Internet Appendix

IA.A Simple Model

Under the physical probability measure $\mathbb{P}$, real cashflow growth is given by

$$\frac{dY_t}{Y_t} = \mu_Y dt + \sigma_Y dW_t, \tag{IA.1}$$

where W is a standard Brownian motion under $\mathbb{P}$. The date t nominal cashflow is given by $X_t = Y_t P_t^{\varphi}$, and so nominal cashflow growth is given by

$$\frac{dX_t}{X_t} = (\mu_Y + \varphi \mu_P)dt + \sigma_Y dW_t. \tag{IA.2}$$

The real SDF is given by

$$\frac{d\pi_t}{\pi_t} = -r dt - \Theta dZ_t, \tag{IA.3}$$

where Z is a standard Brownian motion under $\mathbb{P}$ such that $dZ_t dW_t = \rho dt$. In this section, there is no risk premium associated with sudden shifts in the state of the economy. In contrast with the main model, we therefore assume $\rho > 0$ to ensure that the risk premium is not zero. Consequently, conditional on date t information, the risk-neutral probability of event A occuring at date T is given by

$$E_t^{\mathbb{Q}}[1_A] = E_t\left[\frac{M_T}{M_t} 1_A\right], \tag{IA.4}$$

where M is an exponential martingale under $\mathbb{P}$, defined by

$$\frac{dM_t}{M_t} = -\Theta dZ_t,\ M_0 = 1. \tag{IA.5}$$

The exogenous price index is given by

$$P_t = P_0 e^{\mu_P t}, \tag{IA.6}$$

where μ_P is the constant inflation rate.

The nominal SDF is given by $\pi_t^{\$} = \pi_t / P_t$, and so

$$\frac{d\pi_t^{\$}}{\pi_t^{\$}} = -r^{\$} dt - \Theta dZ_t, \tag{IA.7}$$

where

$$r^{\$} = r + \mu_P. \tag{IA.8}$$

The price-index is not stochastic, so there is no inflation risk premium. We can therefore price risk under the risk-neutral measure $\mathbb{Q}$ with no additional adjustment for an inflation risk premium. If there were a risk premium, we would have to define a different probability measure in order to discount nominal cashflows with the nominal interest rate. Using Girsanov's Theorem, we obtain the evolution of X under $\mathbb{Q}$:

$$\frac{dX_t}{X_t} = (\widehat{\mu}_X + \varphi \mu_P)dt + \sigma_X dW_t^{\mathbb{Q}}. \tag{IA.9}$$

where $W^{\mathbb{Q}}$ is a standard Brownian motion under the risk-neutral measure $\mathbb{Q}$ and $\widehat{\mu}_X = \mu_X - \rho \sigma_X \Theta$ is the risk-neutral expected nominal cash flow growth rate and $\sigma_X = \sigma_Y$.

The date-t nominal after-tax abandonment value of the firm is given by

$$A_t^{\$} = A^{\$}(X_t) = (1-\eta)X_t E_t\left[\int_t^\infty \frac{\pi_u^{\$}}{\pi_t^{\$}}\frac{X_u}{X_t}\right] = (1-\eta)X_t E_t^{\mathbb{Q}}\left[\int_t^\infty e^{-r^{\$}(u-t)}\frac{X_u}{X_t}du\right], \tag{IA.10}$$

where η is the tax rate, which we set to zero in Section 1.

The date-t nominal price of the corporate bond is given by

$$B_t^{\$} = cE_t\left[\int_t^{\tau_D}\frac{\pi_u^{\$}}{\pi_t^{\$}}du\right] + \alpha E_t\left[\frac{\pi_{\tau_D}^{\$}}{\pi_t^{\$}}A^{\$}(X_{\tau_D})du\right] \tag{IA.11}$$

$$= cE_t\left[\int_t^{\tau_D} e^{-r^{\$}(u-t)}du\right] + \alpha E_t\left[e^{-r^{\$}(\tau_D-t)}A^{\$}(X_{\tau_D})du\right], \tag{IA.12}$$

where α is the recovery rate which we set to zero in Section 1. Simplifying the expression for the corporate bond price, we obtain

$$B_t^{\$} = cE_t^{\mathbb{Q}}\left[\int_t^\infty e^{-r^{\$}(u-t)}du - e^{-r^{\$}(\tau_D-t)}E_{\tau_D}^{\mathbb{Q}}\int_{\tau_D}^\infty e^{-r^{\$}(u-\tau_D)}du\right] + \alpha E_t^{\mathbb{Q}}\left[e^{-r^{\$}(\tau_D-t)}A^{\$}(X_{\tau_D})du\right] \tag{IA.13}$$

$$= cE_t^{\mathbb{Q}}\left[\int_t^\infty e^{-r^{\$}(u-t)}du - e^{-r^{\$}(\tau_D-t)}E_{\tau_D}^{\mathbb{Q}}\int_{\tau_D}^\infty e^{-r^{\$}(u-\tau_D)}du\right] + \alpha E_t^{\mathbb{Q}}\left[e^{-r^{\$}(\tau_D-t)}\right]A^{\$}(X_D) \tag{IA.14}$$

$$= \frac{c}{r^{\$}}E_t^{\mathbb{Q}}\left[1-e^{-r^{\$}(\tau_D-t)}\right] + \alpha E_t^{\mathbb{Q}}\left[e^{-r^{\$}(\tau_D-t)}\right]A^{\$}(X_D) \tag{IA.15}$$

since $\tau_D = \inf_{t\geq 0}\{X_t \leq X_D\}$. Therefore

$$B_t^{\$} = \frac{c}{r^{\$}}(1-q_{D,t}^{\$}) + \alpha A^{\$}(X_D)q_{D,t}^{\$}, \tag{IA.16}$$

where

$$q_{D,t}^{\$} = E_t^{\mathbb{Q}}\left[e^{-r^{\$}(\tau_D-t)}\right] \tag{IA.17}$$

is the date-t price of the Arrow-Debreu default claim, which pays off 1 unit of the numeraire (1 dollar) at the time of default τ_D.

From the principle of no arbitrage, the price of the Arrow-Debreu default claim, $q_D^{\$}(x)$, (where $x = \ln X$) satisfies

$$E_t^{\mathbb{Q}}[dq_D^{\$}(x) - q_D^{\$}(x)r^{\$}dt] = 0. \tag{IA.18}$$

Applying Ito's Lemma gives the ordinary differential equation

$$\frac{1}{2}\sigma_Y^2 q_D^{\$\prime\prime}(x) + \left(\widehat{\mu}_Y + \varphi\mu_P - \frac{1}{2}\sigma_Y^2\right)q_D^{\$\prime}(x) - r^{\$}q_D(x) = 0. \tag{IA.19}$$

The general solution of the above ordinary differential equation is given by $q_D^{\$}(x) = k_- e^{a_- x} + k_+ e^{a_+ x}$, where a_- and a_+ are the roots of the following quadratic in a

$$\frac{1}{2}\sigma_Y^2 a^2 + \left(\widehat{\mu}_Y + \varphi\mu_P - \frac{1}{2}\sigma_Y^2\right)a - r^{\$} = 0. \tag{IA.20}$$

It follows that

$$a_\pm = -\frac{\widehat{\mu}_Y + \varphi\mu_P - \frac{1}{2}\sigma_Y^2}{\sigma_Y^2} \pm \sqrt{\left(\frac{\widehat{\mu}_Y + \varphi\mu_P - \frac{1}{2}\sigma_Y^2}{\sigma_Y^2}\right)^2 + \frac{r+\mu_P}{\frac{1}{2}\sigma_Y^2}}. \tag{IA.21}$$

We know that $a_- a_+ = -r^\$$, so a_- and a_+ are of opposite sign if $r^\$ > 0$. We can also see that $a_+ > 0$ if $r^\$ > 0$. Therefore, $a_- < 0$ if $r^\$ > 0$. From the no-bubble condition $\lim_{x\to\infty} |q_D(x)| < \infty$, we see that $c_+ = 0$ and so $q_D^\$(x) = k_- e^{a_- x}$. The boundary condition $q_D(x_D) = 1$ (where $x_D = \ln X_D$) implies that $q_D^\$(x) = e^{a_-(x-x_D)}$, and so

$$q_{D,t}^\$ = e^{-a(\mu_P)(x_t - x_D)}, \tag{IA.22}$$

where

$$a(\mu_P) = \frac{\widehat{\mu}_Y + \varphi\mu_P - \frac{1}{2}\sigma_Y^2}{\sigma_Y^2} + \sqrt{\left(\frac{\widehat{\mu}_Y + \varphi\mu_P - \frac{1}{2}\sigma_Y^2}{\sigma_Y^2}\right)^2 + \frac{r + \mu_P}{\frac{1}{2}\sigma_Y^2}}, \tag{IA.23}$$

and of course $a(\mu_P) > 0$, if $r^\$ > 0$.

We now prove that

$$\frac{\partial s_t}{\partial \mu_P} < 0 \tag{IA.24}$$

if $r^\$ > 0$, where s_t is defined in (5). Observe that

$$\frac{\partial \ln q_{D,t}^\$}{\partial \mu_P} = -(x_t - x_D)\frac{\partial a(\mu_P)}{\partial \mu_P} - a(\mu_P)\frac{\partial (x_t - x_D)}{\partial \mu_P}. \tag{IA.25}$$

We now show that $\frac{\partial a(\mu_P)}{\partial \mu_P} > 0$ if $\varphi > -1/a(\mu_P)$ (provided $r^\$ > 0$). Observe that

$$\frac{1}{2}\sigma_Y^2 a(\mu_P)^2 - \left(\widehat{\mu}_Y + \varphi\mu_P - \frac{1}{2}\sigma_Y^2\right) a(\mu_P) - r^\$ = 0. \tag{IA.26}$$

Differentiating with respect to μ_P gives

$$\sigma_Y^2 \left[a(\mu_P) - \left(\widehat{\mu}_Y + \varphi\mu_P - \frac{1}{2}\sigma_Y^2\right)\right] \frac{\partial a(\mu_P)}{\partial \mu_P} = 1 + \varphi a(\mu_P). \tag{IA.27}$$

Hence

$$\sigma_Y^2 \sqrt{\left(\frac{\widehat{\mu}_Y + \varphi\mu_P - \frac{1}{2}\sigma_Y^2}{\sigma_Y^2}\right)^2 + \frac{r + \mu_P}{\frac{1}{2}\sigma_Y^2}}\, \frac{\partial a(\mu_P)}{\partial \mu_P} = 1 + \varphi a(\mu_P), \tag{IA.28}$$

and so $\frac{\partial a(\mu_P)}{\partial \mu_P} > 0$ if $\varphi > -1/a(\mu_P)$ (provided $r^\$ > 0$). Hence, holding the distance to default, $x_t - x_D$, fixed, the price of the Arrow-Debreu default claim, $q_{D,t}^\$$, decreases with expected inflation if $r^\$ > 0$. For the credit spread, as defined in (5), observe that

$$s_t = \frac{r^\$}{1 - q_{D,t}^\$} - r^\$. \tag{IA.29}$$

Since $1/(1 - q_{D,t}^\$)$ decreases with expected inflation, it follows that

$$\frac{\partial}{\partial \mu_P}\left(\frac{r^\$}{1 - q_{D,t}^\$}\right) < \frac{\partial r^\$}{\partial \mu_P} \tag{IA.30}$$

and hence

$$\frac{\partial s_t}{\partial \mu_P} < 0 \tag{IA.31}$$

if $r^{\$} > 0$.

The date-t value of levered equity (in nominal units) after taxes is given by

$$S_t^{\$} = (1-\eta) E_t^{\mathbb{Q}} \left[\int_t^{\tau_D} e^{-r^{\$}(u-t)} (X_u - c) du \right]. \tag{IA.32}$$

Therefore, we obtain

$$S_t^{\$} = (1-\eta) \left(X_t E_t^{\mathbb{Q}} \left[\int_t^{\infty} e^{-r^{\$}(u-t)} \frac{X_u}{X_t} du - e^{-r^{\$}(\tau_D - t)} \frac{X_D}{X_t} E_{\tau_D}^{\mathbb{Q}} \int_{\tau_D}^{\infty} e^{-r^{\$}(u-\tau_D)} \frac{X_u}{X_D} du \right] \right. \tag{IA.33}$$

$$\left. - \ c E_t^{\mathbb{Q}} \left[\int_t^{\tau_D} e^{-r^{\$}(u-t)} du \right] \right) \tag{IA.34}$$

$$= (1-\eta) X_t \left(\frac{1}{r^{\$} - \widehat{\mu}_X} - E_{\tau_D}^{\mathbb{Q}} [e^{-r^{\$}(\tau_D - t)}] \frac{X_D}{X_t} \frac{1}{r^{\$} - \widehat{\mu}_X} - \frac{c}{r^{\$}} (1 - q_{D,t}^{\$}) \right) \tag{IA.35}$$

$$= (1-\eta) \left(\frac{X_t - X_D q_{D,t}^{\$}}{r^{\$} - \widehat{\mu}_X} - \frac{c}{r^{\$}} (1 - q_{D,t}^{\$}) \right) \tag{IA.36}$$

Within Section 1, we assume an exogenously fixed default boundary, but with an endogenous default policy, the default time τ_D is chosen to maximize the value of levered equity. The smooth pasting condition $\frac{\partial S_t^{\$}}{\partial X_t} |_{X_t = X_D} = 0$ determines the default boundary X_D, i.e.

$$\frac{1 - X_D \left. \frac{\partial q_{D,t}^{\$}}{\partial X_t} \right|_{X_t = X_D}}{r^{\$} - \widehat{\mu}_X} + \frac{c}{r^{\$}} \left. \frac{\partial q_{D,t}^{\$}}{\partial X_t} \right|_{X_t = X_D} = 0, \tag{IA.37}$$

and so we obtain the optimal default policy

$$X_D = c \frac{r + (1-\varphi)\mu_P - \widehat{\mu}_Y}{r + \mu_P} \frac{a(\mu_P)}{1 + a(\mu_P)}. \tag{IA.38}$$

It follows that, for a fixed nominal coupon, c, we have

$$\frac{\partial x_D}{\partial \mu_P} = \frac{(1-\varphi) r - (r - \widehat{\mu}_Y)}{(r + (1-\varphi)\mu_P - \widehat{\mu}_Y)(r + \mu_P)} + \frac{1}{a(\mu_P)(1 + a(\mu_P))} \frac{\partial a(\mu_P)}{\partial \mu_P}. \tag{IA.39}$$

With an endogenous default policy, the distance to default is impacted by inflation. A priori, it is possible that equity holders will choose to default later when when inflation is higher, that is the distance to default will increase. However, for the calibration we have chosen, equityholders default earlier when inflation is higher, because the present value of the coupons they have to pay to bondholders is increased. Even if this were not the case, $\frac{\partial a(\mu_P)}{\partial \mu_P}$ is much larger than $\frac{\partial (x_t - x_D)}{\partial \mu_P}$, so any increase in distance to default would not change the overall sign of $\frac{\partial \ln q_{D,t}^{\$}}{\partial \mu_P}$.

IA.B The Economy

First, we introduce some notation related to jumps in the state of the economy. Suppose that during the small time-interval $[t-\Delta t, t)$ the economy is in state i and that at time t the state changes, so that during the next small time interval $[t, t+\Delta t)$ the economy is in state $j \neq i$. We then define the left-limit of s at time t as

$$s_{t-} = \lim_{\Delta t \to 0} s_{t-\Delta t}, \tag{IA.40}$$

and the right-limit as

$$s_t = \lim_{\Delta t \to 0} s_{t+\Delta t}. \tag{IA.41}$$

Therefore $s_{t-} = i$, whereas $s_t = j$, so the left- and right limits are not equal. If some function E depends on the current state of the economy i.e. $E_t = E(s_t)$, then E is a jump process which is right continuous with left limits, i.e. RCLL. If a jump from state i to $j \neq i$ occurs at date t, then we abuse notation slightly and denote the left limit of E at time t by E_i, where i is the index for the state. i.e. $E_{t-} = \lim_{s\uparrow t} E_s = E_i$. Similarly $E_t = \lim_{s\downarrow t} E_s = E_j$. We shall use the same notation for all processes that jump, because of their dependence on the state of the economy.

Using simple algebra we can write the normalized Kreps-Porteus aggregator in the following compact form:

$$f(c,v) = \beta \left(h^{-1}(v)\right)^{1-\gamma} u\left(c/h^{-1}(v)\right), \tag{IA.42}$$

where

$$\begin{aligned} u(x) &= \frac{x^{1-\frac{1}{\psi}}-1}{1-\frac{1}{\psi}}, \psi > 0, \\ h(x) &= \begin{cases} \frac{x^{1-\gamma}}{1-\gamma}, & \gamma \geq 0, \gamma \neq 1. \\ \ln x, & \gamma = 1. \end{cases} \end{aligned}$$

The representative agent's value function is given by

$$J_t = E_t \int_t^\infty f(C_t, J_t)\, dt. \tag{IA.43}$$

Proposition IA.1 *The SDF of a representative agent with the continuous-time version of Epstein-Zin-Weil preferences is given by*

$$\pi_t = \begin{cases} \left(\beta e^{-\beta t}\right)^{\frac{1-\gamma}{1-\frac{1}{\psi}}} C_t^{-\gamma} \left(p_{C,t} e^{\int_0^t p_{C,s}^{-1} ds}\right)^{-\frac{\gamma-\frac{1}{\psi}}{1-\frac{1}{\psi}}}, & \psi \neq 1 \\ \beta e^{-\beta \int_0^t \left[1+(\gamma-1)\ln(V_s^{-1})\right] ds} C_t^{-\gamma} V_t^{-(\gamma-1)}, & \psi = 1 \end{cases}. \tag{IA.44}$$

When $\psi \neq 1$, the price-consumption ratio in state i, $p_{C,i}$, satisfies the nonlinear equation system:

$$p_{C,i}^{-1} = \overline{r}_i + \gamma\sigma_{C,i}^2 - \mu_{C,i} - \left(1-\frac{1}{\psi}\right)\sum_{j\neq i}\lambda_{ij}\left(\frac{\left(p_{C,j}/p_{C,i}\right)^{\frac{1-\gamma}{1-\frac{1}{\psi}}}-1}{1-\gamma}\right), \; i,\, j \in \{1,\ldots,N\},\, j \neq i. \tag{IA.45}$$

where

$$\overline{r}_i = \beta + \frac{1}{\psi}\mu_{C,i} - \frac{1}{2}\gamma\left(1+\frac{1}{\psi}\right)\sigma_{C,i}^2,\; i \in \{1,\ldots,N\}. \tag{IA.46}$$

When $\psi = 1$, define V_i via

$$J = \ln(CV). \tag{IA.47}$$

Then V_i satisfies the nonlinear equation system:

$$\beta \ln V_i = \mu_{C,i} - \frac{\gamma}{2}\sigma_{C,i}^2 + \sum_{j\neq i} \lambda_{ij} \frac{(V_j/V_i)^{1-\gamma} - 1}{1-\gamma} \; i \in \{1,\ldots,N\},\, j \neq i. \tag{IA.48}$$

IA.C Derivation of the Real SDF

In this section, we derive the state-price density shown in Proposition IA.1.

Proof of Proposition IA.1. Duffie and Skiadas (1994) show that the state-price density for a *general* normalized aggregator f is given by

$$\pi_t = e^{\int_0^t f_v(C_s,J_s)dt} f_c\left(C_t, J_t\right), \tag{IA.49}$$

where $f_c(\cdot,\cdot)$ and $f_v(\cdot,\cdot)$ are the partial derivatives of f with respect to its first and second arguments, respectively, and J is the value function given in (IA.43). The Feynman-Kac Theorem implies

$$f\left(C_t, J_{t-}\right)\big|_{s_{t-}=i} dt + E_t\left[\, dJ_t \right| s_{t-} = i] = 0,\; i \in \{1,\ldots,N\}.$$

Using Ito's Lemma we rewrite the above equation as

$$0 = f\left(C, J_i\right) + CJ_{i,C}g_i + \frac{1}{2}C^2 J_{i,CC}\sigma_{C,i}^2 + \sum_{j\neq i}\lambda_{ij}\left(J_j - J_i\right), \tag{IA.50}$$

for $i, j \in \{1,\ldots,N\}$, $j \neq i$. We guess and verify that $J = h(CV)$, where V_i satisfies the nonlinear equation system

$$0 = \beta u\left(V_i^{-1}\right) + g_i - \frac{1}{2}\gamma\sigma_{C,i}^2 + \sum_{j\neq i}\lambda_{ij}\left(\frac{(V_j/V_i)^{1-\gamma} - 1}{1-\gamma}\right),\; i,j \in \{1,\ldots,N\},\, j \neq i. \tag{IA.51}$$

Substituting (IA.42) into (IA.49) and simplifying gives

$$\pi_t = \beta e^{-\beta\int_0^t \left[1 + \left(\gamma - \frac{1}{\psi}\right)u\left(V_s^{-1}\right)\right]dt} C_t^{-\gamma} V_t^{-\left(\gamma - \frac{1}{\psi}\right)}. \tag{IA.52}$$

When $\psi = 1$, the above equation gives the second expression in (IA.44). We rewrite (IA.51) as

$$\beta\left[1 + \left(\gamma - \frac{1}{\psi}\right)u\left(V_i^{-1}\right)\right] = \overline{r}_i - \left(\gamma - \frac{1}{\psi}\right)\sum_{j\neq i}\lambda_{ij}\left(\frac{(V_j/V_i)^{1-\gamma} - 1}{1-\gamma}\right) - \left[\gamma g_i - \frac{1}{2}\gamma\left(1+\gamma\right)\sigma_{C,i}^2\right],\; i,j \in \{1,\ldots,N\},\, j \neq i, \tag{IA.53}$$

where $\overline{r}_i$ is given in (IA.46). Setting $\psi = 1$ in (IA.53) gives (IA.48). To derive the first expression in (IA.44) from (IA.52) we prove that

$$V_i = \left(\beta p_{C,i}\right)^{\frac{1}{1-\frac{1}{\psi}}},\, \psi \neq 1. \tag{IA.54}$$

We proceed by considering the optimization problem for the representative agent. She chooses her optimal consumption, C^*, and risky asset portfolio, φ, to maximize her expected utility

$$J_t^* = \sup_{C^*,\varphi} E_t \int_t^\infty f\left(C_t^*, J_t^*\right)dt.$$

Observe that J^* depends on optimal consumption-portfolio choice, whereas the J defined previously in (IA.47) depends on exogenous aggregate consumption. The optimization is carried out subject to the dynamic budget constraint, which we now describe. If the agent consumes at the rate, C^*, invests a proportion, φ, of her remaining financial wealth in risky assets, and puts the remainder in the locally risk-free asset, then her financial wealth, W, evolves according to the dynamic budget constraint:

$$\frac{dW_t}{W_{t-}} = \varphi_{t-}\left(dR_{p,t} - r_{t-}dt\right) + r_{t-}dt - \frac{C^*_{t-}}{W_{t-}}dt,$$

where $dR_{p,t}$ is the cum-dividend return on the risky asset portfolio over the time interval $[t, t + dt)$. We define $N_{ij,t}$ as the Poisson process which jumps upward by one whenever the state of the economy switches from i to $j \neq i$. The compensated version of this process is the Poisson martingale

$$N^P_{ij,t} = N_{ij,t} - \lambda_{ij}t.$$

Note that C^* is the consumption to be chosen by the agent, i.e. it is a control, and at this stage we cannot rule out the possibility that it jumps with the state of the economy. In contrast, C is aggregate consumption, and since it is continuous, its left and right limits are equal, i.e. $C_{t-} = C_t$.

The system of Hamilton-Jacobi-Bellman partial differential equations for the agent's optimization problem is

$$\sup_{C^*,\varphi} f\left(C^*_{t-}, J^*_{t-}\right)\Big|_{s_{t-}=i} dt + E_t\left[dJ^*_t \,|\, s_{t-} = i\right] = 0,\ \ i \in \{1, \ldots, N\}.$$

Applying Ito's Lemma to $J^*_t = J^*(W_t, s_t)$ allows us to write the above equation as

$$\begin{aligned} 0 \;=\; & \sup_{C^*_i,\varphi_i} f\left(C^*_i, J^*_i\right) + W_i J^*_{i,W}\left(\varphi_i\left(E_t\left[\frac{dR_{p,t}}{dt}|s_{t-} = i\right] - r_i\right) + r_i - \frac{C^*_i}{W_i}\right) + \frac{1}{2}W_i^2 J^*_{i,WW}\varphi_i^2\frac{1}{dt}E_t[(dR_{p,t})^2|s_{t-} = i] + \\ & + \sum_{j\neq i}\lambda_{ij}\left(J^*_j - J^*_i\right),\ i \in \{1, \ldots, N\},\ j \neq i. \end{aligned}$$

We guess and verify that $J^*_t = h(W_t F_t)$, where F_i satisfies the nonlinear equation system

$$0 = \sup_{C^*_i,\varphi_i} \beta u\left(\frac{C^*_i}{W_i F_i}\right) + \left(\varphi_i\left(E_t\left[\frac{dR_{p,t}}{dt}|s_{t-} = i\right] - r_i\right) + r_i - \frac{C^*_i}{W_i}\right) - \frac{1}{2}\gamma\varphi_i^2 E_t\left[\frac{(dR_{p,t})^2}{dt}|s_{t-} = i\right] + \sum_{j\neq i}\lambda_{ij}\left(\frac{(F_j/F_i)^{1-\gamma} - 1}{1-\gamma}\right),$$

$i \in \{1, \ldots, N\}$, $j \neq i$.

From the first order condition with respect to consumption, we obtain the optimal consumption policy

$$C^*_i \;=\; \beta^{\psi} F_i^{-(\psi-1)} W_i,\ i \in \{1, \ldots, N\},$$

The market for the consumption good must clear, so $W_i = P_i$, $C^*_i = C$ (and thus $J = J^*$). Note that this forces the optimal consumption policy to be continuous. Hence,

$$p_{C,i} = \beta^{-\psi} F_i^{1-\psi}. \tag{IA.55}$$

The above equation implies that for $\psi = 1$, $p_{C,i} = 1/\beta$. The equality, $J = J^*$, implies that $CV_i = WF_i$. Hence, $F_i = p^{-1}_{C,i}V_i$. Using this equation to eliminate F_i from (IA.55) gives (IA.54). Substituting (IA.54) into (IA.52) and (IA.53) gives the expression in (IA.44) for $\psi \neq 1$ and (IA.45). ∎

IA.D The Evolution of the Real SDF

In this section we derive the evolution of the real SDF, as given in (13).

We start by proving that the real SDF satisfies the stochastic differential equation

$$\left.\frac{d\pi_t}{\pi_{t-}}\right|_{s_{t-}=i} = -r_i dt + \left.\frac{dM_t}{M_{t-}}\right|_{s_{t-}=i}, \tag{IA.56}$$

where M is a martingale under $\mathbb{P}$ such that

$$\left.\frac{dM_t}{M_{t-}}\right|_{s_{t-}=i} = -\Theta_i^B dZ_t + \Theta_{ij}^P dN_{ij,t}^P,\ j \in \{1, \ldots, N\},\ j \neq i, \tag{IA.57}$$

r_i is the risk-free rate in state i given by

$$r_i = \overline{r}_i + \sum_{j \neq i} \lambda_{ij} \left[\frac{\gamma - \frac{1}{\psi}}{\gamma - 1} \left(\omega_{ij}^{\frac{\gamma-1}{\gamma-\frac{1}{\psi}}} - 1 \right) - (\omega_{ij} - 1) \right], \tag{IA.58}$$

where

$$\omega_{ij} = \frac{\omega_j}{\omega_i},\ i, j \in \{1, \ldots, N\}, \tag{IA.59}$$

and $\omega_2, \omega_3, \ldots, \omega_N$ are determined by the following system of $N-1$ nonlinear algebraic equations:

$$0 = \omega_j^{-\frac{1-\frac{1}{\psi}}{\gamma-\frac{1}{\psi}}} - \frac{k_1 + \frac{1-\frac{1}{\psi}}{\gamma-1} \sum_{k \neq 1} \lambda_{1k} \left(\omega_k^{\frac{\gamma-1}{\gamma-\frac{1}{\psi}}} - 1 \right)}{k_j + \frac{1-\frac{1}{\psi}}{\gamma-1} \sum_{k \neq j} \lambda_{jk} \left(\frac{\omega_k}{\omega_j}^{\frac{\gamma-1}{\gamma-\frac{1}{\psi}}} - 1 \right)},\ j \in \{2, \ldots, N\},\ \psi \neq 1, \tag{IA.60}$$

$$0 = \ln \omega_j^{\frac{1}{\gamma-1}} - \frac{\mu_{C,i} - \frac{1}{2}\gamma\sigma_{C,i}^2 + \sum_{k \neq i} \lambda_{ik}(\omega_{ik} - 1)}{\mu_{C,j} - \frac{1}{2}\gamma\sigma_{C,j}^2 + \sum_{k \neq j} \lambda_{jk}(\omega_{jk} - 1)},\ j \in \{2, \ldots, N\},\ \psi = 1, \tag{IA.61}$$

where

$$k_i = \overline{r}_i + \gamma\sigma_{C,i}^2 - \mu_{C,i}. \tag{IA.62}$$

Observe that if we define the $N \times N$ matrix $\mathbf{\Omega}$ via $\Omega_{ij} = \omega_{ij}$, the $N \times N$ matrix physical generator matrix for the Markov chain driving the economy, $\mathbf{\Lambda}$, via $\Lambda_{ij} = \lambda_{ij}$, where $\lambda_{ii} = -\sum_{j \neq i} \lambda_{ij}$, then

$$\mathbf{\Omega} = \begin{pmatrix} 1 & \omega_2 & \omega_3 & \cdots & \omega_N \\ \omega_2^{-1} & 1 & \frac{\omega_3}{\omega_2} & \cdots & \frac{\omega_N}{\omega_2} \\ \omega_3^{-1} & \frac{\omega_2}{\omega_3} & 1 & \cdots & \frac{\omega_N}{\omega_3} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \omega_N^{-1} & \frac{\omega_2}{\omega_N} & \frac{\omega_3}{\omega_N} & \cdots & 1 \end{pmatrix} \tag{IA.63}$$

and

$$\mathbf{\Lambda} = \begin{pmatrix} -\sum_{j \neq 1} \lambda_{1j} & \lambda_{12} & \cdots & \lambda_{1N} \\ \lambda_{21} & -\sum_{j \neq 2} \lambda_{1j} & \cdots & \lambda_{2N} \\ \vdots & \vdots & \ddots & \vdots \\ \lambda_{N1} & \lambda_{N2} & \cdots & -\sum_{j \neq N} \lambda_{Nj} \end{pmatrix} \tag{IA.64}$$

Θ_i^B is the market price of risk due to Brownian shocks in state i, given by

$$\Theta_i^B = \gamma \sigma_{C,i},\; i \in \{1, \ldots, N\}, \tag{IA.65}$$

and Θ_{ij}^P is the market price of risk due to Poisson shocks when the economy switches out of state i into state j:

$$\Theta_{ij}^P = \omega_{ij} - 1,\; i, j \in \{1, \ldots, N\},\; j \neq i. \tag{IA.66}$$

We begin the proof by noting that if we define

$$\omega_{ij} = \left. \frac{\pi_t}{\pi_{t-}} \right|_{s_{t-}=i, s_t=j},\; i, j \in \{1, \ldots, N\},\; j \neq i, \tag{IA.67}$$

then (IA.44) implies that

$$\omega_{ij} = \begin{cases} \left(\frac{p_{C,j}}{p_{C,i}}\right)^{-\frac{\gamma - \frac{1}{\psi}}{1 - \frac{1}{\psi}}}, & \psi \neq 1; \\ \left(\frac{V_j}{V_i}\right)^{-(\gamma-1)}, & \psi = 1. \end{cases} \tag{IA.68}$$

Equation (IA.59) follows from the above. We thus see that we need only determine the $N-1$ unknowns $\omega_2, \ldots, \omega_N$.

Using (IA.68) we rewrite (IA.45) and (IA.48) as

$$p_{C,i} = \frac{1}{k_i + \frac{1-\frac{1}{\psi}}{\gamma - 1} \sum_{j \neq i} \lambda_{ij} \left(\omega_{ij}^{\frac{\gamma-1}{\gamma - 1/\psi}} - 1 \right)},\; i, j \in \{1, \ldots, N\},\; j \neq i, \tag{IA.69}$$

where k_i is defined in (IA.62) and

$$\beta \ln V_i = \mu_{C,i} - \frac{1}{2} \gamma \sigma_{C,i}^2 + \sum_{j \neq i} \lambda_{ij} \frac{\omega_{ij} - 1}{1 - \gamma},\; i, j \in \{1, \ldots, N\},\; j \neq i, \tag{IA.70}$$

respectively. Therefore, from (IA.68) and the above two equations it follows that $\omega_2, \ldots, \omega_N$ is the solution of equation system (IA.60) when $\psi \neq 1$ and (IA.61) when $\psi = 1$.

We now derive expressions for the risk-free rate and risk prices. Ito's Lemma implies that the state-price density evolves according to

$$\begin{aligned} \frac{d\pi_t}{\pi_{t-}} &= \frac{1}{\pi_{t-}} \frac{\partial \pi_{t-}}{\partial t} dt + \frac{1}{\pi_{t-}} C_t \frac{\partial \pi_{t-}}{\partial C_t} \frac{dC_t}{C_t} + \frac{1}{2} \frac{1}{\pi_{t-}} C_t^2 \frac{\partial^2 \pi_{t-}}{\partial C_t^2} \left(\frac{dC_t}{C_t} \right)^2 \\ &\quad + \sum_{s_t \neq s_{t-}} \lambda_{s_{t-}, s_t} \frac{\Delta \pi_t}{\pi_{t-}} dt + \frac{\Delta \pi_t}{\pi_{t-}} dN_{s_{t-}, s_t, t}^P, \end{aligned} \tag{IA.71}$$

where $\Delta \pi_t = \pi_t - \pi_{t-}$. The definition (IA.67) implies

$$\frac{\Delta \pi_t}{\pi_{t-}} = \omega_{s_{t-}, s_t} - 1.$$

Together with some standard algebra that allows us to rewrite (IA.71) as

$$\frac{d\pi_t}{\pi_{t-}} = -\left(\kappa_{s_{t-}} + \gamma\mu_{C,s_{t-}} - \frac{1}{2}\gamma\left(1+\gamma\right)\sigma^2_{C,s_{t-}} + \sum_{s_t \neq s_{t-}} \lambda_{s_{t-},s_t}\left(1-\omega_{s_{t-},s_t}\right)\right)dt - \gamma\sigma_{C,s_{t-}}dZ_t + \left(\omega_{s_{t-},s_t} - 1\right)dN^P_{s_{t-},s_t,t}. \tag{IA.72}$$

Comparing the above equation with (IA.56), which is standard in an economy with jumps, gives (IA.65) and (IA.66), in addition to

$$r_i = \kappa_i + \gamma\mu_{C,i} - \frac{1}{2}\gamma\left(1+\gamma\right)\sigma^2_{C,i} + \sum_{j\neq i}\lambda_{ij}\left(1-\omega_{ij}\right),\ i,j \in \{1,\ldots,N\},\ j\neq i,$$

where

$$\kappa_i = \begin{cases} \beta\left[1 + \left(\gamma - \frac{1}{\psi}\right)\frac{\left(\beta p_{C,i}\right)^{-1}-1}{1-\frac{1}{\psi}}\right], & \psi \neq 1,\ i,j\in\{1,\ldots,N\},\ j\neq i; \\ \beta\left[1 + (\gamma-1)\ln\left(V_i^{-1}\right)\right], & \psi = 1,\ i,j\in\{1,\ldots,N\},\ j\neq i, \end{cases} \tag{IA.73}$$

We use Equations (IA.69) and (IA.70) to eliminate $p_{C,i}$ and V_i from (IA.73) to obtain

$$\kappa_i = \begin{cases} \overline{r}_i - \left(\gamma - \frac{1}{\psi}\right)\sum_{j\neq i}\lambda_{ij}\left(\frac{\omega_{ij}^{\frac{\gamma-1}{\gamma-1/\psi}}-1}{1-\gamma}\right) - \left[\gamma\mu_{C,i} - \frac{1}{2}\gamma\left(1+\gamma\right)\sigma^2_{C,i}\right], & \psi\neq 1,\ i,j\in\{1,\ldots,N\},\ j\neq i; \\ \overline{r}_i + \sum_{j\neq i}\lambda_{ij}\left(\omega_{ij}-1\right) - \left[\gamma\mu_{C,i} - \frac{1}{2}\gamma\left(1+\gamma\right)\sigma^2_{C,i}\right], & \psi = 1,\ i,j\in\{1,\ldots,N\},\ j\neq i, \end{cases} \tag{IA.74}$$

so

$$r_i = \begin{cases} \overline{r}_i - \left(\gamma - \frac{1}{\psi}\right)\sum_{j\neq i}\lambda_{ij}\left(\frac{\omega_{ij}^{\frac{\gamma-1}{\gamma-1/\psi}}-1}{1-\gamma}\right) + \sum_{j\neq i}\lambda_{ij}\left(1-\omega_{ij}\right), & \psi\neq 1,\ i,j\in\{1,\ldots,N\},\ j\neq i,; \\ \overline{r}_i, & \psi = 1,\ i\in\{1,\ldots,N\}. \end{cases} \tag{IA.75}$$

Taking the limit of the upper expression in the above equation gives the lower expression, so (IA.58) follows. The total market price of consumption risk in real state i accounts for both Brownian and Poisson shocks, and is thus given by

$$\Theta_i = \sqrt{\left(\Theta_i^B\right)^2 + \sum_{j\neq i}\lambda_{ij}\left(\Theta_{ij}^P\right)^2},\ i,j\in\{1,\ldots,N\},\ j\neq i. \tag{IA.76}$$

Because the Poisson and Brownian shocks in (IA.57) are independent and their respective prices of risk are bounded, M is a martingale under the actual measure $\mathbb{P}$. Thus, M defines the Radon-Nikodym derivative $\frac{d\mathbb{Q}}{d\mathbb{P}}$ via $M_t = E_t\left[\frac{d\mathbb{Q}}{d\mathbb{P}}\right]$. It is a standard result (see Elliott (1982)) that the risk-neutral switching probabilities per unit time are given by

$$\widehat{\lambda}_{ij} = \lambda_{ij}E_t\left[\left.\frac{M_t}{M_{t-}}\right| s_{t-} = i, s_t = j\right],\ j\neq i.$$

The jump component in $d\pi$ comes purely from dM. Thus, using (IA.67), we can simplify the above expression to obtain (IA.77).

The risk-neutral generator matrix for the Markov chain driving the economy is given by the $N\times N$ matrix $\hat{\mathbf{\Lambda}}$, where $\hat{\Lambda}_{ij} = \hat{\lambda}_{ij}$ where $\hat{\lambda}_{ij}dt$ is the risk-neutral probability of switching from state i to state j over a time interval of length dt, where

$$\hat{\lambda}_{ij} = \lambda_{ij}\omega_{ij},\ i\neq j, \tag{IA.77}$$

$$\hat{\lambda}_{ij} = -\sum_{j\neq i}\hat{\lambda}_{ij}. \tag{IA.78}$$

For the special case, where $\lambda_{ij} = pf_j$, $j \neq i$, f_i is the long-run physical probability that the state of the economy is i, and

$$\mathbf{\Lambda} = \begin{pmatrix} -p\sum_{j\neq 1} f_j & pf_2 & \cdots & pf_N \\ pf_1 & -p\sum_{j\neq 2} f_j & \cdots & pf_N \\ \vdots & \vdots & \ddots & \vdots \\ pf_1 & pf_2 & \cdots & -p\sum_{j\neq N} f_N \end{pmatrix} \tag{IA.79}$$

Observe that, $\underline{f} = (f_1, \dots, f_N)^\top$ is the long-run physical distribution of the state of the economy. The state of the economy converges to its long-run physical distribution exponentially at the rate p.

IA.D.1 Prices of Risk: Omega Matrices

For convenience, we show the values used for the omega matrix, (IA.63), within our numerical work. We stress that the omega matrices are endogenous and are stated here for the sake of reference.

In the baseline calibration, given that we assume that real and nominal regimes are independent, we obtain the following omega matrix:

	RL	**RM**	**RH**	**EL**	**EM**	**EH**
RL	1.0000	1.0000	1.0000	0.6959	0.6959	0.6959
RM	1.0000	1.0000	1.0000	0.6959	0.6959	0.6959
RH	1.0000	1.0000	1.0000	0.6959	0.6959	0.6959
EL	1.4369	1.4369	1.4369	1.0000	1.0000	1.0000
EM	1.4369	1.4369	1.4369	1.0000	1.0000	1.0000
EH	1.4369	1.4369	1.4369	1.0000	1.0000	1.0000

(IA.80)

When we relax this assumption, the following omega matrix yields a 25bps uncondtional inflation risk premium:

	RL	**RM**	**RH**	**EL**	**EM**	**EH**
RL	1.0000	0.9387	0.9419	0.6712	0.6472	0.6643
RM	1.0653	1.0000	1.0034	0.7151	0.6895	0.7077
RH	1.0617	0.9966	1.0000	0.7126	0.6871	0.7053
EL	1.4898	1.3985	1.4033	1.0000	0.9642	0.9897
EM	1.5451	1.4504	1.4553	1.0371	1.0000	1.0264
EH	1.5053	1.4130	1.4178	1.0104	0.9742	1.0000

(IA.81)

The following omega matrix yields a 50bps unconditional inflation risk premium:

	RL	**RM**	**RH**	**EL**	**EM**	**EH**
RL	1.0000	0.8719	0.8778	0.6458	0.5954	0.6261
RM	1.1469	1.0000	1.0067	0.7407	0.6829	0.7180
RH	1.1392	0.9933	1.0000	0.7357	0.6783	0.7132
EL	1.5484	1.3501	1.3592	1.0000	0.9219	0.9694
EM	1.6795	1.4644	1.4743	1.0847	1.0000	1.0514
EH	1.5973	1.3927	1.4021	1.0316	0.9511	1.0000

(IA.82)

The following omega matrix yields a 75bps unconditionalinflation risk premium:

	RL	**RM**	**RH**	**EL**	**EM**	**EH**
RL	1.0000	0.8026	0.8106	0.6202	0.5428	0.5835
RM	1.2460	1.0000	1.0100	0.7728	0.6764	0.7270
RH	1.2337	0.9901	1.0000	0.7652	0.6697	0.7198
EL	1.6123	1.2939	1.3068	1.0000	0.8752	0.9407
EM	1.8422	1.4785	1.4932	1.1426	1.0000	1.0748
EH	1.7139	1.3755	1.3892	1.0630	0.9304	1.0000

(IA.83)

The following omega matrix yields a 100bps unconditional inflation risk premium:

	RL	**RM**	**RH**	**EL**	**EM**	**EH**
RL	1.0000	0.7393	0.7488	0.5972	0.4959	0.5428
RM	1.3526	1.0000	1.0128	0.8077	0.6708	0.7342
RH	1.3355	0.9873	1.0000	0.7975	0.6623	0.7249
EL	1.6745	1.2380	1.2539	1.0000	0.8304	0.9090
EM	2.0165	1.4908	1.5099	1.2042	1.0000	1.0946
EH	1.8422	1.3620	1.3795	1.1001	0.9136	1.0000

(IA.84)

Finally, the following omega matrix yields a 125bps unconditional inflation risk premium:

	RL	**RM**	**RH**	**EL**	**EM**	**EH**
RL	1.0000	0.6806	0.6912	0.5757	0.4532	0.5039
RM	1.4692	1.0000	1.0154	0.8459	0.6659	0.7403
RH	1.4469	0.9848	1.0000	0.8330	0.6557	0.7291
EL	1.7369	1.1822	1.2005	1.0000	0.7872	0.8752
EM	2.2064	1.5018	1.5250	1.2703	1.0000	1.1118
EH	1.9845	1.3508	1.3716	1.1426	0.8994	1.0000

(IA.85)

IA.E Nominal SDF

We define the nominal SDF $\pi^{\$}$ by

$$\pi_t^{\$} = \frac{\pi_t}{P_t}. \tag{IA.86}$$

Now we apply Ito's Lemma to obtain

$$\frac{d\pi_t^{\$}}{\pi_t^{\$}} = \frac{d\pi_t}{\pi_t} - \frac{dP_t}{P_t} - \frac{d\pi_t}{\pi_t}\frac{dP_t}{P_t} + \left(\frac{dP_t}{P_t}\right)^2 \tag{IA.87}$$

$$= -r_i dt - \gamma\sigma_{C,i} dZ_t + \sum_{j\neq i}(\omega_{ij} - 1) dN_{ij,t}^P - \mu_{P,i} dt - \sigma_{P,i} dZ_{P,t} + \gamma\rho_{PC,i}\sigma_{P,i}\sigma_{C,i} dt + \sigma_{P,i}^2 dt \tag{IA.88}$$

$$= -(r_i + \mu_{P,i} - \gamma\rho_{PC,i}\sigma_{P,i}\sigma_{C,i} - \sigma_{P,i}^2)dt - \gamma\sigma_{C,i} dZ_t - \sigma_{P,i} dZ_{P,t} + \sum_{j\neq i}(\omega_{ij} - 1) dN_{ij,t}^P. \tag{IA.89}$$

We thus obtain the nominal risk-free rate

$$r_i^{\$} = -E_t\left[\frac{d\pi_t^{\$}}{\pi_t^{\$}}\middle| s_{t-} = i\right] \tag{IA.90}$$

$$= r_i + \mu_{P,i} - \gamma\rho_{PC,i}\sigma_{P,i}\sigma_{C,i} - \sigma_{P,i}^2. \tag{IA.91}$$

We define the exponential martingale (wrt to $\mathbb{P}$) $M^{\$}$ via

$$\frac{dM_t^{\$}}{M_{t-}^{\$}} = -\gamma\sigma_{C,s_{t-}} dZ_t - \sigma_{P,s_{t-}} dZ_{P,t} + \sum_{s_t\neq s_{t-}}(\omega_{s_t,s_{t-}} - 1) dN_{s_t,s_{t-},t}^P,\ M_0^{\$} = 1. \tag{IA.92}$$

We use $M^{\$}$ to define the probability measure $\mathbb{Q}^{\$}$ in the standard way.

IA.F Sticky Cash Flows

Applying Ito's Lemma to (20), we obtain

$$\frac{dX_t}{X_t} = \mu_{X,t}dt + \sigma_Y dW_t + \varphi\sigma_{P,t}dZ_{P,t}, \tag{IA.93}$$

where

$$\mu_{X,t} = \mu_{Y,t} + \varphi(\mu_{P,t} + \rho_{PY,t}\sigma_{Y,t}\sigma_{P,t}) - \frac{1}{2}\varphi(1-\varphi)\sigma_{P,t}^2. \tag{IA.94}$$

We can use the martingale component of $\pi^{\$}$ to define a new probability measure $\mathbb{Q}^{\$}$, under which

$$\hat{\mu}_{X,t}^{\$} = E_t^{\mathbb{Q}^{\$}}\left[\frac{dX_t}{X_t}\right] = \mu_{X,t} - \gamma\sigma_{C,t}(\sigma_Y\rho_{YC,t} + \varphi\sigma_{P,t}\rho_{PC,t}) - \sigma_{P,i}(\sigma_Y\rho_{PY,t} + \varphi\sigma_{P,t}), \tag{IA.95}$$

via Girsanov's Theorem, where $\rho_{YC,t}dt = E_t[dW_t dZ_t]$, $\rho_{PC,t}dt = E_t[dZ_{P,t}dZ_t]$, and $\rho_{PY,t}dt = E_t[dZ_{P,t}dW_t]$.

Defining $x_t = \ln X_t$, under $\mathbb{Q}^{\$}$, we have

$$dx_t = \hat{\mu}_{x,t}^{\$}dt + \sigma_{x,t}dZ_{x,t}^{\mathbb{Q}^{\$}}, \tag{IA.96}$$

where

$$\sigma_{x,t} = \sqrt{\sigma_{Y,t}^2 + 2\varphi\sigma_{Y,t}\sigma_{P,t}\rho_{PY,t} + \varphi^2\sigma_{P,t}^2}, \tag{IA.97}$$

$$\begin{aligned}\hat{\mu}_{x,t}^{\$} &= \hat{\mu}_{X,t}^{\$} - \frac{1}{2}\sigma_{x,t}^2, \\ &= \mu_{X,t} - \gamma\sigma_{C,t}(\sigma_{Y,t}\rho_{YC,t} + \varphi\sigma_{P,t}\rho_{PC,t}) - \sigma_{P,i}(\sigma_Y\rho_{PY,t} + \varphi\sigma_{P,t}) - \frac{1}{2}\sigma_{x,t}^2.\end{aligned} \tag{IA.98}$$

IA.G Liquidation Value

The abandonment or liquidation value of a firm is just its unlevered value, i.e. the present value of future cashflows, ignoring coupon payments to debtholders and default risk. Small, but frequent shocks to a firm's real cashflow growth are modelled by changes in the standard Brownian motion W_t. Small, but frequent shocks to the real SDF are modelled by changes in the standard Brownian motion Z_t. The assumption that $dZ_t dW_t = 0$ means that small, but frequent shocks to cashflow growth are not priced. However, changes in the expected real cashflow growth rate are driven by the same Markov chain as those driving jumps in the SDF. Hence, changes in unlevered firm value driven by changes in the expected real cashflow growth rate will be priced.

Suppose the economy is currently in state i. Then, the risk-neutral probability of the economy switching into a different state $j \neq i$ during a small time interval of length Δt is $\widehat{\lambda}_{ij}\Delta t$ and the risk-neutral probability of not switching is $1 - \widehat{\lambda}_{ij}\Delta t$. We can therefore write the unlevered nominal firm value in state i as

$$A_{i,t}^{\$} = (1-\eta)X_t\Delta t + e^{-(r_i^{\$} - \mu_{X,i}^{\$})\Delta t}\left[\left(1 - \widehat{\lambda}_{ij}\Delta t\right)A_i^{\$} + \sum_{j\neq i}\widehat{\lambda}_{ij}\Delta t A_j^{\$}\right], \; i,j \in \{1,\ldots,N\}, \; j \neq i, \tag{IA.99}$$

where $N = 6$ is the number of states in the economy.

The first term in (IA.99) is the after-tax cash flow received in the next instant and the second term is the discounted continuation value. The discount rate is just the standard discount rate for a perpetuity. Observe that the volatility of cashflow growth does not appear in the discount rate, because $dZ_t dW_t = 0$. The continuation value is the average of $A_{i,t}^{\$}$ and $A_{j,t}^{\$}$, weighted by the risk-neutral probabilities of being in states i and $j \neq i$ a small instant Δt from now. For example, with risk-neutral probability $\widehat{\lambda}_{ij}\Delta t$ the economy will be in state $j \neq i$ and the unlevered nominal firm value will be value will be $A_j^{\$}$. The continuation value is discounted back at a rate reflecting the nominal interest rate rate $r_i^\$$ and the expected nominal earnings growth rate over that instant which is $\mu_{X,i}$ – observe that there is no difference between the physical and risk-neutral nominal earnings growth rates, because $dZ_t dW_t = 0$.

We take the limit of (IA.99) as $\Delta t \to 0$, to obtain

$$0 = (1-\eta)\, X - (r_i^\$ - \mu_{X,i}^\$) A_i^\$ + \sum_{j \neq i} \widehat{\lambda}_{ij} \left(A_j^\$ - A_i^\$ \right), \, i \in \{1, \ldots, N\}, j \neq i.$$

To obtain the solution of the above linear equation system, we define

$$v_{A,i} = \frac{1}{(1-\eta)} \frac{A_i^\$}{X},$$

the before-tax nominal price-earnings ratio in state i. Therefore

$$\left(\operatorname{diag}\left(r_1^\$ - \mu_{X,1}^\$, \ldots, r_N^\$ - \mu_{X,N}^\$ \right) - \widehat{\Lambda} \right) \begin{pmatrix} v_{A,1} \\ \vdots \\ v_{A,N} \end{pmatrix} = 1_{N \times 1}, \tag{IA.100}$$

where $1_{N\times 1}$ is a $N \times 1$ vector of ones, $\operatorname{diag}\left(r_1^\$ - \mu_{X,1}^\$, \ldots, r_N^\$ - \mu_{X,N}^\$ \right)$ is a $N \times N$ diagonal matrix, with the quantities $r_1^\$ - \mu_{X,1}^\$, \ldots, r_N^\$ - \mu_{X,N}^\$$ along the diagonal and $\widehat{\Lambda}$, defined by $[\widehat{\Lambda}]_{ij} = \widehat{\lambda}_{ij}$, $i, j \in \{1, \ldots, N\}$, where

$$\widehat{\lambda}_{ij} = \omega_{ij} \lambda_{ij}, j \neq i \tag{IA.101}$$

$$\widehat{\lambda}_{ii} = -\sum_{j \neq i} \omega_{ij} \lambda_{ij}, j \neq i \tag{IA.102}$$

is the generator matrix of the Markov chain for the combined state of the economy under the risk-neutral measure. Solving (IA.100) gives (27), if $\det \left(\operatorname{diag}\left(r_1^\$ - \mu_{X,1}^\$, \ldots, r_N^\$ - \mu_{X,N}^\$ \right) - \widehat{\Lambda} \right) \neq 0$.

Similarly, we can show that the before-tax value of the claim to the real earnings stream Y, when the current state is i is given by $P_{i,t}^Y = p_i Y_t$, where

$$(p_1, \ldots, p_N)^\top = (\operatorname{diag}\left(r_1 - \mu_{Y,1}, \ldots, r_N - \mu_{Y,N} \right) - \widehat{\Lambda})^{-1} 1_{6 \times 1} \tag{IA.103}$$

Hence, from the basic asset pricing equation

$$E_t \left[\frac{dP_t^Y + Y dt}{P_t^Y} - r_{s_{t-}} dt \middle| s_{t-} = i \right] = -E_t \left[\frac{d\pi_t}{\pi_t} \frac{dP_t^Y}{P_t^Y} \middle| s_{t-} = i \right],$$

we obtain the unlevered risk premium:

$$E_t \left[\frac{dP_t^Y + Y_t dt}{P^Y} - r_{s_{t-}} dt \middle| s_{t-} = i \right] = -\sum_{j \neq i} \left(\widehat{\lambda}_{ij} - \lambda_{ij} \right) \left(\frac{p_j}{p_i} - 1 \right) dt, i, j \in \{1, \ldots, N\}.$$

Applying Ito's Lemma,

$$dP_{i,t}^X = p_i dX_t + \sum_{j \neq i} \lambda_{ij} \left(p_j - p_i \right) dt + \sum_{j \neq i} \left(p_j - p_i \right) dN_{ij,t}^P, \, i, j \in \{1, \ldots, N\},$$

Thus, the volatility of returns on unlevered equity in state i is given by

$$\sigma_{R,i} = \sqrt{\sum_{j \neq i} \lambda_{ij} \left(\frac{p_j}{p_i} - 1 \right)^2}, \; i, j \in \{1, \ldots, N\}.$$

IA.H Arrow-Debreu Securities – Default

The Arrow-Debreu default claim denoted by $q^{\$}_{D,ij}$ is the value of a dollar paid if default occurs in state j and the current state is i. In a static capital structure model, these are the only Arrow-Debreu claims needed. Given the initial state (in which the firm selected its capital structure, there are N^2 such claims: $\{q^{\$}_{D,ij}\}_{i,j \in \{1,\ldots,N\}}$. We assume, without loss of generality, that the regimes are labelled so that the default boundaries respect a monotonic ordering $X_{D,1} > \ldots > X_{D,N}$.

We say that a firm's earnings are in the default region $\mathcal{D}_k$, $k = 0, \ldots, N-1$, when they fall in the interval $(X_{D,k+1}, X_{D,k}]$, assuming that $X_{D,0} \to \infty$. Region $\mathcal{D}_N$ is $(-\infty, X_{D,N}]$.

Proposition IA.2 *Let A_k be*

$$A_k = \begin{pmatrix} 0_{N-k \times N-k} & -I_{N-k \times N-k} \\ 2S^{-1}_{x,k}(\widehat{\Lambda}_k - R^{\$}_k) & 2S^{-1}_{x,k} M_{x,k} \end{pmatrix},$$

where $0_{n \times m} \in \mathbb{R}^{n \times m}$ denotes a matrix of zeros, $I_{n \times n} \in \mathbb{R}^{n \times n}$ denotes the n-dimensional identity matrix, $\widehat{\Lambda}_k$, $R^{\$}_k$, $M_{x,k}$, and $S_{x,k}$ are the $N-k$ by $N-k$ matrices obtained by removing the first k rows and columns of $\widehat{\Lambda}$,

$$R^{\$} = \text{diag}(r^{\$}_1, \ldots, r^{\$}_N), \qquad M_x = \text{diag}(\hat{\mu}^{\$}_{x,1}, \ldots, \hat{\mu}^{\$}_{x,N}), \quad \textit{and} \quad S_x = \text{diag}(\sigma^2_{x,1}, \ldots, \sigma^2_{x,N}),$$

with $\widehat{\mu}^{\$}_{x,i} = \widehat{\mu}^{\$}_{X,i} - \frac{1}{2}\sigma^2_{X,i}$ and $\sigma_{x,i} = \sigma_{X,i}$ the drift and diffusion coefficient of $x = \log X$ under $\mathbb{Q}^{\$}$.

Given the integration constants $h_{i,j}(\omega)$, the default Arrow-Debreu in region $\mathcal{D}_k$ are given by

Region $\mathcal{D}_0$:

$$q^{\$}_{D,ij}(x) = \sum_{l=1}^{N} h_{ij}(\omega_{0,l}) e^{-\omega_{0,l} x}, \tag{IA.104}$$

where $\omega_{0,1} > \ldots > \omega_{0,N} > 0$ are the N positive eigenvalues of A_0.

Region $\mathcal{D}_k, k \in \{1, \ldots, N-1\}$:

$$q^{\$}_{D,ij}(x) = \delta_{ij}, \qquad i \in \{1, \ldots, k\}, j \in \{1, \ldots, N\},$$

$$q^{\$}_{D,ij}(x) = \sum_{l=1}^{2(N-k)} h_{ij}(\omega_l) e^{-\omega_l x} - [A^{-1}_k B_k]_{i-k,j}, \qquad i \in \{k+1, \ldots, N\}, j \in \{1, \ldots, N\}. \tag{IA.105}$$

where $\omega_{k,l}$ are the $2(N-k)$ eigenvalues of A_k and

$$B_k = \begin{pmatrix} 0_{N-k \times k} & 0_{N-k \times N-k} \\ B^{\varnothing}_k & 0_{N-k \times N-k} \end{pmatrix}, \qquad B^{\varnothing}_k = \begin{pmatrix} 2\frac{\widehat{\lambda}_{k+1,1}}{\sigma^2_{k+1}} & 2\frac{\widehat{\lambda}_{k+1,2}}{\sigma^2_{x,k+1}} & \cdots & 2\frac{\widehat{\lambda}_{k+1,k}}{\sigma^2_{x,k+1}} \\ 2\frac{\widehat{\lambda}_{k+2,1}}{\sigma^2_{x,k+2}} & 2\frac{\widehat{\lambda}_{x,k+2,2}}{\sigma^2_{x,k+2}} & \cdots & 2\frac{\widehat{\lambda}_{x,k+2,k}}{\sigma^2_{x,k+2}} \\ \vdots & \vdots & \cdots & \vdots \\ 2\frac{\widehat{\lambda}_{N,1}}{\sigma^2_{x,N}} & 2\frac{\widehat{\lambda}_{N,2}}{\sigma^2_{x,N}} & \cdots & 2\frac{\widehat{\lambda}_{N,k}}{\sigma^2_{x,N}} \end{pmatrix}.$$

Region $\mathcal{D}_N$:

$$q^{\$}_{D,k,ij}(x) = \delta_{ij}, \quad \forall i,j. \tag{IA.106}$$

In each default region, for each ω, the integration constants $h_{k+1,\bullet}(\omega) \equiv [h_{k+1,j}(\omega)]_{j=1,\ldots,N} \in \mathbb{R}^{1\times N}$, are identified by the boundary conditions (Section IA.H.3), and the remaining integration constants

$$H_k(\omega) = \begin{pmatrix} h_{k+2,1}(\omega) & \cdots & h_{k+2,N}(\omega) \\ \vdots & \cdots & \vdots \\ h_{N,1}(\omega) & \cdots & h_{N,N}(\omega) \end{pmatrix} \tag{IA.107}$$

are given by

$$\boldsymbol{H}_k(\omega) = -\boldsymbol{G}_k^{-1}(\omega)\ \underline{g}_{k+\bullet,1}(\omega)\ h_{k+1,\bullet}(\omega) \tag{IA.108}$$

where $\underline{g}_{k+\bullet,k+1}(\omega) \equiv [g_{i,k+1}(\omega)]_{i=k+2,\ldots,N} \in \mathbb{R}^{(N-k-1)\times 1}$ comprises the last $N-k-1$ elements of the first column of

$$\boldsymbol{G}(\omega) = 2S_x^{-1}(\widehat{\Lambda} - R^{\$}) - \omega(2S_x^{-1}M_x - \omega I_{N\times N}). \tag{IA.109}$$

and $G_k(\omega)$ is the $N-k-1$ by $N-k-1$ matrix obtained by removing the first $k+1$ rows and columns of $G(\omega)$, i.e.

$$\boldsymbol{G}_k(\omega) = [\boldsymbol{G}(\omega)]_{i\in\{k+2,\ldots,N\},\, j\in\{k+2,\ldots,N\}},\ for\, k \in \{0,\ldots,N-1\} \tag{IA.110}$$

The next two subsections outline the proof of Proposition IA.2.

IA.H.1 Region $\mathcal{D}_0$: $X_t > X_{D,1}$

We start by analyzing the case where earnings at the current date t are above the highest default boundary, i.e. $X_t > X_{D,1}$. Hence, if earnings hit the boundary $X_{D,j}$ from above for the first time in state j, $\{q^{\$}_{D,ij}\}_{i,j\in\{1,\ldots,N\}}$ will pay one dollar; otherwise, the security expires worthless. Starting from (31), we change the probability measure from $\mathbb{P}$ to $\mathbb{Q}^{\$}$ to obtain

$$q^{\$}_{D,ij,t} = E_t^{\mathbb{Q}^{\$}}\left[e^{-\int_t^{\tau_D} r_u^{\$} du} I_{\{s_{\tau_D}=j\}} \middle| s_t = i \right]. \tag{IA.111}$$

From the Feynman-Kac Theorem, we hence obtain

$$E_t^{\mathbb{Q}^{\$}}[dq^{\$}_{D,ij} - r_i^{\$} q^{\$}_{D,ij} dt] = 0,\ i,j \in \{1,\ldots,N\}. \tag{IA.112}$$

Using Ito's Lemma, the above equation can be rewritten as the following second-order ordinary differential-equation system:[40]

$$\frac{1}{2}\sigma_{x,i}^2 \frac{d^2 q^{\$}_{D,ij}}{dx^2} + \widehat{\mu}^{\$}_{x,i} \frac{dq^{\$}_{D,ij}}{dx} + \sum_{k\neq i} \widehat{\lambda}_{ik} \left(q^{\$}_{D,kj} - q^{\$}_{D,ij} \right) = r_i^{\$} q^{\$}_{D,ij},\ i,j \in \{1,\ldots,N\}, \tag{IA.113}$$

where $\widehat{\mu}^{\$}_{x,i}$ and $\sigma_{x,i}$ are defined in (IA.98) and (IA.97), respectively.

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[40]Note that since the puts are perpetual, $\frac{\partial q^{\$}_{ij,t}}{\partial t} = 0$. Hence, $q^{\$}_{ij,t}$ is a function of the stochastic process $x = \log X$ and the state of the economy.

In order to solve this system of ODEs, define

$$\begin{aligned} z_{ij} &= q_{D,ij}, i,j \in \{1,\ldots,N\} && \text{(IA.114)} \\ z_{N+i,j} &= \frac{dq_{D,ij}}{dx}, i,j \in \{1,\ldots,N\}. && \text{(IA.115)} \end{aligned}$$

Then, we obtain the following first order linear system

$$\begin{aligned} \frac{dz_{ij}}{dx} - z_{N+i,j} &= 0,\ i,j \in \{1,\ldots,N\}, && \text{(IA.116)} \\ \frac{dz_{N+i,j}}{dx} + \frac{2\widehat{\mu}^{\$}_{x,i}}{\sigma^2_{x,i}} z_{N+i,j} + \sum_{k\neq i} \frac{2\widehat{\lambda}_{ik}}{\sigma^2_{x,i}} \left(z_{kj} - z_{ij}\right) - \frac{2r^{\$}_i}{\sigma^2_{x,i}} z_{ij} &= 0,\ i,j \in \{1,\ldots,N\}. \end{aligned}$$

Expressing the above equation system in matrix form gives

$$\boldsymbol{Z}' + \boldsymbol{A}_0\boldsymbol{Z} = \boldsymbol{0}_{2N\times N}, \tag{IA.117}$$

where the ij'th element of the $2N$ by N matrix, $\boldsymbol{Z}$, is

$$[\boldsymbol{Z}]_{ij} = z_{ij}, i \in \{1,\ldots,2N\}, j \in \{1,\ldots,N\}, \tag{IA.118}$$

and $\boldsymbol{Z}' = \frac{d\boldsymbol{Z}}{dx}$.

To solve eq. (IA.117), one first finds the eigenvectors and eigenvalues of $\boldsymbol{A}_0$. Their defining equation is

$$\boldsymbol{A}_0\underline{e}_i = \omega_i\underline{e}_i,\ i \in \{1,\ldots,2N\}, \tag{IA.119}$$

where ω_i is the i'th eigenvalue and $\underline{e}_i$ is the corresponding eigenvector. Note that $\boldsymbol{A}_0$ has N positive and N negative eigenvalues (Jobert and Rogers (2006)).

It follows from (IA.119) that the eigenvalues of A_0 are the roots of its characteristic polynomial; that is, any eigenvalue ω is a solution to the following $2N$'th-order polynomial:

$$\det(\boldsymbol{A}_0 - \omega\boldsymbol{I}) = 0.$$

To simplify the above expression for the characteristic polynomial, we then use the following identity from Silvester: If $\boldsymbol{F} = \begin{pmatrix} \boldsymbol{F}_{11} & \boldsymbol{F}_{12} \\ \boldsymbol{F}_{21} & \boldsymbol{F}_{22} \end{pmatrix}$, where $\boldsymbol{F}_{ij}$, $i,j \in \{1,2\}$ are N by N matrices, any two of which commute with each other, then

$$\det\boldsymbol{F} = \det(\boldsymbol{F}_{11}\boldsymbol{F}_{22} - \boldsymbol{F}_{12}\boldsymbol{F}_{21}). \tag{IA.120}$$

Since

$$\boldsymbol{A}_0 - \omega\boldsymbol{I} = \begin{pmatrix} -\omega\boldsymbol{I} & -\boldsymbol{I} \\ 2\boldsymbol{S}_x^{-1}(\widehat{\boldsymbol{\Lambda}} - \boldsymbol{R}^{\$}) & 2\boldsymbol{S}_x^{-1}\boldsymbol{M}_x - \omega\boldsymbol{I} \end{pmatrix}$$

and diagonal matrices commute with all other matrices of the same size, any pair of the N submatrices in $\boldsymbol{A}_0$ commute. Therefore, one can apply (IA.120) and

$$0 = \det(\boldsymbol{A}_0 - \omega\boldsymbol{I}) = \det\boldsymbol{G}(\omega), \tag{IA.121}$$

where

$$\boldsymbol{G}(\omega) = \omega(\omega\boldsymbol{I} - 2\boldsymbol{S}_x^{-1}\boldsymbol{M}_x) - 2\boldsymbol{S}_x^{-1}(\boldsymbol{R}^{\$} - \widehat{\boldsymbol{\Lambda}}). \tag{IA.122}$$

Observe that the ω's in the N by N matrix $\boldsymbol{G}(\omega)$ appear only along the diagonal. When $N \leq 2$ the polynomial (IA.121) is of order 4 or less and can be solved exactly in closed-form. When $N \geq 3$, it must be solved numerically. From Jobert and Rogers (2006), we know that N of the solutions to (IA.121) lie in the left half-plane of the Argand diagram and N lie in the right half-plane. Once the eigenvalues have been obtained, the eigenvectors are obtained by solving (IA.119). We then define the $2N$ by $2N$ matrix of eigenvectors, E, by stacking the eigenvectors as follows

$$\boldsymbol{E} = (\underline{e}_1, \dots, \underline{e}_{2N}).$$

Hence, the ij'th component of $\boldsymbol{E}$ is the i'th element of the j'th eigenvector, i.e.

$$\boldsymbol{E}_{ij} = (\underline{e}_j)_i.$$

Given the $\boldsymbol{E}$ matrix, we can define the $2N$ by N matrix $\boldsymbol{W}$ via

$$\boldsymbol{E}\boldsymbol{W} = \boldsymbol{Z}. \tag{IA.123}$$

We can then rewrite (IA.117) as

$$\boldsymbol{E}\boldsymbol{W}' + \boldsymbol{A}_0\boldsymbol{E}\boldsymbol{W} = \boldsymbol{0}_{2N\times N}, \tag{IA.124}$$

$$\Leftrightarrow \boldsymbol{E}^{-1}(\boldsymbol{E}\boldsymbol{W}' + \boldsymbol{A}_0\boldsymbol{E}\boldsymbol{W}) = \boldsymbol{W}' + \boldsymbol{E}^{-1}\boldsymbol{A}_0\boldsymbol{E}\boldsymbol{W} = \boldsymbol{0}_{2N\times N}, \tag{IA.125}$$

$$\Leftrightarrow \boldsymbol{W}' + \boldsymbol{D}\boldsymbol{W} = \boldsymbol{0}_{2N\times N}, \tag{IA.126}$$

where

$$\boldsymbol{D} = \boldsymbol{E}^{-1}\boldsymbol{A}_0\boldsymbol{E} = \mathrm{diag}(\omega_1, \dots, \omega_{2N}). \tag{IA.127}$$

The first order differential equation system of eq. (IA.126) is similar to that of eq. (IA.117), with the notable difference that $\boldsymbol{D}$ is a diagonal matrix while $\boldsymbol{A}_0$ is not. Making use of this, we premultiply both sides of eq. (IA.126) by the integrating factor $e^{\boldsymbol{D}x}$, which yields

$$e^{\boldsymbol{D}x}\boldsymbol{W}' + e^{\boldsymbol{D}x}\boldsymbol{D}\boldsymbol{W} = (e^{\boldsymbol{D}x}\boldsymbol{W})' = \boldsymbol{0}_{2N\times N}.$$

Integrating the above equation gives

$$e^{\boldsymbol{D}x}\boldsymbol{W} = \boldsymbol{K},$$

where $\boldsymbol{K}$ is a $2N$ by N matrix of constants of integration. Therefore, the general solution of eq. (IA.126) is

$$\boldsymbol{W} = e^{-\boldsymbol{D}x}\boldsymbol{K},$$

which, given eq. (IA.123), implies

$$\boldsymbol{Z} = \boldsymbol{E}e^{-\boldsymbol{D}x}\boldsymbol{K} = e^{-\boldsymbol{D}x}\boldsymbol{E}\boldsymbol{K}, \tag{IA.128}$$

given that $\boldsymbol{D}$ is $2N \times 2N$ and diagonal, and that $\boldsymbol{E}$ is $2N \times 2N$.

Thus,

$$q^{\$}_{D,ij}(x) = \sum_{l=1}^{2N} h_{ij}(\omega_l)e^{-\omega_l x}, \tag{IA.129}$$

where the $h_{ij}(\omega_l)$ are constants of integration that depend on the eigenvalues. Then, to ensure the finiteness of the Arrow-Debreu prices as $x \to \infty$, focus on the N positive eigenvalues of $\boldsymbol{A}_0$. That is, we obtain $q^{\$}_{D,ij}$, the Arrow-Debreu prices in region $\mathcal{D}_0$, where

$X > X_{D,1}$:

$$q^{\$}_{D,ij}(x) = \sum_{m=1}^{N} h_{ij}(\omega_m) e^{-\omega_{0,m} x}, \tag{IA.130}$$

where, without loss of generality, $\omega_{0,1} > \ldots > \omega_{0,N} > 0$ are the N positive eigenvalues of $\boldsymbol{A}_0$, or equivalently the N positive roots of the characteristic polynomial $\det \boldsymbol{G}(\omega) = 0$

Note that, for any eigenvalue ω of $\boldsymbol{A}_0$, the solution $q^{\$}_{D,ij} = h_{ij}(\omega)e^{-\omega x}$, with $h_{ij}(\omega_l) = 0, \forall \omega_l \neq \omega$, solves (IA.129). Indeed, we then have

$$\boldsymbol{Z} = \begin{bmatrix} \boldsymbol{H}(\omega) \\ -\omega \boldsymbol{H}(\omega) \end{bmatrix} e^{-\omega x}, \quad \text{where} \quad \boldsymbol{H}(\omega) = \begin{pmatrix} h_{11}(\omega) & \cdots & h_{1N}(\omega) \\ \vdots & \ldots & \vdots \\ h_{N1}(\omega) & \cdots & h_{NN}(\omega) \end{pmatrix}$$

and

$$\boldsymbol{Z}' = -\omega \boldsymbol{Z}.$$

Hence, (IA.117) implies that

$$(-\omega \boldsymbol{I}_{2N\times 2N} + \boldsymbol{A}_0)\boldsymbol{Z} = \boldsymbol{0}_{2N\times N},$$

or, equivalently,

$$\begin{pmatrix} -\omega \boldsymbol{I}_{N\times N} & -\boldsymbol{I}_{N\times N} \\ 2\boldsymbol{S}_x^{-1}(\widehat{\boldsymbol{\Lambda}} - \boldsymbol{R}^{\$}) & 2\boldsymbol{S}_x^{-1}\boldsymbol{M}_x - \omega \boldsymbol{I}_{N\times N} \end{pmatrix} \begin{bmatrix} \boldsymbol{H}(\omega) \\ -\omega \boldsymbol{H}(\omega) \end{bmatrix} = \boldsymbol{0}_{2N\times N}. \tag{IA.131}$$

This particular solution is important since it allows us to express $N(N-1)$ of the N^2 integation constants in terms of the first N ones. Indeed, simplifying (IA.131) gives

$$\begin{aligned} -\omega \boldsymbol{I}_{N\times N}\boldsymbol{H}(\omega) + \boldsymbol{I}_{N\times N}\omega \boldsymbol{H}(\omega) &= \boldsymbol{0}_{N\times N}, \\ (2\boldsymbol{S}_x^{-1}(\widehat{\boldsymbol{\Lambda}} - \boldsymbol{R}^{\$}) - \omega(2\boldsymbol{S}_x^{-1}\boldsymbol{M}_x - \omega \boldsymbol{I}_{N\times N}))\boldsymbol{H}(\omega) &= \boldsymbol{0}_{N\times N}, \end{aligned}$$

where the first equation is trivial. To solve the second equation, we first consider

$$\boldsymbol{G}(\omega)\left(h_{1j}(\omega), \ldots, h_{Nj}(\omega)\right)^T = \boldsymbol{0}_{N\times 1}, \tag{IA.132}$$

where $\boldsymbol{G}(\omega)$ is defined in (IA.122). We denote the ij'th element of $\boldsymbol{G}(\omega)$ by $g_{ij}(\omega)$. Observe that only the diagonal elements of $\boldsymbol{G}(\omega)$ depend on ω. We know from (IA.121) that $\det \boldsymbol{G}(\omega) = 0$. Thus, the equations

$$\sum_{k=1}^{N} g_{ik}(\omega) h_{kj}(\omega),\ i \in \{1, \ldots, N\} \tag{IA.133}$$

are linearly dependent. However, the system

$$\sum_{k=1}^{N} g_{ik}(\omega) h_{kj}(\omega),\ i \in \{2, \ldots, N\} \tag{IA.134}$$

is linearly independent, allowing us to solve for $h_{kj}(\omega_{0,m})$, $k \in \{2, \ldots, N\}$ in terms of $h_{1j,m}$, for $j \in \{1, \ldots, N\}$, that is

$$\left(h_{2j}(\omega), \ldots, h_{Nj}(\omega)\right)^\top = -\begin{pmatrix} g_{22}(\omega) & \cdots & g_{2N}(\omega) \\ \vdots & \ldots & \vdots \\ g_{N2}(\omega) & \ldots & g_{NN}(\omega) \end{pmatrix}^{-1} \left(g_{21}, \ldots, g_{N1}\right)^\top h_{1j}(\omega).$$

where $(g_{21},\ldots,g_{N1})^\top$ is independent of ω For ease of notation, define

$$\underline{v}^0(\omega) = -\begin{pmatrix} g_{22}(\omega) & \cdots & g_{2N}(\omega) \\ \vdots & \cdots & \vdots \\ g_{N2}(\omega) & \cdots & g_{NN}(\omega) \end{pmatrix}^{-1} (g_{21},\ldots,g_{N1})^\top \tag{IA.135}$$

We see that $\underline{v}^0(\omega) \in \mathbb{R}^{N-1}$ is column vector, i.e.

$$\underline{v}^0(\omega) = (v^0(\omega)_1,\ldots,v^0(\omega)_{N-1})^\top, \tag{IA.136}$$

and so

$$\Big(h_{2j}(\omega),\ldots,h_{Nj}(\omega)\Big)^\top = (v^0(\omega)_1,\ldots,v^0(\omega)_{N-1})^\top h_{1j}(\omega). \tag{IA.137}$$

Therefore,

$$h_{ij}(\omega) = v^0(\omega)_{i-1} h_{1j}(\omega),\ i \in \{2,\ldots,N\},\ j \in \{1,\ldots,N\}. \tag{IA.138}$$

Now, recalling that ω can take N values, given by the N positive eigenvalues of A_0, define

$$v^0_{m,j} = [\underline{v}^0(\omega_{0,m})]_j,\ j \in \{1,\ldots,N-1\},\ \ m \in \{1,\ldots,N\}. \tag{IA.139}$$

Therefore

$$q^{\$,0}_{D,ij}(x) = \begin{cases} \sum_{m=1}^N h^0_{1j,m} e^{-\omega_{0,m}x},\ i=1 \\ \sum_{m=1}^N h^0_{1j,m} v^0_{m,i-1} e^{-\omega_{0,m}x},\ i \in \{2,\ldots,N\} \end{cases}, \tag{IA.140}$$

where the superscript 0 is used to clarify that this is the solution for region $\mathcal{D}_0$ and

$$h^0_{1j,m} = h_{1j}(\omega_{0,m}). \tag{IA.141}$$

It is helpful to define

$$\boldsymbol{H}^0_m = \boldsymbol{H}(\omega_{0,m}). \tag{IA.142}$$

We can thus see that $h^0_{1j,1},\ldots,h^0_{1j,N}$ are from different matrices $\boldsymbol{H}^0_1,\ldots,\boldsymbol{H}^0_N$ – $h^0_{1j,m}$ is the $1j$'th element of the matrix $\boldsymbol{H}^0_m$.

Writing out the N by N matrix $\boldsymbol{q}^{\$,0}_D(x) = [q^{\$,0}_{D,ij}(x)]_{ij}$, we have

$$\boldsymbol{q}^{\$,0}_D(x) = \begin{pmatrix} \sum_{m=1}^N h^0_{11,m} e^{-\omega_{0,m}x} & \sum_{m=1}^N h^0_{12,m} e^{-\omega_{0,m}x} & \cdots & \sum_{m=1}^N h^0_{1N,m} e^{-\omega_{0,m}x} \\ \sum_{m=1}^N h^0_{11,m} v^0_{m,1} e^{-\omega_{0,m}x} & \sum_{m=1}^N h^0_{12,m} v^0_{m,1} e^{-\omega_{0,m}x} & \cdots & \sum_{m=1}^N h^0_{1N,m} v^0_{m,1} e^{-\omega_{0,m}x} \\ \vdots & \vdots & \cdots & \vdots \\ \sum_{m=1}^N h^0_{11,m} v^0_{m,N-1} e^{-\omega_{0,m}x} & \sum_{m=1}^N h^0_{12,m} v^0_{m,N-1} e^{-\omega_{0,m}x} & \cdots & \sum_{m=1}^N h^0_{1N,m} v^0_{m,N-1} e^{-\omega_{0,m}x} \end{pmatrix} \tag{IA.143}$$

Note that (IA.153) contains N^2 constants of integration $h^0_{1j,m}$, $j,m \in \{1,\ldots,N\}$, which will be identified by the value and smooth pasting conditions (Section IA.H.3).

IA.H.2 Region $\mathcal{D}_k$: $X_{D,k+1} < X \le X_{D,k}$

We now turn to the analysis of Arrow-Debreu securities in region $\mathcal{D}_k$, i.e. when current earnings are above default boundary $k+1$, but below default boundary k. Note that the above analysis in default region $\mathcal{D}_0$ can be seen as a special case of the analysis below, with $X_{D,0} \to \infty$.

First, note that $q^{\$}_{D,ij} = \delta_{ij}, \forall i \leq k$. Indeed, if earnings are currently lower than $X_{D,k} < X_{D,k-1} < \ldots$, and if the current state is $i \leq k$, then the firm is in default and the present value of a dollar when the firm defaults in state j is 1 if $i = j$, and 0 otherwise. In particular, this means that, in region $\mathcal{D}_N$, where $X \leq X_{D,N}$, we have

$$q^{\$}_{D,ij} = \delta_{ij}.$$

Applying (IA.112) to the unknown $q^{\$}_{D,ij}, i > k$, yields the following system of ODEs

$$\begin{aligned}
\frac{dz_{k+1,j}}{dx} - z_{N+k+1,j} &= 0, \\
\frac{dz_{k+2,j}}{dx} - z_{N+k+2,j} &= 0, \\
&\vdots \\
\frac{dz_{N,j}}{dx} - z_{2N,j} &= 0, \\
\frac{dz_{N+k+1,j}}{dx} + \frac{2\widehat{\mu}^{\$}_{x,k+1}}{\sigma^2_{x,k+1}} z_{N+k+1,j} + \sum_{l=1}^{k} \frac{2\widehat{\lambda}_{k+1,l}}{\sigma^2_{x,k+1}} \left(\delta_{lj} - z_{k+1,j}\right) & \\
+ \sum_{l=k+2}^{N} \frac{2\widehat{\lambda}_{k+1,l}}{\sigma^2_{x,k+1}} \left(z_{lj} - z_{k+1,j}\right) - \frac{2r^{\$}_{k+1}}{\sigma^2_{x,k+1}} z_{k+1,j} &= 0, \\
\frac{dz_{N+k+2,j}}{dx} + \frac{2\widehat{\mu}^{\$}_{x,k+2}}{\sigma^2_{x,k+2}} z_{N+k+2,j} + \sum_{l=1}^{k} \frac{2\widehat{\lambda}_{k+2,l}}{\sigma^2_{x,k+2}} \left(\delta_{l,j} - z_{k+2,j}\right) & \\
+ \sum_{l=k+1, l\neq k+2}^{N} \frac{2\widehat{\lambda}_{k+2,l}}{\sigma^2_{x,k+2}} \left(z_{l,j} - z_{k+2,j}\right) - \frac{2r^{\$}_{k+2}}{\sigma^2_{x,k+2}} z_{k+2,j} &= 0 \\
&\vdots \\
\frac{dz_{2N,j}}{dx} + \frac{2\widehat{\mu}^{\$}_{x,N}}{\sigma^2_{x,N}} z_{2N,j} + \sum_{l=1}^{k} \frac{2\widehat{\lambda}_{N,l}}{\sigma^2_{x,N}} \left(\delta_{l,j} - z_{N,j}\right) + \sum_{l=k+1}^{N-1} \frac{2\widehat{\lambda}_{N,l}}{\sigma^2_{x,N}} \left(z_{l,j} - z_{N,j}\right) - \frac{2r^{\$}_{N}}{\sigma^2_{x,N}} z_{N,j} &= 0,
\end{aligned}$$

for $j = \{1, \ldots, N\}$. Rewriting the above equation system in matrix form, we obtain

$$\boldsymbol{Z}'_k + \boldsymbol{A}_k \boldsymbol{Z}_k + \boldsymbol{B}_k \; = \; \boldsymbol{Z}'_k + \boldsymbol{A}_k \left(\boldsymbol{Z}_k + \boldsymbol{A}_k^{-1} \boldsymbol{B}_k\right) \; = \; \widetilde{\boldsymbol{Z}}'_k + \boldsymbol{A}_k \widetilde{\boldsymbol{Z}}_k = 0, \tag{IA.144}$$

where $\widetilde{\boldsymbol{Z}}_k = \left(\boldsymbol{Z}_k + \boldsymbol{A}_k^{-1} \boldsymbol{B}_k\right)$, $\boldsymbol{Z}_k$ is the following $2(N-k)$ by N matrix

$$\boldsymbol{Z}_k = \begin{pmatrix}
z_{k+1,1} & z_{k+1,2} & \cdots & z_{k+1,N} \\
z_{k+2,1} & z_{k+2,2} & \cdots & z_{k+2,N} \\
\vdots & \vdots & \ldots & \vdots \\
z_{N,1} & z_{N,2} & \cdots & z_{N,N} \\
z_{N+k+1,1} & z_{N+k+1,2} & \cdots & z_{N+k+1,N} \\
z_{N+k+2,1} & z_{N+k+2,2} & \cdots & z_{N+k+2,N} \\
\vdots & \vdots & \ldots & \vdots \\
z_{2N,1} & z_{2N,2} & \cdots & z_{2N,N}
\end{pmatrix}.$$

Note that the $\boldsymbol{B}_k$ matrix of constants arises from the δ_{lj}'s appearing in the above differential equations:

(i) These appear only in the last $N-k$ equations. Hence, the first $N-k$ rows of the $\boldsymbol{B}_k$ matrix will comprise of zeros.

(ii) Since the sum in which the δ_{lj}'s appear are from 1 to k, δ_{lj} will be zero for all l whenever $j>k$. Hence, the last $N-k$ columns of the $\boldsymbol{B}_k$ matrix will comprise of zeros.

Thereafter, the development made, in region $\mathcal{D}_0$, between equations (IA.119) and (IA.128) can be applied to $\widetilde{\boldsymbol{Z}}_k$ in (IA.144) to yield

$$\widetilde{\boldsymbol{Z}}_k = e^{-\boldsymbol{D}_k x}\boldsymbol{E}_k\boldsymbol{K}_k, \tag{IA.145}$$

or, equivalently,

$$\boldsymbol{Z}_k = e^{-\boldsymbol{D}_k x}\boldsymbol{E}_k\boldsymbol{K}_k - \boldsymbol{A}_k^{-1}\boldsymbol{B}_k. \tag{IA.146}$$

Therefore,

$$\begin{aligned} q_{D,ij}^{\$,k}(x) &= \delta_{ij}, i\in\{1,\dots,k\}, j\in\{1,\dots,N\} \\ q_{D,ij}^{\$,k}(x) &= \sum_{m=1}^{2(N-k)} h_{ij}(\omega_m)e^{-\omega_m x} - [\boldsymbol{A}_k^{-1}\boldsymbol{B}_k]_{i-k,j}, i\in\{k+1,\dots,N\}, j\in\{1,\dots,N\}. \end{aligned}$$

Note that the $-k$ offset on the rows of the $2(N-k)\times N$ matrix $\boldsymbol{A}_k^{-1}\boldsymbol{B}_k$ simply accounts for the fact the first row of this matrix corresponds to the $(k+1)^{\text{th}}$ Arrow-Debreu security. Once more, for each eigenvalue ω of $\boldsymbol{A}_k$, the particular solution

$$\boldsymbol{Z}_k = \begin{pmatrix} \boldsymbol{H}_k(\omega) \\ -\omega\boldsymbol{H}_k(\omega) \end{pmatrix} e^{-\omega y} - \boldsymbol{A}_k^{-1}\boldsymbol{B}_k, \quad \text{where} \quad \boldsymbol{H}_k(\omega) = \begin{pmatrix} h_{k+1,1}(\omega) & \cdots & h_{k+1,N}(\omega) \\ \vdots & \dots & \vdots \\ h_{N,1}(\omega) & \cdots & h_{N,N}(\omega) \end{pmatrix},$$

can be used to express the elements in final $N-k-1$ rows of $\boldsymbol{H}_k$ but in terms of $h_{k+1,1}(\omega),\dots,h_{k+1,N}(\omega)$ as follows:

$$\Big(h_{k+2,j}(\omega),\dots,h_{Nj}(\omega)\Big)^\top = -\boldsymbol{G}_k^{-1}(\omega)\ (g_{k+2,1},\dots,g_{N,1})^\top h_{k+1,j}(\omega), \tag{IA.147}$$

where the $N-k-1$ by $N-k-1$ matrix $G_k(\omega)$ is given by

$$\boldsymbol{G}_k(\omega) = \begin{pmatrix} g_{k+2,k+2}(\omega) & \cdots & g_{k+2,N}(\omega) \\ \vdots & \dots & \vdots \\ g_{N,k+2}(\omega) & \cdots & g_{N,N}(\omega) \end{pmatrix}. \tag{IA.148}$$

For ease of notation, we define

$$\underline{v}^k(\omega) = -\boldsymbol{G}_k^{-1}(\omega)(g_{k+2,1},\dots,g_{N,1})^\top \tag{IA.149}$$

We see that $\underline{v}^k(\omega)\in\mathbb{R}^{N-k-1}$ is a column vector, i.e.

$$\underline{v}^k(\omega) = (v^k(\omega)_1,\dots,v^k(\omega)_{N-k-1})^\top, \tag{IA.150}$$

and so

$$\Big(h_{k+2,j}(\omega),\dots,h_{N,j}(\omega)\Big)^\top = (v^k(\omega)_1,\dots,v^k(\omega)_{N-k-1})^\top h_{k+1,j}(\omega). \tag{IA.151}$$

Therefore,

$$h_{ij}(\omega) = v^k(\omega)_{i-k-1} h_{k+1,j},\ i \in \{k+2,\dots,N\},\ j \in \{1,\dots,N\}. \tag{IA.152}$$

This leaves us with N free constants for each of the $2(N-k)$ eigenvalues, giving $2N(N-k)$ constants of integration in total for region $\mathcal{D}_k$. Hence,

$$q^{\$,k}_{D,ij}(x) = \begin{cases} \delta_{ij},\ i \in \{1,\dots,k\} \\ \sum_{m=1}^{2(N-k)} h^k_{k+1\,j,m} e^{-\omega_m x} - [\boldsymbol{A}_k^{-1}\boldsymbol{B}_k]_{1,j},\ i = k+1 \\ \sum_{m=1}^{2(N-k)} h^k_{k+1\,j,m} v^k_{m,i-k-1} e^{-\omega_m x} - [\boldsymbol{A}_k^{-1}\boldsymbol{B}_k]_{i-k,j},\ i \in \{k+2,\dots,N\} \end{cases}., \tag{IA.153}$$

where $h^k_{k+1\,j,m} = h_{k+1\,j}(\omega^k_m)$. Observe that $h^k_{k+1\,j,m}$ is the j'th entry in the first row of $\boldsymbol{H}^k_m = \boldsymbol{H}^k(\omega^k_m)$.

In matrix form, we have

$$\boldsymbol{q}^{\$,k}_D(x) = \begin{pmatrix} \boldsymbol{I}_{k\times k} & \boldsymbol{0}_{N-k\times N-k} \\ \boldsymbol{q}^{\$,k}_{D,L}(x) & \boldsymbol{q}^{\$,k}_{D,R}(x) \end{pmatrix} \tag{IA.154}$$

where $\boldsymbol{q}^{\$,k}_{D,L}(x)$ is the following $N-k$ by k matrix function

$$\boldsymbol{q}^{\$,k}_{D,L}(x) = \begin{pmatrix} \sum_{m=1}^{2(N-k)} h^k_{k+1\,1,m} e^{-\omega_m x} - [\boldsymbol{A}_k^{-1}\boldsymbol{B}_k]_{1,1} & \cdots & \sum_{m=1}^{2(N-k)} h^k_{k+1\,k,m} e^{-\omega_m x} - [\boldsymbol{A}_k^{-1}\boldsymbol{B}_k]_{1,k} \\ \sum_{m=1}^{2(N-k)} h^k_{k+1\,1,m} v^k_{m,1} e^{-\omega_m x} - [\boldsymbol{A}_k^{-1}\boldsymbol{B}_k]_{2,1} & \cdots & \sum_{m=1}^{2(N-k)} h^k_{k+1\,k,m} v^k_{m,1} e^{-\omega_m x} - [\boldsymbol{A}_k^{-1}\boldsymbol{B}_k]_{2,k} \\ \vdots & \ddots & \vdots \\ \sum_{m=1}^{2(N-k)} h^k_{k+1\,1,m} v^k_{m,N-k-1} e^{-\omega_m x} - [\boldsymbol{A}_k^{-1}\boldsymbol{B}_k]_{N-k,1} & \cdots & \sum_{m=1}^{2(N-k)} h^k_{k+1\,k,m} v^k_{m,N-k-1} e^{-\omega_m x} - [\boldsymbol{A}_k^{-1}\boldsymbol{B}_k]_{N-k,k} \end{pmatrix}, \tag{IA.155}$$

and $\boldsymbol{q}^{\$,k}_{D,R}(x)$ is the following $N-k$ by $N-k$ matrix function

$$\boldsymbol{q}^{\$,k}_{D,R}(x) = \begin{pmatrix} \sum_{m=1}^{2(N-k)} h^k_{k+1\,k+1,m} e^{-\omega_m x} - [\boldsymbol{A}_k^{-1}\boldsymbol{B}_k]_{1,k+1} & \cdots & \sum_{m=1}^{2(N-k)} h^k_{k+1\,N,m} e^{-\omega_m x} - [\boldsymbol{A}_k^{-1}\boldsymbol{B}_k]_{1,N} \\ \sum_{m=1}^{2(N-k)} h^k_{k+1\,k+1,m} v^k_{m,1} e^{-\omega_m x} - [\boldsymbol{A}_k^{-1}\boldsymbol{B}_k]_{2,k+1} & \cdots & \sum_{m=1}^{2(N-k)} h^k_{k+1\,N,m} v^k_{m,1} e^{-\omega_m x} - [\boldsymbol{A}_k^{-1}\boldsymbol{B}_k]_{2,N} \\ \vdots & \ddots & \vdots \\ \sum_{m=1}^{2(N-k)} h^k_{k+1\,k+1,m} v^k_{m,N-k-1} e^{-\omega_m x} - [\boldsymbol{A}_k^{-1}\boldsymbol{B}_k]_{N-k,k+1} & \cdots & \sum_{m=1}^{2(N-k)} h^k_{k+1\,N,m} v^k_{m,N-k-1} e^{-\omega_m x} - [\boldsymbol{A}_k^{-1}\boldsymbol{B}_k]_{N-k,N} \end{pmatrix}, \tag{IA.156}$$

$\boldsymbol{I}_{k\times k}$ is the k by k identity matrix and $\boldsymbol{0}_{N-k\times N-k}$ is the $N-k$ by $N-k$ matrix of zeros.

IA.H.3 Boundary conditions for Arrow-Debreu default claims

Given the above development, we are left with N^2 free integration constants in region $\mathcal{D}_0$, and $2(N-k)N$ free constants in region $\mathcal{D}_k, k \in \{1,\dots,N-1\}$. Hence, we still have to solve for the N^3 constants,

$$N^2 + \sum_{k=1}^{N-1} 2(N-k)N = N^2 + 2N \sum_{k=1}^{N-1} (N-k) = N^3, \tag{IA.157}$$

that satisfy the N^3 boundary conditions (value matching & smooth pasting) of the problem at hand. For each default boundary $X_{D,k}, k \in \{1,\dots,N\}$, we have that:

(VM) The value of the $N+(N-k)N$ Arrow-Debreu securities with $i \geq k$, must be the same on both sides of the default boundary, i.e.

$$q_{D,ij}^{\$,k-1}(x_{D,k}) = q_{ij}^{\$,k}(x_{D,k}), \text{ where } , i \in \{k,\ldots,N\}, j \in \{1,\ldots,N\}; \tag{IA.158}$$

(SP) The first derivatives of the $(N-k)N$ Arrow-Debreu securities with $i > k$, must be the same on both side of each default boundary, i.e.

$$\left.\frac{dq_{ij}^{\$,k-1}}{dx}\right|_{x_{D,k}} = \left.\frac{dq_{ij}^{\$,k}}{dx}\right|_{x_{D,k}}, \text{ where } , i \in \{k+1,\ldots,N\}, j \in \{1,\ldots,N\}; \tag{IA.159}$$

where the k superscript in q^k notation highlights that the Arrow-Debreu claim is computed in the region $\mathcal{D}_k$.

We can write the N^3 boundary conditions in vector-matrix form as

$$\boldsymbol{T}_N \underline{\phi}_N = \underline{\chi}_N. \tag{IA.160}$$

The N^3 by N^3 sparse matrix $\boldsymbol{T}_N$ is defined by

$$\boldsymbol{T}_N = \begin{pmatrix} \overline{\boldsymbol{L}}_{N,1} & -\overline{\boldsymbol{M}}_{N,1} & & & & \\ & \overline{\boldsymbol{L}}_{N,2} & -\overline{\boldsymbol{M}}_{N,2} & & & \\ & & \overline{\boldsymbol{L}}_{N,3} & -\overline{\boldsymbol{M}}_{N,3} & & \\ & & & \ddots & \ddots & \\ & & & & \overline{\boldsymbol{L}}_{N,N-1} & -\overline{\boldsymbol{M}}_{N,N-1} \\ & & & & & \overline{\boldsymbol{L}}_{N,N} \end{pmatrix}, \tag{IA.161}$$

where the unspecified entries are matrices of zeros and

$$\overline{\boldsymbol{L}}_{N,k} = \text{diag}\,\underbrace{(\boldsymbol{L}_{N,k},\ldots,\boldsymbol{L}_{N,k})}_{N \text{ repetitions}}, \tag{IA.162}$$

$$\overline{\boldsymbol{M}}_{N,k} = \text{diag}\,\underbrace{(\boldsymbol{M}_{N,k},\ldots,\boldsymbol{M}_{N,k})}_{N \text{ repetitions}}, \tag{IA.163}$$

where

$$\boldsymbol{L}_{N,1} = \left(\begin{array}{ccc} e^{-\omega_{0,1}x_{D,1}} & \cdots & e^{-\omega_{0,N}x_{D,1}} \\ \hline v_{1,1}^{0} e^{-\omega_{0,1}x_{D,1}} & \cdots & v_{N,1}^{0} e^{-\omega_{0,N}x_{D,1}} \\ \vdots & \cdots & \vdots \\ v_{1,N-1}^{0} e^{-\omega_{0,1}x_{D,1}} & \cdots & v_{N,N-1}^{0} e^{-\omega_{0,N}x_{D,1}} \\ \hline v_{1,1}^{0}\omega_{0,1} e^{-\omega_{0,1}x_{D,1}} & \cdots & v_{N,1}^{0}\omega_{0,N} e^{-\omega_{0,N}x_{D,1}} \\ \vdots & \cdots & \vdots \\ v_{1,N-1}^{0}\omega_{0,1} e^{-\omega_{0,1}x_{D,1}} & \cdots & v_{N,N-1}^{0}\omega_{0,N} e^{-\omega_{0,N}x_{D,1}} \end{array}\right) \in \mathbb{R}^{1+2(N-1)\times N}, \tag{IA.164}$$

and

$$
\boldsymbol{L}_{N,k} = \left(\begin{array}{ccc}
e^{-\omega_1^{k-1} x_{D,k}} & \dots & e^{-\omega_{2(N-(k-1))}^{k-1} x_{D,k}} \\
\hdashline
v_{1,1}^{k-1} e^{-\omega_1^{k-1} x_{D,k}} & \dots & v_{2(N-(k-1)),1}^{k-1} e^{-\omega_{2(N-(k-1))}^{k-1} x_{D,k}} \\
\vdots & \dots & \vdots \\
v_{1,N-k}^{1} e^{-\omega_1^{k-1} x_{D,k}} & \dots & v_{2(N-(k-1)),N-k}^{k-1} e^{-\omega_{2(N-(k-1))}^{k-1} x_{D,k}} \\
\hdashline
v_{1,1}^{k-1} \omega_1^{k-1} e^{-\omega_1^{k-1} x_{D,k}} & \dots & v_{2(N-(k-1)),1}^{k-1} \omega_{2(N-(k-1))}^{k-1} e^{-\omega_{2(N-(k-1))}^{k-1} x_{D,k}} \\
\vdots & \dots & \vdots \\
v_{1,N-k}^{1} \omega_1^{k-1} e^{-\omega_1^{k-1} x_{D,k}} & \dots & v_{2(N-(k-1)),N-k}^{k-1} \omega_{2(N-(k-1))}^{k-1} e^{-\omega_{2(N-(k-1))}^{k-1} x_{D,k}}
\end{array}\right) \in \mathbb{R}^{[1+2(N-k)]\times 2[N-(k-1)]}, \tag{IA.165}
$$

$$
k \in \{2, \dots, N-1\}, \tag{IA.166}
$$

and

$$
\boldsymbol{L}_{N,N} = \left(\begin{array}{cc} e^{-\omega_1^{N-1} x_{D,N}} & e^{-\omega_2^{N-1} x_{D,N}} \end{array} \right), \tag{IA.167}
$$

and

$$
\boldsymbol{M}_{N,k} = \left(\begin{array}{ccc}
0 & \dots & 0 \\
\hdashline
e^{-\omega_1^{k} x_{D,k}} & \dots & e^{-\omega_{2(N-k)}^{k} x_{D,k}} \\
v_{1,1}^{k} e^{-\omega_1^{k} x_{D,k}} & \dots & v_{2(N-k),1}^{k} e^{-\omega_{2(N-k)}^{k} x_{D,k}} \\
\vdots & \dots & \vdots \\
v_{1,N-k-1}^{1} e^{-\omega_1^{k} x_{D,k}} & \dots & v_{2(N-k),N-k-1}^{k} e^{-\omega_{2(N-k)}^{k} x_{D,k}} \\
\hdashline
\omega_1^{k} e^{-\omega_1^{k} x_{D,k}} & \dots & \omega_{2(N-k)}^{k} e^{-\omega_{2(N-k)}^{k} x_{D,k}} \\
v_{1,1}^{k} \omega_1^{k} e^{-\omega_1^{k} x_{D,k}} & \dots & v_{2(N-k),1}^{k} \omega_{2(N-k)}^{k} e^{-\omega_{2(N-k)}^{k} x_{D,k}} \\
\vdots & \dots & \vdots \\
v_{1,N-k-1}^{1} \omega_1^{k} e^{-\omega_1^{k} x_{D,k}} & \dots & v_{2(N-k),N-k-1}^{k} \omega_{2(N-k)}^{k} e^{-\omega_{2(N-k)}^{k} x_{D,k}}
\end{array}\right) \in \mathbb{R}^{[1+2(N-k)]\times 2(N-k)}, \tag{IA.168}
$$

$$
k \in \{1, \dots, N-1\}. \tag{IA.169}
$$

Observe that

$$
\boldsymbol{M}_{N,N-1} = \left(\begin{array}{cc}
0 & 0 \\
\hdashline
e^{-\omega_1^{N-1} x_{D,N-1}} & e^{-\omega_2^{N-1} x_{D,N-1}} \\
\hdashline
\omega_1^{N-1} e^{-\omega_1^{k} x_{D,N-1}} & \omega_2^{N-1} e^{-\omega_2^{N-1} x_{D,N-1}}
\end{array}\right) \in \mathbb{R}^{3\times 2}. \tag{IA.170}
$$

The N^3-dimensional column vector $\underline{\phi}_N$ contains the N^3 constants of integration as follows:

$$
\underline{\phi}_N = (\underline{h}_{11}^0, \dots, \underline{h}_{1N}^0, \underline{h}_{21}^1, \dots, \underline{h}_{2N}^1, \dots, \underline{h}_{N1}^{N-1}, \dots, \underline{h}_{NN}^{N-1})^\top \in \mathbb{R}^{N^3}. \tag{IA.171}
$$

where

$$
\underline{h}_{1j}^0 = (h_{1j,1}^0, \dots, h_{1j,N}^0)^\top \tag{IA.172}
$$

$$
\underline{h}_{k+1,j}^k = (h_{k+1,j,1}^k, \dots, h_{k+1,j,2(N-k)}^k)^\top,\ k \in \{1, \dots, N-1\}. \tag{IA.173}
$$

The N^3-dimensional column vector $\underline{\chi}_N$ is given by

$$\underline{\chi}_N = (\underline{\chi}(N,1)^\top, \ldots, \underline{\chi}(N,N)^\top)^\top, \tag{IA.174}$$

where

$$\underline{\chi}(N,k) = (\underline{\chi}_1(N,k)^\top, \ldots. \underline{\chi}_N(N,k)^\top, \underline{0}_{1\times N(N-k)})^\top, \tag{IA.175}$$

and

$$\underline{\chi}_j(N,k) = (\chi_{1j}(N,k), \ldots, \chi_{N-k+1,j}(N,k))^\top, \tag{IA.176}$$

where

$$\chi_{1j}(N,1) = \delta_{1j} \tag{IA.177}$$

$$\chi_{ij}(N,1) = -[\boldsymbol{A}_1^{-1}\boldsymbol{B}_1]_{i-1,j},\ i \in \{2,\ldots,N\},\ j \in \{1,\ldots,N\}, \tag{IA.178}$$

and

$$\chi_{1j}(N,k) = [\boldsymbol{A}_{k-1}^{-1}\boldsymbol{B}_{k-1}]_{1j} + \delta_{kj} \tag{IA.179}$$

$$\chi_{ij}(N,k) = [\boldsymbol{A}_{k-1}^{-1}\boldsymbol{B}_{k-1}]_{ij} - [\boldsymbol{A}_k^{-1}\boldsymbol{B}_k]_{i-1,j},\ i \in \{2,\ldots,N-k+1\},\ j \in \{1,\ldots,N\}, \tag{IA.180}$$

and

$$\chi_{1j}(N,N) = [\boldsymbol{A}_{N-1}^{-1}\boldsymbol{B}_{N-1}]_{1j} + \delta_{Nj},\ j \in \{1,\ldots,N\}. \tag{IA.181}$$

To understand how to implement the above scheme, observe that for $N = 3$, we have

$$\boldsymbol{T}_3 \underline{\phi}_3 = \underline{\chi}_3, \tag{IA.182}$$

where

$$\boldsymbol{T}_3 = \begin{pmatrix} \overline{\boldsymbol{L}}_{3,1} & -\overline{\boldsymbol{M}}_{3,1} & \\ & \overline{\boldsymbol{L}}_{3,2} & -\overline{\boldsymbol{M}}_{3,2} \\ & & \overline{\boldsymbol{L}}_{3,3} \end{pmatrix} \tag{IA.183}$$

$$= \left(\begin{array}{ccc|ccc|ccc} \boldsymbol{L}_{3,1} & \boldsymbol{0}_{5\times3} & \boldsymbol{0}_{5\times3} & -\boldsymbol{M}_{3,1} & \boldsymbol{0}_{5\times4} & \boldsymbol{0}_{5\times4} & \boldsymbol{0}_{5\times2} & \boldsymbol{0}_{5\times2} & \boldsymbol{0}_{5\times2} \\ \boldsymbol{0}_{5\times3} & \boldsymbol{L}_{3,1} & \boldsymbol{0}_{5\times3} & \boldsymbol{0}_{5\times4} & -\boldsymbol{M}_{3,1} & \boldsymbol{0}_{5\times4} & \boldsymbol{0}_{5\times2} & \boldsymbol{0}_{5\times2} & \boldsymbol{0}_{5\times2} \\ \boldsymbol{0}_{5\times3} & \boldsymbol{0}_{5\times3} & \boldsymbol{L}_{3,1} & \boldsymbol{0}_{5\times4} & \boldsymbol{0}_{5\times4} & -\boldsymbol{M}_{3,1} & \boldsymbol{0}_{5\times2} & \boldsymbol{0}_{5\times2} & \boldsymbol{0}_{5\times2} \\ \hline \boldsymbol{0}_{3\times3} & \boldsymbol{0}_{3\times3} & \boldsymbol{0}_{3\times3} & \boldsymbol{L}_{3,2} & \boldsymbol{0}_{3\times4} & \boldsymbol{0}_{3\times4} & -\boldsymbol{M}_{3,2} & \boldsymbol{0}_{3\times2} & \boldsymbol{0}_{3\times2} \\ \boldsymbol{0}_{3\times3} & \boldsymbol{0}_{3\times3} & \boldsymbol{0}_{3\times3} & \boldsymbol{0}_{3\times4} & \boldsymbol{L}_{3,2} & \boldsymbol{0}_{3\times4} & \boldsymbol{0}_{3\times2} & -\boldsymbol{M}_{3,2} & \boldsymbol{0}_{3\times2} \\ \boldsymbol{0}_{3\times3} & \boldsymbol{0}_{3\times3} & \boldsymbol{0}_{3\times3} & \boldsymbol{0}_{3\times4} & \boldsymbol{0}_{3\times4} & \boldsymbol{L}_{3,2} & \boldsymbol{0}_{3\times2} & \boldsymbol{0}_{3\times2} & -\boldsymbol{M}_{3,2} \\ \hline \boldsymbol{0}_{1\times3} & \boldsymbol{0}_{1\times3} & \boldsymbol{0}_{1\times3} & \boldsymbol{0}_{1\times4} & \boldsymbol{0}_{1\times4} & \boldsymbol{0}_{1\times4} & \boldsymbol{L}_{3,3} & \boldsymbol{0}_{1\times2} & \boldsymbol{0}_{1\times2} \\ \boldsymbol{0}_{1\times3} & \boldsymbol{0}_{1\times3} & \boldsymbol{0}_{1\times3} & \boldsymbol{0}_{1\times4} & \boldsymbol{0}_{1\times4} & \boldsymbol{0}_{1\times4} & \boldsymbol{0}_{1\times2} & \boldsymbol{L}_{3,3} & \boldsymbol{0}_{1\times2} \\ \boldsymbol{0}_{1\times3} & \boldsymbol{0}_{1\times3} & \boldsymbol{0}_{1\times3} & \boldsymbol{0}_{1\times4} & \boldsymbol{0}_{1\times4} & \boldsymbol{0}_{1\times4} & \boldsymbol{0}_{1\times2} & \boldsymbol{0}_{1\times2} & \boldsymbol{L}_{3,3} \end{array}\right), \tag{IA.184}$$

and

$$\boldsymbol{L}_{3,1} = \begin{pmatrix} e^{-\omega_{0,1} x_{D,1}} & e^{-\omega_{0,2} x_{D,1}} & e^{-\omega_{0,3} x_{D,1}} \\ v_{1,1}^0 e^{-\omega_{0,1} x_{D,1}} & v_{2,1}^0 e^{-\omega_{0,2} x_{D,1}} & v_{3,1}^0 e^{-\omega_{0,3} x_{D,1}} \\ v_{1,2}^0 e^{-\omega_{0,1} x_{D,1}} & v_{2,2}^0 e^{-\omega_{0,2} x_{D,1}} & v_{3,2}^0 e^{-\omega_{0,3} x_{D,1}} \\ v_{1,1}^0 \omega_{0,1} e^{-\omega_{0,1} x_{D,1}} & v_{2,1}^0 \omega_{0,2} e^{-\omega_{0,2} x_{D,1}} & v_{3,1}^0 \omega_{0,3} e^{-\omega_{0,3} x_{D,1}} \\ v_{1,2}^0 \omega_{0,1} e^{-\omega_{0,1} x_{D,1}} & v_{2,2}^0 \omega_{0,2} e^{-\omega_{0,2} x_{D,1}} & v_{3,2}^0 \omega_{0,3} e^{-\omega_{0,3} x_{D,1}} \end{pmatrix}, \tag{IA.185}$$

$$\boldsymbol{M}_{3,1} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ e^{-\omega_1^1 x_{D,1}} & e^{-\omega_2^1 x_{D,1}} & e^{-\omega_3^1 x_{D,1}} & e^{-\omega_4^1 x_{D,1}} \\ v_{1,1}^1 e^{-\omega_1^1 x_{D,1}} & v_{2,1}^1 e^{-\omega_2^1 x_{D,1}} & v_{3,1}^1 e^{-\omega_3^1 x_{D,1}} & v_{4,1}^1 e^{-\omega_4^1 x_{D,1}} \\ \omega_1^1 e^{-\omega_1^1 x_{D,1}} & \omega_2^1 e^{-\omega_2^1 x_{D,1}} & \omega_3^1 e^{-\omega_3^1 x_{D,1}} & \omega_4^1 e^{-\omega_4^1 x_{D,1}} \\ v_{1,1}^1 \omega_1^1 e^{-\omega_1^1 x_{D,1}} & v_{2,1}^1 \omega_2^1 e^{-\omega_2^1 x_{D,1}} & v_{3,1}^1 \omega_3^1 e^{-\omega_3^1 x_{D,1}} & v_{4,1}^1 \omega_4^1 e^{-\omega_4^1 x_{D,1}} \end{pmatrix}, \tag{IA.186}$$

$$\boldsymbol{L}_{3,2} = \begin{pmatrix} e^{-\omega_1^1 x_{D,2}} & e^{-\omega_2^1 x_{D,2}} & e^{-\omega_3^1 x_{D,2}} & e^{-\omega_4^1 x_{D,2}} \\ v_{1,1}^1 e^{-\omega_1^1 x_{D,2}} & v_{2,1}^1 e^{-\omega_2^1 x_{D,2}} & v_{3,1}^1 e^{-\omega_3^1 x_{D,2}} & v_{4,1}^1 e^{-\omega_4^1 x_{D,2}} \\ v_{1,1}^1 \omega_1^1 e^{-\omega_1^1 x_{D,2}} & v_{2,1}^1 \omega_2^1 e^{-\omega_2^1 x_{D,2}} & v_{3,1}^1 \omega_3^1 e^{-\omega_3^1 x_{D,2}} & v_{4,1}^1 \omega_4^1 e^{-\omega_4^1 x_{D,2}} \end{pmatrix}, \tag{IA.187}$$

$$\boldsymbol{M}_{3,2} = \begin{pmatrix} 0 & 0 \\ e^{-\omega_1^2 x_{D,2}} & e^{-\omega_2^2 x_{D,2}} \\ \omega_1^2 e^{-\omega_1^2 x_{D,2}} & \omega_2^2 e^{-\omega_2^2 x_{D,2}} \end{pmatrix}, \tag{IA.188}$$

$$\boldsymbol{L}_{3,1} = \begin{pmatrix} e^{-\omega_1^2 x_{D,3}} & e^{-\omega_2^2 x_{D,3}} \end{pmatrix} \tag{IA.189}$$

$$\underline{\phi}_3 = (h_{11,1}^0, \ldots, h_{11,3}^0, h_{12,1}^0, \ldots, h_{12,3}^0, h_{13,1}^0, \ldots, h_{13,3}^0, h_{21,1}^1, \ldots, h_{21,4}^1, h_{22,1}^1, \ldots, h_{22,4}^1, h_{23,1}^1, \ldots, h_{23,4}^1, h_{31,1}^2, h_{31,2}^2, h_{32,1}^2, h_{32,2}^2, h_{33,1}^2, h_{33,2}^2)^\top \tag{IA.190}$$

The 3^3-dimensional column vector $\underline{\chi}_3$ is given by

$$\underline{\chi}_3 = (\underline{\chi}(3,1)^\top, \underline{\chi}(3,2)^\top, \underline{\chi}(3,3)^\top)^\top, \tag{IA.191}$$

where

$$\underline{\chi}(3,k) = (\underline{\chi}_1(3,k)^\top, \ldots, \underline{\chi}_3(3,k)^\top, \underline{0}_{1\times 3(3-k)})^\top, \tag{IA.192}$$

and

$$\underline{\chi}_j(3,k) = (\chi_{1j}(3,k), \dots, \chi_{3-k+1,j}(3,k))^\top, \tag{IA.193}$$

where

$$\chi_{1j}(3,1) = \delta_{1j} \tag{IA.194}$$

$$\chi_{ij}(3,1) = -[\boldsymbol{A}_1^{-1}\boldsymbol{B}_1]_{i-1,j},\ i \in \{2,3\},\ j \in \{1,\dots,3\}, \tag{IA.195}$$

and

$$\chi_{1j}(3,k) = [\boldsymbol{A}_{k-1}^{-1}\boldsymbol{B}_{k-1}]_{1j} + \delta_{kj} \tag{IA.196}$$

$$\chi_{ij}(3,k) = [\boldsymbol{A}_{k-1}^{-1}\boldsymbol{B}_{k-1}]_{ij} - [\boldsymbol{A}_k^{-1}\boldsymbol{B}_k]_{i-1,j},\ i \in \{2,\dots,3-k+1\},\ j \in \{1,\dots,3\}, \tag{IA.197}$$

and

$$\chi_{1j}(3,3) = [\boldsymbol{A}_2^{-1}\boldsymbol{B}_2]_{1j} + \delta_{3j},\ j \in \{1,\dots,N\}. \tag{IA.198}$$

We can thus visualize the construction of $\underline{\chi}_3 = (\underline{\chi}(3,1), \underline{\chi}(3,2), \underline{\chi}(3,3))^\top$ as follows:

$$\begin{array}{lllll}
 & \underline{\chi}_1(3,1) \rightarrow & (\chi_{11}(3,1), & \chi_{21}(3,1), & \chi_{31}(3,1))^\top \\
\underline{\chi}(3,1) \rightarrow & \underline{\chi}_2(3,1) \rightarrow & (\chi_{12}(3,1), & \chi_{22}(3,1), & \chi_{32}(3,1))^\top \\
 & \underline{\chi}_3(3,1) \rightarrow & (\chi_{13}(3,1), & \chi_{23}(3,1), & \chi_{33}(3,1))^\top \\
 & \quad \underline{0}_{1\times 6} & & & \\
\hline
 & \underline{\chi}_1(3,2) \rightarrow & (\chi_{11}(3,2), & \chi_{21}(3,2))^\top & \\
\underline{\chi}(3,2) \rightarrow & \underline{\chi}_2(3,2) \rightarrow & (\chi_{12}(3,2), & \chi_{22}(3,2))^\top & \\
 & \underline{\chi}_3(3,2) \rightarrow & (\chi_{13}(3,2), & \chi_{23}(3,2))^\top & \\
 & \quad \underline{0}_{1\times 3} & & & \\
\hline
 & \underline{\chi}_1(3,3) \rightarrow & \chi_{11}(3,3) & & \\
\underline{\chi}(3,3) \rightarrow & \underline{\chi}_2(3,2) \rightarrow & \chi_{12}(3,3) & & \\
 & \underline{\chi}_3(3,3) \rightarrow & \chi_{13}(3,3) & &
\end{array}, \tag{IA.199}$$

which can be written more explicitly as

$$\begin{array}{lllll}
 & \underline{\chi}_1(3,1) \rightarrow & (1, & -[\boldsymbol{A}_1^{-1}\boldsymbol{B}_1]_{11}, & -[\boldsymbol{A}_1^{-1}\boldsymbol{B}_1]_{21})^\top \\
\underline{\chi}(3,1) \rightarrow & \underline{\chi}_2(3,1) \rightarrow & (0, & -[\boldsymbol{A}_1^{-1}\boldsymbol{B}_1]_{12}, & -[\boldsymbol{A}_1^{-1}\boldsymbol{B}_1]_{22})^\top \\
 & \underline{\chi}_3(3,1) \rightarrow & (0, & -[\boldsymbol{A}_1^{-1}\boldsymbol{B}_1]_{13}, & -[\boldsymbol{A}_1^{-1}\boldsymbol{B}_1]_{23}^\top \\
 & \quad \underline{0}_{1\times 6} & & & \\
\hline
 & \underline{\chi}_1(3,2) \rightarrow & ([\boldsymbol{A}_1^{-1}\boldsymbol{B}_1]_{11}, & [\boldsymbol{A}_1^{-1}\boldsymbol{B}_1]_{21} - [\boldsymbol{A}_2^{-1}\boldsymbol{B}_2]_{11})^\top & \\
\underline{\chi}(3,2) \rightarrow & \underline{\chi}_2(3,2) \rightarrow & ([\boldsymbol{A}_1^{-1}\boldsymbol{B}_1]_{12} + 1, & [\boldsymbol{A}_1^{-1}\boldsymbol{B}_1]_{22} - [\boldsymbol{A}_2^{-1}\boldsymbol{B}_2]_{12})^\top & \\
 & \underline{\chi}_3(3,2) \rightarrow & ([\boldsymbol{A}_1^{-1}\boldsymbol{B}_1]_{13}, & [\boldsymbol{A}_1^{-1}\boldsymbol{B}_1]_{23} - [\boldsymbol{A}_2^{-1}\boldsymbol{B}_2]_{13})^\top & \\
 & \quad \underline{0}_{1\times 3} & & & \\
\hline
 & \underline{\chi}_1(3,3) \rightarrow & [\boldsymbol{A}_2^{-1}\boldsymbol{B}_2]_{11} & & \\
\underline{\chi}(3,3) \rightarrow & \underline{\chi}_2(3,2) \rightarrow & [\boldsymbol{A}_2^{-1}\boldsymbol{B}_2]_{12} & & \\
 & \underline{\chi}_3(3,3) \rightarrow & [\boldsymbol{A}_2^{-1}\boldsymbol{B}_2]_{13} + 1 & &
\end{array}, \tag{IA.200}$$

and so

$$\underline{\chi}_3 = \begin{array}{c}
(1, -[\boldsymbol{A}_1^{-1}\boldsymbol{B}_1]_{11}, -[\boldsymbol{A}_1^{-1}\boldsymbol{B}_1]_{21}, 0, -[\boldsymbol{A}_1^{-1}\boldsymbol{B}_1]_{12}, -[\boldsymbol{A}_1^{-1}\boldsymbol{B}_1]_{22}, 0, -[\boldsymbol{A}_1^{-1}\boldsymbol{B}_1]_{13}, -[\boldsymbol{A}_1^{-1}\boldsymbol{B}_1]_{23}, \underline{0}_6^\top, -\left[\boldsymbol{A}_1^{-1}\boldsymbol{B}_1\right]_{11}, \\
\left[\boldsymbol{A}_1^{-1}\boldsymbol{B}_1\right]_{21} - \left[\boldsymbol{A}_2^{-1}\boldsymbol{B}_2\right]_{11}, \left[\boldsymbol{A}_1^{-1}\boldsymbol{B}_1\right]_{12} + 1, \left[\boldsymbol{A}_1^{-1}\boldsymbol{B}_1\right]_{22} - \left[\boldsymbol{A}_2^{-1}\boldsymbol{B}_2\right]_{12}, \left[\boldsymbol{A}_1^{-1}\boldsymbol{B}_1\right]_{23} - \left[\boldsymbol{A}_2^{-1}\boldsymbol{B}_2\right]_{13}, \\
0, 0, 0, \left[\boldsymbol{A}_2^{-1}\boldsymbol{B}_2\right]_{11}, \left[\boldsymbol{A}_2^{-1}\boldsymbol{B}_2\right]_{12}, \left[\boldsymbol{A}_2^{-1}\boldsymbol{B}_2\right]_{13} + 1)^\top
\end{array}, \tag{IA.201}$$

where $\underline{0}_6$ is the 6×1 column vector of zeros.

IA.I Modified Arrow-Debreu Default Claims

Arrow-Debreu securities provide the expected value of a \$ 1 cash flow conditional on the state of the world in which they occur. In particular, in region $\mathcal{D}_k$ at time t, we have the Arrow-Debreu default claim price

$$q_{D,ij,t}^{\$,k} = E_t \left[\frac{\pi_{\tau_D}^{\$}}{\pi_t^{\$}} 1_{\{s_{\tau_D}=j\}} \Big| s_t = i \right] \tag{IA.202}$$

$$= E_t^{\mathbb{Q}^{\$}} \left[e^{-\int_t^{\tau_D} r_u^{\$} du} 1_{\{s_{\tau_D}=j\}} \Big| s_t = i \right]. \tag{IA.203}$$

For cash flows that do not depend on the level of earnings when the firm defaults, $X_{\tau_D} = e^{x_{\tau_D}}$, these Arrow-Debreu securities yield a straightforward approach to derive the cash flows' expected values. If a cash flow does depend on X_{τ_D}, the Arrow-Debreu securities may not be as useful.

In a model with no jumps, the earnings always approaches the default boundary from above along a continuous path and default occurs when $X_{\tau_D} = X_D$; that is, there is no uncertainty with respect to the level of earnings upon default and Arrow-Debreu securities can readily be used to compute expected cash flows. In our economy, however, "deep defaults" can occur immediately when the state of the economy jumps from its current state to a worse state.

We order default boundaries such that $X_{D,1} > \ldots > X_{D,N}$; hence, state N is the best state of the economy, state 1 is the worst. When the state of the economy jumps toward a better state, the default boundary decreases as growth opportunities improve; hence, if the firm was not in default, it is even further away from default after the jump. However, if the level of earnings is $X_{\tau_D^-} \in (X_{D,j+1}, X_{D,j}]$ prior to a jump into state j at time τ_D, the firm automatically defaults. The level of earnings X_{τ_D} is then only a fraction of the default boundary $X_{D,j}$, and the firm will thus be able to honor its obligations to the debtholders, for instance, only partially.

We thus introduce "modified" Arrow-Debreu securities

$$\widetilde{q}^{\$}_{D,ij,t} = E_t \left[\frac{\pi^{\$}_{\tau_D}}{\pi^{\$}_t} \frac{X_{\tau_D}}{X_{D,j}} 1_{s_{\tau_D}=j} \Big| s_t = i \right] \tag{IA.204}$$

$$= \frac{X_t}{X_{D,j}} E_t \left[\frac{\pi^{\$}_{\tau_D}}{\pi^{\$}_t} \frac{X_{\tau_D}}{X_t} 1_{s_{\tau_D}=j} \Big| s_t = i \right] \tag{IA.205}$$

$$= e^{x_t - x_{D,j}} E_t \left[\frac{\pi^{\$}_{\tau_D}}{\pi^{\$}_t} \frac{X_{\tau_D}}{X_t} 1_{s_{\tau_D}=j} \Big| s_t = i \right]. \tag{IA.206}$$

We now change the probability measure from $\mathbb{P}$ to $\widetilde{\mathbb{Q}}$, using the exponential martingale $\widetilde{M}$ (with respect to $\mathbb{P}$), given by

$$\left. \frac{d\widetilde{M}_t}{\widetilde{M}_t} \right|_{s_{t-}=i} = \frac{d(\pi^{\$}_t X_t)}{\pi^{\$}_t X_t} - E_t \left[\frac{d(\pi^{\$}_t X_t)}{\pi^{\$}_t X_t} \bigg| s_{t-} = i \right] \tag{IA.207}$$

$$= -\gamma \sigma_{C,i} dZ_t - (\varphi - 1)\sigma_{P,i} dZ_{P,t} + \sigma_{Y,i} dW_t + \sum_{j \neq i} (\omega_{ij} - 1) dN^P_{ij,t}. \tag{IA.208}$$

Therefore

$$\widetilde{q}^{\$}_{D,ij,t} = e^{x_t - x_{D,j}} \overline{q}_{ij,t}, \tag{IA.209}$$

where

$$k_t|_{s_{t-}=i} = r^{\$}_i + \gamma \rho_{YC,i} \sigma_{Y,i} \sigma_{C,i} + \varphi \gamma \rho_{PC,i} \sigma_{P,i} \sigma_{C,i} + \rho_{PY,i} \sigma_{P,i} \sigma_{Y,i} + \varphi \sigma^2_{P,i} - \mu_{X,i} \tag{IA.210}$$

and

$$\overline{q}_{ij,t} = E^{\widetilde{\mathbb{Q}}}_t \left[e^{-\int_t^{\tau_D} k_u du} 1_{s_{\tau_D}=j} \Big| s_t = i \right]. \tag{IA.211}$$

We have N^3 boundary conditions (value matching & smooth pasting) for the modified Arrow-Debreu securities. For each default boundary $X_{D,k}, k \in \{1, \dots, N\}$, we have that:

(VM) The value of the $N + (N - k)N$ modified Arrow-Debreu securities with $i \geq k$, must be the same on both sides of the default boundary, i.e.

$$\widetilde{q}^{k-1}_{D,ij}(x_{D,k}) = \widetilde{q}^{k}_{ij}(x_{D,k}), \text{ where } , i \in \{k, \dots, N\}, j \in \{1, \dots, N\}; \tag{IA.212}$$

(SP) The first derivatives of the $(N - k)N$ modified Arrow-Debreu securities with $i > k$, must be the same on both side of each default boundary, i.e.

$$\left.\frac{d\widetilde{q}^{k-1}_{ij}}{dx}\right|_{x_{D,k}} = \left.\frac{d\widetilde{q}^{k}_{ij}}{dx}\right|_{x_{D,k}}, \text{ where } , i \in \{k+1, \dots, N\}, j \in \{1, \dots, N\}. \tag{IA.213}$$

Observe that the above N^3 boundary conditions imply the following N^3 boundary conditions for $\overline{q}_{ij,t}$:

(VM)

$$\overline{q}^{k-1}_{D,ij}(x_{D,k}) = \overline{q}^{k}_{ij}(x_{D,k}), \text{ where } , i \in \{k, \dots, N\}, j \in \{1, \dots, N\}; \tag{IA.214}$$

(SP)

$$\left.\frac{d\overline{q}^{k-1}_{ij}}{dx}\right|_{x_{D,k}} = \left.\frac{d\overline{q}^{k}_{ij}}{dx}\right|_{x_{D,k}}, \text{ where } , i \in \{k+1, \dots, N\}, j \in \{1, \dots, N\}; \tag{IA.215}$$

We can therefore evaluate $\overline{q}_{ij,t}$ in the same way as $q_{ij,t}$, but replacing $r^{\$}$ with k and $\hat{\mu}^{\$}_{x,i}$ with

$$\hat{\mu}^{\widetilde{\mathbb{Q}}}_{x,i} = \hat{\mu}^{\$}_{x,i} - \left(\sigma^2_{Y,i} + 2\varphi\rho_{PY,i}\sigma_{P,i}\sigma_{Y,i} + \sigma^2_{P,i}\right) \tag{IA.216}$$

Technically, the (standard) Arrow-Debreu securities are special cases of their modified counterparts, with $\frac{x_{\tau_D}}{x_{D,j}} = 1$. Moreover, when in region $\mathcal{D}_0$, deep defaults are irrelevant, because the economy is already in the worst possible state.

not a direct concern, the firm would survive even to a jump to the worse state, state 1. Hence, the general solution in (IA.130) holds. However, in region $\mathcal{D}_k, k > 0$, applying (IA.112) to the unknown $\tilde{q}_{D,ij}, i > k$, accounting for deep defaults, yields the following system of ODEs

$$\begin{aligned} \frac{dz_{i,j}}{dx} - z_{N+i,j} &= 0, \\ \frac{dz_{N+i,j}}{dx} + \frac{2\widehat{\mu}_{x,i}}{\sigma^2_{x,i}} z_{N+i,j} + \sum_{l=1}^{k} \frac{2\widehat{\lambda}_{i,l}}{\sigma^2_{x,i}} \left(e^{x-x_{D,j}}\delta_{lj} - z_{i,j}\right) & \\ + \sum_{l=k+1, l\neq i}^{N} \frac{2\widehat{\lambda}_{i,l}}{\sigma^2_{x,i}} \left(z_{l,j} - z_{i,j}\right) - \frac{2r_i}{\sigma^2_{x,i}} z_{i,j} &= 0, \end{aligned}$$

with $i \in \{k+1, \dots, N\}$ and $j \in \{1, \dots, N\}$. This can be written equivalently in matrix form as

$$Z'_k + A_k Z_k + \widetilde{B}_k = 0, \tag{IA.217}$$

where

$$\widetilde{B}_k = \begin{pmatrix} 0_{N-k\times k} & 0_{N-k\times N-k} \\ \widetilde{B}_k^{\o} & 0_{N-k\times N-k} \end{pmatrix},$$

and

$$\widetilde{B}_k^{\o} = \begin{pmatrix} 2\frac{\widehat{\lambda}_{k+1,1}}{\sigma_{k+1}^2}e^{x-x_{D,1}} & 2\frac{\widehat{\lambda}_{k+1,2}}{\sigma_{k+1}^2}e^{x-x_{D,2}} & \cdots & 2\frac{\widehat{\lambda}_{k+1,k}}{\sigma_{k+1}^2}e^{x-x_{D,k}} \\ 2\frac{\widehat{\lambda}_{k+2,1}}{\sigma_{k+2}^2}e^{x-x_{D,1}} & 2\frac{\widehat{\lambda}_{k+2,2}}{\sigma_{k+2}^2}e^{x-x_{D,2}} & \cdots & 2\frac{\widehat{\lambda}_{k+2,k}}{\sigma_{k+2}^2}e^{x-x_{D,k}} \\ \vdots & \vdots & \cdots & \vdots \\ 2\frac{\widehat{\lambda}_{N,1}}{\sigma_N^2}e^{x-x_{D,1}} & 2\frac{\widehat{\lambda}_{N,2}}{\sigma_N^2}e^{x-x_{D,2}} & \cdots & 2\frac{\widehat{\lambda}_{N,k}}{\sigma_N^2}e^{x-x_{D,k}} \end{pmatrix}.$$

Now, $\widetilde{B}_k$ is not constant with respect to x anymore, but $\widetilde{B}_k' = \widetilde{B}_k$. Hence, letting $\widetilde{Z}_k = Z_k + (A_k + I)^{-1}\widetilde{B}_k$, we have $\widetilde{Z}_k' = Z_k' + (A_k + I)^{-1}\widetilde{B}_k$

$$\widetilde{Z}_k' + A_k\widetilde{Z}_k = Z_k' + (A_k + I)^{-1}\widetilde{B}_k + A_kZ_k + A_k(A_k + I)^{-1}\widetilde{B}_k \tag{IA.218}$$

$$= Z_k' + A_kZ_k + \widetilde{B}_k = 0. \tag{IA.219}$$

Once more, the development made between equations (IA.119) and (IA.128) can be applied to $\widetilde{Z}_k$ in (IA.219) to yield

$$\widetilde{Z}_k = e^{-D_kx}E_kK_k, \tag{IA.220}$$

or, equivalently,

$$Z_k = e^{-D_kx}E_kK_k - (A_k + I)^{-1}\widetilde{B}_k. \tag{IA.221}$$

Therefore,

$$\tilde{q}_{D,ij}(x) = \delta_{ij}e^{x-x_{Dj}} = \delta_{ij}\frac{X}{X_{Di}}, \quad i \in \{1, \ldots, k\}, j \in \{1, \ldots, N\},$$

$$\tilde{q}_{D,i,j}(x) = \sum_{l=1}^{2(N-k)} h_{ij}(\omega_l)e^{-\omega_lx} - [(A_k + I)^{-1}\widetilde{B}_k]_{i-k,j}, \quad i \in \{k+1, \ldots, N\}, j \in \{1, \ldots, N\}.$$

For $1 \le i \le k$, if the earnings at current time t are $X_t < X_{D,i}$ while the current state is i, it must be that the state just jumped to state i at time t^- and the firm is now in (deep) default, hence the first equation in the above system.

IA.J Bond Prices

In this proof it is not necessary to distinguish between the state of the economy at dates $t-$ and t. The central part of our proof consists of proving that

$$E_t\left[\int_t^{\tau_D} \frac{\pi_s^\$}{\pi_t^\$}ds \middle| s_t = i\right] = \frac{1}{r_{P,i}^\$} - \sum_{j=1}^N \frac{q_{D,ij}^\$}{r_{P,j}^\$}, \tag{IA.222}$$

where $r_{P,i}^\$$, the discount rate for a fixed nominal perpetuity, when the economy is in state i, is given by

$$r_{P,i}^\$ = \left(E_t\left[\int_t^\infty \frac{\pi_s^\$}{\pi_t^\$}ds \middle| s_t = i\right]\right)^{-1}, \tag{IA.223}$$

and

$$E_t\left[\left.\frac{\pi^{\$}_{\tau_D}}{\pi^{\$}_t}\alpha A^{\$}_{\tau_D}\left(X_{\tau_D}\right)\right| s_t = i\right] \quad = \quad \sum_{j=1}^{N} \alpha A^{\$}_j\left(X_{D,j}\right)\tilde{q}^{\$}_{D,ij}. \tag{IA.224}$$

To prove (IA.222), we note that

$$E_t\left[\left.\int_t^{\tau_D}\frac{\pi^{\$}_s}{\pi^{\$}_t}ds\right| s_t = i\right] = E_t\left[\left.\int_t^{\infty}\frac{\pi^{\$}_s}{\pi^{\$}_t}ds\right| s_t = i\right] - E_t\left[\left.\frac{\pi^{\$}_{\tau_D}}{\pi^{\$}_t}\int_{\tau_D}^{\infty}\frac{\pi^{\$}_s}{\pi^{\$}_{\tau_D}}ds\right| s_t = i\right],$$

and conditioning on the event $\{s_{\tau_D} = j\}$, we obtain

$$E_t\left[\left.\frac{\pi^{\$}_{\tau_D}}{\pi^{\$}_t}\int_{\tau_D}^{\infty}\frac{\pi^{\$}_s}{\pi^{\$}_{\tau_D}}ds\right| s_t = i\right] = \sum_{j=1}^{N} E_t\left[\left.\Pr\left(s_{\tau_D} = j \right| s_t = i\right)\frac{\pi^{\$}_{\tau_D}}{\pi^{\$}_t}\int_{\tau_D}^{\infty}\frac{\pi^{\$}_s}{\pi^{\$}_{\tau_D}}ds\right| s_t = i\right].$$

Since consumption is Markovian, so is the state-price density, which implies that

$$\begin{aligned}
& E_t\left[\left.\Pr\left(s_{\tau_D} = j \right| s_t = i\right)\frac{\pi^{\$}_{\tau_D}}{\pi^{\$}_t}\int_{\tau_D}^{\infty}\frac{\pi^{\$}_s}{\pi^{\$}_{\tau_D}}ds\right| s_t = i\right] \\
= \quad & E_t\left[\left.\Pr\left(s_{\tau_D} = j \right| s_t = i\right)\frac{\pi^{\$}_{\tau_D}}{\pi^{\$}_t}\right| s_t = i\right] E_t\left[\left.\int_{\tau_D}^{\infty}\frac{\pi^{\$}_s}{\pi^{\$}_{\tau_D}}ds\right| s_{\tau_D} = j\right].
\end{aligned}$$

Therefore

$$\begin{aligned}
E_t\left[\left.\int_t^{\tau_D}\frac{\pi^{\$}_s}{\pi^{\$}_t}ds\right| s_t = i\right] \quad = \quad & E_t\left[\left.\int_t^{\infty}\frac{\pi^{\$}_s}{\pi^{\$}_t}ds\right| s_t = i\right] \\
& -\sum_{j=1}^{N} E_t\left[\left.\Pr\left(s_{\tau_D} = j \right| s_t = i\right)\frac{\pi^{\$}_{\tau_D}}{\pi^{\$}_t}\right| s_t = i\right] E_t\left[\left.\int_{\tau_D}^{\infty}\frac{\pi^{\$}_s}{\pi^{\$}_{\tau_D}}ds\right| s_{\tau_D} = j\right].
\end{aligned} \tag{IA.225}$$

Conditional on being in state i, the value of a claim which pays one risk-free unit of consumption in perpetuity is $E_t\left[\left.\int_t^{\infty}\frac{\pi^{\$}_s}{\pi^{\$}_t}ds\right| s_t = i\right]$, so the discount rate for this perpetuity, $r^{\$}_{P,i}$, is given by (IA.223). Consequently, (IA.225) implies

$$E_t\left[\left.\int_t^{\tau_D}\frac{\pi^{\$}_s}{\pi^{\$}_t}ds\right| s_t = i\right] = \frac{1}{r^{\$}_{P,i}} - \sum_{j=1}^{N}\frac{E_t\left[\left.\Pr\left(s_t = i \right| s_{\tau_D} = j\right)\frac{\pi^{\$}_{\tau_D}}{\pi^{\$}_t}\right| s_t = i\right]}{r^{\$}_{P,j}}. \tag{IA.226}$$

To obtain (IA.222) from the above expression, we note that

$$q^{\$}_{D,ij,t} = E_t\left[\left.\Pr\left(s_{\tau_D} = j \right| s_{\tau_t} = i\right)\frac{\pi^{\$}_{\tau_D}}{\pi^{\$}_t}\right| s_t = i,\right]. \tag{IA.227}$$

To prove (IA.224), we condition on the event $\{s_{\tau_D} = j\}$ to obtain

$$\alpha E_t \left[\frac{\pi^{\$}_{\tau_D}}{\pi^{\$}_t} A^{\$}_{\tau_D} \left(X_{\tau_D} \right) \middle| s_t = i \right] = \alpha \sum_{j=1}^{N} A^{\$}_j \left(X_{D,j} \right) E_t \left[\frac{X_{\tau_D}}{X_{D,j}} \Pr\left(s_{\tau_D} = s_j \middle| s_t = i \right) \frac{\pi^{\$}_{\tau_D}}{\pi^{\$}_t} \middle| s_t = i \right].$$

Using (IA.227) to simplify the above expression we obtain (34).

IA.K Equity Risk Premium

Applying Ito's Lemma to $S^{\$}_{i,t}$ gives

$$\frac{dS^{\$}_{i,t} + (X_t - c)dt}{S^{\$}_{i,t}} = \frac{X_t}{S^{\$}_{i,t}} \frac{\partial S^{\$}_{i,t}}{\partial X_t} \frac{dX_t}{X_t} + \frac{1}{2} \frac{X_t^2}{S_{i,t}} \frac{\partial^2 S^{\$}_{i,t}}{\partial X_t^2} \left(\frac{dX_t}{X_t} \right)^2 + \sum_{j \neq i}^{N} \frac{S^{\$}_{j,t} - S^{\$}_{i,t}}{S^{\$}_{i,t}} dN_{ij,t} + \frac{(X_t - c)dt}{S^{\$}_{i,t}}, \ i, j \in \{1, \ldots, N\}. \quad \text{(IA.228)}$$

Observe that

$$\frac{\partial S^{\$}_{i,t}}{\partial X_t} = (1-\eta) \frac{1}{r^{\$}_{A,i}} - \sum_{j=1}^{N} \left(A^{\$}_j (X_{D,j}) \frac{\partial \tilde{q}^{\$}_{D,ij,t}}{\partial X_t} - (1-\eta) \frac{\partial q_{D,ij,t}}{\partial X_t} \frac{c}{r^{\$}_{P,j}} \right) \quad \text{(IA.229)}$$

$$\frac{\partial^2 S^{\$}_{i,t}}{\partial X_t^2} = - \sum_{j=1}^{N} \left(A^{\$}_j (X_{D,j}) \frac{\partial^2 \tilde{q}^{\$}_{D,ij,t}}{\partial X_t^2} - (1-\eta) \frac{\partial^2 q_{D,ij,t}}{\partial X_t^2} \frac{c}{r^{\$}_{P,j}} \right), \ i, j \in \{1, \ldots, N\}. \quad \text{(IA.230)}$$

Define the date-t conditional nominal expected return

$$\mu^{\$}_{R,i,t} = \frac{1}{dt} E_t \left[\frac{dS^{\$}_{s_{t-},t} + (X_t - c)dt}{S^{\$}_{s_{t-},t}} \middle| s_{t-} = i \right], \ i \in \{1, \ldots, N\}. \quad \text{(IA.231)}$$

The basic asset pricing equation is

$$\mu^{\$}_{R,i,t} - r^{\$}_{i,t} = -\frac{1}{dt} E_t \left[\frac{d\pi^{\$}_t}{\pi^{\$}_t} \frac{dS^{\$}_{s_{t-},t}}{S^{\$}_{s_{t-},t}} \middle| s_{t-} = i \right], \ i \in \{1, \ldots, N\}. \quad \text{(IA.232)}$$

Hence

$$\mu^{\$}_{R,i,t} - r^{\$}_{i,t} = \sum_{j \neq i} (1 - \omega_{ij}) \frac{S^{\$}_j - S^{\$}_i}{S^{\$}_i} \lambda_{ij} + \frac{X_t}{S^{\$}_{i,t}} \frac{\partial S^{\$}_{i,t}}{\partial X_t} E_t \left[\left(\sigma_{Y,t} dW_t + \varphi \sigma_{P,t} dZ_{P,t} \right) \left(\gamma \sigma_{C,i} dZ_t + \sigma_{P,i} dZ_{P,t} \right) \right] \quad \text{(IA.233)}$$

$$= \sum_{j \neq i} (1 - \omega_{ij}) \frac{S^{\$}_j - S^{\$}_i}{S^{\$}_i} \lambda_{ij} + \frac{X_t}{S^{\$}_{i,t}} \frac{\partial S^{\$}_{i,t}}{\partial X_t} \left(\rho_{YC,i} \sigma_{Y,i} \gamma \sigma_{C,i} + \rho_{YP,i} \sigma_{Y,i} \sigma_{P,i} + \gamma \varphi \rho_{PC,i} \sigma_{P,t} \sigma_{C,i} + \varphi \sigma^2_{P,i} \right) \quad \text{(IA.234)}$$

$$, \ i, j \in \{1, \ldots, N\}. \quad \text{(IA.235)}$$

The unexpected nominal stock return in state i is given by

$$\sum_{j \neq i} \sigma^P_{R,ij} dN^P_{ij,t} + \frac{X_t}{S^{\$}_{i,t}} \frac{\partial S^{\$}_{i,t}}{\partial X_t} \left(\sigma_{Y,i} dW_t + \varphi \sigma_{P,i} dZ_{P,t} \right), \ i, j \in \{1, \ldots, N\}, \quad \text{(IA.236)}$$

where

$$\sigma^P_{R,ij} = \frac{S^\$_j}{S^\$_i} - 1,\ i,j \in \{1,\ldots,N\}. \tag{IA.237}$$

If we define

$$S_{i,t} = \frac{S^\$_{i,t}}{P_t}, \tag{IA.238}$$

then Ito's Lemma implies that

$$\frac{dS_{i,t}}{S_{i,t}} = \frac{dS^\$_{i,t}}{S^\$_{i,t}} - \frac{dP_t}{P_t} + \sigma^2_{P,i}dt - \frac{dS^\$_{i,t}}{S^\$_{i,t}}\frac{dP_t}{P_t} \tag{IA.239}$$

$$= -\frac{dP_t}{P_t} + \sigma^2_{P,i}dt - \frac{X_t}{S^\$_{i,t}}\frac{\partial S^\$_{i,t}}{\partial X_t}\frac{dX_t}{X_t}\frac{dP_t}{P_t} \tag{IA.240}$$

$$= \frac{dS^\$_{i,t}}{S^\$_{i,t}} - \frac{dP_t}{P_t} + \sigma^2_{P,i}dt - \frac{X_t}{S^\$_{i,t}}\frac{\partial S^\$_{i,t}}{\partial X_t}(\sigma_{P,i}\sigma_{Y,i}\rho_{PY,i}dt + \varphi\sigma^2_{P,i})dt. \tag{IA.241}$$

Therefore, the real risk premium in state i is

$$\mu_{R,i} - r_i = \mu^\$_{R,i} - \mu_{P,i} + \sigma^2_{P,i} - \frac{X_t}{S^\$_{i,t}}\frac{\partial S^\$_{i,t}}{\partial X_t}(\sigma_{P,i}\sigma_{Y,i}\rho_{PY,i} + \varphi\sigma^2_{P,i})dt - r_i \tag{IA.242}$$

$$= \mu^\$_{R,i} - r^\$_i - \mu_{P,i} + \sigma^2_{P,i} - \frac{X_t}{S^\$_{i,t}}\frac{\partial S^\$_{i,t}}{\partial X_t}(\sigma_{P,i}\sigma_{Y,i}\rho_{PY,i} + \varphi\sigma^2_{P,i})dt + r^\$_i - r_i \tag{IA.243}$$

$$= \mu^\$_{R,i} - r^\$_i - \mu_{P,i} + \sigma^2_{P,i} - \frac{X_t}{S^\$_{i,t}}\frac{\partial S^\$_{i,t}}{\partial X_t}(\sigma_{P,i}\sigma_{Y,i}\rho_{PY,i} + \varphi\sigma^2_{P,i})dt + \mu_{P,i} - \gamma\rho_{PC,i}\sigma_{P,i}\sigma_{C,i} - \sigma^2_{P,i} \tag{IA.244}$$

$$= \mu^\$_{R,i} - r^\$_i - \frac{X_t}{S^\$_{i,t}}\frac{\partial S^\$_{i,t}}{\partial X_t}(\sigma_{P,i}\sigma_{Y,i}\rho_{PY,i} + \varphi\sigma^2_{P,i})dt - \gamma\rho_{PC,i}\sigma_{P,i}\sigma_{C,i} \tag{IA.245}$$

$$= \sum_{j\neq i}(1-\omega_{ij})\frac{S^\$_j - S^\$_i}{S^\$_i}\lambda_{ij} + \frac{X_t}{S^\$_{i,t}}\frac{\partial S^\$_{i,t}}{\partial X_t}\left(\rho_{YC,i}\sigma_{Y,i}\gamma\sigma_{C,i} + \gamma\varphi\rho_{PC,i}\sigma_{P,t}\sigma_{C,i}\right) - \gamma\rho_{PC,i}\sigma_{P,i}\sigma_{C,i}, \tag{IA.246}$$

where

$$\frac{X_t}{S^\$_{i,t}}\frac{\partial S^\$_{i,t}}{\partial X_t} = \frac{X_t}{S^\$_{i,t}}\left[\frac{A^\$_i(X_t)}{X_t} - \sum_{j=1}^N\left(A^\$_j(X_{D,j})\frac{\partial\tilde{q}^\$_{D,ij,t}}{\partial X_t} - (1-\eta)\frac{\partial q^\$_{D,ij,t}}{\partial X_t}v_{B,i}c\right)\right]. \tag{IA.247}$$

Observe that if $\rho_{YC,i} = 0$ and $\rho_{YP,i} = 0$, we obtain (39) and

$$\mu_{R,i} - r_i = \sum_{j\neq i}(1-\omega_{ij})\frac{S^\$_j - S^\$_i}{S^\$_i}\lambda_{ij} + \left(\varphi\frac{X_t}{S^\$_{i,t}}\frac{\partial S^\$_{i,t}}{\partial X_t} - 1\right)\gamma\rho_{PC,i}\sigma_{P,t}\sigma_{C,i}. \tag{IA.248}$$

Similarly, the nominal expected risk premium for a corporate bond is

$$-\frac{1}{dt}E_t\left[\frac{d\pi_t^\$}{\pi_t^\$}\frac{dB_{s_{t-},t}^\$}{B_{s_{t-},t}^\$}\middle| s_{t-}=i\right] \tag{IA.249}$$

$$=\sum_{j\neq i}(1-\omega_{ij})\frac{B_j^\$-B_i^\$}{B_i^\$}\lambda_{ij}+\frac{X_t}{B_{i,t}^\$}\frac{\partial B_{i,t}^\$}{\partial X_t}\left(\sigma_{Y,i}\gamma\sigma_{C,i}\rho_{YC,i}+\sigma_{Y,i}\sigma_{P,i}\rho_{YP,i}+\varphi\sigma_{P,i}\gamma\sigma_{C,i}\rho_{PC,i}+\varphi\sigma_{P,i}^2\right), \tag{IA.250}$$

where

$$\frac{\partial B_{i,t}^\$}{\partial X_t}=-c\sum_{j=1}^{N}\frac{\partial q_{D,ij,t}^\$}{\partial X_t}\frac{1}{r_{P,j}^\$}+\alpha\sum_{j=1}^{N}A_j^\$(X_{D,j})\frac{\partial\hat{q}_{D,ij,t}^\$}{\partial X_t}, \tag{IA.251}$$

and the real expected risk premium for a corporate bond is

$$-\frac{1}{dt}E_t\left[\frac{d\pi_t^\$}{\pi_t^\$}\frac{dB_{s_{t-},t}^\$}{B_{s_{t-},t}^\$}\middle| s_{t-}=i\right]-\gamma\rho_{PC,i}\sigma_{P,i}\sigma_{C,i}-\sigma_{P,i}^2. \tag{IA.252}$$

IA.L Finite Maturity Corporate Bonds

Proposition IA.3 *Conditional on the current state being i, the date-t nominal price of a zero coupon risk-free bond which pays out one USD at date $t+\tau$, is $B_{i,\tau}^\$$, which is the i'th element of the vector*

$$\underline{B}_\tau^\$=(B_{1,\tau}^\$,\ldots,B_{N,\tau}^\$)^\top, \tag{IA.253}$$

where

$$\underline{B}_\tau^\$=\exp[-(\boldsymbol{R}^\$-\hat{\boldsymbol{\Lambda}})\tau]\underline{1}_N, \tag{IA.254}$$

and $\underline{1}_N$ is the $N\times 1$ vector of ones, and

$$\boldsymbol{R}^\$=diag(r_1^\$,\ldots,r_N^\$). \tag{IA.255}$$

Proof of Proposition IA.3. The time-t nominal price of a zero coupon bond paying off 1 USD at time $t+\tau$, conditional on the current state of the economy being i is given by

$$B_{i,\tau}^\$=E_t\left[\frac{\pi_{t+\tau}^\$}{\pi_t^\$}\right] \tag{IA.256}$$

$$=E_t^{\mathbb{Q}^\$}\left[e^{-\int_t^{t+\tau}r_u^\$du}\right]. \tag{IA.257}$$

From the no-arbitrage condition

$$E_t\left[dB_{s_{t-},\tau}^\$-r_{s_{t-}}^\$B_{s_{t-},\tau}^\$dt\right]=-E_t\left[dB_{s_{t-},\tau}^\$\frac{d\pi_t^\$}{\pi_{t-}^\$}\right], \tag{IA.258}$$

applying Ito's Lemma gives the following linear system

$$\sum_{j\neq i}(B_{j,\tau}^\$-B_{i,\tau}^\$)\hat{\lambda}_{ij}-r_i^\$B_{i,\tau}^\$=\frac{\partial}{\partial\tau}B_{i,\tau}^\$,\ i\in\{1,\ldots,N\}, \tag{IA.259}$$

where $B^{\$}_{i,0} = 1$. Defining

$$\underline{B}^{\$}_{\tau} = (B^{\$}_{1,\tau}, \ldots, B^{\$}_{N,\tau})^{\top}, \tag{IA.260}$$

we have

$$\frac{\partial}{\partial \tau} \underline{B}^{\$}_{\tau} = -(\boldsymbol{R}^{\$} - \hat{\boldsymbol{\Lambda}}) \underline{B}^{\$}_{\tau}, \tag{IA.261}$$

where

$$\boldsymbol{R}^{\$} = \text{diag}(r^{\$}_1, \ldots, r^{\$}_N), \tag{IA.262}$$

and $\underline{B}^{\$}_{\tau=0} = \underline{1}_N$. Therefore

$$\underline{B}^{\$}_{\tau} = \exp[-(\boldsymbol{R}^{\$} - \hat{\boldsymbol{\Lambda}})\tau] \underline{1}_N \tag{IA.263}$$

$$= \boldsymbol{E} e^{-\boldsymbol{D}\tau} \boldsymbol{E}^{-1} \underline{1}_N, \tag{IA.264}$$

where $\boldsymbol{E} = [\underline{e}_1, \ldots, \underline{e}_N]$ and $\boldsymbol{D} = \text{diag}(d_1, \ldots, d_N)$, and $\underline{e}_n$, $n \in \{1, \ldots, N\}$, where $\underline{e}_n$ is an eigenvector of $\boldsymbol{R}^{\$} - \hat{\boldsymbol{\Lambda}}$ with corresponding eigenvalue d_n, ordered such that $d_1 \leq \ldots \leq d_N$. Observe that

$$B^{\$}_{i,\tau} = \sum_{k=1}^{N} E_{ik} e^{-d_k \tau} \sum_{j=1}^{N} [E^{-1}]_{kj}. \tag{IA.265}$$

■

Proposition IA.4 *Finite maturity risk-free debt pays a coupon at the rate c until maturity (time T) and the amount P at maturity. The time-t price of finite maturity risk-free debt when the current state is i is given by*

$$B^{\$}_{f,i,\tau} = c \left(\frac{1}{r^{\$}_{P,i}} - B^{\$}_{i,\tau} \sum_{j=1}^{N} \frac{1}{r^{\$}_{P,j}} \hat{\Pr}(s_T = j | s_t = i) \right) + P^{\$} B^{\$}_{i,\tau}. \tag{IA.266}$$

Proof of Proposition IA.4. The date-t value of finite maturity risk-free debt is

$$B^{\$}_{f,i,T-t} = c E_t \left[\int_t^T \frac{\pi^{\$}_s}{\pi^{\$}_t} ds | s_t = i \right] + P^{\$} E_t \left[\frac{\pi^{\$}_T}{\pi^{\$}_t} | s_t = i \right],$$

conditional on the current state being i. Hence,

$$B^{\$}_{f,i,T-t} = c E_t \left[\int_t^{\infty} \frac{\pi^{\$}_s}{\pi^{\$}_t} ds | s_t = i \right] - E_t \left[\int_T^{\infty} \frac{\pi^{\$}_s}{\pi^{\$}_t} ds | s_t = i \right] + P^{\$} E_t \left[\frac{\pi^{\$}_T}{\pi^{\$}_t} | s_t = i \right].$$

We know that

$$E_t \left[\int_t^{\infty} \frac{\pi^{\$}_s}{\pi^{\$}_t} ds | s_t = i \right] = \frac{1}{r^{\$}_{P,i}}.$$

Furthermore,

$$
\begin{aligned}
E_t\left[\int_T^\infty \frac{\pi_s^\$}{\pi_t^\$} ds | s_t = i\right] &= E_t\left[\frac{\pi_T^\$}{\pi_t^\$} | s_t = i\right] \sum_{j=1}^N E_T\left[\int_T^\infty \frac{\pi_s^\$}{\pi_T^\$} ds | s_T = j\right] \hat{\Pr}(s_T = j | s_t = i) \\
&= \sum_{j=1}^N E_T\left[\int_T^\infty \frac{\pi_s^\$}{\pi_T^\$} ds | s_T = j\right] \hat{\Pr}(s_T = j | s_t = i) E_t\left[\frac{\pi_T^\$}{\pi_t^\$} | s_t = i\right] \\
&= \sum_{j=1}^N \frac{1}{r_{P,j}^\$} \hat{\Pr}(s_T = j | s_t = i) B_{i,\tau}^\$,
\end{aligned}
$$

where $\tau = T - t$ and

$$E_t\left[\frac{\pi_T}{\pi_t} | s_t = i\right] = B_{i,\tau}^\$.$$

Equation (IA.266) follows. ■

Proposition IA.5 *Finite maturity corporate debt pays a coupon at the rate c until default (the random time, τ_D) or maturity (time T), whichever is earlier, and the amount $P^\$$ at maturity, if default has not already occurred. The time-t price of finite maturity corporate debt is*

$$\overline{B}_{i,T-t}^\$ = cE_t\left[\int_t^{\min(\tau_D, T)} \frac{\pi_s^\$}{\pi_t^\$} ds | s_t = i\right] + P^\$ E_t\left[\frac{\pi_T^\$}{\pi_t^\$} 1_{\{\tau_D > T\}} | s_t = i\right] + E_t\left[\frac{\pi_{\tau_D}^\$}{\pi_t^\$} 1_{\{\tau_D \le T\}} A_{s_{\tau_D}}^\$ (X_{\tau_D}) | s_t = i\right].$$

Closed-form expressions for $B_{f,ij,T-t}^0$ and $B_{f,i,T-t}^0$ are given in the proposition above, whereas $\hat{p}_{D,i,T-t}$ and $q_{D,ij,T-t}$ are computed by Monte-Carlo simulation.

Proof of Proposition IA.5. The proof follows immediately from the definition of the security. ■

IA.M Dividend Strips

Proposition IA.6 *Conditional on the current state being i, the date-t nominal price of an unlevered dividend strip which pays out the nominal cashflow $X_{t+\tau}$ at date $t + \tau$, is $S_{i,\tau}^\$$, which is the i'th element of the vector*

$$\underline{S}_\tau^\$ = (S_{1,\tau}^\$, \ldots, S_{N,\tau}^\$)^\top, \tag{IA.267}$$

where

$$\underline{S}_\tau^\$ = X_t \exp[-(\boldsymbol{R}^\$ - \widehat{\boldsymbol{M}}^\$ - \hat{\boldsymbol{\Lambda}})\tau] \underline{1}_N, \tag{IA.268}$$

and $\underline{1}_N$ is the $N \times 1$ vector of ones, and

$$\boldsymbol{R}^\$ = diag(r_1^\$, \ldots, r_N^\$) \tag{IA.269}$$

$$\widehat{\boldsymbol{M}}^\$ = diag(\hat{\mu}_{X,1}^\$, \ldots, \hat{\mu}_{X,N}^\$). \tag{IA.270}$$

and $\hat{\mu}_{X,i}^\$$ is defined in (IA.95).

Proof of Proposition IA.6.

The proof follows the same steps as the Proof of Proposition IA.3.

■

IA.N Markov Chain - Statistics

We now compute the conditional mean of expected consumption growth and the conditional covariance of expected consumption growth with expected inflation. Suppose the economy is in state i at date t. The expression for $E_t[\mu_{C,t+u}|s_t = i]$ is:

$$E_t[\mu_{C,t+u}|s_t = i] = \sum_{j=1}^{N} \mu_{C,j} \Pr(s_{t+u} = j|s_t = i), \tag{IA.271}$$

where

$$\Pr(s_{t+u} = j|s_t = i) = [\exp(\Lambda u)]_{ij}, \tag{IA.272}$$

and also for $Cov_t[\mu_{C,t+u}, \mu_{P,t+u}|s_t = i]$:

$$Cov_t[\mu_{C,t+u}, \mu_{P,t+u}|s_t = i] = \sum_{j=1}^{N} (\mu_{C,j} - E_t[\mu_{C,t+u}|s_t = i])(\mu_{P,j} - E_t[\mu_{P,t+u}|s_t = i]) \Pr(s_{t+u} = j|s_t = i). \tag{IA.273}$$

IA.O Sticky Prices

Here, we show that in a continuous-time New Keynesian model with costly price-adjustment, the expected growth rate of nominal profits goes up by less than 1% if expected inflation increases by 1%.

The economy consists of a continuum of firms producing differentiated goods and a continuum of identical households. Financial markets are dynamically complete.

IA.O.1 Households

There is a continuum of households, who make real expenditure flow decisions and decide how many person-hours to work per unit time. We sometimes refer to them as consumer-workers. They obtain utility flows from real consumption expenditure flows and disutility flows from the person-hours they work per unit time. Households are identical, so we maximize the following expected utility function of the representative agent:

$$E_t \int_t^{\infty} e^{-\delta(u-t)} \left(\ln C_u - \frac{N_u^{1+\phi}}{1+\phi} \right) du, \tag{IA.274}$$

where C_t is related to the consumption flows of individual goods via the Dixit-Stiglitz aggregator, i.e.

$$C_t = \left(\int_{f \in [0,1]} C_t(f)^{1-\frac{1}{\epsilon}} \right)^{\frac{1}{1-\frac{1}{\epsilon}}}, \tag{IA.275}$$

where $\epsilon > 1$ is the elasticity of substitution between any two goods.

The underlying reason for differentiation across goods could be because of types of goods, branding or quality.

IA.O.2 Firms, imperfect competition and costly price adjustment

There is a continuum of firms, $f \in [0, 1]$. Firms produce differentiated goods, which households can choose to consume. The goods are differentiated, because the elasticity of substitution between any two goods is finite. Hence, firms have monopoly power, which they can use to extract rents from consumers by setting prices to maximize firm value.[41]

We are assuming that the objective of a firm is to maximize the expected present value of profits as opposed to the welfare of households, who own the firm. This is because we are modelling the agency conflict between managers of firms who have different objectives from the firms' owners, i.e. the households.

Firm f produces good f as follows:

$$O_t(f) = A_t N_t(f), \tag{IA.276}$$

where $O_t(f)$ is time t output flow for firm f and $N_t(f)$ is the labor input flow demanded by firm f. The level of technological progress A is common across firms. Technological change takes place exogenously according to

$$\frac{dA_t}{A_t} = \mu dt + \sigma dZ_t, \tag{IA.277}$$

where Z is a standard Brownian motion under the physical probability measure $\mathbb{P}$.

The price of good f is denoted by $P(f)$ and firms pay wages at the nominal wage rate W_t, and so firm f's real profit flow is

$$\Pi_t(f) = \frac{P_t(f)}{P_t} O_t(f) - \frac{W_t}{P_t} N_t(f). \tag{IA.278}$$

When firms adjust prices they face quadratic adjustment costs, such that they choose a price process to maximize date-t firm value net of adjustment costs, i.e.

$$J_t(f) = \sup_{(P_u(f))_{u \geq t}} E_t \int_t^\infty \frac{\Lambda_u}{\Lambda_t} \left[\Pi_u(f) - \frac{1}{2}\theta \left(\frac{dP_u(f)/du}{P_u(f)} \right)^2 O_u \right] du, \tag{IA.279}$$

where the SDF process, Λ, given by

$$\Lambda_t = e^{-\delta t} C_t^{-1}, \tag{IA.280}$$

is determined in equilibrium. Observe that in the special case $\theta = 0$, there are no price adjustment costs, making prices fully flexible.

IA.O.3 Individual firm's problem

We now solve an individual firm's optimal stochastic control problem. An individual firm seeks to exploit its monopoly power to maximize the expected value if its profits at the expense of householders. Firm f's optimal stochastic control problem is

$$J_t(f) = \sup_{(P_u(f))_{u \geq t}} E_t \int_t^\infty \frac{\Lambda_u}{\Lambda_t} \left[\Pi_u(f) - \frac{1}{2}\theta \left(\frac{dP_u(f)/du}{P_u(f)} \right)^2 O_u \right] du, \tag{IA.281}$$

where the SDF process, Λ, given by

$$\Lambda_t = e^{-\delta t} C_t^{-1}. \tag{IA.282}$$

is determined in equilibrium. We assume that price adjustment costs are redistributed to shareholders, so that the date-t value of firm f is given by

$$E_t \int_t^\infty \frac{\Lambda_u}{\Lambda_t} \Pi_u(f) du. \tag{IA.283}$$

[41] In the limiting case of $\epsilon \to \infty$, goods are no longer differentiated and firms have zero monopoly power.

Assuming that the aggregate price index is locally-free, date-t inflation is given by $\mu_{P,t} = \frac{1}{P_t}\frac{dP_t}{dt}$ and we can obtain a relationship between real profits flows, inflation, and the nominal interest rate, as shown below.

Proposition IA.7 *If the nominal interest rate rule is given by*

$$r_t^{\$} = a_0 + a_1 \mu_{P,t}, \tag{IA.284}$$

then nominal profit flow for firm f is given by $\Pi_t^{\$}(f)$, where

$$d \ln \Pi_t^{\$}(f) = da_t - [(\epsilon - 1)(1 + \phi) - 1](a_0 - r_n)dt + \varphi \mu_{P,t} dt, \tag{IA.285}$$

with

$$\varphi = 1 - [(\epsilon - 1)(1 + \phi) - 1](a_1 - 1)], \tag{IA.286}$$

where r_n is the natural rate of interest given in (IA.344).

If $\epsilon > 1 + \frac{1}{1+\phi}$ and $a_1 > 1$ (i.e. the Taylor principle holds), then $\varphi < 1$, and so, expected nominal profit growth increases by less than one percentage point when inflation increases by one percentage point.

Proof of Proposition IA.7

Firm-level optimal price setting problem

We solve the firm-level optimal price setting problem. We have the standard result that

$$C_t(f)/C_t = \left(\frac{P_t(f)}{P_t}\right)^{-\epsilon}. \tag{IA.287}$$

In equilibrium $C_t(f) = O_t(f)$ and $C_t = O_t$, and so,

$$O_t(f)/O_t = \left(\frac{P_t(f)}{P_t}\right)^{-\epsilon}. \tag{IA.288}$$

Since we have $C_t = O_t$, firm f's optimal stochastic control problem is

$$J_t(f) = O_t \sup_{(P_u(f))_{u \geq t}} E_t \int_t^{\infty} e^{-\delta(u-t)} O_u^{-1} \left[D_u(f) - \frac{1}{2}\theta \left(\frac{dP_u(f)/du}{P_u(f)}\right)^2 O_u \right] du, \tag{IA.289}$$

Now observe that

$$D_t(f) = \left(\frac{P_t(f)}{P_t} - \frac{W_t}{P_t A_t}\right) O_t(f) \tag{IA.290}$$

$$= \left(\frac{P_t(f)}{P_t} - \frac{W_t}{P_t A_t}\right) \left(\frac{P_t(f)}{P_t}\right)^{-\epsilon} O_t \tag{IA.291}$$

$$= \left(\left(\frac{P_t(f)}{P_t}\right)^{1-\epsilon} - \frac{W_t}{P_t A_t} \left(\frac{P_t(f)}{P_t}\right)^{-\epsilon} \right) O_t \tag{IA.292}$$

Therefore

$$J_t(f) = O_t \sup_{(P_u(f))_{u \geq t}} E_t \int_t^{\infty} e^{-\delta(u-t)} \left[\left(\frac{P_u(f)}{P_u}\right)^{1-\epsilon} - \frac{W_u}{P_u A_u} \left(\frac{P_u(f)}{P_u}\right)^{-\epsilon} - \frac{1}{2}\theta \left(\frac{dP_u(f)/du}{P_u(f)}\right)^2 \right] du. \tag{IA.293}$$

The above problem is equivalent to

$$\hat{J}_t(f) = \sup_{(P_u(f))_{u\geq t}} E_t \int_t^{\infty} e^{-\delta(u-t)} \left[\left(\frac{P_u(f)}{P_u}\right)^{1-\epsilon} - \frac{W_u}{P_u A_u}\left(\frac{P_u(f)}{P_u}\right)^{-\epsilon} - \frac{1}{2}\theta \left(\frac{dP_u(f)/du}{P_u(f)}\right)^2 \right] du. \tag{IA.294}$$

We assume that prices are locally risk-free. The firm uses the control $dP_t(f)/dt$ to alter the state variable $P_t(f)$, and so we define

$$\overline{\mu}_{P(f),t} = \frac{dP_t(f)}{dt}. \tag{IA.295}$$

Firm f's problem is to choose the rate of change for the price of good f, i.e. $\mu_{P(f)}$ to maximize

$$E_t \int_t^{\infty} e^{-\delta(u-t)} \left[\left(\frac{P_u(f)}{P_u}\right)^{1-\epsilon} - \frac{W_u}{P_u A_u}\left(\frac{P_u(f)}{P_u}\right)^{-\epsilon} - \frac{1}{2}\theta \left(\frac{\overline{\mu}_{P(f)}}{P_u(f)}\right)^2 \right] du. \tag{IA.296}$$

To solve the individual firm's problem, we apply the stochastic maximum principle. Define the Hamiltonian

$$\mathcal{H}_t = \left(\frac{P_t(f)}{P_t}\right)^{1-\epsilon} - \frac{W_t}{P_t A_t}\left(\frac{P_t(f)}{P_t}\right)^{-\epsilon} - \frac{1}{2}\theta \left(\frac{\overline{\mu}_{P(f),t}}{P_t(f)}\right)^2 + E_t\left[\frac{d\hat{J}_t}{dt}\right]. \tag{IA.297}$$

The HJB equation is

$$\sup_{\overline{\mu}_{P(f),t}} \mathcal{H}_t - \delta \hat{J}_t \tag{IA.298}$$

The FOC is

$$\mathcal{H}_{\overline{\mu}_{P(f),t}} = 0, \tag{IA.299}$$

and so

$$\hat{J}_{P(f)} = \theta \frac{\overline{\mu}_{P(f)}}{P(f)^2} \tag{IA.300}$$

Differentiating the HJB equation with respect to the state variable $P(f)$ gives

$$\mathcal{H}_{P(f)} + \mathcal{H}_{\mu_{P(f)}} \frac{\partial \overline{\mu}_{P(f)}}{\partial P(f)} - \delta \hat{J}_{P(f)} = 0. \tag{IA.301}$$

Using the FOC (IA.299) we obtain

$$\mathcal{H}_{P(f)} - \delta \hat{J}_{P(f)} = 0. \tag{IA.302}$$

Therefore

$$(\epsilon - 1)\frac{1}{P}\left(\frac{P(f)}{P}\right)^{-\epsilon} = \epsilon \frac{W}{PA}\frac{1}{P}\left(\frac{P(f)}{P}\right)^{-\epsilon-1} + \theta \frac{\overline{\mu}_{P(f)}}{P(f)^2} + E_t\left[\frac{d\hat{J}_{P(f)}}{dt}\right] - \delta \hat{J}_{P(f)}. \tag{IA.303}$$

The above equation tells us that at the optimum the marginal loss in revenues from higher prices and the marginal loss from price adjustment costs must be offset by marginal gains from lower labor costs and changes in the present expected value of future profits.

Using the FOC (IA.300), we obtain

$$(\epsilon-1)\frac{1}{P}\left(\frac{P(f)}{P}\right)^{-\epsilon}-\theta\frac{1}{P(f)}\left(\frac{\overline{\mu}_{P(f)}}{P(f)}\right)^{2}=\epsilon\frac{W}{PA}\frac{1}{P}\left(\frac{P(f)}{P}\right)^{-\epsilon-1}+\theta E_t\left[\frac{d(\overline{\mu}_{P(f)}/P(f)^2)}{dt}\right]-\delta\theta\frac{\overline{\mu}_{P(f)}}{P(f)^2}. \tag{IA.304}$$

Define inflation in good f via

$$\mu_{P,t}(f)=\frac{1}{P_t(f)}\frac{dP_t(f)}{dt}=\frac{\overline{\mu}_{P(f),t}}{P_t(f)}. \tag{IA.305}$$

Hence

$$(\epsilon-1)\frac{1}{P}\left(\frac{P(f)}{P}\right)^{-\epsilon}=\epsilon\frac{W}{PA}\frac{1}{P}\left(\frac{P(f)}{P}\right)^{-\epsilon-1}+\theta\frac{1}{P_t(f)}E_t\left[\frac{d\mu_{P,t}(f)}{dt}\right]-\delta\theta\frac{\mu_{P,t}(f)}{P_t(f)}, \tag{IA.306}$$

By symmetry

$$P_t(f)=P_t, \tag{IA.307}$$

and so

$$\mu_{P,t}dt=\mu_{P,t}(f)dt=\frac{dP_t(f)}{P_t(f)}=\frac{dP_t}{P_t}. \tag{IA.308}$$

Therefore, the real wage rate is given by

$$\frac{W_t}{P_t}=A_t\left[\left(1-\frac{1}{\epsilon}\right)-\frac{\theta}{\epsilon}\left(E_t\left[\frac{d\mu_{P,t}}{dt}\right]-\delta\mu_{P,t}\right)\right]. \tag{IA.309}$$

Real profit flow for firm f is hence given by

$$\Pi_t(f)=\left(1-\frac{W_t}{P_tA_t}\right)O_t(f) \tag{IA.310}$$

$$=\left[\frac{1}{\epsilon}+\frac{\theta}{\epsilon}\left(E_t\left[\frac{d\mu_{P,t}}{dt}\right]-\delta\mu_{P,t}\right)\right]O_t(f). \tag{IA.311}$$

Representative household's stochastic optimal control problem

We now derive an expression for O_t and hence $O_t(f)$ in terms of the nominal interest rate i. We need to consider the representative household's stochastic optimal control problem, which is

$$V_t=\sup_{(C_u)_{u\geq t},(N_u)_{u\geq t},(\phi_u)_{u\geq t}}E_t\int_t^\infty e^{-\delta(u-t)}\left(\ln C_u-\frac{N_u^{1+\phi}}{1+\phi}\right)du, \tag{IA.312}$$

subject to the dynamic intertemporal budget constraint

$$dH_t=\left(r_tH_t+\frac{W_t}{P_t}N_{h,t}-C_t\right)dt+H_t\phi_t\left(dP_{D,t}+\Pi_tdt\right), \tag{IA.313}$$

where H_t is the time-t financial wealth of the representative household, $P_{D,t}$ is the time-t real present-value value of future real aggregate profits, i.e.

$$P_{D,t}=E_t\left[\int_t^\infty\frac{\Lambda_u}{\Lambda_t}\Pi_u\right], \tag{IA.314}$$

and ϕ_t is the fraction of financial wealth invested in the claim to future profits, i.e. the risky stock. Define

$$a_t = \ln A_t, \tag{IA.315}$$

$$h_t = \ln H_t - a_t, \tag{IA.316}$$

$$c_t = \ln C_t - a_t. \tag{IA.317}$$

Therefore

$$V_t = E_t \int_t^{\infty} e^{-\delta(u-t)} a_u du + \hat{V}_t, \tag{IA.318}$$

where

$$\hat{V}_t = \sup_{(c_u)_{u\geq t},\,(n_u)_{u\geq t},\,(\phi_u)_{u\geq t},} E_t \int_t^{\infty} e^{-\delta(u-t)} \left(c_u - \frac{e^{(1+\phi)n_u}}{1+\phi} \right) du, \tag{IA.319}$$

and

$$dh_t = \left\{ [r_t + \phi_t(\mu_{R,t} - r)] - e^{c_t - h_t} + \frac{W_t}{A_t P_t} e^{n_t - h_t} - \frac{1}{2}\phi_t^2 \sigma_R^2 - \mu_a \right\} dt + \phi_t \sigma_{R,t} dZ_{R,t} - \sigma dZ_t, \tag{IA.320}$$

where

$$\mu_a = E_t \left[\frac{da_t}{dt} \right] = \mu - \frac{1}{2}\sigma^2, \tag{IA.321}$$

and

$$\mu_{R,t} = \frac{1}{dt} E_t \left[\frac{dP_{D,t} + \Pi_t dt}{P_{D,t}} \right], \tag{IA.322}$$

$$\sigma_{R,t} dZ_{R,t} = \frac{dP_{D,t} + \Pi_t dt}{P_{D,t}} - E_t \left[\frac{dP_{D,t} + \Pi_t dt}{P_{D,t}} \right], \tag{IA.323}$$

$$\sigma_{R,t} = \frac{1}{dt} E_t \left(\frac{dP_{D,t}}{P_{D,t}} \right)^2. \tag{IA.324}$$

The Hamilton-Jacobi-Bellman equation for the household's stochastic optimal control problem is

$$0 = \sup c_t - \frac{e^{(1+\phi)n_t}}{1+\phi} - \delta \hat{V}_t + \hat{V}_{h,t} \left\{ [r_t + \phi_t(\mu_{R,t} - r_t)] - e^{c_t - h_t} + \frac{W_t}{A_t P_t} e^{n_t - h_t} - \frac{1}{2}\phi_t^2 \sigma_{R,t}^2 - \mu_a \right\} \tag{IA.325}$$

$$+ \frac{1}{2}(\phi_t^2 \sigma_{R,t}^2 - 2\rho_{R,t}\sigma\sigma_{R,t} + \sigma^2)\hat{V}_{hh,t} + \text{terms stemming from other state variables which are not controlled,} \tag{IA.326}$$

where

$$\rho_{R,t} dt = E_t[dZ_t dZ_R]. \tag{IA.327}$$

The FOCs are

$$1 = e^{c_t - h_t} \hat{V}_t, \tag{IA.328}$$

$$e^{n_t(1+\phi)} = \frac{W_t}{A_t P_t} e^{n_t - h_t} \hat{V}_{h,t}, \tag{IA.329}$$

$$\phi_t = \frac{\hat{V}_{h,t}(\mu_{R,t} - r_t) - \hat{V}_{hh,t}\rho\sigma_{R,t}\sigma}{\sigma_{R,t}^2 \left(\hat{V}_{h,t} - \hat{V}_{hh,t} \right)}, \text{ if } \sigma_{R,t} \neq 0. \tag{IA.330}$$

The first two FOCs can be combined to yield

$$\frac{W_t}{A_t P_t} e^{n} = e^{c_t} e^{n_t(1+\phi)}. \tag{IA.331}$$

In equilibrium, markets clear and so

$$c_t = n_t. \tag{IA.332}$$

Hence

$$e^{n_t(1+\phi)} = \frac{W_t}{A_t P_t}. \tag{IA.333}$$

Since $\ln O_t = y_t = a_t + n_t$ we have

$$\frac{W_t}{e_t^a P_t} = e^{(1+\phi)(y_t - a_t)}, \tag{IA.334}$$

which is equivalent to

$$\frac{W_t}{P_t A_t} = \left(\frac{O_t}{A_t}\right)^{1+\phi}. \tag{IA.335}$$

Using the Ansatz (standard for logarithmic preferences)

$$\hat{V}_t = a_h h_t + j_t, \tag{IA.336}$$

where j_t is independent of h_t, we see that

$$\phi_t = \frac{\mu_{R,t} - r_t}{\sigma_{R,t}^2}. \tag{IA.337}$$

The three equation model in continuous-time with aggregate risk
In equilibrium, open interest in the bond market is zero, and so $\phi_t = 1$, yielding

$$\mu_{R,t} - r_t = \sigma_{R,t}^2. \tag{IA.338}$$

It follows from (IA.309) that

$$\left(\frac{O_t}{A_t}\right)^{1+\phi} = 1 - \frac{1}{\epsilon} - \frac{\theta}{\epsilon}\left(E_t\left[\frac{d\mu_{P,t}}{dt}\right] - \delta\mu_{P,t}\right) \tag{IA.339}$$

When the cost of adjusting prices is zero, i.e. $\theta = 0$, then

$$N_n^{1+\phi} = \left(\frac{O_{n,t}}{A_t}\right)^{1+\phi} = 1 - \frac{1}{\epsilon}, \tag{IA.340}$$

where we append a subscript n to equilibrium variables. Hence,

$$O_{n,t} = A_t N_n = A_t \left(1 - \frac{1}{\epsilon}\right)^{\frac{1}{1+\phi}}, \tag{IA.341}$$

and so (via (IA.280)), it follows that with $\theta = 0$,

$$\Lambda_t = e^{-\delta t} A_t^{-1} \left(1 - \frac{1}{\epsilon}\right)^{-\frac{1}{1+\phi}}. \tag{IA.342}$$

Applying Ito's Lemma, we obtain

$$\frac{d\Lambda_t}{\Lambda_t} = -r_n dt - \Theta_n dZ_t, \tag{IA.343}$$

where

$$r_n = \delta + \mu - \frac{1}{2}\sigma^2 \tag{IA.344}$$

$$\Theta_n = \sigma. \tag{IA.345}$$

Now, from (IA.339) and (IA.340), we obtain the stochastic and nonlinearised version of the New Keynesian Phillips Curve:

$$e^{(1+\phi)x_t} = 1 - \tau\left(E_t\left[\frac{d\mu_{P,t}}{dt}\right] - \delta\mu_{P,t}\right), \tag{IA.346}$$

where

$$\tau = \frac{\theta}{\epsilon - 1} > 0, \tag{IA.347}$$

and

$$x_t = o_t - o_{n,t} \tag{IA.348}$$

is the output gap, with

$$o_t = \ln O_t, \tag{IA.349}$$

$$o_{n,t} = \ln O_{n,t}. \tag{IA.350}$$

$$\tag{IA.351}$$

Hence,

$$d\mu_{P,t} = \left[\delta\mu_{P,t} - \frac{1}{\tau}\left(e^{(1+\phi)x_t} - 1\right)\right]dt + dM_{\pi,t}, \tag{IA.352}$$

where $E_t[dM_{\pi,t}] = 0$. The transversality condition from the firm's optimization problem is

$$\lim_{T\to\infty} E_t[e^{-\delta(T-t)}\mu_{P,T}] = 0. \tag{IA.353}$$

Solving (IA.352) backwards in time from the date-T, we obtain

$$\mu_{P,t} = \frac{1}{\tau}E_t\int_t^T e^{-\delta(u-t)}\left(e^{(1+\phi)x_u} - 1\right)du + E_t[e^{-\delta(T-t)}\mu_{P,T}]. \tag{IA.354}$$

Letting $T \to \infty$ and imposing the transversality condition (IA.353) gives

$$\mu_{P,t} = \frac{1}{\tau}E_t\left[\int_t^\infty e^{-\delta(u-t)}\left(e^{(1+\phi)x_u} - 1\right)du\right]. \tag{IA.355}$$

From the Feynman-Kac Theorem, we obtain the ode

$$0 = \frac{1}{2}\sigma_{x,t}^2 \mu_P''(x_t) + \mu_{x,t}\mu_P'(x_t) - \delta\mu_P(x_t) + \frac{1}{\tau}\left(e^{(1+\phi)x_t} - 1\right), \tag{IA.356}$$

where

$$\mu_{x,t} = E_t\left[\frac{dx_t}{dt}\right], \tag{IA.357}$$

$$\sigma_{x,t}^2 = E_t\left[\frac{(dx_t)^2}{dt}\right]. \tag{IA.358}$$

The equilibrium SDF is given by

$$\Lambda_t = e^{-\delta t} O_t^{-1}. \tag{IA.359}$$

Therefore, the time-t value of the claim to aggregate output flow is

$$P_{O,t} = O_t E_t\left[\int_t^\infty \frac{\Lambda_u}{\Lambda_t}\frac{O_u}{O_t}\right] \tag{IA.360}$$

$$= \frac{O_t}{\delta}. \tag{IA.361}$$

Time-t labor income flow is given by

$$L_t = \frac{W_t}{P_t} N_t = O_t N_t^{1+\phi}. \tag{IA.362}$$

Therefore, the time-t real present-value of labor income flow is given by

$$P_{L,t} = E_t\left[\int_t^\infty \frac{\Lambda_u}{\Lambda_t} L_u du\right] = O_t E_t\left[\int_t^\infty e^{-\delta(u-t)} N_u^{1+\phi} du\right] \tag{IA.363}$$

$$= O_t E_t\left[\int_t^\infty e^{-\delta(u-t)} N_u^{1+\phi} du\right] \tag{IA.364}$$

$$= \left(1 - \frac{1}{\epsilon}\right) O_t E_t\left[\int_t^\infty e^{-\delta(u-t)} e^{(1+\phi)x_u} du\right] \tag{IA.365}$$

$$= \left(1 - \frac{1}{\epsilon}\right) O_t \left(\tau\mu_{P,t} + \frac{1}{\delta}\right) \tag{IA.366}$$

The present value of real profits is given by $P_{D,t}$, where

$$P_{D,t} + P_{L,t} = P_{O,t} = \frac{O_t}{\delta}. \tag{IA.367}$$

Therefore,

$$P_{D,t} = \frac{1}{\epsilon}\frac{O_t}{\delta} - \tau\left(1 - \frac{1}{\epsilon}\right) Y_t \mu_{P,t} = \frac{Y_t}{\epsilon\delta}(1 - \delta\theta\mu_{P,t}). \tag{IA.368}$$

From, (IA.359), we obtain

$$\ln\Lambda = -\delta t - o_{n,t} - x_t, \tag{IA.369}$$

and so, applying Ito's Lemma gives

$$d \ln \Lambda = -\delta dt - da_t - dx_t, \tag{IA.370}$$

where $a_t = \ln A_t$. Hence

$$d \ln \Lambda = -\delta dt - \left(\mu - \frac{1}{2}\sigma^2\right) dt - \sigma dZ_t - dx_t, \tag{IA.371}$$

We know that

$$E_t[d \ln \Lambda_t] = -\left(r_t + \frac{1}{2}\Theta_t^2\right) dt, \tag{IA.372}$$

where r_t is the equilibrium risk-free rate and Θ_t is the equilibrium market price of risk. Hence

$$E_t[dx_t] = \left(-\delta - \mu + \frac{1}{2}\sigma^2 + r_t + \frac{1}{2}\Theta_t^2\right) dt \tag{IA.373}$$

Therefore, we obtain

$$dx_t = \left(-\delta - \mu + \frac{1}{2}\sigma^2 + r_t + \frac{1}{2}\Theta_t^2\right) dt + dM_{x,t}, \tag{IA.374}$$

where $M_x = (M_x)_{t \in \mathcal{T}}$ is a stochastic process such that

$$E_t[dM_{x,t}] = 0. \tag{IA.375}$$

This just means that M_x is a local martingale. It follows that

$$d \ln \Lambda_t - E_t[d \ln \Lambda_t] = -\sigma dZ_t - dM_{x,t}. \tag{IA.376}$$

If we assume that M_x is a continuous stochastic process, then (via the Martingale Representation Theorem)

$$dM_t = \sigma_{x,t} dZ_{x,t} \tag{IA.377}$$

where $Z_x = (Z_{x,t})_{t \in \mathcal{T}}$ is a standard Brownian motion and $\sigma_{x,t}$ is stochastic process (which has to be adapted to the relevant filtration). Therefore, by making the sole assumption that M is a continuous process, we obtain

$$dx_t = \left(-\delta - \mu + \frac{1}{2}\sigma^2 + r_t + \frac{1}{2}\Theta_t^2\right) dt + \sigma_{x,t} dZ_{x,t}. \tag{IA.378}$$

If we assume that only fundamental shocks can be present, ruling out sunspot equilibria, then $Z_x = Z_t$, and so

$$dx_t = \left(-\delta - \mu + \frac{1}{2}\sigma^2 + r_t + \frac{1}{2}\Theta_t^2\right) dt + \sigma_{x,t} dZ_t. \tag{IA.379}$$

It follows that

$$d \ln \Lambda_t - E_t[d \ln \Lambda_t] = -\Theta_t dZ_t. \tag{IA.380}$$

where

$$-\Theta_t dZ_t = -\sigma dZ_t - \sigma_{x,t} dZ_t. \tag{IA.381}$$

Therefore

$$\Theta_t = \sigma + \sigma_{x,t}, \tag{IA.382}$$

and

$$dx_t = \left(-\delta - \mu + \frac{1}{2}\sigma^2 + r_t + \frac{1}{2}\left(\sigma^2 + 2\sigma\sigma_{x,t} + \sigma_{x,t}^2\right)\right)dt + \sigma_{x,t}dZ_t \tag{IA.383}$$

$$= \left(r_t - r_n + \frac{1}{2}\sigma_{x,t}\left(2\sigma + \sigma_{x,t}\right)\right)dt + \sigma_{x,t}dZ_t, \tag{IA.384}$$

where r_n is the natural rate of interest given in (IA.344). Inflation is locally risk-free, so the nominal interest rate is given by

$$r_t^{\$} = r_t + \mu_{P,t}, \tag{IA.385}$$

and so, given x_0, we obtain the stochastic dynamic investment-savings curve:

$$dx_t = \left(r_t^{\$} - \mu_{P,t} - r_n + \frac{1}{2}\sigma_{x,t}\left(2\sigma + \sigma_{x,t}\right)\right)dt + \sigma_{x,t}dZ_t. \tag{IA.386}$$

Returning to (IA.355), we see that $\mu_{P,t} = \mu_P(x_t)$, and so (IA.352) can be rewritten as

$$d\mu_{P,t} = \left[\delta\mu_{P,t} - \frac{1}{\tau}\left(e^{(1+\phi)x_t} - 1\right)\right]dt + \mu_P'(x_t)\sigma_{x,t}dZ_t. \tag{IA.387}$$

Showing the output gap is locally determinstic

We now show that if the nominal interest rate depends only on the output gap and inflation, then $\sigma_{x,t}$=0. If we apply Ito's Lemma to (IA.368), after some algebra, we see that

$$\frac{dP_{D,t} + \Pi_t dt}{P_{D,t}} = (r_t + \sigma_{R,t}\sigma_{o,t})dt + \sigma_{R,t}dZ_t, \tag{IA.388}$$

where

$$r_t = \delta + \mu_{o,t} - \frac{1}{2}\sigma_{o,t}^2, \tag{IA.389}$$

$$\mu_{o,t} = \mu_a + \mu_{x,t}, \tag{IA.390}$$

$$\sigma_{o,t} = \sigma + \sigma_{x,t}, \tag{IA.391}$$

$$\sigma_{R,t} = \sigma_{o,t} - \frac{\delta\theta\mu_P'(x_t)\sigma_{x,t}}{1 - \delta\theta\mu_P(x_t)}. \tag{IA.392}$$

Observe also that

$$E_t\left[\frac{dP_{D,t} + \Pi_t dt}{P_{D,t}} - r_t dt\right] = -E_t\left[\frac{d\Lambda_t}{\Lambda_t}\frac{dP_{D,t}}{P_{D,t}}\right], \tag{IA.393}$$

and so (IA.388) is consistent with no arbitrage. We can write

$$\mu_{R,t} = r_t + \sigma_{R,t}\sigma_{o,t}, \tag{IA.394}$$

but we also have (IA.338), and so

$$\sigma_{R,t}^2 = \sigma_{R,t}\sigma_{o,t}. \tag{IA.395}$$

Therefore $\sigma_{R,t} = 0$ or $\sigma_{R,t} = \sigma_{o,t}$. The case $\sigma_{R,t} = 0$ is ruled out via (IA.330), and so $\sigma_{R,t} = \sigma_{o,t}$, which implies that $\mu_P'(x_t)\sigma_{x,t} = 0$. Therefore, at least one of the following two equations holds:

$$\mu_P'(x_t) = 0 \tag{IA.396}$$

$$\sigma_{x,t} = 0. \tag{IA.397}$$

Observe that if $\mu_P'(x_t) = 0$, then the ode for inflation, , implies that

$$\mu_P'(x_t) = \frac{1}{\tau\delta}(e^{(1+\phi)x_t} - 1), \tag{IA.398}$$

which can only be valid if x_t is constant, which is not the case, so we have a contradiction. Therefore $\sigma_{x,t} = 0$. Hence, if the nominal interest rate does not depend on any shocks in addition to Z, then x and $\mu_{P,t}$ evolve deterministically, and so (IA.386) and (IA.387) reduce to

$$\frac{dx_t}{dt} = (r_t^{\$} - \mu_{P,t} - r_n), \tag{IA.399}$$

$$\frac{d\mu_{P,t}}{dt} = \delta\mu_{P,t} - \frac{1}{\tau}\left(e^{(1+\phi)x_t} - 1\right). \tag{IA.400}$$

Endogenous sticky cash flows

If $\sigma_{x,t} = 0$, then (IA.311) reduces to

$$\Pi_t(f) = \left[\frac{1}{\epsilon} + \frac{\theta}{\epsilon}\left(\frac{d\mu_{P,t}}{dt} - \delta\mu_{P,t}\right)\right] O_t(f). \tag{IA.401}$$

Therefore,

$$\ln \Pi_t(f) = \ln\left[1 - (\epsilon - 1)(e^{(1+\phi)x_t} - 1)\right] + \ln O_t(f) - \ln\epsilon \tag{IA.402}$$

$$= -(\epsilon - 1)(1+\phi)x_t + \ln O_t(f) - \ln\epsilon, \tag{IA.403}$$

where the last line follows by using an linear approximation. Applying Ito's Lemma gives

$$d\ln\Pi_t(f) = da_t + [1 - (\epsilon - 1)(1+\phi)]dx_t \tag{IA.404}$$

$$= da_t + [1 - (\epsilon - 1)(1+\phi)](r_t^{\$} - \mu_{P,t} - r_n)dt. \tag{IA.405}$$

Now, if the nominal interest rate is given by

$$i_t = a_0 + a_1\mu_{P,t}, \tag{IA.406}$$

then we have

$$d\ln\Pi_t(f) = da_t - [(\epsilon - 1)(1+\phi) - 1][a_0 - r_n + (a_1 - 1)\mu_{P,t}]dt. \tag{IA.407}$$

We therefore see that if inflation increases by one percentage point, then real profit growth decreases by $[(\epsilon - 1)(1+\phi) - 1](a_1 - 1)$ percentage points. Observe that $(\epsilon - 1)(1+\phi) - 1 > 0$ if and only if

$$\epsilon > 1 + \frac{1}{1+\phi}. \tag{IA.408}$$

Defining nominal profit flow for firm f via

$$\Pi_t^{\$}(f) = P_t\Pi_t(f), \tag{IA.409}$$

we obtain

$$d \ln \Pi_t^{\$}(f) = da_t - [(\epsilon - 1)(1 + \phi) - 1][a_0 - r_n]dt + \varphi \mu_{P,t} dt, \tag{IA.410}$$

where

$$\varphi = 1 - [(\epsilon - 1)(1 + \phi) - 1](a_1 - 1)]. \tag{IA.411}$$

Therefore, if $\epsilon > 1 + \frac{1}{1+\phi}$ and $a_1 > 1$, then $\varphi < 1$, and so, expected nominal profit growth increases by less than one percentage point when inflation increases by one percentage point. ■