

THE UNIVERSITY OF CHICAGO

MONODROMY OF SOME FAMILIES OF CUBIC AND K3 SURFACES

A DISSERTATION SUBMITTED TO  
THE FACULTY OF THE DIVISION OF THE PHYSICAL SCIENCES  
IN CANDIDACY FOR THE DEGREE OF  
DOCTOR OF PHILOSOPHY

DEPARTMENT OF MATHEMATICS

BY  
ADÁN MEDRANO MARTÍN DEL CAMPO

CHICAGO, ILLINOIS

AUGUST 2022

Copyright © 2022 by Adán Medrano Martín del Campo  
All Rights Reserved

*To my parents, Adán and Olga*

*To my sister, Olga*

*To my family and friends*

*This life is a dream, a gift we receive*

*To live and to love, we forge the path*

*Our nightmare in birth, our struggle for worth*

*In vain we carry on our mission to become*

*Adapt to this world, it's a chance we must take*

*We'll sing our song, we'll play our hand*

THE PATH - HAKEN, 2013



*We are the fire that whispers our mother's words*

*Help me, love, help me finish it*

*We are the soil that joy gave form*

*We are the dream, and these are my father's hands*

*Help me, love, help me finish it*

*We are the soil that joy gave form*

GRAVES - CALIGULA'S HORSE, 2017

# TABLE OF CONTENTS

LIST OF FIGURES . . . . .	vi
ACKNOWLEDGMENTS . . . . .	vii
ABSTRACT . . . . .	viii
1 INTRODUCTION . . . . .	1
2 CUBIC SURFACES . . . . .	3
2.1 Introduction . . . . .	3
2.2 Preliminaries . . . . .	6
2.3 Restricting $\text{Im}(\rho)$ . . . . .	10
2.4 Explicit computations for $f = y^2z - x^3 + xz^2$ . . . . .	15
2.5 Computing generators of $\text{Im}(\rho)$ . . . . .	19
2.6 Proof of the Main Theorem . . . . .	29
3 QUARTIC SURFACES . . . . .	31
3.1 Introduction . . . . .	31
3.2 Degree 2 del Pezzo Surfaces . . . . .	35
3.3 Computing $\text{Im}(\rho_2)$ . . . . .	37
3.4 Lattices . . . . .	40
3.5 Computing $\text{Im}(\rho_4)$ . . . . .	47
4 SURFACES OVER SEXTIC CURVES . . . . .	61
4.1 Introduction . . . . .	61
4.2 Lattices . . . . .	63
4.3 Restricting $\text{Im}(\rho_d)$ . . . . .	65
4.4 Decompositions of $H^2(X_d; \mathbb{C})$ . . . . .	66
4.5 The moduli of smooth sextic curves . . . . .	71
4.6 Conjectures . . . . .	73
REFERENCES . . . . .	74

## LIST OF FIGURES

2.1	Mapping of the lines in $V(w^3 - f)$ (left) to lines in $V(w^3 - f_H)$ (right) induced by $A'$ . The lines are determined by their <i>position</i> in the diagram, and each labeled line on the left diagram is mapped to the line with the same label on the right diagram. Each box is in correspondance to the inflection points shown in Section 2.4, and lines in each box lie over the same inflection point of $f$ (left) and $f_H$ (right). Moreover, each line has a defining equation which depends on a power of $\omega = e^{2\pi i/3}$ . Within each box, the powers $\omega^n$ in the defining equations of the lines are $1, \omega, \omega^2$ from top to bottom. . . . .	21
2.2	Generators $\gamma_-, \gamma_+$ of $\pi_1(\mathbb{C} \setminus \{\pm 1\}, 0)$ . . . . .	24
2.3	Plots of $\mathcal{X}_N(\gamma_-)$ (left) and $\mathcal{X}_N(\gamma_+)$ (right) for $N = 100$ . . . . .	26
2.4	Plots of $\mathcal{Y}_N(\gamma_-)$ (left) and $\mathcal{Y}_N(\gamma_+)$ (right) for $N = 100$ . . . . .	26
2.5	Permutation of the roots of $R_0$ induced by $\gamma_-$ (left) and $\gamma_+$ (right) . . . . .	27
2.6	Permutation on the $y$ -coordinates of inflection points of $f$ induced by $\gamma_-$ (above) and $\gamma_+$ (below). . . . .	27
3.1	The Schläfli graph (left) and Gosset graph (right). . . . .	32

## ACKNOWLEDGMENTS

I would like to express my most sincere gratitude to my advisor, Benson Farb. I cannot think of a person that better embodies the joy of doing mathematics, and I feel extremely lucky for having shared our journey over the past 5 years. For your invaluable guidance, encouragement and wisdom regarding topics mathematical and not, your wholehearted patience, kindness and support, your everlasting and contagious enthusiasm for mathematics, and for approaching me to the problems contained in this thesis, I thank you.

I also thank my second advisor, Eduard Looijenga, for his guidance, patience and wisdom throughout the trying times during which our interactions took place. Moreover, I am grateful to my friends and colleagues whose helpful conversations and exchanges helped me shape this mathematical work, including Ronno Das, Nathaniel Mayer, Reid Harris, Ishan Banerjee, Peter Huxford, Olga Medrano Martin del Campo, Trevor Hyde, Maxime Bergeron, Nick Salter, Danny Calegary, Dan Margalit, Igor Dolgachev, Daniel Allcock, Madhav Nori, Michael Artin, Curtis McMullen and many others. Further, many of these interactions took place in a space provided by the Jump Trading Mathlab Research Grant.

I am grateful to my family, whose unconditional love and support have been a constant source of motivation. I thank my fellow UChicago colleagues for their cheerful camaraderie, my lifelong friends for being present through both halcyon and tempestuous days, and the Mathematical Olympiad community for the sense of belonging they provided me.

Finally, I would like to thank my former mentor at MIT, Roger Casals, whose enthusiasm for mathematics parallels that of Benson, and who initially pointed me towards him and sparked my curiosity for monodromy, marking the beginning of this journey.

# ABSTRACT

This thesis is a compilation of three papers.

In Chapter 1, we introduce the general setup for these papers, which concern families of branched covers of  $\mathbb{P}^2$  branching over smooth curves of a fixed degree.

In Chapter 2, we consider the family of smooth cubic surfaces which can be realized as threefold-branched covers of  $\mathbb{P}^2$ , with branch locus equal to a smooth cubic curve. This family is parametrized by the space  $\mathcal{U}_3$  of smooth cubic curves in  $\mathbb{P}^2$  and each surface is equipped with a  $\mathbb{Z}/3\mathbb{Z}$  deck group action. We compute the image of the monodromy map  $\rho$  induced by the action of  $\pi_1(\mathcal{U}_3)$  on the 27 lines contained on the cubic surfaces of this family. Due to a classical result, this image is contained in the Weyl group  $W(E_6)$ . Our main result is that  $\rho$  is surjective onto the centralizer of the image of a generator of the deck group. Our proof is mainly computational, and relies on the relation between the 9 inflection points in a cubic curve and the 27 lines contained in the cubic surface branching over it.

In Chapter 3, we study the two families of surfaces which arise from considering cyclic branched covers of  $\mathbb{P}^2$  over smooth quartic curves. These consist of degree 2 del Pezzo surfaces with a  $\mathbb{Z}/2\mathbb{Z}$  action and  $K3$  surfaces with a  $\mathbb{Z}/4\mathbb{Z}$  action. We compute the monodromy groups of both families. In the first case, we obtain the Weyl group  $W(E_7)$ , corresponding to the automorphisms of the 56 lines contained in a degree 2 del Pezzo surface. In the second case we obtain an arithmetic lattice: the unitary group  $U(h_{L_-})$  of a type  $(1, 6)$  quadratic form over  $\mathbb{Z}[i]$  by building on results of Kondo and Allcock, Carlson, Toledo.

In Chapter 4, we study families of surfaces which arise from cyclic branched covers of  $\mathbb{P}^2$  over smooth sextic curves. These consist of surfaces with a  $\mathbb{Z}/d\mathbb{Z}$  action for  $d = 2, 3, 6$ . We bound the monodromy groups of the families corresponding to  $d = 2, 3$ . In doing so, we conjecture equivalent characterizations of the moduli of smooth sextic curves as quotients by building on results of Looijenga and Allcock, Carlson, Toledo.

# CHAPTER 1

## INTRODUCTION

Families of algebraic varieties are ubiquitous in algebraic geometry. A basic but often difficult question that arises for any such family is to determine its monodromy. A classical example is the *universal cubic surface* given by the smooth fiber bundle

$$\begin{array}{ccc}
 S & \hookrightarrow & \mathcal{E}_{3,3} & & \{(S, p) \mid p \in S\} \\
 & & \downarrow & & \downarrow \\
 & & \mathcal{U}_{3,3} & & S
 \end{array}$$

where  $\mathcal{U}_{3,3}$  is the parameter space of smooth cubic surfaces in  $\mathbb{P}^3$ . This bundle induces a monodromy homomorphism  $\rho : \pi_1(\mathcal{U}_{3,3}) \rightarrow \text{Aut}(H^2(S; \mathbb{Z}))$ . Klein and Jordan proved that the Galois group of the equation for the 27 lines on a cubic is precisely  $W(E_6)$ , the Weyl group of  $E_6$ , which coincides with the image of the monodromy homomorphism induced by the universal family  $\mathcal{E}_{3,3} \rightarrow \mathcal{U}_{3,3}$ .

Given a smooth degree  $d$  curve  $C$  in  $\mathbb{P}^2$  and an integer  $k \mid d$ , we may associate to it a degree  $k$  cyclic branch cover  $S$  of  $\mathbb{P}^2$  branching over it. A natural question that arises from this construction is to determine the monodromy representation of the family of surfaces it induces. This has been extensively studied in the case of universal families of a given type of varieties. By considering subfamilies of the universal ones and computing their monodromy representation, one obtains a better global picture of the one associated to the universal family, as well as of its acting fundamental group (see, e.g. [McM13]).

The parameter space of homogeneous degree  $d$  polynomials in variables  $x, y, z$  is given by

$$\mathbb{P}\left(\text{Sym}^d(\mathbb{C}^3)\right) = \mathbb{P}^{N(d)} \quad \text{where } N(d) = \binom{d+2}{2} - 1.$$

The *vanishing locus* of  $f \in \mathbb{P}^{N(d)}$  is defined as the set  $V(f) = \{P \in \mathbb{P}^2 \mid f(P) = 0\}$ . The *discriminant locus* is the subset  $\Delta_d \subset \mathbb{P}^{N(d)}$  consisting of polynomials whose vanishing locus

is singular. The parameter space of *smooth* degree  $d$  plane curves in  $\mathbb{P}^2$  is therefore defined as

$$\mathcal{U}_d = \mathbb{P}^{N(d)} \setminus \Delta_d.$$

Let  $f \in \mathcal{U}_d$  and let  $C = V(f)$  be its vanishing locus. Then  $[C] = d[H] \in H_2(\mathbb{P}^2; \mathbb{Z})$ , where  $[H]$  is the hyperplane class in  $\mathbb{P}^2$ . The curve  $C$  is a complex codimension 1 submanifold of  $\mathbb{P}^2$ , so there exists a cyclic  $k$ -fold branched cover  $X$  of  $\mathbb{P}^2$  with branched locus equal to  $C$  if and only if  $[C]$  is a multiple of  $k$  of  $[H]$  (see [Mor01], Proposition 4.10) and this is equivalent to  $k \mid d$ . It is a classical result (see [Zar29]) that the fundamental group of the complement of a smooth degree  $d$  curve in  $\mathbb{P}^2$  is cyclic of order  $d$ . In light of this, we study the cases corresponding to  $d = 3$  in Chapter 2,  $d = 4$  in Chapter 3 and  $d = 6$  Chapter 4.

The problem studied in this thesis lies within a much more general context, arising from the study of universal families of degree  $d$  cyclic branched covers over smooth hypersurfaces of degree  $n$  in  $\mathbb{P}^N$ , where  $d$  divides  $n$ .

$$\begin{array}{ccc} X_{d,n} & \hookrightarrow & \mathcal{E}_{d,n,N} \\ & & \downarrow \\ & & \mathcal{U}_{n,N} \end{array}$$

In this thesis, we study the family corresponding to the cases  $(d, n, N) = (3, 3, 2)$ ,  $(4, 4, 2)$  and  $(2, 4, 2)$ . The problem of determining the monodromy of these families has been previously studied in lower dimension. In [McM13], McMullen provides a description of the family of degree  $d$  cyclic branched covers branching over  $n$  distinct points in  $\mathbb{P}^1$ , given by a smooth subvariety of degree  $n$  in  $\mathbb{P}^1$ . This corresponds to the case  $(d, n, N) = (d, n, 1)$ . He then determines for which pairs  $(d, n)$  the induced monodromy map is surjective onto the corresponding automorphism group with the natural restrictions that come along with the branch cover structure.

# CHAPTER 2

## CUBIC SURFACES

### 2.1 Introduction

In this chapter we study the subfamily of smooth cubic surfaces which can be realized as cyclic branched covers of  $\mathbb{P}^2$ , branching over a smooth cubic plane curve. To do this we exploit connections between datum associated to cubic surfaces, and its analogue for cubic curves. The most notable example in this chapter is the relation between the 27 lines contained in a cubic surface, and the 9 inflection points of the curve over which it branches.

For  $f \in \mathcal{U}_3$  consider the cyclic 3-fold branched cover

$$\begin{array}{ccc} & X_f & \\ & \downarrow \mathbb{Z}/3\mathbb{Z} & \\ \mathbb{P}^2 & \longleftarrow & V(f) \end{array}$$

The surface  $X_f$  can be embedded into  $\mathbb{P}^3$  as a cubic surface  $V(w^3 - f)$ . In particular,  $\mathcal{U}_3$  parametrizes all such surfaces of the form  $V(w^3 - f)$ . We define the *universal 3-branched cover of  $\mathbb{P}^2$*  as the fiber bundle

$$\begin{aligned} \mathcal{E}_3 &= \left\{ (P, f) \in \mathbb{P}^3 \times \mathcal{U}_3 \mid P \in V(w^3 - f) \right\} \rightarrow \mathcal{U}_3 \\ & (P, f) \mapsto f \end{aligned}$$

where the fiber of  $\mathcal{E}_3 \rightarrow \mathcal{U}_3$  is diffeomorphic to a smooth cubic surface. For the sake of simplicity in calculations that will be carried on further in this chapter, choose the curve  $f = y^2z - x^3 + xz^2$  as a base point in  $\mathcal{U}_3$ . The action of  $\pi_1(\mathcal{U}_3)$  on  $H^2(V(w^3 - f); \mathbb{Z})$  induces a monodromy homomorphism

$$\rho : \pi_1(\mathcal{U}_3) \rightarrow \text{Aut} \left( H^2 \left( V(w^3 - f); \mathbb{Z} \right) \right).$$

In [DL81], Dolgachev-Libgober describe  $\pi_1(\mathcal{U}_3)$  as a central extension

$$1 \rightarrow \mathcal{H}_3(\mathbb{Z}/3\mathbb{Z}) \rightarrow \pi_1(\mathcal{U}_3) \rightarrow SL_2(\mathbb{Z}) \rightarrow 1 \quad (2.1)$$

of  $SL_2(\mathbb{Z})$  by the  $\mathbb{Z}/3\mathbb{Z}$ -points of the 3-dimensional Heissenberg group, defined as

$$\mathcal{H}_3(\mathbb{Z}/3\mathbb{Z}) = \left\{ \begin{pmatrix} 1 & a & c \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix} \mid a, b, c \in \mathbb{Z}/3\mathbb{Z} \right\}.$$

The extension (2.1) is split, so  $\pi_1(\mathcal{U}_3)$  has a semidirect product structure  $\mathcal{H}_3(\mathbb{Z}/3\mathbb{Z}) \rtimes_{\varphi} SL_2(\mathbb{Z})$ , which we describe later in the chapter.

The image of  $\rho$  can be restricted noting that the intersection form in  $H^2(V(w^3 - f); \mathbb{Z})$  remains invariant under the action of the image of  $\rho$ . As explained in Section 2.2.1, this implies that the image of  $\rho$  is contained in the automorphism group of the 27 lines contained in  $V(w^3 - f)$  due to an argument in [Har79]. On the same paper, Harris shows this group is precisely  $W(E_6)$ , the Weyl group of  $E_6$ . Moreover, each fiber of  $\mathcal{E}_3$  has a  $\mathbb{Z}/3\mathbb{Z}$  deck group action induced by its cyclic branched cover structure. A generator  $T$  of this deck group action is given by

$$\begin{aligned} T : V(w^3 - f) &\rightarrow V(w^3 - f) \\ [x : y : z : w] &\mapsto [x : y : z : e^{-2\pi i/3} w] \end{aligned}$$

and  $T$  induces an action  $\Omega$  on  $H^2(V(w^3 - f); \mathbb{Z})$  which commutes with the image of  $\rho$ . As we shall see,  $\Omega$  is realized as the image of a generator of the center of  $\mathcal{H}_3(\mathbb{Z}/3\mathbb{Z})$  inside  $\pi_1(\mathcal{U}_3)$ . Altogether, this shows that one may restrict the monodromy homomorphism  $\rho$  to

the centralizer  $C_{W(E_6)}(\Omega)$  of  $\Omega$  in  $W(E_6)$ , giving:

$$\rho : \pi_1(\mathcal{U}_3) \rightarrow C_{W(E_6)}(\Omega).$$

Our main result in this chapter is the following.

**Theorem 2.1.1.** *The monodromy representation*

$$\rho : \pi_1(\mathcal{U}_3) \rightarrow C_{W(E_6)}(\Omega)$$

*of the universal 3-branched cover of  $\mathbb{P}^2$  is surjective and its image is isomorphic to the semidirect product*

$$\mathcal{H}_3(\mathbb{Z}/3\mathbb{Z}) \rtimes_{\varphi} SL_2(\mathbb{Z}/3\mathbb{Z}).$$

The first step towards the proof of Theorem 2.1.1 is to find geometric representatives of a basis for  $H^2(V(w^3 - f); \mathbb{Z})$  in terms of  $V(f) \subset \mathbb{P}^2$ . The crucial observation is that to each of the 9 inflection points of  $V(f)$ , we can associate 3 lines in  $V(w^3 - f)$ , all of which lie over the given inflection point.

Then we reduce the range of  $\rho$  to  $W(E_6)$ . The action  $\Omega$  induced by the  $\mathbb{Z}/3\mathbb{Z}$  deck group action gives a permutation of the 27 lines in  $V(w^3 - f)$  which permutes the line triples lying over a same inflection point disjointly. The image of  $\rho$  commutes with  $\Omega$  and thus the image of  $\rho$  is contained in  $C_{W(E_6)}(\Omega)$ .

Using the semidirect product structure on  $\pi_1(\mathcal{U}_3)$ , we choose 4 of its elements and compute their action on the geometric datum associated to the chosen basepoint curve

$$f = y^2z - x^3 + xz^2.$$

The key step of the proof is performing these computations explicitly. The group generated by the images of the 4 chosen elements is then shown to be isomorphic to  $\mathcal{H}_3(\mathbb{Z}/3\mathbb{Z}) \rtimes_{\varphi}$

$SL_2(\mathbb{Z}/3\mathbb{Z})$ . The centralizer  $C_{W(E_6)}(\Omega)$  and this subgroup both have order 648, so they must be isomorphic. Since

$$\mathcal{H}_3(\mathbb{Z}/3\mathbb{Z}) \rtimes_{\varphi} SL_2(\mathbb{Z}/3\mathbb{Z}) < \text{Im}(\rho) < C_{W(E_6)}(\Omega)$$

the three groups must be isomorphic, concluding the proof. A public repository containing the Sage [The21] and Mathematica [Inc] computations employed can be found in this link:

Computation Repository

## 2.2 Preliminaries

### 2.2.1 General facts about cubic surfaces

We begin by recalling facts about smooth cubic surfaces which we will employ through the chapter. These facts can be found in e.g. [Har77]. Let  $X \subset \mathbb{P}^3$  be a smooth cubic surface.

A classical result states all smooth cubic surfaces  $X$  are blowups of  $\mathbb{P}^2$  at six points  $p_1, \dots, p_6$  in general position, and moreover  $X$  contains 27 lines. Therefore, the intersection form on  $H^2(X; \mathbb{Z})$  is of type  $(1, 6)$  and we have the decomposition

$$H^2(X; \mathbb{Z}) \cong H^2(X; \mathbb{Z})_+ \oplus H^2(X; \mathbb{Z})_- \cong \mathbb{Z} \oplus \mathbb{Z}^6.$$

The cohomology classes of any six pairwise disjoint lines in  $X$  form a basis for  $H^2(X; \mathbb{Z})_-$ . The intersection pattern of these 27 lines is given by the dual of the Schläfli graph, regarding each line as a vertex, and two lines intersect if and only if their corresponding vertices are joined by an edge. Let

$$\begin{array}{ccc} Bl_{\{p_1, \dots, p_6\}}(\mathbb{P}^2) & \xrightarrow{\cong} & X \\ \downarrow \pi & & \\ \mathbb{P}^2 & & \end{array}$$

be the blowup map from  $X$  to  $\mathbb{P}^2$  at the points  $p_1, \dots, p_6$ . Set  $e_0, e_1, \dots, e_6 \in H^2(X; \mathbb{Z})$  as  $e_0 = \pi^{-1}(\text{PD}[H])$ , the preimage of the Poincaré dual of the hyperplane class  $[H]$  in  $\mathbb{P}^2$ , and  $e_i$  as the class of the exceptional divisor  $L_i$  corresponding to  $p_i$  for  $i = 1, \dots, 6$ . The divisors  $L_1, \dots, L_6$  constitute 6 of the lines contained in  $X$ , and the remaining  $21 = 15 + 6$  lines are given by blowups of

- the 15 lines which pass through  $p_i$  and  $p_j$  for all pairs  $i \neq j$ , and
- the 6 conics determined by 5 of the blowup points  $p_i$ .

The lines  $L_1, \dots, L_6$  are pairwise disjoint, and thus  $e_1, \dots, e_6$  form a basis for  $H^2(X; \mathbb{Z})_-$ . With respect to the basis  $\{e_0, e_1, \dots, e_6\}$  of  $H^2(X; \mathbb{Z})$ , the intersection form is given by

$$(\cdot, \cdot)_X : H^2(X; \mathbb{Z}) \times H^2(X; \mathbb{Z}) \rightarrow \mathbb{Z}$$

$$(\mathbf{a}, \mathbf{b}) \mapsto \mathbf{a}^T \begin{pmatrix} 1 & 0 \\ 0 & -I_6 \end{pmatrix} \mathbf{b}.$$

Let  $L_{i,j}$  be the line in  $X$  which is a blowup of the line passing through  $p_i$  and  $p_j$  and  $L_{i^*}$  be the line in  $X$  which is a blowup of the conic passing through all blowup points except for  $p_i$ . Then

$$[L_{i,j}] = e_0 - e_i - e_j \quad \text{and} \quad [L_{i^*}] = 2e_0 + e_i - (e_1 + \dots + e_6).$$

In [Har79], it is shown that the group of automorphisms of  $H^2(X; \mathbb{Z})$  that preserve the intersection form is isomorphic to the Galois group of  $X$ , and these two groups are isomorphic to  $W(E_6)$ . These automorphisms are determined by the images of the cohomology classes of any six disjoint lines, such as  $e_1, \dots, e_6$ . Since the intersection form is preserved, this is equivalent to a permutation of the 27 lines in  $X$ .

### 2.2.2 Inflection points of $V(f)$ versus lines in $V(w^3 - f)$

The lines on a cubic surface  $X \cong V(w^3 - f)$  have a very particular structure, which can be described in terms of the inflection points of  $f$ . Since  $f$  is a smooth cubic curve, it has 9 inflection points, given by the intersection  $V(f) \cap V(\det \text{Hess}(f))$ , where

$$\text{Hess}(f) = \begin{pmatrix} \frac{\partial^2 f}{\partial x^2} & \frac{\partial^2 f}{\partial x \partial y} & \frac{\partial^2 f}{\partial x \partial z} \\ \frac{\partial^2 f}{\partial y \partial x} & \frac{\partial^2 f}{\partial y^2} & \frac{\partial^2 f}{\partial y \partial z} \\ \frac{\partial^2 f}{\partial z \partial x} & \frac{\partial^2 f}{\partial z \partial y} & \frac{\partial^2 f}{\partial z^2} \end{pmatrix}$$

is the Hessian matrix of  $f$ . The structure of the lines is described in the following proposition.

**Proposition 2.2.1.** Consider a smooth cubic curve  $V(f) \subset \mathbb{P}^2$  and the branched cover

$$\begin{array}{ccc} V(w^3 - f) & & \\ \mathbb{Z}/3\mathbb{Z} \downarrow p & & \\ \mathbb{P}^2 & \longleftrightarrow & V(f) \end{array}$$

Let  $P$  be an inflection point of  $V(f)$  and let  $l_P \subset \mathbb{P}^2$  be the tangent line to  $V(f)$  at  $P$ . Then  $p^{-1}(l_P)$  consists of 3 concurrent lines at  $p^{-1}(P)$  in  $V(w^3 - f)$ . Namely, the 27 lines contained in  $V(w^3 - f)$  lie over each of the 9 inflection points of  $f$  in concurrent triples.

*Proof.* With a change of coordinates,  $f$  can be transformed into its *Hesse normal form*,

$$f = x^3 + y^3 + z^3 - 3\mu xyz$$

for some  $\mu$  such that  $\mu^3 \neq 1$ . We have  $\det \text{Hess}(f) = 216(1 - \mu^3)xyz$ , so the inflection points of  $f$  are

$$[1 : -1 : 0] \quad [1 : -\omega : 0] \quad [1 : -\omega^2 : 0]$$

$$[-1 : 0 : 1] \quad [-\omega : 0 : 1] \quad [-\omega^2 : 0 : 1]$$

$$[0 : 1 : -1] \quad [0 : 1 : -\omega] \quad [0 : 1 : -\omega^2]$$

The tangent line to  $V(f)$  at a point  $P$  is given by the equation

$$\nabla f_P \cdot (x, y, z) = x \frac{\partial f}{\partial x}(P) + y \frac{\partial f}{\partial y}(P) + z \frac{\partial f}{\partial z}(P) = 0$$

and for such an  $f$  we have  $\nabla f = (3x^2 - 3\mu yz, 3y^2 - 3\mu zx, 3z^2 - 3\mu xy)$ . Hence the tangent lines at the inflection points of  $f$  are

$$\begin{aligned} &V(x + y + \mu z) \quad V(x + \omega^2 y + \mu \omega z) \quad V(x + \omega y + \mu \omega^2 z) \\ &V(x + \mu y + z) \quad V(\omega^2 x + \mu \omega y + z) \quad V(\omega x + \mu \omega^2 y + z) \\ &V(\mu x + y + z) \quad V(\mu \omega x + y + \omega^2 z) \quad V(\mu \omega^2 x + y + \omega z) \end{aligned}$$

in correspondence with the inflection points shown above. Consider a tangent line  $L$  at one of the inflection points  $P$ , say  $L = V(\mu x + y + z)$ . We now proceed to determine its preimage  $p^{-1}(L)$  (the preimage for all remaining lines is determined in an analogous manner). Points in  $p^{-1}(L)$  satisfy  $y = -\mu x - z$ . Combining this equation along with  $f$  we obtain

$$w^3 - x^3 - z^3 + (\mu x + z)^3 + 3\mu xz(\mu x + z) = w^3 - (1 - \mu^3)x^3 = 0$$

and thus, letting  $\eta$  be a cube root of  $1 - \mu^3$ , we have that  $p^{-1}(L)$  consists of three lines through  $p^{-1}(P)$  given by

$$V(w - \omega^n \eta x) \cap V(\mu x + y + z) \subset \mathbb{P}^3 \quad \text{for } n = 0, 1, 2.$$

This is analogous for the remaining tangent lines at the inflection points of  $V(f)$ , so to

each inflection point  $P$  of  $V(f)$ , we have associated 3 lines in  $V(w^3 - f)$  passing through  $p^{-1}(P)$ .  $\square$

It should be emphasized that Proposition 2.2.1 is the crucial property that characterises the cubic surfaces of the form  $V(w^3 - f)$ , and this property will be exploited throughout this chapter.

## 2.3 Restricting $\mathbf{Im}(\rho)$

In this section, we show that the image of  $\rho$  is contained in the centralizer of an order 3 element  $\Omega \in W(E_6)$ .

### 2.3.1 The action $\Omega$

Consider the action  $\tau$  of  $\mathcal{H}_3(\mathbb{Z}/3\mathbb{Z})$  on  $V(f)$  coming from the extension given in [DL81], which acts on the inflection points of  $V(f)$ . These inflection points correspond to the 3-torsion points of  $V(f)$ , or those points  $P$  with  $3P = 0$ , given an elliptic curve structure on it. Moreover,  $\tau$  acts by translation on the  $\mathbb{Z}/3\mathbb{Z} \times \mathbb{Z}/3\mathbb{Z}$  lattice formed by these 3-torsion points. Namely, the translation on the  $\mathbb{Z}/3\mathbb{Z} \times \mathbb{Z}/3\mathbb{Z}$  lattice is given by the  $a$  and  $b$  entries of a matrix element of  $\mathcal{H}_3(\mathbb{Z}/3\mathbb{Z})$  as follows:

$$\tau : \mathcal{H}_3(\mathbb{Z}/3\mathbb{Z}) \curvearrowright V(f)(\mathbb{Z}/3\mathbb{Z}) \cong \mathbb{Z}/3\mathbb{Z} \times \mathbb{Z}/3\mathbb{Z}$$

$$\begin{pmatrix} 1 & a & c \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix} \cdot (x, y) = (x + a, y + b).$$

We will use this action in Section 2.5.1, further explained from the Hessian normal form of a cubic curve. The action of an element  $M \in \mathcal{H}_3(\mathbb{Z}/3\mathbb{Z})$  under  $\tau$  comes from a linear transformation in  $\mathbb{P}^2$  [DL81]. This action can be lifted to a linear transformation on  $\mathbb{P}^3$

giving an automorphism of  $V(w^3 - f)$  which induces the element in  $\text{Im}(\rho)$  coming from  $M$ . Let  $Z$  be a generator of the center  $Z(\mathcal{H}_3(\mathbb{Z}/3\mathbb{Z}))$ . Lifting  $Z$  to an action on  $\mathbb{P}^3$  gives an automorphism  $T$  which fixes the inflection points in the curve  $V(f) \cap V(w)$ , and acts on  $V(w^3 - f)$  by multiplication by  $\omega^{-1} = e^{-2\pi i/3}$  on the  $w$ -coordinate:

$$T : [x : y : z : w] \mapsto [x : y : z : \omega^{-1}w] \quad \omega = e^{\frac{2\pi i}{3}}.$$

This is precisely a generator of the  $\mathbb{Z}/3\mathbb{Z}$  deck group action on  $V(w^3 - f)$ . Let  $\Omega$  be the element of  $\text{Im}(\rho)$  induced by  $T$ . We can now prove the following proposition.

**Proposition 2.3.1.**  $\text{Im}(\rho)$  is contained in the centralizer  $C_{\text{Aut}(H^2(V(w^3-f); \mathbb{Z}))}(\Omega)$ .

*Proof.* The homomorphism  $\Phi : SL_2(\mathbb{Z}) \rightarrow \text{Aut}(\mathcal{H}_3(\mathbb{Z}/3\mathbb{Z}))$  induced by the split group extension

$$1 \rightarrow \mathcal{H}_3(\mathbb{Z}/3\mathbb{Z}) \rightarrow \pi_1(\mathcal{U}_3) \rightarrow SL_2(\mathbb{Z}) \rightarrow 1$$

described in [DL81] is given by the composition

$$\begin{array}{ccc} & \Phi & \\ & \curvearrowright & \\ SL_2(\mathbb{Z}) & \xrightarrow{\text{mod } 3} & SL_2(\mathbb{Z}/3\mathbb{Z}) \xrightarrow{\varphi} \text{Aut}(\mathcal{H}_3(\mathbb{Z}/3\mathbb{Z})) \end{array}$$

where the map  $\varphi$  is given by:

$$\begin{aligned} \varphi : SL_2(\mathbb{Z}/3\mathbb{Z}) &\rightarrow \text{Aut}(\mathcal{H}_3(\mathbb{Z}/3\mathbb{Z})) \\ M &\mapsto \left[ \varphi_M : \begin{pmatrix} 1 & a & c \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix} \mapsto \begin{pmatrix} 1 & M_1(a, b) & c \\ 0 & 1 & M_2(a, b) \\ 0 & 0 & 1 \end{pmatrix} \right] \\ M(a, b) &= (M_1(a, b), M_2(a, b)) \end{aligned}$$

for all  $M \in SL_2(\mathbb{Z}/3\mathbb{Z})$ . Since the center of  $\mathcal{H}_3(\mathbb{Z}/3\mathbb{Z})$  is given by

$$Z(\mathcal{H}_3(\mathbb{Z}/3\mathbb{Z})) \cong \left\langle \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \right\rangle$$

it follows that  $\Phi_N$  fixes  $Z(\mathcal{H}_3(\mathbb{Z}/3\mathbb{Z}))$  for every  $N \in SL_2(\mathbb{Z})$ . Therefore,  $Z(\mathcal{H}_3(\mathbb{Z}/3\mathbb{Z}))$  is in the center of  $\pi_1(\mathcal{U}_3)$ . Since  $\Omega \in \rho(Z(\mathcal{H}_3(\mathbb{Z}/3\mathbb{Z})))$  by construction, it follows that  $\Omega$  is in the center of  $\text{Im}(\rho)$ , or equivalently,  $\text{Im}(\rho)$  is contained in the centralizer of  $\Omega$ .  $\square$

### 2.3.2 Restriction to $W(E_6)$

**Proposition 2.3.2.**  $\text{Im}(\rho)$  is contained in  $W(E_6)$ .

*Proof.* The action of any element  $\sigma \in \pi_1(\mathcal{U}_3)$  on  $V(w^3 - f)$  maps  $V(w^3 - f) \cap V(w)$  to itself, inducing a permutation on the set of 9 inflection points of  $V(f)$ . This implies that the preimages of the inflection points under the branched cover  $p$  are permuted as well, and we have shown these are exactly the 27 lines in  $V(w^3 - f)$ . Therefore,  $\sigma$  maps lines to lines, and for any line  $L \subset V(w^3 - f)$  we have

$$\rho(\sigma)([L]) = [\sigma(L)].$$

The incidence of the 27 lines contained in  $V(w^3 - f)$  is therefore preserved. The classes of these lines span  $H^2(V(w^3 - f); \mathbb{Z})$  and its intersection form is non-degenerate, so  $\rho$  preserves said intersection form. In [Har79], it is shown that automorphisms of  $H^2(V(w^3 - f); \mathbb{Z})$  preserving the form lie within the odd orthogonal group  $O_6^-(\mathbb{Z}/2\mathbb{Z})$ , which is isomorphic to the Weyl group  $W(E_6)$ . Hence,  $\text{Im}(\rho) \subset W(E_6)$ .  $\square$

Summarizing, Proposition 2.3.1 and Proposition 2.3.2 give the following corollary.

**Corollary 2.3.3.** We have  $\text{Im}(\rho) \subset C_{W(E_6)}(\Omega)$ .

### 2.3.3 Computing $|C_{W(E_6)}(\Omega)|$

Consider the  $W(E_6)$ -action on itself by conjugation. By the orbit-stabilizer theorem applied to  $\Omega$ , we have

$$51840 = |W(E_6)| = |\text{Orbit}_{W(E_6)}(\Omega)| \cdot |C_{W(E_6)}(\Omega)|$$

and the order of the orbit of  $\Omega$  corresponds to the size of its conjugacy class. We will determine the size of this conjugacy class by looking at the character table of  $W(E_6)$ , found for example in [Fra51]. For this, we must compute the action of  $\Omega$  on the set of lines of our chosen base point, the surface  $V(w^3 - f)$  corresponding to the curve

$$f = y^2z - x^3 + xz^2.$$

With the help of [The21], we enumerate the 27 lines in  $V(w^3 - f)$ . Then, we find a set of six pairwise disjoint lines  $L_1, \dots, L_6 \subset V(w^3 - f)$ , and proceed to compute the intersection pattern of the remaining 21 lines with each  $L_i$  in order to compute their cohomology classes. We obtain that for our choice of lines,  $\Omega$  acts by

$$\begin{array}{lll} L_1 \mapsto L_{5^*} & L_2 \mapsto L_{2,3} & L_3 \mapsto L_{3,6} \\ L_4 \mapsto L_{1^*} & L_5 \mapsto L_{4^*} & L_6 \mapsto L_{2,6} \end{array}$$

following the notation in Section 2.2.1. Therefore, with respect to the basis  $\{e_0, [L_1], \dots, [L_6]\}$  we have

$$\Omega = \begin{pmatrix} 4 & 2 & 1 & 1 & 2 & 2 & 1 \\ -1 & -1 & 0 & 0 & 0 & -1 & 0 \\ -2 & -1 & -1 & 0 & -1 & -1 & -1 \\ -2 & -1 & -1 & -1 & -1 & -1 & 0 \\ -1 & -1 & 0 & 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 & -1 & -1 & 0 \\ -2 & -1 & 0 & -1 & -1 & -1 & -1 \end{pmatrix} \in \text{Aut} \left( H^2 \left( V \left( w^3 - f \right); \mathbb{Z} \right) \right).$$

Now consider the complex  $W(E_6)$ -representation given by

$$\mathbb{C}^7 \cong H^2 \left( V \left( w^3 - f \right); \mathbb{C} \right) \cong H^2 \left( V \left( w^3 - f \right); \mathbb{Z} \right) \otimes \mathbb{C}$$

This representation contains a copy of the trivial representation, since the canonical class is fixed by  $W(E_6)$ , and a copy of an irreducible 6-dimensional representation. Namely,

$$H^2 \left( V \left( w^3 - f \right); \mathbb{C} \right) \cong \mathbb{C}_{\text{triv}} \oplus V_6.$$

Since  $\text{Trace}(\Omega) = -2$ , at the level of characters we have

$$-2 = \chi_{H^2(V(w^3-f);\mathbb{C})}(\Omega) = \chi_{\mathbb{C}_{\text{triv}}}(\Omega) + \chi_{V_6}(\Omega) = 1 + \chi_{V_6}(\Omega)$$

and therefore  $\chi_{V_6}(\Omega) = -3$ . There exist two 6-dimensional irreducible representations of  $W(E_6)$ , both of which have a unique conjugacy class whose character equals  $-3$ . Therefore,

$\Omega$  must belong to this conjugacy class, which has order 80. Thus, we conclude

$$\left| C_{W(E_6)}(\Omega) \right| = \frac{|W(E_6)|}{\left| \text{Orbit}_{W(E_6)}(\Omega) \right|} = \frac{51840}{80} = 648.$$

## 2.4 Explicit computations for $f = y^2z - x^3 + xz^2$

We have constructed via Proposition 2.2.1 a correspondence between the inflection points of a smooth cubic curve  $V(f)$  and triples of lines in its associated surface  $V(w^3 - f)$  lying over the inflection points of  $V(f)$ . The goal of this section is to compute this datum explicitly for a choice of  $f$  serving as a base point in  $\mathcal{U}_3$ . This datum is then manipulated with the aid of computer software ([The21] and [Inc]), and used in Section 2.5 to explicitly compute generators of  $\text{Im}(\rho)$ .

### 2.4.1 Inflection points of $f$

Consider the curve  $f = y^2z - x^3 + xz^2$ . Then

$$\nabla f = \begin{pmatrix} -3x^2 + z^2 \\ 2yz \\ y^2 + 2xz \end{pmatrix} \quad \text{and} \quad \text{Hess}(f) = \begin{pmatrix} -6x & 0 & 2z \\ 0 & 2z & 2y \\ 2z & 2y & 2x \end{pmatrix}.$$

Hence,  $\det \text{Hess}(f) = 8(3x(y^2 - xz) - z^3)$  and the inflection points of  $f$  are given by

$$V(y^2z - x^3 + xz^2) \cap V(3xy^2 - 3x^2z - z^3).$$

Let  $[x : y : z]$  be an inflection point of  $f$ .

- If  $z = 0$ , then  $f = -x^3 = 0$  so  $x = 0$ , and thus  $[x : y : z] = [0 : 1 : 0]$ .
- If  $z \neq 0$ , then one may assume  $z = 1$ . Then  $y^2 = x^3 - x$  and  $3xy^2 - 3x^2z - z^3 = 0$ .

Substituting  $y^2$  in the second equation we obtain

$$R(x) = 3x^4 - 6x^2 - 1 = 0.$$

Hence, the remaining 8 inflection points are  $[\alpha : \pm\sqrt{\alpha^3 - \alpha} : 1]$  with  $\alpha$  a root of  $R(x)$ .

Namely,

$$\alpha \in \left\{ \pm a, \pm ia \frac{(\sqrt{3}-1)^2}{2} \right\} \quad \text{where } a = \sqrt{\frac{3+2\sqrt{3}}{3}}.$$

To describe the  $y$ -coordinate of these inflection points, note that  $\alpha^2 - 1 = \pm \frac{2\sqrt{3}}{3}$ , so

$$y \in \left\{ \pm b, \pm ib, \pm b(1+i)(\sqrt{3}-1), \pm b(1-i)(\sqrt{3}-1) \right\} \quad \text{where } b = \sqrt{a \frac{2\sqrt{3}}{3}}.$$

Therefore, the inflection points of  $f = y^2z - x^3 + xz^2$  are

$$[-a : +ib : 1] \quad \left[ -ia \frac{(\sqrt{3}-1)^2}{2} : -b(1+i)(\sqrt{3}-1) : 1 \right] \quad \left[ +ia \frac{(\sqrt{3}-1)^2}{2} : -b(1-i)(\sqrt{3}-1) : 1 \right]$$

$$[-a : -ib : 1] \quad \left[ +ia \frac{(\sqrt{3}-1)^2}{2} : +b(1-i)(\sqrt{3}-1) : 1 \right] \quad \left[ -ia \frac{(\sqrt{3}-1)^2}{2} : +b(1+i)(\sqrt{3}-1) : 1 \right]$$

$$[0 : 1 : 0]$$

$$[+a : +b : 1]$$

$$[+a : -b : 1]$$

and we present them in this way since these inflection points are in direct correspondence with the inflection points of the Hesse normal form  $f_H$  of  $f = y^2z - x^3 + xz^2$  as shown in the proof of Proposition 2.2.1 after transforming  $V(f)$  to  $V(f_H)$  via the linear map  $A : \mathbb{P}^2 \rightarrow \mathbb{P}^2$  given by

$$A = \begin{pmatrix} 1 & 0 & -a \\ -\frac{\sqrt{3}+1}{2} & -i\sqrt[4]{\frac{3+2\sqrt{3}}{4}} & -a\left(\frac{\sqrt{3}-1}{2}\right) \\ -\frac{\sqrt{3}+1}{2} & i\sqrt[4]{\frac{3+2\sqrt{3}}{4}} & -a\left(\frac{\sqrt{3}-1}{2}\right) \end{pmatrix}.$$

We use this transformation to compute the images of the generators of  $\mathcal{H}_3(\mathbb{Z}/3\mathbb{Z}) < \pi_1(\mathcal{U}_3)$ , as [DL81] describe the action of  $\mathcal{H}_3(\mathbb{Z}/3\mathbb{Z})$  in terms on the Hessian form of cubic curves.

### 2.4.2 Lines in $V(w^3 - f_\lambda)$

Consider a family of variations of  $f$  parametrized by  $\lambda \in \mathbb{C} \setminus \{\pm 1\}$ , given by

$$f_\lambda := y^2z - (x - z)(x + z)(x - \lambda z) = y^2z - x^3 + \lambda x^2z + xz^2 - \lambda z^3$$

where  $f = f_0$ . We now proceed to compute the inflection points of  $V(f_\lambda)$  and the lines in  $V(w^3 - f_\lambda)$ , following Section 2.4.1 and the proof of Proposition 2.2.1, respectively. These calculations are crucial for the computation of  $\rho(SL_2(\mathbb{Z}))$ , as we shall see in Section 2.5.2. To compute the inflections points of  $f_\lambda$ , we use

$$\nabla f_\lambda = \begin{pmatrix} -3x^2 + z^2 + 2\lambda xz \\ 2yz \\ y^2 + 2xz - 3\lambda z^2 \end{pmatrix} \quad \text{and} \quad \text{Hess}(f_\lambda) = \begin{pmatrix} -6x + 2\lambda z & 0 & 2z + 2\lambda x \\ 0 & 2z & 2y \\ 2z + 2\lambda x & 2y & 2x - 6\lambda z \end{pmatrix}.$$

Hence,  $\frac{1}{8} \det \text{Hess}(f) = (3x - \lambda z)(y^2 - xz + 3\lambda z^2) - z(z + \lambda x)^2$  so the inflection points of  $f$  are given by

$$V(y^2z - x^3 + \lambda x^2z + xz^2 - \lambda z^3) \cap V((3x - \lambda z)y^2 - z(x^2(3 + \lambda^2) - 8\lambda xz + z^2(1 + 3\lambda^2))).$$

Let  $[x : y : z]$  be an inflection point.

- If  $z = 0$ , then  $f = -x^3 = 0$  so  $x = 0$ , and thus  $[x : y : z] = [0 : 1 : 0]$ .

- If  $z \neq 0$ , then one may assume  $z = 1$ . Then

$$y^2 = x^3 - \lambda x^2 - x + \lambda$$

$$0 = (3x - \lambda)y^2 - (3 + \lambda^2)x^2 + 8\lambda x - 1 - 3\lambda^2$$

Substituting  $y^2$  in the second equation we obtain

$$R_\lambda(x) = 3x^4 - 4\lambda x^3 - 6x^2 + 12\lambda x - 1 - 4\lambda^2 = 0.$$

Hence, the remaining 8 inflection points are  $\left[ \alpha : \pm\sqrt{\alpha^3 - \lambda\alpha^2 - \alpha + \lambda} : 1 \right]$  with  $\alpha$  a root of  $R_\lambda(x)$ .

We may now proceed to compute the triples of lines in  $V(w^3 - f_\lambda)$  above each inflection point in  $V(f_\lambda)$ .

- At the inflection point  $[0 : 1 : 0]$ , we have the tangent line is given by  $z = 0$ . This gives

$$w^3 - f_\lambda = w^3 + x^3 = 0$$

so the three lines lying above  $[0 : 1 : 0]$  are

$$V(w + \omega^n x) \cap V(z) \quad \text{for } n = 0, 1, 2.$$

- At an inflection point of the form  $P = \left[ \alpha : \sqrt{\alpha^3 - \lambda\alpha^2 - \alpha + \lambda} : 1 \right]$ , the tangent line at  $P$  is given by

$$\left( -3\alpha^2 + 2\lambda\alpha + 1 \right) x + \left( 2\sqrt{\alpha^3 - \lambda\alpha^2 - \alpha + \lambda} \right) y + \left( \alpha^3 - \lambda\alpha^2 + \alpha - 2\lambda \right) z = 0.$$

To compute the three lines lying over  $P$  in  $V(w^3 - f_\lambda)$ , we substitute

$$y^2 = \frac{((-3\alpha^2 + 2\lambda\alpha + 1)x + (\alpha^3 - \lambda\alpha^2 + \alpha - 2\lambda)z)^2}{4(\alpha^3 - \lambda\alpha^2 - \alpha + \lambda)}$$

in  $f_\lambda$ , and using that  $R_\lambda(\alpha) = 0$  to simplify, we obtain

$$w^3 - f_\lambda = w^3 + (x - \alpha z)^3 = 0.$$

Hence, the lines lying over  $P$  are

$$V(w + \omega^n(x - \alpha z)) \cap V\left((-3\alpha^2 + 2\lambda\alpha + 1)x + \left(2\sqrt{\alpha^3 - \lambda\alpha^2 - \alpha + \lambda}\right)y + (\alpha^3 - \lambda\alpha^2 + \alpha - 2\lambda)z\right)$$

for  $n = 0, 1, 2$ .

## 2.5 Computing generators of $\text{Im}(\rho)$

The goal of this section is to use the datum computed in Section 2.4 to determine explicitly the images under  $\rho$  of 4 elements of  $\pi_1(\mathcal{U}_3)$ . The images computed will serve a posteriori as generators of  $\text{Im}(\rho)$ . The 4 elements for which we choose to compute their images come from the semidirect product structure of  $\pi_1(\mathcal{U}_3)$ ,

$$\pi_1(\mathcal{U}_3) \cong \mathcal{H}_3(\mathbb{Z}/3\mathbb{Z}) \rtimes_{\varphi} SL_2(\mathbb{Z})$$

and will, again a posteriori, provide the semidirect product structure of  $\text{Im}(\rho)$ .

### 2.5.1 Images of generators of $\mathcal{H}_3(\mathbb{Z}/3\mathbb{Z})$

The transformation  $A : \mathbb{P}^2 \rightarrow \mathbb{P}^2$  introduced in Section 2.4.1 maps the curve  $f = y^2z - x^3 + xz^2$  to its Hesse normal form,  $f_H = x^3 + y^3 + z^3 - 3\mu xyz$ , where  $\mu = \sqrt{3} + 1$ . The map  $A$

induces a map

$$A' = \begin{pmatrix} A & 0 \\ 0 & -\eta \end{pmatrix} : \mathbb{P}^3 \rightarrow \mathbb{P}^3$$

where  $\eta$  is a cube root of  $1 - \mu^3$ . Since  $\mu$  is real, we can take  $\eta$  to be real for convenience. The map  $A'$  maps the surface  $V(w^3 - f)$  to  $V(w^3 - f_H)$ , and being a linear map, it defines a mapping between the 27 lines of one surface to the other. Namely, we have

$$\begin{array}{ccc} V(w^3 - f) & \xrightarrow{A'} & V(w^3 - f_H) \\ \uparrow & & \uparrow \\ L & \xrightarrow{A'} & A'(L) = L_H \end{array}$$

and with the help of [The21], we compute the mapping of the lines as depicted in Figure 2.1. For each surface, a set of 6 pairwise non-intersecting lines  $\{L_1, \dots, L_6\}$  is found along with their incidences with the remaining 21 lines. This allows us to compute the classes of the 27 lines with respect to the basis  $e_0, [L_1], \dots, [L_6]$  of  $H^2(V(w^3 - f); \mathbb{Z})$  and  $H^2(V(w^3 - f_H); \mathbb{Z})$  respectively.

The purpose of  $A'$  is to determine the  $\mathcal{H}_3(\mathbb{Z}/3\mathbb{Z})$ -action on  $V(f)$  from the corresponding  $\mathcal{H}_3(\mathbb{Z}/3\mathbb{Z})$ -action on  $V(f_H)$ . The later action is explicitly described in [DL81]. It is given by translation on the lattice of inflection points of  $f_H$ , and it is generated by

$$X = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \omega & 0 \\ 0 & 0 & \omega^2 \end{pmatrix} \quad \text{and} \quad Y = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} \quad \text{in } PSL_3(\mathbb{C}).$$

$L_3$	$L_{5,6}$	$L_{4,5}$	$\xrightarrow{A'}$	$L_3$	$L_{3,4}$	$L_{4^*}$
$L_{3,6}$	$L_{3,4}$	$L_5$		$L_{3,6}$	$L_{1,2}$	$L_{4,5}$
$L_{6^*}$	$L_{1,2}$	$L_{4^*}$		$L_{6^*}$	$L_{5,6}$	$L_5$
$L_{1^*}$	$L_{2,6}$	$L_{2,4}$		$L_{1^*}$	$L_6$	$L_{1,6}$
$L_{1,4}$	$L_{2^*}$	$L_{1,6}$		$L_{1,4}$	$L_{2,6}$	$L_{3,5}$
$L_4$	$L_6$	$L_{3,5}$		$L_4$	$L_{2^*}$	$L_{2,4}$
$L_{1,3}$	$L_1$	$L_{3^*}$		$L_{1,3}$	$L_1$	$L_{3^*}$
$L_{2,5}$	$L_{5^*}$	$L_2$		$L_{2,5}$	$L_{5^*}$	$L_2$
$L_{4,6}$	$L_{1,5}$	$L_{2,3}$		$L_{4,6}$	$L_{1,5}$	$L_{2,3}$

Figure 2.1: Mapping of the lines in  $V(w^3 - f)$  (left) to lines in  $V(w^3 - f_H)$  (right) induced by  $A'$ . The lines are determined by their *position* in the diagram, and each labeled line on the left diagram is mapped to the line with the same label on the right diagram. Each box is in correspondance to the inflection points shown in Section 2.4, and lines in each box lie over the same inflection point of  $f$  (left) and  $f_H$  (right). Moreover, each line has a defining equation which depends on a power of  $\omega = e^{2\pi i/3}$ . Within each box, the powers  $\omega^n$  in the defining equations of the lines are  $1, \omega, \omega^2$  from top to bottom.

These can be lifted to automorphisms of  $V(w^3 - f_H)$  as

$$X' = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \omega & 0 & 0 \\ 0 & 0 & \omega^2 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad \text{and} \quad Y' = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad \text{in } PSL_4(\mathbb{C})$$

and these last two maps provide a permutation on the lines of  $V(w^3 - f_H)$ . This can be translated into a permutation of the lines in  $V(w^3 - f)$  with the map  $A'$ . Thus, with the help of Figure 2.1 we obtain:

1. The map  $X'$  induces a permutation  $H_1 \in W(E_6)$  which maps

$$\begin{array}{lll} L_1 \mapsto L_6 & L_2 \mapsto L_{3,5} & L_3 \mapsto L_{1,3} \\ L_4 \mapsto L_{6^*} & L_5 \mapsto L_{2,3} & L_6 \mapsto L_{3,4} \end{array}$$

2. The map  $Y'$  induces a permutation  $H_2 \in W(E_6)$  which maps

$$\begin{array}{lll} L_1 \mapsto L_{3^*} & L_2 \mapsto L_{2,5} & L_3 \mapsto L_{1,2} \\ L_4 \mapsto L_{2,6} & L_5 \mapsto L_3 & L_6 \mapsto L_{2,4} \end{array}$$

and therefore, with respect to our basis  $\{e_0, [L_1], \dots, [L_6]\}$  we obtain the matrices

$$H_1 = \begin{pmatrix} 3 & 0 & 1 & 1 & 2 & 1 & 1 \\ -1 & 0 & 0 & -1 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 & -1 & -1 & 0 \\ -2 & 0 & -1 & -1 & -1 & -1 & -1 \\ -1 & 0 & 0 & 0 & -1 & 0 & -1 \\ -1 & 0 & -1 & 0 & -1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \quad \text{and} \quad H_2 = \begin{pmatrix} 3 & 2 & 1 & 1 & 1 & 0 & 1 \\ -1 & -1 & 0 & -1 & 0 & 0 & 0 \\ -2 & -1 & -1 & -1 & -1 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ -1 & -1 & 0 & 0 & 0 & 0 & -1 \\ -1 & -1 & -1 & 0 & 0 & 0 & 0 \\ -1 & -1 & 0 & 0 & -1 & 0 & 0 \end{pmatrix}$$

where

$$\mathbf{H} := \langle H_1, H_2 \rangle = \rho(\mathcal{H}_3(\mathbb{Z}/3\mathbb{Z})) < W(E_6).$$

With the help of [Inc], we compute the size of the matrix group generated by  $H_1$  and  $H_2$ , and we obtain that  $|\mathbf{H}| = |\mathcal{H}_3(\mathbb{Z}/3\mathbb{Z})| = 27$ . Therefore,  $\mathbf{H} \cong \mathcal{H}_3(\mathbb{Z}/3\mathbb{Z})$ .

### 2.5.2 Images of elements coming from $SL_2(\mathbb{Z})$

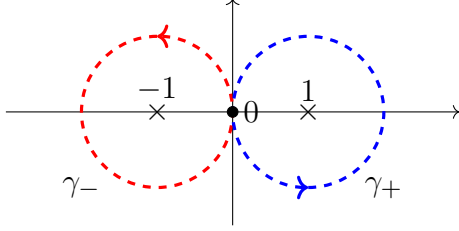
Now we use our results from Section 2.4.2 to compute the image of two elements in  $SL_2(\mathbb{Z}) < \pi_1(\mathcal{U}_3)$ . Recall that by Hurwitz's theorem, every smooth cubic curve is realized as a double branched cover of  $\mathbb{P}^1$ , with branch locus equal to 4 distinct points  $\{a, b, c, d\} \in \mathbb{P}^1$ . With a suitable change of coordinates given by a projective linear transformation in  $\mathbb{P}^2$ , the branch points  $\{a, b, c\}$  may be mapped to  $\{\pm 1, \infty\}$ , and the remaining branch point  $d$  is mapped to some  $\lambda \in \mathbb{C} \setminus \{\pm 1\}$ . Therefore, a cubic curve can be embedded in  $\mathbb{P}^2$  as a curve  $f_\lambda$  in the family defined in Section 2.4.2

$$f_\lambda = y^2 z - (x - z)(x + z)(x - \lambda z) \quad \text{with } \lambda \in \mathbb{C} \setminus \{\pm 1\}$$

which is the projectivization of the elliptic curve  $y^2 = (x - 1)(x + 1)(x - \lambda)$ . The curve  $f = y^2 z - x^3 + xz^2$  is in such form letting  $\lambda = 0$ , and the space  $\mathbb{C} \setminus \{\pm 1\}$  parametrizes a subset of curves in  $\mathcal{U}_3$  containing  $f$ . A loop based at 0 in  $\mathbb{C} \setminus \{\pm 1\}$  naturally parametrizes a loop based at  $f$  in  $\mathcal{U}_3$  via the map  $\lambda \mapsto f_\lambda$ . Since every cubic curve  $f_\lambda$  has an inflection point at  $[0 : 1 : 0]$ , this inflection point is fixed by any automorphism of  $f$  coming from elements in  $\pi_1(\mathcal{U}_3)$  given by loops in  $\mathcal{U}_3$  consisting of curves of the form  $f_\lambda$ . This implies that for such loops, the action of  $\mathcal{H}_3(\mathbb{Z}/3\mathbb{Z}) < \pi_1(\mathcal{U}_3)$  on  $H^2(V(w^3 - f); \mathbb{Z})$  is trivial, as there is no translation on the inflection points of  $f$ . Therefore these loops come from  $SL_2(\mathbb{Z}) < \pi_1(\mathcal{U}_3)$ . In light of this, we consider the generators of  $\pi_1(\mathbb{C} \setminus \{\pm 1\})$  depicted in Figure 2.2 along with the corresponding loops in  $\mathcal{U}_3$  parametrized by these generators. Our goal is now to compute the image under  $\rho$  of these loops in  $\pi_1(\mathcal{U}_3)$ .

A loop  $\gamma \in \pi_1(\mathbb{C} \setminus \{\pm 1\})$  induces a permutation on the 27 lines of  $V(w^3 - f)$  as follows: in Section 2.4 we prove that the 24 of the lines in  $V(w^3 - f_\lambda)$  are given by

$$V(w + \omega^n(x - \alpha z)) \cap V\left((-3\alpha^2 + 2\lambda\alpha + 1)x + \left(2\sqrt{\alpha^3 - \lambda\alpha^2 - \alpha + \lambda}\right)y + (\alpha^3 - \lambda\alpha^2 + \alpha - 2\lambda)z\right)$$



$$\gamma_- : [0, 1] \rightarrow \mathbb{C} \setminus \{\pm 1\} \quad t \mapsto -1 + e^{2\pi it}$$

$$\gamma_+ : [0, 1] \rightarrow \mathbb{C} \setminus \{\pm 1\} \quad t \mapsto 1 - e^{2\pi it}$$

Figure 2.2: Generators  $\gamma_-, \gamma_+$  of  $\pi_1(\mathbb{C} \setminus \{\pm 1\}, 0)$ .

and the remaining 3 are given by  $V(w + \omega^n x) \cap V(z)$ . Letting  $\lambda = \gamma(t)$ ,  $\alpha$  and  $\lambda$  vary continuously in  $t$ . Indeed,  $\alpha$  is a root of  $R_\lambda(x)$ , a quartic polynomial whose coefficients are polynomials in  $\lambda$ . This implies that along the surfaces  $V(w^3 - f_\lambda)$ , these lines vary continuously. For a line  $L \subset V(w^3 - f)$ , let  $L(\gamma, t)$  be the line in  $V(w^3 - f_{\gamma(t)})$  obtained by varying  $L$  along  $\gamma$ . Thus,  $\gamma$  induces a permutation given by

$$L(\gamma, 0) \mapsto L(\gamma, 1).$$

To compute these permutations explicitly, we recall that lines in  $V(w^3 - f_\lambda)$  lie over the inflection points of  $f_\lambda$  in triples. The lines  $V(w + \omega^n x) \cap V(z)$  are fixed along any path  $\gamma$ , as the coefficients of their defining equations are constant functions of  $\lambda$ . Thus, it remains to study what happens to the remaining 8 triples of lines. Let  $p(\gamma, t)$  be the inflection point of  $f_{\gamma(t)}$  over which the line  $L(\gamma, t)$  lies. Then, we have the following proposition.

**Proposition 2.5.1.** The permutation  $L(\gamma, 0) \mapsto L(\gamma, 1)$  induces a permutation  $p(\gamma, 0) \mapsto p(\gamma, 1)$  on the inflection points of  $f$ . Moreover, the permutation  $L(\gamma, 0) \mapsto L(\gamma, 1)$  is completely determined by its induced permutation  $p(\gamma, 0) \mapsto p(\gamma, 1)$ .

*Proof.* Since any loop  $\gamma$  induces an automorphism of  $V(f_\lambda)$ , the permutation  $L(\gamma, 0) \mapsto L(\gamma, 1)$  induces a permutation  $p(\gamma, 0) \mapsto p(\gamma, 1)$  by restriction to the inflection point of  $f$  contained in each line. This implies that the lines over each inflection point  $p(\gamma, 0)$  in  $f$  are mapped to the lines over  $p(\gamma, 1)$ .

Moreover, in the defining equation  $w + \omega^n(x - \alpha z)$  of any of these 24 lines, the  $x$ -

coefficient  $\omega^n$  is a constant function of  $\lambda$ . Therefore this coefficient remains constant in the defining equations of  $L(\gamma, 0)$  and  $L(\gamma, 1)$  for any line  $L$  and any loop  $\gamma$ . The permutation on the lines is simply determined by the defining equation

$$V \left( \left( -3\alpha^2 + 2\lambda\alpha + 1 \right) x \pm \left( 2\sqrt{\alpha^3 - \lambda\alpha^2 - \alpha + \lambda} \right) y + \left( \alpha^3 - \lambda\alpha^2 + \alpha - 2\lambda \right) z \right).$$

In particular, the  $y$ -coordinate of each of the 8 inflection points of  $f$  distinct from  $[0 : 1 : 0]$  distinguishes each inflection point. It is also a scalar multiple of the  $y$ -coefficient in the defining equations of the 24 lines over the inflection points distinct from  $[0 : 1 : 0]$ , and this coefficient distinguishes such defining equations. Therefore the defining equation of our lines changes according to this coefficient, which changes according to the inflection points of  $f$ . This shows that the induced permutation  $L(\gamma, 0) \mapsto L(\gamma, 1)$  is completely determined by its associated permutation on the inflection points  $p(\gamma, 0) \mapsto p(\gamma, 1)$ .  $\square$

Hence it suffices to study the permutation  $p(\gamma, 0) \mapsto p(\gamma, 1)$ . To do so, we simply study the permutation of the  $y$ -coordinates of these inflection points. We compute this with the help of [Inc] as follows:

Given a path  $\gamma$ ,  $y$ -coordinate of the inflection points of  $f_{\gamma(t)}$  distinct from  $[0 : 1 : 0]$  are determined by the roots of the polynomial  $R_{\gamma(t)}$ , which are the  $x$ -coordinates of these inflection points. We plot

$$\begin{aligned} \mathcal{X}_N(\gamma) &:= \left\{ \left( x, \frac{k}{N} \right) \in \mathbb{C} \times [0, 1] \mid k = 0, 1, \dots, N \text{ and } [x : y : 1] \text{ is an inflection point of } f_{\gamma\left(\frac{k}{N}\right)} \right\} \\ \mathcal{Y}_N(\gamma) &:= \left\{ \left( y, \frac{k}{N} \right) \in \mathbb{C} \times [0, 1] \mid k = 0, 1, \dots, N \text{ and } [x : y : 1] \text{ is an inflection point of } f_{\gamma\left(\frac{k}{N}\right)} \right\} \end{aligned}$$

Uniform continuity of the map  $t \mapsto p(\gamma, t)$  guarantees that for a sufficiently large integer  $N$ , the distance between the  $y$ -coordinates corresponding to the inflection points  $p\left(\gamma, \frac{k}{N}\right)$  and  $p\left(\gamma, \frac{k+1}{N}\right)$  is bounded by a fixed  $\varepsilon > 0$  for every  $k$ . Therefore the  $y$ -coordinates of the

points  $p\left(\gamma, \frac{k}{N}\right)$  can be determined from the starting point  $p(\gamma, 0)$ . This is illustrated for the curves  $\gamma_-$  and  $\gamma_+$  in Figure 2.3 and Figure 2.4.

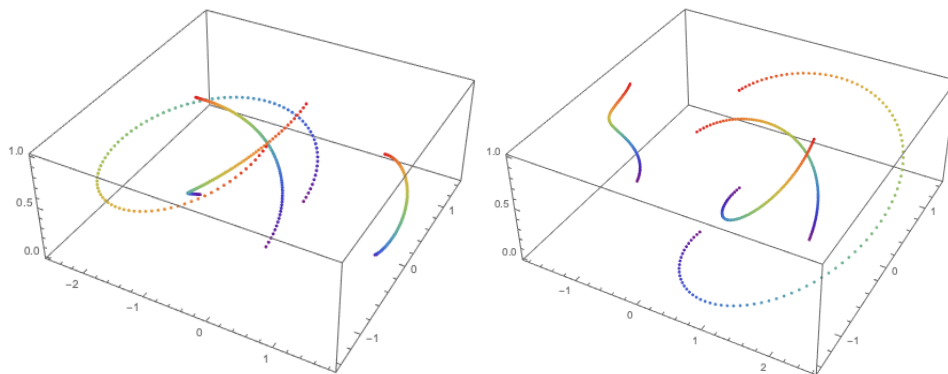


Figure 2.3: Plots of  $\mathcal{X}_N(\gamma_-)$  (left) and  $\mathcal{X}_N(\gamma_+)$  (right) for  $N = 100$ .

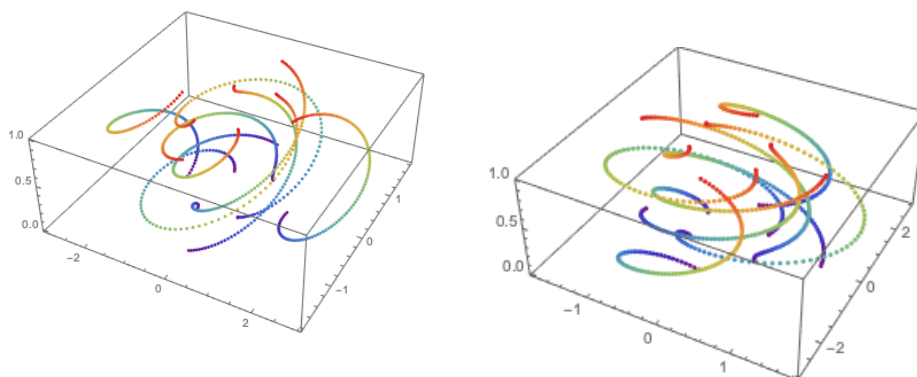


Figure 2.4: Plots of  $\mathcal{Y}_N(\gamma_-)$  (left) and  $\mathcal{Y}_N(\gamma_+)$  (right) for  $N = 100$ .

The permutations on the roots of  $R_0$  induced by  $\gamma_-$  and  $\gamma_+$  are given as in Figure 2.5, and the permutations on  $y$ -coordinates of the inflection points of  $f_0 = f$  induced by  $\gamma_-$  and  $\gamma_+$  are given as in Figure 2.6.

1. The loop  $\gamma_-$  induces a permutation  $G_1 \in W(E_6)$  which maps

$$\begin{array}{lll}
 L_1 \mapsto L_1 & L_2 \mapsto L_2 & L_3 \mapsto L_{4,5} \\
 L_4 \mapsto L_6 & L_5 \mapsto L_{3,4} & L_6 \mapsto L_{3,5}
 \end{array}$$

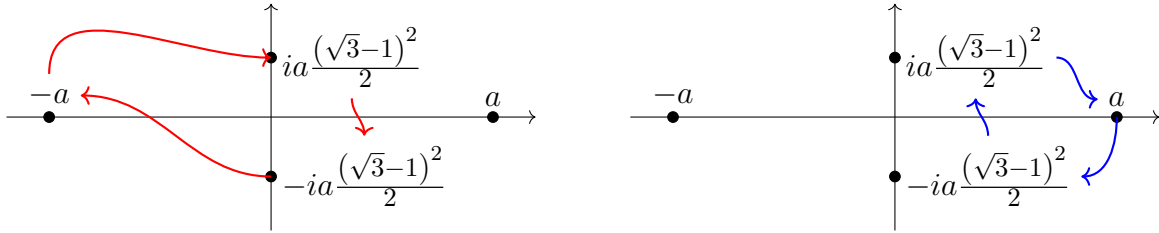


Figure 2.5: Permutation of the roots of  $R_0$  induced by  $\gamma_-$  (left) and  $\gamma_+$  (right)

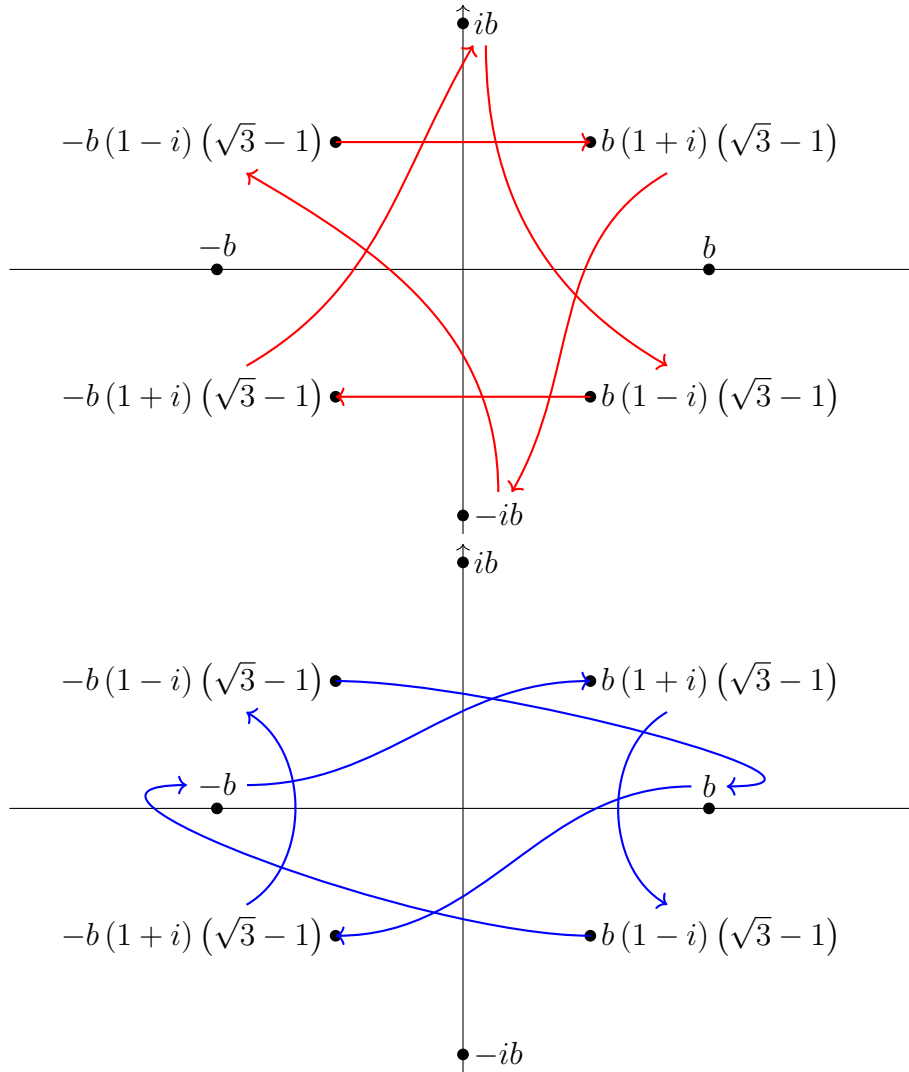


Figure 2.6: Permutation on the  $y$ -coordinates of inflection points of  $f$  induced by  $\gamma_-$  (above) and  $\gamma_+$  (below).

2. The loop  $\gamma_+$  induces a permutation  $G_2 \in W(E_6)$  which maps

$$\begin{array}{lll} L_1 \mapsto L_{5,6} & L_2 \mapsto L_{1,6} & L_3 \mapsto L_3 \\ L_4 \mapsto L_4 & L_5 \mapsto L_2 & L_6 \mapsto L_{1,5} \end{array}$$

and therefore, with respect to our basis  $\{e_0, [L_1], \dots, [L_6]\}$  we obtain the matrices

$$G_1 = \begin{pmatrix} 2 & 0 & 0 & 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & -1 & -1 \\ -1 & 0 & 0 & -1 & 0 & -1 & 0 \\ -1 & 0 & 0 & -1 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \end{pmatrix} \quad \text{and} \quad G_2 = \begin{pmatrix} 2 & 1 & 1 & 0 & 0 & 0 & 1 \\ -1 & 0 & -1 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ -1 & -1 & 0 & 0 & 0 & 0 & -1 \\ -1 & -1 & -1 & 0 & 0 & 0 & 0 \end{pmatrix}$$

where

$$\mathbf{G} := \langle G_1, G_2 \rangle < \rho(SL_2(\mathbb{Z})) < W(E_6).$$

With the help of [Inc], we compute the order of the matrix group generated by  $G_1$  and  $G_2$ , and we obtain  $|\mathbf{G}| = 24 = 2^3 \cdot 3$ . Moreover, we make the following observations about  $\mathbf{G}$ :

- $[G_1, G_2] \neq 1$ , so  $\mathbf{G}$  is not abelian.
- $G_1^3 = G_2^3 = 1$  and  $\langle G_1 \rangle \neq \langle G_2 \rangle$ , so 3-Sylow subgroups of  $\mathbf{G}$  are not normal.
- $\text{ord}(G_1 G_2) = 6$  and  $\text{ord}([G_1, G_2]) = 4$ , so  $\mathbf{G}$  contains elements of order 4 and 6.

Using the classification of groups of order 24 in [Nai11], the first 3 observations reduce our possibilities for  $\mathbf{G}$  to  $SL_2(\mathbb{Z}/3\mathbb{Z})$ ,  $S_4$  and  $A_4 \times \mathbb{Z}/2\mathbb{Z}$ . Since  $S_4$  has no elements of order 6 and  $A_4 \times \mathbb{Z}/2\mathbb{Z}$  has no elements of order 4, we conclude  $\mathbf{G} \cong SL_2(\mathbb{Z}/3\mathbb{Z})$ .

## 2.6 Proof of the Main Theorem

Now that we have computed explicitly  $H_1, H_2, G_1, G_2$  inside  $\text{Im}(\rho)$ , we are ready to conclude.

Recall the groups  $\mathbf{H}, \mathbf{G} < \text{Im}(\rho)$  are defined in Section 2.5 and we have shown

$$\mathbf{H} := \langle H_1, H_2 \rangle \cong \mathcal{H}_3(\mathbb{Z}/3\mathbb{Z}) \quad \text{and} \quad \mathbf{G} := \langle G_1, G_2 \rangle \cong SL_2(\mathbb{Z}/3\mathbb{Z}).$$

Consider the subgroup of  $\text{Im}(\rho)$  generated by  $H_1, H_2, G_1, G_2$ ,

$$\mathbf{I} := \langle H_1, H_2, G_1, G_2 \rangle.$$

We will show  $\mathbf{I} \cong \text{Im}(\rho)$  in order to conclude.

*Proof of Theorem 2.1.1.* With the help of [Inc], we compute the intersection of the matrix groups  $\mathbf{H}$  and  $\mathbf{G}$  and obtain that  $\mathbf{H} \cap \mathbf{G} = \{1\}$ . Conjugating  $H_1$  and  $H_2$  by  $G_1$  and  $G_2$  we obtain

$$\begin{aligned} G_1 H_1 G_1^{-1} &= \Omega H_1 H_2 & G_2 H_1 G_2^{-1} &= H_1 \\ G_1 H_2 G_1^{-1} &= H_2 & G_2 H_2 G_2^{-1} &= \Omega^{-1} H_1^{-1} H_2. \end{aligned}$$

Therefore,  $\mathbf{H}$  is normal in  $\mathbf{I}$ . These two facts together imply that

$$\mathbf{I} \cong \mathbf{H} \rtimes \mathbf{G} \cong \mathcal{H}_3(\mathbb{Z}/3\mathbb{Z}) \rtimes SL_2(\mathbb{Z}/3\mathbb{Z}).$$

The homomorphism  $\varphi : SL_2(\mathbb{Z}/3\mathbb{Z}) \rightarrow \text{Aut}(\mathcal{H}_3(\mathbb{Z}/3\mathbb{Z}))$  described in Section 2.3 completely describes this semidirect product structure, and an explicit isomorphism  $\mathbf{I} \rightarrow \mathcal{H}_3(\mathbb{Z}/3\mathbb{Z}) \rtimes_{\varphi}$

$SL_2(\mathbb{Z}/3\mathbb{Z})$  is given by

$$\begin{aligned}
 H_1 &\mapsto \left( \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, I \right) & G_1 &\mapsto \left( I, \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \right) \\
 H_2 &\mapsto \left( \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}, I \right) & G_2 &\mapsto \left( I, \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix} \right).
 \end{aligned}$$

Since  $|\mathbf{I}| \leq |\text{Im}(\rho)| \leq |C_{W(E_6)}(\Omega)| = 648$  and

$$|\mathbf{I}| = |\mathbf{H}| \cdot |\mathbf{G}| = |\mathcal{H}_3(\mathbb{Z}/3\mathbb{Z})| \cdot |SL_2(\mathbb{Z}/3\mathbb{Z})| = 27 \cdot 24 = 648$$

we have that the equalities must hold. By Corollary 2.3.3, we have that  $\text{Im}(\rho) \subset C_{W(E_6)}(\Omega)$ , so we conclude

$$\mathbf{I} \cong \text{Im}(\rho) \cong C_{W(E_6)}(\Omega) \cong \mathcal{H}_3(\mathbb{Z}/3\mathbb{Z}) \rtimes_{\varphi} SL_2(\mathbb{Z}/3\mathbb{Z}).$$

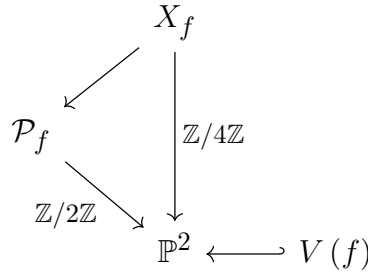
□

# CHAPTER 3

## QUARTIC SURFACES

### 3.1 Introduction

In this chapter we compute the monodromy of branched covers of  $\mathbb{P}^2$  whose branch locus is a smooth quartic curve. For  $f \in \mathcal{U}_4$  consider the cyclic 2-fold and 4-fold covers branched over  $V(f)$ :



The surface  $\mathcal{P}_f$  is a degree 2 del Pezzo surface, and  $X_f$  is a quartic  $K3$  surface. We define the *universal (2,4)-branched and (4,4)-branched covers of  $\mathbb{P}^2$*  as the fiber bundles

$$\mathcal{E}_{2,4} = \{(P, f) \in \mathcal{P}_f \times \mathcal{U}_4 \mid P \in \mathcal{P}_f\} \quad \mathcal{E}_{4,4} = \{(P, f) \in X_f \times \mathcal{U}_4 \mid P \in X_f\}$$

given respectively by the fibrations

$$\begin{array}{ccc}
 \mathcal{P}_f \hookrightarrow \mathcal{E}_{2,4} & (P, f) & \\
 \downarrow & \downarrow & \\
 \mathcal{U}_4 & f & \\
 \end{array}
 \quad
 \begin{array}{ccc}
 X_f \hookrightarrow \mathcal{E}_{4,4} & (P, f) & \\
 \downarrow & \downarrow & \\
 \mathcal{U}_4 & f & \\
 \end{array}$$

where the fibers  $\mathcal{P}_f$  and  $X_f$  are diffeomorphic to a degree 2 del Pezzo surface and a quartic  $K3$  surface, respectively. Fixing a base point  $f \in \mathcal{U}_4$ , we simply refer to these fibers by  $\mathcal{P}$  and  $X$ . The action of  $\pi_1(\mathcal{U}_4)$  on  $H^2(\mathcal{P}; \mathbb{Z})$  and  $H^2(X; \mathbb{Z})$  induces monodromy homomorphisms

$$\rho_2 : \pi_1(\mathcal{U}_4) \rightarrow \text{Aut}\left(H^2(\mathcal{P}; \mathbb{Z})\right) \quad \rho_4 : \pi_1(\mathcal{U}_4) \rightarrow \text{Aut}\left(H^2(X; \mathbb{Z})\right).$$

The images of  $\rho_2$  and  $\rho_4$  can be further restricted by noting that the respective intersection forms in  $H^2(\mathcal{P}; \mathbb{Z})$  and  $H^2(X; \mathbb{Z})$  remain invariant under the actions of  $\rho_2$  and  $\rho_4$ .

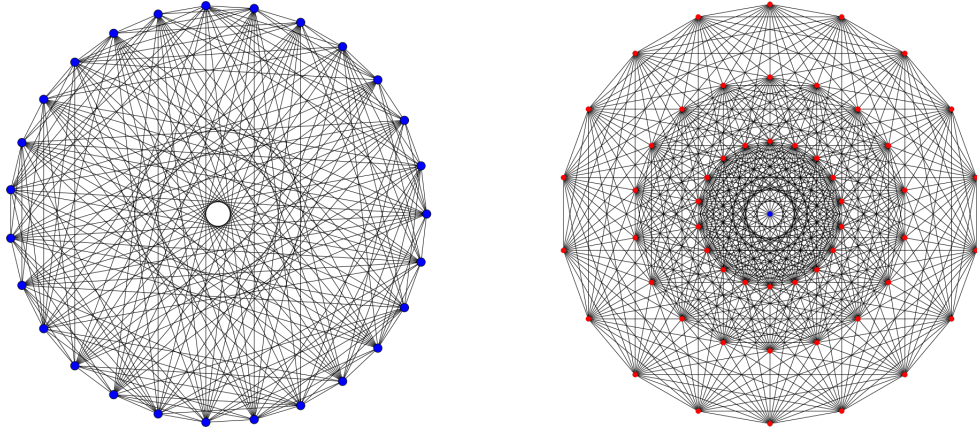


Figure 3.1: The Schläfli graph (left) and Gosset graph (right).

As explained in Section 3.2, this implies that the image of  $\rho_2$  is contained in the automorphism group of the 56 lines contained in  $\mathcal{P}$ . The intersection pattern of these lines is given by the Gosset graph (see Figure 3.1), whose automorphism group is  $W(E_7)$ , the Weyl group of  $E_7$ . This fact restricts the image of  $\rho_2$ , so that:

$$\rho_2 : \pi_1(\mathcal{U}_4) \rightarrow W(E_7).$$

This cannot be further restricted, as this map is onto. Namely, we have the following:

**Theorem 3.1.1.** *The monodromy representation*

$$\rho_2 : \pi_1(\mathcal{U}_4) \rightarrow W(E_7)$$

*of the universal  $(2, 4)$ -branched cover of  $\mathbb{P}^2$  is surjective.*

Each fiber of  $\mathcal{E}_{2,4}$  comes equipped with a  $\mathbb{Z}/2\mathbb{Z}$  deck group action induced by its cyclic branched cover structure. The generator  $\tau$  of this action is referred to as the *Geiser involution* on  $\mathcal{P}$ , it corresponds to the generator of the center of  $W(E_7)$  [Dol12]. As explained in

Section 3.2, the 56 lines contained in  $\mathcal{P}$  lie in pairs over each of the 28 bitangents to the underlying quartic curve  $V(f)$ . The quotient  $W(E_7)/\langle\tau\rangle \cong \mathrm{Sp}_6(\mathbb{Z}/2\mathbb{Z})$  corresponds to the Galois group of these 28 bitangents, as discussed in [Har79].

**Corollary 3.1.2.** The action of  $\tau$  on  $H^2(\mathcal{P};\mathbb{Z})$  is realized as an element of the monodromy group  $\mathrm{Im}(\rho_2)$ .

In order to determine the image of  $\rho_4$ , we make use of the action  $T$  on  $H^2(X;\mathbb{Z})$  induced by its  $\mathbb{Z}/4\mathbb{Z}$  deck group action. This action determines two sublattices of  $H^2(X;\mathbb{Z})$ , namely:

$$L_+ = \left\{ x \in H^2(X;\mathbb{Z}) \mid T^2x = x \right\} \quad L_- = \left\{ x \in H^2(X;\mathbb{Z}) \mid T^2x = -x \right\}.$$

As discussed in Section 3.4, these sublattices have discriminant  $2^8$  and discriminant group (or *glue group* as referred to in [McM09]) isomorphic to  $(\mathbb{Z}/2\mathbb{Z})^{\oplus 8}$  and the action of  $T$  on this group is induced by the Geiser involution on  $\mathcal{P}$ . By considering the branched cover  $X \rightarrow \mathcal{P}$ , we obtain the pullback map

$$H^2(\mathcal{P};\mathbb{Z}) \rightarrow H^2(X;\mathbb{Z})$$

and  $L_+$  is then generated by the pullbacks of generators of  $H^2(\mathcal{P};\mathbb{Z})$ . As discussed in Section 4 of [AS11], the action  $T$  is purely non-symplectic, meaning that  $T\omega = \pm i\omega$  on the invariant line  $H^{2,0}(X;\mathbb{C})$ . The Néron-Severi, or Picard, group  $\mathrm{NS}(X) := H^2(X;\mathbb{Z}) \cap H^{1,1}(X;\mathbb{C})$  of a generic fiber  $X$  is then

$$\mathrm{Pic}(X) \cong \mathrm{NS}(X) \cong L_+ \cong \langle 2 \rangle \oplus A_1^{\oplus 7}.$$

This determines the Hodge decomposition on  $L_{\pm} \otimes \mathbb{C}$  since  $L_+ \otimes \mathbb{C} \subset H^{1,1}(X;\mathbb{C})$  and therefore  $L_- \otimes \mathbb{C}$  contains the lines  $H^{2,0}(X;\mathbb{C}), H^{0,2}(X;\mathbb{C})$ , hence  $L_- \otimes \mathbb{C} \cong \mathbb{C} \oplus \mathbb{C}^{12} \oplus \mathbb{C}$ .

The monodromy map  $\rho_2$  then determines the action of  $\rho_4$  on  $L_+$ , and it remains to study the action of  $\rho_4$  on  $L_-$ . To do this, we observe that the action of  $\rho_4$  commutes with the deck

group action  $T$ , therefore having its image lie in the centralizer  $C_{L_-}(T)$  of  $T$  in  $L_-$ . Since  $L_-$  is equipped with an action  $T$  satisfying  $T^2 + I = 0$ , this centralizer can be regarded as a  $\mathbb{Z}[i]$ -module. Following [All02] and [Kon20], we obtain that as a  $\mathbb{Z}[i]$ -lattice,

$$L_- \cong \left( \mathbb{Z}[i]^7, h_{L_-} \right)$$

where  $h_{L_-}$  is quadratic form of signature  $(1, 6)$  given by (see Proposition 3.5.4)

$$h_{L_-} = -2 \left( |z_0|^2 + |z_1|^2 + |z_2|^2 + |z_3|^2 + |z_4|^2 - \Re(z_1\bar{z}_2 + z_3\bar{z}_4 + z_5\bar{z}_6) - \Im(z_1\bar{z}_2 + z_3\bar{z}_4 + z_5\bar{z}_6) \right)$$

Since  $\rho_4$  preserves the intersection form on  $L_-$ , and therefore the hermitian form  $h_{L_-}$ , it follows that

$$\mathrm{Im} \left( \rho_4 |_{L_-} \right) \subseteq U \left( h_{L_-} \right).$$

Using a result by [Kon00] on the characterization of the moduli space of smooth quartic curves as a ball complement quotient, we show that these two groups coincide. Analogous results have been used to study the monodromy and fundamental group of cubic surfaces (see, e.g. [Lib78]). Furthermore, as explained in Section 3.4.2, the lattices  $L_{\pm}$  are pairwise orthogonal and therefore primitive within  $H^2(X; \mathbb{Z})$ . By virtue of the relation between the actions of  $\rho_4$  on the discriminant groups of  $L_{\pm}$ , we obtain that if  $\rho_4$  acts trivially on  $L_-$  then it follows that the action on  $L_+$  is trivial too. This implies that the monodromy map  $\rho_4$  is completely determined by its action on the sublattice  $L_-$ , namely:

**Theorem 3.1.3.** *The monodromy representation*

$$\rho_4 : \pi_1(\mathcal{U}_4) \rightarrow U \left( h_{L_-} \right)$$

*of the universal  $(4, 4)$ -branched cover of  $\mathbb{P}^2$  is surjective. In particular, its monodromy is arithmetic.*

## 3.2 Degree 2 del Pezzo Surfaces

### 3.2.1 General facts

Degree 2 del Pezzo surfaces are realized as blow-ups of  $\mathbb{P}^2$  at 7 points in general position. Given a degree 2 del Pezzo surface  $\mathcal{P}$ , its anticanonical linear system  $|-K_{\mathcal{P}}|$  defines a rational map  $p : \mathcal{P} \rightarrow \mathbb{P}^2$  which has as branch locus a smooth quartic curve  $V(f)$ . Such surface  $\mathcal{P}$  contains exactly 56 exceptional divisors corresponding to the pullbacks of the 28 bitangents of  $V(f)$ . These 56 divisors can be labeled as follows:

Let  $P_1, P_2, \dots, P_7$  be the seven points at which  $\mathbb{P}^2$  is blown up to obtain  $\mathcal{P}$  and let  $L_1, L_2, \dots, L_7$  be the exceptional divisors corresponding to the blow-ups at these points. Let  $e_i$  be the Poincaré dual of each divisor  $L_i$  and  $e_0 = p^*(\text{PD}[H])$  be the Poincaré dual of the pullback of the hyperplane class  $[H]$  in  $\mathbb{P}^2$ . Then the 56 exceptional divisors contained in  $\mathcal{P}$  have corresponding cohomology classes given by

$$\begin{aligned} [L_i] &= e_i & 1 \leq i \leq 7 \\ [L_i^*] &= 3e_0 - e_i - (e_1 + e_2 + \dots + e_7) \\ [L_{i,j}] &= e_0 - e_i - e_j & 1 \leq i < j \leq 7 \\ [L_{i,j}^*] &= 2e_0 + e_i + e_j - (e_1 + e_2 + \dots + e_7) \end{aligned}$$

These 56 classes span  $H^2(\mathcal{P}; \mathbb{Z})$ , whose intersection form with respect to the basis  $\{e_0, e_1, \dots, e_7\}$  is

$$\langle \cdot, \cdot \rangle_{\mathcal{P}} : H^2(\mathcal{P}; \mathbb{Z}) \times H^2(\mathcal{P}; \mathbb{Z}) \rightarrow \mathbb{Z} \quad (\mathbf{a}, \mathbf{b}) \mapsto \mathbf{a}^T \begin{pmatrix} 1 & 0 \\ 0 & -I_7 \end{pmatrix} \mathbf{b}.$$

### 3.2.2 The Gosset and Schläfli graphs

As explained in Section 8 of [Dol12], the classes of the 56 exceptional divisors of  $\mathcal{P}$  are in correspondence with the vertices of the Gosset graph, where two vertices share an edge if and only if the inner product of the corresponding cohomology classes is equal to 1. Moreover, the induced subgraph given by the vertices adjacent to a given vertex form a graph isomorphic to the Schläfli graph, which gives the intersection pattern of the 27 lines contained in a smooth cubic surface. The automorphism group of the Schläfli and Gosset graphs are the Weyl groups  $W(E_6)$  and  $W(E_7)$  respectively. A depiction of both graphs is shown in Figure 3.1.

This phenomenon is fundamentally related to the following geometric construction in [Har79]. A degree 2 del Pezzo surface  $\mathcal{P}$  may be regarded as a blow-up of a cubic surface  $S$  over a point  $Q$  not contained in any of its 27 lines. The projections of the strict transforms of the 27 lines contained in  $S$  and the exceptional divisor over  $Q$  via  $p : \mathcal{P} \rightarrow \mathbb{P}^2$  are in correspondence with the 28 bitangents of the quartic curve  $V(f) \subset \mathbb{P}^2$ . Moreover, the 56 exceptional divisors contained in  $\mathcal{P}$  lie over each bitangent of  $V(f)$  in pairs. Each pair of such exceptional divisors has inner product 2 with respect to the intersection form in  $H^2(\mathcal{P}; \mathbb{Z})$  and they are precisely the pairs

$$\{([L_i], [L_i^*]) \mid 1 \leq i \leq 7\} \quad \text{and} \quad \left\{ \left( [L_{i,j}], [L_{i,j}^*] \right) \mid 1 \leq i < j \leq 7 \right\}.$$

We call each pair of such lines *dual* and we denote the dual of a line  $L$  by  $L^*$ . Let  $L_Q$  be the exceptional divisor over  $Q$ ,  $\mathcal{S}$  be the set of strict transforms of the 27 lines in the cubic surface  $S$ , and  $\mathcal{S}^*$  be the set of dual lines to those in  $\mathcal{S}$ . A direct computation shows that

$$\langle [L_Q], [L] \rangle_{\mathcal{P}} = \begin{cases} 0 & L \in \mathcal{S} \\ 1 & L \in \mathcal{S}^* \end{cases}$$

Moreover, the intersection pattern of the lines in  $\mathcal{S}$  and in  $\mathcal{S}^*$  is given by the Schläfli graph.

Hence, for any line  $L$  contained in  $\mathcal{P}$  we may define the following sets of 27 lines associated to  $L$ :

$$\mathcal{S}_L = \{L' \subset \mathcal{P} \mid \langle [L], [L'] \rangle_{\mathcal{P}} = 0\} \quad \mathcal{S}_L^* = \{L' \subset \mathcal{P} \mid \langle [L], [L'] \rangle_{\mathcal{P}} = 1\}$$

where  $\mathcal{S}_{L^*} = \mathcal{S}_L^*$ . With this notation, we also have that

$$L \in \mathcal{S}_{L'} \iff L' \in \mathcal{S}_L \quad \text{and} \quad L \in \mathcal{S}_{L'}^* \iff L' \in \mathcal{S}_L^*.$$

### 3.3 Computing $\text{Im}(\rho_2)$

In this section we prove Theorem 3.1.1, namely, we show that

$$\text{Im}(\rho_2 : \pi_1(\mathcal{U}_4) \rightarrow \text{Aut}(H^2(\mathcal{P}; \mathbb{Z}))) \cong W(E_7).$$

The proof consists of 3 main steps: computing the stabilizer of a line  $L$  in  $\mathcal{P}$ , restricting the  $\text{Im}(\rho_2)$  and showing that  $\text{Im}(\rho_2)$  acts transitively on the 56 lines contained in  $\mathcal{P}$ .

#### 3.3.1 Stabilizer of a line $L$

In [Har79], Harris studies the monodromy groups of the 28 bitangents of a smooth quartic and the 27 lines in a smooth cubic surface are studied. It is a classical result of Klein and Jordan (see [Har79] for a proof) that the monodromy group of 27 lines in a cubic surface  $S$  is isomorphic to

$$W(E_6) \cong O_6^-(\mathbb{Z}/2\mathbb{Z}).$$

The monodromy group of the 28 bitangents of a smooth quartic curve  $V(f)$  is isomorphic to

$$W(E_7)^+ \cong O_6(\mathbb{Z}/2\mathbb{Z}) \cong \text{Sp}_6(\mathbb{Z}/2\mathbb{Z})$$

and moreover, we have that  $W(E_7) \cong W(E_7)^+ \times \mathbb{Z}/2\mathbb{Z}$ . It follows that an element of the monodromy group of the 28 bitangents of a quartic curve that fixes one bitangent is determined by its monodromy action on the 27 lines on  $S$  corresponding to the remaining bitangents. It is then shown in [Har79] that automorphisms of the 27 lines in  $S$  preserving a set of 6 of these lines generate  $W(E_6)$  and can be realized as the monodromy action given by a path in the parameter space of smooth quartic curves  $\mathcal{U}_4$ , implying the following key lemma.

**Lemma 3.3.1** (Harris, [Har79], Section II.4). The stabilizer of any bitangent in the monodromy group of the 28 bitangents of a smooth quartic curve is isomorphic to  $W(E_6)$ .

Now we turn to the monodromy action of interest on the 56 lines in  $\mathcal{P}$ , proving the following.

**Proposition 3.3.2.** The stabilizer in  $\text{Im}(\rho_2)$  of any line in  $\mathcal{P}$  is isomorphic to  $W(E_6)$ .

*Proof.* Suppose  $g \in \pi_1(\mathcal{U}_4)$  lies in the stabilizer of a line  $L$  in  $\mathcal{P}$ . Then  $g$  must lie in the stabilizer of its dual line  $L^*$ . Moreover, since the intersection form is preserved by the monodromy action, the set  $\mathcal{S}_L$  must be permuted, and its permutation completely determines that of  $\mathcal{S}_L^*$ , hence the action of  $g$ . The automorphism group of  $\mathcal{S}_L$  is then isomorphic to the automorphism group of their underlying bitangents, and by Lemma 3.3.1 this is precisely the stabilizer of the bitangent lying under  $L$ , which is isomorphic to  $W(E_6)$ .  $\square$

### 3.3.2 Restricting $\text{Im}(\rho_2)$

We proceed now to reduce  $\text{Im}(\rho_2)$  to two possibilities with the aid of the following two lemmas.

**Lemma 3.3.3.**  $\text{Im}(\rho_2)$  is contained in  $W(E_7)$ .

*Proof.* We have seen that the automorphism group of the Gosset graph is isomorphic to  $W(E_7)$  and the intersection pattern of the 56 lines in  $\mathcal{P}$  is given by this graph. Since the

monodromy action preserves the intersection form in  $H^2(\mathcal{P}; \mathbb{Z})$  and the classes of the 56 lines in  $\mathcal{P}$  generate  $H^2(\mathcal{P}; \mathbb{Z})$ , it follows that the monodromy action of any element in  $\pi_1(\mathcal{U}_4)$  is determined by an automorphism of the Gosset graph, thus giving the desired restriction.  $\square$

**Lemma 3.3.4.**  $\text{Im}(\rho_2)$  contains  $W(E_7)^+$ .

*Proof.* The action of  $\text{Im}(\rho_2)$  on the 56 lines in  $\mathcal{P}$  preserves the 28 pairs of lines lying over each bitangent of  $V(f)$ . Recall that the automorphism group of the 28 bitangents of  $V(f)$  is isomorphic to  $W(E_7)^+$ . As noted in Proposition 3.3.2 above, any such automorphism is realized by an element of  $\pi_1(\mathcal{U}_4)$ , which in turn induces an automorphism of the 28 pairs of lines in  $\mathcal{P}$ . This implies the desired contention.  $\square$

Since  $W(E_7)^+$  is a subgroup of index 2 in  $W(E_7)$ , Lemma 3.3.3 and Lemma 3.3.4 imply the following:

**Proposition 3.3.5.**  $\text{Im}(\rho_2)$  is isomorphic either to  $W(E_7)^+$  or to  $W(E_7)$ .

### 3.3.3 Transitivity on the lines contained in $\mathcal{P}$

Finally, we see  $\text{Im}(\rho_2)$  acts transitively on the 56 lines, which will allow us to conclude our proof.

**Lemma 3.3.6.** For any line  $L$  contained in  $\mathcal{P}$ , the group  $\text{Im}(\rho_2)$  acts transitively on  $\mathcal{S}_L$  and on  $\mathcal{S}_L^*$ .

*Proof.* By Proposition 3.3.2, the stabilizer of  $L$  is isomorphic to  $W(E_6)$ , which is the automorphism group of  $\mathcal{S}_L$ . Since the  $W(E_6)$  action on the lines in a smooth cubic surface is transitive, so is the action on  $\mathcal{S}_L$ . Since  $\text{Im}(\rho_2)$  permutes pairs consisting of a line and its dual, transitivity on  $\mathcal{S}_L^*$  follows from that on  $\mathcal{S}_L$ .  $\square$

**Lemma 3.3.7.** The  $\text{Im}(\rho_2)$ -orbit of a line  $L$  in  $\mathcal{P}$  contains lines in both  $\mathcal{S}_L$  and  $\mathcal{S}_L^*$ .

*Proof.* Consider a line  $N \neq L, L^*$  in the orbit of  $L$ , and suppose  $N \in \mathcal{S}_L^*$ . The intersection pattern of the lines in  $\mathcal{S}_N^*$  is given by the Schläfli graph, where each vertex has valence 16. Since  $L \in \mathcal{S}_N^*$ , this implies that  $\mathcal{S}_N^*$  intersects both sets  $\mathcal{S}_L$  and  $\mathcal{S}_L^*$ . By Lemma 3.3.6, the stabilizer of  $N$  acts transitively on  $\mathcal{S}_N^*$ , and thus the result follows. In the case  $N \in \mathcal{S}_L$ , the proof is analogous.  $\square$

**Proposition 3.3.8.**  $\text{Im}(\rho_2)$  acts transitively on the 56 lines of  $\mathcal{P}$ .

*Proof.* For a line  $L$  in  $\mathcal{P}$ , the orbit-stabilizer theorem implies that

$$|\text{Im}(\rho)| = |\text{Orbit}(L)| \cdot |\text{Stab}(L)|.$$

Proposition 3.3.2 and Proposition 3.3.5 imply that

$$\begin{aligned} |\text{Stab}(L)| &= |W(E_6)| = 51840 \\ |\text{Im}(\rho)| &\in \{|W(E_7)|, |W(E_7)^+|\} = \{2903040, 1451520\} \end{aligned}$$

so it follows that  $|\text{Orbit}(L)| \in \{28, 56\}$ . By Lemma 3.3.7, the orbit of  $L$  intersects both  $\mathcal{S}_L$  and  $\mathcal{S}_L^*$  and by Lemma 3.3.6, the  $\text{Im}(\rho_2)$ -action is transitive on each of  $\mathcal{S}_L$  and  $\mathcal{S}_L^*$ , implying that

$$|\text{Orbit}(L)| \geq 1 + 27 + 27 > 28$$

and therefore  $|\text{Orbit}(L)| = 56$ . This implies that  $\text{Im}(\rho_2) \cong W(E_7)$ .  $\square$

### 3.4 Lattices

In this section we delve into the lattice theory of associated to the  $K3$  surface. We will restrict  $\rho_4$  to two sublattices  $L_+$  and  $L_-$  of  $H^2(X; \mathbb{Z})$  associated to the cyclic  $\mathbb{Z}/4\mathbb{Z}$  deck group action on  $X$ . In doing so, our goal is to relate these two restrictions via the commutative diagram given in Proposition 3.4.4.

### 3.4.1 Preliminaries

We recall general facts and definitions about lattices which will be used in the treatment of the monodromy of the family of  $K3$  surfaces of interest in this chapter. This exposition is contained in chapter 2 of [Kon20].

A *lattice*  $(L, \langle, \rangle_L)$  of rank  $r$  is a pair consisting of a free abelian group  $L$  of rank  $r$  together with a symmetric bilinear form  $\langle, \rangle_L : L \times L \rightarrow \mathbb{Z}$ . The quotient

$$A_L = L^*/L$$

denoted as the *discriminant group* of  $L$ . If  $A_L$  is trivial, we say that  $L$  is *unimodular*.

A sublattice  $S \subset L$  is *primitive* if  $L/S$  is torsion-free. If  $L$  is non-degenerate, the orthogonal complement  $S^\perp$  of any sublattice  $S \subset L$  is primitive.

The group of automorphisms of  $L$  is denoted by  $O(L)$  and is called its *orthogonal group*. For an even lattice  $L$ , we define its *discriminant quadratic form* by

$$q_L : A_L \rightarrow \mathbb{Q}/2\mathbb{Z} \quad q_L(x) = \langle x, x \rangle_{L \otimes \mathbb{Q}}$$

where  $\langle, \rangle_{L \otimes \mathbb{Q}}$  is a bilinear extension of  $\langle, \rangle_L$  to  $L \otimes \mathbb{Q}$ . A subgroup  $H \subset A_L$  is called *isotropic* if  $q_L|_H = 0$ . We denote the group of group automorphisms of  $A_L$  preserving  $q_L$  by  $O(q_L)$ . Any automorphism of  $L$  can be extended to  $L^*$ , thus inducing an automorphism of  $A_L$  which respects  $q_L$ . This induces a natural group homomorphism

$$O(L) \rightarrow O(q_L).$$

A lattice  $L$  is called *2-elementary* if  $A_L \cong (\mathbb{Z}/2\mathbb{Z})^{\oplus l}$  for some  $l$ . If  $L$  is an indefinite 2-elementary lattice, then the homomorphism  $O(L) \rightarrow O(q_L)$  is known to be surjective.

Let  $L$  be an even unimodular lattice,  $S \subset L$  a primitive sublattice and  $T = S^\perp$  its

orthogonal complement. There are natural embeddings

$$S \oplus T \subset L \cong L^* \subset S^* \oplus T^*.$$

Then  $L/(S \oplus T)$  is embedded as an isotropic subgroup of  $A_S \oplus A_T$ . This gives the commutative diagram

$$\begin{array}{ccc}
 & & A_S \\
 & \xrightarrow{\cong} & \nearrow \\
 L/(S \oplus T) & \xrightarrow{\iota} & A_S \oplus A_T \\
 & \searrow & \downarrow \gamma_{ST} \\
 & & A_T \\
 & \xrightarrow{\cong} & \nwarrow
 \end{array} \tag{*}$$

where the maps  $p_S \circ \iota$  and  $p_T \circ \iota$  induce group isomorphisms between  $L/(S \oplus T)$  and  $A_S, A_T$ . In particular, there is an isomorphism  $A_S \cong A_T$ . Furthermore, the isomorphism

$$\gamma_{ST} = p_T \circ \left( p_S|_{\iota(L/(S \oplus T))} \right)^{-1} : A_S \rightarrow A_T$$

satisfies that  $q_S = -\gamma_{ST} \circ q_T$ , hence  $O(q_S) \cong O(q_T)$  via the map  $c_{\gamma_{ST}}$  given by conjugation by  $\gamma_{ST}$ . Let  $c_S$  and  $c_T$  be the conjugation map given by

$$\begin{array}{ll}
 c_S : O(q_S) \rightarrow \text{Aut}(L/(S \oplus T)) & c_T : O(q_T) \rightarrow \text{Aut}(L/(S \oplus T)) \\
 \varphi \mapsto (p_S \circ \iota)^{-1} \circ \varphi \circ (p_S \circ \iota) & \varphi \mapsto (p_T \circ \iota)^{-1} \circ \varphi \circ (p_T \circ \iota)
 \end{array}$$

By commutativity of the diagram (\*), the images of  $c_S$  and  $c_T$  coincide, and we denote

$O(L/(S \oplus T))$  by this image. Therefore, the following commutative diagram is induced:

$$\begin{array}{ccc}
 & & O(q_S) \\
 & \nearrow \cong & \downarrow \cong \\
 O(L/(S \oplus T)) & \xrightarrow{c_S} & \\
 & \searrow \cong & \\
 & & O(q_T)
 \end{array}
 \quad (**)$$

### 3.4.2 $K3$ lattices

A  $K3$  surface  $X$  is a simply connected complex surface whose canonical class  $K_X$  is trivial. The  $K3$ -lattice is the lattice given by  $H^2(X; \mathbb{Z})$  together with its intersection pairing given by the cup product. This lattice is isomorphic to the even unimodular lattice of signature type  $(3, 19)$ :

$$U^{\oplus 3} \oplus E_8(-1)^{\oplus 2}$$

where  $U$  is the hyperbolic lattice and  $E_8(-1)$  is the negative  $E_8$  lattice. Namely,

$$U = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad E_8(-1) = \begin{pmatrix} -2 & 1 & & & & & & & \\ & 1 & -2 & 1 & & & & & \\ & & 1 & -2 & 1 & & & & \\ & & & 1 & -2 & 1 & & & \\ & & & & 1 & -2 & 1 & & 1 \\ & & & & & 1 & -2 & 1 & \\ & & & & & & 1 & -2 & \\ & & & & & & & 1 & -2 \end{pmatrix}$$

We now turn to our family of  $K3$  surfaces  $X$  branching over a smooth quartic curve. Before computing the monodromy of this family, we introduce two auxiliary sublattices of  $H^2(X; \mathbb{Z})$  crucial to its computation. These lattices arise from the  $\mathbb{Z}/4\mathbb{Z}$  deck group action

on  $X$ , and are extensively used in [Kon00] in order to describe the moduli space of smooth quartic curves in  $\mathbb{P}^2$  as a ball complement quotient. The deck group action on  $X$  induces an action  $T : H^2(X; \mathbb{Z}) \rightarrow H^2(X; \mathbb{Z})$  which satisfies  $T^4 = I$  and commutes with the action of  $\rho_4$ . Two sublattices associated to  $T$  that arise naturally are

$$L_+ = \left\{ x \in H^2(X; \mathbb{Z}) \mid T^2 x = x \right\} \quad L_- = \left\{ x \in H^2(X; \mathbb{Z}) \mid T^2 x = -x \right\}.$$

**Proposition 3.4.1.** The lattices  $L_+$  and  $L_-$  are the orthogonal complement of each other in  $H^2(X; \mathbb{Z})$ .

*Proof.* We will show that  $L_+^\perp = L_-$  only, as the proof that  $L_-^\perp = L_+$  is analogous. For  $x \in L_+$  and  $y \in L_-$ ,

$$\langle x, y \rangle = \langle T^2 x, T^2 y \rangle = \langle x, -y \rangle = -\langle x, y \rangle$$

and thus  $\langle x, y \rangle = 0$ . This shows that  $L_- \subset L_+^\perp$ . To see the reverse inclusion, suppose  $y \in L_+^\perp$ . Then

$$\langle x, (T^2 + I)y \rangle = 0 \quad \forall x \in L_+$$

We have that  $(T^2 + I)y \in L_+$  since  $T^4 = I$ . Since  $L_+$  is non-degenerate,  $(T^2 + I)y = 0$ , so  $y \in L_-$ .  $\square$

Proposition 3.4.1 implies that  $L_+$  and  $L_-$  are primitive lattices in  $H^2(X; \mathbb{Z})$ . Let  $\mathcal{P}$  be the intermediate double branched cover of  $\mathbb{P}^2$  corresponding to  $X$ . Then  $X$  is also a double branched cover of  $\mathcal{P}$ , and the branched cover map  $p : X \rightarrow \mathcal{P}$  induces the pullback

$$p^* : H^2(\mathcal{P}; \mathbb{Z}) \rightarrow H^2(X; \mathbb{Z})$$

$$e_i \mapsto \tilde{e}_i$$

The lattice  $L_+$  is generated by  $\tilde{e}_0, \tilde{e}_1, \dots, \tilde{e}_7$ . Letting  $H = H^2(X; \mathbb{Z}) / (L_+ \oplus L_-)$ , we have

that

$$H \cong A_{L_+} \cong A_{L_-} \cong (\mathbb{Z}/2\mathbb{Z})^{\oplus 8}$$

and thus  $L_+$  and  $L_-$  are 2-elementary even lattices. In [Nik80], Theorem 3.6.2, 2-elementary even lattices are classified by their rank and minimal number of generators, hence

$$L_+ \cong \langle 2 \rangle \oplus A_1^{\oplus 7} \quad L_- \cong A_1^{\oplus 2} \oplus D_4^{\oplus 2} \oplus U \oplus U(2).$$

In [Kon00] it is shown that  $O(q_{L_+})$  is isomorphic to a split extension of  $\mathrm{Sp}_6(\mathbb{Z}/2\mathbb{Z})$  by  $\mathbb{Z}/2\mathbb{Z}$ , thus a semidirect product. Since  $\mathrm{Aut}(\mathbb{Z}/2\mathbb{Z})$  is trivial, it is a direct product so  $O(q_+) \cong W(E_7)$ . Diagram (\*\*) then gives

$$O(q_{L_+}) \cong O(q_{L_-}) \cong W(E_7).$$

### 3.4.3 A useful diagram

In order to relate  $\rho_4$  to the lattices  $L_+, L_-$ , we will use a commutative diagram consisting of quotient and restriction maps. The key observation lies in the fact that  $\rho_4$  acts on each summand  $L_+$  and  $L_-$ .

**Proposition 3.4.2.** The action of  $\rho_4$  can be restricted from  $H^2(X; \mathbb{Z})$  to  $L_+$  and to  $L_-$ .

*Proof.* Since the action of  $\rho_4$  and  $T$  commute, we have

$$T^2 \rho_4(\gamma) x = \rho_4(\gamma) T^2 x = \pm \rho_4(\gamma) x.$$

for  $x \in L_{\pm}$  and  $\gamma \in \pi_1(\mathcal{U}_4)$ , and hence  $\rho_4(\gamma) x \in L_{\pm}$ . □

We proceed to define the necessary maps for our diagram. Proposition 3.4.2 tells us that

there are well defined restriction maps  $\text{res}(L_+)$  and  $\text{res}(L_-)$  given by

$$\begin{aligned} \text{res}(L_+) : \text{Im}(\rho_4) &\rightarrow O(L_+) & \text{res}(L_-) : \text{Im}(\rho_4) &\rightarrow O(L_-) \\ \rho_4(\gamma) &\mapsto \rho_4(\gamma)|_{L_+} & \rho_4(\gamma) &\mapsto \rho_4(\gamma)|_{L_-}. \end{aligned}$$

We define the composition maps  $\rho_4^+ = \text{res}(L_+) \circ \rho_4$  and  $\rho_4^- = \text{res}(L_-) \circ \rho_4$ . This gives the map

$$\begin{aligned} \text{mod}(L_+ \oplus L_-) : \text{Im}(\rho_4) &\rightarrow O(H) \\ \rho_4(\gamma) &\mapsto \left( \widetilde{\rho_4^+(\gamma)} \pmod{L_+}, \widetilde{\rho_4^-(\gamma)} \pmod{L_-} \right) \Big|_H \end{aligned}$$

which is well defined regarding  $H$  as a subgroup of  $A_{L_+} \oplus A_{L_-}$ . Finally, since  $L_+$  and  $L_-$  are even indefinite 2-elementary lattices, we have the surjective homomorphisms

$$\begin{aligned} \text{mod}(L_+) : O(L_+) &\rightarrow O(q_{L_+}) & \text{mod}(L_-) : O(L_-) &\rightarrow O(q_{L_-}) \\ \varphi &\mapsto \tilde{\varphi} \pmod{L_+} & \varphi &\mapsto \tilde{\varphi} \pmod{L_-} \end{aligned}$$

Before putting these maps together, we compute  $\text{Im}(\rho_4^+)$ .

**Proposition 3.4.3.**  $\text{Im}(\rho_4^+) \cong W(E_7)$ .

*Proof.* The action of  $\rho_2$  on  $H^2(\mathcal{P}; \mathbb{Z})$  completely determines that of  $\rho_4$  on  $L_+$  by conjugation  $c_{p^*}$  with the pullback  $p^* : H^2(\mathcal{P}; \mathbb{Z}) \rightarrow L_+$ . Together with Theorem 3.1.1, this gives the following commutative diagram:

$$\begin{array}{ccc} \pi_1(\mathcal{U}_4) & \xrightarrow{\rho_4} & \text{Im}(\rho_4) \\ \rho_2 \downarrow & \searrow^{\rho_4^+} & \downarrow \text{res}(L_+) \\ W(E_7) & \xrightarrow{c_{p^*}} & \text{Im}(\rho_4^+) \end{array}$$

Since  $\rho_4^+$  is surjective, so is  $c_{p^*}$ . Since the pullback  $p^*$  is a group isomorphism between

$H^2(\mathcal{P}; \mathbb{Z})$  and  $L_+$ , we have that  $c_{p^*}$  is injective. Hence,  $c_{p^*}$  is an isomorphism and  $\text{Im}(\rho_4^+) \cong W(E_7)$ .  $\square$

The restriction of  $\text{mod}(L_+)$  to  $\text{Im}(\rho_4^+)$  is surjective and is thus an isomorphism since  $O(q_{L_+}) \cong W(E_7)$ . Putting together the discussion above, we obtain the following.

**Proposition 3.4.4.** The following diagram commutes:

$$\begin{array}{ccccc}
& & & & O(q_{L_-}) \\
& & \nearrow^{\rho_4^-} & & \uparrow \\
& & O(L_-) & \xrightarrow{\text{mod}(L_-)} & \\
& \nearrow^{\rho_4^-} & \nearrow^{\text{res}(L_-)} & & \nearrow^{c_{L_-}} \\
\pi_1(\mathcal{U}_4) & \xrightarrow{\rho_4} & \text{Im}(\rho_4) & \xrightarrow{\text{mod}(L_+ \oplus L_-)} & O(H) \\
& \searrow_{\rho_4^+} & \searrow_{\text{res}(L_+)} & & \searrow_{c_{L_+}} \\
& & W(E_7) & \xrightarrow[\cong]{\text{mod}(L_+)} & O(q_{L_+}) \\
& & & & \uparrow \cong \\
& & & & c_{\gamma_{L_+ L_-}}
\end{array}$$

### 3.5 Computing $\text{Im}(\rho_4)$

In this section we prove Theorem 3.1.3. To do so, we first show that  $\text{Im}(\rho_4)$  and  $\text{Im}(\rho_4^-)$  are isomorphic. We then proceed to study  $\mathbb{Z}[i]$ -lattice structure on  $L_-$  to describe  $\text{Im}(\rho_4^-)$ . Finally, we use Kondo's description of the moduli space of smooth quartic curves proved in [Kon00] in order to compute  $\text{Im}(\rho_4)$ .

#### 3.5.1 Reduction to $\rho_4^-$

**Proposition 3.5.1.**  $\text{Im}(\rho_4) \cong \text{Im}(\rho_4^-)$ .

*Proof.* First, we show that for any  $\gamma \in \pi_1(\mathcal{U}_4)$ , if  $\rho_4^-(\gamma)$  is trivial, so is  $\rho_4^+(\gamma)$ . Using the

diagram in Proposition 3.4.4, we have that

$$\begin{aligned}
\rho_4^-(\gamma) = 0 &\implies \text{mod}(L_-) \circ \rho_4^-(\gamma) = 0 \\
&\implies c_{\gamma_{L_+L_-}} \circ \text{mod}(L_+) \circ \rho_4^+(\gamma) = 0 \\
(c_{\gamma_{L_+L_-}} \text{ is injective}) &\implies \text{mod}(L_+) \circ \rho_4^+(\gamma) = 0 \\
(\text{mod}(L_+) \text{ is injective}) &\implies \rho_4^+(\gamma) = 0.
\end{aligned}$$

Finally, we show that the map  $\text{res}(L_-)$  is injective. Let  $\rho_4(\gamma) \in \ker(\text{mod}(L_-))$  for some  $\gamma \in \pi_1(\mathcal{U}_4)$ . Then

$$\rho_4^-(\gamma) = 0 \implies \rho_4^+(\gamma) = 0.$$

Hence, the action of  $\rho_4(\gamma)$  on  $L_+ \oplus L_-$  is trivial. Since  $L_+ \oplus L_-$  is a finite index sublattice of  $H^2(X; \mathbb{Z})$ ,  $\rho_4(\gamma)$  acts trivially on  $H^2(X; \mathbb{Z})$ , so  $\rho_4(\gamma) = 0$ , implying the desired injectivity. Altogether,  $\text{res}(L_-)$  is an isomorphism onto its image and the proposition follows.  $\square$

### 3.5.2 $\mathbb{Z}[i]$ -module structure on $L_-$

The action  $T$  endows  $L_-$  with a  $\mathbb{Z}[i]$ -module structure since  $T^2 + I$  acts trivially on  $L_-$ . Furthermore, the Hodge decomposition on  $H^2(X; \mathbb{C})$  provides  $L_-$  a hermitian form induced by its intersection form. We study this structure on  $L_-$  following the technique in [All02].

The action  $T$  induces a decomposition

$$H^2(X; \mathbb{C}) \cong \bigoplus_{\zeta^4=1} H^2(X; \mathbb{C})_\zeta \cong \bigoplus_{\zeta^4=1} V_\zeta$$

where  $V_\zeta = H^2(X; \mathbb{C})_\zeta := \ker(T - \zeta I)$ . The eigenspaces  $V_i, V_{-i}$  are conjugate and  $L_- \otimes_{\mathbb{Z}} \mathbb{C} \cong V_i \oplus V_{-i}$ . Let  $j : L_- \rightarrow V_i$  be the  $\mathbb{Z}[i]$ -linear composition map given by

$$\begin{array}{ccc}
& H^2(X; \mathbb{C}) & \\
\nu \nearrow & & \searrow \text{proj}_{V_i} \\
L_- & \xrightarrow{j} & V_i
\end{array}$$

where  $\nu : H^2(X; \mathbb{Z}) \hookrightarrow H^2(X; \mathbb{C})$  is the natural inclusion and  $\text{proj}_{V_i} : H^2(X; \mathbb{C}) \rightarrow V_i$  is a projection. The bilinear form on  $L_-$  can be extended  $\mathbb{Z}[i]$ -linearly to  $L_- \otimes_{\mathbb{Z}[i]} \mathbb{C}$  and  $V_i$  can be given a hermitian form induced by the intersection pairing, given by  $h(a, b) = 2 \langle a, \bar{b} \rangle$ .

**Proposition 3.5.2.** The map  $j_{\mathbb{C}} : L_- \otimes_{\mathbb{Z}[i]} \mathbb{C} \rightarrow V_i$  is an isometric isomorphism.

*Proof.* Let  $a \in L_-$ . Since  $V_i, V_{-i}$  are conjugate,  $a = j(a) + \overline{j(a)}$ . We then have that

$$\begin{aligned}
\langle a, a \rangle &= \langle j(a) + \overline{j(a)}, j(a) + \overline{j(a)} \rangle \\
&= \langle j(a), j(a) \rangle + \langle \overline{j(a)}, \overline{j(a)} \rangle + 2 \langle j(a), \overline{j(a)} \rangle \\
h(j(a), j(a)) &= 2 \langle j(a), \overline{j(a)} \rangle
\end{aligned}$$

hence  $j_{\mathbb{C}}$  is an isometry. Since  $\dim_{\mathbb{C}} V_i = \dim_{\mathbb{C}} L_- \otimes_{\mathbb{Z}[i]} \mathbb{C} = 7$ , it follows that  $j_{\mathbb{C}}$  is an isomorphism.  $\square$

### 3.5.3 The Hodge structure on $H^2(X; \mathbb{C})$

The Hodge structure on  $H^2(X; \mathbb{C})$  is given by

$$\begin{aligned}
H^2(X; \mathbb{C}) &\cong H^{2,0}(X; \mathbb{C}) \oplus H^{1,1}(X; \mathbb{C}) \oplus H^{0,2}(X; \mathbb{C}) \\
&\cong \mathbb{C} \oplus \mathbb{C}^{20} \oplus \mathbb{C}
\end{aligned}$$

and it is determined by the line  $H^{2,0}$  since  $H^{0,2} \cong \overline{H^{2,0}}$  and  $H^{1,1} \cong (H^{2,0} \oplus H^{0,2})^{\perp}$ . Since  $T$  is of finite order, it is a biholomorphism and therefore respects the Hodge decomposition

on  $H^2(X; \mathbb{C})$ . This further decomposes the spaces  $V_\zeta$  into mutually orthogonal subspaces

$$V_\zeta \cong V_\zeta^{2,0} \oplus V_\zeta^{1,1} \oplus V_\zeta^{0,2}.$$

with respect to the intersection pairing on  $H^2(X; \mathbb{C})$ , given by

$$\langle \alpha, \bar{\beta} \rangle = \int_X \alpha \wedge \bar{\beta}.$$

Let  $h' = |z_0|^2 - |z_1|^2 - \dots - |z_6|^2$  be the standard quadratic form of signature  $(1, 6)$ .

**Proposition 3.5.3.** The hermitian vector space  $(V_i, h)$  is isomorphic to  $\mathbb{C}^{1,6} = (\mathbb{C}^7, h')$ .

*Proof.* We will show that  $h$  is positive-definite on  $V_i^{2,0} \oplus V_i^{0,2}$  and negative-definite on  $V_i^{1,1}$ .

We will also show the former and latter spaces are 1 and 6 dimensional respectively, concluding the proof.

Let  $\omega \in H^{2,0}(X; \mathbb{C})$  be a holomorphic form. Then  $T\omega$  must be holomorphic too. Since  $V_1 \oplus V_{-1}$  is spanned by rational classes,  $V_1 \oplus V_{-1} \subset H^{1,1}(X; \mathbb{C})$  and hence  $T\omega \neq \pm\omega$ . This implies that  $T\omega = \pm i\omega$  and therefore  $T\bar{\omega} = \mp i\bar{\omega}$ , hence

$$V_i^{2,0} \oplus V_i^{0,2} \cong V_{-i}^{2,0} \oplus V_{-i}^{0,2} \cong \mathbb{C}.$$

Moreover, the intersection pairing on  $\omega$  satisfies

$$h(\omega, \omega) = 2 \langle \omega, \bar{\omega} \rangle = 2 \int_X \omega \wedge \bar{\omega} > 0$$

so  $V_i^{2,0} \oplus V_i^{0,2}$  is positive-definite. The Kähler form  $\kappa$  associated to  $X$  lies in  $H^{1,1}(X; \mathbb{C})$  and has positive intersection pairing with itself being an integral cohomology class, so  $\langle \kappa, \bar{\kappa} \rangle > 0$ . Since  $H^2(X; \mathbb{C})$  has signature  $(3, 19)$ , the classes  $\omega, \bar{\omega}, \kappa$  span a maximal positive-definite

subspace of  $H^2(X; \mathbb{C})$ . This implies that  $V_i^{1,1}$  and  $V_{-i}^{1,1}$  are negative-definite and

$$V_i^{1,1} \cong V_{-i}^{1,1} \cong \mathbb{C}^6.$$

□

### 3.5.4 $\mathbb{Z}[i]$ -lattice structure on $L_-$

We now turn to describe  $L_-$  as a  $\mathbb{Z}[i]$ -lattice. A  $\mathbb{Z}[i]$ -lattice structure is induced on  $L_-$  via the action  $T$  by endowing  $L_-$  with the hermitian form

$$\langle x, y \rangle_{\mathbb{Z}[i]} = \langle x, y \rangle - i \langle x, Ty \rangle.$$

**Proposition 3.5.4.** As a  $\mathbb{Z}[i]$ -lattice,  $L_-$  is isomorphic to  $\mathbb{Z}[i]^7$  equipped with the type  $(1, 6)$  quadratic form

$$h_{L_-} = -2 \left( |z_0|^2 + |z_1|^2 + |z_2|^2 + |z_3|^2 + |z_4|^2 - \Re(z_1 \bar{z}_2 + z_3 \bar{z}_4 + z_5 \bar{z}_6) - \Im(z_1 \bar{z}_2 + z_3 \bar{z}_4 + z_5 \bar{z}_6) \right).$$

*Proof.* In Chapter 10 of [Kon20], the action of  $T$  on

$$L_- \cong A_1^{\oplus 2} \oplus D_4^{\oplus 2} \oplus U \oplus U(2)$$

is described. This action is diagonal, acting on  $A_1^{\oplus 2}$ ,  $U \oplus U(2)$  and each copy of  $D_4$  by blocks. This implies that each of these summands carries a  $\mathbb{Z}[i]$ -lattice structure, and in order to determine the quadratic form on  $L_-$ , it suffices to determine the quadratic form on each summand separately.

- $A_1^{\oplus 2}$  : As a  $\mathbb{Z}$ -lattice, it is generated by  $u, v$  and  $T$  acts by  $Tu = v$ . Hence,  $u$  generates  $A_1^{\oplus 2}$  as a  $\mathbb{Z}[i]$ -module. Since  $\langle u, u \rangle_{\mathbb{Z}[i]} = -2$ , the quadratic form on  $A_1^{\oplus 2}$  is given by

the  $(0, 1)$  form

$$h_{A_1^{\oplus 2}} = \langle z, z \rangle_{\mathbb{Z}[i]} = -2|z|^2.$$

- $U \oplus U(2)$  : As a  $\mathbb{Z}$ -lattice, it is generated by  $e, f, e', f'$  where  $e, f$  generate  $U$  and  $e', f'$  generate  $U(2)$ . The action of  $T$  is given by

$$Te = -e - e' \quad Tf = f - f'$$

and thus  $e, f$  generate  $U \oplus U(2)$  as a  $\mathbb{Z}[i]$ -module. We then have

$$\begin{pmatrix} \langle e, e \rangle_{\mathbb{Z}[i]} & \langle e, f \rangle_{\mathbb{Z}[i]} \\ \langle f, e \rangle_{\mathbb{Z}[i]} & \langle f, f \rangle_{\mathbb{Z}[i]} \end{pmatrix} = \begin{pmatrix} 0 & 1 - i \\ 1 + i & 0 \end{pmatrix}$$

hence the quadratic form on  $U \oplus U(2)$  is given by the  $(1, 1)$  form

$$h_{U \oplus U(2)} = \langle (z, w), (z, w) \rangle_{\mathbb{Z}[i]} = 2\Re(z\bar{w}) + 2\Im(z\bar{w}).$$

- $D_4$  : As a  $\mathbb{Z}$ -lattice,  $D_4$  is isomorphic to

$$\left\{ (x_1, x_2, x_3, x_4) \in \mathbb{Z}^4 \mid x_1 + x_2 + x_3 + x_4 \equiv 0 \pmod{2} \right\}.$$

equipped with the negative dot product. The basis

$$\{(1, 1, 0, 0), (-1, 1, 0, 0), (0, -1, 1, 0), (0, 0, -1, 1)\}$$

gives the usual Cartan matrix describing the  $\mathbb{Z}$ -lattice structure of  $D_4$ . The action of  $T$  is given by

$$T(x_1, x_2, x_3, x_4) = (x_2, -x_1, x_4, -x_3).$$

Altogether,  $p = (1, 1, 0, 0)$  and  $q = (0, -1, 1, 0)$  generate  $D_4$  as a  $\mathbb{Z}[i]$ -module. We then have

$$\begin{pmatrix} \langle p, p \rangle_{\mathbb{Z}[i]} & \langle p, q \rangle_{\mathbb{Z}[i]} \\ \langle q, p \rangle_{\mathbb{Z}[i]} & \langle q, q \rangle_{\mathbb{Z}[i]} \end{pmatrix} = \begin{pmatrix} -2 & 1 - i \\ 1 + i & -2 \end{pmatrix}$$

and thus the quadratic form on  $D_4$  is given by the  $(0, 2)$  form

$$h_{D_4} = \langle (z, w), (z, w) \rangle_{\mathbb{Z}[i]} = -2|z|^2 - 2|w|^2 + 2\Re(z\bar{w}) + 2\Im(z\bar{w}).$$

□

### 3.5.5 Realization of the Deck group via monodromy

Before proceeding to compute  $\text{Im}(\rho_4^-)$ , we will show that the action  $T$  on  $H^2(X; \mathbb{Z})$  induced by Deck group is realized via monodromy. By Corollary 3.1.2, there is a loop  $\gamma$  realizing the Geiser involution action on  $H^2(\mathcal{P}; \mathbb{Z})$ . Then the following holds:

**Proposition 3.5.5.**  $\rho_4(\gamma) = T$ .

To prove this, we show that  $\gamma$  realizes the Deck group action on  $\mathcal{P}$ , and consequently on  $X$ . We will see this by first observing that  $\gamma$  fixes  $\mathbb{P}^2$ , as it fixes every bitangent to a base point quartic in  $\mathcal{U}_4$ .

**Lemma 3.5.6.** If  $n$  bitangents to a smooth quartic curve are concurrent, then  $n \leq 4$ .

*Proof.* Suppose  $n$  bitangents are concurrent at a point. We may lift these lines to  $n$  concurrent lines in  $\mathcal{P}$ . Letting  $L_i$  be one of these  $n$  lines, blowing down  $L_i^*$  produces a smooth cubic surface  $S$  and the remaining  $n - 1$  lines are concurrent within  $S$ . Nevertheless, every line in a cubic surface intersects 10 other lines, which can be separated into 2 disjoint sets of 5 pairwise non-intersecting lines. This implies that any 4 lines in  $S$  cannot be pairwise intersecting, and thus  $n - 1 \leq 3$  or  $n \leq 4$ , as we wanted. □

**Lemma 3.5.7.**  $\gamma$  fixes every point in every bitangent to a given smooth quartic curve.

*Proof.* Let  $f \in \mathcal{U}_4$  be a base point curve, and let  $\ell_1, \ell_2, \dots, \ell_{28}$  be the 28 bitangent lines to  $V(f)$  in  $\mathbb{P}^2$ . For each pair of indices  $1 \leq i < j \leq 28$  let

$$P_{ij} = \ell_i \cap \ell_j.$$

Let  $L_i, L_i^*$  be the pair of lines in  $\mathcal{P}$  lying over the bitangent  $\ell_i$ . The lines  $\{L_i, L_i^*\}$  are interchanged by  $\gamma$ , thus leaving each bitangent  $\ell_i$  fixed. Therefore,  $\gamma$  fixes each point  $P_{ij}$  in  $\mathbb{P}^2$ . Any two distinct bitangents to  $V(f)$  must intersect due to Bezout's theorem. Some of these bitangents may be concurrent, but at any concurrence point, at most 4 bitangents may meet by Lemma 3.5.6. Hence,  $\gamma$  fixes at least 9 points within each bitangent  $\ell_i$ , therefore fixing all of  $\ell_i$ .  $\square$

Let  $\varphi_\gamma : V(f) \rightarrow V(f)$  be the automorphism on  $V(f)$  induced by  $\gamma$ . Let  $\Phi_\gamma : \mathbb{P}^2 \rightarrow \mathbb{P}^2$  be an extension of  $\varphi_\gamma$  such that  $\Phi_\gamma|_{V(f)} = \varphi_\gamma$  (see Theorem 4.2 in [Hir05]) and  $\Phi_\gamma$  agrees with the monodromy action of  $\gamma$  on  $V(f)$ . Namely,  $\Phi_\gamma$  fixes each bitangent  $\ell_i$  and  $\Phi_\gamma$  is an isometry of  $\mathbb{P}^2$ .

**Proposition 3.5.8.**  $\Phi_\gamma$  is the identity map on  $\mathbb{P}^2$ .

*Proof.* Note  $\varphi_\gamma$  must be the identity map since  $\gamma$  fixes all bitangents to  $V(f)$ , which completely determine the quartic curve  $V(f)$ . At each point  $P_{ij}$  in  $\mathbb{P}^2$ , the differential

$$D_{P_{ij}}\Phi_\gamma : T_{P_{ij}}\mathbb{P}^2 \rightarrow T_{P_{ij}}\mathbb{P}^2$$

is the identity map, as it is the identity in the directions spanned by  $\ell_i$  and  $\ell_j$  by Lemma 3.5.7. Since  $\Phi_\gamma$  is an isometry and  $\mathbb{P}^2$  is a complete compact Riemannian manifold,  $\Phi_\gamma$  must be the identity as well.  $\square$

Hence,  $\Phi_\gamma$  fixes  $\mathbb{P}^2$  and  $\gamma$  acts as the Deck transformation  $w \mapsto -w$  on  $\mathcal{P} = \{w^2 = f\}$ .

**Proposition 3.5.9.**  $\gamma$  acts as the Deck transformation  $w \rightarrow iw$  on  $X = \{w^4 = f\}$ .

*Proof.* Choose a point  $P$  in  $\mathbb{P}^2$  not lying in  $V(f)$ . The preimages of  $P$  via the branched cover  $\mathcal{P} \rightarrow \mathbb{P}^2$  are

$$P_+ = (w, P)$$

$$P_- = (-w, P)$$

and the preimages of  $P_+, P_-$  via the branched cover and  $X \rightarrow \mathcal{P}$  are, respectively

$$Q_1 = (w, P)$$

$$Q_i = (iw, P)$$

$$Q_{-1} = (-w, P)$$

$$Q_{-i} = (-iw, P)$$

The action of  $\gamma$  maps  $P_+ \rightarrow P_-$  in  $\mathcal{P}$  and thus, without loss of generality,  $\gamma$  maps

$$Q_1 \xrightarrow{\gamma} Q_i$$

$$Q_{-i} \xleftarrow{\gamma} Q_{-1}$$

Let  $t$  be the Deck group action of the branched cover  $X \rightarrow \mathbb{P}^2$ . Then  $t^2$  is the Deck group action of the branched cover  $X \rightarrow \mathcal{P}$ . In particular,  $t^2$  fixes the points  $P_+, P_-$  and therefore  $t^2$  must interchange the pairs of points within the pairs  $Q_1, Q_{-1}$  and  $Q_i, Q_{-i}$ . This implies, without loss of generality, that  $t$  maps the points  $Q_1, Q_i, Q_{-1}, Q_{-i}$  as follows:

$$\begin{array}{ccc} Q_1 & \xrightarrow{t} & Q_i \\ t \uparrow & & \downarrow t \\ Q_{-i} & \xleftarrow{t} & Q_{-1} \end{array}$$

Now suppose  $\gamma$  maps  $Q_i \rightarrow Q_1$  (and therefore  $Q_{-i} \rightarrow Q_{-1}$ ). Observing how  $t$  and  $\gamma$  permute the set of points  $\{Q_1, Q_i, Q_{-1}, Q_{-i}\}$ , we see that  $t$  acts as the 4-cycle (1234) while  $\gamma$  acts as the involution (12)(34). This is a contradiction because the permutations (1234) and

(12)(34) do not commute, while  $t$  and  $\gamma$  do. Hence,  $\gamma$  cannot map  $Q_i \rightarrow Q_1$  and thus it maps the points as

$$\begin{array}{ccc} Q_1 & \xrightarrow{\gamma} & Q_i \\ \gamma \uparrow & & \downarrow \gamma \\ Q_{-i} & \xleftarrow{\gamma} & Q_{-1} \end{array}$$

coinciding with  $t$ . Our initial choice of  $P$  in  $\mathbb{P}^2 \setminus V(f)$  was arbitrary, so  $\gamma$  coincides with  $t$  on all of  $\mathbb{P}^2$ , concluding that the monodromy action of  $\gamma$  is given by that of the Deck transformation  $t$ . □

### 3.5.6 The moduli of smooth quartic curves

The image of  $\rho_4^-$  must lie in the centralizer  $C_{L_-}(T)$  of  $T$  in  $L_-$  since  $\rho_4$  and  $T$  commute. This is equivalent to respecting the  $\mathbb{Z}[i]$ -module structure on  $L_-$ . Moreover,  $\rho_4^-$  is norm-preserving, which translates to the corresponding  $\mathbb{Z}[i]$ -lattice automorphisms to be unitary. Therefore

$$\mathrm{Im}(\rho_4^-) \subset C_{L_-}(T) \cap O(L_-) \cong U(h_{L_-}).$$

For simplicity and consistency with the notation in [Kon00], we let

$$\Gamma = U(h_{L_-}).$$

Our goal is to show that  $\mathrm{Im}(\rho_4^-) \cong \Gamma$ , thus computing the monodromy group of  $\rho_4$  by Proposition 3.5.1. In order to do this, we appeal to the moduli space of smooth quartic curves  $\mathcal{M}_3$ . Theorem 2.5 in [Kon00] characterizes this moduli space as the quotient

$$\mathcal{M}_3 \cong (\mathcal{D}_6 - \mathcal{H}) / \Gamma$$

where  $\mathcal{D}_6$  is a complex 6-dimensional ball and  $\mathcal{H}$  is a locally finite union of hyperplanes in  $\mathcal{D}_6$ . Following a similar treatment to that in [All02], we consider a cover  $\widetilde{\mathcal{M}}_3$  of  $\mathcal{M}_3$  as

follows:

For a given smooth quartic curve  $V(f)$  along with the corresponding  $K3$  surface  $X_f$  lying over it in the fibration  $\mathcal{E}_{4,4}$ , define a *framing*  $\lambda$  on  $V(f)$  as an isomorphism

$$\lambda : \left( \mathbb{Z}[i]^7, h_{L_-} \right) \rightarrow L_- \subset H^2(X_f; \mathbb{Z}).$$

which respects the intersection form on each lattice. Define  $\widetilde{\mathcal{M}}_3$  as the moduli space of framed smooth quartic curves, where  $\mathbb{P}(\Gamma) = \Gamma/\mathbb{Z}[i]^*$  acts on  $\widetilde{\mathcal{M}}_3$  by precomposition. That is,  $\Gamma$  acts on  $\widetilde{\mathcal{M}}_3$  up to scalar equivalence and for  $g \in \Gamma$  we have

$$g \cdot (V(f), \lambda) = \left( V(f), \lambda \circ g^{-1} \right)$$

where framings  $\lambda, \lambda'$  differing by a unit in  $\mathbb{Z}[i]$  are equivalent. The composition  $j_{\mathbb{C}} \circ \lambda_{\mathbb{C}} : \mathbb{C}^{1,6} \rightarrow V_i$  gives a negative-definite hyperplane

$$(j_{\mathbb{C}} \circ \lambda_{\mathbb{C}})^{-1} \left( V_i^{1,1} \right) \subset \mathbb{C}^{1,6}.$$

These hyperplanes are parametrized by the ball  $\mathcal{D}_6$  and provide the period map  $\wp : \widetilde{\mathcal{M}}_3 \rightarrow \mathcal{D}_6$ . In [Kon00] it is shown that those points in  $\mathcal{D}_6$  which are not in the image of  $\wp$  correspond precisely to roots in  $L^-$ , namely, those  $\delta \in L_-$  satisfying  $\langle \delta, \delta \rangle = -2$ . Each root  $\delta \in L_-$  has an associated hyperplane

$$H_{\delta} = \{z \in \mathcal{D}_6 \mid \langle z, \delta \rangle = 0\} \subset \mathcal{D}_6$$

and we denote the union of these hyperplanes by

$$\mathcal{H} = \bigcup_{\langle \delta, \delta \rangle = -2} H_{\delta}.$$

By the Torelli theorem for  $K3$  surfaces ([IR 71]) it follows that the period map  $\wp : \widetilde{\mathcal{M}}_3 \rightarrow \mathcal{D}_6 - \mathcal{H}$  is an isomorphism. This provides a commutative diagram

$$\begin{array}{ccc} \widetilde{\mathcal{M}}_3 & \xrightarrow[\cong]{\wp} & \mathcal{D}_6 - \mathcal{H} \\ \mathbb{P}(\Gamma) \downarrow & & \downarrow \mathbb{P}(\Gamma) \\ \mathcal{M}_3 & \xrightarrow[\tilde{\wp}]{\cong} & (\mathcal{D}_6 - \mathcal{H}) / \Gamma \end{array}$$

### 3.5.7 Proof of the main theorem

A loop in  $\mathcal{U}_4$  may be regarded as one in  $\mathcal{M}_3$ . This induces the map  $\mu : \pi_1(\mathcal{M}_3) \rightarrow \Gamma$  given by  $\mu(\ell) = \rho_4^-(\ell)$ .

**Proposition 3.5.10.** The monodromy map  $\mu : \pi_1(\mathcal{M}_3) \rightarrow \Gamma$  is surjective.

*Proof.* Surjectivity to  $\mathbb{P}(\Gamma)$  follows from the connectedness of  $\widetilde{\mathcal{M}}_3$ . Fix a smooth quartic curve  $V(f)$  and a point  $(V(f), \lambda_0) \in \widetilde{\mathcal{M}}_3$ . For any  $g \in \mathbb{P}(\Gamma)$  there is a path  $\tilde{\ell}$  starting at  $(V(f), \lambda_0)$  and ending at  $(V(f), \lambda_\ell)$  such that  $\lambda_\ell = \lambda_0 \circ g^{-1}$ . Hence,  $\tilde{\ell}$  descends to a loop  $\ell$  in  $\mathcal{M}_3$  based at  $V(f)$  such that

$$\mu(\ell) = \lambda_\ell^{-1} \circ \lambda_0 = g.$$

This shows surjectivity to  $\Gamma$  up to scalar equivalence, so it remains to show scalars are realized by  $\mu$ . Indeed, scalars in  $\Gamma$  correspond to powers of the action  $T$  induced by the Deck transformation  $t$  of the branched cover  $X \rightarrow \mathbb{P}^2$ . By Proposition 3.5.5, these are realized by a path  $\gamma$  inducing the Geiser involution on  $\mathcal{P}$ , and thus surjectivity to  $\Gamma$  follows.  $\square$

It remains to relate  $\mu$  to  $\pi_1(\mathcal{U}_4)$ . Note that  $\mathcal{M}_3$  is given by an  $PGL_3(\mathbb{C})$  quotient  $q : \mathcal{U}_4 \rightarrow \mathcal{M}_3$  as

$$\mathcal{M}_3 \cong \mathcal{U}_4 / PGL_3(\mathbb{C}).$$

Moreover,  $\mathcal{M}_3$  embeds in the moduli space of genus 3 curves  $\mathcal{M}_3$  since every smooth quartic curve is a genus 3 curve. The moduli  $\mathcal{M}_3$  is obtained by removing the moduli  $\mathcal{H}_3$  of

hyperelliptic genus 3 surfaces from  $\mathcal{M}_3$  since every genus 3 curve is either planar quartic or hyperelliptic. That is,

$$\mathcal{M}_3 = \mathcal{M}_3 \cup \mathcal{H}_3.$$

The moduli  $\mathcal{H}_3$  is also referred to as the genus 3 hyperelliptic locus.

**Proposition 3.5.11.** The natural map  $q_* : \pi_1(\mathcal{U}_4) \rightarrow \pi_1(\mathcal{M}_3)$  induced by the quotient  $q$  is surjective.

*Proof.* Consider the subvariety  $\mathcal{O} \subset \mathcal{M}_3$  of genus 3 curves with a nontrivial automorphism group. Then  $\mathcal{O}$  is the union of irreducible components for each topological type of faithful finite group action on a genus 3 surface  $\Sigma_3$ . Using the Riemann-Hurwitz formula, the dimension over  $\mathbb{C}$  of each component is at most 5, with equality only holding for the hyperelliptic locus  $\mathcal{H}_3$ . In particular, the remaining components distinct of  $\mathcal{H}_3$  lie within  $\mathcal{M}_3$  and are of dimension at most 4. Since  $\mathcal{M}_3$  is 6-dimensional, we have that

$$\text{codim}_{\mathcal{M}_3}(\mathcal{O} - \mathcal{H}_3) \geq 2 \quad \text{and} \quad \text{codim}_{\mathcal{U}_4}(q^{-1}(\mathcal{O} - \mathcal{H}_3)) \geq 2.$$

By excising  $q^{-1}(\mathcal{O} - \mathcal{H}_3)$  from  $\mathcal{U}_4$  we obtain a  $PGL_3(\mathbb{C})$  fibration

$$q : \mathcal{U}_4 - q^{-1}(\mathcal{O} - \mathcal{H}_3) \rightarrow \mathcal{M}_3 - (\mathcal{O} - \mathcal{H}_3).$$

The long homotopy exact sequence induced by this fibration is given by

$$\cdots \rightarrow \pi_1(PGL_3(\mathbb{C})) \rightarrow \pi_1(\mathcal{U}_4 - q^{-1}(\mathcal{O} - \mathcal{H}_3)) \rightarrow \pi_1(\mathcal{M}_3 - (\mathcal{O} - \mathcal{H}_3)) \rightarrow \pi_0(PGL_3(\mathbb{C})) = 1$$

implying that the map  $\pi_1(\mathcal{U}_4 - q^{-1}(\mathcal{O} - \mathcal{H}_3)) \rightarrow \pi_1(\mathcal{M}_3 - (\mathcal{O} - \mathcal{H}_3))$  is surjective. Furthermore, we have

$$\pi_1(\mathcal{M}_3) \cong \pi_1(\mathcal{M}_3 - (\mathcal{O} - \mathcal{H}_3)) \quad \text{and} \quad \pi_1(\mathcal{U}_4) \cong \pi_1(\mathcal{U}_4 - q^{-1}(\mathcal{O} - \mathcal{H}_3))$$

since the excised spaces are of codimension at least 2. Therefore,  $q_* : \pi_1(\mathcal{U}_4) \rightarrow \pi_1(\mathcal{M}_3)$  is surjective.  $\square$

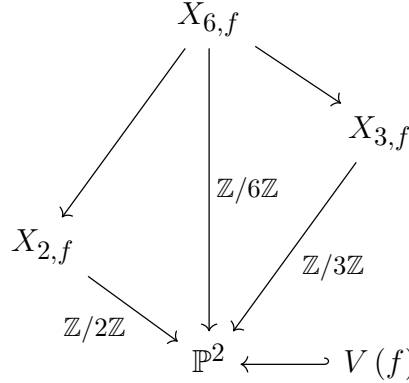
The composition of the maps  $\mu \circ q_*$  gives precisely the monodromy map  $\rho_4^-$ . Proposition 3.5.10 and Proposition 3.5.11 combined give surjectivity of  $\rho_4^-$ , concluding our computation.

# CHAPTER 4

## SURFACES OVER SEXTIC CURVES

### 4.1 Introduction

For  $f \in \mathcal{U}_6$  consider the cyclic 2-fold, 3-fold and 6-fold covers branched over  $V(f)$ :



The surface  $X_{2,f}$  is a  $K3$  surface. For  $d = 2, 3, 6$ , we define the *universal*  $(d, 6)$ -branched cover of  $\mathbb{P}^2$  as the fiber bundle

$$\mathcal{E}_{d,6} = \{(P, f) \in \mathcal{P}_f \times \mathcal{U}_6 \mid P \in X_d\}$$

given by the fibration

$$\begin{array}{ccc}
 X_d & \hookrightarrow & \mathcal{E}_{d,6} & & (P, f) \\
 & & \downarrow & & \downarrow \\
 & & \mathcal{U}_6 & & f
 \end{array}$$

Fixing a base point  $f \in \mathcal{U}_6$ , we simply refer to these fibers by  $X_d$ . The action of  $\pi_1(\mathcal{U}_6)$  on  $H^2(X_d; \mathbb{Z})$  induces monodromy homomorphisms

$$\rho_d : \pi_1(\mathcal{U}_6) \rightarrow \text{Aut}\left(H^2(X_d; \mathbb{Z})\right).$$

The images of each  $\rho_d$  can be further restricted by noting that the respective intersection forms in  $H^2(X_d; \mathbb{Z})$  remain invariant under the actions of  $\rho_d$ . Each surface  $X_d$  is too endowed

with a deck group action, which induces corresponding actions  $T_d$  of order  $d$  on  $H^2(X_d; \mathbb{Z})$  and the actions of  $\rho_d$  and  $T_d$  commute. Following closely the technique used in Chapter 3, we consider orthogonal lattices

$$L_1 = \left\{ x \in H^2(X_d; \mathbb{Z}) \mid (T_d - 1)x = 0 \right\}$$

$$L_{\zeta_d} = \left\{ x \in H^2(X_d; \mathbb{Z}) \mid \left( T_d^{d-1} + \cdots + T_d + I \right) x = 0 \right\}.$$

We then frame the moduli space of smooth sextic curves  $\mathcal{M}_6$ , on which we have an action of the orthogonal transformations of the lattice  $L_{\zeta_d}$  which commute with  $T_d$ . For  $d = 2, 3$  these groups are

$$\Gamma_2^+ = O^+(L_{-1}) \quad \Gamma_3 = U\left(\mathbb{Z}[\omega]^{21}, q_{3,18}\right)$$

and they restrict  $\text{Im}(\rho_d)$ . Via the period map and by virtue of the Torelli theorem for  $K3$  surfaces, we obtain an isomorphism from our framed moduli space to the symmetric domains of type IV (in the sense of [Loo03])  $(\mathcal{D}_{6,2} - \mathcal{H})$  for  $d = 2$ . Together with the connectedness of the period domain, we have the following results:

**Theorem 4.1.1.** *The moduli space of smooth sextic curves  $\mathcal{M}_6$  is isomorphic to the quotient*

$$\mathcal{M}_6 \cong (\mathcal{D}_{6,2} - \mathcal{H}) / \Gamma_2^+$$

**Theorem 4.1.2.** *The image of the monodromy map  $\rho_2$  satisfies  $\mathbb{P}(\Gamma_2^+) \subset \text{Im}(\rho_2) \subset \Gamma_2^+$ .*

We conclude with conjectural remarks regarding analogous statements for the case  $d = 3$ . In this case, the period map involved is not well understood, and we appeal to the treatment of triple covers given in [Mir85] via their associated Tschirnhausen modules to propose an analogue tool that allows for a description of the corresponding period domain.

## 4.2 Lattices

In this section we study the lattice theory of associated to the surfaces  $X_d$ . We will restrict  $\rho_d$  to two sublattices  $L_{\zeta_d}$  and  $L_{\zeta_d}^\perp$  of  $H^2(X_d; \mathbb{Z})$  associated to the cyclic  $\mathbb{Z}/d\mathbb{Z}$  deck group action on  $X$ . In doing so, our goal is to relate these two restrictions via the commutative diagram given in Proposition 4.2.2. We use the same lattice conventions as in Chapter 3.

### 4.2.1 Discriminant and orthogonal groups

We introduce two auxiliary sublattices of  $H^2(X_d; \mathbb{Z})$  crucial to its computation. These lattices arise from the  $\mathbb{Z}/d\mathbb{Z}$  deck group action on  $X_d$ , which induces an action  $T_d : H^2(X; \mathbb{Z}) \rightarrow H^2(X_d; \mathbb{Z})$  which satisfies  $T_d^d = I$  and commutes with the action of  $\rho_d$ . Two sublattices associated to  $T_d$  that arise naturally are

$$\begin{aligned} L_1 &= \left\{ x \in H^2(X_d; \mathbb{Z}) \mid (T_d - 1)x = 0 \right\} \\ L_{\zeta_d} &= \left\{ x \in H^2(X_d; \mathbb{Z}) \mid \left( T_d^{d-1} + \cdots + T_d + I \right) x = 0 \right\}. \end{aligned}$$

**Proposition 4.2.1.** The lattices  $L_1$  and  $L_{\zeta_d}$  are the orthogonal complements in  $H^2(X_d; \mathbb{Z})$ .

*Proof.* Showing  $L_1^\perp = L_{\zeta_d}$  is analogous to  $L_{\zeta_d}^\perp = L_1$ , we do the first. For  $x \in L_1$  and  $y \in L_{\zeta_d}$ ,

$$\langle x, y \rangle = \frac{1}{d} \left\langle \left( T_d^{d-1} + \cdots + T_d + I \right) x, \left( T_d^{d-1} + \cdots + T_d + I \right) y \right\rangle = 0.$$

and thus  $L_{\zeta_d} \subset L_1^\perp$ . To see the reverse inclusion, suppose  $y \in L_1^\perp$ . Then

$$\left\langle x, \left( T_d^{d-1} + \cdots + T_d + I \right) y \right\rangle = 0 \quad \forall x \in L_1$$

We have  $\left( T_d^{d-1} + \cdots + T_d + I \right) y \in L_1$  since  $T_d^d = I$ . Since  $L_1$  is non-degenerate, it follows that  $\left( T_d^{d-1} + \cdots + T_d + I \right) y = 0$ , so  $y \in L_{\zeta_d}$ . □

Proposition 4.2.1 implies that  $L_1$  and  $L_{\zeta_d}$  are primitive lattices in  $H^2(X_d; \mathbb{Z})$ . The branched cover map  $f_d : X_d \rightarrow \mathbb{P}^2$  induces the pullback  $f_d^*$  and the lattice  $L_1$  is generated by  $f_d^*H$ , where  $H$  is the hyperplane class in  $\mathbb{P}^2$ . Letting  $H_d = H^2(X_d; \mathbb{Z}) / (L_1 \oplus L_{\zeta_d})$ ,

$$H_d \cong A_{L_1} \cong A_{L_{\zeta_d}} \cong \mathbb{Z}/d\mathbb{Z}.$$

and  $O(L_1) \cong \mathbb{Z}/2\mathbb{Z}$  since  $\text{rk}(L_1) = 1$ . Now, we consider the cases  $d = 2$  and  $d = 3$ :

- $d = 2$ :  $L_1$  and  $L_{-1}$  are 2-elementary even lattices. In [Nik80], Theorem 3.6.2, these lattices are classified by their rank and minimal number of generators, hence

$$L_1 \cong \langle 2 \rangle \quad L_{-1} \cong A_1 \oplus U^{\oplus 2} \oplus E_8(-1)^{\oplus 2}.$$

Moreover, since  $\text{Aut}(\mathbb{Z}/2\mathbb{Z}) = 1$  the corresponding orthogonal groups are trivial:

$$O(q_{L_1}) \cong O(q_{L_{-1}}) \cong 1.$$

- $d = 3$ : Since  $A_{L_{\zeta_3}} \cong \mathbb{Z}/3\mathbb{Z}$ , and  $q_{L_{\zeta_3}}(x) = \frac{1}{3}$  for  $x \neq 0$ , it follows that

$$O(q_{L_1}) \cong O(q_{L_{\zeta_3}}) \cong \mathbb{Z}/2\mathbb{Z}$$

#### 4.2.2 A useful diagram

We build a diagram similar to that in Section 3.4.3 which will help us restrict  $\text{Im}(\rho_d)$ . Since  $T_d$  acts on each  $L_1$  and  $L_{\zeta_d}$ , there are restriction maps  $\text{res}(L_1)$  and  $\text{res}(L_{\zeta_d})$  given by

$$\begin{array}{ll} \text{res}(L_1) : \text{Im}(\rho_d) \rightarrow O(L_1) & \text{res}(L_{\zeta_d}) : \text{Im}(\rho_d) \rightarrow O(L_{\zeta_d}) \\ \rho_d(\gamma) \mapsto \rho_d(\gamma)|_{L_1} & \rho_d(\gamma) \mapsto \rho_d(\gamma)|_{L_{\zeta_d}} \end{array}$$

We define the compositions  $\rho_d^1 = \text{res}(L_1) \circ \rho_d$  and  $\rho_d^{\zeta_d} = \text{res}(L_{\zeta_d}) \circ \rho_d$ , giving the map

$$\begin{aligned} \text{mod}(L_1 \oplus L_{\zeta_d}) : \text{Im}(\rho_d) &\rightarrow O(H_d) \\ \rho_d(\gamma) &\mapsto \left( \widetilde{\rho_d^1(\gamma)} \pmod{L_1}, \widetilde{\rho_d^{\zeta_d}(\gamma)} \pmod{L_{\zeta_d}} \right) \Big|_{H_d} \end{aligned}$$

which is well defined regarding  $H_d$  as a subgroup of  $A_{L_1} \oplus A_{L_{\zeta_d}}$ . Finally, consider

$$\begin{aligned} \text{mod}(L_1) : O(L_1) &\rightarrow O(q_{L_1}) & \text{mod}(L_{\zeta_d}) : O(L_{\zeta_d}) &\rightarrow O(q_{L_{\zeta_d}}) \\ \varphi &\mapsto \tilde{\varphi} \pmod{L_1} & \varphi &\mapsto \tilde{\varphi} \pmod{L_{\zeta_d}} \end{aligned}$$

Putting together the discussion above, we obtain the following.

**Proposition 4.2.2.** The following diagram commutes:

$$\begin{array}{ccccc} & & & & O(L_{\zeta_d}) \xrightarrow{\text{mod}(L_{\zeta_d})} O(q_{L_{\zeta_d}}) \\ & \nearrow^{\rho_d^{\zeta_d}} & & \nearrow^{c_{L_{\zeta_d}}} & \\ \pi_1(\mathcal{U}_6) & \xrightarrow{\rho_d} & \text{Im}(\rho_d) & \xrightarrow{\text{mod}(L_1 \oplus L_{\zeta_d})} & O(H_d) \\ & \searrow_{\rho_d^1} & & \searrow_{c_{L_1}} & \\ & & O(L_1) & \xrightarrow{\text{mod}(L_1)} & O(q_{L_1}) \end{array}$$

$\begin{array}{c} \cong \\ \uparrow c_{\gamma_{L_1 L_{\zeta_d}}} \\ \cong \end{array}$

### 4.3 Restricting $\text{Im}(\rho_d)$

We apply Proposition 4.2.2 to the cases  $d = 2$  and  $d = 3$ .

- $d = 2$ : The orthogonal groups  $O(q_{L_1})$ ,  $O(q_{L_{-1}})$  are trivial, and moreover, elements of  $\text{Im}(\rho_2)$  must have trivial spinor norm. The generator  $f_2^* H \in L_1$  satisfies  $(f_2^* H)^2 = 2 > 0$ , and thus the reflection  $s$  about it has negative spinor norm. Moreover, the product of the spinor norm of the restrictions of every  $g \in \text{Im}(\rho_2)$  to  $L_1$ ,  $L_{-1}$  is

trivial, as the spinor norm of  $g$  is trivial. This implies that

$$\text{res}(L_{-1})(g) = 1 \implies \text{res}(L_1)(g) = 1$$

and thus,  $\text{Im}(\rho_2)$  is isomorphic to a subgroup of  $O(L_{-1})$ .

- $d = 3$ : Letting  $\omega = \zeta_3$ , we have the orthogonal groups  $O(q_{L_1})$ ,  $O(q_{L_\omega}) \cong \mathbb{Z}/2\mathbb{Z}$  and in this case the map  $\text{mod}(L_\omega) : O(L_\omega) \rightarrow O(q_{L_\omega})$  is an isomorphism. Our commutative diagram then implies that

$$\text{res}(L_\omega)(g) = 1 \implies \text{res}(L_1)(g) = 1$$

and thus,  $\text{Im}(\rho_3)$  is isomorphic to a subgroup of  $O(L_\omega)$ .

## 4.4 Decompositions of $H^2(X_d; \mathbb{C})$

In this section we study the interaction of the Hodge and Eigenvalue decompositions on  $H^2(X_d; \mathbb{C})$ . The description of the eigenspaces associated to each surface  $X_d$  will allow us to the study of the moduli space of sextic curves in the following section.

### 4.4.1 Hodge decomposition

By the Lefschetz hyperplane theorem, the surfaces  $X_d$  are simply connected. Further, these surfaces are connected and orientable, so the cohomological information is contained in their second cohomology group alone. The Hodge structure on  $H^2(X_d; \mathbb{C})$  is given by

$$H^2(X_d; \mathbb{C}) \cong H^{2,0}(X_d; \mathbb{C}) \oplus H^{1,1}(X_d; \mathbb{C}) \oplus H^{0,2}(X_d; \mathbb{C})$$

The key tools we employ to compute the Hodge diamonds of the surfaces  $X_d$  are:

- **Riemann-Hurwitz's formula:**

$$\chi(X_d) = d\chi(\mathbb{P}^2) - (d-1)\chi(V(f))$$

where  $V(f)$  is our branch locus of  $f_d$  in  $\mathbb{P}^2$ .

- **Noether's formula:**

$$\chi_{\text{hol}}(X_d) = \frac{c_2 + c_1^2}{12} = \frac{\chi(X_d) + K_{X_d}^2}{12}$$

where  $c_1, c_2$  are the first and second Chern classes of  $X_d$ , given by its canonical and Euler class.

- **Hirzebruch's signature formulas:**

$$\sigma(X_d) = \frac{1}{3} \left( K_{X_d}^2 - 2\chi(X_d) \right) = d \cdot \sigma(\mathbb{P}^2) - \frac{d^2 - 1}{3d} [V(f)]^2$$

where  $[V(f)]$  is the class of  $V(f)$  in  $\mathbb{P}^2$ .

Using this formulae, we obtain the following:

Surface	$X_2$			$X_3$			$X_6$		
		1			1			1	
		0	0		0	0		0	0
Hodge Diamond	1	20	1	3	37	3	10	86	10
		0	0		0	0		0	0
		1			1			1	
Signature		(3, 19)			(7, 36)			(21, 85)	

#### 4.4.2 Eigenvalue decomposition

The action  $T_d$  induces a decomposition

$$H^2(X_d; \mathbb{C}) \cong \bigoplus_{\zeta^d=1} H^2(X_d; \mathbb{C})_\zeta \cong \bigoplus_{\zeta^d=1} V_\zeta$$

where  $V_\zeta = H^2(X; \mathbb{C})_\zeta := \ker(T_d - \zeta I)$ . The eigenspaces  $V_\zeta, V_{\bar{\zeta}}$  are conjugate and

$$L_{\zeta_d} \otimes_{\mathbb{Z}} \mathbb{C} \cong V_{\zeta_d} \oplus \cdots \oplus V_{\zeta_d^{d-1}}.$$

Let  $j : L_{-\zeta_d} \rightarrow V_{\zeta_d}$  be the composition map given by

$$\begin{array}{ccc} & H^2(X_d; \mathbb{C}) & \\ \nu \nearrow & & \searrow \text{proj}_{V_{\zeta_d}} \\ L_{\zeta_d} & \xrightarrow{j} & V_{\zeta_d} \end{array}$$

where  $\nu : H^2(X_d; \mathbb{Z}) \hookrightarrow H^2(X_d; \mathbb{C})$ ,  $\text{proj}_{V_{\zeta_d}} : H^2(X_d; \mathbb{C}) \rightarrow V_{\zeta_d}$  are the natural inclusion and projection. Since  $T_d$  is of finite order, it is a biholomorphism and therefore respects the Hodge decomposition on  $H^2(X_d; \mathbb{C})$ . This further decomposes the spaces  $V_\zeta$  into mutually orthogonal subspaces

$$V_\zeta \cong V_\zeta^{2,0} \oplus V_\zeta^{1,1} \oplus V_\zeta^{0,2}.$$

with respect to the intersection pairing on  $H^2(X; \mathbb{C})$ , given by

$$\langle \alpha, \bar{\beta} \rangle = \int_X \alpha \wedge \bar{\beta}.$$

#### 4.4.3 Structure of $V_{-1}$

We have  $L_{-1} \otimes \mathbb{C} \cong V_{-1}$  and  $L_{-1} \cong A_1 \oplus U^{\oplus 2} \oplus E_8(-1)^{\oplus 2}$ , where  $U$  is the hyperbolic lattice and  $E_8(-1)$  is the negative  $E_8$  lattice, with  $A_1$  and  $E_8$  having the intersection form



hence  $j_{\mathbb{C}}$  is an isometry. Since  $\dim_{\mathbb{C}} V_{\omega} = \dim_{\mathbb{C}} L_{\omega} \otimes_{\mathbb{Z}[i]} \mathbb{C} = 21$ , it follows that  $j_{\mathbb{C}}$  is an isomorphism.  $\square$

Let  $h_{3,18}$  be the standard quadratic form of signature  $(3, 18)$ .

**Proposition 4.4.2.** The hermitian vector space  $(V_{\zeta_d}, h)$  is isomorphic to  $\mathbb{C}^{3,18} = (\mathbb{C}^{21}, h_{3,18})$ .

*Proof.* We will show that  $h$  is positive-definite on  $V_{\omega}^{2,0} \oplus V_{\omega}^{0,2}$  and negative-definite on  $V_{\omega}^{1,1}$ . We will also show the former and latter spaces are 3 and 18 dimensional respectively, concluding the proof.

Let  $\alpha \in H^{2,0}(X_3; \mathbb{C})$  be a holomorphic form. Then  $T_3\alpha$  must be holomorphic too. Since  $V_1$  is spanned by a rational class,  $V_1 \subset H^{1,1}(X_3; \mathbb{C})$  and hence  $T_3\alpha \neq \alpha$ . This implies that  $T_3\alpha = \omega^k\alpha$  for  $k = \pm 1$  and therefore  $T_3\bar{\alpha} = \omega^{-k}\bar{\alpha}$ , hence

$$V_{\omega}^{2,0} \cong V_{\bar{\omega}}^{0,2}.$$

Similarly,  $V_{\omega}^{0,2} \cong V_{\bar{\omega}}^{2,0}$ . The intersection pairing on  $\omega$  satisfies

$$h(\omega, \omega) = 2 \langle \omega, \bar{\omega} \rangle = 2 \int_X \omega \wedge \bar{\omega} > 0$$

so  $V_{\omega}^{2,0} \oplus V_{\omega}^{0,2}$  and  $V_{\bar{\omega}}^{2,0} \oplus V_{\bar{\omega}}^{0,2}$  are positive-definite. The Kähler form  $\kappa$  associated to  $X$  lies in  $H^{1,1}(X_3; \mathbb{C})$  and has positive intersection pairing with itself being an integral cohomology class, so  $\langle \kappa, \bar{\kappa} \rangle > 0$ . Since  $H^2(X_3; \mathbb{C})$  has signature  $(7, 36)$ , a basis of  $V_{\omega} \oplus V_{\bar{\omega}}$  along with  $\kappa$  span a maximal positive-definite subspace of  $H^2(X_3; \mathbb{C})$ . This implies that  $V_{\omega}^{1,1}$  and  $V_{\bar{\omega}}^{1,1}$  are negative-definite and

$$V_{\omega}^{1,1} \cong V_{\bar{\omega}}^{1,1} \cong \mathbb{C}^{18}.$$

$\square$

## 4.5 The moduli of smooth sextic curves

The image of  $\rho_d^{\zeta_d}$  must lie in the centralizer  $C_{L_{\zeta_d}}(T_d)$  of  $T_d$  in  $L_{\zeta_d}$  since  $\rho_d$  and  $T_d$  commute.

For  $d = 2$ , this doesn't produce any further restrictions and

$$\mathrm{Im}(\rho_2^{-1}) \subset O(L_{-1})$$

For  $d = 3$ , this is equivalent to respecting the  $\mathbb{Z}[\omega]$ -module structure on  $L_\omega$ . Moreover,  $\rho_3^\omega$  is norm-preserving, which translates to the corresponding  $\mathbb{Z}[\omega]$ -lattice automorphisms to be unitary. Therefore

$$\mathrm{Im}(\rho_3^\omega) \subset C_{L_\omega}(T_3) \cap O(L_\omega) \cong U(\mathbb{Z}[\omega]^{21}, q_{3,18}).$$

For simplicity, we denote

$$\Gamma_2 = O(L_{-1}) \quad \Gamma_3 = U(\mathbb{Z}[\omega]^{21}, q_{3,18}).$$

We conjecture that  $\mathrm{Im}(\rho_d) \cong \Gamma_d$  for  $d = 2$  and  $d = 3$ , and we appeal to the moduli space of smooth sextic curves  $\mathcal{M}_6$  to make partial progress towards this conjecture. Following a similar treatment to that in [All02], we consider covers  $\widetilde{\mathcal{M}}_6$  of  $\mathcal{M}_6$ , referred to as framings. Through the section, we fix the lattices

$$\begin{aligned} \Lambda &= A_1 \oplus U^{\oplus 2} \oplus E_8(-1)^{\oplus 2} \\ \Omega &= (\mathbb{Z}[\omega]^{21}, q_{3,18}). \end{aligned}$$

### 4.5.1 Framings

For a given smooth sextic curve  $V(f)$  along with the corresponding surface  $X_{f,d}$  lying over it in the fibration  $\mathcal{E}_{d,6}$ , define a *framing*  $\lambda$  on  $V(f)$  as an isomorphism

$$\lambda : \Lambda \rightarrow L_{-1} \quad \tau : \Omega \rightarrow L_\omega$$

for  $d = 2$  and  $d = 3$  respectively, which respects the intersection form on each lattice. Define  $\widetilde{\mathcal{M}}_{6,d}$  as the moduli space of  $d$ -framed smooth sextic curves. Here  $\mathbb{P}(\Gamma_2) = \Gamma_2/\mathbb{Z}^*$  and  $\mathbb{P}(\Gamma_3) = \Gamma_3/\mathbb{Z}[\omega]^*$  act respectively on  $\widetilde{\mathcal{M}}_{6,2}$  and  $\widetilde{\mathcal{M}}_{6,3}$  by precomposition. That is,  $\Gamma_d$  acts on  $\widetilde{\mathcal{M}}_{6,d}$  up to scalar equivalence and for  $g \in \Gamma_d$ ,

$$g \cdot (V(f), \nu) = (V(f), \nu \circ g^{-1})$$

where framings  $\nu, \nu'$  differing by a unit are equivalent.

- $d = 2$ : The composition  $j_{\mathbb{C}} \circ \lambda_{\mathbb{C}} : \mathbb{C}^{2,19} \rightarrow V_{-1}$  gives a negative-definite 19-dimensional subspace

$$(j_{\mathbb{C}} \circ \lambda_{\mathbb{C}})^{-1} (V_{-1}^{1,1}) \subset \mathbb{C}^{2,19}.$$

- $d = 3$ : The composition  $j_{\mathbb{C}} \circ \tau_{\mathbb{C}} : \mathbb{C}^{3,18} \rightarrow V_\omega$  gives a negative-definite 18-dimensional subspace

$$(j_{\mathbb{C}} \circ \tau_{\mathbb{C}})^{-1} (V_\omega^{1,1}) \subset \mathbb{C}^{3,18}.$$

### 4.5.2 Period maps

For  $d = 2$ , the surfaces  $X_2$  are  $K3$  surfaces. The subspaces given by each framing  $\lambda$  are parametrized by the Grassmanian  $\text{Gr}_{\mathbb{C}}^-(19, 21)$  of negative definite 19-dimensional subspaces of  $V_{-1}$ . Letting

$$\mathcal{D}_{6,2} \subset \mathbb{P}(x \in V_{-1} \mid \langle x, x \rangle = 0, \langle x, \bar{x} \rangle > 0)$$

be a connected component and  $\Gamma_2^+ \subset \Gamma_2$  the index 2 subgroup leaving  $\mathcal{D}_{6,2}$  invariant, these negative-definite subspaces, together with the perpendicularity condition to the pullback  $f_2^*H$  provide the period map

$$\wp : \widetilde{\mathcal{M}}_{6,2}^+ \rightarrow \mathcal{D}_{6,2}$$

where  $\widetilde{\mathcal{M}}_{6,2}^+$  is given by restricting the  $\mathbb{P}(\Gamma_2)$ -action on  $\widetilde{\mathcal{M}}_{6,2}$  to  $\mathbb{P}(\Gamma_2^+)$ . In [Loo03] it is shown that those points in  $\mathcal{D}_{6,2}$  which are not in the image of  $\wp$  correspond to the set  $\mathcal{H}$  of roots in  $L_{-1}$ , namely, those  $\delta \in L_{-1}$  satisfying  $\langle \delta, \delta \rangle = -2$ . By the Torelli theorem for  $K3$  surfaces ([IR 71]) it follows that the period map  $\wp : \widetilde{\mathcal{M}}_{6,2}^+ \rightarrow \mathcal{D}_{6,2} - \mathcal{H}$  is an isomorphism. This provides a commutative diagram

$$\begin{array}{ccc} \widetilde{\mathcal{M}}_{6,2}^+ & \xrightarrow[\cong]{\wp} & \mathcal{D}_{6,2} - \mathcal{H} \\ \mathbb{P}(\Gamma_2^+) \downarrow & & \downarrow \mathbb{P}(\Gamma_2^+) \\ \mathcal{M}_6 & \xrightarrow[\cong]{\tilde{\wp}} & (\mathcal{D}_{6,2} - \mathcal{H}) / \Gamma_2^+ \end{array}$$

thus giving a description of the moduli space  $\mathcal{M}_6$  as a quotient of a type IV domain.

## 4.6 Conjectures

We conjecture that a similar characterization of  $\mathcal{M}_6$  can be given via the period corresponding to triple covers with a  $\mathbb{Z}/3\mathbb{Z}$  action coinciding with that of the Deck group transformation of triple covers over  $\mathbb{P}^2$  branching over a smooth sextic curve.

We conjecture that Tschirnhausen modules of Galois triple covers over  $\mathbb{P}^2$  in the sense of [Mir85] could work as a candidate to provide an analogue surjectivity and injectivity argument as the one the Torelli theorem for  $K3$  surfaces provides in the case  $d = 2$ .

In particular, connectivity of the period domains gives rise to a partial computation of the monodromy groups for the families of surfaces in the case  $d = 2$  following Proposition 3.5.10, showing that  $\mathbb{P}(\Gamma_2^+) \subset \text{Im}(\rho_2) \subset \Gamma_2^+$ . We conjecture that a similar argument could be replicated by describing a connected period domain for triple covers over  $\mathbb{P}^2$ .

## REFERENCES

- [All02] Toledo Allcock Carlson. “The complex hyperbolic geometry of the moduli space of cubic surfaces”. In: *J. Algebraic Geom.* 11 (2002), pp. 659–724.
- [AS11] Michela Artebani and Alessandra Sarti. “Symmetries of order four on K3 surfaces”. In: *Journal of the Mathematical Society of Japan* 67 (Feb. 2011). DOI: 10.2969/jmsj/06720503.
- [DL81] Igor Dolgachev and Anatoly S. Libgober. “On the fundamental group of the complement to a discriminant variety”. In: *Lecture Notes in Mathematics*. Vol. 862. 1981, pp. 1–25.
- [Dol12] Igor V. Dolgachev. *Classical Algebraic Geometry: A Modern View*. Cambridge University Press, 2012. DOI: 10.1017/CB09781139084437.
- [Fra51] J. S. Frame. “The classes and representations of the groups of 27 lines and 28 bitangents”. In: *Annali di Matematica* 32.83 (1951). DOI: <https://doi.org/10.1007/BF02417955>.
- [Har77] Robin Hartshorne. *Algebraic Geometry*. 1977. Chap. V, pp. 395–409.
- [Har79] Joe Harris. “Galois Groups of Enumerative Problems”. In: *Duke Mathematical Journal* 46.4 (1979).
- [Hir05] Susumu Hirose. “Surfaces in the complex projective plane and their mapping class groups”. In: *Algebraic and Geometric Topology* 5 (June 2005), pp. 577–613. DOI: <https://doi.org/10.2140/agt.2005.5.577>.
- [Inc] Wolfram Research Inc. *Mathematica, Version 12.2*. Champaign, IL, 2020. URL: <https://www.wolfram.com/mathematica>.
- [IR 71] I. Shafarevich I.R. Piatetski-Shapiro. “A Torelli theorem for algebraic surfaces of type K3”. In: *Math. USSR Izv* 5 (1971), pp. 547–587.
- [Kon00] Shigeyuki Kondo. “A complex hyperbolic structure for the moduli space of curves of genus 3”. In: *J. reine angew. Math.* 525 (2000), pp. 219–232.
- [Kon20] Shigeyuki Kondo. *K3 Surfaces*. EMS Tracts in Mathematics 32, 2020. Chap. II.
- [Lib78] Anatoly Libgober. “On the Fundamental Group of the Space of Cubic Surfaces”. In: *Mathematische Zeitschrift* 162 (1978), pp. 63–67. DOI: 10.1007/bf01437823.
- [Loo03] Eduard Looijenga. “Compactifications defined by arrangements. II. Locally symmetric varieties of type IV”. In: *Duke Math* 119 (2003), pp. 527–588.

- [McM09] Curtis McMullen. “K3 surfaces, entropy and glue”. In: *Journal für die Reine und Angewandte Mathematik* 658 (Sept. 2009). DOI: 10.1515/CRELLE.2011.048.
- [McM13] Curtis McMullen. “Braid groups and Hodge theory”. In: *Math. Ann.* 355 (2013), pp. 893–946.
- [Mir85] Rick Miranda. “Triple Covers in Algebraic Geometry”. In: *American Journal of Mathematics* 107 (1985), pp. 1123–1158.
- [Mor01] Shigeyuki Morita. *Geometry of Characteristic Classes*. Iwanami Series in Modern Mathematics 199. American Mathematical Society, 2001. Chap. 4, p. 152.
- [Nai11] Vipul Naik. *Classification of groups of order 24*. 2011. URL: [https://groupprops.subwiki.org/wiki/Classification\\_of\\_groups\\_of\\_order\\_24](https://groupprops.subwiki.org/wiki/Classification_of_groups_of_order_24).
- [Nik80] V.V. Nikulin. “Integral symmetric bilinear forms and its applications”. In: *Math. USSR Izv* 14 (1980), pp. 103–167.
- [The21] The Sage Developers. *SageMath, the Sage Mathematics Software System (Version 9.2)*. 2021.
- [Zar29] O. Zariski. “On the problem of existence of algebraic functions of two variables possessing a given branch curve”. In: *Amer. J. Math.* 51 (1929), pp. 305–328.