Received: 5 November 2021 | Revised: 10 July 2022 | Accepted: 23 August 2022

DOI: 10.1112/plms.12492

RESEARCH ARTICLE

Proceedings of the London Mathematical Society

Uniqueness of the critical and supercritical Liouville quantum gravity metrics

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Funding information
NSF, Grant/Award Numbers: DMS-1757479, DMS-1953848; Clay mathematics institute

Abstract

We show that for each $\mathbf{c}_{\mathrm{M}} \in [1, 25)$, there is a unique metric associated with Liouville quantum gravity (LQG) with matter central charge $\mathbf{c}_{\mathrm{M}}$. An earlier series of works by Ding–Dubédat–Dunlap–Falconet, Gwynne–Miller, and others showed that such a metric exists and is unique in the subcritical case $\mathbf{c}_{\mathrm{M}} \in (-\infty, 1)$, which corresponds to coupling constant $\gamma \in (0, 2)$. The critical case $\mathbf{c}_{\mathrm{M}} = 1$ corresponds to $\gamma = 2$ and the supercritical case $\mathbf{c}_{\mathrm{M}} \in (1, 25)$ corresponds to $\gamma \in \mathbb{C}$ with $|\gamma| = 2$. Our metric is constructed as the limit of an approximation procedure called Liouville first passage percolation, which was previously shown to be tight for $\mathbf{c}_{\mathrm{M}} \in [1, 25)$ by Ding and Gwynne (2020). In this paper, we show that the subsequential limit is uniquely characterized by a natural list of axioms. This extends the characterization of the LQG metric proven by Gwynne and Miller (2019) for $\mathbf{c}_{\mathrm{M}} \in (-\infty, 1)$ to the full parameter range $\mathbf{c}_{\mathrm{M}} \in (-\infty, 25)$. Our argument is substantially different from the proof of the characterization of the LQG metric for $\mathbf{c}_{\mathrm{M}} \in (-\infty, 1)$. In particular, the core part of the argument is simpler and does not use confluence of geodesics.

MSC 2020
60D05, 60G60 (primary)

Contents

1 | INTRODUCTION

1.1 | Overview

Liouville quantum gravity (LQG) is a one-parameter family of random fractal surfaces which originated in the physics literature in the 1980s [7, 16, 37] as a class of canonical models of random geometry in two dimensions. One possible choice of parameter is the *matter central charge*

Phase	LFPP parameter	LFPP exponent	Coupling constant	Matter central charge	Topology
Subcritical	$\xi \in (0, \xi_{\text{crit}})$	$Q > 2$	$\gamma \in (0, 2)$	$\mathbf{c}_{\text{M}} \in (-\infty, 1)$	Bi-Hölder w.r.t. Euclidean
Critical	$\xi = \xi_{\text{crit}}$	$Q = 2$	$\gamma = 2$	$\mathbf{c}_{\text{M}} = 1$	Euclidean topology, not Hölder
Supercritical	$\xi > \xi_{\text{crit}}$	$Q \in (0, 2)$	γ complex, $\lvert\gamma\rvert = 2$.	$\mathbf{c}_{\text{M}} \in (1, 25)$	$\exists$ singular points

FIGURE 1 Comparison of the different phases of LQG This paper proves that the LQG metric is unique in the critical and supercritical phases. The bi-Hölder continuity with respect to to the Euclidean metric in the subcritical phase is proven in [17]. The statement that the critical LQG metric induces the Euclidean topology, but is not Hölder continuous, is proven in [13].

$\mathbf{c}_{\text{M}} \in (-\infty, 25)$. Heuristically speaking, for an open domain $U \subset \mathbb{C}$, an LQG surface with matter central charge $\mathbf{c}_{\text{M}}$ is a sample from 'the uniform measure on Riemannian metric tensors g on U, weighted by $(\det \Delta_g)^{-\mathbf{c}_{\text{M}}/2}$', where Δ_g denotes the Laplace–Beltrami operator. This definition is far from rigorous, for example, because the space of Riemannian metric tensors on U is infinite-dimensional, so there is not an obvious notion of a uniform measure on this space. However, there are various ways of defining LQG surface rigorously, as we discuss just below.

Definition 1.1. We refer to LQG with $\mathbf{c}_{\text{M}} \in (-\infty, 1)$, $\mathbf{c}_{\text{M}} = 1$, and $\mathbf{c}_{\text{M}} \in (1, 25)$ as the *subcritical*, *critical*, and *supercritical* phases, respectively.

See Figure 1 for a summary of the three phases. One way to define LQG rigorously in the subcritical and critical phases is via the *David–Distler–Kawai (DDK) ansatz*. The DDK ansatz states that for $\mathbf{c}_{\text{M}} \in (-\infty, 1]$, the Riemannian metric tensor associated with an LQG surface takes the form

$$g = e^{\gamma h}(dx^2 + dy^2), \quad \text{where } \gamma \in (0, 2] \text{ satisfies } \quad \mathbf{c}_{\text{M}} = 25 - 6\left(\frac{2}{\gamma} + \frac{\gamma}{2}\right)^2. \tag{1.1}$$

Here, $dx^2 + dy^2$ denotes the Euclidean metric tensor on U and h is a variant of the Gaussian free field (GFF) on U, the most natural random generalized function on U. We refer to [5, 41, 43] for more background on the GFF.

The Riemannian metric tensor in (1.1) is still not well-defined since the GFF is not a function, so $e^{\gamma h}$ does not make literal sense. Nevertheless, it is possible to rigorously define various objects associated with (1.1) using regularization procedures. To do this, one considers a family of continuous functions $\{h_\varepsilon\}_{\varepsilon > 0}$ which approximate h, then takes an appropriate limit of objects defined using h_ε in place of h. Objects which have been constructed in this manner include the LQG area and length measures [18, 31, 39], Liouville Brownian motion [4, 19], the correlation functions for the random 'fields' $e^{\alpha h}$ for $\alpha \in \mathbb{R}$ [32], and the distance function (metric) associated with (1.1), at least for $\mathbf{c}_{\text{M}} < 1$ [8, 27].

LQG in the subcritical and critical phases is expected, and in some cases proven, to describe the scaling limit of various types of random planar maps. For example, in keeping with the above heuristic definition, LQG with $\mathbf{c}_{\text{M}} \in (-\infty, 1]$ should describe the scaling limit of random planar maps sampled with probability proportional to $(\det \Delta)^{-\mathbf{c}_M/2}$, where Δ is the discrete Laplacian. We refer to [5, 20, 23] for expository articles on subcritical and critical LQG.

The supercritical phase $\mathbf{c}_M \in (1, 25)$ is much more mysterious than the subcritical and critical phases, even from the physics perspective. In this case, the DDK ansatz does not apply. In fact, the parameter γ from (1.1) is complex with $|\gamma| = 2$, so attempting to directly analytically continue formulae from the subcritical case to the supercritical case often gives nonsensical complex answers. It is expected that supercritical LQG still corresponds in some sense to a random geometry related to the GFF. However, until very recently there have been few mathematically rigorous results for supercritical LQG. See [22] for an extensive discussion of the physics literature and various conjectures concerning LQG with $\mathbf{c}_M \in (1, 25)$.

The purpose of this paper is to show that in the critical and supercritical phases, that is, when $\mathbf{c}_M \in [1, 25)$, there is a canonical metric (distance function) associated with LQG. This was previously established in the subcritical phase $\mathbf{c}_M \in (-\infty, 1)$ in the series of papers [8, 17, 24, 25, 27]. Our results resolve [27, Problems 7.17 and 7.18], which ask for a metric associated with LQG for $\mathbf{c}_M \in [1, 25)$.

This paper builds on [11], which proved the tightness of an approximation procedure for the metric when $\mathbf{c}_M \in [1, 25)$ (using [15] and some estimates from [8] which also work for the critical/supercritical cases), and [36], which proved various properties of the subsequential limits. The analogs of these works in the subcritical case are [8] and [17], respectively. We will also use one preliminary lemma which was proven in [12] (Lemma 2.12), but we will not need the main result of [12], that is, the confluence of geodesics property.

Our results are analogous to those of [27], which proved uniqueness of the subcritical LQG metric. We will prove that the subsequential limiting metrics in the critical and supercritical cases are uniquely characterized by a natural list of axioms. However, our proof is very different from the argument of [27], for two main reasons.

- A key input in [27] is *confluence of geodesics*, which says that two LQG geodesics with the same starting point and different target points typically coincide for a non-trivial initial interval of time [24]. We replace the core part of the argument in [27], which corresponds to [27, section 4], by a simpler argument which does not use confluence of geodesics (Section 4). Instead, our argument is based on counting the number of events of a certain type which occur. Confluence of geodesics was proven for the critical and supercritical LQG metrics in [12], but it is not needed in this paper.
- There are many additional difficulties in our proof, especially in Section 5, arising from the fact that the metrics we work with are not continuous with respect to the Euclidean metric, or even finite-valued.

The first point reduces the complexity of this paper as compared to [27], whereas the second point increases it. The net effect is that our argument is overall longer than [27], but conceptually simpler and requires less external input. We note that all of our arguments apply in the subcritical phase as well as the critical and supercritical phases, so this paper also gives a new proof of the results of [27].

1.2 | Convergence of Liouville first passage percolation

For concreteness, throughout this paper we will restrict attention to the whole-plane case. We let h be the whole-plane GFF with the additive constant chosen so that its average over the unit circle is zero. Once the LQG metric for h is constructed, it is straightforward to construct metrics associated with variants of the GFF on other domains via restriction and/or local absolute continuity; see [27, Remark 1.5]. As in the subcritical case, the construction of our metric uses an approximation procedure called *Liouville first passage percolation* (LFPP). To define LFPP, we first introduce a family of continuous functions which approximate h. For $s > 0$ and $z \in \mathbb{C}$, let $p_s(z) = \frac{1}{2\pi s}\exp(-\frac{|z|^2}{2s})$ be the heat kernel. For $\varepsilon > 0$, we define a mollified version of the GFF by

$$h_\varepsilon^*(z) := (h * p_{\varepsilon^2/2})(z) = \int_{\mathbb{C}} h(w) p_{\varepsilon^2/2}(z-w)\,dw, \quad \forall z \in \mathbb{C}, \tag{1.2}$$

where the integral is interpreted in the sense of distributional pairing. We use $p_{\varepsilon^2/2}$ instead of p_ε so that the variance of $h_\varepsilon^*(z)$ is $\log \varepsilon^{-1} + O_\varepsilon(1)$.

We now consider a parameter $\xi > 0$, which will shortly be chosen to depend on the matter central charge $\mathbf{c}_{\mathrm{M}}$ (see (1.6)). LFPP with parameter ξ is the family of random metrics $\{D_h^\varepsilon\}_{\varepsilon>0}$ defined by

$$D_h^\varepsilon(z,w) := \inf_{P: z \to w} \int_0^1 e^{\xi h_\varepsilon^*(P(t))} |P'(t)|\,dt, \quad \forall z, w \in \mathbb{C}, \tag{1.3}$$

where the infimum is over all piecewise continuously differentiable paths $P : [0,1] \to \mathbb{C}$ from z to w. To extract a non-trivial limit of the metrics D_h^ε, we need to re-normalize. We (somewhat arbitrarily) define our normalizing factor by

$$\mathfrak{a}_\varepsilon := \text{median of } \inf\left\{ \int_0^1 e^{\xi h_\varepsilon^*(P(t))} |P'(t)|\,dt \,:\, P \text{ is a left–right crossing of } [0,1]^2 \right\}, \tag{1.4}$$

where a left–right crossing of $[0,1]^2$ is a piecewise continuously differentiable path $P : [0,1] \to [0,1]^2$ joining the left and right boundaries of $[0,1]^2$. We do not know the value of $\mathfrak{a}_\varepsilon$ explicitly. The best currently available estimates are given in [14, Theorem 1.11].

More generally, the definition (1.3) of LFPP also makes sense when h is a *whole-plane GFF plus a bounded continuous function*, that is, a random distribution of the form $\widetilde{h} + f$, where $\widetilde{h}$ is a whole-plane GFF and f is a (possibly random and $\widetilde{h}$-dependent) bounded continuous function.

In terms of LFPP, the main result of this paper gives the convergence of the metrics $\mathfrak{a}_\varepsilon^{-1} D_h^\varepsilon$ for each $\xi > 0$. For values of ξ corresponding to the supercritical case $\mathbf{c}_{\mathrm{M}} \in (1, 25)$, the limiting metric is not continuous with respect to the Euclidean metric. Hence, we cannot expect convergence with respect to the uniform topology. Instead, as in [11], we will work with the topology of the following definition.

Definition 1.2. Let $X \subset \mathbb{C}$. A function $f : X \times X \to \mathbb{R} \cup \{-\infty, +\infty\}$ is *lower semicontinuous* if whenever $(z_n, w_n) \in X \times X$ with $(z_n, w_n) \to (z, w)$, we have $f(z,w) \leqslant \liminf_{n\to\infty} f(z_n, w_n)$. The *topology on lower semicontinuous functions* is the topology whereby a sequence of such functions $\{f_n\}_{n\in\mathbb{N}}$ converges to another such function f if and only if

(i) whenever $(z_n, w_n) \in X \times X$ with $(z_n, w_n) \to (z, w)$, we have $f(z,w) \leqslant \liminf_{n\to\infty} f_n(z_n, w_n)$;
(ii) for each $(z,w) \in X \times X$, there exists a sequence $(z_n, w_n) \to (z, w)$ such that $f_n(z_n, w_n) \to f(z,w)$.

It follows from [3, Lemma 1.5] that the topology of Definition 1.2 is meterizable (see [11, section 1.2]). Furthermore, [3, Theorem 1(a)] shows that the metric inducing this topology can be taken to be separable.

Theorem 1.3. *Let h be a whole-plane GFF, or more generally a whole-plane GFF plus a bounded continuous function. For each $\xi > 0$, the re-scaled LFPP metrics $\mathfrak{a}_\varepsilon^{-1} D_h^\varepsilon$ converge in probability with respect to the topology on lower semicontinuous functions on $\mathbb{C} \times \mathbb{C}$ (Definition 1.2). The limit D_h is a random metric on $\mathbb{C}$, except that it is allowed to take on infinite values.*

To make the connection between Theorem 1.3 and the LQG metric, we need to discuss the LFPP distance exponent Q. It was shown in [11, Proposition 1.1] that for each $\xi > 0$, there exists $Q = Q(\xi) > 0$ such that

$$\mathfrak{a}_\varepsilon = \varepsilon^{1-\xi Q + o_\varepsilon(1)}, \quad \text{as} \quad \varepsilon \to 0. \tag{1.5}$$

The existence of Q is proven via a subadditivity argument, so the exact relationship between Q and ξ is not known. However, it is known that $Q \in (0, \infty)$ for all $\xi > 0$ and Q is a continuous, non-increasing function of ξ [11, 15]. See also [1, 28] for bounds for Q in terms of ξ.

As we will discuss in more detail below, LFPP with parameter ξ is related to LQG with matter central charge

$$\mathbf{c}_{\mathrm{M}} = \mathbf{c}_{\mathrm{M}}(\xi) = 25 - 6Q(\xi)^2. \tag{1.6}$$

The function $\xi \mapsto Q(\xi)$ is continuous and $Q(\xi) \to \infty$ as $\xi \to 0$ and $Q(\xi) \to 0$ as $\xi \to \infty$ [11, Proposition 1.1]. So, the formula (1.6) shows that there is a value of ξ corresponding to each $\mathbf{c}_{\mathrm{M}} \in (-\infty, 25)$. Furthermore, $\xi \mapsto Q(\xi)$ is strictly decreasing on (0,0.7), so the function $\xi \mapsto \mathbf{c}_{\mathrm{M}}(\xi)$ is injective on this interval. We expect that it is in fact injective on all of $(0, \infty)$, which would mean that there is a one-to-one correspondence between ξ and $\mathbf{c}_{\mathrm{M}}$.[†]

The relation between ξ and $\mathbf{c}_{\mathrm{M}}$ in (1.6) is not explicit since the dependence of Q on ξ is not known explicitly. The only exact relation between $\mathbf{c}_{\mathrm{M}}$ and ξ which we know is that $\mathbf{c}_{\mathrm{M}} = 0$ corresponds to $\xi = 1/\sqrt{6}$. This is equivalent to the fact that the Hausdorff dimension of LQG with $\gamma = \sqrt{8/3}$ is 4. See [10] for details.

From (1.6), we see that $Q(\xi) = 2$ corresponds to the critical value $\mathbf{c}_{\mathrm{M}} = 1$, which motivates us to define

$$\xi_{\mathrm{crit}} := \inf\{\xi > 0 : Q(\xi) = 2\}. \tag{1.7}$$

It follows from [11, Proposition 1.1] that ξ_{crit} is the unique value of ξ for which $Q(\xi) = 2$ and from [28, Theorem 2.3] that $\xi_{\mathrm{crit}} \in [0.4135, 0.4189]$. We have $Q > 2$ for $\xi < \xi_{\mathrm{crit}}$ and $Q \in (0, 2)$ for $\xi > \xi_{\mathrm{crit}}$.

[†] One way to prove the injectivity of $\xi \mapsto \mathbf{c}_{\mathrm{M}}(\xi)$ would be to show that if ξ and $\mathbf{c}_{\mathrm{M}}$ are related as in (1.6), then ξ is the distance exponent for the dyadic subdivision model in [22] with parameter $\mathbf{c}_{\mathrm{M}}$: indeed, this would give an inverse to the function $\xi \mapsto \mathbf{c}_{\mathrm{M}}(\xi)$. We expect that this can be proven using similar arguments to the ones used to related LFPP and Liouville graph distance in [10], see also the discussion of LFPP in [22, section 2.3].

Definition 1.4. We refer to LFPP with $\xi < \xi_{\text{crit}}$, $\xi = \xi_{\text{crit}}$, and $\xi > \xi_{\text{crit}}$ as the *subcritical*, *critical*, and *supercritical* phases, respectively.

By (1.6), the three phases of LFPP correspond exactly to the three phases of LQG in Definition 1.1.

Theorem 1.3 has already been proven in the subcritical phase $\xi < \xi_{\text{crit}}$ (but this paper simplifies part of the proof). Indeed, it was shown by Ding, Dubédat, Dunlap, and Falconet [8] that in this case the re-scaled LFPP metrics $\mathfrak{a}_\varepsilon^{-1} D_h^\varepsilon$ are tight with respect to the topology of uniform convergence on compact subsets of $\mathbb{C} \times \mathbb{C}$, which is a stronger topology than the one in Definition 1.2. Subsequently, it was shown by Gwynne and Miller [27], building on [17, 24, 25], that the subsequential limit is unique. This was done by establishing an axiomatic characterization of the limiting metric.

The limiting metric in the subcritical phase induces the same topology on $\mathbb{C}$ as the Euclidean metric, but has very different geometric properties. This metric can be thought of as the Riemannian distance function associated with the Riemannian metric tensor (1.1), where $\mathbf{c}_{\text{M}} \in (-\infty, 1)$ and ξ are related as in (1.6). The relation between $\mathbf{c}_{\text{M}}$ and ξ can equivalently be expressed as $\gamma = \xi d(\xi)$, where $\gamma \in (0, 2)$ is as in (1.1) and $d(\xi) > 2$ is the Hausdorff dimension of the limiting metric [10, 29]. See [9] for a survey of results about the subcritical LQG metric (and some previous results in the critical and supercritical cases).

In the critical and supercritical cases, Theorem 1.3 is new. We previously showed in [11] that for all $\xi > 0$, the metrics $\{\mathfrak{a}_\varepsilon^{-1} D_h^\varepsilon\}_{\varepsilon>0}$ are tight with respect to the topology on lower semicontinuous functions. The contribution of the present paper is to show that the subsequential limit is unique. We will do this by proving that the limiting metric is uniquely characterized by a list of axioms analogous to the one in [27] (see Theorems 1.8 and 1.13).

In the critical case $\xi = \xi_{\text{crit}}$, the limiting metric D_h induces the same topology as the Euclidean metric [13], and can be thought of as the Riemannian distance function associated with critical ($\gamma = 2$) LQG. We refer to [38] for a survey of results concerning the critical LQG *measure*.

In the supercritical case $\xi > \xi_{\text{crit}}$, the limiting metric in Theorem 1.3 does not induce the Euclidean topology on $\mathbb{C}$. Rather, almost surely there exists an uncountable, Euclidean-dense set of *singular points* $z \in \mathbb{C}$ such that

$$D_h(z, w) = \infty, \quad \forall w \in \mathbb{C} \setminus \{z\}. \tag{1.8}$$

However, for each fixed $z \in \mathbb{C}$, almost surely z is not a singular point, so the set of singular points has zero Lebesgue measure. Moreover, any two non-singular points lie at finite D_h-distance from each other [11]. One can think of singular points as infinite 'spikes' which D_h-rectifiable paths must avoid.

If we let $\{h_\varepsilon\}_{\varepsilon>0}$ be the circle average process for the GFF [18, section 3.1], then the set of singular points is (almost) the same as the set of points $z \in \mathbb{C}$ which have *thickness* greater than Q, in the sense that

$$\limsup_{\varepsilon \to 0} \frac{h_\varepsilon(z)}{\log \varepsilon^{-1}} > Q. \tag{1.9}$$

See [36, Proposition 1.11] for a precise statement. It is shown in [30] that almost surely

$$\limsup_{\varepsilon \to 0} h_\varepsilon(z) / \log \varepsilon^{-1} \in [-2, 2], \quad \forall z \in \mathbb{C},$$

which explains why ξ_{crit} (which corresponds to $Q = 2$) is the critical threshold for singular points to exist.

Remark 1.5 (Conjectured random planar map connection). In the subcritical case, the LQG metric is conjectured to describe the scaling limit of various types of random planar maps, equipped with their graph distance, with respect to the Gromov–Hausdorff topology (see [27, section 1.3]). This conjecture naturally extends to the critical case. In particular, the critical LQG metric should be the Gromov–Hausdorff scaling limit of random planar maps sampled with probability proportional to the partition function of, for example, the discrete GFF, the O(2) loop model, the critical 4-state Potts model, or the critical Fortuin–Kasteleyn model with parameter $q = 4$ [2, 23, 42]. A naive guess in the supercritical case is that the LQG metric for $\mathbf{c}_{\mathrm{M}} \in (1, 25)$ should describe the scaling limit of random planar maps sampled with probability proportional to $(\det \Delta)^{-\mathbf{c}_{\mathrm{M}}/2}$, where Δ is the discrete Laplacian. This guess appears to be false, however, since numerical simulations and heuristics suggest that such planar maps converge in the scaling limit to trees (see [22, section 2.2] and the references therein). Rather, in order to get supercritical LQG in the limit, one should consider planar maps sampled with probability proportional to $(\det \Delta)^{-\mathbf{c}_{\mathrm{M}}/2}$ which are in some sense 'allowed to have infinitely many vertices'. We do not know how to make sense of such maps rigorously. However, [22] defines a random planar map which should be in the same universality class: it is the adjacency graph of a dyadic tiling of $\mathbb{C}$ by squares which all have the same '$\mathbf{c}_{\mathrm{M}}$-LQG size' with respect to an instance of the GFF. See [22] for further discussion.

1.3 | Characterization of the LQG metric

Since we already know that LFPP is tight for all $\xi > 0$ [11], in order to prove Theorem 1.3 we need to show that the subsequential limit is unique. To accomplish this, we will prove that for each $\xi > 0$, there is a unique (up to multiplication by a deterministic positive constant) metric satisfying certain axioms. That is, we will extend the characterization result of [27] to the supercritical case. To state our axioms, we first need some preliminary definitions.

Definition 1.6. Let (X, d) be a metric space, with d allowed to take on infinite values.

- A *curve (also known as a path)* in (X, d) is a continuous function $P : [a, b] \to X$ for some interval $[a, b]$.
- For a curve $P : [a, b] \to X$, the *d-length* of P is defined by

$$\mathrm{len}(P; d) := \sup_T \sum_{i=1}^{\#T} d(P(t_i), P(t_{i-1})),$$

  where the supremum is over all partitions $T : a = t_0 < \cdots < t_{\#T} = b$ of $[a, b]$. Note that the d-length of a curve may be infinite. In particular, the d-length of P is infinite if there are times $s, t \in [a, b]$ such that $d(P(s), P(t)) = \infty$.
- We say that (X, d) is a *length space* if for each $x, y \in X$ and each $\varepsilon > 0$, there exists a curve of d-length at most $d(x, y) + \varepsilon$ from x to y. If $d(x, y) < \infty$, a curve from x to y of d-length *exactly* $d(x, y)$ is called a *geodesic*.

- For $Y \subset X$, the *internal metric of* d on Y is defined by

$$d(x,y;Y) := \inf_{P \subset Y} \operatorname{len}(P;d), \quad \forall x,y \in Y, \tag{1.10}$$

  where the infimum is over all curves P in Y from x to y. Note that $d(\cdot,\cdot;Y)$ is a metric on Y, except that it is allowed to take infinite values.
- If $X \subset \mathbb{C}$, we say that d is a *lower semicontinuous metric* if the function $(x,y) \to d(x,y)$ is lower semicontinuous with respect to the Euclidean topology. We equip the set of lower semicontinuous metrics on X with the topology on lower semicontinuous functions on $X \times X$, as in Definition 1.2, and the associated Borel σ-algebra.

The axioms which characterize our metric are given in the following definition.

Definition 1.7 (LQG metric). Let $\mathcal{D}'$ be the space of distributions (generalized functions) on $\mathbb{C}$, equipped with the usual weak topology. For $\xi > 0$, a *(strong) LQG metric with parameter* ξ is a measurable function $h \mapsto D_h$ from $\mathcal{D}'$ to the space of lower semicontinuous metrics on $\mathbb{C}$ with the following properties.[†] Let h be a *GFF plus a continuous function* on $\mathbb{C}$: that is, h is a random distribution on $\mathbb{C}$ which can be coupled with a random continuous function f in such a way that $h - f$ has the law of the whole-plane GFF. Then the associated metric D_h satisfies the following axioms.

I. **Length space.** Almost surely, $(\mathbb{C}, D_h)$ is a length space.

II. **Locality.** Let $U \subset \mathbb{C}$ be a deterministic open set. The D_h-internal metric $D_h(\cdot,\cdot;U)$ is almost surely given by a measurable function of $h|_U$.

III. **Weyl scaling.** For a continuous function $f : \mathbb{C} \to \mathbb{R}$, define

$$(e^{\xi f} \cdot D_h)(z,w) := \inf_{P : z \to w} \int_0^{\operatorname{len}(P;D_h)} e^{\xi f(P(t))}\, dt, \quad \forall z,w \in \mathbb{C}, \tag{1.11}$$

where the infimum is over all D_h-rectifiable paths from z to w in $\mathbb{C}$ parameterized by D_h-length (we use the convention that $\inf \emptyset = \infty$). Then almost surely $e^{\xi f} \cdot D_h = D_{h+f}$ for every continuous function $f : \mathbb{C} \to \mathbb{R}$.

IV. **Scale and translation covariance.** Let Q be as in (1.5). For each fixed deterministic $r > 0$ and $z \in \mathbb{C}$, almost surely

$$D_h(ru + z, rv + z) = D_{h(r\cdot+z)+Q\log r}(u,v), \quad \forall u,v \in \mathbb{C}. \tag{1.12}$$

V. **Finiteness.** Let $U \subset \mathbb{C}$ be a deterministic, open, connected set and let $K_1, K_2 \subset U$ be disjoint, deterministic, compact, connected sets which are not singletons. Almost surely, $D_h(K_1, K_2; U) < \infty$.

Definition 1.7 is nearly identical to the analogous definition in the subcritical case [27, section 1.2], except we only require the metric to be lower semicontinuous, rather than requiring it to

[†] We do not care how D is defined on any subset of $\mathcal{D}'$ which has probability zero for the distribution of any whole-plane GFF plus a continuous function.

induce the Euclidean topology. Because we allow D_h to take infinite values, we need to include a finiteness condition (Axiom V) to rule out metrics which assign infinite distance to too many pairs of points. For example, if we defined D_h for every distribution h by $D_h(z,w) = 0$ if $z = w$ and $D_h(z,w) = \infty$ if $z \neq w$, then $h \mapsto D_h$ would satisfy all of the conditions of Definition 1.7 except for Axiom V.

Axioms I, II, and III are natural from the heuristic that the LQG metric should be given by 'integrating $e^{\xi h}$ along paths, then taking an infimum over paths'. We remark that if h is a GFF plus a continuous function and D_h is a weak LQG metric, then almost surely the Euclidean metric is continuous with respect to D_h [36, Proposition 1.10] (but D_h is not continuous with respect to the Euclidean metric if $\xi > \xi_{\mathrm{crit}}$). Consequently, almost surely every path of finite D_h-length is Euclidean continuous.

Axiom IV is the metric analog of the LQG coordinate change formula from [18, section 2], but restricted to translation and scaling. Following [18], we can think of the pairs $(\mathbb{C}, D_h)$ and $(\mathbb{C}, h(r\cdot+z) + Q\log r)$ as representing two different parameterizations of the same LQG surface. Axiom IV implies that the metric is an intrinsic function of the LQG surface, that is, it is invariant under changing coordinates to a different parameterization. We do not assume that the metric is covariant with respect to rotations in Definition 1.7: this turns out to be a consequence of the other axioms (see Proposition 1.9).

The following theorem extends [27, Theorem 1.2] to the critical and supercritical phases.

Theorem 1.8. *For each $\xi > 0$, there is an LQG metric D with parameter ξ such that the limiting metric of Theorem 1.3 is almost surely equal to D_h whenever h is a whole-plane GFF plus a bounded continuous function. Furthermore, this LQG metric is unique in the following sense. If D and $\widetilde{D}$ are two LQG metrics with parameter ξ, then there is a deterministic constant $C > 0$ such that almost surely $\widetilde{D}_h = CD_h$ whenever h is a whole-plane GFF plus a continuous function.*

Theorem 1.8 tells us that for every $\mathbf{c}_{\mathrm{M}} \in (-\infty, 25)$, there is an essentially unique† metric associated with LQG with matter central charge $\mathbf{c}_{\mathrm{M}}$ (recall the non-explicit relation between ξ and $\mathbf{c}_{\mathrm{M}}$ from (1.6)). The deterministic positive constant C from Theorem 1.8 can be fixed in various ways. For example, we can require that the median of the D_h-distance between the left and right sides of the unit square is 1 in the case when h is a whole-plane GFF normalized so that its average over the unit circle is 0. Due to (1.4), the limit of LFPP has this normalization.

Theorem 1.8 implies that the LQG metric is covariant with respect to rotation, not just scaling and translation. See [27, Remark 1.6] for a heuristic discussion of why we do not need to assume rotational invariance in Definition 1.7.

Proposition 1.9. *Let $\xi > 0$ and let D be an LQG metric with parameter ξ. Let h be a whole-plane GFF plus a continuous function and let $\omega \in \mathbb{C}$ with $|\omega| = 1$. Almost surely,*

$$D_h(u,v) = D_{h(\omega\cdot)}(\omega^{-1}u, \omega^{-1}v), \quad \forall u,v \in \mathbb{C}. \tag{1.13}$$

† Strictly speaking, we only show that there is a unique LQG metric with parameter ξ for each $\xi \in (0,\infty)$. To deduce that the metric with central charge $\mathbf{c}_{\mathrm{M}}$ is unique we would need to know that $\xi \mapsto \mathbf{c}_{\mathrm{M}}(\xi)$ is injective. We expect that this injectivity is not hard to prove, but a proof of has so far only been written down for $\xi \in (0, 0.7)$. See the discussion just after (1.6).

Proof. Define $D_h^{(\omega)}(u,v) := D_{h(\omega\cdot)}(\omega^{-1}u, \omega^{-1}v)$. It is easily verified that $D^{(\omega)}$ satisfies the conditions of Definition 1.7, so Theorem 1.8 implies that there is a deterministic constant $C > 0$ such that almost surely $D_h^{(\omega)} = CD_h$ whenever h is a whole-plane GFF plus a continuous function. To check that $C = 1$, consider the case when h is a whole-plane GFF h normalized so that its average over the unit circle is 0. Then the law of h is rotationally invariant, so $\mathbb{P}[D_h(0,\partial\mathbb{D}) > R] = \mathbb{P}[D_h^{(\omega)}(0,\partial\mathbb{D}) > R]$ for every $R > 0$. Therefore, $C = 1$. □

Proposition 1.9 implies that D_h is covariant with respect to complex affine maps. It is natural to expect that D_h is also covariant with respect to general conformal maps, in the following sense. Let $U, \widetilde{U} \subset \mathbb{C}$ be open and let $\phi : U \to \widetilde{U}$ be a conformal map. Then it should be the case that almost surely

$$D_h(\phi(u), \phi(v); \widetilde{U}) = D_{h\circ\phi + Q\log|\phi'|}(u, v; U), \quad \forall u, v \in U. \tag{1.14}$$

In the subcritical case, the coordinate change relation (1.14) was proven in [26]. We expect that the proof there can be adapted to treat the critical and supercritical cases as well.

Various properties of the LQG metric D_h for $\mathbf{c}_{\mathrm{M}} \in [1, 25)$ have already been established in the literature. For example, for $\mathbf{c}_{\mathrm{M}} \in (1, 25)$ almost surely each D_h-metric ball $\mathcal{B}$ centered at a non-singular point is not D_h-compact [29, Proposition 1.14], but the boundaries of the connected components of $\mathbb{C} \setminus \mathcal{B}$ are D_h-compact and are Jordan curves [12, Theorem 1.4]. Furthermore, one has a confluence property for LQG geodesics [12, Theorem 1.6] and a version of the Knizhnik–Polyakov–Zamolodchikov (KPZ) formula, which relates Hausdorff dimensions with respect to D_h and the Euclidean metric [36, Theorem 1.15]. Simulations of supercritical LQG metric balls and geodesics can be found in [9, 11, 12].

There are many open problems related to the LQG metric for $\mathbf{c}_{\mathrm{M}} \in [1, 25)$. A list of open problems concerning LQG with $\mathbf{c}_{\mathrm{M}} \in (1, 25)$ can be found in [22, section 6]. Moreover, most of the open problems for the LQG metric with $\mathbf{c}_{\mathrm{M}} \in (-\infty, 1)$ from [27, section 7] are also interesting for $\mathbf{c}_{\mathrm{M}} \in [1, 25)$. Here, we mention one open problem which has not been discussed elsewhere.

Problem 1.10. Let $D_h^{(\xi)}$ denote the LQG metric with parameter ξ. Does $D_h^{(\xi)}$, appropriately re-scaled, converge in some topology as $\xi \to \infty$ (equivalently, $\mathbf{c}_{\mathrm{M}} \to 25$)? Even if one does not have convergence of the whole metric, can anything be said about the limits of $D_h^{(\xi)}$-metric balls, geodesics, and so on?

1.4 | Weak LQG metrics

In this subsection, we will introduce a notion of weak LQG metric for general $\xi > 0$ (Definition 1.12), which is similar to Definition 1.7 but with Axiom IV replaced by a weaker condition. Our notion of a weak LQG metric first appeared in [36]. We will then state a uniqueness theorem for weak LQG metrics (Theorem 1.13) and explain why our other main theorems (Theorems 1.3 and 1.8) follow from this theorem. A similar notion of weak LQG metrics was used in the proof of uniqueness of the subcritical LQG metric [17, 27].

To motivate the definition of weak LQG metrics, we first observe that every possible subsequential limit of the re-scaled LFPP metrics $\mathfrak{a}_\varepsilon^{-1} D_h^\varepsilon$ satisfies Axioms I, II, and III in Definition 1.7. This is intuitively clear from the definition, and not too hard to check rigorously (see [36, section 2]). It is also easy to see that every possible subsequential limit of LFPP satisfies Axiom V for $r = 1$ (that is, it satisfies the coordinate change formula for translations). However, it is far from obvious that the subsequential limits satisfy Axiom V when $r \neq 1$. The reason is that re-scaling space changes the value of ε in (1.3): for $\varepsilon, r > 0$, one has [17, Lemma 2.6]

$$D_h^\varepsilon(rz, rw) = r D_{h(r\cdot)}^{\varepsilon/r}(z, w), \quad \forall z, w \in \mathbb{C}.$$

So, since we only have subsequential limits of $\mathfrak{a}_\varepsilon^{-1} D_h^\varepsilon$, we cannot directly deduce that the subsequential limit satisfies an exact spatial scaling property.

Because of the above issue, we do not know how to check Axiom IV for subsequential limits of LFPP directly. Instead, we will prove a stronger uniqueness statement than the one in Theorem 1.8, under a weaker list of axioms which can be checked for subsequential limits of LFPP. We will then deduce from this stronger uniqueness statement that the weaker list of axioms implies the axioms in Definition 1.7 (Lemma 1.15).

An *annular region* is a bounded open set $A \subset \mathbb{C}$ such that A is homeomorphic to an open, closed, or half-open Euclidean annulus. If A is an annular region, then ∂A has two connected components, one of which disconnects the other from ∞. We call these components the outer and inner boundaries of A, respectively.

Definition 1.11 (Distance across and around annuli). Let d be a length metric on $\mathbb{C}$. For an annular region $A \subset \mathbb{C}$, we define $d(\text{across } A)$ to be the d-distance between the inner and outer boundaries of A. We define $d(\text{around } A)$ to be the infimum of the d-lengths of paths in A which disconnect the inner and outer boundaries of A.

Note that both $d(\text{across } A)$ and $d(\text{around } A)$ are determined by the internal metric of d on A. Distances around and across Euclidean annuli play a similar role to 'hard crossings' and 'easy crossings' of 2×1 rectangles in percolation theory. One can get a lower bound for the d-length of a path in terms of the d-distances across the annuli that it crosses. On the other hand, one can 'string together' paths around Euclidean annuli to get upper bounds for d-distances. The following is (almost) a re-statement of [36, Definition 1.6].

Definition 1.12 (Weak LQG metric). Let $\mathcal{D}'$ be as in Definition 1.12. For $\xi > 0$, a *weak LQG metric with parameter* ξ is a measurable function $h \mapsto D_h$ from $\mathcal{D}'$ to the space of lower semicontinuous metrics on $\mathbb{C}$ which satisfies properties I (length metric), II (locality), and III (Weyl scaling) from Definition 1.7 plus the following two additional properties.

IV′. **Translation invariance.** For each deterministic point $z \in \mathbb{C}$, almost surely $D_{h(\cdot+z)} = D_h(\cdot + z, \cdot + z)$.

V′. **Tightness across scales.** Suppose that h is a whole-plane GFF and let $\{h_r(z)\}_{r>0, z\in\mathbb{C}}$ be its circle average process. Let $A \subset \mathbb{C}$ be a deterministic Euclidean annulus. In the notation of Definition 1.11, the random variables

$$r^{-\xi Q} e^{-\xi h_r(0)} D_h(\text{across } rA) \quad \text{and} \quad r^{-\xi Q} e^{-\xi h_r(0)} D_h(\text{around } rA)$$

and the reciprocals of these random variables for $r > 0$ are tight.

We think of Axiom V′as a substitute for Axiom IV of Definition 1.7. Indeed, Axiom V′does not give an exact spatial scaling property, but it still allows us to get estimates for D_h which are uniform across different Euclidean scales.

It was shown in [36, Theorem 1.7] that every subsequential limit of the re-scaled LFPP metrics $\mathfrak{a}_\varepsilon^{-1} D_h^\varepsilon$ is a weak LQG metric in the sense of Definition 1.12. Actually, [36] allows for a general family of scaling constants $\{\mathfrak{c}_r\}_{r>0}$ in Axiom V′in place of $r^{\xi Q}$, but it was shown in [14, Theorem 1.9] that one can always take $\mathfrak{c}_r = r^{\xi Q}$. So, our definition is equivalent to the one in [36].

From the preceding paragraph and the tightness of $\mathfrak{a}_\varepsilon^{-1} D_h^\varepsilon$ [11], we know that there exists a weak LQG metric for each $\xi > 0$. Most of this paper is devoted to the proof of the uniqueness of the weak LQG metric.

Theorem 1.13. *For each $\xi > 0$, the weak LQG metric is unique in the following sense. If D and $\widetilde{D}$ are two weak LQG metrics with parameter ξ, then there is a deterministic constant $C > 0$ such that almost surely $D_h = C\widetilde{D}_h$ whenever h is a whole-plane GFF plus a continuous function.*

Let us now explain why Theorem 1.13 is sufficient to establish our main results, Theorems 1.3 and 1.8. We first observe that every strong LQG metric is a weak LQG metric.

Lemma 1.14. *For each $\xi > 0$, each strong LQG metric (Definition 1.7) is a weak LQG metric (Definition 1.12).*

Proof. Let D be a strong LQG metric. It is immediate from Axiom V of Definition 1.7 with $r = 1$ that D satisfies translation invariance (Axiom IV′). We need to check Axiom V′. To this end, let h be a whole-plane GFF normalized so that $h_1(0) = 0$. Weyl scaling (Axiom III) together with conformal covariance (Axiom IV) gives

$$r^{-\xi Q} e^{-\xi h_r(0)} D_h(r\cdot, r\cdot) = D_{h(r\cdot)-h_r(0)}(\cdot,\cdot) \stackrel{d}{=} D_h(\cdot,\cdot), \tag{1.15}$$

where the equality in law is due to the scale invariance of the law of h, modulo additive constant.

To get tightness across scales, it therefore suffices to show that for each fixed Euclidean annulus A, almost surely $D_h(\text{across } A)$ and $D_h(\text{around } A)$ are finite and positive. Our finiteness condition Axiom V easily implies that these two quantities are almost surely finite. To see that they are almost surely positive, it suffices to show that for any two deterministic, disjoint, Euclidean-compact sets $K_1, K_2 \subset \mathbb{C}$, almost surely $D_h(K_1, K_2) > 0$. Indeed, on the event $\{D_h(K_1, K_2) = 0\}$ we can find sequences of points $z_n \in K_1$ and $w_n \in K_2$ such that $D_h(z_n, w_n) \to 0$. After possibly passing to a subsequence, we can arrange that $z_n \to z \in K_1$ and $w_n \to w \in K_2$. By the lower semicontinuity of D_h, we get $D_h(z, w) = 0$. Since z and w are distinct and D_h is a metric (not a pseudometric) this implies that $\mathbb{P}[D_h(K_1, K_2) = 0] = 0$. □

Theorem 1.13 implies that one also has the converse to Lemma 1.14.

Lemma 1.15. *For each $\xi > 0$, every weak LQG metric is a strong LQG metric in the sense of Definition 1.7.*

Proof of Lemma 1.15 *assuming Theorem* 1.13. Let D be a weak LQG metric. It is clear that z satisfies Axioms I, II, III, and V of Definition 1.7. To show that D is a strong LQG metric, we need to check

Axiom IV of Definition 1.7 in the case when $z = 0$ (note that we already have translation invariance from Definition 1.12). To this end, for $b > 0$ let

$$D_h^{(b)}(\cdot,\cdot) := D_{h(b\cdot)+Q\log b}(\cdot/b, \cdot/b). \tag{1.16}$$

If h is a whole-plane GFF with $h_1(0) = 0$ then by the scale invariance of the law of h, modulo additive constant, we have $h(b\cdot) - h_b(0) \stackrel{d}{=} h$. Consequently, if h is a whole-plane GFF plus a continuous function, then $h(b\cdot) + Q\log b$ is also a whole-plane GFF plus a continuous function. Hence, $D_h^{(b)}$ is well-defined.

We need to show that almost surely $D_h^{(b)} = D_h$. We will prove this using Theorem 1.13. We first claim that $D_h^{(b)}$ is a weak LQG metric. It is easy to check that $D^{(b)}$ satisfies Axioms I, II, III, and IV′in Definition 1.12. To check Axiom V′, we use Weyl scaling (Axiom III) to get that

$$r^{-\xi Q} e^{-\xi h_r(0)} D_h^{(b)}(r\cdot, r\cdot) = e^{-\xi(h_r(0)-h_{r/b}(0))} e^{\xi h_b(0)} \times (r/b)^{-\xi Q} e^{-\xi h_{r/b}(0)} D_{h(b\cdot)-h_b(0)}((r/b)\cdot, (r/b)\cdot).$$

In the case when h is a whole-plane GFF, the random variables $h_r(0) - h_{r/b}(0)$ and $h_b(0)$ are each centered Gaussian with variance $\log\max\{b, 1/b\}$ [18, section 3.1]. Tightness across scales (Axiom V′) for D applied with $h(b\cdot) - h_b(0) \stackrel{d}{=} h$ in place of h and r/b in place of r therefore implies tightness across scales for $D^{(b)}$.

Hence, we can apply Theorem 1.13 with $\widetilde{D} = D^{(b)}$ to get that for each $b > 0$, there is a deterministic constant $\mathfrak{k}_b > 0$ such that whenever h is a whole-plane GFF plus a continuous function, almost surely

$$D_h^{(b)} = \mathfrak{k}_b D_h.$$

It remains to show that $\mathfrak{k}_b = 1$.

For $b_1, b_2 > 0$, we have $D^{(b_1 b_2)} = (D^{(b_1)})^{(b_2)}$, which implies that almost surely $D_h^{(b_1 b_2)} = \mathfrak{k}_{b_2} D_h^{(b_1)} = \mathfrak{k}_{b_1}\mathfrak{k}_{b_2} D_h$. Therefore,

$$\mathfrak{k}_{b_1 b_2} = \mathfrak{k}_{b_1}\mathfrak{k}_{b_2}. \tag{1.17}$$

It is also easy to see that $\mathfrak{k}_b$ is a Lebesgue measurable function of b. Indeed, by Weyl scaling (Axiom III) and since $h(b\cdot) - h_b(0) \stackrel{d}{=} h$,

$$\mathfrak{k}_b e^{-\xi h_b(0)} D_h(b\cdot, b\cdot) = e^{-\xi h_b(0)} D_h^{(b)}(b\cdot, b\cdot) = b^{\xi Q} D_{h(b\cdot)-h_b(0)}(\cdot,\cdot) \stackrel{d}{=} b^{\xi Q} D_h(\cdot,\cdot). \tag{1.18}$$

The function $b \mapsto b^{-\xi Q} e^{-\xi h_b(0)}$ is continuous and D_h is lower semicontinuous. Hence, the metrics $b^{-\xi Q} e^{-\xi h_b(0)} D_h(b\cdot, b\cdot)$ depend continuously on b with respect to the topology on lower semicontinuous functions. Therefore, the law of $\mathfrak{k}_b^{-1} D_h$ depends continuously on b with respect to the topology on lower semicontinuous functions. It follows that $\mathfrak{k}_b$ is continuous, hence Lebesgue measurable.

The relation (1.17) and the measurability of $b \mapsto \mathfrak{k}_b$ imply that $\mathfrak{k}_b = b^\alpha$ for some $\alpha \in \mathbb{R}$. By (1.18), we have $b^{\alpha-\xi Q} e^{-\xi h_b(0)} D_h(b\cdot, b\cdot) \stackrel{d}{=} D_h(\cdot,\cdot)$ for each $b > 0$. In particular, Axiom V′, holds for D with $\xi Q - \alpha$ in place of ξQ. Hence, $\alpha = 0$. ☐

Proof of Theorem 1.3, *assuming Theorem* 1.13. By [11, Theorem 1.2], if h is a whole-plane GFF plus a bounded continuous function, then for each $\xi > 0$, the re-scaled LFPP metrics $\mathfrak{a}_\varepsilon^{-1} D_h^\varepsilon$ are tight with respect to the topology of Definition 1.2. In fact, by [36, Theorem 1.7], for any sequence of positive ε values tending to zero there is a weak LQG metric D and a subsequence $\varepsilon_n \to 0$ such that whenever h is a whole-plane GFF plus a continuous functions, the metrics $\mathfrak{a}_{\varepsilon_n}^{-1} D_h^{\varepsilon_n}$ converge in probability to D_h with respect to this topology. By Theorem 1.13, if D and $\widetilde{D}$ are two weak LQG metrics arising as subsequential limits in this way, then there is a deterministic $C > 0$ such that almost surely $\widetilde{D}_h = C D_h$ whenever h is a whole-plane GFF plus a continuous function.

If h is a whole-plane GFF normalized so that $h_1(0) = 0$, then by the definition of $\mathfrak{a}_\varepsilon$ in (1.4), the median $\mathfrak{a}_\varepsilon^{-1} D_h^\varepsilon$-distance between the left and right sides of $[0,1]^2$ is 1. By passing this through to the limit, we get that the constant C above must be equal to 1. Therefore, almost surely $D_h = \widetilde{D}_h$ whenever h is a whole-plane GFF plus a continuous function, so the subsequential limit of $\mathfrak{a}_\varepsilon^{-1} D_h^\varepsilon$ is unique. □

Proof of Theorem 1.8, *assuming Theorem* 1.13. The uniqueness of the strong LQG metric follows from Theorem 1.13 and Lemma 1.15. The existence follows from the existence of the limit in Theorem 1.3, [36, Theorem 1.7] (which says that the limit is a weak LQG metric), and Lemma 1.15. □

1.5 | Outline

As explained in section 1.4, to establish our main results we only need to prove Theorem 1.13. To this end, let h be a whole-plane GFF and let D_h and $\widetilde{D}_h$ be two weak LQG metrics as in Definition 1.12. We need to show that there is a deterministic constant $C > 0$ such that almost surely $\widetilde{D}_h = C D_h$. In this subsection, we will give an outline of the proof of this statement. Throughout this outline and the rest of the paper, we will frequently use without comment the following fact, which is [36, Proposition 1.12].

Lemma 1.16 [36]. *Almost surely, the metric D_h is complete and finite-valued on $\mathbb{C} \setminus \{\text{singular points}\}$. Moreover, every pair of points in $\mathbb{C} \setminus \{\text{singular points}\}$ can be joined by a D_h-geodesic (Definition 1.6).*

1.5.1 | Optimal bi-Lipschitz constants

By [14, Theorem 1.10], the metrics D_h and $\widetilde{D}_h$ are almost surely bi-Lipschitz equivalent, so in particular almost surely they have the same set of singular points. We define the optimal upper and lower bi-Lipschitz constants

$$\mathfrak{c}_* := \inf\left\{ \frac{\widetilde{D}_h(u,v)}{D_h(u,v)} \,:\, u,v \in \mathbb{C} \setminus \{\text{singular points}\},\, u \neq v \right\} \quad \text{and}$$
$$\mathfrak{C}_* := \sup\left\{ \frac{\widetilde{D}_h(u,v)}{D_h(u,v)} \,:\, u,v \in \mathbb{C} \setminus \{\text{singular points}\},\, u \neq v \right\}. \tag{1.19}$$

Lemma 1.17. *Each of $\mathfrak{c}_*$ and $\mathfrak{C}_*$ is almost surely equal to a deterministic, positive, finite constant.*

Proof. By the bi-Lipschitz equivalence of D_h and $\widetilde{D}_h$, almost surely $\mathfrak{c}_*$ and $\mathfrak{C}_*$ are positive and finite. We know from [36, Lemma 3.12] that almost surely for each $z \in \mathbb{C}$, we have $\lim_{R\to\infty} D_h(z, \partial B_R(z)) = \infty$. With this fact in hand, the lemma follows from exactly the same elementary tail triviality argument as in the subcritical case [27, Lemma 3.1]. □

We henceforth replace $\mathfrak{c}_*$ and $\mathfrak{C}_*$ by their almost sure values in Lemma 1.17, so that each of $\mathfrak{c}_*$ and $\mathfrak{C}_*$ is a deterministic constant depending only on the laws of D_h and $\widetilde{D}_h$ and almost surely

$$\mathfrak{c}_* D_h(u,v) \leqslant \widetilde{D}_h(u,v) \leqslant \mathfrak{C}_* D_h(u,v), \quad \forall u,v \in \mathbb{C}. \tag{1.20}$$

1.5.2 | Main idea of the proof

To prove Theorem 1.13, it suffices to show that $\mathfrak{c}_* = \mathfrak{C}_*$. In the rest of this subsection, we will give an outline of the proof of this fact. There are many subtleties in our proof which we will gloss over in this outline in order to focus on the key ideas. So, the statements in the rest of this subsection should not be taken as mathematically precise.

At a very broad level, the basic strategy of our proof is similar to the proof of the uniqueness of the subcritical LQG metric in [27]. However, the details in Sections 3 and 5 are substantially different from the analogous parts of [27], and the argument in Section 4 is completely different from anything in [27].

We now give a very rough explanation of the main idea of our proof. Assume by way of contradiction that $\mathfrak{c}_* < \mathfrak{C}_*$. We will show that for any $\mathfrak{c}' \in (\mathfrak{c}_*, \mathfrak{C}_*)$, there are many 'good' pairs of distinct non-singular points $u, v \in \mathbb{C}$ such that $\widetilde{D}_h(u,v) \leqslant \mathfrak{c}' D_h(u,v)$ (Section 3). In fact, we will show that the set of such points is large enough that every D_h-geodesic P has to get $\widetilde{D}_h$-close to each of u and v for many 'good' pairs of points u, v (Sections 4 and 5). For each of these good pairs of points, we replace a segment of P by the concatenation of a $\widetilde{D}_h$-geodesic from a point of P to u, a $\widetilde{D}_h$-geodesic from u to v, and a $\widetilde{D}_h$-geodesic from v to a point of P. This gives a new path with the same endpoints as P.

By our choice of good pairs of points u, v, the $\widetilde{D}_h$-length of each of the replacement segments is at most a constant slightly larger than $\mathfrak{c}'$ times its D_h-length. Furthermore, by the definition of $\mathfrak{C}_*$ the $\widetilde{D}_h$-length of each segment of P which was not replaced is at most $\mathfrak{C}_*$ times its D_h-length. Morally, we would like to say that this implies that there exists $\mathfrak{c}'' \in (\mathfrak{c}', \mathfrak{C}_*)$ such that almost surely

$$\widetilde{D}_h(z,w) \leqslant \mathfrak{c}'' D_h(z,w), \quad \forall z,w \in \mathbb{C}. \tag{1.21}$$

The bound (1.21) contradicts the fact that $\mathfrak{C}_*$ is the optimal upper bi-Lipschitz constant (recall (1.19)). In actuality, what we will prove is a bit more subtle: assuming that $\mathfrak{c}_* < \mathfrak{C}_*$, we will establish for 'many' small values of $r > 0$ and each $\delta > 0$ an upper bound for

$$\mathbb{P}\left[\widetilde{D}_h(z,w) \leqslant (\mathfrak{C}_* - \delta) D_h(z,w),\ \forall z,w \in \overline{B}_r(0) \text{ satisfying certain conditions}\right]. \tag{1.22}$$

See Proposition 1.21 for a somewhat more precise statement. This upper bound will be incompatible with a lower bound for the same probability (Proposition 1.18), which will lead to our desired contradiction.

In the rest of this subsection, we give a more detailed, section-by-section outline of the proof.

1.5.3 | Section 2: Preliminary estimates

We will fix some notation, then record several basic estimates for the LQG metric which are straightforward consequences of results in the existing literature (mostly [36]).

1.5.4 | Section 3: Quantitative estimates for optimal bi-Lipschitz constants

Let $\mathfrak{C}' \in (\mathfrak{c}_*, \mathfrak{C}_*)$. By the definition (1.19) of $\mathfrak{c}_*$ and $\mathfrak{C}_*$, it holds with positive probability that there exists non-singular points $u, v \in \mathbb{C}$ such that $\widetilde{D}_h(u,v) \geqslant \mathfrak{C}' D_h(u,v)$. The purpose of Section 3 is to prove a quantitative version of this statement. The argument of Section 3 is similar to the argument of [27, section 3], but many of the details are different due to the fact that our metrics do not induce the Euclidean topology.

The following is a simplified version of the main result of Section 3 (see Proposition 3.5 for a precise statement).

Proposition 1.18. *There exists $p \in (0,1)$, depending only on the laws of D_h and $\widetilde{D}_h$, such that for each $\mathfrak{C}' \in (0, \mathfrak{C}_*)$ and each sufficiently small $\varepsilon > 0$ (depending on $\mathfrak{C}'$ and the laws of D_h and $\widetilde{D}_h$), there are at least $\frac{3}{4}\log_8 \varepsilon^{-1}$ values of $r \in [\varepsilon^2, \varepsilon] \cap \{8^{-k}\}_{k\in\mathbb{N}}$ such that*

$$\mathbb{P}\left[\exists \text{ a `regular' pair of points } u, v \in \overline{B}_r(0) \text{ such that } \widetilde{D}_h(u,v) \geqslant \mathfrak{C}' D_h(u,v)\right] \geqslant p. \tag{1.23}$$

The statement that u and v are 'regular' in (1.23) means that these points satisfy several regularity conditions which are stated precisely in Definition 3.2. These conditions include an upper bound on $D_h(u,v)$ (so in particular u and v are non-singular) and a lower bound on $|u-v|$ in terms of r. We emphasize that the parameter p in Proposition 1.18 does not depend on $\mathfrak{C}'$. This will be crucial for our purposes, see the discussion just after Proposition 1.21.

We will prove Proposition 1.18 by contradiction. In particular, we will assume that there are arbitrarily small values of $\varepsilon > 0$ for which there are at least $\frac{1}{4}\log_8 \varepsilon^{-1}$ values of $r \in [\varepsilon^2, \varepsilon] \cap \{8^{-k}\}_{k\in\mathbb{N}}$ such that

$$\mathbb{P}\left[\widetilde{D}_h(u,v) < \mathfrak{C}' D_h(u,v), \forall \text{ `regular' pairs of points } u, v \in \overline{B}_r(0)\right] \geqslant 1-p. \tag{1.24}$$

If p is small enough (depending only on the laws of D_h and $\widetilde{D}_h$), then we can use the assumption (1.24) together with the near-independence of the restrictions of the GFF to disjoint concentric annuli (Lemma 2.1) and a union bound to get the following. For any bounded open set $U \subset \mathbb{C}$, it holds with high probability that U can be covered by balls $B_r(z)$ for $z \in U$ and $r \in [\varepsilon^2, \varepsilon] \cap \{8^{-k}\}_{k\in\mathbb{N}}$ such that the event in (1.24) occurs.

We will then work on the high-probability event that we have such a covering of U. Consider points $\mathbbm{z}, \mathbbm{w} \in U$ such that there exists a D_h-geodesic P from $\mathbbm{z}$ to $\mathbbm{w}$ which is contained in U. We will replace several segments of P between pairs of 'regular' points u, v as in (1.24) by $\widetilde{D}_h$-geodesics from u to v. The $\widetilde{D}_h$-length of each of these geodesics is at most $\mathfrak{C}' D_h(u,v)$. Furthermore, by (1.19), the D_h-length of each segment of P which we did not replace is at most $\mathfrak{C}_*$ times its D_h-length. We thus obtain a path from $\mathbbm{z}$ to $\mathbbm{w}$ with $\widetilde{D}_h$-length at most $\mathfrak{C}'' D_h(u,v)$, where $\mathfrak{C}'' \in (\mathfrak{C}', \mathfrak{C}_*)$ is a constant depending only on $\mathfrak{C}'$ and the laws of D_h and $\widetilde{D}_h$. With high probability, this works for any D_h-geodesic contained in U. So, by taking U to be arbitrarily large, we contradict the definition of $\mathfrak{C}_*$. This yields Proposition 1.18.

By the symmetry in our hypotheses for D_h and $\widetilde{D}_h$, we also get the following analog of Proposition 1.18 with the roles of D_h and $\widetilde{D}_h$ interchanged.

Proposition 1.19. *There exists $p \in (0,1)$, depending only on the laws of D_h and $\widetilde{D}_h$, such that for each $\mathfrak{c}' > \mathfrak{c}_*$ and each sufficiently small $\varepsilon > 0$ (depending on $\mathfrak{c}'$ and the laws of D_h and $\widetilde{D}_h$), there are at least $\frac{3}{4}\log_8 \varepsilon^{-1}$ values of $r \in [\varepsilon^2, \varepsilon] \cap \{8^{-k}\}_{k\in\mathbb{N}}$ for which*

$$\mathbb{P}\left[\exists \text{ a 'regular' pair of points } u, v \in \overline{B}_r(0) \text{ such that } \widetilde{D}_h(u,v) \leqslant \mathfrak{c}' D_h(u,v)\right] \geqslant p. \tag{1.25}$$

1.5.5 | Section 4: The core argument

The idea of the rest of the proof of Theorem 1.13 is to show that if $\mathfrak{c}_* < \mathfrak{C}_*$, then Proposition 1.19 implies a contradiction to Proposition 1.18.

The core part of the proof is given in Section 4, where we will prove Theorem 1.13 conditional on the existence of events and bump functions satisfying certain specified properties. The needed events and bump functions will be constructed in Section 5. Section 4 plays a role analogous to [27, sections 4 and 6], but the proof is completely different.

We will consider a set of admissible radii $\mathcal{R} \subset (0,1)$, which will eventually be taken to be equal to $\rho^{-1}\mathcal{R}_0$, where ρ is a constant and $\mathcal{R}_0$ is the set of $r \in \{8^{-k}\}_{k\in\mathbb{N}}$ for which (1.25) holds. We also fix a constant $\mathbb{p} \in (0,1)$, which will eventually be chosen to be close to 1, in a manner depending only on the laws of D_h and $\widetilde{D}_h$, and we set

$$\mathfrak{c}' := \frac{\mathfrak{c}_* + \mathfrak{C}_*}{2}, \quad \text{so that} \quad \mathfrak{c}' \in (\mathfrak{c}_*, \mathfrak{C}_*) \quad \text{if} \quad \mathfrak{c}_* < \mathfrak{C}_*.$$

We will assume that for each $r \in \mathcal{R}$ and each $z \in \mathbb{C}$, we have defined an event $\mathsf{E}_{z,r}$ and a deterministic function $\mathsf{f}_{z,r}$ satisfying the following properties.

- $\mathsf{E}_{z,r}$ is determined by $h|_{B_{4r}(z)\setminus B_r(z)}$, viewed modulo additive constant, and $\mathbb{P}[\mathsf{E}_{z,r}] \geqslant \mathbb{p}$.
- $\mathsf{f}_{z,r}$ is smooth, non-negative, and supported on the annulus $B_{3r}(z) \setminus B_r(z)$.
- Assume that $\mathsf{E}_{z,r}$ occurs and P' is a $D_{h-\mathsf{f}_{z,r}}$-geodesic between two points of $\mathbb{C} \setminus B_{4r}(z)$ which spends 'enough' time in the support of $\mathsf{f}_{z,r}$. Then there are times $s < t$ such that $P'([s,t]) \subset B_{4r}(z)$ and

$$\widetilde{D}_{h-\mathsf{f}_{z,r}}(P'(s), P'(t)) \leqslant \mathfrak{c}'(t-s). \tag{1.26}$$

The precise list of properties that we need is stated in Subsection 4.1.

Roughly speaking, the support of $\mathsf{f}_{z,r}$ will be a long narrow tube contained in a small neighborhood of $\partial B_{2r}(0)$. On the event $\mathsf{E}_{z,r}$, there will be many 'good' pairs of non-singular points u, v in the support of $\mathsf{f}_{z,r}$ such that $\widetilde{D}_h(u,v) \leqslant \mathfrak{c}'_0 D_h(u,v)$ and the $\widetilde{D}_h$-geodesic from u to v is contained in the support of $\mathsf{f}_{z,r}$, where $\mathfrak{c}'_0 \in (\mathfrak{c}_*, \mathfrak{c}')$ is fixed. See Figure 2 for an illustration. We will show that $\mathsf{E}_{z,r}$ occurs with high probability for $r \in \mathcal{R}$ using Proposition 1.19 (with $\mathfrak{c}'_0$ instead of $\mathfrak{c}'$) and a long-range independence statement for the GFF (Lemma 2.3).

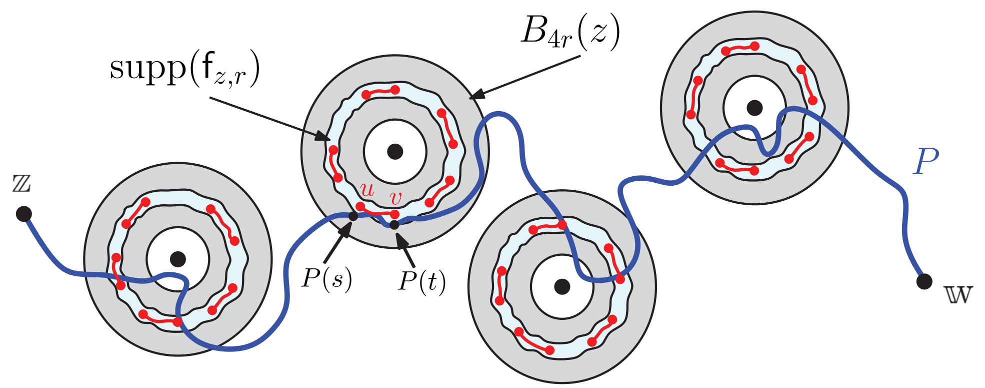

FIGURE 2 Illustration of three 'good' balls (that is, ones for which $\mathsf{E}_{z,r}$ occurs) and one 'very good' ball (that is, one for which $\mathsf{E}_{z,r}(h+\mathsf{f}_{z,r})$ occurs) which are hit by the D_h-geodesic P. Each of the 'good' balls contains several pairs of non-singular points u, v in the support of $\mathsf{f}_{z,r}$ (light blue) for which $\widetilde{D}_h(u,v) \leqslant \mathfrak{c}_0' D_h(u,v)$. These points and the $\widetilde{D}_h$-geodesics joining them are shown in red. For the 'very good' ball (the labeled ball in the figure), P gets $D_{h-\mathsf{f}_{z,r}}$-close to each of u and v for one of the aforementioned pairs of points u, v. To prove Proposition 1.20, we will show that there are lots of 'very good' balls for which P spends a lot of time in the support of $\mathsf{f}_{z,r}$.

The function $\mathsf{f}_{z,r}$ will be very large on most of its support. So, by Weyl scaling (Axiom III), a $D_{h-\mathsf{f}_{z,r}}$-geodesic which enters the support of $\mathsf{f}_{z,r}$ will tend to spend a long time in the support of $\mathsf{f}_{z,r}$. This will force the $D_{h-\mathsf{f}_{z,r}}$-geodesic to get $D_{h-\mathsf{f}_{z,r}}$-close to each of u and v for one of the aforementioned 'good' pairs of points u, v. The estimate (1.26) will follow from this and the triangle inequality. Most of Section 4 is devoted to proving an estimate (Proposition 4.3) which roughly speaking says the following.

Proposition 1.20. *Assume that $\mathfrak{c}_* < \mathfrak{C}_*$ and we have defined events $\mathsf{E}_{z,r}$ and functions $\mathsf{f}_{z,r}$ satisfying the above properties. As $\delta \to 0$, it holds uniformly over all $\mathbb{z}, \mathbb{w} \in \mathbb{C}$ that*

$$\mathbb{P}\left[\widetilde{D}_h(\mathbb{z},\mathbb{w}) > (\mathfrak{C}_* - \delta) D_h(\mathbb{z},\mathbb{w}),\ \textit{regularity conditions}\right] = O_\delta(\delta^\mu), \quad \forall \mu > 0. \tag{1.27}$$

We think of a ball $B_{4r}(z)$ as 'good' if the event $\mathsf{E}_{z,r}$ occurs and 'very good' if the event $\mathsf{E}_{z,r}(h+\mathsf{f}_{z,r})$, which is defined in the same manner as $\mathsf{E}_{z,r}$ but with $h+\mathsf{f}_{z,r}$ instead of h, occurs. By definition, if $B_{4r}(z)$ is'good' for h, then $B_{4r}(z)$ is 'very good' for $h-\mathsf{f}_{z,r}$.

Let P be the D_h-geodesic from $\mathbb{z}$ to $\mathbb{w}$ (which is almost surely unique, see Lemma 2.7). Recall that $\mathbb{P}[\mathsf{E}_{z,r}] \geqslant \mathbb{p}$, which is close to 1, and $\mathsf{E}_{z,r}$ is determined by $h|_{B_{4r}(z)\setminus B_r(z)}$, viewed modulo additive constant. From this, it is easy to show using the near-independence of the restrictions of h to disjoint concentric annuli (Lemma 2.1) that P has to hit $B_r(z)$ for lots of 'good' balls $B_{4r}(z)$.

To prove Proposition 1.20, it suffices to show that with high probability, there are many 'very good' balls $B_{4r}(z)$ such that the D_h-geodesic P from $\mathbb{z}$ to $\mathbb{w}$ spends 'enough' time in the support of the bump function $\mathsf{f}_{z,r}$. Indeed, the condition (1.26) (with $h+\mathsf{f}_{z,r}$ instead of h) will then give us lots of pairs of points s, t such that $\widetilde{D}_h(P(s),P(t)) \leqslant \mathfrak{c}'(t-s)$, which in turn will show that $\widetilde{D}_h(\mathbb{z},\mathbb{w})$ is bounded away from $\mathfrak{C}_* D_h(\mathbb{z},\mathbb{w})$ (see Proposition 4.6).

In [27], it was shown that P hits many 'very good' balls by using confluence of geodesics (which was proven in [24]) to get an approximate Markov property for P. In this paper, we will instead show this using a simpler argument based on counting the number of events of a certain type which occur. More precisely, for $r \in \mathcal{R}$ and a finite collection of points Z such that the balls $B_{4r}(z)$ for $z \in Z$ are disjoint, we will let $F_{Z,r}$ be (roughly speaking) the event that the following is true.

- Each ball $B_{4r}(z)$ for $z \in Z$ is 'good'.
- The D_h-geodesic P from $\mathbb{z}$ to $\mathbb{w}$ hits $B_r(z)$ for each $z \in Z$.
- With $\mathsf{f}_{Z,r} := \sum_{z \in Z} \mathsf{f}_{z,r}$, the $D_{h-\mathsf{f}_{Z,r}}$-geodesic from $\mathbb{z}$ to $\mathbb{w}$ spends 'enough' time in the support of $\mathsf{f}_{z,r}$ for each $z \in Z$.

We also let $F'_{Z,r}$ be defined in the same manner as $F_{Z,r}$ but with $h + \mathsf{f}_{Z,r}$ in place of h, that is, $F'_{Z,r}$ is the event that the following is true.

- Each $B_{4r}(z)$ for $z \in Z$ is 'very good'.
- The $D_{h+\mathsf{f}_{Z,r}}$-geodesic from $\mathbb{z}$ to $\mathbb{w}$ hits $B_r(z)$ for each $z \in Z$.
- The D_h-geodesic P from $\mathbb{z}$ to $\mathbb{w}$ spends 'enough' time in the support of $\mathsf{f}_{z,r}$ for each $z \in Z$.

Using a basic Radon–Nikodym derivative for the GFF, one can show that there is a constant $C > 0$ depending only on the laws of D_h and $\widetilde{D}_h$ such that

$$C^{-k}\mathbb{P}[F_{Z,r}] \leqslant \mathbb{P}[F'_{Z,r}] \leqslant C^k\mathbb{P}[F_{Z,r}], \quad \text{whenever } \#Z \leqslant k \tag{1.28}$$

(see Lemma 4.4). We will eventually take k to be a large constant, independent of $r, \mathbb{z}, \mathbb{w}$, depending on the number μ in (1.27). So, the relation (1.28) suggests that the number of sets Z such that $\#Z \leqslant k$ and $F_{Z,r}$ occurs should be comparable to the number of such sets for which $F'_{Z,r}$ occurs.

Furthermore, one can show that if ε is small enough, then for each $r \in [\varepsilon^2, \varepsilon]$, the number of sets Z with $\#Z \leqslant k$ such that $F_{Z,r}$ occurs grows like a positive power of ε^{-k} (Proposition 4.5). Indeed, as explained above, there are many sets Z_0 such that for each $z \in Z_0$, the ball $B_{4r}(z)$ is good and the ball $B_r(z)$ is hit by P. We need to produce many sets Z for which these properties hold and also that $D_{h-\mathsf{f}_{Z,r}}$-geodesic spends enough time in the support of $\mathsf{f}_{z,r}$ for each $z \in Z$. To do this, we start with a set Z_0 as above and iteratively remove the 'bad' points $z \in Z_0$ such that the $D_{h-\mathsf{f}_{Z_0,r}}$-geodesic from $\mathbb{z}$ to $\mathbb{w}$ does not spend very much time in the support of $\mathsf{f}_{z,r}$. By doing so, we obtain a set $Z \subset Z_0$ such that $F_{Z,r}$ occurs and $\#Z$ is not too much smaller than $\#Z_0$. See Subsection 4.3 for details.

By combining the preceding two paragraphs with an elementary calculation (see the end of Subsection 4.2), we infer that with high probability there are lots of sets Z with $\#Z \leqslant k$ such that $F'_{Z,r}$ occurs. In particular, there must be lots of 'very good' balls $B_{4r}(z)$ for which P spends a lot of time in the support of $\mathsf{f}_{z,r}$. As explained above, this gives Proposition 1.20.

Once Proposition 1.20 is established, one can take a union bound over many pairs of points $\mathbb{z}, \mathbb{w} \in B_r(0)$ to get, roughly speaking, the following (see Lemma 4.20 for a precise statement).

Proposition 1.21. *Assume that $\mathfrak{c}_* < \mathfrak{C}_*$. For each sufficiently small $\varepsilon > 0$ (depending only on the laws of D_h and $\widetilde{D}_h$), there are at least $\frac{3}{4}\log_8 \varepsilon^{-1}$ values of $r \in [\varepsilon^2, \varepsilon] \cap \{8^{-k}\}_{k\in\mathbb{N}}$ for which*

$$\lim_{\delta\to 0}\mathbb{P}\left[\exists \text{ a 'regular' pair } \mathbb{z}, \mathbb{w} \in \overline{B}_r(0) \text{ such that } \widetilde{D}_h(\mathbb{z},\mathbb{w}) \geqslant (\mathfrak{C}_* - \delta)D_h(\mathbb{z},\mathbb{w})\right] = 0, \tag{1.29}$$

uniformly over the choices of ε and r.

Proposition 1.21 is incompatible with Proposition 1.18 since the parameter p in Proposition 1.18 does not depend on $\mathfrak{C}'$. We thus obtain a contradiction to the assumption that $\mathfrak{c}_* < \mathfrak{C}_*$, so we conclude that $\mathfrak{c}_* = \mathfrak{C}_*$ and hence Theorem 1.13 holds.

1.5.6 | Section 5: Constructing events and bump functions

In Section 5, we will construct the events $\mathsf{E}_{z,r}$ and the bump functions $\mathsf{f}_{z,r}$ described just before Proposition 1.20. This part of the argument has some similarity to [27, section 5], which gives a roughly similar construction in the subcritical case. But, the details are very different. The main reason for this is as follows.

Recall that we want to force a $D_{h-\mathsf{f}_{z,r}}$-geodesic P' to get $D_{h-\mathsf{f}_{z,r}}$-close to each of u and v, where u, v are non-singular points in the support of $\mathsf{f}_{z,r}$ such that $\widetilde{D}_h(u,v) \leqslant \mathfrak{c}_0' D_h(u,v)$. We will do this in two steps: first we force P' to get Euclidean-close to each of u and v, then we force P' to get $D_{h-\mathsf{f}_{z,r}}$-close to each of u and v. In the subcritical phase, the metric D_h is Euclidean-continuous, so the second step is straightforward. However, this is not the case in the supercritical phase, so a substantial amount of work is needed to force P' to get $D_{h-\mathsf{f}_{z,r}}$-close to each of u and v. Because of this, we will define the events $\mathsf{E}_{z,r}$ in a significantly different way as compared to [27]. We refer to Subsection 5.1 for a more detailed outline.

2 | PRELIMINARIES

In this subsection, we first establish some standard notational conventions (Subsection 2.1). We then record several lemmas about a weak LQG metric D_h which are either proven elsewhere (that is, in [12, 36]) or are straightforward consequences of statements which are proven elsewhere. The reader may wish to skim this section on a first read and refer back to the various lemmas as needed.

2.1 | Notational conventions

We write $\mathbb{N} = \{1, 2, 3, \dots\}$ and $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$.

For $a < b$, we define the discrete interval $[a, b]_{\mathbb{Z}} := [a, b] \cap \mathbb{Z}$.

If $f : (0, \infty) \to \mathbb{R}$ and $g : (0, \infty) \to (0, \infty)$, we say that $f(\varepsilon) = O_\varepsilon(g(\varepsilon))$ (respectively, $f(\varepsilon) = o_\varepsilon(g(\varepsilon))$) as $\varepsilon \to 0$ if $f(\varepsilon)/g(\varepsilon)$ remains bounded (respectively, tends to zero) as $\varepsilon \to 0$. We similarly define $O(\cdot)$ and $o(\cdot)$ errors as a parameter goes to infinity.

Let $\{E^\varepsilon\}_{\varepsilon>0}$ be a one-parameter family of events. We say that E^ε occurs with

- *polynomially high probability* as $\varepsilon \to 0$ if there is a $\mu > 0$ (independent from ε and possibly from other parameters of interest) such that $\mathbb{P}[E^\varepsilon] \geqslant 1 - O_\varepsilon(\varepsilon^\mu)$;
- *superpolynomially high probability* as $\varepsilon \to 0$ if $\mathbb{P}[E^\varepsilon] \geqslant 1 - O_\varepsilon(\varepsilon^\mu)$ for every $\mu > 0$.

For $z \in \mathbb{C}$ and $r > 0$, we write $B_r(z)$ for the open Euclidean ball of radius r centered at z. More generally, for $X \subset \mathbb{C}$ we write $B_r(X) = \bigcup_{z \in X} B_r(z)$. We also define the open annulus

$$\mathbb{A}_{r_1,r_2}(z) := B_{r_2}(z) \setminus \overline{B_{r_1}(z)}, \quad \forall 0 < r_r < r_2 < \infty. \tag{2.1}$$

Topological concepts such as 'open', 'closed', 'boundary', and so on, are always defined with respect to the Euclidean topology unless otherwise stated. For $X \subset \mathbb{C}$, we write $\overline{X}$ for its Euclidean closure and ∂X for its Euclidean boundary.

We will typically use the symbols r and $\mathbbm{r}$ for Euclidean radii. Many of our estimates for weak LQG metrics are required to be uniform over different values of r (or $\mathbbm{r}$). The reason why we need to include this condition is that we only have tightness across scales (Axiom V′) instead of exact scale invariance (Axiom IV), so estimates are not automatically uniform across different Euclidean scales.

2.2 | Some remarks on internal metrics

Throughout the rest of this section, we let h be a whole-plane GFF and D_h be a weak LQG metric as in Definition 1.12.

Let $X \subset \mathbb{C}$ (not necessarily open or closed) and recall from Definition 1.6 that $D_h(\cdot,\cdot;X)$ is the D_h-internal metric on X, which is a metric on X except that it is allowed to take on infinite values. It is easy to check (see, for example, [6, Proposition 2.3.12]) that the $D_h(\cdot,\cdot;X)$-length of any D_h-rectifiable path contained in X (and hence also every $D_h(\cdot,\cdot;X)$-rectifiable path) is the same as its D_h-length.

The notion of a $D_h(\cdot,\cdot;X)$-geodesic between points of X is well-defined by Definition 1.6: it is simply a path in X whose D_h-length is the same as the $D_h(\cdot,\cdot;X)$-distance between its endpoints, provided this distance is finite. Such a geodesic may not exist for every pair of points in X. However, such geodesics exist for some pairs of points: for example, if $z, w \in X$ and there is a D_h-geodesic P from z to w which is contained in X, then P is a $D_h(\cdot,\cdot;X)$-geodesic.

We will most often consider internal metrics on open sets (which appear in the locality assumption Axiom II for D_h). But, we will sometimes also have occasion to consider internal metrics on the closures of open sets. Recall that for an open set $U \subset \mathbb{C}$, $h|_U$ is the random distribution on U obtained by restricting the distributional pairing $f \mapsto (h, f)$ to functions which are supported on U. Following, for example, [40, section 3.3], for a closed set $K \subset \mathbb{C}$, we define

$$\sigma(h|_K) := \bigcap_{\varepsilon>0} \sigma\left(h|_{B_\varepsilon(K)}\right), \tag{2.2}$$

where $B_\varepsilon(K)$ is the Euclidean ε-neighborhood of K.

We say that a random variable is almost surely determined by $h|_K$ if it is almost surely equal to a random variable which is measurable with respect to $\sigma(h|_K)$. Similarly, we say that a random variable is almost surely determined by $h|_K$, viewed modulo additive constant, if it is almost surely equal to a random variable which is measurable with respect to $\sigma((h+c)|_K)$ for any possibly random $c \in \mathbb{R}$.

The metric $D_h(\cdot,\cdot;K)$ is equal to the internal metric of $D_h(\cdot,\cdot;B_\varepsilon(K))$ on K for any $\varepsilon > 0$. So, by locality (Axiom II) and (2.2), the metric $D_h(\cdot,\cdot;K)$ is measurable with respect to $\sigma(h|_K)$.

2.3 | Independence for the GFF

The following lemma is a consequence of the fact that the restrictions of the GFF to disjoint concentric annuli, viewed modulo additive constant, are nearly independent. See [25, Lemma 3.1] for a slightly more general statement.

Lemma 2.1 [25]. *Fix $0 < s_1 < s_2 < 1$. Let $\{r_k\}_{k\in\mathbb{N}}$ be a decreasing sequence of positive numbers such that $r_{k+1}/r_k \leqslant s_1$ for each $k \in \mathbb{N}$ and let $\{E_{r_k}\}_{k\in\mathbb{N}}$ be events such that $E_{r_k} \in$*

$\sigma((h - h_{r_k}(0))|_{\mathbb{A}_{s_1 r_k, s_2 r_k}(0)})$ for each $k \in \mathbb{N}$. For $K \in \mathbb{N}$, let $N(K)$ be the number of $k \in [1, K]_{\mathbb{Z}}$ for which E_{r_k} occurs.

(1) *For each $a > 0$ and each $b \in (0,1)$, there exists $p = p(a, b, s_1, s_2) \in (0,1)$ and $c = c(a, b, s_1, s_2) > 0$ (independent of the particular choice of $\{r_k\}$ and $\{E_{r_k}\}$) such that if*

$$\mathbb{P}\left[E_{r_k}\right] \geqslant p, \quad \forall k \in \mathbb{N}, \tag{2.3}$$

then

$$\mathbb{P}[N(K) < bK] \leqslant ce^{-aK}, \quad \forall K \in \mathbb{N}. \tag{2.4}$$

(2) *For each $p \in (0,1)$, there exists $a = a(p, s_1, s_2) > 0$, $b = b(p, s_1, s_2) \in (0,1)$, and $c = c(p, s_1, s_2) > 0$ (independent of the particular choice of $\{r_k\}$ and $\{E_{r_k}\}$) such that if (2.3) holds, then (2.4) holds.*

Lemma 2.1 still applies if we require that $E_{r_k} \in \sigma((h - h_{r_k}(0))|_{\overline{\mathbb{A}}_{s_1 r_k, s_2 r_k}(0)})$ (that is, we consider a closed annulus rather than an open annulus). This is an immediate consequence of the definition of the σ-algebra generated by the restriction of h to a closed set (2.2). We will use this fact without comment several times in what follows.

For the proof of Lemma 4.18, we will need a minor variant of Lemma 2.1 where we do not require that the annuli are concentric.

Lemma 2.2. *Fix $0 < s_1 < s_2 < 1$ and $s_0 \in (0, \min\{s_1, 1 - s_2\})$. Let $\{r_k\}_{k \in \mathbb{N}}$ be a decreasing sequence of positive real numbers and let $\{z_k\}_{k \in \mathbb{N}}$ be a sequence of points in $\mathbb{C}$ such that*

$$r_{k+1}/r_k \leqslant s_1 - s_0 \quad \text{and} \quad |z_k| \leqslant s_0 r_k, \quad \forall k \in \mathbb{N}. \tag{2.5}$$

Let $\{E_{r_k}(z_k)\}_{k \in \mathbb{N}}$ be events such that for each $k \in \mathbb{N}$, the event $E_{r_k}(z_k)$ is almost surely determined by $h|_{\overline{\mathbb{A}}_{s_1 r_k, s_2 r_k}(z_k)}$, viewed modulo additive constant. For $K \in \mathbb{N}$, let $N(K)$ be the number of $k \in [1, K]_{\mathbb{Z}}$ for which $E_{r_k}(z_k)$ occurs.

(1) *For each $a > 0$ and each $b \in (0,1)$, there exists $p = p(a, b, s_0, s_1, s_2) \in (0,1)$ and $c = c(a, b, s_0, s_1, s_2) > 0$ (independent of the particular choice of $\{r_k\}$, $\{z_k\}$, and $\{E_{r_k}(z_k)\}$) such that if*

$$\mathbb{P}\left[E_{r_k}(z_k)\right] \geqslant p, \quad \forall k \in \mathbb{N}, \tag{2.6}$$

then

$$\mathbb{P}[N(K) < bK] \leqslant ce^{-aK}, \quad \forall K \in \mathbb{N}. \tag{2.7}$$

(2) *For each $p \in (0,1)$, there exists $a = a(p, s_0, s_1, s_2) > 0$, $b = b(p, s_0, s_1, s_2) \in (0,1)$, and $c = c(p, s_0, s_1, s_2) > 0$ (independent of the particular choice of $\{r_k\}$, $\{z_k\}$, and $\{E_{r_k}(z_k)\}$) such that if (2.6) holds, then (2.7) holds.*

Proof. Since $|z_k| \leqslant s_0 r_k$,

$$\mathbb{A}_{s_1 r_k, s_2 r_k}(z_k) \subset \mathbb{A}_{(s_1-s_0)r_k,(s_2+s_0)r_k}(0).$$

Hence, $E_{r_k}(z_k)$ is almost surely determined by $h|_{\overline{\mathbb{A}}_{(s_1-s_0)r_k,(s_2+s_0)r_k}(0)}$, viewed modulo additive constant. Since $0 < s_1 - s_0 < s_2 + s_0 < 1$ and by (2.5), we can apply Lemma 2.1 with $s_1 - s_0$ in place of s_1 and $s_2 + s_0$ in place of s_2 to obtain the lemma statement. □

We will also need an estimate which comes from the fact that the restrictions of the GFF to small disjoint Euclidean balls are nearly independent. See [27, Lemma 2.7] for a proof.

Lemma 2.3 [27]. *Let h be a whole-plane GFF and fix $s > 0$. Let $n \in \mathbb{N}$ and let $\mathcal{Z}$ be a collection of $\#\mathcal{Z} = n$ points in $\mathbb{C}$ such that $|z - w| \geqslant 2(1+s)$ for each distinct $z, w \in \mathcal{Z}$. For $z \in \mathcal{Z}$, let E_z be an event which is determined by $(h - h_{1+s}(z))|_{B_1(z)}$. For each $p, q \in (0,1)$, there exists $n_* = n_*(s,p,q) \in \mathbb{N}$ such that if $\mathbb{P}[E_z] \geqslant p$ for each $z \in \mathcal{Z}$, then*

$$\mathbb{P}\left[\bigcup_{z\in\mathcal{Z}} E_z\right] \geqslant q, \quad \forall n \geqslant n_*.$$

2.4 | Basic facts about weak LQG metrics

In this subsection, we will record some facts about our weak LQG metric D_h which are mostly proven elsewhere and which will be used frequently in what follows. Similar results are proven in the subcritical case in [17, 33].

Remark 2.4. Many of the estimates in [12, 36] involve 'scaling constants' $\mathfrak{c}_r$ for $r > 0$. It was shown in [14, Theorem 1.9] that one can take $\mathfrak{c}_r = r^{\xi Q}$. We will use this fact without comment whenever we cite results from [12, 36].

It was shown in [36, Lemma 3.1] that one has the following stronger version of Axiom V′.

Lemma 2.5 [36]. *Let $U \subset \mathbb{C}$ be open and let $K_1, K_2 \subset U$ be two disjoint, deterministic compact sets (allowed to be singletons). The re-scaled internal distances $r^{-1} e^{-\xi h_r(0)} D_h(rK_1, rK_2; rU)$ and their reciprocals as r varies are tight (recall the notation from Definition 1.6).*

The following proposition, which is [36, Proposition 1.8], is a more quantitative version of Lemma 2.5 in the case when K_1, K_2 are connected and are not singletons.

Lemma 2.6 [36]. *Let $U \subset \mathbb{C}$ be an open set (possibly all of $\mathbb{C}$) and let $K_1, K_2 \subset U$ be two disjoint, deterministic, connected, compact sets which are not singletons. For each $r > 0$, it holds with superpolynomially high probability as $R \to \infty$, at a rate which is uniform in the choice of r, that*

$$R^{-1} r^{\xi Q} e^{\xi h_r(0)} \leqslant D_h(rK_1, rK_2; rU) \leqslant R r^{\xi Q} e^{\xi h_r(0)}.$$

Suppose that $A \subset \mathbb{C}$ is a deterministic bounded open set which has the topology of a Euclidean annulus and whose inner and outer boundaries are not singletons. Recall the notation for

D_h-distance across and around Euclidean annuli from Definition 1.11. It is easy to see from Lemma 2.6 that with superpolynomially high probability as $R \to \infty$, uniformly in the choice of r,

$$R^{-1} r^{\xi Q} e^{\xi h_r(0)} \leqslant D_h(\text{around } A) \leqslant R r^{\xi Q} e^{\xi h_r(0)},$$

and the same is true for $D_h(\text{across } A)$.

Recall from Lemma 1.16 that almost surely any two non-singular points z, w for D_h can be joined by a D_h-geodesic, that is, a path of D_h-length $D_h(z, w)$. In the subcritical case, it was shown in [33, Theorem 1.2] that for a *fixed* choice of z and w, almost surely this geodesic is unique (see also [9, Lemma 4.2] for a simplified proof). The same proof also works in the critical and supercritical cases. We will need a slightly more general statement than the uniqueness of geodesics between fixed points. For two sets $K_1, K_2 \subset \mathbb{C}$, a D_h-*geodesic* from K_1 to K_2 is a path from a point of K_1 to a point of K_2 such that

$$\text{len}(P; D_h) = D_h(K_1, K_2) := \inf_{z \in K_1, w \in K_2} D_h(z, w). \tag{2.8}$$

Lemma 2.7. *Let $K_1, K_2 \subset \mathbb{C}$ be deterministic disjoint Euclidean-compact sets. Almost surely, there is a unique D_h-geodesic from K_1 to K_2.*

Proof. For existence, choose sequences of points $u_n \in K_1$ and $v_n \in K_2$ such that $\lim_{n\to\infty} D_h(u_n, v_n) = D_h(K_1, K_2)$. Since K_1 and K_2 are Euclidean-compact, after possibly passing to a subsequence we can find $u \in K_1$ and $v \in K_2$ such that $|u_n - u| \to 0$ and $|v_n - v| \to 0$. By the lower semicontinuity of D_h,

$$D_h(u, v) \leqslant \liminf_{n\to\infty} D_h(u_n, v_n) = D_h(K_1, K_2).$$

Hence, $D_h(u, v) = D_h(K_1, K_2)$ and a D_h-geodesic from u to v (which exists by Lemma 1.16) is also a D_h-geodesic from K_1 to K_2.

The uniqueness of the D_h-geodesic from K_1 to K_2 follows from the same argument as in the case when K_1 and K_2 are singletons, see [33, section 3] or [9, Lemma 4.2]. □

2.5 | Estimates for distances in disks and annuli

In this subsection, we will prove some basic estimates for D_h which are straightforward consequences of the concentration bounds for LQG distances established in [36]. We begin with a uniform comparison of distances around and across Euclidean annuli with different center points and radii.

Lemma 2.8. *Fix $\zeta > 0$. Let $U \subset \mathbb{C}$ be a bounded open set and let $b > a > 0$ and $d > c > 0$. For each $\mathbb{r} > 0$, it holds with superpolynomially high probability as $\delta_0 \to 0$ (at a rate which depends on ζ, U, a, b, c, d and the law of D_h, but is uniform in $\mathbb{r}$) that*

$$D_h\big(\text{around } \mathbb{A}_{a\delta\mathbb{r}, b\delta\mathbb{r}}(z)\big) \leqslant \delta^{-\zeta} D_h\big(\text{across } \mathbb{A}_{c\delta\mathbb{r}, d\delta\mathbb{r}}(z)\big), \quad \forall z \in \mathbb{r}U, \quad \forall \delta \in (0, \delta_0]. \tag{2.9}$$

Proof. Basically, this follows from Lemma 2.6 and a union bound. A little care is needed to discretize things so that we only have to take a union bound over polynomially many events.

Fix $a_1, a_2, b_1, b_2 > 0$ and $c_1, c_2, d_1, d_2 > 0$ such that

$$a < a_2 < a_1 < b_1 < b_2 < b \quad \text{and} \quad c < c_2 < c_1 < d_1 < d_2 < d.$$

By Lemma 2.6, for each $z \in \mathbb{C}$ it holds with superpolynomially high probability as $\delta \to 0$ (at a rate depending only on $\zeta, a_1, b_1, c_1, d_1$, and the law of D_h) that

$$D_h\Big(\text{around } \mathbb{A}_{a_1\delta\mathbb{r}, b_1\delta\mathbb{r}}(z)\Big) \leqslant \delta^{\xi Q - \zeta/2}\mathbb{r}^{\xi Q} e^{\xi h_{\delta\mathbb{r}}(z)} \quad \text{and}$$
$$D_h\Big(\text{across } \mathbb{A}_{c_1\delta\mathbb{r}, d_1\delta\mathbb{r}}(z)\Big) \geqslant \delta^{\xi Q + \zeta/2}\mathbb{r}^{\xi Q} e^{\xi h_{\delta\mathbb{r}}(z)}. \tag{2.10}$$

Let $s > 0$ be much smaller than $\min\{a_1 - a_2, b_2 - b_1, c_1 - c_d, d_2 - d_1\}$. By a union bound, it holds with superpolynomially high probability as $\delta \to 0$ that the bound (2.10) holds for all $z \in (s\delta\mathbb{r}\mathbb{Z}^2) \cap B_{\mathbb{r}}(\mathbb{r}U)$.

For each $z \in \mathbb{r}U$, there exists $z' \in (s\delta\mathbb{r}\mathbb{Z}^2) \cap B_{\mathbb{r}}(\mathbb{r}U)$ such that

$$\mathbb{A}_{a_1\delta\mathbb{r}, b_1\delta\mathbb{r}}(z') \subset \mathbb{A}_{a_2\delta\mathbb{r}, b_2\delta\mathbb{r}}(z) \quad \text{and} \quad \mathbb{A}_{c_1\delta\mathbb{r}, d_1\delta\mathbb{r}}(z') \subset \mathbb{A}_{c_2\delta\mathbb{r}, d_2\delta\mathbb{r}}(z).$$

For this choice of z',

$$D_h\Big(\text{around } \mathbb{A}_{a_2\delta\mathbb{r}, b_2\delta\mathbb{r}}(z)\Big) \leqslant D_h\Big(\text{around } \mathbb{A}_{a_1\delta\mathbb{r}, b_1\delta\mathbb{r}}(z')\Big) \quad \text{and}$$
$$D_h\Big(\text{across } \mathbb{A}_{c_2\delta\mathbb{r}, d_2\delta\mathbb{r}}(z)\Big) \geqslant D_h\Big(\text{across } \mathbb{A}_{c_1\delta\mathbb{r}, d_1\delta\mathbb{r}}(z')\Big).$$

By (2.10) with z' in place of z, we infer that with superpolynomially high probability as $\delta \to 0$,

$$D_h\Big(\text{around } \mathbb{A}_{a_2\delta\mathbb{r}, b_2\delta\mathbb{r}}(z)\Big) \leqslant \delta^{-\zeta} D_h\Big(\text{across } \mathbb{A}_{c_2\delta\mathbb{r}, c_2\delta\mathbb{r}}(z)\Big), \quad \forall z \in \mathbb{r}U. \tag{2.11}$$

To upgrade to an estimate which holds for all $\delta \in (0, \delta_0]$ simultaneously, let

$$q \in \Big(1, (\min\{a_2/a, b/b_2, c_2/c, d/d_2\})^{1/100}\Big).$$

By a union bound over integer powers of q, we infer that with superpolynomially high probability as $\delta_0 \to 0$, the estimate (2.11) holds for all $\delta \in (0, \delta_0] \cap \{q^{-k} : k \in \mathbb{N}\}$. By our choice of q, for each $\delta \in (0, \delta_0]$, there exists $k \in \mathbb{N}$ such that $q^{-k} \in (0, \delta_0]$ and for each $z \in \mathbb{C}$,

$$\mathbb{A}_{a_2 q^{-k}\mathbb{r}, b_2 q^{-k}\mathbb{r}}(z) \subset \mathbb{A}_{a\delta\mathbb{r}, b\delta\mathbb{r}}(z) \quad \text{and} \quad \mathbb{A}_{c_2 q^{-k}\mathbb{r}, d_2 q^{-k}\mathbb{r}}(z) \subset \mathbb{A}_{c\delta\mathbb{r}, d\delta\mathbb{r}}(z).$$

Hence (2.11) for δ follows from (2.11) with q^{-k} in place of δ. □

Our next estimate gives a moment bound for the LQG distance from the center point of a closed disk to a point on its boundary, along paths which are contained in the disk.

Lemma 2.9. *For each $p \in (0, 2Q/\xi)$, there exists $C_p > 0$, depending only on p and the law of D_h, such that*

$$\mathbb{E}\Big[\Big(r^{-\xi Q} e^{-\xi h_r(0)} D_h\Big(w, 0; \overline{B}_r(0)\Big)\Big)^p\Big] \leqslant C_p, \quad \forall w \in \partial B_r(0). \tag{2.12}$$

Proof. Fix $w \in \partial B_r(0)$. All of our estimates are required to be uniform in the choice of w. The idea of the proof is to string together countably many D_h-rectifiable loops centered at points on the segment $[0, w]$, with geometric Euclidean sizes.

For $\varepsilon \in (0, r)$, define

$$w_\varepsilon := \left(1 - \frac{\varepsilon}{r}\right) w \quad \text{and} \quad A_\varepsilon := \mathbb{A}_{\varepsilon/2,\varepsilon}(w_\varepsilon)$$

and note that $A_\varepsilon \subset B_r(0)$.

By Lemma 2.6, for each $q > 0$,

$$\mathbb{E}\left[\left(\varepsilon^{-\xi Q} e^{-\xi h_\varepsilon(w_\varepsilon)} D_h(\text{around } A_\varepsilon)\right)^q\right] \preceq 1, \quad \forall \varepsilon > 0, \tag{2.13}$$

with the implicit constant depending only on q and the law of D_h. By Hölder's inequality, for each $p > 0$ and each $q > 1$,

$$\begin{aligned}
&\mathbb{E}\left[\left(r^{-\xi Q} e^{-\xi h_r(0)} D_h(\text{around } A_\varepsilon)\right)^p\right] \\
&\quad \leqslant \left(\frac{\varepsilon}{r}\right)^{\xi Q p} \mathbb{E}\left[\left(\varepsilon^{-\xi Q} e^{-\xi h_\varepsilon(w_\varepsilon)} D_h(\text{around } A_\varepsilon)\right)^{\frac{qp}{1-q}}\right]^{1-1/q} \\
&\qquad \times \mathbb{E}\left[e^{qp\xi(h_\varepsilon(w_\varepsilon) - h_r(0))}\right]^{1/q} \\
&\quad \preceq \left(\frac{\varepsilon}{r}\right)^{\xi Q p} \mathbb{E}\left[e^{qp\xi(h_\varepsilon(w_\varepsilon) - h_r(0))}\right]^{1/q},
\end{aligned} \tag{2.14}$$

where in the last line we used (2.13). The random variable $h_\varepsilon(w_\varepsilon) - h_r(0)$ is centered Gaussian with variance at most $\log(r/\varepsilon)$ plus a universal constant. We therefore infer from (2.14) that for each $p > 0$ and each $q > 1$,

$$\mathbb{E}\left[\left(r^{-\xi Q} e^{-\xi h_r(0)} D_h(\text{around } A_\varepsilon)\right)^p\right] \preceq \left(\frac{\varepsilon}{r}\right)^{\xi Q p - q p^2 \xi^2/2} \tag{2.15}$$

with the implicit constant depending only on p, q.

Let

$$w'_\varepsilon := \frac{\varepsilon}{r} w \quad \text{and} \quad A'_\varepsilon := \mathbb{A}_{\varepsilon/2,\varepsilon}(w'_\varepsilon),$$

which is contained in $B_r(0)$ for $\varepsilon \in (0, r/2]$. Via a similar argument to the one leading to (2.15), we also have that for each $p > 0$ and each $q > 1$,

$$\mathbb{E}\left[\left(r^{-\xi Q} e^{-\xi h_r(0)} D_h\left(\text{around } A'_\varepsilon\right)\right)^p\right] \preceq \left(\frac{\varepsilon}{r}\right)^{\xi Q p - q p^2 \xi^2/2}. \tag{2.16}$$

For $k \in \mathbb{N}$, let $\varepsilon_k := 2^{-k} r$. Suppose that π_k is a path in A_{ε_k} which disconnects the inner and outer boundaries and π'_k is a path in A'_{ε_k} which disconnects the inner and outer boundaries of A'_{ε_k}. Then the union of the paths π_k and π'_k for $k \in \mathbb{N}$ is connected and contained in $B_r(0)$ and its closure contains both 0 and w. From this, we see that the union of these paths and $\{0, w\}$ contains a path from 0 to w which is contained in $\overline{B}_r(0)$. Hence,

$$D_h\Big(w, 0; \overline{B}_r(0)\Big) \leqslant \sum_{k=0}^{\infty} D_h\Big(\text{around } A_{\varepsilon_k}\Big) + \sum_{k=0}^{\infty} D_h\Big(\text{around } A'_{\varepsilon_k}\Big). \tag{2.17}$$

Assume now that $p \in (0, \min\{1, 2Q/\xi\})$. Since the function $x \mapsto x^p$ is concave, hence subadditive, we can take pth moments of both sides of (2.17), then apply (2.15) and (2.16), to get

$$\begin{aligned}
&\mathbb{E}\Big[\Big(r^{-\xi Q}e^{-\xi h_r(0)}D_h\Big(w, 0; \overline{B}_r(0)\Big)\Big)^p\Big] \\
&\leqslant \sum_{k=0}^{\infty} \mathbb{E}\Big[\Big(r^{-\xi Q}e^{-\xi h_r(0)}D_h\Big(\text{around } A_{\varepsilon_k}\Big)\Big)^p\Big] \\
&\quad + \sum_{k=0}^{\infty} \mathbb{E}\Big[\Big(r^{-\xi Q}e^{-\xi h_r(0)}D_h\Big(\text{around } A'_{\varepsilon_k}\Big)\Big)^p\Big] \\
&\preceq \sum_{k=0}^{\infty} \Big(\frac{\varepsilon_k}{r}\Big)^{\xi Q p - q p^2 \xi^2/2} \\
&\preceq \sum_{k=0}^{\infty} 2^{-k(\xi Q p - q p^2 \xi^2/2)}.
\end{aligned} \tag{2.18}$$

Since $p < 2Q/\xi$, if $q > 1$ is sufficiently close to 1, we have $\xi Q p - q p^2 \xi^2/2 > 0$. Hence, this last sum is finite. This gives (2.12) for $p < 1$. For $p \geqslant 1$, we obtain (2.12) via the same argument, but with the triangle inequality for the L^p norm used in place of the subadditivity of $p \mapsto x^p$. □

Using Lemma 2.9 and Markov's inequality, we obtain the following estimate, which says that with high probability 'most' points on a circle are not too LQG-far from the center point. Note that (unlike for subcritical LQG) we cannot say that this is the case for *all* points on the circle, for example, because there could be singular points on the circle.

Lemma 2.10. *For each $R > 1$,*

$$\mathbb{E}\Big[\Big|\Big\{w \in \partial B_r(0) : D_h\Big(w, 0; \overline{B}_r(0)\Big) > R r^{\xi Q} e^{\xi h_r(0)}\Big\}\Big|\Big] \leqslant R^{-2Q/\xi + o_R(1)} r, \tag{2.19}$$

where $|\cdot|$ denotes one-dimensional Lebesgue measure and the rate of convergence of the $o_R(1)$ depends only on the law of D_h.

Proof. This follows from Lemma 2.9 and Markov's inequality. □

We will also need a lemma to ensure that all of the D_h-geodesics between points in a specified Euclidean-compact set are contained in a larger compact set.

Lemma 2.11. *There exists $\mu > 0$, depending only on the law of D_h, such that the following is true. Let $K \subset \mathbb{C}$ be compact. For each $\mathfrak{r} > 0$, it holds with probability $1 - O_R(R^{-\mu})$ as $R \to \infty$ (at a rate depending only on K and the law of D_h) that each D_h-geodesic between two points of $\mathfrak{r}K$ is contained in $B_{R\mathfrak{r}}(0)$.*

Proof. Fix $\mathfrak{r} > 0$ and for $s > 0$, let

$$E_s := \left\{D_h\left(\text{around } \mathbb{A}_{s\mathfrak{r},2s\mathfrak{r}}(0)\right) < D_h\left(\text{across } \mathbb{A}_{2s\mathfrak{r},3s\mathfrak{r}}(0)\right)\right\}.$$

Using tightness across scales (Axiom V′) and a basic absolute continuity argument (see, for example, the proof of [21, Lemma 6.1]), we can find a $p \in (0,1)$, depending only on the law of D_h, such that $\mathbb{P}[E_s] \geqslant p$ for all $s, \mathfrak{r} > 0$.

Let $\rho > 0$ be chosen so that $K \subset B_\rho(0)$. By assertion 2 of Lemma 2.1 (applied to logarithmically many radii $r_k \in [\rho\mathfrak{r}, R\mathfrak{r}/3]$), we can find $\mu > 0$ as in the lemma statement such that for with probability $1 - O_R(R^{-\mu})$, there exists $s \in [\rho, R/3]$ such that E_s occurs.

On the other hand, it is easily seen that if E_s occurs, then no D_h-geodesic P between two points of $B_{s\mathfrak{r}}(0)$ can exit $B_{3s\mathfrak{r}}(0)$. Indeed, otherwise we could replace a segment of P by a segment of a path in $\mathbb{A}_{s\mathfrak{r},2s\mathfrak{r}}(0)$ which disconnects the inner and outer boundaries to get a path with the same endpoints as P but strictly shorter D_h-length than P. □

2.6 | Regularity of geodesics

The following lemma is (almost) a re-statement of [12, Corollary 3.7]. Roughly speaking, the lemma states that every point in an LQG geodesic is surrounded by a loop of small Euclidean diameter whose D_h-length is much shorter than the D_h-length of the geodesic. A similar lemma also appears in [36, section 2.4].

Lemma 2.12. *For each $\chi \in (0,1)$, there exists $\theta > 0$, depending only on χ and the law of D_h, such that for each Euclidean-bounded open set $U \subset \mathbb{C}$ and each $\mathfrak{r} > 0$, it holds with polynomially high probability as $\varepsilon_0 \to 0$, uniformly over the choice of $\mathfrak{r}$, that the following is true for each $\varepsilon \in (0, \varepsilon_0]$. Suppose $z \in \mathfrak{r}U$, $x, y \in \mathbb{C} \setminus B_{\varepsilon^\chi \mathfrak{r}}(z)$, and $s > 0$ such that there is a D_h-geodesic P from x to y with $P(s) \in B_{\varepsilon\mathfrak{r}}(z)$. Then*

$$D_h\left(\textit{around } \mathbb{A}_{\varepsilon\mathfrak{r},\varepsilon^\chi\mathfrak{r}}(z)\right) \leqslant \varepsilon^\theta s. \tag{2.20}$$

Proof. [12, Corollary 3.7] shows that with polynomially high probability as $\varepsilon_0 \to 0$, the condition in the lemma statement holds for $\varepsilon = \varepsilon_0$. The statement for all $\varepsilon \in (0, \varepsilon_0]$ follows from the statement for $\varepsilon = \varepsilon_0$ (applied with χ replaced by χ' slightly larger than χ) together with a union bound over dyadic values of ε. □

As explained in [12, 36], Lemma 2.12 functions as a substitute for the fact that in the supercritical case, D_h is not locally Hölder continuous with respect to the Euclidean metric. It says that the D_h-distance around a small Euclidean annulus centered at a point on a D_h-geodesic is small. A path of near-minimal length around this annulus can be linked up with various other paths to get upper bounds for D_h-distances in terms of Euclidean distances.

We will need the following generalization of Lemma 2.12, which follows from exactly the same proof. The lemma statement differs from Lemma 2.12 in that we consider a $D_{h-f}(\cdot,\cdot;\mathfrak{r}\overline{U})$-geodesic,

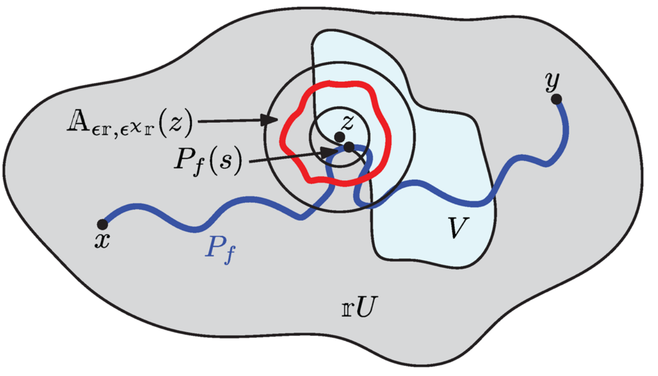

FIGURE 3 Illustration of the statement of Lemma 2.13 in the case where $s = \inf\{t > 0 : P_f(t) \in V\}$ (which is the main case that we will use). The path P_f is a $D_{h-f}(\cdot,\cdot;\mathbb{r}U)$-geodesic and the set V is the support of f. The lemma gives us an upper bound for $D_h(\text{around } \mathbb{A}_{\varepsilon\mathbb{r},\varepsilon^\chi\mathbb{r}}(z))$.

for a possibly random non-negative bump function f, instead of a D_h-geodesic (recall the discussion of geodesics for internal metrics from Subsection 2.2). See Figure 3 for an illustration of the lemma statement.

Lemma 2.13. *For each $\chi \in (0,1)$, there exists $\theta > 0$ depending only on χ and the law of D_h, such that for each Euclidean-bounded open set $U \subset \mathbb{C}$ and each $\mathbb{r} > 0$, it holds with polynomially high probability as $\varepsilon_0 \to 0$, uniformly over the choice of $\mathbb{r}$, that the following is true for each $\varepsilon \in (0,\varepsilon_0]$. Let $V \subset \mathbb{r}U$ and let $f : \mathbb{C} \to [0,\infty)$ be a non-negative continuous function which is identically zero outside of V. Let $z \in \mathbb{r}[U \setminus B_{\varepsilon^\chi}(\partial U)]$, $x, y \in (\mathbb{r}\overline{U}) \setminus (V \cup B_{\varepsilon^\chi\mathbb{r}}(z))$, and $s > 0$ such that there is a $D_{h-f}(\cdot,\cdot;\mathbb{r}\overline{U})$-geodesic P_f from x to y with $P_f(s) \in B_{\varepsilon\mathbb{r}}(z)$. Assume that*

$$s \leqslant \inf\{t > 0 : P_f(t) \in V\}. \tag{2.21}$$

Then

$$D_h\big(\text{around } \mathbb{A}_{\varepsilon\mathbb{r},\varepsilon^\chi\mathbb{r}}(z)\big) \leqslant \varepsilon^\theta s. \tag{2.22}$$

The statement of Lemma 2.13 holds with polynomially high probability for all possible choices of V, f, x, y, z, s, P_f. In particular, these objects are allowed to be random and/or ε-dependent. We also emphasize that the time s in (2.21) is allowed to be equal to $\inf\{t > 0 : P_f(t) \in V\}$, in which case $P_f(s) \in \partial V$. In fact, this is the main setting in which we will apply Lemma 2.13.

In the setting of Lemma 2.13, since f is non-negative, we have $D_{h-f}(u,v;\mathbb{r}\overline{U}) \leqslant D_h(u,v;\mathbb{r}\overline{U})$ for all $u, v \in \mathbb{r}U$. Furthermore, the condition (2.21) implies that the D_{h-f}-length of $P_f|_{[0,s]}$ is the same as its D_h-length. These two facts allow us to apply the proof of Lemma 2.12 (as given in [12, section 3.2]) essentially verbatim to obtain Lemma 2.13.

Out next lemma tells us that an LQG geodesic cannot trace a deterministic curve. Just like in Lemma 2.13, we will consider not just a D_h-geodesic but a $D_{h-f}(\cdot,\cdot;\mathbb{r}\overline{U})$-geodesic for a possible random continuous function f.

Lemma 2.14. *For each $M > 0$, there exists $\nu > 0$, depending only on M and the law of D_h, such that the following is true. Let $U \subset \mathbb{C}$ be a deterministic open set and let $\eta : [0,T] \to U \setminus B_{\varepsilon^{1/2}}(\partial U)$*

be a deterministic parameterized curve. For each $\mathbb{r} > 0$*, it holds with probability* $1 - O_\varepsilon(\varepsilon^\nu)$ *as* $\varepsilon \to 0$ *(the implicit constant depends only on* M *and the law of* D_h*) that the following is true. Let* $f : \mathbb{C} \to [-M, M]$ *be a continuous function and let* P_f *be a* $D_{h-f}(\cdot, \cdot; \mathbb{r}\overline{U})$*-geodesic between two points of* $\mathbb{r}[U \setminus B_{\varepsilon^{1/2}}(\eta)]$*. Then*

$$|\{t \in [0, T] : P_f \cap B_{\varepsilon\mathbb{r}}(\mathbb{r}\eta(t)) \neq \emptyset\}| \leqslant \varepsilon^\nu T, \tag{2.23}$$

where $|\cdot|$ *denotes one-dimensional Lebesgue measure.*

We emphasize that, as in Lemma 2.13, the function f and the geodesic P_f in Lemma 2.14 are allowed to be random and ε-dependent (but η is fixed).

Proof of Lemma 2.14. The idea of the proof is that (by Lemma 2.1) for a 'typical' time $t \in [0, T]$, there is a loop in $\mathbb{A}_{\varepsilon\mathbb{r}, \varepsilon^{1/2}\mathbb{r}}(\mathbb{r}\eta(t))$ which disconnects the inner and outer boundaries and whose D_h-length is much shorter than the D_h-distance from the loop to $B_{\varepsilon\mathbb{r}}(\mathbb{r}\eta(t))$. The existence of such a loop prevents a D_{h-f}-geodesic from hitting $B_{\varepsilon\mathbb{r}}(\mathbb{r}\eta(t))$.

For $k \in \mathbb{N}$, let

$$r_k := 4^k \varepsilon \mathbb{r}.$$

For $t \in [0, T]$, define the event

$$E_k(t) := \left\{ D_h\Big(\text{around } \mathbb{A}_{2r_k, 3r_k}(\mathbb{r}\eta(t))\Big) \leqslant \frac{1}{2} e^{-2\xi M} D_h\Big(\text{across } \mathbb{A}_{r_k, 2r_k}(\mathbb{r}\eta(t))\Big) \right\}. \tag{2.24}$$

By locality and Weyl scaling (Axioms II and V′), the event $E_k(t)$ is almost surely determined by $h|_{\mathbb{A}_{r_k, 3r_k}(\mathbb{r}\eta(t))}$, viewed modulo additive constant. By adding a bump function to h and using absolute continuity together with tightness across scales (see, for example, the proof of [21, Lemma 6.1]), we see that there exists $p > 0$ (depending only on M and the law of D_h) such that $\mathbb{P}[E_k(t)] \geqslant p$ for each $k \in \mathbb{N}$ and $t \in [0, T]$. Consequently, assertion 2 of Lemma 2.1 implies that there exists $\nu > 0$ depending only on M and the law of D_h such that

$$\mathbb{P}\left[\exists k \in [1, \log_4 \varepsilon^{-1/2} - 1]_{\mathbb{Z}} \text{ such that } E_k(t) \text{ occurs}\right] \geqslant 1 - O_\varepsilon(\varepsilon^{2\nu}), \tag{2.25}$$

with the implicit constant in the $O_\varepsilon(\cdot)$ depending only on M and the law of D_h.

Say that $t \in [0, T]$ is *good* if $E_k(t)$ occurs for some $k \in [1, \log_4 \varepsilon^{-1/2} - 1]_{\mathbb{Z}}$, and that t is *bad* otherwise. By (2.25),

$$\mathbb{E}[|\{t \in [0, T] : t \text{ is bad}\}|] \leqslant O_\varepsilon(\varepsilon^{2\nu}) T.$$

By Markov's inequality, it holds with probability $1 - O_\varepsilon(\varepsilon^\nu)$ that

$$|\{t \in [0, T] : t \text{ is bad}\}| \leqslant \varepsilon^\nu T. \tag{2.26}$$

To prove (2.23), it remains to show that if t is good and f is as in the lemma statement, then no $D_{h-f}(\cdot, \cdot; \mathbb{r}\overline{U})$-geodesic between two points of $\mathbb{r}[U \setminus B_{\varepsilon^{1/2}}(\eta)]$ can hit $B_{\varepsilon\mathbb{r}}(\mathbb{r}\eta(t))$. To see this, let P_f be such a geodesic and choose $k \in [1, \log_4 \varepsilon^{-1/2} - 1]_{\mathbb{Z}}$ such that $E_k(t)$ occurs. By (2.24), there is a path π in $\mathbb{A}_{2r_k,3r_k}(\mathbb{r}\eta(t))$ which disconnects the inner and outer boundaries of this annulus such that

$$\mathrm{len}(\pi; D_h) < e^{-2\xi M} D_h\Big(\text{across } \mathbb{A}_{r_k,2r_k}(\mathbb{r}\eta(t))\Big).$$

By Weyl scaling (Axiom III) and since f takes values in $[-M, M]$,

$$\mathrm{len}\big(\pi; D_{h-f}\big) < D_{h-f}\Big(\text{across } \mathbb{A}_{r_k,2r_k}(\mathbb{r}\eta(t))\Big). \tag{2.27}$$

Since $\varepsilon\mathbb{r} \leqslant r_k \leqslant \frac{1}{2}\varepsilon^{1/2}\mathbb{r}$ and the endpoints of P are at Euclidean distance at least $\varepsilon^{1/2}\mathbb{r}$ from $\mathbb{r}\eta$, we see that if P_f hits $B_{\varepsilon\mathbb{r}}(\mathbb{r}\eta(t))$ then the following is true. There are times $0 < \tau < \sigma < \mathrm{len}(P; D_{h-f})$ such that $P(\tau), P(\sigma) \in \pi$ and P crosses between the inner and outer boundaries of $\mathbb{A}_{r_k,2r_k}(\mathbb{r}\eta(t))$ between times τ and σ. Since $\eta \subset U \setminus B_{\varepsilon^{1/2}}(\partial U)$, we have $\pi \subset \mathbb{r}U$. By (2.27), we can obtain a path in $\mathbb{r}\overline{U}$ with the same endpoints as P_f which is D_{h-f}-shorter than P_f by replacing $P_f|_{[\tau,\sigma]}$ by a segment of the path π. This contradicts the fact that P_f is a $D_{h-f}(\cdot,\cdot;\mathbb{r}\overline{U})$-geodesic, so we conclude that P_f cannot hit $B_{\varepsilon\mathbb{r}}(\mathbb{r}\eta(t))$, as required. □

3 | QUANTIFYING THE OPTIMALITY OF THE OPTIMAL BI-LIPSCHITZ CONSTANTS

3.1 | Events for the optimal bi-Lipschitz constants

Let h be a whole-plane GFF and let D_h and $\widetilde{D}_h$ be two weak LQG metrics. We define the optimal upper and lower bi-Lipschitz constants $\mathfrak{c}_*$ and $\mathfrak{C}_*$ as in Subsection 1.5.1, so that $\mathfrak{c}_*$ and $\mathfrak{C}_*$ are deterministic and almost surely (1.20) holds. Recall from Subsection 1.5 that we aim to prove by contradiction that $\mathfrak{c}_* = \mathfrak{C}_*$. For this purpose, we will need several estimates which have non-trivial content only if $\mathfrak{c}_* < \mathfrak{C}_*$.

From the optimality of $\mathfrak{c}_*$ and $\mathfrak{C}_*$, we know that for every $\mathfrak{C}' < \mathfrak{C}_*$,

$$\mathbb{P}\big[\exists \text{ non-singular } u, v \in \mathbb{C} \text{ such that } \widetilde{D}_h(u, v) \geqslant \mathfrak{C}' D_h(u, v)\big] > 0. \tag{3.1}$$

A similar statement holds for every $\mathfrak{c}' > \mathfrak{c}_*$. The goal of this section is to prove various quantitative versions of (3.1), which include regularity conditions on u and v and which are required to hold uniformly over different Euclidean scales.

Our results will be stated in terms of two events, which are defined in Definitions 3.1 and 3.2. In this subsection, we will prove some basic facts about these events and state the main estimates we need for them (Propositions 3.3 and 3.10). Then, in Subsection 3.2, we will prove our main estimates.

Definition 3.1. For $r > 0$, $\beta > 0$, and $\mathfrak{C}' > 0$, we let $G_r(\beta, \mathfrak{C}')$ be the event that there exist $z, w \in \overline{B}_r(0)$ such that

$$\widetilde{D}_h\big(B_{\beta r}(z), B_{\beta r}(w)\big) \geqslant \mathfrak{C}' D_h(z, w).$$

The event $G_r(\beta, \mathfrak{C}')$ is a slightly stronger version of the event in (3.1). Our other event has a more complicated definition, and includes several regularity conditions on u and v. See Figure 4 for an illustration.

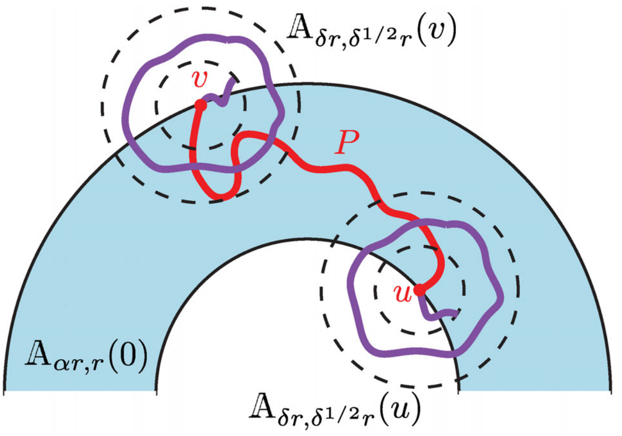

FIGURE 4 Illustration of the event $H_r(\alpha, \mathfrak{C}')$ of Definition 3.2. The last condition (iv) says that for each $\delta > 0$, there exist purple paths as in the figure whose D_h-lengths are at most $\delta^\theta D_h(u, v)$. The figure is not shown to scale — in actuality we will take α to be close to 1, so the light blue annulus will be quite narrow.

Definition 3.2. For $r > 0$, $\alpha \in (3/4, 1)$, and $\mathfrak{C}' > 0$, we let $H_r(\alpha, \mathfrak{C}')$ be the event that there exist non-singular points $u \in \partial B_{\alpha r}(0)$ and $v \in \partial B_r(0)$ such that

$$\widetilde{D}_h(u, v) \geqslant \mathfrak{C}' D_h(u, v) \tag{3.2}$$

and a D_h-geodesic P from u to v such that the following is true.

(i) $P \subset \overline{\mathbb{A}}_{\alpha r,r}(0)$.
(ii) The Euclidean diameter of P is at most $r/100$.
(iii) $D_h(u, v) \leqslant (1 - \alpha)^{-1} r^{\xi Q} e^{\xi h_r(0)}$.
(iv) Let $\theta > 0$ be as in Lemma 2.13 with $\chi = 1/2$. For each $\delta \in (0, (1 - \alpha)^2]$,

$$\max\left\{D_h(u, \partial B_{\delta r}(u)), D_h\left(\text{around } \mathbb{A}_{\delta r,\delta^{1/2}r}(u)\right)\right\} \leqslant \delta^\theta D_h(u, v) \tag{3.3}$$

and the same is true with the roles of u and v interchanged.

The main result of this section, which will be proven in Subsection 3.2, tells us that (for appropriate values of $\beta, \mathfrak{C}'', \alpha, \mathfrak{C}'$) if $\mathbb{P}[G_{\mathbbm{r}}(\beta, \mathfrak{C}'')] \geqslant \beta$, then there are lots of 'scales' $r < \mathbbm{r}$ for which $\mathbb{P}[H_r(\alpha, \mathfrak{C}')]$ is bounded below by a constant which does not depend on r or $\mathfrak{C}'$.

Proposition 3.3. *There exist $\alpha \in (3/4, 1)$ and $p \in (0, 1)$, depending only on the laws of D_h and $\widetilde{D}_h$, such that for each $\mathfrak{C}' \in (0, \mathfrak{C}_*)$, there exists $\mathfrak{C}'' = \mathfrak{C}''(\mathfrak{C}') \in (\mathfrak{C}', \mathfrak{C}_*)$ such that for each $\beta \in (0, 1)$, there exists $\varepsilon_0 = \varepsilon_0(\beta, \mathfrak{C}') > 0$ with the following property. If $\mathbbm{r} > 0$ and $\mathbb{P}[G_{\mathbbm{r}}(\beta, \mathfrak{C}'')] \geqslant \beta$, then the following is true for each $\varepsilon \in (0, \varepsilon_0]$.*

(A) There are at least $\frac{3}{4}\log_8 \varepsilon^{-1}$ values of $r \in [\varepsilon^2\mathbbm{r}, \varepsilon\mathbbm{r}] \cap \{8^{-k}\mathbbm{r} : k \in \mathbb{N}\}$ for which $\mathbb{P}[H_r(\alpha, \mathfrak{C}')] \geqslant p$.

We emphasize that in Proposition 3.3, the parameters α and p do *not* depend in $\mathfrak{C}'$. This will be crucial for our argument in Subsection 4.5.

In the remainder of this subsection, we will prove some basic lemmas about the events of Definitions 3.1 and 3.2, some of which are consequences of Proposition 3.3. In order for Proposition 3.3 to have non-trivial content, one needs a lower bound for $\mathbb{P}[G_{\mathbb{r}}(\beta, \mathfrak{C}')]$. It is straightforward to check that one has such a lower bound if $\mathbb{r} = 1$ and β is small enough.

Lemma 3.4. *For each $\mathfrak{C}' < \mathfrak{C}_*$, there exists $\beta > 0$, depending on $\mathfrak{C}'$ and the laws of D_h and $\widetilde{D}_h$, such that $\mathbb{P}[G_1(\beta, \mathfrak{C}')] > 0$.*

Proof. We will prove the contrapositive. Let $\mathfrak{C}' > 0$ and assume that

$$\mathbb{P}\big[G_1(\beta, \mathfrak{C}')\big] = 0, \quad \forall \beta > 0. \tag{3.4}$$

We will show that $\mathfrak{C}' \geqslant \mathfrak{C}_*$. The assumption (3.4) implies that almost surely

$$\widetilde{D}_h\big(B_\beta(z), B_\beta(w)\big) < \mathfrak{C}' D_h(z, w), \quad \forall z, w \in \overline{B}_1(0), \quad \forall \beta > 0. \tag{3.5}$$

By lower semicontinuity, for each $z, w \in B_1(0)$,

$$\widetilde{D}_h(z, w) \leqslant \liminf_{\beta \to 0} \widetilde{D}_h\big(B_\beta(z), B_\beta(w)\big),$$

so (3.5) implies that almost surely

$$\widetilde{D}_h(z, w) \leqslant \mathfrak{C}' D_h(z, w), \quad \forall z, w \in \overline{B}_1(0). \tag{3.6}$$

By the translation invariance property of D_h (Axiom IV′) and the translation invariance of the law of h, viewed modulo additive constant, (3.6) implies that almost surely

$$\widetilde{D}_h(z, w) \leqslant \mathfrak{C}' D_h(z, w), \quad \forall z, w \in \mathbb{C} \text{ such that } |z - w| \leqslant 1. \tag{3.7}$$

For a general pair of non-singular points $z, w \in \mathbb{C}$, we can apply (3.7) to finitely pairs of points along a D_h-geodesic from z to w to get that almost surely $\widetilde{D}_h(z, w) \leqslant \mathfrak{C}' D_h(z, w)$ for all $z, w \in \mathbb{C}$. By the minimality of $\mathfrak{C}_*$, this shows that $\mathfrak{C}' \geqslant \mathfrak{C}_*$, as required. □

By combining Proposition 3.3 and Lemma 3.4, we get the following.

Proposition 3.5. *There exist $\alpha \in (3/4, 1)$ and $p \in (0, 1)$, depending only on the laws of D_h and $\widetilde{D}_h$, such that for each $\mathfrak{C}' \in (0, \mathfrak{C}_*)$ and each sufficiently small $\varepsilon > 0$ (depending on $\mathfrak{C}'$ and the laws of D_h and $\widetilde{D}_h$), there are at least $\frac{3}{4} \log_8 \varepsilon^{-1}$ values of $r \in [\varepsilon^2, \varepsilon] \cap \{8^{-k}\}_{k \in \mathbb{N}}$ for which $\mathbb{P}[H_r(\alpha, \mathfrak{C}')] \geqslant p$.*

Proof. Let $\alpha \in (3/4, 1)$ and $p \in (0, 1)$ (depending only on the laws of D_h and $\widetilde{D}_h$) and $\mathfrak{C}'' \in (\mathfrak{C}', \mathfrak{C}_*)$ (depending only on $\mathfrak{C}'$ and the laws of D_h and $\widetilde{D}_h$) be as in Proposition 3.3. By Lemma 3.4 (applied with $\mathfrak{C}''$ instead of $\mathfrak{C}'$), there exists $\beta > 0$, depending only on $\mathfrak{C}'$ and the laws of D_h and $\widetilde{D}_h$, such that $\mathbb{P}[G_1(\beta, \mathfrak{C}'')] \geqslant \beta$. By Proposition 3.3 applied with $\mathbb{r} = 1$, we now obtain the proposition statement. □

We will also need an analog of Proposition 3.5 with the events $G_r(\beta, \mathfrak{C}')$ in place of the events $H_r(\alpha, \mathfrak{C}')$, which strengthens Lemma 3.4.

Proposition 3.6. *For each $\mathfrak{C}' \in (0, \mathfrak{C}_*)$, there exists $\beta > 0$, depending on $\mathfrak{C}'$ and the laws of D_h and $\widetilde{D}_h$, such that for each small enough $\varepsilon > 0$ (depending on $\mathfrak{C}'$ and the laws of D_h and $\widetilde{D}_h$), there are at least $\frac{3}{4}\log_8 \varepsilon^{-1}$ values of $r \in [\varepsilon^2, \varepsilon] \cap \{8^{-k}\}_{k\in\mathbb{N}}$ for which $\mathbb{P}[G_r(\beta, \mathfrak{C}')] \geqslant \beta$.*

We will deduce Proposition 3.6 from Proposition 3.5 and the following elementary relation between the events $H_r(\cdot,\cdot)$ and $G_r(\cdot,\cdot)$.

Lemma 3.7. *If $\alpha \in (3/4, 1)$ and $\zeta \in (0,1)$, there exists $\beta > 0$, depending only on α, ζ, and the laws of D_h and $\widetilde{D}_h$, such that the following is true. For each $r > 0$ and each $\mathfrak{C}' > 0$, if $H_r(\alpha, \mathfrak{C}')$ occurs, then $G_r(\beta, \mathfrak{C}' - \zeta)$ occurs.*

Proof. Assume that $H_r(\alpha, \mathfrak{C}')$ occurs and let u and v be as in Definition 3.2 of $H_r(\alpha, \mathfrak{C}')$. By Definition 3.1 of $G_r(\beta, \mathfrak{C}' - \zeta)$, it suffices to find $\beta > 0$ as in the lemma statement such that

$$\widetilde{D}_h\big(B_{\beta r}(u), B_{\beta r}(v)\big) \geqslant (\mathfrak{C}' - \zeta) D_h(u,v). \tag{3.8}$$

To this end, let $\delta > 0$ and suppose that P^δ is a path from $B_{\delta r}(u)$ to $B_{\delta r}(v)$; P_u^δ and P_v^δ are paths from u and v to $\partial B_{\delta^{1/2} r}(u)$ and $\partial B_{\delta^{1/2} r}(v)$, respectively; and π_u^δ and π_v^δ are paths in $\mathbb{A}_{\delta r, \delta^{1/2} r}(u)$ and $\mathbb{A}_{\delta r, \delta^{1/2} r}(u)$, respectively, which disconnect the inner and outer boundaries. Then the union $P^\delta \cup P_u^\delta \cup P_v^\delta \cup \pi_u^\delta \cup \pi_v^\delta$ contains a path from u to v. From this observation followed by (3.3) of Definition 3.2 and the definition (1.19) of $\mathfrak{C}_*$, we get that if $\delta \in (0, (1-\alpha)^4]$ then

$$\begin{aligned}
\widetilde{D}_h(u,v) &\leqslant \widetilde{D}_h(B_{\delta r}(u), B_{\delta r}(v)) + \sum_{w\in\{u,v\}} \widetilde{D}_h\big(w, \partial B_{\delta^{1/2} r}(w)\big) \\
&\qquad + \sum_{w\in\{u,v\}} \widetilde{D}_h\big(\text{around } \mathbb{A}_{\delta r, \delta^{1/2} r}(w)\big) \\
&\leqslant \widetilde{D}_h(B_{\delta r}(u), B_{\delta r}(v)) + \mathfrak{C}_* \sum_{w\in\{u,v\}} D_h\big(w, \partial B_{\delta^{1/2} r}(w)\big) \\
&\qquad + \mathfrak{C}_* \sum_{w\in\{u,v\}} D_h\big(\text{around } \mathbb{A}_{\delta r, \delta^{1/2} r}(w)\big) \\
&\leqslant \widetilde{D}_h(B_{\delta r}(u), B_{\delta r}(v)) + 2\mathfrak{C}_*\Big(\delta^{\theta/2} + \delta^\theta\Big) D_h(u,v).
\end{aligned} \tag{3.9}$$

By (3.2) and (3.9), we obtain

$$\widetilde{D}_h(B_{\delta r}(u), B_{\delta r}(v)) \geqslant \Big[\mathfrak{C}' - 2\mathfrak{C}_*\Big(\delta^{\theta/2} + \delta^\theta\Big)\Big] D_h(u,v). \tag{3.10}$$

We now obtain (3.8) by choosing $\delta \in (0, (1-\alpha)^4]$ to be sufficiently small, depending on ζ and $\mathfrak{C}_*$, and setting $\beta = \delta$. ☐

Proof of Proposition 3.6. Let $\alpha \in (3/4, 1)$ and $p \in (0,1)$ (depending only on the laws of D_h and $\widetilde{D}_h$) be as in Proposition 3.5. Also let $\mathfrak{C}'' := (\mathfrak{C}' + \mathfrak{C}_*)/2 \in (\mathfrak{C}', \mathfrak{C}_*)$. By Proposition 3.5 (applied with $\mathfrak{C}''$ instead of $\mathfrak{C}'$), for each small enough $\varepsilon > 0$, there are at least $\frac{3}{4}\log_8 \varepsilon^{-1}$ values of $r \in [\varepsilon^2, \varepsilon] \cap \{8^{-k}\}_{k\in\mathbb{N}}$ for which $\mathbb{P}[H_r(\alpha, \mathfrak{C}'')] \geqslant p$. By Lemma 3.7, applied with $\mathfrak{C}''$ in place of $\mathfrak{C}'$ and $\zeta = \mathfrak{C}'' - \mathfrak{C}'$, we see that there exists $\beta > 0$, depending only on $\mathfrak{C}'$ and the laws of D_h and $\widetilde{D}_h$, such that if $H_r(\alpha, \mathfrak{C}'')$ occurs, then $G_r(\beta, \mathfrak{C}')$ occurs. Combining the preceding two sentences gives the proposition statement with $p \wedge \beta$ in place of β. □

Since our assumptions on the metrics D_h and $\widetilde{D}_h$ are the same, all of the results above also hold with the roles of D_h and $\widetilde{D}_h$ interchanged. For ease of reference, we will record some of these results here.

Definition 3.8. For $r > 0, \beta > 0$, and $\mathfrak{c}' > 0$, we let $\widetilde{G}_r(\beta, \mathfrak{c}')$ be the event that the event $G_r(\beta, 1/\mathfrak{c}')$ of Definition 3.1 occurs with the roles of D_h and $\widetilde{D}_h$ interchanged. That is, $\widetilde{G}_r(\beta, \mathfrak{c}')$ is the event that there exists $z, w \in \overline{B}_r(0)$ such that

$$\widetilde{D}_h(z, w) \leqslant \mathfrak{c}' D_h\big(B_{\beta r}(z), B_{\beta r}(w)\big).$$

Definition 3.9. For $r > 0$, $\alpha \in (3/4, 1)$, and $\mathfrak{c}' > 0$, we let $\widetilde{H}_r(\alpha, \mathfrak{c}')$ be the event that the event $H_r(\alpha, 1/\mathfrak{c}')$ of Definition 3.2 occurs with the roles of D_h and $\widetilde{D}_h$ interchanged. That is, $\widetilde{H}_r(\alpha, \mathfrak{c}')$ is the event that there exist non-singular points $u \in \partial B_{\alpha r}(0)$ and $v \in \partial B_r(0)$ such that

$$\widetilde{D}_h(u, v) \leqslant \mathfrak{c}' D_h(u, v) \tag{3.11}$$

and a $\widetilde{D}_h$-geodesic $\widetilde{P}$ from u to v such that the following is true.

(i) $\widetilde{P} \subset \overline{\mathbb{A}}_{\alpha r, r}(0)$.
(ii) The Euclidean diameter of $\widetilde{P}$ is at most $r/100$.
(iii) $\widetilde{D}_h(u, v) \leqslant (1-\alpha)^{-1} r^{\xi Q} e^{\xi h_r(0)}$.
(iv) Let $\theta > 0$ be as in Lemma 2.13 with $\chi = 1/2$. For each $\delta \in (0, (1-\alpha)^2]$,

$$\max\big\{\widetilde{D}_h(u, \partial B_{\delta r}(u)), \widetilde{D}_h\big(\text{around } \mathbb{A}_{\delta r, \delta^{1/2} r}(u)\big)\big\} \leqslant \delta^\theta \widetilde{D}_h(u, v) \tag{3.12}$$

and the same is true with the roles of u and v interchanged.

We have the following analog of Proposition 3.3.

Proposition 3.10. *There exist $\alpha \in (3/4, 1)$ and $p \in (0, 1)$, depending only on the laws of D_h and $\widetilde{D}_h$, such that for each $\mathfrak{c}' > \mathfrak{c}_*$, there exists $\mathfrak{c}'' = \mathfrak{c}''(\mathfrak{c}') \in (\mathfrak{c}_*, \mathfrak{c}')$ such that for each $\widetilde{\beta} \in (0, 1)$, there exists $\varepsilon_0 = \varepsilon_0(\widetilde{\beta}, \mathfrak{c}') > 0$ with the following property. If $\mathbb{r} > 0$ and $\mathbb{P}[\widetilde{G}_{\mathbb{r}}(\widetilde{\beta}, \mathfrak{c}'')] \geqslant \widetilde{\beta}$, then the following is true for each $\varepsilon \in (0, \varepsilon_0]$.*

(A') There are at least $\frac{3}{4}\log_8 \varepsilon^{-1}$ values of $r \in [\varepsilon^2 \mathbb{r}, \varepsilon \mathbb{r}] \cap \{8^{-k}\mathbb{r} : k \in \mathbb{N}\}$ for which $\mathbb{P}[\widetilde{H}_r(\alpha, \mathfrak{c}')] \geqslant p$.

We will also need the following analog of Proposition 3.6.

Proposition 3.11. *For each $\mathfrak{c}' > \mathfrak{c}_*$, there exists $\widetilde{\beta} > 0$, depending on $\mathfrak{c}'$ and the laws of D_h and $\widetilde{D}_h$, such that for each small enough $\varepsilon > 0$ (depending on $\mathfrak{c}'$ and the laws of D_h and $\widetilde{D}_h$), there are at least $\frac{3}{4}\log_8 \varepsilon^{-1}$ values of $r \in [\varepsilon^2, \varepsilon] \cap \{8^{-k}\}_{k \in \mathbb{N}}$ for which $\mathbb{P}[\widetilde{G}_r(\widetilde{\beta}, \mathfrak{c}')] \geqslant \widetilde{\beta}$.*

3.2 | Proof of Proposition 3.3

To prove Proposition 3.3, we will prove the contrapositive, as stated in the following proposition.

Proposition 3.12. *There exists* $\alpha \in (3/4, 1)$ *and* $p \in (0,1)$*, depending only on the laws of* D_h *and* $\widetilde{D}_h$*, such that for each* $\mathfrak{C}' \in (0, \mathfrak{C}_*)$*, there exists* $\mathfrak{C}'' = \mathfrak{C}''(\mathfrak{C}') \in (\mathfrak{C}', \mathfrak{C}_*)$ *such that for each* $\beta \in (0,1)$*, there exists* $\varepsilon_0 = \varepsilon_0(\beta, \mathfrak{C}') > 0$ *with the following property. If* $\mathbb{r} > 0$ *and there exists* $\varepsilon \in (0, \varepsilon_0]$ *satisfying the condition (B) just below, then* $\mathbb{P}[G_{\mathbb{r}}(\beta, \mathfrak{C}'')] < \beta$.

(B) There are at least $\frac{1}{4}\log_8 \varepsilon^{-1}$ *values of* $r \in [\varepsilon^2 \mathbb{r}, \varepsilon \mathbb{r}] \cap \{8^{-k}\mathbb{r} : k \in \mathbb{N}\}$ *for which* $\mathbb{P}[H_r(\alpha, \mathfrak{C}')] < p$.

Note that the second-to-last last sentence of Proposition 3.12 (that is, the one just before condition (B)) is the contrapositive of the second-to-last sentence of Proposition 3.3 (that is, the one just before condition (A)). The proof of Proposition 3.12 is similar to the argument in [27, section 3.2], but the definitions of the events involved are necessarily different due to the existence of singular points.

The basic idea of the proof is as follows. If we assume that (B) holds for a small enough (universal) choice of $p \in (0,1)$, then we can use Lemma 2.1 (independence across concentric annuli) and a union bound to cover space by Euclidean balls of the form $B_{r/2}(z)$ for $r \in [\varepsilon^2\mathbb{r}, \varepsilon\mathbb{r}]$ with the following property. For each $u \in \partial B_{\alpha r}(z)$ and each $v \in \partial B_r(z)$ which are joined by a geodesic P satisfying the numbered conditions in Definition 3.2, we have $\widetilde{D}_h(u,v) \leqslant \mathfrak{C}' D_h(u,v)$.

By considering the times when a D_h-geodesic between two fixed points $\mathbb{z}, \mathbb{w} \in \mathbb{C}$ crosses the annulus $\mathbb{A}_{\alpha r, r}(z)$ for such a z and r, we will be able to show that $\widetilde{D}_h(B_\beta(\mathbb{z}), B_\beta(\mathbb{w})) \leqslant \mathfrak{C}'' D_h(\mathbb{z}, \mathbb{w})$ for a suitable constant $\mathfrak{C}'' \in (\mathfrak{C}', \mathfrak{C}_*)$. Applying this to an appropriate β-dependent collection of pairs of points $(\mathbb{z}, \mathbb{w})$ will show that $\mathbb{P}[G_{\mathbb{r}}(\beta, \mathfrak{C}'')] < \beta$. The reason why we need to make α close to 1 is to ensure that the events we consider depend on h in a sufficiently 'local' manner (see the proof of Lemma 3.13).

Let us now define the events to which we will apply Lemma 2.1. See Figure 5 for an illustration of the definition. We will discuss the purpose of each condition in the event just below.

For $z \in \mathbb{C}$, $r > 0$, and parameters $\delta_0 \in (0, 1/100)$, $\alpha \in (1-\delta_0, 1)$, and $A > 1$, let $E_r(z) = E_r(z; \delta_0, \alpha, A, \mathfrak{C}')$ be the event that the following is true.

(1) (*Regularity along geodesics*) For each $D_h(\cdot,\cdot;\overline{\mathbb{A}}_{r/2,2r}(z))$-geodesic P between two points of $\partial\mathbb{A}_{r/2,2r}(z)$, each $\delta \in (0, \delta_0]$, and each $x \in \mathbb{A}_{3r/4,3r/2}(z)$ such that $P \cap B_{\delta r}(x) \neq \emptyset$,

$$D_h\big(\text{around } \mathbb{A}_{\delta r, \delta^{1/2} r}(x)\big) \leqslant \delta^\theta D_h\big(\text{across } \mathbb{A}_{\delta r, \delta^{1/2} r}(x)\big), \tag{3.13}$$

where (as in Definition 3.2) θ is as in Lemma 2.13 with $\chi = 1/2$.

(2) (*Distance around* $\mathbb{A}_{3r/2,2r}(z)$) We have

$$D_h\big(\text{around } \mathbb{A}_{3r/2,2r}(z)\big) \leqslant \min\left\{ (1-\alpha)^{-1} r^{\xi Q} e^{\xi h_r(z)}, \frac{\mathfrak{c}_*}{2\mathfrak{C}_*}\delta_0^{-\theta} D_h\big(\mathbb{A}_{3r/4,3r/2}(z), \partial\mathbb{A}_{r/2,2r}(z)\big) \right\}. \tag{3.14}$$

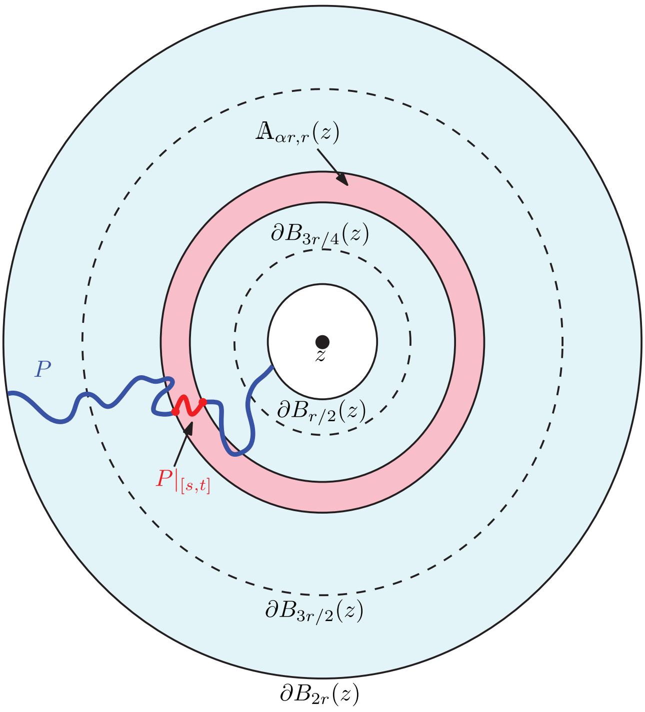

FIGURE 5 Illustration of the definition of $E_r(z)$. We have shown the annuli involved in the definition and an example of a $D_h(\cdot,\cdot;\mathbb{A}_{r/2,2r}(z))$-geodesic P between two points of $\partial\mathbb{A}_{r/2,2r}(z)$, which appears in several of the conditions. Condition 1 allows us to compare distances around and across small annuli surrounding points of $\mathbb{A}_{3r/4,3r/2}(z)$ which are hit by P. Condition 2 provides an upper bound for the D_h-distance around the outer annulus $\mathbb{A}_{3r/2,2r}(z)$. Condition 3 gives an upper bound for the Euclidean diameters of segments of P which are contained in the pink annulus $\mathbb{A}_{\alpha r,r}(z)$, such as the red segment in the figure. Condition 4 gives an upper bound for the D_h-distance around $\mathbb{A}_{\alpha r,r}(z)$. Finally, condition 5 will allow us to show that the $\widetilde{D}_h$-length of a red segment like $P|_{[s,t]}$ is at most $\mathfrak{C}'(t-s)$.

(3) (*Euclidean length of geodesic segments in* $\mathbb{A}_{\alpha r,r}(z)$) For each $D_h(\cdot,\cdot;\overline{\mathbb{A}}_{r/2,2r}(z)$-geodesic P between two points of $\partial\mathbb{A}_{r/2,2r}(z)$ and any two times $t>s>0$ such that $P([s,t])\subset\overline{\mathbb{A}}_{\alpha r,r}(z)$, we have

$$|P(t)-P(s)|\leqslant\delta_0 r. \tag{3.15}$$

(4) (*Distance around* $\mathbb{A}_{\alpha r,r}(z)$) We have

$$D_h\big(\text{around }\mathbb{A}_{\alpha r,r}(z)\big)\leqslant AD_h\big(\text{across }\mathbb{A}_{\alpha r,r}(z)\big). \tag{3.16}$$

(5) (*Converse of* $H_r(\alpha,\mathfrak{C}')$) Let $u\in\partial B_{\alpha r}(z)$ and $v\in\partial B_r(z)$ such that $|u-v|\leqslant\delta_0 r$ and

$$D_h\Big(\text{around }\mathbb{A}_{\delta_0 r,\delta_0{}^{1/2}r}(v)\Big)\leqslant\frac{\mathfrak{c}_*}{2\mathfrak{C}_*}D_h\big(\mathbb{A}_{3r/4,3r/2}(z),\partial\mathbb{A}_{r/2,2r}(z)\big). \tag{3.17}$$

Assume that there is a D_h-geodesic P' from u to v such that the numbered conditions in Definition 3.2 of $H_r(\alpha, \mathfrak{C}')$ occur but with z in place of 0, that is,

(i) $P' \subset \overline{\mathbb{A}}_{\alpha r, r}(z)$;

(ii) the Euclidean diameter of P' is at most $r/100$;

(iii) $D_h(u, v) \leqslant (1-\alpha)^{-1} r^{\xi Q} e^{\xi h_r(z)}$;

(iv) for each $\delta \in (0, (1-\alpha)^2]$,

$$\max\left\{D_h(u, \partial B_{\delta r}(u)), D_h\left(\text{around } \mathbb{A}_{\delta r, \delta^{1/2} r}(u)\right)\right\} \leqslant \delta^{\theta} D_h(u, v) \tag{3.18}$$

and the same is true with the roles of u and v interchanged.

Then $\widetilde{D}_h(u, v) \leqslant \mathfrak{C}' D_h(u, v)$.

The most important condition in the definition of $E_r(z)$ is condition 5. By Definition 3.2 and the translation invariance of the law of h, modulo additive constant, if $\mathbb{P}[H_r(\alpha, \mathfrak{C}')]$ is small, then the probability of condition 5 is large. The extra condition (3.17) on u and v is included in order to prevent D_h-geodesics or $\widetilde{D}_h$-geodesics between u and v from exiting $\mathbb{A}_{r/2,2r}(z)$. This is needed to ensure that $E_r(z)$ is determined by $h|_{\mathbb{A}_{r/2,2r}(z)}$, which in turn is needed to apply Lemma 2.1. See Lemma 3.13.

We will eventually consider a D_h-geodesic P which enters $B_{r/2}(z)$ and apply condition 5 to the D_h-geodesic $P' = P|_{[s,t]}$ from $u = P(s)$ to $v = P(t)$, where s and t are suitably chosen times such that $P(s) \in \partial B_{\alpha r}(z)$ and $P(t) \in \partial B_r(z)$. The first three conditions in the definition of $E_r(z)$ will allow us to do so (see Lemma 3.16). In particular, condition 1 will allow us to check (3.18) for $u = P(s)$ and $v = P(t)$. Condition 2 will be used in conjunction with condition 1 to check (3.17). Condition 3 will be used to upper-bound the Euclidean diameter of $P|_{[s,t]}$.

Condition 4 will be used to show that the intervals $[s, t]$ as above for varying choices of r and z such that $E_r(z)$ occurs and P enters $B_{r/2}(z)$ cover a uniformly positive fraction of the time interval on which P is defined. See Lemma 3.18.

Let us now explain why we can apply Lemma 2.1 to the events $E_r(z)$. For the statement, recall the definition of the restriction of the GFF to a closed set from (2.2).

Lemma 3.13. *The event $E_r(z)$ is almost surely determined by $h|_{\overline{\mathbb{A}}_{r/2,2r}(z)}$, viewed modulo additive constant.*

Proof. It is immediate from Weyl scaling (Axiom III) that adding a constant to h does not affect the occurrence of $E_r(z)$. Therefore, $E_r(z)$ is almost surely determined by h viewed modulo additive constant. We need to show that $E_r(z)$ is almost surely determined by $h|_{\overline{\mathbb{A}}_{r/2,r}(z)}$.

Each of conditions 1, 2, 3, and 4 in the definition of $E_r(z)$ depends only on $D_h(\cdot, \cdot; \overline{\mathbb{A}}_{r/2,r}(z))$. By locality (Axiom II; see also Subsection 2.2), we get that each of these four conditions is almost surely determined by $h|_{\overline{\mathbb{A}}_{r/2,2r}(z)}$.

We still need to treat condition 5. To this end, we claim that if $u \in \partial B_{\alpha r}(z)$ and $v \in \partial B_r(z)$ such that $|u - v| \leqslant \delta_0 r$ and (3.17) holds (as in condition 5), then every D_h-geodesic and every $\widetilde{D}_h$-geodesic from u to v is contained in $\mathbb{A}_{r/2,2r}(z)$. The claim implies that the set of $D_h(\cdot, \cdot; \mathbb{A}_{r/2,2r}(z))$-geodesics from u to v is the same as the set of D_h-geodesics from u to v, and similarly with $\widetilde{D}_h$ in place of D_h. This, in turn, implies that condition 5 is equivalent to the analogous condition where we require that P' is a $D_h(\cdot, \cdot; \mathbb{A}_{r/2,2r}(z))$-geodesic instead of a D_h-geodesic and we replace $D_h(u, v)$ and $\widetilde{D}_h(u, v)$ by $D_h(u, v; \mathbb{A}_{r/2,2r}(z))$ and $\widetilde{D}_h(u, v; \mathbb{A}_{r/2,2r}(z))$, respectively. It then follows from locality (Axiom II) that $E_r(z)$ is almost surely determined by $h|_{\overline{\mathbb{A}}_{r/2,2r}(z)}$, viewed modulo additive constant.

It remains to prove the claim in the preceding paragraph. Let u and v be as above and let P be path from u to v which exits $\mathbb{A}_{r/2,2r}(z)$. We need to show that P is neither a D_h-geodesic nor a $\widetilde{D}_h$-geodesic. By (3.17), there is a path $\pi \subset \mathbb{A}_{\delta_0 r, \delta_0{}^{1/2} r}(v)$ such that

$$\operatorname{len}(\pi; D_h) < \frac{\mathfrak{c}_*}{\mathfrak{C}_*} D_h\big(\mathbb{A}_{3r/4,3r/2}(z), \partial\mathbb{A}_{r/2,2r}(z)\big). \tag{3.19}$$

By the bi-Lipschitz equivalence of D_h and $\widetilde{D}_h$, this implies that also

$$\operatorname{len}\big(\pi; \widetilde{D}_h\big) < \widetilde{D}_h\big(\mathbb{A}_{3r/4,3r/2}(z), \partial\mathbb{A}_{r/2,2r}(z)\big). \tag{3.20}$$

Since $u, v \in B_{\delta_0 r}(v)$, the path P must hit π before the first time it crosses from $\mathbb{A}_{3r/4,3r/2}(z)$ to $\partial\mathbb{A}_{r/2,2r}(z)$ and after the last time that it does so. Therefore, (3.19) implies that we can replace a segment of P with a segment of π to get a path with the same endpoints and shorter D_h-length. Hence, P is not a D_h-geodesic. Similarly, (3.20) implies that P is not a $\widetilde{D}_h$-geodesic. □

We now check that $E_r(z)$ occurs with high probability if the parameters are chosen appropriately.

Lemma 3.14. *For each $p \in (0,1)$, there exist parameters $\delta_0 \in (0, 1/100)$, $\alpha \in (1-\delta_0, 1)$, and $A > 1$, depending only on p and the laws of D_h and $\widetilde{D}_h$, such that the following is true. Let $\mathfrak{C}' \in (0, \mathfrak{C}_*)$ and $\mathbb{r} > 0$ and assume that (B) holds for our given choice of α and p. Then there are at least $\frac{1}{4}\log_8 \varepsilon^{-1}$ values of $r \in [\varepsilon^2\mathbb{r}, \varepsilon\mathbb{r}] \cap \{8^{-k}\}_{k\in\mathbb{N}}$ such that $\mathbb{P}[E_r(z)] \geqslant 1 - 2p$ for each $z \in \mathbb{C}$.*

Proof. By the translation invariance of the law of h, viewed modulo additive constant, and Axiom IV′, it suffices to prove the lemma in the case when $z = 0$.

By Lemma 2.13 (applied with $f \equiv 0$), we can find $\delta_0 \in (0, 1/100)$ depending only on p and the laws of D_h and $\widetilde{D}_h$ such that for each $r > 0$, the probability of condition 1 in the definition of $E_r(0)$ is at least $1 - p/4$. By tightness across scales (Axiom V′), after possibly shrinking δ_0, we can find $\alpha \in (1-\delta_0, 1)$ depending only on the laws of D_h and $\widetilde{D}_h$ such that the probability of condition 2 is also at least $1 - p/4$.

By Lemma 2.14 (applied with $f \equiv 0$ and η the unit-speed parameterization of $\partial B_1(0)$), after possibly shrinking α, in a manner depending on δ_0, we can arrange that for each $r > 0$, it holds with probability at least $1 - p/4$ that the following is true. For each $D_h(\cdot,\cdot;\overline{\mathbb{A}}_{r/2,2r}(0))$-geodesic P from a point of $\partial B_{r/2}(0)$ to a point of $\partial B_r(0)$, the one-dimensional Lebesgue measure of the set

$$\big\{x \in \partial B_r(0) \,:\, P \cap B_{100(1-\alpha)r}(x) \neq \emptyset\big\} \tag{3.21}$$

is at most $\delta_0 r$. If $t > s > 0$ such that $P([s,t]) \subset \overline{\mathbb{A}}_{\alpha r, r}(0)$, then the one-dimensional Lebesgue measure of the set (3.21) is at least the Euclidean diameter of $P([s,t])$. This shows that condition 3 in the definition of $E_r(0)$ occurs with probability at least $1 - p/4$.

By tightness across scales (Axiom V′), we can find $A > 1$ (depending on α) such that for each $r > 0$, condition 4 in the definition of $E_r(0)$ occurs with probability at least $1 - p/4$. By (B) and the Definition 3.2 of $H_r(\alpha, \mathfrak{C}')$, there are at least $\frac{1}{4}\log_8 \varepsilon^{-1}$ values of $r \in [\varepsilon^2\mathbb{r}, \varepsilon\mathbb{r}] \cap \{8^{-k}\}_{k\in\mathbb{N}}$ such that condition 5 in the definition of $E_r(0)$ occurs with probability at least $1 - p$. We note that the requirement (3.17) does not show up in (B), but including the requirement (3.17) makes the condition weaker, so makes the probability of the condition larger.

Taking a union bound over the five conditions in the definition of $E_r(0)$ now concludes the proof. □

With Lemmas 3.13 and 3.14 in hand, we can now apply Lemma 2.1 to obtain the following.

Lemma 3.15. *There exist parameters $p_* \in (0,1)$, $\delta_0 \in (0, 1/100)$, $\alpha \in (1-\delta_0, 1)$, and $A > 1$, depending only on the laws of D_h and $\widetilde{D}_h$, such that the following is true. Let $\mathfrak{C}' \in (0, \mathfrak{C}_*)$ and $\mathbb{r} > 0$ and assume that (B) holds for our given choice of α and with $p = p_*$. For each fixed bounded open set $U \subset \mathbb{C}$, it holds with probability tending to 1 as $\varepsilon \to 0$ (at a rate depending only on U) that for each $z \in \mathbb{r}U$, there exists $r \in [\varepsilon^2\mathbb{r}, \varepsilon\mathbb{r}]$ and $w \in B_{r/2}(z)$ such that $E_r(w)$ occurs.*

Proof. By Lemma 2.1, there exists a universal constant $p_* \in (0,1)$ such that the following is true. Let $\mathbb{r} > 0$, let $\varepsilon \in (0,1)$, let $K \geqslant \frac{1}{4}\log_8 \varepsilon^{-1}$, and let $r_1, \dots, r_K \in [\varepsilon^2\mathbb{r}, \varepsilon\mathbb{r}] \cap \{8^{-k}\}_{k\in\mathbb{N}}$ be distinct. If $z \in \mathbb{C}$ and $F_{r_k}(z)$ for $k = 1, \dots, K$ is an event which is almost surely determined by $h|_{\overline{\mathbb{A}}_{r_j/2, 2r_j}(z)}$, viewed modulo additive constant, and has probability at least $1 - 2p_*$, then

$$\mathbb{P}\left[\exists k \in [1,K]_{\mathbb{Z}} \text{ such that } F_{r_k} \text{ occurs}\right] \geqslant 1 - O_\varepsilon(\varepsilon^{100}),$$

with the implicit constant in the $O_\varepsilon(\cdot)$ universal.

We now choose δ_0, α, A as in Lemma 3.14 with $p = p_*$. For $\mathfrak{C}' \in (0, \mathfrak{C}_*)$ and $\mathbb{r} > 0$, we apply the above statement to the radii $r \in [\varepsilon^2\mathbb{r}, \varepsilon\mathbb{r}] \cap \{8^{-k}\}_{k\in\mathbb{N}}$ from Lemma 3.14, which are chosen so that $\mathbb{P}[E_r(w)] \geqslant 1 - 2p_*$ for all $w \in \mathbb{C}$. By Lemma 3.14, if (B) holds with $p = p_*$, then there are at least $\frac{1}{4}\log_8 \varepsilon^{-1}$ such radii. Hence, if (B) holds, then

$$\mathbb{P}\left[\exists r \in [\varepsilon^2\mathbb{r}, \varepsilon\mathbb{r}] \text{ such that } E_r(w) \text{ occurs}\right] \geqslant 1 - O_\varepsilon(\varepsilon^{100}), \quad \forall z \in \mathbb{C}, \tag{3.22}$$

with the implicit constant in the $O_\varepsilon(\cdot)$ universal.

The lemma statement now follows by applying (3.22) to each of the $O_\varepsilon(\varepsilon^{-2})$ points $w \in B_{\mathbb{r}}(\mathbb{r}U) \cap (\frac{\varepsilon\mathbb{r}}{100}\mathbb{Z}^2)$, then taking a union bound. □

Henceforth, fix p_*, δ_0, α, and A as in Lemma 3.15. Also fix

$$\mathfrak{C}'' \in \left(\mathfrak{C}' + \frac{A}{A+1}(\mathfrak{C}_* - \mathfrak{C}'), \mathfrak{C}_*\right), \tag{3.23}$$

and note that we can choose $\mathfrak{C}''$ in a manner depending only on $\mathfrak{C}'$ and the laws of D_h and $\widetilde{D}_h$ (since A depends only on the laws of D_h and $\widetilde{D}_h$).

We will show that for each $\beta > 0$, there exists $\varepsilon_0 = \varepsilon_0(\beta, \mathfrak{C}') > 0$ such that if $\mathbb{r} > 0$, $\varepsilon \in (0, \varepsilon_0]$, and (B) holds for the above values of $\mathbb{r}, \varepsilon, p_*, \alpha$, then with probability greater than $1 - \beta$,

$$\widetilde{D}_h\big(B_{\beta\mathbb{r}}(\mathbb{z}), B_{\beta\mathbb{r}}(\mathbb{w})\big) \leqslant \mathfrak{C}'' D_h(\mathbb{z}, \mathbb{w}) \quad \forall \mathbb{z}, \mathbb{w} \in B_{\mathbb{r}}(0). \tag{3.24}$$

By Definition 3.1, the bound (3.24) implies that $\mathbb{P}[G_{\mathbb{r}}(\beta, \mathfrak{C}'')^c] > 1 - \beta$, which is what we aim to show in Proposition 3.12.

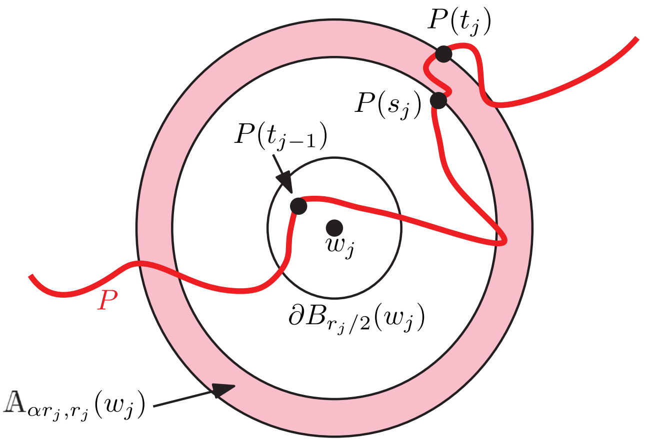

FIGURE 6 Illustration of the definition of the times s_j and t_j and the balls $B_{r_j}(w_j)$

By Lemma 2.11, there is some large bounded open set $U \subset \mathbb{C}$ (depending only on β and the law of D_h) such that for each $\mathbb{r} > 0$, it holds with probability at least $1 - \beta/2$ that each D_h-geodesic between two points of $\overline{B}_{\mathbb{r}}(0)$ is contained in $\mathbb{r}U$. For $\varepsilon > 0$, let $F_{\mathbb{r}}^\varepsilon$ be the event that this is the case and for each $z \in \mathbb{r}U$, there exists $r \in [\varepsilon^2\mathbb{r}, \varepsilon\mathbb{r}]$ and $w \in B_{r/2}(z)$ such that $E_r(w)$ occurs. By Lemma 3.15, if (B) holds then

$$\mathbb{P}[F_{\mathbb{r}}^\varepsilon] \geqslant 1 - \beta/2 - o_\varepsilon(1), \tag{3.25}$$

where the rate of convergence of the $o_\varepsilon(1)$ depends only on U, hence only on β and the law of D_h.

We henceforth assume that $F_{\mathbb{r}}^\varepsilon$ occurs. We will show that if ε is small enough, then (3.24) holds. Let $\mathbb{z}, \mathbb{w} \in B_{\mathbb{r}}(0)$ and let $P : [0, D_h(\mathbb{z}, \mathbb{w})] \to \mathbb{C}$ be a D_h-geodesic from $\mathbb{z}$ to $\mathbb{w}$. We assume that

$$\varepsilon \leqslant \frac{1}{4}\beta \quad \text{and} \quad |\mathbb{z} - \mathbb{w}| \geqslant \beta\mathbb{r}. \tag{3.26}$$

The reason why we can make these assumptions is that ε_0 is allowed to depend on β and (3.24) holds vacuously if $|\mathbb{z} - \mathbb{w}| \leqslant \beta\mathbb{r}$. We will inductively define a sequence of times

$$0 = t_0 < s_1 < t_1 < s_2 < t_2 < \cdots < s_J < t_J \leqslant D_h(\mathbb{z}, \mathbb{w}).$$

See Figure 6 for an illustration.

Let $t_0 = 0$. Inductively, assume that $j \in \mathbb{N}$ and t_{j-1} has been defined. By the definition of $F_{\mathbb{r}}^\varepsilon$, we have $P(t_{j-1}) \in \mathbb{r}U$ and there exists $r_j \in [\varepsilon^2\mathbb{r}, \varepsilon\mathbb{r}]$ and $w_j \in B_{r_j/2}(P(t_{j-1}))$ such that $E_{r_j}(w_j)$ occurs. Fix (in some arbitrary manner) a particular choice of r_j and w_j with these properties.

Let t_j be the first time $t \geqslant t_{j-1}$ for which $P(t) \notin B_{r_j}(w_j)$, or let $t_j = D_h(\mathbb{z}, \mathbb{w})$ if no such time exists. If $t_j < D_h(\mathbb{z}, \mathbb{w})$, we also let s_j be the last time before t_j at which P hits $\partial B_{\alpha r_j}(w_j)$, so that $s_j \in [t_{j-1}, t_j]$ and $P([s_j, t_j]) \subset \overline{\mathbb{A}}_{\alpha r_j, r_j}(w_j)$.

Finally, define

$$\underline{J} := \max\left\{ j \in \mathbb{N} : |\mathbb{z} - P(t_{j-1})| < 2\varepsilon\mathbb{r} \right\} \quad \text{and}$$
$$\overline{J} := \min\left\{ j \in \mathbb{N} : |\mathbb{w} - P(t_{j+1})| < 2\varepsilon\mathbb{r} \right\}. \tag{3.27}$$

The reason for the definitions of $\underline{J}$ and $\overline{J}$ is that $\mathbbm{z}, \mathbbm{w} \notin B_{r_j}(w_j)$ for $j \in [\underline{J}, \overline{J}]_{\mathbb{Z}}$ (since $r_j \leqslant \varepsilon\mathbbm{r}$ and $P(t_j) \in B_{r_j}(w_j)$). Whenever $|\mathbbm{w} - P(t_{j-1})| \geqslant \varepsilon\mathbbm{r}$, we have $t_j < D_h(\mathbbm{z}, \mathbbm{w})$ and $|P(t_{j-1}) - P(t_j)| \leqslant 2\varepsilon\mathbbm{r}$. Therefore,

$$P(t_{\underline{J}}) \in B_{4\varepsilon\mathbbm{r}}(\mathbbm{z}) \quad \text{and} \quad P(t_{\overline{J}}) \in B_{4\varepsilon\mathbbm{r}}(\mathbbm{w}). \tag{3.28}$$

The most important estimate that we need for the times s_j and t_j is the following lemma.

Lemma 3.16. *For each* $j \in [\underline{J}, \overline{J}]_{\mathbb{Z}}$,

$$\widetilde{D}_h\big(P(s_j), P(t_j)\big) \leqslant \mathfrak{C}'(t_j - s_j) \quad \textit{and} \quad \widetilde{D}_h\big(P(t_{j-1}), P(s_j)\big) \leqslant \mathfrak{C}_*(s_j - t_{j-1}). \tag{3.29}$$

The second inequality in (3.29) is immediate from the definition (1.19) of $\mathfrak{C}_*$. We will prove the first inequality in (3.29) by applying condition 5 in the definition of $E_{r_j}(w_j)$ with $u = P(s_j)$ and $v = P(t_j)$. The following lemma will be used in conjunction with condition 1 in the definition of $E_{r_j}(w_j)$ to check the requirement (3.17) from condition 5.

Lemma 3.17. *For each* $j \in [\underline{J}, \overline{J}]_{\mathbb{Z}}$, *we have*

$$t_j - s_j \leqslant (1-\alpha)^{-1} r_j^{\xi Q} e^{\xi h_{r_j}(w_j)} \tag{3.30}$$

and

$$D_h\Big(\textit{across } \mathbb{A}_{\delta_0 r_j, \delta_0{}^{1/2} r_j}(P(t_j))\Big) \leqslant \frac{\mathfrak{c}_*}{2\mathfrak{C}_*}\delta^{-\theta} D_h\Big(\mathbb{A}_{3r_j/4, 3r_j/2}(z), \partial\mathbb{A}_{r_j/2, 2r_j}(z)\Big). \tag{3.31}$$

Proof. See Figure 7 for an illustration. Let s_j' be the first time that P enters $B_{3r_j/2}(w_j)$ and let t_j' be the last time that P exits $B_{3r_j/2}(w_j)$. Then $s_j' < s_j < t_j < t_j'$. The definitions (3.27) of $\underline{J}$ and $\overline{J}$ show that the endpoints $\mathbbm{z}, \mathbbm{w}$ of P are not in $B_{2r_j}(w_j)$, so P must cross between the inner and outer boundaries of the annulus $\mathbb{A}_{3r_j/2, 2r_j}(w_j)$ before time s_j' and after time t_j'. By considering the segment of P between two consecutive times when it hits a path around $\mathbb{A}_{3r_j/2, 2r_j}(w_j)$ of near-minimal length and using the fact that P is a D_h-geodesic, we see that

$$t_j' - s_j' \leqslant D_h\Big(\text{around } \partial\mathbb{A}_{3r_j/2, 2r_j}(z)\Big). \tag{3.32}$$

By (3.32), followed by condition 2 in the definition of $E_{r_j}(w_j)$, we obtain

$$t_j - s_j \leqslant t_j' - s_j' \leqslant D_h\Big(\text{around } \partial\mathbb{A}_{3r_j/2, 2r_j}(z)\Big) \leqslant (1-\alpha)^{-1} r_j^{\xi Q} e^{\xi h_{r_j}(w_j)},$$

which is (3.30).

The path P must cross between the inner and outer boundaries of the annulus $\mathbb{A}_{\delta_0 r_j, \delta_0{}^{1/2} r_j}(P(t_j))$ between times t_j' and s_j'. By (3.32) followed by condition 2 in the definition of $E_{r_j}(w_j)$,

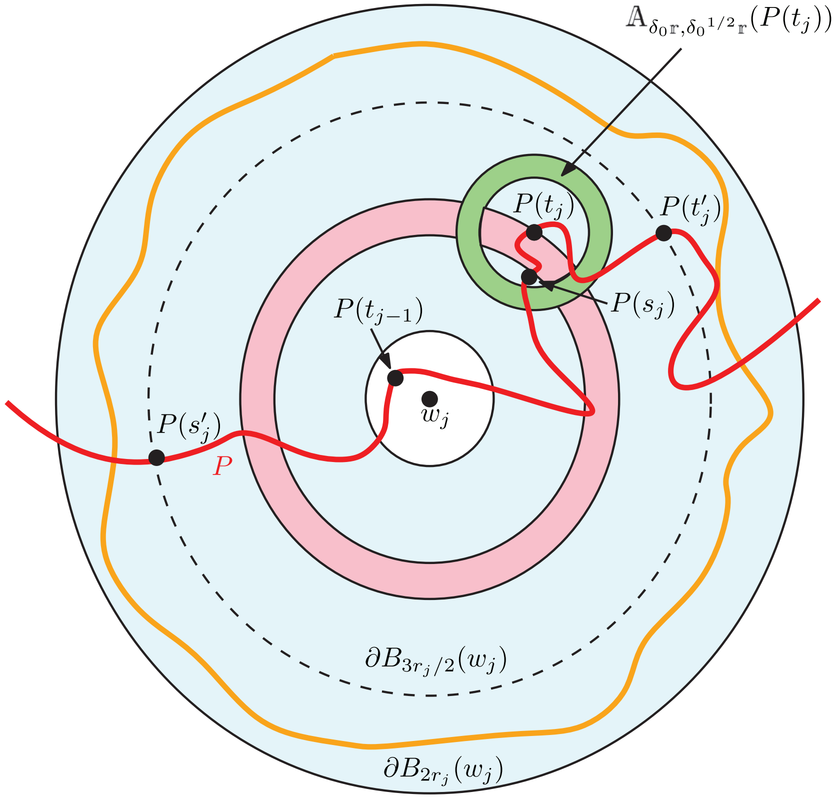

FIGURE 7 Illustration of the proof of Lemma 3.17. We upper-bound $t_j - s_j$ and $D_h(\text{across } \mathbb{A}_{\delta_0 r_j, \delta_0{}^{1/2} r_j}(P(t_j)))$ in terms of $t_j' - s_j'$, upper-bound $t_j' - s_j'$ in terms of the D_h-length of the orange loop, and upper-bound the D_h-length of the orange loop using condition 2 in the definition of $E_{r_j}(w_j)$. Note that the picture is not to scale. For example, in actuality the inner radius of $\mathbb{A}_{\delta_0 r_j, \delta_0{}^{1/2} r_j}(P(t_j))$ is much smaller than its outer radius.

$$\begin{aligned} D_h\Big(\text{across } \mathbb{A}_{\delta_0 r_j, \delta_0{}^{1/2} r_j}(P(t_j))\Big) &\leqslant t_j' - s_j' \\ &\leqslant D_h\Big(\text{around } \partial \mathbb{A}_{3r_j/2, 2r_j}(z)\Big) \\ &\leqslant \frac{\mathfrak{c}_*}{2\mathfrak{C}_*}\delta^{-\theta} D_h\Big(\mathbb{A}_{3r_j/4, 3r_j/2}(z), \partial \mathbb{A}_{r_j/2, 2r_j}(z)\Big). \end{aligned}$$

This gives (3.31). □

Proof of Lemma 3.16. The second inequality in (3.29) is immediate from the definition (1.19) of $\mathfrak{C}_*$. To get the first inequality, we want to apply condition 5 in the definition of $E_{r_j}(w_j)$ to the points $u = P(s_j) \in \partial B_{\alpha r_j}(w_j)$ and $v = P(t_j) \in \partial B_{r_j}(w_j)$. To do this, we need to check the hypotheses of condition 5 in the definition of $E_{r_j}(w_j)$.

To this end, let σ_j be the last time before s_j at which P enters $\mathbb{A}_{r_j/2, 2r_j}(w_j)$ and let τ_j be the first time after t_j at which P exits $\mathbb{A}_{r_j/2, 2r_j}(w_j)$. Then $P|_{[\sigma_j, \tau_j]}$ is a $D_h(\cdot, \cdot; \overline{\mathbb{A}}_{r_j/2, 2r_j}(w_j))$-geodesic between two points of $\partial \mathbb{A}_{r_j/2, 2r_j}(w_j)$ and $\sigma_j < s_j < t_j < \tau_j$. By the definitions of s_j and t_j, we have

$$P|_{[s_j,t_j]} \subset \overline{\mathbb{A}}_{\alpha r_j, r_j}(w_j). \tag{3.33}$$

By (3.33) and condition 3 in the definition of $E_{r_j}(w_j)$,

$$\big(\text{Euclidean diameter of } P([s_j,t_j])\big) \leqslant \delta_0 r_j \leqslant \frac{r_j}{100}. \tag{3.34}$$

By condition 1 in the definition of $E_{r_j}(w_j)$,

$$D_h\Big(\text{around } \mathbb{A}_{\delta r_j, \delta^{1/2} r_j}(P(t_j))\Big) \leqslant \delta^\theta D_h\Big(\text{across } \mathbb{A}_{\delta r_j, \delta^{1/2} r_j}(P(t_j))\Big), \quad \forall \delta \in (0,\delta_0]; \tag{3.35}$$

and the same is true with $P(s_j)$ in place of $P(t_j)$. By definition, $|P(t_j) - P(s_j)| \geqslant (1-\alpha) r_j$ so for each $\delta \in (0,(1-\alpha)^2]$, the path $P|_{[s_j,t_j]}$ crosses between the inner and outer boundaries of the annuli $\mathbb{A}_{\delta r_j, \delta^{1/2} r_j}(P(s_j))$ and $\mathbb{A}_{\delta r_j, \delta^{1/2} r_j}(P(t_j))$. Since $1-\alpha < \delta_0$, (3.35) implies that

$$D_h\Big(\text{around } \mathbb{A}_{\delta r_j, \delta^{1/2} r_j}(P(t_j))\Big) \leqslant \delta^\theta (t_j - s_j) = \delta^\theta D_h(P(s_j), P(t_j)), \quad \forall \delta \in (0,(1-\alpha)^2]; \tag{3.36}$$

and the same is true with $P(s_j)$ in place of $P(t_j)$ on the left side.

By (3.36), for each $\zeta > 0$ and each $\delta \in (0,(1-\alpha)^2]$ we can find a path π_δ in $\mathbb{A}_{\delta r_j, \delta^{1/2} r_j}(P(t_j))$ which disconnects the inner and outer boundaries and has D_h-length at most $(\delta^\theta + \zeta)(t_j - s_j)$. If we let a_δ (respectively, b_δ) be the first (respectively, last) time that P hits π_δ, then $a_\delta \leqslant t_j \leqslant b_\delta$ and since P is a D_h-geodesic we must have $b_\delta - a_\delta \leqslant \operatorname{len}(\pi_\delta; D_h)$. Furthermore, the segment $P|_{[t_j,b_\delta]}$ hits $\partial B_{\delta \mathbb{r}}(P(t_j))$, so for each $\delta \in (0,(1-\alpha)^2]$,

$$D_h\big(P(t_j), \partial B_{\delta \mathbb{r}}(P(t_j))\big) \leqslant b_\delta - t_j \leqslant b_\delta - a_\delta \leqslant \operatorname{len}(\pi_\delta; D_h) \leqslant (\delta^\theta + \zeta)(t_j - s_j). \tag{3.37}$$

Sending $\zeta \to 0$ and recalling that P is a D_h-geodesic gives

$$D_h\big(P(t_j), \partial B_{\delta \mathbb{r}}(P(t_j))\big) \leqslant \delta^\theta D_h\big(P(s_j), P(t_j)\big), \quad \forall \delta \in (0,(1-\alpha)^2]. \tag{3.38}$$

We similarly obtain (3.38) with the roles of $P(s_j)$ and $P(t_j)$ interchanged.

Finally, by Lemma 3.17 and (3.35) (with $\delta = \delta_0$),

$$D_h\Big(\text{around } \mathbb{A}_{\delta_0 r_j, \delta_0{}^{1/2} r_j}(P(t_j))\Big) \leqslant \frac{\mathfrak{c}_*}{2\mathfrak{C}_*} D_h\Big(\mathbb{A}_{3r_j/4, 3r_j/2}(z), \partial \mathbb{A}_{r_j/2, 2r_j}(z)\Big). \tag{3.39}$$

We are now ready to explain why we can apply condition 5 with $u = P(s_j)$ and $v = P(t_j)$. The hypothesis (5i) follows from (3.33). The condition (3.17) and the hypothesis (5ii) for the Euclidean diameter of $P|_{[s_j,t_j]}$ follow from (3.34). The needed upper bound (5iii) for $D_h(P(s_j), P(t_j))$ follows from (3.30) The hypothesis (5iv) follows from (3.36) and (3.38). The hypothesis (3.18) follows from (3.39). Hence, we can apply condition 5 in the definition of $E_{r_j}(w_j)$ to $P|_{[s_j,t_j]}$ to get $\widetilde{D}_h(P(s_j),P(t_j)) \leqslant \mathfrak{C}'(t_j - s_j)$, as required. □

The last lemma we need for the proof of Proposition 3.12 tells us that the time intervals $[s_j, t_j]$ occupy a positive fraction of the total D_h-length of the path P.

Lemma 3.18. *For each* $j \in [\underline{J}, \overline{J}]_{\mathbb{Z}}$,

$$s_j - t_{j-1} \leqslant \frac{A}{A+1}(t_j - t_{j-1}). \tag{3.40}$$

Proof. By the definition of r_j and the definitions of $\underline{J}$ and $\overline{J}$ in (3.27), for $j \in [\underline{J}, \overline{J}]_{\mathbb{Z}}$ we have $r_j \leqslant \varepsilon\mathbb{r}$ and $|P(t_j) - \mathbb{z}| \wedge |P(t_j) - \mathbb{w}| \geqslant 2\varepsilon\mathbb{r}$. Since $P(t_{j-1}) \in B_{r_j/2}(w_j)$ and $P(s_j) \in \partial B_{\alpha r_j}(w_j)$, we infer that the D_h-geodesic P must cross between the inner and outer boundaries of the annulus $\mathbb{A}_{\alpha r_j, r_j}(w_j)$ at least once before time t_{j-1} and at least once after time s_j. By condition 4 in the definition of $E_{r_j}(w_j)$, there is a path in $\mathbb{A}_{\alpha r_j, r_j}(w_j)$ disconnecting the inner and outer boundaries of this annulus with D_h-length arbitrarily close to $AD_h(\partial B_{\alpha r_j}(w_j), \partial B_{r_j}(w_j))$. The geodesic P must hit this path at least once before time t_{j-1} and at least once after time s_j. Since P is a D_h-geodesic and $P(s_j) \in \partial B_{\alpha r_j}(w_j)$, $P(t_j) \in \partial B_{r_j}(w_j)$, it follows that

$$s_j - t_{j-1} \leqslant AD_h\Big(\partial B_{\alpha r_j}(w_j), \partial B_{r_j}(w_j)\Big) \leqslant A(t_j - s_j).$$

Adding $A(s_j - t_{j-1})$ to both sides of this inequality, then dividing by $A + 1$, gives (3.40). □

Proof of Proposition 3.12. Our above estimates show that if the event $F_{\mathbb{r}}^{\varepsilon}$ of (3.25) occurs, then we have the following string of inequalities:

$$\begin{aligned}
&\widetilde{D}_h(B_{4\varepsilon\mathbb{r}}(\mathbb{z}), B_{4\varepsilon\mathbb{r}}(\mathbb{w})) \\
&\quad\leqslant \sum_{j=\underline{J}+1}^{\overline{J}} \left[\widetilde{D}_h\big(P(t_{j-1}), P(s_j)\big) + \widetilde{D}_h\big(P(s_j), P(t_j)\big)\right] \quad \text{(by (3.28))} \\
&\quad\leqslant \sum_{j=\underline{J}+1}^{\overline{J}} \left[\mathfrak{C}_*(s_j - t_{j-1}) + \mathfrak{C}'(t_j - s_j)\right] \quad \text{(by Lemma 3.16)} \\
&\quad= \sum_{j=\underline{J}+1}^{\overline{J}} \left[\mathfrak{C}'(t_j - t_{j-1}) + (\mathfrak{C}_* - \mathfrak{C}')(s_j - t_{j-1})\right] \\
&\quad\leqslant \left(\mathfrak{C}' + \frac{A}{A+1}(\mathfrak{C}_* - \mathfrak{C}')\right) \sum_{j=\underline{J}+1}^{\overline{J}} (t_j - t_{j-1}) \quad \text{(by Lemma 3.18)} \\
&\quad\leqslant \left(\mathfrak{C}' + \frac{A}{A+1}(\mathfrak{C}_* - \mathfrak{C}')\right) D_h(\mathbb{z}, \mathbb{w}) \quad \text{(since } P \text{ is a } D_h\text{-geodesic)} \\
&\quad\leqslant \mathfrak{C}'' D_h(\mathbb{z}, \mathbb{w}) \quad \text{(by 3.23)).}
\end{aligned} \tag{3.41}$$

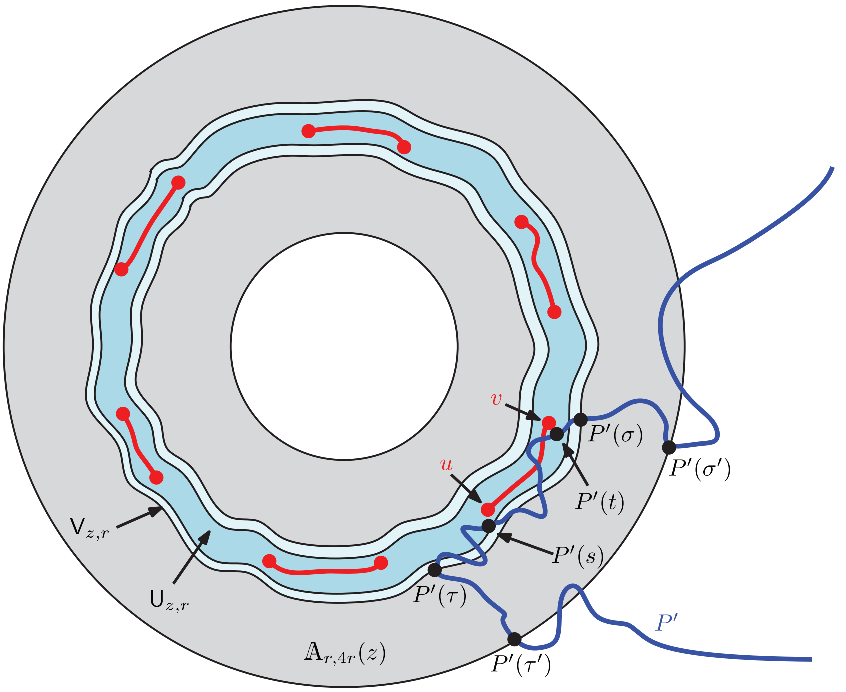

FIGURE 8 Illustration of the objects defined in Subsection 4.1. The bump function $\mathsf{f}_{z,r}$ is supported on $\mathsf{V}_{z,r}$ and identically equal to M on $\mathsf{U}_{z,r}$. The figure shows a $D_{h-\mathsf{f}_{z,r}}$-geodesic P' (blue) and a $(B_{4r}(z), \mathsf{V}_{z,r})$-excursion $(\tau', \tau, \sigma, \sigma')$ for P'. On the event $\mathsf{E}_{z,r}$, there are many 'good' pairs of points $u, v \in \mathsf{U}_{z,r}$ such that $\widetilde{D}_h(u,v) \leqslant \mathfrak{c}' D_h(u,v)$ and there is a $\widetilde{D}_h$-geodesic from u to v which is contained in $\mathsf{U}_{z,r}$ (several such geodesics are shown in red). We obtain hypothesis C for $\mathsf{E}_{z,r}$ by forcing P' to get close to u and v for one such 'good' pair of points.

By (3.25), we have $\mathbb{P}[F_{\mathbb{r}}^{\varepsilon}] \geqslant 1 - \beta/2 - o_\varepsilon(1)$, with the rate of convergence of the $o_\varepsilon(1)$ uniform in the choice of $\mathbb{r}$. Hence, we can choose $\varepsilon_0 = \varepsilon_0(\beta, \mathfrak{C}') > 0$ small enough so that $4\varepsilon_0 \leqslant \beta$ and $\mathbb{P}[F_{\mathbb{r}}^{\varepsilon}] > 1 - \beta$ for each $\varepsilon \in (0, \varepsilon_0]$. By (3.41) and Definition 3.1 of $G_{\mathbb{r}}(\beta, \mathfrak{C}'')$, we see that for $\varepsilon \in (0, \varepsilon_0]$, the condition (B) implies that $\mathbb{P}[G_{\mathbb{r}}(\beta, \mathfrak{C}'')] < \beta$, as required. □

4 | THE CORE ARGUMENT

4.1 | Properties of events and bump functions

In this section, we will assume the existence of events and smooth bump functions which satisfy certain conditions. We will then use these objects to prove Theorem 1.13. The objects will be constructed in Section 5 and are illustrated in Figure 8.

To state the conditions which our events and bump functions need to satisfy, we define the optimal upper and lower bi-Lipschitz constants $\mathfrak{C}_*$ and $\mathfrak{c}_*$ as in Section 3 and we set

$$\mathfrak{c}' := \frac{\mathfrak{c}_* + \mathfrak{C}_*}{2}, \tag{4.1}$$

which belongs to $(\mathfrak{c}_*, \mathfrak{C}_*)$ if $\mathfrak{c}_* < \mathfrak{C}_*$.

We will consider a set of admissible radii $\mathcal{R} \subset (0,1)$ which is required to satisfy

$$r'/r \geqslant 8, \quad \forall r, r' \in \mathcal{R} \quad \text{such that} \quad r' > r. \tag{4.2}$$

The reason for restricting attention to a set of radii as in (4.2) is that in Section 5, we will need to use Proposition 3.10 in order to construct our events.

We also fix a number $\mathbb{p} \in (0,1)$, which we will choose later in a manner depending only on D_h and $\widetilde{D}_h$ (the parameter $\mathbb{p}$ is chosen in Lemma 4.18).

Finally, we fix numbers $\mathsf{M}, \mathsf{a}, \mathsf{A}, \mathsf{K}, \mathsf{b}, \mathsf{c}, \mathsf{L} > 0$, which we require to satisfy the relations

$$\mathsf{A} > \mathsf{a} \quad \text{and} \quad \mathsf{a} - 4e^{-\xi \mathsf{M}}\mathsf{L} > \frac{2\mathsf{A}}{\mathsf{a}}\mathsf{b}. \tag{4.3}$$

We henceforth refer to these numbers as the *parameters*. Most constants in our proofs will be allowed to depend on the parameters. The parameters will be chosen in Section 5, in a manner depending only on $\mathbb{p}$ and the laws of D_h and $\widetilde{D}_h$ (see also Proposition 4.2).

Throughout this section, we will assume that for each $r \in \mathcal{R}$ and each $z \in \mathbb{C}$, we have defined the following objects.

- An event $\mathsf{E}_{z,r} = \mathsf{E}_{z,r}(h)$ such that $\mathsf{E}_{z,r}$ is almost surely determined by $h|_{\overline{\mathbb{A}}_{r,4r}(z)}$, viewed modulo additive constant (recall (2.2)), $\mathbb{P}[\mathsf{E}_{z,r}] \geqslant \mathbb{p}$, and $\mathsf{E}_{z,r}$ satisfies the three hypotheses listed just below.
- Deterministic open sets $\mathsf{U}_{z,r}, \mathsf{V}_{z,r} \subset \mathbb{A}_{r,3r}(z)$, each of which has the topology of an open Euclidean annulus and disconnects the inner and outer boundaries of $\mathbb{A}_{r,3r}(z)$, such that $\overline{\mathsf{U}}_{z,r} \subset \mathsf{V}_{z,r}$ and $\overline{\mathsf{V}}_{z,r} \subset \mathbb{A}_{r,3r}(z)$.
- A deterministic smooth function $\mathsf{f}_{z,r} : \mathbb{C} \to [0, \mathsf{M}]$ such that $\mathsf{f}_{z,r} \equiv \mathsf{M}$ on $\mathsf{U}_{z,r}$ and $\mathsf{f}_{z,r} \equiv 0$ on $\mathbb{C} \setminus \mathsf{V}_{z,r}$.

To state the needed hypotheses for the event $\mathsf{E}_{z,r}$, we make the following definition.

Definition 4.1. Let $P : [0,T] \to \mathbb{C}$ be a path and let $O, V \subset \mathbb{C}$ be open sets with $\overline{V} \subset O$. A (O,V)-*excursion* of P is a 4-tuple of times $(\tau', \tau, \sigma, \sigma')$ such that

$$P(\tau'), P(\sigma') \in \partial O, \quad P((\tau', \sigma')) \subset O,$$

τ is the first time after τ' that P enters $\overline{V}$, and σ is the last time before σ' at which P exits $\overline{V}$.

An (O,V) excursion is illustrated in Figure 8. We assume that on the event $\mathsf{E}_{z,r}$, the following is true.

(A) We have

$$D_h(\mathsf{V}_{z,r}, \partial \mathbb{A}_{r,3r}(z)) \geqslant \mathsf{a} r^{\xi Q} e^{\xi h_r(z)},$$

$$D_h(\text{around } \mathbb{A}_{3r,4r}(z)) \leqslant \mathsf{A} r^{\xi Q} e^{\xi h_r(z)}, \quad \text{and}$$

$$D_h(\text{around } \mathsf{U}_{z,r}) \leqslant \mathsf{L} r^{\xi Q} e^{\xi h_r(z)}.$$

(B) The Radon–Nikodym derivative of the law of $h + \mathsf{f}_{z,r}$ with respect to the law of h, with both distributions viewed modulo additive constant, is bounded above by K and below by $1/\mathsf{K}$.

(C) Let $P' : [0, T] \to \mathbb{C}$ be a $D_{h-\mathsf{f}_{z,r}}$-geodesic between two points which are not in $B_{4r}(z)$, parameterized by its $D_{h-\mathsf{f}_{z,r}}$-length. Assume that (in the terminology of Definition 4.1), there is a $(B_{4r}(z), \mathsf{V}_{z,r})$-excursion $(\tau', \tau, \sigma, \sigma')$ for P' such that

$$D_h\big(P'(\tau), P'(\sigma); B_{4r}(z)\big) \geqslant \mathsf{b} r^{\xi Q} e^{\xi h_r(z)}. \tag{4.4}$$

Then there are times $\tau \leqslant s < t \leqslant \sigma$ such that

$$t - s \geqslant \mathsf{c} r^{\xi Q} e^{\xi h_r(z)} \quad \text{and} \quad \widetilde{D}_{h-\mathsf{f}_{z,r}}\big(P'(s), P'(t); B_{4r}(z)\big) \leqslant \mathfrak{c}'(t - s). \tag{4.5}$$

Constructing objects which satisfy the above conditions (especially hypothesis C) will require a lot of work. The proof of the following proposition will occupy all of Section 5.

Proposition 4.2. *Assume that $\mathfrak{c}_* < \mathfrak{C}_*$. For each $\mathrm{p} \in (0,1)$, there exist $\mathfrak{c}'' \in (\mathfrak{c}_*, \mathfrak{c}')$ and a set of radii $\mathcal{R}$ as in (4.2), depending only on p and the laws of D_h and $\widetilde{D}_h$, with the following properties.*

- *There is a choice of parameters depending only on p and the laws of D_h and $\widetilde{D}_h$, such that for each $r \in \mathcal{R}$ and each $z \in \mathbb{C}$, there exist an event $\mathsf{E}_{z,r}$, open sets $\mathsf{U}_{z,r}, \mathsf{V}_{z,r}$, and a function $\mathsf{f}_{z,r}$ satisfying the above hypotheses.*
- *For each $\widetilde{\beta} > 0$, there exists $\varepsilon_0 > 0$, depending only on p, $\widetilde{\beta}$, and the laws of D_h and $\widetilde{D}_h$, such that the following holds for each $\varepsilon \in (0, \varepsilon_0]$. If $\mathrm{r} > 0$ and that the event of Definition 3.8 satisfies $\mathbb{P}[\widetilde{G}_{\mathrm{r}}(\widetilde{\beta}, \mathfrak{c}'')] \geqslant \widetilde{\beta}$, then the cardinality of $\mathcal{R} \cap [\varepsilon^2 \mathrm{r}, \varepsilon \mathrm{r}]$ is at least $\frac{5}{8} \log_8 \varepsilon^{-1}$.*

The proof of Proposition 4.2 in Section 5 will be via an intricate explicit construction. To give the reader some intuition, we will now explain roughly what is involved in this construction, without any quantitative estimates. The reader may want to look at Figure 8 while reading the explanation.

The set $\mathsf{U}_{z,r}$ where $\mathsf{f}_{z,r}$ attains its maximal possible value will be a long narrow 'tube' which disconnects the inner and outer boundaries of $\mathbb{A}_{r,3r}(z)$ and is contained in a small Euclidean neighborhood of $\partial B_{2r}(z)$. The set $\mathsf{V}_{z,r}$ where $\mathsf{f}_{z,r}$ is supported will be a slightly larger tube containing $\mathsf{U}_{z,r}$. The event $\mathsf{E}_{z,r}$ corresponds, roughly speaking, to the event that there are many 'good' pairs of non-singular points $u, v \in \mathsf{U}_{z,r}$ with the following properties (plus a long list of regularity conditions).

- $\widetilde{D}_h(u, v) \leqslant \mathfrak{c}_0' D_h(u, v)$, where $\mathfrak{c}_0' \in (\mathfrak{c}_*, \mathfrak{c}')$ is fixed.
- $|u - v|$ is bounded below by a constant times r.
- There is a $\widetilde{D}_h$-geodesic from u to v which is contained in $\mathsf{U}_{z,r}$.

Hypotheses A and B for $\mathsf{E}_{z,r}$ will be immediate consequences of the regularity conditions in the definition of $\mathsf{E}_{z,r}$. Hypothesis C will be obtained as follows. Suppose that P' is a $D_{h-\mathsf{f}_{z,r}}$-geodesic as in hypothesis C. Since the bump function $\mathsf{f}_{z,r}$ is very large on $\mathsf{U}_{z,r}$, we infer that if $x, y \in \mathsf{V}_{z,r}$, then the $D_{h-\mathsf{f}_{z,r}}$-length of any path between x and y which spends a lot of time outside of $\mathsf{U}_{z,r}$ is much greater than the $D_{h-\mathsf{f}_{z,r}}$-length of a path between x and y which spends most of its time in $\mathsf{U}_{z,r}$. By applying this with $x = P'(\tau)$ and $y = P'(\sigma)$, we find that $P'|_{[\tau,\sigma]}$ has to spend most of its time in $\mathsf{U}_{z,r}$.

This will allow us to find a 'good' pair of points $u, v \in \mathsf{U}_{z,r}$ as above such that $P'|_{[\tau,\sigma]}$ gets very $D_{h-\mathsf{f}_{z,r}}$-close to each of u and v. Since the $\widetilde{D}_h$-geodesic between u and v is contained in $\mathsf{U}_{z,r}$ and $\mathsf{f}_{z,r}$

attains its maximal possible value on $\mathsf{U}_{z,r}$, subtracting $\mathsf{f}_{z,r}$ from h reduces $\widetilde{D}_h(u,v)$ by at least as much as $D_h(u,v)$. Consequently, one has $\widetilde{D}_{h-\mathsf{f}_{z,r}}(u,v) \leqslant \mathfrak{c}_0' D_{h-\mathsf{f}_{z,r}}(u,v)$. We will then obtain (4.5) by choosing s and t such that $P'(s)$ and $P'(t)$ are close to u and v, respectively, and applying the triangle inequality.

To produce lots of 'good' pairs of points $u,v \in \mathsf{U}_{z,r}$, we will apply Proposition 3.10 together with a local independence argument based on Lemma 2.3 (to upgrade from a single pair of points with positive probability to many pairs of points with high probability). This application of Proposition 3.10 is the reason why we need to assume that $\mathbb{P}[\widetilde{G}_{\mathbb{r}}(\widetilde{\beta},\mathfrak{c}'')] \geqslant \widetilde{\beta}$ in the second part of Proposition 4.2; and why we need to restrict to a set of admissible radii $\mathcal{R}$, instead of defining our events for every $r > 0$.

4.2 | Estimate for ratios of D_h and $\widetilde{D}_h$ distances

We now state the main estimate which we will prove using the events $\mathsf{E}_{z,r}$. In particular, we will show that the probability of a certain 'bad' event, which we now define, is small. For $\mathbb{r} > 0$, $\varepsilon > 0$, and disjoint compact sets $K_1, K_2 \subset B_{2\mathbb{r}}(0)$, let $\mathcal{G}_{\mathbb{r}}^{\varepsilon} = \mathcal{G}_{\mathbb{r}}^{\varepsilon}(K_1,K_2)$ be the event that the following is true.

(1) $\widetilde{D}_h(K_1,K_2) \geqslant \mathfrak{C}_* D_h(K_1,K_2) - \frac{1}{2}\varepsilon^{2\xi(Q+3)}\mathbb{r}^{\xi Q} e^{\xi h_{\mathbb{r}}(0)}$.
(2) For each $z \in B_{3\mathbb{r}}(0)$ and each $r \in [\varepsilon^2\mathbb{r}, \varepsilon\mathbb{r}] \cap \mathcal{R}$, we have

$$r^{\xi Q} e^{\xi h_r(z)} \in \left[\varepsilon^{2\xi(Q+3)}\mathbb{r}^{\xi Q} e^{\xi h_{\mathbb{r}}(0)}, \varepsilon^{\xi(Q-3)}\mathbb{r}^{\xi Q} e^{\xi h_{\mathbb{r}}(0)}\right].$$

(3) For each $z \in B_{3\mathbb{r}}(0)$, there exists $r \in \mathcal{R} \cap [\varepsilon^2\mathbb{r}, \varepsilon\mathbb{r}]$ and $w \in (\frac{r}{100}\mathbb{Z}^2) \cap B_{r/25}(z)$ such that $\mathsf{E}_{w,r}$ occurs.

The most important condition in the definition of $\mathcal{G}_{\mathbb{r}}^{\varepsilon}$ is condition 1. We want to show that if $\mathfrak{c}_* < \mathfrak{C}_*$, then this condition is extremely unlikely. The motivation for this is that it will eventually be used in Subsection 4.5 to derive a contradiction to Proposition 3.5. Indeed, Proposition 3.5 gives a *lower* bound for the probability that there exist points $u,v \in \overline{B}_{\mathbb{r}}(0)$ satisfying certain conditions such that $\widetilde{D}_h(u,v)$ is 'close' to $\mathfrak{C}_* D_h(u,v)$. We will show that this lower bound is incompatible with our upper bound for the probability of condition 1 in the definition of $\mathcal{G}_{\mathbb{r}}^{\varepsilon}$.

Conditions 2 and 3 in the definition of $\mathcal{G}_{\mathbb{r}}^{\varepsilon}$ are global regularity conditions. We will show in Lemma 4.18 that Proposition 4.2 implies that these two conditions occur with high probability. This, in turn, means that an upper bound for $\mathbb{P}[\mathcal{G}_{\mathbb{r}}^{\varepsilon}]$ implies an upper bound for the probability of condition 1. The next three subsections are devoted to the proof of the following proposition.

Proposition 4.3. *Assume that $\mathfrak{c}_* < \mathfrak{C}_*$ and we have constructed a set of admissible radii $\mathcal{R}$ as in (4.2) and events $\mathsf{E}_{z,r}$, sets $\mathsf{U}_{z,r}$ and $\mathsf{V}_{z,r}$, and bump functions $\mathsf{f}_{z,r}$ for $z \in \mathbb{C}$ and $r \in \mathcal{R}$ which satisfy the conditions of Subsection 4.1. Let $\eta \in (0,1)$ and $\mathbb{r} > 0$. Also let $K_1, K_2 \subset B_{2\mathbb{r}}(0)$ be disjoint compact sets such that* $\operatorname{dist}(K_1,K_2) \geqslant \eta\mathbb{r}$ *and* $\operatorname{dist}(K_1,\partial B_{\mathbb{r}}(0)) \geqslant \eta\mathbb{r}$, *where* dist *denotes Euclidean distance.*[†]

[†] The reason why we require that $\operatorname{dist}(K_1,\partial B_{\mathbb{r}}(0)) \geqslant \eta\mathbb{r}$ in Proposition 4.3 is as follows. Our events involve the circle average $h_{\mathbb{r}}(0)$. We only want to add to or subtract from h functions of the form $\mathsf{f}_{z,r}$ whose supports are disjoint from $\partial B_{\mathbb{r}}(0)$, so that adding or subtracting $\mathsf{f}_{z,r}$ does not change $h_{\mathbb{r}}(0)$. The condition that $\operatorname{dist}(K_1,\partial B_{\mathbb{r}}(0)) \geqslant \eta\mathbb{r}$ ensures that there is a segment of the D_h-geodesic from K_1 to K_2 of Euclidean length at least $\eta\mathbb{r}$ which is disjoint from $\partial B_{\mathbb{r}}(0)$. We will eventually

Then

$$\mathbb{P}\left[\mathcal{G}_{\mathbb{r}}^{\varepsilon}(K_1, K_2)\right] = O_{\varepsilon}(\varepsilon^{\mu}), \quad \forall \mu > 0 \tag{4.6}$$

with the implicit constant in the $O_\varepsilon(\cdot)$ depending only on μ, η, and the parameters (not on $\mathbb{r}, K_1, K_2$).

It is crucial for our purposes that the implicit constant in the $O_\varepsilon(\cdot)$ in (4.6) does not depend on $\mathbb{r}, K_1, K_2$. This is because we will eventually take K_1 and K_2 to be Euclidean balls whose radii are a power of ε times $\mathbb{r}$ (see Lemma 4.19). Proposition 4.2 is not needed for the proof of Proposition 4.3. Rather, all we need is the statement that $\mathsf{E}_{z,r}, \mathsf{U}_{z,r}, \mathsf{V}_{z,r}$, and $\mathsf{f}_{z,r}$ exist and satisfy the required properties for each $r \in \mathcal{R}$ (we do not care how large $\mathcal{R}$ is). Proposition 4.2 is just needed to check that the auxiliary condition 3 in the definition $\mathcal{G}_{\mathbb{r}}^{\varepsilon}$ occurs with high probability.

We will now explain how to prove Proposition 4.3 conditional on two propositions (Propositions 4.5 and 4.6) whose proofs will occupy most of this section. The proof will be based on counting the number of events of a certain type which occur. Let us now define these events.

Assume that $\mathfrak{c}_* < \mathfrak{C}_*$. Also fix $\mathbb{r} > 0$ and disjoint compact sets $K_1, K_2 \subset B_{2\mathbb{r}}(0)$. For $r \in \mathcal{R}$ (which we will eventually take to be much smaller than $\mathbb{r}$), let $\mathcal{Z}_r = \mathcal{Z}_r^{\mathbb{r}}(K_1, K_2)$ be the set of non-empty subsets $Z \subset \frac{r}{100}\mathbb{Z}^2$ such that[‡]

$$B_{4r}(z) \cap B_{4r}(z') = \emptyset \quad \text{and} \quad B_{4r}(z) \cap (K_1 \cup K_2 \cup \partial B_{\mathbb{r}}(0)) = \emptyset, \quad \forall \text{ distinct } z, z' \in Z. \tag{4.7}$$

For a set $Z \in \mathcal{Z}_r$, we define

$$\mathsf{f}_{Z,r} = \sum_{z \in Z} \mathsf{f}_{z,r}.$$

By Lemma 2.7, almost surely there is a unique D_h-geodesic from K_1 to K_2. Since the laws of h and $h - \mathsf{f}_{Z,r}$ are mutually absolutely continuous [34, Proposition 3.4], for each $r \in \mathcal{R}$ and each $Z \in \mathcal{Z}_r$, almost surely there is a unique $D_{h-\mathsf{f}_{Z,r}}$-geodesic from K_1 to K_2. Hence, the following definition makes sense. For $Z \in \mathcal{Z}_r$ and $q > 0$ we define $F_{Z,r}^{q,\mathbb{r}} = F_{Z,r}^{q,\mathbb{r}}(h; K_1, K_2)$ to be the event that the following is true.

(1) $\widetilde{D}_h(K_1, K_2) \geqslant \mathfrak{C}_* D_h(K_1, K_2) - q\mathbb{r}^{\xi Q} e^{\xi h_{\mathbb{r}}(0)}$.
(2) The event $\mathsf{E}_{z,r}(h)$ occurs for each $z \in Z$.
(3) We have

$$r^{\xi Q} e^{\xi h_r(z)} \in \left[q\mathbb{r}^{\xi Q} e^{\xi h_{\mathbb{r}}(0)}, 2q\mathbb{r}^{\xi Q} e^{\xi h_{\mathbb{r}}(0)}\right], \quad \forall z \in Z.$$

(4) For each $z \in Z$, the D_h-geodesic from K_1 to K_2 hits $B_r(z)$.

choose to subtract functions $\mathsf{f}_{z,r}$ whose supports are close to such a segment, see the proof of Proposition 4.5 at the end of Subsection 4.3.

[‡] The reason why we require that $B_{4r}(z) \cap \partial B_{\mathbb{r}}(0) = \emptyset$ in (4.7) is to ensure that adding or subtracting the function $\mathsf{f}_{z,r}$ for $z \in Z$ (which is supported on $B_{4r}(z)$) does not change the circle average $h_{\mathbb{r}}(0)$ (cf. Footnote). This fact is used in the proof of Lemma 4.15.

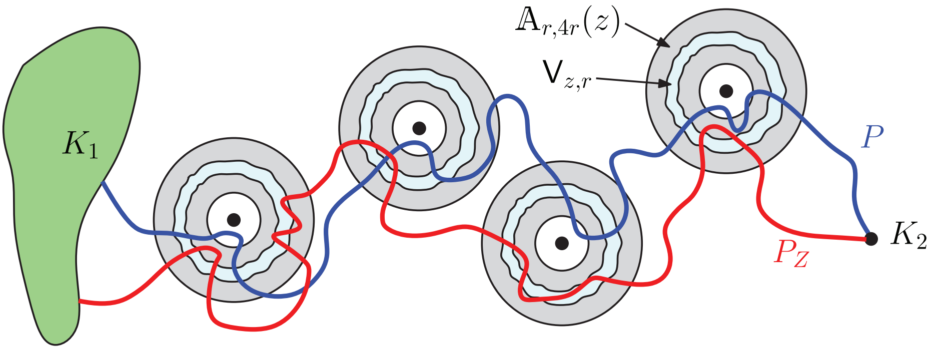

FIGURE 9 Illustration of the definition of $F^{q,\mathbb{r}}_{Z,r}$. Here, we have shown K_1 as a non-singleton set and K_2 as a point, but K_1 and K_2 can be any disjoint compact sets. The set Z consists of the four center points of the annuli in the figure. For each of these points, we have shown the set $\mathsf{V}_{z,r}$ (that is, the support of $\mathsf{f}_{z,r}$) in light blue and the annulus $\mathbb{A}_{r,4r}(z)$ in gray. On $F^{q,\mathbb{r}}_{Z,r}$, the D_h-geodesic from K_1 to K_2 (blue) hits each of the balls $B_r(z)$ for $z \in Z$. Moreover, the $D_{h-\mathsf{f}_{Z,r}}$-geodesic from K_1 to K_2 (red) has a 'large' $(B_{4r}(z), \mathsf{V}_{z,r})$-excursion for each $z \in Z$.

(5) For each $z \in Z$, the $D_{h-\mathsf{f}_{Z,r}}$-geodesic P_Z from K_1 to K_2 has a $(B_{4r}(z), \mathsf{V}_{z,r})$-excursion $(\tau_z', \tau_z, \sigma_z, \sigma_z')$ such that

$$D_h(P_Z(\tau_z), P_Z(\sigma_z); B_{4r}(z)) \geqslant \mathsf{b} r^{\xi Q} e^{\xi h_r(z)}.$$

See Figure 9 for an illustration of the definition. Condition 1 for $F^{q,\mathbb{r}}_{Z,r}$ is closely related to the main condition 1 in the definition of $\mathcal{G}^{\varepsilon}_{\mathbb{r}}$. The purpose of conditions 2 and 4 is to allow us to apply our hypotheses for $\mathsf{E}_{z,r}$ to study D_h-distances on the event $F^{q,\mathbb{r}}_{Z,r}$. Condition 3 provides up-to-constants comparisons of the 'LQG sizes' of different balls $B_r(z)$ for $z \in Z$. Finally, condition 5 will enable us to apply hypothesis C for $\mathsf{E}_{z,r}$ to each $z \in Z$.

Proposition 4.3 will turn out to be a straightforward consequence of three estimates for the events $F^{q,\mathbb{r}}_{z,r}$, which we now state. Our first estimate follows from a standard formula for the Radon–Nikodym derivative between the laws of h and $h + \mathsf{f}_{Z,r}$.

Lemma 4.4. *For $r \in \mathcal{R}$, $Z \in \mathcal{Z}_r$, and $q > 0$, let $F^{q,\mathbb{r}}_{Z,r}(h + \mathsf{f}_{Z,r})$ be the event $F^{q,\mathbb{r}}_{Z,r}(h)$ defined with $h + \mathsf{f}_{Z,r}$ in place of h. For each $Z \subset \mathcal{Z}_r$,*

$$\mathsf{K}^{-\#Z}\mathbb{P}\left[F^{q,\mathbb{r}}_{Z,r}(h)\right] \leqslant \mathbb{P}\left[F^{q,\mathbb{r}}_{Z,r}(h + \mathsf{f}_{Z,r})\right] \leqslant \mathsf{K}^{\#Z}\mathbb{P}\left[F^{q,\mathbb{r}}_{Z,r}(h)\right]. \tag{4.8}$$

Proof. By Weyl scaling (Axiom III) and the fact that $\mathsf{E}_{z,r}(h)$ is almost surely determined by h, viewed modulo additive constant, we get that the event $F^{q,\mathbb{r}}_{Z,r}(h)$ is almost surely determined by h, viewed modulo additive constant. By a standard calculation for the GFF (see, for example, the proof of [34, Proposition 3.4]), the Radon–Nikodym derivative of the law of $h + \mathsf{f}_{Z,r}$ with respect to the law of h, with both distributions viewed modulo additive constant, is equal to

$$\exp\left((h, \mathsf{f}_{Z,r})_\nabla - \frac{1}{2}(\mathsf{f}_{Z,r}, \mathsf{f}_{Z,r})_\nabla\right),$$

where $(f,g)_\nabla = \int_{\mathbb{C}} \nabla f(z)\cdot\nabla g(z)\,d^2z$ denotes the Dirichlet inner product. Recall that each $\mathsf{f}_{z,r}$ for $z\in Z$ is supported on the annulus $\mathbb{A}_{r,4r}(z)$. Since $Z\in\mathcal{Z}_r$, the definition (4.7) shows that the balls $B_{4r}(z)$ for $z\in Z$ are disjoint. Hence, the random variables $(h,\mathsf{f}_{Z,r})_\nabla$ are independent, so the above Radon–Nikodym derivative factors as the product

$$\prod_{z\in Z}\exp\Big((h,\mathsf{f}_{z,r})_\nabla - \frac{1}{2}(\mathsf{f}_{z,r},\mathsf{f}_{z,r})_\nabla\Big). \tag{4.9}$$

By condition 2 in the definition of $F^{q,\mathbb{r}}_{Z,r}(h)$, on this event $\mathsf{E}_{z,r}(h)$ occurs for each $z\in Z$. Consequently, hypothesis B for $\mathsf{E}_{z,r}(h)$ shows that on $F^{q,\mathbb{r}}_{z,r}(h)$, each of the factors in the product (4.9) is bounded above by K and below by K^{-1}. This implies (4.8). □

Our next estimate tells us that on $\mathcal{G}^\varepsilon_{\mathbb{r}}$, there are many choices of Z for which $F^{q,\mathbb{r}}_{Z,r}(h)$ occurs.

Proposition 4.5. *There exists $c_1>0$, depending only on the parameters and η, such that for each $k\in\mathbb{N}$, there exists $\varepsilon_*>0$, depending only on k, the parameters, and η, such that the following is true for each $\mathbb{r}>0$ and each $\varepsilon\in(0,\varepsilon_*]$. Assume that $\mathrm{dist}(K_1,K_2)\geqslant\eta\mathbb{r}$ and $\mathrm{dist}(K_1,\partial B_{\mathbb{r}}(0))\geqslant\eta\mathbb{r}$. If $\mathcal{G}^\varepsilon_{\mathbb{r}}(K_1,K_2)$ occurs, then there exists a random $r\in[\varepsilon^2\mathbb{r},\varepsilon\mathbb{r}]$ and a random $q\in[\frac{1}{2}\varepsilon^{2\xi(Q+3)},\varepsilon^{\xi(Q-3)}]\cap\{2^{-l}\}_{l\in\mathbb{N}}$ such that*

$$\#\Big\{Z\in\mathcal{Z}_r : \#Z\leqslant k \text{ and } F^{q,\mathbb{r}}_{Z,r}(h)\text{ occurs}\Big\}\geqslant\varepsilon^{-c_1k}. \tag{4.10}$$

Proposition 4.5 will be proven in Subsection 4.3. Our final estimate gives an unconditional upper bound for the number of Z for which $F^{q,\mathbb{r}}_{Z,r}(h+\mathsf{f}_{Z,r})$ occurs.

Proposition 4.6. *There is a constant $C_2>0$, depending only on the parameters, such that the following is true. For each $r\in\mathcal{R}$, each $q>0$, and each $k\in\mathbb{N}$, almost surely*

$$\#\Big\{Z\in\mathcal{Z}_r \ : \ \#Z\leqslant k \text{ and } F^{q,\mathbb{r}}_{Z,r}(h+\mathsf{f}_{Z,r})\text{ occurs}\Big\}\leqslant C_2^k. \tag{4.11}$$

We will give the proof of Proposition 4.6 in Subsection 4.4. The proofs of Propositions 4.5 and 4.6 are both via elementary deterministic arguments based on the hypotheses for $\mathsf{E}_{z,r}$ and the definition of $\mathsf{F}^{q,\mathbb{r}}_{Z,r}$. See the beginnings of Subsections 4.3 and 4.4 for overviews of the proofs.

Let us now explain how to deduce Proposition 4.3 from the above three estimates.

Proof of Proposition 4.3. Throughout the proof, all implicit constants are required to depend only on ξ and the parameters. Fix $\mathbb{r}>0$ and disjoint compact sets $K_1,K_2\subset B_{2\mathbb{r}}(0)$ such that $\mathrm{dist}(K_1,K_2)\geqslant\eta\mathbb{r}$ and $\mathrm{dist}(K_1,\partial B_{\mathbb{r}}(0))\geqslant\eta\mathbb{r}$. For $\varepsilon>0$, let

$$\mathbf{R}_\varepsilon := \mathcal{R}\cap[\varepsilon^2\mathbb{r},\varepsilon\mathbb{r}] \quad\text{and}\quad \mathbf{Q}_\varepsilon := \Big[\frac{1}{2}\varepsilon^{2\xi(Q+3)},\varepsilon^{\xi(Q-3)}\Big]\cap\{2^{-l}\}_{l\in\mathbb{N}}.$$

The cardinality of $\mathbf{R}_\varepsilon\times\mathbf{Q}_\varepsilon$ is at most a ξ-dependent constant times $(\log\varepsilon^{-1})^2$. By interchanging the order of summation and expectation, then applying Proposition 4.6 and Lemma 4.4, we get that for each $k \in \mathbb{N}$,

$$\begin{aligned}
(\log \varepsilon^{-1})^2 &\geq \sum_{r \in \mathbf{R}_\varepsilon} \sum_{q \in \mathbf{Q}_\varepsilon} \sum_{\substack{Z \in \mathcal{Z}_r \\ \#Z \leqslant k}} \mathbb{E}\left[\frac{\mathbb{1}_{F_{Z,r}^{q,\mathbb{r}}(h + \mathsf{f}_{Z,r})}}{\#\{Z' \in \mathcal{Z}_r \,:\, \#Z' \leqslant k,\ F_{Z',r}^{q,\mathbb{r}}(h + \mathsf{f}_{Z',r}) \text{ occurs}\}} \right] \\
&\geq C_2^{-k} \sum_{r \in \mathsf{R}_\varepsilon} \sum_{q \in \mathsf{Q}_\varepsilon} \sum_{\substack{Z \in \mathcal{Z}_r \\ \#Z \leqslant k}} \mathbb{P}\left[F_{Z,r}^{q,\mathbb{r}}(h + \mathsf{f}_{Z,r}) \right] \qquad \text{(Proposition 4.6)} \\
&\geq C_2^{-k} \mathsf{K}^{-k} \sum_{r \in \mathsf{R}_\varepsilon} \sum_{q \in \mathsf{Q}_\varepsilon} \sum_{\substack{Z \in \mathcal{Z}_r \\ \#Z \leqslant k}} \mathbb{P}\left[F_{Z,r}^{q,\mathbb{r}}(h) \right] \qquad \text{(Lemma 4.4)} \\
&= C_2^{-k} \mathsf{K}^{-k} \mathbb{E}\left[\sum_{r \in \mathsf{R}_\varepsilon} \sum_{q \in \mathsf{Q}_\varepsilon} \#\left\{ Z \in \mathcal{Z}_r \,:\, \#Z \leqslant k,\ F_{Z,r}^{q,\mathbb{r}}(h) \text{ occurs} \right\} \right]. \qquad (4.12)
\end{aligned}$$

By Proposition 4.5, for each small enough $\varepsilon > 0$ (how small depends on k) on the event $\mathcal{G}_{\mathbb{r}}^{\varepsilon}(K_1, K_2)$ the double sum inside the expectation in the last line of (4.12) is at least $\varepsilon^{-c_1 k}$. Hence, for each small enough $\varepsilon > 0$ (depending on k),

$$(\log \varepsilon^{-1})^2 \geq C_2^{-k} \mathsf{K}^{-k} \varepsilon^{-c_1 k} \mathbb{P}\left[\mathcal{G}_{\mathbb{r}}^{\varepsilon}(K_1, K_2) \right]. \qquad (4.13)$$

Re-arranging this inequality and choosing k to be slightly larger than μ / c_1 yields (4.6). □

4.3 | Proof of Proposition 4.5

Fix $\mathbb{r} > 0$ and compact sets $K_1, K_2 \subset B_{\mathbb{r}}(0)$ such that $\operatorname{dist}(K_1, K_2) \geqslant \eta \mathbb{r}$ and $\operatorname{dist}(K_1, \partial B_{\mathbb{r}}(0)) \geqslant \eta \mathbb{r}$. It is straightforward to show from the definition of $\mathcal{G}_{\mathbb{r}}^{\varepsilon}$ that if $\mathcal{G}_{\mathbb{r}}^{\varepsilon}$ occurs, then there are many 3-tuples (Z, r, q) with $r \in \mathcal{R} \cap [\varepsilon \mathbb{r}, \varepsilon^2 \mathbb{r}]$, $q \in [\varepsilon^{2\xi(Q+3)}/2, \varepsilon^{\xi(Q-3)}] \cap \{2^{-l}\}_{l \in \mathbb{N}}$, and $Z \in \mathcal{Z}_r$ for which all of the conditions in the definition of $F_{Z,r}^{q,\mathbb{r}}$ occur except possibly condition 5, that is, the event of the following definition occurs.

Definition 4.7. For $r \in \mathcal{R}$, $Z \in \mathcal{Z}_r$, and $q > 0$, we define $\overline{F}_{Z,r}^{q,\mathbb{r}}(h) = \overline{F}_{Z,r}^{q,\mathbb{r}}(h; K_1, K_2)$ to be the event that all of the conditions in the definition of $F_{Z,r}^{q,\mathbb{r}}(h)$ occur except possibly condition 5, that is, $\overline{F}_{Z,r}^{q,\mathbb{r}}(h)$ is the event that the following is true.

(1) $\widetilde{D}_h(K_1, K_2) \geqslant \mathfrak{C}_* D_h(K_1, K_2) - q \mathbb{r}^{\xi Q} e^{\xi h_{\mathbb{r}}(0)}$.
(2) The event $\mathsf{E}_{z,r}$ occurs for each $z \in Z$.
(3) We have

$$r^{\xi Q} e^{\xi h_r(z)} \in \left[q \mathbb{r}^{\xi Q} e^{\xi h_{\mathbb{r}}(0)}, 2 q \mathbb{r}^{\xi Q} e^{\xi h_{\mathbb{r}}(0)} \right], \quad \forall z \in Z.$$

(4) For each $z \in Z$, the D_h-geodesic from K_1 to K_2 hits $B_r(z)$.

Recall that condition 5 asserts that for each $z \in Z$, the $D_{h-\mathsf{f}_{Z,r}}$-geodesic P_Z from K_1 to K_2 has a $(B_{4r}(z), \mathsf{V}_{z,r})$-excursion $(\tau_z', \tau_z, \sigma_z, \sigma_z')$ such that $D_h(P_Z(\tau_z), P_Z(\sigma_z); B_{4r}(z)) \geqslant \mathsf{b} r^{\xi Q} e^{\xi h_r(z)}$. The difficulty with checking condition 5 is that the $D_{h-\mathsf{f}_{Z,r}}$-geodesic from K_1 to K_2 could potentially spend a very small amount of time in $\mathsf{V}_{z,r}$ for some of the points $z \in Z$, or possibly even avoid some of the sets $\mathsf{V}_{z,r}$ altogether. To deal with this, we will show that if $Z \in \mathcal{Z}_r$ and $\overline{F}_{Z,r}^{q,\mathbb{r}}$ occurs, then there is a subset $Z' \subset Z$ such that $\#Z'$ is at least a constant times $\#Z$ and $F_{Z',r}^{q,\mathbb{r}}$ occurs (Lemma 4.13).

The idea for constructing Z' is as follows. In Lemma 4.8 we show that $D_{h-\mathsf{f}_{Z,r}}(K_1, K_2)$ is smaller than $D_h(K_1, K_2)$ minus a constant times $q\mathbb{r}^{\xi Q} e^{\xi h_{\mathbb{r}}(0)} \#Z$. Intuitively, subtracting $\mathsf{f}_{Z,r}$ substantially reduces the distance from K_1 to K_2. Since $\mathsf{f}_{Z,r}$ is supported on $\bigcup_{z\in Z} \mathsf{V}_{z,r}$, this implies that the $D_{h-\mathsf{f}_{Z,r}}$-geodesic P_Z from K_1 to K_2 has to spend at least a constant times $q\mathbb{r}^{\xi Q} e^{\xi h_{\mathbb{r}}(0)} \#Z$ units of time in $\bigcup_{z\in Z} \mathsf{V}_{z,r}$ (otherwise, its length would have to be larger than $D_{h-\mathsf{f}_{Z,r}}(K_1, K_2)$). We then iteratively remove the 'bad' points $z \in Z$ for which there does *not* exist a $(B_{4r}(z), \mathsf{V}_{z,r})$-excursion $(\tau_z', \tau_z, \sigma_z, \sigma_z')$ for P_Z such that

$$D_h(P_Z(\tau_z), P_Z(\sigma_z)) \geqslant \mathsf{b} r^{\xi Q} e^{\xi h_r(z)}.$$

For each of the above 'bad' points $z \in Z$, the intersection of P_Z with $\mathsf{V}_{z,r}$ is in some sense small. Since the function $\mathsf{f}_{z,r}$ is supported on $V_{z,r}$, removing the 'bad' points from Z does not increase $D_{h-\mathsf{f}_{Z,r}}(K_1, K_2)$ by very much. Consequently, at each stage of the iterative procedure it will still be the case that $D_{h-\mathsf{f}_{Z,r}}(K_1, K_2)$ is substantially smaller than $D_h(K_1, K_2)$. As above, this implies that P_Z spends a substantial amount of time in $\bigcup_{z\in Z} \mathsf{V}_{z,r}$. We show in Lemma 4.12 that the amount of time that P_Z spends in each $\mathsf{V}_{z,r}$ is at most a constant times $q\mathbb{r}^{\xi Q} e^{\xi h_{\mathbb{r}}(0)}$. This allows us to show that the iterative procedure has to terminate before we have removed too many points from Z.

To begin the proof, we establish an upper bound for $D_{h-\mathsf{f}_{Z,r}}(K_1, K_2)$ in terms of $D_h(K_1, K_2)$ on the event $\overline{F}_{Z,r}^{q,\mathbb{r}}(h)$. The reason why this bound holds is that the D_h-geodesic from K_1 to K_2 has to cross the regions $\mathsf{U}_{z,r}$ for $z \in Z$. Since $\mathsf{f}_{Z,r}$ is very large on $\mathsf{U}_{z,r}$ and by hypothesis A for $\mathsf{E}_{z,r}$, the $D_{h-\mathsf{f}_{Z,r}}$-distances around the regions $\mathsf{U}_{z,r}$ for $z \in Z$ is small. This allows us to find $\#Z$ 'shortcuts' along the D_h-geodesic with small $D_{h-\mathsf{f}_{Z,r}}$-length.

Lemma 4.8. *There is a constant $C_3 > 2\mathsf{A}\mathsf{b}/\mathsf{a}$, depending only on the parameters, such that the following is true. Let $r \in \mathcal{R}$, $Z \subset \mathcal{Z}_r$, and $q > 0$ and assume that $\overline{F}_{Z,r}^{q,\mathbb{r}}(h)$ occurs. Then*

$$D_{h-\mathsf{f}_{Z,r}}(K_1, K_2) \leqslant D_h(K_1, K_2) - C_3 q\mathbb{r}^{\xi Q} e^{\xi h_{\mathbb{r}}(0)} \#Z. \tag{4.14}$$

Proof. See Figure 10 for an illustration. By condition 2 in the definition of $\overline{F}_{Z,r}^{q,\mathbb{r}}(h)$, the event $\mathsf{E}_{z,r}(h)$ occurs for each $z \in Z$. So, by hypothesis A for $\mathsf{E}_{z,r}$ and condition 3 in the definition of $\overline{F}_{Z,r}^{q,\mathbb{r}}(h)$, we can find for each $z \in Z$ a path π_z in $\mathsf{U}_{z,r}$ which disconnects the inner and outer boundaries of $\mathsf{U}_{z,r}$ such that

$$\mathrm{len}(\pi_z; D_h) \leqslant 2D_h\left(\text{around } \mathsf{U}_{z,r}\right) \leqslant 4\mathsf{L} q\mathbb{r}^{\xi Q} e^{\xi h_{\mathbb{r}}(0)}. \tag{4.15}$$

By condition 4 in the definition of $\overline{F}_{Z,r}^{q,\mathbb{r}}(h)$, the D_h-geodesic P from K_1 to K_2 hits $B_r(z)$ for each $z \in Z$. Furthermore, $B_{4r}(z) \cap (K_1 \cup K_2) = \emptyset$ for each $z \in Z$ (recall (4.7)) and π_z disconnects the inner and outer boundaries of $\mathbb{A}_{r,4r}(z)$ for each $z \in Z$. It follows that for each $z \in Z$, we can find times $s_z < t_z$ such that $P(s_z), P(t_z) \in \pi_z$, the path $P|_{[s_z,t_z]}$ hits $B_r(z)$, and $P((s_z, t_z))$ lies in the open

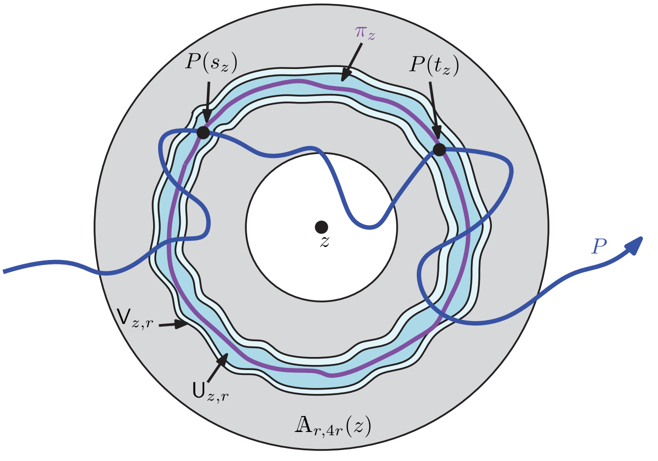

FIGURE 10 Illustration of the proof of Lemma 4.8. Since $\mathsf{f}_{z,r}$ is very large on $\mathsf{U}_{z,r}$, the $D_{h-\mathsf{f}_{Z,r}}$-length of the purple path π_z is very short. By replacing the segment $P|_{[s_z,t_z]}$ by a segment of π_z for each $z \in Z$, we obtain a new path from K_1 to K_2 whose $D_{h-\mathsf{f}_{Z,r}}$-length is substantially smaller than $D_h(K_1, K_2)$.

region which is disconnected from ∞ by π_z. Since the balls $B_{4r}(z)$ for $z \in Z$ are disjoint (again by (4.7)), the time intervals $[s_z, t_z]$ for $z \in Z$ are disjoint.

The path P must cross from $\mathsf{V}_{z,r}$ to $\partial B_r(z)$ between times s_z and t_z, so by hypothesis A for $\mathsf{E}_{z,r}$ and condition 3 in the definition of $\overline{F}^{q,\mathbb{r}}_{Z,r}(h)$,

$$t_z - s_z \geqslant D_h\big(\mathsf{V}_{z,r}, \partial B_r(z)\big) \geqslant \mathsf{a} q \mathbb{r}^{\xi Q} e^{\xi h_{\mathbb{r}}(0)}. \tag{4.16}$$

Let P' be the path obtained from P by excising each segment $P|_{[s_z,t_z]}$ and replacing it by a segment of π_z with the same endpoints. Since $\mathsf{f}_{Z,r}$ is non-negative, Weyl scaling (Axiom III) shows that

$$\begin{aligned}
\mathrm{len}\Big(P' \setminus \bigcup_{z\in Z} \pi_z; D_{h-\mathsf{f}_{Z,r}}\Big) &\leqslant \mathrm{len}\Big(P' \setminus \bigcup_{z\in Z} \pi_z; D_h\Big) \\
&= \mathrm{len}(P; D_h) - \sum_{z\in Z}(t_z - s_z) \\
&\leqslant D_h(K_1, K_2) - \mathsf{a} q \mathbb{r}^{\xi Q} e^{\xi h_{\mathbb{r}}(0)} \# Z \quad \text{(by 4.16)).}
\end{aligned} \tag{4.17}$$

Furthermore, since $\mathsf{f}_{Z,r}$ is identically equal to M on each of the sets $\mathsf{U}_{z,r}$ for $z \in Z$ (which contains π_z) we get from (4.15) that

$$\mathrm{len}\big(\pi_z; D_{h-\mathsf{f}_{Z,r}}\big) \leqslant 4e^{-\xi \mathsf{M}} \mathsf{L} q \mathbb{r}^{\xi Q} e^{\xi h_{\mathbb{r}}(0)}. \tag{4.18}$$

Combining (4.17) and (4.18) shows that

$$D_{h-\mathsf{f}_{Z,r}}(K_1, K_2) \leqslant \mathrm{len}\big(P'; D_{h-\mathsf{f}_{Z,r}}\big) \leqslant D_h(K_1, K_2) - \big(\mathsf{a} - 4e^{-\xi \mathsf{M}} \mathsf{L}\big) q \mathbb{r}^{\xi Q} e^{\xi h_{\mathbb{r}}(0)} \# Z.$$

This gives (4.14) with $C_3 = \mathsf{a} - 4e^{-\xi \mathsf{M}} \mathsf{L}$. We note that $C_3 > 2\mathsf{A}\mathsf{b}/\mathsf{a}$ due to (4.3). □

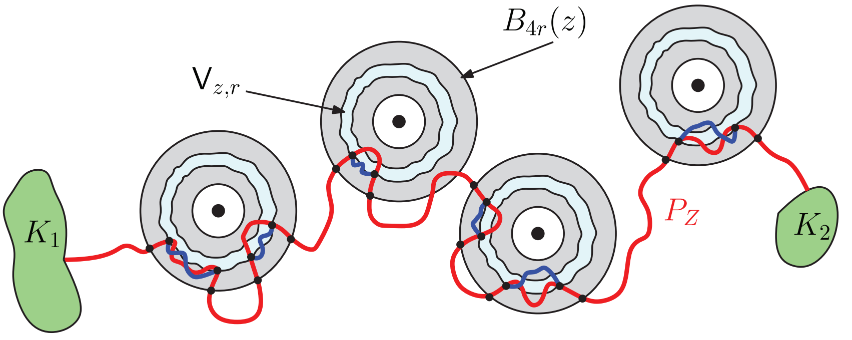

FIGURE 11 Illustration of the proof of Lemma 4.9. The set Z consists of the four center points of the annuli in the figure. For each $z \in Z$, we have indicated each of the points $P_Z(\tau'), P_Z(\tau), P_Z(\sigma), P_Z(\sigma')$ for the $(B_{4r}(z), \mathsf{V}_{z,r})$-excursions $(\tau', \tau, \sigma, \sigma') \in \mathcal{T}_{z,r}(P_Z)$ with a black dot. The proof proceeds by replacing each of the segments $P_Z|_{[\tau,\sigma]}$ by a D_h-geodesic with the same endpoints (shown in blue).

We next establish an inequality in the opposite direction from the one in Lemma 4.8, that is, an upper bound for $D_h(K_1, K_2)$ in terms of $D_{h-\mathsf{f}_{Z,r}}(K_1, K_2)$. This latter estimate holds unconditionally (that is, we do not need to truncate on any event).

Lemma 4.9. *Let $r \in \mathcal{R}$ and $Z \in \mathcal{Z}_r$. Let P_Z be the $D_{h-\mathsf{f}_{Z,r}}$-geodesic from K_1 to K_2. For $z \in Z$, let $\mathcal{T}_{z,r}(P_Z)$ be the set of $(B_{4r}(z), \mathsf{V}_{z,r})$-excursions of P_Z (Definition 4.1). Then*

$$D_h(K_1, K_2) \leqslant D_{h-\mathsf{f}_{Z,r}}(K_1, K_2) + \sum_{z\in Z} \sum_{(\tau',\tau,\sigma,\sigma')\in\mathcal{T}_{z,r}(P_Z)} D_h(P_Z(\tau), P_Z(\sigma)). \tag{4.19}$$

Proof. See Figure 11 for an illustration. By the definition (4.7) of $\mathcal{Z}_r$, we have $B_{4r}(z) \cap (K_1 \cup K_2) = \emptyset$ for each $z \in Z$. From this and Definition 4.1, we see that for each $z \in Z$, the set $P_Z^{-1}(\mathsf{V}_{z,r})$ is contained in the union of the excursion intervals $[\tau, \sigma]$ for $(\tau', \tau, \sigma, \sigma') \in \bigcup_{z\in Z} \mathcal{T}_{z,r}(P_Z)$. Furthermore, since the balls $B_{4r}(z)$ for $z \in Z$ are disjoint, it follows that the excursion intervals $[\tau, \sigma]$ for $(\tau', \tau, \sigma, \sigma') \in \bigcup_{z\in Z} \mathcal{T}_{z,r}(P_Z)$ are disjoint. Since P_Z is continuous, there are only finitely many such intervals.

Let P_Z' be the path from K_1 to K_2 obtained from P_Z by replacing each of the segments $P_Z|_{[\sigma,\tau]}$ for $(\tau', \tau, \sigma, \sigma') \in \bigcup_{z\in Z} \mathcal{T}_{z,r}(P_Z)$ by a D_h-geodesic from $P_Z(\tau)$ to $P_Z(\sigma)$. The function $\mathsf{f}_{Z,r}$ is supported on $\bigcup_{z\in Z} \mathsf{V}_{z,r}$ and the path P_Z does not hit $\bigcup_{z\in Z} \mathsf{V}_{z,r}$ except during the above excursion intervals $[\sigma, \tau]$. Hence, the D_h-length of each of the segments of P_Z which are not replaced when we construct P_Z' is the same as its $D_{h-\mathsf{f}_{Z,r}}$-length. From this, we see that the D_h-length of P_Z' is at most $\mathrm{len}(P_Z; D_{h-\mathsf{f}_{Z,r}})$ plus the sum of the D_h-lengths of the replacement segments. In other words, $\mathrm{len}(P_Z'; D_h)$ is at most the right side of (4.19). □

If we assume that $\bigcap_{z\in Z} \mathsf{E}_{z,r}$ occurs, then we can replace the second sum on the right side of (4.19) by a maximum.

Lemma 4.10. *Let $r \in \mathcal{R}$ and $Z \in \mathcal{Z}_r$. Assume that $\bigcap_{z\in Z} \mathsf{E}_{z,r}$ occurs and let P_Z be the $D_{h-\mathsf{f}_{Z,r}}$-geodesic from K_1 to K_2. For $z \in Z$, let $\mathcal{T}_{z,r}(P_Z)$ be as in Lemma 4.9. Then*

$$D_h(K_1, K_2) \leqslant D_{h-\mathsf{f}_{Z,r}}(K_1, K_2) + \frac{\mathsf{A}}{\mathsf{a}} \sum_{z\in Z} \max_{(\tau',\tau,\sigma,\sigma')\in\mathcal{T}_{z,r}(P_Z)} D_h(P_Z(\tau), P_Z(\sigma)). \tag{4.20}$$

For the proof of Lemma 4.10, we will need an upper bound for the amount of time that P_Z can spend in $\mathsf{V}_{Z,r}$. This upper bound is a straightforward consequence of the upper bound for $D_h(\text{around } \mathbb{A}_{3r,4r}(z))$ from hypothesis A for $\mathsf{E}_{z,r}$.

Lemma 4.11. *Let $r \in \mathcal{R}$, let $Z \subset \mathcal{Z}_r$, and assume that $\bigcap_{z\in Z} \mathsf{E}_{z,r}$ occurs. Let P_Z be the $D_{h-\mathsf{f}_{Z,r}}$-geodesic from K_1 to K_2. For $z \in Z$ such that $P_Z \cap V_{z,r} \neq \emptyset$, let S_z (respectively, T_z) be the first time that P_Z enters $\overline{\mathsf{V}}_{z,r}$ (respectively, the last time that P_Z exits $\mathsf{V}_{z,r}$). Then*

$$T_z - S_z \leqslant \mathsf{A} r^{\xi Q} e^{\xi h_r(z)}. \tag{4.21}$$

Proof. By hypothesis A for $\mathsf{E}_{z,r}$, for each $\zeta > 0$ there is a path π_z in $\mathbb{A}_{3r,4r}(z)$ which disconnects the inner and outer boundaries of $\mathbb{A}_{3r,4r}(z)$ such that

$$\mathrm{len}(\pi_z; D_h) \leqslant (\mathsf{A} + \zeta) r^{\xi Q} e^{\xi h_r(z)}. \tag{4.22}$$

Since $\mathsf{f}_{Z,r}$ is non-negative, the $D_{h-\mathsf{f}_{Z,r}}$-length of π_z is at most its D_h-length.

Since $B_{4r}(z) \cap (K_1 \cup K_2) = \emptyset$ (recall (4.7)), the path P_Z must hit π_z before time S_z and again after time T_z. Since P_Z is a $D_{h-\mathsf{f}_{Z,r}}$-geodesic, the $D_{h-\mathsf{f}_{Z,r}}$-length of the segment of P_Z between any two times when it hits π_z is at most the $D_{h-\mathsf{f}_{Z,r}}$-length of π_z (otherwise, concatenating two segments of P_Z with a segment of π_z would produce a path with the same endpoints as P_Z which is $D_{h-\mathsf{f}_{Z,r}}$-shorter than P_Z). Therefore, (4.22) gives

$$T_z - S_z \leqslant \mathrm{len}\left(\pi_z; D_{h-\mathsf{f}_{Z,r}}\right) \leqslant \mathrm{len}(\pi_z; D_h) \leqslant (\mathsf{A} + \zeta) r^{\xi Q} e^{\xi h_r(z)}. \tag{4.23}$$

Sending $\zeta \to 0$ now concludes the proof. □

Proof of Lemma 4.10. In light of Lemma 4.9, it suffices to show that for each $z \in Z$, the number of $(B_{4r}(z), \mathsf{V}_{z,r})$-excursions satisfies

$$\#\mathcal{T}_{z,r}(P_Z) \leqslant \frac{\mathsf{A}}{\mathsf{a}}. \tag{4.24}$$

To obtain (4.24), we first note that for each $(\tau', \tau, \sigma, \sigma') \in \mathcal{T}_{z,r}(P_Z)$, the path P_Z crosses between $\partial B_{3r}(z)$ and $\mathsf{V}_{z,r}$ during each of the time intervals $[\tau', \tau]$ and $[\sigma, \sigma']$. Since $\mathsf{f}_{Z,r}$ vanishes in $B_{3r}(z) \setminus \mathsf{V}_{z,r}$ and by hypothesis A for $\mathsf{E}_{z,r}$,

$$\min\{\tau - \tau', \sigma' - \sigma\} \geqslant D_{h-\mathsf{f}_{Z,r}}(\partial B_{3r}(z), \mathsf{V}_{z,r}) \geqslant D_h(\partial B_{3r}(z), \mathsf{V}_{z,r}) \geqslant \mathsf{a} r^{\xi Q} e^{\xi h_r(z)}. \tag{4.25}$$

Let S_z and T_z be the first time that P_Z enters $V_{z,r}$ and the last time that P_Z exits $V_{z,r}$, as in Lemma 4.11. If $(\tau_0', \tau_0, \sigma_0, \sigma_0') \in \mathcal{T}_{z,r}(P_Z)$ and $(\tau_1', \tau_1, \sigma_1, \sigma_1') \in \mathcal{T}_{z,r}(P_Z)$ are the first and last excursions in chronological order, then $S_z = \tau_0$ and $T_z = \sigma_1$. Hence, for each excursion $(\tau', \tau, \sigma, \sigma') \in \mathcal{T}_{z,r}(P_Z)$ which is not the first (respectively, last) excursion in chronological order, the time interval $[\tau', \tau]$ (respectively, $[\sigma, \sigma']$) is contained in $[S_z, T_z]$. Furthermore, these time intervals for different excursions are disjoint. By summing the estimate (4.25) over all elements of $\mathcal{T}_{z,r}(P_Z)$, we get that if $\#\mathcal{T}_{z,r}(P_Z) \geqslant 2$, then

$$T_z - S_z \geqslant \mathsf{a} r^{\xi Q} e^{\xi h_r(z)} \#\mathcal{T}_{z,r}(P_Z). \tag{4.26}$$

Combining (4.26) and (4.21) gives (4.24) in the case when $\#\mathcal{T}_{z,r}(P_Z) \geqslant 2$. If $\#\mathcal{T}_{z,r}(P_Z) \leqslant 1$, then (4.24) holds vacuously since $\mathsf{A}/\mathsf{a} \geqslant 1$. □

For the proof of Proposition 4.5, we will need a slightly different upper bound for the amount of time that the $D_{h-\mathsf{f}_{Z,r}}$-geodesic can spend in $\mathsf{V}_{z,r}$ as compared to the one in Lemma 4.11.

Lemma 4.12. *There is a constant $C_4 > 0$, depending only on the parameters, such that the following is true. Let $r \in \mathcal{R}$, $Z \subset \mathcal{Z}_r$, and $q > 0$ and assume that $\overline{F}^{q,\mathbb{r}}_{Z,r}(h)$ occurs. Let P_Z be the $D_{h-\mathsf{f}_{Z,r}}$-geodesic from K_1 to K_2. For each $z \in Z$,*

$$\max\left\{\sup_{u,v\in P_Z\cap \mathsf{V}_{z,r}} D_h(u,v), \operatorname{len}\big(P_Z \cap \mathsf{V}_{z,r}; D_h\big)\right\} \leqslant C_4 q \mathbb{r}^{\xi Q} e^{\xi h_{\mathbb{r}}(0)}. \tag{4.27}$$

Proof. By condition 2 in the definition of $\overline{F}^{q,\mathbb{r}}_{Z,r}(h)$, the event $\bigcap_{z\in Z} \mathsf{E}_{z,r}$ occurs. The bound (4.27) holds vacuously if $P_Z \cap \mathsf{V}_{z,r} = \emptyset$, so assume that $P_Z \cap \mathsf{V}_{z,r} \neq \emptyset$. For $z \in Z$, let S_z (respectively, T_z) be the first time that P_Z enters $\overline{\mathsf{V}}_{z,r}$ (respectively, the last time that P_Z exits $\overline{\mathsf{V}}_{z,r}$), as in Lemma 4.11. By Lemma 4.11 followed by condition 3 in the definition of $\overline{F}_{Z,r}(h)$,

$$T_z - S_z \leqslant \mathsf{A} r^{\xi Q} e^{\xi h_r(z)} \leqslant 2\mathsf{A} q \mathbb{r}^{\xi Q} e^{\xi h_{\mathbb{r}}(0)}$$

Furthermore, $P_Z^{-1}(\mathsf{V}_{Z,r}) \subset [S_z, T_z]$, so

$$\begin{aligned}\max\left\{\sup_{u,v\in P_Z\cap \mathsf{V}_{z,r}} D_{h-\mathsf{f}_{Z,r}}(u,v), \operatorname{len}\Big(P_Z \cap \mathsf{V}_{z,r}; D_{h-\mathsf{f}_{Z,r}}\Big)\right\} &\leqslant T_z - S_z \\ &\leqslant 2\mathsf{A} q \mathbb{r}^{\xi Q} e^{\xi h_{\mathbb{r}}(0)}.\end{aligned}$$

Since $\mathsf{f}_{Z,r} \leqslant \mathsf{M}$, the bound (4.14) combined with Weyl scaling (Axiom III) gives (4.27) with $C_4 = 2e^{\xi \mathsf{M}}\mathsf{A}$. □

The following lemma is the main input in the proof of Proposition 4.5. It allows us to produce configurations Z for which $F^{q,\mathbb{r}}_{Z,r}(h)$, instead of just $\overline{F}^{q,\mathbb{r}}_{Z,r}(h)$, occurs.

Lemma 4.13. *There is a constant $c_5 > 0$, depending only on the parameters, such that the following is true. Let $r \in \mathcal{R}$, $Z \in \mathcal{Z}_r$, and $q > 0$ and assume that $\overline{F}^{q,\mathbb{r}}_{Z,r}(h)$ occurs. There exists $Z' \subset Z$ such that $F^{q,\mathbb{r}}_{Z',r}(h)$ occurs and $\#Z' \geqslant c_5 \#Z$.*

Proof. Step 1: Iteratively removing 'bad' points. It is immediate from Definition 4.7 that if $\overline{F}^{q,\mathbb{r}}_{Z,r}(h)$ occurs and $Z' \subset Z$ is non-empty, then $Z' \in \mathcal{Z}_r$ and $\overline{F}^{q,\mathbb{r}}_{Z',r}(h)$ occurs. So, we need to produce a set $Z' \subset Z$ such that $\#Z'$ is at least a constant times $\#Z$ and condition 5 in the definition of $F^{q,\mathbb{r}}_{Z',r}(h)$ occurs. Since $D_h(u,v; B_{4r}(z)) \geqslant D_h(u,v)$ for all $u, v \in \mathbb{C}$, it suffices to find $Z' \subset Z$ such that if $P_{Z'}$ is the $D_{h-\mathsf{f}_{Z',r}}$-geodesic from K_1 to K_2 and $\mathcal{T}_{z,r}(P_{Z'})$ denotes the set of $(B_{4r}(z), \mathsf{V}_{z,r})$-excursions for $P_{Z'}$, then

$$\max_{(\tau',\tau,\sigma,\sigma')\in\mathcal{T}_{z,r}(P_{Z'})} D_h(P_{Z'}(\tau), P_{Z'}(\sigma)) \geqslant \mathsf{b} r^{\xi Q} e^{\xi h_r(z)}. \tag{4.28}$$

We will construct Z' by iteratively removing the 'bad' points $z \in Z'$ such that the condition of (4.28) does not hold. To this end, let $Z_0 := Z$. Inductively, suppose that $m \in \mathbb{N}_0$ and $Z_m \subset Z$ has been defined. Let P_{Z_m} be the $D_{h-\mathsf{f}_{Z_m,r}}$-geodesic from K_1 to K_2 and let Z_{m+1} be the set of $z \in Z_m$ such that

$$\max_{(\tau',\tau,\sigma,\sigma')\in\mathcal{T}_{z,r}(P_{Z_m})} D_h\Big(P_{Z_m}(\tau), P_{Z_m}(\sigma)\Big) \geqslant \mathsf{b} r^{\xi Q} e^{\xi h_r(z)}. \tag{4.29}$$

If $Z_{m+1} = Z_m$, then (4.28) holds with $Z' = Z_m$, so the event $F^{q,\mathbb{r}}_{Z_m,r}(h)$ occurs. So, to prove the lemma it suffices to show that the above procedure stabilizes before $\#Z_m$ gets too much smaller than $\#Z$. More precisely, we will show that there exists $c_5 > 0$ as in the lemma statement such that

$$\#Z_m \geqslant c_5 \#Z, \quad \forall m \in \mathbb{N}. \tag{4.30}$$

Since $Z_{m+1} \subset Z_m$ for each $m \in \mathbb{N}_0$ and Z_0 is finite, it follows that there must be some $m \in \mathbb{N}$ such that $Z_m = Z_{m+1}$. We know that $F^{q,\mathbb{r}}_{Z_m,r}(h)$ occurs for any such m, so (4.30) implies the lemma statement.

It remains to prove (4.30). The idea of the proof is as follows. At each step of our iterative procedure, we only remove points $z \in Z_m$ for which $P_{Z_m} \cap \mathsf{V}_{z,r}$ is small, in a certain sense. Using this, we can show that $D_{h-\mathsf{f}_{Z_{m+1},r}}(K_1,K_2)$ is not too much bigger than $D_{h-\mathsf{f}_{Z_m,r}}(K_1,K_2)$ (see (4.32)). Iterating this leads to an upper bound for $D_{h-\mathsf{f}_{Z_m,r}}(K_1,K_2)$ in terms of $D_{h-\mathsf{f}_{Z,r}}(K_1,K_2)$ (see (4.33)). We then use the fact that $D_{h-\mathsf{f}_{Z,r}}(K_1,K_2)$ has to be substantially smaller than $D_h(K_1,K_2)$ (Lemma 4.8) together with our upper bound for the amount of time that P_{Z_m} spends in each of the $\mathsf{V}_{z,r}$'s (Lemma 4.12) to obtain (4.30).

Step 2: Comparison of $D_{h-\mathsf{f}_{Z_m,r}}(K_1,K_2)$ and $D_h(K_1,K_2)$. Let us now proceed with the details. Let $m \in \mathbb{N}_0$. By the definition (4.29) of Z_{m+1} and condition 3 in the definition of $\overline{F}^{q,\mathbb{r}}_{Z,r}(h)$,

$$\max_{(\tau',\tau,\sigma,\sigma')\in\mathcal{T}_{z,r}(P_{Z_m})} D_h\Big(P_{Z_m}(\tau), P_{Z_m}(\sigma)\Big) \leqslant 2\mathsf{b} q \mathbb{r}^{\xi Q} e^{\xi h_{\mathbb{r}}(0)}, \quad \forall z \in Z_m \setminus Z_{m+1}. \tag{4.31}$$

We have $Z_m \setminus Z_{m+1} \in \mathcal{Z}_r$ and $h - \mathsf{f}_{Z_m,r} = h - \mathsf{f}_{Z_{m+1},r} - \mathsf{f}_{Z_m\setminus Z_{m+1},r}$. Since we are assuming that $\overline{F}^{q,\mathbb{r}}_{Z,r}(h)$ occurs and $Z_m \setminus Z_{m+1} \subset Z$, condition 2 of Definition 4.7 implies that $\bigcap_{z\in Z_m\setminus Z_{m+1}} \mathsf{E}_{z,r}$ occurs. Since $\mathsf{E}_{z,r}$ depends only on $h|_{\overline{\mathbb{A}}_{r,4r}(z)}$ and the support of $\mathsf{f}_{Z_{m+1},r}$ is disjoint from $\overline{\mathbb{A}}_{r,4r}(z)$ for $z \in Z_m \setminus Z_{m+1}$, we get that $\bigcap_{z\in Z_m\setminus Z_{m+1}} \mathsf{E}_{z,r}$ also occurs with $h - \mathsf{f}_{Z_{m+1},r}$ in place of h. We may therefore apply Lemma 4.10 with $h - \mathsf{f}_{Z_{m+1},r}$ in place of h and $Z_m \setminus Z_{m+1}$ in place of Z to get that

$$\begin{aligned} &D_{h-\mathsf{f}_{Z_{m+1},r}}(K_1,K_2) \\ &\quad\leqslant D_{h-\mathsf{f}_{Z_m,r}}(K_1,K_2) \\ &\qquad + \frac{\mathsf{A}}{\mathsf{a}} \sum_{z\in Z_m\setminus Z_{m+1}} \max_{(\tau',\tau,\sigma,\sigma')\in\mathcal{T}_{z,r}(P_{Z_m})} D_{h-\mathsf{f}_{Z_{m+1},r}}\Big(P_{Z_m}(\tau), P_{Z_m}(\sigma)\Big) \\ &\qquad\qquad \text{(by Lemma 4.10)} \\ &\quad\leqslant D_{h-\mathsf{f}_{Z_m,r}}(K_1,K_2) \end{aligned}$$

$$+\frac{\mathsf{A}}{\mathsf{a}}\sum_{z\in Z_m\setminus Z_{m+1}}\max_{(\tau',\tau,\sigma,\sigma')\in\mathcal{T}_{z,r}(P_{Z_m})} D_h\Big(P_{Z_m}(\tau),P_{Z_m}(\sigma)\Big)$$

$$(\text{since } \mathsf{f}_{Z_{m+1},r}\geqslant 0)$$

$$\leqslant D_{h-\mathsf{f}_{Z_m,r}}(K_1,K_2)+\frac{2\mathsf{A}\mathsf{b}}{\mathsf{a}}q\mathbb{r}^{\xi Q}e^{\xi h_{\mathbb{r}}(0)}(\#Z_m-\#Z_{m+1})\quad(\text{by (4.31)}).\tag{4.32}$$

Iterating the inequality (4.32) m times, then applying Lemma 4.8 to $Z=Z_0\in\mathcal{Z}_r$ gives

$$\begin{aligned}D_{h-\mathsf{f}_{Z_m,r}}(K_1,K_2)&\leqslant D_{h-\mathsf{f}_{Z,r}}(K_1,K_2)+\frac{2\mathsf{A}\mathsf{b}}{\mathsf{a}}q\mathbb{r}^{\xi Q}e^{\xi h_{\mathbb{r}}(0)}(\#Z-\#Z_m)\\&\leqslant D_h(K_1,K_2)-\left(C_3-\frac{2\mathsf{A}\mathsf{b}}{\mathsf{a}}\right)q\mathbb{r}^{\xi Q}e^{\xi h_{\mathbb{r}}(0)}\#Z\\&\qquad-\frac{2\mathsf{A}\mathsf{b}}{\mathsf{a}}q\mathbb{r}^{\xi Q}e^{\xi h_{\mathbb{r}}(0)}\#Z_m\\&\leqslant D_h(K_1,K_2)-\left(C_3-\frac{2\mathsf{A}\mathsf{b}}{\mathsf{a}}\right)q\mathbb{r}^{\xi Q}e^{\xi h_{\mathbb{r}}(0)}\#Z.\end{aligned}\tag{4.33}$$

Note that in the last line, we simply dropped a negative term.

Step 3: Conclusion. By Lemma 4.10 (with Z_m in place of Z), followed by (4.33),

$$\begin{aligned}\frac{\mathsf{A}}{\mathsf{a}}\sum_{z\in Z_m}\max_{(\tau',\tau,\sigma,\sigma')\in\mathcal{T}_{z,r}(P_{Z_m})} D_h\Big(P_{Z_m}(\tau),P_{Z_m}(\sigma)\Big)&\geqslant D_h(K_1,K_2)-D_{h-\mathsf{f}_{Z_m,r}}(K_1,K_2)\\&\geqslant\left(C_3-\frac{2\mathsf{A}\mathsf{b}}{\mathsf{a}}\right)q\mathbb{r}^{\xi Q}e^{\xi h_{\mathbb{r}}(0)}\#Z.\end{aligned}\tag{4.34}$$

As explained above, since $Z_m\subset Z$ we know that $\overline{F}^{q,\mathbb{r}}_{Z_m,r}(z)$ occurs. Hence, we can apply Lemma 4.12 (with Z_m in place of Z), then sum over all $z\in Z_m$, to get

$$\sum_{z\in Z_m}\max_{(\tau',\tau,\sigma,\sigma')\in\mathcal{T}_{z,r}(P_{Z_m})} D_h\Big(P_{Z_m}(\tau),P_{Z_m}(\sigma)\Big)\leqslant C_4q\mathbb{r}^{\xi Q}e^{\xi h_{\mathbb{r}}(0)}\#Z_m,\quad\forall z\in Z_m.\tag{4.35}$$

Combining (4.34) and (4.35) yields

$$\#Z_m\geqslant c_5\#Z\quad\text{with}\quad c_5=\frac{\mathsf{a}}{\mathsf{A}C_4}\left(C_3-\frac{2\mathsf{A}\mathsf{b}}{\mathsf{a}}\right).\tag{4.36}$$

That is, (4.30) holds with this choice of c_5. Note that $c_5>0$ since $C_3>2\mathsf{A}\mathsf{b}/\mathsf{a}$ (Lemma 4.8). □

Proof of Proposition 4.5. Fix $\mathbb{r}>0$ and compact sets $K_1,K_2\in B_{2\mathbb{r}}(0)$ with $\mathrm{dist}(K_1,K_2)\geqslant\eta\mathbb{r}$. Assume that $\mathcal{G}^{\varepsilon}_{\mathbb{r}}=\mathcal{G}^{\varepsilon}_{\mathbb{r}}(K_1,K_2)$ occurs and let P be the D_h-geodesic from K_1 to K_2. We first produce an $r\in\mathcal{R}\cap[\varepsilon^2\mathbb{r},\varepsilon\mathbb{r}]$, a $q>0$, and a large collection of sets $Z\in\mathcal{Z}_r$ for which $\overline{F}^{q,\mathbb{r}}_{Z,r}(h)$ occurs.

To this end, let T be the first exit time of P from $B_{3\mathbb{r}}(0)$, or $T=D_h(K_1,K_2)$ if $P\subset B_{3\mathbb{r}}(0)$ (the reason why we consider T is that conditions 2 and 3 in the definition of $\mathcal{G}^{\varepsilon}_{\mathbb{r}}$ are only required to hold on $B_{3\mathbb{r}}(0)$). By condition 3 in the definition of $\mathcal{G}^{\varepsilon}_{\mathbb{r}}$, for each point $w\in P([0,T])$ there exists $r\in\mathcal{R}\cap[\varepsilon^2\mathbb{r},\varepsilon\mathbb{r}]$ and $z\in(\frac{r}{100}\mathbb{Z}^2)\cap B_{3\mathbb{r}}(0)$ such that $\mathsf{E}_{z,r}$ occurs and $w\in B_{r/25}(z)$.

Since $\mathrm{dist}(K_1, K_2) \geqslant \eta \mathbb{r}$ and $\mathrm{dist}(K_1, \partial B_{3\mathbb{r}}(0)) \geqslant \mathbb{r}$, it follows that $P([0,T])$ is a connected set of Euclidean diameter at least $\eta\mathbb{r}$. Furthermore, since $\mathrm{dist}(K_1, \partial B_{\mathbb{r}}(0)) \geqslant \eta\mathbb{r}$, there must be a segment of $P|_{[0,T]}$ of Euclidean diameter at least $\eta\mathbb{r}$ which is disjoint from $\partial B_{\mathbb{r}}(0)$.

Hence, we can find a constant $x > 0$, depending only on η, with the following property. There are at least $\lfloor x/\varepsilon \rfloor$ pairs $(z_1, r_1), \dots, (z_{\lfloor x/\varepsilon\rfloor}, r_{\lfloor x/\varepsilon \rfloor})$, each consisting of a radius $r_j \in \mathcal{R} \cap [\varepsilon^2 \mathbb{r}, \varepsilon\mathbb{r}]$ and a point $z_j \in (\frac{r}{100}\mathbb{Z}^2) \cap B_{3\mathbb{r}}(0)$, such that the following is true.

(i) The balls $B_{4r_j}(z_j)$ for $j = 1, \dots, \lfloor x/\varepsilon \rfloor$ are disjoint and none of these balls intersects $K_1 \cup K_2 \cup \partial B_{\mathbb{r}}(0)$.
(ii) E_{z_j, r_j} occurs for each $j = 1, \dots, \lfloor x/\varepsilon \rfloor$.
(iii) The path P hits $B_{r_j/25}(z_j)$ for each $j = 1, \dots, \lfloor x/\varepsilon \rfloor$.

By condition 2 in the definition of $\mathcal{G}^\varepsilon_{\mathbb{r}}$, for each $j \in [1, \lfloor x/\varepsilon \rfloor]_{\mathbb{Z}}$ there exists $q \in [\varepsilon^{2\xi(Q+3)}/2, \varepsilon^{\xi(Q-3)}] \cap \{2^{-l}\}_{l\in\mathbb{N}}$ such that $r_j^{\xi Q} e^{\xi h_{r_j}(z_j)} \in [q\mathbb{r}^{\xi Q} e^{\xi h_{\mathbb{r}}(0)}, 2q\mathbb{r}^{\xi Q} e^{\xi h_{\mathbb{r}}(0)}]$. The cardinality of the set

$$\big(\mathcal{R} \cap [\varepsilon^2\mathbb{r}, \varepsilon\mathbb{r}]\big) \times \left(\left[\frac{1}{2}\varepsilon^{2\xi(Q+3)}, \varepsilon^{\xi(Q-3)}\right] \cap \{2^{-l}\}_{l\in\mathbb{N}} \right)$$

is at most a constant (depending only on ξ) times $(\log \varepsilon^{-1})^2$. So, there must be some $r \in \mathcal{R} \cap [\varepsilon^2\mathbb{r}, \varepsilon\mathbb{r}]$ and $q \in [\varepsilon^{2\xi(Q+3)}/2, \varepsilon^{\xi(Q-3)}] \cap \{2^{-l}\}_{l\in\mathbb{N}}$ such that

$$\#\mathcal{J} \succeq \frac{1}{\varepsilon(\log\varepsilon^{-1})^2}, \quad \text{where}$$
$$\mathcal{J} := \left\{ j \in [1, \lfloor x\varepsilon^{-1} \rfloor]_{\mathbb{Z}} \;:\; r_j = r,\; r_j^{\xi Q} e^{\xi h_{r_j}(z_j)} \in \left[q\mathbb{r}^{\xi Q} e^{\xi h_{\mathbb{r}}(0)}, 2q\mathbb{r}^{\xi Q} e^{\xi h_{\mathbb{r}}(0)} \right] \right\} \tag{4.37}$$

with the implicit constant depending only on x (hence only on η). Henceforth, fix such an r and q and let $\mathcal{J}$ be as in (4.37). Also define

$$\mathcal{S} := \{z_j : j \in \mathcal{J}\}, \quad \text{so that} \quad \#\mathcal{S} \succeq \frac{1}{\varepsilon(\log\varepsilon^{-1})^2}. \tag{4.38}$$

If $Z \subset \mathcal{S}$, then property (iii) above implies that $Z \in \mathcal{Z}_r$, where $\mathcal{Z}_r$ is defined as in (4.7). Furthermore, since $q \geqslant \varepsilon^{2\xi(Q+3)}/2$, condition 1 in the definition of $\mathcal{G}^\varepsilon_{\mathbb{r}}$ implies that $\widetilde{D}_h(\mathrm{dist}(K_1, K_2)) \geqslant \mathfrak{C}_* D_h(\mathrm{dist}(K_1, K_2)) - q\mathbb{r}^{\xi Q} e^{\xi h_{\mathbb{r}}(0)}$. From this together with properties (ii) and (iii) above and our choice of $\mathcal{J}$ in (4.37), we see that the event $\overline{F}^{q,\mathbb{r}}_{Z,r}(h)$ of Definition 4.7 occurs.

By Lemma 4.13, for each $Z \subset \mathcal{S}$ there exists $Z' \subset Z$ such that $F^{q,\mathbb{r}}_{Z',r}(h)$ occurs and $\#Z' \geqslant c_5 \#Z$. Fix (in some arbitrary manner) a choice of Z' for each Z, so that $Z \mapsto Z'$ is a function from subsets of $\mathcal{S}$ to subsets of $\mathcal{S}$ for which $F^{q,\mathbb{r}}_{Z',r}(h)$ occurs. We will now lower-bound the cardinality of the set

$$\{Z' : \#Z = k\}. \tag{4.39}$$

To this end, consider a set $\widetilde{Z} \subset \mathcal{S}$ for which $F^{q,\mathbb{r}}_{\widetilde{Z},r}(h)$ occurs and $\#\widetilde{Z} \in [c_5 k, k]$ (that is, $\widetilde{Z}$ is a possible choice of the set Z' when $\#Z = k$). Since $Z' \subset Z$ for each $Z \subset \mathcal{S}$, the number of $Z \subset \mathcal{S}$ such that $\#Z = k$ and $Z' = \widetilde{Z}$ is at most the number of possibilities for the set $Z \setminus \widetilde{Z}$ (subject to $\#Z = k$ and $Z' = \widetilde{Z}$), which is at most

$$\binom{\#\mathcal{S}}{k - \#\widetilde{Z}} \leqslant \binom{\#\mathcal{S}}{\lfloor (1-c_5)k \rfloor}.$$

On the other hand, for each $k \in \mathbb{N}$, the number of sets $Z \subset \mathcal{S}$ such that $\#Z = k$ is $\binom{\#\mathcal{S}}{k}$.

The cardinality of the set (4.39) is least the number of $Z \subset \mathcal{S}$ with $\#Z = k$, divided by the maximal cardinality of the pre-image of a set $\widetilde{Z}$ under $Z \mapsto Z'$. Hence, by combining the two counting formulae from the previous paragraph, we get that the cardinality of the set in (4.39), and hence the number of sets $\widetilde{Z} \subset \mathcal{S}$ for which $F^{q,\mathbb{r}}_{\widetilde{Z},r}(h)$ occurs and $\#\widetilde{Z} \in [c_5 k, k]$, is at least

$$\binom{\#\mathcal{S}}{k}\binom{\#\mathcal{S}}{\lfloor (1-c_5)k\rfloor}^{-1} \succeq (\#\mathcal{S})^{c_5 k} \succeq \varepsilon^{-c_5 k}(\log \varepsilon^{-1})^{-2c_5 k}$$

with the implicit constant depending only on the parameters and k (in the last inequality we used (4.38)). This gives (4.10) for c_1 slightly smaller than c_5. □

4.4 | Proof of Proposition 4.6

The proof of Proposition 4.6 is based on counting the number of points $z \in \frac{r}{100}\mathbb{Z}^2$ which could possibly be an element of some $Z \in \mathcal{Z}_r$ for which $F^{q,\mathbb{r}}_{Z,r}(h + \mathsf{f}_{Z,r})$ occurs. To this end, we make the following definition.

Definition 4.14. For $r \in \mathcal{R}$ and $q > 0$, we say that $z \in \frac{r}{100}\mathbb{Z}^2$ is *r, q-good* if the following conditions are satisfied.

(i) The event $\mathsf{E}_{z,r}(h + \mathsf{f}_{z,r})$ occurs.
(ii) $r^{\xi Q} e^{\xi h_r(z)} \in [q\mathbb{r}^{\xi Q} e^{\xi h_{\mathbb{r}}(0)}, 2q\mathbb{r}^{\xi Q} e^{\xi h_{\mathbb{r}}(0)}]$.
(iii) Let P be the D_h-geodesic from K_1 to K_2. There is a $(B_{4r}(z), \mathsf{V}_{z,r})$-excursion $(\tau'_z, \tau_z, \sigma_z, \sigma'_z)$ for P such that

$$D_{h+\mathsf{f}_{z,r}}(P(\tau_z), P(\sigma_z); B_{4r}(z)) \geqslant \mathsf{b} r^{\xi Q} e^{\xi h_r(z)}. \tag{4.40}$$

Lemma 4.15. *Let $r \in \mathcal{R}$ and $q > 0$. If $Z \in \mathcal{Z}_r$ and $F^{q,\mathbb{r}}_{Z,r}(h + \mathsf{f}_{Z,r})$ occurs, then every $z \in Z$ is r, q-good.*

Proof. Let $z \in Z$ and assume that $F^{q,\mathbb{r}}_{Z,r}(h + \mathsf{f}_{Z,r})$ occurs. By condition 2 in the definition of $F^{q,\mathbb{r}}_{Z,r}(h + \mathsf{f}_{Z,r})$, the event $\mathsf{E}_{z,r}(h + \mathsf{f}_{Z,r})$ occurs. Since $\mathsf{E}_{z,r}(h + \mathsf{f}_{z,r})$ depends only on $(h + \mathsf{f}_{z,r})|_{\mathbb{A}_{r,4r}(z)}$ and $\mathsf{f}_{Z,r} - \mathsf{f}_{z,r} \equiv 0$ outside of $B_{4r}(z)$, it follows that $\mathsf{E}_{z,r}(h + \mathsf{f}_{Z,r}) = \mathsf{E}_{z,r}(h + \mathsf{f}_{z,r})$. This gives condition (i) in Definition 4.14.

Condition (ii) in Definition 4.14 follows from condition 3 in the definition of $F^{q,\mathbb{r}}_{Z,r}(h + \mathsf{f}_{Z,r})$ and the fact that the support of $\mathsf{f}_{Z,r}$ is disjoint from $\partial B_{\mathbb{r}}(0)$ and from $\partial B_r(z)$ for each $z \in Z$ (recall (4.7)). By condition 5 in the definition of $F^{q,\mathbb{r}}_{Z,r}(h + \mathsf{f}_{Z,r})$, we get that z satisfies condition (iii) of Definition 4.14 but with $D_{h+\mathsf{f}_{Z,r}}$ instead of $D_{h+\mathsf{f}_{z,r}}$ in (4.40). Since the support of $\mathsf{f}_{Z,r} - \mathsf{f}_{z,r}$ is disjoint from $B_{4r}(z)$, the internal distances of $D_{h+\mathsf{f}_{Z,r}}$ and $D_{h+\mathsf{f}_{z,r}}$ on $B_{4r}(z)$ are identical. Hence, condition (iii) holds. □

In light of Lemma 4.15, we seek to upper-bound the number of r, q-good points $z \in \frac{r}{100}\mathbb{Z}^2$. When doing so, we can assume without loss of generality that $F^{q,\mathbb{r}}_{Z_0,r}(h + \mathsf{f}_{Z_0,r})$ occurs for some

$Z_0 \in \mathcal{Z}_r$ with $\#Z_0 \leqslant k$ (otherwise, the proposition statement is vacuous). The main input in the proof of Proposition 4.6 is the following lemma.

Lemma 4.16. *There is a constant $C_6 > 0$, depending only on the parameters and the laws of D_h and $\widetilde{D}_h$, such that the following is true. Let $r \in \mathcal{R}$ and let $Z_0, Z_1 \in \mathcal{Z}_r$. Assume that the event $F^{q,\mathbb{r}}_{Z_0,r}(h + \mathsf{f}_{Z_0,r})$ occurs, each $z \in Z_1$ is r, q-good, and each ball $B_{4r}(z)$ for $z \in Z_1$ is disjoint from $\bigcup_{z' \in Z_0} B_{4r}(z')$ (equivalently, $Z_0 \cup Z_1 \in \mathcal{Z}_r$). Then*

$$\#Z_1 \leqslant C_6 \#Z_0.$$

We now explain the idea of the proof of Lemma 4.16. By condition 1 in the definition of $F^{q,\mathbb{r}}_{Z_0,r}(h + \mathsf{f}_{Z_0,r})$, on this event,

$$\widetilde{D}_{h+\mathsf{f}_{Z_0,r}}(K_1, K_2) \geqslant \mathfrak{C}_* D_{h+\mathsf{f}_{Z_0,r}}(K_1, K_2) - q\mathbb{r}^{\xi Q} e^{\xi h_{\mathbb{r}}(0)}. \tag{4.41}$$

We will show that if $\#Z_1$ is too much larger than $\#Z_0$, then (4.41) cannot hold. The reason for this is as follows. Let P be the D_h-geodesic from K_1 to K_2. By condition (iii) in Definition 4.14, each $z \in Z_1$ satisfies the condition of hypothesis C for the event $\mathsf{E}_{z,r}(h + \mathsf{f}_{z,r})$. Hypothesis C therefore gives us a pair of times $s_z, t_z \in P^{-1}(B_{4r}(z))$ such that $t_z - s_z \geqslant \mathsf{c}q\mathbb{r}^{\xi Q} e^{\xi h_{\mathbb{r}}(0)}$ and

$$\widetilde{D}_h(P(s_z), P(t_z); B_{4r}(z)) \leqslant \mathfrak{c}'(t_z - s_z) = \mathfrak{c}' D_h(P(s_z), P(t_z)). \tag{4.42}$$

Since $\mathsf{f}_{Z_0,r}$ vanishes on $B_{4r}(z)$ for each $z \in Z_1$ and $\mathsf{f}_{Z_0,r}$ is non-negative, the relation (4.42) implies that also

$$\widetilde{D}_{h+\mathsf{f}_{Z_0,r}}(P(s_z), P(t_z); B_{4r}(z)) \leqslant \mathfrak{c}' D_{h+\mathsf{f}_{Z_0,r}}(P(s_z), P(t_z)).$$

In other words, we have at least $\#Z_1$ 'shortcuts' along P where the $\widetilde{D}_{h+\mathsf{f}_{Z_0,r}}$-distance is at most $\mathfrak{c}'$ times the $D_{h+\mathsf{f}_{Z_0,r}}$-distance. By following P and taking these shortcuts, we obtain a path from K_1 to K_2 whose $\widetilde{D}_{h+\mathsf{f}_{Z_0,r}}$-length is at most $\mathfrak{C}_*$ times the $D_{h+\mathsf{f}_{Z_0,r}}$-length of P minus a positive constant times $q\mathbb{r}^{\xi Q} e^{\xi h_{\mathbb{r}}(0)} \#Z_1$ (see (4.49)). We then use Lemma 4.17 to upper-bound the $D_{h+\mathsf{f}_{Z_0,r}}$-length of P in terms of $\#Z_0$. This leads to an upper bound for $\widetilde{D}_{h+\mathsf{f}_{Z_0,r}}(K_1, K_2)$ which is inconsistent with (4.41) unless $\#Z_1$ is bounded above by a constant times $\#Z_0$.

We need the following lemma for the proof of Lemma 4.16.

Lemma 4.17. *Let $C_4 > 0$ be as in Lemma 4.12. Let $r \in \mathcal{R}$, $Z \in \mathcal{Z}_r$, and $q > 0$ and assume that $F^{q,\mathbb{r}}_{Z,r}(h + \mathsf{f}_{Z,r})$ occurs. Then the D_h-geodesic P from K_1 to K_2 satisfies*

$$\operatorname{len}\left(P; D_{h+\mathsf{f}_{Z,r}}\right) \leqslant D_h(K_1, K_2) + C_4 q\mathbb{r}^{\xi Q} e^{\xi h_{\mathbb{r}}(0)} \#Z. \tag{4.43}$$

Proof. The function $\mathsf{f}_{Z,r}$ is supported on $\bigcup_{z \in Z} \mathsf{V}_{z,r}$. By Weyl scaling (Axiom III),

$$\operatorname{len}\left(P \setminus \bigcup_{z \in Z} \mathsf{V}_{z,r}; D_{h+\mathsf{f}_{Z,r}}\right) = \operatorname{len}\left(P \setminus \bigcup_{z \in Z} \mathsf{V}_{z,r}; D_h\right) \leqslant D_h(K_1, K_2). \tag{4.44}$$

By Lemma 4.12, applied with $h + \mathsf{f}_{Z,r}$ in place of h,

$$\operatorname{len}\left(P \cap \mathsf{V}_{z,r}; D_{h+\mathsf{f}_{Z,r}}\right) \leqslant C_4 q \mathbb{r}^{\xi Q} e^{\xi h_{\mathbb{r}}(0)}, \quad \forall z \in Z. \tag{4.45}$$

Combining (4.44) and (4.45) yields (4.43). □

Proof of Lemma 4.16. Let P be the D_h-geodesic from K_1 to K_2. By conditions (i) and (iii) in Definition 4.14 together with hypothesis C for the event $\mathsf{E}_{z,r}(h + \mathsf{f}_{z,r})$, for each $z \in Z_1$, there are times $0 < s_z < t_z < D_h(K_1, K_2)$ such that $P([s_z, t_z]) \subset B_{4r}(z)$,

$$t_z - s_z \geqslant \mathsf{c} r^{\xi Q} e^{\xi h_r(z)} \geqslant \mathsf{c} q \mathbb{r}^{\xi Q} e^{\xi h_{\mathbb{r}}(0)}, \quad \text{and} \quad \widetilde{D}_h(P(s_z), P(t_z); B_{4r}(z)) \leqslant \mathfrak{c}'(t_z - s_z). \tag{4.46}$$

Note that to get $r^{\xi Q} e^{\xi h_r(z)} \geqslant q \mathbb{r}^{\xi Q} e^{\xi h_{\mathbb{r}}(0)}$, we used condition (ii) from Definition 4.14 and to get that $P([s_z, t_z]) \subset B_{4r}(z)$, we used Definition 4.1.

If $z \in Z_1$, then by hypothesis $B_{4r}(z)$ is disjoint from $\bigcup_{z' \in Z_0} B_{4r}(z')$. Hence, $B_{4r}(z)$ and $P([s_z, t_z])$ are disjoint from the support of $\mathsf{f}_{Z_0,r}$. We can therefore deduce from (4.46) and Weyl scaling (Axiom III) that for each $z \in Z_1$,

$$\begin{aligned}
&\operatorname{len}\left(P|_{[s_z,t_z]}; D_{h+\mathsf{f}_{Z_0,r}}\right) = t_z - s_z \geqslant \mathsf{c} q \mathbb{r}^{\xi Q} e^{\xi h_{\mathbb{r}}(0)} \quad \text{and} \\
&\widetilde{D}_{h+\mathsf{f}_{Z_0,r}}(P(s_z), P(t_z); B_{4r}(z)) \leqslant \mathfrak{c}'(t_z - s_z) \leqslant \mathfrak{c}' D_{h+\mathsf{f}_{Z_0,r}}(P(s_z), P(t_z)).
\end{aligned} \tag{4.47}$$

Let $N = \#Z_1$ and let $z_1, \dots, z_N$ be the elements of Z_1, ordered so that

$$s_{z_1} < t_{z_1} < s_{z_2} < t_{z_2} < \cdots < s_{z_N} < t_{z_N}.$$

Such an ordering is possible since $P([s_z, t_z]) \subset B_{4r}(z)$, so these path increments are disjoint. For notational simplicity, we also define $t_{z_0} = 0$ and $s_{z_{N+1}} = D_h(K_1, K_2)$, so that $P(t_{z_0}) \in K_1$ and $P(t_{z_{N+1}}) \in K_2$.

By the bi-Lipschitz equivalence of D_h and $\widetilde{D}_h$ (1.20) and Weyl scaling,

$$\widetilde{D}_{h+\mathsf{f}_{Z_0,r}}(P(t_{z_n}), P(s_{z_{n+1}})) \leqslant \mathfrak{C}_* D_{h+\mathsf{f}_{Z_0,r}}(P(t_{z_n}), P(s_{z_{n+1}})), \quad \forall n \in [0, N]_{\mathbb{Z}}. \tag{4.48}$$

We now have the following estimate:

$$\begin{aligned}
&\widetilde{D}_{h+\mathsf{f}_{Z_0,r}}(K_1, K_2) \\
&\quad \leqslant \sum_{n=1}^{N} \widetilde{D}_{h+\mathsf{f}_{Z_0,r}}(P(s_{z_n}), P(t_{z_n})) + \sum_{n=0}^{N} \widetilde{D}_{h+\mathsf{f}_{Z_0,r}}(P(t_{z_n}), P(s_{z_{n+1}})) \\
&\qquad\qquad \text{(triangle inequality)} \\
&\quad \leqslant \mathfrak{c}' \sum_{n=1}^{N} D_{h+\mathsf{f}_{Z_0,r}}(P(s_{z_n}), P(t_{z_n})) + \mathfrak{C}_* \sum_{n=0}^{N} D_{h+\mathsf{f}_{Z_0,r}}(P(t_{z_n}), P(s_{z_{n+1}})) \\
&\qquad\qquad \text{(by (4.47) and (4.48))}
\end{aligned}$$

$$
\begin{aligned}
&= \mathfrak{C}_* \left[\sum_{n=1}^{N} D_{h+\mathsf{f}_{Z_0,r}}(P(s_{z_n}), P(t_{z_n})) + \sum_{n=0}^{N} D_{h+\mathsf{f}_{Z_0,r}}(P(t_{z_n}), P(s_{z_{n+1}})) \right] \\
&\qquad - (\mathfrak{C}_* - \mathfrak{c}') \sum_{n=1}^{N} D_{h+\mathsf{f}_{Z_0,r}}(P(s_{z_n}), P(t_{z_n})) \\
&\leqslant \mathfrak{C}_* \operatorname{len}\left(P; D_{h+\mathsf{f}_{Z_0,r}}\right) - (\mathfrak{C}_* - \mathfrak{c}')\mathfrak{c} q \mathbb{r}^{\xi Q} e^{\xi h_{\mathbb{r}}(0)} \# Z_1 \quad \text{(by (4.47))} \\
&\leqslant \mathfrak{C}_* D_h(K_1, K_2) + \mathfrak{C}_* C_4 q \mathbb{r}^{\xi Q} e^{\xi h_{\mathbb{r}}(0)} \# Z_0 - (\mathfrak{C}_* - \mathfrak{c}')\mathfrak{c} q \mathbb{r}^{\xi Q} e^{\xi h_{\mathbb{r}}(0)} \# Z_1 \\
&\qquad \text{(by Lemma 4.17)} \\
&\leqslant \mathfrak{C}_* D_{h+\mathsf{f}_{Z_0,r}}(K_1, K_2) + \mathfrak{C}_* C_4 q \mathbb{r}^{\xi Q} e^{\xi h_{\mathbb{r}}(0)} \# Z_0 - (\mathfrak{C}_* - \mathfrak{c}')\mathfrak{c} q \mathbb{r}^{\xi Q} e^{\xi h_{\mathbb{r}}(0)} \# Z_1 \\
&\qquad \text{(since } \mathsf{f}_{Z_0,r} \geqslant 0\text{).} \qquad (4.49)
\end{aligned}
$$

By combining (4.41) and (4.49), we obtain

$$
(\mathfrak{C}_* - \mathfrak{c}')\mathfrak{c} q \# Z_1 - \mathfrak{C}_* C_4 q \mathbb{r}^{\xi Q} e^{\xi h_{\mathbb{r}}(0)} \# Z_0 \leqslant q \mathbb{r}^{\xi Q} e^{\xi h_{\mathbb{r}}(0)} \leqslant q \mathbb{r}^{\xi Q} e^{\xi h_{\mathbb{r}}(0)} \# Z_0
$$

$$
\text{which implies} \quad \# Z_1 \leqslant C_6 \# Z \quad \text{where} \quad C_6 := \frac{1 + \mathfrak{C}_* C_4}{(\mathfrak{C}_* - \mathfrak{c}')\mathfrak{c}}.
$$

□

Proof of Proposition 4.6. We can assume that there exists some $Z_0 \in \mathcal{Z}_r$ with $\# Z_0 \leqslant k$ such that $F_{Z_0,r}^{q,\mathbb{r}}(h + \mathsf{f}_{Z_0,r})$ occurs (otherwise, (4.11) holds vacuously). Let $Z_1 \in \mathcal{Z}_r$ be a set such that each $z \in Z_1$ is r, q-good (Definition 4.14) and each $B_{4r}(z)$ for $z \in Z_1$ is disjoint from $\bigcup_{z' \in Z_0} B_{4r}(z')$. We assume that $\# Z_1$ is maximal among all subsets of $\mathcal{Z}_r$ with this property. By Lemma 4.16, we have $\# Z_1 \leqslant C_6 k$.

Now let $Z \in \mathcal{Z}_r$ such that $F_{Z,r}^{q,\mathbb{r}}(h + \mathsf{f}_{Z,r})$ occurs. We claim that for each $z \in Z$, the ball $B_{4r}(z)$ intersects $B_{4r}(z')$ for some $z' \in Z_0 \cup Z_1$. Indeed, by Lemma 4.15, each $z \in Z$ is r, q-good. So, if there is a $z \in Z$ such that $B_{4r}(z)$ is disjoint from $B_{4r}(z')$ for each $z' \in Z_0 \cup Z_1$, then $Z_1 \cup \{z\}$ satisfies the conditions in the definition of Z_1. This contradicts the maximality of $\# Z_1$.

Each $z \in Z$ belongs to $\frac{r}{100}\mathbb{Z}^2$. Hence, for each $z' \in Z_0 \cup Z_1$, the number of $z \in Z$ for which $B_{4r}(z) \cap B_{4r}(z') \neq \emptyset$ is at most some universal constant R. By the preceding paragraph, any $Z \in \mathcal{Z}_r$ such that $F_{Z,r}^{q,\mathbb{r}}(h + \mathsf{f}_{Z,r})$ occurs can be obtained by the following procedure. For each $z' \in Z_0 \cup Z_1$, we either choose a point $z \in \frac{r}{100}\mathbb{Z}^2$ such that $B_{4r}(z) \cap B_{4r}(z') \neq \emptyset$; or we choose no point (so we have at most $R + 1$ choices for each $z' \in Z_0 \cup Z_1$). Then, we take Z to be the set of points that we have chosen. Therefore,

$$
\begin{aligned}
\#\left\{ Z \in \mathcal{Z}_r \ : \ \# Z \leqslant k \text{ and } F_{Z,r}^{q,\mathbb{r}}(h + \mathsf{f}_{Z,r}) \text{ occurs} \right\} &\leqslant (R+1)^{\# Z_0 + \# Z_1} \\
&\leqslant (R+1)^{(C_6+1)k}. \qquad (4.50)
\end{aligned}
$$

This gives (4.11) with $C_2 = (R+1)^{C_6+1}$. □

4.5 | Proof of uniqueness assuming Proposition 4.2

In this subsection, we will prove Theorem 1.13, which asserts the uniqueness of weak LQG metrics, assuming Proposition 4.2. As explained in Subsection 1.5.1, it suffices to show that the optimal bi-Lipschitz constants satisfy $\mathfrak{c}_* = \mathfrak{C}_*$. To accomplish this, we will assume by way of contradiction that $\mathfrak{c}_* < \mathfrak{C}_*$. We also assume the conclusion of Proposition 4.2 (whose proof has been postponed). Throughout this subsection, we fix $\mathbb{p} \in (0,1)$ (which will be chosen in Lemma 4.18) and we let $\mathfrak{c}'' \in (\mathfrak{c}_*, \mathfrak{C}_*)$ and $\mathcal{R} \subset (0,1)$ be as in Proposition 4.2 for this choice of $\mathbb{p}$. We also assume that the parameters of Subsection 4.1 have been chosen as in Proposition 4.2 for our given choice of $\mathbb{p}$.

We first check that the auxiliary conditions in the definition of the event $\mathcal{G}_{\mathbb{r}}^{\varepsilon}(K_1, K_2)$ of Subsection 4.2 occur with high probability when ε is small, which together with Proposition 4.3 leads to an upper bound for the probability of the main condition

$$\widetilde{D}_h(K_1, K_2) \geqslant \mathfrak{C}_* D_h(K_1, K_2) - \frac{1}{2}\varepsilon^{2\xi(Q+3)}\mathbb{r}^{\xi Q} e^{\xi h_{\mathbb{r}}(0)}.$$

We note that the auxiliary conditions do not depend on K_1 and K_2.

Lemma 4.18. *There is a universal choice of the parameter $\mathbb{p} \in (0,1)$ such that the following is true. Let $\widetilde{\beta} > 0$ and let $\mathbb{r} > 0$ such that $\mathbb{P}[\widetilde{G}_{\mathbb{r}}(\widetilde{\beta}, \mathfrak{c}'')] \geqslant \widetilde{\beta}$. It holds with probability tending to 1 as $\varepsilon \to 0$ (at a rate depending only on $\widetilde{\beta}$ and the laws of D_h and $\widetilde{D}_h$, not on $\mathbb{r}$) that conditions 2 and 3 in the definition of $\mathcal{G}_{\mathbb{r}}^{\varepsilon}$ occur, that is,*

(2) *for each $z \in B_{3\mathbb{r}}(0)$ and each $r \in [\varepsilon^2\mathbb{r}, \varepsilon\mathbb{r}] \cap \mathcal{R}$, we have*

$$r^{\xi Q} e^{\xi h_r(z)} \in \left[\varepsilon^{2\xi(Q+3)}\mathbb{r}^{\xi Q} e^{\xi h_{\mathbb{r}}(0)}, \varepsilon^{\xi(Q-3)}\mathbb{r}^{\xi Q} e^{\xi h_{\mathbb{r}}(0)}\right];$$

(3) *for each $z \in B_{3\mathbb{r}}(0)$, there exist $r \in \mathcal{R} \cap [\varepsilon^2\mathbb{r}, \varepsilon\mathbb{r}]$ and $w \in (\frac{r}{100}\mathbb{Z}^2) \cap B_{r/25}(z)$ such that $\mathsf{E}_{w,r}$ occurs.*

Proof. By a standard estimate for the circle average process of the GFF (see, for example, [35, Proposition 2.4]), it holds with polynomially high probability as $r \to 0$ that $|h_r(z)| \leqslant 3\log r^{-1}$ for all $z \in B_3(0)$. By the scale invariance of the law of h, modulo additive constant, we get that with polynomially high probability as $r \to 0$ (at a universal rate) we have $|h_r(z) - h_{\mathbb{r}}(0)| \leqslant 3\log(\mathbb{r}/r)$ for all $z \in B_{3\mathbb{r}}(0)$. By a union bound over logarithmically many values of $r \in \mathcal{R} \cap [\varepsilon^2\mathbb{r}, \varepsilon\mathbb{r}]$, we get that with probability tending to 1 as $\varepsilon \to 0$,

$$\begin{aligned} |h_r(z) - h_{\mathbb{r}}(0)| \leqslant 3\log(\mathbb{r}/r) &\in [3\log\varepsilon^{-2}, 3\log\varepsilon^{-1}], \\ \forall r \in \mathcal{R} \cap [\varepsilon^2\mathbb{r}, \varepsilon\mathbb{r}], &\quad \forall z \in B_{3\mathbb{r}}(0). \end{aligned} \tag{4.51}$$

The bound (4.51) immediately implies condition 2 in the lemma statement.

We now turn our attention to condition 3. By the properties of the events $\mathsf{E}_{z,r}$, we know that $\mathsf{E}_{z,r}$ is almost surely determined by $h|_{\overline{\mathbb{A}}_{r,4r}(z)}$, viewed modulo additive constant, and $\mathbb{P}[\mathsf{E}_{z,r}] \geqslant \mathbb{p}$ for each $z \in \mathbb{C}$ and $r \in \mathcal{R}$. Furthermore, by Proposition 4.2 our hypothesis that $\mathbb{P}[\widetilde{G}_{\mathbb{r}}(\widetilde{\beta}, \mathfrak{c}'')] \geqslant \widetilde{\beta}$ implies that for each small enough $\varepsilon > 0$ (how small depends only on $\widetilde{\beta}$ and the laws of D_h and $\widetilde{D}_h$),

$$\#\big(\mathcal{R}\cap[\varepsilon^2\mathbbm{r},\varepsilon\mathbbm{r}]\big) \geqslant \frac{5}{8}\log_8\varepsilon^{-1}.$$

We may therefore apply Lemma 2.2 with the radii $r_k \in \mathcal{R}\cap[\varepsilon^2\mathbbm{r},\varepsilon\mathbbm{r}]$, the points $z_k \in \frac{r_k}{100}\mathbb{Z}^2$ chosen so that $|z-z_k| \leqslant r_k/50$, and the events $E_{r_k}(z_k) = \mathsf{E}_{z_k,r_k}$. From Lemma 2.2, we obtain that if $\mathbbm{p}$ is chosen to be sufficiently close to 1, in a universal manner, then for each $z\in\mathbb{C}$, it holds with probability at least $1-O_\varepsilon(\varepsilon^5)$ (at a rate depending only on the laws of D_h and $\widetilde{D}_h$) that there exist $r\in\mathcal{R}\cap[\varepsilon^2\mathbbm{r},\varepsilon\mathbbm{r}]$ and $w\in(\frac{r}{100}\mathbb{Z}^2)\cap B_{r/50}(z)$ such that $\mathsf{E}_{w,r}$ occurs.

By a union bound, it holds with probability tending to 1 as $\varepsilon\to 0$ (at a rate depending only on $\widetilde{\beta}$ and the laws of D_h and $\widetilde{D}_h$) that for each $z\in(\frac{\varepsilon^2\mathbbm{r}}{100}\mathbb{Z}^2)\cap B_{3\mathbbm{r}}(0)$, there exist $r\in\mathcal{R}\cap[\varepsilon^2\mathbbm{r},\varepsilon\mathbbm{r}]$ and $w\in(\frac{r}{100}\mathbb{Z}^2)\cap B_{r/50}(z)$ such that $\mathsf{E}_{w,r}$ occurs. Henceforth, assume that this is the case. For a general choice of $z\in B_{3\mathbbm{r}}(0)$, we choose $z'\in(\frac{\varepsilon^2\mathbbm{r}}{100}\mathbb{Z}^2)\cap B_{3\mathbbm{r}}(0)$ such that $|z-z'|\leqslant\varepsilon^2\mathbbm{r}/50$, then we choose $r\in\mathcal{R}\cap[\varepsilon^2\mathbbm{r},\varepsilon\mathbbm{r}]$ and $w\in(\frac{r}{100}\mathbb{Z}^2)\cap B_{r/50}(z')$ such that $\mathsf{E}_{w,r}$ occurs. Then $|w-z'|\leqslant(\varepsilon^2\mathbbm{r}+r)/50\leqslant r/25$. Hence, condition 3 in the lemma statement holds with probability tending to 1 as $\varepsilon\to 0$. □

We henceforth assume that the parameter $\mathbbm{p}$ is chosen as in Lemma 4.18. By combining Proposition 4.3 with Lemma 4.18, we obtain the following.

Lemma 4.19. *Let $\widetilde{\beta}>0$ and let $\mathbbm{r}>0$ such that $\mathbb{P}[\widetilde{G}_{\mathbbm{r}}(\widetilde{\beta},\mathfrak{c}'')]\geqslant\widetilde{\beta}$. Also let $\nu>0$ and $\beta>0$. It holds with probability tending to 1 as $\delta\to 0$ (at a rate depending only on $\nu,\widetilde{\beta},\beta$ and the laws of D_h and $\widetilde{D}_h$) that*

$$\begin{aligned}\widetilde{D}_h(B_{\delta^\nu\mathbbm{r}}(z),B_{\delta^\nu\mathbbm{r}}(w)) &\leqslant \mathfrak{C}_* D_h(B_{\delta^\nu\mathbbm{r}}(z),B_{\delta^\nu\mathbbm{r}}(w)) - \delta\mathbbm{r}^{\xi Q}e^{\xi h_{\mathbbm{r}}(0)},\\ \forall z,w\in\left(\frac{\delta^\nu\mathbbm{r}}{100}\mathbb{Z}^2\right)&\cap B_{\mathbbm{r}}(0)\quad \textit{such that}\quad |z-w|\geqslant\beta\mathbbm{r}\\ \textit{and}\quad &\operatorname{dist}(z,\partial B_{\mathbbm{r}}(0))\geqslant\beta\mathbbm{r}.\end{aligned}\tag{4.52}$$

Proof. Fix $\nu'>0$ to be chosen later, in a manner depending only on ν and ξ. By Proposition 4.3 (applied with $\eta=\beta/2$) and a union bound, it holds with superpolynomially high probability as $\varepsilon\to 0$ that the event $\mathcal{G}^\varepsilon_{\mathbbm{r}}(\overline{B}_{\varepsilon^{\nu'}\mathbbm{r}}(z),\overline{B}_{\varepsilon^{\nu'}\mathbbm{r}}(w))$ does not occur for any pair of points $z,w\in(\frac{\varepsilon^{\nu'}\mathbbm{r}}{100}\mathbb{Z}^2)\cap B_{\mathbbm{r}}(0)$ with $|z-w|\geqslant\beta\mathbbm{r}$ and $\operatorname{dist}(z,\partial B_{\mathbbm{r}}(0))\geqslant\beta\mathbbm{r}$. By combining this with Lemma 4.18 and recalling the definition of $\mathcal{G}^\varepsilon_{\mathbbm{r}}$ (in particular, condition 1), we get that with probability tending to 1 as $\varepsilon\to 0$,

$$\begin{aligned}\widetilde{D}_h\big(B_{\varepsilon^{\nu'}\mathbbm{r}}(z),B_{\varepsilon^{\nu'}\mathbbm{r}}(w)\big) &\leqslant \mathfrak{C}_* D_h\big(B_{\varepsilon^{\nu'}\mathbbm{r}}(z),B_{\varepsilon^{\nu'}\mathbbm{r}}(w)\big) - \varepsilon^{2\xi(Q+3)}\mathbbm{r}^{\xi Q}e^{\xi h_{\mathbbm{r}}(0)},\\ \forall z,w\in\left(\frac{\varepsilon^{\nu'}\mathbbm{r}}{100}\mathbb{Z}^2\right)&\cap B_{\mathbbm{r}}(0)\quad \text{such that}\quad |z-w|\geqslant\beta\mathbbm{r}\\ \text{and}\quad &\operatorname{dist}(z,\partial B_{\mathbbm{r}}(0))\geqslant\beta\mathbbm{r}.\end{aligned}\tag{4.53}$$

We now conclude the proof by applying the above estimate with $\varepsilon=\varepsilon(\delta)>0$ chosen so that $\varepsilon^{2\xi(Q+3)}=\delta$ and with $\nu'=\nu/(2\xi(Q+3))$. □

Recall the definition of the event $H_{\mathbb{r}}(\alpha, \mathfrak{C}')$ from Definition 3.2, which says that there is a point $u \in \partial B_{\alpha\mathbb{r}}(0)$ and a point $v \in \partial B_{\mathbb{r}}(0)$ satisfying certain conditions such that $\widetilde{D}_h(u,v) \leqslant \mathfrak{C}' D_h(u,v)$. From Lemma 4.19 and a geometric argument, we obtain the following, which will eventually be used to get a contradiction to Proposition 3.5.

Lemma 4.20. *Let $\widetilde{\beta} > 0$ and let $\mathbb{r} > 0$ such that $\mathbb{P}[\widetilde{G}_{\mathbb{r}}(\widetilde{\beta}, \mathfrak{c}'')] \geqslant \widetilde{\beta}$. For each $\alpha \in (3/4, 1)$, we have*

$$\lim_{\delta \to 0} \mathbb{P}[H_{\mathbb{r}}(\alpha, \mathfrak{C}_* - \delta)] = 0$$

at a rate depending only on $\alpha, \widetilde{\beta}$, and the laws of D_h and $\widetilde{D}_h$.

Proof. Let $\nu > 0$ to be chosen later, in a manner depending only on the laws of D_h and $\widetilde{D}_h$. By Lemma 4.19 applied with $\beta = (1-\alpha)/2$, it holds with probability tending to 1 as $\delta \to 0$ that

$$\begin{aligned} &\widetilde{D}_h(B_{\delta^\nu \mathbb{r}}(z), B_{\delta^\nu \mathbb{r}}(w)) \leqslant \mathfrak{C}_* D_h(B_{\delta^\nu \mathbb{r}}(z), B_{\delta^\nu \mathbb{r}}(w)) - \delta \mathbb{r}^{\xi Q} e^{\xi h_{\mathbb{r}}(0)}, \\ &\forall z, w \in \left(\frac{\delta^\nu \mathbb{r}}{100} \mathbb{Z}^2\right) \cap B_{\mathbb{r}}(0) \quad \text{such that} \quad |z-w| \geqslant \frac{1-\alpha}{2}\mathbb{r} \\ &\text{and} \quad \operatorname{dist}(z, \partial B_{\mathbb{r}}(0)) \geqslant \frac{1-\alpha}{2}\mathbb{r}. \end{aligned} \tag{4.54}$$

Henceforth, assume that that (4.54) holds.

Recalling Definition 3.2, we consider points $u \in \partial B_{\alpha\mathbb{r}}(0)$ and $v \in \partial B_{\mathbb{r}}(0)$ such that

- $D_h(u,v) \leqslant (1-\alpha)^{-1} \mathbb{r}^{\xi Q} e^{\xi h_{\mathbb{r}}(0)}$; and
- for each $\delta \in (0, (1-\alpha)^2]$, we have

$$\max\left\{D_h(u, \partial B_{\delta\mathbb{r}}(u)), D_h\left(\text{around } \mathbb{A}_{\delta\mathbb{r}, \delta^{1/2}\mathbb{r}}(u)\right)\right\} \leqslant \delta^\theta D_h(u,v) \tag{4.55}$$

  and the same is true with the roles of u and v interchanged.

We will show that if ν is chosen to be large enough (depending only on the laws of D_h and $\widetilde{D}_h$), then for each small enough $\delta > 0$ (depending only on $\alpha, \widetilde{\beta}$, and the laws of D_h and $\widetilde{D}_h$), we have

$$\widetilde{D}_h(u,v) \leqslant \left(\mathfrak{C}_* - \frac{1-\alpha}{4}\delta\right) D_h(u,v), \quad \forall u, v \text{ satisfying the above conditions.} \tag{4.56}$$

By Definition 3.2, the relation (4.56) implies that $H_{\mathbb{r}}(\alpha, \mathfrak{C}_* - \frac{1-\alpha}{4}\delta)$ does not occur. Since δ can be made arbitrarily small, this implies the lemma statement.

See Figure 12 for an illustration of the proof of (4.56). Let $z \in (\frac{\delta^\nu \mathbb{r}}{100}\mathbb{Z}^2) \cap B_{\delta^\nu \mathbb{r}}(u)$ and $w \in (\frac{\delta^\nu \mathbb{r}}{100}\mathbb{Z}^2) \cap B_{\delta^\nu \mathbb{r}}(v)$. If δ is small enough, then $|z-w| \geqslant (1-\alpha)\mathbb{r}/2$ and $\operatorname{dist}(z, \partial B_{\mathbb{r}}(0)) \geqslant (1-\alpha)\mathbb{r}/2$. By (4.54), there is a path P^δ from $B_{\delta^\nu \mathbb{r}}(z)$ to $B_{\delta^\nu \mathbb{r}}(w)$ such that

$$\begin{aligned} \operatorname{len}\left(P^\delta; \widetilde{D}_h\right) &\leqslant \mathfrak{C}_* D_h(B_{\delta^\nu \mathbb{r}}(z), B_{\delta^\nu \mathbb{r}}(w)) - \frac{\delta}{2}\mathbb{r}^{\xi Q} e^{\xi h_{\mathbb{r}}(0)} \\ &\leqslant \mathfrak{C}_* D_h(u,v) - \frac{\delta}{2}\mathbb{r}^{\xi Q} e^{\xi h_{\mathbb{r}}(0)} \quad (\text{since } u \in B_{\delta^\nu \mathbb{r}}(z) \text{ and } v \in B_{\delta^\nu \mathbb{r}}(w)) \end{aligned}$$

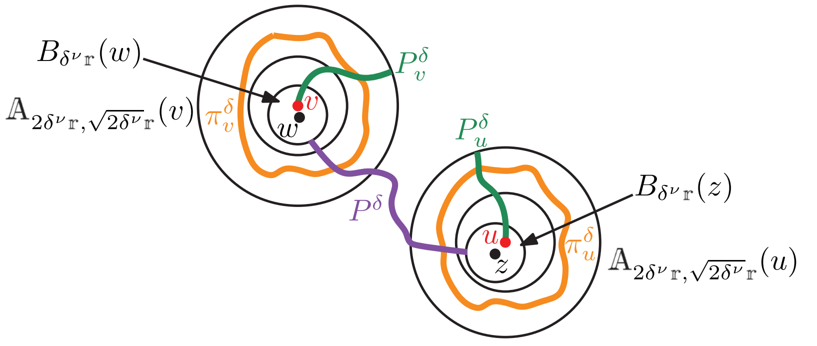

FIGURE 12 Illustration of the five paths used to get an upper bound for $\widetilde{D}_h(u,v)$ in the proof of Lemma 4.20. The $\widetilde{D}_h$-length of P^δ is bounded above using (4.54) and the $\widetilde{D}_h$-lengths of the other four paths are bounded above using (4.55).

$$\leqslant \left(\mathfrak{C}_* - \frac{1-\alpha}{2}\delta\right) D_h(u,v) \quad (\text{since } D_h(u,v) \leqslant (1-\alpha)^{-1}\mathfrak{r}^{\xi Q} e^{\xi h_{\mathfrak{r}}(0)}). \tag{4.57}$$

By (4.55) (applied with $\sqrt{2\delta^\nu}$ in place of δ), if δ is small enough (depending on α) then there are paths P_u^δ and P_v^δ from u and v to $\partial B_{\sqrt{2\delta^\nu}\mathfrak{r}}(u)$ and $\partial B_{\sqrt{2\delta^\nu}\mathfrak{r}}(v)$, respectively, such that

$$\max\left\{\text{len}(P_u^\delta; D_h), \text{len}(P_v^\delta; D_h)\right\} \leqslant 2^{\theta/2}\delta^{\nu\theta/2} D_h(u,v). \tag{4.58}$$

Furthermore, by (4.55) applied with $2\delta^\nu$ in place of δ, there are paths π_u^δ and π_v^δ in $\mathbb{A}_{2\delta^\nu \mathfrak{r}, \sqrt{2\delta^\nu}\mathfrak{r}}(u)$ and $\mathbb{A}_{2\delta^\nu \mathfrak{r}, \sqrt{2\delta^\nu}\mathfrak{r}}(u)$, respectively, which disconnect the inner and outer boundaries and satisfy

$$\max\left\{\text{len}(\pi_u^\delta; D_h), \text{len}(\pi_v^\delta; D_h)\right\} \leqslant 2^{\theta}\delta^{\nu\theta} D_h(u,v). \tag{4.59}$$

Since $\max\{|z-u|, |w-v|\} \leqslant \delta^\nu \mathfrak{r}$, the union $P^\delta \cup P_u^\delta \cup P_v^\delta \cup \pi_u^\delta \cup \pi_v^\delta$ contains a path from u to v. Therefore, combining (4.57), (4.58), and (4.59), then using the bi-Lipschitz equivalence of D_h and $\widetilde{D}_h$ (1.20) gives

$$\widetilde{D}_h(u,v) \leqslant \left(\mathfrak{C}_* - \frac{1-\alpha}{2}\delta\right) D_h(u,v) + \sum_{x \in \{u,v\}} \left(\text{len}(P_x^\delta; \widetilde{D}_h) + \text{len}(\pi_x^\delta; \widetilde{D}_h)\right)$$

$$\leqslant \left(\mathfrak{C}_* - \frac{1-\alpha}{2}\delta + 2^{\theta/2+1}\mathfrak{C}_*\delta^{\nu\theta/2} + 2^{\theta+1}\mathfrak{C}_*\delta^{\nu\theta}\right) D_h(u,v).$$

If $\nu > 2/\theta$ and δ is small enough, then this implies (4.56). □

Proof of Theorem 1.13. By Proposition 3.5, there exist $\alpha \in (3/4, 1)$ and $p \in (0,1)$, depending only on the laws of D_h and $\widetilde{D}_h$, such that for each $\delta > 0$ and each small enough $\varepsilon > 0$ (depending only on δ and the laws of D_h and $\widetilde{D}_h$), there are at least $\frac{3}{4}\log_8 \varepsilon^{-1}$ values of $r \in [\varepsilon^2, \varepsilon] \cap \{8^{-k}\}_{k \in \mathbb{N}}$ such that

$$\mathbb{P}[H_r(\alpha, \mathfrak{C}_* - \delta)] \geqslant p. \tag{4.60}$$

Let $\mathfrak{c}''$ be as in Proposition 4.2, so that $\mathfrak{c}''$ depends only on the laws of D_h and $\widetilde{D}_h$. By Proposition 3.11 (applied with $\mathfrak{c}''$ in place of $\mathfrak{c}'$), there exist $\widetilde{\beta} > 0$ and $\varepsilon_0 > 0$ (depending only on the laws of D_h and $\widetilde{D}_h$) such that for each $\varepsilon \in (0, \varepsilon_0]$, there are at least $\frac{3}{4}\log_8 \varepsilon^{-1}$ values of $r \in [\varepsilon^2, \varepsilon] \cap \{8^{-k}\}_{k\in\mathbb{N}}$ for which $\mathbb{P}[\widetilde{G}_r(\widetilde{\beta}, \mathfrak{c}'')] \geqslant \widetilde{\beta}$. By combining this with Lemma 4.20, we get that if α and p are as in (4.60), then there exists $\delta > 0$, depending only on α, p, and the laws of D_h and $\widetilde{D}_h$, such that for each $\varepsilon \in (0, \varepsilon_0]$, there are at least $\frac{3}{4}\log_8 \varepsilon^{-1}$ values of $r \in [\varepsilon^2, \varepsilon] \cap \{8^{-k}\}_{k\in\mathbb{N}}$ for which

$$\mathbb{P}[H_r(\alpha, \mathfrak{C}_* - \delta)] \leqslant \frac{p}{2}. \tag{4.61}$$

The total number of radii $r \in [\varepsilon^2, \varepsilon] \cap \{8^{-k}\}_{k\in\mathbb{N}}$ is at most $\log_8 \varepsilon^{-1}$, so there cannot be at least $\frac{3}{4}\log_8 \varepsilon^{-1}$ values of $r \in [\varepsilon^2, \varepsilon] \cap \{8^{-k}\}_{k\in\mathbb{N}}$ for which (4.60) holds and at least $\frac{3}{4}\log_8 \varepsilon^{-1}$ values of $r \in [\varepsilon^2, \varepsilon] \cap \{8^{-k}\}_{k\in\mathbb{N}}$ for which (4.61) holds. We thus have a contradiction, so we conclude that $\mathfrak{c}_* = \mathfrak{C}_*$. □

5 | CONSTRUCTING EVENTS AND BUMP FUNCTIONS

5.1 | Setup and outline

The goal of this section is to prove Proposition 4.2. Extending (4.1), we define

$$\mathfrak{c}' := \frac{\mathfrak{c}_* + \mathfrak{C}_*}{2} \quad \text{and} \quad \mathfrak{c}'_0 := \frac{\mathfrak{c}_* + \mathfrak{c}'}{2}, \tag{5.1}$$

so that if $\mathfrak{c}_* < \mathfrak{C}_*$, then $\mathfrak{c}_* < \mathfrak{c}'_0 < \mathfrak{c}' < \mathfrak{C}_*$.

Throughout this section, we fix $\mathbb{p} \in (0,1)$ as in Proposition 4.2. Note that $\mathbb{p}$ is allowed to be arbitrarily close to 1. We seek to construct a set of radii $\mathcal{R} \subset (0,1)$ and, for each $z \in \mathbb{C}$ and $r \in \mathcal{R}$, open sets $\mathsf{U}_{z,r} \subset \mathsf{V}_{z,r} \subset \mathbb{A}_{r,4r}(z)$, a smooth bump function $\mathsf{f}_{z,r}$ supported on $\mathsf{V}_{z,r}$, and an event $\mathsf{E}_{z,r}$ with $\mathbb{P}[\mathsf{E}_{z,r}] \geqslant \mathbb{p}$ which satisfy the conditions in Subsection 4.1.

For simplicity, for most of this section we will take $z = 0$ and remove z from the notation, so we will call our objects $\mathsf{U}_r, \mathsf{V}_r, \mathsf{f}_r, \mathsf{E}_r$. At the very end of the proof, we will define objects for a general choice of z by translating space.

Let $\alpha \in (3/4, 1)$ and $p_0 = p \in (0,1)$ be as in Proposition 3.10, so that α and p_0 depend only on the laws of D_h and $\widetilde{D}_h$. We define our initial set of 'good' radii

$$\mathcal{R}_0 := \left\{ r \in \{8^{-k}\}_{k\in\mathbb{N}} : \mathbb{P}[\widetilde{H}_r(\alpha, \mathfrak{c}'_0)] \geqslant p_0 \right\}. \tag{5.2}$$

By Proposition 3.10, there exists $\mathfrak{c}'' > 0$, depending only on the laws of D_h and $\widetilde{D}_h$, such that if $\mathbb{r} > 0$ and $\widetilde{\beta} > 0$ such that $\mathbb{P}[\widetilde{G}_{\mathbb{r}}(\widetilde{\beta}, \mathfrak{c}'')] \geqslant \widetilde{\beta}$, then for each small enough $\varepsilon > 0$ (how small is independent of $\mathbb{r}$),

$$\#\left(\mathcal{R}_0 \cap [\varepsilon^2 \mathbb{r}, \varepsilon \mathbb{r}]\right) \geqslant \frac{3}{4}\log_8 \varepsilon^{-1}.$$

We will eventually establish Proposition 4.2 with the set of admissible radii given by $\mathcal{R} = \rho^{-1}\mathcal{R}_0$, where $\rho \in (0,1)$ is a constant depending only on $\mathbb{p}$ and the laws of D_h and $\widetilde{D}_h$.

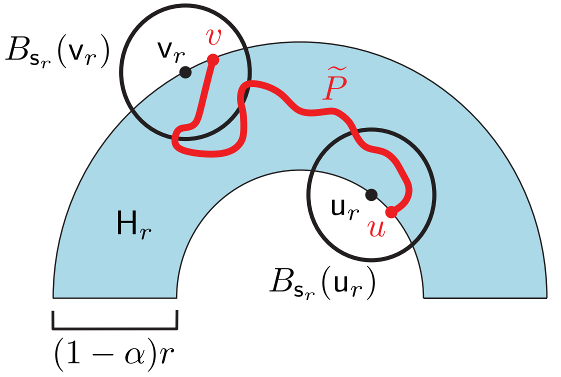

FIGURE 13 Illustration of the objects involved in Lemma 5.2

Recall the basic idea of the construction as explained just after Proposition 4.2. We will take U_r to be a narrow 'tube' with the topology of a Euclidean annulus which is contained in a small neighborhood of $\partial B_{2r}(0)$, and V_r to be a small Euclidean neighborhood of U_r. We will then take E_r to be the event that there are many 'good' pairs of points $u, v \in \mathsf{U}_r$ such that $\widetilde{D}_h(u,v) \leqslant \mathfrak{c}_0' D_h(u,v)$, plus a long list of regularity conditions. The idea for checking hypothesis C for E_r is that by Weyl scaling (Axiom III), the $D_{h-\mathsf{f}_r}$-lengths of paths contained in U_r tend to be much shorter than the $D_{h-\mathsf{f}_r}$-lengths of paths outside of V_r. We will use this fact to force a $D_{h-\mathsf{f}_r}$-geodesic P_r to get $D_{h-\mathsf{f}_r}$-close to each of u and v for one of our good pairs of points u, v. We will then apply the triangle inequality to find times s, t such that $\widetilde{D}_{h-\mathsf{f}_r}(P_r(s), P_r(t)) \leqslant \mathfrak{c}'(t-s)$. Note that the application of the triangle inequality here is the reason why we need to require that $\widetilde{D}_h(u,v) \leqslant \mathfrak{c}_0' D_h(u,v)$ for $\mathfrak{c}_0' < \mathfrak{c}'$.

The broad ideas of this section are similar to those of [27, section 5], which performs a similar construction in the subcritical case. However, the details are quite different from [27, section 5], for three reasons. First, the conditions which we need our event to satisfy are slightly different from the ones needed in [27] since our argument in Section 4 is completely different from the argument of [27, section 4]. Second, we make some minor simplifications to various steps of the construction as compared to [27]. Third, and most importantly, we want to treat the supercritical case so there are a number of additional difficulties arising from the fact that the metric does not induce the Euclidean topology. These difficulties necessitate additional conditions on the events and additional arguments as compared to the subcritical case. Especially, many of the conditions in the definition of E_r and all of arguments of Subsection 5.10 can be avoided in the subcritical case. We will now give a more detailed outline of our construction.

In Subsection 5.2, we will consider an event for a single 'good' pair of points u, v and show that for $r \in \mathcal{R}_0$, the probability of this event is bounded below by a constant p depending only on the laws of D_h and $\widetilde{D}_h$. See Lemma 5.2 for a precise statement and Figure 13 for an illustration of the event.

The event we consider is closely related to the event $\widetilde{H}_r(\alpha, \mathfrak{c}_0')$ of Definition 3.9. We require that there is a point $u \in \partial B_{\alpha r}(0)$ and a point $v \in \partial B_r(0)$ such that $\widetilde{D}_h(u,v) \leqslant \mathfrak{c}_0' D_h(u,v)$ and a $\widetilde{D}_h$-geodesic $\widetilde{P}$ from u to v which is contained in a specified deterministic half-annulus $\mathsf{H}_r \subset \mathbb{A}_{\alpha r, r}(0)$. We also impose two additional constraints on u and v which will be important later.

(i) We require that u is contained in a certain small *deterministic* ball $B_{\mathsf{s}_r}(\mathsf{u}_r)$ centered at a point $\mathsf{u}_r \subset \partial B_{\alpha r}(0)$ and v is contained in a small deterministic ball $B_{\mathsf{s}_r}(\mathsf{v}_r)$ centered at a point

$\mathsf{v}_r \in \partial B_r(0)$, where s_r is deterministic number which is comparable to a small constant times r. The reason for this condition is that we will eventually define our set U_r so that it has a 'bottleneck' at several translated and scaled copies of the balls $B_{\mathsf{s}_r}(\mathsf{u}_r)$ and $B_{\mathsf{s}_r}(\mathsf{v}_r)$ (that is, removing these balls disconnects U_r; see Figure 15), and we need U_r to be deterministic. We will show that this condition happens with positive probability by considering finitely many possible choices for the balls $B_{\mathsf{s}_r}(\mathsf{u}_r)$ and $B_{\mathsf{s}_r}(\mathsf{v}_r)$ and using a pigeonhole argument.

(ii) We require that the internal distance $D_h(u, x; \overline{B}_{\mathsf{s}_r}(\mathsf{u}_r))$ is small for 'most' points $x \in \partial B_{\mathsf{s}_r}(\mathsf{u}_r)$, and we impose a similar condition for v. The purpose of this condition is to upper-bound the $D_{h-\mathsf{f}_r}$-distance from a $D_{h-\mathsf{f}_r}$-geodesic to u, once we have forced it to get Euclidean-close to u. The condition will be shown to occur with high probability using Lemma 2.10.

In Subsection 5.3, we will define $\mathsf{F}_{z,r}$ for $z \in \mathbb{C}$ and $r \in \mathcal{R}_0$ to be the event of Subsection 5.2, but translated so that we are working with annuli centered at z rather than 0. We will then show that $\mathsf{F}_{z,r}$ is locally determined by h (Lemma 5.7).

In Subsection 5.4, we will introduce several parameters to be chosen later, including the parameter $\rho \in (0,1)$ mentioned above. We will then define the open sets U_r and V_r and the bump function f_r for $r \in \rho^{-1}\mathcal{R}_0$ in terms of these parameters. More precisely,

- the set U_r will be the union of a large finite number of disjoint sets of the form $\mathsf{H}_{\rho r} \cup B_{\mathsf{s}_{\rho r}}(\mathsf{u}_{\rho r}) \cup B_{\mathsf{s}_{\rho r}}(\mathsf{v}_{\rho r}) + z$ for $z \in \partial B_{2r}(0)$ (that is, the sets appearing in the definition of $\mathsf{F}_{z,\rho r}$), together with long narrow 'tubes' linking these sets together into an annular region. See Figure 15 for an illustration;
- the set V_r will be a small Euclidean neighborhood of U_r;
- the function f_r will attain its maximal value at each point of U_r and will be supported on V_r.

The reason for our definition of U_r is as follows. Since $r \in \rho^{-1}\mathcal{R}_0$, for each of the sets $\mathsf{H}_{\rho r} \cup B_{\mathsf{s}_{\rho r}}(\mathsf{u}_{\rho r}) \cup B_{\mathsf{s}_{\rho r}}(\mathsf{v}_{\rho r}) + z$ in the definition of U_r, there is a positive chance that the event $\mathsf{F}_{z,\rho r}$ of Subsection 5.3 occurs. Hence, by the long-range independence properties of the GFF (Lemma 2.3), it is very likely that $\mathsf{F}_{z,\rho r}$ occurs for many of the points z. This gives us the desired large collection of 'good' pairs of points $u, v \in \mathsf{U}_r$. See Lemma 5.13.

In Subsection 5.5, we will define the event E_r. The event E_r includes the condition that $\mathsf{F}_{z,\rho r}$ occurs for many of the points $z \in \partial B_{2r}(0)$ involved in the definition of U_r (condition 4), plus a large number of additional high-probability regularity conditions. Then, in Subsection 5.6, we will show that we can choose the parameters of Subsection 5.4 in such a way that E_r occurs with probability at least $\mathbb{p}$ (Proposition 5.9). We will also show that E_r satisfies hypotheses A and B of Subsection 4.1 (Proposition 5.17). In Subsection 5.7, we will explain how to conclude the proof of Proposition 4.2 assuming that our objects also satisfy hypothesis C of Subsection 4.1.

The rest of the section is then devoted to checking that our objects satisfy hypothesis C of Subsection 4.1 (Proposition 5.18). Recalling the statement of hypothesis C, we will assume that E_r occurs and consider a $D_{h-\mathsf{f}_r}$-geodesic P_r between two points of $\mathbb{C} \setminus B_{4r}(0)$. We will further assume that P_r has a $(B_{4r}(0), \mathsf{V}_r)$-excursion $(\tau', \tau, \sigma, \sigma')$ such that $D_h(P_r(\tau), P_r(\sigma); B_{4r}(0))$ is bounded below by an appropriate constant times $r^{\xi Q} e^{\xi h_r(0)}$ (recall Definition 4.1). We aim to find times $s < t$ for P_r such that $t - s$ is not too small and $\widetilde{D}_{h-\mathsf{f}_{z,r}}(P_r(s), P_r(t); B_{4r}(0)) \leqslant \mathfrak{c}'(t-s)$.

In Subsection 5.8, we will show that the *Euclidean* distance between the points $P_r(\tau), P_r(\sigma) \in \partial\mathsf{V}_r$ is bounded below by a constant times r (Lemma 5.20) and that $P_r|_{[\tau,\sigma]}$ is contained in a small Euclidean neighborhood of V_r (Lemma 5.22). These statements are proven using the regularity conditions in the definition of E_r. In particular, the lower bound for $|P_r(\tau) - P_r(\sigma)|$ comes from the regularity of D_h-distances along a geodesic (Lemma 2.13). The statement that $P_r|_{[\tau,\sigma]}$ is contained in a small Euclidean neighborhood of V_r is proven as follows. Since f_r is very large on U_r, we know that $D_{h-\mathsf{f}_r}$-distances inside U_r are very small, which leads to a very small upper bound for $\sigma - \tau = D_{h-\mathsf{f}_r}(P_r(\tau), P_r(\sigma))$ (Lemma 5.21). Since f_r is supported on V_r, the $D_{h-\mathsf{f}_r}$-length of any segment of P_r which is disjoint from V_r is the same as its D_h-length, which will be larger than our upper bound for $\sigma - \tau$ unless the Euclidean diameter of the segment is very small.

In Subsection 5.9, we will use the results of Subsection 5.8 and the definition of U_r to show that the following is true. There is a point $z \in \partial B_{2r}(0)$ as in the definition of U_r such that $\mathsf{F}_{z,\rho r}$ occurs and P_r gets Euclidean-close to each of the 'good' points u and v in the definition of $\mathsf{F}_{z,\rho r}$ (Lemma 5.23). The reason why this is true is that, by the results of Subsection 5.8, $P_r([\tau,\sigma])$ is contained in a small neighborhood of U_r and has Euclidean diameter of order r, and the definition of U_r implies that removing small neighborhoods of the points u and v disconnects U_r (see Figure 15).

Showing that P_r gets Euclidean-close to u and v is not enough for our purposes since $D_{h-\mathsf{f}_r}$ is not Euclidean-continuous, so it is possible for two points to be Euclidean-close but not $D_{h-\mathsf{f}_r}$-close. Therefore, further arguments are needed to show that P_r gets $D_{h-\mathsf{f}_r}$-close to each of u and v. We remark that this is one of the main reasons why the argument in this section is more difficult than the analogous argument in the subcritical case [27, section 5].

In Subsection 5.10, we will show that there are times s and t for P_r such that $D_{h-\mathsf{f}_r}(P_r(t), u)$ and $D_{h-\mathsf{f}_r}(P_r(s), v)$ are each much smaller than $D_{h-\mathsf{f}_r}(u, v)$ (Lemma 5.26). The key tool which allows us to do this is the condition in the definition of $\mathsf{F}_{z,\rho r}$ which says that $D_h(u, x; \overline{B}_{\mathsf{s}_{\rho r}}(\mathsf{u}_{\rho r}) + z)$ is small for 'most' points of $\partial B_{\mathsf{s}_{\rho r}}(\mathsf{u}_{\rho r}) + z$ (recall point (ii) in the summary of Subsection 5.2). However, this condition is not sufficient for our purposes since it is possible that the 'Euclidean size' of $P_r \cap (B_{\mathsf{s}_{\rho r}}(\mathsf{u}_{\rho r}) + z)$ is small, and hence P_r manages not to hit a geodesic from u to x for any of the 'good' points $x \in \partial B_{\mathsf{s}_{\rho r}}(\mathsf{u}_{\rho r}) + z$ such that $D_h(u, x; B_{\mathsf{s}_{\rho r}}(\mathsf{u}_{\rho r}) + z)$ is small. To avoid this difficulty, we will need to carry out a careful analysis of, roughly speaking, the 'excursions' that P_r makes in and out of the ball $B_{\mathsf{s}_{\rho r}}(\mathsf{u}_{\rho r}) + z$.

In Subsection 5.11, we will conclude the proof that E_r satisfies hypothesis C using the result of Subsection 5.10 and the triangle inequality.

5.2 | Existence of a shortcut with positive probability

Throughout the rest of this section, we let

$$\lambda \in \left(0, 10^{-100} \min\left\{\mathfrak{c}_*, 1/\mathfrak{C}_*, (\mathfrak{c}_*/\mathfrak{C}_*)^2\right\}\right) \tag{5.3}$$

be a small constant to be chosen later, in a manner depending only on the laws of D_h and $\widetilde{D}_h$ (not on $\mathbb{p}$). We will frequently use λ in the definitions of events and other objects when we need a small constant whose particular value is unimportant.

In this subsection, we will prove that for each $r \in \mathcal{R}_0$, it holds with positive probability (uniformly in $r \in \mathcal{R}_0$) that there is a 'good' pair of non-singular points $u, v \in \overline{B}_r(0)$ such that $\widetilde{D}_h(u, v) \leqslant \mathfrak{c}_0' D_h(u, v)$ and certain regularity conditions hold. In later subsections, we will use the long-range independence of the GFF to say that with high probability, there are many such pairs of points contained in our open set U_r. To state our result, we need the following definition.

Definition 5.1. Let $z \in \mathbb{C}$ and $b > a > 0$. A *horizontal or vertical half-annulus* $H \subset \mathbb{A}_{a,b}(z)$ is the intersection of $\mathbb{A}_{a,b}(z)$ with one of the four half-planes

$$\{w \in \mathbb{C} : \operatorname{Re} w > \operatorname{Re} z\}, \quad \{w \in \mathbb{C} : \operatorname{Re} w < \operatorname{Re} z\},$$
$$\{w \in \mathbb{C} : \operatorname{Im} w > \operatorname{Im} z\}, \quad \text{or} \quad \{w \in \mathbb{C} : \operatorname{Im} w < \operatorname{Im} z\}.$$

Lemma 5.2. *Let α and $\mathcal{R}_0$ be as in (5.2). There exists $\mathsf{t} \in (0, \lambda(1-\alpha)^2]$, $\mathsf{S} > 3$, and $\mathsf{p} \in (0,1)$ (depending only on λ and the laws of D_h and $\widetilde{D}_h$) such that for each $r \in \mathcal{R}_0$, there exists a deterministic horizontal or vertical half-annulus $\mathsf{H}_r \subset \mathbb{A}_{\alpha r, r}(0)$, a deterministic radius $\mathsf{s}_r \in [\mathsf{t}r, \mathsf{t}^{1/2}r] \cap \{4^{-k}r\}_{k\in\mathbb{N}}$, and deterministic points*

$$\mathsf{u}_r \in \partial \mathsf{H}_r \cap \left\{\alpha r e^{i\lambda \mathsf{t} k} : k \in [1, 2\pi\lambda^{-1}\mathsf{t}^{-1}]_{\mathbb{Z}}\right\} \quad \textit{and}$$
$$\mathsf{v}_r \in \partial \mathsf{H}_r \cap \left\{r e^{i\lambda \mathsf{t} k} : k \in [1, 2\pi\lambda^{-1}\mathsf{t}^{-1}]_{\mathbb{Z}}\right\} \tag{5.4}$$

such that with probability at least p, the following is true. There exist non-singular points $u \in \partial B_{\alpha r}(0) \cap B_{\mathsf{s}_r/2}(\mathsf{u}_r)$ and $v \in \partial B_r(0) \cap B_{\mathsf{s}_r/2}(\mathsf{v}_r)$ with the following properties.

(1) $\widetilde{D}_h(u, v) \leqslant \mathfrak{c}_0' D_h(u, v)$.
(2) *There is a $\widetilde{D}_h$-geodesic $\widetilde{P}$ from u to v which is contained in $\overline{\mathsf{H}}_r$.*
(3) *The one-dimensional Lebesgue measure of the set*

$$\left\{x \in \partial B_{\mathsf{s}_r}(\mathsf{u}_r) : D_h\left(x, u; \overline{B}_{\mathsf{s}_r}(\mathsf{u}_r)\right) > \lambda \widetilde{D}_h(u, v)\right\}$$

is at most $(\lambda/2)\mathsf{s}_r$. Moreover, the same is true with v and v_r in place of u and u_r.
(4) *There exists $t \in [3r, \mathsf{S}r]$ such that*

$$D_h\left(\text{around } \mathbb{A}_{t,2t}(0)\right) \leqslant \lambda D_h\left(\text{across } \mathbb{A}_{2t,3t}(0)\right).$$

See Figure 13 for an illustration of the statement of Lemma 5.2. Most of this subsection is devoted to the proof of Lemma 5.2. Before discussing the proof, we will first discuss the motivation for the various conditions in the lemma statement.

In Subsection 5.4, we will consider a small but fixed constant $\rho \in (0,1)$. To build the set $\mathsf{U}_r = \mathsf{U}_{0,r}$ appearing in Section 4, we will use long narrow tubes to 'link up' several sets of the form $\mathsf{H}_{\rho r} \cup B_{\mathsf{s}_{\rho r}}(\mathsf{u}_{\rho r}) \cup B_{\mathsf{s}_{\rho r}}(\mathsf{v}_{\rho r}) + z$, for varying choices of $z \in \partial B_{2r}(0)$. We need U_r to be deterministic, which is why we need to make a deterministic choice of the half-annulus H_r, the radius s_r, and the points u_r and v_r in Lemma 5.2. Furthermore, we want there to be only finitely many possibilities for the set $r^{-1}\mathsf{U}_r$, which allows us to get certain estimates for U_r trivially by taking a maximum over the possibilities. This is why we require that H_r is a vertical or horizontal half-annulus and why we require that the points u_r and v_r belong to the finite sets in (5.4).

Our set U_r will have 'bottlenecks' at the balls $B_{\mathsf{s}_{\rho r}}(\mathsf{u}_{\rho r}) + z$ and $B_{\mathsf{s}_{\rho r}}(\mathsf{v}_{\rho r}) + z$, so that any path which travels more than a constant-order Euclidean distance inside the set U_r will have to enter many of these balls. The requirement that $u \in B_{\mathsf{s}_{\rho r}/2}(\mathsf{u}_{\rho r})$ and $v \in B_{\mathsf{s}_{\rho r}/2}(\mathsf{v}_{\rho r})$ is needed to force a path which spends a lot of time in U_r to get close to u and v. The requirement that $\widetilde{P} \subset \overline{\mathsf{H}}_r$ in condition 2 is needed to ensure that subtracting from h a large bump function which attains its maximal value at each point of U_r decreases $\widetilde{D}_h(u,v)$ by at least as much as $D_h(u,v)$, so the condition $\widetilde{D}_h(u,v) \leqslant \mathfrak{c}_0' D_h(u,v)$ is preserved.

Condition 3 in Lemma 5.2 is needed to upper-bound the LQG distance from a path to each of u and v, once we know that it gets Euclidean-close to u and v (this is done in Subsection 5.10). The reason why our distance bound is in terms of $\widetilde{D}_h(u,v)$ is that we eventually want to show that the $\widetilde{D}_{h-\mathsf{f}_r}$-distance from a $D_{h-\mathsf{f}_r}$-geodesic to each of u and v is at most a small constant times $\widetilde{D}_{h-\mathsf{f}_r}(u,v)$. We will then use condition 1 in Lemma 5.2 and the triangle inequality to deduce hypothesis C. Note that condition 3 includes a bound on D_h-distances, but this immediately implies a bound for $\widetilde{D}_h$-distances due to the bi-Lipschitz equivalence of D_h and $\widetilde{D}_h$ (1.20).

The only purpose of condition 4 is to ensure that the event in the lemma statement depends locally on h (see Lemma 5.7). This local dependence is not automatically true since a D_h-geodesic from u to v could get very Euclidean-far away from u and v.

We now turn our attention to the proof of Lemma 5.2. To this end, let us first record what we get from the Definition 3.9 of $\widetilde{H}_r(\alpha, \mathfrak{c}_0')$ and the Definition (5.2) of $\mathcal{R}_0$.

Lemma 5.3. *For each $r \in \mathcal{R}_0$, there is a deterministic horizontal or vertical half-annulus $\mathsf{H}_r \subset \mathbb{A}_{\alpha r, r}(0)$ such that with probability at least $p_0/4$, there exist non-singular points $u \in \partial B_{\alpha r}(0)$ and $v \in \partial B_r(0)$ with the following properties.*

(1) $\widetilde{D}_h(u,v) \leqslant \mathfrak{c}_0' D_h(u,v)$.
(2) *There is a $\widetilde{D}_h$-geodesic $\widetilde{P}$ from u to v which is contained in $\overline{\mathsf{H}}_r$.*
(3) *With $\theta = \theta(1/2)$ as in Lemma 2.13, for each $\delta \in (0,(1-\alpha)^2]$,*

$$\max\left\{\widetilde{D}_h(u, \partial B_{\delta r}(u)), \widetilde{D}_h(v, \partial B_{\delta r}(v))\right\} \leqslant \delta^\theta \widetilde{D}_h(u,v).$$

Proof. By Definition 3.9 of $\widetilde{H}_r(\alpha, \mathfrak{c}_0')$ and the definition (5.2) of $\mathcal{R}_0$, for each $r \in \mathcal{R}_0$ it holds with probability at least p_0 that there exist $u \in \partial B_{\alpha r}(0)$ and $v \in \partial B_r(0)$ such that conditions 1 and 3 in the lemma statement hold and there is a $\widetilde{D}_h$-geodesic $\widetilde{P}$ from u to v which is contained in $\overline{\mathbb{A}}_{\alpha r, r}(0)$ and has Euclidean diameter at most $r/100$. Since $\widetilde{P} \subset \overline{\mathbb{A}}_{\alpha r, r}(0)$ and $\widetilde{P}$ has Euclidean diameter at most $r/100$, trivial geometric considerations show that $\widetilde{P}$ must be contained in the closure of one of the four horizontal or vertical half-annuli of $\mathbb{A}_{\alpha r, r}(0)$. Hence, we can choose one such half-annulus H_r in a deterministic manner such that with probability at least $p_0/4$, conditions 1 and 3 in the lemma statement hold and $\widetilde{P} \subset \overline{\mathsf{H}}_r$, that is, condition 2 holds. □

Lemma 5.3 gives us a pair of points u, v satisfying conditions 1 and 2 in Lemma 5.2. We still need to check conditions 3 and 4. Condition 3 will require the most work. To get this condition, we want to apply Lemma 2.10. However, the points u and v are random, so we cannot just apply the lemma directly. Instead, we will apply Lemma 2.10 in conjunction with Lemma 2.1 (independence across concentric annuli) and a union bound to cover space by balls where an event occurs which is closely related to the one in Lemma 2.10. Then, we will use a geometric argument based on condition 3 of Lemma 5.3 to transfer from an estimate for balls containing u and v to an estimate for u and v themselves.

Let us now define the event to which we will apply Lemma 2.1. For $z \in \mathbb{C}$, $s > 0$, and $R > 0$, let $G_s(z; R)$ be the event that the following is true.

(1) The one-dimensional Lebesgue measure of the set of $x \in \partial B_s(z)$ for which

$$\widetilde{D}_h\left(x, \partial B_{s/2}(z); \overline{\mathbb{A}}_{s/2,s}(z)\right) > R s^{\xi Q} e^{\xi h_s(z)}$$

is at most $(\lambda/2)s$.

(2) $\widetilde{D}_h(\text{around } \mathbb{A}_{s/2,s}(z)) \leqslant R s^{\xi Q} e^{\xi h_s(z)}$.

(3) $\widetilde{D}_h(\text{across } \mathbb{A}_{s/2,s}(z)) \geqslant (1/R) s^{\xi Q} e^{\xi h_s(z)}$.

Since the event $G_s(z;R)$ involves only internal distances in $\overline{\mathbb{A}}_{s/2,s}(z)$, the locality property (Axiom II; see also Subsection 2.2) implies that $G_s(z;R)$ is almost surely determined by $h|_{\overline{\mathbb{A}}_{s/2,s}(z)}$. Furthermore, by Weyl scaling (Axioms III), the occurrence of $G_s(z;R)$ is unaffected by adding a constant to h. Therefore,

$$G_s(z;R) \in \sigma\left((h - h_{2s}(z))|_{\overline{\mathbb{A}}_{s/2,s}(z)}\right). \tag{5.5}$$

We can also arrange that the probability of $G_s(z;R)$ is close to 1 by making R large.

Lemma 5.4. *For each $p \in (0,1)$, there exists $R > 0$, depending only on p, λ and the law of $\widetilde{D}_h$, such that for each $z \in \mathbb{C}$ and each $s > 0$, we have $\mathbb{P}[G_s(z;R)] \geqslant p$.*

Proof. By Lemma 2.10 (and the fact that a path from $x \in \partial B_s(z)$ to z must hit $\partial B_{s/2}(z)$), if R is chosen to be sufficiently large, depending only on p and the law of $\widetilde{D}_h$, then the first condition in the definition of $G_s(z;R)$ has probability at least $1 - p/3$. By tightness across scales (Axiom V′), after possibly increasing R we can arrange that the other two conditions in the definition of $G_s(z;R)$ also have probability at least p. □

Let us now apply Lemma 2.1 to get the following.

Lemma 5.5. *There exists $R > 0$, depending only on λ and the law of $\widetilde{D}_h$, such that for each $r > 0$, it holds with polynomially high probability as $\varepsilon \to 0$ (at a rate depending only on λ and the law of $\widetilde{D}_h$) such that the following is true. For each point*

$$z \in \left\{\alpha r e^{i\lambda\varepsilon k} \,:\, k \in [1, 2\pi\lambda^{-1}\varepsilon^{-1}]_{\mathbb{Z}}\right\} \cup \left\{r e^{i\lambda\varepsilon k} \,:\, k \in [1, 2\pi\lambda^{-1}\varepsilon^{-1}]_{\mathbb{Z}}\right\}, \tag{5.6}$$

we have

$$\#\left\{k \in \left[\frac{1}{2}\log_4 \varepsilon^{-1}, \log_4 \varepsilon^{-1}\right]_{\mathbb{Z}} \,:\, G_{4^{-k}r}(z;R) \text{ occurs}\right\} \geqslant \frac{3}{8}\log_4 \varepsilon^{-1}. \tag{5.7}$$

Proof. By (5.5) and Lemma 5.4 (applied with p sufficiently close to 1), we can apply Lemma 2.1 (independence across concentric annuli) to get the following. There exists $R > 0$ as in the lemma statement such that for each $z \in \mathbb{C}$ and each $r > 0$,

$$\mathbb{P}\left[\#\left\{k \in \left[\frac{1}{2}\log_4 \varepsilon^{-1}, \log_4 \varepsilon^{-1}\right]_{\mathbb{Z}} \,:\, G_{4^{-k}r}(z;R) \text{ occurs}\right\} \geqslant \frac{3}{8}\log_4 \varepsilon^{-1}\right] \geqslant 1 - O_\varepsilon(\varepsilon^2).$$

The lemma follows from this and a union bound over the $O_\varepsilon(\varepsilon^{-1})$ points in the set (5.6). □

The following lemma is the main step in the proof of Lemma 5.2.

Lemma 5.6. *There exist* $\mathsf{t} \in (0, \lambda(1-\alpha)^2]$ *and* $\mathsf{p} \in (0,1)$ *(depending only on* λ *and the laws of* D_h *and* $\widetilde{D}_h$*) such that for each* $r \in \mathcal{R}_0$*, there exist a deterministic vertical or horizontal half-annulus* $\mathsf{H}_r \subset \mathbb{A}_{\alpha r, r}(0)$*, a deterministic radius* $\mathsf{s}_r \in [\mathsf{t}r, \mathsf{t}^{1/2} r] \cap \{4^{-k} r\}_{k \in \mathbb{N}}$*, and deterministic points* $\mathsf{u}_r, \mathsf{v}_r \in \partial \mathsf{H}_r$ *as in (5.4) such that with probability at least* $2\mathsf{p}$*, the following is true. There exist non-singular points* $u \in \partial B_{\alpha r}(0) \cap B_{\mathsf{s}_r}(\mathsf{u}_r)$ *and* $v \in \partial B_r(0) \cap B_{\mathsf{s}_r}(\mathsf{v}_r)$ *such that conditions 1, 2, and 3 from Lemma 5.2 hold.*

Proof. Step 1: Setup. Let α and p_0 be as in the definition of $\mathcal{R}_0$ from (5.2). Let the half-annulus H_r for $r \in \mathcal{R}_0$ be as in Lemma 5.3 and let $R > 0$ be as in Lemma 5.5. Also let $\mathsf{t} > 0$ be small enough so that the event of Lemma 5.5 with t in place of ε occurs with probability at least $1 - p_0/8$. We can arrange that t is small enough so that

$$\mathsf{t} \leqslant \lambda(1-\alpha)^2 \quad \text{and} \quad (2R^2+1)(2\mathsf{t})^\theta \leqslant \lambda^2, \tag{5.8}$$

where θ is as in Lemma 5.3. Then with probability at least $p_0/8$, the event of Lemma 5.3 and the event of Lemma 5.5 with $\varepsilon = \mathsf{t}$ both occur. Henceforth, assume that these two events occur.

Let $\widetilde{P}$ be the $\widetilde{D}_h$-geodesic from u to v which is contained in $\overline{\mathsf{H}}_r$, as in Lemma 5.3. By the conditions in Lemma 5.3, the conditions 1 and 2 in the statement of Lemma 5.2 hold for this choice of u, v, and $\widetilde{P}$. It remains to deal with condition 3.

Step 2: Reducing to a statement for a random radius and pair of points. We can choose random points

$$z_1 \in \partial \mathsf{H}_r \cap \left\{ \alpha r e^{i\lambda \mathsf{t} k} : k \in [1, 2\pi \lambda^{-1} \mathsf{t}^{-1}]_{\mathbb{Z}} \right\} \quad \text{and}$$
$$z_2 \in \partial \mathsf{H}_r \cap \left\{ r e^{i\lambda \mathsf{t} k} : k \in [1, 2\pi \lambda^{-1} \mathsf{t}^{-1}]_{\mathbb{Z}} \right\}$$

such that

$$|u - z_1| \leqslant \mathsf{t}r/50 \quad \text{and} \quad |v - z_2| \leqslant \mathsf{t}r/50. \tag{5.9}$$

The event of Lemma 5.5 (with $\varepsilon = \mathsf{t}$) implies that for each $i \in \{1,2\}$, there are at least $\frac{3}{8}\log_4 \mathsf{t}^{-1}$ values of $k \in [\frac{1}{2}\log_4 \mathsf{t}^{-1}, \log_4 \mathsf{t}^{-1}]_{\mathbb{Z}}$ such that $G_{4^{-k}r}(z_i; R)$ occurs. Since the number of choices for k is at most $\frac{1}{2}\log_4 \mathsf{t}^{-1}$, there must be some (random) $k_* \in [\frac{1}{2}\log_4 \mathsf{t}^{-1}, \log_4 \mathsf{t}^{-1}]_{\mathbb{Z}}$ such that $G_{4^{-k_*}r}(z_1; R) \cap G_{4^{-k_*}r}(z_2; R)$ occurs. We pick one such value of k_* in a measurable manner and set

$$s := 4^{-k_*} r, \quad \text{so that} \quad s \in [\mathsf{t}r, \mathsf{t}^{1/2} r] \cap \{4^{-k} r\}_{k \in \mathbb{N}}. \tag{5.10}$$

We claim that condition 3 in Lemma 5.2 holds with s in place of s_r and z_1, z_2 in place of $\mathsf{u}_r, \mathsf{v}_r$. Once the claim has been proven, we have that with probability at least $p_0/8$, the conditions in the lemma statement hold with the random variables s, z_1, z_2 in place of the deterministic parameters $\mathsf{s}_r, \mathsf{u}_r, \mathsf{v}_r$. The number of possible choices for s is at most $\frac{1}{2}\log_4 \mathsf{t}^{-1}$ and the number of possible choices for each of z_1, z_2 is at most a constant (depending only on λ and the laws of D_h and $\widetilde{D}_h$) times t^{-1}. Therefore, our claim implies that there is some constant $\mathsf{p} > 0$ (which depends only on p_0 and t, hence only on the laws of D_h and $\widetilde{D}_h$) and a *deterministic* choice of parameters $\mathsf{s}_r, \mathsf{u}_r$, and v_r such that with probability at least $2\mathsf{p}$, the conditions of the lemma statement hold for $\mathsf{s}_r, \mathsf{u}_r$, and v_r.

Step 3: Estimates for distances in $B_s(z_1)$ and $B_s(z_2)$. It remains to prove the claim in the preceding paragraph. By our choices of z_1, z_2 (5.9) and s (5.10),

$$u \in B_{s/2}(z_1) \subset B_s(z_1) \subset B_{2\mathfrak{t}^{1/2}r}(u) \quad \text{and} \quad v \in B_{s/2}(z_2) \subset B_s(z_2) \subset B_{2\mathfrak{t}^{1/2}r}(v). \tag{5.11}$$

From this, condition 3 from Lemma 5.3 (with $\delta = 2\mathfrak{t}^{1/2}$), and the definition of $G_s(z_i; R)$, we obtain

$$\begin{aligned}(2\mathfrak{t}^{1/2})^{\theta}\widetilde{D}_h(u,v) &\geqslant \max\left\{\widetilde{D}_h\left(u, \partial B_{2\mathfrak{t}^{1/2}r}(u)\right), \widetilde{D}_h\left(v, \partial B_{2\mathfrak{t}^{1/2}r}(v)\right)\right\} \quad \text{(by Lemma 5.3)}\\ &\geqslant \max\left\{\widetilde{D}_h(u, \partial B_s(z_1)), \widetilde{D}_h(v, \partial B_s(z_2))\right\} \quad \text{(by (5.11))}\\ &\geqslant \max_{i\in\{1,2\}} \widetilde{D}_h\left(\text{across } \mathbb{A}_{s/2,s}(z_i)\right)\\ &\qquad\qquad (\text{since } u \in B_{s/2}(z_1) \text{ and } v \in B_{s/2}(z_2))\\ &\geqslant \frac{1}{R}\max_{i\in\{1,2\}} s^{\xi Q}e^{\xi h_s(z_i)} \quad (\text{by condition 3 for } G_s(z_i;R)).\end{aligned} \tag{5.12}$$

We now apply (5.12) to upper-bound the quantities $s^{\xi Q}e^{\xi h_s(z_i)}$ appearing in conditions 1 and 2 in the definition of $G_s(z_i; R)$. Upon doing so, we obtain the following observations for $i = 1, 2$.

(i) The one-dimensional Lebesgue measure of the set of $x \in \partial B_s(z_i)$ for which

$$\widetilde{D}_h\left(x, \partial B_{s/2}(z_i); \overline{B}_s(z_i)\right) > R^2(2\mathfrak{t}^{1/2})^{\theta}\widetilde{D}_h(u,v)$$

is at most $(\lambda/2)s$.

(ii) We have

$$\widetilde{D}_h\left(\text{around } \mathbb{A}_{s/2,s}(z_i)\right) \leqslant R^2(2\mathfrak{t}^{1/2})^{\theta}\widetilde{D}_h(u,v). \tag{5.13}$$

Step 4: Checking condition 3. If $x \in \partial B_s(z_1)$, then the union of any path from x to $\partial B_{s/2}(z_1)$, any path in $\mathbb{A}_{s/2,s}(z_1)$ which disconnects the inner and outer boundaries of $\mathbb{A}_{s/2,s}(z_i)$, and any path from u to $\partial B_s(z_1)$ must contain a path from u to x (see Figure 14). By (5.13) and the second inequality in (5.12), we therefore have

$$\begin{aligned}\widetilde{D}_h\left(x, u; \overline{B}_s(z_1)\right) &\leqslant \widetilde{D}_h\left(x, \partial B_{s/2}(z_1); \overline{B}_s(z_1)\right) + \widetilde{D}_h\left(\text{around } \mathbb{A}_{s/2,s}(z_1)\right) + \widetilde{D}_h(u, \partial B_s(z_1))\\ &\leqslant \widetilde{D}_h\left(x, \partial B_{s/2}(z_1); \overline{B}_s(z_1)\right) + \left(R^2+1\right)(2\mathfrak{t}^{1/2})^{\theta}\widetilde{D}_h(u,v).\end{aligned} \tag{5.14}$$

By combining (5.14) with observation (i) above, we get that for all $x \in \partial B_s(z_1)$ except on a set of one-dimensional Lebesgue measure at most $(\lambda/2)s$,

$$\widetilde{D}_h\left(x, u; \overline{B}_s(z_1)\right) \leqslant (2R^2+1)(2\mathfrak{t})^{\theta}\widetilde{D}_h(u,v). \tag{5.15}$$

By (5.15) and our choice of $\mathfrak{t}$ in (5.8), we get that for all $x \in \partial B_s(z_1)$ except on a set of one-dimensional Lebesgue measure at most $(\lambda/2)s$,

$$\widetilde{D}_h\left(x, u; \overline{B}_s(z_1)\right) \leqslant \lambda^2\widetilde{D}_h(u,v). \tag{5.16}$$

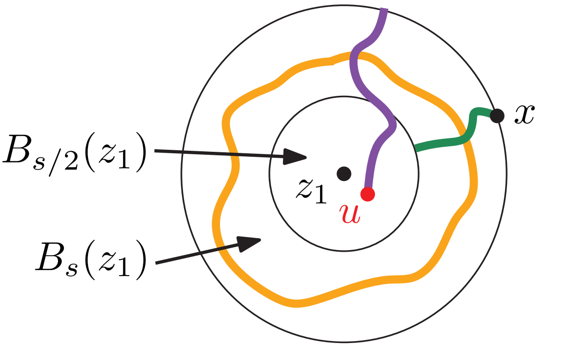

FIGURE 14 Illustration of the proof of condition 3 in Lemma 5.2 with (s, z_1) in place of $(\mathsf{s}_r, \mathsf{u}_r)$. The concatenation of the purple, orange, and green paths in the figure contains a path from u to x. The $\widetilde{D}_h$-length of the purple path can be bounded above in terms of $\widetilde{D}_h(u, v)$ by condition 3 from Lemma 5.3. The $\widetilde{D}_h$-length of the orange path can be bounded above in terms of $\widetilde{D}_h(u, v)$ using (5.13), which in turn is proven using conditions 2 and 3 in the definition of $G_s(z_1; R)$. For most points $x \in \partial B_s(z_1)$, the $\widetilde{D}_h$-length of the green path can be bounded above in terms of $\widetilde{D}_h(u, v)$ by condition 1 in the definition of $G_s(z_1; R)$.

Since $\lambda < \mathfrak{c}_*$, the estimate (5.16) together with the bi-Lipschitz equivalence of D_h and $\widetilde{D}_h$ implies that

$$D_h\left(x, u; \overline{B}_s(z_1)\right) \leqslant \lambda \widetilde{D}_h(u, v). \tag{5.17}$$

This gives condition 3 in Lemma 5.2 with z_1 in place of u_r and s in place of s_r. The analogous bound with z_2 in place of v_r and s in place of s_r is proven similarly. □

Proof of Lemma 5.2. Let p be as in Lemma 5.6. In light of Lemma 5.6, it suffices to find $\mathsf{S} > 3$ such that with probability at least $1 - \mathsf{p}$, condition 4 in the lemma statement holds, that is, there exists $t \in [3r, \mathsf{S}r]$ such that

$$D_h\left(\text{around } \mathbb{A}_{t,2t}(0)\right) \leqslant \lambda D_h\left(\text{across } \mathbb{A}_{2t,3t}(0)\right). \tag{5.18}$$

One can easily check using a 'subtracting a bump function' argument and Weyl scaling (Axiom III) that there exists $q \in (0, 1)$ (depending only on λ and the law of D_h) such that for each fixed $t > 0$, the probability of the event in (5.18) is at least q. See [21, Lemma 6.1] for similar argument. We can then apply assertion 2 of Lemma 2.1 to a collection of logarithmically many evenly spaced radii $t_k \in [3r, \mathsf{S}r]$ to find that the probability that there does not exist $t \in [3r, \mathsf{S}r]$ such that (5.18) holds decays like a negative power of S as $\mathsf{S} \to \infty$, at a rate which depends only on the laws of D_h and $\widetilde{D}_h$. We can therefore choose S large enough so that this probability is at most p, as required. □

5.3 | Building block event

We will use Lemma 5.2 to define an event which will be the 'building block' for the event $\mathsf{E}_r = \mathsf{E}_{0,r}$. Let the parameters $\mathsf{S}, \mathsf{p} > 0$, the half-annulus $\mathsf{H}_r \subset \mathbb{A}_{\alpha r, r}(0)$, the radius $\mathsf{s}_r \in [\mathsf{t}r, \mathsf{t}^{1/2} r] \cap \{4^{-k} r\}_{k \in \mathbb{N}}$, and the points

$$\mathsf{u}_r \in \partial \mathsf{H}_r \cap \left\{\alpha r e^{i\lambda \mathsf{t} k} : k \in [1, 2\pi\lambda^{-1}\mathsf{t}^{-1}]_{\mathbb{Z}}\right\} \quad \text{and}$$

$$\mathsf{v}_r \in \partial \mathsf{H}_r \cap \left\{ r e^{i\lambda \mathsf{t} k} \,:\, k \in [1, 2\pi\lambda^{-1}\mathsf{t}^{-1}]_{\mathbb{Z}} \right\}$$

be as in Lemma 5.2.

For $z \in \mathbb{C}$, let

$$\begin{aligned} \mathsf{H}_{z,r} &:= \mathsf{H}_r + z \subset \mathbb{A}_{\alpha r, r}(z), \\ \mathsf{u}_{z,r} &:= \mathsf{u}_r + z \in \partial \mathsf{H}_{z,r} \cap \partial B_{\alpha r}(z), \quad \text{and} \\ \mathsf{v}_{z,r} &:= \mathsf{v}_r + z \in \partial \mathsf{H}_{z,r} \cap \partial B_r(z). \end{aligned}$$

We also let $\mathsf{F}_{z,r}$ be the event of Lemma 5.2 with the translated field $h(\cdot - z)$ in place of h. That is, $\mathsf{F}_{z,r}$ is the event that there exist non-singular points $u \in \partial B_{\alpha r}(z) \cap B_{\mathsf{s}_r/2}(\mathsf{u}_{z,r})$ and $v \in \partial B_r(z) \cap B_{\mathsf{s}_r/2}(\mathsf{v}_{z,r})$ with the following properties.

(1) $\widetilde{D}_h(u,v) \leqslant \mathfrak{c}_0' D_h(u,v)$.
(2) There is a $\widetilde{D}_h$-geodesic $\widetilde{P}$ from u to v which is contained in $\overline{\mathsf{H}}_{z,r}$.
(3) The one-dimensional Lebesgue measure of the set

$$\left\{ x \in \partial B_{\mathsf{s}_r}(\mathsf{u}_{z,r}) \,:\, D_h\left(x, u; \overline{B}_{\mathsf{s}_r}(\mathsf{u}_{z,r})\right) > \lambda \widetilde{D}_h(u,v) \right\}$$

is at most $(\lambda/2)\mathsf{s}_r$ and the same is true with v and $\mathsf{v}_{z,r}$ in place of u and $\mathsf{u}_{z,r}$.
(4) There exists $t \in [3r, \mathsf{S}r]$ such that

$$D_h\left(\text{around } \mathbb{A}_{t,2t}(z)\right) \leqslant \lambda D_h\left(\text{across } \mathbb{A}_{2t,3t}(z)\right).$$

By Lemma 5.2, the translation invariance of the law of h, viewed modulo additive constant, and the translation invariance of D_h and $\widetilde{D}_h$ (Axiom IV′), we have

$$\mathbb{P}[\mathsf{F}_{z,r}] \geqslant \mathsf{p}, \quad \forall z \in \mathbb{C}, \quad \forall r \in \mathcal{R}_0. \tag{5.19}$$

The other property of $\mathsf{F}_{z,r}$ which we need is that it depends locally on h.

Lemma 5.7. *The event $\mathsf{F}_{z,r}$ is almost surely determined by the restriction of h to $B_{3\mathsf{S}r}(z)$, viewed modulo additive constant.*

Proof. It is clear from Weyl scaling (Axiom III) that adding a constant to h does not affect the occurrence of $\mathsf{F}_{z,r}$, so $\mathsf{F}_{z,r}$ is almost surely determined by h, viewed modulo additive constant. It therefore suffices to show that $\mathsf{F}_{z,r}$ is almost surely determined by $h|_{B_{3\mathsf{S}r}(z)}$.

To this end, we first observe that by locality (Axiom II), the condition 4 in the definition of $\mathsf{F}_{z,r}$ is almost surely determined by $h|_{B_{3\mathsf{S}r}(z)}$. We claim that if this condition holds, then

$$D_h(x,y) = D_h(x,y; B_{3\mathsf{S}r}(z)), \quad \forall x,y \in B_{3r}(z); \tag{5.20}$$

and the same is true with $\widetilde{D}_h$ in place of D_h.

Indeed, it is clear that (5.20) holds if $x = y$ or if either x or y is a singular point. Hence, we can assume that $x \neq y$ and that x and y are not singular points. To prove (5.20), it suffices to show that each D_h-geodesic from x to y is contained in $B_{3\mathsf{S}r}(z)$. To see this, let P be a path from x to

y which exits $B_{3\mathsf{S}r}(z)$. Let $t \in [3r, \mathsf{S}r]$ be as in condition 4 in the definition of $\mathsf{F}_{z,r}$. We can find a path $\pi \subset \mathbb{A}_{t,2t}(z)$ which disconnects the inner and outer boundaries of $\mathbb{A}_{t,2t}(z)$ such that

$$\mathrm{len}(\pi; D_h) < D_h\big(\text{across } \mathbb{A}_{2t,3t}(z)\big).$$

Since $x, y \in B_{3r}(z)$ and P exists $B_{3t}(z)$, the path P must hit π, then cross between the inner and outer boundaries of $\mathbb{A}_{2t,3t}(z)$, then subsequently hit π again. This means that there are two points of $P \cap \pi$ such that D_h-length of the segment of P between the two points is at least $D_h(\text{across } \mathbb{A}_{2t,3t}(z))$. The D_h-distance between these two points is at most the D_h-length of π, which by our choice of π is strictly less than $D_h(\text{across } \mathbb{A}_{2t,3t}(z))$. Hence, P cannot be a D_h-geodesic. We therefore obtain (5.20) for D_h.

To prove (5.20) with $\widetilde{D}_h$ in place of D_h, we observe that if t is as in condition 4 in the definition of $\mathsf{F}_{z,r}$, then

$$\begin{aligned}\widetilde{D}_h\big(\text{around } \mathbb{A}_{t,2t}(z)\big) \leqslant \mathfrak{C}_* D_h\big(\text{around } \mathbb{A}_{t,2t}(z)\big) &\leqslant \lambda \mathfrak{C}_* D_h\big(\text{across } \mathbb{A}_{2t,3t}(z)\big)\\ &\leqslant \lambda(\mathfrak{C}_*/\mathfrak{c}_*)\widetilde{D}_h\big(\text{across } \mathbb{A}_{2t,3t}(z)\big).\end{aligned}$$

We have $\lambda(\mathfrak{C}_*/\mathfrak{c}_*) < 1$, so we can now prove (5.20) with $\widetilde{D}_h$ in place of D_h via exactly the same argument given above.

Due to (5.20), the definition of $\mathsf{F}_{z,r}$ is unaffected if we require that $\widetilde{P}$ is a $\widetilde{D}_h(\cdot,\cdot; B_{3\mathsf{S}r}(z))$-geodesic instead of a $\widetilde{D}_h$-geodesic and we replace D_h-distances and $\widetilde{D}_h$-distances by $D_h(\cdot,\cdot; B_{3\mathsf{S}r}(z))$-distances and $\widetilde{D}_h(\cdot,\cdot; B_{3\mathsf{S}r}(z))$-distances throughout. It then follows from locality (Axiom II) that $\mathsf{F}_{z,r}$ is almost surely determined by $h|_{B_{3\mathsf{S}r}(z)}$, as required. □

5.4 | Definitions of U_r, V_r, and f_r

The definitions of $\mathsf{E}_r, \mathsf{U}_r, \mathsf{V}_r$, and f_r will depend on parameters

$$1 > \mathsf{a}_1 > \frac{1}{\mathsf{A}_2} > \mathsf{a}_3 > \mathsf{a}_4 > \mathsf{a}_5 > \mathsf{a}_6 > \frac{1}{\mathsf{A}_7} > \frac{1}{\mathsf{A}_8} > \mathsf{a}_9 > \frac{1}{\mathsf{A}_{10}}, \tag{5.21}$$

which will be chosen in Subsection 5.5 in a manner depending only on $\mathbb{p}$, λ, and the laws of D_h and $\widetilde{D}_h$. The parameters are listed in (5.21) in the order in which they will be chosen. Each parameter will be allowed to depend on the earlier parameters as well as the number λ from (5.3) (which is allowed to depend only on the laws of D_h and $\widetilde{D}_h$, not on $\mathbb{p}$). Each parameter will also be allowed to depend on the numbers $\alpha, \mathsf{t}, \mathsf{S}, \mathsf{p}$ appearing in Lemma 5.2 (which have already been fixed, in a manner depending only on λ and the laws of D_h and $\widetilde{D}_h$).

Also let $\rho \in (0,1)$ be a small parameter which will also be chosen in Subsection 5.5 in a manner depending only on λ and the laws of D_h and $\widetilde{D}_h$. We will have

$$\mathsf{a}_4 > \rho > \mathsf{a}_5, \tag{5.22}$$

and ρ will be allowed to depend on $\lambda, \mathsf{a}_1, \mathsf{A}_2, \mathsf{a}_3, \mathsf{a}_4$ and the numbers appearing in Lemma 5.2.

In the rest of this subsection, we will give the definition of the open sets U_r and V_r and the bump function f_r in terms of ρ and the parameters from (5.21). See Figure 15 for an illustration.

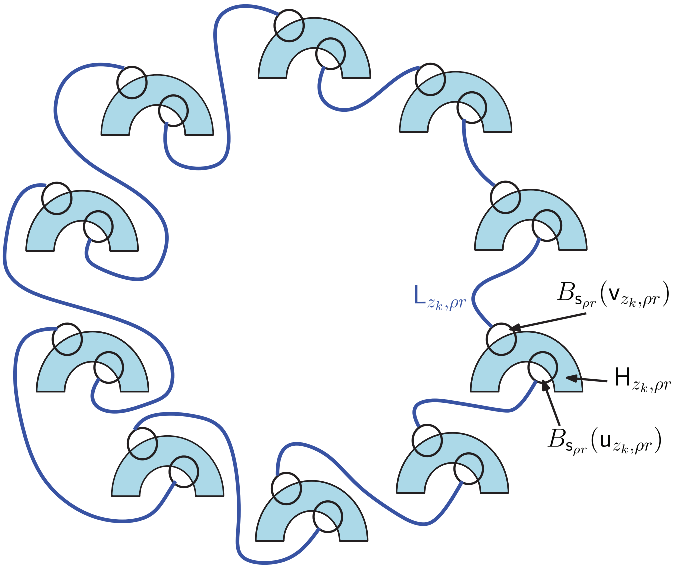

FIGURE 15 The figure shows the sets $\mathsf{H}_{z,\rho r}$, $B_{\mathsf{s}_{\rho r}}(\mathsf{u}_{z,\rho r})$, $B_{\mathsf{s}_{\rho r}}(\mathsf{v}_{z,\rho r})$, and $\mathsf{L}_{z,\rho r}$ for $z \in \mathsf{Z}_r$. We define U_r to be the union of $\mathsf{H}_{z,\rho r}$, $B_{\mathsf{s}_{\rho r}}(\mathsf{u}_{z,\rho r})$, $B_{\mathsf{s}_{\rho r}}(\mathsf{v}_{z,\rho r})$ and $B_{\lambda \mathsf{t} \rho r}(\mathsf{L}_{z,\rho r})$ for $z \in \mathsf{Z}_r$. We define $\mathsf{V}_r := B_{\mathsf{a}_9 r}(\mathsf{U}_r)$. The bump function f_r is supported on V_r and attains its maximal value A_8 at every point of U_r.

For $r \in \rho^{-1}\mathcal{R}_0$, let

$$\mathsf{K}_\rho := \left\lceil \frac{\lambda}{\mathsf{S}\rho} \right\rceil, \tag{5.23}$$

where S is as in Lemma 5.2. We define the set of 'test points'

$$\mathsf{Z}_r = \mathsf{Z}_r(\rho) := \left\{ 2r \exp\left(2\pi i k/\mathsf{K}_\rho\right) : k \in \left[1, \mathsf{K}_\rho\right]_{\mathbb{Z}} \right\} \subset \partial B_{2r}(0). \tag{5.24}$$

The event E_r will include the condition that the event $\mathsf{F}_{z,\rho r}$ of Subsection 5.3 occurs for 'many' of the points $z \in \mathsf{Z}_r$.

Recall the half-annuli $\mathsf{H}_{z,\rho r}$ and the balls $B_{\mathsf{s}_{\rho r}}(\mathsf{u}_{z,\rho r})$ and $B_{\mathsf{s}_{\rho r}}(\mathsf{v}_{z,\rho r})$ from the definition of $\mathsf{F}_{z,\rho r}$. We emphasize that by Lemma 5.2, the number of possible choices for the half-annulus $(\rho r)^{-1}[\mathsf{H}_{z,\rho r} - z]$ and the balls $(\rho r)^{-1}[B_{\mathsf{s}_{\rho r}}(\mathsf{u}_{z,\rho r}) - z]$ and $(\rho r)^{-1}[B_{\mathsf{s}_{\rho r}}(\mathsf{v}_{z,\rho r}) - z]$ is at most a constant depending only on λ and the laws of D_h and $\widetilde{D}_h$.

We will now construct a 'tube' which links up the sets $\mathsf{H}_{z,\rho r} \cup B_{\mathsf{s}_{\rho r}}(\mathsf{u}_{z,\rho r}) \cup B_{\mathsf{s}_{\rho r}}(\mathsf{v}_{z,\rho r})$ for $z \in \mathsf{Z}_r$. For $k \in [1, \mathsf{K}_\rho]_{\mathbb{Z}}$, let $z_k := 2r\exp(2\pi i k/\mathsf{K}_r)$ be the kth element of Z_r. We also set $z_{\mathsf{K}_\rho+1} := z_1$. We choose for each $k \in [1, \mathsf{K}_\rho]_{\mathbb{Z}}$ a smooth simple path $\mathsf{L}_{z_k,\rho r}$ from the point of $B_{\mathsf{s}_{\rho r}}(\mathsf{v}_{z_k,\rho r})$ which is furthest from $\mathsf{H}_{z_k,\rho r}$ to the point of $B_{\mathsf{s}_{\rho r}}(\mathsf{u}_{z_{k+1},\rho r})$ which is furthest from $\mathsf{H}_{z_{k+1},\rho r}$. We can arrange that these paths have the following properties.

(i) Each $\mathsf{L}_{z_k,\rho r}$ is contained in the $10\rho r$-neighborhood of $\partial B_{2r}(0)$.
(ii) The Euclidean distance from $\mathsf{L}_{z_k,\rho r}$ to each of the half-annuli $\mathsf{H}_{z_k,\rho r}$ and $\mathsf{H}_{z_{k+1},\rho r}$ is at least $\mathsf{s}_{\rho r}/2$.
(iii) The Euclidean distance from $\mathsf{L}_{z_k,\rho r}$ to each of the following sets is at least $(1-\alpha)\rho r/4$:

- the sets $\mathsf{H}_{w,\rho r}$ for $w \in \mathsf{Z}_r \setminus \{z_k, z_{k+1}\}$;
- the sets $\mathsf{L}_{w,\rho r}$ for $w \in \mathsf{Z}_r \setminus \{z_k\}$;
- the sets $B_{\mathsf{s}_{\rho r}}(\mathsf{v}_{w,\rho r})$ for $w \in \mathsf{Z}_r \setminus \{z_k\}$;
- the sets $B_{\mathsf{s}_{\rho r}}(\mathsf{u}_{w,\rho r})$ for $w \in \mathsf{Z}_r \setminus \{z_{k+1}\}$.

(iv) The number of possibilities for the path $(\rho r)^{-1}(\mathsf{L}_{z_k,\rho r} - z_k)$ is at most a constant depending only on ρ, λ, and the laws of D_h and $\widetilde{D}_h$.

With t as in Lemma 5.2, we define

$$\mathsf{U}_r = \mathsf{U}_r(\rho) := \bigcup_{z \in \mathsf{Z}_r(\rho)} \left[\mathsf{H}_{z,\rho r} \cup B_{\mathsf{s}_{\rho r}}(\mathsf{u}_{z,\rho r}) \cup B_{\mathsf{s}_{\rho r}}(\mathsf{v}_{z,\rho r}) \cup B_{\lambda \mathsf{t} \rho r}(\mathsf{L}_{z,\rho r})\right] \tag{5.25}$$

and

$$\mathsf{V}_r = \mathsf{V}_r(\mathsf{U}_r, \mathsf{a}_9) := B_{\mathsf{a}_9 r}(\mathsf{U}_r). \tag{5.26}$$

We emphasize that V_r is determined by U_r and a_9 and (once a_9 is fixed) the number of possible choices for the set $r^{-1}\mathsf{U}_r$ is at most a finite constant depending only on ρ, λ, and the laws of D_h and $\widetilde{D}_h$. We cannot take $r^{-1}\mathsf{U}_r$ to be independent from r since the radius $\mathsf{s}_{\rho r}$ and the half-annulus $\mathsf{H}_{\rho r}$ from Lemma 5.2 are allowed to depend on ρr. This is a consequence of the fact that we only have tightness across scales, not exact scale invariance. However, a constant upper bound for the number of possibilities for $r^{-1}\mathsf{U}_r$ will be enough for our purposes.

Let

$$\mathsf{f}_r : \mathbb{C} \to [0, \mathsf{A}_8] \tag{5.27}$$

be a smooth bump function which is identically equal to A_8 on U_r and which is supported on V_r. We can choose f_r in such a way that $\mathsf{f}_r(r\cdot)$ depends only on $r^{-1}\mathsf{U}_r$, which means that the number of possible choices for $\mathsf{f}_r(r\cdot)$ is at most a finite constant depending only on $\mathsf{t}, \rho, \lambda$, and the laws of D_h and $\widetilde{D}_h$.

5.5 | Definition of E_r

We will now define the event $\mathsf{E}_r = \mathsf{E}_{0,r}$ appearing in Subsection 4.1. Recall the parameters from (5.21) and (5.22). For $r \in \rho^{-1}\mathcal{R}_0$, let E_r be the event that the following is true. We will discuss the purpose of each condition just after the definition.

(1) *(Bound for distance across)* We have

$$\min\left\{D_h\left(\text{across } \mathbb{A}_{r,1.5r}(0)\right),\ D_h\left(\text{across } \mathbb{A}_{2.5r,3r}(0)\right)\right\} \geqslant \mathsf{a}_1 r^{\xi Q} e^{\xi h_r(0)}.$$

(2) *(Bound for distance around)* We have

$$D_h\left(\text{around } \mathbb{A}_{3r,4r}(0)\right) \leqslant \mathsf{A}_2 r^{\xi Q} e^{\xi h_r(0)}.$$

(3) *(Regularity along geodesics)* The event of Lemma 2.13 occurs with $U = \mathbb{A}_{1,4}(0)$, $\chi = 1/2$, and $\varepsilon_0 = \mathsf{a}_3$. That is, for each $\varepsilon \in (0, \mathsf{a}_3]$, the following is true. Let $V \subset \mathbb{A}_{r,4r}(0)$ and let

$f : \mathbb{C} \to [0,\infty)$ be a non-negative continuous function which is identically zero outside of V. Let $z \in \mathbb{A}_{r+\varepsilon^{1/2},4r-\varepsilon^{1/2}}(0)$, $x, y \in \overline{\mathbb{A}}_{r,4r}(0) \setminus (V \cup B_{\varepsilon^{1/2}r}(z))$, and $s > 0$ such that there is a $D_{h-f}(\cdot,\cdot;\overline{\mathbb{A}}_{r,4r}(0))$-geodesic P_f from x to y with $P_f(s) \in B_{\varepsilon r}(z)$. Assume that $s \leqslant \inf\{t > 0 : P_f(t) \in V\}$. Then with $\theta = \theta(1/2) > 0$ as in Lemma 2.13,

$$D_h\big(\text{around } \mathbb{A}_{\varepsilon r,\varepsilon^{1/2}r}(z)\big) \leqslant \varepsilon^{\theta} s. \tag{5.28}$$

(4) *(Existence of shortcuts)* Let Z_r be the set of test points as in (5.24). For each connected circular arc $I \subset \partial B_{2r}(0)$ with Euclidean length at least $\mathsf{a}_4 r/2$, there exists $z \in I \cap \mathsf{Z}_r$ such that the event $\mathsf{F}_{z,\rho r}$ of Subsection 5.3 occurs.

(5) *(Comparison of distances in small annuli)* For each $z \in \mathbb{A}_{1.5r,3r}(0)$ and each $\delta \in (0,\mathsf{a}_5]$,

$$D_h\big(\text{around } \mathbb{A}_{\delta r/4,\delta r/2}(z)\big) \leqslant \delta^{-1/4} D_h\big(\text{across } \mathbb{A}_{2\delta r,3\delta r}(z)\big). \tag{5.29}$$

(6) *(Reverse Hölder continuity)* For each $z, w \in \mathbb{A}_{1.5r,3r}(0)$ with $|z-w| \leqslant \lambda^{-1}\mathsf{a}_5 r$,

$$D_h\big(z,w;\mathbb{A}_{r,4r}(0)\big) \geqslant \left(\frac{|z-w|}{r}\right)^{\xi(Q+3)} r^{\xi Q} e^{\xi h_r(0)}.$$

(7) *(Internal distance in U_r)* We have

$$D_h(\text{around } \mathsf{U}_r) \leqslant \mathsf{A}_7 r^{\xi Q} e^{\xi h_r(0)}. \tag{5.30}$$

More strongly, there is a path $\Pi \subset \mathsf{U}_r$ which disconnects the inner and outer boundaries of U_r and has D_h-length at most $\mathsf{A}_7 r^{\xi Q} e^{\xi h_r(0)}$ such that each point of the outer boundary[†] of U_r lies at Euclidean distance at most $\mathsf{a}_6 r$ from Π.

(8) *(Intersections of geodesics with a small neighborhood of the boundary)* Let $f : \mathbb{C} \to [0,\mathsf{A}_8]$ be a continuous function and let P_f be a $D_{h-f}(\cdot,\cdot;\overline{\mathbb{A}}_{r,4r}(0))$-geodesic between two points of $\partial B_{4r}(0)$. The one-dimensional Lebesgue measure of the set of $x \in \partial\mathsf{U}_r$ such that $P_f \cap B_{2\mathsf{a}_9 r}(x) \neq \emptyset$ is at most $\lambda t \rho r$. Moreover, the same is true with $\partial\mathsf{U}_r$ replaced by each of the circles $\partial B_{\mathsf{s}_{\rho r}}(\mathsf{u}_{z,\rho r})$ and $\partial B_{\mathsf{s}_{\rho r}}(\mathsf{v}_{z,\rho r})$ for $z \in \mathsf{Z}_r$.

(9) *(Radon–Nikodym derivative bound)* The Dirichlet inner product of h with f_r satisfies

$$|(h,\mathsf{f}_r)_\nabla| \leqslant \mathsf{A}_{10}. \tag{5.31}$$

We will eventually show that E_r satisfies the hypotheses for $\mathsf{E}_{0,r}$ listed in Subsection 4.1. Before beginning the proof of this fact, we discuss the various conditions in the definition of E_r.

Conditions 1 and 2 occur with high probability due to tightness across scales (Axiom V′). These conditions are needed to ensure that hypothesis A from Subsection 4.1 is satisfied. Condition 2 is also useful for upper-bounding the amount of time that a D_h-geodesic or a $D_{h-\mathsf{f}_r}$-geodesic between points outside of $B_{4r}(0)$ can spend in V_r. Indeed, if π is a path in $\mathbb{A}_{3r,4r}(0)$ which disconnects the inner and outer boundaries of near-minimal D_h-length (equivalently, near-minimal $D_{h-\mathsf{f}_r}$-length since $\mathsf{V}_r \cap \mathbb{A}_{3r,4r}(0) = \emptyset$), then any such geodesic must hit π both before and after hitting

† The set U_r has the topology of a Euclidean annulus, so its boundary has two connected components, one of which disconnects the other from ∞. The outer boundary is the outer of these two components.

V_r. The length of the geodesic segment between these hitting times is at most the length of π. See Lemma 5.12 for an application of this argument.

Condition 3 holds with high probability due to Lemma 2.13. This condition will eventually be applied with $V = \mathsf{V}_r$ and $f = \mathsf{f}_r$. We allow a general choice of V and f in the condition statement since we will choose the parameter a_3 in condition 3 before we choose the parameters $\rho, \mathsf{A}_8, \mathsf{a}_9$ involved in the definitions of V_r and f_r. The condition will be used in two places: to lower-bound the Euclidean distance between two points on a $D_{h-\mathsf{f}_r}$-geodesic in terms of their D_h-distance (Lemma 5.11); and to link up a point on a $D_{h-\mathsf{f}_r}$-geodesic which is close to $\partial \mathsf{U}_r$ with a path in U_r (Lemma 5.21).

Condition 4 is in some sense the most important condition in the definition of E_r. Due to the definition of $\mathsf{F}_{z,\rho r}$ from Subsection 5.3, this condition provides a large collection of 'good' pairs of points $u, v \in \mathsf{U}_r$ such that $\widetilde{D}_h(u,v) \leqslant \mathfrak{c}_0' D_h(u,v)$. The fact that we consider the event $\mathsf{F}_{z,\rho r}$ in this condition is the reason why we need to require that $r \in \rho^{-1}\mathcal{R}_0$. We will need to make ρ small in order to make the set of test points $z \in \mathsf{Z}_r$ of (5.24) large, so that we can apply a long-range independence result for the GFF (Lemma 2.3) to say that condition 4 occurs with high probability. See Lemma 5.13.

Condition 5 has high probability due to Lemma 2.8, and will be used in Subsection 5.10. More precisely, we will consider a segment of a $D_{h-\mathsf{f}_r}$-geodesic which is contained in a small Euclidean neighborhood of the ball $B_{\mathsf{s}_{\rho r}}(\mathsf{u}_{z,\rho r})$ in the definition of $\mathsf{F}_{z,\rho r}$. We will use the paths around annuli provided by condition 5 to 'link up' this geodesic segment to a short path from u to the boundary of this ball, as provided by condition 3 in the definition of $\mathsf{F}_{z,\rho r}$ (see Lemma 5.34).

Condition 6 has high probability due to the local reverse Hölder continuity of D_h with respect to the Euclidean metric [36, Proposition 3.8]. This condition will be used in several places, for example, to force a $D_{h-\mathsf{f}_r}$-geodesic between two points of $\partial \mathsf{V}_r$ to stay in a small Euclidean neighborhood of V_r (Lemma 5.22). See also the summary of Subsection 5.8 in Subsection 5.1. The requirement that $|z-w| \leqslant \lambda^{-1}\mathsf{a}_5 r$ is needed to make the condition occur with high probability (cf. [36, Proposition 3.8]).

Condition 7 has high probability due to a straightforward argument based on tightness across scales and the fact that there are only finitely many possibilities for $r^{-1}\mathsf{U}_r$ (see Lemma 5.15). This condition will be used to check the condition on D_h(around U_r) in hypothesis A for E_r. The reason why we need to require that each point of the outer boundary of U_r is close to the path Π is as follows. In the proof of Lemma 5.21, we will consider a $D_{h-\mathsf{f}_r}$-geodesic P_r and times $\tau < \sigma$ at which it hits $\partial \mathsf{V}_r$. We will upper-bound $\sigma - \tau = D_{h-\mathsf{f}_r}(P_r(\tau), P_r(\sigma))$ by concatenating a segment of Π with segments of small loops surrounding $P_r(\tau)$ and $P_r(\sigma)$ which are provided by condition 3. The condition on Π is needed to ensure that these small loops actually intersect Π.

Recall that $\mathsf{f}_r : \mathbb{C} \to [0, \mathsf{A}_8]$. Condition 8 has high probability due to Lemma 2.14. We will eventually apply this condition with $f = \mathsf{f}_r$ in order to say that a $D_{h-\mathsf{f}_r}$-geodesic cannot spend much time in the region $\mathsf{V}_r \setminus \mathsf{U}_r$ where f_r takes values strictly between 0 and A_8 (see Lemmas 5.28 and 5.32). The reason why we allow a general choice of f in the condition statement is that $\mathsf{V}_r = B_{\mathsf{a}_9 r}(\mathsf{U}_r)$, and hence also f_r, depends on the parameter a_9, which needs to be made small enough to make the probability of condition 8 close to 1.

The purpose of condition 9 is to check the Radon–Nikodym derivative hypothesis B from Subsection 4.1, see Proposition 5.17. This condition occurs with high probability due to the scale invariance of the law of h, modulo additive constant, and the fact that there are only finitely many possibilities for $\mathsf{f}_r(r\cdot)$ (Lemma 5.16).

5.6 | Properties of E_r

We first check that E_r satisfies an appropriate measurability condition.

Lemma 5.8. *The event E_r is almost surely determined by $h|_{\overline{\mathbb{A}}_{r,4r}(0)}$, viewed modulo additive constant.*

Proof. By Weyl scaling (Axiom III) that the occurrence of E_r is unaffected by adding a constant to h, so E_r is almost surely determined by h viewed modulo additive constant. It is immediate from locality (Axiom II; see also Subsection 2.2) that each condition in the definition of E_r except possibly condition 4 is almost surely determined by $h|_{\overline{\mathbb{A}}_{r,4r}(0)}$. Lemma 5.7 implies that condition 4 is almost surely determined by $h|_{\overline{\mathbb{A}}_{r,4r}(0)}$ as well. □

Most of the rest of this subsection is devoted to proving the following.

Proposition 5.9. *For each $\mathbb{p} \in (0,1)$, we can choose the parameters in (5.21) and (5.22) in such a way that*

$$\mathbb{P}[\mathsf{E}_r] \geqslant \mathbb{p}, \quad \forall r \in \rho^{-1}\mathcal{R}_0. \tag{5.32}$$

To prove Proposition 5.9, we will treat the conditions in the definition of E_r in order. For each condition, we will choose the parameters involved in the condition, in a manner depending only on $\mathbb{p}, \lambda$, and the laws of D_h and $\widetilde{D}_h$, in such a way that the condition occurs with high probability. For some of the conditions, we will impose extra constraints on the parameters beyond just the numerical ordering in (5.21) and (5.22). These constraints will be stated and referenced as needed in the later part of the proof.

Lemma 5.10. *There exists $\mathsf{a}_1 > 1/\mathsf{A}_2 > \mathsf{a}_3 > 0$ depending only on $\mathbb{p}, \lambda$, and the laws of D_h and $\widetilde{D}_h$ such that for each $r > 0$, the probability of each of conditions 1, 2, and 3 in the definition of E_r is at least $1 - (1-\mathbb{p})/10$.*

Proof. By tightness across scales (Axiom V′), we can choose $\mathsf{a}_1, \mathsf{A}_2 > 0$ such that the probabilities of conditions 1 and 2 are each at least $1-(1-\mathbb{p})/10$. By Lemma 2.13, we can choose $\mathsf{a}_3 > 0$ such that the probability of condition 3 is at least $1-(1-\mathbb{p})/10$. □

We henceforth fix $\mathsf{a}_1, \mathsf{A}_2, \mathsf{a}_3$ as in Lemma 5.10. Our next task is to make an appropriate choice of the parameter a_4 appearing in condition 4.

Lemma 5.11. *Let $r > 0$ and assume that conditions 1, 2, and 3 in the definition of E_r occur. Let $V \subset \mathbb{A}_{r,3r}(0)$ and let $f : \mathbb{C} \to [0,\infty)$ be a non-negative continuous function which is identically zero outside of V. Also let P_f be a $D_{h-f}(\cdot,\cdot;\overline{\mathbb{A}}_{r,4r}(0))$-geodesic between two points of $\partial B_{4r}(0)$ and define the times*

$$\tau := \inf\{t > 0 : P_f(t) \in V\} \quad \text{and} \quad \sigma := \sup\{t > 0 : P_f(t) \in V\}. \tag{5.33}$$

There exists $\mathsf{a}_4 > 0$ depending only on $\mathbb{p}, \lambda$, and the laws of D_h and $\widetilde{D}_h$ such that the following is true. If

$$D_h\big(P_f(\tau), P_f(\sigma); B_{4r}(0)\big) \geqslant \frac{\mathsf{a}_1{}^2}{4\mathsf{A}_2} r^{\xi Q} e^{\xi h_r(0)}, \tag{5.34}$$

then

$$|P_f(\tau) - P_f(\sigma)| \geqslant \mathsf{a}_4 r. \tag{5.35}$$

The motivation for our choice of a_4 comes from hypothesis C for E_r from Subsection 4.1. We will eventually apply Lemma 5.11 with $V = \mathsf{V}_r$, $f = \mathsf{f}_r$, and P_f equal to a $(B_{4r}(0), \mathsf{V}_r)$-excursion of a $D_{h-\mathsf{f}_r}$-geodesic between two points of $\mathbb{C} \setminus B_{4r}(0)$ (recall Definition 4.1). The assumption (5.34) is closely related to the condition (4.4) from hypothesis C. The lower bound for $|P_f(\tau) - P_f(\sigma)|$ from (5.35) will eventually be combined with condition 4 in the definition of E_r to ensure that there is a $z \in \mathsf{Z}_r$ such that $\mathsf{F}_{z,r}$ occurs and our $D_{h-\mathsf{f}_r}$-geodesic gets Euclidean-close to each of the points u, v appearing in the definition of $\mathsf{F}_{z,r}$ (see Subsection 5.9).

For the proof of Lemma 5.11, we need the following lemma.

Lemma 5.12. *Assume we are in the setting of Lemma 5.11 and let V, f, P_f, τ, and σ be as in that lemma. For each $\varepsilon \in (0, \mathsf{a}_3]$, one has*

$$\max\big\{D_h\big(\text{around } \mathbb{A}_{\varepsilon r, \varepsilon^{1/2} r}(P_f(\tau))\big), D_h\big(\text{around } \mathbb{A}_{\varepsilon r, \varepsilon^{1/2} r}(P_f(\sigma))\big)\big\}$$
$$\leqslant 2\mathsf{A}_2 \varepsilon^{\theta} r^{\xi Q} e^{\xi h_r(0)}. \tag{5.36}$$

Proof. Let τ_0 (respectively, σ_0) be the last time before τ (respectively, the first time after σ) at which P_f hits $\partial B_{3r}(0)$. By condition 2 in the definition of E_r, there is a path $\Pi \subset \mathbb{A}_{3r,4r}(0)$ with D_h-length at most $2\mathsf{A}_2 r^{\xi Q} e^{\xi h_r(0)}$ which disconnects the inner and outer boundaries of $\mathbb{A}_{3r,4r}(0)$. Since f is supported on $\mathbb{A}_{r,3r}(0)$, the D_{h-f}-length of Π is the same as its D_h-length. The path P_f must hit Π before time τ_0 and after time σ_0. Since P_f is a $D_{h-f}(\cdot,\cdot; \overline{\mathbb{A}}_{r,4r}(0))$-geodesic, we infer that

$$\sigma_0 - \tau_0 \leqslant \operatorname{len}\big(\Pi; D_{h-f}\big) \leqslant 2\mathsf{A}_2 r^{\xi Q} e^{\xi h_r(0)}. \tag{5.37}$$

Indeed, otherwise we could replace a segment of P_f by a segment of Π to get a path in $\overline{\mathbb{A}}_{r,4r}(0)$ with the same endpoints as P_f but shorter D_{h-f}-length.

By condition 3 in the definition of E_r applied to the $D_{h-f}(\cdot,\cdot; \overline{\mathbb{A}}_{r,4r}(0))$-geodesic $P_f|_{[\tau_0,\sigma_0]}$ and with $z = P_f(\tau)$ and $s = \tau - \tau_0$, for each $\varepsilon \in (0, \mathsf{a}_3]$,

$$D_h\big(\text{around } \mathbb{A}_{\varepsilon r, \varepsilon^{1/2} r}(P_f(\tau))\big) \leqslant \varepsilon^{\theta}(\tau - \tau_0) \leqslant \varepsilon^{\theta}(\sigma_0 - \tau_0) \leqslant 2\varepsilon^{\theta} \mathsf{A}_2 r^{\xi Q} e^{\xi h_r(0)}, \tag{5.38}$$

where the last inequality is by (5.37). The analogous bound with σ in place of τ follows from the same argument applied with P_f replaced by its time reversal. □

Proof of Lemma 5.11. See Figure 16 for an illustration. By Lemma 5.12, for each $\varepsilon \in (0, \mathsf{a}_3]$ there is a path $\pi_\varepsilon \subset \mathbb{A}_{\varepsilon r, \varepsilon^{1/2} r}(P_f(\tau))$ such that

$$\operatorname{len}(\pi_\varepsilon; D_h) \leqslant 4\varepsilon^{\theta} \mathsf{A}_2 r^{\xi Q} e^{\xi h_r(0)}. \tag{5.39}$$

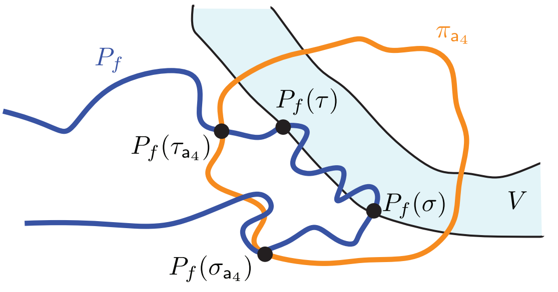

FIGURE 16 Illustration of the proof of Lemma 5.11. If $|P_f(\tau) - P_f(\sigma)| < a_4 r$, then the union of the orange loop π_{a_4} and the segments $P_f|_{[\tau_{a_4},\tau]}$ and $P_f|_{[\sigma,\sigma_{a_4}]}$ contains a path from $P_f(\tau)$ to $P_f(\sigma)$ of D_{h-f}-length less than $\frac{a_1{}^2}{4A_2} r^{\xi Q} e^{\xi h_r(0)}$. This yields the contrapositive of the lemma statement.

Let $a_4 \in (0, a_3]$ be chosen so that

$$4a_4{}^\theta A_2 < \frac{a_1{}^2}{16A_2}. \tag{5.40}$$

By (5.39) and since f is non-negative,

$$\operatorname{len}\left(\pi_{a_4}; D_{h-f}\right) \leqslant \operatorname{len}\left(\pi_{a_4}; D_h\right) < \frac{a_1{}^2}{16A_2} r^{\xi Q} e^{\xi h_r(0)}. \tag{5.41}$$

We will prove the contrapositive of the lemma statement with this choice of a_4, that is, we will show that if $|P_f(\tau) - P_f(\sigma)| < a_4 r$, then $D_h(P_f(\tau), P_f(\sigma); B_{4r}(0)) < \frac{a_1{}^2}{4A_2} r^{\xi Q} e^{\xi h_r(0)}$.

If $|P_f(\tau) - P_f(\sigma)| < a_4 r$, then $P_f(\sigma) \in B_{a_4 r}(P_f(\tau))$. Since the endpoints of P_f lie in $\partial B_{4r}(0)$, which is disjoint from $B_{a_4{}^{1/2} r}(P_f(\tau))$, it follows that P_f hits π_{a_4} before time τ and after time σ. Let τ_{a_4} (respectively, σ_{a_4}) be the last time before time τ (respectively, the first time after time σ) at which P_f hits π_{a_4}. Since P_f is a $D_{h-f}(\cdot,\cdot; \overline{\mathbb{A}}_{r,4r}(0))$-geodesic,

$$\sigma_{a_4} - \tau_{a_4} \leqslant \operatorname{len}\left(\pi_{a_4}; D_{h-f}\right) < \frac{a_1{}^2}{16A_2} r^{\xi Q} e^{\xi h_r(0)}.$$

By the definitions (5.33) of τ and σ, the path segments $P_f|_{[\tau_{a_4},\tau]}$ and $P_f|_{[\sigma,\sigma_{a_4}]}$ are disjoint from the support of f. So, the D_{h-f}-lengths of these segments are the same as their D_h-lengths. Consequently,

$$\begin{aligned}\operatorname{len}\left(P_f|_{[\tau_{a_4},\tau]}; D_h\right) + \operatorname{len}\left(P_f|_{[\sigma,\sigma_{a_4}]}; D_h\right) &\leqslant \operatorname{len}\left(P_f|_{[\tau_{a_4},\sigma_{a_4}]}; D_{h-f}\right) \\ &= \sigma_{a_4} - \tau_{a_4} < \frac{a_1{}^2}{16A_2} r^{\xi Q} e^{\xi h_r(0)}.\end{aligned} \tag{5.42}$$

The union of $P_f([\tau_{a_4}, \tau])$, $P_f([\sigma, \sigma_{a_4}])$, and π_{a_4} contains a path from $P_f(\tau)$ to $P_f(\sigma)$. Since $V \subset B_{3r}(0)$, this path is contained in $B_{4r}(0)$. We therefore infer from (5.41) and (5.42) that

$$D_h\big(P_f(\tau), P_f(\sigma); B_{4r}(0)\big) \leqslant \frac{3\mathsf{a}_1{}^2}{16\mathsf{A}_2} r^{\xi Q} e^{\xi h_r(0)} < \frac{\mathsf{a}_1{}^2}{4\mathsf{A}_2} r^{\xi Q} e^{\xi h_r(0)}$$

as required. □

Henceforth, fix a_4 as in Lemma 5.11. We will now choose ρ so that condition 4 in the definition of E_r occurs with high probability.

Lemma 5.13. *There exists $\rho \in (0, \lambda \mathsf{a}_4)$, depending only on $\mathbb{p}, \lambda$, and the laws of D_h and $\widetilde{D}_h$, such that*

$$\rho^\theta \mathsf{A}_2 \leqslant \lambda \mathsf{a}_1 \tag{5.43}$$

and the following is true. For each $r \in \rho^{-1}\mathcal{R}_0$, it holds with probability at least $1-(1-\mathbb{p})/10$ that condition 4 in the definition of E_r occurs.

Proof. By the definition of K_ρ in (5.23) and the definition of $\mathsf{Z}_r(\rho)$ in (5.24), there is a constant $c > 0$ depending only on S, a_4, and λ (hence only on $\mathbb{p}, \lambda$, and the laws of D_h and $\widetilde{D}_h$) such that for each $\rho \in (0, \lambda/\mathsf{S})$ and each $r \in \rho^{-1}\mathcal{R}_0$, the set $\mathsf{Z}_r = \mathsf{Z}_r(\rho)$ satisfies the following properties.

(i) We have $|z - w| \geqslant 50\mathsf{S}\rho r$ for each distinct $z, w \in \mathsf{Z}_r(\rho)$ (note that λ is much smaller than $1/50$, see (5.3)).
(ii) Each connected circular arc $J \subset \partial B_{2r}(0)$ with Euclidean length at least $\mathsf{a}_4 r/4$ contains at least $\lfloor c\rho^{-1} \rfloor$ points of $\mathsf{Z}_r(\rho)$.

Furthermore, there is a constant $C > 0$ depending only on a_4 and a deterministic collection $\mathcal{J}$ of arcs $J \subset \partial B_{2r}(0)$ such that $\#\mathcal{J} \leqslant C$, each $J \in \mathcal{J}$ has Euclidean length $\mathsf{a}_4 r/4$, and each arc $I \subset \partial B_{2r}(0)$ with Euclidean length at least $\mathsf{a}_4 r/2$ contains some $J \in \mathcal{J}$.

By (5.19), for each $r \in \rho^{-1}\mathcal{R}_0$ and each $z \in \mathsf{Z}_r(\rho)$, we have $\mathbb{P}[\mathsf{F}_{z,\rho r}] \geqslant \mathsf{p}$. By Lemma 5.7, each $\mathsf{F}_{z,\rho r}$ is almost surely determined by $h|_{B_{3\rho r}(z)}$, viewed modulo additive constant. Therefore, we can apply Lemma 2.3 with h replaced by the re-scaled field $h(r\cdot)$, which agrees in law with h modulo additive constant, and $\mathcal{Z} = r^{-1}(J \cap \mathsf{Z}_r)$ to get the following. If ρ is chosen to be sufficiently small (depending on p and C, hence only on $\mathbb{p}, \lambda$, and the laws of D_h and $\widetilde{D}_h$), then

$$\mathbb{P}\left[\bigcup_{z \in \mathsf{Z}_r \cap J} \mathsf{F}_{z,\rho r}\right] \geqslant 1 - \frac{1-\mathbb{p}}{10C}, \quad \forall J \in \mathcal{J}.$$

By a union bound over all $J \in \mathcal{J}$, we get that with probability at least $1-(1-\mathbb{p})/10$, each $J \in \mathcal{J}$ contains a point $z \in \mathsf{Z}_r(\rho)$ such that $\mathsf{F}_{z,\rho r}$ occurs. By the defining property of $\mathcal{J}$, this concludes the proof. □

We next deal with conditions 5 and 6 in the definition of E_r, which amounts to citing some already-proven lemmas.

Lemma 5.14. *There exists $\mathsf{a}_5 \in (0, \lambda(1-\alpha)\mathsf{t}\rho]$ (where t is as in Lemma 5.2), depending only on $\mathbb{p}, \lambda$, and the laws of D_h and $\widetilde{D}_h$, such that for each $r > 0$, the probability of each of conditions 5 and 6 in the definition of E_r is at least $1-(1-\mathbb{p})/10$.*

Proof. The existence of $\mathsf{a}_5 \in (0, \lambda \mathsf{t}\rho]$ such that condition 5 in the definition of E_r each occur with probability at least $1 - (1 - \mathbb{p})/10$ follows from Lemma 2.8. By the local reverse Hölder continuity of D_h with respect to the Euclidean metric [36, Proposition 3.8], after possibly shrinking a_5 we can arrange that condition 6 also occurs with probability at least $1 - (1 - \mathbb{p})/10$. □

We henceforth fix a_5 as in Lemma 5.14. We also let $\mathsf{a}_6 \in (0, \min\{\lambda \mathsf{a}_3, \mathsf{a}_5\})$ be chosen (in a manner depending only on $\mathbb{p}\lambda$, and the laws of D_h and $\widetilde{D}_h$) so that

$$(2\mathsf{a}_6)^\theta \mathsf{A}_2 \leqslant \lambda \mathsf{a}_5{}^{\xi(Q+3)}. \tag{5.44}$$

The particular choice of a_6 from (5.44) will be important in the proof of Lemma 5.21.

Lemma 5.15. *There exists $\mathsf{A}_7 > 1/\mathsf{a}_6$, depending only on $\mathbb{p}, \lambda$, and the laws of D_h and $\widetilde{D}_h$, such that for each $r \in \rho^{-1}\mathcal{R}_0$, the probability of condition 7 in the definition of E_r is at least $1 - (1 - \mathbb{p})/10$.*

Proof. The set U_r has the topology of a Euclidean annulus and its boundary consists of two piecewise smooth Jordan loops. Write $\partial^{\text{out}}\mathsf{U}_r$ for the outer boundary of U_r, that is, the outer of the two loops. If $r \in \rho^{-1}\mathcal{R}_0$ is fixed, then as $\varepsilon \to 0$ the Euclidean Hausdorff distance between the following two sets tends to zero: $\partial^{\text{out}}\mathsf{U}_r$ and $\partial B_{\varepsilon r}(\partial^{\text{out}}\mathsf{U}_r) \cap \mathsf{U}_r$ (that is, the intersection with U_r of the boundary of the Euclidean ε-neighborhood of $\partial^{\text{out}}\mathsf{U}_r$).

Since we have already chosen ρ in a manner depending only on $\mathbb{p}, \lambda$, and the laws of D_h and $\widetilde{D}_h$, the number of possible choices for $r^{-1}\mathsf{U}_r$ is at most a constant depending only on $\mathbb{p}, \lambda$, and the laws of D_h and $\widetilde{D}_h$. By combining this with the preceding paragraph, we find that there exists $\varepsilon > 0$, depending only on $\mathbb{p}, \lambda$, and the laws of D_h and $\widetilde{D}_h$, such that for each $r \in \rho^{-1}\mathcal{R}_0$, the Euclidean Hausdorff distance between $\partial^{\text{out}}\mathsf{U}_r$ and $\partial B_{\varepsilon r}(\partial^{\text{out}}\mathsf{U}_r) \cap \mathsf{U}_r$ is at most $\mathsf{a}_6 r$.

By tightness across scales (in the form of Lemma 2.5) and the fact that there are only finitely many possibilities for $r^{-1}\mathsf{U}_r$, there exists $\mathsf{A}_7 > 0$ such that for each $r \in \rho^{-1}\mathcal{R}_0$, it holds with probability at least $1 - (1 - \mathbb{p})/10$ that the following is true. There is a path $\Pi \subset B_{\varepsilon r}(\partial^{\text{out}}\mathsf{U}_r) \cap \mathsf{U}_r$ which disconnects $\partial^{\text{out}}\mathsf{U}_r$ from $\partial B_{\varepsilon r}(\partial^{\text{out}}\mathsf{U}_r) \cap \mathsf{U}_r$ and has D_h-length at most $\mathsf{A}_7 r^{\xi Q} e^{\xi h_r(0)}$.

The path Π disconnects the inner and outer boundaries of U_r, so the existence of Π immediately implies (5.30). Furthermore, by our choice of ε, each point $x \in \partial^{\text{out}}\mathsf{U}_r$ lies at Euclidean distance at most $\mathsf{a}_6 r$ from a point of $\partial B_{\varepsilon r}(\partial^{\text{out}}\mathsf{U}_r) \cap \mathsf{U}_r$. Since Π disconnects $\partial^{\text{out}}\mathsf{U}_r$ from $\partial B_{\varepsilon r}(\partial^{\text{out}}\mathsf{U}_r) \cap \mathsf{U}_r$, the line segment from x to this point of $\partial B_{\varepsilon r}(\partial^{\text{out}}\mathsf{U}_r) \cap \mathsf{U}_r$ intersects Π. Consequently, the Euclidean distance from x to Π is at most $\mathsf{a}_6 r$. □

We henceforth fix A_7 as in Lemma 5.15 and define

$$\mathsf{A}_8 := \frac{1}{\xi} \max\left\{ \log \frac{\mathsf{A}_7}{\lambda \mathsf{a}_5{}^{\xi(Q+3)}}, \log \frac{\mathsf{A}_7}{\lambda \mathsf{a}_1} \right\}. \tag{5.45}$$

Recall from (5.27) that A_8 is the maximal value attained by f_r. We now treat the remaining two conditions in the definition of f_r.

Lemma 5.16. *There exists $\mathsf{a}_9 \in (0, \lambda/\mathsf{A}_8)$ and $\mathsf{A}_{10} > 1/\mathsf{a}_9$, depending only on $\mathbb{p}, \lambda$, and the laws of D_h and $\widetilde{D}_h$, such that for each $r \in \rho^{-1}\mathcal{R}_0$, the probability of each of conditions 8 and 9 in the definition of E_r is at least $1 - (1 - \mathbb{p})/10$.*

Proof. Since we have already chosen ρ in a manner depending only on $\mathbb{p}, \lambda$, and the laws of D_h and $\widetilde{D}_h$, the number of possible choices for $r^{-1}\mathsf{U}_r$ is at most a constant depending only on $\mathbb{p}, \lambda$, and the laws of D_h and $\widetilde{D}_h$. The set U_r has the topology of a Euclidean annulus and its boundary consists of two piecewise smooth Jordan loops. By the preceding sentence, the Euclidean length of each of the two boundary loops of U_r is at most a constant (depending only on $\mathbb{p}, \lambda$, and the laws of D_h and $\widetilde{D}_h$) times r. We can therefore apply Lemma 2.14 with $M = \mathsf{A}_8$ and the curve η given by each of the two boundary loops of U_r, parameterized by its Euclidean length. This shows that there exists $\mathsf{a}_9 \in (0, \lambda/\mathsf{A}_8)$ depending only on $\mathbb{p}, \lambda$, and the laws of D_h and $\widetilde{D}_h$ such that the event of condition 8 in the definition of E_r for the set $\partial \mathsf{U}_r$ occurs with probability at least $1 - (1 - \mathbb{p})/20$.

By a union bound over at most a universal constant times $(\lambda \mathsf{t} \rho)^{-1}$ points $z \in \mathsf{Z}_r$, after possibly decreasing a_9 we can also arrange that with probability at least $1 - (1 - \mathbb{p})/20$, the event of condition 8 occurs for each of the circles $\partial B_{\mathsf{s}_{\rho r}}(\mathsf{u}_{z,\rho r})$ and $\partial B_{\mathsf{s}_{\rho r}}(\mathsf{v}_{z,\rho r})$ for $z \in \mathsf{Z}_r$. Combining this with the preceding paragraph shows that condition 8 has probability at least $1 - (1 - \mathbb{p})/10$.

The number of possible choices for the function $\mathsf{f}_r(r\cdot)$ is at most a constant depending only on $\mathbb{p}, \lambda$, and the laws of D_h and $\widetilde{D}_h$. By the conformal invariance of the Dirichlet inner product and the scale invariance of the law of h, viewed modulo additive constant,

$$(h, \mathsf{f}_r)_\nabla = (h(r\cdot), \mathsf{f}_r(r\cdot))_\nabla \stackrel{d}{=} (h, \mathsf{f}_r(r\cdot))_\nabla.$$

Therefore, we can find $\mathsf{A}_{10} > 1/\mathsf{a}_9$ depending only on $\mathbb{p}, \lambda$, and the laws of D_h and $\widetilde{D}_h$ such that the probability of condition 9 is at least $1 - (1 - \mathbb{p})/10$. □

Proof of Proposition 5.9. Combine Lemmas 5.10, 5.13, 5.14, 5.15, and 5.16. □

We can also easily check the first two of the three hypotheses for E_r from Subsection 4.1.

Proposition 5.17. *Let $r \in \rho^{-1}\mathcal{R}_0$. On the event E_r, hypotheses A and B in Subsection 4.1 hold for $\mathsf{E}_{0,r} = \mathsf{E}_r$ with*

$$\mathsf{a} = \mathsf{a}_1, \quad \mathsf{A} = \mathsf{A}_2, \quad \mathsf{L} = \mathsf{A}_7, \tag{5.46}$$

and an appropriate choice of $\mathsf{K} > 0$ depending only on the parameters from (5.21) and (5.22) (hence only on $\mathbb{p}, \lambda$, and the laws of D_h and $\widetilde{D}_h$). That is, on E_r, the following is true.

(A) We have

$$D_h(\mathsf{V}_r, \partial \mathbb{A}_{r,3r}(0)) \geqslant \mathsf{a}_1 r^{\xi Q} e^{\xi h_r(0)},$$

$$D_h(\text{around } \mathbb{A}_{3r,4r}(0)) \leqslant \mathsf{A}_2 r^{\xi Q} e^{\xi h_r(0)}, \quad \text{and}$$

$$D_h(\text{around } \mathsf{U}_r) \leqslant \mathsf{A}_7 r^{\xi Q} e^{\xi h_r(0)}.$$

(B) There is a constant $\mathsf{K} > 0$, depending only on the parameters from (5.21) and (5.22), such that the Radon–Nikodym derivative of the law of $h + \mathsf{f}_r$ with respect to the law of h, with both distributions viewed modulo additive constant, is bounded above by K and below by K^{-1}.

Proof. We have $\mathsf{V}_r \subset \mathbb{A}_{1.5r,2.5r}(0)$, so hypothesis A follows immediately from conditions 1, 2, and 7 in the definition of E_r. By a standard calculation for the GFF (see, for example, the proof of [34, Proposition 3.4]), the Radon–Nikodym derivative of the law of $h + \mathsf{f}_r$ with respect to the law of h, with both distributions viewed modulo additive constant, is equal to

$$\exp\Big((h, \mathsf{f}_r)_\nabla - \frac{1}{2}(\mathsf{f}_r, \mathsf{f}_r)_\nabla\Big),$$

where $(\cdot,\cdot)_\nabla$ is the Dirichlet inner product. Since the number of possibilities for $\mathsf{f}_r(r\cdot)$ is at most a constant depending only on $\mathbb{p}$, λ, and the laws of D_h and $\widetilde{D}_h$, we infer that $(\mathsf{f}_r, \mathsf{f}_r)_\nabla$ is bounded above by a constant C depending only on $\mathbb{p}$, λ, and the laws of D_h and $\widetilde{D}_h$ (cf. the proof of Lemma 5.16). By combining this with condition 9 in the definition of E_r, we get that on E_r, we have the Radon–Nikodym derivative bounds

$$\exp\Big(-\mathsf{A}_{10} - \frac{1}{2}C\Big) \leqslant \exp\Big((h, \mathsf{f}_r)_\nabla - \frac{1}{2}(\mathsf{f}_r, \mathsf{f}_r)_\nabla\Big) \leqslant \exp(\mathsf{A}_{10}).$$

This gives hypothesis B with $\mathsf{K} = \exp(\mathsf{A}_{10} + C/2)$. □

Most of the rest of this section is devoted to checking hypothesis C of Subsection 4.1 for the events E_r.

Proposition 5.18. *Fix $\mathfrak{c}' > \mathfrak{c}'_0$. If λ is chosen to be small enough (in a manner depending only on the laws of D_h and $\widetilde{D}_h$) and the parameters from (5.21) and (5.22) are chosen appropriately, subject to the constraints stated in the discussion around (5.21) and (5.22), then hypothesis C holds for the events E_r with*

$$\mathsf{b} := \frac{{\mathsf{a}_1}^2}{4\mathsf{A}_2} \quad \text{and} \quad \mathsf{c} := {\mathsf{a}_5}^{\xi(Q+3)} e^{-\xi \mathsf{A}_8}. \tag{5.47}$$

That is, let $r \in \rho^{-1}\mathcal{R}_0$ and assume that E_r occurs. Let P_r be a $D_{h-\mathsf{f}_r}$-geodesic between two points of $\mathbb{C} \setminus B_{4r}(0)$, parameterized by its $D_{h-\mathsf{f}_r}$-length. Assume that there is a $(B_{4r}(0), \mathsf{V}_r)$-excursion $(\tau', \tau, \sigma, \sigma')$ for P_r (Definition 4.1) such that

$$D_h(P_r(\tau), P_r(\sigma); B_{4r}(0)) \geqslant \mathsf{b} r^{\xi Q} e^{\xi h_r(0)}. \tag{5.48}$$

There exist times $\tau \leqslant s < t \leqslant \sigma$ such that

$$t - s \geqslant \mathsf{c} r^{\xi Q} e^{\xi h_r(0)} \quad \text{and} \quad \widetilde{D}_{h-\mathsf{f}_r}\big(P_r(s), P_r(t); \mathbb{A}_{r,4r}(0)\big) \leqslant \mathfrak{c}'(t-s). \tag{5.49}$$

The proof of Proposition 5.18 will occupy Subsections 5.8 through 5.11.

5.7 | Proof of Proposition 4.2 assuming Proposition 5.18

In this subsection, we will assume Proposition 5.18 and deduce Proposition 4.2. As explained in Section 4, this gives us a proof of our main results modulo Proposition 5.18.

Assume that the parameters from (5.21) and (5.22) are chosen so that the conclusions of Propositions 5.9 and 5.18 are satisfied. Let $\mathcal{R}_0$ be as in (5.2) and let $\mathcal{R} := \rho^{-1}\mathcal{R}_0$. Since $\mathcal{R}_0 \subset \{8^{-k}\}_{k\in\mathbb{N}}$, we have $r'/r \geqslant 8$ whenever $r, r' \in \mathcal{R}$ with $r' > r$, so (4.2) holds.

The event E_r is defined for each $r \in \mathcal{R}$. By Lemma 5.8, the event E_r is almost surely determined by $h|_{\overline{\mathbb{A}}_{r,4r}(0)}$, viewed modulo additive constant. By Proposition 5.9, $\mathbb{P}[\mathsf{E}_r] \geqslant \mathbb{p}$ for each $r \in \mathcal{R}$. By the definitions in Subsection 5.4, the sets U_r and V_r and the functions f_r satisfy the requirements for $\mathsf{U}_{0,r}$, $\mathsf{V}_{0,r}$, and $\mathsf{f}_{0,r}$ from Subsection 4.1, with the maximal value of f_r given by $\mathsf{M} = \mathsf{A}_8$. By Propositions 5.17 and 5.18, the event E_r satisfies hypotheses A, B, and C from Subsection 4.1 for $z = 0$, with the parameters $\mathsf{a}, \mathsf{A}, \mathsf{L}, \mathsf{K}, \mathsf{b}, \mathsf{c}$ depending on the parameters from (5.21) and (5.22).

To check the needed parameter relation (4.3), we observe that Proposition 5.17 gives $\mathsf{a} = \mathsf{a}_1$, $\mathsf{A} = \mathsf{A}_2$, and $\mathsf{L} = \mathsf{A}_7$. By (5.21), we immediately get $\mathsf{A} \geqslant \mathsf{a}$. Furthermore, by (5.47),

$$\frac{2\mathsf{A}}{\mathsf{a}}\mathsf{b} = \frac{2\mathsf{A}_2}{\mathsf{a}_1} \times \frac{\mathsf{a}_1{}^2}{4\mathsf{A}_2} = \frac{\mathsf{a}_1}{2}. \tag{5.50}$$

Moreover, by (5.45),

$$\mathsf{a} - 4e^{-\xi \mathsf{M}}\mathsf{L} = \mathsf{a}_1 - 4e^{-\xi \mathsf{A}_8}\mathsf{A}_7 \geqslant \mathsf{a}_1 - 4\lambda \mathsf{a}_1 > \frac{\mathsf{a}_1}{2}. \tag{5.51}$$

Combining (5.50) and (5.51) gives the second inequality in (4.3).

For $r \in \mathcal{R}$ and $z \in \mathbb{C}$, we define $\mathsf{E}_{z,r}$ to be the event E_r of Subsection 5.5 with the translated field $h(\cdot - z) - h_1(-z) \stackrel{d}{=} h$ in place of h. We also define $\mathsf{U}_{z,r} := \mathsf{U}_r + z$, $\mathsf{V}_{z,r} := \mathsf{V}_r + z$, and $\mathsf{f}_{z,r}(\cdot) := \mathsf{f}_r(\cdot - z)$. By the translation invariance property of weak LQG metrics (Axiom IV′), the objects $\mathsf{E}_{z,r}, \mathsf{U}_{z,r}, \mathsf{V}_{z,r}$, and $\mathsf{f}_{z,r}$ satisfy the hypotheses of Subsection 4.1.

It remains to prove the asserted lower bound for $\#(\mathcal{R} \cap [\varepsilon^2 \mathbb{r}, \varepsilon \mathbb{r}])$ under the assumption that $\mathbb{P}[\widetilde{G}_{\mathbb{r}}(\widetilde{\beta}, \mathfrak{c}'')] \geqslant \widetilde{\beta}$. By Proposition 3.10 (applied with $\mathfrak{c}_0'$ instead of $\mathfrak{c}'$), the definition (5.2), of $\mathcal{R}_0$, and our choice of α and p_0 immediately preceding (5.2), there exists $\mathfrak{c}'' \in (\mathfrak{c}_*, \mathfrak{C}_*)$ depending only on $\mathfrak{c}_0'$ and the laws of D_h and $\widetilde{D}_h$ such that the following is true. For each $\widetilde{\beta} > 0$ there exists $\varepsilon_1 > 0$, depending only on $\mathbb{p}, \widetilde{\beta}$, and the laws of D_h and $\widetilde{D}_h$, such that for each $\varepsilon \in (0, \varepsilon_1]$ and each $\mathbb{r} > 0$ such that $\mathbb{P}[\widetilde{G}_{\mathbb{r}}(\widetilde{\beta}, \mathfrak{c}'')] \geqslant \widetilde{\beta}$, the cardinality of $\mathcal{R}_0 \cap [\varepsilon^2 \mathbb{r}, \varepsilon \mathbb{r}]$ is at least $\frac{3}{4}\log_8 \varepsilon^{-1}$. This implies that if $\varepsilon \in (0, \varepsilon_1]$,

$$\begin{aligned}
\#\big(\mathcal{R} \cap [\varepsilon^2 \mathbb{r}, \varepsilon \mathbb{r}]\big) &= \#\big(\mathcal{R}_0 \cap [\rho\varepsilon^2 \mathbb{r}, \rho\varepsilon \mathbb{r}]\big) \quad (\text{since } \mathcal{R} = \rho^{-1}\mathcal{R}_0) \\
&\geqslant \#\big(\mathcal{R}_0 \cap [(\rho\varepsilon)^2 \mathbb{r}, \rho\varepsilon \mathbb{r}]\big) - \#\big(\mathcal{R}_0 \cap [(\rho\varepsilon)^2 \mathbb{r}, \rho\varepsilon^2 \mathbb{r}]\big) \\
&\geqslant \#\big(\mathcal{R}_0 \cap [(\rho\varepsilon)^2 \mathbb{r}, \rho\varepsilon \mathbb{r}]\big) - \log_8 \rho^{-1} \quad (\text{since } \mathcal{R}_0 \subset \{8^{-k}\}_{k \in \mathbb{N}}) \\
&\geqslant \frac{3}{4}\log_8 \varepsilon^{-1} - \log_8 \rho^{-1} \qquad (\text{since } \rho\varepsilon \leqslant \varepsilon_1) \\
&\geqslant \frac{5}{8}\log_8 \varepsilon^{-1} \qquad (\text{for small enough } \varepsilon > 0, \text{ depending on } \rho).
\end{aligned}$$

Thus, Proposition 4.2 has been proven. □

5.8 | Initial estimates for a geodesic excursion

To prove our main results, it remains to prove Proposition 5.18. In the rest of this section, we will assume that we are in the setting of Proposition 5.18, that is, we assume that E_r occurs, P_r

is a $D_{h-\mathsf{f}_r}$-geodesic between two points of $\mathbb{C} \setminus B_{4r}(0)$, and $(\tau', \tau, \sigma, \sigma')$ is a $(B_{4r}(0), \mathsf{V}_r)$-excursion satisfying (5.48). It follows from Definition 4.1 that

$$P_r(\tau'), P_r(\sigma') \in \partial B_{4r}(0), \quad P_r(\tau), P_r(\sigma) \in \partial \mathsf{V}_r, \quad P_r((\tau', \sigma')) \subset B_{4r}(0),$$
$$\text{and} \quad P_r((\tau', \tau)) \cup P_r((\sigma, \sigma')) \subset B_{4r}(0) \setminus \overline{\mathsf{V}}_r. \tag{5.52}$$

We will prove (5.49) via a purely deterministic argument. We first check the following lemma, which will enable us to apply conditions 3 and 8 in the definition of E_r to $P_r|_{[\tau', \sigma']}$.

Lemma 5.19. *The path $P_r|_{[\tau', \sigma']}$ is contained in $\overline{\mathbb{A}}_{r,4r}(0)$ and is a $D_{h-\mathsf{f}_r}(\cdot, \cdot; \overline{\mathbb{A}}_{r,4r}(0))$-geodesic between two points of $\partial B_{4r}(0)$.*

Proof. We have $P_r|_{(\tau', \sigma')} \subset B_{4r}(0)$ and $P_r(\tau'), P_r(\sigma') \in \partial B_{4r}(0)$ by (5.52). We claim that P_r does not enter $B_r(0)$. Assume the claim for the moment. Then $P_r|_{(\tau', \sigma')} \subset \overline{\mathbb{A}}_{r,4r}(0)$. Since P_r is a $D_{h-\mathsf{f}_r}$-geodesic, the $D_{h-\mathsf{f}_r}$-length of $P_r|_{[\tau', \sigma']}$ is the same as the $D_{h-\mathsf{f}_r}$-distance between its endpoints. We conclude that $P_r|_{(\tau', \sigma')}$ is a path in $\overline{\mathbb{A}}_{r,4r}(0)$ whose $D_{h-\mathsf{f}_r}$-length is the same as the $D_{h-\mathsf{f}_r}$-distance between its endpoints, which is at most the $D_{h-\mathsf{f}_r}(\cdot, \cdot; \overline{\mathbb{A}}_{r,4r}(0))$-distance between its endpoints. Hence, $P_r|_{[\tau', \sigma']}$ is a $D_{h-\mathsf{f}_r}(\cdot, \cdot; \overline{\mathbb{A}}_{r,4r}(0))$-geodesic.

It remains to show that P_r does not enter $B_r(0)$. Assume by way of contradiction that $P_r \cap B_r(0) \neq \emptyset$. By condition 7 (internal distance in U_r) in the definition of E_r, there is a path Π in U_r which disconnects the inner and outer boundaries of U_r such that

$$\text{len}(\Pi; D_h) \leqslant 2\mathsf{A}_7 r^{\xi Q} e^{\xi h_r(0)}.$$

Let τ_0 (respectively, σ_0) be the first (respectively, last) time that P_r hits Π.

Since P_r is a $D_{h-\mathsf{f}_r}$-geodesic and $\mathsf{f}_r \equiv \mathsf{A}_8$ on U_r,

$$\sigma_0 - \tau_0 = D_{h-\mathsf{f}_r}(P_r(\tau_0), P_r(\sigma_0)) \leqslant \text{len}\Big(\Pi; D_{h-\mathsf{f}_r}\Big) \leqslant 2e^{-\xi \mathsf{A}_8} \mathsf{A}_7 r^{\xi Q} e^{\xi h_r(0)}. \tag{5.53}$$

On the other hand, since $\mathsf{U}_r \subset \mathbb{A}_{1.5r,2.5r}(0)$ and we are assuming that P_r hits $B_r(0)$, it follows that P_r must cross between the inner and outer boundaries of the annulus $\mathbb{A}_{r,1.5r}(0)$ between time τ_0 and time σ_0. Since $\mathsf{f}_r \equiv 0$ on $\mathbb{A}_{r,1.5r}(0)$ and by condition 1 (lower bound for distance across) in the definition of E_r,

$$\sigma_0 - \tau_0 = \text{len}\Big(P_r|_{[\tau_0, \sigma_0]}; D_{h-\mathsf{f}_r}\Big) \geqslant D_h\big(\text{across } \mathbb{A}_{r,1.5r}(0)\big) \geqslant \mathsf{a}_1 r^{\xi Q} e^{\xi h_r(0)}. \tag{5.54}$$

By our choice of A_8 in (5.45), the right side of (5.53) is smaller than the right side of (5.54), which supplies the desired contradiction. □

From Lemma 5.11, we now obtain the following.

Lemma 5.20. *We have*

$$|P_r(\sigma) - P_r(\tau)| \geqslant \mathsf{a}_4 r.$$

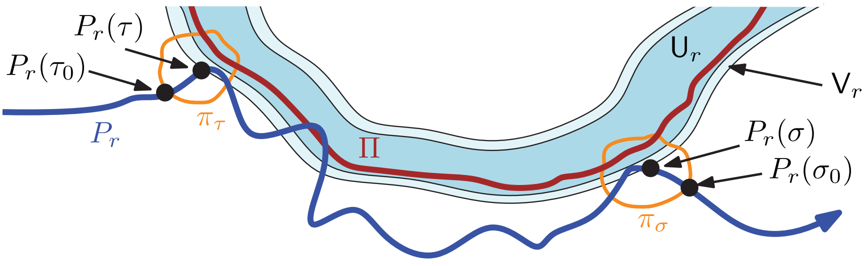

FIGURE 17 Illustration of the proof of Lemma 5.21. We obtain a path from a point of $P_r([\tau',\tau])$ to a point of $P_r([\sigma,\sigma'])$ whose $D_{h-\mathsf{f}_r}$-length is at most the right side of (5.55) by concatenating segments of π_τ, Π, and π_σ. This implies an upper bound for $\sigma-\tau$ since P_r is a $D_{h-\mathsf{f}}$-geodesic.

Proof. Due to Lemma 5.19 and (5.48), this follows from Lemma 5.11 applied with $V=\mathsf{V}_r$, $f=\mathsf{f}_r$, and P_f equal to the $D_{h-\mathsf{f}_r}$-geodesic $P_r|_{[\tau',\sigma']}$. $\square$

By (5.52), we have $P_r^{-1}(\overline{\mathsf{V}}_r)\subset[\tau,\sigma]$. We will now establish an upper bound for the length of this time interval.

Lemma 5.21. *We have*

$$\sigma-\tau\leqslant\frac{1}{2}\mathsf{a}_5{}^{\xi(Q+3)}r^{\xi Q}e^{\xi h_r(0)}. \tag{5.55}$$

Proof. See Figure 17 for an illustration. Let $\mathsf{a}_6\in(0,\lambda\mathsf{a}_3]$ be as in (5.44). By Lemma 5.19, we can apply Lemma 5.12 (with $\varepsilon=2\mathsf{a}_6$) to the $D_h(\cdot,\cdot;\overline{\mathbb{A}}_{r,4r}(0))$-geodesic $P_r|_{[\tau',\sigma']}$ to get that there are paths $\pi_\tau\subset\mathbb{A}_{2\mathsf{a}_6r,(2\mathsf{a}_6)^{1/2}r}(P_r(\tau))$ and $\pi_\sigma\subset\mathbb{A}_{2\mathsf{a}_6r,(2\mathsf{a}_6)^{1/2}r}(P_r(\sigma))$ which disconnect the inner and outer boundaries of their respective annuli such that

$$\max\{\mathrm{len}(\pi_\tau;D_h),\mathrm{len}(\pi_\sigma;D_h)\}\leqslant(2\mathsf{a}_6)^\theta\mathsf{A}_2r^{\xi Q}e^{\xi h_r(0)}\leqslant\lambda\mathsf{a}_5{}^{\xi(Q+3)}r^{\xi Q}e^{\xi h_r(0)}, \tag{5.56}$$

where the last inequality is by (5.44). Let τ_0 be the last time before τ that P_r hits π_τ and let σ_0 be the first time after σ that P_r hits π_σ. Then $\tau_0\in[\tau',\tau]$ and $\sigma_0\in[\sigma,\sigma']$.

By condition 7 (internal distance in U_r) in the definition of E_r, there is a path $\Pi\subset\mathsf{U}_r$ which disconnects the inner and outer boundaries of U_r, has D_h-length at most $\mathsf{A}_7r^{\xi Q}e^{\xi h_r(0)}$, and such that each point of the outer boundary of U_r lies at Euclidean distance at most a_6r from Π. We have $P_r(\tau)\in\partial\mathsf{V}_r=\partial B_{\mathsf{a}_9r}(\mathsf{U}_r)$ and $P([\tau',\tau])$ is contained in the unbounded connected component of $\mathbb{C}\setminus\mathsf{U}_r$. Hence, $P_r(\tau)$ lies at Euclidean distance at most a_9r from the outer boundary of U_r. Therefore, the Euclidean distance from $P_r(\tau)$ to Π is at most $(\mathsf{a}_9+\mathsf{a}_6)r\leqslant2\mathsf{a}_6r$, where we use that $\mathsf{a}_9\leqslant\mathsf{a}_6$ by definition.

Since $\pi_\tau\subset\mathbb{A}_{2\mathsf{a}_6r,(2\mathsf{a}_6)^{1/2}r}(P_r(\tau))$ and π_τ disconnects the inner and outer boundaries of $\mathbb{A}_{2\mathsf{a}_6r,(2\mathsf{a}_6)^{1/2}r}(P_r(\tau))$, it follows from the preceding paragraph that π_τ intersects Π. Similarly, π_σ intersects Π. Hence, the union of the loops Π, π_τ, and π_σ contains a path from $P_r(\tau_0)$ to $P_r(\sigma_0)$. Therefore,

$$\begin{aligned}\sigma-\tau\leqslant\sigma_0-\tau_0&=D_{h-\mathsf{f}_r}(P_r(\tau_0),P_r(\sigma_0))\\&\leqslant\mathrm{len}\left(\pi_\tau;D_{h-\mathsf{f}_r}\right)+\mathrm{len}\left(\pi_\sigma;D_{h-\mathsf{f}_r}\right)+\mathrm{len}\left(\Pi;D_{h-\mathsf{f}_r}\right)\end{aligned} \tag{5.57}$$

Let us now bound the right side of (5.57). Since f_r is non-negative, the $D_{h-\mathsf{f}_r}$-length of each of π_τ and π_σ is at most the right side of (5.56). Since $\mathsf{f}_r \equiv \mathsf{A}_8$ on U_r,

$$\mathrm{len}\left(\Pi; D_{h-\mathsf{f}_r}\right) = e^{-\xi \mathsf{A}_8}\, \mathrm{len}(\Pi; D_h) \leqslant e^{-\xi \mathsf{A}_8} \mathsf{A}_7 r^{\xi Q} e^{\xi h_r(0)} \leqslant \lambda \mathsf{a}_5{}^{\xi(Q+3)} r^{\xi Q} e^{\xi h_r(0)}, \tag{5.58}$$

where the last inequality uses the definition (5.45) of A_8. Plugging these estimates into (5.57) gives

$$\sigma - \tau \leqslant 3\lambda \mathsf{a}_5{}^{\xi(Q+3)} r^{\xi Q} e^{\xi h_r(0)}, \tag{5.59}$$

which is stronger than (5.55). □

Combining Lemma 5.21 with condition 6 (reverse Hölder continuity) in the definition of E_r allows us to show that any segment of $P_r|_{[\tau,\sigma]}$ which is disjoint from V_r must have small Euclidean diameter.

Lemma 5.22. *Each segment of $P_r|_{[\tau,\sigma]}$ which is disjoint from V_r has Euclidean diameter at most $\mathsf{a}_5 r$. In particular,*

$$P_r([\tau,\sigma]) \subset B_{\mathsf{a}_5 r}(\mathsf{V}_r).$$

Proof. Suppose by way of contradiction that there is a segment $P_r|_{[t,s]}$ for times $\tau \leqslant t < s \leqslant \sigma$ which is disjoint from V_r and has Euclidean diameter larger than $\mathsf{a}_5 r$. By (5.52), $P_r([\tau,\sigma])$ intersects $\overline{\mathsf{V}}_r$. Hence, by possibly replacing $P_r|_{[t,s]}$ by a segment of P_r which travels from $\partial \mathsf{V}_r$ to $\partial B_{\mathsf{a}_5 r}(\mathsf{V}_r)$, we can assume without loss of generality that $P_r([t,s])$ is contained in $B_{\mathsf{a}_5 r}(\mathsf{V}_r)$, which in turn is contained in $\mathbb{A}_{1.5r,3r}(0)$ by the definition of V_r (Subsection 5.4). By the reverse Hölder continuity condition 6 in the definition of E_r, the D_h-length of $P_r|_{[t,s]}$ is at least $\mathsf{a}_5{}^{\xi(Q+3)} r^{\xi Q} e^{\xi h_r(0)}$. Since f_r is supported on V_r, the $D_{h-\mathsf{f}_r}$-length of $P_r|_{[t,s]}$ is equal to its D_h-length, so is also at least $\mathsf{a}_5{}^{\xi(Q+3)} r^{\xi Q} e^{\xi h_r(0)}$. Since $P_r|_{[\tau,\sigma]}$ is a $D_{h-\mathsf{f}_r}$-geodesic, we therefore have

$$\sigma - \tau \geqslant s - t \geqslant \mathsf{a}_5{}^{\xi(Q+3)} r^{\xi Q} e^{\xi h_r(0)}. \tag{5.60}$$

This contradicts Lemma 5.21. □

5.9 | Forcing a geodesic to enter balls centered at $\mathsf{u}_{z,\rho r}$ and $\mathsf{v}_{z,\rho r}$

Recall the balls $B_{\mathsf{s}_{\rho r}}(\mathsf{u}_{z,\rho r})$ and $B_{\mathsf{s}_{\rho r}}(\mathsf{v}_{z,\rho r})$ appearing in the definition of the 'building block' event $\mathsf{F}_{z,\rho r}$ from Subsection 5.3. On $\mathsf{F}_{z,\rho r}$, there are points $u \in B_{\mathsf{s}_{\rho r}}(\mathsf{u}_{z,\rho r})$ and $v \in B_{\mathsf{s}_{\rho r}}(\mathsf{v}_{z,\rho r})$ which satisfy $\widetilde{D}_h(u,v) \leqslant \mathfrak{c}_0' D_h(u,v)$, plus several other conditions. To prove Proposition 5.18, we want to force P_r to get $D_{h-\mathsf{f}_r}$-close to each of u and v for one of these pairs of points u, v, then apply the triangle inequality. To do this, the first step is to force P_r to get close to the balls $B_{\mathsf{s}_{\rho r}}(\mathsf{u}_{z,\rho r})$ and $v \in B_{\mathsf{s}_{\rho r}}(\mathsf{v}_{z,\rho r})$ for some $z \in \mathsf{Z}_r$ such that $\mathsf{F}_{z,\rho r}$ occurs. We will carry out this step in this subsection. Our goal is to prove the following lemma.

Lemma 5.23. *Let $\mathsf{Z}_r \subset \partial B_{2r}(0)$ be as in (5.24). There exists $z \in \mathsf{Z}_r$ such that $\mathsf{F}_{z,\rho r}$ occurs and the following is true. Let $\mathsf{s}_{\rho r}$, $\mathsf{u}_{z,\rho r}$, and $\mathsf{v}_{z,\rho r}$ be the radius and points as in the definition of $\mathsf{F}_{z,\rho r}$. There*

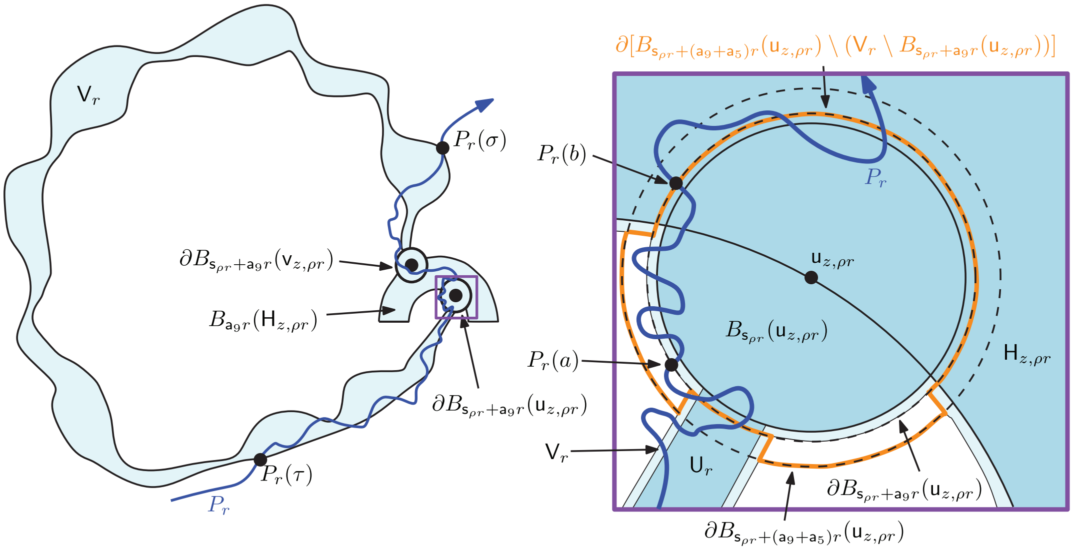

FIGURE 18 Illustration of the statement of Lemma 5.23. Left: The set V_r (light blue) and the path segment $P_r|_{[\tau,\sigma]}$. For simplicity, we have not drawn the details of V_r except in the $\mathsf{a}_9 r$-neighborhood of the set $\mathsf{H}_{z,\rho r}\cup B_{\mathsf{s}_{\rho r}}(\mathsf{u}_{z,\rho r})\cup B_{\mathsf{s}_{\rho r}}(\mathsf{v}_{z,\rho r})$. The set U_r is not shown. Right: The left panel zoomed in on the purple box. We have shown a subset of U_r (light blue) and a subset of $\mathsf{V}_r\setminus\mathsf{U}_r$ (lighter blue). By (5.62), the path segment $P_r|_{[a,b]}$ is required to stay region outlined in orange.

exist times $\tau\leqslant a<b\leqslant\sigma$ which satisfy the following conditions:

$$P_r(a),P_r(b)\in\partial B_{\mathsf{s}_{\rho r}+\mathsf{a}_9 r}(\mathsf{u}_{z,\rho r}),\quad |P_r(b)-P_r(a)|\geqslant \mathsf{s}_{\rho r}/8,\quad \textit{and} \tag{5.61}$$

$$P_r([a,b])\subset B_{\mathsf{s}_{\rho r}+(\mathsf{a}_9+\mathsf{a}_5)r}(\mathsf{u}_{z,\rho r})\setminus\left(\mathsf{V}_r\setminus B_{\mathsf{s}_{\rho r}+\mathsf{a}_9 r}(\mathsf{u}_{z,\rho r})\right). \tag{5.62}$$

Moreover, the same is true with $\mathsf{v}_{z,\rho r}$ in place of $\mathsf{u}_{z,\rho r}$.

See Figure 18 for an illustration of the statement of Lemma 5.23. Before discussing the proof, we make some comments on the statement. The ball $B_{\mathsf{s}_{\rho r}+\mathsf{a}_9 r}(\mathsf{u}_{z,\rho r})$ appearing in Lemma 5.23 is significant because, by the definition of $\mathsf{V}_{\rho r}$ in (5.26), this is the largest Euclidean ball centered at $\mathsf{u}_{z,\rho r}$ which is contained in $\mathsf{V}_{\rho r}$. The significance of the ball $B_{\mathsf{s}_{\rho r}+(\mathsf{a}_9+\mathsf{a}_5)r}(\mathsf{u}_{z,\rho r})$ appearing in (5.62) is that by Lemma 5.22, the path $P_r|_{[\tau,\sigma]}$ cannot exit the $\mathsf{a}_5 r$-neighborhood of V_r. We note that $\mathsf{s}_{\rho r}\geqslant \mathsf{t}\rho r$ (Lemma 5.2), which is much larger than $\mathsf{a}_5 r$ (Lemma 5.14), which in turn is much larger than $\mathsf{a}_9 r$ (recall the discussion surrounding (5.21)). So, the balls in (5.61) and (5.62) are only slightly larger than $B_{\mathsf{s}_{\rho r}}(\mathsf{u}_{z,\rho r})$.

Lemma 5.23 will be a consequence of Lemmas 5.20 and 5.22 (which give a lower bound for $|P_r(\tau)-P_r(\sigma)|$ and an upper bound for the Euclidean diameter of any segment of P_r which is disjoint from V_r), condition 4 in the definition of E_r (which gives lots of points $z\in\mathsf{Z}_r$ for which $\mathsf{F}_{z,\rho r}$ occurs), and some basic geometric arguments based on the definition of U_r from Subsection 5.4.

We encourage the reader to look at Figure 19 while reading the proof. Let us start by explaining why we can apply condition 4 in the definition of E_r. We have $P_r(\tau),P_r(\sigma)\in\partial\mathsf{V}_r$ by (5.52) and $|P_r(\sigma)-P_r(\tau)|\geqslant \mathsf{a}_4 r$ by Lemma 5.20. Moreover, by the definition of V_r in Subsection 5.4, the Euclidean distance from each point of V_r to $\partial B_{2r}(0)$ is at most $100\rho r$, which by our choice

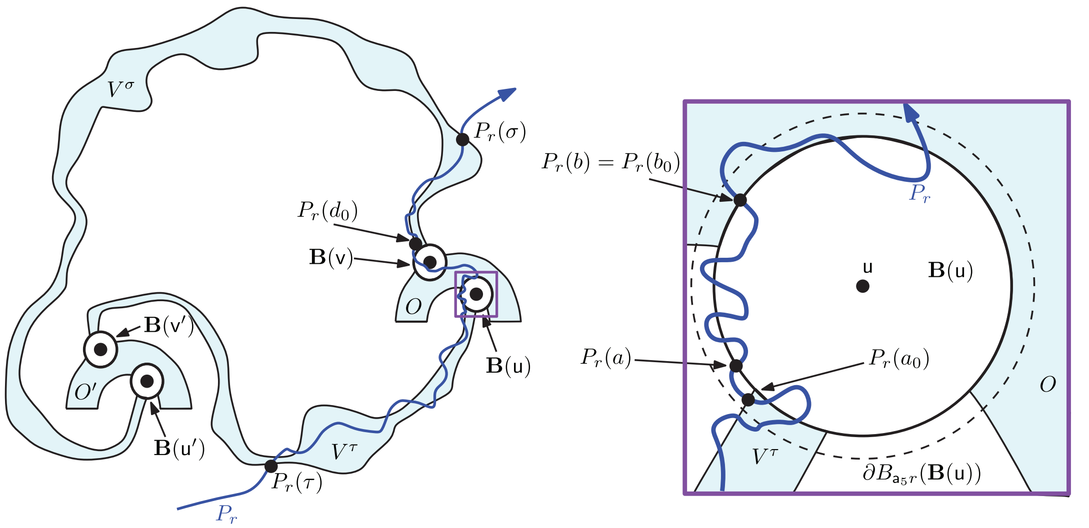

FIGURE 19 Left: The connected components V^τ, V^σ, O, O' of $\mathsf{V}_r \setminus [\mathbf{B}(\mathsf{u}) \cup \mathbf{B}(\mathsf{v}) \cup \mathbf{B}(\mathsf{v}') \cup \mathbf{B}(\mathsf{u}')]$ and the point $P_r(d_0)$ where P_r first enters V^σ. For simplicity we have drawn V^τ and V^σ as 'blobs' rather than showing the details of how V_r is defined in Subsection 5.4 (cf. Figure 15). Right: A zoomed-in view in the purple box from the left figure. Here b_0 is the first time that P_r hits O, a_0 is the last time before b_0 at which P_r exits V^τ, a is the first time after a_0 at which P_r hits $\partial\mathbf{B}(\mathsf{u})$, and b is the last time before b_0 at which P_r exits $\mathbf{B}(\mathsf{u})$. In the figure, we have $a \neq a_0$ and $b = b_0$, but any combination of $a = a_0$ or $a \neq a_0$ and/or $b = b_0$ or $b \neq b_0$ is possible.

of ρ in Lemma 5.13 is at most $100\lambda \mathsf{a}_4 r \leqslant \mathsf{a}_4 r/100$. Therefore, the set $\partial B_{2r}(0) \setminus [\overline{B}_{100\rho r}(P_r(\tau)) \cup \overline{B}_{100\rho r}(P_r(\sigma))]$ consists of two disjoint connected arcs of $\partial B_{2r}(0)$ which each have Euclidean length at least $\mathsf{a}_4 r/2$. Let J (respectively, J') be the one of these two arcs which goes in the counterclockwise (respectively, clockwise) direction from $\overline{B}_{100\rho r}(P_r(\tau))$ to $\overline{B}_{100\rho r}(P_r(\sigma))$.

By condition 4 in the definition of E_r, there exist $z \in J \cap \mathsf{Z}_r$ and $z' \in J' \cap \mathsf{Z}_r$ such that $\mathsf{F}_{z,\rho r}$ and $\mathsf{F}_{z',\rho r}$ both occur. To lighten notation, we write

$$\mathsf{u} := \mathsf{u}_{z,\rho r}, \quad \mathsf{v} := \mathsf{v}_{z,\rho r}, \quad \mathsf{u}' := \mathsf{u}_{z',\rho r}, \quad \mathsf{v}' := \mathsf{v}_{z',\rho r}$$

and

$$\mathbf{B}(w) := B_{\mathsf{s}_{\rho r} + \mathsf{a}_9 r}(w), \quad \forall w \in \{\mathsf{u}, \mathsf{v}, \mathsf{u}', \mathsf{v}'\}. \tag{5.63}$$

The set $\mathsf{V}_r \setminus [\mathbf{B}(\mathsf{u}) \cup \mathbf{B}(\mathsf{v}) \cup \mathbf{B}(\mathsf{v}') \cup \mathbf{B}(\mathsf{u}')]$ consists of exactly four connected components which each lie at Euclidean distance at least $\mathsf{s}_{\rho r}/4$ from each other. We call these connected components V^τ, V^σ, O, O'. We can choose the labeling so that with $\mathsf{H}_{z,\rho r}$ and $\mathsf{H}_{z',\rho r}$ the half-annuli as in the definitions of $\mathsf{F}_{z,\rho r}$ and $\mathsf{F}_{z',\rho r}$,

$$P_r(\tau) \in \partial V^\tau, \quad P_r(\sigma) \in \partial V^\sigma, \quad O \subset B_{\mathsf{a}_9 r}(\mathsf{H}_{z,\rho r}) \quad \text{and} \quad O' \subset B_{\mathsf{a}_9 r}(\mathsf{H}_{z',\rho r}). \tag{5.64}$$

We note that the boundary of each of these connected components intersects exactly two of the boundaries of the balls $\mathbf{B}(w)$ for $w \in \{\mathsf{u}, \mathsf{v}, \mathsf{u}', \mathsf{v}'\}$. See Figure 19, left, for an illustration.

Let d_0 be the first time that $P_r|_{[\tau,\sigma]}$ hits $\overline{V}^\sigma$ (this time is well-defined since we know that $P_r(\sigma) \in \partial V^\sigma$). By Lemma 5.22, each segment of $P_r|_{[\tau,\sigma]}$ which is disjoint from V_r has Euclidean diameter at most $\mathsf{a}_5 r$, which is much smaller than $\mathsf{s}_{\rho r}/4$. It follows that either $P_r(d_0) \in B_{\mathsf{a}_5 r}(\mathbf{B}(\mathsf{v})) \cap \overline{V}^{\sigma}$ or $P_r(d_0) \in B_{\mathsf{a}_5 r}(\mathbf{B}(\mathsf{u}')) \cap \overline{V}^{\sigma}$. For simplicity, we henceforth assume that

$$P_r(d_0) \in B_{\mathsf{a}_5 r}(\mathbf{B}(\mathsf{v})) \cap \overline{V}^{\sigma}; \tag{5.65}$$

the other case can be treated in an identical manner.

Most of the rest of the proof will focus on what happens near $\mathbf{B}(\mathsf{u})$. See Figure 19, right, for an illustration. We first define a time b_0 such that $P_r(b_0)$ will be Euclidean-close to the point $P_r(b)$ from Lemma 5.23.

Lemma 5.24. *Let b_0 be the smallest $t \geqslant \tau$ for which $P_r(b_0) \in \overline{O}$. Then $b_0 < d_0$ and $P_r(b_0) \in \partial O \cap B_{\mathsf{a}_5 r}(\mathbf{B}(\mathsf{u}))$.*

Proof. The path $P_r|_{[\tau, d_0]}$ travels from ∂V^{τ} to $B_{\mathsf{a}_5 r}(\mathbf{B}(\mathsf{v})) \cap \overline{V}^{\sigma}$ and does not enter V^{σ}. The set $\mathsf{V}_r \setminus (V^{\sigma} \cup O)$ has two connected components which lie at Euclidean distance at least $(1-\alpha)\rho r/2 \geqslant \mathsf{a}_5 r$ (recall our choice of a_5 from Lemma 5.14) from each other, one of which contains $\mathbf{B}(\mathsf{v})$ and the other of which contains V^{τ}. By Lemma 5.22, $P_r|_{[\tau, d_0]}$ cannot travel Euclidean distance more than $\mathsf{a}_5 r$ without hitting V_r. Hence, $P_r|_{[\tau, d_0]}$ must hit O before it hits $\overline{V}^{\sigma}$. Therefore, $b_0 < d_0$ and $P_r(b_0) \in \partial O$. Furthermore, since $\mathbf{B}(\mathsf{v})$ and V^{τ} are contained in different connected components of $\mathsf{V}_r \setminus (V^{\sigma} \cup O)$ and by the definitions of b_0 and d_0, we have $P_r([\tau, b_0]) \cap (V^{\sigma} \cup O \cup \mathbf{B}(\mathsf{v})) = \emptyset$.

We need to show that $P_r(b_0) \in B_{\mathsf{a}_5 r}(\mathbf{B}(\mathsf{u}))$. Indeed, since $P_r|_{[\tau, b_0]}$ cannot hit $V^{\sigma} \cup O \cup \mathbf{B}(\mathsf{v})$ and cannot travel Euclidean distance more than $\mathsf{a}_5 r$ outside of V_r, it must be the case that

$$P_r(b_0) \in B_{\mathsf{a}_5 r}\big(V^{\tau} \cup O' \cup \mathbf{B}(\mathsf{u}) \cup \mathbf{B}(\mathsf{u}') \cup \mathbf{B}(\mathsf{v}')\big).$$

The sets V^{τ}, O', $\mathbf{B}(\mathsf{u}')$, and $\mathbf{B}(\mathsf{v}')$ each lie at Euclidean distance larger than $\mathsf{a}_5 r$ from O, so since $P_r(b_0) \in \partial O$ we must have $P_r(b_0) \in B_{\mathsf{a}_5 r}(\mathbf{B}(\mathsf{u}))$. ☐

Next, we define a time a_0 such that $P_r(a_0)$ will be Euclidean-close to the point $P_r(a)$ from Lemma 5.23.

Lemma 5.25. *Let a_0 be the last time t before b_0 for which $P_r(t) \in \overline{V}^{\tau}$. Then*

$$|P_r(b_0) - P_r(a_0)| \geqslant \mathsf{s}_{\rho r}/4 \quad \textit{and} \quad P_r([a_0, b_0]) \subset B_{\mathsf{a}_5 r}(\mathbf{B}(\mathsf{u})) \setminus (\mathsf{V}_r \setminus \mathbf{B}(\mathsf{u})). \tag{5.66}$$

Proof. Since $P_r(b_0) \in \partial O$ and the Euclidean distance from V^{τ} to O is at least $\mathsf{s}_{\rho r}/4$, we immediately obtain that $|P_r(b_0) - P_r(a_0)| \geqslant \mathsf{s}_{\rho r}/4$. It remains to prove the inclusion in (5.66).

By definition, the set $P_r([a_0, b_0])$ is disjoint from $V^{\tau} \cup O$. Furthermore, by Lemma 5.22, each segment of $P_r|_{[a_0, b_0]}$ which is not contained in V_r has Euclidean diameter at most $\mathsf{a}_5 r$. Therefore,

$$P_r([a_0, b_0]) \subset B_{\mathsf{a}_5 r}\big(V^{\sigma} \cup O' \cup \mathbf{B}(\mathsf{u}) \cup \mathbf{B}(\mathsf{v}) \cup \mathbf{B}(\mathsf{v}') \cup \mathbf{B}(\mathsf{u}')\big). \tag{5.67}$$

The set on the right side of (5.67) has two connected components, one of which is equal to $B_{\mathsf{a}_5 r}(\mathbf{B}(\mathsf{u}))$ and the other of which contains the other five sets in the union. Since $P_r(b_0) \in B_{\mathsf{a}_5 r}(\mathbf{B}(\mathsf{u}))$ (Lemma 5.24), we get that $P_r([a_0, b_0]) \subset B_{\mathsf{a}_5 r}(\mathbf{B}(\mathsf{u}))$ and $P_r([a_0, b_0])$ is disjoint from

$V^\sigma \cup O' \cup \mathbf{B}(\mathsf{v}) \cup \mathbf{B}(\mathsf{v}') \cup \mathbf{B}(\mathsf{u}')$. Since we already know that $P_r([a_0, b_0])$ is disjoint from $V^\tau \cup O$, we obtain the inclusion in (5.66). □

Proof of Lemma 5.23. Let a be the first time $t \geqslant a_0$ such that $P_r(t) \in \overline{\mathbf{B}}(\mathsf{u})$ and let b be the last time $t \leqslant b_0$ such that $P_r(t) \in \overline{\mathbf{B}}(\mathsf{u})$. Note that we might have $a = a_0$ and/or $b = b_0$ (see Figure 19, right). By (5.66), $P_r|_{[a_0,b_0]}$ cannot hit $\mathsf{V}_r \setminus \mathbf{B}(\mathsf{u})$. By this and Lemma 5.22, $P_r|_{[a_0,b_0]}$ cannot travel Euclidean distance more than $\mathsf{a}_5 r$ without entering $\mathbf{B}(\mathsf{u})$. Consequently, the times a and b are well-defined and

$$\max\{|P_r(a) - P_r(a_0)|, |P_r(b) - P_r(b_0)|\} \leqslant \mathsf{a}_5 r. \tag{5.68}$$

By (5.66) and (5.68) and the triangle inequality,

$$|P_r(b) - P_r(a)| \geqslant \mathsf{s}_{\rho r}/4 - 2\mathsf{a}_5 r, \tag{5.69}$$

which is at least $\mathsf{s}_{\rho r}/8$ since $\mathsf{s}_{\rho r} \geqslant \mathsf{t}\rho r \geqslant \lambda \mathsf{a}_5$ (by our choice of $\mathsf{s}_{\rho r}$ in Lemma 5.2 and our choice of a_5 in Lemma 5.14). By the definitions of a and b, we have $P_r(a), P_r(b) \in \partial \mathbf{B}(\mathsf{u})$. Since $a, b \in [a_0, b_0]$ and by Lemma 5.25, we also have the inclusion (5.62).

This gives the lemma statement for $\mathsf{u} = \mathsf{u}_{z,\rho r}$. The statement with $\mathsf{v} = \mathsf{v}_{z,\rho r}$ in place of u follows by repeating Lemma 5.25 and the argument above with d_0 used in place of b_0. □

5.10 | Forcing a geodesic to get close to u and v

We henceforth fix $z \in \mathsf{Z}_r$ and times $a, b \in [\tau, \sigma]$ as in Lemma 5.23. We also let u and v be as in the definition of $\mathsf{F}_{z,\rho r}$, so that $u \in B_{\mathsf{s}_{\rho r}/2}(\mathsf{u}_{z,\rho r})$, $v \in B_{\mathsf{s}_{\rho r}/2}(\mathsf{v}_{z,\rho r})$, and $\widetilde{D}_h(u, v) \leqslant \mathfrak{c}_0' D_h(u, v)$. Recall that we are trying to force the path P_r to get $D_{h-\mathsf{f}_r}$-close to each of u and v.

Lemma 5.23 tells us that P_r gets Euclidean-close to each of u and v, but this is not sufficient for our purposes since in the supercritical case D_h is not continuous with respect to the Euclidean metric. To ensure that P_r gets $D_{h-\mathsf{f}_r}$-close to each of u and v, we will need a careful argument involving several of the conditions in the definitions of $\mathsf{F}_{z,\rho r}$ and E_r. The main result of this subsection is the following lemma.

Lemma 5.26. *There is a constant $C > 0$, depending only on ξ, such that the following is true. Almost surely, there exists $t \in [\tau, \sigma]$ such that*

$$P_r(t) \in B_{\mathsf{s}_{\rho r} + (3\mathsf{a}_5 + \mathsf{a}_9) r}(\mathsf{u}_{z,\rho r}) \quad \textit{and} \tag{5.70}$$

$$D_{h-\mathsf{f}_r}\left(P_r(t), u; \mathbb{A}_{r,4r}(0)\right) \leqslant C \lambda e^{-\xi \mathsf{A}_8} \widetilde{D}_h(u, v). \tag{5.71}$$

Moreover, the same is true with v and $\mathsf{v}_{z,\rho r}$ in place of u and $\mathsf{u}_{z,\rho r}$.

We will eventually choose λ to be much smaller than $1/C$, so that the right side of (5.71) is much smaller than $e^{-\xi \mathsf{A}_8} \widetilde{D}_h(u, v)$. We will only prove Lemma 5.26 for u; the statement with v in place of u is proven in an identical manner.

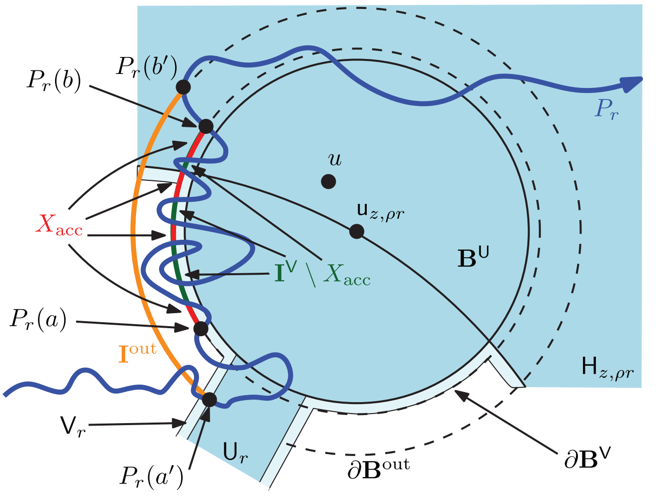

FIGURE 20 Illustration of several of the objects involved in Subsection 5.10. The arc $\mathbf{I}^{\vee} \subset \partial\mathbf{B}^{\vee}$ is the union of the red set X_{acc} consisting of points which are accessible from $\mathbf{I}^{\mathrm{out}}$ in $\overline{\mathbf{B}^{\mathrm{out}}} \setminus (\mathbf{B}^{\vee} \cup P_r([a',b'])$ and the green set $\mathbf{I}^{\vee} \setminus X_{\mathrm{acc}}$. Note that a connected component of $\mathbf{I}^{\vee} \setminus X_{\mathrm{acc}}$ can contain points of $P_r([a',b'])$ in its interior (relative to $\mathbf{I}^{\vee}$).

5.10.1 | Setup

Before proceeding with the proof of Lemma 5.26, we introduce some notation. See Figure 20 for an illustration. We define the Euclidean balls

$$\mathbf{B}^{\cup} := B_{\mathsf{s}_{\rho r}}(\mathsf{u}_{z,\rho r}), \quad \mathbf{B}^{\vee} := B_{\mathsf{s}_{\rho r}+\mathsf{a}_9 r}(\mathsf{u}_{z,\rho r}), \quad \text{and} \quad \mathbf{B}^{\mathrm{out}} := B_{\mathsf{s}_{\rho r}+(3\mathsf{a}_5+\mathsf{a}_9)r}(\mathsf{u}_{z,\rho r}). \tag{5.72}$$

The reason why we care about $\mathbf{B}^{\cup}$ and $\mathbf{B}^{\vee}$ is that by the definitions of U_r and V_r, the ball $\mathbf{B}^{\cup}$ (respectively, $\mathbf{B}^{\vee}$) is the largest Euclidean ball centered at $\mathsf{u}_{z,\rho r}$ which is contained in U_r (respectively, V_r). The reason why we care about $\mathbf{B}^{\mathrm{out}}$ is that by Lemma 5.23, $P_r|_{[a,b]}$ cannot exit the ball $B_{\mathsf{s}_{\rho r}+(\mathsf{a}_5+\mathsf{a}_9)r}(\mathsf{u}_{z,\rho r}) \subset \mathbf{B}^{\mathrm{out}}$. We need $\mathbf{B}^{\mathrm{out}}$ to have a slightly larger radius than $\mathsf{s}_{\rho r} + (\mathsf{a}_5 + \mathsf{a}_9)r$ for the purposes of Lemma 5.34.

We also define

$$a' := \sup\{t \leqslant a : P_r(t) \in \partial\mathbf{B}^{\mathrm{out}}\} \quad \text{and} \quad b' := \inf\{t \geqslant b : P_r(t) \in \partial\mathbf{B}^{\mathrm{out}}\}. \tag{5.73}$$

Then $a' < a < b < b'$. Furthermore, Lemma 5.23 implies that $P_r([a,b]) \subset \mathbf{B}^{\mathrm{out}}$, so the definitions of a' and b' show that $P_r([a',b']) \subset \overline{\mathbf{B}^{\mathrm{out}}}$ and $P_r((a',b')) \subset \mathbf{B}^{\mathrm{out}}$.

Recall that the point u appearing in Lemma 5.26 is contained in $\mathbf{B}^{\cup}$. Lemma 5.26 holds vacuously if $u \in P_r([a',b'])$, so we can assume without loss of generality that

$$u \notin P_r([a',b']). \tag{5.74}$$

The set $\partial\mathbf{B}^{\mathrm{out}} \setminus \{P_r(a'), P_r(b')\}$ consists of two disjoint arcs. Since $P_r|_{[a',b']}$ is a simple curve in $\overline{\mathbf{B}^{\mathrm{out}}}$ which intersects $\partial\mathbf{B}^{\mathrm{out}}$ only at its endpoints, it follows that exactly one of these two arcs is disconnected from u by $P_r|_{[a',b']}$. We assume without loss of generality that the clockwise arc of

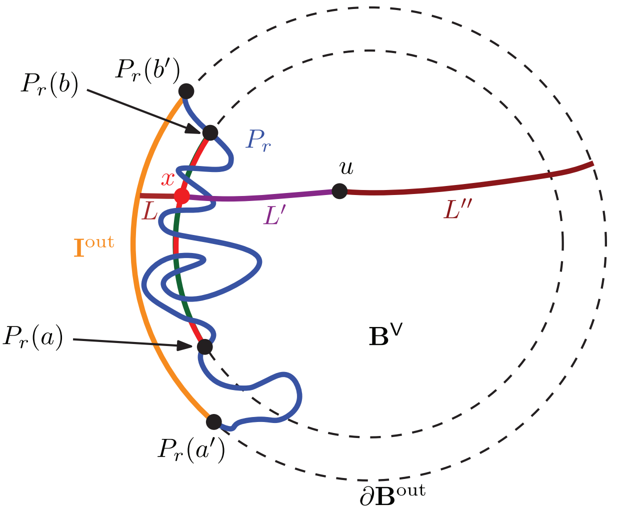

FIGURE 21 Illustration of the proof of Lemma 5.27. The path $P_r|_{[a',b']}$ must intersect $L \cup L' \cup L''$. By our choices of L and L'', it must in fact intersect L'.

$\partial\mathbf{B}^{\text{out}}$ from $P_r(a')$ to $P_r(b')$ is disconnected from u. Let

$$\mathbf{I}^{\text{out}} := \{\text{open clockwise arc of } \partial\mathbf{B}^{\text{out}} \text{ from } P_r(a') \text{ to } P_r(b')\}$$
$$\mathbf{I}^{\vee} := \{\text{open clockwise arc of } \partial\mathbf{B}^{\vee} \text{ from } P_r(a) \text{ to } P_r(b)\}. \tag{5.75}$$

Note that $P_r([a',b'])$ disconnects $\mathbf{I}^{\text{out}}$ from u in $\mathbf{B}^{\text{out}}$, but does not necessarily disconnect $\mathbf{I}^\vee$ from u in $\mathbf{B}^{\text{out}}$. By Lemma 5.23, we have $|P_r(b) - P_r(a)| \geqslant \mathfrak{s}_{\rho r}/8$, so the Euclidean length of $\mathbf{I}^\vee$ satisfies

$$|\mathbf{I}^\vee| \geqslant \mathfrak{s}_{\rho r}/8. \tag{5.76}$$

We say that $x \in \mathbf{I}^\vee$ is *accessible from* $\mathbf{I}^{\text{out}}$ in $\overline{\mathbf{B}^{\text{out}}} \setminus (\mathbf{B}^\vee \cup P_r([a',b']))$ if there is a path in $\overline{\mathbf{B}^{\text{out}}} \setminus (\mathbf{B}^\vee \cup P_r([a',b']))$ from x to a point of $\mathbf{I}^{\text{out}}$. Let

$$X_{\text{acc}} := \left\{x \in \mathbf{I}^\vee : x \text{ is accessible from } \mathbf{I}^{\text{out}} \text{ in } \overline{\mathbf{B}^{\text{out}}} \setminus (\mathbf{B}^\vee \cup P_r([a',b']))\right\}. \tag{5.77}$$

See Figure 20 for an illustration. One of the main reasons why we are interested in the set X_{acc} is the following elementary topological fact.

Lemma 5.27. *If $x \in X_{\text{acc}}$, then every path in $\overline{\mathbf{B}^{\text{out}}}$ from u to x hits $P_r([a',b'])$.*

Proof. See Figure 21 for an illustration. Recall that $\mathbf{I}^{\text{out}}$ and $\partial\mathbf{B}^{\text{out}} \setminus \overline{\mathbf{I}^{\text{out}}}$ are the open clockwise and counterclockwise arcs of $\partial\mathbf{B}^{\text{out}}$ from $P_r(a')$ to $P_r(b')$, respectively. By the assumption made just before (5.76), $P_r|_{[a',b']}$ disconnects $\mathbf{I}^{\text{out}}$ but not $\partial\mathbf{B}^{\text{out}} \setminus \overline{\mathbf{I}^{\text{out}}}$ from u in $\mathbf{B}^{\text{out}}$.

By the definition (5.77) of X_{acc}, there is a path L from x to a point of $\mathbf{I}^{\text{out}}$ in $\overline{\mathbf{B}^{\text{out}}}$ which is disjoint from $\mathbf{B}^\vee \cup P_r([a',b'])$. Furthermore, since $P_r|_{[a',b']}$ does not disconnect $\partial\mathbf{B}^{\text{out}} \setminus \overline{\mathbf{I}^{\text{out}}}$ from u in $\overline{\mathbf{B}^{\text{out}}}$, there is a path from u to a point of $\partial\mathbf{B}^{\text{out}} \setminus \overline{\mathbf{I}^{\text{out}}}$ in $\overline{\mathbf{B}^{\text{out}}}$ which is disjoint from $P_r([a',b'])$.

Now consider a path L' in $\overline{\mathbf{B}^{\mathrm{out}}}$ from u to x. The union $L \cup L' \cup L''$ contains a path in $\overline{\mathbf{B}^{\mathrm{out}}}$ joining the two arcs of $\partial \mathbf{B}^{\mathrm{out}} \setminus \{P_r(a'), P_r(b')\}$. Since $P_r|_{[a',b']}$ is a path in $\overline{\mathbf{B}^{\mathrm{out}}}$, topological considerations show that $P_r|_{[a',b']}$ must hit $L \cup L' \cup L''$. Since $P_r|_{[a',b']}$ cannot hit L or L'' by definition, we get that $P_r|_{[a',b']}$ must hit L'. □

For $x \in \mathbf{I}^{\vee}$, we define

$$x' := \frac{\mathsf{s}_{\rho r}}{\mathsf{s}_{\rho r} + \mathsf{a}_9 r}(x - \mathsf{u}_{z,\rho r}) + \mathsf{u}_{z,\rho r} \in \partial \mathbf{B}^{\cup}, \tag{5.78}$$

so that x' is the unique point of $\partial \mathbf{B}^{\cup}$ which lies on the line segment from the center point $\mathsf{u}_{z,\rho r}$ to x. We also let

$$X_{\mathrm{dist}} := \left\{ x \in \mathbf{I}^{\vee} \,:\, D_h\left(x', u; \overline{\mathbf{B}^{\cup}}\right) \leqslant \lambda \widetilde{D}_h(u,v) \right\}. \tag{5.79}$$

By condition 3 in the definition of $\mathsf{F}_{z,\rho r}$, the set $\{x' \in \partial \mathbf{B}^{\cup} : x \notin X_{\mathrm{dist}}\}$ has one-dimensional Lebesgue measure at most $(\lambda/2)\mathsf{s}_{\rho r}$. By scaling, we therefore have

$$|X_{\mathrm{dist}}| \geqslant |\mathbf{I}^{\vee}| - \lambda \mathsf{s}_{\rho r}. \tag{5.80}$$

5.10.2 | Proof of Lemma 5.26 assuming that the accessible set is not too small

The following lemma tells us that the conclusion of Lemma 5.26 is satisfied provided X_{acc} is not too small relative to $\mathsf{s}_{\rho r}$.

Lemma 5.28. *If the one-dimensional Lebesgue measure of X_{acc} satisfies $|X_{\mathrm{acc}}| > 3\lambda \mathsf{s}_{\rho r}$, then there is a time $t \in [a', b'] \subset [\tau, \sigma]$ such that*

$$D_{h - \mathfrak{f}_r}\left(P_r(t), u; \overline{\mathbf{B}^{\cup}}\right) \leqslant 2\lambda e^{-\xi \mathsf{A}_8} \widetilde{D}_h(u,v). \tag{5.81}$$

We note that Lemma 5.28 implies that if $|X_{\mathrm{acc}}| > 3\lambda \mathsf{s}_{\rho r}$, then the conclusion of Lemma 5.26 holds with $C = 2$. This is because $P_r([a',b']) \subset \mathbf{B}^{\mathrm{out}}$ and $\overline{\mathbf{B}^{\cup}} \subset \mathbb{A}_{r,4r}(0)$.

The idea of the proof of Lemma 5.28 is that if $|X_{\mathrm{acc}}| > 3\lambda \mathsf{s}_{\rho r}$, then by (5.80) there must be a point $x \in X_{\mathrm{acc}} \cap X_{\mathrm{dist}}$. By Lemma 5.27, every path in $\mathbf{B}^{\mathrm{out}}$ from u to x must hit $P_r([a',b'])$. We then want to use the definition (5.79) of X_{dist} to upper-bound the $D_{h-\mathfrak{f}_r}$-distance from u to the intersection point. There is a minor technicality arising from the fact that (5.79) only gives a bound for the distance from u to $x' \in \partial \mathbf{B}^{\cup}$, rather than from u to x. To deal with this technicality, we will use condition 8 (intersections of geodesics with a small neighborhood of the boundary) in the definition of E_r to say that there are not very many points $x \in \mathbf{I}^{\vee}$ for which P_r hits the segment $[x, x']$.

Proof of Lemma 5.28. Define $x' \in \partial \mathbf{B}^{\cup}$ for $x \in \mathbf{I}^{\vee}$ as in (5.78). Let

$$Y := \left\{ x \in X_{\mathrm{acc}} \,:\, P_r([a',b']) \cap [x, x'] \neq \emptyset \right\}. \tag{5.82}$$

If $x \in Y$, then x' lies at Euclidean distance at most $\mathsf{a}_9 r$ from $P_r([a',b'])$. By condition 8 in the definition of E_r (in particular, we use the last sentence of the condition), the one-dimensional

Lebesgue measure of the set $\{x' \in \partial \mathbf{B}^{\cup} : x \in Y\}$ is at most $\lambda t \rho r \leqslant \lambda \mathsf{s}_{\rho r}$. By scaling, we get that the one-dimensional Lebesgue measure of Y is at most $2\lambda \mathsf{s}_{\rho r}$.

Hence, if $|X_{\text{acc}}| > 3\lambda \mathsf{s}_{\rho r}$, then $|X_{\text{acc}} \setminus Y| > \lambda \mathsf{s}_{\rho r}$. By (5.80), this implies that the one-dimensional Lebesgue measure of $X_{\text{dist}} \cap (X_{\text{acc}} \setminus Y)$ is positive, so there exists $x \in X_{\text{dist}} \cap (X_{\text{acc}} \setminus Y)$.

Since $x \in X_{\text{dist}}$, the definition (5.79) implies that there is a path L in $\overline{\mathbf{B}^{\cup}}$ from u to x' such that

$$\text{len}(L; D_h) \leqslant 2\lambda \widetilde{D}_h(u, v).$$

The union of L and $[x, x']$ gives a path in $\overline{\mathbf{B}^{\vee}}$ from u to x. Since $x \in X_{\text{acc}}$, Lemma 5.27 implies that the path $P_r|_{[a',b']}$ must hit $L \cup [x, x']$. Since $x \notin Y$, the path $P_r|_{[a',b']}$ does not hit $[x, x']$.

Therefore, $P_r|_{[a',b']}$ must hit L. Since $L \subset \overline{\mathbf{B}^{\cup}}$ is a path started from u of D_h-length at most $2\lambda \widetilde{D}_h(u, v)$, we get that

$$D_h\left(P_r(t), u; \overline{\mathbf{B}^{\cup}}\right) \leqslant 2\lambda \widetilde{D}_h(u, v), \tag{5.83}$$

where $t \in [a', b']$ is chosen so that $P_r(t) \in L$.

Since f_r attains its maximum value A_8 at each point of $\overline{\mathsf{U}}_r \supset \overline{\mathbf{B}^{\cup}}$, we infer from Weyl scaling (Axiom III) that

$$D_{h-\mathsf{f}_r}(P_r(t), u; \overline{\mathbf{B}^{\cup}}) = e^{-\xi \mathsf{A}_8} D_h\left(P_r(t), u; \overline{\mathbf{B}^{\cup}}\right).$$

Combining this with (5.83) gives (5.81). □

5.10.3 | The set of arcs of $\mathbf{I}^{\vee} \setminus \overline{X}_{\text{acc}}$

In light of Lemma 5.28, for the rest of the proof of Lemma 5.26 we can assume that

$$|X_{\text{acc}}| \leqslant 3\lambda \mathsf{s}_{\rho r}. \tag{5.84}$$

Intuitively, we do not expect (5.84) to be the typical situation since it implies that $P_r([a', b'])$ disconnects 'most' points of $\mathbf{I}^{\vee}$ from $\mathbf{I}^{\text{out}}$ (recall (5.77)). This, in turn, means that a large portion of $P_r([a', b'])$ is outside of V_r. This is unexpected since P_r is a $D_{h-\mathsf{f}_r}$-geodesic and f_r is non-negative and supported on V_r, so $P_r|_{[a',b']}$ should want to spend most of its time in V_r. However, we are not able to easily rule out (5.84). We note that Lemma 5.22 does not rule out (5.84) since it could be that $P_r|_{[a',b']}$ has many small excursions outside of V_r, each of Euclidean diameter at most $\mathsf{a}_5 r$.

Hence, we need to prove Lemma 5.26 under the assumption (5.84). This will require a finer analysis of the structure of the set X_{acc}.

The set $\mathbf{I}^{\vee} \setminus \overline{X}_{\text{acc}}$ is a countable union of disjoint open arcs of $\mathbf{I}^{\vee}$. Let $\mathcal{I}$ be the set of all such arcs and for $I \in \mathcal{I}$, write $|I|$ for its Euclidean length (equivalently, its one-dimensional Lebesgue measure). The elements of $\mathcal{I}$ are the green arcs in Figure 20.

We now give an outline of the proof of Lemma 5.26 subject to the assumption (5.84). As a consequence of (5.84), we get that 'most' points of $\mathbf{I}^{\vee}$ are contained in $\mathbf{I}^{\vee} \setminus \overline{X}_{\text{acc}}$, so $\sum_{I \in \mathcal{I}} |I|$ is close to $|\mathbf{I}^{\vee}|$ (Lemma 5.29). From this and (5.80), we see that 'most' of the arcs $I \in \mathcal{I}$ intersect X_{dist} (Lemma 5.33). From condition 5 (comparison of distances in small annuli) in the definition of E_r (applied with $\delta = |I|/r$) and a geometric argument, we get the following. If $I \in \mathcal{I}$ and y_I is one of the endpoints of I, then there is a loop in $\mathbb{A}_{2|I|,3|I|}(y_I)$ which disconnects the inner and outer boundaries and whose D_h-length (hence also its $D_{h-\mathfrak{f}_r}$-length) is bounded above by $(|I|/r)^{-1/4}$ times (roughly speaking) the D_h-length of the segment of P_r joining the endpoints of I. By concatenating this loop with a path in $\overline{\mathbf{B}^\cup}$ from u to x', for a point $x' \in I \cap X_{\text{dist}}$, we obtain an upper bound for $D_{h-\mathfrak{f}_r}(u, P_r([a', b']))$ in terms of $|I|$ and the D_h-length of the segment of P_r joining the endpoints of I (Lemma 5.34). We will then use a pigeonhole argument to say that there exists $I \in \mathcal{I}$ for which this last quantity is much smaller than $e^{-\xi \mathsf{A}_8} \widetilde{D}_h(u, v)$.

Let us now give the details. We start with a lower bound for the sum of the Lebesgue measures of the arcs in $\mathcal{I}$.

Lemma 5.29. *The total one-dimensional Lebesgue measure of the arcs in $\mathcal{I}$ satisfies*

$$\sum_{I \in \mathcal{I}} |I| = |\mathbf{I}^\vee \setminus \overline{X}_{\text{acc}}| \geqslant |\mathbf{I}^\vee| - 3\lambda \mathsf{s}_{\rho r}. \tag{5.85}$$

Proof. We first claim that each point of $\overline{X}_{\text{acc}} \setminus X_{\text{acc}}$ belongs to $P_r([a', b']) \cap \mathbf{I}^\vee$. Indeed, suppose $x \in \overline{X}_{\text{acc}}$ and $x \notin P_r([a', b'])$. We need to show that $x \in X_{\text{acc}}$. Since $P_r([a', b'])$ is a Euclidean-closed set, x lies at positive Euclidean distance from $P_r([a', b'])$. Since $x \in \overline{X}_{\text{acc}}$, there exists $y \in X_{\text{acc}}$ such that the arc of $\mathbf{I}^\vee$ between x and y is disjoint from $P_r([a', b'])$. By the definition of X_{acc} (5.77), there is a path from a point of $\mathbf{I}^{\text{out}}$ to y which is contained in $\overline{\mathbf{B}^{\text{out}}} \setminus (\mathbf{B}^\vee \cup P_r([a', b']))$. The union of this path and the arc of $\mathbf{I}^\vee$ between x and y gives a path from $\mathbf{I}^{\text{out}}$ to x which is contained in $\overline{\mathbf{B}^{\text{out}}} \setminus (\mathbf{B}^\vee \cup P_r([a', b']))$.

By, for example, Lemma 2.14 (applied to the unit-speed parameterization of the circle $\partial \mathbf{B}^\vee$), almost surely the set $P_r([a', b']) \cap \mathbf{I}^\vee$ has zero one-dimensional Lebesgue measure. By this, the previous paragraph, and our assumption (5.84),

$$\sum_{I \in \mathcal{I}} |I| = |\mathbf{I}^\vee \setminus \overline{X}_{\text{acc}}| = |\mathbf{I}^\vee \setminus X_{\text{acc}}| \geqslant |\mathbf{I}^\vee| - 3\lambda \mathsf{s}_{\rho r}. \qquad \square$$

We will also need the following elementary topological fact.

Lemma 5.30. *For each $I \in \mathcal{I}$, there is a segment of $P_r|_{[a,b]}$ joining the two endpoints of I which is contained in $\mathbf{B}^{\text{out}} \setminus \mathbf{B}^\vee$.*

Proof. See Figure 22 for an illustration. Let $R \subset \mathbf{B}^{\text{out}} \setminus \overline{\mathbf{B}^\vee}$ be the open region bounded by $\mathbf{I}^{\text{out}}$, $\mathbf{I}^\vee$, and the segments $P_r([a', a])$ and $P_r([b, b'])$. Then R has the topology of the open unit disk and $I \subset \partial R$. By the definition (5.77) of X_{acc} and since $I \subset \mathbf{I}^\vee \setminus X_{\text{acc}}$, there is no path in $\overline{R}$ from I to $\mathbf{I}^{\text{out}}$ which is disjoint from $P_r([a', b'])$. Hence, $P_r([a', b'])$ disconnects I from $\mathbf{I}^{\text{out}}$ in R.

Since $P_r([a', a]) \cup P_r([b, b']) \subset \partial R$ and $P_r([a, b]) \cap \partial \mathbf{B}^{\text{out}} = \emptyset$, the set $P_r([a', b']) \cap R$ consists of countably many disjoint segments of $P_r|_{[a,b]}$ with endpoints in $\mathbf{I}^\vee$. Since P_r is continuous, these segments accumulate only at points of $\mathbf{I}^\vee$. Since I is connected and $P_r([a', b'])$ disconnects I from $\mathbf{I}^{\text{out}}$ in R, there are times $c, d \in [a, b]$ with $c < d$ such that $P_r(c), P_r(d) \in \mathbf{I}^\vee$, $P_r((c, d)) \subset R$, and $P_r([c, d])$ disconnects I from $\mathbf{I}^{\text{out}}$ in R.

Let $\widehat{I}$ be the set of points of $\mathbf{I}^\vee$ which are disconnected from $\mathbf{I}^{\text{out}}$ in R by $P_r([c, d])$ (not including the endpoints of $P_r([c, d])$). Equivalently, $\widehat{I}$ is the segment of $\mathbf{I}^\vee$ between $P_r(c)$ and $P_r(d)$. Then $\widehat{I}$ is a connected open arc of $\mathbf{I}^\vee$ which contains I. Moreover, every path from $\widehat{I}$ to $\mathbf{I}^{\text{out}}$ in $\overline{\mathbf{B}^{\text{out}}} \setminus \mathbf{B}^\vee$ either hits $P_r([c, d])$ or exits R (in which case it must intersect either $P_r([a', a])$ or $P_r([b, b'])$). Hence, no such path can be disjoint from $P_r([a', b'])$. So, by the definition (5.77) of X_{acc}, we have

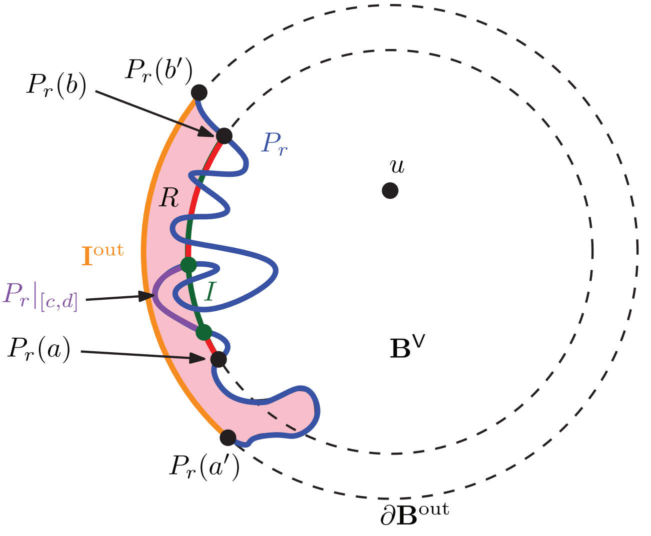

FIGURE 22 Illustration of the proof of Lemma 5.30. The region R is shown in pink and the desired segment $P_r|_{[c,d]}$ of P is shown in purple.

$\widehat{I} \subset \mathbf{I}^\vee \setminus X_{\text{acc}}$. Since $\widehat{I}$ is an open arc of $\mathbf{I}^\vee$, also $\widehat{I} \subset \mathbf{I}^\vee \setminus \overline{X}_{\text{acc}}$. Since I is a connected component of $\mathbf{I}^\vee \setminus \overline{X}_{\text{acc}}$, it follows that $\widehat{I} = I$. □

5.10.4 | Regularity of arcs in $\mathcal{I}$

We will next record some bounds for the sizes of the individual arcs in $\mathcal{I}$, starting with an upper bound.

Lemma 5.31. *For each $I \in \mathcal{I}$, we have $|I| \leqslant \mathsf{a}_5 r$.*

Proof. By Lemma 5.30, for each $I \in \mathcal{I}$ there is a segment of $P_r|_{[a,b]}$ joining the endpoints of I which is contained in $\mathbf{B}^{\text{out}} \setminus \mathbf{B}^\vee$. By Lemma 5.23, $P_r|_{[a,b]}$ does not hit $\mathsf{V}_r \setminus \mathbf{B}^\vee$, so this segment of $P_r|_{[a,b]}$ is disjoint from V_r. The Euclidean diameter of this segment is at least $|I|$. By Lemma 5.22, the Euclidean diameter of the segment is at most $\mathsf{a}_5 r$, so we get $|I| \leqslant \mathsf{a}_5 r$, as required. □

We do not have a uniform lower bound for the sizes of the arcs in $\mathcal{I}$. But, using condition 8 (intersections of geodesics with a small neighborhood of the boundary) in the definition of E_r, we can say that the small arcs make a negligible contribution to the total one-dimensional Lebesgue measure of $\mathcal{I}$.

Lemma 5.32. *Define the set of small arcs*

$$\mathcal{I}_{\text{small}} := \{I \in \mathcal{I} : |I| \leqslant \mathsf{a}_9 r\}. \tag{5.86}$$

Then

$$\sum_{I \in \mathcal{I}_{\text{small}}} |I| \leqslant 2\lambda \mathsf{s}_{\rho r}. \tag{5.87}$$

Proof. By Lemma 5.30, for each $I \in \mathcal{I}$ the endpoints of I are hit by $P_r|_{[a',b']}$. Hence, the Euclidean distance from each point of I to $P_r([a',b'])$ is at most $|I|$. In particular, if $I \in \mathcal{I}_{\text{small}}$, then the Euclidean distance from each point of I to $P_r([a',b'])$ is at most $\mathsf{a}_9 r$. This implies that the Euclidean distance from $P_r([a',b'])$ to each point of the arc $I' := \{x' : x \in I\} \subset \partial \mathbf{B}^{\cup}$ is at most $2\mathsf{a}_9 r$, where here we use the notation (5.78).

The arcs I' for $I \in \mathcal{I}_{\text{small}}$ are disjoint and we have $|I'| \geqslant |I|/2$. Therefore, the one-dimensional Lebesgue measure of the set of points $x' \in \partial \mathbf{B}^{\cup}$ which lie at Euclidean distance at most $2\mathsf{a}_9 r$ from $P_r([a',b'])$ is at least

$$\frac{1}{2} \sum_{I \in \mathcal{I}_{\text{small}}} |I|.$$

By condition 8 in the definition of E_r (in particular, we use the last sentence of the condition), the one-dimensional Lebesgue measure of the set of $x' \in \partial \mathbf{B}^{\cup}$ which lie at Euclidean distance at most $2\mathsf{a}_9 r$ from $P_r([a',b'])$ is at most $\lambda \mathsf{t} \rho r$, so

$$\frac{1}{2} \sum_{I \in \mathcal{I}_{\text{small}}} |I| \leqslant \lambda \mathsf{t} \rho r \leqslant \lambda \mathsf{s}_{\rho r}, \tag{5.88}$$

where the last inequality comes from the definition of $\mathsf{s}_{\rho r}$ (recall Lemma 5.2). □

We will now consider a certain ‘good’ subset of $\mathcal{I}$, and show that the arcs in this subset cover most of $\mathbf{I}^{\vee}$. Let

$$\mathcal{I}^* := \{I \in \mathcal{I} : |I| \geqslant \mathsf{a}_9 r \text{ and } I \cap X_{\text{dist}} \neq \emptyset\}. \tag{5.89}$$

Lemma 5.33. *The total one-dimensional Lebesgue measure of the arcs in $\mathcal{I}^*$ satisfies*

$$\sum_{I \in \mathcal{I}^*} |I| \geqslant |\mathbf{I}^{\vee}| - 6\lambda \mathsf{s}_{\rho r}. \tag{5.90}$$

Proof. Let $\mathcal{I}_{\text{small}}$ be as in (5.86). We can write $\mathbf{I}^{\vee}$ as the disjoint union of X_{acc}, the arcs in $\mathcal{I}_{\text{small}}$, and the arcs in $\mathcal{I}$ with $|I| \geqslant \mathsf{a}_9 r$. By the definition (5.89) of $\mathcal{I}^*$,

$$X_{\text{dist}} \subset \overline{X}_{\text{acc}} \cup \bigcup_{I \in \mathcal{I}_{\text{small}}} I \cup \bigcup_{I \in \mathcal{I}^*} I. \tag{5.91}$$

We therefore have the following string of inequalities:

$$\begin{aligned} |\mathbf{I}^{\vee}| - \lambda \mathsf{s}_{\rho r} &\leqslant |X_{\text{dist}}| \quad \text{(by (5.80))} \\ &\leqslant |\overline{X}_{\text{acc}}| + \sum_{I \in \mathcal{I}_{\text{small}}} |I| + \sum_{I \in \mathcal{I}^*} |I| \quad \text{(by (5.91))} \\ &\leqslant 3\lambda \mathsf{s}_{\rho r} + 2\lambda \mathsf{s}_{\rho r} + \sum_{I \in \mathcal{I}^*} |I| \quad \text{(by Lemmas 5.29 and 5.32)}. \end{aligned} \tag{5.92}$$

Re-arranging gives (5.90). □

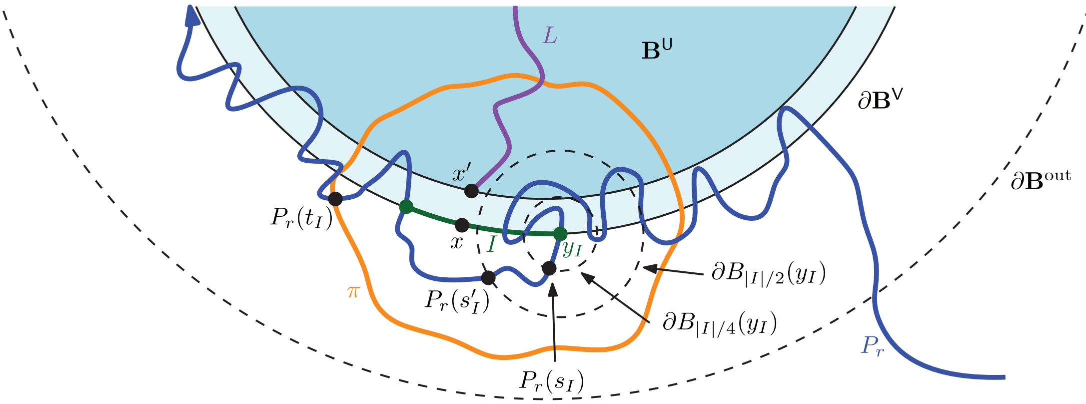

FIGURE 23 Illustration of the proof of Lemma 5.34. The orange loop π has D_h-length at most $2(|I|/r)^{-1/4} D_h(\text{across } \mathbb{A}_{|I|/4,|I|/2}(y_I))$, and is provided by condition 5 (comparison of distance in small annuli) in the definition of E_r. The point x belongs to $I \cap X_{\text{dist}}$. The purple path L goes from u (not pictured) to x', has D_h-length at most $2\lambda \widetilde{D}_h(u,v)$, and is provided by the definition (5.79) of X_{dist}. The bound (5.94) is obtained by concatenating a segment of π with a segment of L, then bounding $D_h(\text{across } \mathbb{A}_{|I|/4,|I|/2}(y_I))$ in terms of $t_I - s_I$.

5.10.5 | Building a path from a point of P_r to u

The following lemma is the main quantitative estimate needed for the proof of Lemma 5.26.

Lemma 5.34. *Let $I \in \mathcal{I}^*$ and let y_I be the initial endpoint of I. There are times $a' < s_I < t_I < b'$ such that*

$$P_r([s_I, t_I]) \subset B_{3|I|}(y_I), \quad t_I - s_I \geqslant \left(\frac{|I|}{4r}\right)^{\xi(Q+2)+1/4} r^{\xi Q} e^{\xi h_r(0)}, \quad \text{and} \tag{5.93}$$

$$D_{h-\mathfrak{f}_r}\big(P_r(t_I), u; \mathbb{A}_{r,4r}(0)\big) \leqslant 2\lambda e^{-\xi \mathsf{A}_8} \widetilde{D}_h(u,v) + 2(|I|/r)^{-1/4}(t_I - s_I). \tag{5.94}$$

We will eventually deduce Lemma 5.26 from Lemma 5.34 by showing that there exists an $I \in \mathcal{I}^*$ for which $2|I|^{-1/4}(t_I - s_I)$ is much smaller than $e^{-\xi \mathsf{A}_8} \widetilde{D}_h(u,v)$.

Proof of Lemma 5.34. See Figure 23 for an illustration. Throughout the proof we fix $I \in \mathcal{I}^*$.

Step 1: Definition of s_I and t_I. By Lemma 5.31, we have $|I| \leqslant \mathsf{a}_5 r$. Hence, we can apply condition 5 (comparison of distances in small annuli) in the definition of E_r with $\delta = |I|/r$ to get that there is a path $\pi \subset \mathbb{A}_{2|I|,3|I|}(y_I)$ such that

$$\text{len}(\pi; D_h) \leqslant 2(|I|/r)^{-1/4} D_h\big(\text{across } \mathbb{A}_{|I|/4,|I|/2}(y_I)\big). \tag{5.95}$$

We have $y_I \in \partial\mathbf{B}^{\vee}$ and $P_r(b') \in \partial\mathbf{B}^{\text{out}}$. The Euclidean distance from $\partial\mathbf{B}^{\text{out}}$ to $\partial\mathbf{B}^{\vee}$ is $3\mathsf{a}_5 r \geqslant 3|I|$. Therefore, the path P_r must hit both $\partial B_{|I|/4}(y_I)$ and π between the (unique) time when it hits y_I and the time b'. Let s_I (respectively, t_I) be the first time that P_r hits $\partial B_{|I|/4}(y_I)$ (respectively, π) after the time when it hits y_I. Then $a' < s_I < t_I < b'$ and (since P_r cannot travel from y_I to $\partial B_{3|I|}(y_I)$ without hitting π),

$$P_r([s_I, t_I]) \subset B_{3|I|}(y_I).$$

We will check the other conditions in the lemma statement for this choice of t_I and s_I.

Step 2: Upper-bound for $D_{h-\mathsf{f}_r}(P_r(t_I), u; \mathbb{A}_{r,4r}(0))$ *in terms of* $D_h(\text{across } \mathbb{A}_{|I|/4,|I|/2}(y_I))$. By the definition (5.89) of $\mathcal{I}^*$, there exists $x \in I \cap X_{\text{dist}}$. By the definition (5.79) of X_{dist}, if we let $x' \in \partial \mathbf{B}^{\mathsf{U}}$ be the point corresponding to x as in (5.78), then there is a path L from u to x' in $\overline{\mathbf{B}^{\mathsf{U}}}$ such that

$$\text{len}(L; D_h) \leqslant 2\lambda \widetilde{D}_h(u, v).$$

Since L is contained in $\overline{\mathbf{B}^{\mathsf{U}}}$, which is contained in $\overline{\mathsf{U}}_r$, and $\mathsf{f}_r \equiv \mathsf{A}_8$ on $\overline{\mathsf{U}}_r$,

$$\text{len}\left(L; D_{h-\mathsf{f}_r}\right) \leqslant 2\lambda e^{-\xi \mathsf{A}_8} \widetilde{D}_h(u, v). \tag{5.96}$$

The definition (5.89) of $\mathcal{I}^*$ gives $|I| \geqslant \mathsf{a}_9 r$, so

$$|x' - y_I| \leqslant |I| + |x - x'| = |I| + \mathsf{a}_9 r \leqslant 2|I|.$$

Since $\pi \subset \mathbb{A}_{2|I|,3|I|}(y_I)$, it follows that π intersects L and (since $3|I| \leqslant 3\mathsf{a}_5 r$) also $\pi \subset \mathbf{B}^{\text{out}}$. Since $P_r(t_I) \in \pi$, the path $\pi \cup L$ contains a path from u to $P_r(t_I)$. We have $\pi \cup L \subset \mathbf{B}^{\text{out}} \subset \mathbb{A}_{r,4r}(0)$. By (5.95) (and the fact that f_r is non-negative) and (5.96),

$$\begin{aligned} D_{h-\mathsf{f}_r}&\left(P_r(t_I), u; \mathbb{A}_{r,4r}(0)\right) \\ &\leqslant \text{len}\left(L; D_{h-\mathsf{f}_r}\right) + \text{len}\left(\pi; D_{h-\mathsf{f}_r}\right) \\ &\leqslant 2\lambda e^{-\xi \mathsf{A}_8} \widetilde{D}_h(u, v) + 2(|I|/r)^{-1/4} D_h\left(\text{across } \mathbb{A}_{|I|/4,|I|/2}(y_I)\right). \end{aligned} \tag{5.97}$$

Step 3: Comparing $t_I - s_I$. *to* $D_h(\text{across } \mathbb{A}_{|I|/4,|I|/2}(y_I))$. We claim that

$$t_I - s_I \geqslant D_h\left(\text{across } \mathbb{A}_{|I|/4,|I|/2}(y_I)\right). \tag{5.98}$$

Once (5.98) is established, the bound (5.97) immediately gives (5.94). Furthermore, the lower bound for $t_I - s_I$ in (5.93) also follows from (5.98) and the reverse Hölder continuity condition 6 in the definition of E_r (applied with $z \in \partial B_{|I|/4}(y_I)$ and $w \in \partial B_{|I|/2}(y_I)$), which gives

$$D_h\left(\text{across } \mathbb{A}_{|I|/4,|I|/2}(y_I)\right) \geqslant \left(\frac{|I|}{4r}\right)^{\xi(Q+2)+1/4} r^{\xi Q} e^{\xi h_r(0)}.$$

Hence, it remains to prove (5.98). Let s_I' be the first time after s_I at which P_r exits $B_{|I|/2}(y_I)$. Then $P_r|_{[s_I, s_I']}$ is a path between the inner and outer boundaries of $\mathbb{A}_{|I|/4,|I|/2}(y_I)$. We claim that

$$P_r([s_I, s_I']) \cap \mathsf{V}_r = \emptyset. \tag{5.99}$$

Since f_r vanishes outside of V_r, (5.99) implies that

$$\begin{aligned} t_I - s_I \geqslant s_I' - s_I = \text{len}\left(P_r|_{[s_I, s_I']}; D_{h-\mathsf{f}_r}\right) &= \text{len}\left(P_r|_{[s_I, s_I']}; D_h\right) \\ &\geqslant D_h\left(\text{across } \mathbb{A}_{|I|/4,|I|/2}(y_I)\right), \end{aligned} \tag{5.100}$$

which is (5.98).

To prove (5.99), we first note that by Lemma 5.30, the path P_r does not enter $\mathbf{B}^{\mathsf{V}}$ between the time when it hits y_I and the time when it hits the other endpoint of I. Since the Euclidean distance between the endpoints of I is at least $|I|/2$, s'_I must be smaller than the time when P_r hits the other endpoint of I. Hence, $P_r([s_I, s'_I]) \cap \mathbf{B}^\vee = \emptyset$. In particular, Lemma 5.30 implies that $[s_I, s'_I] \subset [a, b]$. By Lemma 5.21, $P_r|_{[a,b]}$ does not hit $\mathsf{V}_r \setminus \mathbf{B}^\vee$. Therefore, (5.99) holds. □

5.10.6 | Pigeonhole arguments

In light of Lemma 5.34, we seek an arc $I \in \mathcal{I}^*$ for which $t_I - s_I$ is much smaller than $(|I|/r)^{1/4}\widetilde{D}_h(u,v)$. To find such an arc, we will partition the set $\mathcal{I}^*$ based on the Euclidean sizes of the arcs. Let

$$\underline{K} := \lfloor \log_2(1/\mathsf{a}_5) \rfloor \quad \text{and} \quad \overline{K} := \lceil \log_2(1/\mathsf{a}_9) \rceil - 1. \tag{5.101}$$

For $k \in [\underline{K}, \overline{K}]_{\mathbb{Z}}$, let

$$\mathcal{I}_k^* := \left\{ I \in \mathcal{I}^* : |I| \in [2^{-k-1}r, 2^{-k}r) \right\}. \tag{5.102}$$

By Lemma 5.31 and the definition (5.89) of $\mathcal{I}^*$, we have $\mathsf{a}_9 r \leqslant |I| \leqslant \mathsf{a}_5 r$ for each $I \in \mathcal{I}^*$. Hence, $\mathcal{I}^*$ is the disjoint union of $\mathcal{I}_k^*$ for $k \in [\underline{K}, \overline{K}]_{\mathbb{Z}}$.

The proof that there exists an arc $I \in \mathcal{I}^*$ for which $t_I - s_I$ is small is based on a pigeonhole argument. Lemma 5.33 implies that the total Euclidean length of the arcs in $\mathcal{I}^*$ is close to $|\mathbf{I}^\vee|$. Hence, there must be some $k \in [\underline{K}, \overline{K}]_{\mathbb{Z}}$ for which $\#\mathcal{I}_k^*$ is larger than a constant times $r^{-1}2^{k/2}|\mathbf{I}^\vee|$: otherwise, the sum of the lengths of the arcs in $\mathcal{I}^*$ would be too small (Lemma 5.35). In the proof of Lemma 5.26, we will then use an argument based on Lemma 5.34 and Markov's inequality to show that there must be an $I \in \mathcal{I}_k^*$ for which $t_I - s_I$ is sufficiently small.

Let us start with the pigeonhole argument for the Euclidean lengths of the arcs in $\mathcal{I}^*$.

Lemma 5.35. *Let* $\mathsf{t} > 0$ *be the constant appearing in Lemma 5.2, so that the radius of* $\mathbf{B}^\cup$ *satisfies* $\mathsf{s}_{\rho r} \in [\mathsf{t}\rho r, \mathsf{t}^{1/2}\rho r]$. *Almost surely, there exist a random* $k \in [\underline{K}, \overline{K}]_{\mathbb{Z}}$ *and a collection of arcs* $\mathcal{I}_k^{**} \subset \mathcal{I}_k^*$ *such that* $\#\mathcal{I}_k^{**} \succeq 2^{k/2}\mathsf{t}\rho$, *with a deterministic universal implicit constant, and the balls* $B_{3|I|}(y_I)$ *for* $I \in \mathcal{I}_k^{**}$ *are disjoint (here* y_I *is the first endpoint of* I *hit by* P_r, *as in Lemma 5.34).*

Proof. We have

$$\begin{aligned}
|\mathbf{I}^\vee|/2 &\leqslant |\mathbf{I}^\vee| - 6\lambda \mathsf{s}_{\rho r} \quad (\text{since } |\mathbf{I}^\vee| \geqslant \mathsf{s}_{\rho r}/8 \text{ by (5.76)}) \\
&\leqslant \sum_{I \in \mathcal{I}^*} |I| \quad (\text{by Lemma 5.33}) \\
&\leqslant \sum_{k=\underline{K}}^{\overline{K}} \sum_{I \in \mathcal{I}_k^*} |I| \quad (\text{since } \mathcal{I}^* = \textstyle\bigcup_{k=\underline{K}}^{\overline{K}} \mathcal{I}_k^*) \\
&\leqslant r \sum_{k=\underline{K}}^{\overline{K}} 2^{-k} \#\mathcal{I}_k^* \quad (\text{by (5.102)}).
\end{aligned} \tag{5.103}$$

We claim that there exists $k \in [\underline{K}, \overline{K}]_{\mathbb{Z}}$ such that $\#\mathcal{I}_k^* \geqslant 2^{k/2} r^{-1} |\mathbf{I}^\vee|$. Indeed, if this is not the case then (5.103) gives

$$|\mathbf{I}^\vee|/2 \leqslant |\mathbf{I}^\vee| \sum_{k=\underline{K}}^{\overline{K}} 2^{-k/2} \quad \Rightarrow \quad 1/2 \leqslant \frac{1}{1-2^{-1/2}} 2^{-\underline{K}/2}$$

which is not true since $2^{-\underline{K}/2} \leqslant 2\mathsf{a}_5{}^{1/2}$, which is much smaller than $(1-2^{-1/2})/2$.

Henceforth, fix $k \in [\underline{K}, \overline{K}]_{\mathbb{Z}}$ such that $\#\mathcal{I}_k^* \geqslant 2^{k/2} r^{-1} |\mathbf{I}^\vee|$. The arcs in $\mathcal{I}_k^*$ are disjoint and have lengths in $[2^{-k-1} r, 2^{-k} r)$. Hence, for each $I \in \mathcal{I}_k^*$, the number of arcs in $\mathcal{I}_k^*$ which are contained in $B_{3|I|}(y_I)$ is at most some universal constant. It follows that we can find a subcollection $\mathcal{I}_k^{**} \subset \mathcal{I}_k^*$ such that $\#\mathcal{I}_k^{**} \succeq 2^{k/2} r^{-1} |\mathbf{I}^\vee|$ and the balls $B_{3|I|}(y_I)$ for $I \in \mathcal{I}_k^{**}$ are disjoint. We conclude by noting that by (5.76) and our choice of $\mathsf{s}_{\rho r}$ in Lemma 5.2,

$$r^{-1} |\mathbf{I}^\vee| \succeq r^{-1} \mathsf{s}_{\rho r} \geqslant \mathsf{t}\rho.$$

□

Proof of Lemma 5.26. Throughout the proof, all implicit constants are required to be deterministic and depend only on ξ.

Let $k \in [\underline{K}, \overline{K}]_{\mathbb{Z}}$ and $\mathcal{I}_k^{**} \subset \mathcal{I}_k^*$ be as in Lemma 5.35, so that $\#\mathcal{I}_k^{**} \succeq 2^{k/2} \mathsf{t}\rho$. For $I \in \mathcal{I}_k^{**}$, let $a' < s_I < t_I < b'$ be as in Lemma 5.34. Lemma 5.34 tells us that $P_r([s_I, t_I]) \subset B_{3|I|}(y_I)$. Lemma 5.35 implies that the balls $B_{3|I|}(y_I)$ are disjoint for different choices of $I \in \mathcal{I}_k^{**}$. Hence, the intervals $[s_I, t_I]$ for $I \in \mathcal{I}_k^{**}$ are disjoint.

In light of Lemma 5.34, we seek $I \in \mathcal{I}_k^{**}$ for which $t_I - s_I$ is much smaller than $(|I|/r)^{1/4}$. To find such an I, we will first choose a sub-collection of $\mathcal{I}_k^{**}$, which is not too much smaller than $\mathcal{I}_k^{**}$, such that the increments $t_I - s_I$ for $I \in \mathcal{I}_k^{**}$ are all comparable (step 1). We will then use Lemma 5.34 to upper bound the sum of the increments $t_I - s_I$ over all arcs I in this collection (step 2). Finally, we will use a pigeonhole argument to find an I for which $t_I - s_I$ is small (step 3).

Step 1: Finding a sub-collection on which $t_I - s_I$ is controlled. We seek a collection of distinct arcs $I_1, \dots, I_N \in \mathcal{I}_k^{**}$ such that N is not too much smaller than $\#\mathcal{I}_k^{**}$ and the geodesic time increments $t_{I_j} - s_{I_j}$ for $j = 1, \dots, N$ are all comparable. We will find such a collection via a pigeonhole argument.

The bound (5.93) of Lemma 5.34 followed by the definition (5.102) of $\mathcal{I}_k^*$ shows that for $I \in \mathcal{I}_k^{**}$,

$$t_I - s_I \geqslant \left(\frac{|I|}{4r}\right)^{\xi(Q+3)} r^{\xi Q} e^{\xi h_r(0)} \geqslant 2^{-(k+2)\xi(Q+3)} r^{\xi Q} e^{\xi h_r(0)}. \tag{5.104}$$

By combining this with the crude bound $t_I - s_I \leqslant \sigma - \tau$ and Lemma 5.21, we get that for $I \in \mathcal{I}_k^{**}$,

$$\begin{aligned} t_I - s_I &\in [2^{-(k+2)\xi(Q+3)} r^{\xi Q} e^{\xi h_r(0)}, \mathsf{a}_5{}^{\xi(Q+3)} r^{\xi Q} e^{\xi h_r(0)}] \\ &\subset [2^{-(k+2)\xi(Q+3)} r^{\xi Q} e^{\xi h_r(0)}, r^{\xi Q} e^{\xi h_r(0)}]. \end{aligned} \tag{5.105}$$

The number of intervals of the form $[q, 2q]$ for $q > 0$ needed to cover $[2^{-(k+2)\xi(Q+3)} r^{\xi Q} e^{\xi h_r(0)}, r^{\xi Q} e^{\xi h_r(0)}]$ is at most a constant (depending only on ξ) times k.

Consequently, we can find a random $q > 0$, an integer

$$N \succeq k^{-1}\#\mathcal{I}_k^{**} \succeq k^{-1}2^{k/2}\mathfrak{t}\rho, \tag{5.106}$$

and intervals $I_1, \dots, I_N \in \mathcal{I}_k^{**}$ such that $t_{I_j} - s_{I_j} \in [q, 2q]$ for each $j \in [1, N]_{\mathbb{Z}}$.

Since the intervals $[s_{I_j}, t_{I_j}]$ for $j \in [1, N]_{\mathbb{Z}}$ are disjoint, we can choose our numbering so that

$$s_{I_1} < t_{I_1} < s_{I_2} < t_{I_2} < \cdots < s_{I_N} < t_{I_N}. \tag{5.107}$$

Step 2: Bounding q. We will now use the estimate (5.94) from Lemma 5.34 to show that the number q from the preceding paragraph must be small relative to $\widetilde{D}_h(u,v)$. For each $j \in [1, N]_{\mathbb{Z}}$, we have $|I_j| \in [2^{-k-1}r, 2^{-k}r]$ and $t_{I_j} - s_{I_j} \in [q, 2q]$. By plugging these bounds into (5.94), we get

$$D_{h-\mathfrak{f}_r}\left(P_r(t_{I_j}), u; \mathbb{A}_{r,4r}(0)\right) \preceq \lambda e^{-\xi \mathsf{A}_8}\widetilde{D}_h(u,v) + 2^{k/4}q, \quad \forall j \in [1, N]_{\mathbb{Z}} \tag{5.108}$$

with a universal implicit constant.

By (5.108) (with $j = 1$ and $j = N$) and the triangle inequality for the points $P(t_{I_1}), u, P(t_{I_N})$,

$$t_{I_N} - t_{I_1} = D_{h-\mathfrak{f}_r}\left(P_r(t_{I_1}), P_r(t_{I_N}); \mathbb{A}_{r,4r}(0)\right) \preceq \lambda e^{-\xi \mathsf{A}_8}\widetilde{D}_h(u,v) + 2^{k/4}q. \tag{5.109}$$

On the other hand, (5.107) and our choices of N and q around (5.106) shows that

$$t_{I_N} - t_{I_1} \geqslant \sum_{j=2}^{N}(t_{I_j} - s_{I_j}) \geqslant (N-1)q \succeq k^{-1}2^{k/2}\mathfrak{t}\rho q. \tag{5.110}$$

Combining (5.109) and (5.110) gives

$$k^{-1}2^{k/2}\mathfrak{t}\rho q \preceq \lambda e^{-\xi \mathsf{A}_8}\widetilde{D}_h(u,v) + 2^{k/4}q \tag{5.111}$$

which re-arranges to give

$$q \preceq \frac{\lambda}{k^{-1}2^{k/2}\mathfrak{t}\rho - R2^{k/4}} e^{-\xi \mathsf{A}_8}\widetilde{D}_h(u,v) \tag{5.112}$$

for a constant $R > 0$ which depends only on ξ.

Step 3: Conclusion. We have $2^k \geqslant 2^{\underline{K}} \geqslant 1/(2\mathsf{a}_5)$, which can be taken to be as large as we would like as compared to $1/(\mathfrak{t}\rho)$ (recall from the discussion surrounding (5.22) that a_5 is chosen after ρ and the parameters from Lemma 5.2). Hence, we can arrange that $k^{-1}2^{k/2}\mathfrak{t}\rho q \geqslant 2R2^{k/4}$. Therefore, (5.112) gives

$$q \preceq \frac{k2^{-k/2}}{\mathfrak{t}\rho} e^{-\xi \mathsf{A}_8}\widetilde{D}_h(u,v). \tag{5.113}$$

Plugging (5.113) into (5.108) shows that for each $j \in [1, N]_{\mathbb{Z}}$,

$$D_{h-\mathfrak{f}_r}\left(P_r(t_{I_j}), u; \mathbb{A}_{r,4r}(0)\right) \preceq \left(\lambda + \frac{k2^{-k/4}}{\mathfrak{t}\rho}\right) e^{-\xi \mathsf{A}_8}\widetilde{D}_h(u,v). \tag{5.114}$$

Since $k \geqslant \underline{K} \geqslant \log_2(1/\mathsf{a}_5) - 1$, the coefficient on the right side of (5.114) can be made to be smaller than 2λ provided the parameters are chosen appropriately. This yields (5.71) for an

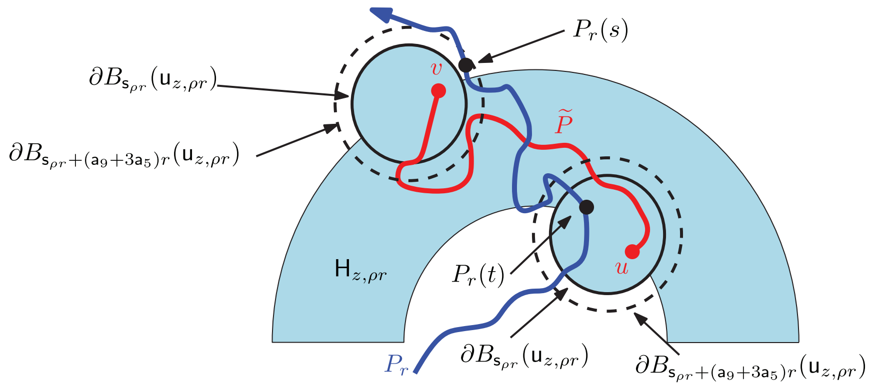

FIGURE 24 Illustration of the proof of Proposition 5.18. We consider a $z \in \mathsf{Z}_r$ for which $\mathsf{F}_{z,\rho r}$ occurs as in Lemma 5.23. We look at the corresponding pair of points u, v such that $\widetilde{D}_h(u,v) \leqslant \mathfrak{c}_0' D_h(u,v)$ and there is a $\widetilde{D}_h$-geodesic $\widetilde{P}$ from u to v which is contained in $\overline{\mathsf{H}}_{z,\rho r} \subset \mathsf{U}_r$. Lemma 5.26 tells us that there are times s, t for P_r such that $D_h(P_r(t), u)$ and $D_h(P_r(s), v)$ are each much smaller than $e^{-\xi \mathsf{A}_8} \widetilde{D}_h(u,v) = \widetilde{D}_{h-\mathsf{f}_r}(u,v)$. We then use the triangle inequality to show that $\widetilde{D}_h(P_r(t), P_r(s)) \leqslant \mathfrak{c}'|s-t|$.

appropriate choice of C. The inclusion (5.70) holds since $t_I \in [a', b']$ and $P_r([a',b']) \subset \mathbf{B}^{\text{out}}$ by definition (5.73). □

5.11 | Proof of Proposition 5.18

Step 1: Choice of s and t. See Figure 24 for an illustration. Let $z \in \mathsf{Z}_r$ and $u, v \in \partial \mathsf{H}_{z,\rho r}$ be as in Subsection 5.10, so that $\mathsf{F}_{z,\rho r}$ occurs and u, v are as in the definition of $\mathsf{F}_{z,\rho r}$. In particular,

$$\widetilde{D}_h(u,v) \leqslant \mathfrak{c}_0' D_h(u,v). \tag{5.115}$$

By Lemma 5.26, almost surely there exists $t \subset [\tau, \sigma]$ such that

$$P_r(t) \in B_{\mathsf{s}_{\rho r}+(3\mathsf{a}_5+\mathsf{a}_9)r}(\mathsf{u}_{z,\rho r}) \quad \text{and} \quad D_{h-\mathsf{f}_r}\left(P_r(t), u; \mathbb{A}_{r,4r}(0)\right) \leqslant C\lambda e^{-\xi \mathsf{A}_8} \widetilde{D}_h(u,v). \tag{5.116}$$

By the definition of $\mathsf{F}_{z,\rho r}$, we have $u \in B_{\mathsf{s}_{\rho r}/2}(\mathsf{u}_{z,\rho r})$. By this, (5.116), and the triangle inequality,

$$|P_r(t) - u| \leqslant \mathsf{s}_{\rho r} + (3\mathsf{a}_5 + \mathsf{a}_9)r + \frac{\mathsf{s}_{\rho r}}{2} \leqslant 2\mathsf{t}^{1/2}\rho r, \tag{5.117}$$

where the second inequality comes from the fact that $\mathsf{s}_{\rho r} \leqslant \mathsf{t}^{1/2}\rho r$ (Lemma 5.2) and the fact that each of a_5 and a_9 can be chosen to be much smaller than t.

By Lemma 5.26 with $\mathsf{v}_{z,\rho r}$ and v in place of $\mathsf{u}_{z,\rho r}$ and u, there exists $s \in [\tau, \sigma]$ such that

$$D_{h-\mathsf{f}_r}\left(P_r(s), v; \mathbb{A}_{r,4r}(0)\right) \leqslant C\lambda e^{-\xi \mathsf{A}_8} \widetilde{D}_h(u,v) \quad \text{and} \quad |P_r(s) - v| \leqslant 2\mathsf{t}^{1/2}\rho r. \tag{5.118}$$

We will check the conditions of (5.49) for this choice of s and t (possibly with the order of s and t interchanged).

Step 2: Lower bound for $|s-t|$. Recall that the points u and v lie on the inner and outer boundaries, respectively, of the annulus $\mathbb{A}_{\alpha\rho r,\rho r}(z)$. From this, the inequalities for Euclidean distances in (5.117) and (5.118), and the triangle inequality, we get

$$|P_r(t) - P_r(s)| \geqslant (1-\alpha)\rho r - 4\mathsf{t}^{1/2}\rho r \geqslant \frac{1-\alpha}{2}\rho r, \tag{5.119}$$

where in the last inequality we use that $\mathsf{t}^{1/2}$ is much smaller than $1-\alpha$ (Lemma 5.2).

This right side of (5.119) is at least $\mathsf{a}_5 r$, so the reverse Hölder continuity condition 6 in the definition of E_r gives

$$D_h\big(P_r(t), P_r(s); \mathbb{A}_{r,4r}(0)\big) \geqslant \mathsf{a}_5{}^{\xi(Q+3)} r^{\xi Q} e^{\xi h_r(0)}. \tag{5.120}$$

By Lemma 5.19, $P_r|_{[\tau',\sigma']}$ is a $D_{h-\mathsf{f}_r}(\cdot,\cdot;\overline{\mathbb{A}}_{r,4r}(0))$-geodesic. In fact, since $P_r([s,t]) \subset \mathbb{A}_{r,4r}(0)$, we have that $P_r|_{[s,t]}$ is a $D_{h-\mathsf{f}_r}(\cdot,\cdot;\mathbb{A}_{r,4r}(0))$-geodesic. Since $\mathsf{f}_r \leqslant \mathsf{A}_8$, we get from (5.120) that

$$\begin{aligned} |s-t| &= D_{h-\mathsf{f}_r}\big(P_r(t), P_r(s); \mathbb{A}_{r,4r}(0)\big) \\ &\geqslant e^{-\xi \mathsf{A}_8} D_h\big(P_r(t), P_r(s); \mathbb{A}_{r,4r}(0)\big) \\ &\geqslant \mathsf{a}_5{}^{\xi(Q+3)} e^{-\xi \mathsf{A}_8} r^{\xi Q} e^{\xi h_r(0)} \end{aligned} \tag{5.121}$$

which gives the first inequality in (5.49).

Step 3: upper bound for $\widetilde{D}_{h-\mathsf{f}_r}(P_r(t), P_r(s); \mathbb{A}_{r,4r}(0))$. We now prove the second inequality in (5.49). From the bi-Lipschitz equivalence of D_h and $\widetilde{D}_h$ and Weyl scaling (Axiom III), we get that $D_{h-\mathsf{f}_r}$ and $\widetilde{D}_{h-\mathsf{f}_r}$ are also bi-Lipschitz equivalent, with the same lower and upper bi-Lipschitz constants $\mathfrak{c}_*$ and $\mathfrak{C}_*$. Therefore, (5.116) and (5.118) imply that

$$\max\Big\{\widetilde{D}_{h-\mathsf{f}_r}\big(P_r(t), u; \mathbb{A}_{r,4r}(0)\big), \widetilde{D}_{h-\mathsf{f}_r}\big(P_r(s), v; \mathbb{A}_{r,4r}(0)\big)\Big\} \leqslant \mathfrak{C}_* C\lambda e^{-\xi \mathsf{A}_8} \widetilde{D}_h(u,v). \tag{5.122}$$

Let $\widetilde{P}$ be the $\widetilde{D}_h$-geodesic from u to v which is contained in $\overline{\mathsf{H}}_{z,\rho r}$, as in condition 2 in the definition of $\mathsf{F}_{z,\rho r}$. Since $\widetilde{P}$ is a $\widetilde{D}_h$-geodesic, $\widetilde{P} \subset \mathsf{U}_r$, and f_r attains its maximal value A_8 everywhere on U_r,

$$\widetilde{D}_{h-\mathsf{f}_r}\big(u, v; \mathbb{A}_{r,4r}(0)\big) = e^{-\xi \mathsf{A}_8} \widetilde{D}_h(u,v). \tag{5.123}$$

By (5.122), (5.123), and the triangle inequality, followed by (5.115),

$$\begin{aligned} \widetilde{D}_{h-\mathsf{f}_r}\big(P_r(t), P_r(s); \mathbb{A}_{r,4r}(0)\big) &\leqslant (1 + 2\mathfrak{C}_* C\lambda) e^{-\xi \mathsf{A}_8} \widetilde{D}_h(u,v) \\ &\leqslant (1 + 2\mathfrak{C}_* C\lambda) \mathfrak{c}_0' e^{-\xi \mathsf{A}_8} D_h(u,v). \end{aligned} \tag{5.124}$$

On the other hand, since $\mathsf{f}_r \leqslant \mathsf{A}_8$, Weyl scaling gives

$$D_{h-\mathsf{f}_r}(u,v) \geqslant e^{-\xi \mathsf{A}_8} D_h(u,v). \tag{5.125}$$

Hence,

$$\begin{aligned} |s-t| &= D_{h-\mathsf{f}_r}(P_r(t), P_r(s)) \quad \text{(since } P_r \text{ is a } D_{h-\mathsf{f}_r}\text{-geodesic)} \\ &\geqslant D_{h-\mathsf{f}_r}(u,v) - D_{h-\mathsf{f}_r}(P_r(t), u) - D_{h-\mathsf{f}_r}(P_r(s), v) \quad \text{(triangle inequality)} \\ &\geqslant e^{-\xi \mathsf{A}_8} D_h(u,v) - 2C\lambda e^{-\xi \mathsf{A}_8} \widetilde{D}_h(u,v) \quad \text{(by (5.116), (5.118), and (5.125))} \end{aligned}$$

$$\geqslant e^{-\xi \mathsf{A}_8} D_h(u,v) - 2C\lambda e^{-\xi \mathsf{A}_8} \mathfrak{C}_* D_h(u,v) \quad \text{(bi-Lipschitz equivalence)}$$
$$= (1 - 2\mathfrak{C}_* C\lambda) e^{-\xi \mathsf{A}_8} D_h(u,v). \tag{5.126}$$

Combining (5.124) and (5.126) gives

$$\widetilde{D}_{h-\mathfrak{f}_r}\left(P_r(t), P_r(s); \mathbb{A}_{r,4r}(0)\right) \leqslant \frac{1 + 2\mathfrak{C}_* C\lambda}{1 - 2\mathfrak{C}_* C\lambda} \mathfrak{c}_0' |s - t|. \tag{5.127}$$

Since $\mathfrak{c}_0' < \mathfrak{c}'$ and $\mathfrak{c}_0', \mathfrak{c}'$ depend on the laws of D_h and $\widetilde{D}_h$ (recall (5.1)), we can choose λ to be small enough, in a manner depending only on laws of D_h and $\widetilde{D}_h$, so that

$$\frac{1 + 2\mathfrak{C}_* C\lambda}{1 - 2\mathfrak{C}_* C\lambda} \mathfrak{c}_0' \leqslant \mathfrak{c}'. \tag{5.128}$$

Then (5.127) gives the second inequality in (5.49). □

ACKNOWLEDGEMENTS

We thank an anonymous referee for helpful comments on an earlier version of this article. Jian Ding was partially supported by NSF grants DMS-1757479 and DMS-1953848. Ewain Gwynne was partially supported by a Clay research fellowship.

JOURNAL INFORMATION

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