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# THE MEMORY EFFECT, SYMMETRIES AND INFRARED FINITE SCATTERING THEORY IN QUANTUM FIELD THEORY AND QUANTUM GRAVITY

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To my parents and teachers

*nanos gigantum humeris insidentes*

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# ABSTRACT

This thesis, based on two papers by Satishchandran and Wald [1, 2], investigates the behavior of classical and quantum fields in scattering theory in asymptotically flat spacetimes. It has been known that the presence of massless fields will give rise to a “memory effect” in four dimensions. At order $1/r$ a massless field generically will not return to the same value at late retarded times as it had at early retarded times. This memory effect is deeply connected to the asymptotic symmetry group of an asymptotically flat spacetime as well as the infrared divergences encountered in quantum field theory and quantum gravity. The full scope of this thesis is to fully understand the relationship between these seemingly disparate phenomena and develop an infrared finite scattering theory in QFT and quantum gravity. To understand the origin of these relationships we investigate the behavior of massless scalar, electromagnetic, and gravitational perturbations near null infinity in all dimensions greater than or equal to 4 assuming that they admit an expansion in $1/r$. We consider the gravitational memory effect and show that in even dimensions, the memory effect is at Coulombic order and can be decomposed into null and ordinary memory. In odd dimensions, the memory effect vanishes. In 4 dimensions, there is a close relationship between memory and the supertranslations charge/flux relations. We then show that the vanishing of memory of at radiative order is responsible for the lack of IR divergences in higher than 4 dimensions but is directly responsible for IR divergences in 4 dimensions. IR divergences are artifacts of trying to represent states with memory in the standard Fock space. For a well-defined S-matrix, it is necessary to define in/out Hilbert spaces with memory. Such a construction was given by Faddeev and Kulish (FK) for QED. Their construction "dresses" momentum states of the charged particles by pairing them with memory states of the electromagnetic field to produce states of vanishing large gauge charges at spatial infinity. However, in massless QED, due to collinear divergences, the "dressing" has an infinite energy flux so these states are unphysical. In Yang-Mills theory the "soft particles" used for dressing also contribute to the current

flux, invalidating the FK procedure. In quantum gravity, the analogous FK construction would attempt to produce a Hilbert space of eigenstates of supertranslation charges at spatial infinity. However, we prove that there are no eigenstates of supertranslation charges except the vacuum. Thus, the FK construction fails in quantum gravity. We investigate some alternatives to FK constructions but find that these also do not work. We believe that to treat scattering at a fundamental level in quantum gravity - as well as in massless QED and YM theory - it is necessary to take an algebraic viewpoint rather than shoehorn the in/out states into some fixed Hilbert space. We outline the framework of such an IR finite scattering theory.

# CHAPTER 1

# INTRODUCTION

In both classical and quantum scattering theory one is often interested in the description of the "outgoing" field at large distances from the "source". At asymptotically late times, any radiative degrees of freedom propagate away from sources and the fields are approximately free. In a scattering process such as, for example, the classical merger of black holes or the quantum scattering of charged particles in QED, the outgoing fields encode the details of the scattering event which can be measured by LIGO or the LHC. The prediction of the "outgoing" field at asymptotically late times given "incoming" initial data at asymptotically early times is known as "scattering theory". In the quantum theory, this approach embodies the "$S$-matrix formalism" of quantum field theory and is widely viewed as a fundamental approach to quantum gravity.

In this thesis, we examine fundamental aspects of classical and quantum scattering theory. In particular, over the past half-century, there are three seemingly disparate insights and issues in scattering theory in four spacetime dimensions that have been argued to be deeply related [3, 4]. (I) The discovery that the group of symmetries of scattering theory is actually an infinite dimensional group of "asymptotic symmetries" [5]. (II) The well-known issue of "infrared divergences" in the standard quantum $S$-matrix in the presences of massless particles due to the emission of an infinite number of low-frequency massless particles in any generic scattering process [6, 7, 8, 9]. (III) The existence of the memory effect which, in the gravitational case, is the permanent relative displacement of an arrangement of test particles due to the passage of a burst of gravitational radiation [10]. In Maxwell theory, the "electromagnetic memory effect" corresponds to a permanent "momentum kick" of a test charge due to the passage of a burst of electromagnetic radiation [11]. While the gravitational memory effect is expected to be experimentally detected by LIGO or LISA [12, 13, 14], it also plays a fundamental role in both classical and quantum scattering theory. Indeed, as we explain below, the existence of the memory effect in four dimensions implies the enlargement of the asymptotic symmetry group and is the obstruction to defining the standard $S$-matrix in the quantum theory [3, 15].

This thesis is divided into two parts. In chapter 2, we first analyze the classical memory effect in *all* spacetime dimensions. An important aim of this analysis is to clarify the origin of the relationship between memory, symmetries and infrared divergences. In chapter 3, we then consider the quantum implications of the memory effect where we argue that one must go well beyond the traditional $S$-matrix formulation of scattering theory in order include memory as a quantum observable and have a well-defined, infrared finite scattering theory in quantum field theory and quantum gravity.

We now turn to the description of this work.

## 1 *The asymptotic behavior of massless fields in all spacetime dimensions*

The classical memory effect is a radiative effect defined at asymptotically large distances from the source. Therefore an analysis of memory requires a precise analysis of the asymptotic behavior of massless fields. This analysis was first carried out in four dimensions by Bondi and collaborators in the 1960's [16, 17, 18] by assuming a $1/r$ expansion of the metric near "null infinity" ($r \to \infty$ at fixed retarded time $u$). The primary aim of sec. 2 of chapter 2 is to extend this analysis to all $d \geq 4$ spacetime dimensions. To achieve this we assume an expansion in powers of $1/r$ as an ansatz in a similar spirit to the original analysis by Bondi and collaborators. The generality of this ansatz is shown in Appendix A where we shown that, for $d$ even, our ansatz is compatible with smoothness of the metric at $\mathscr{I}^+$.

The main results of sec. 2 of chapter 2 are given in Props. 2.1 and 2.2 as well as Theorems 1–

3. These results can be summarized as follows: we show that if one can choose a gauge compatible with our ansatz in which the massless fields satisfy a wave equation near null infinity, then a recursive solution can be straightforwardly obtained order by order in $1/r$ given "free data", up to constraints, at "radiative order" $O(1/r^{d/2-1})$ or "Coulombic order" $O(1/r^{d-3})$. For electromagnetic fields or linearized gravitational fields the relevant gauge in which these propositions apply is the "Lorentz gauge" whereas in fully nonlinear general relativity the relevant gauge is the Harmonic gauge. We prove in Props. 2.3–2.5 that the only obstruction to putting the fields in these gauges is in $d = 4$ if there is a non-vanishing flux of charge current to $\mathscr{I}^+$ in the electromagnetic case or a non-vanishing energy flux in the gravitational case. This is clearly far too restrictive in general relativity and therefore in $d = 4$ one must use the "Bondi gauge" conditions. However, in higher dimensions this approach yields a simpler and more straightforward method for analyzing the behavior of massless fields near null infinity in higher dimensions.

## *2 The memory effect in all spacetime dimensions*

Given this asymptotic analysis of sec. 2 of sec. 2 we then straightforwardly analyze the gravitational memory effect for any scattering scenario where the metric is stationary at leading order in $1/r$ at early and late times. The precise stationarity conditions at early and late times are spelled out in sec. 3.1 of chapter 2. We focus particularly on the gravitational memory effect, but exactly similar conclusions shall hold in the electromagnetic case. In four dimensions, it can be shown that the memory effect naturally decomposes into "null memory" and "ordinary memory". "Null memory", which was originally discovered by Christodoulou, is associated with a flux of energy to null infinity [19]. Ordinary memory, which was originally discovered by Zel'dovich and Polnarev [10], can be associated with the metric being non-stationary at one order faster than "Coulombic order" which, in four dimensions, is at order

$1/r$. This will generically occur if there is a flux of matter stress-energy at past or future timelike infinity. In sec. 3 of chapter 2 we generalize this result to all spacetime dimensions. More precisely, we prove that the memory effect vanishes at all fall-off slower than Coulombic order and is first non-vanishing at Coulombic order $O(1/r^{d-3})$. In even dimensions we prove that the Coulombic order memory admits a decomposition into "ordinary memory" and "null memory" in a manner similar to the decomposition in four dimensions. However in odd spacetime dimensions, we show that the total memory effect at Coulombic order vanishes.

Given this decomposition we then investigate the general relationship between memory and symmetries in sec. 3.4 of chapter 2. In four spacetime dimensions, there are two ways in which memory is related to asymptotic symmetries. The first way is when memory can be expressed as a diffeomorphism. The memory effect is physically the displacement of test masses due to the passage of radiation. In four dimensions, it has been previously noted that this displacement can correspond to a diffeomorphism. This diffeomorphism is an asymptotic symmetry known as a "supertranslation" which is an "angle dependant" time translation and does not lie in the usual Poincaré group. Physically this implies that, due to the flux of radiation, the early and late time frames in the stationary eras are related by (at least) a supertranslation which is not a degeneracy of the symplectic form [20]. Therefore, the existence of memory implies the enlargement of the asymptotic symmetry group to include all supertranslations in four dimensions. The second way memory is related to symmetries is through "charge-flux" relations. Asymptotic supertranslations can be used to generate corresponding "supertranslation charges" whose total flux through all of $\mathscr{I}^+$ is determined by the memory and the null memory [21]. The change in the supertranslation charge between asymptotically early and late times is equivalent to the ordinary memory effect and so this "charge-flux" relation can be equivalently viewed as a rearrangement of the memory formula expressed in terms of ordinary and null memory. These charge-flux relations can be similarly defined at past null infinity. The matching of supertranslation charges at spatial infinity [22, 23, 24, 25, 26] relates the incoming charges, memory and null memory to the corresponding outgoing quantities at future infinity and plays an important role in the quantum scattering theory in chapter 3.

While these arguments illustrate the necessity of the enlargement of the asymptotic symmetry group in $d = 4$ we show that memory is *not* related to any symmetries for $d > 4$. To see this, we first show that the memory effect can always be decomposed into scalar, vector and tensor parts on the $(d-2)$-sphere. The null memory is always of scalar type but the ordinary memory can be of scalar, vector or tensor type. Furthermore, we show that all types of memory can be sourced by physically reasonable matter distributions which satisfy the dominant energy condition. We show that memories of vector and tensor type cannot be described by a diffeomorphism. Memories of scalar type can be described by a diffeomorphism but this diffeomorphism is pure gauge for $d > 4$ and is precisely a supertranslation in $d = 4$. We give an explicit example in four dimensions of a shell of matter with vector stresses which gives rise to a vector ordinary memory effect. Since vector memory is not related to diffeomorphisms, this shows that even in four dimensions there is no general correspondence between memory and symmetries. We also consider the relationship between the Coulombic memory effect to charges and fluxes in higher dimensions. The ordinary memory effect in higher dimensions can again be interpreted in terms of the difference of "charges" and therefore its "flux" can be expressed in terms of the higher dimensional memory and null memory. However, these "charges" are not generated by any local symmetries on $\mathscr{I}^+$.[1] For these reasons, one can consistently reduce the asymptotic symmetry group to the finite dimensional Poincaré group in $d > 4$ spacetime dimensions whereas one cannot consistently do so in $d = 4$.

We finally consider, in sec. 3.7 of chapter 2, the relationship between memory and "infrared

1. One could instead attempt to define supertranslation charges following the methods of [21]. However, the charges associated to supertranslations in higher dimensions diverge at null infinity [20, 27]

divergences" in quantum scattering theory. As is well-known, there are no infrared divergences in dimensions $d > 4$. The main purpose of this subsection was to understand this statement in terms of memory and its relation to the existence of the Weinberg soft graviton theorem which holds in all dimensions and implies the factorization of $S$-matrix amplitudes into "hard" (high frequency) and "soft" (low frequency) parts [28]. The full algebra of asymptotic observables and its corresponding quantization is given in detail in chapter 3 however, to see the essential differences between four and higher dimensions we considered the simple model of linearized quantum gravity coupled to a classical stress energy. We show that if the classical stress energy produces classical radiation with no memory then the "out" graviton state is in the standard Fock representation. However, if the classical stress tensor produces a classical field with memory then the "out" graviton state has an *infinite* number of low-frequency gravitons and cannot be represented in the standard Fock representation. This state with memory, however, can be expressed as a state in an inequivalent Fock representation. Given an infrared cutoff, we show that one can factor the state into "hard" part (which lies in the standard Fock space) and "soft" part (which lies in an inequivalent "memory Fock representation"). We argue that this factorization should hold more generally and is equivalent to the Weinberg soft graviton theorem. In higher dimensions, we show that "out" states with non-vanishing higher dimensional memory lie in the standard "out" Fock space. This is due to the fact that the memory occurs at "Coulombic order" and not "radiative order" and therefore does not affect the quantization of the radiative gravitons. Additionally, one can do a similar 'hard/soft" splitting of the "out" state but there is no necessity to do so in order to describe the "out" state as a Fock space state. In this regard, the $S$-matrix as a map on the standard Fock space is perfectly well-defined in $d > 4$ dimensions but one runs into serious issues in defining the $S$-matrix $d = 4$ spacetime dimensions. More precisely, in the full quantum scattering theory one will have non-vanishing amplitudes for the "out" state to lie in different memory Fock representations. Therefore, a well-defined scattering theory in four dimensions must include all of the uncountably infinite number of memory Fock representations in both the "in" and "out" Hilbert spaces. Whether such a Hilbert space exists in quantum field theory or quantum gravity is considered in chapter 3.

In summary, in four spacetime dimensions, the memory effect is intimately linked to the enlargement of the asymptotic symmetry group to the BMS group as well as the infrared divergences found in the standard $S$-matrix formulation of scattering theory. In higher dimensions, the memory occurs at Coulombic order $O(1/r^{d-3})$ whereas radiation decays as $O(1/r^{d/2-1})$. Therefore, for $d > 4$, the memory no longer occurs at radiative order and this is ultimately the reason for the reduction of the asymptotic symmetry group to the Poincaré group as well as the lack of infrared divergences in higher dimensions.

## *3 Infrared finite scattering theory in quantum field theory and quantum gravity*

In chapter 3 we investigate the long-standing problem of defining an "infrared finite" scattering theory in quantum field theory and quantum gravity. As described above, the outgoing state will generically have memory and cannot be represented as a state in the standard Fock representation. The most common way of dealing with such infrared divergences is by introducing an infrared cut-off so that the "out" state is expressible as an ordinary Fock state and then once can calculate "inclusive cross sections" which yields the probability for any "hard" process given the emission of any "soft" quanta [9, 28]. This procedure is extremely successful in obtaining observables relevant for collider physics. However, this infrared cutoff removes the memory effect as a quantum observable. Furthermore the "soft" radiation also results in an enormous amount of decoherence of the "hard" particles [29, 30] and this decoherence is observable in *finite* time interference experiments [31, 2]. Finally, this situation is highly unsatisfactory if one wishes to view the $S$-matrix as a fundamental quantity in the formulation of quantum field theory and quantum gravity since the $S$-matrix itself is undefined.

As explained previously, if one starts with an "in" state in the standard Fock space with zero memory then the "out" state will generically have memory and will not lie in the standard Fock space. Therefore, to have a well-defined $S$-matrix one must include states with memory into a single, separable Hilbert space. However, there are an uncountably infinite number of inequivalent memory Fock representations and the memory is not conserved (i.e. the "in" memory is not generically equivalent to the "out" memory). There is a priori no clear way to assemble these Fock spaces together in a manner which will be preserved under scattering. Furthermore, one will encounter infrared divergences if one chooses the incorrect "out" representation.

In chapter 3 we investigate the existence of any suitably representations for scattering theory in quantum electrodynamics (QED) with a massive or massless charged fields, Yang-Mills theories or in full quantum gravity. To study this problem it is essential to define the theory at asymptotically early and late times prior to a choice of Hilbert space representation. This is achieved by the algebraic approach to quantum field theory which we review in sec. 2 and 3 of chapter 3. In this approach, the asymptotic quantum theory is defined by specifying the algebra of observables and states then correspond to a class of positive linear maps on the algebra to the complex numbers. More concretely, a state is determined by specifying the "$n$-point correlation functions" of the field where we further require all states to have the same singular structure as the vacuum. While this notion of a state is equivalent, by the GNS construction, to the more usual definition as an element of a prescribed Hilbert space, this notion of a state allows one to simultaneously consider states which live in *different* representations. In other words, states with or without memory are treated with equal footing in this approach. In sec. 4.1, 5.1, 5.5 and 6.1 of chapter 3 we take the algebraic approach and define the algebra of asymptotic observables and states for each theory that we consider. We then extend these algebras in sec. 4.2, 5.2 and 6.2 to include “charges” and “fluxes” which correspond to generators of the asymptotic symmetry group in each respective theory.

Given this reformulation of the theory in terms of the algebra of observables and the corresponding scattering states, we then consider the consider the problem of finding a suitable Hilbert space representation for scattering theory which yields an infrared finite $S$-matrix. Remarkably, this was accomplished in QED with massive charged particles by Faddeev and Kulish in the 1970’s [32]. In sec. 4.4 of chapter 3 we recast their construction in terms of the extended algebra of observables (i.e. charges and memory) of QED. In QED, the “incoming” electromagnetic memory effect can also be decomposed into an “ordinary memory effect” and a “null memory effect”. The ordinary electromagnetic memory effect is expressible in terms of the difference of “large gauge charges” defined at past timelike infinity and spatial infinity which are charges generated by asymptotic “angle-dependent” gauge transformations. The null electromagnetic memory effect corresponds to the angular distribution of the incoming current-flux through $\mathscr{I}^-$ due to the flux of any incoming massless charged particles. In QED with massive charged particles, the null memory effect vanishes and one only gets a contribution to memory from the ordinary memory effect. Furthermore, the incoming charge at timelike infinity is completely determined by the incoming massive matter and the incoming memory is determined by the incoming electromagnetic radiation. Therefore, the large gauge charge at spatial infinity is uniquely determined from the initial data.

We now summarize the construction of Faddeev-Kulish states. A more detailed overview which contains more of the mathematical details can be found in the introduction of chapter 3 and the full construction can be found in sec. 4.4. The Faddeev-Kulish construction corresponds to constructing a Hilbert space of eigenstates of the charge at spatial infinity. Since charge at spatial infinity satisfies a “matching condition” [33] the Hilbert space of “in” states will evolve into a similarly defined “out” eigenspace of the large gauge charges at future infinity. To construct these eigenstates we note that the eigenstates of the large gauge charge at past timelike infinity are (improper) plane wave states of the incoming electron where the eigenvalue corresponds to the asymptotic Liénard–Wiechert solution of a sum of charged particles and antiparticles with momenta determined by the incoming plane wave. Given a definite value of the large gauge charge at spatial infinity, the required incoming memory is uniquely determined by the incoming charge at timelike infinity and the definite charge at spatial infinity. Therefore, the eigenstates of the large gauge charge correspond to wave packets where each plane wave state of the electron is highly correlated with incoming electromagnetic radiation with memory uniquely determined by the incoming plane wave state. Such states are referred to as "dressed electrons" where the incoming electron is highly correlated with a "cloud of soft photons" (i.e. radiation states with memory).

While the resulting Hilbert space of states of definite charges at spatial infinity should evolve into a corresponding "out" Hilbert space of definite charge, the charges at spatial infinity are not Lorentz invariant unless all of the charges vanish — including the total electric charge. Therefore, the Hilbert spaces of definite charge will not have a continuous action of the Lorentz group unless all of the charges are set to zero. One must then pair an incoming "dressed" electron" with a "dressed" positron in such a way that all of the charges at spatial infinity vanish. This construction provides a Hilbert space of states which has a continuous action of the Poincaré group and yields a well-defined, IR-finite $S$-matrix. However, there are clearly a number of unpleasant features of this construction. The first is the vanishing of total electric charge. This may seem like a severe restriction since one may, for instance, wish to consider the scattering of two electrons. As was argued by [34], one could still achieve this within the Faddeev and Kulish framework by hiding two positrons "behind the moon" (i.e. incoming states which do not interact strongly with the electrons). Therefore, in principle, this restriction does not fully invalidate the construction. Another unpleasant feature is that the incoming state of the electron and the incoming state of the photons are independent degrees of freedom, however the Faddeev-Kulish construction forces a high degree of entanglement between the photons and the electrons resulting in a complete decoherence of the electrons [30, 35, 29, 2]. One cannot consider, for example, “undressed” incoming electrons and so the Faddeev-Kulish Hilbert space artificially excludes a large class of physical states. Nevertheless, this class of states do yield a genuine, infrared finite $S$-matrix.

However, we show in sec. 5.4 and 5.5 that a similar construction does *not* work in a satisfactory way for QED with massless charged particles and for Yang-Mills theories. The analog of the “dressing procedure” is to now pair eigenstates of the charge-current flux in massless QED or the gluon color-flux in Yang Mill theories with the appropriate memory. However, the memories obtained in this procedure contain “collinear divergences” in the angular behavior of the photon or gluon fields which implies that the incoming field must have infinite energy. Therefore, one must attempt to pair the incoming massless particles or gluons with radiation states with infinite energy. In the Yang-Mills case the situation is worse because, due to the non-linearities in Yang-Mills theory, the color-flux of the dressing will now further contribute to the charge at spatial infinity and due to the collinear divergences the additional contribution to the color-flux is also infinite. Therefore, while one can do the dressing procedure (with infinite energy dressing) in massless QED, there is no analog of this procedure in Yang-Mills. Nevertheless, one could consider other means of obtaining eigenstates of the charge. We prove that the only eigenstates are states whose correlation functions are Casimirs of the Lie algebra. For example, the one-point function must vanish and the two-point function must be proportional to the Cartan-Killing metric of the Lie algebra. While there exist states which satisfy these conditions, this is clearly far too restrictive to represent all “hard” scattering processes.

The situation in quantum gravity is even worse. First, in order to have a well-defined action of the Lorentz group one needs to set all of the supertranslation charges to vanish —

including the total four-momentum. The only state which satisfies this is the vacuum state and so the Faddeev-Kulish construction fails at this elementary step. One could forgo having a well-defined action of Lorentz and consider eigenstates with non-vanishing supertranslation charges. This is possible in linearized gravity with massive or massless sources by an analog dressing procedure outlined above. Furthermore, due to the lack of "collinear divergences" in quantum gravity [36] there is no fundamental issue in dressing both massive and massless fields. However, there is a serious issue in extending this to full quantum gravity. In this case, the incoming flux of gravitational radiation energy (i.e. the "null memory") contributes to the supertranslation charges at spatial infinity. These nonlinearities, as in the Yang Mills case, imply that any attempt to pair the incoming graviton energy flux with incoming memory will also introduce more incoming energy and spoil the eigenstate condition we wish to achieve. One could hope to obtain these eigenstates by some other procedure besides dressing, however the only eigenstates of the supertranslation charge are those that are invariant under all supertranslations. We prove in Theorem 5 that the only such state is the vacuum state.

In sec. 7 we explore alternative constructions involving direct integrals with respect to Gaussian measures of the memory Fock representations and find that these also do not work. While our analysis does not exhaust all possibilities, we believe that in all cases except for QED with massive charged fields, there is no satisfactory Hilbert space construction for "in" and "out" states which yields a well-defined, infrared finite $S$-matrix.

We believe that the inherent issue is simply in demanding that all physical states of interest lie in a single Hilbert space representation. As already emphasized, the algebra of observables and the space of correlation functions is perfectly well-defined. However, as we have argued, there is in general no preferred Hilbert space of states which evolves into itself in massless QED, Yang-Mills theories and quantum gravity. Therefore, in sec. 8, we advocate that if one wishes to have a well-defined $S$-matrix one should take an algebraic point of view on scattering theory. In particular, one can choose any "in" state (defined as a list of correlation functions) that one wishes — without any a priori "dress requirements" — and evolve this state to an "out" state defined as a list of correlation functions. At no stage would we demand that the "in" and "out" states lie in any pre-chosen Hilbert space and we would encounter infrared divergences if we select the wrong one. We believe that one must adopt this manifestly infrared-finite approach to scattering theory if one wishes to treat scattering at a fundamental level.

## *Notation and conventions*

We work in natural units ($G = c = \hbar = 1$) and will use the notation and sign conventions of [37]. In particular, our metric signature is "mostly positive" and our sign convention for curvature is such that the scalar curvature of a round sphere is positive. Greek indices $(\mu, \nu, \dots)$ correspond to tensors in the "bulk" of the spacetime.[2]. We now spell out our notational conventions in chapter 2 and chapter 3 which differ slightly. The most important difference between chapter 2 and chapter 3 is that in chapter 2 we work in the physical asymptotically flat spacetime whereas in chapter 3 it will be far more convenient to work in the unphysical, conformally compactified spacetime.

In chapter 2 we will consider the expansion of tensors in terms of powers of $1/r$ in the physical, $d$-dimensional, asymptotically flat spacetime. In this chapter, Greek indices are raised and lowered with respect to the "background" Minkowski metric $\eta_{\mu\nu}$. In this chapter, capital Latin indices $(A, B, C, \dots)$ will be used to denote tensors on the $(d-2)-$sphere. We will also use capital Latin indices to denote coordinates, $x^A$, on the sphere and components in this coordinate basis. (We do not feel that the potential confusion resulting from using the same notation for a tensor on a sphere and its components in a coordinate basis is sufficient to justify introducing another alphabet into our notation.) When we expand a

2. We also will use Greek indices in several places in sec. 2 of chapter 3 to denote tensors on phase space.

scalar field $\phi$ in powers of $1/r$, $\phi^{(n)}$ will denote the coefficient of $1/r^n$. When we expand a tensor field $t_{a_1 \dots a_k}$ in powers of $1/r$, the quantity $t^{(n)}_{a_1 \dots a_k}$ will denote the coefficient of $1/r^n$ in a *normalized* basis. In particular, for a co-vector field, $t_a$, the quantity $t^{(n)}_A$ is such that its action on the normalized basis element $\frac{1}{r}\frac{\partial}{\partial x^A}$ falls as $1/r^n$. This differs from a much more common convention [38, 39, 40] where $t^{(n)}_A$ would be such that its action on $\frac{\partial}{\partial x^A}$ falls as $1/r^n$. Our conventions thereby avoid a spurious mixing of orders, and the orders we assign to components do not depend on whether we are using Cartesian or spherical coordinates.

In chapter 3, we will generally use the symbol $\mathscr{A}$ to denote a $*$-algebra of observables, $\omega$ to denote a state on the algebra, $\mathscr{H}$ to denote a Hilbert space and $\mathscr{F}$ to denote a Fock space. Algebras of local field observables in the asymptotic past and future will be denoted as $\mathscr{A}_{\text{in}}$ and $\mathscr{A}_{\text{out}}$, respectively. We will append a superscript "in" or "out" on various other quantities to distinguish between quantities defined in the asymptotic past or future, but we will omit this superscript when the context is clear. We will use superscripts "EM", "KG", "KG0", "YM", and"GR" on quantities to distinguish between the particular cases of the electromagnetic, massive Klein-Gordon, massless Klein-Gordon, Yang-Mills, and gravitational fields, respectively. Thus, for example, $\mathscr{A}^{\text{EM}}_{\text{in}}$ denotes the algebra of local electromagnetic field observables in the asymptotic past. We will append subscripts "$Q$" and "$P$" to denote the extensions of algebras of local field observables to include large gauge charges and Poincare generators respectively. Thus, for example $\mathscr{A}^{\text{KG}}_{\text{in},Q}$ denotes the extension of the algebra of local field observables of a massive scalar field in the asymptotic past to include large gauge charges.

Quantum observables will be denoted by the boldfaced version of the symbol for the corresponding classical observable; for example, the quantum observable corresponding to a classical scalar field $\phi$ is denoted by $\boldsymbol{\phi}$.

We will work with the Penrose conformal completion (see, e.g., [37, 41]) of flat spacetime (for QED and Yang-Mills theory) and asymptotically flat spacetimes (for gravity). The conformal boundaries $\mathscr{I}^\pm$ denote future/past null infinity, $i^0$ denotes spatial infinity and $i^\pm$ denotes future/past timelike infinity. The conformal factor will be denoted by $\Omega$ and without loss of generality we impose the Bondi condition $\nabla_\mu \nabla_\nu \Omega = 0$ at null infinity $\mathscr{I}^\pm$. The null normal to $\mathscr{I}^\pm$ will be denote $n^\mu = \nabla^\mu \Omega$.

We will frequently encounter down index tensors on $\mathscr{I}^\pm$ that are orthogonal to $n^\mu$ in each index. We will denote such tensors with capital Latin letters $(A, B, \dots)$. Again, we do not believe the potential confusion arising from the conflict with the notation in chapter 2 warrants introducing another alphabet into our notation. For example, the pullback of the electric field $E_\mu = F_{\mu\nu} n^\nu$ to $\mathscr{I}^\pm$ is such a tensor and it will be denoted as $E_A$. Similarly, the (degenerate) metric on $\mathscr{I}^\pm$ (obtained from the pullback of the conformal spacetime metric) will be denoted as $q_{AB}$. We also will use capital Latin letters to denote equivalence classes of "up" index tensors on $\mathscr{I}^\pm$, where two such tensors are equivalent if they differ by a multiple of $n^\mu$ in any index. (Such "up" index tensors are dual to the corresponding down index tensors.) The metric $q_{AB}$ acts non-degenerately on such equivalence classes of vectors, so it has an inverse, which we will denote as $q^{AB}$. We will use $q_{AB}$ and $q^{AB}$ to lower and raise capital Latin indices. Most of our analysis will be done with incoming fields on past null infinity $\mathscr{I}^-$ and we will use coordinates $x = (v, x^A)$ on $\mathscr{I}^-$, where $v$ is the advanced Bondi time coordinate and $x^A$ are arbitrary coordinates on a 2-sphere. Note that the index on the coordinates $x^A$ should not be confused with a tensor index as described above.

# CHAPTER 2

# THE ASYMPTOTIC BEHAVIOR OF MASSLESS FIELDS AND THE MEMORY EFFECT IN ALL SPACETIME DIMENSIONS

## 1 Introduction

In the early 1960's, Bondi and collaborators [16, 17, 18] performed a general analysis of the asymptotic behavior of the metric near "null infinity" ($r \to \infty$ at fixed retarded time $u$) for asymptotically flat spacetimes. They assumed an expansion of the metric in powers of $1/r$ and obtained a recursive algorithm for solving the Einstein equations near null infinity. Several years later, Penrose [42] gave an elegant, geometric reformulation of the Bondi ansatz via conformal compactification. A similar analysis of higher even-dimensional, asymptotically flat spacetimes can be given using conformal compactification [27]. However, such a conformal compactification is not possible for odd dimensional spacetimes with gravitational radiation [43].

In sec. 2 of this chapter, we will analyze the asymptotic behavior of massless scalar, electromagnetic, and linearized gravitational fields near null infinity in Minkowski spacetimes with $d \geq 4$. We will then analyze asymptotically flat, nonlinear general relativity near null infinity. Since we wish to treat odd dimensions as well as even dimensions, we will not use conformal compactification but, instead, will assume an expansion in powers of $1/r$ as an ansatz. For $d$ even with $d > 4$, our ansatz is precisely equivalent to smoothness[1] at $\mathscr{I}^+$ in the conformally compactified spacetime, whereas we will see in Appendix A that for $d = 4$ it is slightly weaker, i.e., we allow a small class of additional solutions that would not be

1. It should be noted that our analysis will be primarily concerned with behavior of fields at $1/r^{d-3}$ and slower fall off, so for our main results, "smoothness" can be replaced by differentiability to the corresponding order.

allowed by smoothness at $\mathscr{I}^+$. Our fields will be allowed to have arbitrary interior sources, i.e., only the field equations near null infinity will be used. Near null infinity the fall-off of the sources is required to be rapid enough to ensure that there is a finite flux through spheres near null infinity.

In sec. 3 of this chapter, we will give a thorough analysis of the memory effect in nonlinear general relativity in all dimensions $d \geq 4$. An important aim of our analysis is to extend and clarify the work of Strominger and collaborators [44, 4, 45, 46, 47, 38].

We begin our analysis in sec. 2.1 by considering a massless scalar field, $\phi$, in $d$-dimensional Minkowski spacetime. We show that the wave equation gives a recursion relation that relates different coefficients in an expansion of the field in powers of $1/r$. This recursion relation motivates an expansion in integer steps, with the slowest fall-off being $1/r^{d/2-1}$ ("radiative order"). In odd dimensions, integer powers starting at $1/r^{d-3}$ ("Coulombic order") must also be allowed. The "free data" needed to specify a solution is characterized in sec. 2.2.

We then consider an electromagnetic field, $A_\mu$, in sec. 2.3. It is very convenient to put $A_\mu$ in Lorenz gauge, $\partial^\mu A_\mu = 0$, since then many of the results for the scalar field can be directly taken over. In order to put the electromagnetic field in Lorenz gauge, we need to solve the scalar wave equation with a source. We show that when $d > 4$, this can be done in a manner compatible with our $1/r$ expansion ansatz. However, when $d = 4$ we cannot do this if there is a nonvanishing flux of charge to null infinity. In Lorenz gauge, each Cartesian component of $A_\mu$ satisfies the same recursion relations as the scalar wave equation, but there also are additional conditions ("constraints") arising from the Lorenz gauge condition itself. It is convenient to write the recursion relations and constraints in terms of the components $A_u, A_r, A_A$ in coordinates $(u, r, x^A)$ where $u$ is the retarded time and $x^A$ denotes coordinates on the $(d-2)$-sphere. We do this explicitly in sec. 2.3. The "free data" is then characterized.

Gravitational perturbations, $h_{\mu\nu}$, are considered in sec. 2.4. In order to put $h_{\mu\nu}$ in Lorenz gauge, $\partial^\mu \bar{h}_{\mu\nu} = 0$ (with $\bar{h}_{\mu\nu} \equiv h_{\mu\nu} - 1/2\eta_{\mu\nu}h$ and $h \equiv \eta^{\mu\nu}h_{\mu\nu}$), we need to solve the vector wave equation with a source. Again, we find that when $d > 4$, this can be done in a manner compatible with our $1/r$ expansion ansatz. However, when $d = 4$ we cannot do this if there is a nonvanishing flux of matter stress-energy to null infinity. We give the recursion relations and constraints explicitly in terms of the components $h_{uu}, h_{ur}, h_{rr}, h_{uA}, h_{rA}, h_{AB}$ and identify the "free data."

It might be thought that the full, nonlinear Einstein equation would be much more difficult to analyze. However, as we shall see in sec. 2.5, the nonlinear terms first enter Einstein's equation at order $1/r^{d-2}$ and they first affect the behavior of the metric at Coulombic order $1/r^{d-3}$. Similarly, the nonlinear terms in the harmonic gauge condition first affect the metric at Coulombic order. Thus, under our ansatz concerning the expansion of the metric in powers of $1/r$, the analysis of the nonlinear Einstein equation coincides with the linearized analysis until Coulombic order, and the differences at Coulombic order can be taken into account in a relatively straightforward manner.

In sec. 3, we turn our attention to the memory effect, i.e., the permanent relative displacement of an arrangement of test particles near null infinity that are initially at rest. We assume that the metric initially is stationary to Coulombic order, goes through a non-stationary epoch, and again becomes stationary to Coulombic order. The precise stationarity assumptions and the motivation for them are spelled out in sec. 3.1. We obtain general properties of the memory tensor in sec. 3.2. In sec. 3.3, we calculate the memory tensor for all $d \geq 4$. We show that the memory tensor vanishes at all fall-off slower than Coulombic, i.e., it vanishes at order $1/r^n$ for all $n < d - 3$. In even dimensions, the memory tensor at Coulombic order can be nonvanishing [38, 48] and we also show that it naturally decomposes into "null memory" and "ordinary memory," in a manner similar to the known decomposition in 4-dimensions [39]. "Null memory" is associated with a flux of energy to null infinity, whereas we show that "ordinary memory" is associated with the metric being non-stationary at one order faster fall-off than Coulombic, as will generically occur if there is a flux of matter stress-energy moving inertially in from infinity or out to infinity at less than the speed of light. In odd dimensions, we show that the total memory effect vanishes near null infinity at Coulombic order.

As discussed in sec. 3.4, in all dimensions, the memory effect can be decomposed into scalar, vector and tensor parts on the $(d-2)$-sphere. Null memory is always of scalar type, but ordinary memory can be of any type. We give an explicit example in linearized gravity in $d = 4$ dimensions involving a shell of matter with vector stresses that gives rise to vector (i.e., "magnetic parity") ordinary memory at order $1/r$. In sec. 3.5, we show that scalar memory can be characterized by a diffeomorphism. This diffeomorphism is an asymptotic symmetry in $d = 4$ dimensions, but it is gauge for $d > 4$. Vector and tensor memory cannot be described by a diffeomorphism.

We then consider the relationship of memory to charges and conservation laws in eq. (1.9). In $d = 4$ dimensions, we show in sec. 3.6.1 how the charges and fluxes associated with supertranslations can be used to derive the formula for scalar memory. Although memory cannot be associated with an asymptotic symmetry when $d > 4$, similar expressions are obtained from our general formulas for memory in sec. 3.3. In sec. 3.6.2 we provide some arguments in favor of "antipodal matching" of solutions between future and past null infinity, and show that under the assumption of antipodal matching, we obtain expressions that can be interpreted as representing conservation laws relating charges and fluxes at past and future null infinity.

Finally, in sec. 3.7 we show that in $d = 4$ dimensions, the presence of a nontrivial memory effect at future null infinity is intimately related to infrared divergences in the "out" state in quantum field theory. The factorization of the "out" state vector into a product of "hard" and "soft" parts is shown for the case of quantum linearized gravity with a classical source, and is argued to hold generally.

## 2 The general behavior of fields near null infinity

Consider $d$-dimensional Minkowski spacetime with $d \geq 4$. In terms of global inertial coordinates $(t, x^1, \ldots, x^{d-1})$, the metric takes the form

$$\eta = -dt^2 + \sum_{\mu=1}^{d-1} (dx^\mu)^2. \tag{2.1}$$

Let $r = (\sum (x^\mu)^2)^{1/2}$, let $u \equiv t - r$, and let $x^A$ be arbitrary coordinates on the spheres of constant $r$ and $u$. In the coordinates $(u, r, x^A)$, the Minkowski metric $\eta$ takes the form

$$\eta = -du^2 - 2dudr + r^2 q_{AB} dx^A dx^B \tag{2.2}$$

where $q_{AB}$ is the metric on the round unit $(d-2)$-sphere. Let

$$K^\mu = (\partial/\partial r)^\mu \tag{2.3}$$

$$l^\mu = (\partial/\partial u)^\mu - \frac{1}{2}(\partial/\partial r)^\mu \tag{2.4}$$

so that $K^\mu$ and $l^\mu$ are the future-directed, radially outgoing and ingoing null vector fields, which satisfy

$$K^\mu l_\mu = -1\,. \tag{2.5}$$

Let $q_{\mu\nu}$ denote the spacetime tensor field whose pullback to spheres of constant $u$ and $r$ is $q_{AB}$ and $K^\mu q_{\mu\nu} = 0 = l^\mu q_{\mu\nu}$. The metric can be written as

$$\eta_{\mu\nu} = -2K_{(\mu} l_{\nu)} + r^2 q_{\mu\nu}. \tag{2.6}$$

We will be concerned in the following with the behavior of fields near "null infinity" in this spacetime, i.e., the limit as $r \to \infty$ at fixed $(u, x^A)$.

## 1 *Ansatz for the massless scalar field*

Consider a massless Klein-Gordon field $\phi$ satisfying

$$\Box\phi = 0 \tag{2.7}$$

where $\Box \equiv \eta^{\mu\nu}\partial_\mu\partial_\nu$. (In the next subsection, we will allow a source term $S$, i.e., we will consider $\Box\phi = S$.) We assume, as a *preliminary ansatz*, that near null infinity, $\phi$ can be expanded as a series in $1/r$ as follows:

$$\phi \sim \sum_{j=0}^{\infty} \frac{1}{r^{\alpha+j}} \phi^{(j)}(u, x^A) \tag{2.8}$$

where $\alpha \in (0, 1]$. Here, the meaning of the "$\sim$" in eq. (2.8) is as follows: We do *not* require that the series on the right side of this equation converges (even for large $r$) but require that for any $N \geq 0$ we have

$$\phi - \sum_{j=0}^{N} \frac{1}{r^{\alpha+j}} \phi^{(j)}(u, x^A) = O(1/r^{\alpha+N+1}) \tag{2.9}$$

as $r \to \infty$, i.e., we require this series to be asymptotic. We further require that all partial derivatives of the left side of eq. (2.9) with respect to $u$ and $x^A$ are also $O(1/r^{\alpha+N+1})$, whereas $k$ partial derivatives with respect to $r$ are $O(1/r^{\alpha+N+1+k})$. For convenience, we have taken the upper limit in the sum in eq. (2.8) to be $\infty$, but all of our results will require eq. (2.9) to hold only for finite $N$ (with the precise value of $N$ needed depending on the result).

We now substitute eq. (2.8) into eq. (2.7) and collect the terms that fall off as $1/r^{\alpha+j+1}$. We thereby obtain the following recursion relations for the coefficients appearing in eq. (2.8)

$$[\mathscr{D}^2 + (\alpha + j - 1)(\alpha + j - d + 2)]\phi^{(j-1)} + (2\alpha + 2j - d + 2)\partial_u \phi^{(j)} = 0 \tag{2.10}$$

Here, $\mathscr{D}^2 = \mathscr{D}_A \mathscr{D}^A$ is the Laplacian on the unit sphere, where $\mathscr{D}_A$ is the derivative operator associated with $q_{AB}$ and sphere indices are lowered and raised with $q_{AB}$ and $q^{AB}$.

It follows immediately from eq. (2.10) that if, for some $i \geq 0$, $\phi^{(i)}$ has nonpolynomial dependence on $u$, then for even $d$, no solution of the form eq. (2.8) exists unless $\alpha = 1$, whereas for odd $d$, no solution of the form eq. (2.8) exists unless $\alpha = 1/2$. To see this, we note that unless the coefficient of the $\partial_u \phi^{(j)}$ term vanishes for some $j$, the nonpolynomial dependence of $\phi^{(i)}$ will propagate to $\phi^{(i-1)}$ and thence to $\phi^{(i-2)}$, etc. This will result in an inconsistency in eq. (2.10) at the lowest nontrivial order, $j = 0$, since the first term in that equation is then absent. Thus, the coefficient of $\partial_u \phi^{(j)}$ in eq. (2.10) must vanish for some $j$. For $d$ even, this requires $\alpha = 1$, in which case the coefficient vanishes for $j = d/2 - 2$. For $d$ odd, this requires $\alpha = 1/2$, in which case the coefficient vanishes for $j = (d-3)/2$.

However, in the odd dimensional case, eq. (2.8) with $\alpha = 1/2$ is not adequate for several reasons. First, eq. (2.8) with $\alpha = 1/2$ does not admit static solutions, since static solutions satisfy Laplace's equation and fall off as integral powers of $1/r$, starting at order, $1/r^{d-3}$. Second, when a source term $S$ is considered in eq. (2.7), it is natural to allow $S$ to fall off with integral powers of $1/r$. In particular, in order to have a nonvanishing, finite source flux at null infinity, it will be necessary to have $S$ fall off as $1/r^{d-2}$. Such source terms will generate terms in $\phi$ that fall off as integral powers of $1/r$, again starting at order $1/r^{d-3}$. Third, even if one does not consider sources, for nonlinear equations such as Einstein's equation, quadratic and higher order even powers of the field will generate terms that fall off as integral powers of $1/r$. This will lead to inconsistencies unless one also includes integral powers of

$1/r$ in the fall-off of the field, again starting at order $1/r^{d-3}$.

Thus, in odd dimensions, we must allow integral powers of $1/r$ starting at least at order $1/r^{d-3}$. However, in odd dimensions, the coefficient of a term that falls as $1/r^p$ for integer $p < d-3$ must have polynomial dependence in $u$ of degree $< p$ in order for the recursion relations to terminate. (Source terms and nonlinear terms will not enter the recursion relations at these orders.) Such solutions do not appear to be of any physical interest, and we will exclude them from our ansatz.

Thus, we adopt the following as the final form of our ansatz:

$$\phi \sim \sum_{n=d/2-1}^{\infty} \frac{1}{r^n}\phi^{(n)}(u,x^A) \qquad d \text{ even} \tag{2.11}$$

$$\phi \sim \sum_{n=d/2-1}^{\infty} \frac{1}{r^n}\phi^{(n)}(u,x^A) + \sum_{p=d-3}^{\infty} \frac{1}{r^p}\tilde{\phi}^{(p)}(u,x^A) \qquad d \text{ odd} \tag{2.12}$$

where the meaning of "$\sim$" is as explained below eq. (2.8). Note that in eq. (2.12), $n$ runs over half-integer values rather than integer values (as in eq. (2.11)). We have done this (rather than insert $\alpha = 1/2$ and keep integer values) so that the superscript "$(n)$" is always associated with $1/r^n$ fall-off and so that we can write the recursion in the same form

$$[\mathscr{D}^2 + (n-1)(n-d+2)]\phi^{(n-1)} + (2n-d+2)\partial_u\phi^{(n)} = 0 \tag{2.13}$$

in both even and odd dimensions. In both even and odd dimensions, we refer to the leading (slowest fall-off) term $n = d/2-1$ as *radiative order*, and we refer to the term with $1/r^{d-3}$ fall-off as *Coulombic order*. In odd dimensions, the $\tilde{\phi}^{(p)}$ satisfy separate recursion relations of the same form

$$[\mathscr{D}^2 + (p-1)(p-d+2)]\tilde{\phi}^{(p-1)} + (2p-d+2)\partial_u\tilde{\phi}^{(p)} = 0. \tag{2.14}$$

In the source free case, $\tilde{\phi}^{(p)}$ must have polynomial dependence in $u$ with degree no higher than $p-d+3$ in order for the expansion to terminate at order $d-3$. However, this restriction will not apply when source terms or nonlinear terms are present.

*Remark* 2.1. *Note that the lower limit of the sum in (2.11) was taken to be radiative order, $n = d/2 - 1$. However, the ansatz would not be changed if we allowed the lower limit of the sum to extend to $n = 1$ for $d > 4$ because the recursion relation eq.* (2.13) *at $n = d/2 - 1$ yields*

$$[\mathscr{D}^2 - (d/2-2)(d/2-1)]\phi^{(d/2-2)} = 0 \tag{2.15}$$

*which implies $\phi^{(d/2-2)} = 0$. The recursion relations at smaller $n$ then successively yield $\phi^{(n)} = 0$ for all $n < d/2 - 1$. Similarly, the lower limit of the first sum in (2.12) could be taken to be $n = 1/2$ without affecting the ansatz. The upper limit of the sums appearing in (2.11) and (2.12) were taken to be $\infty$ for convenience. Most of our analysis will concern the behavior of fields at Coulombic order and slower fall-off and only a small number of derivatives will be taken, so the asymptotic expansion need hold only to the corresponding order.*

Finally, we address the issue of the reasonableness of our ansatz, i.e., what classes of solutions to eq. (2.7) satisfy our ansatz. In Minkowski spacetime of both even and odd[2] dimensions, there is an alternative criterion of smoothness of the conformally rescaled field $\overline{\phi} = \Omega^{-(d/2-1)}\phi$ at future null infinity, $\mathscr{I}^+$, in the conformally completed spacetime. Since $\Omega = 1/r$ is a suitable conformal factor for Minkowski spacetime, it is easily seen that smoothness of $\overline{\phi}$ at $\Omega = 0$ is equivalent to our asymptotic expansion eq. (2.11) in even dimensions and our asymptotic expansion eq. (2.12) without the integer power terms in odd dimensions. By the argument[3] of Prop. 11.1.1 of [37], smoothness at $\mathscr{I}^+$ holds for all

2. Future null infinity does not exist for an odd dimensional radiating spacetime [43], but it exists for odd dimensional Minkowski spacetime.

3. Prop. 11.1.1 of [37] is stated for $d = 4$ but is easily generalized to Minkowski spacetime of arbitrary

solutions to eq. (2.7) with smooth initial data of compact support. Thus, all solutions with initial data of compact support satisfy our ansatz. Furthermore, static, asymptotically flat solutions satisfy the asymptotic expansion eq. (2.11) in even dimensions and the asymptotic expansion eq. (2.12) with only the integer power terms in odd dimensions. It follows that in both even and odd dimensions, all solutions to eq. (2.7) with smooth initial that corresponds to a static asymptotically flat solution outside of a compact region satisfy our ansatz.

## 2 Solutions to the scalar wave recursion relations

We now consider the scalar wave equation with smooth source $S$

$$\Box\phi = S. \tag{2.16}$$

We assume that $S$ also has an expansion in powers of $1/r$. In order that the flux of $S$ through a sphere near null infinity be finite in the limit as $r \to \infty$, we must have $S = O(1/r^{d-2})$. We take as our ansatz for $S$

$$S \sim \sum_{n=d-2}^{\infty} \frac{1}{r^n} S^{(n)}(u, x^A). \tag{2.17}$$

In even dimensions, the sum ranges over integer $n$. In odd dimensions, we could also allow half-integral powers of $1/r$ in the expansion of $S$, beginning at order $1/r^{d-5/2}$. Indeed, for nonlinear equations, half-integral powers would appear as an effective source generated by cubic and higher order terms in the field, although these terms would first enter only at order $1/r^{3(d/2-1)}$. However, we will be primarily interested in the behavior of solutions $\phi$ at fall-off ranging from radiative ($1/r^{d/2-1}$) to Coulombic ($1/r^{d-3}$) orders. In odd dimensions, only the leading order source term $S^{(d-2)}/r^{d-2}$ will enter our analysis. Therefore, for notational simplicity, we will take the sum in eq. (2.17) to range only over integer values of $n$ in both

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dimension

even and odd dimensions. Note that our asymptotic expansion takes account only of sources "near null infinity." Sources that go out to infinity along, e.g., timelike inertial trajectories do not contribute at all to the asymptotic expansion of $S$.

In even dimensions, under the ansatz eq. (2.11), the recursion relations eq. (2.13) are modified by the source term to become

$$[\mathscr{D}^2 + (n-1)(n-d+2)]\phi^{(n-1)} + (2n-d+2)\partial_u \phi^{(n)} = S^{(n+1)}. \tag{2.18}$$

In odd dimensions, under the ansatz eq. (2.12), eq. (2.13) is unmodified, but eq. (2.14) is modified to become

$$[\mathscr{D}^2 + (p-1)(p-d+2)]\tilde{\phi}^{(p-1)} + (2p-d+2)\partial_u \tilde{\phi}^{(p)} = S^{(p+1)}. \tag{2.19}$$

It should be noted that when $d = 4$, eq. (2.18) for $n = 1$ yields $S^{(2)} = 0$. Thus, for $d = 4$ there is an inconsistency with our ansatz eq. (2.11) when $S^{(2)} \neq 0$, i.e., when there is nonvanishing flux of the source through spheres near null infinity. This could be accommodated by modifying the ansatz in $d = 4$ to allow an additional series of terms that fall as $\ln r/r^n$. This issue will arise in the next subsections when we consider whether the Lorenz gauge condition can be imposed on electromagnetic fields and linearized gravitational perturbations, and we will see that a non-vanishing flux of charge current or stress energy will provide an obstruction to imposing the Lorenz gauge in $d = 4$ in a manner compatible with our ansatz. Similarly, in full, nonlinear general relativity, we will find that a non-vanishing flux of stress energy or Bondi news will provide an obstruction to imposing the harmonic gauge in $d = 4$ in a manner compatible with our ansatz. Rather than include any such additional $\ln r$ terms in these cases, we will simply not impose the Lorenz and harmonic gauges in $d = 4$ when these obstructions exist. For the analysis of this subsection, we will simply restrict consideration to the case that $S^{(2)} = 0$ when $d = 4$, so that our ansatz can be imposed.

We now consider two procedures for solving the above recursion relations. The first procedure is as follows: Consider, first, the even dimensional case, where we must solve eq. (2.18) with integral $n$. By our ansatz for $\phi$ and $S$, this equation automatically holds for $n = d/2 - 1$, since $\phi^{(d/2-2)} = S^{(d/2)} = 0$ and the coefficient of $\partial_u \phi^{(d/2-1)}$ vanishes. (Here, when $d = 4$, we have assumed that $S^{(2)} = 0$.) Thus, we may specify $\phi^{(d/2-1)}(u, x^A)$ arbitrarily. The $n = d/2$ equation then yields

$$2\partial_u \phi^{(d/2)} = S^{(d/2+1)} - [\mathscr{D}^2 + (n-1)(n-d+2)]\phi^{(d/2-1)}. \tag{2.20}$$

The right side is "known," so this equation can be straightforwardly integrated to obtain $\phi^{(d/2)}$. The solution is unique up to the arbitrary specification of $\phi_0^{(d/2)}(x^A) = \phi^{(d/2)}(u_0, x^A)$ at the retarded time $u = u_0$. This procedure can then be iterated indefinitely to solve for $\phi^{(n)}$ for all $n > d/2 - 1$ up to the arbitrary specification of $\phi_0^{(n)}(x^A) = \phi^{(n)}(u_0, x^A)$.

In odd dimensions, we must solve eq. (2.13) with half-integral $n$ as well as eq. (2.19). To solve eq. (2.13), we may again, specify $\phi^{(d/2-1)}(u, x^A)$ arbitrarily. We may then again uniquely solve for $\phi^{(n)}$ for all $n > d/2 - 1$ up to the arbitrary specification of $\phi_0^{(n)}(x^A) = \phi^{(n)}(u_0, x^A)$. Similarly, we can uniquely solve eq. (2.19) with $p = d - 3$ for $\tilde{\phi}^{(d-3)}$, up to the arbitrary specification of $\tilde{\phi}_0^{(d-3)}(x^A) = \tilde{\phi}^{(d-3)}(u_0, x^A)$. We can then perform a similar iteration to obtain $\tilde{\phi}^{(p)}$ for all $p > d - 3$, up to the arbitrary specification of $\tilde{\phi}_0^{(p)}(x^A) = \tilde{\phi}^{(p)}(u_0, x^A)$.

We summarize these results in the following proposition.

**Proposition 2.1.** *Let $\phi$ be given by the asymptotic expansion eq. (2.11)-(2.12) and let $S$ be given by the asymptotic expansion eq.* (2.17). *Assume further that for $d = 4$ we have $S^{(2)} = 0$. Then, in even dimensions, a unique solution to the recursion relations eq.* (2.18) *is obtained by arbitrarily specifying* $\phi^{(d/2-1)}(u, x^A)$ *(i.e., specifying* $\phi$ *at "radiative order") and arbitrarily specifying* $\phi^{(n)}(u_0, x^A)$ *for all* $n > d/2 - 1$ *at some initial time* $u_0$*. Similarly, in odd dimensions, a unique solution to the recursion relations eq.* (2.13) *and eq.* (2.19) *is obtained by arbitrarily specifying* $\phi^{(d/2-1)}(u, x^A)$ *(i.e., specifying* $\phi$ *at "radiative order") and arbitrarily specifying both* $\phi^{(n)}(u_0, x^A)$ *for all* $n > d/2 - 1$ *and* $\tilde{\phi}^{(p)}(u_0, x^A)$ *for all* $p \geq d - 3$ *at some initial time* $u_0$*.*

The second procedure involves solving the recursion relations in the reverse order. Suppose that, for some $n > d/2-1$, we specify $\phi^{(n)}(u, x^A)$ arbitrarily. We can then try to solve eq. (2.18) for $\phi^{(n-1)}$. In order to do so, we must invert the angular operator $\mathscr{D}^2 + (n-1)(n-d+2)$. A unique inverse of this operator exists whenever $-(n-1)(n-d+2)$ is not an eigenvalue of the Laplacian, $\mathscr{D}^2$. Since the eigenvalues of $\mathscr{D}^2$ are $-\ell(\ell + d - 3)$ for $\ell = 0, 1, \ldots$, it can be seen that this operator is invertible at every order in odd dimensions, where $n$ is half-integer. On the other hand in even dimensions, this operator is invertible when $n \leq d - 3$, but it is not invertible when $n > d - 3$. Thus, in even dimensions, we can specify $\phi^{(d-3)}(u, x^A)$ arbitrarily and then uniquely solve for $\phi^{(d-4)}(u, x^A)$ by inverting the angular operator in eq. (2.18). Iterating this process, we uniquely obtain $\phi^{(n)}(u, x^A)$ for all $n < d - 3$. We then can solve for $\phi^{(n)}(u, x^A)$ for all $n > d - 3$ as before, with the freedom to arbitrarily specify $\phi^{(n)}(u_0, x^A)$. In odd dimensions, we can similarly arbitrarily specify $\phi^{(n_0)}(u, x^A)$ for *any* half-integer $n_0 \geq d/2 - 1$. We can then uniquely solve for $\phi^{(n)}(u, x^A)$ for all $n < n_0$ by inversion of the angular operators, and then solve for $\phi^{(n)}(u, x^A)$ for all $n > n_0$ as before, with the freedom to arbitrarily specify $\phi^{(n)}(u_0, x^A)$. This can be summarized as follows:

**Proposition 2.2.** *Let* $\phi$ *be given by the asymptotic expansion eq. (2.11)-(2.12) and let* $S$ *be given by the asymptotic expansion eq.* (2.17)*. Assume further that for* $d = 4$ *we have* $S^{(2)} = 0$*. Then, in even dimensions, a unique solution to the recursion relations eq.* (2.18) *is obtained by arbitrarily specifying* $\phi^{(d-3)}(u, x^A)$ *(i.e., specifying* $\phi$ *at "Coulombic order") and arbitrarily specifying* $\phi^{(n)}(u_0, x^A)$ *for all* $n > d - 3$ *at some initial time* $u_0$*. Similarly, in odd* *dimensions, a unique solution to the recursion relations eq.* (2.13) *and eq.* (2.19) *is obtained by arbitrarily specifying* $\phi^{(n_0)}(u, x^A)$ *for any half-integral* $n_0$*, and, for some initial time* $u_0$*, arbitrarily specifying* $\phi^{(n)}(u_0, x^A)$ *for all* $n > n_0$ *and* $\tilde{\phi}^{(p)}(u_0, x^A)$ *for all* $p \geq d - 3$*.*

An important corollary of the argument leading to Proposition 2.2 is the following:

**Corollary 2.1.** *Suppose for* $d$ *even we have* $\partial_u \phi^{(n_0)} = 0$ *for some* $n_0 < d-3$*. Then* $\phi^{(n)} = 0$ *for all* $n < n_0$*. Similarly, if* $\partial_u \phi^{(d-3)} = 0$ *and* $S^{(d-2)} = 0$*, then* $\phi^{(n)} = 0$ *for all* $n < d - 3$*. For* $d$ *odd, if* $\partial_u \phi^{(n_0)} = 0$ *for some half-integral* $n_0$ *(without restriction), then* $\phi^{(n)} = 0$ *for all* $n < n_0$*.*

Finally, it is worth noting that for $n > d - 3$, the spherical harmonic $Y_{n-d+2,m}$ is in the kernel of $\mathscr{D}^2 + (n-1)(n-d+2)$. It follows immediately that in the source-free case, for $d$ even we have that

$$\alpha^d_{nm} \equiv \int_{\mathbb{S}^2} d\Omega \; Y_{n-d+2,m} \phi^{(n)} \tag{2.21}$$

is a constant of motion for all $n > d - 3$ [49, 50], i.e., $\partial_u \alpha^d_{nm} = 0$, where $d\Omega$ is the measure on the $(d-2)-$sphere. Similarly, in the source free case, for $d$ odd we have that

$$\tilde{\alpha}^d_{pm} \equiv \int_{\mathbb{S}^2} d\Omega \; Y_{p-d+2,m} \tilde{\phi}^{(p)} \tag{2.22}$$

is a constant of motion for all $p > d - 3$.

## *3 Maxwell's equations*

Consider Maxwell's Equations with vector potential $A_a$ and charge-current $j_a$ on $d-$dimensional Minkowski spacetime

$$\Box A_\mu - \partial_\mu \partial^\nu A_\nu = -4\pi j_\mu \tag{2.23}$$

where $\partial^\mu j_\mu = 0$. In analogy with the scalar field ansatz (2.11) and (2.12), we assume as an ansatz that there exists a choice of gauge for $A_a$ such that it admits an asymptotic expansion of the form

$$A_\mu \sim \sum_{n=d/2-1}^{\infty} \frac{1}{r^n} A_\mu^{(n)}(u, x^A) \qquad d \text{ even} \tag{2.24}$$

$$A_\mu \sim \sum_{n=d/2-1}^{\infty} \frac{1}{r^n} A_\mu^{(n)}(u, x^A) + \sum_{p=d-3}^{\infty} \frac{1}{r^p} \tilde{A}_\mu^{(p)}(u, x^A) \qquad d \text{ odd.} \tag{2.25}$$

We further assume, in analogy with eq. (2.17) that $j_a$ admits an asymptotic expansion of the form

$$j_\mu \sim \sum_{n=d-2}^{\infty} \frac{1}{r^n} j_\mu^{(n)}(u, x^A). \tag{2.26}$$

In addition, we require that $j_\mu^{(d-2)}(u, x^A) \to 0$ as $u \to -\infty$, i.e. there is no current flux to future null infinity at asymptotically early times. Here, as already mentioned at the end of the Introduction, $A_\mu^{(n)}$, $\tilde{A}_\mu^{(n)}$, and $j_\mu^{(n)}$ are defined so that their *normalized* basis components are independent of $r$—in contrast to a more common convention where the orders of the expansion would denote the powers of $1/r$ occurring in the expansion of coordinate basis components of $A_\mu$ in the coordinates of eq. (2.2). Thus, in our convention, $A_r^{(n)}$, $A_u^{(n)}$, and $A_A^{(n)}$ all contribute to the *physical* fall off rate of $1/r^n$, i.e., $A_A^{(n)}$ is the $1/r^n$ part of $1/r(\partial/\partial x^A)^\mu A_\mu$, not the $1/r^n$ part of $(\partial/\partial x^A)^\mu A_\mu$. Our convention avoids a spurious "mixing of orders" in equations due to the different behavior of the coordinate basis elements. Again, our assumption that upper limits in the above asymptotic expansions run to $\infty$ is for convenience, as only finitely many orders will be needed for our main results.

In even[4] dimensions, we now compare our ansatz eq. (2.24) to what would be obtained

4. We are not aware of any smoothness at $\mathscr{I}^+$ criterion for $A_\mu$ that can be formulated in odd dimensions, since $A_\mu$ itself cannot be smooth at $\mathscr{I}^+$ for radiating solutions and giving $A_\mu$ a conformal weight would not appear to be of any use since Maxwell's equations are not conformally invariant when $d \neq 4$.

by requiring that $A_\mu$ (with no conformal weight) be smooth at $\mathscr{I}^+$. Since $\Omega = 1/r$ is a suitable conformal factor for Minkowski spacetime, the necessary and sufficient condition for smoothness of $A_\mu$ at $\mathscr{I}^+$ is that its components, $(A_u, A_\Omega, A_A)$, defined by

$$A = A_u du + A_\Omega d\Omega + A_A dx^A \tag{2.27}$$

be smooth functions of $(u, \Omega, x^A)$ at $\Omega = 0$. For $d = 4$, it is easily seen that this smoothness criterion differs from the asymptotic expansion eq. (2.24) only in that the smoothness criterion (i) allows a 0th order term, $A_u^{(0)}$, in $A_u$ and (ii) requires $A_r^{(1)} = 0$. It is easily seen that $A_u^{(0)}$ can be set to zero by a gauge transformation, so smoothness at $\mathscr{I}^+$ implies that our ansatz eq. (2.24) holds. Conversely, we show in Appendix 0.1 that starting from our ansatz eq. (2.24), one can set $A_r^{(1)} = 0$ by a gauge transformation if and only if[5] $j_r^{(3)} = 0$. Thus, for $d = 4$ our ansatz eq. (2.24) is slightly weaker than smoothness at $\mathscr{I}^+$ in that it admits additional solutions with $j_r^{(3)} \neq 0$.

In higher even dimensional spacetimes, eq. (2.24) requires strictly faster fall-off than needed for smoothness of $A_a$ (with no conformal weighting) at $\mathscr{I}^+$. Thus, eq. (2.24) is nominally stronger than the condition of smoothness of $A_\mu$ at $\mathscr{I}^+$. However, we show in Appendix 0.1 that the Lorenz gauge condition can be imposed when $d > 4$ within a slower fall-off ansatz. As explained in Remark 2.2, the slower fall-off solutions excluded by eq. (2.24) are therefore pure gauge. Thus, in even dimensional spacetimes with $d > 4$, our ansatz is exactly equivalent to smoothness of $A_\mu$ (in some gauge) at $\mathscr{I}^+$.

In the following, we will focus on the even dimensional case, and then indicate how the arguments can be modified to accommodate the odd dimensional case. Just as in the scalar case, Maxwell's equations give rise to recursion relations for the coefficients of the asymptotic

5. $j_r^{(3)}$ must be independent of $u$ by conservation of current. A nonvanishing $j_r^{(3)}$ would correspond to having an *ingoing* null current near $\mathscr{I}^+$.

expansions eq. (2.24) and eq. (2.26). In the even dimensional case, these recursion relations are explicitly

$$\Big[\mathscr{D}^2 + (n-1)(n-d+2)\Big]A_u^{(n-1)} + (2n-d+2)\partial_u A_u^{(n)} - \partial_u \psi^{(n+1)} = -4\pi j_u^{(n+1)} \quad (2.28)$$

$$\Big[\mathscr{D}^2+n(n-d+1)\Big]A_r^{(n-1)}+(d-2)A_u^{(n-1)}+(2n-d+2)\partial_u A_r^{(n)}-2\mathscr{D}^A A_A^{(n-1)}+n\psi^{(n)} = -4\pi j_r^{(n+1)} \quad (2.29)$$

$$\Big[\mathscr{D}^2+(n-1)(n-d+2)-1\Big]A_A^{(n-1)}-2\mathscr{D}_A\Big(A_u^{(n-1)}-A_r^{(n-1)}-\frac{\psi^{(n)}}{2}\Big)+(2n-d+2)\partial_u A_A^{(n)} = -4\pi j_A^{(n+1)} \quad (2.30)$$

where $n$ takes integer values. Here, we have defined

$$\psi \equiv \partial^\mu A_\mu \quad (2.31)$$

so

$$\psi^{(n)} = \mathscr{D}^A A_A^{(n-1)} + (d-n-1)(A_r^{(n-1)} - A_u^{(n-1)}) - \partial_u A_r^{(n)}. \quad (2.32)$$

It would be very convenient to put $A_\mu$ in Lorenz gauge, $\psi = 0$. On general grounds, we know that $A_\mu$ can always be put in the Lorenz gauge, but it is not obvious *a priori* whether it can be put in Lorenz gauge in such a way that the form of the asymptotic expansions, eq. (2.24) is maintained. We now investigate this issue.

Under a gauge transformation, we have

$$A_\mu \to A_\mu - \partial_\mu \phi. \quad (2.33)$$

Thus, in order to put $A_\mu$ in Lorenz gauge, we must solve

$$\Box \phi = \psi. \quad (2.34)$$

Thus, the equation that we must solve is of the same form as eq. (2.16), which we analyzed in the previous section. However, there are two key differences: (i) From its definition, *a priori*, $\psi$ may fall off as slowly as $1/r^{d/2-1}$ rather than $1/r^{d-2}$. (ii) We do not require that $\phi$ satisfy the ansatz eq. (2.11) but rather that $\partial_\mu \phi$ satisfy the ansatz eq. (2.24). Therefore, we may take the ansatz for $\phi$ to be

$$\phi \sim \sum_{n=d/2-2}^{\infty} \frac{1}{r^n} \phi^{(n)}(u, x^A) \tag{2.35}$$

where $\partial_u \phi^{(d/2-2)} = 0$. In $d = 4$ dimensions, we may also add the term $c \ln r$ to the ansatz for $\phi$, where $c$ is a constant.

We first note that it follows immediately from $\partial^\mu j_\mu = 0$ that $\partial_u j_r^{(d-2)} = 0$. Hence, if $j_a^{(d-2)} \to 0$ as $u \to -\infty$ as we have assumed in our ansatz above, we have

$$j_r^{(d-2)} = 0. \tag{2.36}$$

Thus, the $r$-component of $j_\mu$ falls off at least one power of $1/r$ faster than required by the ansatz eq. (2.26). Since $d/2 \leq d-2$ for all $d \geq 4$, it follows immediately from eq. (2.29) with $n = d/2 - 1$ that

$$\psi^{(d/2-1)} = 0, \tag{2.37}$$

i.e., Maxwell's equations require $\psi$ to fall off at least one power of $1/r$ faster than implied by the ansatz (2.24). To proceed further, we must separately consider the cases $d > 4$ and $d = 4$.

When $d > 4$ all components of $j_a$ vanish at order $n = d/2$. It follows from eq. (2.28) with $n = d/2 - 1$ that

$$\partial_u \psi^{(d/2)} = 0. \tag{2.38}$$

We now can solve the scalar recursion relation eq. (2.18) at order $n = d/2 - 1$ by allowing a nonvanishing $\phi^{(d/2-2)}$ given by

$$\phi^{(d/2-2)} = [\mathscr{D}^2 - (d/2-2)^2]^{-1}\psi^{(d/2)}. \tag{2.39}$$

Although $\phi^{(d/2-2)}$ falls off more slowly than allowed by the ansatz eq. (2.11), since $\partial_u\phi^{(d/2-2)} = 0$ the gradient of $\phi^{(d/2-2)}/r^{d/2-2}$ will be compatible with the ansatz eq. (2.24). Furthermore, since $\partial_u\phi^{(d/2-2)} = 0$, the scalar recursion relations imply that all slower fall-off terms vanish. We may now specify $\phi^{(d/2-1)}$ arbitrarily and solve the recursion relations for the faster fall-off terms in the same manner as in Proposition 2.1. Thus, when $d > 4$, there is no difficulty in putting $A_a$ in the Lorenz gauge in a manner compatible with the ansatz eq. (2.24).

When $d = 4$, we still have $\psi^{(1)} = 0$ but we now have

$$\partial_u\psi^{(2)} = 4\pi j_u^{(2)}. \tag{2.40}$$

The scalar recursion relation eq. (2.18) at order $n = 1$ (with the term $c\ln r$ added to the ansatz for $\phi$) yields

$$c + \mathscr{D}^2\phi^{(0)} = \psi^{(2)}. \tag{2.41}$$

However, $\phi^{(0)}$ has to be $u$-independent in order that $\partial_a\phi$ satisfy the ansatz eq. (2.24). This requires $\partial_u\psi^{(2)}$ to vanish and hence $j_u^{(2)} = 0$, i.e., there can be no flux of charge to infinity.[6] Conversely, if $j_u^{(2)} = 0$, then $\psi^{(2)}$ is $u$-independent. We can choose $c$ to cancel the $\ell = 0$ part of $\psi^{(2)}$. We can then invert $\mathscr{D}^2$ to solve for $\phi^{(0)}$. Thus, for $d = 4$, we can solve eq. (2.41) if and only if $j_u^{(2)} = 0$. We may then choose $\phi^{(1)}$ arbitrarily and solve the remaining recursion relations for the faster fall-off terms in the same manner as in Proposition 2.1. Thus, for $d = 4$, $A_\mu$ can be put in the Lorenz gauge in a manner compatible with the ansatz eq. (2.24) if and only if $j_u^{(2)} = 0$.

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6. The Lorenz gauge can be imposed with $j_u^{(2)} \neq 0$ by adding a series with terms of the form $\ln r/r^n$ [51].

We now describe the modifications to the above results for odd dimensions. The recursion relations for $A_\mu^{(n)}$ take the form eqs. (2.28)-(2.30) with $n$ half-integral and with the current source terms absent, whereas the recursion relations for $\tilde{A}_\mu^{(p)}$ take the same form as eqs. (2.28)-(2.30) with $n$ replaced by $p$, with $p$ an integer. The ansatz for $\phi$ is taken to be

$$\phi \sim \sum_{n=d/2-2}^{\infty} \frac{1}{r^n} \phi^{(n)}(u, x^A) + \sum_{p=d-3}^{\infty} \frac{1}{r^p} \tilde{\phi}^{(p)}(u, x^A) \tag{2.42}$$

with $\partial_u \phi^{(d/2-2)} = 0$. The analysis of imposing the Lorenz gauge then proceeds in close parallel to the even dimensional case for $d > 4$. We find that the Lorenz gauge can always be imposed in a manner compatible with the ansatz eq. (2.25).

We summarize our above results on the imposition of the Lorenz gauge in the following proposition:

**Proposition 2.3.** *In Minkowski spacetime of dimension $d \geq 4$, suppose that in some gauge the vector potential $A_\mu$ satisfies our ansatz eq. (2.24) (for $d$ even) or our ansatz eq. (2.25) (for $d$ odd). Suppose further that the charge-current $j_\mu$ satisfies eq.* (2.26) *and that $j_\mu^{(d-2)}(u, x^A) \to 0$ as $u \to -\infty$. Then for all $d > 4$, $A_\mu$ can be put in the Lorenz gauge in such a way that it continues to satisfy our ansatz. In $d = 4$ the Lorenz gauge condition can be imposed within the ansatz eq. (2.24) if and only if $j_u^{(2)} = 0$, i.e., if and only if the flux of charge to null infinity vanishes.*

*Remark* 2.2. *We show in Appendix 0.1 that if, for $d > 4$, we had allowed the sum in eq. (2.24) to extend to $n = 1$ and the sum in eq. (2.25) to extend to $n = 1/2$, our proof that the Lorenz gauge condition can be imposed within the revised ansatz would still go through. Since $\Box A_\mu = -4\pi j_\mu$ in the Lorenz gauge and $j_\mu^{(n)} = 0$ for $n < d - 2$, it follows from Remark 2.1 that in Lorenz gauge, we have $A_\mu^{(n)} = 0$ for all $n < d/2 - 1$. Thus, the only solutions excluded by starting the sums at $n = d/2 - 1$ in eqs. (2.24) and (2.25) (rather than at $n = 1$ and $n = 1/2$) are pure gauge.*

*Remark* 2.3. *Suppose that $A_\mu$ satisfies the ansatz eq. (2.24) and is stationary at all orders $n \leq m$ where $m \leq d-2$. Suppose further that $j_\mu^{(n)} = 0$ for all $n \leq m+1$. By conservation of $j_\mu$, we obtain $\partial_u j_r^{(n)} = 0$ for all $n \leq m+2$. It follows directly from its definition, eq.* (2.32)*, that $\psi^{(n)}$ must be stationary for all $n \leq m$. However, using eq.* (2.29) *and the stationarity of $j_r^{(n)}$ for all $n \leq m+2$, we obtain the stronger result that $\psi^{(n)}$ actually must be stationary for all $n \leq m+1$. We then may solve the recursion relation eq.* (2.18) *for all $n \leq m-1$ by setting*

$$\phi^{(n-1)} = [\mathscr{D}^2 + (n-1)(n-d+2)]^{-1}\psi^{(n+1)}. \tag{2.43}$$

*We may then set $\phi^{(m-1)} = 0$ and solve the recursion relations for $\phi^{(n)} = 0$ for $n \geq m$ as in Proposition 2.1. The resulting gauge transformation will put $A_\mu$ in the Lorenz gauge satisfying the ansatz eq. (2.24) and maintaining stationarity at all orders $n \leq m$. In particular, if a solution with $j_\mu = 0$ is stationary in some gauge to order $m \leq d-2$, then it is stationary in a Lorenz gauge to the same order.*

*Remark* 2.4. *Let $d = 4$ and suppose $j_u^{(2)} = 0$. Suppose, further, that $j_r^{(3)} = 0$ so that, as shown in Appendix 0.1, our ansatz is equivalent to smoothness of $A_\mu$ at $\mathscr{I}^+$ in some gauge. Although, by Prop. 2.3, the Lorenz gauge can be imposed within our ansatz eq. (2.24), it need not be the case that $A_r^{(1)} = 0$ in the Lorenz gauge, in which case $A_\mu$ in the Lorenz gauge will not be smooth at $\mathscr{I}^+$. In other words, in $d = 4$ when $j_u^{(2)} = 0$, the Lorenz gauge is compatible with our ansatz but it need not be compatible with smoothness of $A_\mu$ at $\mathscr{I}^+$.*

When $A_\mu$ is in Lorenz gauge—as, by Proposition 2.3 we may assume for $d > 4$ and for $d = 4$ when $j_u^{(2)} = 0$—it satisfies

$$\Box A_\mu = -4\pi j_\mu \tag{2.44}$$

$$\partial^\mu A_\mu = 0. \tag{2.45}$$

The recursion relations arising from $\Box A_\mu = -4\pi j_\mu$ are just eqs. (2.28)-(2.30) with $\psi = 0$ in even dimensions. (They are modified as described above in odd dimensions.) The recursion relations arising from $\partial^\mu A_\mu = 0$ are just $\psi^{(n)} = 0$ where $\psi^{(n)}$ is given by eq. (2.32). However, it is more convenient to work with a linear combination of this equation and the other equations so as to eliminate all $u$-derivatives. This can be achieved by defining

$$\omega = K^\mu[\Box A_\mu + 4\pi j_\mu] - 2K^\mu \partial_\mu \psi - (d-2)\psi/r \tag{2.46}$$

where $K^\mu = (\partial/\partial r)^\mu$. When eq. (2.44) holds, the vanishing of $\omega$ is equivalent to the vanishing of $\psi$. The relation $\omega^{(n+2)} = 0$ yields

$$[\mathscr{D}^2 - (n-d+2)(n-d+3)]A_r^{(n)} + (2n-d+2)(n-d+3)A_u^{(n)} + (2n-d+2)\mathscr{D}^A A_A^{(n)} = -4\pi j_r^{(n+2)} \tag{2.47}$$

which contains no $u$-derivatives (and therefore also does not mix different orders).

We now consider the analogs of Propositions 2.1 and 2.2 for Maxwell's equations in Lorenz gauge. By eq. (2.44) each Cartesian component of $A_\mu$ satisfies the scalar wave equation. Therefore, we may directly apply Propositions 2.1 and 2.2 to determine the data needed to uniquely determine a solution to eq. (2.44) alone. Thus, the remaining task is to specify this data in such a way that eq. (2.45) holds. However, if eq. (2.44) holds we have

$$\Box\psi = \Box\partial^\mu A_\mu = \partial^\mu \Box A_\mu = -4\pi\partial^\mu j_\mu = 0. \tag{2.48}$$

Thus, $\psi$ satisfies the homogeneous scalar wave equation, and we can ensure that $\psi = 0$ by choosing data for $A_\mu$ so as to ensure that the corresponding data for $\psi$ yields the solution $\psi = 0$. Again, we can determine this using Propositions 2.1 and 2.2, and also using the fact that when eq. (2.44) holds, the vanishing of $\psi^{(n)}$ is equivalent to the vanishing of $\omega^{(n+1)}$. Putting all of the above statements together, it follows using Proposition 2.1 that a unique solution to Maxwell's equations in Lorenz gauge can be determined by specifying $A_\mu^{(d/2-1)}$

subject to eq. (2.47) for $n = d/2 - 1$, and then specifying $A_\mu^{(n)}(u_0)$ for all $n > d/2 - 1$ subject to eq. (2.47) holding at $u = u_0$ (see exercise 2 of [47] for the case $d = 4$ with $j_\mu = 0$). In odd dimensions, we also must similarly specify data for $\tilde{A}_\mu^{(p)}$ at $u = u_0$ subject to the constraint for all $p$.

Alternatively, in even dimensions, using Proposition 2.2, a solution can be uniquely determined by specifying data at Coulombic order, $A_\mu^{(d-3)}$. However, in this case, the constraint eq. (2.47) at $n = d - 3$ ensures that $\psi^{(d-2)} = 0$ but this does not quite suffice to ensure that $\psi$ vanishes at all slower fall-off. This is because the recursion relation eq. (2.13) for $n = d - 2$ yields

$$\mathscr{D}^2\psi^{(d-3)} = -(d-4)\partial_u\psi^{(d-2)} = 0 \tag{2.49}$$

which does not imply that the $\ell = 0$ part of $\psi^{(d-3)}$ must vanish. Hence, the condition

$$[\psi^{(d-3)}]|_{\ell=0} = 0 \tag{2.50}$$

must be imposed separately. Using eq. (2.44), we may write this condition purely in terms of the Coulombic order data as

$$\partial_u\mathcal{Q}(u) = -\mathcal{A}_d j_u^{(d-2)}|_{\ell=0} \tag{2.51}$$

where

$$\mathcal{Q}(u) = \frac{\mathcal{A}_d}{4\pi}[A_r^{(d-3)} + (d-4)A_u^{(d-3)}]|_{\ell=0} \qquad d \text{ even (in Lorenz gauge)} \tag{2.52}$$

and $\mathcal{A}_d$ is the area of a unit $(d-2)-$sphere

$$\mathcal{A}_d = \frac{2\pi^{\frac{d-1}{2}}}{\Gamma(\frac{d-1}{2})}. \tag{2.53}$$

Using the Lorenz gauge condition, it can be verified that $\mathcal{Q}(u)$ is the total electric charge at time $u$, defined by

$$\mathcal{Q}(u) \equiv \frac{1}{4\pi} \int_{\mathbb{S}^2} d\Omega \; F_{ur}^{(d-2)} \tag{2.54}$$

with $F_{\mu\nu} = 2\partial_{[\mu} A_{\nu]}$. Thus, eq. (2.51) expresses conservation of charge. Note that the formula eq. (2.52) for $\mathcal{Q}(u)$ holds only in the Lorenz gauge and thus cannot be used in $d = 4$ when $j_u^{(2)} \neq 0$. In odd dimensions, we do not obtain a similar additional constraint, but eq. (2.51) follows directly from the recursion relation for $\tilde{A}_a^{(d-3)}$ corresponding to eq. (2.28) with $p = d-3$ as well as the Lorenz gauge condition given by eq. (2.32) with $p = d-3$, where the charge is now given by

$$\mathcal{Q}(u) = \frac{\mathcal{A}_d}{4\pi} [\tilde{A}_r^{(d-3)} + (d-4)\tilde{A}_u^{(d-3)}]|_{\ell=0} \qquad d \text{ odd (in Lorenz gauge).} \tag{2.55}$$

We summarize our results as follows:

**Theorem 1.** *Suppose $d > 4$ or $d = 4$ and $j_u^{(2)} = 0$, so that the Lorenz gauge condition can be imposed. Then a unique solution to the recursion relations and constraints for Maxwell's equations in the Lorenz gauge is obtained by specifying data in either of the following two ways:*

*(1)* Radiative Order Data: *Specify $A_\mu^{(d/2-1)}(u, x^A)$ subject to the constraint eq.* (2.47) *at $n = d/2 - 1$. Specify $A_\mu^{(n)}(u = u_0, x^A)$ for all $n > d/2 - 1$ subject to the constraint eq.* (2.47) *at $u = u_0$. In odd dimensions, also specify $\tilde{A}_\mu^{(p)}(u = u_0, x^A)$ for all $p \geq d - 3$, subject to the constraint eq.* (2.47) *at $u = u_0$.*

*(2)* Coulombic Order Data: *In even dimensions, specify $A_\mu^{(d-3)}(u, x^A)$ subject to the constraint eq.* (2.47) *at $n = d - 3$ and the additional constraint eq.* (2.51)*; specify $A_\mu^{(n)}(u = u_0, x^A)$ for all $n > d - 3$ subject to the constraint eq.* (2.47) *at $u = u_0$. In odd dimensions, specify $A_\mu^{(m)}(u, x^A)$ for any half-integer $m \geq d/2 - 1$, subject to the*

*constraint eq. (2.47) at $n = m$, specify $A_\mu^{(n)}(u = u_0, x^A)$ for all $n > m$ subject to the constraint eq. (2.47) at $u = u_0$; specify $\tilde{A}_\mu^{(p)}(u = u_0, x^A)$ for all $p \geq d - 3$, subject to the constraint eq. (2.47) at $u = u_0$.*

## *4 Linearized Einstein equation*

We consider the linearized Einstein equation on $d$-dimensional Minkowski spacetime for a metric perturbation $h_{\mu\nu}$ with stress-energy source $T_{\mu\nu}$

$$-2\delta G_{\mu\nu} \equiv \Box \overline{h}_{\mu\nu} - 2\partial_{(\mu}\partial^\rho \overline{h}_{\nu)\rho} + \eta_{\mu\nu}\partial^\rho\partial^\sigma \overline{h}_{\rho\sigma} = -16\pi T_{\mu\nu} \tag{2.56}$$

where $\delta G_{\mu\nu}$ is the linearized Einstein tensor and

$$\overline{h}_{\mu\nu} \equiv h_{\mu\nu} - \frac{1}{2} h \eta_{\mu\nu} \tag{2.57}$$

with $h \equiv \eta^{\mu\nu} h_{\mu\nu}$. Our ansatz for $h_{\mu\nu}$ is

$$h_{\mu\nu} \sim \sum_{n=d/2-1}^{\infty} \frac{1}{r^n} h_{\mu\nu}^{(n)}(u, x^A) \qquad d \text{ even} \tag{2.58}$$

$$h_{\mu\nu} \sim \sum_{n=d/2-1}^{\infty} \frac{1}{r^n} h_{\mu\nu}^{(n)}(u, x^A) + \sum_{p=d-3}^{\infty} \frac{1}{r^p} \tilde{h}_{\mu\nu}^{(p)}(u, x^A) \qquad d \text{ odd} \tag{2.59}$$

where our conventions for labeling the orders in this expansion is as in the electromagnetic case. Our ansatz for $T_{\mu\nu}$ is

$$T_{\mu\nu} \sim \sum_{n=d-2}^{\infty} \frac{1}{r^n} T_{\mu\nu}^{(n)}(u, x^A). \tag{2.60}$$

In addition, we require that $T_{\mu\nu}$ satisfy the dominant energy condition and that $T_{\mu\nu}^{(d-2)}(u, x^A) \to 0$ as $u \to -\infty$, i.e. there is no stress energy flux to future null infinity at asymptotically early times. In odd dimensions, it would be reasonable to also allow terms in the expansion of $T_{\mu\nu}$ that fall as half-integral powers of $1/r$—and when we consider the full Einstein's equation, nonlinearities will effectively generate such terms in the equations. However, our analysis will mainly be concerned with the terms in $h_{\mu\nu}$ with fall-off ranging from radiative ($1/r^{d/2-1}$) to Coulombic ($1/r^{d-3}$) orders, for which only the leading order terms in the expansion of $T_{\mu\nu}$ will contribute, so for simplicity, we do not include half-integral powers of $1/r$ in the ansatz for $T_{\mu\nu}$ in odd dimensions.

In even dimensions, we can compare our ansatz eq. (2.58) to what would be obtained by requiring that $\Omega^2 h_{\mu\nu}$ with $\Omega = 1/r$ be smooth at $\mathscr{I}^+$, i.e., at $\Omega = 0$. For $d = 4$, if one assumes smoothness at $\mathscr{I}^+$ in some gauge, then, by a further choice of gauge (see [52] or p.280 of [37]), one can ensure that $h_{\mu\nu}$ satisfies our ansatz eq. (2.58). Conversely, if $h_{\mu\nu}$ satisfies our ansatz, then $\Omega^2 h_{\mu\nu}$ will be smooth at $\mathscr{I}^+$ if and only if $h^{(1)}_{rr}$ vanishes. In Appendix 0.2, we show that we can set $h^{(1)}_{rr} = 0$ by a gauge transformation provided that[7] $T^{(3)}_{ur} = T^{(3)}_{rr} = T^{(3)}_{rA} = 0$. Thus, for $d = 4$, our ansatz is slightly weaker than smoothness of $\Omega^2 h_{\mu\nu}$ at $\mathscr{I}^+$ in that we allow additional solutions with $T^{(3)}_{ra} \neq 0$.

For even dimensional spacetimes with $d > 4$, our ansatz eq. (2.58) requires faster fall-off than what is needed for smoothness of $\Omega^2 h_{\mu\nu}$ at $\mathscr{I}^+$. However, starting with smoothness at $\mathscr{I}^+$ and choosing the conformal Gaussian null gauge, it was shown in [52] that the fall-off given by our ansatz holds; we also will show in Appendix 0.2 that, starting with smoothness of $\Omega^2 h_{\mu\nu}$ at $\mathscr{I}^+$, the Lorenz gauge condition can be imposed, which also implies the faster fall-off given by our ansatz. Thus, in even dimensional spacetimes with $d > 4$, our ansatz is precisely equivalent to smoothness of $\Omega^2 h_{\mu\nu}$ at $\mathscr{I}^+$ in some gauge.

In even dimensions, where $n$ is integer, Einstein's equation gives rise to the following

7. $T^{(3)}_{ur}$, $T^{(3)}_{rr}$, and $T^{(3)}_{rA}$ are independent of $u$ by conservation and the dominant energy condition. These quantities vanish identically if the stress-energy is produced by a scalar or electromagnetic field satisfying our ansatz for those fields.

system of recursion relations:

$$\Big[\mathscr{D}^2+(n-1)(n-d+2)\Big]\overline{h}^{(n-1)}_{uu}+(2n-d+2)\partial_u\overline{h}^{(n)}_{uu}-(d-n-2)(\chi^{(n)}_r-\chi^{(n)}_u)$$
$$-2\partial_u\chi^{(n+1)}_u-\mathscr{D}^A\chi^{(n)}_A+\partial_u\chi^{(n+1)}_r=-16\pi T^{(n+1)}_{uu} \tag{2.61}$$

$$\Big[\mathscr{D}^2+n(n-d+1)\Big]\overline{h}^{(n-1)}_{ur}+(d-2)\overline{h}^{(n-1)}_{uu}+(2n-d+2)\partial_u\overline{h}^{(n)}_{ur}-2\mathscr{D}^A\overline{h}^{(n-1)}_{uA}-\mathscr{D}^A\chi^{(n)}_A$$
$$-(d-n-2)\chi^{(n)}_r+(d-2)\chi^{(n)}_u=-16\pi T^{(n+1)}_{ur} \tag{2.62}$$

$$\Big[\mathscr{D}^2+(n-1)(n-d+2)-1\Big]\overline{h}^{(n-1)}_{uA}-2\mathscr{D}_A(\overline{h}^{(n-1)}_{uu}-\overline{h}^{(n-1)}_{ur})+(2n-d+2)\partial_u\overline{h}^{(n)}_{uA}$$
$$-\mathscr{D}_A\chi^{(n)}_u-\partial_u\chi^{(n+1)}_A=-16\pi T^{(n+1)}_{uA} \tag{2.63}$$

$$\Big[\mathscr{D}^2+(n-1)(n-d+2)-2(d-2)]\overline{h}^{(n-1)}_{rr}+2(d-2)\overline{h}^{(n-1)}_{ur}+(2n-d+2)\partial_u\overline{h}^{(n)}_{rr}$$
$$+2q^{AB}\overline{h}^{(n-1)}_{AB}-4\mathscr{D}^A\overline{h}^{(n-1)}_{Ar}+2n\chi^{(n)}_r=-16\pi T^{(n+1)}_{rr} \tag{2.64}$$

$$\Big[\mathscr{D}^2+(n-1)(n-d+2)-d-1\Big]\overline{h}^{(n-1)}_{rA}+d\overline{h}^{(n-1)}_{uA}-2\mathscr{D}_A\overline{h}^{(n-1)}_{ur}+(2n-d+2)\partial_u\overline{h}^{(n)}_{rA}$$
$$+2\mathscr{D}_A\overline{h}^{(n-1)}_{rr}-2\mathscr{D}^B\overline{h}^{(n-1)}_{BA}-\mathscr{D}_A\chi^{(n)}_r+(n+1)\chi^{(n)}_A=-16\pi T^{(n+1)}_{rA} \tag{2.65}$$

$$\Big[\mathscr{D}^2+(n-1)(n-d+2)-2\Big]\overline{h}^{(n-1)}_{AB}+2\Big(\overline{h}^{(n-1)}_{rr}-2\overline{h}^{(n-1)}_{ur}+\overline{h}^{(n-1)}_{uu}\Big)q_{AB}$$
$$-4\mathscr{D}_{(A}\Big(\overline{h}^{(n-1)}_{B)u}-\overline{h}^{(n-1)}_{B)r}\Big)+(2n-d+2)\partial_u\overline{h}^{(n)}_{AB}-2\left(\mathscr{D}_{(A}\chi^{(n)}_{B)}-\frac{q_{AB}}{2}\mathscr{D}^C\chi^{(n)}_C\right)$$
$$+(d-n-4)(\chi^{(n)}_r-\chi^{(n)}_u)q_{AB}-q_{AB}\partial_u\chi^{(n+1)}_r=-16\pi T^{(n+1)}_{AB}. \tag{2.66}$$

Here we have defined

$$\chi_a = \partial^b \overline{h}_{ab} \tag{2.67}$$

so that

$$\chi_u^{(n)} = \mathscr{D}^A \overline{h}^{(n-1)}_{Au} + (d-n-1)\Big(\overline{h}^{(n-1)}_{ur} - \overline{h}^{(n-1)}_{uu}\Big) - \partial_u \overline{h}^{(n)}_{ur} \tag{2.68}$$

$$\chi_r^{(n)} = \mathscr{D}^A \overline{h}^{(n-1)}_{Ar} + (d-n-1)\Big(\overline{h}^{(n-1)}_{rr} - \overline{h}^{(n-1)}_{ur}\Big) - q^{AB}\overline{h}^{(n-1)}_{AB} - \partial_u \overline{h}^{(n)}_{rr} \tag{2.69}$$

$$\chi_A^{(n)} = \mathscr{D}^B \overline{h}^{(n-1)}_{AB} + (d-n)\Big(\overline{h}^{(n-1)}_{rA} - \overline{h}^{(n-1)}_{uA}\Big) - \partial_u \overline{h}^{(n)}_{rA}. \tag{2.70}$$

In odd dimensions, where $n$ is half-integral, eqs. (2.61)-(2.66) hold with $T_{\mu\nu} = 0$, whereas the recursion relations for $\tilde{h}^{(p)}_{\mu\nu}$ are the same as eqs. (2.61)-(2.66).

In the electromagnetic case, the current $j_a$ is subject only to the conservation law $\partial^\mu j_\mu = 0$. This gave rise to the condition $K^\mu j^{(d-2)}_\mu = 0$, where $K^\mu = (\partial/\partial r)^\mu$. The stress-energy tensor $T_{\mu\nu}$ is also subject to the conservation law $\partial^\mu T_{\mu\nu} = 0$. This gives rise to the condition

$$K^\mu T^{(d-2)}_{\mu\nu} = 0. \tag{2.71}$$

However, in the gravitational case, we have the further requirement that the stress-energy tensor satisfy the dominant energy condition. The only way eq. (2.71) can be compatible with the dominant energy condition is if

$$T^{(d-2)}_{\mu\nu} = \alpha K_\mu K_\nu \tag{2.72}$$

for some function $\alpha(u, x^A)$. Thus, all components of $T^{(d-2)}_{\mu\nu}$ must vanish except for $T^{(d-2)}_{uu}$.

It is of interest to examine the gauge dependence of the radiative order metric $h^{(d/2-1)}_{\mu\nu}$ and the gauge invariant quantities that can be constructed from $h^{(d/2-1)}_{\mu\nu}$. Under a gauge transformation, we have

$$h_{\mu\nu} \to h_{\mu\nu} - \partial_{(\mu}\xi_{\nu)} \tag{2.73}$$

so

$$h_{\mu\nu}^{(d/2-1)} \to h_{\mu\nu}^{(d/2-1)} - [\partial_{(\mu}\xi_{\nu)}]^{(d/2-1)} \tag{2.74}$$

where

$$\begin{aligned}[\partial_{(\alpha}\xi_{\beta)}]^{(d/2-1)} =& \mathscr{D}_{(A}\xi_{B)}^{(d/2-2)} - \mathscr{D}_{(A}\Big(K_{\beta)}\xi_u^{(d/2-2)} - r_{\beta)}\xi_r^{(d/2-2)}\Big) + \xi_r^{(d/2-2)} q_{\alpha\beta} \\ & - \xi_u^{(d/2-2)} q_{\alpha\beta} - r_{(\alpha}\xi_{B)}^{(d/2-2)} - \left(\frac{d}{2}-2\right) r_{(\alpha}\xi_{\beta)}^{(d/2-2)} - K_{(\alpha}\partial_u\xi_{\beta)}^{(d/2-1)}.\end{aligned} \tag{2.75}$$

Here $\xi_\alpha^{(d/2-2)}$ must be stationary in order to maintain our ansatz eqs.(2.58) and (2.59). It is clear from eq. (2.75) that $\xi_\alpha^{(d/2-1)}$ can always be used to set $h_{uu}^{(d/2-1)}, h_{ur}^{(d/2-1)}$ and $h_{uA}^{(d/2-1)}$ to zero. It also is clear from eq. (2.75) that the remaining components can be changed only by a stationary transformation. It follows immediately that $\partial_u h_{\mu\nu}^{(d/2-1)}$ is gauge invariant for all $\mu, \nu \neq u$. However, using the linearized Einstein equation, it can be shown[8] that $\partial_u h_{rr}^{(d/2-1)} = \partial_u h_{rA}^{(d/2-1)} = \partial_u (q^{AB} h_{AB}^{(d/2-1)}) = 0$. Therefore, the only nontrivial gauge invariant quantity that can be constructed from $\partial_u h_{\mu\nu}^{(d/2-1)}$ is

$$N_{\mu\nu} \equiv \left(q_\mu{}^\rho q_\nu{}^\sigma - \frac{1}{d-2} q_{\mu\nu} q^{\rho\sigma}\right) \partial_u h_{\rho\sigma}^{(d/2-1)}. \tag{2.76}$$

We may view $N_{\alpha\beta}$ as a tensor on the sphere, denoted $N_{AB}$. $N_{AB}$ is called the *Bondi news tensor.*

We now seek to put $h_{\mu\nu}$ in Lorenz gauge,

$$\partial^\mu \overline{h}_{\mu\nu} = 0 \tag{2.77}$$

8. The vanishing of $\partial_u h_{rr}^{(d/2-1)}$, $\partial_u h_{rA}^{(d/2-1)}$ and $\partial_u (q^{AB} h_{AB}^{(d/2-1)})$ follows from eq. (2.84) below, together with eqs. (2.68)-(2.70) for $n = d/2 - 1$.

while preserving the form of the ansatz eqs. (2.58) or (2.59). Under a gauge transformation, $h_{\mu\nu}$ changes by eq. (2.73). Thus, we can put $h_{\mu\nu}$ into Lorenz gauge if and only if we can solve

$$\Box \xi_\mu = 2\chi_\mu. \tag{2.78}$$

Thus, the equations we must solve take the same basic form as the scalar wave equation, and we can analyze them in close parallel to the electromagnetic case. We take our ansatz for $\xi_\mu$ to be

$$\xi_\mu \sim \sum_{n=d/2-2}^{\infty} \frac{1}{r^n}\xi_\mu^{(n)}(u,x^A) \qquad d \text{ even} \tag{2.79}$$

$$\xi_\mu \sim \sum_{n=d/2-2}^{\infty} \frac{1}{r^n}\xi_\mu^{(n)}(u,x^A) + \sum_{p=d-3}^{\infty} \frac{1}{r^p}\tilde{\xi}_\mu^{(p)}(u,x^A) \qquad d \text{ odd} \tag{2.80}$$

where it is required in both of these expressions that $\partial_u \xi_\mu^{(d/2-2)} = 0$. When $d = 4$, we may also add a term $c(\partial/\partial u)_\mu \ln r$ to $\xi_\mu$, where $c$ is a constant.

When $d > 4$, the stress-energy terms in eqs. (2.61)-(2.66) do not enter at radiative order $n = d/2-1$. The $ur$,$rr$ and $rA$ components of these equations yield, respectively

$$-(d/2-1)\chi_r^{(d/2-1)} + 2(d/2-1)\chi_u^{(n)} - \mathscr{D}^A\chi_A^{(d/2-1)} = 0 \tag{2.81}$$

$$(d-2)\chi_r^{(d/2-1)} = 0 \tag{2.82}$$

$$(d/2)\chi_A^{(d/2-1)} - \mathscr{D}_A\chi_r^{(d/2-1)} = 0. \tag{2.83}$$

Thus, we have

$$\chi_\mu^{(d/2-1)} = 0. \tag{2.84}$$

The $uu$, $uA$ and $AB$ components yield, respectively

$$-2\partial_u \chi_u^{(d/2)} + \partial_u \chi_r^{(d/2)} = 0 \tag{2.85}$$

$$\partial_u \chi_A^{(d/2)} = 0 \tag{2.86}$$

$$q_{AB} \partial_u \chi_r^{(d/2)} = 0 \tag{2.87}$$

which implies

$$\partial_u \chi_a^{(d/2)} = 0. \tag{2.88}$$

As in the electromagnetic case, equations (2.84) and (2.88) ensure that we can solve eq. (2.78) within the ansatz.

However, when $d = 4$, we still have that $\chi_\mu^{(1)}$ vanishes but eq. (2.61) for $n = 1$ yields

$$\partial_u \chi_u^{(2)} = 8\pi T_{uu}^{(2)}. \tag{2.89}$$

As in the electromagnetic case, this will give rise to an obstruction to solving eq. (2.78) within the ansatz if and only if $T_{uu}^{(2)}$ is nonvanishing. Thus, for $d = 4$, the necessary and sufficient condition for imposing the Lorenz gauge within our ansatz is that $T_{uu}^{(2)}$ vanish identically.

We summarize these results in the following proposition:

**Proposition 2.4.** *For all $d > 4$, any $h_{\mu\nu}$ that satisfies our ansatz eq. (2.58) (for $d$ even) or ansatz eq. (2.59) (for $d$ odd) can be put in the Lorenz gauge in such a way that it continues to satisfy our ansatz. In $d = 4$ the Lorenz gauge condition can be imposed within the ansatz if and only if $T_{uu}^{(2)} = 0$.*

*Remark* 2.5. *As in the electromagnetic case, for $d > 4$ we show in Appendix 0.2 that the Lorenz gauge condition could still be imposed if we weakened the fall-off conditions to $1/r$ fall-off in even dimensions and $1/\sqrt{r}$ fall-off in odd dimensions. As in Remark 2.2, this justifies our* *taking the lower limit of the sum in eq. (2.58) and eq. (2.59) to start at* $n = d/2 - 1$*. Also, as in the electromagnetic case, it follows that if a solution is stationary in some gauge for all* $n \leq m$ *with* $m \leq d-2$ *and if* $T^{(n)}_{\mu\nu} = 0$ *for all* $n \leq m+1$*, then it is stationary in a Lorenz gauge for all* $n \leq m$*.*

*Remark* 2.6. *Let* $d = 4$ *and* $T^{(2)}_{uu} = 0$*. Suppose further that* $T^{(3)}_{r\nu} = 0$ *so that our ansatz is equivalent to smoothness of* $\Omega^2 h_{\mu\nu}$ *at* $\mathscr{I}^+$ *in some gauge. Although, by Prop. 2.4, the Lorenz gauge can be imposed within our ansatz eq. (2.58), it need not be the case that* $h^{(1)}_{rr} = 0$ *in the Lorenz gauge, in which case* $\Omega^2 h_{\mu\nu}$ *in the Lorenz gauge will not be smooth at* $\mathscr{I}^+$*, i.e., the Lorenz gauge need not be compatible with smoothness at* $\mathscr{I}^+$*.*

When $h_{\mu\nu}$ is in Lorenz gauge—as, by Proposition 2.4 we may assume for $d > 4$ and for $d = 4$ when $T^{(2)}_{uu} = 0$—it satisfies

$$\Box \bar{h}_{\mu\nu} = -16\pi T_{\mu\nu} \tag{2.90}$$

$$\partial^\mu \bar{h}_{\mu\nu} = 0. \tag{2.91}$$

The recursion relations for eq. (2.90) are eqs. (2.61)-(2.66) with $\chi_\mu = 0$. The recursion relations arising from eq. (2.91) are just eqs. (2.68)-(2.70) with $\chi_\mu = 0$. Again, it is useful to eliminate the terms in eqs. (2.68)-(2.70) with $u$-derivatives using eqs. (2.61)-(2.66). This can be achieved by defining

$$\tau_\mu = K^\nu[\Box \bar{h}_{\mu\nu} + 16\pi T_{\mu\nu}] - 2K^\nu \partial_\nu \chi_\mu - (d-2)\chi_\mu / r. \tag{2.92}$$

When eq. 2.90 holds, the vanishing of $\tau_\mu$ is equivalent to the vanishing of $\chi_\mu$. The relation $\tau^{(n+2)}_\mu = 0$ yields

$$[\mathscr{D}^2 - (n-d+2)(n-d+3)]\bar{h}^{(n)}_{ru} + (n-d+3)(2n-d+2)\bar{h}^{(n)}_{uu} - (2n-d+2)\mathscr{D}^A \bar{h}^{(n)}_{uA} = -16\pi T^{(n+2)}_{ru} \tag{2.93}$$

$$[\mathscr{D}^2 - ((n-d+2)^2+n)]\overline{h}^{(n)}_{rr} + (d-2+(n-d+3)(2n-d+2))\overline{h}^{(n)}_{ur}$$
$$- (2n-d+2)q^{AB}\overline{h}^{(n)}_{AB} + (2n-d)\mathscr{D}^A\overline{h}^{(n)}_{Ar} = -16\pi T^{(n+2)}_{rr} \tag{2.94}$$

$$[\mathscr{D}^2 - (n-d+3)(n-d+2) + (2n-d+1)]\overline{h}^{(n)}_{rA} + (2n-d+2)(n-d+2)\overline{h}^{(n)}_{uA}$$
$$+ 2\mathscr{D}_A(\overline{h}^{(n)}_{rr} - \overline{h}^{(n)}_{ur}) + (2n-d+2)\mathscr{D}^B\overline{h}^{(n)}_{AB} = -16\pi T^{(n+2)}_{rA}. \tag{2.95}$$

Equations (2.93)–(2.95) reduce to the "constraint equations" given by [38] if one applies the additional gauge conditions that they impose.

The analysis of the appropriate data for solutions to eq. (2.90) and (2.91) follows in exact parallel with the electromagnetic case. We solve the wave equation given by eqs. (2.61)-(2.66) with $\chi_\mu = 0$, subject to the constraints eqs. (2.93)-(2.95). We can specify data at radiative order subject to the constraints and solve for the faster fall-off terms exactly as in the electromagnetic case. We also can specify data at Coulombic order and solve for slower fall-off terms. In exact parallel with the electromagnetic case, in even dimensions, in addition to the Coulombic order constraints, the Coulombic order data must satisfy

$$\partial_u \mathcal{M} = -\mathcal{A}_d T^{(d-2)}_{uu}|_{\ell=0} \tag{2.96}$$

where

$$\mathcal{M} = \frac{1}{16\pi}\mathcal{A}_d[\overline{h}^{(d-3)}_{ur} + (d-4)\overline{h}^{(d-3)}_{uu}]|_{\ell=0} \qquad d \text{ even (in Lorenz gauge)} \tag{2.97}$$

with $\mathcal{A}_d$ given by eq. (2.53). Thus, $\mathcal{M}$ satisfies the same flux relation as the linearized Bondi mass in linearized gravity, and thus it can differ from the linearized Bondi mass only by a constant. To show that $\mathcal{M}$ is, indeed, the linearized Bondi mass, it suffices to show that it agrees with the Bondi mass in the stationary case, where $t^\mu = (\partial/\partial u)^\mu$, is a Killing field. In the stationary case, it can be verified that $\mathcal{M}$ agrees with the Komar mass formula

$$\mathcal{M} = -\frac{1}{16\pi}\frac{(d-2)}{(d-3)}\int_{\mathbb{S}^2} \boldsymbol{\epsilon}_{\mu\nu\rho\sigma}\nabla^{\mu}t^{\nu} \tag{2.98}$$

where $\boldsymbol{\epsilon}_{\mu\nu\rho\sigma}$ is the volume form and the integral is taken over a sphere near infinity. Since the Komar mass agrees with the Bondi mass in the stationary case [53], it follows that $\mathcal{M}$ is, indeed, the linearized Bondi mass[9]. In odd dimensions, we do not obtain a similar additional constraint, but the recursion relation eq. (2.61) with $p = d-3$ as well as the Lorentz gauge constraint eq. (2.68) with $p = d-3$ implies that eq. (2.96) holds where the linearized Bondi mass is given by

$$\mathcal{M} = \frac{1}{16\pi}\mathcal{A}_d[\bar{\tilde{h}}^{(d-3)}_{ur} + (d-4)\bar{\tilde{h}}^{(d-3)}_{uu}]|_{\ell=0} \qquad d \text{ odd (in Lorenz gauge).} \tag{2.99}$$

We summarize our results on solutions to the linearized Einstein equation in Lorenz gauge with the following theorem:

**Theorem 2.** *Suppose $d > 4$ or $d = 4$ and $T^{(2)}_{uu} = 0$, so that the Lorenz gauge condition can be imposed. Then a unique solution to the recursion relations and constraints for the linearized Einstein equation in Lorenz gauge is obtained by specifying data in either of the following two ways:*

*(1)* Radiative Order Data: *Specify $h^{(d/2-1)}_{\mu\nu}(u, x^A)$ subject to the constraints eqs. (2.93)-(2.95) at $n = d/2-1$. Specify $h^{(n)}_{\mu\nu}(u = u_0, x^A)$ for all $n > d/2-1$ subject to the constraints eqs. (2.93)-(2.95) at $u = u_0$. In odd dimensions, also specify $\tilde{h}^{(p)}_{\mu\nu}(u = u_0, x^A)$ for all $p \geq d-3$, subject to the constraint eqs. (2.93)-(2.95) at $u = u_0$.*

*(2)* Coulombic Order Data: *In even dimensions, specify $h^{(d-3)}_{\mu\nu}(u, x^A)$ subject to the con-*

9. We caution the reader that eq. (2.97) holds only in the Lorenz gauge, which cannot be imposed for $d = 4$ when $T^{(2)}_{uu} \neq 0$.

*straints eqs. (2.93)-(2.95) at $n = d - 3$ and the additional constraint eq.* (2.96)*; specify $h^{(n)}_{\mu\nu}(u = u_0, x^A)$ for all $n > d-3$ subject to the constraints eqs. (2.93)-(2.95) at $u = u_0$. In odd dimensions, specify $h^{(m)}_{\mu\nu}(u, x^A)$ for any $m \geq d/2 - 1$, subject to the constraints eqs. (2.93)-(2.95) at $n = m$, specify $h^{(n)}_{\mu\nu}(u = u_0, x^A)$ for all $n > m$ subject to the constraints eqs. (2.93)-(2.95) at $u = u_0$; specify $\tilde{h}^{(p)}_{\mu\nu}(u = u_0, x^A)$ for all $p \geq d - 3$, subject to the constraints eqs. (2.93)-(2.95) at $u = u_0$.*

## 5 Nonlinear Einstein equation

For the nonlinear Einstein equation, we write $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$ and we assume the same ansatz for $h_{\mu\nu}$ as in linearized gravity (see sec. 2.4). For $d$ even with $d > 4$, our ansatz eq. (2.58) is equivalent to smoothness of $\Omega^2 h_{\mu\nu}$ (and, therefore, smoothness of $\Omega^2 g_{\mu\nu} = \Omega^2 \eta_{\mu\nu} + \Omega^2 h_{\mu\nu}$) at $\mathscr{I}^+$ by the same arguments as for the linearized case. For $d = 4$ our ansatz eq. (2.58) in linearized gravity was slighter weaker than smoothness of $\Omega^2 h_{\mu\nu}$ at $\mathscr{I}^+$ in that it admitted additional solutions for which $h^{(1)}_{rr}$ cannot be set to zero by a gauge transformation within our ansatz. However, we show in Appendix 0.3 that, if the Bondi news is nonvanishing at all angles at any time, such additional solutions do not exist in the nonlinear theory. Thus, for $d = 4$ our ansatz eq. (2.58) is also equivalent to smoothness at $\mathscr{I}^+$ for spacetimes in which $N_{AB}$ is nonvanishing everywhere on some cross-section.

The nonlinear Einstein equation is far more complex than the linearized Einstein equation. However, since the slowest fall-off of $h_{\mu\nu}$ is $1/r^{d/2-1}$, the nonlinear terms first enter at order $(1/r^{d/2-1})^2 = 1/r^{d-2}$. Consequently, for $n < d - 3$, the recursion relations for the full Einstein equation are identical to eqs. (2.61)-(2.66) in the linearized case. For $n = d - 3$, the equations are modified by terms of the form $(\partial_u h^{(d/2-1)}_{\mu\nu})^2$ and $h^{(d/2-1)}_{\mu\nu} \partial_u^2 h^{(d/2-1)}_{\rho\sigma}$, which are the only types of nonlinear terms that can contribute at this order. At higher orders, the nonlinear correction terms are far more complicated, but they always involve adding terms arising from metric components of slower fall-off.

We define the non-linear part of the Einstein tensor $\mathcal{G}_{\mu\nu}$ as

$$\mathcal{G}_{\mu\nu} \equiv G_{\mu\nu} - \delta G_{\mu\nu} \tag{2.100}$$

where $G_{\mu\nu}$ is the Einstein tensor and $\delta G_{\mu\nu}$ is the linearized Einstein tensor defined in eq. (2.56). Our ansatz then implies an asymptotic expansion of $\mathcal{G}_{\mu\nu}$ in integer powers of $1/r$ in even dimensions and both integer and half-integer powers in odd dimensions. In both even and odd dimensions, the expansion starts at order $1/r^{d-2}$. In all dimensions, Einstein's equations give rise to the same set of recursion relations as in the linearized case with the replacement

$$8\pi T^{(n)}_{\mu\nu} \to 8\pi T^{(n)}_{\mu\nu} - \mathcal{G}^{(n)}_{\mu\nu} \text{ for } n \geq d-2 \tag{2.101}$$

where $n$ is an integer in even dimensions and takes on both integer and half integer values in odd dimensions. By a direct calculation, we find that the leading order contribution to $\mathcal{G}_{\mu\nu}$ is given by

$$\mathcal{G}^{(d-2)}_{\mu\nu} = -\frac{1}{4} N^{\rho\sigma} N_{\rho\sigma} K_\mu K_\nu + \frac{1}{2}\partial_u \Big( q^{\rho\sigma} q^{\kappa\delta} c_{\rho\kappa} N_{\sigma\delta} K_\mu K_\nu + q^{\rho\sigma} c_{r\rho} N_{\sigma(\mu} K_{\nu)} + c_{rr} N_{\mu\nu} \Big) \tag{2.102}$$

where $c_{\mu\nu} \equiv h^{(d/2-1)}_{\mu\nu}$ and $N_{\mu\nu}$ is the Bondi news tensor as defined in eq. (2.76). In writing eq. (2.102), we have used the fact that, as in the linearized case (see eq. (2.84)), the recursion relations imply that $\chi^{(d/2-1)}_\mu = 0$, where $\chi_\mu \equiv \partial^\nu \bar{h}_{\mu\nu}$.

We wish to determine whether the metric $g_{\mu\nu}$ can be put in the harmonic gauge while maintaining our $1/r$ expansion ansatz. To put the metric in harmonic gauge, we must find coordinate functions $x^\mu$ such that

$$\Box_g x^\mu = 0 \tag{2.103}$$

where $\Box_g \equiv g^{\mu\nu}\nabla_\mu\nabla_\nu$ and $\nabla_\mu$ is the derivative operator compatible with $g_{\mu\nu}$. Let

$$x^\mu = \mathring{x}^\mu + \xi^\mu \tag{2.104}$$

where $\mathring{x}^\mu$ are global inertial coordinates of $\eta_{\mu\nu}$, satisfying $\partial_\alpha \mathring{x}^\mu = \delta_\alpha{}^\mu$. Applying $\Box_g$ to eq. (2.104) we obtain

$$\Box\xi^\mu = -\frac{1}{\sqrt{-g}}\partial_\alpha(\sqrt{-g}g^{\alpha\mu}) - H^{\alpha\beta}\partial_\alpha\partial_\beta\xi^\mu - \frac{1}{\sqrt{-g}}\partial_\alpha(\sqrt{-g}g^{\alpha\beta})\partial_\beta\xi^\mu \tag{2.105}$$

where, again, $\Box \equiv \eta^{ab}\partial_a\partial_b$, and

$$H^{\alpha\beta} \equiv g^{\alpha\beta} - \eta^{\alpha\beta}\,. \tag{2.106}$$

Here we have used the fact that, for any function $f$,

$$\Box_g f = \frac{1}{\sqrt{-g}}\partial_\alpha(\sqrt{-g}g^{\alpha\beta}\partial_\beta f). \tag{2.107}$$

In parallel with the analysis of imposition of the Lorenz gauge condition in linearized gravity, we will be able to put the metric in harmonic gauge in nonlinear gravity while maintaining our expansion ansatz if we can solve eq. (2.105) via the ansatz

$$\xi^\mu \sim \sum_{n=d/2-2}^{\infty} \frac{1}{r^n}\xi^{\mu\,(n)}(u, x^A), \tag{2.108}$$

with $\partial_u\xi^{\mu\,(d/2-2)} = 0$. In odd dimensions, the sum in eq. (2.108) is allowed to run over integer values (starting at $d-3$) as well as half-integer values. For $d = 4$, in the case of a stationary spacetime with Killing field $\partial/\partial u$, we may also add a term $c(\partial/\partial u)^\mu \ln r$ to $\xi^\mu$, where $c$ is a constant.

To analyze existence of solutions to eq. (2.105) of the form eq. (2.108), we note that eq. (2.105) is of the form

$$\Box\xi^{\mu} = \chi^{\mu} + L^{\mu}(h, \xi) \tag{2.109}$$

where, again, $\chi^{\mu} \equiv \partial_{\alpha}\overline{h}^{\alpha\mu}$, and where $L^{\mu}$ is composed of terms that are (i) quadratic and higher order in $h_{\mu\nu}$ or (ii) linear in $\xi^{\mu}$ and linear or higher order in $h_{\mu\nu}$. The leading order contribution of $L^{\mu}$ to this equation arises at order $1/r^{d-2}$.

Consider, first, the case $d > 4$. As noted previously, the non-linear contributions to Einstein's equation enter at order $1/r^{d-2}$, so the recursion relations derived for the linearized Einstein's equation given by eqs. (2.61)–(2.66) are equivalent to the recursion relations for the full, non-linear Einstein's equation for $n \leq d-3$. As already noted above, these equations imply that $\chi_{\mu}^{(d/2-1)}$ must vanish. It also follows that $\chi_{\mu}^{(d/2)}$ is stationary. It then follows that we can solve eq. (2.109) at order $1/r^{d/2}$ by a choice of $\xi^{\mu\,(d/2-2)}$ that is stationary. We may then specify $\xi_{\mu}^{(d/2-1)}$ arbitrarily and recursively solve eq. (2.105) with the ansatz eq. (2.108) for all of the faster fall-off terms, in the same manner as in Prop. 2.1. The source $L^{\mu}$ plays an innocuous role in this procedure since it is obtained from $\xi^{\mu}$ at orders that have already been solved for and thus is a "known" source term.

For the case $d = 4$, we still have that $\chi_{\mu}^{(1)} = 0$. In addition, since $\partial_u h_{rA}^{(1)} = 0$ (as follows from eqs. (2.84) and (2.70)), we may perform a gauge transformation of the form eq. (2.75) to set $h_{rA}^{(1)} = 0$. We then find that $\chi_{r}^{(2)}$ and $\chi_{A}^{(2)}$ are stationary. However, $\chi_{u}^{(2)}$ now satisfies

$$\partial_u\chi_u^{(2)} = 8\pi T_{uu}^{(2)} - \mathcal{G}_{uu}^{(2)}. \tag{2.110}$$

Using eq. (2.102) together with $h_{rA}^{(1)} = 0$, we obtain

$$\mathcal{G}_{uu}^{(2)} = -\frac{1}{4}N^{CD}N_{CD} + \frac{1}{2}\partial_u\left(C^{CD}N_{CD}\right) \tag{2.111}$$

where $C_{AB}$ is the trace free part of the projection of $h_{\alpha\beta}^{(1)}$ onto the sphere. However, eq. (2.109) implies the $u$-component of the leading order term $\xi^{\mu\,(0)}$ satisfies

$$\mathscr{D}^2\xi_u^{(0)} = \chi_u^{(2)} + N_{AB}C^{AB} \tag{2.112}$$

and hence

$$\mathscr{D}^2\left(\partial_u\xi_u^{(0)}\right) = 8\pi T_{uu}^{(2)} + \frac{1}{4}N^{CD}N_{CD} + \frac{1}{2}\partial_u\left(N_{AB}C^{AB}\right). \tag{2.113}$$

Since $T_{uu}^{(2)} \geq 0$, if we assume that $N_{AB}$ vanishes as $u \to \pm\infty$, it is easily seen that we cannot have $\partial_u\xi_u^{(0)} = 0$ at all $u$ as required unless both $T_{uu}^{(2)}$ and $N_{AB}$ vanish identically. Thus, we cannot impose the harmonic gauge condition within our ansatz[10] if $T_{uu}^{(2)} \neq 0$ or $N_{AB} \neq 0$. On the other hand, if the spacetime is stationary—i.e. if it admits a timelike Killing field $t^a$—then $T_{uu}^{(2)} = 0$ and $N_{AB} = 0$. Using the fact that the equations for $\xi_r^{(0)}$ and $\xi_A^{(0)}$ contain only "source terms" that are stationary—it can be seen that we can solve eq. (2.112) by choosing $\xi^{\mu(0)}$ to be stationary (provided that we again add the term $cg_{\mu\nu}t^\nu\ln(r)$ to our ansatz to solve the $\ell = 0$ part of eq. (2.112)). The recursion relations for all faster fall-off can then be solved as in the case $d > 4$ so the harmonic gauge condition can be imposed[11].

We summarize these results in the following proposition:

**Proposition 2.5.** *For all $d > 4$, any $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$ that satisfies our ansatz eq. (2.58) (for $d$ even) or our ansatz eq. (2.59) (for $d$ odd) can be put in the harmonic gauge in such a way that it continues to satisfy our ansatz. In $d = 4$ the harmonic gauge condition cannot be imposed within the ansatz if $T_{uu}^{(2)} \neq 0$ or $N_{AB} \neq 0$.*

*Remark* 2.7. *In linearized gravity, the restriction $T_{uu}^{(2)} = 0$ in $d = 4$ allows all vacuum solutions as well as all solutions with a stress-energy source that has vanishing flux at null*

10. We could impose the harmonic gauge condition for nonvanishing $T_{uu}^{(2)}$ or $N_{AB}$ for $d = 4$ if we modified our ansatz to allow additional series involving terms of the form $(\ln r)^k/r^n$ [54].

11. If the spacetime is non-stationary and $T_{uu}^{(2)} = 0 = N_{AB}$ then we do not believe that the metric can be put in harmonic gauge within our ansatz, but we have not proven this.

*infinity. Thus, the Lorenz gauge can be imposed in linearized gravity within our ansatz in a wide variety of circumstances of interest. However, in nonlinear general relativity, the harmonic gauge cannot be imposed within our ansatz in $d = 4$ if—in addition to $T^{(2)}_{uu} \neq 0$—the Bondi news is also nonvanishing, i.e., in $d = 4$ the harmonic gauge cannot be imposed within our ansatz in any spacetime with gravitational radiation. In particular, for $d = 4$ we cannot use the harmonic gauge when considering the memory effect in the next section, so we will have to treat the case $d = 4$ separately.*

When $g_{\mu\nu}$ is in the harmonic gauge, it satisfies

$$G^H_{\mu\nu} = 8\pi T_{\mu\nu} \tag{2.114}$$

$$H^\nu \equiv \frac{1}{\sqrt{-g}}\partial_\nu[\sqrt{-g}g^{\mu\nu}] = 0 \tag{2.115}$$

where $G^H_{\mu\nu}$ is the Einstein tensor in the harmonic gauge

$$G^H_{\mu\nu} = G_{\mu\nu} + g_{\rho(\mu}\partial_{\nu)}H^\rho - \frac{1}{2}g_{\mu\nu}\partial_\rho H^\rho. \tag{2.116}$$

We now turn to the issue of whether these equations can be solved recursively within our ansatz. We restrict consideration to $d > 4$, since, as just remarked above, the harmonic gauge can be imposed only in trivial cases when $d = 4$.

Taking the divergence of eq. (2.116) with respect to $\nabla_\mu$ and using the Bianchi identity we find that when $G^H_{\mu\nu} = 8\pi T_{\mu\nu}$, we have

$$\Box H_\mu = W_\mu(h, H) \tag{2.117}$$

where $W_\mu$ is linear in $H_\mu$ and its first derivative and is quadratic and higher order in $h_{\mu\nu}$ and its first derivative. It follows that if $H^{(d/2-1)}_\mu = 0$ for all $u$ and $H^{(n)}_\mu = 0$ for $n > d/2 - 1$

at some $u = u_0$, then $H_\mu = 0$. Namely, if we inductively assume that $H_\mu^{(n)} = 0$ for all $n \leq k$, then the source term arising from $W_\mu$ that appears in the recursion equation for $H_\mu^{(k+1)}$ will vanish. It then follows from the same arguments as used to prove Prop. 2.1 that $H_\mu^{(k+1)} = 0$.

It is convenient to replace $H^\mu$ by

$$\tau'_\mu = K^b[-2G_{\mu\nu}^{(H)} + 16\pi T_{\mu\nu}] + 2K^\nu \partial_\nu H_\mu + (d-2)H_\mu/r \tag{2.118}$$

where the form of $\tau'_\mu$ has been chosen so that, for $n < d-3$, ${\tau'}_\mu^{(n+2)}$ can be expressed purely in terms of $h_{\mu\nu}^{(n)}$, with no $u$-derivatives of $h_{\mu\nu}$ appearing. When eq. (2.114) holds, the vanishing of ${\tau'}_\mu^{(n+1)}$ implies the vanishing of $H_\mu^{(n)}$. Thus we obtain a solution to eqs. (2.114) and (2.115) if we can solve eq. (2.114) in such a way that we also obtain $\tau'_\mu = 0$.

The recursion relations for eq. (2.114) for $n < d-3$ are identical to eqs. (2.61)–(2.66) with $\chi_\mu = 0$. In addition, we have ${\tau'}_\mu^{(n+2)} = \tau_\mu^{(n+2)}$ for $n < d-3$, where $\tau_\mu$ is the corresponding quantity in linearized gravity given by eq. (2.92). Thus, for $n < d-3$, the recursion relations and constraints are identical to the linearized case. It follows that if one specifies data at radiative order, one may solve the recursion relations for $h_{\mu\nu}^{(n)}$ for all $n < d-3$ exactly as in the linearized case. The recursion relations and constraints needed to solve for $h_{\mu\nu}^{(n)}$ for $n \geq d-3$ receive nonlinear corrections relative to the linearized equations. However, the nonlinear terms entering the equations will be of the form of products of metric perturbations arising at lower orders. Consequently, the non-linear terms can be effectively treated as source terms in our recursive analysis and they pose no difficulties in solving for $h_{\mu\nu}^{(n)}$ for $n \geq d-3$. We thereby obtain the following theorem:

**Theorem 3.** *Suppose $d > 4$ so that, by Prop. 2.5, the harmonic gauge condition can be imposed. Then a unique solution to the recursion relations and constraints for the Einstein's equation in the harmonic gauge is obtained by the following specification of data: Specify $h_{\mu\nu}^{(d/2-1)}(u, x^A)$ subject to the constraints ${\tau'}_\mu^{(d/2+1)} = 0$ (which are identical to eqs. (2.93)-*

*(2.95) at $n = d/2-1$). Specify $h^{(n)}_{\mu\nu}(u=u_0, x^A)$ for all $n > d/2-1$ subject to the constraints $\tau'^{(n+2)}_{\mu} = 0$ at $u = u_0$. In odd dimensions, also specify $\tilde{h}^{(p)}_{\mu\nu}(u=u_0, x^A)$ for all $p \geq d-3$ subject to the constraint $\tau'^{(p+2)}_{\mu} = 0$ at $u = u_0$.*

Note that there is no analog of the "Coulombic order data specification" method for getting a solution of the recursion relations in nonlinear general relativity, since the Bondi news enters the equations for the metric at Coulombic order. Thus, we need to know the solution at radiative order before we can determine whether $h^{(d-3)}_{\mu\nu}(u, x^A)$ is a solution to the recursion relations and constraints.

Finally, it is worth noting that the analog of eq. (2.96) in nonlinear general relativity for $d > 4$ is

$$\partial_u M = -\mathcal{A}_d T^{(d-2)}_{uu}|_{\ell=0} - \frac{1}{32\pi}\mathcal{A}_d N^{AB} N_{AB}|_{\ell=0} \tag{2.119}$$

where in even dimensions

$$M = \frac{1}{16\pi}\mathcal{A}_d[\bar{h}^{(d-3)}_{ur} + (d-4)\bar{h}^{(d-3)}_{uu} - C^{AB}N_{AB}]|_{\ell=0} \quad d \text{ even (in harmonic gauge)}, \tag{2.120}$$

and in odd dimensions

$$M = \frac{1}{16\pi}\mathcal{A}_d[\bar{\tilde{h}}^{(d-3)}_{ur} + (d-4)\bar{\tilde{h}}^{(d-3)}_{uu} - C^{AB}N_{AB}]|_{\ell=0} \quad d \text{ odd (in harmonic gauge)} \tag{2.121}$$

where $\mathcal{A}_d$ is the area of a unit $(d-2)-$sphere given by eq. (2.53). By the same arguments as given in the linearized case, $M$ is the Bondi mass. Again, the above formulas for $M$ apply only in harmonic gauge and thus cannot be applied when $d = 4$ if $T^{(2)}_{uu} \neq 0$ or $N_{AB} \neq 0$. A gauge invariant expression for the Bondi mass in all even dimensions $d \geq 4$ was given in [27]. Positivity of the Bondi mass in even dimensions was proven in [52].

# 3 The memory effect

We now turn our attention to the analysis of the memory effect in nonlinear general relativity in $d \geq 4$ dimensions. In physical terms, the memory effect can be described as the permanent relative displacement resulting from the passage of a "burst of gravitational radiation" of a system of test particles that are initially at rest. The relative displacement of test particles is governed by the geodesic deviation equation

$$(v^\mu \nabla_\mu)^2 \xi^\nu = -R_{\mu\rho\sigma}{}^\nu v^\mu v^\sigma \xi^\rho \tag{3.1}$$

where $v^\mu$ is the tangent the worldline of the test particle, $\xi^\mu$ is the deviation vector and $R_{\mu\nu\rho\sigma}$ is the Riemann tensor. In our case, we will be interested in test particles near future null infinity and wish to determine the leading order memory effect in a $1/r$ expansion.

We note that there are closely analogous "memory effects" for electromagnetic and scalar fields [11, 55, 56]. For the electromagnetic field or the scalar field, the memory effect would correspond to a charged particle with electric or scalar charge, originally at rest, getting a momentum kick after the passage of a burst of electromagnetic or scalar radiation. However, since we now have fully developed the machinery for the gravitational case, we will bypass the analysis of these other cases and go directly to the analysis of the memory effect in general relativity.

## *1 Stationarity conditions at early and late retarded times*

Our first task in analyzing the memory effect is to define more precisely what we mean by a "burst of gravitational radiation," i.e., to specify the stationarity conditions that we will assume hold at early and late retarded times.

We wish to consider spacetimes where there is significant gravitational radiation near future null infinity only over some finite range of retarded time. We envision this radiation as arising from "localized event" in the interior of the spacetime involving the interaction of matter and/or black holes and/or gravitational waves—although our entire analysis will be done near future null infinity and will not make any assumptions about the source of the gravitational radiation. Thus, we wish to consider a situation where the metric is (nearly) stationary at early retarded times and again becomes (nearly) stationary at late retarded times.

However, it would be much too strong a condition to demand that the metric becomes stationary at early and late retarded times at all orders in $1/r$. This is because we wish to allow for the presence of bodies of matter (or black holes) that move inertially from/towards infinity at early/late retarded times. To see the implications of this, we note that a static multipole of angular order $\ell$ will decay as $r \to \infty$ at fixed global inertial time $t$ as $1/r^{\ell+d-3}$. However, for inertially moving bodies, the $\ell$th multipole moment will grow with time as $t^{\ell}$. Thus, near future null infinity, there will be contributions from the $\ell$th multipole solution that result in $h_{\mu\nu}$ behaving as[12]

$$h_{\mu\nu} \sim \frac{t^{\ell}}{r^{\ell+d-3}} = \frac{(u+r)^{\ell}}{r^{\ell+d-3}} = \frac{1}{r^{d-3}} + \frac{\ell u}{r^{d-2}} + \dots \tag{3.2}$$

Thus, the leading order behavior of $h_{\mu\nu}$ is Coulombic—but note that $h_{\mu\nu}$ is *not* spherically symmetric near null infinity at Coulombic order. Although $h_{\mu\nu}$ is stationary at Coulombic order, it is, in general, non-stationary for $\ell \geq 1$ at order $1/r^{d-2}$. This non-stationarity can be removed for $\ell = 1$ by Lorentz boosting to a frame where the center of mass of the matter is at rest, but $h_{\mu\nu}$ will, in general, be genuinely non-stationary at order $1/r^{d-2}$ for $\ell \geq 2$.

12. The $\ell$th multipole solution with leading order time dependence eq. (3.2) will also have terms that behave as $t^{\ell-2k}/r^{\ell-2k+d-3}$ with $k$ integer and $2k \leq \ell$, which also will contribute to the field at future null infinity in the same manner as indicated in eq. (3.2).

The late time behavior near null infinity in curved spacetime with matter (or black holes) inertially moving to infinity along timelike trajectories cannot be expected to satisfy a stronger stationarity condition than would hold for inertially moving bodies in Minkowski spacetime. Indeed, as we shall see in the next subsection, if we were to require stationarity at order $1/r^{d-2}$ at both late and early retarded times, we would entirely exclude the "ordinary memory" effect. On the other hand, we do not believe that we would exclude any interesting phenomena by assuming that the metric becomes stationary at Coulombic order at early and late retarded times.

We will therefore adopt as our stationarity condition that, in some gauge within our ansatz, the metric becomes stationary at Coulombic order and slower fall-off at early and late retarded times. More precisely, in even dimensions we require that there exist a gauge in which

$$\partial_u h^{(n)}_{\mu\nu} \to 0 \quad \text{as} \quad u \to \pm\infty \text{ for } n \leq d-3, \tag{3.3}$$

and in odd dimensions we require that there exist a gauge in which

$$\partial_u h^{(n)}_{\mu\nu} \to 0 \quad \text{as} \quad u \to \pm\infty \text{ for } n < d-3 \tag{3.4}$$

$$\partial_u \tilde{h}^{(d-3)}_{\mu\nu} \to 0 \quad \text{as} \quad u \to \pm\infty. \tag{3.5}$$

It follows immediately from these conditions that in the stationary eras, the nonlinear terms in Einstein's equation are $O(1/r^{2(d-2)})$ and will not enter the equations to the orders to which we will work. In addition, stationarity at Coulombic order implies that the Bondi mass (which can be defined in any gauge) is time independent, which implies that $T^{(d-2)}_{uu}|_{\ell=0}$ vanish in the stationary eras. However, positivity of $T^{(d-2)}_{uu}$ then implies that $T^{(d-2)}_{uu} = 0$ and the dominant energy condition then implies that $T^{(d-2)}_{\mu\nu} = 0$. We can then apply Remark 2.5 to conclude that, for $d > 4$, without loss of generality, the fall-off conditions eq. (3.3) or eq. (3.4) can be assumed to hold in a harmonic gauge, as we shall assume in the following. It then follows from Cor. 2.1 that in even dimensions we have $h^{(n)}_{\mu\nu} = 0$ for all $n < d-3$, and in odd dimensions, $h^{(n)}_{\mu\nu} = 0$ for all $n < d-2$.

Finally, we note that Mädler and Winicour [57] have imposed a "weak stationarity condition" in their treatment of the memory effect in linearized gravity in 4 dimensions. Their condition effectively requires the metric to be stationary at order $1/r^2$, i.e., one order faster fall-off than Coulombic. Thus, their condition is stronger than ours. As we shall see in the next subsection, this stronger condition rules out all "ordinary memory" effects.

## *2 The memory tensor and its properties at Coulombic order and slower fall-off*

As discussed in the previous subsection, we wish to consider a spacetime where the metric near future null infinity is stationary at Coulombic order, $1/r^{d-3}$, at early and late retarded times. We consider an array of test particles near null infinity whose tangents $v^\mu$ initially point in the $(\partial/\partial u)^\mu$ direction. We wish to compute the memory effect for such test particles at all orders $n \leq d-3$. Since the metric differs from the Minkowski metric only at order $1/r^{d/2-1}$, the geodesic determined by $v^\mu$ will differ from the corresponding integral curve of $(\partial/\partial u)^\mu$ beginning only at order $1/r^{d/2-1}$, and $u$ will differ from an affine parametrization also beginning only at this order. Since the curvature also falls off as $r^{d/2-1}$, it can be seen that the deviations of $v^\mu$ from $(\partial/\partial u)^\mu$ in eq. (3.1) can affect $\xi^\mu$ only at order $r^{d-2}$ and faster fall-off. Since we consider only the memory effect at orders $n \leq d-3$, we may therefore replace $v^\mu$ in eq. (3.1) with $(\partial/\partial u)^\mu$, i.e., we may replace eq. (3.1) with

$$\frac{\partial^2}{\partial u^2}\xi^\mu = -R_{u\nu u}{}^{\mu}\xi^\nu. \tag{3.6}$$

Since, by our ansatz, $T_{\mu\nu} = O(1/r^{d-2})$, it follows immediately from Einstein's equation that the Ricci tensor vanishes at Coulombic order and slower fall off. Consequently, we may replace the Riemann tensor in eq. (3.6) with the Weyl tensor. We also may replace $\xi^\nu$ on the right side of eq. (3.6) with its initial value, $\xi^\nu_0$, since $\xi^\nu - \xi^\nu_0 = O(1/r^{d/2-1})$, so this difference cannot contribute to the right side at Coulombic and slower fall-off. Thus, at Coulombic and slower fall-off, we have

$$\frac{\partial^2}{\partial u^2}\xi^\mu = -C_{u\nu u}{}^\mu \xi^\nu_0 . \tag{3.7}$$

Now suppose that the metric is stationary at Coulombic order and slower fall-off for $u \to \pm\infty$, as discussed in the previous subsection. Integrating eq. (3.7) twice, we obtain

$$\xi^{(n)\mu}\Big|_{u=-\infty}^{u=\infty} = \Delta^{(n)\mu}{}_\nu\, \xi^\nu_0 \qquad \text{for } n \le d-3 \tag{3.8}$$

where

$$\Delta^{(n)}_{\mu\nu} \equiv - \int_{-\infty}^{\infty} du' \int_{-\infty}^{u'} du'' C^{(n)}_{u\nu u\mu} . \tag{3.9}$$

We refer to $\Delta^{(n)}_{\mu\nu}$ as the $n$-th order *memory tensor*. It characterizes the memory effect at order $1/r^n$. We note that the Weyl tensor at these orders is equivalent to the linearized Weyl tensor and is gauge invariant. Therefore, the memory effect at these orders is manifestly gauge invariant.

It follows immediately from its definition, eq. (3.9), that for all $n \le d-3$ the memory tensor, $\Delta^{(n)}_{\mu\nu}$, is symmetric, trace-free, and has vanishing $u$-components,

$$\Delta^{(n)}_{\mu\nu} = \Delta^{(n)}_{\nu\mu}\,, \quad \Delta^{(n)\mu}{}_\mu = 0\,, \quad \Delta^{(n)}_{u\nu} = 0 \qquad \text{for all}\, n \le d-3. \tag{3.10}$$

Obviously, from its definition, $\Delta^{(n)}_{\mu\nu}$ does not depend on $u$, so we also have $\partial_u \Delta^{(n)}_{\mu\nu} = 0$.

Additional properties of $\Delta^{(n)}_{\mu\nu}$ follow from the Bianchi identity. We remind the reader that the uncontracted Bianchi identity is

$$\nabla_{[\mu}R_{\nu\rho]\sigma\kappa} = 0. \tag{3.11}$$

Contracting over $\mu$ and $\sigma$ yields

$$g^{\mu\sigma}\nabla_\mu R_{\nu\rho\sigma\kappa} = 2\nabla_{[\nu}R_{\rho]\kappa}. \tag{3.12}$$

Applying $g^{\mu\delta}\nabla_\delta$ to eq. (3.11) we obtain

$$\Box_g R_{\nu\rho\sigma\kappa} + g^{\lambda\mu}\nabla_\lambda\nabla_\nu R_{\rho\mu\sigma\kappa} + g^{\lambda\mu}\nabla_\lambda\nabla_\rho R_{\mu\nu\sigma\kappa} = 0. \tag{3.13}$$

Commuting the derivatives in the second and third terms of eq. (3.13) and using eq. (3.12) we obtain

$$\begin{aligned}\Box_g R_{\nu\rho\sigma\kappa} =& 4\nabla_{[\nu}\nabla_{|[\sigma}R_{\kappa]|\rho]} - 2g^{\mu\lambda}g^{\delta\pi}R_{\lambda[\nu\rho]\delta}R_{\pi\mu\sigma\kappa} - 2g^{\delta\pi}R_{\delta[\nu}R_{\rho]\pi\sigma\kappa} \\ &- 2g^{\mu\lambda}g^{\delta\pi}R_{\sigma\delta\lambda[\nu}R_{\rho]\mu\pi\kappa} - 2g^{\mu\lambda}g^{\delta\pi}R_{\delta\kappa\lambda[\nu}R_{\rho]\mu\sigma\pi}.\end{aligned} \tag{3.14}$$

We also remind the reader that the Riemann tensor is related to the Weyl tensor by

$$R_{\mu\nu\rho\sigma} = C_{\mu\nu\rho\sigma} + \frac{4}{d-2}g_{[\mu|[\rho}R_{\sigma]|\nu]} - \frac{2}{(d-1)(d-2)}Rg_{\mu[\rho}g_{\sigma]\nu}. \tag{3.15}$$

In *linearized gravity* with $R_{\mu\nu} = 0$, the above relations imply

$$\partial^\mu C_{\mu\nu\rho\sigma} = 0 \qquad \text{(linearized gravity)} \tag{3.16}$$

and

$$\Box C_{\mu\nu\rho\sigma} = 0 \qquad \text{(linearized gravity)}. \tag{3.17}$$

These relations, of course, do not hold in nonlinear general relativity, and they also do not hold in linearized gravity when $R_{\mu\nu} \neq 0$. However, let

$$E_{\mu\nu} \equiv C_{u\nu u\mu} \tag{3.18}$$

so that $E_{\mu\nu}$ is the "electric part" of the Weyl tensor. Define $\mathcal{T}_\mu$ by

$$\mathcal{T}_\mu = K^\nu \Box E_{\mu\nu} - 2K^\nu \partial_\nu \partial^\alpha E_{\alpha\mu} - (d-2)\partial^\alpha E_{\alpha\mu}/r \tag{3.19}$$

where $K^\mu = (\partial/\partial r)^\mu$. In linearized gravity with $R_{\mu\nu} = 0$, we have $\mathcal{T}_\mu = 0$. Remarkably, we find[13] that in nonlinear general relativity with our ansatz for $h_{\mu\nu}$ and $T_{\mu\nu}$, we have $\mathcal{T}_\mu^{(n+2)} = 0$ for all $n \leq d-3$. Now, the formula eq. (3.19) defining $\mathcal{T}_\mu$ is exactly the same as the formula eq. (2.92) defining $\tau_\mu^{(n+2)}$ under the substitution $h_{\mu\nu} \to E_{\mu\nu}$ and $T_{\mu\nu} \to 0$. Thus, $E_{\mu\nu}$ satisfies eqs. (2.94) and (2.95) with vanishing right side for all $n \leq d-3$. (Equation (2.93) is trivial since the $u$-components of $E_{\mu\nu}$ vanish.) Integrating this equation twice with respect to $u$, we find that for $n \leq d-3$, $\Delta_{\mu\nu}^{(n)}$ satisfies

$$[\mathscr{D}^2 - (n-d+1)(n-d+2)]\Delta_{rr}^{(n)} + (2n-d)\mathscr{D}^A \Delta_{Ar}^{(n)} = 0 \tag{3.20}$$

$$[\mathscr{D}^2 - (n-d+3)(n-d+2) + (2n-d+1)]\Delta_{rA}^{(n)} + 2\mathscr{D}_A \Delta_{rr}^{(n)} + (2n-d+2)\mathscr{D}^B \Delta_{AB}^{(n)} = 0 \tag{3.21}$$

where we used the fact that the trace of $\Delta_{\mu\nu}$ vanishes to relate $\Delta_{rr}$ to $q^{AB}\Delta_{AB}$. We note that eqs. (3.20) and (3.21) have nothing to do with the harmonic gauge condition and hold for $d = 4$ as well as $d > 4$.

13. The peeling properties of the Weyl tensor [58] (which are a consequence of our ansatz) were used to show this.

These relations will be used in sec. 3.4 below. They also have the following important consequence. The spherically symmetric ($\ell = 0$) part of $\Delta_{\mu\nu}$ automatically has $\Delta_{rA} = 0$ and $\Delta_{AB} \propto q_{AB}$, since no vector on the sphere can be spherically symmetric and $q_{AB}$ is the only tensor of this index type that is spherically symmetric. Consequently, eq. (3.20) implies that the spherically symmetric part of $\Delta^{(n)}_{\mu\nu}$ vanishes for $n \leq d-3$. Similar arguments also show that eqs. (3.20) and (3.21) imply that the $\ell = 1$ part of $\Delta^{(n)}_{\mu\nu}$ vanishes for all $n \leq d-3$. This implies that

$$[\Delta^{(n)}_{\mu\nu}]|_{\ell=0,1} = 0 \qquad \text{for all } n \leq d-3. \tag{3.22}$$

In addition, in $d = 4$ dimensions, eqs. (3.20) and (3.21) imply

$$\Delta^{(1)}_{r\nu} = 0 \tag{3.23}$$

and, similarly, in $d = 6$ dimensions, we obtain

$$\Delta^{(3)}_{rr} = 0. \tag{3.24}$$

However, in higher dimensions, all components of the Coulombic order memory tensor (other than $u$ components and the trace) may be nonvanishing. These results in $d = 4$ and $d = 6$ dimensions also follow directly from the peeling properties of the Weyl tensor in these dimensions [58].

## *3 Evaluation of the memory tensor at Coulombic order and slower fall-off*

We now evaluate $\Delta^{(n)}_{\mu\nu}$ for all $n \leq d-3$. We separately consider the cases (1) $d > 4$ and even, (2) $d$ odd, and (3) $d = 4$. For $d > 4$, we impose the harmonic gauge condition to greatly simplify the analysis.

## 1 $d$ even, $d > 4$

For $n \leq d-3$, the relevant components of the $n$th order Weyl tensor take the form

$$C^{(n)}_{u\alpha u\beta} = \alpha^{(n)}_{\alpha\beta}{}^{\rho\sigma} h^{(n-2)}_{\rho\sigma} + \beta^{(n)}_{\alpha\beta}{}^{\rho\sigma} \partial_u h^{(n-1)}_{\rho\sigma} + \gamma^{(n)}_{\alpha\beta}{}^{\rho\sigma} \partial^2_u h^{(n)}_{\rho\sigma} \tag{3.25}$$

Here $\alpha^{(n)}_{\alpha\beta}{}^{\rho\sigma}, \beta^{(n)}_{\alpha\beta}{}^{\rho\sigma}, \gamma^{(n)}_{\alpha\beta}{}^{\rho\sigma}$ are given by

$$\alpha^{(n)}_{\alpha\beta}{}^{\rho\sigma} = -\frac{1}{2}(n-1)(n-2) r_\alpha r_\beta n^\rho n^\sigma + (n-2) n^\rho n^\sigma r_{(\alpha}\mathscr{D}_{\beta)} - \frac{1}{2} n^\rho n^\sigma \mathscr{D}_\alpha \mathscr{D}_\beta + \frac{1}{2}(n-2) q_{\alpha\beta} n^\rho n^\sigma, \tag{3.26}$$

$$\begin{aligned} \beta^{(n)}_{\alpha\beta}{}^{\rho\sigma} =& -(n-1) r_\alpha r_\beta n^{(\rho} K^{\sigma)} + n^{(\rho} K^{\sigma)} r_{(\alpha}\mathscr{D}_{\beta)} - n r_{(\alpha} n^{(\rho} q_{\beta)}{}^{\sigma)} + n^\rho q_{(\beta}{}^\sigma \mathscr{D}_{\alpha)} \\ &+ \frac{1}{2} q_{\alpha\beta}(2 n^{(\rho} K^{\sigma)} - n^{c\rho} n^\sigma), \end{aligned} \tag{3.27}$$

$$\gamma^{(n)}_{\alpha\beta}{}^{\rho\sigma} = -\frac{1}{2} r_\alpha r_\beta K^\rho K^\sigma - r_{(\alpha} K^{(\rho} q_{\beta)}{}^{\sigma)} - \frac{1}{2} q_\alpha{}^\rho q_\beta{}^\sigma \tag{3.28}$$

where $K^\mu = (\partial/\partial r)^\mu$, $n^\mu = (\partial/\partial u)^\mu$ and $r_\mu = (dr)_\mu$.

We now use the recursion relations to eliminate $h^{(n-2)}_{\mu\nu}$ and $h^{(n-1)}_{\mu\nu}$ in favor of $h^{(n)}_{\mu\nu}$ in eq. (3.25). We consider, first, the case $n < d-3$; we will treat the case $n = d-3$ after we have completed the analysis for $n < d-3$.

For $n < d-3$, the relevant recursion relations do not contain any nonlinear terms in $h_{\mu\nu}$ and are thus given by eqs. (2.61)-(2.70) with $\chi^\mu = 0$. In addition, the stress-energy tensor does not appear in any equations at the orders relevant to this analysis. It is clear from the arguments that led to Theorem 2 that it must be possible to eliminate $h^{(n-2)}_{\mu\nu}$ and $h^{(n-1)}_{\mu\nu}$ in favor of $h^{(n)}_{\mu\nu}$, but it is useful to have an explicit construction, which we now give.

First, we can directly invert the angular operator appearing in eq. (2.61) to solve for

$\overline{h}_{uu}^{(n-1)}$ in terms of $\overline{h}_{uu}^{(n)}$. Explicitly, we have

$$\overline{h}_{uu}^{(n-1)} = -(2n-d+2)\Big[\mathscr{D}^2 + (n-1)(n-d+2)\Big]^{-1}\partial_u \overline{h}_{uu}^{(n)}. \tag{3.29}$$

Note that $\overline{h}_{uu}^{(n)}$ appears in this solution only in the form $\partial_u \overline{h}_{uu}^{(n)}$. We then iterate this procedure to obtain $\overline{h}_{uu}^{(n-2)}$ in terms of $\overline{h}_{uu}^{(n-1)}$ and thence $\overline{h}_{uu}^{(n)}$, thereby expressing $\overline{h}_{uu}^{(n-2)}$ in terms of inverse angular operators applied to $\partial_u^2 \overline{h}_{uu}^{(n)}$. Next, we eliminate $\mathscr{D}^A \overline{h}_{Au}^{(n-1)}$ using eq. (2.68) (with $\chi_u^{(n)} = 0$) and substitute into eq. (2.62). The resulting equation can then be solved for $\overline{h}_{ur}^{(n-1)}$ in terms of $\partial_u \overline{h}_{ur}^{(n)}$ and $\partial_u \overline{h}_{uu}^{(n)}$. Iterating, we obtain $\overline{h}_{ur}^{(n-2)}$ in terms of $\partial_u^2 \overline{h}_{ur}^{(n)}$ and $\partial_u^2 \overline{h}_{uu}^{(n)}$. We then similarly invert eq. (2.63) to solve for $\overline{h}_{uA}^{(n-1)}$ and then $\overline{h}_{uA}^{(n-2)}$.

Thus far, we have shown how to write the $uu$, $ur$, and $uA$ components of $\overline{h}_{\mu\nu}$ at orders $n-2$ and $n-1$ in terms of these components at $n$-th order. To proceed further, we note that $\overline{h} \equiv \overline{h}^{\mu}{}_{\mu} = -2\overline{h}_{ur} + \overline{h}_{rr} + q^{AB}\overline{h}_{AB}$ satisfies the ordinary scalar wave equation. Hence, we can recursively solve for $\overline{h}^{(n-1)}$ and $\overline{h}^{(n-2)}$ in terms of $\partial_u \overline{h}^{(n)}$ and $\partial_u^2 \overline{h}^{(n)}$ respectively. Then one can use eq. (2.64) and eq. (2.69) to obtain

$$\Big[\mathscr{D}^2 + (n-d+1)(n-2)]\overline{h}_{rr}^{(n-1)} = 2(d-2n+2)\overline{h}_{ur}^{(n-1)} + 2\overline{h}^{(n-1)} - (2n-d-2)\partial_u \overline{h}_{rr}^{(n)}. \tag{3.30}$$

This equation can be used to solve for $\overline{h}_{rr}^{(n-1)}$ and $\overline{h}_{rr}^{(n-2)}$ in terms of $n$th order quantities. We can then use eq. (2.70) and eq. (2.65) to solve for $\overline{h}_{rA}^{(n-1)}$ and $\overline{h}_{rA}^{(n-2)}$ in terms of $n$th order quantities. Finally, we solve (2.66) to obtain $\overline{h}_{AB}^{(n-1)}$ and $\overline{h}_{AB}^{(n-2)}$ in terms of $n$th order quantities.

The above results show explicitly that we can write $h_{\mu\nu}^{(n-2)}$ as an operator (composed of inverses of angular operators and angular derivatives) applied to $\partial_u^2 h_{\mu\nu}^{(n)}$. Similarly, we can write $h_{\mu\nu}^{(n-1)}$ as such an operator applied to $\partial_u h_{\mu\nu}^{(n)}$. Substituting this result in eq. (3.25), we see that for all $n < d-3$, the $n$th order Weyl tensor takes the form

$$C^{(n)}_{u\alpha u\beta} = O^{(n)}_{\alpha\beta}{}^{\rho\sigma}\partial_u^2 \bar{h}^{(n)}_{\rho\sigma} \tag{3.31}$$

where the operator $O$ is constructed of inverses of angular operators and angular derivatives. It follows immediately from eq. (3.9) that for $n < d-3$ the memory tensor takes the form

$$\Delta^{(n)}_{\mu\nu} = P^{(n)}{}_{\mu\nu}{}^{\rho\sigma}[\Delta\bar{h}^{(n)}_{\rho\sigma}] \qquad \text{for } n < d-3 \tag{3.32}$$

where

$$\Delta\bar{h}^{(n)}_{\mu\nu} \equiv \bar{h}^{(n)}_{\mu\nu}(u \to \infty) - \bar{h}^{(n)}_{\mu\nu}(u \to -\infty) \tag{3.33}$$

and $P^{(n)}{}_{\mu\nu}{}^{\rho\sigma}$ is a linear operator constructed from inverses of angular operators and angular derivatives. However, as already remarked below eq. (3.5), we have $h^{(n)}_{\mu\nu} = 0$ for all $n < d-3$ when the metric is stationary at Coulombic order. Thus, the memory tensor vanishes at slower than Coulombic fall-off

$$\Delta^{(n)}_{\mu\nu} = 0 \qquad \text{for } n < d-3. \tag{3.34}$$

In particular, for $d > 4$ the memory tensor vanishes at radiative order [20]. Now consider the case $n = d-3$. The calculation $\Delta^{(d-3)}_{\mu\nu}$ differs from the above calculation for $n < d-3$ only in that (i) $\Delta\bar{h}^{(d-3)}_{\mu\nu}$ need not vanish and (ii) the recursion relations eqs. (2.61)-(2.66) used to solve for $h^{(d-4)}_{\mu\nu}$ will now contain the additional terms $T^{(d-2)}_{\mu\nu}$ and $\mathcal{G}^{(d-2)}_{\mu\nu}$ (see eq. (2.101)). With regard to these additional terms the only nonvanishing component of $T^{(d-2)}_{\mu\nu}$ is $T^{(d-2)}_{uu}$. Similarly, it can be seen from eq. (2.102) that all of the components of $\mathcal{G}^{(d-2)}_{\mu\nu}$ except $\mathcal{G}^{(d-2)}_{uu}$ are $u$-derivatives of quantities that vanish in stationary eras. It is not difficult to show that the total $u$-derivative terms do not contribute to $\Delta^{(d-3)}_{\mu\nu}$ under our stationarity conditions. Thus, the terms involving $T^{(d-2)}_{\mu\nu}$ and $\mathcal{G}^{(d-2)}_{\mu\nu}$ give rise to additional terms in the memory tensor that are proportional to the integral of the total flux, $F$, of matter and gravitational energy to null infinity

$$F \equiv T_{uu}^{(d-2)} + \frac{1}{32\pi} N^{AB} N_{AB}. \tag{3.35}$$

Carrying through the calculation of $\Delta_{\mu\nu}^{(d-3)}$ in the manner described above, we obtain the final formula

$$\Delta_{\mu\nu}^{(d-3)} = P_{\mu\nu}[\Delta \overline{h}_{\rho\sigma}^{(d-3)}]_{\ell>1} + \int_{-\infty}^{\infty} du L_{\mu\nu}[F]_{\ell>1} \tag{3.36}$$

where

$$\begin{aligned}
P_{\mu\nu}[\Delta\overline{h}_{\rho\sigma}^{(d-3)}] = {} & \frac{1}{2} r_\mu r_\nu \Bigg[ (d-3)(d-4)^2(d-6)\mathscr{D}_5^{-2}\mathscr{D}_4^{-2}\Delta\overline{h}_{uu}^{(d-3)} + \Delta\overline{h}_{rr}^{(d-3)} \\
& + 4\mathscr{D}_{(\mu}\mathscr{D}_3^{-2}\mathscr{D}_{\nu)}\mathscr{D}_4^{-2} + \frac{(d-4)^2}{d-2}\Big((d-5)(d-6)D_5^{-2} - 2\Big)D_4^{-2}\Delta\overline{h}^{(d-3)} \Bigg] \\
& - (d-4)^2(d-6) r_{(\mu}\mathscr{D}_{\nu)}\mathscr{D}_5^{-2}\mathscr{D}_4^{-2}\Delta\overline{h}_{uu}^{(d-3)} - 2(d-3)(d-4)\mathscr{D}_3^{-2} r_{(\mu}\mathscr{D}_{\nu)}\mathscr{D}_4^{-2}\Delta\overline{h}_{uu}^{(d-3)} \\
& - 2(d-3)(d-4)(d-6)\mathscr{D}_3^{-2} r_{(\mu}\mathscr{D}_{\nu)}\mathscr{D}_5^{-2}\mathscr{D}_4^{-2}\Delta\overline{h}_{uu}^{(d-3)} - (d-3)(d-4) r_{(\mu}q_{\nu)}{}^\rho \Delta\overline{h}_{\rho u}^{(d-3)} \\
& - \frac{d-4}{d-2} r_{(\mu}\mathscr{D}_{\nu)}\Big((d-5)(d-6)\mathscr{D}_5^{-2} - 1\Big)\mathscr{D}_4^{-2}\Delta\overline{h}^{(d-3)} + (d-6) r_{(\mu}\mathscr{D}_{\nu)}\mathscr{D}_5^{-2}\Delta\overline{h}_{ru}^{(d-3)} \\
& + 2(d-6)(d-3)\mathscr{D}_3^{-2} r_{(\mu}\mathscr{D}_{\nu)}\mathscr{D}_5^{-2}\Delta\overline{h}_{ru}^{(d-3)} + \frac{1}{2}(d-4)\Big( -(d-6)\mathscr{D}_\mu\mathscr{D}_\nu\mathscr{D}_5^{-2} + q_{\mu\nu} \\
& + 4\mathscr{D}_{(\mu}\mathscr{D}_3^{-2}\mathscr{D}_{\nu)}\mathscr{D}_4^{-2} - 4(d-6)\mathscr{D}_{(\mu}\mathscr{D}_3^{-2}\mathscr{D}_{\nu)}\mathscr{D}_5^{-2} + (d-6)(d-7)q_{\mu\nu}\mathscr{D}_5^{-2}\Big)\mathscr{D}_4^{-2}\Delta\overline{h}_{uu}^{(d-3)} \\
& + \frac{1}{2}\frac{d-4}{d-2}\Big( -(d-6)\mathscr{D}_\mu\mathscr{D}_\nu\mathscr{D}_5^{-2} - q_{\mu\nu} + (d-5)(d-6)q_{\mu\nu}\mathscr{D}_5^{-2}\Big)\mathscr{D}_4^{-2}\Delta\overline{h}^{(d-3)} \\
& - \left(2(d-6)\mathscr{D}_{(\mu}\mathscr{D}_3^{-2}\mathscr{D}_{\nu)}\mathscr{D}_5^{-2} - (d-6)q_{\mu\nu}\mathscr{D}_5^{-2} - \frac{1}{d-2}q_{\mu\nu}\right)\Delta\overline{h}_{ru}^{(d-3)} - \frac{q_{\mu\nu}}{d-2}\Delta\overline{h}_{rr}^{(d-3)} \\
& + (d-4)\mathscr{D}_{(\mu}q_{\nu)}{}^\rho\Delta\overline{h}_{\rho u}^{(d-3)} + r_{(\mu}q_{\nu)}{}^\rho\Delta\overline{h}_{\rho r}^{(d-3)} + \frac{1}{2}\left(q_\mu{}^\rho q_\nu{}^\sigma - \frac{1}{d-2}q_{\mu\nu}q^{\rho\sigma}\right)\Delta\overline{h}_{\rho\sigma}^{(d-3)}
\end{aligned} \tag{3.37}$$

and

$$L_{\mu\nu} = 8\pi\Big[r_\mu r_\nu(d-3)(d-4)(d-6)\mathscr{D}_5^{-2}\mathscr{D}_4^{-2} - 2\Big((d-4)(d-6)r_{(\mu}\mathscr{D}_{\nu)}\mathscr{D}_5^{-2}$$
$$- 2(d-3)(d-6)\mathscr{D}_3^{-2}r_{(\mu}\mathscr{D}_{\nu)}\mathscr{D}_5^{-2}\Big)\mathscr{D}_4^{-2} + \Big(-(d-6)\mathscr{D}_\mu\mathscr{D}_\nu\mathscr{D}_5^{-2}$$
$$- 4(d-6)\mathscr{D}_{(\mu}\mathscr{D}_3^{-2}\mathscr{D}_{\nu)}\mathscr{D}_5^{-2} + q_{\mu\nu} + (d-6)(d-7)q_{\mu\nu}\mathscr{D}_5^{-2}\Big)\mathscr{D}_4^{-2}$$
$$+ 4\mathscr{D}_{(\mu}\mathscr{D}_3^{-2}\mathscr{D}_{\nu)}\mathscr{D}_4^{-2}\mathscr{D}_4^{-2} + 4(d-3)\mathscr{D}_3^{-2}r_{(\mu}\mathscr{D}_{\nu)}\mathscr{D}_4^{-2}\Big]F. \tag{3.38}$$

Here, in order to write these equations in a more compact form, we have introduced the notation

$$\mathscr{D}_3^2 \equiv [\mathscr{D}^2 - (d-3)] \tag{3.39}$$
$$\mathscr{D}_4^2 \equiv [\mathscr{D}^2 - (d-4)] \tag{3.40}$$
$$\mathscr{D}_5^2 \equiv [\mathscr{D}^2 - 2(d-5)]. \tag{3.41}$$

The notation $[\cdot]_{\ell>1}$ in eq. (3.36) means that only the $\ell > 1$ part of the quantity is to be taken. The memory tensor $\Delta^{(d-3)}_{\mu\nu}$ has only $\ell > 1$ spherical harmonic parts (see eq. (3.22)). However, $\Delta\bar{h}^{(d-3)}_{\rho\sigma}$ and $F$ have $\ell = 0, 1$ parts. The $\ell = 0, 1$ parts of $\Delta\bar{h}^{(d-3)}_{\rho\sigma}$ and $F$ should be excluded from eqs. (3.37) and (3.38) for the computation of ordinary and null memory.

Equation (3.36) naturally splits the memory tensor into a "null memory" piece associated with the flux $F$ of stress-energy and/or Bondi news to null infinity, and an "ordinary memory" piece associated with the change in the metric in harmonic gauge at Coulombic order. The ordinary memory piece can be rewritten in terms of $\Delta E^{(d-1)}_{\mu\nu} = \Delta C^{(d-1)}_{\mu u \nu u}$ as follows[14]:

$$
\begin{aligned}
&P_{\mu\nu}[\Delta \bar{h}^{(d-3)}_{\rho\sigma}] = -r_\mu r_\nu (d-4)(d-6)\mathscr{D}_5^{-2}\mathscr{D}_4^{-2}\Delta E^{(d-1)}_{rr} \\
&+ 2(d-4)r_{(\mu}\mathscr{D}_{\nu)}(\mathscr{D}^2-2)\mathscr{D}^{-2}\mathscr{D}_5^{-2}\mathscr{D}_4^{-2}\Delta E^{(d-1)}_{rr} \\
&+ d(d-2)q_\mu{}^\rho q_\nu{}^\sigma(\mathscr{D}^2-2)^{-1}\mathscr{D}_{-4}^{-2}\Delta E^{(d-1)}_{\rho\sigma} + \frac{(d-4)(d-6)}{d-2}q_{\mu\nu}\mathscr{D}_5^{-2}\mathscr{D}_4^{-2}\Delta E^{(d-1)}_{rr} \\
&+ 2d(d-2)(\mathscr{D}^2-2)^{-1}\mathscr{D}_{-4}^{-2}\mathscr{D}_{(\mu}\mathscr{D}_{-3}^{-2}\mathscr{D}_{\nu)}\mathscr{D}^{-2}\mathscr{D}^\lambda\mathscr{D}^\kappa\Delta E^{(d-1)}_{\lambda\kappa} \\
&- \frac{d(d-2)^2}{d-3}(\mathscr{D}^2-2)^{-1}\mathscr{D}_{-4}^{-2}\left(\mathscr{D}_\mu\mathscr{D}_\nu - \frac{1}{d-2}q_{\mu\nu}\mathscr{D}^2\right)\mathscr{D}_{-2}^{-2}\mathscr{D}^{-2}\mathscr{D}^\lambda\mathscr{D}^\kappa\Delta E^{(d-1)}_{\lambda\kappa} \\
&+ (\mathscr{D}^2-2)^{-1}\mathscr{D}_{-4}^{-2}\left(2d\mathscr{D}_{(\mu}\mathscr{D}_{-3}^{-2}\mathscr{D}_{\nu)} + d\left(\mathscr{D}_\mu\mathscr{D}_\nu - \frac{1}{d-2}q_{\mu\nu}\mathscr{D}^2\right)\mathscr{D}_{-2}^{-2} + dq_{\mu\nu}\right)\Delta E^{(d-1)}_{rr} \\
&+ 2(d-2)\mathscr{D}_{(\mu}\mathscr{D}_{-3}^{-2}\mathscr{D}_{-5}^{2}\mathscr{D}_3^{-2}(\mathscr{D}^2-1)^{-1}(q_{\nu)}{}^\lambda - \mathscr{D}_{\nu)}\mathscr{D}^{-2}\mathscr{D}^\lambda)\Delta E^{(d-1)}_{r\lambda} \\
&+ \frac{1}{d-3}\left(\mathscr{D}_\mu\mathscr{D}_\nu - \frac{1}{d-2}q_{\mu\nu}\mathscr{D}^2\right)[(d-6) - 2(d-4)(\mathscr{D}^2-2)\mathscr{D}^{-2}]\mathscr{D}_5^{-2}\mathscr{D}_4^{-2}\Delta E^{(d-1)}_{rr} \\
&- 2(d-2)(d-4)\mathscr{D}_3^{-2}(\mathscr{D}^2-1)^{-1}r_{(\mu}(q_{\nu)}{}^\lambda - \mathscr{D}_{\nu)}\mathscr{D}^{-2}\mathscr{D}^\lambda)\Delta E^{(d-1)}_{r\lambda} \\
&- 2d(d-2)(\mathscr{D}^2-2)^{-1}\mathscr{D}_{-4}^{-2}\mathscr{D}_{(\mu}\mathscr{D}_{-3}^{-2}\mathscr{D}^\lambda\Delta E^{(d-1)}_{\nu)\lambda}
\end{aligned}
\tag{3.42}
$$

where

$$\mathscr{D}^2_{-4} \equiv [\mathscr{D}^2 + (d-4)] \tag{3.43}$$

$$\mathscr{D}^2_{-5} \equiv [\mathscr{D}^2 + (d-5)] \tag{3.44}$$

$$\mathscr{D}^2_{-3} \equiv [\mathscr{D}^2 + (d-3)] \tag{3.45}$$

$$\mathscr{D}^2_{-2} \equiv [\mathscr{D}^2 + (d-2)]. \tag{3.46}$$

Again, the The $\ell = 0, 1$ parts of $\Delta E^{(d-1)}_{\rho\sigma}$ should be excluded from eq. (3.42) for the computation of memory.

14. It should be possible to derive eqs. (3.38) and (3.42) directly from eqs. (3.12) and (3.14), bypassing the need to introduce the harmonic gauge. We have shown that such a derivation can be given in linearized gravity with sources.

Since $F$ is gauge invariant, null memory is manifestly gauge invariant. Since $C^{(d-1)}_{\alpha\beta\gamma\delta}$ is gauge invariant in stationary eras, ordinary memory is also manifestly gauge invariant when expressed in the form of eq. (3.42). We shall see in sec. 3.3.3 that eqs. (3.38) and (3.42) also hold in $d = 4$.

We now consider the effects on the memory tensor at Coulombic order of placing stronger stationarity conditions than those imposed by eq. (3.3) on the metric at early and late times in even dimensions. Specifically, suppose we were to require that

$$\partial_u h^{(k)}_{\mu\nu} \to 0 \quad \text{as} \quad u \to \pm\infty \text{ for } k \leq d-2, \tag{3.47}$$

i.e., suppose that we require stationarity at one order faster fall-off than Coulombic. Suppose that, in addition, we require

$$T^{(d-1)}_{\mu\nu} \to 0 \quad \text{as} \quad u \to \pm\infty\,. \tag{3.48}$$

In the stationary eras, the nonlinear terms in Einstein's equation are $O(1/r^{2(d-2)})$ and will not enter our analysis to the orders we consider. It then follows from Remark 2.5 that $\bar{h}^{(n)}_{\mu\nu}$ can be put in harmonic gauge such that in the stationary eras, we have $\partial_u h^{(n)}_{\mu\nu} = 0$ for all $n \leq d-2$. It further follows from Cor. 2.1 that, in the stationary eras, $\bar{h}^{(n)}_{\mu\nu} = 0$ for all $n < d-3$. Furthermore, $h^{(d-3)}_{\mu\nu}$ satisfies eqs. (2.61)-(2.66) for $n = d-2$ with all terms involviing $\chi_\mu$, $u$-derivatives, and stress-energy put to zero. In addition, $h^{(d-3)}_{\mu\nu}$ satisfies eqs. (2.93)-(2.95) for $n = d-3$ with vanishing stress-energy terms. It is not difficult to show that the unique solution to these equations is

$$\bar{h}^{(d-3)}_{uu} = \bar{h}^{(d-3)}_{ur} = \bar{h}^{(d-3)}_{rr} = const. \tag{3.49}$$

with all other components vanishing. This corresponds to the Schwarzschild solution in harmonic gauge at Coulombic order. Thus, with the stronger stationarity conditions eq. (3.47) and eq. (3.48), the solution approaches the Schwarzschild solution (possibly with different masses) at early and late retarded times. Thus, $\Delta h^{(d-3)}_{\mu\nu}$ has only an $\ell = 0$ part, and cannot contribute to memory by eq. (3.22). Thus, if the stronger stationarity conditions eq. (3.47) and eq. (3.48) hold at early and late retarded times, then ordinary memory vanishes (but a nonvanishing null memory effect may still occur).

## 2 $d$ odd

For $d$ odd, the analysis of the memory effect for $n < d-3$—where $n$ is now half-integral—follows the even dimensional case exactly, and we find that

$$\Delta^{(n)}_{\mu\nu} = 0 \qquad \text{for } n < d-3. \tag{3.50}$$

Since $\tilde{h}^{(d-3)}_{\mu\nu}$ is the leading order term in the integer power part of the expansion of $h_{\mu\nu}$ (see eq. (2.59)), the only contribution to $C^{(d-3)}_{u\mu u\nu}$ is

$$C^{(d-3)}_{u\mu u\nu} = \gamma^{(d-3)\rho\sigma}_{\mu\nu}\partial_u^2\tilde{h}^{(d-3)}_{\rho\sigma} \tag{3.51}$$

where the $\gamma^{(d-3)\rho\sigma}_{\mu\nu}$ is given by eq. (3.28) with $n = d-3$. Einstein's equation in harmonic gauge yields

$$(d-4)\partial_u\bar{\tilde{h}}^{(d-3)}_{\mu\nu} = -16\pi T^{(d-2)}_{\mu\nu} + 2\mathcal{G}^{(d-2)}_{\mu\nu}. \tag{3.52}$$

However, we have

$$T^{(d-2)}_{\mu\nu} = T^{(d-2)}_{uu}K_\mu K_\nu \tag{3.53}$$

and

$$\mathcal{G}^{(d-2)}_{\mu\nu} = -\frac{1}{4}N^{AB}N_{AB}K_\mu K_\nu + \partial_u B_{\mu\nu} \tag{3.54}$$

where $B_{\mu\nu}$ vanishes in stationary eras. From eq. (3.28), it is easily seen that $\gamma^{(d-3)\rho\sigma}_{\mu\nu} K_\rho K_\sigma = 0$. It can also be seen immediately from eq. (3.9) and eq. (3.51) that $B_{\mu\nu}$ cannot contribute to $\Delta_{\mu\nu}$. Thus, we find that for $d$ odd,

$$\Delta^{(d-3)}_{\mu\nu} = 0 \qquad \text{for } d \text{ odd.} \tag{3.55}$$

and thus the memory effect vanishes at Coulombic order (as well as slower fall-off) in odd dimensions.

At first sight, it may seem paradoxical that there is a major difference between odd and even dimensions in the memory effect at Coulombic order: First, in odd dimensions there is a flux of energy to null infinity at order $1/r^{d-2}$ in exact parallel with the even dimensional case, so why isn't there a null memory contribution at Coulombic order? Second, if one considers, e.g., the scattering of timelike particles, one would expect that the retarded solution at early and late times should behave like eq. (3.2) at late and early times, potentially giving rise to a nonvanishing $\Delta h_{\mu\nu}$ at Coulombic order in odd dimensions. Why doesn't this give rise to an ordinary memory effect?

The answer to the first question is that the key difference that occurs in odd dimensions—as compared with even dimensions with $d > 4$—is that terms with integer power fall-off slower than $1/r^{d-3}$ are not permitted. In even dimensions with $d > 4$, the possible presence of a nonvanishing $h^{(d-4)}_{\mu\nu}$ and $h^{(d-5)}_{\mu\nu}$ effectively makes the null and ordinary memory independent. In odd dimensions, there can, indeed, be a null memory effect, but it is always exactly canceled by ordinary memory.

The answer to the second question is more subtle and has to do with the manner in which the retarded solution approaches a solution of the form eq. (3.2) at late times for particles moving on inertial trajectories. To see this, it is illuminating to consider the concrete example of the retarded solution, $\phi$, to the scalar wave equation eq. (2.16) with source corresponding to the creation of a scalar particle with scalar charge $q$ at time $t = 0$ at the origin in $5-$dimensional Minkowksi spacetime

$$S = q\theta(t)\delta^{(4)}(\vec{x}). \tag{3.56}$$

The exact retarded solution for such a source is

$$\phi = q\frac{\theta(u)}{(2\pi r)^2}\frac{r+u}{\sqrt{u(2r+u)}}. \tag{3.57}$$

For $r \gg u$, eq. (3.57) admits an expansion in half-integer powers of $1/r$ fully consistent with our ansatz eq. (2.12)

$$\phi = \frac{q}{2\sqrt{\pi}(2\pi r)^{3/2}}\frac{\theta(u)}{\sqrt{u}} + \frac{3\sqrt{\pi}q}{4(2\pi r)^{5/2}}\theta(u)\sqrt{u} + O(1/r^{7/2}). \tag{3.58}$$

No integer powers of $1/r$ occur. In particular, at all retarded times, the scalar field vanishes at Coulombic order. In addition, as $u \to \infty$, we have $\phi^{(3/2)} \to 0$, so $\phi$ vanishes at late retarded time at Coulombic and slower fall-off. On the other hand, if we fix $r$ and take the limit of the exact solution eq. (3.57) as $u \to \infty$, we obtain the Coulomb solution

$$\lim_{u\to\infty} \phi = \frac{q}{4\pi^2 r^2}. \tag{3.59}$$

In other words, $\phi$ approaches the Coulomb solution at *timelike infinity* ($u \to \infty$ at fixed $r$), but does *not* approach the Coulomb solution at *null infinity* ($r \to \infty$ at fixed $u$) even if $u$ is then taken to be arbitrarily large. In other words, the Coulomb solution eq. (3.59) will not be evident to an observer unless he waits a time much longer than the light travel time to the source.

Similarly, in the gravitational case in any odd dimension $d \geq 5$, consider classical particle scattering wherein the particles move on timelike, inertial trajectories at early and late times. Then at early retarded times, the retarded solution at Coulombic order, $\tilde{h}^{(d-3)}_{\mu\nu}$, will have the multipolar structure corresponding to the incoming particles, as in eq. (3.2). However, except for $h_{uu}$, this multipolar structure will not change with $u$ and will remain the same as $u \to \infty$, i.e., $\Delta\tilde{h}^{(d-3)}_{\mu\nu} = 0$ except for $\mu = \nu = u$. The ordinary memory effect that may result from a nonvanishing $\Delta\tilde{h}^{(d-3)}_{uu}$ will be exactly canceled by the null memory effect.

Thus, the total memory effect vanishes at Coulombic order in odd dimensions. However, the above considerations suggest that it may be possible to define a notion of a memory effect at timelike infinity that would be nonvanishing.

## 3 $d = 4$

In dimension $d = 4$, radiative and Coulombic order coincide, since $d/2 - 1 = d - 3 = 1$. Our analysis for $d > 4$ was based upon the imposition of the harmonic gauge, so it cannot be applied[15] in nonlinear gravity when $d = 4$ if $T^{(2)}_{\mu\nu} \neq 0$ or $N_{AB} \neq 0$. Thus, we cannot impose the harmonic gauge, i.e., we cannot set $\chi_\mu = 0$ in eqs. (2.61)-(2.70), nor can we use the corresponding simplifications in calculating the nonlinear terms in Einstein's equation. Nevertheless, the properties of the metric perturbation at radiative order described in the paragraph below eq. (2.72) still apply. In particular, the Bondi news tensor

$$N_{AB} = \left(q_A{}^C q_B{}^D - \frac{1}{2} q_{AB} q^{CD}\right) \partial_u h^{(1)}_{CD} \tag{3.60}$$

15. However, our harmonic gauge analysis can be applied in linearized gravity to the case where $T^{(2)}_{\mu\nu} = 0$, in which case ordinary memory is possible.

is gauge invariant. Furthermore, at radiative (= Coulombic) order, the only components of the Weyl tensor that can be nonvanishing are the $uAuB$ components, which are given by

$$C^{(1)}_{uAuB} = -\frac{1}{2}\partial_u N_{AB} = -\frac{1}{2}\partial_u^2 h^{(1)}_{AB}. \tag{3.61}$$

Integrating this equation twice, we immediately obtain the following extremely simple formula for the memory tensor:

$$\Delta^{(1)}_{AB} = \frac{1}{2}\Delta h^{(1)}_{AB} \tag{3.62}$$

where $\Delta h^{(1)}_{AB}$ denotes the difference between $h^{(1)}_{AB}$ in the initial and final stationary eras. Equation (3.62) holds in any gauge compatible with our ansatz.

To proceed further we use Einstein's equations eqs. (2.61)-(2.70) with $\chi_a$ *not* put to zero and with $T^{(2)}_{ab}$ replaced by $T^{(2)}_{ab} - \mathcal{G}^{(2)}_{ab}/8\pi$. These equations can be simplified significantly by restricting consideration to the case $h^{(1)}_{rr} = 0$ (see Appendix 0.3), in which case we can impose the Bondi gauge conditions $h_{rr} = h_{rA} = 0$ and $\partial_r(\det(h_{AB})) = 0$. Einstein's equations do not directly yield an equation for $\partial_u h^{(1)}_{AB}$, but they do yield an equation for $\partial_u \mathscr{D}^B h^{(1)}_{AB}$, which can be integrated to obtain $\mathscr{D}^B \Delta^{(1)}_{AB}$ and thence $\Delta^{(1)}_{AB}$. We will not carry out the analysis here, as it has already been done by many authors[16] [10, 19, 40, 39]. The final result is that the memory tensor in 4 dimensions can be expressed as [19, 40, 39]

$$\Delta^{(1)}_{AB} = [P_{AB}]_{\ell>1} + \int_{-\infty}^{\infty} du L_{AB}[F]_{\ell>1}. \tag{3.63}$$

16. References [10, 39] worked in the context of linearized gravity whereas [40, 19] analyzed the memory effect in four dimensions in full, nonlinear general relativity. References [39, 40, 19] considered contributions to memory from null sources whereas [10] did not. References [10, 19, 40, 39] all considered ordinary memory effects in which $\Delta Q = 0$ in eq. (3.64).

Here the “ordinary memory” (the first term in eq. (3.63)) is given by

$$P_{AB} = -2\left(\mathscr{D}_A\mathscr{D}_B - \frac{1}{2}q_{AB}\mathscr{D}^2\right)\mathscr{D}^{-2}(\mathscr{D}^2+2)^{-1}\Delta P + 2\epsilon_{(A}{}^{C}\mathscr{D}_{B)}\mathscr{D}_C\mathscr{D}^{-2}(\mathscr{D}^2+2)^{-1}\Delta Q \tag{3.64}$$

where

$$P \equiv C^{(3)}_{urur}, \tag{3.65}$$

$$Q \equiv \frac{1}{2}\epsilon^{\mu\nu}C^{(3)}_{\mu\nu ru}, \tag{3.66}$$

and $\Delta P$ and $\Delta Q$ correspond to the difference in these quantities at early and late retarded times. Only the $\ell > 1$ parts of $\Delta P$ and $\Delta Q$ enter the formula for memory. The contributions to ordinary memory of $\Delta P$ and $\Delta Q$ are usually referred to as its “electric parity" and “magnetic parity" parts[17], respectively. The “null memory” (the second term in eq. (3.63)) is given by [19, 40, 39]

$$L_{AB}[F] = 16\pi\left(\mathscr{D}_A\mathscr{D}_B - \frac{1}{2}q_{AB}\mathscr{D}^2\right)\mathscr{D}^{-2}(\mathscr{D}^2+2)^{-1}F \tag{3.67}$$

where $F$ is the total flux of matter and gravitational energy to null infinity—given by eq. (3.35) with $d = 4$—and only the $\ell > 1$ part is taken. Equations (3.64) and (3.67) agree with eqs. (3.38) and (3.42) with $d$ set equal to 4.

Finally, suppose that we were to impose the strong stationarity conditions eq. (3.47) and eq. (3.48) at early and late retarded times. Our analysis for $d > 4$ used the harmonic gauge, which we cannot assume here. However, the gauge freedom for the metric at order $1/r$ that preserves strong stationarity is given by eq. (2.75) with $d = 4$, with the requirement that $\xi^{(0)}_\mu$ is stationary and $\xi^{(1)}_\mu$ vanishes. We can use up the full gauge freedom of $h^{(1)}_{\mu\nu}$ by setting $\eta^{\mu\nu}h^{(1)}_{\mu\nu} = 0$ and $h^{(1)}_{AB} = 0$. One then can show that Einstein’s equations with these gauge

17. As we shall see in the next subsection, the “magnetic parity” part is the same as the “vector part” in a spherical tensor decomposition.

conditions imply that when eq. (3.47) and eq. (3.48) hold the metric at Coulombic order (i.e. order $1/r$) must be Schwarzschild. The stronger stationarity conditions together with the field equations also imply that $h^{(2)}_{\mu\nu}$ and $h^{(3)}_{\mu\nu}$ do not contribute to $P$ or $Q$ as defined in eqs. (3.65) and (3.66). Since $h^{(1)}_{\mu\nu}$ is spherically symmetric it follows that $Q = 0$ and $P$ is spherically symmetric. Hence, as was the case for $d > 4$, we find that when $d = 4$, the ordinary memory vanishes if the stronger stationarity conditions eq. (3.47) and eq. (3.48) are imposed.

We summarize the main results of this subsection in the following theorem:

**Theorem 4.** *Suppose $d \geq 4$ and the metric satisfies the stationarity condition eq.* (3.3) *(for even dimensions) or eqs.* (3.4) *and* (3.5) *(for odd dimensions) at early and late retarded times. Then the memory tensor, defined by eq.* (3.9)*, has the following properties:*

*(1) In odd dimensions, $\Delta^{(n)}_{\mu\nu} = 0$ for all $n \leq d - 3$.*

*(2) In even dimensions, $\Delta^{(n)}_{\mu\nu} = 0$ for all $n < d - 3$. For $n = d - 3$, the memory tensor can be decomposed into "ordinary memory" and "null memory" as in eq.* (3.36)*. For $d > 4$, the ordinary and null memory are given, respectively, by eq.* (3.37) *and eq.* (3.38)*. For $d = 4$, the ordinary and null memory are given, respectively, by eq.* (3.64) *(or eq.* (3.42)*) and eq.* (3.67)*. If one imposes the stronger stationarity conditions eq.* (3.47) *and eq.* (3.48) *at early and late retarded times, then the ordinary memory vanishes at Coulombic order (but null memory may still be nonvanishing at Coulombic order).*

## 4 Non-scalar memory

As proven in [59] (see Propositions 2.1 and 2.2 of that reference), any (co)-vector field, $v_A$, on a sphere in $(d-2)$-dimensions can be decomposed into its vector and scalar parts via

$$w_A = W_A + \mathscr{D}_A W \tag{3.68}$$

where $\mathscr{D}^A W_A = 0$. Any symmetric tensor field, $x_{AB}$ on the sphere can be decomposed into its tensor, vector, and scalar parts via

$$x_{AB} = X_{AB} + \mathscr{D}_{(A}X_{B)} + \left(\mathscr{D}_A\mathscr{D}_B - \frac{1}{d-2}q_{AB}\mathscr{D}^2\right)X + \frac{1}{d-2}q_{AB}Y \tag{3.69}$$

where $\mathscr{D}^A X_{AB} = 0 = q^{AB}X_{AB}$ and $\mathscr{D}^A X_A = 0$. Any rotationally invariant operator (such as $\mathscr{D}^2$) acting on $w_A$ or $x_{AB}$ maps the scalar, vector, and tensor parts into themselves, i.e., rotationally invariant operations cannot "mix" these different parts.

Thus, the Coulombic order memory tensor $\Delta^{(d-3)}_{\mu\nu}$ may be decomposed into its scalar, vector, and tensor parts via

$$\Delta^{(d-3)}_{rr} = -U \tag{3.70}$$

$$\Delta^{(d-3)}_{rA} = R_A + \mathscr{D}_A R \tag{3.71}$$

$$\Delta^{(d-3)}_{AB} = S_{AB} + \mathscr{D}_{(A}S_{B)} + \left(\mathscr{D}_A\mathscr{D}_B - \frac{1}{d-2}q_{AB}\mathscr{D}^2\right)T + \frac{1}{d-2}q_{AB}U \tag{3.72}$$

where $\mathscr{D}^A R_A = 0 = \mathscr{D}^A S_A$ and $\mathscr{D}^A S_{AB} = 0 = q^{AB}S_{AB}$. Note that the fact that $\Delta^{(d-3)}_{\mu\nu}$ is traceless was used to relate $\Delta^{(d-3)}_{rr}$ to the scalar function $U$ appearing in eq. (3.72). In $d = 4$ dimensions, the tensor part, $S_{AB}$, in eq. (3.72) vanishes, since there are no divergence-free, trace-free, symmetric, rank$-2$ tensors on $\mathbb{S}^2$. Furthermore, on $\mathbb{S}^2$, the vector part $S_A$ can always be written as $S_A = \epsilon_{AB}\mathscr{D}^B S$. Thus, in $d = 4$ dimensions, the "vector part" can be replaced by a "magnetic parity scalar" part. In addition, since $\Delta^{(1)}_{r\mu} = 0$ in 4 dimensions (see eq. (3.23)), we also have $U = R_A = R = 0$ when $d = 4$. For $d = 6$, we have $U = 0$ (see eq. (3.24)).

We shall refer to $U$, $R$, and $T$ as "scalar memory", $R_A$ and $S_A$ as "vector memory," and $S_{AB}$ as "tensor memory". The scalar functions $U$, $R$, and $T$ are not independent because $\Delta^{(d-3)}_{\mu\nu}$ must satisfy the "constraint equations" eq. (3.20) and eq. (3.21) with $n = d-3$. This yields

$$[\mathscr{D}^2 - 2]U - (d-6)\mathscr{D}^2 R = 0 \tag{3.73}$$

and

$$[\mathscr{D}^2 + 2(d-4)]R + \frac{1}{2}(d-4)[\mathscr{D}^2 + 2(d-3)]T - \frac{d}{d-2}U = 0. \tag{3.74}$$

Note that for $d = 4$, this implies that $U = R = 0$, so scalar memory takes the form

$$[\Delta^{(1)}_{AB}]_{\text{scalar}} = \left(\mathscr{D}_A\mathscr{D}_B - \frac{1}{2}q_{AB}\mathscr{D}^2\right)T \qquad \text{for } d = 4. \tag{3.75}$$

The vector part of eq. (3.20) vanishes, but eq. (3.21) implies that $R_A$ and $S_A$ must satisfy

$$[\mathscr{D}^2 + (d-5)]R_A + \frac{1}{2}(d-4)[\mathscr{D}^2 + (d-3)]S_A = 0. \tag{3.76}$$

The constraint equations (3.20) and (3.21) do not give any restrictions on $S_{AB}$.

We can use eqs. (3.73) and (3.74) to solve for $U$ and $R$ in terms of $T$ and we can use eq. (3.76) to solve for $R_A$ in terms of $S_A$. Thus, the memory tensor is fully characterized by $T$, $S_A$, and $S_{AB}$, i.e., the trace-free part of the angle-angle components of the memory tensor.

No other obvious restrictions on $\Delta^{(d-3)}_{\mu\nu}$ arise from Einstein's equations near null infinity for $d$ even—of course, we have already shown that $\Delta^{(d-3)}_{\mu\nu} = 0$ for $d$ odd. This suggests that—in addition to scalar memory—magnetic parity memory may be possible for $d = 4$, and vector and tensor memory may be possible for $d > 4$ (for $d$ even). We now investigate whether this is possible for physically reasonable solutions.

Consider, first, null memory. The null memory part of $\Delta^{(d-3)}_{\mu\nu}$ is constructed from a rotationally invariant operator $L_{\mu\nu}$ (see eq. (3.38) and eq. (3.67)) acting on the integrated flux $F$. Since $F$ is a scalar function on the sphere, it follows immediately that null memory is always of purely scalar type.

The analysis of ordinary memory requires that we know that the Coulombic order solution $h^{(d-3)}_{\mu\nu}$ at early and late retarded times. This is not feasible in nonlinear general relativity but can be analyzed in linearized gravity. Consider, first, classical particle scattering, as treated in [55]. For classical particle scattering, the solution at early and late retarded times is a sum of boosted linearized Schwarzschild solutions. It is easily checked that for boosted, linearized Schwarzschild solutions, $h^{(d-3)}_{\mu\nu}$ is of purely scalar type. Since ordinary memory is obtained by applying a rotationally invariant operator to $\Delta h^{(d-3)}_{\mu\nu}$, it follows that ordinary memory is of purely scalar type for particle scattering in linearized gravity.

However, it is not difficult to show that vector and tensor ordinary memory can occur in linearized gravity for the retarded solution arising from other kinds of ingoing or outgoing matter stress-energy satisfying the dominant energy condition. In particular, magnetic parity (i.e, vector) ordinary memory can be produced in $d = 4$. To see this, consider a stress-energy tensor (for $t = u + r > 0$)

$$T_{\mu\nu} = \frac{1}{r^2}[\rho u_\mu u_\nu + l_{\mu\nu}]\delta(r - vt) \tag{3.77}$$

where $\rho > 0$ is a constant, $u^\mu$ corresponds to a radially outward 4-velocity with velocity $1 > v > 0$, and the components of $l_{\mu\nu}$ in a Cartesian basis (or normalized spherical basis) are independent of $t$ and $r$ and, on the unit sphere, are given by

$$l_{\mu\nu} = -\epsilon_{(\mu}{}^{\lambda}[u_{\nu)}(\mathscr{D}^2 + 1) - \gamma v \mathscr{D}_{\nu)}]\mathscr{D}_\lambda \alpha. \tag{3.78}$$

Here $\alpha$ is a time independent, arbitrary function on the sphere (containing multipoles $l > 1$) and $\gamma \equiv (1 - v^2)^{-1/2}$. For $\alpha = 0$, eq. (3.77) would correspond to an outgoing spherical dust shell and its stress-energy would be conserved. The $l_{\mu\nu}$ term has been constructed so that it is purely of magnetic parity (i.e., vector) type and is conserved by itself, so its addition to the stress-energy tensor does not affect conservation. By choosing $\rho$ sufficiently large, we can ensure that $T_{\mu\nu}$ satisfies the dominant energy condition. Thus, we see no principle that would imply that a stress-energy tensor of the form eq. (3.77) is not physically possible.

Since we are considering linearized gravity and there is no stress energy flux to null infinity, we may work in the Lorenz gauge. In a Cartesian basis, each component of $\overline{h}_{\mu\nu}$ satisfies the ordinary scalar wave equation with source. At radiative order, the contribution of $l_{\mu\nu}$ to the retarded solution for $u > 0$ is independent of $u$ and is given by

$$\overline{h}^{(1)}_{\mu\nu}(x^A) = 8\pi \int_{\mathbb{S}^2} d\Omega' \frac{l_{\mu\nu}(x'^A)}{1 - v\hat{r}(x^A) \cdot \hat{r}(x'^A)} \tag{3.79}$$

where the integral is taken over a sphere and $\hat{r}$ denotes the unit radial vector (with parallel transport in Euclidean space is understood in taking the dot product of vectors at different points on the sphere). It can be seen that $h^{(1)}_{AB}$ is, in general, nonvanishing. It must be of purely vector type since the source is of purely vector type and the retarded Green's function is rotationally invariant.

Now suppose one starts in the distant past with a static laboratory and no incoming gravitational radiation. At retarded time $u = 0$, a laboratory assistant launches a shell with stress energy of the form eq. (3.77). This shell then continues to move radially outward with velocity $v$ forever. Then, $h^{(1)}_{AB}$ has no magnetic parity part at early retarded times, but it has a nonvanishing magnetic parity part at late times. By eq. (3.62), this yields a nonvanishing magnetic parity memory tensor.

We note that Mädler and Winicour [57] have shown that under the stronger stationarity condition that they impose, magnetic parity memory cannot occur. This result is consistent with our results because, as we have already shown, their stronger stationarity condition rules out all ordinary memory, and null memory is always of scalar type. Bieri [60] has shown that magnetic parity memory cannot occur for vacuum solutions with "small data" in nonlinear general relativity. This result also is consistent with our results.

Finally, we comment that examples with tensor ordinary memory can be obtained for $d > 4$ by choosing a shell stress-energy tensor[18]

$$T_{\mu\nu} = \frac{1}{r^{d-2}}[\rho u_\mu u_\nu + S_{\mu\nu}]\delta(r - vt) \tag{3.80}$$

where $S_{\mu\nu}$ has vanishing $u$ and $r$ components and its angle-angle components are of purely tensor type.

In summary, *null memory is always of scalar type in linear and nonlinear general relativity. Ordinary memory also is of scalar type for classical particle scattering in linearized gravity. However, ordinary memory need not be of scalar type in general. In particular, we have constructed explicit examples with outgoing shells of matter in linearized gravity that give rise to magnetic parity (= vector) ordinary memory in 4 dimensions and tensor ordinary memory in higher even dimensions.*

18. We can, of course, also construct sources with vector memory for $d > 4$ in a similar manner to eqs. (3.77) and (3.78).

### 5 *Memory as a diffeomorphism*

In this subsection, we consider the issue of whether the memory tensor up to Coulombic order can be written as an infinitesimal diffeomorphism, i.e., whether there exists a vector field $\xi^\mu$ such that

$$\Delta^{(n)}_{\mu\nu} = [\nabla_{(\mu}\xi_{\nu)}]^{(n)} \tag{3.81}$$

for all $n \leq d-3$. One reason why this question is of some interest can be seen from the following considerations.

We introduce the following new gauge: For $d > 4$, start in the harmonic gauge in the early time stationary era $u < u_0$. For $d = 4$, start in an arbitrary gauge compatible with our ansatz and stationarity assumption for $u < u_0$. Then, for $u < u_0$, we have $h^{(n)}_{\mu\nu} = 0$ for all $n < d-3$ and $\partial_u h^{(d-3)}_{\mu\nu} = 0$. By a further gauge transformation of the form $\psi^a = uf(x^A)/r^{d-3}(\partial/\partial u)^\mu$, we may, in addition, set $h^{(d-3)}_{uu} = 0$ for $u < u_0$. Now, define coordinates for $u \geq u_0$ by fixing the $(r, x^A)$ coordinates along each geodesic determined by the initial tangent $\partial/\partial u$ and taking the $u$ coordinate to be given by the affine parameter along each geodesic. This agrees with proper time up to and including order $1/r^{d-3}$. Thus, the new coordinates are essentially Gaussian normal coordinates, except that the initial surface $u = u_0$ is not orthogonal to $(\partial/\partial u)^\mu$. By the same argument as for Gaussian normal coordinates, we have $\partial_u g_{u\mu} = 0$ (and, hence $\partial_u h_{u\mu} = 0$) at all times at Coulombic order and slower fall-off. Note that the new coordinates will *not*, in general, be harmonic in the radiative era or the final stationary era.

For $u \geq u_0$, the coordinate vector fields $\partial/\partial r$ and $\partial/\partial x^A$ are deviation vectors for the timelike geodesic congruence with tangent field $u^\mu = (\partial/\partial u)^\mu$. We have

$$\begin{aligned}\frac{\partial^2 h_{\mu\nu}}{\partial u^2} &= \frac{\partial^2 g_{\mu\nu}}{\partial u^2} = \frac{\partial^2}{\partial u^2}\left[g_{ab}\left(\frac{\partial}{\partial x^\mu}\right)^a\left(\frac{\partial}{\partial x^\nu}\right)^b\right] \\ &= u^d\nabla_d\, u^c\nabla_c\left[g_{ab}\left(\frac{\partial}{\partial x^\mu}\right)^a\left(\frac{\partial}{\partial x^\nu}\right)^b\right] \\ &= g_{ab}\, u^d\nabla_d\, u^c\nabla_c\left[\left(\frac{\partial}{\partial x^\mu}\right)^a\left(\frac{\partial}{\partial x^\nu}\right)^b\right]. \end{aligned} \qquad (3.82)$$

This equation holds to all orders in $1/r$ in our coordinates. The derivatives of the term in brackets on the right side of eq. (3.82) yield terms where $u^\rho\nabla_\rho u^\sigma\nabla_\sigma$ acts on a single coordinate vector field and terms where one derivative each acts on each of the two coordinate vector fields. The terms where two derivatives act on a single coordinate vector field can be evaluated from the geodesic deviation equation. The terms where one derivative acts on each of the coordinate vector fields are $O(1/r^{d-2})$. Thus, we obtain in our gauge

$$\frac{\partial^2 h^{(n)}_{\mu\nu}}{\partial u^2} = -2C^{(n)}_{u\mu u\nu} \qquad (3.83)$$

for all $n \leq d-3$. It follows immediately from the definition, eq. (3.9), of the memory tensor that in our gauge we have

$$\Delta^{(n)}_{\mu\nu} = \frac{1}{2}\Delta h^{(n)}_{\mu\nu} \qquad (3.84)$$

for all $n \leq d-3$. Note that the right side of eq. (3.84) is the full memory tensor, including null memory. This expression is compatible with our previous expression eq. (3.36) for $d > 4$ because that expression held in harmonic gauge whereas eq. (3.84) is valid only in the gauge we have defined above. Equation (3.84) also is compatible with eq. (3.62) for $d = 4$.

Now, suppose we start with an array of geodesic test particles that are initially "at rest" at early times and consider their final configuration at late times. If eq. (3.81) holds, then $\Delta h^{(n)}_{\mu\nu}$ is "pure gauge" for all $n \leq d-3$. This means that if we displace the test particles by

$\xi^\mu$ at late times, they will go back to their original relative configuration at Coulombic and slower fall-off. In other words, at Coulombic order, the final spacetime *geometry* is the same as the initial geometry. On the other hand, if eq. (3.81) does not hold, then it is impossible to displace the particles so that they go back to their original relative configuration. A genuine change in the geometry at Coulombic order has occurred.

We now turn to the analysis of whether one can find a $\xi^\mu$ so that eq. (3.81) holds. It is clear that in order for $[\nabla_{(\mu}\xi_{\nu)}]^{(n)}$ to vanish for $n < d-3$ and be $u$-independent at $n = d-3$, we must choose $\xi^\mu$ to be such that $\xi_\mu^{(n)} = 0$ for $n < d-4$ whereas

$$\xi_\mu^{(d-4)} = J_\mu(x^A)\,, \qquad \xi_\mu^{(d-3)} = uB_\mu(x^A). \tag{3.85}$$

Decomposing $J_\mu(x^A)$ and $B_\mu(x^A)$ into their scalar, vector, and tensor parts, we see that we have 6 scalar functions on the sphere, 2 divergence-free vector fields on the sphere, and no transverse, traceless tensors. On the other hand, the decomposition of a general symmetric tensor, $t_{\mu\nu}$, on the sphere yields 7 scalar functions, 3 divergence-free vector fields, and 1 transverse, traceless tensor (for $d > 4$). Thus, *a priori*, we are one free scalar, one free vector, and one free tensor (for $d > 4$) short of being able to express a general tensor on the sphere in the form we seek.

However, $\Delta_{\mu\nu}$ is not a general tensor on the sphere. It has vanishing $u$-components, is trace-free, and its scalar and vector parts satisfy the constraint eqs. (3.73), (3.74) and (3.76). The symmetrized derivative of $\xi_a$ at order $1/r^{d-3}$ is

$$\begin{aligned}[\nabla_{(\mu}\xi_{\nu)}]^{(d-3)} =& q_{(\mu}{}^{\sigma}\mathscr{D}_{\nu)}J_\sigma + (J_r - J_u)q_{\mu\nu} + r_{(\mu}\mathscr{D}_{\nu)}J_r - K_{(\mu}\mathscr{D}_{\nu)}J_u - q_{(\mu}{}^{\sigma}r_{\nu)}J_\sigma \\ &- (d-4)r_{(\mu}J_{\nu)} - K_{(\mu}B_{\nu)}.\end{aligned} \tag{3.86}$$

It is clear from this equation that we may choose $B_\nu$ such that the $u$ components of eq. (3.86) vanish, so we need only consider whether $J_\mu$ can be chosen so as to make the non-$u$ components of the right side of eq. (3.86) match $\Delta^{(d-3)}_{\mu\nu}$. We may separately consider the scalar, vector, and tensor parts. The scalar parts of $J_\mu$ are $J_r$, $J_u$, and $J$, where $J$ denotes the scalar part of $J_A$. Equating the scalar part of eq. (3.86) to the scalar part of $\Delta^{(d-3)}_{\mu\nu}$ (see eqs. (3.70)-)3.72)), we obtain the following equations

$$(d-4)J_r = U \tag{3.87}$$

$$J_r - (d-3)J = 2R \tag{3.88}$$

$$J = T \tag{3.89}$$

$$\mathscr{D}^2 J + (d-2)(J_r - J_u) = U. \tag{3.90}$$

This is an overdetermined system for $J_r$, $J_u$, and $J$. The necessary and sufficient condition for a solution to exist is that $U$, $R$, and $T$ satisfy

$$\frac{U}{(d-4)} - (d-3)T = 2R. \tag{3.91}$$

However, it can be shown that this equation is implied by the constraint equations eqs. (3.73) and (3.74). Thus, the scalar part of $\Delta_{\mu\nu}$ can always be written in the form eq. (3.81) for a $\xi^a$ of the form eq. (3.85). Thus, scalar memory at Coulombic order is always given by a diffeomorphism [38]. In particular, as is well known, the scalar memory eq. (3.75) for $d = 4$ is of the form of a supertranslation. However, a similar calculation shows that no such miracles occur for vector memory, and vector memory can *never* be written in the form eq. (3.81). Tensor memory, of course, also can never be written in the form eq. (3.81).

In summary, *scalar memory at Coulombic order always can be written as a diffeomorphism, but this never holds for vector and tensor memory.*

## 1 Charges and memory

In $d = 4$ dimensions, it is well known [21] that all asymptotic symmetries at future null infinity give rise to associated charges and fluxes. In this sub-subsection, we will show that the charges and fluxes associated with supertranslations are intimately related to the memory effect in 4 dimensions, and, indeed, we will derive the formula for scalar memory in $d = 4$ from the supertranslation charges and fluxes. We will then obtain corresponding results for $d > 4$. Since the derivations and formulas of [21] apply only to the vacuum case, in the following two paragraphs we will restrict to the case where $T_{\mu\nu} = 0$ in a neighborhood of null infinity. We will then restore $T_{\mu\nu}$ in our formulas.

Consider a supertranslation, i.e., a diffeomorphism belonging to the gauge equivalence class of

$$\psi^\mu = f(x^A)\left(\frac{\partial}{\partial u}\right)^\mu - f(x^A)\left(\frac{\partial}{\partial r}\right)^\mu - q^{BC}\mathscr{D}_B f(x^A)\frac{1}{r}\left(\frac{\partial}{\partial x^C}\right)^\mu + \dots \tag{3.92}$$

where the $\dots$ stand for a vector field that vanishes as $r \to \infty$ for fixed $u$ and $x^A$. From general considerations [21] arising from the Lagrangian formulation of general relativity, a charge $\mathcal{Q}_f^+$, and flux, $\mathscr{F}_f^+$, can be associated with $\psi^\mu$ such that for any $u_0, u_1$, we have

$$\mathcal{Q}_f^+(u_1) - \mathcal{Q}_f^+(u_0) = \int_{u_0}^{u_1} du \int_{\mathbb{S}^2} d\Omega\ \mathscr{F}_f^+. \tag{3.93}$$

An explicit formula for $\mathcal{Q}_f^+$ (originally due to Geroch [41]) is given in eq.(98) of [21], and an explicit formula for $\mathscr{F}_f^+$ is given in eq.(82) of [21]. Here we have inserted a superscript "+" to distinguish these charges and fluxes from similar quantities at past null infinity, which will be considered later. The flux is evaluated to be

$$\mathscr{F}^{+}_{f} = -\frac{1}{32\pi}(fN^{AB}N_{AB} - 2N^{AB}\mathscr{D}_A\mathscr{D}_B f). \tag{3.94}$$

The formula for the charge is considerably more complicated, but this formula simplifies considerably in stationary eras, when $N_{AB} = 0$. From eq.(98) of [21], we find that in stationary eras we have

$$\mathscr{Q}^{+}_{f}\Big|_{\text{stationary}} = -\frac{1}{8\pi}\int_{\mathbb{S}^2} d\Omega\ fC^{(3)}_{urur}. \tag{3.95}$$

Thus, if we impose the stationarity conditions of sec. 3.1 and we let $u_0 \to -\infty$ and $u_1 \to +\infty$ in eq. (3.93), we obtain

$$\mathscr{Q}^{+}_{f}(+\infty) - \mathscr{Q}^{+}_{f}(-\infty) = \int_{\mathscr{I}^+} dud\Omega\ \mathscr{F}^{+}_{f}. \tag{3.96}$$

The flux integral can be rewritten as

$$\begin{aligned}\int_{\mathscr{I}^+} dud\Omega\ \mathscr{F}^{+}_{f} &= -\int_{\mathscr{I}^+} dud\Omega\ fF + \frac{1}{16\pi}\int_{\mathbb{S}^2} d\Omega \mathscr{D}^A\mathscr{D}^B f \int_{-\infty}^{\infty} du\ N_{AB} \\ &= -\int_{\mathscr{I}^+} dud\Omega\ fF + \frac{1}{8\pi}\int_{\mathbb{S}^2} d\Omega\ \left(\mathscr{D}^A\mathscr{D}^B f\right)\Delta^{(1)}_{AB} \\ &= -\int_{\mathscr{I}^+} dud\Omega\ fF + \frac{1}{8\pi}\int_{\mathbb{S}^2} d\Omega f\mathscr{D}^A\mathscr{D}^B\Delta^{(1)}_{AB}\end{aligned} \tag{3.97}$$

where $F = \frac{1}{32\pi}N^{AB}N_{AB}$ is the Bondi flux, and we used eq. (3.62) in the second line. The contribution to $\int_{\mathscr{I}^+} dud\Omega\ \mathscr{F}^{+}_{f}$ arising from the term $fF$ is often referred to as the "hard" integrated flux (or "hard charge") whereas the term involving $\Delta^{(1)}_{AB}$ is called the "soft" integrated flux (or "soft charge"). The terms $\mathscr{Q}^{+}_{f}(-\infty)$ and $\mathscr{Q}^{+}_{f}(+\infty)$ can be viewed as the contributions to "hard charge" coming from the asymptotic past (spatial infinity) and future (timelike infinity). From eqs. (3.95) - (3.97), we obtain,

$$\int_{\mathbb{S}^2} d\Omega \; f C^{(3)}_{urur}\Big|_{+\infty} - \int_{\mathbb{S}^2} d\Omega \; f C^{(3)}_{urur}\Big|_{-\infty} - 8\pi \int_{\mathscr{I}^+} dud\Omega \; f F = - \int_{\mathbb{S}^2} d\Omega \; f \mathscr{D}^A \mathscr{D}^B \Delta^{(1)}_{AB} \quad (3.98)$$

which relates the hard charges to the soft charge. Note that if $f$ is an $\ell = 0$ or $\ell = 1$ spherical harmonic (in which case $\psi^a$ is a translation), the term in $\Delta^{(1)}_{AB}$ does not contribute, and this equation corresponds to the integrated conservation law for Bondi 4-momentum.

Since eq. (3.98) holds for all $f$, this equation must hold pointwise on the sphere. Therefore, we obtain

$$- \mathscr{D}^A \mathscr{D}^B \Delta^{(1)}_{AB} = C^{(3)}_{urur}\Big|_{+\infty} - C^{(3)}_{urur}\Big|_{-\infty} - 8\pi \int_{-\infty}^{\infty} du \; F. \quad (3.99)$$

It is easily seen that vector memory makes no contribution to $\mathscr{D}^A \mathscr{D}^B \Delta^{(1)}_{AB}$. On the other hand, substituting the form eq. (3.75) of scalar memory, we obtain

$$- \frac{1}{2} \mathscr{D}^2 (\mathscr{D}^2 + 2) T = C^{(3)}_{urur}\Big|_{+\infty} - C^{(3)}_{urur}\Big|_{-\infty} - 8\pi \int_{-\infty}^{\infty} duF. \quad (3.100)$$

Solving for $T$ and substituting back in eq. (3.75), we obtain a formula for scalar memory that agrees with the scalar part of eq. (3.63).

In the above two paragraphs, we have restricted to the case where $T_{\mu\nu} = 0$ in a neighborhood of null infinity in order to use the formulas given in [21]. However, eq. (3.63) holds when $T_{\mu\nu} \neq 0$. This shows that when $T_{\mu\nu} \neq 0$, eq. (3.98) is modified merely by the simple substitution $F = \frac{1}{32\pi} N^{AB} N_{AB} \to \frac{1}{32\pi} N^{AB} N_{AB} + T^{(2)}_{uu}$.

We now consider the case $d > 4$. As we have seen in the previous subsection, in $d > 4$ dimensions, scalar memory is still given by a diffeomorphism. However, this diffeomorphism is now pure gauge, i.e., it has vanishing symplectic product with all asymptotically flat perturbations. Thus, nontrivial charges and fluxes cannot be associated with these diffeomorphisms via the Lagrangian formalism. Nevertheless, our general memory formula eq. (3.36) can be interpreted as a charge/flux formula. Namely, we may write this formula in the form

$$P_{\mu\nu}[\bar{h}^{(d-3)}_{\rho\sigma}]\Big|_{\infty} - P_{\mu\nu}[\bar{h}^{(d-3)}_{\rho\sigma}]\Big|_{-\infty} + \int\limits_{-\infty}^{\infty} du\; L_{\mu\nu}[F] = \Delta^{(d-3)}_{\mu\nu}. \tag{3.101}$$

Now for arbitrary scalar field $f$ on the sphere, *define* the scalar charge, $\mathcal{Q}^+_f$ during a stationary era, by[19]

$$\mathcal{Q}^+_f = \int\limits_{\mathbb{S}^2} d\Omega P_{AB}[\bar{h}^{(d-3)}_{\rho\sigma}]\left(\mathscr{D}^A\mathscr{D}^B - \frac{1}{d-2}q^{AB}\mathscr{D}^2\right)f. \tag{3.102}$$

Using eq. (3.42), we can rewrite the right side of eq. (3.101) in terms of $\Delta E^{(d-1)}_{rr}$. It then can be seen that eq. (3.102) corresponds to eq.(5.21) of [38], but with different angular weights, i.e. our $f$ is related to their $f$ by angular operators. Multiplying eq. (3.101) by $(\mathscr{D}^A\mathscr{D}^B - \frac{1}{d-2}q^{AB}\mathscr{D}^2)f$ and integrating over a sphere, we obtain

$$\mathcal{Q}^+_f\Big|_{\infty} - \mathcal{Q}^+_f|_{-\infty} + \int\limits_{\mathscr{I}^+} f\left(\mathscr{D}^A\mathscr{D}^B - \frac{q^{AB}}{d-2}\mathscr{D}^2\right)L_{AB}[F] = \int\limits_{\mathbb{S}^2} d\Omega f\left(\mathscr{D}^A\mathscr{D}^B - \frac{q^{AB}}{d-2}\mathscr{D}^2\right)\Delta^{(d-3)}_{AB} \tag{3.103}$$

which is closely analogous to eq. (3.98) and can be given an interpretation in terms of "hard" and "soft" charges.

Similarly, during stationary eras we can define the vector charge, $\mathcal{Q}^+_{\beta_A}$, associated with a

19. An important difference between $d > 4$ and $d = 4$ is that the scalar charge for $d > 4$ is defined *only* during stationary eras, whereas in $d = 4$ a local, gauge invariant scalar charge can be defined at all times (even though we gave the formula eq. (3.95) for scalar charge in $d = 4$ only during a stationary era). The existence of local, gauge invariant charge during radiative eras in $d = 4$ traces back, by the considerations of [21], to its association with an asymptotic symmetry. Since there is no such association in $d > 4$, we see no reason to believe that a local, gauge invariant scalar charge corresponding to eq. (3.102) can be defined during radiative eras for $d > 4$.

divergence free vector field $\beta_A$ on the sphere by the formula

$$\mathcal{Q}^{+}_{\beta_A} = \int_{\mathbb{S}^2} d\Omega\ \beta^B \mathscr{D}^A P_{AB}[\bar{h}^{(d-3)}_{\rho\sigma}]. \tag{3.104}$$

We then obtain

$$\mathcal{Q}^{+}_{\beta_A}\Big|_{\infty} - \mathcal{Q}^{+}_{\beta_A}\Big|_{-\infty} = \int_{\mathbb{S}^2} d\Omega\ \beta^B \mathscr{D}^A \Delta^{(d-3)}_{AB}. \tag{3.105}$$

No contribution from $F$ appears in this equation since $L_{AB}[F]$ cannot have a vector part. Finally, for any divergence-free, trace-free tensor field $\gamma_{AB}$ on the sphere, we can define the tensor charge $\mathcal{Q}^{+}_{\gamma_{AB}}$ during a stationary era by

$$\mathcal{Q}^{+}_{\gamma_{AB}} = \int_{\mathbb{S}^2} d\Omega\ \gamma^{AB} P_{AB}[\bar{h}^{(d-3)}_{\rho\sigma}] \tag{3.106}$$

and obtain

$$\mathcal{Q}^{+}_{\gamma_{AB}}\Big|_{\infty} - \mathcal{Q}^{+}_{\gamma_{AB}}\Big|_{-\infty} = \int_{\mathbb{S}^2} d\Omega\ \gamma^{AB} \Delta^{(d-3)}_{AB}. \tag{3.107}$$

Of course, there is no information contained in eq. (3.103), eq. (3.105), and eq. (3.107) than that which already appeared in eq. (3.36).

## 2 Conservation laws

Thus far, the analysis of this chapter has been concerned solely with the behavior of fields near future null infinity. Of course, the same analysis could be applied to past null infinity. In this sub-subsection, we wish to consider the relationship between quantities at past and future null infinity. Under the assumptions specified below, we will obtain a conservation law relating past and future null infinity.

Consider, first, the case of a scalar field $\phi$ in Minkowski spacetime with $d$ even and $d \geq 4$, with source $S = 0$ in a neighborhood of future null infinity. We restrict attention to solutions, $\phi_{\ell m}$, whose angular dependence is given by a single spherical harmonic, $Y_{\ell m}$. (A general solution, of course, can be expressed as a superposition of such solutions.) Suppose that at Coulombic order, $\phi_{\ell m}$ is stationary at early retarded times, $\partial_u \phi^{(d-3)}_{\ell m} = 0$, so that at early times,

$$\phi^{(d-3)}_{\ell m} = c Y_{\ell m}(x^A) \tag{3.108}$$

where $c$ is a constant. In the recursion relations eq. (2.13), we may replace $\mathscr{D}^2$ by $-\ell(\ell+d-3)$, so we have

$$(2n - d + 2)\partial_u \phi^{(n)}_{\ell m} = [\ell(\ell + d - 3) - (n - 1)(n - d + 2)]\phi^{(n-1)}_{\ell m}. \tag{3.109}$$

Thus, as usual, we obtain $\phi^{(n)}_{\ell m} = 0$ for $n < d - 3$. For $d - 3 \leq n < \ell + d - 2$, we see that $\phi^{(n)}_{\ell m}$ is a polynomial, $\mathcal{P}_n(u)$, in $u$ of degree $n - d + 3$, with the coefficients of the polynomials at the different orders related by eq. (3.109). For $n = \ell + d - 2$, we obtain $\partial_u \phi^{(\ell+d-2)}_{\ell m} = 0$, so we may terminate the series by setting $\phi^{(n)}_{\ell m} = 0$ for $n \geq \ell + d - 2$. We thereby obtain an *exact solution* of the form

$$\phi_{\ell m} = \sum_{n=\ell+d-3}^{d-3} \frac{\mathcal{P}_n(u)}{r^n} Y_{\ell m}(x^A). \tag{3.110}$$

This solution is of direct physical interest, since it corresponds to the $Y_{\ell m}$ part of the retarded solution with source corresponding to matter in inertial motion (e.g., classical incoming particles on inertial timelike trajectories). The general solution with $Y_{\ell m}$ angular dependence that is stationary at Coulomb order is eq. (3.110) plus a solution with an asymptotic expansion whose slowest fall-off term is at order $1/r^{\ell+d-2}$, and with the coefficients of the higher powers of $1/r^n$ being polynomials in $u$ of degree $n - (\ell + d - 2)$. This series cannot terminate.

We consider, now, the exact solution eq. (3.110). The highest power of $u$ in eq. (3.110) appears as the term $Cu^\ell/r^{\ell+d-3}$, where $C$ is related to the coefficient of the Coulombic order coefficient $c$ by an $\ell$ fold product of the numerical factors arising from successively solving eq. (3.109). Now, consider the behavior of the solution eq. (3.110) near past null infinity. We can determine this behavior by writing $u = v - 2r$ and re-expanding in $1/r$. It is immediately clear that the highest power of $v$ occurring in this solution will be the term $C'v^\ell/r^{\ell+d-3}$. The coefficient $C$ is related to the Coulombic order coefficient $C'$ at past null infinity by a set of recursion relations. The recursion relations at past null infinity are the same as the recursion relations at future null infinity except for the following important difference: $\partial/\partial r$ is now past directed, which gives rise to a change in the sign of the $\partial/\partial u$ term in each of the recursion relations. Thus, we end up with $\ell$ sign flips by the time we reach Coulombic order. We thereby obtain $C' = (-1)^\ell C$, i.e., we have

$$\phi^{(d-3)}_{\ell m}\Big|_{\mathscr{I}^-} = (-1)^\ell C Y_{\ell m}(x^A). \tag{3.111}$$

Since $(-1)^\ell Y_{\ell m}(x^A) = Y_{\ell m}(-x^A)$, this means that the solution eq. (3.110) at Coulombic order has an "antipodal matching" between $\mathscr{I}^+$ and $\mathscr{I}^-$ [47].

The antipodal matching eq. (3.111) has been shown only for the exact solutions eq. (3.110) that terminate at order $1/r^{\ell+d-3}$. However, since the additional terms in the asymptotic series of more general solutions behave no worse than $u^k/r^{k+\ell+d-2}$ for $k \geq 0$, these individual terms would not contribute at Coulombic order at $\mathscr{I}^-$. Of course, the series composed of these terms is merely an asymptotic series near $\mathscr{I}^+$, and we clearly cannot determine the behavior of solutions near $\mathscr{I}^-$ from an asymptotic expansion near $\mathscr{I}^+$. Nevertheless, it seems not implausible that the antipodal matching may hold for a much more general class of solutions than the exact solutions eq. (3.110). In any case, since the antipodal matching holds for eq. (3.110) for all $\ell, m$ and the retarded solution corresponding to incoming inertial particles is a sum of such solutions, the antipodal matching holds for the retarded solution for incoming inertial particles—as can be verified directly from the explicit form of the solution [47].

Similar antipodal matching results hold for Maxwell's equations and for linearized gravity [44, 4, 47, 33]. The situation in nonlinear general relativity is less clear. Even for a solution that is stationary at Coulombic order, nonlinear terms will enter Einstein's equation at order $2(d-2)$. However, even in the linear case above, the behavior at $\mathscr{I}^-$ at Coulombic order depends on the form of the solution at order $n = \ell + d - 3$ near $\mathscr{I}^+$. Thus, for large $l$, the nonlinear terms in Einstein's equation cannot be ignored. Nevertheless, it remains not implausible that the antipodal matching may continue to hold in quite general circumstances. Indeed, the matching of supertranslation charges was shown under the Ashtekar-Hansen [61] asymptotic flatness conditions at spatial infinity together with an additional null regularity condition at spatial infinity [26, 62]. However, a full proof of the matching conditions would require an extension of the Christodoulou and Klainerman analysis [63] for initial data with non-vanishing supertranslation charges at spatial infinity.

In any case, we will now *assume* that we have a solution to Einstein's equation for which the antipodal matching holds at Coulombic order and consider the consequences. The key point is that the matching of the Coulombic order metrics implies a corresponding matching of the charges of the previous subsection, since the charges are constructed out of the Coulombic order metric. In particular, in $d = 4$ dimensions, we have

$$\mathcal{Q}^+_f|_{u=-\infty} = \mathcal{Q}^-_{\tilde{f}}|_{v=+\infty} \tag{3.112}$$

where $\mathcal{Q}^-_{\tilde{f}}$ denotes the charge at $\mathscr{I}^-$ associated to the supertranslation $\tilde{\psi}^a$ with $\tilde{f}$ antipodally matched to $f$. Since, in analogy to eq. (3.96), we have

$$\mathcal{Q}^-_{\tilde{f}}|_{v=+\infty} - \mathcal{Q}^-_{\tilde{f}}|_{v=-\infty} = \int\limits_{\mathscr{I}^-} dv d\Omega\, \mathcal{F}^-_{\tilde{f}} \tag{3.113}$$

where

$$\mathcal{F}^-_{\tilde{f}} = \frac{1}{32\pi}(\tilde{f}N^{AB}N_{AB} + 2N^{AB}\mathscr{D}_A\mathscr{D}_B\tilde{f}) \tag{3.114}$$

we obtain the conservation law [44, 4, 47]

$$\begin{aligned} &\mathcal{Q}^+_f|_{u=+\infty} + \int_{\mathscr{I}^+} fF - \frac{1}{8\pi}\int_{\mathbb{S}^2} d\Omega\, f\mathscr{D}^A\mathscr{D}^B\Delta^{(1)}_{AB}|_{\mathscr{I}^+} \\ &= \mathcal{Q}^-_{\tilde{f}}|_{v=-\infty} + \int_{\mathscr{I}^-} \tilde{f}F + \frac{1}{8\pi}\int_{\mathbb{S}^2} d\Omega\, \tilde{f}\mathscr{D}^A\mathscr{D}^B\Delta^{(1)}_{AB}|_{\mathscr{I}^-}. \end{aligned} \tag{3.115}$$

This may be interpreted as saying that the ingoing hard charge plus the integrated hard and soft fluxes at $\mathscr{I}^-$ are equal to the corresponding quantities at $\mathscr{I}^+$.

Similarly, in $d > 4$ dimensions, we get a similar antipodal matching of the scalar, vector, and tensor charges defined by eq. (3.102), eq. (3.104), and eq. (3.106), which leads to similar conservation laws.

## *7 Memory and infrared divergences in four and higher dimensions*

In $d = 4$ dimensions, there is a very close relationship between the memory effect and infrared divergences that occur in quantum field theory. This follows directly from the fact that, by eq. (3.62), the memory tensor is just the change in $h^{(1)}_{AB}$ between late and early retarded times. Thus, if $\Delta^{(1)}_{AB} \neq 0$, then $h^{(1)}_{AB}(u, x^A)$ cannot vanish at future null infinity at both $u \to -\infty$ and $u \to +\infty$. It follows that the Fourier transform of $h^{(1)}_{AB}$ with respect to $u$ will diverge at small $\omega$ as $1/\omega$. As we shall now explain, this behavior gives rise to infrared divergences in quantum field theory. Exactly similar behavior occurs in the scalar and electromagnetic cases, but we will restrict our discussion here to the gravitational case.

Let $d \geq 4$, with $d$ allowed to be odd as well as even. The Lagrangian formulation of general relativity gives rise to a conserved symplectic current density $w^\mu$ constructed out of a background solution $g_{\mu\nu}$ and two perturbations $h_{\mu\nu}$ and $h'_{\mu\nu}$. Consider the symplectic flux $(\partial/\partial u)^\mu w_\mu = w_u$ near future null infinity. Only the leading order term $w_u^{(d-2)}$ can contribute to this flux. However, only the radiative order parts of $h_{\mu\nu}$ and $h'_{\mu\nu}$ can contribute to $w_u^{(d-2)}$, and the deviation of $g_{\mu\nu}$ from the flat metric $\eta_{\mu\nu}$ cannot contribute at all. We obtain

$$w_u^{(d-2)}(h'_{AB}, h_{CD}) = \frac{1}{32\pi}(C^{AB}N'_{AB} - C'^{AB}N_{AB}) \tag{3.116}$$

where $N_{AB}$ is the Bondi news tensor, eq. (2.76) and $C_{AB}$ is the trace free part of the projection of $h_{AB}^{(d/2-1)}$ onto the sphere. In writing eq. (3.116), we have imposed the gauge conditions $h_{rA}^{(1)} = h_{uu}^{(1)} = \eta^{\mu\nu}h_{\mu\nu}^{(1)} = 0$ in $d = 4$ and we have imposed the harmonic gauge for $d > 4$. The integrated symplectic flux can be used to define a symplectic form $\Omega(h'_{AB}, h_{CD})$ at future null infinity

$$\Omega(h'_{AB}, h_{CD}) = \int_{-\infty}^{\infty} du \int_{\mathbb{S}^2} d\Omega \; w_u^{(d-2)}(h'_{AB}, h_{CD}). \tag{3.117}$$

Equation (3.117) gives us the necessary structure to define a Fock space of "outgoing graviton" states. We define the "one-particle outgoing Hilbert space" $\mathscr{H}_{\rm out}$ as the space of radiative order trace-free $\psi_{AB}$ that are purely positive frequency with respect to $u$, with inner product given by

$$\langle \psi'_{AB} | \psi_{CD} \rangle = -i\Omega(\psi'^*_{AB}, \psi_{CD}) \tag{3.118}$$

where "$*$" denotes complex conjugation. More precisely, we define $\mathscr{H}_{\rm out}$ by starting with smooth positive frequency $\psi_{AB}$ with fast fall-off in $u$, defining the inner product eq. (3.118) on such $\psi_{AB}$, and taking the Cauchy completion. The inner product eq. (3.118) is positive definite, as can be seen from the fact that in Fourier transform space, it is given by

$$\langle \psi'_{AB} | \psi_{CD} \rangle = \frac{1}{16\pi} \int_{\mathbb{S}^2} d\Omega \int_0^{\infty} \omega d\omega \; \hat{\psi}'^*_{AB} \hat{\psi}^{AB} \tag{3.119}$$

where the "hat" denotes the Fourier transform. A classical solution $h_{\mu\nu}$ can be associated with a state in $\mathscr{H}_{\text{out}}$ via $h_{\mu\nu} \to h^{(1)}_{+AB}$—where the subscript "+" denotes the positive frequency part—provided, of course, that $h^{(1)}_{+AB} \in \mathscr{H}_{\text{out}}$. Given $\mathscr{H}_{\text{out}}$, one may then define the corresponding Fock space $\mathscr{F}(\mathscr{H}_{\text{out}})$. A free field operator, $\mathbf{h}^{\text{out}}_{\mu\nu}$, on $\mathscr{F}(\mathscr{H}_{\text{out}})$ can then be defined in the usual manner in terms of annihilation and creation operators. Note that this construction is well defined even if the quantum gravity theory has not been defined in the interior spacetime [64].

However, this space, $\mathscr{F}(\mathscr{H}_{\text{out}})$, of outgoing graviton states need not be adequate to describe all physically relevant outgoing states. This is most easily seen by considering the theory of *linearized* quantum gravity (i.e., a massless, spin-2 field) with a *classical* stress energy source, i.e., the stress-energy operator is taken to be $T_{ab}\mathbf{I}$ where $T_{ab}$ is a classical stress energy and $\mathbf{I}$ is the identity operator. This is a well defined, mathematically consistent theory that can be solved exactly. After analyzing this theory, we will discuss the implications for a full theory in which the stress-energy is fully quantum and the nonlinear effects of gravity are taken into account.

The Heisenberg equations of motion for the field operator $\mathbf{h}_{\mu\nu}$ for linearized gravity with a classical stress-energy source are easily solved to yield

$$\mathbf{h}_{\mu\nu} = \mathbf{h}^{\text{in}}_{\mu\nu} + h^{\text{ret}}_{\mu\nu}\mathbf{I} \tag{3.120}$$

where $\mathbf{h}^{\text{in}}_{\mu\nu}$ is the free field operator corresponding to the "in" field and $h^{\text{ret}}_{\mu\nu}$ is the classical retarded solution with classical source $T_{ab}$. Suppose we consider the state $|0_{\text{in}}\rangle$, corresponding to the vacuum state of $\mathbf{h}^{\text{in}}_{\mu\nu}$. If we *assume* that this state corresponds to some state $\Psi \in \mathscr{F}(\mathscr{H}_{\text{out}})$, then it follows from eq. (3.120) that for any one particle state $\psi_{AB}$, we have

$$\mathbf{a}_{\text{out}}(\psi_{AB})\Psi = -\langle\psi_{AB}|h^{\text{ret}}_{+AB}\rangle\Psi. \tag{3.121}$$

The solution to this equation is the coherent state associated with $h^{\rm ret}_{+AB}$, namely

$$\Psi \propto \exp\left[-\mathbf{a}^{\dagger}_{\rm out}(h^{\rm ret}_{+AB})\right]|0_{\rm out}\rangle \tag{3.122}$$

Equation (3.122) was derived under the assumption that $\Psi \in \mathscr{F}(\mathscr{H}_{\rm out})$. If $h^{\rm ret}_{+AB}$ has finite norm in the inner product eq. (3.118), then the right side of eq. (3.122) defines a state in $\mathscr{F}(\mathscr{H}_{\rm out})$, and this state corresponds to $|0_{\rm in}\rangle$. However, if $h^{\rm ret}_{+AB}$ does not have finite norm in the inner product eq. (3.118), then the right side of eq. (3.122) does not define a state in $\mathscr{F}(\mathscr{H}_{\rm out})$. It follows that $|0_{\rm in}\rangle$ cannot correspond to a state in $\mathscr{F}(\mathscr{H}_{\rm out})$. This should not be a cause of any distress. The Heisenberg state $|0_{\rm in}\rangle$ is well defined everywhere as a state on the algebra of local field observables. It is similarly well defined on the algebra of asymptotic field observables near future null infinity. All of its correlation functions are well defined at future null infinity. If we wish to represent this state as a vector in a Hilbert space, $\tilde{\mathscr{H}}_{\rm out}$, carrying a representation of the "out" field observables, we may always do so via the GNS construction. However, if $h^{\rm ret}_{+AB}$ does not have finite Klein-Gordon norm, the representation of the field observables on $\tilde{\mathscr{H}}_{\rm out}$ cannot be unitarily equivalent to its representation on $\mathscr{F}(\mathscr{H}_{\rm out})$ (see [64], Section V.A of [65]).

Now let $d = 4$ and consider the case where the classical source $T_{ab}$ is such that the corresponding retarded solution $h^{\rm ret}_{ab}$ has a nonvanishing memory tensor $\Delta^{(1)}_{AB} \neq 0$. Then, as already noted in the first paragraph of this subsection, the Fourier transform of of $h^{\rm ret\,(1)}_{AB}$ with respect to $u$ will diverge at small $\omega$ as $1/\omega$. But by eq. (3.119), we then have

$$\left\|\hat{h}^{\rm ret\,(1)}_{+AB}\right\|^2 = \frac{1}{16\pi}\int d\Omega \int_0^\infty d\omega\omega|\hat{h}^{\rm ret\,(1)}_{+AB}|^2 = \infty \tag{3.123}$$

on account of the "infrared divergence" as $\omega \to 0$. Thus, the "out" state corresponding to $|0_{\rm in}\rangle$—or, for than matter, any other state in $\mathscr{F}(\mathscr{H}_{\rm in})$—does not live in $\mathscr{F}(\mathscr{H}_{\rm out})$, and one would have to work with a different representation to represent this state as a vector in a Hilbert space. Exactly analogous results hold in the scalar and electromagnetic cases for $d = 4$.

We have just shown that in linearized gravity with a classical source for which a nontrivial memory effect is present in the classical retarded solution—as would occur generically in classical particle scattering—the "out" state $\Psi$ is not a state in $\mathscr{F}(\mathscr{H}_{\text{out}})$. However, the infrared divergence described in the previous paragraph is sufficiently innocuous that one can, in effect, proceed as though one were dealing with a state in $\mathscr{F}(\mathscr{H}_{\text{out}})$. To see this, consider, first, the case where no infrared divergences occur and $h^{\text{ret}}_{+AB}$ has finite Klein-Gordon norm, so eq. (3.122) defines a state in $\mathscr{F}(\mathscr{H}_{\text{out}})$. Choose a frequency $\omega_0 > 0$ and decompose $\mathscr{H}_{\text{out}}$ into the direct sum of its "hard" and "soft" graviton spaces

$$\mathscr{H}_{\text{out}} = \mathscr{H}^H_{\text{out}} \oplus \mathscr{H}^S_{\text{out}} \tag{3.124}$$

where $\mathscr{H}^H_{\text{out}}$ is spanned by trace-free $\psi_{AB}$ composed of frequencies $\omega \geq \omega_0$ and $\mathscr{H}^S_{\text{out}}$ is spanned by trace-free $\psi_{AB}$ composed of frequencies $\omega_0 > \omega \geq 0$. The Fock space $\mathscr{F}(\mathscr{H}_{\text{out}})$ then factorizes as

$$\mathscr{F}(\mathscr{H}_{\text{out}}) = \mathscr{F}(\mathscr{H}^H_{\text{out}}) \otimes \mathscr{F}(\mathscr{H}^S_{\text{out}}). \tag{3.125}$$

Now decompose $h^{\text{ret}}_{+AB}$ into its "hard" and "soft" parts,

$$h^{\text{ret}}_{+AB} = [h^{\text{ret}}_{+AB}]^H + [h^{\text{ret}}_{+AB}]^S. \tag{3.126}$$

The creation operator $\mathbf{a}^\dagger_{\text{out}}(h^{\text{ret}}_{+AB})$ appearing in eq. (3.122) and be written as the sum of creation operators for the hard and soft parts of $h^{\text{ret}}_{+AB}$. Since these operators commute, $\Psi$ factorizes as

$$\Psi = \Psi^H \otimes \Psi^S \tag{3.127}$$

where $\Psi^H \in \mathscr{F}(\mathscr{H}_{\rm out}^H)$ is the coherent state associated with $[h^{\rm ret}_{+AB}]^H$ and $\Psi^S \in \mathscr{F}(\mathscr{H}_{\rm out}^S)$ is the coherent state associated with $[h^{\rm ret}_{+AB}]^S$. The factorization in eq. (3.127) implies that if we are interested solely in the "hard part" of the outgoing state, we may effectively put in an "infrared cutoff" at $\omega = \omega_0$ and work with the state $\Psi^H$ in the Fock space $\mathscr{F}(\mathscr{H}_{\rm out}^H)$. In particular, the probability that $\Psi^H \in \mathscr{F}(\mathscr{H}_{\rm out}^H)$ contains a specified number of "hard gravitons" in specified modes is the same as the sum of the probabilities that $\Psi \in \mathscr{F}(\mathscr{H}_{\rm out})$ contains these "hard gravitons" and any number of "soft gravitons." This is the essential content of the "soft theorems" [28]. In perturbation theory, the fact that inclusion of the effects of "soft gravitons" does not affect the calculation of "hard graviton" probabilities manifests itself in a cancelation of the contributions of "real soft gravitons" and "virtual soft gravitons."

The above discussion assumed that $h^{\rm ret\,(1)}_{AB}$ does not have infrared divergences, in which case there is no need to decompose the "out" state into "hard" and "soft" parts. Now consider the case where a memory effect is present and $h^{\rm ret\,(1)}_{AB}$ does have an infrared divergence. Then, as discussed above, $\Psi \notin \mathscr{F}(\mathscr{H}_{\rm out})$. Nevertheless, we may still write

$$\Psi = \Psi^H \otimes \tilde{\Psi}^S \tag{3.128}$$

where $\Psi^H \in \mathscr{F}(\mathscr{H}_{\rm out}^H)$ is the coherent state associated with $[h^{\rm ret}_{+AB}]^H$ and $\tilde{\Psi}^S \in \tilde{\mathscr{F}}^S_{\rm out}$ is the "soft graviton" state written as a vector in the Hilbert space $\tilde{\mathscr{F}}^S_{\rm out}$ in the representation to which it belongs. Although the "soft graviton" content of $\Psi$ near future null infinity is ill defined (since $\Psi^S \notin \mathscr{F}(\mathscr{H}_{\rm out}^S)$), the "hard graviton" content of $\Psi$ is well defined (since $\Psi^H \in \mathscr{F}(\mathscr{H}_{\rm out}^H)$). Thus, if we are interested only in the "hard particle" content of $\Psi$ near future null infinity, we may, in effect, put in an infrared cutoff and treat $\Psi$ as an ordinary Fock space state $\Psi^H \in \mathscr{F}(\mathscr{H}_{\rm out}^H)$ [9].

All of the above discussion beginning with eq. (3.120) holds for the rather trivial theory of linearized gravity with a classical stress-energy source. It is quite a leap to go from this theory to the case of quantum, interacting matter and quantum, nonlinear general relativity, especially since a quantum theory of nonlinear general relativity is not in hand. Nevertheless, let us consider a scattering situation where, by assumption, we have non-interacting ingoing "hard" particles at early times and non-interacting outgoing "hard" particles late times. Consider the "soft" content of the outgoing state, associated with $\omega < \omega_0$, where $\omega_0$ is much less than any inverse length or time scale associated with the interaction. Then, it seems plausible that the dominant contributions to this "soft" content will come from asymptotically early and late times, where the "hard" particles are non-interacting and effectively can be treated classically. If so, then a factorization similar to eq. (3.128) should occur, but with the following important difference: If we fix an "in" state consisting of "hard" particles in momentum eigenstates, then the hard content of the "out" state should have a nonvanishing amplitude for "hard" particles in many different momentum eigenstates. (Of course, total energy-momentum is conserved.)

But this means that there also should be nonvanishing amplitudes for different memory tensors. However, there are an uncountably infinite set of inequivalent memory Fock representations. How should these representations be put together into a single separable Hilbert space such that the $S$-matrix is well-defined (i.e. infrared finite) and has a continuous action of the asymptotic symmetry group? This question shall be directly addressed in chapter 3.

Finally, we note that essentially all of our discussion above also applies to $d > 4$. For $d$ even, the memory tensor is first nonvanishing only at Coulombic order. However, since $C^{(d/2-1)}_{u\mu u\nu}$ can be expressed as inverse angular operators acting on $\partial^{d/2-2}C^{(d-3)}_{u\mu u\nu}/\partial u^{d/2-2}$, it can be seen that the Fourier transform of $h^{(d/2-1)}_{AB}(u, x^A)$ behaves as $\omega^{d/2-3}$ as $\omega \to 0$. Thus, there are no infrared divergences for $d > 4$. This result holds in odd dimensions as well. Thus, although one can still factorize states into "hard" and "soft" parts, there is no necessity to do so in order to describe the "out" state as a Fock space state.[20]

20. However, there may be other considerations that indicate the utility of factorization of the out state into "hard" and "soft" parts (see [66, 67]).

# CHAPTER 3

# INFRARED FINITE SCATTERING THEORY IN QUANTUM FIELD THEORY AND QUANTUM GRAVITY

## 1 Introduction

In this chapter, we consider the problem of infrared divergences in quantum scattering theory. The seminal work of Lehmann, Symanzik and Zimmerman (LSZ) [68], Haag and Ruelle [69, 70], and others established that conventional scattering theory should be well-defined in the case of massive quantum fields. In particular, for massive fields, it should be possible to obtain a unitary $S$-matrix relating the standard "in" and "out" Fock spaces of asymptotic states. However, in four spacetime dimensions, when one has massless quantum fields, one encounters severe difficulties in carrying out this program [6, 7, 8, 9]. Classical massless fields that interact with massive fields or undergo suitable self-interactions will generically undergo a *memory effect* wherein, at order $1/r$ in null directions, the field at late retarded times will not return to the value it had at early retarded times.[1] Thus, at order $1/r$, the Fourier transform of a solution with memory will diverge as $1/\omega$ at low frequencies. In the quantum theory, the one-particle norm of the positive frequency part of such a solution is infinite. Consequently, if one tries to express a quantum state corresponding to a classical solution with memory as a vector in the standard Fock representation, it will have an infinite number of "soft" (i.e. arbitrarily low frequency) massless quanta and its norm will be "infrared divergent." In other words, although states with memory are entirely legitimate quantum field states that necessarily arise in scattering processes, they cannot be accommodated in

1. In spacetime dimension $d$, the memory effect occurs at Coulombic order, $1/r^{d-3}$, whereas radiation decays as $1/r^{d/2-1}$ [1]. For $d = 4$, both occur at order $1/r$, so memory directly affects the quantization of the "in" and "out" radiation. For $d > 4$, the memory effect does not lead to infrared divergences in the quantized radiation. The discussion of this chapter is restricted to $d = 4$.

the standard Fock space. Consequently, the $S$-matrix cannot be defined as a map taking "in" states in the standard Fock representation to "out" states in the standard Fock representation, and infrared (IR) divergences will arise if one attempts to do so.

The most common way of dealing with such infrared divergences is to initially impose an infrared cutoff (so that the "out" state can be expressed as an ordinary Fock space vector), calculate inclusive processes that sum over all possible states of the low frequency massless quanta in the cutoff state, and then remove the cutoff [9, 28, 71]. As a practical matter, this procedure works quite successfully if one is interested in obtaining typical quantities of direct relevance for accelerator experiments, such as (inclusive) cross-sections for the scattering of "hard" particles. However, the infrared cutoff removes the memory effect, so one cannot even ask questions about memory as a quantum observable, as has been of particular recent interest (see e.g. [72, 15, 65] and references therein). Furthermore, even if one is interested only in "hard" particles, this approach cannot properly deal with issues such as the entanglement of "hard" and "soft" particles, which should result in decoherence of the "hard" particles [29, 30]. More significantly, this approach is highly unsatisfactory if one wishes to view the $S$-matrix as a fundamental quantity in the formulation of quantum field theory and quantum gravity, since the $S$-matrix itself is undefined.[2]

In order have a well-defined $S$-matrix, it clearly is necessary to construct Hilbert spaces of "in" and "out" states such that the "in" states evolve to the "out" states. As we have just indicated, this is not the case if one takes the "in" and "out" Hilbert spaces to be the standard Fock spaces, since a generic state "in" Fock space state will evolve to an "out"

2. We note that there are at least two notions of an "infrared finite" $S$-matrix in the literature. The notion that we are concerned with in this chapter is to construct appropriate Hilbert spaces of "in" and "out" states and to obtain the $S$-matrix as a well-defined map between these Hilbert spaces. An alternative notion is to develop a procedure for rendering the standard (infrared divergent) $S$-matrix amplitude finite (see e.g. [73, 74]). While such a procedure then can be used to calculate "inclusive quantities" or determine formal properties of the $S$-matrix amplitudes [66, 75], there is no actual "out" state (with memory) constructed by this procedure.

state with a nonvanishing probability for nonzero memory, which cannot be accommodated in the "out" Fock space. Thus, if we wish to have a well-defined $S$-matrix, we must make alternative choices of the "in" and "out" Hilbert spaces that contain states with nonvanishing memory. In order to have a satisfactory scattering theory, these "in" and "out" Hilbert space constructions should satisfy the following properties:

(1) The "in" and "out" Hilbert spaces are obtained by the "same construction." More precisely, if we identify the algebra of "out" field observables with the algebra of "in" field observables via a change of the time orientation of the bulk spacetime, we require that the "in" and "out" Hilbert space representations of these algebras be unitarily equivalent.

(2) Dynamical evolution maps all "in" states to "out" states and vice-versa, so that one has a unitary $S$-matrix.

(3) The "in" and "out" Hilbert spaces should admit a natural, continuous action of the Poincaré group.[3]

(4) The "in" Hilbert space should be large enough to contain incoming states representing all "hard" scattering processes.

(5) The "in" and "out" Hilbert spaces should be separable, so that they are not "too large".[4]

For the case of quantum electrodynamics (QED) with a massive charged field, a satisfactory construction of "in" and "out" Hilbert spaces was given many years ago by Faddeev and Kulish [32] based on the earlier work of [80, 81, 82]. However, the main purpose of this

3. In the gravitational case we require that the "in" and "out" Hilbert spaces should admit a natural, continuous action of the BMS group.

4. A nonseparable Hilbert space was previously studied in [76, 77, 78, 79] which considered the direct sum over all memory representations. We discuss the deficiencies of this direct sum in sec. 7.

chapter is to show that a similar construction does *not* work in a satisfactory way for QED with massless charged particles and for Yang-Mills theory. Furthermore, we will show that such a construction does not work at all in quantum gravity. We argue that in these cases, at a fundamental level, scattering theory should be formulated at the level of algebraic states, without attempting to "shoehorn" all the states into a single, separable Hilbert space.

Since many of our arguments and constructions require a considerable amount of technical machinery, we now provide a sketch of all of the key results of the chapter, so that a reader can obtain the gist of our arguments without having to delve into the details that we will provide in due course in the body of the chapter. This overview will be more detailed than the brief overview given in sec. 0.3 of chapter 1 and a more quantitative emphasis will placed on the construction of the Faddeev-Kulish Hilbert space.

We begin by describing the Faddeev-Kulish construction for QED with a massive charged field. In order to understand the relevant ingredients of their construction, it is necessary to reformulate it in the language of the memory effect and the related symmetries and charges. In this section, for ease of explanation, we will work in the bulk spacetime — introducing "null coordinates" $(u, r, x^A)$, where $u = t - r$ and $x^A$ denotes angular coordinates on the sphere — and work to appropriate orders in $1/r$. However, in the remainder of this chapter it will be more convenient and conceptually clearer to express both the classical and quantum theory in terms of the conformal completion of Minkowski spacetime.

The classical memory effect at future null infinity for an electromagnetic field corresponds to having the angular components, $A_A^{(1)}(u, x^A)$, of the vector potential at order $1/r$ asymptote to different values at early and late retarded times, $u \to \pm\infty$. Since the electric field at order $1/r$ is given by

$$E_A^{(1)} = -\partial_u A_A^{(1)} \tag{1.1}$$

it follows that there will be a nontrivial memory effect if and only if at order $1/r$ the electric field satisfies $\int\limits_{-\infty}^{\infty} du\ E_A^{(1)} \neq 0$. Since $\int\limits_{-\infty}^{\infty} du\ E_A^{(1)}$ is proportional to the integrated force on a test particle placed at a large distance from the source of radiation, this fact allows one to give a physical interpretation of the memory effect in terms of a charged test particle receiving a net momentum kick at order $1/r$ due to the passage of the radiation [83, 11]. Since we assume that $E_A^{(1)} \to 0$ at early and late retarded times, $A_A^{(1)}$ is "pure gauge" at early and late retarded times, but the electromagnetic memory

$$\Delta_A^{\text{out}} := -\int\limits_{-\infty}^{\infty} du\ E_A^{(1)} = A_A^{(1)}|_{u=+\infty} - A_A^{(1)}|_{u=-\infty} \tag{1.2}$$

is gauge invariant, as is obvious from the fact that it is given by an integral of the electric field. In eq. (1.2), we have appended the superscript "out" to $\Delta_A^{\text{out}}$ to distinguish the electromagnetic memory of the outgoing radiation from the electromagnetic memory, $\Delta_A^{\text{in}}$, of incoming radiation.

The gauge transformations relevant for changing the angular components of the vector potential at order $1/r$ are the so-called "large gauge transformations"

$$A_\mu \to A_\mu + \nabla_\mu \lambda \tag{1.3}$$

where $\lambda = \lambda(x^A)$ is a function of purely the angular coordinates $x^A$. Under such a gauge transformation, we have

$$A_A^{(1)} \to A_A^{(1)} + \mathscr{D}_A \lambda \tag{1.4}$$

where $\mathscr{D}_A$ is the derivative operator on the unit sphere. In fact, the "gauge transformations" eq. (1.3) are actually "symmetries" in the sense that they have nonvanishing symplectic product with other solutions, i.e., they are not degeneracies of the symplectic form. There are charges and fluxes associated with these symmetries. The charge $\mathcal{Q}_u(\lambda)$ associated with the symmetry $\lambda$ at retarded time $u$ is given by

$$\mathcal{Q}_u(\lambda) = \frac{1}{4\pi} \int\limits_{S(u)} d\Omega \; \lambda(x^A) F^{(2)}_{ur}(u, x^A) \tag{1.5}$$

where $S(u) \cong \mathbb{S}^2$ is an asymptotic sphere at fixed retarded time $u$, $F_{\mu\nu}$ is the electromagnetic field tensor, and the superscript "(2)" denotes the order $1/r^2$ part of the field as $r \to \infty$ at the given value of $u$. The difference of the charge $\mathcal{Q}_u(\lambda)$ at two retarded times $u_1$ and $u_2$ is determined by a corresponding flux between these retarded times associated with the symmetry $\lambda$

$$\mathcal{Q}_{u_2}(\lambda) - \mathcal{Q}_{u_1}(\lambda) = \int\limits_{u_1}^{u_2} du \int\limits_{\mathbb{S}^2} d\Omega \; \lambda(x^A) \left( J^{(2)}_u(u, x^A) - \frac{1}{4\pi} \mathscr{D}^A E^{(1)}_A(u, x^A) \right) \tag{1.6}$$

where $J^{(2)}_\mu$ is the charge-current at order $1/r^2$, which can be nonvanishing only if there are massless charged fields. In the limit as $u_1 \to -\infty$ and $u_2 \to +\infty$, this first term corresponds to the total flux of charge-current, which we denote as

$$\mathcal{J}^{\text{out}}(\lambda) := \int\limits_{-\infty}^{\infty} du \int\limits_{\mathbb{S}^2} d\Omega \; \lambda(x^A) J^{(2)}_u(u, x^A) \tag{1.7}$$

where, again, the "out" corresponds to the "outgoing" flux of massless charge-current. The second term on the right-hand-side of eq. (1.6) in this limit is proportional to the divergence of memory $\mathscr{D}^A \Delta^{\text{out}}_A(x^A)$ smeared with $\lambda(x^A)$ on the sphere. Finally, in this limit, the charges $\mathcal{Q}_{u_2}(\lambda)$ and $\mathcal{Q}_{u_1}(\lambda)$ approach future time-like infinity $i^+$ and spatial infinity $i^0$, respectively. Therefore, in the case where $u_1 = -\infty$ and $u_2 = +\infty$, eq. (1.6) yields that the charges given by eq. (1.5) and the flux of null charge-current given by eq. (1.7) are related to the electromagnetic memory by[5]

$$\frac{1}{4\pi}\int_{\mathbb{S}^2} d\Omega\ \Delta_A^{\text{out}}(x^A)\mathscr{D}^A\lambda = \mathcal{Q}_{i^+}(\lambda) - \mathcal{Q}_{i^0}(\lambda) + \mathcal{J}^{\text{out}}(\lambda) \tag{1.8}$$

where the charges are defined as limits as $u \to \pm\infty$. The difference of charges on the right hand side of eq. (1.8) is known as the "ordinary memory effect" and the contribution due to the total charge-current flux of massless charged fields is known as the "null memory effect" [11].

Similar charges and fluxes associated with the symmetry $\lambda$ can be defined at past null infinity, wherein we replace retarded time $u$ in the above formulas by advanced time $v = t + r$. In a scattering situation, there is, in general, no direct relation between the memory $\Delta_A^{\text{out}}$ of the electromagnetic field at future null infinity and the memory $\Delta_A^{\text{in}}$ at past null infinity. Indeed, as we have already indicated, if the incoming electromagnetic field has vanishing memory, the outgoing electromagnetic field will generically have nonvanishing memory. However, there is a matching of the incoming and outgoing charges as one approaches spatial infinity [22, 23, 24, 33, 25]. Specifically, we have

$$\mathcal{Q}_{i^0}^{\text{out}}(\lambda) = \mathcal{Q}_{i^0}^{\text{in}}(\lambda \circ \Upsilon) \tag{1.9}$$

where we have used "in/out" to denote that the limit is taken from past/future null infinity to spatial infinity and $\Upsilon$ is the antipodal map on a sphere, so that $(\lambda \circ \Upsilon)(\theta, \varphi) = \lambda(\pi - \theta, \varphi + \pi)$.

The "conservation law" eq. (1.9) is the key to enabling one to define "in" and "out" Hilbert

5. Memory can be decomposed into electric and magnetic parity parts via $\Delta_A = \mathscr{D}_A\alpha + \epsilon_A{}^B\mathscr{D}_B\beta$. Equation (1.8) only involves the electric part, $\alpha$. Magnetic parity charges similar to eq. (1.5) can be defined (although they are not associated with large gauge transformations) and an analog of eq. (1.8) (without the current term) then holds. The constructions of this chapter of the thesis could be straightforwardly extended to include magnetic parity memory. However, magnetic parity memory does not arise in usual scattering processes starting with states of vanishing memory — although it can occur in certain processes (see [1] for an example in the gravitational case). We will focus entirely on electric parity memory in this chapter.

spaces satisfying conditions (1)–(5) in the case of QED with massive charged fields. If we restrict all of the incoming states to have fixed, definite large gauge charges at spatial infinity for all $\lambda$, then the outgoing states will have the corresponding charges given by eq. (1.9). Hence, if we can construct "in" and "out" Hilbert spaces of definite values of all charges at spatial infinity, it should be possible to satisfy properties (1) and (2) above. However, since these charges are not invariant under Lorentz transformations, it will not be possible to have the Poincaré group have a continuous action on a space of incoming or outgoing states of definite charges except in the case where all charges (including the ordinary total electric charge) vanish[6] at spatial infinity [85, 84, 86, 34]. Thus, in order to satisfy property (3), we restrict the incoming states — and, therefore, the outgoing states by eq. (1.9) — to have vanishing charges at spatial infinity. It may appear that the requirement of vanishing total electric charge will violate condition (4) of our above requirements on the "in" Hilbert space, since it will allow us to consider only scattering processes with an equal number of charged particles and antiparticles. However, if we wish to consider the scattering of, say, two electrons, we can simply add two positrons "behind the moon", i.e., incoming states where they don't interact significantly with the electrons or with each other [34]. Thus, arguably, the restriction to states of vanishing charges does not preclude having representatives of all "hard" scattering processes.

As discussed in more technical detail in sec. 4, a separable Hilbert space of "in" and "out" states of vanishing charges for QED with massive charged particles can be constructed as follows. For the construction of the "in" Hilbert space, we note that the charges at past timelike infinity, $\mathcal{Q}_{i^-}(\lambda)$, are determined by the incoming state of the massive charged particles. The (improper) incoming Fock space state $|p_1 \dots p_n; q_1 \dots q_n\rangle$ consisting of $n$

6. One could start with a Hilbert space with nonvanishing charges and obtain a new space that admits an action of the Poincaré group by taking the direct sum of the continuous family of Hilbert spaces with charges equal to the action of the Lorentz group on the original charges. However, this direct sum Hilbert space would be non-separable. Furthermore, there would be no infinitesimal action of the Lorentz group on the direct sum Hilbert space, so, in particular, the angular momentum operator would not be defined [84, 85].

incoming charged particles and $n$ incoming antiparticles with definite momenta $p_1, \ldots, p_n$ and $q_1, \ldots, q_n$, respectively, has vanishing total ordinary electric charge and can be seen to be an eigenstate of the charge operator $\boldsymbol{\mathcal{Q}}_{i^-}(\lambda)$ for all $\lambda$. We denote its eigenvalue as $\mathcal{Q}_{i^-}(\lambda; p_1 \ldots q_n)$. By the corresponding version of eq. (1.8) for past null infinity with $\mathcal{J}^{\text{in}} = 0$ (since we are considering only massive charged particles) we can obtain an (improper) state for which all large gauge charges vanish at spatial infinity ($v \to +\infty$) by pairing $|p_1 \ldots p_n; q_1 \ldots q_n\rangle$ with any incoming electromagnetic field state that lies in the representation with memory $\Delta_A^{\text{in}}$ determined by[7]

$$\frac{1}{4\pi} \int_{\mathbb{S}^2} d\Omega \; \Delta_A^{\text{in}} \mathscr{D}^A \lambda = -\mathcal{Q}_{i^-}(\lambda; p_1, \ldots, q_n) \tag{1.10}$$

for all $\lambda(x^A)$. In other words, we can take the tensor product of the one-dimensional Hilbert space spanned by the improper state $|p_1 \ldots p_n; q_1 \ldots q_n\rangle$ with the Fock space representation of the electromagnetic field with memory $\Delta_A^{\text{in}}$ given by eq. (1.10). The pairing of the charged particle state $|p_1 \ldots p_n; q_1 \ldots q_n\rangle$ with electromagnetic states in the representation $\Delta_A^{\text{in}}$ is usually referred to as "dressing" the charged particles with a corresponding "cloud of soft photons."[8] We can then obtain a Hilbert space with arbitrary proper Fock space states of $n$ charged particles, $n$ antiparticles and arbitrary "hard" photon states by taking a direct integral over $p_1, \ldots, p_n, q_1, \ldots, q_n$. We then take the direct sum over $n$. This yields a separable Hilbert space that has representatives of all incoming states of the massive charged particles with vanishing total electric charge and all incoming "hard" photon states. This construction is equivalent to the one given by Faddeev and Kulish. All states in this "in" Hilbert space are eigenstates with eigenvalue 0 of all of the large gauge charges at spatial

7. This relation uniquely determines the electric parity memory (see footnote 5).

8. Note that one does not have to "dress" the charged particles with a specific state , i.e., *any* electromagnetic state with the required memory is allowed. The "cloud of soft photons" refers to any state in the representation with the required memory. A charged particle "dressed" with an infrared "cloud of soft photons" is sometimes referred to as an "infraparticle" [87, 84].

infinity.[9] By conservation of charges at spatial infinity, these states should evolve to states in the similarly constructed "out" Hilbert space. Indeed, finiteness of the Faddeev-Kulish "S-matrix amplitudes" has been verified to all orders in perturbation theory [89]. These results are also supported by recent, rigorous analyses of perturbative QED [90] as well as non-perturbative studies of nonrelativistic QED [91, 92].[10] Consequently, all of the above properties (1)–(5) should be satisfied.

Thus, the Faddeev-Kulish construction provides definitions of "in" and "out" Hilbert spaces that enable one to have a well-defined $S$-matrix. However, it should be noted that this construction has a number of unpleasant features. First, it allows only states of vanishing total ordinary electric charge. As already mentioned above, this can be dealt with by putting any excess charges "behind the moon." A more unpleasant feature is that it requires the incoming massive charged particles to be "dressed" with incoming electromagnetic states with the corresponding memory. This dressing is quite unnatural, since — although the incoming massive charged field and incoming electromagnetic radiative field are completely independent degrees of freedom — it requires the incoming electromagnetic radiative state to "know" the exact state of the incoming charged field. Furthermore, since each state in the Faddeev-Kulish Hilbert space has an extremely high degree of entanglement between the state of the massive charged field and the state of the electromagnetic field, one cannot have a coherent superposition of incoming charged particle states of different momenta [29]. Thus, the Faddeev-Kulish Hilbert space appears to artificially exclude many states that one might wish to consider. Nevertheless, by restricting consideration to the states in the Faddeev-Kulish Hilbert space, one should obtain a genuine $S$-matrix, with no infrared divergences.

9. The relationship between the Faddeev and Kulish "dressed states" and eigenstates of $\boldsymbol{\mathcal{Q}}_{i^0}(\lambda)$ has been previously discussed in [66, 88].

10. Similar analyses have also been done in the case of infrared divergences arising from the scattering of a massless scalar field coupled to a massive scalar field (sometimes referred to as the "Nelson model") [93, 94, 95].

We turn now to differences that occur if we consider QED with a massless charged field, as will be discussed in detail in sec. 5. Since there are no incoming massive particles, the charges $\mathcal{Q}_{i^-}(\lambda)$ at timelike infinity vanish in the massless case. However, since there are incoming massless particles, the charge-current flux $\mathcal{J}^{\text{in}}(\lambda)$ at null infinity will not vanish. One can perform a construction of "in" and "out" Hilbert spaces that is completely analogous to the Faddeev-Kulish construction as follows: In the massless case, the (improper) incoming Fock space states $|p_1 \dots p_n; q_1 \dots q_n\rangle$ of the charged field are eigenstates of the charge-current operator $\boldsymbol{\mathcal{J}}^{\text{in}}(\lambda)$ for all $\lambda$. Therefore, one can again pair $|p_1 \dots p_n; q_1 \dots q_n\rangle$ with the incoming electromagnetic field states that lie in the representation with memory $\Delta_A^{\text{in}}$ chosen so as to give vanishing charges at spatial infinity. In this case, by eq. (1.8), the required $\Delta_A^{\text{in}}$ is determined by

$$\frac{1}{4\pi}\int_{\mathbb{S}^2} d\Omega\ \Delta_A^{\text{in}}\mathscr{D}^A\lambda = -\mathcal{J}^{\text{in}}(\lambda; p_1, \dots, q_n)\,. \tag{1.11}$$

where $\mathcal{J}^{\text{in}}(\lambda; p_1, \dots, q_n)$ denotes the eigenvalue of $\boldsymbol{\mathcal{J}}^{\text{in}}(\lambda)$ in the state $|p_1 \dots p_n; q_1 \dots q_n\rangle$. By taking a direct integral over $p_1, \dots, p_n, q_1, \dots, q_n$ and a direct sum over $n$, we will again get a separable Hilbert space of states with vanishing charges at spatial infinity, so the "in" Hilbert space should unitarily map to the similarly constructed "out" Hilbert space under dynamical evolution. This yields a direct analog for massless charged fields of the Faddeev-Kulish construction for massive charged fields.

However, although the Faddeev-Kulish construction can be carried out in close analogy with the massive case, a truly significant difference arises in the nature of the resulting states. For massive charged fields, the charges $\mathcal{Q}_{i^-}(\lambda; p_1 \dots q_n)$ for the state $|p_1 \dots p_n; q_1 \dots q_n\rangle$ correspond to a smooth function on the sphere. Consequently, the corresponding memory $\Delta_A^{\text{in}}$ determined by eq. (1.10) is smooth, and the corresponding memory representation of the electromagnetic field has a dense set of nonsingular states. By contrast, in the massless case, the flux $\mathcal{J}^{\text{in}}(\lambda; p_1, \dots, q_n)$ for the state $|p_1 \dots p_n; q_1 \dots q_n\rangle$ has $\delta$-function angular singularities in each of the directions $x_i^A$ of the momenta $p_1, \dots, p_n, q_1, \dots, q_n$. It follows that the memory $\Delta_A^{\text{in}}(x^B)$ determined by eq. (1.11) will have angular singularities of the form $1/|x^B - x_i^B|$ in the vicinity of $x_i^B$. These additional angular singularities occurring in the massless case correspond to what are referred to as *collinear divergences*. If one is interested in calculating inclusive cross-sections, they merely give rise to an additional nuisance in that one must introduce a further angular cutoff in addition to the usual infrared cutoff when performing calculations [96, 97]. But they give rise to a fatal difficulty for the usefulness of the "in" and "out" Hilbert spaces constructed above. The angular singularities in the memory are such that the memory is not square integrable over a sphere. This implies that the expected electric field $\langle E_A^{\text{in}} \rangle$ in any state in the memory representation paired with $|p_1 \dots p_n; q_1 \dots q_n\rangle$ cannot be square integrable over null infinity. By further arguments, it can be seen that the total energy flux of the electromagnetic field in any state in this memory representation is infinite. In other words, in the massless case, the required "soft photon dressing" of the charged particles always carries infinite energy. Thus, *all* of the allowed states of the electromagnetic field in this construction are physically unacceptable. Although we should be able to obtain a well-defined scattering theory between states in the "in" and "out" Hilbert spaces, none of the scattering states are of any physical relevance.

We now turn to Yang-Mills theory with a compact, semi-simple Lie group, which will be discussed in more detail in sec. 5.5. The Yang-Mills fields occurring in nature are strongly coupled to other fields and do not behave as free fields at asymptotically early and late times. However, we can consider the scattering theory of "pure" Yang-Mills theory (with no coupling to other fields) as a toy model that has features similar to both electromagnetism and gravity. Collinear divergences similar to massless QED occur in Yang-Mills theory. Consequently, as in massless QED, the "dressing" required by the Faddeev-Kulish construction will be singular. However, an additional — and, in some respects, even more serious — difficulty arises in the Yang-Mills case, due to the fact that the Yang-Mills field acts as its own source. The analog of eq. (1.8) in the Yang-Mills case is

$$\frac{1}{4\pi}\int_{\mathbb{S}^2} d\Omega\ \Delta^{\mathrm{YM,out}}_{A,j}\mathscr{D}^A\lambda^j = \mathcal{Q}^{\mathrm{YM}}_{i^+}(\lambda) - \mathcal{Q}^{\mathrm{YM}}_{i^0}(\lambda) + \mathcal{J}^{\mathrm{YM,out}}(\lambda) \tag{1.12}$$

where $j$ denotes a Lie algebra index, such indices are lowered and raised with the Cartan-Killing metric, and the charges are defined by a natural generalization of eq. (1.5) where the Lie algebra valued field strength is now integrated with $\lambda^i(x^A)$. Since there are no massive sources, the charges at timelike infinity vanish, $\mathcal{Q}^{\mathrm{YM}}_{i^+}(\lambda) = 0$. The charge-current flux in the Yang-Mills case is

$$\mathcal{J}^{\mathrm{YM,out}}(\lambda) = \frac{1}{2\pi}\int_{-\infty}^{\infty} du \int_{\mathbb{S}^2} d\Omega\ c^i{}_{jk} q^{AB}\lambda_i A^{(1)j}_A E^{(1)k}_B \tag{1.13}$$

where $c^i{}_{jk}$ denote the structure constants of the Lie algebra. Similar charges and fluxes associated to the symmetry $\lambda$ can be defined at past null infinity. The analog of eq. (1.11) for obtaining eigenstates of vanishing charge[11] at spatial infinity for the Yang-Mills field is

$$\frac{1}{4\pi}\int_{\mathbb{S}^2} d\Omega\ \Delta^{\mathrm{YM,in}}_{A,j}\mathscr{D}^A\lambda^j = -\mathcal{J}^{\mathrm{YM,in}}(\lambda)\,. \tag{1.14}$$

The key difference with massless QED is that the "hard" and "soft" quanta now correspond to the same field. Thus, we must use "soft" Yang-Mills quanta to "dress" (via the memory, $\Delta^{\mathrm{YM,in}}_{A,j}$) the "hard" Yang-Mills quanta. But these "soft" quanta will then make additional contributions to the current flux, so we will not get an eigenstate of charges at spatial infinity by choosing the memory to satisfy eq. (1.14), with $\mathcal{J}^{\mathrm{YM,in}}(\lambda)$ the flux of the "hard" Yang-Mills quanta. Thus—in addition to the fact that, as in the case of massless QED, this soft "dressing" is singular and therefore the corresponding Yang-Mills current flux is

11. The charges at spatial infinity satisfy the commutation relations $\left[\mathcal{Q}^{\mathrm{YM}}_{i^0}(\lambda_1), \mathcal{Q}^{\mathrm{YM}}_{i^0}(\lambda_2)\right] = \mathcal{Q}^{\mathrm{YM}}_{i^0}([\lambda_1,\lambda_2])$ where $[\lambda_1,\lambda_2]^i = c^i{}_{jk}\lambda_1^j\lambda_2^k$. For a semisimple Lie group it is impossible to have an eigenstate of all charges unless all of the charge eigenvalues vanish.

infinite—one cannot get states of vanishing charges at spatial infinity by attempting to pair flux eigenstates with corresponding memory representations.

Thus, in order to obtain an analog of the Faddeev-Kulish Hilbert space in the Yang-Mills case, one must find some other means to obtain a suitable Hilbert space of eigenstates of vanishing charges at spatial infinity. However, there are insufficiently many such states, as can be seen from the fact that the charge $\mathcal{Q}^{\rm YM}_{i^0}(\lambda)$ at spatial infinity acts on the "in" and "out" states as an infinitesimal generator of the large gauge transformation associated with $\lambda^j$. Thus, an "in" state with vanishing charges at spatial must be gauge invariant with respect to all large gauge transformations, which is a strong constraint on the $n$-point functions of the Yang-Mills electric field. In particular this implies that the 1-point function must vanish, the 2-point function must be proportional to the Cartan-Killing metric and, more generally, all $n$-point functions must be proportional to Casimirs of the Lie algebra. Although there exist states that satisfy these conditions, these conditions are far too restrictive to allow one to satisfy condition (4) of our requirements on the "in" Hilbert space.

We now turn to general relativity, which will be considered in detail in sec. 6. We introduce Bondi coordinates $(u, r, x^A)$, and let $C_{\mu\nu}(u, x^A)$ denote the deviation of the spacetime metric from the asymptotic Minkowski metric $\eta_{\mu\nu}$ at leading order in $1/r$ in these coordinates.[12] The classical memory effect at future null infinity in the gravitational case corresponds to having the angular components, $C_{AB}$, at order $1/r$ asymptote to different values at early and late retarded times, $u \to \pm\infty$. This will occur if and only if at order $1/r$ the Bondi News $N_{AB} = -\partial_u C_{AB}$ satisfies $\int_{-\infty}^{\infty} du\ N_{AB} \neq 0$. In the gravitational case, the presence of memory physically corresponds to an array of test particles initially at rest receiving a permanent relative displacement at order $1/r$ due to the passage of gravitational radiation [10].

If the Bondi News goes to zero at early and late retarded times, $C_{AB}$ will be "pure gauge"

12. Again, we state our main results in this section in Bondi coordinates in the bulk spacetime, but in sec. 6 we will work at null infinity in the conformally completed spacetime.

at early and late retarded times, but the gravitational memory

$$\Delta_{AB}^{\text{GR,out}} := \frac{1}{2}\int_{-\infty}^{\infty} du\ N_{AB} = -\frac{1}{2}\Big(C_{AB}|_{u=+\infty} - C_{AB}|_{u=-\infty}\Big) \tag{1.15}$$

is gauge invariant. The relevant gauge transformations in the gravitational case are the supertranslations whose infinitesimal action is given by

$$C_{AB} \to C_{AB} - fN_{AB} - 2\left(\mathscr{D}_A\mathscr{D}_B f - \frac{1}{2}q_{AB}\mathscr{D}^C\mathscr{D}_C f\right) \tag{1.16}$$

where $f = f(x^A)$ is an arbitrary function on the sphere and $q_{AB}$ is the metric on the unit sphere. The supertranslations are, in fact, "symmetries," i.e., they are not degeneracies of the symplectic form. Again, there are charges and fluxes associated with these symmetries. The charge $\mathcal{Q}_u^{\text{GR}}(f)$ associated with the supertranslation $f$ at retarded time $u$ is given by

$$\mathcal{Q}_u^{\text{GR}}(f) = -\frac{1}{8\pi}\int_{S(u)} d\Omega\ f(x^A)\left[C^{(3)}_{urur}(u,x^A) - \frac{1}{4}N^{AB}C_{AB}\right] \tag{1.17}$$

where $S(u) \cong \mathbb{S}^2$ is an asymptotic sphere at fixed retarded time $u$, $C_{\mu\nu\rho\sigma}$ is the Weyl tensor and the superscript "(3)" denotes the order $1/r^3$ part as $r \to \infty$ at fixed $u$. The difference of the charge $\mathcal{Q}_u^{\text{GR}}(f)$ at two retarded times $u_1$ and $u_2$ is determined by a corresponding flux between these retarded times associated with the symmetry $f$

$$\mathcal{Q}_{u_2}^{\text{GR}}(f) - \mathcal{Q}_{u_1}^{\text{GR}}(f) = -\frac{1}{32\pi}\int_{u_1}^{u_2} du \int_{\mathbb{S}^2} d\Omega\ f\Big(N^{AB}N_{AB} + 2\mathscr{D}^A\mathscr{D}^B N_{AB}\Big). \tag{1.18}$$

If other massless fields are present and if their stress energy $T_{\mu\nu}$ satisfies the dominant energy condition then eq. (1.18) is modified by the substitution $N^{AB}N_{AB} \to N^{AB}N_{AB} + 32\pi T^{(2)}_{uu}$ where the "(2)" denotes the order $1/r^2$ part as $r \to \infty$ at fixed $u$. In the gravitational case, eq. (1.18) with $u_1 \to -\infty$ and $u_2 \to +\infty$ directly yields an analogous formula to eq. (1.8) relating charges, fluxes and memory[13]

$$-\frac{1}{8\pi}\int_{\mathbb{S}^2} d\Omega\ \Delta_{AB}^{\text{GR,out}}\mathscr{D}^A\mathscr{D}^B f = \mathcal{Q}_{i^+}^{\text{GR}}(f) - \mathcal{Q}_{i^0}^{\text{GR}}(f) + \mathcal{J}^{\text{GR,out}}(f). \tag{1.19}$$

The terms on the right hand side of eq. (1.19) involving the difference of charges is referred to as "ordinary memory" [10] and the term involving the flux is usually referred to as "null memory" or "nonlinear memory" [39, 19] given by

$$\mathcal{J}^{\text{GR,out}}(f) = \frac{1}{32\pi}\int_{-\infty}^{\infty} du \int_{\mathbb{S}^2} d\Omega\ f N^{AB}N_{AB}. \tag{1.20}$$

Similar formulas hold for the "in" memory in terms of difference of charges at past timelike infinity and spatial infinity as well as incoming null memory. As in the electromagnetic case, there is a matching of the incoming and outgoing charges as one approaches spatial infinity as originally conjectured by Strominger [22]

$$\mathcal{Q}_{i^0}^{\text{GR,out}}(f) = \mathcal{Q}_{i^0}^{\text{GR,in}}(f \circ \Upsilon). \tag{1.21}$$

As is well known, significant difficulties arise in the formulation of a quantum theory of gravity in the bulk spacetime. However, as Ashtekar has emphasized, no such difficulties arise in the asymptotic quantization of the radiative degrees of freedom of the gravitational field at null infinity [64, 3, 65]. Thus, the notion of asymptotic states of the quantum gravitational field in asymptotically flat spacetimes is well-defined, irrespective of the details of the bulk theory of gravity. In view of the classical memory effect, it is not possible that "in" states

13. This equation determines the electric parity part of the memory. The "magnetic parity" part of the memory is determined by $\epsilon^{CA}\mathscr{D}_C\mathscr{D}^B\Delta_{AB}^{\text{GR}}$, which can be expressed in terms of the difference of magnetic parity charges [1] with no null memory contribution. All of the analysis of this chapter could be straightforwardly generalized to include magnetic parity memory. However, as in the electromagnetic case (see footnote 5), we shall focus entirely on electric parity memory in this chapter.

with vanishing memory (i.e., states in the standard "in" Fock representation) will generically evolve to "out" states with vanishing memory. Thus, infrared divergences similar to those occurring in QED must arise if one attempts to define an $S$-matrix with the conventional choices of "in" and "out" Hilbert spaces. One may ask whether there exist alternative choices satisfying conditions (1)–(5) above.

In linearized gravity with matter sources, massive fields will contribute to ordinary memory and massless fields will contribute to null memory, in close analogy with QED. In the case of QED, the vanishing of the charges $\mathcal{Q}_{i^0}(\lambda)$ at spatial infinity – including the ordinary electric charge – was required to have a Lorentz group action. As discussed above, in QED, it can be argued that the requirement of vanishing total electric charge is not a problem for obtaining representatives of all "hard" scattering processes because one can always put additional charges "behind the moon". However, in linearized gravity with massive/massless sources, the analogous requirement of vanishing total 4-momentum is a serious problem, since the ordinary vacuum state is the only state that satisfies this requirement — there is no way to "cancel" the 4-momentum if a state by adding particles. Therefore the Faddeev-Kulish construction fails for this elementary reason at this initial stage.

Nevertheless, one could give up on having a well-defined action of the Lorentz transformations and attempt to construct states of definite, non-vanishing charges $\mathcal{Q}^{\mathrm{GR}}_{i^0}(f)$ at spatial infinity [98]. For linearized gravity with a massive field source, one can straightforwardly carry out an analog of the construction of sec. 4.4 for massive QED. For linearized gravity with a massless field source, one can carry out an analog of the construction of sec. 5.4 for massless QED. Indeed, the situation for linearized gravity with a massless field source is somewhat better than massless QED in that the singularities of the memory are less severe [36]. In linearized gravity, for incoming momentum eigenstates of massless particles, the corresponding flux in eq. (1.19) will again have $\delta$-function angular singularities. However, on account of the presence of two derivatives on the left side of eq. (1.19) (as compared to the one derivative in eq. (1.11)), the corresponding collinear divergence singularities of $\Delta^{\text{GR}}_{AB}$ will be of the form $\log|x^A - x^A_i|$. Although still singular, this is square integrable and does not imply an infinite energy flux of soft gravitons. Thus, arguably, in linearized gravity, the "dressed states" in the analogously constructed Faddeev-Kulish Hilbert space are physically acceptable, although since there is no a well-defined action of the Lorentz group, the states obtained in this construction do not have a well-defined angular momentum.

However, we show in sec. 6.3 that the Faddeev-Kulish type of construction fails catastrophically in nonlinear gravity. The fundamental problem is that, as in the Yang-Mills case, the "soft gravitons" that must be used to dress the "hard gravitons" will contribute their own flux, thereby invalidating any attempt to pair flux eigenstates of hard gravitons with corresponding memory representations. Thus, as in the Yang-Mills case, in order to obtain an analog of the Faddeev-Kulish Hilbert space, one must find some other means to obtain eigenstates of the supertranslation charges. In the Yang-Mills case, charge eigenstates must have vanishing charges and thus the $n$-point functions of any eigenstate of all large gauge charges must be invariant under all large gauge transformations. Although this is a highly restrictive condition, there do exist some invariant states besides the vacuum state. However, as we show in Theorem 5 of sec. 6.3, the corresponding condition in quantum gravity is that the $n$-point functions of the news must be invariant under supertranslations. However, this requirement is incompatible with the fall-off requirements on states. Thus, apart from the vacuum state, there are no eigenstates whatsoever of the supertranslation charges. Thus there is no analog of the Faddeev-Kulish Hilbert space in nonlinear gravity.

Thus, if one is to obtain "in" and "out" Hilbert spaces in quantum gravity that satisfy properties (1)–(5), one will have to do so by a very different means than by the Faddeev-Kulish construction. We explore some possibilities in sec. 7 involving direct integrals with respect to Gaussian measures of Fock representations with memory. We find that these also do not work. Of course, our analysis does not exhaust all possibilities, but we do not see any further avenues of approach that appear promising. Thus, we believe that for gravity (as well as for Yang-Mills theory and massless QED), no satisfactory Hilbert space construction of “in” and “out” states can be given.

What does this mean for scattering theory? There is no problem defining “in” and “out” states that should accommodate all scattering processes, allowing arbitrary incoming and outgoing “hard” particle states and arbitrary memory. The difficulties arise entirely from the attempt to “shoehorn” all states relevant to scattering theory into a single, separable Hilbert space. It is our view that there is no need to try to do this. An “in” state can be defined in the algebraic viewpoint as a positive linear function on the algebra of “in” observables. In this viewpoint one would specify an “in” state by giving the complete list of the correlation functions of the “in” fields — where this list must satisfy positivity requirements. Any state in any Hilbert space construction gives rise to a state in this sense, since one can compute all the correlation functions and they will automatically satisfy the positivity requirement. Conversely, the Gel’fand-Naimark-Segal (GNS) construction shows that any state in the algebraic sense can be realized as a vector in some Hilbert space, so one does not get entirely new objects by considering states in the algebraic sense. But by considering states in the algebraic sense, one is freed from the necessity of choosing in advance a particular Hilbert space in which it lies. Thus, one may consider any “in” state that one wishes, without placing any “dress requirements” on the state. If one evolves the chosen “in” state through the bulk, one will get some “out” state, defined, again, as a list of correlation functions of the “out” fields. There is no reason to impose an a priori restriction as to which Hilbert space this “out” state will lie in — and one will get infrared divergences if one selects the wrong one. As discussed in sec. 8, we see no difficulty of principle in describing scattering theory in this framework. Of course, if one is interested in calculating quantities relevant to collider physics, we are not suggesting that there would be any advantage to taking such an approach over the usual approach of working in the standard Fock space and imposing infrared cutoffs.

However, if one wishes to treat scattering at a fundamental level, we believe it is necessary to approach it from such an algebraic viewpoint on "in" and "out" states.

The structure of the remainder of the chapter is as follows. In sec. 2 we briefly review the classical phase space of a free scalar field in an asymptotically flat spacetime and give a precise notion of "observable" on this phase space. We also define local field observables at null infinity and determine their Poisson brackets. In sec. 3, we review the algebraic viewpoint on quantization and the formulation of free field theory in this framework. We briefly review the notion of Hadamard states in the bulk and consider their limit to null infinity in the case of massless fields. Although we are, of course, interested in the scattering theory of interacting fields, these interacting fields are assumed to behave like free fields in the asymptotic past and future, so the results of this section provide the tools needed to define the asymptotic quantization of interacting fields. In sec. 4 we consider QED with a massive, charged Klein-Gordon field. In sec. 4.1 we construct the asymptotic algebra and Hadamard states of the massive scalar field at timelike infinity and the electromagnetic field at null infinity. In sec. 4.2 we consider the extension of the field algebras to include charges and Poincaré generators. In sec. 4.3 we obtain Fock representations of the field algebras. The standard Fock representation of the massive Klein-Gordon field provides all of the necessary asymptotic states of that field, but we need all of the memory representations of the electromagnetic field to have an adequate supply of asymptotic electromagnetic states for scattering theory. In sec. 4.4 we use these representations to construct the Faddeev-Kulish representation in massive QED by pairing momentum eigenstates of the Klein-Gordon field with corresponding memory representations of the electromagnetic field, thereby "dressing" the charged particles. In sec. 5 we consider massless scalar QED. In sec. 5.1 we construct the asymptotic algebra of field observables at null infinity for a massless, charged Klein-Gordon field, and we extend this algebra to include charges and Poincaré generators in sec. 5.2. In sec. 5.3 we obtain the Fock representations of the massless scalar field. In sec. 5.4 we construct the analog of the Faddeev-Kulish representations for massless QED and point out the serious problems arising from the singular nature of the required memory representations. In sec. 5.5 we consider source-free Yang-Mills theory and discuss the new serious difficulty that arises from the fact that the Faddeev-Kulish "dressing" also contributes to the charge-current flux. In sec. 6 we consider general relativity. In sec. 6.1 we provide the asymptotic algebra of observables in vacuum gravity. This algebra is extended in sec. 6.2 to include the BMS charges. In sec. 6.3 we prove the non-existence of Faddeev-Kulish representations in quantum gravity. Some alternatives to Faddeev-Kulish representations are explored in sec. 7 but none are found to be satisfactory. Finally, in sec. 8 we advocate for the development of an "algebraic scattering theory," wherein one does not attempt to "shoehorn" all of the asymptotic states of scattering theory into pre-chosen "in" and "out" Hilbert spaces. Such a formulation of scattering theory would be manifestly infrared finite.

## 2 Classical phase space: Observables and asymptotic description

Our interest in this chapter of the thesis is in interacting quantum field theories, specifically, QED, Yang-Mills theory, and quantum gravity. However, we will be concerned only with the description of states at asymptotically early and late times, where it will be assumed that the states correspond to states of "in" and "out" free field theories. Thus, in essence, for the considerations of this chapter, we need only be concerned with the structure of free field theory. The quantum theory of a free field is based on the phase space structure of the classical theory. In this section, we will review the phase space structure relevant for our considerations and explain the notion of "observable" that we shall use. For the case of a massless field, we will relate the "bulk" description of the field to its asymptotic description at null infinity.

Since the phase space of a field theory is infinite dimensional, it would take some effort to define a mathematically precise Fréchet space or other structure of phase space (see [99]) that would properly incorporate the smoothness and fall-off conditions of the fields and provide a suitable topology on these fields. We believe that this could be done but we shall not attempt to do so here. Thus, we will freely use terms like "smooth vector field on phase space" in our discussion below without attempting to give a mathematically precise meaning to such terms.

The basic structure of the classical phase space of a linear field theory is well illustrated by the case of a real scalar field $\phi$. Since we want to apply our constructions to the asymptotic behavior of the gravitational field in general relativity, it would not be reasonable to assume more structure than would be present on a globally hyperbolic, asymptotically flat, curved spacetime. Thus, we will take as our model system a real scalar field $\phi$ on a globally hyperbolic spacetime $(\mathcal{M}, g)$, with Lagrangian

$$\mathcal{L} = -\frac{1}{2}\left[\nabla^\mu\phi\nabla_\mu\phi + m^2\phi^2 + \xi R\phi^2\right] \tag{2.1}$$

where $m$ denotes the mass, $\xi$ is an arbitrary constant, and $R$ is the Ricci scalar. Then $\phi$ satisfies

$$\left(\Box - m^2 - \xi R\right)\phi = 0. \tag{2.2}$$

As discussed in detail in [100, 101] and many other references, the Lagrangian eq. (2.1) endows the theory with a symplectic form, which thereby provides the space of initial data for solutions with a phase space structure. For the scalar field eq. (2.1), the points of phase space $\mathcal{P}$ can be taken to be the quantities $(\phi, n^\mu\nabla_\mu\phi)$ on a spacelike Cauchy surface $\Sigma$, where $n^\mu$ denotes the unit normal to $\Sigma$. In general, the symplectic form, $\Omega$, is a 2-form on $\mathcal{P}$, i.e., at each point of $\mathcal{P}$ it maps a pair of tangent vectors into a number. However, in the case of a linear theory as considered here, $\mathcal{P}$ has a vector space (or, more generally, an affine space[14])

14. As we shall see, in electromagnetism and gravity, the presence of large gauge transformations implies that there are many points of phase space that have vanishing gauge invariant field strengths. Any of these

structure, and we can identify tangent vectors with points of $\mathcal{P}$. Consequently, we can view $\Omega$ as a bilinear map on $\mathcal{P}$. The symplectic product of two solutions $\phi_1, \phi_2$ is given by

$$\Omega_\Sigma^{\mathrm{KG}}(\phi_1, \phi_2) = \int_\Sigma \sqrt{h} d^3x \; \left[\phi_1 n^\mu \nabla_\mu \phi_2 - \phi_2 n^\mu \nabla_\mu \phi_1\right] \tag{2.3}$$

where $\sqrt{h}d^3x$ is the proper volume element on $\Sigma$. This symplectic product is conserved, i.e., it is independent of the choice of Cauchy surface $\Sigma$.

If $\mathcal{P}$ were finite dimensional, the nondegeneracy of the symplectic form $\Omega_{\alpha\beta}$ would imply that it has an inverse $\Omega^{\alpha\beta}$, where the Greek indices here represent tensor indices on phase space. A classical observable $F$ on $\mathcal{P}$ could then be taken to be an arbitrary smooth map $F : \mathcal{P} \to \mathbb{R}$. The inverse symplectic form would then allow us to define the Poisson bracket of any two such observables $F_1, F_2$ to be the observable on phase space given by

$$\left\{F_1, F_2\right\} = \Omega^{\alpha\beta} \nabla_\alpha F_1 \nabla_\beta F_2. \tag{2.4}$$

However, on an infinite dimensional phase space, the symplectic form is only weakly nondegenerate and its inverse will not be defined on all one-forms on the phase space. Thus, we cannot use eq. (2.4) to define the Poisson bracket of arbitrary smooth functions on phase space.

Nevertheless, on a general phase space, we can define the Poisson bracket on a particular class of smooth functions $F$. Namely, suppose $F$ is such that there is a smooth vector field $X^\alpha$ on $\mathcal{P}$ with the property that for all smooth curves $z(\alpha)$ on phase space, we have

$$\delta F = \Omega(\delta z, X) \tag{2.5}$$

points could serve as an “origin”.

where

$$\delta z := \frac{d}{d\alpha} z(\alpha)\bigg|_{\alpha=0} , \quad \delta F := \frac{d}{d\alpha} F(z(\alpha))\bigg|_{\alpha=0} . \tag{2.6}$$

Formally, eq. (2.5) corresponds to $X^\alpha = \Omega^{\alpha\beta}\nabla_\beta F$, but eq. (2.5) is expressed in a way that avoids the introduction of the inverse symplectic form. If eq. (2.5) holds, we say that the function $F$ *generates* the vector field $X^\alpha$. Given two functions $F_1$ and $F_2$ on phase space that generate vector fields $X_1^\alpha$ and $X_2^\alpha$, respectively, we can define the Poisson bracket of $F_1$ and $F_2$ by

$$\Big\{F_1, F_2\Big\} := -\Omega(X_1, X_2). \tag{2.7}$$

Formally, this corresponds to eq. (2.4) because, formally, $\Omega(X_1, X_2) = \Omega_{\alpha\beta} X_1^\alpha X_2^\beta = \Omega_{\alpha\beta}\Omega^{\alpha\gamma}\nabla_\gamma F_1 \Omega^{\beta\eta}\nabla_\eta F_2 = \Omega^{\eta\gamma}\nabla_\gamma F_1 \nabla_\eta F_2$. However, eq. (2.7) avoids introducing the inverse symplectic form and is well-defined. For the case of an infinite dimensional phase space, we define an *observable* to be a smooth function $F$ on phase space that satisfies eq. (2.5). By construction, the Poisson bracket of any two observables is well-defined. It is only for classical observables in this sense that we can hope/expect to have quantum representatives.

The situation with regard to obtaining observables simplifies considerably in the case of a phase space $\mathcal{P}$ with vector space structure, as considered here. Consider a vector field $X$ corresponding to an infinitesimal displacement at each $\phi$ of the form of an affine transformation

$$\phi \mapsto \phi + \epsilon(L\phi + \chi_0) \tag{2.8}$$

where $\chi_0$ is a constant ($\phi$-independent) displacement and $L$ is a linear map on phase space. Suppose that $L$ satisfies

$$\Omega(\psi, L\phi) = -\Omega(L\psi, \phi) \tag{2.9}$$

for all $\phi, \psi \in \mathcal{P}$. Then it is straightforward to verify that the function $F : \mathcal{P} \to \mathbb{R}$ defined by

$$F(\phi) = \Omega(\phi, \chi_0) + \frac{1}{2}\Omega(\phi, L\phi) \tag{2.10}$$

satisfies eq. (2.5). Thus, any function $F$ on phase space of the form eq. (2.10) with $L$ satisfying eq. (2.9) is an observable on phase space.

An important class of observables on $\mathcal{P}$ are the local field observables. Let $f : \mathcal{M} \to \mathbb{R}$ be a test function on spacetime, i.e., a smooth function of compact support. Let $F_f$ be the linear function on phase space given by

$$F_f(\phi) = \phi(f) := \int \sqrt{-g}d^4y\ \phi(y)f(y). \tag{2.11}$$

where $y$ denotes arbitrary coordinates on $\mathcal{M}$. Then $F_f$ can be shown to be an observable as follows. Let

$$Ef = Af - Rf \tag{2.12}$$

where $Af$ denotes the advanced solution to eq. (2.2) with source $f$ and $Rf$ denotes the retarded solution with source $f$. Then $Ef$ a smooth, source-free solution to eq. (2.2) with initial data of compact support, so it corresponds to a point in $\mathcal{P}$. By lemma 3.2.1 of [102], for any solution $\phi$ we have[15]

$$\phi(f) = \Omega_\Sigma^{\text{KG}}(\phi, Ef). \tag{2.13}$$

Thus, $F_f$ is of the form eq. (2.10) with $\chi_0 = Ef$ and $L = 0$. Thus, the "smeared fields" $\phi(f)$ are observables.[16] It is not difficult to see that the Poisson bracket of smeared fields is given

15. Note that our convention for the symplectic form in eq. (2.3) has the opposite sign compared to the one used in [102].

16. Note that the field evaluated at a point, $\phi(x)$, is too singular to be considered to be an observable. The associated vector field $X^\alpha$ would correspond to an infinitesimal displacement in the direction of the singular solution given by the advanced minus retarded solution with delta function source at $x$, which does not lie in the phase space.

by

$$\left\{\phi(f_1), \phi(f_2)\right\} = E(f_1, f_2)1 \tag{2.14}$$

where "1" denotes the constant function on $\mathcal{P}$ that maps all points to 1, and

$$E(f_1, f_2) = \int \sqrt{-g} d^4y \ f_1(y) E f_2(y) \, . \tag{2.15}$$

For any Cauchy surface $\Sigma$ and any test function $s$ on $\Sigma$, we may define the linear function $F_{\Sigma,s}$ on phase space by

$$F_{\Sigma,s}(\phi) = \phi_\Sigma(s) := \int_\Sigma \sqrt{h} d^3x \ \phi(x) s(x). \tag{2.16}$$

We may similarly define $(n^\mu \nabla_\mu \phi)_\Sigma(s)$. These "3-smeared" fields are also observables, which can be seen from eq. (2.13) to be equivalent to the "4-smeared" observables eq. (2.11). Namely, we have

$$\phi_\Sigma(s) = \phi(f) \tag{2.17}$$

where $f$ is a test function on spacetime such that the initial data for $Ef$ on $\Sigma$ is $[Ef]_\Sigma = 0$ and $[n^\mu \nabla_\mu (Ef)]_\Sigma = s$. A similar formula holds for $(n^\mu \nabla_\mu \phi)_\Sigma(s)$.

Our main interest in this chapter is to characterize the states of quantum fields in asymptotically flat spacetimes at asymptotically early and late times. It therefore will be important to have a description of the phase space and observables that characterizes the behavior of the field at asymptotically early and late times. We shall now explain how this can be done for massless fields. The corresponding asymptotic quantization will be described in the next section. The classical and quantum asymptotic description of massive fields will be given in sec. 4.

For massless fields, we assume that past null infinity, $\mathscr{I}^-$, and future null infinity, $\mathscr{I}^+$, can be treated as Cauchy surfaces, so that initial data at $\mathscr{I}^-$ or $\mathscr{I}^+$ uniquely determines a solution.[17] For a massless field with the Cauchy surface taken to be $\mathscr{I}^-$, initial data for solutions consists of the specification of the conformally weighted scalar field, $\Phi$, on $\mathscr{I}^-$

$$\Phi := \lim_{\mathscr{I}^-} \Omega^{-1}\phi \tag{2.18}$$

where $\Omega$ is a conformal factor, which, in Bondi coordinates, can be chosen to be $\Omega = 1/r$. We assume that the solutions in $\mathcal{P}$ are such that $\partial\Phi/\partial v = O(1/|v|^{1+\epsilon})$ for some $\epsilon > 0$ as $v \to \pm\infty$. This will ensure that all integrals below will converge. Note, however, that we do *not* assume that $\Phi \to 0$ as $v \to \pm\infty$, as this would exclude the memory effect. Although we could, of course, restrict consideration to initial data at $\mathscr{I}^-$ satisfying $\Phi \to 0$ as $v \to \pm\infty$, if interactions occur in the bulk, such initial data will generically evolve to fields at $\mathscr{I}^+$ that do not satisfy $\Phi \to 0$ as $u \to \infty$. Since we wish to treat $\mathscr{I}^-$ and $\mathscr{I}^+$ on an equal footing in scattering theory, we do not require $\Phi \to 0$ as $v \to \pm\infty$ at $\mathscr{I}^-$.

In terms of the initial data eq. (2.18), the symplectic product eq. (2.3) is given by

$$\Omega^{\text{KG0}}_{\Sigma}(\phi_1, \phi_2) = \int_{\mathscr{I}^-} dv d\Omega \left[\Phi_1 \frac{\partial \Phi_2}{\partial v} - \Phi_2 \frac{\partial \Phi_1}{\partial v}\right] \tag{2.19}$$

where we have inserted an extra "0" in the superscript "KG0" on $\Omega$ to indicate that this formula holds only for the case of a massless scalar field. It is convenient to define

$$\Pi := \partial_v \Phi \tag{2.20}$$

on $\mathscr{I}^-$, since this quantity will arise in many formulas below. It follows from eq. (2.19) that

17. This is true in Minkowski spacetime but is an assumption in a general asymptotically flat spacetime. It would not hold in spacetimes with a black hole or white hole, but one could presumably then supplement the asymptotic description of states at null infinity by including states on the horizon of the black hole or white hole.

for any test function $s$ on $\mathscr{I}^-$, we have

$$\Pi(s) := \int_{\mathscr{I}^-} dv d\Omega \, \frac{\partial \Phi}{\partial v}(v, x^A) s(v, x^A) = \frac{1}{2} \Omega_\Sigma^{\mathrm{KG0}}(Ef, \Phi) = -\frac{1}{2}\phi(f) \tag{2.21}$$

where $f$ is a function on spacetime such that on $\mathscr{I}^-$ we have $\lim\limits_{\mathscr{I}^-} \Omega^{-1} Ef = s$. Thus, the smeared field quantities $\Pi(s)$ on $\mathscr{I}^-$ are observables on phase space that are essentially equivalent[18] to the bulk field observables $\phi(f)$. The Poisson brackets of these observables at $\mathscr{I}^-$ are given by

$$\begin{aligned} \left\{\Pi(s_1), \Pi(s_2)\right\} &= \frac{1}{4}\left\{\phi(f_1), \phi(f_2)\right\} \\ &= \frac{1}{4} E(f_1, f_2) 1 \\ &= \frac{1}{4} \Omega_\Sigma^{\mathrm{KG0}}(Ef_1, Ef_2) 1 \\ &= \frac{1}{4} \int_{\mathscr{I}^-} dv d\Omega \left( s_2 \frac{\partial s_1}{\partial v} - s_1 \frac{\partial s_2}{\partial v} \right) 1. \end{aligned} \tag{2.22}$$

Here, the third line was obtained by writing

$$E(f_1, f_2) = \int \sqrt{-g} d^4 y \; f_1(y) E f_2(y) = -\Omega_\Sigma^{\mathrm{KG0}}(Ef_2, Ef_1) \tag{2.23}$$

where eq. (2.13) with $f = f_1$ and $\phi = Ef_2$ was used.

Finally, note that if $w$ is a test function of the form $w = \partial s / \partial v$ for some test function $s$,

18. We say "essentially equivalent" because if $f$ is of compact support on spacetime, then in a curved spacetime — where Huygens' principle does not hold for the wave equation (2.2) — $Ef$ will not be of compact support on $\mathscr{I}^-$ and vice-versa, so the test function spaces do not align precisely. We will ignore this issue here. Except for the case of nonlinear gravity, our applications are to Minkowski spacetime, where Huygens' principle does hold and the correspondence is exact.

then

$$\begin{aligned}\Phi(w) &:= \int_{\mathscr{I}^-} dv d\Omega \; \Phi(v, x^A) w(v, x^A) = \int_{\mathscr{I}^-} dv d\Omega \; \Phi \frac{\partial s}{\partial v} \\ &= -\int_{\mathscr{I}^-} dv d\Omega \; \frac{\partial \Phi}{\partial v} s = -\Pi(s). \end{aligned} \tag{2.24}$$

Thus, $\Phi(w)$ for $w = \partial s/\partial v$ is equal to $-\Pi(s)$ and hence is well-defined and corresponds to a local observable in the bulk. However, if $w$ is not of this form — i.e., if $\int dv \; w(v, x^A) \neq 0$ for some $x^A$ — then $\Phi(w)$ does not correspond to a local observable in the bulk.

# 3 Algebraic viewpoint: Quantization of free fields and asymptotic quantization of massless interacting fields

Since we are concerned in this chapter with the possible choices of a Hilbert space of "in" and "out" states in scattering theory, it is essential to have a notion of the structure of the theory prior to a choice of Hilbert space. The algebraic approach provides such a notion. The purpose of this section is to review the key ideas ideas in the algebraic approach and describe the asymptotic quantization of massless fields corresponding to the asymptotic characterization of phase space given at the end of the previous section. For further discussion of the algebraic viewpoint we refer the reader to [102, 103, 104].

In the algebraic approach, one assumes that the quantum field observables have the structure of a $*$-algebra $\mathscr{A}$. States are then defined as positive linear functions on the algebra, i.e., a state, $\omega$, is simply a linear map $\omega : \mathscr{A} \to \mathbb{C}$ such that $\omega(a^* a) \geq 0$ for all $a \in \mathscr{A}$. If we take $\mathscr{A}$ to be generated by local (smeared) field observables, then an arbitrary element $a \in \mathscr{A}$ would be a sum of products of local field observables, so a specification of $\omega$ would be equivalent to providing the complete list of the correlation functions of the field observables.

Our interest in this chapter is in interacting quantum field theories, specifically, QED, Yang-Mills theory, and quantum gravity. There are many nontrivial and still unanswered questions about the formulation of interacting quantum field theories. However, in this chapter, we will be concerned only with the behavior of these fields at asymptotically early and late times. As is normally done in scattering theory, we will simply *assume* that states of the theory behave at asymptotically early and late times like states of the corresponding "in" and "out" free field theories, i.e., that the interactions can be neglected at asymptotically early and late times. Of course, the determination of the relationship between the "in" and "out" states requires knowledge of the interacting quantum field theory, but our analysis in this chapter will be exclusively concerned with the nature of "in" and "out" states and whether suitable Hilbert spaces of such states can be defined. Thus, as previously stated at the beginning of sec. 2, for the considerations of this chapter, we need only be concerned with the structure of free field theory.

The structure of the quantum theory of a free field is well illustrated by the case of a real scalar field $\phi$, eq. (2.1). The classical phase space structure of the real scalar field was described in sec. 2. The quantum theory of $\phi$ is defined by specifying an algebra, $\mathscr{A}$, of quantum observables. We obtain $\mathscr{A}$ by starting with the free algebra of the smeared fields $\boldsymbol{\phi}(f)$, their formal adjoints $\boldsymbol{\phi}(f)^*$ and an identity $\mathbf{1}$ where $f$ is a real-valued, smooth function on $\mathcal{M}$ with compact support. The algebra $\mathscr{A}$ is then obtained by factoring this free algebra by the following relations:

(A.I) $\boldsymbol{\phi}(c_1 f_1 + c_2 f_2) = c_1 \boldsymbol{\phi}(f_1) + c_2 \boldsymbol{\phi}(f_2)$ for any $f_1, f_2$ and any $c_1, c_2 \in \mathbb{R}$, i.e., the smeared field is linear in the test function

(A.II) $\boldsymbol{\phi}((\Box - m^2 - \xi R) f) = 0$ for all $f$, i.e., $\boldsymbol{\phi}$ satisfies the field equation in the distributional sense

(A.III) $\boldsymbol{\phi}(f)^* = \boldsymbol{\phi}(f)$ for all $f$, i.e., the field is Hermitian

(A.IV) $[\boldsymbol{\phi}(f_1), \boldsymbol{\phi}(f_2)] = iE(f_1, f_2)\mathbf{1}$, i.e., the field satisfies canonical commutation relations (see eq. (2.14))

As already mentioned above, a state is a linear map $\omega : \mathscr{A} \to \mathbb{C}$ that satisfies $\omega(a^* a) \geq 0$ for all algebra elements $a \in \mathscr{A}(\mathcal{M}, g)$. We further require the normalization condition $\omega(\mathbf{1}) = 1$. A state is thus determined by specifying its smeared "$n$-point correlation functions" $\omega(\boldsymbol{\phi}(f_1) \ldots \boldsymbol{\phi}(f_n))$. If we have a Hilbert space $\mathscr{H}$ on which the smeared fields are represented as operators satisfying (A.I)–(A.IV), then any normalized vector $|\Psi\rangle \in \mathscr{H}$ gives rise to a state via $\omega(a) = \langle\Psi|\pi(a)|\Psi\rangle$ for all $a \in \mathscr{A}$, where $\pi(a)$ is the operator representative of $a$. More generally, any normalized density matrix $\rho$ on $\mathscr{H}$ gives rise to a state via $\omega(a) = \mathrm{tr}(\rho\pi(a))$. Conversely, by a remarkably simple construction due to Gel'fand, Naimark and Segal (GNS), given an algebraic state $\omega : \mathscr{A} \to \mathbb{C}$, one can obtain a representation, $\pi$, of $\mathscr{A}$ on a Hilbert space $\mathscr{H}$ and a vector $|\Psi\rangle \in \mathscr{H}$ such that $\omega(a) = \langle\Psi|\pi(a)|\Psi\rangle$ for all $a \in \mathscr{A}$. The GNS construction consists of starting with the vector space $\mathscr{A}$ and using $\omega$ to define an inner product on $\mathscr{A}$. One then completes $\mathscr{A}$ in this inner product and factors out any degenerate elements to get a Hilbert space $\mathscr{H}$. By construction, $\mathscr{H}$ contains a dense set of vectors $|a\rangle$ corresponding to elements $a \in \mathscr{A}$. We obtain a representation, $\pi$, of $\mathscr{A}$ on $\mathscr{H}$ by the formula $\pi(a)\,|b\rangle = |ab\rangle$ for all $a, b \in \mathscr{A}$. The vector $|\Psi\rangle \in \mathscr{H}$ corresponding to $\omega$ is simply $|\mathbf{1}\rangle$. Note that $|\mathbf{1}\rangle$ is cyclic, i.e., the action of $\pi(a)$ on $|\mathbf{1}\rangle$ for all $a \in \mathscr{A}$ generates a dense subspace of states. Note further that this construction uses only the $*$-algebra structure of $\mathscr{A}$.

A state is called pure if it cannot be written as a sum of two other states with positive coefficients; otherwise the state is referred to as mixed. The GNS construction will represent a mixed state as a vector (rather than density matrix) in $\mathscr{H}$, but for a mixed state the GNS representation will be reducible. In particular, for a state that corresponds to a density matrix on a Hilbert space $\tilde{\mathscr{H}}$ that carries an irreducible representation of $\mathscr{A}$, the GNS construction will suitably enlarge $\tilde{\mathscr{H}}$ to a Hilbert space $\mathscr{H}$ on which the state is represented as a vector.[19]

An important class of states are known as "Gaussian states" (also referred to as "quasi-free states" or "vacuum states"). By definition, for Gaussian states, the $n$-point functions for $n > 2$ are given by formulas in terms of the 1- and 2-point functions that are analogous to the formulas for the $n$-th moments of a Gaussian probability distribution. This can be described by saying that the "connected $n$-point functions" (also known as "truncated $n$-point functions") vanish for all $n > 2$ (see e.g. [103]). For example, for a Gaussian state of the Klein-Gordon field the 3-point function is given by

$$\begin{aligned}\omega(\boldsymbol{\phi}(y_1)\boldsymbol{\phi}(y_2)\boldsymbol{\phi}(y_3)) =&\omega(\boldsymbol{\phi}(y_1)) \cdot \omega(\boldsymbol{\phi}(y_2)\boldsymbol{\phi}(y_3)) + \omega(\boldsymbol{\phi}(y_2)) \cdot \omega(\boldsymbol{\phi}(y_3)\boldsymbol{\phi}(y_1)) \\ &+ \omega(\boldsymbol{\phi}(y_3)) \cdot \omega(\boldsymbol{\phi}(y_1)\boldsymbol{\phi}(y_3)) - 2\omega(\boldsymbol{\phi}(y_1)) \cdot \omega(\boldsymbol{\phi}(y_2)) \cdot \omega(\boldsymbol{\phi}(y_3))\end{aligned} \tag{3.1}$$

where all "unsmeared" formulas here and below should be interpreted as holding distributionally. The GNS Hilbert space of a Gaussian state $\omega$ has a natural Fock space structure

$$\mathscr{F}(\mathscr{H}_1) = \mathbb{C} \oplus \left[ \bigoplus_{n\geq 1} \underbrace{\left(\mathscr{H}_1 \otimes_S \cdots \otimes_S \mathscr{H}_1\right)}_{n \text{ times}} \right]. \tag{3.2}$$

where $\otimes_S$ is the symmetrized tensor product, and the inner product on the "one-particle Hilbert space" $\mathscr{H}_1$ is determined[20] by the 2-point function $\omega(\boldsymbol{\phi}(y_1)\boldsymbol{\phi}(y_2))$. In Minkowski spacetime, the Poincaré invariant vacuum state $|0\rangle$ is a Gaussian state and the Fock space eq. (3.2) is the standard choice of Hilbert space for free field theory.

The general definition of a state given above admits many states with singular ultraviolet behavior — too singular for nonlinear field observables to be defined. It is therefore necessary

19. For example, the GNS construction represents a thermal state as a vector in its "thermofield double".

20. More precisely on the space of smooth functions $f$ of compact support we define the inner product $\langle f_1 | f_2 \rangle = \omega(\boldsymbol{\phi}(f_1)^* \boldsymbol{\phi}(f_2))$ (see [105] for details).

to impose an additional restriction on the short distance behavior of states. In most treatments of quantum field theory in Minkowski spacetime, this issue is not highlighted because the vacuum state $|0\rangle$ has the required ultraviolet behavior, as do all states in the corresponding Fock space eq. (3.2) with smooth $n$-particle "mode functions". Thus, the states that are normally considered in usual treatments satisfy the required condition on ultraviolet behavior. However, in this chapter, we seek alternative choices of Hilbert spaces — since, as explained in sec. 1, the standard Fock space of "in" and "out" states cannot accommodate the states that arise in scattering processes — so it is essential that we explicitly impose the condition that states have the required ultraviolet behavior. This additional restriction on states is given by the Hadamard condition, which requires that the short distance behavior of the 2-point function of any allowed state be of the form

$$\omega(\boldsymbol{\phi}(y_1)\boldsymbol{\phi}(y_2)) = \frac{1}{4\pi^2}\frac{U(y_1,y_2)}{\sigma + i0^+T} + V(y_1,y_2)\log(\sigma + i0^+T) + W(y_1,y_2). \tag{3.3}$$

Here $\sigma$ is the squared geodesic distance between $y_1$ and $y_2$, $T = t(y_1) - t(y_2)$ with $t$ a global time function on spacetime, $U$ and $V$ are smooth, symmetric functions that are locally constructed via the Hadamard recursion relations [106], and $W$ is also smooth and symmetric. The Hadamard condition can be very usefully reformulated in terms of microlocal spectral conditions on the distribution $\omega(\boldsymbol{\phi}(y_1)\boldsymbol{\phi}(y_2))$ [107], but we shall not need this reformulation here. The Hadamard condition eq. (3.3) together with the positivity condition on states implies that the connected $n$-point functions for $n \neq 2$ of a Hadamard state are smooth and symmetric [108].

We conclude this section by giving the asymptotic quantization of $\boldsymbol{\phi}$ in the massless case. Again, we assume that past null infinity, $\mathscr{I}^-$, and future null infinity, $\mathscr{I}^+$, can be treated as Cauchy surfaces. We cannot proceed by starting with the bulk theory and taking limits of correlation functions to $\mathscr{I}^-$ or $\mathscr{I}^+$, since the quantum fields are distributional on spacetime and cannot straightforwardly be restricted to a lower dimensional surfaces such as $\mathscr{I}^-$ or $\mathscr{I}^+$. However, we can proceed by working with the asymptotic description of the classical phase space given at the end of the previous section.

For the asymptotic quantization on $\mathscr{I}^-$, we take the observables on phase space to be $\mathbf{\Pi}(s)$ (eq. (2.21)), where $s$ is an arbitrary test function on $\mathscr{I}^-$ with conformal weight $-1$. We define the algebra $\mathscr{A}_{\text{in}}$ by starting with the free algebra generated by $\mathbf{\Pi}(s)$, $\mathbf{\Pi}(s)^*$ and $\mathbf{1}$ and factoring it by relations corresponding to (A.I)–(A.IV). Conditions (A.I) and (A.III) translate straightforwardly to $\mathscr{A}_{\text{in}}$. There is no condition corresponding to condition (A.II) since we are now smearing $\mathbf{\Pi}$ with free data for solutions. The commutation relation (A.IV) translates to

$$[\mathbf{\Pi}(x_1), \mathbf{\Pi}(x_2)] = \frac{i}{2}\delta'(v_1, v_2)\delta_{\mathbb{S}^2}(x_1^A, x_2^A)\mathbf{1} \tag{3.4}$$

(see eq. (2.22)) where $x = (v, x^A)$ are coordinates on $\mathscr{I}^-$ and this equation is to be understood as a distributional relation on $\mathscr{I}^-$. This completes our specification of the algebra $\mathscr{A}_{\text{in}}$. The algebra $\mathscr{A}_{\text{out}}$ is defined similarly.

The algebra $\mathscr{A}_{\text{in}}$ constructed in this manner is essentially equivalent[21] to the bulk free field algebra $\mathscr{A}$. For an interacting theory, the bulk algebra, of course, is no longer a free field algebra, but the central assumption of scattering theory is that states on the bulk algebra asymptote to states on the free field algebras $\mathscr{A}_{\text{in}}$ and $\mathscr{A}_{\text{out}}$ at early and late times, respectively.

We now impose regularity conditions on states on $\mathscr{A}_{\text{in}}$. For the bulk theory, we imposed the Hadamard condition eq. (3.3) on states on $\mathscr{A}$, and we wish to express this condition as a corresponding condition on states on $\mathscr{A}_{\text{in}}$. For the conformally invariant case ($\xi = 1/6$) in a spacetime with a regular timelike infinity, it has been shown [109] that Hadamard states on

21. We say "essentially equivalent" for the reason stated in footnote 18.

$\mathscr{A}$ correspond to states on $\mathscr{A}_{\text{in}}$ whose 2-point function is of the form[22]

$$\omega\left(\mathbf{\Pi}(x_1)\mathbf{\Pi}(x_2)\right) = -\frac{1}{\pi}\frac{\delta_{\mathbb{S}^2}(x_1^A, x_2^A)}{(v_1 - v_2 - i0^+)^2} + S(x_1, x_2) \tag{3.5}$$

where $S(x_1, x_2)$ is a smooth function on $\mathscr{I}^- \times \mathscr{I}^-$. In particular, this result holds in Minkowski spacetime (for an arbitrary $\xi$, since $\xi$ does not enter the equations of motion in that case). We assume that this form holds generally for massless fields, i.e., with no restriction to $\xi = 1/6$ or to spacetimes with a regular timelike infinity. Thus, we impose eq. (3.5) as the ultraviolet regularity condition on states on $\mathscr{A}_{\text{in}}$. Note that in Minkowski spacetime, the 2-point function of the Poincaré invariant vacuum state $\omega_0$ takes the form eq. (3.5) with $S = 0$, i.e.,

$$\omega_0\left(\mathbf{\Pi}(x_1)\mathbf{\Pi}(x_2)\right) = -\frac{1}{\pi}\frac{\delta_{\mathbb{S}^2}(x_1^A, x_2^A)}{(v_1 - v_2 - i0^+)^2}. \tag{3.6}$$

In addition to the ultraviolet regularity condition on states, we impose the following decay conditions on states, analogous to the classical decay conditions mentioned below eq. (2.18): We require that $S$ and all connected $n$-point functions for $n \neq 2$ decay for any set of $|v_i| \to \infty$ as $O((\sum_i v_i^2)^{-1/2-\epsilon})$ for some $\epsilon > 0$.

In the subsequent sections, we will assume that the quantization of the "in" and "out" electromagnetic and gravitational fields are given by a direct analog of our construction of $\mathscr{A}_{\text{in}}$ above, and we will impose ultraviolet regularity (Hadamard) conditions on states given by the direct analog of eq. (3.5), as well as the analogous decay conditions.

Finally, we note that we have included only observables that are linear in the field $\phi$ in our algebras, $\mathscr{A}$ and $\mathscr{A}_{\text{in}}$, of local field observables. For the case of the bulk theory, $\mathscr{A}$ can be extended to include smeared polynomial quantities ("Wick polynomials") in the field by a

22. A similar result holds for any field (including massive fields) on a Killing horizon [105].

"Hadamard normal ordering" procedure (see [103]). However, an analogous procedure does *not* work for $\mathscr{A}_{\text{in}}$, as Hadamard normal ordering produces quantities that are too singular in the angular directions. Thus, we cannot extend $\mathscr{A}_{\text{in}}$ to include polynomial local field observables. Nevertheless, quantities that are quadratic in the fields can be defined as quadratic forms by Hadamard subtraction, using eq. (3.6) for the subtraction. In particular, for any Hadamard state $\omega$, we may define the expected value of $\mathbf{\Pi}^2$ by

$$\begin{aligned}\omega\left(\mathbf{\Pi}^2(x)\right) &= \lim_{x'\to x}\left[\omega\left(\mathbf{\Pi}(x)\mathbf{\Pi}(x')\right) - \omega_0\left(\mathbf{\Pi}(x)\mathbf{\Pi}(x')\right)\right] \\ &= S(x,x).\end{aligned} \tag{3.7}$$

We can use this notion to define expected values of observables that are quadratic in the fields. However, higher powers of $\mathbf{\Pi}(x)$ cannot even be defined as quadratic forms. In particular, since the stress-energy flux through $\mathscr{I}^-$ is $T_{vv} = \Pi^2$ this implies that the local energy flux cannot be defined as an operator and is only well-defined as a quadratic form (i.e. only its expected value is well-defined). This result is in accord with arguments given in [110].

## 4 QED with a massive, charged Klein-Gordon field

In this section we consider massive scalar QED, i.e., the theory of a Maxwell field $A_\mu$, coupled to a charged massive complex Klein-Gordon scalar field $\varphi$ in Minkowski spacetime. The Lagrangian for this theory is

$$\mathcal{L} = -\frac{1}{4}F^{\mu\nu}F_{\mu\nu} - \frac{1}{2}D^\mu\overline{\varphi}D_\mu\varphi - \frac{1}{2}m^2\overline{\varphi}\varphi \tag{4.1}$$

where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ and $D_\mu$ is the gauge covariant derivative operator

$$D_\mu\varphi := \partial_\mu\varphi - iqA_\mu\varphi\,, \quad D_\mu\overline{\varphi} := \overline{D_\mu\varphi}\,. \tag{4.2}$$

The theory is invariant under the action of gauge transformations

$$A_\mu \mapsto A_\mu + \partial_\mu \lambda\,, \quad \varphi \mapsto e^{iq\lambda}\varphi\,. \tag{4.3}$$

In sec. 4.1, we give the asymptotic quantization of the massive Klein-Gordon and electromagnetic fields. In sec. 4.2, we extend the algebra of asymptotic observables to include large gauge charges and Poincaré generators. In sec. 4.3, we construct Fock representations of the extended algebra of asymptotic observables with arbitrary choices of memory. Finally, in sec. 4.4 we construct the Faddeev-Kulish Hilbert space.

## *1 Asymptotic quantization of QED with a massive Klein-Gordon field*

We wish to provide a characterization of the states in QED in terms of free field states in the asymptotic past and asymptotic future. For definiteness, we will focus upon the asymptotic past; exactly the same procedure is used for the asymptotic future. We assume that in the asymptotic past, classical solutions approach solutions to the free Klein-Gordon and Maxwell equations.[23] Correspondingly, in the asymptotic past, states in QED should approach "free field 'in' states," i.e., states on the tensor product

$$\mathscr{A}_{\text{in}} = \mathscr{A}_{\text{in}}^{\text{EM}} \otimes \mathscr{A}_{\text{in}}^{\text{KG}}\,. \tag{4.4}$$

of the asymptotic algebra, $\mathscr{A}_{\text{in}}^{\text{EM}}$, of the free electromagnetic field with the asymptotic algebra, $\mathscr{A}_{\text{in}}^{\text{KG}}$, of the free massive Klein-Gordon field. Thus, our task in this subsection is to obtain the free field algebras $\mathscr{A}_{\text{in}}^{\text{EM}}$ and $\mathscr{A}_{\text{in}}^{\text{KG}}$.

The strategy for obtaining $\mathscr{A}_{\text{in}}^{\text{EM}}$ was presented in the previous section. The electromag-

23. This assumption is supported by rigorous studies of the classical behavior of the QED fields [111, 112, 113, 114].

netic field is conformally invariant, so, classically, for solutions with appropriate fall-off at spatial infinity, one can choose a gauge[24] so that the vector potential $A_\mu$ extends smoothly to $\mathscr{I}^-$. We may further choose a gauge for which $n^\mu A_\mu|_{\mathscr{I}^-} = 0$ where $n^\mu := \partial_v$ is the null normal. In this gauge, the pullback of $A_\mu$ to $\mathscr{I}^-$ is a down index tensor on $\mathscr{I}^-$ that is orthogonal to $n^\mu$, so in accord with the notational conventions stated at the end of sec. 0.3, we denote it as $A_A$.

The points of the classical phase space are given by the specification of $A_A$ on $\mathscr{I}^-$. This is the analog of the specification of $\Phi$ on $\mathscr{I}^-$ in the scalar field case. The analog of the observable $\Pi = \partial_v \Phi$ on $\mathscr{I}^-$ is the *electric field*

$$E_A = -\pounds_n A_A = -\partial_v A_A \tag{4.5}$$

which is the pullback to $\mathscr{I}^-$ of $F_{\mu\nu}n^\nu$. Note that $E_A$ is gauge invariant. The symplectic form is given by

$$\Omega^{\mathrm{EM}}_{\mathscr{I}}(A_1, A_2) = -\frac{1}{4\pi}\int\limits_{\mathscr{I}^-} d^3x \left[E_{1A}A_2^A - E_{2A}A_1^A\right]. \tag{4.6}$$

The local field observables for the Maxwell field on $\mathscr{I}^-$ are

$$E(s) = \int\limits_{\mathscr{I}^-} d^3x \; E_A(x)s^A(x) \tag{4.7}$$

where $s^A$ is test vector field on $\mathscr{I}^-$, with no conformal weight, and the capital Latin index is in accord with the notational conventions stated at the end of sec. 0.3 because eq. (4.7) depends only on the equivalence class of the vector field. Note that the observable $E(s)$ generates the infinitesimal affine transformation $A_A \to A_A - 2\pi\epsilon s_A$. The Poisson brackets

24. Note that the vector potential $A_\mu$ is *not* smooth at null infinity in the Lorenz gauge when there is a non-vanishing total charge (see Remark 4 and eq. (52) of [1]). Nevertheless one make other gauge choices that yield a smooth $A_\mu$.

are

$$\{E(s_1), E(s_2)\} = -4\pi^2 \Omega^{\text{EM}}_{\mathscr{I}}(s_1, s_2) 1 \tag{4.8}$$

where, for test functions $s_1^A, s_2^A$ we have that

$$\Omega^{\text{EM}}_{\mathscr{I}}(s_1, s_2) = -\frac{1}{4\pi} \int_{\mathscr{I}} d\Omega dv \left[s_1^A \partial_v s_{2A} - s_2^A \partial_v s_{1A}\right] = -\frac{1}{2\pi} \int_{\mathscr{I}^-} d\Omega dv \ s_1^A \partial_v s_{2A}. \tag{4.9}$$

In exact parallel with the asymptotic quantization of the massless scalar field given in sec. 3, the algebra $\mathscr{A}^{\text{EM}}_{\text{in}}$ is defined to be the free algebra generated by the smeared fields $\boldsymbol{E}(s)$, their formal adjoints $\boldsymbol{E}(s)^*$, and an identity $\mathbf{1}$ — where $s^A(x)$ are real test vector fields on $\mathscr{I}^-$ — factored by the following relations:

(B.I) $\boldsymbol{E}(c_1 s_1 + c_2 s_2) = c_1 \boldsymbol{E}(s_1) + c_2 \boldsymbol{E}(s_2)$ for any $s_1^A$, $s_2^A$ and any $c_1, c_2 \in \mathbb{R}$

(B.II) $\boldsymbol{E}(s)^* = \boldsymbol{E}(s)$ for all $s^A$

(B.III) $[\boldsymbol{E}(s_1), \boldsymbol{E}(s_2)] = -i4\pi^2 \Omega^{\text{EM}}_{\mathscr{I}}(s_1, s_2)\mathbf{1}$ for any $s_1^A$, $s_2^A$

We shall denote states on the algebra $\mathscr{A}^{\text{EM}}_{\text{in}}$ as $\omega^{\text{EM}}$. The Hadamard regularity condition on asymptotic states of the electromagnetic field analogous to eq. (3.5) is that the 2-point function has the form

$$\omega^{\text{EM}}(\boldsymbol{E}_A(x_1)\boldsymbol{E}_B(x_2)) = -\frac{q_{AB}\delta_{\mathbb{S}^2}(x_1^A, x_2^A)}{(v_1 - v_2 - i0^+)^2} + S_{AB}(x_1, x_2) \tag{4.10}$$

where $S_{AB}$ is a (state-dependent) smooth bi-tensor on $\mathscr{I}^-$ that is symmetric under the simultaneous interchange of $x_1, x_2$ and the indices $A, B$. Furthermore, we require that the connected $n$-point functions for $n \neq 2$ of a Hadamard state on $\mathscr{I}^-$ are smooth.[25] The 2-point function of the Poincaré invariant vacuum state $\omega_0^{\text{EM}}$ is given by eq. (4.10) with $S_{AB} = 0$.

25. It is possible that, in analogy with the bulk theory [108], this requirement actually a consequence of eq. (4.10). However, we have not investigated whether this is the case.

Finally, we impose a decay condition on states to ensure that all fluxes are well-defined. We require states to be such that $S_{AB}$ and all connected $n$-point functions for $n \neq 2$ decay for any set of $|v_i| \to \infty$ as $O((\sum_i v_i^2)^{-1/2-\epsilon})$ for some $\epsilon > 0$. This completes the specification of $\mathscr{A}_{\text{in}}^{\text{EM}}$ and the allowed states on $\mathscr{A}_{\text{in}}^{\text{EM}}$.

We turn now to the asymptotic quantization of a massive complex scalar field $\varphi$ in Minkowski spacetime. We follow the same basic strategy of finding an appropriate asymptotic surface that can be treated as a Cauchy surface. We then obtain the asymptotic description of the classical phase space by finding appropriate initial data on the asymptotic surface and we express the symplectic form in terms of this initial data. We then use eq. (2.13) to obtain observables involving this initial data that correspond to local observables in the bulk, and we obtain the Poisson brackets of these observables. This enables us to define $\mathscr{A}_{\text{in}}^{\text{KG}}$.

For massive fields, the appropriate asymptotic surface is an asymptotic hyperboloid $\mathcal{H}^-$ rather than $\mathscr{I}^-$. To see this, we introduce a coordinate system as follows [115] (see also [116, 117]). Let

$$\tau^2 := t^2 - r^2 , \quad \rho := \tanh^{-1} r/t. \tag{4.11}$$

where $t, r$ are the standard Minkowski time and radial coordinates. These coordinates foliate the interiors of the future/past light cone of an arbitrary choice of origin in Minkowski spacetime by a family of Riemannian hyperboloids with $\tau =$ constant (see fig. 3.1). The induced metric on the 3-dimensional unit-hyperboloid $\mathcal{H}$ (with $\tau^2 = 1$) is given by

$$ds^2_{\mathcal{H}} = \frac{d\rho^2}{1+\rho^2} + \rho^2 s_{AB} dx^A dx^B \tag{4.12}$$

where $s_{AB}$ is the metric on the unit 2-sphere and $x^A$ are coordinates on the sphere. The metric on a hyperboloid with $\tau^2 \neq 1$ is just $\tau^2 ds^2_{\mathcal{H}}$. Note that any point on $\mathcal{H}$ can also be thought of as a unit-normalized timelike vector $p$ in Minkowski spacetime, and we will use the notation $p = (\rho, x_p^A)$ to denote points on $\mathcal{H}$. The induced volume element on $\mathcal{H}$ is then

$$d^3p := \frac{\rho^2}{\sqrt{1+\rho^2}} d\rho d\Omega. \tag{4.13}$$

The stationary phase method suggests that as $\tau \to -\infty$ at fixed $p$, there exists a gauge[26] such that (up to a constant phase factor) the leading order asymptotic behaviour of $\varphi$ is given by [115] (see also [116])

$$\varphi \sim \frac{\sqrt{m}}{2(2\pi\tau)^{3/2}} \left[c(p)e^{-im\tau} + i\bar{b}(p)e^{im\tau}\right] \tag{4.14}$$

where $p$ denotes a future-directed unit-normalized momentum and thus, a point $p = (\rho, x_p^A)$ on the unit-hyperboloid $\mathcal{H}^-$ in the tangent space at past timelike infinity. Note that although each hyperboloid of constant $\tau$ extends to past null infinity, the hyperboloid $\mathcal{H}^-$ corresponds to taking the limit $\tau \to -\infty$ at fixed $p = (\rho, x_p^A)$ and thus gives a representation of unit-timelike directions at past timelike infinity closely analogous to the description of spatial infinity given by Ashtekar and Hansen [61].[27] We will assume that the asymptotic behavior of $\varphi$ is given by eq. (4.14) and that $\mathcal{H}^-$ can be treated as a Cauchy surface.

The initial data on $\mathcal{H}^-$ of a solution consists of the complex functions $b(p)$ and $c(p)$ appearing in eq. (4.14). The symplectic form on this initial data can be written as [115][28]

$$\Omega_{i^-}^{\text{KG}}(\varphi_1, \varphi_2) = -\frac{im^2}{4(2\pi)^3} \int_{\mathcal{H}^-} d^3p \, \left[b_1(p)\bar{b}_2(p) + c_1(p)\bar{c}_2(p) - (1 \leftrightarrow 2)\right]. \tag{4.15}$$

---

26. In the Lorenz gauge for the electromagnetic vector potential, the scalar field in eq. (4.14) would have an additional overall phase $e^{iq\log\tau}$ in its asymptotic behavior (see e.g. [115] or Ch. IV of [118]). However this logarithmic "Coulomb phase" can be eliminated by a different choice of gauge. The vector potential in the Lorenz gauge is also badly behaved at null infinity (see footnote 24).

27. A similar analysis at timelike infinity can be found in [116, 117].

28. Our convention for the symplectic form differs from that in [115] by a factor of $-1/2$.

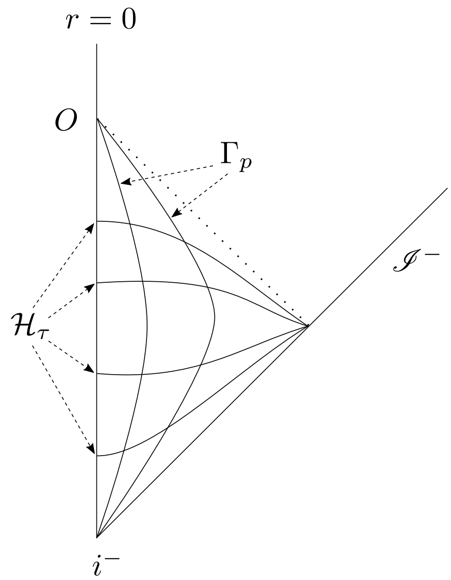


Figure 3.1: A schematic picture (with angular dimensions suppressed) of the family of hyperboloids $\mathcal{H}_\tau$ used to take the limits to timelike infinity $i^-$. The vertical line labeled $r = 0$ is the axis of rotational symmetry, $O$ is an arbitrary choice of origin in Minkowski spacetime with past light cone depicted by a dotted line, and $\mathscr{I}^-$ denotes past null infinity. The $\mathcal{H}_\tau$ are 3-dimensional hyperboloids of $\tau =$ constant with $\tau \to -\infty$ corresponding to the limiting hyperboloid $\mathcal{H}^-$ at $i^-$. The $\Gamma_p$ denote curves of constant $p = (\rho, x_p^A)$ along which the limit to past timelike infinity is taken.

The symplectic form is a real-bilinear map on the real and imaginary parts of $b$ and $c$, but it is not complex-bilinear (or complex-bi-antilinear) in $b$ and $c$. On the complex plane, it is often convenient to treat $z = x + iy$ and $\overline{z} = x - iy$ as though they were independent quantities, imposing that they are conjugates only at the end of any calculation. For similar reasons, it is convenient to treat $\overline{b}$ and $\overline{c}$ as though they are quantities independent of $b$ and $c$ on phase space, imposing that they be conjugates of $b$ and $c$ at the end of any calculation. Thus, we will take a point in the asymptotic description of the classical phase space to be represented as the quadruple $(b(p), \overline{b}(p), c(p), \overline{c}(p))$. The symplectic form is then a complex-bilinear function of its variables.

Local observables are obtained by smearing the fields with a *complex* test function $w(p)$ on $\mathcal{H}^-$. The smeared fields $b(w)$ and $\overline{b}(w)$ are defined by

$$b(w) := \int_{\mathcal{H}^-} d^3p\; b(p)\overline{w}(p)\,, \qquad \overline{b}(w) := \int_{\mathcal{H}^-} d^3p\; \overline{b}(p)w(p). \tag{4.16}$$

Note that we take $b(w)$ to be anti-linear in $w$ whereas $\overline{b}(w)$ is linear in $w$, so $\overline{b(w)} = \overline{b}(w)$. Note also that the Hamiltonian vector fields corresponding to these observables are given by

$-\frac{4(2\pi)^3}{im^2}(0, \overline{w}(p), 0, 0)$ and $\frac{4(2\pi)^3}{im^2}(w(p), 0, 0, 0)$ respectively. We similarly define the smeared local observables

$$c(w) := \int_{\mathcal{H}^-} d^3p \; c(p)w(p) \,, \qquad \overline{c}(w) := \int_{\mathcal{H}^-} d^3p \; \overline{c}(p)\overline{w}(p) \tag{4.17}$$

where we now take $c(w)$ to be linear and $\overline{c}(w)$ to be anti-linear in $w$. The nontrivial Poisson brackets are

$$\left\{b(w_1), \overline{b}(w_2)\right\} = -i\frac{4(2\pi)^3}{m^2} \langle w_1, w_2 \rangle_{\mathcal{H}^-} 1 \,, \quad \left\{c(w_1), \overline{c}(w_2)\right\} = -i\frac{4(2\pi)^3}{m^2} \langle w_2, w_1 \rangle_{\mathcal{H}^-} 1. \tag{4.18}$$

Here, the inner product $\langle w_1, w_2 \rangle_{\mathcal{H}^-}$ is the ordinary $L^2$ inner product on $\mathcal{H}^-$ with the volume element eq. (4.13), which is antilinear in its first argument and is linear in its second argument.

The asymptotic quantization algebra, $\mathscr{A}^{\text{KG}}_{\text{in}}$, for the massive complex scalar field is then defined by starting with the free algebra generated by the smeared fields $\boldsymbol{b}(w)$, $\boldsymbol{c}(w)$, their formal adjoints $\boldsymbol{b}(w)^*, \boldsymbol{c}(w)^*$ and an identity $\mathbf{1}$. We note that the adjoint operators $\boldsymbol{b}(w)^*$ and $\boldsymbol{c}(w)^*$ correspond to the complex conjugate observables $\overline{b}(w)$ and $\overline{c}(w)$ respectively. We then factor this algebra by the analog of the linearity condition (B.I), the commutation relations

$$[\boldsymbol{b}(w_1), \boldsymbol{b}(w_2)^*] = \frac{4(2\pi)^3}{m^2} \langle w_1, w_2 \rangle_{\mathcal{H}^-} \mathbf{1} \,, \quad [\boldsymbol{c}(w_1), \boldsymbol{c}(w_2)^*] = \frac{4(2\pi)^3}{m^2} \langle w_2, w_1 \rangle_{\mathcal{H}^-} \mathbf{1} \tag{4.19}$$

and vanishing commutators for all other fields. This completes the specification of the algebra $\mathscr{A}^{\text{KG}}_{\text{in}}$ of local observables of the massive, charged Klein-Gordon field.

The Hadamard condition for states $\omega^{\text{KG}}$ on $\mathscr{A}^{\text{KG}}_{\text{in}}$, is that the 2-point functions given by $\omega^{\text{KG}}(\boldsymbol{b}(p_1)\boldsymbol{b}(p_2))$, $\omega^{\text{KG}}(\boldsymbol{c}(q_1)\boldsymbol{c}(q_2))$, $\omega^{\text{KG}}(\boldsymbol{b}(p_1)\boldsymbol{c}(q_2))$, and $\omega^{\text{KG}}(\boldsymbol{b}(p_1)^*\boldsymbol{c}(q_1))$ are smooth, whereas the remaining 2-point functions have the form

$$\omega^{\text{KG}}(\boldsymbol{b}(p_1)\boldsymbol{b}(p_2)^*) = \frac{4(2\pi)^3}{m^2}\delta_{\mathcal{H}}(p_1, p_2) + B(p_1, p_2) \tag{4.20a}$$

$$\omega^{\text{KG}}(\boldsymbol{c}(q_1)\boldsymbol{c}(q_2)^*) = \frac{4(2\pi)^3}{m^2}\delta_{\mathcal{H}}(q_2, q_1) + C(q_2, q_1) \tag{4.20b}$$

where it is understood that $\delta_{\mathcal{H}}$ is to be smeared with a complex conjugate test function $\overline{w}$ in its first argument and a test function $w$ in its second argument, and the functions $B$ and $C$ are (state-dependent) smooth functions on $\mathcal{H} \times \mathcal{H}$. Furthermore, the connected $n$-point functions for $n \neq 2$ of $\omega^{\text{KG}}$ are required to be smooth. Note that the 2-point function of the Poincaré invariant vacuum state $\omega_0^{\text{KG}}$ is given by eq. (4.20) with $B = C = 0$.

In addition, we impose the following decay condition on states: We require that $B$, $C$ and all the connected $n$-point functions for $n \neq 2$ of $\omega^{\text{KG}}$ decay for any set of $|p_i| \to \infty$ as $O((\sum_i p_i^2)^{-1/2-\epsilon})$ for some $\epsilon > 0$. This completes our specification of the regularity conditions on states.

## 2 *Extension of the asymptotic quantization algebra to include charges and Poincaré generators*

The algebra $\mathscr{A}_{\text{in}}$ that we have defined in the previous subsection was generated by the local field observables of the asymptotic "in" fields. Thus, the only observables represented in $\mathscr{A}_{\text{in}}$ are the local fields. However, there are additional observables of interest, where, here and elsewhere in this chapter, we use the term "observables" in the precise sense explained in sec. 2. In this section, we will extend $\mathscr{A}_{\text{in}}$ to the algebra $\mathscr{A}_{\text{in,Q}}$ by the addition of generators of large gauge transformations (i.e. "charges"). We will then further extend this algebra to an algebra $\mathscr{A}_{\text{in,QP}}$ that includes the generators of Poincaré symmetries. We will construct these algebras by obtaining observables on the classical phase space that generate large gauge transformations and Poincaré symmetries. These observables automatically have well-defined Poisson brackets with themselves and with the local fields. We then will obtain $\mathscr{A}_{\text{in,Q}}$ by starting with the free algebra generated by $\mathscr{A}_{\text{in}}$ together with the observables that generate large gauge transformations and then factoring by the commutation relations obtained from the Poisson brackets. We will then further enlarge this algebra to $\mathscr{A}_{\text{in,QP}}$ by including the observables that generate Poincaré symmetries.

We first consider the large gauge charges. As stated above, QED has an invariance under eq. (4.3). The transformations eq. (4.3) with $\lambda$ vanishing at infinity are genuine gauge transformations in the sense that the infinitesimal versions of these transformations are degeneracies of the symplectic form. In order to construct a phase space with a nondegenerate symplectic form, one must pass to the space of gauge orbits [100], so fields that differ by a gauge transformation correspond to the same point of phase space. However, as previously noted in sec. 1, the transformations eq. (4.3) with $\lambda = \lambda(x^A)$ are not degeneracies of the symplectic form. Such "large gauge transformations" must be treated as symmetries and they act nontrivially on the classical phase space. The infinitesimal version of these symmetries defines a vector field on phase space. We will show that this vector field on phase space is generated by a classical observable, which will be referred to as a "charge." Consequently, we can expect that the quantum algebra $\mathscr{A}_{\text{in}}$ can be extended to include quantum representatives of the charges.

Since the asymptotic description of phase space is the Cartesian product of the Klein-Gordon and Maxwell phase spaces, we can separately consider the action of large gauge transformations on the Klein-Gordon and Maxwell fields separately. We will thereby obtain two "charges": (i) a charge $\mathcal{Q}_{i^-}$ that generates large gauge transformations on the Klein-Gordon field and (ii) a "memory" quantity that generates large gauge transformations on the Maxwell field. The sum of these two, denoted $\mathcal{Q}_{i^0}$, generates large gauge transformations on the full phase space. The reason for the use of "$i^0$" in the notation for the total charge will be explained below.

We first consider the action of the large gauge transformations on the classical Klein-Gordon phase space, i.e., on the asymptotic fields on $\mathcal{H}^-$. The large gauge transformations are parametrized by a smooth function $\lambda(x^A)$ on $\mathbb{S}^2$, which describes the asymptotic behavior of the transformation eq. (4.3) on the scalar field as $\rho \to \infty$. It is useful to pick a unique representative of this transformation throughout $\mathcal{H}^-$ as follows. Let $\lambda_{\mathcal{H}}(p)$ be the unique function on $\mathcal{H}^-$ which satisfies

$$\triangle_{\mathcal{H}} \lambda_{\mathcal{H}}(p) = 0\,, \quad \lim_{\rho\to\infty} \lambda_{\mathcal{H}}(p) = \lambda(x^A) \tag{4.21}$$

where $\triangle_{\mathcal{H}}$ is the Laplace operator on $\mathcal{H}^-$. The solution $\lambda_{\mathcal{H}}(p)$ can be expressed in terms of the boundary value $\lambda(x^A)$ using a Green's function as [119]

$$\lambda_{\mathcal{H}}(p) = \int_{\mathbb{S}^2} d\Omega\ G_{\mathcal{H}}(p, x^A)\lambda(x^A)\,, \quad G_{\mathcal{H}}(p, x^A) = \frac{1}{4\pi(\sqrt{1+\rho^2} - \rho\hat{p}\cdot\hat{r})^2}. \tag{4.22}$$

Here, $\hat{r}$ is the unit vector in $\mathbb{R}^3$ corresponding to the point $x^A$ on the unit 2-sphere, and $\hat{p}$ denotes the projection of the point $p \in \mathcal{H}^-$ onto the unit 2-sphere, also represented as a unit vector in $\mathbb{R}^3$. The Euclidean dot product $\hat{p}\cdot\hat{r}$ of these unit vectors is the cosine of the geodesic distance between two points on $\mathbb{S}^2$ with respect to the unit 2-sphere metric. Note that this Green's function satisfies

$$\int_{\mathbb{S}^2} d\Omega\ G_{\mathcal{H}}(p, x^A) = 1. \tag{4.23}$$

In terms of $\lambda_{\mathcal{H}}$, the action of the large gauge transformations on the asymptotic scalar field is given by

$$b(p) \mapsto b(p)e^{-iq\lambda_{\mathcal{H}}(p)}, \quad \bar{b}(p) \mapsto \bar{b}(p)e^{iq\lambda_{\mathcal{H}}(p)}$$
$$c(p) \mapsto c(p)e^{iq\lambda_{\mathcal{H}}(p)}, \quad \bar{c}(p) \mapsto \bar{c}(p)e^{-iq\lambda_{\mathcal{H}}(p)}. \tag{4.24}$$

The infinitesimal action of large gauge transformations on phase space is given by

$$(b(p), \bar{b}(p), c(p), \bar{c}(p)) \to (b(p), \bar{b}(p), c(p), \bar{c}(p)) + iq\lambda_{\mathcal{H}}(p)\epsilon\,(-b(p), \bar{b}(p), c(p), -\bar{c}(p)). \tag{4.25}$$

This transformation is of the form eq. (2.8) with $\chi_0 = 0$ and

$$L(b, \bar{b}, c, \bar{c}) = (b', \overline{b'}, c', \overline{c'}) \tag{4.26}$$

with $b'(p) = -iq\lambda_{\mathcal{H}}(p)b(p)$, $c'(p) = iq\lambda_{\mathcal{H}}(p)c(p)$. The linear map $L$ satisfies eq. (2.9) so we obtain the observable

$$\mathcal{Q}_{i^-}(\lambda) := \frac{1}{2}\Omega^{\text{KG}}_{i^-}((b, \bar{b}, c, \bar{c}), L(b, \bar{b}, c, \bar{c})) = \frac{qm^2}{4(2\pi)^3}\int\limits_{\mathcal{H}^-} d^3p\ \lambda_{\mathcal{H}}(p)\left[b(p)\bar{b}(p) - c(p)\bar{c}(p)\right]. \tag{4.27}$$

Note that the integrand on the right-hand-side of eq. (4.27) corresponds to the asymptotic limit to $\mathcal{H}^-$ of $J_\mu\tau^\mu$, where $\tau^\mu$ is the unit normal to the surfaces of constant $\tau$ and the charge-current vector $J^\mu$ of the scalar field is given by

$$J^\mu = -\frac{iq}{2}[\overline{\varphi}D^\mu\varphi - \varphi D^\mu\overline{\varphi}]\,. \tag{4.28}$$

Thus, for $\lambda(x^A) = \text{constant}$, $\mathcal{Q}_{i^-}$ is the total ordinary electric charge of the massive scalar field [115].

Since the observables $\mathcal{Q}_{i^-}(\lambda)$ generate the large gauge transformations eq. (4.25), it is straightforward to compute their Poisson brackets. The Poisson brackets of the charges with themselves vanish

$$\left\{\mathcal{Q}_{i^-}(\lambda_1), \mathcal{Q}_{i^-}(\lambda_2)\right\} = 0 \tag{4.29}$$

The Poisson brackets of the charges with the smeared fields are

$$\begin{aligned}\left\{\mathcal{Q}_{i^-}(\lambda), b(w)\right\} = -iqb(\lambda_{\mathcal{H}}w)\,, \quad \left\{\mathcal{Q}_{i^-}(\lambda), \bar{b}(w)\right\} = iq\bar{b}(\lambda_{\mathcal{H}}w) \\ \left\{\mathcal{Q}_{i^-}(\lambda), c(w)\right\} = iqc(\lambda_{\mathcal{H}}w)\,, \quad \left\{\mathcal{Q}_{i^-}(\lambda), \bar{c}(w)\right\} = -iq\bar{c}(\lambda_{\mathcal{H}}w)\,.\end{aligned} \tag{4.30}$$

Since $\mathcal{Q}_{i^-}(\lambda)$ is an observable on the Klein-Gordon phase space, it has vanishing Poisson bracket with all electromagnetic observables.

We now consider the action of large gauge transformations on the Maxwell phase space. The large gauge transformation $\lambda = \lambda(x^A)$ acts on the Maxwell phase space by

$$A_A \mapsto A_A + \mathscr{D}_A\lambda\,, \quad E_A \mapsto E_A. \tag{4.31}$$

This affine transformation is generated by $\frac{1}{4\pi}\Delta(\lambda)$ where $\Delta(\lambda)$ is defined by

$$\Delta(\lambda) := -\int_{\mathscr{I}^-} dvd\Omega\ E_A(v, x^B)\mathscr{D}^A\lambda(x^B). \tag{4.32}$$

Thus, $\Delta(\lambda)$ is an observable on the Maxwell phase space. We refer to $\Delta(\lambda)$ as the *memory* of the Maxwell field associated with the large gauge transformation $\lambda$. Since the local electromagnetic field observables $E(s)$ (see eq. (4.7)) are invariant under eq. (4.31), $\Delta(\lambda)$ has vanishing Poisson bracket with all local electromagnetic field observables. Since $\Delta(\lambda)$ is an observable on the Maxwell phase space, it also has vanishing Poisson bracket with all Klein-Gordon observables.

The generator of gauge transformations on the full phase space of the Klein-Gordon and

Maxwell fields is given by the sum of eq. (4.27) and eq. (4.32)

$$\mathcal{Q}_{i^0}(\lambda) := \mathcal{Q}_{i^-}(\lambda) + \frac{1}{4\pi}\Delta(\lambda)\,. \tag{4.33}$$

The Poisson brackets of $\mathcal{Q}_{i^0}(\lambda)$ with all local field observables are the same as those of $\mathcal{Q}_{i^-}(\lambda)$. The subscript "$i^0$" has been placed on $\mathcal{Q}_{i^0}(\lambda)$ because its value can be computed by taking limits of surface integrals of the electric field as one approaches spatial infinity, $i^0$, along $\mathscr{I}^-$. This can be shown by the following lengthy argument.

First, we show that the charge $\mathcal{Q}_{i^-}(\lambda)$ can be computed as a bulk limit of the electric field. In the bulk spacetime, the massive scalar is coupled to the electromagnetic field via the Maxwell equation

$$\frac{1}{4\pi}\nabla_\nu F^{\nu\mu} = J^\mu \tag{4.34}$$

with $J^\mu$ given by eq. (4.28). We assume that the limit to $\mathcal{H}^-$ of the "electric field"

$$\mathcal{E}_\mu(p) = \lim_{\tau\to-\infty} \tau^2 h_\mu{}^\nu F_{\nu\sigma}\tau^\sigma \tag{4.35}$$

exists and defines a smooth tensor field $\mathcal{E}_a$ on $\mathcal{H}^-$, where $h_{\mu\nu} = g_{\mu\nu} + \tau_\mu\tau_\nu$ is the induced metric on the hyperboloids of constant $\tau$. From the Maxwell equation (4.34) and the falloff of the scalar field, it follows that there exists an electric potential $V(p)$ on $\mathcal{H}^-$ so that $\mathcal{E}_a(p) = \mathcal{D}_a V(p)$, which satisfies

$$\frac{1}{4\pi}\triangle_{\mathcal{H}} V(p) = \frac{qm^2}{4(2\pi)^3}\left[b(p)\bar{b}(p) - c(p)\bar{c}(p)\right] \tag{4.36}$$

where $\mathscr{D}_a$ denotes the derivative operator on $\mathcal{H}^-$ and $\triangle_{\mathcal{H}}$ again denotes the Laplacian on

$\mathcal{H}^-$. By Green's identity, for any large gauge transformation $\lambda$, we have

$$\begin{aligned} \frac{1}{4\pi}\mathscr{D}^a\left(\lambda_{\mathcal{H}}\mathscr{D}_a V - V\mathscr{D}_a\lambda_{\mathcal{H}}\right) &= \frac{1}{4\pi}(\lambda_{\mathcal{H}}\Delta_{\mathcal{H}}V - V\Delta_{\mathcal{H}}\lambda_{\mathcal{H}}) \\ &= \frac{qm^2}{4(2\pi)^3}\left[b(p)\bar{b}(p) - c(p)\bar{c}(p)\right] \end{aligned} \tag{4.37}$$

Integrating this equation over $\mathcal{H}^-$ and applying Gauss' theorem to the left side, we obtain

$$\begin{aligned} \lim_{\rho\to\infty}\frac{1}{4\pi}\int_{\mathbb{S}^2} d\Omega\ \lambda(x^A)(\cosh\rho)^2\rho^a\mathcal{E}_a &= \frac{qm^2}{4(2\pi)^3}\int_{\mathcal{H}^-} d^3p\ \lambda_{\mathcal{H}}(p)\left[b(p)\bar{b}(p) - c(p)\bar{c}(p)\right] \\ &= \mathcal{Q}_{i^-}(\lambda) \end{aligned} \tag{4.38}$$

where $\rho^a$ is the unit-spacelike-normal to the $\rho =$ constant cross-sections of $\mathcal{H}^-$. Thus, as we desired to show, the charge $\mathcal{Q}_{i^-}(\lambda)$ can be obtained as an asymptotic surface integral as $\rho \to \infty$ of the electric field $\mathcal{E}_a$ on $\mathcal{H}^-$, which itself is obtained as the bulk limit eq. (4.35) as $\tau \to -\infty$. For $\lambda(x^A) =$ constant, eq. (4.38) corresponds to the usual Gauss law formula for charge.

Next, we assume that the analogue of the "null regularity" condition imposed at spatial infinity in [33] holds at timelike infinity. This yields

$$\lim_{\rho\to\infty}(\cosh\rho)^2\rho^a\mathcal{E}_a \quad (\text{along } \mathcal{H}^-) = \lim_{v\to-\infty} F_{\mu\nu}l^\mu n^\nu \quad (\text{along } \mathscr{I}^-) \tag{4.39}$$

where $l^\mu$ is a vector field at $\mathscr{I}$ satisfying $l_\mu l^\mu = 0$ and $l^\mu n_\mu = -1$. This quantity is not a function on the electromagnetic phase space, i.e., it depends on non-radiative (Coulombic) information at $\mathscr{I}^-$ that is obtained from bulk limits. It follows that

$$\mathcal{Q}_{i^-}(\lambda) = \lim_{v\to-\infty}\frac{1}{4\pi}\int_{\mathbb{S}^2} d\Omega\ \lambda(x^A)F_{\mu\nu}l^\mu n^\nu. \tag{4.40}$$

Now, Maxwell's equations imply that on $\mathscr{I}^-$ we have

$$\pounds_n(F_{\mu\nu}l^\mu n^\nu) = \mathscr{D}^A E_A. \tag{4.41}$$

It follows immediately that

$$\begin{aligned}\Delta(\lambda) &= \frac{1}{4}\int\limits_{\mathscr{I}^-} dv d\Omega\ \lambda(x^A)\frac{\partial(F_{\mu\nu}l^\mu n^\nu)}{\partial v} \\ &= \lim_{v\to+\infty}\frac{1}{4}\int\limits_{\mathbb{S}^2} d\Omega\ \lambda(x^A)F_{\mu\nu}l^\mu n^\nu - \lim_{v\to-\infty}\frac{1}{4}\int\limits_{\mathbb{S}^2} d\Omega\ \lambda(x^A)F_{\mu\nu}l^\mu n^\nu.\end{aligned} \tag{4.42}$$

Thus, we obtain our desired result

$$\mathcal{Q}_{i^0}(\lambda) = \lim_{v\to+\infty}\frac{1}{4\pi}\int\limits_{\mathbb{S}^2} d\Omega\ \lambda(x^A)F_{\mu\nu}l^\mu n^\nu \tag{4.43}$$

which shows that $\mathcal{Q}_{i^0}(\lambda)$ can be computed in terms of a limit of $F_{\mu\nu}l^\mu n^\nu$ as one approaches $i^0$ along $\mathscr{I}^-$. However, it should be kept in mind that $F_{\mu\nu}l^\mu n^\nu$ is not an observable on the Klein-Gordon and Maxwell phase spaces. The definition of $\mathcal{Q}_{i^0}(\lambda)$ as an observable is given by eqs. (4.27), (4.32) and (4.33).

It is worth noting that on any cross-section $S \cong \mathbb{S}^2$ of $\mathscr{I}$, we can define the Maxwell charge associated with the large gauge transformation $\lambda$ by

$$\mathcal{Q}_S(\lambda) = \frac{1}{4\pi}\int\limits_S d\Omega\ \lambda(x^A)F_{\mu\nu}l^\mu n^\nu. \tag{4.44}$$

This notion of "large gauge charge at a finite advanced time" may be useful for a number of purposes. However, unlike $\mathcal{Q}_{i^-}(\lambda)$ and $\mathcal{Q}_{i^0}(\lambda)$, the quantity $\mathcal{Q}_S(\lambda)$ does *not* correspond to an observable on phase space. Thus, $\mathcal{Q}_S(\lambda)$ does not have well-defined Poisson brackets with local field observables and we cannot expect $\mathcal{Q}_S(\lambda)$ to have a well-defined counterpart in quantum field theory.[29]

We now turn to the extension of the algebra, $\mathscr{A}_{\text{in}}$, of local quantum observables to an algebra $\mathscr{A}_{\text{in,Q}}$ that includes the large gauge charges $\mathcal{Q}_{i^-}(\lambda)$ and $\Delta(\lambda)$ (and thereby, $\mathcal{Q}_{i^0}(\lambda)$). We start with the free algebra generated by the smeared local field observables together with observables labeled as $\boldsymbol{\mathcal{Q}}_{i^-}(\lambda)$ and $\boldsymbol{\Delta}(\lambda)$ and their adjoints for all large gauge transformations $\lambda(x^A)$. We then factor this algebra by linearity in the test functions and $\lambda$ as well as Hermitian conditions for $\boldsymbol{E}$, $\boldsymbol{\mathcal{Q}}_{i^-}$, and $\boldsymbol{\Delta}$. Finally, we factor the algebra by the commutation relations corresponding to all of the Poisson bracket relations we have obtained above. This defines the desired extended algebra $\mathscr{A}_{\text{in,Q}}$.

However, since $\boldsymbol{\Delta}(\lambda)$ commutes with all observables[30] in $\mathscr{A}_{\text{in,Q}}$, it follows that in any representation of $\mathscr{A}_{\text{in,Q}}$, a shift of $\boldsymbol{\Delta}(\lambda)$ by a multiple of the identity — with no corresponding shift of $\boldsymbol{E}$ — would also yield a representation of the algebra. This implies that there are many states on the extended algebra where the value of the memory observable is not related to value of the electric field by eq. (4.32). In order to exclude such states, we require as a further condition on states $\omega$ (in addition to the Hadamard condition and the fall-off conditions of sec. 4.1) that[31]

$$\omega(\boldsymbol{\Delta}(\lambda)) = -\int_{\mathscr{I}^-} dv d\Omega\, \omega(\boldsymbol{E}_A(v, x^B))\mathscr{D}^A\lambda(x^B). \tag{4.45}$$

We have a similar multiple of the identity ambiguity for operator representatives of

29. The nonexistence of operators corresponding to $\mathcal{Q}_S(\lambda)$ in quantum field theory is in agreement with the arguments given in [110].

30. Note, however, that this will not be the case after we further extend the algebra to $\mathscr{A}_{\text{in,QP}}$ by including Poincaré generators, as we shall do below.

31. We could also impose conditions on higher $n$-point functions of memory, but we will not need these, so we shall not impose them here.

$\boldsymbol{\mathcal{Q}}_{i^-}(\lambda)$. We could similarly impose the additional requirement on states that

$$\omega(\boldsymbol{\mathcal{Q}}_{i^-}(\lambda)) = \frac{m^2}{4(2\pi)^3} q \int_{\mathcal{H}^-} d^3p \; \lambda_{\mathcal{H}}(p) \; \omega(\boldsymbol{b}(p)^* \boldsymbol{b}(p) - \boldsymbol{c}(p)^* \boldsymbol{c}(p)) \tag{4.46}$$

where the expected value of quadratic quantities was defined at the end of sec. 3. However, for the massive scalar field, we will only be interested in states in the standard Fock space (see sec. 4.3.1). For such states, instead of demanding eq. (4.46), we can more simply demand that the charge $\boldsymbol{\mathcal{Q}}_{i^-}(\lambda)$ annihilates the Fock vacuum state, since this removes the multiple of the identity ambiguity in this representation and implies that eq. (4.46) holds for all states in this representation.

We now follow the same strategy to further extend the algebra $\mathscr{A}_{\text{in,Q}}$ to an algebra $\mathscr{A}_{\text{in,QP}}$ that also includes observables corresponding to the generators of Poincaré transformations. The first step is to write down the action of the Poincaré group on the classical phase space and show that its infinitesimal action is generated by an observable on the classical phase space. Again, we may consider the Poincaré action on the Maxwell phase space and the Klein-Gordon phase space separately.

Poincaré transformations correspond to a particular class of diffeomorphisms of $\mathscr{I}^-$, and these act naturally on the fields $A_A$ and $E_A$, so it is straightforward to determine their action on the Maxwell phase space. Lorentz transformations (with origin taken to be that used to define the hyperboloids of eq. (4.11)) similarly correspond to isometries of $\mathcal{H}^-$ and thus have a natural action on the asymptotic Klein-Gordon fields. Thus, we only need to explain the action of translations. In terms of the asymptotic coordinates used above, it can be shown that, at leading order, a translation corresponds to the transformation $\tau \mapsto \tau + f_{\mathcal{H}}(p)$ where the function $f_{\mathcal{H}}(p)$ satisfies[32]

$$(\triangle_{\mathcal{H}} - 3) f_{\mathcal{H}}(p) = 0\,, \quad \lim_{\rho\to\infty} \rho^{-1} f_{\mathcal{H}}(p) = f(x^A) \tag{4.47}$$

with $f(x^A)$ is a smooth function on $\mathbb{S}^2$ supported only on the $\ell = 0, 1$ spherical harmonics. The solution can again be written in terms of a Green's function as [119]

$$f_{\mathcal{H}}(p) = \int_{\mathbb{S}^2} d\Omega\ \tilde{G}_{\mathcal{H}}(p, x^A) f(x^A)\,, \quad \tilde{G}_{\mathcal{H}}(p, x^A) = \frac{1}{4\pi(\sqrt{1+\rho^2} - \rho\hat{p}\cdot\hat{r})^3} \tag{4.48}$$

where the notation $\hat{p}$ and $\hat{r}$ is as explained below eq. (4.22). The action of these translations on the asymptotic fields is given by

$$b(p) \mapsto b(p) e^{-imf_{\mathcal{H}}(p)}\,, \quad \bar{b}(p) \mapsto \bar{b}(p) e^{imf_{\mathcal{H}}(p)}\,, \quad c(p) \mapsto c(p) e^{-imf_{\mathcal{H}}(p)}\,, \quad \bar{c}(p) \mapsto \bar{c}(p) e^{imf_{\mathcal{H}}(p)}. \tag{4.49}$$

The infinitesimal action of an arbitrary Poincaré transformation on both the Maxwell and Klein-Gordon phase spaces is thus given by a linear transformation, $P$. It can be verified that $P$ satisfies eq. (2.9). We thereby obtain an observable, $F_P$, on phase space corresponding to an arbitrary infinitesimal Poincaré transformation $P$ by the formula,

$$F_P(\phi) = \frac{1}{2}\Omega(\phi, P\phi) \tag{4.50}$$

where "$\phi$" stands for a point in the Cartesian product of the Maxwell and Klein-Gordon phase spaces. For a given choice of origin in the bulk spacetime, we can write an arbitrary infinitesimal Poincaré transformation as $P = T + X$ where $T$ is a translation and $X$ is a Lorentz transformation. We denote the corresponding observables as $F_T$ and $F_X$. The

32. One can also represent BMS supertranslations at timelike infinity by considering solutions $f_{\mathcal{H}}(p)$ to eq. (4.47) on $\mathcal{H}^-$ with the boundary value $f(x^A)$ now being allowed to be any smooth function on $\mathbb{S}^2$.

Poisson bracket of these observables with the local asymptotic massive scalar field observables are given by

$$\{F_T, b(w)\} = ib(f_{\mathcal{H}} w)\,, \quad \{F_X, b(w)\} = b(\pounds_X w) \tag{4.51}$$

where $\pounds_X w$ is the Lie derivative of the complex test function $w(p)$ with respect to the Killing vector field on $\mathcal{H}$ representing the Lorentz transformation $X$. Analogous formulae also hold for the observable $c(w)$. The Poisson bracket of the Poincaré obervables with the local electromagnetic field observables on $\mathscr{I}^-$ are given by

$$\{F_T, E(s)\} = E(\pounds_T s)\,, \quad \{F_X, E(s)\} = E(\pounds_X s) \tag{4.52}$$

where, now, the Poincaré translation $T$ is represented by the vector field $fn^\mu$ on $\mathscr{I}^-$ with $f$ supported on the $\ell = 0, 1$ spherical harmonics and the Lorentz transformation is a conformal Killing vector field $X^A$ on $\mathbb{S}^2$. Further, the Poisson brackets of the Poincaré observables with memory and charges are given by

$$\{F_T, \Delta(\lambda)\} = 0\,, \quad \{F_T, \mathcal{Q}_{i^-/i^0}(\lambda)\} = 0 \tag{4.53a}$$

$$\{F_X, \Delta(\lambda)\} = \Delta(\pounds_X \lambda)\,, \quad \{F_X, \mathcal{Q}_{i^-/i^0}(\lambda)\} = \mathcal{Q}_{i^-/i^0}(\pounds_X \lambda). \tag{4.53b}$$

Finally the brackets of the Poincaré generators with themselves are

$$\{F_{T_1}, F_{T_2}\} = 0\,, \quad \{F_{X_1}, F_{X_2}\} = F_{[X_1, X_2]} \tag{4.54a}$$

$$\{F_X, F_T\} = F_{T'} \tag{4.54b}$$

where in the above, if $T$ is represented by a function $f(x^A)$ then $T'$ is represented by $f' = \pounds_X f - 1/2(\mathscr{D}_A X^A) f$. It should be noted that eq. (4.53b) shows that memory is not Lorentz invariant unless it vanishes. Similarly, the charges at timelike and spatial infinity are not Lorentz invariant unless all of the charges (including the ordinary total electric charge) vanish. In addition, the nonvanishing of the Poisson brackets of $F_X$ with $\mathcal{Q}_{i0}$ shows that the observables $F_X$ are not gauge invariant unless all of the charges vanish, as expected from the considerations given in [120].

The extended algebra $\mathscr{A}_{\text{in,QP}}$ is now obtained by adding Hermitian elements $\boldsymbol{F}_P$ for each Poincaré generator $P$ to $\mathscr{A}_{\text{in,Q}}$ and factoring by commutation relations corresponding to all of the above Poisson bracket relations.

## *3 Fock representations*

In the previous sections, we have constructed the asymptotic local field algebra $\mathscr{A}_{\text{in}}$ and we have extended it to the algebras $\mathscr{A}_{\text{in,Q}}$ and $\mathscr{A}_{\text{in,QP}}$ that include large gauge charges and Poincaré generators. The ordinary Minkowski vacuum state, $\omega_0 := \omega_0^{\text{KG}} \otimes \omega_0^{\text{EM}}$, is the Gaussian state on $\mathscr{A}_{\text{in}}$ with vanishing 1-point function and with 2-point function given by eqs. (4.10) and (4.20) with $S_{AB} = B = C = 0$. We can extend its action to $\mathscr{A}_{\text{in,QP}}$ such that for all $a \in \mathscr{A}_{\text{in,QP}}$ we have $\omega_0(a\boldsymbol{\Delta}) = \omega_0(\boldsymbol{\Delta}a) = \omega_0(a\boldsymbol{\mathcal{Q}}_{i^-}) = \omega_0(\boldsymbol{\mathcal{Q}}_{i^-}a) = \omega_0(a\boldsymbol{F}_P) = \omega_0(\boldsymbol{F}_P a) = 0$, i.e., such that $\omega_0$ is an eigenstate with eigenvalue zero of all the charges and Poincaré generators. The GNS representation of $\omega_0$ will yield the usual Fock space of incoming particle states. However, all states in this Fock space will be eigenstates of memory with vanishing eigenvalue. The corresponding construction of an "out" Hilbert space will similarly contain only states with vanishing memory. Consequently, as discussed at length in sec. 1, this choice of Hilbert space is not adequate for scattering theory, since it does not contain the states with memory that arise from the scattering processes.

The purpose of this subsection is to construct a large supply of states — including states with memory — that later can be reassembled into a Hilbert space satisfying properties (1)–(5) of sec. 1. We will do so by constructing the ordinary Fock representation of the asymptotic Klein-Gordon scalar field and the asymptotic electromagnetic field. We will then construct corresponding "memory representations" of the electromagnetic field by shifting the electromagnetic field by the identity multiplied by a classical electromagnetic field with the desired memory. We will thereby obtain states of the electromagnetic field with arbitrary memory.

## 1 Fock representation of the massive field algebra

In this subsection, we construct the standard Fock representation of the asymptotic Klein-Gordon scalar field. This will give us an ample supply of incoming states of the Klein-Gordon field.

The vacuum state on the extended asymptotic Klein-Gordon algebra $\mathscr{A}^{\mathrm{KG}}_{\mathrm{in,QP}}$ is the Gaussian algebraic state $\omega_0^{\mathrm{KG}}$ with vanishing 1-point function and 2-point function given by (see eq. (4.20))

$$\omega_0^{\mathrm{KG}}(\boldsymbol{b}(w_1)\boldsymbol{b}(w_2)^*) = \frac{4(2\pi)^3}{m^2}\langle w_1, w_2\rangle_{\mathcal{H}} \ , \quad \omega_0^{\mathrm{KG}}(\boldsymbol{c}(w_1)\boldsymbol{c}(w_2)^*) = \frac{4(2\pi)^3}{m^2}\langle w_2, w_1\rangle_{\mathcal{H}} \quad (4.55)$$

for all test functions $w_1(p), w_2(p)$, where $\langle\,,\rangle_{\mathcal{H}}$ is the $L^2$ inner product on $\mathcal{H}^-$. Furthermore, $\omega_0^{\mathrm{KG}}$ is an eigenstate of eigenvalue zero of $\boldsymbol{\mathcal{Q}}_{i^-}(\lambda)$ and all of the Poincaré generators.

The GNS construction for $\omega_0^{\mathrm{KG}}$ yields a Hilbert space $\mathscr{F}^{\mathrm{KG}}$ with a natural Fock space structure. Concretely this construction is obtained as follows. On the space of complex test functions on $\mathcal{H}^-$ we define the inner product

$$\langle w_1|w_2\rangle := \omega_0^{\mathrm{KG}}(\boldsymbol{b}(w_1)\boldsymbol{b}(w_2)^*) = \frac{4(2\pi)^3}{m^2}\langle w_1, w_2\rangle_{\mathcal{H}} \quad (4.56)$$

using the 2-point function in eq. (4.55). This is a non-degenerate, positive, Hermitian inner product. Let $\mathscr{H}^{\mathrm{KG}}$ be the completion of the space of test functions in this inner product, i.e.,

$\mathscr{H}^{\text{KG}}$ is the Hilbert space of square-integrable functions (i.e. wave packets) of the timelike momentum $p$ represented as points on $\mathcal{H}^-$. This Hilbert space serves as the "one particle" Hilbert space for particles in the Fock space. The Fock space of particles is given by

$$\mathscr{F}^{\text{KG}}_{\text{particles}} = \mathbb{C} \oplus \bigoplus_{n\geq 1} \underbrace{\left(\mathscr{H}^{\text{KG}} \otimes_S \cdots \otimes_S \mathscr{H}^{\text{KG}}\right)}_{n \text{ times}} . \tag{4.57}$$

where $\otimes_S$ is the symmetrized tensor product. On this Fock space the $\boldsymbol{b}(w)^*$ acts as a creation operator and $\boldsymbol{b}(w)$ acts as the annihilation operator for a particle wave packet $w(p)$. An identical construction with the inner product

$$\omega_0^{\text{KG}}(\boldsymbol{c}(w_1)\boldsymbol{c}(w_2)^*) = \frac{4(2\pi)^3}{m^2} \langle w_2, w_1 \rangle_{\mathcal{H}} \tag{4.58}$$

gives the "one antiparticle" Hilbert space $\overline{\mathscr{H}}^{\text{KG}}$, and the corresponding Fock space $\mathscr{F}^{\text{KG}}_{\text{antiparticles}}$ on which $\boldsymbol{c}(w)^*$ acts as a creation operator and $\boldsymbol{c}(w)$ acts as the annihilation operator for an antiparticle wave packet $\overline{w}(p)$. Since the operators $\boldsymbol{b}(w)$ and $\boldsymbol{b}(w)^*$ commute with $\boldsymbol{c}(w)$ and $\boldsymbol{c}(w)^*$, the full Fock space representation is given by the tensor product of the particle and antiparticle Fock spaces

$$\mathscr{F}^{\text{KG}} = \mathscr{F}^{\text{KG}}_{\text{particles}} \otimes \mathscr{F}^{\text{KG}}_{\text{antiparticles}} . \tag{4.59}$$

The algebraic state $\omega_0^{\text{KG}}$ corresponds to the vacuum state of the Fock space, which we denote as $|\omega_0^{\text{KG}}\rangle \in \mathscr{F}^{\text{KG}}$. We have

$$\boldsymbol{b}(w)\,|\omega_0^{\text{KG}}\rangle = \boldsymbol{c}(w)\,|\omega_0^{\text{KG}}\rangle = \boldsymbol{\mathcal{Q}}_{i^-}(\lambda)\,|\omega_0^{\text{KG}}\rangle = 0\,, \quad \text{for all } w(p), \lambda(x^A)\,. \tag{4.60}$$

A dense set of Hadamard states in this Fock space is generated by the linear span of the vacuum $|\omega_0^{\text{KG}}\rangle$ and symmetric tensor products of the particle and antiparticle wave packet states with test functions $w(p)$.

The large gauge transformations and Poincaré transformations have a strongly continuous unitary action on $\mathscr{F}^{\mathrm{KG}}$. The vacuum state $|\omega_0^{\mathrm{KG}}\rangle$ is invariant under these transformations. The large gauge transformations and translations act on the one-particle/antiparticle spaces as multiplication by a phase. The Lorentz group acts on the one-particle/antiparticle spaces by its natural action on $L^2(\mathcal{H}^-)$. The action on the full Fock space is immediately obtained by extending the action to symmetric tensor products of the one-particle/antiparticle spaces [121].

In the construction of the Faddeev-Kulish representations (see sec. 4.4) it will be useful to work with "improper states" of definite particle/antiparticle momenta. Formally, these states correspond to applying the point-wise creation operators $\boldsymbol{b}(p)^*$ and $\boldsymbol{c}(q)^*$ to the vacuum

$$|p_1, \dots, p_n\rangle = \frac{1}{\sqrt{n!}}\boldsymbol{b}(p_1)^* \dots \boldsymbol{b}(p_n)^* |\omega_0^{\mathrm{KG}}\rangle \text{ and } |q_1, \dots, q_m\rangle = \frac{1}{\sqrt{m!}}\boldsymbol{c}(q_1)^* \dots \boldsymbol{c}(q_m)^* |\omega_0^{\mathrm{KG}}\rangle \tag{4.61}$$

where $p, q \in \mathcal{H}^-$ correspond to the momenta of particles and antiparticles respectively. Although eq. (4.61) is well-defined if we smear with test functions in all variables, the definite momentum states $|p_1, \dots, p_n\rangle$ and $|q_1, \dots, q_m\rangle$ themselves have infinite norm and are not genuine states in $\mathscr{F}^{\mathrm{KG}}$. However, we can make mathematical sense of these improper states and their relationship to $\mathscr{F}^{\mathrm{KG}}$ in a precise way as follows. Let $\mathscr{H}_p^{\mathrm{KG}} \cong \mathbb{C}$, be the one-dimensional complex Hilbert spaces spanned by the symbol $|p\rangle$. Thus, $|p\rangle$ is a genuine state in $\mathscr{H}_p^{\mathrm{KG}}$. Similarly, for all $p_1, \dots, p_n$ we define the one-dimensional complex Hilbert space $\mathscr{H}_{p_1 \cdots p_n}^{\mathrm{KG}}$ by

$$\mathscr{H}_{p_1 \cdots p_n}^{\mathrm{KG}} = \mathscr{H}_{p_1}^{\mathrm{KG}} \otimes_S \cdots \otimes_S \mathscr{H}_{p_n}^{\mathrm{KG}} \tag{4.62}$$

where the symmetric tensor product symbol indicates here that we identify the Hilbert spaces that differ by a permutation of $p_1, \dots, p_n$. Then $\mathscr{H}_{p_1 \cdots p_n}^{\mathrm{KG}}$ is spanned by $|p_1, \dots, p_n\rangle := |p_1\rangle \otimes_S \cdots \otimes_S |p_n\rangle$. We similarly define the $n$-antiparticle Hilbert spaces of definite momenta. The Fock space $\mathscr{F}^{\mathrm{KG}}$ is then given by the direct sum of the direct integral of these Hilbert spaces

$$\mathscr{F}^{\rm KG} \cong \bigoplus_{n,m\geq 0} \int_{\mathcal{H}^{n+m}} d^3p_1 \dots d^3p_n d^3q_1 \dots d^3q_m \; \mathscr{H}^{\rm KG}_{p_1\cdots p_n} \otimes \overline{\mathscr{H}}^{\rm KG}_{q_1\cdots q_m} \tag{4.63}$$

where $d^3p$ and $d^3q$ denote the Lorentz invariant measure eq. (4.13) on the hyperboloid. States in the Fock space are thus given by expressions of the form

$$|\Psi\rangle = \sum_{n,m} \int_{\mathcal{H}^{n+m}} d^3p_1 \dots d^3p_n d^3q_1 \dots d^3q_m \; \psi(p_1, \dots p_n, q_1, \dots, q_m) \, |p_1 \dots p_n\rangle \otimes |q_1 \dots q_m\rangle \tag{4.64}$$

where $\psi$ is a complex $L^2$-function of $p_i$ and $q_i$ that is invariant under permutations of the $p_i$ and permutations of the $q_i$. The quantities $|p_1 \dots p_n\rangle$ and $|q_1 \dots q_m\rangle$ appearing in this equation are the mathematically well-defined basis elements of $\mathscr{H}^{\rm KG}_{p_1\cdots p_n}$ and $\overline{\mathscr{H}}^{\rm KG}_{q_1\cdots q_m}$.

The action of the charge operator $\boldsymbol{\mathcal{Q}}_{i^-}(\lambda)$ on $\mathscr{F}^{\rm KG}$ can be expressed very conveniently in this representation of the Fock space, since the direct integral decomposition eq. (4.63) corresponds to the spectral decomposition of the operator $\boldsymbol{\mathcal{Q}}_{i^-}(\lambda)$. Formally, the states $|p_1, \dots, p_n\rangle$ and $|q_1, \dots, q_m\rangle$ are eigenstates of $\boldsymbol{\mathcal{Q}}_{i^-}(\lambda)$ with eigenvalues given by

$$\begin{aligned} \boldsymbol{\mathcal{Q}}_{i^-}(\lambda) \, |p_1, \dots, p_n\rangle &= q\left(\sum_{i=1}^{n} \lambda_{\mathcal{H}}(p_i)\right) |p_1, \dots, p_n\rangle \\ \boldsymbol{\mathcal{Q}}_{i^-}(\lambda) \, |q_1, \dots, q_n\rangle &= -q\left(\sum_{i=1}^{n} \lambda_{\mathcal{H}}(q_i)\right) |q_1, \dots, q_n\rangle \end{aligned} \tag{4.65}$$

where $\lambda_{\mathcal{H}}$ is given by eq. (4.22). Note that the unsmeared version of eq. (4.65) is

$$\begin{aligned} \boldsymbol{\mathcal{Q}}_{i^-}(x^A) \, |p_1, \dots, p_n\rangle &= q\left(\sum_{i=1}^{n} G_{\mathcal{H}}(p_i, x^A)\right) |p_1, \dots, p_n\rangle \\ \boldsymbol{\mathcal{Q}}_{i^-}(x^A) \, |q_1, \dots, q_n\rangle &= -q\left(\sum_{i=1}^{n} G_{\mathcal{H}}(q_i, x^A)\right) |q_1, \dots, q_n\rangle \, . \end{aligned} \tag{4.66}$$

where $G_{\mathcal{H}}(p, x^A) = (4\pi)^{-1}(\sqrt{1+\rho^2} - \rho\hat{p}\cdot\hat{r})^{-2}$ (see eq. (4.22)). For a state $|\Psi\rangle$ in the Fock space lying in the subspace of $n$ particles and $m$ antiparticles, the action of the charge operator is given by

$$
\begin{aligned}
\boldsymbol{\mathcal{Q}}_{i^-}(\lambda)\,|\Psi\rangle = q \int\limits_{\mathcal{H}^{n+m}} & d^3p_1 \dots d^3p_n d^3q_1 \dots d^3q_m \;\psi(p_1, \dots p_n, q_1, \dots, q_m) \times \\
& \left(\sum_{i=1}^{n} \lambda_{\mathcal{H}}(p_i) - \sum_{i=1}^{m} \lambda_{\mathcal{H}}(q_i)\right) |p_1 \dots p_n\rangle \otimes |q_1 \dots q_m\rangle \,.
\end{aligned} \tag{4.67}
$$

All states in this subspace are eigenstates of the total charge operator $\boldsymbol{\mathcal{Q}}_{i^-}(1)$

$$
\boldsymbol{\mathcal{Q}}_{i^-}(1)\,|\Psi\rangle = q(n-m)\,|\Psi\rangle \;. \tag{4.68}
$$

However, for non-constant $\lambda$, there are no proper eigenstates of $\boldsymbol{\mathcal{Q}}_{i^-}(\lambda)$ apart from the vacuum state.

## 2 Fock representations of the Maxwell field algebra

The vacuum state on the extended asymptotic algebra $\mathscr{A}^{\mathrm{EM}}_{\mathrm{in,QP}}$ of the electromagnetic field is the Gaussian algebraic state $\omega_0^{\mathrm{EM}}$ with vanishing 1-point function and 2-point function given by (see eq. (4.10))

$$
\omega_0^{\mathrm{EM}}(\boldsymbol{E}(s_1)\boldsymbol{E}(s_2)) = -\int\limits_{\mathbb{R}^2\times\mathbb{S}^2} dv_1 dv_2 d\Omega\; \frac{q_{AB}s_1^A(v_1, x^A)s_2^B(v_2, x^A)}{(v_1 - v_2 - i0^+)^2}. \tag{4.69}
$$

Furthermore, $\omega_0^{\mathrm{EM}}$ is an eigenstate of eigenvalue zero of memory, $\boldsymbol{\Delta}(\lambda)$, and all of the Poincaré generators.

The GNS construction for $\omega_0^{\mathrm{EM}}$ yields a Hilbert space $\mathscr{F}_0^{\mathrm{EM}}$ with a natural Fock space structure. Concretely this construction is obtained as follows. On the space of positive frequency Schwartz test functions on $\mathscr{I}^-$ we define the inner product

$$\langle s_1|s_2\rangle_0 := \omega_0^{\mathrm{EM}}(\boldsymbol{E}(s_1)^*\boldsymbol{E}(s_2)) = 2\pi\int_0^\infty \omega d\omega \int_{\mathbb{S}^2} d\Omega\ \overline{\hat{s}_1^A(\omega, x^A)}\hat{s}_{A,2}(\omega, x^A) \tag{4.70}$$

where the "hat" denotes the Fourier transform, and the final equality above is the Fourier space representation of eq. (4.69). We define the one-particle Hilbert space, $\mathscr{H}_0^{\mathrm{EM}}$, to be the completion of the space of positive frequency test functions in this inner product. The GNS Fock space associated with the vacuum state is then given by

$$\mathscr{F}_0^{\mathrm{EM}} = \mathbb{C} \oplus \bigoplus_{n\geq 1} \underbrace{\left(\mathscr{H}_0^{\mathrm{EM}} \otimes_S \cdots \otimes_S \mathscr{H}_0^{\mathrm{EM}}\right)}_{n \text{ times}}. \tag{4.71}$$

For real $s^A$, the smeared electric field operator $\boldsymbol{E}_0(s) := \pi_0^{\mathrm{EM}}[\boldsymbol{E}(s)]$ in this representation is given by

$$\boldsymbol{E}_0(s) = \boldsymbol{a}_0(s^-) + \boldsymbol{a}_0^\dagger(s^+) \tag{4.72}$$

where $\boldsymbol{a}_0, \boldsymbol{a}_0^\dagger$ are the usual annihilation and creation operators on the Fock space and superscripts "$\pm$" denote the positive/negative frequency parts, respectively. There is a dense subspace of Hadamard states given by the span of the vacuum $|\omega_0^{\mathrm{EM}}\rangle$ and any finite products of $\boldsymbol{a}_0^\dagger(s_i^+)$ applied to the vacuum, with $s_i$ an arbitrary test function. The Poincaré transformations act on $\mathscr{H}_0^{\mathrm{EM}}$ by their natural action on $\mathscr{I}^-$, which gives rise to a strongly continuous action on $\mathscr{F}_0^{\mathrm{EM}}$. The vacuum state, $|\omega_0^{\mathrm{EM}}\rangle$, is invariant under these transformations.

The Fock space $\mathscr{F}_0^{\mathrm{EM}}$ constructed above is the usual choice of Hilbert space for the "in" radiative states of the Maxwell fields. However, all states in this Hilbert space are eigenstates of memory, $\boldsymbol{\Delta}(\lambda)$, with eigenvalue zero — as follows immediately from the fact that the vacuum is an eigenstate of $\boldsymbol{\Delta}(\lambda)$ with eigenvalue zero, and $\boldsymbol{\Delta}(\lambda)$ commutes with $\boldsymbol{E}(s)$ and hence with $\boldsymbol{a}_0^\dagger(s^+)$. However, even in the classical theory, memory is not conserved between "in" and "out" states in generic scattering processes. Thus, even if we restrict to the zero-memory Fock space $\mathscr{F}_0^{\mathrm{EM}}$ for the Maxwell in-states, the out states obtained will not have zero memory and hence will not live in the zero memory out-Fock space $\mathscr{F}_0^{\mathrm{EM}}$. One is thus forced to consider states which have non-vanishing memory to describe scattering. Thus, the states in $\mathscr{F}_0^{\mathrm{EM}}$ do *not* give us an ample supply of states to use in scattering theory.

However, we can construct different Fock representations containing states with nonvanishing memory as follows (see [3]). Choose a smooth classical electric field $e_A(x)$ on $\mathscr{I}^-$ that satisfies our decay conditions but is such that the corresponding classical memory

$$\Delta(e, \lambda) = -\int_{\mathscr{I}^-} dv d\Omega \; e_A(v, x^B)\mathscr{D}^A\lambda(x^B) \tag{4.73}$$

is non-vanishing. Consider the algebra automorphism $\mathfrak{a}_e : \mathscr{A}_{\mathrm{in,Q}}^{\mathrm{EM}} \to \mathscr{A}_{\mathrm{in,Q}}^{\mathrm{EM}}$ determined by

$$\mathfrak{a}_e[\boldsymbol{E}(s)] = \boldsymbol{E}(s) + e(s)\mathbf{1}\,, \quad \mathfrak{a}_e[\boldsymbol{\Delta}(\lambda)] = \boldsymbol{\Delta}(\lambda) + \Delta(e,\lambda)\mathbf{1}. \tag{4.74}$$

This is easily seen to define an automorphism, since the commutation relations are unaffected by shifting the operators by a multiple of $\mathbf{1}$. Using this automorphism we define a new algebraic state $\omega_e^{\mathrm{EM}}$ by

$$\omega_e^{\mathrm{EM}}(\boldsymbol{O}) := \omega_0^{\mathrm{EM}}(\mathfrak{a}_e[\boldsymbol{O}]) \quad \text{ for all } \boldsymbol{O} \in \mathscr{A}_{\mathrm{in,Q}}^{\mathrm{EM}}. \tag{4.75}$$

Then $\omega_e^{\mathrm{EM}}$ is a Gaussian, Hadamard state on $\mathscr{A}_{\mathrm{in,Q}}^{\mathrm{EM}}$ that satisfies eq. (4.45) and, for each $\lambda$, is an eigenstate of $\boldsymbol{\Delta}(\lambda)$ with eigenvalue $\Delta(e, \lambda)$. The GNS construction for $\omega_e^{\mathrm{EM}}$ yields a Hilbert space $\mathscr{F}_e^{\mathrm{EM}}$ with a Fock space structure and a vacuum state $|\omega_e^{\mathrm{EM}}\rangle$ corresponding to $\omega_e^{\mathrm{EM}}$. Every state in $\mathscr{F}_e^{\mathrm{EM}}$ is an eigenstate of $\boldsymbol{\Delta}(\lambda)$ with eigenvalue $\Delta(e, \lambda)$. Thus, this construction — for the various different choices of classical electric field $e_A$ — gives an ample supply of states with any desired memory.

It should be noted that if $e_A$ and $e'_A$ are smooth and satisfy our decay conditions, then the

Fock representations obtained by the above GNS construction will be unitarily equivalent[33] if and only if $\Delta(e,\lambda) = \Delta(e',\lambda)$ for all $\lambda$. Thus, there are as many unitarily inequivalent constructions as there are choices of memory one-form $\Delta_A(x^A)$ on $\mathbb{S}^2$. In particular, there are uncountably many such constructions. If $e_A$ and $e'_A$ are such that $\Delta(e,\lambda) = \Delta(e',\lambda)$ (so that they give rise to unitarily equivalent representations), then the state $\omega^{\text{EM}}_{e'}$ — which corresponds the the vacuum state in $\mathscr{F}^{\text{EM}}_{e'}$ — corresponds in $\mathscr{F}^{\text{EM}}_{e}$ to the coherent state associated with the classical solution $e'_A - e_A$. Thus, the representations with nonvanishing memory do not have a "preferred" vacuum state, i.e., the vacuum state of the Fock representation depends on the choice of representative classical electric field $e_A$. Nevertheless, the unitary equivalence class of the Fock representations $\mathscr{F}^{\text{EM}}_{e}$ correspond to all smooth $e_A$ with memory $\Delta_A$. Thus, the "memory representations" can be labeled by the memory of the representative — i.e., as $\mathscr{F}^{\text{EM}}_{\Delta}$ rather than $\mathscr{F}^{\text{EM}}_{e}$ — and we shall do so in the following.

It also should be noted that for any given choice of memory $\Delta_A(x^A)$ on $\mathbb{S}^2$ and any given choice of frequency $\omega_0 > 0$ one can find a representative classical electric field $e_A(v, x^B)$ with memory equal to $\Delta_A(x^A)$ such that the Fourier transform of $e_A$ is nonvanishing only for frequencies $\omega < \omega_0$. Thus, the states in $\mathscr{F}^{\text{EM}}_{\Delta}$ can be viewed as differing from the states in $\mathscr{F}^{\text{EM}}_{0}$ only in the (arbitrarily) far infrared. However, for $\Delta_A(x^A) \neq 0$, if one tries to formally express a normalized state in $\mathscr{F}^{\text{EM}}_{\Delta}$ as a state in $\mathscr{F}^{\text{EM}}_{0}$, one will find that it has infinitely many "soft photons" and cannot be normalized. Thus, the states in $\mathscr{F}^{\text{EM}}_{\Delta}$ are genuinely different from states in $\mathscr{F}^{\text{EM}}_{0}$.

The above construction yields representations of $\mathscr{A}^{\text{EM}}_{\text{in,Q}}$ with any desired memory. We now consider whether these representations can be extended to representations of $\mathscr{A}^{\text{EM}}_{\text{in,QP}}$, i.e., whether one can define an action of Poincaré generators on $\mathscr{F}^{\text{EM}}_{\Delta}$ such that the commutation

33. However, if $e_A$ and $e'_A$ are not smooth, the norm (defined in eq. (4.70)) of the positive frequency part of $e_A - e'_A$ need not be finite even when they have the same memory. In that case, the Fock space constructions will not be unitarily equivalent. This point will be relevant to the considerations of sec. 5.4.

relations corresponding to eqs. (4.51)–(4.54b) hold. Consider, first, a translation. The natural action of a finite translation on the classical electric field $e_A$ maps it into an electric field $e'_A$ with the same memory. As noted above, the representation obtained from $e'_A$ is therefore unitarily equivalent to the representation obtained from $e_A$. It follows that the natural action of finite translations can be represented by a unitary map on $\mathscr{F}^{\mathrm{EM}}_{\Delta}$. This map is strongly continuous in the translation parameter, so we get a self-adjoint operator on $\mathscr{F}^{\mathrm{EM}}_{\Delta}$ representing an arbitrary translation generator $T$, which satisfies all of the required commutation relations.[34] Thus, the above Fock representations of $\mathscr{A}^{\mathrm{EM}}_{\mathrm{in,Q}}$ with nonvanishing memory can be extended to include Poincaré translations.

However, the Fock representations $\mathscr{F}^{\mathrm{EM}}_{\Delta}$ with nonvanishing memory *cannot* be extended to include the action of the Lorentz generators [122, 3]. As noted above, for all $\lambda$ all states in $\mathscr{F}^{\mathrm{EM}}_{\Delta}$ are eigenstates of $\boldsymbol{\Delta}(\lambda)$ with eigenvalue $\Delta(e, \lambda)$. Thus, for all $\lambda$, the memory operator commutes with all operators on $\mathscr{F}^{\mathrm{EM}}_{\Delta}$. However, if $\Delta(e, \lambda)$ is nonvanishing for some $\lambda$, then by eq. (4.53b) some Lorentz generator $X$ must have a nonvanishing commutator with $\boldsymbol{\Delta}(\lambda)$. Thus, the above Fock representations of $\mathscr{A}^{\mathrm{EM}}_{\mathrm{in,Q}}$ with nonvanishing memory *cannot* be extended to representations of $\mathscr{A}^{\mathrm{EM}}_{\mathrm{in,QP}}$.

## 4 *Faddeev-Kulish representation*

We turn now to the issue of whether we can find Hilbert spaces of incoming and outgoing states that satisfy properties (1)–(5) listed in sec. 1. The standard choice of "in" Hilbert space $\mathscr{F}_{\mathrm{in}} = \mathscr{F}^{\mathrm{KG}} \otimes \mathscr{F}^{\mathrm{EM}}_{0}$ and correspondingly constructed standard "out" Hilbert space $\mathscr{F}_{\mathrm{out}}$ of the "out" algebra does not work, since all states in $\mathscr{F}_{\mathrm{in}}$ and $\mathscr{F}_{\mathrm{out}}$ have vanishing memory, but scattering takes states with vanishing memory to states with nonvanishing memory. As

34. We emphasize that the spectrum of the energy operator corresponding to time translations is bounded below by zero but does not achieve the value zero for *any* state with memory.

we shall discuss further in sec. 7, one could attempt to allow memory by replacing $\mathscr{F}_0^{\mathrm{EM}}$ with a direct sum, $\oplus_\Delta \mathscr{F}_\Delta^{\mathrm{EM}}$, over all unitarily inequivalent memory Fock spaces. However, since there are uncountably many such memory Fock spaces, this would give a non-separable Hilbert space, in violation of property (5). Furthermore, each state in such a direct sum would have a nonvanishing probability for only a countable number of discrete values of memory. However, scattering with an "in" state of this sort surely does not produce an "out" state of this sort, so property (4) also will not be satisfied by this choice of the "in" and "out" Hilbert spaces. Finally, by eq. (4.53b), since the memory is not Lorentz invariant there cannot be continuous action of Lorentz on the direct sum — in violation of property (3) — so the angular momentum cannot be defined. A more promising possibility would be to take some sort of direct integral of memory representation Hilbert spaces. However, as we shall discuss further in sec. 7, the natural Lorentz invariant Gaussian measure on memory has support on memories that are too singular to be admissible, and there does not appear to any other choices of measure for a direct integral construction that have the prospect of satisfying properties (3) or (4).

Nevertheless, it is possible to give a construction, due to Faddeev and Kulish [32], of "in" and "out" Hilbert spaces that satisfy (1)–(5). The construction involves taking a direct integral over the memory Fock spaces of the electromagnetic field but correlating these Fock spaces with (improper) momentum eigenstates of the massive Klein-Gordon field so as to produce states with vanishing charges $\mathcal{Q}_{i^0}(\lambda)$ at spatial infinity. This is a useful construction because of the fact that, as shown in [23, 24, 33, 25], for solutions to the Maxwell equations that are suitably regular at spatial infinity, the charges $\mathcal{Q}_{i^0}^{\mathrm{in}}(\lambda)$ obtained from the limit along past null infinity are matched antipodally to the similarly defined charges $\mathcal{Q}_{i^0}^{\mathrm{out}}(\lambda)$ obtained from the limit along future null infinity,

$$\mathcal{Q}_{i^0}^{\mathrm{in}}(\lambda) = \mathcal{Q}_{i^0}^{\mathrm{out}}(\lambda \circ \Upsilon) \tag{4.76}$$

where $\Upsilon$ is the antipodal map on $\mathbb{S}^2$. Thus, any "in" state that is an eigenstate of $\mathcal{Q}^{\text{in}}_{i^0}(\lambda)$ for all $\lambda$ should evolve to an "out" state that is an eigenstate of $\mathcal{Q}^{\text{out}}_{i^0}(\lambda \circ \Upsilon)$ of the same eigenvalue. However, since by eq. (4.53b) the Lorentz group generators have nontrivial commutators with the charges at spatial infinity, the Lorentz group generators cannot act on a Hilbert space of states of definite charges *except* in the case where all of the charges vanish, $\mathcal{Q}_{i^0}(\lambda) = 0$ for all $\lambda$ [85, 84, 86, 34]. Therefore, we seek to construct "in" and "out" Hilbert spaces composed of states that are eigenstates of eigenvalue zero of all of the large gauge charges (including the total electric charge) at spatial infinity.

To construct an "in" Hilbert space with $\mathcal{Q}_{i^0}(\lambda) = 0$ for all $\lambda$, we make use of the relation

$$\boldsymbol{\mathcal{Q}}_{i^0}(\lambda) = \boldsymbol{\mathcal{Q}}_{i^-}(\lambda) + \frac{1}{4\pi}\boldsymbol{\Delta}(\lambda) \tag{4.77}$$

(see eq. (4.33)). We start with the one-dimensional Hilbert space $\mathscr{H}^{\text{KG}}_{p_1\dots p_n} \otimes \overline{\mathscr{H}}^{\text{KG}}_{q_1\dots q_n}$ of $n$ incoming particles and $n$ incoming antiparticles in momentum states $p_1, \dots, p_n$ and $q_1, \dots, q_n$, respectively (see eq. (4.62)). This state has vanishing total electric charge and large gauge charges

$$\mathcal{Q}_{i^-}(\lambda) = q\sum_{i=1}^{n}\left(\lambda_{\mathcal{H}}(p_i) - \lambda_{\mathcal{H}}(q_i)\right) \tag{4.78}$$

(see eq. (4.65)), with $\lambda_{\mathcal{H}}(p)$ given by eq. (4.22). Therefore, we can obtain a state with $\boldsymbol{\mathcal{Q}}_{i^0}(\lambda) = 0$ for all $\lambda$ if we can find a memory representation $\mathscr{F}^{\text{EM}}_{\Delta}$ such that for all $\lambda$ we have

$$\Delta(\lambda; p_1, \dots, q_n) = -4\pi q\sum_{i=1}^{n}\left(\lambda_{\mathcal{H}}(p_i) - \lambda_{\mathcal{H}}(q_i)\right). \tag{4.79}$$

This will be the case if for any $p$ we can solve

$$\mathscr{D}^A\Delta_A(x^A; p) = 4\pi q(G_{\mathcal{H}}(p, x^A) - 1) \tag{4.80}$$

with $G_{\mathcal{H}}$ given by eq. (4.22). If we can solve eq. (4.80), then the memory representation

$\mathscr{F}^{\mathrm{EM}}_{\Delta}$ obtained from any $e_A$ such that

$$-\int_{-\infty}^{\infty} dv\ e_A(v, x^A) = \sum_{i=1}^{n} \Big(\Delta_A(x^A; p_i) - \Delta_A(x^A; q_i)\Big) \tag{4.81}$$

will have memory satisfying eq. (4.79).

We can decompose any one-form on $\mathbb{S}^2$ such as $\Delta_A$ into its electric and magnetic parts as

$$\Delta_A = \mathscr{D}_A \alpha + \epsilon_A{}^B \mathscr{D}_B \beta \tag{4.82}$$

The magnetic part[35] will not contribute to $\mathscr{D}^A \Delta_A$, so eq. (4.80) becomes

$$\mathscr{D}^2 \alpha = 4\pi q (G_{\mathcal{H}}(p, x^A) - 1) \tag{4.83}$$

By eq. (4.23), the right side is orthogonal to the $\ell = 0$ spherical harmonic, so this equation can be uniquely solved. Since $G_{\mathcal{H}}(p, x^A)$ is smooth, it follows that $\Delta_A(x^A; p)$ is smooth and, hence, $e_A(v, x^A)$ can be chosen to be smooth.

We now have all of the ingredients needed for the Faddeev-Kulish construction. As described in the previous subsection, the standard Fock space $\mathscr{F}^{\mathrm{KG}}$ for the Klein-Gordon field can be obtained by taking a direct sum of direct integrals of the one-dimensional Hilbert spaces $\mathscr{H}^{\mathrm{KG}}_{p_1 \cdots p_n} \otimes \overline{\mathscr{H}}^{\mathrm{KG}}_{q_1 \cdots q_m}$ of momentum eigenstates (see eq. (4.63)). As stated above, the standard "in" Hilbert space is then obtained by taking the tensor product of this Klein-Gordon Fock space with the standard (zero memory) Fock space $\mathscr{F}^{\mathrm{EM}}_0$ for the electromagnetic field. The Faddeev-Kulish construction modifies this procedure as follows. Prior to taking the direct integral, we pair the state $|p_1 \dots p_n\rangle \otimes |q_1 \dots q_n\rangle$ with the Fock representation $\mathscr{F}^{\mathrm{EM}}_{\Delta(p_1,\dots,q_n)}$ obtained from an electric field $e_A$ on $\mathscr{I}^-$ satisfying eq. (4.81).

35. As previously explained (see footnote 5), we restrict consideration in any case to purely electric parity memory.

All electromagnetic states in this representation have memory given by eq. (4.79), so the states in $\mathscr{H}^{\text{KG}}_{p_1\cdots p_n} \otimes \overline{\mathscr{H}}^{\text{KG}}_{q_1\cdots q_n} \otimes \mathscr{F}^{\text{EM}}_{\Delta(p_1,\ldots,q_n)}$ have $\mathcal{Q}_{i^0}(\lambda) = 0$ for all $\lambda$. We now take the direct integral of these Hilbert spaces over $p_1, \ldots, q_n$ and the direct sum over $n$ to obtain the Faddeev-Kulish "in" Hilbert space

$$\mathscr{H}^{\text{FK}}_{\text{in}} := \bigoplus_{n=0}^{\infty} \int_{\mathcal{H}^{2n}} d^3p_1 \ldots d^3p_n d^3q_1 \ldots d^3q_n \; \mathscr{H}^{\text{KG}}_{p_1\cdots p_n} \otimes \overline{\mathscr{H}}^{\text{KG}}_{q_1\cdots q_n} \otimes \mathscr{F}^{\text{EM}}_{\Delta(p_1,\ldots,q_n)} \tag{4.84}$$

All states in $\mathscr{H}^{\text{FK}}_{\text{in}}$ are eigenstates of $\boldsymbol{\mathcal{Q}}^{\text{in}}_{i^0}(\lambda)$ with eigenvalue zero for all $\lambda$. The "out" Hilbert space $\mathscr{H}^{\text{FK}}_{\text{out}}$ is constructed similarly.

It should be noted that $\mathscr{H}^{\text{FK}}_{\text{in}}$ does not carry a representation of the algebra $\mathscr{A}_{\text{in,QP}}$ or even of the unextended algebra $\mathscr{A}_{\text{in}}$. The massive field operators $\boldsymbol{b}(w)$, $\boldsymbol{c}(w)$ have nontrivial commutators with with $\boldsymbol{\mathcal{Q}}_{i^0}(\lambda)$ (see eq. (4.30)) and cannot be made to act on $\mathscr{H}^{\text{FK}}_{\text{in}}$. However, all gauge invariant observables in $\mathscr{A}_{\text{in,QP}}$ commute with $\boldsymbol{\mathcal{Q}}_{i^0}(\lambda)$, and therefore $\mathscr{H}^{\text{FK}}_{\text{in}}$ carries a representation of the subalgebra of gauge invariant observables.

The Faddeev-Kulish "in" and "out" Hilbert spaces can be seen to satisfy requirements (1)–(5) of sec. 1 as follows. Requirement (1) is automatically satisfied, since $\mathscr{H}^{\text{FK}}_{\text{out}}$ is obtained by the same construction as $\mathscr{H}^{\text{FK}}_{\text{in}}$. Satisfaction of requirement (2) follows from conservation of the large gauge charges, eq. (4.76), which implies that any state in $\mathscr{H}^{\text{FK}}_{\text{in}}$ must evolve to an eigenstate of eigenvalue zero of $\boldsymbol{\mathcal{Q}}^{\text{out}}_{i^0}(\lambda)$ for all $\lambda$, which, presumably, must lie in $\mathscr{H}^{\text{FK}}_{\text{out}}$. With regard to requirement (3), the translation group acts naturally on both $\mathscr{H}^{\text{KG}}_{p_1\cdots p_n} \otimes \overline{\mathscr{H}}^{\text{KG}}_{q_1\cdots q_n}$ and $\mathscr{F}^{\text{EM}}_{\Delta}$ so there is no problem obtaining its action on $\mathscr{H}^{\text{FK}}_{\text{in}}$ [86]. A Lorentz transformation $\Lambda$ maps $\mathscr{H}^{\text{KG}}_{p_1\cdots p_n} \otimes \overline{\mathscr{H}}^{\text{KG}}_{q_1\cdots q_n}$ to $\mathscr{H}^{\text{KG}}_{\Lambda p_1\ldots\Lambda p_n} \otimes \overline{\mathscr{H}}^{\text{KG}}_{\Lambda q_1\ldots\Lambda q_n}$ and maps $\mathscr{F}^{\text{EM}}_{\Delta}$ to $\mathscr{F}^{\text{EM}}_{\Lambda\Delta}$. However, since $\Lambda[e_A(p_1 \ldots q_n)]$ defines the same memory Fock space as $e_A(\Lambda p_1 \ldots \Lambda q_n)$, there is no problem obtaining an action of the Lorentz group on $\mathscr{H}^{\text{FK}}_{\text{in}}$, so requirement (3) is satisfied [86]. Note that there would be a problem with obtaining Lorentz group action if we had similarly constructed a Hilbert space of eigenstates of $\boldsymbol{\mathcal{Q}}_{i^0}(\lambda)$ with nonvanishing eigenvalues

[85, 84]. With regard to requirement (4) since, as discussed above, $e_A$ can be chosen to be of arbitrarily low frequency, each $\mathscr{F}_\Delta^{\rm EM}$ contains representatives of any desired "hard" photon state. It is clear that $\mathscr{H}_{\rm in}^{\rm FK}$ contains states of arbitrary momenta of the charged particles and antiparticles provided that the number of particles and antiparticles are equal. As discussed in sec. 1, although this equality of particle and antiparticle number yields a genuine restriction on the allowed states, one can deal with this in the consideration of scattering by putting any extra/unwanted particles "behind the moon." Thus, arguably, requirement (4) is satisfied. Finally, it is straightforward to show that requirement (5) is satisfied.

The states in $\mathscr{H}_{\rm in}^{\rm FK}$ correspond to incoming particles/antiparticles together with incoming photons in states whose memory is highly correlated with the momenta of the particles and antiparticles. As mentioned above, we may view the states in any memory Fock space $\mathscr{F}_\Delta^{\rm EM}$ as corresponding to a state in $\mathscr{F}_0^{\rm EM}$ together with infinitely many "soft photons." Thus, we may view the memory associated with $e_A(p_1 \dots q_n)$ as "dressing" the incoming charged particle state $|p_1 \dots p_n\rangle \otimes |q_1 \dots q_n\rangle$ with a "soft photon cloud." In $\mathscr{H}_{\rm in}^{\rm FK}$, all charged particle states must be "dressed" in this manner.

The "dress requirements" imposed by $\mathscr{H}_{\rm in}^{\rm FK}$ have a number of unpleasant consequences. Most notably, one cannot consider a coherent superposition of charged particle states of different momenta, since charged particle states with different momenta are required to be dressed with soft photon clouds corresponding to different representations of the electromagnetic field. These orthogonal soft photon clouds will preclude any interference effects arising from superposing charged particle states of different momenta. Nevertheless, as we have argued above, $\mathscr{H}_{\rm in}^{\rm FK}$ contains a supply of states that is adequate for analyzing many scattering processes of interest.

# 5 QED with a massless, charged Klein-Gordon field

In this section, we consider QED with the massive charged Klein-Gordon field of sec. 4 replaced by a massless charged Klein-Gordon field. Thus, we consider the theory defined by the Lagrangian eq. (4.1) with $m = 0$. Most of the analysis carries through in close parallel with the massive case. However, as we shall see, a significant difference arises in the construction of the Faddeev-Kulish Hilbert space due to the fact that the memory representations of the electromagnetic field needed in the construction are singular. In sec. 5.5, we shall consider the source-free Yang-Mills case. In addition to the problems of massless QED, a new problem arises from the fact that the "soft dressing" contributes to the Yang-Mills charge-current flux, thereby invalidating the construction of eigenstates of large gauge charges via "dressing." Although one can obtain charge eigenstates by other means, there are insufficiently many eigenstates to obtain Hilbert spaces for scattering.

## *1 Asymptotic quantization algebra*

The asymptotic quantization of the electromagnetic field was already given in sec. 4.1, so we need only give the asymptotic quantization of the massless charged Klein-Gordon field. As discussed in sec. 2, the asymptotic behavior of a massless scalar field in the asymptotic past is described by

$$\Phi(x) := \lim_{\mathscr{I}^-} \Omega^{-1}\varphi \tag{5.1}$$

(see eq. (2.18)). The symplectic form is given by

$$\Omega^{\mathrm{KG0}}_{\mathscr{I}}((\Phi_1, \overline{\Phi}_1), (\Phi_2, \overline{\Phi}_2)) = -\frac{1}{2} \int\limits_{\mathscr{I}^-} d^3x \left[\Phi_1 \partial_v \overline{\Phi}_2 + \overline{\Phi}_1 \partial_v \Phi_2 - (1 \leftrightarrow 2)\right] , \tag{5.2}$$

where the superscript "KG0" denotes that this is the symplectic form of a massless scalar field. The above symplectic form differs from eq. (2.19) only in that we are now considering a complex, rather than real, scalar field. For the same reasons as indicated below eq. (4.15), it is convenient to treat $\Phi$ and $\overline{\Phi}$ as though they were independent quantities and to take the asymptotic phase space to consist of the pairs $(\Phi, \overline{\Phi})$. Then $\Omega^{\text{KG0}}_{\mathscr{I}}$ is a complex-bilinear function of its variables. It is convenient, as in eq. (2.21) of sec. 2, to write $\Pi = \partial_v \Phi$ and $\overline{\Pi} = \partial_v \overline{\Phi}$.

In parallel with eq. (2.21), the local scalar field observables on $\mathscr{I}^-$ are

$$\Pi(s) := \int_{\mathscr{I}} d^3x \; \Pi(x) s(x) \,, \quad \overline{\Pi}(s) := \int_{\mathscr{I}} d^3x \; \overline{\Pi}(x) \overline{s}(x) \tag{5.3}$$

where $s(x)$ is a smooth complex function on $\mathscr{I}^-$ with conformal weight $-1$. Note that we take $\Pi(s)$ to be linear in $s$ while $\overline{\Pi}(s)$ is antilinear in the test function $s(x)$. The Hamiltonian vector fields for these observables are given by the pairs $(0, s)$ and $(\overline{s}, 0)$, respectively. The only nonvanishing Poisson brackets are

$$\left\{ \overline{\Pi}(s_1), \Pi(s_2) \right\} = -\Omega^{\text{KG0}}_{\mathscr{I}}((\overline{s}_1, 0), (0, s_2)) 1 = \frac{1}{2} \int_{\mathscr{I}^-} d^3x \left[ \overline{s}_1 \partial_v s_2 - s_2 \partial_v \overline{s}_1 \right] . \tag{5.4}$$

The additional factor of 2 in the above formula relative to eq. (2.22) arises because we are now working with a complex scalar field.

The asymptotic quantization algebra, $\mathscr{A}^{\text{KG0}}_{\text{in}}$, for the massless charged Klein-Gordon field is defined by starting with the free, unital $*$-algebra generated by $\mathbf{\Pi}(s)$, its formal adjoint $\mathbf{\Pi}(s)^*$, and the identity $\mathbf{1}$. We then factor this algebra by the linearity condition[36] (A.I) and

36. Since $\mathbf{\Pi}(s)$ is a complex scalar field, the scalar multiplication in the linearity condition (A.I) must be extended to $\mathbb{C}$.

the commutation relation

$$[\mathbf{\Pi}(s_1)^*, \mathbf{\Pi}(s_2)] = -i\Omega^{\mathrm{KG0}}_{\mathscr{I}}((\bar{s}_1, 0), (0, s_2))\mathbf{1} \tag{5.5}$$

together with vanishing commutators for $\mathbf{\Pi}(s_1)$ and $\mathbf{\Pi}(s_2)$.

The Hadamard condition on states $\omega^{\mathrm{KG0}}$ on $\mathscr{A}^{\mathrm{KG0}}_{\mathrm{in}}$ is that the 2-point function $\omega^{\mathrm{KG0}}(\mathbf{\Pi}(x_1)\mathbf{\Pi}(x_2))$ is smooth, whereas

$$\omega^{\mathrm{KG0}}(\mathbf{\Pi}(x_1)^*\mathbf{\Pi}(x_2)) = -\frac{1}{\pi}\frac{\delta_{\mathbb{S}^2}(x_1^A, x_2^A)}{(v_1 - v_2 - i0^+)^2} + P(x_1, x_2) \tag{5.6}$$

where $P$ is a (state dependent) smooth function on $\mathscr{I}^- \times \mathscr{I}^-$ with $P(x_1, x_2) = \overline{P}(x_2, x_1)$. In addition, the connected $n$-point functions for $n \neq 2$ of $\omega^{\mathrm{KG0}}$ are required to be smooth. The 2-point function of the Poincaré invariant vacuum state $\omega^{\mathrm{KG0}}_0$ is given by eq. (5.6) with $P = 0$.

Finally, we require that $P$ and all connected $n$-point functions of $\omega^{\mathrm{KG0}}$ for $n \neq 2$ decay for any set of $|v_i| \to \infty$ as $O((\sum_i v_i^2)^{-1/2-\epsilon})$ for some $\epsilon > 0$.

## *2 Extension to include charges and Poincaré generators*

We have already given the extension of $\mathscr{A}^{\mathrm{EM}}_{\mathrm{in}}$ to $\mathscr{A}^{\mathrm{EM}}_{\mathrm{in,Q}}$ and $\mathscr{A}^{\mathrm{EM}}_{\mathrm{in,QP}}$ in sec. 4.2, so we need only obtain the charge and Poincaré observables for the massless Klein-Gordon field to obtain the desired extensions of the algebra of observables for massless QED.

Classically, the action of the large gauge transformations parametrized by the smooth function $\lambda(x^A)$ on $\mathbb{S}^2$ is given by

$$\Phi(x) \to e^{iq\lambda}\Phi(x) \tag{5.7}$$

where $q$ is the charge of the Klein-Gordon field. The observables $\Pi$ and $\overline{\Pi}$ transform as

$$\Pi(s) \mapsto \Pi(e^{iq\lambda}s)\,, \quad \overline{\Pi}(s) \mapsto \overline{\Pi}(e^{iq\lambda}s). \tag{5.8}$$

The vector field field on the asymptotic Klein-Gordon phase space associated with infinitesimal gauge transformation is thus $iq\lambda(-\Phi, \overline{\Phi})$. This is the Hamiltonian vector field of the observable

$$\mathcal{J}(\lambda) = -\frac{iq}{2}\int_{\mathscr{I}^-} d^3x\ \lambda(x^A)\left[\Phi(x)\overline{\Pi}(x) - \overline{\Phi}(x)\Pi(x)\right]. \tag{5.9}$$

Thus, $\mathcal{J}(\lambda)$ is the infinitesimal generator of the gauge transformations eq. (5.7), i.e., it is the contribution of the KG field to the "charge". (However, we use the letter "$\mathcal{J}$" rather than "$\mathcal{Q}$" since the right side of eq. (5.9) corresponds to the integrated Klein-Gordon charge-current flux $J_\mu n^\mu$ through $\mathscr{I}^-$.) The Poisson brackets $\mathcal{J}(\lambda)$ with $\Pi(s)$ and $\overline{\Pi}(s)$ are

$$\left\{\mathcal{J}(\lambda), \Pi(s)\right\} = q\Pi(i\lambda s)\,, \quad \left\{\mathcal{J}(\lambda), \overline{\Pi}(s)\right\} = q\overline{\Pi}(i\lambda s) \tag{5.10}$$

whereas $\{\mathcal{J}(\lambda), \mathcal{J}(\lambda')\} = 0$. Of course, $\mathcal{J}(\lambda)$ has vanishing Poisson brackets with the electromagnetic field observables.

As previously found in sec. 4.2, the generator of large gauge transformations on the asymptotic Maxwell phase space is the memory, eq. (4.32). Thus, the observable that generates large gauge transformations on the full Klein-Gordon-Maxwell phase space is[37]

$$\mathcal{Q}_{i^0}(\lambda) = \mathcal{J}(\lambda) + \frac{1}{4\pi}\Delta(\lambda). \tag{5.11}$$

By arguments similar to those given in the massive case in sec. 4.2, it can be seen that $\mathcal{Q}_{i^0}(\lambda)$ can be obtained by taking limits of surface integrals of the electric field as one approaches $i^0$

37. If a massive charged Klein-Gordon also is present, then the additional term $\mathcal{Q}_{i^-}(\lambda)$ would also be present on the right side of eq. (5.11).

along $\mathscr{I}^-$, so the subscript "$i^0$" is appropriate.

In parallel with the massive case, the algebra $\mathscr{A}_{\rm in}^0 := \mathscr{A}_{\rm in}^{\rm KG0} \otimes \mathscr{A}_{\rm in}^{\rm EM}$ can now be extended to an algebra $\mathscr{A}_{\rm in,Q}^0$ by including the algebra elements $\boldsymbol{\mathcal{J}}(\lambda)$ and $\boldsymbol{\Delta}(\lambda)$ (and, hence, $\boldsymbol{\mathcal{Q}}_{i^0}(\lambda)$) satisfying commutation relations corresponding to the above Poisson bracket relations.

The Poincaré transformations act naturally on $\mathscr{I}^-$. As in the massive case, each infinitesimal Poincaré transformation $P$ is generated by an observable $F_P$ on phase space. Writing $P = T + X$ where $T$ is a translation and $X$ is a Lorentz transformation, the Poisson brackets of the Poincaré generators with the Klein-Gordon observables are

$$\{F_T, \Pi(s)\} = \Pi(\pounds_T s)\,, \quad \{F_X, \Pi(s)\} = \Pi(\pounds_X s + \tfrac{1}{2} s \mathscr{D}_A X^A) \tag{5.12a}$$

$$\{F_T, \mathcal{J}(\lambda)\} = 0\,, \quad \{F_X, \mathcal{J}(\lambda)\} = \mathcal{J}(\pounds_X \lambda) \tag{5.12b}$$

and the Poisson brackets of the Poincaré generators with memory, charges at spatial infinity and themselves are given by eqs. (4.53a)–(4.54b).

As in the massive case, the algebra $\mathscr{A}_{\rm in,Q}^0$ can be further extended to an algebra $\mathscr{A}_{\rm in,QP}^0$ by including algebra elements associated with these observables satisfying commutation relations corresponding to these Poisson bracket relations.

## *3 Fock representations*

In analogy with the massive case, we now construct the Fock representation of $\mathscr{A}_{\rm in}^{\rm KG0}$ based upon the Poincaré invariant vacuum state.[38] The Fock representations of $\mathscr{A}_{\rm in}^{\rm EM}$ of interest

38. In a similar manner to the electromagnetic case, there exists a memory effect for the massless Klein-Gordon field as well as a "scalar charge" at spatial infinity relating the "in" and "out" memories (see e.g. [123] and sec. F.2 of [1]). In the absence of a "source" for the massless scalar field, the scalar memory is conserved in scattering. Therefore in massless QED, we may restrict attention to "in" states with zero "scalar memory" for simplicity.

were already constructed in sec. 4.3.2.

The Poincaré invariant vacuum state is the Gaussian state determined by a vanishing 1-point function and 2-point function given by (see eq. (5.6))

$$\omega_0^{\mathrm{KG0}}(\mathbf{\Pi}(s_1)^*\mathbf{\Pi}(s_2)) = -\frac{1}{\pi}\int\limits_{\mathbb{R}^2\times\mathbb{S}^2} dv_1 dv_2 d\Omega\, \frac{\overline{s_1(v_1,x^A)}s_2(v_2,x^A)}{(v_1-v_2-i0^+)^2} \tag{5.13}$$

for all test functions $s_1(x)$ and $s_2(x)$. The 2-point function gives rise to the inner product

$$\langle s_1|s_2\rangle := \omega_0^{\mathrm{KG0}}(\mathbf{\Pi}(s_1)^*\mathbf{\Pi}(s_2)) = 2\int\limits_0^\infty \omega d\omega \int\limits_{\mathbb{S}^2} d\Omega\, \overline{\hat{s}_1(\omega,x^A)}\hat{s}_2(\omega,x^A). \tag{5.14}$$

on complex-valued test functions on $\mathscr{I}^-$, where in the last equality we have rewritten eq. (5.13) in terms of the positive frequency parts of the Fourier transform of the test functions. The completion of the space of test functions yields the "one-particle" Hilbert space $\mathscr{H}^{\mathrm{KG0}}$. The corresponding Fock space for particles is

$$\mathscr{F}^{\mathrm{KG0}}_{\text{particles}} = \mathbb{C}\oplus\bigoplus_{n\geq 1}\underbrace{\left(\mathscr{H}^{\mathrm{KG0}}\otimes_S\cdots\otimes_S\mathscr{H}^{\mathrm{KG0}}\right)}_{n\text{ times}}. \tag{5.15}$$

An identical construction with the inner product $\overline{\langle s_1|s_2\rangle}$ yields the "one antiparticle" Hilbert space $\overline{\mathscr{H}}^{\mathrm{KG0}}$ and corresponding Fock space $\mathscr{F}^{\mathrm{KG0}}_{\text{antiparticles}}$. One can decompose $\mathbf{\Pi}(s)$ and $\mathbf{\Pi}(s)^*$ into creation and annihilation operators for particles and antiparticles as in eq. (4.72) but we shall not need to do so here. The full Fock space of particles and antiparticles is then

$$\mathscr{F}^{\mathrm{KG0}} = \mathscr{F}^{\mathrm{KG0}}_{\text{particles}}\otimes\mathscr{F}^{\mathrm{KG0}}_{\text{antiparticles}}. \tag{5.16}$$

The algebraic state $\omega_0^{\mathrm{KG0}}$ is the vacuum state of the Fock space which we denote as $|\omega_0^{\mathrm{KG0}}\rangle\in\mathscr{F}^{\mathrm{KG0}}$. A dense set of Hadamard states in this Fock space is generated by the linear span of the vacuum $|\omega_0^{\mathrm{KG0}}\rangle$ and symmetric tensor products of particle and antiparticle wave packet states.

As in the massive case, the large gauge transformations and Poincaré transformations have a strongly continuous unitary action on $\mathscr{F}^{\mathrm{KG0}}$, so $\mathscr{F}^{\mathrm{KG0}}$ carries a representation of $\mathscr{A}^{\mathrm{KG0}}_{\mathrm{in,QP}}$. The vacuum state $|\omega_0^{\mathrm{KG0}}\rangle$ is invariant under these transformations. The large gauge transformations act on the one-particle/antiparticle spaces as multiplication by a phase. The Poincaré group acts on the one-particle/antiparticle spaces by its natural action on $\mathscr{I}^-$.

As in the massive case, it will be useful to express $\mathscr{F}^{\mathrm{KG0}}$ as a direct integral over improper momentum eigenstates. It is useful to parametrize the plane wave solution of 4-momentum $p$ by $p = (\omega, x_p^A)$, where $\omega$ is the frequency of the wave and $x_p^A \in \mathbb{S}^2$ is the direction of the plane wave. As in the massive case, we define $\mathscr{H}_p^{\mathrm{KG0}}$ to be the one-complex-dimensional Hilbert space for particles spanned by $|p\rangle$ and we define

$$\mathscr{H}^{\mathrm{KG0}}_{p_1\cdots p_n} = \mathscr{H}^{\mathrm{KG0}}_{p_1} \otimes_S \cdots \otimes_S \mathscr{H}^{\mathrm{KG0}}_{p_n}, \tag{5.17}$$

which is spanned by $|p_1, \ldots, p_n\rangle$. We similarly define $\overline{\mathscr{H}}_q^{\mathrm{KG0}}$ and $\overline{\mathscr{H}}^{\mathrm{KG0}}_{q_1\cdots q_n}$ for antiparticles. The Fock space can then be written as

$$\mathscr{F}^{\mathrm{KG0}} \cong \bigoplus_{n,m\geq 0} \int_{(C^+)^{n+m}} d^3p_1 \ldots d^3p_n d^3q_1 \ldots d^3q_m \; \mathscr{H}^{\mathrm{KG0}}_{p_1\cdots p_n} \otimes \overline{\mathscr{H}}^{\mathrm{KG0}}_{q_1\cdots q_m} \tag{5.18}$$

where $d^3p$ denotes the Lorentz invariant measure $d^3p = \omega d\omega d\Omega$ on the positive frequency "cone" $C^+ := \{(\omega, x_p^A) \mid \omega > 0\}$. An arbitrary state $|\Psi\rangle \in \mathscr{F}^{\mathrm{KG0}}$ can be expressed as

$$|\Psi\rangle = \sum_{n,m} \int_{(C^+)^{n+m}} d^3p_1 \ldots d^3p_n d^3q_1 \ldots d^3q_m \; \psi_{nm}(p_1, \ldots, q_m) \, |p_1 \ldots p_n\rangle \otimes |q_1 \ldots q_m\rangle \tag{5.19}$$

where $\psi_{nm}$ is a complex, square-integrable function invariant under permutations of $p_i$ and permutations of $q_i$ and supported on non-negative frequencies.

Again, the Fock space decomposition eq. (5.18) corresponds to the spectral decomposition of the charge-current flux operator $\boldsymbol{\mathcal{J}}(\lambda)$. Formally, we have

$$\boldsymbol{\mathcal{J}}(\lambda)\,|p_1\dots p_n\rangle\otimes|q_1\dots q_m\rangle = q\bigg(\sum_{i=1}^{n}\lambda(x^A_{p_i})-\sum_{i=1}^{m}\lambda(x^A_{q_i})\bigg)\,|p_1\dots p_n\rangle\otimes|q_1\dots q_m\rangle\ . \tag{5.20}$$

The formal "unsmeared" action of $\boldsymbol{\mathcal{J}}(x^A)$ on plane wave states is the sum of $\delta$-functions on $\mathbb{S}^2$ whose support is determined by the momenta of the plane waves

$$\boldsymbol{\mathcal{J}}(x^A)\,|p_1\dots p_n\rangle = q\bigg(\sum_{i=1}^{n}\delta_{\mathbb{S}^2}(x^A_{p_i},x^A)\bigg)\,|p_1\dots p_n\rangle \tag{5.21}$$

$$\boldsymbol{\mathcal{J}}(x^A)\,|q_1\dots q_n\rangle = -q\bigg(\sum_{i=1}^{n}\delta_{\mathbb{S}^2}(x^A_{q_i},x^A)\bigg)\,|q_1\dots q_n\rangle\,. \tag{5.22}$$

The action of $\boldsymbol{\mathcal{J}}(\lambda)$ on a proper state $|\Psi\rangle\in\mathscr{F}^{\rm KG0}$ lying in the subspace of $n$ particles and $m$ antiparticles is given by

$$\boldsymbol{\mathcal{J}}(\lambda)\,|\Psi\rangle = \int d^3p_1\dots d^3p_n d^3q_1\dots d^3q_m\psi(p_1,\dots,p_n,q_1,\dots,q_m)\times \tag{5.23}$$

$$q\bigg(\sum_{i=1}^{n}\lambda(x^A_{p_i})-\sum_{i=1}^{m}\lambda(x^A_{q_i})\bigg)\,|p_1\dots p_n\rangle\otimes|q_1\dots q_m\rangle\,. \tag{5.24}$$

All states in this subspace are eigenstates of the total charge operator $\boldsymbol{\mathcal{J}}(1)$

$$\boldsymbol{\mathcal{J}}(1)\,|\Psi\rangle = q(n-m)\,|\Psi\rangle\,. \tag{5.25}$$

However, for non-constant $\lambda$, there are no proper eigenstates of $\boldsymbol{\mathcal{J}}(\lambda)$ apart from the vacuum state.

We now turn to the construction of the analog for massless QED of the Faddeev-Kulish Hilbert space for massive QED given in sec. 4.4. Again, the key idea is to make use of conservation of large gauge charge at spatial infinity, eq. (4.76), and construct "in" and "out" Hilbert spaces composed of states that are eigenstates of all the large gauge charges at spatial infinity (including total electric charge).

The large gauge charges at spatial infinity are now given by

$$\boldsymbol{\mathcal{Q}}_{i^0}(\lambda) = \boldsymbol{\mathcal{J}}(\lambda) + \frac{1}{4\pi}\boldsymbol{\Delta}(\lambda). \tag{5.26}$$

In parallel with the massive case, to obtain states with vanishing charge, we start with the one-dimensional Hilbert space $\mathscr{H}^{\mathrm{KG0}}_{p_1\cdots p_n} \otimes \overline{\mathscr{H}}^{\mathrm{KG0}}_{q_1\cdots q_n}$, which has vanishing total electric charge and charge-current flux given by

$$\mathcal{J}(\lambda) = q\sum_{i=1}^{n}\left(\lambda(x^A_{p_i}) - \lambda(x^A_{q_i})\right). \tag{5.27}$$

We wish to pair this state with the memory Fock space of the electromagnetic field, with memory given by

$$\Delta(\lambda; p_1, \dots, q_n) = -4\pi q\sum_{i=1}^{n}\left(\lambda(x^A_{p_i}) - \lambda(x^A_{q_i})\right) \tag{5.28}$$

for all $\lambda$. If, for all $p_1, \dots, q_n$ we can find a classical, smooth electromagnetic field $e_A(p_1, \dots, q_n)$ that has this memory satisfying eq. (5.28), then the Faddeev-Kulish Hilbert space

$$\mathscr{H}^{\mathrm{FK0}}_{\mathrm{in}} := \bigoplus_{n=0}^{\infty} \int d^3p_1 \dots d^3p_n d^3q_1 \dots d^3q_n \; \mathscr{H}^{\mathrm{KG0}}_{p_1\cdots p_n} \otimes \overline{\mathscr{H}}^{\mathrm{KG0}}_{q_1\cdots q_n} \otimes \mathscr{F}^{\mathrm{EM}}_{\Delta(p_1,\dots,q_n)} \tag{5.29}$$

should satisfy the desired conditions (1)–(5) of sec. 1. Thus, the key issue is whether we can obtain an acceptable solution to eq. (5.28).

In parallel with the massive case (see eq. (4.83)), we will be able to solve eq. (5.28) if and only if we can solve

$$\mathscr{D}^A \Delta_A = \mathscr{D}^2 \alpha = q[4\pi \delta_{\mathbb{S}^2}(x^A, x^A_{p_i}) - 1]. \tag{5.30}$$

In contrast to eq. (4.83), the right side of eq. (5.30) is not smooth. The general solution to eq. (5.30) is

$$\alpha = q \log(1 - \hat{r} \cdot \hat{p}_i) + \text{const.} \tag{5.31}$$

where the dot product is defined by viewing $x^A, x^A_{p_i} \in \mathbb{S}^2$ as unit vectors $\hat{r}, \hat{p}_i$ in $\mathbb{R}^3$, respectively, and taking their Euclidean inner product. Thus,

$$\Delta_A(x^A; p) = q \mathscr{D}_A \log(1 - \hat{r} \cdot \hat{p}_i) \tag{5.32}$$

is the unique solution to eq. (5.30), and

$$\Delta_A(x^A; p_1, \dots, p_n, q_1, \dots, q_n) = \sum_{i=1}^{n} \left( \Delta_A(x^A, p_i) - \Delta_A(x^A, q_i) \right) \tag{5.33}$$

will yield a solution to eq. (5.28) via

$$\Delta(\lambda; p_1, \dots, p_n, q_1, \dots, q_n) = \int_{\mathbb{S}^2} d\Omega \ \mathscr{D}^A \lambda \ \Delta_A(x^A; p_1, \dots, p_n, q_1, \dots, q_n). \tag{5.34}$$

Thus, for massless QED we can solve eq. (5.28) and perform the Faddeev-Kulish Hilbert space construction eq. (5.29). However, there is now a very significant difference from the massive case. As can be seen from eq. (5.32), the required memory diverges as $1/|x^A - x^A_{p_i}|$ at each particle and antiparticle momentum and, hence, is not square integrable on $\mathbb{S}^2$. Consequently, any classical electric field $e_A(v, x^A; p_1, \dots, q_n)$ that gives rise to the required memory and satisfies our required fall-off conditions in $v$ cannot be smooth and, indeed, cannot be square integrable on all spheres. It follows that the states in $\mathscr{F}^{\text{EM}}_{e(p_1,\dots,q_n)}$ cannot be Hadamard.[39] Furthermore, the failure of $e_A(v, x^A; p_1, \ldots, q_n)$ to be square integrable on spheres implies that its classical energy flux through $\mathscr{I}^-$ diverges

$$\int_{\mathscr{I}^-} d^3x\, T_{vv} = \infty\,. \tag{5.35}$$

It follows that all states in $\mathscr{F}^{\text{EM}}_{e(p_1,\ldots,q_n)}$ have infinite expected energy flux through $\mathscr{I}^-$.

Thus, although each charged particle and antiparticle can be "dressed with soft photons" in a manner similar to the massive case, we find that in massless QED this "dressing" has nontrivial angular singularities. These angular singularities correspond to the "collinear divergences" that arise in perturbative scattering calculations in massless QED when working with momentum eigenstates. If one chooses to ignore the physical effects of the soft photons and calculate only probabilities for inclusive "hard" processes, the collinear divergences can be dealt with by imposing an angular cutoff. Indeed the "Kinoshita-Lee-Nauenberg (KLN) theorem" states that, in addition to imposing a frequency cutoff, if one imposes an angular cut-off one can again obtain (inclusive) cross-sections by summing over all low frequency and small angle quanta in the cutoff state and then removing the cutoffs [96, 97]. However, the whole point of the Faddeev-Kulish construction is to take the states in the Faddeev-Kulish Hilbert space eq. (5.29) seriously as exact "in" and "out" states of the quantum field, so that one gets a genuine $S$-matrix relating them. The angular singularities represent genuine singularities in the physical properties of these states. Thus, although a Faddeev-Kulish Hilbert space construction can be carried out in massless QED in close parallel with massive QED, all of the states in the resulting Hilbert space in massless QED are singular and are not of physical relevance.

39. The failure of $e_A(v, x^A; p_1, \ldots, q_n)$ to be square integrable implies that two different choices of "dressing" $e$ and $e'$ with memory eq. (5.33) will, in general, yield unitarily *inequivalent* Fock representations (see footnote 33). Therefore we must label these singular representations by the choice of dressing, $e$, rather than the memory, $\Delta$.

In this subsection, we consider the scattering of a source-free Yang-Mills field. As we shall see, this provides a simple model that has features similar to both massless QED as well as the gravitational case to be considered in sec. 6.

A Yang-Mills gauge field $A^j_\mu$ is a one-form field valued in a Lie-algebra $\mathfrak{g}$ of a compact, semi-simple group $G$. The Lagrangian for this theory is

$$\mathcal{L} = -\frac{1}{4} F^{\mu\nu,i} F_{\mu\nu.i} \tag{5.36}$$

where the Yang-Mills field strength tensor is defined by

$$F^i_{\mu\nu} := \partial_\mu A^i_\nu - \partial_\nu A^i_\mu + c^i{}_{jk} A^j_\mu A^k_\nu \tag{5.37}$$

where $c^i{}_{jk}$ is the structure tensor of the Lie algebra $\mathfrak{g}$ and Lie algebra indices are raised and lowered with the (positive-definite) Cartan-Killing metric

$$k_{ij} := -c^l{}_{ik} c^k{}_{jl} \,. \tag{5.38}$$

This theory is invariant under the action of the Yang-Mills gauge transformations

$$A^i_\mu \mapsto A^i_\mu + \partial_\mu \lambda^i + c^i{}_{jk} A^j_\mu \lambda^k. \tag{5.39}$$

We assume that in the asymptotic past and future, the nonlinear interactions of the Yang-Mills field with itself become negligible, and the Yang-Mills field behaves as a free field.[40] In that case, the Yang-Mills field behaves asymptotically at $\mathscr{I}^-$ like a collection of

40. Of course, this property does not hold for the Yang-Mills fields occurring in nature on account of their interactions with other fields, which do not become negligible in the asymptotic past and future.

decoupled electromagnetic fields. The points of the incoming classical phase space are again given by the specification of the pullback of $A^i_\mu$ to $\mathscr{I}^-$. We again choose a gauge where $n^\mu A^i_\mu|_{\mathscr{I}^-} = 0$ and denote the pullback of $A^i_\mu$ to $\mathscr{I}^-$ in our chosen gauge as $A^i_A$.

The local field observables on phase space are again the smeared electric fields

$$E(s) = \int_{\mathscr{I}^-} d^3x \; E^i_A(x) s^A_i(x) \tag{5.40}$$

where $s^A_i$ is a Lie-algebra valued test vector field on $\mathscr{I}^-$ and

$$E^i_A = -\pounds_n A^i_A = -\partial_v A^i_A \tag{5.41}$$

is the pullback of $F^i_{\mu\nu} n^\nu$ to $\mathscr{I}^-$. Note that $E(s)$ generates the infinitesimal affine transformation $A^i_A \to A^i_A - 2\pi\epsilon s^i_A$ on phase space.

In exact parallel to the electromagnetic case, the algebra $\mathscr{A}^{\text{YM}}_{\text{in}}$ is defined to be the free algebra generated by the smeared field $\boldsymbol{E}(s)$ satisfying (B.I)–(B.III) in sec. 4.1 where the symplectic form of the Yang-Mills field on $\mathscr{I}^-$ is

$$\Omega^{\text{YM}}_{\mathscr{I}}(A_1, A_2) = -\frac{1}{4\pi} \int_{\mathscr{I}^-} dv d\Omega \; \left[ E^{A,i}_1 A_{2A,i} - E^{A,i}_2 A_{1A,i} \right]. \tag{5.42}$$

The corresponding Hadamard regularity condition on the asymptotic states of the Yang-Mills field is that the 2-point function has the form

$$\omega(\boldsymbol{E}^i_A(x_1) \boldsymbol{E}^j_A(x_2)) = -\frac{k^{ij} q_{AB} \delta_{\mathbb{S}^2}(x^A_1, x^A_2)}{(v_1 - v_2 - i0^+)^2} + S^{ij}_{AB}(x_1, x_2) \tag{5.43}$$

where $S^{ij}_{AB}$ is a (state-dependent) smooth bi-tensor on $\mathscr{I}^-$ that is symmetric under the simultaneous interchange of $x_1, x_2$ and the indices $A, B$ and $i, j$. Additionally, the connected $n$-point functions for $n \neq 2$ are smooth. We also impose the same decay requirements of $S^{ij}_{AB}$

and all connected $n$-point functions for $n \neq 2$ as in the electromagnetic case in sec. 4.1. The 2-point function of the vacuum state $\omega_0$ takes the form of eq. (5.43) with $S^{ij}_{AB} = 0$.

Apart from the extra Lie algebra index, there is no difference between Yang-Mills theory and electromagnetism in the above construction of the algebra of asymptotic local field observables and the regularity conditions on states. However, a significant difference with electromagnetism arises when we consider the extension of the algebra to include large gauge charges. In the Yang-Mills case, the infinitesimal action of a large gauge transformation is given by

$$A^i_A \mapsto A^i_A + \epsilon \left[\mathscr{D}_A \lambda^i + c^i{}_{jk} A^j_A \lambda^k\right], \qquad E^i_A \mapsto E^i_A + \epsilon c^i{}_{jk} E^j_A \lambda^k. \tag{5.44}$$

In particular, the electric field $E^i_A$ is no longer gauge invariant. The "charge" that generates this infinitesimal gauge transformation is

$$\mathcal{Q}^{\mathrm{YM}}_{i^0}(\lambda) := -\frac{1}{4\pi} \int_{\mathscr{I}^-} d^3x \left(2c^i{}_{jk} \lambda_i A^{A,j} E^k_A + \lambda^i \mathscr{D}^A E_{A,i}\right) \tag{5.45}$$

where the subscript "$i^0$" again has been inserted to indicate that — assuming that no additional fields with Yang-Mills charge are present — $\mathcal{Q}^{\mathrm{YM}}_{i^0}(\lambda)$ can be obtained by taking limits of surface integrals of the Yang-Mills electric field as one approaches $i^0$ along $\mathscr{I}^-$, as can be shown by arguments similar to the massive and massless QED cases.

It is useful to separate the contributions to $\mathcal{Q}^{\mathrm{YM}}_{i^0}(\lambda)$ into their linear and nonlinear parts. The linear part is the *memory* of the Yang-Mills field associated with large gauge transformation $\lambda$:

$$\Delta^{\mathrm{YM}}(\lambda) := -\int_{\mathscr{I}^-} dv d\Omega\; E^i_A(v, x^B) \mathscr{D}^A \lambda_i(x^B). \tag{5.46}$$

Although the memory is no longer the generator of large gauge transformations, it is still an observable on the asymptotic phase space, since $\frac{1}{4\pi}\Delta^{\text{YM}}(\lambda)$ generates the affine transformation

$$A^i_A \mapsto A^i_A + \epsilon \mathscr{D}_A \lambda^i \,, \qquad E^i_A \mapsto E^i_A. \tag{5.47}$$

The nonlinear part of $\mathcal{Q}^{\text{YM}}_{i^0}(\lambda)$ is the *Yang-Mills charge-current flux* observable

$$\mathcal{J}^{\text{YM}}(\lambda) := \frac{1}{2\pi} \int\limits_{\mathscr{I}^-} dv d\Omega \; c^i{}_{jk} \lambda_i A^{A,j} E^k_A \,. \tag{5.48}$$

By definition, we have

$$\mathcal{Q}^{\text{YM}}_{i^0}(\lambda) = \mathcal{J}^{\text{YM}}(\lambda) + \frac{1}{4\pi}\Delta^{\text{YM}}(\lambda). \tag{5.49}$$

This is closely analogous to eq. (5.11) except that now, the "null memory" $\mathcal{J}^{\text{YM}}(\lambda)$ arises from the Yang-Mills field itself, not some additional massless charged field.

The Poisson brackets of $\mathcal{Q}^{\text{YM}}_{i^0}(\lambda)$ and $\Delta^{\text{YM}}(\lambda)$ with themselves and with the local fields $E(s)$ can be computed using eq. (2.7) from the above phase space transformations that they generate. We obtain

$$\{\mathcal{Q}^{\text{YM}}_{i^0}(\lambda), E(s)\} = E([\lambda, s]) \,, \quad \{\Delta^{\text{YM}}(\lambda), E(s)\} = 0 \tag{5.50a}$$

$$\{\Delta^{\text{YM}}(\lambda_1), \Delta^{\text{YM}}(\lambda_2)\} = 0 \,, \quad \{\mathcal{Q}^{\text{YM}}_{i^0}(\lambda_1), \Delta^{\text{YM}}(\lambda_2)\} = \Delta^{\text{YM}}([\lambda_1, \lambda_2]) \tag{5.50b}$$

$$\{\mathcal{Q}^{\text{YM}}_{i^0}(\lambda_1), \mathcal{Q}^{\text{YM}}_{i^0}(\lambda_2)\} = \mathcal{Q}^{\text{YM}}_{i^0}([\lambda_1, \lambda_2]) \tag{5.50c}$$

where the bracket denotes the Lie bracket $[X, Y]^i = c^i{}_{jk} X^j Y^k$ between any two elements $X, Y$ of the Lie-algebra $\mathfrak{g}$. Finally, the action of the infinitesimal Poincaré tranformations $F_P$ with $E(s), \Delta^{\text{YM}}(\lambda), \mathcal{Q}_{i^0}(\lambda)$ and themselves are again given by eqs. (4.52)–(4.54b).

In exact parallel with sec. 4.2 and 5.2, the algebra $\mathscr{A}^{\text{YM}}_{\text{in}}$ can be extended to $\mathscr{A}^{\text{YM}}_{\text{in,QP}}$ by including $\boldsymbol{\mathcal{Q}}^{\text{YM}}_{i^0}(\lambda)$, $\boldsymbol{\Delta}^{\text{YM}}(\lambda)$ and the Poincaré generators $\boldsymbol{F}_P$ in the algebra, with commutation relations corresponding to the above Poisson bracket relations. In addition, as before, we impose the further condition on states

$$\omega(\boldsymbol{\Delta}^{\rm YM}(\lambda)) = -\int\limits_{\mathscr{I}^-} dv d\Omega\ \omega(\boldsymbol{E}^i_A(v,x^B))\mathscr{D}^A\lambda_i(x^B)\,, \tag{5.51}$$

which ensures that the expectation value of the memory observable corresponds to eq. (5.46).

The Fock representations $\mathscr{F}^{\rm YM}_\Delta$ of $\mathscr{A}^{\rm YM}_{\rm in}$ can be constructed in direct analogy to electromagnetic case. The GNS construction based upon the vacuum state $\omega_0$ again yields the standard Fock space $\mathscr{F}^{\rm YM}_0$, for which every state is an eigenstate of $\boldsymbol{\Delta}^{\rm YM}(\lambda)$ with vanishing eigenvalue. Representations of nonvanishing memory can be constructed in the same manner as discussed in sec. 4.3.2. The representation of $\mathscr{A}^{\rm YM}_{\rm in}$ on zero-memory Fock space $\mathscr{F}^{\rm YM}_0$ can be extended to a representation of $\mathscr{A}^{\rm YM}_{\rm in,QP}$. However, this is not possible for the representations of nonvanishing memory on account of the nontrivial commutation relations of $\boldsymbol{\Delta}^{\rm YM}(\lambda)$ with both $\boldsymbol{\mathcal{Q}}^{\rm YM}_{i^0}(\lambda)$ and with Lorentz generators.

As in electromagnetism, the zero-memory Fock space $\mathscr{F}^{\rm YM}_0$ can be represented as a direct integral of "improper" plane wave states. As before, an arbitrary state $|\Psi\rangle \in \mathscr{F}^{\rm YM}_0$ can be expressed as

$$|\Psi\rangle = \sum_n \int\limits_{(C^+)^n} d^3p_1 \dots d^3p_n\ \psi^{A_1\dots A_n}_{(n)i_1,\dots,i_n}(p_1,\dots,p_n)\varepsilon^{i_1}_{A_1}\dots\varepsilon^{i_n}_{A_n}\,|p_1,\dots,p_n\rangle \tag{5.52}$$

where $\psi_{(n)}$ is a complex $L^2$ tensor-field and $\varepsilon^i_A$ denotes Lie-algebra valued "polarization vectors" which satisfy

$$k_{ij}\varepsilon^i_A\varepsilon^j_B = \frac{1}{2}q_{AB}\,, \quad q^{AB}\varepsilon^i_A\varepsilon^j_B = \frac{1}{n}k^{ij} \tag{5.53}$$

where $n$ is the dimension of the group $G$. The corresponding Fock space decomposition corresponds to the spectral decomposition the "null memory operator" $\boldsymbol{\mathcal{J}}^{\rm YM}(\lambda)$. Formally we have,

$$\boldsymbol{\mathcal{J}}^{\rm YM}(\lambda)\varepsilon^{i_1}_{A_1}\dots\varepsilon^{i_n}_{A_n}\left|p_1,\dots,p_n\right\rangle = \\ \left(c^{i_1}{}_{jk}\lambda^j(x^A_{p_1})\varepsilon^k_{A_1}\dots\varepsilon^{i_n}_{A_n}+\dots+c^{i_n}{}_{jk}\lambda^j(x^A_{p_n})\varepsilon^{i_1}_{A_1}\dots\varepsilon^k_{A_n}\right)\left|p_1,\dots,p_n\right\rangle \tag{5.54}$$

In the "unsmeared" form, the formal action of $\boldsymbol{\mathcal{J}}^{\rm YM}(x^A)$ on plane wave states is given by

$$\boldsymbol{\mathcal{J}}^{\rm YM}_j(x^A)\varepsilon^{i_1}_{A_1}\dots\varepsilon^{i_n}_{A_n}\left|p_1,\dots,p_n\right\rangle = \\ \left(\delta_{\mathbb{S}^2}(x^A,x^A_{p_1})c^{i_1}{}_{jk}\varepsilon^k_{A_1}\dots\varepsilon^{i_n}_{A_n}+\dots+\delta_{\mathbb{S}^2}(x^A,x^A_{p_n})c^{i_n}{}_{jk}\varepsilon^{i_1}_{A_1}\dots\varepsilon^k_{A_n}\right)\left|p_1,\dots,p_n\right\rangle. \tag{5.55}$$

We turn now to the issue of whether analogs of the Faddeev-Kulish "in" and "out" Hilbert spaces can be constructed.[41] As the in the case of Maxwell fields, for solutions to the Yang-Mills equations which are suitably regular at spatial infinity, the charges obtained from the limit along past and future null infinity to spatial infinity match by

$$\mathcal{Q}^{\rm YM,in}_{i^0}(\lambda)=\mathcal{Q}^{\rm YM,out}_{i^0}(\lambda\circ\Upsilon) \tag{5.56}$$

where $\Upsilon$ is the antipodal map on $\mathbb{S}^2$. Therefore, we are again led to seek "in" and "out" Hilbert spaces composed of eigenstates of the charge observable

$$\boldsymbol{\mathcal{Q}}^{\rm YM}_{i^0}(\lambda)=\boldsymbol{\mathcal{J}}^{\rm YM}(\lambda)+\frac{1}{4\pi}\boldsymbol{\Delta}^{\rm YM}(\lambda). \tag{5.57}$$

It should be noted first, that, in contrast to the abelian case, the charge operator now satisfies the nontrivial commutation relation $[\boldsymbol{\mathcal{Q}}^{\rm YM}_{i^0}(\lambda_1),\boldsymbol{\mathcal{Q}}^{\rm YM}_{i^0}(\lambda_2)]=\boldsymbol{\mathcal{Q}}^{\rm YM}_{i^0}([\lambda_1,\lambda_2])$. For a semisimple Lie group as considered here, it follows that there cannot exist any eigenstate of $\boldsymbol{\mathcal{Q}}^{\rm YM}_{i^0}(\lambda)$ for all $\lambda^i$ unless the eigenvalues vanish for all $\lambda^i$. (In massive and massless

41. Dressed states in non-abelian gauge theories have been previously considered in, e.g. [124, 125, 126].

QED, eigenstates of $\mathcal{Q}^{\mathrm{YM}}_{i0}(\lambda)$ of nonzero eigenvalue exist, although we restricted to vanishing eigenvalue in order to have an infinitesimal action of the Lorentz group.) In the case of massive and massless QED in sec. 4.4 and 5.4, the analog of $\mathcal{J}^{\mathrm{YM}}(\lambda)$ was played by the charge or current flux of an additional scalar field. In those cases, we can choose an (improper) eigenstate of this scalar field observable and then "dress" it with electromagnetic field states belonging to a corresponding memory representation. However, in the present case, $\mathcal{J}^{\mathrm{YM}}(\lambda)$ and $\boldsymbol{\Delta}^{\mathrm{YM}}(\lambda)$ arise from the same Yang-Mills field, so we cannot independently choose the eigenstate of $\mathcal{J}^{\mathrm{YM}}(\lambda)$ and the memory representation to which it belongs.

To gain insight into the nature of the difficulty caused by the fact that $\mathcal{J}^{\mathrm{YM}}(\lambda)$ and $\boldsymbol{\Delta}^{\mathrm{YM}}(\lambda)$ arise from the same field, let us attempt to construct eigenstates of $\mathcal{Q}^{\mathrm{YM}}_{i0}(\lambda)$ with vanishing eigenvalue by following a similar procedure to that used in massless QED. As in massless QED, we can start with an improper plane wave momentum eigenstate $\varepsilon^{i_1}_{A_1} \dots \varepsilon^{i_n}_{A_n} |p_1 \dots p_n\rangle$ in the zero memory incoming Fock space $\mathscr{F}^{\mathrm{YM}}_0$. This state is an eigenstate of $\mathcal{J}^{\mathrm{YM}}(\lambda)$ with eigenvalue given by eq. (5.55). We now wish to "dress" this state with "soft YM particles" belonging to the memory representation with $-\Delta^{\mathrm{YM}}(\lambda)/4\pi$ equal to this eigenvalue, so as to produce an eigenstate of $\mathcal{Q}^{\mathrm{YM}}_{i0}(\lambda)$ with vanishing eigenvalue. As in massless QED, on account of the $\delta$-functions on $\mathbb{S}^2$ appearing in eq. (5.55), the required memory will be singular, and the dressed states will have infinite expected total energy flux. Thus, as in massless QED, the states constructed in this manner will be unphysical. However, a further major difficulty occurs in the Yang-Mills case because the "dressing" now also contributes to the Yang-Mills charge-current flux. Since the dressing is singular the Yang-Mills charge-current flux of the "dressing" is infinite and so the resulting "dressed state" cannot be defined. Furthermore, even if the dressing could be defined, the resulting state would no longer be an eigenstate of $\mathcal{J}^{\mathrm{YM}}(\lambda)$ and, hence, is not an eigenstate of $\mathcal{Q}^{\mathrm{YM}}_{i0}(\lambda)$. Thus, the states produced by the Faddeev-Kulish "dressing" procedure are not only unphysical but they do even yield the desired eigenstate property that motivated their construction.

Thus, in order to implement the strategy for constructing "in" and "out" Hilbert spaces based upon the conservation law eq. (5.56), we must seek eigenstates of $\boldsymbol{\mathcal{Q}}^{\rm YM}_{i0}(\lambda)$ of vanishing eigenvalue by some procedure other than "dressing." To see the nature of the restrictions on states imposed by the eigenstate condition, we note that, by definition, for any eigenstate $\omega$ of $\boldsymbol{\mathcal{Q}}^{\rm YM}_{i0}(\lambda)$ with vanishing eigenvalue, we have

$$\omega(\boldsymbol{\mathcal{Q}}^{\rm YM}_{i0}(\lambda)\boldsymbol{E}(s)) = \omega(\boldsymbol{E}(s)\boldsymbol{\mathcal{Q}}^{\rm YM}_{i0}(\lambda)) = 0. \tag{5.58}$$

However, the commutation relation eq. (5.50a) then implies

$$c^{i}{}_{jk}\omega(\boldsymbol{E}^{j}_{A}(x))\lambda^{k} = 0 \text{ for all } \lambda^{k}(x^{A}). \tag{5.59}$$

which, for a semi-simple Lie algebra, implies, in turn, that

$$\omega(\boldsymbol{E}^{j}_{A}(x)) = 0\,. \tag{5.60}$$

Thus, the 1-point function of any eigenstate must vanish. Note that it then follows from eq. (5.51) that — in contrast with massless QED — the expected memory must vanish. It also then follows that the expected Yang-Mills charge-current flux must vanish. By similar arguments, the 2-point must satisfy

$$c^{i_1}{}_{jk}\omega(\boldsymbol{E}^{k}_{A_1}(x_1)\boldsymbol{E}^{i_2}_{A_2}(x_2)) + c^{i_2}{}_{jk}\omega(\boldsymbol{E}^{i_1}_{A_1}(x_1)\boldsymbol{E}^{k}_{A_2}(x_2)) = 0. \tag{5.61}$$

This implies that $\omega(\boldsymbol{E}^{i_1}_{A_1}(x_1)\boldsymbol{E}^{i_2}_{A_2}(x_2))$ must be proportional to $k^{i_1 i_2}$. This condition is satisfied by choices of $S^{ij}_{AB}(x_1, x_2)$ in eq. (5.43) of the form $S^{ij}_{AB}(x_1, x_2) = S'_{AB}(x_1, x_2)k^{ij}$. Nevertheless, this is an extremely restrictive condition on the 2-point function. More generally, the $n$-point correlation functions of $\omega$ must be proportional to Casimirs[42] of the Lie algebra $\mathfrak{g}$.

42. For a semi-simple Lie algebra $\mathfrak{g}$, the number of independent Casimirs is finite and is equal to the rank

Thus, although there exist nontrivial algebraic eigenstates of $\mathcal{Q}^{\mathrm{YM}}_{i^0}(\lambda)$, it is clear that there are insufficiently many states to obtain a Hilbert large enough to carry representatives of all "hard" scattering processes.

In summary, in Yang-Mills theory, the Faddeev-Kulish "dressing" procedure fails to produce eigenstates of $\mathcal{Q}^{\mathrm{YM}}_{i^0}(\lambda)$. Although eigenstates of $\mathcal{Q}^{\mathrm{YM}}_{i^0}(\lambda)$ do exist, there are insufficiently many of them for scattering theory. Thus, the attempt to construct "in" and "out" Hilbert spaces composed of eigenstates of charges fails. In the next section, we will see that in the gravitational case, this failure is even more dramatic, since there are no eigenstates of the large gauge (i.e., supertranslation) charges at all except for the vacuum state.

# 6 Vacuum general relativity

In this section we turn our attention to the asymptotic quantum theory of full nonlinear general relativity at null infinity. There are many nontrivial, unresolved issues concerning the formulation of quantum gravity in the bulk. However, as has been emphasized by Ashtekar [64, 3], in asymptotically flat spacetimes the asymptotic phase space of general relativity at null infinity is an affine manifold similar to that of electromagnetism. Consequently, one can quantize the asymptotic degrees of freedom in exact parallel with the electromagnetic case. The asymptotic symmetries of general relativity are the BMS transformations, which enlarge the Poincaré group by the inclusion of supertranslations. The supertranslations play a role in the asymptotic quantization of general relativity that is closely analogous to the role played by large gauge transformations in electromagnetism.

We present the asymptotic quantization algebra of local field observables for general relativity in sec. 6.1. The extension of this algebra to include the charges that generate BMS

of $\mathfrak{g}$. For $\mathfrak{g} = \mathfrak{su}(n)$, the number of independent Casimirs is $n-1$.

transformations is given in sec. 6.2. We then show in sec. 6.3 that an analog of Faddeev-Kulish "in" and "out" Hilbert spaces does not exist in quantum gravity.

## 1 *Asymptotic quantization of general relativity*

As discussed in [99], the points in the asymptotic phase space of general relativity at past null infinity can be specified by an equivalence class of derivative operators intrinsic to $\mathscr{I}^-$. For our purposes, it is convenient to instead adopt the following equivalent formulation. Choose a Bondi advanced time coordinate $v$ and consider the foliation of $\mathscr{I}^-$ by the cross-sections with $v =$ constant. This foliation determines a unique null vector $l^\mu$ at $\mathscr{I}^-$ which is normal to the cross-sections and at $\mathscr{I}^-$ satisfies

$$l^\mu n_\mu = -1\,, \quad l^\mu l_\mu = 0. \tag{6.1}$$

Then, the points of the asymptotic phase space are specified by the *shear* of $l^\mu$ which is defined by[43]

$$\sigma_{\mu\nu} = (q_\mu{}^\alpha q_\nu{}^\beta - \tfrac{1}{2} q_{\mu\nu} q^{\alpha\beta}) \nabla_\alpha l_\beta \tag{6.2}$$

where $q_{\mu\nu}$ is the metric on the cross-sections. Since $\sigma_{\mu\nu}$ is orthogonal to $l^\mu$ and $n^\mu$, we can write it as $\sigma_{AB}$. In general relativity, $\sigma_{AB}$ is the analogue of the vector potential $\mathcal{A}_A$ in the electromagnetic case. The analogue of the electric field $E_A$ is given by the *News tensor* $N_{AB}$ defined as

$$N_{AB} := 2\pounds_n \sigma_{AB} = 2\partial_v \sigma_{AB}. \tag{6.3}$$

43. Alternatively, one can define a symmetric trace free tensor, $C_{AB}$, as defined below eq. (2.111), which correspond to the angular components of the physical metric at order $1/r$ in Bondi coordinates in the bulk spacetime. This is related to the shear that we have defined by $C_{AB} = -2\sigma_{AB}$; see [127].

The asymptotic symplectic form is then given by

$$\Omega^{\rm GR}_{\mathscr{I}}(\sigma_1,\sigma_2) = \frac{1}{16\pi}\int\limits_{\mathscr{I}^-} d^3x\ \left[N_1^{AB}\sigma_{2AB} - N_2^{AB}\sigma_{1AB}\right]. \tag{6.4}$$

The local field observables are the smeared News

$$N(s) := \int\limits_{\mathscr{I}^-} d^3x\ N_{AB}(x)s^{AB}(x)\,. \tag{6.5}$$

where $s^{AB}$ is a real test tensor field. The smeared News generates the affine transformation $\sigma_{AB} \mapsto \sigma_{AB} + \epsilon 8\pi s_{AB}$ on phase space. The Poisson brackets of the smeared News are computed to be

$$\left\{N(s_1), N(s_2)\right\} = -64\pi^2\Omega^{\rm GR}_{\mathscr{I}}(s_1,s_2)1 = 8\pi\int\limits_{\mathscr{I}^-} d^3x\ \left[s_{1AB}\partial_v s_2^{AB} - s_{2AB}\partial_v s_1^{AB}\right]. \tag{6.6}$$

In exact parallel with electromagnetism, the asymptotic quantization algebra of local field observables, $\mathscr{A}^{\rm GR}_{\rm in}$, is defined to be the unital $*$-algebra generated by the elements $\boldsymbol{N}(s)$, $\boldsymbol{N}(s)^*$ and $\mathbf{1}$, factored by the following relations:

(C.I) $\boldsymbol{N}(c_1s_1 + c_2s_2) = c_1\boldsymbol{N}(s_1) + c_2\boldsymbol{N}(s_2)$ for any $s_1^{AB}, s_2^{AB}$ and any $c_1, c_2 \in \mathbb{R}$

(C.II) $\boldsymbol{N}(s)^* = \boldsymbol{N}(s)$ for all $s^{AB}$

(C.III) $[\boldsymbol{N}(s_1), \boldsymbol{N}(s_2)] = -64\pi^2 i\Omega^{\rm GR}_{\mathscr{I}}(s_1,s_2)\mathbf{1}$

The Hadamard regularity condition on asymptotic states $\omega$ on the News algebra $\mathscr{A}^{\rm GR}_{\rm in}$ analogous to eq. (3.5) is that the 2-point function has the form

$$\omega(\boldsymbol{N}_{AB}(x_1)\boldsymbol{N}_{CD}(x_2)) = -8\frac{\left(q_{A(C}q_{D)B} - {}^1\!/_2 q_{AB}q_{CD}\right)\delta_{\mathbb{S}^2}(x_1^A, x_2^A)}{(v_1 - v_2 - i0^+)^2} + S_{ABCD}(x_1,x_2) \tag{6.7}$$

where $S_{ABCD}$ is a (state-dependent) bi-tensor on $\mathscr{I}^-$ that is symmetric in $A, B$ and in $C, D$, satisfies $q^{AB} S_{ABCD} = q^{CD} S_{ABCD} = 0$ and is symmetric under the simulataneous interchange of $x_1$ with $x_2$ and the pair of indices $A, B$ with the pair $C, D$. As before, we also require that $S_{ABCD}$ and the connected $n$-point functions for $n \neq 2$ of a Hadamard state on $\mathscr{I}^-$ are smooth and decay as $O((\sum_i v_i^2)^{-1/2-\epsilon})$ for some $\epsilon > 0$.

## *2 Extension of the asymptotic quantization algebra to include BMS charges*

The gauge symmetries of general relativity are the diffeomorphisms on spacetime. However, the transformations induced by diffeomorphisms that preserve the asymptotic structure of spacetime but do not vanish at null infinity are not degeneracies of the symplectic form and must be treated as symmetries. The group of such diffeomorphisms is known as the Bondi-Metzner-Sachs (BMS) group. For a given choice of Bondi advanced time coordinate $v$ on $\mathscr{I}^-$, the vector field $\xi^\mu$ that generates an arbitrary infinitesimal BMS transformation takes the form

$$\xi^\mu = (f + \tfrac{1}{2} v \mathscr{D}_A X^A) n^\mu + X^\mu \,. \tag{6.8}$$

Here $f(x^A)$ is an arbitrary smooth function on $\mathbb{S}^2$ with conformal weight $+1$, and $X^\mu$ is a vector field tangent to the cross-sections of $\mathscr{I}$ that is a conformal Killing vector field on the 2-sphere. By a slight abuse of the notational conventions[44] stated at the end of sec. 0.3, we will denote this vector field as $X^A$. The transformations with $X^A = 0$ are referred to as *supertranslations* and the supertranslations with $f$ given by a linear combination of $\ell = 0, 1$ spherical harmonics are the ordinary *translations*. The transformations generated by $X^A$ are Lorentz transformations. However, it should be noted that the decomposition of $\xi^\mu$ into a supertranslation and a Lorentz transformation depends on the choice of Bondi advanced

44. By the conventions of sec. 0.3, $X^A$ would denote an equivalence class of vector fields $X^\mu$ modulo multiples of $n^\mu$. Here, since we have made a choice of Bondi advanced time coordinate $v$, we use $X^A$ to denote the particular representative that is tangent to the cross-sections of constant $v$.

time coordinate $v$, i.e., if $\xi^\mu$ is a "pure Lorentz transformation" ($f = 0$) for one choice of $v$, it would correspond to a Lorentz transformation (with the same $X^A$) plus a supertranslation for other choices of $v$.

The action of an infinitesimal BMS transformation of the form eq. (6.8) on phase space is given by the following affine transformation

$$\begin{aligned}\sigma_{AB} &\mapsto \sigma_{AB} + \epsilon\Big[\tfrac{1}{2}(f + \tfrac{1}{2}v\mathscr{D}_C X^C)N_{AB} + (\mathscr{D}_A\mathscr{D}_B - \tfrac{1}{2}q_{AB}\mathscr{D}^2)f + \pounds_X\sigma_{AB} \\ &\qquad - \tfrac{1}{2}(\mathscr{D}_C X^C)\sigma_{AB}\Big] \\ N_{AB} &\mapsto N_{AB} + \epsilon\left[(f + \tfrac{1}{2}v\mathscr{D}_C X^C)\partial_v N_{AB} + \pounds_X N_{AB}\right].\end{aligned}$$

Note that $N_{AB}$ is *not* invariant under BMS symmetries. The charge observable that generates this infinitesimal BMS transformation is given by (see [127])

$$\begin{aligned}\mathcal{Q}^{\mathrm{GR}}_{i^0}(f, X) = \frac{1}{16\pi}\int_{\mathscr{I}^-} dv d\Omega\; N^{AB}\Big[&\tfrac{1}{2}(f + \tfrac{1}{2}v\mathscr{D}_C X^C)N_{AB} + \mathscr{D}_A\mathscr{D}_B f \\ &+ \pounds_X\sigma_{AB} - \tfrac{1}{2}(\mathscr{D}_C X^C)\sigma_{AB}\Big].\end{aligned} \tag{6.9}$$

As we shall now explain, we have inserted the subscript "$i^0$" on $\mathcal{Q}^{\mathrm{GR}}_{i^0}(f, X)$ for reasons analogous our use of this notation in massive and massless QED. As shown in [21, 127], the right-hand-side of eq. (6.9) can be written as[45]

$$\begin{aligned}&\frac{1}{16\pi}\int_{\mathscr{I}^-} dv d\Omega\; N^{AB}\left[\tfrac{1}{2}(f + \tfrac{1}{2}v\mathscr{D}_C X^C)N_{AB} + \mathscr{D}_A\mathscr{D}_B f + \pounds_X\sigma_{AB} - \tfrac{1}{2}(\mathscr{D}_C X^C)\sigma_{AB}\right] \\ &= \lim_{v\to\infty}\mathcal{Q}^{\mathrm{GR}}_v(f, X) - \lim_{v\to-\infty}\mathcal{Q}^{\mathrm{GR}}_v(f, X)\end{aligned} \tag{6.10}$$

45. Note that eq. (6.10) has a relative overall sign compared to eq. (1.18) in sec. 1 due to the fact that we are now working at $\mathscr{I}^-$ rather than $\mathscr{I}^+$.

with

$$\mathcal{Q}_v^{\mathrm{GR}}(f,X) = \frac{1}{8\pi} \int\limits_{S(v)} d\Omega \, \Big[\tfrac{1}{2}(f + \tfrac{1}{2}v\mathscr{D}_A X^A)\sigma^{AB}N_{AB} + X^A \sigma_{AB}\mathscr{D}_C\sigma^{BC} - \tfrac{1}{4}\sigma^2\mathscr{D}_C X^C + \Omega^{-1}C_{\mu\nu\lambda\rho}\xi^\mu l^\nu n^\lambda l^\rho\Big] \tag{6.11}$$

where $\sigma^2 := \sigma_{AB}\sigma^{AB}$, the integral on the right side is taken over the cross-section, $S(v) \cong \mathbb{S}^2$, of advanced time $v$ on $\mathscr{I}^-$, and $C_{\mu\nu\lambda\rho}$ is the Weyl tensor of the conformally-completed spacetime. If massive fields (or black/white holes) are present, they would, in general, contribute to $\lim\limits_{v\to-\infty} \mathcal{Q}_v^{\mathrm{GR}}(f,X)$ in a manner similar to massive QED. We will assume, for simplicity, that this is not the case and thus that[46] $\lim\limits_{v\to-\infty} \mathcal{Q}_v^{\mathrm{GR}}(f,X) = 0$. In that case, eq. (6.10) shows that $\mathcal{Q}_{i^0}^{\mathrm{GR}}(f,X)$ can be obtained as a limit of a surface integral of local quantities as one approaches spatial infinity, $i^0$, along $\mathscr{I}^-$. Thus, the subscript "$i^0$" is appropriate in eq. (6.9).

As in the Yang-Mills case, it is useful to separate the contributions to $\mathcal{Q}_{i^0}^{\mathrm{GR}}(f,X)$ into their linear and nonlinear parts. The linear term arises only for supertranslations and defines the gravitational memory observable

$$\Delta^{\mathrm{GR}}(f) := \frac{1}{2} \int\limits_{\mathscr{I}^-} dv d\Omega \; N_{AB}(v,x^C)\mathscr{D}^A\mathscr{D}^B f(x^C) \tag{6.12}$$

which, we note, vanishes if $f$ is a linear combination of $\ell = 0, 1$ spherical harmonics. On the asymptotic phase space, $\frac{1}{8\pi}\Delta^{\mathrm{GR}}(f)$ generates the affine transformation

$$\begin{aligned} \sigma_{AB} &\mapsto \sigma_{AB} + \epsilon(\mathscr{D}_A\mathscr{D}_B - \tfrac{1}{2}q_{AB}\mathscr{D}^2)f \\ N_{AB} &\mapsto N_{AB}. \end{aligned} \tag{6.13}$$

46. We emphasize that the conclusions of sec. 6.3 and, in particular, Theorem 5 do not depend upon this assumption.

For $X^A = 0$, the supertranslation charge $\mathcal{Q}^{\rm GR}_{i^0}(f)$, which generates supertranslations by eq. (6.9) with $X^A = 0$, is given by

$$\mathcal{Q}^{\rm GR}_{i^0}(f) = \mathcal{J}^{\rm GR}(f) + \frac{1}{8\pi}\Delta^{\rm GR}(f) \tag{6.14}$$

where

$$\mathcal{J}^{\rm GR}(f) := \frac{1}{32\pi}\int_{\mathscr{I}^-} dv d\Omega\ f N^{AB} N_{AB} \tag{6.15}$$

is called the *null memory*. If massless fields with stress energy $T_{\mu\nu}$ are present they will, in general contribute to the null memory by the simple substitution $N^{AB}N_{AB} \to N^{AB}N_{AB} + 32\pi\Omega^{-2}T_{\mu\nu}n^\mu n^\nu$ in eq. (6.15). However, for simplicity, we shall consider only case of vacuum gravitational fields in this section.

The Poisson bracket of $\mathcal{Q}^{\rm GR}_{i^0}(f, X)$ with the local News observable is given by

$$\left\{\mathcal{Q}^{\rm GR}_{i^0}(f, X), N(s)\right\} = N(s') \tag{6.16}$$

where $s'_{AB} = (f + \frac{1}{2}v\mathscr{D}_C X^C)\partial_v s_{AB} + \pounds_X s_{AB} - \frac{1}{2}(\mathscr{D}_C X^C)s_{AB}$. We also have

$$\left\{\mathcal{Q}^{\rm GR}_{i^0}(f_1), \mathcal{Q}^{\rm GR}_{i^0}(f_2)\right\} = 0\,, \quad \left\{\mathcal{Q}^{\rm GR}_{i^0}(X_1), \mathcal{Q}^{\rm GR}_{i^0}(X_2)\right\} = \mathcal{Q}^{\rm GR}_{i^0}([X_1, X_2]), \tag{6.17a}$$

$$\left\{\mathcal{Q}^{\rm GR}_{i^0}(X), \mathcal{Q}^{\rm GR}_{i^0}(f)\right\} = \mathcal{Q}^{\rm GR}_{i^0}(\pounds_X f - {}^1\!/\!_2(\mathscr{D}_A X^A)f) \tag{6.17b}$$

where $[X_1, X_2]$ is the Lie bracket of $X_1^A$ and $X_2^A$. The memory observable has vanishing Poisson brackets with the News and with itself

$$\left\{\Delta^{\rm GR}(f), N(s)\right\} = 0, \qquad \left\{\Delta^{\rm GR}(f_1), \Delta^{\rm GR}(f_2)\right\} = 0. \tag{6.18}$$

Finally, we have

$$\left\{\mathcal{Q}^{\text{GR}}_{i^0}(f_1), \Delta^{\text{GR}}(f_2)\right\} = 0, \qquad \left\{\mathcal{Q}^{\text{GR}}_{i^0}(X), \Delta^{\text{GR}}(f)\right\} = \Delta^{\text{GR}}(\pounds_X f - {}^1\!/\!_2(\mathscr{D}_A X^A)f). \quad (6.19)$$

Thus, the memory observable is supertranslation-invariant but not Lorentz-invariant.

In exact parallel with massive and massless QED and Yang-Mills theory, we now can extend the algebra, $\mathscr{A}^{\text{GR}}_{\text{in}}$, of asymptotic local field observables to an algebra $\mathscr{A}^{\text{GR}}_{\text{in,Q}}$ by including $\boldsymbol{\mathcal{Q}}^{\text{GR}}_{i^0}(f, X)$ and $\boldsymbol{\Delta}^{\text{GR}}(f)$ in the algebra, with commutation relations corresponding to the above Poisson bracket relations. (The Poincaré generators are, of course, already included in the BMS charges $\boldsymbol{\mathcal{Q}}^{\text{GR}}_{i^0}(f, X)$, so there is no need for a further extension of the algebra.) In parallel with eqs. (4.45) and (5.51) we impose on states the condition

$$\omega(\boldsymbol{\Delta}^{\text{GR}}(f)) = \frac{1}{2}\int\limits_{\mathscr{I}^-} dv d\Omega \; \omega(\boldsymbol{N}_{AB}(v, x^A))\mathscr{D}^A\mathscr{D}^B f. \qquad (6.20)$$

The Fock representations of $\mathscr{A}^{\text{GR}}_{\text{in}}$ can be constructed in direct analogy to electromagnetic case. The GNS construction based upon the vacuum state $\omega_0$ again yields the standard Fock space, $\mathscr{F}^{\text{GR}}_0$, for which every state is an eigenstate of $\boldsymbol{\Delta}^{\text{GR}}(f)$ with vanishing eigenvalue. Again, representations of nonvanishing memory, $\mathscr{F}^{\text{GR}}_{\Delta}$, can be constructed in the same manner as discussed in sec. 4.3.2. The representation of $\mathscr{A}^{\text{GR}}_{\text{in}}$ on the zero-memory Fock space can be extended to a representation of $\mathscr{A}^{\text{GR}}_{\text{in,Q}}$. The representations of $\mathscr{A}^{\text{GR}}_{\text{in}}$ on the Fock spaces of nonzero memory can be extended to include representatives of the generators of supertranslations, $\boldsymbol{\mathcal{Q}}^{\text{GR}}_{i^0}(f)$. However, the representations of nonzero memory cannot be extended to include representatives of the generators of infinitesimal Lorentz transformations, $\boldsymbol{\mathcal{Q}}^{\text{GR}}_{i^0}(X)$, on account of the nontrivial commutation relation of $\boldsymbol{\Delta}^{\text{GR}}(f)$ with $\boldsymbol{\mathcal{Q}}^{\text{GR}}_{i^0}(X)$. In particular, angular momentum is not well-defined on the Fock spaces of nonzero memory.

For solutions of the vacuum Einstein equation that satisfy the Ashtekar-Hansen [61] asymptotic flatness conditions together with an additional null regularity condition at spatial infinity, it was shown in [26, 62] that the charges $\mathcal{Q}^{\rm GR}_{i^0}(f, X)$ obtained from the limit along past null infinity are matched antipodally to the similarly defined charges obtained from the limit along future null infinity.[47] In particular the supertranslation charges satisfy

$$\mathcal{Q}^{\rm GR,in}_{i^0}(f) = \mathcal{Q}^{\rm GR,out}_{i^0}(f \circ \Upsilon) \tag{6.21}$$

where, as before, $\Upsilon$ is the antipodal map on $\mathbb{S}^2$. This is an exact analog of eq. (4.76) in electromagnetism. As previously explained in sec. 4.4, this conservation law provides a potential means of constructing "in" and "out" Hilbert spaces satisfying the desired properties (1)–(5) given in sec. 1. Namely, if we can construct an "in" Hilbert space composed entirely of eigenstates of the supertranslation charges, it will evolve to an "out" Hilbert space composed of eigenvectors of corresponding eigenvalue. In order to have a continuous action of the Lorentz group, we must choose the eigenvalues of all of the charges to vanish. If such "in" and "out" Hilbert spaces of vanishing charges are separable and contain sufficiently many states to account for all "hard" scattering processes, then properties (1)–(5) should hold.

In massive QED, this strategy was successfully implemented by the Faddeev-Kulish construction described in sec. 4.4. In this construction, one "dresses" each momentum eigenstate of the incoming massive charged particles with an electromagnetic state belonging to the memory representation whose memory cancels the large gauge charges of the incoming charged particle state, so as to produce an eigenstate of vanishing eigenvalue of all of the total large gauge charges. As shown in sec. 5.4, in massless QED, the same "dressing" construction

47. For linearized gravity around a Minkowski background, the matching of the supertranslation charges is also shown in [128, 25] (see also [129]).

can be given. However, the required memory in this case is singular — so singular that the "soft photon dressing" has infinite energy flux. As shown in sec. 5.5, the situation is worse in Yang-Mills theory. Not only is a singular "dressing" required, but the "dressing" does not provide eigenstates of charge. Although eigenstates of charge can be constructed by other means, there are insufficiently many of them for scattering theory. We turn now to the analysis of the situation in quantum gravity.

First, as already mentioned above, in order to have a continuous action of the Lorentz group on the Hilbert space, it is necessary to restrict to eigenstates of the charges of vanishing eigenvalue. In massive QED, this required the vanishing of all large gauge charges, including the total ordinary electric charge. The vanishing of large gauge charges for $\ell \geq 1$ is achieved by the "dressing," but the vanishing of the total ordinary electric charge is an unwanted restriction on the scattering states. Nevertheless, arguably, this is not a genuine restriction since one could always put additional particles "behind the moon" to make the total charge vanish. However, in the gravitational case, the corresponding requirement for a continuous action of the Lorentz group is for all of the supertranslation charges $\mathcal{Q}^{\mathrm{GR}}_{i^0}(f)$ to vanish, include the charges associated with ordinary translations. In other words, the states must be eigenstates of 4-momentum of eigenvalue zero. But the vacuum state is the only such state; one cannot cancel the 4-momentum of a state of interest by putting additional particles "behind the moon." Thus, the Faddeev-Kulish construction fails for this elementary reason at this initial stage.

Nevertheless, one could give up on having a continuous action of the Lorentz group and seek eigenstates of $\mathcal{Q}^{\mathrm{GR}}_{i^0}(f)$ of nonzero eigenvalue. It is instructive to see what happens if one attempts to construct such eigenstates by a "dressing" procedure.

First, consider *linearized gravity* with an additional massless quantum field source, where the null memory is due to the massless source rather than to gravitational radiation. As previously noted below eq. (6.15), the massless field will contribute a null memory of the form

$$\mathcal{J}^{\mathrm{GR,source}}(f) = \int_{\mathscr{I}^-} dv d\Omega \; f(\Omega^{-2} T_{\mu\nu} n^\mu n^\nu) \, . \tag{6.22}$$

In a manner similar to massless QED and Yang-Mills theory, the (improper) plane wave states of the massless source field are formal eigenstates of null memory. In order to produce an eigenstate of $\boldsymbol{\mathcal{Q}}^{\mathrm{GR}}_{i^0}(f)$ with eigenvalue $\mathcal{Q}^{\mathrm{GR}}_{i^0}(f)$ for all $f$ in linearized gravity, we must "dress" a source particle momentum eigenstate $|p\rangle$ by choosing a memory representation of the gravitational field such that

$$\mathscr{D}^A \mathscr{D}^B \Delta^{\mathrm{GR}}_{AB}(x^A; p) = 8\pi\omega \delta_{\mathbb{S}^2}(x^A, x^A_p) - 8\pi \mathcal{Q}^{\mathrm{GR}}_{i^0}(x^A) \tag{6.23}$$

where $\omega$ is the frequency associated with the null momentum $p = (\omega, x^A_p)$. This equation differs from the corresponding equation (5.30) in massless QED in that now there are two angular derivatives of memory rather than one. This difference results in milder angular singularities in the solution, namely, $|\Delta^{\mathrm{GR}}_{AB}| \sim |\log(|x^A - x^A_p|)|$ rather than $|\Delta_A| \sim 1/|x^A - x^A_p|$ as in massless QED. In other words, the collinear divergences in quantum gravity are less severe than in massless QED and Yang-Mills theory. Although the required memories are still singular, they are square integrable, and the corresponding "dressed states" — which were previously constructed in [36] — do not have an infinite energy flux. In this respect, the situation in linearized gravity is *better* than in massless QED and Yang-Mills theory, although since we cannot construct eigenstates of vanishing charges, the action of the Lorentz group is undefined and therefore the angular momentum is undefined for all such "dressed" states in linearized gravity.

However, in *nonlinear gravity*, as in Yang-Mills theory, the "dressing" will now contribute to the null memory, so the resulting dressed state is no longer an eigenstate of $\boldsymbol{\mathcal{J}}^{\mathrm{GR}}(f)$ and hence is not an eigenstate[48] of $\boldsymbol{\mathcal{Q}}_{i^0}^{\mathrm{GR}}(f)$. Thus, the "dressing" construction does not yield the desired eigenstate property that motivated the procedure. In order to implement the strategy for constructing "in" and "out" Hilbert spaces based upon the conservation law eq. (6.21), we must seek eigenstates of the supertranslation charges $\boldsymbol{\mathcal{Q}}_{i^0}^{\mathrm{GR}}(f)$ by some other means. In Yang-Mills theory, we were able to find some eigenstates of large gauge charges, but insufficiently many to do scattering theory. However, one of the key results of this chapter is that in gravity, there are no nontrivial eigenstates at all.[49] This is shown by the following theorem:

**Theorem 5.** *Let $f$ be any smooth function on $\mathbb{S}^2$ whose support is all of $\mathbb{S}^2$, i.e., $f$ does not vanish identically on any open subset of $\mathbb{S}^2$. Suppose that the state $\omega$ is Hadamard, satisfies our decay conditions, and is an eigenstate of the supertranslation charge $\boldsymbol{\mathcal{Q}}_{i^0}^{GR}(f)$. Then $\omega = \omega_0$, where $\omega_0$ is the BMS-invariant vacuum state.*

*Proof.* Since $\omega$ is an eigenstate of $\boldsymbol{\mathcal{Q}}_{i^0}^{\mathrm{GR}}(f)$, we have

$$\omega(\boldsymbol{\mathcal{Q}}_{i^0}^{\mathrm{GR}}(f)\boldsymbol{N}(s)) = \omega(\boldsymbol{N}(s)\boldsymbol{\mathcal{Q}}_{i^0}^{\mathrm{GR}}(f)) = \kappa\, \omega(\boldsymbol{N}(s)). \tag{6.24}$$

where $\kappa$ denotes the eigenvalue, which is real since since $\boldsymbol{\mathcal{Q}}_{i^0}^{\mathrm{GR}}(f)$ is self-adjoint. Thus, we have

$$0 = \omega([\boldsymbol{\mathcal{Q}}_{i^0}^{\mathrm{GR}}(f), \boldsymbol{N}(s)]) = i\omega(\boldsymbol{N}(f\partial_v s)) \tag{6.25}$$

where the commutation relation corresponding to eq. (6.16) was used. Since this holds for all

48. In contrast to the Yang-Mills case, the dressing contribution to the null memory is finite and so the "dressed state" can have a well-defined expected charge. Nevertheless, it cannot be an eigenstate of $\boldsymbol{\mathcal{Q}}_{i^0}^{\mathrm{GR}}(f)$.

49. Note that there is no state in any non-zero memory representation that has vanishing null memory eq. (6.15); zero is merely the lower bound of the continuous spectrum of the null memory operator, as emphasized by Ashtekar [3]. Consequently, in contrast to claims in [130], memory vacua are not eigenstates of the charges $\boldsymbol{\mathcal{Q}}_{i^0}^{\mathrm{GR}}(f)$ at spatial infinity.

$s^{AB}$, we have for all $x = (v, x^A)$

$$f(x^A)\frac{\partial}{\partial v}\left[\omega(\boldsymbol{N}_{AB}(x))\right] = 0 \tag{6.26}$$

Since $f$ does not vanish on open sets, it follows that $\omega(\boldsymbol{N}_{AB}(x))$ is constant in $v$. The decay conditions then imply that the 1-point function $\omega(\boldsymbol{N}_{AB}(x))$ vanishes identically.

By similar arguments starting with

$$\omega(\boldsymbol{\mathcal{Q}}^{\mathrm{GR}}_{i^0}(f)\boldsymbol{N}(s_1)\dots\boldsymbol{N}(s_n)) = \omega(\boldsymbol{N}(s_1)\dots\boldsymbol{N}(s_n)\boldsymbol{\mathcal{Q}}^{\mathrm{GR}}_{i^0}(f)) = \kappa\,\omega(\boldsymbol{N}(s_1)\dots\boldsymbol{N}(s_n)) \tag{6.27}$$

we find that the $n$-point functions satisfy

$$\sum_{i=1}^{n}\partial_{v_i}\omega(\boldsymbol{N}_{A_1B_1}(x_1)\dots\boldsymbol{N}_{A_nB_n}(x_n)) = 0. \tag{6.28}$$

It then follows that $S_{ABCD}$ in eq. (6.7) and the truncated $n$-point functions also satisfy this equation. But $S_{ABCD}$ and the truncated $n$-point functions are required to decay as $O((\sum_i v_i^2)^{-1/2-\epsilon})$. It follows that $S_{ABCD}$ and all truncated $n$-point functions of $\omega$ vanish, i.e., $\omega = \omega_0$. $\square$

We emphasize that the implications of Theorem 5 are quite strong in that constructions based upon the use of eq. (6.21) require eigenstates of $\boldsymbol{\mathcal{Q}}^{\mathrm{GR}}_{i^0}(f)$ for all $f$. Note that eq. (6.28) is in close parallel to eq. (5.61) in the Yang-Mills case. However, nontrivial solutions to eq. (5.61) do exist, whereas the vacuum state is the only state that satisfies eq. (6.28). Thus, in the gravitational case, the attempt to construct "in" and "out" Hilbert spaces by using charge eigenstates fails in a much more catastrophic manner.

# 7 Non-Faddeev-Kulish representations

As we have just seen, the Faddeev-Kulish "dressing" procedure cannot be used to construct a Hilbert space of "in" and "out" states in quantum gravity, nor is there any other procedure that can produce eigenstates of the supertranslation charges $\mathcal{Q}^{\mathrm{GR}}_{i^0}$. Nevertheless, as described at the end of sec. 6.2, there is an ample supply of "in" and "out" states given by the memory Fock spaces $\mathscr{F}^{\mathrm{GR}}_{\Delta}$, with $\Delta^{\mathrm{GR}}_{AB}$ an arbitrary smooth, symmetric trace free tensor on $\mathbb{S}^2$. Is there is some other way of assembling these states into Hilbert spaces in such a way that the desired conditions (1)–(5) given in sec. 1 can be satisfied? In this section, we explore this possibility.

An obvious candidate for the "in" Hilbert space would be the direct sum over all of the "in" memory Fock spaces $\mathscr{F}^{\mathrm{GR,in}}_{\Delta}$, i.e.,

$$\mathscr{F}^{\mathrm{in}}_{\mathrm{DS}} = \bigoplus_{\Delta} \mathscr{F}^{\mathrm{GR,in}}_{\Delta} \tag{7.1}$$

where $\Delta^{\mathrm{GR}}_{AB}$ ranges over all (say, smooth) symmetric trace free tensors on $\mathbb{S}^2$, where the "DS" subscript stands for "direct sum." $\mathscr{F}^{\mathrm{out}}_{\mathrm{DS}}$ would then be defined similarly. Clearly, $\mathscr{F}^{\mathrm{in}}_{\mathrm{DS}}$ and $\mathscr{F}^{\mathrm{out}}_{\mathrm{DS}}$ would then allow all possible memories. However, this choice has many serious deficiencies.[50] First, since there are uncountably many choices of $\Delta^{\mathrm{GR}}_{AB}$, this Hilbert space is clearly nonseparable. Second, although the BMS group acts naturally on $\mathscr{F}^{\mathrm{in}}_{\mathrm{DS}}$, Lorentz transformations act nontrivially on memory and a "small" Lorentz transformation will map a vector in the sector $\mathscr{F}^{\mathrm{GR}}_{\Delta}$ into an entirely different sector $\mathscr{F}^{\mathrm{GR}}_{\Delta'}$. Since all states in different memory sectors are orthogonal to each other, Lorentz transformations do not act in a strongly

50. The first two of these deficiencies are analogous to what would occur if attempted to take the Hilbert space of one-dimensional Schrödinger quantum mechanics to be $\oplus_{x\in\mathbb{R}} H_x$ where $H_x$ is a one dimensional Hilbert space representing an eigenstate of the position operator with eigenvalue $x$. This Hilbert space is nonseparable and does not admit a strongly continuous action of translations — so the momentum operator cannot be defined.

continuous manner on $\mathscr{F}_{\rm DS}^{\rm in}$. Thus, infinitesimal generators of Lorentz transformations — in particular, angular momentum — cannot be defined. However, by far the most serious deficiency is that it is clear that states in $\mathscr{F}_{\rm DS}^{\rm in}$ will not evolve to states in the similarly defined "out" Hilbert space $\mathscr{F}_{\rm DS}^{\rm out}$, so condition (2) of the sec. 1 will not be satisfied. To see this, we note that — since the norm of the direct sum is the sum over the norms in each $\mathscr{F}_{\Delta}^{\rm GR,in}$ and any uncountable sum of strictly positive numbers is infinite — for any vector in $\mathscr{F}_{\rm DS}^{\rm in}$ the probability of having a given value of memory can be nonvanishing for only a countable number of memories. Thus, the possible memories of any state in $\mathscr{F}_{\rm DS}^{\rm in}$ are discrete. However, it seems clear that most states in $\mathscr{F}_{\rm DS}^{\rm in}$ will evolve to "out" states where the memory is continuously distributed. Such states cannot lie in $\mathscr{F}_{\rm DS}^{\rm out}$.

A more promising candidate would be to take a direct integral of the Fock spaces $\mathscr{F}_{\Delta}^{\rm GR,in}$ with respect to a measure that is continuously distributed in $\Delta_{AB}^{\rm GR}$. To do so, we first need to make a precise choice of the space, $\mathscr{M}$, of memories. Then we need to specify a $\sigma$-algebra of measurable subsets of $\mathscr{M}$. Then, we need to define a measure on $\mathscr{M}$, i.e., a map, $\mu$, from measurable subsets to nonnegative real numbers such that $\mu(\varnothing) = 0$ and $\mu$ is "countably additive", that is, for any countable collection of disjoint measurable sets $\{\mathscr{O}_i\}$ we have that $\mu(\cup_i \mathscr{O}_i) = \sum_i \mu(\mathscr{O}_i)$. Given such a measure, $\mu$, we can construct a direct integral Hilbert space $\mathscr{F}_{\rm DI}^{\rm in}(\mu)$ from the memory Fock spaces $\mathscr{F}_{\Delta}^{\rm GR,in}$ as follows: A vector $|\Psi\rangle \in \mathscr{F}_{\rm DI}^{\rm in}(\mu)$ consists of the specification of a measurable family of vectors $|\psi(\Delta)\rangle \in \mathscr{F}_{\Delta}^{\rm GR,in}$ for all $\Delta_{AB}^{\rm GR}$, where $|\Psi\rangle$ and $|\Psi'\rangle$ are considered equivalent if $|\psi(\Delta)\rangle$ and $|\psi'(\Delta)\rangle$ differ only on a set of measure zero. The states in $\mathscr{F}_{\rm DI}^{\rm in}(\mu)$ are required to have finite norm

$$\| \Psi \|^2 = \int_{\mathscr{M}} d\mu \ \| \psi(\Delta) \|^2 < \infty \tag{7.2}$$

and the inner product of two states is then defined by

$$\langle \Psi_1 | \Psi_2 \rangle = \int_{\mathscr{M}} d\mu \ \langle \psi_1(\Delta) | \psi_2(\Delta) \rangle \, . \tag{7.3}$$

The direct sum Hilbert space $\mathscr{F}^{\text{in}}_{\text{DS}}$ is a special case of the direct integral Hilbert space wherein the $\sigma$-algebra is taken to be all subsets of $\mathscr{M}$ and $\mu$ is taken to be the "discrete measure" that assigns unit measure to any subset consisting of a single point. As already stated above, this yields a Hilbert space that is nonseparable and has other unacceptable properties. However, choices of "continuous measures" can yield a separable Hilbert space and have the possibility of satisfying the other properties desired for scattering theory. If $\mathscr{M}$ were a finite dimensional vector space, there is an essentially unique notion of Lebesgue measure and this would provide a natural choice of measure. However, $\mathscr{M}$ is infinite dimensional, so there is no notion of Lebesgue measure (see sec. A.4 of [131]). Since there is a direct correspondence between memories and supertranslations [20, 1] and the supertranslations comprise a group, one might try to use an "invariant Haar measure" on memories. However, the supertranslation group is not locally compact and therefore the Haar measure does not exist. Nevertheless, there are well-defined notions of Gaussian measures.[51] We can obtain a natural class of Gaussian measures in the following manner (see, e.g., [132, 133, 134] for further details).

We start with the topological vector space of smooth functions $f$ with conformal weight 1 on $\mathbb{S}^2$ with the nuclear topology. The trace-free part of $\mathscr{D}_A \mathscr{D}_B f$ for such $f$ provides a space of test tensor fields for memory (see eq. (6.12)). We take $\mathscr{M}$ to be the topological

51. In path integral formulations of Euclidean QFT of some field $\phi$, it is common to write the measure in the path integral as $D\phi \, e^{-S_0(\phi)}$ where $S_0(\phi)$ is the Euclidean action of a free field and $D\phi$ is a "Lebesgue measure" on the space of fields. However, $D\phi$ does not really exist and it is the full quantity $D\phi \, e^{-S_0(\phi)}$ which is a genuine Gaussian measure; the covariance of this measure is the Euclidean Green's function determined by the action $S_0$. This is also true in path integral formulations of quantum mechanics where the measure over the space of paths is the (Gaussian) Wiener measure [131].

dual space. We choose the $\sigma$-algebra of subsets to be generated by the "cylindrical sets" (see [132, 134]). By the Bochner-Minlos theorem (see sec. A.6 of [131]), a Gaussian measure on $\mathscr{M}$ (centered at zero) is then determined by specifying any positive, symmetric, bilinear map $K(f_1, f_2)$ (the "covariance matrix") on the space of test functions. A necessary condition for two Gaussian measures with covariance $K$ and $K'$, respectively, to be equivalent (i.e., such that they agree on which subsets of $\mathscr{M}$ have measure zero) is that they define equivalent norms on the space of test functions. In other words, $K$ and $K'$ are equivalent if there is a positive constant $c$ such that

$$c^{-1}K(f,f) \leq K'(f,f) \leq cK(f,f) \tag{7.4}$$

for all test functions $f$. Thus, there exists a very large class of inequivalent Gaussian measures that can be constructed by this procedure.

Thus, the key issue for the construction of a Gaussian measure on the space, $\mathscr{M}$, of memories — and, thereby, direct integral Hilbert spaces of "in" and "out" states — is the choice of covariance matrix $K$. A key criterion is that $K$ be Lorentz invariant, since the resulting Gaussian measure $\mu$ will then be Lorentz invariant, and the Lorentz group will then act naturally on $\mathscr{F}^{\text{in}}_{\text{DI}}(\mu)$. However, there is a unique choice of Lorentz invariant covariance matrix $K$ (see Ch. III.4 of [135]). This covariance matrix is simply the $L^2$ inner product on the space of smooth test tensors $\mathscr{D}_A\mathscr{D}_B f$ on $\mathbb{S}^2$

$$K(f_1, f_2) = \int\limits_{\mathbb{S}^2\times\mathbb{S}^2} d\Omega_1 d\Omega_2 \; K_{ABCD}(x_1^A, x_2^B)\mathscr{D}^A\mathscr{D}^B f_1(x_1^A)\mathscr{D}^C\mathscr{D}^D f_2(x_2^B) \tag{7.5}$$

with integral kernel

$$K_{ABCD}(x_1^A, x_2^B) = \delta_{\mathbb{S}^2}(x_1^A, x_2^B)\left(q_{A(C}q_{D)B} - \tfrac{1}{2}q_{AB}q_{CD}\right). \tag{7.6}$$

The direct integral Hilbert space $\mathscr{F}_{\mathrm{DI}}^{\mathrm{in}}(\mu)$ obtained from the Gaussian measure determined by $K$ is a separable Hilbert space.

However, there is a very serious problem with attempting to use this Hilbert space for scattering theory. For any Gaussian measure constructed in the manner described above using a covariance matrix $K$ there is a subset of $\mathscr{M}$, determined by $K$, known as the "Cameron-Martin space" on which $\mu$ has zero measure (see theorem 2.4.7 in [132]). For the case of a Gaussian measure with covariance given by eq. (7.6), the Cameron-Martin space is the space of square-integrable memories. This means that "almost all" of the memories, $\Delta_{AB}^{\mathrm{GR}}$, that contribute to $\mathscr{F}_{\mathrm{DI}}^{\mathrm{in}}(\mu)$ fail to be square-integrable.[52] However, as argued in sec. 5.4, states with a non-square-integrable memory that satisfy our fall-off conditions cannot be Hadamard and have divergent expected total energy flux. Thus, all of the states in $\mathscr{F}_{\mathrm{DI}}^{\mathrm{in}}(\mu)$ are unphysical.

Thus, the direct integral Hilbert space obtained from the Gaussian measure constructed from the Lorentz invariant covariance matrix eq. (7.6) does not yield an acceptable candidate for "in" and "out" Hilbert spaces. One could try instead using a covariance matrix $K'$ corresponding, e.g., to a Sobolev norm, so that the states in $\mathscr{F}_{\mathrm{DI}}^{\mathrm{in}}(\mu')$ would have physically acceptable memories.[53] However, one would then have to give up on having a natural Lorentz group action. More significantly, there would be no reason to expect that states in $\mathscr{F}_{\mathrm{DI}}^{\mathrm{in}}(\mu')$ would evolve to states in the similarly constructed $\mathscr{F}_{\mathrm{DI}}^{\mathrm{out}}(\mu')$. Of course, we also could make different choices of the precise specification of $\mathscr{M}$, different choices of the $\sigma$-algebra, and one could also try to use non-Gaussian measures. We certainly have not proven that no such choice could work. But we see no reason to believe that there is any such choice that would

52. This is analogous to the statement that in the case of Wiener measure, the differentiable paths are of measure zero, while in the Euclidean path integral the field configurations with finite action are of measure zero.

53. See [136] for a construction of a direct integral Hilbert space in the electromagnetic case with respect to Gaussian measures defined on square integrable memories. Such representations have a well-defined action of translations, but the action of Lorentz is not well-defined.

work to construct “in” and “out” Hilbert spaces with the desired properties for scattering theory.

## 8 Algebraic scattering theory

As we have seen in sec. 4.4, the Faddeev-Kulish construction in massive QED gives a basically satisfactory way of defining “in” and “out” Hilbert spaces in such a way that a genuine $S$-matrix should exist. However, as we found in sec. 5.4, the analogous construction in massless QED does not work, as the required “soft photon dressing” gives all states an infinite expected total energy flux due to collinear divergences. As discussed in sec. 5.5, these problems persist in Yang-Mills theory, but an additional serious difficulty arises in that case due to the fact that the “soft dressing” itself will carry a large gauge charge-current flux, which will spoil the property that the “dressing” is designed to achieve. As we found in sec. 6, in the gravitational case the problems caused by collinear divergences are not as severe, but the problem arising from the fact that any soft graviton dressing will contribute to supertranslation fluxes is much more severe, and we proved in sec. 6.3 that no analog of the Faddeev-Kulish “in” and “out” Hilbert spaces can exist in quantum gravity. Finally, we explored alternatives to the Faddeev-Kulish construction in sec. 7 and found that several natural attempts do not work. It is our strong belief that in the gravitational case, no definition of “in” and “out” Hilbert spaces will satisfy conditions (1)–(5) of sec. 1.

It should be emphasized that there is no difficulty in the construction of “in” and “out” states. As we found, we can construct Fock space representations of $\mathscr{A}^{\mathrm{GR}}_{\mathrm{in}}$ for the “in” and “out” states with arbitrary choices of memory $\Delta^{\mathrm{GR}}_{AB}$. As we have noted in sec. 6.3, the representations with non-vanishing memory cannot be extended as representations of the full algebra $\mathscr{A}^{\mathrm{GR}}_{\mathrm{in,Q}}$. However, one can obtain “in” states by starting with any choice of smooth memory $\Delta^{\mathrm{GR}}(f)$ and considering its “Lorentz orbit” i.e., the space of all memories obtained by acting by Lorentz transformations on chosen memory $\Delta^{\mathrm{GR}}(f)$. This "orbit space" is *finite* dimensional and can be equipped with a Lorentz invariant measure (see, e.g. a similar analysis for supertranslation charges by McCarthy [137]). A direct integral of the memory Fock spaces over this orbit space yields a representation of $\mathscr{A}^{\mathrm{GR}}_{\mathrm{in,Q}}$. Applying this procedure to all smooth memories yields an enormous supply of physically acceptable states. This supply of states is certainly ample enough to encompass all of the "in" states that one might wish to consider, and we expect that it also would be ample enough to encompass all of the "out" states that arise from the dynamical evolution of these "in" states. Thus, the difficulties that we have elucidated in this chapter do not arise from any problems with constructing "in" and "out" states nor do they arise from any problems with dynamical evolution through the bulk. They arise solely from the attempt to assemble all of the "in" and "out" states of interest into a single (separable) Hilbert space.

However, there is no reason to try to force the "in" and "out" states to live in a single Hilbert space. The algebra of asymptotic observables is entirely well-defined. As reviewed in sec. 3, in the algebraic viewpoint, a state is simply a positive, linear map on the algebra of observables. There is no need to specify a Hilbert space in order to define a state. The regularity conditions that we have imposed upon asymptotic states — namely the Hadamard condition and decay conditions — also do not require the specification of a Hilbert space. However, given a state, the GNS construction allows us to represent that state as a vector in a Hilbert space representation of the algebra. Thus, Hilbert spaces of asymptotic states may be viewed as somewhat analogous to coordinate patches on a manifold.[54] Given any point of a manifold, one can choose a coordinate patch in which it lies, and it is often very convenient to do so. Similarly, given any state on an algebra, one can choose a Hilbert space

54. However, it should be kept in mind that this analogy is not perfect in that Hilbert space representations are much more rigid than coordinate patches. In particular, it is important that coordinate patches have nontrivial overlap regions, whereas irreducible Hilbert space representations will not overlap unless they coincide.

in which it lies, and it is often very convenient to do so. However, in the case of a manifold of nontrivial topology, it would not be reasonable to demand that a single coordinate patch represent all points of interest in the manifold. Similarly, in the case of scattering theory, it does not appear reasonable to demand that a single Hilbert space represent all scattering states of interest.

What would scattering theory look like in a framework where no "in" and "out" Hilbert spaces are specified at the outset? In the algebraic viewpoint, one would specify an "in" state $\omega_{\rm in}$ as a positive linear map on the "in" algebra of asymptotic observables. This would consist of specifying the correlation functions of all of these observables. Of course, there is nothing stopping one from considering an "in" state that corresponds to a vector in the standard zero memory Fock representation $\mathscr{F}_0^{\rm in}$ — but one would not be forced to do so in this framework. Similarly, in massive QED, one would be allowed to "dress" the incoming charged particles with incoming electromagnetic states in the corresponding memory representation as in the Faddeev-Kulish construction, but it also would be allowed to consider "bare" incoming charged particles. Given $\omega_{\rm in}$, one then computes the corresponding outgoing state $\omega_{\rm out}$ by obtaining all of its correlation functions of the "out" observables. Of course, this is much easier said than done, since one would not have the simplicity of the LSZ reduction, which relies, in particular, on the ability to express any "in" or "out" state in terms of local field operators acting on the Poincaré invariant vacuum state — which would not be the case for states of nonzero memory. Nevertheless, if one wishes to know any particular "out" correlation function, it seems clear that to any finite order in perturbation theory, it must be possible to evolve this correlation function backwards into the past and express it in terms of "in" correlation functions, all of which would have been given in the specification of $\omega_{\rm in}$. In this manner, we should, in principle, be able to determine a convex-linear[55] superscattering matrix

55. Given two algebraic states $\omega$ and $\omega'$ any convex linear combination $\lambda\omega + (1-\lambda)\omega'$ where $0 \leq \lambda \leq 1$, gives a new state.

$\$$ such that

$$\omega_{\text{out}} = \$\omega_{\text{in}}. \tag{8.1}$$

Here, we have adopted the terminology “superscattering matrix” and the notation $\$$ from Hawking [138] even though there are substantial differences in our motivation and framework from his. Hawking was concerned with generalizing the usual framework of scattering theory to allow pure states to evolve to mixed states (“information loss”), but he was not concerned with infrared issues and he assumed that all states lie in the folium of a single Hilbert space representation containing the Poincaré invariant vacuum. We are not concerned here with information loss but are similarly generalizing the framework so as to allow $\$$ to map between all regular algebraic states, not necessarily belonging to the folium of a single Hilbert space representation. In this framework, conservation of probability would be expressed by the requirement that if $\omega_{\text{in}}$ is any normalized “in” state (i.e., $\omega_{\text{in}}(\mathbf{1}) = 1$), then $\omega_{\text{out}} = \$\omega_{\text{in}}$ also is normalized. If there is no information loss, then $\$$ would take any pure algebraic “in” state to a pure algebraic “out” state.

We note that the notion of an algebraic state is, in principal, sufficient to answer all physical questions regarding the field observables. In particular, the specification of a state $\omega$ yields the expected value of all powers of $\boldsymbol{N}(s)$. Since the conditions of the Hamburger moment problem (see e.g. [139, 140]) hold for a free field,[56] these moments determine the probability distribution for observing the values of this field observable. Therefore, despite the absence of a pre-chosen Hilbert space, one can determine the probability distribution of field observables.

Of course, if one is interested only in calculating the types of quantities that might be measured in collider experiments, there is no need to develop a new framework for scattering

56. However, nonlinear observables such as the stress tensor in the bulk do not satisfy the required conditions on moments. Determining the probability distribution for general, nonlinear observables remains an open problem.

theory that properly treats infrared effects, since quantities like inclusive cross-sections surely can be calculated much more efficiently by present means than in a framework in which one takes proper account of the far infrared degrees of freedom. Nevertheless, we believe it would be of interest to further develop the "algebraic scattering" framework that we have sketched above.

# APPENDIX A

# RELATIONSHIP OF OUR ANSATZ TO SMOOTHNESS AT $\mathscr{I}^+$ IN $D = 4$

In this appendix, we address the relationship between our ansatz in chapter 2 for the asymptotic fall-off of the vector potential and metric to their smoothness at $\mathscr{I}^+$. As noted in sec. 2.3–2.5, it is easily seen that in $d = 4$ that smoothness of $A_\mu$ at $\mathscr{I}^+$ implies that our ansatz (2.24) holds, and smoothness of $\Omega^2 h_{\mu\nu}$ at $\mathscr{I}^+$ implies that our ansatz (2.58) holds. However, for $d = 4$ the ansatz (2.24) implies smoothness of $A_\mu$ at $\mathscr{I}^+$ only under the additional condition that $A_r^{(1)} = 0$, and the ansatz (2.58) implies smoothness of $\Omega^2 h_{\mu\nu}$ at $\mathscr{I}^+$ only under the additional condition that $h_{rr}^{(1)} = 0$. In this Appendix, we investigate the conditions under which these additional restrictions can be imposed as gauge conditions. We show that this is possible in electromagnetism when $j_r^{(3)} = 0$ and in linearized gravity when $T_{ur}^{(3)} = T_{rr}^{(3)} = T_{rA}^{(3)} = 0$. However, when these quantities are nonvanishing, there are solutions within our ansatz that are not smooth at $\mathscr{I}^+$. Nevertheless, in nonlinear gravity, we show that $h_{rr}^{(1)} = 0$ if the Bondi news is nonvanishing everywhere on one cross-section, in which case our ansatz in $d = 4$ is equivalent to smoothness at $\mathscr{I}^+$.

## 1 *Electromagnetism*

The $\ell \neq 0$ part of $A_r^{(1)}$ is gauge invariant within our ansatz, so if it is nonvanishing, it cannot be set to zero by a gauge transformation. By eq. (2.37), we have $\psi^{(1)} = 0$. Equation (2.32) with $n = 1$ then yields $\partial_u A_r^{(1)} = 0$, so $A_r^{(1)}$ is independent of $u$. The $r$-component of Maxwell's equations given by eq. (2.29) in four dimensions with $n = 3$ gives that

$$\mathscr{D}^2 A_r^{(1)} = -4\pi j_r^{(3)}. \tag{0.1}$$

This equation implies that the $\ell = 0$ part of $j_r^{(3)}$ must vanish. It also implies that $\partial_u j_r^{(3)} = 0$, as also can be proven directly from current conservation and $\psi^{(1)} = 0$. However, if the $\ell \neq 0$ part of $j_r^{(3)}$ is nonvanishing, we will obtain solutions within our ansatz such that $A_r^{(1)} \neq 0$. Such solutions are not smooth at $\mathscr{I}^+$ in any gauge.

Conversely, if $j_r^{(3)} = 0$, then the $\ell \neq 0$ part of $A_r^{(1)}$ vanishes by eq. (0.1). The $\ell = 0$ part of $A_r^{(1)}$ can then be set to zero within our ansatz by a gauge transformation of the form $\phi = c\ln(r)$. Thus, if $j_r^{(3)} = 0$, all solutions within our ansatz are smooth at $\mathscr{I}^+$ in some gauge.

## *2 Linearized Gravity*

The $\ell > 1$ part of $h_{rr}^{(1)}$ is gauge invariant within our ansatz, so if it is nonvanishing, it cannot be set to zero by a gauge transformation. From $\chi_r^{(1)} = 0$ and eq. (2.69), we obtain $\partial_u h_{rr}^{(1)} = 0$. The $ur$ and $rr$ components of the linearized Einstein's equation given by eqs. (2.62) and (2.64) with $n = 2$ yield, respectively,

$$\mathscr{D}^2 \overline{h}_{ur}^{(1)} - \mathscr{D}^A \chi_A^{(2)} = -16\pi T_{ur}^{(3)} \tag{0.2}$$

$$[\mathscr{D}^2 - 2]\overline{h}_{rr}^{(1)} + 2\overline{h}_{ur}^{(1)} - 2\mathscr{D}^A \overline{h}_{Ar}^{(1)} + 2\chi_r^{(2)} = -16\pi T_{rr}^{(3)}. \tag{0.3}$$

The angular divergence of the $rA$ component, eq. (2.65), yields

$$\mathscr{D}^2 \mathscr{D}^A \overline{h}_{rA}^{(1)} - 2\mathscr{D}^2 \overline{h}_{ur}^{(1)} + 2\mathscr{D}^2 \overline{h}_{rr}^{(1)} - \mathscr{D}^2 \chi_r^{(2)} + \mathscr{D}^A \chi_A^{(2)} = -16\pi \mathscr{D}^A T_{rA}^{(3)}. \tag{0.4}$$

Applying $\mathscr{D}^2$ to eq. (0.3) and taking a linear combination of the above equations, we obtain

$$\mathscr{D}^2[\mathscr{D}^2 + 2]h_{rr}^{(1)} = -16\pi(\mathscr{D}^2 T_{rr}^{(3)} + 2\mathscr{D}^A T_{rA}^{(3)} + 2T_{ur}^{(3)}) \tag{0.5}$$

where we used the fact that $\overline{h}_{rr} = h_{rr}$. The $\ell = 0$ and $\ell = 1$ parts of the right side must therefore vanish, and the right side must be stationary. Indeed, using conservation of stress energy and the dominant energy condition it can be shown that $T^{(3)}_{ur}, T^{(3)}_{rr}$ and $T^{(3)}_{rA}$ are stationary. However, the $\ell > 1$ part of the right side can be nonvanishing, and, if it is, we obtain a solution within our ansatz such that $h^{(1)}_{rr} \neq 0$. Such solutions are not smooth at $\mathscr{I}^+$ in any gauge.

Conversely, if $T^{(3)}_{ur} = T^{(3)}_{rr} = T^{(3)}_{rA} = 0$, then eq. (0.5) implies that $h^{(1)}_{rr}$ is a linear combination of an $l = 0$ and an $l = 1$ spherical harmonic. Let

$$X^a = c\left(\frac{\partial}{\partial u}\right)^a + f(x^A)\left(\frac{\partial}{\partial u}\right)^a - f(x^A)\left(\frac{\partial}{\partial r}\right)^a - q^{BC}\mathscr{D}_B f(x^A)\frac{1}{r}\left(\frac{\partial}{\partial x^C}\right)^a \tag{0.6}$$

where $c$ is a constant and $f(x^A)$ is a linear combination of $\ell = 1$ spherical harmonics, so that $X^a$ is a translational Killing field of the background Minkowksi spacetime. By a gauge transformation of the form

$$\xi^a = X^a \ln(r) \tag{0.7}$$

we can set the $\ell = 0, 1$ parts of $h^{(1)}_{rr}$ to zero within our ansatz. Thus, we can set $h^{(1)}_{rr} = 0$ and the solution is smooth at $\mathscr{I}^+$.

## *3 Nonlinear Gravity*

Again, we obtain $\partial_u h^{(1)}_{rr} = 0$. But there now is a new, nontrivial equation containing $h^{(1)}_{rr} = 0$. The $AB$-components of the Einstein equation given by eq. (2.66) with $n = 1$ where the right hand side of eq. (2.66) now picks up an additional nonlinear contribution $\mathcal{G}^{(2)}_{AB}$ given by eq. (2.102) with $d = 4$. We obtain

$$-q_{AB}\partial_u \chi^{(2)}_r = 2\partial_u(h^{(1)}_{rr} N_{AB}). \tag{0.8}$$

The right side is traceless whereas the left side is pure trace, so the only way this equation can hold is if both sides vanish. Thus, using $\partial_u h^{(1)}_{rr} = 0$, we obtain

$$h^{(1)}_{rr} \partial_u N_{AB} = 0. \tag{0.9}$$

This equation has no analog in the linearized theory. Since $N_{AB} \to 0$ as $u \to \pm\infty$, it implies that if the Bondi news is nonvanishing at angle $x^A$ at any $u$, then $h^{(1)}_{rr}(x^A) = 0$ at all $u$ (since $h^{(1)}_{rr}$ is independent of $u$). Thus, in particular, if the Bondi news is nonvanishing everywhere on one cross-section of $\mathscr{I}^+$, then $h^{(1)}_{rr} = 0$, and our ansatz in $d = 4$ is equivalent to smoothness at $\mathscr{I}^+$.

# APPENDIX B

# APPLYING THE LORENZ GAUGE WITH A SLOWER FALL-OFF ANSATZ FOR $D > 4$

In our ansatz eqs. (2.24)-(2.25) for $A_\mu$ and our ansatz eqs. (2.58)-(2.59) for $h_{\mu\nu}$, the slowest fall-off term was assumed to be at radiative order, $n = d/2 - 1$. However, in even dimensions with $d > 4$, the conditions of smoothness of $A_\mu$ and $\Omega^2 h_{\mu\nu} = r^{-2} h_{\mu\nu}$ at $\mathscr{I}^+$ would, *a priori*, allow terms with slower fall-off than permitted by our ansatz. This suggests a danger that our ansatz might exclude some solutions of physical interest. In this Appendix, we show that this is not the case by weakening our ansatz to permit slower fall-off, allowing the integer powers in even dimensions to start at order $1/r$ and allowing the half-integer powers in odd dimensions to start at order $1/\sqrt{r}$ for all $d > 4$. We will show that the Lorenz gauge can still be imposed within the context of this weaker ansatz. Since the Cartesian components of $A_\mu$ and $h_{\mu\nu}$ satisfy the scalar wave equation in Lorenz gauge, it follows from Remark 2.1 that the only additional solutions allowed by our weaker fall-off ansatz vanish in Lorenz gauge. Thus, the only new solutions allowed by the weaker ansatz are pure gauge. This justifies our stronger choice of ansatz eqs. (2.24)-(2.25) and eqs. (2.58)-(2.59)

## 1 *Electromagnetism*

We take our slower fall-off ansatz for the vector potential $A_a$ for $d > 4$ to be

$$A_\mu \sim \sum_{n=1}^{\infty} \frac{1}{r^n} A_\mu^{(n)}(u, x^A) \qquad d \text{ even} \tag{0.1}$$

$$A_\mu \sim \sum_{n=1/2}^{\infty} \frac{1}{r^n} A_\mu^{(n)}(u, x^A) + \sum_{p=d-3}^{\infty} \frac{1}{r^p} \tilde{A}_\mu^{(p)}(u, x^A) \qquad d \text{ odd.} \tag{0.2}$$

As discussed in sec. 2.3, in order to impose the Lorenz gauge we must solve the scalar wave equation (2.34) for a gauge scalar field $\phi$ with source $\psi$.

Consider, first, the case of $d$ even. We seek to solve eq. (2.34) with the ansatz

$$\phi \sim c \ln r + \sum_{n=0}^{\infty} \frac{1}{r^n} \phi^{(n)}(u, x^A) \tag{0.3}$$

where $c$ is a constant, and we require $\partial_u \phi^{(0)} = 0$ in order that $\partial_a \phi = O(1/r)$. The recursion relations for $\phi^{(n)}$ are given by eq. (2.18) with $\psi$ replacing $S$. Although $S = O(1/r^{d-2})$, *a priori* we have $\psi = O(1/r)$. However, an analysis similar to the proof of Prop. 2.3 shows that $\psi^{(1)}$ vanishes and

$$\partial_u \psi^{(2)} = -(d-4)\partial_u A_u^{(1)} \tag{0.4}$$

To solve eq. (2.18), we start with the radiative order recursion relation ($n = d/2 - 1$ in eq. (2.18)), which yields

$$\left[\mathscr{D}^2 - (d/2-2)(d/2-1)\right] \phi^{(d/2-2)} = \psi^{(d/2)}. \tag{0.5}$$

This angular operator is invertible, so we may uniquely solve for $\phi^{(d/2-2)}$. There is no difficulty in solving the recursion relations at faster fall-off, since we may then specify $\phi^{(d/2-1)}$ arbitrarily and solve for $\phi^{(n)}$ with $n > d/2 - 1$ as in Prop. 2.1. To obtain $\phi^{(n)}$ with $n < d/2 - 2$ we proceed iteratively by inverting the angular operators in the slower fall-off recursion relations. This works without any difficulty until we get to eq. (2.18) with $n = 1$.

$$c + \mathscr{D}^2 \phi^{(0)} = \psi^{(2)} + (d-4)\partial_u \phi^{(1)} \tag{0.6}$$

If the right side of this equation were not stationary, $\phi^{(0)}$ could not be stationary and the desired gauge transformation would not exist. However, we now shall show that the right side of eq. (0.6) is indeed stationary.

To show this, let

$$\gamma \equiv A_u - \partial_u \phi. \tag{0.7}$$

By Maxwell's equations, when $j_a = 0$, we have

$$\Box A_u = \partial_u \psi. \tag{0.8}$$

Thus, if $\phi$ satisfies eq. (2.34) and if $j_a = 0$, then $\gamma$ satisfies $\Box\gamma = 0$. Of course, $j_a$ need not be zero and we have not yet obtained a solution, $\phi$, to eq. (2.34). However, we have $j_a^{(n)} = 0$ for all $n < d-2$, and we have constructed above a solution to the recursion relations eq. (2.18) to solve for $\phi^{(n)}$ for all $n > 0$. Therefore, we obtain quantities $\gamma^{(n)}$ that satisfy the homogeneous recursion relations eq. (2.13) for all $1 < n < d-2$. In parallel with the argument of the previous paragraph, at radiative order, $n = d/2 - 1$, these relations imply that $\gamma^{(d/2-2)} = 0$. It then follows that $\gamma^{(n)} = 0$ for all $1 \leq n \leq d/2-2$. For $n = 1$, we obtain

$$\partial_u \phi^{(1)} = A_u^{(1)}. \tag{0.9}$$

But the Maxwell equation eq. (0.8) yields

$$(d-4)\partial_u A_u^{(1)} = -\partial_u \psi^{(2)}. \tag{0.10}$$

Thus the right side of eq. (0.6) is indeed stationary, as we desired to show. In parallel with solving eq. (2.41) when $d = 4$, we can choose $c$ so as to cancel the $l = 0$ part of the right side. We may then invert eq. (0.6) to obtain $\phi^{(0)}$. Thus, in even dimensions, the Lorenz gauge can be imposed within the weakened ansatz (0.1).

We now turn to the odd dimensional case. We take the scalar field $\phi$ to have the following expansion in powers of $1/r$

$$\phi \sim \sum_{n=-1/2}^{\infty} \frac{1}{r^n}\phi^{(n)}(u, x^A) + \sum_{p=d-3}^{\infty} \frac{1}{r^p}\tilde{\phi}^{(p)}(u, x^A). \tag{0.11}$$

Note that we allow a term, $\phi^{(-1/2)}$, that *grows* with $r$ as $r^{1/2}$. In order that $\partial_a\phi$ be consistent with our ansatz (0.2), it is necessary and sufficient that $\partial_u\phi^{(-1/2)} = 0$.

There is no difficulty in solving the recursion relations for $\tilde{\phi}^{(p)}$. There also is no difficulty in solving the recursion relations for $\phi^{(n)}$ for $n \geq 1/2$ in the manner specified in Prop. 2.2. However, there is a potential difficulty that arises when one attempts to solve the recursion relation for $\phi^{(-1/2)}$

$$[\mathscr{D}^2 + \frac{1}{4}(2d-5)]\phi^{(-1/2)} = (d-3)\partial_u\phi^{(1/2)} + \psi^{(3/2)}. \tag{0.12}$$

This equation can be uniquely solved for $\phi^{(-1/2)}$, but $\phi^{(-1/2)}$ will be stationary as required if and only if the right side be stationary. However, the stationarity of the right side can be proven in the same manner as done above for the even dimensional case. Thus, in odd dimensions, the Lorenz gauge can be imposed within the weakened ansatz (0.2).

## 2 Linearized Gravity

We take the slower fall-off ansatz for the metric perturbation $h_{\mu\nu}$ to be

$$h_{\mu\nu} \sim \sum_{n=1}^{\infty} \frac{1}{r^n} h_{\mu\nu}^{(n)}(u, x^A) \qquad d \text{ even} \tag{0.13}$$

$$h_{\mu\nu} \sim \sum_{n=1/2}^{\infty} \frac{1}{r^n} h_{\mu\nu}^{(n)}(u, x^A) + \sum_{p=d-3}^{\infty} \frac{1}{r^p} \tilde{h}_{\mu\nu}^{(p)}(u, x^A) \qquad d \text{ odd.} \tag{0.14}$$

We seek a gauge vector field, $\xi_\mu$, satisfying eq. (2.78). We take our ansatz for $\xi_\mu$ to be

$$\xi_\mu \sim c(\partial/\partial u)_\mu \ln r + \sum_{n=0}^{\infty} \frac{1}{r^n} \xi_\mu^{(n)}(u, x^A) \qquad d \text{ even} \tag{0.15}$$

$$\xi_\mu \sim \sum_{n=-1/2}^{\infty} \frac{1}{r^n} \xi_\mu^{(n)}(u, x^A) + \sum_{p=d-3}^{\infty} \frac{1}{r^p} \tilde{\xi}_\mu^{(p)}(u, x^A) \qquad d \text{ odd} \tag{0.16}$$

where, in even dimensions, $\partial_u \xi_\mu^{(0)} = 0$, and, in odd dimensions, $\partial_u \xi_\mu^{(-1/2)} = 0$.

In even dimensions, we can solve the recursion relations in parallel with the electromagnetic case. The only potential difficulty arises showing that $\partial_u \xi_\mu^{(0)} = 0$. This requires showing that in the recursion relation for $\xi_u^{(0)}$

$$c + \mathscr{D}^2 \xi_u^{(0)} = -2\chi_u^{(2)} + (d-4)\partial_u \xi_u^{(1)}, \tag{0.17}$$

the right side must be stationary. However, stationarity can be proven in close parallel with the electromagnetic case by defining

$$\Gamma \equiv -\bar{h}_{uu} + \frac{1}{d-2}\bar{h} - \partial_u \xi_u \tag{0.18}$$

and showing $\Gamma^{(n)} = 0$ for all $1 \leq n \leq d/2 - 2$, from which it can then can be shown that the right side of eq. (0.17) is stationary. We then can solve eq. (0.17) to obtain a stationary $\xi_u^{(0)}$. The equations for $\xi_r^{(0)}$ and $\xi_A^{(0)}$ can then be solved, and these quantities are stationary. Thus, in even dimensions, the Lorenz gauge can be imposed within the weakened ansatz (0.13).

The odd dimensional case mirrors the analysis of the electromagnetic case in odd dimensions, with the substitution of the argument of the previous paragraph to prove stationarity of $\xi_\mu^{(-1/2)} = 0$

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