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Testing for parameter change epochs in GARCH time series

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Summary: We develop a uniform test for detecting and dating the integrated or mildly explosive behaviour of a strictly stationary generalized autoregressive conditional heteroskedasticity (GARCH) process. Namely, we test the null hypothesis of a globally stable GARCH process with constant parameters against the alternative that there is an 'abnormal' period with changed parameter values. During this period, the parameter-value change may lead to an integrated or mildly explosive behaviour of the volatility process. It is assumed that both the magnitude and the timing of the breaks are unknown. We develop a double-supreme test for the existence of breaks, and then provide an algorithm to identify the periods of changes. Our theoretical results hold under mild moment assumptions on the innovations of the GARCH process. Technically, the existing properties for the quasi-maximum likelihood estimation in the GARCH model need to be reinvestigated to hold uniformly over all possible periods of change. The key results involve a uniform weak Bahadur representation for the estimated parameters, which leads to weak convergence of the test statistic to the supreme of a Gaussian process. Simulations in the Appendix show that the test has good size and power for reasonably long time series. We apply the test to the conventional early-warning indicators of both the financial market and a representative of the emerging Fintech market, i.e., the Bitcoin returns.

Keywords: *GARCH*, *IGARCH*, *change-point analysis*, *concentration inequalities*, *uniform test*.

JEL codes: *C01 econometrics*, *C58 Financial Econometrics*, *G17 - Financial Forecasting and Simulation*.

1. INTRODUCTION

Volatility is an important indicator for economic and financial stability. There is growing evidence of the unstable behaviour of the historical volatility of numerous micro- and macro-level data,

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such as individual asset returns, VIX (the Chicago Board Options Exchange volatility index), inflation, and unemployment. Bloom (2007) documents the unstable behaviour of the higher moments of many economic variables, such as research and development (R&D) rates related to the uncertainty about future productivity. It is understood that the nature of uncertainty is the unpredictability of any model to the future path of a time series. Therefore, it may be connected with a change of the parameter values in the underlying data-generating process. Ignoring parameter change may thus lead to biased analysis in policymaking and forecasting. This motivates us to consider a general method of testing parameter constancy for models of volatility.

For modeling the volatility processes, the highly celebrated autoregressive conditional heteroskedasticity (ARCH) model proposed by Engle (1982) is important for describing the pervasive phenomena of heteroskedasticity presented in many time series. One key generalization of ARCH is the GARCH model, i.e.,

$$X_i^2 = \zeta_i^2 \sigma_i^2,$$
$$\sigma_i^2 = \alpha_0 + \sum_{j=1}^{r} \alpha_j X_{i-j}^2 + \sum_{k=1}^{s} \beta_k \sigma_{i-k}^2, \tag{1.1}$$

where the conditional variance σ_i^2 depends on the past observations X_{i-j}^2, but also on the historical conditional variance σ_{i-k}^2. ζ_i assumed to be i.i.d. innovations; see Paolella (2019) for more details of the model.

Hillebrand (2005) points out that neglecting parameter changes in GARCH models leads to biased parameter fitting. Thus a change-point analysis should be conducted before reporting a parameter fit of a GARCH model. Among various possible changes of parameters of the underlying process, moving from the covariance stationarity to the infinite variance has come to the centre of our focus for its potential use of detecting periods of economic uncertainty. In addition to the case of integrated GARCH, we refer to the volatility process behaving more explosive after the change as a 'mildly explosive' one, which can be considered as an analogue of a mildly explosive unit-root return process. The name 'mildly explosive' follows Lee and Hansen (1994), who refer to a GARCH(1,1) model with $\alpha_1 + \beta_1 > 1$ as a 'mildly explosive' one.

Is there empirical evidence of the existence of mildly explosive region of a GARCH model with fitted parameters? One often sees sudden, integrated, or mildly explosive behaviour in the second moment of the process which bounces back after a while. Figure 1 shows a rolling window fit of parameter values of a GARCH(1,1) model using Bitcoin data. We can see clear signs of time-varying parameters. In particular, there are regions of the estimated parameters falling out of the covariance stationary regime ($\hat{\alpha}_1 + \hat{\beta}_1 \geq 1$). Such kind of data phenomena suggest that the underlying processes have time-varying parameters, calling for a rigorous quantitative treatment for detection of change periods and making corresponding inference.

The aim of our paper is to develop a generalized uniform test for GARCH models that is able to detect exuberant behaviour periods (periods with integrated or mildly explosive parameter values) associated with the empirical phenomena of mild explosiveness in the second moment. The test is constructed by looking at the supreme of Wald-type test statistics over all possible intervals with changing parameters. Numerous estimation methods for the parameters of GARCH models have been proposed, and their consistency and asymptotic normality have been carefully studied in the literature. A conventional estimation approach is the quasi-maximum likelihood estimation (QMLE), e.g., Bollerslev and Wooldridge (1992). Also Fan et al. (2014) study QMLE of GARCH

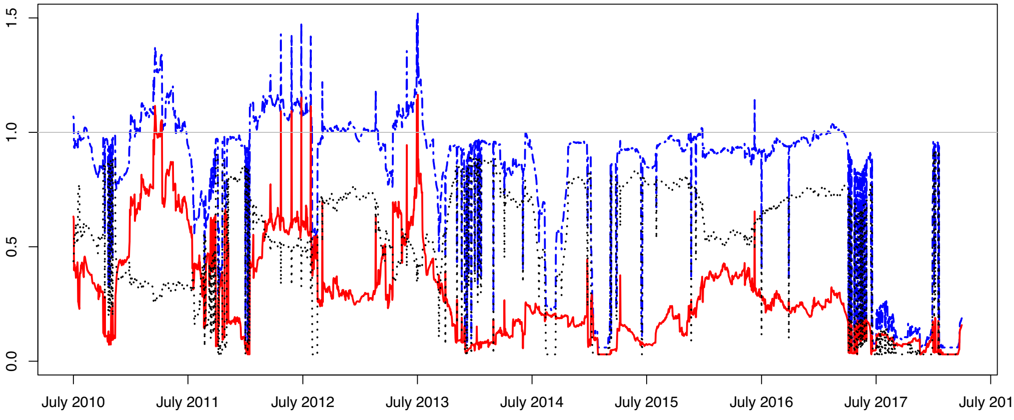

Figure 1. A plot of estimated GARCH(1,1) parameters using the Bitcoin data over a rolling window of size 200. $\hat{\alpha}_1 + \hat{\beta}_1$ estimate persistence parameter (dash line), $\hat{\alpha}_1$ (solid line), $\hat{\beta}_1$ (dotted line), threshold of mild explosiveness ($\alpha_1 + \beta_1 = 1$).

models with heavy-tailed likelihoods. Peng and Yao (2003) propose a least absolute deviation estimator. Jensen and Rahbek (2004) establish consistency and asymptotic normality of the QMLE in the linear ARCH model. It is well known that under the assumption of the strict stationarity of a GARCH model, there is still a region of parameter values allowing for realizations with unstable volatility behaviour. The leading case is the 'IGARCH' process. Nelson (1990) looks at the behaviour of an integrated GARCH (IGARCH) process, and it is known that the unconditional mean of the IGARCH's conditional variance is not finite, which implies infinite second or higher moments (i.e., eruptive behaviour). Lee and Hansen (1994) provide an asymptotic theory for a strictly stationary GARCH(1,1) Quasi maximum likelihood estimator (QMLE) estimator allowing for the case of IGARCH, and mildly explosive conditional variance and even nonstationarity. Jensen and Rahbek (2004) consider asymptotic inference for a nonstationary GARCH model.

Despite the rich empirical literature which suggests the existence of an unstable moment period of a GARCH process, there is only sparse literature on determining and testing the period of integrated/mild explosiveness in an uniform manner. Francq and Zakoïan (2012) provide important estimation results on nonstationary GARCH models, and they also provide a test for parameter constancy of a GARCH(1,1) process without assuming strict stationarity. Complementary to their study, our focus is on the integrated/mildly explosive parameter region and we extend the test to a uniform context. There is also a large and extensive literature on testing for mild explosiveness, and dating the period of instability in the price or dividend processes of a financial asset using a supreme unit-root test for bubbles. See, for example, Phillips et al. (2011) for a left-tailed, augmented Dickey–Fuller test (ADF) for the mildly explosive behaviour in the 1990s Nasdaq. Hafner (2020) considers such bubble tests for cryptocurrencies. Harvey et al. (2019) investigate a bubble test with a smooth time-varying volatility function. The underlying models focus usually on unit-root or mildly explosive autoregressive (AR) processes to test the change of the AR(1) coefficient. Often, the variance of the errors stays the same or varies smoothly after the explosion, which means that the volatility increase is mostly driven by the increase of the AR parameter. In our model, we choose a different approach to model a mild explosion of volatility: we describe the evolution of the data-generating process by a GARCH process, and therefore link the source of a change in the volatility to a change of the parameters in the volatility recursion.

In addition, there is literature on break detection for multiple break points for nonlinear time series, cf. Berkes et al. (2004), Davis et al. (2008), Bardet et al. (2012), Fryzlewicz and Subba Rao (2014), and Chen and Hong (2016). In particular, Bardet et al. (2012) derive a breakpoint detection procedure for general recursively defined time series via a penalized maximum likelihood method and prove its consistency. Their formulation is rather general, leading to the restriction that the considered time series have to be covariance stationary. Davis et al. (2008) propose a model with piece-wise stationary time series with independent segments. Fryzlewicz and Subba Rao (2014) invent a novel method to find break points, and test for covariance stationary ARCH processes using CUSUM (cumulative sum) statistics. Chen and Hong (2016) impose smoothly varying GARCH parameters and estimate them locally over some window. They provide a likelihood ratio approach to test if the global GARCH estimates significantly deviate from the local parameter estimates. This allows them to detect if there is a change, but it does not allow us to find specific breakpoints. However, their approach is more appropriate than ours if the parameter values vary smoothly over time. The work most connected to ours is Berkes et al. (2004), where a sequential change-point testing in GARCH(p, q) models is discussed. They consider testing for a change of the whole parameter vector θ based on a CUSUM-type statistic by plugging in estimates of θ into the likelihood of future steps. It is not straightforward to adapt their approach to testing for linear hypotheses in θ, in particular testing for mild explosiveness. Comparatively, our test allows for several multivariate extensions which may be interesting in change-point analysis. In sum, our test is different, but complementary, to the above study, as we propose breakpoint detectors for GARCH models in the noncovariance stationary regime, and provide a solid theoretical backup via a uniform testing procedure for the presence of breakpoints.

It is worth noting that, unlike a bubble test for an AR process, it is quite debatable to link a direct cause of the bursting behaviour to the volatility process; see Jurado et al. (2015). On the contrary, volatility bursting can also be related to time-varying risk aversion, sentiment, bubbles, or uncertainty. Nevertheless, we are trying to establish a rigorous theoretical framework of testing for the mildly explosive interval using a GARCH model for the volatility process. It should be stressed that we focus on one aspect of the parameter; namely, changes in the parameters driving the volatility over time. We do not claim that our method can directly identify the cause of this behaviour. In sum, we develop a change-point test for detecting possible unstable behaviour of a strictly stationary GARCH(r, s) process. The null hypothesis is a GARCH process with globally constant parameters, while the alternative is the existence of a period in which the parameter values change to another (higher) values. This increase potentially leads to a period of mildly explosive volatility.

Assuming that no information on the period and the change itself is available, we develop a test statistic based on supremes which searches over all possible sub-windows of the data. We prove asymptotic consistency and provide a limit distribution of our test statistic. It is important that the test is not of unit-root type, since hypothesis and the alternative are still in the regime where the GARCH process is strictly stationary. The theoretical contributions are extending the existing theoretical results on GARCH QML estimators to uniform consistency statements over an arbitrary observation period. Besides, a uniform weak Bahadur representation and the corresponding uniform distributional limit results are shown. For the proofs, we carve out the essential analytical properties of the likelihood functions and use new concentration inequalities from Zhang and Wu (2017), leading to mild moment assumptions. Empirically, we find that our test is useful for the early identification of the critical periods of financial crisis for two important early-warning indicator of the economic condition.

We introduce some notations we use throughout the paper. For $q > 0$ and vector $v = (v_1, \ldots, v_d)^\top \in \mathbb{R}^d$, let $|v|_1 := \sum_{i=1}^d |v_i|$. For matrices $A \in \mathbb{R}^{d\times d}$, we similarly use $|A|_1 := \sum_{i,j=1}^d |A_{i,j}|$. We denote by $|A|_2 = \max_{|v|=1} |Av|_2$ the spectral norm of A. We use $Z_n \xrightarrow{d} Z$ and $Z_n \xrightarrow{p} Z$ to denote convergence in distribution and convergence in probability for random variables Z_n, Z. For some sequences (a_n) and (b_n) of positive numbers, we write $a_n = O(b_n)$ or $a_n = o(b_n)$ if there exists a positive constant C such that $a_n/b_n \le C$ or $a_n/b_n \to 0$, respectively. For two sequences of random variables (X_n) and (Y_n), we write $X_n = o_p(Y_n)$ (resp. $X_n = O_p(Y_n)$) if $X_n/Y_n \to 0$ in probability (X_n/Y_n is bounded in probability). For some nonnegative real number x, let $\lfloor x \rfloor$ denote the flooring operator, i.e., the largest integer smaller than or equal to x.

Our text is organized as follows. Section 2 provides the results of the important GARCH(1,1) model and the corresponding test procedure. Section 3 concerns the estimation and theoretical results in a general GARCH(r, s) model. In particular, Section 3.1 introduces the framework of the QMLE and its consistency; Section 3.2 presents the theoretical foundations of our uniform test; Section 3.3 discusses the estimation of the covariance matrix of the QMLE appearing in the test statistic; Section 3.4 extends the results to a general parameter constancy test. Section 4 discusses the behaviour of the test in examples from practice. Section 5 concludes. The technical proofs and simulations are delegated to the online Appendix.

2. A UNIFORM MILD EXPLOSIVENESS TEST FOR GARCH(1,1)

In this section, we introduce our model by starting with a simple testing framework for the GARCH(1,1) model. Then we will provide a rigorous theoretical treatment by starting with a more general GARCH(r, s) model in the following section. We consider first of all the baseline GARCH(1,1) model over the whole sample period with possible time-varying parameters,

$$\begin{aligned} X_i &= \zeta_i \sigma_i, \\ \sigma_i^2 &= \alpha_0(i) + \alpha_1(i) X_{i-1}^2 + \beta_1(i)\sigma_{i-1}^2, \quad i \in \mathbb{Z}, \end{aligned} \tag{2.1}$$

where ζ_i is an i.i.d. sequence of random variables with $\mathbb{E}\zeta_1 = 0$, $\mathbb{E}\zeta_1^2 = 1$, and $\alpha_0(i), \alpha_1(i), \beta_1(i) > 0$ are the underlying parameters at each time point. We collect data of this model at time points $1, \ldots, n$.

We summarize the parameters into $\theta(i) = (\alpha_0(i), \alpha_1(i), \beta_1(i))'$. In the case that the parameters are constant, i.e., $\theta(i) \equiv \theta = (\alpha_0', \alpha_1', \beta_1')'$, the top Lyapunov exponent associated with this model, according to Bougerol and Picard (1992b), is

$$\gamma(\theta) = \mathbb{E}\log(\alpha_1 \zeta_1^2 + \beta_1).$$

In particular, it is shown that, for example in Francq and Zakoïan (2012), if $\gamma(\theta) < 0$, then the conditional volatility σ_i converges almost surely to $\sigma_{i,\infty}$ as $i \to \infty$, with $\sigma_{i,\infty} = \lim_{n\to\infty} \alpha_0^*\{1 + \sum_{k=1}^{n-1} \log(\alpha_1^* \zeta_{t-1}^2 + \beta_1^*) \cdots \log(\alpha_1^* \zeta_{t-k}^2 + \beta_1^*)\}$. It is worth noting that $\gamma(\theta) < 0$ allows (for instance) the IGARCH case, i.e., $\alpha_1 + \beta_1 = 1$. We illustrate in Figure 2 the region of parameters corresponding to the case that the volatility process is noncovariance stationary, but still strictly stationary (integrated or mildly explosive).

The aim of this paper is to construct a test that is able to detect if there exists a period where the parameters of the GARCH model have changed. The theory developed in this paper is theo-

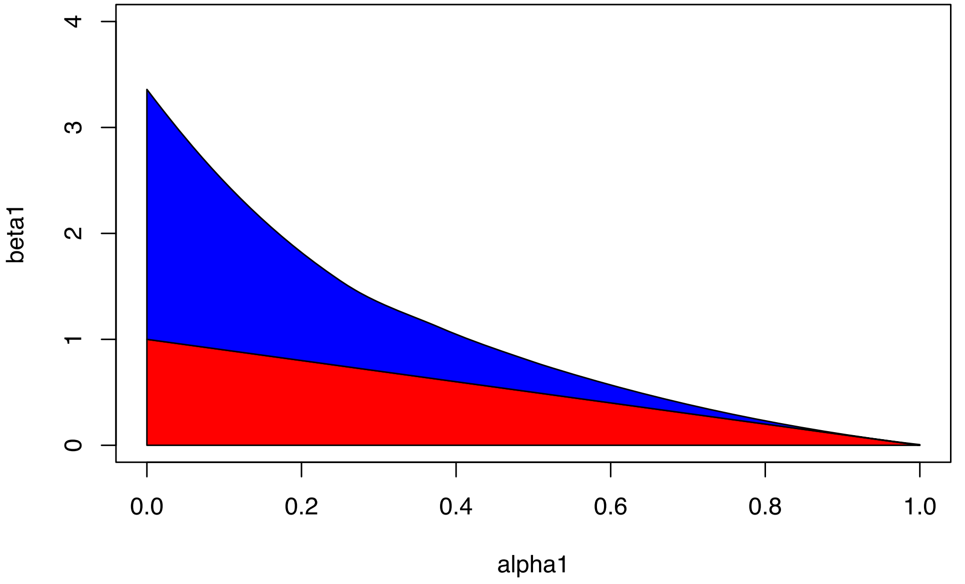

Figure 2. The plot of the feasible parameter region with a standard normally distributed ζ_1, where the down region corresponds to covariance stationarity and the blue region corresponds to the strictly stationary mildly explosive region; X axis, α_1, Y axis, β_1.

retically supported for null hypotheses in the regime of strict stationarity (see the definition of Θ below). *However, the alternative hypotheses of our test includes GARCH processes which are not strictly stationary.* Moreover, due to the monotonicity of the test statistics developed in this paper, we conjecture that the test also works if the null hypothesis lies in the nonstationary regime. Formally, we would like to test whether there exists a period $\{n_1, \ldots, n_2\}$ (with $1 < n_1 < n_2 < n$), in which the parameter values in (2.1) change their values compared with $\{1, \ldots, n\}$. The task breaks into two parts: First, one has to check for the existence of a change, for which a uniform test is needed. Second, one has to identify the period of the change and to estimate the corresponding parameters. Furthermore, we certainly would like to make inference on our estimated parameters.

2.1. Hypotheses and the likelihood function

In this subsection, we provide our test hypotheses, and parameter estimators. For our studies, let

$$\Theta = \{\theta = (\alpha_0, \alpha_1, \beta_1) \in \mathbb{R}^3 : \gamma(\theta) < 0, \alpha_0, \alpha_1, \beta_1 > 0\}$$

be the parameter space which contains all possible configurations of $\theta = (\alpha_0, \alpha_1, \beta_1)$.

Let $\theta(i) = (\alpha_0(i), \alpha_1(i), \beta_1(i))'$ denote the true parameter in the baseline model, which equals $\theta^* = (\alpha_0^*, \alpha_1^*, \beta_1^*) \in \Theta$ at the beginning, and possibly has a period of significant change in $\{\lfloor n\tau_1^*\rfloor + 1, \ldots, \lfloor n\tau_2^*\rfloor\}$ (where $\tau_1^*, \tau_2^* \in [0, 1]$, $\tau_1^* < \tau_2^*$) with magnitude $\Delta^* > 0$ and direction

$H \in \mathbb{R}^3$. Namely,

$$\theta(i) = \begin{cases} \theta^*, & i \leq \lfloor n\tau_1^* \rfloor, \\ \theta^* + H\Delta^*, & \lfloor n\tau_1^* \rfloor + 1 \leq i \leq \lfloor n\tau_2^* \rfloor, \\ < \theta^* + H\Delta^*, & i > \lfloor n\tau_2^* \rfloor. \end{cases} \tag{2.2}$$

It is worth noting that we do not expect that the parameter values after the explosive period return to the original ones. We only assume that they drop back to a lower value after this period, i.e., $\theta(i) < \theta^* + H\Delta^*$ for $i > \lfloor n\tau_2^* \rfloor$. We will give a formal proof of the consistency of our test if $\theta^* + H\Delta^* \in \Theta$. Recall that Θ is the parameter region of strict stationary. It should be noted that the space of allowed parameter configurations can be relaxed even further by sacrificing the estimation accuracy of the constant term $\alpha_0^*(i)$ (cf. Francq and Zakoïan (2012)).

An interesting question is to test whether the process is stable, i.e., $\alpha_1(i) + \beta_1(i) < 1$ for all time points $i = 1, \ldots, n$ versus the hypothesis that there exists a period of begin integrated or mild explosive, in which $\alpha_1(i) + \beta_1(i) \geq 1$ for some i. $\alpha_1(i) + \beta_1(i)$ is referred to as the persistence parameter in our setting. Graphically, this corresponds to the question of whether or not there exists regions where the process leaves the variance-stationary regime (i.e., the variance explodes).

It is therefore natural to formulate the hypotheses in the following way: Let $H = (0, 1, 1)'$. We want to test if $\theta(i)$ changes in direction of $\alpha_1^* + \beta_1^*$, that is, with some fixed value of $c := H\theta^*$, $c := \alpha_1^* + \beta_1^* < 1$, we want to test

$$H_0^{pre} : \Delta^* < 0 \quad \text{v.s.} \quad H_1^{pre} : \Delta^* \geq 0. \tag{2.3}$$

To transfer the setting to the one of change-point tests, we modify (2.3) as follows:

$$H_0 : \Delta^* = 0 \quad \text{v.s.} \quad H_1 : \Delta^* > 0. \tag{2.4}$$

Since the statistical behaviour of X_i is continuous with respect to Δ^*, a test procedure for (2.4) will automatically yield a reasonable test for (2.3). We will discuss the connection between (2.3) and (2.4) in Remark 3.5.

Our method is a way to test the parameter constancy for GARCH processes. For example, in practice, a useful choice for c may be obtained from $c = \hat{\alpha}_1 + \hat{\beta}_1$, where $\hat{\alpha}_1, \hat{\beta}_1$ are obtained from fitting a global model with all observations. We illustrate this with VIX in our empirical study (cf. Section 4). To construct a test, we first derive estimators for the parameters. For a fixed period $\lfloor n\tau_1 \rfloor + 1, \ldots, \lfloor n\tau_2 \rfloor$, we can use a standard QMLE approach. It is not hard to see from (2.1) that, in the case of the constant parameters $\theta(i) \equiv \theta$,

$$\sigma_i^2 = \alpha_0/(1 - \beta_1) + \alpha_1 \sum_{k=1}^{\infty} \beta_1^k X_{i-1-k}^2 \quad \text{a.s.}$$

The truncated version that can be calculated from a sample is

$$\sigma_i^{2c} = \alpha_0/(1 - \beta_1) + \alpha_1 \sum_{k=1}^{i-2} \beta_1^k X_{i-1-k}^2.$$

The quasi-likelihood approach is to use the negative log likelihood function

$$L_{n,\tau_1,\tau_2}^c(\theta) := \frac{1}{n} \sum_{i=\lfloor n\tau_1 \rfloor + 1}^{\lfloor n\tau_2 \rfloor} \ell(X_i^2, Y_i^c, \theta),$$

where $Y_i^c := (X_{i-1}^2, \ldots, X_1^2, 0, 0, ...)$ and

$$\ell(X_i^2, Y_i^c, \theta) := \frac{1}{2}\Big(\frac{X_i^2}{\sigma_i^{2c}} + \log \sigma_i^{2c}\Big). \tag{2.5}$$

The estimated parameter with observations during any given period $\lfloor n\tau_1 \rfloor + 1, \ldots, \lfloor n\tau_2 \rfloor$ is defined as

$$\hat{\theta}_{n,\tau_1,\tau_2} = \operatorname{argmin}_{\theta \in \Theta} L_{n,\tau_1,\tau_2}^c(\theta). \tag{2.6}$$

REMARK 2.1. It is worth noting that $L_{n,\tau_1,\tau_2}^c(\theta)$ also includes observations X_i from earlier time points $i \le \lfloor n\tau_1 \rfloor$ through σ_i^{2c}. One might wonder if the accuracy of the likelihood is affected by using observations before the change point $\lfloor n\tau_1 \rfloor$. The impact of the terms X_i with $i \le \lfloor n\tau_1 \rfloor$ decays geometrically, and we see in Proposition 3.1 that it is theoretically negligible even under the alternative. Moreover, it should be noted that any other initialization of σ_i^{2c} (for instance with $X_i = 1$ for $i \le \lfloor n\tau_1 \rfloor$) may be even more inaccurate in a finite sample perspective.

2.2. *Test statistics and an algorithm*

In this subsection the descriptions of the test statistics and the algorithm is provided. Under the null H_0 and further regularity conditions (cf. Section 3), $\hat{\theta}_{n,\tau_1,\tau_2}$ is asymptotically normal with covariance matrix

$$\Sigma = V(\theta^*)^{-1} I(\theta^*) V(\theta^*)^{-1}, \tag{2.7}$$

where $Y_i = (X_j : -\infty < j \le i-1)$ contains the whole past and

$$V(\theta) := \mathbb{E}[\nabla_\theta^2 \ell(X_i^2, Y_i, \theta)], \qquad I(\theta) := \mathbb{E}[\nabla_\theta \ell(X_i^2, Y_i, \theta) \cdot \nabla_\theta \ell(X_i^2, Y_i, \theta)'].$$

Estimation of Σ is done based on the observations $1, \ldots, \lfloor n\tau_1 \rfloor$ as follows: $\hat{\theta}_{n,0,\tau_1}$ is the estimator of θ^* in the stationary regime (using the notation (2.6)), and

$$\bar{\Sigma}_{n,\tau_1} := \bar{V}_{n,\tau_1}(\hat{\theta}_{n,0,\tau_1})^{-1} \bar{I}_{n,\tau_1}(\hat{\theta}_{n,0,\tau_1}) \bar{V}_{n,\tau_1}(\hat{\theta}_{n,0,\tau_1})^{-1}, \tag{2.8}$$

where

$$\bar{V}_{n,\tau_1}(\theta) := \frac{1}{\tau_1} \nabla_\theta^2 L_{n,0,\tau_1}^c(\theta), \qquad \bar{I}_{n,\tau_1}(\theta) := \frac{1}{n\tau_1} \sum_{i=1}^{\lfloor n\tau_1 \rfloor} \nabla_\theta \ell(X_i^2, Y_i^c, \theta) \nabla_\theta \ell(X_i^2, Y_i^c, \theta)'.$$

For given $\tau_1 < \tau_2$, the test statistic associated with our hypothesis H_0 of interest is

$$\hat{B}_n(\tau_1, \tau_2) := \sqrt{n(\tau_2 - \tau_1)} (H' \bar{\Sigma}_{n,\tau_1} H)^{-1/2} \{H' \hat{\theta}_{n,\tau_1,\tau_2} - H'\theta^*\}, \tag{2.9}$$

where $\hat{\alpha}_{1,n,\tau_1,\tau_2}, \hat{\beta}_{1,n,\tau_1,\tau_2}$ are the second and third components of $\hat{\theta}_{n,\tau_1,\tau_2}$, and $\bar{\Sigma}_{n,\tau_1,\tau_2}$ is an estimator of Σ using observations outside of $\{\lfloor n\tau_1 \rfloor, \ldots, \lfloor n\tau_2 \rfloor\}$. For instance, we can set $\bar{\Sigma}_{n,\tau_1,\tau_2}$ to be the standard covariance matrix estimator obtained by replacing V, I with their empirical counterparts with observations outside $\{\lfloor n\tau_1 \rfloor, \ldots, \lfloor n\tau_2 \rfloor\}$.

The feasible search set for explosive periods is defined to be

$$R_{\kappa,\kappa'} := \{(\tau_1, \tau_2) \in [0,1]^2 : \kappa' \le \tau_1 < \tau_2, \tau_2 - \tau_1 \ge \kappa\} \tag{2.10}$$

(with some $\kappa, \kappa' > 0$, for instance $\kappa = \kappa' = 0.1$), ensuring proper estimation of the variance-covariance matrix of the estimated parameters Σ due to $\tau_1 \ge \kappa'$ and a change detection based on enough samples due to $\tau_2 - \tau_1 \ge \kappa$. The uniformity test is thus taken on the set $R_{\kappa,\kappa'}$ to be

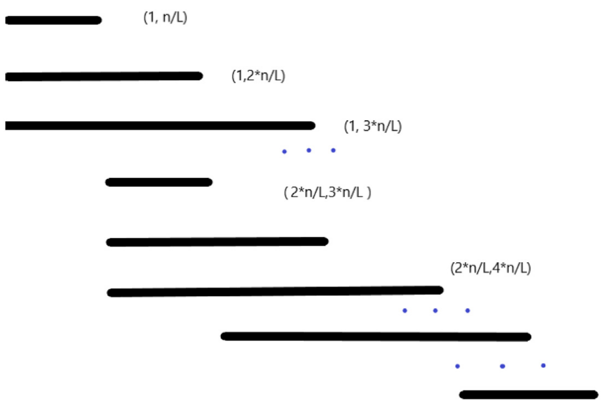

Figure 3. The plot of the windows where the supreme is calculated.

any combination of τ_1, τ_2 with $\kappa < |\tau_1 - \tau_2| \leq 1 - \kappa'$. The supreme of $\hat{B}_n(\tau_1, \tau_2)$ with respect to $R_{\kappa,\kappa'}$ converges asymptotically to the supreme of a Gaussian process, namely $\left\{\frac{B(\tau_2)-B(\tau_1)}{(\tau_2-\tau_1)^{1-1/2}}\right\}$, where $B(\cdot)$ is a 1-dimensional Brownian motion. We show this formally in Theorem 3.3 in Section 3.2.

Empirically, we cannot exhaust all the values $(\tau_1, \tau_2) \in R_{\kappa,\kappa'}$. For the ease of implementation and derivation, we define our feasible search set to be $\kappa, \kappa' := \{(\tau_1, \tau_2) \in [0, 1]^2 : \tau_1 < \tau_2, 1 - \kappa' \geq \tau_2 - \tau_1 \geq \kappa\}$, where κ, κ' are the bound on the distance between (0,1). We therefore need to restrict the calculation of the supreme to a set of grid points as an approximation of our supreme test statistics. We summarize the test procedure for a given acceptance rate $\delta \in (0, 1)$ (typically, $\delta = 0.9$ or $\delta = 0.95$) in the following context.

Algorithm 1

Step 0 Choose some $L > 0$ (the number of grid points associated with detection accuracy). The corresponding grid points are $\mathcal{G} = \left\{\frac{j}{L} : j = 0, \ldots, L\right\}$ on the time line.

Step 1 Let H denote the direction in which a change of parameters should be checked, cf. (2.2). Fix some baseline value $H'\theta^*$.

Step 2 Choose values for $\kappa, \kappa' \in (0, 1)$. We suggest setting $\kappa = 0.1, \kappa' = 0.1$.

Step 3 For each given interval $(\tau_1, \tau_2) \in R_{\kappa,\kappa'} \cap \mathcal{G}^2$, determine the associated QLME $\hat{\theta}_{n,\tau_1,\tau_2}$ defined in (2.6) and calculate $\bar{\Sigma}_{n,\tau_1,\tau_2}$ as in (2.8). Then determine $\hat{B}_n(\tau_1, \tau_2)$ via (2.9). Figure 3 shows how one calculates the supreme test statistic over different windows associated with the grid points.

Step 4 For the critical value of this test, we can approximate the quantile of the test statistic via simulation of the limiting Gaussian process under the null hypothesis H_0: for large N (e.g. $N = 10{,}000$), generate for each $k \in \{1, \ldots, N\}$ i.i.d. $\varepsilon_i^{[k]} \sim N(0, 1), i = 1, \ldots, n$ and calculate

$$\hat{\mu}_{n,k} := \sup_{(\tau_1,\tau_2)\in R_{\kappa,\kappa'}\cap\mathcal{G}^2} \frac{1}{\sqrt{n(\tau_2 - \tau_1)}} \sum_{i=\lfloor n\tau_1\rfloor+1}^{\lfloor n\tau_2\rfloor} \varepsilon_i^{[k]}.$$

We define $\hat{q}_{W,\delta} := \hat{\mu}_{n,[\lfloor N\cdot\delta\rfloor]}$, where $\hat{\mu}_{n,[1]}, \ldots, \hat{\mu}_{n,[N]}$ are the order statistics of $\hat{\mu}_{n,1}, \ldots, \hat{\mu}_{n,N}$.

Step 5 We can now make a test decision based on the critical values from the previous steps. If

$$\hat{B}_n := \sup_{(\tau_1,\tau_2)\in R_{\kappa,\kappa'}\cap\mathcal{G}^2} \hat{B}_n(\tau_1, \tau_2) > \hat{q}_{W,\delta}, \tag{2.11}$$

there is a significant shock in the parameter values. In this case, one can estimate the true shock period as $[\tau_1^*, \tau_2^*]$ by

$$(\hat{\tau}_{1,n}, \hat{\tau}_{2,n}) \in \operatorname{argmax}_{(\tau_1,\tau_2)\in R_{\kappa,\kappa'}\cap R_{\kappa,\kappa'}} \hat{B}_n(\tau_1, \tau_2).$$

If instead (2.11) does not hold, we conclude that there is no evidence for a period of parameter change.

Step 6 In case of the significance of our uniform test in Step 5, we re-estimate the parameter $\hat{\theta}_{n,\hat{\tau}_1,\hat{\tau}_2}$, and produce the confidence interval based on Theorem 3.3.

We name this the GARCH Supreme Richter-Wang-Wu (GSRWW) test. The procedure depends on some tuning parameters. We have experimented with various choices and make our suggestions as follows.

Remark 2.2 (Choices of tuning parameters L, κ, κ').

- (*i*) We suggest making L as large as possible so that the calculation on the machine is still done within an acceptable time. In principle, $L = n$ is optimal, but may lead to an infeasible duration of computation in practice. Lower choices of L will decrease detection accuracy of the break points τ_1^*, τ_2^*, and may decrease the power of the test since short change periods (with small $\tau_2^* - \tau_1^*$) with small impacts Δ^* may naturally be overseen.
- (*ii*) The theoretical results hold for all fixed choices of κ, κ'. The consistency results we derive rely on the fact that $\tau_2 - \tau_1 \geq \kappa$ is bounded from below so that $L^c_{n,\tau_1,\tau_2}(\theta)$ contains a number of observations, which is formally proportional to n. Similarly, $\tau_1 \geq \kappa'$ is needed to ensure that the estimate $\bar{\Sigma}_{n,\tau_1}$ of the true covariance matrix Σ is uniformly consistent. We conjecture that the restriction $\tau_2 - \tau_1 \geq \kappa$ may be discarded when using a much more sophisticated theoretical discussion. The main difficulty might be of technical nature; namely, one has to derive the result under *minimal moment assumptions* on the GARCH process. Therefore, in practice, κ should not be regarded as a tuning parameter, but can be chosen as small as possible $\big(\text{i.e., } \kappa = \frac{1}{L}\big)$.
- (*iii*) In practice, the choice of κ' is a trade-off between a good estimation of Σ in the test statistics and finding a break point near the boundary. One cannot expect from a test to detect a change if it had not seen enough 'normal' data before. Therefore, choosing $\kappa' = 0.1$ seems quite reasonable to us, but in principle, smaller values may be chosen.

As it follows from our discussion, the grid accuracy L should be investigated in more detail. We will consider this empirically in the simulation section, i.e., Appendix A in the Supplementary Material.

2.3. *Generalizations*

In this section, we discuss several directions of generalizations of our test statistics.

(a) **Testing for other directions of parameter changes.** It shall be noted that H does not have to be fixed as $H = (0, 1, 1)'$. Instead one can choose any direction of parameter change by choosing the corresponding $H \in \mathbb{R}^3$. For instance, it is possible to check separately for a change of α_0^* or α_1^* by choosing $H = (1, 0, 0)'$ or $H = (0, 1, 0)'$, respectively.

(b) **Testing for more than one direction.** Based on our weak convergence result Theorem 3.3 below, it is easily possible to elaborate tests which *simultaneously* check for more than one parameter change. To this end, we simply choose a full rank matrix $H \in \mathbb{R}^{3\times p}$, where $p \in \mathbb{N}$ corresponds to the number of different directions one wants to test, e.g., a simultaneous change in α_0 and α_1 would lead to $H = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}$. In this case, one simply has to modify the definition of $\hat{B}_n$ in (2.11) to

$$\hat{B}_n^{simult} := \sup_{(\tau_1,\tau_2)\in R_{\kappa,\kappa'}\cap\mathcal{G}^2} \max\left\{\hat{B}_n(\tau_1, \tau_2)_j : j = 1, \ldots, p\right\}.$$

Correspondingly, the critical value $\hat{q}_{W,1-\delta}^{simult}$ is defined based on i.i.d. $\varepsilon_i^{[k]} \sim N(0, I_{p\times p})$, $i = 1, \ldots, n$ and

$$\hat{\mu}_{n,k} := \sup_{(\tau_1,\tau_2)\in R_{\kappa,\kappa'}\cap\mathcal{G}^2} \max\left\{\frac{1}{\sqrt{n(\tau_2-\tau_1)}} \sum_{i=\lfloor n\tau_1\rfloor+1}^{\lfloor n\tau_2\rfloor} \varepsilon_{i,j}^{[k]} : j = 1, \ldots, p\right\}.$$

We can also adopt other types of test statistics. For example, for a two-sided test, one may consider the Euclidean norm $|\cdot|_2$ instead of taking $\max\{\cdot : j = 1, \ldots, p\}$.

(c) **Detection of multiple change intervals.** Our method can be directly modified to detect multiple change points. The basic idea is taken from Jeng et al. (2013). Let $T := \{1, \ldots, n\}$ denote the active training set. Let $m = 0$ denote the counter of changes, and $\tau_{1,n}^{(0)} = \tau_{2,n}^{(0)} = 0$. We shall repeat the following steps until T is empty:

   (1) Perform Step 0–Step 6 from Algorithm 1 as Subsection 2.2 based on the training set $X_i, i \in T$ (in particular, with $n = |T|$).

   (2) If no change was detected, stop. Otherwise, increase m by 1 and put $(\hat{\tau}_{1,n}^{(m)}, \hat{\tau}_{2,n}^{(m)}) \subset (0, 1)$ as the interval of change (with respect to the original observation interval (0,1) corresponding to (0,1)).

   (3) Delete the region with explosive behaviour from the training set; that is, update

$$T = T\backslash\{\lfloor n\hat{\tau}_{1,n}^{(m)}\rfloor + 1, \lfloor n\hat{\tau}_{2,n}^{(m)}\rfloor\}.$$

   If T is not empty, go back to Step 1. If T is empty, return the $\left\{\left(\hat{\tau}_{1,n}^{(k)}, \hat{\tau}_{2,n}^{(k)}\right) : k = 1, \ldots, m\right\}$.

   The collection of multiple selected intervals is then given by $\left\{\left(\hat{\tau}_{1,n}^{(k)}, \hat{\tau}_{2,n}^{(k)}\right) : k = 1, \ldots, m\right\}$.

(d) **Generalization to GARCH(r, s) models.** The theoretical results are developed for general GARCH(r, s) models. The test procedure can be used for GARCH(r, s) models presented in (1.1) and according parameter vector $\theta^* = (\alpha_0^*, \ldots, \alpha_r^*, \beta_1^*, \ldots, \beta_s^*)'$. The only change in the testing procedure is the adaptation of the likelihood function in (2.5) to the one for GARCH(r, s) models given in (3.2) below.

(e) **A general parameter constancy test.** When one would just like to test for the constancy of $H'\theta^*$ without setting a specific baseline value $c = H'\theta^*$, we can modify our test statistics $\hat{B}_n$ as follows:
We define

$$\hat{B}_n^{cp}(\tau_1, \tau_2) := \sqrt{n(\tau_2 - \tau_1)\frac{\tau_1}{\tau_2}}(H'\bar{\Sigma}_{n,\tau_1}H)^{-1/2}\{H'\hat{\theta}_{n,\tau_1,\tau_2} - H'\hat{\theta}_{n,0,\tau_1}\},$$

and

$$\hat{B}_n^{cp} := \sup_{(\tau_1,\tau_2)\in R_{\kappa,\kappa'}\cap\mathcal{G}^2} \hat{B}_n^{cp}(\tau_1, \tau_2).$$

Based on Theorem 3.4, the critical value $\hat{q}_{W,\delta}^{cp}$ needs to be redefined based on the distribution of $\hat{\mu}_{n,k}^{cp}$ as follows:
Let $\varepsilon_i^{[k]} \sim N(0, 1)$, $i = 1, \ldots, n$ and

$$\hat{\mu}_{n,k}^{cp} := \sup_{(\tau_1,\tau_2)\in R_{\kappa,\kappa'}\cap\mathcal{G}^2} \frac{1}{\sqrt{n\tau_2}}\left\{\sqrt{\frac{\tau_1}{\tau_2 - \tau_1}}\sum_{i=\lfloor n\tau_1\rfloor+1}^{\lfloor n\tau_2\rfloor}\varepsilon_i^{[k]} - \sqrt{\frac{\tau_2 - \tau_1}{\tau_1}}\sum_{i=1}^{\lfloor n\tau_1\rfloor}\varepsilon_i^{[k]}\right\}.$$

The theoretical properties of the statistics are briefly discussed in Subsection 3.4.

3. THEORETICAL RESULTS FOR GENERAL GARCH(r, s) MODEL

In this section we derive the theoretical properties of the GSRWW test and provide the necessary definition of the estimators in the general GARCH(r, s) model. We also formulate the GSRWW test and provide the necessary theoretical results in a general GARCH(r, s) model. For $r, s \in \mathbb{N}$, $\theta(i) = (\alpha_0(i), \alpha_1(i), \ldots, \alpha_r(i), \beta_1(i), \ldots, \beta_s(i))'$, we consider the GARCH($r$, s) model (1.1)

$$X_i^2 = \zeta_i^2\sigma_i^2,$$

$$\sigma_i^2 = \alpha_0(i) + \sum_{j=1}^{r}\alpha_j(i)X_{i-j}^2 + \sum_{k=1}^{s}\beta_k(i)\sigma_{i-k}^2.$$

Here, ζ_i are i.i.d. innovations with $\mathbb{E}\zeta_1 = 0$ and $\mathbb{E}\zeta_1^2 = 1$.

Recall our change point setting in (2.2), which is now defined with some fixed direction $H \in \mathbb{R}^{r+s+1}$ (instead of $H \in \mathbb{R}^3$ for GARCH(1,1)). Recall the hypotheses as in (2.4). We first analyse the model under the null hypothesis of constant parameters, i.e., $\theta(i) \equiv \theta^* = (\alpha_0^*, \alpha_1^*, \ldots, \alpha_r^*, \beta_1^*, \ldots, \beta_s^*)'$. Following Francq and Zakoïan (2004), we now present the set of assumptions to ensure the existence of a unique stationary solution to our model in (1.1). Define $f(\theta) = (\alpha_1, \ldots, \alpha_r, \beta_1, \ldots, \beta_s)'$ and let $e_j = (0, \ldots, 0, 1, 0, \ldots, 0)' \in \mathbb{R}^{r+s}$ be the unit column vector with the jth element being 1, $1 \leq j \leq r + s$. Define the $(r + s) \times (r + s)$-matrix as

$$A_i(\theta) = (f(\theta)\zeta_i^2, e_1, \ldots, e_{r-1}, f(\theta), e_{r+1}, \ldots, e_{r+s-1})'.$$

Recall that $|A|_2$ is the spectral norm of a quadratic matrix A. Define the top Lyapunov exponent of $A_i(\theta)$ as

$$\gamma(\theta) := \lim_{i\to\infty} \frac{1}{i} \log |A_i(\theta)A_{i-1}(\theta)\ldots A_1(\theta)|_2.$$

This exists if $\mathbb{E}|\zeta_0|^a < \infty$ for some $a > 0$ (cf. Francq and Zakoïan, 2004).

ASSUMPTION 3.1. *Suppose that*

(A1) ζ_0^2 *has a nondegenerate distribution with* $\mathbb{E}\zeta_0^2 = 1$.
(A2) Let $\alpha_{min} > 0$, *and*

$$\tilde{\Theta} = \left\{\theta \in \mathbb{R}_{\geq 0}^{r+s+1} : \alpha_0 \geq \alpha_{min}, \gamma(\theta) < 0 \; a.s., \sum_{j=1}^{s} \beta_j < 1\right\}. \tag{3.1}$$

Let $\Theta \subset \tilde{\Theta}$ *be compact. Assume that* $\theta^* \in int(\Theta)$.
(A3) Let $\mathcal{A}_\theta(z) := \sum_{i=1}^{r} \alpha_i z^i$, $\mathcal{B}_\theta(z) := 1 - \sum_{j=1}^{s} \beta_j z^j$. *If* $s > 0$, $\mathcal{A}_{\theta^*}(z)$ *and* $\mathcal{B}_{\theta^*}(z)$ *have no common root,* $\mathcal{A}_{\theta^*}(1) \neq 0$ *and* $\alpha_r^* + \beta_s^* \neq 0$.

Condition (A2), i.e., $\gamma(\theta^*) < 0$, guarantees the strict stationarity of the GARCH process. Note that this includes parameter values corresponding to IGARCH or mildly explosive GARCH with $\sum_j \alpha_j^* + \sum_k \beta_k^* > 1$. From Francq and Zakoïan (2004) and Proposition B.1 in the online Appendix B, we see that Assumption 3.1 implies existence of a solution of (1.1) which has geometric decay of dependence.

3.1. QMLE in $GARCH(r, s)$ and its consistency under null and alternative

In this subsection, we describe the QMLE, and formulate a theorem that yields its uniform consistency under the null and the alternative. Since we are providing the theory for GARCH(r, s) models, we now have to define the corresponding more general likelihood function involved in the estimation procedure. For estimation of $\theta^* \in \Theta$, we consider the following QML approach. We denote by $Y_i^c := (X_{i-1}^2, X_{i-2}^2, \ldots, X_1^2, 0, 0, ...)$ the observed data until time $i - 1$. For $0 \leq \tau_1 < \tau_2 \leq 1$,

$$L_{n,\tau_1,\tau_2}^c(\theta) := \frac{1}{n} \sum_{i=\lfloor n\tau_1\rfloor+1}^{\lfloor n\tau_2\rfloor} \ell(X_i^2, Y_i^c, \theta),$$

where

$$\ell(x, y, \theta) := \frac{1}{2}\Big(\frac{x}{\sigma^2(y, \theta)} + \log \sigma^2(y, \theta)\Big) \tag{3.2}$$

and $\sigma^2(y, \theta)$ follows the recursion

$$\sigma^2(y, \theta) = \alpha_0 + \sum_{j=1}^{r} \alpha_j y_j + \sum_{k=1}^{s} \beta_k \sigma^2((y_{k+1}, y_{k+2}, ...), \theta). \tag{3.3}$$

The analytic definition of the recursion of $\sigma^2(y, \theta)$ is formulated in a forward way (using $y_1, y_2, ...$ instead of $y_{-1}, y_{-2}, ...$) because we plug in $y = Y_i^c$, which is formulated in a backward way, leading to the usual quasi-likelihood approach for GARCH models. Note that $\sigma^2(Y_i^c, \theta)$ in (3.3) terminates after a finite number of steps due to zeros in Y_i^c. Morevoer, instead of using the truncated version $Y_i^c = (X_{i-1}^2, X_{i-2}^2, \ldots, X_1^2, 0, \ldots, 0)$ which corresponds to assuming that all initial values $X_0^2 = X_{-1}^2 = \ldots = 0$, one can also use different initial values like $X_0^2 = X_{-1}^2 = \ldots = \alpha_0$ or $X_0^2 = X_{-1}^2 = \ldots = X_1^2$ as investigated in Francq and Zakoïan (2004). For a discussion of different initial values, consider Bougerol and Picard (1992a) (in the case of strict stationarity).

Let $\sigma_t^2 = \alpha_0/\big(1 - \sum_{k=1}^s \beta_k\big) + \sum_{j=1}^r \alpha_j X_{t-j}^2 + \sum_{j=1}^r \alpha_j \sum_{k=1}^\infty \sum_{j_1=1}^s \sum_{j_2=1}^s \cdots \sum_{j_k=1}^s \beta_{j_1}\beta_{j_2} \cdots \beta_{j_k} X_{t-i-j_1-\ldots-j_k}^2$. With the defined likelihood function, for $0 \le \tau_1 < \tau_2 \le 1$, an estimator $\hat{\theta}_{n,\tau_1,\tau_2}$ of θ in the observation interval $i = \lfloor n\tau_1 \rfloor + 1, \ldots, \lfloor n\tau_2 \rfloor$ is obtained as in (2.6) via

$$\hat{\theta}_{n,\tau_1,\tau_2} := \mathrm{argmin}_{\theta\in\Theta} L_{n,\tau_1,\tau_2}^c(\theta). \tag{3.4}$$

With these definitions, we obtain the following uniform consistency under the null hypothesis of no parameter change.

THEOREM 3.1 (UNIFORM CONSISTENCY OF $\hat{\theta}_{n,\tau_1,\tau_2}$ *under the null* H_0). *Let Assumption 3.1 and H_0 hold. Then for each $\kappa > 0$,*

$$\sup_{0\le\tau_1<\tau_2\le 1, |\tau_1-\tau_2|\ge\kappa} |\hat{\theta}_{n,\tau_1,\tau_2} - \theta^*|_1 \xrightarrow{p} 0.$$

Additionally, we have the following result under the alternative.

PROPOSITION 3.1 (CONVERGENCE OF THE STATISTICS UNDER THE ALTERNATIVE H1) *Let Assumption 3.1 and H_1 hold, where $\theta^* + H\Delta^* \in int(\Theta)$. Then,*

$$|\hat{\theta}_{n,\tau_1^*,\tau_2^*} - (\theta^* + H\Delta^*)|_1 \xrightarrow{p} 0.$$

Note that this result is important and remarkable in the following sense: Even though the whole past of the process is used in the calculation of $L_{n,\tau_1,\tau_2}^c(\theta)$ (in particular realizations with $i \le \lfloor n\tau_1^* \rfloor$, which follow a model with parameters θ^* instead of $\theta^* + H\Delta^*$), $\hat{\theta}_{n,\tau_1^*,\tau_2^*}$ converges to the value $\theta^* + H\Delta^*$ in the alternative. The reason is that the past values only have a small impact on the whole likelihood due to the geometric decay of the coefficients in ℓ, cf. Lemma B.3 in online Appendix B.

3.2. Limiting distribution of the test statistics

Given the consistency of our QMLE in a GARCH(r, s) model, we provide a distribution theorem for $\hat{\theta}_{n,\tau_1,\tau_2}$ that allows us to obtain critical values for the uniform test defined in Section 2 and more general tests. Recall the change point setting (2.2) with $H \in \mathbb{R}^{r+s+1}$. We analyse the limiting distribution of the test statistics given in (2.11) and (2.9) under the null @ H_0 of no parameter change. Recall

$$\hat{B}_n = \sup_{(\tau_1,\tau_2)\in R_{\kappa,\kappa'}} \hat{B}_n(\tau_1, \tau_2),$$

where

$$\hat{B}_n(\tau_1, \tau_2) = \sqrt{n(\tau_2 - \tau_1)}(H'\bar{\Sigma}_{n,\tau_1} H)^{-1/2}\{H'\hat{\theta}_{n,\tau_1,\tau_2} - H'\theta^*\}.$$

First, we approximate the difference $\hat{\theta}_{n,\tau_1,\tau_2} - \theta^*$ by a simple linear form uniformly in τ_1, τ_2. The following type of theoretical result is also known as a weak Bahadur representation of $\hat{\theta}_{n,\tau_1,\tau_2}$.

For $\kappa \in (0,1)$, we define

$$R_\kappa := \{(\tau_1, \tau_2) \in [0,1]^2 : \tau_1 < \tau_2, |\tau_1 - \tau_2| \geq \kappa\}.$$

THEOREM 3.2 (WEAK BAHADUR REPRESENTATION). *Let Assumption 3.1 and H_0 hold. Assume that for some $a > 0$, $\mathbb{E}|\zeta_0|^{4+a} < \infty$. Then for each $\kappa > 0$,*

$$\sup_{(\tau_1,\tau_2)\in R_\kappa} \left|\{\hat{\theta}_{n,\tau_1,\tau_2} - \theta^*\} + ((\tau_2 - \tau_1)V(\theta^*))^{-1} \cdot \nabla_\theta L_{n,\tau_1,\tau_2}(\theta^*)\right| = O_p(\log(n)^3 n^{-1}),$$

where $L_{n,\tau_1,\tau_2}(\theta) := \frac{1}{n}\sum_{i=\lfloor n\tau_1\rfloor+1}^{\lfloor n\tau_2\rfloor} \ell(X_i^2, Y_i, \theta)$.

This linearization result allows to transfer the properties of the sum $\nabla_\theta L_{n,\tau_1,\tau_2}$ to the difference $\hat{\theta}_{n,\tau_1,\tau_2} - \theta^*$; especially we obtain a limit distribution of $\hat{\theta}_{n,\tau_1,\tau_2}$ uniformly in $(\tau_1,\tau_2) \in R_\kappa$ under H_0 by using Gaussian approximation results from Wu and Zhou (2011). The functional limit distribution then naturally implies the pointwise convergence results from Francq and Zakoïan (2004) and it is much stronger, as it can be used as a starting point to apply theorems from empirical process theory (such as the continuous mapping theorem). Let $\ell^\infty(T)$ denote the space of bounded functions $f : T \to \mathbb{R}$, (cf. Van der Vaart, 1998, Section 18, Example 18.5). As a direct consequence of the uniform Bahadur representation, we can derive the distribution of the difference of the estimator and the true value θ^* under the null.

THEOREM 3.3 (ASYMPTOTIC DISTRIBUTION OF THE TEST STATISTICS) *Suppose that Assumption 3.1 and H_0 holds. Suppose that there exists $a' > 0$ such that $\mathbb{E}|\zeta_0|^{4+a'} < \infty$. Fix $\kappa > 0$ and suppose that H_0 is true. Then on $\ell^\infty(R_\kappa)^{r+s+1}$,*

$$\sqrt{n(\tau_2 - \tau_1)}\left\{\hat{\theta}_{n,\tau_1,\tau_2} - \theta^*\right\} \xrightarrow{d} \Sigma^{1/2}\left\{\frac{B(\tau_2) - B(\tau_1)}{\sqrt{\tau_2 - \tau_1}}\right\},$$

where $B(\cdot)$ is a standard $(r+s+1)$-dimensional Brownian motion, and

$$\Sigma = V(\theta^*)^{-1} I(\theta^*) V(\theta^*)^{-1}$$

is from (2.7). where $\mu_4 := \mathbb{E}\zeta_0^4$.

As a direct corollary, we obtain with the continuous mapping theorem the limit distribution of $\hat{B}_n$ with a known covariance matrix Σ. To obtain the critical values of our test, we need to derive quantiles for the test statistics $\hat{B}_n$, which can be inferred by its limit distribution.

COROLLARY 3.1. *Suppose that Assumption 3.1 and H_0 holds. Suppose that there exists $a' > 0$ such that $\mathbb{E}|\zeta_0|^{4+a'} < \infty$. Fix $\kappa > 0$. Let $H \in \mathbb{R}^{r+s+1}$. Let $\Sigma_H := H'\Sigma H$. Then,*

$$\sup_{(\tau_1,\tau_2)\in R_\kappa} \sqrt{n(\tau_2 - \tau_1)}\Sigma_H^{-1/2}\left\{H'\hat{\theta}_{n,\tau_1,\tau_2} - H'\theta^*\right\}$$

$$\xrightarrow{d} \sup_{(\tau_1,\tau_2)\in R_\kappa} \left\{\frac{B(\tau_2) - B(\tau_1)}{\sqrt{\tau_2 - \tau_1}}\right\},$$

where $B(\cdot)$ is a standard 1-dimensional Brownian motion.

REMARK 3.1. Note that this result could easily be generalized to $H \in \mathbb{R}^{(r+s+1)\times d}$, which allows us to detect more than one deviation from a 'stable' state, as remarked in Section 2.3.

If a process behaves mildly explosive in the second moment and has constant parameters in all time periods, our theoretical results are still valid. The reason is that this case belongs to the null hypothesis, as we only restrict θ^* to lie in the parameter region of strict stationarity.

REMARK 3.2. It is worth noting that, different from Francq and Zakoïan (2004), we have a slightly stronger moment assumption, i.e., $\mathbb{E}|\zeta_0|^{4+a'} < \infty$. The reason is that proving a uniform convergence as in (3.1) typically needs a high-level Bahadur-type approximation result which incorporates uniform approximation of the likelihood towards its expectation. To do so, we use concentration inequalities from Zhang and Wu (2017), and Gaussian approximation results from Wu and Zhou (2011), which need the summands to have a little more than two moments. Here, the likelihood of having more than two moments corresponds to ζ_0 having more than four moments.

Since Σ_H is unknown in practice, we next discuss an analogue of Corollary 3.1 where Σ_H is replaced by a consistent estimator.

3.3. *Estimation of Σ and statistical properties with the estimated Σ.*

In this subsection we show that the proposed estimator $\bar{\Sigma}_{n,\tau_1}$ of Σ in (2.8) is a uniformly consistent estimator for Σ. In addition, we show that the limit distribution of the test statistics remains the same with the plugged variance-covariance estimator. It is well known that the following alternative representation holds (cf. Proposition B.2 in online Appendix B):

$$\begin{aligned}\Sigma &= V(\theta^*)^{-1} I(\theta^*) V(\theta^*)^{-1} \\ &= \frac{\mu_4 - 1}{2} \cdot V(\theta^*)^{-1}.\end{aligned} \tag{3.5}$$

However, here we restrict ourselves to the estimation of Σ via the representation of (3.5) to avoid estimating μ_4 separately. To coincide with a typical change-point test and to obtain a high power, $\bar{\Sigma}_{n,\tau_1}$ in (2.8) was defined only with observations from the null hypothesis $i = 1, \ldots, \lfloor n\tau_1 \rfloor$. We have the following result.

PROPOSITION 3.2 (UNIFORM CONSISTENCY OF THE COVARIANCE ESTIMATOR) *Suppose that Assumption 3.1 and H_0 holds. Suppose that there exists $a' > 0$ such that $\mathbb{E}|\zeta_0|^{4+a'} < \infty$. Fix $\kappa' > 0$. Then:*

(i) $\sup_{\tau_1 \geq \kappa'} |\bar{V}_{n,\tau_1}(\hat{\theta}_{n,0,\tau_1}) - V(\theta^*)|_1 \xrightarrow{p} 0$.

(ii) If additionally $\mathbb{E}|\zeta_0|^{8+a'} < \infty$, $\sup_{\tau_1 \geq \kappa'} |\bar{I}_n(\hat{\theta}_{n,0,\tau_1}) - I(\theta^*)| \xrightarrow{p} 0$, and

$$\sup_{\tau_1 \geq \kappa'} \left|\bar{\Sigma}_{n,\tau_1} - \Sigma\right|_2 \xrightarrow{p} 0.$$

As a corollary of Theorem 3.3 and Proposition 3.2, we now obtain the limit distribution of $\hat{B}_n$ with Slutzky's lemma. Recall $R_{\kappa,\kappa'}$ from (2.10).

COROLLARY 3.2. *Suppose that Assumption 3.1 and H_0 holds. Suppose that there exists $a' > 0$ such that $\mathbb{E}|\zeta_0|^{8+a'} < \infty$. Let $H \in \mathbb{R}^{r+s+1}$. Then,*

$$\begin{aligned}\hat{B}_n &= \sup_{(\tau_1,\tau_2)\in R_{\kappa,\kappa'}} \sqrt{n(\tau_2 - \tau_1)}(H'\bar{\Sigma}_{n,\tau_1} H)^{-1/2}\left\{H'\hat{\theta}_{n,\tau_1,\tau_2} - H'\theta^*\right\} \\ &\xrightarrow{d} \sup_{(\tau_1,\tau_2)\in R_{\kappa,\kappa'}} \left\{\frac{B(\tau_2) - B(\tau_1)}{\sqrt{\tau_2 - \tau_1}}\right\} =: W,\end{aligned}$$

where $B(\cdot)$ is a standard 1-dimensional Brownian motion.

REMARK 3.3 (MODIFICATION OF THE TEST AND THE HYPOTHESES).

(i) Let $q_{W,\delta}$ denote the $(1-\delta)$ quantile of W. Then $\mathbb{1}_{\{\hat{B}_n > q_{W,\delta}\}}$ is also a level δ test for the extended hypotheses

$$H_0 : \Delta^* \leq 0 \quad vs. \quad H_1 : \Delta^* > 0. \tag{3.6}$$

The reason being that $\Delta^* < 0$ in connection with the uniform consistency of Theorem 3.1 only produces smaller values of the test statistics $\hat{B}_n$.

(ii) For any fixed $\theta^* \in \text{int}(\Theta)$, the power function $\beta(\Delta^*) := \mathbb{P}_{\Delta^*}(\hat{B}_n > q_{W,\delta})$ is continuous around $\Delta^* = 0$ since the process X_i from (1.1) depends continuously on Δ^* through $\theta^*(i)$. Therefore, $\mathbb{1}_{\{\hat{B}_n > q_{W,\delta}\}}$ is also a level δ test for

$$H_0 : \Delta^* < 0 \quad vs. \quad H_1 : \Delta^* \geq 0. \tag{3.7}$$

Intuitively, one can argue that (3.7) is nearly the same as testing

$$H_0 : \Delta^* \leq \epsilon$$

with some arbitrarily small $\epsilon > 0$, which again is nearly the same as testing (3.6).

If the significance of $\hat{B}_n$ is detected, τ_1^*, τ_2^* can be estimated by the choice

$$(\hat{\tau}_{1,n}, \hat{\tau}_{2,n}) \in \text{argmax}_{(\tau_1,\tau_2)\in R_{\kappa,\kappa'}} \hat{B}_n(\tau_1, \tau_2).$$

This result shows that the test provided in Algorithm 1 in Section 2 is a test with asymptotic size δ. Based on the consistency result from Proposition 3.1, we obtain that the test based on Algorithm 1 also has asymptotic power 1, which is shown in the following corollary.

COROLLARY 3.3. *Let Assumption 3.1 and H_1 hold, where $\theta^* + H\Delta^* \in int(\Theta)$ and $(\tau_1^*, \tau_2^*) \in R_{\kappa,\kappa'}$. Then,*

$$\hat{B}_n \xrightarrow{p} \infty.$$

REMARK 3.4. We conjecture that this result can be extended even to nonstationary alternatives where $\theta^* + H\Delta^* \notin \Theta$ as long as $H'(1, 0, \ldots, 0) = 0$. The reason for this restriction is that Francq and Zakoïan (2012) discovered that one cannot expect $\hat{\alpha}_0$ to be consistently estimated in the nonstationary regime.

3.4. Theoretical results for a general parameter constancy test

Finally, we provide theoretical results for the generalization case of (d) from Section 2.3. Namely, recall that

$$\hat{B}_n^{cp} = \sup_{(\tau_1,\tau_2)\in R_{\kappa,\kappa'}} \hat{B}_n^{cp}(\tau_1, \tau_2),$$

where

$$\hat{B}_n^{cp}(\tau_1, \tau_2) = \sqrt{n(\tau_2 - \tau_1)\frac{\tau_1}{\tau_2}}(H'\bar{\Sigma}_{n,\tau_1} H)^{-1/2}\big\{H'\hat{\theta}_{n,\tau_1,\tau_2} - H'\hat{\theta}_{n,0,\tau_1}\big\}.$$

In opposite to $\hat{B}_n$, we do not focus on testing a particular c. The normalization with τ_1, τ_2 in $\hat{B}_n^{cp}(\tau_1, \tau_2)$ is chosen such that its limit distribution for fixed τ_1, τ_2 has an $N(0, 1)$ distribution. In the following, the theorem ensures the asymptotic performance of the generalized test statistics under the null and alternative hypotheses.

THEOREM 3.4. *Suppose that Assumption 3.1 holds. Suppose that there exists $a' > 0$ such that $\mathbb{E}|\zeta_0|^{8+a'} < \infty$. Let $H \in \mathbb{R}^{r+s+1}$. Then under H_0,*

$$\hat{B}_n^{cp} \xrightarrow{d} \sup_{(\tau_1,\tau_2)\in R_{\kappa,\kappa'}} \frac{1}{\sqrt{\tau_2}} \left\{ \sqrt{\frac{\tau_1}{\tau_2-\tau_1}} \{B(\tau_2) - B(\tau_1)\} - \sqrt{\frac{\tau_2-\tau_1}{\tau_1}} B(\tau_1) \right\},$$

where $B(\cdot)$ is a standard 1-dimensional Brownian motion. If instead H_1 holds with $\theta^* + H\Delta^* \in$ int(Θ) and $(\tau_1^*, \tau_2^*) \in R_{\kappa,\kappa'}$, then

$$\hat{B}_n^{cp} \xrightarrow{p} \infty.$$

4. REAL DATA APPLICATION

In this section, we apply our test to real data. We first consider two commonly used financial risk indicators. One is the VIX, and the other is the Treasury-EuroDollar (TED) rate spread. The VIX is a weighted combination of prices for a range of options on the S&P 500 index, which reflects the market expectation of the volatility level. The TED spread is the difference between the 3-Month London Interbank Offered Rate (LIBOR) based on US dollars and the 3-Month Treasury Bill, which typically measures the liquidity among the inter-bank money market. The VIX is available from Yahoo Finance (2019), and the TED spread (Federal Reserve Economic Data, 2019) is downloaded from the following address: https://fred.stlouisfed.org/series/TEDRATE. We adopt a daily frequency for the TED (the VIX) for the time span 01/07/2004–09/05/2018 (05/01/2004–05/09/2018).

The VIX is often regarded as a measure of the market fear of stock investors, which is related to the cost of purchasing insurance against market downturns. We usually see that the VIX will be high in a bearish market and low in a bullish market. The TED spread represents the credit risk in the general economy. It signals how banks are willing to lend to each other, which is related to the liquidity of the markets. A high level of TED spread is a sign of low liquidity and high risk of default on inter-bank loans.

Both the VIX and the TED spread are often considered early-warning indicators of market stress. Namely, when market uncertainty is high, a temporary shock to the financial system leads to increased default or otherwise adverse effect to the global financial market; see, for example, as described in González-Hermosillo and Hesse (2011). Abrupt changes of the parameter values of the underlying processes are likely to be associated with this type of sudden changes of market conditions. The goal of our analysis is to discover the existence of periods of unstable behaviour of the underlying volatility process. This can be helpful to decide if a government should perform an intervention based on the estimated underlying parameter values. Figure 4 shows a plot of the following adjusted series: the TED spread Y_i, the log returns $L_i := \log\left(\frac{Y_i}{Y_{i-1}}\right)$, and absolute log returns of the TED spread. From the plot, we observe that the returns fluctuate around the zero. During the years 2008–2009, there is a period of high volatility. We divide the data into a sequence of consecutive windows of 1,000 days each. The log returns L_i are stationary in all the windows (suggested by the ADF test) and serial correlation is taken out by fitting an ARMA(p, q)

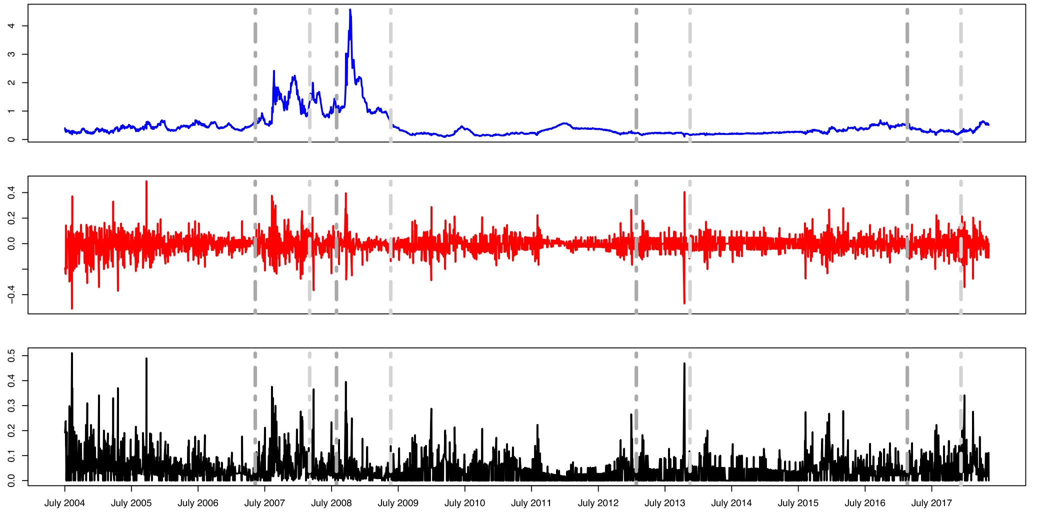

Figure 4. The plot of TED spread in percentage (upper panel), the log difference of TED spread (middle panel) and the absolute value of TED spread (lower panel). The dates of change are marked with grey lines. (Starting line: dark grey, ending line: light grey.).

Table 1. The detected significant break periods for the TED spread, the corresponding persistence parameter ($\hat{\alpha}_1 + \hat{\beta}_1$) and the test statistics; (***) means significant at both 0.95,0.90. Parameter estimation for $\hat{\alpha}_1$, $\hat{\beta}_1$ in brackets. The null hypothesis is $\alpha_1 + \beta_1 = 0.95$.

	$\hat{\tau}_1$	$\hat{\tau}_2$	In	Out	Test statistics
1	2007-05-09	2008-02-29	1.02 (0.14, 0.88)	0.95 (0.08, 0.87)	4.22(***)
2	2008-07-25	2009-05-20	1.08 (0.44, 0.64)	0.95 (0.05, 0.90)	19.74(***)
3	2013-01-24	2013-11-14	1.05 (0.28, 0.77)	0.95 (0.04, 0.91)	7.41(***)
4	2017-02-16	2017-12-07	1.00 (0.05, 0.95)	0.95 (0.04, 0.91)	3.32(***)

process of the form

$$L_i = \sum_{j=1}^{p} \alpha_j L_{i-j} + \sum_{k=1}^{q} \beta_k \varepsilon_{i-k} + \varepsilon_i$$

in advance and the following analysis is done on the estimated residuals after the QMLE fitting,

$$\hat{X}_i := \hat{\varepsilon}_i.$$

From the histogram and Q-Q plot of the time series in Figure 5, we observe a strong evidence of leptokurtic behaviour. We follow the suggestions as in Section 2 for the choices of tuning parameters, and grid size L is chosen to be $L = 100$ throughout this section. Figure 6 shows the moving window fitting results. In Table 1 we present the detected periods of the mildly explosive behaviour. We have adopted our tested with extension to a multiple change-point algorithm as discussed in Section 2.3 (c). Therefore, multiple significant periods of change can be detected. The GSRWW test identifies the major financial crises such as the US subprime mortgage crisis

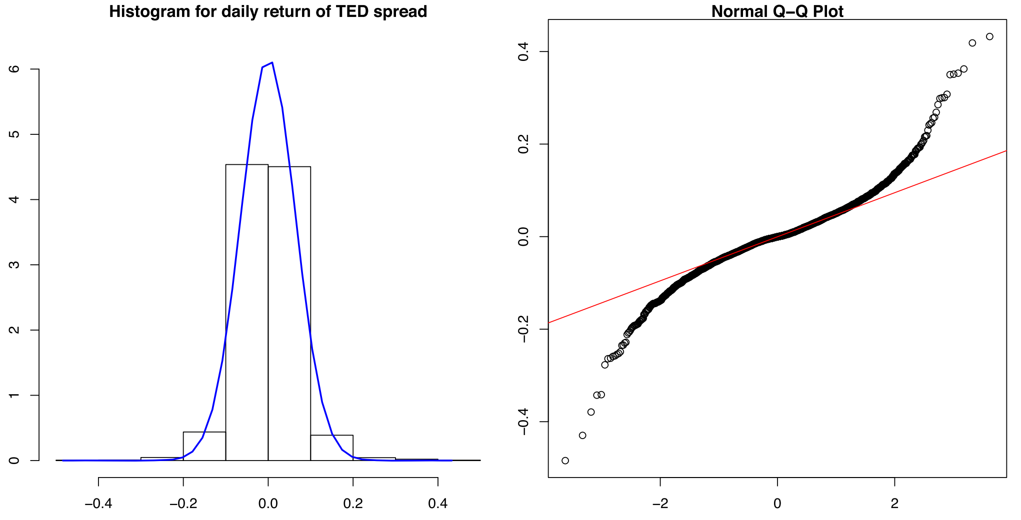

Figure 5. Q-Q plot and the histogram for the daily TED spread.

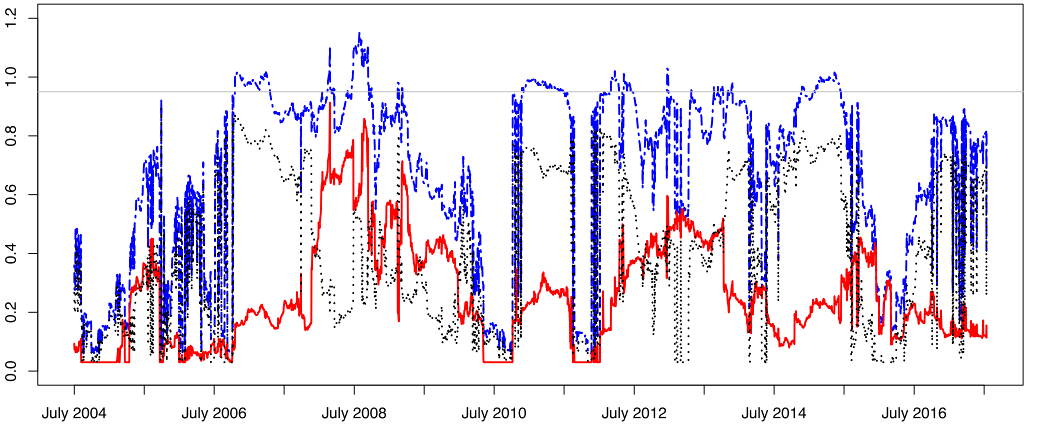

Figure 6. A plot of estimated GARCH(1,1) parameters using the TED data over a rolling window of size 200. $\hat{\alpha}_1 + \hat{\beta}_1$ estimate persistence parameter (blue dash line), $\hat{\alpha}_1$ (solid line), $\hat{\beta}_1$ (dotted line), threshold ($\alpha_1 + \beta_1 = 0.95$).

as early as May 2007, and lasts until February 2008. Furthermore, the test can detect some short-lived instability early; such as, in October 2013 the TED spread dropped due to the worries of a potential default on the US debt.

The corresponding time series of the VIX is plotted in Figure 7. We observe that the index value increases sharply during the subprime crisis. A similar leptokurtic behaviour of the series can be found in Figure 8. Figure 9 presents the moving window parameter fitting results. We cannot detect any significant intervals against the null hypothesis of $H_0 : \alpha_1 + \beta_1 = 1$. Instead we fit a global model first using the whole sample and test against the fitted value of the global

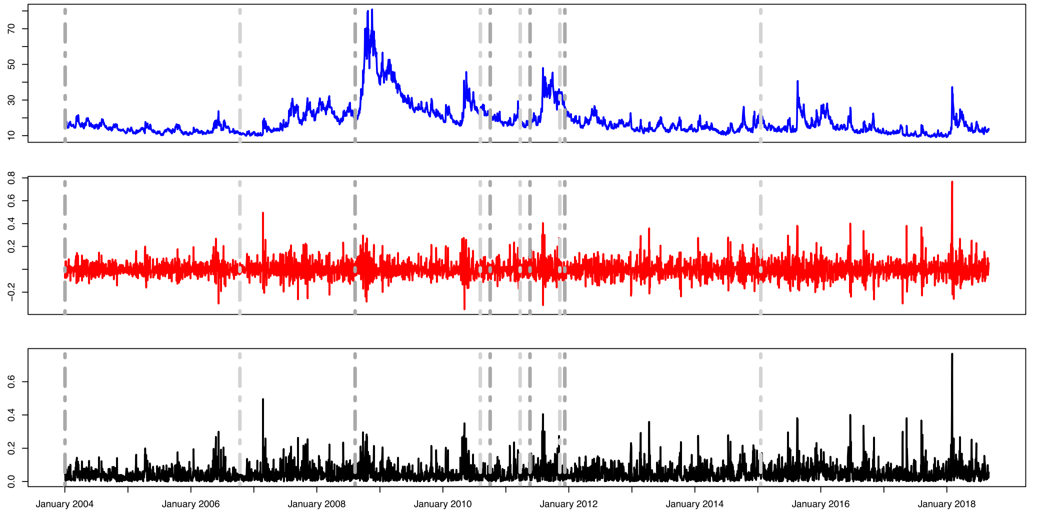

Figure 7. The plot of VIX (upper panel), the log difference of VIX (middle panel), and the absolute value of VIX (lower panel). The dates of change are marked with grey lines. (Starting line: dark grey, ending line: light grey.).

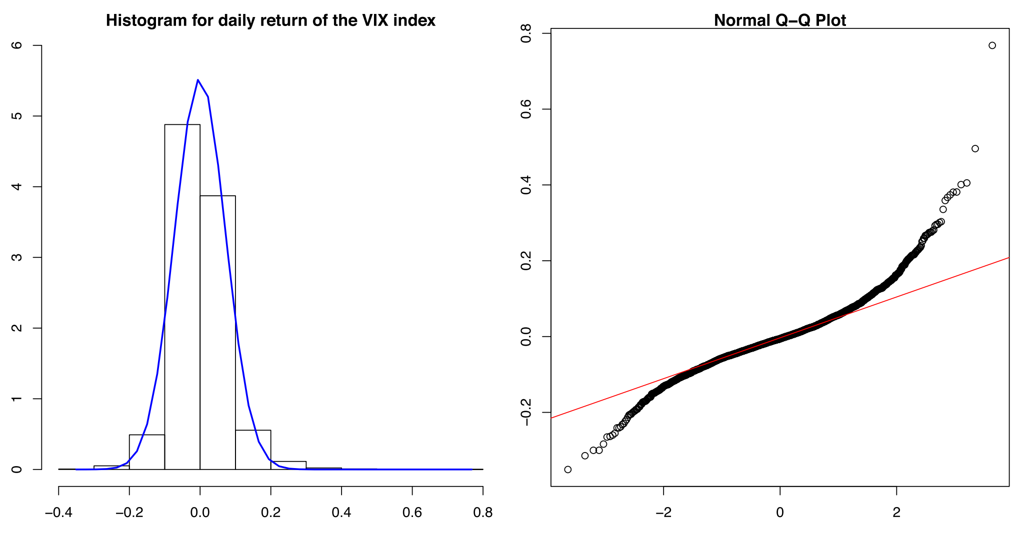

Figure 8. Q-Q plot and the histogram for the daily VIX index.

model, i.e., $\alpha_1 + \beta_1 = c := 0.95$. We have detected five intervals of change points, as listed in Table 2. In particular, the period end in October 2006 signifies the early warning of the subprime mortgage crisis. The period starting on $2011 - 05 - 24$ corresponds to the Euro debt crisis. In sum, our test can pick up the critical periods of financial crises early for both the VIX and the

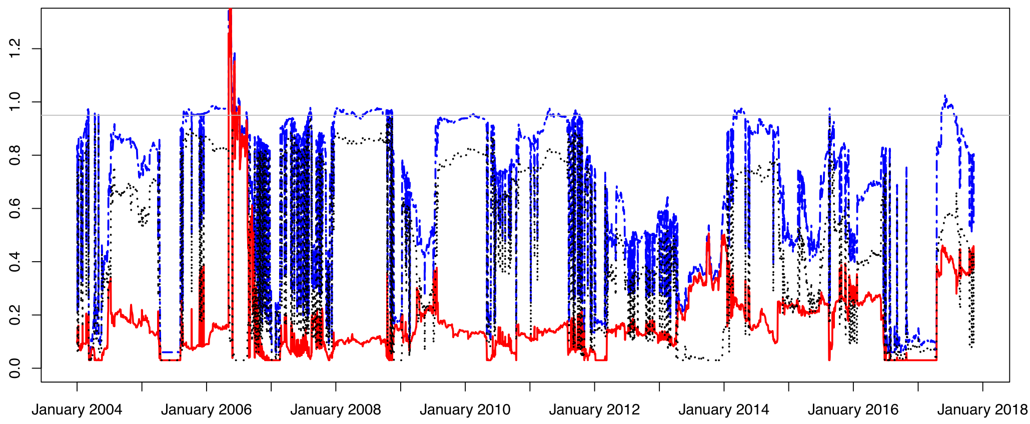

Figure 9. A plot of estimated GARCH(1,1) parameters using the VIX data over a rolling window of size 200. $\hat{\alpha}_1 + \hat{\beta}_1$ estimate persistence parameter (dash line), $\hat{\alpha}_1$ (solid line), $\hat{\beta}_1$ (dotted line), threshold $(\alpha_1 + \beta_1 = 0.95)$.

Table 2. The detected significant break periods for the VIX. The corresponding persistence parameter $(\hat{\alpha}_1 + \hat{\beta}_1)$ and the test statistics. The null hypothesis is $\alpha_1 + \beta_1 = 0.95$; $(^{***})$ means significant at both 0.95,0.90. Parameter estimation for $\hat{\alpha}_1$, $\hat{\beta}_1$ are in brackets.

	$\hat{\tau}_1$	$\hat{\tau}_2$	In	Out	Test statistics
1	2004-01-05	2006-10-13	1.01 (0.11, 0.89)	0.95 (0.05, 0.90)	10.94$(^{***})$
2	2010-10-05	2011-03-28	1.33 (0.82, 0.52)	0.95 (0.11, 0.84)	14.11$(^{***})$
3	2008-08-13	2010-08-09	1.00 (0.12, 0.88)	0.95 (0.05, 0.90)	7.72$(^{***})$
4	2011-05-24	2011-11-11	1.04 (0.23, 0.81)	0.95 (0.08, 0.87)	4.20$(^{***})$
5	2011-12-12	2015-01-21	1.01 (0.15, 0.86)	0.95 (0.05,0.90)	10.37$(^{***})$

TED spread. Besides, it can also successfully signify small periods of turbulence in the volatility processes of the the early-warning indicators.

Next, we test our methodology on the recent emerging Fintech markets. We gather the Bitcoin price series from 19 July 2010 to 05 November 2018 at a daily frequency. The data (CoinMarketCap, 2019) source is https://coinmarketcap.com/currencies/bitcoin/historical-data/. We show the returns and the absolute returns for the Bitcoin price series in Figure 10. We can see that there are several high-volatility periods. The volatility level is higher before 2013 followed by a stable period. Recently, the market volatility increased. The Q-Q plots and histograms in Figure 11 indicate the heavy-tailedness of the underlying distribution. We present the test results with consecutive windows of 1,000 days in Table 3. Again the log returns are stationary in all the windows (by results of ADF tests) and serial correlation is taken out by fitting an ARMA process in advance. We apply our test to the obtained residuals, which indicates the presence of multiple market 'euphoria' episodes in the series. The GSSWW identifies the most significant high-volatility period, including the period covering the June 2016 crash, the crashes during summer 2017, and the fear of market regulation in October 2017. We have chosen Bitcoin as

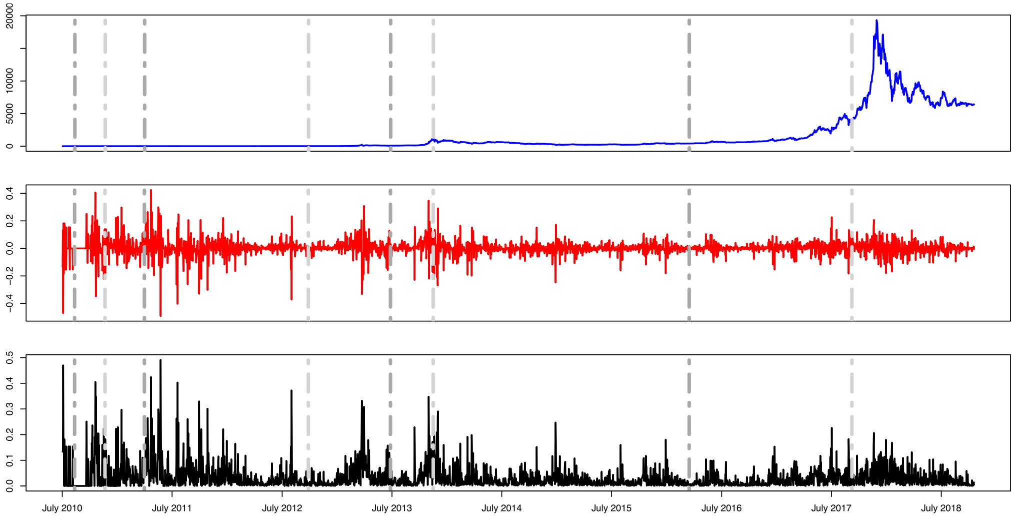

Figure 10. The plot of Bitcoin price (upper panel), the log difference of Bitcoin (middle panel) and the absolute returns (lower panel). The dates of change are marked with grey lines. (Starting line: dark grey, ending line: light grey.).

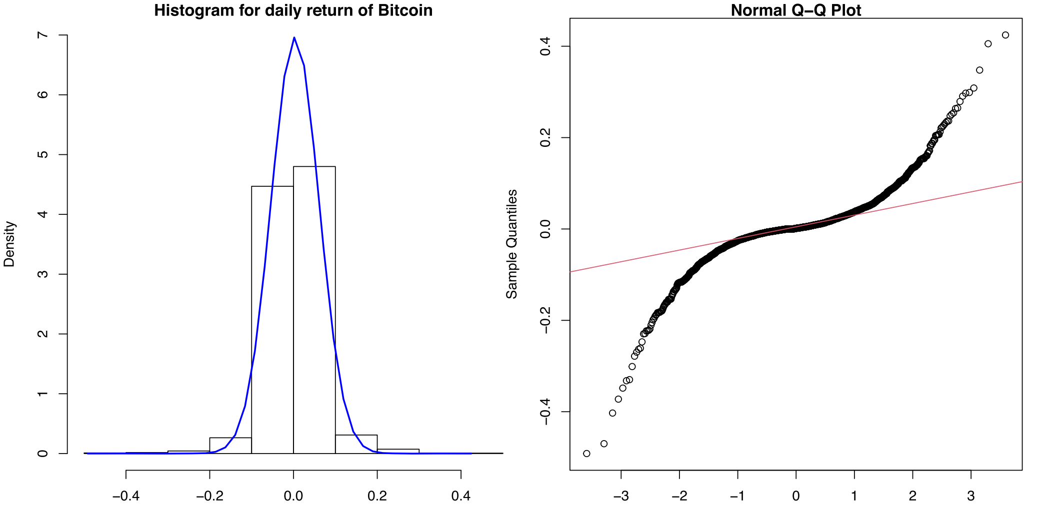

Figure 11. Q-Q plot and the histogram for the Bitcoin returns.

an important additional study, as the Fintech markets are known to behave independently with respect to the conventional financial market. Bitcoin is not controlled by any government, and speculators can use our test results for abnormal regimes of Bitcoin as indicators of the market sentiment.

Table 3. The detected significant break periods for the Bitcoin log returns, the corresponding persistence parameter $(\hat{\alpha}_1 + \hat{\beta}_1)$, and the test statistics. $(^{***})$ means significant at both 0.95,0.90. Parameter estimation for $\hat{\alpha}_1$, $\hat{\beta}_1$ in brackets. Testing the corresponding persistence parameter $(\alpha_1 + \beta_1 = 1)$.

	$\hat{\tau}_1$	$\hat{\tau}_2$	In	Out	Test statistics
1	2010-08-28	2010-12-07	1.33 (0.84, 0.49)	1.00 (0.37, 0.63)	3.92$(^{***})$
2	2011-04-17	2012-10-13	1.49 (1.02, 0.47)	1.00 (0.28, 0.72)	18.44$(^{***})$
3	2013-07-13	2013-12-02	1.71 (1.46, 0.25)	0.97 (0.19, 0.78)	16.11$(^{***})$
4	2016-04-01	2017-09-24	1.42 (1.01, 0.41)	0.96 (0.10, 0.86)	23.45$(^{***})$

5. CONCLUSION

In this paper, we propose a uniform test for a mildly explosive GARCH process with double-supreme statistics. Theoretical results about the uniform parameter consistency and asymptotic distribution of the test statistics are provided. Our test is easy to implement, and can help to effectively identify mildly explosive periods with good sizes and power. The quality of the test is discussed via a simulation study in the online Appendix. We applied our procedure to real data time series as the TED spread, the VIX, and the Bitcoin price series, and tracked their corresponding volatile periods. Further work may extend the algorithm to online procedures, allowing for real time detection of breaks.

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SUPPORTING INFORMATION

Additional Supporting Information may be found in the online version of this article at the publisher's website:

Online Appendix
Replication Package

Co-editor Dennis Kristensen handled this manuscript.