z_{\alpha_0} = S_{\bar{k}}$. Let V_{α_0} be an open subset of S as in 9.4, then ${}^{\mathfrak{p}}\mathrm{H}^n(f_*\mathcal{F})$ is a local system $\mathcal{K}^n_{\alpha_0}[\dim S]$ when restricted to V_{α_0}. As in 9.4, we have a short exact sequence of Tate modules on V_{α_0}

$$0 \longrightarrow \mathrm{T}_{\overline{\mathbb{Q}}_\ell}(R_{\alpha_0}) \longrightarrow \mathrm{T}_{\overline{\mathbb{Q}}_\ell}(P|_{V_{\alpha_0}}) \longrightarrow \mathrm{T}_{\overline{\mathbb{Q}}_\ell}(A_{\alpha_0}) \longrightarrow 0.$$

The action of $\mathrm{T}_{\overline{\mathbb{Q}}_\ell}(P|_{V_{\alpha_0}})$ factors through $\mathrm{T}_{\overline{\mathbb{Q}}_\ell}(A_{\alpha_0})$ because of weights. By [Ngô10, Lemme 7.4.11] (cf. Remark 9.4.2), we may choose a geometric point $u_{\alpha_0} \in V_{\alpha_0}$, defined over a finite field, so that we can use Corollary 9.5.2 to deduce that $\mathrm{H}^\bullet(M_{u_{\alpha_0}}, \mathcal{F}|_{M_{u_{\alpha_0}}})$ is a free $\Lambda_{A_{\alpha_0,u_{\alpha_0}}}$-module. The spectral sequence (9.6.1) is especially simple in this case and only consists of terms $\mathcal{K}^n_{\alpha_0,u_{\alpha_0}}[-n + \dim S]$ for $n \in \mathbb{Z}$. In other words, we have a (non-canonical) isomorphism of $\Lambda_{A_{\alpha_0,u_{\alpha_0}}}$-modules

$$\mathrm{H}^\bullet(M_{u_{\alpha_0}}, \mathcal{F}|_{M_{u_{\alpha_0}}}) \cong \bigoplus_n \mathcal{K}^n_{\alpha_0,u_{\alpha_0}}[-n + \dim S].$$

So we proved Proposition 9.4.1 in the base case.

Next, let $\alpha \in \Sigma$ and suppose Proposition 9.4.1 is proved for all $\alpha' \in \Sigma$ such that $Z_\alpha \subset Z_{\alpha'}$. Let $u_\alpha \in V_\alpha$ be a geometric point defined over a finite field, and S_α be the strict Henselization of S at u_α. As in 9.6, we have actions

$$\mathrm{T}_{\overline{\mathbb{Q}}_\ell}(P_{u_\alpha}) \boxtimes {}^{\mathfrak{p}}\mathrm{H}^n(f_*\mathcal{F})|_{S_\alpha} \longrightarrow {}^{\mathfrak{p}}\mathrm{H}^{n-1}(f_*\mathcal{F})|_{S_\alpha}.$$

These actions can be represented by a diagonal matrix using decomposition over supports Σ. This means that we have a canonical isomorphism of graded Λ_{P,u_α}-modules

$$\bigoplus_n {}^{\mathfrak{p}}\mathrm{H}^n(f_*\mathcal{F})|_{S_\alpha}[-n] \simeq \bigoplus_{\alpha,n} K_\alpha^n[-n].$$

Using a quasi-lifting $A_{u_\alpha} \to P_{u_\alpha}$, one has induces Λ_{A,u_α}-module structure that is compatible with Galois action and, when restricted to V_α, is the same as the Λ_{A,u_α}-module structure on $\mathcal{K}_\alpha$ defined in 9.4 using factorization.

Using inductive hypothesis and the fact that $\mathrm{T}_{\overline{\mathbb{Q}}_\ell}(P)$ is polarizable (one of the axioms for weak abelian fibrations), we have the following result [Ngô10, Proposition 7.7.4]:

Proposition 9.7.1. *With notation above, we have that for any $m \in \mathbb{Z}$,*

$$\bigoplus_n \mathrm{H}^m(K^n_{\alpha',u_\alpha})[-n]$$

is a free graded Λ_{A,u_α}-module.

Proof. We have by inductive hypothesis and [Ngô10, Lemme 7.4.11] an isomorphism of $\Lambda_{A_{\alpha'}}$-modules on $V_{\alpha'}$

$$\mathcal{K}_{\alpha'} \cong \Lambda_{A_{\alpha'}} \otimes E_{\alpha'}$$

for some graded local system $E_{\alpha'}$ on $V_{\alpha'}$. Restricting to $V_{\alpha'} \cap S_\alpha$ and pick a geometric point $\bar{y}_{\alpha'}$ over the generic point $y_{\alpha'}$ of $V_{\alpha'} \cap S_\alpha$. Using polarizability assumption, we can show that the specialization map $\mathrm{T}_{\overline{\mathbb{Q}}_\ell}(A_{u_\alpha}) \to \mathrm{T}_{\overline{\mathbb{Q}}_\ell}(A_{\bar{y}_{\alpha'}})$ is injective, hence a direct summand as $\mathrm{Gal}(\bar{y}_{\alpha'}/y_{\alpha'})$-representations. So the stalk $\Lambda_{A,\bar{y}_{\alpha'}}$ is a free Λ_{A,u_α}-module, hence so is $\mathcal{K}_{\alpha',\bar{y}_{\alpha'}}$. Using [Ngô10, Lemme 7.4.11] again, we have that

$$\mathcal{K}_{\alpha'}|_{V_{\alpha'}\cap S_\alpha} \cong \Lambda_{A,u_\alpha} \boxtimes E'_{\alpha'}$$

as Λ_{A,u_α}-modules for some graded local system $E'_{\alpha'}$. Taking intermediate extension and then stalk at u_α, and since these operations commute with external tensor product, we have that K_{α',u_α} is the external tensor product of Λ_{A,u_α} with another complex, hence the same is true for its m-th cohomology groups. ∎

We can now finish the inductive proof using a key property of $\overline{\mathbb{Q}}_\ell$-algebra Λ_{A,u_α}: its projective modules are also injective. The spectral sequence (9.6.1) degenerates at E_2-page, and we have a decreasing filtration of

$$H \coloneqq \bigoplus_n \mathrm{H}^n(M_{u_\alpha}, \mathcal{F}|_{M_{u_\alpha}})[-n]$$

whose m-th graded part is

$$\bigoplus_n \mathrm{H}^m({}^{\mathfrak{p}}\mathrm{H}^n(f_*\mathcal{F})_{u_\alpha})[-m-n] = \mathrm{H}^m\left(\bigoplus_n {}^{\mathfrak{p}}\mathrm{H}^n(f_*\mathcal{F})_{u_\alpha}[-n]\right)[-m],$$

all compatible with Λ_{A,u_α}-action. It can be further decomposed using supports into

$$\mathrm{H}^m\left(\bigoplus_{\alpha',n} K^n_{\alpha',u_\alpha}[-n]\right)[-m]$$

as Λ_{A,u_α}-modules. If $\alpha' \neq \alpha$, then $K^n_{\alpha',u_\alpha} \neq 0$ only if $Z_{\alpha'} \supset Z_\alpha$, and if $\alpha' = \alpha$, $\mathrm{H}^m(K^n_{\alpha,u_\alpha}) \neq 0$ only if $m = -\dim Z_\alpha$. Let $H'' = F^{-\dim Z_\alpha}H$ and H' be the preimage of

$$\mathrm{H}^{-\dim Z_\alpha}\left(\bigoplus_{\alpha'\neq\alpha,n} K^n_{\alpha',u_\alpha}[-n]\right)[\dim Z_\alpha]$$

in H''. Then we have a filtration

$$H' \subset H'' \subset H,$$

in which H is a free Λ_{A,u_α}-module by Corollary 9.5.2, and so is H' and H/H'' by inductive hypothesis. Then since H and H/H'' are free, so is H'' because Λ_{A,u_α} is local (by Kaplansky's theorem, which applies to non-commutative rings). Consider exact sequence

$$0 \longrightarrow H' \longrightarrow H'' \longrightarrow H''/H' \longrightarrow 0.$$

As H' is free hence also injective (using the special property of exterior algebra Λ_{A,u_α}), the sequence splits. So $H''/H' = \mathcal{K}_{\alpha,u_\alpha}$ is projective hence free. This finishes the proof of Proposition 9.4.1.

9.8 Statement of Support Theorem

Now we are ready to state and prove the main theorem of this chapter. The common assumptions in this section are as follows:

(1) Let S be a Deligne-Mumford stack of finite type over k.

(2) Let $f\colon M \to S$ be a proper morphism of Deligne-Mumford stacks of relative dimension d together with an action of a smooth commutative Deligne-Mumford group stack $g\colon P \to S$, such that (f, g) is a weak abelian fibration (see Remark 9.1.3).

(3) The geometric fibers of f are homeomorphic to projective varieties.

(4) Let $\mathcal{F} \in \mathrm{D}^{\mathrm{b}}_{\mathrm{c}}(M, \overline{\mathbb{Q}}_\ell)$ be a self-dual P-equivariant complex, such that $f_*\mathcal{F}$ has cohomological degrees bounded above by $-\dim M + 2d$.

Theorem 9.8.1. *With the assumptions above, if K is a geometrically simple summand in the decomposition of some ${}^{\mathfrak{p}}\mathrm{H}^n(f_*\mathcal{F})$ with support Z. Let δ_Z be the minimal value of δ*

on Z, then we have

$$\mathrm{codim}_S(Z) \leq \delta_Z.$$

Moreover, if the equality holds, then there exists an open subset $U \subset S_{\bar{k}}$ such that $U \cap Z \neq \emptyset$, and a local system L on $U \cap Z$ such that i_*L (where i is the closed embedding $U \cap Z \to U$) is a direct summand of the top cohomology $\mathbf{R}^{-\dim M+2d} f_*\mathcal{F}$ restricted to U.

Proof. Although f and g are not morphism of schemes, the argument for Proposition 9.4.1 still applies. More precisely, everything except Lemma 9.5.1 works verbatim for general Deligne-Mumford stacks, and Lemma 9.5.1 only potentially breaks because *a priori* $[M_s/P_s]$ for a geometric point $s \in S$ can be a 2-stack instead of an algebraic stack. However, since we also assume that M_s is homeomorphic to a projective scheme (and P_s is always homeomorphic to a smooth group scheme), the argument in Lemma 9.5.1 still works.

By Proposition 9.4.1, $K|_{U\cap Z}$ is a direct summand of a free $(\Lambda_P)|_{U\cap Z}$-module, whose top cohomology is denoted by L. Moreover, the intermediate extension of L to Z is also a simple perverse summand of $f_*\mathcal{F}$. Then the argument in § 9.2 proves both claims of the theorem. ∎

Corollary 9.8.2. *If in Theorem 9.8.1 (f, g) is also δ-regular, then the equality* $\mathrm{codim}_S(Z) = \delta_Z$ *holds.*

Proof. Clear since the definition of δ-regularity gives the inequality in the opposite direction. ∎

9.8.3 Let $\pi_0(P)$ be a sheaf of finite abelian groups over S such that for any geometric point $s \in S$ its stalk is the group of connected components $\pi_0(P_s)$. Suppose $\pi_0(P)$ is a quotient of some constant sheaf $\mathbf{X}$ where $\mathbf{X}$ is a finite abelian group. Since $\mathcal{F}$ is P-equivariant, P canonically acts on $\mathcal{F}$ by (9.3.2). The induced action of P on ${}^{\mathfrak{p}}\mathrm{H}^\bullet(f_*\mathcal{F})$

factors through $\pi_0(P)$ by *Lemme d'homotopie* [LN08, Lemme 3.2.3], hence $\mathbf{X}$ acts on ${}^{\mathfrak{p}}\mathrm{H}^n(f_*\mathcal{F})$ for each n. Let ${}^{\mathfrak{p}}\mathrm{H}^n(f_*\mathcal{F})_\kappa$ be the κ-isotypic direct summand for any character $\kappa\colon \mathbf{X}\to\overline{\mathbb{Q}}_\ell^\times$. Similarly we have κ-isotypic summand of ordinary cohomology $\mathrm{H}^n(f_*\mathcal{F})_\kappa$. By [LN08, Lemme 3.2.5], there exists an integer $N>0$ and a decomposition

$$f_*\mathcal{F}=\bigoplus_{\kappa\in\mathbf{X}^*}(f_*\mathcal{F})_\kappa,$$

such that for any $\alpha\in\mathbf{X}$, the restriction of $(\alpha-\kappa(\alpha)\mathrm{id})^N$ on $(f_*\mathcal{F})_\kappa$ is zero, and we have

$$\begin{aligned}{}^{\mathfrak{p}}\mathrm{H}^n(f_*\mathcal{F})_\kappa&={}^{\mathfrak{p}}\mathrm{H}^n((f_*\mathcal{F})_\kappa),\\ \mathrm{H}^n(f_*\mathcal{F})_\kappa&=\mathrm{H}^n((f_*\mathcal{F})_\kappa).\end{aligned}$$

Proposition 9.8.4. *Theorem 9.8.1 and Corollary 9.8.2 hold if we replace* ${}^{\mathfrak{p}}\mathrm{H}^n(f_*\mathcal{F})$ *by* ${}^{\mathfrak{p}}\mathrm{H}^n(f_*\mathcal{F})_\kappa$ *and* $\mathrm{H}^{-\dim M+2d}(f_*\mathcal{F})$ *by* $\mathrm{H}^{-\dim M+2d}(f_*\mathcal{F})_\kappa$*.*

Proof. Apply the same proof to $(f_*\mathcal{F})_\kappa$. ∎

9.9 Application to mH-fibrations

In this section, we apply Theorem 9.8.1 to mH-fibration $h_X\colon \mathcal{M}_X\to\mathcal{A}_X$, or more precisely a subset of the map $\tilde{h}_X^\natural\colon\tilde{\mathcal{M}}_X^\natural\to\tilde{\mathcal{A}}_X^\natural$. There is an action of Picard stack $\tilde{p}_X^\natural\colon\tilde{\mathcal{P}}_X^\natural\to\tilde{\mathcal{A}}_X^\natural$ on $\tilde{h}_X^\natural$, and both are relative Deligne-Mumford stacks of finite type by Proposition 8.1.1. By Proposition 8.1.2, $\tilde{h}_X^\natural$ is also proper. If we restrict to G-very ample locus $\tilde{\mathcal{A}}_\gg^\natural$, by Proposition 6.7.1, the Tate modules of $\tilde{\mathcal{P}}_\gg^\natural$ are polarizable. Therefore $(\tilde{h}_\gg^\natural,\tilde{p}_\gg^\natural)$ is a weak abelian fibration.

In general, $(\tilde{h}_\gg^\natural,\tilde{p}_\gg^\natural)$ is not δ-regular, but we can still find an open subset over which the fibration is δ-regular. This subset is dense in $\tilde{\mathcal{A}}_\gg^\natural$ because over G-very ample locus the $\delta=0$ stratum is dense by Proposition 6.3.13. According to Proposition 7.4.5, for any fixed $\delta\in\mathbb{N}$, all but finitely many irreducible components of $\tilde{\mathcal{A}}_\delta^\natural$ is contained in this

δ-regular locus.

Let $\tilde{\mathcal{A}}^\dagger \subset \tilde{\mathcal{A}}^\natural_\gg$ be the largest open subset satisfying the following conditions:

(1) The restriction of (h_X, p_X) is δ-regular;

(2) The local model of singularity as in Theorem 6.10.2 holds.

Let $\mathcal{Q}$ be the intersection complex of $\tilde{\mathcal{M}}^\dagger$. Then by Proposition 8.4.3, we have that over any connected component $\mathcal{U}$ of $\tilde{\mathcal{A}}^\dagger$, the cohomological degrees of $\tilde{h}^\dagger_*\mathcal{Q}$ is bounded above by $-\dim \mathcal{M}_\mathcal{U} + 2d$, where $\mathcal{M}_\mathcal{U}$ is the preimage of $\mathcal{U}$ in $\tilde{\mathcal{M}}^\dagger$, and d is the relative dimension of $\tilde{h}^\dagger$ over $\mathcal{U}$. Finally, the fibers of $\tilde{h}^\dagger$ are homeomorphic to projective varieties by Corollary 6.9.2.

Definition 9.9.1. A closed subset $Z \subset \tilde{\mathcal{A}}^\dagger$ is said to be *inductive* if there exists an open subset $\mathcal{U} \subset \tilde{\mathcal{A}}^\dagger$ such that $\mathcal{U} \cap Z \neq \varnothing$ has multiplicity-free boundary divisors in the sense of Proposition 5.1.25, and for any $a \in (\mathcal{U} \cap Z)(\bar{k})$ and any $\bar{v} \in X(\bar{k})$, one of the following cases is true

(1) The boundary divisor is 0 at $\bar{v}$, and the discriminant valuation $d_{\bar{v}+}(a) \leq 1$, or

(2) a is both unramified and ν-regular semisimple at $\bar{v}$.

9.9.2 We are now able to prove one of the main results of this paper:

Theorem 9.9.3 (Support Theorem)**.** *Let K be a simple perverse summand of ${}^\mathfrak{p}\mathrm{H}^\bullet(\tilde{h}^\dagger_*\mathcal{Q})_{\mathrm{st}}$, and Z is the support of K, then Z is inductive. In particular, Z must be δ-critical.*

Proof. Applying Corollary 9.8.2 to $(\tilde{h}^\dagger, \tilde{p}^\dagger)$, we see that Z must be the closure of a δ-critical stratum. So by the description of δ-critical strata Corollary 7.4.7, we only need to show the following: if $Z' \subset \tilde{\mathcal{A}}^\dagger$ is an irreducible δ-critical closed subset but not inductive, then Z' cannot be the support of any simple perverse summand K.

Indeed, using Proposition 7.5.5 (and the discussions before it), we may find a closed subset $Z \subset \tilde{\mathcal{A}}^{\dagger}$ containing Z' such that both the boundary divisor and all the local Newton points stays locally constant, and Z is inductive. We may even assume that in Definition 9.9.1 for Z we also have $\mathcal{U} \cap Z' \neq \emptyset$. By Proposition 8.5.15, the top cohomology of $(\tilde{h}^{\dagger}_{*}\mathcal{Q})_{\mathrm{st}}$ is a local system L on $\mathcal{U}$. However, if Z' supports a perverse summand, then L contains a direct summand supported on proper closed subset $\mathcal{U} \cap Z'$, which is impossible. This finishes the proof. ∎

9.9.4 The case of κ-constituent is more complicated. The most notable issue is that *a priori* an endoscopic stratum may not be contained in $\tilde{\mathcal{A}}^{\dagger}$, hence we cannot use Proposition 7.4.5 to deduce δ-regularity. Nevertheless, we do not need to prove δ-regularity in an open neighborhood around the endoscopic strata, but only need to show δ-regularity *within* those strata. This can be achieved with the help of (6.11.5).

Since we are not confined to $\tilde{\mathcal{A}}^{\dagger}$, we consider a larger open subset $\tilde{\mathcal{A}}^{\ddagger}$ defined similar to $\tilde{\mathcal{A}}^{\dagger}$ but with δ-regularity condition removed. Let $(\kappa, \vartheta^{\bullet}_{\kappa,\xi})$ be an endoscopic datum with endoscopic group $H = H_{\xi}$. There is a canonical finite map

$$\tilde{\nu}_{\mathcal{A}}\colon \tilde{\mathcal{A}}^{\kappa}_{H,X} \longrightarrow \tilde{\mathcal{A}}_{X}.$$

By (6.11.5), the difference $r^{G}_{H} = \delta - \delta_{H}$ is locally constant on $\tilde{\mathcal{A}}^{\kappa}_{H,X}$. We would like to restrict to the locus $\tilde{\mathcal{A}}^{\kappa,\dagger}_{H}$, which is, analogous to $\tilde{\mathcal{A}}^{\dagger}$, defined to be the open subset of $\tilde{\mathcal{A}}^{\kappa,\natural}_{H,X}$ that satisfies ampleness condition for Proposition 7.4.5 and has a local model of singularity as in Theorem 6.10.2. Therefore $(\tilde{h}^{\kappa,\dagger}_{H}, \tilde{p}^{\kappa,\dagger}_{H})$ is δ_{H}-regular.

For any irreducible component $\mathcal{U}_{H}$ of $\tilde{\mathcal{A}}^{\kappa,\dagger}_{H}$, let $\mathcal{U} = \tilde{\nu}_{\mathcal{A}}(\mathcal{U}_{H})$ and $r_{\mathcal{U}} \in \mathbb{N}$ be the value of r^{G}_{H} on $\mathcal{U}_{H}$. If we can show that

$$\operatorname{codim}_{\tilde{\mathcal{A}}_{X}}(\mathcal{U}) = r_{\mathcal{U}},$$

then for any $\delta \geq r_{\mathcal{U}}$, we will have

$$\operatorname{codim}_{\tilde{\mathcal{A}}_X}(\mathcal{U}_\delta) = r_{\mathcal{U}} + \operatorname{codim}_{\tilde{\mathcal{A}}_{H,X}}(\mathcal{U}_{H,\delta-r_{\mathcal{U}}}) \geq r_{\mathcal{U}} + (\delta - r_{\mathcal{U}}) = \delta.$$

In other words, δ-regularity holds for those δ-strata with non-trivial intersection with $\mathcal{U}$ as long as it holds for the generic stratum of $\mathcal{U}$.

9.9.5 Due to the insufficiency in ampleness, direct estimate of the codimension of $\mathcal{U}$ seems difficult. See § 6.8, but see also Remark 6.11.8. Moreover, we also need Theorem 6.10.2 to hold, which depends on some cohomological condition that is essentially the same problem as the dimension estimate.

On the other hand, if we assume Theorem 6.10.2 holds, although dimension estimate looks like a condition needed to apply Theorem 9.8.1, which will then be used to prove Theorem 8.3.4, the logic can actually be partially reversed, and the codimension estimate will be a *conclusion* instead of a condition, which we now explain. This does not solve the issue itself, but does allow us to make the statement of Theorem 8.3.4 cleaner by not including another technical condition.

Irrespective of δ-regularity, the freeness statement Proposition 9.4.1 still holds. For any $a_H \in \mathcal{U}_H$ mapping to $a \in \mathcal{U}$, there is a canonical map of the Picard stacks

$$\mathcal{P}_a \longrightarrow \mathcal{P}_{H,a_H},$$

which identifies their Néron models

$$\mathcal{P}_a^\flat \simeq \mathcal{P}_{H,a_H}^\flat.$$

This identifies the abelian part of the Tate modules

$$\Lambda_{A_a,a} \simeq \Lambda_{A_{H,a_H},a_H}.$$

Let $\mathcal{Q}_H^\kappa$ be the intersection complex on $\tilde{\mathcal{M}}_H^{\kappa,\dagger}$, then we may apply Theorem 9.8.1 to $(\tilde{h}_H^{\kappa,\dagger}, \tilde{p}_H^{\kappa,\dagger})$, so generically over $\mathcal{U}_H$, the sheaf ${}^{\mathfrak{p}}\mathrm{H}^\bullet(\tilde{h}_{H,*}^{\kappa,\dagger}\mathcal{Q}_H^\kappa)_{\mathrm{st}}$ is isomorphic to a free $\mathrm{T}_{\overline{\mathbb{Q}}_\ell}(\mathcal{P}_H)$-module of rank 1. In particular, using Grothendieck-Lefschetz trace formula, it is not hard to count k'-points on fibers $\mathcal{M}_{H,a_H}$ (modulo Picard action and weighted by the sheaf $\mathcal{Q}_H^\kappa$ to be precise) for any finite extension k'/k and a general point $a_H \in \mathcal{U}_H(\bar{k})$.

On the other hand, $\mathcal{Q}$-weighted point-counting on $\mathcal{M}_a$ is not hard either if we only consider a general enough point $a \in \mathcal{U}(\bar{k})$: indeed, due to how transfer map $\tilde{\nu}_{\mathcal{A}}$ is induced by the absolute transfer map $\nu_{\mathbf{H}}$ in (2.5.1), we shall see later that it boils down to counting points on some very simple multiplicative affine Springer fibers. Therefore by Chebotarev's density theorem, suppose for any sufficiently general point $a \in \mathcal{U}(\bar{k})$ and any sufficiently large finite extension k'/k we have equality in Frobenius trace

$$\mathrm{Tr}(\sigma_{k'}, \mathrm{H}^\bullet(\mathcal{M}_a, \mathcal{Q})_\kappa) = \sum_{a_H \mapsto a} \mathrm{Tr}(\sigma_{k'}, \mathrm{H}^\bullet(\mathcal{M}_{H,a_H}^\kappa, \mathcal{Q}_H^\kappa)_{\mathrm{st}}), \tag{9.9.1}$$

then we have generically over $\mathcal{U}$

$$(\tilde{h}_*^\ddagger \mathcal{Q})_\kappa \cong (\tilde{h}_{H,*}^{\kappa,\dagger}\mathcal{Q}_H^\kappa)_{\mathrm{st}} \tag{9.9.2}$$

up to semisimplification with respect to Frobenius action.

Let $\mathcal{U}_0 \subset \mathcal{U}$ be the open locus where the above equality holds, and let $\mathcal{A}$ be the irreducible component of $\tilde{\mathcal{A}}^\ddagger$ containing $\mathcal{U}$. Let d be the dimension of $\mathcal{M}_a$, and r' be the codimension of $\mathcal{U}$ in $\tilde{\mathcal{A}}_X$. Then $\mathcal{M}_{H,a_H}$ has dimension $d - r_{\mathcal{U}}$. By (9.9.2), we know that the perverse amplitude of $(\tilde{h}_*^\ddagger \mathcal{Q})_\kappa$ on $\mathcal{U}_0$ is from $-d + r_{\mathcal{U}}$ to $d - r_{\mathcal{U}}$, which translate to ordinary cohomological degrees from $-\dim\mathcal{U} - d + r_{\mathcal{U}}$ to $-\dim\mathcal{U} + d - r_{\mathcal{U}}$ over $\mathcal{U}_0$ as it is geometrically a graded local system on $\mathcal{U}_0$. Note that

$$-\dim\mathcal{U} + d - r_{\mathcal{U}} = -\dim\mathcal{A} + d - (r_{\mathcal{U}} - r') \leq -\dim\mathcal{A} + d,$$

the last inequality is due to the improved Goresky-MacPherson inequality in the discussion of § 9.2. However, $-\dim \mathcal{A} + d$ is the top ordinary cohomological degree of $(\tilde{h}^{\ddagger}_{*}\mathcal{Q})_a$ given by irreducible components of $\mathcal{M}_a$, and we know that each κ-isotypic summand does contribute non-trivially to the top ordinary cohomologies. Thus it forces the inequality above to be an equality and so $r_{\mathcal{U}} = r'$. Thus we have the following theorem.

Theorem 9.9.6. *Let ${}^{\mathfrak{p}}\mathrm{H}^{\bullet}(\tilde{h}^{\ddagger}_{*}\mathcal{Q})_{\kappa}$ be the κ-isotypic summand in ${}^{\mathfrak{p}}\mathrm{H}^{\bullet}(\tilde{h}^{\ddagger}_{*}\mathcal{Q})$ and K is a geometrically simple perverse summand in ${}^{\mathfrak{p}}\mathrm{H}^{\bullet}(\tilde{h}^{\ddagger}_{*}\mathcal{Q})_{\kappa}$. Let Z be the support of K. Then there exists a pointed endoscopic datum $(\kappa, \vartheta^{\bullet}_{\kappa})$ with endoscopic group H such that Z is contained in $\tilde{\nu}_{\mathcal{A}}(\tilde{\mathcal{A}}^{\kappa}_{H,X})$, where $\tilde{\nu}_{\mathcal{A}} \colon \tilde{\mathcal{A}}^{\kappa}_{H,X} \to \tilde{\mathcal{A}}_X$ is the finite endoscopic transfer map.*

Moreover, if we can find an open subset $\mathcal{U} \subset \tilde{\mathcal{A}}^{\ddagger}$ such that $\mathcal{U} \cap Z \neq \emptyset$ and $\tilde{\nu}_{\mathcal{A}}^{-1}(\mathcal{U}) \subset \tilde{\mathcal{A}}^{\dagger,\kappa}_{H}$, and for any $a \in (\mathcal{U} \cap Z)(\bar{k})$, the equality (9.9.1) *holds for any sufficiently large finite extension k'/k, then we have that $\tilde{\nu}_{\mathcal{A}}^{-1}(Z)$ equals the union of closed subsets that appear as the potential support of ${}^{\mathfrak{p}}\mathrm{H}^{\bullet}(\tilde{h}^{\kappa,\dagger}_{H,*}\mathcal{Q}^{\kappa}_{H})_{\mathrm{st}}$ given by Theorem 9.9.3.*

Proof. The assumptions of the theorem and the discussions above ensures δ-regularity on Z. Replacing Proposition 8.5.15 by Proposition 8.5.16, the same argument for Theorem 9.9.3 then applies. ∎

CHAPTER 10

COUNTING POINTS

In this chapter we review general facts about counting k-points on k-groupoids. We will then connect point counting on multiplicative affine Springer fibers with both its cohomology and the orbital integrals. Similarly, we will also apply the general results to mH-fibrations. Most of the results are slight generalization of those in [Ngô10, § 8] with some small improvements. Combined with the results in § 9.9, we will be able to prove both Theorem 8.3.4 and Theorem 2.6.11.

10.1 Generalities on Counting Points

We first review some general facts about counting points following [Ngô10, § 8.1]. In this paper, we need more general coefficients than constant sheaf $\overline{\mathbb{Q}}_\ell$, and because there has been some significant development on the algebro-geometric tools used for these purposes, we will base our discussions on those frameworks as well.

10.1.1 First we let $f\colon \mathscr{X} \to \operatorname{Spec} k$ be a k-variety, or more generally an algebraic stack of finite type over k, and $\mathcal{F} \in \mathrm{D}^{\mathrm{b}}_{\mathrm{c}}(\mathscr{X}, \overline{\mathbb{Q}}_\ell)$ a bounded constructible lisse-étale complex on $\mathscr{X}$ with coefficients in $\overline{\mathbb{Q}}_\ell$. Then for any k-point $x \in \mathscr{X}(k)$ and a fixed geometric point $\bar{x} = \operatorname{Spec} \bar{k}$ over x, we have a continuous representation of $\mathrm{Gal}(\bar{k}/k)$ on stalk $\mathcal{F}_{\bar{x}}$. Let Frobenius element $\sigma_k \in \mathrm{Gal}(\bar{k}/k)$ be the one defined by taking $q = p^m$-th root, where q is the cardinality of k, then we may define trace function as

$$\begin{aligned}\mathrm{Tr}_{\mathcal{F}}\colon \mathscr{X}(k) &\longrightarrow \overline{\mathbb{Q}}_\ell\\ x &\longmapsto \sum_{i\in\mathbb{Z}} (-1)^i \,\mathrm{Tr}(\sigma_k, \mathrm{H}^i(\mathcal{F}_{\bar{x}})).\end{aligned}$$

If the complex $\mathcal{F}$ is not bounded, then the above construction still makes sense, except that the trace function now takes values only as infinite series, but one can impose convergence condition if so desired. For our purposes, we are mostly interested in Deligne-Mumford stacks of finite types, therefore boundedness condition is preserved by any six-functor operation, so we do not worry about potential issues caused by unboundedness.

A classical result is Grothendieck-Lefschetz trace formula, which states that

$$\#_{\mathcal{F}}\, \mathscr{X}(k) \coloneqq \sum_{x \in \mathscr{X}(k)} \mathrm{Tr}_{\mathcal{F}}(x) = \mathrm{Tr}_{f_! \mathcal{F}}\,. \tag{10.1.1}$$

This is a special case of the relative version of the formula

$$\#_{\mathcal{F}}\, \mathscr{X}(k) = \#_{f_! \mathcal{F}}\, \mathscr{Y}(k) \tag{10.1.2}$$

for a morphism $f\colon \mathscr{X} \to \mathscr{Y}$ between k-schemes of finite types.

A perhaps more common form of (10.1.1) is the following: if k has $q = p^m$ elements, let σ be the m-fold iteration of the absolute Frobenius on $\mathscr{X}$, then σ is defined over k. It is a (non-trivial) fact that there is a functorial isomorphism $\iota\colon \mathcal{F} \simeq \sigma^* \mathcal{F}$ for any $\mathcal{F}$, see [Sta22, Tag 03SL]. The morphism σ is finite, hence proper, so we have adjunction map

$$\mathcal{F} \longrightarrow \sigma_* \sigma^* \mathcal{F} \simeq \sigma_! \sigma^* \mathcal{F}.$$

Pushing forward using $f_!$, we see that σ induces a map on cohomological groups

$$\sigma \otimes_k \bar{k}\colon \mathrm{H}^i_{\mathrm{c}}(\mathscr{X}_{\bar{k}}, \mathcal{F}) \longrightarrow \mathrm{H}^i_{\mathrm{c}}(\mathscr{X}_{\bar{k}}, \sigma^* \mathcal{F}) \simeq \mathrm{H}^i_{\mathrm{c}}(\mathscr{X}_{\bar{k}}, \mathcal{F}),$$

where the second isomorphism is defined using the canonical identification ι above. It turns out that this map is the same map induced by σ_k-action on $f_! \mathcal{F}$, and so we have

$$\#_{\mathcal{F}}\, \mathscr{X}(k) = \sum_{i \in \mathbb{Z}} (-1)^i\, \mathrm{Tr}\big(\sigma \otimes_k \mathrm{id}_{\bar{k}}, \mathrm{H}^i_{\mathrm{c}}(\mathscr{X}_{\bar{k}}, \mathcal{F})\big). \tag{10.1.3}$$

Remark 10.1.2. The notions of the plethora of Frobenii are sometimes confusing, especially for the so-called "geometric Frobenius". The Frobenius $\sigma \otimes_k \bar{k}$ is called geometric Frobenius in some literature, while a different construction, also bearing the name of geometric Frobenius, is the map $\mathrm{id}_{\mathscr{X}} \otimes_k \sigma_k$. The latter is the one we used to define $\mathrm{Tr}_{\mathcal{F}}$, and both induce the same action on $\bar{k}$-points and cohomologies. Since we only care about point-counting and cohomological actions, we shall not distinguish the two and simply use σ in the remaining sections.

10.1.3 In [Beh93], Behrend generalizes (10.1.2) to the case where $\mathscr{X}$ is a smooth Deligne-Mumford stack of finite type over k, and $\mathcal{F}$ is the constant sheaf. In the stack case, the $\mathcal{F}$-weighted point counting is defined as

$$\#_{\mathcal{F}}\, \mathscr{X}(k) \coloneqq \sum_{x \in \mathscr{X}(k)/\sim} \frac{\mathrm{Tr}_{\mathcal{F}}(x)}{\#\,\mathrm{Aut}_{\mathscr{X}}(x)(k)},$$

where x ranges over isomorphism classes of $\mathscr{X}(k)$. Note that since Aut_x is a k-group scheme of finite type and k is a finite field, the above definition makes sense even for Artin stacks. The result is further expanded to smooth Artin stacks and arbitrary complexes in [Beh03], assuming certain technical boundedness or convergence condition is met.

In [LO08a,LO08b], Laszlo and Olsson developed six-functor formalism for Artin stacks, and the smoothness assumption in Behrend's results is removed under these frameworks by Sun in [Sun12] because of the newly-available duality. As noted by Sun in [Sun12], an important difference between [Beh03] and [Sun12] is that the former uses arithmetic Frobenius and ordinary cohomology (probably due to lack of duality results at the time), while in the latter the geometric version is used. Because in [Sun12] the author considers a very general setup, some complicated "stratifiablility" condition and a convergence condition are used. However, for our purposes, we only consider bounded complexes on Deligne-Mumford stacks, so those conditions are automatically met. Therefore, to summarize, for any morphism $f\colon \mathscr{X} \to \mathscr{Y}$ of Deligne-Mumford stacks of finite types over k

and any $\mathcal{F} \in D^b_c(\mathscr{X}, \overline{\mathbb{Q}}_\ell)$, the trace formula (10.1.2) always holds.

10.1.4 In [LN08, Appendice A.3] and [Ngô10, § 8.1], an equivariant version of (10.1.3) is developed with specific applications to affine Springer fibers and Hitchin fibrations in mind. Here we reformulate those results using the general trace formula (10.1.2) for Deligne-Mumford stacks.

Suppose $\mathcal{M}$ is a Deligne-Mumford stack of finite type over k together with an action of a commutative Deligne-Mumford group stack $\mathcal{P}$ of finite type over k. The groupoids $\mathcal{M}(\bar{k})$ and $\mathcal{P}(\bar{k})$ carry natural actions of geometric Frobenius σ, and the 2-categorical quotient $\mathscr{X}(\bar{k})$ is the 2-category where:

(1) the objects are the objects $m \in \mathcal{M}(\bar{k})$;

(2) the 1-morphisms $m \to m'$ are pairs (p, f) where $p \in \mathcal{P}(\bar{k})$ and f is an element in $\mathrm{Hom}_{\mathcal{M}(\bar{k})}(p(m), m')$;

(3) the 2-morphisms $(p, f) \Rightarrow (p', f')$ is an element $j \in \mathrm{Hom}_{\mathcal{P}(\bar{k})}(p, p')$ such that the composition of $j\colon p(m) \to p'(m)$ with $f'\colon p'(m) \to m'$ is equal to $f\colon p(m) \to m'$.

According to [Ngô06, Lemme 4.7], such 2-category is equivalent to a groupoid if and only if for any $m \in \mathcal{M}(\bar{k})$, the homomorphism induced by $\mathcal{P}$ action

$$\mathrm{Aut}_{\mathcal{P}(\bar{k})}(1_{\mathcal{P}}) \longrightarrow \mathrm{Aut}_{\mathcal{M}(\bar{k})}(m)$$

is injective. In view of product formula Proposition 6.9.6, this condition is always met for the anisotropic part of mH-fibrations. When this condition is met, then a 1-morphism (p, f) has only the trivial 2-automorphism, thus $\mathscr{X}(\bar{k})$ is equivalent to the groupoid where:

(1) the objects are the objects $m \in \mathcal{M}(\bar{k})$;

(2) the morphisms $m \to m'$ are isomorphism classes of pairs (p, f).

In this case, we omit f and simply write a morphism as $p\colon m \to m'$, or equivalently we have canonical isomorphism $p(m) \simeq m'$.

We will now always assume that $\mathscr{X}(\bar{k})$ is equivalent to a groupoid. The category $\mathscr{X}(k)$ of fixed points under the action of σ is as follows:

(1) the objects are triples (m, p, f) such that $f\colon \sigma(m) \to p(m)$ is an isomorphism;

(2) a morphism $h\colon (m, p, f) \to (m', p', f')$ is a pair (h, ϕ) where $h \in \mathcal{P}(\bar{k})$ and $\phi\colon hm \to m'$ is an isomorphism, such that $\sigma(h)ph^{-1} = p'$ and $\sigma(\phi)\colon \sigma(hm) \to \sigma(m')$ is equal to the map $f'^{-1} \circ (p'\phi) \circ (\sigma(h)f)$;

(3) the sets of 2-morphisms are guaranteed either empty or singletons by assumption, making this category a groupoid.

Remark 10.1.5. If we use f to identify $\sigma(m)$ and $p(m)$ and similarly for f', then the last condition on (h, ϕ) can be simplified to $\sigma(\phi) = p'(\phi)$. If $p = p'$, then the conditions further simplify to $h \in \mathcal{P}(k)/\sim$ and $\sigma(\phi) = p(\phi)$. In this case, ϕ and ϕ' are equivalent if $\phi' = \alpha\phi$ for some $\alpha \in \mathrm{Aut}_{\mathcal{P}}(1_{\mathcal{P}})$.

Since $\mathscr{X}(k)$ has only finitely many isomorphism classes, and each object has only finitely many automorphisms, if τ is a $\overline{\mathbb{Q}}_\ell$-valued function on the set of isomorphism classes $\mathscr{X}(k)/\sim$, the following sum makes sense

$$\#_\tau\, \mathscr{X}(k) = \sum_{x \in \mathscr{X}(k)/\sim} \frac{\tau(x)}{\#\,\mathrm{Aut}_{\mathscr{X}(k)}(x)}.$$

10.1.6 Let $x = (m, p) \in \mathscr{X}(k)$ as above, then the σ-conjugacy class of p is determined by the isomorphism class of x. Let P be the coarse space of $\mathcal{P}$, then P is a commutative group scheme, and we have isomorphism between geometric connected components

$$\pi_0(\mathcal{P}) \simeq \pi_0(P).$$

According to Lang's theorem, the σ-conjugacy classes of $P(\bar{k})$ can be identified with

$$P_\sigma \simeq \mathrm{H}^1(k, P) \simeq \mathrm{H}^1(k, \pi_0(P)).$$

This shows that $\mathcal{P}_\sigma \simeq \mathrm{H}^1(k, \pi_0(\mathcal{P}))$ as well, the former denoting the σ-conjugacy classes of the objects in $\mathcal{P}(\bar{k})$. Therefore, given $x \in \mathscr{X}(k)$, we define $\mathrm{cl}(x)$ be the corresponding σ-conjugacy class of $\pi_0(\mathcal{P})$ also viewed as an element in $\mathrm{H}^1(k, \pi_0(\mathcal{P}))$. The class $\mathrm{cl}(x)$ depends only on the isomorphism class of x. For any σ-invariant character $\kappa\colon \pi_0(\mathcal{P}) \to \overline{\mathbb{Q}}_\ell^\times$, we have a pairing

$$\langle \mathrm{cl}(x), \kappa \rangle = \kappa(\mathrm{cl}(x)) \in \overline{\mathbb{Q}}_\ell^\times.$$

For any $\overline{\mathbb{Q}}_\ell$-valued function τ on $\mathscr{X}(k)/\sim$, we define (τ, κ)-weighted point counting

$$\#_\tau \mathscr{X}(k)_\kappa = \sum_{x \in \mathscr{X}(k)/\sim} \frac{\langle \mathrm{cl}(x), \kappa \rangle \tau(x)}{\#\mathrm{Aut}_{\mathscr{X}(k)}(x)}.$$

10.1.7 Let $\mathcal{F}$ now be $\mathcal{P}$-equivariant, that is, we are supplied with a fixed isomorphism $\psi\colon a^*\mathcal{F} \simeq \overline{\mathbb{Q}}_\ell \boxtimes \mathcal{F}$ on $\mathcal{P} \times \mathcal{M}$, where a is the action map $\mathcal{P} \times \mathcal{M} \to \mathcal{M}$, and usual cocycle and identity axioms of equivariance is met for ϕ. In this way, the objects of $\mathcal{P}(\bar{k})$ acts on the sheaf $\mathcal{F}|_{\mathcal{M}_{\bar{k}}}$, hence also on any cohomology groups.

According to *Lemme d'homotopie* [LN08, Lemme 3.2.3], the induced action on $\mathrm{H}_c^\bullet(\mathcal{M}_{\bar{k}}, \mathcal{F})$ factors through $\pi_0(\mathcal{P})$. Note that the statement in *loc. cit.* is about perverse cohomologies, but the proof still works for ordinary cohomologies since the main ingredients in the proof being smooth base change and exactness of pullback functor still hold, and by duality it works for cohomology with compact support. The group $\mathcal{P}$ does not need to be smooth because we always have $\mathcal{P}^{\mathrm{red}}$ to be smooth. Thus, if κ is a σ-invariant character of $\pi_0(\mathcal{P})$, then σ acts on the κ-isotypic subspace $\mathrm{H}_c^\bullet(\mathcal{M}, \mathcal{F})_\kappa$.

Finally, by equivariance, the trace function $\mathrm{Tr}_{\mathcal{F}}$ descends to a function on groupoid

$\mathscr{X}(k)$, still denoted by $\mathrm{Tr}_{\mathcal{F}}$, as follows: for any pair (m,p) such that $\sigma(m)\simeq p(m)$, the equivariance structure ψ identifies the stalk of $\mathcal{F}$ at m with that at $\sigma(m)$, hence defining an action of σ on that stalk, and we can take the trace. Now we have the following variant of Grothendieck-Lefschetz trace formula, which is a generalization of [LN08, Proposition A.3.1] (see also [Ngô10, Proposition 8.1.6]).

Proposition 10.1.8. *Let $\mathcal{M}$ be a Deligne-Mumford stack of finite type over k with an action of a commutative Deligne-Mumford group stack $\mathcal{P}$ of finite type over k, and suppose $\mathscr{X}(\bar{k})=[\mathcal{M}(\bar{k})/\mathcal{P}(\bar{k})]$ is equivalent to a groupoid. Let $\mathcal{F}$ be a bounded constructible $\mathcal{P}$-equivariant complex. Then for any σ-invariant character $\kappa\colon \pi_0(\mathcal{P})\to\overline{\mathbb{Q}}_\ell^\times$ we have equality*

$$\#\mathcal{P}_0(k)\,\#_{\mathcal{F}}\,\mathscr{X}(k)_\kappa=\sum_{i\in\mathbb{Z}}(-1)^i\,\mathrm{Tr}(\sigma,\mathrm{H}_c^i(\mathcal{M},\mathcal{F})_\kappa),$$

where $\mathcal{P}_0$ is the neutral component of $\mathcal{P}$.

Proof. Using Fourier transform on finite group $\pi_0(\mathcal{P})$, we have that

$$\sum_{i\in\mathbb{Z}}(-1)^i\,\mathrm{Tr}(\sigma,\mathrm{H}_c^i(\mathcal{M},\mathcal{F})_\kappa)=\frac{1}{\#\pi_0(\mathcal{P})}\sum_{p\in\pi_0(\mathcal{P})}\kappa(p)\,\mathrm{Tr}(p^{-1}\circ\sigma,\mathrm{H}_c^i(\mathcal{M},\mathcal{F})).$$

Note that if a character χ is not σ-invariant, then its isotypic space has no contribution to the right-hand side, which is then equal to

$$\frac{1}{\#\pi_0(\mathcal{P})_\sigma}\sum_{p\in\pi_0(\mathcal{P})_\sigma}\kappa(p)\,\mathrm{Tr}(\dot{p}^{-1}\circ\sigma,\mathrm{H}_c^i(\mathcal{M},\mathcal{F})),\tag{10.1.4}$$

where $\dot{p}$ is any choice of representative of σ-conjugacy class p in $\pi_0(\mathcal{P})$.

In the proof of [LN08, Proposition A.3.1], another variant of (10.1.3) by Deligne and Lusztig is used (see [DL76]). The idea is that if we assume that $\mathcal{M}$ is a quasi-projective scheme, then $p^{-1}\circ\sigma$ is the Frobenius map of another k-model of $\mathcal{M}_{\bar{k}}$. Here $\mathcal{M}$ is a Deligne-Mumford stack, but we can still descend $\mathcal{M}_{\bar{k}}$ to a k-model such that $\dot{p}^{-1}\circ\sigma$ is the induced Frobenius. Then using (10.1.3) for Deligne-Mumford stacks, (10.1.4) is equal to

$$\frac{1}{\#\pi_0(\mathcal{P})_\sigma}\sum_{p\in\pi_0(\mathcal{P})_\sigma}\sum_{\substack{m\in\mathcal{M}(\bar{k})/\sim\\ \sigma(m)=\dot{p}(m)}}\frac{\kappa(p)\operatorname{Tr}_{\mathcal{F}}(m)}{\#\left\{\phi\in\operatorname{Aut}_{\mathcal{M}(\bar{k})}(m)\;\middle|\;\sigma(\phi)=\dot{p}(\phi)\right\}}.$$

Let P be the coarse space of $\mathcal{P}$. According to Remark 10.1.5, we have

$$\#P(k)\,\#\left\{\phi\in\operatorname{Aut}_{\mathcal{M}(\bar{k})}(m)\;\middle|\;\sigma(\phi)=\dot{p}(\phi)\right\}=\#\operatorname{Aut}_{\mathcal{P}}(1_{\mathcal{P}})(k)\,\#\operatorname{Aut}_{\mathscr{X}(k)}(m,\dot{p}).$$

Note that $\pi_0(\mathcal{P})_\sigma$ and $\pi_0(\mathcal{P})^\sigma=\pi_0(\mathcal{P})(k)$ have the same cardinality, and $P(k)\to\pi_0(\mathcal{P})(k)$ is surjective by Lang's theorem. So we obtain

$$\sum_{i\in\mathbb{Z}}(-1)^i\operatorname{Tr}(\sigma,\mathrm{H}^i_c(\mathcal{M},\mathcal{F})_\kappa)=\#\mathcal{P}_0(k)\sum_{p\in\pi_0(\mathcal{P})_\sigma}\sum_{\substack{m\in\mathcal{M}(\bar{k})/\sim\\ \sigma(m)=\dot{p}(m)}}\frac{\kappa(p)\operatorname{Tr}_{\mathcal{F}}(m)}{\#\operatorname{Aut}_{\mathscr{X}(k)}(m,\dot{p})}.$$

Finally, note that the set of pairs $(m,\dot{p})$ in the summations above is in bijection with the isomorphism classes in $\mathscr{X}(k)$ by definition, thus we reach the desired equality

$$\sum_{i\in\mathbb{Z}}(-1)^i\operatorname{Tr}(\sigma,\mathrm{H}^i_c(\mathcal{M},\mathcal{F})_\kappa)=\#\mathcal{P}_0(k)\sum_{(m,p)\in\mathscr{X}(k)/\sim}\frac{\kappa(p)\operatorname{Tr}_{\mathcal{F}}(m)}{\#\operatorname{Aut}_{\mathscr{X}(k)}(m,p)}.$$

This finishes the proof. ∎

10.1.9 So far we only considered the case where $\mathcal{M}$ and $\mathcal{P}$ are of finite types, but to apply our results to multiplicative affine Springer fibers, we need another variant for locally finite type cases. Here we no longer need to consider Deligne-Mumford stacks but only schemes. Let M be a k-scheme with an action of a commutative k-group scheme P, both locally of finite types. We assume they satisfies the following assumptions:

(1) The $\bar{k}$-points of the group of connected components $\pi_0(P)$ is a finitely generated

abelian group.

(2) The stabilizer of any geometric point of M in P is of finite type over k

(3) We can find a torsion-free discrete subgroup $\Lambda \subset P$ such that both P/Λ and M/Λ are of finite types.

Note that in the above conditions since stabilizers of points in M are of finite type, the action of Λ on M is necessarily free. Note also that such Λ always exists if we only require P/Λ to be of finite type. Indeed, let Λ_0 be the largest free quotient of $P^{\mathrm{red}}_{\bar{k}}$, then the kernel

$$P^{\mathrm{ft}} = \ker(P^{\mathrm{red}}_{\bar{k}} \longrightarrow \Lambda_0)$$

is of finite type. Since Λ_0 is free, we may pick an arbitrary lifting $\gamma\colon \Lambda_0 \to P_{\bar{k}}$, which is necessarily defined over some finite extension k'/k.

Since P^{ft} is of finite type, the group $P^{\mathrm{ft}}(k')$ is finite. This means that when N is divisible by $\#P^{\mathrm{ft}}(k')$, the restriction of γ to $\Lambda = N\Lambda_0$ is σ-equivariant hence defined over k, and P/Λ is clearly of finite type. The condition that M/Λ is finite type is clearly independent of the choice of Λ, so the third condition above is equivalent to saying M/Λ is of finite type for any specific choice of Λ.

From now on, since we only care about counting points, we will use M and P to denote the respective sets of their $\bar{k}$-points. Because the points of M and P have no automorphisms, this notation will not cause any confusion.

10.1.10 Consider quotient stack $\mathscr{X} = [M/P]$, whose groupoid of k-points is as follows:

(1) the objects are pairs (m, p) where $m \in M$ and $p \in P$ such that $p(m) = \sigma(m)$;

(2) the morphisms $(m, p) \to (m', p')$ are $h \in P$ such that $hm = m'$ and $p' = \sigma(h)ph^{-1}$.

Given $x = (m, p) \in \mathscr{X}(k)$, the σ-conjugacy class $\mathrm{cl}(x)$ of p depends only on the isomorphism class of x. Since $p(m) = \sigma(m)$ and m is defined over some finite extension k'/k, the class $\mathrm{cl}(x)$ is necessarily torsion, hence it lies in the torsion subgroup of P_σ, which is identified with $\mathrm{H}^1(k, P)$. By [Ngô10, Lemme 8.1.12], any character κ of $\mathrm{H}^1(k, P)$ can be extended to a torsion character $\tilde{\kappa}$ of P_σ.

Since we have equivalence of quotient stacks

$$\mathscr{X} = [M/P] = [(M/\Lambda)/(P/\Lambda)],$$

we know that $\mathscr{X}(k)$ has only finitely many isomorphism classes, and the automorphism group of each object is also finite. Therefore for any character $\kappa\colon \mathrm{H}^1(k, P) \to \overline{\mathbb{Q}}_\ell^\times$, and any $\overline{\mathbb{Q}}_\ell$-valued function τ on the isomorphism classes of $\mathscr{X}(k)$, we may consider sum

$$\#_\tau \mathscr{X}(k)_\kappa = \sum_{x \in \mathscr{X}(k)/\sim} \frac{\langle \mathrm{cl}(x), \kappa \rangle \tau(x)}{\#\mathrm{Aut}_{\mathscr{X}(k)}(x)}.$$

Let $\tilde{\kappa}$ be an extension of κ to $P(\bar{k})_\sigma$, which inflates to a σ-invariant character of P. Since $\tilde{\kappa}$ has finite order, we can choose Λ so that the restriction of $\tilde{\kappa}$ to Λ is trivial. So $\tilde{\kappa}$ descends to a character of P/Λ, and if $\Lambda' \subset \Lambda$ is a σ-stable sublattice of finite index, $\tilde{\kappa}$ induces a character of P/Λ' too.

Let $\mathcal{F}$ be a bounded locally constructible complex on M that is P-equivariant, and still denote by $\mathcal{F}$ its descent to M/Λ. As before, $\mathrm{Tr}_{\mathcal{F}}$ descends to a function on $\mathscr{X}(k)$, also denoted by $\mathrm{Tr}_{\mathcal{F}}$. Let $\mathrm{H}_c^\bullet(M/\Lambda, \mathcal{F})_{\tilde{\kappa}}$ be the $\tilde{\kappa}$-isotypic direct summand of $\mathrm{H}_c^\bullet(M/\Lambda, \mathcal{F})$. The following is a generalization of [Ngô10, Proposition 8.1.13].

Proposition 10.1.11. *We have the following equality*

$$\# P_0(k) \#_{\mathcal{F}} \mathscr{X}(k)_\kappa = \sum_{i \in \mathbb{Z}} (-1)^i \,\mathrm{Tr}(\sigma, \mathrm{H}_c^i(M/\Lambda, \mathcal{F})_{\tilde{\kappa}}).$$

Moreover, if $\Lambda' \subset \Lambda$ is a σ-stable sublattice of finite index, then we have for each i a *canonical isomorphism*

$$\mathrm{H}^i_c(M/\Lambda, \mathcal{F})_{\tilde{\kappa}} \xrightarrow{\sim} \mathrm{H}^i_c(M/\Lambda', \mathcal{F})_{\tilde{\kappa}}.$$

Proof. Since Λ is free, we have $P_0 \simeq (P/\Lambda)_0$. Then the first formula is a consequence of Proposition 10.1.8. If we replace k by a finite extension k' of degree e, then for any $x' \in \mathscr{X}(k')$ we have a σ^e-conjugacy class in P. Since κ, Λ, etc. are σ-invariant, they are also σ^e-invariant, so we also have

$$\# P_0(k') \#_{\mathcal{F}} \mathscr{X}(k')_\kappa = \sum_{i \in \mathbb{Z}} (-1)^i \operatorname{Tr}\big(\sigma^e, \mathrm{H}^i_c(M/\Lambda, \mathcal{F})_{\tilde{\kappa}}\big).$$

Since e is arbitrary, it means that the map

$$\bigoplus_i (-1)^i \mathrm{H}^i_c(M/\Lambda, \mathcal{F})_{\tilde{\kappa}} \longrightarrow \bigoplus_i (-1)^i \mathrm{H}^i_c(M/\Lambda', \mathcal{F})_{\tilde{\kappa}}$$

is an isomorphism of σ-modules which also respects grading i. So we have the second claim. ∎

Corollary 10.1.12. *Let $\mathscr{X} = [M/P]$ (resp. $\mathscr{X}' = [M'/P']$) and $\kappa\colon P \to \overline{\mathbb{Q}}_\ell^\times$ (resp. $\kappa'\colon P' \to \overline{\mathbb{Q}}_\ell^\times$) be a σ-invariant character of finite order. Let $\mathcal{F}$ (resp. $\mathcal{F}'$) be a P-equivariant (resp. P'-equivariant) bounded locally constructible complex on M (resp. M'). Suppose there exists some N such that for all $e > N$ and k'/k a finite extension of degree e we always have*

$$\# P_0(k') \#_{\mathcal{F}} \mathscr{X}(k')_\kappa = \# P'_0(k') \#_{\mathcal{F}'} \mathscr{X}'(k')_{\kappa'},$$

then we have

$$\# P_0(k) \#_{\mathcal{F}} \mathscr{X}(k)_\kappa = \# P'_0(k) \#_{\mathcal{F}'} \mathscr{X}'(k)_{\kappa'}.$$

Proof. After choosing appropriate lattices $\Lambda \subset P$ and $\Lambda' \subset P'$, the assumptions imply that $\mathrm{H}_c^\bullet(M/\Lambda), \mathcal{F})_{\bar{\kappa}}$ and $\mathrm{H}_c^\bullet(M'/\Lambda'), \mathcal{F}')_{\bar{\kappa}'}$ are isomorphic σ-modules up to semisimplification. Taking the traces of σ and we get the result. ∎

10.2 Counting Points on Multiplicative Affine Springer Fibers

In this section, we apply the general results from the previous section to multiplicative affine Springer fibers. We use the notations from Chapter 4 which we now review. Let $X_v = \operatorname{Spec}\mathcal{O}_v$ be the formal disc around a closed point $v \in |X|$, and $X_v^\bullet = \operatorname{Spec} F_v$ is the punctured disc. Let π be a fixed choice of uniformizer in $\mathcal{O}_v$. Let $\breve{X}_v = \operatorname{Spec}\breve{\mathcal{O}}_v$ where $\breve{\mathcal{O}}_v = \mathcal{O}_v \hat{\otimes}_k \bar{k}$ and similarly $\breve{X}_v^\bullet = \operatorname{Spec}\breve{F}_v$. For any embedding $\bar{v}\colon k_v \to \bar{k}$, we let $\breve{X}_{\bar{v}} = \operatorname{Spec}\breve{\mathcal{O}}_{\bar{v}}$ be the component of $\breve{X}_v$ containing $\bar{v}$. If $k_v = k$, then $\breve{X}_{\bar{v}} = \breve{X}_v$.

10.2.1 Let $a \in \mathfrak{C}_{\mathfrak{M}}(\mathcal{O}_v) \cap \mathfrak{C}_{\mathfrak{M}}^\times(F_v)^{\mathrm{rs}}$, then there exists some $\gamma_{\mathfrak{M}} \in \mathfrak{M}^\times(F_v)$ with $\chi_{\mathfrak{M}}(\gamma_{\mathfrak{M}}) = a$ by Theorem 2.4.24. We have multiplicative affine Springer fiber $\mathcal{M}_v(a)$ defined using $\gamma_{\mathfrak{M}}$, whose set of $\bar{k}$-points is

$$\left\{g \in G(F_v)/G(\mathcal{O}_v) \;\middle|\; \mathrm{Ad}_g^{-1}(\gamma_{\mathrm{ad}}) \in G^{\mathrm{ad}}(\mathcal{O}_v)\pi^{\lambda_v}G^{\mathrm{ad}}(\mathcal{O}_v)\right\},$$

where γ_{ad} is the image of $\lambda_{\mathfrak{M}}$ in G^{ad}, and λ_v is the boundary divisor of a, which may also be viewed as an F_v-rational dominant cocharacter in $\check{\mathbb{X}}(T^{\mathrm{ad}})$ through map

$$\mathfrak{A}_{\mathfrak{M}} \to \mathfrak{A}_{\mathrm{Env}(G^{\mathrm{sc}})}.$$

We denote the Newton point of γ_{ad} by ν_v. Since we only care about point-counting, we will replace $\mathcal{M}_v(a)$ with its reduced subfunctor and still use notation $\mathcal{M}_v(a)$ for simplicity. We know that the reduced functor $\mathcal{M}_v(a)$ is represented by a k-scheme locally of finite type.

10.2.2 Following [Ngô10], let $\mathfrak{J}_a$ be the regular centralizer at a, and $\mathfrak{J}'_a$ be a smooth group scheme over X_v with connected fibers and a homomorphism $\mathfrak{J}'_a \to \mathfrak{J}_a$ such that its restriction to $X_v^\bullet$ is an isomorphism. Note that $\mathfrak{J}'_a$ is necessarily commutative.

Consider reduced commutative group scheme over k

$$\mathcal{P}'_v(a)^{\mathrm{red}} \coloneqq \mathsf{Gr}^{\mathrm{red}}_{\mathfrak{J}'_a} = (\mathbb{L}\mathfrak{J}'_a/\mathbb{L}^+\mathfrak{J}'_a)^{\mathrm{red}}.$$

The second equality above is due to the fact that $\mathfrak{J}'_a$ has connected special fiber. This is a group scheme locally of finite type over k, and we use $\mathcal{P}'_v(a)$ for simplicity. Similarly, we have $\mathcal{P}_v(a)$ for $\mathfrak{J}_a$, and we have homomorphism

$$\mathcal{P}_v(a)' \longrightarrow \mathcal{P}_v(a).$$

The action of $\mathcal{P}_v(a)$ on $\mathcal{M}_v(a)$ induces an action of $\mathcal{P}'_v(a)$ on $\mathcal{M}_v(a)$. By Proposition 4.4.7, this action satisfies conditions laid out in the beginning of § 10.1.9, and so we may express different κ-weighted point-countings using trace formula on cohomologies. As in § 10.1.9, we also use $\mathcal{M}_v(a)$, $\mathcal{P}_v(a)$, etc. to denote their respective sets of $\bar{k}$-points.

10.2.3 We now try to connect point-countings on stack $\mathscr{X}' = [\mathcal{M}_v(a)/\mathcal{P}'_v(a)]$ with local orbital integrals. First, the following lemma relates the κ in orbital integrals with κ in § 10.1:

Lemma 10.2.4 ([Ngô10, Lemme 8.2.4])**.** *Assuming $\mathfrak{J}'_a$ has connected special fiber, then we have canonical isomorphism*

$$\mathrm{H}^1(F_v, \mathfrak{J}_a) \simeq \mathrm{H}^1(k, \mathcal{P}'_v(a)).$$

By this lemma, any character $\kappa\colon \mathrm{H}^1(F_v, \mathfrak{J}_a) \to \overline{\mathbb{Q}}_\ell^\times$ induces a character of $\mathrm{H}^1(k, \mathcal{P}'_v(a))$

and vice versa, and we still use κ to denote the character of the latter. Let $\mathcal{P}'_v(a)_\sigma$ be the group of σ-conjugacy classes of $\mathcal{P}'_v(a)$, and $\mathrm{H}^1(k, \mathcal{P}'_v(a))$ is the subgroup of torsion elements (due to continuity requirement). Given $x = (m, p) \in \mathscr{X}'(k)$, we have associated class given by the σ-conjugacy class of p:

$$\mathrm{cl}(x) \in \mathrm{H}^1(k, \mathcal{P}'_v(a)) \simeq \mathrm{H}^1(F_v, \mathfrak{J}_a).$$

The element $\gamma_{\mathfrak{M}}$ is regular (as an F_v-point), so we have canonical isomorphism $\mathfrak{J}_a|_{F_v} \simeq I_{\gamma_{\mathfrak{M}}} \subset G$. If a section of $\chi_{\mathfrak{M}}$ exists (e.g., when a Steinberg quasi-section exists), then $\gamma_{\mathfrak{M}}$ can be chosen to be in $\mathfrak{M}(\mathcal{O}_v)^{\mathrm{reg}}$, and in this case we have $\mathfrak{J}_a \simeq I_{\gamma_{\mathfrak{M}}}$. In either case, $\mathrm{cl}(x)$ may also be regarded as an element of $\mathrm{H}^1(F_v, I_{\gamma_{\mathfrak{M}}})$.

If $g \in G(\breve{F}_v)$ is a representative of m and $j \in I_{\gamma_{\mathfrak{M}}}(F_v)$ is a representative of p, then $p(m) = \sigma(m)$ implies that $\sigma(g)^{-1}jg \in G(\breve{\mathcal{O}}_v)$, hence j is σ-conjugate to 1 in $G(\breve{F}_v)$. This shows that the image of $\mathrm{cl}(x)$ in $\mathrm{H}^1(F_v, G)$ is trivial. Note that here we are dependent on a choice of $\gamma_{\mathfrak{M}}$, otherwise we cannot relate $\mathrm{H}^1(F_v, \mathfrak{J}_a)$ with $\mathrm{H}^1(F_v, G)$.

We let $\mathsf{Gr}^{\leq \lambda_v}$ be the affine Schubert variety of the adjoint group G^{ad}. The map of groupoids

$$\mathcal{M}_v(a)(\bar{k}) \longrightarrow \left[\mathbb{L}^+ G^{\mathrm{ad}} \backslash \mathsf{Gr}^{\leq \lambda_v}\right](\bar{k})$$

is σ-equivariant and $\mathcal{P}'_v(a)(\bar{k})$-invariant, so any function τ on the set of isomorphism classes of

$$\left[\mathbb{L}^+ G^{\mathrm{ad}} \backslash \mathsf{Gr}^{\leq \lambda_v}\right](k)$$

induces a function on $\mathscr{X}'(k)$ by pullback. Note that since G has connected fibers, the said isomorphism classes are in bijection with double cosets

$$G(\mathcal{O}_v)\pi^\mu G(\mathcal{O}_v)$$

where $\mu \in \check{\mathbb{X}}(T^{\mathrm{ad}})$ is an F_v-rational dominant cocharacter with $\mu \le \lambda_v$. In particular, the function τ can be the trace function $\mathrm{Tr}_{\mathcal{F}}$ where $\mathcal{F}$ is bounded and constructible $\mathbb{L}^+G^{\mathrm{ad}}$-equivariant complex on $\mathsf{Gr}^{\le\lambda_v}$.

Proposition 10.2.5. *Assuming $\mathfrak{J}'_a$ has connected special fiber, and let κ be a character of $\mathrm{H}^1(F_v, \mathfrak{J}_a)$. Then for any function τ on $\left[\mathbb{L}^+G^{\mathrm{ad}}\backslash\mathsf{Gr}^{\le\lambda_v}\right](k)$, we have equality*

$$\#_\tau\, \mathscr{X}'(k)_\kappa = \mathrm{vol}(\mathfrak{J}'_a(\mathcal{O}_v), \mathrm{d}t_v)\, \mathbf{O}_a^\kappa(\tau, \mathrm{d}t_v),$$

where $\mathrm{d}t_v$ is any Haar measure on $\mathfrak{J}_a(F_v)$.

Proof. For any $x \in \mathscr{X}'(k)$, the class $\mathrm{cl}(x)$ can be identified with an element in the kernel of the map

$$\mathrm{H}^1(F_v, I_{\gamma\mathfrak{m}}) \longrightarrow \mathrm{H}^1(F_v, G).$$

So we can decompose $\mathscr{X}'(k)$ into disjoint full subcategories

$$\mathscr{X}'(k) = \coprod_\xi \mathscr{X}'_\xi(k),$$

where ξ ranges over the said kernel above, and $\mathscr{X}'(k)_\xi$ consists of objects x with $\mathrm{cl}(x) = \xi$.

For a fixed class ξ, we pick a representative $j_\xi \in I_{\gamma\mathfrak{m}}(\breve{F}_v)$. Let $x = (m, p) \in \mathscr{X}'_\xi(k)$, then we can find $h \in \mathcal{P}'_v(a)$ such that $\sigma(h)ph^{-1} = j_\xi$. Replacing (m, p) with an isomorphic object (hm, j_ξ), we may always assume $p = j_\xi$. In other words, we may restrict to an equivalent full subcategory of $\mathscr{X}'_\xi(k)$ consisting of objects (m, j_ξ), and the morphisms from (m, j_ξ) to (m', j_ξ) are elements in $h \in \mathcal{P}'_v(a)(k)$ such that $hm = m'$. Since $\mathfrak{J}'_a$ has connected fibers, we have

$$\mathcal{P}'_v(a)(k) = \mathfrak{J}_a(F_v)/\mathfrak{J}'_a(\mathcal{O}_v) \simeq I_{\gamma_{\mathfrak{M}}}(F_v)/\mathfrak{J}'_a(\mathcal{O}_v).$$

For an object (m, j_ξ), pick a representative $g \in G(\breve{F}_v)$, then $j_\xi(m) = \sigma(m)$ implies that $\sigma(g)^{-1} j_\xi g \in G(\breve{\mathcal{O}}_v)$. Thus we may pick g so that we actually have $\sigma(g)^{-1} j_\xi g = 1$, and if g and g' are two such choices, we must have $g' = gg_0$ for some $g_0 \in G(\mathcal{O}_v)$. So an object (m, j_ξ) determines a unique element in quotient set $G(\breve{F}_v)/G(\mathcal{O}_v)$. Thus, the category $\mathscr{X}'_\xi(k)$ is equivalent to the following category O_ξ:

(1) the objects are elements $g \in G(\breve{F}_v)/G(\mathcal{O}_v)$, such that $\sigma(g)^{-1} j_\xi g = 1$ and $\mathrm{Ad}_g^{-1}(\gamma_{\mathfrak{M}})$ is contained in $\mathfrak{M}(\breve{\mathcal{O}}_v)$;

(2) a morphism $g \to g_1$ is an element $h \in \mathcal{P}_v(a)(k)$ such that $hg = g_1$, where the action of $\mathcal{P}'_v(a)(k)$ is induced by the isomorphism $\mathfrak{J}_a \simeq I_{\gamma_{\mathfrak{M}}}$ over $X_v^\bullet$.

Choose $g_\xi \in G(\breve{F}_v)$ such that $\sigma(g_\xi)^{-1} j_\xi g_\xi = 1$, and let

$$\gamma_\xi = \mathrm{Ad}_{g_\xi}^{-1}(\gamma_{\mathfrak{M}}),$$

then we necessarily have $\gamma_\xi \in \mathfrak{M}^\times(F_v)$. One can verify that the $G(F_v)$-conjugacy class of γ_ξ does not depend on the choice of either j_ξ or g_ξ, but only on the cohomology class $\xi \in \mathrm{H}^1(F_v, I_{\gamma_{\mathfrak{M}}})$ (hence also implicitly depends on the choice of $\gamma_{\mathfrak{M}}$).

Let $g' = g_\xi^{-1} g$, then O_ξ is equivalent to the following category O'_ξ: the objects are $g' \in G(F_v)/G(\mathcal{O}_v)$, such that $\mathrm{Ad}_{g'}^{-1}(\gamma_\xi) \in \mathfrak{M}(\mathcal{O}_v)$, and a morphism $g' \to g'_1$ is an element $h \in \mathcal{P}'_v(a)(k)$ such that $hg' = g'_1$. Here the action of $\mathcal{P}'_v(a)(k)$ is induced by the isomorphism $\mathfrak{J}_a \simeq I_{\gamma_\xi}$ over $X_v^\bullet$.

Therefore, the isomorphism classes of $\mathscr{X}'_\xi(k)$ are in bijection with double cosets

$$g' \in I_{\gamma_\xi}(F_v)\backslash G(F_v)/G(\mathcal{O}_v)$$

such that $\mathrm{Ad}_{g'}^{-1}(\gamma_\xi) \in \mathfrak{M}(\mathcal{O}_v)$, and automorphism group of g' is

$$(I_{\gamma_\xi}(F_v) \cap g'G(\mathcal{O}_v)(g')^{-1})/\mathfrak{J}'_a(\mathcal{O}_v).$$

As a result, for any Haar measure $\mathrm{d}t_v$ on $\mathfrak{J}_a(F_v) \simeq I_{\gamma_\xi}(F_v)$ and any $\overline{\mathbb{Q}}_\ell$-valued function τ on $\mathscr{X}'_\xi(k)$, we have

$$\#_\tau\,\mathscr{X}'_\xi(k) = \sum_{g'} \frac{\tau(g')\,\mathrm{vol}(\mathfrak{J}'_a(\mathcal{O}_v),\mathrm{d}t_v)}{\mathrm{vol}(I_{\gamma_\xi}(F_v) \cap g'G(\mathcal{O}_v)(g')^{-1})},$$

where g' ranges over the double cosets above. This implies that

$$\#_\tau\,\mathscr{X}'_\xi(k) = \mathrm{vol}(\mathfrak{J}'_a(\mathcal{O}_v),\mathrm{d}t_v)\,\mathbf{O}_{\gamma_\xi}(\tau,\mathrm{d}t_v).$$

Summing over all classes ξ, we have our result. ∎

In general, $\mathfrak{J}_a$ may have disconnected special fiber, and recall we have open subgroup $\mathfrak{J}_a^0$ of fiberwise neutral component. We will connect the point-counting of $\mathscr{X}(k) = [\mathcal{M}_v(a)/\mathcal{P}_v(a)](k)$ with that of $\mathscr{X}^0(k) = \left[\mathcal{M}_v(a)/\mathcal{P}_v^0(a)\right](k)$, where $\mathcal{P}_v^0(a)$ is the (reduced) affine Grassmannian of $\mathfrak{J}_a^0$ (not to be confused with the neutral component of $\mathcal{P}_v(a)$, which we denote by $\mathcal{P}_v(a)_0$).

We have homomorphism

$$\mathrm{H}^1(F_v,\mathfrak{J}_a) \simeq \mathrm{H}^1(k,\mathcal{P}_v^0(a)) \longrightarrow \mathrm{H}^1(k,\mathcal{P}_v(a)),$$

so a character κ of $\mathrm{H}^1(k,\mathcal{P}_v(a))$ induces a character of $\mathrm{H}^1(F_v,\mathfrak{J}_a)$, still denoted by κ.

Proposition 10.2.6. *Let τ be a function on $\left[\mathbb{L}^+G^{\mathrm{ad}}\backslash\mathsf{Gr}^{\leq\lambda_v}\right](k)$. If we have a character $\kappa\colon \mathrm{H}^1(F_v,\mathfrak{J}_a)\to\overline{\mathbb{Q}}_\ell^\times$ that is induced by a character of $\mathrm{H}^1(k,\mathcal{P}_v(a))$, then we have equality*

$$\#_\tau\,\mathscr{X}(k)_\kappa=\mathrm{vol}(\mathfrak{J}_a^0(\mathcal{O}_v),\mathrm{d}t_v)\,\mathbf{O}_a^\kappa(\tau,\mathrm{d}t_v),$$

where $\mathrm{d}t_v$ is any Haar measure on $\mathfrak{J}_a(F_v)$. If κ is not induced by a character of $\mathrm{H}^1(k,\mathcal{P}_v(a))$, then

$$\mathbf{O}_a^\kappa(\tau,\mathrm{d}t_v)=0.$$

Proof. We already know by Proposition 10.2.5 that

$$\#_\tau\,\mathscr{X}^0(k)_\kappa=\mathrm{vol}(\mathfrak{J}_a^0(\mathcal{O}_v),\mathrm{d}t_v)\,\mathbf{O}_a^\kappa(\tau,\mathrm{d}t_v),$$

if κ is a character of $\mathrm{H}^1(k,\mathcal{P}_v^0(a))$. So we need to prove that if κ is induced by a character of $\mathrm{H}^1(k,\mathcal{P}_v(a))$, we have

$$\#_\tau\,\mathscr{X}^0(k)_\kappa=\#_\tau\,\mathscr{X}(k)_\kappa,$$

and $\#_\tau\,\mathscr{X}^0(k)_\kappa=0$ otherwise.

Let $\Pi\coloneqq\pi_0(\mathfrak{J}_{a,v})$ be the kernel of $\mathcal{P}_v^0(a)\to\mathcal{P}_v(a)$. It is the group of connected components of the special fiber of $\mathfrak{J}_a$. For each isomorphism class in $\mathscr{X}(k)$, we fix a representative $x=(m,p)$.

We consider a full subcategory $\mathscr{X}^0(k)_x$ of $\mathscr{X}^0(k)$ consisting of objects lying over x for each representative x. If x and x' are representatives of two isomorphism classes in $\mathscr{X}(k)$, then no object in $\mathscr{X}^0(k)_x$ is isomorphic to any object in $\mathscr{X}^0(k)_{x'}$. On the other hand, since $\mathcal{P}_v^0(a)\to\mathcal{P}_v(a)$ is surjective, any object in $\mathscr{X}^0(k)$ is isomorphic to an object of $\mathscr{X}^0(k)_x$ for some x. Thus we may replace $\mathscr{X}^0(k)$ with disjoint union of full subcategories

$$\coprod_{x} \mathscr{X}^0(k)_x.$$

For a given $x = (m, p)$, the category $\mathscr{X}^0(k)_x$ is described as follows:

(1) the objects are (m, p_0) such that p_0 maps to p (in particular, $\mathscr{X}^0(k)_x$ is non-empty);

(2) the morphism from (m, p_0) to (m, p_0') is a element $h \in \mathcal{P}_v^0(\mathfrak{J}_a)$ with $hm = m$ and $\sigma(h)p_0h^{-1} = p_0'$.

In particular, if we fix $x_0 = (m, p_0) \in \mathscr{X}^0(k)_x$, then any (m, p_0') may be written as (m, p_0p') for some $p' \in \Pi$, and the automorphism of (m, p_0) is the group

$$H_0 = \left\{h \in \mathcal{P}_v^0(a) \,\middle|\, hm = m, \sigma(h)h^{-1} \in \Pi\right\}.$$

Therefore $\mathscr{X}^0(k)_x$ is equivalent the categorical quotient $[\Pi/H_0]$, where H_0 acts on Π through map

$$\begin{aligned} \alpha\colon H_0 &\longrightarrow \Pi \\ h &\longmapsto \sigma(h)h^{-1}. \end{aligned}$$

The isomorphism classes are represented by the $\operatorname{coker}(\alpha)$, and automorphism groups are isomorphic to $\ker(\alpha)$. It also implies that H_0 is finite.

Let κ be a character of $\mathrm{H}^1(k, \mathcal{P}_v^0(a))$, the latter is identified with the torsion part of $\mathcal{P}_v^0(a)_\sigma$. We still denote the restriction to Π_σ or Π by κ. The restriction of κ to $\alpha(H_0) \in \Pi$ is trivial by definition, so it induces a character on $\operatorname{coker}(\alpha)$, again still denoted by κ. The restriction of function τ to $\mathscr{X}^0(k)_x$ is constant with value $\tau(x)$. So we have equality of summations

$$\sum_{x'\in\mathscr{X}^0(k)_x/\sim}\frac{\langle \mathrm{cl}(x')\tau(x'),\kappa\rangle}{\#\operatorname{Aut}_{\mathscr{X}^0}(x')}=\langle \mathrm{cl}(x_0),\kappa\rangle\tau(x)\sum_{z\in\operatorname{coker}(\alpha)}\frac{\langle z,\kappa\rangle}{\#\ker(\alpha)}.$$

If κ is not induced by a character of $\mathrm{H}^1(k,\mathcal{P}_v(a))$, that is, non-trivial on Π, then the right-hand side is 0. Summing over x, we have in this case

$$\#_\tau\,\mathscr{X}^0(k)_\kappa=0.$$

If κ is trivial on Π, then the same summation above is equal to

$$\langle \mathrm{cl}(x_0),\kappa\rangle\tau(x)\frac{\#\operatorname{coker}(\alpha)}{\#\ker(\alpha)}=\langle \mathrm{cl}(x_0),\kappa\rangle\tau(x)\frac{\#\Pi}{\# H_0}.$$

If $h_0\in H_0$ and let h be its image in $\mathcal{P}_v(a)$, then $h\in\mathcal{P}_v(a)(k)$, and we have short exact sequence

$$1\longrightarrow\Pi\longrightarrow H_0\longrightarrow\operatorname{Aut}_{\mathscr{X}(k)}(x)\longrightarrow 1.$$

As a result, we have $\# H_0=\#\Pi\#\operatorname{Aut}_{\mathscr{X}(k)}(x)$, and summing over x of the sums above we have

$$\#_\tau\,\mathscr{X}(k)_\kappa=\#_\tau\,\mathscr{X}^0(k)_\kappa.$$

This finishes the proof. ∎

Finally, using the connection between point-counting on $\mathscr{X}(k)$ and Frobenius trace on cohomologies, we can give a cohomological interpretation of orbital integrals. According to Proposition 4.4.7 and § 10.1.9, we may find a σ-stable torsion-free subgroup $\Lambda\subset\mathcal{P}_v^0(a)$ such that Λ acts freely on $\mathcal{M}_v(a)$, and both $\mathcal{P}_v^0(a)/\Lambda$ and $\mathcal{M}_v(a)$ are of finite types over k.

Moreover, $\mathcal{M}_v(a)/\Lambda$ is proper, so its cohomologies with compact support is canonically isomorphic to ordinary cohomologies. Since the kernel of $\mathcal{P}_v^0(a) \to \mathcal{P}_v(a)$ is of finite type, Λ maps isomorphically to its image in $\mathcal{P}_v(a)$, so we may also treat Λ as a subgroup of $\mathcal{P}_v(a)$.

Corollary 10.2.7. *Let $\mathcal{F}$ be a bounded constructible $\mathbb{L}^+G^{\mathrm{ad}}$-equivariant complex on $\mathsf{Gr}^{\le\lambda_v}$ and κ be a character of $\mathrm{H}^1(F_v, \mathfrak{J}_a)$. Let $\mathfrak{J}_a^{\flat,0}$ be the open subgroup scheme of the Néron model of $\mathfrak{J}_a$ of fiberwise neutral components. Then for any Λ as above, we have*

$$\sum_{i\in\mathbb{Z}}(-1)^i\operatorname{Tr}(\sigma, \mathrm{H}^i([\mathcal{M}_v(a)/\Lambda], \mathcal{F})) = \mathrm{vol}(\mathfrak{J}_a^{\flat,0}, \mathrm{d}t_v)\,\mathbf{O}_a^\kappa(\mathrm{Tr}_{\mathcal{F}}, \mathrm{d}t_v).$$

Proof. Apply Proposition 10.1.11 and Proposition 10.2.5, to $\mathfrak{J}_a^0$, we have

$$\sum_{i\in\mathbb{Z}}(-1)^i\operatorname{Tr}(\sigma, \mathrm{H}^i([\mathcal{M}_v(a)/\Lambda], \mathcal{F})) = \#\mathcal{P}_v^0(a)_0(k)\,\mathrm{vol}(\mathfrak{J}_a^0, \mathrm{d}t_v)\,\mathbf{O}_a^\kappa(\mathrm{Tr}_{\mathcal{F}}, \mathrm{d}t_v),$$

where $\mathcal{P}_v^0(a)_0$ is the neutral component of $\mathcal{P}_v^0(a)$. The injective map $\mathfrak{J}_a^0 \to \mathfrak{J}_a^{\flat,0}$ induces short exact sequence

$$1 \longrightarrow \mathfrak{J}_a^{\flat,0}(\breve{\mathcal{O}}_v)/\mathfrak{J}_a^0(\breve{\mathcal{O}}_v) \longrightarrow \mathcal{P}_v^0(a) \longrightarrow \mathcal{P}_v^{\flat,0}(a) \longrightarrow 1.$$

The group $\mathcal{P}_v^{\flat,0}(a)$ is discrete, so the $\bar{k}$-point of the neutral component $\mathcal{P}_v^0(a)_0$ can be identified with $\mathfrak{J}_a^{\flat,0}(\breve{\mathcal{O}}_v)/\mathfrak{J}_a^0(\breve{\mathcal{O}}_v)$, compatible with σ-action. Since both $\mathfrak{J}_a^{\flat,0}$ and $\mathfrak{J}_a^0$ have connected fibers, we have

$$\mathcal{P}_v^0(a)_0(k) \simeq \mathfrak{J}_a^{\flat,0}(\mathcal{O}_v)/\mathfrak{J}_a^0(\mathcal{O}_v).$$

This implies that

$$\#\mathcal{P}_v^0(a)_0(k)\,\mathrm{vol}(\mathfrak{J}_a^0(\mathcal{O}_v), \mathrm{d}t_v) = \mathrm{vol}(\mathfrak{J}_a^{\flat,0}(\mathcal{O}_v), \mathrm{d}t_v)$$

for any Haar measure on $\mathfrak{J}_a(F_v)$. Combining it with the equality above, we have our result. ∎

10.3 Counting Points on mH-fibrations

In this section, we fix $a \in \mathcal{A}_X^\natural(k)$ and consider point-counting problems on $\mathcal{M}_a$. Let $\mathfrak{J}_a' \to \mathfrak{J}_a$ be a morphism of smooth commutative groups schemes on X that is an isomorphism over $U = X - \mathfrak{D}_a$. Suppose $\mathfrak{J}_a'$ has connected fibers. For example, we may let $\mathfrak{J}_a' = \mathfrak{J}_a^0$. Let $\mathcal{P}_a'$ be the Picard stack classifying $\mathfrak{J}_a'$-torsors on X, then the $\mathcal{P}_a$-action on $\mathcal{M}_a$ induces an action of $\mathcal{P}_a'$. Unlike [Ngô10], we do not need to choose such $\mathfrak{J}_a'$ that $\mathcal{P}_a'$ is a scheme, because in Proposition 10.1.8 we allow $\mathcal{P}$ to be a Deligne-Mumford stack.

10.3.1 By Proposition 6.9.6, we have $\mathrm{Gal}(\bar{k}/k)$-equivariant equivalence of groupoids

$$[\mathcal{M}_a/\mathcal{P}_a] = \prod_{v\in|X-U|} [\mathcal{M}_v(a)/\mathcal{P}_v(a)],$$

and similarly if we replace $\mathcal{P}_a$ (resp. $\mathcal{P}_v(a)$) by $\mathcal{P}_a'$ (resp. $\mathcal{P}_v'(a)$). In particular, we have equivalence of groupoids of k-points

$$[\mathcal{M}_a/\mathcal{P}_a'](k) = \prod_{v\in|X-U|} [\mathcal{M}_v(a)/\mathcal{P}_v'(a)](k).$$

A constructible complex or a function on the left-hand side is called *factorizable* if it is an exterior tensor of the same on the right-hand side. In this case, we write $\mathcal{F} = \boxtimes_{v\in|X-U|}\mathcal{F}_v$ or $\tau = \boxtimes_{v\in|X-U|}\tau_v$ respectively.

Since a is anisotropic, the group $\pi_0(\mathcal{P}_a)$ is finite with σ-action. For any $v \in |X-U|$, the map $\mathcal{P}_v'(a) \to \mathcal{P}_a'$ induces map of connected components

$$\pi_0(\mathcal{P}_v'(a)) \longrightarrow \pi_0(\mathcal{P}_a').$$

Any σ-invariant character

$$\kappa\colon \pi_0(\mathcal{P}_a)_\sigma \longrightarrow \overline{\mathbb{Q}}_\ell^\times$$

induces characters on $\pi_0(\mathcal{P}'_a)$ and $\pi_0(\mathcal{P}'_v(a))$. On the other hand, if a Steinberg quasi-section exists for $\chi_{\mathfrak{M}}$ and $x \in [\mathcal{M}_a/\mathcal{P}_a](k)$ corresponds to a tuple of points $x_v \in [\mathcal{M}_v(a)/\mathcal{P}'_v(a)](k)$, then we have

$$\langle \mathrm{cl}(x), \kappa \rangle = \prod_{v \in |X-U|} \langle \mathrm{cl}(x_v), \kappa \rangle.$$

Here we implicitly assume that each $\mathcal{M}_v(a)$ is defined using the section induced by the same Steinberg quasi-section. Note that if $\kappa = 1$, the above equality still holds even if Steinberg quasi-section does not exist. Thus we have the following result:

Proposition 10.3.2. *Suppose a Steinberg quasi-section exists for $\chi_{\mathfrak{M}}$, then for any σ-invariant character κ of $\pi_0(\mathcal{P}_a)$ and any $\overline{\mathbb{Q}}_\ell$-valued factorizable function τ on $[\mathcal{M}_a/\mathcal{P}'_a](k)$, we have*

$$\#_\tau[\mathcal{M}_a/\mathcal{P}'_a](k)_\kappa = \prod_{v \in |X-U|} \#_{\tau_v}[\mathcal{M}_v(a)/\mathcal{P}'_v(a)](k)_\kappa.$$

If $\kappa = 1$, the same equality holds even if Steinberg quasi-section does not exist.

Corollary 10.3.3. *For any σ-invariant character κ of $\pi_0(\mathcal{P}_a)$, and $\mathcal{F}$ a bounded, constructible and factorizable $\mathcal{P}_a$-equivariant complex on $\mathcal{M}_a$, suppose either a Steinberg quasi-section exists or $\kappa = 1$, we have*

$$\sum_{n \in \mathbb{Z}} (-1)^n \operatorname{Tr}(\sigma, \mathrm{H}^n(\mathcal{M}_{a,\bar{k}}, \mathcal{F})_\kappa) = \#(\mathcal{P}'_a)_0(k) \prod_{v \in |X-U|} \#_{\mathcal{F}_v}[\mathcal{M}_v(a)/\mathcal{P}'_v(a)](k)_\kappa.$$

Proof. This follows from Propositions 10.1.8 and 10.3.2 and that $\mathcal{M}_a$ is proper by Proposition 8.1.2. ∎

10.3.4 Finally, we express $\#_\tau[\mathcal{M}_v(a)/\mathcal{P}_v(a)](k)_\kappa$ as orbital integrals. For that purpose, we choose at each point $v \in |X - U|$:

(1) a trivialization of the $Z_{\mathfrak{M}}$-torsor $\mathcal{L}$ on the formal disc X_v;

(2) a Haar measure $\mathrm{d}t_v$ on $\mathfrak{J}_a(F_v)$.

In this way we have equality by Proposition 10.2.5:

$$\#_\tau[\mathcal{M}_v(a)/\mathcal{P}'_v(a)](k)_\kappa = \mathrm{vol}(\mathfrak{J}'_a(\mathcal{O}_v), \mathrm{d}t_v)\, \mathbf{O}^\kappa_{a,v}(\tau_v, \mathrm{d}t_v).$$

Therefore if either a Steinberg quasi-section exists or $\kappa = 1$, we have

$$\sum_{n\in\mathbb{Z}} (-1)^n \operatorname{Tr}(\sigma, \mathrm{H}^n(\mathcal{M}_{a,\bar{k}}, \mathcal{F})_\kappa) = \#(\mathcal{P}'_a)_0(k) \prod_{v\in|X-U|} \mathrm{vol}(\mathfrak{J}'_a(\mathcal{O}_v), \mathrm{d}t_v)\, \mathbf{O}^\kappa_{a,v}(\mathrm{Tr}_{\mathcal{F}_v}, \mathrm{d}t_v).$$

10.4 Stabilization over $\tilde{\mathcal{A}}_H^{\kappa,\dagger}$

Now we go back to the situation in § 9.9 and prove Theorem 8.3.4 over the locus $\tilde{\mathcal{A}}_H^{\kappa,\dagger}$. More precisely, let $\mathcal{U} \subset \tilde{\mathcal{A}}^\ddagger$ be the largest open subset over which $\tilde{\nu}_{\mathcal{A}}^{-1}(\mathcal{U}) \subset \tilde{\mathcal{A}}_H^{\kappa,\dagger}$, then we are going to prove Theorem 8.3.5 over $\mathcal{U}$.

10.4.1 Indeed, by Theorem 9.9.3, the set of supports of ${}^{\mathfrak{p}}\mathrm{H}^\bullet(\tilde{h}_{H,*}^{\kappa,\dagger}\mathcal{Q}_H^\kappa)_{\mathrm{st}}$ only contains inductive subsets in $\tilde{\mathcal{A}}_H^{\kappa,\dagger}$. Pick a point $a \in \tilde{\nu}_{\mathcal{A}}(\tilde{\mathcal{A}}_H^{\kappa,\dagger}) \cap \mathcal{U}(\bar{k})$ and without loss of generality we may assume it is defined over k and so is the endoscopic datum (otherwise we can base change to some k' and replace k by k'). Make a general enough so that any $a_H \in \tilde{\nu}_{\mathcal{A}}^{-1}(a)$ has $\delta_{H,a_H} = 0$. It is possible by our assumptions on $\mathcal{U}$.

Recall on $\mathfrak{C}_{\mathfrak{M},H}$ we have equality of principal divisors

$$\nu_H^*\mathfrak{D}_{\mathfrak{M}} = \mathfrak{D}_{\mathfrak{M},H} + 2\mathfrak{R}_H^G,$$

where both $\mathfrak{D}_{\mathfrak{W},H}$ and $\mathfrak{R}_H^G$ are reduced divisors. Similar to Proposition 6.3.13, we may find a so that every a_H intersects with $\mathfrak{D}_{\mathfrak{W},H} + \mathfrak{R}_H^G$ transversally and $a_H^*(\mathfrak{D}_{\mathfrak{W},H} + \mathfrak{R}_H^G)$ does not collide with the boundary divisor. Let $\mathcal{U}'$ be the open dense subset of $\mathcal{U}$ consisting of such a. Without loss of generality, we may also assume $\mathcal{U}'$ to be irreducible by looking at each irreducible component.

For any $a \in \tilde{\nu}_{\mathcal{A}}(\tilde{\mathcal{A}}_H^{\kappa,\dagger}) \cap \mathcal{U}'$ and $\bar{v} \in X(\bar{k})$ such that the boundary divisor is not 0 at $\bar{v}$, since $\delta_{\bar{v}}(a) = 0$, the local Newton point equals the boundary divisor at $\bar{v}$. This implies that there is a unique point a_H mapping to a (see the proof of Proposition 6.11.2). In other words, the map

$$\tilde{\nu}_{\mathcal{A}}^{-1}(\mathcal{U}') \longrightarrow \mathcal{U}'$$

is a closed embedding. Moreover, if we let $\mathcal{M}' = \tilde{h}_X^{-1}(\mathcal{U}')$ (and with $\mathcal{U}'$ being irreducible), the intersection complex $\mathcal{Q}|_{\mathcal{M}'}$ is isomorphic to $\overline{\mathbb{Q}}_\ell[\dim \mathcal{M}'](\dim \mathcal{M}'/2)$, and similarly for the H-side.

The natural homomorphism $\mathfrak{J}_a \to \mathfrak{J}_{H,a_H}$ induces homomorphism $\mathcal{P}_a \to \mathcal{P}_{H,a_H}$ and similarly the local analogues $\mathcal{P}_v(a) \to \mathcal{P}_{H,v}(a_H)$. So we have an action of $\mathcal{P}_v(a)$ on $\mathcal{M}_{H,v}(a_H)$ too.

Lemma 10.4.2. *With the assumptions above, for any closed point $v \in |X|$, we have equality*

$$\#_{\overline{\mathbb{Q}}_\ell}[\mathcal{M}_v(a)/\mathcal{P}_v(a)](k)_\kappa = q^{\deg(v) r_{H,v}^G(a_H)}\, \#_{\overline{\mathbb{Q}}_\ell}[\mathcal{M}_{H,v}(a_H)/\mathcal{P}_v(a)](k)_{\mathrm{st}},$$

where $\deg(v)$ *is the degree of v over k and $r_{H,v}^G(a_H)$ is the degree of $a_H^*\mathfrak{R}_H^G$ at v. Moreover, it is a non-zero rational number.*

Proof. The proof is the same as [Ngô10, Lemme 8.5.7] so we only give a sketch and some other necessary input from the group case. If we let $\bar{v}$ is a geometric point lying over v, then we have three possibilities:

(1) $d_{H,\bar{v}+}(a_H) = d_{\bar{v}+}(a) = r^G_{H,\bar{v}}(a_H) = 0$, or

(2) $d_{H,\bar{v}+}(a_H) = d_{\bar{v}+}(a) = 1$ and $r^G_{H,\bar{v}}(a_H) = 0$, or

(3) $d_{H,\bar{v}+}(a_H) = 0$, $d_{\bar{v}+}(a) = 2$, and $r^G_{H,\bar{v}}(a_H) = 1$.

In the last two cases the boundary divisor vanishes at $\bar{v}$. The first two cases works verbatim as in the Lie algebra case. In the third case, one also reduces to groups of semisimple rank 1, and can easily show that geometrically $\mathcal{M}_{\bar{v}}(a)$ is a union of infinite chain of $\mathbb{P}^1$ as in the case of [Ngô10, § 8.3]. One can also deduce the computation from the Lie algebra case and the fact that $\mathrm{Env}(\mathrm{SL}_2) \cong \mathrm{Mat}_2 \cong \mathfrak{gl}_2$ compatible with GL_2-conjugation. After that, since we also assumed existence of a Steinberg quasi-section, the remaining computations for Lie algebras also carry through. This finishes the proof. ∎

10.4.3 Let $\deg(a_H^*\mathfrak{R}_H^G)$ be the sum of $\deg(v)r^G_{H,v}(a_H)$. By Lemma 6.3.4 and Corollary 6.8.4 and (6.11.5), we have equalities

$$\deg(a_H^*\mathfrak{R}_H^G) = r_H^G(a_H) = \dim \mathcal{P}_a - \dim \mathcal{P}_{H,a_H} = \dim \mathcal{M}_a - \dim \mathcal{M}_{H,a_H},$$

and this number does not depend on a or a_H. Using Corollary 10.3.3, we have

$$\sum_{n\in\mathbb{Z}} (-1)^n \operatorname{Tr}(\sigma, \mathrm{H}^n(\mathcal{M}_{a,\bar{k}}, \mathcal{Q})_\kappa) = q^{-\dim \mathcal{M}'/2} \#(\mathcal{P}_a)_0(k) \prod_{v\in|X-U|} \#_{\overline{\mathbb{Q}}_\ell}[\mathcal{M}_v(a)/\mathcal{P}_v(a)](k)_\kappa,$$

and similarly (with $\mathcal{P}_a$-action instead of $\mathcal{P}_{H,a_H}$-action on $\mathcal{M}_{H,a_H}$)

$$\sum_{n\in\mathbb{Z}} (-1)^n \operatorname{Tr}(\sigma, \mathrm{H}^n(\mathcal{M}_{H,a_H,\bar{k}}, \mathcal{Q}_H^\kappa)_{\mathrm{st}}) = q^{-\dim \mathcal{M}_H^{\kappa,\prime}/2} \#(\mathcal{P}_a)_0(k) \prod_{v\in|X-U|} \#_{\overline{\mathbb{Q}}_\ell}[\mathcal{M}_{H,v}(a_H)/\mathcal{P}_v(a)](k).$$

Combining with Lemma 10.4.2, we see that

$$\sum_{n\in\mathbb{Z}}(-1)^n \operatorname{Tr}(\sigma, \mathrm{H}^n(\mathcal{M}_{a,\bar{k}}, \mathcal{Q})_\kappa)$$
$$= q^{(\dim \mathcal{M}_H^{\kappa,\prime} - \dim \mathcal{M}')/2 + r_H^G(a_H)} \sum_{n\in\mathbb{Z}}(-1)^n \operatorname{Tr}(\sigma, \mathrm{H}^n(\mathcal{M}_{H,a_H,\bar{k}}, \mathcal{Q}_H^\kappa)_{\mathrm{st}}).$$

Since $(\dim \mathcal{M}_H^{\kappa,\prime} - \dim \mathcal{M}')/2 + r_H^G(a_H)$ depends only on $\mathcal{U}'$ not on a_H, we see that the local systems ${}^{\mathfrak{p}}\mathrm{H}^\bullet(\tilde{h}_*^\ddagger \mathcal{Q})_\kappa|_{\mathcal{U}'}$ and $\tilde{\nu}_{\mathcal{A},*}({}^{\mathfrak{p}}\mathrm{H}^\bullet(\tilde{h}_{H,*}^\dagger \mathcal{Q}_H^\kappa)_{\mathrm{st}})|_{\mathcal{U}'}$ are isomorphic up to a Tate twist and semisimplification with respect to σ. However, we also know that both are local systems of pure weight 0, therefore the Tate twist must be trivial, in other words, we must have

$$(\dim \mathcal{M}_H^{\kappa,\prime} - \dim \mathcal{M}')/2 + r_H^G(a_H) = 0,$$

and so

$$\sum_{n\in\mathbb{Z}}(-1)^n \operatorname{Tr}(\sigma, \mathrm{H}^n(\mathcal{M}_{a,\bar{k}}, \mathcal{Q})_\kappa) = \sum_{n\in\mathbb{Z}}(-1)^n \operatorname{Tr}(\sigma, \mathrm{H}^n(\mathcal{M}_{H,a_H,\bar{k}}, \mathcal{Q}_H^\kappa)_{\mathrm{st}}).$$

We also deduce that

$$\dim \mathcal{U} - \dim \tilde{\nu}_{\mathcal{A}}(\tilde{\mathcal{A}}_H^{\kappa,\dagger}) \cap \mathcal{U} = r_H^G(a_H).$$

It is the same process preceding Theorem 9.9.6, and it is simpler here because the computation is more refined.

10.4.4 As the result of previous discussion, we may apply Theorem 9.9.6 to $\mathcal{U}$ and it implies that the set of supports of ${}^{\mathfrak{p}}\mathrm{H}^\bullet(\tilde{h}_*^\ddagger \mathcal{Q})_\kappa$ only contains δ-critical subsets whose preimages in $\tilde{\mathcal{A}}_H^{\kappa,\dagger}$ are inductive. For convenience, if $Z \subset \tilde{\nu}_{\mathcal{A}}(\tilde{\mathcal{A}}_H^{\kappa,\dagger}) \cap \mathcal{U}$ is an irreducible closed δ-critical subset whose preimage in $\tilde{\mathcal{A}}_H^{\kappa,\dagger}$ is inductive, then we call such subset *relevant.*

Suppose $K_Z \subset {}^{\mathfrak{p}}\mathrm{H}^\bullet(\tilde{h}^\ddagger_* \mathcal{Q})_\kappa$ is the largest direct summand with support Z, then by the proof of Theorem 9.9.6, we may find an open subset $Z' \subset Z$ such that $K_Z|_{Z'}$ is a local system. Shrinking Z' if necessary, we may also assume that the abelian part of the Tate module $\Lambda_{A,Z}$ is also a local system and K_Z is a free graded $\Lambda_{A,Z}$-module generated by the top degree.

Similarly, if $K_{H,Z} \subset \tilde{\nu}_{\mathcal{A},*}({}^{\mathfrak{p}}\mathrm{H}^\bullet(\tilde{h}^\dagger_{H,*}\mathcal{Q}^\kappa_H)_{\mathrm{st}})$ is the largest direct summand with support Z, then further shrinking Z' if necessary, we may also assume $K_{H,Z}|_{Z'}$ is a free graded Λ_{H,A_H}-module generated be the top cohomology. Moreover, the natural maps $\mathcal{P}_a \to \mathcal{P}_{H,a_H}$ induces natural isomorphism $\tilde{\nu}^*_{\mathcal{A}}\Lambda_{A,Z} \simeq \Lambda_{H,A_H,\tilde{\nu}^{-1}_{\mathcal{A}}(Z)}$ over $\tilde{\nu}^{-1}_{\mathcal{A}}(Z')$.

By (8.4.1), the top cohomologies of ${}^{\mathfrak{p}}\mathrm{H}^\bullet(\tilde{h}^\ddagger_* \mathcal{Q})_\kappa$ and $\tilde{\nu}_{\mathcal{A},*}({}^{\mathfrak{p}}\mathrm{H}^\bullet(\tilde{h}^\dagger_{H,*}\mathcal{Q}^\kappa_H)_{\mathrm{st}})$, at stalk level, are isomorphic as σ-modules, which means that they are isomorphic as local systems after taking semisimplification with respect to σ. We also know that both K_Z and $K_{H,Z}$ are geometrically semisimple for any Z. Therefore we can deduce Theorem 8.3.5 hence also Theorem 8.3.4 over $\mathcal{U}$ using inductive argument below.

Indeed, there is a partial order on all relevant subsets Z by inclusion. The unique maximal element is just $Z_0 = \tilde{\nu}_{\mathcal{A}}(\tilde{\mathcal{A}}^{\kappa,\dagger}_H) \cap \mathcal{U}$. We have already established that the σ-semisimplifications $K^{\mathrm{ss}}_{Z_0}$ and K^{ss}_{H,Z_0} are isomorphic over some open subset $Z'_0 \subset Z_0$. Although the functor of intermediate extension is not exact in general, it is still exact on geometrically semisimple local systems. Therefore $K^{\mathrm{ss}}_{Z_0}$ and K^{ss}_{H,Z_0} are isomorphic. Over a sufficiently small open subset Z' in an arbitrary Z, ${}^{\mathfrak{p}}\mathrm{H}^\bullet(\tilde{h}^\ddagger_* \mathcal{Q})_\kappa$ decomposes into $K_Z|_{Z'}$ and $K_W|_{Z'}$ for all relevant W containing Z, and similarly for $\tilde{\nu}_{\mathcal{A},*}({}^{\mathfrak{p}}\mathrm{H}^\bullet(\tilde{h}^\dagger_{H,*}\mathcal{Q}^\kappa_H)_{\mathrm{st}})$. By inductive hypothesis, all K^{ss}_W and $K^{\mathrm{ss}}_{H,W}$ are isomorphic, and in particular it is true for their top cohomologies. This implies that the top cohomologies of $K_Z|^{\mathrm{ss}}_{Z'}$ and $K_{H,Z}|^{\mathrm{ss}}_{Z'}$ are also isomorphic. Since $K_Z|_{Z'}$ is generated by its top cohomology over $\Lambda_{A,Z}$, and similarly for $K_{H,Z}|_{Z'}$, we see that $K_Z|^{\mathrm{ss}}_{Z'}$ and $K_{H,Z}|^{\mathrm{ss}}_{Z'}$ are isomorphic, hence so are K^{ss}_Z and $K^{\mathrm{ss}}_{H,Z}$. This finishes the proof of Theorem 8.3.5 over $\mathcal{U}$.

10.5 Fundamental Lemma for Adjoint Groups

In this section we are going to prove the fundamental lemma for adjoint groups $G = G^{\mathrm{ad}}$ and its elliptic endoscopic groups H.

10.5.1 Let $X_v = \operatorname{Spec} \mathcal{O}_v$ where $\mathcal{O}_v = k[[\pi_v]]$ and $X_v^\bullet = \operatorname{Spec} F_v$ where $F_v = k((\pi_v))$. Let G_v be a reductive group scheme over X_v obtained by a $\mathrm{Out}(\mathbf{G})$-torsor $\vartheta_{G,v}$. Let $(\kappa, \vartheta_{\kappa,v})$ be an endoscopic datum with endoscopic group H_v. Assume G_v is of adjoint type, then it has no center, and since H_v is elliptic, in this case it is necessarily true that H_v has the same semisimple rank as G_v, in other words, $r = r_{H_v}$.

Since G_v is of adjoint type, then all its simple factors of types A_{2m} form a direct factor G_v' of G_v. In this case, since H_v has the same semisimple rank as G_v, by looking at the Dynkin diagrams, we conclude that H_v also contains G_v' as direct factors. Therefore, upon replacing G_v and H_v with G_v/G_v' and H_v/G_v' respectively, we may assume it has no simple factor of type A_{2m}. As a result, we may assume that a Steinberg quasi-section exists for G_v.

10.5.2 We need an approximation result for multiplicative affine Springer fibers, which is valid for all G_v (not just adjoint groups).

Proposition 10.5.3. *Suppose a Steinberg quasi-section exists for G_v (not necessarily of adjoint type) over k. Then for a fixed $a \in \mathfrak{C}_{\mathfrak{M}}(\mathcal{O}_v) \cap \mathfrak{C}_{\mathfrak{M}}^{\times,\mathrm{rs}}(F_v)$, there exists some integer N such that for any $a' \in \mathfrak{C}_{\mathfrak{M}}(\mathcal{O}_v)$ with $a \equiv a' \bmod \pi_v^N$, we have isomorphisms $\mathcal{M}_v(a) \cong \mathcal{M}_v(a')$ and $\mathcal{P}_v(a) \cong \mathcal{P}_v(a')$ compatible with the action of $\mathcal{P}_v(a)$ (resp. $\mathcal{P}_v(a')$) on $\mathcal{M}_v(a)$ (resp. $\mathcal{M}_v(a')$). For arbitrary group G_v potentially without a Steinberg quasi-section over k, the same holds after base change to $\bar{k}$.*

Proof. Clearly we only need to prove for $\mathfrak{M} = \mathrm{Env}(G^{\mathrm{sc}})$. The case after base changing to $\bar{k}$ is proved in [Chi19, Theorem 5.1.1], whose proof is an adaptation of the argument for

[Ngô10, Proposition 3.5.1]. The key is that if $\gamma \in \mathfrak{M}(\mathcal{O}_v)$ is the point given by the section induced by a Steinberg quasi-section, then one may find some point $g \in G^{\mathrm{sc}}(\mathcal{O}_v)$ (the proof in *loc. cit.* uses the group $\mathfrak{M}^\times$ but G^{sc} also works) such that $g \equiv 1 \bmod \pi_v^N$ and that if we replace γ by $g^{-1}\gamma g$, then the resulting multiplicative affine Springer fiber (of invariant a) will be isomorphic to that of a', together with local Picard action. This part of the proof works rationally over k as long as a Steinberg quasi-section is defined over k. After taking the image of g in $G(\mathcal{O}_v)$ we obtain the desired result. ∎

Although we cannot guarantee the existence of Steinberg quasi-section for H_v, it does not matter for our purposes because we only care about stable orbital integrals of H_v. Indeed, since we still have the same approximation result over $\bar{k}$, and the stable orbital integral depends only on the geometric conjugacy class over $F_{\bar{v}}$, we will obtain the same stable orbital integral from both a_H and a'_H.

10.5.4 Pick a smooth projective and geometrically connected curve X over k together with two distinct k-points v and ∞ and a $\pi_0(\kappa)$-torsor ϑ_κ with a trivialization at ∞ (in other words, a $\pi_0(\kappa)$-torsor $\vartheta_\kappa^\bullet$ pointed over ∞), such that we have an isomorphism between the completion of X at v and X_v above and an isomorphism $\vartheta_\kappa|_{X_v} \cong \vartheta_{\kappa,v}$. Let $G = G^{\mathrm{ad}}$ and H be the corresponding twists of $\mathbf{G} = \mathbf{G}^{\mathrm{ad}}$ and $\mathbf{H}$ respectively, then $G|_{X_v}$ is isomorphic to G_v and similarly for H_v.

Pick any monoid $\mathfrak{M} \in \mathcal{FM}_0(G^{\mathrm{sc}})$ such that $\mathfrak{A}_{\mathfrak{M}}$ is of standard type and let $\mathfrak{M}_H$ be the corresponding endoscopic monoid. Let $a_v \in \mathfrak{C}_{\mathfrak{M}}(\mathcal{O}_v) \cap \mathfrak{C}_{\mathfrak{M}}^{\times,\mathrm{rs}}(F_v)$, viewed as a point in $[\mathfrak{C}_{\mathfrak{M}}/Z_{\mathfrak{M}}](\mathcal{O}_v)$ and suppose $a_{H,v,1}, \dots, a_{H,v,e}$ be all the liftings to $\left[\mathfrak{C}_{\mathfrak{M},H}/Z_{\mathfrak{M}}^\kappa\right](\mathcal{O}_v)$. Then we can always find $Z_{\mathfrak{M}}^\kappa$-torsors $\mathcal{L}_1^\kappa, \dots, \mathcal{L}_e^\kappa$ and sections $a_{H,i} \in \mathfrak{C}_{\mathfrak{M},H,\mathcal{L}_i^\kappa}$ such that over $\mathcal{O}_v$ they induce $a_{H,v,i}$, and they all induce the same $Z_{\mathfrak{M}}$-torsor $\mathcal{L}$. By adding more cocharacters at points other than v, we can ensure each $\mathcal{L}_i^\kappa$ is very (H,N)-ample for arbitrarily large N. If the added cocharacters are the same for every $\mathcal{L}_i^\kappa$, then we also make sure that they still induce the same $Z_{\mathfrak{M}}$-torsor $\mathcal{L}$. Moreover, $\mathcal{L}$ can also be made very (G,N)-ample for arbitrarily large N. Let $c = |\mathbf{Z}^{\mathrm{sc}}|$, then we can also ensure that $\mathcal{L}$ is a c-th power.

To summarize, we may make the following choices:

(1) A very (G,N)-ample irreducible component of $\mathcal{B}_X$ such that any (and every) $Z_{\mathfrak{M}}$-torsor in its image in $\mathrm{Bun}_{Z_{\mathfrak{M}}}$ is a c-th power. Here N is as in Proposition 10.5.3. Its preimage in $\mathcal{A}_X$ is simply denoted by $\mathcal{A}$. In particular, a Steinberg quasi-section exists over $\mathcal{A}$.

(2) There exists some $a \in \mathcal{A}(k)$ contained in $\nu_{\mathcal{A}}(\mathcal{A}_{H,X}^{\kappa})$, such that its restriction to X_v gives a_v, and its preimage $\nu_{\mathcal{A}}^{-1}(a)$ consists entirely of very (H,N)-ample points.

(3) For any $a_{H,i} \in \nu_{\mathcal{A}}^{-1}(a)$, $a_H(X)$ intersects with divisor $\mathfrak{D}_{\mathfrak{M},H} + \mathfrak{R}_H^G$ transversally outside of v, and the boundary divisor and $\mathfrak{D}_{\mathfrak{M},H} + \mathfrak{R}_H^G$ does not collide outside v. In addition, $a_{H,i}$ is very $(H,\delta_{a_{H,i}})$-ample and very $(H,N(\delta_{a_{H,i}}))$-ample , where $N(\delta_{a_{H,i}})$ is as required in Proposition 7.4.5.

Replace $\mathcal{A}$ by an open dense subset such that its preimage $\mathcal{A}_H \coloneqq \nu_{\mathcal{A}}^{-1}(\mathcal{A})$ is entirely H-ample enough in the sense as listed above. By Theorem 6.10.2, local model of singularity exists for H over $\mathcal{A}_H$. This is due to the fact that

$$\mathrm{H}^0\left(\check{X}, (\mathrm{Lie}(\mathfrak{J}_{H,a_H}^{\flat})/\mathfrak{z}_H)^*\right) = 0$$

for all $a_H \in \mathcal{A}_H$. Because H is semisimple we have $\mathfrak{z}_H = 0$, and since we have isomorphism of Néron models $\mathfrak{J}_a^{\flat} \simeq \mathfrak{J}_{H,a_H}^{\flat}$ for any a_H mapping to a, local model of singularity holds for G over $\mathcal{A}$ as well. By Corollary 6.11.4, we may assume that the anisotropic locus is not empty and its complement has codimension larger than $\delta_{a_{H,i}}$ for every i above. Replace $\mathcal{A}$ and $\mathcal{A}_H$ by their respective anisotropic loci.

10.5.5 Let $h\colon \mathcal{M} \to \mathcal{A}$ and $h_H\colon \mathcal{M}_H \to \mathcal{A}_H$ be the associated mH-fibrations of G and H respectively, and we have étale open subset $\tilde{\mathcal{A}} \to \mathcal{A}$ and $\tilde{\mathcal{A}}_H \to \mathcal{A}_H$. Now over $\tilde{\mathcal{A}}$ and

$\tilde{\mathcal{A}}_H = \tilde{\nu}_{\mathcal{A}}^{-1}(\tilde{\mathcal{A}})$ we have already proved stabilization theorem in § 10.4. Let $Z \subset \nu_{\mathcal{A}}(\mathcal{A}_H)$ be the locally closed subset such that for all $a' \in Z(\bar{k})$ we have a' has the same image as a in $[\mathfrak{C}_{\mathfrak{M}}/Z_{\mathfrak{M}}](\mathcal{O}_v/\pi_v^N)$, and in addition all $a'_H \in \nu_{\mathcal{A}}^{-1}(a')$, $a'_H(\check{X})$ intersects with $\mathfrak{D}_{\mathfrak{M},H} + \mathfrak{R}_H^G$ transversally outside of v. Then we automatically have a bijection $\nu_{\mathcal{A}}^{-1}(a') \cong \nu_{\mathcal{A}}^{-1}(a)$ such that $a'_{H,i}$ has the same image in $[\mathfrak{C}_{\mathfrak{M},H}/Z_{\mathfrak{M}}^{\kappa}](\mathcal{O}_v/\pi_v^N)$ as $a_{H,i}$.

Let $\tilde{Z}$ be the preimage of Z in $\tilde{\mathcal{A}}$. The endoscopic datum is pointed at ∞, meaning at ∞ the group G is split. Clearly there exists an integer m such that for all field extension k'/k of degrees at least m, the set $\bar{\mathbf{T}}_{\mathbf{M}}^{\mathrm{rs}}(k')$ is non-empty. As a result, by slightly increasing ampleness if necessary, we have $\tilde{Z}(k') \neq \varnothing$. Let $\tilde{a}' \in \tilde{Z}(k')$ and $\tilde{\nu}_{\mathcal{A}}^{-1}(a') = \{\tilde{a}'_{H,1}, \dots, \tilde{a}'_{H,e}\}$. Let a' (resp. $a'_{H,i}$) be the image of $\tilde{a}'$ in Z (resp. $\nu_{\mathcal{A}}^{-1}(Z)$). Then by Proposition 10.5.3, the multiplicative affine Springer fiber of a' at v is k'-isomorphic to that of a (viewed as a k'-point) at v, together with local Picard actions. Similar result also holds for each $a'_{H,i}$ and $a_{H,i}$ except only over $\bar{k}$, but such deficiency does not affect stable point-counting.

10.5.6 Since $\mathcal{Q}$ can be described using local model of singularity, its induced function on $[\mathcal{M}_a/\mathcal{P}_a](k)$ is clearly factorizable into local factors, but not in a unique way since the Tate twists can be adjusted among local factors. Here for convenience we use the convention that at any geometric point $\bar{v}' \in X - \{v\}$, the generic fiber of $\mathcal{Q}_{\bar{v}'}$ is isomorphic to $\overline{\mathbb{Q}}_\ell$, and at $\bar{v}$ over v the generic fiber of $\mathcal{Q}_{\bar{v}}$ is isomorphic to $\overline{\mathbb{Q}}_\ell[\dim \mathcal{M}](\dim \mathcal{M}/2)$. This completely determines the factorization.

Using stabilization theorem over $\tilde{\mathcal{A}}_H$ and Corollary 10.3.3, we have

$$\begin{aligned}
\#(\mathcal{P}_{a'})_0(k') & \prod_{v' \in |(X-U)_{k'}|} \#_{\mathcal{Q}_{v'}}[\mathcal{M}_{v'}(a')/\mathcal{P}_{v'}(a')](k')_\kappa \\
&= \#(\mathcal{P}_{a'})_0(k') \sum_{i=1}^{e} \prod_{v' \in |(X-U)_{k'}|} \#_{\mathcal{Q}_{H,v'}^{\kappa}}[\mathcal{M}_{H,v'}(a'_{H,i})/\mathcal{P}_{v'}(a')](k') \\
&= \#(\mathcal{P}_{a'})_0(k') \left(\sum_{i=1}^{e} \#_{\mathcal{Q}_{H,v}^{\kappa}}[\mathcal{M}_{H,v}(a'_{H,i})/\mathcal{P}_v(a')](k') \right)
\end{aligned}$$

$$\prod_{v\neq v'\in|(X-U)_{k'}|} \#_{\mathcal{Q}^{\kappa}_{H,v'}}[\mathcal{M}_{H,v'}(a'_{H,i})/\mathcal{P}_{v'}(a')](k'),$$

where the last equality is due to our assumption on $a'_{H,i}$ at places other than v. Note that here each individual term in the summation may not make sense but their sum does.

Apply Lemma 10.4.2 to points other than v and cancel out the (necessarily non-zero) terms in the above equality, we have

$$\#_{\mathcal{Q}_v}[\mathcal{M}_v(a')/\mathcal{P}_v(a')](k')_\kappa = \sum_{i=1}^{e}(q')^{-r_H^G(a'_{H,i})+r_{H,v}^G(a'_{H,i})}\#_{\mathcal{Q}^{\kappa}_{H,v}}[\mathcal{M}_{H,v}(a'_{H,i})/\mathcal{P}_v(a')](k'),$$

where $q' = |k'|$. It is true for all k'/k of degrees greater than or equal to m, so the following is true over k:

$$\#_{\mathcal{Q}_v}[\mathcal{M}_v(a)/\mathcal{P}_v(a)](k)_\kappa = \sum_{i=1}^{e} q^{-r_H^G(a_{H,i})+r_{H,v}^G(a_{H,i})}\#_{\mathcal{Q}^{\kappa}_{H,v}}[\mathcal{M}_{H,v}(a_{H,i})/\mathcal{P}_v(a)](k).$$

By Proposition 10.2.6, the left-hand side is equal to

$$\mathrm{vol}(\mathfrak{J}^0_a(\mathcal{O}_v),\mathrm{d}t_v)\,\mathbf{O}^{\kappa}_a(\mathrm{Tr}_{\mathcal{Q}_v},\mathrm{d}t_v)$$

for some Haar measure $\mathrm{d}t_v$ on $\mathfrak{J}_a(F_v)$. By our assumptions, the function $\mathrm{Tr}_{\mathcal{Q}_v}$ is none other than the IC-function f^{λ_v} induced by the intersection complex on $\mathsf{Gr}^{\le\lambda_v}_{G,v}$ and scaled by $q^{-\dim\mathcal{M}/2+\langle\rho,\lambda_v\rangle}$. Similar results holds on the H-side. Therefore combining these, and noting that $(\dim\mathcal{M}-\dim_{a_H}\mathcal{M}_H)/2 = r_H^G(a_H)$, we reach equality of orbital integrals

$$q^{\langle\rho,\lambda_v\rangle}\,\mathbf{O}^{\kappa}_a(f^{\lambda_v},\mathrm{d}t_v) = \sum_{i=1}^{e} q^{r_{H,v}^G(a_{H,i})+\langle\rho_H,\lambda_{H,v,i}\rangle}\,\mathbf{SO}_{a_{H,i}}(f_H^{\lambda_{H,v,i}},\mathrm{d}t_v),$$

where again, each individual term on the right-hand side may not make sense but their sum does.

10.5.7 Now we translate back to the group setting. Given $\gamma \in G(F_v)$, $\gamma_H \in H(F_v)$ and λ as in Theorem 2.6.11, we may find $\mathfrak{M}$ and $a_{\mathfrak{M}} \in [\mathfrak{C}_{\mathfrak{M}}/Z_{\mathfrak{M}}](\mathcal{O}_v)$ and $a_{\mathfrak{M},H} \in [\mathfrak{C}_{\mathfrak{M},H}/Z_{\mathfrak{M}}^{\kappa}](F_v)$ using the process in § 4.6. Choose our $a_{H,i}$ so that over F_v is isomorphic to $a_{\mathfrak{M},H}$ and we have the corresponding a. To avoid confusion, we denote the stable conjugacy class of γ (denoted by a in Theorem 2.6.11) by a_G and similarly a_H for γ_H (it would not cause confusion since the global objects $a_{H,i}$ are additionally indexed by i). Note that we have equality

$$r_{H,v}^{G}(a_{H,i}) + \langle \rho_H, \lambda_{H,v,i}\rangle - \langle \rho, \lambda_v\rangle = d(a_G)/2 - d_H(a_H)/2,$$

where d and d_H are the (non-extended) discriminant valuation of G and H respectively. Thus, we finally reach the equality of orbital integrals for groups G and H over $\mathcal{O}_v$:

$$q^{-d(a_G)/2}\, \mathbf{O}_{a_G}^{\kappa}(f^{\lambda}, \mathrm{d}t_v) = q^{-d_H(a_H)/2}\, \mathbf{SO}_{a_H}\left(\sum_{i=1}^{e} f_H^{\lambda_{H,i}}, \mathrm{d}t_v\right).$$

This finishes the proof of Theorem 2.6.11.

10.5.8 Reverse the local argument and global argument in § 10.4, we now proved Theorem 8.3.5 hence also Theorem 8.3.4.

10.5.9 When the condition of Theorem 2.6.11 is not satisfies, the proof fails, but not in a very serious way. First, when H is not large enough, i.e. $r_H < r$, we essentially only need to strengthen Theorem 6.10.2 which we already noted as doable in Remark 6.10.15 but we have not completely worked out the details. Secondly, if a Steinberg quasi-section does not exists, it boils down to a more detailed analysis of the effect of κ-twisting. In other words, we need to understand the definition of transfer factor better. To the author's knowledge, it is not really done in any significant capacity yet, therefore opens the door for a future project.

CHAPTER 11

COMMENTS ON FUTURE DEVELOPMENT

We close this paper by commenting on what project is being or can be done at the time of writing, as well as some guesses of what looks relevant and promising. In particular, we would like to propose a conjectural strengthened statement of our support theorem for mH-fibrations.

11.1 More on Endoscopy

The original motivation behind the current paper is to prove the yet open twisted-weighted fundamental lemma. At the time of writing, the author is aware of some ongoing effort to prove that result using the traditional Lie algebra method. So far the fundamental lemmas for both standard and twisted endoscopy are proved, essentially by [Ngô10] combined with a series of works by Waldspurger, and the non-twisted weighted case is known for split groups by [CL10, CL12]. Therefore the effort would be extending the latter proof to non-split cases.

11.1.1 Although the Lie algebra route does look promising, we believe the group method provides some unique advantage. For one, it would allow us to provide a unified geometric framework for proving similar results in general. The fundamental lemma for Lie algebra being useful is partially due to its connection to the group case, which is done by analytic method. Moreover, the twisted fundamental lemma itself does not seem to have a Lie algebra version. Instead, people have to combine Waldspurger's "non-standard" fundamental lemma and the standard one to deduce the twisted case. This means that by using groups directly, we could understand the twisted cases even better.

The twisted analogue of mH-fibration is currently under construction and a substantial amount of work has been done and we plan to release them in a future paper. The weighted cases involve studying stability condition for mH-fibrations and twisted mH-fibrations, and the project is still at an early stage.

11.1.2 There is a persistent theme in group theory over local fields, namely the statement involving groups (in other words, the statement we "care *more* about") is usually a lot messier than the analogue for Lie algebras, frequently caused by isogenies. One of the main reasons for this, in the author's opinion, is that such statements usually try to avoid using categories with automorphisms as much as possible. However, it is clear that sometimes notions like stacks and gerbes can really help to conceptualize statements. One small example of this principle is our construction of global Schubert schemes in § 5.3 where some stackiness is allowed and a much cleaner definition can be given compared to those in the literature.

The more important example, however, would be the transfer factors. The original definition of transfer factor is extremely complicated, and the transfer factor for Lie algebras, though considerably simpler, is still quite involved. Luckily, due to the existence of Kostant section, the natural gerbe under regular centralizer can be trivialized and the transfer factor at those sections can be explicitly computed. In group case, the gerbe under the regular centralizer is not necessarily trivial since a Steinberg quasi-section may not exist. We believe this gerbe has close connection with the definition of transfer factors and by understanding this gerbe better we may achieve a conceptual definition of the transfer factors.

11.2 Inductive Support and Beyond Endoscopy

The support theorems we are able to prove for mH-fibrations are not entirely satisfactory, because it only provides an upper bound of supports. In an astonishing paper [MS18], the authors proved a sort of support theorem in a very general setting, from which they were able to deduce Ngô's original support theorem. It is possible that their method can be adapted to deduce our support theorem as well, but the problem is their result would not be any stronger even if that is the case, because the result in *loc. cit.* is also an upper bound, and that upper bound, without further input, is weaker than ours.

Largely speaking, all kinds of support theorems known so far boil down to some kind of dimension counting, and dimension is related to weights and duality in cohomologies. Therefore it seems these methods are inherently unable to deal with anything related to connected components, unless those components happen to have symmetry under some finite group (as in endoscopy). The extra components in mH-fibrations, however, do not look like they come from a group but rather some sort of correspondence. In § 8.5, we defined a new sort of Hecke stack, but we never really studied the correspondence generated by that stack. A preliminary dimension counting shows that those Hecke stacks are in general not graph-like in the sense of [Yun11] and they contain some exotic components that are closely related to the inductive strata. Unfortunately, we are currently unable to deduce anything concrete about the correspondence given by those exotic components, but we do suspect they play a role in determining the precise supports.

11.2.1 For simplicity, we will give a conjecture about the supports in a very special case. The reader should be able to easily infer what the conjecture would look like in the general case if proper combinatorial notations are taken care of.

Let $G = G^{\mathrm{sc}}$ is split and λ is the unique dominant short coroot. In this case, V_λ is the unique quasi-minuscule representation of $\check{G}$ in which the weights are the ones in the W-orbit of λ and 0. The multiplicity $m_{\lambda,0}$ is equal to the number of simple short coroots.

Let $h\colon \mathcal{M} \to \mathcal{A}$ be the *restricted* mH-fibration with boundary divisor $D \cdot \lambda$, where $D = x_1 + \cdots + x_d$ is a multiplicity-free effective divisor of degree $d \gg 0$ on X. Let $D' = x_2 + \cdots + x_d$, and $h'\colon \mathcal{M}' \to \mathcal{A}'$ be the restricted mH-fibration with boundary divisor $D' \cdot \lambda$. Then we have linear embedding $\iota\colon \mathcal{A}' \to \mathcal{A}$ of codimension $\langle \rho, \lambda \rangle$, and the image is an maximal element among the inductive subsets of $\mathcal{A}$. For a general point

$a \in \mathcal{A}(\bar{k})$, the mH-fiber $\mathcal{M}_a$ is a torsor under $\mathcal{P}_a$, while for a general point $a' \in \iota(\mathcal{A}')$, $\mathcal{M}_{a'}$ has exactly $m_{\lambda,0}$ irreducible components modulo $\mathcal{P}_{a'}$. There is an open dense subset in $\mathcal{A}$ over which the top ordinary cohomology of $\mathcal{M}$, after taking the stable constituent, is isomorphic to $\overline{\mathbb{Q}}_\ell$, and similarly over $\iota(\mathcal{A}')$ it is generically isomorphic to a local system of rank $m_{\lambda,0}$. Similarly, let D_e be a subdivisor of D by taking away e-points, and $\iota_e\colon \mathcal{A}_e \to \mathcal{A}$ be the obvious analogue of ι.

Conjecture 11.2.2. *Let K_e be the perverse summand supported on $\iota_e(\mathcal{A}_e)$ in the stable constituent of the cohomology $(h_*\mathrm{IC}_{\mathcal{M}})_{\mathrm{st}}$. Then the contribution of K_e to the top ordinary cohomology over an open dense subset of $\iota(\mathcal{A}_e)$ is a local system of rank $(m_{\lambda,0}-1)^e$.*

11.2.3 The significance, as pointed out to the author by Ngô, is that the support seems sensitive to the boundary divisor, and the smaller the inductive subset is, the bigger the multiplicity in their contribution. In some sense it seems to align pretty well with the general gist of so-called Beyond Endoscopy program.

On the other hand, we do not believe mH-fibration is the right tool to tackle problems in Beyond Endoscopy. Rather, we speculate that there is a limit version, which we temporarily call the Ran-mH-fibrations, that might be more relevant. Essentially, mH-fibrations are a bunch of fibrations over a base moduli whose components are basically symmetric powers of curves. However, this means that at points outside of divisors we are still using the standard special function being the characteristic function on $G(\mathcal{O})$, which seems completely contrary to the setting of Beyond Endoscopy. Therefore, it is only right if we do not treat those symmetric powers of curves as a disjoint union, but integrate them in a meaningful way. This leads to the notion of Ran space and the Ran-mH-fibration should be the analogue of mH-fibration defined over an appropriate Ran space.

We should caution readers that this part is highly speculative. Nevertheless, even if it is completely unrelated to Beyond Endoscopy, we still think the notion of Ran-mH-fibration will be very worth exploring by itself.

11.2.4 Another more concrete and accessible goal "beyond endoscopy" is perhaps the Hitchin-type fibrations in general, namely we simply consider the mapping stack related to a group G acting on a space M. There has been some on going joint effort by B. Morrissey and B. C. Ngô on this front with some encouraging results. A less general but more arithmetically inspired case would be geometrization of relative trace formulae. After the unifying framework proposed in [SV17], it would be very interesting to see how multiplicative Hitchin-type fibrations can be used to tackle fundamental lemmas in those settings.

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